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LA BRECHA DE MASA Y LA CURVATURA DE LOS CAMPOS CUÁNTICOS

THE MASS GAP AND THE CURVATURE OF QUANTUM FIELDS

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La brecha de masa y la curvatura de los campos cuánticos

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RESUMEN

En recientes artículos científicos, este investigador, ha reformulado las ecuaciones de Yang-Mills, introducidas en 1954, las mismas que comportan una generalización no conmutativa de la electrodinámica cuántica (QED), en la medida en que, las ecuaciones de Yang-Mills, no solamente se reducen a la QED cuando las partículas portadoras del campo no tienen masa, sino que también, se reducen a la QED cuando las partículas portadoras del campo, tienen masa, en combinación con las ecuaciones de Higgs y otros principios y lineamientos de orden relativistas, que explican también la curvatura geométrica de los campos cuánticos. De este modo, se plantea una teoría que unifica de manera satisfactoria, la teoría electrodébil y la cromodinámica cuántica, ésta última, la cual regula las interacciones fuertes. En consecuencia, a través de los modelos matemáticos que han sido propuestos por este investigador en artículos científicos recientes, se ha demostrado que para todo grupo simple compacto G, hay una teoría de Yang-Mills en $\mathbb{R}^4$ que comporta un grupo gauge y que además, comporta una "*brecha de masa*" (mass gap). La brecha de masa equivale a que no pueden existir excitaciones con energía arbitrariamente pequeña, sino que hay un valor mínimo superior a cero.

Palabras clave: física de partículas, campos de gauge, teorías de calibre, campos de Einstein, campos de Higgs

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The mass gap and the curvature of quantum fields

ABSTRACT

In recent scientific articles, this researcher has reformulated the Yang-Mills equations, introduced in 1954, which involve a non-commutative generalization of quantum electrodynamics (QED), insofar as the Yang-Mills equations are not only reduced to QED when the particles carrying the field have no mass, but also, they are reduced to QED when the particles carrying the field have mass, in combination with the Higgs equations and other principles and relativistic order guidelines, which also explain the geometric curvature of quantum fields. In this way, a theory is proposed that satisfactorily unifies the electroweak theory and quantum chromodynamics, the latter, which regulates strong interactions.Consequently, through the mathematical models that have been proposed by this researcher in recent scientific articles, it has been shown that for every simple compact group G, there is a Yang-Mills theory in $\mathbb{R}^4$, that involves a gauge group and that also entails a "mass gap". The mass gap means that there can be no excitations with arbitrarily small energy, but that there is a minimum value greater than zero.

Keywords: particle physics, gauge fields, caliber theories, Einstein fields, Higgs fields

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INTRODUCCIÓN

Forman parte del modelo estándar de física de partículas, las teorías de campo de gauge, las mismas que advierten, que un campo cuántico específico, demuestra una simetría interna, conocida como invariancia de gauge, que se describe a través de una función de interacción compleja sin acción de un grupo de Lie. Un cambio de gauge significa un cambio de factor de fase, así:

$$\Lambda_{\nu\mu}^{\mu\nu} \equiv \sum_{\hbar^4}^{\triangleq} \mathfrak{H}^{\circledast} \wr \boxtimes^4 \frac{i}{\hbar}\int_{-\infty}^{+\infty} \mathcal{R}$$

$$\doteqdot \frac{\partial^2 \overleftrightarrow{\Delta^4}}{\partial^4 \overleftrightarrow{\nabla^2}} + \frac{\partial^4 \overleftrightarrow{\Delta^2}}{\partial^2 \overleftrightarrow{\nabla^4}} + \frac{\partial^2 \psi^4}{\overbrace{\partial^2\hbar^4}^{\mathfrak{R}} \underbrace{\mathcal{H}^2}_{\mathfrak{I}}} - \frac{\partial^4 \psi^2}{\overbrace{\partial^4\hbar^2}^{\mathfrak{I}} \underbrace{\mathcal{H}^4}_{\mathfrak{R}}} \langle \mathrm{t}-\mathrm{t}_0|x-x_0\rangle \mathcal{K}(x_{n-1}\to x_n;\Delta \mathrm{t}_n) - \mathfrak{m}\mathfrak{c}\gamma = \frac{1}{2\varpi\hbar}\mathrm{e}^{-\frac{i}{\hbar}\mathfrak{V}_n \Delta \mathrm{t}_n}$$

$$\cong \int_{-\infty}^{+\infty} \exp\left(i p_n \Delta \mathfrak{x}_n/\hbar - i\Delta \mathrm{t}_n/\hbar(1/2m(\Delta \mathfrak{x}_n/\Delta \mathrm{t}_n)^2 - \mathfrak{V}_n \wp_n^2/2m)\right)$$

$$= \sqrt{\frac{m}{2\pi i\hbar \Delta \mathrm{t}_n}} \times \int \frac{\left(\frac{i\Delta \mathrm{t}_n}{\hbar}\right) \mathfrak{dp}_{\mathfrak{N}}}{2\varpi\hbar} \exp\left(i/\hbar \sum_{n=1}^{\mathfrak{N}} (p_n\, x_n - x_{n-1}/\Delta \mathrm{t}_n\right.$$

$$\left. - \mathfrak{E}_n)\Delta \mathrm{t}\right)$$

$$\times \int_{t_0}^{t_f} \mathcal{D}\left(x(t)\right) \mathrm{e}^{i\mathfrak{S}(x(t))/\hbar} \int \mathfrak{dx}_{\mathfrak{N}-1} \left(\frac{m}{2\pi i\hbar \Delta \mathrm{t}_n}\right)^{\mathfrak{N}/2} \sum_{\alpha\beta\gamma\delta\varepsilon\epsilon\zeta\eta\theta\vartheta\iota\kappa\lambda\mu\nu}^{\xi o\rho\varrho\sigma\varsigma\tau\upsilon\varphi\phi\omega\psi} \int \mathfrak{K}_\eta \to \Delta \mathrm{t} \left\langle \phi^{\alpha\beta} \middle| \psi^{\gamma\delta} \middle| \varphi^{\varepsilon\zeta} \right\rangle \Psi^{\sqrt[2]{\iota\hbar g}} \Upsilon^{-i\mathfrak{E}\Delta \mathrm{t}/\hbar}$$

$$+ \mathcal{O}(\Delta \mathrm{t}^4)\langle \rho|\psi\rangle \frac{\partial^2 \mathbb{R}^4}{\partial^2 \delta^{-1}} \wr \frac{\partial^4 \psi^2}{\partial^\dagger \psi^\star} \left\langle \phi^{\alpha\beta} \middle| \psi^{\gamma\delta} \middle| \varphi^{\varepsilon\zeta} \right\rangle$$

$$\wr \int_{\xi\omega}^{\mu\nu} (\wp^\tau)^4 \frac{\partial^2 \kappa^4}{\partial^2 \lambda^4}$$

$$\times \int \frac{\partial^2 \Psi^*}{\partial^4 \Phi^\Omega} - \frac{\partial^4 \Gamma^2}{\partial^2 \Gamma^4} + \frac{1}{4\mathfrak{M}^4 \mathfrak{c}^4} \mathfrak{dp} \frac{1}{\sqrt{2g\mathfrak{m}^4\mathfrak{c}^4}} \langle \rho|\psi\rangle \mathfrak{dp} \frac{1}{\sqrt{2\varpi\hbar}\mathrm{e}^{\iota\wp\frac{\lambda}{\hbar}}} \langle \sigma|\psi\rangle \risingdotseq \int_{\xi\omega}^{\mu\nu} (\wp^\tau)^2\, \partial^4 \kappa^2$$

$$/\partial^4 \lambda^2 \risingdotseq \int_{\sigma\upsilon}^{\rho\varrho} (\wp^\tau)^4\, \partial^2 \kappa^4/\partial^2 \lambda^4 \risingdotseq \int_{\sigma\upsilon}^{\rho\varrho} (\wp^\tau)^2\, \partial^4 \kappa^2/\partial^4 \lambda^2$$

$$\times \int \frac{\partial^2 \Psi^*}{\partial^4 \Phi^\Omega} - \frac{\partial^4 \Gamma^2}{\partial^2 \Gamma^4} + \frac{1}{4\mathfrak{M}^4 \mathfrak{c}^4} \langle \varphi|\psi\rangle \mathfrak{dp} \frac{1}{\sqrt{2g\mathfrak{m}^4\mathfrak{c}^4}} \mathfrak{dp} \frac{1}{\sqrt{2\varpi\hbar}\mathrm{e}^{\iota\wp\frac{\lambda}{\hbar}}} \langle \phi|\psi\rangle$$

indistintamente si se tratan de partículas con o sin masa. Dado que ψ puede depender de x, y, s y t, el factor de fase relativo de ψ en dos puntos diferentes del espacio-tiempo, no es por lo tanto, necesariamente arbitrario. En otras palabras, la arbitrariedad en la elección del factor de fase al ser de carácter local, equivale a una brecha de masa desde la formulación de Higgs. En consecuencia, la invariancia de gauge desde las coordenadas cuánticas x, y, z, y bajo criterios de unificación, se expresan de la siguiente manera:

$$\mathcal{A}_{\mu}^{\prime} = \mathcal{A}_{\mu} + \frac{\frac{1}{e\partial\alpha}}{\partial\chi_{\mu}}$$

$$\mathcal{A}_{\nu}^{\prime} = \mathcal{A}_{\nu} + \frac{\frac{1}{e\partial\beta}}{\partial\chi_{\nu\upsilon}}$$

$$\mathcal{A}_{\mu}^{\prime} = \mathcal{A}_{\mu} + \frac{\frac{1}{e\partial\alpha}}{\partial\gamma_{\mu}}$$

$$\mathcal{A}_{\nu}^{\prime} = \mathcal{A}_{\nu} + \frac{\frac{1}{e\partial\beta}}{\partial\gamma_{\nu\upsilon}}$$

$$\mathcal{A}_{\mu}^{\prime} = \mathcal{A}_{\mu} + \frac{\frac{1}{e\partial\alpha}}{\partial\mathcal{Z}_{\mu}}$$

$$\mathcal{A}_{\nu}^{\prime} = \mathcal{A}_{\nu} + \frac{\frac{1}{e\partial\beta}}{\partial\mathcal{Z}_{\nu\upsilon}}$$

De este modo, se demuestra, que sea cual fuere el campo de gauge, las partículas con o sin masa, alcanzan un comportamiento cuántico semejante y superior a cero, esto es, a la luz de los instantones y simetría quiral.

Más adelante, en el apartado de Resultados y Discusión, se abordará la formulación matemática de la unificación de la cromodinámica cuántica, la fuerza electrodébil, la brecha de masa desde la formulación de Higgs y los campos cuánticos y su relación con las ecuaciones de campo einstenianas, a propósito de la siguiente constante universal que explica la simetría de las relaciones cuánticas en un grupo de gauge y que se traduce a lo que sigue:

$\mu := \inf \mathrm{Spec}(\hat{H}) \backslash 0 > 0$ $= \xi_{\lambda\Omega\psi}^{\sigma\varsigma\zeta}$Σ∫∫∫∫ħȸꞨꞬǮҖҴҜҩӜꜪϚ$\pi mc\mathbb{R}^4$

O expresada de otra manera:

$$
\begin{aligned}
&\mu := \inf \mathrm{Spec}(\hat{H}) \backslash 0 > 0 \\
&= \mathcal{L} \sum_{\bowtie} \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \left\langle \mathrm{e}^4 \left| \sqrt[4]{\frac{1}{4\mathcal{R}} \Delta^4 \partial^2 \mathfrak{R}\text{7א} - \sum \langle \alpha\beta\gamma\delta\varepsilon\epsilon\zeta\eta\theta | \vartheta\iota\kappa\lambda\mu\nu\xi o\rho\varrho\sigma \rangle . \|\text{ძ}\tau^2\|} \right| \frac{\partial^2\varphi}{\partial^2\phi} . \partial^2\psi - \partial^2\omega . \partial^2\Psi \right\rangle \xi^{\sigma\varsigma\zeta}_{\lambda\Omega\psi} \prod_{\nu\upsilon}^{\mu} \frac{\partial\Delta^2}{\partial\phi^2} \\
&+ \partial\varphi^4 \bigvee_{\mu}^{\upsilon\nu} \partial\Delta^2\partial \\
&/\partial\phi^2 - \partial\varphi^4 \iiint_{ijk}^{abc} \frac{\langle \mathfrak{H}^2 | \mathfrak{m}^4 c^4 | \hbar^\dagger \rangle \sum_i \mathcal{M}^2 \mathcal{H}^4 \wp^{\pitchfork}}{\boxdot^4 \langle \alpha|\beta\rangle\langle\psi|\phi\rangle \left\| \frac{\partial\Psi^4}{\mathcal{H}^{\mathfrak{I}}} \right\|} \circledast \frac{\partial^2 \check{\mathfrak{Z}}}{\partial^2 \text{Ꙗ}} . \frac{\partial^2 \text{Җ}}{\partial \mathfrak{H}^4} \\
&\trianglerighteq \uplus \frac{\partial^2 \text{ДК}}{\partial^2 \text{ɰ}} \star \partial^2 \text{Ӝ} - \sum_{\triangleq}^{\bowtie} \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \|\otimes^4\| \langle \circledcirc^4 \mid \boxtimes^4 \rangle \|\odot^4\| + \sum_{\mathfrak{i}} \mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu} - \sum_{\alpha\beta} \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \partial^2 \mathcal{D}^{\lambda\kappa} / \mathcal{D}_\zeta + \sum_{\phi\varphi} \int \psi^{\omega\sigma\rho} \int \frac{\Lambda^\Psi}{\partial\Psi^2} \\
&- \partial^4 \wp_{\frac{\hbar}{\mu\nu\sqrt{-\mathfrak{g}}}} / \langle 8\varpi^{\rho\delta} | 1/4\kappa^\varepsilon \rangle \equiv \frac{\langle \text{Ꭓ}\varsigma | m^4 | c^4 \rangle}{\langle \partial\rho^4 | \partial\sigma^4 | \partial\delta^4 \rangle} / \mathbb{R}^4 \\
&+ \mathfrak{G}_{\mu\nu} \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \left\langle \frac{16\pi\mathfrak{G}\rho^4\nabla^4}{\partial^2\phi_\psi} \mathfrak{R}_{00} - \mathcal{R}h_{\alpha\beta} \middle| c^4 \mathfrak{T}_{\mu\nu} - \frac{1}{2\mathcal{R}\mathfrak{g}_{\mu\nu}} + \Lambda_{\mu\nu} - \mathfrak{R}\hbar_{\alpha\beta} \right\rangle \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \frac{\mathcal{D}^2\psi(\mathfrak{x}\mathfrak{y}\mathfrak{z})}{\mathfrak{D}(\mathfrak{x}\mathfrak{y}\mathfrak{z})^2\phi} \\
&- \mathfrak{B}\varphi \int \Lambda_{\mu\nu}\Lambda^{\mu\nu} \left\langle 2\mathfrak{m}/\hbar^2 \middle| \wp_4 / e^{\sqrt{\frac{2\mathfrak{m}}{\hbar^2(\mathcal{V}-\mathcal{E})\langle\phi|\psi\rangle}}(\varphi+\theta)} \right\rangle - \frac{1}{4\kappa G g^2} / \Lambda_{\mu\nu}\Lambda^{\mu\nu}
\end{aligned}
$$

Donde:

Ꙗ= Constante electrodébil.

$$D_\mu(\cdot) = (\partial_\mu + igA^j_\mu \boldsymbol{\tau}_j)(\cdot) \qquad \rightarrow \qquad D_\mu \boldsymbol{\Psi} = \partial_\mu \boldsymbol{\Psi} + igA^j_\mu \boldsymbol{\tau}_j \boldsymbol{\Psi} \quad \mathbf{A}^j(\mathbf{x}) = A^j_\alpha \, dx^\alpha, \qquad A^j_\alpha \in \mathfrak{g}_s$$

$$T_g(D_\mu \boldsymbol{\Psi}) = U_{g(\mathbf{x})} D_\mu \boldsymbol{\Psi} \qquad T_g(A^j_\mu \boldsymbol{\tau}_j) = U_g(A^j_\mu \boldsymbol{\tau}_j) U_g^{-1} + \frac{i}{g}(\partial_\mu U_g) U_g^{-1}$$

$$\mathcal{L}_{\mathrm{cg}} = \frac{1}{4} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$\mathcal{L}_{\text{fer-cg}} = \mathrm{i}\bar{\boldsymbol{\Psi}}_L \gamma^\mu \left(\partial_\mu - \frac{g'Y}{4} B_\mu \boldsymbol{\tau}^3 - \frac{g}{4} \boldsymbol{\tau}^a W^a_\mu \right) \boldsymbol{\Psi}_L + \mathrm{i}\bar{\boldsymbol{\Psi}}_R \gamma^\mu \left(\partial_\mu - \frac{g'Y}{4} B_\mu \boldsymbol{\tau}^3 \right) \boldsymbol{\Psi}_R \quad \mathcal{L}_{EW} = \mathcal{L}_{cg} + \mathcal{L}_{fer-cg}$$

$$[\boldsymbol{\tau}_i, \boldsymbol{\tau}_j] = f_{ijk} \boldsymbol{\tau}_k \qquad T_g(A^j_\mu) \approx A^j_\mu + \frac{1}{g}(\partial_\mu \epsilon^j) + f_{jkl} \epsilon^k A^l_\mu$$

Җ= Constante estándar de partículas.

$$\begin{pmatrix} e^+ \\ \bar{\nu}_e \end{pmatrix}_{\circlearrowleft}, \begin{pmatrix} \mu^+ \\ \bar{\nu}_\mu \end{pmatrix}_{\circlearrowleft}, \begin{pmatrix} \tau^+ \\ \bar{\nu}_\tau \end{pmatrix}_{\circlearrowleft} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_{\circlearrowleft}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_{\circlearrowleft}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_{\circlearrowleft}$$

$$e^-_{\circlearrowleft}, \mu^-_{\circlearrowleft}, \tau^-_{\circlearrowleft} \qquad e^+_{\circlearrowleft}, \mu^+_{\circlearrowleft}, \tau^+_{\circlearrowleft}$$

$$
\begin{aligned}&+(\bar{\nu}_L\ \bar{e}_L)\tilde{\sigma}^\mu i\partial_\mu \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\\ &+\bar{e}_R\sigma^\mu i\partial_\mu e_R\end{aligned}
\quad
\begin{aligned}&+(\bar{u}_L\ \bar{d}_L)\tilde{\sigma}^\mu i\partial_\mu \begin{pmatrix}u_L\\ d_L\end{pmatrix}\\ &+\bar{u}_R\sigma^\mu\partial_\mu u_R\\ &+\bar{d}_R\sigma^\mu\partial_\mu d_R\end{aligned}
\quad
\begin{aligned}&-\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ &-\tfrac{1}{8}tr(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu})\end{aligned}
-\frac{1}{2}tr(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu})
$$

$$
\begin{aligned}&-\tfrac{\sqrt{2}}{v}\left[(\bar{u}_L\ \bar{d}_L)\phi M^d d_R+\bar{d}_R\bar{M}^d\bar{\phi}\begin{pmatrix}u_L\\ d_L\end{pmatrix}\right]\\ &-\tfrac{\sqrt{2}}{v}\left[(-\bar{d}_L\ \bar{u}_L)\phi^* M^u u_R+\bar{u}_R\bar{M}^u\bar{\phi}^T\begin{pmatrix}-d_L\\ u_L\end{pmatrix}\right]\end{aligned}
-\tfrac{\sqrt{2}}{v}\left[(\bar{\nu}_L\ \bar{e}_L)\phi M^e e_R+\bar{e}_R\bar{M}^e\bar{\phi}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\right]
$$

$$
\begin{aligned}&(\bar{u}_L\ \bar{d}_L)\sigma^\mu i[-\tfrac{ig_1}{6}B_\mu+\tfrac{ig_2}{2}\mathbf{W}_\mu]\begin{pmatrix}u_L\\ d_L\end{pmatrix}\\ &+\bar{u}_R\sigma^\mu\tfrac{2}{3}ig_1B_\mu u_R\\ &-\bar{d}_R\sigma^\mu\tfrac{1}{3}ig_1B_\mu d_R\end{aligned}
\quad
\begin{aligned}&(\bar{\nu}_L\ \bar{e}_L)\tilde{\sigma}^\mu i[-\tfrac{ig_1}{2}B_\mu+\tfrac{ig_2}{2}\mathbf{W}_\mu]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\\ &-\bar{e}_R\sigma^\mu ig_1B_\mu e_R\end{aligned}
$$

$$
\begin{aligned}&+(\bar{u}_L\ \bar{d}_L)\tilde{\sigma}^\mu ig\mathbf{G}_\mu\begin{pmatrix}u_L\\ d_L\end{pmatrix}\\ &+\bar{u}_R\sigma^\mu ig\mathbf{G}_\mu u_R\\ &+\bar{d}_R\sigma^\mu ig\mathbf{G}_\mu d_R\end{aligned}
$$

$$
\mathcal{L}_{Higgs}=\overline{\left([\partial_\mu+\frac{1}{2}ig_1B_\mu+\frac{1}{2}ig_2\mathbf{W}_\mu]\phi\right)}\left([\partial_\mu+\frac{1}{2}ig_1B_\mu+\frac{1}{2}ig_2\mathbf{W}_\mu]\phi\right)-\frac{m_H^2\left(\bar{\phi}\phi-\frac{v^2}{2}\right)^2}{2v^2}
$$

ДК= Constante de interacción fuerte.

$$
\mathcal{L}_{\mathrm{QCD}}=\bar{q}i\gamma^\mu\partial_\mu q-\bar{q}mq-g\bar{q}\gamma^\mu T_a q G^a_\mu-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a
$$

$$
\begin{aligned}\mathcal{L}_{\mathrm{QCD}}&=\bar{q}\left(i\gamma^\mu D_\mu-m\right)q-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\\ &=\bar{q}\left(i\gamma^\mu(\partial_\mu+igT_aG^a_\mu)-m\right)q-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\\ &=\bar{q}\left(i\gamma^\mu\partial_\mu-m\right)q-g\left(\bar{q}\gamma^\mu T_a q\right)G^a_{\mu\nu}-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\end{aligned}
$$

$$
G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\sum_{b,c=1}^{8}f^{abc}A^b_\mu A^c_\nu,\quad \mu,\nu\in\{0,1,2,3\}\quad \partial_\mu\mathbf{G}^{\mu\nu}+[\mathbf{A}_\mu,\mathbf{G}^{\mu\nu}]=-\mathbf{J}^\nu
$$

$$
\mathcal{D}_\mu=\partial_\mu+\mathrm{i}\frac{\mathrm{g}'}{2}\vec{\tau}\cdot\vec{W}_\mu+\mathrm{ig}\frac{Y}{2}\mathrm{B}_\mu\quad \mathcal{L}_{EW}=\mathcal{L}_{bos.}+\mathcal{L}_{ferm.}\quad \mathbf{Q}=\mathbf{T}_3+\frac{Y}{2}\mathbf{I}
$$

$$
\begin{aligned}&\mathcal{L}_{\mathrm{bos.}}=\frac{1}{4}W_{\mu\nu}W^{\mu\nu}-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ &\mathcal{L}_{\mathrm{ferm.}}=\bar{\psi}_L\gamma^\mu\left(\mathrm{i}\partial_\mu-\mathrm{g}'\frac{Y}{2}B_\mu-\mathrm{g}\frac{1}{2}\vec{\tau}\cdot\vec{W}_\mu\right)\psi_L+\bar{\psi}_R\gamma^\mu\left(\mathrm{i}\partial_\mu-\mathrm{g}'\frac{Y}{2}B_\mu\right)\psi_R\end{aligned}
$$

Ж= Constante relativista.

$$\nu_{rec} = \nu_{em} e^{-\Phi} \quad \frac{\text{ciclos}}{\Delta t_{obs}} = \frac{\text{ciclos}}{\Delta t_{em}} e^{-\Phi} \quad \begin{array}{l} \Delta t_{em} = \Delta t_{obs} e^{-\Phi} \\ \Delta t_{obs} = \Delta t_{em} e^{\Phi} \end{array} \quad \begin{array}{l} E_{obs} = E_{con} e^{-\Phi} \\ h\nu_{rec} = h\nu_{em} e^{-\Phi} \\ \nu_{rec} = \nu_{em} e^{-\Phi} \end{array}$$

$$\begin{array}{l} \nabla_\beta \vec{u} = (\partial_\beta u^\alpha + \Gamma^\alpha_{\mu\beta} u^\mu) \vec{e}_\alpha \\ \nabla_\beta u^\alpha = \partial_\beta u^\alpha + \Gamma^\alpha_{\mu\beta} u^\mu \end{array} \quad \nabla_\beta \vec{u} = (\partial_\beta u^\alpha) \vec{e}_\alpha + u^\alpha \Gamma^\mu_{\alpha\beta} \vec{e}_\mu \quad \nabla_\beta \vec{u} = (\partial_\beta u^\alpha) \vec{e}_\alpha + u^\alpha (\partial_\beta \vec{e}_\alpha) \quad \nabla_\beta \vec{u} = \partial_\beta (u^\alpha \vec{e}_\alpha)$$

$$0 = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\beta} u^\mu u^\beta$$

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\beta} u^\mu u^\beta \qquad \Gamma^\alpha_{\beta\mu} = \frac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\sigma\beta} + \partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\beta\mu})$$

$$\frac{dx^\beta}{d\tau} \nabla_\beta u^\alpha = \partial_\beta u^\alpha \frac{dx^\beta}{d\tau} + \Gamma^\alpha_{\mu\beta} u^\mu \frac{dx^\beta}{d\tau}$$

$$\nabla_{\vec{u}} u^\alpha = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\beta} u^\mu u^\beta$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \Gamma^\mu_{\beta\alpha} A_\mu - \partial_\beta A_\alpha + \Gamma^\mu_{\alpha\beta} A_\mu$$

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha} \qquad \partial_\alpha T^{\alpha\beta} = 0 \rightarrow \nabla_\alpha T^{\alpha\beta} = 0 \qquad \partial_\beta u^\alpha = 0 \rightarrow \nabla_\beta u^\alpha = 0$$

$$\frac{d^2 \xi^\alpha}{d\tau^2} = R^\alpha_{\beta\mu\nu} u^\beta \xi^\mu u^\nu \quad d^2 \xi^\alpha = R^\alpha_{\beta\mu\nu} dx^\beta \xi^\mu dx^\nu$$

$$R^{\alpha\beta} = \frac{4\pi G}{c^2} T^{\alpha\beta} \quad \partial_t g_{\alpha\beta} = -2R_{\alpha\beta} \quad R^{00} = \frac{d^2 \Pi_0}{d(x^0)^2} \quad \Rightarrow \quad R^{00} = \nabla^2 V$$

$$-R = G^\alpha_\alpha = \frac{8\pi G}{c^4} T$$

$$R_{\alpha\beta} = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) \quad \nabla_\beta G^{\alpha\beta} = \nabla_\beta \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0, \qquad G^{\alpha\beta} = kT^{\alpha\beta}$$

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

$$\Box^2 \Phi = 4\pi G\rho \rightarrow \Phi(x,t) = \int_V \frac{G\rho(x', t - \frac{r}{c})}{r} dV \quad \nabla^2 \Phi = 4\pi G\rho \rightarrow \Phi(x,t) = \int_V \frac{G\rho(x', t)}{r} dV$$

$$a = -\nabla(\phi + \frac{2\phi^2}{c^2} + \psi) - \frac{1}{c} \frac{\partial \zeta}{\partial t} + \frac{v}{c} \times (\nabla \times \zeta) + \frac{3}{c^2} v \frac{\partial \phi}{\partial t} + \frac{4}{c^2} v(v \cdot \nabla)\phi - \frac{v^2}{c^2} \nabla \phi$$

$$a = -\nabla \phi + \eta$$

$$\eta = -\nabla(\frac{2\phi^2}{c^2} + \psi) - \frac{1}{c} \frac{\partial \zeta}{\partial t} + \frac{v}{c} \times (\nabla \times \zeta) + \frac{3}{c^2} v \frac{\partial \phi}{\partial t} + \frac{4}{c^2} v(v \cdot \nabla)\phi - \frac{v^2}{c^2} \nabla \phi$$

$$F_f(r) = -\frac{GMm}{r^2} + \frac{L^2}{mr^3} - \frac{3GML^2}{mc^2r^4} \qquad U_f(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{GML^2}{mc^2r^3}$$

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}(\gamma m v)}{\mathrm{d}t} = m\gamma^3 a = \frac{m\,a}{[1-(v^2/c^2)]^{3/2}} \qquad v = c\sqrt{1 - \frac{m^2c^4}{(mc^2+K)^2}}$$

Ϫ=Constante de Gauge en lagrangiano.

$$S_{YM} = \int_{\mathcal{M}} \mathcal{L}_{YM}(\mathbf{F}(\mathbf{x}), \mathbf{x}) \left(\sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n\right) = \frac{1}{4g^2}\int_{\mathcal{M}} Tr[*\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{x})]\, d^4\mathbf{x}$$

$$m\frac{dx^\mu}{d\tau} = p^\mu, \qquad \frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = gQ^a F_a^{\mu\nu} u_\nu \quad F^a_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + g f_{abc} A^b_\mu A^c_\nu$$

$$\frac{dQ^a}{d\tau} = -g f^{abc} A^\mu_b u_\mu Q_c \quad F^\mu = m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu \quad \Rightarrow \mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E})$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t, \mathbf{x}) = 0 \qquad T^{\mu\nu} = \frac{g^{\mu\nu}}{4} F^{\rho\sigma}_a F^a_{\rho\sigma} + F^a_{\mu\rho} F^{\nu}_{\rho\ a}$$

$$\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\, g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}\right)\right] + \frac{m^2c^2}{\hbar^2}\phi = 0 \quad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \sum_\nu \partial_\nu \partial^\nu, \qquad \mu = \frac{mc}{\hbar}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* - \frac{1}{2}\mu^2\phi^*\phi \quad \hat{\phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(\hat{a}_\mathbf{p} e^{-\frac{i}{\hbar}E_\mathbf{p}t} e^{\frac{i}{\hbar}\mathbf{px}} + \hat{a}^\dagger_\mathbf{p} e^{\frac{i}{\hbar}E_\mathbf{p}t} e^{-\frac{i}{\hbar}\mathbf{px}}\right)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$$

METODOLOGÍA

La teorización desplegada en el presente manuscrito, resulta de la aplicación de una metodología de investigación integral, esto es, bajo un enfoque híbrido, tanto desde el punto de vista cualitativo como en su dimensión cuantitativa. El tipo de investigación que ha sido desarrollado a lo largo del presente Artículo Científico, es esencialmente predictivo, a la luz de la física teórica, más no, acusa carácter empírico o experimental. Por otro lado, las líneas de investigación adoptadas para la formulación del estado del arte, se ajustan al constructivismo. Cabe indicar, que no existe población de estudio en la medida en que el presente artículo científico, no es de carácter sociológico o social, más aun, en mérito a su impacto en la realidad de transformación. Tampoco se han implementado técnicas de recolección de información, tales como encuestas, entrevistas, etc, salvo revisión bibliográfica, a razón del campo de investigación abordado. Adicionalmente a lo antes expuesto, es perciso resaltar, que el material de apoyo es meramente bibliográfico. La técnica metodológica, dada la complejidad de la temática escrutada, es deductiva, pues la teorización en sentido estricto, ha sido desarrollada desde principios y premisas generales que son inherentes a la física de partículas en sentido lato. Finalmente, para efectos de construir y desarrollar las ecuaciones constantes en el presente artículo científico, se ha tomado en consideración el Modelo Estándar de Física de Partículas, muy especialmente, en tratándose de los campos de Yang – Mills, sin perjuicio de los demás sistemas de recalibración deducidos y esbozados a lo largo del presente Artículo Científico.

RESULTADOS Y DISCUSIÓN

Los resultados planteados en el presente trabajo, para discusión, quedan expresados en los siguientes puntos:

1. Transformación de Gauge desde las teorías de Yang – Mills:

Trabajando en dimensión $\mathbb{R}^4$, se deducen las siguientes ecuaciones a partir de un campo de Yang – Mills:

$$\mathrm{F} = \divideontimes \mathrm{F}$$

$$\mathrm{F} = \divideontimes \mathrm{F}$$

En consecuencia, los instantones se reducen a lo que sigue:

$$D_A \star F = D_A \pm F = \mp D_A F > 0$$

Por lo tanto, la transformación de gauge ψ es igual a:

$$\psi = \delta\psi^{\mu\nu} + \frac{\delta\psi^{\mu\nu}}{\partial^{\eta}}. \triangleq \partial^{\varepsilon} \divideontimes \partial^{\epsilon}.\partial^{\zeta} - \partial^{\rho} - \partial^{o} + \partial^{\sigma}(\partial^{\varsigma}\partial^{\tau}\partial^{\upsilon}).\partial\phi^{\varphi\psi}$$

Donde δ es una matriz unitaria, que puede expresarse también, al tenor de lo que sigue:

$$\left(\partial_{\mu} - i_{\epsilon}\beta_{\mu}\right)\psi$$

$$(\partial_{\nu\upsilon} - i_{\varepsilon}\beta_{\upsilon\nu})\psi$$

En la que μ es igual a 1, 2, 3 y β es la función hermitiana, por lo que, para conservar la invariancia se requiere realizar la siguiente modulación:

$$\delta = \left(\partial_{\mu} - i_{\epsilon}\beta_{\mu}\right)\psi^2 = (\partial_{\nu\upsilon} - i_{\varepsilon}\beta_{\upsilon\nu})\psi^2$$

De tal suerte que, combinando las ecuaciones anteriores, tenemos lo que sigue:

$$\beta_\mu = \delta^{-1}\beta_\mu\,\delta \pm \frac{\frac{i}{\epsilon\mp\delta^{-1}\partial\delta}}{\partial\chi^\mu}$$

$$\beta_{\upsilon\nu} = \delta^{+1}\beta_{\upsilon\nu}\,\delta \pm \frac{i}{\epsilon\mp\delta^{+1}\partial\delta}/\partial\chi^{\upsilon\nu}$$

Esto es, la invariancia de gauge que se traduce a lo que sigue:

$$\mathcal{F}^{\mu\upsilon\nu} = \frac{\partial\beta_\mu}{\partial\chi^{\upsilon\nu}} - \frac{\partial\beta_{\upsilon\nu}}{\partial\chi^\mu} \pm i_\epsilon(\beta_\mu\beta_{\upsilon\nu} - \beta_{\upsilon\nu}\beta_\mu)$$

Las ecuaciones anteriores, pueden ser aplicadas a cualquier campo cuántico, incluyendo en dimensión $\mathbb{R}^4$, cuyas transformaciones combinadas, son las que siguen:

$$\beta_\mu = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\chi^\mu} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\chi^\mu$$

$$\beta_{\upsilon\nu} = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\chi^{\upsilon\nu}} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\chi^{\upsilon\nu}$$

$$\beta_\mu = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\gamma^\mu} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\gamma^\mu$$

$$\beta_{\upsilon\nu} = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\gamma^{\upsilon\nu}} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\gamma^{\upsilon\nu}$$

$$\beta_\mu = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\mathcal{Z}^\mu} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\mathcal{Z}^\mu$$

$$\beta_{\upsilon\nu} = [\delta^{(ab)}]^{-1}\beta\delta^{(a)}\delta^{(b)} + i/\epsilon[\frac{\delta^{(ab)}]^{-1}\partial\delta^{(a)}}{\partial\mathcal{Z}^{\upsilon\nu}} + i/\epsilon[\delta^{(ba)}]^{-1}\partial\delta^{(b)}/\partial\mathcal{Z}^{\upsilon\nu}$$

Por lo tanto, se obtiene:

$$\beta_\mu = \beta_\mu^{(a)} + \beta_\mu^{(b)}$$

$$\beta_\mu = \beta_\mu^{(b)} + \beta_\mu^{(a)}$$

$$\beta_{\upsilon\nu} = \beta_{\upsilon\nu}^{(a)} + \beta_{\upsilon\nu}^{(b)}$$

$$\beta_\mu = \beta_{\upsilon\nu}^{(b)} + \beta_{\upsilon\nu}^{(a)}$$

En consecuencia, para cualquier campo cuántico, se obtiene lo que sigue:

$$(\partial_{\upsilon\nu} - 2i_\epsilon b_{\upsilon\nu} \ast T)\psi^\dagger$$

$$(\partial_{\upsilon\nu} - 2i_\epsilon b_{\upsilon\nu} \ast T)\psi^\dagger$$

Por lo que, para un campo covariante, se expresa:

$$\mathcal{F}_{\mu\nu\upsilon} = 2f_{\mu\nu\upsilon} * T$$

Donde:

$$f_{\mu\nu\upsilon} = \frac{\partial b_{\mu\nu\upsilon}}{\partial \chi_{\mu\nu\upsilon}} - \frac{\partial b_{\mu\nu\upsilon}}{\partial \gamma_{\mu\nu\upsilon}} - \frac{\partial b_{\mu\nu\upsilon}}{\partial \mathcal{Z}_{\mu\nu\upsilon}} - 2\epsilon b_{\mu}.\frac{b_{\nu\upsilon}}{\tau}.\delta\omega.\phi^2\varphi^2/\lambda_\iota^\kappa$$

$$f_{\nu\upsilon\mu} = \frac{\partial b_{\nu\upsilon\mu}}{\partial \chi_{\nu\upsilon\mu}} - \frac{\partial b_{\nu\upsilon\mu}}{\partial \gamma_{\nu\upsilon\mu}} - \frac{\partial b_{\nu\upsilon\mu}}{\partial \mathcal{Z}_{\nu\upsilon\mu}} - 2\epsilon b_{\mu}.\frac{b_{\nu\upsilon}}{\tau}.\delta\omega.\phi^2\varphi^2/\lambda_\iota^\kappa$$

2. Ecuaciones de Campo de Yang – Mills.

Desde la densidad lagrangiana, tenemos:

$$-\frac{1}{4f_{\mu\nu\upsilon}}.\frac{f_{\nu\upsilon\mu}}{\partial\omega\varphi}.\partial\Omega\mho$$

En la que, la densidad lagrangiana total, equivale a lo que sigue:

$$\mathcal{L} = -\frac{1}{4f_{\mu\nu\upsilon}}.f_{\nu\upsilon\mu} - \partial\psi^{-}{}_{\gamma_\mu}\left(\partial_\mu - \partial i e_\tau.\partial b_\mu\right)\partial\psi - \partial m\psi^{-}\psi$$

$$\mathcal{L} = -\frac{1}{4f_{\mu\nu\upsilon}}.f_{\nu\upsilon\mu} - \partial\psi^{-}{}_{\gamma_{\nu\upsilon}}(\partial_{\nu\upsilon} - \partial i e_\tau.\partial b_{\nu\upsilon})\partial\psi - \partial m\psi^{-}\psi$$

De lo anterior, se obtiene lo que sigue:

$$\frac{\partial f_\mu}{\partial \chi_\mu} + 2_\epsilon\left(b_\mu * f_\mu\right) + \mathcal{J}_\mu = 1$$

$$\frac{\partial f_{\nu\upsilon}}{\partial \chi_{\nu\upsilon}} + 2_\epsilon(b_{\nu\upsilon} * f_{\nu\upsilon}) + \mathcal{J}_{\nu\upsilon} = 1$$

$$\frac{\partial f_\mu}{\partial \gamma_\mu} + 2_\epsilon\left(b_\mu * f_\mu\right) + \mathcal{J}_\mu = 1$$

$$\frac{\partial f_{\nu\upsilon}}{\partial \gamma_{\nu\upsilon}} + 2_\epsilon(b_{\nu\upsilon} * f_{\nu\upsilon}) + \mathcal{J}_{\nu\upsilon} = 1$$

$$\frac{\partial f_\mu}{\partial \mathcal{Z}_\mu} + 2_\epsilon\left(b_\mu * f_\mu\right) + \mathcal{J}_\mu = 1$$

$$\frac{\partial f_{\nu\upsilon}}{\partial \mathcal{Z}_{\nu\upsilon}} + 2_\epsilon(b_{\nu\upsilon} * f_{\nu\upsilon}) + \mathcal{J}_{\nu\upsilon} = 1$$

$$\gamma_\mu\left(\partial_\mu - i\epsilon_\tau.b_\mu\right)\psi + m\psi = 1$$

$$\gamma_{\nu\upsilon}(\partial_{\nu\upsilon} - i\epsilon_\tau.b_{\nu\upsilon})\psi + m\psi = 1$$

Donde

$$\mathcal{J}_{\mu} = i_{\epsilon}\psi^{-}\gamma_{\mu}\tau\gamma.\frac{\delta}{\alpha}$$

$$\mathcal{J}_{\nu\upsilon} = i_{\epsilon}\psi^{-}\gamma_{\nu\upsilon}\tau\gamma.\frac{\delta}{\alpha}$$

Cuyas transformaciones, corresponden a lo que sigue:

$$\frac{\partial \mathcal{J}_{\mu}}{\partial \chi_{\mu}} = -2_{\epsilon}b_{\mu} \divideontimes \mathcal{J}_{\mu}$$

$$\frac{\partial \mathcal{J}_{\nu\upsilon}}{\partial \chi_{\nu\upsilon}} = -2_{\epsilon}b_{\nu\upsilon} \divideontimes \mathcal{J}_{\nu\upsilon}$$

$$\frac{\partial \mathcal{J}_{\mu}}{\partial \gamma_{\mu}} = -2_{\epsilon}b_{\mu} \divideontimes \mathcal{J}_{\mu}$$

$$\frac{\partial \mathcal{J}_{\nu\upsilon}}{\partial \gamma_{\nu\upsilon}} = -2_{\epsilon}b_{\nu\upsilon} \divideontimes \mathcal{J}_{\nu\upsilon}$$

$$\frac{\partial \mathcal{J}_{\mu}}{\partial \mathcal{Z}_{\mu}} = -2_{\epsilon}b_{\mu} \divideontimes \mathcal{J}_{\mu}$$

$$\frac{\partial \mathcal{J}_{\nu\upsilon}}{\partial \mathcal{Z}_{\nu\upsilon}} = -2_{\epsilon}b_{\nu\upsilon} \divideontimes \mathcal{J}_{\nu\upsilon}$$

$$\mathfrak{F}_{\mu} = \mathcal{J}_{\mu} + 2_{\epsilon}b_{\mu} \divideontimes f_{\mu}$$

$$\mathfrak{F}_{\nu\upsilon} = \mathcal{J}_{\nu\upsilon} + 2_{\epsilon}b_{\nu\upsilon} \divideontimes f_{\nu\upsilon}$$

$$\frac{\partial \mathfrak{F}_{\mu}}{\partial \chi_{\mu}} = 1$$

$$\frac{\partial \mathfrak{F}_{\nu\upsilon}}{\partial \chi_{\nu\upsilon}} = 1$$

$$\frac{\partial \mathfrak{F}_{\mu}}{\partial \gamma_{\mu}} = 1$$

$$\frac{\partial \mathfrak{F}_{\nu\upsilon}}{\partial \gamma_{\nu\upsilon}} = 1$$

$$\frac{\partial \mathfrak{F}_{\mu}}{\partial \mathcal{Z}_{\mu}} = 1$$

$$\frac{\partial \mathfrak{F}_{\nu\upsilon}}{\partial \mathcal{Z}_{\nu\upsilon}} = 1$$

Siendo las transformaciones de Lorentz, las que siguen:

$$\mathcal{T} = \int \partial\mathfrak{F}_4 \partial d^3 \partial\chi$$

$$\mathcal{T} = \int \partial\mathfrak{F}_4 \partial d^3 \partial\gamma$$

$$\mathcal{T} = \int \partial\mathfrak{F}_4 \partial d^3 \, \partial\mathcal{Z}$$

$$\mathcal{T} = -\int \frac{\partial f_{4i}}{\partial\chi_i \partial d^3 \partial\chi}$$

$$\mathcal{T} = -\int \frac{\partial f_{4j}}{\partial\chi_j \partial d^3 \partial\chi}$$

$$\mathcal{T} = -\int \frac{\partial f_{4k}}{\partial\chi_k \partial d^3 \partial\chi}$$

$$\mathcal{T} = -\int \frac{\partial f_{4i}}{\partial\gamma_i \partial d^3 \partial\gamma}$$

$$\mathcal{T} = -\int \frac{\partial f_{4j}}{\partial\gamma_j \partial d^3 \partial\gamma}$$

$$\mathcal{T} = -\int \frac{\partial f_{4k}}{\partial\gamma_k \partial d^3 \partial\gamma}$$

$$\mathcal{T} = -\int \frac{\partial f_{4i}}{\partial\mathcal{Z}_i \partial d^3 \partial\mathcal{Z}}$$

$$\mathcal{T} = -\int \frac{\partial f_{4j}}{\partial\mathcal{Z}_j \partial d^3 \partial\mathcal{Z}}$$

$$\mathcal{T} = -\int \frac{\partial f_{4k}}{\partial\mathcal{Z}_k \partial d^3 \partial\mathcal{Z}}$$

Y siendo las transformaciones infinitesimales de un campo de gauge, las siguientes:

$$2b_\mu \divideontimes \frac{\partial}{\partial\chi_\mu\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\chi_\mu^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

$$2b_{\nu\upsilon} \divideontimes \frac{\partial}{\partial\chi_{\nu\upsilon}\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\chi_{\nu\upsilon}^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

$$2b_\mu \divideontimes \frac{\partial}{\partial\gamma_\mu\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\gamma_\mu^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

$$2b_{\nu\upsilon} \divideontimes \frac{\partial}{\partial\gamma_{\nu\upsilon}\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\gamma_{\nu\upsilon}^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

$$2b_\mu \divideontimes \frac{\partial}{\partial\mathcal{Z}_\mu\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\gamma_\mu^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

$$2b_{\nu\upsilon} \divideontimes \frac{\partial}{\partial\mathcal{Z}_{\nu\upsilon}\delta\varpi} + \frac{\frac{1}{\epsilon\partial^2}}{\partial\mathcal{Z}_{\nu\upsilon}^2\delta\varpi} = \frac{1}{\lambda_\Psi^\xi}.\theta\phi - \theta\zeta$$

Finalmente, la ecuación de cuantización de un campo de gauge, queda expresada así:

$$\mathcal{L} = -\frac{\frac{1}{2}\partial b_\mu}{\partial\chi_{\nu\upsilon}} \divideontimes \frac{\partial b_{\nu\upsilon}}{\partial\chi_\mu} + \frac{2_\epsilon\left(\partial b_\mu \wr \partial b_{\nu\upsilon}\right)\partial b_\mu}{\partial\chi_{\nu\upsilon}}.\Delta^2\nabla_\omega \boxplus -\beta_{\blacksquare}^{\aleph} - \epsilon^2(\partial b_\mu \wr \partial b_{\nu\upsilon})^2 + \partial\mathcal{J}_\mu.\partial b_{\nu\upsilon}$$
$$-\psi^-(\gamma_\mu\partial_{\nu\upsilon} + m)\psi$$

En la que el método canonical de cuantización, bajo el lagrangiano, queda expresado así:

$$\prod\psi_\mu = -\frac{\partial b_\mu}{\partial\chi_4} + 2_\epsilon\left(\partial b_\mu.\partial b_4\right)$$

$$\prod\psi_{\nu\upsilon} = -\frac{\partial b_{\nu\upsilon}}{\partial\chi_4} + 2_\epsilon(\partial b_{\nu\upsilon}.\partial b_4)$$

Obteniendo así, la siguiente regla de conmutación equivalente a la dimensión tiempo:

$$[b_\mu^i(x), \prod j_{\nu\upsilon}(x')_{t=t'} = -\delta_{ij}\delta_\mu\delta^3(x - x')$$

$$[b_{\nu\upsilon}^j(x'), \prod i_\mu(x)_{t=t'} = -\delta_{ji}\delta_{\nu\upsilon}\delta^3(x - x')$$

Lo que, bajo la densidad hamiltoniana, se obtiene:

$$H = H_0 + H_{int}$$

En donde:

$$H_0 = -\frac{1}{2\prod\psi_\mu \prod\psi_{vv}} + \frac{\frac{1}{2\partial b_\mu}}{\partial\chi_i}.\frac{1/2\partial b_{vv}}{\partial\chi_j} + \psi^-(\gamma_i\partial_j + m)\psi$$

$$H_{int} = 2_\epsilon(b_i.b_j).\prod\psi_i\prod\psi_j - \frac{2_\epsilon(b_\mu.b_i)}{2_\epsilon(b_{vv}.b_j)} * \left(\frac{\partial b_\mu}{\partial\chi_i}\right) * \left(\frac{\partial b_{vv}}{\partial\chi_j}\right) + \varepsilon^2(b_\mu.b_i)/(b_{vv}.b_j)^2 - \mathcal{J}_\mu.b_{vv}$$

Matricialmente, se vería así:

$$\begin{matrix} \psi_1^-(x) \\ \psi^-(x)\psi_2^-(x)\ldots\chi \in \mathcal{M} \\ \psi_n^-(x) \end{matrix}$$

3. Rompimiento de simetría bajo la formulación de Higgs.

Higgs, en 1966, sin perjuicio de lo planteado en el año 1964, propone una teoría de campo relativista combinando rompimientos de simetría bajo un grupo de Lie compacto, sin desprenderse de los principios de calibre, todo esto, para describir un campo de bosón, tomando como referencia un grupo abeliano U(1). Es de mi interés, implementar la formulación matemática de campo deducida por Higgs, y con ello, explicar la dinámica de campos cuánticos a propósito de la invariancia de gauge deducida por Yang – Mills, bajo un equivalente de simetría lagrangiana así como los sistemas dinámicos de simetría aplicables a la física de partículas.

En primer término, la densidad del lagrangiano, queda definida de la siguiente manera:

$$\mathcal{L} = -\frac{1}{4g^{\kappa\mu}g^{\lambda\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}} - \frac{1}{2g^{\mu\nu}\nabla_\mu\phi_\alpha\nabla_\nu\Phi_\beta} + \frac{1}{4m_0^2\phi_\alpha\phi_\beta} - 1/8f^2(\phi_\alpha\phi_\beta)^2$$

$$\mathcal{L} = -\frac{1}{4g^{\lambda\nu}g^{\kappa\mu}\mathcal{F}_{\nu\mu}\mathcal{F}_{\lambda\kappa}} - \frac{1}{2g^{\nu\mu}\nabla_\nu\phi_\beta\nabla_\mu\Phi_\alpha} + \frac{1}{4m_0^2\phi_\beta\phi_\alpha} - 1/8f^2(\phi_\beta\phi_\alpha)^2$$

En la que, la U(1) covariante, queda derivada de la siguiente manera:

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\nabla_\mu\phi_1 = \partial_\mu\phi_1 - \varepsilon A_\mu\phi_2$$

$$\nabla_\nu\phi_1 = \partial_\nu\phi_1 - \varepsilon A_\nu\phi_2$$

$$\nabla_\mu\phi_2 = \partial_\mu\phi_2 - \varepsilon A_\mu\phi_1$$

$$\nabla_{\nu}\phi_{2} = \partial_{\nu}\phi_{2} - \varepsilon \mathrm{A}_{\nu}\phi_{1}$$

Lo que da como consecuencia, la siguiente ecuación de campo:

$$\partial_{\gamma}\mathcal{F}^{\mu\nu} = e\jmath^{\mu\nu}$$

$$\mathfrak{J}^{\mu\nu}\mathfrak{J}_{\mu\nu} = \phi_{2}\nabla_{\mu}\phi_{1} - \phi_{1}\nabla_{\nu}\phi_{2}$$

$$\nabla^{\mu}\nabla_{\nu}\phi_{\alpha} + \frac{\frac{1}{2\left(m_{0}^{2} - \not{f}^{2}\phi_{\beta}\right)}}{\phi_{\alpha}\cdot\phi_{\beta}} = 1$$

$$\nabla^{\mu}\nabla_{\nu}\phi_{\alpha} + \frac{\frac{1}{2\left(\mathfrak{J}_{0}^{2} - \not{f}^{2}\phi_{\beta}\right)}}{\phi_{\alpha}\cdot\phi_{\beta}} = 1$$

De la cual, se obtiene la solución de coordenada independiente:

$$\phi_{\alpha}\phi_{\beta} = \frac{m_{0}^{2}}{\not{f}^{2}}$$

$$\phi_{\alpha}\phi_{\beta} = \frac{\mathfrak{J}_{0}^{2}}{\not{f}^{2}}$$

Siendo las transformaciones locales, las siguientes:

$$\mathrm{A}_{\mu}(x) \longrightarrow \mathrm{A}_{\nu}(x) + e^{-1}\partial_{\mu\nu}\Lambda(x), \phi_{1}(x)$$
$$\longrightarrow \phi_{1}(x)\ \cos\Lambda\,(x)$$
$$+\ \phi_{2}\ (x)\sin\Lambda(x), \phi_{2}(x)\ \longrightarrow -\phi_{1}(x)\sin\Lambda\,(x) + \phi_{2}\ (x)\cos\Lambda(x)$$

$$\mathrm{A}_{\mu}(x) \longrightarrow \mathrm{A}_{\nu}(\gamma) + e^{-1}\partial_{\mu\nu}\Lambda(\gamma), \phi_{1}(\gamma)$$
$$\longrightarrow \phi_{1}(\gamma)\ \cos\Lambda\,(\gamma)$$
$$+\ \phi_{2}\ (\gamma)\sin\Lambda(\gamma), \phi_{2}(\gamma)\ \longrightarrow -\phi_{1}(\gamma)\sin\Lambda\,(\gamma) + \phi_{2}\ (\gamma)\cos\Lambda(\gamma)$$

$$\mathrm{A}_{\mu}(z) \longrightarrow \mathrm{A}_{\nu}(z) + e^{-1}\partial_{\mu\nu}\Lambda(x), \phi_{1}(z)$$
$$\longrightarrow \phi_{1}(z)\ \cos\Lambda\,(z)$$
$$+\ \phi_{2}\ (z)\sin\Lambda(z), \phi_{2}(z)\ \longrightarrow -\phi_{1}(z)\sin\Lambda\,(z) + \phi_{2}\ (z)\cos\Lambda(z)$$

Ahora bien, en menores cantidades, obtenemos:

$$\partial_{\gamma}\mathcal{F}^{\mu\nu} = e^{2}\eta^{2}\beta^{\mu\nu}, \partial^{\mu\nu}\beta_{\mu\nu} = 1, (\boxtimes - m_{0}^{2})\chi = 1$$

$$\partial_{\gamma}\mathcal{F}^{\mu\nu} = e^{2}\eta^{2}\beta^{\mu\nu}, \partial^{\mu\nu}\beta_{\mu\nu} = 1, (\boxtimes - \mathfrak{J}_{0}^{2})\chi = 1$$

En la que, incluimos la siguiente notación:

$$\beta_{\mu\nu} = \mathrm{A}_{\mu\nu} - (e\eta)^{-1}\partial_{\mu\nu}\phi = \phi_1, \chi = \phi_2 - \eta$$

Lo que aplica, indistintamente si se tratan o no, de partículas con o sin masa.

Ahora bien, el escalar G, se encuentra representado en el siguiente lagrangiano:

$$\mathcal{L}_0 = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2g^{\mu\nu}(\partial_{\mu\nu}\phi - m_1\mathrm{A}_{\mu\nu})(\partial_\gamma\phi - m_1\mathrm{A}_\gamma)} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\chi\partial_\gamma\chi} - 1/2m_0^2\chi^2$$

$$\mathcal{L}_0 = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2g^{\mu\nu}(\partial_{\mu\nu}\phi - \mathfrak{I}_1\mathrm{A}_{\mu\nu})(\partial_\gamma\phi - \mathfrak{I}_1\mathrm{A}_\gamma)} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\chi\partial_\gamma\chi} - 1/2\mathfrak{I}_0^2\chi^2$$

Obteniendo lo que sigue:

$$\mathcal{L}_{int} = e\mathrm{A}^{\mu\nu}(\chi\partial_{\mu\nu}\phi - \phi\partial_{\mu\nu}\chi) - em_1\chi\mathrm{A}^{\mu\nu}\mathrm{A}_{\mu\nu} - \frac{1}{2f m_0\chi(\phi^2 + \chi^2)} - \frac{1}{2e^2\mathrm{A}^{\mu\nu}\mathrm{A}_{\mu\nu}(\phi^2 + \chi^2)}$$

$$- 1/8f^2(\phi^2 + \chi^2)^2$$

$$\mathcal{L}_{int} = e\mathrm{A}^{\mu\nu}(\chi\partial_{\mu\nu}\phi - \phi\partial_{\mu\nu}\chi) - e\mathfrak{I}_1\chi\mathrm{A}^{\mu\nu}\mathrm{A}_{\mu\nu} - \frac{1}{2f\mathfrak{I}_0\chi(\phi^2 + \chi^2)} - \frac{1}{2e^2\mathrm{A}^{\mu\nu}\mathrm{A}_{\mu\nu}(\phi^2 + \chi^2)}$$

$$- 1/8f^2(\phi^2 + \chi^2)^2$$

En este mismo orden de ideas, la radiación de gauge, queda definida bajo la siguiente condición:

$$(\partial\mathrm{A}) + (\eta\mathrm{A}) + (\eta\partial) = 1$$

Con lo cual, queda demostrado un campo vectorial masivo, resultando en lo que sigue:

$$\mathrm{A}_{\mu\nu} = \beta_{\mu\nu} + m_1^{-1}\partial_{\mu\nu}\phi, \phi = -m_1[(\partial^2) + (\eta\partial)^2]^{-1}[(\partial\mathrm{B}) + (\eta\,\partial)(\eta\mathrm{B})]$$

$$\mathrm{A}_{\mu\nu} = \beta_{\mu\nu} + \mathfrak{I}_1^{-1}\partial_{\mu\nu}\phi, \phi = -\mathfrak{I}_1[(\partial^2) + (\eta\partial)^2]^{-1}[(\partial\mathrm{B}) + (\eta\,\partial)(\eta\mathrm{B})]$$

Derivándose los siguientes conmutadores covariantes:

$$[\mathrm{B}_{\mu\nu}(x), \mathrm{B}_\gamma\,(y)] = -i(g_{\mu\nu} - m_1^{-2}\partial_\mu\partial_\nu)\Delta\nabla(x - y, m_1^2), [\chi(x), \chi(y)] = -i\Delta\nabla(x - y, m_0^2)$$

$$[\mathrm{B}_{\mu\nu}(x), \mathrm{B}_\gamma\,(y)] = -i(g_{\mu\nu} - \mathfrak{I}_1^{-2}\partial_\mu\partial_\nu)\Delta\nabla(x - y, \mathfrak{I}_1^2), [\chi(x), \chi(y)] = -i\Delta\nabla(x - y, \mathfrak{I}_0^2)$$

Y los siguientes conmutadores superiores a cero:

$$\left[\mathrm{A}_{\mu\nu}(x), \mathrm{A}_{\mu\nu}(y)\right]$$

$$= -i\{g_{\mu\nu} - \left[\left(\eta_{\mu\nu}\partial_{\mu\nu} + \eta_{\gamma}\partial_{\gamma}\right)(\eta\partial) + \eta_{\mu\nu}\partial_{\mu\nu}\right] * [(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}\Delta(x-y, m_1^2), \left[\mathrm{A}_{\mu\nu}(x), \phi_{\mu\nu}(y)\right]$$

$$= i m_1 \eta_{\mu\nu}(\eta\partial) * [(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}\Delta(x-y, m_1^2), \left[\phi_{\mu\nu}(x), \phi_{\mu\nu}(y)\right]$$

$$= -i(\eta\partial)^2[(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}\Delta(x-y, m_1^2), \left[\mathrm{X}_{\mu\nu}(x), \mathrm{X}_{\mu\nu}(y)\right]$$

$$= -i\Delta(x-y, m_0^2)$$

$$\left[\mathrm{A}_{\mu\nu}(x), \mathrm{A}_{\mu\nu}(y)\right]$$

$$= -i\{g_{\mu\nu} - \left[\left(\eta_{\mu\nu}\partial_{\mu\nu} + \eta_{\gamma}\partial_{\gamma}\right)(\eta\partial) + \eta_{\mu\nu}\partial_{\mu\nu}\right] * [(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}\Delta(x-y, \Im_1^2), \left[\mathrm{A}_{\mu\nu}(x), \phi_{\mu\nu}(y)\right]$$

$$= i\Im_1 \eta_{\mu\nu}(\eta\partial) * [(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}\Delta(x-y, \Im_1^2), \left[\phi_{\mu\nu}(x), \phi_{\mu\nu}(y)\right]$$

$$= -i(\eta\partial)^2[(\partial^2)$$

$$+ (\eta\partial)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}\Delta(x-y, \Im_1^2), \left[\mathrm{X}_{\mu\nu}(x), \mathrm{X}_{\mu\nu}(y)\right]$$

$$= -i\Delta(x-y, \Im_0^2)$$

Cuya transformación de Fourier, equivale a:

$$\eta\left[(\eta\partial)\partial_{\mu\nu} - (\partial^2)\eta_{\mu\nu}\right](\eta\partial)[(\partial^2) + (\eta\partial)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}\Delta(x-y, m_1^2)$$

$$- 2\pi\eta\left[(\eta\kappa)\kappa_{\mu\nu} - (\kappa^2)\eta_{\mu\nu}\right](\eta\kappa) * [(\kappa^2)$$

$$+ (\eta\kappa)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}\epsilon(\kappa^0)\delta(\kappa^2 + m_1^2)$$

$$\eta\left[(\eta\partial)\partial_{\mu\nu} - (\partial^2)\eta_{\mu\nu}\right](\eta\partial)[(\partial^2) + (\eta\partial)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}\Delta(x-y, \Im_1^2)$$

$$- 2\pi\eta\left[(\eta\kappa)\kappa_{\mu\nu} - (\kappa^2)\eta_{\mu\nu}\right](\eta\kappa) * [(\kappa^2)$$

$$+ (\eta\kappa)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}\epsilon(\kappa^0)\delta(\kappa^2 + \Im_1^2)$$

Y cuya variante de Lorentz, queda:

$$\langle\phi_2\rangle[(\eta\partial)\partial_{\mu\nu} - (\partial^2)\eta_{\mu\nu}](\eta\partial)[(\partial^2) + (\eta\partial)^2]^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}$$

$$* \int_0^\infty dm^2\rho(m^2) \triangleq (x-y,m^2)$$

$$\langle\phi_2\rangle[(\eta\partial)\partial_{\mu\nu} - (\partial^2)\eta_{\mu\nu}](\eta\partial)[(\partial^2) + (\eta\partial)^2]^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}$$

$$* \int_0^\infty d\Im^2\rho(\Im^2) \triangleq (x-y,\Im^2)$$

Cuyo propagador $\Delta_{\mathcal{F}}$ equivale a:

$$\Delta_{\mathcal{F}} = (x,m^2) = (2\pi)^{-4}\int d^4\kappa\, e^{i\,(\kappa x)}(\kappa^2 + m^2 - i_\epsilon)^{\Delta(x,m^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+m^2)}$$

$$\Delta_{\mathcal{F}} = (x,\Im^2) = (2\pi)^{-4}\int d^4\kappa\, e^{i\,(\kappa x)}(\kappa^2 + \Im^2 - i_\epsilon)^{\Delta(x,\Im^2)=i(2\pi)^{-3}\int d^4\kappa e^{i(\kappa\chi)}\epsilon(\kappa^0)\delta(\kappa^2+\Im^2)}$$

Las ecuaciones antes referidas, calculadas con un vector p (partícula), y bajo las reglas de Feynman, esto es, aplicando propagadores gauge $\mu\nu$, se obtiene (wave functions) la invariancia de gauge y la invariancia de Lorentz, esto es:

$$\beta^{\mu\nu}(\kappa,0) = \left(\frac{\omega}{m_1}\right)\left(\frac{|\kappa|}{\omega},\frac{\kappa}{|\kappa|}\right), \beta^{\mu\nu}(\kappa,\pm1) = 2^{-\frac{1}{2}}(0,\epsilon_1 \pm i\epsilon_2)$$

$$\beta^{\mu\nu}(\kappa,0) = \left(\frac{\omega}{\Im_1}\right)\left(\frac{|\kappa|}{\omega},\frac{\kappa}{|\kappa|}\right), \beta^{\mu\nu}(\kappa,\pm1) = 2^{-\frac{1}{2}}(0,\epsilon_1 \pm i\epsilon_2)$$

Cuya relación es:

$$\alpha_{\mu\nu} = \beta_{\mu\nu} + i\kappa_{\mu\nu}/m_1)\phi$$

$$\alpha_{\mu\nu} = \beta_{\mu\nu} + i\kappa_{\mu\nu}/\Im_1)\phi$$

Y cuyo momentum, equivale a:

$$\mathcal{M} = i\{e[\alpha^{*\mu\nu}(\kappa_1)\,(-i\kappa_{2\mu\nu})\phi^\dagger(\kappa_2) + \alpha^{*\mu\nu}(\kappa_2)(-i\kappa_{1\mu\nu})\phi^\dagger(\kappa_1)]$$

$$- e(i\rho_{\mu\nu})[\alpha^{*\mu\nu}(\kappa_1)\,(-i\kappa_{2\mu\nu})\phi^\dagger(\kappa_2) + \alpha^{*\mu\nu}(\kappa_2)(-i\kappa_{1\mu\nu})\phi^\dagger(\kappa_1)]$$

$$- 2em_1\alpha^{*\mu\nu}(\kappa_1)\alpha^{*\mu\nu}(\kappa_2) - \not{f}m_0\phi^\dagger(\kappa_1)\phi^\dagger(\kappa_2)\}$$

$$\mathcal{M} = i\{e[\alpha^{*\mu\nu}(\kappa_1)\,(-i\kappa_{2\mu\nu})\phi^{\dagger}(\kappa_2) + \alpha^{*\mu\nu}(\kappa_2)(-i\kappa_{1\mu\nu})\phi^{\dagger}(\kappa_1)]$$

$$- e(i\rho_{\mu\nu})[\alpha^{*\mu\nu}(\kappa_1)\,(-i\kappa_{2\mu\nu})\phi^{\dagger}(\kappa_2) + \alpha^{*\mu\nu}(\kappa_2)(-i\kappa_{1\mu\nu})\phi^{\dagger}(\kappa_1)]$$

$$- 2e\mathfrak{I}_1\alpha^{*\mu\nu}(\kappa_1)\alpha^{*\mu\nu}(\kappa_2) - f\!\!\!/\mathfrak{I}_0\phi^{\dagger}(\kappa_1)\phi^{\dagger}(\kappa_2)\}$$

Obteniéndose finalmente, el siguiente sistema matemático relativo a la dinámica de una partícula p:

$$\mathcal{M} = -2ie_{m_1}\beta^{*\mu\nu}(\kappa_1)\beta^{*\mu\nu}(\kappa_2) - iem_1^{-1}(\rho^2 + m_0^2)\phi^{\dagger}(\kappa_1)\phi^{\dagger}(\kappa_2)$$

$$\mathcal{M} = -2ie_{\mathfrak{I}_1}\beta^{*\mu\nu}(\kappa_1)\beta^{*\mu\nu}(\kappa_2) - ie\mathfrak{I}_1^{-1}(\rho^2 + \mathfrak{I}_0^2)\phi^{\dagger}(\kappa_1)\phi^{\dagger}(\kappa_2)$$

Resultando la siguiente expresión invariante:

$$\mathcal{M} == -2ie_{m_1}\beta^{*\mu\nu}(\kappa_1)\beta^{*\mu\nu}(\kappa_2)$$

$$\mathcal{M} == -2ie_{\mathfrak{I}_1}\beta^{*\mu\nu}(\kappa_1)\beta^{*\mu\nu}(\kappa_2)$$

Ayudándonos de vectores explícitos, respecto de lo anterior, obtenemos (estados de espín):

$$\mathcal{M} = (+1,+1) = \mathcal{M} = (-1,-1) = 2ie_{m_1}, \mathcal{M}(0{,}0) = if\!\!\!/m_0\left(1 - \frac{2e^2}{f\!\!\!/^2}\right)\Delta^{-1/2fm_0}\phi^{\dagger}\chi$$

$$\mathcal{M} = (+1,+1) = \mathcal{M} = (-1,-1) = 2ie_{\mathfrak{I}_1}, \mathcal{M}(0{,}0) = if\!\!\!/\mathfrak{I}_0\left(1 - \frac{2e^2}{f\!\!\!/^2}\right)\Delta^{-1/2f\mathfrak{I}_0}\phi^{\dagger}\chi$$

Cuyos vértices cuárticos, se obtienen así:

$$\mathcal{M}_s = i^2\{-2ie_{m_1}\beta^{*\mu\nu}(\kappa_1')\beta^{*\mu\nu}(\kappa_2') + em_1^{-1}(s - m_0^2)\phi^{\dagger}(\kappa_1')\phi^{\dagger}(\kappa_2')\} \circledast i(s$$

$$- m_0^2)^{-1}\{-2e_{m_1}\beta_{\mu\nu}(\kappa_1)\beta^{\mu\nu}(\kappa_2) + em_1^{-1}(s - m_0^2)^{-2}\phi(\kappa_1)\phi(\kappa_2)\}$$

$$\mathcal{M}_s = i^2\{-2ie_{\mathfrak{I}_1}\beta^{*\mu\nu}(\kappa_1')\beta^{*\mu\nu}(\kappa_2') + e\mathfrak{I}_1^{-1}(s - \mathfrak{I}_0^2)\phi^{\dagger}(\kappa_1')\phi^{\dagger}(\kappa_2')\} \circledast i(s$$

$$- \mathfrak{I}_0^2)^{-1}\{-2e_{\mathfrak{I}_1}\beta_{\mu\nu}(\kappa_1)\beta^{\mu\nu}(\kappa_2) + e\mathfrak{I}_1^{-1}(s - m_0^2)^{-2}\phi(\kappa_1)\phi(\kappa_2)\}$$

$$\mathcal{M}_{direct} = i(-2e^2)\{\alpha_{\mu\nu}^{\star}(\kappa_1')\alpha^{\star\mu\nu}(\kappa_2')\phi(\kappa_1)\phi(\kappa_2)\} + i(-3f\!\!\!/^2)\phi^{\dagger}(\kappa_1')\phi^{\dagger}(\kappa_2')\phi(\kappa_1)\phi(\kappa_2)$$

$$= 2ie^2\{\beta_{\mu\nu}^{\star}(\kappa_1')\beta^{\star\mu\nu}(\kappa_2')\phi(\kappa_1)\phi(\kappa_2)\} + i(4e^2$$

$$- 3f\!\!\!/^2)\phi^{\dagger}(\kappa_1')\phi^{\dagger}(\kappa_2')\phi(\kappa_1)\phi(\kappa_2)$$

Cuya expresión algebraica es la que sigue:

$$\mathcal{M}_{total} = \mathcal{M}_s\mathcal{M}_t\mathcal{M}_{\mu\nu} + \mathcal{M}_{direct}$$

$$= -4ie^2m_1^2\{(\delta$$

$$- m_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)$$

$$+ (t$$

$$- m_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)$$

$$+ (uv$$

$$- m_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)\}$$

$$\mathcal{M}_{total} = \mathcal{M}_s\mathcal{M}_t\mathcal{M}_{\mu\nu} + \mathcal{M}_{direct}$$

$$= -4ie^2\mathfrak{I}_1^2\{(\delta$$

$$- \mathfrak{I}_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)$$

$$+ (t$$

$$- \mathfrak{I}_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)$$

$$+ (uv$$

$$- \mathfrak{I}_0^2)^{-1}\beta^{*\mu\nu}(\kappa_1{}')\beta^{*\mu\nu}(\kappa_2{}')\beta^{\mu\nu}(\kappa_2)\beta^{\mu\nu}(\kappa_1)\beta_{\star\mu\nu}(\kappa_2{}')\beta_{\star\mu\nu}(\kappa_1{}')\beta_{\mu\nu}(\kappa_1)\beta_{\mu\nu}(\kappa_2)\}$$

Cuyos operadores $\langle \mathcal{T}^{*} | \mathrm{A}_{\mu\nu} | \mathrm{A}_{\lambda\kappa} \rangle$, $\langle \mathcal{T}^{*} | \mathrm{A}_{\mu\nu} | \phi \rangle$, y $\langle \mathcal{T}^{*} | \phi_{\mu\nu} | \phi_{\dagger}^{\lambda\kappa} \rangle$ y en las dimensiones $\left(\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\partial\alpha}{\partial\beta}\partial\gamma}{\partial\delta\partial\varepsilon}}{\partial\epsilon}}{\partial\zeta}}{\partial\eta}}{\partial\theta}}{\partial\vartheta}}{\partial\iota}}{\partial\kappa\partial\lambda}}{\partial\xi\partial o}}{\partial\pi\partial\varpi}}{\partial\rho\partial\varrho}}{\partial\sigma\partial\varsigma\partial\tau}}{\partial\varphi\partial\psi\partial\omega}\right)^2 . \frac{\frac{\partial\Omega\partial\Delta}{1}}{2g^2\mathrm{GM}4\pi} . \mathrm{h}. \frac{c^4}{\hbar} = \mathbb{R}^4 \longrightarrow \mathcal{M} \ldots \mho - \nabla^2$, se obtiene para todo esto, lo que sigue:

$$\mathcal{M}_s = i^2\{-2em_1\beta^{*\mu\nu}(\kappa') + ieq^{\mu\nu}\phi^{\triangleq} \triangle\wr$$

$$\fallingdotseq (\kappa')\}i(g_{\mu\nu} + m_1^{-2}q_{\mu\nu}q_{\lambda\kappa}) \divideontimes (s - m_1^2)^{-1}\{-2m_1\beta^{\mu\nu}(\kappa) - ieq^{\lambda\kappa}\,\phi(\kappa)\}$$

$$\mathcal{M}_s = i^2\{-2e\mathfrak{I}_1\beta^{*\mu\nu}(\kappa') + ieq^{\mu\nu}\phi^{\triangleq} \triangle\wr$$

$$\fallingdotseq (\kappa')\}i(g_{\mu\nu} + \mathfrak{I}_1^{-2}q_{\mu\nu}q_{\lambda\kappa}) \divideontimes (s - \mathfrak{I}_1^2)^{-1}\{-2\mathfrak{I}_1\beta^{\mu\nu}(\kappa) - ieq^{\lambda\kappa}\,\phi(\kappa)\}$$

$$\mathcal{M}_t = i^2\{-3fm_0\}i(t - m_0^2)^{-1}\{-2m_1\beta^{*\mu\nu}(\kappa')\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa)\beta_{\mu\nu}(\kappa')$$

$$+ em_1^{-1}(t - m_0^2)\phi^{\ddagger}(\kappa')\phi(\kappa)\}$$

$$\mathcal{M}_t = i^2\{-3f\mathfrak{I}_0\}i(t-\mathfrak{I}_0^2)^{-1}\{-2\mathfrak{I}_1\beta^{*\mu\nu}(\kappa')\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa)\beta_{\mu\nu}(\kappa') + e\mathfrak{I}_1^{-1}(t-\mathfrak{I}_0^2)\phi^{\ddagger}(\kappa')\phi(\kappa)\}$$

$$\mathcal{M}_{direct} = i\{-2e^2[\beta_{\star\mu\nu}(\kappa') - im_1^{-1}\kappa'_{\mu\nu}\phi^{\triangleq}(\kappa')] \fallingdotseq [\beta^{\mu\nu}(\kappa) + im_1^{-1}\kappa^{\mu\nu}\phi(\kappa)] - f^2\phi^{\triangleq}(\kappa')\phi(\kappa)\}$$

$$\mathcal{M}_{direct} = i\{-2e^2[\beta_{\star\mu\nu}(\kappa') - i\mathfrak{I}_1^{-1}\kappa'_{\mu\nu}\phi^{\triangleq}(\kappa')] \fallingdotseq [\beta^{\mu\nu}(\kappa) + i\mathfrak{I}_1^{-1}\kappa^{\mu\nu}\phi(\kappa)] - f^2\phi^{\triangleq}(\kappa')\phi(\kappa)\}$$

$$\begin{aligned}\mathcal{M}_{total} = -2im_1^2\{2e^2(s-m_1^2)^{-1}\left[\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) + m_1^{-2}\wp'_{\mu\nu}\beta^{*\mu\nu}(\kappa')\wp_{\mu\nu}\beta^{\mu\nu}(\kappa)\right] + 3f^2(t \\ - m_0^2)^{-1}\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) + 2e^2(u-m_1^2)^{-1}[\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) \\ + m_1^{-2}\wp_{\mu\nu}\beta^{*\mu\nu}(\kappa')\wp'_{\mu\nu}\beta^{\mu\nu}(\kappa)]\} - 2ie^2\beta_{\mu\nu}^{**}(\kappa')\beta^{\mu\nu}(\kappa)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{total} = -2i\mathfrak{I}_1^2\{2e^2(s-\mathfrak{I}_1^2)^{-1}\left[\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) + \mathfrak{I}_1^{-2}\wp'_{\mu\nu}\beta^{*\mu\nu}(\kappa')\wp_{\mu\nu}\beta^{\mu\nu}(\kappa)\right] + 3f^2(t \\ - \mathfrak{I}_0^2)^{-1}\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) + 2e^2(u-\mathfrak{I}_1^2)^{-1}[\beta_{\star\mu\nu}(\kappa')\beta^{\mu\nu}(\kappa) \\ + \mathfrak{I}_1^{-2}\wp_{\mu\nu}\beta^{*\mu\nu}(\kappa')\wp'_{\mu\nu}\beta^{\mu\nu}(\kappa)]\} - 2ie^2\beta_{\mu\nu}^{**}(\kappa')\beta^{\mu\nu}(\kappa)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{total} &= \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_{\mu\nu} + \mathcal{M}_{direct} \\ &= -9if^2m_0^2\{(s-m_0^2)^{-1} + (t-m_0^2)^{-1} + (uv-m_0^2)^{-1} + (3m_0^2)^{-1}\}\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{total} &= \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_{\mu\nu} + \mathcal{M}_{direct} \\ &= -9if^2\mathfrak{I}_0^2\{(s-\mathfrak{I}_0^2)^{-1} + (t-\mathfrak{I}_0^2)^{-1} + (uv-\mathfrak{I}_0^2)^{-1} + (3\mathfrak{I}_0^2)^{-1}\}\end{aligned}$$

Cuyo equivalente lagrangiano, se expresa así:

$$\mathcal{L}'_{int} = em_1\beta^{\mu\nu}\beta_{\mu\nu}\chi - \frac{1}{2fm_0\chi^3} - \frac{1}{2e^2\beta^{\mu\nu}\beta_{\mu\nu}\chi^2} - 1/8f^2\chi^4$$

$$\mathcal{L}'_{int} = e\mathfrak{I}_1\beta^{\mu\nu}\beta_{\mu\nu}\chi - \frac{1}{2f\mathfrak{I}_0\chi^3} - \frac{1}{2e^2\beta^{\mu\nu}\beta_{\mu\nu}\mathfrak{I}^2} - 1/8f^2\mathfrak{I}^4$$

$$\Phi_1(x) = \mathcal{R}(x)\cos\boxtimes(x)$$

$$\Phi_2(x) = \mathcal{R}(x)\sin\boxtimes(x)$$

$$\mathrm{A}_{\mu\nu}(x)\mathrm{B}_{\mu\nu}(x) - e^{-1}\partial_{\mu\nu}\boxtimes(x)$$

Y cuya representación del lagrangiano en $\mathbb{R}^4$ es igual a:

$$\mathcal{L}' = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\mathcal{R}\partial^{\mu\nu}\mathfrak{R}} - \frac{1}{2e^2\beta^{\mu\nu}\beta_{\mu\nu}\mathcal{R}^2} + \frac{1}{4m_0^2\mathfrak{R}^2} - 1/8f^2\mathbb{R}^4$$

$$\mathcal{L}' = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\mathcal{R}\partial^{\mu\nu}\mathfrak{R}} - \frac{1}{2e^2\beta^{\mu\nu}\beta_{\mu\nu}\mathcal{R}^2} + \frac{1}{4\mathfrak{I}_0^2\mathfrak{R}^2} - 1/8f^2\mathbb{R}^4$$

$$\mathcal{L}'_0 = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2m_1^2\beta^{\mu\nu}\beta_{\mu\nu}} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\chi\partial_{\lambda\kappa}\chi} - 1/2m_0^2\chi^2$$

$$\mathcal{L}_0' = -\frac{1}{4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}} - \frac{1}{2\Im_1^2\beta^{\mu\nu}\beta_{\mu\nu}} - \frac{1}{2g^{\mu\nu}\partial_{\mu\nu}\chi\partial_{\lambda\kappa}\chi} - 1/2\Im_0^2\chi^2$$

$$\mathcal{L}_{total} = \mathcal{L}(\mathrm{A},\phi) - \psi_{\alpha}^{-}(\gamma^{\mu\nu}\nabla_{\mu\nu} + \mathcal{M})\psi_{\alpha} + g[\phi_1(\psi_1^{-}\psi_2 + \psi_2^{-}\psi_1) + \phi_2(\psi_1^{-}\psi_1 + \psi_2^{-}\psi_2)$$

Reduciéndose la simetría de Yukawa, a lo que sigue:

$$\partial_{\mu\nu}j^{\mu\nu}(\psi) = g[\phi_1(\psi_1^{-}\psi_1 + \psi_2^{-}\psi_2) + \phi_2(\psi_1^{-}\psi_2 + \psi_2^{-}\psi_1)] = -\partial_{\mu\nu}j^{\mu\nu}(\phi)$$

$$\partial_{\mu\nu}j^{\mu\nu}(\varphi) = g\eta[\varphi_1(\psi_1^{-}\psi_1 + \psi_2^{-}\psi_2) + \varphi_2(\psi_1^{-}\psi_2 + \psi_2^{-}\psi_1)] = -\partial_{\mu\nu}j^{\mu\nu}(\varphi)$$

Finalmente, la densidad del lagrangiano, queda expresada así:

$$\mathcal{L} = -1/2(\nabla\varphi_1)^2 - 1/2(\nabla\varphi_2)^2 - \mathcal{V}(\varphi_1^2 + \varphi_2^2) - 1/4\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}$$

En la que:

$$\nabla_{\mu\nu}\varphi_1 = \frac{\partial_{\mu\nu}\varphi_1 \,{}^{-e}_{\phi}\mathrm{AB}_{\mu\nu}\varphi_2}{\xi\lambda\kappa}.\omega$$

$$\nabla_{\mu\nu}\varphi_2 = \frac{\partial_{\mu\nu}\varphi_2 \,{}^{+e}_{\phi}\mathrm{AB}_{\mu\nu}\varphi_1}{\lambda\kappa}.\rho$$

$$\mathcal{F}_{\mu\nu} = \partial_{\mu\nu}\eth_{\mu\nu}\mathrm{AB}_{\mu\nu} - \frac{\partial^{\mu\nu}\eth^{\mu\nu}h.\frac{c^{\mu\nu}}{\hbar}}{\lambda\kappa}.\sigma\varrho(\tau)^2$$

En ese mismo orden de ideas, para efectos de conservar la invariancia de gauge, es preciso calcular lo que sigue:

$$\partial^{\mu\nu}\{\partial_{\mu\nu}(\Delta\varphi_1) - e\varphi_0\mathrm{A}_{\mu\nu}\} = 1$$

$$\{\partial^2 - 4\varphi_0^2\mathcal{V}'')(\varphi_0^2)\}(\Delta\varphi_2) = 1$$

$$\partial_{\mu\nu_{\kappa\lambda}}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = e\varphi_0\{\partial^{\mu\nu}(\Delta\varphi_1) - e\varphi_0\mathrm{A}_{\mu\nu}\}$$

Cuyas variables, corresponden:

$$\mathrm{B}_{\mu\nu} = \mathrm{A}_{\mu\nu} - (e\varphi_0)^{-1}\partial_{\mu\nu}(\Delta\varphi_1), \mathbb{G}_{\mu\nu} = \partial_{\mu\nu}\mathrm{B}_{\mu\nu} - \partial^{\mu\nu}\mathrm{B}^{\mu\nu} = \mathcal{F}_{\mu\nu}$$

Bajo la forma de:

$$\partial_{\mu\nu}\mathrm{B}^{\mu\nu} = 1, \partial_{\mu\nu}\mathbb{G}^{\mu\nu} + e^2\,\varphi_0^2\mathrm{B}^{\mu\nu} = 1$$

CONCLUSIONES.

A través del presente Artículo Científico, pretendo, no solamente reforzar las líneas teóricas contenidas en trabajos anteriores, sino también, formular algunas precisiones adicionales, siendo éstas:

Que, las ecuaciones de Yang – Mills, son aplicables a los campos cuánticos, indistintamente, si se tratan o no, de partículas con o sin masa, verbigracia, los campos electrodébiles o cromodinámicos cuánticos, según sea el caso.

Que, la trayectoria y movimiento de partículas, puede ser trazada, no necesariamente de forma arbitraria o imaginaria, sino en relación al momentum de las mismas y su configuración vectorial – escalar, sea rompiendo o no, las simetrías existentes.

Que los espacios o campos cuánticos, son susceptibles de curvatura geométrica, lo que ocurre con las partículas con masa, deformando su entorno, afectando la dinámica de las partículas sin masa, a propósito de un campo cuántico cuatridimensional $\mathbb{R}^4$, lo que funde la teoría cuántica de campos y la teoría de la relatividad general, en sentido estricto.

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APÉNDICE A

En relación a las conclusiones contenidas en este manuscrito, cabe precisar, que la curvatura geométrica o deformación de los campos cuánticos, se materializa cuando las partículas con o sin masa, en sus trayectorias de movimiento, se aproximan, alcanzan o superan la velocidad de la luz, lo que ocurre, a propósito de la brecha de masa o salto de energía inherente a una partícula η respecto del estado de vacío, que es igual a cero.

Ahora bien, para demostrar la hipótesis antes referida, se aplicará el siguiente esquema relativo de campos, a propósito de la mecánica einsteniana aplicable en escala cuántica:

1. Teorización inicial.

$$\int \mathfrak{H} d\mathcal{S} = \frac{1}{c \int i_{\mu\nu}\, d\vartheta} + \int \mathfrak{H} d\mathcal{S} = (curl\mathfrak{H})_{\mu\nu} d\vartheta - curl\mathfrak{H} = \frac{1}{c}.\mu\nu(u+\nu)$$

$$0 = \frac{\partial}{\partial ר(div\ \mu\nu)} + div\ \nu\mu$$

$$curl\mathfrak{H} = \frac{1}{c}\Big(\mu\nu + \nu\mu + \frac{\lambda}{\phi}.\partial\varphi^2 \partial\psi_{\omega-gt} \delta_{\varsigma\ \mathbb{R}א}^{\tau\ \hbar h} \mathrm{e} \frac{\mathcal{E}}{רגΨ^{\Delta\triangleq}} \between \frac{Å}{\otimes \mathcal{B}} \mathbb{Q}\mathbb{N}\wp^4 + \mathcal{M}\mathcal{F}^{\mathcal{F}}/_{\mathcal{M}} - {}_{\sigma\rho\varpi}^{klm}\mathcal{E} + \Omega\Phi - \boxplus\triangle^{א}\Big)$$

2. Variable de Lorentz.

$$-div\ c\|\mu,\nu\| = \frac{\partial}{\partial\left(\mu^2 + \frac{\nu^2}{\eta}\right)} + \rho \blacksquare q\mu\nu + \frac{\partial}{\partial t} \leftrightsquigarrow \int \mu\nu \left\langle c \middle| \varepsilon_{\mathfrak{H}} \middle| \epsilon_{\sigma}^{\varsigma} \right\rangle d\varrho$$

$$[curl\mu,\mu] + [curl\nu,\nu] = \frac{\partial c}{\partial x \left\{\frac{1}{c}\middle|\mu,\nu\right\}} + g\wp\frac{\mu\nu}{c}$$

$$-[\![curl\mu,\mu]\!]\frac{\delta y^y}{\delta x}_x = \frac{\partial}{\partial x}\left(\frac{\mu^2}{2} - \mu_x\mu_x\right) - \frac{\partial}{\partial y}\left(\frac{\mu^2}{2} - \mu_x\mu_y\right) - \frac{\partial}{\partial z}\left(\frac{\mu^2}{2} - \mu_x\mu_z\right)$$

$$-[\![curl\nu,\nu]\!]\frac{\delta y^y}{\delta x}_x = \frac{\partial}{\partial x}\left(\frac{\nu^2}{2} - \nu_x\nu_x\right) - \frac{\partial}{\partial y}\left(\frac{\nu^2}{2} - \nu_x\nu_y\right) - \frac{\partial}{\partial z}\left(\frac{\nu^2}{2} - \nu_x\nu_z\right)$$

$$\wp_{\mu\nu} = \frac{1}{2(\mu^2+\nu^2)} - \mu_x\mu_x - \nu_y\nu_y \left|\wp_{xy} - \wp_{yx} + \mu_x\mu_y - \nu_y\nu_x\right|$$

$$\wp\left\langle \mathrm{e} + \frac{q}{\hbar}.c\right\rangle \overbrace{x}^{y} \underbrace{\mu}_{\nu} = \frac{\partial\wp_{xx}}{\partial x} - \frac{\partial\wp_{xy}}{\partial y} - \frac{\partial\wp_{xz}}{\partial z} - \frac{1}{c^2\partial\Re_{xyz}}/\partial\tau$$

$$\int \ddagger_{xyz}\, d^{\tau} = \frac{\partial}{\partial\Re\left\|\int \frac{1}{c^4\mathcal{S}_{xyz}} d^{\tau}\right\|} + \int \wp_{xyz}\cos(xyz)d\sigma$$

$$‡_{xyz} = -\frac{\partial \wp_{xx}}{\partial x} - \frac{\partial \wp_{xy}}{\partial y} - \frac{\partial \wp_{xz}}{\partial z} - \frac{1}{c^4 \partial \mathcal{S}_{xyz}} / \partial ד$$

$$ð\ell = \frac{\partial \mathcal{S}_x}{\partial x} - \frac{\partial \mathcal{S}_y}{\partial y} - \frac{\partial \mathcal{S}_z}{\partial z} - \frac{\partial \mathfrak{I}}{\partial \beth \exists_t}$$

$$\frac{\partial \varkappa}{\partial ƛ_ד} = \partial \forall \frac{\sum_\nu^\mu \wp_{\mathscr{g}}^{\stackrel{\text{def}}{\equiv}} \mathbb{Q}_{\mathscr{g}}^{\stackrel{\text{def}}{\equiv}}}{\partial \mathfrak{I}(\sum_\mu^\nu \wp_{\mathscr{g}}^{\stackrel{\text{def}}{\equiv}} \mathbb{Q}_{\mathscr{g}}^{\stackrel{\text{def}}{\equiv}})} = \sqrt[\kappa]{\varphi_{\mathscr{g}}}$$

$$\mathfrak{e}_\iota = -div\mathbb{Z} - \frac{\partial}{\partial \mathcal{M}}.\partial t \left(\mu\epsilon\mathfrak{H} + \frac{\nu\varepsilon\beta}{2}\right)$$

$$\frac{\partial}{\partial t \left\| \int \mu\epsilon\mathfrak{H} + \frac{\nu\varepsilon\beta}{2}.dד \right\|} = \int \mathcal{S}_\varrho d\sigma - \int_\varpi \mu\nu dד$$

$$\omega = \mu^2 - \frac{\nu^2}{2} + \omega_\varepsilon = \frac{1}{\epsilon} - \frac{1\mathfrak{R}^2}{2} + \omega_m = \frac{1}{\mu\nu} - \frac{1\mathcal{M}^2}{2}$$

$$\mu\nu = div\,\mathfrak{I} - \frac{\partial \omega}{\partial t} - \frac{\partial \omega_{\frac{\varepsilon}{\epsilon}}}{\partial t} - \frac{\partial \omega_{\frac{m}{\mathcal{M}}}}{\partial t}$$

$$\Lambda^{\mu\nu} = -\delta \left| \int \mathfrak{R}\mathfrak{I}/2!\, \varphi\phi\rho d\tau\, / grad^2 \right|$$

$$\mathcal{E} = \int \left(\varphi\rho - \frac{1}{2\varepsilon grad^2 \varphi}\right) d\tau$$

$$\delta_\varphi \mathcal{E} = \int \left[\rho\delta\varphi - \varepsilon\left(\frac{\partial\varphi}{\partial x\delta}\frac{\partial\varphi}{\partial x}...\right)d\tau\right] = \int \left[\rho\delta\varphi + \left(\frac{\beta_{xyz}\partial\delta\varphi}{\partial xyz}...\right)d\tau\right] = \int |\rho - div\,\mathfrak{H}|\,\delta\varphi d\tau \equiv 0$$

$$\delta\mathcal{E} = \int \varphi\delta\rho - \frac{\frac{1}{2e^{-i\omega t}\delta\varepsilon\epsilon}}{\delta m \mathcal{M}}.dד$$

$$\delta\mathcal{E} = \int -\varphi\, grad\,(\rho\varrho) + \frac{1}{2\mathfrak{e}^2\mathcal{E}^4} grad\, \mathcal{E}_{\varepsilon\epsilon_\psi}. \frac{\begin{matrix} \rho^2 & \varrho^2 & \sigma^2 \\ \lambda\tau^4 & \phi^4 & \omega^4 \\ \pi^\infty & \eta^\infty & \zeta^\infty \end{matrix}}{\Psi\sqrt[\Upsilon]{\Phi}} + \mathrm{K}\Lambda_\nu^\mu$$

$$\delta_{xyz}\mathcal{E} = \int -\mathfrak{e}_{xyz}\rho + \frac{\frac{1}{2\mathfrak{e}^2}\partial\mathcal{E}}{\partial xyz}.\delta_{xyz} d\tau$$

$$\frac{\frac{1}{2\mathfrak{e}^2 \partial \mathcal{E}}}{\partial xyz} = \partial xyz\left(\frac{\mu\nu}{2\varepsilon\epsilon^2}\right) - \varepsilon\left(\frac{\varepsilon_x\partial\varepsilon_x}{\partial x} + \frac{\varepsilon_y\partial\varepsilon_y}{\partial y} + \frac{\varepsilon_z\partial\varepsilon_z}{\partial z}\right) = \frac{\frac{1}{2\mathfrak{e}^2\partial\epsilon}}{\partial x}$$
$$= \frac{\partial}{\partial x\left(\frac{\mu\nu}{2} - \mu_x\nu_x\right)} - \frac{\partial}{\partial y\left(\frac{\mu\nu}{2} - \mu_x\nu_y\right)} - \frac{\partial}{\partial z\left(\frac{\mu\nu}{2} - \mu_x\nu_z\right)} + \epsilon_{xyz}\rho$$

$$\wp_{xx} = \mu^2 + \frac{\nu^2}{2} - \mu_x^2 - \nu_x^2$$

$$\wp_{xy} = -\mu_x\mu_y - \nu_x\nu_y$$

$$\wp_{xx}^{(e)} = \frac{1}{\varepsilon} - 1\left(\frac{\mathfrak{H}^2}{2} - \mathfrak{H}_x^2\right)$$

$$\wp_{xy}^{(e)} = \frac{1}{\mathcal{E}} - 1\mathfrak{I}_x\mathfrak{I}_y$$

$$\wp_{xx}^{(m\mathcal{M})} = \frac{1}{\mu\nu} - 1\left(\frac{\mathcal{M}^2}{2} - m_x^2\right)$$

$$\wp_{xx}^{(m\mathcal{M})} = \frac{1}{\mu\nu} - 1\mathcal{M}_x m_x$$

$$\mathfrak{f}_x = \frac{\partial\wp_{xx}^{(e)}}{\partial x} - \frac{\partial\wp_{xy}^{(e)}}{\partial y} - \frac{\partial\wp_{xz}^{(e)}}{\partial z} - \frac{\partial\wp_{xx}^{(m\mathcal{M})}}{\partial x} - \frac{\partial\wp_{xy}^{(m\mathcal{M})}}{\partial y} - \frac{\partial\wp_{xz}^{(m\mathcal{M})}}{\partial z} - \frac{1}{c^4\partial\delta_{xyz}}/\partial\tau$$

$$\dagger = \mathrm{e}\wp + \frac{1}{c}(\mu,\nu) - \frac{1}{2}grad\ (\xi g) + (g\nabla)\mu\nu + \frac{1}{c(\xi g)} + (g\Delta) - \frac{1}{2grad}\ (\nu,\mu) + (\mathcal{M}m\nabla)\hbar - 1/c(\nu,\mu) + (\mathcal{M}m\nabla)\hbar$$

$$curl\ \mathfrak{H} = \frac{1}{c\left(\mathcal{E}\mathcal{E} \pm \sum \mathbb{Q}_g\rho_g + \sum \mathbb{Q}_i\rho_i\right)} div\ \mathcal{E}\mathcal{E} = \sum \mathbb{Q}_g\rho_g + \sum \mathbb{Q}_i\rho_i$$

$$\wp = (\mathcal{E} - 1)\ (\mathcal{E}\mathrm{e} + \frac{1}{c(\mu,\nu)})$$

$$\mathcal{M}m = (\mu\nu - 1)\ (\lambda\hbar - \frac{1}{c(\mu,\nu)})$$

$$\imath = (\mathcal{L}\lambda - 1)\ (\mathcal{E}\mathrm{e} + \frac{1}{c(\mu,\nu)})$$

3. Variable de Fizeau.

$$\mathcal{V}_x = \mathcal{V}_o + \varrho_i\left(1 - \frac{1}{\eta^2}\right)$$

$$\wp_y = (\mathcal{E}\mathcal{E} - 1)\left(1 - \frac{\alpha\varrho_x}{c}\right)\zeta$$

$$\mathcal{M}m_y = (\mu\nu - 1)\left(1 - \frac{\alpha\varrho_x}{c}\right)\zeta$$

$$\mathbb{C}\alpha = \mathcal{V}_e - \frac{\mathcal{V}(\mathcal{E}\mathcal{E} - 1)\alpha\varrho_x}{c} + \varrho_x(\mathcal{E}\mathcal{E} - 1)$$

$$\mathbb{C} = \mathcal{V}\mu\alpha\frac{(\mu\nu - 1)\varrho_x}{c} + \varrho_x(\mu\nu - 1)\alpha$$

$$\mathcal{V} = \mathcal{V}_o + \varrho_x\left(1 - \frac{1}{\eta^2}\right)$$

4. Variable de la velocidad de la luz.

$$\Delta\varphi - \frac{\frac{1}{c^2\partial^2\varphi}}{\partial\tau^2} = \frac{1}{\underset{c\ \leftrightarrow}{\mathrm{e}}} = curl\ \hbar = c\ curl\ (curl\ \mathrm{e}) = -c\left(\Delta\Gamma\varepsilon + grad\ (div\ \mathrm{e})\right) = \mathbb{C}\alpha\mathrm{e}$$

5. Principio de Relatividad.

$$\frac{\varpi_{\mu\nu} d^2 \tau_{\mu\nu}}{dt^2} = \mathbb{R}^{\mu\nu} \mathfrak{R}_{\mu\nu} / \mathcal{R}\mathrm{e}$$

$$\mathrm{X}_{\mu}^{\triangleq} \mathrm{X}_{\nu}^{\triangleq} = \mathrm{X}_{\mu}^{\overset{\mathrm{m}}{=}} \mathrm{X}_{\nu}^{\overset{\mathrm{m}}{=}}$$

$$\frac{dx_{\mu}^{\triangleq}}{dt_{\nu}^{\triangleq}} = dx_{\mu}^{\overset{\mathrm{m}}{=}} dt_{\nu}^{\overset{\mathrm{m}}{=}} - \mathcal{V}$$

$$\frac{d^2 x_{\mu}^{\triangleq}}{d^2 t_{\nu}^{\triangleq}} = d^2 x_{\mu}^{\overset{\mathrm{m}}{=}} d^2 t_{\nu}^{\overset{\mathrm{m}}{=}} - \mathcal{V}$$

$$V = \left\| \mathcal{V}_o + \varrho_i \left(1 - \frac{1}{\eta^2}\right) \right\| - \varrho_i = \mathcal{V}_o - \frac{\varrho_i}{\eta^2}$$

6. Transformaciones de Lorentz.

$$\lambda^2(\chi^2 + y^2 + z^2 - c^2 t^2) = \left(\chi^{\overset{\mathrm{def}}{=} 2} + y^{\overset{\mathrm{def}}{=} 2} + z^{\overset{\mathrm{def}}{=} 2} - c^{\overset{\mathrm{def}}{=} 2} t^{\overset{\mathrm{def}}{=} 2}\right) = \alpha_{1\mu 1\nu}^2 + \alpha_{2\mu 2\nu}^2 + \alpha_{3\mu 3\nu}^2 + \alpha_{\eta\mu\eta\nu}^2 = 1$$

$$\alpha_\eta = \frac{1}{\sqrt{1-\beta^2}} + \frac{i\beta}{\sqrt{1-\beta^2}} - \frac{-1}{\sqrt{1-\beta^2}} + \frac{-i\beta}{\sqrt{1-\beta^2}}$$

$$x' = x + \frac{i\beta_{\mu\nu}}{\sqrt{1-\beta^2}} + \mu\nu' = \frac{\mu\nu - i\beta_{xyz}}{\sqrt{1-\beta^2}} + x' = \frac{x - \beta_{\mathrm{ct}}}{\sqrt{1-\beta^2}} = \mathfrak{t} = \mathfrak{t} - \frac{\beta}{c} x / \sqrt{1-\beta^2}$$

$$x' = x - \frac{\mathfrak{v}\mathfrak{t}}{\sqrt{1 - \frac{\mathfrak{v}^2}{\mathfrak{c}^2}}}$$

$$\mu\nu' = \mathfrak{t} - \frac{\frac{\mathfrak{v}}{\mathfrak{c}^2} x}{\sqrt{1 - \frac{\mathfrak{v}^2}{\mathfrak{c}^2}}}$$

$$\frac{\mathfrak{X}^2}{\sqrt[\mathfrak{R}]{1 - \frac{\mathfrak{v}^2}{\mathfrak{c}^2}}} + \frac{\mathfrak{Y}^2}{\mathfrak{R}^2} + \frac{\mathfrak{Z}^2}{\mathfrak{R}^2} = 1$$

$$\mathfrak{B}_{\mathfrak{v}} = \mathfrak{B} / \sqrt{1 - \frac{\mathfrak{q}^2}{\mathfrak{c}^2}}$$

$$\Delta \mathrm{t} = \Delta \mathrm{t}' \frac{\aleph}{\sqrt{1 - \frac{\mathfrak{v}^2}{\mathfrak{c}^2}}}$$

$$\sum \Delta \mathrm{t} = \sum \Delta \mathrm{t}' \frac{\aleph}{\sqrt{1 - \frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}$$

$$\sum \Delta \mathrm{t}' = \sum \Delta \mathrm{t} . \sqrt{1 - \frac{\mathfrak{q}^2}{\mathfrak{c}^2}}$$

$$x' = \lambda(\mu v)x - vt/\sqrt{1-\frac{v^2}{c^2}}$$

$$t = \lambda(\mu v)t - v/c^2x/\sqrt{1-\frac{v^2}{c^2}}$$

7. Teorema de Velocidades.

$$\mathfrak{q}_{\mathfrak{x}} = \mathfrak{q}'_{\mathfrak{x}} + \frac{\mathfrak{v}}{1} + \mathfrak{q}'_{\mathfrak{x}}\mathfrak{v}/\mathfrak{c}^2$$

$$\mathfrak{q}_{\mathfrak{y}} = \frac{\sqrt{1-\frac{v^2}{c^2}}\,\mathfrak{q}'_{\mathfrak{y}}}{1} + \mathfrak{q}'_{\mathfrak{x}}\mu v/\mathfrak{c}^2$$

$$\mathfrak{q}_{\mathfrak{z}} = \frac{\sqrt{1-\frac{v^2}{c^2}}\,\mathfrak{q}'_{\mathfrak{z}}}{1} + \mathfrak{q}'_{\mathfrak{x}}\mu v/\mathfrak{c}^2$$

$$\mathfrak{q}^2 = \mathfrak{q}'^2 + \mathfrak{v}^2 + 2\mathfrak{q}'^{\mu v}cos\theta' - (\frac{\mathfrak{q}'^{\mu v}}{\mathfrak{c}}sin\theta'\frac{)^2}{1} + \frac{\mathfrak{q}'^{\mu v}}{\mathfrak{c}^2}cos\theta')^2$$

$$\frac{\mathfrak{q}^2}{\mathfrak{c}^2} = 1 + \frac{\mathfrak{q}'^{\mu v}}{\mathfrak{c}}cos\theta')^2 - \frac{\left(1-\frac{\mathfrak{q}'^2}{\mathfrak{c}^2}\right)\left(1-\frac{\mathfrak{v}^2}{\mathfrak{c}^2}\right)}{(1+\frac{\mathfrak{q}'^{\mu v}}{\mathfrak{c}}cos\theta')^2}$$

$$\mathfrak{q} = \mathfrak{q}' + \frac{\mathfrak{v}}{1} + \frac{\mathfrak{q}'^{\mu v}}{\mathfrak{c}^2}$$

$$\mathcal{V} = \mathcal{V}_o + \frac{\mathfrak{q}}{1+\frac{\mathcal{V}_o\mathfrak{q}}{\mathfrak{c}^2}}$$

$$\mathcal{V} = \mathcal{V}_o + q\left(1-\frac{1}{\mathfrak{n}^2}\right)$$

$$sin\omega\left(t - lx + my + \frac{nz}{c}\right)$$

$$sin\omega'\left(t' - l'x' + m'y' + \frac{n'z'}{c'}\right)$$

$$\omega' = \omega 1 - \frac{l\frac{lv}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$l' = l - \frac{\frac{v}{c}}{1} - l\frac{lv}{c}$$

$$m' = \frac{m}{1-\frac{lv}{c}}\sqrt{1-\frac{v^2}{c^2}}$$

$$n' = \frac{n}{1-\frac{lv}{c}}\sqrt{1-\frac{v^2}{c^2}}$$

$$\frac{\partial}{\partial x} = b\left(\frac{\partial}{\partial x} - \frac{v}{c^2\frac{\partial}{\partial t'}}\right) + \frac{\partial}{\partial y} = b\left(\frac{\partial}{\partial y} - \frac{v}{c^2\frac{\partial}{\partial t'}}\right) + \frac{\partial}{\partial z} = b\left(\frac{\partial}{\partial z} - \frac{v}{c^2\frac{\partial}{\partial t'}}\right)$$

$$\rho' = \mathfrak{b}(1-\frac{\mathfrak{v}\mathfrak{q}_{\mathfrak{x}}}{\mathfrak{c}^2})\rho$$

$$q'_x = \mathfrak{q}_{\mathfrak{x}} - \frac{\mathfrak{v}}{1-\frac{\mathfrak{q}_{\mathfrak{x}}\mathfrak{v}}{\mathfrak{c}^2}}$$

$$q'_{\mathfrak{y}} = \mathfrak{q}_{\mathfrak{y}}/\mathfrak{b}\left(1-\frac{\mathfrak{q}_{\mathfrak{x}}\mathfrak{v}}{\mathfrak{c}^2}\right)$$

$$q'_{\mathfrak{z}} = \mathfrak{q}_{\mathfrak{z}}/\mathfrak{b}\left(1-\frac{\mathfrak{q}_{\mathfrak{x}}\mathfrak{v}}{\mathfrak{c}^2}\right)$$

$$\rho = \rho'/\sqrt{1-\frac{v^2}{c^2}}$$

$$d\tau_0 = d\tau' = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\rho_0 = \rho/\sqrt{1-\frac{\mathfrak{q}^2}{c^2}}$$

$$\phi = \omega\left(\mathfrak{t} - \mathfrak{l}\mathfrak{x} + \mathfrak{m}\mathfrak{y} + \frac{\mathfrak{n}\mathfrak{z}}{\mathfrak{c}}\right)$$

$$\phi' = \omega'\left(\mathfrak{t}' - \mathfrak{l}'\mathfrak{x}' + \mathfrak{m}'\mathfrak{y}' + \frac{\mathfrak{n}'\mathfrak{z}'}{\mathfrak{c}'}\right) sin\phi$$

$$\mathfrak{A}' = \mathfrak{A}1 - \frac{\frac{\mathfrak{v}}{\mathfrak{c}}cos\varphi}{\sqrt{1-\frac{v^2}{c^2}}}$$

8. Ecuaciones de Movimiento.

$$\mathfrak{d}\mathfrak{q}' = \frac{\mathfrak{d}\mathfrak{q}}{1} - \frac{\mathfrak{q}\mathfrak{v}}{\mathfrak{c}^2} + \frac{q\frac{v}{\mathfrak{c}^2}\mathfrak{d}\mathfrak{q}}{(1-\mathfrak{q}\mathfrak{v}/\mathfrak{c}^2)^2} = \mathfrak{d}\mathfrak{q}/(1-\mathfrak{q}\mathfrak{v}/\mathfrak{c}^2)^2$$

$$\mathfrak{d}\mathfrak{t} = \mathfrak{d}\mathfrak{t}' + \frac{\frac{\mathfrak{v}}{\mathfrak{c}^2\mathfrak{d}\mathfrak{x}'}}{\sqrt{1-\frac{v^2}{c^2}}} = \mathfrak{d}\mathfrak{t}'/\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}$$

$$\mathfrak{m}.\frac{\mathfrak{d}\mathfrak{q}}{\mathfrak{d}\mathfrak{t}}/(\sqrt{1-\frac{\mathfrak{q}^2}{c^2}})^{\frac{3}{2}} = \varepsilon\mathfrak{e}_{\mathfrak{x}}\mathcal{E}_{\mathfrak{P}}\mathfrak{E}_{\mathfrak{v}}^{\mathfrak{u}}$$

$$\frac{\mathfrak{d}\mathfrak{q}}{\mathfrak{d}\mathfrak{t}}/(\sqrt{1-\frac{\mathfrak{q}^2}{c^2}})^{\frac{3}{2}} = \frac{d}{dt\left\|\frac{\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right\|} = \frac{d}{dt\left|\frac{\mathfrak{m}\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right|} = \mathfrak{f}_{xyz}$$

$$\frac{d}{dt\,\langle\frac{\mathfrak{m}\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\rangle} = \mathfrak{k}\kappa$$

$$\mathfrak{k}\kappa\mathfrak{q} = \mathfrak{q}.\frac{d}{dt}\left|\frac{\mathfrak{m}\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right| = \frac{d}{dt\left|\frac{\mathfrak{m}\mathcal{q}^2}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right|} - \frac{\mathfrak{m}\mathfrak{q}\mathcal{q}}{\sqrt{\mathfrak{J}}} = \frac{d}{dt\left|\frac{\mathfrak{m}\mathcal{q}^2}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right|} + \mathfrak{m}\mathfrak{c}^2\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}$$

$$\mathfrak{k}\kappa\mathfrak{q} = d/dt\left|\frac{\mathfrak{m}\mathfrak{c}^2}{\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}}\right|$$

$$\mathcal{E}_{\stackrel{\text{def}}{=}} = \mathfrak{m}\mathfrak{c}^2/\sqrt{1-\frac{\mathfrak{q}^2}{\mathfrak{c}^2}}$$

$$\mathcal{E}_{\stackrel{\text{def}}{=}} = \mathfrak{m}\mathfrak{c}^2 + \mathcal{M}/2\mathfrak{q}^2$$

9. Energía Inercial.

$$\mathrm{x} = \mathrm{ct} + \alpha + \mathfrak{l}'/\mathfrak{b}\left(1+\frac{\mathfrak{v}}{\mathrm{c}}\right)$$

$$\mathfrak{l}_1 = \mathfrak{l}'\sqrt{1-\frac{\frac{v}{c}}{1}-\frac{v}{c}}$$

$$\mathfrak{l}_2 = \mathfrak{l}'\sqrt{1+\frac{\frac{v}{c}}{1}-\frac{v}{c}}$$

$$\mathfrak{A}_1 = \mathfrak{A}'\sqrt{1+\frac{\frac{v}{c}}{1}-\frac{v}{c}}$$

$$\mathfrak{A}_2 = \mathfrak{A}'\sqrt{1-\frac{\frac{v}{c}}{1}+\frac{v}{c}}$$

$$\eta_1 = \frac{\mathfrak{f}\mathfrak{t}_1 1}{2\mathrm{A}_1^2} = \frac{1}{2\mathfrak{f}\mathfrak{t}'}A'^2\sqrt{1+\frac{\frac{v}{c}}{1}-\frac{v}{c}}$$

$$\eta_2 = \frac{\mathfrak{f}\mathfrak{t}_2 1}{2\mathrm{A}_2^2} = \frac{1}{2\mathfrak{f}\mathfrak{t}'}A'^2\sqrt{1-\frac{\frac{v}{c}}{1}+\frac{v}{c}}$$

$$\eta_1 + \eta_2 = 2\eta'/\sqrt{1 - \frac{v^2}{c^2}}$$

$$\mathfrak{l}_1 = \frac{\mathfrak{k}\mathfrak{t}_1 1}{2\mathfrak{c}\mathfrak{A}_1^2} = \frac{1}{2\mathfrak{c}\mathfrak{k}\mathfrak{t}'}A'^2\sqrt{1 + \frac{\frac{v}{c}}{1} - \frac{v}{c}}$$

$$\mathfrak{l}_2 = \frac{\mathfrak{k}\mathfrak{t}_2 1}{2\mathfrak{c}\mathfrak{A}_2^2} = \frac{1}{2\mathfrak{c}\mathfrak{k}\mathfrak{t}'}A'^2\sqrt{1 - \frac{\frac{v}{c}}{1} + \frac{v}{c}}$$

$$\mathfrak{l}_1 - \mathfrak{l}_2 = 2\eta'/\,c^2.\mathcal{V}/\sqrt{1 - \frac{v^2}{c^2}}$$

$$\mathcal{E}_{\underline{\underline{\text{def}}}} = \mathfrak{m}\mathfrak{c}^2/\sqrt{1 - \frac{v^2}{c^2}}$$

$$\Delta\mathcal{E}_{\underline{\underline{\text{def}}}} = \Delta\mathcal{E}_{\underline{\underline{\text{def}}}}'/\sqrt{1 - \frac{v^2}{c^2}}$$

$$(\mathcal{E}_{\underline{\underline{\text{def}}}} + \Delta\mathcal{E}_{\underline{\underline{\text{def}}}}) = [\![\mathcal{M} + \Delta\mathcal{E}_{\underline{\underline{\text{def}}}}'/c^2]\!]/\sqrt{1 - \frac{v^2}{c^2}}$$

$$\beta = \frac{\mathcal{M}\mathfrak{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta\beta = (\mathfrak{l}_1 - \mathfrak{l}_2) = \frac{\Delta\mathcal{E}'_{\mathfrak{v}}}{c^2}/\sqrt{1 - \frac{v^2}{c^2}}$$

$$(\beta + \Delta\beta) = [\mathcal{M} + \Delta\mathcal{E}_{\underline{\underline{\text{def}}}}'/c^2]\mathfrak{v}/\sqrt{1 - \frac{v^2}{c^2}}$$

10. Variable de cuatro dimensiones de Minkowski:

$$A'_{xyz} = A_{xyz} + i\beta A_{\mu\nu}/\sqrt{1 - \beta^2}$$

$$A'_{\mu\nu} = A_{\mu\nu} - i\beta A_{xyz}/\sqrt{1 - \beta^2}$$

$$\mathfrak{X}'_{\mu\nu} = \sum \alpha_{\mu\nu}\,\mathfrak{X}_{\mu\nu}$$

$$\mathfrak{A}'_{\mu\nu} = \sum_{\mu\nu} \alpha_{\mu\nu}\mathfrak{A}_{\mu\nu}$$

$$\mathfrak{T}'_{\mu\nu} = \sum_{\sigma\tau} \alpha_{\mu\sigma}\mathfrak{A}_{\nu\tau}\,\mathfrak{T}_{\mu\nu}$$

$$\mathfrak{T}'_{\mathfrak{r}_1 \ldots \mathfrak{r}_\mathfrak{n}} = \sum_{\sigma_1 \ldots \sigma_\eta} \alpha_{\mathfrak{r}_1\sigma_1} \alpha_{\mathfrak{r}_2\sigma_2} \ldots \alpha_{\mathfrak{r}_\eta\sigma_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

11. Variable tensorial.

$$\left(\mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}\right) \pm \left(\mathfrak{U}_{\sigma_1 \ldots \sigma_\eta}\right) = \left(\mathfrak{T}_{\sigma_1 \ldots \sigma_\eta} \pm \mathfrak{U}_{\sigma_1 \ldots \sigma_\eta}\right)$$

$$\mathfrak{T}_{\sigma_1 \ldots \sigma_\eta} \mathfrak{U}_{\tau_1 \ldots \tau_\eta}$$

$$\mathfrak{T}'_{\delta_1 \ldots \delta_\eta} = \sum_{\sigma_1 \ldots \sigma_\eta} \alpha_{\delta_1\sigma_1} \alpha_{\delta_2\sigma_2} \ldots \alpha_{\delta_\eta\sigma_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}$$

$$\mathfrak{U}'_{\mathfrak{t}_1 \ldots \mathfrak{t}_\eta} = \sum_{\tau_1 \ldots \tau_\eta} \alpha_{\mathfrak{t}_1\tau_1} \alpha_{\mathfrak{t}_2\tau_2} \ldots \alpha_{\mathfrak{t}_\eta\tau_\eta} \mathfrak{U}_{\tau_1 \ldots \tau_\eta}$$

$$\mathfrak{T}'_{\delta_1 \ldots \delta_\eta} \mathfrak{U}'_{\mathfrak{t}_1 \ldots \mathfrak{t}_\eta} = \sum_{\sigma_1 \ldots \sigma_\eta \tau_1 \ldots \tau_\eta} \alpha_{\delta_1\sigma_1} \ldots \alpha_{\delta_\eta\sigma_\eta} \alpha_{\mathfrak{t}_1\tau_1} \ldots \alpha_{\mathfrak{t}_\eta\tau_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta} \mathfrak{U}_{\tau_1 \ldots \tau_\eta}$$

$$\left(\mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}\right)\left(\mathfrak{U}_{\tau_1 \ldots \tau_\eta}\right) = \left(\mathfrak{T}_{\sigma_1 \ldots \sigma_\eta} \mathfrak{U}_{\tau_1 \ldots \tau_\eta}\right)$$

$$\sum_{\sigma_1 \ldots \sigma_\eta} \mathfrak{U}_{\sigma_1 \ldots \sigma_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta} = \mathfrak{V}_{\sigma_{\eta+1} \ldots \sigma_\mathfrak{m}}$$

$$\mathfrak{U}'_{\delta_1 \ldots \delta_\eta} = \sum_{\tau_1 \ldots \tau_\eta} \alpha_{\delta_1\tau_1} \alpha_{\delta_2\tau_2} \ldots \alpha_{\delta_\eta\tau_\eta} \mathfrak{U}'_{\tau_1 \ldots \tau_\eta}$$

$$\mathfrak{T}'_{\sigma_1 \ldots \sigma_\eta} = \sum_{\sigma_1 \ldots \sigma_\eta} \alpha_{\delta_1\sigma_1} \alpha_{\delta_2\sigma_2} \ldots \alpha_{\delta_\eta\sigma_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}$$

$$\mathfrak{V}'_{\delta_{\eta+1} \ldots \delta_\mathfrak{m}} = \sum_{\delta_1 \ldots \delta_\eta \tau_1 \ldots \tau_\eta \sigma_1 \ldots \sigma_\eta} \alpha_{\delta_1\tau_1} \ldots \alpha_{\delta_\eta\tau_\eta} \alpha_{\delta_1\sigma_1} \ldots \alpha_{\delta_\eta\sigma_\eta} \mathfrak{U}_{\tau_1 \ldots \tau_\eta} \mathfrak{T}_{\sigma_1 \ldots \sigma_\eta}$$

$$\mathfrak{V}'_{\delta_{\eta+1} \ldots \delta_\mathfrak{m}} = \sum_{\sigma_{\eta+1} \ldots \sigma_\mathfrak{m}} \alpha_{\delta_{\eta+1}\sigma_{\eta+1}} \ldots \alpha_{\delta_\mathfrak{m}\sigma_\mathfrak{m}} \mathfrak{U}'_{\sigma_{\eta+1} \ldots \sigma_\mathfrak{m}}$$

$$\left(\mathfrak{U}'_{\sigma_1 \ldots \sigma_\mathfrak{n}}\right)\left(\mathfrak{T}'_{\sigma_1 \ldots \sigma_\mathfrak{n}}\right) = \left(\mathfrak{V}'_{\sigma_{\mathfrak{n}+1} \ldots \sigma_\mathfrak{m}}\right)$$

$$\left(\mathfrak{T}_{\mu\nu}\right)\left(\mathfrak{U}_{\mu\nu}\right) = (\mathfrak{V})$$

$$\mathfrak{V} = \frac{\dfrac{\sum_\mathfrak{u} \mathfrak{v} \dfrac{\sum_{\mu\nu}(\mathfrak{T}_{\mu\nu} \mathfrak{U}_{\mu\nu} \mathfrak{S}_{\mu\nu} \mathfrak{T}^2_{\mu\nu} \mathfrak{U}^2_{\mu\nu} \mathfrak{S}^2_{\mu\nu})^\rho 1}{2} \sum_{\mathfrak{i}\mathfrak{k}} \mathfrak{l}\mathfrak{m}\, 1}{6} 1}{24} (\mathfrak{V}_{\mathfrak{i}\mathfrak{k}\mathfrak{l}\mathfrak{m}})(\mathfrak{E}_{\mathfrak{i}\mathfrak{k}\mathfrak{l}\mathfrak{m}})$$

$$\mathfrak{x}_{\mu\nu} = \sum_{\mu\nu} \alpha_{\mu\nu} \mathfrak{x}'_{\mu\nu}$$

$$\frac{\partial}{\mathfrak{X}'_{\mu\nu}} = \sum_{\mu\nu} \alpha_{\mu\nu} \frac{\partial}{\mathfrak{X}'_{\nu\mu}}$$

$$\left(\frac{\partial}{\partial_{\chi_\tau}}\right)(\mathfrak{T}_{\sigma_1\ldots\sigma_n}) = \|\partial\mathfrak{T}_{\sigma_1\ldots\sigma_n}/\partial_{\chi_\tau}\|$$

$$\left(\frac{\partial}{\partial_{\chi_{\sigma_{\mu\nu}}}}\right)(\mathfrak{T}_{\sigma_1\ldots\sigma_n}) = \sum_{\sigma_{\mu\nu}} \frac{\partial}{\partial_{\chi_{\sigma_{\mu\nu}}}}\left(\mathfrak{T}_{\sigma_{\mu\nu}\ldots\sigma_{\mu\nu-1}\sigma_{\nu\mu+1}\ldots\sigma_{\nu\mu}}\right)$$

$$\left(\frac{\partial}{\partial_{\mu\nu}}\right)\mathfrak{T} = \left(\frac{\partial\mathfrak{T}}{\partial_{\chi_{\mu\nu}}}\right)$$

$$\left(\frac{\partial}{\partial_{\chi_{\mu\nu}}}\right)(\mathfrak{T}_{\mu\nu}) = \sum_{\mu\nu}\left(\frac{\partial\mathfrak{T}_{\mu\nu}}{\partial_{\chi_{\mu\nu}}}\right)$$

$$\left(\frac{\partial}{\partial \mathrm{X}_\tau}\right)\left|\left(\frac{\partial}{\partial \mathrm{X}_\tau}\right)(\mathfrak{T}_{\sigma_1\ldots\sigma_n})\right| = \left(\frac{\partial}{\partial \mathrm{X}_\tau}\right)\left(\frac{\partial\mathfrak{T}_{\sigma_1\ldots\sigma_n}}{\partial \mathrm{X}_\tau}\right) = \left(\frac{\sum_\tau \partial^2\,\mathfrak{T}_{\sigma_1\ldots\sigma_n}}{\partial \mathrm{X}_\tau^2}\right)(\boxdot\,\mathfrak{T}_{\sigma_1\ldots\sigma_n})$$

12. Estado de vacío.

$$\frac{1}{2\left\|\frac{1}{v} - V + \frac{1}{v} + V\right\|\partial\tau}\partial \mathrm{t} = \frac{\partial\tau}{\partial \mathrm{X}^\Gamma} + \frac{1}{v} - V\frac{\partial\tau}{\partial \mathrm{t}}, \frac{\partial\tau}{\partial \mathrm{X}^\Gamma} + \frac{v}{\sqrt{V^2 - v^2}\frac{\partial\tau}{\partial \mathrm{t}}} = 0$$

$$\tau = \alpha\left(\mathrm{t} - \frac{\mathfrak{v}}{V^2} - v^2 x'\right)$$

$$\xi = V\tau$$

$$\xi = \alpha V\left(t - \frac{v}{V^2} - v^2 x'\right)$$

$$x'/v - V = t$$

$$\xi = \alpha.V^2/\sqrt{\frac{V^2 - v^2}{x'}}$$

$$\eta = V_\tau = \alpha V\left(t - \frac{v}{V^2} - v^2\, x'\right)$$

$$\beta = \frac{1}{1} - (v/V)^2$$

$$\varphi X^2/(\sqrt{1-(v/V)^2})^2 + y^2 + z^2 = \mathcal{R}^2$$

$$\tau = 1/\sqrt{1-(v/V)^2}|t - v/V^2 x|$$

$$\tau = t\sqrt{1-(v/V)^2} = t - \langle 1 - \sqrt{1-(v/V)^2}\rangle\, t$$

$$x = \omega_\xi + \frac{v}{1} + \frac{v\omega_\xi}{V^2 t}$$

$$y = \frac{\sqrt{1-(v/V)^2}}{1} + \frac{v\omega_\xi}{V^2}\omega_{\eta^\dagger}$$

$$\mathcal{U}^2 = (\frac{dx}{dt)^2} + \frac{dy}{dt)^2}$$

$$\omega^2 = \omega_\xi^2 + \omega_\eta^2$$

$$\alpha = arctg\omega_y/\omega_x$$

$$\mathcal{U} = \frac{\sqrt{v^2 + \omega^2 + 2v\omega\cos\alpha - (v\omega\sin\frac{a}{V})^2}}{1 + v\omega\cos\alpha\,/V^2}$$

$$U = v + \frac{\omega}{1} + v\omega/V^2$$

$$U = V.2V - \kappa - \frac{\lambda}{2V} - \kappa - \lambda + \frac{\kappa\lambda}{V} \lll V$$

$$U = V + \frac{\omega}{1} + \frac{\omega}{v} = V$$

$$v + \frac{\omega}{1} + \frac{v\omega}{V^2}$$

$$\frac{\frac{1}{V\partial X}}{\partial\tau} = \partial\beta\left(N - \frac{v}{V}Y\right) - \frac{\partial\beta\left(M + \frac{v}{V}Z\right)1}{V}\frac{\partial\beta\left(Y - \frac{v}{V}N\right)}{\partial\tau} = \frac{\partial L}{\partial\zeta} - \frac{\partial\beta\left(N - \frac{v}{V}Y\right)}{\partial\zeta}$$

$$\beta = \frac{1}{\sqrt{1 - (v/V)^2}}$$

$$(\beta\xi - \frac{\alpha\beta v}{V}\xi)^2 + (\eta - \frac{b\beta v}{V\xi)^2} + (\zeta - \frac{c\beta v}{V)^2} = \mathcal{R}^2$$

$$S'/S = \sqrt{1 - (\frac{\frac{v}{V)^2}}{1} - v/Vcos\varphi}$$

$$\frac{E'}{E} = \frac{A'^2}{16\pi}S'\frac{A^2}{16\pi}S = 1\ -\frac{v}{V}cos\varphi/\sqrt{1 - (v/V)^2}$$

$$\frac{E'}{E} = \sqrt{1 - \frac{\frac{v}{V}}{1} + \frac{v}{V}}$$

$$A^t = A.1 - \frac{\frac{v}{V}\cos\varphi}{\sqrt{1 - (v/V)^2}}$$

$$\cos\varphi^t = \cos\varphi - \frac{\frac{v}{V}}{1} - \frac{v}{V}\cos\varphi$$

$$v^t = v.1 - \frac{v}{V}\cos\frac{\varphi}{\sqrt{1 - (v/V)^2}}$$

$$A^m = A^n.1 + \frac{v}{V}\cos\varphi^n/\sqrt{1 - (v/V)^2} = A.1 - \frac{2.v}{V}\cos\varphi + \left(\frac{\frac{v}{V)^2}}{1} - \left(\frac{v}{V)^2}\right)\right)$$

$$\cos\varphi^m = \cos\varphi^n + \frac{\frac{v}{V}}{1} + \frac{v}{V}\cos\varphi^n = -(1 + \left(\frac{v}{V)^2}\right)\cos\varphi - \frac{2.\frac{v}{V}}{1} - 2.\frac{v}{V}\cos\varphi + (\frac{v}{V})^2$$

$$v^m = v^n.1 + \frac{v}{V}\cos\frac{\varphi^n}{\sqrt{1-(v/V)^2}} = v.1 - 2.\frac{v}{V}\cos\varphi + (\frac{\frac{v}{V})^2}{1} - \left(\frac{v}{V})^2\right)$$

$$P = 2.\frac{A^2}{16\pi}(\cos\varphi - \frac{\frac{v}{V})^2}{1} - (\frac{v}{V})^2/\cos\varphi^2$$

$$\frac{1}{V\left(\mu_\xi\eta\zeta\rho^\dagger + \frac{\partial X^\dagger}{\partial\tau}\right)} = \frac{\partial N^\dagger}{\partial\eta} - \frac{\partial M^\dagger}{\partial\zeta}$$

$$\frac{\mu d^2x}{dt^2} = \epsilon(X,Y,Z)$$

$$\frac{\mu d^2\xi\eta\zeta}{d\tau^2} = \frac{\epsilon}{\mu} - \frac{\frac{1}{\beta^3(X^\dagger,Y^\dagger,Z^\dagger)}}{\frac{v}{V(M,N)}}$$

$$\mathcal{L}m = \mathfrak{u}/(\sqrt{1-v/V)^2})^3$$

$$\mathcal{T}m = \frac{\mathfrak{u}}{1} - (v/V)^2$$

$$\mathcal{W} = \int \epsilon X dx = \int_0^v \beta^3\, \nu d\mu = \mu\nu V^2(1/\sqrt{1-(v/V)^2} - 1)$$

$$P = \int X dx = \frac{\mu\nu}{\epsilon}.V^2\left(\frac{1}{\sqrt{1-(\frac{v}{V})^2}} - 1\right)$$

$$-\frac{d^2y}{dt^2} = \frac{\mathfrak{v}^2}{\mathfrak{R}} = \frac{\epsilon}{\mu}.\frac{v}{V}.N.\sqrt{1-(\frac{v}{V})^2}$$

$$\mathfrak{R} = V^2.\frac{\mu}{\epsilon}.\frac{\frac{v}{V}}{\sqrt{1-(\frac{v}{V})^2}}.1/N$$

13. Estado Fundamental de Vacío.

$$\left(\frac{\partial}{\partial\mathfrak{X}_{\mu\nu}}\right)(\mathfrak{F}_{\mu\nu}) = (\mathfrak{T}_{\mu\nu})\left(\mathfrak{F}^\star{}_{\mu\nu}\right) = 0$$

$$\frac{1}{2(\mathfrak{F}_{\mu\nu})}(\mathfrak{F}_{\mu\nu}) = \mathfrak{h}^2 - \mathfrak{e}^2$$

$$\frac{1}{2\mathfrak{i}(\mathfrak{F}_{\mu\nu})\left(\mathfrak{F}^\star{}_{\mu\nu}\right)} = (\mathfrak{e}\mathfrak{h})$$

$$-(\mathfrak{F}_{\mu\nu})(\mathfrak{F}_{\mu\nu}) = \rho^2\left(1 - \frac{\varrho^2}{\mathfrak{c}^2}\right) = \mathfrak{p}_0^2$$

$$(\mathfrak{F}_{\mu\nu})\,(\mathfrak{I}_{\mu\nu}) = (\mathfrak{K}_{\mu\nu})$$

$$(\mathfrak{K}_{\mu\nu}) = -\left(\frac{\partial}{\partial \mathfrak{X}_{\mu\nu}}\right)(\mathfrak{T}_{\mu\nu}) = 1/2\|(\mathfrak{F}_{u\sigma})(\mathfrak{F}_{\nu\sigma}) - (\mathfrak{F}^{*}{}_{u\sigma})(\mathfrak{F}^{*}{}_{\nu\sigma})\|$$

14. Movimiento Isotrópico.

$$(\mathfrak{S}_{\mu\nu}) = \left(\frac{dx_{\mu\nu}}{\sqrt{-\sum dx_{\sigma}^{2}}}\right)$$

$$\left(\mathfrak{F}_{\mu\nu}^{(e)}\right) = (\mathfrak{F}_{\mu\nu})(\mathfrak{S}_{\mu\nu})$$

$$\mathfrak{F}_{\eta}^{(e)} = \frac{\frac{1}{\sqrt{1-\frac{\varrho^{2}}{c^{2}}\mathfrak{e}_{x}}}\mathfrak{q}_{x}}{\mathfrak{c}}.\mathfrak{e}_{x} + \frac{\mathfrak{q}_{\mathfrak{y}}}{\mathfrak{c}}.\mathfrak{e}_{\mathfrak{y}} + \frac{\mathfrak{q}_{\mathfrak{z}}}{\mathfrak{c}}.\mathfrak{e}_{z}$$

$$\left(\mathfrak{F}_{\mu\nu}^{(\mathfrak{m})}\right) = -\mathfrak{i}\left(\mathfrak{F}^{*}{}_{\mu\nu}\right)(\mathfrak{S}_{\mu\nu})$$

$$\mathfrak{F}_{\eta}^{(\mathfrak{m})} = \mathfrak{i}/\sqrt{1-\frac{\varrho^{2}}{c^{2}}.\frac{\mathfrak{q}_{x}}{\mathfrak{c}}.\mathfrak{h}_{\mathfrak{x}} + \frac{\mathfrak{q}_{\mathfrak{y}}}{\mathfrak{c}}.\mathfrak{h}_{\mathfrak{y}}} + \frac{\mathfrak{q}_{\mathfrak{z}}}{\mathfrak{c}}.\mathfrak{h}_{\mathfrak{z}}$$

$$\mathfrak{J}_{\mathfrak{i}}^{\mathfrak{k}} = \frac{\frac{\rho_{0}}{\sqrt{1-\frac{\varrho^{2}}{c^{2}}\mathfrak{q}_{x}}}}{\mathfrak{c}} = \rho.\mathfrak{q}_{x}/\mathfrak{c}$$

$$\left(\frac{\partial}{\partial \mathfrak{X}_{\mu\nu}}\right)(\mathfrak{F}_{\mu\nu} + \mathfrak{P}_{\mu\nu}) = \frac{1}{\mathfrak{c}(\mathfrak{F}_{\mu\nu}^{\mathfrak{ij}})} + \mathfrak{p}_{0}(\mathfrak{S}_{\mu\nu}) - \left(\frac{\partial}{\partial \mathfrak{X}_{\mu\nu}}\right)\left(\mathfrak{F}^{*}{}_{\mu\nu} + \mathfrak{i}\mathfrak{W}_{\mu\nu}\right) = 0$$

$$(\mathfrak{S}_{\mu\nu})\left(\frac{\partial(\mathfrak{F}_{\mu\nu} + \mathfrak{P}_{\mu\nu})}{\partial \mathfrak{X}_{\mu\nu}}\right) = (\mathfrak{J}_{\mu}^{\mathfrak{k}})(\mathfrak{S}_{\mu\nu}) = \rho_{0}(\mathfrak{S}_{\mu\nu})(\mathfrak{S}_{\mu\nu}) = -\rho_{0}$$

$$\rho_{0} = -(\mathfrak{S}_{\mu\nu})\left(\partial(\mathfrak{F}_{\mu\nu} + \mathfrak{P}_{\mu\nu})/\partial \mathfrak{X}_{\mu\nu}\right)$$

15. Tensor de Energía.

$$(\mathfrak{T}_{\mu\nu}^{0}) = (\frac{1}{2(\mathfrak{F}_{\mu\sigma})(\mathfrak{F}_{\nu\sigma})} - \left(\mathfrak{F}^{\dagger}{}_{\mu\sigma}\right)(\mathfrak{F}^{*}{}_{\nu\sigma}))$$

$$(\mathfrak{T}_{\mu\nu}^{\mathfrak{e}}) = \frac{1}{\varepsilon} - 1\left((\mathfrak{T}_{\mu\sigma})(\mathfrak{T}_{\nu\sigma}) - \frac{1}{4(\delta_{\mu\sigma})(\delta_{\nu\sigma})(\mathfrak{T}_{\sigma\tau})(\mathfrak{T}_{\sigma\tau})}\right)$$

$$\mathfrak{f}^{0} = -\mathfrak{e}div\,\mathfrak{p}^{**} - \mathfrak{h}div\,\mathfrak{m}^{**} + \mathfrak{e}\left(\frac{1}{\mathfrak{c}}\left(\mathfrak{i},\frac{\mathfrak{q}}{\mathfrak{c}}\right) + \rho\right) + \left|\frac{\mathfrak{i}}{\mathfrak{c}} + \frac{\mathfrak{q}}{\mathfrak{c}}.\mathfrak{p},\mathfrak{h}\right| + \left\|\frac{\rho^{**}}{\mathfrak{c}} - rot\rho^{*},\mathfrak{h}\right\| - \left\|\frac{\mathfrak{m}^{**}}{\mathfrak{c}} - rot\mathfrak{m}^{*},\mathfrak{e}\right\|$$

$$\varphi^{0} = \mathfrak{e}(\mathfrak{i} + \rho\mathfrak{q}) + \mathfrak{c}\left(\mathfrak{e},\frac{\rho^{**}}{\mathfrak{c}} - rot\,\rho^{*}\right) + \mathfrak{c}\left(\mathfrak{h},\frac{\mathfrak{m}^{**}}{\mathfrak{c}} - rot\,\mathfrak{m}^{*}\right)$$

$$\omega = \sigma,\varphi,\phi(\mathfrak{e}^{*2} - \left(\frac{\mathfrak{q}}{\mathfrak{c}},\mathfrak{e}^{*})^{2}\right)\sqrt{1-\mathfrak{q}^{2}/\mathfrak{c}^{2}}$$

$$-\mathfrak{f}_x = \frac{\partial \rho}{\partial x} + \frac{\frac{\mu^x \lambda \mathfrak{q}_x}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial x} + \frac{\frac{\mu^x \lambda \mathfrak{q}_y}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial y} + \frac{\frac{\mu^x \lambda \mathfrak{q}_z}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial z} + \frac{\frac{\mu^x \lambda \mathfrak{q}_x}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial \mathrm{t}}$$

$$+ \mathfrak{q}_x \left(\frac{\partial}{\partial x} \left(\frac{\frac{\mu^x \lambda \mathfrak{q}_x}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\frac{\mu^x \lambda \mathfrak{q}_y}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\frac{\mu^x \lambda \mathfrak{q}_z}{\mathfrak{w}^2 \partial \mathfrak{q}_x}}{\partial z} \right) + \frac{\partial}{\partial \mathrm{t}} \left(\frac{\mu^x}{\mathfrak{w}^2} \right) \right) - \frac{\mathfrak{i}}{\mathfrak{c}} . \eta$$

$$= -\frac{\mathfrak{i}}{\mathfrak{c}} \frac{\partial \rho}{\partial \mathrm{t}} - \mathfrak{f}_x$$

$$= \frac{\partial \rho}{\partial x} + \frac{\mu^*}{1} - \frac{\mathfrak{q}^2}{\mathfrak{c}^2 \left(\frac{\lambda \mathfrak{q}_x \partial \mathfrak{q}_x}{\partial x} + \frac{\lambda \mathfrak{q}_y \partial \mathfrak{q}_x}{\partial y} + \frac{\lambda \mathfrak{q}_z \partial \mathfrak{q}_x}{\partial z} + \frac{\partial \mathfrak{q}_x}{\partial \mathrm{t}} \right)} + \lambda \mathfrak{q}_x \mathfrak{A} - \frac{1}{\mathfrak{c}^2} . \eta . \kappa_4$$

$$= -\frac{1}{\mathfrak{c}^2} \frac{\partial \rho}{\partial \mathrm{t}} . \kappa_4 + \mathcal{A}$$

16. Dinámica del punto de masa.

$$-\frac{\int \mathfrak{K}_{\mu\nu} dx_1 \ldots dx_4 = \int \sum \partial\, \mathfrak{T}_{\mu\nu}}{\partial \mathfrak{X}_{\mu\nu} dx_1} \ldots dx_4 = \frac{\partial \mathfrak{T}_{\mu\nu 4}}{\partial \mathfrak{X}_{\mu\nu 4} dx_1} \ldots dx_4$$

$$\int dx_4 \frac{\partial}{\partial x_4 (\int \mathfrak{T}_{\mu\nu 4}\ \int dx_1 dx_2 dx_3)}$$

$$-\int_{\overrightarrow{\mathfrak{K}_{\mu\nu 4}}} dx_4 = |\mathfrak{T}_{\mu\nu 4}| \left\| \mathfrak{G}^{\mathfrak{x}^{\dagger\ddagger}{}_4}_{\mathfrak{x}^{\dagger}{}_4} \right\| \overrightarrow{}_{\mathfrak{K}_{\mu\nu 4}} = \frac{d\mathfrak{T}_{\mu\nu 4}}{dx_4} \int \frac{\mathfrak{i}}{\mathfrak{c}} \varphi \mathfrak{f}_x \mathfrak{g}_x \eta$$

17. Ecuaciones einstenianas de campo.

$$\sum_{\mu\nu} \partial / \partial \mathcal{X}_{\mu\nu} (\sqrt{-\mathfrak{g} \gamma_{\sigma\mu} \mathcal{T}_{\mu\nu}}) = \frac{\frac{1}{2 \sum_{\mu\nu} \sqrt{-\mathfrak{g}} . \partial \gamma_{\mu\nu}}}{\partial \mathcal{X}_\sigma \mathcal{T}_{\mu\nu}} = \frac{1}{2 \sum_{\mu\nu} \frac{\partial}{\partial \mathcal{X}_{\mu\nu}} (\sqrt{-\mathfrak{g} \mathfrak{g}_{\sigma\mu} \Theta_{\mu\nu}})}$$

$$= 1/2 \sum_{\mu\nu} \sqrt{-\mathfrak{g} . \partial \mathfrak{g}_{\mu\nu} / \partial \mathcal{X}_\sigma . \Theta_{\mu\nu}}$$

$$-2x\mathfrak{t}_{\sigma\gamma} = \sqrt{-\mathfrak{g}} (\frac{\frac{\sum_{\rho\varrho\tau} \gamma_{\beta\mu\nu} \partial \mathfrak{g}_{\varrho\tau}}{\partial \mathcal{X}_\sigma} \partial \gamma_{\varrho\tau}}{\partial \mathcal{X}_\rho} - 1/2 \sum_{\alpha\rho\varrho\tau} \frac{\frac{\delta_{\sigma\mu\nu} \gamma_{\alpha\tau} \partial \mathfrak{g}_{\varrho\tau}}{\partial \mathcal{X}_\alpha} \partial \gamma_{\varrho\tau}}{\partial \mathcal{X}_\rho}$$

17.1. Variable hamiltoniana.

$$\int (\delta \mathcal{H} - 2x \sum_{\mu\nu} \sqrt{-\mathfrak{g}}\, \mathcal{T}_{\mu\nu} \delta \gamma_{\mu\nu}) d\tau = 0$$

$$\mathcal{H} = \frac{\frac{1}{2\sqrt{-\mathfrak{g}} \sum_{\alpha\beta\rho\varrho\sigma} \gamma_{\varsigma\varpi o}\, \partial \mathfrak{g}_{\alpha\beta\rho\varrho\sigma}}}{dx_\varphi} \partial \mathcal{X}_{\varsigma\varpi o}$$

$$\delta (\sqrt{-\mathfrak{g}}) = -\frac{1}{2 \sum_{\mu\nu} -\mathfrak{g} \mathfrak{g}_{\mu\nu} \delta \gamma_{\mu\nu}}$$

$$\delta\left(\frac{\partial \mathfrak{g}_{\mu\nu}}{\partial \mathcal{X}_{\mu\nu}}\right) = \frac{\partial}{\partial \mathcal{X}_{\alpha}(\delta \mathfrak{g}_{\mu\nu})} = -\sum_{\mu\nu} \partial/\partial \mathcal{X}_{\alpha}(\mathfrak{g}_{\mu\nu}\delta\gamma_{\mu\nu})\delta(\partial\gamma_{\mu\nu}/\partial\mathcal{X}_{\mu\nu}) = \partial/\partial\mathcal{X}_{\alpha}(\delta\gamma_{\mu\nu})$$

$$\int \delta\mathcal{H}d\tau = \sum_{\mu\nu}(-\frac{\partial}{\partial\mathcal{X}_{\alpha}}\left(\frac{\sqrt{-g\gamma_{\mu\nu}}\partial g_{\mu\nu}}{\partial\mathcal{X}_{\beta}}\right) + \sqrt{-\frac{g\gamma_{\mu\nu}g\gamma_{\nu\mu}\partial g_{\mu\nu}}{\partial\mathcal{X}_{\beta}}}_{\mu\nu\nu\mu} + \frac{1}{2\sqrt{-g.\frac{\partial g_{\mu\nu}}{\partial\mathcal{X}_{\alpha}}\frac{\partial g_{\mu\nu}}{\partial\mathcal{X}_{\beta}}}}$$

$$-1/4 g_{\mu\nu}\gamma_{\nu\mu}\frac{\partial g_{\mu\nu}}{\partial\mathcal{X}_{\alpha}}\frac{\partial g_{\mu\nu}}{\partial\mathcal{X}_{\beta}})\delta\gamma_{\mu\nu}.dt$$

17.2. Sistema de Coordenadas.

$$\mathcal{J}' = \int\sqrt{-g}\sum_{\substack{\mu\nu\\ \nu\mu\\ \alpha\beta\gamma\delta\\ \varepsilon\epsilon\zeta\eta\\ \rho\varrho\sigma\varsigma}} \pi_{\mu\nu}\pi_{\nu\mu}\gamma_{\mu\nu}\gamma_{\nu\mu}\mathfrak{P}_{\mu\nu}\mathfrak{P}_{\nu\mu}\partial/\partial\mathcal{X}_{\mu\nu}(\pi_{\mu\nu}\pi_{\nu\mu}\gamma_{\mu\nu}\gamma_{\nu\mu}\mathfrak{P}_{\mu\nu}\mathfrak{P}_{\nu\mu})\mathfrak{P}_{\mu\nu}\mathfrak{P}_{\nu\mu}\partial/\partial\mathcal{X}_{\nu\mu})d\tau$$

$$\rho_{\mu\nu}\int -4\int +4\int\sum_{\mu\nu}\int \alpha\beta\gamma\delta\varepsilon\epsilon\zeta\eta\theta\vartheta\iota\kappa\lambda\xi o\pi\varpi\rho\varrho\sigma\varsigma\tau\upsilon\varphi\phi\psi\omega = \frac{\partial\mathcal{X}_{\mu\nu}}{\partial\mathcal{X}_{\nu\mu}} = \delta\gamma_{\mu\nu} - \frac{\partial(\Delta\mathcal{X}_{\mu\nu})}{\partial\mathcal{X}_{\mu\nu}}$$

$$= \delta\gamma_{\nu\mu} - \frac{\partial(\Delta\mathcal{X}_{\nu\mu})}{\partial\mathcal{X}_{\nu\mu}}.d\tau - \frac{\partial^2(\Delta\mathcal{X}_{\mu\nu})}{\partial\mathcal{X}_{\nu\mu}}$$

17.3. Formalización de las ecuaciones einstenianas de campo.

$$\delta\left|\int \mathfrak{ds} = 0\right|$$

$$\mathfrak{ds}^2 = \sum_{\mu\nu}\mathfrak{dx}_{\mu\nu}^2$$

$$\mathfrak{ds}^2 = \sum_{\mu\nu}\mathfrak{g}_{\mu\nu}\,\mathfrak{dx}_{\mu}\mathfrak{dx}_{\nu}$$

$$\mathfrak{ds}^2 = \sum_{\mu\nu}\mathfrak{g}_{\mu\nu}\,\mathfrak{dx}_{\mu}\mathfrak{dx}_{\nu} = -\sum_{\mu\nu}\mathfrak{dx}_{\mu\nu}^2$$

$$\sum_{\mu\nu}\mathfrak{A}_{\mu\nu}\mathfrak{dx}_{\mu\nu} = \phi$$

$$\sum_{\mu\nu}\mathfrak{A}'_{\mu\nu}\mathfrak{dx}'_{\mu\nu} = \sum_{\alpha}\mathfrak{A}'_{\alpha}\mathfrak{dx}'_{\alpha} = \sum_{\alpha\mu\nu}\frac{\mathfrak{A}'_{\alpha\mu\nu}.\partial\mathfrak{x}'_{\alpha}}{\partial\mathfrak{x}'_{\mu\nu}}.\mathfrak{dx}_{\mu\nu\alpha} = \phi$$

$$\sum_{\mu\nu}\mathrm{A}_{\mu\nu}\,\mathfrak{d}_{\mu}^{(1)}\mathfrak{d}_{\nu}^{(2)} = \Phi$$

$$\sum_{\mu\nu}\mathrm{A}'_{\mu\nu}\,\mathfrak{d}_{\mu}^{(1)'}\mathfrak{d}_{\nu}^{(2)'} = \sum_{\mu\nu}\mathrm{A}_{\alpha\beta}\,\mathfrak{d}_{\alpha}^{(1)}\mathfrak{d}_{\beta}^{(2)} = \frac{\sum_{\mu\nu\alpha\beta}\partial\mathfrak{x}_{\alpha}}{\partial\mathfrak{x}'_{\mu}}.\partial\mathfrak{x}_{\beta}/\partial\mathfrak{x}'_{\nu}\mathrm{A}_{\alpha\beta}\mathfrak{d}_{\mu}^{(1)'}\mathfrak{d}_{\nu}^{(2)'}$$

$$\sum_{\mu\nu}\mathrm{A}'_{\mu\nu}\,\mathfrak{d}'_{\mu}\mathfrak{d}'_{\nu} = \sum_{\mu\nu}\mathrm{A}_{\alpha\beta}\,\mathfrak{d}_{\alpha}^{\beta}\mathfrak{d}_{\beta}^{\alpha} = \frac{\sum_{\mu\nu\alpha\beta}\partial\mathfrak{x}_{\alpha}}{\partial\mathfrak{x}'_{\mu}}.\partial\mathfrak{x}_{\beta}/\partial\mathfrak{x}'_{\nu}\mathrm{A}_{\alpha\beta}\mathfrak{d}'_{\mu}\mathfrak{d}'_{\nu}$$

$$A'_{\mu\nu} = A'_{\mu}B'_{\nu} = \frac{\sum_{\alpha\beta}\frac{\partial\mathfrak{x}_{\alpha}}{\partial\mathfrak{x}'_{\mu}}\partial\mathfrak{x}_{\beta}}{\partial\mathfrak{x}'_{\nu}}A'_{\alpha}B'_{\beta} = \sum_{\alpha\beta}\frac{\partial\mathfrak{x}_{\alpha}}{\partial\mathfrak{x}'_{\mu}}.\frac{\partial\mathfrak{x}_{\beta}}{\partial\mathfrak{x}'_{\nu}}.A_{\alpha\beta}\,A^{\alpha\beta}$$

$$\sum_{\substack{\alpha\beta\gamma\rho\varrho\sigma\\ \varsigma\tau\upsilon\mu\nu}}\left\|A^{\alpha\beta\gamma\rho\varrho\sigma}_{\varsigma\tau\upsilon\mu\nu}\,B^{\alpha\beta\gamma\rho\varrho\sigma}_{\varsigma\tau\upsilon\mu\nu}\right\| = \Gamma^{\alpha\beta\gamma\rho\varrho\sigma}_{\varsigma\tau\upsilon\mu\nu}$$

$$\left|\sum_{\alpha\beta\mu\nu}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\mu\nu\alpha\beta}\right|.\left\|\mathfrak{d}\xi_{\mu\nu}\mathfrak{d}\xi_{\alpha\beta}\mathfrak{d}\xi^{\mu\nu\alpha\beta}\right\|.\langle\delta_{\mu\nu}\delta_{\alpha\beta}\delta^{\nu\alpha\beta}\rangle = 1$$

$$|\mathfrak{g}_{\mu\nu}| = \left|\sum_{\sigma}(\alpha_{\sigma\mu}\alpha_{\sigma\nu})\right| = (\alpha_{\mu\nu\sigma})^{2} - \sqrt{\mathfrak{g}\delta_{\varepsilon\epsilon\zeta\vartheta\iota\kappa}\mathfrak{d}\tau}.\frac{\mathfrak{d}\tau_{0}}{\sum_{\alpha\beta\lambda\mu}\mathbb{C}_{\alpha\beta\lambda\mu}}.g^{\mu\nu}g^{\rho\varrho}g^{\sigma\varsigma}$$

$$= \mathbb{C}_{\mu\nu\rho\varrho\sigma\varsigma}.\sqrt{gV}/\mathfrak{d}\mathfrak{x}^{\eta}{}^{\mu\nu\lambda\kappa}_{\sigma\varsigma\tau\varrho}$$

$$\mathbb{C}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}{}_{\mu\nu\rho\varrho\sigma\varsigma} = \sum_{\substack{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa\\ \mu\nu\rho\varrho\sigma\varsigma}}\sqrt{\mathfrak{g}\delta_{\mu\nu\rho\varrho\sigma\varsigma}}\,\mathfrak{g}^{\mu\nu\rho\varrho\sigma\varsigma}\mathfrak{g}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}$$

$$= \sum_{\substack{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa\\ \mu\nu\rho\varrho\sigma\varsigma}} 1/\sqrt{\mathfrak{g}\delta_{\mu\nu\rho\varrho\sigma\varsigma}}\,\mathfrak{g}^{\mu\nu\rho\varrho\sigma\varsigma}\mathfrak{g}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa} = \Gamma^{\mu\nu\rho\varrho\sigma\varsigma}_{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}$$

$$\Gamma^{\mu\nu\rho\varrho\sigma\varsigma}_{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa} = \sum_{\substack{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa\\ \mu\nu\rho\varrho\sigma\varsigma}}\Gamma^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}_{\mu\nu\rho\varrho\sigma\varsigma}\,\mathfrak{g}^{\mu\nu\rho\varrho\sigma\varsigma}\mathfrak{g}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}$$

$$F^{\mu\nu} = \frac{1}{2\hbar\sum_{\alpha\beta}\mathfrak{G}^{\mu\nu}_{\alpha\beta}\,F^{\alpha\beta}} = \frac{1}{2\sum_{\mu\nu\alpha\beta}\mathfrak{G}^{\mu\nu}_{\alpha\beta}\,F^{\mu\nu}}$$

$$= \frac{1}{4\omega\psi\varphi\sum_{\alpha\beta\mu\nu}\mathfrak{G}^{\mu\nu\lambda}_{\alpha\beta\sigma}\,\mathfrak{G}^{\delta\varepsilon\epsilon\tau\upsilon\omega}_{\kappa\lambda\xi\rho\sigma\varsigma\varrho}}$$

$$- \frac{\sum_{\mu\nu}\mathfrak{G}^{\mu\nu\lambda}_{\alpha\beta\sigma}\mathfrak{G}^{\delta\varepsilon\epsilon\tau\upsilon\omega}_{\kappa\lambda\xi\rho\sigma\varsigma\varrho} = \sum_{\substack{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa\\ \mu\nu\rho\varrho\sigma\varsigma}}\sqrt{\mathfrak{g}\delta_{\mu\nu\rho\varrho\sigma\varsigma}}\,\mathfrak{g}^{\mu\nu\rho\varrho\sigma\varsigma}\mathfrak{g}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}.1}{\sqrt{\mathfrak{g}\delta_{\mu\nu\rho\varrho\sigma\varsigma}}\,\mathfrak{g}^{\mu\nu\rho\varrho\sigma\varsigma}\mathfrak{g}^{\alpha\beta\gamma\varepsilon\epsilon\zeta\vartheta\iota\kappa}}$$

$$= 2(\delta^{\mu\nu\lambda}\delta_{\alpha\beta\hbar\sigma\tau} - \delta^{\alpha\beta\hbar}\delta_{\mu\nu\lambda\sigma\tau}$$

$$\delta\omega = 1/\omega\left\|\frac{\frac{1}{2\sum_{\mu\nu\sigma}\frac{\partial\mathfrak{g}_{\mu\nu}}{\partial\mathfrak{x}_{\sigma}}.\mathfrak{d}\mathfrak{x}_{\mu\nu}}}{\mathfrak{d}\lambda}.\frac{\mathfrak{d}\mathfrak{x}_{\alpha\beta}}{\mathfrak{d}\lambda} + \sum_{\mu\nu}\mathfrak{g}_{\mu\nu}.\frac{\mathfrak{d}\mathfrak{x}_{\mu\nu}}{\mathfrak{d}\lambda}\delta\left(\frac{\mathfrak{d}\mathfrak{x}_{\nu\mu}}{\mathfrak{d}\lambda}\right)\right\|$$

$$\frac{\sum_{\mu\nu}\mathfrak{g}_{\mu\nu\sigma}.\mathfrak{d}^{2}\,\mathfrak{x}_{\mu\nu}}{\mathfrak{d}\mathfrak{s}^{2}} + \frac{\sum_{\mu\nu}{\mu\nu \atop \sigma}\mathfrak{d}\mathfrak{x}_{\mu\nu}}{\mathfrak{d}\mathfrak{s}}.\frac{\mathfrak{d}\mathfrak{x}_{\alpha\beta\sigma}}{\mathfrak{d}\mathfrak{s}}$$

$${\mu\nu \atop \sigma} = 1/2(\frac{\partial\mathfrak{x}_{\mu\sigma}}{\partial\mathfrak{x}_{\mu\nu}} + \frac{\partial\mathfrak{x}_{\nu\sigma}}{\partial\mathfrak{x}_{\mu\nu}} - \frac{\partial\mathfrak{g}_{\mu\nu\sigma}}{\partial\mathfrak{x}_{\sigma}} - \frac{\partial\mathfrak{g}_{\mu\nu\tau}}{\partial\mathfrak{x}_{\tau\varsigma}} = \partial\phi\int\mathfrak{J}\psi\varphi\,\mathfrak{d}\omega.\mathfrak{d}\phi$$

$$\frac{\mathfrak{d}^{2}\phi}{\mathfrak{d}\mathfrak{s}^{2}} = \frac{\sum_{\mu\nu}\frac{\partial^{2}\phi}{\partial\mathfrak{x}_{\mu}\partial\mathfrak{x}_{\nu}}.\mathfrak{d}\mathfrak{x}_{\mu}}{\mathfrak{d}\mathfrak{s}}.\frac{\mathfrak{d}\mathfrak{x}_{\nu}}{\mathfrak{d}\mathfrak{s}} + \frac{\sum_{\tau}\partial\phi}{\partial\chi_{\tau}}.\mathfrak{d}^{2}\chi_{\tau}/\mathfrak{d}\mathfrak{s}^{2}$$

$$\frac{\mathfrak{d}^{2}\phi}{\mathfrak{d}\mathfrak{s}^{2}} = \sum\nolimits_{\mu\nu}(\frac{\partial^{2}\phi}{\partial\mathfrak{x}_{\mu}\partial\mathfrak{x}_{\nu}} - \sum_{\tau}{\mu\nu \atop \tau}\partial\phi/\partial\chi_{\tau}).\mathfrak{d}\mathfrak{x}_{\mu}/\mathfrak{d}\mathfrak{s}.\frac{\mathfrak{d}\mathfrak{x}_{\nu}}{\mathfrak{d}\mathfrak{s}}$$

$$\frac{\partial}{\partial \mathfrak{x}_\nu \left(\psi . \frac{\partial\phi}{\partial \mathfrak{x}_\mu}\right)} - \sum_\tau \genfrac{}{}{0pt}{}{\mu\nu}{\tau} \left(\psi . \frac{\partial \phi}{\partial \mathfrak{x}_\tau}\right)$$

$$\frac{1}{\mathscr{g}} . \frac{\partial \mathscr{g}}{\partial \mathfrak{x}_\alpha} = \sum\nolimits_{\mu\nu} \frac{\partial \mathfrak{g}_{\mu\nu}}{\partial \mathfrak{x}_\alpha} - \mathfrak{g}^{\mu\nu} = \frac{2}{\sqrt{\mathscr{g}}} . \frac{\partial\sqrt{\mathscr{g}}}{\partial \mathfrak{x}_\alpha}$$

$$\sum_\tau \genfrac{}{}{0pt}{}{\mu\nu}{\tau} = \sum_\tau \genfrac{}{}{0pt}{}{\tau}{\mu\nu} = \frac{1}{2} \sum\nolimits_{\tau\alpha} \mathfrak{g}^{\tau\alpha} \frac{\partial \mathscr{g}_{\tau\alpha}}{\partial \mathfrak{x}_{\mu\nu}} = \frac{1}{\sqrt{\mathscr{g}}} . \frac{\partial \sqrt{\mathscr{g}}}{\partial \mathfrak{x}_\mu}$$

$$A_\sigma = \frac{1}{\sqrt{\mathscr{g}}} \left(\sum\nolimits_{\mu\nu} \mathscr{g}_{\mu\nu\sigma} . \frac{\partial\left(A_{\mu\nu}\sqrt{\mathscr{g}}\right)}{\partial \mathfrak{x}_\nu} + \sqrt{\mathscr{g}} \sum_{\tau\mu\nu} \genfrac{}{}{0pt}{}{\tau\upsilon}{\sigma} A^{\tau\upsilon}\right)$$

$$= 1/\sqrt{\mathscr{g}} \, (\sum_{\mu\nu} \partial \left(\frac{\mathscr{g}_{\mu\nu\sigma} A^{\mu\nu}\sqrt{\mathscr{g}}}{\partial \mathfrak{x}_\nu} + \frac{1}{2\sqrt{\mathscr{g}}\sum_{\mu\nu}(\frac{\partial \mathscr{g}_{\mu\nu\sigma}}{\partial \mathfrak{x}_\mu} + \frac{\partial \mathscr{g}_{\nu\mu\sigma}}{\partial \mathfrak{x}_\nu}} - \frac{\partial \mathscr{g}_{\mu\nu}}{\partial \mathfrak{x}_\sigma}\right) A^{\mu\nu})$$

$$A_\sigma = \frac{1}{\sqrt{\mathscr{g}}} \left(\sum\nolimits_{\mu\nu} \mathscr{g}_{\mu\nu\sigma} . \frac{\partial\left(A_{\mu\nu}\sqrt{\mathscr{g}}\right)}{\partial \mathfrak{x}_\nu} - \sqrt{\mathscr{g}} \sum_{\tau\mu\nu} \genfrac{}{}{0pt}{}{\tau\upsilon}{\sigma} A^{\tau\upsilon}\right)$$

$$= 1/\sqrt{\mathscr{g}} \, (\sum_{\mu\nu} \partial \left(\frac{\mathscr{g}_{\mu\nu\sigma} A^{\mu\nu}\sqrt{\mathscr{g}}}{\partial \mathfrak{x}_\nu} - \frac{1}{2\sqrt{\mathscr{g}}\sum_{\mu\nu}(\frac{\partial \mathscr{g}_{\mu\nu\sigma}}{\partial \mathfrak{x}_\mu} + \frac{\partial \mathscr{g}_{\nu\mu\sigma}}{\partial \mathfrak{x}_\nu}} - \frac{\partial \mathscr{g}_{\mu\nu}}{\partial \mathfrak{x}_\sigma}\right) A^{\mu\nu})$$

$$A_{\mu\nu\lambda} = \frac{\partial^2 A_\mu}{\partial \mathfrak{x}_\mu \partial \mathfrak{x}_\lambda} - \frac{\sum_\tau \genfrac{}{}{0pt}{}{\mu\lambda}{\tau} \partial A_\tau}{\partial \mathfrak{x}_\nu} + \frac{\genfrac{}{}{0pt}{}{\mu\nu}{\tau}\partial A_\tau}{\partial \mathfrak{x}_\lambda} - \sum_\tau \frac{\genfrac{}{}{0pt}{}{\nu\lambda}{\tau}\partial A_\mu}{\partial \mathfrak{x}_\tau} + \sum_{\sigma\tau} \frac{\genfrac{}{}{0pt}{}{\nu\lambda\tau\mu}{\tau\ \sigma}\partial A_\sigma}{\partial \mathfrak{x}_{\sigma\tau\lambda\nu}} - \sum_\sigma \frac{\genfrac{}{}{0pt}{}{\mu\nu}{\sigma}\partial}{\partial \mathfrak{x}_\lambda} - \sum_\tau \frac{\genfrac{}{}{0pt}{}{\mu\lambda\nu\tau}{\tau\ \sigma}\partial A_\sigma}{\partial \mathfrak{x}_{\sigma\tau\lambda\nu}}$$

$$\mathfrak{E}_\sigma = \sum\nolimits_{\mu\nu} \frac{\partial \mathfrak{E}^{\mu\nu}_\sigma}{\partial \mathfrak{x}_\mu} - \frac{\frac{1}{2\sum_{\mu\tau\nu} \mathscr{g}^{\tau\mu}\,\partial \mathscr{g}_{\mu\nu}}}{\partial \mathfrak{x}_\lambda} \mathfrak{E}^{\mu\nu}_\sigma$$

$$\sum_{\mu\nu} \partial \mathfrak{I}^{\mu\nu}_{\lambda\sigma} / \partial \mathfrak{H}_\mu = \frac{\frac{1}{2\hbar\sum_{\mu\tau\nu}\mathscr{g}^{\tau\mu}}\,\partial \mathscr{g}_{\mu\nu}}{\partial \mathfrak{x}_\lambda} . \mathfrak{I}^{\mu\nu}_{\lambda\sigma} + \mathfrak{R}_{\lambda\sigma}$$

$$\begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$$\sum\nolimits_{\mu\nu} \partial(\mathfrak{I}^{\mu\nu}_{\lambda\sigma} + \mathfrak{t}^{\mu\nu}_{\lambda\sigma}) \frac{\partial \mathfrak{x}_\lambda}{\mathbb{R}^{\alpha\beta\gamma\delta\epsilon\varepsilon\zeta\eta\iota\kappa\lambda\mu\nu\xi}_{o\pi\varpi\rho\varrho\sigma\varsigma\tau\upsilon\phi\varphi\psi\omega}} = 0$$

$$\Gamma^\Lambda = \int \frac{\tau\upsilon\sigma}{d\omega^2} (\frac{1}{2dl^2} \triangleq \frac{\sum_{\mu\nu} \mathcal{G}\Delta^{\mu\nu}\,\partial \mathscr{g}_{\mu\nu}}{\partial \mathfrak{x}_\sigma} . d\tau^2/ds^2$$

$$ds^2 = \sum\nolimits_{\mu\nu}\int \mathfrak{g}_{\mu\nu}\, d\chi_\mu d\chi_\nu = -d\xi_1^2 - d\xi_2^2 - d\xi_3^2 - d\xi_\eta^2/\sqrt{-g}$$

$$\frac{d}{d\chi_4} = \left(m\sum\nolimits_{\mu\nu} g_{\mu\nu\sigma} . \frac{d\chi_\mu}{ds}\right) = \sum_{\upsilon\tau} \Gamma^\tau_{\upsilon\sigma} . \frac{d\chi_\nu}{d\chi_4} m \sum\nolimits_{\mu\nu} \mathscr{g}_{\tau\upsilon} . \frac{d\chi_\mu}{ds} + \int d\nabla \mathfrak{R}_{\lambda\sigma} d\upsilon$$

$$-\delta_{\sigma\lambda}^{\mu\nu}\sqrt{-g}.\frac{\partial\rho}{\partial\chi_\sigma}+\frac{\rho}{\rho_{o\,\stackrel{\text{def}}{\equiv}}}+\mathfrak{I}_{\lambda\sigma}^{\mu\nu}+\sum\nolimits_{\mu\nu}\frac{\partial}{\partial\chi_\upsilon}\left(\rho * g_{\dagger\sigma\upsilon}.\frac{d\chi_\omega}{ds}.\frac{d\chi_\psi}{ds}\right)$$

$$=\frac{\dfrac{1}{2\sum_{\mu\nu}\rho \circledast \partial g_{\mu\nu\sigma}}}{\mathfrak{R}_{\lambda\sigma}}\partial\chi_\sigma\frac{d\chi_\omega}{ds}.\frac{d\chi_\psi}{ds}+\mathfrak{R}_{\lambda\sigma}=1$$

$$\sum_{\mu\nu}\partial(\mathfrak{I}_{\lambda\sigma}^{\mu\nu}-\mathfrak{B}_{\epsilon}^{\mu\nu})/\partial\chi_\sigma=\rho_\varepsilon.d\chi_\mu/ds+\mathfrak{T}^{\mu\nu}$$

$$\sum_{\mu\nu}\partial(\mathfrak{I}_{\lambda\sigma}^{\mu\nu\dagger}-\mathfrak{B}_{(m)}^{\mu\nu\circledast})/\partial\chi_\sigma=\rho_{(m)}.d\chi_\mu/ds+\mathfrak{T}^{\mu\nu}$$

$$\left\langle \mathfrak{B}_{\epsilon}^{\mu\nu}=\sigma_\epsilon\sum\nolimits_{\alpha\beta}\frac{\dfrac{\mathcal{g}_{\alpha\beta}\mathfrak{F}^{\mu\nu}d\chi_\beta}{ds}}{\mathfrak{B}_{\epsilon\dagger}^{\mu\nu*}=\mathfrak{B}_{\epsilon}^{\mu\nu}\sum\nolimits_{\alpha\beta}\mathfrak{B}_{\epsilon}^{\mu\nu}\dfrac{\mathcal{g}_{\alpha\beta}\mathfrak{F}^{\mu\nu}d\chi_{\mu\nu}}{ds}}_{e^{-i\omega t}}\right|\mathfrak{B}_{(m)}^{\mu\nu}=\sigma_{(m)}\sum\nolimits_{\alpha\beta}\frac{\dfrac{\mathcal{g}_{\alpha\beta}\mathfrak{F}^{\mu\nu}d\chi_\beta}{ds}}{\mathfrak{B}_{(m)\dagger}^{\mu\nu*}=\mathfrak{B}_{(m)}^{\mu\nu}\sum\nolimits_{\alpha\beta}\mathfrak{B}_{(m)}^{\mu\nu}\dfrac{\mathcal{g}_{\alpha\beta}\mathfrak{F}^{\mu\nu}d\chi_{\mu\nu}}{ds}}_{e^{-i\omega t}}\left|\mathfrak{Q}^{\mu\nu}=-\lambda\sum\nolimits_{\alpha\beta}\mathcal{g}_{\alpha\beta}\mathfrak{F}^{\mu\nu\sigma\lambda}.\frac{d\chi_\beta}{ds}\right\rangle$$

$$\Delta\mathcal{g}_{\sigma}^{\mu\nu}=\sum_{\alpha}\frac{\partial}{\partial\chi_\sigma}(\mathcal{g}^{\mu\sigma}.\frac{\partial\Delta\chi_\mu}{\partial\chi_\alpha}+\mathcal{g}^{\nu\sigma}.\frac{\partial\Delta\chi_\nu}{\partial\chi_\tau})-\frac{\partial\mathcal{g}^{\mu\nu}}{\partial\chi_\alpha}\frac{\partial\Delta\chi_\alpha}{\partial\chi_\sigma}$$

$$-\frac{\frac{1}{2\Delta\mathcal{H}}\sum_{\mu\nu\sigma\alpha}\dfrac{\mathfrak{G}_{\mu\nu}{}^{\sigma\tau}}{\partial^2}\Delta\mathcal{g}_{\sigma}^{\mu\nu}\int d\tau.\partial\mathcal{H}\sqrt{-g}}{\Delta\mathfrak{I}}.\partial^2\Delta\chi_\nu$$

$$\mathfrak{F}=\int d\tau\sum_{\mu\nu\alpha\sigma}\frac{\partial^2}{\partial\mathfrak{f}_\alpha}\partial\chi_\sigma\left(\frac{\dfrac{\mathcal{g}^{\nu\alpha}\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\sigma}^{\mu\nu}}\partial\Delta\chi_{\mu\nu}}{\partial\chi_\sigma}-\frac{\partial}{\partial\chi_\tau}\left(\frac{\mathcal{g}^{\nu\sigma}\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\alpha}^{\mu\nu}}\right)\Delta\chi_\tau\right)$$

$$\mathfrak{d}\mathfrak{J}=\delta\left(\int\mathcal{H}\sqrt{-g}\,d\tau\right)=\frac{\int d\tau\sum_{\mu\nu\sigma}(\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}^{\mu\nu}}\delta\mathcal{g}^{\mu\nu}+\partial\big(\mathcal{H}\sqrt{-g}\big)/\partial\mathcal{g}_{\sigma}^{\mu\nu}\delta\mathcal{g}_{\sigma}^{\mu\nu}$$

$$\mathfrak{G}_{\mu\nu}=\frac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}^{\mu\nu}}-\sum\nolimits_{\mu\nu\sigma}\frac{\partial^2}{\partial\chi_\sigma}(\frac{\partial\mathcal{H}\sqrt{-g}\mathcal{S}_{\sigma\tau}^{\mu\nu}}{\partial\mathcal{g}_{\sigma\tau}^{\mu\nu}}-\frac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\alpha\beta}^{\mu\nu}}-\frac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}^{\kappa}}$$

$$\mathcal{S}_{\sigma}^{\mu\nu}=\frac{\partial}{\partial\chi_\upsilon}(\mathcal{g}^{\tau\nu}\mathfrak{G}_{\sigma\tau})+\frac{\dfrac{1}{2\sum_{\mu\nu}\partial\mathcal{g}^{\mu\nu}}}{\partial\chi_\sigma}.\mathfrak{G}_{\mu\nu}$$

$$=\frac{\sum_{\mu\nu\tau}(\mathcal{g}^{\upsilon\tau}.\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\sigma\tau}^{\mu\nu}}+\mathcal{g}_{\sigma\tau}^{\mu\nu}.\frac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\mu\nu}^{\sigma\tau}}+\mathcal{g}_{\alpha\beta}^{\mu\nu}.\frac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\mu\nu}^{\alpha\beta}}+\frac{1}{2\mathcal{S}_{\sigma\tau}^{\mu\nu}\mathcal{H}\sqrt{-g}}$$

$$-\frac{1}{4\mathcal{g}_{\sigma\tau}^{\mu\nu}}.\frac{\dfrac{\partial\mathcal{H}\sqrt{-g}}{\partial\mathcal{g}_{\mu\nu}^{\alpha\beta}}}{\dfrac{dg^{\mu\nu}}{\dfrac{dg^{\sigma\tau}}{d^2g^{\mu\nu\sigma\tau}}}}.\mathcal{g}_{\rho\alpha\beta}^{\mu\nu}-\frac{\partial g_\kappa}{\partial\chi_\rho}.\frac{1}{{}^{\mu\nu}_{\sigma}\mathfrak{L}\kappa^{\alpha\beta\gamma\delta\varepsilon\epsilon\zeta\eta\vartheta\iota\lambda\mu\nu\xi}_{o\varpi\rho\varrho\sigma\varsigma\tau\upsilon\varphi\phi\chi\psi\omega}}-\frac{\sqrt{-g}}{4\kappa}+\frac{1}{2\mathcal{S}_{\sigma\tau}^{\mu\nu}\mathcal{g}_{\mu\nu}^{\sigma\tau}\Gamma_{\rho\omega r}^{\alpha\beta}}-$$

$$\boxplus\,\hbar_{\sigma\tau.}^{\mu\nu}{}_{\rho\kappa}\kappa\rho_o-\frac{\dfrac{1}{2\sum_\zeta\partial\hbar_{\sigma\tau\;\rho\kappa}^{\mu\nu}}}{\partial\chi_\tau}$$

$$\begin{matrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{matrix}$$

17.4. Ecuaciones relativistas aplicables a campos cuánticos curvos o con deformación geométrica.

$$\mathfrak{G}_{ijkl} = \mathfrak{G}^{ijkl} = \frac{1}{2\sqrt{-\mathfrak{g}}} . \delta_{ijkl}$$

$$\mathfrak{G}_{ijkl} = \mathfrak{G}^{ijkl} = \frac{1}{2\sqrt{-\mathfrak{g}'}} . \delta_{ijkl}$$

$$\mathcal{A}^{\alpha_i \ldots \alpha_j} = \frac{\sum_s \partial \mathcal{A}^{\alpha_i \ldots \alpha_s^j}}{\partial ד_s} + \sum_{sד} \binom{\mathbb{Z}ד}{ג} \mathcal{A}^{\lambda_i \ldots \alpha_s^ד}$$

$$\sum_{בד} \binom{sב}{ג} = \frac{1}{2\hbar \sum_{\alpha s} \mathcal{g}^{\mathfrak{H}\varkappa} \left(\frac{\partial \mathcal{g}_{\alpha\mathfrak{s}}}{\partial ד_\lambda} + \frac{\partial \mathcal{g}_{\alpha\lambda}}{\partial ד_s} - \frac{\partial \mathcal{g}_{\alpha\aleph}}{\partial ד_\alpha} \right)} = \frac{1}{2 \sum \mathcal{g}^{\alpha\mathfrak{s}}} . \frac{\partial \mathcal{g}_{\alpha\mathfrak{s}}}{\partial ד_\lambda} = \partial \left(\frac{\ell \mathcal{g} \sqrt{-\mathfrak{g}}}{\partial ד_s} \right)$$

$$\Phi = 1/\sqrt{-\mathfrak{g}} \sum_{\mu\nu} \partial/\partial ד_{\mu\nu} \, (\sqrt{-\mathfrak{g}} \mathrm{A}^{\mu\nu}$$

$$\mathrm{A}_\sigma = \sum_{\mu\nu} \partial \mathrm{A}_\sigma^{\mu\nu} / \partial \mathfrak{R}_{\mu\nu} - \frac{1}{2 \sum_{\mu\nu\tau} \mathcal{g}^{\tau\mu\nu}} . \frac{\partial \mathcal{g}_{\mu\nu}}{\partial \mathfrak{R}_\sigma} . \mathrm{A}_\sigma^{\mu\nu}$$

$$(ik, lm) = \frac{1}{2} \left(\frac{\partial^2 \mathcal{g}_{im}}{\partial ד_{ik} \partial ד_{lm}} + \frac{\partial^2 \mathcal{g}_{ik}}{\partial ד_{il} \partial ד_{kl}} - \frac{\partial^2 \mathcal{g}_{il}}{\partial ד_{im} \partial ד_{il}} - \frac{\partial^2 \mathcal{g}_{kiml}}{\partial ד_{lmi} \partial ד_{klm}} \right) + \sum_{\rho\sigma} \mathcal{g}^{\rho\sigma} \left(\left\| \begin{matrix} im \\ \rho \end{matrix} \right\| \left\| \begin{matrix} kl \\ \sigma \end{matrix} \right\| - \left\| \begin{matrix} il \\ \rho \end{matrix} \right\| \left\| \begin{matrix} km \\ \sigma \end{matrix} \right\| \right)$$

$$(ik, lm) = \sum_\rho \mathcal{g}^{\kappa\rho} (i\rho, lm) = \frac{\partial \left\| \begin{matrix} il \\ \kappa \end{matrix} \right\|}{\partial ד_m} - \frac{\partial \left\| \begin{matrix} im \\ \kappa \end{matrix} \right\|}{\partial ד_i} + \sum_{\rho\sigma} \mathcal{g}^{\rho\sigma} \left(\left\| \begin{matrix} im \\ \rho \end{matrix} \right\| \left\| \begin{matrix} kl \\ \sigma \end{matrix} \right\| - \left\| \begin{matrix} il \\ \rho \end{matrix} \right\| \left\| \begin{matrix} km \\ \sigma \end{matrix} \right\| \right)$$

$$\sum_{\mu\nu} \partial \Lambda_{\rho\sigma}^{\mu\nu} = \frac{1}{2 \sum_{\mu\nu\tau} \mathcal{g}^{\mu\nu\tau}} . \frac{\partial \mathcal{g}_{\mu\nu}}{\partial \mathfrak{R}_\sigma \Gamma_{\rho\sigma}^{\mu\nu}} + \mathrm{K}_\Lambda$$

$$\Gamma_{\rho\sigma}^{\mu\nu} = - \left\| \begin{matrix} \mu\nu \\ \sigma \end{matrix} \right\| = - \sum_\alpha \mathcal{g}^{\sigma\alpha} \left\| \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\| = -1/2 \sum_\alpha \mathcal{g}^{\sigma\alpha} \left(\frac{\partial \mathcal{g}_{\mu\alpha}}{\partial \mathfrak{R}_{\mu\nu}} + \frac{\partial \mathcal{g}_{\nu\alpha}}{\partial \mathfrak{R}_{\nu\mu}} - \frac{\partial \mathcal{g}_{\mu\nu}}{\partial \mathfrak{R}_\alpha} \right)$$

$$\frac{\sum_\alpha \partial \Lambda_\sigma^\alpha}{\partial \mathfrak{R}_\alpha} = - \sum_{\alpha\beta} \Gamma_{\sigma\beta}^\alpha \Gamma_\alpha^\beta$$

$$\frac{d^2 \chi_\tau}{d\mathfrak{s}^2} = \sum_{\mu\nu} \Gamma_{\mu\nu}^\tau . \frac{d\chi_\mu}{ds} \frac{d\chi_\nu}{ds}$$

$$\mathfrak{R}_{\mu\nu} = -\kappa ד_{\mu\nu} = \frac{\sum_\alpha \partial \Gamma_{\mu\nu}^{\alpha\sigma}}{\partial \mathfrak{R}_\alpha} + \sum_{\alpha\beta} \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta = - \kappa ד_{\mu\nu}$$

$$\delta \left\| \int \mathfrak{G} - \kappa \sum_{\mu\nu} \mathcal{g}^{\mu\nu} ד_{\mu\nu} \right\| d\tau$$

$$\mathfrak{G} = \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\beta}_{\tau\alpha}$$

$$\sum_{\alpha}\frac{\partial}{\partial\mathfrak{R}_{\alpha}}\left(\frac{\partial\mathfrak{G}}{\partial g^{\alpha\sigma}_{\mu\nu}}\right) - \frac{\partial\mathfrak{G}}{\partial g^{\alpha\sigma}_{\mu\nu}} = -\kappa ℸ_{\mu\nu}\frac{\partial\mathfrak{G}}{\partial g^{\alpha\sigma}_{\mu\nu}} = -\sum_{\alpha\beta}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \frac{\partial\mathfrak{G}}{\partial g^{\alpha\sigma}_{\mu\nu}} = \Gamma^{\alpha}_{\mu\nu}$$

$$\sum_{\sigma\mu\nu}\frac{\partial}{\partial\mathfrak{R}_{\alpha}}\partial\mathfrak{R}_{\alpha}\left(g^{\alpha\sigma}_{\mu\nu}.\frac{\partial\mathfrak{G}}{g^{\alpha\sigma}_{\mu\nu}}\right) - \frac{\partial\mathfrak{G}}{\partial\mathfrak{R}_{\sigma}} = -\kappa\sum_{\mu\nu}ℸ_{\mu\nu}\,g^{\alpha\sigma}_{\mu\nu}$$

$$\sum_{\lambda}\frac{\partial ℸ^{\lambda}_{\sigma}}{\partial\mathfrak{R}_{\lambda}} = \frac{\frac{1}{2\sum_{\mu\nu}\partial g^{\mu\nu}}}{\partial\mathfrak{R}_{\sigma}ℸ_{\mu\nu}}$$

$$\sum_{\lambda}\frac{\partial ℸ^{\lambda}_{\sigma}}{\partial\mathfrak{R}_{\lambda}} = \sum_{\mu\nu}\Gamma^{\mu}_{\sigma\nu}ℸ^{\nu}_{\mu}$$

$$\sum_{\lambda}\frac{\partial}{\partial\mathfrak{R}_{\lambda}}(ℸ^{\lambda}_{\sigma} + \mathfrak{t}^{\lambda}_{\sigma})$$

$$\kappa\mathfrak{t}^{\lambda}_{\sigma} = \frac{1}{2}\delta^{\lambda}_{\sigma}\sum_{\mu\nu\alpha\beta} g^{\mu\nu}\,\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \sum_{\mu\nu\alpha} g^{\mu\nu}\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\nu\alpha}$$

$$\mathfrak{t}^{\lambda}_{\sigma} = \frac{1}{2\kappa\left(\mathfrak{G}\delta^{\lambda}_{\sigma} - \sum_{\mu\nu} g^{\sigma}_{\mu\nu}\frac{\partial\mathfrak{G}}{g^{\mu\nu}_{\lambda}}\right)}$$

$$ℸ^{\lambda}_{\sigma} = \frac{1}{2\delta^{\lambda}_{\sigma}\sum_{\mu\nu\alpha\beta} g^{\mu\nu}\,\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}} - \sum_{\mu\nu\alpha} g^{\mu\nu}\,\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\nu\alpha}$$

$$\frac{\sum_{\alpha\beta}\partial^{2} g^{\alpha\beta}}{\partial\mathfrak{R}_{\alpha}\partial\mathfrak{R}_{\beta}} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\beta}_{\tau\alpha} + \sum\nolimits_{\alpha\beta}\partial/\partial\mathfrak{R}_{\alpha}(g^{\alpha\beta}\partial lg\sqrt{-\mathfrak{g}}/\partial\mathfrak{R}_{\beta}) = -\kappa\sum_{\sigma}ℸ^{\rho}_{\sigma}$$

$$\sum_{\alpha\nu}\partial/\partial\mathfrak{R}_{\alpha}(g^{\nu\lambda}\,\Gamma^{\alpha}_{\mu\nu}) - \sum_{\alpha\beta\nu} g^{\nu\beta}\Gamma^{\alpha}_{\nu\mu}\Gamma^{\lambda}_{\beta\alpha} = -\kappa\,ℸ^{\tau}_{\mu\nu}$$

$$\sum_{\alpha\nu}\partial/\partial\mathfrak{R}_{\alpha}(g^{\nu\lambda}\,\Gamma^{\alpha}_{\mu\nu}) - 1/2\delta^{\lambda}_{\mu}\sum_{\alpha\beta\mu\nu} g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\,(ℸ^{\lambda}_{\mu} + \mathfrak{t}^{\lambda}_{\mu})$$

$$\frac{\partial}{\partial\mathfrak{R}_{\mu\nu}}\left\|\sum_{\alpha\beta}\frac{\partial^{2} g^{\alpha\beta}}{\partial\mathfrak{R}_{\alpha}\partial\mathfrak{R}_{\beta}} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\beta}_{\tau\alpha}\right\| = \frac{\sum_{\alpha\beta}\partial}{\partial\mathfrak{R}_{\alpha}}\left(g^{\alpha\beta}\frac{\partial lg\sqrt{-\mathfrak{g}}}{\partial\mathfrak{R}_{\beta}}\right) = -\kappa\sum\nolimits_{\sigma}ℸ^{\rho}_{\sigma}$$

$$ℸ^{\mu\nu} = \sqrt{-\mathfrak{g}}\rho_{0}\,.\frac{d\chi_{\mu}}{ds}\frac{d\chi_{\nu}}{ds}$$

$$\sum_{\mu}ℸ^{\mu}_{\nu} = \sum_{\mu\nu} g_{\mu\nu}\,ℸ^{\mu\nu} = \rho_{0}\sqrt{-\mathfrak{g}}$$

$$\mathfrak{G}_{im} = \sum_{l}(il, lm) = \mathfrak{R}_{im} + \mathfrak{S}_{im}$$

$$\mathfrak{R}_{im} = -\sum_{l}\frac{\partial\begin{Vmatrix} im \\ l\end{Vmatrix}}{\partial\mathfrak{R}_{l}} + \sum\nolimits_{\rho l}\begin{vmatrix} il \\ \rho\end{vmatrix}\begin{vmatrix}\rho m \\ l\end{vmatrix}$$

$$\mathfrak{R}_{im} = \frac{\sum_{l}\partial\Gamma^{l}_{im}}{\partial\mathfrak{R}_{l}} + \sum_{\rho l}\Gamma^{l}_{i\rho}\Gamma^{\rho}_{ml}$$

$$\mathfrak{R}_{im} = \sum_{l} \partial \Gamma_{im}^{l} / \partial \mathfrak{R}_{l} + \sum_{\rho l} \Gamma_{i\rho}^{l} \Gamma_{ml}^{\rho} = -\kappa \left(\daleth_{im} - \frac{1}{2 g_{im} \mathfrak{J}} \right)$$

$$\mathfrak{S}_{im} = \sum_{l} \frac{\partial \left\| \begin{matrix} im \\ l \end{matrix} \right\|}{\partial \mathfrak{R}_{m}} - \sum_{\rho l} \left| \begin{matrix} im \\ \rho \end{matrix} \right| \left| \begin{matrix} \rho l \\ l \end{matrix} \right|$$

$$\mathfrak{S}_{im} = -\kappa \left(\daleth_{im} - \frac{1}{2 g_{im} \mathfrak{J}} \right)$$

$$\sum_{\varrho\sigma} g^{\varrho\sigma} \daleth_{\varrho\sigma} = \sum_{\sigma} \daleth_{\sigma}^{\varrho} = \mathrm{T}$$

$$\mathfrak{G}_{\mu\nu} = -\kappa \daleth_{\mu\nu}$$

$$\mathfrak{R}_{\mu\nu} = -\kappa \daleth_{\mu\nu}$$

$$\sum_{\alpha\beta} \frac{\partial}{\partial \mathfrak{R}_{\alpha}} \left(\frac{g^{\alpha\beta} \partial lg\sqrt{-\mathfrak{g}}}{\partial \mathfrak{R}_{\beta}} \right) = -\kappa \sum_{\sigma} \daleth_{\sigma}^{\rho}$$

$$\frac{1}{2 \sum_{im} g_{im}} = \frac{\partial g^{im}}{\partial \mathfrak{R}_{\sigma}} = -\frac{\partial lg\sqrt{-\mathfrak{g}}}{\partial \mathfrak{R}_{\sigma}}$$

$$\partial / \partial \mathfrak{R}_{\mu\nu} \left[\!\left[\sum_{\alpha\beta} \frac{\partial^{2} g^{\alpha\beta}}{\partial \mathfrak{R}_{\alpha} \partial \mathfrak{R}_{\beta}} - \kappa (\daleth + \mathfrak{t}) \right]\!\right]$$

$$\mathbb{A}^{'\sigma\tau} = \frac{\partial \mathfrak{R}_{\sigma}'}{\partial \mathfrak{R}_{\mu}} \frac{\partial \mathfrak{R}_{\tau}'}{\partial \mathfrak{R}_{\nu}} \mathbb{A}^{\mu\nu} = \frac{\partial \mathfrak{R}_{\sigma}'}{\partial \mathfrak{R}_{\mu}} \frac{\partial \mathfrak{R}_{\tau}'}{\partial \mathfrak{R}_{\nu}} \mathbb{A}^{\nu\mu} = \frac{\partial \mathfrak{R}_{\sigma}'}{\partial \mathfrak{R}_{\nu}} \frac{\partial \mathfrak{R}_{\tau}'}{\partial \mathfrak{R}_{\mu}} \mathbb{A}^{\mu\nu} = \mathbb{A}^{'\tau\sigma}$$

$$\mathbb{G} = \left| \frac{\partial \mathfrak{R}_{\mu}}{\partial \mathfrak{R}_{\sigma}'} \right| \cdot \left| \frac{\partial \mathfrak{R}_{\nu}}{\partial \mathfrak{R}_{\tau}'} \right| \cdot |\mathfrak{g}_{\mu\nu}| = \left(\frac{\partial \mathfrak{R}_{\mu}}{\partial \mathfrak{R}_{\sigma}'} \right)^{2} \mathfrak{g}, \sqrt{-\mathfrak{g}'} = \left| \frac{\partial \mathfrak{R}_{\mu}}{\partial \mathfrak{R}_{\sigma}'} \right| \sqrt{-\mathfrak{g}}$$

$$d\tau' = \left| \frac{\partial \mathfrak{R}_{\mu}}{\partial \mathfrak{R}_{\sigma}'} \right| . d\tau$$

$$\delta\omega = 1/\omega \left| \frac{\dfrac{\dfrac{\frac{1}{2} \partial g^{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} d\chi_{\mu}}{d\lambda} d\chi_{\nu}}{d\lambda} \delta\chi_{\sigma} + \frac{g^{\mu\nu} d\chi_{\mu}}{d\lambda} \delta \left(\frac{d\chi_{\nu}}{d\lambda} \right) = \frac{d}{d\lambda} (\delta\chi_{\nu}) \right|$$

$$\kappa\omega = \frac{d}{d\lambda} \left(\frac{g^{\mu\nu}}{\omega} \frac{d\chi_{\mu}}{d\lambda} \right) - \frac{\frac{1}{2\omega} \partial g^{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} \frac{d\chi_{\mu}}{d\lambda} \frac{d\chi_{\nu}}{d\lambda}$$

$$g^{\mu\nu} = \frac{d^{2}\chi_{\mu}}{ds^{2}} + \frac{\partial g^{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} \frac{d\chi_{\sigma}}{ds} \frac{d\chi_{\mu}}{ds} - \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} \frac{d\chi_{\mu}}{ds} \frac{d\chi_{\nu}}{ds}$$

$$[\mu\nu, \sigma] = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial \mathfrak{R}_{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial \mathfrak{R}_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} \right)$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial \mathfrak{R}_{\mu}} \frac{d\chi_{\mu}}{ds}$$

$$\psi = \frac{\dfrac{\partial \phi}{d\chi_{\mu}} \dfrac{d\chi_{\mu}}{ds}}{d\psi}$$

$$\chi = \frac{\partial^{2}\phi}{\partial \mathfrak{R}_{\mu} \partial \mathfrak{R}_{\nu}} \frac{d\chi_{\mu}}{ds} \frac{d\chi_{\nu}}{ds} + \frac{\partial \phi}{\partial \mathfrak{R}_{\mu}} \frac{d^{2}\chi_{\mu}}{ds^{2}} \left(\frac{\partial^{2}\phi}{\partial \mathfrak{R}_{\mu} \partial \mathfrak{R}_{\nu}} - [\mu\nu, \sigma] \frac{\partial \phi}{\partial \mathfrak{R}_{\tau}} \right) \frac{d\chi_{\mu}}{ds} \frac{d\chi_{\nu}}{ds}$$

$$\psi \frac{\partial^{2}\phi}{\partial \mathfrak{R}_{\mu} \partial \mathfrak{R}_{\nu}} - [\mu\nu, \tau] \psi \frac{\partial \phi}{\partial \mathfrak{R}_{\tau}}$$

$$\frac{\partial}{\partial \mathfrak{R}_{\nu}} \left(\psi \frac{\partial \phi}{\partial \mathfrak{R}_{\mu}} \right) - [\mu\nu, \tau] \left(\psi \frac{\partial \phi}{\partial \mathfrak{R}_{\tau}} \right)$$

$$\mathbb{A}_{\mu\nu\sigma} = \frac{\partial \mathrm{A}_{\mu\nu}}{\partial \Re_\sigma} - [\sigma\mu,\tau]\mathbb{A}_{\tau\nu} - [\sigma\nu,\tau]\mathbb{A}_{\mu\tau}$$

$$\frac{\frac{1}{\sqrt{-\mathfrak{g}}}\partial\sqrt{-\mathfrak{g}}}{\partial \Re_\sigma} = \frac{\frac{1}{2}\partial \log(-\mathfrak{g})}{\partial \Re_\sigma} = \frac{\frac{\frac{1}{2\mathscr{g}^{\mu\nu}}}{\partial \mathscr{g}_{\mu\nu}}}{\partial \Re_\sigma} = \frac{1}{2\mathscr{g}_{\mu\nu}}\partial \mathscr{g}^{\mu\nu}/\partial \Re_\tau$$

$$\begin{Bmatrix} \mathscr{g}_{\mu\sigma} d\mathscr{g}^{\nu\sigma} = -\mathscr{g}^{\nu\sigma} d\mathscr{g}_{\mu\sigma} \\ \frac{\mathscr{g}_{\mu\sigma}\partial \mathscr{g}^{\nu\sigma}}{\partial \Re_\lambda} = \frac{-\mathscr{g}^{\nu\sigma}\partial \mathscr{g}_{\mu\sigma}}{\partial \Re_\lambda} \end{Bmatrix}$$

$$\begin{Bmatrix} d\mathscr{g}^{\mu\nu} = -\mathscr{g}^{\mu\alpha}\mathscr{g}^{\nu\beta} d\mathscr{g}_{\alpha\beta} \\ \frac{\partial \mathscr{g}^{\mu\nu}}{\partial \Re_\sigma} = \frac{-\mathscr{g}^{\mu\alpha}\mathscr{g}^{\nu\beta}\partial \mathscr{g}_{\alpha\beta}}{\partial \Re_\sigma} \end{Bmatrix}$$

$$\frac{\partial \mathscr{g}_{\alpha\beta}}{\partial \Re_\sigma} = [\alpha\sigma,\beta] + [\beta\sigma,\alpha]$$

$$\frac{\partial \mathscr{g}^{\mu\nu}}{\partial \Re_\sigma} = -\mathscr{g}^{\mu\tau}[\tau\sigma,\nu] - \mathscr{g}^{\nu\tau}[\tau\sigma,\mu]$$

$$\frac{\frac{1}{\sqrt{-\mathfrak{g}}}\partial\sqrt{-\mathfrak{g}}}{\partial \Re_\sigma} = [\mu\sigma,\mu]$$

$$\frac{\partial}{\partial \Re_\nu\left(\mathscr{g}^{\mu\nu}\mathbb{A}_{\mu\nu}\right)} - \frac{\mathbb{A}_{\mu\nu}\partial \mathscr{g}^{\mu\nu}}{\partial \Re_\nu} - \frac{1}{2}\,\mathscr{g}^{\tau\alpha}\left(\frac{\partial \mathscr{g}_{\mu\alpha}}{\partial \Re_\nu} + \frac{\partial \mathscr{g}_{\nu\alpha}}{\partial \Re_\mu} - \frac{\partial \mathscr{g}_{\mu\nu}}{\partial \Re_\alpha}\right)\mathscr{g}^{\mu\nu}\mathbb{A}_{\mu\nu}$$

$$\frac{\frac{1}{2}\partial \mathscr{g}^{\tau\nu}}{\partial \Re_\nu}\mathbb{A}_\tau + \frac{\frac{1}{2\partial \mathscr{g}^{\tau\mu}}}{\partial \Re_\mu \mathbb{A}_\tau} + \frac{\frac{1}{\sqrt{-\mathfrak{g}}}\partial\sqrt{-\mathfrak{g}}}{\partial \Re_\sigma}\mathscr{g}^{\mu\nu}\mathbb{A}_\tau$$

$$\Phi = \frac{\frac{1}{\sqrt{-\mathfrak{g}}}\partial}{\partial \Re_\nu(\sqrt{-\mathfrak{g}}\mathbb{A}^{\mu\nu})}$$

$$\mathbb{B}_{\mu\nu\sigma} = \mathbb{A}_{\mu\nu\sigma} + \mathbb{A}_{\nu\sigma\mu} + \mathbb{A}_{\sigma\mu\nu} = \frac{\partial \mathbb{A}_{\mu\nu}}{\partial \Re_\sigma} + \frac{\partial \mathbb{A}_{\nu\sigma}}{\partial \Re_\mu} + \frac{\partial \mathbb{A}_{\mu\sigma}}{\partial \Re_\nu}$$

$$\frac{\partial}{\partial \Re_\sigma}\left(\mathscr{g}^{\mu\alpha}\mathscr{g}^{\nu\beta}\mathbb{A}_{\mu\nu}\right) - \frac{\mathscr{g}^{\mu\alpha}\partial \mathscr{g}^{\nu\beta}}{\partial \Re_\sigma}\mathbb{A}_{\mu\nu} - \frac{\mathscr{g}^{\nu\beta}\partial \mathscr{g}^{\mu\alpha}}{\partial \Re_\sigma}\mathbb{A}_{\mu\nu}$$

$$\sqrt{-\mathfrak{g}}\mathbb{A}_\mu = \frac{\partial\left(\sqrt{-\mathfrak{g}}\mathbb{A}_\mu^\sigma\right)}{\partial \Re_\sigma} - \frac{1}{2}\frac{\partial \mathscr{g}_{\rho\sigma}}{\partial \Re_\mu}\sqrt{-\mathfrak{g}}\mathbb{A}^{\rho\sigma}$$

$$\sqrt{-\mathfrak{g}}\mathbb{A}_\mu = \frac{\partial\left(\sqrt{-\mathfrak{g}}\mathbb{A}_\mu^\sigma\right)}{\partial \Re_\sigma} + \frac{1}{2}\frac{\partial \mathscr{g}^{\rho\sigma}}{\partial \Re_\mu}\sqrt{-\mathfrak{g}}\mathbb{A}_{\rho\sigma}$$

$$\mathbb{A}_{\mu\sigma\tau} = \frac{\partial^2 \mathbb{A}_\mu}{\partial \Re_\sigma \partial \Re_\tau} - \frac{[\mu\sigma,\rho]\partial \mathbb{A}_\rho}{\partial \Re_\tau} - \frac{[\mu\tau,\rho]\partial \mathbb{A}_\rho}{\partial \Re_\sigma} - \frac{[\sigma\tau,\rho]\partial \mathbb{A}_\mu}{\partial \Re_\rho}$$
$$+ \left[\!\!\left[\frac{\partial}{\partial \Re_\tau}[\mu\sigma,\rho] + [\mu\tau,\alpha][\alpha\sigma,\rho] + [\sigma\tau,\alpha][\sigma\mu,\rho]\right]\!\!\right]\mathbb{A}_\rho$$

$$\mathbb{B}_{\mu\sigma\tau}^{\rho} = \frac{\partial}{\partial \Re_\tau}\,[\mu\sigma,\rho] + \frac{\partial}{\partial \Re_\sigma}[\mu\tau,\rho] - [\mu\sigma,\alpha][\alpha\tau,\rho] + [\mu\tau,\alpha][\alpha\sigma,\rho]$$

$$\mathcal{R}_{\mu\nu} = -\frac{\partial}{\partial \Re_\alpha[\mu\nu,\alpha]} + [\mu\alpha,\beta][\nu\beta,\alpha]$$

$$\mathbb{S}_{\mu\nu} = \partial^2 \log\frac{\sqrt{-\mathfrak{g}}}{\partial \Re_\mu \partial \Re_\nu} - \frac{[\mu\nu,\alpha]\partial \log\sqrt{-\mathfrak{g}}}{\partial \Re_\alpha}$$

$$\langle\delta\int \begin{matrix} \mathcal{H}d\tau = 0 \\ \mathcal{H} = \mathscr{g}^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta}\sqrt{-\mathfrak{g}} = 1 \end{matrix}\rangle$$

$$\delta\mathcal{H} = \Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta}\delta \mathscr{g}^{\mu\nu} + 2\mathscr{g}^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} = -\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} + 2\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta}\delta\left(\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta}\right)$$

$$\delta\left(g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right)=\frac{1}{2}\delta\left(g^{\mu\nu}g^{\beta\lambda}\left(\frac{\partial g_{\nu\lambda}}{\partial\Re_{\alpha}}+\frac{\partial g_{\alpha\lambda}}{\partial\Re_{\nu}}+\frac{\partial g_{\alpha\nu}}{\partial\Re_{\lambda}}\right)\right)$$

$$\delta\mathcal{H}=-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu}-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\delta g^{\mu\beta}_{\alpha}$$

$$\begin{Bmatrix}\dfrac{\partial\mathcal{H}}{\partial g^{\mu\nu}}=-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\\ \dfrac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\sigma}}=\Gamma^{\sigma}_{\mu\nu}\end{Bmatrix}$$

$$\frac{\partial}{\partial\Re_{\alpha}}\left(\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\alpha}}\right)-\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}}=g^{\mu\nu}_{\sigma}\frac{\partial}{\partial\Re_{\alpha}}\left(\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\alpha}}\right)=\frac{\partial}{\partial\Re_{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\alpha}}\right)-\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\alpha}}\frac{\partial g^{\mu\nu}_{\alpha}}{\partial\Re_{\sigma}}-2\kappa t^{\alpha}_{\sigma}\frac{\partial\mathcal{H}}{\partial g^{\mu\nu}_{\alpha}}-\delta^{\alpha}_{\sigma}\mathcal{H}$$

$$\kappa t^{\alpha}_{\sigma}=\frac{1}{2\delta^{\alpha}_{\sigma}g^{\mu\nu}\Gamma^{\lambda}_{\mu\beta}\Gamma^{\beta}_{\nu\lambda}}-g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}$$

$$\frac{g^{\nu\sigma}\partial\Gamma^{\alpha}_{\mu\nu}}{\partial\Re_{\alpha}}=\frac{\partial}{\partial\Re_{\alpha}\left(g^{\nu\sigma}g^{\nu\beta}\partial\Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\beta}\right)}-\frac{\partial g^{\nu\sigma}g^{\sigma\beta}}{\partial\Re_{\alpha}\Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\beta}}$$

$$\frac{\partial}{\partial\Re_{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right)=-\kappa\left(t^{\sigma}_{\mu}-\frac{1}{2\delta^{\sigma}_{\mu}t}\right)\sqrt{-g}=1$$

$$\frac{\partial}{\partial\Re_{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right)=-\kappa\left(\left(t^{\sigma}_{\mu}-\frac{1}{2\delta^{\sigma}_{\mu}t}\right)-\frac{1}{2}\delta^{\sigma}_{\mu}(t+ד)\right)\sqrt{-g}=1$$

$$\frac{\partial}{\partial\Re_{\alpha}}\Gamma^{\alpha}_{\mu\nu}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}=-\kappa\left(ד_{\mu\nu}-\frac{1}{2g_{\mu\nu}ד}\right)\sqrt{-g}=1$$

$$\frac{\partial}{\partial\Re_{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}-\frac{1}{2\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}}\right)=-\kappa\left(t^{\sigma}_{\mu}+ד^{\sigma}_{\mu}\right)$$

$$\frac{\partial^{2}}{\partial\Re_{\alpha}\partial\Re_{\sigma}\left(g^{\sigma}\Gamma^{\alpha}_{\beta\mu}\right)}=-\frac{\frac{1}{2}\partial^{2}}{\partial\Re_{\alpha}\partial\Re_{\sigma}}\left(g^{\sigma\beta}g^{\alpha\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial\Re_{\beta}}+\frac{\partial g_{\beta\lambda}}{\partial\Re_{\mu}}+\frac{\partial g_{\mu\beta}}{\partial\Re_{\lambda}}\right)\right)$$

$$\Re_{\mu\nu}+\mathfrak{S}_{\mu\nu}=-\kappa\left(ד_{\mu\nu}+\frac{1}{2g_{\mu\nu}ד}\right)$$

$$\Re_{\mu\nu}=-\sum_{\alpha}\frac{\partial}{\partial\Re_{\alpha}\begin{vmatrix}\mu\nu\\ \alpha\end{vmatrix}}+\sum_{\alpha\beta}\begin{vmatrix}\mu\alpha\\ \beta\end{vmatrix}\begin{vmatrix}\nu\beta\\ \alpha\end{vmatrix}$$

$$\mathfrak{S}_{\mu\nu}=\frac{\partial^{2}\log\sqrt{g}}{\partial\Re_{\mu}\partial\Re_{\nu}}-\frac{\sum_{\alpha}\begin{vmatrix}\mu\nu\\ \alpha\end{vmatrix}\partial\log\sqrt{g}}{\partial\Re_{\alpha}}$$

$$\sum_{\alpha}\frac{\partial^{2}\gamma_{\mu\omega}}{\partial\Re_{\nu}\partial\Re_{\alpha}}+\sum_{\alpha}\frac{\partial^{2}\gamma_{\nu\alpha}}{\partial\Re_{\mu}\partial\Re_{\alpha}}\partial\Re_{\mu}\partial\Re_{\alpha}-\sum_{\alpha}\frac{\partial^{2}\gamma_{\mu\nu}}{\partial\Re^{2}_{\alpha}}-\frac{\partial^{2}}{\partial\Re_{\mu}\partial\Re_{\nu}}\left(\sum_{\alpha}\gamma_{\alpha\alpha}\right)$$
$$=-2\kappa(ד_{\mu\nu}-\frac{1}{2\delta_{\mu\nu}\sum_{\alpha}ד_{\alpha\alpha}}+\frac{2\partial^{2}\psi}{\partial\Re_{\mu}\partial\Re_{\nu}}-\delta_{\mu\nu}\sum_{\alpha}\frac{\partial^{2}\psi}{\partial\Re^{2}_{\alpha}}-4\frac{\partial^{2}\psi}{\partial\Re_{\mu}\partial\Re_{\nu}}+\frac{\partial g^{\rho\sigma}}{\partial\Re_{\alpha}}$$

$$-\frac{\partial\sqrt{-g}ד^{\sigma}_{\rho}}{\partial\Re_{\mu}\partial\Re_{\nu}}=\frac{\dfrac{\sum_{\alpha\beta\tau}\frac{\left(\partial\gamma^{\dagger}_{\alpha\beta}\right)}{\partial\Re_{\Lambda}d\tau}}{\partial\Delta\Re_{\Lambda}}}{\partial\Re_{\mu}\partial\Re_{\nu}}-\frac{\partial^{2}\Delta\Re_{\Lambda}}{\sqrt{-g}\Delta}-\int\frac{1}{2}ðא^{*}-\int-\frac{1}{4}\otimes g^{\mu\nu}_{\alpha\beta}.\,\delta\partial\Delta g^{\mu\nu}$$

17.5. Perturbaciones de los campos cuánticos curvos por el movimiento acelerado de partículas masivas (ondas a escala cuántica).

$$\mathfrak{g}_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$$

$$-\sum\nolimits_{\alpha} \frac{\partial}{\partial \chi_\alpha} \begin{matrix}\mu\nu\\ \alpha\end{matrix} + \sum\nolimits_{\alpha} \frac{\partial}{\partial \chi_\nu} \begin{matrix}\mu\alpha\\ \alpha\end{matrix} + \sum_{\alpha\beta} \begin{matrix}\mu\alpha\\ \beta\end{matrix} \begin{matrix}\nu\beta\\ \alpha\end{matrix} - \sum_{\alpha\beta} \begin{matrix}\mu\nu\\ \alpha\end{matrix} \begin{matrix}\alpha\beta\\ \beta\end{matrix} = -\kappa\left(\mathfrak{T}_{\mu\nu} - \frac{1}{2\mathfrak{g}_{\mu\nu}\mathfrak{T}}\right)$$

$$\frac{\sum_\alpha(\frac{\partial^2\gamma_{\mu\nu}}{\partial\chi_\alpha^2} + \frac{\partial^2\gamma_{\alpha\alpha}}{\partial\chi_\mu\partial\chi_\nu} - \frac{\partial^2\gamma_{\mu\alpha}}{\partial\chi_\nu\partial\chi_\alpha} - \partial^2\gamma_{\nu\alpha}}{\partial\chi_\mu\partial\chi_\alpha}) = 2\kappa\left(\mathfrak{T}_{\mu\nu} - \frac{1}{2\mathfrak{g}_{\mu\nu}}\sum_\alpha \mathfrak{T}_{\alpha\alpha}\right)$$

$$\frac{\sum_\alpha(\frac{\partial^2\gamma'_{\mu\nu}}{\partial\chi_\alpha^2} + \frac{\partial^2\gamma'_{\alpha\beta}}{\partial\chi_\nu\partial\chi_\alpha} - \frac{\partial^2\gamma'_{\mu\alpha}}{\partial\chi_\mu\partial\chi_\alpha} - \partial^2\gamma'_{\nu\alpha}}{\partial\chi_\alpha\partial\chi_\beta}) = 2\kappa\left(\mathfrak{T}_{\mu\nu} - \frac{1}{2\mathfrak{g}_{\mu\nu}}\sum_\alpha \mathfrak{T}_{\alpha\alpha}\right)$$

$$\gamma'_{\mu\nu} = -\kappa/2\pi \int \frac{\mathfrak{T}_{\mu\nu}(\chi_O y_o z_0, t - r)}{r} d\mathfrak{V}_0$$

$$\frac{\sum_\sigma \partial\mathfrak{T}_\mu^\sigma}{\partial\chi_\sigma} + \frac{1}{2\sum_{\rho\sigma}\frac{\partial\mathfrak{g}^{\rho\sigma}}{\partial\chi_\mu\mathfrak{T}_{\rho\sigma}}}$$

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \frac{1}{2\delta_{\mu\nu}\sum_\alpha\gamma'_{\alpha\alpha}} = \gamma'_{\mu\nu} - \frac{1}{2\delta_{\mu\nu}\gamma'}$$

$$\sum_\alpha \partial\mathfrak{t}_{\mu\sigma}/\partial\chi_\sigma = \sum_\alpha \partial/\partial\chi_\sigma \left(\frac{1}{4\kappa}\left(\sum_{\alpha\beta}\left(\frac{\frac{\partial\gamma'_{\alpha\beta}}{\partial\chi_\mu\partial\gamma'_{\alpha\beta}}}{\partial\chi_\sigma}\right) - \frac{1}{2}\frac{\partial\gamma'}{\partial\chi_\mu}\frac{\partial\gamma'}{\partial\chi_\sigma}\right)\right.$$

$$\left. - \frac{1}{8\kappa\delta_{\mu\sigma}\left(\sum_{\alpha\beta\lambda}(\frac{\partial\gamma'_{\alpha\beta}}{\partial\chi_\lambda})^2 - \frac{1}{2\sum_\lambda(\partial\gamma'\frac{\hbar}{\mathfrak{H}}/\partial\chi_\lambda)^2}\right)}\right)$$

$$4\kappa\daleth_{\mu\sigma} = \left(\frac{\sum_{\alpha\beta}\left(\frac{\frac{\partial\gamma'_{\alpha\beta}}{\partial\chi_\mu\partial\gamma'_{\alpha\beta}}}{\partial\chi_\sigma}\right) - 1}{2\frac{\partial\gamma'}{\partial\chi_\mu}\frac{\partial\gamma'}{\partial\chi_\sigma}}\right) - \frac{1}{2\delta_{\mu\sigma}\left(\sum_{\alpha\beta\lambda}(\frac{\partial\gamma'_{\alpha\beta}}{\partial\chi_\lambda})^2 - \frac{1}{\frac{\frac{\hbar}{\mathfrak{H}}}{2\sum_\lambda(\partial\gamma'\frac{\varkappa}{\mathcal{R}}/\partial\chi_\lambda)^2}}\right)}$$

$$\kappa\mathfrak{I}_{\sigma}^{\alpha} = \frac{1}{2\delta_{\sigma}^{\alpha}\sum_{\mu\nu\lambda\beta}\mathfrak{g}^{\mu\nu}\begin{matrix}\mu\lambda\nu\beta\\ \beta\ \ \lambda\end{matrix}} - \sum_{\mu\nu\lambda}\mathfrak{g}^{\mu\nu}\begin{matrix}\mu\lambda\nu\sigma\\ \alpha\ \ \lambda\end{matrix}$$

$$S = -1/4(\sum_{\mu}\mathrm{A}_{\mu\mu})^2 + \frac{1}{2\sum_{\mu}\mathrm{A}_{\mu\mu}}\sum_{\rho\sigma}\mathrm{A}_{\rho\sigma}\,\alpha_{\rho}\alpha_{\sigma} + \frac{1}{4}(\sum_{\rho\sigma}\mathrm{A}_{\rho\sigma}\alpha_{\rho}\alpha_{\sigma})^2 + \frac{1}{2\sum_{\mu\nu}\mathrm{A}_{\mu\nu}^2} - \sum_{\mu\sigma\tau}\mathrm{A}_{\mu\sigma}\mathrm{A}_{\mu\tau}\alpha_{\sigma}\alpha_{\tau}$$

17.6. Puente Einstein – Rosen en un campo cuántico geométricamente deformado o curvo, generado por una partícula masiva acelerada, explica (1) la superposición cuántica; (2) la aniquilación de partículas y antipartículas; y, (3) el entrelazamiento cuántico.

$$\mathfrak{d}\mathfrak{s}^2 = \mathfrak{d}\chi_1^2 - \mathfrak{d}\chi_2^2 - \mathfrak{d}\chi_3^2 + \alpha^2\chi_1^2\mathfrak{d}\chi_4^2$$

$$\mathcal{R}^{i\,k}_{\ lm} = \mathfrak{g}^2\mathcal{R}^{kl}\mathfrak{R}_{kl} = \mathbb{R}_{ik} - \frac{1}{2\mathfrak{g}_{ik}\daleth} = -\mathfrak{I}_{ik}$$

$$\mathfrak{I}_{ik} = \frac{1}{4\mathfrak{g}_{ik}\varphi_{\alpha\beta}\varphi^{\alpha\beta}} - \varphi_{\iota\alpha}\varphi_{\lambda}^{\kappa}/{}_{\sigma}^{\mu}\varrho^{\nu\sigma}$$

$$\mathfrak{d}\mathfrak{s}^2 = -\mathfrak{d}\chi_1^2 - \mathfrak{d}\chi_2^2 - \mathfrak{d}\chi_3^2 + (\alpha^2\chi_1^2 + \sigma)\mathfrak{d}\chi_4^2$$

$$\mathfrak{d}\mathfrak{s}^2 = -\frac{1}{1} - \frac{2m}{r}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left(1 - \frac{2m}{r}\right)dt^2$$

$$\mathfrak{d}\mathfrak{s}^2 = -4(\mu^2 + 2m)d\mu^2 + (\mu^2 + 2m)^2\,(d\theta^2 + sin^2\theta d\phi^2) + \frac{\mu^2}{\mu^2 + 2m}\,dt^2$$

$$\mathfrak{d}\mathfrak{s}^2 = -\frac{1}{1} - \frac{2m}{r - \varepsilon^2/2\mathfrak{r}^2}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \left(1 - \frac{2m}{r} - \varepsilon^2/2\mathfrak{r}^2\right)dt^2$$

$$\varphi_{\mu\nu} = \varphi_{\mu,\nu} - \varphi_{\nu,\mu} + \mathfrak{g}^2\varphi_{\mu\nu}{}_{\sigma}^{\mu}\varrho^{\nu\sigma} = \mathfrak{g}^2\left(\mathfrak{R}_{ik} + \varphi_{\alpha\beta}\varphi^{\alpha\beta}\right)$$

$$\frac{\sum_{\alpha}\partial^2}{\partial\mathfrak{R}_{\alpha}^2\left(\gamma\acute{}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\gamma\acute{}_{\alpha\alpha}\right)} = 2\kappa\mathfrak{I}_{\mu\nu}\left(\mathfrak{I}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\mathfrak{T}_{\alpha\alpha}\right)$$

$$\gamma_{\mu\nu} = \gamma\acute{}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\gamma\acute{}_{\alpha\alpha}$$

$$\gamma\acute{}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\gamma_{\alpha\alpha}$$

$$\gamma\acute{}_{\mu\nu} = -\frac{\frac{\kappa}{\overline{2\pi\int\mathfrak{I}_{\mu\nu}\,(\chi_O y_o z_0, t - r)}}}{r}.d\mathfrak{B}_0$$

$$\sum_{\alpha}\frac{\partial}{\partial\mathfrak{R}_{\alpha}}\left(\frac{\frac{\partial\gamma\acute{}_{\mu\nu}}{\partial\mathfrak{R}_{\sigma}}\partial\gamma\acute{}_{\mu\nu}}{\partial\mathfrak{R}_{\alpha}}\frac{1}{2}\delta_{\alpha\sigma}\sum_{\mu\nu\beta}\left(\frac{\partial\gamma\acute{}_{\mu\nu}}{\partial\mathfrak{R}_{\beta})^2}\right)\right.$$

$$\sum_{\sigma} \frac{\partial \sqrt{-\mathfrak{g}} \mathfrak{J}_{\mu}^{\sigma}}{\partial \mathfrak{R}_{\sigma}} + \frac{\frac{1}{2 \sum_{\rho\sigma} \partial \mathfrak{g}^{\rho\sigma}}}{\partial \mathfrak{R}_{\mu}} \sqrt{-\mathfrak{g}} \mathfrak{J}_{\sigma}^{\rho} \mathfrak{J}^{\rho\sigma} \mathfrak{J}_{\rho\sigma} - 4\kappa \sum_{\nu} \frac{\partial \mathfrak{J}_{\rho\sigma}}{\partial \mathfrak{R}_{\nu}}$$

$$\mathfrak{t}_{\mu\nu} = \frac{1}{4\kappa} \left(\frac{\frac{\sum_{\alpha\beta} \partial \gamma'_{\alpha\beta}}{\partial \mathfrak{R}_{\mu}} \partial \gamma'_{\alpha\beta}}{\partial \mathfrak{R}_{\nu}} - \frac{\frac{1}{2} \delta_{\mu\nu} \sum_{\alpha\beta\tau} (\partial \gamma'_{\alpha\beta}}{\partial \mathfrak{R}_{\tau})^2} \right)$$

$$\frac{\partial}{\partial \mathfrak{R}_{\alpha} \left(\frac{\partial \mathfrak{G}^*}{\partial \mathfrak{g}_{\alpha}^{\mu\nu}} \right)} - \frac{\partial \mathfrak{G}^*}{\partial \mathfrak{g}^{\mu\nu}} = \frac{\partial \mathfrak{M}}{\partial \mathfrak{g}^{\mu\nu}} = \frac{\partial}{\partial \mathfrak{R}_{\alpha}} \left(\frac{\partial \mathfrak{M}}{\partial \mathfrak{q}_{(p)_{\alpha}}} \right) - \partial \mathfrak{M} \partial \mathfrak{q}_{(p)_{\beta}}$$

$$\mathfrak{G}^* = \sqrt{-\mathfrak{g}} \mathfrak{g}^{\mu\nu} \left(\binom{\mu\alpha}{\beta} \binom{\mu\beta}{\alpha} - \binom{\mu\nu}{\alpha} \binom{\alpha\beta}{\beta} \right)$$

$$\Delta \mathfrak{g}^{\mu\nu} = \frac{\mathfrak{g}^{\mu\alpha} \partial \Delta \chi_{\nu}}{\partial \mathfrak{R}_{\alpha}} + \frac{\mathfrak{g}^{\nu\alpha} \partial \Delta \chi_{\mu}}{\partial \mathfrak{R}_{\alpha}} + \Delta \mathfrak{g}_{\sigma}^{\mu\nu\rho} = \frac{\partial \Delta \mathfrak{g}^{\mu\nu}}{\partial \mathfrak{R}_{\sigma}} - \frac{\mathfrak{g}_{\sigma}^{\mu\nu\rho} \partial \Delta \chi_{\alpha}}{\partial \mathfrak{R}_{\sigma}}$$

$$\sqrt{-\mathfrak{g}} \Delta \left(\frac{\mathfrak{G}^*}{\sqrt{-\mathfrak{g}}} \right) = \delta_{\sigma}^{\nu} \frac{\partial \Delta \chi_{\sigma}}{\partial \mathfrak{R}_{\nu}} + 2 \frac{\partial \mathfrak{G}^*}{\partial \mathfrak{g}_{\alpha}^{\mu\nu}} + \mathfrak{G}^* \frac{\mathfrak{g}^{\mu\nu} \partial^2 \Delta \chi_{\sigma}}{\partial \mathfrak{R}_{\nu} \partial \mathfrak{R}_{\alpha}} + \delta_{\sigma}^{\nu}$$
$$= \frac{2 \partial \mathfrak{G}^*}{\partial \mathfrak{g}^{\mu\sigma} \mathfrak{g}^{\mu\nu}} + \frac{2 \partial \mathfrak{G}^*}{\partial \mathfrak{g}_{\alpha}^{\mu\sigma}} + \frac{1}{2} \mathfrak{G}^* \delta_{\sigma}^{\nu} - \partial \mathfrak{G}^* / \partial \mathfrak{g}_{\nu}^{\mu\alpha} \mathfrak{g}_{\sigma}^{\mu\alpha}$$

$$\mathfrak{G}_{\mu\nu} = -\kappa \left(\mathfrak{T}_{\mu\nu} - \frac{1}{2 \mathfrak{g}^{\mu\nu} \mathfrak{T}} \right)$$

$$\mathfrak{G}_{\mu\nu} = - \frac{\partial}{\partial \mathfrak{R}_{\alpha} (\mu\nu, \alpha)} + (\mu\alpha, \beta)(\nu\beta, \alpha) + \partial^2 log \frac{\sqrt{-\mathfrak{g}}}{\partial \mathfrak{R}_{\mu} \partial \mathfrak{R}_{\nu}} - \frac{(\mu\nu, \alpha) \partial log \sqrt{-\mathfrak{g}}}{\partial \mathfrak{R}_{\alpha}} \lambda \mathfrak{g}_{\mu\nu}$$

Apéndice B.

Formalización matemática relativa a los espacios cuánticos curvos, como un intento por reconciliar la relatividad general y especial y la mecánica cuántica.

1. Operador Lindblad en espacios cuánticos curvos.

$$\varrho(z, t + \delta t) = \int dz' \sum_{\mu\nu} \Lambda^{\mu\nu} (z|z', \delta t) \mathcal{L}_{\mu} \varrho(z', t) \mathcal{L}_{\nu}^{\dagger} \int dz \Lambda^{\mu\nu} (z|z', \delta t) \mathcal{L}_{\nu}^{\dagger} \mathcal{L}_{\mu}, \Lambda^{\mu\nu} (z|z', \delta t)$$
$$= \delta_0^{\mu} \delta_0^{\gamma} \delta(z', z) + \mathcal{W}^{\mu\nu}(z|z') \delta t + \mathcal{O}(\delta t)^2, \frac{\partial \varrho(z, t)}{\partial t}$$
$$= \int dz' \, \mathcal{W}^{\mu\nu}(z|z') \mathcal{L}_{\mu} \varrho(z', t) \mathcal{L}_{\nu}^{\dagger} - \frac{1}{2 \mathcal{W}^{\mu\nu}(z) \{ \mathcal{L}_{\nu}^{\dagger} \mathcal{L}_{\mu} \varrho \}_{+}}, \mathcal{W}^{\mu\nu}(z)$$
$$= \int dz' \, \mathcal{W}^{\mu\nu}(z|z'), \Lambda^{\mu\nu}(z|z', \delta t)$$
$$= \begin{bmatrix} \delta(z', z) + \delta t \mathcal{W}^{00}(z|z') & \delta t \mathcal{W}^{\beta 0}(z|z') \\ \delta t \mathcal{W}^{\alpha 0}(z|z') & \delta t \mathcal{W}^{\alpha\beta}(z|z') \end{bmatrix} + \mathcal{O}(\delta t)^2 \mathcal{L}_{\mu}^{\dagger}$$

2. Modelo Kramers–Moyal en espacios cuánticos curvos.

$$\mathcal{D}^{\mu\nu}_{\eta,\iota_1\otimes\iota_\eta}(z') = \frac{1}{n!\int dz\,\mathcal{W}^{\mu\nu}(z|z')(z-z')_{\iota_1}\otimes(z-z')_{\iota_\eta}}, \Lambda^{\mu\nu}(z|z',\delta t)\delta^{\mu}_{0}\delta^{\nu}_{0}\delta(z',z)$$
$$+\ \delta t\sum_{\eta=0}^{\infty}\mathcal{D}^{\mu\nu}_{\eta,\iota_1\otimes\iota_\eta}(z')\left(\partial^{\eta}/\partial z'_{\iota_1}\otimes\partial z'_{\iota_\eta}\right)\delta(z,z') + \mathcal{O}(\delta t)^2, \frac{\partial\varrho(z,t)}{\partial t}$$
$$= \sum_{\eta=1}^{\infty}(-1)^{\eta}\left(\partial^{\eta}/\partial z'_{\iota_1}\otimes\partial z'_{\iota_\eta}\right)\left(\mathcal{D}^{00}_{\eta,\iota_1\otimes\iota_\eta}(z)\varrho(z,t)\right) - i[\mathcal{H}(z),\varrho(z)]$$
$$+\,\mathcal{D}^{\alpha\beta}_{\eta,\iota_1\otimes\iota_\eta}(z)\{\mathcal{L}^{\dagger}_{\alpha}\mathcal{L}_{\beta}\varrho(z)\}_{+}$$
$$+\sum_{\mu\nu\neq 00}\sum_{\eta=1}^{\infty}(-1)^{\eta}\left(\frac{\partial^{\eta}}{\partial z'_{\iota_1}\otimes\partial z'_{\iota_\eta}}\right)\left(\mathcal{D}^{\mu\nu}_{\eta,\iota_1\otimes\iota_\eta}(z)\mathcal{L}_{\mu}\varrho(z,t)\mathcal{L}^{\dagger}_{\nu}\right), \frac{d\langle\mathcal{O}\rangle}{dt}$$
$$= \int dz Tr\left[\frac{\mathcal{O}(z)\partial\varrho}{\partial t}\right]$$
$$= \int dz Tr\varrho\left[-i[\mathcal{O}(z),\mathcal{H}(z)] + \mathcal{D}^{\alpha\beta}_{\eta,\iota_1\otimes\iota_\eta}(z)\mathcal{L}^{\dagger}_{\beta}\mathcal{O}(z)\mathcal{L}_{\alpha}\right.$$
$$\left.-\frac{1}{2}\mathcal{D}^{\alpha\beta}_{\eta,\iota_1\otimes\iota_\eta}(z)\left\{\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\mathcal{O}(z)\right\}_{+}\sum_{\eta=1}^{\infty}\mathcal{D}^{\mu\nu}_{\eta,\iota_1\otimes\iota_\eta}(z')\left(\frac{\partial^{\eta}}{\partial z'_{\iota_1}\otimes\partial z'_{\iota_\eta}}\right)\left(\mathcal{L}^{\dagger}_{\nu}\mathcal{L}_{\mu}\mathcal{O}(z,t)\right)_{+}\right], \frac{d\langle z_{\iota}\rangle}{dt}$$
$$= \int dz\,\mathcal{D}^{\mu\nu}_{1,\iota}Tr\left[\mathcal{L}^{\dagger}_{\nu}\mathcal{L}_{\mu}\varrho(z,t)\right]$$

$$\frac{\partial\rho(z,t)}{\partial t} = \{\mathcal{H}_c,\rho(z,t)\} + tr(\{\mathcal{H}_l(z)\varrho(z)\}), \frac{d\sigma^2_{z_{\iota_1},z_2}}{dt}$$
$$= 2\,\langle\mathcal{D}^{\alpha\beta}_{2,\iota_1,\iota_2}\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\rangle + \langle z_2\mathcal{D}^{\alpha\beta}_{1,z_{\iota_1}}\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\rangle - \langle z_{\iota_2}\rangle\langle\mathcal{D}^{\alpha\beta}_{1,z_{\iota_1}}\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\rangle + \langle z_{\iota_1}\mathcal{D}^{\alpha\beta}_{1,z_{\iota_2}}\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\rangle$$
$$-\,\langle z_{\iota_1}\rangle\langle\mathcal{D}^{\alpha\beta}_{1,z_{\iota_2}}\mathcal{L}^{\dagger}_{\beta}\mathcal{L}_{\alpha}\rangle, \langle\alpha\left|\frac{\partial\varrho}{\partial t}\right|\beta\rangle = -i\langle\alpha|\mathcal{H}(z),\varrho|\beta\rangle - \frac{1}{2}\mathcal{D}_0\big(\mathcal{L}(\alpha)-\mathcal{L}(\beta)\big)^2\langle\alpha|\varrho|\beta\rangle$$

$$\mathcal{M}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)$$
$$= \int\mathcal{D}z\Lambda^{\mu\nu}(z|z',\chi,\gamma,\delta t)(z-z')_{i_1}(\omega_1)\otimes(z-z')_{i_\eta}(\omega_\eta), \mathcal{M}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)$$
$$= \delta^{\mu}_{0}\delta^{\nu}_{0}$$
$$+\,\delta t\eta!\,\mathcal{D}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t), \mathcal{C}^{\mu\nu}(\mu,z',\chi,\gamma)\int\mathcal{D}z\,e^{i\int d\omega\mu(\omega)\otimes(z(\omega)-z'(\omega))}\Lambda^{\mu\nu}(z|z',\chi,\gamma),$$

$$\mathcal{C}^{\mu\nu}(\mu,z',\chi,\gamma) = \frac{\sum_{\eta=0}^{\infty}\int d\omega_1\otimes d\omega_\eta\mu_{i_1}(\omega_1)\otimes\mu_{i_\eta}(\omega_\eta)}{\eta!\,\mathcal{M}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)}$$

$$\Lambda^{\mu\nu}(z|z',\chi,\gamma,\delta t)$$
$$= \frac{\sum_{\eta=0}^{\infty}\int d\omega_1\otimes d\omega_\eta\,\mathcal{M}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)}{\eta!}\delta^{\eta}/\delta z'_{i_1}(\omega_1)\otimes\delta z'_{i_\eta}(\omega_\eta)\delta(z,z')$$

$$\frac{\partial\varrho(z,\delta t)}{\partial t}=\sum_{\eta=1}^{\infty}\int d\omega_1\otimes d\omega_\eta\,(-1)^\eta\delta^\eta/\,\delta z'_{i_1}(\omega_1)\otimes\delta z'_{i_\eta}(\omega_\eta)\left(\mathcal{D}^{00}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)\varrho(Z)\right)$$
$$-i[\mathcal{H},\varrho(Z)]+\int d\chi d\gamma\,\mathcal{D}_0^{\alpha\beta}(z,\chi,\gamma)\mathcal{L}_\beta^\dagger(\gamma)\varrho(z)\mathcal{L}_\alpha(\chi)$$
$$-\frac{1}{2}\mathcal{D}_0^{\alpha\beta}(z,\chi,\gamma)\left\{\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\varrho\right\}$$
$$+\sum_{\mu\nu\neq 00}\sum_{\eta=0}^{\infty}\int d\chi d\gamma d\omega_1\otimes d\omega_\eta(-1)^\eta\delta^\eta$$
$$/\delta z'_{i_1}(\omega_1)\otimes\delta z'_{i_\eta}(\omega_\eta)\left(\mathcal{D}^{\mu\nu}_{\eta,i_1\otimes i_\eta}(z',\omega_1\otimes\omega_\eta,\chi,\gamma,\delta t)\mathcal{L}_\nu^\dagger(\gamma)\varrho(z)\mathcal{L}_\mu(\chi)\right)$$

$$\int d\chi d\gamma 2\beta_\mu^{i^*}(\chi)\mathcal{D}^{\mu\nu}_{2,ij}(\chi,\gamma)\beta_j^\gamma(\gamma)+\beta_\mu^{i^*}(\chi)\mathcal{D}^{\mu\beta}_{1,i}(\chi,\gamma)\alpha_\beta(\gamma)+\alpha_\alpha^{\circledcirc}(\chi)\mathcal{D}^{\alpha\mu}_{1,i}(\chi,\gamma)\beta_i^\mu(\gamma)$$
$$+\alpha_\alpha^{\circledcirc}(\chi)\mathcal{D}_0^{\alpha\beta}(\chi,\gamma)\alpha_\beta(\gamma)\int d\chi d\gamma\left\langle\left|[\beta^\star,\alpha^\star]\begin{bmatrix}2\mathcal{D}_2(\chi,\gamma) & \mathcal{D}_1^{br\dagger}(\chi,\gamma)\\ \mathcal{D}_1^{br}(\chi,\gamma) & \mathcal{D}_0^{-1}(\chi,\gamma)\end{bmatrix}\begin{bmatrix}\beta(\gamma)\\ \alpha(\gamma)\end{bmatrix}\right|-\mathbb{I}\right\rangle$$

3. Modelo Cauchy-Schwartz en espacios cuánticos curvos.

$$Tr\left[\iint dzdz'\Lambda^{\mu\nu}(z|z',\delta t)\mathcal{O}_\mu(z,z')\rho(z')\mathcal{O}_\nu^\dagger(z,z')\right],\langle\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2\rangle$$
$$=\int dzdz'Tr\left[\Lambda^{\mu\nu}(z|z',\delta t)\mathcal{O}_{1\mu}\varrho(z')\mathcal{O}_{2\nu}^\dagger\right],\|\bar{\mathcal{O}}\|^2=\|(\|\bar{\mathcal{O}}\|^2\bar{\mathcal{O}}_1-\langle\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2\rangle\bar{\mathcal{O}}_1)\|^2$$
$$=\|\bar{\mathcal{O}}\|^2(\|\bar{\mathcal{O}}_1\|^2\|\bar{\mathcal{O}}_2\|^2$$
$$-\langle|\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2|\rangle^2)\int dzTr\left[2\beta^{\iota\circledast}\mathcal{D}^{\mu\nu}_{2,ij}\beta^j\mathcal{L}_\mu\varrho(z)\mathcal{L}_\nu^\dagger\right]\int dzTr\left[\mathcal{D}_0^{\alpha\beta}\mathcal{L}_\alpha\varrho(z)\mathcal{L}_\beta^\dagger\right]$$
$$\geq\left|\iint dzTr\left[\beta^{\iota\circledast}\mathcal{D}^{\mu\alpha}_{1,\iota}\mathcal{L}_\mu\varrho(z)\mathcal{L}_\alpha^\dagger\right]\right|^2,\langle\mathcal{D}_0\rangle$$
$$=\langle\mathcal{D}_1^{br}\rangle\langle\mathcal{D}_2^{br}\rangle^\dagger\inf_{\varrho(z)}\int dzTr\left[\mathcal{D}_0^{\alpha\beta}\mathcal{L}_\alpha\varrho(z)\mathcal{L}_\beta^\dagger\right],\langle\mathcal{D}_1^{br}\rangle_\iota\int dzTr\left[\mathcal{D}^{\mu\alpha}_{1,\iota}\mathcal{L}_\mu\varrho(z)\mathcal{L}_\alpha^\dagger\right],\langle\mathcal{D}_2\rangle_{ij}$$
$$=\int dzTr\left[\mathcal{D}^{\mu\nu}_{2,ij}\mathcal{L}_\mu\varrho(z)\mathcal{L}_\nu^\dagger\right]$$

$$Tr\left[\iint dzdz'^{\Lambda^{\mu\nu}(z|z',\chi,\gamma)\mathcal{O}_\mu(z,z',\chi)}\rho(z')\mathcal{O}_\nu^\dagger(z,z',y)\right],\langle\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2\rangle$$
$$=\int dzdz'Tr\left[\Lambda^{\mu\nu}(z|z',\chi,\gamma)\mathcal{O}_{1\mu}(\chi)\varrho(z')\mathcal{O}_{2\nu}^\dagger(\gamma)\right],\|\bar{\mathcal{O}}_1\|^2\|\bar{\mathcal{O}}_2\|^2-|\langle\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2\rangle|^2,\|\bar{\mathcal{O}}_1\|^2$$
$$=\int dzd\chi d\gamma Tr\left[\mathcal{D}_0^{\alpha\beta}(z,\chi,\gamma)\mathcal{L}_\alpha(\chi)\varrho(z)\mathcal{L}_\beta^\dagger(\gamma)\right]\langle\mathcal{D}_0\rangle,\|\bar{\mathcal{O}}_2\|^2$$
$$=2\int dzd\chi d\gamma Tr\left[\beta^{j\circledast}(\tilde{\chi})\mathcal{D}^{\mu\nu}_{2,ij}(\widehat{z,\chi,\gamma})\mathcal{L}_\mu(\chi)\varrho(z)\mathcal{L}_\nu^\dagger(\gamma)\beta^{i\circledast}(\hat{\gamma})\right],|\langle\bar{\mathcal{O}}_1,\bar{\mathcal{O}}_2\rangle|^2$$
$$=\left|\iint dzd\chi Tr\left[\beta^{i\circledast}(\bar{\chi})\mathcal{D}^{\alpha\nu}_{1,i}(z,\chi)\mathcal{L}_\alpha(\chi)\varrho(z)\mathcal{L}_\beta^\dagger(\gamma)\right]\right|^2\left|\langle\int d\chi\beta^{i\circledast}(\bar{\chi})\mathcal{D}^{br}_{1,i}(\widehat{z,\chi,\gamma})\rangle^\dagger\right|^2$$

$$\frac{\partial\varrho}{\partial t}=\{\mathcal{H}_c,\varrho\}-i[\widehat{\mathcal{H}}^{(m)},\varrho]+\frac{1}{2}[\widehat{\mathcal{H}}^{(m)},\varrho]-\frac{1}{2}[\widehat{\mathcal{H}}^{(m)},\varrho]+\left\{\rho^i,\{\rho^i,\mathcal{D}_{2,ij}\varrho\}\right\}$$
$$+\frac{1}{2}\mathcal{D}_{0,ij}\left[\frac{\partial\widehat{\mathcal{H}}^{(m)}}{\partial q_i},\left[\varrho\frac{\partial\widehat{\mathcal{H}}^{(m)}}{\partial q_j}\right]\right],\langle\mathcal{D}_1^{\mathcal{T}}\rangle_{ij}^{\dagger}$$
$$=\sum_{\mu\nu\neq 00}\int dzTr[\mathcal{D}_{1,i}^{\mu\nu}\mathcal{L}_\mu\varrho(z)\mathcal{L}_\nu^{\dagger}],\langle\frac{\omega\otimes\partial\mathcal{H}_l}{\partial\vec{z}}\rangle\langle\frac{\omega\otimes\partial\mathcal{H}_l}{\partial\vec{z}}\rangle^{\dagger},\rho(z,t)$$
$$=\int dz'd\chi d\gamma\Lambda^{\mu\nu}(z|z',t,\chi,\gamma)\,\mathcal{L}_\mu(\chi,z,z')\varrho(z',0)\mathcal{L}_\nu^{\dagger}(\gamma,z,z')$$

$$\int dz'd\chi d\gamma\mathrm{A}_\mu^{\circledast}(\chi,z,z')\Lambda^{\mu\nu}(z|z',\chi,\gamma)\,\mathrm{A}_\nu(\gamma,z,z')\int d\chi d\gamma[\beta^{\circledast}(\chi),\alpha^{\circledast}(\chi)]\begin{bmatrix}2\mathcal{D}_2(\chi,\gamma) & \mathcal{D}_1^{br}(\chi,\gamma)\\ \mathcal{D}_1^{br}(\chi,\gamma) & \mathcal{D}_0(\chi,\gamma)\end{bmatrix}\begin{bmatrix}\beta(\gamma)\\ \alpha(\gamma)\end{bmatrix}$$

$$\int d\chi d\gamma\alpha_\nu^{i\circledast}(\chi)\mathcal{D}_{1,i}^{\mu\alpha}(\chi)(\mathcal{D}_0^{-1})_{\alpha\beta}(\chi,\gamma)\mathcal{D}_{1,j}^{\beta\nu}(\chi')\alpha_\nu^{i}(\chi')\int d\chi d\gamma 2\alpha_\mu^{i\circledast}(\chi)\mathcal{D}_{2,ij}^{\mu\nu}(\chi,\gamma)\alpha_\nu^{j}(\gamma)$$

$$16\langle\mathcal{D}_2(\overline{\chi,\chi})\rangle\langle\mathcal{D}_0\rangle,\succcurlyeq\langle\mathcal{D}_1^{\mathcal{T}}(\bar{\chi})\rangle\langle\mathcal{D}_1^{\mathcal{T}}(\bar{\chi})\rangle^{\dagger}\int d\chi d\gamma\,\langle\mathcal{D}_2(\chi,\gamma)\rangle\succcurlyeq\langle\mathcal{D}_1^{br}(\bar{\chi})\rangle\langle\mathcal{D}_1^{br}(\bar{\chi})\rangle^{\dagger}$$
$$\succcurlyeq\langle\int d\chi\mathcal{D}_1^{br}(\bar{\chi})\rangle\langle\int d\chi\mathcal{D}_1^{\mathcal{T}}(\bar{\chi})\rangle^{\dagger}\langle\mathcal{D}_2(\chi,\gamma)\rangle_{ij}\int dzTr\left[\mathcal{D}_{2,ij}^{\mu\nu}(\widehat{z,\chi,\gamma})\mathcal{L}_\mu(\chi)\varrho(z)\mathcal{L}_\nu^{\dagger}(\gamma)\right]$$

$$\int d\chi d\gamma\,\langle\mathcal{D}_2(\chi,\gamma)\rangle\geq\frac{\mathcal{M}^4}{32\lambda},\frac{16\int d\chi d\gamma\,\langle\mathcal{D}_2(\chi,\gamma)\rangle}{\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}\langle\mathcal{D}_0\rangle}$$
$$\succcurlyeq\frac{\langle\int\frac{d\chi\mathcal{D}_1^{br}(\bar{\chi})}{\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}}\rangle\langle\int\frac{d\chi\mathcal{D}_1^{\mathcal{T}}(\bar{\chi})}{\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}}\rangle^{\dagger}\int d\chi d\gamma\,\langle\mathcal{D}_{2,\pi_\Phi\pi_\Phi}(\chi,\gamma)\rangle}{\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}}$$
$$\succcurlyeq\frac{\left|\int d\chi\,\langle\frac{\widehat{m}(\chi)}{\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}}\rangle\right|^2}{32\lambda}=\left|\widehat{\Phi}(\bar{\chi})\right|^2/32\mathfrak{G}^2\lambda$$

4. Gravedad cuántica einsteniana por espacios curvos.

$$\mathcal{H}_l(\Phi)=\int d^3\chi\phi(\chi)\langle\widehat{m}(\chi)\rangle,\dot{\pi}_\phi=\frac{\nabla^2\Phi}{4\pi\mathfrak{G}}-m(\chi),Tr=[\{\mathcal{H}_l,\varrho\}]=Tr\left[\int\frac{d^3\hat{\chi}m(\chi)\delta\varrho}{\delta\pi_\phi}\right]$$
$$=-\frac{\sum_{\mu\nu\neq 00}Tr\int d^3\chi\mathcal{D}_{1,\pi_\phi}^{\mu\nu}(\phi,\pi_\phi,\chi)\,\mathcal{L}_\mu(\chi)\delta\varrho}{\delta\pi_\phi\mathcal{L}_\nu^{\dagger}(\chi)},\dot{\pi}_\phi$$
$$=\frac{\nabla^2\Phi}{4\pi\mathfrak{G}}-\widehat{m}(\chi)+\mu(\phi,\widehat{m})\mathcal{J}(t,x),\mathbb{E}_{m,\phi}[\mu\mathcal{J}(x,t)\mu\mathcal{J}(y,t')]\langle\mathcal{D}_2(x,y,\phi)\rangle\delta(t,t'),\sigma_{\mathcal{F}}^2$$
$$=\frac{2\mathfrak{G}^2}{\frac{\Delta\Gamma\int d^3\chi d^3\gamma d^3\chi'd^3\gamma'm(\chi)m(\gamma)\overline{(\chi-\vec{\chi})\otimes(\gamma-\vec{\gamma})}}{|\chi-\chi'|^3|\gamma-\gamma'|^3\langle\mathcal{D}_2(\chi',\gamma',\phi)\rangle}}$$

$$\lambda=\frac{1}{2}\int d\chi d\gamma\mathcal{D}_0^{\alpha\beta}(\chi,\gamma)\left(\langle\mathcal{L}\middle|\mathcal{L}_\beta^{\dagger}(\gamma)\mathcal{L}_\alpha(\chi)\middle|\mathcal{L}\rangle+\langle\mathcal{R}\middle|\mathcal{L}_\beta^{\dagger}(\gamma)\mathcal{L}_\alpha(\chi)\middle|\mathcal{R}\rangle\right)$$

$$\frac{\sigma_\alpha^2\mathcal{N}r_{\mathcal{N}}^4\Delta\Gamma}{\mathcal{V}_\beta\mathfrak{G}^2}\geq\mathcal{D}_2\geq\frac{\mathcal{N}_\lambda\mathcal{M}_\lambda^2}{\mathcal{V}_\lambda\lambda},\sigma_\alpha^2\sim\frac{\mathcal{D}_2\mathfrak{G}^2}{r_{\mathcal{N}}^4\mathcal{N}\Delta\Gamma\int d^3\chi'^{\mathcal{D}_2(\phi_\beta)}},\frac{\sigma_\alpha^2\mathcal{N}r_{\mathcal{N}}^4\Delta\Gamma}{m_{\mathcal{N}}\mathfrak{G}^2}\geq\frac{\ell_\rho^3}{m_\rho\mathcal{D}_2}\geq\frac{\mathcal{M}_\lambda}{\lambda},\sigma_\alpha^2\mathcal{N}r_{\mathcal{N}}^3\Delta\Gamma/\mathfrak{G}^2$$
$$\geq\ell_\rho^2\mathcal{D}_2\geq\mathcal{N}_\lambda\mathcal{M}_\lambda^2/\mathcal{R}_\lambda\lambda$$

$$\int dz \mathrm{A}_{\mu}^{\star}(z,z')\Lambda^{\mu\nu}(z|z',\delta t)\,\mathrm{A}_{\nu}(z,z')\int dzdz'^{\lambda^{\mu}(z|z')\int dzdz'}\,\lambda^{\mu}(z|z')\mathcal{P}_{\mu}(z,z'),\Lambda^{\mu\nu}(z|z')$$
$$= \mathcal{U}_{\rho}^{\mu\dagger}(z|z')\lambda^{\rho}(z|z')\mathcal{U}_{\sigma}^{\nu\dagger}(z|z')\int dz\mathrm{A}_{\mu}^{\star}(z,z')\Lambda^{\mu\nu}(z|z',\delta t)\mathrm{A}_{\nu}(z,z')$$
$$= \int dz(\mathcal{U}\mathcal{A})_{\mu}^{\dagger}(z|z')\lambda^{\mu}(z|z')(\mathcal{U}\mathcal{A})_{\mu}(z,z')\int dz\left|(\mathcal{U}\mathcal{A})_{\mu}\right|^{2}(z,z')\lambda^{\mu}(z|z'),$$

$$Tr\left[\int dz\Lambda^{\mu\nu}(z|z')\mathcal{O}_{\mu}(z,z')\rho(z')\,\mathcal{O}_{\nu}^{\dagger}(z,z')\right]$$

$$2\beta_{\mu}^{i*}\mathcal{D}_{2,ij}^{\mu\nu}\beta_{j}^{\gamma}+\beta_{\mu}^{i*}\mathcal{D}_{1,i}^{\mu\beta}\alpha_{\beta}+\alpha_{\alpha}^{\circledcirc}\mathcal{D}_{1,i}^{\alpha\mu}\beta_{i}^{\mu}+\alpha_{\alpha}^{\circledcirc}\mathcal{D}_{0}^{\alpha\beta}\alpha_{\beta}[\beta^{\star},\alpha^{\star}]\left\langle\begin{matrix}2\mathcal{D}_{2} & \mathcal{D}_{1}^{br\dagger}\\ \mathcal{D}_{1}^{br} & \mathcal{D}_{0}^{-1}\end{matrix}\right\rangle\left\langle\begin{matrix}\beta\\ \alpha\end{matrix}\right\rangle$$

$$\rho'^{(z)} = \int dzd\chi d\gamma\,\Lambda^{\mu\nu}(z|z',\chi,\gamma)\mathcal{L}_{\mu}(\chi,z,z')\varrho(z')\mathcal{L}_{\nu}^{\dagger}(\gamma,z,z')$$

$$\int dzd\chi d\gamma\,\mathrm{A}_{\mu}^{\star}(\chi,z,z')\Lambda^{\mu\nu}(z|z',\chi,\gamma)\mathrm{A}_{\nu}(\gamma,z,z')$$

$$\mathfrak{G}^{\mu\nu} = \frac{16\pi\mathfrak{G}}{c^{4}m^{4}}\langle\hat{\mathcal{T}}^{\mu\nu}\rangle,\partial\hat{\sigma}_{\mathcal{M}} \approx -i\left[\hat{\sigma}_{\mathcal{M}},\widehat{\mathcal{H}}_{\mathcal{M}}\right]\otimes\langle\widehat{\mathcal{H}}_{\mathcal{G}}\rangle,\frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$= -i\left[\widehat{\mathcal{H}}(z),\rho(z,t)\right]+\frac{1}{2}\left(\left\{\widehat{\mathcal{H}}(z),\hat{\rho}(z,t)\right\}-\left\{\hat{\rho}(z,t),\widehat{\mathcal{H}}(z)\right\}\right)\int dzTr\hat{\rho}\,(z,t),\frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$= -i\left|\widehat{\mathcal{H}}(z),\hat{\rho}(z,t)\right|+\sum_{\alpha,z'}\mathcal{W}^{\alpha}(z|z')\hat{\mathcal{L}}_{\alpha}\hat{\rho}\,(z',t)\hat{\mathcal{L}}_{\alpha}^{\dagger}$$
$$-\frac{1}{2}\mathcal{W}^{\alpha}(z)\left\{\hat{\mathcal{L}}_{\alpha}^{\dagger}\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z,t)\right\}_{+}\frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$= \{\mathcal{H}_{0}(z),\hat{\rho}(z,t)\}-i\left[\widehat{\mathcal{H}}(q),\hat{\rho}(z,t)\right]+\frac{1}{2}\left(\left\{\widehat{\mathcal{H}}(q),\hat{\rho}(z,t)\right\}-\left\{\hat{\rho}(z,t),\widehat{\mathcal{H}}(q)\right\}\right)$$
$$+\frac{1}{2}\int dxdx'\left\{q_{i}(x'),\left\{q_{j}(x'),\mathcal{D}_{2}^{ij}(x,x')\hat{\rho}(z,t)\right\}\right\}$$
$$+\frac{1}{2}\int dxdx'\,\mathcal{D}_{0,ij}(x,x')\otimes\left[\left\{\widehat{\mathcal{H}}(q),\rho^{i}(x)\right\},\left[\hat{\rho}(z,t),\left\{\widehat{\mathcal{H}}(q),\rho^{j}(x')\right\}\right]\right]$$

$$\hat{\rho}_{cq} = \int dz\rho(z,t)|z\rangle\langle z|\otimes\hat{\sigma}\,(z,t), \hat{\rho}(z,t) = \rho(z,t)\hat{\sigma}(z,t), \frac{\partial\hat{\sigma}(t)}{\partial t}$$
$$= \frac{i}{\hbar}\left[\widehat{\mathcal{H}}(t), \hat{\sigma}(t)\right] - \lambda^{\alpha\beta}(t)\left[\hat{\mathcal{L}}_\alpha\hat{\sigma}(t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha, \hat{\sigma}(t)\right\}_+\right], \rho(z,t)$$
$$= \int dz\mathcal{P}(z|z',t)\rho(z',0), \frac{d\rho(z,t)}{dt} = \int dz'\,\mathcal{W}(z|z',t)\rho(z',t)$$
$$- \mathcal{W}(z,t)\rho(z,t), \mathcal{W}(z,t) = \int dz'\,\mathcal{W}\left(z'^{|z,t}\right), \frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$= -i\left[\widehat{\mathcal{H}}(z,t), \hat{\rho}(z,t)\right] + \lambda^{\alpha\beta}(z,t)\left(\hat{\mathcal{L}}_\alpha\hat{\rho}(z,t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha, \hat{\rho}(z,t)\right\}_+\right)$$
$$+ \frac{\partial}{\partial t}\int dz'\mathcal{W}^{\alpha\beta}\,(z|z',t)\hat{\mathcal{L}}_\alpha, \hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\mathcal{W}^{\alpha\beta}(z,t)\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha, \hat{\rho}(z,t)\right\}_+, \frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$= -i\left[\widehat{\mathcal{H}}(z), \hat{\rho}(z,t) + \frac{\mathcal{W}^{\alpha\beta}}{\tau}\left[\hat{\mathcal{L}}_\alpha, \hat{\rho}(z,t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha, \hat{\rho}(z,t)\right\}_+\right]\right]$$
$$- \hat{\mathcal{L}}_\alpha\chi_\hbar^{\alpha\beta}(z)\otimes\nabla\hat{\rho}(z,t)\hat{\mathcal{L}}_\beta^\dagger + \tau\hat{\mathcal{L}}_\alpha\nabla\otimes\left[\gamma_\hbar^{\alpha\beta}(z)\right]\hat{\rho}(z,t)\hat{\mathcal{L}}_\beta^\dagger$$
$$+ \tau\hat{\mathcal{L}}_\alpha\nabla\otimes\mathcal{D}^{\alpha\beta}(z)\hat{\rho}(z,t)\otimes\overleftarrow{\nabla}\hat{\mathcal{L}}_\beta^\dagger, \widehat{\mathcal{H}}(z) = \hbar^{\alpha\beta}(z)\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha, \Omega\begin{bmatrix} 0 & \mathcal{I}_\eta \\ -\mathcal{I}_\eta & 0 \end{bmatrix}, \chi_\hbar^{\alpha\beta}(z)$$
$$= \Omega\nabla\hbar^{\alpha\beta}(z), \frac{\partial\hat{\rho}(z',t)}{\partial t} = Tr\left\{\widehat{\mathcal{H}}(z), \hat{\rho}(z,t)\right\}, \hat{\rho}(z) = \frac{1}{2}\rho_{\mathcal{L}}(z)\hat{\sigma}_{\mathcal{L}} + \frac{1}{2}\rho_{\mathcal{R}}(z)\hat{\sigma}_{\mathcal{R}}, \frac{\partial\hat{\rho}(z')}{\partial t}$$
$$= \frac{1}{2}\left\{Tr\widehat{\mathcal{H}}(z)\hat{\sigma}_{\mathcal{L}}, \rho_{\mathcal{L}}(z)\right\} + \frac{1}{2}\left\{Tr\widehat{\mathcal{H}}(z)\hat{\sigma}_{\mathcal{R}}, \rho_{\mathcal{R}}(z)\right\}, \hat{\sigma}(t) = \mathcal{E}\left(\hat{\sigma}(0)\right)$$
$$= \sum_\mu \mathcal{K}_\mu\,\hat{\sigma}(0)\mathcal{K}_\mu^\dagger, \mathcal{K}_\mu^\dagger\mathcal{K}_\mu\hat{\rho}(z,t) = \int dz'\Lambda_{z|z',t}\,\hat{\rho}(z',0)$$

$$\lambda_{z',t} = \int dz\Lambda_{z|z',t}\,, \hat{\rho}(z,t)$$
$$= \int dz'\sum_\mu \mathcal{K}_\mu(z|z',t)\hat{\rho}(z',0)\mathcal{K}_\mu(z|z',t)^\dagger \sum_\mu \int dz\mathcal{K}_\mu(z|z',t)^\dagger\,\mathcal{K}_\mu(z|z',t), \hat{\rho}_{cq}(t)$$
$$= \int dzdz'\sum_\mu |z\rangle\langle z'|\otimes\mathcal{K}_\mu(z|z',t)\,\hat{\rho}_{cq}(0)\left|z'\right\rangle\langle z|\otimes\mathcal{K}_\mu(z|z',t)^\dagger, \Lambda_t$$
$$= \exp\left(\int_0^t d\delta\mathcal{L}_\delta\right)\mathcal{L}_t\hat{\rho}(z,t) = \lim_{\delta t\to 0}\hat{\rho}(z,t+\delta t) - \frac{\hat{\rho}(z,t)}{\delta t}, \hat{\rho}(z,t)$$
$$= \mathcal{K}(t)\hat{\rho}(z',t), \hat{\rho}(z',t)$$
$$= \sum_{\alpha\beta=0}\int dz'\Lambda^{\alpha\beta}\,(z|z',t)\hat{\mathcal{L}}_\alpha\hat{\rho}(z',0)\hat{\mathcal{L}}_\beta^\dagger \sum_{\alpha\beta}\int dz\Lambda^{\alpha\beta}\,(z|z',t)\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha$$

$$\Lambda^{00}(z|z',t+\delta t)$$
$$= \Lambda^{00}(z|z',t) - \delta(z-z')\gamma^{00}(z',t)\delta t + \mathcal{W}^{00}(z|z',t)\delta t + \mathcal{O}(\delta t^2), \Lambda^{\alpha\beta}(z|z',t+\delta t)$$
$$= \Lambda^{\alpha\beta}(z|z',t) - \delta(z-z')\lambda^{\alpha\beta}(z',t)\delta t + \mathcal{W}^{\alpha\beta}(z|z',t)\delta t + \mathcal{O}(\delta t^2)$$

$$\Lambda^{00}(z|z',t+\delta t) = \delta(z-z')\left(1-\delta t\gamma^{00}(z',t)\right) + \mathcal{W}^{00}(z|z',t)\delta t + \mathcal{O}(\delta t^2), \Lambda^{\alpha\beta}(z|z',t+\delta t)$$
$$= \delta(z-z')\lambda^{\alpha\beta}(z',t)\delta t + \mathcal{W}^{\alpha\beta}(z|z',t)\delta t + \mathcal{O}(\delta t^2),$$

$$\hat{\rho}(z,t+\delta t)=\int dz'\,\delta(z-z')\left(\left(1-\delta t\gamma^{00}(z',t)\right)\right)\hat{\rho}(z',t)+\delta t\lambda^{\alpha 0}(z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)$$
$$+\delta t\lambda^{0\beta}(z',t)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}+\delta t\lambda^{\alpha\beta}(z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$
$$+\sum_{\alpha\beta}dz'\delta t\,\mathcal{W}^{\alpha\beta}(z|z',t)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$

$$\frac{\partial\hat{\rho}(z',t)}{\partial t}=-\gamma^{00}(z',t)\hat{\rho}(z',t)+\lambda^{\alpha 0}(z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)+\lambda^{0\beta}(z',t)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$
$$+\lambda^{\alpha\beta}(z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}+\sum_{\alpha\beta}dz'\,\mathcal{W}^{\alpha\beta}(z|z',t)\hat{\mathcal{L}}_{\alpha}\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$

$$\widehat{\mathcal{H}}(z,t)=\frac{1}{2i}\left(\lambda^{0\beta}\hat{\mathcal{L}}_{\beta}^{\dagger}-\lambda^{\alpha 0}\hat{\mathcal{L}}_{\alpha}\right),\widehat{\mathcal{N}}(z,t)=-\gamma^{00}(z',t)+\frac{1}{2}\left(\lambda^{0\beta}\hat{\mathcal{L}}_{\beta}^{\dagger}+\lambda^{\alpha 0}\hat{\mathcal{L}}_{\alpha}\right),\frac{\partial\hat{\rho}(z,t)}{\partial t}$$
$$=-i\left[\widehat{\mathcal{H}}(z',t),\hat{\rho}(z',t)\right]+\lambda^{\alpha\beta}(z',t)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$
$$+\int dz'\,\mathcal{W}^{\alpha\beta}(z|z',t)\hat{\mathcal{L}}_{\alpha},\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}$$
$$+\left\{\widehat{\mathrm{N}}(z',t),\hat{\rho}(z',t)\right\}_{+}\int dzTr\frac{\partial\hat{\rho}(z',t)}{\partial t},\widehat{\mathrm{N}}(z',t)$$
$$=\frac{1}{2}\left(\sum_{\alpha\beta}\lambda^{\alpha\beta}\,(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}\hat{\mathcal{L}}_{\alpha}+\int dz\sum_{\alpha\beta}\mathcal{W}^{\alpha\beta}\,(z|z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}\hat{\mathcal{L}}_{\alpha}\right)$$

$$\nabla^{00}\hat{\rho}(z',t)=\int dz'\,\mathcal{W}^{00}(z|z')\hat{\rho}(z',t)-\mathcal{W}^{00}(z)\hat{\rho}(z',t),\nabla^{00}\hat{\rho}(z',t)=\chi_{\hbar}^{00}\otimes\nabla\hat{\rho}(z',t),\chi_{\hbar}^{00}$$
$$=\Omega\nabla\mathcal{H},\chi_{\hbar}^{00}=\left(\frac{\partial\hbar^{00}}{\partial\rho}-\frac{\partial\hbar^{00}}{\partial\varrho}\right)^{\tau}$$

$$\mathcal{W}(z|z',t)=\left\langle\begin{matrix}\sum_{\eta=0}^{4}\mathcal{M}_{\eta}^{00}\,(z',t)/\eta!\nabla_{z'}^{\otimes\eta}\delta(z-z') & \sum_{\eta=0}^{4}\mathcal{M}_{\eta}^{\alpha\beta}\,(z',t)/\eta!\nabla_{z'}^{\otimes\eta}\delta(z-z')\\ \sum_{\eta=0}^{4}\mathcal{M}_{\eta}^{\alpha 0}\,(z',t)/\eta!\nabla_{z'}^{\otimes\eta}\delta(z-z') & \mathcal{M}_{0}^{\alpha\beta}(z',t)\delta(z-z')\end{matrix}\right\rangle-\mathbb{I}\chi_{\hbar}^{\alpha\beta}$$
$$=\left(\frac{\partial\hbar^{\alpha\beta}}{\partial\rho}-\frac{\partial\hbar^{\alpha\beta}}{\partial\varrho}\right)^{\tau}$$

$$\mathcal{W}^{\alpha\beta}(z+\Delta-\Gamma|z'-\Delta,t)\hat{\rho}(z'-\Delta,t)=\mathcal{W}^{\alpha\beta}(z+\Delta-\Gamma|z'-\Delta,t)\hat{\rho}(z'-\Delta,t)$$
$$=\sum_{\eta=0}^{\infty}(-\Delta\otimes\nabla)^{\eta}/\eta!\left[\mathcal{W}^{\alpha\beta}(z+\Delta|z',t)\hat{\rho}(z',t)\right]\int d\Delta\,\mathcal{W}^{\alpha\beta}(z|z'-\Delta,t)\hat{\rho}(z'-\Delta,t)$$
$$=\sum_{\eta=0}^{\infty}(-\nabla)^{\otimes\eta}\times\left[\frac{\mathcal{M}_{\eta}^{\alpha\beta}(z,t)}{\eta!}\hat{\rho}(z',t)\right]$$

$$\mathcal{M}_{0}^{\alpha\beta}=\int d\Delta\mathcal{W}^{\alpha\beta}\,(z+\Delta|z',t),\mathcal{M}_{1}^{\alpha\beta}=\int d\Delta\Delta\mathcal{W}^{\alpha\beta}(z+\Delta|z',t)\,,\chi_{\hbar}^{\alpha\beta}(z)$$
$$=\Omega\nabla\hbar^{\alpha\beta}(z)\int d\Delta\Delta\mathcal{W}_{\hbar}^{\alpha\beta}(z+\Delta|z')\approx\chi_{\hbar}^{\alpha\beta}(z)-\gamma_{\hbar}^{\alpha\beta}(z),\mathcal{D}^{\alpha\beta}(z)$$
$$=\frac{1}{2}\int d\Delta\Delta\otimes\Delta\mathcal{W}_{\hbar}^{\alpha\beta}\,(z+\Delta|z')\int d\Delta\,\mathcal{W}^{\alpha\beta}\left(z'^{|z'-\Delta}\right)\hat{\rho}(z'-\Delta,t)$$
$$=\mathcal{W}_{\hbar}(z)\hat{\rho}(z',t)-\chi_{\hbar}^{\alpha\beta}(z)\otimes\nabla\hat{\rho}(z',t)+\nabla\boxtimes\chi_{\hbar}^{\alpha\beta}\hat{\rho}(z',t)+\nabla\otimes\mathcal{D}^{\alpha\beta}(z)\hat{\rho}(z',t)\otimes\overleftarrow{\nabla}$$

$$\frac{\partial\hat{\rho}(z',t)}{\partial t} = i\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right] + \left|\lambda^{\alpha\beta}(z) + \mathcal{M}_0^{\alpha\beta}(z)\right|\left[\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right]$$
$$- \hat{\mathcal{L}}_\alpha\chi_\hbar^{\alpha\beta}(z)\otimes\nabla\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger + \hat{\mathcal{L}}_\alpha\nabla\otimes\left[\gamma_\hbar^{\alpha\beta}(z)\hat{\rho}(z',t)\right]\hat{\mathcal{L}}_\beta^\dagger$$
$$+ \hat{\mathcal{L}}_\alpha\nabla\otimes\mathcal{D}^{\alpha\beta}(z)\hat{\rho}(z',t)\otimes\overleftarrow{\nabla}\hat{\mathcal{L}}_\beta^\dagger$$

$$\frac{\partial\hat{\rho}(z',t)}{\partial t} = i\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right] + \frac{\mathcal{W}^{\alpha\beta}(z)}{\tau}\left[\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right]\left|\mathcal{M}_0^{\alpha\beta}(z)\right|$$
$$- \hat{\mathcal{L}}_\alpha\chi_\hbar^{\alpha\beta}(z)\otimes\nabla\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger + \frac{\tau\hat{\mathcal{L}}_\alpha\,\partial}{\partial\rho\otimes\left[\gamma_\hbar^{\alpha\beta}(z)\hat{\rho}(z',t)\right]\hat{\mathcal{L}}_\beta^\dagger} + \frac{\dfrac{\tau\hat{\mathcal{L}}_\alpha\,\partial}{\partial\rho\otimes\mathcal{D}^{\alpha\beta}(z)\hat{\rho}(z',t)\otimes\partial}}{\overleftarrow{\partial}\rho\hat{\mathcal{L}}_\beta^\dagger}$$
$$- \chi_\hbar^{00}(z)\otimes\nabla\psi$$

$$\left|\hat{\rho}(z',t)\rangle_{\alpha\beta} = \sum_i\sqrt{\rho_i(z)}\,|i\rangle_\alpha\otimes|i\rangle_\beta, \frac{\partial\hat{\rho}(z',t)\rangle_{\alpha\beta}}{\partial t}\right.$$
$$= \left[-i\widehat{\mathcal{H}}(z,t)\otimes\mathbb{I} + i\mathbb{I}\otimes\widehat{\mathcal{H}}^\tau(z) + \left[\lambda^{\alpha\beta}(z) + \mathcal{W}^{\alpha\beta}(z)\right]\right]\left(\hat{\mathcal{L}}_\alpha\otimes\hat{\mathcal{L}}_\beta^\circledast - \frac{1}{2}\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\otimes\mathbb{I}\right.$$
$$\left.- \frac{1}{2}\mathbb{I}\otimes\hat{\mathcal{L}}_\alpha^\mathcal{T}\hat{\mathcal{L}}_\beta^\circledast\right)$$
$$- \left|\hat{\mathcal{L}}_\alpha\otimes\hat{\mathcal{L}}_\beta^\circledast\chi_\hbar^{\alpha\beta}\otimes\nabla + \hat{\mathcal{L}}_\alpha\otimes\hat{\mathcal{L}}_\beta^\circledast\nabla\otimes\gamma_\hbar^{\alpha\beta}(z) + \hat{\mathcal{L}}_\alpha\otimes\hat{\mathcal{L}}_\beta^\circledast\nabla^{\otimes 2}\otimes\mathcal{D}^{\alpha\beta}(z)\right|\hat{\rho}(z',t)\rangle_{\alpha\beta}e^{\mathcal{L}t}$$

$$\mathcal{W}^{\alpha\beta}(z,z'-\Delta) \approx \frac{1}{\tau\delta^{(2\eta)}\left[\Delta - \tau\chi_\hbar^{\alpha\beta}(z)\right]}\int d\Delta\,\mathcal{W}^{\alpha\beta}(z,z'-\Delta)\hat{\rho}(z'-\Delta,t)$$
$$= \frac{1}{\tau}e^{\tau\{\hbar^{\alpha\beta}\otimes\varphi\}}\hat{\rho}(z',t)e^{\tau\{\hbar^{\alpha\beta}\otimes\varphi\}}\hat{\rho}(q',\rho,0) = \hat{\rho}\left(q - \int_0^\tau dt\frac{\partial\hbar}{\partial\rho},\rho + \int_0^\tau dt\frac{\partial\hbar}{\partial\rho},\varrho\right)$$
$$\approx \hat{\rho}\left(q - \tau\frac{\partial\hbar}{\partial\rho},\rho + \tau\frac{\partial\hbar}{\partial\rho}\varrho\right)$$

$$\nabla\hat{\rho}(q,\rho) = \frac{1}{\tau}e^{\tau\{\hbar^{\alpha\beta}\otimes\varphi\}}\hat{\mathcal{L}}^\dagger\hat{\rho}\hat{\mathcal{L}} - \frac{1}{2\tau\left\{\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}\hat{\rho}\right\}_+},\nabla_{\alpha\beta}\hat{\rho}(z',t)e^{\alpha\beta}$$
$$= \frac{1}{\tau}\left[\int d\Delta\,\mathcal{W}_\hbar^{\alpha\beta}(z|z'-\Delta)\hat{\mathcal{L}}_\alpha\hat{\rho}(z'-\Delta,t)\hat{\mathcal{L}}_\beta^\dagger\right.$$
$$\left.- \frac{\mathcal{W}_\hbar^{\alpha\beta}(z)1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right],\nabla_{\alpha\beta}\hat{\rho}(z',t)e^{\alpha\beta}$$
$$= \hat{\mathcal{L}}_\alpha e^{\tau\{\hbar^{\alpha\beta}\otimes\varphi\}}\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger - \frac{\frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+}{\tau},\nabla_{\alpha\beta}\hat{\rho}(z',t)e^{\alpha\beta}$$
$$= \frac{1}{\tau}\left[\hat{\mathcal{L}}_\alpha\hat{\rho}\left(z' - \tau\chi_\hbar^{\alpha\beta}\right)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right],\chi_\hbar^{\alpha\beta}\nabla_{\alpha\beta}\hat{\rho}(z',t)$$
$$\approx \frac{\chi_\hbar^{\alpha\beta}}{\tau}\left[\hat{\mathcal{L}}_\alpha\hat{\rho}(z'-\tau)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right]$$

$$\frac{\partial\hat{\rho}(z',t)}{\partial t} = \frac{i}{\hbar}\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right] + \nabla_{\alpha\beta}\hat{\rho}(z',t)e^{\alpha\beta}$$

$$= -\frac{i}{\hbar}\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right] + \frac{1}{\tau\left[\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(z',t)\right\}_+\right]}$$

$$- \chi_\hbar^{\alpha\beta}\otimes\hat{\mathcal{L}}_\alpha\nabla\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger, \frac{\partial\hat{\rho}(z',t)}{\partial t}$$

$$= -i\left[\widehat{\mathcal{H}}^{(m)}(q,\rho)\hat{\rho}(q,\rho)\right] + \{\hbar^{00}\hat{\rho}(q,\rho)\}$$

$$+ \sum_{\alpha,\beta}\frac{1}{\tau}\left[\hat{\mathcal{L}}_\alpha\hat{\rho}\left(q - \tau\frac{\partial\hbar^{\alpha\beta}}{\partial\rho}, \rho + \tau\frac{\partial\hbar^{\alpha\beta}}{\partial\rho}\right)\hat{\mathcal{L}}_\beta^\dagger - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\rho}(q,\rho)\right\}_+\right]$$

$$= i\beta q\left[\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}\hat{\rho}(q,\rho)\right] + \left\{\frac{\rho^2}{2m}\hat{\rho}(q,\rho)\right\}$$

$$+ \frac{\omega}{\tau}|\uparrow\rangle\left\langle\downarrow|\hat{\rho}(q,\rho+\tau\beta)|\uparrow\right\rangle\left\langle\downarrow\right| + \frac{\omega}{\tau}|\downarrow\rangle\left\langle\uparrow|\hat{\rho}(q,\rho+\tau\beta)|\downarrow\right\rangle\left\langle\uparrow\right| - \frac{\omega}{2\tau}\{\mathbb{I},\hat{\rho}(q,\rho)\}_+$$

$$\frac{\partial\mu_\uparrow(q,\rho)}{\partial t} = -\frac{\rho}{m}\frac{\partial\mu_\uparrow(q,\rho)}{\partial q} + \frac{\omega}{\tau}[\mu_\uparrow(q,\rho+\tau\beta) - \mu_\uparrow(q,\rho)], \frac{\partial\mu_\downarrow(q,\rho)}{\partial t}$$

$$= -\frac{\rho}{m}\frac{\partial\mu_\downarrow(q,\rho)}{\partial q} + \frac{\omega}{\tau}[\mu_\downarrow(q,\rho+\tau\beta) - \mu_\downarrow(q,\rho)], \frac{\partial\mu(q,\rho)}{\partial t}$$

$$= -\frac{\rho}{m}\frac{\partial\mu(q,\rho)}{\partial\chi} \pm \omega\beta\frac{\partial\mu(q,\rho)}{\partial\rho} + \frac{\omega\beta^2\tau}{2}\partial^2\mu(q,\rho)/\partial\rho^2$$

$$\alpha(q,\rho) = \rho\left(q - \frac{\rho}{m}t\right)e^{-2i\beta\omega\tau\left(q-\frac{\rho\tau}{2m}\right)-\omega t/\tau}$$

$$\frac{\partial\hat{\rho}(z',t)}{\partial t} = -i\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right] + \int dx d^3 y \mathcal{D}\Delta\,\mathcal{W}_\hbar^{\alpha\beta}(z|z'-\Delta,x,y)\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z'-\Delta,t)\hat{\mathcal{L}}_\beta^\dagger(y)$$

$$-\frac{1}{2}\mathcal{W}_\hbar^{\alpha\beta}(z,x,y)\left\{\hat{\mathcal{L}}_\beta^\dagger(y)\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z',t)\right\}_+$$

$$= -i\left[\widehat{\mathcal{H}}(z,t)\hat{\rho}(z',t)\right]$$

$$+ \int dx d^3 y\mathcal{W}_\hbar^{\alpha\beta}(z,x,y)\left[\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger(y) - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger(y)\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z',t)\right\}_+\right]$$

$$- \int dx\left[\chi_\hbar^{\alpha\beta}(z,x)\hat{\mathcal{L}}_\alpha(x)\otimes\nabla\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger(x)\right.$$

$$\left.- \nabla\otimes\mathcal{D}_\hbar^{\alpha\beta}(z,x,y)\otimes\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger(y)\overleftarrow{\nabla}\right]$$

$$- \int dx\nabla\otimes\left[\gamma_\hbar^{\alpha\beta}(z,x,y)\hat{\mathcal{L}}_\alpha(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_\beta^\dagger(y)\right]\int\mathcal{D}\Delta\,dx_1dx_2\otimes\mathcal{W}^{\alpha_1\beta_1,\alpha_2\beta_2}(z|z'$$

$$- \Delta,x_1,x_2)\otimes\hat{\mathcal{L}}_{\alpha_2}(x_2)\hat{\mathcal{L}}_{\alpha_1}(x_1)\hat{\rho}(z'-\Delta,t)\hat{\mathcal{L}}_{\beta_1}^\dagger(x_1)\hat{\mathcal{L}}_{\beta_2}^\dagger(x_2)$$

$$\widehat{\mathcal{H}}(x) = \int dx\hbar^{\alpha\beta}(z',x)\hat{\mathcal{L}}_\beta^\dagger(x)\hat{\mathcal{L}}_\alpha(x)\chi_\hbar^{\alpha\beta}(z,x) = \delta\hbar^{\alpha\beta}(z,x)/\delta z(x)\,,\widehat{\mathcal{H}}(x)$$

$$= \int dx\hbar^{0\alpha}(z',x)\mathbb{I}\hat{\mathcal{L}}_\alpha(x)\begin{bmatrix}\mathcal{D}_\hbar^{00}(x,y) & \delta(x,y)\chi_\hbar(z,y)\\ \chi_\hbar^\dagger(z,x)\delta(x,y) & \mathcal{W}_\hbar(z,x,y)\end{bmatrix} \succcurlyeq \epsilon_{\mathcal{M}_\mathfrak{B}}$$

$$\pi^{\alpha\beta} = \sqrt{g}\left(\kappa^{\alpha\beta} - g^{\alpha\beta}\kappa\right), \mathcal{H}_{\mathfrak{ADM}}[\mathcal{N}, \vec{\mathcal{N}}]$$
$$= \int dx \left(\mathcal{N}\mathcal{H}^{(tot)} + \mathcal{N}^{\alpha}\mathcal{H}_{\alpha}^{(tot)}\right) + \oint ds \bigotimes \mathcal{H}^{(boundary)}, \widehat{\mathcal{H}}^{(tot)}$$
$$= \widehat{\mathcal{H}}^{(gr)} + \widehat{\mathcal{H}}^{(m)}, \mathcal{H}^{(gr)} = \mathfrak{G}_{abcd}\pi^{\alpha\beta}\pi^{cd} - g^{\frac{1}{2}}\mathfrak{R}, \mathfrak{G}_{abcd}$$
$$= \frac{1}{2}g^{-\frac{1}{2}}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}), \mathcal{H}_{\alpha}^{(tot)} = \mathcal{H}_{\alpha}^{(gr)} + \mathcal{H}_{\alpha}^{(m)}, \mathcal{H}^{(gr)}[\mathcal{N}, \vec{\mathcal{N}}]$$
$$= \int dx \left(\mathcal{N}\mathcal{H}^{(gr)} + \mathcal{N}^{(\alpha)}\mathcal{H}_{\alpha}^{(gr)}\right), \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]$$
$$= \int dx \left(\mathcal{N}\mathcal{H}^{(m)} + \mathcal{N}^{(\alpha)}\mathcal{H}_{\alpha}^{(m)}\right), \frac{\partial g_{ab}}{\partial t} = \frac{\delta \mathcal{H}_{\mathfrak{ADM}}[\mathcal{N}, \vec{\mathcal{N}}]}{\delta \pi^{\alpha\beta}}, \frac{\partial \pi^{\alpha\beta}}{\partial t}$$
$$= \delta \mathcal{H}_{\mathfrak{ADM}}[\mathcal{N}, \vec{\mathcal{N}}]/\delta g_{ab}$$

$$\frac{\partial \rho(g,\pi,t)}{\partial t} = \{\mathcal{H}_{\mathfrak{ADM}}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}$$
$$= \{\mathcal{H}^{(gr)}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}_g + \{\mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}_m + \{\mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}_g, \{\mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}_g$$
$$= \int dx \left(\frac{\frac{\delta \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]}{\delta g_{ab}} \delta\rho}{\delta \pi^{\alpha\beta}} - \frac{\delta \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]}{\delta \pi^{\alpha\beta}} \delta\rho/\delta g_{ab} \right), \{\mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]\rho\}_m$$
$$= \int dx \left(\frac{\frac{\delta \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]}{\delta \phi} \delta\rho}{\delta \pi_{\phi}} - \frac{\delta \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]}{\delta \pi_{\phi}} \delta\rho/\delta\phi \right)$$

$$\widehat{\mathcal{H}}^{(tot)} = \mathcal{H}^{(gr)}\mathbb{I} + \hbar^{\alpha\beta}(g,\pi)\hat{\mathcal{L}}_{\beta}^{\dagger}\hat{\mathcal{L}}_{\alpha}, \widehat{\mathcal{H}}^{(m)}$$
$$= \frac{1}{2}g^{-\frac{1}{2}}\pi_{\phi}^{2} + \sqrt{g}g^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi + \sqrt{g}m^{4}\phi(x)^{2} - 4\sqrt{g}\Lambda_{cc}, \mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]$$
$$= \frac{1}{2}\int \frac{d^{3}p}{(2\pi)^{3}\omega_{\vec{p}}\left(\hat{\alpha}_{\vec{p}}^{\dagger}(g)\hat{\alpha}_{\vec{p}}(g) + \hat{\alpha}_{\vec{p}}(g)\hat{\alpha}_{\vec{p}}^{\dagger}(g)\right)}, \mathcal{H}_{\mathfrak{ADM}}[\mathcal{N}, \vec{\mathcal{N}}]$$
$$= \int dx \left(\mathcal{N}\mathcal{H}^{(tot)} + \mathcal{N}^{\alpha}\widehat{\mathcal{H}}_{\alpha}^{(tot)}\right)$$

$$\frac{\partial \rho(z,t)}{\partial t} = \int dx\mathcal{N}\{\mathcal{H}^{(gr)}\hat{\rho}(z',t)\} + \int dx\, \mathcal{N}^{\alpha}\left\{\mathcal{H}_{\alpha}^{(gr)}\hat{\rho}(z',t)\right\} - i[\mathcal{H}^{(m)}[\mathcal{N}, \vec{\mathcal{N}}]\hat{\rho}(z',t)]$$
$$+ \int dxdy\mathcal{D}\Delta\left[\mathcal{N}\mathcal{W}_{\hbar}^{\alpha\beta}(z|z'-\Delta,x,y)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z'-\Delta)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right.$$
$$\left. - \frac{1}{2}\mathcal{N}\mathcal{W}_{\hbar}^{\alpha\beta}(z,x,y)\left\{\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right\}_{+}\right]$$
$$+ \int dxdy\mathcal{D}\Delta\left[\mathcal{N}_{\alpha}\mathcal{W}_{\rho^{\alpha}}^{\alpha\beta}(z|z'-\Delta,x,y)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z'-\Delta)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right.$$
$$\left. - \frac{1}{2}\mathcal{N}_{\alpha}\mathcal{W}_{\rho^{\alpha}}^{\alpha\beta}(z,x,y)\left\{\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right\}_{+}\right]$$

$$\frac{\partial \rho(g,\pi,t)}{\partial t} = \mathcal{L}_{\mathcal{H}_{\mathfrak{ADM}}}\rho(g,\pi,t) = \int dx[\mathcal{N}\mathcal{L}\hat{\rho}(g,\pi,t) + \mathcal{N}^{\alpha}\mathcal{L}_{\alpha}\hat{\rho}(g,\pi,t)]$$

$$\frac{\partial\hat{\rho}(z,t)}{\partial t} = \int dx \left[\mathcal{N}\{\mathcal{H}^{(gr)}\hat{\rho}(z',t)\} + \mathcal{N}^{\alpha}\left\{\mathcal{H}_{\alpha}^{(gr)}\hat{\rho}(z',t)\right\} - i\left[\mathcal{H}^{(m)}[\mathcal{N},\vec{\mathcal{N}}]\hat{\rho}(z',t)\right]\right]$$
$$+ \int dxdy\mathcal{N}\mathcal{D}\Delta\left[\mathcal{W}_{\hbar}^{\alpha\beta}(z|z'-\Delta,x,y)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z'-\Delta,t)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right.$$
$$\left.-\frac{1}{2}\mathcal{W}_{\hbar}^{\alpha\beta}(z,x,y)\left\{\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)\right\}_{+}\right]\int dy\mathcal{D}\Delta\mathcal{N}\,\mathcal{W}_{\hbar}^{\alpha\beta}(z|z'-\Delta,x,y)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z'$$
$$-\Delta,t)\hat{\mathcal{L}}_{\beta}^{\dagger}(y)$$
$$= \frac{\mathcal{N}}{\tau}\exp[\tau(\hbar^{-1})(z,x)\{\hbar(z,x)\otimes\varphi\}]_{\gamma}^{\alpha}\otimes\hbar^{\beta\gamma}(z,x)\hat{\mathcal{L}}_{\alpha}(x)\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}(x)\int\mathcal{W}_{\hbar}^{\alpha\beta}(z|z'$$
$$-\Delta)\hat{\rho}(z-\Delta,t)\,\mathcal{D}\Delta$$
$$= \frac{1}{\tau}\hbar^{\alpha\beta}(g_{\alpha\beta})\hat{\rho}(z',t) + \frac{\frac{\delta\hbar^{\alpha\beta}(z)}{\delta g_{ab}}\delta\hat{\rho}(z,t)}{\delta\pi^{\alpha\beta}} - \frac{\frac{\delta\hbar^{\alpha\beta}(z)}{\delta\pi^{\alpha\beta}}\delta\hat{\rho}(z,t)}{\delta g_{ab}}$$
$$+ \tau\mathcal{D}^{abcd,\alpha\beta}(g)\delta^{2}\hat{\rho}(z',t)/\,\delta\pi^{\alpha\beta}\delta\pi^{cd}$$

$$\frac{\partial\hat{\rho}(z,t)}{\partial t} = \int dx \left[\mathcal{N}\{\mathcal{H}^{(gr)}\hat{\rho}(z',t)\} + \mathcal{N}^{\alpha}\left\{\mathcal{H}_{\alpha}^{(gr)}\hat{\rho}(z',t)\right\} - i\left[\mathcal{H}^{(m)}[\mathcal{N},\vec{\mathcal{N}}]\hat{\rho}(z',t)\right]\right]$$
$$+ \int dx\frac{1}{\tau}\mathcal{N}\hbar^{\alpha\beta}(z)\left[\hat{\mathcal{L}}_{\alpha}\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger} - \frac{1}{2}\left\{\hat{\mathcal{L}}_{\alpha}\hat{\rho}(z',t)\hat{\mathcal{L}}_{\beta}^{\dagger}\right\}_{+}\right]$$
$$+ \int dx\mathcal{N}\Gamma^{00}\hat{\rho}(z',t) + \int dx\mathcal{N}\left[\frac{\frac{\delta\hbar^{\alpha\beta}}{\delta g_{ab}}\hat{\mathcal{L}}_{\alpha}\delta\hat{\rho}(z,t)}{\delta\pi^{\alpha\beta}}\hat{\mathcal{L}}_{\beta}^{\dagger} + \mathcal{D}_{\pi}^{abcd,\alpha\beta}(g)\delta^{2}/\delta\pi^{\alpha\beta}\delta\pi^{cd}\right]$$

$$\frac{\partial\hat{\rho}(z,t)}{\partial t} = \int dx\mathcal{N}\{\widehat{\mathcal{H}}^{(gr)}\hat{\rho}\} + \int\mathcal{N}^{\alpha}dx\left\{\mathcal{H}_{\alpha}^{(gr)}\hat{\rho}\right\}$$
$$+ \int dx\sqrt{g}Tr\mathcal{N}\{\widehat{\mathcal{H}}^{(m)}\hat{\rho}\}\frac{\partial\hat{\rho}}{\partial t}\int dx\mathcal{N}\{\mathcal{H},\hat{\rho}\}$$
$$+ \int dx\mathcal{N}^{\alpha}\{\mathcal{H}_{\alpha},\hat{\rho}\} - i\{\widehat{\mathcal{H}}^{(m)}[\mathcal{N},\vec{\mathcal{N}}]\hat{\rho}\} + \frac{1}{2}\int dx\left(\mathcal{N}\{\widehat{\mathcal{H}}^{(m)}\hat{\rho}\} - \{\hat{\rho},\widehat{\mathcal{H}}^{(m)}\}\right)$$
$$+ \frac{1}{2}dxdy\mathcal{N}\lambda_{\mu\nu\hat{\sigma}\tau}(g,x,y)\left[\hat{\mathcal{L}}^{\mu\nu}(x),\left[\hat{\rho},\hat{\mathcal{L}}^{\hat{\sigma}\tau}(y)\right]\right]$$
$$+ \frac{1}{2}dxdy\mathcal{N}\left\{\mathcal{D}_{2}^{\alpha\beta}(x,y)\mathcal{J}_{\alpha}'(g,x),\{\mathcal{J}_{\beta}'(g,y)\}\right\}$$

$$\hat{\rho}_{\alpha\beta} = \sum_{\mu\nu} |g,\pi\rangle_\alpha \otimes |g,\pi\rangle_\beta \otimes |\hat{\rho}_{materia}(g,\pi)\rangle_{\alpha\beta}, \frac{\partial\hat{\sigma}(t)}{\partial t} - i[\widehat{\mathcal{H}}^{(m)}\hat{\sigma}(t)]$$

$$+ \frac{\int dxdx'\mathcal{D}_0^{\alpha\beta}(x-x')\left(\hat{\mathcal{L}}_\alpha(x)\hat{\sigma}(t)\hat{\mathcal{L}}_\beta^\dagger(x') - \frac{1}{2}\left\{\hat{\mathcal{L}}_\beta^\dagger(x')\hat{\mathcal{L}}_\alpha(x)\hat{\sigma}(t)\right\}_+\right) d\,\widehat{\mathcal{H}}^{(m)}}{dt} \int dxdx'\,\mathcal{D}_0(x - x')\left[\hat{\mathcal{L}}_\alpha(x), \left[\widehat{\mathcal{H}}^{(m)}, \hat{\mathcal{L}}_\beta^\dagger(x')\right]\right] \lim_{x-x'\to\ell} \mathcal{D}_2^{ijkl}(x,x')$$

$$= \frac{\frac{\mathcal{D}_2\sqrt{g}\mathcal{N}}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2\beta g^{ij}g^{jk})d\widehat{\mathcal{H}}^{(gr)}[\mathcal{N},\vec{\mathcal{N}}]}{dt}$$

$$= \{\widehat{\mathcal{H}}^{(gr)}[\mathcal{N},\vec{\mathcal{N}}], \mathcal{H}_{\mathfrak{ADM}}[\mathcal{N},\vec{\mathcal{N}}]\} + \frac{1}{2}\int dxdx'\,\mathcal{D}_2^{ijkl}(x,x')\frac{\delta^2}{\delta\pi^{ij}(x)\delta\pi^{kl}(x')\widehat{\mathcal{H}}^{(gr)}[\mathcal{N},\vec{\mathcal{N}}]}$$

$$= \widehat{\mathcal{H}}^{(gr)}[\mathcal{N},\vec{\mathcal{N}}]\widehat{\mathcal{H}}^{(m)}[\mathcal{N},\vec{\mathcal{N}}] + \frac{\int dx\mathcal{N}^2\mathcal{D}_2\sqrt{g}1}{2(g^{ik}g^{jl} + g^{il}g^{jk} - 2\beta g^{ij}g^{jk})\mathfrak{G}_{ijkl}}$$

$$= \widehat{\mathcal{H}}^{(gr)}[\mathcal{N},\vec{\mathcal{N}}]\widehat{\mathcal{H}}^{(m)}[\mathcal{N},\vec{\mathcal{N}}] + \left(d^2 - \frac{d}{2}\right)(1+\beta)\mathcal{D}_2\int dx\mathcal{N}^2\sum_z Tr\mathcal{E}_{z|z'}\otimes\mathbb{I}(\mathcal{P}_r(z')) = 1$$

$$\frac{d}{dt}\int dzTr\hat{\mathcal{A}}_\delta(z,t)\hat{\rho}_\delta(z) = \frac{d}{dt}\int dzTr\,\hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\rho}_\mathcal{H}(z)$$

$$\frac{d\hat{\mathcal{A}}_\mathcal{H}(z,t)}{dt} = \frac{\partial\hat{\mathcal{A}}_\mathcal{H}(z,t)}{\partial t} + i[\widehat{\mathcal{H}}(z), \hat{\mathcal{A}}_\mathcal{H}(z,t)] + \int d\Delta\mathcal{W}^{\alpha\beta}(z+\Delta|z)\hat{\mathcal{L}}_\beta^\dagger\,\hat{\mathcal{A}}_\mathcal{H}(z+\Delta,t)\hat{\mathcal{L}}_\alpha - \frac{1}{2}\mathcal{W}^{\alpha\beta}(z)\left\{\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{L}}_\alpha\hat{\mathcal{A}}_\mathcal{H}(z,t)\right\}_+$$
$$= \frac{\partial\hat{\mathcal{A}}_\mathcal{H}(z,t)}{\partial t} + i[\widehat{\mathcal{H}}(z), \hat{\mathcal{A}}_\mathcal{H}(z,t)] + \mathcal{W}^{\alpha\beta}(z)\left[\hat{\mathcal{L}}_\beta^\dagger\hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\alpha - \frac{1}{2}\left\{\hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\beta^\dagger(x)\hat{\mathcal{L}}_\alpha\right\}_+\right] - \hat{\mathcal{L}}_\beta^\dagger\left\{\hbar^{\alpha\beta}, \hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\alpha\int dx\,\sqrt{g}\hat{\mathcal{A}}_\delta(z,x)\right\}$$

$$\frac{d\hat{\mathcal{A}}_\mathcal{H}(z,t)}{dt} = \frac{\partial\hat{\mathcal{A}}_\mathcal{H}(z,t)}{\partial t} + i[\widehat{\mathcal{H}}(z), \hat{\mathcal{A}}_\mathcal{H}(z,t)] + \int dx\left[d\Delta\mathcal{W}^{\alpha\beta}(z+\Delta|z)\hat{\mathcal{L}}_\beta^\dagger(y)\,\hat{\mathcal{A}}_\mathcal{H}(z+\Delta,t)\hat{\mathcal{L}}_\alpha(x) - \frac{1}{2}\mathcal{W}^{\alpha\beta}(z)\left\{\hat{\mathcal{L}}_\beta^\dagger(y)\hat{\mathcal{L}}_\alpha(x)\hat{\mathcal{A}}_\mathcal{H}(z,t)\right\}_+\right]$$
$$= \frac{\partial\hat{\mathcal{A}}_\mathcal{H}(z,t)}{\partial t} + i[\widehat{\mathcal{H}}(z), \hat{\mathcal{A}}_\mathcal{H}(z,t)] + \int \mathcal{W}^{\alpha\beta}(z,x,y)\left[\hat{\mathcal{L}}_\beta^\dagger(y)\hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\alpha(x) - \frac{1}{2}\left\{\hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\beta^\dagger(y)\hat{\mathcal{L}}_\alpha(x)\right\}_+\right] - \int dx\,\hat{\mathcal{L}}_\beta^\dagger(x)\left\{\hbar^{\alpha\beta}(z,x), \hat{\mathcal{A}}_\mathcal{H}(z,t)\hat{\mathcal{L}}_\alpha(x)\int dx\,\sqrt{g}\hat{\mathcal{A}}_\delta(z,x)\right\}$$

5. Agujeros Negros Cuánticos en espacios curvos.

$$ds^2 = -(\mathcal{N}cdt)^2 + g_{ij}(dx^i + \mathcal{N}^i cdt)(dx^j + \mathcal{N}^j cdt), \delta$$

$$= \int d^4x \left(\frac{\pi^{ij}\partial g_{ij}}{\partial t} + \frac{\pi_m \partial \phi_m}{\partial t} - \mathcal{N}\mathcal{H} - \mathcal{N}^i \mathcal{H}_i \right), \mathcal{H}$$

$$\equiv \left[-\frac{c^4}{32\pi\mathfrak{G}} g^{\frac{1}{2}}\mathcal{R} + \frac{\frac{32\pi\mathfrak{G}}{c^4} 1}{g^{\frac{1}{2}}(g_{ik}g_{jl}\pi^{ij}\pi^{kl})} - \frac{1}{2}\pi^2 \right] + \mathcal{H}^{(m)}, \mathcal{H}_i$$

$$\equiv \frac{c^4}{16\pi\mathfrak{G} g_{ij}\nabla_k \pi^{jk}} + \mathcal{H}_i^{(m)}, \pi_{ij}$$

$$= -\frac{c^4}{32\pi\mathfrak{G}} g^{\frac{1}{2}}(\mathcal{K}_{ij} - \mathcal{K}g_{ij})\delta^{ij}\sqrt{g}\mathrm{N}^2\Gamma^{\mu\nu}(1 + 2\Phi/c^4)\pi_\phi$$

$$\delta = \frac{\int d^4x \left(\frac{\pi_\phi \partial\phi}{\partial t} + \frac{\pi_m \partial \phi_m}{\partial t} \right) \int d^4x \left(-\frac{2\mathfrak{G}\pi c^4}{3}\pi_\Phi^2 + \frac{(\nabla\Phi)^2}{16\pi\mathfrak{G}} \right) \int d^4x \Phi(\chi) m^4 c^4(\chi) + \delta^{ij}\rho_i \rho_j}{2m}$$

$$\dot{\phi} = \frac{4\pi\mathfrak{G}c^4}{3}\pi_\phi, \dot{\pi}_\phi = \frac{\nabla^4\phi}{4\pi\mathfrak{G}} - \frac{m^4(\chi)\partial\rho}{\partial t}$$

$$= \left\{ \mathcal{H}_c + \mathcal{H}_0^{(m)}\rho \right\} - \frac{\partial_i \Phi(\chi)\partial\rho}{\partial \rho_i}$$

$$+ \int \frac{d^4x m^4(\chi)\delta\rho}{\delta\pi_\phi(\chi)}, \langle \mathcal{D}_{1,\pi_\phi}^{br}(\chi) \rangle = \langle -m^4(\chi) \rangle, Tr[\{\mathcal{H}_l, \varrho\}]$$

$$= \int d^4x Tr\left[\widehat{m}\chi\delta\rho/\delta\pi_\phi(\chi) \right]$$

$$\frac{\partial\varrho}{\partial t} \approx \{\mathcal{H}_c(\phi), \varrho\} - i\left[\mathcal{H}_0^{(m)}, \varrho\right] + \frac{\int d^4x \left[\frac{\widehat{m}(\chi)\delta\varrho}{\delta\pi_\phi} + \frac{\delta\varrho}{\delta\pi_\phi}\widehat{m(\chi)} \right] + d^3x\, d^3y \delta^2}{\delta\pi_\phi(\chi)\delta\pi_\phi(\chi')(\mathcal{D}_2(\phi, \pi_\phi, \chi, \gamma)\varrho)}$$

$$+ \frac{1}{2}\int d^3x d^\gamma\, \mathcal{D}_0(\Phi, \chi, \chi')([\widehat{m}(\chi), [\varrho, \widehat{m}(\gamma)]]), \mathcal{F}(\varrho)$$

$$= \frac{\mathcal{D}_\mathcal{F} 1}{2}\int d\chi d\chi' d\gamma \left\{ \mathrm{N}(\chi)\sqrt{g(\chi)}\left\{ \sqrt{g(\chi')}\epsilon(\chi - \chi')\mathcal{H}(\gamma) \right\}\varrho \right\}, \mathcal{F}(\varrho)$$

$$= \frac{\int d\chi d\chi'\, \delta}{\delta\pi_\phi(\chi)(\pi_\phi(\chi')\varrho)}, \frac{\partial\varrho}{\partial t}$$

$$\approx \{\mathcal{H}_c(\phi), \varrho\} - i\left[\mathcal{H}_0^{(m)}, \varrho\right]$$

$$+ c^4$$

$$/\hbar\tau \int d^3x \left[e^{\frac{\hbar\tau}{c^4}\int d\gamma\epsilon(\chi-\gamma)(1+2\phi(\chi)/c^4)\delta/\delta\pi_\phi(\gamma)}(1 - 2\phi(\chi)/c^4)\psi(\chi)\varrho\psi^\dagger(\chi) \right.$$

$$\left. - \frac{1}{2}\{\widehat{m}^4(\chi), \varrho\}_+ \right]$$

$$\frac{\partial\varrho}{\partial t} \approx \{\mathcal{H}_c(\phi), \varrho\} - i\left[\mathcal{H}_0^{(m)}, \varrho\right] + c^4/\hbar\tau \int d^3x \left[e^{\frac{\hbar\tau}{c^4}\int d\gamma\epsilon(\chi-\gamma)\delta/\delta\pi_\phi(\gamma)}\psi(\chi)\varrho\psi^\dagger(\chi) - \frac{1}{2}\{\widehat{m}^4(\chi), \varrho\}_+ \right]$$

$$ds_{FLRW}^2 = -dt^2 + \alpha^2(t)(dr^2 + r^2 d\Omega^2), ds_{\mathcal{f}i}^2 = -d\tau_v^2 + \alpha_v^2(\tau_\omega)(d\eta_\omega^2 + \eta_\omega^2 d\Omega^2), ds_{\mathcal{D}_v}^2$$

$$= -d\tau_v^2 + \alpha_v^2(\tau_v)(d\eta_v^2 + \sin\hbar^2(\eta_v)d\Omega^2), \bar{\Omega}_\mathcal{M} \equiv \frac{16\pi\mathfrak{G}\bar{\rho}_{\mathcal{M}_0}\bar{\alpha}_0^3}{3\bar{\mathcal{H}}^2\bar{\alpha}^3}, \bar{\Omega}_\mathcal{R}$$

$$\equiv \frac{16\pi\mathfrak{G}\bar{\rho}_{\mathcal{R}_0}\bar{\alpha}_0^4}{3\bar{\mathcal{H}}^2\bar{\alpha}^4}, \bar{\Omega}_\mathcal{K} \equiv \frac{-\kappa_v f_{vi}^{\frac{2}{3}} f_v^{\frac{1}{3}}}{\bar{\alpha}^2\bar{\mathcal{H}}^2}, \bar{\Omega}_\mathcal{Q} \equiv \frac{-\dot{f}_v^2}{18 f_v(1 - f_v)\bar{\mathcal{H}}^2}$$

$$\Delta\mu_{\mathrm{B}} \equiv (m-\mathcal{M}) - \left(m_{\mathrm{B}}^{\odot} - \mathcal{M}_{\mathrm{B}}\right)$$

$$= \alpha\chi_1 - \beta c \left\langle \begin{matrix} (\Delta m_{\mathrm{B}})^2 \Sigma_{m_{\mathrm{B}},\chi_1} \Sigma_{m_{\mathrm{B}},c} \\ \Sigma_{m_{\mathrm{B}},\chi_1} (\Delta\chi_1)^2 \Sigma_{\chi_1,c} \\ \Sigma_{m_{\mathrm{B}},c} \Sigma_{\chi_1,c} (\Delta c)^2 \end{matrix} \right\rangle + \begin{vmatrix} \sigma_z^2 + \sigma_{lens}^2 & & \\ & 0 & \\ & & 0 \end{vmatrix} \mathfrak{C}_{syst}^{ij} \sum_{\psi} \sigma_\psi \partial_\psi \mu^i \partial_\psi \mu^j$$

$$\mathcal{L} \equiv \prod_{i=1}^{\mathrm{N}} Pr\left[\left(\widehat{m}_{\mathrm{B}}^{\otimes\varphi}, \hat{\chi}_1, \hat{c}\right)_i \middle| \mathcal{H}\right]$$

$$= \prod_{i=1}^{\mathrm{N}} \int Pr\left[\left(\widehat{m}_{\mathrm{B}}^{\otimes\varphi}, \hat{\chi}_1, \hat{c}\right)_i \middle| \left(m_{\mathrm{B}}^{\otimes\varphi}, x_1, c\right)_i, \mathcal{H}\right] \otimes Pr\left[\left(m_{\mathrm{B}}^{\otimes\varphi}, x_1, c\right) \middle| \mathcal{H}\right] d\mathcal{M}_{\mathrm{B}} dx_1 dc$$

$$Pr(\gamma|\Theta) \equiv det\left(2\pi\xi_j\zeta_i\right)^{-1/2} exp\left(-\frac{1}{2}(\gamma-\gamma_0)^{\tau}\xi_l^{-1}(\gamma-\gamma_0)\partial^2\zeta/\partial t\right)$$

$$\mathcal{L}_{\Delta\mu_0} \equiv Pr\left(\hat{Z}_{\Delta\mu_0}|\gamma,\Theta\right)d\gamma$$

$$= \det\left[2\pi\left(\xi_d + \Lambda\xi_j\Lambda^{\tau}\right)\zeta\right]^{-\frac{1}{2}} \exp\otimes\left[-\frac{1}{2}\left(\hat{Z}_{\Delta\mu_0} - \gamma_0\Lambda\right)^{\tau}\left(\xi_d + \Lambda\xi_j\Lambda^{\tau}\right)^{-1}\zeta\otimes\left(\hat{Z}_{\Delta\mu_0} - \gamma_0\Lambda\right)\right], \mathcal{L} = 1/12\sigma_\gamma \int_{-3\sigma_\gamma}^{3\sigma_\gamma} \mathcal{L}_{\Delta\mu_0}(\gamma + \Delta\mu_0)\, d\Delta\mu_0$$

$$d_{\mathcal{L}} = \frac{(1+z)c}{\mathcal{H}_0\sqrt{|\Omega_{\kappa 0}|} sin\eta\left(\sqrt{|\Omega_{\kappa 0}|}\int_{\frac{1}{(1+z)}}^{1}\frac{d\gamma}{\mathcal{H}(\gamma)}\right)}, \mathcal{H}(\gamma) \equiv \sqrt{\Omega_{\kappa 0} + \Omega_{\mathcal{M}0}\gamma + \Omega_{\kappa 0}\gamma^2 + \Omega_{\Lambda 0}\gamma^4} sin\eta(\chi)$$

$$\equiv \begin{cases} sin\,\hbar(\chi), \Omega_{\kappa 0} > 0 \\ \chi, \Omega_{\kappa 0} = 0 \\ sin\,(\chi), \Omega_{\kappa 0} < 0 \end{cases}, d\mathcal{L} = (1+z)^2 d_{\Lambda}, d_{\Lambda}$$

$$= ct^{2/3}\int_t^{t_0}\frac{2dt'}{\left(2 + f_\nu(t')(t')^{\frac{2}{3}}\right)} = ct^{2/3}[\mathcal{F}(t_0) - \mathcal{F}(t)]$$

$$\mathcal{F}(t) \equiv 2t^{\frac{1}{3}} + \frac{\frac{\beta^{\frac{1}{3}}}{12}\ln\left(\frac{\left(t^{\frac{1}{3}} + \beta^{\frac{1}{3}}\right)^2}{t^{\frac{2}{3}}} - \beta^{\frac{1}{3}}t^{\frac{1}{3}} + \beta^{\frac{2}{3}}\right) + \beta^{\frac{1}{3}}}{\sqrt{3}tan^{-1}\left(2t^{\frac{1}{3}} - \frac{\beta^{\frac{1}{3}}}{\sqrt{3}\beta^{\frac{1}{3}}}\right)}, z + 1 = \frac{(2+f_\nu)f_\nu^{\frac{1}{3}}}{3f_{\nu_0}^{\frac{1}{3}}\bar{\mathcal{H}}_0 t}$$

$$= \frac{2^{\frac{4}{3}}t^{\frac{1}{3}}(t+b)}{f_{\nu_0}^{\frac{1}{3}}\bar{\mathcal{H}}_0 t(2t+3b)^{\frac{4}{3}}}, f_\nu(t) = 3f_{\nu_0}\bar{\mathcal{H}}_0 t + \left(1 - f_{\nu_0}\right)\left(2 + f_{\nu_0}\right)\mu_0(z)\langle\mu_{\mathcal{TS},\Lambda\mathcal{CDM}}\Omega_{\mathcal{M}_0}\rangle$$

$$\hat{d}_{\mathcal{L}}(\hat{z}) = 1 - \hat{z}/1 + z\, d_{\mathcal{L}}(z) = (1+\hat{z})\mathcal{D}(z), 1 + \hat{z}$$

$$= (1+\hat{z})\left(1 + z_{obs}^{pec}\right)\left(1 + z_{obs}^{\phi}\right)\left(1 + z_{em}^{pec}\right)\left(1 + z_{em}^{\phi}\right)$$

6. Saturación de Kernels en espacios cuánticos curvos.

$$\mathcal{D}_0^{\alpha\beta}\|\chi,\gamma\| = \frac{\lambda^{\alpha\beta} r_0^3}{m_0^4 \mathscr{g}_{\mathrm{N}}\|\chi,\gamma\|}, \mathcal{D}_{2,ij}^{\mu\nu}|\chi,\gamma| = \frac{\frac{1}{2}\left(\mathcal{D}_1^{br}\right)_i^{\mu\alpha}\langle\chi\rangle m_0^4}{r_0^3\lambda} \mathscr{g}_{\mathrm{N}}^{-1}|\chi,\gamma| \left(\mathcal{D}_1^{br\circledast}\right)_i^{\mu\alpha}\langle\gamma\rangle, \mathcal{F}|\chi,\gamma|$$

$$= \prod_{i=1}^{d}\sum_{\eta=0}^{\mathrm{N}} c_\eta \langle r_0\rangle \mathcal{H}_{2\eta}\left|\chi_i - \frac{\gamma_i}{r_0}\right|, \mathcal{D}_2|\chi,\gamma|$$

$$= \frac{\dfrac{\frac{1}{2}\left(\mathcal{D}_1^{br}\right)_i^{\mu\alpha}(\chi) m_0^4}{r_0^3\lambda\mathcal{F}|\chi,\gamma|\mathscr{g}_{\mathrm{N}}|\chi,\gamma|\left(\mathcal{D}_1^{br\circledast}\right)_i^{\mu\alpha}\langle\gamma\rangle\mathcal{D}_0^{\alpha\beta}}}{\|\chi,\gamma\|} - \frac{1}{4\pi\nabla_\chi^2\left(\dfrac{1}{|\chi-\gamma|}\right)\delta|\chi,\gamma|}, (\mathcal{D}_0^{-1})_{\alpha\beta}\|\chi,\gamma\|$$

$$= \frac{(\mathcal{D}_0^{-1})_{\alpha\beta}}{4\varpi}\nabla_\gamma^2\delta|\chi,\gamma|, \mathcal{D}_{2,ij}^{\mu\nu}|\chi,\gamma| = \frac{1}{2}, \frac{\mathcal{D}_{1,i}^{\mu\alpha}(\chi)(\mathcal{D}_0^{-1})_{\alpha\beta}}{4\varpi\nabla_\gamma^2\delta|\chi,\gamma|\mathcal{D}_{1,j}^{\beta\nu}(\gamma)}\mathcal{D}_2^{ijkl}\|\chi,\chi'\|$$

$$= -\frac{1}{8}\mathcal{D}\sqrt{\mathscr{g}(\chi)}\mathrm{N}(\chi)\mathscr{g}^{ij}\mathscr{g}^{kl}\Delta_{\chi'}\delta|\chi,\chi'|, \mathcal{D}_2^{ijkl}\|\chi,\chi'\|$$

$$= -\frac{1}{8}\mathcal{D}\delta^{ij}\delta^{kl}\left(1+\Phi(\chi)\right)\Delta_{\chi'}\delta(\chi,\chi'), \mathcal{D}_{0,ijkl}\|\chi,\chi'\|$$

$$= 1/2d^2\mathcal{D}\sqrt{\mathscr{g}(\chi)}\mathrm{N}(\chi')\mathscr{g}_{ij}(\chi)\mathscr{g}_{kl}(\chi')\mathfrak{G}\langle\chi,\chi'\rangle$$

$$\varrho\{\phi,\pi_\phi,t\} = \begin{bmatrix}\mu_{\mathcal{L}}(\phi,\pi_\phi,t) & \alpha(\phi,\pi_\phi,t)\\ \alpha^\star(\phi,\pi_\phi,t) & \mu_{\mathcal{R}}(\phi,\pi_\phi,t)\end{bmatrix}, \frac{\rho_{\mathcal{Q}}\int\mathcal{D}\phi\mathcal{D}\pi_\phi\varrho(\phi,\pi_\phi,t), \partial\rho_{\mathcal{Q}}}{\partial t}\int\mathcal{D}\phi\mathcal{D}\pi_\phi$$
$$- i\left[\mathcal{H}(\phi,\pi_\phi),\varrho(\phi,\pi_\phi)\right]$$
$$+ \int\mathcal{D}\phi\mathcal{D}\pi_\phi\int d\chi d\gamma\left[\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\mathcal{L}_\beta^\dagger(\gamma)\right.$$
$$\left.-\frac{1}{2}\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\left\{\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\right\}\right]$$

$$\int\mathcal{D}\phi\mathcal{D}\pi_\phi\int d\chi d\gamma\left[\langle\mathcal{L}\left|\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\mathcal{L}_\beta^\dagger(\gamma)\right|\mathcal{R}\rangle\right.$$
$$\left.-\frac{1}{2}\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\right|\mathcal{R}\rangle\right]$$

$$\langle\mathcal{L}\left|\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\mathcal{L}_\beta^\dagger(\gamma)\right|\mathcal{R}\rangle$$
$$\sim\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)f_\mathcal{L}(\chi)f_\mathcal{R}(\gamma)$$
$$-\frac{1}{2}\int\mathcal{D}\phi\mathcal{D}\pi_\phi\int d\chi d\gamma\,\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\,\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi,t)\right|\mathcal{R}\rangle,\varrho(\phi,\pi_\phi,t)$$
$$=\mathcal{U}_\mathcal{L}(\phi,\pi_\phi,t)|\mathcal{L}\rangle\langle\mathcal{L}|+\mathcal{U}_\mathcal{R}(\phi,\pi_\phi,\chi,\gamma)|\mathcal{R}\rangle\langle\mathcal{R}|+\alpha(\phi,\pi_\phi,t)|\mathcal{L}\rangle\langle\mathcal{R}|$$
$$+\alpha^\star(\phi,\pi_\phi,t)|\mathcal{R}\rangle\langle\mathcal{L}|$$
$$-\frac{1}{2}\int\mathcal{D}\phi\mathcal{D}\pi_\phi\,\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\left(\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{L}\rangle+\langle\mathcal{R}\left|\mathcal{L}_\alpha(\chi)\mathcal{L}_\beta^\dagger(\gamma)\right|\mathcal{R}\rangle\right)$$
$$-\frac{1}{2}\mathcal{D}_0^{\alpha\beta}(\chi,\gamma)\left(\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{L}\rangle+\langle\mathcal{R}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{R}\rangle\langle\mathcal{L}|\rho_\mathcal{Q}|\mathcal{R}\rangle\right),\lambda$$
$$=\frac{1}{2}\int d\chi d\gamma\mathcal{D}_0^{\alpha\beta}(\chi,\gamma)\left(\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{L}\rangle+\langle\mathcal{R}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{R}\rangle\right),\langle\mathcal{D}_0\rangle$$
$$=\int\mathcal{D}\phi\mathcal{D}\pi_\phi\int d\chi d\gamma Tr\left[\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\varrho(\phi,\pi_\phi)\right]$$
$$\leq 2\lambda,\int\mathcal{D}\phi\mathcal{D}\pi_\phi\int d\chi d\gamma\mathcal{D}_0^{\alpha\beta}(\phi,\pi_\phi,\chi,\gamma)\left(\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{L}\rangle\,\mathcal{U}_\mathcal{L}(\phi,\pi_\phi,t)\right.$$
$$\left.+\langle\mathcal{R}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{R}\rangle\,\mathcal{U}_\mathcal{R}(\phi,\pi_\phi,t)\right)\int d\chi d\gamma\mathcal{D}_0^{\alpha\beta}(\chi,\gamma)\,\langle\mathcal{L}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{L}\rangle\,\langle\mathcal{L}|\rho_\mathcal{Q}|\mathcal{L}\rangle$$
$$+\langle\mathcal{R}\left|\mathcal{L}_\beta^\dagger(\gamma)\mathcal{L}_\alpha(\chi)\right|\mathcal{R}\rangle\,\langle\mathcal{R}|\rho_\mathcal{Q}|\mathcal{R}\rangle$$

$$\frac{\partial\rho_\mathcal{Q}}{\partial t}=\frac{1}{2}\int d^3\chi d^3\gamma\mathcal{D}_0(\chi,\gamma)\left(\left[\widehat{m}^4(\chi),\left[\rho_\mathcal{Q},\widehat{m}^4(\gamma)\right]\right]\right)\left\langle\mathcal{L}\left|\frac{\partial\rho_\mathcal{Q}}{\partial t}\right|\mathcal{R}\right\rangle$$
$$=-\int d^3\chi d^3\gamma\,\mathcal{D}_0(\chi,\gamma)\big(m_\mathcal{L}(\chi)-m_\mathcal{R}(\chi)\big)\big(m_\mathcal{L}(\gamma)-m_\mathcal{R}(\gamma)\big)\langle\mathcal{L}|\rho_\mathcal{Q}|\mathcal{R}\rangle,\lambda$$
$$=\int d^3\chi d^3\gamma\,\mathcal{D}_0(\chi,\gamma)\big(m_\mathcal{L}(\chi)-m_\mathcal{R}(\chi)\big)\big(m_\mathcal{L}(\gamma)-m_\mathcal{R}(\gamma)\big),\lambda$$
$$=\int\frac{d^3\chi d^3\gamma\mathcal{D}_0}{|\chi-\gamma|}\big(m_\mathcal{L}(\chi)-m_\mathcal{R}(\chi)\big)\big(m_\mathcal{L}(\gamma)-m_\mathcal{R}(\gamma)\big),\lambda$$
$$=\int\frac{d^3\chi d^3\gamma\mathcal{D}_0}{|\chi-\gamma|}\big(m_\mathcal{L}(\chi)-m_\mathcal{L}(\chi)\big)\big(m_\mathcal{R}(\gamma)-m_\mathcal{R}(\gamma)\big),\lambda$$
$$=\int\frac{d^3\chi d^3\gamma\mathcal{D}_0}{|\chi-\gamma|}\left(\sum_{i,j}m_{\mathcal{L},i}(\chi)m_{\mathcal{L},j}(\gamma)+\sum_{i,j}m_{\mathcal{R},i}(\chi)m_{\mathcal{R},j}(\gamma)\right)$$

$$\nabla^2\Phi=4\pi\mathfrak{G}[m(\chi,t)+\mu(\phi,\widetilde{m})\mathcal{J}(\chi',t)]\mathbb{E}_m,\phi(\chi,t)=-\frac{\mathfrak{G}\int d^3\chi'\,[m(\chi,t)-\mu(\phi,\widetilde{m})\mathcal{J}(\chi',t)]}{|\chi-\chi'|},\overrightarrow{\mathcal{F}_{tot}}$$
$$=-\int d^3\chi m(\chi)\nabla\phi\,,\overrightarrow{\mathcal{F}_{tot}}$$
$$=-\frac{\mathfrak{G}\int\frac{d^3\chi d^3\chi' m(\chi)\left(\overrightarrow{\chi-\chi'}\right)}{|\chi-\chi'|^3[m(\chi',t)-\mathcal{J}(\chi',t)]},\sigma_\mathcal{F}^2=\frac{1}{\Delta\mathcal{T}}2\mathfrak{G}^2\int d^3\chi d^3\gamma d^3\chi'^{d^3}\gamma'\,m(\chi)m(\gamma)\left(\overrightarrow{\chi-\chi'}\right)\otimes\left(\overrightarrow{\gamma-\gamma'}\right)}{|\chi-\chi'|^3|\gamma-\gamma'|^3\langle\mathcal{D}_2(\chi',\gamma',\phi')\rangle},\sigma_\mathcal{F}^2$$
$$=\frac{\overline{\dfrac{2\mathfrak{G}^2}{\Delta\mathcal{T}\sum_{ij}\int d^3\chi d^3\gamma\chi'^{m_i(\chi)}\,m_j(\gamma)\left(\overrightarrow{\chi-\chi'}\right)\otimes\left(\overrightarrow{\gamma-\gamma'}\right)}}}{|\chi-\chi'|^3|\gamma-\gamma'|^3\langle\mathcal{D}_2(\chi',\phi_\beta)\rangle},\sigma_\mathcal{F}^2\sim\mathfrak{N}\mathfrak{G}^2\rho^2r_\mathrm{N}^2\int d^3\chi'^{\langle\mathcal{D}_2(\phi_\beta)\rangle},\sigma_\mathcal{F}^2$$
$$\sim\mathcal{D}_2\frac{\mathfrak{N}\mathfrak{G}^2\rho^2r_\mathrm{N}^2\mathcal{V}_\beta}{\Delta\mathcal{T}},\mathcal{D}_2\leq\frac{\sigma_\alpha^2\mathrm{N}r_\mathrm{N}^4\Delta\mathcal{T}}{\mathcal{V}_\beta\mathfrak{G}^2},\lambda\sim\mathcal{N}_\lambda\mathcal{M}_\lambda^2/\mathcal{V}_\lambda\mathcal{D}_2$$

$$\frac{\sigma_\alpha^2 \mathrm{N} r_\mathrm{N}^4 \Delta\mathcal{T}}{\mathcal{V}_\beta \mathfrak{G}^2} \geq \mathcal{D}_2 \geq \frac{\mathcal{N}_\lambda \mathcal{M}_\lambda^2}{\mathcal{V}_\lambda \lambda}, \frac{\sigma_\alpha^2 \mathrm{N} r_\mathrm{N}^4 \Delta\mathcal{T} m_\rho}{m_\mathrm{N} \mathfrak{G}^2 \ell_\rho^3} \geq \mathcal{D}_2, \frac{\sigma_\alpha^2 \mathrm{N} r_\mathrm{N}^4 \Delta\mathcal{T}}{m_\mathrm{N} \mathfrak{G}^2} \geq \frac{\ell_\rho^3 \mathcal{D}_2}{m_\rho} \geq \frac{\mathcal{M}_\lambda}{\lambda}, \sigma_\mathcal{F}^2 \sim \frac{\ell_\rho^4 \mathfrak{G}^2 m_\mathrm{N}^4 \mathrm{N} \mathcal{D}_2}{\Delta\mathcal{T} r_\mathrm{N}^4}, \mathcal{D}_2 \leq \frac{\Delta\mathcal{T} \ell_\rho^4 \sigma_\alpha^2 \mathrm{N} r_\mathrm{N}^4}{\mathfrak{G}^2}, \lambda \sim \frac{\mathcal{N}_\lambda \mathcal{M}_\lambda^2}{\ell_\rho^4 \mathcal{D}_2 \mathcal{R}_\lambda}, \Delta\mathcal{T} \sigma_\alpha^2 \mathrm{N} r_\mathrm{N}^4 / \mathfrak{G}^2 \geq \ell_\rho^4 \mathcal{D}_2 \geq \mathcal{N}_\lambda \mathcal{M}_\lambda^2 / \mathcal{R}_\lambda \lambda$$

7. Dualidad Onda – Partícula en espacios cuánticos curvos.

$$\mathcal{H}_{min}(Z|\mathcal{B}) + \min_{\mathcal{W}} \mathcal{H}_{max}(\mathcal{W}) \geq \log_2 \eta, \mathcal{H}_{min}(Z) + \min_{\mathcal{W}} \mathcal{H}_{max}(\mathcal{W}) \geq \log_2 \eta, \mathcal{H}_{min}(Z) = -\log_2 1 + \frac{\mathcal{D}}{2}, \mathcal{D} = \frac{1}{2}(\mathcal{D}_1 + \mathcal{D}_2)\mathcal{D}_{1(2)} = \left[\frac{|\rho_1 + \rho_2|}{\rho_1} + \rho_2\right] \min_{\mathcal{W}} \mathcal{H}_{max}(\mathcal{W}) = \log_2\left(1 + \sqrt{1 - \mathcal{V}^2}\right), \mathcal{V} = \rho_j^{max} - \frac{\rho_j^{min}}{\rho_j^{max}} + \rho_j^{min}, \mathcal{H}_{max}(\mathcal{P}) = 2\log_2 \sum_j \sqrt{\rho_j}, \mathcal{H}_{min}(\mathcal{P}) = -\log_2 \max_j \rho_j, \mathcal{H}_{min}(\mathcal{A}|\mathcal{B}) = -\log_2 \rho_{guess}(\mathcal{A}|\mathcal{B}), \rho_{guess}(\mathcal{A}|\mathcal{B}) = \max_{\mathcal{M}_\alpha} \sum_\alpha \rho_\alpha Tr\left[\mathbb{M}_\alpha \rho_\beta^\alpha\right], \mathcal{H}_{min}(\mathcal{P}) = -\log_2 \max_j \rho_j, \mathcal{H}_{max}(\mathcal{A}|\mathcal{B}) = \log_2 \rho_{secr}(\mathcal{A}|\mathcal{B}), \rho_{secr}(\mathcal{A}|\mathcal{B}) = \max_{\sigma_\beta} \mathcal{F}\left(\rho_{\alpha\beta} \mathbb{I} \otimes \sigma_\beta\right), \mathcal{H}_{max}(\mathcal{P}) = 2\log_2 \sum_j \sqrt{\rho_j},$$

$$\mathcal{H}_{max}(\mathcal{A}|\mathcal{B}) \leq \log_2\left\{1 + \sqrt{(\eta - 1)^2 - \left[\eta \rho_{guess}(\mathcal{A}|\mathcal{B}) - 1\right]^2}\right\}, \mathcal{V} = \eta \rho_{guess}^{max(\phi_\kappa)}(\mathcal{W}) - \frac{1}{\eta - 1}, \mathcal{D}(Z|\mathcal{B}) = \eta \rho_{guess}(Z|\mathcal{B}) - 1/\eta - 1$$

$$\mathcal{BS}1 = \frac{1}{\sqrt{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}, \mathcal{BS}2 = 1/\sqrt{2}\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} \mathcal{PM}1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_\chi} \end{pmatrix}, \mathcal{PM}2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_\delta} \end{pmatrix}$$

$$\Big|\psi\rangle = 1/2\sqrt{2} \frac{\{i\left[\left(1 + e^{i\phi_\delta}\right) + e^{i\phi_\chi}\left(1 - e^{i\phi_\delta}\right)\right]|\mathcal{D}_1\rangle}{+\left[\left(e^{i\phi_\delta} - 1\right) - e^{i\phi_\chi}\left(1 + e^{i\phi_\delta}\right)\right]|\mathcal{D}_2\rangle\}}, \rho_1 = |\langle \mathcal{D}_1|\psi\rangle|^2 = 2 - \cos(\phi_\chi + \phi_\delta) + \cos\frac{(\phi_\chi - \phi_\delta)}{4}, \rho_2 = |\langle \mathcal{D}_2|\psi\rangle|^2 = 2 + \cos(\phi_\chi + \phi_\delta) - \cos\frac{(\phi_\chi - \phi_\delta)}{4}, \min_{\mathcal{W}} \mathcal{H}_{max}(\mathcal{W}) = \min_{\phi_\chi} \log_2\left(1 + \sqrt{1 - \sin\phi_\delta^2 \sin\phi_\chi^2}\right) = \log_2\left(1 + \sqrt{1 - \sin\phi_\delta^2}\right)$$

$$|\psi\rangle = \frac{1}{2}\left[i\left(1 + e^{i\phi_\delta}\right)|\mathcal{D}_1\rangle + \left(1 - e^{i\phi_\delta}\right)|\mathcal{D}_2\rangle\right], \mathcal{H}_{min}(Z) = \log_2 cos^2\left(\frac{\phi_\delta}{2}\right) = -\log_2(1 + \cos\phi_\delta/2)$$

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FE DE ERRATAS – 27 de diciembre del 2024.

En todas las ecuaciones constantes en este manuscrito, en las que consten los símbolos . . o , según corresponda, se los reemplazará por cualquiera de los siguientes símbolos ·· ⊗ ⊠⊛⊡⁎⋆∗×, según sea el caso. La misma regla de corrección, aplica a todos los artículos científicos de mi autoría.

En todas las ecuaciones constantes en este manuscrito, en las que conste el símbolo ′ se lo aplicará según corresponda a la operación matemática de que se trate. La misma regla de corrección, aplica a todos los artículos científicos de mi autoría.

APÉNDICE C.

Modelo Yang – Mills para espacios cuánticos geométricamente curvos y agujeros negros cuánticos – Formalización Matemática.

1. Relatividad General.

$$d_{\mathfrak{E}}: \daleth(\Gamma): \mathfrak{a} \mapsto d_{\mathfrak{E}}\mathfrak{a} \coloneqq d\mathfrak{a} - ic_{\mathfrak{G}}[\mathfrak{U}_{\mathfrak{E}}, \mathfrak{a}]_{\wedge}\mathfrak{U}_{\mathfrak{E}} \wedge \mathfrak{a} - (-1)^{\rho}\mathfrak{a}\wedge\mathfrak{U}_{\mathfrak{E}}$$

$$\mathfrak{F}_{\mathfrak{G}} \coloneqq d\mathfrak{U}_{\mathfrak{E}} - ic_{\mathfrak{G}}\mathfrak{U}_{\mathfrak{E}}\wedge\mathfrak{U}_{\mathfrak{E}} = \sum_{\mathcal{J}}\big(d\mathfrak{U}_{\mathfrak{G}}^{\mathcal{J}} + \frac{c_{\mathfrak{G}}}{2}\sum_{\mathcal{J},\mathcal{K}} f_{\mathcal{J}\mathcal{K}}^{\mathcal{J}}\, \mathfrak{U}_{\mathfrak{G}}^{\mathcal{J}}\wedge\mathfrak{U}_{\mathfrak{G}}^{\mathcal{K}}\big)\, \tau_{\mathcal{J}}$$

$$d_{\mathfrak{E}}\phi = d\phi - ic_{\mathfrak{G}}[\mathfrak{U}_{\mathfrak{E}}, \phi]_{\wedge} = \sum_{\mathcal{J}}\left(d\phi^{\mathcal{J}} + c_{\mathfrak{G}}\sum_{\mathcal{J},\mathcal{K}} f_{\mathcal{J}\mathcal{K}}^{\mathcal{J}}\, \mathfrak{U}_{\mathfrak{G}}^{\mathcal{J}}\phi^{\mathcal{K}}\right)\tau_{\mathcal{J}} \in \Gamma(\mathcal{M}, \mathcal{V}(\mathfrak{E}), \mathfrak{G})$$

2. Métrica inercial.

$$\Gamma_{\mu\nu}^{\lambda} \coloneqq \frac{1}{2}g^{\lambda\sigma}\big(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\big)\big(\mathcal{M}_{\mathfrak{G}}\otimes\mathcal{M}_{\mathfrak{L}}, \pi_{\mathcal{J}}, \mathcal{M}_{\mathfrak{G}}, \mathcal{SO}(1,3)\big), \pi_{\mathcal{J}}: \mathcal{M}_{\mathfrak{G}}\otimes\mathcal{M}_{\mathfrak{L}} \longrightarrow \mathcal{M}_{\mathfrak{G}}: \mathcal{M}_{\mathfrak{L}}(\rho) \longmapsto \rho$$

$$\pi_{\mathcal{J}}^{\#}:\Omega^1(\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{G}}) \longmapsto \Gamma\left(\mathcal{M}_{\mathfrak{L}},\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{L}},\mathcal{SO}(1,3)\right): d\chi^\mu \longmapsto \mathrm{e}^\alpha \coloneqq \varepsilon_\mu^\alpha(\rho) d\chi^\mu \varepsilon_\mu^\alpha(\chi) \coloneqq \left.\frac{\partial\xi^\alpha(\chi,\gamma)}{\partial\gamma^\mu}\right|_{\gamma=\chi}$$

$$\mathfrak{v} \coloneqq \frac{1}{4!}\epsilon_{\circ\circ\circ\circ}\mathrm{e}^\circ \wedge \mathrm{e}^\circ \wedge \mathrm{e}^\circ \wedge \mathrm{e}^\circ = \det[\varepsilon]^2\, d\chi^0 \wedge d\chi^1 \wedge d\chi^2 \wedge d\chi^3$$

$$g_{\mu\nu} = [\varepsilon^t \eta_{\mathcal{L}} \varepsilon]_{\mu\nu} = \eta_{\mathcal{L}\circ\circ}\varepsilon_\mu^\circ \varepsilon_\nu^\circ - \det[\varepsilon]^2$$

$$\mathfrak{v} = \sqrt{-\det[g]}\ d\chi^0 \wedge d\chi^1 \wedge d\chi^2 \wedge d\chi^3$$

$$\mathfrak{G}_{\alpha\beta} \coloneqq \frac{1}{2}\epsilon_{\alpha\beta\circ\circ}\mathrm{e}^\circ \wedge \mathrm{e}^\circ, \mathfrak{V}_\alpha \coloneqq \frac{1}{3!}\epsilon_{\alpha\beta\circ\circ}\mathrm{e}^\circ \wedge \mathrm{e}^\circ \wedge \mathrm{e}^\circ$$

$$\mathfrak{w}^{\alpha\beta} = \omega_\mu^{\alpha\beta}\, d\chi^\mu \in \mathcal{V}^2(\mathcal{T}\mathcal{M}_{\mathfrak{L}})\otimes\Omega^1(\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{L}}) \otimes \mathcal{A}\mathfrak{d}(g_{\mathcal{SO}(1,3)})$$

$$d_{\mathfrak{w}}\mathfrak{a}^\alpha \coloneqq d\mathfrak{a}^\alpha + c_{gr}\eta_{\mathcal{L}\circ\circ}\mathfrak{w}^{\alpha\circ} \wedge \mathfrak{a}^\circ, \mathfrak{J}^\alpha \coloneqq d_{\mathfrak{w}}\mathrm{e}^\alpha \in \mathcal{V}^1(\mathcal{T}\mathcal{M}_{\mathfrak{L}})\otimes\Omega^2(\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{L}})$$

$$\mathfrak{a} = \alpha_{j_1\otimes j_q}^{i_1\otimes i_\rho}\left(\partial_{i_1} \bigotimes \cdots \otimes \partial_{i_\rho}\right)\left(\mathrm{e}^{j_1} \wedge \cdots \wedge \mathrm{e}^{j_q}\right)$$

$$\langle\mu,\nu\rangle_{\mathcal{L}} = \mu^t \cdot \eta_{\mathcal{L}} \cdot \nu = \eta_{\mathcal{L}\circ\circ}\mu^\circ\nu^\circ, \mathfrak{G}_{\mathcal{SO}}:\mathrm{e} \longmapsto \mathfrak{G}_{\mathcal{SO}}(\mathrm{e}) = \mathrm{e}' = g_{\mathcal{L}} \cdot \mathrm{e}, \mathfrak{G}_{\mathcal{SO}}:\mathfrak{w} \longmapsto \mathfrak{G}_{\mathcal{SO}}(\mathfrak{w}) = \mathfrak{w}' = g_{\mathcal{L}} \cdot \mathfrak{w} \cdot g_{\mathcal{L}}^{-1} + c_{gr}^{-1} g_{\mathcal{L}} \cdot d g_{\mathcal{L}}^{-1}$$

$$\mathcal{SO}^\uparrow(1,3) \coloneqq \{g_{\mathcal{L}} \in \mathcal{SO}(1,3) | [g_{\mathcal{L}}]_0^0 > 0\}, \mathcal{SO}^\downarrow(1,3) \coloneqq \{g_{\mathcal{L}} \in \mathcal{SO}(1,3) | [g_{\mathcal{L}}]_0^0 < 0\}$$

$$\mathfrak{R}^{\alpha\beta} \coloneqq d\mathfrak{w}^{\alpha\beta} + c_{gr}\ \mathfrak{w}_\circ^\alpha \wedge \mathfrak{w}^{\beta\circ} \in \mathcal{V}^2(\mathcal{T}\mathcal{M}_{\mathfrak{L}})\otimes\Omega^2(\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{L}}) \otimes \mathcal{A}\mathfrak{d}(g_{\mathcal{SO}(1,3)})$$

$$\mathfrak{R}^{\alpha\beta} = \sum_{c<d}\mathcal{R}_{cd}^{\alpha\beta}\,\mathrm{e}^c \wedge \mathrm{e}^d = \frac{1}{2}\mathcal{R}_{\circ\circ}^{\alpha\beta}\mathrm{e}^\circ \wedge \mathrm{e}^\circ, \mathcal{R}_{\alpha\beta} \coloneqq \eta_{\mathcal{L}\alpha\star}\mathcal{R}_{\circ\beta}^{\star\circ}\mathcal{R}_{\circ\star}^{\star\circ}, d_{\mathfrak{w}}(d_{\mathfrak{w}}\mathrm{e}^\alpha) = c_{gr}\eta_{\mathcal{L}\circ\circ}\mathfrak{R}^{\alpha\circ} \wedge \mathrm{e}^\circ$$

$$\mu,\nu \in \mathcal{V}(\mathcal{T}\mathcal{M}_{\mathfrak{E}}), \langle\mu,\nu\rangle_{\mathfrak{E}} = \mu^t \cdot \eta_{\mathfrak{E}} \cdot \nu = \eta_{\mathfrak{E}\circ\circ}\mu^\circ\nu^\circ|_{\mathfrak{L}\to\mathfrak{E}} \Rightarrow \mathfrak{v}_{\mathfrak{E}} \coloneqq \det[\varepsilon_{\mathfrak{E}}]\ d\chi^0 \wedge d\chi^1 \wedge d\chi^2 \wedge d\chi^3 - \det[\varepsilon]^2$$

$$\mathfrak{v}_{\mathfrak{E}} = \sqrt{\det[g_{\mathfrak{E}}]}\ d\chi^0 \wedge d\chi^1 \wedge d\chi^2 \wedge d\chi^3$$

3. Métrica Poincaré.

$$\mathcal{ISO}(1,3) = \mathcal{SO}(1,3) \ltimes \mathcal{T}^4[\mathcal{P}_\alpha,\mathcal{P}_\beta][\mathcal{J}_{\alpha\beta},\mathcal{P}_c] = -\eta_{\alpha c}\mathcal{P}_\beta + \eta_{\beta c}\mathcal{P}_\alpha[\mathcal{J}_{\alpha\beta},\mathcal{J}_{cd}] = \eta_{\alpha c}\mathcal{J}_{\beta d} + \eta_{\beta c}\mathcal{J}_{\alpha d} - \eta_{\beta d}\mathcal{J}_{\alpha c} + \eta_{\alpha d}\mathcal{J}_{\beta c}(\mathcal{M}_{\mathfrak{G}}\otimes\mathcal{M}_{\mathfrak{L}},\pi_{\mathcal{J}},\mathcal{M}_{\mathfrak{G}},\mathfrak{G}_{c\mathcal{P}}), [\Theta_{\mathcal{J}}]_{\alpha\beta} \coloneqq \begin{cases}\mathcal{P}_{\alpha\beta} & \mathcal{J}=1\\ \mathcal{J}_{\alpha\beta} & \mathcal{J}=2\end{cases}[\Theta_{\mathcal{I}},\Theta_{\mathcal{J}}] \coloneqq \mathcal{F}_{\mathcal{I}\mathcal{J}}^{\mathcal{K}}\Theta_{\mathcal{K}}$$

$$\begin{cases} \mathcal{F}_{11}^1 = \mathcal{F}_{11}^2 = \mathcal{F}_{12}^2 = \mathcal{F}_{21}^2 = \mathcal{F}_{22}^2 = 0 \\ [\mathcal{F}_{12}^1]_{\alpha\beta;cd}^{ef} = [\mathcal{F}_{21}^1]_{cd;\alpha\beta}^{ef} = \eta_{\alpha c}\delta_\beta^e\delta_d^f - \eta_{\beta c}\delta_\alpha^e\delta_d^f \\ [\mathcal{F}_{22}^2]_{\alpha\beta;cd}^{ef} = -\eta_{\alpha c}\delta_\beta^e\delta_d^f + \eta_{\beta c}\delta_\alpha^e\delta_d^f - \eta_{\beta d}\delta_\alpha^e\delta_c^f + \eta_{\alpha d}\delta_\beta^e\delta_c^f \end{cases}$$

$$\mathfrak{U}_{c\mathcal{P}}^{\mathcal{I}} = \begin{cases} \mathcal{J}_{\circ\circ} \otimes \mathfrak{m}^{\circ\circ} & \mathcal{I} = 1 \\ \mathcal{P}_{\circ\circ} \otimes \mathfrak{G}^{\circ\circ} & \mathcal{I} = 2 \end{cases} \in \Omega^1(\mathcal{T}^{\circledast}\mathcal{M}) \otimes \mathcal{A}\mathfrak{d}(\mathcal{g}_{c\mathcal{P}})$$

$$\mathfrak{F}_{c\mathcal{P}}^{\mathcal{I}} = \begin{cases} \mathcal{J}_{\circ\circ} \otimes \mathfrak{R}^{\circ\circ} & \mathcal{I} = 1 \\ \mathcal{P}_{\circ\circ} \otimes d\mathfrak{w}\mathfrak{G}^{\circ\circ} & \mathcal{I} = 2 \end{cases} \in \Omega^2(\mathcal{T}^{\circledast}\mathcal{M}) \otimes \mathcal{A}\mathfrak{d}(\mathcal{g}_{c\mathcal{P}})$$

4. Métrica dual de Hodge.

$$\mathfrak{a} := \frac{1}{\rho!}\mathfrak{a}_{i_1\otimes i_\rho}\mathfrak{e}^{i_1} \wedge\cdots\wedge \mathfrak{e}^{i_\rho} \in \Omega^\rho(\mathcal{T}^{\circledast}\mathcal{M}_\blacksquare), \mathfrak{b} := \frac{1}{\rho!}\mathfrak{b}_{i_1\otimes i_\rho}\mathfrak{e}^{i_1} \wedge\cdots\wedge \mathfrak{e}^{i_\rho} \in \Omega^\rho(\mathcal{T}^{\circledast}\mathcal{M}_\blacksquare)$$

$$\langle\mathfrak{a},\mathfrak{b}\rangle_\blacksquare := \frac{1}{\rho!}\eta_\blacksquare^{i_1j_1}\cdots\eta_\blacksquare^{i_\rho j_\rho}\mathfrak{a}_{i_1\otimes i_\rho}\mathfrak{b}_{j_1\otimes j_\rho}\|\mathfrak{a}\|_\blacksquare^2 := \langle\mathfrak{a},\mathfrak{a}\rangle_\blacksquare$$

$$\mathfrak{a} \wedge \widehat{\mathcal{H}}_\blacksquare(\mathfrak{b}) := \det[\eta_\blacksquare]^{\frac{1}{2}}\langle\mathfrak{a},\mathfrak{b}\rangle_\blacksquare\mathfrak{v}_\blacksquare$$

$$\widehat{\mathcal{H}}_\blacksquare : \Omega^\rho \longrightarrow \Omega^{\eta-\rho} : \mathfrak{b} \longmapsto \hat{\mathfrak{b}} := \widehat{\mathcal{H}}_\blacksquare(\mathfrak{b}) = \frac{\det[\eta_\blacksquare]^{\frac{1}{2}}}{\rho!\,(\eta-\rho)!}\mathfrak{b}_{i_1\otimes i_\rho}[\epsilon]_{i_{\rho+1}\otimes i_\eta}^{i_1\otimes i_\rho}\mathfrak{e}^{i_{\rho+1}}\wedge\cdots\mathfrak{e}^{i_\eta}, [\epsilon]_{i_{\rho+1}\otimes i_\eta}^{i_1\otimes i_\rho}$$

$$:= \frac{1}{\rho!}\eta_\blacksquare^{i_1j_1}\cdots\eta_\blacksquare^{i_\rho j_\rho}\mathfrak{e}_{j_1\otimes j_\rho i_{\rho+1}\otimes i_\eta}$$

$$\widehat{\mathcal{H}}_\blacksquare : \mathfrak{b}_{i_1\otimes i_\rho} \mapsto \hat{\mathfrak{b}}_{i_{\rho+1}\otimes i_\eta} := \frac{\det[\eta_\blacksquare]^{\frac{1}{2}}}{\rho!}\mathfrak{b}_{i_1\otimes i_\rho}[\epsilon]_{i_{\rho+1}\otimes i_\eta}^{i_1\otimes i_\rho}, \hat{\mathfrak{b}} = \frac{1}{(\eta-\rho)!}\hat{\mathfrak{b}}_{i_1\otimes i_{\eta-\rho}}\mathfrak{e}^{i_1}\wedge\cdots\wedge\mathfrak{e}^{i_{\eta-\rho}}$$

$$\widehat{\mathcal{H}}_\blacksquare \circ \widehat{\mathcal{H}}_\blacksquare(\mathfrak{a}) = \det[\eta_\blacksquare]\,(-1)^{\rho(\eta-\rho)}\mathfrak{a} \Longrightarrow \widehat{\mathcal{H}}_\blacksquare^{-1}(\mathfrak{a}) = \det[\eta_\blacksquare]^{-1}\,(-1)^{\rho(\eta-\rho)}\widehat{\mathcal{H}}_\blacksquare(\mathfrak{a})$$

$$\widehat{\mathcal{H}}_\blacksquare(1) = \det[\eta_\blacksquare]^{\frac{1}{2}}\mathfrak{e}^0 \wedge\cdots\wedge \mathfrak{e}^{\eta-1} = \det[\eta_\blacksquare]^{\frac{1}{2}}\mathfrak{v} = \det[\eta_\blacksquare]\det[\varepsilon]\ d\chi^0 \wedge d\chi^1 \wedge d\chi^2 \wedge d\chi^3$$

$$\hat{d} : \Omega^\rho(\mathcal{T}^{\circledast}\mathcal{M}_\blacksquare) \longrightarrow \Omega^{\rho-1}(\mathcal{T}^{\circledast}\mathcal{M}_\blacksquare) : \mathfrak{a} \mapsto \hat{d}\mathfrak{a} := (-1)^{\rho(\eta-\rho)}\widehat{\mathcal{H}}_\blacksquare^{-1}\left(d\widehat{\mathcal{H}}_\blacksquare(\mathfrak{a})\right)$$

$$= \det[\eta_\blacksquare]^{-1}\,(-1)^{\rho(\eta-\rho+1)}\widehat{\mathcal{H}}_\blacksquare\left(d\widehat{\mathcal{H}}_\blacksquare(\mathfrak{a})\right)$$

$$\int\langle d\mathfrak{a},\mathfrak{b}\rangle_\blacksquare\mathfrak{v}_\blacksquare = \det[\eta_\blacksquare]^{-\frac{1}{2}}\int d\mathfrak{a}\wedge\widehat{\mathcal{H}}_\blacksquare(\mathfrak{b}) = \det[\eta_\blacksquare]^{-\frac{1}{2}}(-1)^{\rho(\eta-\rho)}\int \mathfrak{a}\wedge d\widehat{\mathcal{H}}_\blacksquare(\mathfrak{b})$$

$$= \det[\eta_\blacksquare]^{-\frac{1}{2}}\int \mathfrak{a}\wedge\widehat{\mathcal{H}}_\blacksquare\left((-1)^{\rho(\eta-\rho)}\widehat{\mathcal{H}}_\blacksquare^{-1}\left(d\widehat{\mathcal{H}}_\blacksquare(\mathfrak{b})\right)\right) = \int\langle\mathfrak{a},\hat{d}\mathfrak{b}\rangle_\blacksquare\mathfrak{v}_\blacksquare$$

$$\Delta_\blacksquare := d\hat{d} + \hat{d}d, \langle\Delta_\blacksquare\mathfrak{a},\mathfrak{b}\rangle_\blacksquare = \langle d\hat{d}\mathfrak{a},\mathfrak{b}\rangle_\blacksquare + \langle\hat{d}d\mathfrak{a},\mathfrak{b}\rangle_\blacksquare = \langle\mathfrak{a},d\hat{d}\mathfrak{b}\rangle_\blacksquare + \langle\mathfrak{a},\hat{d}d\mathfrak{b}\rangle_\blacksquare = \langle\mathfrak{a},\Delta_\blacksquare,\mathfrak{b}\rangle_\blacksquare$$

$$\widehat{\mathcal{H}}_\blacksquare \circ \widehat{\mathcal{H}}_\blacksquare(\mathfrak{a}) = (-1)^{\frac{\eta^2}{4}}\det[\eta_\blacksquare]\,\mathfrak{a} \in \Omega^\rho(\mathcal{T}^{\circledast}\mathcal{M}_\blacksquare)$$

$$\tilde{\mathfrak{S}}^{\mathcal{I}} := e^{i_{\rho(1}} \wedge\cdots\wedge e^{i_{\frac{\eta}{2})}}, m := \frac{\eta!}{\left(\left(\frac{\eta}{2}\right)!\right)^2}\widehat{\mathcal{H}}_\blacksquare(\tilde{\mathfrak{S}}^{\mathcal{I}}) = (-1)^{\frac{\eta}{2}}\tilde{\mathfrak{S}}^{\mathcal{I}}, \tilde{\mathfrak{S}}^\pm := \tilde{\mathfrak{S}}^{\mathcal{I}} \pm (-1)^{\frac{\eta}{2}}\tilde{\mathfrak{S}}^{\mathcal{I}}, \widehat{\mathcal{H}}_\blacksquare(\tilde{\mathfrak{S}}^+)$$

$$= (-1)^{\frac{\eta}{2}}(\tilde{\mathfrak{S}}^{\mathcal{I}} + \tilde{\mathfrak{S}}^{\mathcal{I}}) = +(-1)^{\frac{\eta}{2}}\tilde{\mathfrak{S}}^+ \in \mathbb{V}_{\widehat{\mathcal{H}}}^+\left(\Omega^{\frac{\eta}{2}}\right), \widehat{\mathcal{H}}_\blacksquare(\tilde{\mathfrak{S}}^-) = (-1)^{\frac{\eta}{2}}(\tilde{\mathfrak{S}}^{\mathcal{I}} - \tilde{\mathfrak{S}}^{\mathcal{I}})$$

$$= -(-1)^{\frac{\eta}{2}}\tilde{\mathfrak{S}}^- \in \mathbb{V}_{\widehat{\mathcal{H}}}^-(\Omega^{\eta/2})$$

$$\mathbb{V}\left(\Omega^{\frac{\eta}{2}}\right) = \mathbb{V}^+\left(\Omega^{\frac{\eta}{2}}\right) \otimes \mathbb{V}^-\left(\Omega^{\frac{\eta}{2}}\right), \mathbb{P}^{\pm}_{\widehat{\mathcal{H}}} \coloneqq \frac{1}{2}\left(1 \pm \widehat{\mathcal{H}}_{\blacksquare}\right), \mathbb{P}^{\pm}_{\widehat{\mathcal{H}}}\mathfrak{a} = \mathfrak{a}^{\pm}\widehat{\mathcal{H}}_{\blacksquare}(\mathfrak{a}^{\pm}) \in \mathbb{V}^{\pm}_{\widehat{\mathcal{H}}}\left(\Omega^2(\mathcal{T}^{\circledast}\mathcal{M}_{\blacksquare})\right)$$

5. Métrica Euclídea.

$$\widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^1_{\mathfrak{E}}\right) = \mathrm{e}^2_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}\right) = -\mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}\right) = \mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}, \widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}\right) = \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}\right) = -\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}, \widehat{\mathcal{H}}_{\mathfrak{E}}\left(\mathrm{e}^2_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}\right) = \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^1_{\mathfrak{E}}$$

$$\mathfrak{S}^{+\alpha}_{\mathfrak{E}} \coloneqq \left\{\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^1_{\mathfrak{E}} + \mathrm{e}^2_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}} - \mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}} + \mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}\right\}$$
$$\mathfrak{S}^{-\alpha}_{\mathfrak{E}} \coloneqq \left\{\mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^1_{\mathfrak{E}} - \mathrm{e}^2_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}} + \mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}}, \mathrm{e}^0_{\mathfrak{E}} \wedge \mathrm{e}^3_{\mathfrak{E}} - \mathrm{e}^1_{\mathfrak{E}} \wedge \mathrm{e}^2_{\mathfrak{E}}\right\}$$

$$\widehat{\mathcal{H}}_{\mathbb{E}} = \left(\mathfrak{S}^{\pm}_{\mathfrak{E}}\right) = \pm\mathfrak{S}^{\pm}_{\mathfrak{E}} \in \mathbb{V}^{\pm}_{\widehat{\mathcal{H}}}, \mathbb{P}^{\pm}_{\widehat{\mathcal{H}}}\left(\mathfrak{S}^{\pm}_{\mathfrak{E}}\right) = \mathfrak{S}^{\pm}_{\mathfrak{E}}, \mathbb{P}^{\pm}_{\widehat{\mathcal{H}}}\left(\mathfrak{S}^{\pm}_{\mathfrak{E}}\right) = 1$$
$$\widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^1_{\mathfrak{L}}) = i\mathrm{e}^2_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}) = -i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}) = i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}$$
$$\widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}) = -i\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}) = i\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}, \widehat{\mathcal{H}}_{\mathbb{L}}(\mathrm{e}^2_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}) = -i\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^1_{\mathfrak{L}}$$

$$\mathfrak{S}^{+\alpha}_{\mathfrak{L}} \coloneqq \{\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^1_{\mathfrak{L}} + i\mathrm{e}^2_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}} - i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}} + i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}\}$$
$$\mathfrak{S}^{-\alpha}_{\mathfrak{L}} \coloneqq \{\mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^1_{\mathfrak{L}} - i\mathrm{e}^2_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}} + i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}}, \mathrm{e}^0_{\mathfrak{L}} \wedge \mathrm{e}^3_{\mathfrak{L}} - i\mathrm{e}^1_{\mathfrak{L}} \wedge \mathrm{e}^2_{\mathfrak{L}}\}$$

$$\mathfrak{F}^{\pm}_{\mathfrak{L}} \coloneqq \mathbb{P}^{\pm}_{\widehat{\mathcal{H}}}\mathfrak{F}_{\mathfrak{L}}, \widehat{\mathcal{H}}_{\mathbb{L}}\left(\mathfrak{F}^{\pm}_{\mathfrak{L}}\right) = \pm\mathfrak{F}^{\pm}_{\mathfrak{L}} \in \mathbb{V}^{\pm}_{\widehat{\mathcal{H}}}(\Omega^2_{\mathbb{L}})$$

$$\mathcal{S}^3 \coloneqq \{\mathbb{Z}_1, \mathbb{Z}_2 \in \mathfrak{C} \mid |\mathbb{Z}_1|^2 + |\mathbb{Z}_2|^2 = 1\}$$

$$\mathcal{M} \coloneqq \begin{pmatrix} \mathbb{Z}_1 & -\mathbb{Z}_2 \\ \mathbb{Z}_2^{\circledast} & \mathbb{Z}_1^{\circledast} \end{pmatrix} \det[\mathcal{M}]\, \mathcal{M}^{\dagger} \boxdot \mathcal{M} = 1_2, \mathbb{V}^{\mathfrak{E}}_{\mathfrak{S}} \ni \lambda_{\delta}(\mathbb{Z}_1, \mathbb{Z}_2) = \sum_{0 \le \kappa \le 2\delta} c_{\kappa} z_1^{\kappa} z_2^{2\delta-\kappa}\, c_{\kappa}$$
$$\in \mathbb{R}, \{\mathbb{Z}_1, \mathbb{Z}_2\} \in \delta^3(\pi_{\delta}(g_{\mathfrak{E}}) \circ \lambda_{\delta})(\mathbb{Z}_1, \mathbb{Z}_2) = \lambda_{\delta}(\mathbb{Z}_1', \mathbb{Z}_2')\begin{pmatrix} \mathbb{Z}_1' \\ \mathbb{Z}_2' \end{pmatrix}$$
$$= g_{\mathbb{E}}^{-1}\begin{pmatrix} \mathbb{Z}_1 \\ \mathbb{Z}_2 \end{pmatrix} \dot{\pi}_{\delta}(g_{\mathfrak{E}}) \circ \lambda_{\delta} \coloneqq \frac{d}{dt}\lambda_{\delta}(e^{t\mathfrak{g}}\mathbb{Z}_1, e^{t\mathfrak{g}}\mathbb{Z}_2)\Big|_{t=0} \langle \lambda_{\delta}', \lambda_{\delta}\rangle$$
$$\coloneqq \frac{1}{\pi^2}\int_{\mathbb{C}^2}^{\infty} \lambda_{\delta}'^{\circledast}\, \lambda_{\delta} e^{-(|\mathbb{Z}_1|^2+|\mathbb{Z}_2|^2)} d\chi_1 d\gamma_1 d\chi_2 d\gamma_2, \mathbb{Z}_{\alpha} \coloneqq \chi_{\alpha} + i\gamma_{\alpha} > \infty\|\lambda_{\delta}\|$$
$$\coloneqq \langle \lambda_{\delta}, \lambda_{\delta}\rangle^{1/2}$$

$$c_{\kappa} = (\kappa!\,(2\delta - \kappa)!)^{-\frac{1}{2}}\lambda_{1/2}(\mathbb{Z}_1, \mathbb{Z}_2) = \tilde{c}^t \cdot z = (\tilde{c}_1, \tilde{c}_2)\begin{pmatrix} \mathbb{Z}_1 \\ \mathbb{Z}_2 \end{pmatrix}$$
$$= \tilde{c}_1\mathbb{Z}_1 + \tilde{c}_2\mathbb{Z}_2\left(\pi_{\frac{1}{2}}(g_{\mathfrak{E}}) \circ \lambda_{\frac{1}{2}}\right)(\mathbb{Z}_1, \mathbb{Z}_2) = \lambda_{\frac{1}{2}}(\mathbb{Z}_1', \mathbb{Z}_2') = \tilde{c}^t\left(g_{\mathfrak{E}}^{\dagger} \cdot z\right)$$
$$= (g_{\mathfrak{E}} \cdot \tilde{c})^{\dagger} \cdot z$$
$$\pi^{\mathfrak{E}}_{\delta_1 \otimes \delta_\eta} \coloneqq \left(\pi_{\delta_1}(g_{\mathfrak{E}}) \otimes \pi_{\delta_2}(g_{\mathfrak{E}}), \mathbb{V}^{\mathfrak{E}}_{\delta_1} \otimes \mathbb{V}^{\mathfrak{E}}_{\delta_2}\right)\pi_{\frac{1}{2}}(g_{\mathfrak{L}}) = g_{\mathfrak{L}} = (g_{\mathfrak{L}}^t)^{-1}\overline{\pi_{\frac{1}{2}}}(g_{\mathfrak{L}})^{\circledast} = g_{\mathfrak{L}}^{\circledast} = \left(g_{\mathfrak{L}}^{\dagger}\right)^{-1}$$
$$\dot{\pi}^{\tilde{\mathfrak{E}}}_{\delta_1 \otimes \bar{\delta}_\eta} \coloneqq \dot{\pi}_{\delta_1}\widehat{(g_{\mathfrak{L}})} \otimes \dot{\pi}_{\bar{\delta}_2}\widehat{(g_{\mathfrak{L}})} \Longrightarrow \dot{\pi}^{\tilde{\mathfrak{L}}}_{\delta_1 \otimes \bar{\delta}_\eta} \coloneqq \left(\pi_{\delta_1}\widehat{(g_{\mathfrak{L}})} \otimes \pi_{\widehat{\delta_2}}\widehat{(g_{\mathfrak{L}})}\right)\mathbb{V}^{\mathfrak{L}}_{\frac{1}{2},1/2} \coloneqq \mathbb{V}^{\mathfrak{L}}_{1/2} \otimes \mathbb{V}^{\mathfrak{L}}_{\widehat{1/2}}$$

6. Métrica de Lorentz.

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{so}(1{,}3) \otimes \mathfrak{C} = \mathfrak{sl}(2, \mathbb{C}) \oplus \overline{\mathfrak{sl}(2, \mathbb{C})}$$

$$\mathfrak{so}(3) = \mathfrak{su}(2), \mathfrak{so}(3) \otimes \mathbb{C} = \mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$$

$$Spin\,(4) = \mathcal{SU}(2) \otimes \mathcal{SU}(2), Spin\,(1{,}3) = \mathcal{SL}(2,\mathbb{C}), Spin\,(4) = \mathcal{SO}(4) \times \mathbb{Z}_2 \mathcal{U}(1), Spin\,(1{,}3) =, \mathcal{SO}\,(1{,}3) \times \mathbb{Z}_2 \mathcal{U}(1)$$

7. Métrica de Clifford.

$$\zeta,\chi \in \mathcal{C\ell} \boxplus (\mathcal{V}_\eta), \zeta\chi + \chi\zeta = 2\langle\zeta,\chi\rangle_{\mathcal{C\ell}}, \langle\zeta,\chi\rangle_{\mathcal{C\ell}} \coloneqq \zeta^t \otimes \eta \otimes \chi = \eta_{\circ\circ} \rtimes \zeta^\circ \rtimes \chi^\circ \curlywedge \gamma : \mathcal{V}_\eta \to \mathcal{C\ell} \ominus (\mathcal{V}_\eta): e^\alpha \mapsto \gamma^\alpha, Spin(\mathcal{V}_\eta)$$

$$\coloneqq \left\{\gamma^{**} \in \mathcal{C\ell}_0 \boxtimes (\mathcal{V}_\eta) \middle| \overbrace{\gamma}^{\circledast} \gamma^\dagger = \pm 1_{\delta\rho}, \overbrace{\gamma}^{\circledast} \upsilon \underbrace{\gamma}_{\triangleq} \in \mathcal{V}_\eta \forall \upsilon \in \mathcal{V}_\eta \right\} \tau^{\stackrel{\circ}{=}}(\gamma)(\upsilon)$$

$$\coloneqq \gamma\upsilon \overbrace{\gamma}^{\ddagger} \langle \tau(\gamma)(\upsilon), \tau(\gamma)(\mu) \rangle^{\circledcirc} = \langle \overbrace{\gamma}^{\circledast} \upsilon \underbrace{\gamma}_{\triangleq} \overbrace{\gamma}^{\circledast} \mu \underbrace{\gamma}_{\triangleq} \rangle = \left(sign \left[\overbrace{\gamma}^{\circledast} \gamma^\dagger \right] \right) \langle \mu, \upsilon \rangle = \widehat{\langle \mu, \upsilon \rangle}$$

$$Spin(\mathcal{V}_\eta) \ni \Gamma_\delta \coloneqq \sqrt{\frac{1}{\det[\eta]}} \gamma^0\gamma^1\gamma^2\gamma^3 \det[\eta] = \begin{cases} +1 & \eta = \eta_{\mathfrak{E}} \\ -1 & \eta = \eta_{\mathfrak{L}} \end{cases} \mathcal{P}_\delta^\pm \coloneqq \frac{1}{2}(1_{\delta\rho} \pm \Gamma_\delta)$$

$$\in End\,(\mathbb{V}_\delta^\circ) \mathcal{P}_\delta^\pm \hat{\mathcal{P}}_\delta^\pm = \langle \mathcal{P}_\delta^\pm \rangle^2 \mathcal{P}_\delta^\mp \mathbb{V}_\delta = \mathbb{V}_\delta^+ \oplus \mathbb{V}_\delta^- \xi_\delta^\pm \coloneqq \mathcal{P}_\delta^\pm \xi \in \mathbb{V}_\delta^\pm \coloneqq Im\,(\mathcal{P}_\delta^\pm) = Ker(\mathcal{P}_\delta^\mp)\Gamma_\delta : End\,(\mathbb{V}_\delta^\pm): \xi_\delta^\pm \mapsto \Gamma_\delta \xi_\delta^\pm = \pm \xi_\delta^\pm$$

$$\gamma_{\delta\varepsilon} = \{\gamma_{\delta\varepsilon}^0 \gamma_{\delta\varepsilon}^1 \gamma_{\delta\varepsilon}^2 \gamma_{\delta\varepsilon}^3\}^t = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \right\}^\tau i^2 = j^2 = \kappa^2 = ij + ji = j\kappa + \kappa j = \kappa i + i\kappa = 1$$

$$\Gamma_{\delta\varepsilon} = \sqrt{\frac{1}{\det[\eta_\varepsilon]}} \gamma_{\delta\varepsilon}^0 \gamma_{\delta\varepsilon}^1 \gamma_{\delta\varepsilon}^2 \gamma_{\delta\varepsilon}^3 = \gamma_{\delta\varepsilon}^0, \gamma_{\delta\varepsilon}^1, \gamma_{\delta\varepsilon}^2, \gamma_{\delta\varepsilon}^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{P}_{\delta\varepsilon}^+ \coloneqq \frac{1 + \Gamma_{\delta\varepsilon}}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{P}_{\delta\varepsilon}^-$$

$$\coloneqq \frac{1 - \Gamma_{\delta\varepsilon}}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Sigma_\varepsilon : \delta\rho(1) \longrightarrow \mathcal{SU}(2): \{1, i, j, \kappa\} \mapsto \sigma_\varepsilon \coloneqq \{1_2, -i\sigma\} \Sigma_\varepsilon : \gamma_{\delta\varepsilon} \mapsto \gamma_{\nu\varepsilon} = (\gamma_{\nu\varepsilon}^0 \gamma_{\nu\varepsilon}^1 \gamma_{\nu\varepsilon}^2 \gamma_{\nu\varepsilon}^3)^t$$

$$= \left(\begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_2 & -i\sigma^1 \\ i\sigma^1 & 0_2 \end{pmatrix} \begin{pmatrix} 0_2 & -i\sigma^2 \\ i\sigma^2 & 0_2 \end{pmatrix} \begin{pmatrix} 0_2 & -i\sigma^3 \\ i\sigma^3 & 0_2 \end{pmatrix} \right)^\tau \gamma_{\nu\varepsilon}^\alpha \gamma_{\nu\varepsilon}^{\alpha\dagger}$$

$$= 1_2 \det[\gamma_{\nu\varepsilon}^\alpha] \in \mathcal{SU}(2) \Sigma_\varepsilon : \mathcal{P}_{\delta\varepsilon}^\pm \mapsto \mathcal{P}_{\nu\varepsilon}^\pm = \mathcal{P}_{\delta\varepsilon}^\pm \big|_{0 \mapsto 0_2, 1 \mapsto 1_2}$$

$$\Sigma_{\mathfrak{L}}^\pm : \delta\rho(1) \mapsto \mathcal{H}(2) \cap \delta\mathcal{L}(2,\mathbb{C}): \{1, i, j, k\} \mapsto \sigma_{\mathfrak{L}}^\pm \coloneqq \{1_2, \pm\sigma\}$$

$$\gamma_{\nu\mathcal{L}} = (\gamma_{\nu\mathcal{L}}^0 \gamma_{\nu\mathcal{L}}^1 \gamma_{\nu\mathcal{L}}^2 \gamma_{\nu\mathcal{L}}^3)^t \left(\begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix} \begin{pmatrix} 0_2 & \sigma^1 \\ -\sigma^1 & 0_2 \end{pmatrix} \begin{pmatrix} 0_2 & \sigma^2 \\ -\sigma^2 & 0_2 \end{pmatrix} \begin{pmatrix} 0_2 & \sigma^3 \\ -\sigma^3 & 0_2 \end{pmatrix} \right)^\tau \gamma_{\nu\varepsilon}^\alpha$$

$$= \gamma_{\nu\varepsilon}^{\alpha\dagger} \det[\gamma_{\nu\mathcal{L}}^\alpha] \Longrightarrow \gamma_{\nu\mathcal{L}}^\alpha \in \mathcal{H}(2) \cap \delta\mathcal{L}(2,\mathbb{C}) \Gamma_{\nu\mathcal{L}} = \sqrt{\frac{1}{\det[\eta_{\mathcal{L}}]}} \gamma_{\nu\mathcal{L}}^0 \gamma_{\nu\mathcal{L}}^1 \gamma_{\nu\mathcal{L}}^2 \gamma_{\nu\mathcal{L}}^3$$

$$= \begin{pmatrix} -1_2 & 0_2 \\ 0_2 & 1_2 \end{pmatrix} \Sigma_{\mathfrak{L}}^\pm : \mathcal{P}_{\delta\varepsilon}^\pm \mapsto \mathcal{P}_{\nu\varepsilon}^\pm = \mathcal{P}_{\delta\varepsilon}^\pm \big|_{0 \mapsto 0_2, 1 \mapsto 1_2} \mathcal{P}_{\nu\mathfrak{L}}^\pm = \mathcal{P}_{\nu\varepsilon}^\pm$$

$$= \gamma_{\nu\mathcal{L}} \begin{cases} \gamma_{\delta\mathbb{L}} \coloneqq \{\gamma_{\delta\mathbb{E}}^0, \sqrt{-1}\gamma_{\delta\mathbb{E}}^1, \sqrt{-1}\gamma_{\delta\mathbb{E}}^2, \sqrt{-1}\gamma_{\delta\mathbb{E}}^3\}^\tau \\ \bar{\gamma}_{\delta\mathbb{L}} \coloneqq \{\gamma_{\delta\mathbb{E}}^0, -\sqrt{-1}\gamma_{\delta\mathbb{E}}^1, -\sqrt{-1}\gamma_{\delta\mathbb{E}}^2, -\sqrt{-1}\gamma_{\delta\mathbb{E}}^3\}^\tau \end{cases} = \mathcal{O}^{\mathcal{P}} \otimes \gamma_{\delta\mathcal{L}} \mathfrak{sl}(2,\mathbb{C})$$

$$\alpha_i \in \mathbb{C}, \beta_i \in \mathbb{H} \longmapsto (\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = \alpha_1\alpha_2 \otimes \beta_1\beta_2, \alpha \in \mathbb{C}, \beta \in \mathbb{H}(2) \longmapsto (\alpha \otimes \beta)^\dagger$$
$$= \alpha^* \otimes \beta^\dagger \left(\sqrt{-1} \otimes 1\right) \otimes (1 \otimes i) = \sqrt{-1} \otimes i \neq -1 \overset{\delta \otimes \hbar \otimes \eta}{\longrightarrow} \sqrt{-1} i \neq -1$$

$$\Gamma_{\delta\mathcal{L}} = \sqrt{-1}\gamma^0_{\delta\mathcal{L}}\gamma^1_{\delta\mathcal{L}}, \gamma^2_{\delta\mathcal{L}}\gamma^3_{\delta\mathcal{L}} = \Gamma_{\delta\varepsilon} = -\bar{\Gamma}_{\delta\mathcal{L}}\mathcal{P}^+_{\delta\mathbb{L}} = \mathcal{P}^-_{\delta\mathbb{L}} = \mathcal{P}^+_{\delta\mathbb{E}}\mathcal{P}^-_{\delta\mathbb{L}} = \bar{\mathcal{P}}^+_{\delta\mathbb{L}} = \mathcal{P}^-_{\delta\mathbb{E}}(\gamma^\alpha_{\blacksquare\mathcal{L}})^\dagger$$
$$= \gamma^0_{\blacksquare\mathcal{L}}\gamma^\alpha_{\blacksquare\mathcal{L}}\gamma^0_{\blacksquare\mathcal{L}}\alpha \in \{\eta\}$$

8. Métrica de Higgs.

$$\lambda_{1/2}(\phi_\mathcal{U}\phi_\mathcal{D}) \coloneqq \phi_\mathcal{W} = \begin{pmatrix}\phi_\mathcal{U}\\ \phi_\mathcal{D}\end{pmatrix} \in \mathbb{V}^{\mathbb{E}}_{\frac{1}{2}} \in \mathbb{C}\Gamma_{\delta\varepsilon}\phi^\mathcal{U}_\mathcal{W} = -\phi^\mathcal{U}_\mathcal{W} \in \mathbb{V}^{\mathbb{EU}}_{\frac{1}{2}} \Longrightarrow \mathcal{P}^-_{\delta\mathbb{E}}\phi_\mathcal{W} = \phi^\mathcal{U}_\mathcal{W}$$
$$= \begin{pmatrix}\phi_\mathcal{U}\\ 0\end{pmatrix}\Gamma_{\delta\varepsilon}\phi^\mathcal{D}_\mathcal{W} = +\phi^\mathcal{D}_\mathcal{W} \in \mathbb{V}^{\mathbb{ED}}_{\frac{1}{2}} \Longrightarrow \mathcal{P}^+_{\delta\mathbb{E}}\phi_\mathcal{W} = \phi^\mathbb{D}_\mathcal{W} = \begin{pmatrix}0\\ \phi_\mathcal{D}\end{pmatrix}\phi_\mathcal{H} \in \{\phi^\mathcal{U}_\mathcal{W}\phi^\mathcal{D}_\mathcal{W}\}$$
$$\subset \mathbb{V}^{\mathbb{E}}_{\frac{1}{2}} = \mathbb{V}^{\mathbb{EU}}_{\frac{1}{2}} \otimes \mathbb{V}^{\mathbb{ED}}_{\frac{1}{2}}\hat{\phi}^\mathcal{U}_\mathcal{W} \coloneqq \frac{\phi^\mathcal{U}_\mathcal{W}}{\phi_\mathcal{U}} = \begin{pmatrix}1\\0\end{pmatrix}\hat{\phi}^\mathcal{D}_\mathcal{W} \coloneqq \frac{\phi^\mathbb{D}_\mathcal{W}}{\phi_\mathcal{D}} = \begin{pmatrix}0\\1\end{pmatrix}\phi'_\mathrm{A}$$
$$= \left[\mathcal{g}^\dagger_\mathfrak{C}\right]_\mathrm{A}^{\mathrm{B}}\phi_\mathrm{B}\pi_{\frac{1}{2}}[\mathcal{g}_r] = \mathcal{g}^\alpha_r(\varphi) \coloneqq e^{-i\left(\frac{\varphi}{2}\right)\sigma^\alpha}$$
$$= \begin{cases} \begin{pmatrix}\cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2}\\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}\alpha = 1 \\ \begin{pmatrix}\cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2}\\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}\alpha = 2 \quad \langle\phi_\mathcal{W}\varphi_\mathcal{W}\rangle \coloneqq \phi^*_\mathrm{A}\delta^{\mathrm{AB}}\varphi_\mathrm{B} = (\phi^\mathrm{A})^\star\varphi_\mathrm{A} \\ \begin{pmatrix}e^{-i\left(\frac{\varphi}{2}\right)\sigma^\alpha} & 0\\ 0 & e^{i\left(\frac{\varphi}{2}\right)\sigma^\alpha}\end{pmatrix}\alpha = 3 \end{cases}$$
$$= \phi^\dagger_w \bigotimes \varphi_\omega\,[1_2]^\mathrm{A}_\mathrm{B}Spin\,(4) = \mathcal{SU}_\mathcal{R}(2) \otimes \mathcal{SU}_\mathcal{L}(2) = \mathcal{SO}(4) \bigotimes \{\mathcal{R}, \mathcal{L}\}$$

9. Métrica Weyl.

$$\lambda_{1/2}(\xi_{\mathcal{U}}\xi_{\mathcal{D}}) \coloneqq \xi = \begin{pmatrix}\xi_{\mathcal{U}}\\ \xi_{\mathcal{D}}\end{pmatrix} \in \mathbb{V}_{\frac{1}{2}}^{\mathbb{L}} \in \mathbb{C}\lambda_{\frac{1}{2}}(\widehat{\xi_{\mathcal{U}}\xi_{\mathcal{D}}}) \coloneqq \dot{\xi} = \begin{pmatrix}\dot{\xi}_{\mathcal{U}}\\ \dot{\xi}_{\mathcal{D}}\end{pmatrix} \in \mathbb{V}_{\frac{1}{2}}^{\mathbb{L}} \in \mathbb{C}, \xi' = \pi_{\frac{1}{2}}^{\mathbb{L}}(g_{\mathcal{L}})\xi$$
$$= g_{\mathcal{L}}^{\dagger}(\varphi)\xi[\xi']_{\mathrm{A}} = [g_{\mathcal{L}}(\varphi)]_{\mathrm{B}}^{\mathrm{A}}[\xi]_{\mathrm{B}}\dot{\xi}' = \pi_{\frac{1}{2}}^{\mathbb{L}}(g_{\mathcal{L}})\hat{\xi} = g_{\mathcal{L}}^{\circledast}(\varphi)\tilde{\xi}[\bar{\bar{\xi}}']_{\vec{\mathrm{A}}}$$
$$= [g_{\mathcal{L}}^{\circledast}(\varphi)]_{\bar{\mathrm{B}}}^{\vec{\mathrm{A}}}[\ddot{\xi}]_{\ddot{\mathrm{B}}}\xi^{\mathrm{A}} = \epsilon_2^{\mathrm{AB}}\xi_{\mathrm{B}}, \dot{\xi}_{\mathrm{A}} = \epsilon_2^{\dot{\mathrm{A}}\hat{\mathrm{B}}}\dot{\xi}_{\dot{\mathrm{B}}}[\epsilon_2]^{\mathrm{AB}} = [\epsilon_2]^{\widehat{\mathrm{AB}}} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$
$$= -[\epsilon_2]_{\mathrm{AB}} = -[\epsilon_2]_{\widehat{\mathrm{AB}}}\langle\xi,\zeta\rangle \coloneqq \xi_{\mathrm{A}}\epsilon_2^{\mathrm{AB}}\zeta_{\mathrm{B}} = \xi_{\mathcal{U}}\zeta_{\mathcal{D}}\overline{\langle\xi,\zeta\rangle} \coloneqq \dot{\xi}_{\mathrm{A}}\epsilon_2^{\dot{\mathrm{A}}\mathrm{B}}\dot{\zeta}_{\mathrm{B}}$$
$$= \xi_{\dot{\mathcal{U}}}\dot{\zeta}_{\dot{\mathcal{D}}} - \xi_{\dot{\mathcal{D}}}\dot{\zeta}_{\dot{\mathcal{U}}}\langle\xi',\zeta'\rangle = \langle g_{\mathcal{L}}\xi, g_{\mathcal{L}}\zeta\rangle = [g_{\mathcal{L}}\xi]_{\mathrm{A}}\epsilon_2^{\mathrm{AB}}[g_{\mathcal{L}}\zeta]_{\mathrm{B}}$$
$$= \xi_{\mathrm{A}}[g_{\mathcal{L}}^{t}]_{C}^{\mathrm{A}}\epsilon_2^{\mathrm{CD}}[g_{\mathcal{L}}]_{D}^{B}\zeta_{\mathrm{B}} = \xi^{\mathrm{A}}\zeta_{\mathrm{A}} = \langle\xi,\zeta\rangle[g_{\mathcal{L}}^{t}]_{C}^{A}\epsilon_2^{\mathrm{CD}}[g_{\mathcal{L}}]_{D}^{B} = \epsilon_2^{\mathrm{AB}}[\xi^{\mathcal{T}}]^{\mathrm{B}} \coloneqq \xi_{\mathrm{A}}\epsilon_2^{\mathrm{AB}}$$
$$= \xi^{\mathrm{B}}[\dot{\xi}^{\mathcal{T}}]^{\dot{\mathrm{B}}} \coloneqq \widehat{\xi_{\mathrm{A}}}\epsilon_2^{\widehat{\mathrm{AB}}} = \dot{\xi}^{\dot{\mathrm{B}}} \Rightarrow \langle\xi,\zeta\rangle = \xi^{\mathcal{T}} \bigotimes \zeta\langle\dot{\xi},\dot{\zeta}\rangle = \dot{\xi}^{\mathcal{T}} \bigotimes \dot{\zeta}\,\pi_{\frac{1}{2}}(g_{\beta})$$
$$= g_{\beta}^{\alpha}(\chi) \coloneqq e^{-i\left(\frac{\chi}{2}\right)\sigma^{\alpha}} = \begin{cases}\begin{pmatrix}\cosh\frac{\chi}{2} & -i\sinh\frac{\chi}{2}\\ -i\sinh\frac{\chi}{2} & \cosh\frac{\chi}{2}\end{pmatrix}\alpha = 1\\ \begin{pmatrix}\cosh\frac{\chi}{2} & -\sinh\frac{\chi}{2}\\ \sinh\frac{\chi}{2} & \cosh\frac{\chi}{2}\end{pmatrix}\alpha = 2\\ \begin{pmatrix}e^{-i\left(\frac{\chi}{2}\right)\sigma^{\alpha}} & 0\\ 0 & e^{i\left(\frac{\chi}{2}\right)\sigma^{\alpha}}\end{pmatrix}\alpha = 3\end{cases} \in \mathcal{H}(2)\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}$$
$$= \mathbb{V}_{\frac{1}{2}}^{\mathcal{UL}} \bigoplus \mathbb{V}_{\frac{1}{2}}^{\mathcal{LD}}\,\Gamma_{\delta\mathcal{L}}\xi_{\mathcal{U}} = -\xi_{\mathcal{U}} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{LR}} \Rightarrow \mathcal{P}_{\delta\mathbb{L}}^{-}\xi = \xi_{\mathcal{U}} = \begin{pmatrix}\xi_{\mathcal{U}}\\ 0\end{pmatrix}\Gamma_{\delta\mathcal{L}}\xi_{\mathcal{D}} = -\xi_{\mathcal{D}}$$
$$\in \mathbb{V}_{\frac{1}{2}}^{\mathcal{LD}} \Rightarrow \mathcal{P}_{\delta\mathbb{L}}^{+}\xi = \xi_{\mathcal{D}} = \begin{pmatrix}0\\ \xi_{\mathcal{D}}\end{pmatrix}\xi \in \{\xi_{\mathcal{U}},\xi_{\mathcal{D}}\} \subset \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}} = \mathbb{V}_{\frac{1}{2}}^{\mathcal{LU}} \bigoplus \mathbb{V}_{\frac{1}{2}}^{\mathcal{LD}}$$

$$\bar{\Gamma}_{\delta\mathcal{L}}\dot{\xi}_{\mathcal{U}} = +\dot{\xi}_{\mathcal{U}} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{LU}} \Rightarrow \bar{\mathcal{P}}_{\delta\mathbb{L}}^{+}\dot{\xi} = \dot{\xi}_{\mathcal{U}} = \begin{pmatrix}\dot{\xi}_{\mathcal{U}}\\ 0\end{pmatrix}\bar{\Gamma}_{\delta\mathcal{L}}\dot{\xi}_{\mathcal{D}} = -\dot{\xi}_{\mathcal{D}} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{LD}} \Rightarrow \bar{\mathcal{P}}_{\delta\mathbb{L}}^{-}\dot{\xi} = \dot{\xi}_{\mathcal{D}} = \begin{pmatrix}0\\ \check{\xi}_{\mathcal{D}}\end{pmatrix}\dot{\xi}$$
$$\in \{\widehat{\xi_{\mathcal{U}},\xi_{\mathcal{D}}}\} \subset \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}} = \mathbb{V}_{\frac{1}{2}}^{\mathcal{LU}} \bigoplus \mathbb{V}_{\frac{1}{2}}^{\mathcal{LD}}\,\mathfrak{g}_{Spin(1,3)} = \mathfrak{sl}(2,\mathbb{C}) \bigoplus \overline{\mathfrak{sl}(2,\mathbb{C})}$$
$$= \mathfrak{so}(1,3) \otimes \{\mathcal{R},\mathcal{L}\}$$

10. Métrica de Dirac.

$$\psi \coloneqq \begin{pmatrix}\dot{\xi}\\ \xi\end{pmatrix} \in \mathbb{V}_{\frac{1}{2}\otimes\frac{1}{2}}^{\mathcal{L}} \coloneqq \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}} \otimes \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\xi \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\,\dot{\xi} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\psi^{\mathcal{L}} \coloneqq \mathcal{P}_{\mathcal{VL}}^{-}\psi = \begin{pmatrix}\dot{\xi}\\ 0_{\delta\mathcal{L}}\end{pmatrix}\Gamma_{\mathcal{VL}}\psi^{\mathcal{L}} = -\psi^{\mathcal{L}}\psi^{\mathcal{R}}$$
$$\coloneqq \mathcal{P}_{\mathcal{VL}}^{+}\psi = \begin{pmatrix}\dot{0}_{\delta\mathcal{L}}\\ \xi\end{pmatrix}\Gamma_{\mathcal{VL}}\psi^{\mathcal{R}} = +\psi^{\mathcal{R}}$$

11. Métrica Vectorial.

11.1 Métrica vectorial euclidiana.

$$\Sigma_{\mathfrak{E}}: \delta^3 \to \delta\mathcal{U}(2): \hat{v}_{\mathcal{V}\mathfrak{E}} \mapsto \hat{v}_{\delta\mathfrak{E}} \coloneqq \sigma_{\mathfrak{E}}^t \bigotimes \hat{v}_{\mathcal{V}\mathfrak{E}} = \begin{pmatrix} \hat{v}^0 - i\hat{v}^3 & -i\hat{v}^1 - \hat{v}^2 \\ -i\hat{v}^1 + \hat{v}^2 & \hat{v}^0 + i\hat{v}^3 \end{pmatrix} \mathbb{V}^{\mathfrak{E}}_{\frac{1}{2},\frac{1}{2}}$$
$$\ni \begin{pmatrix} \phi_{\mathcal{U}} \\ \phi_{\mathcal{D}} \end{pmatrix} \bigotimes \begin{pmatrix} \phi_{\mathcal{U}} \\ \phi_{\mathcal{D}} \end{pmatrix}^{*} \mapsto \begin{pmatrix} \phi_{\mathcal{U}} & -\phi_{\mathcal{D}}^{*} \\ \phi_{\mathcal{D}} & \phi_{\mathcal{U}}^{*} \end{pmatrix} \langle \hat{\mu}_{\delta\varepsilon}, \hat{v}_{\delta\varepsilon} \rangle$$
$$\coloneqq \frac{1}{2}(Tr[\hat{\mu}_{\delta\varepsilon}]Tr[\hat{v}_{\delta\varepsilon}] - Tr[\hat{\mu}_{\delta\varepsilon}\hat{v}_{\delta\varepsilon}]) = \hat{\mu}^0\hat{v}^0 + \hat{\mu}^1\hat{v}^1 + \hat{\mu}^2\hat{v}^2 + \hat{\mu}^3\hat{v}^3 \|\hat{\mu}_{\delta\varepsilon}\|^2$$
$$\coloneqq \langle \hat{\mu}_{\delta\varepsilon}, \hat{v}_{\delta\varepsilon} \rangle = \det[\hat{\mu}_{\delta\varepsilon}] = 1$$

$$\mu_{\delta\varepsilon} = \|\mu_{\delta\varepsilon}\| \hat{v}_{\delta\varepsilon} \|\mu_{\delta\varepsilon}\|^2 > 0, \Sigma_{\mathfrak{E}}: \mathcal{O}(4) \to \delta\mathcal{U}(2) \bigotimes \mathbb{R}^4: v_{\nu\varepsilon} \mapsto v_{\delta\varepsilon}\hat{v}'_{\delta\varepsilon} = \tau\left(\mathcal{g}_r^{\alpha}(\varphi)\right)(\hat{v}_{\delta\varepsilon})$$
$$= \sigma_{\varepsilon}^t \begin{cases} \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1 \\ \hat{v}^2 \cos\varphi - \hat{v}^3 \sin\varphi \\ \hat{v}^2 \sin\varphi - \hat{v}^3 \cos\varphi \end{pmatrix} \alpha = 1 \\ \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1 \cos\varphi + \hat{v}^3 \sin\varphi \\ \hat{v}^2 \\ -\hat{v}^1 \sin\varphi + \hat{v}^3 \cos\varphi \end{pmatrix} \alpha = 2 \langle \tau(\mathcal{g}_r)(v_{\delta\varepsilon}) \coloneqq \mathcal{g}_r v_{\delta\varepsilon} \mathcal{g}_r^{\dagger} \rangle \\ \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1 \cos\varphi - \hat{v}^2 \sin\varphi \\ \hat{v}^1 \sin\varphi + \hat{v}^2 \cos\varphi \\ \hat{v}^3 \end{pmatrix} \alpha = 3 \end{cases}$$
$$\mathcal{g}: \mathbb{R} \mapsto \mathfrak{G}: \varphi \mapsto \mathcal{g}(\varphi)\mathcal{g}(\vartheta + \varphi) = \mathcal{g}(\vartheta)\mathcal{g}(\varphi) = 1_{id}$$

11.2. Métrica Vectorial de Lorentz.

$$\Sigma_{\mathcal{L}}^{\pm}: \mathcal{H}yp^3 \mapsto \mathcal{H}(2) \cap (\delta\mathcal{L}(2,\mathbb{C}) \otimes \{-1,1\}): \hat{\mu}_{\nu\mathcal{L}} \mapsto \hat{\mu}_{\delta\mathcal{L}}^{\pm} \coloneqq \left(\sigma_{\mathcal{L}}^{\pm}\right)^{\tau} \otimes \|\hat{\mu}_{\nu\mathcal{L}}\|_{\mathcal{L}} =$$
$$\begin{pmatrix} \hat{\mu}^0 \pm \hat{\mu}^3 & \pm\hat{\mu}^1 \mp i\hat{\mu}^2 \\ \pm\hat{\mu}^1 \pm i\hat{\mu}^2 & \hat{\mu}^0 \mp \hat{\mu}^3 \end{pmatrix} \mathbb{V}^{\mathcal{L}}_{\frac{1}{2}\otimes\frac{\bar{1}}{2}} \ni \begin{pmatrix} \xi_{\mathcal{U}} \\ \xi_{\mathcal{D}} \end{pmatrix} \otimes \begin{pmatrix} \xi_{\mathcal{U}} \\ \xi_{\mathcal{D}} \end{pmatrix}^{*} \mapsto \begin{pmatrix} \frac{(\xi_{\mathcal{U}} + \xi_{\mathcal{U}}^{*})}{2} & \xi_{\mathcal{D}}^{*} \\ \xi_{\mathcal{D}} & i\frac{(\xi_{\mathcal{U}}^{*} - \xi_{\mathcal{U}})}{2} \end{pmatrix} \langle \hat{\mu}_{\delta\mathcal{L}}, \hat{v}_{\delta\mathcal{L}} \rangle \coloneqq$$
$$\frac{1}{2}(Tr[\hat{\mu}_{\delta\mathcal{L}}]Tr[\hat{v}_{\delta\mathcal{L}}] - Tr[\hat{\mu}_{\delta\mathcal{L}}\hat{v}_{\delta\mathcal{L}}]) = \hat{\mu}^0\hat{v}^0 - \hat{\mu}^1\hat{v}^1 - \hat{\mu}^2\hat{v}^2 - \hat{\mu}^3\hat{v}^3 \|\hat{\mu}_{\delta\mathcal{L}}\|^2 \coloneqq \langle \hat{\mu}_{\delta\mathcal{L}} \hat{v}_{\delta\mathcal{L}} \rangle =$$
$$(\hat{\mu}^0)^2 - (\hat{\mu}^1)^2 - (\hat{\mu}^2)^2 - (\hat{\mu}^3)^2 = \det[\hat{\mu}_{\delta\mathcal{L}}] \Sigma_{\mathcal{L}}^{\pm}: \mathcal{O}(1,3) \otimes \mathfrak{R} \mapsto \mathcal{H}(2) \otimes \mathfrak{R}: \mu_{\nu\mathcal{L}} \mapsto \mu_{\delta\mathcal{L}} \mu_{\delta\mathcal{L}}^{\pm'} =$$
$$\begin{cases} \tau\left(g_{\beta}^{\alpha}(\chi)\right)(\mu_{\delta\mathcal{L}}^{+}) \\ \left(g_{\beta}^{\alpha\dagger}(\chi)\right)(\mu_{\delta\mathcal{L}}^{-}) \end{cases} = \sigma_{\mathcal{L}}^{\pm^t} \begin{cases} \begin{pmatrix} \mu^0 \cosh\chi + \mu^1 \sinh\chi \\ \mu^0 \sinh\chi + \mu^1 \cosh\chi \\ \mu^2 \\ \mu^3 \end{pmatrix} \alpha = 1 \\ \begin{pmatrix} \mu^0 \cosh\chi + \mu^2 \sinh\chi \\ \mu^1 \\ \mu^0 \sinh\chi + \mu^2 \cosh\chi \\ \mu^3 \end{pmatrix} \alpha = 2 \langle \tau(\mathcal{g}_{\beta})(v_{\delta\mathcal{L}}) \coloneqq \mathcal{g}_{\beta} v_{\delta\mathcal{L}} \mathcal{g}_{\beta}^{\dagger} \rangle \\ \begin{pmatrix} \mu^0 \cosh\chi + \mu^3 \sinh\chi \\ \mu^1 \\ \mu^2 \\ \mu^0 \sinh\chi + \mu^3 \cosh\chi \end{pmatrix} \alpha = 3 \end{cases}$$

$$\mathscr{g}_{\mathcal{L}}^{+}(\varphi,\chi) \coloneqq \mathscr{g}_{r}(\varphi) \bigotimes \mathscr{g}_{\beta}(\varphi)\, \mathscr{g}_{\mathcal{L}}^{-}(\varphi,\chi)$$

$$\coloneqq \mathscr{g}_{r}(\varphi) \bigotimes \mathscr{g}_{r}^{*}(\chi)\mathscr{g}_{r}(\varphi) \circ \mathscr{g}_{\beta}^{(*)}(\chi) = \mathscr{g}_{\beta}^{(*)}(\chi) \circ \mathscr{g}_{r}(\varphi)$$

12. Espacios cuánticos geométricamente curvos.

12.1. Curvatura de Riemann.

$$\Omega^2\left(\mathcal{T}^{\circledast}\mathcal{M}\right) \bigotimes \mathcal{V}^2\,(\mathcal{TM}) \ni \mathfrak{R} = \frac{1}{2}\mathfrak{R}^{\alpha\beta}\left(\partial_\alpha \times \partial_\beta\right) = \frac{1}{4}\mathcal{R}_{cd}^{\alpha\beta}\left(\mathrm{e}^c \wedge \mathrm{e}^d\right)\left(\partial_\alpha \times \partial_\beta\right)\mathcal{R}_{cd}^{\alpha\beta}$$

$$\Longrightarrow \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline\end{array} \otimes \begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \; \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$$

$$= \widehat{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}} \oplus \widetilde{\square\;\;\square} \oplus \blacksquare \oplus \overline{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \; \\ \cline{1-1}\end{array}} \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$$

$$\mathcal{R}_{cd}^{\alpha\beta} = \mathcal{W}_{cd}^{\alpha\beta} + \frac{2}{3}\delta_{[c}^{[\alpha}\eta^{\beta]\circ}\hat{\mathcal{R}}_{d]\circ} + \frac{1}{12}\left(\delta_c^\alpha\delta_d^\beta + \delta_d^\alpha\delta_c^\beta\right)\hat{\mathcal{R}}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{4}\eta_{\alpha\beta}\mathcal{R}$$

$$\Omega^1\left(\mathcal{T}^{\circledast}\mathcal{M}_{\mathfrak{G}}\right) \bigotimes \mathcal{V}^{-1}\,(\mathcal{TM}_{\mathcal{L}}) \ni e = e^\alpha\partial_\alpha = \varepsilon_\mu^\alpha(\chi \in \mathcal{M}_{\mathfrak{G}})d\chi^\mu\partial_\alpha g_{\mu\nu} = \eta_{\circ\circ}\varepsilon_\mu^\circ\varepsilon_\nu^\circ \Longrightarrow \boxed{\mu}\otimes\boxed{\nu}$$

$$= \square\square \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline\end{array} = \widetilde{\square\;\;\square} \bigoplus \square\,\blacksquare \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline\end{array} g_{\mu\nu}(\chi \in \mathcal{M}_{\mathfrak{G}})$$

$$= \eta_{\mu\nu} + \hbar_{\mu\nu}(\chi \in \mathcal{M}_{\mathfrak{G}})\hbar_{\mu\nu}\left(\overset{\forall}{_{\mu\nu}}\chi \in \mathcal{M}_{\mathfrak{G}}\right)\gamma_{\mu\nu}(\chi)$$

$$\coloneqq \hbar_{\mu\nu}(\chi) - \frac{1}{\eta\cdot\eta}\eta_{\mu\nu}\left(\hbar_{\rho\sigma}(\chi)\eta^{\rho\sigma}\right) \Longrightarrow \eta^{\mu\nu}\gamma_{\mu\nu}(\chi)$$

$$\Omega^1\left(\mathcal{T}^{\circledast}\mathcal{M}\right) \bigotimes \mathcal{V}^2\,(\mathcal{TM}) \ni \mathfrak{w} = \frac{1}{2}\mathfrak{w}^{\alpha\beta}\left(\partial_\alpha \times \partial_\beta\right) = \frac{1}{2}\omega_c^{\alpha\beta}\mathrm{e}^c\left(\partial_\alpha \times \partial_\beta\right)\omega_c^{\alpha\beta} \Longrightarrow \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline\end{array} \otimes \boxed{c}$$

$$= \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array} \oplus \square \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array} \mathrm{e}^\alpha\partial_\beta = \varepsilon_\mu^\alpha\varepsilon_\nu^\beta\, d\chi^\mu\left(\frac{\partial}{\partial\chi^\nu}\right) = \delta_\beta^\alpha$$

12.2. Curvatura de Dirac.

$$\partial\colon \mathcal{V}^\pm \mapsto \mathcal{V}^\mp\colon \xi^\pm \mapsto \partial\xi^\pm \in \mathcal{V}^\mp, d \coloneqq \iota_\gamma d\colon \Gamma(\mathcal{M},\Omega^1) \overset{\iota_\gamma}{\longmapsto} \Gamma(\mathcal{M},\Omega^0)\xi \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}} \mapsto d\xi \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\dot{\xi} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}$$

$$\mapsto d\dot{\xi} \in \mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}$$

$$\partial_{\delta\mathcal{L}} \coloneqq \sigma_\tau^\varepsilon \otimes \partial = 1_2\partial_0 - i\sigma^1\partial_1 - i\sigma^2\partial_2 - i\sigma^3\partial_3 = \begin{pmatrix} \partial_0 - i\partial_3 & -i\partial_1 - \partial_2 \\ -i\partial_1 + \partial_2 & \partial_0 + i\partial_3 \end{pmatrix}, \partial_{\delta\mathcal{L}} \coloneqq \sigma_\varepsilon^{+^\tau} \otimes \partial$$

$$= 1_2\partial_0 + \sigma^1\partial_1 + \sigma^2\partial_2 + \sigma^3\partial_3 = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} \langle \partial_{\delta_\blacksquare}, v_{\delta_\blacksquare} \rangle$$

$$= \partial_0 v^0 \pm \partial_1 v^1 \pm \partial_2 v^2 \pm \partial_3 v^3$$

12.3. Curvatura euclidiana.

$$\iota_\gamma d_{\delta\varepsilon} = \partial_{\delta\varepsilon} \coloneqq \gamma_{\delta\varepsilon}^{\odot}\partial_{\odot} = \begin{pmatrix} 0 & \hbar_{\mathfrak{E}}^{\odot}\partial_{\odot} \\ \bar{\hbar}_{\mathfrak{E}}^{\odot}\partial_{\odot} & 0 \end{pmatrix} \hbar_{\mathfrak{E}}^{\alpha} \coloneqq (1, i, j, \kappa)\hat{\hbar}_{\mathfrak{E}}^{\alpha}$$
$$\coloneqq (-1, -i, -j, -\kappa)\mathcal{P}_{\delta}^{\pm}\partial_{\delta\varepsilon}\mathcal{P}_{\delta}^{\pm}\mathcal{P}_{\delta}^{\mp}\partial_{\delta\varepsilon}\mathcal{P}_{\delta}^{\mp} = \partial_{\delta\varepsilon}\mathcal{P}_{\delta}^{\pm}\mathcal{P}_{\delta}^{\mp}$$

12.4. Curvatura de Lorentz.

$$d_{\delta\mathfrak{L}}\xi \coloneqq \iota_\gamma d_{\delta\mathfrak{L}}\xi = \gamma_{\delta\mathcal{L}}^{\odot}(\partial_{\odot}\xi) \coloneqq \partial_{\delta\mathcal{L}}, \bar{d}_{\delta\mathfrak{L}}\dot{\xi} \coloneqq \iota_{\bar{\gamma}}\bar{d}_{\delta\mathfrak{L}}\dot{\xi} = \bar{\gamma}_{\delta\mathcal{L}}^{\odot}(\partial_{\odot}\dot{\xi}) \coloneqq \bar{\partial}_{\delta\mathcal{L}}\xi$$
$$d_{\nu\mathcal{L}}\psi = \iota_\gamma d\psi = \iota_\gamma\left((\partial_j\psi)\mathrm{e}^j\right) = \gamma_{\nu\mathcal{L}}^{\odot}\partial_0\psi \coloneqq \partial_{\nu\mathcal{L}}\psi\psi^{\pm} \in \mathbb{V}_{\frac{1}{2}\otimes\frac{\bar{1}}{2}}^{\mathcal{L}^{\pm}} \Longrightarrow \partial_{\nu\mathcal{L}}\psi^{\pm}\left(\frac{1_{\delta\rho} \pm \Gamma_{\nu\mathcal{L}}}{2}\psi\right)$$
$$= \frac{1_{\delta\rho} \pm \Gamma_{\nu\mathcal{L}}}{2}(\partial_{\nu\mathcal{L}}\psi) \in \mathbb{V}_{\frac{1}{2}\otimes\frac{\bar{1}}{2}}^{\mathcal{L}^{\mp}}\partial_{\nu\mathcal{L}}^2 = ((\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2)1_{\delta\rho}$$
$$\coloneqq \Delta_{\nu\mathcal{L}}d_{\nu\varepsilon} = \iota_\gamma d = \iota_\gamma\left((\partial_j)\mathrm{e}^j\right) = \gamma_{\nu\varepsilon}^{\odot}\partial_0 \coloneqq \partial_{\nu\varepsilon}\partial_{\nu\varepsilon}^2$$
$$= ((\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2)1_{\delta\rho} \coloneqq \Delta_{\nu\varepsilon}$$

12.5. Curvatura de Higgs.

$$d_{\delta\mathcal{L}}\phi_{\mathcal{H}} = \partial_{\delta\mathcal{L}}\phi_{\mathcal{H}}\partial_{\delta\mathcal{L}}(\mathcal{P}_{\delta\varepsilon}^{\pm}\phi_{\mathcal{H}}) = \mathcal{P}_{\delta\varepsilon}^{\mp}(\partial_{\delta\mathcal{L}}\phi_{\mathcal{H}})i\delta_{\delta\mathcal{L}}$$
$$\coloneqq i d_{\delta\mathcal{L}}^{\dagger}\langle i\delta_{\delta\mathcal{L}}\phi_{\mathcal{H}}, \phi_{\mathcal{H}}\rangle\langle\phi_{\mathcal{H}}, i d_{\delta\mathcal{L}}\phi_{\mathcal{H}}\rangle \int_{\mathcal{M}_{\mathcal{L}}}^{\square} \langle i\delta_{\delta\mathcal{L}}\phi_{\mathcal{H}}, \phi_{\mathcal{H}}\rangle$$
$$= -i\int_{\mathcal{M}_{\mathcal{L}}}^{\square}\left(\gamma_{\delta\mathcal{L}}^{\odot^{\dagger}}\partial_0\phi_{\mathcal{H}}\right)^{\dagger}\phi_{\mathcal{H}} = -i\int_{\mathcal{M}_{\mathcal{L}}}^{\square}(\partial_0\phi_{\mathcal{H}}^{\dagger})\gamma_{\delta\mathcal{L}}^{\odot^{\dagger}}\phi_{\mathcal{H}}$$
$$= -i\int_{\mathcal{M}_{\mathcal{L}}}^{\square}\partial_0\left(\phi_{\mathcal{H}}^{\dagger}\gamma_{\delta\mathcal{L}}^{\odot^{\dagger}}\phi_{\mathcal{H}}\right) + i\int_{\mathcal{M}_{\mathcal{L}}}^{\square}\phi_{\mathcal{H}}^{\dagger}\gamma_{\delta\mathcal{L}}^{\odot^{\dagger}}(\partial_0\phi_{\mathcal{H}})$$
$$= \int_{\mathcal{M}_{\mathcal{L}}}^{\square}\langle\phi_{\mathcal{H}}, i d_{\delta\mathcal{L}}, \phi_{\mathcal{H}}\rangle d_{\delta\mathcal{L}}\langle\phi_{\mathcal{H}}, \phi_{\mathcal{H}}\rangle \coloneqq \langle\delta_{\delta\mathcal{L}}\phi_{\mathcal{H}}, \phi_{\mathcal{H}}\rangle + \langle\phi_{\mathcal{H}}, d_{\delta\mathcal{L}}\phi_{\mathcal{H}}\rangle$$
$$= 2\langle\phi_{\mathcal{H}}, d_{\delta\mathcal{L}}\phi_{\mathcal{H}}\rangle$$

13. Métrica Utiyama – Yang - Mills.

$$\left(\mathcal{M}\otimes\left(\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\oplus\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\right),\pi_{\delta\rho},\mathcal{M},\mathfrak{G}_{c\mathcal{P}}\otimes Spin(1,3)\right)\tau_{cov}: Spin(1,3)\mapsto \delta\mathcal{O}(1,3)\otimes\{\mathcal{R},\mathcal{L}\}\pi_{\delta\rho}$$
$$:=\frac{\tau_{cov}}{\{\mathcal{R},\mathcal{L}\}}:\mathcal{M}\otimes\left(\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\oplus\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\right)\mapsto\frac{\mathcal{M}}{\{\mathcal{R},\mathcal{L}\}}:\xi|_{\rho}\mapsto\rho\in\mathcal{M},\bar{\mathfrak{A}}_{\delta\rho}:=Tr_{\mathcal{CL}}[\mathfrak{m}\bar{\delta}]$$
$$=\frac{i}{2}\mathfrak{m}^{\odot\odot}\bar{\delta}_{\odot\odot},\mathfrak{A}_{\delta\rho}:=Tr_{\mathcal{CL}}[\mathfrak{m}\delta]=\frac{i}{2}\mathfrak{m}^{\odot\odot}\delta_{\odot\odot}\bar{\delta}^{\alpha\beta}:=i\left[\frac{\bar{\gamma}^{\alpha}}{2},\frac{\bar{\gamma}^{\beta}}{2}\right],\delta^{\alpha\beta}$$
$$:=i\left[\frac{\gamma^{\alpha}}{2},\frac{\gamma^{\beta}}{2}\right]\bar{d}_{\delta\rho}\dot{\xi}:=d\dot{\xi}-ic_{gr}\bar{\mathfrak{A}}_{\delta\rho}\dot{\xi},d_{\delta\rho}\xi:=d\xi-ic_{gr}\mathfrak{A}_{\delta\rho}\xi\bar{\mathfrak{F}}_{\delta\rho}$$
$$:=d\bar{\mathfrak{A}}_{\delta\rho}+ic_{gr}\bar{\mathfrak{A}}_{\delta\rho}\wedge\bar{\mathfrak{A}}_{\delta\rho}=\frac{1}{2}\bar{\delta}_{\odot\odot}\mathcal{R}^{\odot\odot},\mathfrak{F}_{\delta\rho}:=d\mathfrak{A}_{\delta\rho}+ic_{gr}\mathfrak{A}_{\delta\rho}\wedge\mathfrak{A}_{\delta\rho}$$
$$=\frac{1}{2}\delta_{\odot\odot}\mathcal{R}^{\odot\odot}\bar{d}_{\delta\rho}:=\iota_{\gamma}\bar{d}_{\delta\rho}=(\bar{\partial}-ic_{gr}\bar{\mathfrak{A}}_{\delta\rho})d_{\delta\rho}:=\iota_{\gamma}d_{\delta\rho}$$
$$=\left(\partial-ic_{gr}\mathfrak{A}_{\delta\rho}\right)\left(\mathcal{M}\otimes\mathbb{V}_{\frac{1}{2}}^{\varepsilon},\pi_{\delta\mu},\mathcal{M},\delta\mathcal{U}_{\omega}(2)\right)$$

$$\mathfrak{G}_{\delta\mu}:End(\Gamma_{\omega}):\phi_{\omega}\mapsto\mathfrak{G}_{\delta\mu}(\phi_{\omega}):=g_{\delta\mu}\phi_{\omega}\left[\mathfrak{G}_{\delta\mu}(\phi_{\omega})\right]_{\mathbb{I}}=\left[g_{\delta\mu}\right]_{\mathcal{J}}^{\mathcal{J}}[\phi_{\omega}]_{\mathcal{J}}\mathcal{J},\mathcal{J}\{\mathcal{U},\mathcal{D}\},\mathfrak{G}_{\delta\mu}(\mathfrak{A}_{\delta\mu})$$
$$=g_{\delta\mu}^{-1}\mathfrak{A}_{\delta\mu}g_{\delta\mu}+ic_{\delta\mu}^{-1}g_{\delta\mu}^{-1}dg_{\delta\mu}=g_{\delta\mu}^{-1}\mathfrak{A}_{\delta\mu}g_{\delta\mu}\mathfrak{G}_{\delta o}:End\left(\Omega^{1}(\mathcal{T}^{\circledast}\mathcal{M})\right):\mathfrak{A}_{\delta\mu}$$
$$\mapsto\mathfrak{G}_{\delta o}(\mathfrak{A}_{\delta\mu})=\Lambda\mathfrak{A}_{\delta\mu}\mathfrak{F}_{\delta\mu}=\mathfrak{F}_{\delta\mu}^{\mathcal{J}}-ic_{\delta\mu}\mathfrak{A}_{\delta\mu}\wedge\mathfrak{A}_{\delta\mu}$$
$$=\left(d\mathfrak{A}_{\delta\mu}^{\mathcal{J}}+\frac{c_{\delta\mu}}{2}f_{\mathcal{J}\mathcal{K}}^{\mathcal{J}}\mathfrak{A}_{\delta\mu}^{\mathcal{J}}\wedge\mathfrak{A}_{\delta\mu}^{\mathcal{K}}\right)\tau_{\mathcal{J}}\in\Omega^{2}(\mathcal{T}^{\circledast}\mathcal{M})\bigotimes Ad(\mathfrak{su}(2))\,d_{g}\mathfrak{a}$$
$$:=1_{\delta\mu}d\mathfrak{a}-\frac{i}{2}c_{\delta\mu}\left[\mathfrak{A}_{\delta\mu},\mathfrak{a}\right]_{\wedge}=1_{\delta\mu}d\mathfrak{a}-ic_{\delta\mu}\mathfrak{A}_{\delta\mu}\wedge\mathfrak{a}\left(d_{g}\wedge d_{g}\right)\mathfrak{a}^{\alpha}$$
$$=-ic_{\delta\mu}\mathfrak{F}_{\delta\mu}\wedge\mathfrak{a}^{\alpha}d_{g}\mathfrak{F}_{\delta\mu}\mathfrak{A}_{\delta\mu}=\mathfrak{A}_{\delta\mu}^{\mathcal{J}}\tau_{\mathcal{J}}:=\mathcal{A}_{\alpha}^{\mathcal{J}}\mathrm{e}^{\alpha}\tau_{\mathcal{J}}\mathfrak{F}_{\delta\mu}=\mathfrak{F}_{\delta\mu}^{\mathcal{J}}\tau_{\mathcal{J}}$$
$$:=\frac{1}{2}\mathcal{F}_{\alpha\beta}^{\mathcal{J}}\mathrm{e}^{\alpha}\wedge\mathrm{e}^{\beta}\tau_{\mathcal{J}}\Longrightarrow\mathcal{F}_{\alpha\beta}^{\mathcal{J}}=\partial_{\alpha}\mathcal{A}_{\beta}^{\mathcal{J}}-\partial_{\beta}\mathcal{A}_{\alpha}^{\mathcal{J}}+c_{\delta\mu}f_{\mathcal{J}\mathcal{K}}^{\mathcal{J}}\mathcal{A}_{\alpha}^{\mathcal{J}}\mathcal{A}_{\beta}^{\mathcal{K}}d_{g}\phi_{\omega}$$
$$:=(1_{\delta\mu}\partial-ic_{\delta\mu}\mathfrak{A}_{\delta\mu})\phi_{\omega}\mathfrak{A}_{\delta\mu}=\gamma^{\odot}\mathcal{A}_{\odot}^{\mathcal{J}}\tau_{\mathcal{J}}[\tau_{\mathcal{J}}\phi_{\mathcal{H}}]_{\mathrm{A}}:=[\tau_{\mathcal{J}}]_{\mathrm{A}}^{\mathrm{B}}\phi_{\mathrm{B}}\mathrm{A},\mathrm{B}\in\{\mathcal{U},\mathcal{D}\}$$

$$(\mathcal{M}\otimes\left(\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\oplus\mathbb{V}_{\frac{1}{2}}^{\mathcal{L}}\right)\otimes\mathbb{V}_{\frac{1}{2}}^{\varepsilon},\pi_{\delta\rho}\oplus\pi_{\delta\mu},\mathcal{M},\mathfrak{G}_{c\mathcal{P}}\otimes Spin(1,3)\otimes\delta\mathcal{U}_{\omega}(2)\mathfrak{A}_{\delta g}$$
$$=\mathfrak{A}_{\delta\mu}\bigotimes 1_{\delta\rho}+1_{\delta\mu}\bigotimes\mathfrak{A}_{\delta\rho}\,\mathfrak{F}_{\delta g}=\mathfrak{F}_{\delta\mu}\bigotimes 1_{\delta\rho}+1_{\delta\mu}\bigotimes\mathfrak{F}_{\delta\rho}\,d_{\delta g}$$
$$:=\left(1_{\delta\mu}\bigotimes 1_{\delta\rho}\right)d-ic_{\delta\mu}\left(\mathfrak{A}_{\delta\mu}\bigotimes 1_{\delta\rho}\right)-ic_{gr}\left(1_{\delta\mu}\bigotimes\mathfrak{A}_{\delta\rho}\right))$$

$$\Xi^{1}=\xi^{1}\otimes\phi_{\omega}^{1},\Xi^{2}=\xi^{2}\otimes\phi_{\omega}^{2}\Longrightarrow\begin{cases}\langle\Xi^{1},\Xi^{2}\rangle=\langle\xi^{1},\xi^{2}\rangle\langle\phi_{\omega}^{1},\phi_{\omega}^{2}\rangle\\ \langle\xi^{1},\Xi^{2}\rangle=\quad\langle\xi^{1},\xi^{2}\rangle\phi_{\omega}^{2}\\ \langle\Xi^{1},\xi^{2}\rangle=\quad\langle\xi^{1},\xi^{2}\rangle\phi_{\omega}^{1}\\ \langle\Xi^{1},\phi_{\omega}^{2}\rangle=\quad\langle\phi_{\omega}^{1},\phi_{\omega}^{2}\rangle\xi^{1}\\ \langle\phi_{\omega}^{1},\Xi^{2}\rangle=\quad\langle\phi_{\omega}^{1},\phi_{\omega}^{2}\rangle\xi^{2}\end{cases}\quad\xi\bigotimes\phi_{\omega}^{1}+\xi\bigotimes\phi_{\omega}^{2}$$
$$=\xi\bigotimes\langle\phi_{\omega}^{1}+\phi_{\omega}^{2}\rangle\xi^{1}\bigotimes\phi_{\omega}+\xi^{2}\bigotimes\phi_{\omega}=\langle\xi^{1}+\xi^{2}\rangle\bigotimes\phi_{\omega}$$

$$\Xi_\omega \coloneqq \xi^1 \bigotimes \hat{\phi}^\mu_\omega + \xi^2 \bigotimes \hat{\phi}^D_\omega \in \Gamma\left(\mathcal{TM}_\mathcal{L}\mathbb{V}^\mathcal{L}_{\frac{1}{2}} \bigotimes \mathbb{V}^\varepsilon_{\frac{1}{2}}\,Spin(1,3) \bigotimes \delta\mathcal{U}_\omega(2)\right)\dot{\Xi}_\omega$$
$$\coloneqq \dot{\xi}^1 \bigotimes \hat{\phi}^\mu_\omega + \dot{\xi}^2 \bigotimes \hat{\phi}^D_\omega$$
$$\in \Gamma\left(\mathcal{TM}_\mathcal{L}\mathbb{V}^\mathcal{L}_{\frac{1}{2}} \bigotimes \mathbb{V}^\varepsilon_{\frac{1}{2}}\,Spin(1,3) \bigotimes \delta\mathcal{U}_\omega(2)\right)\langle\dot{\Xi}_\omega,\dot{\Xi}_\omega\rangle$$
$$= \langle\xi^1,\dot{\xi}^1\rangle\langle\hat{\phi}^\mu_\omega,\hat{\phi}^\mu_\omega\rangle + \langle\dot{\xi}^2,\xi^1\rangle\langle\hat{\phi}^D_\omega,\hat{\phi}^\mu_\omega\rangle + \langle\dot{\xi}^1,\xi^2\rangle\langle\hat{\phi}^\mu_\omega,\hat{\phi}^D_\omega\rangle + \langle\dot{\xi}^2,\xi^2\rangle\langle\hat{\phi}^D_\omega,\hat{\phi}^D_\omega\rangle$$
$$= \langle\xi^1,\dot{\xi}^1\rangle + \langle\dot{\xi}^2,\xi^2\rangle\pi_{\frac{1}{2}}(g_\varepsilon)\dot{\Xi}_\omega = \dot{\xi}^1 \bigotimes (g^\dagger_\varepsilon\hat{\phi}^\mu_\omega) + \dot{\xi}^2 \bigotimes (g^\dagger_\varepsilon\hat{\phi}^D_\omega)$$
$$= [g^\dagger_\varepsilon]^1_1\dot{\xi}^1 \bigotimes \binom{1}{0} + [g^\dagger_\varepsilon]^1_2\dot{\xi}^1 \bigotimes \binom{0}{1} + [g^\dagger_\varepsilon]^2_1\dot{\xi}^2 \bigotimes \binom{1}{0}$$
$$+ [g^\dagger_\varepsilon]^2_2\dot{\xi}^2 \bigotimes \binom{0}{1}\left([g^\dagger_\varepsilon]^1_1\dot{\xi}^1 + [g^\dagger_\varepsilon]^2_1\dot{\xi}^2\right) \bigotimes \binom{1}{0}$$
$$+ \left([g^\dagger_\varepsilon]^1_2\dot{\xi}^1 + [g^\dagger_\varepsilon]^2_2\dot{\xi}^2\right) \bigotimes \binom{0}{1}[g^\star_\varepsilon]^1_\mathrm{A}[\dot{\xi}^\mathrm{A}]_1 \bigotimes \hat{\phi}^\mu_\omega + [g^\star_\varepsilon]^2_\mathrm{A}[\dot{\xi}^\mathrm{A}]_2 \bigotimes \hat{\phi}^D_\omega$$

$$\langle\dot{\Xi}'_\omega,\dot{\Xi}'_\omega\rangle = \langle\dot{\Xi}'_\omega,\dot{\Xi}'_\omega\rangle = \pi_{\frac{1}{2}}(g_\varepsilon)\dot{\Xi}_\omega\tau_\mathcal{J} \cdot \dot{\Xi}_\omega \coloneqq \dot{\xi}^1 \bigotimes \tau_\mathcal{J} \cdot \hat{\phi}^\mu_\omega + \dot{\xi}^2 \bigotimes \tau_\mathcal{J} \cdot \hat{\phi}^D_\omega = \tau_\mathcal{J}\gamma^\alpha\Xi$$
$$= (\gamma^\alpha_{\delta\mathcal{L}}\xi) \bigotimes \phi_\omega \Longrightarrow \langle\dot{\Xi}^*_\omega,\gamma^\alpha_{\delta\mathcal{L}}\Xi_\omega\rangle = \langle\dot{\xi}^1,\gamma^\alpha_{\delta\mathcal{L}}\xi^1\rangle + \langle\dot{\xi}^2,\gamma^\alpha_{\delta\mathcal{L}}\xi^2\rangle$$
$$= \dot{\xi}^{1\dagger_\epsilon}\gamma^\alpha_{\delta\mathcal{L}}\xi^1 + \dot{\xi}^{2\dagger_\epsilon}\gamma^\alpha_{\delta\mathcal{L}}\xi^2\bar{d}_{\delta g}\colon End\left(\mathbb{V}^\mathcal{L}_{\frac{1}{2}}\oplus\mathbb{V}^\mathcal{L}_{\frac{1}{2}}\right)\colon \dot{\Xi}_\omega \mapsto \bar{d}_{\delta g}\dot{\Xi}_\omega \coloneqq (\iota_{\bar{\gamma}}d_{\bar{\delta}g})\dot{\Xi}_\omega$$
$$= \partial\dot{\Xi}_\omega - ic_{gr}\left((\overline{\mathfrak{A}}_{\delta\rho}\dot{\xi}^1) \bigotimes \hat{\phi}^\mu_\omega + (\overline{\mathfrak{A}}_{\delta\rho}\dot{\xi}^2) \bigotimes \hat{\phi}^D_\omega\right)$$
$$- ic_{\delta\mu}\left(\dot{\xi}^1 \bigotimes (\mathfrak{A}_{\delta\mu}\hat{\phi}^\mu_\omega) + \dot{\xi}^2 \bigotimes (\mathfrak{A}_{\delta\mu}\hat{\phi}^D_\omega)\right)$$
$$= (\partial\dot{\xi}^1) \bigotimes \hat{\phi}^\mu_\omega + (\partial\dot{\xi}^2) \bigotimes \hat{\phi}^D_\omega$$
$$- ic_{gr}\left((\overline{\mathfrak{A}}_{\delta\rho}\dot{\xi}^1) \bigotimes \hat{\phi}^\mu_\omega + (\overline{\mathfrak{A}}_{\delta\rho}\dot{\xi}^2) \bigotimes \hat{\phi}^D_\omega\right)$$
$$- ic_{\delta\mu}\mathcal{A}^\mathcal{J}_\circledcirc\bar{\gamma}^\circledcirc\left(\dot{\xi}^1 \bigotimes (\tau_\mathcal{J} \cdot \hat{\phi}^\mu_\omega) + \dot{\xi}^2 \bigotimes (\tau_\mathcal{J} \cdot \hat{\phi}^D_\omega)\right)$$

14. Métrica Utiyama – Yang – Mills en lagrangiano.

14.1. Bosones y calibre gravitacional.

$$\mathcal{L}_{\mathcal{YMU}} = \mathcal{L}_{\mathcal{YM}} + \mathcal{L}_{\mathfrak{GR}} + \mathcal{L}_{\mathfrak{FM}} + \mathcal{L}_{\mathfrak{H}} + \mathcal{L}_{\mathfrak{FM}-\mathfrak{H}}\mathfrak{I}_\blacksquare$$
$$\coloneqq \int_{\mathcal{T}^\circledcirc\mathcal{M}_\mathcal{L}}^{\square} \mathcal{L}_\blacksquare \;\boxdot \in \{\mathcal{YM},\mathfrak{GR},\mathfrak{FM},\mathfrak{H},\mathfrak{FM}-\mathfrak{H}\}$$

$$\mathcal{L}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) \coloneqq \mathcal{C}_{DIM} Tr_{\mathcal{G}}[\mathfrak{F}_{\mathfrak{G}} \wedge \widehat{\mathfrak{F}}_{\mathfrak{G}}] = i\mathcal{C}_{DIM} Tr_{\mathcal{G}}[\|\mathfrak{F}_{\mathfrak{G}}\|^2]\mathfrak{v} \frac{\delta \mathcal{J}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})}{\delta \mathfrak{A}_{\mathfrak{G}}} \Longrightarrow \hat{d}_{\mathfrak{G}} \mathfrak{F}_{\mathfrak{G}}$$

$$\coloneqq \hat{d}\mathfrak{F}_{\mathfrak{G}} - ic_{\mathfrak{G}}[\widehat{\mathfrak{A}}_{\mathfrak{G}}, \mathfrak{F}_{\mathfrak{G}}]_{\wedge} \widehat{\mathfrak{A}}_{\mathfrak{G}} \wedge \mathfrak{a}$$

$$\coloneqq \det[\eta]^{-1}(-1)^{\rho(\eta-\rho+1)} \widehat{\mathcal{H}}\left(\mathfrak{A}_{\mathfrak{G}} \wedge \widehat{\mathcal{H}}(\mathfrak{a})\right)$$

$$\Longrightarrow \int \langle d_{\mathfrak{G}} a, b \rangle \mathfrak{v} = \int \langle \mathfrak{a}, \hat{d}_{\mathfrak{G}} \mathfrak{b} \rangle \frac{1}{i\mathcal{C}_{DIM}} \frac{\delta \mathcal{J}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})}{\delta \mathfrak{A}_{\mathfrak{G}}}$$

$$\coloneqq \frac{1}{ic_{\mathfrak{G}}} \lim_{t \mapsto 0} \frac{d}{dt} \mathcal{J}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}} + t\mathfrak{a}) = \lim_{t \mapsto 0} \frac{d}{dt} \int \|\mathfrak{F}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}} + t\mathfrak{a})\|^2 \mathfrak{v}$$

$$= 2\lim_{t \mapsto 0} \int \langle \frac{d}{dt} \mathfrak{F}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}} + t\mathfrak{a}), \mathfrak{F}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}} + t\mathfrak{a}) \rangle \, \mathfrak{v} = 2\lim_{t \mapsto 0} \int \langle d_{\mathfrak{G}} \mathfrak{a}, \mathfrak{F}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) \rangle \mathfrak{v}$$

$$= 2\lim_{t \mapsto 0} \int \langle \mathfrak{a}, \hat{d}_{\mathfrak{G}} \mathfrak{F}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) \rangle \mathfrak{v}$$

$$\mathcal{L}_{\mathcal{YM}}(\mathfrak{A}_{\delta\mu}) \coloneqq iTr_{\delta\mu}\left[\left\|\mathfrak{F}_{\delta g}(\mathfrak{A}_{\delta\mu})\right\|^2\right]\mathfrak{v}$$

$$= Tr_{\delta\mu}[\mathfrak{F}_{\delta g}(\mathfrak{A}_{\delta\mu})$$

$$\wedge \widehat{\mathfrak{F}}_{\delta\mu}(\widehat{\mathfrak{A}}_{\delta\mu})] \frac{\delta \mathcal{J}_{\mathcal{YM}}(\mathfrak{A}_{\delta\mu})}{\delta \mathfrak{A}_{\delta\mu}} \hat{d}_{\delta\mu} \mathfrak{F}_{\delta\mu} \frac{1}{4} \mathcal{F}_{\mathcal{J}}^{\odot\odot} \mathcal{F}_{\odot\odot}^{\mathcal{J}} \mathfrak{v} \mathcal{F}_{\alpha\beta}^{\mathcal{J}} \eta_{\mathcal{L}}^{\odot\odot} (\partial_0 \mathcal{F}_{\odot\alpha}^{\mathcal{J}}$$

$$+ c_{\delta\mu} f_{\mathcal{J}\mathcal{K}}^{\mathcal{J}} \mathcal{A}_{\odot}^{\mathcal{J}} \mathfrak{F}_{\odot\alpha}^{\mathcal{K}}) \tau_{\mathcal{J}} [\partial] = [\mathcal{A}] = [\mathcal{T}] = \mathcal{L}^{-1}[\mathfrak{v}] = \mathcal{L}^4$$

$$\Longrightarrow [\mathcal{L}_{\mathcal{YM}}(\mathfrak{A}_{\delta\mu})] \mathcal{L}_{\mathfrak{G}\mathfrak{R}}(\mathfrak{m}, \mathfrak{e}) \coloneqq -\frac{i}{\hbar\kappa_{\varepsilon}} Tr_{c\mathcal{P}}[\|\mathfrak{F}_{c\mathcal{P}}\|^2]\mathfrak{v}$$

$$= -\frac{1}{\hbar\kappa_{\varepsilon}} Tr_{c\mathcal{P}}[\mathfrak{F}_{c\mathcal{P}} \wedge \widehat{\mathfrak{F}}_{c\mathcal{P}}] \hbar\kappa_{\varepsilon} \mathcal{L}_{\mathfrak{G}\mathfrak{R}} = -Tr_{c\mathcal{P}}[\mathfrak{F}_{c\mathcal{P}} \wedge \widehat{\mathfrak{F}}_{c\mathcal{P}}]$$

$$= \mathcal{F}_{\mathcal{J}\mathcal{J}}^{\mathcal{K}} Tr_{\delta o}[\mathfrak{F}_{c\mathcal{P}}^{\mathcal{J}} \wedge \widehat{\mathfrak{F}}_{c\mathcal{P}}^{\mathcal{J}}] \Theta_{\kappa} = Tr_{\delta o}[\mathfrak{R}(\mathfrak{m}) \wedge \mathfrak{S}(\mathfrak{e})] = \frac{1}{2} \mathfrak{R}^{\odot\odot}(\mathfrak{m}) \wedge \mathfrak{S}_{\odot\odot}(\mathfrak{e})$$

$$= \frac{1}{\hbar\kappa_{\varepsilon}} Tr_{\delta o}[\mathfrak{R}(\mathfrak{m}) \wedge \mathfrak{S}(\mathfrak{e})] = \frac{1}{\hbar\kappa_{\varepsilon}} \mathfrak{R}\mathfrak{v}[\mathcal{R}] = [\hbar\kappa_{\varepsilon}]^{-1} = \mathcal{L}^{-2}[\mathfrak{v}] = \mathcal{L}^4$$

$$\Longrightarrow [\mathcal{L}_{\mathfrak{G}\mathfrak{R}}] \frac{\delta \mathcal{J}_{\mathfrak{G}\mathfrak{R}}(\mathfrak{m}, \mathfrak{e})}{\delta \mathfrak{e}} \Longrightarrow \frac{1}{\kappa_{\varepsilon}} \mathfrak{G}^{\alpha} \coloneqq \frac{1}{2} \epsilon_{\alpha\odot\odot\odot} \mathcal{R}^{\odot\odot} \wedge \mathfrak{e}^{\odot} \frac{\delta \mathcal{J}_{\mathfrak{G}\mathfrak{R}}(\mathfrak{m}, \mathfrak{e})}{\delta \mathfrak{m}}$$

$$\Longrightarrow \frac{1}{\hbar\kappa_{\varepsilon}} d_{\mathfrak{m}} \mathfrak{e}^{\alpha} = \frac{1}{\hbar\kappa_{\varepsilon}} \mathfrak{I}^{\alpha}$$

$$\frac{1}{\hbar\kappa_{\varepsilon}} \mathfrak{G}_{\alpha} \Longrightarrow \frac{1}{\hbar\kappa_{\varepsilon}} \mathfrak{G}_{\varepsilon}^{\alpha\beta} \coloneqq \mathcal{R}^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} \mathcal{R} \frac{1}{\hbar\kappa_{\varepsilon}} \mathfrak{I}^{\alpha} \Longrightarrow \frac{1}{2\hbar\kappa_{\varepsilon}} \left(\partial_{\mu} \varepsilon_{\nu}^{\alpha} + c_{gr} \omega_{\mu\odot}^{\alpha} \varepsilon_{\nu}^{\odot} - (\mu \leftrightarrow \nu)\right)$$

14.2. Fermiones y calibre gravitacional.

$$\mathcal{L}_{\mathfrak{F}\mathfrak{M}} \coloneqq \mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathcal{R}} + \mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathfrak{L}} = (\mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathcal{R}} + \mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathfrak{L}}) \mathfrak{v} \mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathcal{R}}(\xi^e, \xi^{e\dagger_{\epsilon}}) \coloneqq \langle \xi^{e\circledast} i d_{\delta\rho} \xi^e \rangle =$$

$$i\xi^{e\dagger_{\epsilon}}(d_{\delta\rho}\xi^e) \frac{\delta \mathcal{J}_{\mathcal{F}\mathcal{M}}^{\mathcal{R}}(\xi^e, \xi^{e\dagger_{\epsilon}})}{\delta \xi^{e\dagger_{\epsilon}}} \Longrightarrow i d_{\delta\rho} \xi^e \Longrightarrow \gamma^{\odot}\left(i\partial_0 - \frac{1}{2} c_{gr} \omega_{\odot}^{\star\star} \delta_{\star\star}\right) \xi^e \mathcal{L}_{\mathfrak{F}\mathfrak{M}}^{\mathfrak{L}}(\dot{\Xi}_{\omega}, \dot{\Xi}_{\omega}^{\dagger_{\epsilon}}) \coloneqq$$

$$\langle \dot{\Xi}_{\omega}^{\circledast}, i\bar{d}_{\delta g} \dot{\Xi}_{\omega} \rangle = i\dot{\Xi}_{\omega}^{\dagger_{\epsilon}}(\bar{d}_{\delta g} \dot{\Xi}_{\omega}) \frac{\delta \mathcal{J}_{\mathcal{F}\mathcal{M}}^{\mathcal{R}}(\dot{\Xi}_{\omega}, \dot{\Xi}_{\omega}^{\dagger_{\epsilon}})}{\delta \dot{\Xi}_{\omega}^{\dagger_{\epsilon}}} \left(i\partial_0 - \frac{1}{2} c_{gr} \omega_{\odot}^{\star\star} \bar{\delta}_{\star\star}\right) + c_{\delta\mu} \mathcal{A}_{\odot}^{\mathcal{J}} \tau_{\mathcal{J}} \dot{\Xi}_{\omega}$$

$$\mathcal{L}_{\mathcal{F}\mathcal{M}}^{m_e} = m_e \langle \dot{\xi}^{e\circledast}, \xi^e \rangle = m_e (\dot{\xi}^e)^{\dagger_{\epsilon}} \cdot \xi^e$$

$$= m_e [\dot{\xi}^{e\circledast}]_{\mathrm{A}} \epsilon_2^{\mathrm{AB}} [\xi^e]_{\mathrm{B}} \langle \hat{\xi}_{\mu}^{e\circledast} \xi_D^e \rangle - \langle \hat{\xi}_D^{e\circledast} \xi_{\mu}^e \rangle \langle \dot{\Xi}_{\omega} \Phi_{\omega} \rangle \langle \phi_{\omega}^{\mu} \hat{\phi}_{\omega}^{\mu} \rangle \dot{\xi}^{\nu}$$

$$+ \langle \phi_{\omega}^D \hat{\phi}_{\omega}^D \rangle \|\dot{\xi}^e \phi_{\mu}\| \langle \dot{\xi}^{\nu} \phi_D \dot{\xi}^e \lambda \phi_{\mu} \lambda \phi_D \rangle$$

14.3. Métrica de Higgs y calibre gravitacional.

$$\frac{\delta \mathcal{I}_{\mathcal{H}}(\phi_{\mathcal{H}}, \phi_{\mathcal{H}}^{\dagger})}{\delta \phi_{\mathcal{H}}^{\dagger}} \Longrightarrow \left(\left(id_{\delta\mu} \right)^2 - \frac{\delta \mathcal{V}(\phi_{\mathcal{H}}, \phi_{\mathcal{H}}^{\dagger})}{\delta \phi_{\mathcal{H}}^{\dagger}} \right) \phi_{\mathcal{H}} \int \langle 1_{\delta\mu} \Delta \mathfrak{A}_{\delta\mu} ic_{\delta\mu} d\phi_{\mathcal{H}} \rangle \langle d\phi_{\mathcal{H}}^{\dagger} (\iota_{\gamma} d_{\delta\mu}) \partial \mathfrak{A}_{\delta\mu} \rangle \xi^{e\dagger\epsilon} \gamma^{\circ} \gamma^{\circ} \tau_{\mathcal{J}}$$

$$\frac{1}{2} \epsilon_{\alpha\beta\circledcirc\circledcirc} \mathfrak{I}_{\blacksquare}^{\circ} \wedge e^{\alpha\beta} \left\| \frac{\delta \mathcal{L}_{\blacksquare}}{\delta(dm^{\alpha\beta})} \frac{\delta \mathcal{L}_{\blacksquare}}{\delta(de^{\alpha\beta})} \right\|^{\hbar\Phi_{\blacksquare}} \langle \hbar\kappa_{\varepsilon} \rangle \mathfrak{G}_{\alpha}^{\blacksquare} \delta_{\alpha\beta} \xi$$

15. Curvatura de Yang – Mills.

$$\frac{1}{2}\frac{d}{dt}\mathcal{A}'_0 \int_{\beta}^{\square} \nabla\varphi^2 |\mathcal{F}^+|^2 = \int_{\mathcal{M}}^{\square} \langle \nabla\varphi'^2 \mathcal{F}^+, \partial_t(\mathcal{F}^+) \rangle$$
$$= \int_{\tau_\eta}^{\square} \langle \nabla\varphi^2 \mathcal{F}^+, \nabla\varphi' \mathcal{D}\mathcal{D}^{\circledast} \mathcal{F}(t) dt \rangle \int_0^{\tau_\eta} \langle \nabla'\varphi' \rtimes \nabla\mathcal{F}^+ \rangle \int \mathcal{H}^{\frac{1}{2}+\varepsilon, q-2\varepsilon}$$
$$\cap \nabla\psi' \left| \nabla\Psi' \frac{\partial\varphi}{\partial t} \right.$$
$$+ \nabla\Omega' \frac{\partial\psi}{\partial t} \mathcal{D}_{ref\varphi} \left| \langle \partial\Omega', \partial\Upsilon', \partial\Psi' \rangle \prod_{i=1}^{j} \langle \nabla_{ref}^{\circledast} \nabla_{ref}^{(\ell_i)} \nabla_{ref}^{\circledast} \rangle_{\mathrm{A}} \int ds \sum_{\ell_i > r-1-\frac{\eta}{2}} \left\| \frac{\eta - 2(r-1-\ell_i)}{2\hbar} \right\|_{\frac{\eta}{\ell_i+1} \otimes \frac{2\eta}{\eta - 2(r-1-\ell_i)}} \langle \mathcal{C}_r, \Omega_1 \rangle \right.$$

$$\frac{1}{2}\frac{d}{dt}\mathcal{A}_0 \int_{\beta}^{\square} \nabla\varphi^2 |\mathcal{F}^+|^2 = \int_{\mathcal{M}}^{\square} \langle \nabla\varphi^2 \mathcal{F}^+, \partial_t(\mathcal{F}^+) \rangle = \int_{\tau_\eta}^{\square} \langle \nabla\varphi^2 \mathcal{F}^+, \nabla\varphi \mathcal{D}\mathcal{D}^{\circledast} \mathcal{F}(t) dt \rangle$$
$$\int_0^{\tau_\eta} \langle \nabla\varphi \rtimes \nabla\mathcal{F}^+ \rangle \int \mathcal{H}^{\frac{1}{2}+\varepsilon, q-2\varepsilon} \cap \psi \left| \nabla\Psi \frac{\partial\varphi}{\partial t} + \nabla\Omega \frac{\partial\psi}{\partial t} \mathcal{D}_{ref\varphi} \right| \langle \partial^2\Omega, \partial^2\Upsilon, \partial^2\Psi \rangle \prod_{\iota=1}^{J} \langle \nabla_{ref}^{\circledast} \nabla_{ref}^{(\ell_\iota)} \nabla_{ref}^{\circledast} \rangle_{\mathrm{A}}$$
$$\int ds \sum_{\ell_\iota > r-1-\frac{\eta}{2}} \left\| \frac{\eta - 2(r-1-\ell_\iota)}{2\hbar} \right\|_{\frac{\eta}{\ell_\iota+1} \frac{\eta}{\ell_\iota+1} \otimes \frac{\widetilde{2\eta}}{\eta - 2(r-1-\ell_\iota)}} \langle \mathcal{C}_r, \Omega_1 \rangle$$

$$\|\mathfrak{G}\|_{\mathcal{H}_{\mathcal{A}_\eta}^{\mathcal{J}}} \coloneqq \mathcal{A}'_0 \sqrt{\|\nabla_{\mathrm{A}}\beta'\|^2 + \|\beta\|^2} \mathcal{V}(\mathcal{A}')$$
$$\coloneqq \sup_{\beta' \epsilon \mathcal{H}^1 \left(\mathcal{T}^{\circledast}\mathcal{M} \bigotimes g_e \right) \beta} \frac{\left| \int \langle \mathcal{F}'_{\mathcal{A}}, \mathcal{D}'_{\mathcal{A}}\beta \rangle \right|}{\|\mathfrak{G}'\|_{\mathcal{H}_{\mathcal{A}}^{\mathcal{J}}}} \frac{1+\epsilon}{1-\epsilon} \|\mathfrak{G}'\|_{\frac{2\eta}{\eta}-2} \|\mathrm{A}_1$$
$$- \mathrm{A}_2\|_\eta \left\|\nabla_{\mathrm{A}_1}\beta\right\|^2 - \left\|\nabla_{\mathrm{A}_2}\beta\right\|^2 \frac{\left| \int \langle \mathcal{F}'_{\mu\nu(\mathcal{A})}, \mathcal{D}'_{\mu\nu(\mathcal{A})}\beta \rangle \right|}{\left(\left\|\nabla_{\mu\nu(\mathcal{A})}\beta\right\|^2 + \|\beta\|^2 \right)^{\frac{1}{2}}} \langle \nabla_{\mu\nu(\mathcal{A})} \mu^{-1} \rangle |\mathcal{M}\nabla\Psi|$$
$$+ \left\| \frac{1}{2\mathcal{C}} \right\| \langle \xi\psi_i \textstyle\prod_{(\alpha)} \mathcal{Y}\mathcal{M} \rangle |\partial_i \mathcal{F}|^{1-\vartheta} \|d\delta d\psi\|_{\mathcal{H}-1} \langle \delta_\infty^\theta \rangle \inf \left| \frac{1}{4\pi^2 \eta_\kappa} \right|_{\xi}^{\zeta} \left| \mathcal{F}_{\mathrm{B}_\rho}^{+} \right|^2$$
$$- \left| \mathcal{F}_{\mathrm{A}_\sigma}^{-} \right|^2 \bigotimes \left| \mathcal{F}_{\mathrm{B}_\sigma}^{-} \right|^2 - \left| \mathcal{F}_{\mathrm{A}_\rho}^{+} \right|^2$$

$$\|\mathfrak{G}\|_{\mathcal{H}^{\mathcal{J}}_{\mathcal{A}_\eta}} := \mathcal{A}_0\sqrt{\|\nabla_{\mathrm{A}}\beta\|^2 + \|\beta\|^2}\mathcal{V}(\mathcal{A}) := \underset{\beta\epsilon\mathcal{H}^1\left(\mathcal{T}^{\circledast}\mathcal{M}\bigotimes g_e\right)}{sup}\beta$$

$$\frac{|\int\langle\mathcal{F}_{\mathcal{A}},\mathcal{D}_{\mathcal{A}}\beta\rangle|}{\|\mathfrak{G}\|_{\mathcal{H}^{\mathcal{J}}_{\mathcal{A}}}}\frac{1+\epsilon}{1-\epsilon}\|\mathfrak{G}\|_{\frac{2\eta}{\eta}-2}\|\mathrm{A}_1-\mathrm{A}_2\|_\eta\left\|\nabla_{\mathrm{A}_1}\beta\right\|^2 - \left\|\nabla_{\mathrm{A}_2}\beta\right\|^2\frac{\left|\int\langle\mathcal{F}_{\mu\nu(\mathcal{A})},\mathcal{D}_{\mu\nu(\mathcal{A})}\beta\rangle\right|}{\left(\left\|\nabla_{\mu\nu(\mathcal{A})}\beta\right\|^2+\|\beta\|^2\right)^{\frac{1}{2}}}$$

$$\langle\nabla_{\mu\nu(\mathcal{A})}\mu^{-1}\rangle|\mathcal{M}\nabla\Psi| + \left\|\frac{1}{2\mathcal{C}}\right\|\langle\xi\psi_\iota\prod\nolimits_{(\alpha)}\mathcal{YM}\rangle\,|\partial_\iota\mathcal{F}|^{1-\vartheta}\|d\delta d\psi\|_{\mathcal{H}-1}\langle\delta^\theta_\infty\rangle$$

$$\inf\left|\frac{1}{4\pi^2\eta_\kappa}\right|_\xi^\zeta\left|\mathcal{F}^+_{\mathrm{B}_\rho}\right|^2 - \left|\mathcal{F}^-_{\mathrm{A}_\sigma}\right|^2\bigotimes\left|\mathcal{F}^-_{\mathrm{B}_\sigma}\right|^2 - \left|\mathcal{F}^+_{\mathrm{A}_\rho}\right|^2$$

$$\frac{\partial\mathcal{F}_{\mathrm{A}}}{\partial t} + \Delta_{\mathrm{A}}\mathcal{F}_{\mathrm{A}}\frac{\partial_\alpha}{\partial t} - \mathcal{D}^{\otimes}_{\nabla_{ref+\alpha}}\Omega\frac{\partial\Omega}{\partial t} - \Delta_{\nabla_{ref+\alpha}}\Omega\left(\mathcal{R}^{ij}_{kl}\frac{\mathcal{R}}{2(m+2)}\mathcal{R}^{\mathbb{HP}^m}_{ijkl} + \mathcal{R}^{hyp}_{ijkl} - 2\omega^{ij}_{kl}\right)\langle\nabla_{\mathrm{A}\omega}\nabla^{\circledast}_{\mathrm{A}\omega}\rangle\left[\!\left[\mathcal{F}^+_{\mathrm{A}_\rho},\omega\right]\!\right]\left[\!\left[\mathcal{F}^-_{\mathrm{A}_\sigma},\omega\right]\!\right]\langle\mathcal{G}_{\mathfrak{E}}\rangle\mathrm{A}_{\mathfrak{E}}\mathrm{B}_{\mathfrak{E}}$$

$$\frac{\partial\mathcal{F}_{\mathrm{A}}}{\partial t} + \Delta_{\mathrm{A}}\mathcal{F}_{\mathrm{A}}\frac{\partial_\alpha}{\partial t} - \mathcal{D}^{\otimes}_{\nabla_{ref+\alpha}}\Omega\frac{\partial\Omega}{\partial t} - \Delta_{\nabla_{ref+\alpha}}\Omega\left(\mathcal{R}^{ij}_{kl}\widetilde{\frac{\mathcal{R}}{2(m+2)}\mathcal{R}^{\mathbb{HP}^m}_{ijkl}} + \mathcal{R}^{hyp}_{ijkl} - 2\omega^{ij}_{kl}\right)\langle\nabla_{\mathrm{A}\omega}\nabla^{\circledast}_{\mathrm{A}\omega}\rangle\left[\!\left[\mathcal{F}^+_{\mathrm{A}_\rho},\omega\right]\!\right]\left[\!\left[\mathcal{F}^-_{\mathrm{A}_\sigma},\omega\right]\!\right]\langle\mathcal{G}_{\mathfrak{E}}\rangle\mathrm{A}_{\mathfrak{E}}\mathrm{B}_{\mathfrak{E}}$$

$$\Lambda^2\mathcal{T}^{\circledast}\mathcal{M} = \Lambda^2_{\mathfrak{sp}(m)} \oplus \Lambda^2_{\mathfrak{sp}(1)} \oplus \Lambda^2_{(\mathfrak{sp}(m)\oplus\mathfrak{sp}(1))^\dagger}\kappa(\rho) + \delta\frac{1}{2}\int_{\mathcal{M}}^{\square}|\mathcal{F}_{\mathrm{A}}|^2 d\mathcal{V}$$

16. Agujero negro cuántico de Yang – Mills.

$$\delta = \frac{1}{2}\sqrt{-g}d^4\chi[\mathcal{R} - \mathcal{F}_{\mathcal{M}} - \mathcal{F}^{\mathcal{YM}}_\rho]\mathfrak{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{T}_{\mu\nu}, ds^2$$
$$= -f(r)dt^2 + f(r)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)\frac{d}{dt}\big(r(f)\big)$$
$$= 1\frac{Q^2}{r^2} - \frac{2^\rho q^{2\rho}_{\mathcal{YM}}}{r^{4\rho-2}}f(r)$$
$$= 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2} + \frac{Q_{\mathcal{YM}}}{r^{4\rho-2}}\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right)\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right)$$
$$- 2iqg^{\mu\nu}\mathrm{A}_\mu\partial_\nu\Phi - \mathrm{q}^2 g^{\mu\nu}\mathrm{A}_\mu\mathrm{A}_\nu\Phi - m^4\Phi(t,r,\theta,\phi)$$
$$= \sum_{m'\ell}\frac{\varphi(r)}{r}\Upsilon_{\ell m'}(\theta,\phi)e^{-i\omega t}\left(\frac{d^2}{dr^2} + \omega^2 - \mathcal{V}(r)\right)\varphi(r)r_{\odot}$$
$$= r - \frac{1}{2\kappa_{\hbar_-}}\ln\left(\frac{r}{r_{\hbar_-}} - 1\right) + \frac{1}{2\kappa_+\ln\left(\frac{r}{r_{\hbar_+}} - 1\right)}$$

$$\kappa_{\hbar_+} = \left|\frac{1}{2}f'(r_{\hbar_+})\right|, \kappa_{\hbar_-} = \left|\frac{1}{2}f'(r_{\hbar_-})\right|\mathcal{V}_{eff-1}(r) = f(r)\left[\frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r}\right]\mathcal{V}_{eff-2}(r)$$
$$= \omega^2 - f(r)\left[\frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r}\right] + \frac{\mathrm{q}^2Q^2}{r^2} - \frac{2\omega qQ}{r} - m^4 f(r)$$

$$\varphi(r) \sim \begin{cases} e^{-i\left(\omega - \frac{qQ}{r_{\hbar_+}}\right) r_*} & r_* \mapsto -\infty \\ e^{i\omega r} & r_* \mapsto \infty \end{cases} \varphi \sim \exp(-\mathbb{I}m\omega\mu)\varphi_0 \left|\varphi_{r_\hbar}\right|^2$$

$$\sim \exp(\kappa_i\mu)|\varphi_0|^2\beta = -\frac{\mathbb{I}m\omega}{\kappa_i} i\kappa - \left(\eta + \frac{1}{2}\right) - \Lambda(\eta) = \Omega(\eta)\kappa = \frac{\mathcal{V}_0}{\sqrt{2\mathcal{V}_0^{(2)}}\omega^2}$$

$$= \left[\mathcal{V}_0 + \left(-2\mathcal{V}_0^{(2)}\right)^{\frac{1}{2}}\widetilde{\Lambda}(\eta)\right]$$

$$- i\left(\eta + \frac{1}{2}\right)\left(-2\mathcal{V}_0^{(2)}\right)^{\frac{1}{2}}\left[1 + \widetilde{\Omega}(\eta)\right]\left(4\frac{\partial^2}{\partial_\mu \partial_\nu} + \mathcal{V}(\mu,\nu)\right)\varphi(\mu,\nu)\varphi(\mathrm{N})$$

$$= \varphi(\mathcal{W}) + \varphi(\mathfrak{E}) - \varphi(\delta) - \frac{\hbar^2}{16}\mathcal{V}(\delta)\big(\varphi(\mathcal{W}) + \varphi(\mathfrak{E})\big)\varphi(t) = \sum_{j=1}^{\rho} \mathcal{C}_j e^{i\omega_j t}\chi_\eta$$

$$= \varphi(\eta\hbar) = \sum_{j=1}^{\rho} \mathcal{C}_j e^{i\omega_j \eta\hbar} = \sum_{j=1}^{\rho} \mathcal{C}_j z_j^{\eta}\mathrm{A}(z) = \prod_{j=1}^{\rho}(z - z_j)$$

$$= \sum_{m=0}^{\rho} \alpha_m z^{\rho-m} \sum_{m=0}^{\rho} \alpha_m \chi_{\eta-m}$$

$$= \sum_{m=0}^{\rho} \alpha_m \sum_{j=1}^{\rho} \mathcal{C}_j z_j^{\eta-m} = \sum_{j=1}^{\rho} \mathcal{C}_j z_j^{\eta-\rho}\mathrm{A}(z_j) - \chi_\eta\omega_j = \frac{i}{\hbar}\ln(z_j)$$

$$\frac{|q|}{m} > \left.\frac{|Q|}{\mathcal{M}}\right|_{ext} m^4 r_{\hbar_+}^2 \gg l(l+1)m^4 r_{\hbar_+}^2 \gg 2\kappa_{r_{\hbar_+}} r_{\hbar_+} r_0 = \frac{\mathrm{q}^2 Q^2}{qQ\omega - m^4 r_{\hbar_+}^2 \kappa_{r_{\hbar_+}}}\mathcal{K}$$

$$\simeq \frac{\kappa_{r_{\hbar_+}}^2 m^4 r_{\hbar_+}^2 qQ}{2f_{r_0}\left(\kappa_{r_{\hbar_+}} m^4 r_{\hbar_+}^2 - qQ\omega\right)^2}\frac{q}{m} \geq \frac{r_{\hbar_+}}{Q}\mathcal{M} = \frac{-2Q_{\mathcal{YM}}\rho r^3 + r^{4\rho+1}}{r^{4\rho}}Q$$

$$= \sqrt{\frac{-4Q_{\mathcal{YM}}\rho r^4 + 3Q_{\mathcal{YM}}r^4 + r^{4\rho+2}}{r^{4\rho}}}\, r_{\hbar_\pm} = \frac{\mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2(1 + Q_{\mathcal{YM}})}}{1 + Q_{\mathcal{YM}}}$$

$$Q_{all-exe}^2 = Q^2(1 + Q_{\mathcal{YM}}) = Q^2\left(1 - \frac{\sqrt{2}}{2}q_{\mathcal{YM}}\right) = r^2\left(1 - \frac{\sqrt{2}}{2}q_{\mathcal{YM}}\right)^2\beta$$

$$\simeq 2f_{r_0}\frac{m^4}{q^2}\left(1 - \frac{\sqrt{2}}{2}q_{\mathcal{YM}}\right)^{-2}\frac{q}{m} > \left(1 - \frac{\sqrt{2}}{2}q_{\mathcal{YM}}\right)^{-1} 2\sqrt{f_{r_0}}\left(1 - \frac{\sqrt{2}}{2}q_{\mathcal{YM}}\right)^{-1}$$

$$> \frac{q}{m}$$

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APÉNDICE D.

Deformación del tejido espacio – tiempo en campos cuánticos curvos.

1. Interacción gravitacional.

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + \mathcal{V}\psi\partial_t\rho + \nabla\cdot\partial_t(\rho\vec{\upsilon}) + \nabla\cdot\left(\rho\vec{\upsilon}\bigotimes\vec{\upsilon}+\rho\iota\right) - \rho\nabla\mathcal{V}\partial_t\xi + \nabla$$

$$\cdot\,[\vec{\upsilon} + (\xi+\rho)] = -\rho\vec{\upsilon}\cdot\nabla^2\mathcal{V} - \mathrm{\rho} + \mathrm{\rho}_{\mathfrak{FDM}}(\gamma-1)\rho\varepsilon\kappa^{1+1/\eta}$$

2. Deformabilidad – materia oscura y microagujeros negros en espacios cuánticos curvos.

$$\delta = \int d^4\chi\sqrt{-g}\left[\frac{\mathcal{R}}{16\pi} - \nabla_\alpha\bar{\phi}\nabla^\alpha\phi - \mathcal{V}(\bar{\phi}\phi) + \mathcal{L}_m\right]j_\mu i\left(\bar{\phi}\nabla_\mu\phi - \phi\nabla_\mu\bar{\phi}\right)\mathrm{N}_\beta$$

$$\equiv \int d^3\chi\,\sqrt{-g}g^{0\mu}j_\mu$$

$$\mathcal{T}_{\mu\nu}^{(\phi)} = -g_{\mu\nu}\left(\partial_\alpha\bar{\phi}\,\partial^\alpha\phi + \mathcal{V}(\bar{\phi}\phi)\right) + \partial_\alpha\bar{\phi}\,\partial_\nu\phi + \partial_\alpha\phi\,\partial_\nu\bar{\phi}\nabla_\mu\nabla^\mu\bar{\phi} = \phi\mathcal{V}'^{(\bar{\phi}\phi)} \coloneqq \frac{d\mathcal{V}}{d|\phi|^2}$$

$$\mathcal{T}_{\mu\nu}^{(\mathrm{N}\delta)} = [\rho(1+\epsilon)+\mathcal{P}]u_\mu u_\nu + \mathcal{P}g_{\mu\nu}\mathrm{N}_{\mathcal{F}}\int d^3\chi\,\sqrt{-g}g^{0\mu}\rho u_\mu g_{\mu\nu}$$

$$= diag\left(-e^{\nu(r)}, e^{\mu(r)}, r^2, r^2, sin^2\theta\right)\phi(t,r) = \phi_0(r)e^{-i\omega t}\phi_0''$$

$$= e^\mu(\mathcal{V}'(\phi_0^2) - \omega^2 e^{-\nu})\phi_0 + \left(\frac{\mu'-\nu'}{2} - \frac{2}{r}\right)\phi_0'$$

$$\mu' = 8\pi r e^\mu[\omega^2\phi_0^2 e^{-\nu} + \mathcal{V}(\bar{\phi}\phi) + e^{-\mu}\phi_0'^2 + \rho(1+\epsilon)] - \frac{e^\mu - 1}{r}$$

$$\nu' = 8\pi r e^\mu[\omega^2\phi_0^2 e^{-\nu} + \mathcal{V}(\bar{\phi}\phi) + e^{-\mu}\phi_0'^2 + \mathcal{P}] + \frac{e^\mu - 1}{r}$$

$$\mathcal{P}' = -[\rho(1+\epsilon)+\mathcal{P}]\frac{\nu'}{2}$$

$$\mathrm{N}_\beta = 8\pi\int_0^\infty drr^2 e^{(\mu-\nu)/2}\omega\,\phi_0^2$$

$$\mathrm{N}_{\mathcal{F}} = 4\pi\int_0^{\mathcal{R}_f} drr^2 e^{\mu/2}\rho$$

$$\mathcal{M}_{tot} = \lim_{r\mapsto\infty} \frac{r}{2}\left(1 - e^{-\mu(r)}\right)\frac{d\mathrm{N}_{\mathcal{F}}}{d\sigma} \propto \frac{d\mathrm{N}_{\beta}}{d\sigma} - \frac{\partial \mathcal{M}_{tot}}{\partial \rho_c}\frac{\partial \mathrm{N}_{\mathcal{F}}}{\partial \phi_c} + \frac{\partial \mathcal{M}_{tot}}{\partial \phi_c}\frac{\partial \mathrm{N}_{\mathcal{F}}}{\partial \rho_c}$$

$$g_{tt} = -1 + \frac{2\mathcal{M}_{tot}}{r}$$

$$- \xi_{ij}\chi^i\chi^j\left(1 + \frac{3\lambda_{tidal}}{r^5}\right)\hbar_{\mu\nu}\Upsilon_{\Lambda}(\theta,\varphi)$$

$$\times\, diag\left(-e^{\nu(r)}\mathcal{H}_0(r), e^{\mu(r)}\mathcal{H}_2(r)r^2\kappa(r), r^2\kappa(r), sin^2\theta\right)\delta\phi(t,r,\theta,\varphi)$$

$$= \phi_1(r)\frac{e^{-i\omega t}}{r}\Upsilon_{\Lambda}(\theta,\varphi)\phi_1''\frac{\mu' - \nu'}{2}\phi_1'$$

$$+ \left[-2\phi_0' - r\phi_0'' + \frac{\mu' + \nu'}{2}r\phi_0' + \omega^2 r\phi_0 e^{\mu-\nu}\right]\mathcal{H}_0$$

$$+ \left[\frac{6e^{\mu}}{r^2} + \frac{\mu' + \nu'}{2r} + 16\pi\phi_0'^2 + e^{\mu}\left(\mathcal{V}'(\phi_0^2) + 2\phi_0^2\mathcal{V}''(\phi_0^2) - \omega^2 e^{-\nu}\right)\right]\phi_1\mathcal{H}_0''$$

$$+ \left[\frac{\mu' - \nu'}{2} + \frac{2}{r}\right]\mathcal{H}_0'$$

$$+ \left[-8\pi\frac{1 + 3c_{\delta}^2}{c_{\delta}^2}\phi_0'^2 + 8\pi\omega^2 e^{\mu-\nu}\frac{c_{\delta}^2 - 1}{c_{\delta}^2}\phi_0^2 - \frac{\mu'\nu' + \nu'^2}{2} + \nu'' + \frac{3\mu' + 7\nu'}{2r}\right.$$

$$\left. + \frac{\mu' + \nu'}{2rc_{\delta}^2} - \frac{6}{r}e^{\mu}\right]\mathcal{H}_0$$

$$= \left[-\frac{16\pi}{r}\frac{1 + 3c_{\delta}^2}{c_{\delta}^2}\phi_0'' + \frac{8\pi}{r}\left(3\mu' + \nu' + \frac{\mu' - \nu'}{c_{\delta}^2} - \frac{4}{r}\frac{1 + 3c_{\delta}^2}{c_{\delta}^2}\right)\phi_0'\right.$$

$$\left. + \frac{16\pi}{r}e^{\mu}\left(\mathcal{V}'(\phi_0^2)\frac{c_{\delta}^2 + 1}{c_{\delta}^2} + \omega^2 e^{-\nu}\frac{c_{\delta}^2 - 1}{c_{\delta}^2}\right)\phi_0\right]\phi_1$$

$$\nu'' = 8\pi e^{\mu}(r\mathcal{P}\mu' + r\mathcal{P}' + \mathcal{P}) + 16\pi r\phi_0'\phi_0'' + 8\pi\phi_0'^2 + 16\pi r e^{\mu}\left(-\mathcal{V}'(\phi_0^2) + \omega^2 e^{-\nu}\right)\phi_0\phi_0'$$

$$- 8\pi e^{\mu}\mathcal{V}(\phi_0^2)(r\mu' + 1) + 8\pi\omega^2 e^{\mu}[r(\mu' - \nu')e^{-\mu} + e^{-\nu}]\phi_0^2 + e^{\mu}\frac{r\mu' - 1}{r^2}$$

$$+ \frac{1}{r^2}$$

$$\mathcal{H}_0'' + \left(\frac{2}{r} + e^{\mu}\frac{2\mathcal{M}}{r^2}\right)\mathcal{H}_0' - \left(\frac{6e^{\mu}}{r^2} + e^{2\mu}\frac{4\mathcal{M}^2}{r^4}\right)\mathcal{H}_0$$

$$\approx c_1\mathcal{Q}_2^2\left(\frac{r}{\mathcal{M}} - 1\right)$$

$$+ c_2\mathcal{P}_2^2\left(\frac{r}{\mathcal{M}}\right.$$

$$\left. - 1\right)\langle(1-2\mathfrak{C})^2 2\mathfrak{C}(\gamma$$

$$- 1)(1-\mathfrak{C})^2\log(1-2\mathfrak{C})(1+\gamma)\rangle^{-1}\phi_1(r)\sum_i \phi_{1,i}r^i\,\phi_1(r)\phi_{1,3}r^3$$

$$+ \mathcal{O}(r^5)\mathcal{H}_0(r)\mathcal{H}_{0,2}r^2 + \mathcal{O}(r^4)\lim_{r\mapsto\infty}\phi_1\ \mathcal{V}(\bar{\phi}\phi)$$

$$= m^4\bar{\phi}\phi$$

$$+ \frac{\lambda}{2}(\bar{\phi}\phi)^2\pi\Lambda_{int}^2 m\frac{\lambda^2}{64\pi m^3}\frac{\sigma}{m}\mathcal{P}\frac{4}{9}\rho_0\left[\left(1+\frac{3}{4}\frac{e}{\rho_0}\right)^{\frac{1}{2}} - 1\right]^2 e_{ff}(\phi)2m^4\phi^4$$

$$+ \frac{3}{2}\lambda\phi^3$$

$$ds^2 = -\alpha(r)^2dt^2 + \alpha(r)^2dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\phi_0''\left[-\frac{\omega^2\alpha^2}{\alpha^2} + \alpha^2\mathcal{V}'\right]\phi_0$$

$$+ \left[\frac{\alpha'}{\alpha} - \frac{\sigma'}{\sigma} - \frac{2}{r}\right]\phi_0'\frac{\alpha}{2}\left[\frac{1-\alpha^2}{r}\right.$$

$$\left. + 8\pi r\alpha^2\left(\frac{\omega^2\phi^2}{\alpha^2} + \mathcal{V} + \frac{\phi_0'^2}{\alpha^2} + \rho(1+\epsilon)\right)\right]\frac{\alpha}{2}\left[\frac{\alpha^2-1}{r}\right.$$

$$\left. + 8\pi r\alpha^2\left(\frac{\omega^2\phi^2}{\alpha^2} - \mathcal{V} + \frac{\phi_0'^2}{\alpha^2} + \mathcal{P}'\right)\right]$$

$$- [\rho(1+\epsilon) + \mathcal{P}]\frac{\alpha'}{\alpha}\hbar_{\mu\nu}\Upsilon_{\Lambda}(\theta,\varphi)$$

$$\times\, diag(-\alpha(r)^2\mathcal{H}_0(r), \alpha(r)^2\mathcal{H}_2(r)r^2\kappa(r), r^2\kappa(r), sin^2\theta)$$

$$\mathcal{H}_0'' - \left[\frac{\alpha'}{\alpha} - \frac{\sigma'}{\sigma} - \frac{2}{r}\right]\mathcal{H}_0'$$

$$- \left[8\pi\omega^2\phi_0^2\frac{\alpha'}{\alpha^2}\frac{1-c_\delta^2}{c_\delta^2} + 8\pi\phi_0'^2\frac{1+3c_\delta^2}{c_\delta^2} - 2\frac{\alpha'\alpha'}{\alpha\alpha} + 4\frac{\alpha'^2}{\alpha^2} - \frac{\alpha'\alpha'}{r\alpha}\frac{1+3c_\delta^2}{c_\delta^2}\right.$$

$$\left.- \frac{\alpha'}{r\alpha}\frac{1+7c_\delta^2}{c_\delta^2} + 6\frac{\alpha^2}{r^2}\right]\mathcal{H}_0$$

$$= 16\varpi\left[\omega^2\phi_0\frac{\alpha^2}{r\alpha^2}\frac{c_\delta^2-1}{c_\delta^2} + \phi_0\mathcal{V}'\frac{\alpha^2}{r}\frac{1+c_\delta^2}{c_\delta^2} - \frac{\phi_0''}{r}\frac{1+3c_\delta^2}{c_\delta^2} + \phi_0'\frac{\alpha'}{r\alpha}\frac{c_\delta^2-1}{c_\delta^2}\right.$$

$$\left.- 2\frac{\phi_0'}{r^2}\frac{1+3c_\delta^2}{c_\delta^2}\right]$$

$$\phi_1'' = \left[\frac{\alpha'}{\alpha} - \frac{\sigma'}{\sigma}\right]\phi_1' + \left[-\omega^2\frac{\alpha^2}{\sigma^2} + 32\pi\phi_0'^2 + 2\phi_0^2\alpha^2\mathcal{V}'' + \alpha^2\mathcal{V}' - \frac{\alpha'}{r\alpha} + \frac{\alpha'\alpha'}{r\alpha} + 6\frac{\alpha^2}{r^2}\right]\phi_1$$

$$+ \left[\omega^2 r\phi_0\frac{\alpha^2}{\sigma^2} - r\phi_0'' + \left(r\frac{\alpha'}{\alpha} + r\frac{\sigma'}{\sigma} - 2\right)\phi_0'\right]\mathcal{H}_0$$

$$\alpha'' = 4\pi\omega^2[2r\phi_0^2\alpha\alpha' + 2r\phi_0\alpha^2\phi_0' + \phi_0^2\alpha^2]\frac{1}{\alpha}$$

$$+ \left[4\pi r\alpha^2\left(-\frac{\omega^2\phi_0^2}{\alpha^2} + \mathcal{P} - \mathcal{V} + \frac{\phi_0'^2}{\alpha^2}\right) + \frac{\alpha^2-1}{2r}\right]\alpha'$$

$$+ \left[4\pi r(2\mathcal{P}\alpha\alpha' - 2\mathcal{V}\alpha\alpha' - 2\phi_0\alpha^2\phi_0'\mathcal{V}' + \alpha^2\mathcal{P}' + 2\phi_0'\phi_0'') + 4\pi\alpha^2(\mathcal{P} - \mathcal{V})\right.$$

$$\left.+ 4\pi\phi_0'^2 + \frac{\alpha\alpha'}{r} + \frac{1-\alpha^2}{2r^2}\right]\alpha$$

3. Tejido espacio tiempo cuántico curvo cíclico.

$$\mathcal{M}^{\delta\hookrightarrow\delta'}\left(\rho^{\mathbb{Im}\setminus\{\delta'\}}\right) := \sum_{i,j}\langle i|^{\delta}\mathcal{M}\left(\rho^{\mathbb{Im}\setminus\{\delta'\}}\bigotimes|i\rangle\langle j|^{\delta'}\right)|j\rangle^{\delta}$$

$$\left(\mathcal{M}_1\bigotimes\mathcal{M}_2\right)^{\mathcal{O}_1\hookrightarrow\mathfrak{I}_2}(\rho) = \mathcal{M}_2\big(\mathcal{M}_1(\rho)\big)\forall\rho\epsilon\mathcal{L}(\ln(\mathcal{M}_1))$$

$$\mathfrak{N}^{comp} = \{\delta \hookrightarrow \delta' | \delta \epsilon \mathbb{Out}(\mathfrak{N}^{maps}) \delta \epsilon \mathbb{In}(\mathfrak{N}^{maps}) \mathcal{H}^{\delta} \cong \mathcal{H}^{\delta'}\} \mathcal{N}$$

$$:= \left(\bigotimes_{\mathcal{M} \epsilon \mathfrak{N}^{maps}} \mathcal{M} \right)^{\{\delta \hookrightarrow \delta' \epsilon \mathfrak{N}^{comp}\}} \mathfrak{N}^{sys}$$

$$:= \ln(\mathcal{N}) \cup \left(\bigcup_{\delta \hookrightarrow \delta' \epsilon \mathfrak{N}^{comp}} \delta \right) Tr_{\mathbb{Out} \setminus \delta_{\odot}} \circ \mathcal{M} \circ \mathcal{N}^{\delta_{\mathcal{J}}} \delta_1$$

$$\mapsto \delta_2 \ln \mathcal{G}^{sig}\, \mathcal{M}Dec \circ \mathcal{M}^f \circ Enc$$

$$\mathcal{P}(\chi_\delta | \chi_1 \cdots \chi_\eta | \alpha_1 \cdots \alpha_\eta) = \mathcal{P}\left(\mathcal{M}^{\Lambda_1}_{\chi_1|\alpha_1} \cdots \mathcal{M}^{\Lambda_\eta}_{\chi_\eta|\alpha_\eta}\right)$$

$$= Tr\left[\left(\mathcal{M}^{\Lambda_1^{\mathcal{J}}\Lambda_1^{\mathcal{O}}}_{\chi_1|\alpha_1} \bigotimes \mathcal{M}^{\Lambda_\eta^{\mathcal{J}}\Lambda_\eta^{\mathcal{O}}}_{\chi_\eta|\alpha_\eta}\right) \mathcal{W}\right] q \mathcal{W}^{\mathrm{A}<\mathrm{B}}$$

$$+ (1-q) \mathcal{W}^{\mathrm{B}>\mathrm{A}} \mathcal{P}(\chi\gamma|\alpha\beta) q \mathcal{P}^{\mathrm{A}<\mathrm{B}}(\chi\gamma|\alpha\beta) + (1-q) \mathcal{P}^{\mathrm{B}<\mathrm{A}}(\chi\gamma|\alpha\beta)$$

$$\mathcal{M}^{\Lambda}_{\chi|\alpha}\left(\rho^{\Lambda^{\mathfrak{J}}}\right) := Tr_{\Lambda^{\odot}}\left[\left(|\chi\rangle\langle i|^{\Lambda^{\odot}} \bigotimes \mathfrak{J}^{\Lambda^{\mathcal{O}}}\right)\left[\mathcal{M}^{\Lambda}\left(|\alpha\rangle\langle\alpha|^{\Lambda^{\delta}} \bigotimes \rho^{\Lambda^{\mathfrak{J}}}\right)\right]\right] \mathfrak{N}^{maps}_{\mathcal{W},\mathcal{N}}$$

$$:= \mathcal{W} \triangle \{\mathcal{M}^{\Lambda_\kappa}\}_{\kappa=1}^{\mathcal{N}} \mathfrak{N}^{comp}_{\mathcal{W},\mathcal{N}} := \{\Lambda^{\iota}_{\kappa} \hookrightarrow \Lambda'^{\mathcal{J}}_{\kappa}, \Lambda^{\mathcal{O}}_{\kappa} \hookrightarrow \Lambda'^{\mathcal{O}}_{\kappa}\}_{\kappa=1}^{\mathcal{N}} \mathcal{P}(\chi_1 \cdots \chi_\eta | \alpha_1 \cdots \alpha_\eta)$$

$$= \frac{Tr\left[\prod \square_{\chi_1} \otimes \prod \square_{\chi_\eta} \left(\mathcal{P}_{\mathcal{W}}(|\alpha_1\rangle\langle\alpha_1| \otimes \alpha_\eta\rangle\langle\alpha_\eta|)\right)\right]}{\sum_{\chi_1 \cdots \chi_\eta} Tr\left[\prod \square_{\chi_1} \otimes \prod \square_{\chi_\eta} \left(\mathcal{P}_{\mathcal{W}}(|\alpha_1\rangle\langle\alpha_1| \otimes \alpha_\eta\rangle\langle\alpha_\eta|)\right)\right]} \bar{\mathcal{W}}\left(\mathcal{M}^{\Lambda_j^{\mathcal{J}}\Lambda_j^{\mathcal{O}}}_{\alpha_j}\right)$$

$$:= \left[\left(\mathbb{I}^{\Lambda_1^{\mathcal{J}}\Lambda_1^{\mathcal{O}}} \bigotimes \mathcal{M}^{\Lambda_j^{\mathcal{J}}\Lambda_j^{\mathcal{O}}}_{\alpha_j} \bigotimes \mathbb{I}^{\Lambda_\eta^{\mathcal{J}}\Lambda_\eta^{\mathcal{O}}}\right) \mathcal{W}\right] (\alpha|0\rangle^{\mathfrak{C}} + \beta|0\rangle^{\mathfrak{C}}) \bigotimes |\psi\rangle^{\tau}$$

$$\mapsto \alpha|0\rangle^{\mathfrak{C}} \bigotimes \mathcal{V}^{\mathrm{B}} \mathcal{U}^{\mathrm{A}} |\psi\rangle^{\tau}$$

$$+ \beta|1\rangle^{\mathfrak{C}} \bigotimes \mathcal{U}^{\mathrm{A}} \mathcal{V}^{\mathrm{B}} |\psi\rangle^{\tau} \mathcal{U}_i^{\Lambda,f} |\Omega\rangle^{\Lambda_i^{\mathcal{J}}} |\Omega\rangle^{\Lambda_i^{\mathcal{O}}} \mathcal{U}_i^{\Lambda,f} |\Omega\rangle^{\Lambda_i^{\mathcal{J}}} \mathcal{U}_i^{\mathrm{A}} |\Omega\rangle^{\Lambda_i^{\mathcal{O}}} (\alpha|0\rangle^{\mathfrak{C}}$$

$$+ \beta|1\rangle^{\mathfrak{C}}) \bigotimes |\psi\rangle^{\tau} \mapsto \alpha|0\rangle^{\mathfrak{C}} \bigotimes \mathcal{V}_2^{\mathrm{B}} \mathcal{U}_1^{\mathrm{A}} |\psi\rangle^{\tau} + \beta|1\rangle^{\mathfrak{C}} \bigotimes \mathcal{U}_1^{\mathrm{A}} \mathcal{V}_2^{\mathrm{B}} |\psi\rangle^{\tau}$$

$$\mathcal{M}\bigotimes\mathcal{N}$$

$$=\left(\mathcal{J}^{\mathbb{O}\mathrm{ut}(\mathcal{M})}\bigotimes\mathcal{N}\right)$$

$$\circ\left(\mathcal{M}\bigotimes\mathcal{J}^{\mathbb{I}\mathrm{m}(\mathcal{N})}\right)Tr_{\delta\Re}\left[|\phi^+\rangle\langle\phi^+|^{\delta\Re}\left(\mathcal{M}\bigotimes\mathcal{J}^{\mathcal{R}}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|\phi^+\rangle\langle\phi^+|^{\delta'\Re}\right)\right)|\phi^+\rangle\langle\phi^+|^{\delta\Re}\right]$$

$$=\langle\phi^+|^{\delta\mathcal{R}}\left(\mathcal{M}\bigotimes\mathcal{J}^{\mathcal{R}}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|\phi^+\rangle\langle\phi^+|^{\delta'\Re}\right)\right)|\phi^+\rangle^{\delta\mathcal{R}}\langle\phi^+|^{\delta\mathcal{R}}\left(\mathcal{M}\bigotimes\mathcal{J}^{\mathcal{R}}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|\phi^+\rangle\langle\phi^+|^{\delta'\Re}\right)\right)|\phi^+\rangle^{\delta\mathcal{R}}$$

$$=\sum_{i,j,k,\ell}\langle ii|^{\delta\mathcal{R}}\left(\mathcal{M}\bigotimes\mathcal{J}^{\mathcal{R}}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|jj\rangle\langle\kappa\kappa|^{\delta'\Re}\right)\right)|\ell\ell\rangle^{\delta\mathcal{R}}$$

$$=\sum_{i,j,k,\ell}\langle i|^{\delta}\left(\mathcal{M}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|j\rangle\langle\kappa|^{\delta'}\right)\right)|\ell\rangle^{\delta}\,\langle i|^{\mathcal{R}}\mathcal{J}^{\mathcal{R}}|j\rangle\langle\kappa|^{\mathcal{R}}|l\rangle^{\mathcal{R}}=\sum_{i,k}\langle i|^{\delta}\left(\mathcal{M}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\bigotimes|i\rangle\langle\kappa|^{\delta'}\right)\right)|\kappa\rangle^{\delta}$$

$$=\mathcal{M}^{\delta\hookrightarrow\delta'}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\delta'\}}\right)$$

$$\mathcal{M}^{\{\delta\hookrightarrow\delta',\mathcal{R}\hookrightarrow\mathcal{R}'\}}:=\left(\mathcal{M}^{\delta\hookrightarrow\delta'}\right)^{\mathcal{R}\hookrightarrow\mathcal{R}'}=\left(\mathcal{M}^{\mathcal{R}\hookrightarrow\mathcal{R}'}\right)^{\delta\hookrightarrow\delta'}$$

$$\left(\mathcal{M}^{\delta\hookrightarrow\delta'}\right)^{\mathcal{R}\hookrightarrow\mathcal{R}'}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\right)$$

$$=\sum_{\kappa,\ell}\sum_{i,j}\langle\kappa|^{\mathcal{R}}\langle i|^{\delta}\left(\mathcal{M}\left(|\kappa\rangle\langle\ell|^{\mathcal{R}'}\bigotimes\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\bigotimes|i\rangle\langle j|^{\delta'}\right)\right)|j\rangle^{\delta}|\ell\rangle^{\mathcal{R}}$$

$$\left(\mathcal{M}^{\mathcal{R}\hookrightarrow\mathcal{R}'}\right)^{\delta\hookrightarrow\delta'}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\right)$$

$$=\sum_{i,j}\sum_{\kappa,\ell}\langle\kappa|^{\mathcal{R}}\langle i|^{\delta}\left(\mathcal{M}\left(|\kappa\rangle\langle\ell|^{\mathcal{R}'}\bigotimes\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\bigotimes|i\rangle\langle j|^{\delta'}\right)\right)|j\rangle^{\delta}|\ell\rangle^{\mathcal{R}}$$

$$\left(\mathcal{M}^{\delta\hookrightarrow\delta'}\right)^{\mathcal{R}\hookrightarrow\mathcal{R}'}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\right)=\left(\mathcal{M}^{\mathcal{R}\hookrightarrow\mathcal{R}'}\right)^{\delta\hookrightarrow\delta'}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\right)$$

$$=\sum_{i,j,k,\ell}\langle\kappa|^{\mathcal{R}}\langle i|^{\delta}\left(\mathcal{M}\left(|\kappa\rangle\langle\ell|^{\mathcal{R}'}\bigotimes\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\bigotimes|i\rangle\langle j|^{\delta'}\right)\right)|j\rangle^{\delta}|\ell\rangle^{\mathcal{R}}$$

$$:=\mathcal{M}^{\{\delta\hookrightarrow\delta',\mathcal{R}\hookrightarrow\mathcal{R}'\}}\left(\rho^{\mathbb{I}\mathrm{m}\setminus\{\mathcal{R}',\delta'\}}\right)$$

$$\mathcal{M}^{\{\delta_\kappa \hookrightarrow \delta'_\kappa, \epsilon \mathfrak{M}\}} := \left(\left(\left(\left(\mathcal{M}^{\delta_{\pi(1)} \hookrightarrow \delta'_{\pi(1)}}\right)^{\delta_{\pi(2)} \hookrightarrow \delta'_{\pi(2)}}\right)^{\dagger}\right)^{\delta_{\pi(\kappa)} \hookrightarrow \delta'_{\pi(\kappa)}}\right)$$

$$= \left(\left(\left(\left(\mathcal{M}^{\delta_{\sigma(1)} \hookrightarrow \delta'_{\sigma(1)}}\right)^{\delta_{\sigma(2)} \hookrightarrow \delta'_{\sigma(2)}}\right)^{\triangleq}\right)^{\delta_{\sigma(\kappa)} \hookrightarrow \delta'_{\sigma(\kappa)}}\right) \quad \zeta_{past}(\mathcal{R}^{\delta})$$

$$:= \{\mathcal{P} \epsilon \mathcal{T}^{\mathcal{G}} : \exists \mathcal{Q} \epsilon \mathcal{R}^{\delta}, \mathcal{P} \prec \mathcal{Q}\} \delta_{\mathbb{I}}$$

$$:= \{\delta \epsilon \ln : \xi(\delta) \epsilon \, \zeta_{past}(\mathcal{R}^{\delta_{\odot}}) \Gamma^{\mu\nu} \Gamma_{\mu\nu}\} Tr_{\mathfrak{Out} \backslash \delta_{\odot}} \star \mathcal{N}$$

$$= Tr_{\mathfrak{Out} \backslash \delta_{\odot}} \star \mathcal{N} \boxtimes Tr^{\rho}_{\mathfrak{Out} \backslash \delta_{\triangleq}} \mathcal{N}^{\mathcal{R}^{\delta_{\odot}}} \star Tr^{\rho}_{\mathcal{T}^{\mathcal{G}} \backslash \chi(\mathcal{R}^{\delta_{\odot}})}$$

$$\left|\mathcal{W}^{\mathcal{QS}}\right\rangle = |1\rangle\rangle^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{T}} \Lambda^{\mathcal{I}}} |1\rangle\rangle^{\Lambda^{\odot} \mathrm{B}^{\mathcal{I}}} |1\rangle\rangle^{\mathcal{B}^{\odot} \mathcal{D}^{\mathcal{I}}_{\mathfrak{T}}} |00\rangle\rangle^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{T}} \mathcal{D}^{\mathcal{I}}_{\mathfrak{C}}}$$

$$+ |1\rangle\rangle^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{T}} \mathcal{B}^{\mathcal{I}}} |1\rangle\rangle^{\mathcal{B}^{\odot} \Lambda^{\mathcal{I}}} |1\rangle\rangle^{\Lambda^{\odot} \mathcal{D}^{\mathcal{I}}_{\mathfrak{T}}} |11\rangle\rangle^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{T}} \mathcal{D}^{\mathcal{I}}_{\mathfrak{C}}} \Big((\alpha \langle 0|$$

$$+ \beta \langle 1|)^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{C}}} \bigotimes \langle \psi^{*}|^{\mathfrak{C}^{\mathfrak{D}}_{\mathfrak{T}}} \bigotimes \left(\langle \mathcal{U}^{\Lambda^{*}} |^{\Lambda^{\mathcal{I}} \Lambda^{\odot}} \bigotimes \left(\langle \mathcal{V}^{\mathcal{B}^{*}} |^{\mathcal{B}^{\mathcal{I}} \mathcal{B}^{\odot}} \right)^{\dagger} \right) \otimes |\mathcal{W}^{\mathcal{QS}}\rangle$$

$$= \alpha |0\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{C}}} \bigotimes (\mathcal{V}^{\mathrm{B}} \mathcal{U}^{\mathrm{A}} |\psi\rangle)^{\mathcal{D}^{\mathcal{I}}_{\mathfrak{T}}} + \beta |1\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{C}}} \bigotimes (\mathcal{U}^{\mathrm{A}} \mathcal{V}^{\mathrm{B}} |\psi\rangle)^{\mathcal{D}^{\mathcal{I}}_{\mathfrak{T}}}$$

$$\mathcal{W}_{\mathcal{QS}} : |0\rangle^{\mathfrak{C}^{\odot}_{\mathfrak{C}}} |j\rangle^{\mathfrak{C}^{\odot}_{\mathfrak{T}}} |\kappa\rangle^{\Lambda^{\odot}} |\ell\rangle^{\mathcal{B}^{\odot}} \mapsto |0\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{C}}} |\ell\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{T}}} |j\rangle^{\Lambda^{\mathcal{I}}} |\kappa\rangle^{\mathcal{B}^{\mathcal{I}}}$$

$$\mathcal{W}_{\mathcal{QS}} : |1\rangle^{\mathfrak{C}^{\odot}_{\mathfrak{C}}} |j\rangle^{\mathfrak{C}^{\odot}_{\mathfrak{T}}} |\kappa\rangle^{\Lambda^{\odot}} |\ell\rangle^{\mathcal{B}^{\odot}} \mapsto |1\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{C}}} |\ell\rangle^{\mathfrak{D}^{\mathfrak{I}}_{\mathfrak{T}}} |j\rangle^{\Lambda^{\mathcal{I}}} |\kappa\rangle^{\mathcal{B}^{\mathcal{I}}}$$

$$\mathcal{W} = \frac{1}{d_{\Lambda^{\mathcal{I}}} d_{\mathcal{B}^{\mathcal{I}}}} \left(\mathbb{I}^{\Lambda^{\mathcal{I}} \Lambda^{\odot} \mathrm{B}^{\mathcal{I}} \mathcal{B}^{\odot}} + \sigma^{\mathrm{B} \preccurlyeq \mathrm{A}} + \sigma^{\mathrm{A} \preccurlyeq \mathrm{B}} + \sigma^{\mathrm{A} \not\preccurlyeq \not\succcurlyeq \mathrm{B}} \right) \sigma^{\mathrm{B} \preccurlyeq \mathrm{A}}$$

$$:= \sum_{i,j>0} c_{ij} \, \sigma_i^{\Lambda^{\mathcal{I}}} \bigotimes \mathbb{I}^{\Lambda^{\odot}} \bigotimes \mathbb{I}^{\mathrm{B}^{\mathcal{I}}} \bigotimes \sigma_j^{\mathcal{B}^{\odot}}$$

$$+ \sum_{i,j,\kappa>0} d_{ij\kappa} \, \sigma_i^{\Lambda^{\mathcal{I}}} \bigotimes \mathbb{I}^{\Lambda^{\odot}} \bigotimes \sigma_j^{\mathrm{B}^{\mathcal{I}}} \bigotimes \sigma_\kappa^{\mathcal{B}^{\odot}}$$

$$\sigma^{\mathrm{A} \preccurlyeq \mathrm{B}} := \sum_{i,j>0} e_{ij} \mathbb{I}^{\Lambda^{\mathcal{I}}} \bigotimes \sigma_i^{\Lambda^{\odot}} \bigotimes \sigma_j^{\mathrm{B}^{\mathcal{I}}} \bigotimes \mathbb{I}^{\mathcal{B}^{\odot}}$$

$$+ \sum_{i,j,\kappa>0} f_{ij\kappa} \, \sigma_i^{\Lambda^{\mathcal{I}}} \bigotimes \sigma_j^{\Lambda^{\odot}} \bigotimes \sigma_\kappa^{\mathrm{B}^{\mathcal{I}}} \bigotimes \mathbb{I}^{\mathcal{B}^{\odot}}$$

$$\sigma^{A\nless\ngtr B} := \sum_{i>0} \mathcal{V}_i \sigma_i^{\Lambda^{\jmath}} \bigotimes \mathbb{I}^{\Lambda^{\odot}} \bigotimes \mathbb{I}^{B^{\jmath}} \bigotimes \mathbb{I}^{\mathcal{B}^{\odot}} + \sum_{i>0} \chi_i \, \mathbb{I}^{\Lambda^{\jmath}} \bigotimes \mathbb{I}^{\Lambda^{\odot}} \bigotimes \sigma_i^{B^{\jmath}} \bigotimes \mathbb{I}^{\mathcal{B}^{\odot}}$$

$$+ \sum_{i,j>0} g_{ij} \sigma_i^{\Lambda^{\jmath}} \bigotimes \mathbb{I}^{\Lambda^{\odot}} \bigotimes \sigma_j^{B^{\jmath}} \bigotimes \mathbb{I}^{\mathcal{B}^{\odot}} \, Tr_{\Lambda^{\jmath}\Lambda^{\odot}} \mathcal{W} \bigotimes \frac{\mathbb{I}^{\Lambda^{\odot}}}{d_{\Lambda^{\odot}}} Tr_{\Lambda^{\jmath}}$$

$$\diamond \, \mathcal{W}\left(\rho^{\Lambda^{\odot}} \bigotimes \sigma^{\mathcal{B}^{\odot}}\right) \not\equiv Tr_{\Lambda^{\jmath}} \diamond \mathcal{W}\left(\tilde{\rho}^{\Lambda^{\odot}} \bigotimes \sigma^{\mathcal{B}^{\odot}}\right)$$

$$\mathcal{U}_{1,2}^{\Lambda,f} = \mathcal{U}_1^{\Lambda,f} \bigotimes \mathcal{U}_2^{\Lambda,f}$$

$$\mathcal{V}_{1,2}^{\mathcal{B},f} = \mathcal{V}_1^{\mathcal{B},f} \bigotimes \mathcal{V}_2^{\mathcal{B},f}$$

$$\mathcal{U}_1^{\Lambda,f} |\Omega\rangle^{\Lambda_i^{\jmath}} = |\Omega\rangle^{\Lambda_i^{\mathcal{O}}} \mathcal{U}_i^{\Lambda} |\psi\rangle^{\Lambda_i^{\mathcal{O}}}$$

$$\zeta_{Enc_{QS}} := |0\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\kappa\rangle^{\Lambda^{\odot}} |\ell\rangle^{\mathcal{B}^{\odot}} \mapsto |0\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\kappa\rangle^{\Lambda_1^{\odot}} |\Omega\rangle^{\Lambda_2^{\odot}} |\Omega\rangle^{B_1^{\odot}} |\ell\rangle^{B_2^{\odot}}$$

$$\zeta_{Enc_{QS}} := |1\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\kappa\rangle^{\Lambda^{\odot}} |\ell\rangle^{\mathcal{B}^{\odot}} \mapsto |1\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\Omega\rangle^{\Lambda_1^{\odot}} |\kappa\rangle^{\Lambda_2^{\odot}} |\ell\rangle^{B_1^{\odot}} |\Omega\rangle^{B_2^{\odot}}$$

$$\left|\psi_{0jk\ell}^{f}\right\rangle^{\mathbb{O}\mathrm{ut}} := |0\rangle^{\mathcal{D}_{\mathfrak{C}}^{\jmath}} |j\rangle^{\mathcal{D}_{\mathfrak{T}}^{\jmath}} |\kappa\rangle^{\Lambda_1^{\jmath}} |\Omega\rangle^{\Lambda_2^{\jmath}} |\Omega\rangle^{\mathcal{B}_1^{\jmath}} |\ell\rangle^{\mathcal{B}_2^{\jmath}}$$

$$\left|\psi_{1jk\ell}^{f}\right\rangle^{\mathbb{O}\mathrm{ut}} := |1\rangle^{\mathcal{D}_{\mathfrak{C}}^{\jmath}} |j\rangle^{\mathcal{D}_{\mathfrak{T}}^{\jmath}} |\Omega\rangle^{\Lambda_1^{\jmath}} |\kappa\rangle^{\Lambda_2^{\jmath}} |\ell\rangle^{\mathcal{B}_1^{\jmath}} |\Omega\rangle^{\mathcal{B}_2^{\jmath}}$$

$$\mathcal{W}_{QS}^{f} := |0\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\kappa\rangle^{\Lambda_1^{\odot}} |\Omega\rangle^{\Lambda_2^{\odot}} |\Omega\rangle^{B_1^{\odot}} |\ell\rangle^{B_2^{\odot}} \mapsto |0\rangle^{\mathcal{D}_{\mathfrak{C}}^{\jmath}} |\ell\rangle^{\mathcal{D}_{\mathfrak{T}}^{\jmath}} |j\rangle^{\Lambda_1^{\jmath}} |\Omega\rangle^{\Lambda_2^{\jmath}} |\Omega\rangle^{\mathcal{B}_1^{\jmath}} |\kappa\rangle^{\mathcal{B}_2^{\jmath}}$$

$$\mathcal{W}_{QS}^{f} := |1\rangle^{\mathfrak{C}_{\mathfrak{C}}^{\odot}} |j\rangle^{\mathfrak{C}_{\mathfrak{T}}^{\odot}} |\Omega\rangle^{\Lambda_1^{\odot}} |\kappa\rangle^{\Lambda_2^{\odot}} |\ell\rangle^{B_1^{\odot}} |\Omega\rangle^{B_2^{\odot}} \mapsto |1\rangle^{\mathcal{D}_{\mathfrak{C}}^{\jmath}} |\kappa\rangle^{\mathcal{D}_{\mathfrak{T}}^{\jmath}} |\Omega\rangle^{\Lambda_1^{\jmath}} |\ell\rangle^{\Lambda_2^{\jmath}} |j\rangle^{\mathcal{B}_1^{\jmath}} |\Omega\rangle^{\mathcal{B}_2^{\jmath}}$$

$$\mathcal{P}_{\mathcal{W}}\left(|\alpha\rangle\langle\alpha|^{\Lambda_\delta} \bigotimes |\beta\rangle\langle\beta|^{\mathcal{B}_\delta}\right)$$

$$= \sum_{ijst} \sum_{\kappa\cdots r} \langle\kappa moq|^{\Lambda^{\jmath}\Lambda^{\odot}B^{\jmath}\mathcal{B}^{\odot}} \langle\kappa|^{\Lambda^{\jmath}} \langle\eta|j\rangle\langle j|^{\Lambda^{\odot}} \langle o|^{B^{\jmath}} \langle r|\delta\rangle\langle\delta|^{B^{\odot}} (\cdot\cdot$$

$$\cdot) i|^{\Lambda^{\jmath}} \langle i|\ell\rangle^{\Lambda^{\jmath}} \langle m|^{\Lambda^{\odot}} \langle\tau|^{B^{\jmath}} \langle\tau|\rho\rangle^{B^{\jmath}} \langle q|^{B^{\odot}} \langle\kappa moq|^{\Lambda^{\jmath}\Lambda^{\odot}B^{\jmath}\mathcal{B}^{\odot}} \begin{pmatrix} \mathcal{M}_{\alpha}^{\Lambda}\left(|\kappa\rangle\langle\ell|^{\Lambda'^{\jmath}}\right) \\ \bigotimes \mathcal{M}_{\beta}^{\mathcal{B}}\left(|o\rangle\langle\rho|^{\mathcal{B}'^{\jmath}}\right) \\ \bigotimes \mathcal{W}\left(|mq\rangle\langle\eta r|^{\Lambda'^{\mathcal{O}}\mathcal{B}'^{\mathcal{O}}}\right) \end{pmatrix} \langle\ell\eta\rho r|^{\Lambda^{\jmath}\Lambda^{\odot}B^{\jmath}\mathcal{B}^{\odot}}$$

$$= \sum_{\kappa\cdots r} \langle\kappa\eta or|^{\Lambda^{\jmath}\Lambda^{\odot}B^{\jmath}\mathcal{B}^{\odot}} \left(\left[\mathcal{M}_{\alpha}^{\Lambda}(|\kappa\rangle\langle\ell|)\right]^{\tau} \bigotimes \left[\mathcal{M}_{\beta}^{\mathcal{B}}(|o\rangle\langle\rho|)\right]^{\tau} \bigotimes \mathcal{W}(|mq\rangle\langle\eta r|)\right) \langle\ell m\rho q|^{\Lambda^{\jmath}\Lambda^{\odot}B^{\jmath}\mathcal{B}^{\odot}}$$

$$= \sum_{ijst} \sum_{\kappa\cdots r} \langle i|^{\Lambda^{\jmath}} \langle\ell|\kappa\rangle^{\Lambda^{\jmath}} \langle j|^{\Lambda^{\odot}} \langle\tau|^{B^{\jmath}} \langle\rho|o\rangle^{B^{\jmath}} \langle\delta|^{B^{\odot}} (\cdots) i|^{\Lambda^{\jmath}} \langle m|\eta\rangle^{\Lambda^{\odot}} \langle j|^{\Lambda^{\odot}} \langle\tau|^{B^{\jmath}} \langle q|r\rangle^{B^{\odot}} \langle\delta|^{B^{\odot}}$$

$$= Tr_{\Lambda^{\jmath}\Lambda^{\circledcirc}B^{\jmath}\mathcal{B}^{\circledcirc}}\left[\sum_{\kappa\cdots r}\langle\ell|\kappa\rangle^{\Lambda^{\jmath}}\bigotimes\langle\rho|o\rangle^{B^{\jmath}}(\cdots)\langle m|\eta\rangle^{\Lambda^{\circledcirc}}\bigotimes\langle q|r\rangle^{B^{\circledcirc}}\right]$$

$$= Tr_{\Lambda^{\jmath}\Lambda^{\circledcirc}B^{\jmath}\mathcal{B}^{\circledcirc}}\left[\begin{array}{c}\left(\sum_{\kappa\ell}\langle\ell|\kappa\rangle^{\Lambda^{\jmath}}\bigotimes\left[\mathcal{M}_{\alpha}^{\Lambda}\left(|\kappa\rangle\langle\ell|^{\Lambda^{\jmath}}\right)\right]^{\tau}\right)\bigotimes\left(\sum_{o\rho}\langle\rho|o\rangle^{B^{\jmath}}\bigotimes\left[\mathcal{M}_{\beta}^{\mathcal{B}}(|o\rangle\langle\rho|)\right]^{\tau}\right)\\ \bigotimes\left(\sum_{m\eta qr}|mq\rangle\langle\eta r|^{\Lambda^{\circledcirc}B^{\circledcirc}}\right)\bigotimes\mathcal{W}\left(|mq\rangle\langle\eta r|^{\Lambda^{\circledcirc}B^{\circledcirc}}\right)\end{array}\right]\mathcal{P}(\chi\gamma|\alpha\beta)$$

$$= Tr\left[\left(\mathcal{M}_{\chi|\alpha}^{\Lambda^{\jmath}\Lambda^{\circledcirc}}\bigotimes\mathcal{M}_{\gamma|\beta}^{B^{\jmath}B^{\circledcirc}}\right)\mathcal{W}\right]\sum_{m,\eta}\langle m|\eta\rangle^{B^{\circledcirc}}\bigotimes\mathcal{N}_{\mathcal{W},\mathcal{A}}\left(\langle m|\eta\rangle^{B^{\circledcirc}}\right)$$

$$= \sum_{i,j,\kappa,\ell}\langle i|^{\Lambda^{\jmath}}\langle\kappa|^{\Lambda^{\circledcirc}}\left(\mathcal{M}_{\alpha}^{\Lambda}\bigotimes\mathcal{W}\right)\left(|i\rangle\langle j|^{\Lambda'^{\jmath}}\right)\bigotimes|\kappa m\rangle\langle\ell\eta|^{\Lambda'^{\circledcirc}B^{\circledcirc}}|j\rangle^{\Lambda^{\jmath}}\langle\ell|^{\Lambda^{\circledcirc}}$$

$$= \sum_{i,j,\kappa,\ell}\langle\kappa|^{\Lambda^{\circledcirc}}\mathcal{M}_{\alpha}^{\Lambda}\left(|i\rangle\langle j|^{\Lambda'^{\jmath}}\right)|\ell\rangle^{\Lambda^{\circledcirc}}\bigotimes\langle i|^{\Lambda^{\jmath}}\mathcal{W}\left(|\kappa m\rangle\langle\ell\eta|^{\Lambda'^{\circledcirc}B^{\circledcirc}}\right)|j\rangle^{\Lambda^{\jmath}}$$

$$\sum_{i,j,\kappa,\ell,m,\eta}|m\rangle\langle\eta|^{B^{\circledcirc}}\bigotimes\langle\kappa|^{\Lambda^{\circledcirc}}\mathcal{M}_{\alpha}^{\Lambda}\left(|i\rangle\langle j|^{\Lambda'^{\jmath}}\right)|\ell\rangle^{\Lambda^{\circledcirc}}$$

$$\bigotimes\langle i|^{\Lambda^{\jmath}}\mathcal{W}\left(|\kappa m\rangle\langle\ell\eta|^{\Lambda'^{\circledcirc}B^{\circledcirc}}\right)|j\rangle^{\Lambda^{\jmath}} = \sum_{i,j,\kappa,\ell,m,\eta}|m\rangle\langle\eta|^{B^{\circledcirc}}\bigotimes\langle\ell|^{\Lambda^{\circledcirc}}\left[\mathcal{M}_{\alpha}^{\Lambda}\left(|i\rangle\langle j|^{\Lambda'^{\jmath}}\right)\right]^{\tau}|\kappa\rangle^{\Lambda^{\circledcirc}}$$

$$\bigotimes\langle i|^{\Lambda^{\jmath}}\mathcal{W}\left(|\kappa m\rangle\langle\ell\eta|^{\Lambda'^{\circledcirc}B^{\circledcirc}}\right)|j\rangle^{\Lambda^{\jmath}} =$$

$$= \sum_{i,j,\kappa,\ell,m,\eta,\rho,q}|m\rangle\langle\eta|^{B^{\circledcirc}}\bigotimes\langle\ell|^{\Lambda^{\circledcirc}}|\rho\rangle\langle\rho|^{\Lambda^{\circledcirc}}\left[\mathcal{M}_{\alpha}^{\Lambda}(|i\rangle\langle j|)\right]^{\tau}|\kappa\rangle^{\Lambda^{\circledcirc}}$$

$$\bigotimes\langle i|^{\Lambda^{\jmath}}|q\rangle\langle q|^{\Lambda^{\jmath}}\mathcal{W}(|\kappa m\rangle\langle\ell\eta|)|j\rangle^{\Lambda^{\jmath}}$$

$$= \sum_{i,j,\kappa,\ell,m,\eta,\rho,q}|m\rangle\langle\eta|^{B^{\circledcirc}}\bigotimes\langle\rho|^{\Lambda^{\circledcirc}}\left[\mathcal{M}_{\alpha}^{\Lambda}(|i\rangle\langle j|)\right]^{\tau}|\kappa\rangle^{\Lambda^{\circledcirc}}\langle\ell|\rho\rangle^{\Lambda^{\circledcirc}}$$

$$\bigotimes\langle q|^{\Lambda^{\jmath}}\mathcal{W}(|\kappa m\rangle\langle\ell\eta|)|j\rangle^{\Lambda^{\jmath}}\langle i|q\rangle^{\Lambda^{\jmath}}$$

$$= \sum_{\rho.q}\langle\rho q|^{\Lambda^{\circledcirc}\Lambda^{\jmath}}\left(\mathbb{I}^{\mathcal{B}^{\circledcirc}\mathcal{B}^{\jmath}}\bigotimes\sum_{i,j}|j\rangle\langle i|\bigotimes\left[\mathcal{M}_{\alpha}^{\Lambda}(|i\rangle\langle j|)\right]^{\tau}\right)$$

$$\left(\sum_{\kappa,\ell,m,\eta}|\kappa m\rangle\langle\ell\eta|\bigotimes\mathcal{W}(|\kappa m\rangle\langle\ell\eta|)\right)|\rho q\rangle^{\Lambda^{\circledcirc}\Lambda^{\jmath}} = Tr_{\Lambda^{\jmath}\Lambda^{\circledcirc}}\left(\left(\mathbb{I}^{\mathcal{B}^{\circledcirc}\mathcal{B}^{\jmath}}\bigotimes\mathcal{M}_{\alpha}^{\Lambda^{\jmath}\Lambda^{\circledcirc}}\right)\mathcal{W}\right)$$

$$= \overline{\mathcal{W}}\left(\mathcal{M}_{\alpha}^{\Lambda^{\jmath}\Lambda^{\circledcirc}}\right)$$

$$\alpha|0\rangle^{\mathcal{D}_{\mathcal{C}}^{\mathcal{J}}}|\psi_{\mathcal{B}}\rangle^{\mathcal{D}_{\tau}^{\mathcal{J}}}|\psi_{\tau}\rangle^{\Lambda^{\mathcal{J}}}|\psi_{\Lambda}\rangle^{\mathrm{B}^{\mathcal{J}}}+\beta|1\rangle^{\mathcal{D}_{\mathcal{C}}^{\mathcal{J}}}|\psi_{\Lambda}\rangle^{\mathcal{D}_{\tau}^{\mathcal{J}}}|\psi_{\mathrm{B}}\rangle^{\Lambda^{\mathcal{J}}}|\psi_{\tau}\rangle^{\mathrm{B}^{\mathcal{J}}}$$

$$\coloneqq|\phi\rangle^{\mathcal{D}_{\mathcal{C}}^{\mathcal{J}}\mathcal{D}_{\tau}^{\mathcal{J}}\Lambda^{\mathcal{J}}\mathrm{B}^{\mathcal{J}}}Tr_{\mathcal{D}_{\mathcal{C}}^{\mathcal{J}}\mathcal{D}_{\tau}^{\mathcal{J}}\Lambda^{\mathcal{J}}\mathrm{B}^{\mathcal{J}}}\left[\mathcal{W}_{\mathcal{QS}}\left(\left|\psi\rangle\langle\psi\right|^{\mathcal{C}_{\mathcal{C}}^{\mathcal{O}}\mathcal{C}_{\tau}^{\mathcal{O}}\Lambda^{\odot}\mathcal{B}^{\odot}}\right)\right]$$

$$=|\alpha|^2\left|\psi_{\Lambda}\rangle\langle\psi_{\Lambda}\right|^{\mathrm{B}^{\mathcal{J}}}+|\beta|^2\left|\psi_{\tau}\rangle\langle\psi_{\tau}\right|^{\mathrm{B}^{\mathcal{J}}}+Tr_{\mathcal{D}_{\mathcal{C}}^{\mathcal{J}}\mathcal{D}_{\tau}^{\mathcal{J}}\mathrm{B}^{\mathcal{J}}}\left[\mathcal{W}_{\mathcal{QS}}\left(\left|\psi\rangle\langle\psi\right|^{\mathcal{C}_{\mathcal{C}}^{\mathcal{O}}\mathcal{C}_{\tau}^{\mathcal{O}}\Lambda^{\odot}\mathcal{B}^{\odot}}\right)\right]$$

$$=|\alpha|^2\left|\psi_{\tau}\rangle\langle\psi_{\tau}\right|^{\Lambda^{\mathcal{J}}}+|\beta|^2\left|\psi_{\mathrm{B}}\rangle\langle\psi_{\mathrm{B}}\right|^{\Lambda^{\mathcal{J}}}$$

4. Formalización cuántica.

$$(\mathcal{U},\mathcal{V})\mapsto\mathcal{U}_{\tau}\mathcal{V}_{\tau}\bigotimes|0\rangle\langle 0|_{\mathfrak{C}}+\mathcal{V}_{\tau}\mathcal{U}_{\tau}\bigotimes|1\rangle\langle 1|_{\mathfrak{C}}\,\mathcal{M}_{\alpha}^{\Lambda_{\mathcal{J}}\Lambda_{\mathcal{O}}}\coloneqq\mathfrak{T}\bigotimes\mathcal{M}_{\alpha}^{\Lambda}\left(|1\rangle\rangle\langle\langle 1|\right)$$

$$\in\mathcal{H}^{\Lambda_{\mathcal{J}}}\bigotimes\mathcal{H}^{\Lambda_{\mathcal{O}}}\,\mathcal{P}\left(\mathcal{M}_{\alpha_1|\chi_1}^{\Lambda^1}\cdots\mathcal{M}_{\alpha_\eta|\chi_\eta}^{\Lambda^\eta}\right)$$

$$=tr\left[\left(\mathcal{M}_{\alpha_1|\chi_1}^{\Lambda_{\mathcal{J}}^1\Lambda_{\mathcal{O}}^1}\bigotimes\mathcal{M}_{\alpha_\eta|\chi_\eta}^{\Lambda_{\mathcal{J}}^\eta\Lambda_{\mathcal{O}}^\eta}\right)\mathcal{W}\right]\left|\omega_{switch}\right\rangle\frac{1}{\sqrt{2}}\left(\left|\omega^{\mathrm{A}\preccurlyeq\mathrm{B}}\right\rangle|0\rangle^{\mathfrak{C}}\right.$$

$$\left.+\left|\omega^{\mathrm{B}\preccurlyeq\mathrm{A}}\right\rangle|1\rangle^{\mathfrak{C}}\right)\left|\omega^{\mathrm{A}\preccurlyeq\mathrm{B}}\right\rangle=|1\rangle\rangle^{\mathcal{H}^{in}\Lambda_{\mathcal{J}}}|1\rangle\rangle^{\Lambda_{\mathcal{O}}\mathrm{B}_{\mathcal{J}}}|1\rangle\rangle^{\mathrm{B}_{\mathcal{O}}\mathcal{H}^{out}}$$

5. Modelo Hubbard en espacios cuánticos curvos.

$$\Gamma(\kappa,\omega)=\frac{1}{\pi}\Im m\underbrace{\frac{1}{\omega-\varepsilon_{\kappa}-\mathcal{R}e\Sigma(\omega)+i\delta}}_{\equiv\mathfrak{G}(\kappa,\omega)}$$

$$\mathcal{H}=\sum_{ij\sigma}\tau_{ij}\hat{c}_{i\sigma}^{\dagger}\hat{c}_{j\sigma}+\mathcal{U}\sum_{i}\hat{\eta}_{i\uparrow}\hat{\eta}_{i\downarrow}$$

6. Entrelazamiento subatómico.

$$\langle\alpha|\rho_{\mathrm{A}}|\tilde{\alpha}\rangle=\frac{1}{\mathcal{Z}}\left\langle\alpha\left|Tr_{\mathrm{B}}e^{-\beta\mathcal{H}}\right|\tilde{\alpha}\right\rangle=\frac{1}{\mathcal{Z}}\sum_{\xi}\left\langle\alpha,\xi\left|e^{-\beta\mathcal{H}}\right|\tilde{\alpha},\xi\right\rangle\varepsilon^{\beta/f}\ln\left\|\rho_{\mathrm{A}}^{\tau_1^{\beta/f}}\right\|_1$$

$$\mathcal{H}=-\mathfrak{J}\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z-\hbar\sum_{i}\sigma_i^{\chi}$$

$$\mathcal{H} = -\tau \sum_{\langle i,j \rangle} \left(e^{i\theta_{i,j}} c_i^\dagger c_j + \hbar.c. \right) + \mathcal{V} \sum_{\langle i,j \rangle} \left(\eta_i - \frac{1}{2} \right) \left(\eta_j - \frac{1}{2} \right) \varepsilon^f \mathrm{A} \tau^{-2\eta_r} \frac{d\mathcal{M}}{d\hbar} \bigg|_{\hbar \to h_c}$$

$$= \alpha |\hbar - h_c|^{\Delta_\epsilon \nu - 1} \frac{d\mathcal{W}_1}{d\hbar} \bigg|_{\hbar \to h_c}$$

$$= -\alpha \ln|\hbar - h_c| \, \langle \alpha | \rho_\mathrm{A} | \tilde{\alpha} \rangle \frac{1}{\mathcal{Z}} \sum_\xi \langle \alpha, \xi | e^{-\beta\mathcal{H}} | \tilde{\alpha}, \xi \rangle$$

$$= \frac{1}{\mathcal{Z}} \sum_\xi \sum_\eta \frac{\beta^\eta}{\eta!} \langle \alpha, \xi | \mathcal{H}^\eta | \tilde{\alpha}, \xi \rangle \frac{\langle \alpha | \rho_\mathrm{A} | \tilde{\alpha} \rangle}{\langle \alpha' | \rho_\mathrm{A} | \tilde{\alpha}' \rangle} \frac{\mathcal{P}(\alpha, \tilde{\alpha})}{\mathcal{P}(\alpha', \tilde{\alpha}')}$$

$$\mathcal{W}(\hbar) = \max_\chi f(\chi; \hbar) \frac{\partial f}{\partial \chi_i} (\chi^*) \forall i \, \mathcal{W}(h)$$

$$= f(\chi^*(h), \hbar) \frac{d\mathcal{W}}{d\hbar} \sum_i \frac{\partial f(\chi^*)}{\partial \chi_i} \frac{\partial \chi_i^*}{\partial h}$$

$$+ \frac{\partial f}{\partial h} \frac{d\mathcal{W}(\hbar)}{d\hbar} \frac{\partial f(\chi^*(h), \hbar)}{\partial h} |g(\tilde{\rho})| \tilde{\rho} \mathcal{U}(\chi) \rho(h) \mathcal{U}^\dagger(\chi) \frac{d\mathcal{W}_1}{d\hbar} \frac{\partial g}{\partial \hbar}$$

$$\sum_{ij} \frac{\partial g}{\partial \tilde{\rho}_{ij}} \frac{\partial \tilde{\rho}_{ij}}{\partial \hbar} \frac{\partial \tilde{\rho}}{\partial \hbar} \mathcal{U}^\star(\chi^*) \frac{\partial \rho(\hbar)}{\partial h} \mathcal{U}^\dagger(\chi^*) \frac{d\mathcal{W}_1}{d\hbar} \sum_{ij} \frac{\frac{\partial g}{\partial \tilde{\rho}_{ij}} \left[\mathcal{U}^\dagger(\chi^*) \frac{\partial \rho(\hbar)}{\partial h} \mathcal{U}^\star(\chi^*) \right]_{ij}}{\frac{d\mathcal{W}_1}{d\hbar} \sum_{ij} \frac{\partial g}{\partial \tilde{\rho}_{ij}} \left[\mathcal{U}_c \frac{\partial \rho(\hbar)}{\partial h} \mathcal{U}_c^\dagger \right]_{ij}}$$

7. Efecto Mpemba en espacios cuánticos curvos.

$$\mathcal{L}(\rho) = -i[\mathcal{H}, \rho]$$

$$+ \sum_\alpha \mathcal{J}_\alpha \rho \mathcal{J}_\alpha^\dagger - \frac{1}{2} \{ \mathcal{J}_\alpha^\dagger \mathcal{J}_\alpha, \rho \} \rho(\tau) e^{\mathcal{L}_\tau} \rho_{\delta\delta}$$

$$+ \sum_{i=1}^{d^2-1} c_i e^{\lambda_i \tau} \mathcal{R}_i \, Tr[\mathcal{L}_1 |\delta\mathcal{M}\mathfrak{E}\rangle\langle\delta\mathcal{M}\mathfrak{E}|] c_1 \mathcal{D}(\rho(\tau), \rho_{\delta\delta}) = \|\rho(\tau) - \rho_{\delta\delta}\|$$

$$\mathcal{U} = \exp[-i\delta(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)] \, |\phi_1\rangle\langle 0|$$

$$\mathrm{A} = \begin{pmatrix} \frac{\alpha}{\sqrt{|\alpha|^2 - |\beta|^2}} & \frac{\beta^\dagger}{\sqrt{|\alpha|^2 - |\beta|^2}} & 0 \\ \frac{\beta}{\sqrt{|\alpha|^2 - |\beta|^2}} & \frac{\beta^\dagger}{\sqrt{|\alpha|^2 - |\beta|^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{B} = \begin{pmatrix} \alpha' & 0 & c^{\triangleq} \\ 0 & 1 & 0 \\ c & 0 & -\alpha'^{\star} \end{pmatrix} \mathrm{A}$$

$$= \mathcal{P}_{01}(\alpha)\mathcal{Z}_{01}(\beta)\mathcal{R}_{01}(\gamma, \pi/2)\mathcal{Z}_{01}(\delta)$$

$$= \mathcal{P}_{01}(\alpha)\mathcal{R}_{01}^{z}(\beta + \delta)\mathcal{R}_{01}(\gamma, \pi/2 - 2\delta)\mathrm{B}$$

$$= \mathcal{P}_{02}(\alpha')\mathcal{Z}_{02}(\beta')\mathcal{R}_{02}(\gamma', \pi/2)\mathcal{Z}_{02}(\delta')$$

$$= \mathcal{P}_{02}(\alpha')\mathcal{Z}_{02}(\beta' + \delta')\mathcal{R}_{02}(\gamma', \pi/2 - 2\delta')$$

$$\mathcal{P}_{01}(\chi) = \begin{pmatrix} e^{i\chi} & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{P}_{02}(\chi) = \begin{pmatrix} e^{i\chi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\chi} \end{pmatrix}, \mathcal{Z}_{01}(\chi) = \begin{pmatrix} e^{-i\chi} & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{Z}_{02}(\chi)$$

$$= \begin{pmatrix} e^{-i\chi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\chi} \end{pmatrix}, \mathcal{R}_{01}(\chi, \gamma)$$

$$= \begin{pmatrix} \cos\left(\frac{\chi}{2}\right) & -ie^{-i\gamma}\sin\left(\frac{\chi}{2}\right) & 0 \\ -ie^{i\gamma}\sin\left(\frac{\chi}{2}\right) & \cos\left(\frac{\chi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{R}_{02}(\chi, \gamma)$$

$$= \begin{pmatrix} \cos\left(\frac{\chi}{2}\right) & 0 & -ie^{-i\gamma}\sin\left(\frac{\chi}{2}\right) \\ 0 & 1 & 0 \\ -ie^{i\gamma}\sin\left(\frac{\chi}{2}\right) & 0 & \cos\left(\frac{\chi}{2}\right) \end{pmatrix} \arg\langle \alpha' - \frac{\beta'}{\gamma'} + \delta' \rangle$$

$$\mathcal{U} = \mathrm{AB} = \mathcal{P}_{01}(\alpha)\mathcal{Z}_{01}(\beta + \delta)\mathcal{R}_{01}(\gamma, \pi/2 - 2\delta) = \mathcal{P}_{02}(\alpha')\mathcal{Z}_{02}(\beta' + \delta')\mathcal{R}_{02}(\gamma', \pi/2\delta')$$

$$= \mathcal{P}_{01}(\alpha)\mathcal{P}_{02}(\alpha')\mathcal{Z}_{01}(\beta + \delta)\mathcal{Z}_{02}(\beta' + \delta') = \mathcal{R}_{01}(\gamma, \phi)\mathcal{R}_{02}(\gamma', \phi')$$

$$\mathcal{L}(\Omega_1,\Omega_2) \mapsto \mathcal{L}\left(\Omega_1 e^{i\phi},\Omega_2 e^{i\phi'}\right)\frac{d\rho}{dt}\mathcal{L}\rho(\tau)$$

$$\coloneqq -i[\mathcal{H},\rho] + \sum_j \left(2\mathcal{J}_j\rho\mathcal{J}_j^\dagger - \left\{\mathcal{J}_j^\dagger\mathcal{J}_j,\rho\right\}\right)\rho(\tau)e^{\mathcal{L}_\tau}\rho_{in}\rho_{\delta\delta} + \sum_{i=1}^{d^2-1} c_i e^{\lambda_i\tau}\mathcal{R}_i\,\rho(\tau)$$

$$-\rho_{\delta\delta} \simeq c_1 e^{\lambda_1\tau}\mathcal{R}_1\rho(\tau) - \rho_{\delta\delta} \simeq c_1 e^{\lambda_1\tau}\mathcal{R}_1 + c_1^{\star} e^{\lambda_1^{*}\tau}\mathcal{R}_1^\dagger$$

$$= |c_1| e^{\mathcal{R}e[\lambda_1]\tau}\left(e^{i(\omega_1\tau+\delta_1)}\mathcal{R}_1 + e^{-i(\omega_1\tau+\delta_1)}\mathcal{R}_1^\dagger\right)$$

$$= |c_1| e^{\mathcal{R}e[\lambda_1]\tau}(\cos(\omega_1\tau+\delta_1)\mathcal{R}_1' + \sin(\omega_1\tau+\delta_1)\mathcal{R}_2')$$

$$= c_1(\tau)\mathcal{R}_1' + c_2(\tau)\mathcal{R}_2'\frac{d\rho}{dt}\mathcal{L}\rho(\tau)$$

$$\coloneqq -i[\mathcal{H},\rho] + \left(2\mathcal{J}\rho\mathcal{J}^\dagger - \left\{\mathcal{J}^\dagger\mathcal{J},\rho\right\}\right)\dot{\rho} - \gamma\rho_{\varrho\varrho} - i\frac{\Omega}{2}\left(\rho_{1_\varrho} - \rho_{\varrho_1}\right) - \gamma\dot{\rho}_{\varrho\varrho}$$

$$-\frac{\gamma}{2\rho_{1_\varrho}}\left\| i\frac{\Omega}{2}\left(\rho_{1_\varrho} - \rho_{\varrho_1}\right)\right\|$$

$$\mathrm{A} = \begin{pmatrix} 0 & 0 & -\Omega_\rho/2 \\ 0 & -\gamma & \Omega_\rho/2 \\ \Omega_\rho/2 & -\Omega_\rho/2 & -\gamma/2 \end{pmatrix}\left(\gamma - \sqrt{\gamma^2 - 4\Omega_\rho^2}\right)/2 \;\approx \Omega_\rho^2/\gamma$$

8. Estructuras cuánticas en espacios curvos.

$$\phi_{\mathrm{A}}(\rho) := \sum_i \Lambda_i \rho \Lambda_i \sum_i \rho_{\alpha_i} \Lambda_i \, \mathfrak{J}_{\mathrm{A}}(\rho) := \delta(\rho \| \phi_{\mathrm{A}}(\rho) \|) = \delta\big(\phi_{\mathrm{A}}(\rho)\big) - \delta(\rho) \phi_{\mathrm{A}}(\rho)$$

$$= Tr_{\varepsilon} \left[\mathcal{U} \left(\rho \bigotimes |\epsilon_0\rangle \langle \epsilon_0| \right) \mathcal{U}^{\dagger} \right] \rho_{\vec{r}} \frac{1}{d} (\mathbb{I} + \mathfrak{C}_d \vec{r} \times \vec{\Lambda}) \Lambda_i \frac{1}{d} (\mathbb{I}$$

$$+ \mathfrak{C}_d \vec{r} \times \vec{\Lambda}) \rho_{\alpha_i} Tr\, (\Lambda_i \rho_{\vec{r}})$$

$$= \frac{1}{d} [1 + (d-1) \vec{\alpha}_i \times \vec{r}] \phi_{\mathrm{A}}(\rho_{\vec{r}}) \frac{1}{d} (\mathbb{I}$$

$$+ \mathfrak{C}_d \mathcal{P}_{\mathrm{A}} \vec{r} \times \vec{\Lambda}) \frac{d-1}{d} \sum_{i=1}^{d} (\vec{\alpha}_i \times \vec{r}) \vec{\alpha}_i \, |\delta(\rho) - \delta(\sigma)|$$

$$\leq d^{\frac{1}{4}} \left(1 + \frac{\ln(d-1)}{\sqrt{2}} \right) \sqrt{\|\rho - \sigma\|_2} \left| \delta\big(\rho_{\vec{r}_2}\big) - \delta\big(\rho_{\vec{r}_1}\big) \right|$$

$$\leq g(d) \sqrt{\|\vec{r}_1 - \vec{r}_2\|} \|\vec{r}_\eta\| (\mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{A}})^{\eta} \vec{r} = \left(\prod_{\kappa=1}^{\eta} \epsilon_\kappa \right) \|\vec{r}\| (\forall \eta \in \mathbb{N}_{>0}) \mathfrak{J}_{\chi} \left(\rho_{\vec{r}_\eta} \right)$$

$$\leq g(d) \sqrt{\|\vec{r}_\eta - \mathcal{P}_\chi \vec{r}_\eta\|} = g(d) \sqrt{\frac{\|(1 - \mathcal{P}_\chi)(\mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{A}})^{\eta} \vec{r}\|}{\|(\mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{A}})^{\eta} \vec{r}\|}} \sqrt{\|(\mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{A}})^{\eta} \vec{r}\|}$$

$$= g(d) \sqrt{\|(1 - \mathcal{P}_\chi) \hat{r}_{\mathrm{B}}\|} \sqrt{\|(\mathcal{P}_{\mathcal{B}} \mathcal{P}_{\mathcal{A}})^{\eta} \vec{r}\|}$$

$$\delta = g(d) \sqrt{\|(1 - \mathcal{P}_\chi) \hat{r}_{\mathrm{B}}\|} \sqrt{[\mathcal{O}(\epsilon)]^{\eta_{min}} \|\vec{r}\|} \eta_{min}$$

$$= 2 \frac{\ln \left(\frac{\delta}{g(d) \sqrt{\|\vec{r}\| \|(1 - \mathcal{P}_\chi) \hat{r}_{\mathrm{B}}\|}} \right)}{\ln[\mathcal{O}(\epsilon)]} \mathcal{H}_{bin} \left(\frac{1 + \mu\lambda}{2} \right) - \mathcal{H}_{bin} \left(\frac{1 + \lambda}{2} \right)$$

$$\leq \lambda^2 (1 - \mu^4) \ln 2 \, \mathfrak{J}_{\chi} \left(\rho_{\vec{r}_\eta} \right) = \mathcal{H}_{bin} \left(\frac{1 + \|\mathcal{P}_\chi \vec{r}_\eta\|}{2} \right) - \mathcal{H}_{bin} \left(\frac{1 + \|\vec{r}_\eta\|}{2} \right)$$

$$\leq (\hat{\alpha} \cdot \vec{r})^2 \big(\hat{\alpha} \cdot \hat{\beta} \big)^{2(2\eta - 1)} \left[1 - \big(\hat{\chi} \cdot \hat{\beta} \big)^4 \ln 2 \right] \mathfrak{J}_{\chi} \left(\rho_{\vec{r}_\eta} \right)$$

$$\leq \max_{\{\chi\}} \mathfrak{J}_{\chi} \left(\rho_{\vec{r}_\eta} \right) = \ln d - \delta(\rho_{\vec{r}}) =: \mathbb{I}(\rho_{\vec{r}})$$

9. Redes Cuánticas en espacios cuánticos curvos.

$$\eta = \sum_{\delta=1}^{\infty} \mathrm{N}(\delta) \frac{\delta(\delta-1)}{\mathrm{N}(\mathrm{N}-1)} = \frac{\langle\delta\rangle - 1}{\mathrm{N}-1} \alpha^{\star} \simeq \frac{3-\tau}{7-2\tau} \zeta\left(\frac{3}{2}\right) \sqrt{\mathrm{N}} \rho_{\mathcal{UST}}(\ell|\mathcal{N})$$

$$\simeq (\ell|\sigma^2) e^{-\ell^2/(2\sigma^2)} \int d\ell(\ell+1) \rho_{\mathcal{UST}}(\ell|\mathcal{N}) = \sqrt{\frac{\pi}{2}\sigma + 1} \sim \sqrt{\mathrm{N}} \rho_{\mathcal{UST}}(\delta)$$

$$\sim \delta^{3/2} r(\delta|\delta') \sim (\delta')^{1/2} \delta^{3/2}$$

$$\mathrm{N}v(\delta, \tau+1) = \mathrm{N}v(\delta, \tau) - [\langle\delta\rangle - 1]^{-1} \delta(\delta-1) v(\delta, \tau)$$

$$+ [\langle\delta\rangle - 1]^{-1} \sum_{\delta'=\delta+1}^{\infty} \delta'^{(\delta'-1)} v(\delta', \tau) r(\delta|\delta')$$

$$+ \alpha \sum_{\delta'=1}^{\delta-1} \delta' v(\delta', \tau)(\delta - \delta') \frac{\mathrm{N}v(\delta-\delta', \tau) - \delta(\delta - 2\delta')}{\mathrm{N} - \delta'}$$

$$- \alpha \sum_{\delta'=1}^{\delta-1} \delta' v(\delta', \tau) \left(\delta - (s' - s) + \delta \frac{\mathrm{N}v(\delta, \tau) - \delta(\delta - \delta')}{\mathrm{N} - \delta'}\right) \mathrm{N}v(\delta, \tau+1)$$

$$= \mathrm{N}v(\delta, \tau) - \langle\delta\rangle^{-1} \delta^2 v(\delta, \tau) + \langle\delta\rangle^{-1} \sum_{\delta'=\delta+1}^{\infty} (\delta')^2 v(\delta', \tau) r(\delta|\delta')$$

$$+ \alpha \sum_{\delta'=1}^{\delta-1} \delta' v(\delta', \tau)(\delta - \delta') v(\delta - \delta', \tau) - 2\alpha\delta v(\delta, \tau) f(\delta)$$

$$= \begin{cases} \delta v(\delta, \infty) & \delta > 0 \\ 0 & \delta \le 0 \end{cases} \delta f(\delta) - \kappa \delta^{-\frac{3}{2}} \sum_{\delta'=\delta+1}^{\infty} (\delta')^{\frac{3}{2}} f(\delta')$$

$$= \alpha\langle\delta\rangle \left(\sum_{\delta'=1}^{\delta-1} f(\delta') f(\delta' - \delta) - 2f(\delta) \right) g(\delta)$$

$$= \begin{cases} \delta^{-3/2} \sum_{\delta'=\delta}^{\infty} (\delta')^{3/2} f(\delta') & \delta > 0 \\ 0 & \delta \le 0 \end{cases}$$

$$\chi(\tau) = \sum_{\delta=1}^{\infty} f(\delta) e^{\delta\tau} = \sum_{\delta=-\infty}^{\infty} f(\delta) z^{-\delta}$$

$$\mathcal{Z}(\tau) = \sum_{\delta=0}^{\infty} g(\delta) e^{\delta\tau} = \sum_{\delta=-\infty}^{\infty} g(\delta) z^{-\delta}$$

$$\chi'^{(\tau)} + \kappa\chi(\tau) - \kappa\mathcal{Z}(\tau) = \alpha\chi'(0)\big(\chi^2(\tau) - 2\chi(\tau)\big)(1 - e^{-\tau})\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\mathcal{Z}(\tau)$$

$$= \mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi(\tau) - \mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi(0)\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi(\tau)$$

$$= \sum_{\delta=0}^{\infty}\delta^{3/2}f(\delta)e^{\delta\tau}\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\mathcal{Z}(\tau) = \sum_{\delta=0}^{\infty}\delta^{3/2}g(\delta)e^{\delta\tau}$$

$$\chi(0) = 1$$

$$\chi'(0) = \langle\delta\rangle$$

$$\chi''(0) = \langle\delta^2\rangle$$

$$\chi'''(0) = \langle\delta^3\rangle$$

$$\chi'(0) + \kappa\chi(0) - \kappa\mathcal{Z}(0) = -\alpha\chi'(0)$$

$$\chi''(0) + \kappa\chi'^{(0)} - \kappa\mathcal{Z}'^{(0)} = 0$$

$$\chi'''(0) + \kappa\chi''(0) - \kappa\mathcal{Z}''(0) = 2\alpha[\chi'(0)]^3$$

$$\mathcal{Z}(0) = \sum_{\delta=1}^{\infty}\delta^{-3/2}\sum_{\delta'=\delta}^{\infty}(\delta')^{3/2}f(\delta') = \sum_{\delta'=1}^{\infty}\left(\sum_{\delta'=1}^{\delta}\delta^{-3/2}\right)(\delta')^{3/2}f(\delta')$$

$$= \sum_{\delta'=1}^{\infty}\mathcal{H}_{\delta'}^{(3/2)}(\delta')^{3/2}f(\delta') \simeq \sum_{\delta'=1}^{\infty}\left(\zeta\left(\frac{3}{2}\right)(\delta')^{3/2} - 2\delta' + \frac{1}{2}\right)f(\delta')$$

$$\simeq \zeta\left(\frac{3}{2}\right)\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi(0) - 2\chi'(0)$$

$$\mathcal{Z}'^{(0)} = \sum_{\delta'=1}^{\infty}\mathcal{H}_{\delta'}^{\left(\frac{1}{2}\right)}(\delta')^{\frac{3}{2}}f(\delta') \simeq 2\chi''(0) + \zeta\left(\frac{1}{2}\right)\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi(0) + \mathcal{Z}''(0)$$

$$= \sum_{\delta'=1}^{\infty}\mathcal{H}_{\delta'}^{\left(-\frac{1}{2}\right)}(\delta')^{\frac{3}{2}}f(\delta') \simeq \frac{2}{3}\chi'''(0) + \zeta\left(\frac{1}{2}\right)\mathcal{D}_\tau^{\left(\frac{3}{2}\right)}\chi'''(0)$$

$$f(\delta)=\begin{cases}\delta^{-\tau+1}/\mathcal{H}_{\delta_{max}}^{(\tau-1)} & 0<\delta\le\delta_{max}\\ 0 & otherwise\end{cases}\mathcal{D}_{\tau}^{\left(\frac{3}{2}\right)}\chi(0)=\frac{\mathcal{H}_{\delta_{max}}^{(\tau-5/2)}}{\mathcal{H}_{\delta_{max}}^{(\tau-1)}}\chi'(0)=\frac{\mathcal{H}_{\delta_{max}}^{(\tau-2)}}{\mathcal{H}_{\delta_{max}}^{(\tau-1)}}\chi''(0)$$

$$=\frac{\mathcal{H}_{\delta_{max}}^{(\tau-3)}}{\mathcal{H}_{\delta_{max}}^{(\tau-1)}}\kappa\frac{\chi''(0)}{\mathcal{Z}'(0)-\chi'(0)}$$

$$=\frac{\mathcal{H}_{\delta_{max}}^{(\tau-3)}}{2\mathcal{H}_{\delta_{max}}^{(\tau-3)}+\zeta\left(\frac{1}{2}\right)\mathcal{H}_{\delta_{max}}^{(\tau-5/2)}-\mathcal{H}_{\delta_{max}}^{(\tau-2)}}\alpha^{*}\kappa\frac{\mathcal{Z}(0)-\chi(0)}{\chi'(0)}-$$

$$=\kappa\frac{\zeta\left(\frac{1}{2}\right)\mathcal{H}_{\delta_{max}}^{(\tau-5/2)}-2\mathcal{H}_{\delta_{max}}^{(\tau-3)}-\mathcal{H}_{\delta_{max}}^{(\tau-1)}}{2\mathcal{H}_{\delta_{max}}^{(\tau-2)}}$$

$$\kappa\simeq\frac{1}{2}-\frac{1}{2}\frac{4-\tau}{7-2\tau}\zeta\left(\frac{1}{2}\right)\frac{1}{\sqrt{\delta_{max}}}+\alpha^{*}$$

$$\simeq\frac{3-\tau}{7-2\tau}\zeta\left(\frac{3}{2}\right)\sqrt{\delta_{max}}+\left[-2-\frac{(4-\tau)(3-\tau)}{(7-2\tau)^{2}}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\right]\chi'''(0)$$

$$+\kappa\chi''(0)-\kappa\mathcal{Z}''(0)\sim\delta_{max}^{5-\tau}2\alpha[\chi'(0)]^{3}\sim\begin{cases}(\delta_{max}^{3-\tau})^{3} & \alpha\sim\delta_{max}^{0}\\ \delta_{max}^{1/2}(\delta_{max}^{3-\tau})^{3} & \alpha\sim\delta_{max}^{1/2}\end{cases}$$

$$\rho(\ell)=\frac{\sum_{\delta}\delta(\delta-1)\upsilon(\delta)\rho_{\mathcal{UST}}(\ell|\delta)}{\sum_{\delta}\delta(\delta-1)\upsilon(\delta)}\bar{\ell}\int_{0}^{\infty}\ell\rho(\ell)d\ell$$

$$=\frac{\sum_{\delta}\delta(\delta-1)\upsilon(\delta)\sqrt{\pi/2}\sqrt{\delta}}{\sum_{\delta}\delta(\delta-1)\upsilon(\delta)}\sqrt{\frac{\pi}{2}}\sum_{\delta=1}^{\delta_{max}}\delta^{\frac{3}{2}-\tau}(\delta-1)\approx\alpha^{*}\sum_{\delta=1}^{\delta_{max}}\delta^{1-\tau}(\delta-1)$$

$$\approx\frac{3-\tau}{7-2\tau}\sqrt{2\pi}\sqrt{\delta_{max}}$$

10. Dinámica orbital en espacios cuánticos curvos.

$$\nu_{3N}^{(\alpha)}=\sum_{i\neq j\neq\kappa}\nu^{(\alpha)}(\tau_i,\tau_j,\tau_\kappa)\,\omega^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa,\varrho_i,\varrho_j)=\omega_{pro}^{(\alpha)}(\varrho_i,\varrho_j)\sum_{\lambda}\mathcal{O}_{\lambda}^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa,\hat{\varrho}_i,\hat{\varrho}_j)$$

$$=\mathrm{A}_{\lambda}\left[\mathcal{M}_{\lambda}^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa)\bigotimes\mathcal{N}_{\lambda}^{(\alpha)}(\hat{\varrho}_i,\hat{\varrho}_j)\right]_{00}$$

$$v_{3N}^{(c\tau)} = \frac{c_\xi}{2f_\pi^4\Lambda_\chi}\sum_{i\neq j\neq\kappa}\tau_i\cdot\tau_j\, v_{3N}^{(1\pi)} - \frac{g_A c_D}{8f_\pi^4\Lambda_\chi}\sum_{i\neq j\neq\kappa}\tau_i\cdot\tau_j\frac{(\sigma_j\cdot\varrho_j)(\sigma_i\cdot\varrho_j)}{\varrho_j^2+m_\pi^4}v_{3N}^{(c_1)}$$

$$-\frac{g_A^2 c_1 m_\pi^4}{2f_\pi^4}\sum_{i\neq j\neq\kappa}\tau_i\cdot\tau_j\frac{(\sigma_i\cdot\varrho_i)(\sigma_j\cdot\varrho_j)}{(\varrho_i^2+m_\pi^4)(\varrho_j^2+m_\pi^4)}v_{3N}^{(c_3)}$$

$$-\frac{g_A^2 c_3}{4f_\pi^4}\sum_{i\neq j\neq\kappa}\tau_i\cdot\tau_j\frac{(\sigma_i\cdot\varrho_i)(\sigma_j\cdot\varrho_j)}{(\varrho_i^2+m_\pi^4)(\varrho_j^2+m_\pi^4)}\varrho_i\cdot\varrho_j v_{3N}^{(c_4)}$$

$$-\frac{g_A^2 c_4}{8f_\pi^4}\sum_{i\neq j\neq\kappa}(\tau_i\cdot\tau_j)\cdot\tau_\kappa\times\frac{(\sigma_i\cdot\varrho_i)(\sigma_j\cdot\varrho_j)}{(\varrho_i^2+m_\pi^4)(\varrho_j^2+m_\pi^4)}(\varrho_i\cdot\varrho_j)\times\sigma_\kappa$$

$$\alpha\times\beta = -\sqrt{3}[\alpha_1\otimes\beta_1]_{00}(\alpha\times\beta)_\mu = -i\sqrt{2}(\alpha\times\beta)_{1_\mu}[\mathcal{M}_\ell\otimes\mathcal{N}_{\ell'}]_{\lambda_\mu}$$

$$=\sum_{mm'}(\ell m\ell'|\lambda_\mu)\mathcal{M}_{\ell m}\mathcal{N}_{\ell' m'}\,\alpha_{\ell m}\sqrt{\frac{4\pi}{3}}\alpha\Upsilon_{\ell m}(\hat{\alpha})\omega^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa,\varrho_i,\varrho_j)$$

$$=\omega_{pro}^{(\alpha)}(\varrho_i,\varrho_j)\sum_\lambda\mathcal{O}_\lambda^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa,\hat{\varrho}_i,\hat{\varrho}_j)$$

$$=\mathrm{A}_\lambda\left[\mathcal{M}_\lambda^{(\alpha)}(\sigma_i,\sigma_j,\sigma_\kappa)\bigotimes\mathcal{N}_\lambda^{(\alpha)}(\hat{\varrho}_i,\hat{\varrho}_j)\right]_{00}\omega^{(c\tau)}(\sigma_i,\sigma_j,\sigma_\kappa,\varrho_i,\varrho_j)$$

$$=\frac{c_\xi}{2f_\pi^4\Lambda_\chi}\omega^{(1\pi)}(\sigma_i,\sigma_j,\sigma_\kappa,\varrho_i,\varrho_j)$$

$$=\frac{g_A c_D}{8f_\pi^4\Lambda_\chi}\frac{\varrho_j^2}{\varrho_j^2+m_\pi^4}\sum_\lambda\mathcal{O}_\lambda^{(1\pi)}(\sigma_i,\sigma_j,\sigma_\kappa,\hat{\varrho}_i,\hat{\varrho}_j)$$

$$=\sqrt{4\pi}(1010|\lambda_0)\left[[\sigma_1(i)\otimes\sigma_1(j)]_\lambda\otimes\Psi_\lambda(\hat{\varrho}_j)\right]_{00}\mathcal{O}_\lambda^{(1\pi)}(\sigma_i,\sigma_j,\sigma_\kappa,\hat{\varrho}_i,\hat{\varrho}_j)$$

$$=\begin{cases}\frac{1}{3}\sigma_i\cdot\sigma_j & (\lambda=0)\\ \frac{1}{3}\delta_{ij}(\hat{\varrho}_j) & (\lambda=2)\end{cases}\quad\delta_\zeta(\hat{\varrho}) = 3(\sigma_1\cdot\hat{\varrho})(\sigma_2\cdot\hat{\varrho})-\sigma_1\cdot\sigma_2$$

$$\omega^{(c_1)}(\sigma_i, \sigma_j, \sigma_\kappa, \varrho_i, \varrho_j)$$

$$= -\frac{g_{\mathrm{A}}^2 c_1 m_\pi^4}{2 f_\pi^4} \frac{\varrho_i \varrho_j}{(\varrho_i^2 + m_\pi^4)(\varrho_j^2 + m_\pi^4)}$$

$$\times \sum_{\lambda=0}^{2} \mathcal{O}_\lambda^{(c_1)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= \frac{4\pi}{3} \hat{\lambda} \left[[\sigma_1(i) \otimes \sigma_1(j)]_\lambda \otimes \left[\Psi_1(\hat{\varrho}_i) \otimes \Psi_1(\hat{\varrho}_j)\right]_\lambda\right]_{00} \mathcal{O}_\lambda^{(c_1)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= \begin{cases} \frac{1}{3}(\sigma_i \cdot \sigma_j)(\hat{\varrho}_i \cdot \hat{\varrho}_j) & (\lambda = 0) \\ \frac{1}{2}(\sigma_i \cdot \sigma_j) \times (\hat{\varrho}_i \cdot \hat{\varrho}_j) & (\lambda = 1)\, \mathcal{T}_\Gamma(\hat{\varrho} \cdot \hat{\varrho}') \\ \frac{1}{3}\mathcal{T}_{ij}(\hat{\varrho}_i \cdot \hat{\varrho}_j) & (\lambda = 3) \end{cases}$$

$$= 3/2[(\sigma_1 \cdot \hat{\varrho})(\sigma_2 \cdot \hat{\varrho}') + (\sigma_2 \cdot \hat{\varrho})(\sigma_1 \cdot \hat{\varrho}')] - (\sigma_1 \cdot \sigma_2)(\hat{\varrho} \cdot \hat{\varrho}')$$

$$v_{3\mathrm{N}}^{(c_3)}(\sigma_i, \sigma_j, \sigma_\kappa, \varrho_i, \varrho_j) = \frac{g_{\mathrm{A}}^2 c_3}{4 f_\pi^4} \frac{\varrho_i^2 \varrho_j^2}{(\varrho_i^2 + m_\pi^4)(\varrho_j^2 + m_\pi^4)} \times \sum_{\lambda=0}^{2} \mathcal{O}_\lambda^{(c_3)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= 4\pi(-)^{\lambda+1} \hat{\lambda} \sum_{\ell_i \ell_j} (1010|\ell_i 0)(1010|\ell_j 0) \begin{Bmatrix} \ell_i & \ell_j & \lambda \\ 1 & 1 & 1 \end{Bmatrix}$$

$$\times [\sigma_1(i) \otimes \sigma_1(j)]_\lambda \bigotimes \left[\left[\Psi_{\ell_i}(\hat{\varrho}_i) \otimes \Psi_{\ell_j}(\hat{\varrho}_j)\right]_\lambda\right]_{00} \mathcal{O}_\lambda^{(c_3)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= \begin{cases} \frac{1}{3}(\sigma_i \cdot \sigma_j)(\hat{\varrho}_i \cdot \hat{\varrho}_j)^2 & (\lambda = 0) \\ \frac{1}{2}(\sigma_i \cdot \sigma_j) \times (\hat{\varrho}_i \cdot \hat{\varrho}_j)(\hat{\varrho}_i \times \hat{\varrho}_j) & (\lambda = 1) \\ \frac{1}{3}(\hat{\varrho}_i \cdot \hat{\varrho}_j)\mathcal{T}_{ij}(\hat{\varrho}_i \cdot \hat{\varrho}_j) & (\lambda = 3) \end{cases}$$

$$v_{3\mathrm{N}}^{(c_4)}(\sigma_i, \sigma_j, \sigma_\kappa, \varrho_i, \varrho_j) = \frac{g_{\mathrm{A}}^2 c_4}{8 f_\pi^4} \frac{\varrho_i^2 \varrho_j^2}{(\varrho_i^2 + m_\pi^4)(\varrho_j^2 + m_\pi^4)} \times \sum_{\lambda=0}^{3} \mathcal{O}_\lambda^{(c_4)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= 4\pi i \sqrt{6} \hat{\lambda} \sum_{\ell_i \ell_j \delta} \hat{\delta} (1010|\ell_i 0)(1010|\ell_j 0) \begin{Bmatrix} \ell_i & \ell_j & \lambda \\ 1 & 1 & \delta \\ 1 & 1 & 1 \end{Bmatrix}$$

$$\times \left[\left[[\sigma_1(i) \otimes \sigma_1(j)]_\delta \bigotimes \sigma_1(\kappa)\right]_\lambda \bigotimes \left[\Psi_{\ell_i}(\hat{\varrho}_i) \otimes \Psi_{\ell_j}(\hat{\varrho}_j)\right]_\lambda\right]_{00}$$

$$\mathcal{O}_0^{(c_4)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j) = \frac{1}{6}[(\sigma_i \cdot \sigma_j) \times \sigma_\kappa]\left[1 - (\hat{\varrho}_i \cdot \hat{\varrho}_j)^2\right]$$

$$\mathcal{O}_1^{(c_4)}(\sigma_i, \sigma_j, \sigma_\kappa, \hat{\varrho}_i, \hat{\varrho}_j)$$

$$= \frac{1}{10}\left[4(\sigma_i \cdot \sigma_j)\sigma_\kappa - (\sigma_j \cdot \sigma_\kappa)\sigma_i - (\sigma_\kappa \cdot \sigma_i)\sigma_j\right]$$

$$\cdot (\hat{\varrho}_i \times \hat{\varrho}_j) \times \mathcal{T}_{ij}(\hat{\varrho}_i \cdot \hat{\varrho}_j)\left[1 - (\hat{\varrho}_i \cdot \hat{\varrho}_j)^2\right]$$

$$\tilde{\epsilon}_i = \epsilon_i + \sum_j v_{ij}^{(mon)} \langle \widehat{\mathrm{N}}_j \rangle = \frac{\sum_{\mathcal{J}}(2\mathcal{J}+1)\left\langle ij;\mathcal{J}\middle|\hat{\mathcal{V}}\middle|ij;\mathcal{J}\right\rangle}{\sum_{\mathcal{J}}(2\mathcal{J}+1)}$$

11. Ondas cuánticas desplazándose en espacios cuánticos curvos.

$$\rho(\theta|d,\mathcal{M}) = \frac{\pi(\theta|\mathcal{M})\mathcal{L}(d|\theta,\mathcal{M})}{\mathcal{Z}_{\mathcal{M}}}$$

$$\propto \exp\left(-\sum_{i=1}^{\mathrm{N}} \frac{2|d(f_i) - \hbar(f_i;\theta)|^2}{\mathrm{N}\delta_\eta(f_i)}\right)\pi(\theta|\mathcal{M})\prod_{\rho=1}^{\wp}\pi(\theta_\rho|\mathcal{M})\mathrm{A}(\theta,\theta+\Delta\theta)$$

$$= \min\left[1\frac{\pi(\theta+\Delta\theta)}{\pi(\theta)}\left(\frac{\mathcal{L}(d|\theta+\Delta\theta)}{\mathcal{L}(d|\theta)}\right)^\beta\right]$$

$$\mathcal{W} = \mathcal{R}\mathcal{V}^\dagger\mathcal{B}^\dagger\delta\mathcal{F}\mathcal{B}\mathcal{V}\left|\psi(\mathfrak{L})\right\rangle \coloneqq \mathcal{W}_{\mathfrak{L}} \cdots \mathcal{W}_2\mathcal{W}_1\left|\psi^{(0)}\right\rangle|\theta\rangle_\delta \coloneqq \sum_{\chi\in\Theta(\delta)} \mathfrak{C}_\chi\, |\chi\rangle_\delta$$

$$\delta_\hbar(s) \coloneqq \left\{\gamma \in \Theta(\delta) : \left|\mathfrak{C}_\gamma\right|^2 \geq \alpha \max_{\chi\in\Theta(\delta)}\left|\mathfrak{C}_\chi\right|^2\right\}\delta' \coloneqq \max[\mathcal{P}, \min(\lceil\log_2|\delta_\hbar(s)|\rceil, \delta - \mathcal{P})]$$

$$\mathbb{E}(\theta_\rho) \coloneqq \sum_{\chi\in\theta_\rho(\delta=\mathcal{P})} \left|\mathfrak{C}_\chi\right|^2\chi$$

$$\mathbb{V}(\theta_\rho) \coloneqq \sqrt{\sum_{\chi\in\theta_\rho(\delta=\mathcal{P})} \left|\mathfrak{C}_\chi\right|^2 \left(\chi - \mathbb{E}(\theta_\rho)\right)^2}\, \theta_{\rho,(min,max)} = \mathbb{E}(\theta_\rho) \mp \lambda\mathbb{V}(\theta_\rho)$$

$$d(t) - \hbar(t;\theta) = \eta(t)\mathcal{F}_+\hbar_+(t) + \mathcal{F}_\chi\hbar_\chi(t)\eta(f_i)\eta(f_j) = \frac{\tau}{2}\delta_\eta(f_i)\delta_{ij}$$

$$\mathcal{V}|0\rangle_{\mathcal{D}}|0\rangle_{\mathfrak{C}} = \frac{1}{\sqrt{2\rho}}\sum_{i=0}^{\rho-1}|i\rangle_{\mathcal{D}}\sum_{j\in\{0,1\}}|j\rangle_{\mathfrak{C}} = \frac{1}{\sqrt{2\rho}}\left[|0\rangle_{\mathcal{D}} + |1\rangle_{\mathcal{D}} + |\rho-1\rangle_{\mathcal{D}}\right]\bigotimes\left[|0\rangle_{\mathfrak{C}} + |1\rangle_{\mathfrak{C}}\right]$$

$$\mathcal{B}|\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|\Lambda(\theta,\theta+\Delta\theta_i)\rangle_\Lambda|\varphi\rangle_{\mathfrak{C}} = |\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|\Lambda(\theta,\theta+\Delta\theta_i)\rangle_\Lambda\mathcal{U}(\vartheta)|\varphi\rangle_{\mathfrak{C}}$$

$$\mathcal{F}|\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|\varphi\rangle_{\mathfrak{C}} = \begin{cases} |\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|0\rangle_{\mathfrak{C}} & |\varphi\rangle_{\mathfrak{C}} = |0\rangle_{\mathfrak{C}} \\ |\theta\rangle_\delta|i\rangle_{\mathcal{D}}|-\Delta\theta_i\rangle_{\mathfrak{C}}|1\rangle_{\mathfrak{C}} & |\varphi\rangle_{\mathfrak{C}} = |0\rangle_{\mathfrak{C}} \end{cases}$$

$$\delta|\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|\varphi\rangle_{\mathfrak{C}} = \begin{cases} |\theta\rangle_\delta|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{C}}|0\rangle_{\mathfrak{C}} & |\varphi\rangle_{\mathfrak{C}} = |0\rangle_{\mathfrak{C}} \\ |\theta\rangle_\delta|i\rangle_{\mathcal{D}}|-\Delta\theta_i\rangle_{\mathfrak{C}}|1\rangle_{\mathfrak{C}} & |\varphi\rangle_{\mathfrak{C}} = |0\rangle_{\mathfrak{C}} \end{cases}$$

$$\mathcal{R}|i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{E}}|\varphi\rangle_{\mathfrak{C}} = \begin{cases} -|0\rangle_{\mathcal{D}}|0\rangle_{\mathfrak{E}}|0\rangle_{\mathfrak{C}} & (i,\Delta\theta_i,\varphi) = (0,0,0) \\ |i\rangle_{\mathcal{D}}|\Delta\theta_i\rangle_{\mathfrak{E}}|\varphi\rangle_{\mathfrak{C}} & otherwise \end{cases}$$

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APÉNDICE E.

1. Modelo abeliano de Higgs para interacción sistémica tanto de partículas y antipartículas supermasivas como de partículas y antipartículas masivas en espacios cuánticos curvos.

$$d\sigma(p_e) \coloneqq \sigma(\partial p) \coloneqq \sum_{e\in\partial p} \sigma(e) \coloneqq \sigma(e_1) + \sigma(e_2) + \sigma(e_3) + \sigma(e_4)$$

$$\begin{aligned}\mathcal{S}_{\mathrm{N},\beta,\kappa,\zeta}(\sigma,\phi,r) \\ \coloneqq -\beta \sum_{p\in\mathfrak{C}_2(\mathrm{B_N})} \rho\big(d\sigma(p_e)\big) - \kappa - \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \\ - \big(\sigma(e)\big) \sum_{\substack{e\in\mathfrak{C}_1(\mathrm{B_N}):\\ \partial e=y-x}} r(x)r(y)\rho\big(\sigma(e)-\phi(\partial e)\big) + \zeta \sum_{x\in\mathfrak{C}_0(\mathrm{B_N})} (r(x)^2-1)^2 \\ + \sum_{x\in\mathfrak{C}_0(\mathrm{B_N})} r(x)^2\end{aligned}$$

$$\begin{aligned}d\mu_{\mathrm{N},\beta,\kappa,\zeta}(\sigma,\phi,r) \\ = \mathcal{Z}^{-1}_{\mathrm{N},\beta,\kappa,\zeta} e^{-\mathcal{S}_{\mathrm{N},\beta,\kappa,\zeta(\sigma,\phi,r)}} \prod_{e\in\mathfrak{C}_1(\mathrm{B_N})^+} d\mu_{\mathcal{G}}\,\big(\sigma(e)\big) \prod_{x\in\mathfrak{C}_0(\mathrm{B_N})^+} d\mu_{\mathcal{G}}\big(\phi(x)\big)d\mu_{\mathbb{R}_+}\big(r(x)\big)\end{aligned}$$

$$\mathcal{S}_{\mathrm{N},\beta,\kappa,\zeta}(\sigma,\phi) \coloneqq -\beta \sum_{p\in\mathfrak{C}_2(\mathrm{B_N})} \rho\big(d\sigma(p)\big) - \kappa \sum_{\substack{e\in\mathfrak{C}_1(\mathrm{B_N}):\\ \partial e=y-x}} \rho\big(\sigma(e)-\phi(\partial e)\big)$$

$$\langle f(\sigma,\phi)\rangle_{\beta,\kappa,\zeta} \coloneqq \lim_{\mathrm{N}\mapsto\infty} \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\,[f(\sigma,\phi)]$$

$$\mathcal{L}_\gamma(\sigma,\phi) \coloneqq \rho\big(\sigma(\gamma)-\phi(\partial\gamma)\big)\sigma \in \Omega^1(\mathrm{B_N},\mathcal{G})\phi \in \Omega^0(\mathrm{B_N},\mathcal{G})$$

$$\big|\langle\mathcal{L}_\gamma(\sigma,\phi)\rangle_{\beta,\kappa,\zeta} - \Theta'_{\beta,\kappa}(\gamma)\mathcal{H}_\kappa(\gamma)\big| \le \mathcal{K}_0\big(e^{-4(\beta+\kappa/6)} + |supp\ \gamma|^{-1/2}\big)^{1/4}$$

$$\Theta'_{\beta,\kappa}(\gamma) = e^{-2|supp\ \gamma|e^{-24\beta-4\kappa}}\big(1+(e^{8\kappa}-1)\langle\mathcal{L}_e(\sigma,\phi)\rangle_{\zeta,\kappa,\zeta}\big)$$

$$\mathcal{H}_\kappa(\gamma) \coloneqq \langle\mathcal{L}_\gamma(\sigma,\phi)\rangle_{\zeta,\kappa,\zeta}$$

$$\mathcal{K}_0 \le 2\cdot 18^3 + |supp\ \gamma|^{-\frac{1}{2}}e^{-4\kappa}\big(18^2(2+e^{-4\kappa})\big)^{min(\ell_1,\ell_2)} + \mathcal{O}_\kappa(1)$$

$$\langle\mathcal{L}_\gamma(\sigma,\phi)\rangle_{\beta,\kappa,\zeta} = \Theta'_{\beta,\kappa}(\gamma)\mathcal{H}_\kappa(\gamma) + \mathcal{O}_\beta(1) + \mathcal{O}_{|supp\ \gamma|}(1)$$

$$\frac{\langle\mathcal{L}_{\gamma'}\,(\sigma,\phi)\rangle_{\beta,\kappa,\zeta}\langle\mathcal{L}_{\gamma-\gamma'}(\sigma,\phi)\rangle_{\beta,\kappa,\zeta}}{\langle\mathcal{W}_\gamma\,(\sigma)\rangle_{\beta,\kappa,\zeta}}$$

$$\langle\mathcal{L}_{\gamma'}(\sigma,\phi)\rangle_{\beta,\kappa,\zeta}\langle\mathcal{L}_{\gamma-\gamma'}(\sigma,\phi)\rangle_{\beta,\kappa,\zeta} \simeq \langle\mathcal{W}_\gamma\,(\sigma)\rangle_{\beta,\kappa,\zeta} f(r)$$

$$\langle \mathcal{L}_{\gamma'}(\sigma,\phi)\rangle_{\beta,\kappa,\zeta}\langle \mathcal{L}_{\gamma-\gamma'}(\sigma,\phi)\rangle_{\beta,\kappa,\zeta} = \langle \mathcal{W}_{\gamma}(\sigma)\rangle_{\beta,\kappa,\zeta}\mathcal{H}_{\kappa}(\gamma')^2 + \mathcal{O}_{|supp\,\gamma|}(1) + \mathcal{O}_{\beta+6\kappa}(1)$$

$$\frac{\partial}{\partial x^{j_{\sigma(1)}}}\Big|_{\alpha} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_{\sigma(k)}}}\Big|_{\alpha} \coloneqq sgn\,(\sigma)\frac{\partial}{\partial x^{j_1}}\Big|_{\alpha} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}\Big|_{\alpha}$$

$$-\frac{\partial}{\partial x^{j_{\sigma(1)}}}\Big|_{\alpha} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_{\sigma(k)}}}\Big|_{\alpha} \coloneqq -sgn\,(\sigma)\frac{\partial}{\partial x^{j_1}}\Big|_{\alpha} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}\Big|_{\alpha}$$

$$q + q' \coloneqq \sum_{e\in\mathfrak{C}_k^+(\mathcal{L})} (q[c] + q'[c])\,c$$

$$supp\,q \coloneqq \{c \in \mathfrak{C}_k^+(\mathcal{L}) : q[c] \neq 0\}$$

$$\partial c \coloneqq \sum_{k'\in\{1\cdots k\}}\Bigg((-1)^{\kappa'}\frac{\partial}{\partial x^{j_1}}\Big|_{\alpha} \wedge\cdots\wedge \frac{\partial}{\partial x^{j_{k'-1}}}\Big|_{\alpha} \wedge \frac{\partial}{\partial x^{j_{k'+1}}}\Big|_{\alpha} \wedge\cdots\wedge \frac{\partial}{\partial x^{j_k}}\Big|_{\alpha} + (-1)^{\kappa'+1}\frac{\partial}{\partial x^{j_1}}\Big|_{\alpha+e_{j_{k'}}} \wedge\cdots$$
$$\wedge \frac{\partial}{\partial x^{j_{k'-1}}}\Big|_{\alpha+e_{j_{k'}}} \wedge \frac{\partial}{\partial x^{j_{k'+1}}}\Big|_{\alpha+e_{j_{k'}}} \wedge\cdots\wedge \frac{\partial}{\partial x^{j_k}}\Big|_{\alpha+e_{j_{k'}}}\Bigg)$$

$$\hat{\partial} c = \sum_{c'\in\mathfrak{C}_{k+1}(\mathcal{L})} (\partial c'[c])c'$$

$$\omega(q) = \omega\left(\sum \alpha_i c_i\right) = \sum \alpha_i \omega(c_i) \quad \sum_{1\le j_1<\cdots<j_k\le m} \omega_{j_1\cdots j_k}\, dx^{j_1} \wedge\cdots\wedge dx^{j_\kappa}$$

$$\omega_{j_1\cdots j_k}(\alpha) = \omega\left(\frac{\partial}{\partial x^{j_1}}\Big|_{\alpha} \wedge\cdots\wedge \frac{\partial}{\partial x^{j_k}}\Big|_{\alpha}\right) dx^{j_{\sigma(1)}} \wedge\cdots\wedge dx^{j_{\sigma(k)}} \coloneqq sgn(\sigma) dx^{j_1} \wedge\cdots\wedge dx^{j_\kappa}$$

$$\partial_i \hbar(\alpha) \coloneqq \hbar(\alpha + e_i) - \hbar(\alpha)$$

$$d\omega = \sum_{1\le j_1<\cdots<j_k\le m}\sum_{i=1}^{m} \partial_i \omega_{j_1\cdots j_k}\, dx^i \wedge \left(dx^{j_1} \wedge\cdots\wedge dx^{j_\kappa}\right)$$

$$\begin{cases} \sigma(e) \mapsto \eta(x) + \sigma(e) + \eta(y) & e = (x,y) \in \mathfrak{C}_1(\mathrm{B_N}) \\ \phi(x) \mapsto \phi(x) + \eta(x) & x \in \mathfrak{C}_0(\mathrm{B_N}) \end{cases}$$

$$\mu_{\mathrm{N},\beta,\kappa}(\sigma) \coloneqq \mathcal{Z}_{\mathrm{N},\beta,\kappa}^{-1}\exp\left(\beta\sum_{p\in\mathfrak{C}_2(\mathrm{B_N})}\rho\big(d\sigma(p)\big) + \kappa\sum_{e\in\mathfrak{C}_1(\mathrm{B_N})}\rho\big(\sigma(e)\big)\right)$$

$$\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[f(\sigma,\phi)] = \mathbb{E}_{\mathrm{N},\beta,\kappa}[f(\sigma,1)]$$

$$\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\big[\mathcal{L}_{\gamma}(\sigma,\phi)\big] = \mathbb{E}_{\mathrm{N},\beta,\kappa}\big[\mathcal{L}_{\gamma}(\sigma,1)\big] = \mathbb{E}_{\mathrm{N},\beta,\kappa}\big[\rho\big(\sigma(\gamma)\big)\big]$$

$$\mathcal{H}_{\kappa}(\gamma) = \lim_{\mathrm{N}\mapsto\infty}\mathcal{Z}_{\mathrm{N},\kappa}^{-1}\sum_{\eta\in\Omega^0(\mathrm{B_N},\mathcal{G})}\rho\big(\eta(-x_1)\big)\rho\big(\eta(-x_2)\big)\, e^{-\kappa\sum_{\in\mathfrak{C}_1(\mathrm{B_N})}\rho(\eta(\partial e))}$$

$$\mathcal{Z}_{\mathrm{N},\kappa} \coloneqq \sum_{\eta\in\Omega^0(\mathrm{B_N},\mathcal{G})} e^{\kappa\sum_{\in\mathfrak{C}_1(\mathrm{B_N})}\rho(\eta(\partial e))}$$

$$\mathcal{H}_\kappa(\gamma) = \langle \mathcal{L}_\gamma(\sigma,\phi)\rangle_{\zeta,\kappa,\zeta} = \lim_{\mathrm{N}\mapsto\infty} \langle \mathcal{L}_\gamma(\sigma,\phi)\rangle_{\mathrm{N},\zeta,\kappa,\zeta} = \lim_{\mathrm{N}\mapsto\infty} \mathbb{E}_{\mathrm{N},\zeta,\kappa,\zeta}\langle \mathcal{L}_\gamma(\sigma,\phi)\rangle$$

$$= \lim_{\mathrm{N}\mapsto\infty} \mathbb{E}_{\mathrm{N},\zeta,\kappa}\langle \mathcal{L}_\gamma(\sigma,1)\rangle = \lim_{\mathrm{N}\mapsto\infty} \mathcal{Z}^{-1}_{\mathrm{N},\zeta,\kappa,\zeta} \sum_{\sigma\in\Omega_0^1(\mathrm{B_N},\mathcal{G})} \rho(\sigma(\gamma)) e^{\kappa\sum_{\in\mathfrak{C}_1(\mathrm{B_N})}\rho(\sigma(e))}$$

$$\left\| \mathcal{L}_\gamma\left(\sum_{\hat{\hat{\sigma}}'\in\Sigma'} \hat{\hat{\sigma}}\right) - \mathcal{L}_\gamma(\hat{\hat{\sigma}}')\right\| = \prod_{e\in\gamma} \prod_{e\in(\gamma-\gamma_c)-\gamma'} \prod_{e\in(\gamma-\gamma_c)-\gamma'''} \sum_{\hat{\hat{\sigma}}'\in\hat{\hat{\Sigma}}'\setminus\Sigma'} \hat{\hat{\sigma}}\,\rho(d\hat{\sigma})(p_e)$$

$$\varphi_r(g) \coloneqq e^{r\Re(\rho(g)-\rho(0))}$$

$$\varphi_\infty(g) \coloneqq \begin{cases} 1 & if\ g = 0 \\ 0 & if\ g\in\mathcal{G}\setminus\{0\} \end{cases}$$

$$\theta'_{\beta,\kappa}(\hat{g}) \coloneqq \frac{\sum_{g\in\mathcal{G}} \rho(g)\varphi_{\mathrm{B}}(g)\varphi_{\kappa}(g+\hat{g})^2}{\sum_{g\in\mathcal{G}} \varphi_{\mathrm{B}}(g)\varphi_{\kappa}(g+\hat{g})^2}$$

$$\theta'_{\mathrm{N},\beta,\kappa}(\gamma) \coloneqq \mathbb{E}_{\mathrm{N},\zeta,\kappa}\left[\prod_{e\in\gamma} \theta'_{\beta,\kappa}(\sigma(e))\right], \theta'_{\mathrm{N},\beta,\kappa}(\xi) \coloneqq \mathbb{E}_{\mathrm{N},\zeta,\kappa}\left[\prod_{e\in\xi} \theta'_{\beta,\kappa}(\sigma(e))\right]$$

$$\alpha_0(r) \coloneqq \sum_{g\in\mathcal{G}\setminus\{0\}} \varphi_{\mathrm{r}}(g)^2, \alpha_1(r) \coloneqq \max_{g\in\mathcal{G}\setminus\{0\}} \varphi_{\mathrm{r}}(g)^2$$

$$\alpha_2(\beta,\kappa) \coloneqq \alpha_0(\beta)\alpha_0(\kappa)^{\frac{1}{6}}, \alpha_3(\beta,\kappa) \coloneqq \left|1-\theta_{\beta,\kappa}(0)\right|, \alpha_4(\beta,\kappa)$$

$$\left[\theta_{\mathrm{N},\beta,\kappa}(\sigma(e))\middle| \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \max_{g\in\mathcal{G}} \mathbb{E}_{\mathrm{N},(\beta,\kappa),(\zeta,\kappa)} \prod_{e\in(\gamma-\gamma_c)-\gamma'} \prod_{e\in\gamma} \prod_{e\in\gamma\gamma-\gamma':\sigma(e)-d\sigma(p_e)} \prod_{e\in\gamma\gamma\setminus\gamma':\sigma'(e)} \prod_{e\in\gamma_c:\sigma'(e)} \langle \theta_{\mathrm{N},\beta,\kappa}(\sigma'^{(e)}) - (g) \right.$$

$$\left. - d\sigma(p_e)\rangle \left|\left\|1-\theta_{\beta,\kappa}(0)^{|supp\ \gamma_c|}\right\|\right\|1-\theta_{\beta,\kappa}(0)^{|supp\ \gamma'|}\left\|\right| \middle| \Theta_{\mathrm{N},\beta,\kappa,\zeta}\right]$$

$$\varepsilon^2\mu_{\mathrm{N},\beta,\kappa,\zeta}\Upsilon_{\beta,\kappa}(e')\sum_{e,e'\in\gamma} Cov_{\ \mathrm{N},\beta,\kappa,\zeta}\left(\log\theta_{\beta,\kappa}\left(\sigma'(e)\right)\right)\left(\log\theta_{\beta,\kappa}\left(\sigma(e')\right)\right)$$

$$\langle \sum_{e\in supp\ \gamma} \frac{Var_{\mathrm{N},\beta,\kappa,\zeta}\left(e^{\sum_{e\in\gamma} dist_1(supp\ \gamma\partial\mathfrak{C}_1(\mathrm{B_N}))}\log\theta_{\beta,\kappa}\left(\sigma(e)\right)(\beta,\kappa)\right) + 2\left\|\log\theta_{\beta,\kappa}\right\|_\infty^2}{\sum_{e\in supp\ \gamma: e\neq e'} \left(\sigma(e')\right)^{dist_0(e,e')}} - e^{|supp\ \gamma_c|\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\Upsilon_{\mathrm{N},\beta,\kappa,\zeta}(\gamma')}\rangle$$

$$\left|\log\frac{1-\varphi_\beta\varphi_\kappa}{1-\varphi_\beta\varphi_\kappa}\right| \log\left|e^{-2\varphi_\beta\varphi_\kappa}\right| \sum_{g\in\mathcal{G}} \log\theta_{\beta,\kappa}(g)\,|supp\ \gamma_c|^{-1} \sum_{e\in\gamma} \Upsilon_{\mathrm{N},\beta,\kappa,\zeta}(e')\delta_{\mathrm{N},\beta,\kappa,\zeta}(\gamma')\left[1_{\sigma(e)=g}\right]$$

$$\hat{\Theta}'_{\mathrm{N},\beta,\kappa,\zeta} = \langle \sqrt{\left[\hat{\theta}'_{\mathrm{N},\beta,\kappa}(\sigma(e)) \middle| \begin{matrix} \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \max_{g\in\mathcal{G}} \mathbb{E}_{\mathrm{N},(\beta,\kappa),(\zeta,\kappa)} \prod_{e\in(\gamma-\gamma_c)-\gamma'} \prod_{e\in\gamma} \prod_{e\in\gamma\gamma-\gamma':\sigma(e)-d\sigma(p_e)} \prod_{e\in\gamma\gamma\setminus\gamma':\sigma'(e)} \prod_{e\in\gamma_c:\sigma'(e)} \langle \begin{matrix}\hat{\theta}'_{\mathrm{N},\beta,\kappa}(\sigma'^{(e)}) \\ -(g)-d\sigma(p_e)\end{matrix}\rangle \\ \left|\left\|1-\theta_{\beta,\kappa}(\psi)^{|supp\ \gamma_c|}\right\|\right\|1-\theta_{\beta,\kappa}(\phi)^{|supp\ \gamma'|}\left\|\hat{\Theta}'_{\mathrm{N},\beta,\kappa,\zeta}\right| \end{matrix} \middle| d\xi\right]}\rangle$$

$$\left[\!\left[\hat{\Theta}'_{\mathrm{N},\beta,\kappa,\zeta} = \left[\hat{\theta}'_{\mathrm{N},\beta,\kappa}(\sigma(e)) \middle| \begin{matrix} \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \max_{g\in\mathcal{G}} \mathbb{E}_{\mathrm{N},(\beta,\kappa),(\zeta,\kappa)} \prod_{e\in(\gamma-\gamma_c)-\gamma'} \prod_{e\in\gamma} \prod_{e\in\gamma\gamma-\gamma':\sigma(e)-d\sigma(p_e)} \prod_{e\in\gamma\gamma\setminus\gamma':\sigma'(e)} \prod_{e\in\gamma_c:\sigma'(e)} \langle \begin{matrix}\hat{\theta}'_{\mathrm{N},\beta,\kappa}(\sigma'^{(e)}) \\ -(g)-d\sigma(p_e)\end{matrix}\rangle \\ \left|\left\|1-\theta_{\beta,\kappa}(\psi)^{|supp\ \gamma_c|}\right\|\right\|1-\theta_{\beta,\kappa}(\phi)^{|supp\ \gamma'|}\left\|\hat{\Theta}'_{\mathrm{N},\beta,\kappa,\zeta}\right| \end{matrix} \middle| d\xi\right]\right]\!\right]$$

$$\varepsilon^2 \mu_{\mathrm{N},\beta,\kappa,\zeta} \Upsilon_{\beta,\kappa}(e') \sum_{e,e' \in \gamma} Cov_{\,\mathrm{N},\beta,\kappa,\zeta} \left(\log \theta_{\beta,\kappa}\left(\sigma'(e)\right)\right)\left(\log \theta_{\beta,\kappa}\left(\sigma(e')\right)\right)$$

$$\langle \sum_{e \in supp\, \gamma} \genfrac{}{}{0pt}{}{Var_{\mathrm{N},\beta,\kappa,\zeta}\left(e^{\sum_{e\in\gamma} dist_{\varrho}(e,e')} \log \theta_{\beta,\kappa}\left(\sigma(e)\right)\right) + 2\left\|\log \theta_{\beta,\kappa}\right\|_{\infty}^{2}}{\sum_{e \in supp\, \gamma : e \neq e'} \left(\sigma(e')\right)^{dist_0(e,e')}} - e^{|supp\, \gamma| \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \Upsilon_{\mathrm{N},\beta,\kappa,\zeta}(\gamma')} \rangle$$

$$\left|\log \frac{1-\varphi_\beta \varphi_\kappa}{1-\varphi_\beta \varphi_\kappa}\right| \log\left|e^{-2\varphi_\beta \varphi_\kappa}\right| \sum_{g \in \mathcal{G}} \log \theta_{\beta,\kappa}(g) \, |supp\, \gamma|^{-1} \sum_{e \in \gamma} \Upsilon_{\mathrm{N},\beta,\kappa,\zeta}(e') \delta_{\mathrm{N},\beta,\kappa,\zeta}(\gamma') \left[1_{\sigma(e)=g}\right]$$

$$\alpha_\eta(\beta,\kappa) := \min_{g_1, g_2 \cdots g_\eta \in \mathcal{G}} \left(1 - \frac{\sum_{g\in\mathcal{G}} \rho(g) \left(\prod_{\kappa=1}^{\eta} \varphi_{\mathrm{B}}\left(g+g_\kappa\right)^2 \varphi_\kappa(g)^2\right)}{\sum_{g\in\mathcal{G}} \left(\prod_{\kappa=1}^{\eta} \varphi_{\mathrm{B}}\left(g+g_\kappa\right)^2 \varphi_\kappa(g)^2\right)}\right)$$

$$\varphi_r(g) = e^{r\left(\Re\rho(g) - \rho(0)\right)} = e^{r\beta\left(\Re\rho(g) - \rho(0)\right)} = \varphi_r(-g)$$

$$\varphi_r(g)^a = \varphi_{ar}(g)$$

$$\varphi_r(\sigma) := \prod_{e \in \mathfrak{C}_1(\mathrm{B}_\mathrm{N})} \varphi_r\left(\sigma(e)\right)$$

$$\varphi_r(\sigma) := \prod_{p \in \mathfrak{C}_2(\mathrm{B}_\mathrm{N})} \varphi_r\left(\omega(p)\right)$$

$$\varphi_{\beta,\kappa}(\sigma) := \varphi_\kappa(\sigma) \varphi_\beta(d\sigma)$$

$$\mu_{\mathrm{N},\beta,\kappa}(\sigma) = \frac{\varphi_{\beta,\kappa}(\sigma)}{\sum_{\sigma' \in \Omega^1(\mathrm{B}_\mathrm{N},\mathcal{G})} \varphi_{\beta,\kappa}(\sigma')}$$

$$\varphi_{\beta,\kappa}(\sigma) = \varphi_{\beta,\kappa}(\sigma') \varphi_{\beta,\kappa}(\sigma - \sigma')$$

$$\mu_{\mathrm{N},\beta,\kappa}\left(\left\{\sigma \in \Omega^1(\mathrm{B}_\mathrm{N},\mathcal{G}) : \sigma' \leq \sigma\right\}\right) \leq \varphi_{\beta,\kappa}(\sigma')$$

$$\sigma'' \leq \sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e)$$

$$supp\, \sigma'' \subseteq \mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}(e)$$

$$\left(\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e)\right)\Big|_{supp\, \sigma''} = \sigma|_{supp\, \sigma''} = \sigma''$$

$$d\left(\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e)\right)\Big|_{supp\, d\sigma''} = d\sigma''$$

$$d\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e)(p) = d\sigma(p)$$

$$d\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e)(p) = d\sigma''(p)$$

$$\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e) = \left(\sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(\xi)\right)\Big|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(e) \leq \sigma|_{\mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}}(\xi)$$

$$\xi' := supp\, \sigma|_{\mathfrak{C}_1(\mathrm{B}_\mathrm{N}) \setminus \mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}(\xi)} \cup supp\, \sigma'|_{\mathfrak{C}_1(\mathrm{B}_\mathrm{N}) \setminus \mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}(\xi)}$$

$$\sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}(\xi)} = \sigma|_{\mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}(\xi')}$$

$$d\sigma|_{\xi(p)} = \sigma|_{\xi(\partial p)} = \sigma(\partial p) = d\sigma(p)$$

$$d\left(\sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi}\right) = d\left(\sigma - \sigma|_{\xi}\right) = d\sigma - d\left(\sigma|_{\xi}\right) = d\sigma - d\sigma = 0$$

$$\begin{cases} \sigma \coloneqq \hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} \\ \sigma' \coloneqq \hat{\sigma}' \end{cases}$$

$$\sigma \coloneqq \hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}$$

$$\hat{\sigma}'|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}$$

$$\begin{aligned} d\sigma &= d\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right) \\ &= d\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right)_{\xi_0} - \left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right)_{\xi_1} + d\left(\hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right) - d\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right) = d\hat{\sigma} \end{aligned}$$

$$\left\{e \in supp\ \hat{\sigma}: d\hat{\sigma}|_{\pm supp\ \partial e} \neq 0\right\} \subseteq \left\{e \in supp\ \sigma: d\sigma|_{\pm supp\ \partial e} \neq 0\right\}$$

$$\left\{e \in supp\ \hat{\sigma}': d\hat{\sigma}'|_{\pm supp\ \partial e} \neq 0\right\} \subseteq \left\{e \in supp\ \sigma': d\sigma'|_{\pm supp\ \partial e} \neq 0\right\}$$

$$\begin{aligned} \xi_{\xi_{0,\hat{\sigma},\hat{\sigma}'}} &= \mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}\left(\xi_0 \cup \left\{e \in supp\ \hat{\sigma}: d\hat{\sigma}|_{\pm supp\ \partial e} \neq 0\right\} \cup \left\{e \in supp\ \hat{\sigma}': d\hat{\sigma}'|_{\pm supp\ \partial e} \neq 0\right\}\right) \\ &\subseteq \mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}\left(\xi_0 \cup \left\{e \in supp\ \sigma: d\sigma|_{\pm supp\ \partial e} \neq 0\right\} \cup \left\{e \in supp\ \sigma': d\sigma'|_{\pm supp\ \partial e} \neq 0\right\}\right) \\ &= \xi_{\xi_{0,\sigma,\sigma'}} \end{aligned}$$

$$\begin{aligned} &\varphi_{\beta_1,\kappa}(\hat{\sigma})\varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right) \\ &\quad = \varphi_{\beta_1,\kappa}(\hat{\sigma}) \sum_{\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\xi_{\phi,\sigma,\sigma'}}}\right. \\ &\quad \left. + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\sigma,\sigma'}}}\right) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\xi_{\phi,\sigma,\sigma'}}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\sigma,\sigma'}}}\right) \end{aligned}$$

$$\begin{aligned} &\varphi_{\beta_1,\kappa}(\hat{\sigma}')\varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}'|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right) \\ &\quad = \varphi_{\beta_1,\kappa}(\hat{\sigma}') \sum_{\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\xi_{\phi,\sigma,\sigma'}}}\right. \\ &\quad \left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\sigma,\sigma'}}}\right) \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\xi_{0,\sigma,\sigma'}}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\sigma,\sigma'}}}\right) \end{aligned}$$

$$\mu_{\mathrm{N},\beta,\kappa} \otimes \mu_{\mathrm{N},\beta,\kappa}\left(\left\{\left(\hat{\sigma},\hat{\sigma}'\right) \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \otimes \Omega_0^1(\mathrm{B_N},\mathcal{G}): \hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right\}\right) = \mu_{\mathrm{N},\beta,\kappa}(\sigma)$$

$$\sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right) \bigotimes 1\left(\hat{\sigma}'\Big|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right) = \varphi_\kappa(\sigma) \sum_{\sigma' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})} \varphi_\kappa(\sigma')$$

$$\sum_{\substack{\hat{\sigma}\in\Omega_0^1(\mathrm{B_N},\mathcal{G})\\ \hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right)1\left(\hat{\sigma}|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right)_{\xi_0}$$

$$= \varphi_\kappa(\sigma)\sum_{\sigma'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}\varphi_\kappa(\sigma')\sum_{\substack{\hat{\sigma}\in\Omega_0^1(\mathrm{B_N},\mathcal{G})\\ \hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}} 1\left(\hat{\sigma} = \sigma'|_{\xi_{\xi_{\psi,\sigma,\sigma'}}}\right.$$

$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{0,\sigma,\sigma'}}}\right)_{\xi_0} \bigotimes 1\left(\hat{\sigma}' = \sigma\big|_{\xi_{\xi_{\psi,\sigma,\sigma'}}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{0,\sigma,\sigma'}}}\right)_{\xi_0}$$

$$\sum_{\substack{\hat{\sigma}\in\Omega_0^1(\mathrm{B_N},\mathcal{G})\\ \hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right)1\left(\hat{\sigma}|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right)_{\xi_1}$$

$$= \varphi_\kappa(\sigma)\sum_{\sigma'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}\varphi_\kappa(\sigma')\sum_{\substack{\hat{\sigma}\in\Omega_0^1(\mathrm{B_N},\mathcal{G})\\ \hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G})}} 1\left(\hat{\sigma} = \sigma'|_{\xi_{\xi_{\psi,\sigma,\sigma'}}}\right.$$

$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\sigma,\sigma'}}}\right)_{\xi_1} \bigotimes 1\left(\hat{\sigma}' = \sigma\big|_{\xi_{\xi_{\psi,\sigma,\sigma'}}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\sigma,\sigma'}}}\right)_{\xi_1}$$

$$\left|\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[f_0(\sigma)f_1(\sigma)] - \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[f_0(\sigma)]\,\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[f_1(\sigma)]\right|$$
$$\leq 2\|f_0\|_\zeta\|f_1\|_\zeta\sum_{e\in\xi_1}\mu^{\xi_0}_{\mathrm{N},(\zeta,\kappa),(\zeta,\kappa)}\left(\left\{\left(\hat{\sigma},\hat{\sigma}'\right)\in\Omega_0^1(\mathrm{B_N}\mathcal{G})\otimes\Omega_0^1(\mathrm{B_N}\mathcal{G}): e\in\xi_{\xi_{0,\hat{\sigma},\hat{\sigma}'}}\right\}\right)$$

$$\left|\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[\mathcal{F}(\sigma)] = \mathbb{E}^{\xi_\phi}_{\mathrm{N},(\zeta,\kappa),(\zeta,\kappa)}[\mathcal{F}(\sigma)]\right| = \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\bigotimes\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right)\right]$$

$$= \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\bigotimes\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right)\right]\bigotimes 1\left(\xi_1\cap\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}\right.$$
$$\left.=\phi\right)$$
$$+ \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\bigotimes\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}}\right)\right]\bigotimes 1\left(\xi_1\cap\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}\right.$$
$$\left.\neq\phi\right)$$

$$\left|\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[\mathcal{F}(\sigma)] = \mathbb{E}^{\xi_\psi}_{\mathrm{N},(\zeta,\kappa),(\zeta,\kappa)}[\mathcal{F}(\sigma)]\right| = \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}}\right)\right]$$

$$= \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}}\right)\right] \bigotimes 1\left(\xi_1 \cap \xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} = \psi\right)$$

$$+ \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}}\right)\right] \bigotimes 1\left(\xi_1 \cap \xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} \neq \psi\right)$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma})\varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right)_{\xi_0}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}) \sum_{\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\xi_{\phi,\sigma,\sigma'}}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\sigma,\sigma'}}}\right)_{\xi_0} \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\xi_{\phi,\sigma,\sigma'}}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\sigma,\sigma'}}}\right)_{\xi_0}$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma}')\varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}'|_{\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right)_{\xi_1}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}') \sum_{\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\xi_{\psi,\sigma,\sigma'}}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\sigma,\sigma'}}}\right)_{\xi_1} \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\xi_{\psi,\sigma,\sigma'}}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\psi,\sigma,\sigma'}}}\right)_{\xi_1}$$

$$\mu_{\mathrm{N},(\beta,\kappa),(\zeta,\kappa)}\left(\hat{\sigma},\hat{\sigma}'\right)$$
$$:= \mu_{\mathrm{N},\beta,\kappa}\otimes\mu_{\mathrm{N},\beta,\kappa}\left(\left\{\left(\hat{\sigma},\hat{\sigma}'\right) \in \Omega^1(\mathrm{B_N},\mathcal{G})\otimes\Omega^1_0(\mathrm{B_N},\mathcal{G}) : \sigma = \hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right\}\right) = \mu_{\mathrm{N},\beta,\kappa}(\sigma)$$

$$\begin{cases} \sigma := \hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}} \\ \sigma' := \hat{\sigma}' \end{cases}$$

$$\sigma := \hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}}$$

$$\hat{\sigma}'|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}}$$

$$d\sigma = d\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}}\right) = d\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}}\right)_{\xi_0} - \left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}}\right)_{\xi_1} + d\left(\hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}}\right) - d\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}}\right)$$
$$= d\hat{\sigma}$$

$$\xi_{0,\sigma,\sigma'} = \mathfrak{C}_{\mathcal{G}(\hat{\sigma},\sigma')}\left(\xi_0 \cup \left\{e \in supp\ \hat{\sigma}: d\hat{\sigma}|_{\pm supp\ \partial e} \neq 0\right\} \cup \left\{e \in supp\ \hat{\sigma}': d\hat{\sigma}'|_{\pm supp\ \partial e} \neq 0\right\}\right)$$
$$\subseteq \mathfrak{C}_{\mathcal{G}(\sigma,\sigma')}\left(\xi_0 \cup \left\{e \in supp\ \sigma: d\sigma|_{\pm supp\ \partial e} \neq 0\right\} \cup \left\{e \in supp\ \sigma': d\sigma'|_{\pm supp\ \partial e} \neq 0\right\}\right)$$
$$= \xi_{0,\sigma,\sigma'}$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma})\varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right)$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}) \sum_{\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\sigma,\sigma'}}\right.$$
$$\left. + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\sigma,\sigma'}}\right) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\phi,\sigma,\sigma'}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\phi,\sigma,\sigma'}}\right)$$

$$\mu_{\mathrm{N},\beta,\kappa} \otimes \mu_{\mathrm{N},\beta,\kappa}\left(\left\{\left(\hat{\sigma},\hat{\sigma}'\right) \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \otimes \Omega_0^1(\mathrm{B_N},\mathcal{G}): \hat{\sigma}|_{\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\xi_{\phi,\hat{\sigma},\hat{\sigma}'}}} = \sigma\right\}\right) = \mu_{\mathrm{N},\beta,\kappa}(\sigma)$$

$$\sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right) \bigotimes 1\left(\hat{\sigma}'\Big|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right) = \varphi_\kappa(\sigma) \sum_{\sigma' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})} \varphi_\kappa(\sigma')$$

$$\sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right) 1\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_0}$$
$$= \varphi_\kappa(\sigma) \sum_{\sigma' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})} \varphi_\kappa(\sigma') \sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} 1\left(\hat{\sigma} = \sigma'|_{\xi_{\sigma,\sigma'}}\right.$$
$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\sigma,\sigma'}}\right)_{\xi_0} \bigotimes 1\left(\hat{\sigma}' = \sigma\Big|_{\xi_{\sigma,\sigma'}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\sigma,\sigma'}}\right)_{\xi_0}$$

$$\sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} \varphi_\kappa(\hat{\sigma})\varphi_\kappa\left(\hat{\sigma}'\right) 1\left(\hat{\sigma}|_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_1}$$
$$= \varphi_\kappa(\sigma) \sum_{\sigma' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})} \varphi_\kappa(\sigma') \sum_{\substack{\hat{\sigma} \in \Omega_0^1(\mathrm{B_N},\mathcal{G}) \\ \hat{\sigma}' \in \Omega_0^1(\mathrm{B_N},\mathcal{G})}} 1\left(\hat{\sigma} = \sigma'|_{\xi_{\sigma,\sigma'}}\right.$$
$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\sigma,\sigma'}}\right)_{\xi_1} \bigotimes 1\left(\hat{\sigma}' = \sigma\Big|_{\xi_{\psi,\sigma,\sigma'}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\backslash\xi_{\psi,\sigma,\sigma'}}\right)_{\xi_1}$$

$$\left|\mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}[\mathcal{F}(\sigma)] = \mathbb{E}^{\xi_\sigma}_{\mathrm{N},(\zeta,\kappa),(\zeta,\kappa)}[\mathcal{F}(\sigma)]\right| = \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\hat{\sigma},\hat{\sigma}'}}\right)\right]$$

$$= \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\hat{\sigma},\hat{\sigma}'}}\right)\right] \bigotimes 1\left(\xi_1 \cap \xi_{\hat{\sigma},\hat{\sigma}'} = \sigma\right)$$

$$+ \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta} \bigotimes \mathbb{E}_{\mathrm{N},\beta,\kappa,\zeta}\left[\mathcal{F}\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'\Big|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\hat{\sigma},\hat{\sigma}'}}\right)\right] \bigotimes 1\left(\xi_1 \cap \xi_{\hat{\sigma},\hat{\sigma}'} \neq \sigma\right)$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma})\varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_0}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}) \sum_{\sigma'\in\Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\phi,\sigma,\sigma'}}\right.$$
$$\left. + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\phi,\sigma,\sigma'}}\right)_{\xi_0} \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\phi,\sigma,\sigma'}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\phi,\sigma,\sigma'}}\right)_{\xi_0}$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma})\varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma}|_{\xi_{\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_1}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}) \sum_{\sigma'\in\Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}') \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\phi,\sigma,\sigma'}}\right.$$
$$\left. + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\phi,\sigma,\sigma'}}\right)_{\xi_1} \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\phi,\sigma,\sigma'}} + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\phi,\sigma,\sigma'}}\right)_{\xi_1}$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma}')\varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}'|_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_0}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}') \sum_{\sigma'\in\Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\psi,\sigma,\sigma'}}\right.$$
$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\sigma,\sigma'}}\right)_{\xi_0} \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\psi,\sigma,\sigma'}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\sigma,\sigma'}}\right)_{\xi_0}$$

$$\varphi_{\beta_1,\kappa}(\hat{\sigma}')\varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}'|_{\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} + \hat{\sigma}|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\hat{\sigma},\hat{\sigma}'}} = \sigma\right)_{\xi_1}$$
$$= \varphi_{\beta_1,\kappa}(\hat{\sigma}') \sum_{\sigma'\in\Omega^1(\mathrm{B_N},\mathcal{G})} \varphi_{\beta_2,\kappa}(\hat{\sigma}) \bigotimes 1\left(\hat{\sigma}' = \sigma'|_{\xi_{\psi,\sigma,\sigma'}}\right.$$
$$\left. + \sigma|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\sigma,\sigma'}}\right)_{\xi_1} \bigotimes 1\left(\hat{\sigma} = \sigma|_{\xi_{\psi,\sigma,\sigma'}} + \sigma'|_{\mathfrak{C}_1(\mathrm{B_N})\setminus\xi_{\psi,\sigma,\sigma'}}\right)_{\xi_1}$$

$$dist_1(e,\xi_\sigma) \coloneqq \frac{1}{2}\min\left\{\left|\mathfrak{C}_{\mathcal{G}_{\hat{\sigma},\hat{\sigma}'}}(e)\right| : \sigma,\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G}), \mathfrak{C}_{\mathcal{G}_{\hat{\sigma},\hat{\sigma}'}}(e) \cap \xi_\sigma = \phi\right\}$$
$$= \frac{1}{2}\min\left\{\left|\mathfrak{C}_{\mathcal{G}_{\hat{\sigma}}}(e)\right| : \sigma \in \Omega^1(\mathrm{B_N},\mathcal{G}), \mathfrak{C}_{\mathcal{G}_{\hat{\sigma}}}(e) \cap \xi_\sigma \neq \phi\right\}$$

$$dist_1(e,\xi_\sigma) \coloneqq \frac{1}{2}\min\left\{\left|\mathfrak{C}_{\mathcal{G}_{\hat{\sigma},\hat{\sigma}'}}(e)\right| : \sigma,\sigma' \in \Omega^1(\mathrm{B_N},\mathcal{G}), \mathfrak{C}_{\mathcal{G}_{\hat{\sigma},\hat{\sigma}'}}(e) \cap \xi_\sigma = \psi\right\}$$
$$= \frac{1}{2}\min\left\{\left|\mathfrak{C}_{\mathcal{G}_{\hat{\sigma}}}(e)\right| : \sigma \in \Omega^1(\mathrm{B_N},\mathcal{G}), \mathfrak{C}_{\mathcal{G}_{\hat{\sigma}}}(e) \cap \xi_\sigma \neq \psi\right\}$$

$$\mu_{\mathrm{N},\beta,\kappa_1}\otimes\mu_{\mathrm{N},\beta,\kappa_2}\left(\left\{\left(\hat{\sigma},\hat{\sigma}'\right)\in\Omega^1(\mathrm{B_N},\mathcal{G})\otimes\Omega_0^1(\mathrm{B_N},\mathcal{G}):\hat{\sigma}|_{\mathfrak{C}_{\mathcal{G}_{\left(\hat{\sigma},\hat{\sigma}'\right)}}(e)}\geq 2\mathcal{M}\left|supp\ d\left(\hat{\sigma}|_{\mathfrak{C}_{\mathcal{G}_{\left(\hat{\sigma},\hat{\sigma}'\right)}}(e)}\right)\right|\geq 2\mathcal{M}'\right\}\right)$$

$$\mathcal{K}_1=(\kappa_1,\kappa_2):=\left(\alpha_0(\kappa_1)+\alpha_0(\kappa_2)+\alpha_0(\kappa_1)\alpha_0(\kappa_2)\right)^{-1}\alpha_1(\beta)^{dist_1(e,\partial\mathfrak{C}_1(\mathrm{B_N}))}$$

$$\mu_{\mathrm{N},\beta,\kappa}\left(\left\{\hat{\sigma}\in\Omega^1(\mathrm{B_N},\mathcal{G}):d\hat{\sigma}(p)\neq 0\right\}\right)\leq\mathcal{K}_2\alpha_2(\beta,\kappa)$$

$$\mu_{\mathrm{N},\beta,\kappa}\left(\left\{\sigma\in\Omega^1(\mathrm{B_N},\mathcal{G}):\left|\mathfrak{C}_{\mathcal{G}_{(\sigma)}}(\xi_\sigma)\right|\geq 2\mathcal{M}\left|supp\ d\left(\sigma|_{\mathfrak{C}_{\mathcal{G}_{(\sigma)}}(\xi_\sigma)}\right)\right|\geq 2\mathcal{M}'\right\}\right)^{\mathcal{M}'}$$

$$\sum_{\hat{\sigma}\in\Omega^1(\mathrm{B_N},\mathcal{G}),\hat{\sigma}'\in\Omega^1(\mathrm{B_N},\mathcal{G}):(supp\ \hat{\sigma}\cup supp\hat{\sigma}')^+=\xi}\ \prod_{e'\in\mathfrak{C}_1(\mathrm{B_N})^+}\varphi_{\kappa_1}\left(\hat{\hat{\sigma}}(e')\right)^2\prod_{e''\in\mathfrak{C}_1(\mathrm{B_N})^+}\varphi_{\kappa_2}\left(\hat{\hat{\sigma}}'(e'')\right)^2$$
$$\leq\prod_{e'\in\xi}\left\{\varphi_{\kappa_1}(0)^2\left(\sum_{\hat{\hat{\sigma}}'(e')\in\mathcal{G}\backslash\{0\}}\varphi_{\kappa_2}\left(\hat{\hat{\sigma}}'(e')\right)^2\right)+\left(\sum_{\hat{\hat{\sigma}}(e')\in\mathcal{G}\backslash\{0\}}\varphi_{\kappa_1}\left(\hat{\hat{\sigma}}(e')\right)^2\right)\varphi_{\kappa_2}(0)^2\right.$$
$$\left.+\left(\sum_{\hat{\hat{\sigma}}(e')\in\mathcal{G}\backslash\{0\}}\varphi_{\kappa_1}\left(\hat{\hat{\sigma}}(e')\right)^2\right)+\left(\sum_{\hat{\hat{\sigma}}'(e')\in\mathcal{G}\backslash\{0\}}\varphi_{\kappa_2}\left(\hat{\hat{\sigma}}'(e')\right)^2\right)\right\}$$
$$=\prod_{e'\in\xi}\left(\alpha_0(\kappa_1)+\alpha_0(\kappa_2)+\alpha_0(\kappa_1)\alpha_0(\kappa_2)\right)=\left(\alpha_0(\kappa_1)+\alpha_0(\kappa_2)+\alpha_0(\kappa_1)\alpha_0(\kappa_2)\right)^{|\xi|}$$

$$\sum_{\xi\subseteq\mathfrak{C}_1^+(\mathrm{B_N},\mathcal{G}):e\in\xi,|\xi|\geq\mathcal{M}|\bar{\mathcal{G}}|_\xi}\ \sum_{\hat{\sigma}\in\Omega^1(\mathrm{B_N},\mathcal{G}),\hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G}):(supp\ \hat{\sigma}\cup supp\hat{\sigma}')^+=\xi|supp\ d\hat{\sigma}|\geq 2\mathcal{M}'}\mu_{\mathrm{N},\beta,\kappa_1}\otimes\mu_{\mathrm{N},\beta,\kappa_2}\left\{(\hat{\sigma},\hat{\sigma}')\right.$$
$$\left.\in\Omega^1(\mathrm{B_N},\mathcal{G})\otimes\Omega_0^1(\mathrm{B_N},\mathcal{G}):\langle\hat{\sigma}|_{\mathfrak{C}_{\mathcal{G}_{(\hat{\sigma},\hat{\sigma}')}}(e)}=\hat{\hat{\sigma}}\rangle\langle\hat{\sigma}'|_{\mathfrak{C}_{\mathcal{G}_{(\hat{\sigma},\hat{\sigma}')}}(e)}=\hat{\hat{\sigma}}'\rangle\right\}$$

$$\mu_{\mathrm{N},\beta,\kappa_1}\otimes\mu_{\mathrm{N},\beta,\kappa_2}\left(\left\{(\hat{\sigma},\hat{\sigma}')\in\Omega^1(\mathrm{B_N},\mathcal{G})\otimes\Omega_0^1(\mathrm{B_N},\mathcal{G}):\langle\hat{\sigma}|_{\mathfrak{C}_{\mathcal{G}_{(\hat{\sigma},\hat{\sigma}')}}(e)}=\hat{\hat{\sigma}}\rangle\langle\hat{\sigma}'|_{\mathfrak{C}_{\mathcal{G}_{(\hat{\sigma},\hat{\sigma}')}}(e)}=\hat{\hat{\sigma}}'\rangle\right\}\right)$$
$$\leq\mu_{\mathrm{N},\beta,\kappa_1}\otimes\mu_{\mathrm{N},\beta,\kappa_2}\left(\left\{(\hat{\sigma},\hat{\sigma}')\in\Omega^1(\mathrm{B_N},\mathcal{G})\otimes\Omega_0^1(\mathrm{B_N},\mathcal{G}):\left|\hat{\hat{\sigma}}\leq\hat{\sigma}\right|\left|\hat{\hat{\sigma}}'\leq\hat{\sigma}'\right|\right\}\right)$$
$$=\mu_{\mathrm{N},\beta,\kappa_1}\left(\left\{\hat{\sigma}\in\Omega^1(\mathrm{B_N},\mathcal{G}):\left|\hat{\hat{\sigma}}\leq\hat{\sigma}\right|\right\}\right)\mu_{\mathrm{N},\beta,\kappa_2}\left(\left\{\hat{\sigma}'\in\Omega_0^1(\mathrm{B_N},\mathcal{G}):\left|\hat{\hat{\sigma}}'\leq\hat{\sigma}'\right|\right\}\right)$$
$$\leq\varphi_{\beta,\kappa_1}\left(\hat{\hat{\sigma}}\right)\varphi_{\beta,\kappa_2}\left(\hat{\hat{\sigma}}'\right)\sqrt{\frac{|supp\ \gamma|}{|supp\ y_c|}}\bigotimes\mathcal{K}_2\alpha_2\left(\beta,\kappa\right)^2\Theta'_{\beta,\kappa}(\gamma)^{1+2|supp\ \gamma|/|supp\ (\gamma-y_c)|}\langle\sigma_e\rangle$$

2. Vórtice Cuántico en espacios curvos.

$$\mathcal{F}_{\mathcal{T}}=\rho_s\kappa\beta_{ind}\left[s'\left(\xi+\frac{\Delta\xi}{2}\right)-s'\left(\xi-\frac{\Delta\xi}{2}\right)\right]$$

$$\mathcal{F}_{\mathcal{M}}=\rho_s\kappa s'\otimes\left(\upsilon_{\mathcal{L}}-\upsilon_{s,nl}\right)\Delta\xi$$

$$\mathcal{F}_{\mathcal{D}}=\left[\gamma_0 s'\otimes s'\otimes\left(\upsilon_{\mathcal{L}}-\upsilon_\eta\right)-\gamma_0' s'\otimes\left(\upsilon_{\mathcal{L}}-\upsilon_\eta\right)\right]\Delta\xi$$

$$\frac{ds}{dt}=\upsilon_{\mathcal{L}}=\upsilon_{\mathcal{S}_0}+\alpha s'\bigotimes\left(\upsilon_{\mathcal{L}}-\upsilon_{\mathcal{S}_0}\right)-\alpha' s'\bigotimes\left[s'\bigotimes\left(\upsilon_\eta-\upsilon_{\mathcal{S}_0}\right)\right]$$

$$\upsilon_{\mathcal{S}_0}=\frac{\kappa}{4\pi}\int_{\mathcal{L}}\frac{s'(\xi_1)\otimes\left(s(\xi_1)-s(\xi)\right)}{|s(\xi_1)-s(\xi)|^3}d\xi_1$$

3. Modelo Abeliano Yang – Mills – Higgs en espacios cuánticos curvos.

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \left(\partial_\mu\phi^\star - ie\mathcal{A}_\mu\phi^\star\right)(\partial^\mu\phi + ie\mathcal{A}^\mu\phi) - \frac{\lambda}{2}(\varphi^\star\varphi - v^2)^2$$

$$\phi = \rho e^{i\theta}$$

$$\mathcal{L} = -\frac{1}{4}\widetilde{\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}} + e^2\rho^2\left(\widetilde{\mathcal{A}_\mu} + \frac{1}{e}\overline{\partial_\mu}\theta\right)\left(\widetilde{\mathcal{A}^\mu} + \frac{1}{e}\overline{\overline{\partial^\mu}}\theta\right) + \overline{\partial_\mu}\rho\overline{\overline{\partial^\mu}}\ddot{\rho} - \frac{\lambda}{2}(\rho^2 - v^2)^2$$

$$\frac{d^2\hbar}{d\omega^2}(\omega) - \frac{4e^2}{\lambda}\tan\hbar^2(\omega)\,h(\omega)$$

$$(1-z^2)\frac{d^2\hbar}{dz^2}(z) - 2z\frac{d\hbar}{dz}(z) + \left[l(l+1) - \frac{m^2}{(1-z^2)}\right]h(z)$$

$$l = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{4e^2}{\lambda}}$$

$$h(z) = \alpha\mathcal{P}_l^m(z) + \beta\mathcal{Q}_l^m(z)$$

$$\mathcal{P}_l^m(z) = \frac{1}{\Gamma(1-m)}\left(\frac{1+z}{1-z}\right)^{\frac{m}{2}}\mathcal{F}\left(-l, l+1; 1-m; \frac{1-z}{2}\right)$$

$$\mathcal{Q}_l^m(z) = \frac{\pi}{2\sin(m\pi)}\left[\cos(m\pi)\,\mathcal{P}_l^m(z) - \frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}\mathcal{P}_l^{-m}(z)\right]$$

$$\phi = \pm v\tan\hbar(\omega)e^{i\theta}$$

$$\mathcal{A}_\mu = -\frac{1}{e}\partial_\mu\theta + \varepsilon_\mu[\alpha\mathcal{P}_l^m(\tan\hbar(\omega)) + \beta\mathcal{Q}_l^m(\tan\hbar(\omega))]$$

$$p^2 = -\frac{\lambda v^2}{2}\varepsilon^\mu\varepsilon_\mu = p_\mu\varepsilon^\mu$$

$$\partial_\mu\partial^\mu\rho + \lambda\rho(\rho^2 - v^2) = 0$$

$$\frac{d^2\rho}{d\omega^2} + \frac{\lambda}{p^2}\rho(\rho^2 - v^2)$$

$$\left(\frac{d\rho}{d\omega}\right)^2 = -\frac{\lambda}{2p^2}\rho^2(\rho^2 - 2v^2) + \alpha v^2$$

$$\left(\frac{d\gamma}{d\omega}\right)^2 = \frac{\lambda v^2\varepsilon^2}{2p^2}\gamma^4 + \frac{\lambda v^2}{p^2}\gamma^2 + \frac{\alpha}{\varepsilon^2}$$

$$\alpha = \varepsilon^2 = \frac{2m^2}{1+m^2}$$

$$p^2 = -\frac{\lambda v^2}{1+m^2}$$

$$\left(\frac{d\gamma}{d\omega}\right)^2 = (1-\gamma^2)(1-m^2\gamma^2)$$

$$\rho(\omega) = \pm v\sqrt{\frac{2m^2}{1+m^2}}sin(\omega+d,m)\,sin(\omega+d,1) = \tan\hbar(\omega+d)$$

$$\frac{d^2\hbar}{dz^2}(\omega)+\frac{2e^2}{p^2}\rho^2(\omega)\hbar(\omega)$$

$$\hbar(\omega)=\hbar(\mathcal{Z}),\mathcal{Z}=\frac{1}{\tau^2}\rho^2(\omega)$$

$$\frac{d^2\rho}{d\omega^2}=-\frac{\lambda}{p^2}\rho(\rho^2-\upsilon^2)$$

$$\left(\frac{d\rho}{d\omega}\right)^2=-\frac{\lambda}{2p^2}\rho^2(\rho^2-2\upsilon^2)+\alpha\upsilon^2$$

$$\frac{d^2\hbar}{d\mathcal{Z}^2}+\left(\frac{\gamma}{\mathcal{Z}}-\frac{\delta}{1-\mathcal{Z}}-\frac{\epsilon\kappa^2}{1-\kappa^2\mathcal{Z}}\right)\frac{d\hbar}{d\mathcal{Z}}+\frac{(s+\alpha\beta\kappa^2\mathcal{Z})}{\mathcal{Z}(1-\mathcal{Z})(1-\kappa^2\mathcal{Z})}\hbar$$

$$\tau^2=\frac{2\kappa^2}{(1+\kappa^2)}\upsilon^2$$

$$\gamma+\delta+\epsilon=\alpha_\pm+\beta_\mp+1$$

$$p^2=-\frac{2\lambda\upsilon^2\kappa^2}{\alpha(1+\kappa^2)^2}$$

$$\hbar(\mathcal{Z})=\mathcal{C}_1Hn\left(\kappa^2,0;\alpha_+,\beta_-,\frac{1}{2},\frac{1}{2};\mathcal{Z}\right)+\mathcal{C}_2Hn\left(\kappa^2,0;\alpha_-,\beta_+,\frac{1}{2},\frac{1}{2};\mathcal{Z}\right)$$

$$\mathcal{Z}=\frac{1}{\tau^2}\rho^2(\omega)=\frac{(1+\kappa^2)}{2\kappa^2}\frac{1}{\upsilon^2}\rho^2(\omega)$$

$$\alpha=\frac{2\kappa^2(1+m^2)}{(1+\kappa^2)^2}=\frac{2m^2}{(1+m^2)}\Longrightarrow\kappa^2=1/m^2$$

$$\phi=\pm\upsilon\sqrt{\frac{2m^2}{1+m^2}}sin(\omega+d,m)e^{i\theta}$$

$$\mathcal{A}_\mu=-\frac{1}{e}\partial_\mu\theta+\varepsilon_\mu\left[\mathcal{C}_1Hn\left(\kappa^2,0;\alpha_+,\beta_-,\frac{1}{2},\frac{1}{2};sin^2(\omega+d,m)\right)\right.$$
$$\left.+\,\mathcal{C}_2Hn\left(\kappa^2,0;\alpha_-,\beta_+,\frac{1}{2},\frac{1}{2};sin^2(\omega+d,m)\right)\right]$$

$$p^2=-\frac{\lambda\upsilon^2}{1+m^2}\varepsilon^\mu\varepsilon_\mu=p_\mu\varepsilon^\mu$$

$$\begin{cases}Hn(1,s;\alpha,\beta,\gamma,\delta;z)=(1-z)^r & \mathcal{F}(r+\alpha,r+\beta;\gamma;z)\\ r=\xi-\sqrt{\xi^2-\alpha\beta-s} & \xi=\frac{1}{2}(\gamma-\alpha-\beta)\end{cases}$$

$$\phi=\pm\upsilon\,pq(\omega+d,m)e^{i\theta}$$

$$\mathcal{A}_\mu=-\frac{1}{e}\partial_\mu\theta+\varepsilon_\mu\left[\mathcal{C}_1Hn\left(\kappa^2,0;\alpha_+,\beta_-,\frac{1}{2},\frac{1}{2};\mathcal{Z}\right)+\mathcal{C}_2Hn\left(\kappa^2,0;\alpha_-,\beta_+,\frac{1}{2},\frac{1}{2};\mathcal{Z}\right)\right]$$

$$\mathcal{Z}=\varepsilon^2\frac{(1+\kappa^2)}{2\kappa^2}\,pq^2(\omega+d,m)$$

$$\omega=p_\mu\chi^\mu+\omega_0,\varepsilon^\mu\varepsilon_\mu=p_\mu\varepsilon^\mu$$

$$(d\wp/dz)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

$$\left(\frac{d\gamma}{d\omega}\right)^2 = -\frac{\lambda\upsilon^2\varepsilon^2}{2p^2}\gamma^4 + \frac{\lambda\upsilon^2}{p^2}\gamma^2 + \frac{\alpha}{\varepsilon^2}$$

$$\left(\frac{d\not{f}}{d\omega}\right)^2 = -\frac{2\lambda\upsilon^2\varepsilon^2}{2p^2}\not{f}^3 + \frac{4\lambda\upsilon^2}{p^2}\not{f}^2 + \frac{4\alpha}{\varepsilon^2}\not{f}$$

$$\not{f}(\omega) = g(\omega) + \frac{2}{3}\frac{1}{\varepsilon^2}$$

$$\left(\frac{dg}{d\omega}\right)^2 = -\frac{2\lambda\upsilon^2\varepsilon^2}{2p^2}g^3 + \frac{4}{\varepsilon^2}\left(\frac{2}{3}\frac{\lambda\upsilon^2}{p^2} + \alpha\right)g + \frac{8}{3}\frac{1}{\varepsilon^4}\left(\frac{4\lambda\upsilon^2}{9p^2} + \alpha\right)$$

$$p^2 = -\varepsilon^2\frac{\lambda\upsilon^2}{2}$$

$$\begin{aligned}\left(\frac{dg}{d\omega}\right)^2 &= 4g^3 + \frac{4}{\varepsilon^4}\left(\varepsilon^2\alpha - \frac{4}{3}\right)g + \frac{8}{3}\frac{1}{\varepsilon^6}\left(\varepsilon^2\alpha - \frac{8}{9}\right) \\ &= 4\left[g - \frac{1}{\varepsilon^2}\left(\frac{1}{3} + \sqrt{1-\varepsilon^2\alpha}\right)\right]\left[g - \frac{1}{\varepsilon^2}\left(\frac{1}{3} - \sqrt{1-\varepsilon^2\alpha}\right)\right]\left[g + \frac{2}{3}\frac{1}{\varepsilon^2}\right]\end{aligned}$$

$$g(\omega) = \wp\left(\omega + d; -\frac{4}{\varepsilon^4}\left(\varepsilon^2\alpha - \frac{4}{3}\right), \frac{8}{3}\frac{1}{\varepsilon^6}\left(\varepsilon^2\alpha - \frac{8}{9}\right)\right)$$

$$\begin{aligned}\rho^2(\omega) &= \upsilon^2\left[\frac{2}{3} + \varepsilon^2\wp\left(\omega + d; -\frac{4}{\varepsilon^4}\left(\varepsilon^2\alpha - \frac{4}{3}\right), \frac{8}{3}\frac{1}{\varepsilon^6}\left(\varepsilon^2\alpha - \frac{8}{9}\right)\right)\right] \\ &= \upsilon^2\left[\frac{2}{3} + \wp\left(\frac{1}{\varepsilon}(\omega + d); -4\left(\varepsilon^2\alpha - \frac{4}{3}\right), -\frac{8}{3}\left(\varepsilon^2\alpha - \frac{8}{9}\right)\right)\right]\end{aligned}$$

$$\varepsilon^2\alpha = \frac{4\kappa^2}{(1+\kappa^2)^2}$$

$$\left(\frac{d\gamma}{d\omega}\right)^2 = -\frac{\lambda\upsilon^2\varepsilon^2}{2p^2}\gamma^4 + \frac{\lambda\upsilon^2}{p^2}\gamma^2 + \frac{\alpha}{\varepsilon^2}$$

$$\frac{d^2\rho}{d\omega^2} + \frac{\lambda}{p^2}\rho(\rho^2 - \upsilon^2)$$

$$\rho(\omega) = \pm\upsilon\,\mathrm{sin}[\tau(\omega+\alpha)+\beta, 1] = \pm\upsilon\tan\hbar\,[\tau(\omega+\alpha)+\beta], p^2 = -\frac{\lambda\upsilon^2}{2\tau^2}$$

$$\rho(\omega) = \pm\upsilon\,\mathrm{ns}[\tau(\omega+\alpha)+\beta, 1] = \frac{\pm\upsilon}{\tan\hbar\,[\tau(\omega+\alpha)+\beta]}, p^2 = -\frac{\lambda\upsilon^2}{2\tau^2}$$

$$\rho(\omega) = \pm\sqrt{2}\upsilon\,\mathrm{cn}[\tau(\omega+\alpha)+\beta, 1] = \frac{\pm\upsilon}{\cos\hbar\,[\tau(\omega+\alpha)+\beta]}, p^2 = \frac{\lambda\upsilon^2}{\tau^2}$$

$$\rho(\omega) = \pm\sqrt{2}\upsilon\,\mathrm{ds}[\tau(\omega+\alpha)+\beta, 0] = \frac{\pm\sqrt{2}\upsilon}{\sin[\tau(\omega+\alpha)+\beta]}, p^2 = -\frac{\lambda\upsilon^2}{\tau^2}$$

$$\rho(\omega) = \pm\sqrt{2}\upsilon\,\mathrm{dc}[\tau(\omega+\alpha)+\beta, 0] = \frac{\pm\sqrt{2}\upsilon}{\cos[\tau(\omega+\alpha)+\beta]}, p^2 = -\frac{\lambda\upsilon^2}{\tau^2}$$

$$\rho(\omega) = \pm\tan\hbar\,[\tau(\omega+\alpha)+\beta] = \pm\upsilon\frac{\mu+\tan\hbar\,[\tau(\omega+\alpha)+\beta]}{1+\mu\tan\hbar\,[\tau(\omega+\alpha)+\beta]}, \mu = \tan\hbar(\beta)$$

4. Cuantización morfológica relativa a las partículas y antipartículas supermasivas, a las partículas y antipartículas masivas e hiperpartículas.

$$i\hbar = \frac{\partial\psi}{\partial t} = \mathcal{H}\psi, \mathrm{A}^{OUT} = \mathcal{U}\mathrm{A}^{IN}\mathcal{U}^{\dagger}, \mathcal{U} = \exp\left[-i\frac{t}{\hbar}\mathcal{H}\right], |\psi^{(t)}\rangle = |\mathcal{U}|\psi^{(t)}\rangle, \psi^{(l)} \mapsto \phi^{(l)}, \varrho^{(l)} \mapsto \rho^{(l)}, \mathcal{H}$$

$$= i\frac{\hbar}{\tau}\ln\mathcal{U}, \mathcal{F} = \sum_{j,j'=0}^{\mathcal{D}-1}\sum_{\kappa,\kappa'=0}^{n-1}\mathcal{U}_{j\kappa}^{*}\delta_{j\kappa;j'\kappa'}\mathcal{U}_{j'\kappa'}, \mathcal{F} = \sum_{l=1}^{\mathcal{M}}\omega^{(l)}\left|\langle\phi^{(l)}|\mathcal{U}|\psi^{(l)}\rangle\right|^{2}, \delta_{j\kappa;j'\kappa'}$$

$$= \sum_{l=1}^{\mathcal{M}}\omega^{(l)}\phi_{j}^{(l)*}\phi_{j'}^{(l)}\psi_{\kappa}^{(l)*}\psi_{\kappa'}^{(l)}, \delta_{j\kappa;j'\kappa'} = \sum_{l=1}^{\mathcal{M}}\omega^{(l)}\varrho_{jj'}^{\sim(l)*}\rho_{\kappa\kappa'}^{\sim(l)*}, \delta_{j\kappa;j'\kappa'}$$

$$= \sum_{l=1}^{\mathcal{M}}\omega^{(l)}\varrho_{jj'}^{\sim(l+1)*}\rho_{\kappa\kappa'}^{\sim(l)*}, \langle\psi|\phi\rangle = \sum_{j,\kappa=0}^{\mathrm{N}-1}\psi_{j}^{*}\mathcal{Q}_{j\kappa}\phi_{\kappa}, \langle\psi|\phi\rangle = \sum_{i=0}^{\mathrm{N}-1}\psi_{i}^{*}\phi_{i}, 1$$

$$= \sum_{i=0}^{\mathrm{N}-1}|\psi_{i}|^{2}, \mathcal{G}_{ij} = \sum_{\kappa=0}^{n-1}\mathcal{U}_{j\kappa}\mathcal{U}_{j\kappa}^{*}, \delta_{ij} = \mathcal{G}_{ij'}\delta_{\kappa q} = \sum_{j=0}^{\mathcal{D}-1}\mathcal{U}_{j\kappa}^{*}\mathcal{U}_{jq}, \mathrm{A}^{OUT}$$

$$= \sum_{\delta=0}^{\mathrm{N}_{\delta}-1}\mathcal{B}_{\delta}\mathrm{A}^{IN}\mathcal{B}_{\delta}^{\dagger}, \mathcal{D} = \sum_{j=0}^{\mathcal{D}-1}\sum_{\kappa=0}^{n-1}|\mathcal{U}_{j\kappa}|^{2}, \mathcal{F} = \sum_{i,j=0}^{\mathrm{N}-1}\psi_{i}^{*}\mathcal{H}_{ij}\psi_{j}$$

$$\mathcal{L} = \sum_{j,j'=0}^{\mathcal{D}-1}\sum_{\kappa,\kappa'=0}^{n-1}\mathcal{U}_{j\kappa}^{*}\delta_{j\kappa;j'\kappa'}\mathcal{U}_{j'\kappa'} + \sum_{j,j'=0}^{\mathcal{D}-1}\lambda_{jj'}\left[\delta_{jj'} - \sum_{\kappa=0}^{n-1}\mathcal{U}_{j\kappa}^{*}\mathcal{U}_{j'\kappa}\right]\vec{\mu}\max\delta\mathcal{U} = \lambda\mathcal{U}$$

$$\delta\mathcal{U}^{[\delta]} = \lambda^{[\delta]}\mathcal{U}^{[\delta]}$$

$$\langle\mathcal{U}^{[\delta']}|\lambda^{[\delta']\dagger}|\mathcal{U}^{[\delta]}\rangle = \langle\mathcal{U}^{[\delta']}|\lambda^{[\delta]}|\mathcal{U}^{[\delta]}\rangle\sum_{j,j'=0}^{\mathcal{D}-1}\sum_{\kappa,\kappa'=0}^{n-1}\left(\lambda_{jj'}^{[\delta']}, \mathcal{U}_{j'\kappa}^{[\delta']}\right)^{*}\mathcal{U}_{j\kappa}^{[\delta]} = \sum_{j,j'=0}^{\mathcal{D}-1}\sum_{\kappa,\kappa'=0}^{n-1}\mathcal{U}_{j\kappa}^{[\delta']*}\lambda_{jj'}^{[\delta]}\mathcal{U}_{j'\kappa}^{[\delta]}$$

$$\mathcal{F}_{\delta\delta'} = \langle\mathcal{U}^{[\delta]}|\delta|\mathcal{U}^{[\delta']}\rangle = \sum_{j,j'=0}^{\mathcal{D}-1}\sum_{\kappa,\kappa'=0}^{n-1}\mathcal{U}_{j\kappa}^{[\delta]*}\delta_{j\kappa;j'\kappa'}\mathcal{U}_{j'\kappa'}^{[\delta']}$$

$$\delta \approx \sum_{\delta=0}^{\mathrm{N}_{\delta}-1}\frac{1}{\mathcal{F}_{\delta\delta}}|\delta|\mathcal{U}^{[\delta]}\rangle\langle\mathcal{U}^{[\delta]}|\delta|, \mathcal{V} = \sum_{\delta=0}^{\mathrm{N}_{\delta}-1}\omega_{\delta}\mathcal{U}^{[\delta]}, \omega_{\delta} = \frac{\langle\mathcal{V}|\delta|\mathcal{U}^{[\delta]}\rangle}{\mathcal{F}_{\delta\delta}}$$

$$\langle\mathcal{V}|\delta|\mathcal{V}\rangle = \sum_{\delta=0}^{\mathrm{N}_{\delta}-1}|\omega_{\delta}|^{2}\langle\mathcal{U}^{[\delta]}|\delta|\mathcal{U}^{[\delta]}\rangle$$

$$\mathrm{A}^{OUT} = \sum_{\delta=0}^{\mathrm{N}_{\delta}-1}|\omega_{\delta}|^{2}\mathcal{U}^{[\delta]}\mathrm{A}^{IN}\mathcal{U}^{[\delta]\dagger}$$

$$\mathcal{U} = \mathfrak{U}^{\dagger}\mathfrak{U}\mathcal{U}\mathfrak{B}^{\dagger}\mathfrak{B}, \tilde{\mathcal{U}} = \mathfrak{U}\mathcal{U}\mathfrak{B}^{\dagger}$$

$$\tilde{U}_{j\kappa} = \sum_{i=0}^{\mathcal{D}-1} \sum_{q=0}^{n-1} \mathfrak{U}_{ji} \mathcal{U}_{iq} \mathfrak{B}^*_{\kappa q}$$

$$\mathcal{F} = \langle \mathcal{U}|\delta|\mathcal{U}\rangle = \langle \widetilde{\mathcal{U}|\delta|\mathcal{U}}\rangle, \tilde{\delta}_{j\kappa;j'\kappa'} = \sum_{i,i'=0}^{\mathcal{D}-1} \sum_{q,q'=0}^{n-1} \mathfrak{U}_{ji} \mathfrak{B}^*_{\kappa q}\, \delta_{iq;i'q'} \mathfrak{U}^*_{j'i'} \mathfrak{B}_{\kappa' q'}$$

$$\mathcal{R}_{ex} = \langle \psi|\mathcal{R}|\psi\rangle, \mathcal{R}_{ex} = \langle \mathcal{U}|\mathcal{R}|\mathcal{U}\rangle = \sum_{j,j'=0}^{\mathcal{D}-1} \sum_{\kappa,\kappa'=0}^{n-1} \mathcal{U}^*_{j\kappa}\, \mathcal{R}_{j\kappa;j'\kappa'} \mathcal{U}_{j'\kappa'}, \mathcal{R}_{ex} = Tr\Upsilon\mathcal{R}$$

$$= \sum_{\delta=0}^{\mathrm{N}_\delta - 1} \mathcal{P}^{[\delta]} \langle \mathcal{U}^{[\delta]} | \mathcal{R} | \mathcal{U}^{[\delta]} \rangle$$

$$\lambda_{ji} = Herm \sum_{m=0}^{\mathcal{D}-1} \sum_{\kappa,q=0}^{n-1} \mathcal{U}^*_{i\kappa}\, \delta_{j\kappa;mq} \mathcal{U}_{mq} i\hbar \frac{\partial \mathcal{U}}{\partial t} = \delta\mathcal{U}$$

$$\mathcal{U}^{[t]} = \exp\left[-i\frac{t}{\hbar}\delta\right] \mathcal{U}^0, \mathcal{U}^{(t)} = U^{(t)} \cdots U^{(1)} \mathcal{U}^{(0)} \mathcal{V}^{(1)\dagger} \cdots \mathcal{V}^{(t)\dagger}, \psi^{(t)} = \mathcal{U}^{(t)} \cdots U^{(1)} \psi^{(0)}$$

$$\delta_{j\kappa;j'\kappa'} \approx \lambda_{jj'} \delta_{\kappa\kappa'} + \delta_{jj'} \nu_{\kappa\kappa'}, \delta\mathcal{U} = \lambda\mathcal{U} + \mathcal{U}\nu$$

$$\mathcal{U}^{(t)} = \exp\left[-i\frac{t}{\hbar}\lambda\right] \mathcal{U}^0 \exp\left[-i\frac{t}{\hbar}\nu\right]$$

$$|\mathcal{H}|\psi^{(\delta)}\rangle = |\lambda^{[\delta]}|\psi^{(\delta)}\rangle, \lambda^{[\delta]} = \frac{\langle \phi|\mathcal{H}|\phi\rangle}{\langle \phi|\phi\rangle} \vec{\phi} \max\langle \phi|\phi^{[\delta']}\rangle \langle \mathcal{U}^{[\delta']}|\lambda^{[\delta']}|\mathcal{U}\rangle \langle \mathcal{U}^{[\delta']}|\mathcal{U}\rangle \langle \mathcal{U}^{[\delta']}|\delta|\mathcal{U}\rangle$$

$$\sum_{j,j'=0}^{\mathcal{D}-1} \sum_{\kappa,\kappa'=0}^{n-1} \left(\lambda^{[\delta']}_{jj'}, \mathcal{U}^{[\delta']}_{j'\kappa}\right)^* \mathcal{U}_{j\kappa} = \sum_{j,j'=0}^{\mathcal{D}-1} \sum_{\kappa,\kappa'=0}^{n-1} \mathcal{U}^{[\delta']}_{j\kappa} \lambda^{[\delta']}_{jj'} \mathcal{U}_{j'\kappa}$$

5. Vórtice cuántico bajo el modelo abeliano de Higgs en espacios cuánticos curvos.

$$\mathcal{L}_{(\phi,\mathcal{V}_\alpha)} = -\frac{1}{4\mu\left(\frac{|\phi|}{\upsilon}\right)} \mathcal{V}_{\alpha\beta} \mathcal{V}^{\alpha\beta} + \nabla_\alpha \phi^* \nabla^\alpha \phi - \frac{\lambda}{2}\mu\left(\frac{|\phi|}{\upsilon}\right)(|\phi|^2 - \upsilon^2)^2$$

$$\mathcal{L} = -\frac{1}{4\mu\left(\frac{|\phi|}{\upsilon}\right)} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} + \mathcal{D}_\alpha \phi^* \mathcal{D}^\alpha \phi - \frac{\lambda}{2q^2}\mu(|\phi|)(|\phi|^2 - 1)^2$$

$$\mathfrak{E} = \int d^2\chi \left\{ \frac{1}{2\mu(|\phi|)} \mathcal{F}^2_{12} + \mathcal{D}_k \varphi^* \mathcal{D}_k \varphi + \frac{\lambda}{2q^2} \mu(|\phi|)(|\phi|^2 - 1)^2 \right\}$$

$$\partial_k \left[\frac{1}{\mu(|\phi|)} \mathcal{F}^{kj} \right] = -i\left(\varphi^* \mathcal{D}^j \varphi - \varphi \mathcal{D}^j \varphi^*\right), \mathcal{D}_k \mathcal{D}^k \varphi$$

$$= -\frac{\partial}{\partial \varphi^*} \left\{ \frac{1}{2\mu(|\phi|)} \mathcal{F}^2_{12} + \frac{\lambda}{2q^2} \mu(|\phi|)(|\phi|^2 - 1)^2 \right\}$$

$$\mathfrak{E} = \int d^2\chi \left\{ \frac{1}{2\mu(|\phi|)} \left(\mathcal{F}_{12} \pm \frac{\sqrt{\lambda}}{q} \mu(|\phi|)(|\phi|^2 - 1) \right)^2 + |\mathcal{D}_1\varphi \pm i\mathcal{D}_2\varphi|^2 \right\}$$

$$\pm \int d^2\chi \mathcal{F}_{12} \left(|\varphi|^2 \left(1 - \frac{\sqrt{\lambda}}{q} \right) + 1 \right)$$

$$\mathcal{F}_{12} = \pm\, \mu(|\phi|)(1 - |\varphi|^2)$$

$$\mathfrak{E} = \pm \int d^2\chi \mathcal{F}_{12} = \pm 2\pi\eta, \frac{1}{r}\frac{d\alpha}{dr} = \pm\mu(g)(g^2 - 1)\frac{dg}{dr} \pm \frac{\alpha g}{r}$$

$$\mathcal{H} = \frac{1}{2\mu(|g|)r^2}\left(\frac{d\alpha}{dr}\right)^2 + \left(\frac{dg}{dr}\right)^2 + \left(\frac{\alpha g}{r}\right)^2 + \frac{1}{2}\mu(g)(g^2-1)^2$$

$$\frac{d}{dr}\left(\frac{1}{r\mu(g)}\frac{d\alpha}{dr}\right) = \frac{2\alpha g^2}{r}\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)$$
$$= \frac{\alpha^2 g}{r^2} - \frac{1}{4\mu^2(|g|)r^2}\left(\frac{d\alpha}{dr}\right)^2 \frac{d\mu}{dg} + \frac{1}{4}\frac{d}{dg}(\mu(g)(g^2-1)^2)$$

$$\alpha = r\frac{d\mu}{dr}, r^2\frac{d^2\mu}{dr^2} + r\frac{d\mu}{dr} - r^2\mu(u)(e^{2\mu} - 1), \mu(|\phi|) = \frac{\ln|\varphi|}{|\varphi|^2 - 1}, \mathcal{V}(|\phi|)$$
$$= \frac{1}{2}\ln|\varphi|(|\varphi|^2 - 1)\,, r^2\frac{d^2\mu}{dr^2} + r\frac{d\mu}{dr} - r^2\mu$$

$$\mathcal{H}(r) = n\big[\mathcal{K}_0(r)$$
$$- e^{-2n\mathcal{K}_0(r)}\left(\mathcal{K}_0(r) - 2n\mathcal{K}_1^2(r)\right)\big] \int_0^\infty drr\big[\mathcal{K}_0(r)$$
$$- e^{-2n\mathcal{K}_0(r)}\left(\mathcal{K}_0(r) - 2n\mathcal{K}_1^2(r)\right)\big] \int_0^\infty drr\,\mathcal{K}_0(r) - e^{-2n\mathcal{K}_0(r)}$$
$$= -r\mathcal{K}_1(r)e^{-2n\mathcal{K}_0(r)}\Big|_0^\infty + 2n\int_0^\infty \mathcal{K}_1^2(r)e^{-2n\mathcal{K}_0(r)} \lim_{r\mapsto 0} r\mathcal{K}_1(r)\, e^{-2n\mathcal{K}_0(r)}$$
$$= \lim_{r\mapsto\infty} \mathcal{K}_1^2(r)\, e^{-2n\mathcal{K}_0(r)}$$

$$g(r) = \frac{e^{\gamma_\xi}}{2}r + \frac{e^{\gamma_\xi}}{8}\left[\gamma_\xi - 1 + \ln\left(\frac{r}{2}\right)\right]r^3$$

$$\alpha(r) = 1 + \frac{1}{4}\left[2\gamma_\xi - 1 + \ln\left(\frac{r}{2}\right)\right]r^2$$

$$g(r) = \zeta r - \frac{\zeta}{4}r^3, \alpha(r) = 1 - \frac{1}{2}r^2 + \frac{\zeta^2}{4}r^4$$

$$g(r) \simeq 1 - \mathcal{K}_0(\varrho r) \simeq 1 - \mathcal{O}\left(\sqrt{\frac{\pi}{2\varrho r}}e^{-\varrho r}\right)$$

$$\alpha(r) \simeq r\mathcal{K}_1(\varrho r) \simeq \mathcal{O}\left(\sqrt{\frac{\pi r}{2r}}e^{-\varrho r}\right)$$

$$\begin{aligned}\mathrm{A}_1 = \partial_2\mu + n\partial_1\theta, \mathrm{A}_2 = -\partial_1\mu + n\partial_2\theta, \mathcal{F}_{12} &= -(\partial_1^2 + \partial_2^2)\mu + n\varepsilon_{ij}\partial_i\partial_j\theta \\ &= -(\partial_1^2 + \partial_2^2)\mu + 2\pi n\delta^{(2)}(\vec{r}), (\partial_1^2 + \partial_2^2 - 1)\mu \\ &= 2\pi n\delta^{(2)}(\vec{r}), (\partial_1^2 + \partial_2^2 - 1)\mathcal{K}_0(r) = -2\pi\delta^{(2)}(\vec{r}), \mathcal{F}_{12} \\ &= \beta\mathcal{K}_0(r) + 2\pi(\eta - \beta)\delta^{(2)}(\vec{r})\end{aligned}$$

$$\begin{aligned}\mathfrak{E} = 2\pi\beta + \frac{1}{2}\int_{r\leq\epsilon} d^2\chi \frac{\mathcal{F}_{12}^2}{\mu(|\phi|)}\frac{1}{2}\int_{r\leq\epsilon} d^2\chi \frac{\mathcal{F}_{12}^2}{\mu(|\phi|)} \\ = -\frac{1}{2}\int_{r\leq\epsilon} d^2\chi \frac{\beta^2\ln^2 r - 4\pi(\eta-\beta)\beta\ln r\,\delta^{(2)}(\vec{r}) + 4\pi^2(\eta-\beta)^2\left(\delta^{(2)}(\vec{r})\right)^2}{\beta\ln r} \\ = \frac{1}{2}\int_{r\leq\epsilon} d^2\chi\left(\beta\ln r - 4\pi(\eta-\beta)\delta^{(2)}(\vec{r}) + 4\pi^2\frac{(\eta-\beta)^2}{\beta}\frac{\delta^{(2)}(\vec{r})}{\ln r}\delta^{(2)}(\vec{r})\right)\end{aligned}$$

$$\begin{aligned}\mathfrak{E} = 2\pi\beta + 2\pi(\eta-\beta) = 2\pi\eta, \mu(|\phi|) = \chi\frac{\ln|\phi|}{|\phi|^2 - 1}, \mu(r) = -\eta\mathcal{K}_0\left(\sqrt{\chi}r\right), \alpha(r) \\ = \eta\sqrt{\chi}\mathcal{K}_1\left(\sqrt{\chi}r\right), \mathcal{F}_{12}(r) = \eta\chi\mathcal{K}_0\left(\sqrt{\chi}r\right)\end{aligned}$$

$$\begin{aligned}\varphi(\vec{r}) = \exp[\mu(\vec{r}) + i\Omega(\vec{r})], \Omega(\vec{r}) = \sum_{k=1}^{n}\theta(\vec{r} - \vec{r}_k), \mathrm{A}_1 = \partial_2\mu + \partial_1\Omega, \mathrm{A}_2 \\ = -\partial_1\mu + \partial_2\Omega, (\partial_1^2 + \partial_2^2 - 1)\mu = 0\end{aligned}$$

$$\partial_j\mathcal{K}_0(|\vec{r}|) = -\frac{\chi^j}{|\vec{r}|}\mathcal{K}_1(|\vec{r}|), \partial_j^2\mathcal{K}_0(|\vec{r}|) = \frac{\left(\chi^j\right)^2}{|\vec{r}|^2}\mathcal{K}_0(|\vec{r}|) + \frac{\left(\chi^j\right)^2 - \left(\varepsilon_{jk}\chi^k\right)^2}{|\vec{r}|^3}\mathcal{K}_1(|\vec{r}|)$$

$$\mu|\vec{r}| = -\sum_{k=1}^{n}\mathcal{K}_0(|\vec{r} - \vec{r}_k|), \varphi(\vec{r}) = \prod_{k=1}^{n}e^{-\mathcal{K}_0(|\vec{r}-\vec{r}_k|)}e^{i\theta(\vec{r}-\vec{r}_k)}$$

$$\begin{aligned}\mathrm{A}_i|\vec{r}| = -\varepsilon_{ij}\sum_{k=1}^{n}\left[\frac{\chi^j - \chi_k^j}{|\vec{r} - \vec{r}_k|^2} - \frac{\chi^j - \chi_k^j}{|\vec{r} - \vec{r}_k|}\mathcal{K}_1(|\vec{r} - \vec{r}_k|)\right], \mathrm{A}_i^{(1)}|\vec{r}| \\ = -\frac{\chi^2 - \chi_1^2}{|\vec{r} - \vec{r}_k|^2} - \frac{\chi^2 - \chi_1^2}{|\vec{r} - \vec{r}_k|}\mathcal{K}_1(|\vec{r} - \vec{r}_k|) = -\frac{\sin\theta_1}{r_1}\left(1 - r_1\mathcal{K}_1(r_1)\right), \mathrm{A}_2^{(1)}|\vec{r}| \\ = -\frac{\chi^1 - \chi_1^1}{|\vec{r} - \vec{r}_k|^2} - \frac{\chi^1 - \chi_1^1}{|\vec{r} - \vec{r}_k|}\mathcal{K}_1(|\vec{r} - \vec{r}_k|) = -\frac{\cos\theta_1}{r_1}\left(1 - r_1\mathcal{K}_1(r_1)\right)\end{aligned}$$

$$\begin{aligned}\partial_i\mathrm{A}_j = \sum_{k=1}^{n}\Bigg[\varepsilon_{ij}\Bigg(\frac{1}{|\vec{r} - \vec{r}_k|^2} - 2\frac{\left(\chi^i - \chi_k^i\right)^2}{|\vec{r} - \vec{r}_k|^4} - \frac{1}{|\vec{r} - \vec{r}_k|}\mathcal{K}_1(|\vec{r} - \vec{r}_k|) \\ + \frac{\left(\chi^i - \chi_k^i\right)^2}{|\vec{r} - \vec{r}_k|^4}\mathcal{K}_2(|\vec{r} - \vec{r}_k|)\Bigg)\Bigg]\end{aligned}$$

$$\mathcal{F}_{12}=\sum_{k=1}^{n}\left[-\frac{2}{|\vec{r}-\vec{r}_k|}\mathcal{K}_1(|\vec{r}-\vec{r}_k|)+\mathcal{K}_2(|\vec{r}-\vec{r}_k|)\right]$$

$$\phi(\gamma)\mapsto e^{i\omega(\gamma)}\phi(\gamma),\mathcal{V}_\alpha(\gamma)\mapsto\mathcal{V}_\alpha(\gamma)+\frac{1}{q}\frac{\partial\omega(\gamma)}{\partial\gamma^\alpha}$$

$$\mathcal{L}_{(\phi,\mathcal{V}_\alpha)}=-\frac{1}{4\mu\left(\frac{\rho}{v}\right)}\mathcal{V}_{\alpha\beta}\mathcal{V}^{\alpha\beta}+\frac{\partial\rho}{\partial\gamma^\alpha}\frac{\partial\rho}{\partial\gamma_\alpha}+\mathrm{q}^2\mathcal{V}_\alpha\mathcal{V}^\alpha\rho^2-\frac{\lambda}{2}\mu\left(\frac{\rho}{v}\right)(\rho^2-v^2)^2$$

$$\rho(\gamma)=v+\frac{1}{\sqrt{2}}\eta(\gamma),\mathcal{V}_\alpha=\sqrt{\mu(1)}\mathrm{B}_\alpha$$

$$\mathcal{L}_{(\phi,\mathcal{V}_\alpha)}=\mathcal{L}^{(2)}_{(\phi,\mathcal{V}_\alpha)}+\mathcal{L}^{int}_{(\phi,\mathcal{V}_\alpha)}$$

$$\mathcal{L}^{(2)}_{(\phi,\mathcal{V}_\alpha)}=-\frac{1}{4}\mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}+\frac{1}{2}\frac{\partial\eta}{\partial\gamma^\alpha}\frac{\partial\eta}{\partial\gamma_\alpha}+\mathrm{q}^2v^2\mu(1)\mathcal{B}_\alpha\mathcal{B}^\alpha-\lambda\mu(1)v^2\eta^2,\mathcal{L}^{int}_{(\phi,\mathcal{V}_\alpha)}$$
$$=\sqrt{2}\mu(1)q^2v\eta\mathcal{B}_\alpha\mathcal{B}^\alpha+\frac{1}{2}\mu(1)q^2\eta^2\mathcal{B}_\alpha\mathcal{B}^\alpha-\sum_{p=1}^{\infty}\beta_p\eta^p\,\mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}-\sum_{p=3}^{\infty}\gamma_p\eta^p$$

$$\beta_p=\frac{\mu(1)\mathcal{H}^{(p)}(1)}{2^{\frac{p}{2}+2}p!\,v^p},\gamma_p=\left(\lambda_{p-2}+\frac{1}{\sqrt{2}}\lambda_{p-3}+\frac{1}{8}\lambda_{p-4}\right)v^{4-p},\lambda_p=\frac{\lambda\mu^{(p)}(1)}{2^{\frac{p}{2}}p!}$$

$$\mathcal{L}^{int(3)}_{(\phi,\mathcal{V}_\alpha)}=\frac{1}{\sqrt{2}}q^2v\eta\mathcal{B}_\alpha\mathcal{B}^\alpha-\frac{1}{4\sqrt{2}v}\eta\mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}$$

$$\mathcal{L}^{int(4)}_{(\phi,\mathcal{V}_\alpha)}=\frac{1}{4}q^2\eta^2\mathcal{B}_\alpha\mathcal{B}^\alpha-\frac{1}{48v^2}\eta\mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}-\frac{\lambda}{48}\eta^4$$

$$\mathcal{L}_{(\Psi,\mathcal{V}_\alpha)}=\overline{\Psi}\left(i\gamma^\alpha\nabla^3_\alpha-\mathcal{M}_\mathcal{F}\right)\Psi,\mathcal{L}_{(\Psi,\mathcal{V}_\alpha)}=q^2v^4\{\overline{\Psi}\left(i\gamma^\alpha\mathfrak{D}^3_\alpha-m\right)\Psi\}$$

$$\mathcal{H}=-\frac{1}{2m}\sum_{k=1}^{2}(\partial_k-i\mathfrak{Z}\mathcal{A}_k)^2-\frac{g\mathfrak{Z}\mathcal{F}_{12}}{4m}\sigma_3,\mathcal{H}_+=\frac{1}{2m}\mathcal{D}\mathcal{D}^\dagger-(g-2)\frac{\mathfrak{Z}\mathcal{F}_{12}}{4m},\mathcal{H}_-$$
$$=\frac{1}{2m}\mathcal{D}^\dagger\mathcal{D}+(g-2)\frac{\mathfrak{Z}\mathcal{F}_{12}}{4m}$$

$$\mathcal{D}=e^{-i\theta}\left[\partial_r-\frac{i}{r}(\partial_\theta-i\mathfrak{Z}\mathrm{A}_\theta)\right],\mathcal{D}^\dagger=-e^{i\theta}\left[\partial_r+\frac{i}{r}(\partial_\theta-i\mathfrak{Z}\mathrm{A}_\theta)\right],v_l(r,\theta)$$
$$=\mathrm{N}_l r^{\mathfrak{Z}\eta-l}e^{\mathfrak{Z}\eta\mathcal{K}_0(r)}e^{il\theta}2\pi\mathrm{N}_l^2\int_0^\infty r^{2(\mathfrak{Z}\eta-l)+1}e^{2\mathfrak{Z}\eta\mathcal{K}_0(r)}=1$$

$$\mathfrak{E}_l=\frac{(g-2)\mathcal{Z}}{4m}\int d^2\chi\,\mathcal{F}_{12}|v_l|^2$$
$$=\frac{\pi(g-2)\mathfrak{Z}\eta}{2m}\mathrm{N}_l^2\int_0^\infty drr^{2(\mathfrak{Z}\eta-l)+1}\mathcal{K}_0(r)e^{2\mathfrak{Z}\eta\mathcal{K}_0(r)}\left[\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}(\partial_\theta-i\mathfrak{Z}\mathrm{A}_\theta)^2\right.$$
$$\left.+\frac{g\mathfrak{Z}}{2r}\frac{d\mathrm{A}_\theta}{dr}\sigma_3+k^2\right]\psi k=\sqrt{2m\xi}$$

$$\psi(r,\theta)=\xi_l(r)e^{il\theta}r^2\frac{d^2\xi_l}{dr^2}+r\frac{d\xi_l}{dr}+\left[k^2r^2-(l-\mathfrak{Z}\mathrm{A}_\theta)^2+\frac{1}{2}gZr\frac{d\mathrm{A}_\theta}{dr}\sigma_3\right]\xi_l=0$$

$$\psi_l(r,\theta)=\left(\alpha_l\mathcal{J}_{|l-Zn|}(kr)+\beta_l\mathcal{Y}_{|l-Zn|}(kr)\right)e^{il\theta},\mathcal{J}_m(z)\simeq\sqrt{\frac{2}{\pi z}}\cos\left(z-m\frac{\pi}{2}-\frac{\pi}{4}\right),\mathcal{Y}_m(z)$$

$$\simeq\sqrt{\frac{2}{\pi z}}\sin\left(z-m\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$\psi_l(r,\theta)=\frac{1}{\sqrt{2\pi k}}\left(\frac{e^{ikr}}{\sqrt{r}}e^{-i\frac{\pi}{2}|l-Zn|}e^{i\frac{\pi}{4}}(\alpha_l-i\beta_l)+\frac{e^{-ikr}}{\sqrt{r}}e^{-i\frac{\pi}{2}|l-Zn|}e^{i\frac{\pi}{4}}(\alpha_l+i\beta_l)\right)e^{il\theta}$$

$$\psi_l^{free}(r,\theta)=\frac{1}{\sqrt{2\pi k}}\left(\frac{e^{ikr}}{\sqrt{r}}e^{-i\frac{\pi}{2}|l|}e^{-i\frac{\pi}{4}}+\frac{e^{-ikr}}{\sqrt{r}}e^{i\frac{\pi}{2}|l|}e^{i\frac{\pi}{4}}\right)$$

$$\psi_l(r,\theta)=\mathrm{A}\left(\frac{e^{i(kr+2\delta_l)}}{\sqrt{r}}e^{-i\frac{\pi}{2}|l|}e^{-i\frac{\pi}{4}}+\frac{e^{-ikr}}{\sqrt{r}}e^{i\frac{\pi}{2}|l|}e^{i\frac{\pi}{4}}\right)e^{il\theta}$$

$$\delta_l=\frac{\pi}{2}(|l|-|l-Zn|)-\arctan\left(\frac{\alpha_l}{\beta_l}\right)$$

6. Agujeros negros cuánticos puros, provocados por partículas y antipartículas supermasivas e hiperpartículas.

$$\mathcal{I}_{\mathcal{QT}}=\frac{1}{32\pi\mathcal{G}}\int d^{\mathcal{D}}\chi\sqrt{|g|}\left[\mathcal{R}+\sum_{\eta=2}^{\eta_{max}}\alpha_\eta\mathcal{Z}_\eta\right]$$

$$ds^2=-\mathcal{N}(r)^2f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega^2_{\mathcal{D}-2}$$

$$\frac{d\mathcal{N}}{dr}\frac{d}{dr}[r^{\mathcal{D}-1}\hbar(\psi)],\hbar(\psi)\equiv\psi+\sum_{\eta=2}^{\eta_{max}}\alpha_\eta\Psi^\eta,\psi\equiv\frac{1-f(r)}{r^2},\hbar(\psi)=\frac{m}{r^{\mathcal{D}-1}},f$$

$$=1\left(\frac{m}{\alpha_{\mathcal{N}}}\right)^{\frac{1}{\mathcal{N}}}r^{2-\frac{(\mathcal{D}-1)}{\mathcal{N}}}+\alpha_\eta\geq0\forall\eta\lim_{\eta\mapsto\infty}(\alpha_\eta)^{\frac{1}{\eta}}=\mathfrak{C}>0$$

$$\alpha_\eta\frac{\alpha^{\eta-1}}{\eta}\eta\alpha^{\eta-1}\frac{(1-(-1)^\eta)}{2}\alpha^{\eta-1}\frac{(1-(-1)^\eta)\Gamma\left(\frac{\eta}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{\eta+1}{2}\right)}\alpha^{\eta-1}$$

$$\hbar(\psi)\frac{\psi}{1-\alpha\psi}-\log\frac{(1-\alpha\psi)}{\alpha}\frac{\psi}{(1-\alpha\psi)^2}\frac{\psi}{1-\alpha^2\psi^2}\frac{\psi}{\sqrt{1-\alpha^2\psi^2}}$$

$$f(r) = 1 - \frac{mr^2}{r^{\mathcal{D}-1} + \alpha m} \otimes \frac{r^2}{\alpha}\left(1 - e^{\frac{\alpha m}{r^{\mathcal{D}-1}}}\right) \otimes \frac{2mr^2}{r^{\mathcal{D}-1} + 2\alpha m + \sqrt{r^{2(\mathcal{D}-1)} + 4\alpha m r^{\mathcal{D}-1}}} \otimes \frac{2mr^2}{r^{\mathcal{D}-1} + \sqrt{r^{2(\mathcal{D}-1)} + 4\alpha^2 m^2}} \bigotimes \frac{mr^2}{\sqrt{r^{2(\mathcal{D}-1)} + \alpha^2 m^2}}$$

$$\frac{\psi}{1-\alpha\psi} = \frac{m}{r^{\mathcal{D}-1}}, f = 1 - \frac{mr^2}{r^{\mathcal{D}-1} + \alpha m}$$

$$\mathcal{M} = \frac{(\mathcal{D}-2)\Omega_{\mathcal{D}-2} r_+^{\mathcal{D}-1}}{32\pi\mathcal{G}} \hbar(\psi_+), \mathcal{T} = \frac{1}{4\pi r_+}\left[\frac{(\mathcal{D}-1)r_+^2 \hbar(\psi_+)}{\hbar'(\psi_+)} - 2\right]$$

$$\delta = -\frac{(\mathcal{D}-2)\Omega_{\mathcal{D}-2}}{16\mathfrak{G}} \int \frac{\hbar'(\psi_+)}{\psi_+^{\frac{\mathcal{D}}{2}}} d\psi_+, d\mathcal{M} = \mathcal{T} d\mathcal{S}, \mathcal{G}\mathcal{M} = \frac{3\pi r_+^4}{8(r_+^2 - \alpha)}, \mathcal{T} = \frac{r_+^2 - 2\alpha}{2\pi r_+^3}, \mathcal{G}\mathcal{S}$$

$$= \frac{\pi^2 r_+^3}{2} + 3\pi^2 \alpha r_+ - \frac{3\pi^2\alpha^2 r_+}{4(r_+^2 - \alpha)} - \frac{15\pi^2 \alpha^{\frac{3}{2}}}{4} \arctan \hbar\left(\frac{\sqrt{\alpha}}{r_+}\right)$$

$$\mathcal{P}_a^{cde}\mathcal{R}_{bcde} - \frac{1}{2} g_{ab}\mathcal{L} - 2\nabla^c\nabla^d\mathcal{P}_{abcd}$$

$$\tilde{\mathcal{Z}}_{\eta+5} = \frac{3(n+3)}{2(n+1)\mathcal{D}(\mathcal{D}-1)} \tilde{\mathcal{Z}}_1 \tilde{\mathcal{Z}}_{\eta+4} + \frac{3(n+4)}{2n\mathcal{D}(\mathcal{D}-1)} \tilde{\mathcal{Z}}_2 \tilde{\mathcal{Z}}_{\eta+3} - \frac{(n+3)(n+4)}{2n(n+1)\mathcal{D}(\mathcal{D}-1)} \tilde{\mathcal{Z}}_3 \tilde{\mathcal{Z}}_{\eta+2}$$

$$\mathcal{Z}_\eta = -\frac{1}{(\mathcal{D}-2\eta)} \tilde{\mathcal{Z}}_\eta$$

$$\begin{aligned} \tilde{\mathcal{Z}}_1 = -\mathcal{R}, \tilde{\mathcal{Z}}_2 = -\frac{1}{(\mathcal{D}-2)(\mathcal{D}-3)} & [\mathcal{R}^2 - 4\mathcal{R}^{ab}\mathcal{R}_{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd}] \langle \mathcal{R}_{ac}^{bd}\mathcal{R}_{bd}^{ef}\mathcal{R}_{ef}^{ac}\rangle \mathcal{R}_{abcd}\mathcal{R}^{abcd}\mathcal{R} |\mathcal{R}_a^c\mathcal{R}_c^a\mathcal{R}| |\mathcal{R}_{abcd}\mathcal{R}_e^{acb}\mathcal{R}^{de}| \\ & - (\mathcal{R}_{acbd}\mathcal{R}^{ab}\mathcal{R}^{cd}) \|\mathcal{R}_a^c\mathcal{R}_c^b\mathcal{R}_b^a\| \langle \mathcal{R}_b^a\mathcal{R}_a^c\mathcal{R}_c^d\mathcal{R}_d^b\mathcal{R}_{ab}\mathcal{R}^{ab}\mathcal{R}_{cd}\mathcal{R}^{cd}\mathcal{R}\mathcal{R}_a^c\mathcal{R}_c^b\mathcal{R}_b^a\mathcal{R}^2\mathcal{R}_{ab}\mathcal{R}^{ab}\mathcal{R}^4\mathcal{R}\mathcal{R}^{ab}\mathcal{R}^{cd}\mathcal{W}_{abcd}\rangle \\ & - \langle \mathcal{R}^2\mathcal{W}_{abcd}\mathcal{W}^{abcd}\mathcal{R}\mathcal{R}_b^a\mathcal{W}_{ac}^{de}\mathcal{W}_{de}^{bc}\mathcal{R}_a^c\mathcal{R}^{ab}\mathcal{R}^{de}\mathcal{W}_{bdce}\mathcal{R}^{ab}\mathcal{R}^{cd}\mathcal{W}_{ac}^{ef}\mathcal{W}_{bdef}\mathcal{R}\mathcal{W}_{ab}^{ef}\mathcal{W}^{abcd}\mathcal{W}_{cdef}\rangle \\ & + \langle \mathcal{R}_a^c\mathcal{R}^{ab}\mathcal{W}_b^{def}\mathcal{W}_{cdef}\mathcal{R}^{ab}\mathcal{W}_a^{cde}\mathcal{W}_{bc}^{fg}\mathcal{W}_{defg}\mathcal{W}_{ab}^{cd}\mathcal{W}_{cd}^{ef}\mathcal{W}_{ef}^{gh}\mathcal{W}_{gh}^{ab}\rangle \end{aligned}$$

$$\bigotimes \|\mathcal{W}_b^a\mathcal{W}_a^c\mathcal{W}_c^d\mathcal{W}_d^b\mathcal{W}_{ab}\mathcal{W}^{ab}\mathcal{W}_{cd}\mathcal{W}^{cd}\mathcal{W}\mathcal{W}_a^c\mathcal{W}_c^b\mathcal{W}_b^a\mathcal{W}^2\mathcal{W}_{ab}\mathcal{W}^{ab}\mathcal{W}^4\mathcal{R}\mathcal{W}^{ab}\mathcal{W}^{cd}\|$$

$$ds^2 = -\mathcal{N}(r)^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{\mathcal{D}-2}^2$$

$$\begin{aligned} \mathcal{I}_{Q\mathcal{T}} = [\mathcal{N}, f] = \Omega_{\mathcal{D}-2} \int dt dr \mathrm{N}(r) \left[r^{\mathcal{D}-1}\hbar(\psi)\right]' \frac{1}{\Omega_{\mathcal{D}-2} r^{\mathcal{D}-2}} \frac{\delta\mathcal{I}[\mathcal{N}, f]}{\delta\mathcal{N}} \\ = \frac{2\varepsilon_{tt}}{f\mathcal{N}^2}, \frac{1}{\Omega_{\mathcal{D}-2} r^{\mathcal{D}-2}} \frac{\delta\mathcal{I}[\mathcal{N}, f]}{\delta f} = \frac{2\varepsilon_{tt}}{\mathcal{N} f^2} + \mathcal{N}\varepsilon_{rr}, \frac{\delta\mathcal{I}[\mathcal{N}, f]}{\delta\mathcal{N}} = 0 \Longrightarrow \hbar'(\psi)\mathcal{N}'(r) \\ = 0, \frac{\delta\mathcal{I}[\mathcal{N}, f]}{\delta\mathcal{N}} = 0 \Longrightarrow \left[r^{\mathcal{D}-1}\hbar(\psi)\right]' = 0 \end{aligned}$$

$$\mathcal{L}_n^{(i)}\Big|_{\delta\delta\delta} = \alpha_i \mathcal{F}(\mathcal{N}, f)^\eta$$

$$\kappa^t = \frac{c_t}{\sqrt{\sigma f(r)}}, \kappa^r = c_r\sqrt{\sigma f(r)}, \kappa^\theta = \frac{c_\theta}{r}, \kappa^\phi = \frac{c_\phi}{r\sin\theta}, \kappa^\phi = \frac{c_\psi}{r\sin\theta\cos\phi} c_r^2 + \sigma\left(c_\theta^2 + c_\phi^2 + c_\psi^2\right) = c_t^2$$

$$\mathcal{G}_{\mu\nu}\kappa^\mu\kappa^\nu = \frac{\left(c_\theta^2 + c_\phi^2 + c_\psi^2\right)[r^2 f'' + rf' - 4(f(r) - 1)]}{2r^2} \mathcal{G}_{\mu\nu}\kappa^\mu\kappa^\nu \propto \frac{64\alpha m^2 r^6}{(r^4 + \alpha m)^3}$$

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Apéndice F.

1. Modelo de gravedad de Yang – Mills, basado en supersimetrías de gauge en campos cuánticos relativistas.

$$\mathring{\Gamma}^{\mu}{}_{\sigma\nu} = \frac{1}{2} g^{\mu\rho}\left(\partial_\sigma g_{\rho\nu} + \partial_\nu g_{\rho\sigma} - \partial_\rho g_{\sigma\nu}\right)$$

$$\mathring{\Gamma}^{\sigma}{}_{\rho\sigma} = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}).$$

$$\mathring{\Gamma}^{\mu}{}_{\sigma\nu} = \mathring{e}_a^{\mu}\partial_\nu \mathring{e}^{a}_{\sigma} = -\mathring{e}^{a}_{\sigma}\partial_\nu \mathring{e}_a^{\mu}.$$

$$\mathring{T}^{\rho}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} - \mathring{\Gamma}^{\rho}{}_{\nu\mu} = 0$$

$$\mathring{K}^{\mu\nu\rho} = \frac{1}{2}\left(\mathring{T}^{\nu}\mu\rho + \mathring{T}^{\mu\rho\nu} - \mathring{T}^{\rho\nu\mu}\right) = 0$$

$$\mathring{\Gamma}^{\mu}{}_{\sigma\nu} = \overset{\blacksquare}{\Gamma}^{\mu}{}_{\sigma\nu} - \overset{\blacksquare}{K}^{\mu}{}_{\sigma\nu}$$

$$\dot{\Gamma}^{\mu}{}_{\sigma\nu} = \overset{\blacksquare}{e}_a^{\mu}\partial_\nu \dot{e}^{a}{}_{\sigma} = -\overset{\blacksquare}{e}^{a}_{\sigma}{}_{\sigma}\partial_\nu \overset{\blacksquare}{e}_a{}^{\mu},$$

$$\overset{\blacksquare}{T}^{\rho}{}_{\mu\nu} = \overset{\blacksquare}{\Gamma}^{\rho}{}_{\mu\nu} - \overset{\blacksquare}{\Gamma}^{\rho}{}_{\nu\mu} = \overset{\blacksquare}{e}^{\rho}_{a}\partial_\mu \dot{e}^{a}{}_{\nu} - \overset{\blacksquare}{e}^{\rho}_{a}\partial_\nu \overset{\blacksquare}{e}^{a}{}_{\mu}.$$

$$\overset{\blacksquare}{K}^{\mu\nu\rho} = \frac{1}{2}\left(\overset{\blacksquare}{T}^{\nu\mu\rho} + \overset{\blacksquare}{T}^{\mu\rho\nu} - \overset{\blacksquare}{T}^{\rho\nu\mu}\right)$$

$$\begin{aligned}
&\widetilde{\nabla}_\rho V^{\mu_1\ldots\rho\ldots\mu_n}\nu_{\nu_1\ldots\nu_m} \\
&= \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}V^{\mu_1\ldots\rho\cdots\mu_n}\nu_{\nu_1\ldots\nu_m}\right) \\
&= \partial_\rho V^{\mu_1\ldots\rho\ldots\mu_n}\nu_{\nu_1\ldots\nu_m} + \mathring{\Gamma}\sigma_{\rho\sigma}V^{\mu_1\ldots\rho\ldots\mu_n}\nu_{\nu_1\ldots\nu_m}.
\end{aligned}$$

$$\begin{aligned}
&\mathring{\nabla}_\rho V^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} = \partial_\rho V^{\mu_1\ldots\mu_n}\nu_{\nu_1\ldots\nu_m} \\
&+\mathring{\Gamma}^{\mu_1}\sigma\rho V^{\sigma\mu_2\ldots\mu_n}\nu_1\ldots\nu_m + \cdots + \mathring{\Gamma}^{\mu_n}{}_{\sigma\rho}V^{\mu_1\ldots\mu_{n-1}\sigma}{}_{\nu_1\ldots\nu_m} \\
&-\mathring{\Gamma}^{\sigma}\nu_{\nu_1\rho}V^{\mu_1\ldots\mu_n}{}_{\sigma\nu_2\ldots\nu_m} - \cdots - \mathring{\Gamma}^{\sigma}{}_{\nu_m\rho}V^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_{m-1}\sigma}.
\end{aligned}$$

$$\begin{aligned}
&\overset{\blacksquare}{\nabla}_\rho V^{\mu_1\ldots\mu_n}\nu_1\ldots\nu_m = \partial_\rho V^{\mu_1\ldots\mu_n}\nu_{\nu_1\ldots\nu_m} \\
&+\dot{\Gamma}^{\mu_1}{}_{\sigma\rho}V^{\sigma\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \cdots + \dot{\Gamma}^{\mu_n}\sigma\rho V^{\mu_1\ldots\mu_{n-1}\sigma}{}_{\nu_1\ldots\nu_m} \\
&-\dot{\Gamma}^{\sigma}{}_{\nu_1\rho}V^{\mu_1\ldots\mu_n}{}_{\sigma\nu_2\ldots\nu_m} - \cdots - \dot{\Gamma}^{\sigma}{}_{\nu_m\rho}V^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_{m-1}\sigma}.
\end{aligned}$$

$$\vec{D}_\nu = \vec{\partial}_\nu + i\frac{q_{\mathrm{e}}}{\hbar}A_\nu, \overleftarrow{D} = \overleftarrow{\partial}_\nu - i\frac{q_{\mathrm{e}}}{\hbar}A_\nu$$

$$\mathrm{e}_0 = [0,0,0,0,1,0,0,0]^T,$$

$$\mathrm{e}_x = [0,1,0,0,0,0,0,0]^T,$$

$$\mathrm{e}_y = [0,0,1,0,0,0,0,0]^T,$$

$$\mathrm{e}_z = [0,0,0,1,0,0,0,0]^T.$$

$$\mathrm{e}^a = \eta^{ab}\mathrm{e}_b = -\gamma_{\mathrm{B}}^{a}\mathrm{e}_0,$$

$$\bar{\mathrm{e}}^a = \eta^{ab}\bar{\mathrm{e}}_b = -\bar{\mathrm{e}}_0\gamma_{\mathrm{B}}^{a}.$$

$$\Theta = \sqrt{\frac{\varepsilon_0}{2}}cA^a\mathrm{e}_a = \sqrt{\frac{\varepsilon_0}{2}}[0, cA^x, cA^y, cA^z, \phi_{\mathrm{e}}, 0,0,0]^T.$$

$$\begin{aligned}
\Psi &= -(\mathbf{I}_8 + \mathrm{e}_0\bar{\mathrm{e}}_0)\gamma_{\mathrm{B}}^{a}\partial_a\Theta \\
&= \sqrt{\frac{\varepsilon_0}{2}}[0, E^x, E^y, E^z, 0, icB^x, icB^y, icB^z]^T
\end{aligned}$$

$$\psi_8 = \psi\mathrm{e}_0 = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \psi, \mathbf{0}, \mathbf{0}, \mathbf{0}]^T$$

$$\Phi = \frac{q_e}{\sqrt{2\epsilon_0}}\bar{\psi}(\gamma_F)\psi = \sqrt{\frac{\varepsilon_0}{2}}\mu_0 c J_e^a \mathrm{e}_a$$

$$= \sqrt{\frac{\varepsilon_0}{2}}\left[0, \mu_0 c J_e^x, \mu_0 c J_e^y, \mu_0 c J_e^z, \rho_e/\varepsilon_0, 0,0,0\right]^T.$$

$$\boldsymbol{\gamma}_F = \boldsymbol{\gamma}_F^a \mathrm{e}_a = \left[\mathbf{0}, \boldsymbol{\gamma}_F^x, \boldsymbol{\gamma}_F^y, \boldsymbol{\gamma}_F^z, \boldsymbol{\gamma}_F^0, \mathbf{0}, \mathbf{0}, \mathbf{0}\right]^T,$$

$$\vec{\partial} = -\mathrm{e}^a \vec{\partial}_a = \left[0, \vec{\partial}_x, \vec{\partial}_y, \vec{\partial}_z, -\vec{\partial}_0, 0,0,0\right]^T,$$

$$\vec{D} = -\mathrm{e}^a \vec{D}_a = \left[0, \vec{D}_x, \vec{D}_y, \vec{D}_z, -\vec{D}_0, 0,0,0\right]^T.$$

$$\vec{D} = \vec{\partial} - i\frac{q_e\sqrt{2/\varepsilon_0}}{\hbar c}\Theta, \overleftarrow{D} = \overleftarrow{\partial} + i\frac{q_e\sqrt{2/\varepsilon_0}}{\hbar c}\bar{\Theta}$$

$$\bar{\Psi}\Psi = -\frac{1}{4\mu_0}F_{ab}F^{ab} = \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 - \frac{1}{\mu_0}\mathbf{B}^2\right)$$

$$i\bar{\Psi}\gamma_B^5\Psi = \frac{1}{4\mu_0}F_{ab}\tilde{F}^{ab} = -\varepsilon_0 c \mathbf{E}\cdot\mathbf{B}$$

$$F^{ab} = \partial^a A^b - \partial^b A^a$$

$$= \begin{bmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & -B^z & B^y \\ E^y/c & B^z & 0 & -B^x \\ E^z/c & -B^y & B^x & 0 \end{bmatrix}.$$

$$\tilde{F}^{ab} = \frac{1}{2}\varepsilon^{abcd}F_{cd} = \begin{bmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & E^z/c & -E^y/c \\ B^y & -E^z/c & 0 & E^x/c \\ B^z & E^y/c & -E^x/c & 0 \end{bmatrix}.$$

$$\mathbf{t}^a = \left(\gamma_B^0\gamma_B^5\gamma_B^a\right)^*.$$

Propiedades restringuidas

$$\bar{\Phi}_{L2}\mathrm{t}^a\Phi_{L1} = 0, \quad \text{for all } \Phi_{L1}, \Phi_{L2} \in V_L$$

Relaciones mutuales de conmutación

$$\left[\mathbf{t}^0, \mathbf{t}^i\right] = \mathbf{0}, \qquad \left[\mathbf{t}^i, \mathbf{t}^j\right] = 2i\varepsilon^{ijk}\mathbf{t}^k$$

Matrices de conmutatividad

$$\left[\mathbf{t}^a, \boldsymbol{\gamma}_B^5\right] = \mathbf{0}, \qquad \left[\mathbf{t}^a, \boldsymbol{\gamma}_B^b\boldsymbol{\gamma}_B^c\right] = \mathbf{0}$$

Hermiticidad y unitariedad

$$\mathbf{t}^{a\dagger} = \mathbf{t}^a, \qquad (\mathbf{t}^a)^{-1} = \mathbf{t}^{a\dagger}$$

Otras propiedades

$$\mathrm{Tr}(\mathbf{t}^a) = 0, \qquad \mathrm{Tr}\left(\mathbf{t}^a\mathbf{t}^b\right) = 8\delta^{ab}$$

$$\vec{\partial}_\nu \mathbf{I}_g^{(a)} = -ig_g\left(\partial_\nu X_{(a)}\right)\mathbf{t}^{(a)}\mathbf{I}_g^{(a)},$$

$$\left[\mathbf{t}^{(a)}, \mathbf{I}_g^{(a)}\right] = \mathbf{0}, \mathbf{I}_g^{(a)\dagger}\mathbf{I}_g^{(a)} = \mathbf{I}_8/g_g.$$

$$\mathbf{I}_{\mathrm{g}} = \begin{bmatrix} \mathbf{I}_{\mathrm{g}}^{0} \\ \mathbf{I}_{\mathrm{g}}^{x} \\ \mathbf{I}_{\mathrm{g}}^{y} \\ \mathbf{I}_{\mathrm{g}}^{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{0}X_{0}} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{x}X_{x}} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{y}X_{y}} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{z}X_{z}} \end{bmatrix}.$$

$$\partial_{\nu}X_{a} = \partial_{\nu}x_{a}$$

$$g_{\mathrm{g}} = \frac{E_{\mathrm{g}}}{\hbar c}$$

$$E_{\mathrm{g}} = c\sqrt{p^{2}}$$

$$m_{\mathrm{e}}' = m_{\mathrm{e}}, g_{\mathrm{g}}' = g_{\mathrm{g}}$$

$$\begin{aligned}\mathcal{L}_{0} =& \Big[\frac{\hbar c}{4}\bar{\psi}_{8}\big(\bar{\gamma}_{\mathrm{F}}\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_{\nu}\mathbf{I}_{\mathrm{g}}\vec{D} - \overleftarrow{D}\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_{\nu}\mathbf{I}_{\mathrm{g}}\gamma_{\mathrm{F}}\big)\psi_{8} \\ &+ \frac{im_{\mathrm{e}}'c^{2}}{2}\bar{\psi}_{8}\mathrm{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_{\nu}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger}\psi_{8} - (2m_{\mathrm{e}}' - m_{\mathrm{e}})c^{2}\bar{\psi}_{8}\psi_{8} \\ &+ i\bar{\Psi}\mathbf{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_{\nu}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger}\Psi + \bar{\Psi}\Psi\Big]\sqrt{-g}.\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{0} =& \Big[\frac{i\hbar c}{4}\eta_{ab}\bar{\psi}_{8}\big(\overleftarrow{D}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{b}\mathbf{t}^{a}\gamma_{\mathrm{F}} - \bar{\gamma}_{\mathrm{F}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{b}\mathbf{t}^{a}\vec{D}\big)\psi_{8} \\ &+ \frac{m_{\mathrm{e}}'c^{2}}{2}\eta_{ab}\bar{\psi}_{8}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{b}\bar{\mathbf{t}}^{a}\psi_{8} - (2m_{\mathrm{e}}' - m_{\mathrm{e}})c^{2}\bar{\psi}_{8}\psi_{8} \\ &+ \eta_{ab}\bar{\Psi}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{b}\bar{\mathbf{t}}^{a}\Psi + \bar{\Psi}\Psi\Big]\sqrt{-g}.\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{0} =& \Big[\frac{i\hbar c}{2}\bar{\psi}\big(\gamma_{\mathrm{F}}^{\nu}\vec{D}_{\nu} - \overleftarrow{D}_{\nu}\gamma_{\mathrm{F}}^{\nu}\big)\psi - m_{\mathrm{e}}c^{2}\bar{\psi}\psi - \frac{1}{4\mu_{0}}F_{\mu\nu}F^{\mu\nu}\Big]\sqrt{-g} \\ =& \Big[\frac{i\hbar c}{2}\bar{\psi}\big(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu} - \overleftarrow{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\big)\psi - m_{\mathrm{e}}c^{2}\bar{\psi}\psi - J_{\mathrm{e}}^{\nu}A_{\nu} \\ &- \frac{1}{4\mu_{0}}F_{\mu\nu}F^{\mu\nu}\Big]\sqrt{-g}.\end{aligned}$$

$$J_{\mathrm{e}}^{\nu} = q_{\mathrm{e}}c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi$$

$$\mathbf{I}_{\mathrm{g}} \to \mathbf{U}\mathbf{I}_{\mathrm{g}}, \mathbf{U} = \bigotimes_{a}\mathbf{U}_{a}, \mathbf{U}_{a} = e^{i\phi_{(a)}\mathbf{t}^{(a)}}$$

$$\mathbf{I}_{\mathrm{g}} \to \mathbf{U}\mathbf{I}_{\mathrm{g}} = \begin{bmatrix} \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{0}(X_{0}-\phi_{0}/g_{\mathrm{g}})} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{x}(X_{x}-\phi_{x}/g_{\mathrm{g}})} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{y}(X_{y}-\phi_{y}/g_{\mathrm{g}})} \\ \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-ig_{\mathrm{g}}\mathbf{t}^{z}(X_{z}-\phi_{z}/g_{\mathrm{g}})} \end{bmatrix}$$

$$X_{a} \to X_{a} - \phi_{a}/g_{\mathrm{g}}$$

$$\mathbf{I}_{\mathrm{g}} \to \tilde{\mathbf{U}}\mathbf{I}_{\mathrm{g}}, \tilde{\mathbf{U}} = \bigotimes_{a}\tilde{\mathbf{U}}_{a}, \tilde{\mathbf{U}}_{a} = e^{i\theta_{(a)}\gamma_{\mathrm{B}}^{5}\mathbf{t}^{(a)}}$$

$$x^{a} \to \Lambda_{b}^{a}(\mathrm{x})x^{b}$$

$$\mathbf{t}^{a} \to \mathbf{\Lambda}_{\mathrm{J}}(\mathrm{x})\mathbf{t}^{a}\mathbf{\Lambda}_{\mathrm{J}}^{-1}(x) = \mathbf{t}^{a}$$

$$\delta\mathbf{I}_{\mathrm{g}}^{a} = i\mathbf{t}^{(a)}\mathbf{I}_{\mathrm{g}}^{a}\delta\phi_{(a)}$$

$$
\begin{aligned}
&\delta\mathcal{L}_0\\
&=\sum_a\Bigg[\frac{\hbar c}{4}\bar{\psi}_8\Big(\bar{\gamma}_{\mathrm{F}}\overline{\delta\mathbf{I}_{\mathrm{g}}^{(a)}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\mathbf{I}_{\mathrm{g}}^{(a)}\vec{D}+\bar{\gamma}_{\mathrm{F}}\overline{\mathbf{I}_{\mathrm{g}}^{(a)}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\delta\mathbf{I}_{\mathrm{g}}^{(a)}\vec{D}\\
&-\overleftarrow{D}\overline{\delta\mathbf{I}_{\mathrm{g}}^{(a)}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\mathbf{I}_{\mathrm{g}}^{(a)}\gamma_{\mathrm{F}}-\overleftarrow{D}\overline{\mathbf{I}_{\mathrm{g}}^{(a)}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\delta\mathbf{I}_{\mathrm{g}}^{(a)}\gamma_{\mathrm{F}}\Big)\psi_8\\
&+\frac{im_{\mathrm{e}}'c^2}{2}\bar{\psi}_8\left(\delta\mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\overline{\mathbf{I}_{\mathrm{g}}}^{\dagger}+\mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\overline{\delta\mathbf{I}_{\mathrm{g}}^{(\boldsymbol{a})}}^{\dagger}\right)\psi_8\\
&+i\bar{\Psi}\left(\delta\mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\overline{\mathbf{I}_{\mathrm{g}}(a)}^{\dagger}+\mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\vec{\partial}_\nu\overline{\delta\mathbf{I}_{\mathrm{g}}^{(\boldsymbol{a})}}^{\dagger}\right)\Psi\Bigg]\sqrt{-g}\\
&=\frac{\sqrt{-g}}{g_{\mathrm{g}}}\Bigg[\frac{i\hbar c}{4}\bar{\psi}_8\left(\bar{\gamma}_{\mathrm{F}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^a\vec{D}-\overleftarrow{D}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^a\gamma_{\mathrm{F}}\right)\psi_8\\
&-\frac{m_{\mathrm{e}}'c^2}{2}\bar{\psi}_8\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\bar{\mathbf{t}}^a\psi_8-\bar{\Psi}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\bar{\mathbf{t}}^a\Psi\Bigg]\vec{\partial}_\nu\delta\phi_a\\
&=\frac{\sqrt{-g}}{g_{\mathrm{g}}}\Bigg[\frac{i\hbar c}{4}\bar{\psi}_8\left(\bar{\gamma}_{\mathrm{F}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^a\vec{D}-\overleftarrow{D}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^a\gamma_{\mathrm{F}}\right)\psi_8\\
&+\frac{m_{\mathrm{e}}'c^2}{2}\bar{\psi}_8\mathrm{t}^a\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^5\psi_8+\bar{\Psi}\mathrm{t}^a\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^5\Psi\Bigg]\vec{\partial}_\nu\delta\phi_a\\
&=\frac{\sqrt{-g}}{g_{\mathrm{g}}}T_{\mathrm{m}}^{a\nu}\vec{\partial}_\nu\delta\phi_a.
\end{aligned}
$$

$$
\begin{aligned}
T_{\mathrm{m}}^{\mu\nu}&=T_{\mathrm{D}}^{\mu\nu}+T_{\mathrm{em}}^{\mu\nu},\\
T_{\mathrm{D}}^{\mu\nu}&=\frac{i\hbar c}{4}\bar{\psi}_8(\bar{\gamma}_{\mathrm{F}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{\mu}\vec{D}-\overleftarrow{D}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{\mu}\gamma_{\mathrm{F}})\psi_8\\
&\quad+\frac{m_{\mathrm{e}}'c^2}{2}\bar{\psi}_8\mathrm{t}^{\mu}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^5\psi_8\\
&=\frac{c}{2}P^{\mu\nu,\rho\sigma}\left[i\hbar\bar{\psi}(\gamma_{\mathrm{F}\rho}\vec{D}_\sigma-\overleftarrow{D}_\rho\gamma_{\mathrm{F}\sigma})\psi-m_{\mathrm{e}}'cg_{\rho\sigma}\bar{\psi}\psi\right]\\
&=\frac{i\hbar c}{4}\bar{\psi}(\gamma_{\mathrm{F}}^{\mu}\vec{D}^{\nu}+\gamma_{\mathrm{F}}^{\nu}\vec{D}^{\mu}-\overleftarrow{D}^{\nu}\gamma_{\mathrm{F}}^{\mu}-\overleftarrow{D}^{\mu}\gamma_{\mathrm{F}}^{\nu})\psi\\
&\quad-\frac{1}{2}g^{\mu\nu}\left[\frac{i\hbar c}{2}\bar{\psi}(\gamma_{\mathrm{F}}^{\rho}\vec{D}_\rho-\overleftarrow{D}_\rho\gamma_{\mathrm{F}}^{\rho})\psi-m_{\mathrm{e}}'c^2\bar{\psi}\psi\right],\\
T_{\mathrm{em}}^{\mu\nu}&=\bar{\Psi}\mathrm{t}^{\mu}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^5\Psi=\frac{1}{\mu_0}\left(F_{\rho}^{\mu}F^{\rho\nu}+\frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\\
&=\frac{1}{\mu_0}P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_\rho A_\sigma\partial_\eta A_\lambda.
\end{aligned}
$$

$$
P^{\mu\nu,\rho\sigma}=\frac{1}{2}(g^{\mu\sigma}g^{\rho\nu}+g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma})
$$

$$
I_{\rho\sigma}^{\mu\nu}=\frac{1}{2}(\delta_\rho^{\mu}\delta_\sigma^{\nu}+\delta_\sigma^{\mu}\delta_\rho^{\nu})
$$

$$
\begin{aligned}
P^{\mu\nu,\rho\sigma,\eta\lambda}&=g^{\eta\sigma}g^{\lambda\mu}g^{\nu\rho}-g^{\eta\mu}g^{\lambda\sigma}g^{\nu\rho}-g^{\eta\rho}g^{\lambda\mu}g^{\nu\sigma}+g^{\eta\mu}g^{\lambda\rho}g^{\nu\sigma}\\
&\quad-g^{\mu\sigma}g^{\nu\lambda}g^{\rho\eta}+g^{\mu\sigma}g^{\nu\eta}g^{\rho\lambda}+g^{\mu\rho}g^{\nu\lambda}g^{\sigma\eta}-g^{\mu\rho}g^{\nu\eta}g^{\sigma\lambda}\\
&\quad+\frac{1}{2}g^{\mu\nu}(g^{\eta\rho}g^{\lambda\sigma}-g^{\eta\sigma}g^{\lambda\rho}-g^{\rho\lambda}g^{\sigma\eta}+g^{\rho\eta}g^{\sigma\lambda}).
\end{aligned}
$$

$$
\delta\mathcal{L}=0
$$

$$\delta S_0$$

$$= \int \delta \mathcal{L}_0 d^4x = \int \frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{a\nu} \partial_\nu \delta\phi_a d^4x$$

$$= \int \partial_\nu \left(\frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{a\nu} \delta\phi_a \right) d^4x - \int \partial_\nu \left(\frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{a\nu} \right) \delta\phi_a d^4x$$

$$= -\int \mathring{e}_a^{\mu} \partial_\nu \left(\frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{a\nu} \right) \delta\phi_\mu d^4x$$

$$= -\int \left[\partial_\nu \left(8_a^{\mu} \sqrt{-g} g_{\mathrm{g}} \right. \right.$$

$$\left. \left. T_{\mathrm{m}}^{a\nu} \right) - \frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{a\nu} \left(\partial_\nu \ell_a^{\mu} \right) \right] \delta\phi_\mu d^4x$$

$$= -\int \left[\partial_\nu \left(\frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{\mu\nu} \right) - \frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{\sigma\nu} \left(\mathring{e}^{a}{}_{\sigma} \partial_\nu \mathring{e}_a^{\mu} \right) \right] \delta\phi_\mu d^4x$$

$$= -\int \left[\partial_\nu \left(\frac{\sqrt{-g}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{\mu\nu} \right) + \frac{\sqrt{-g}}{g_{\mathrm{g}}} \mathring{\Gamma}^{\mu}{}_{\sigma\nu} T_{\mathrm{m}}^{\sigma\nu} \right] \delta\phi_\mu d^4x$$

$$= -\int \frac{\sqrt{-g}}{g_{\mathrm{g}}} \left(\partial_\nu T_{\mathrm{m}}^{\mu\nu} + \mathring{\Gamma}^{\mu}{}_{\sigma\nu} T_{\mathrm{m}}^{\sigma\nu} + \mathring{\Gamma}^{\nu}{}_{\sigma\nu} T_{\mathrm{m}}^{\mu\sigma} \right) \delta\phi_\mu d^4x$$

$$= -\int \frac{\sqrt{-g}}{g_{\mathrm{g}}} \left(\mathring{\nabla}_\nu T_{\mathrm{m}}^{\mu\nu} \right) \delta\phi_\mu d^4x$$

$$\mathring{\nabla}_\nu T_{\mathrm{m}}^{\mu\nu} = 0$$

$$T_{\mathrm{m}}{}^{\nu}{}_{\nu} = (2m_{\mathrm{e}}' - m_{\mathrm{e}}) c^2 \bar{\psi}\psi$$

$$\vec{\mathcal{D}}_\nu = \vec{\partial}_\nu - i g_{\mathrm{g}}' H_{a\nu} \mathbf{t}^a$$

$$H_{a\nu} \to H_{a\nu} + \frac{1}{g_{\mathrm{g}}'} \partial_\nu \phi_a$$

$$\mathbf{H}_{\mu\nu} = H_{a\mu\nu} \mathbf{t}^a, H_{a\mu\nu} = \partial_\mu H_{a\nu} - \partial_\nu H_{a\mu}.$$

$$H_{\rho\mu\nu} = e_\rho^a H_{a\mu\nu}$$

$$\widetilde{H}_{\sigma\lambda}^{a} = \frac{1}{2} \varepsilon_{\sigma\lambda\mu\nu} S^{a\mu\nu}$$

$$S^{a\mu\nu} = e^{a}{}_{\rho} \left[\frac{1}{2} \left(H^{\nu\mu\rho} + H^{\mu\rho\nu} - H^{\rho\nu\mu} \right) + g^{\rho\mu} H^{\sigma\nu}{}_{\sigma} - g^{\rho\nu} H^{\sigma\mu}{}_{\sigma} \right].$$

$$\mathcal{L}_{\mathrm{g,kin}} = \frac{1}{8\kappa} H_{a\mu\nu} \widetilde{H}_{\sigma\lambda}^{a} \varepsilon^{\mu\nu\sigma\lambda} \sqrt{-g} = \frac{1}{4\kappa} H_{a\mu\nu} S^{a\mu\nu} \sqrt{-g}.$$

$$H_{a\alpha\beta} \widetilde{H}_{\sigma\lambda}^{a} \varepsilon^{\alpha\beta\sigma\lambda} = \frac{1}{2} H_{a\alpha\beta} S^{a\mu\nu} \varepsilon_{\sigma\lambda\mu\nu} \varepsilon^{\alpha\beta\sigma\lambda}$$

$$= H_{a\mu\nu} S^{a\mu\nu} - H_{a\nu\mu} S^{a\mu\nu} = 2 H_{a\mu\nu} S^{a\mu\nu}.$$

$$\mathcal{L} = \left[\frac{\hbar c}{4} \bar{\psi}_8 \left(\bar{\gamma}_{\mathrm{F}} \bar{\mathbf{I}}_{\mathrm{g}} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^{\nu} \vec{\mathcal{D}}_\nu \mathbf{I}_{\mathrm{g}} \vec{D} - \grave{D}_{\mathrm{I}}^{\mathrm{g}} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^{\nu} \vec{\mathcal{D}}_\nu \mathbf{I}_{\mathrm{g}} \gamma_{\mathrm{F}} \right) \psi_8 \right.$$

$$+ \frac{i m_{\mathrm{e}}' c^2}{2} \bar{\psi}_8 \mathbf{I}_{\mathrm{g}}^{\dagger} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^{\nu} \overline{\vec{\mathcal{D}}}_\nu^{\dagger} \bar{\mathbf{I}}_{\mathrm{g}}^{\dagger} \psi_8 - (2m_{\mathrm{e}}' - m_{\mathrm{e}}) c^2 \bar{\psi}_{\mathrm{g}} \psi_8$$

$$\left. + i \bar{\Psi} \mathbf{I}_{\mathrm{g}}^{\dagger} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^{\nu} \overline{\mathcal{D}}_\nu^{\dagger} \bar{\mathbf{I}}_{\mathrm{g}}^{\dagger} \Psi + \bar{\Psi}\Psi + \frac{1}{4\kappa} H_{a\mu\nu} S^{a\mu\nu} \right] \sqrt{-g}.$$

$$\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\vec{\mathcal{D}}_{\nu}\mathbf{I}_{\mathrm{g}} = \sum_{a} \bar{\mathbf{I}}_{\mathrm{g}}^{(a)}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\left(\vec{\partial}_{\nu} - ig_{\mathrm{g}}'H_{(a)\mathrm{t}}\mathbf{t}^{(a)}\right)\mathbf{I}_{\mathrm{g}}^{(a)}$$

$$= -ig_{\mathrm{g}}\sum_{a}\left(\partial_{\nu}X_{(a)} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{(a)\nu}\right)\bar{\mathbf{I}}_{\mathrm{g}}^{(a)}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{(a)}\mathbf{I}_{\mathrm{g}}^{(a)}$$

$$= -i\left(\partial_{\nu}X_{a} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{a\nu}\right)\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\mathbf{t}^{a}.$$

$$\mathbf{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\bar{\vec{\mathcal{D}}}_{\nu}^{\dagger}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger} = \sum_{a} \mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\left(\vec{\partial}_{\nu} - ig_{\mathrm{g}}'H_{(a)\nu}\bar{\mathbf{t}}^{(a)}\right)\bar{\mathbf{I}}_{\mathrm{g}}^{(a)\dagger}$$

$$= -ig_{\mathrm{g}}\sum_{\nu}\left(\partial_{\nu}X_{(a)} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{(a)\nu}\right)\mathbf{I}_{\mathrm{g}}^{(a)\dagger}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\bar{\mathbf{t}}^{(a)}\bar{\mathbf{I}}_{\mathrm{g}}^{(a)\dagger}$$

$$= -i\left(\partial_{\nu}X_{a} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{a\nu}\right)\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\bar{\mathbf{t}}^{a}$$

$$= i\left(\partial_{\nu}X_{a} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{a\nu}\right)\mathbf{t}^{a}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^{5}.$$

$$\mathcal{L} = \Bigg\{\left(\partial_{\nu}X_{a} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{a\nu}\right)$$
$$\times\left[\frac{i\hbar c}{4}\bar{\psi}_{8}\left(\overleftarrow{\mathcal{D}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{a}\gamma_{\mathrm{F}} - \bar{\gamma}_{\mathrm{F}}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{a}\vec{D}\right)\psi_{8}\right.$$
$$\left.-\frac{m_{\mathrm{e}}'c^{2}}{2}\bar{\psi}_{8}\mathrm{t}^{a}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^{5}\psi_{8} - \bar{\Psi}\mathrm{t}^{a}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^{5}\Psi\right]$$
$$-(2m_{\mathrm{e}}' - m_{\mathrm{e}})c^{2}\bar{\psi}_{8}\psi_{8} + \bar{\Psi}\Psi + \frac{1}{4\kappa}H_{a\mu\nu}S^{a\mu\nu}\Bigg\}\sqrt{-g}$$
$$= \left[-\left(\partial_{\nu}X_{a} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{a\nu}\right)T_{\mathrm{m}}^{a\nu} - (2m_{\mathrm{e}}' - m_{\mathrm{e}})c^{2}\bar{\psi}_{8}\psi_{8} + \bar{\Psi}\Psi\right.$$
$$\left.+\frac{1}{4\kappa}H_{a\mu\nu}S^{a\mu\nu}\right]\sqrt{-g}.$$

$$H_{a\nu} \rightarrow \sqrt{\kappa}H_{a\nu}'$$
$$H_{a\mu\nu} \rightarrow \sqrt{\kappa}H_{a\mu\nu}',$$
$$\tilde{H}_{\sigma\lambda}^{a} \rightarrow \sqrt{\kappa}\tilde{H}_{\sigma\lambda}'^{a}.$$

$$\mathcal{L}_{\mathrm{g,kin}} \rightarrow \frac{1}{8}H_{a\mu\nu}'\tilde{H}_{\sigma\lambda}'^{a}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}.$$

$$E_{\mathrm{g}}' = E_{\mathrm{g}}\sqrt{\frac{\kappa}{\hbar c}} = \sqrt{8\pi\alpha_{\mathrm{g}}}$$

$$\alpha_{\mathrm{g}} = \frac{\kappa c p^{2}}{8\pi\hbar} = \frac{Gp^{2}}{\hbar c^{3}}$$

$$\alpha_{\mathrm{g}} = \frac{\kappa m_{\mathrm{e}}^{2}c^{3}}{8\pi\hbar} = \frac{Gm_{\mathrm{e}}^{2}}{\hbar c}$$

$$\vec{\mathcal{D}}_{\nu} \rightarrow \vec{\partial}_{\nu} - i\frac{E_{\mathrm{g}}'}{\sqrt{\hbar c}}H_{a\nu}'\mathbf{t}^{a}$$

$$\mathring{e}_{\mu}^{\nu} = \partial^{\nu}x_{\mu} = \partial^{\nu}X_{\mu} = \delta_{\mu}^{\nu}$$
$$\eta_{\mu\nu} = \partial_{\nu}x_{\mu} = \partial_{\nu}X_{\mu}$$

$$\int e^{iS[\bar{\psi},\psi,A,H]/(\hbar c)}\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\mathcal{D}H$$

$$S[\bar{\psi},\psi,A,H] = \int \mathcal{L}d^{4}x$$

$$C_{\mathrm{em}}(A) \equiv \partial_{\nu}A^{\nu} = \sqrt{2\mu_{0}}\bar{\mathbf{e}}_{0}\gamma_{\mathrm{B}}^{\nu}\partial_{\nu}\Theta$$

$$1 = \int \delta\left[C_{\text{em}}\left(A^{(\theta)}\right)\right]\det\left[\frac{\delta C_{\text{em}}\left(A^{(\theta)}\right)}{\delta\theta}\right]\mathcal{D}\theta$$

$$A_{\nu}^{(\theta)} = A_{\nu} - \frac{\hbar}{e}\partial_{\nu}\theta$$

$$\int \delta[C_{\text{em}}(A)]\det\left(-\frac{\hbar}{e}\partial^{2}\right)$$
$$\times e^{iS[\bar{\psi},\psi,A,H]/(\hbar c)}\mathcal{D}\theta\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\mathcal{D}H.$$

$$\delta[C_{\text{em}}(A)]$$
$$= N_{\text{em}}\int \exp\left(\frac{-i}{2\mu_{0}\hbar c\xi_{\text{e}}}\int w^{2}d^{4}x\right)\delta[C_{\text{em}}(A) - w]\mathcal{D}w$$
$$= N_{\text{em}}\exp\left(\frac{-i}{2\mu_{0}\hbar c\xi_{\text{e}}}\int [C_{\text{em}}(A)]^{2}d^{4}x\right)$$
$$= N_{\text{em}}\exp\left(\frac{i}{\hbar c}\int \mathcal{L}_{\text{em,gf}}d^{4}x\right)$$

$$\mathcal{L}_{\text{em,gf}} = -\frac{1}{2\mu_{0}\xi_{\text{e}}}[C_{\text{em}}(A)]^{2} = -\frac{1}{2\mu_{0}\xi_{\text{e}}}(\partial_{\nu}A^{\nu})^{2}$$
$$= -\frac{1}{\xi_{\text{e}}}\bar{\Theta}\bar{\partial}_{\rho}\gamma_{\text{B}}^{\rho}\mathrm{e}_{0}\bar{\mathrm{e}}_{0}\gamma_{\text{B}}^{\sigma}\vec{\partial}_{\sigma}\Theta.$$

$$\det\left(-\frac{\hbar}{e}\partial^{2}\right)$$
$$= \int \exp\left(i\int \bar{c}_{\text{em}}\partial^{2}c_{\text{em}}d^{4}x\right)\mathcal{D}c_{\text{em}}\mathcal{D}\bar{c}_{\text{em}}$$
$$= \int \exp\left(\frac{i}{\hbar c}\int \mathcal{L}_{\text{em,ghost}}d^{4}x\right)\mathcal{D}c_{\text{em}}\mathcal{D}\bar{c}_{\text{em}}$$

$$\mathcal{L}_{\text{em,ghost}} = \hbar c\bar{c}_{\text{em}}\partial^{2}c_{\text{em}} = -\hbar c\bar{c}_{\text{em8}}\partial^{2}c_{\text{em8}}$$

$$C_{\text{g}}^{\mu}(H) \equiv \partial_{\rho}H^{\mu\rho} + \partial_{\rho}H^{\rho\mu} - \partial^{\mu}H^{\rho}{}_{\rho}$$
$$= 2P^{\alpha\beta,\rho\mu}\partial_{\rho}H_{\alpha\beta} = 0.$$

$$1 = \prod_{\mu}\int \delta\left[C_{\text{g}}^{(\mu)}\left(H^{(\phi)}\right)\right]\det\left[\frac{\delta C_{\text{g}}^{(\mu)}\left(H^{(\phi)}\right)}{\delta\phi^{(\mu)}}\right]\mathcal{D}\phi_{(\mu)}$$

$$H_{\mu\nu}^{(\phi)} = H_{\mu\nu} + \frac{1}{g_{\text{g}}'}\partial_{\nu}\phi_{\mu}$$

$$\int \left\{\prod_{\mu}\delta[C_{\text{g}}^{\mu}(H)]\det\left(-\frac{1}{g_{\text{g}}'}\partial^{2}\right)\right\}$$
$$\times e^{iS[\bar{\psi},\psi,A,H]/(\hbar c)}\mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\mathcal{D}H.$$

$$\prod_{\mu}\delta\left[C_{\text{g}}^{\mu}(H)\right] = N_{\text{g}}\prod_{\mu}\int \exp\left(\frac{i}{4\kappa\hbar c\xi_{\text{g}}}\int w^{(\mu)}w_{(\mu)}d^{4}x\right)$$
$$\times \delta\left[C_{\text{g}}^{(\mu)}(H) - w^{(\mu)}\right]\mathcal{D}w^{(\mu)}$$
$$= N_{\text{g}}\exp\left(\frac{i}{4\kappa\hbar c\xi_{\text{g}}}\int C_{\text{g}}^{\mu}(H)C_{\text{g}\mu}(H)d^{4}x\right)$$
$$= N_{\text{g}}\exp\left(\frac{i}{\hbar c}\int \mathcal{L}_{\text{g,gf}}d^{4}x\right).$$

$$\mathcal{L}_{\mathrm{g,gf}} = \frac{1}{4\kappa\xi_{\mathrm{g}}} C_{\mathrm{g}}^{\mu}(H) C_{\mathrm{g}\mu}(H)$$

$$= \frac{1}{4\kappa\xi_{\mathrm{g}}} \left(\partial_{\rho} H^{\mu\rho} + \partial_{\rho} H^{\rho\mu} - \partial^{\mu} H_{\rho}^{\rho}\right)$$

$$\times \left(\partial_{\eta} H_{\mu}{}^{\eta} + \partial_{\eta} H_{\mu}^{\eta} - \partial_{\mu} H^{\eta}{}_{\eta}\right)$$

$$= \frac{1}{\kappa\xi_{\mathrm{g}}} \eta_{\gamma\delta} P^{\alpha\beta,\lambda\gamma} P^{\rho\sigma,\eta\delta} \partial_{\lambda} H_{\alpha\beta} \partial_{\eta} H_{\rho\sigma}.$$

$$\prod_{\mu} \det\left(\frac{1}{g_{\mathrm{g}}'} \partial^2\right)$$

$$= \int \exp\left(-i \int \bar{c}_{\mathrm{g}} \partial^2 c_{\mathrm{g}} d^4x\right) \mathcal{D}c_{\mathrm{g}} \mathcal{D}\bar{c}_{\mathrm{g}}$$

$$= \int \exp\left(\frac{i}{\hbar c} \int \mathcal{L}_{\mathrm{g,ghost}} d^4x\right) \mathcal{D}c_{\mathrm{g}} \mathcal{D}\bar{c}_{\mathrm{g}}.$$

$$\mathcal{L}_{\mathrm{g,\,ghost}} = -\hbar c \bar{c}_{\mathrm{g}} \partial^2 c_{\mathrm{g}} = \hbar c \bar{c}_{\mathrm{g}8} \partial^2 c_{\mathrm{g}8}$$

$$\mathcal{L}_{\mathrm{FP}} = \mathcal{L} + \mathcal{L}_{\mathrm{em,gf}} + \mathcal{L}_{\mathrm{em,ghost}} + \mathcal{L}_{\mathrm{g,gf}} + \mathcal{L}_{\mathrm{g,ghost}}$$

$$\psi \to e^{i\theta' c_{\mathrm{em}} Q} \psi$$

$$A_{\nu} \to A_{\nu} - \frac{\hbar}{e} \theta' \vec{\partial}_{\nu} c_{\mathrm{em}}$$

$$\bar{c}_{\mathrm{em}} \to \bar{c}_{\mathrm{em}} - \frac{1}{\mu_0 c e \xi_{\mathrm{e}}} \theta' C_{\mathrm{em}}(A)$$

$$c_{\mathrm{em}} \to c_{\mathrm{em}}$$

$$\mathbf{I}_{\mathrm{g}} \to \left(\bigotimes_{\mu} e^{i\phi' c_{\mathrm{g}(\mu)} \mathbf{t}^{(\mu)}}\right) \mathbf{I}_{\mathrm{g}}$$

$$X_{\mu} \to X_{\mu} - \frac{1}{g_{\mathrm{g}}} \phi' c_{\mathrm{g}\mu}$$

$$H_{\mu\nu} \to H_{\mu\nu} + \frac{1}{g_{\mathrm{g}}'} \phi' \vec{\partial}_{\nu} c_{\mathrm{g}\mu}$$

$$\bar{c}_{\mathrm{g}}^{\mu} \to \bar{c}_{\mathrm{g}}^{\mu} - \frac{1}{\kappa\hbar c g_{\mathrm{g}}' \xi_{\mathrm{g}}} \phi' C_{\mathrm{g}}^{\mu}(H)$$

$$c_{\mathrm{g}}^{\mu} \to c_{\mathrm{g}}^{\mu}$$

$$\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial c_{\mathrm{em}}} - \partial_{\rho}\left[\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial(\partial_{\rho} c_{\mathrm{em}})}\right] + \partial_{\rho}\partial_{\sigma}\left[\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial(\partial_{\rho}\partial_{\sigma} c_{\mathrm{em}})}\right] = 0$$

$$\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial \bar{c}_{\mathrm{em}}} - \partial_{\rho}\left[\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial(\partial_{\rho} \bar{c}_{\mathrm{em}})}\right] + \partial_{\rho}\partial_{\sigma}\left[\frac{\partial \mathcal{L}_{\mathrm{em,ghost}}}{\partial(\partial_{\rho}\partial_{\sigma} \bar{c}_{\mathrm{em}})}\right] = 0.$$

$$\partial^2 \bar{c}_{\mathrm{em}} = 0, \partial^2 c_{\mathrm{em}} = 0$$

$$\partial^2 \bar{c}_{\mathrm{g}} = 0, \partial^2 c_{\mathrm{g}} = 0$$

$$\mathcal{L}_{\mathrm{UGM}} = \mathcal{L}_{\mathrm{D,kin}} + \mathcal{L}_{\mathrm{em,kin}} + \mathcal{L}_{\mathrm{em,int}} + \mathcal{L}_{\mathrm{g,kin}} + \mathcal{L}_{\mathrm{g,\,int}}$$
$$+ \mathcal{L}_{\mathrm{em\,,gf}} + \mathcal{L}_{\mathrm{g,gf}}.$$

$$\mathcal{L}_{\mathrm{D,kin}} = \frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu} - \bar{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)\psi - m_{\mathrm{e}}c^{2}\bar{\psi}\psi,$$

$$\mathcal{L}_{\mathrm{em,kin}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = \bar{\Psi}\Psi$$

$$= \bar{\Theta}\bar{\partial}_{\rho}\gamma_{\mathrm{B}}^{\rho}(\mathbf{I}_8 + \mathrm{e}_0\bar{\mathrm{e}}_0)^2\gamma_{\mathrm{B}}^{\sigma}\vec{\partial}_{\sigma}\Theta,$$

$$\mathcal{L}_{\mathrm{em,int}} = -J_{\mathrm{e}}^{\nu}A_{\nu} = -q_{\mathrm{e}}c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi A_{\nu} = \bar{\Phi}\Theta + \bar{\Theta}\Phi,$$

$$\mathcal{L}_{\mathrm{g,kin}} = \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu},$$

$$\mathcal{L}_{\mathrm{g,int}} = -\frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}T_{\mathrm{m}}^{\mu\nu}H_{\mu\nu}$$

$$= -\frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}\left\{\frac{c}{2}P^{\mu\nu,\rho\sigma}\left[i\hbar\bar{\psi}\left(\gamma_{\mathrm{F}\rho}\vec{\partial}_{\sigma} - \bar{\partial}_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi\right.\right.$$

$$\left. -q_{\mathrm{e}}\bar{\psi}\left(\gamma_{\mathrm{F}\rho}A_{\sigma} + A_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi - m_{\mathrm{e}}'cg_{\rho\sigma}\bar{\psi}\psi\right]$$

$$\left. + \frac{1}{\mu_0}P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_{\rho}A_{\sigma}\partial_{\eta}A_{\lambda}\right\}H_{\mu\nu},$$

$$\mathcal{L}_{\mathrm{em,gf}} = -\frac{1}{2\mu_0\xi_{\mathrm{e}}}(\partial_{\nu}A^{\nu})^2 = -\frac{1}{\xi_{\mathrm{e}}}\bar{\Theta}\bar{\partial}_{\rho}\gamma_{\mathrm{B}}^{\rho}\mathrm{e}_0\bar{\mathbf{e}}_0\gamma_{\mathrm{B}}^{\sigma}\vec{\partial}_{\sigma}\Theta,$$

$$\mathcal{L}_{\mathrm{g,gf}} = \frac{1}{\kappa\xi_{\mathrm{g}}}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_{\lambda}H_{\alpha\beta}\partial_{\eta}H_{\rho\sigma}.$$

$$\Psi' = -\gamma_{\mathrm{B}}^{\rho}\partial_{\rho}\Theta$$

$$\mathcal{L}_{\mathrm{em,kin}}^{\prime(\xi_{\mathrm{e}}=1)} = \bar{\Psi}'\Psi' = \mathcal{L}_{\mathrm{em,kin}} + \mathcal{L}_{\mathrm{em,gf}}^{(\xi_{\mathrm{e}}=1)}$$

$$= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\mu_0}(\partial_{\nu}A^{\nu})^2.$$

$$\mathcal{L}_{\mathrm{em,kin}}' = \mathcal{L}_{\mathrm{em,kin}} + \mathcal{L}_{\mathrm{em,gf}}$$

$$= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\mu_0\xi_{\mathrm{e}}}(\partial_{\nu}A^{\nu})^2.$$

$$\mathcal{L}_{\mathrm{g,kin}}^{\prime(\xi_{\mathrm{g}}=1)} = \mathcal{L}_{\mathrm{g,kin}} + \mathcal{L}_{\mathrm{g,gf}}^{(\xi_{\mathrm{g}}=1)}$$

$$= \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu} + \frac{1}{\kappa}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_{\lambda}H_{\alpha\beta}\partial_{\eta}H_{\rho\sigma}.$$

$$\mathcal{L}_{\mathrm{g,kin}}' = \mathcal{L}_{\mathrm{g,kin}} + \mathcal{L}_{\mathrm{g,gf}}$$

$$= \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu} + \frac{1}{\kappa\xi_{\mathrm{g}}}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_{\lambda}H_{\alpha\beta}\partial_{\eta}H_{\rho\sigma}.$$

$$\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial H_{\mu\nu}} - \partial_{\rho}\left[\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\left(\partial_{\rho}H_{\mu\nu}\right)}\right] = 0$$

$$-P^{\mu\nu,\rho\sigma}\partial^2 H_{\rho\sigma} = \kappa T_{\mathrm{m}}^{\mu\nu}.$$

$$\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\bar{\Theta}} - \partial_{\rho}\left[\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\left(\partial_{\rho}\bar{\Theta}\right)}\right] = 0$$

$$\partial^2\Theta = \Phi + \gamma_{\mathrm{B}}^{\rho}(\mathbf{I}_8 + \mathrm{e}_0\bar{\mathrm{e}}_0)\mathrm{t}^{\mu}\gamma_{\mathrm{B}}^{\nu}\gamma_{\mathrm{B}}^{5}(\mathbf{I}_8 + \mathrm{e}_0\bar{\mathrm{e}}_0)$$

$$\times\partial_{\rho}\left(H_{\mu\nu}\gamma_{\mathrm{B}}^{\sigma}\partial_{\sigma}\Theta\right) + \frac{1}{2}\gamma_{\mathrm{B}}^{5}\gamma_{\mathrm{B}}^{\nu}\mathrm{t}^{\mu}\Phi H_{\mu\nu}$$

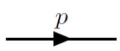
$$\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial A_{\sigma}} - \partial_{\rho}\left[\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\left(\partial_{\rho}A_{\sigma}\right)}\right] = 0$$

$$\partial^2 A^{\sigma} = \mu_0 J_{\mathrm{e}}^{\sigma} - 2P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_{\rho}\left(H_{\mu\nu}\partial_{\eta}A_{\lambda}\right)$$

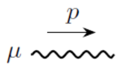
$$-\mu_0 P^{\mu\nu,\rho\sigma}J_{\mathrm{e}\rho}H_{\mu\nu}$$

$$\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\bar{\psi}} - \partial_{\rho}\left[\frac{\partial\mathcal{L}_{\mathrm{UGM}}}{\partial\left(\partial_{\rho}\bar{\psi}\right)}\right] = 0$$

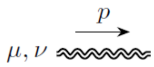
$$i\hbar c\gamma_{\mathrm{F}}^{\rho}\vec{\partial}_{\rho}\psi - m_{\mathrm{e}}c^{2}\psi = q_{\mathrm{e}}c\gamma_{\mathrm{F}}^{\rho}\psi A_{\rho}$$
$$+P^{\mu\nu,\rho\sigma}\left(i\hbar c\gamma_{\mathrm{F}\sigma}\vec{\partial}_{\rho}\psi - \frac{m_{\mathrm{e}}c^{2}}{2}\eta_{\rho\sigma}\psi + \frac{i\hbar c}{2}\gamma_{\mathrm{F}\sigma}\psi\vec{\partial}_{\rho}\right.$$
$$\left.-q_{\mathrm{e}}c\gamma_{\mathrm{F}\sigma}\psi A_{\rho} + \frac{q_{\mathrm{e}}c}{4}\eta_{\rho\sigma}\gamma_{\mathrm{F}}^{\lambda}A_{\lambda}\psi\right)H_{\mu\nu}.$$
$$\int e^{iS_{\mathrm{D,kin}}[\bar{\psi},\psi]/(\hbar c)}\mathcal{D}\bar{\psi}\mathcal{D}\psi$$
$$S_{\mathrm{D,kin}}[\bar{\psi},\psi] = \int \mathcal{L}_{\mathrm{D,kin}}d^{4}x = c\int \bar{\psi}(x)(i\hbar\,\partial\!\!\!/ - m_{\mathrm{e}}c)\psi(x)d^{4}x.$$
$$(i\hbar\,\partial\!\!\!/ - m_{\mathrm{e}}c)D_{\mathrm{D}}(x_{1}-x_{2}) = i\mathbf{I}_{4}\delta^{4}(x_{1}-x_{2})$$
$$(p\!\!\!/ - m_{\mathrm{e}}c\mathbf{I}_{4})D_{\mathrm{D}}(p) = i\mathbf{I}_{4}.$$
$$\tilde{D}_{\mathrm{D}}(p) = \frac{i(p\!\!\!/ + m_{\mathrm{e}}c\mathbf{I}_{4})}{p^{2}-m_{\mathrm{e}}^{2}c^{2}+i\epsilon}$$
$$\int e^{iS'_{\mathrm{em,kin}}(\xi_{\mathrm{e}}=1)}[A]/(\hbar c)\mathcal{D}A$$
$$S'^{(\xi_{\mathrm{e}}=1)}_{\mathrm{em,kin}}[A] = \int \mathcal{L}'^{(\xi_{\mathrm{e}}=1)}_{\mathrm{em,kin}}d^{4}x$$
$$= \frac{1}{2\mu_{0}\hbar^{2}}\int A_{\mu}(x)(\eta^{\mu\nu}\hbar^{2}\partial^{2})A_{\nu}(x)d^{4}x.$$



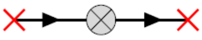
$$= \frac{i(p\!\!\!/ + m_{\mathrm{e}}c\mathbf{I}_{4})}{p^{2}-m_{\mathrm{e}}^{2}c^{2}+i\epsilon}$$



$$= \frac{-i}{p^{2}+i\epsilon}\times\left[\eta_{\mu\nu} - (1-\xi_{\mathrm{e}})\frac{p_{\mu}p_{\nu}}{p^{2}+i\epsilon}\right]$$



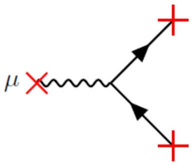
$$= \frac{i}{p^{2}+i\epsilon}\times\left[P^{(D)}_{\mu\nu,\rho\sigma} - \frac{1-\xi_{\mathrm{g}}}{p^{2}+i\epsilon}(\eta_{\alpha\rho}p_{\beta}p_{\sigma} + \eta_{\alpha\sigma}p_{\beta}p_{\rho} + \eta_{\beta\rho}p_{\alpha}p_{\sigma} + \eta_{\beta\sigma}p_{\alpha}p_{\rho})\right]$$



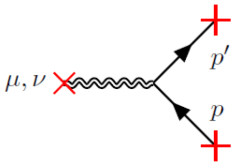
$$= i[(Z_{\psi}-1)p\!\!\!/ - (Z_{\psi}Z_{\mathrm{m}}-1)m_{\mathrm{e}}c\mathbf{I}_{4}]$$

$$= -i(Z_{\mathrm{A}}-1)(p^{2}\eta^{\mu\nu} - p^{\mu}p^{\nu})$$

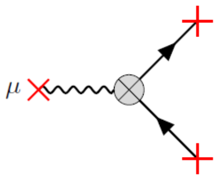
$$= i(Z_{H}-1)p^{2}\hat{P}^{\alpha\beta\eta\lambda}_{1,2,1}$$



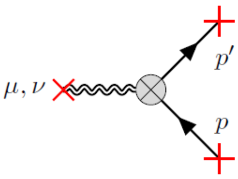
$$= \frac{-iq_{\mathrm{e}}\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_{0}\hbar c}}$$



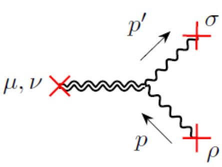
$$= -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\rho\sigma}\times[\gamma_{\mathrm{F}\rho}(p+p')_{\sigma} - m'_{\mathrm{e}}c\eta_{\rho\sigma}\mathbf{I}_{4}]$$



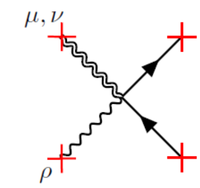
$$= (Z_{\psi}-1)\frac{-iq_{\mathrm{e}}\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_{0}\hbar c}}$$



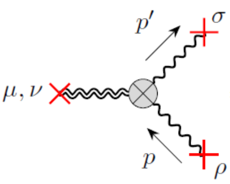
$$= -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\rho\sigma}[(Z_{\mathrm{g}\psi}-1)\gamma_{\mathrm{F}\rho}(p+p')_{\sigma} - (Z_{\mathrm{g}\psi}Z_{\mathrm{gm}}-1)m'_{\mathrm{e}}c\eta_{\rho\sigma}\mathbf{I}_{4}]$$



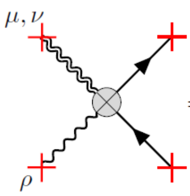
$$= -i\sqrt{\frac{\kappa c}{\hbar}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\sigma\eta,\rho\lambda}p'_{\eta}p_{\lambda}$$



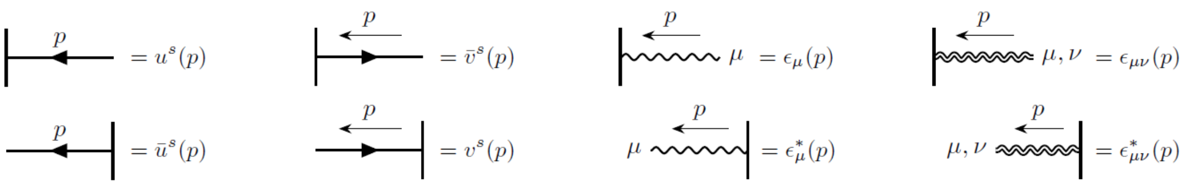
$$= \frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_{0}}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\rho\sigma}\gamma_{\mathrm{F}\sigma}$$



$$= -i(Z_{\mathrm{gA}}-1)\sqrt{\frac{\kappa c}{\hbar}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\sigma\eta,\rho\lambda}p'_{\eta}p_{\lambda}$$



$$= (Z_{\mathrm{g}\psi}-1)\frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_{0}}}\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}}P^{\mu\nu,\rho\sigma}\gamma_{\mathrm{F}\sigma}$$



$$\eta^{\mu\nu}\hbar^2\partial^2 D_{\nu\rho}^{(\text{em},\xi_\text{e}=1)}(x_1-x_2)=i\delta_\rho^\mu\delta^4(x_1-x_2)$$

$$-\eta^{\mu\nu}p^2\tilde{D}_{\nu\rho}^{(\text{em},\xi_\text{e}=1)}(p)=i\delta_\rho^\mu$$

$$\tilde{D}_{\nu\rho}^{(\text{em},\xi_\text{e}=1)}(p)=\frac{-i\eta_{\nu\rho}}{p^2+i\epsilon}$$

$$\tilde{D}_{\nu\rho}^{(\text{em})}(p)=\frac{-i}{p^2+i\epsilon}\left[\eta_{\nu\rho}-(1-\xi_\text{e})\frac{p_\nu p_\rho}{p^2+i\epsilon}\right]$$

$$\int e^{iS_{\text{g,kin}}'^{(\xi_\text{e}=1)}[H]/(\hbar c)}\mathcal{D}H$$

$$S_{\text{g,kin}}'^{(\xi_\text{o}=1)}[H]=\int \mathcal{L}_{\text{g, kin}}'^{(\xi_\text{o}=1)}d^4x$$

$$=\frac{1}{\kappa\hbar^2}\int H_{\mu\nu}(x)\left(-P^{\mu\nu,\alpha\beta}\hbar^2\partial^2\right)H_{\alpha\beta}(x)d^4x.$$

$$-P^{\mu\nu,\alpha\beta}\hbar^2\partial^2 D_{\alpha\beta,\rho\sigma}^{(\text{g},\xi_\text{g}=1)}(x_1-x_2)=iI_{\rho\sigma}^{\mu\nu}\delta^4(x_1-x_2)$$

$$P^{\mu\nu,\alpha\beta}p^2\tilde{D}_{\alpha\beta,\rho\sigma}^{(\text{g},\xi_\text{g}=1)}(p)=iI_{\rho\sigma}^{\mu\nu}$$

$$\tilde{D}_{\alpha\beta,\rho\sigma}^{(\text{g},\xi_\text{g}=1)}(p)=\frac{iP_{\alpha\beta,\rho\sigma}^{(D)}}{p^2+i\epsilon}$$

$$P_{\mu\nu,\rho\sigma}^{(D)}=\frac{1}{2}\left(\eta_{\mu\sigma}\eta_{\rho\nu}+\eta_{\mu\rho}\eta_{\nu\sigma}-\frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)$$

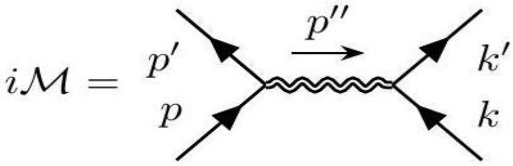
$$\tilde{D}_{\alpha\beta,\rho\sigma}^{(\text{g})}(p)=\frac{i}{p^2+i\epsilon}\Big[P_{\alpha\beta,\rho\sigma}^{(D)}-\frac{1-\xi_\text{g}}{p^2+i\epsilon}(\eta_{\alpha\rho}p_\beta p_\sigma$$

$$+\eta_{\alpha\sigma}p_\beta p_\rho+\eta_{\beta\rho}p_\alpha p_\sigma+\eta_{\beta\sigma}p_\alpha p_\rho)\Big].$$

$$\mathcal{L}_{\text{em,int}}=-J_\text{e}^\mu A_\mu=-i\sqrt{\frac{\hbar c}{\mu_0}}\bar{\psi}\left(\frac{-iq_\text{e}\gamma_\text{F}^\mu}{\sqrt{\varepsilon_0\hbar c}}\right)\psi A_\mu$$

$$=-i\tilde{u}_0^2\tilde{A}_0\sqrt{\frac{\hbar c}{\mu_0}}\bar{u}(p')\left(\frac{-iq_\text{e}\gamma_\text{F}^\mu}{\sqrt{\varepsilon_0\hbar c}}\right)u(p)\epsilon_\mu(k).$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{g,\,int}} &= -\frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} T_{\mathrm{m}}^{\mu\nu} H_{\mu\nu} \\
&= -i\sqrt{\frac{\hbar c}{\kappa}} \left\{ -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\rho\sigma} \right. \\
&\times \left[i\hbar\bar{\psi}\gamma_{\mathrm{F}\rho}(\vec{\partial}_\sigma - \bar{\partial}_\sigma)\psi - m'_{\mathrm{e}} c \eta_{\rho\sigma}\bar{\psi}\psi \right] \Big\} H_{\mu\nu} \\
&- \frac{i\hbar}{\sqrt{\mu_0\kappa}} \bar{\psi} \left\{ \frac{iq_{\mathrm{e}}}{\hbar} \sqrt{\frac{\kappa}{\varepsilon_0} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\rho\sigma} \gamma_{\mathrm{F}\sigma}} \right\} \psi A_\rho H_{\mu\nu} \\
&- i\epsilon_0 \hbar c \sqrt{\frac{c}{\kappa\hbar}} \left\{ -i\sqrt{\frac{\kappa c}{\hbar}} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\sigma\eta,\rho\lambda} \partial_\eta A_\sigma \partial_\lambda A_\rho \right\} H_{\mu\nu} \\
&= -i\tilde{u}_0^2 \tilde{H}_0 \sqrt{\frac{\hbar c}{\kappa}} \bar{u}(p') \left\{ -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\rho\sigma} \right. \\
&\times \left[\gamma_{\mathrm{F}\rho}(p+p')_\sigma - m'_{\mathrm{e}} c \eta_{\rho\sigma} \mathbf{I}_4 \right] \Big\} u(p) \epsilon_{\mu\nu}(q) \\
&- \frac{i\hbar\tilde{u}_0^2 \tilde{A}_0 \tilde{H}_0}{\sqrt{\mu_0\kappa}} \bar{u}(p') \left\{ \frac{iq_{\mathrm{e}}}{\hbar} \sqrt{\frac{\kappa}{\varepsilon_0}} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\rho\sigma} \gamma_{\mathrm{F}\sigma} \right\} u(p) \epsilon_\rho(k) \epsilon_{\mu\nu}(q) \\
&- i\tilde{A}_0^2 \tilde{H}_0 \sqrt{\frac{\epsilon_0^2 c^3}{\kappa\hbar^3}} \epsilon_\sigma^*(k') \left\{ -i\sqrt{\frac{\kappa c}{\hbar}} \frac{g'_{\mathrm{g}}}{g_{\mathrm{g}}} P^{\mu\nu,\sigma\eta,\rho\lambda} k'_\eta k_\lambda \right\} \\
&\times \epsilon_\rho(k) \epsilon_{\mu\nu}(q).
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{iP_{\alpha\mu,\beta\nu}}{(p-p')^2 + i\epsilon} \\
&\times \bar{u}(p') \left\{ -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}} P^{\alpha\mu,}{}_{\rho\sigma} \left[\gamma_{\mathrm{F}}^\rho (p+p')^\sigma - m_{\mathrm{e}} c \eta^{\rho\sigma} \mathbf{I}_4 \right] \right\} u(p) \\
&\times \bar{u}(k') \left\{ -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}} P^{\beta\nu,}{}_{\lambda\eta} \left[\gamma_{\mathrm{F}}^\lambda (k+k')^\eta - m_{\mathrm{e}} c \eta^{\lambda\eta} \mathbf{I}_4 \right] \right\} u(k) \\
&= -\frac{i\kappa c}{4\hbar} \frac{P_{\rho\sigma,\lambda\eta}}{(p-p')^2 + i\epsilon} \bar{u}(p') \left[\gamma_{\mathrm{F}}^\rho (p+p')^\sigma - m_{\mathrm{e}} c \eta^{\rho\sigma} \mathbf{I}_4 \right] u(p) \\
&\times \bar{u}(k') \left[\gamma_{\mathrm{F}}^\lambda (k+k')^\eta - m_{\mathrm{e}} c \eta^{\lambda\eta} \mathbf{I}_4 \right] u(k).
\end{aligned}
$$

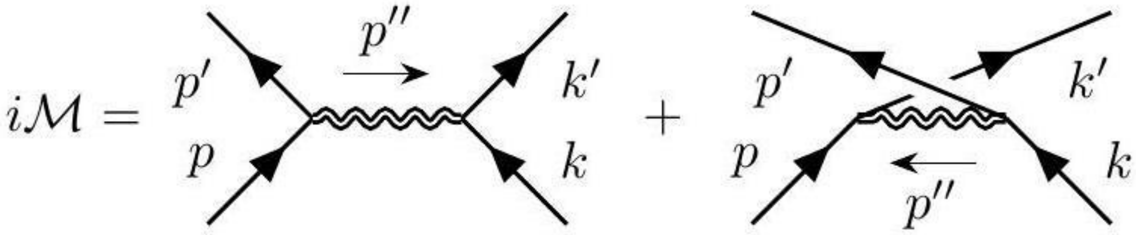
$$
\begin{gathered}
\bar{u}^{s'}(p') \gamma_{\mathrm{F}}^\rho u^s(p) \to 2m_{\mathrm{e}} c \delta^{ss'} \delta^{\rho 0}, \bar{u}^{s'}(p') u^s(p) \to 2m_{\mathrm{e}} c \delta^{ss'}, \\
\bar{u}^{r'}(k') \gamma_{\mathrm{F}}^\lambda u^r(k) \to 2m_{\mathrm{e}} c \delta^{rr'} \delta^{\lambda 0}, \bar{u}^{r'}(k') u^r(k) \to 2m_{\mathrm{e}} c \delta^{rr'}. \\
(p+p')^\sigma = (k+k')^\sigma = 2m_{\mathrm{e}} c \delta^{\sigma 0} + \mathcal{O}(|\mathbf{p}''|), \\
(p-p')^2 = (k'-k)^2 = -|\mathbf{p}''|^2 + \mathcal{O}(|\mathbf{p}''|^4).
\end{gathered}
$$

$$
\begin{aligned}
i\mathcal{M} &= \frac{i\kappa m_{\mathrm{e}}^4 c^5}{\hbar} \frac{\delta^{ss'} \delta^{rr'} P_{\rho\sigma,\lambda\eta}}{|\mathbf{p}''|^2 - i\epsilon} \left(2\delta^{\rho 0}\delta^{\sigma 0} - \eta^{\rho\sigma} \right) \left(2\delta^{\lambda 0}\delta^{\eta 0} - \eta^{\lambda\eta} \right) \\
&= \frac{2i\kappa m_{\mathrm{e}}^4 c^5}{\hbar} \frac{\delta^{ss'} \delta^{rr'}}{|\mathbf{p}''|^2 - i\epsilon}.
\end{aligned}
$$

$$
\tilde{V}_{\mathrm{g}}(\mathbf{p}'') = -\frac{\hbar^3}{4m_{\mathrm{e}}^2 c} \sum_{s',r'} \mathcal{M} = -\frac{\kappa\hbar^2 m_{\mathrm{e}}^2 c^4}{2(|\mathbf{p}''|^2 - i\epsilon)}
$$

$$V_{\mathrm{g}}(\mathbf{r}) = \int \tilde{V}_{\mathrm{g}}(\mathbf{p}'')e^{i\mathbf{p}''\cdot\mathbf{r}/\hbar}\frac{d^3p''}{(2\pi\hbar)^3}$$
$$= -\frac{\kappa m_{\mathrm{e}}^2 c^4}{2(2\pi)^3\hbar}\int \frac{e^{i\mathbf{p}''\cdot\mathbf{r}/\hbar}}{|\mathbf{p}''|^2 - i\epsilon}d^3p''$$
$$= -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi|\mathbf{r}|}e^{i|\mathbf{r}|\sqrt{i\epsilon}/\hbar}.$$

$$V_{\mathrm{g}}(\mathbf{r}) = -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi|\mathbf{r}|} = -\frac{G m_{\mathrm{e}}^2}{|\mathbf{r}|} = -\frac{\hbar c \alpha_{\mathrm{g}}}{|\mathbf{r}|}.$$



$$=\frac{iP_{\alpha\mu,\beta\nu}}{(p-p')^2}$$
$$\times \bar{u}(p')\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\mu,}\left[\gamma^{\rho}_{\rho\sigma}(p+p')^{\sigma} - m_{\mathrm{e}}c\eta^{\rho\sigma}\mathbf{I}_4\right]\right\}u(p)$$
$$\times \bar{u}(k')\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\beta\nu,\,\lambda\eta}[\left[\gamma^{\lambda}_{\mathrm{F}}(k+k')^{\eta} - m_{\mathrm{e}}c\eta^{\lambda\eta}\mathbf{I}_4\right]\right\}u(k)$$
$$-\frac{iP_{\alpha\mu,\beta\nu}}{(p-k')^2}$$
$$\times \bar{u}(k')\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\mu,}{}_{\rho\sigma}\left[\gamma^{\rho}_{\mathrm{F}}(p+k')^{\sigma} - m_{\mathrm{e}}c\eta^{\rho\sigma}\mathbf{I}_4\right]\right\}u(p)$$
$$\times \bar{u}(p')\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\beta\nu,\,\lambda\eta}\left[\gamma^{\lambda}_{\mathrm{F}}(k+p')^{\eta} - m_{\mathrm{e}}c\eta^{\lambda\eta}\mathbf{I}_4\right]\right\}u(k)$$
$$= -\frac{i\kappa c}{4\hbar}P_{\rho\sigma,\lambda\eta}\left\{\frac{1}{t}\bar{u}(p')\left[\gamma^{\rho}_{\mathrm{F}}(p+p')^{\sigma} - m_{\mathrm{e}}c\eta^{\rho\sigma}\mathbf{I}_4\right]u(p)\right.$$
$$\times \bar{u}(k')\left[\gamma^{\lambda}_{\mathrm{F}}(k+k')^{\eta} - m_{\mathrm{e}}c\eta^{\lambda\eta}\mathbf{I}_4\right]u(k)$$
$$-\frac{1}{u}\bar{u}(k')\left[\gamma^{\rho}_{\mathrm{F}}(p+k')^{\sigma} - m_{\mathrm{e}}c\eta^{\rho\sigma}\mathbf{I}_4\right]u(p)$$
$$\left.\times \bar{u}(p')\left[\gamma^{\lambda}_{\mathrm{F}}(k+p')^{\eta} - m_{\mathrm{e}}c\eta^{\lambda\eta}\mathbf{I}_4\right]u(k)\right\}.$$

$$s = (p+k)^2 = (p'+k')^2 = E_{\mathrm{cm}}^2/c^2,$$
$$t = (p-p')^2 = (k'-k)^2 = -2p_{\mathrm{r}}^2(1-\cos\,\theta_{\mathrm{r}}),$$
$$u = (p-k')^2 = (p'-k)^2 = -2p_{\mathrm{r}}^2(1+\cos\,\theta_{\mathrm{r}}).$$
$$p^{\mu} = (E_{\mathrm{r}}/c, 0,0, p_{\mathrm{r}}),$$
$$k^{\mu} = (E_{\mathrm{r}}/c, 0,0, -p_{\mathrm{r}}),$$
$$p^{\mu} = (E_{\mathrm{r}}/c, p_{\mathrm{r}}\sin\,\theta_{\mathrm{r}}, 0, p_{\mathrm{r}}\cos\,\theta_{\mathrm{r}}),$$
$$k'^{\mu} = (E_{\mathrm{r}}/c, -p_{\mathrm{r}}\sin\,\theta_{\mathrm{r}}, 0, -p_{\mathrm{r}}\cos\,\theta_{\mathrm{r}}).$$

$$
\begin{aligned}
|\mathcal{M}|^2 = &\frac{\kappa^2 c^2}{16\hbar^2} P_{\rho\sigma,\lambda\eta} P_{\alpha\beta,\gamma\delta} \\
&\times \Big(\frac{1}{t^2} \mathrm{Tr}\{[\gamma_\mathrm{F}^\rho (p+p')^\sigma - m_\mathrm{e} c \eta^{\rho\sigma} \mathbf{I}_4] \mathbf{u}_p \\
&\times [\gamma_\mathrm{F}^\alpha (p+p')^\beta - m_\mathrm{e} c \eta^{\alpha\beta} \mathbf{I}_4] \mathbf{u}_{p'}\} \\
&\times \mathrm{Tr}\{[\gamma_\mathrm{F}^\lambda (k+k')^\eta - m_\mathrm{e} c \eta^{\lambda\eta} \mathbf{I}_4] \mathbf{u}_k \\
&\times [\gamma_\mathrm{F}^\gamma (k+k')^\delta - m_\mathrm{e} c \eta^{\gamma\delta} \mathbf{I}_4] \mathbf{u}_{k'}\} \\
&+ \frac{1}{u^2} \mathrm{Tr}\{[\gamma_\mathrm{F}^\rho (p+k')^\sigma - m_\mathrm{e} c \eta^{\rho\sigma} \mathbf{I}_4] \mathbf{u}_p \\
&\times [\gamma_\mathrm{F}^\alpha (p+k')^\beta - m_\mathrm{e} c \eta^{\alpha\beta} \mathbf{I}_4] \mathbf{u}_{k'}\} \\
&\times \mathrm{Tr}\{[\gamma_\mathrm{F}^\lambda (k+p')^\eta - m_\mathrm{e} c \eta^{\lambda\eta} \mathbf{I}_4] \mathbf{u}_k \\
&\times [\gamma_\mathrm{F}^\gamma (k+p')^\delta - m_\mathrm{e} c \eta^{\gamma\delta} \mathbf{I}_4] \mathbf{u}_{p'}\} \\
&- \frac{1}{tu} \mathrm{Tr}\{[\gamma_\mathrm{F}^\rho (p+p')^\sigma - m_\mathrm{e} c \eta^{\rho\sigma} \mathbf{I}_4] \mathbf{u}_p \\
&\times [\gamma_\mathrm{F}^\alpha (p+k')^\beta - m_\mathrm{e} c \eta^{\alpha\beta} \mathbf{I}_4] \mathbf{u}_{k'} \\
&\times [\gamma_\mathrm{F}^\lambda (k+k')^\eta - m_\mathrm{e} c \eta^{\lambda\eta} \mathbf{I}_4] \mathbf{u}_k \\
&\times [\gamma_\mathrm{F}^\gamma (k+p')^\delta - m_\mathrm{e} c \eta^{\gamma\delta} \mathbf{I}_4] \mathbf{u}_{p'}\} \\
&- \frac{1}{tu} \mathrm{Tr}\{[\gamma_\mathrm{F}^\rho (p+k')^\sigma - m_\mathrm{e} c \eta^{\rho\sigma} \mathbf{I}_4] \mathbf{u}_p \\
&\times [\gamma_\mathrm{F}^\alpha (p+p')^\beta - m_\mathrm{e} c \eta^{\alpha\beta} \mathbf{I}_4] \mathbf{u}_{p'} \\
&\times [\gamma_\mathrm{F}^\lambda (k+p')^\eta - m_\mathrm{e} c \eta^{\lambda\eta} \mathbf{I}_4] \mathbf{u}_k \\
&\times [\gamma_\mathrm{F}^\gamma (k+k')^\delta - m_\mathrm{e} c \eta^{\gamma\delta} \mathbf{I}_4] \mathbf{u}_{k'}\} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 \\
&= \frac{\kappa^2 c^2}{64\hbar^2 t^2 u^2} \{(s-t)^2 t^2 (5s^2 - 6st + 5t^2) \\
&-tu(-9s^4 + 12s^3 t - 4st^3 + t^4) \\
&+u^2(5s^4 - 12s^3 t + 8s^2 t^2 + 4st^3 - 3t^4) \\
&+2u^3(-8s^3 + 2st^2 + t^3) + u^4(22s^2 + 4st - 3t^2) \\
&-u^5(16s + t) + 5u^6 + 4m_\mathrm{e}^2 c^2 [-2s^3(t^2 + 7tu + u^2) \\
&-(t+u)(t^2+u^2)(2t^2 - 7tu + 2u^2) \\
&+s^2(t+u)(2t^2 + 31tu + 2u^2) \\
&+2s(t^4 - 12t^3 u - 13t^2 u^2 - 12tu^3 + u^4)] \\
&-16 m_\mathrm{e}^4 c^4 [-s(2t-3u)(3t-2u)(t+u) \\
&+(t^2 + tu + u^2)(3s^2 + 3t^2 - 11tu + 3u^2)] \\
&-64 m_\mathrm{e}^6 c^6 tu[5(t+u) - 8s] + 256 m_\mathrm{e}^8 c^8 (t^2 - tu + u^2)\}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{d\sigma}{d\Omega}\right)_\mathrm{cm} = \frac{\hbar^2 c^2}{64\pi^2 E_\mathrm{cm}^2} \langle |\mathcal{M}|^2 \rangle \\
&= \\
&= \frac{\hbar^2 \alpha_\mathrm{g}^2}{4 m_\mathrm{e}^4 c^2 E_\mathrm{cm}^2 p_\mathrm{r}^4 \sin^4 \theta_\mathrm{r}} [16(m_\mathrm{e}^4 c^4 + 8 m_\mathrm{e}^2 c^2 p_\mathrm{r}^2 + 8 p_\mathrm{r}^4)^2 \\
&-4(3 m_\mathrm{e}^8 c^8 + 53 m_\mathrm{e}^6 c^6 p_\mathrm{r}^2 + 270 m_\mathrm{e}^4 c^4 p_\mathrm{r}^4 + 456 m_\mathrm{e}^2 c^2 p_\mathrm{r}^6 \\
&+240 p_\mathrm{r}^8) \sin^2 \theta_\mathrm{r} + p_\mathrm{r}^4 (61 m_\mathrm{e}^4 c^4 + 240 m_\mathrm{e}^2 c^2 p_\mathrm{r}^2 \\
&+186 p_\mathrm{r}^4) \sin^4 \theta_\mathrm{r} - p_\mathrm{r}^8 \sin^6 \theta_\mathrm{r}].
\end{aligned}
$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{cm}} = \frac{\hbar^2 \alpha_{\mathrm{g}}^2 m_{\mathrm{e}}^4 c^6}{E_{\mathrm{cm}}^2 p_{\mathrm{r}}^4 \sin^4\ \theta_{\mathrm{r}}}(4 - 3\sin^2\ \theta_{\mathrm{r}})$$

$$\begin{aligned}
\psi_{\mathrm{bare}} &= \sqrt{Z_\psi}\psi, & A^\mu_{\mathrm{bare}} &= \sqrt{Z_A}A^\mu \\
m_{\mathrm{e,\,bare}} &= Z_m m_{\mathrm{e}}, & e_{\mathrm{bare}} &= Z_e e, \\
\xi_{\mathrm{e,\,bare}} &= Z_{\xi\mathrm{e}}\xi_{\mathrm{e}}, & H^{\mu\nu}_{\mathrm{bare}} &= \sqrt{Z_H}H^{\mu\nu} \\
m'_{\mathrm{e,\,bare}} &= Z_{gm} m'_{\mathrm{e}}, & g'_{\mathrm{g,\,bare}} &= Z_{\mathrm{g}} g'_{\mathrm{g}} \\
\xi_{\mathrm{g,\,bare}} &= Z_{\xi\mathrm{g}}\xi_{\mathrm{g}}.
\end{aligned}$$

$$Z_e = \frac{Z_{e\psi}}{Z_\psi\sqrt{Z_A}}, Z_{\mathrm{g}} = \frac{Z_{\mathrm{g}\psi}}{Z_\psi\sqrt{Z_H}} = \frac{Z_{\mathrm{g}A}}{Z_A\sqrt{Z_H}}$$

$$Z_{e\psi} = Z_\psi$$

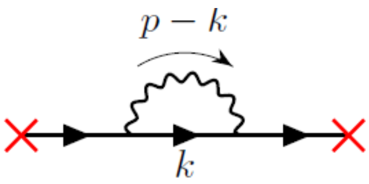
$$Z_{\xi\mathrm{e}} = Z_A, Z_{\xi\mathrm{g}} = Z_H$$

$$Z_i = 1 + \lambda\delta Z_i^{(1)} + \lambda^2\delta Z_i^{(2)} + \cdots$$

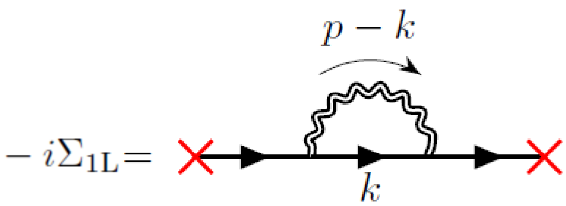
Renormalización	$\delta Z_i^{(1)}$	$\delta Z_i^{(1)}$
$Z_\psi = 1 + \delta Z_\psi^{(1)} + \cdots$	$-\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$	$-\frac{\kappa c p^2}{64\pi^2\hbar}\left[\frac{7}{\epsilon_{\mathrm{UV}}} - \frac{4}{\epsilon_{\mathrm{IR}}} + 10 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$
$Z_m = 1 + \delta Z_m^{(1)} + \cdots$	$-\frac{3\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{4}{3} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_e^2 c^2}\right)\right]$	$\frac{\kappa c p^2}{16\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + 1 + \log\left(\frac{4\pi\mu^2 e e^{-\gamma}}{m_e^2 c^2}\right)\right]$
$Z_A = 1 + \delta Z_A^{(1)} + \cdots$	$-\frac{\alpha_{\mathrm{e}}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 \mathrm{c}^2}\right)\right]$	$-\frac{\kappa c p^2}{24\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{1}{6} + \log(4\pi\mu^2 e^{-\gamma})\right]$
$Z_{\mathrm{g}\psi} = 1 + \delta Z_{\mathrm{g}\psi}^{(1)} + \cdots$	$-\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$	$\frac{\kappa c p^2}{192\pi^2\hbar}\left[\frac{11}{\epsilon_{\mathrm{UV}}} + \frac{12}{\epsilon_{\mathrm{IR}}} + \frac{172}{3} + 23\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$
$Z_{\mathrm{g}m} = 1 + \delta Z_{\mathrm{g}m}^{(1)} + \cdots$	$-\frac{3\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{4}{3} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 \mathrm{C}^2}\right)\right]$	$\frac{5\kappa c p^2}{192\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{31}{30} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$
$Z_{\mathrm{g}A} = 1 + \delta Z_{\mathrm{g}A}^{(1)} + \cdots$	$-\frac{\alpha_{\mathrm{e}}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 \mathrm{c}^2}\right)\right]$	$\frac{\kappa c p^2}{8\pi^2\hbar}\left[\frac{1}{\epsilon\epsilon_{\mathrm{UV}}} + \frac{43}{12} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{(m_{\mathrm{e}}^2 c^2)^{4/3}}\right)\right]$
Renormalización	$\delta Z_H^{(1)}$	$\delta Z_H^{(1)}$

$$Z_H = 1 + \delta Z_H^{(1)} + \cdots \qquad -\frac{\kappa c c^2}{192\pi^2\hbar}\left\{\frac{37}{15} - \frac{4m_{\mathrm{e}^2}c^2}{p^2} - \frac{24m_{\mathrm{e}}^4c^4}{p^4} + \log\left(m_{\mathrm{e}}^2c^2\right) + \left(1 - \frac{4m_{\mathrm{e}}^2c^2}{p^2} - \frac{16m_{\mathrm{e}}^2c^2}{p^4}\right)\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right]\right\} \qquad -\frac{\kappa c p^2}{96\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{29}{30} + \log\left(4\pi\mu^2e^{-\gamma}\right)\right]$$

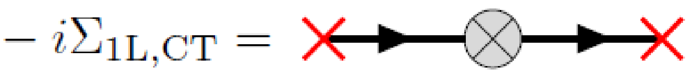
$$\mathcal{L}_{\mathrm{D,kin}} = \frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu} - \bar{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)\psi - m_{\mathrm{e}}c^2\bar{\psi}\psi$$
$$+\left(Z_{\psi} - 1\right)\frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu} - \bar{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)\psi$$
$$-\left(Z_{\psi}Z_m - 1\right)m_{\mathrm{e}}^2c^2\bar{\psi}\psi,$$
$$\mathcal{L}_{\mathrm{em,kin}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \left(Z_A - 1\right)\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu},$$
$$\mathcal{L}_{\mathrm{em,\,int}} = -q_{\mathrm{e}}c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi A_{\nu} - \left(Z_{\psi} - 1\right)q_{\mathrm{e}}c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi A_{\nu},$$
$$\mathcal{L}_{\mathrm{g,kin}} = \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu} + \left(Z_H - 1\right)\frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu},$$
$$\mathcal{L}_{\mathrm{g,\,int}} = -\frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}\left\{\frac{c}{2}P^{\mu\nu,\rho\sigma}\left[i\hbar\bar{\psi}\left(\boldsymbol{\gamma}_{\mathrm{F}\rho}\vec{\partial}_{\sigma} - \bar{\partial}_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi\right.\right.$$
$$\left.-q_{\mathrm{e}}\bar{\psi}\left(\gamma_{\mathrm{F}\rho}A_{\sigma} + A_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi - m_{\mathrm{e}}'c\eta_{\rho\sigma}\bar{\psi}\psi\right]$$
$$\left.+\frac{1}{\mu_0}P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_{\rho}A_{\sigma}\partial_{\eta}A_{\lambda}\right\}H_{\mu\nu}$$
$$-\frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}\left\{\frac{c}{2}P^{\mu\nu,\rho\sigma}\left[\left(Z_{\mathrm{g}\psi} - 1\right)i\hbar\bar{\psi}\left(\boldsymbol{\gamma}_{\mathrm{F}\rho}\vec{\partial}_{\sigma} - \bar{\partial}_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi\right.\right.$$
$$-\left(Z_{\mathrm{g}\psi} - 1\right)q_{\mathrm{e}}\bar{\psi}\left(\gamma_{\mathrm{F}\rho}A_{\sigma} + A_{\rho}\gamma_{\mathrm{F}\sigma}\right)\psi$$
$$\left.-\left(Z_{\mathrm{g}\psi}Z_{\mathrm{g}m} - 1\right)m_{\mathrm{e}}'c\eta_{\rho\sigma}\bar{\psi}\psi\right]$$
$$\left.+\left(Z_{\mathrm{g}A} - 1\right)\frac{1}{\mu_0}P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_{\rho}A_{\sigma}\partial_{\eta}A_{\lambda}\right\}H_{\mu\nu},$$
$$\mathcal{L}_{\mathrm{g,gf}} = \frac{1}{\kappa\xi_{\mathrm{g}}}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_{\lambda}H_{\alpha\beta}\partial_{\eta}H_{\rho\sigma}.$$
$$\mathcal{L}_{\mathrm{em,gf}} = -\frac{1}{2\mu_0\xi_{\mathrm{e}}}\left(\partial_{\nu}A^{\nu}\right)^2,$$

$-i\Sigma_{1\mathrm{L}}=$ 

$$= \int \frac{-i\eta_{\mu\nu}}{(p-k)^2}\left(\frac{ie\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_0\hbar c}}\right)\frac{i(k + m_{\mathrm{e}}c\mathbf{I}_4)}{k^2 - m_{\mathrm{e}}^2c^2}\left(\frac{ie\gamma_{\mathrm{F}}^{\nu}}{\sqrt{\varepsilon_0\hbar c}}\right)\frac{d^Dk}{(2\pi)^D}$$
$$= \frac{i\alpha_{\mathrm{e}}}{4\pi}\left\{\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right]\left(\not{p} - 4m_{\mathrm{e}}c\mathbf{I}_4\right)\right.$$
$$+\left(1 + \frac{m_{\mathrm{e}}^2c^2}{p^2}\right)\not{p} - 6m_{\mathrm{e}}c\mathbf{I}_4 + \left(1 - \frac{m_{\mathrm{e}}^2c^2}{p^2}\right)\log\left(\frac{m_{\mathrm{e}}^2c^2}{m_{\mathrm{e}}^2c^2 - p^2}\right)$$
$$\left.\times\left[\left(1 + \frac{m_{\mathrm{e}}^2c^2}{p^2}\right)\not{p} - 4m_{\mathrm{e}}c\mathbf{I}_4\right]\right\},$$



$$= \int \frac{iP^{(D)}_{\alpha\beta,\eta\lambda}}{(p-k)^2}\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\beta}_{\rho\sigma}\left[\gamma^{\rho}_{\mathrm{F}}(p+k)^{\sigma}-\eta^{\rho\sigma}m_{\mathrm{e}}c\mathbf{I}_4\right]\right\}$$
$$\cdot\frac{i(\not{k}+m_{\mathrm{e}}c\mathbf{I}_4)}{k^2-m_{\mathrm{e}}^2c^2}\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\eta\lambda,}_{\mu\nu}\left[\gamma^{\mu}_{\mathrm{F}}(p+k)^{\nu}\right.\right.$$
$$\left.\left.-\eta^{\mu\nu}m_{\mathrm{e}}c\mathbf{I}_4\right]\right\}\frac{d^Dk}{(2\pi)^D}$$
$$= -\frac{i\kappa c p^2}{64\pi^2\hbar}\left\{\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right](pp-5m_{\mathrm{e}}c\mathbf{I}_4)\right.$$
$$+\left(1+\frac{m_{\mathrm{e}}^2c^2}{p^2}+\frac{2m_{\mathrm{e}}^4c^4}{p^4}\right)\not{p}-8m_{\mathrm{e}}c\mathbf{I}_4$$
$$+\left(1-\frac{m_{\mathrm{e}}^2c^2}{p^2}\right)\log\left(\frac{m_{\mathrm{e}}^2c^2}{m_{\mathrm{e}}^2c^2-p^2}\right)\left[\left(1+\frac{m_{\mathrm{e}}^2c^2}{p^2}+\frac{2m_{\mathrm{e}}^4c^4}{p^4}\right)\not{p}\right.$$
$$\left.\left.-\left(5+\frac{3m_{\mathrm{e}}^2c^2}{p^2}\right)m_{\mathrm{e}}c\mathbf{I}_4\right]\right\},$$



$$= i\left[\delta Z^{(1)}_{\psi}pp-\left(\delta Z^{(1)}_{\psi}+\delta Z^{(1)}_{\mathrm{m}}\right)m_{\mathrm{e}}c\mathbf{I}_4\right]$$

$$\Sigma_{1\,\mathrm{L}}=\Sigma_{1\,\mathrm{L,\,partícula}}+\Sigma_{1\,\mathrm{L,\,partícula\text{-}supermasiva}}+\Sigma_{1\,\mathrm{L,CT}}$$

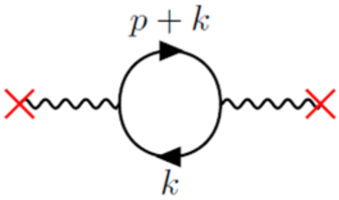
$$\Sigma_{1\,\mathrm{L}}\big|_{\not{p}=m_{\mathrm{e}}c\mathbf{I}_4}=\mathbf{0},\ \frac{d\Sigma_{1\,\mathrm{L}}}{d\not{p}}\bigg|_{\not{p}=m_{\mathrm{e}}c\mathbf{I}_4}=\mathbf{0}$$

$$\delta Z^{(1)}_{\mathrm{m,\,partícula}}=-\frac{3\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\frac{4}{3}+\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right]$$

$$\delta Z^{(1)}_{\psi,\,\mathrm{partícula}}=-\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\frac{2}{\epsilon_{\mathrm{IR}}}+4+3\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right]$$

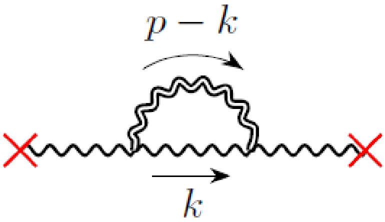
$$\delta Z^{(1)}_{\mathrm{m,\,partícula\text{-}supermasiva}}=\frac{\kappa c p^2}{16\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}}+1+\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right],$$

$$\delta Z^{(1)}_{\psi,\,\mathrm{partícula\text{-}supermasiva}}=-\frac{\kappa c p^2}{64\pi^2\hbar}\left[\frac{7}{\epsilon_{\mathrm{UV}}}-\frac{4}{\epsilon_{\mathrm{IR}}}+10+3\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right].$$

$i\Pi^{\mu\nu}_{1\mathrm{L}} =$ 

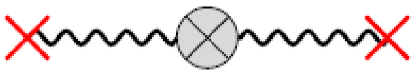
$$= -\int \mathrm{Tr}\left[\frac{i(\not{k} + m_{\mathrm{e}}c\mathbf{I}_4)}{k^2 - m_{\mathrm{e}}^2c^2}\left(\frac{ie\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_0\hbar c}}\right) \cdot \frac{i(\not{p} + \not{k} + m_{\mathrm{e}}c\mathbf{I}_4)}{(p+k)^2 - m_{\mathrm{e}}^2c^2}\left(\frac{ie\gamma_{\mathrm{F}}^{\nu}}{\sqrt{\varepsilon_0\hbar c}}\right)\right]\frac{d^Dk}{(2\pi)^D}$$

$$= -\frac{i\alpha_{\mathrm{e}}}{3\pi}\left\{\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right) + \frac{5}{3} + \frac{4m_{\mathrm{e}}^2c^2}{p^2} + \left(1 + \frac{2m_{\mathrm{e}}^2c^2}{p^2}\right)\sqrt{1 - \frac{4m_{\mathrm{e}}^2c^2}{p^2}} \times \log\left[1 + \frac{p^2}{2m_{\mathrm{e}}^2c^2}\left(\sqrt{1 - \frac{4m_{\mathrm{e}}^2c^2}{p^2}} - 1\right)\right]\right\} \times (p^2\eta^{\mu\nu} - p^{\mu}p^{\nu}),$$

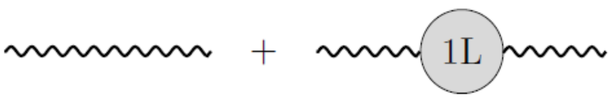
$i\Pi^{\mu\nu}_{1\mathrm{L}} =$ 

$$= \int \frac{iP^{(D)}_{\alpha\beta,\eta\lambda}}{(p-k)^2}\frac{-i\eta_{\rho\sigma}}{k^2}\left(-i\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\beta,\nu\xi,\rho\kappa}p_{\xi}k_{\kappa}\right) \times \left(-i\sqrt{\frac{\kappa c}{\hbar}}P^{\eta\lambda,\sigma\zeta,\mu\delta}k_{\zeta}p_{\delta}\right)\frac{d^Dk}{(2\pi)^D}$$

$$= -\frac{i\kappa c p^2}{24\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{1}{6} + \log\left(-\frac{4\pi\mu^2e^{-\gamma}}{p^2}\right)\right] \times (p^2\eta^{\mu\nu} - p^{\mu}p^{\nu})$$

$i\Pi^{\mu\nu}_{1\mathrm{L,CT}} =$  $=$

$$-i\delta Z_{\mathrm{A}}^{(1)}(p^2\eta^{\mu\nu} - p^{\mu}p^{\nu})$$



$$= \frac{-i\eta_{\mu\nu}}{p^2} + \frac{-i\eta_{\mu\mu'}}{p^2}i\Pi^{\mu'\nu'}_{1\,\mathrm{L}}\frac{-i\eta_{\nu'\nu}}{p^2}$$

$$= (1 + \Pi_{1\,\mathrm{L}})\frac{-i\eta_{\mu\nu}}{p^2} + i\Pi_{1\,\mathrm{L}}\frac{p_{\mu}p_{\nu}}{p^4}.$$

$$\Pi_{1\text{ L}} = \Pi_{1\text{ L, partícula}} + \Pi_{1\text{ L, partícula-supermasiva}} + \Pi_{1\text{ L,CT}},$$

$$\Pi_{1\text{ L, partícula}} = -\frac{\alpha_{\text{e}}}{3\pi}\left\{\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\text{e}}^2 c^2}\right) + \frac{5}{3} + \frac{4m_{\text{e}}^2 c^2}{p^2} + \left(1 + \frac{2m_{\text{e}}^2 c^2}{p^2}\right)\sqrt{1 - \frac{4m_{\text{e}}^2 c^2}{p^2}} \times \log\left[1 + \frac{p^2}{2m_{\text{e}}^2 c^2}\left(\sqrt{1 - \frac{4m_{\text{e}}^2 c^2}{p^2}} - 1\right)\right]\right\},$$

$$\Pi_{1\text{ L, partícula-supermasiva}} = -\frac{\kappa c p^2}{24\pi^2\hbar}\left[\frac{1}{\epsilon_{\text{UV}}} + \frac{1}{6} + \log\left(-\frac{4\pi\mu^2 e^{-\gamma}}{p^2}\right)\right]$$

$$\Pi_{1\text{ L,CT}} = \Pi_{1\text{ L,CT, partícula}} + \Pi_{1\text{ L,CT, partícula-supermasiva}},$$

$$\Pi_{1\text{ L,CT, partícula}} = -\delta Z^{(1)}_{A,\text{ partícula}},$$

$$\Pi_{1\text{ L,CT, partícula-supermasiva}} = -\delta Z^{(1)}_{A,\text{ partícula-supermasiva}}.$$

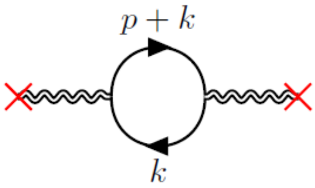
$$\Pi_{1L}\big|_{p^2=0} = 0,\ \frac{\Pi_{1\text{ L}}}{p^4}\bigg|_{p^2=\infty} = 0$$

$$\delta Z^{(1)}_{A,\text{ partícula}} = -\frac{\alpha_{\text{e}}}{3\pi}\left[\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\text{e}}^2 c^2}\right)\right]$$

$$\delta Z^{(1)}_{A,\text{ partícula-supermasiva}} = -\frac{\kappa c p^2}{24\pi^2\hbar}\left[\frac{1}{\epsilon_{\text{UV}}} + \frac{1}{6} + \log\left(4\pi\mu^2 e^{-\gamma}\right)\right].$$

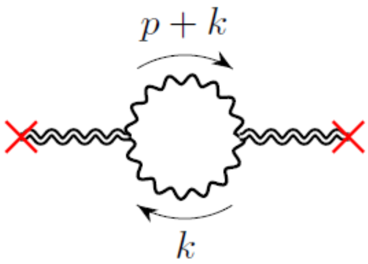
$$\Pi^{(\text{r})}_{1\text{ L, partícula}} = \Pi_{1\text{ L, partícula}} + \Pi_{1\text{ L,CT, partícula}} = -\frac{\alpha_{\text{e}}}{3\pi}\left\{\frac{5}{3} + \frac{4m_{\text{e}}^2 c^2}{p^2} + \left(1 + \frac{2m_{\text{e}}^2 c^2}{p^2}\right)\sqrt{1 - \frac{4m_{\text{e}}^2 c^2}{p^2}} \times \log\left[1 + \frac{p^2}{2m_{\text{e}}^2 c^2}\left(\sqrt{1 - \frac{4m_{\text{e}}^2 c^2}{p^2}} - 1\right)\right]\right\}$$

$$\Pi^{(\text{r})}_{1\text{ L, partícula-supermasiva}} = \Pi_{1\text{ L, partícula-supermasiva}} + \Pi_{1\text{ L,CT, partícula-supermasiva}} = \frac{\kappa c p^2}{24\pi^2\hbar}\log\left(-p^2\right)$$



$$i\Xi^{\alpha\beta,\eta\lambda}_{1\,\mathrm{L},} = -\int \mathrm{Tr}\left[\frac{i(k+m_{\mathrm{e}}c\mathbf{I}_4)}{k^2-m_{\mathrm{e}}^2c^2}\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\beta,}{}_{\kappa\delta}\left[\gamma_{\mathrm{F}}^{\kappa}(p+2k)^{\delta} -\eta^{\kappa\delta}m_{\mathrm{e}}c\mathbf{I}_4\right]\right\}\frac{i(p+k+m_{\mathrm{e}}c\mathbf{I}_4)}{(p+k)^2-m_{\mathrm{e}}^2c^2} \cdot\left\{-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\eta\lambda,}{}_{\rho\sigma}\left[\gamma_{\mathrm{F}}^{\rho}(p+2k)^{\sigma}-\eta^{\rho\sigma}m_{\mathrm{e}}c\mathbf{I}_4\right]\right\}\right]\frac{d^Dk}{(2\pi)^D}$$

$$=\frac{i\kappa c p^4}{480\pi^2\hbar}\left\{\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{A}}+\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_{\mathrm{e}}^2c^2}\right)\right]\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{B}} +\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p^2}\right)^{3/2}\log\left[1+\frac{p^2}{2m_{\mathrm{e}}^2c^2}\left(\sqrt{1-\frac{4m_{\mathrm{e}}^2c^2}{p^2}}-1\right)\right] \times\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{C}}\right\},$$



$$i\Xi^{\alpha\beta,\eta\lambda}_{1\,\mathrm{L},} = \frac{1}{2}\int\frac{-i\eta^{\mu\nu}}{k^2}\frac{-i\eta^{\rho\sigma}}{(p+k)^2}\left(-i\sqrt{\frac{\kappa c}{\hbar}}P^{\alpha\beta,\mu\kappa,\rho\xi}k_{\kappa}(p+k)_{\xi}\right) \times\left(-i\sqrt{\frac{\kappa c}{\hbar}}P^{\eta\lambda,\sigma\delta,\nu\zeta}(p+k)_{\delta}k_{\zeta}\right)\frac{d^Dk}{(2\pi)^D}$$

$$=\frac{i\kappa c p^4}{240\pi^2\hbar}\left\{\hat{P}^{\alpha\beta,\eta\lambda}_{81,94,30}+\left[\frac{1}{\epsilon_{\mathrm{UV}}}+\log\left(-\frac{4\pi\mu^2e^{-\gamma}}{p^2}\right)\right]\hat{P}^{\alpha\beta,\eta\lambda}_{3,2,1}\right\},$$

$$i\Xi^{\alpha\beta,\eta\lambda}_{1\mathrm{L,CT}} =$$  $$= i(Z_H-1)p^2\hat{P}^{\alpha\beta,\eta\lambda}_{1,2,1}$$

$$P^{\mu\nu,\rho\sigma}_{a,b,c} = \frac{1}{2c}\left(a\eta^{\mu\sigma}\eta^{\rho\nu}+a\eta^{\mu\rho}\eta^{\nu\sigma}-b\eta^{\mu\nu}\eta^{\rho\sigma}\right)$$

$$\hat{P}^{\alpha\beta,\eta\lambda}_{a,b,c} = \frac{1}{2c}\left(a\hat{\eta}^{\alpha\lambda}\hat{\eta}^{\beta\eta}+a\hat{\eta}^{\alpha\eta}\hat{\eta}^{\beta\lambda}-b\hat{\eta}^{\alpha\beta}\hat{\eta}^{\eta\lambda}\right)$$

$$\hat{\eta}^{\alpha\beta} = \eta^{\alpha\beta}-\frac{p^{\alpha}p^{\beta}}{p^2}.$$

$$\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{A}} = 4\hat{P}^{\alpha\beta,\eta\lambda}_{27,23,15}-\frac{2m_{\mathrm{e}}^2c^2}{p^2}\hat{P}^{\alpha\beta,\eta\lambda}_{19,86,3} -\frac{m_{\mathrm{e}}^4c^4}{p^4}\left(64\hat{P}^{\alpha\beta,\eta\lambda}_{1,-1,1}+45P^{\alpha\beta,\eta\lambda}_{1,0,1}\right),$$

$$\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{B}} = \hat{P}^{\alpha\beta,\eta\lambda}_{3,2,1}-\frac{10m_{\mathrm{e}}^2c^2}{p^2}\hat{P}^{\alpha\beta,\eta\lambda}_{1,2,1}-\frac{30m_{\mathrm{e}}^4c^4}{p^4}P^{\alpha\beta,\eta\lambda}_{1,0,1},$$

$$\hat{P}^{\alpha\beta,\eta\lambda}_{\mathrm{C}} = \hat{P}^{\alpha\beta,\eta\lambda}_{3,2,1}+\frac{8m_{\mathrm{e}}^2c^2}{p^2}\hat{P}^{\alpha\beta,\eta\lambda}_{1,-1,1}.$$

$$= \frac{iP_{\alpha\beta,\eta\lambda}}{p^2} + \frac{iP_{\alpha\beta\alpha'\beta'}}{p^2} i\Xi_{1\,\mathrm{L}}^{\alpha'\beta'\eta'\lambda'} \frac{iP_{\eta'\lambda'\eta\lambda}}{p^2}$$

$$= (1 + \Xi_{1\,\mathrm{L}}) \frac{iP_{\alpha\beta,\eta\lambda}}{p^2}.$$

$$\Xi_{1\,\mathrm{L}} = \Xi_{1\,\mathrm{L,\,partícula}} + \Xi_{1\,\mathrm{L,\,partícula}} + \Xi_{1\,\mathrm{L,CT}}$$

$$\Xi_{1L,\,\mathrm{partícula}} = -\frac{\kappa c p^2}{32\pi^2\hbar}\left\{\frac{37}{15} - \frac{20m_e^2c^2}{3p^2} - \frac{40m_e^4c^4}{p^4} + \left(1 - \frac{4m_e^2c^2}{p^2} - \frac{16m_e^4c^4}{p^4}\right)\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_e^2c^2}\right)\right] + \left(1 - \frac{2m_e^2c^2}{p^2} - \frac{8m_e^4c^4}{p^4}\right)\sqrt{1 - \frac{4m_e^2c^2}{p^2}} \times \log\left[1 + \frac{p^2}{2m_e^2c^2}\left(\sqrt{1 - \frac{4m_e^2c^2}{p^2}} - 1\right)\right]\right\}$$

$$\Xi_{1\,\mathrm{L,\,partícula}} = -\frac{\kappa c p^2}{16\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{29}{30} + \log\left(-\frac{4\pi\mu^2e^{-\gamma}}{p^2}\right)\right]$$

$$\Xi_{1\,\mathrm{L,CT}} = \Xi_{1\,\mathrm{L,CT,\,partícula}} + \Xi_{1\,\mathrm{L,CT,\,partícula}},$$

$$\Xi_{1\,\mathrm{L,CT,\,partícula}} = -6\delta Z_{\mathrm{H,\,partícula}}^{(1)},$$

$$\Xi_{1\,\mathrm{L,CT,\,partícula}} = -6\delta Z_{\mathrm{H,\,partícula}}^{(1)}.$$

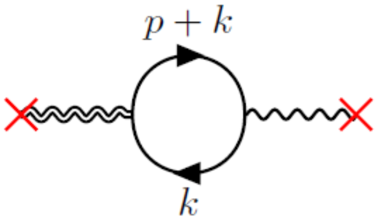
$$\Xi_{1\,\mathrm{L}}\big|_{p^2=0} = 0,\ \left.\frac{\Xi_{1\,\mathrm{L}}}{p^4}\right|_{p^2=\infty} = 0.$$

$$\delta Z_{\mathrm{H,\,partícula}}^{(1)} = -\frac{\kappa c p^2}{192\pi^2\hbar}\left\{\frac{37}{15} - \frac{4m_e^2c^2}{p^2} - \frac{24m_e^4c^4}{p^4} + \log\left(m_e^2c^2\right) + \left(1 - \frac{4m_e^2c^2}{p^2} - \frac{16m_e^2c^2}{p^4}\right)\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_e^2c^2}\right)\right]\right\}$$

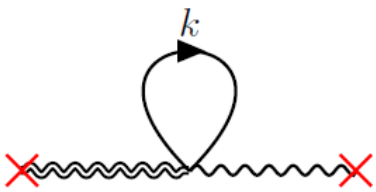
$$\delta Z_{\mathrm{H,partícula}}^{(1)} = -\frac{\kappa c p^2}{96\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{29}{30} + \log\left(4\pi\mu^2e^{-\gamma}\right)\right].$$

$$\Xi_{1L,\,\mathrm{partícula}}^{(\mathrm{r})} = \Xi_{1\,\mathrm{L,\,partícula}} + \Xi_{1\,\mathrm{L,CT,\,partícula}} = -\frac{\kappa c p^2}{32\pi^2\hbar}\left\{-\frac{8m_e^2c^2}{3p^2}\left(1 + \frac{6m_e^2c^2}{p^2}\right) - \log\left(m_e^2c^2\right) + \left(1 - \frac{2m_e^2c^2}{p^2} - \frac{8m_e^4c^4}{p^4}\right)\sqrt{1 - \frac{4m_e^2c^2}{p^2}} \times \log\left[1 + \frac{p^2}{2m_e^2c^2}\left(\sqrt{1 - \frac{4m_e^2c^2}{p^2}} - 1\right)\right]\right\}$$

$$\Xi_{1\,\mathrm{L,\,partícula}}^{(\mathrm{r})} = \Xi_{1\,\mathrm{L,\,partícula}} + \Xi_{1\,\mathrm{L,CT,\,partícula}} = \frac{\kappa c p^2}{16\pi^2\hbar}\log\left(-p^2\right)$$

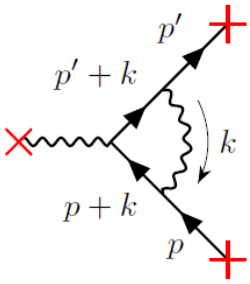


$$= 0,$$



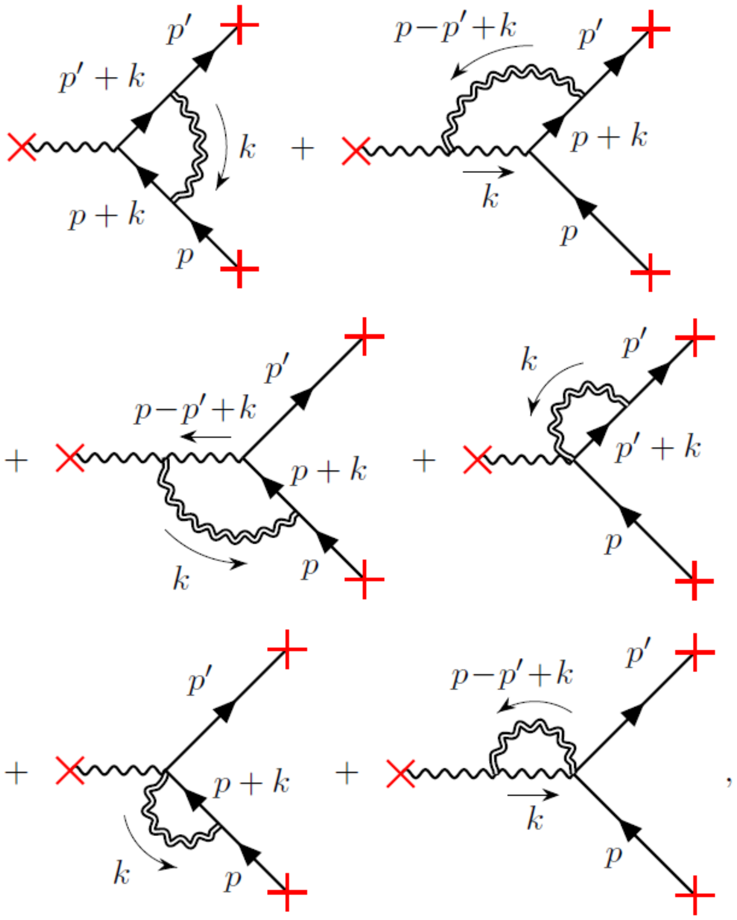
$$= 0.$$

$$\frac{-iq_{\mathrm{e}}\Gamma^{\mu}_{1\ \mathrm{L,\ partícula}}}{\sqrt{\varepsilon_0 \hbar c}} =$$

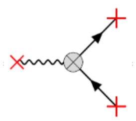


,

$$\frac{-iq_{\mathrm{e}}\Gamma^{\mu}_{1\ \mathrm{L,\ partícula\text{-}supermasiva}}}{\sqrt{\varepsilon_0\hbar c}} =$$



$$\frac{-iq_{\mathrm{e}}\Gamma^{\mu}_{1\ \mathrm{L,CT}}}{\sqrt{\varepsilon_0\hbar c}} =$$



$$= \delta Z_{\psi}^{(1)} \frac{-iq_{\mathrm{e}}\gamma^{\mu}_{\mathrm{F}}}{\sqrt{\varepsilon_0\hbar c}}.$$

$$\Gamma^{\mu}_{1\ \mathrm{L}} = \Gamma^{\mu}_{1\ \mathrm{L,\ partícula}} + \Gamma^{\mu}_{1\ \mathrm{L,\ partícula\text{-}supermasiva}} + \Gamma^{\mu}_{1\ \mathrm{L,CT}}.$$

$$\bar{u}(p')\Gamma^{\mu}_{1\ \mathrm{L}}u(p)\Big|_{p=p'} = 0$$

$$\bar{u}(p')\Gamma^{\mu}_{1\ \mathrm{L}}u(p)$$

$$= \bar{u}(p')\left[F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L}}(q^2)\gamma^{\mu}_{\mathrm{F}} + F^{(\mathrm{e}\gamma)}_{2,1\ \mathrm{L}}(q^2)\frac{iq_{\nu}}{\hbar m_{\mathrm{e}}c}\hat{S}^{\mu\nu}_{\mathrm{F}}\right]u(p).$$

$$F^{(\mathrm{e}\gamma)}_{1}(q^2) = F^{(\mathrm{e}\gamma)}_{1,0\ \mathrm{L}} + F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L}}(q^2) + \cdots,$$

$$F^{(\mathrm{e}\gamma)}_{2}(q^2) = F^{(\mathrm{e}\gamma)}_{2,0\ \mathrm{L}} + F^{(\mathrm{e}\gamma)}_{2,1\ \mathrm{L}}(q^2) + \cdots$$

$$F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,\ partícula}} = F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,\ partícula}} + F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,CT,\ partícula}} = 0,$$

$$F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} + F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,CT,\ partícula\text{-}supermasiva}} = 0$$

$$F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,\ partícula}} = \frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right],$$

$$F^{(\mathrm{e}\gamma)}_{1,1\ \mathrm{L,CT,\ partícula}} = -\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right].$$

$$F^{(\mathrm{e,\ diag\ })}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = -\frac{\alpha_{\mathrm{g}}}{8\pi}\left[\frac{7}{\epsilon_{\mathrm{UV}}} + \frac{4}{\epsilon_{\mathrm{IR}}} + 18 + 11\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right],$$

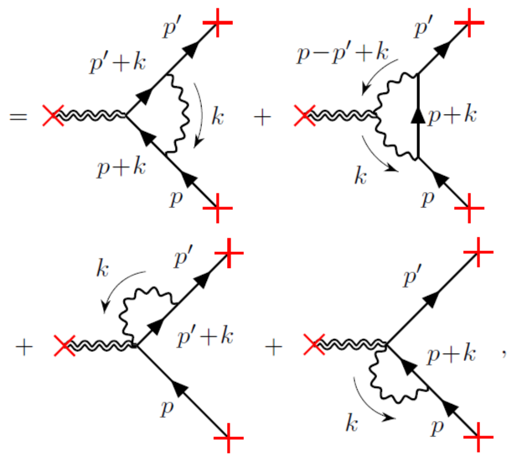
$$F^{(\mathrm{e}\gamma,\ \mathrm{diag2\ })}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = F^{(\mathrm{e}\gamma,\ \mathrm{diag3\ })}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = F^{(\mathrm{e,\ diag\ })}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = 0,$$

$$F^{(\mathrm{er\ diag4\ })}_{1,1\ \mathrm{L,\ partícula\text{-}supermasiva}} = F^{(\mathrm{er,diag5\ })}_{1,1\ \mathrm{L,partícula-supermasiva}}$$

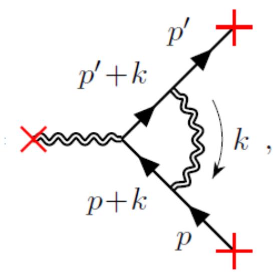
$$= \frac{7\alpha_{\mathrm{g}}}{8\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + 2 + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right],$$

$$F^{(\mathrm{e})}_{1,1\ \mathrm{L,CT,\ partícula\text{-}supermasiva}} = -\frac{\alpha_{\mathrm{g}}}{8\pi}\left[\frac{7}{\epsilon_{\mathrm{UV}}} - \frac{4}{\epsilon_{\mathrm{IR}}} + 10 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right].$$

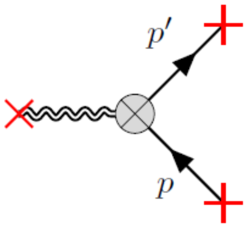
$$-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}\,\Gamma^{\mu\nu}_{1\ \mathrm{L,\ partícula}}$$



$$-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}\,\Gamma^{\mu\nu}_{1\ \mathrm{L,\ partícula\text{-}supermasiva}} =$$



$$-\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}\Gamma^{\mu\nu}_{1\,\mathrm{L,CT}}$$



$$= -\frac{i}{2}\sqrt{\frac{\kappa c}{\hbar}}P^{\mu\nu,\rho\sigma}\left[\delta Z^{(1)}_{\mathrm{g}\psi}\gamma_{\mathrm{F}\rho}(p+p')_\sigma - \left(\delta Z^{(1)}_{\mathrm{g}\psi}+\delta Z^{(1)}_{\mathrm{gm}}\right)m_\mathrm{e}c\eta_{\rho\sigma}\mathbf{I}_4\right]$$

$$=\frac{i}{4}\sqrt{\frac{\kappa c}{\hbar}}\Big\{\delta Z^{(1)}_{\mathrm{g}\psi}\eta^{\mu\nu}(pp+p') - 2\left(\delta Z^{(1)}_{\mathrm{g}\psi}+\delta Z^{(1)}_{\mathrm{gm}}\right)\eta^{\mu\nu}m_\mathrm{e}c\mathbf{I}_4 - \delta Z^{(1)}_{\mathrm{g}\psi}\left[\gamma^{\mu}_{\mathrm{F}}(p+p')^\nu + \gamma^{\nu}_{\mathrm{F}}(p+p')^\mu\right]\Big\}.$$

$$\Gamma^{\mu\nu}_{1\,\mathrm{L}} = \Gamma^{\mu\nu}_{1\,\mathrm{L,\,partícula}} + \Gamma^{\mu\nu}_{1\,\mathrm{L,\,partícula\text{-}supermasiva}} + \Gamma^{\mu\nu}_{1\,\mathrm{L,CT}}.$$

$$\bar{u}(p')\Gamma^{\mu\nu}_{1,1\,\mathrm{L}}u(p)\Big|_{p=p'} = 0.$$

$$\bar{u}(p')\Gamma^{\mu\nu}_{1,1\,\mathrm{L}}u(p) =$$

$$= \bar{u}(p')\Bigg\{F^{(\mathrm{eg})}_{1,1\,\mathrm{L}}m_\mathrm{e}c\eta^{\mu\nu}\mathbf{I}_4 + F^{(\mathrm{eg})}_{2,1\,\mathrm{L}}\frac{(p+p')^\mu(p+p')^\nu}{2m_\mathrm{e}c}\mathbf{I}_4 + F^{(\mathrm{eg})}_{3,1\,\mathrm{L}}\frac{iq_\rho}{\hbar m_\mathrm{e}c}\left[(p+p')^\mu\hat{S}^{\nu\rho}_{\mathrm{F}} + (p+p')^\nu\hat{S}^{\mu\rho}_{\mathrm{F}}\right] + F^{(\mathrm{eg})}_{4,1\,\mathrm{L}}q^\mu q^\nu\mathbf{I}_4\Bigg\}u(p)$$

$$F^{(\mathrm{eg})}_{1,1\,\mathrm{L,\,partícula}} = F^{(\mathrm{eg})}_{1,1\,\mathrm{L,\,partícula}} + F^{(\mathrm{eg})}_{1,1\,\mathrm{L,CT,\,partícula}} = 0,$$

$$F^{(\mathrm{eg})}_{2,1\,\mathrm{L,\,partícula}} = F^{(\mathrm{eg})}_{2,1\,\mathrm{L,\,partícula}} + F^{(\mathrm{eg})}_{2,1\,\mathrm{L,CT,\,partícula}} = 0,$$

$$F^{(\mathrm{eg})}_{1,1\,\mathrm{L,\,partícula\text{-}supermasiva}} = F^{(\mathrm{eg})}_{1,1\,\mathrm{L,\,partícula\text{-}supermasiva}} + F^{(\mathrm{eg})}_{1,1\,\mathrm{L,CT,\,partícula\text{-}supermasiva}} = 0,$$

$$F^{(\mathrm{eg})}_{2,1\,\mathrm{L,\,partícula\text{-}supermasiva}} = F^{(\mathrm{eg})}_{2,1\,\mathrm{L,\,partícula\text{-}supermasiva}} + F^{(\mathrm{eg})}_{2,1\,\mathrm{L,CT,\,partícula\text{-}supermasiva}} = 0.$$

$$F^{(\mathrm{eg,\,diag1})}_{1,1\,\mathrm{L,\,partícula}} = \frac{\alpha_\mathrm{e}}{12\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{22}{3} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right],$$

$$F^{(\mathrm{eg\ diag2})}_{1,1\,\mathrm{L,\,partícula}} = -\frac{\alpha_\mathrm{e}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{3} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right],$$

$$F^{(\mathrm{eg,\,diag3})}_{1,1\,\mathrm{L,\,partícula}} = F^{(\mathrm{eg,diag4})}_{1,1\,\mathrm{L,\,partícula}} = \frac{\alpha_\mathrm{e}}{2\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{3}{2} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right].$$

$$F^{(\mathrm{eg,\,diag1})}_{2,1\,\mathrm{L,\,partícula}} = \frac{\alpha_\mathrm{e}}{12\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{6}{\epsilon_{\mathrm{IR}}} + \frac{56}{3} + 7\log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right]$$

$$F^{(\mathrm{eg,diag2})}_{2,1\,\mathrm{L,\,partícula}} = \frac{2\alpha_\mathrm{e}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{17}{12} + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right],$$

$$F^{(\mathrm{eg,diag3})}_{2,1\,\mathrm{L,\,partícula}} = F^{(\mathrm{eg,diag4})}_{2,1\,\mathrm{L,\,partícula}} = -\frac{\alpha_\mathrm{e}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + 3 + \log\left(\frac{4\pi\mu^2e^{-\gamma}}{m_\mathrm{e}^2c^2}\right)\right].$$

$$F^{(\mathrm{eg})}_{1,1\ \mathrm{L,CT,\ partícula}} = -\frac{3\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{4}{3} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right],$$

$$F^{(\mathrm{eg})}_{2,1\ \mathrm{L,CT,\ partícula}} = -\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right].$$

$$\delta Z^{(1)}_{gm,\ \mathrm{partícula}} = -\frac{3\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{4}{3} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right],$$

$$\delta Z^{(1)}_{g\psi,\ \mathrm{partícula}} = -\frac{\alpha_{\mathrm{e}}}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 4 + 3\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

$$F^{(\mathrm{eg})}_{1,1\ \mathrm{L,partícula-supermasiva}} = -\frac{5\kappa c p^2}{192\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{31}{30} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

$$F^{(\mathrm{eg})}_{2,1\ \mathrm{L,\ partícula\text{-}supermasiva}}$$
$$= -\frac{3\kappa c p^2}{576\pi^2\hbar}\left[\frac{11}{\epsilon_{\mathrm{UV}}} + \frac{12}{\epsilon_{\mathrm{IR}}} + \frac{172}{3} + 23\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right].$$

$$F^{(\mathrm{eg})}_{1,1\ \mathrm{L,CT,\ partícula\text{-}supermasiva}} = \frac{5\kappa c p^2}{192\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{31}{30} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

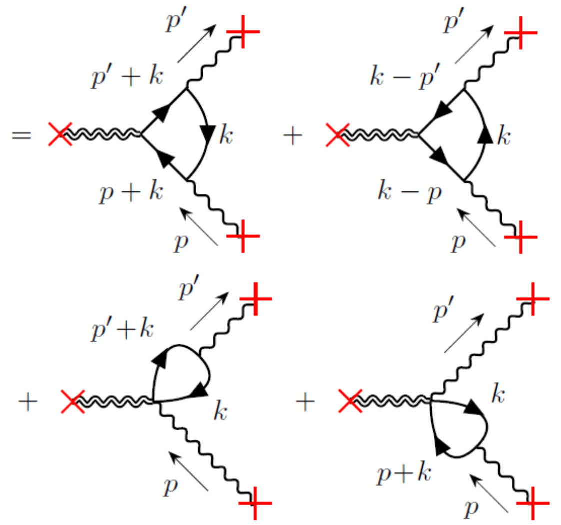
$$F^{(\mathrm{eg})}_{2,1\ \mathrm{L,CT,partícula-supermasiva}}$$
$$= \frac{3\kappa c p^2}{576\pi^2\hbar}\left[\frac{11}{\epsilon_{\mathrm{UV}}} + \frac{12}{\epsilon_{\mathrm{IR}}} + \frac{172}{3} + 23\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right].$$

$$\delta Z^{(1)}_{\mathrm{gm,partícula-supermasiva}} = \frac{5\kappa c p^2}{192\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{31}{30} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

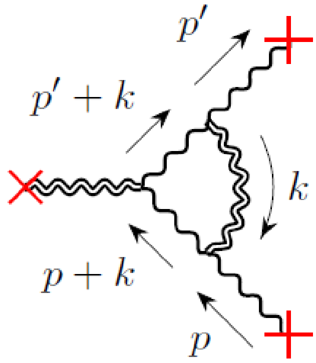
$$\delta Z^{(1)}_{g\psi,\ \mathrm{partícula\text{-}supermasiva}}$$
$$= \frac{\kappa c p^2}{192\pi^2\hbar}\left[\frac{11}{\epsilon_{\mathrm{UV}}} + \frac{12}{\epsilon_{\mathrm{IR}}} + \frac{172}{3} + 23\log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

$$\delta Z^{(1)}_{gA,\ \mathrm{partícula}} = \delta Z^{(1)}_{A,\ \mathrm{partícula}}$$
$$= -\frac{\alpha_{\mathrm{e}}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right]$$

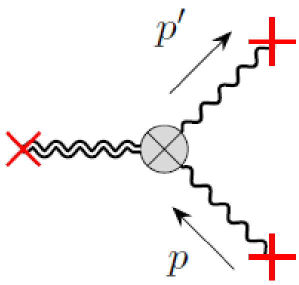
$$\delta Z^{(1)}_{gA,\mathrm{partícula-supermasiva}} = \delta Z^{(1)}_{A,\ \mathrm{partícula\text{-}supermasiva}} + \delta Z^{(1)}_{g\psi,\ \mathrm{partícula\text{-}supermasiva}} - \delta Z^{(1)}_{\psi,\ \mathrm{partícula\text{-}supermasiva}}$$
$$= \frac{\kappa c p^2}{8\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{43}{12} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{(m_{\mathrm{e}}^2 c^2)^{4/3}}\right)\right]$$



$$-i\sqrt{\frac{\kappa c}{\hbar}}\,\Gamma^{\mu\nu,\sigma,\rho}_{1\ \mathrm{L,\ partícula}}$$



$$-i\sqrt{\frac{\kappa c}{\hbar}}\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L, partícula-supermasiva}} =$$



$$-i\sqrt{\frac{\kappa c}{\hbar}}\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L,CT}} =$$

$$= -i\delta Z^{(1)}_{gA}\sqrt{\frac{\kappa c}{\hbar}}P^{\mu\nu,\sigma\eta,\rho\lambda}p'_{\eta}p_{\lambda}.$$

$$\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L}} = \Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L, partícula}} + \Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L, partícula-supermasiva}} + \Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L,CT}}.$$

$$\epsilon^{*}_{\sigma}(p')\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L}}\epsilon_{\rho}(p)\Big|_{p=p'} = 0$$

$$\Gamma^{(\text{diag1 })\mu\nu,\sigma,\rho}_{1\text{ L,partícula}}\Big|_{p=p'} = \Gamma^{(\text{diag2 })\mu\nu,\sigma,\rho}_{1\text{ L, partícula}}\Big|_{p=p'}$$

$$= -\frac{\alpha_{\text{e}}}{12\pi}\left[\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{4\pi\mu^{2}e^{-\gamma}}{m_{\text{e}}^{2}c^{2}}\right)\right](4p^{\mu}p^{\nu}\eta^{\rho\sigma}$$

$$-p^{\mu}p^{\rho}\eta^{\nu\sigma} - p^{\nu}p^{\rho}\eta^{\mu\sigma} - p^{\mu}p^{\sigma}\eta^{\nu\rho} - p^{\nu}p^{\sigma}\eta^{\mu\rho})$$

$$+\frac{\alpha_{\text{e}}}{15\pi m_{\text{e}}^{2}c^{2}}p^{\mu}p^{\nu}p^{\rho}p^{\sigma}$$

$$\Gamma^{(\text{diag3 })\mu\nu,\sigma,\rho}_{1\text{ L, partícula}}\Big|_{p=p'}$$

$$= \frac{\alpha_{\text{e}}}{6\pi}\left[\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{4\pi\mu^{2}e^{-\gamma}}{m_{\text{e}}^{2}c^{2}}\right)\right]$$

$$\times p^{\rho}(p^{\mu}\eta^{\nu\sigma} + p^{\nu}\eta^{\mu\sigma} - p^{\sigma}\eta^{\mu\nu})$$

$$\Gamma^{(\text{diag })\mu\nu,\sigma,\rho}_{1\text{ L, partícula}}\Big|_{p=p'}$$

$$=$$

$$\frac{\alpha_{\text{e}}}{6\pi}\left[\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{4\pi\mu^{2}e^{-\gamma}}{m_{\text{e}}^{2}c^{2}}\right)\right]$$

$$\times p^{\sigma}(p^{\mu}\eta^{\nu\rho} + p^{\nu}\eta^{\mu\rho} - p^{\rho}\eta^{\mu\nu})$$

$$\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L, partícula}}\Big|_{p=p'}$$

$$= \frac{\alpha_{\mathrm{e}}}{3\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_{\mathrm{e}}^2 c^2}\right)\right] P^{\mu\nu,\sigma\eta,\rho\lambda} p'_{\eta} p_{\lambda}$$

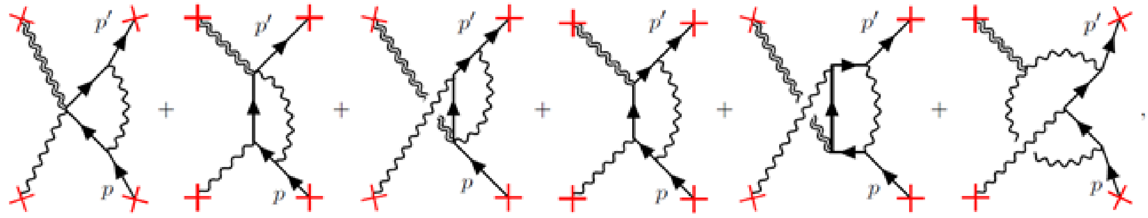
$$+ \frac{\alpha_{\mathrm{e}}}{15\pi m_{\mathrm{e}}^2 c^2} p^{\mu} p^{\nu} p^{\rho} p^{\sigma}.$$

$$\Gamma^{\mu\nu,\sigma,\rho}_{1\text{ L,partícula−supermasiva}}\Big|_{p=p'}$$

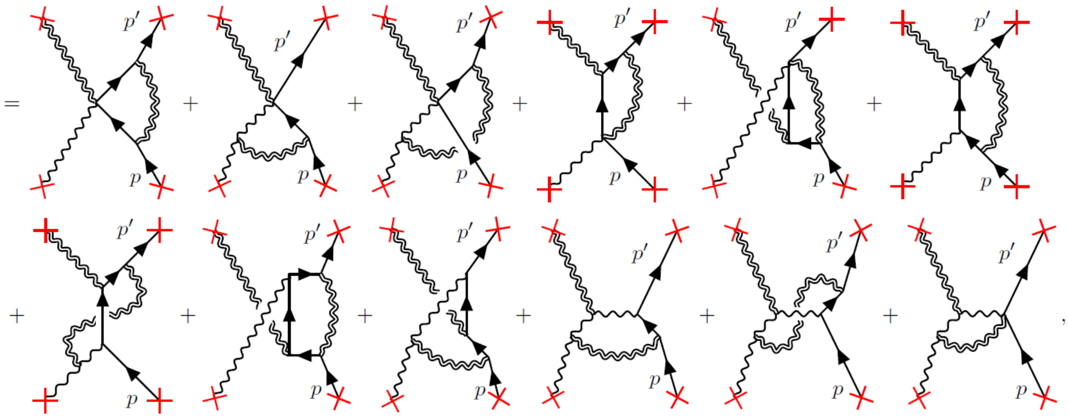
$$= -\frac{\kappa c}{8\pi^2\hbar}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{11}{6} + \log\left(-\frac{4\pi\mu^2 e^{-\gamma}}{p^2}\right)\right] p^{\mu} p^{\nu} p^{\rho} p^{\sigma}.$$

$$\frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_0}}\Gamma^{\mu\nu,\rho}_{1\text{ L, partícula}}$$

=

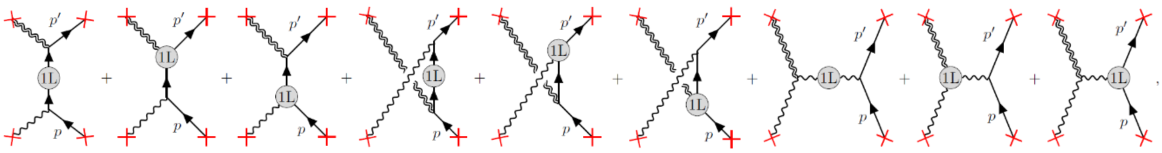


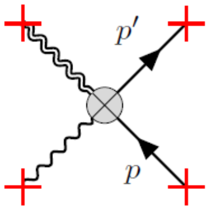
$$\frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_0}}\Gamma^{\mu\nu,\rho}_{1\text{ L,partícula−supermasiva}}$$



$$\frac{iq_{\mathrm{c}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_0}}\Gamma^{\mu\nu,\rho}_{1\text{ L,2P\&3P}}$$

=



$$\frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_0}}\Gamma^{\mu\nu,\rho}_{1\,\mathrm{L,CT}} =$$ 

$$= \left(Z_{g\psi}-1\right)\frac{iq_{\mathrm{e}}}{\hbar}\sqrt{\frac{\kappa}{\varepsilon_0}}P^{\mu\nu,\rho\sigma}\gamma_{\mathrm{F}\sigma}$$

$$\Gamma^{\mu}_{1\,\mathrm{L}} = \Gamma^{\mu\nu,\rho}_{1\,\mathrm{L,\,partícula}} + \Gamma^{\mu\nu,\rho}_{1\,\mathrm{L,\,partícula\text{-}supermasiva}} + \Gamma^{\mu\nu,\rho}_{1\,\mathrm{L,CT}}$$
$$\bar{u}(p')\Gamma^{\mu\nu,\rho}_{1,1\,\mathrm{L}}u(p)\epsilon_{\mu\nu}(q')\epsilon_{\rho}(q)\Big|_{\substack{s=u=m_{\mathrm{e}}^2c^2\\ t=0}} = 0.$$

$$\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(p^2\right) \approx \Pi^{(\mathrm{r})}_{\mathrm{LL,\,partícula}}\left(-|\mathbf{p}|^2\right)$$
$$= -\frac{\alpha_{\mathrm{e}}}{3\pi}\left\{\frac{5}{3}-\frac{4m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}+\left(1-\frac{2m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}\right)\sqrt{1+\frac{4m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}}\right.$$
$$\left.\times \log\left[1-\frac{|\mathbf{p}|^2}{2m_{\mathrm{e}}^2c^2}\left(\sqrt{1+\frac{4m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}}-1\right)\right]\right\},$$

$$\Pi^{(\mathrm{r})}_{\mathrm{iL,\,partícula\text{-}supermasiva}}\left(p^2\right) \approx \Pi^{(\mathrm{r})}_{\mathrm{iLL,partícula-supermasiva}}\left(-|\mathbf{p}|^2\right)$$
$$= -\frac{\kappa c|\mathbf{p}|^2}{24\pi^2\hbar}\log\left(|\mathbf{p}|^2\right).$$

$$\tilde{V}_{\mathrm{e,1\,L,partícula}}(\mathbf{p}) = \frac{\hbar^2e^2}{\varepsilon_0\left(|\mathbf{p}|^2-i\epsilon\right)}\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(-|\mathbf{p}|^2\right),$$

$$\tilde{V}_{\mathrm{e,1\,L,\,partícula\text{-}supermasiva}}(\mathbf{p}) = \frac{\hbar^2e^2}{\varepsilon_0\left(|\mathbf{p}|^2-i\epsilon\right)}\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula\text{-}supermasiva}}\left(-|\mathbf{p}|^2\right).$$

$$V_{\mathrm{e,1\,L,\,partícula}}(\mathbf{r})$$
$$= \int \tilde{V}_{\mathrm{e,1\,L,\,partícula}}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\frac{d^3p}{(2\pi\hbar)^3}$$
$$= \frac{\hbar^2e^2}{(2\pi\hbar)^3\varepsilon_0}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{e^{ip_{\mathrm{r}}r\cos\,\theta_{\mathrm{r}}/\hbar}}{\left(p_{\mathrm{r}}^2-i\epsilon\right)}\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(-p_{\mathrm{r}}^2\right)$$
$$\times\, p_{\mathrm{r}}^2\sin\,\theta_{\mathrm{r}}d\phi_{\mathrm{r}}d\theta_{\mathrm{r}}dp_{\mathrm{r}}$$
$$= \frac{\hbar^2e^22\pi\hbar}{(2\pi\hbar)^3\varepsilon_0\,r}\int_0^{\infty}\left|_0^{\pi}\frac{ip_{\mathrm{r}}e^{ip_{\mathrm{r}}r\cos\,\theta_{\mathrm{r}}/\hbar}}{p_{\mathrm{r}}^2-i\epsilon}\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(-p_{\mathrm{r}}^2\right)dp_{\mathrm{r}}\right.$$
$$= \frac{e^2}{4\pi^2\varepsilon_0 r}\int_0^{\infty}\frac{ip_{\mathrm{r}}\left(e^{-ip_{\mathrm{r}}r/\hbar}-e^{ip_{\mathrm{r}}r/\hbar}\right)}{p_{\mathrm{r}}^2-i\epsilon}\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(-p_{\mathrm{r}}^2\right)dp_{\mathrm{r}}$$
$$= \frac{e^2}{8\pi^2\varepsilon_0 r}\int_{-\infty}^{\infty}\frac{ip_{\mathrm{r}}\left(e^{-ip_{\mathrm{r}}r/\hbar}-e^{ip_{\mathrm{r}}r/\hbar}\right)}{p_{\mathrm{r}}^2-i\epsilon}\Pi^{(\mathrm{r}}_{1\,\mathrm{L,\,partícula}}\left(-p_{\mathrm{r}}^2\right)dp_{\mathrm{r}}.$$

$$\mathrm{Im}\left[\Pi^{(\mathrm{r})}_{1\,\mathrm{L,\,partícula}}\left(-p_{\mathrm{r}}^2\right)\right] = \pm\frac{\alpha_{\mathrm{e}}}{3}\sqrt{1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}}\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right).$$

$$V_{e,1\text{ L, partícula}}(\mathbf{r})$$

$$=\frac{e^2}{8\pi^2\varepsilon_0 r}\left\{-2\pi i \operatorname*{Res}_{p_r=-\sqrt{i\epsilon}}\left[ip_r\frac{e^{-ip_r r/h}}{p_r^2-i\epsilon}\Pi^{(r)}_{1L,\text{ partícula}}(-p_r^2)\right]\right.$$

$$-2\pi i\text{Res}_{p_r=\sqrt{i\epsilon}}ip_r\frac{e^{ip_r r/h}}{p_r^2-i\epsilon}\Pi^{(r)}_{1\text{ L, partícula}}(-p_r^2)\Big]$$

$$-2\int_{\epsilon-i\infty}^{\epsilon-2im_e c}\frac{ip_r e^{-ip_r r/h}}{(p_r^2-i\epsilon)}i\text{Im}\left[\Pi^{(r)}_{1\text{ L, partícula}}(-p_r^2)\right]dp_r$$

$$\left.+2\int_{-\epsilon+2im_e c}^{-\epsilon+i\infty}\frac{ip_r e^{ip_r r/h}}{(p_r^2-i\epsilon)}i\text{Im}\left[\Pi^{r)}_{1\text{ L, partícula}}(-p_r^2)\right]dp_r\right\}$$

$$=\frac{e^2}{8\pi^2\varepsilon_0 r}\left\{-2\pi i\left[\frac{1}{2}e^{ir\sqrt{\epsilon\epsilon}/h}\Pi^{(r)}_{1\text{ L, partícula}}(-i\epsilon)\right]\right.$$

$$-2\pi i\left[\frac{1}{2}e^{ir\sqrt{i\epsilon}/h}\Pi^{(r)}_{1\text{ L, partícula}}(-i\epsilon)\right]$$

$$+\frac{2\alpha_e}{3}\int_{\epsilon-i\infty}^{\epsilon-2im_e c}\frac{2p_r e^{-ip_r r/h}}{(p_r^2-i\epsilon)}\sqrt{1-\frac{4m_e^2c^2}{p_{ri}^2}}\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_r$$

$$-\epsilon+i\infty p_r e^{ip_r r/h}$$

$$(p_r^2-i\epsilon)\sqrt{1-\frac{4m_e^2c^2}{p_{ri}^2}}\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_r\Bigg\}.$$

$$V_{e,1\text{ L, partícula}}(\mathbf{r})=\frac{e^2}{8\pi^2\varepsilon_0 r}$$

$$\times\left[\frac{2\alpha_e}{3}\int_{-\infty}^{-2m_o c}\frac{e^{p_{ri}r/h}}{p_{ri}}\sqrt{1-\frac{4m_e^2c^2}{p_{ri}^2}}\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_{ri}\right.$$

$$\left.+\frac{2\alpha_e}{3}\int_{2m_e c}^{\infty}\frac{e^{-p_{ri}r/h}}{p_{ri}}\sqrt{1-\frac{4m_e^2c^2}{p_{ri}^2}}\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_{ri}\right]$$

$$=\frac{2\hbar c\alpha_e^2}{3\pi r}\int_{2m_e c}^{\infty}\frac{e^{-p_{ri}r/h}}{p_{ri}}\sqrt{1-\frac{4m_e^2c^2}{p_{ri}^2}}\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_{ri}$$

$$=\frac{2\hbar c\alpha_e^2}{3\pi r}\int_{2m_e c}^{\infty}\frac{e^{-p_{ri}r/h}}{p_{ri}^2}\sqrt{(p_{ri}-2m_e c)(p_{ri}+2m_e c)}$$

$$\times\left(1+\frac{2m_e^2c^2}{p_{ri}^2}\right)dp_{ri}.$$

$$V_{e,1\text{ L, partícula}}(\mathbf{r})$$

$$\approx\frac{\hbar c\alpha_e^2}{\pi r}\int_0^{\infty}\frac{e^{-(p_t+2m_e c)r/h}}{4m_e^2c^2}\sqrt{4m_e c p_t}dp_t$$

$$=\frac{\hbar c\alpha_e^2 e^{-2m_e cr/h}}{2\pi r(m_e c)^{3/2}}\int_0^{\infty}e^{-p_t r/h}\sqrt{p_t}dp_t$$

$$=\frac{\hbar c\alpha_e^2 e^{-2m_e cr/h}}{2\pi r(m_e c)^{3/2}}\frac{\sqrt{\pi}}{2(r/\hbar)^{3/2}}$$

$$=\frac{\hbar c\alpha_e^2 e^{-2m_e c|\mathbf{r}|/h}}{4\sqrt{\pi}|\mathbf{r}|(m_e c|\mathbf{r}|/\hbar)^{3/2}}.$$

$$V_{\mathrm{e,1\ L,partícula-supermasiva}}(\mathbf{r})$$
$$= \int \tilde{V}_{\mathrm{e,1\ L,partícula-supermasiva}}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \frac{d^3p}{(2\pi\hbar)^3}$$
$$= -\frac{\alpha_{\mathrm{e}}\kappa c^2}{48\pi^4\hbar} \int \frac{|\mathbf{p}|^2 \log(|\mathbf{p}|^2)}{|\mathbf{p}|^2 - i\epsilon} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3p$$
$$= \frac{\alpha_{\mathrm{e}}\kappa\hbar^2 c^2}{12\pi^2|\mathbf{r}|^3} e^{i|\mathbf{r}|\sqrt{i\epsilon}/\hbar}$$
$$V_{\mathrm{e,1\ L,\ partícula\text{-}supermasiva}}(\mathbf{r}) = \frac{\kappa\hbar^2 c^2 \alpha_{\mathrm{e}}}{12\pi^2|\mathbf{r}|^3} = \frac{2G\hbar^2\alpha_{\mathrm{e}}}{3\pi c^2|\mathbf{r}|^3}$$
$$V_{\mathrm{e}}(\mathbf{r}) = V_{\mathrm{e}}(\mathbf{r}) + V_{\mathrm{e,1\ L,\ partícula}}(\mathbf{r}) + V_{\mathrm{e,1\ L,\ partícula\text{-}supermasiva}}(\mathbf{r})$$
$$= \frac{\hbar c\alpha_{\mathrm{e}}}{|\mathbf{r}|} + \frac{\hbar c\alpha_{\mathrm{e}}^2 e^{-2m_{\mathrm{e}}c|\mathbf{r}|/\hbar}}{4\sqrt{\pi}|\mathbf{r}|(m_{\mathrm{e}}c|\mathbf{r}|/\hbar)^{3/2}} + \frac{2G\hbar^2\alpha_{\mathrm{e}}}{3\pi c^2|\mathbf{r}|^3}$$
$$= \frac{\hbar c\alpha_{\mathrm{e}}}{|\mathbf{r}|}\left[1 + \frac{\alpha_{\mathrm{e}} e^{-2m_{\mathrm{e}}c|\mathbf{r}|/\hbar}}{4\sqrt{\pi}(m_{\mathrm{e}}c|\mathbf{r}|/\hbar)^{3/2}} + \frac{2G\hbar}{3\pi c^3|\mathbf{r}|^2}\right].$$
$$\Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(p^2) \approx \Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-|\mathbf{p}|^2)$$
$$= \frac{\kappa c|\mathbf{p}|^2}{32\pi^2\hbar}\left\{\frac{8m_{\mathrm{e}}^2c^2}{3|\mathbf{p}|^2}\left(1 - \frac{6m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}\right) - \log(m_{\mathrm{e}}^2c^2) + \left(1 - \frac{2m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}\right)\left(1 + \frac{4m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}\right)^{3/2} \times \log\left[1 - \frac{|\mathbf{p}|^2}{2m_{\mathrm{e}}^2c^2}\left(\sqrt{1 + \frac{4m_{\mathrm{e}}^2c^2}{|\mathbf{p}|^2}} - 1\right)\right]\right\},$$
$$\Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(p^2) \approx \Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-|\mathbf{p}|^2) = -\frac{\kappa c|\mathbf{p}|^2}{16\pi^2\hbar}\log(|\mathbf{p}|^2).$$

$$\tilde{V}_{\mathrm{g,1\ L,\ partícula}}(\mathbf{p}) = -\frac{\kappa\hbar^2 m_{\mathrm{e}}^2 c^4}{2(|\mathbf{p}|^2 - i\epsilon)} \Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-|\mathbf{p}|^2)$$
$$\tilde{V}_{\mathrm{g,1\ L,\ partícula}}(\mathbf{p}) = -\frac{\kappa\hbar^2 m_{\mathrm{e}}^2 c^4}{2(|\mathbf{p}|^2 - i\epsilon)} \Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-|\mathbf{p}|^2)$$
$$V_{\mathrm{g,1\ L,\ partícula}}(\mathbf{r})$$
$$= \int \tilde{V}_{\mathrm{g,1\ L,\ partícula}}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \frac{d^3p}{(2\pi\hbar)^3}$$
$$= -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi^2 r} \int_{-\infty}^{\infty} \frac{ip_{\mathrm{r}}\left(e^{-ip_{\mathrm{r}}r/\hbar} - e^{ip_{\mathrm{r}}r/\hbar}\right)}{p_{\mathrm{r}}^2 - i\epsilon} \Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-p_{\mathrm{r}}^2) dp_{\mathrm{r}}.$$
$$\mathrm{Im}\left[\Xi^{(\mathrm{r})}_{\mathrm{1\ L,\ partícula}}(-p_{\mathrm{r}}^2)\right] = \pm\frac{\kappa c p_{\mathrm{ri}}^2}{32\pi\hbar}\left(1 + \frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1 - \frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}$$

$$
\begin{aligned}
&V_{\mathrm{g,1\ L,\ partícula}}(\mathrm{r}) \\
&= -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi^2 r}\Bigg\{-2\pi i \mathrm{Res}_{p_{\mathrm{r}}=-\sqrt{i\epsilon}}\left[ip_{\mathrm{r}}\frac{e^{-ip_{\mathrm{r}}r/\hbar}}{p_{\mathrm{r}}^2 - i\epsilon}\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}\left(-p_{\mathrm{r}}^2\right)\right] \\
&-2\pi i \mathrm{Res}_{p_{\mathrm{r}}=\sqrt{i\epsilon}}\left[ip_{\mathrm{r}}\frac{e^{ip_{\mathrm{r}}r/\hbar}}{p_{\mathrm{r}}^2 - i\epsilon}\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}\left(-p_{\mathrm{r}}^2\right)\right] \\
&-2\int_{\epsilon-i\infty}^{\epsilon-2im_{\mathrm{e}}c}\frac{ip_{\mathrm{r}}e^{-ip_{\mathrm{r}}r/\hbar}}{(p_{\mathrm{r}}^2 - i\epsilon)}i\mathrm{Im}\left[\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}\left(-p_{\mathrm{r}}^2\right)\right]dp_{\mathrm{r}} \\
&+2\int_{-\epsilon+2im_{\mathrm{e}}c}^{-\epsilon+i\infty}\frac{ip_{\mathrm{r}}e^{ip_{\mathrm{r}}r/\hbar}}{p_{\mathrm{r}}^2 - i\epsilon)}i\mathrm{Im}\left[\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}\left(-p_{\mathrm{r}}^2\right)\right]dp_{\mathrm{r}}\Bigg\} \\
&= -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi^2 r}\Bigg\{-2\pi i\left[\frac{1}{2}e^{ir\sqrt{i\epsilon}/\hbar}\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}(-i\epsilon)\right] \\
&-2\pi i\left[\frac{1}{2}e^{ir\sqrt{i\epsilon}/\hbar}\Xi_{1\ \mathrm{L,\ partícula}}^{(\mathrm{r})}(-i\epsilon)\right] \\
&+\frac{\kappa c}{16\pi\hbar}\int_{\epsilon-i\infty}^{\epsilon-2im_{\mathrm{e}}c}\frac{p_{\mathrm{r}}^3 e^{-ip_{\mathrm{r}}r/\hbar}}{(p_{\mathrm{r}}^2 - i\epsilon)} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}dp_{\mathrm{r}} \\
&+\frac{\kappa c}{16\pi\hbar}\int_{-\epsilon+2im_{\mathrm{e}}c}^{-\epsilon+i\infty}\frac{p_{\mathrm{r}}^3 e^{ip_{\mathrm{r}}r/\hbar}}{(p_{\mathrm{r}}^2 - i\epsilon)} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}dp_{\mathrm{r}}\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
&V_{\mathrm{g,1\ L,partícula}}(\mathrm{r}) = -\frac{\kappa m_{\mathrm{e}}^2 c^4}{8\pi^2 r}\Bigg[\frac{\kappa c}{16\pi\hbar}\int_{-\infty}^{-2m_{\mathrm{e}}c}p_{\mathrm{ri}}e^{p_{\mathrm{ri}}r/\hbar} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}dp_{\mathrm{ri}} \\
&+\frac{\kappa c}{16\pi\hbar}\int_{2m_{\mathrm{e}}c}^{\infty}p_{\mathrm{ri}}e^{-p_{\mathrm{ri}}r/\hbar} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}dp_{\mathrm{ri}}\Bigg] \\
&= -\frac{\kappa^2 m_{\mathrm{e}}^2 c^5}{64\pi^3\hbar r}\int_{2m_{\mathrm{e}}c}^{\infty}p_{\mathrm{ri}}e^{-p_{\mathrm{ri}}r/\hbar} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)\left(1-\frac{4m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)^{3/2}dp_{\mathrm{ri}} \\
&= -\frac{\kappa^2 m_{\mathrm{e}}^2 c^5}{64\pi^3\hbar r}\int_{2m_{\mathrm{e}}c}^{\infty}\frac{e^{-p_{\mathrm{ri}}r/\hbar}}{p_{\mathrm{ri}}^2}\left[(p_{\mathrm{ri}}-2m_{\mathrm{e}}c)(p_{\mathrm{ri}}+2m_{\mathrm{e}}c)\right]^{3/2} \\
&\times\left(1+\frac{2m_{\mathrm{e}}^2c^2}{p_{\mathrm{ri}}^2}\right)dp_{\mathrm{ri}}.
\end{aligned}
$$

$$V_{\text{g,1 L, partícula}}(\mathbf{r})$$
$$\approx -\frac{3\kappa^2 m_e^2 c^5}{128\pi^3 \hbar r}\int_0^\infty \frac{e^{-(p_t+2m_e c)r/\hbar}}{4m_e^2c^2}(4m_e c p_t)^{3/2} dp_t$$
$$= -\frac{3\kappa^2 m_e^2 c^5 e^{-2m_e cr/\hbar}}{64\pi^3\hbar\sqrt{m_e cr}}\int_0^\infty e^{-p_t r/\hbar} p_t^{3/2} dp_t$$
$$= -\frac{3\kappa^2 m_e^2 c^5 e^{-2m_e cr/\hbar}}{64\pi^3\hbar\sqrt{m_e cr}}\frac{3\sqrt{\pi}}{4(r/\hbar)^{5/2}}$$
$$= -\frac{9\kappa^2 c^3 (m_e c\hbar)^{3/2} e^{-2m_e c|\mathbf{r}|/\hbar}}{256\pi^2\sqrt{\pi}|\mathbf{r}|^{7/2}}$$
$$= -\frac{9G^2 (m_e c\hbar)^{3/2} e^{-2m_e c|\mathbf{r}|/\hbar}}{4\sqrt{\pi}c^5|\mathbf{r}|^{7/2}}$$

$$V_{\text{g,1 L,partícula}}(\mathbf{r})$$
$$= \int \tilde{V}_{\text{g,1 L,partícula}}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\frac{d^3p}{(2\pi\hbar)^3}$$
$$= \frac{\kappa^2 m_e^2 c^5}{256\pi^5\hbar^2}\int \frac{|\mathbf{p}|^2 \log(|\mathbf{p}|^2)}{|\mathbf{p}|^2 - i\epsilon} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3p$$
$$= -\frac{\kappa^2 \hbar m_e^2 c^5}{64\pi^3|\mathbf{r}|^3} e^{i|\mathbf{r}|\sqrt{i\epsilon}/\hbar}$$

$$V_{\text{g,1 L, partícula}}(\mathbf{r}) = -\frac{\kappa^2\hbar m_e^2 c^5}{64\pi^3|\mathbf{r}|^3} = -\frac{G^2\hbar m_e^2}{\pi c^3|\mathbf{r}|^3}$$

$$V_g(\mathbf{r}) = V_g(\mathbf{r}) + V_{\text{g,1 L, partícula}}(\mathbf{r}) + V_{\text{g,1 L, partícula}}(\mathbf{r})$$
$$= -\frac{Gm_e^2}{|\mathbf{r}|} - \frac{G^2\hbar m_e^2}{\pi c^3|\mathbf{r}|^3} - \frac{9G^2(m_e c\hbar)^{3/2} e^{-2m_e c|\mathbf{r}|/\hbar}}{4\sqrt{\pi}c^5|\mathbf{r}|^{7/2}}$$
$$= -\frac{Gm_e^2}{|\mathbf{r}|}\left(1 + \frac{G\hbar}{\pi c^3|\mathbf{r}|^2} + \frac{9G\hbar e^{-2m_e c|\mathbf{r}|/\hbar}}{4c^3\sqrt{\pi m_e c/\hbar}|\mathbf{r}|^{5/2}}\right).$$

$$F_{2,1\text{ L, partícula}}^{e\gamma} = \frac{\alpha_e}{2\pi}$$

$$F_{2,1\text{ L, partícula-supermasiva}}^{e\gamma} = \frac{7\alpha_g}{4\pi}$$

$$F_{2,1\text{ L, partícula-supermasiva}}^{(e\gamma,\text{ diag1}} = -\frac{\alpha_g}{6\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{61}{6} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_e^2c^2}\right)\right]$$

$$F_{2,1\text{ L, partícula-supermasiva}}^{(e\gamma,\text{iag2})} = F_{2,1\text{ L, partícula-supermasiva}}^{(e,\text{ diag })}$$
$$= \frac{\alpha_g}{\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{7}{2} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_e^2c^2}\right)\right]$$

$$F_{2,1\text{ L, partícula-supermasiva}}^{(e,\text{ diag4}} = F_{2,1\text{ L, partícula-supermasiva}}^{(e\gamma,\text{ iag5 })}$$
$$= -\frac{33\alpha_g}{36\pi}\left[\frac{1}{\epsilon_{UV}} + \frac{64}{33} + \log\left(\frac{4\pi\mu^2 e^{-\gamma}}{m_e^2c^2}\right)\right]$$

$$F_{2,1\text{ L, partícula-supermasiva}}^{(e,\text{ diag })} = 0$$

$$\dot{e}_{av} = \partial_v x_a + H_{av}$$
$$g_{\mu\nu} = \eta^{ab}\dot{e}_{a\mu}\dot{e}_{b\nu}$$

$$\mathcal{L}_{\mathrm{TEGRW}} = \Bigg\{ \dot{e}_{a\nu} \Bigg[\frac{i\hbar c}{4} \bar{\psi}_8 \left(\overleftarrow{D} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^\nu \mathrm{t}^a \gamma_{\mathrm{F}} - \bar{\gamma}_{\mathrm{F}} \gamma_{\mathrm{B}}^5 \gamma_{\mathrm{B}}^\nu \mathrm{t}^a \overrightarrow{D} \right) \psi_8 - \frac{m_{\mathrm{e}} c^2}{2} \bar{\psi}_8 \mathrm{t}^a \gamma_{\mathrm{B}}^\nu \gamma_{\mathrm{B}}^5 \psi_8 - \bar{\Psi} \mathbf{t}^a \gamma_{\mathrm{B}}^\nu \gamma_{\mathrm{B}}^5 \Psi \Bigg] - m_{\mathrm{e}} c^2 \bar{\psi}_8 \psi_8 + \bar{\Psi}\Psi + \frac{1}{4\kappa} H_{a\mu\nu} S^{a\mu\nu} \Bigg\} \sqrt{-g}.$$

$$\mathcal{L}_{\mathrm{TEGRW}} = \Bigg[\frac{i\hbar c}{2} \bar{\psi} \left(\bar{\gamma}_{\mathrm{F}} \vec{\partial} - \overleftarrow{\partial} \gamma_{\mathrm{F}} \right) \psi - m_{\mathrm{e}} c^2 \bar{\psi}\psi + \bar{\Phi}\Theta + \bar{\Theta}\Phi + \bar{\Theta} \overleftarrow{\partial}_\rho \gamma_{\mathrm{B}}^\rho \left(\mathbf{I}_8 + \mathfrak{e}_0 \bar{\mathfrak{e}}_0 \right)^2 \gamma_{\mathrm{B}}^\sigma \vec{\partial}_\sigma \Theta + \frac{1}{4\kappa} H_{\rho\mu\nu} S^{\rho\mu\nu} \Bigg] \sqrt{-g}$$

$$\mathcal{L}_{\mathrm{TEGRW}} = \Bigg[\frac{i\hbar c}{2} \bar{\psi} \left(\gamma_{\mathrm{F}}^\nu \vec{\partial}_\nu - \overleftarrow{\partial}_\nu \gamma_{\mathrm{F}}^\nu \right) \psi - m_{\mathrm{e}} c^2 \bar{\psi}\psi - J_{\mathrm{e}}^\nu A_\nu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\kappa} H_{\rho\mu\nu} S^{\rho\mu\nu} \Bigg] \sqrt{-g}$$

$$\mathcal{L}_{\mathrm{TEGRW}} = \Bigg\{ \frac{i\hbar c}{4} \bar{\psi} \left[\overset{\blacksquare}{e}{}_b^\nu \left(\gamma_{\mathrm{F}}^b \overrightarrow{D}_\nu + \gamma_{\mathrm{F}\nu} \overrightarrow{D}^b \right) - \left(\overleftarrow{D}_\nu \gamma_{\mathrm{F}}^b + \overleftarrow{D}^b \gamma_{\mathrm{F}\nu} \right) \overset{\blacksquare}{e}{}_b^\nu \right] \psi - m_{\mathrm{e}} c^2 \bar{\psi}\psi - \frac{1}{4\mu_0} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{4\kappa} H_{\rho\mu\nu} S^{\rho\mu\nu} \Bigg\} \sqrt{-g}.$$

$$\widetilde{\nabla}_\rho S^{a\nu\rho} = \overset{\blacksquare}{\nabla}_\rho S^{a\nu\rho} = \kappa T_{\mathrm{TEGRW}}^{a\nu}$$

$$T_{\mathrm{TEGRW}}^{\mu\nu} = T_{\mathrm{D}}^{\mu\nu} + T_{\mathrm{D,diff}}^{\mu\nu} + T_{\mathrm{em}}^{\mu\nu} + T_{\mathrm{g}}^{\mu\nu},$$

$$T_{\mathrm{D}}^{\mu\nu} = \frac{i\hbar c}{4} \bar{\psi} \left(\gamma_{\mathrm{F}}^\mu \overrightarrow{D}^\nu + \gamma_{\mathrm{F}}^\nu \overrightarrow{D}^\mu - \overleftarrow{D}^\nu \gamma_{\mathrm{F}}^\mu - \overleftarrow{D}^\mu \gamma_{\mathrm{F}}^\nu \right) \psi - \frac{1}{2} g^{\mu\nu} \left[\frac{i\hbar c}{2} \bar{\psi} \left(\gamma_{\mathrm{F}}^\rho \overrightarrow{D}_\rho - \overleftarrow{D}_\rho \gamma_{\mathrm{F}}^\rho \right) \psi - m_{\mathrm{e}} c^2 \bar{\psi}\psi \right]$$

$$T_{\mathrm{D,diff}}^{\mu\nu} = -\frac{1}{2} g^{\mu\nu} \left[\frac{i\hbar c}{2} \bar{\psi} \left(\gamma_{\mathrm{F}}^\rho \overrightarrow{D}_\rho - \overleftarrow{D}_\rho \gamma_{\mathrm{F}}^\rho \right) \psi - m_{\mathrm{e}} c^2 \bar{\psi}\psi \right],$$

$$T_{\mathrm{em}}^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\rho\mu}{}_\rho F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right),$$

$$T_{\mathrm{g}}^{\mu\nu} = \frac{1}{\kappa} \left(H_{\sigma\rho}{}^{\mu} S^{\sigma\rho\nu} - \frac{1}{4} g^{\mu\nu} H_{\rho\sigma\lambda} S^{\rho\sigma\lambda} \right).$$

$$\widetilde{\nabla}_\nu T_{\mathrm{TEGRW}}^{a\nu} = \mathring{\nabla}_\nu T_{\mathrm{TEGRW}}^{a\nu} = 0$$

$$\widetilde{\nabla}_\rho S^{\mu\nu\rho} = \kappa \mathfrak{T}_{\mathrm{TEGRW}}^{\mu\nu}$$

$$\mathfrak{T}_{\mathrm{TEGRW}}^{\mu\nu} = T_{\mathrm{TEGRW}}^{\mu\nu} + \frac{1}{\kappa} S^{\sigma\rho\nu} \overset{\blacksquare}{\Gamma}{}^{\mu}{}_{\sigma\rho}.$$

$$\widetilde{\nabla}_\nu \tilde{T}_{\mathrm{TEGRW}}^{\mu\nu} = 0$$

$$\widetilde{\nabla}_\rho \left(\gamma_{\mathrm{B}}^\rho \Psi \right) = \gamma_{\mathrm{B}}^\rho \left(\vec{\partial}_\rho + \overset{\blacksquare}{e}{}_\rho^a \widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}_a^\sigma \right) \Psi = -\Phi$$

$$\widetilde{\nabla}_\rho F^{\rho\nu} = \mathring{\nabla}_\rho F^{\rho\nu} = \mu_0 J_{\mathrm{e}}^\nu,$$

$$\mathring{\nabla}_\rho F_{\mu\nu} + \mathring{\nabla}_\mu F_{\nu\rho} + \mathring{\nabla}_\nu F_{\rho\mu} = 0.$$

$$\widetilde{\nabla}_\nu J_{\mathrm{e}}^\nu = \mathring{\nabla}_\nu J_{\mathrm{e}}^\nu = 0.$$

$$i\hbar c \gamma_{\mathrm{F}}^\rho \left(\overrightarrow{D}_\rho + \frac{1}{2} \overset{\blacksquare}{e}{}_\rho^a \widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}_a^\sigma \right) \psi - m_{\mathrm{e}} c^2 \psi = 0$$

$$q = \begin{bmatrix} q^r \\ q^g \\ q^b \end{bmatrix}$$

$$Q_{\mathrm{L}}^{i} \in \left\{\begin{bmatrix} u_{\mathrm{L}} \\ d_{\mathrm{L}} \end{bmatrix}, \begin{bmatrix} c_{\mathrm{L}} \\ s_{\mathrm{L}} \end{bmatrix}, \begin{bmatrix} t_{\mathrm{L}} \\ b_{\mathrm{L}} \end{bmatrix}\right\},$$

$$L_{\mathrm{L}}^{i} \in \left\{\begin{bmatrix} \nu_{e\,\mathrm{L}} \\ e_{\mathrm{L}} \end{bmatrix}, \begin{bmatrix} \nu_{\mu\mathrm{L}} \\ \mu_{\mathrm{L}} \end{bmatrix}, \begin{bmatrix} \nu_{\tau\mathrm{L}} \\ \tau_{\mathrm{L}} \end{bmatrix}\right\},$$

$$u_{\mathrm{R}}^{i} \in \{u_{\mathrm{R}}, c_{\mathrm{R}}, t_{\mathrm{R}}\}, d_{\mathrm{R}}^{i} \in \{d_{\mathrm{R}}, s_{\mathrm{R}}, b_{\mathrm{R}}\},$$

$$e_{\mathrm{R}}^{i} \in \{e_{\mathrm{R}}, \mu_{\mathrm{R}}, \tau_{\mathrm{R}}\}, \nu_{\mathrm{R}}^{i} \in \{\nu_{e\mathrm{R}}, \nu_{\mu\mathrm{R}}, \nu_{\tau\mathrm{R}}\}$$

$$\psi_{j}^{i} \in \{Q_{\mathrm{L}}^{i}, u_{\mathrm{R}}^{i}, d_{\mathrm{R}}^{i}, L_{\mathrm{L}}^{i}, \nu_{\mathrm{R}}^{i}, e_{\mathrm{R}}^{i}\}$$

$$\psi_{8j}^{i} = \psi_{j}^{i}\mathfrak{e}_0 = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \psi_{j}^{i}, \mathbf{0}, \mathbf{0}, \mathbf{0}]^{T}$$

$$\vec{\mathbf{D}}_{\nu} = \vec{\partial}_{\nu} - i\frac{g_{\mathrm{s}}}{\sqrt{\hbar c}}G_{l\nu}\frac{\lambda^{l}}{2} - i\frac{g_{\mathrm{ew}}}{\sqrt{\hbar c}}W_{i\nu}\frac{\sigma_{\mathrm{F}}^{i}}{2} - i\frac{g_{\mathrm{ew}}'}{\sqrt{\hbar c}}B_{\nu}\frac{Y_{\mathrm{w}}}{2},$$

$$\overleftarrow{\mathbf{D}}_{\nu} = \overleftarrow{\partial}_{\nu} + i\frac{g_{\mathrm{s}}}{\sqrt{\hbar c}}G_{l\nu}\frac{\lambda^{l}}{2} + i\frac{g_{\mathrm{ew}}}{\sqrt{\hbar c}}W_{i\nu}\frac{\sigma_{\mathrm{F}}^{i}}{2} + i\frac{g_{\mathrm{ew}}'}{\sqrt{\hbar c}}B_{\nu}\frac{Y_{\mathrm{w}}}{2}.$$

$$\mathbf{G}_{\nu} = G_{l\nu}\frac{\boldsymbol{\lambda}^{l}}{2}, \mathbf{W}_{\nu} = W_{i\nu}\frac{\boldsymbol{\sigma}_{\mathrm{F}}^{i}}{2}$$

$$\mathbf{G}_{\mu\nu} = \partial_{\mu}\mathbf{G}_{\nu} - \partial_{\nu}\mathbf{G}_{\mu} - ig_{s}[\mathbf{G}_{\mu}, \mathbf{G}_{\nu}] = G_{l\mu\nu}\frac{\lambda^{l}}{2}$$

$$G_{l\mu\nu} = \partial_{\mu}G_{l\nu} - \partial_{\nu}G_{l\mu} + g_{\mathrm{s}}(f_{\mathrm{s}})_{l}^{mn}G_{m\mu}G_{n\nu}$$

$$\mathbf{W}_{\mu\nu} = \partial_{\mu}\mathbf{W}_{\nu} - \partial_{\nu}\mathbf{W}_{\mu} - ig_{\mathrm{ew}}[\mathbf{W}_{\mu}, \mathbf{W}_{\nu}] = W_{i\mu\nu}\frac{\sigma_{\mathrm{F}}^{i}}{2}$$

$$W_{i\mu\nu} = \partial_{\mu}W_{i\nu} - \partial_{\nu}W_{i\mu} + g_{\mathrm{ew}}(f_{\mathrm{w}})_{i}{}^{jk}W_{j\mu}W_{k\nu}$$

$$\mathcal{G}_{l} = [0, G_{l0x}, G_{l0y}, G_{l0z}, 0, iG_{lzy}, iG_{lxz}, iG_{lyx}]^{T}$$

$$\mathcal{W}_{i} = [0, W_{i0x}, W_{i0y}, W_{i0z}, 0, iW_{izy}, iW_{ixz}, iW_{iyx}]^{T},$$

$$\mathcal{B} = [0, B_{0x}, B_{0y}, B_{0z}, 0, iB_{zy}, iB_{xz}, iB_{yx}]^{T}.$$

$$\Psi_{i} \in \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_8, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{B}\}$$

$$G_{l} = [0, G_{l}{}^{x}, G_{l}{}^{y}, G_{l}{}^{z}, G_{l}{}^{0}, 0,0,0]^{T}$$
$$= [0, -G_{lx}, -G_{ly}, -G_{lz}, G_{l0}, 0,0,0]^{T},$$

$$W_{i} = [0, W_{i}^{x}, W_{i}^{y}, W_{i}^{z}, W_{i}^{0}, 0,0,0]^{T}$$
$$= [0, -W_{ix}, -W_{iy}, -W_{iz}, W_{i0}, 0,0,0]^{T},$$

$$B = [0, B^{x}, B^{y}, B^{z}, B^{0}, 0,0,0]^{T},$$
$$= [0, -B_{x}, -B_{y}, -B_{z}, B_{0}, 0,0,0]^{T}.$$

$$\vec{\mathbf{D}} = \vec{\partial} - i\frac{g_{\mathrm{s}}}{\sqrt{\hbar c}}G_{l}\frac{\lambda^{l}}{2} - i\frac{g_{\mathrm{ew}}}{\sqrt{\hbar c}}W_{i}\frac{\boldsymbol{\sigma}_{\mathrm{F}}^{i}}{2} - i\frac{g_{\mathrm{ew}}'}{\sqrt{\hbar c}}B\frac{Y_{\mathrm{w}}}{2}$$
$$= [\mathbf{0}, \vec{\mathbf{D}}_{x}, \vec{\mathbf{D}}_{y}, \vec{\mathbf{D}}_{z}, -\vec{\mathbf{D}}_{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}]^{T},$$

$$\overleftarrow{\mathbf{D}} = \overleftarrow{\partial} + i\frac{g_{\mathrm{s}}}{\sqrt{\hbar c}}G_{l}\frac{\lambda^{l}}{2} + i\frac{g_{\mathrm{ew}}}{\sqrt{\hbar c}}W_{i}\frac{\boldsymbol{\sigma}_{\mathrm{F}}^{i}}{2} + i\frac{g_{\mathrm{ew}}'}{\sqrt{\hbar c}}B\frac{Y_{\mathrm{w}}}{2}$$
$$= [\mathbf{0}, \overleftarrow{\mathbf{D}}_{x}, \overleftarrow{\mathbf{D}}_{y}, \overleftarrow{\mathbf{D}}_{z}, \overleftarrow{\mathbf{D}}_{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}].$$

$$\varphi = \begin{bmatrix} \varphi^{+} \\ \varphi^{0} \end{bmatrix}$$

$$\varphi_{8} = \varphi\mathfrak{e}_0 = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}]^{T}$$

$$\mathcal{L} = \mathcal{L}_{S=0} + \mathcal{L}_{S=\frac{1}{2}} + \mathcal{L}_{S=1} + \mathcal{L}_{S=2} + \mathcal{L}_{\mathrm{pot}} + \mathcal{L}_{\mathrm{Yukawa}}$$

$$\mathcal{L}_{S=0} = \hbar c\left(i\bar{\varphi}_8\overleftarrow{\mathbf{D}}\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overrightarrow{\mathcal{D}}_\nu\mathbf{I}_{\mathrm{g}}\overrightarrow{\mathbf{D}}\varphi_8 - \bar{\varphi}_8\overleftarrow{\mathbf{D}}\overrightarrow{\mathbf{D}}\varphi_8\right)\sqrt{-g},$$

$$\mathcal{L}_{S=\frac{1}{2}} = \sum_{i,j}\frac{\hbar c}{4}\bar{\psi}_{8j}^{i}\left(\bar{\gamma}_{\mathrm{F}}\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overrightarrow{\mathcal{D}}_\nu\mathbf{I}_{\mathrm{g}}\overrightarrow{\mathbf{D}}\right.$$

$$\left.-\overleftarrow{\mathbf{D}}\bar{\mathbf{I}}_{\mathrm{g}}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overrightarrow{\mathcal{D}}_\nu\mathbf{I}_{\mathrm{g}}\gamma_{\mathrm{F}}\right)\psi_{8j}^{i}\sqrt{-g},$$

$$\mathcal{L}_{S=1} = \sum_{i}\left(i\bar{\Psi}_i\mathbf{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overline{\overline{\overrightarrow{\mathcal{D}}}}_\nu^{\dagger}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger}\Psi_i + \bar{\Psi}_i\Psi_i\right)\sqrt{-g},$$

$$\mathcal{L}_{S=2} = \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu}\sqrt{-g}.$$

$$\begin{aligned}\mathcal{L}_{\mathrm{Yukawa}} = \sum_{i,j}\Big[&-\frac{i}{2}(Y_{\mathrm{u}}')_{ij}(\bar{Q}_{\mathrm{L8}})_i\tilde{\varphi}\mathbf{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overline{\overline{\mathcal{D}}}_\nu^{\dagger}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger}(u_{\mathrm{R8}})_j\\ &-\frac{i}{2}(Y_{\mathrm{d}}')_{ij}(\bar{Q}_{\mathrm{L8}})_i\varphi\mathbf{I}_{\mathbf{g}}^{\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overline{\overline{\mathcal{D}}}_\nu^{\dagger}\bar{\mathbf{I}}_{\mathrm{g}}^{\dagger}(d_{\mathrm{R8}})_j\\ &-\frac{i}{2}(Y_{\mathrm{e}}')_{ij}(\bar{L}_{\mathrm{L8}})_i\varphi\mathbf{I}_{\mathrm{g}}^{\dagger}\gamma_{\mathrm{B}}^5\gamma_{\mathrm{B}}^{\nu}\overline{\mathcal{D}}_\nu^{\dagger}\bar{\mathrm{I}}_{\mathrm{g}}^{\dagger}(e_{\mathrm{R8}})_j\\ &+(2Y_{\mathrm{u}}' - Y_{\mathrm{u}})_{ij}(\bar{Q}_{\mathrm{L8}})_i\tilde{\varphi}(u_{\mathrm{R8}})_j\\ &+(2Y_{\mathrm{d}}' - Y_{\mathrm{d}})_{ij}(\bar{Q}_{\mathrm{L8}})_i\varphi(d_{\mathrm{R8}})_j\\ &+(2Y_{\mathrm{e}}' - Y_{\mathrm{e}})_{ij}(\bar{L}_{\mathrm{L8}})_i\varphi(e_{\mathrm{R8}})_j + h.c.\Big]\sqrt{-g}.\end{aligned}$$

Teoría	Gravedad unificada	QED	QCD	TEGRW
Simetría de Gauge	$4\times\mathrm{U}(1)$ of $\mathbf{I}_{\mathrm{g}}$	Partícula U(1)	partícula SU(3)	Translaciones de espacio – tiempo.
Dimensión de simetría	4	1	8	∞
Simetría compacta	Compacto	Compacto	Compacto	No compacto
Transformación de simetría	$\otimes_a e^{i\phi(a)}\mathbf{t}^{(a)}$ of $\mathbf{I}_{\mathbf{g}}$	$e^{i\theta Q}$	$e^{i\theta_l\lambda^l/2}$	$e^{\xi^a\vec{\partial}_a}$
Geneadores de simetría	$\mathbf{t}^a$	Q	$\boldsymbol{\lambda}^l/2$	$\vec{\partial}_a$
Generador dimensional de simetría	Dimensión 1	Dimensión 2	Dimensión 3	Dimensión 4
Constante de acoplamiento.	$E_{\mathrm{g}}' = E_{\mathrm{g}}\sqrt{\frac{\kappa}{\hbar c}}$	$e' = e\sqrt{\frac{\mu_0 c}{\hbar}}$	g_{s}	$k = \sqrt{\kappa\hbar\%}$
Constante de acoplamiento en dimensión	Dimensión 1	Dimensión 2	Dimensión 3	Dimensión 4
Campo de Gauge gravitacional	$H_{a\nu}'$	A_ν'	$G_{l\nu}$	$B'^{a}{}_{\nu}$
Derivada covariante de gauge.	$\vec{\partial}_\nu - i\frac{E_{\mathrm{g}}'}{\sqrt{\hbar c}}H_{a\nu}'\mathbf{t}^a$	$\vec{\partial}_\nu + i\frac{e'}{\sqrt{\hbar c}}A_\nu' Q$	$\vec{\partial}_\nu - i\frac{g_{\mathrm{s}}}{\sqrt{\hbar c}}G_{l\nu}\frac{\boldsymbol{\lambda}^l}{2}$	$\vec{\partial}_\nu + i\frac{k}{\sqrt{\hbar c}}B'{}_{\nu}\vec{\partial}_a$
Métrica del campo de gauge.	$H_{a\mu\nu}' = \partial_\mu H_{a\nu}' - \partial_\nu H_{a\mu}'$	$F_{\mu\nu}' = \partial_\mu A_\nu' - \partial_\nu A_\mu'$	$G_{l\mu\nu} = \partial_\mu G_{l\nu} - \partial_\nu G_{l\mu} +$ $g_{\mathrm{s}}(f_{\mathrm{s}})_l^{mn}G_{m\mu}G_{n\nu}$	$\mathrm{T}'{}_{\mu\nu}$ $= \partial_\mu B'^{a}{}_{\nu} - \partial_\nu B'{}_{\mu}$
Densidad lagrangiana	$\frac{1}{8}H_{a\mu\nu}'\widetilde{H}'^{a}{}_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}$	$-\frac{1}{8}F_{\mu\nu}'\tilde{F}_{\sigma\lambda}'\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}$	$-\frac{1}{8}G_{l\mu\nu}\tilde{G}^{l}{}_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}$	$\frac{1}{8}\mathbf{T}_{a\mu\nu}'\tilde{\tilde{\mathrm{T}}}'^{a}{}_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}$

$$\boldsymbol{\gamma}_{\mathrm{B}}^{0}=\begin{bmatrix}\mathbf{I}_4 & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}_4\end{bmatrix}, \boldsymbol{\gamma}_{\mathrm{B}}^{i}=\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}_{\mathrm{B}}^{i}\\ -\boldsymbol{\sigma}_{\mathrm{B}}^{i} & \mathbf{0}\end{bmatrix},$$

$$\boldsymbol{\gamma}_{\mathrm{B}}^{5}=\frac{i}{4!}\varepsilon_{abcd}\boldsymbol{\gamma}_{\mathrm{B}}^{a}\boldsymbol{\gamma}_{\mathrm{B}}^{b}\boldsymbol{\gamma}_{\mathrm{B}}^{c}\boldsymbol{\gamma}_{\mathrm{B}}^{d}=i\boldsymbol{\gamma}_{\mathrm{B}}^{0}\boldsymbol{\gamma}_{\mathrm{B}}^{x}\boldsymbol{\gamma}_{\mathrm{B}}^{y}\boldsymbol{\gamma}_{\mathrm{B}}^{z}.$$

$$\boldsymbol{\sigma}_{\mathrm{B}}^{i}=\mathbf{K}_{\mathrm{boost}}^{i}+i\mathbf{K}_{\mathrm{rot}}^{i}.$$

$$\mathbf{K}_{\mathrm{boost}}^{x}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \mathbf{K}_{\mathrm{boost}}^{y}=\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\mathbf{K}_{\mathrm{boost}}^{z}=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix},$$

$$\mathbf{K}_{\mathrm{rot}}^{x}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}, \mathbf{K}_{\mathrm{rot}}^{y}=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix},$$

$$\mathbf{K}_{\mathrm{rot}}^{z}=\begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}.$$

$$\mathbf{t}^{0}=\begin{bmatrix}0 & -\mathbf{I}_4\\ -\mathbf{I}_4 & 0\end{bmatrix},$$

$$\mathbf{t}^{i}=\begin{bmatrix}-\mathbf{K}_{\mathrm{boost}}^{i}+i\mathbf{K}_{\mathrm{rot}}^{i} & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}_{\mathrm{boost}}^{i}+i\mathbf{K}_{\mathrm{rot}}^{i}\end{bmatrix}.$$

$$\mathcal{L}_{\mathrm{QED},0}=\left[\frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu}-\overleftarrow{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)\psi-m_{\mathrm{e}}c^{2}\bar{\psi}\psi\right]\sqrt{-g}$$

$$\psi\rightarrow U_{\mathrm{e}}\psi,\ \text{donde } U_{\mathrm{e}}=e^{i\theta Q}$$

$$\delta\psi=iQ\psi\delta\theta$$

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{QED},0}=&\Big[\frac{i\hbar c}{2}(\delta\bar{\psi})\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu}-\overleftarrow{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)\psi\\&+\frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\mathrm{F}}^{\nu}\vec{\partial}_{\nu}-\overleftarrow{\partial}_{\nu}\gamma_{\mathrm{F}}^{\nu}\right)(\delta\psi)\\&-m_{\mathrm{e}}c^{2}(\delta\bar{\psi})\psi-m_{\mathrm{e}}c^{2}\bar{\psi}(\delta\psi)\Big]\sqrt{-g}\\&=-\sqrt{-g}Q\hbar c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi\partial_{\nu}\delta\theta\\&=-\frac{\sqrt{-g}\hbar}{e}J_{\mathrm{e}}^{\nu}\partial_{\nu}\delta\theta\end{aligned}$$

$$J_{\mathrm{e}}^{\nu}=q_{\mathrm{e}}c\bar{\psi}\gamma_{\mathrm{F}}^{\nu}\psi$$

$$\begin{aligned}\delta S_{\mathrm{QED},0}=\int\ \delta\mathcal{L}_{\mathrm{QED},0}d^{4}x=-\int\ \frac{\sqrt{-g}\hbar}{e}J_{\mathrm{e}}^{\nu}\partial_{\nu}\delta\theta d^{4}x\\=-\int\ \partial_{\nu}\left(\frac{\sqrt{-g}\hbar}{e}J_{\mathrm{e}}^{\nu}\delta\theta\right)d^{4}x\\+\int\ \partial_{\nu}\left(\frac{\sqrt{-g}\hbar}{e}J_{\mathrm{e}}^{\nu}\right)\delta\theta d^{4}x\\=\int\ \frac{\sqrt{-g}\hbar}{e}\left(\mathring{\nabla}_{\nu}J_{\mathrm{e}}^{\nu}\right)\delta\theta d^{4}x.\end{aligned}$$

$$\widetilde{\nabla}_{\nu}J_{\mathrm{e}}^{\nu}=\mathring{\nabla}_{\nu}J_{\mathrm{e}}^{\nu}=0$$

$$\vec{D}_{\nu}=\vec{\partial}_{\nu}+i\frac{e}{\hbar}A_{\nu}Q$$

$$A_\nu \to A_\nu - \frac{\hbar}{e}\partial_\nu \theta$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{\text{em,kin}} = -\frac{1}{8\mu_0} F_{\mu\nu}\tilde{F}_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\sqrt{-g}.$$

$$\mathcal{L}_{\text{QED}}$$
$$= \left[\frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\text{F}}^{\nu}\vec{D}_\nu - \overleftarrow{D}_\nu\gamma_{\text{F}}^{\nu}\right)\psi - m_{\text{e}}c^2\bar{\psi}\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\right]\sqrt{-g}$$
$$= \left[\frac{i\hbar c}{2}\bar{\psi}\left(\gamma_{\text{F}}^{\nu}\vec{\partial}_\nu - \overleftarrow{\partial}_\nu\gamma_{\text{F}}^{\nu}\right)\psi - m_{\text{e}}c^2\bar{\psi}\psi - J_{\text{e}}^{\nu}A_\nu\right.$$
$$\left.- \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\right]\sqrt{-g}.$$

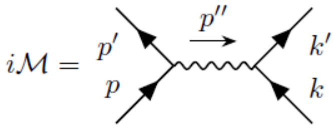
$$A_\nu \to \sqrt{\mu_0}A'_\nu$$

$$F_{\mu\nu} \to \sqrt{\mu_0}F'_{\mu\nu}$$

$$\mathcal{L}_{\text{em,kin}} \to -\frac{1}{8}F'_{\mu\nu}\tilde{F}'_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}$$

$$e' = e\sqrt{\frac{\mu_0 c}{\hbar}} = \sqrt{4\pi\alpha_{\text{e}}}, \alpha_{\text{e}} = \frac{e^2}{4\pi\varepsilon_0\hbar c}.$$

$$\vec{D}_\nu \to \vec{\partial}_\nu + i\frac{e'}{\sqrt{\hbar c}}A'_\nu Q$$



$$= \frac{-i\eta_{\mu\nu}}{(p-p')^2 + i\epsilon}\left[\bar{u}(p')\frac{ie\gamma_{\text{F}}^{\mu}}{\sqrt{\varepsilon_0\hbar c}}u(p)\right]\left[\bar{u}(k')\frac{ie\gamma_{\text{F}}^{\nu}}{\sqrt{\varepsilon_0\hbar c}}u(k)\right]$$

$$= \frac{ie^2}{\varepsilon_0\hbar c}\frac{\eta_{\mu\nu}}{(p-p')^2 + i\epsilon}\bar{u}(p')\gamma_{\text{F}}^{\mu}u(p)\bar{u}(k')\gamma_{\text{F}}^{\nu}u(k).$$

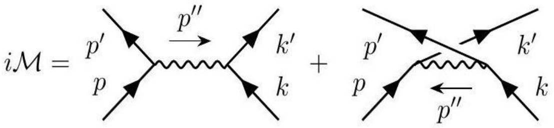
$$\bar{u}^{s'}(p')\gamma_{\text{F}}^{0}u^{s}(p) \to 2m_{\text{e}}c\delta^{ss'}, \bar{u}^{s'}(p')\boldsymbol{\gamma}_{\text{F}}^{i}u^{s}(p) \to 0.$$

$$i\mathcal{M} = -\frac{ie^2}{\varepsilon_0\hbar c}\frac{4m_{\text{e}}^2c^2\delta^{ss'}\delta^{rr'}}{|\mathbf{p}''|^2 - i\epsilon}$$

$$\tilde{V}_{\text{e}}(\mathbf{p}'') = -\frac{\hbar^3}{4m_{\text{e}}^2 c}\sum_{s',r'}\mathcal{M} = \frac{\hbar^2 e^2}{\varepsilon_0(|\mathbf{p}''|^2 - i\epsilon)}$$

$$V_{\text{e}}(\mathbf{r}) = \int \tilde{V}_{\text{e}}(\mathbf{p}'')e^{i\mathbf{p}''\cdot\mathbf{r}/\hbar}\frac{d^3p''}{(2\pi\hbar)^3}$$
$$= \frac{e^2}{(2\pi)^3\varepsilon_0\hbar}\int\frac{e^{i\mathbf{p}''\cdot\mathbf{r}/\hbar}}{|\mathbf{p}''|^2 - i\epsilon}d^3p''$$
$$= \frac{e^2}{4\pi\varepsilon_0|\mathbf{r}|}e^{i|\mathbf{r}|\sqrt{i\epsilon}/\hbar}.$$

$$V_{\text{e}}(\mathbf{r}) = \frac{e^2}{4\pi\varepsilon_0|\mathbf{r}|} = \frac{\hbar c\alpha_{\text{e}}}{|\mathbf{r}|}$$



$$=\frac{-i\eta_{\mu\nu}}{(p-p')^2}\left[\bar{u}(p')\frac{ie\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_0\hbar c}}u(p)\right]\left[\bar{u}(k')\frac{ie\gamma_{\mathrm{F}}^{\nu}}{\sqrt{\varepsilon_0\hbar c}}u(k)\right]$$
$$-\frac{-i\eta_{\mu\nu}}{(p-k')^2}\left[\bar{u}(k')\frac{ie\gamma_{\mathrm{F}}^{\mu}}{\sqrt{\varepsilon_0\hbar c}}u(p)\right]\left[\bar{u}(p')\frac{ie\gamma_{\mathrm{F}}^{\nu}}{\sqrt{\varepsilon_0\hbar c}}u(k)\right]$$
$$=\frac{ie^2}{\varepsilon_0\hbar c}\eta_{\mu\nu}\left[\frac{1}{t}\bar{u}(p')\gamma_{\mathrm{F}}^{\mu}u(p)\bar{u}(k')\gamma_{\mathrm{F}}^{\nu}u(k)\right.$$
$$\left.-\frac{1}{u}\bar{u}(k')\gamma_{\mathrm{F}}^{\mu}u(p)\bar{u}(p')\gamma_{\mathrm{F}}^{\nu}u(k)\right].$$

$$|\mathcal{M}|^2=\frac{e^4}{\varepsilon_0^2\hbar^2c^2}\eta_{\mu\nu}\eta_{\alpha\beta}\left\{\frac{1}{t^2}\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\mu}\mathbf{u}_p\gamma_{\mathrm{F}}^{\alpha}\mathbf{u}_{p'}\right)\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\nu}\mathbf{u}_k\gamma_{\mathrm{F}}^{\beta}\mathbf{u}_{k'}\right)\right.$$
$$+\frac{1}{u^2}\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\mu}\mathbf{u}_p\gamma_{\mathrm{F}}^{\alpha}\mathbf{u}_{k'}\right)\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\nu}\mathbf{u}_k\gamma_{\mathrm{F}}^{\beta}\mathbf{u}_{p'}\right)$$
$$-\frac{1}{tu}\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\mu}\mathbf{u}_p\gamma_{\mathrm{F}}^{\alpha}\mathbf{u}_{k'}\gamma_{\mathrm{F}}^{\nu}\mathbf{u}_k\gamma_{\mathrm{F}}^{\beta}\mathbf{u}_{p'}\right)$$
$$\left.-\frac{1}{tu}\mathrm{Tr}\left(\gamma_{\mathrm{F}}^{\mu}\mathbf{u}_p\gamma_{\mathrm{F}}^{\alpha}\mathbf{u}_{p'}\gamma_{\mathrm{F}}^{\nu}\mathbf{u}_k\gamma_{\mathrm{F}}^{\beta}\mathbf{u}_{k'}\right)\right\}.$$

$$\langle|\mathcal{M}|^2\rangle=\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2$$
$$=\frac{2e^4}{\varepsilon_0^2\hbar^2c^2t^2u^2}\{s^2(t+u)^2+t^4+u^4$$
$$-4m_{\mathrm{e}}^2c^2[s(t^2+3tu+u^2)+t^2(t-2u)+u^2(u-2t)]$$
$$+8m_{\mathrm{e}}^4c^4(t^2+tu+u^2)\}.$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{cm}}=\frac{\hbar^2c^2}{64\pi^2E_{\mathrm{cm}}^2}\left\langle|\mathcal{M}|^2\right\rangle$$
$$=\frac{\hbar^2c^2\alpha_{\mathrm{e}}^2}{E_{\mathrm{cm}}^2p_{\mathrm{r}}^4\sin^4\ \theta_{\mathrm{r}}}\{4(m_{\mathrm{e}}^2c^2+2p_{\mathrm{r}}^2)^2$$
$$+[4p_{\mathrm{r}}^4-3(m_{\mathrm{e}}^2c^2+2p_{\mathrm{r}}^2)^2]\sin^2\ \theta_{\mathrm{r}}+p_{\mathrm{r}}^4\sin^4\ \theta_{\mathrm{r}}\}.$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{cm}}=\frac{\hbar^2\alpha_{\mathrm{e}}^2m_{\mathrm{e}}^4c^6}{E_{\mathrm{cm}}^2p_{\mathrm{r}}^4\sin^4\ \theta_{\mathrm{r}}}(4-3\sin^2\ \theta_{\mathrm{r}}).$$

$$x^a\to U_{\mathrm{TEGRW}}x^a,\ \text{where } U_{\mathrm{TEGRW}}=e^{\xi^a\vec{\partial}_a}.$$
$$\mathcal{D}_\nu=\vec{\partial}_\nu+B^a{}_\nu\vec{\partial}_a.$$
$$B_\nu^a\to B_\nu^a-\partial_\nu\xi^a.$$
$$\overset{\blacksquare}{T}{}^a_{\mu\nu}=\partial_\mu B_\nu^a-\partial_\nu B_\mu^a.$$
$$\mathcal{L}_{\mathrm{TEGRW,\,g}}=\frac{1}{8\kappa}\overset{\blacksquare}{T}_{a\mu\nu}\overset{\widetilde{\blacksquare}}{T}{}^a_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}..$$
$$B_\nu^a\to\sqrt{\kappa}B'{}^a_\nu$$
$$\overset{\blacksquare}{T}{}^a{}_{\mu\nu}\to\sqrt{\kappa}\overset{\blacksquare}{T}{}'{}_{\mu\nu}.$$
$$\mathcal{L}_{\mathrm{TEGRW,\,g}}\to\frac{1}{8}\overset{\blacksquare}{T}{}'_{a\mu\nu}\overset{\widetilde{\blacksquare}}{T}{}'^a_{\sigma\lambda}\varepsilon^{\mu\nu\sigma\lambda}\sqrt{-g}.$$
$$k=\sqrt{\kappa\hbar c}.$$
$$\mathcal{D}_\nu\to\vec{\partial}_\nu+i\frac{k}{\sqrt{\hbar c}}B'^a_\nu\vec{\partial}_a$$
$$\partial_\rho F^{\nu\rho}=(\eta^{\rho\nu}\partial^2-\partial^\rho\partial^\nu)A_\rho=\mu_0J_{\mathrm{e}}^\nu$$
$$\partial_\rho F^{\nu\rho}+\partial^\nu\left(\partial_\rho A^\rho\right)=\partial^2A^\nu=\mu_0J_{\mathrm{e}}^\nu$$

$$\partial_\rho S^{\mu\nu\rho} = \Big[\frac{1}{2}(\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\sigma}\partial^\mu\partial^\rho - \eta^{\mu\rho}\eta^{\nu\sigma}\partial^2 - \eta^{\nu\rho}\eta^{\mu\sigma}\partial^2) - \eta^{\rho\sigma}\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\mu\nu}\eta^{\rho\sigma}\partial^2\Big]H_{\rho\sigma} = \kappa T_{\mathrm{m}}^{\mu\nu}.$$

$$-P^{\mu\nu,\rho\sigma}\partial^2 H_{\rho\sigma} = \kappa T_{\mathrm{m}}^{\mu\nu}$$

$$\frac{\partial\mathcal{L}_{\mathrm{TEGRW}}}{\partial H_{a\nu}} - \partial_\rho\left[\frac{\partial\mathcal{L}_{\mathrm{TEGRW}}}{\partial(\partial_\rho H_{a\nu})}\right] = 0$$

$$\frac{\partial \overset{\blacksquare}{e}_{b\mu}}{\partial H_{a\nu}} = \delta_b^a\delta_\mu^\nu$$

$$\frac{\partial \overset{\blacksquare}{e}^{b\mu}}{\partial H_{a\nu}} = -\overset{\blacksquare}{e}^{b\nu}\overset{\blacksquare}{e}^{a\mu},$$

$$\frac{\partial \overset{\blacksquare}{e}_\mu^b}{\partial H_{a\nu}} = \eta^{ab}\delta_\mu^\nu$$

$$\frac{\partial \overset{\blacksquare}{e}_b^\mu}{\partial H_{a\nu}} = -\overset{\blacksquare}{e}_b^\nu\overset{\blacksquare}{e}^{a\mu}$$

$$\frac{\partial\sqrt{-g}}{\partial H_{a\nu}} = \sqrt{-g}\,\overset{\blacksquare}{e}^{a\nu}$$

$$\frac{\partial g_{\sigma\lambda}}{\partial H_{a\nu}} = \delta_\sigma^\nu\overset{\blacksquare}{e}_\lambda^a + \delta_\lambda^\nu\overset{\blacksquare}{e}_\sigma^a$$

$$\frac{\partial g^{\sigma\lambda}}{\partial H_{a\nu}} = -g^{\sigma\nu}\overset{\blacksquare}{e}^{a\lambda} - g^{\lambda\nu}\overset{\blacksquare}{e}^{a\sigma}.$$

$$\mathcal{L}_{\mathrm{TEGRW,\,g}} = \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu}\sqrt{-g}.$$

$$H_{\rho\mu\nu}S^{\rho\mu\nu} = \frac{1}{2}H_{\rho\mu\nu}H^{\rho\mu\nu} + H_{\rho\mu\nu}H^{\mu\rho\nu} - 2H_{\mu\nu}^{\nu}H_\rho^{\rho\mu}.$$

$$\frac{\partial}{\partial H_{a\nu}}\left(H_{\rho\mu\sigma}H^{\rho\mu\sigma}\right) = 4H_{\sigma\rho}{}^{a}H^{\sigma\nu\rho},$$

$$\frac{\partial}{\partial H_{a\nu}}\left(H_{\rho\mu\sigma}H^{\mu\rho\sigma}\right) = -2H_{\sigma\rho}{}^{a}(H^{\nu\rho\sigma} + H^{\rho\sigma\nu}),$$

$$\frac{\partial}{\partial H_{a\nu}}\left(H_{\mu\sigma}^{\sigma}H_\rho^{\rho\mu}{}_\rho\right) = 2H_{\sigma\rho}{}^{a}\left(g^{\sigma\rho}H^{\lambda\nu}{}_\lambda - g^{\sigma\nu}H_\lambda^{\lambda\rho}\right).$$

$$\frac{\partial}{\partial H_{a\nu}}\left(\frac{1}{4\kappa}H_{\rho\mu\sigma}S^{\rho\mu\sigma}\right) = -\frac{1}{\kappa}H_{\sigma\rho}^{a}S^{\sigma\rho\nu}.$$

$$\frac{\partial}{\partial H_{a\nu}}\left(\frac{1}{4\kappa}H_{\rho\mu\sigma}S^{\rho\mu\sigma}\sqrt{-g}\right)$$

$$= \sqrt{-g}\frac{\partial}{\partial H_{a\nu}}\left(\frac{H_{\rho\mu\sigma}S^{\rho\mu\sigma}}{4\kappa}\right) + \frac{H_{\rho\mu\sigma}S^{\rho\mu\sigma}}{4\kappa}\frac{\partial\sqrt{-g}}{\partial H_{a\nu}}$$

$$= -\sqrt{-g}\left[\frac{1}{\kappa}\left(H_{\mu\sigma}{}^{a}S^{\mu\sigma\nu} - \frac{1}{4}\dot{e}^{a\nu}H_{\rho\mu\sigma}S^{\rho\mu\sigma}\right)\right].$$

$$\frac{\partial\mathcal{L}_{\mathrm{TEGRW,\,g}}}{\partial H_{a\nu}} = -\sqrt{-g}\,T_{\mathrm{g}}^{a\nu}$$

$$T_{\mathrm{g}}^{a\nu} = \frac{1}{\kappa}\overset{\blacksquare}{e}_\lambda^a\left(H_{\mu\sigma}{}^{\lambda}S^{\mu\sigma\nu} - \frac{1}{4}g^{\lambda\nu}H_{\rho\mu\sigma}S^{\rho\mu\sigma}\right).$$

$$\frac{\partial}{\partial(\partial_\rho H_{av})}\left(H_{\lambda\mu\sigma}H^{\lambda\mu\sigma}\right) = 4H^{a\rho\nu},$$

$$\frac{\partial}{\partial(\partial_\rho H_{av})}\left(H_{\lambda\mu\sigma}H^{\mu\lambda\sigma}\right) = 2(H^{\nu\rho a} - H^{\rho\nu a}),$$

$$\frac{\partial}{\partial(\partial_\rho H_{av})}\left(H^{\sigma}_{\mu\sigma}H^{\lambda\mu}_{\lambda}\right) = 2\left(\overset{\blacksquare}{e}{}^{a\nu}H^{\lambda\rho}_{\lambda} - \overset{\blacksquare}{e}{}^{a\rho}H^{\lambda\nu}{}_{\lambda}\right).$$

$$\frac{\partial}{\partial(\partial_\rho H_{av})}\left(\frac{\sqrt{-g}}{4\kappa}H_{\lambda\mu\sigma}S^{\lambda\mu\sigma}\right) = -\frac{\sqrt{-g}}{\kappa}S^{a\nu\rho}.$$

$$-\partial_\rho\left[\frac{\partial\mathcal{L}_{\mathrm{TEGRW,\,g}}}{\partial(\partial_\rho H_{av})}\right] = \frac{1}{\kappa}\partial_\rho\left(\sqrt{-g}S^{a\nu\rho}\right).$$

$$\frac{\partial\mathcal{L}_{\mathrm{TEGRW,\,g}}}{\partial H_{av}} - \partial_\rho\left[\frac{\partial\mathcal{L}_{\mathrm{TEGRW,\,g}}}{\partial(\partial_\rho H_{av})}\right]$$

$$= \frac{1}{\kappa}\partial_\rho\left(\sqrt{-g}S^{a\nu\rho}\right) - \sqrt{-g}T^{a\nu}_{\mathrm{g}}.$$

$$\mathcal{L}_{\mathrm{TEGRW,em}} = -\frac{1}{4\mu_0}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\sqrt{-g}$$

$$\frac{\partial\mathcal{L}_{\mathrm{TEGRW,em}}}{\partial H_{av}} = -\sqrt{-g}T^{a\nu}_{\mathrm{em}}$$

$$T^{a\nu}_{\mathrm{em}} = \overset{\blacksquare}{e}{}^{a}{}_{\mu}\left[\frac{1}{\mu_0}\left(F^{\mu}_{\rho}F^{\rho\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\right].$$

$$\mathcal{L}_{\mathrm{TEGRW,D}} = \left\{\frac{i\hbar c}{4}\overline{\psi}\left[\overset{\blacksquare}{e}{}^{\nu}_{b}\left(\gamma^{b}_{\mathrm{F}}\overrightarrow{D}_{\nu} + \gamma_{\mathrm{F}\nu}\overrightarrow{D}^{b}\right)\right.\right.$$
$$\left.\left. -\left(\overleftarrow{D}_{\nu}\gamma^{b}_{\mathrm{F}} + \overleftarrow{D}^{b}\gamma_{\mathrm{F}\nu}\right)\overset{\blacksquare}{e}{}^{\nu}_{b}\right]\psi - m_{\mathrm{e}}c^2\bar{\psi}\psi\right\}\sqrt{-g}.$$

$$\frac{\partial\mathcal{L}_{\mathrm{TEGRW,D}}}{\partial H_{av}} = -\sqrt{-g}\left(T^{a\nu}_{\mathrm{D}} + T^{a\nu}_{\mathrm{D,diff}}\right)$$

$$T^{a\nu}_{\mathrm{D}} = \dot{e}^{a}{}_{\mu}\left\{\frac{i\hbar c}{4}\bar{\psi}\left(\gamma^{\mu}_{\mathrm{F}}\overrightarrow{D}^{\nu} + \gamma^{\nu}_{\mathrm{F}}\overrightarrow{D}^{\mu} - \overleftarrow{D}^{\nu}\gamma^{\mu}_{\mathrm{F}} - \overleftarrow{D}^{\mu}\gamma^{\nu}_{\mathrm{F}}\right)\psi\right.$$
$$\left. -\frac{1}{2}g^{\mu\nu}\left[\frac{i\hbar c}{2}\bar{\psi}\left(\gamma^{\rho}_{\mathrm{F}}\overrightarrow{D}_{\rho} - \overleftarrow{D}_{\rho}\gamma^{\rho}_{\mathrm{F}}\right)\psi - m_{\mathrm{e}}c^2\bar{\psi}\psi\right]\right\},$$

$$T^{a\nu}_{\mathrm{D,diff}} = \dot{e}^{a}{}_{\mu}\left\{-\frac{1}{2}g^{\mu\nu}\left[\frac{i\hbar c}{2}\bar{\psi}\left(\gamma^{\rho}_{\mathrm{F}}\overrightarrow{D}_{\rho} - \overleftarrow{D}_{\rho}\gamma^{\rho}_{\mathrm{F}}\right)\psi - m_{\mathrm{e}}c^2\bar{\psi}\psi\right]\right\}.$$

$$\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}S^{a\nu\rho}\right) - \kappa T^{a\nu}_{\mathrm{TEGRW}} = 0$$

$$\widetilde{\nabla}_\rho S^{a\nu\rho} = \kappa T^{a\nu}_{\mathrm{TEGRW}}$$

$$\mathring{\nabla}_\rho S^{a\nu\rho} = \partial_\rho S^{a\nu\rho} + \mathring{\Gamma}^{\nu}{}_{\sigma\rho}S^{a\sigma\rho} + \mathring{\Gamma}^{\rho}{}_{\sigma\rho}S^{a\nu\sigma}$$
$$= \partial_\rho S^{a\nu\rho} + \mathring{\Gamma}^{\rho}{}_{\sigma\rho}S^{a\nu\sigma}$$
$$= \widetilde{\nabla}_\rho S^{a\nu\rho}.$$

$$\overset{\blacksquare}{e}{}^{\mu}_{a}\mathring{\nabla}_\rho S^{a\nu\rho} = \kappa T^{\mu\nu}_{\mathrm{TEGRW}}$$

$$\overset{\blacksquare}{e}_a^{\ \mu}\overset{\circ}{\nabla}_\rho S^{a\nu\rho} = \overset{\blacksquare}{e}_a^{\ \mu}\left(\partial_\rho S^{a\nu\rho} + \overset{\circ}{\Gamma}{}^{\nu}{}_{\sigma\rho} S^{a\sigma\rho} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\rho} S^{a\nu\sigma}\right)$$

$$= \left[\partial_\rho\left(\overset{\blacksquare}{e}_a^{\ \mu} S^{a\nu\rho}\right) + \overset{\circ}{\Gamma}{}^{\nu}{}_{\sigma\rho}\overset{\blacksquare}{e}_a^{\ \mu} S^{a\sigma\rho}\right.$$

$$\left. + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\rho}\overset{\blacksquare}{e}_a^{\ \mu} S^{a\nu\sigma}\right] - S^{a\nu\rho}\partial_\rho \overset{\blacksquare}{e}_a^{\ \mu}$$

$$= \left(\partial_\rho S^{\mu\nu\rho} + \overset{\circ}{\Gamma}{}^{\nu}{}_{\sigma\rho} S^{\mu\sigma\rho} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\rho} S^{\mu\nu\sigma}\right.$$

$$\left. + \overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} S^{\sigma\nu\rho}\right) - \overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} S^{\sigma\nu\rho} - S^{\sigma\nu\rho}\overset{\blacksquare}{e}{}^{a}{}_{\sigma}\partial_\rho \overset{\blacksquare}{e}_a^{\ \mu}$$

$$= \overset{\circ}{\nabla}{}_\rho S^{\mu\nu\rho} - S^{\sigma\nu\rho}\left(\overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} + \overset{\blacksquare}{e}{}^{a}{}_{\sigma}\partial_\rho \overset{\blacksquare}{e}_a^{\ \mu}\right)$$

$$= \overset{\circ}{\nabla}{}_\rho S^{\mu\nu\rho} + S^{\sigma\nu\rho}\overset{\circ}{K}{}^{\mu}{}_{\sigma\rho}.$$

$$\overset{\circ}{\nabla}_\rho S^{\mu\nu\rho} = \partial_\rho S^{\mu\nu\rho} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} S^{\sigma\nu\rho} + \overset{\circ}{\Gamma}{}^{\nu}{}_{\sigma\rho} S^{\mu\sigma\rho} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\rho} S^{\mu\nu\sigma}$$

$$= \partial_\rho S^{\mu\nu\rho} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\rho} S^{\mu\nu\sigma} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} S^{\sigma\nu\rho}$$

$$= \tilde{\nabla}_\rho S^{\mu\nu\rho} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} S^{\sigma\nu\rho}.$$

$$\overset{\blacksquare}{e}_a^{\ \mu}\overset{\circ}{\nabla}_\rho S^{a\nu\rho} = \widetilde{\nabla}_\rho S^{\mu\nu\rho} + \overset{\blacksquare}{\Gamma}{}^{\mu}_{\mu} S^{\sigma\nu\rho}.$$

$$\widetilde{\nabla}_\rho S^{\mu\nu\rho} = \kappa \mathfrak{T}^{\mu\nu}$$

$$\mathfrak{T}^{\mu\nu}_{\text{TEGRW}} = T^{\mu\nu}_{\text{TEGRW}} + \frac{1}{\kappa} S^{\sigma\rho\nu}\overset{\blacksquare}{\Gamma}{}^{\mu}_{\sigma\rho}$$

$$\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial \bar{\Theta}} - \partial_\rho\left[\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial(\partial_\rho \bar{\Theta})}\right] = 0$$

$$\widetilde{\nabla}_\rho\left(\gamma_{\text{B}}^{\rho}\Psi\right) = -\Phi$$

$$\gamma_{\text{B}}^{\rho}\left(\vec{\partial}_\rho + \overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}^{\sigma}_{a}\right)\Psi = -\Phi$$

$$\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial A_\nu} - \partial_\rho\left[\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial(\partial_\rho A_\nu)}\right] = 0$$

$$\widetilde{\nabla}_\rho F^{\rho\nu} = \overset{\circ}{\nabla}_\rho F^{\rho\nu} = \mu_0 J_{\text{e}}^{\nu}$$

$$\overset{\circ}{\nabla}_\rho F_{\mu\nu} + \overset{\circ}{\nabla}_\mu F_{\nu\rho} + \overset{\circ}{\nabla}_\nu F_{\rho\mu} = 0$$

$$\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial \bar{\psi}} - \partial_\rho\left[\frac{\partial \mathcal{L}_{\text{TEGRW}}}{\partial(\partial_\rho \bar{\psi})}\right] = 0$$

$$\frac{i}{2}\hbar c\boldsymbol{\gamma}_{\text{F}}^{\rho}\vec{\partial}_\rho\psi - q_{\text{e}}c\boldsymbol{\gamma}_{\text{F}}^{\rho}A_\rho\psi - m_{\text{e}}c^2\psi + \frac{i}{2}\hbar c\widetilde{\nabla}_\rho\left(\boldsymbol{\gamma}_{\text{F}}^{\rho}\psi\right) = 0$$

$$i\hbar c\gamma_{\text{F}}^{\rho}\left(\vec{D}_\rho + \frac{1}{2}\overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}^{\sigma}_{a}\right)\psi - m_{\text{e}}c^2\psi = 0$$

$$-i\hbar c\psi^{\dagger}\left(\overleftarrow{D}_\rho + \frac{1}{2}\overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}^{\sigma}_{a}\right)\gamma_{\text{F}}^{\rho\dagger} - m_{\text{e}}c^2\psi^{\dagger} = 0$$

$$-i\hbar c\bar{\psi}\left(\overleftarrow{D}_\rho + \frac{1}{2}\overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}^{\sigma}_{a}\right)\gamma_{\text{F}}^{\rho} - m_{\text{e}}c^2\bar{\psi} = 0$$

$$i\hbar c\bar{\psi}\left(\gamma_{\text{F}}^{\rho}\overleftarrow{D}_\rho + \frac{1}{2}\overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}{}^{\sigma}_{a}\gamma_{\text{F}}^{\rho}\right)\psi - m_{\text{e}}c^2\bar{\psi}\psi = 0$$

$$-i\hbar c\bar{\psi}\left(\overleftarrow{D}_\rho\gamma_{\text{F}}^{\rho} + \frac{1}{2}\overset{\blacksquare}{e}{}^{a}_{\rho}\widetilde{\nabla}_\sigma \overset{\blacksquare}{e}_{a}{}^{\sigma}\gamma_{\text{F}}^{\rho}\right)\psi - m_{\text{e}}c^2\bar{\psi}\psi = 0$$

$$\left[\frac{i\hbar c}{2}\bar{\psi}(\gamma_{\text{F}}^{\rho}\vec{D}_\rho - \overleftarrow{D}_\rho\gamma_{\text{F}}^{\rho})\psi - m_{\text{e}}c^2\bar{\psi}\psi\right]\sqrt{-g} = 0$$

$$\mathcal{L}_{\text{TEGRW,D}} = 0$$

$$\mathcal{L}^{(\text{ref})}_{\text{TEGRW,}} = \left[\frac{i\hbar c}{2}\bar{\psi}\left(\overset{\blacksquare}{e}{}^{\nu}_{b}\gamma_{\text{F}}^{b}\vec{D}_\nu - \overleftarrow{D}_\nu\gamma_{\text{F}}^{b}\overset{\blacksquare}{e}{}^{\nu}_{b}\right)\psi - m_{\text{e}}c^2\bar{\psi}\psi\right]\sqrt{-g}.$$

$$\mathcal{L}^{(\text{ref})}_{\text{TEGRW,em}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\sqrt{-g}$$

$$\mathcal{L}_{\text{TEGRW, g}}^{(\text{ref})} = \frac{1}{4\kappa} H_{\rho\mu\nu} S^{\rho\mu\nu} \sqrt{-g}.$$

$$\mathcal{L}_{\text{TEGRW, g}} = \mathcal{L}_{\text{GR, g}} - \partial_\mu \left(\frac{1}{\kappa} H^{\nu\mu}{}_{\nu} \sqrt{-g} \right)$$

$$\mathcal{L}_{\text{GR, g}} = -\frac{1}{2\kappa} R \sqrt{-g}$$

$$\int \frac{e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}}{|\mathbf{p}|^2 - i\epsilon} d^3p$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{ip_\text{r} r\cos\,\theta_\text{r}/\hbar}}{p_\text{r}^2 - i\epsilon} p_\text{r}^2 \sin\,\theta_\text{r} d\phi_\text{r} d\theta_\text{r} dp_\text{r}$$

$$=$$

$$= \frac{2i\pi\hbar}{r} \int_0^\infty \Big|_0^\pi \frac{p_\text{r} e^{ip_\text{r} r\cos\,\theta_\text{r}/\hbar}}{p_\text{r}^2 - i\epsilon} dp_\text{r}$$

$$= \frac{2i\pi\hbar}{r} \int_0^\infty p_\text{r} \frac{e^{-ip_\text{r} r/\hbar} - e^{ip_\text{r} r/\hbar}}{p_\text{r}^2 - i\epsilon} dp_\text{r}$$

$$= \frac{i\pi\hbar}{r} \left(\int_{-\infty}^\infty p_\text{r} \frac{e^{-ip_\text{r} r/\hbar}}{p_\text{r}^2 - i\epsilon} dp_\text{r} - \int_{-\infty}^\infty p_\text{r} \frac{e^{ip_\text{r} r/\hbar}}{p_\text{r}^2 - i\epsilon} dp_\text{r} \right)$$

$$= \frac{i\pi\hbar}{r} \left[-2\pi i \text{Res}_{p_\text{r} = -\sqrt{i\epsilon}} \left(p_\text{r} \frac{e^{-ip_\text{r} r/\hbar}}{p_\text{r}^2 - i\epsilon} \right) \right.$$

$$-2\pi i \text{Res}(p_\text{r}$$

$$p_\text{r} = \sqrt{i\epsilon}$$

$$\left. = \frac{e^{ip_\text{r} r/\hbar}}{p_\text{r}^2 - i\epsilon} \right) \Big]$$

$$=$$

$$= \frac{2\pi\pi^2\hbar}{|\mathbf{r}|} e^{i|\mathbf{r}|\sqrt{i\epsilon}/\hbar}.\, 2\pi i \left(\frac{1}{2} e^{ir\sqrt{i\epsilon}/\hbar} \right) - 2\pi i \left(\frac{1}{2} e^{ir\sqrt{i\epsilon}/\hbar} \right) \Big]$$

$$\int \frac{|\mathbf{p}|^2 \log\,(|\mathbf{p}|^2)}{|\mathbf{p}|^2 - i\epsilon} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3p$$

$$= -\frac{\hbar^2}{|\mathbf{r}|^2} \int \nabla_\mathbf{p}^2 \left[\frac{|\mathbf{p}|^2 \log\,(|\mathbf{p}|^2)}{|\mathbf{p}|^2 - i\epsilon} \right] e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3p$$

$$= -\frac{\hbar^2}{|\mathbf{r}|^2} \int \frac{|\mathbf{p}|^2}{|\mathbf{p}|^2 - i\epsilon} \nabla_\mathbf{p}^2 [\log\,(|\mathbf{p}|^2)] e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3p$$

$$= -\frac{2\hbar^2}{|\mathbf{r}|^2} \int \frac{e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}}{|\mathbf{p}|^2 - i\epsilon} d^3p$$

$$= -\frac{4\pi^2\hbar^3}{|\mathbf{r}|^3} e^{i\mathbf{r}|\sqrt{i\epsilon}/\hbar}.$$

$$\Box\,\psi - m^2\psi = 0$$

$$j_\mu := -i\left(\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*\right)$$

$$\nabla_\mu j^\mu = 0$$

$$\rho \equiv j^0 = i(\psi^* \partial_t \psi - \psi \partial_t \psi^*)$$

$$\hat{H}\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi.$$

$$\hat{H}^2\psi = (\hat{\mathbf{p}}^2 + m^2)\psi.$$

$$\hat{H}^2 = \sum_i \alpha_i^2 \hat{p}_i^2 + \sum_{i>j} \left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)\hat{p}_i\hat{p}_j + \sum_i m(\alpha_i\beta + \beta\alpha_i)\hat{p}_i + \beta^2 m^2$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$$
$$\alpha'_k = U\alpha_k U^{-1}, \beta' = U\beta U^{-1}$$
$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$\hat{H} = i\partial_t, \hat{P}_i = -i\partial_i$$
$$i\partial_t \psi = -i\alpha^i \partial_i \psi + \beta m \psi$$
$$\partial_t \psi = -\alpha^i \partial_i \psi - i\beta m \psi$$
$$i\beta\partial_t \psi + i\beta\alpha^i \partial_i \psi - m\psi = 0$$
$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$
$$\gamma^0 := \beta, \gamma^k := \beta\alpha^k.$$
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$
$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.$$
$$(i\not\partial - m)\psi = 0.$$
$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu}$$
$$(\gamma^0)^2 = I, \left(\gamma^k\right)^2 = -I$$
$$\gamma^0 \gamma^k + \gamma^k \gamma^0 = 0$$
$$\gamma^{0^\dagger} = \gamma^0, \gamma^{k\dagger} = -\gamma^k$$
$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$$
$$\gamma^5 := i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$
$$\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu = \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta + 2\left(\eta^{\alpha\mu} \gamma^\beta \gamma^\nu - \eta^{\beta\mu} \gamma^\alpha \gamma^\nu + \eta^{\alpha\nu} \gamma^\mu \gamma^\beta - \eta^{\beta\nu} \gamma^\mu \gamma^\alpha\right).$$
$$i\gamma^0 \partial_t \psi + i\gamma^k \partial_k \psi - m\psi = 0$$
$$-i\left(\partial_t \psi^\dagger\right)\gamma^0 - i\left(\partial_k \psi^\dagger\right)\left(-\gamma^k\right) - m\psi^\dagger = 0$$
$$i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu + m\bar{\psi} = 0$$
$$\bar{\psi}\left(\gamma^\mu \partial_\mu \psi\right) + \left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi = \partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0.$$
$$j^\mu = \bar{\psi}\gamma^\mu \psi$$
$$\rho = \bar{\psi}\gamma^0 \psi = \psi^\dagger \psi = \sum_{i=1}^{4} |\psi_i|^2$$
$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + im\gamma^\nu \partial_\nu \psi = 0,$$
$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + m^2 \psi = 0$$
$$(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu \psi = 2\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi$$
$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi = -\eta^{\mu\nu} \partial_\nu \psi = -\Box\, \psi$$
$$\Box\, \psi - m^2 \psi = 0$$
$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta,$$
$$\Lambda_{\alpha\beta} = \eta_{\alpha\mu} \Lambda^\mu{}_\beta, \Lambda^{\alpha\beta} = \eta^{\beta\mu} \Lambda^\alpha_\mu, \Lambda_\alpha{}^\beta = \eta_{\alpha\mu} \eta^{\beta\nu} \Lambda^\mu_\nu.$$
$$\eta^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta^{\mu\nu} = \Lambda^{\alpha\nu} \Lambda^\beta{}_\nu,$$
$$\Lambda^{\alpha\nu} \Lambda_{\beta\nu} = \Lambda^\alpha{}_\nu \Lambda_\beta{}^\nu = \delta^\alpha_\beta.$$
$$x^\alpha = (\Lambda^{-1})^\alpha{}_\beta x'^\beta,$$
$$(\Lambda^{-1})^\alpha{}_\mu \Lambda^\mu{}_\beta = \delta^\alpha_\beta, \Lambda^\alpha_\mu (\Lambda^{-1})^\mu{}_\beta = \delta^\alpha_\beta.$$
$$(\Lambda^{-1})^\alpha{}_\beta = \Lambda_\beta{}^\alpha,$$
$$x^\alpha = \Lambda_\beta{}^\alpha x'^\beta.$$
$$\Lambda^{\mu\alpha} \Lambda_{\mu\beta} = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta^\alpha_\beta.$$
$$v'^\alpha = \Lambda^\alpha{}_\beta v^\beta, q'_\alpha = \Lambda_\alpha{}^\beta q_\beta,$$
$$\vec{e}'_\alpha = \Lambda_\alpha{}^\beta \vec{e}_\beta.$$
$$\left[\det\left(\Lambda^\alpha_\beta\right)\right]^2 = 1 \implies \left[\det\left(\Lambda^\alpha_\beta\right)\right] = \pm 1$$
$$\Lambda^\alpha_\beta = \delta^\alpha{}_\beta + \lambda^\alpha{}_\beta,$$

$$\Lambda_{\alpha\beta} = \eta_{\alpha\beta} + \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} = \eta^{\alpha\beta} + \lambda^{\alpha\beta}$$
$$\lambda^{\alpha\beta} + \lambda^{\beta\alpha} = 0$$
$$(M^{\rho\sigma})^{\alpha\beta} = -\eta^{\rho\alpha}\eta^{\sigma\beta} + \eta^{\sigma\alpha}\eta^{\rho\beta}.$$
$$\lambda^{\alpha\beta} = \frac{1}{2}C_{\rho\sigma}(M^{\rho\sigma})^{\alpha\beta}$$
$$\lambda^{\alpha\beta} = -C^{\alpha\beta}$$
$$[M^{\rho\sigma}, M^{\mu\nu}] = \eta^{\rho\mu}M^{\sigma\nu} - \eta^{\rho\nu}M^{\sigma\mu} + \eta^{\sigma\nu}M^{\rho\mu} - \eta^{\sigma\mu}M^{\rho\nu}$$
$$\Lambda^{\alpha}_{\beta} = \exp\left(\frac{1}{2}C_{\rho\sigma}(M^{\rho\sigma})^{\alpha}{}_{\beta}\right).$$
$$(B_i)^{\alpha}_{\beta} := \left(M^{0i}\right)^{\alpha}_{\beta}, (R_i)^{\alpha}{}_{\beta} = \frac{1}{2}\epsilon_{ijk}\left(M^{jk}\right)^{\alpha}_{\beta},$$
$$B_1 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix},$$
$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
$$[J_i, J_j] = i\epsilon^k_{ij}J_k, [J_i, K_j] = i\epsilon^k_{ij}K_k, [K_i, K_j] = -i\epsilon^k_{ij}J_k$$
$$\Lambda = \exp\left(\vec{\theta}\cdot\vec{R}_l - \vec{\varphi}\cdot\vec{B}_l\right) = \exp\left(-i\vec{\theta}\cdot\vec{J} + i\vec{\varphi}\cdot\vec{K}\right)$$
$$(B_1)^2 = +I_{01}, \quad (B_2)^2 = +I_{02}, \quad (B_2)^2 = +I_{02},$$
$$(R_1)^2 = -I_{23}, \quad (R_2)^2 = -I_{13}, \quad (R_3)^2 = -I_{12}.$$
$$\Lambda = \exp(-\varphi B_1) = \begin{pmatrix} \cosh(\varphi) & -\sinh(\varphi) & 0 & 0 \\ -\sinh(\varphi) & \cosh(\varphi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\Lambda = \exp(\theta R_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$\sigma^{\mu\nu} := \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}] = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \eta^{\mu\nu})$$
$$[\sigma^{\mu\nu}, \gamma^{\rho}] = \eta^{\mu\rho}\gamma^{\nu} - \eta^{\nu\rho}\gamma^{\mu}$$
$$[\sigma^{\rho\sigma}, \sigma^{\mu\nu}] = \eta^{\rho\mu}\sigma^{\sigma\nu} - \eta^{\rho\nu}\sigma^{\sigma\mu} + \eta^{\sigma\nu}\sigma^{\rho\mu} - \eta^{\sigma\mu}\sigma^{\rho\nu}$$
$$s = \frac{1}{2}C_{\rho\sigma}\sigma^{\rho\sigma}$$
$$S = \exp\left(\frac{1}{2}C_{\rho\sigma}\sigma^{\rho\sigma}\right)$$
$$v'^{\alpha} = \Lambda^{\alpha}{}_{\beta}v^{\beta},$$
$$\psi'^{A} = S^{A}_{B}\psi^{B}$$
$$\sigma^{ij} = \frac{1}{2}\gamma^{i}\gamma^{j} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix},$$
$$S = \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}$$
$$S = \begin{pmatrix} e^{+i(\theta/2)\sigma_1} & 0 \\ 0 & e^{+i(\theta/2)\sigma_1} \end{pmatrix} = \cos(\theta/2)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + i\sin(\theta/2)\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}.$$
$$\sigma^{0i} = \frac{1}{2}\gamma^{0}\gamma^{i} = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$
$$S = \begin{pmatrix} 0 & e^{-\vec{\varphi}\cdot\vec{\sigma}/2} \\ e^{-\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \end{pmatrix}.$$
$$\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}^2 = I$$

$$S=\begin{pmatrix}0 & e^{-(\varphi/2)\sigma_1}\\ e^{-(\varphi/2)\sigma_1} & 0\end{pmatrix}=\cosh(\varphi/2)\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}-\sinh(\varphi/2)\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}$$

$$\sigma^{0i}=\frac{1}{2}\gamma^0\gamma^i=\frac{1}{2}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix},$$

$$S=\begin{pmatrix}e^{+\vec{\varphi}\cdot\vec{\sigma}/2} & 0\\ 0 & e^{-\vec{\varphi}\cdot\vec{\sigma}/2}\end{pmatrix}$$

$$S=\begin{pmatrix}e^{(\varphi/2)\sigma_1} & 0\\ 0 & e^{-(\varphi/2)\sigma_1}\end{pmatrix}=\cosh(\varphi/2)\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}+\sinh(\varphi/2)\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{pmatrix}$$

$$S^{-1}\gamma^\mu S=\Lambda^\mu_\nu\gamma^\nu$$

$$\Lambda\simeq I+\frac{1}{2}C_{\rho\sigma}M^{\rho\sigma}, S\simeq I+\frac{1}{2}C_{\rho\sigma}\sigma^{\rho\sigma}$$

$$\Lambda^\mu_\nu\gamma^\nu\simeq\gamma^\mu+\frac{1}{2}C_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu\gamma^\nu$$

$$S^{-1}\gamma^\mu S\simeq\gamma^\mu-\frac{1}{2}C_{\rho\sigma}(\sigma^{\rho\sigma}\gamma^\mu-\gamma^\mu\sigma^{\rho\sigma})$$

$$[\sigma^{\rho\sigma},\gamma^\mu]=-(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu.$$

$$(M^{\rho\sigma})^\mu_\nu\gamma^\nu=-\eta^{\rho\mu}\gamma^\sigma+\eta^{\sigma\mu}\gamma^\rho,$$

$$S\gamma^\mu S^{-1}=\Lambda_\nu{}^\mu\gamma^\nu$$

$$i\gamma^\mu\partial_\mu\psi-m\psi=0$$

$$i\gamma^\mu\partial'_\mu\psi'-m\psi'=0$$

$$\psi'=S\psi$$

$$i\Lambda_\mu{}^\nu\gamma^\mu S\partial_\nu\psi-mS\psi=0$$

$$i\Lambda^\nu_\mu(S^{-1}\gamma^\mu S)\partial_\nu\psi-m\psi=0$$

$$\Lambda^\nu_\mu(S^{-1}\gamma^\mu S)=\gamma^\nu$$

$$S^{-1}\gamma^\mu S=\Lambda^\mu_\nu\gamma^\nu$$

$$\bar{\psi}'=\psi'^\dagger\gamma^0=\psi^\dagger S^\dagger\gamma^0.$$

$$(\sigma^{\mu\nu})^\dagger=-\gamma^0\sigma^{\mu\nu}\gamma^0,$$

$$S^\dagger=\gamma^0S^{-1}\gamma^0$$

$$\bar{\psi}'=\psi^\dagger\gamma^0S^{-1}=\bar{\psi}S^{-1}$$

$$\vec{e}_A\cdot\vec{e}_B=g_{\mu\nu}e^\mu_A e^\nu_B=\eta_{AB},$$

$$e_{\mu A}=g_{\mu\nu}e^\nu_A, e^{\mu A}=\eta^{AB}e^\mu_B$$

$$e_{\mu A}e^\mu_B=\eta_{AB}$$

$$\tilde{e}^A(\vec{e}_B)=e^A_\mu e^\mu_B=\delta^A_B$$

$$v^A=v^\mu e^A_\mu, v_A=v^\mu e_{\mu A}$$

$$\vec{v}=v^A\vec{e}_A$$

$$v^\mu=v^A e^\mu_A$$

$$\vec{v}\cdot\vec{u}=(v^A\vec{e}_A)\cdot(u^B\vec{e}_B)=v^Au^B\eta_{AB}=v^Au_A$$

$$\vec{z}_\mu=z^A_\mu\vec{e}_A$$

$$\vec{z}_\mu\cdot\vec{z}_\nu=z_{\mu A}z^A_\nu=\left(e_{\lambda A}z^\lambda_\mu\right)(e^A_\sigma z^\sigma_\nu)=\left(e_{\lambda A}\delta^\lambda_\mu\right)(e^A_\sigma\delta^\sigma_\nu)=e_{\mu A}e^A_\nu$$

$$e_{\mu A}e^A_\nu=g_{\mu\nu}$$

$$g=\det\left(e^A_\mu e^B_\nu\eta_{AB}\right)=\left[\det\left(e^A_\mu\right)\right]^2\det(\eta_{AB})=-\left[\det\left(e^A_\mu\right)\right]^2,$$

$$|g|^{1/2}=\det\left(e^A_\mu\right).$$

$$v^\mu=\left(\frac{\partial x^\mu}{\partial X^A}\right)v^A=e^\mu_A v^A$$

$$T_{AB}:=e^\mu_A e^\nu_B T_{\mu\nu}$$

$$\omega_{\mu B\nu}:=\nabla_\nu e_{\mu B}\equiv e_{\mu B;\nu}$$

$$\omega_{AB\nu}=e^\lambda_A\omega_{\lambda B\nu}, \omega_{ABC}=e^\lambda_A e^\sigma_C\omega_{\lambda B\sigma}$$

$$\omega_{\mu\lambda\nu}=e^B_\lambda\omega_{\mu B\nu}$$

$$\partial_\mu\vec{e}_A\equiv\left(e^\nu_{A;\mu}\right)\vec{z}_\nu=\omega^\nu_{A\mu}\vec{z}_\nu=\omega^B{}_{A\mu}\vec{e}_B.$$

$$0 = \eta_{AB;\lambda} = g_{\mu\nu}\left(e^{\mu}_{A} e^{\nu}_{B;\lambda} + e^{\nu}_{B} e^{\mu}_{A;\lambda}\right)$$
$$= e_{\nu A} e^{\nu}_{B;\lambda} + e_{\mu B} e^{\mu}_{A;\lambda} = \omega_{AB\lambda} + \omega_{BA\lambda}$$
$$\omega_{AB\alpha} = -\omega_{BA\alpha}$$
$$\nabla_{\mu} g_{\alpha\beta} = \left(e_{\alpha A} e^{A}_{\beta}\right)_{;\mu} = e_{\alpha A} e^{A}_{\beta;\mu} + e^{A}_{\beta} e_{\alpha A;\mu}$$
$$= e_{\alpha A} \omega^{A}_{\beta\ \mu} + e^{A}_{\beta} \omega_{\alpha A\mu} = \omega_{\beta\alpha\mu} + \omega_{\alpha\beta\mu} = 0$$
$$\delta e^{\mu}_{A} = e^{\mu}_{A;\nu} \delta x^{\nu} = \omega^{\mu}{}_{A\nu} \delta x^{\nu} = (\omega^{B}{}_{A\nu} \delta x^{\nu}) e^{\mu}_{B}$$
$$\delta e^{\mu}_{A} = \Lambda^{B}{}_{A} e^{\mu}_{B}.$$
$$T^{\mu}_{A|\nu} = \left(T^{\lambda}_{A} e^{B}_{\lambda}\right)_{;\nu} e^{\mu}_{B} = T^{B}_{A;\nu} e^{\mu}_{B} = T^{B}_{A,\nu} e^{\mu}_{B}$$
$$e^{\mu}_{A|\nu} = \left(e^{\lambda}_{A} e^{B}_{\lambda}\right)_{;\nu} e^{\mu}_{B} = (\delta^{B}_{A})_{;\nu} e^{\mu}_{B} = 0$$
$$e^{B}_{\mu} T^{\mu}_{A|\nu} = T^{B}_{A|\nu}, e^{A}_{\lambda} T^{\mu}_{A|\nu} = T^{\mu}_{\lambda|\nu}$$
$$T^{\mu}_{A|\nu} = T^{\mu}_{A;\nu} + \omega_{\sigma}{}^{\mu}{}_{\nu} T^{\sigma}_{A}$$
$$T^{\mu}_{A|C} = e^{\nu}_{C} T^{\mu}_{A|\nu} = T^{\mu}_{A;C} + \omega_{\sigma}{}_{C} T^{\sigma}_{A}$$
$$T^{B}_{A|C} = e^{B}_{\mu} T^{\mu}_{A;C} + \omega^{B}_{\sigma\ C} T^{\sigma}_{A} = e^{B}_{\mu} T^{\mu}_{A;C} + \omega_{D}{}^{B}{}_{C} T^{D}_{A}$$
$$T^{A}_{B|\nu} = T^{A}_{B,\nu}$$
$$T_{\mu A|\nu} = T_{\mu A;\nu} - \omega_{\mu}{}^{\sigma}_{\nu} T_{\sigma A}$$
$$T^{\mu}_{\lambda A|\nu} = T^{\mu}_{\lambda A;\nu} + \omega_{\sigma}{}^{\mu}{}_{\nu} T^{\sigma}_{\lambda A} - \omega_{\lambda}{}^{\sigma}_{\nu} T^{\mu}_{\sigma A}$$
$$T^{\mu}_{A\cdot\nu} = \left(T^{\mu}_{B} e^{B}_{\lambda}\right)_{;\nu} e^{\lambda}_{A} = T^{\mu}_{\lambda;\nu} e^{\lambda}_{A}$$
$$e^{\mu}_{A\cdot\nu} = \left(e^{\mu}_{B} e^{B}_{\lambda}\right)_{;\nu} e^{\lambda}_{A} = \left(\delta^{\lambda}_{\mu}\right)_{;\nu} e^{\lambda}_{A} = 0$$
$$e^{A}_{\lambda} T^{\mu}_{A\cdot\nu} = T^{\mu}_{\lambda\cdot\nu}, e^{B}_{\mu} T^{\mu}_{A\cdot\nu} = T^{B}_{A\cdot\nu}$$
$$T^{\mu}_{\lambda\cdot\nu} = T^{\mu}_{\lambda;\nu}$$
$$T^{A}_{B\cdot C} = e^{A}_{\alpha} e^{\beta}_{B} e^{\nu}_{C} T^{\alpha}_{\beta;\nu}$$
$$T^{\alpha}_{\beta;\nu} = e^{\alpha}_{A} e^{B}_{\beta} e^{C}_{\nu} T^{A}_{B\cdot C}$$
$$\partial_{\mu}(\mathbf{T}) = \partial_{\mu}(T^{\nu}_{A} \vec{z}_{\nu} \vec{e}^{A}) = T^{\nu}_{A\cdot\mu} \vec{z}_{\nu} \vec{e}^{A}$$
$$T^{\mu A}_{\cdot\nu} = T^{\mu A}_{;\nu} + \omega^{A}_{B\nu} T^{\mu B}$$
$$T^{\mu}_{A\cdot\nu} = T^{\mu}_{A;\nu} - \omega^{B}{}_{A\nu} T^{\mu}_{B}.$$
$$T^{\mu A}_{B\cdot\nu} = T^{\mu A}_{B;\nu} + \omega^{A}_{C\nu} T^{\mu C}_{B} - \omega^{C}_{B\nu} T^{\mu A}_{C}$$
$$v^{A}_{\cdot B} = e^{\mu}_{B} v^{A}_{\cdot\mu} = e^{\mu}_{B}\left(\partial_{\mu} v^{A} + \omega^{A}{}_{C\mu} v^{C}\right) = \partial_{B} v^{A} + \omega^{A}{}_{CB} v^{C},$$
$$v_{A\cdot B} = \partial_{B} v_{A} - \omega^{C}{}_{AB} v_{C}$$
$$T^{A}_{B\cdot C} = \partial_{C} T^{A}_{B} + \omega^{A}{}_{DC} T^{D}_{B} - \omega^{D}{}_{BC} T^{A}{}_{D}$$
$$e_{\mu A\cdot\alpha} = e_{\mu A;\alpha} - \omega^{C}_{A\alpha} e_{\mu C} = \omega_{\mu A\alpha} - \omega_{\mu A\alpha} = 0$$
$$v^{\lambda}_{;\mu} = \partial_{\mu} v^{\lambda} + \Gamma^{\lambda}_{\nu\mu} v^{\nu}$$
$$v^{\lambda}_{;\mu} = e^{\lambda}_{B} e^{A}_{\mu} (v^{B}_{\cdot A}) = e^{\lambda}_{B} e^{A}_{\mu} (\partial_{A} v^{B} + \omega^{B}{}_{CA} v^{C})$$
$$= e^{\lambda}_{B} \partial_{\mu} v^{B} + \omega^{\lambda}{}_{\nu\mu} v^{\nu} = \partial_{\mu}\left(e^{\lambda}_{B} v^{B}\right) - v^{B} \partial_{\mu} e^{\lambda}_{B} + \omega^{\lambda}{}_{\nu\mu} v^{\nu}$$
$$= \partial_{\mu} v^{\lambda} + \left(\omega^{\lambda}{}_{\nu\mu} - e^{B}_{\nu} \partial_{\mu} e^{\lambda}_{B}\right) v^{\nu}.$$
$$\Gamma^{\lambda}_{\nu\mu} = \omega^{\lambda}_{\nu\mu} - e^{B}_{\nu} \partial_{\mu} e^{\lambda}_{B}$$
$$\omega^{\lambda}_{\nu\mu} = \Gamma^{\lambda}_{\nu\mu} + e^{B}_{\nu} \partial_{\mu} e^{\lambda}_{B}$$
$$\omega^{\lambda}_{\nu\mu} = \Gamma^{\lambda}_{\nu\mu} - e^{\lambda}_{B} \partial_{\mu} e^{B}_{\nu}$$
$$\omega_{AB\mu} = e^{\nu}_{B}\left(e_{\lambda A} \Gamma^{\lambda}_{\nu\mu} - \partial_{\mu} e_{\nu A}\right)$$
$$\omega_{ABC} = -\frac{1}{2}[(f_{ABC} + f_{ACB} + f_{CAB}) - A \leftrightarrow B]$$
$$f_{ABC} := (\partial_{A} e_{\alpha B}) e^{\alpha}_{C}$$
$$R_{AB\mu\nu} = \partial_{\mu} \omega_{AB\nu} - \partial_{\nu} \omega_{AB\mu} + \omega_{AC\mu} \omega^{C}_{B\nu} - \omega_{AC\nu} \omega^{C}_{B\mu}$$
$$R^{\alpha}{}_{\beta\mu\nu} := \partial_{\mu} \Gamma^{\alpha}_{\beta\nu} - \partial_{\nu} \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\sigma\mu} \Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu} \Gamma^{\sigma}_{\beta\mu}.$$
$$\{\gamma^{A}, \gamma^{B}\} = -2\eta^{AB}$$
$$\gamma^{\mu} := \gamma^{A} e_{A}{}^{\mu}$$

$$\{\gamma^\mu,\gamma^\nu\}=\{\gamma^A,\gamma^B\}e_A^\mu e_B^\nu=-2\eta^{AB}e_A^\mu e_B^\nu$$
$$\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$$
$$\gamma_\mu\gamma^\mu=\gamma_A\gamma^A=-4I$$
$$\psi'=S(x)\psi$$
$$\partial_\mu\psi'=\partial_\mu(S\psi)=S(\partial_\mu\psi)+(\partial_\mu S)\psi$$
$$\mathcal{D}_\mu\psi:=\partial_\mu\psi+\Gamma_\mu\psi$$
$$\mathcal{D}_\mu\bar\psi:=\partial_\mu\bar\psi+\bar\psi\bar\Gamma_\mu$$
$$\mathcal{D}_\mu(\bar\psi\psi)=(\mathcal{D}_\mu\bar\psi)\psi+\bar\psi(\mathcal{D}_\mu\psi)=(\partial_\mu\bar\psi)\psi+\bar\psi(\partial_\mu\psi)+\bar\psi(\bar\Gamma_\mu+\Gamma_\mu)\psi$$
$$\mathcal{D}_\mu(\bar\psi\psi)=\partial_\mu(\bar\psi\psi)=(\partial_\mu\bar\psi)\psi+\bar\psi(\partial_\mu\psi)$$
$$\bar\Gamma_\mu=-\Gamma_\mu$$
$$\mathcal{D}_\mu\bar\psi:=\partial_\mu\bar\psi-\bar\psi\Gamma_\mu$$
$$i\gamma^\mu\mathcal{D}_\mu\psi-m\psi=0$$
$$i(\mathcal{D}_\mu\bar\psi)\gamma^\mu+m\bar\psi=0$$
$$\mathcal{D}_\mu(\bar\psi Q^\alpha\psi)=(\mathcal{D}_\mu\bar\psi)Q^\alpha\psi+\bar\psi(\mathcal{D}_\mu Q^\alpha)\psi+\bar\psi Q^\alpha(\mathcal{D}_\mu\psi)$$
$$=(\partial_\mu\bar\psi)Q^\alpha\psi+\bar\psi Q^\alpha(\partial_\mu\psi)+\bar\psi(\mathcal{D}_\mu Q^\alpha-\Gamma_\mu Q^\alpha+Q^\alpha\Gamma_\mu)\psi.$$
$$\mathcal{D}_\mu(\bar\psi Q^\alpha\psi)=\nabla_\mu(\bar\psi Q^\alpha\psi)=(\partial_\mu\bar\psi)Q^\alpha\psi+\bar\psi Q^\alpha(\partial_\mu\psi)+\bar\psi(\nabla_\mu Q^\alpha)\psi$$
$$\bar\psi(\mathcal{D}_\mu Q^\alpha-\Gamma_\mu Q^\alpha+Q^\alpha\Gamma_\mu)\psi=\bar\psi(\nabla_\mu Q^\alpha)\psi$$
$$\mathcal{D}_\mu Q^\alpha=\nabla_\mu Q^\alpha+[\Gamma_\mu,Q^\alpha]$$
$$\mathcal{D}_\mu\gamma^\nu=0$$
$$\nabla_\mu\gamma^\nu+[\Gamma_\mu,\gamma^\nu]=0$$
$$\Gamma_\mu=-\frac{1}{4}\omega_{AB\mu}\gamma^A\gamma^B=-\frac{1}{2}\omega_{AB\mu}\sigma^{AB}$$
$$\sigma^{AB}:=\frac{1}{4}[\gamma^A,\gamma^B]=\frac{1}{2}(\gamma^A\gamma^B+\eta^{AB}).$$
$$\mathcal{D}_\mu e_A^\nu=0$$
$$\vec e_A(x+dx)-\vec e_A\|(x+dx)=\lambda_A{}^B\vec e_B(x+dx)\simeq\lambda_A{}^B\vec e_B(x),$$
$$e^\mu_{A;\nu}dx^\nu=\lambda_A{}^B e_B^\mu.$$
$$e_{\mu C}e^\mu_{A;\nu}dx^\nu=\lambda_A{}^B e_B^\mu e_{\mu C}\Longrightarrow\lambda_{AC}=e_{\mu C}e^\mu_{A;\nu}dx^\nu$$
$$\lambda_{AC}=e_{\mu C}\omega^\mu{}_{A\nu}dx^\nu=\omega_{CA\nu}dx^\nu=-\omega_{AC\nu}dx^\nu$$
$$\delta\psi=-\frac{1}{2}\lambda_{AB}\sigma^{AB}\psi$$
$$\mathcal{D}\psi=\psi(x+dx)-\psi^{\|}(x+dx)=\psi(x+dx)-(\psi(x)$$
$$=\partial_\nu\psi dx^\nu-\delta\psi=\partial_\nu\psi dx^\nu+\frac{1}{2}\lambda_{AB}\sigma^{AB}\psi$$
$$=\left(\partial_\nu\psi dx^\nu-\frac{1}{2}\omega_{AB\nu}\sigma^{AB}\psi\right)dx^\nu\equiv\mathcal{D}_\nu\psi dx^\nu$$
$$\Gamma_\mu=-\frac{1}{2}\omega_{AB\mu}\sigma^{AB}$$
$$[\mathcal{D}_\mu,\mathcal{D}_\nu]\psi=-\frac{1}{2}R_{AB\mu\nu}\sigma^{AB}\psi$$
$$\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu+\Gamma_\mu\Gamma_\nu-\Gamma_\nu\Gamma_\mu=-\frac{1}{2}R_{AB\mu\nu}\sigma^{AB}$$
$$\mathcal{D}_\mu\mathcal{D}_\nu\psi=\partial_\mu(\mathcal{D}_\nu\psi)+\Gamma_\mu(\mathcal{D}_\nu\psi)-\Gamma^\alpha_{\mu\nu}(\mathcal{D}_\alpha\psi)$$
$$=\partial_\mu(\partial_\nu\psi+\Gamma_\nu\psi)+\Gamma_\mu(\partial_\nu\psi+\Gamma_\nu\psi)-\Gamma^\alpha_{\mu\nu}(\partial_\alpha\psi+\Gamma_\alpha\psi)$$
$$=(\partial_\mu\partial_\nu\psi-\Gamma^\alpha_{\mu\nu}\partial_\alpha\psi)+\Gamma_\mu\partial_\nu\psi+\Gamma_\nu\partial_\mu\psi+(\partial_\mu\Gamma_\nu-\Gamma^\alpha_{\mu\nu}\Gamma_\alpha+\Gamma_\mu\Gamma_\nu)\psi,$$
$$\mathcal{D}_\mu\mathcal{D}_\nu\psi=\nabla_\mu\nabla_\nu\psi+\Gamma_\mu\partial_\nu\psi+\Gamma_\nu\partial_\mu\psi+(\nabla_\mu\Gamma_\nu+\Gamma_\mu\Gamma_\nu)\psi$$
$$\mathcal{D}'_\mu\psi':=S\mathcal{D}_\mu\psi$$
$$\mathcal{D}'_\mu\psi'=\partial_\mu\psi'+\Gamma'_\mu\psi'$$
$$\Gamma'_\mu=S\Gamma_\mu S^{-1}-(\partial_\mu S)S^{-1}$$

$$\Lambda_B^A \simeq \delta_B^A + \lambda_B^A, \lambda_B^A := \frac{1}{2} C_{CD} (M^{CD})_B^A$$
$$S_B^A \simeq \delta_B^A + s_B^A, s_B^A := \frac{1}{2} C_{CD} (\sigma^{CD})_B^A$$
$$(\Lambda^{-1})^A{}_B \simeq \delta_B^A - \lambda_B^A, (S^{-1})^A{}_B \simeq \delta_B^A - s_B^A$$
$$\Gamma'_\mu = \Gamma_\mu + [s, \Gamma_\mu] - \partial_\mu s$$
$$\Gamma'_\mu = \Gamma_\mu + \frac{1}{2} C_{CD} [\sigma^{CD}, \Gamma_\mu] - \frac{1}{2} (\partial_\mu C_{CD}) \sigma^{CD}$$
$$\Gamma_\mu = -\frac{1}{2} B_{AB\mu} \sigma^{AB}$$
$$B'_{AB\mu} \sigma^{AB} = B_{AB\mu} \sigma^{AB} + \frac{1}{2} C_{AB} [\sigma^{AB}, B_{CD\mu} \sigma^{CD}] + \frac{1}{2} (\partial_\mu C_{AB}) \sigma^{AB}$$
$$= (B_{AB\mu} + \partial_\mu C_{AB}) \sigma^{AB} + \frac{1}{2} C_{AB} B_{CD\mu} [\sigma^{AB}, \sigma^{CD}],$$
$$B'_{AB\mu} \sigma^{AB} = [B_{AB\mu} + \partial_\mu C_{AB} - (B_{AC\mu} C_B^C + B_{CB\mu} C_A^C)] \sigma^{AB},$$
$$B'_{AB\mu} = B_{AB\mu} + \partial_\mu C_{AB} - (B_{AC\mu} C_B^C + B_{CB\mu} C_A^C)$$
$$\omega'_{AB\mu} = e'^{\lambda}_A \omega'_{\lambda B\mu} = e'^{\lambda}_A e'_{\lambda B;\mu} = \Lambda_A^C e_C^\lambda (\Lambda_B{}^D e_{\lambda D})_{;\mu}$$
$$= \Lambda_A{}^C \Lambda_B{}^D e_C^\lambda e_{\lambda D;\mu} + \Lambda_A{}^C e_C^\lambda e_{\lambda D} \partial_\mu \Lambda_B{}^D.$$
$$\omega'_{AB\mu} = \Lambda_A^C \Lambda_B^D \omega_{CD\mu} + \Lambda_A^C \partial_\mu \Lambda_{BC}$$
$$\omega'_{AB\mu} = \omega_{AB\mu} + \omega_{AC\mu} \lambda_B^C + \omega_{CB\mu} \lambda_A^C - \partial_\mu \lambda_{AB}$$
$$\omega'_{AB\mu} = \omega_{AB\mu} + \partial_\mu C_{AB} - (\omega_{AC\mu} C_B^C + \omega_{CB\mu} C_A^C)$$
$$\not{\mathcal{D}}^2 \psi = \gamma^\mu \gamma^\nu \mathcal{D}_\mu \mathcal{D}_\nu \psi = \frac{1}{2} (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) \mathcal{D}_\mu \mathcal{D}_\nu \psi$$
$$= -g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \psi + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \mathcal{D}_\mu \mathcal{D}_\nu \psi$$
$$= -\mathcal{D}^\mu \mathcal{D}_\mu \psi + \frac{1}{4} [\gamma^\mu, \gamma^\nu] [\mathcal{D}_\mu, \mathcal{D}_\nu] \psi$$
$$= -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{2} \sigma^{\mu\nu} R_{CD\mu\nu} \sigma^{CD} \psi$$
$$= -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{2} R_{ABCD} \sigma^{AB} \sigma^{CD} \psi$$
$$\not{\mathcal{D}}^2 \psi = -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{8} R_{ABCD} \gamma^A \gamma^B \gamma^C \gamma^D \psi$$
$$R = -\frac{1}{2} R_{ABCD} \gamma^A \gamma^B \gamma^C \gamma^D$$
$$\not{\mathcal{D}}^2 \psi = \left(-\mathcal{D}^\mu \mathcal{D}_\mu + \frac{R}{4}\right) \psi$$
$$i \gamma^\mu \mathcal{D}_\mu (i \gamma^\mu \mathcal{D}_\mu \psi - m\psi) = 0$$
$$\Rightarrow \; -D^2 \psi - i m \gamma^\mu \mathcal{D}_\mu \psi = 0$$
$$\Rightarrow \; \not{\mathcal{D}}^2 \psi + m^2 \psi = 0,$$
$$\left(\mathcal{D}^\mu \mathcal{D}_\mu - \frac{R}{4} - m^2\right) \psi = 0$$
$$\mathcal{D}^\mu \mathcal{D}_\mu \psi = \Box\, \psi + 2 \Gamma^\mu \partial_\mu \psi + (\nabla_\mu \Gamma^\mu + \Gamma_\mu \Gamma^\mu) \psi$$
$$R_{\mu\nu} \gamma^\mu \gamma^\nu = R_{AB} \gamma^A \gamma^B = \frac{1}{2} R_{AB} (\gamma^A \gamma^B + \gamma^B \gamma^A) = -R_{AB} \eta^{AB} = -R$$

$$R_{AB}\gamma^A\gamma^B = R_{ACBD}\eta^{CD}\gamma^A\gamma^B = R_{ACBD}\gamma^A\eta^{CD}\gamma^B$$
$$= -\frac{1}{2}R_{ACBD}\gamma^A(\gamma^C\gamma^D + \gamma^D\gamma^C)\gamma^B$$
$$= \frac{1}{2}(R_{ACDB}\gamma^A\gamma^C\gamma^D\gamma^B - R_{ACBD}\gamma^A\gamma^D\gamma^C\gamma^B)$$
$$= \frac{1}{2}(R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - R_{ACBD}\gamma^A\gamma^D\gamma^C\gamma^B),$$
$$R_{AB}\gamma^A\gamma^B = \frac{1}{2}[R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D + (R_{ADCB} + R_{ABDC})\gamma^A\gamma^D\gamma^C\gamma^B]$$
$$= R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D + \frac{1}{2}R_{ABDC}\gamma^A\gamma^D\gamma^C\gamma^B$$
$$\gamma^D\gamma^C\gamma^B = \gamma^B\gamma^D\gamma^C + 2\eta^{BD}\gamma^C - 2\eta^{BC}\gamma^D.$$
$$R_{ABDC}\gamma^A\gamma^D\gamma^C\gamma^B = R_{ABDC}\gamma^A(\gamma^B\gamma^D\gamma^C + 2\eta^{BD}\gamma^C - 2\eta^{BC}\gamma^D)$$
$$= R_{ABDC}\gamma^A\gamma^B\gamma^D\gamma^C + 2R_A{}^B{}_{BC}\gamma^A\gamma^C - 2R_A{}^B{}_{DB}\gamma^A\gamma^D$$
$$= R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - 4R_{AB}\gamma^A\gamma^B,$$
$$R_{AB}\gamma^A\gamma^B = \frac{3}{2}R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - 2R_{AB}\gamma^A\gamma^B$$
$$R_{AB}\gamma^A\gamma^B = \frac{1}{2}R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D$$
$$R = -\frac{1}{2}R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D$$
$$L = \bar{\psi}(i\gamma^\mu\mathcal{D}_\mu - m)\psi$$
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\mathcal{D}_\mu - m)\psi|g|^{1/2}$$
$$S = \int \mathcal{L}|g|^{1/2}d^4x = \int \bar{\psi}(i\gamma^\mu\mathcal{D}_\mu - m)\psi|g|^{1/2}d^4x$$
$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\mathcal{D}_\mu - m)\psi|g|^{1/2}, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0$$
$$i\gamma^\mu\mathcal{D}_\mu\psi - m\psi = 0$$
$$\mathcal{L} = [i\bar{\psi}\gamma^\mu(\partial_\mu\psi + \Gamma_\mu\psi) - m\psi]|g|^{1/2}$$
$$\frac{\partial\mathcal{L}}{\partial\psi} = \bar{\psi}(i\gamma^\mu\Gamma_\mu - m)|g|^{1/2}, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu|g|^{1/2}$$
$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = i\partial_\mu(\bar{\psi}\gamma^\mu|g|^{1/2})$$
$$= i\left[(\partial_\mu\bar{\psi})\gamma^\mu + \bar{\psi}\partial_\mu\gamma^\mu + \frac{1}{2|g|}\bar{\psi}\gamma^\mu\partial_\mu|g|\right]|g|^{1/2}.$$
$$i\left[(\partial_\mu\bar{\psi})\gamma^\mu + \bar{\psi}\partial_\mu\gamma^\mu + \frac{1}{2|g|}\bar{\psi}\gamma^\mu\partial_\mu|g|\right] - \bar{\psi}(i\gamma^\mu\Gamma_\mu - m) = 0$$
$$i(\mathcal{D}_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} + i\bar{\psi}\left[\partial_\mu\gamma^\mu + \frac{1}{2}\gamma^\mu\partial_\mu\ln|g| + [\Gamma_\mu, \gamma^\mu]\right] = 0$$
$$i(\mathcal{D}_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0$$
$$L = \frac{i}{2}[\bar{\psi}\gamma^\mu(\mathcal{D}_\mu\psi) - (\mathcal{D}_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi$$
$$\mathcal{L} = \left\{\frac{i}{2}[\bar{\psi}\gamma^\mu(\mathcal{D}_\mu\psi) - (\mathcal{D}_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi\right\}|g|^{1/2}$$
$$\psi \to e^{-i\theta}\psi, \bar{\psi} \to e^{+i\theta}\bar{\psi}$$

$$j^{\mu} = \left(\frac{\partial L}{\partial(\partial_\mu \psi)}\right)(-i\psi) + (i\bar{\psi})\left(\frac{\partial L}{\partial(\partial_\mu \bar{\psi})}\right)$$
$$= \frac{i}{2}\bar{\psi}\gamma^{\mu}(-i\psi) - \frac{i}{2}(i\bar{\psi})\gamma^{\mu}\psi$$
$$= \bar{\psi}\gamma^{\mu}\psi$$
$$L = K - V,$$
$$K = \frac{i}{2}[\bar{\psi}\gamma^{\mu}(\mathcal{D}_\mu\psi) - (\mathcal{D}_\mu\bar{\psi})\gamma^{\mu}\psi]$$
$$= \frac{i}{2}[\bar{\psi}\gamma^{\mu}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^{\mu}\psi + \bar{\psi}(\gamma^{\mu}\Gamma_\mu + \Gamma_\mu\gamma^{\mu})\psi]$$
$$V = m\bar{\psi}\psi$$
$$\Gamma_\mu = -\frac{1}{2}\omega_{AB\mu}\sigma^{AB} = -\frac{1}{2}e_\mu^C\omega_{ABC}\sigma^{AB}$$
$$\gamma^{\mu}\Gamma_\mu + \Gamma_\mu\gamma^{\mu} = -\frac{1}{2}\omega_{ABC}\gamma^{CAB}$$
$$\gamma^{CAB} = -\gamma^{CBA} = -\gamma^{ACB} = -\gamma^{BAC}$$
$$K = \frac{i}{2}[\bar{\psi}\gamma^{\mu}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^{\mu}\psi] - \frac{i}{4}\bar{\psi}(\omega_{ABC}\gamma^{CAB})\psi.$$
$$K = \frac{i}{2}[\bar{\psi}\gamma^{\mu}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^{\mu}\psi] + \frac{i}{4}\bar{\psi}(f_{ABC}\gamma^{CAB})\psi$$
$$S = \int L|g|^{1/2}d^4x$$
$$L = \frac{i}{2}[\bar{\psi}\gamma^{\mu}(\mathcal{D}_\mu\psi) - (\mathcal{D}_\mu\bar{\psi})\gamma^{\mu}\psi] - m\bar{\psi}\psi$$
$$T_{\mu\nu} = -2\frac{\partial L}{\partial g^{\mu\nu}} + g_{\mu\nu}L$$
$$T_{\mu\nu} = -\frac{1}{2}\left(e_{\mu D}\frac{\delta L}{\delta e_D^{\nu}} + e_{\nu D}\frac{\delta L}{\delta e_D^{\mu}}\right) + g_{\mu\nu}L$$
$$T_{\mu\nu} = \frac{i}{2}[(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\mathcal{D}_{\nu)}\psi)].$$
$$T_\mu^{\mu} = -m\bar{\psi}\psi = -m(\psi^{\dagger}\gamma^{T}\psi) = -m(|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2).$$
$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j$$
$$e_T^{\mu} = n^{\mu}, e_I^{\mu} = E_I^{\mu}$$
$$e_T^{\mu} = (1/\alpha, -\beta^i/\alpha), e_{\mu T} = (-\alpha, 0)$$
$$E_I^0 = 0, E_{0I} = \beta^m E_{mI}$$
$$E_{mI}E_J^m = \delta_{IJ}, E_m^I E_{nI} = \gamma_{mn}$$
$$P_\nu^{\mu} := \delta_\nu^{\mu} + n^{\mu}n_\nu$$
$$\omega_{AB\mu} = e_B^{\nu}(e_{\lambda A}\Gamma_{\nu\mu}^{\lambda} - \partial_\mu e_{\nu A})$$
$$\Gamma_{00}^0 = (\partial_t\alpha + \beta^m\partial_m\alpha - \beta^m\beta^n K_{mn})/\alpha$$
$$\Gamma_{0i}^0 = (\partial_i\alpha - \beta^m K_{im})/\alpha$$
$$\Gamma_{ij}^0 = -K_{ij}/\alpha$$
$$\Gamma_{00}^l = \alpha\partial^l\alpha - 2\alpha\beta^m K_m^l - \beta^l(\partial_t\alpha + \beta^m\partial_m\alpha - \beta^m\beta^n K_{mn})/\alpha + \partial_t\beta^l + \beta^m D_m\beta^l,$$
$$\Gamma_{m0}^l = -\beta^l(\partial_m\alpha - \beta^n K_{mn})/\alpha - \alpha K_m^l + D_m\beta^l,$$
$$\Gamma_{ij}^l = {}^{(3)}\Gamma_{ij}^l + \beta^l K_{ij}/\alpha$$
$$\omega_{TI0} = -\omega_{IT0} = -E_I^m(\partial_m\alpha - \beta^n K_{mn})$$
$$\omega_{TIm} = -\omega_{ITm} = E_I^n K_{nm}$$
$$\omega_{IJ0} = -\omega_{JI0} = -E_J^m[\partial_t E_{mI} - E_I^n(-\alpha K_{mn} + D_m\beta_n)]$$
$$\partial_t\gamma_{mn} = -2\alpha K_{mn} + D_m\beta_n + D_n\beta_m$$
$$\omega_{IJm} = -E_J^n D_m E_{nI}$$
$$\omega_{IJm}^{(3)} := E_I^n D_m E_{nJ}$$

$$\omega_{IJm} = -\omega_{JIm} = E_I^n D_m E_{nJ} \equiv \omega_{IJm}^{(3)}.$$
$$\omega_{ITT} = -\omega_{TIT} = E_I^m \partial_m \alpha/\alpha \equiv \partial_I \alpha/\alpha$$
$$\omega_{TIJ} = -\omega_{ITJ} = E_I^m E_J^n K_{mn} \equiv K_{IJ}$$
$$K_{mn} = -\frac{1}{2\alpha}(\partial_t \gamma_{mn} - D_m \beta_n - D_n \beta_m).$$
$$\omega_{TIJ} = -\omega_{ITJ} = -\frac{E_I^m E_J^n}{2\alpha}(\partial_t \gamma_{mn} - D_m \beta_n - D_n \beta_m).$$
$$\omega_{IJT} = -\omega_{JIT} = -\frac{1}{\alpha}\left[E_J^m \left(\partial_t E_{mI} - \pounds_{\vec{\beta}} E_{mI}\right) + \alpha K_{IJ}\right],$$
$$\omega_{IJK} = -\omega_{JIK} = E_I^n E_K^m D_m E_{nJ} \equiv \omega_{IJK}^{(3)}$$
$$u^\mu \nabla_\mu v_\nu = (v^\mu a_\mu) u_\nu - (u^\mu v_\mu) a_\nu = 0$$
$$n^\mu \nabla_\mu E_{\nu I} = (E_I^\mu a_\mu) n_\nu$$
$$\begin{aligned}
E_I^\mu a_\mu &= E_I^\mu n^\nu \nabla_\nu n_\mu = E_I^\mu n^\nu (\partial_\nu n_\mu - \Gamma_{\mu\nu}^\lambda n_\lambda) \\
&= E_I^\mu \left[\frac{1}{\alpha}(\partial_t n_\mu - \Gamma_{\mu 0}^\lambda n_\lambda) - \frac{\beta^m}{\alpha}(\partial_m n_\mu - \Gamma_{\mu m}^\lambda n_\lambda)\right] \\
&= -\frac{E_I^{\boldsymbol{\beta}^0}}{\alpha}(\partial_t \alpha - \beta^m \partial_m \alpha) + E_I^\mu (\Gamma_{\mu 0}^0 - \beta^m \Gamma_{\mu m}^0) = E_I^n (\Gamma_{n0}^0 - \beta^m \Gamma_{nm}^0) \\
&= E_I^n \left[\frac{1}{\alpha}(\partial_n \alpha - \beta^m \mid K_{mn}) + \frac{\beta^m}{\alpha} K_{nm}\right] = \frac{\partial_I \alpha}{\alpha}.
\end{aligned}$$
$$n^\mu \nabla_\mu E_{\nu I} = n_\nu (\partial_I \ln\ \alpha).$$
$$n^\mu \nabla_\mu E_{mI} = \frac{1}{\alpha}\left\{\partial_t E_{mI} - \pounds_{\vec{\beta}} E_{mI} + \alpha K_{mI}\right\}$$
$$\partial_t E_{mI} - \pounds_{\vec{\beta}} E_{mI} + \alpha K_{mI} = 0$$
$$\partial_t E_{mI} = \pounds_{\vec{\beta}} E_{mI} - \alpha K_{mI}$$
$$\omega_{IJT} = -\omega_{JIT} = -\frac{E_J^m}{\alpha}\left[\partial_t E_{mI} - \pounds_{\vec{\beta}} E_{mI} + \alpha K_{mI}\right]$$
$$\omega_{IJT} = 0$$
$$\partial_t E_{mI} = \pounds_{\vec{\beta}} E_{mI} - \alpha K_{mI} + \alpha Q_{mI}$$
$$\omega_{IJT} = -Q_{JI} = +Q_{IJ},$$
$$\partial_t E_{mI} = \frac{1}{2} E_I^n \partial_t \gamma_{mn}$$
$$\partial_t E_I^m = -\frac{1}{2} E_I^l \gamma^{mn} \partial_t \gamma_{ln}$$
$$\begin{aligned}
\partial_t(\delta_{IJ}) &= \partial_t(E_I^m E_{mJ}) = E_I^m \partial_t E_{mJ} + E_{mJ} \partial_t E_I^m \\
&= \frac{1}{2}(E_I^m E_J^n \partial_t \gamma_{mn} - E_{mJ} E_I^l \gamma^{mn} \partial_t \gamma_{ln}) = 0,
\end{aligned}$$
$$\begin{aligned}
\partial_t(E_m^I E_{nI}) &= \delta^{IJ} \partial_t(E_{mJ} E_{nI}) = \delta^{IJ}(E_{mJ} \partial_t E_{nI} + E_{nI} \partial_t E_{mJ}) \\
&= \frac{1}{2}(E_m^I E_I^a \partial_t \gamma_{an} + E_n^I E_J^a \partial_t \gamma_{am}) = \frac{1}{2}(\delta_m^a \partial_t \gamma_{an} + \delta_n^a \partial_t \gamma_{am}) \\
&= \partial_t \gamma_{mn}.
\end{aligned}$$
$$Q_{mn} = \frac{1}{\alpha}\left[\frac{1}{2}(D_n \beta_m - D_m \beta_n) - \beta^l (E_n^I D_l E_{mI})\right]$$
$$Q_{mn} = -\frac{1}{2\alpha}[(\partial_m \beta_n - \partial_n \beta_m) - \beta^l (E_m^K D_l E_{nK} - E_n^K D_l E_{mK})]$$
$$Q_{IJ} = -\frac{1}{2\alpha}\left[E_I^m E_J^n (\partial_m \beta_n - \partial_n \beta_m) + \beta^l (E_I^m D_l E_{mJ} - E_J^m D_l E_{mI})\right]$$
$$\Gamma_\mu = -\frac{1}{4}\omega_{AB\mu}\gamma^A \gamma^B$$
$$\Gamma_C = -\frac{1}{4}\omega_{ABC}\gamma^A \gamma^B$$

$$\Gamma_T = -\frac{1}{4}\omega_{ABT}\gamma^A\gamma^B = -\frac{1}{4}\left[2\omega_{TIT}\gamma^T\gamma^I + \omega_{IJT}\gamma^I\gamma^J\right]$$
$$\Gamma_T = \frac{1}{4}\left[2\left(\frac{\partial_I\alpha}{\alpha}\right)\gamma^T\gamma^I + \frac{E_J^m}{\alpha}\left(\partial_t E_{mI} - \pounds_{\vec{\beta}}E_{mI} + \alpha K_{mI}\right)\gamma^I\gamma^J\right]$$
$$= \left(\frac{\partial_I\alpha}{2\alpha}\right)\gamma^T\gamma^I - \frac{1}{4}Q_{IJ}\gamma^I\gamma^J,$$
$$\gamma^T\Gamma_T = \left(\frac{\partial_I\alpha}{2\alpha}\right)(\gamma^T)^2\gamma^I - \frac{1}{4}Q_{IJ}\gamma^T\gamma^I\gamma^J = \left(\frac{\partial_I\alpha}{2\alpha}\right)\gamma^I - \frac{1}{4}Q_{IJ}\gamma^T\gamma^I\gamma^J$$
$$\gamma^T\Gamma_T = \left(\frac{\partial_I\alpha}{2\alpha}\right)\gamma^I$$
$$\Gamma_I = -\frac{1}{4}\omega_{ABI}\gamma^A\gamma^B = -\frac{1}{4}\left[2\omega_{TJI}\gamma^T\gamma^J + \omega_{JKI}\gamma^J\gamma^K\right]$$
$$\Gamma_I = -\frac{1}{2}K_{IJ}\gamma^T\gamma^J - \frac{1}{4}\left(E_J^n E_I^m D_m E_{nK}\right)\gamma^J\gamma^K$$
$$= -\frac{1}{2}K_{IJ}\gamma^T\gamma^J - \frac{1}{4}\omega_{JKI}^{(3)}\gamma^J\gamma^K = -\frac{1}{2}K_{IJ}\gamma^T\gamma^J + \Gamma_I^{(3)},$$
$$\omega_{JKI}^{(3)} := E_J^n E_I^m D_m E_{nK},$$
$$\Gamma_I^{(3)} := -\frac{1}{4}\omega_{JKI}^{(3)}\gamma^J\gamma^K.$$
$$\gamma^I\Gamma_I = -\frac{1}{2}K_{IJ}\gamma^I\gamma^T\gamma^J + \gamma^I\Gamma_I^{(3)}$$
$$-\frac{1}{2}K_{IJ}\gamma^I\gamma^T\gamma^J = \frac{1}{2}\left(K_{IJ}\gamma^I\gamma^J\right)\gamma^T = \frac{1}{2}\left(\sum_I K_{II}(\gamma^I)^2 + \sum_{I\neq J} K_{IJ}\gamma^I\gamma^J\right)\gamma^T$$
$$K_{IJ}\gamma^I\gamma^J = -\sum_I K_{II} = -K$$
$$\gamma^I\Gamma_I = -\left(\frac{K}{2}\right)\gamma^T + \gamma^I\Gamma_I^{(3)}$$
$$\gamma^A\Gamma_A = \gamma^\mu\Gamma_\mu = \left[\left(\frac{\partial_I\alpha}{2\alpha}\right)\gamma^I - \frac{1}{4}Q_{IJ}\gamma^T\gamma^I\gamma^J - \left(\frac{K}{2}\right)\gamma^T + \gamma^I\Gamma_I^{(3)}\right]$$
$$\Gamma_t = e_t^A\Gamma_A = \alpha\Gamma_T + \beta^I\Gamma_I = \left(\frac{\partial_I\alpha}{2}\right)\gamma^T\gamma^I - \frac{\alpha}{4}Q_{IJ}\gamma^I\gamma^J - \beta^I\left(\frac{K_{IJ}}{2}\gamma^T\gamma^J - \Gamma_I^{(3)}\right)$$
$$\Gamma_m = e_m^A\Gamma_A = E_m^I\Gamma_I = -\frac{K_{mJ}}{2}\gamma^T\gamma^J + \Gamma_m^{(3)}$$
$$\Gamma_t - \beta^m\Gamma_m = \left(\frac{\partial_I\alpha}{2}\right)\gamma^T\gamma^I - \frac{\alpha}{4}Q_{IJ}\gamma^I\gamma^J = \alpha\Gamma_T$$
$$i\gamma^\mu\mathcal{D}_\mu\psi - m\psi = 0$$
$$\left(\gamma^\mu\partial_\mu + \gamma^\mu\Gamma_\mu + im\right)\psi = 0$$
$$(\gamma^t\partial_t + \gamma^m\partial_m)\psi = -\left(\gamma^\mu\Gamma_\mu + im\right)\psi$$
$$\gamma^t = e_A^t\gamma^A = e_T^t\gamma^T + e_T^t\gamma^{\gamma^*} = \left(\frac{1}{\alpha}\right)\gamma^T,$$
$$\gamma^m = e_A^m\gamma^A = e_T^m\gamma^T + e_I^m\gamma^I = -\left(\frac{\beta^m}{\alpha}\right)\gamma^T + E_I^m\gamma^I$$
$$\gamma^T(\partial_t - \beta^m\partial_m)\psi = -\alpha\left(\lambda^m\partial_m + \gamma^\mu\Gamma_\mu + im\right)\psi,$$
$$\gamma^m = e_A^m\gamma^A = e_T^m\gamma^T + e_I^m\gamma^I = -\left(\frac{\beta^m}{\alpha}\right)\gamma^T + E_I^m\gamma^I = -\left(\frac{\beta^m}{\alpha}\right)\gamma^T + \lambda^m$$
$$\lambda^\mu := P_\nu^\mu\gamma^\nu$$
$$\lambda^m\lambda^n + \lambda^n\lambda^m = -2\gamma^{mn}$$
$$\gamma^m T_m = \left(\lambda^m - \frac{\beta^m}{\alpha}\right)T_m \neq \lambda^m T_m = \lambda^I T_I = \gamma^I T_I$$
$$(\partial_t - \beta^m\partial_m)\psi = -\alpha\gamma^T\left(\lambda^m\partial_m + \gamma^\mu\Gamma_\mu + im\right)\psi$$

$$(\partial_t - \beta^m\partial_m)\psi = -\alpha\gamma^T\left\{\lambda^m\partial_m + \left[\left(\frac{\partial_I\alpha}{2\alpha}\right)\gamma^I - \frac{1}{4}Q_{IJ}\gamma^T\gamma^I\gamma^J - \left(\frac{K}{2}\right)\gamma^T + \gamma^I\Gamma_I^{(3)}\right] + im\right\}\psi$$
$$\gamma^I\partial_I = \gamma^I(e_I^\mu\partial_\mu) = \gamma^I E_I^m\partial_m = \lambda^m\partial_m$$
$$(\partial_t - \beta^m\partial_m)\psi = \alpha\left[-\gamma^T\lambda^m\left(\partial_m + {}^{(3)}\Gamma_m + \frac{\partial_m\alpha}{2\alpha}\right) + \left(\frac{K}{2} - im\gamma^T\right) + \frac{1}{4}Q_{IJ}\lambda^I\lambda^J\right]\psi$$
$$D_m\psi := \partial_m\psi + \Gamma_m^{(3)}\psi$$
$$(\partial_t - \beta^m\partial_m)\psi = -\alpha\gamma^T\left[\lambda^m\left(D_m + \frac{\partial_m\alpha}{2\alpha}\right) + im\right]\psi + \alpha\left(\frac{K}{2} + \frac{1}{4}Q_{mn}\lambda^m\lambda^n\right)\psi$$
$$(\partial_t - \beta^m\partial_m)\psi = -\alpha\gamma^T(\lambda^m D_m\psi + im\psi) + \alpha\left(\frac{K}{2} - \Gamma_T\right)\psi$$
$$\Gamma_T = \gamma^T\lambda^m\left(\frac{\partial_m\alpha}{2\alpha}\right) - \frac{1}{4}Q_{mn}\lambda^m\lambda^n$$
$$\Gamma_T = \gamma^T\lambda^m\left(\frac{\partial_m\alpha}{2\alpha}\right)$$
$$\gamma^T = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$
$$\gamma^t = e_A^t\gamma^A = n_T^t\gamma_T + E_I^{t^0}\gamma_I = \frac{\gamma^T}{\alpha}$$
$$n_\mu\gamma^\mu = -\alpha\gamma^t = -\gamma^T$$
$$(\partial_t - \beta^m\partial_m)\bar{\psi} = -\alpha\left((D_m\bar{\psi})\lambda^m - im\bar{\psi}\right)\gamma^T + \alpha\bar{\psi}\left(\frac{K}{2} + \Gamma_T\right)$$
$$\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*).$$
$$\rho_p := -n_\mu j^\mu = \alpha j^t = \alpha\bar{\psi}\gamma^t\psi = \bar{\psi}\gamma^T\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2,$$
$$j^i = P_\nu^i j^\nu = P_\nu^i(\bar{\psi}\gamma^\nu\psi) = \bar{\psi}(P_\nu^i\lambda^\nu)\psi = \bar{\psi}\lambda^i\psi$$
$$\Pi = \frac{1}{\alpha}(\partial_t\psi - \beta^i\partial_i\psi) + \Gamma_T\psi.$$
$$\bar{\Pi} = \frac{1}{\alpha}(\partial_t\bar{\psi} - \beta^i\partial_i\bar{\psi}) - \bar{\psi}\Gamma_T$$
$$\rho_E = \frac{i}{2}n^\mu n^\mu\left[(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\mathcal{D}_{\nu)}\psi)\right] = \frac{i}{2}[\bar{\psi}\gamma^T\Pi - \bar{\Pi}\gamma^T\psi]$$
$$\Pi^\dagger = \frac{1}{\alpha}(\partial_t\psi^\dagger - \beta^i\partial_i\psi^\dagger) + \psi^\dagger\Gamma_T^\dagger$$
$$\Pi^\dagger\gamma^T = \frac{1}{\alpha}(\partial_t\bar{\psi} - \beta^i\partial_i\bar{\psi}) + \psi^\dagger\Gamma_T^\dagger\gamma^T = \frac{1}{\alpha}(\partial_t\bar{\psi} - \beta^i\partial_i\bar{\psi}) - \bar{\psi}\Gamma_T$$
$$\rho_E = \frac{i}{2}[\psi^\dagger\Pi - \Pi^\dagger\psi]$$
$$\rho_E = \frac{i}{2}[(\psi_1^*\Pi_1 + \psi_2^*\Pi_2 + \psi_3^*\Pi_3 + \psi_4^*\Pi_4) - c.c.]$$
$$\tilde{\Pi} := n^\mu\partial_\mu\psi = \frac{1}{\alpha}(\partial_t\psi - \beta^i\partial_i\psi)$$
$$\rho_E = \frac{i}{2}\left[(\tilde{\psi}^\dagger\Pi - \tilde{\Pi}^\dagger\psi) + \psi^\dagger(\Gamma_T - \Gamma_T^\dagger)\psi\right]$$
$$\Gamma_T - \Gamma_T^\dagger = -\frac{1}{2}Q_{mn}\lambda^m\lambda^n$$
$$\rho_E = \frac{i}{2}\left[(\psi^\dagger\tilde{\Pi} - \tilde{\Pi}^\dagger\psi) - \frac{1}{2}\psi^\dagger(Q_{mn}\lambda^m\lambda^n)\psi\right]$$
$$\rho_E = \frac{i}{2}[\psi^\dagger\tilde{\Pi} - \tilde{\Pi}^\dagger\psi].$$
$$\begin{aligned}
J_i &= -\frac{i}{2}n^\mu P_i^\nu\left[(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\mathcal{D}_{\nu)}\psi)\right] \\
&= -\frac{i}{4}n^\mu P_i^\nu\left[\{(\mathcal{D}_\mu\bar{\psi})\gamma_\nu + (\mathcal{D}_\nu\bar{\psi})\gamma_\mu]\psi - \bar{\psi}[\gamma_\mu(\mathcal{D}_\nu\psi) + \gamma_\nu(\mathcal{D}_\mu\psi)\}\right] \\
&= -\frac{i}{4}[\bar{\Pi}\lambda_i\psi - \bar{\psi}\lambda_i\Pi + \bar{\psi}\gamma^T(P_i^\nu\mathcal{D}_\nu\psi) - (P_i^\nu\mathcal{D}_\nu\bar{\psi})\gamma^T\psi],
\end{aligned}$$

$$P_i^\nu \mathcal{D}_\nu \psi = \mathcal{D}_i \psi = \partial_i \psi + \Gamma_i \psi = \partial_i \psi + \left(\Gamma_i^{(3)} - \frac{K_{im}}{2}\gamma^T \lambda^m\right)\psi = D_i\psi - \frac{K_{im}}{2}\gamma^T\lambda^m\psi$$

$$P_i^\nu \mathcal{D}_\nu \bar{\psi} = \mathcal{D}_i \bar{\psi} = D_i\bar{\psi} + \frac{K_{im}}{2}\bar{\psi}\gamma^T\lambda^m$$

$$\bar{\psi}\gamma^T(P_i^\nu \mathcal{D}_\nu \psi) - (P_i^\nu \mathcal{D}_\nu \bar{\psi})\gamma^T\psi = \bar{\psi}\gamma^T(D_i\psi) - (D_i\bar{\psi})\gamma^T\psi - \frac{K_{im}}{2}\bar{\psi}(\lambda^m + \gamma^T\lambda^m\gamma^T)\psi$$
$$= \psi^\dagger(D_i\psi) - (D_i\psi^\dagger)\psi,$$

$$J_i = -\frac{i}{4}\left[\bar{\Pi}\lambda_i\psi - \bar{\psi}\lambda_i\Pi + \psi^\dagger(D_i\psi) - (D_i\psi^\dagger)\psi\right]$$

$$S_{ij} = \frac{i}{2}\left[(\mathcal{D}_{(i}\bar{\psi})\gamma_{j)}\psi - \bar{\psi}\gamma_{(i}(\mathcal{D}_{j)}\psi)\right]$$
$$= \frac{i}{2}\left[(D_{(i}\bar{\psi})\lambda_{j)}\psi - \bar{\psi}\lambda_{(i}(D_{j)}\psi) + \frac{1}{2}\bar{\psi}(K_{m(i}\gamma^T\lambda^m\lambda_{j)} + \lambda_{(i}K_{j)m}\gamma^T\lambda^m)\psi\right]$$
$$= \frac{i}{2}\left[(D_{(i}\bar{\psi})\lambda_{j)}\psi - \bar{\psi}\lambda_{(i}(D_{j)}\psi) + \frac{1}{2}\bar{\psi}K_{m(i}(\gamma^T\lambda^m\lambda_{j)} + \lambda_{j)}\gamma^T\lambda^m)\psi\right]$$
$$= \frac{i}{2}\left[(D_{(i}\bar{\psi})\lambda_{j)}\psi - \bar{\psi}\lambda_{(i}(D_{j)}\psi) + \frac{1}{2}(\bar{\psi}\gamma^T)K_{m(i}(\lambda^m\lambda_{j)} - \lambda_{j)}\lambda^m)\psi\right].$$

$$K_{mi}(\lambda^m\lambda_j - \lambda_j\lambda^m) = K_{mi}\gamma_{jn}(\lambda^m\lambda^n - \lambda^n\lambda^m) = -2K_{mi}\gamma_{jn}(\lambda^n\lambda^m + \gamma^{nm})$$
$$= -2(K_{ji} + K_{mi}\lambda_j\lambda^m)$$

$$S_{ij} = \frac{i}{2}\left[(D_{(i}\bar{\psi})\lambda_{j)}\psi - \bar{\psi}\lambda_{(i}(D_{j)}\psi) - \psi^\dagger(K_{ij} + K_{m(i}\lambda_{j)}\lambda^m)\psi\right]$$

$$S \equiv S^m{}_m = \frac{i}{2}\left[(D_m\bar{\psi})\lambda^m\psi - \bar{\psi}\lambda^m(D_m\psi) - \psi^\dagger(K + K_{mn}\lambda^m\lambda^n)\psi\right]$$

$$S = \frac{i}{2}\left[(D_m\bar{\psi})\lambda^m\psi - \bar{\psi}\lambda^m(D_m\psi)\right]$$

$$\Pi = -\gamma^T(\lambda^m(D_m\psi) + im\psi) + \frac{K}{2}\psi$$

$$\bar{\Pi} = -\left((D_m\bar{\psi})\lambda^m - im\bar{\psi}\right)\gamma^T + \frac{K}{2}\bar{\psi}$$

$$S = \frac{i}{2}\left[\psi^\dagger\Pi - \Pi^\dagger\psi\right] - m\bar{\psi}\psi$$

$$S = \rho_E - m\bar{\psi}\psi$$

$$ds^2 = (-\alpha^2 + \beta_r\beta^r)dt^2 + 2\beta_r dr dt + a^2dr^2 + b^2r^2d\Omega^2$$

$$e_T^\mu = n^\mu = (1/\alpha, -\beta^r/\alpha, 0, 0)$$

$$e_R^\mu = (0, 1/a, 0, 0), e_\Theta^\mu = (0, 0, 1/rb, 0), e_\Phi^\mu = (0, 0, 0, 1/rb\sin\theta)$$

$$e_{\mu T} = (-\alpha, 0, 0, 0), e_{\mu R} = (a\beta^r, a, 0, 0), e_{\mu\Theta} = (0, 0, rb, 0), e_{\mu\Phi} = (0, 0, 0, rb\sin\theta)$$

$$E_R^i = (1/a, 0, 0), E_\Theta^i = (0, 1/rb, 0), E_\Phi^i = (0, 0, 1/rb\sin\theta)$$

$$E_{iR} = (a, 0, 0), E_{i\Theta} = (0, rb, 0), E_{i\Phi} = (0, 0, rb\sin\theta)$$

$$\gamma^T = \gamma^0, \gamma^R = \gamma^3, \gamma^\Theta = \gamma^2, \gamma^\Phi = \gamma^1$$

$$\gamma^t = \frac{\gamma^T}{\alpha}, \gamma^r = \frac{\gamma^R}{a} - \frac{\beta^r\gamma^T}{\alpha}, \gamma^\theta = \frac{\gamma^\Theta}{rb}, \gamma^\varphi = \frac{\gamma^\Phi}{rb\sin\theta}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$$

$$\lambda^R = \gamma^3, \lambda^\Theta = \gamma^2, \lambda^\Phi = \gamma^1$$

$$\lambda^r = \frac{\gamma^R}{a}, \lambda^\theta = \frac{\gamma^\Theta}{rb}, \lambda^\varphi = \frac{\gamma^\Phi}{rb\sin\theta}$$

$$\lambda^m\lambda^n + \lambda^n\lambda^m = -2\gamma^{mn}$$

$$\omega_{R\Theta\Theta}^{(3)} = \omega_{R\Phi\Phi}^{(3)} = -\frac{1}{a}\left(\frac{1}{r} + \frac{\partial_r b}{b}\right), \omega_{\Theta\Phi\Phi}^{(3)} = -\frac{\cot\theta}{rb}.$$

$$\sigma^{R\Theta} = -\sigma^{23} = +\frac{i}{2}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix},$$
$$\sigma^{R\Phi} = -\sigma^{13} = -\frac{i}{2}\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix},$$
$$\sigma^{\Theta\Phi} = -\sigma^{12} = +\frac{i}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},$$
$$\Gamma_R^{(3)} = 0$$
$$\Gamma_\Theta^{(3)} = -\frac{i}{2}\omega_{R\Theta\Theta}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$$
$$\Gamma_\Phi^{(3)} = +\frac{i}{2}\omega_{R\Phi\Phi}\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix} - \frac{i}{2}\omega_{\Theta\Phi\Phi}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},$$
$$\lambda^I\Gamma_I^{(3)} = \lambda^m\Gamma_m^{(3)} = \frac{1}{2}\begin{pmatrix}0 & 0 & +M_1 & +M_2\\ 0 & 0 & -M_2 & -M_1\\ -M_1 & -M_2 & 0 & 0\\ +M_2 & +M_1 & 0 & 0\end{pmatrix}$$
$$M_1 := \frac{2}{a}\left(\frac{1}{r} + \frac{\partial_r b}{b}\right), M_2 := -i\frac{\cot\,\theta}{rb}$$
$$(\partial_t - \beta^r\partial_r)\psi = -\alpha\gamma^T\left[\lambda^r\left(\partial_r + \frac{\partial_r\alpha}{2\alpha}\right) + \lambda^m\Gamma_m^{(3)} + im\right]\psi + \left(\frac{\alpha K}{2}\right)\psi$$
$$\partial_t\psi_1 - \beta^r\partial_r\psi_1 = \alpha\left[-\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_3 + \frac{i}{rb}\left(\partial_\theta + \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_4 + \left(\frac{K}{2} - im\right)\psi_1\right],$$
$$\partial_t\psi_2 - \beta^r\partial_r\psi_2 = \alpha\left[+\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_4 - \frac{i}{rb}\left(\partial_\theta - \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_3 + \left(\frac{K}{2} - im\right)\psi_2\right],$$
$$\partial_t\psi_3 - \beta^r\partial_r\psi_3 = \alpha\left[-\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_1 + \frac{i}{rb}\left(\partial_\theta + \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_2 + \left(\frac{K}{2} + im\right)\psi_3\right],$$
$$\partial_t\psi_4 - \beta^r\partial_r\psi_4 = \alpha\left[+\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_2 - \frac{i}{rb}\left(\partial_\theta - \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_1 + \left(\frac{K}{2} + im\right)\psi_4\right]$$
$$\psi_\pm^I := \psi_1 \pm \psi_3, \psi_\pm^{II} := \psi_4 \mp \psi_2$$
$$\partial_t\psi_\pm^I - \beta^r\partial_r\psi_\pm^I = \alpha\left[\mp\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_\pm^I + \frac{i}{rb}\left(\partial_\theta + \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_\mp^{II} + \frac{K}{2}\psi_\pm^I - im\psi_\mp^I\right]$$
$$\partial_t\psi_\pm^{II} - \beta^r\partial_r\psi_\pm^{II} = \alpha\left[\mp\frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)\psi_\pm^{II} - \frac{i}{rb}\left(\partial_\theta - \frac{i}{\sin\,\theta}\partial_\varphi + \frac{\cot\,\theta}{2}\right)\psi_\mp^I + \frac{K}{2}\psi_\pm^{II} + im\psi_\mp^{II}\right]$$
$$\rho_p = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2.$$
$$f_r = \bar{\psi}\lambda_r\psi = a(\psi^\dagger\gamma^T\lambda^R)\psi = a[(\psi_1\psi_3^* - \psi_2\psi_4^*) + c.c.],$$
$$f_\theta = \bar{\psi}\lambda_\theta\psi = rb(\psi^\dagger\gamma^T\lambda^\Theta\psi) = \mathrm{irb}[(\psi_1\psi_4^* - \psi_2\psi_3^*) - c.c.]$$
$$f_\varphi = \bar{\psi}\lambda_\varphi\psi = rb\sin\,\theta(\psi^\dagger\gamma^T\lambda^\Phi\psi) = rb\sin\,\theta[(\psi_1\psi_4^* + \psi_2\psi_3^*) + c.c.]$$
$$\Gamma_T = \gamma^T\lambda^r\left(\frac{\partial_r\alpha}{2\alpha}\right)$$
$$\rho_E = \frac{i}{2}\left[\psi^\dagger\tilde{\Pi} - \tilde{\Pi}^\dagger\psi\right] = \frac{i}{2}\left[(\psi_1^*\tilde{\Pi}_1 + \psi_2^*\tilde{\Pi}_2 + \psi_3^*\tilde{\Pi}_3 + \psi_4^*\tilde{\Pi}_4) - c.c.\right]$$
$$\tilde{\Pi}_i = \frac{1}{\alpha}(\partial_t\psi_i - \beta^r\partial_r\psi_i)$$
$$J_r = -\frac{i}{4}\left[\bar{\Pi}\lambda_r\psi - \bar{\psi}\lambda_r\Pi + \psi^\dagger(\partial_r\psi) - (\partial_r\psi^\dagger)\psi\right]$$
$$\bar{\Pi}\lambda_r\psi - \bar{\psi}\lambda_r\Pi = \Pi^\dagger\gamma^T\lambda_r\psi - \psi^\dagger\gamma^T\lambda_r\Pi = \tilde{\Pi}^\dagger\gamma^T\lambda_r\psi - \psi^\dagger\gamma^T\lambda_r\tilde{\Pi} + \psi^\dagger(\Gamma_T^\dagger\gamma^T\lambda_r - \gamma^T\lambda_r\Gamma_T)\psi$$
$$\Gamma_T^\dagger\gamma^T\lambda_i - \gamma^T\lambda_i\Gamma_T = \frac{\partial_r\alpha}{\alpha}(\lambda_i\lambda^r + \delta_i^r)$$
$$J_r = -\frac{i}{4}\left[\tilde{\Pi}^\dagger\gamma^T\lambda_r\psi - \psi^\dagger\gamma^T\lambda_r\tilde{\Pi} + \psi^\dagger(\partial_r\psi) - (\partial_r\psi^\dagger)\psi\right]$$
$$J_r = -\frac{i}{4}\left[a(\psi_1\tilde{\Pi}_3^* - \psi_2\tilde{\Pi}_4^* + \psi_3\tilde{\Pi}_1^* - \psi_4\tilde{\Pi}_2^*)\right.$$
$$\left. + (\psi_1^*\partial_r\psi_1 + \psi_2^*\partial_r\psi_2 + \psi_3^*\partial_r\psi_3 + \psi_4^*\partial_r\psi_4) - c.c.\right]$$
$$S_{ii} = \frac{i}{2}\left[(D_i\bar{\psi})\lambda_i\psi - \bar{\psi}\lambda_i(D_i\psi)\right] = \frac{i}{2}\left[(\partial_i\bar{\psi})\lambda_i\psi - \bar{\psi}\lambda_i(\partial_i\psi) - \bar{\psi}\left(\Gamma_i^{(3)}\lambda_i + \lambda_i\Gamma_i^{(3)}\right)\psi\right]$$

$$S_{ii}=\frac{i}{2}[(\partial_i\bar{\psi})\lambda_i\psi-\bar{\psi}\lambda_i(\partial_i\psi)]$$

$$S_{rr}\ =\frac{i}{2}[(\partial_r\bar{\psi})\lambda_r\psi-\bar{\psi}\lambda_r(\partial_r\psi)]=\frac{ia}{2}[(\psi_1\partial_r\psi_3^*-\psi_2\partial_r\psi_4^*+\psi_3\partial_r\psi_3^*-\psi_4\partial_r\psi_2^*)-c.c.]$$

$$S_{\theta\theta}\ =\frac{i}{2}[(\partial_\theta\bar{\psi})\lambda_\theta\psi-\bar{\psi}\lambda_\theta(\partial_\theta\psi)]=\frac{rb}{2}[(\psi_2\partial_\theta\psi_3^*+\psi_4\partial_\theta\psi_1^*-\psi_1\partial_\theta\psi_4^*-\psi_3\partial_\theta\psi_2^*)+c.c.]$$

$$S_{\varphi\varphi}\ =\frac{i}{2}\left[(\partial_\varphi\bar{\psi})\lambda_\varphi\psi-\bar{\psi}\lambda_\varphi(\partial_\varphi\psi)\right]=\frac{irb\sin\,\theta}{2}\left[(\psi_1\partial_\varphi\psi_4^*+\psi_2\partial_\varphi\psi_3^*+\psi_3\partial_\varphi\psi_2^*+\psi_4\partial_\varphi\psi_1^*)-c.c.\right]$$

$$\psi_i=R_i(t,r)T_i(\theta,\varphi)$$

$$\frac{rb}{R_4}\left[\frac{T_1}{T_3}\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_1+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_3\right]=+\frac{i}{T_3}\left[\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_4,$$

$$\frac{rb}{R_3}\left[\frac{T_2}{T_4}\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_2-\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_4\right]=-\frac{i}{T_4}\left[\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_3,$$

$$\frac{rb}{R_2}\left[\frac{T_3}{T_1}\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_3+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_1\right]=+\frac{i}{T_1}\left[\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_2,$$

$$\frac{rb}{R_1}\left[\frac{T_4}{T_2}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_4-\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_2\right]=-\frac{i}{T_2}\left[\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_1.$$

$$\frac{rb}{R_4}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_1+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_3\right]$$
$$=-\frac{rb}{R_2}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_3+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_1\right],$$

$$\frac{rb}{R_3}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_2-\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_4\right]$$
$$=-\frac{rb}{R_1}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_4-\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_2\right].$$

$$R_2=iR_1,R_4=iR_3,T_3=T_1,T_4=-T_2$$

$$\frac{rb}{R_3}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_1+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_3\right]=+\frac{1}{T_1}\left[\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_2$$

$$\frac{rb}{R_3}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_1+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_3\right]=-\frac{1}{T_2}\left[\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_1$$

$$\frac{rb}{R_1}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_3+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_1\right]=-\frac{1}{T_1}\left[\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_2$$

$$\frac{rb}{R_1}\left[\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_3+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_1\right]=+\frac{1}{T_2}\left[\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right]T_1$$

$$f_1(t,r)=+g_1(\theta,\varphi)$$
$$f_1(t,r)=-g_2(\theta,\varphi)$$
$$f_2(t,r)=-g_1(\theta,\varphi)$$
$$f_2(t,r)=+g_2(\theta,\varphi)$$

$$\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)+im-\frac{K}{2}\right)R_1+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_3=+\frac{kR_3}{rb}$$

$$\left(\frac{1}{\alpha}(\partial_t-\beta^r\partial_r)-im-\frac{K}{2}\right)R_3+\frac{1}{a}\left(\partial_r+\frac{\partial_r\alpha}{2\alpha}+\frac{\partial_r b}{b}+\frac{1}{r}\right)R_1=-\frac{kR_1}{rb}$$

$$\left(\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right)T_1=-kT_2$$

$$\left(\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi+\frac{\cot\,\theta}{2}\right)T_2=+kT_1$$

$$\eth_s^+:=-\partial_\theta-\frac{i}{\sin\,\theta}\partial_\varphi+s\cot\,\theta$$

$$\eth_s^-:=-\partial_\theta+\frac{i}{\sin\,\theta}\partial_\varphi-s\cot\,\theta$$

$$ {}_sY^{l,m} := \begin{cases} \left[\dfrac{(l-s)!}{(l+s)!}\right]^{1/2} \eth^+_{s-1}\cdots\eth^+_0\left(Y^{l,m}\right), & +l \geq s \geq 0 \\ (-1)^s\left[\dfrac{(l-|s|)!}{(l+|s|)!}\right]^{1/2} \eth^-_{s-1}\cdots\phi^-_0\left(Y^{l,m}\right), & -l \leq s \leq 0 \\ 0, & |s| > l \end{cases} $$

$$ {}_{\pm1}Y^{l,m} = \pm\left[\frac{(l-1)!}{(l+1)!}\right]^{1/2} \eth^{\pm}_0 Y^{l,m} $$

$$ = \mp\left[\frac{(l-1)!}{(l+1)!}\right]^{1/2}\left(\partial_\theta \pm \frac{i}{\sin\,\theta}\partial_\varphi\right)Y^{l,m}, $$

$$ {}_{\pm2}Y^{l,m} = \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \eth^{\pm}_1\eth^{\pm}_0 Y^{l,m}, $$

$$ = \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\left(\partial^2_\theta - \cot\,\theta\partial_\theta \pm \frac{2i}{\sin\,\theta}(\partial_\theta - \cot\,\theta)\partial_\varphi - \frac{1}{\sin^2\,\theta}\partial^2_\varphi\right)Y^{l,m} $$

$$ \eth^+_s\left({}_sY^{l,m}\right) = +[(l-s)(l+s+1)]^{1/2}\,{}_{s+1}Y^{l,m}, $$

$$ \eth^-_s\left({}_sY^{l,m}\right) = -[(l+s)(l-s+1)]^{1/2}\,{}_{s-1}Y^{l,m}, $$

$$ \eth^-_{s+1}\eth^+_s\left({}_sY^{l,m}\right) = -[l(l+1) - s(s+1)]_sY^{l,m} $$

$$ \eth^+_{s-1}\eth^-_s\left({}_sY^{l,m}\right) = -[l(l+1) - s(s-1)]_sY^{l,m} $$

$$ L^2 f = \frac{1}{\sin\,\theta}\partial_\theta(\sin\,\theta\partial_\theta f) + \frac{1}{\sin^2\,\theta}\partial^2_\varphi f = \partial^2_\theta f + \cot\,\theta\partial_\theta f + \frac{1}{\sin^2\,\theta}\partial^2_\varphi f $$

$$ \eth^-_{3/2}\eth^+_{1/2} f = \partial^2_\theta f + \cot\,\theta\partial_\theta f + \frac{1}{\sin^2\,\theta}\left(\partial^2_\varphi f + i\cos\,\theta\partial_\varphi f\right) - \frac{1}{4}\left(\frac{1}{\sin^2\,\theta} - 3\right)f $$

$$ \eth^+_{-1/2}\phi^-_{1/2} = \partial^2_\theta f + \cot\,\theta\partial_\theta f + \frac{1}{\sin^2\,\theta}\left(\partial^2_\varphi f + i\cos\,\theta\partial_\varphi f\right) - \frac{1}{4}\left(\frac{1}{\sin^2\,\theta} + 1\right)f $$

$$ /\,1/2_{-}\;\eth^+_{-1/2} f = \partial^2_\theta f + \cot\,\theta\partial_\theta f + \frac{1}{\sin^2\,\theta}\left(\partial^2_\varphi f - i\cos\,\theta\partial_\varphi f\right) - \frac{1}{4}\left(\frac{1}{\sin^2\,\theta} + 1\right)f, $$

$$ \eth^+_{-3/2}\eth^-_{-1/2} = \partial^2_\theta f + \cot\,\theta\partial_\theta f + \frac{1}{\sin^2\,\theta}\left(\partial^2_\varphi f - i\cos\,\theta\partial_\varphi f\right) - \frac{1}{4}\left(\frac{1}{\sin^2\,\theta} - 3\right)f $$

$$ \eth^-_{+1/2}T_1 = kT_2, \eth^+_{-1/2}T_2 = -kT_1 $$

$$ T_1 = {}_{+1/2}Y^{l,m}, T_2 = {}_{-1/2}Y^{l,m} $$

$$ \pm 1/2 Y^{1/2,1/2} = \left(\frac{1}{\sqrt{4\pi}}\right)e^{+i\varphi/2}y_{\pm}(\theta) $$

$$ \pm 1/2 Y^{1/2,-1/2} = \pm\left(\frac{1}{\sqrt{4\pi}}\right)e^{-i\varphi/2}y_{\mp}(\theta) $$

$$ \left(\frac{1}{\alpha}(\partial_t - \beta^r\partial_r) + im - \frac{K}{2}\right)R_1 + \frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)R_3 = -\frac{R_3}{rb} $$

$$ \left(\frac{1}{\alpha}(\partial_t - \beta^r\partial_r) - im - \frac{K}{2}\right)R_3 + \frac{1}{a}\left(\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\right)R_1 = +\frac{R_1}{rb} $$

$$ \partial_t R_1 = \beta^r\partial_r R_1 - \frac{\alpha}{a}\left[\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\left(1 + \frac{a}{b}\right)\right]R_3 + \alpha\left(\frac{K}{2} - im\right)R_1 $$

$$ \partial_t R_3 = \beta^r\partial_r R_3 - \frac{\alpha}{a}\left[\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\left(1 - \frac{a}{b}\right)\right]R_1 + \alpha\left(\frac{K}{2} + im\right)R_3 $$

$$ \partial_t F = \beta^r\partial_r F - \frac{\alpha}{a}\left[\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\left(1 + \frac{a}{b}\right)\right]G + \alpha\left(\frac{K}{2} - im\right)F $$

$$ \partial_t G = \beta^r\partial_r G - \frac{\alpha}{a}\left[\partial_r + \frac{\partial_r\alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r}\left(1 - \frac{a}{b}\right)\right]F + \alpha\left(\frac{K}{2} + im\right)G $$

$$ \psi_{\pm} = \frac{e^{\pm i\varphi/2}}{(4\pi)^{1/2}}\begin{pmatrix} F(t,r)y_{\pm}(\theta) \\ \pm iF(t,r)y_{\mp}(\theta) \\ G(t,r)y_{\pm}(\theta) \\ \mp iG(t,r)y_{\mp}(\theta) \end{pmatrix} $$

$$F=\sum_{n=0}^{\infty} F_n(t)r^n, G=\sum_{n=0}^{\infty} G_n(t)r^n$$
$$a \simeq a_0(t)+a_2(t)r^2+\mathcal{O}(r^4)$$
$$b \simeq b_0(t)+b_2(t)r^2++\mathcal{O}(r^4)$$
$$\alpha \simeq \alpha_0(t)+\alpha_2(t)r^2+\mathcal{O}(r^4)$$
$$\beta^r \simeq \beta_1(t)r+\mathcal{O}(r^3)$$
$$\frac{G}{r}\left(1+\frac{a}{b}\right)\simeq\frac{2G_0}{r}, \qquad \frac{F}{r}\left(1-\frac{a}{b}\right)=\frac{F}{rb}(b-a)\simeq\frac{rF}{b_0}(b_2-a_2)$$
$$\sum_{n=0}^{\infty}(\dot{F}_n+imF_n)r^n+\sum_{n=0}^{\infty}(n+2)G_nr^{n-1}=0$$
$$\sum_{n=0}^{\infty}(\dot{G}_n-imG_n)r^n+\sum_{n=0}^{\infty}nF_nr^{n-1}=0$$
$$\sum_{n=0}^{\infty}(\dot{F}_n+imF_n)r^n+\sum_{n=1}^{\infty}(n+2)G_nr^{n-1}=0$$
$$\sum_{n=0}^{\infty}(\dot{G}_n-imG_n)r^n+\sum_{n=1}^{\infty}nF_nr^{n-1}=0$$
$$\sum_{n=0}^{\infty}(\dot{F}_n+imF_n+(n+3)G_{n+1})r^n=0$$
$$\sum_{n=0}^{\infty}\left(\dot{G}_n-imG_n+(n+1)F_{n+1}\right)r^n=0$$
$$G_{n+1}=-\frac{\dot{F}_n+imF_n}{n+3}, F_{n+1}=-\frac{\dot{G}_n-imG_n}{n+1}$$
$$F=F_0(t)+F_2(t)r^2+\cdots, G=G_1(t)r+G_3(t)r^3+\cdots$$
$$j_\mu^{\mathrm{Tot}}=\left(j_\mu\right)_+ +\left(j_\mu\right)_- =\bar{\psi}_+\gamma_\mu\psi_+ +\bar{\psi}_-\gamma_\mu\psi_-$$
$$\rho_p^{\mathrm{Tot}}=\frac{1}{2\pi}(|F|^2+|G|^2).$$
$$f_r^{\mathrm{Tot}}=\frac{a}{2\pi}(FG^*+GF^*).$$
$$T_{\mu\nu}^{\mathrm{Tot}}=T_{\mu\nu_+}+T_{\mu\nu_-}.$$
$$\rho_E^{\mathrm{Tot}}=\frac{i}{4\pi}\left(F^*\tilde{\Pi}_F+G^*\tilde{\Pi}_G-c.c.\right)$$
$$\tilde{\Pi}_F:=\frac{1}{\alpha}(\partial_tF-\beta^r\partial_rF), \tilde{\Pi}_G:=\frac{1}{\alpha}(\partial_tG-\beta^r\partial_rG).$$
$$\rho_E^{\mathrm{Tot}}=\frac{1}{2\pi}\left[\mathrm{Im}\left(\frac{1}{a}(F^*\partial_rG+G^*\partial_rF)+\frac{2}{rb}F^*G\right)+m(|F|^2-|G|^2)\right]$$
$$J_r^{\mathrm{Tot}}=\frac{1}{4\pi}\mathrm{Im}\left[F^*\partial_rF+G^*\partial_rG-a\left(F^*\tilde{\Pi}_G+G^*\tilde{\Pi}_F\right)\right]$$
$$=\frac{1}{2\pi}\mathrm{Im}[F^*\partial_rF+G^*\partial_rG],$$
$$S_{rr}^{\mathrm{Tot}}=\frac{a}{2\pi}\mathrm{Im}[F^*\partial_rG+G^*\partial_rF]$$
$$S_{\theta\theta}^{\mathrm{Tot}}=\frac{rb}{2\pi}\mathrm{Im}[F^*G]$$
$$S_{\varphi\varphi}^{\mathrm{Tot}}=(\sin^2\theta)S_{\theta\theta}^{\mathrm{Tot}}$$
$$\left(T^\mu{}_\mu\right)^{\mathrm{Tot}}=\left(S^i{}_i\right)^{\mathrm{Tot}}-\rho_E^{\mathrm{Tot}}=(S^r{}_r)^{\mathrm{Tot}}+2\left(S^\theta{}_\theta\right)^{\mathrm{Tot}}-\rho_E^{\mathrm{Tot}}=-\frac{m}{2\pi}(|F|^2-|G|^2),$$
$$ds^2=-\alpha^2dt^2+a^2dr^2+r^2d\Omega^2$$

$$\psi_\pm = \frac{e^{\pm i\varphi/2}}{(4\pi)^{1/2}} \begin{pmatrix} F(t,r)y_\pm(\theta) \\ \pm iF(t,r)y_\mp(\theta) \\ G(t,r)y_\pm(\theta) \\ \mp iG(t,r)y_\mp(\theta) \end{pmatrix}$$

$$F(r,t) = f(r)e^{-i\omega t}, G(r,t) = ig(r)e^{-i\omega t}$$

$$\rho_p = \frac{1}{2\pi}(f^2 + g^2), f_r = 0$$

$$\rho_E = \frac{1}{2\pi}\left[\frac{1}{a}(fg' - f'g) + \frac{2fg}{r} + m(f^2 - g^2)\right]$$

$$J_r = 0$$

$$S_r^r = \frac{1}{2\pi a}(fg' - f'g)$$

$$S_\theta^\theta = S_\varphi^\varphi = \frac{fg}{2\pi r}$$

$$T_\mu^\mu = -\rho_E + S_r^r + 2S_\theta^\theta = -\frac{m}{2\pi}(f^2 - g^2),$$

$$\rho_E = \frac{\omega}{2\pi\alpha}(f^2 + g^2).$$

$$\omega f = +\frac{\alpha}{a}\left[g' + g\left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1 + a)\right)\right] + \alpha mf$$

$$\omega g = -\frac{\alpha}{a}\left[f' + f\left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1 - a)\right)\right] - \alpha mg$$

$$f' = -f\left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1 - a)\right) - ag\left(m + \frac{\omega}{\alpha}\right),$$

$$g' = -g\left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1 + a)\right) - af\left(m - \frac{\omega}{\alpha}\right).$$

$$a' = \frac{a}{2}\left(\frac{1 - a^2}{r} + 8\pi r a^2 \rho_E\right)$$

$$\alpha' = \alpha\left(\frac{a^2 - 1}{2r} + 4\pi r a^2 S_r^r\right).$$

$$\partial_r a = \frac{a}{2}\left(\frac{1 - a^2}{r} + 8\pi r a^2 \rho_E\right),$$

$$\partial_r \alpha = \alpha\left(\frac{a^2 - 1}{2r} + 4\pi r a^2 S^r{}_r\right),$$

$$\partial_r f = -f\left(\frac{\partial_r \alpha}{2\alpha} + \frac{1}{r}(1 - a)\right) - ag\left(m + \frac{\omega}{\alpha}\right),$$

$$\partial_r g = -g\left(\frac{\partial_r \alpha}{2\alpha} + \frac{1}{r}(1 + a)\right) - af\left(m - \frac{\omega}{\alpha}\right),$$

$$S^r{}_r = \frac{1}{2\pi a}(fg' - f'g) = \rho_E - \frac{1}{\pi}\left(\frac{fg}{r} + \frac{m}{2}(f^2 - g^2)\right)$$

$$\partial_r f \simeq -g(m + \omega), \partial_r g \simeq -f(m - \omega)$$

$$\partial_r^2 f \simeq f(m^2 - \omega^2)$$

$$a \simeq 1 + \mathcal{O}(r^2)$$

$$\alpha \simeq \alpha_0 + \mathcal{O}(r^2)$$

$$\alpha \to k\alpha, \omega \to k\omega$$

$$f \simeq f_0 + \mathcal{O}(r^2), g \simeq g_1 r + \mathcal{O}(r^3)$$

$$\partial_r a|_{r=0} = 0,\ \partial_r \alpha|_{r=0} = 0,\ \partial_r f|_{r=0} = 0$$

$$\partial_r g|_{r=0} = g_1$$

$$\delta_g S = \delta \int L|g|^{1/2} d^4x$$

$$\delta g_{\mu\nu} = \eta_{AB}\left(e_\mu^A \delta e_\nu^B + e_\nu^B \delta e_\mu^A\right)$$

$$\delta^\pm e_\mu^A = \frac{1}{2}\left(\delta e_\mu^A \mp \eta^{AB} g_{\mu\nu} \delta e_B^\nu\right), \delta e_\mu^A = \delta^+ e_\mu^A + \delta^- e_\mu^A$$

$$\delta^\pm g_{\mu\nu} = \eta_{AB}\left(e_\mu^A \delta^\pm e_\nu^B + e_\nu^B \delta^\pm e_\mu^A\right) = \frac{\eta_{AB}}{2}\left[e_\mu^A\left(\delta e_\nu^B \mp \eta^{BC} g_{\nu\lambda} \delta e_C^\lambda\right) + e_\nu^B\left(\delta e_\mu^A \mp \eta^{AC} g_{\mu\lambda} \delta e_C^\lambda\right)\right]$$

$$= \frac{\eta_{AB}}{2}\left[e_\mu^A \delta e_\nu^B + e_\nu^B \delta e_\mu^A\right] \mp \frac{1}{2}\left[g_{\nu\lambda} e_\mu^A \delta e_A^\lambda + g_{\mu\lambda} e_\nu^A \delta e_A^\lambda\right]$$

$$= \frac{\delta g_{\mu\nu}}{2} \pm \frac{1}{2}\left[g_{\nu\lambda} e_A^\lambda \delta e_\mu^A + g_{\mu\lambda} e_A^\lambda \delta e_\nu^A\right]$$

$$= \frac{\delta g_{\mu\nu}}{2} \pm \frac{1}{2}\left[e_{\nu A} \delta e_\mu^A + e_{\mu A} \delta e_\nu^A\right] = \frac{\delta g_{\mu\nu}}{2} \pm \frac{\delta g_{\mu\nu}}{2}$$

$$\delta^+ g_{\mu\nu} = \delta g_{\mu\nu}, \delta^- g_{\mu\nu} = 0$$

$$\eta^{AB} e_B^\nu \delta g_{\mu\nu} = \eta^{AB} e_B^\nu\left(e_\nu^C \delta e_{\mu C} + e_{\mu C} \delta e_\nu^C\right) = \delta e_\mu^A + \eta^{AB} e_{\mu C} e_B^\nu \delta e_\nu^C$$

$$= \delta e_\mu^A - \eta^{AB} e_{\mu C} e_\nu^C \delta e_B^\nu = \delta e_\mu^A - \eta^{AB} g_{\mu\nu} \delta e_B^\nu = 2\delta^+ e_\mu^A,$$

$$\delta^+ e_\mu^A = \frac{1}{2}\eta^{AB} e_B^\nu \delta g_{\mu\nu} = \frac{1}{2} e^{\nu A} \delta g_{\mu\nu}$$

$$\delta^+ e_A^\mu = \frac{1}{2} e_{\nu A} \delta g^{\mu\nu} = -\frac{1}{2} g^{\mu\alpha} e_A^\beta \delta g_{\alpha\beta}$$

$$\delta_g S = \frac{1}{2}\int T^{\alpha\beta} \delta g_{\alpha\beta} |g|^{1/2} d^4x$$

$$\delta_g S = \int \left[|g|^{1/2}\left(\frac{\delta L}{\delta e_A^\mu}\right)\delta^+ e_A^\mu + L\delta|g|^{1/2}\right] d^4x$$

$$L\delta|g|^{1/2} = \frac{L}{2|g|^{1/2}}\delta|g| = \frac{L}{2|g|^{1/2}}|g| g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{L}{2}|g|^{1/2} g^{\alpha\beta}\delta g_{\alpha\beta}$$

$$|g|^{1/2}\left(\frac{\delta L}{\delta e_A^\mu}\right)\delta^+ e_A^\mu = -\frac{1}{2}|g|^{1/2}\left(\frac{\delta L}{\delta e_A^\mu}\right) g^{\mu\alpha} e_A^\beta \delta g_{\alpha\beta} = -\frac{1}{4}|g|^{1/2}\left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu}\right)\delta g_{\alpha\beta}$$

$$\delta_g S = \frac{1}{2}\int \left[-\frac{1}{2}\left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu}\right) + g^{\alpha\beta} L\right]\delta g_{\alpha\beta}|g|^{1/2} d^4x$$

$$T^{\alpha\beta} = -\frac{1}{2}\left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu}\right) + g^{\alpha\beta} L$$

$$T_{\mu\nu} = -\frac{1}{2}\left(e_{\mu D}\frac{\delta L}{\delta e_D^\nu} + e_{\nu D}\frac{\delta L}{\delta e_D^\mu}\right) + g_{\mu\nu} L$$

$$T_{\mu\nu} = -\frac{1}{2}\left(e_{\mu D}\frac{\delta K}{\delta e_D^\nu} + e_{\nu D}\frac{\delta K}{\delta e_D^\mu}\right)$$

$$K_1 = \frac{i}{2}\left[e_A^\beta \bar{\psi}\gamma^A(\partial_\beta \psi) - e_A^\alpha(\partial_\alpha \bar{\psi})\gamma^A \psi\right]$$

$$K_2 = \frac{i}{4}\bar{\psi}\left(e_A^\alpha e_C^\beta \partial_\alpha e_{\beta B}\right)\gamma^{CAB}\psi$$

$$\left(T_{\mu\nu}\right)_1 = \frac{i}{2}\left[(\partial_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\partial_{\nu)}\psi)\right]$$

$$\frac{\delta K_2}{\delta e_D^\mu} = \frac{\partial K_2}{\partial e_D^\mu} - \partial_\lambda\left(\frac{\partial K_2}{\partial(\partial_\lambda e_D^\mu)}\right)$$

$$f_{ABC} = e_A^\alpha e_C^\beta \partial_\alpha e_{\beta B}$$

$$f_{ABC} = -e_A^\alpha e_C^\beta e_\beta^D e_{\sigma B}\partial_\alpha e_D^\sigma = -e_{\sigma B} e_A^\alpha \partial_\alpha e_C^\sigma$$

$$\frac{\partial f_{ABC}}{\partial e_D^\nu} = -\frac{\partial(e_{\sigma B} e_A^\alpha)}{\partial e_D^\nu}\partial_\alpha e_C^\sigma = -\left(e_{\sigma B}\frac{\partial e_A^\alpha}{\partial e_D^\nu} + e_A^\alpha \frac{\partial e_{\sigma B}}{\partial e_D^\nu}\right)\partial_\alpha e_C^\sigma$$

$$= -(e_{\sigma B}\delta_\nu^\alpha \delta_A^D - e_A^\alpha e_\sigma^D e_{\nu B})\partial_\alpha e_C^\sigma = -e_{\sigma B}\delta_A^D \partial_\nu e_C^\sigma + e_\sigma^D e_{\nu B}\partial_A e_C^\sigma$$

$$P_{\mu\nu} := \frac{i}{8}\bar{\psi}[e_{\mu D}(e_{\sigma B}\delta_A^D \partial_\nu e_C^\sigma - e_\sigma^D e_{\nu B}\partial_A e_C^\sigma) + \mu \leftrightarrow \nu]\gamma^{CAB}\psi$$
$$= \frac{i}{8}\bar{\psi}[e_{\mu A}e_{\sigma B}\partial_\nu e_C^\sigma - g_{\mu\sigma}e_{\nu B}\partial_A e_C^\sigma + \mu \leftrightarrow \nu]\gamma^{CAB}\psi.$$
$$P_{IJ} = e_I^\mu e_J^\nu P_{\mu\nu} = \frac{i}{8}\bar{\psi}[\eta_{IA}e_{\sigma B}\partial_J e_C^\sigma - \eta_{JB}e_{\sigma I}\partial_A e_C^\sigma + I \leftrightarrow J]\gamma^{CAB}\psi$$
$$= \frac{i}{8}\bar{\psi}[-\eta_{IA}f_{JBC} + \eta_{JB}f_{AIC} + I \leftrightarrow J]\gamma^{CAB}\psi$$
$$= -\frac{i}{8}\bar{\psi}[(f_{JAB} + f_{AJB})\gamma_I^{AB} + (f_{IAB} + f_{AIB})\gamma_J^{AB}]\psi,$$
$$-\partial_\lambda\left(\frac{\partial f_{ABC}}{\partial(\partial_\lambda e_D^\nu)}\right) = -\partial_\lambda\left(\frac{\partial(e_{\sigma B}e_A^\alpha \partial e_C^\sigma)}{\partial(\partial_\lambda e_D^\nu)}\right) = -\partial_\lambda\left(e_{\sigma B}e_A^\alpha \delta_\alpha^\lambda \delta_\nu^\sigma \delta_C^D\right)$$
$$= -\delta_C^D \partial_\lambda\left(e_A^\lambda e_{\nu B}\right) = -\delta_C^D\left(e_A^\lambda \partial_\lambda e_{\nu B} + e_{\nu B}\partial_\lambda e_A^\lambda\right).$$
$$Q_{\mu\nu} := -\frac{i}{8}\bar{\psi}[e_{\mu C}\left(\partial_A e_{\nu B} + e_{\nu B}\partial_\lambda e_A^\lambda\right) + \mu \leftrightarrow \nu]\gamma^{CAB}\psi.$$
$$Q_{IJ} = -e_I^\mu e_J^\nu Q_{\mu\nu}$$
$$= -\frac{i}{8}\bar{\psi}[(\eta_{IC}e_J^\nu + \eta_{JC}e_I^\nu)\partial_A e_{\nu B} + (\eta_{IC}\eta_{JB} + \eta_{JC}\eta_{IB})\partial_\lambda e_A^\lambda]\gamma^{CAB}\psi.$$
$$Q_{IJ} = -\frac{i}{8}\bar{\psi}[\eta_{IC}e_J^\nu \partial_A e_{\nu B} + \eta_{JC}e_I^\nu \partial_A e_{\nu B}]\gamma^{CAB}\psi$$
$$= -\frac{i}{8}\bar{\psi}[e_J^\nu \partial_A e_{\nu B}\gamma_I^{AB} + e_I^\nu \partial_A e_{\nu B}\gamma_J^{AB}]\psi$$
$$= -\frac{i}{8}\bar{\psi}[f_{ABJ}\gamma_I^{AB} + f_{ABI}\gamma_J^{AB}]\psi.$$
$$P_{IJ} + Q_{IJ} = -\frac{i}{8}\bar{\psi}[(f_{ABJ} + f_{JAB} + f_{AJB})\gamma_I{}^{AB} + I \leftrightarrow J]\psi$$
$$= \frac{i}{8}\bar{\psi}[\omega_{ABJ}\gamma_I^{AB} + \omega_{ABI}\gamma_J{}^{AB}]\psi.$$
$$(T_{\mu\nu})_2 = \frac{i}{8}\bar{\psi}[\omega_{AB\mu}\gamma_\nu^{AB} + \omega_{AB\nu}\gamma_\mu^{AB}]\psi$$
$$= \frac{i}{8}\bar{\psi}[\omega_{AB\mu}\{\gamma_\nu, \sigma^{AB}\} + \omega_{AB\nu}\{\gamma_\mu, \sigma^{AB}\}]\psi$$
$$= \frac{i}{8}\bar{\psi}[\{\gamma_\nu, \omega_{AB\mu}\sigma^{AB}\} + \{\gamma_\mu, \omega_{AB\nu}\sigma^{AB}\}]\psi$$
$$= -\frac{i}{4}\bar{\psi}[\{\gamma_\nu, \Gamma_\mu\} + \{\gamma_\mu, \Gamma_\nu\}]\psi = -\frac{i}{2}\bar{\psi}\{\gamma_{(\mu}, \Gamma_{\nu)}\}\psi.$$
$$T_{\mu\nu} = (T_{\mu\nu})_1 + (T_{\mu\nu})_2 = \frac{i}{2}[(\partial_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\partial_{\nu)}\psi) - \bar{\psi}\{\gamma_{(\mu}, \Gamma_{\nu)}\}\psi]$$
$$= \frac{i}{2}[(\partial_{(\mu}\bar{\psi} - \bar{\psi}\Gamma_{(\mu})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\partial_{\nu)}\psi + \Gamma_{\nu)}\psi)],$$
$$T_{\mu\nu} = \frac{i}{2}[(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\mathcal{D}_{\nu)}\psi)].$$
$$\mathcal{D}^\mu T_{\mu\nu} \propto \mathcal{D}^\mu[(\mathcal{D}_\mu\bar{\psi})\gamma_\nu\psi + (\mathcal{D}_\nu\bar{\psi})\gamma_\mu\psi - \bar{\psi}\gamma_\mu(\mathcal{D}_\nu\psi) - \bar{\psi}\gamma_\nu(\mathcal{D}_\mu\psi)]$$
$$= (\mathcal{D}^\mu\mathcal{D}_\mu\bar{\psi})\gamma_\nu\psi + (\mathcal{D}_\mu\bar{\psi})\gamma_\nu(\mathcal{D}^\mu\psi) + (\mathcal{D}^\mu\mathcal{D}_\nu\bar{\psi})\gamma_\mu\psi + (\mathcal{D}_\nu\bar{\psi})\gamma_\mu(\mathcal{D}^\mu\psi)$$
$$-(\mathcal{D}^\mu\bar{\psi})\gamma_\mu(\mathcal{D}_\nu\psi) - \bar{\psi}\gamma_\mu(\mathcal{D}^\mu\mathcal{D}_\nu\psi) - (\mathcal{D}^\mu\bar{\psi})\gamma_\nu(\mathcal{D}_\mu\psi) - \bar{\psi}\gamma_\nu(\mathcal{D}^\mu\mathcal{D}_\mu\psi)$$
$$= (\mathcal{D}^\mu\mathcal{D}_\mu\bar{\psi})\gamma_\nu\psi + (\mathcal{D}^\mu\mathcal{D}_\nu\bar{\psi})\gamma_\mu\psi + (\mathcal{D}_\nu\bar{\psi})\gamma_\mu(\mathcal{D}^\mu\psi)$$
$$-(\mathcal{D}^\mu\bar{\psi})\gamma_\mu(\mathcal{D}_\nu\psi) - \bar{\psi}\gamma_\mu(\mathcal{D}^\mu\mathcal{D}_\nu\psi) - \bar{\psi}\gamma_\nu(\mathcal{D}^\mu\mathcal{D}_\mu\psi),$$
$$(\mathcal{D}_\nu\bar{\psi})\gamma_\mu(\mathcal{D}^\mu\psi) - (\mathcal{D}^\mu\bar{\psi})\gamma_\mu(\mathcal{D}_\nu\psi) = -im[(\mathcal{D}_\nu\bar{\psi})\psi + \bar{\psi}(\mathcal{D}_\nu\psi)]$$
$$= -im\mathcal{D}_\nu(\bar{\psi}\psi)$$
$$\mathcal{D}^\mu T_{\mu\nu} \propto (\mathcal{D}^\mu\mathcal{D}_\nu\bar{\psi})\gamma_\mu\psi - \bar{\psi}\gamma_\mu(\mathcal{D}^\mu\mathcal{D}_\nu\psi) - im\mathcal{D}_\nu(\bar{\psi}\psi)$$

$$
\begin{aligned}
&(\mathcal{D}^\mu \mathcal{D}_\nu \bar{\psi})\gamma_\mu \psi - \bar{\psi}\gamma_\mu(\mathcal{D}^\mu \mathcal{D}_\nu \psi) = g^{\mu\lambda}\left[(\mathcal{D}_\lambda \mathcal{D}_\nu \bar{\psi})\gamma_\mu \psi - \bar{\psi}\gamma_\mu(\mathcal{D}_\lambda \mathcal{D}_\nu \psi)\right] \\
&= \left(\mathcal{D}_\nu \mathcal{D}_\lambda \bar{\psi} - \frac{1}{2} R_{AB\lambda\nu}\sigma^{AB}\bar{\psi}\right)\gamma^\lambda \psi - \bar{\psi}\gamma^\lambda\left(\mathcal{D}_\nu \mathcal{D}_\lambda \psi - \frac{1}{2} R_{AB\lambda\nu}\sigma^{AB}\psi\right) \\
&= \mathcal{D}_\nu\left[(\mathcal{D}_\lambda \bar{\psi})\gamma^\lambda\right]\psi - \bar{\psi}\mathcal{D}_\nu\left(\gamma^\lambda \mathcal{D}_\lambda \psi\right) \\
&= \mathrm{im}\left[(\mathcal{D}_\nu \bar{\psi})\psi + \bar{\psi}(\mathcal{D}_\nu \psi)\right] \\
&= \mathrm{imD}_\nu(\bar{\psi}\psi)
\end{aligned}
$$

$$\mathcal{D}^\mu T_{\mu\nu} = 0$$

$$\mu \geqslant \frac{|m|a}{2Mr_+}\sqrt{1 + \frac{2M}{r_+}}$$

$$r_+ := M + \sqrt{M^2 - a^2}.$$

$$(u')'(t) + iBu'(t) + Au(t) = 0$$

$$A - \lambda B - \lambda^2$$

$$\frac{|m|a}{2Mr_+}.$$

$$ma\mathrm{Re}(\omega) - 2Mr_+|\omega|^2 > 0.$$

$$\Omega_1 := \left(\mathbb{C} \setminus B_\mu(0)\right) \cap ((0,\infty) \times \mathbb{R})$$

$$M > 0, 0 < a < M, \mu > 0, m \in \mathbb{Z}, l \in \{|m|, |m| + 1, \dots\}.$$

$$\lambda \neq -\frac{1}{2Mr_+}\left[ma + ik(M^2 - a^2)^{1/2}\right]$$

$$
\begin{aligned}
z^4 + &\frac{(2n+1)(M^2 - a^2)^{1/2} + ima}{M(2r_+ + M)} z^3 + \frac{c_l - 2M^2\mu^2}{M(2r_+ + M)} z^2 \\
&+ \frac{(2n+1)(M^2 - a^2)^{1/2} + ima}{M(2r_+ + M)} \mu^2 z + \frac{M}{2r_+ + M}\mu^4 = 0,
\end{aligned}
$$

$$\mu^2 \leqslant \frac{l(l+1)}{2M(r_+ + 2M)}$$

$$m \in \mathbb{Z}^*$$

$$
\begin{aligned}
P(z) := z^4 + &\frac{(2n+1)(M^2 - a^2)^{1/2} + ima}{M(2r_+ + M)} z^3 + \frac{c_l - 2M^2\mu^2}{M(2r_+ + M)} z^2 \\
&+ \frac{(2n+1)(M^2 - a^2)^{1/2} + ima}{M(2r_+ + M)} \mu^2 z + \frac{M}{2r_+ + M}\mu^4
\end{aligned}
$$

$$
\begin{aligned}
\frac{P(z)}{\mu^4} &= \left(i\frac{z}{\mu}\right)^4 - \frac{ma - i(2n+1)(M^2 - a^2)^{1/2}}{\mu M(2r_+ + M)}\left(i\frac{z}{\mu}\right)^3 \\
&+ \frac{2M^2\mu^2 - l(l+1)}{\mu^2 M(2r_+ + M)}\left(i\frac{z}{\mu}\right)^2 \\
&+ \frac{ma - i(2n+1)(M^2 - a^2)^{1/2}}{\mu M(2r_+ + M)}\left(i\frac{z}{\mu}\right) + \frac{M}{2r_+ + M}
\end{aligned}
$$

$$P(z) = \mu^4 f\left(i\frac{z}{\mu}\right)$$

$$
\begin{aligned}
f(u) := u^4 - &\frac{ma - i(2n+1)(M^2 - a^2)^{1/2}}{\mu M(2r_+ + M)} u^3 + \frac{2M^2\mu^2 - l(l+1)}{\mu^2 M(2r_+ + M)} u^2 \\
&+ \frac{ma - i(2n+1)(M^2 - a^2)^{1/2}}{\mu M(2r_+ + M)} u + \frac{M}{2r_+ + M},
\end{aligned}
$$

$$\lambda = -\frac{\mu}{2u}(1 + u^2) = -\frac{\mu}{2}\left(u + \frac{1}{u}\right),$$

$$\left(U_1(0) \cap (\mathbb{R} \times (-\infty, 0)) \to (\mathbb{C} \setminus B_1(0)) \cap (\mathbb{R} \times (0,\infty)), u \mapsto \frac{1}{u}\right)$$

$$f(w^{-1}) = w^{-4} - \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M(2r_+ + M)} w^{-3} + \frac{2M^2\mu^2 - l(l+1)}{\mu^2 M(2r_+ + M)} w^{-2}$$
$$+ \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M(2r_+ + M)} w^{-1} + \frac{M}{2r_+ + M}$$
$$= w^{-4}\left[1 - \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M(2r_+ + M)} w + \frac{2M^2\mu^2 - l(l+1)}{\mu^2 M(2r_+ + M)} w^2 \right.$$
$$\left. + \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M(2r_+ + M)} w^3 + \frac{M}{2r_+ + M} w^4\right]$$
$$p(w) := w^4 + \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M^2} w^3$$
$$+ \frac{2M^2\mu^2 - l(l+1)}{\mu^2 M^2} w^2 - \frac{ma - i(2n+1)(M^2-a^2)^{1/2}}{\mu M^2} w + \frac{2r_+ + M}{M}$$
$$= w^4 + \frac{\frac{ma}{M} - i(2n+1)\sqrt{1 - \frac{a^2}{M^2}}}{\mu M} w(w^2-1)$$
$$+ \left[2 - \frac{l(l+1)}{\mu^2 M^2}\right] w^2 + 3 + 2\sqrt{1 - \frac{a^2}{M^2}}$$
$$= w^4 + \frac{m - m\left(1 - \frac{a}{M}\right) - i(2n+1)\sqrt{1 - \frac{a^2}{M^2}}}{\mu M} w(w^2-1)$$
$$+ \left[2 - \frac{l(l+1)}{\mu^2 M^2}\right] w^2 + 3 + 2\sqrt{1 - \frac{a^2}{M^2}}$$
$$= w^4 + \frac{m}{\mu M} w(w^2-1) + \left[2 - \frac{l(l+1)}{\mu^2 M^2}\right] w^2 + 3$$
$$- \frac{m\left(1 - \frac{a}{M}\right) + i(2n+1)\sqrt{1 - \frac{a^2}{M^2}}}{\mu(w^2-1) + 2\sqrt{1 - \frac{a^2}{M^2}}} \mu M$$
$$= p_e(w) + \delta(w),$$
$$p_e(w) := w^4 + \frac{m}{\mu M} w(w^2-1) + \left[2 - \frac{l(l+1)}{\mu^2 M^2}\right] w^2 + 3$$

$$\delta(w) := -\frac{m\left(1 - \frac{a}{M}\right)}{\mu M} w(w^2-1) + 2\sqrt{1 - \frac{a^2}{M^2}}\left[1 - i\frac{2n+1}{2\mu M} w(w^2-1)\right]$$
$$\lambda = -\frac{\mu}{2w}(1 + w^2) = -\frac{\mu}{2}\left(w + \frac{1}{w}\right)$$
$$\Omega_2 := U_1(0) \cap (\mathbb{R} \times (-\infty, 0))$$
$$p_e(w) = w^4 + \alpha w^3 + (\beta - 4)w^2 - \alpha w + 3$$
$$\alpha = \frac{m}{\mu M}, \beta = 6 - \frac{l(l+1)}{\mu^2 M^2},$$
$$\frac{l(l+1)}{6M^2} \geqslant \frac{l(l+1)}{2M\left(3M + \sqrt{M^2 - a^2}\right)} = \frac{l(l+1)}{2M(r_+ + 2M)}$$
$$\mu^2 > \frac{l(l+1)}{6M^2}$$
$$\mu^2 > \frac{l(l+1)}{2M(r_+ + 2M)}.$$

$$q_{\alpha,\beta}(w) := w^4 + \alpha w^3 + (\beta - 4)w^2 - \alpha w + 3$$

$$\alpha = \frac{m}{\mu M}, \beta = 6 - \frac{l(l+1)}{\mu^2 M^2},$$

$$p_e = q_{\alpha,\beta}$$

$$4 < \alpha^2, 0 < \beta$$

$$\begin{aligned}
q_{\alpha,\beta}(w) =& w^4 + \alpha w^3 + (\beta - 4)w^2 - \alpha w + 3 \\
q^*_{\alpha,\beta}(w) =& 3w^4 - \alpha w^3 + (\beta - 4)w^2 + \alpha w + 1, \\
(Tq_{\alpha,\beta})(w) =& 3q_{\alpha,\beta}(w) - q^*_{\alpha,\beta}(w) \\
=& 3w^4 + 3\alpha w^3 + 3(\beta - 4)w^2 - 3\alpha w + 9 \\
& -[3w^4 - \alpha w^3 + (\beta - 4)w^2 + \alpha w + 1] \\
=& 4\alpha w^3 + 2(\beta - 4)w^2 - 4\alpha w + 8 \\
(Tq_{\alpha,\beta})^*(w) =& 8w^3 - 4\alpha w^2 + 2(\beta - 4)w + 4\alpha \\
(T^2 q_{\alpha,\beta})(w) =& 8(Tq_{\alpha,\beta})(w) - 4\alpha(Tq_{\alpha,\beta})^*(w) \\
=& 8[4\alpha w^3 + 2(\beta - 4)w^2 - 4\alpha w + 8] \\
& -4\alpha[8w^3 - 4\alpha w^2 + 2(\beta - 4)w + 4\alpha] \\
=& 16(\alpha^2 + \beta - 4)w^2 - 8\alpha[4 + (\beta - 4)]w + 16(4 - \alpha^2) \\
=& 16(\alpha^2 + \beta - 4)w^2 - 8\alpha\beta w + 16(4 - \alpha^2), \\
(T^2 q_{\alpha,\beta})^*(w) =& 16(4 - \alpha^2)w^2 - 8\alpha\beta w + 16(\alpha^2 + \beta - 4), \\
(T^3 q_{\alpha,\beta})(w) =& 16(4 - \alpha^2)[16(\alpha^2 + \beta - 4)w^2 - 8\alpha\beta w + 16(4 - \alpha^2)] \\
& -16(\alpha^2 + \beta - 4)[16(4 - \alpha^2)w^2 - 8\alpha\beta w + 16(\alpha^2 + \beta - 4)] \\
=& 128\alpha\beta[2(\alpha^2 - 4) + \beta]w + 256[(4 - \alpha^2)^2 - (\alpha^2 - 4 + \beta)^2] \\
=& 128\alpha\beta[2(\alpha^2 - 4) + \beta]w + 256\beta[2(4 - \alpha^2) - \beta] \\
=& 128\beta[2(\alpha^2 - 4) + \beta](\alpha w - 2), \\
(T^3 q_{\alpha,\beta})^*(w) =& 128\beta[2(\alpha^2 - 4) + \beta](-2w + \alpha), \\
(T^4 q_{\alpha,\beta})(w) =& -256\beta[2(\alpha^2 - 4) + \beta] \cdot 128\beta[2(\alpha^2 - 4) + \beta](\alpha w - 2) \\
& -128\beta\alpha[2(\alpha^2 - 4) + \beta] \cdot 128\beta[2(\alpha^2 - 4) + \beta](-2w + \alpha) \\
=& -128 \cdot 256\beta^2[2(\alpha^2 - 4) + \beta]^2(\alpha w - 2) \\
& -128^2\beta^2\alpha[2(\alpha^2 - 4) + \beta]^2(-2w + \alpha) \\
=& 256^2\beta^2[2(\alpha^2 - 4) + \beta]^2 - 128^2\beta^2\alpha^2[2(\alpha^2 - 4) + \beta]^2 \\
=& 128^2(4 - \alpha^2)\beta^2[2(\alpha^2 - 4) + \beta]^2, \\
\gamma_1 =& 8, \\
\gamma_2 =& -16(\alpha^2 - 4), \\
\gamma_3 =& -2 \cdot 128\beta[2(\alpha^2 - 4) + \beta], \\
\gamma_4 =& -128^2(\alpha^2 - 4)\beta^2[2(\alpha^2 - 4) + \beta]^2.
\end{aligned}$$

$$\alpha^2 > 4 \wedge \beta > 0$$

$$\gamma_1 > 0, \gamma_2 < 0, \gamma_3 < 0, \gamma_4 < 0$$

$$k_1 = 2, k_2 = 3, k_3 = 4$$

$$\sum_{j=1}^{3} (-1)^{j-1}(4 + 1 - k_j) = 5 - k_1 - (5 - k_2) + 5 - k_3 = 5 - 2 - (5 - 3) + 5 - 4$$

$$= 3 - 2 + 1 = 2$$

$$4 < \alpha^2 < 6, \frac{4}{100} < \beta < \frac{165}{100}$$

$$\triangle = 4\alpha^6 + \alpha^4\beta^2 - 80\alpha^4\beta + 16\alpha^4 - 16\alpha^2\beta^3 + 432\alpha^2\beta^2 - 960\alpha^2\beta - 320\alpha^2$$
$$+48\beta^4 - 768\beta^3 + 3456\beta^2 - 3072\beta + 768$$
$$= 768 - 320\alpha^2 + 16\alpha^4 + 4\alpha^6 + (-3072 - 960\alpha^2 - 80\alpha^4)\beta$$
$$+(3456 + 432\alpha^2 + \alpha^4)\beta^2 + (-768 - 16\alpha^2)\beta^3 + 48\beta^4$$
$$= 4(\alpha^2 - 4)^2(12 + \alpha^2) - 16(192 + 60\alpha^2 + 5\alpha^4)\beta + (3456 + 432\alpha^2 + \alpha^4)\beta^2$$
$$-16(48 + \alpha^2)\beta^3 + 48\beta^4.$$
$$\triangle < 288 - 8192\beta + 6084\beta^2 - 832\beta^3 + 48\beta^4$$
$$= 4(72 - 2048\beta + 1521\beta^2 - 208\beta^3 + 12\beta^4) = h(\beta),$$
$$h(x) := 4(72 - 2048x + 1521x^2 - 208x^3 + 12x^4),$$
$$h''(x) = 12168 - 4992x + 576x^2 = 24(507 - 208x + 24x)$$
$$= 24\left[\left(\sqrt{24}x - \frac{104}{\sqrt{24}}\right)^2 + \frac{169}{3}\right] = (24x - 104)^2 + 1352 > 0.$$
$$h\left(\frac{4}{100}\right) < 0, h\left(\frac{165}{100}\right) < 0$$
$$h(x) < 0,$$
$$x \in \left(\frac{4}{100}, \frac{165}{100}\right).$$
$$4 < \alpha^2 < 6, 0 < \beta < 2$$
$$q_{\alpha,\beta}(w) := w^4 + \alpha w^3 + (\beta - 4)w^2 - \alpha w + 3$$
$$= w^4 + \alpha w(w^2 - 1) + (\beta - 4)w^2 + 3$$
$$q_{\alpha,\beta}(-1) = q_{\alpha,\beta}(1) = \beta > 0$$
$$q_{\alpha,\beta}(t) = w^4 + \alpha w(w^2 - 1) + (\beta - 4)w^2 + 3$$
$$= w^4 - 4w^2 + 3 + \alpha w(w^2 - 1) + \beta w^2$$
$$\leqslant w^4 - 4w^2 + 3 + \alpha w(w^2 - 1) + 2w^2$$
$$= w^4 - 2w^2 + 3 + \alpha w(w^2 - 1)$$
$$q_{\alpha,\beta}(-2) = 16 - 8 + 3 + \alpha(-2)(4 - 1) = 11 + 6(-\alpha) < 11 - 12 = -1 < 0.$$
$$q_{\alpha,\beta}(2) = 16 - 8 + 3 + \alpha \cdot 2(4 - 1) = 11 + 6\alpha < 11 - 12 = -1 < 0$$
$$4 < \alpha^2 < 6, \frac{4}{100} < \beta < \frac{165}{100}$$
$$\alpha = \frac{m}{\mu M}, \beta = 6 - \frac{l(l+1)}{\mu^2 M^2},$$
$$4 < \alpha^2 < 6, \frac{4}{100} < \beta < \frac{165}{100}.$$
$$U_1(0) \cap (\mathbb{R} \times (-\infty, 0)).$$
$$C := \partial[U_1(0) \cap (\mathbb{R} \times (-\infty, 0))].$$
$$|\delta(w)| = \left| -\frac{m\left(1 - \frac{a}{M}\right)}{\mu M} w(w^2 - 1) + 2\sqrt{1 - \frac{a^2}{M^2}}\left[1 - i\frac{2n+1}{2\mu M}w(w^2 - 1)\right] \right|$$
$$\leqslant |\alpha| \cdot |w|(|w|^2 + 1)\left(1 - \frac{a}{M}\right) + 2\sqrt{1 - \frac{a^2}{M^2}}\left[1 + \frac{2n+1}{2\mu M}|w|(|w|^2 + 1)\right]$$
$$\leqslant 2\sqrt{6}\left(1 - \frac{a}{M}\right) + 2\sqrt{1 - \frac{a^2}{M^2}}\left(1 + \frac{2n+1}{\mu M}\right).$$
$$\left\| \delta|_{B_1(0)} \right\|_\infty \leqslant 2\sqrt{6}\left(1 - \frac{a}{M}\right) + 2\sqrt{1 - \frac{a^2}{M^2}}\left(1 + \frac{2n+1}{\mu M}\right).$$
$$\left.\frac{1}{|g_e|}\right|_C$$

$$\frac{1}{|g_e(w)|} \leqslant \varepsilon$$

$$|g_e(w)| \geqslant \frac{1}{\varepsilon},$$

$$2\sqrt{6}\left(1 - \frac{a}{M}\right) + 2\sqrt{1 - \frac{a^2}{M^2}\left(1 + \frac{2n+1}{\mu M}\right)} < \frac{1}{\varepsilon}$$

$$|\delta(w)| < |g_e(w)|,$$

$$\frac{25}{149} l(l+1) < \mu^2 M^2 < \frac{20}{87} m^2,$$

$$4 < \alpha^2 < 6, \frac{4}{100} < \beta < \frac{165}{100},$$

$$\alpha = \frac{m}{\mu M}, \beta = 6 - \frac{l(l+1)}{\mu^2 M^2}.$$

$$\left(\frac{25}{149} l(l+1), \frac{20}{87} m^2\right)$$

$$|m| > \frac{435}{161}(k+1) + \sqrt{\left(\frac{435}{161}\right)^2 (k+1)^2 + \frac{435}{161} k(k+1)}.$$

$$|m| > 2\frac{435}{161} = \frac{870}{161} \approx 5.40373.$$

$$\alpha = \frac{m}{\mu M}, \beta = 6 - \frac{l(l+1)}{\mu^2 M^2}$$

$$4 < \alpha^2 < 6 \wedge \frac{4}{100} < \beta < \frac{165}{100}$$

$$4 < \frac{m^2}{\mu^2 M^2} < 6 \wedge \frac{4}{100} < 6 - \frac{l(l+1)}{\mu^2 M^2} < \frac{165}{100}$$

$$\frac{4}{100} < 6 - \frac{l(l+1)}{\mu^2 M^2} < \frac{165}{100} \Leftrightarrow -\frac{4}{100} > -6 + \frac{l(l+1)}{\mu^2 M^2} > -\frac{165}{100}$$

$$6 - \frac{4}{100} > \frac{l(l+1)}{\mu^2 M^2} > 6 - \frac{165}{100} \Leftrightarrow \frac{149}{25} > \frac{l(l+1)}{\mu^2 M^2} > \frac{87}{20}$$

$$\frac{25}{149} < \frac{\mu^2 M^2}{l(l+1)} < \frac{20}{87} \Leftrightarrow \frac{25}{149} l(l+1) < \mu^2 M^2 < \frac{20}{87} l(l+1)$$

$$4 < \frac{m^2}{\mu^2 M^2} < 6 \Leftrightarrow \frac{1}{6} < \frac{\mu^2 M^2}{m^2} < \frac{1}{4} \Leftrightarrow \frac{m^2}{6} < \mu^2 M^2 < \frac{m^2}{4}$$

$$\frac{25}{149} l(l+1) < \mu^2 M^2 < \min\left\{\frac{m^2}{4}, \frac{20}{87} l(l+1)\right\} = \min\left\{\frac{m^2}{4}, \frac{l(l+1)}{4.35}\right\}$$

$$\frac{25}{149} l(l+1) \geqslant \frac{25}{149} |m|^2 \geqslant \frac{25}{150} |m|^2 = \frac{m^2}{6}$$

$$\frac{m^2}{4.35} = \min\left\{\frac{m^2}{4}, \frac{m^2}{4.35}\right\} \leqslant \min\left\{\frac{m^2}{4}, \frac{l(l+1)}{4.35}\right\}$$

$$\frac{25}{149} l(l+1) < \mu^2 M^2 < \frac{20}{87} m^2$$

$$\left(\frac{25}{149} l(l+1), \frac{20}{87} m^2\right)$$

$$\frac{20}{87}m^2 > \frac{25}{149}(|m|+k)(|m|+k+1)$$
$$\frac{20}{87}m^2 > \frac{25}{149}(|m|+k)(|m|+k+1) = \frac{25}{149}[m^2+2|m|(k+1)+k(k+1)]$$
$$\frac{805}{12963}m^2 > \frac{25}{149}[2|m|(k+1)+k(k+1)]$$
$$m^2 > \frac{435}{161}[2|m|(k+1)+k(k+1)]$$
$$m^2 - \frac{435}{161}[2|m|(k+1)+k(k+1)] > 0$$
$$\left[|m| - \frac{435}{161}(k+1)\right]^2 - \left(\frac{435}{161}\right)^2 (k+1)^2 - \frac{435}{161}k(k+1) > 0$$
$$\left[|m| - \frac{435}{161}(k+1)\right]^2 > \left(\frac{435}{161}\right)^2 (k+1)^2 + \frac{435}{161}k(k+1)$$
$$|m| > \frac{435}{161}(k+1) + \sqrt{\left(\frac{435}{161}\right)^2 (k+1)^2 + \frac{435}{161}k(k+1)}.$$
$$h(z) := \frac{1-z}{1+z}$$
$$h: U_1(0) \cap (\mathbb{R} \times (-\infty, 0)) \to (0,\infty)^2$$
$$h^{-1}: (0,\infty)^2 \to U_1(0) \cap (\mathbb{R} \times (-\infty, 0))$$
$$h^{-1}(u) = \frac{1-u}{1+u}$$
$$\frac{1-z}{1+z} = \frac{1-x-iy}{1+x+iy} = \frac{(1-x-iy)(1+x-iy)}{(1+x+iy)(1+x-iy)}$$
$$= \frac{1-x^2-y^2-2iy}{(1+x)^2+y^2} \in (0,\infty)^2$$
$$h(z) := \frac{1-z}{1+z}$$
$$h: U_1(0) \cap (\mathbb{R} \times (-\infty, 0)) \to (0,\infty)^2$$
$$\frac{1-u}{1+u} = \frac{1-u_1^2-u_2^2-2iu_2}{(1+u_1)^2+u_2^2} \in \mathbb{R} \times (-\infty, 0)$$
$$\left[\frac{1-u_1^2-u_2^2}{(1+u_1)^2+u_2^2}\right]^2 + \left[\frac{-2u_2}{(1+u_1)^2+u_2^2}\right]^2$$
$$= \frac{(1-u_1^2-u_2^2)^2+4u_2^2}{[(1+u_1)^2+u_2^2]^2} = \frac{(1-u_1^2-u_2^2)^2+4u_2^2}{(1+u_1^2+u_2^2+2u_1)^2}$$
$$= \frac{(1+u_1^2+u_2^2)^2-4(u_1^2+u_2^2)+4u_2^2}{(1+u_1^2+u_2^2)^2+4u_1(1+u_1^2+u_2^2)+4u_1^2}$$
$$= \frac{(1+u_1^2+u_2^2)^2-4u_1^2}{(1+u_1^2+u_2^2)^2+4u_1(1+u_1^2+u_2^2)+4u_1^2}$$
$$< \frac{(1+u_1^2+u_2^2)^2-4u_1^2}{(1+u_1^2+u_2^2)^2} < \frac{(1+u_1^2+u_2^2)^2}{(1+u_1^2+u_2^2)^2} = 1$$
$$\frac{1-u}{1+u} \in U_1(0).$$
$$g(u) := \frac{1-u}{1+u},$$
$$g: (0,\infty)^2 \to U_1(0) \cap (\mathbb{R} \times (-\infty, 0))$$

$$g(h(z)) = g\left(\frac{1-z}{1+z}\right) = \frac{1-\frac{1-z}{1+z}}{1+\frac{1-z}{1+z}} = \frac{1+z-1+z}{1+z+1-z} = \frac{2z}{2} = z$$

$$h(g(u)) = h\left(\frac{1-u}{1+u}\right) = \frac{1-\frac{1-u}{1+u}}{1+\frac{1-u}{1+u}} = \frac{1+u-1+u}{1+u+1-u} = \frac{2u}{2} = u$$

$$\begin{aligned} p(w) =& w^4 + \frac{\frac{ma}{M} - i(2n+1)\sqrt{1-\frac{a^2}{M^2}}}{\mu M} w(w^2-1) \\ &+ \left[2 - \frac{l(l+1)}{\mu^2 M^2}\right] w^2 + 3 + 2\sqrt{1-\frac{a^2}{M^2}} \\ =& w^4 + \alpha w(w^2-1) + (\beta-4)w^2 + 3 + \epsilon, \end{aligned}$$

$$\alpha := \frac{\frac{ma}{M} - i(2n+1)\sqrt{1-\frac{a^2}{M^2}}}{\mu M}, \beta := 6 - \frac{l(l+1)}{\mu^2 M^2}, \epsilon := 2\sqrt{1-\frac{a^2}{M^2}} (> 0).$$

$$\alpha = \alpha_1 - i\alpha_2$$

$$\alpha_1 = \frac{ma}{\mu M^2}, \alpha_2 = \frac{2n+1}{\mu M}\sqrt{1-\frac{a^2}{M^2}} (> 0).$$

$$\begin{aligned} &(p \circ h^{-1})(z) \\ &= \frac{1}{(1+z)^4}[(\beta+\epsilon)z^4 + 4(2+\alpha+\epsilon)z^3 + 2(16-\beta+3\epsilon)z^2 + 4(2-\alpha+\epsilon)z + 1] \end{aligned}$$

$$q(z) := z^4 + \frac{4z}{\beta+\epsilon}[(2+\epsilon)(z^2+1) + \alpha(z^2-1)] + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon}z^2 + 1$$

$$\lambda = -\mu\frac{1+z^2}{1-z^2}$$

$$\alpha_1 \geqslant 0 \wedge 0 \leqslant \beta \leqslant 16 + 3\epsilon$$

$$\lim_{x\to\infty} \arctan\left(\frac{\operatorname{Im}(q(x))}{\operatorname{Re}(q(x))}\right) = 0$$

$$\begin{aligned} q(x) =& x^4 + \frac{4x}{\beta+\epsilon}[(\epsilon+2)(x^2+1) + \alpha(x^2-1)] + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon}x^2 + 1 \\ =& x^4 + \frac{4x}{\beta+\epsilon}[(\epsilon+2)(x^2+1) + \alpha_1(x^2-1)] + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon}x^2 + 1 \\ &+ i\frac{4\alpha_2}{\beta+\epsilon}x(1-x^2) \\ =& x^4 + \frac{4x}{\beta+\epsilon}\{(\alpha_1+\epsilon+2)x^2 - [\alpha_1 - (\epsilon+2)]\} + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon}x^2 + 1 \\ &+ i\frac{4\alpha_2}{\beta+\epsilon}x(1-x^2), \end{aligned}$$

$$\begin{aligned} &\operatorname{Re}(q(0)) = 1 \neq 0, \\ &\operatorname{Re}(q(-1)) = \operatorname{Re}(q(1)) \\ &= 1 + \frac{4}{\beta+\epsilon}\{(\alpha_1+\epsilon+2) - [\alpha_1 - (\epsilon+2)]\} + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon} + 1 \\ &= 2 + \frac{8(\epsilon+2)}{\beta+\epsilon} + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon} = \frac{2(\beta+\epsilon)}{\beta+\epsilon} + \frac{8(\epsilon+2)}{\beta+\epsilon} + 2\frac{16-\beta+3\epsilon}{\beta+\epsilon} \\ &= \frac{2(\beta+\epsilon) + 8(\epsilon+2) + 2(16-\beta+3\epsilon)}{\beta+\epsilon} = 16\frac{3+\epsilon}{\beta+\epsilon} \neq 0. \end{aligned}$$

$$\alpha_1 \geqslant 0 \wedge 0 \leqslant \beta \leqslant 16 + 3\epsilon$$

$$\operatorname{Re}(q(x)) \geqslant x^4 + 1, \text{ if } x > 1$$
$$\operatorname{Im}(q(x)) = \frac{4\alpha_2}{\beta+\epsilon} x(1-x^2) < 0, \text{ if } x > 1$$
$$0 > \frac{\operatorname{Im}(q(x))}{\operatorname{Re}(q(x))} \geqslant \frac{4\alpha_2}{\beta+\epsilon}\frac{x(1-x^2)}{x^4+1} \text{ if } x > 1$$
$$\lim_{x\to\infty} \frac{\operatorname{Im}(q(x))}{\operatorname{Re}(q(x))} = 0, \lim_{x\to\infty} \arctan\left(\frac{\operatorname{Im}(q(x))}{\operatorname{Re}(q(x))}\right) = 0$$
$$\alpha_1 > 2+\epsilon \vee \alpha_1 < -(2+\epsilon),$$
$$\alpha_1 > 2+\epsilon$$
$$h_2 := (\mathbb{R} \to \mathbb{R}, y \mapsto \operatorname{Im}(q(iy)))$$
$$h_2((0,\infty)) \subset (-\infty, 0).$$
$$\alpha_1 < -(2+\epsilon)$$
$$h_2((0,\infty)) \subset (0,\infty)$$
$$0 < \beta < 8+\epsilon$$
$$h_1 := (\mathbb{R} \to \mathbb{R}, y \mapsto \operatorname{Re}(q(iy)))$$
$$\begin{cases} h_1(y) > 0 \text{ para } 0 \leqslant y < \bar{y}_0 \\ h_1(\bar{y}_0) = 0 \\ h_1(y) < 0 \text{ para } \bar{y}_0 < y < \bar{y}_1 \\ h_1(\bar{y}_1) = 0 \\ h_1(y) > 0 \text{ para } y > \bar{y}_1 \end{cases}$$
$$\begin{aligned} q(iy) =& y^4 - 2\frac{16-\beta+3\epsilon}{\beta+\epsilon}y^2 + 1 - \frac{4\alpha_2}{\beta+\epsilon}y(y^2+1) \\ &-i\frac{4y}{\beta+\epsilon}[(\epsilon+2)(y^2-1) + \alpha_1(y^2+1)] \\ =& \left(y^2 - \frac{16-\beta+3\epsilon}{\beta+\epsilon}\right)^2 - \frac{8(8+\epsilon-\beta)(4+\epsilon)}{(\beta+\epsilon)^2} - \frac{4\alpha_2}{\beta+\epsilon}y(y^2+1) \\ &-i\frac{4y}{\beta+\epsilon}[(\alpha_1+\epsilon+2)y^2 + (\alpha_1-(\epsilon+2))] \end{aligned}$$
$$\alpha_1 > 2+\epsilon \vee \alpha_1 < -(2+\epsilon)$$
$$\operatorname{Re}(q(0)) = 1 \neq 0.$$
$$\alpha_1 > 2+\epsilon$$
$$h_2 := (\mathbb{R} \to \mathbb{R}, y \mapsto \operatorname{Im}(q(iy)))$$
$$h_2'(y) = -\frac{4}{\beta+\epsilon}\{[\alpha_1 - (2+\epsilon)] + 3[\alpha_1 + (2+\epsilon)]y^2\} < 0$$
$$h_2(y) = \operatorname{Im}(q(iy)) = -\frac{4y}{\beta+\epsilon}[(\alpha_1+\epsilon+2)y^2 + (\alpha_1-(\epsilon+2))] < 0.$$
$$\alpha_1 < -(2+\epsilon)$$
$$h_2'(y) = -\frac{4}{\beta+\epsilon}\{[\alpha_1 - (2+\epsilon)] + 3[\alpha_1 + (2+\epsilon)]y^2\} > 0$$
$$h_2(y) = \operatorname{Im}(q(iy)) = -\frac{4y}{\beta+\epsilon}[(\alpha_1+\epsilon+2)y^2 + (\alpha_1-(\epsilon+2))] > 0.$$
$$0 < \beta < 8+\epsilon,$$
$$h_1 := (\mathbb{R} \to \mathbb{R}, y \mapsto \operatorname{Re}(q(iy)))$$
$$I_1 := \left(0, \sqrt{\frac{16-\beta+3\epsilon}{\beta+\epsilon}}\right)$$
$$h_1'(y) = 4y\left(y^2 - \frac{16-\beta+3\epsilon}{\beta+\epsilon}\right) - \frac{4\alpha_2(3y^2+1)}{\beta+\epsilon} < 0$$
$$\sqrt{\frac{16-\beta+3\epsilon}{\beta+\epsilon}} = \sqrt{\frac{16+4\epsilon-(\beta+\epsilon)}{\beta+\epsilon}} = \sqrt{\frac{16+4\epsilon}{\beta+\epsilon} - 1} > \sqrt{\frac{16+4\epsilon}{8+2\epsilon} - 1} = 1$$

$$16-\beta+3\epsilon>2\sqrt{2}\sqrt{(4+\epsilon)(8-\beta+\epsilon)}$$
$$(16-\beta+3\epsilon)^2>8(4+\epsilon)(8-\beta+\epsilon)$$
$$[8-\beta+\epsilon+2(4+\epsilon)]^2>8(4+\epsilon)(8-\beta+\epsilon),$$
$$(8-\beta+\epsilon)^2+4(4+\epsilon)(8-\beta+\epsilon)+4(4+\epsilon)^2>8(4+\epsilon)(8-\beta+\epsilon)$$
$$(8-\beta+\epsilon)^2-4(4+\epsilon)(8-\beta+\epsilon)+4(4+\epsilon)^2>0,$$
$$[8-\beta+\epsilon-2(4+\epsilon)]^2>0.$$
$$y_j^4-2\frac{16-\beta+3\epsilon}{\beta+\epsilon}y_j^2+1=0$$
$$(0<)y_0:=\sqrt{\frac{16-\beta+3\epsilon-2\sqrt{2}\sqrt{(4+\epsilon)(8-\beta+\epsilon)}}{\beta+\epsilon}}<\sqrt{\frac{16-\beta+3\epsilon}{\beta+\epsilon}}$$
$$y_1:=\sqrt{\frac{16-\beta+3\epsilon+2\sqrt{2}\sqrt{(4+\epsilon)(8-\beta+\epsilon)}}{\beta+\epsilon}}>\sqrt{\frac{16-\beta+3\epsilon}{\beta+\epsilon}}$$
$$h_1(y_0)=-\frac{4\alpha_2}{\beta+\epsilon}y_0(y_0^2+1)<0, h_1(y_1)=-\frac{4\alpha_2}{\beta+\epsilon}y_1(y_1^2+1)<0.$$
$$h_1(\bar{y}_0)=0.$$
$$\begin{cases} h_1(y)>0 & \text{para } 0\leqslant y<\bar{y}_0 \\ & h_1(\bar{y}_0)=0 \\ h_1(y)<0 & \text{para } y\in I_1 \text{entonces t } y>\bar{y}_0 \end{cases}$$
$$h_1(y)=y^4-2\frac{16-\beta+3\epsilon}{\beta+\epsilon}y^2+1-\frac{4\alpha_2}{\beta+\epsilon}y(y^2+1)$$
$$h_1(\bar{y}_1)=0.$$
$$\triangle=\frac{4096}{(\beta+\epsilon)^6}(8+\epsilon-\beta+2\alpha_2)(8+\epsilon-\beta-2\alpha_2)(8\beta+8\epsilon+2\beta\epsilon+2\epsilon^2+\alpha_2^2)^2$$
$$\alpha_2<\frac{1}{2}(8+\epsilon-\beta),$$
$$\triangle>0$$
$$h_1(-y)=y^4-2\frac{16-\beta+3\epsilon}{\beta+\epsilon}y^2+1+\frac{4\alpha_2}{\beta+\epsilon}y(y^2+1)$$
$$\alpha_2=\frac{1}{2}(8+\epsilon-\beta),$$
$$\triangle=0$$
$$\alpha_2<\frac{1}{2}(8+\epsilon-\beta),$$
$$\triangle<0$$
$$\begin{cases} h_1(y)>0 \ \text{ para } 0\leqslant y<\bar{y}_0 \\ h_1(\bar{y}_0)=0 \\ h_1(y)<0 \ \text{ para } \bar{y}_0<y<\bar{y}_1 \\ h_1(\bar{y}_1)=0 \\ h_1(y)>0 \ \text{ para } y>\bar{y}_1 \end{cases}$$
$$\alpha_1>2+\epsilon\wedge 0<\beta<8+\epsilon,$$

$$q\left(Re^{i\theta}\right) =\left(Re^{i\theta}\right)^4 + \frac{4Re^{i\theta}}{\beta+\epsilon}\left\{(\alpha_1+\epsilon+2)\left(Re^{i\theta}\right)^2 - [\alpha_1-(\epsilon+2)]\right\}$$
$$+2\frac{16-\beta+3\epsilon}{\beta+\epsilon}\left(Re^{i\theta}\right)^2+1$$
$$=R^4e^{4i\theta} + \frac{4Re^{i\theta}}{\beta+\epsilon}\left[(\alpha_1+\epsilon+2)R^2e^{2i\theta} - [\alpha_1-(\epsilon+2)]\right]$$
$$+2\frac{16-\beta+3\epsilon}{\beta+\epsilon}R^2e^{2i\theta}+1$$
$$=R^4\left\{e^{4i\theta} + \frac{4e^{i\theta}}{R(\beta+\epsilon)}\left[(\alpha_1+\epsilon+2)e^{2i\theta} - \frac{1}{R^2}[\alpha_1-(\epsilon+2)]\right]\right.$$
$$\left.+\frac{2}{R^2}\frac{16-\beta+3\epsilon}{\beta+\epsilon}e^{2i\theta}+\frac{1}{R^4}\right\}$$
$$([0,R]\to\mathbb{C}, x\mapsto q(x))$$
$$\left([0,\pi/2]\to\mathbb{C}, \theta\mapsto q\left(Re^{i\theta}\right)\right)$$
$$([0,R]\to\mathbb{C}, y\mapsto q(i(R-y)))$$
$$m\geqslant 2k+1+\sqrt{6k^2+6k+1}$$
$$1>\frac{a}{M}>\frac{2\sqrt{6}}{5}\frac{\sqrt{1+\frac{1}{m}\left[2k+1+\frac{k(k+1)}{m}\right]}}{1+\frac{2/5}{m}\left[2k+1+\frac{k(k+1)}{m}\right]}$$
$$I:=\left(\sqrt{\frac{1}{6}l(l+1)},\frac{\frac{ma}{M}}{2\left(1+\sqrt{1-\frac{a^2}{M^2}}\right)}\right)$$
$$\frac{2\sqrt{6}}{5}\approx 0.979796$$
$$\alpha_1=\frac{ma}{\mu M^2}>2+\epsilon\wedge 0<\beta=6-\frac{l(l+1)}{\mu^2M^2}<8+\epsilon,$$
$$0<6-\frac{l(l+1)}{\mu^2M^2}<8+\epsilon\Leftrightarrow 0>-6+\frac{l(l+1)}{\mu^2M^2}>-(8+\epsilon)$$
$$6>\frac{l(l+1)}{\mu^2M^2}>-(2+\epsilon)\Leftrightarrow 6>\frac{l(l+1)}{\mu^2M^2},\mu^2M^2>\frac{1}{6}l(l+1),$$
$$2+\epsilon<\frac{ma}{\mu M^2}\Leftrightarrow \mu M<\frac{ma}{(2+\epsilon)M}\Leftrightarrow \mu^2M^2<\frac{m^2a^2}{(2+\epsilon)^2M^2}\wedge m\geqslant 0.$$
$$\frac{1}{6}l(l+1)<\mu^2M^2<\frac{(a/M)^2}{(2+\epsilon)^2}m^2\wedge m\geqslant 0.$$
$$m\geqslant 2k+1+\sqrt{6k^2+6k+1},$$

$$\frac{1}{6}l(l+1) < \frac{(a/M)^2}{(2+\epsilon)^2}m^2$$

$$\Leftrightarrow \frac{l(l+1)}{6m^2} < \frac{(a/M)^2}{(2+\epsilon)^2} = \frac{\frac{a^2}{M^2}}{4\left(1+\sqrt{1-\frac{a^2}{M^2}}\right)^2}$$

$$\Leftrightarrow \frac{2l(l+1)}{3m^2}\left(1+\sqrt{1-\frac{a^2}{M^2}}\right)^2 < \frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{2l(l+1)}{3m^2}\left(2+2\sqrt{1-\frac{a^2}{M^2}}-\frac{a^2}{M^2}\right) < \frac{a^2}{M^2}$$

$$\Leftrightarrow 2+2\sqrt{1-\frac{a^2}{M^2}}-\frac{a^2}{M^2} < \frac{3m^2}{2l(l+1)}\frac{a^2}{M^2}$$

$$\Leftrightarrow 2\sqrt{1-\frac{a^2}{M^2}} < \left[1+\frac{3m^2}{2l(l+1)}\right]\frac{a^2}{M^2}-2$$

$$\Leftrightarrow 4\left(1-\frac{a^2}{M^2}\right) < \left\{\left[1+\frac{3m^2}{2l(l+1)}\right]\frac{a^2}{M^2}-2\right\}$$

$$\Leftrightarrow -4\frac{a^2}{M^2} < \left[1+\frac{3m^2}{2l(l+1)}\right]^2\frac{a^4}{M^4}-4\left[1+\frac{3m^2}{2l(l+1)}\right]\frac{a^2}{M^2}$$

$$\Leftrightarrow 0 < \left[1+\frac{3m^2}{2l(l+1)}\right]^2\frac{a^4}{M^4}-\frac{6m^2}{l(l+1)}\frac{a^2}{M^2}$$

$$\Leftrightarrow 0 < \left\{\left[1+\frac{3m^2}{2l(l+1)}\right]^2\frac{a^2}{M^2}-\frac{6m^2}{l(l+1)}\right\}\frac{a^2}{M^2}$$

$$\Leftrightarrow 0 < \left[1+\frac{3m^2}{2l(l+1)}\right]^2\frac{a^2}{M^2}-\frac{6m^2}{l(l+1)}$$

$$\Leftrightarrow \frac{6m^2}{l(l+1)} < \left[1+\frac{3m^2}{2l(l+1)}\right]^2\frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{6m^2}{l(l+1)} < \left[\frac{3m^2+2l(l+1)}{2l(l+1)}\right]^2\frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{6m^2}{l(l+1)}\left[\frac{2l(l+1)}{3m^2+2l(l+1)}\right]^2 < \frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{24m^2l(l+1)}{[3m^2+2l(l+1)]^2} < \frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{24m^2(m+k)(m+k+1)}{[3m^2+2(m+k)(m+k+1)]^2} < \frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{24\left(1+\frac{k}{m}\right)\left(1+\frac{k+1}{m}\right)}{\left[3+2\left(1+\frac{k}{m}\right)\left(1+\frac{k+1}{m}\right)\right]^2} < \frac{a^2}{M^2}$$

$$\Leftrightarrow \frac{24}{25}\frac{1+\frac{1}{m}\left[2k+1+\frac{k(k+1)}{m}\right]}{\left\{1+\frac{2/5}{m}\left[2k+1+\frac{k(k+1)}{m}\right]\right\}^2} < \frac{a^2}{M^2}$$

$$\frac{6m^2}{3m^2+2l(l+1)} \geqslant 1 \Leftrightarrow 6m^2 \geqslant 3m^2+2l(l+1)$$
$$\Leftrightarrow 3m^2 \geqslant 2l(l+1) = 2(m+k)(m+k+1) \Leftrightarrow m^2 \geqslant 2(2k+1)m + 2k(k+1)$$
$$\Leftrightarrow m^2 - 2(2k+1)m - 2k(k+1) \geqslant 0 \Leftrightarrow [m-(2k+1)]^2 - (2k+1)^2 - 2k(k+1) \geqslant 0$$
$$\Leftrightarrow [m-(2k+1)]^2 \geqslant 6k^2+6k+1 \Leftrightarrow m \geqslant 2k+1+\sqrt{6k^2+6k+1}$$
$$m \geqslant 2k+1+\sqrt{6k^2+6k+1}$$
$$\frac{a^2}{M^2} > \frac{24m^2 l(l+1)}{[3m^2+2l(l+1)]^2} \Leftrightarrow \frac{a^2}{M^2} > \frac{6m^2}{3m^2+2l(l+1)} \frac{4l(l+1)}{3m^2+2l(l+1)}$$
$$\Rightarrow \frac{a^2}{M^2} > \frac{4l(l+1)}{3m^2+2l(l+1)} \Leftrightarrow \left[1+\frac{3m^2}{2l(l+1)}\right]\frac{a^2}{M^2} - 2 > 0$$

2. Modelo morfológico de las superpartículas (partículas estrella) en campos cuánticos relativistas y colapsos gravitacionales.

$$\mathcal{L}_P = -\frac{1}{8\pi}\left(W_{\mu\nu}\bar{W}^{\mu\nu} + 2m^2 X_\mu \bar{X}^\mu\right)$$
$$\mathcal{S} = \int \left(\frac{R}{16\pi} + \mathcal{L}_P\right)\sqrt{-g}\,d^4x$$
$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$
$$T_{\mu\nu} = \frac{1}{4\pi}\left[-W_{\lambda(\mu}\bar{W}^{\lambda}_{\nu)} - \frac{g_{\mu\nu}}{4} W_{\alpha\beta}\bar{W}^{\alpha\beta} + m^2\left(X_{(\mu}\bar{X}_{\nu)} - \frac{g_{\mu\nu}}{2} X_\lambda \bar{X}^\lambda\right)\right]$$
$$\nabla_\mu W^{\mu\nu} = m^2 X^\nu$$
$$\nabla_\mu X^\mu = 0$$
$$j^\mu := \frac{i}{8\pi}\left[\bar{W}^{\mu\nu} X_\nu - W^{\mu\nu}\bar{X}_\nu\right]$$
$$\nabla_\mu j^\mu = 0$$
$$ds^2 = \left(-\alpha^2 dt^2 + \beta_i \beta^i\right) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$$
$$n^\mu = \left(1/\alpha, -\beta^i/\alpha\right), n_\mu = (-\alpha, 0,0,0)$$
$$\phi := -n^\mu X_\mu, a_\mu := \gamma_\mu^\nu X_\nu$$
$$X_\mu = a_\mu + n_\mu \phi,$$
$$\mathcal{E}_\mu := n^\nu W_{\mu\nu}, \mathcal{B}_\mu := n^\nu W^*_{\mu\nu}$$
$$W^*_{\alpha\beta} := -\frac{1}{2} E_{\alpha\beta\mu\nu} W^{\mu\nu}$$
$$W_{\mu\nu} = n_\mu \mathcal{E}_\nu - n_\nu \mathcal{E}_\mu + E_{\mu\nu\lambda}\mathcal{B}^\lambda$$
$$W^*_{\mu\nu} = n_\mu \mathcal{B}_\nu - n_\nu \mathcal{B}_\mu - E_{\mu\nu\lambda}\mathcal{E}^\lambda$$
$$\frac{d}{dt}\phi = \alpha\phi K - D_i\left(\alpha a^i\right)$$
$$\frac{d}{dt} a_i = -\alpha \mathcal{E}_i - \partial_i(\alpha\phi)$$
$$\frac{d}{dt}\mathcal{E}^i = \alpha\left(K\mathcal{E}^i + m^2 a^i\right) + [D \times (\alpha\mathcal{B})]^i$$
$$\frac{d}{dt}\mathcal{B}^i = \alpha K \mathcal{B}^i - [D \times (\alpha\mathcal{E})]^i$$
$$D_i \mathcal{E}^i + m^2\phi = 0,$$
$$D_i \mathcal{B}^i = 0.$$
$$\rho := n^\mu n^\nu T_{\mu\nu} = \frac{1}{8\pi}\left[\mathcal{E}^2 + \mathcal{B}^2 + m^2(\phi^2 + a^2)\right]$$
$$\mathcal{E}^2 := \mathcal{E}_\alpha \overline{\mathcal{E}}^\alpha = \mathcal{E}_i \overline{\mathcal{E}}^i, \mathcal{B}^2 := \mathcal{B}_\alpha \overline{\mathcal{B}}^\alpha = \mathcal{B}_i \overline{\mathcal{B}}^i, \phi^2 := \phi\bar{\phi}, a^2 := a_i \bar{a}^i.$$

$$J^i := -\gamma^{i\mu} n^\nu T_{\mu\nu} = \frac{1}{8\pi}\left[E^i{}_{jk}\bar{\mathcal{E}}^j \mathcal{B}^k + m^2 a^i \bar{\phi} + \text{ c.c.}\right],$$

$$S_{ij} := \gamma_i^\mu \gamma_j^\nu T_{\mu\nu} = \frac{1}{8\pi}\{\gamma_{ij}(\mathcal{E}^2 + \mathcal{B}^2) - (\mathcal{B}_i \bar{\mathcal{B}}_j + \mathcal{E}_i \bar{\mathcal{E}}_j + \text{ c.c.})$$

$$+m^2[(a_i \bar{a}_j + \text{ c.c.}) - \gamma_{ij}(a^2 - \phi^2)]\}$$

$$\rho_Q := -n_\mu j^\mu = \alpha j^0 = \frac{i}{8\pi}\left[a_k \bar{\mathcal{E}}^k - \text{ c.c.}\right],$$

$$j_Q^i := \gamma_\mu^i j^\mu = \frac{i}{8\pi}\left[\phi \bar{\mathcal{E}}^i + E^{ijk} a_j \bar{\mathcal{B}}_k - \text{ c.c.}\right].$$

$$ds^2 = -\alpha^2 dt^2 + \psi^4(A dr^2 + r^2 B d\Omega^2),$$

$$\phi(r,t) = \varphi(r) e^{-i\omega t}$$

$$a_r(r,t) = i a(r) e^{-i\omega t}$$

$$\mathcal{E}^r(r,t) = e(r) e^{-i\omega t}$$

$$J_r = \frac{m^2}{8\pi}[a_r \bar{\phi} + c.c] = \frac{m^2}{8\pi}[i a \varphi + c.c] = 0$$

$$j_Q^r = \frac{i}{8\pi}\left[\phi \bar{\mathcal{E}}^r - c.c.\right] = \frac{i}{8\pi}[\varphi e - c.c.] = 0$$

$$\partial_r F = -\omega a - \alpha e A.$$

$$e = -\frac{\alpha m^2 a}{\omega A}$$

$$\partial_r F = \omega a \left(\frac{\alpha^2 m^2}{\omega^2} - 1\right)$$

$$\partial_r a = \frac{\omega \varphi A}{\alpha} - a\left(\frac{2}{r} - \frac{\partial_r A}{2A} + \frac{\partial_r \alpha}{\alpha}\right).$$

$$\partial_r e = -e\left(\frac{2}{r} + \frac{\partial_r A}{2A}\right) - m^2 \varphi$$

$$\partial_r A = A\left[\frac{(1-A)}{r} + 8\pi r A \rho\right]$$

$$\partial_r \alpha = \alpha\left[\frac{(A-1)}{2r} + 4\pi r A S_r^r\right]$$

$$\rho = +\frac{1}{8\pi}[A e^2 + m^2(\varphi^2 + a^2/A)].$$

$$S_r^r = -\frac{1}{8\pi}[A e^2 - m^2(\varphi^2 + a^2/A)].$$

$$\partial_r a = \frac{A\omega F}{\alpha^2} - a\left[\frac{1}{r}(1+A) + 4\pi r A(S_r^r - 2\rho)\right].$$

$$A(r=0) = 1,\ \partial_r A|_{r=0} = 0.$$

$$\lim_{r\to\infty} \alpha = 1,\ \partial_r \alpha|_{r=0} = 0$$

$$\lim_{r\to\infty} \varphi = 0,\ \partial_r \varphi|_{r=0} = 0$$

$$a(r=0) = 0,\ \lim_{r\to\infty} a = 0$$

$$\partial_r \varphi = \frac{\omega a}{\alpha_\infty}\left(\frac{\alpha_\infty^2 m^2}{\omega^2} - 1\right), \partial_r a = \frac{\omega \varphi}{\alpha_\infty}$$

$$\partial_r^2 \varphi = \left(m^2 - \frac{\omega^2}{\alpha_\infty^2}\right)\varphi,$$

$$\varphi = e^{\pm(m^2 - \omega^2/\alpha_\infty^2)^{1/2} r}$$

$$k_a = \frac{\omega \varphi_0}{3\alpha_0}$$

$$k_A = \frac{8\pi}{3}\rho_0$$

$$M = 4\pi \int_0^\infty \rho r^2 dr$$

$$Q := \int \rho_Q \gamma^{1/2} d^3x = 4\pi \int_0^\infty \rho_Q A^{1/2} r^2 dr$$

$$\rho_Q = -\frac{ae}{4\pi} = \frac{\alpha m^2 a^2}{\omega A}$$

$$U := M - mQ$$

$$\partial_t \phi = \alpha K \phi - \frac{1}{A\psi^4}\partial_r(\alpha a_r) + \frac{\alpha a_r}{A\psi^4}\left(\frac{\partial_r A}{2A} - \frac{\partial_r B}{B} - 2\frac{\partial_r \psi}{\psi} - \frac{2}{r}\right)$$

$$\partial_t a_r = -\alpha A \psi^4 \mathcal{E}^r - \partial_r(\alpha\phi)$$

$$\partial_t \mathcal{E}^r = \alpha\left(K\mathcal{E}^r + \frac{m^2 a^r}{A\psi^4}\right)$$

$$\partial_t \alpha = -2\alpha K$$

$$m \to \lambda m, \omega \to \lambda\omega, r \to r/\lambda, e \to \lambda e.$$

$$\mathcal{L} = \frac{R}{16\pi} - \frac{1}{2}\sum_{m=-\ell}^{\ell} \left(\nabla_\mu \Phi_{\ell m} \nabla^\mu \Phi_{\ell m}^* + \mu^2 |\Phi_{\ell m}|^2\right)$$

$$\Phi_{\ell m}(t, r, \vartheta, \varphi) = e^{i\omega t}\psi_\ell(r) Y^{\ell m}(\vartheta, \varphi)$$

$$T_{\mu\nu} = \frac{1}{2}\sum_{m=-\ell}^{\ell} \left[\nabla_\mu \Phi_{\ell m}^* \nabla_\nu \Phi_{\ell m} + \nabla_\mu \Phi_{\ell m} \nabla_\nu \Phi_{\ell m}^* - g_{\mu\nu}(\nabla_\alpha \Phi_{\ell m}^* \nabla^\alpha \Phi_{\ell m} + \mu^2 \Phi_{\ell m}^* \Phi_{\ell m})\right]$$

$$ds^2 = -\alpha^2(r)dt^2 + \gamma^2(r)dr^2 + r^2 d\Omega^2, \gamma^2(r) := \frac{1}{1 - \frac{2M(r)}{r}}$$

$$M' = \frac{\kappa_\ell r^2}{2}\left[\frac{\psi_\ell'^2}{\gamma^2} + \left(\mu^2 + \frac{\omega^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right] = 4\pi r^2 \rho$$

$$\frac{(\alpha\gamma)'}{\alpha\gamma^3} = \kappa_\ell r\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2\right] = 4\pi r(\rho + p_r),$$

$$\frac{1}{r^2\alpha\gamma}\left(\frac{r^2\alpha}{\gamma}\psi_\ell'\right)' = \left(\mu^2 - \frac{\omega^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell,$$

$$\rho := -T^t{}_t = \frac{\kappa_\ell}{8\pi}\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 + \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right],$$

$$p_r := T^r{}_r = \frac{\kappa_\ell}{8\pi}\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right],$$

$$p_T := T^\theta{}_\theta = T^\varphi{}_\varphi = \frac{\kappa_\ell}{8\pi}\left[-\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \mu^2\psi_\ell^2\right].$$

$$C_{99} := \frac{M_T}{R_{99}}$$

$$C_m := \max_{r>0}\left\{\frac{M(r)}{r}\right\} =: \frac{M_m}{R_m}$$

$$fa = \frac{p_r - p_T}{p_r}.$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{\alpha^2\gamma^2} - \frac{1}{\gamma^2}\left(\delta + \frac{L^2}{r^2}\right)$$

$$V_{\text{eff}}(r) := \alpha^2\left(\delta + \frac{L^2}{r^2}\right)$$

$$\alpha - r\alpha' = 0$$

$$E = \sqrt{\frac{\alpha^3}{\alpha - r\alpha'}}, L = \sqrt{\frac{r^3\alpha'}{\alpha - r\alpha'}}.$$

$$v(r) := r\frac{d\phi}{dt} = \sqrt{\frac{r\alpha'(r)}{\alpha(r)}}$$

$$\mu M_T \approx 0.50\ell + 0.82$$

$$\mu M_m \approx 0.50\ell + 0.55$$

$$\mu R_{99} \approx 2.2\ell + 8.7$$

$$\mu R_m \approx 2.2\ell + 6.7$$

$$y := \frac{r - \ell x_0}{\ell^a}$$

$$M_*(y) := \frac{M(r)}{\ell}, \alpha_*(y) := \alpha(r)\ \gamma_*(y) := \gamma(r)\ \psi_*(y) := \ell^{1+\frac{a}{2}}\psi_\ell(r)$$

$$\gamma_*^{-2}(y) = 1 - \frac{2M_*(y)}{x_0}\frac{1}{1 + \ell^{a-1}\frac{y}{x_0}},$$

$$\frac{dM_*}{dy} = x_0^2\left(1 + \ell^{a-1}\frac{y}{x_0}\right)^2 \rho_*$$

$$\frac{1}{\gamma_*^2\alpha_*}\frac{d\alpha_*}{dy} = x_0\left(1 + \ell^{a-1}\frac{y}{x_0}\right)p_{r*} + \frac{M_*}{x_0^2}\frac{\ell^{a-1}}{\left(1 + \ell^{a-1}\frac{y}{x_0}\right)^2}$$

$$\frac{1}{\alpha_*\gamma_*}\frac{d}{dy}\left(\frac{\alpha_*}{\gamma_*}\frac{d\psi_*}{dy}\right) + \frac{2}{x_0}\frac{\ell^{a-1}}{1 + \ell^{a-1}\frac{y}{x_0}}\frac{1}{\gamma_*^2}\frac{d\psi_*}{dy} = -\ell^{2a}\left[\frac{\omega^2}{\alpha_*^2} - \mu^2 - \frac{1}{x_0^2}\frac{1 + \frac{1}{\ell}}{\left(1 + \ell^{a-1}\frac{y}{x_0}\right)^2}\right]\psi_*$$

$$\rho_*(y) := 4\pi\ell^{1+a}\rho(r) = \left(1 + \frac{1}{2\ell}\right)\left[\ell^{-2a}\frac{1}{\gamma_*^2}\left(\frac{d\psi_*}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_*^2} + \mu^2 + \frac{1}{x_0^2}\frac{1 + \frac{1}{\ell}}{\left(1 + \ell^{a-1}\frac{y}{x_0}\right)^2}\right)\psi_*^2\right]$$

$$p_{r*}(y) := 4\pi\ell^{1+a}p_r(r) = \left(1 + \frac{1}{2\ell}\right)\left[\ell^{-2a}\frac{1}{\gamma_*^2}\left(\frac{d\psi_*}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_*^2} - \mu^2 - \frac{1}{x_0^2}\frac{1 + \frac{1}{\ell}}{\left(1 + \ell^{a-1}\frac{y}{x_0}\right)^2}\right)\psi_*^2\right].$$

$$\frac{dM_\infty}{dy} = x_0^2\rho_\infty, \rho_\infty = \frac{1}{\gamma_\infty^2}\left(\frac{d\psi_\infty}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_\infty^2} + \mu_0^2\right)\psi_\infty^2$$

$$\frac{1}{\gamma_\infty^2\alpha_\infty}\frac{d\alpha_\infty}{dy} = x_0 p_{r\infty}, p_{r\infty} = \frac{1}{\gamma_\infty^2}\left(\frac{d\psi_\infty}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_\infty^2} - \mu_0^2\right)\psi_\infty^2$$

$$\frac{1}{\alpha_\infty\gamma_\infty}\frac{d}{dy}\left(\frac{\alpha_\infty}{\gamma_\infty}\frac{d\psi_\infty}{dy}\right) = -\left(\frac{\omega^2}{\alpha_\infty^2} - \mu_0^2\right)\psi_\infty$$

$$\alpha_\infty\gamma_\infty p_{r\infty} = \text{const},$$

$$\frac{dM_\infty}{dy} = x_0^2\rho_\infty, \rho_\infty = \left(\frac{\omega^2}{\alpha_\infty^2} + \mu_0^2\right)\psi_\infty^2$$

$$\frac{1}{\gamma_\infty^2\alpha_\infty}\frac{d\alpha_\infty}{dy} = x_0 p_{r\infty}, p_{r\infty} = \left(\frac{\omega^2}{\alpha_\infty^2} - \mu_0^2\right)\psi_\infty^2$$

$$\alpha_\infty = \frac{\omega}{\mu_0}$$

$$\frac{dM_\infty}{dy} = 2x_0^2\mu_0^2\psi_\infty^2 = 2(1 + \mu^2 x_0^2)\psi_\infty^2$$

$$\psi_*(y) = \psi_\infty(y) + \varepsilon\psi_1(y) + \mathcal{O}(\varepsilon^2)$$

$$\alpha_*(y) = \alpha_\infty[1 + \ell^{a-1}\delta(y) + \mathcal{O}(\ell^{-1})]$$

$$2\ell^{3a-1}\left(\mu_0^2\delta-\frac{y}{x_0^3}\right)\psi_\infty$$

$$\frac{1}{\gamma_\infty^2}\frac{d\delta}{dy}=x_0\left[\frac{1}{\gamma_\infty^2}\left(\frac{d\psi_\infty}{dy}\right)^2-2\left(\mu_0^2\delta-\frac{y}{x_0^3}\right)\psi_\infty^2\right]+\frac{M_\infty}{x_0^2}$$

$$\frac{1}{\gamma_\infty}\frac{d}{dy}\left(\frac{1}{\gamma_\infty}\frac{d\psi_\infty}{dy}\right)=2\left(\mu_0^2\delta-\frac{y}{x_0^3}\right)\psi_\infty$$

$$d\psi_*/dy\approx\left(\sqrt{-2y/x_0^3}-1/(4y)\right)\psi_*.$$

$$p_{T*}(y):=4\pi\ell^{1+a}p_T=\left(1+\frac{1}{2\ell}\right)\left[-\ell^{-2a}\frac{1}{\gamma_*^2}\left(\frac{d\psi_*}{dy}\right)^2+\left(\frac{\omega^2}{\alpha_*^2}-\mu^2\right)\psi_*^2\right]$$

$$p_{T\infty}=\left(\frac{\omega^2}{\alpha_\infty^2}-\mu^2\right)\psi_\infty^2=\frac{\psi_\infty^2}{x_0^2}$$

$$C_{x_0}:=\frac{M_{\infty T}}{x_0}$$

$$V_{\text{eff}}(r)=L^2\frac{\alpha^2}{r^2}=\frac{L^2}{\ell^2}\frac{\alpha_\infty^2}{x_0^2}\left[1+\frac{2}{\ell^{2/3}}\left(\delta(y)-\frac{y}{x_0}\right)+\mathcal{O}\left(\frac{1}{\ell}\right)\right].$$

$$V_1(y):=\ell^{2/3}\left(\frac{x_0^2}{\alpha_\infty^2}\frac{\ell^2}{L^2}V_{\text{eff}}(r)-1\right)=2\left(\delta(y)-\frac{y}{x_0}\right)+\mathcal{O}\left(\frac{1}{\ell^{1/3}}\right)$$

$$\begin{aligned}\mu&\mapsto\lambda\mu\\(\alpha,\gamma,\psi_\ell)&\mapsto(\alpha,\gamma,\psi_\ell)\\u_0&\mapsto\lambda^\ell u_0\\\omega&\mapsto\lambda\omega\\(r,M)&\mapsto\lambda^{-1}(r,M)\\(\rho,p_r,p_T)&\mapsto\lambda^2(\rho,p_r,p_T)\end{aligned}$$

$$\frac{1}{r^2}(r^2\psi_{\text{in}}')'=\left(\mu^2-\omega^2+\frac{\ell(\ell+1)}{r^2}\right)\psi_{\text{in}}$$

$$\psi_{\text{in}}(r)=C_1\frac{J_{\ell+\frac{1}{2}}\left(\sqrt{\omega^2-\mu^2}r\right)}{\sqrt{r}}+C_2\frac{Y_{\ell+\frac{1}{2}}\left(\sqrt{\omega^2-\mu^2}r\right)}{\sqrt{r}}$$

$$\psi_{\text{in}}(r)=u_0\frac{2^{\left(\ell+\frac{1}{2}\right)}\Gamma\left(\ell+\frac{3}{2}\right)}{\left(\sqrt{\omega^2-\mu^2}\right)^{\ell+\frac{1}{2}}}\frac{J_{\ell+\frac{1}{2}}\left(\sqrt{\omega^2-\mu^2}r\right)}{\sqrt{r}}$$

$$\psi_\ell=\left(\frac{\bar{r}}{r_0}\right)^\ell\phi_0\exp\left(-\frac{\bar{r}^2-r_0^2}{\sigma^2}\right)$$

$$\alpha=-(1-\alpha_0)\exp\left(-\bar{r}^2\right)+1$$

$$\Psi=1$$

$$ds^2=-\left(\alpha^2-\beta_j\beta^j\right)dt^2+2\beta_i dt dx^i+\gamma_{ij}dx^i dx^j$$

$$(\Box-m_0^2)\phi=0,$$

$$j^\mu(\phi_1,\phi_2):=-i[\phi_1(\nabla^\mu\phi_2^*)-(\nabla^\mu\phi_1)\phi_2^*],$$

$$(\phi_1,\phi_2):=\int_{\Sigma_t}j^\mu n_\mu d\gamma=-i\int_{\Sigma_t}[\phi_1(\pounds_n\phi_2^*)-(\pounds_n\phi_1)\phi_2^*]d\gamma$$

$$(\phi,\phi)=2\text{Im}\int_{\Sigma_t}\phi(\pounds_n\phi^*)d\gamma$$

$$[\hat{\phi}(t,\vec{x}),\hat{\pi}(t,\vec{y})]=i\delta^{(3)}(\vec{x}-\vec{y}),[\hat{\phi}(t,\vec{x}),\hat{\phi}(t,\vec{y})]=[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{y})]=0$$

$$\hat{a}(f) := (\hat{\phi}, f), f \in X$$
$$\left[\hat{a}(f), \hat{a}(g)^{\dagger}\right] = (g, f), [\hat{a}(f), \hat{a}(g)] = -(g^{*}, f)$$
$$X = X_{+} \oplus X_{+}^{*},$$
$$\hat{a}_{I} := \hat{a}(f_{I}), \hat{a}_{I}^{\dagger} := \hat{a}^{\dagger}(f_{I}),$$
$$\left[\hat{a}_{I}, \hat{a}_{J}^{\dagger}\right] = \delta_{IJ}, \left[\hat{a}_{I}, \hat{a}_{J}\right] = 0$$
$$\hat{\phi}(x) = \sum_{I} \left[\hat{a}_{I} f_{I}(x) + \hat{a}_{I}^{\dagger} f_{I}^{*}(x)\right],$$
$$|N_1, N_2, \dots\rangle = \frac{\left(\hat{a}_1^{\dagger}\right)^{N_1}}{\sqrt{N_1!}} \frac{\left(\hat{a}_2^{\dagger}\right)^{N_2}}{\sqrt{N_2!}} \dots |0\rangle$$
$$\hat{a}_K^{\dagger}|N_1, \dots N_K, \dots\rangle = \sqrt{N_K + 1}|N_1, \dots, N_K + 1, \dots\rangle,$$
$$\hat{a}_K|N_1, \dots, N_K, \dots\rangle = \sqrt{N_K}|N_1, \dots, N_K - 1, \dots\rangle.$$
$$|\psi\rangle = \sum_{N_1, N_2, \dots = 0}^{\infty} \left(C_{N_1 N_2 \dots}\right)|N_1, N_2, \dots\rangle,$$
$$\hat{a}_K|\alpha_1, \alpha_2, \dots\rangle = \alpha_K|\alpha_1, \alpha_2, \dots\rangle$$
$$G_{\mu\nu} = 8\pi G\langle\hat{T}_{\mu\nu}\rangle$$
$$\hat{T}_{\mu\nu} = (\nabla_{\mu}\hat{\phi})(\nabla_{\nu}\hat{\phi}) - \frac{1}{2} g_{\mu\nu}\left[(\nabla_{\alpha}\hat{\phi})(\nabla^{\alpha}\hat{\phi}) + m_0^2 \hat{\phi}\hat{\phi}\right].$$
$$\hat{T}_{\mu\nu} = \frac{1}{2} \sum_{I,J} \left[\hat{a}_I \hat{a}_J T_{\mu\nu}(f_I, f_J) + \hat{a}_I^{\dagger} \hat{a}_J T_{\mu\nu}(f_I^{*}, f_J) + \text{ H.c.}\right]$$
$$T_{\mu\nu}(f_I, f_J) := (\nabla_{\mu} f_I)(\nabla_{\nu} f_J) + (\nabla_{\nu} f_I)(\nabla_{\mu} f_J) - g_{\mu\nu}\left[(\nabla_{\alpha} f_I)(\nabla^{\alpha} f_J) + m_0^2 f_I f_J\right]$$
$$\langle\alpha_1, \alpha_2, \dots|\hat{T}_{\mu\nu}|\alpha_1, \alpha_2, \dots\rangle = \frac{1}{2} T_{\mu\nu}(\phi_{\text{cl}}, \phi_{\text{cl}})$$
$$\langle N_1, N_2, \dots|\hat{T}_{\mu\nu}|N_1, N_2, \dots\rangle = \sum_{I} N_I T_{\mu\nu}(f_I, f_I^{*})$$
$$\hat{\rho} = \sum_{N_1, N_2, \dots} \left(p_{N_1 N_2 \dots} \dots\right)|N_1, N_2, \dots\rangle\langle N_1, N_2, \dots|$$
$$\text{Tr}\left(\hat{\rho}\hat{T}_{\mu\nu}\right) = \sum_{I} \langle N_I\rangle_{\text{stat}} T_{\mu\nu}(f_I, f_I^{*})$$
$$\langle N_I\rangle_{\text{stat}} := \sum_{N_1, N_2, \dots} \left(p_{N_1 N_2 \dots}\right) N_I,$$
$$ds^2 = -\alpha^2(\vec{x})dt^2 + \gamma_{ij}(\vec{x})dx^i dx^j,$$
$$\partial_t^2\phi - \alpha D^i(\alpha D_i\phi) + \alpha^2 m_0^2\phi = 0,$$
$$f_I(t, \vec{x}) = \frac{1}{\sqrt{2\omega_I}} e^{-i\omega_I t} u_I(\vec{x})$$
$$H u_I := -\alpha D^j\left(\alpha D_j u_I\right) + \alpha^2 m_0^2 u_I = \omega_I^2 u_I.$$
$$\langle u_1, u_2\rangle := \int_{\Sigma} u_1^{*}(\vec{x}) u_2(\vec{x}) \frac{d\gamma}{\alpha(\vec{x})}, u_1, u_2 \in Y$$
$$\langle u, Hu\rangle = \int_{\Sigma} \left(|Du(\vec{x})|^2 + m_0^2|u(\vec{x})|^2\right)\alpha(\vec{x})d\gamma$$
$$\langle u_I, u_J\rangle = \delta_{IJ}, I, J = 1,2, \dots$$
$$(f_I, f_J) = -\left(f_I^{*}, f_J^{*}\right) = \delta_{IJ}, \left(f_I, f_J^{*}\right) = 0$$
$$R^{(3)} = 16\pi G\rho$$
$$R_{ij}^{(3)} - \frac{1}{\alpha} D_i D_j \alpha = 4\pi G\left[\gamma_{ij}(\rho - S) + 2S_{ij}\right]$$
$$\rho := n^{\mu} n^{\nu}\langle\hat{T}_{\mu\nu}\rangle$$
$$S_{ij} := (\delta_i{}^{\mu} + n_i n^{\mu})(\delta_j{}^{\nu} + n_j n^{\nu})\langle\hat{T}_{\mu\nu}\rangle$$

$$j_i := (\delta_i^\mu + n_i n^\mu) n^\nu \langle \hat{T}_{\mu\nu} \rangle$$

$$\rho = \sum_I \frac{N_I}{2\omega_I}\left[|Du_I|^2 + \left(\frac{\omega_I^2}{\alpha^2} + m_0^2\right)|u_I|^2\right]$$

$$j_k = \sum_I \frac{N_I}{2}\frac{i}{\alpha}[(D_k u_I)u_I^* - u_I(D_k u_I^*)]$$

$$S_{ij} = \sum_I \frac{N_I}{2\omega_I}\left\{(D_i u_I)(D_j u_I^*) + (D_j u_I)(D_i u_I^*) - \gamma_{ij}\left[|D_i u_I|^2 - \left(\frac{\omega_I^2}{\alpha^2} - m_0^2\right)|u_I|^2\right]\right\}$$

$$R^{(3)} = 8\pi G \sum_I \frac{N_I}{\omega_I}\left[|Du_I|^2 + \left(\frac{\omega_I^2}{\alpha^2} + m_0^2\right)|u_I|^2\right]$$

$$\frac{D^j D_j \alpha}{\alpha} = 8\pi G \sum_I \frac{N_I}{\omega_I}\left[\left(\frac{\omega_I^2}{\alpha^2} - \frac{m_0^2}{2}\right)|u_I|^2\right]$$

$$\left[R_{ij}^{(3)} - \frac{1}{\alpha}D_i D_j \alpha\right]^{\text{tf}} = 4\pi G \sum_I \frac{N_I}{\omega_I}\left[(D_i u_I)(D_j u_I^*) + (D_j u_I)(D_i u_I^*)\right]^{\text{tf}}$$

$$\gamma_{ij} dx^i dx^j = \gamma^2 dr^2 + r^2 d\Omega^2, \gamma = \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

$$u_I(\vec{x}) = v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi), I = (n\ell m)$$

$$-\frac{\alpha}{\gamma r^2}\left(\frac{\alpha r^2}{\gamma} v_{n\ell}'\right)' + \alpha^2\left[\frac{\ell(\ell+1)}{r^2} + m_0^2\right] v_{n\ell} = (\omega_{n\ell})^2 v_{n\ell}$$

$$\int_0^\infty v_{n\ell}(r) v_{n'\ell}^*(r) \frac{\gamma(r)}{\alpha(r)} r^2 dr = \delta_{nn'}$$

$$\int_\Sigma \left\{|Du_I|^2 + \left[m_0^2 - \frac{\omega_I^2}{\alpha^2}\right]|u_I|^2\right\}\alpha d\gamma = 0$$

$$N_{n,\ell,-\ell} = N_{n,\ell,-(\ell-1)} = \cdots = N_{n,\ell,(\ell-1)} = N_{n,\ell,\ell}$$

$$\frac{2GM'}{r^2} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}}\left[\frac{|v_{n\ell}'|^2}{\gamma^2} + \left(\frac{(\omega_{n\ell})^2}{\alpha^2} + m_0^2 + \frac{\ell(\ell+1)}{r^2}\right)|v_{n\ell}|^2\right]$$

$$\frac{1}{\alpha\gamma r^2}\left(\frac{r^2\alpha'}{\gamma}\right)' = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}}\left[\left(2\frac{(\omega_{n\ell})^2}{\alpha^2} - m_0^2\right)|v_{n\ell}|^2\right]$$

$$\frac{(\alpha\gamma)'}{r\alpha\gamma^3} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}}\left[\frac{|v_{n\ell}'|^2}{\gamma^2} + \frac{(\omega_{n\ell})^2}{\alpha^2}|v_{n\ell}|^2\right]$$

$$\hat{\phi}(x) = \sum_{n\ell m} \frac{1}{\sqrt{2\omega_{n\ell}}}\left[\hat{a}_I e^{-i\omega_{n\ell} t} v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi) + \text{ H.c.}\right]$$

$$\psi_{n\ell}'' = -\left[\gamma^2 + 1 - (2\ell+1) r^2\gamma^2\left(\frac{\ell(\ell+1)}{r^2} + m_0^2\right)(\psi_{n\ell})^2\right]\frac{\psi_{n\ell}'}{r} - \left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - m_0^2\right)\gamma^2 \psi_{n\ell}$$

$$\gamma' = \sum_{n\ell} \frac{2\ell+1}{2} r\gamma\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + m_0^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi_{n\ell}')^2\right] - \left(\frac{\gamma^2 - 1}{2r}\right)\gamma$$

$$\alpha' = \sum_{n\ell} \frac{2\ell+1}{2} r\alpha\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - m_0^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi_{n\ell}')^2\right] + \left(\frac{\gamma^2 - 1}{2r}\right)\alpha$$

$$\psi_{n\ell} = \sqrt{\frac{N_{n\ell m}}{\omega_{n\ell}}} v_{n\ell}$$

$$N_{n\ell m} = \omega_{n\ell}\int_0^\infty (\psi_{n\ell})^2 \frac{\gamma}{\alpha} r^2 dr$$

$$\psi_{n\ell}(r) = \frac{\psi^0_{n\ell}}{2\ell+1} r^\ell$$
$$\psi'_{n\ell}(r) = \frac{\ell\psi^0_{n\ell}}{2\ell+1} r^{\ell-1}$$
$$\alpha(r) = 1$$
$$\gamma(r) = 1$$
$$m_0 \mapsto \lambda m_0, \omega_{n\ell} \mapsto \lambda\omega_{n\ell}, r \mapsto \lambda^{-1} r$$

$$\rho = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}}\Big\{\langle\hat{a}_I\hat{a}_J\rangle\left[\left(-\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I u_J + (D_k u_I)(D^k u_J)\right]e^{-i(\omega_I+\omega_J)t}$$
$$+\langle\hat{a}_I^\dagger\hat{a}_J\rangle\left[\left(+\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I^* u_J + (D_k u_I^*)(D^k u_J)\right]e^{+i(\omega_I-\omega_J)t} + c.c.\Big\},$$
$$j_k = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}}\frac{i}{\alpha}\Big\{\langle\hat{a}_I\hat{a}_J\rangle[-\omega_J(D_k u_I)u_J - \omega_I u_I(D_k u_J)]e^{-i(\omega_I+\omega_J)t}$$
$$+\langle\hat{a}_I^\dagger\hat{a}_J\rangle[-\omega_J(D_k u_I^*)u_J + \omega_I u_I^*(D_k u_J)]e^{+i(\omega_I-\omega_J)t} - c.c.\Big\},$$
$$S_{ij} = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}}$$
$$\times\Big\{\langle\hat{a}_I\hat{a}_J\rangle\left[(D_i u_I)(D_j u_J) + (D_j u_I)(D_i u_J) - \gamma_{ij}\left(\left(+\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I u_J + (D_k u_I)(D^k u_J)\right)\right]e^{-i(\omega_I+\omega_J)t}$$
$$+\langle\hat{a}_I^\dagger\hat{a}_J\rangle\left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij}\left(\left(-\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I^* u_J + (D_k u_I^*)(D^k u_J)\right)\right]e^{+i(\omega_I-\omega_J)t}$$
$$+ \text{c.c.}\Big\},$$
$$\hat{H} := \int_{\Sigma_t} \hat{T}_{\mu\nu} k^\mu n^\nu d\gamma$$
$$\hat{H} = \sum_I \hat{N}_I \omega_I$$
$$\hat{\phi}(x) = \sum_I \left[\hat{a}_I f_I(x) + \hat{b}_I^\dagger f_I^*(x)\right]$$
$$\hat{T}_{\mu\nu} = (\nabla_\mu\hat{\phi})(\nabla_\nu\hat{\phi})^\dagger + (\nabla_\nu\hat{\phi})(\nabla_\mu\hat{\phi})^\dagger - g_{\mu\nu}\left[(\nabla_\alpha\hat{\phi})(\nabla^\alpha\hat{\phi})^\dagger + m_0^2\hat{\phi}\hat{\phi}^\dagger\right]$$
$$\hat{T}_{\mu\nu} = \sum_{I,J}\left[\hat{a}_I\hat{a}_J^\dagger T_{\mu\nu}(f_I, f_J^*) + \hat{a}_I\hat{b}_J T_{\mu\nu}(f_I, f_J) + \hat{b}_I^\dagger\hat{a}_J^\dagger T_{\mu\nu}(f_I^*, f_J^*) + \hat{b}_I^\dagger\hat{b}_J T_{\mu\nu}(f_I^*, f_J)\right]$$
$$\rho = \sum_{I,J} \frac{1}{2\sqrt{\omega_I\omega_J}}\Big\{\langle\hat{a}_I^\dagger\hat{a}_J\rangle\left[\left(+\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I^* u_J + (D_k u_I)(D^k u_J^*)\right]e^{+i(\omega_I-\omega_J)t}$$
$$+\langle\hat{a}_I\hat{b}_J\rangle\left[\left(-\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I u_J + (D_k u_I)(D^k u_J)\right]e^{-i(\omega_I+\omega_J)t}$$
$$+\langle\hat{b}_I^\dagger\hat{a}_J^\dagger\rangle\left[\left(-\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I^* u_J^* + (D_k u_I^*)(D^k u_J^*)\right]e^{+i(\omega_I+\omega_J)t}$$
$$+\langle\hat{b}_I^\dagger\hat{b}_J\rangle\left[\left(+\frac{\omega_I\omega_J}{\alpha^2}+m_0^2\right)u_I^* u_J + (D_k u_I^*)(D^k u_J)\right]e^{+i(\omega_I-\omega_J)t}\Big\},$$
$$j_k = \sum_{I,J} \frac{1}{2\sqrt{\omega_I\omega_J}}\frac{i}{\alpha}\Big\{\langle\hat{a}_I^\dagger\hat{a}_J\rangle[-\omega_J(D_k u_I^*)u_J + \omega_I u_I^*(D_k u_J)]e^{-i(\omega_I-\omega_J)t}$$
$$+\langle\hat{a}_I\hat{b}_J\rangle[-\omega_J(D_k u_I)u_J - \omega_I u_I(D_k u_J)]e^{-i(\omega_I+\omega_J)t}$$
$$+\langle\hat{b}_I^\dagger\hat{a}_J^\dagger\rangle[+\omega_J(D_k u_I^*)u_J^* + \omega_I u_I^*(D_k u_J^*)]e^{+i(\omega_I+\omega_J)t}$$
$$+\langle\hat{b}_I^\dagger\hat{b}_J\rangle[-\omega_J(D_k u_I^*)u_J + \omega_I u_I^*(D_k u_J)]e^{+i(\omega_I-\omega_J)t}\Big\},$$

$$S_{ij} = \sum_{I,J} \frac{1}{2\sqrt{\omega_I \omega_J}}$$

$$\times \left\{ \langle \hat{a}_I^\dagger \hat{a}_J \rangle \left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij} \left(\left(-\frac{\omega_I \omega_J}{\alpha^2} + m_0^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right) \right] e^{+i(\omega_I - \omega_J)t} \right.$$

$$+ \langle \hat{a}_I \hat{b}_J \rangle \left[(D_i u_I)(D_j u_J) + (D_j u_I)(D_i u_J) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0^2 \right) u_I u_J + (D_k u_I)(D^k u_J) \right) \right] e^{-i(\omega_I + \omega_J)t}$$

$$+ \langle \hat{b}_I^\dagger \hat{a}_J^\dagger \rangle \left[(D_i u_I^*)(D_j u_J^*) + (D_j u_I^*)(D_i u_J^*) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0^2 \right) u_I^* u_J^* + (D_k u_I^*)(D^k u_J^*) \right) \right] e^{+i(\omega_I + \omega_J)t}$$

$$\left. + \langle \hat{b}_I^\dagger \hat{b}_J \rangle \left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right) \right] e^{-i(\omega_I - \omega_J)t} \right\}$$

$$M \propto (p - p^*)^\gamma$$

$$T = -\ln(\tau^* - \tau)$$

$$\tau \propto -\gamma \ln |p - p^*|$$

$$\Phi(t, r) = \varphi(r) e^{i\omega t}$$

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - \frac{1}{2} \left(\nabla^\mu \Phi \nabla_\mu \Phi^* + m^2 \Phi \Phi^* \right) \right]$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

$$\nabla^\mu \nabla_\mu \Phi - m^2 \Phi = 0$$

$$T_{\mu\nu} = \frac{1}{2} \left[\left(\nabla_\mu \Phi \nabla_\nu \Phi^* + \nabla_\nu \Phi \nabla_\mu \Phi^* \right) - g_{\mu\nu} (\nabla^\alpha \Phi \nabla_\alpha \Phi^* + m^2 \Phi \Phi^*) \right].$$

$$ds^2 = -\alpha^2 dt^2 + \psi^4 (A dr^2 + r^2 B d\Omega^2),$$

$$\Pi := \frac{\partial_t \Phi}{\alpha}, \chi := \partial_r \Phi$$

$$\partial_t \Phi = \alpha \Pi$$

$$\partial_t \chi = \alpha \partial_r \Pi + \Pi \partial_r \alpha$$

$$\partial_t \Pi = \frac{\alpha}{A\psi^4} \left[\partial_r \chi + \chi \left(\frac{2}{r} - \frac{\partial_r A}{2A} + \frac{\partial_r B}{B} + 2\partial_r \ln \psi \right) \right] + \frac{\chi \partial_r \alpha}{A\psi^4} + \alpha K \Pi - \alpha m^2 \phi$$

$$T^{\mu\nu} = S^{\mu\nu} + J^\mu n^\nu + n^\mu J^\nu + \rho n^\mu n^\nu$$

$$\rho = \frac{1}{2} \left(|\Pi|^2 + \frac{|\chi|^2}{A\psi^4} + m^2 |\Phi|^2 \right)$$

$$J_r = -\frac{1}{2} (\chi \Pi^* + \Pi \chi^*)$$

$$S_r^r = \frac{1}{2} \left(|\Pi|^2 + \frac{|\chi|^2}{A\psi^4} - m^2 |\Phi|^2 \right)$$

$$S_\theta^\theta = \frac{1}{2} \left(|\Pi|^2 - \frac{|\chi|^2}{A\psi^4} - m^2 |\Phi|^2 \right)$$

$$\Phi(t = 0, r) = \Phi_0 e^{-r^2/\sigma^2}$$

$$\Pi(t = 0, r) = i\kappa \Phi_0 e^{-r^2/\sigma^2}$$

$$\partial_r^2 \psi + \frac{2}{r} \partial_r \psi + 2\pi \psi^5 \rho = 0$$

$$\rho = \frac{1}{2} \left[|\Pi|^2 + \frac{|\partial_r \Phi|^2}{\psi^4} + m^2 |\Phi|^2 \right].$$

$$\psi(r)|_{r \to \infty} = 1$$

$$\partial_r \psi = \frac{1 - \psi}{r}$$

$$\partial_r \psi|_{r=0} = 0$$

$$\partial_t \alpha = -2\alpha K$$

$$\delta \Phi = \frac{\Phi_c - \Phi_d}{\Phi_d}$$

$$\frac{1}{\psi^2 \sqrt{A}} \left(\frac{2}{r} + \frac{\partial_r B}{B} + 4 \frac{\partial \psi}{\psi} \right) - 2K_\theta^\theta = 0$$

$$C(t, r) = \frac{M(t, r)}{R(t, r)}$$

$$K^A = \epsilon^{AB} \partial_B R$$

$$\mathcal{S}^\mu = T^{\mu\nu} K_\nu$$

$$\partial_\mu\left(\sqrt{-g}\mathcal{S}^\mu\right) = 0$$
$$M(t,r) := \int_{\text{sphere}} \mathcal{S}^t \alpha\sqrt{\gamma}dx^3$$
$$M(t,r) := 4\pi\int_0^r \alpha\mathcal{S}^t r^2\psi^6 A^{1/2}Bdr$$
$$R_{\max} \approx |\Phi_0^* - \Phi_0|^{-2\gamma}$$
$$\ln\, R_{\max} = c - 2\gamma\ln|\Phi_0^* - \Phi_0| + f(\ln|\Phi_0^* - \Phi_0|)$$
$$\omega = \Delta/2\gamma$$
$$f(x) = a_0\sin(\omega x + \varphi)$$
$$\ln\, R_{\max} = c - 2\gamma\ln|\Phi_0^* - \Phi_0| + a_0\sin(\omega\ln|\Phi_0^* - \Phi_0| + \varphi)$$
$$\tau^* = \frac{\tau_n\tau_{m+1} - \tau_{n+1}\tau_m}{\tau_n - \tau_{n+1} - \tau_m + \tau_{m+1}}$$
$$\Delta = 2\ln\left(\frac{\tau^* - \tau_n}{\tau^* - \tau_{n+1}}\right)$$
$$\frac{dr}{d\tilde{r}} = \frac{1}{1 + e^{\beta r^2 + \delta}}$$
$$S = \int\left(\frac{R}{16\pi} - \frac{1}{2}\left[(\mathcal{D}_\mu\phi)^*(\mathcal{D}^\mu\phi) + m^2|\phi|^2\right] - \frac{1}{16\pi}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right)\sqrt{-g}dx^4$$
$$\mathcal{F}_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\mathcal{D}_\mu := \nabla_\mu + iqA_\mu$$
$$\phi \to e^{iq\theta(x^\alpha)}\phi, A_\mu \to A_\mu - \partial_\mu\theta(x^\alpha)$$
$$j_\mu = \frac{iq}{2}\left[\phi^*\mathcal{D}_\mu\phi - \phi(\mathcal{D}_\mu\phi)^*\right]$$
$$= \frac{q}{2}\left[i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) - 2qA_\mu|\phi|^2\right].$$
$$j_\mu = \frac{iq}{2}\left[\phi^*\mathcal{D}_\mu\phi - \phi(\mathcal{D}_\mu\phi)^*\right]$$
$$= \frac{q}{2}\left[i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) - 2qA_\mu|\phi|^2\right]$$
$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
$$\nabla_\mu\mathcal{F}^{\mu\nu} = -4\pi j^\nu, \nabla_\mu\mathcal{F}^{*\mu\nu} = 0$$
$$(\mathcal{D}^\mu\mathcal{D}_\mu - m^2)\phi = 0$$
$$\phi(t,r) = \phi_0(r)e^{i\omega t}$$
$$ds^2 = -\alpha(r)^2dt^2 + A(r)dr^2 + r^2d\Omega^2$$
$$n^\mu = (1/\alpha, 0,0,0), n_\mu = (-\alpha, 0,0,0).$$
$$E^\mu = -n_\nu\mathcal{F}^{\mu\nu}, B^\mu = -n_\nu\mathcal{F}^{*\mu\nu}.$$
$$D_iE^i = 4\pi e,$$
$$e := -n^\mu j_\mu = -\frac{j_0}{\alpha}$$
$$= -\frac{q}{2\alpha}\left[i(\phi^*\partial_t\phi - \phi\partial_t\phi^*) - 2qA_0|\phi|^2\right].$$
$$\frac{dE}{dr} = 4\pi q\phi_0^2\left(\frac{\omega - qF}{\alpha}\right) - \left(\frac{1}{2A}\frac{dA}{dr} + \frac{2}{r}\right)E,$$
$$\frac{dF}{dr} = -\alpha AE.$$
$$\hat{\Pi} := n^\mu\mathcal{D}_\mu\phi^*, \hat{\chi}_i := P_i^\mu\mathcal{D}_\mu\phi,$$
$$\hat{\Pi} = i\left(\frac{\omega - qF}{\alpha}\right)\phi_0e^{i\omega t}, \hat{\chi}_i = (\chi e^{i\omega t}, 0,0)$$
$$\chi = d\phi_0/dr$$
$$\frac{d\chi}{dr} = -\chi\left(\frac{1}{\alpha}\frac{d\alpha}{dr} + \frac{1}{2A}\frac{dA}{dr} + \frac{2}{r}\right) + A\phi_0\left(m^2 - \left(\frac{\omega - qF}{\alpha}\right)^2\right)$$
$${}^{(3)}R = 16\pi\rho,$$
$$\frac{dA}{dr} = A\left\{\frac{1-A}{r} + r(AE)^2 + 4\pi rA\left[\phi_0^2\left(\frac{\omega - qF}{\alpha}\right)^2 + \frac{\chi^2}{A} + (m\phi_0)^2\right]\right\}$$
$$\frac{d\alpha}{dr} = \alpha\left(\frac{A-1}{r} + \frac{1}{2A}\frac{dA}{dr} - r(AE)^2 - 4\pi rA(m\phi_0)^2\right)$$
$$M = 4\pi\int_0^\infty \rho r^2dr$$
$$\rho = \frac{1}{2}\left(\hat{\Pi}^*\hat{\Pi} + \frac{\chi_i^*\chi^i}{A} + m^2\phi^2\right) + \frac{AE^2}{8\pi}$$
$$= \frac{1}{2}\left[\left(\frac{(\omega - qF)^2}{\alpha^2} + m^2\right)\phi^2 + \frac{\chi^2}{A}\right] + \frac{AE^2}{8\pi}.$$

$$Q = \int \ j^0\sqrt{-\gamma}dx^3$$
$$Q = 4\pi q \int_0^\infty \frac{(\omega - qF)}{\alpha}\phi_0^2 A^{1/2} r^2 dr.$$
$$E_B := M - mN = M - (m/q)Q.$$
$$A(r) \rightarrow \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}$$
$$M = \lim_{r\rightarrow\infty}\left[\frac{r}{2}\left(1 + \frac{Q^2}{r^2} - \frac{1}{A}\right)\right].$$
$$\alpha(0) = 1, \qquad \partial_r\alpha(0) = 0$$
$$A(0) = 1, \qquad \partial_r A(0) = 0$$
$$\phi_0(0) = k, \quad \partial_r\phi_0(0) = 0$$
$$F(0) = 0, \qquad \partial_r F(0) = 0$$
$$m \geq (\omega - qF_\infty)/\alpha_\infty$$
$$\alpha \rightarrow \alpha/C_1, \omega \rightarrow \omega/C_1, F \rightarrow F/C_1$$
$$\phi \rightarrow \phi e^{iqC_2 t}, F \rightarrow F + C_2$$
$$\omega \rightarrow \omega + qC_2.$$
$$m \rightarrow \lambda m, q \rightarrow \lambda q, \omega \rightarrow \lambda\omega,$$
$$r \rightarrow r/\lambda, \chi \rightarrow \lambda\chi, E \rightarrow \lambda E,$$
$$S = \int \ d^4x\sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{16\pi}\sum_{m=1}^{N}\ \left[(W_m)_{\mu\nu}(\bar{W}_m)^{\mu\nu} + 2\mu^2(X_m)_\mu(\bar{X}_i)^\mu\right]\right)$$
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$
$$T_{\mu\nu} = \frac{1}{4\pi}\sum_{m=1}^{N}\ \{-(W_m)_{\lambda(\mu}(\bar{W}_m)_{\nu)}\Big)^\lambda - \frac{g_{\mu\nu}}{4}(W_m)_{\alpha\beta}(\bar{W}_m)^{\alpha\beta}$$
$$+\mu^2\left[(X_m)_{(\mu}(\bar{X}_m)_{\nu)}\right] - \frac{g_{\mu\nu}}{2}(X_m)_\lambda(\bar{X}_m)^\lambda\Big]\Big\}.$$
$$\nabla_\mu(W_m)^{\mu\nu} - \mu^2(X_m)^\nu = 0$$
$$\nabla_\nu(X_m)^\nu = 0$$
$$\Phi_m := -n^\mu(X_m)_\mu, (a_m)_i := \gamma_i^\mu(X_m)_\mu$$
$$(\mathcal{E}_m)_i := -n^\mu\gamma_i^\nu(W_m)_{\mu\nu}, (\mathcal{B}_m)_i := -n^\mu\gamma_i^\nu(W_m^*)_{\mu\nu}$$
$$\rho = \ \sum_{m=1}^{N}\ \frac{1}{8\pi}\left\{(\mathcal{E}_m)_i\left(\bar{\mathcal{E}}_m\right)^i + (\mathcal{B}_m)_i\left(\bar{\mathcal{B}}_m\right)^i + \mu^2\left[\Phi_m\bar{\Phi}_m + (a_m)_i(\bar{a}_m)^i\right]\right\}$$
$$j^i = \ \sum_{m=1}^{N}\ \frac{1}{8\pi}\left\{E_{jk}^i\left(\bar{\mathcal{E}}_m\right)^j(\mathcal{B}_m)^k + \mu^2(a_m)^i\bar{\Phi}_m + c.c.\right\}$$
$$S_{ij} = \ \sum_{m=1}^{N}\ \frac{1}{8\pi}\left\{\gamma_{ij}\left[(\mathcal{E}_m)_k\left(\bar{\mathcal{E}}_m\right)^k + (\mathcal{B}_m)_k\left(\bar{\mathcal{B}}_m\right)^k\right] - \left[(\mathcal{B}_m)_i\left(\bar{\mathcal{B}}_m\right)_j + (\mathcal{E}_m)_i\left(\bar{\mathcal{E}}_m\right)_j + c.c.\right]\right.$$
$$\left.+\mu^2\left[\left((a_m)_i(\bar{a}_m)_j + c.c.\right) - \gamma_{ij}\left((a_m)_k(\bar{a}_m)^k - \Phi_m\bar{\Phi}_m\right)\right]\right\}$$
$$\Phi_m(x) \ = \phi_\ell(r,t)Y^{\ell m}(\theta,\varphi)$$
$$(a_m)_i(x) \ = \left(\aleph_\ell(r,t)Y^{\ell m}, \beth_\ell(r,t)\partial_\theta Y^{\ell m}, \beth_\ell(r,t)\partial_\varphi Y^{\ell m}\right)$$
$$(\mathcal{E}_m)_i(x) \ = \left(\epsilon_\ell(r,t)Y^{\ell m}, \xi_\ell(r,t)\partial_\theta Y^{\ell m}, \xi_\ell(r,t)\partial_\varphi Y^{\ell m}\right)$$
$$(\mathcal{B}_m)^i(x) \ = \left(0, \zeta_\ell(r,t)\frac{\partial_\varphi Y^{\ell m}}{\sin\theta}, -\zeta_\ell(r,t)\frac{\partial_\theta Y^{\ell m}}{\sin\theta}\right)$$
$$\sum_{m=-\ell}^{\ell}\ \left|Y^{\ell m}(\theta,\varphi)\right|^2 \ = 1$$
$$\sum_{m=-\ell}^{\ell}\ \bar{Y}^{\ell m}\partial_\theta Y^{\ell m} = \sum_{m=-\ell}^{\ell}\ \bar{Y}^{\ell m}\partial_\varphi Y^{\ell m} \ = 0$$
$$\sum_{m=-\ell}^{\ell}\ \left(\partial_\theta Y^{\ell m}\partial_\theta\bar{Y}^{\ell m} + \frac{1}{\sin^2\theta}\partial_\varphi Y^{\ell m}\partial_\varphi\bar{Y}^{\ell m}\right) \ = \ell(\ell+1)$$
$$\sum_{m=-\ell}^{\ell}\ \partial_\theta Y^{\ell m}\partial_\varphi\bar{Y}^{\ell m} = \sum_{m=-\ell}^{\ell}\ \partial_\varphi Y^{\ell m}\partial_\theta\bar{Y}^{\ell m} \ = 0$$
$$\sum_{m=-\ell}^{\ell}\ \partial_\theta Y^{\ell m}\partial_\theta\bar{Y}^{\ell m} = \frac{1}{\sin^2\theta}\sum_{m=-\ell}^{\ell}\ \partial_\varphi Y^{\ell m}\partial_\varphi\bar{Y}^{\ell m} \ = \frac{\ell(\ell+1)}{2}.$$
$$ds^2 = -\alpha^2(t,r)dt^2 + \psi^4(t,r)(A(t,r)dr^2 + r^2B(t,r)d\Omega^2),$$
$$T_{\mu\nu} = n_\mu n_\nu\rho + n_\mu j_\nu + j_\nu n_\nu + S_{\mu\nu}$$
$$\rho = \frac{1}{8\pi}\left\{\frac{1}{\psi^4 A}|\epsilon_\ell|^2 + \mu^2\left[\frac{1}{\psi^4 A}|\aleph_\ell|^2 + |\phi_\ell|^2\right]\right.$$
$$\left.+\ell(\ell+1)\left[\frac{1}{\psi^4 Br^2}(|\xi_\ell|^2 + \mu^2|\beth_\ell|^2) + \psi^4 Br^2|\zeta_\ell|^2\right]\right\}$$

$$j_r = \frac{1}{8\pi}\{-\ell(\ell+1)A^{1/2}\psi^2\bar{\xi}_\ell\zeta_\ell + \mu^2\aleph_\ell\bar{\phi}_\ell + c.c.\},$$
$$S_{rr} = \frac{\psi^4 A}{8\pi}\left\{-\frac{1}{\psi^4 A}|\epsilon_\ell|^2 + \mu^2\left[\frac{1}{\psi^4 A}|\aleph_\ell|^2 + |\phi_\ell|^2\right]\right.$$
$$\left.+\ell(\ell+1)\left[\frac{1}{\psi^4 Br^2}(|\xi_\ell|^2 - \mu^2|\beth_\ell|^2) + \psi^4 Br^2|\zeta_\ell|^2\right]\right\}$$
$$S_{\theta\theta} = \frac{\psi^4 Br^2}{8\pi}\left\{\frac{1}{\psi^4 A}|\epsilon_\ell|^2 - \mu^2\left[\frac{1}{\psi^4 A}|\aleph_\ell|^2 - |\phi_\ell|^2\right]\right\}$$
$$S_{\varphi\varphi} = \frac{\psi^4 Br^2\sin^2\theta}{8\pi}\left\{\frac{1}{\psi^4 A}|\epsilon_\ell|^2 - \mu^2\left[\frac{1}{\psi^4 A}|\aleph_\ell|^2 - |\phi_\ell|^2\right]\right\}$$
$$\zeta_\ell = \frac{(\aleph_\ell - \partial_r\beth_\ell)}{A^{1/2}B\psi^6 r^2}.$$
$$\partial_t\phi_\ell = -\frac{\alpha}{A\psi^4}\left[(\partial_r\aleph_\ell) + \aleph_\ell\left(\frac{2}{r} - \frac{\partial_r A}{2A} + \frac{\partial_r B}{B} + \frac{2\partial_r\psi}{\psi} + \frac{\partial_r\alpha}{\alpha}\right)\right] + \alpha\frac{\ell(\ell+1)}{B\psi^4 r^2}\beth_\ell + \alpha K\phi_\ell$$
$$\partial_t\aleph_\ell = -\alpha\epsilon_\ell - \partial_r(\alpha\phi_\ell)$$
$$\partial_t\beth_\ell = -\alpha\xi_\ell - \alpha\phi_\ell$$
$$\partial_t\epsilon_\ell = \alpha\frac{\ell(\ell+1)}{B\psi^4 r^2}(\aleph_\ell - \partial_r\beth_\ell) + \alpha[\mu^2\aleph_\ell + (K - K_r^r)\epsilon_\ell]$$
$$\partial_t\xi = \frac{\alpha}{A\psi^4}\left[\partial_r(\aleph_\ell - \partial_r\beth_\ell) + (\aleph_\ell - \partial_r\beth_\ell)\left(\frac{\partial_r\alpha}{\alpha} - \frac{\partial_r A}{2A} - \frac{2\partial_r\psi}{\psi}\right)\right] + \alpha\left[\mu^2\beth_\ell + \left(K - K_\theta^\theta\right)\xi_\ell\right]$$
$$0 = \frac{1}{A\psi^4}\left[\partial_r\epsilon_\ell + \epsilon_\ell\left(\frac{2}{r} - \frac{\partial_r A}{2A} + \frac{\partial_r B}{B} + \frac{2\partial_r\psi}{\psi}\right)\right] - \frac{\ell(\ell+1)}{B\psi^4 r^2}\xi_\ell + \mu^2\phi_\ell.$$
$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta Y^{\ell m}\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2 Y^{\ell m} = -\ell(\ell+1)Y^{\ell m},$$
$$\phi_\ell(r,t) = \varphi_\ell(r)e^{-i\omega t}, \aleph_\ell(r,t) = ia_\ell(r)e^{-i\omega t}, \beth_\ell(r,t) = ib_\ell(r)e^{-i\omega t}$$
$$\epsilon_\ell(r,t) = e_\ell(r)e^{-i\omega t}, \xi_\ell(r,t) = d_\ell(r)e^{-i\omega t}$$
$$\omega\varphi_\ell = \frac{1}{A^{1/2}B\psi^6 r^2}\left(\alpha\frac{\psi^2 Br^2}{A^{1/2}}a_\ell\right)' - \alpha\frac{\ell(\ell+1)}{B\psi^4 r^2}b_\ell$$
$$\omega a_\ell = -\alpha e_\ell - (\alpha\varphi_\ell)'$$
$$\omega b_\ell = -\alpha d_\ell - \alpha\varphi_\ell$$
$$\omega e_\ell = -\alpha\frac{\ell(\ell+1)}{B\psi^4 r^2}(a_\ell - b_\ell') - \alpha\mu^2 a_\ell$$
$$\omega d_\ell = -\frac{1}{A^{1/2}\psi^2}\left(\alpha\frac{(a_\ell - b_\ell')}{A^{1/2}\psi^2}\right)' - \alpha\mu^2 b_\ell$$
$$0 = \frac{1}{A^{1/2}B\psi^6 r^2}\left(\frac{\psi^2 Br^2}{A^{1/2}}e_\ell\right)' - \frac{\ell(\ell+1)}{B\psi^4 r^2}d_\ell + \mu^2\varphi_\ell$$
$$e_\ell = -\frac{1}{\alpha}(F_\ell' + \omega a_\ell), d_\ell = -\frac{1}{\alpha}(F_\ell + \omega b_\ell).$$
$$X_m = e^{-i\omega t}\left[-F_\ell(r)Y^{\ell m}dt + ia_\ell(r)Y^{\ell m}dr + ib_\ell(r)\left(\partial_\theta Y^{\ell m}d\theta + \partial_\varphi Y^{\ell m}d\varphi\right)\right]$$
$$ds^2 = -\alpha^2(r)dt^2 + \psi^4(r)(dr^2 + r^2 d\Omega^2)$$
$$\nabla_\mu\nabla^\mu(X_m)_\nu - R_{\mu\nu}(X_m)^\mu - \mu^2(X_m)_\nu = 0$$
$$0 = F_\ell'' + \frac{2F_\ell'}{r} - \frac{\ell(\ell+1)}{r^2}F_\ell + \frac{2\psi'}{\psi}F_\ell' - \frac{\alpha'}{\alpha}(F_\ell' + 2\omega a_\ell) - \psi^4\left(\mu^2 - \frac{\omega^2}{\alpha^2}\right)F_\ell$$
$$0 = a_\ell'' + \frac{2a_\ell'}{r} - \frac{2a_\ell}{r^2} - \frac{\ell(\ell+1)}{r^2}a_\ell + \frac{2\ell(\ell+1)}{r^3}b_\ell - \frac{2a_\ell}{r}\left(\frac{\alpha'}{\alpha} + \frac{6\psi'}{\psi}\right) + \frac{4\ell(\ell+1)}{r^2}\frac{\psi'}{\psi}b_\ell + \left(\frac{\alpha'}{\alpha} - \frac{2\psi'}{\psi}\right)a_\ell'$$
$$+\frac{2\psi^4}{\alpha^2}\frac{\alpha'}{\alpha}\omega F_\ell - a_\ell\left[10\left(\frac{\psi'}{\psi}\right)^2 + \frac{6\alpha'\psi'}{\alpha\psi} + \left(\frac{\alpha'}{\alpha}\right)^2\right] - \psi^4\left(\mu^2 - \frac{\omega^2}{\alpha^2}\right)a_\ell + 4\pi\psi^4 a_\ell S$$
$$0 = b_\ell'' - \frac{\ell(\ell+1)}{r^2}b_\ell + \frac{2a_\ell}{r} + \left(\frac{\alpha'}{\alpha} - \frac{2\psi'}{\psi}\right)b_\ell' + \frac{4\psi'}{\psi}a_\ell - \psi^4\left(\mu^2 - \frac{\omega^2}{\alpha^2}\right)b_\ell$$
$$0 = \psi'' + \frac{2\psi'}{r} + 2\pi\psi^5\rho$$
$$0 = \alpha'' + \frac{2\alpha'}{r} + \frac{2\alpha'\psi'}{\psi} - 4\pi\alpha\psi^4(S + \rho)$$
$$\rho = \frac{1}{8\pi}\left\{\frac{(F_\ell' + \omega a_\ell)^2}{\alpha^2\psi^4} + \mu^2\left[\frac{F_\ell^2}{\alpha^2} + \frac{a_\ell^2}{\psi^4}\right] + \frac{\ell(\ell+1)}{\psi^4 r^2}\left[\frac{(F_\ell + \omega b_\ell)^2}{\alpha^2} + \mu^2 b_\ell^2 + \frac{(a_\ell - b_\ell')^2}{\psi^4}\right]\right\}$$
$$S = \frac{1}{8\pi}\left\{\frac{(F_\ell' + \omega a_\ell)^2}{\alpha^2\psi^4} + \mu^2\left[\frac{3F_\ell^2}{\alpha^2} - \frac{a_\ell^2}{\psi^4}\right] + \frac{\ell(\ell+1)}{\psi^4 r^2}\left[\frac{(F_\ell + \omega b_\ell)^2}{\alpha^2} - \mu^2 b_\ell^2 + \frac{(a_\ell - b_\ell')^2}{\psi^4}\right]\right\}$$

$$\alpha = \alpha_0 + \alpha_2 r^2 + \mathcal{O}(r^4),$$
$$\psi = \psi_0 + \psi_2 r^2 + \mathcal{O}(r^4)$$
$$\frac{\psi_2}{\psi_0} = -\frac{\pi}{3}\psi_0^4\rho_0, \frac{\alpha_2}{\alpha_0} = \frac{2\pi}{3}\psi_0^4(S_0 + \rho_0)$$
$$F_\ell'' + \frac{2F_\ell'}{r} + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right) - \frac{\ell(\ell+1)}{r^2}\right] F_\ell \approx 0$$
$$a_\ell'' + \frac{2a_\ell'}{r} + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right) - 4\left(\frac{\alpha_2}{\alpha_0} + \frac{6\psi_2}{\psi_0}\right) + 4\pi\psi_0^4 S_0 - \frac{(2+\ell(\ell+1))}{r^2}\right] a_\ell \approx -2\left[\frac{1}{r^2} + 4\left(\frac{\psi_2}{\psi_0}\right)\right]\frac{\ell(\ell+1)}{r} b_\ell,$$
$$b_\ell'' + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right) - \frac{\ell(\ell+1)}{r^2}\right] b_\ell \approx -\frac{2a_\ell}{r}.$$
$$F_\ell'' + \frac{2F_\ell'}{r} + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right) - \frac{\ell(\ell+1)}{r^2}\right] F_\ell \approx 0$$
$$a_\ell'' + \frac{4a_\ell'}{r} + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2 + \frac{4\pi}{3}S_0\right) + \frac{(2-\ell(\ell+1))}{r^2}\right] a_\ell \approx \frac{2\psi_0^4\omega}{\alpha_0^2}\frac{F_\ell}{r}$$
$$b_\ell'' + \left[\psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right) - \frac{\ell(\ell+1)}{r^2}\right] b_\ell \approx -\frac{2a_\ell}{r}$$
$$F_\ell = c_1 r^\ell - 2\left(\frac{\kappa_1}{2}\right)^2 c_1 \frac{r^{\ell+2}}{(2\ell+3)} + O\left(r^{\ell+4}\right)$$
$$a_\ell = \ell c_2 r^{\ell-1} + \left[\frac{\psi_0^4\omega}{\alpha_0^2} c_1 - 2\ell\left(\frac{\kappa_2}{2}\right)^2 c_2\right]\frac{r^{\ell+1}}{(2\ell+3)} + O\left(r^{\ell+3}\right)$$
$$b_\ell = c_2 r^\ell - \left[\frac{\psi_0^4\omega}{\alpha_0^2} c_1 + 2(2\ell+3)\left(\frac{\kappa_1}{2}\right)^2 c_2 - 2\ell\left(\frac{\kappa_2}{2}\right)^2 c_2\right]\frac{r^{\ell+2}}{(\ell+1)(2\ell+3)} + O\left(r^{\ell+4}\right)$$
$$\kappa_1^2 = \psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2\right), \kappa_2^2 = \psi_0^4\left(\frac{\omega^2}{\alpha_0^2} - \mu^2 + \frac{4\pi}{3}S_0\right)$$
$$F_{\ell=0} = c_1 + \frac{c_1}{6}\left(\mu^2 - \frac{\omega^2}{\alpha_0^2}\right) r^2 + \mathcal{O}(r^4), a_{\ell=0} = c_1\left(\frac{\psi_0^4\omega}{3\alpha_0^2}\right) r + \mathcal{O}(r^3).$$
$$\frac{a_\ell - b_\ell'}{r^2} = \frac{1}{r^2}\left[\left(\ell c_2 r^{\ell-1} + O\left(r^{\ell+1}\right)\right) - \left(\ell c_2 r^{\ell-1} + O\left(r^{\ell+1}\right)\right] = O\left(r^{\ell-1}\right)\right.$$
$$F_\ell(r\to\infty) = 0, a_\ell(r\to\infty) = 0, b_\ell(r\to\infty) = 0, \alpha(r\to\infty) = 1, \psi(r\to\infty) = 1$$
$$\alpha = \alpha_0 + \mathcal{O}(r^2), \psi = \psi_0 + \mathcal{O}(r^2)$$
$$F_\ell \propto 1, a_\ell \propto r$$
$$F_\ell \propto r^\ell, a_\ell \propto r^{\ell-1}, b_\ell \propto r^\ell$$
$$M_{\mathrm{ADM}} = -2\lim_{r\to\infty}\left(r^2\frac{d\psi}{dr}\right), M_{\mathrm{Komar}} = \lim_{r\to\infty}\left(r^2\frac{d\alpha}{dr}\right)$$
$$\mu \mapsto \lambda\mu, \omega \mapsto \lambda\omega, r \mapsto \lambda^{-1}r, b_\ell \mapsto \lambda^{-1}b_\ell$$
$$S[g_{ab}, \phi] = \int \left[\frac{f(\phi)}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]\sqrt{-g}\,d^4x + S_{matt}$$
$$T_{ab}^f := \nabla_a(f'\nabla_b\phi) - g_{ab}\nabla_c(f'\nabla^c\phi),$$
$$T_{ab}^\phi := (\nabla_a\phi)(\nabla_b\phi) - g_{ab}\left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right],$$
$$T_{ab}^{\mathrm{fluid}} := (\mu + p/c^2)u_a u_b + p g_{ab},$$
$$\frac{p}{c^2} = \bar{\kappa} m_b n_0\left(\frac{n_b}{n_0}\right)^\gamma$$
$$\mu = m_b n_b + \frac{p/c^2}{\gamma - 1}$$
$$\omega := -\lim_{r\to\infty}\frac{1}{4\pi\sqrt{G_0}}\int_S s_a\nabla^a\phi\,ds$$
$$ds_3^2 = \psi^4[a(t,r)dr^2 + r^2 b(t,r)d\Omega^2]$$

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Apéndice G.

1. Cromodinámica cuántica en espacios cuánticos relativistas o curvos. Formalización matemática.

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\left(F_{\mu\nu}^2\right) + \sum_{i=1}^{N_f} \bar{q}_i\left(\gamma^\mu D_\mu + m_i\right)q_i$$

$$D_\mu = \partial_\mu + A_\mu, F_{\mu\nu} = \left[D_\mu, D_\nu\right]$$

$$\mathrm{SU}\left(N_f\right)_L \times \mathrm{SU}\left(N_f\right)_R \overset{\mathrm{RES}}{\rightarrow} \mathrm{SU}\left(N_f\right)_V$$

$$U(1/T, \vec{x}) = zU(0, \vec{x}), z = e^{i2\pi n/N_c}$$

$$L(T) := \langle\mathcal{P}(\vec{x}, T)\rangle = \left\langle\frac{1}{N_c}\mathrm{tr}_c\mathcal{T}\left(e^{-\int_0^{1/T} dx_0 A_0(\vec{x}, x_0)}\right)\right\rangle$$

$$e^{-F_q(\vec{x})/T} = \langle\mathcal{P}(\vec{x}, T)\rangle$$

$$e^{-F_{\bar{q}q}(\vec{x}-\vec{y})/T} = \langle\mathcal{P}(\vec{x}, T)\mathcal{P}^\dagger(\vec{y}, T)\rangle$$

$$\mathrm{Tr}\hat{A} \equiv \int d^D x \mathrm{tr}\langle x|\hat{A}|x\rangle$$

$$-c\mathrm{Trlog}\,(\mathbf{K}) = c\int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}e^{-\tau\mathbf{K}} = c\int_0^\infty \frac{d\tau}{\tau}\int d^D x \mathrm{tr}\langle x|e^{-\tau\mathbf{K}}|x\rangle$$

$$S_E[\phi] = \frac{1}{2}\int d^D x\phi(x)\left(-D_\mu^2 + m^2\right)\phi(x)$$

$$D_0 = \partial_0 - i\mu, D_i = \partial_i$$

$$Z = \int \mathcal{D}\phi e^{-S_E[\phi]} = \left(\det\left(-D_\mu^2 + m^2\right)\right)^{-1}$$

$$\Gamma = \log\,\det\left(-D_\mu^2 + m^2\right) = \mathrm{Trlog}\,\left(-D_\mu^2 + m^2\right) = -\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau}\langle x|e^{-\tau\left(-D_\mu^2+m^2\right)}|x\rangle$$

$$\Gamma = N\int \frac{d^d x d^d k}{(2\pi)^d}\left[\log\left(1 - e^{-\beta(\omega_k-\mu)}\right) + \log\left(1 - e^{-\beta(\omega_k+\mu)}\right)\right]$$

$$\hat{f} = f\left(M, D_\mu\right)$$

$$D_\mu = \partial_\mu + A_\mu(x)$$

$$\langle x \mid p\rangle = e^{ipx}, \langle p \mid p'\rangle = (2\pi)^D\delta(p - p')$$

$$\mathbf{1} = \int \frac{d^D p}{(2\pi)^D}|p\rangle\langle p|$$

$$\langle x \mid 0\rangle = 1, \hat{p}_\mu|0\rangle = \langle 0|\hat{p}_\mu = 0, \langle 0 \mid 0\rangle = \int d^D x$$

$$\langle x|f(M,D_\mu)|x\rangle = \int \frac{d^Dp}{(2\pi)^D}\langle x|f(M,D_\mu)|p\rangle\langle p \mid x\rangle$$
$$= \int \frac{d^Dp}{(2\pi)^D}\langle p \mid x\rangle\langle x|e^{ip\hat{x}}e^{-ip\hat{x}}f(M,D_\mu)e^{ip\hat{x}}e^{-ip\hat{x}}|p\rangle$$

$$e^{-ip\hat{x}}D_\mu e^{ip\hat{x}} = D_\mu + ip_\mu, e^{-ip\hat{x}}M(x)e^{ip\hat{x}} = M(x)$$

$$e^{-ip\hat{x}}f(M,D_\mu)e^{ip\hat{x}} = f(M,D_\mu + ip_\mu)$$

$$\langle x|f(M,D_\mu)|x\rangle = \int \frac{d^Dp}{(2\pi)^D}\langle x|f(M,D_\mu + ip_\mu)|0\rangle$$

$$D_\mu \to D_\mu + a_\mu$$

$$\mathbf{K} = M(x) - D_\mu^2$$

$$\langle x|e^{-\tau\mathbf{K}}|x\rangle = \frac{1}{(4\pi\tau)^{D/2}}\sum_{n=0}^{\infty} a_n(x)\tau^n$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \int \frac{d^Dp}{(2\pi)^D}\langle x|e^{-\tau\left(M-(D_\mu+ip_\mu)^2\right)}|0\rangle$$
$$= \int \frac{d^Dp}{(2\pi)^D}e^{-\tau p^2}\langle x|e^{-\tau(M-D_\mu^2-2ip_\mu D_\mu)}|0\rangle$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \int \frac{d^Dp}{(2\pi)^D}e^{-\tau p^2}\langle x|\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \cdots|0\rangle$$

$$\Delta_0 = 1$$
$$\Delta_1 = 2i\tau p_\mu D_\mu$$
$$\Delta_2 = -\tau(M - D_\mu^2) - 2\tau^2 p_\mu p_\nu D_\mu D_\nu$$
$$\Delta_3 = -i\tau^2 p_\mu\left(\{D_\mu, M\} - \{D_\mu, D_\nu^2\}\right) - i\frac{4}{3}\tau^3 p_\mu p_\nu p_\alpha D_\mu D_\nu D_\alpha$$
$$\Delta_4 = \frac{\tau^2}{2}\left(M^2 - \{D_\mu^2, M\} + D_\mu^4\right)$$
$$-\frac{\tau^3}{3}p_\mu p_\nu\left(\{M, D_\mu D_\nu\} + D_\mu M D_\nu - \{D_\alpha^2, D_\mu D_\nu\} - D_\mu D_\alpha^2 D_\nu\right)$$
$$+\frac{2}{3}\tau^4 p_\mu p_\nu p_\alpha p_\beta D_\mu D_\nu D_\alpha D_\beta.$$

$$\int \frac{d^Dp}{(2\pi)^D}e^{-\tau p^2}p_{i_1}\cdots p_{i_{2n}} \equiv \frac{1}{(4\pi\tau)^{D/2}}\frac{1}{(2\tau)^n}\delta_{i_1 i_2\cdots i_{2n-1}i_{2n}}$$
$$= \frac{1}{(4\pi\tau)^{D/2}}\frac{1}{(2\tau)^n}\left(\delta_{i_1 i_2}\cdots\delta_{i_{2n-1}i_{2n}} + (\,\zeta_{\text{permutaciones}}\,)\right)$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \frac{1}{(4\pi\tau)^{D/2}}\langle x|1 - \tau M$$
$$+\tau^2\left(\frac{1}{2}M^2 - \frac{2}{3}\{D_\mu^2, M\} - \frac{1}{6}D_\mu M D_\mu + D_\mu^4 + \frac{1}{6}\left(D_\mu D_\nu\right)^2 + \frac{1}{3}D_\mu D_\nu^2 D_\mu\right)$$
$$+\mathcal{O}(6)|0\rangle$$

$$\langle x|\hat{h}|0\rangle = h(x)$$

$$\{D_\mu^2, M\} = \left[D_\mu, [D_\mu, M]\right] + 2[D_\mu, M]D_\mu + 2MD_\mu^2$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \frac{1}{(4\pi\tau)^{D/2}}\left(1-\tau M+\tau^2\left(\frac{1}{2}M^2-\frac{1}{6}M_{\mu\mu}+\frac{1}{12}F_{\mu\nu}^2\right)+\mathcal{O}(\tau^3)\right)$$

$$\begin{aligned}
a_0 &= 1\\
a_1 &= -M\\
a_2 &= \frac{1}{2}M^2-\frac{1}{6}M_{\mu\mu}+\frac{1}{12}F_{\mu\nu}^2\\
a_3 &= -\frac{1}{6}M^3+\frac{1}{12}\{M,M_{\mu\mu}\}+\frac{1}{12}M_\mu^2-\frac{1}{60}M_{\mu\mu\nu\nu}-\frac{1}{60}[F_{\mu\mu\nu},M_\nu]-\frac{1}{30}\{M,F_{\mu\nu}^2\}\\
&\quad -\frac{1}{60}F_{\mu\nu}MF_{\mu\nu}+\frac{1}{45}F_{\mu\nu\alpha}^2-\frac{1}{30}F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}+\frac{1}{180}F_{\mu\mu\nu}^2+\frac{1}{60}\{F_{\mu\nu},F_{\alpha\alpha\mu\nu}\}
\end{aligned}$$

$$\mathrm{Tr}\left(e^{-\tau(M-D_\mu^2)}\right)=\frac{1}{(4\pi\tau)^{D/2}}\sum_{n=0}^{\infty}\int d^D x\,\mathrm{tr}(b_n(x))\tau^n$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle=-\frac{1}{\tau}\frac{\delta}{\delta M(x)}\mathrm{Tr}\left(e^{-\tau(M-D_\mu^2)}\right)$$

$$a_n(x)=-\frac{\delta}{\delta M(x)}\mathrm{tr}b_{n+1}(x)$$

$$\begin{aligned}
b_0 &= 1\\
b_1 &= -M\\
b_2 &= \frac{1}{2}M^2+\frac{1}{12}F_{\mu\nu}^2\\
b_3 &= -\frac{1}{6}M^3-\frac{1}{12}M_\mu^2-\frac{1}{12}F_{\mu\nu}MF_{\mu\nu}-\frac{1}{60}F_{\mu\mu\nu}^2+\frac{1}{90}F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}
\end{aligned}$$

$$\langle x\mid p\rangle=e^{ipx},\langle p\mid p'\rangle=\beta\delta_{p_0p_0'}(2\pi)^d\delta(\vec{p}-\vec{p}\,')$$

$$\mathbf{1}=\frac{1}{\beta}\sum_{p_0}\int\frac{d^dp}{(2\pi)^d}|p\rangle\langle p|$$

$$\langle x|f(M,D_\mu)|x\rangle=\frac{1}{\beta}\sum_{p_0}\int\frac{d^dp}{(2\pi)^d}\langle x|f(M,D_\mu+ip_\mu)|0\rangle$$

$$\vec{A}(x)=0,A_0=A_0(x_0),M(x)=m^2,[m^2,]=0$$

$$\begin{aligned}
\langle x|e^{-\tau\mathbf{K}}|x\rangle &= \frac{1}{\beta}\sum_{p_0}\int\frac{d^dp}{(2\pi)^d}\langle x|e^{-\tau(m^2+\vec{p}^2-(D_0+ip_0)^2)}|0\rangle\\
&= \frac{e^{-\tau m^2}}{(4\pi\tau)^{d/2}}\frac{1}{\beta}\sum_{p_0}\langle x|e^{\tau(D_0+ip_0)^2}|0\rangle
\end{aligned}$$

$$\sum_{n=-\infty}^{\infty}F(n)=\sum_{m=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}dxF(x)e^{i2\pi xm}\right\}$$

$$\frac{1}{\beta}\sum_{p_0}e^{\tau(D_0+ip_0)^2}=\frac{1}{(4\pi\tau)^{1/2}}\sum_{k\in\mathbb{Z}}(\pm)^k e^{-k\beta D_0}e^{-k^2\beta^2/4\tau}$$

$$e^{\beta\partial_0}e^{-\beta D_0}=\Omega(x)$$

$$\Omega(x) = \mathcal{T}\exp\left(-\int_{x_0}^{x_0+\beta} A_0(x_0', \vec{x})dx_0'\right)$$

$$e^{\beta\partial_0} = \mathbf{1}$$

$$e^{-\beta D_0} = \Omega(x)$$

$$\frac{1}{\beta}\sum_{p_0} e^{\tau(D_0+ip_0)^2} = \frac{1}{(4\pi\tau)^{1/2}}\sum_{k\in\mathbb{Z}} (\pm)^k\Omega^k e^{-k^2\beta^2/4\tau}$$

$$\sum_{p_0} f(ip_0 + D_0) = \sum_{p_0} f\left(ip_0 - \frac{1}{\beta}\log(\Omega)\right)$$

$$Q = ip_0 + D_0 = ip_0 - \frac{1}{\beta}\log(\Omega)$$

$$\langle x|e^{-\tau K}|x\rangle = \frac{1}{(4\pi\tau)^{d/2}}e^{-\tau m^2}\frac{1}{\beta}\sum_{p_0} e^{\tau Q^2} = \frac{1}{(4\pi\tau)^{(d+1)/2}}e^{-\tau m^2}\varphi_0(\Omega)$$

$$\varphi_n(\Omega;\tau/\beta^2) = (4\pi\tau)^{1/2}\frac{1}{\beta}\sum_{p_0} \tau^{n/2}Q^n e^{\tau Q^2}, Q = ip_0 - \frac{1}{\beta}\log(\Omega)$$

$$\frac{1}{\beta}\sum_{p_0} \overset{\beta\to\infty}{\to} \int_{-\infty}^{\infty}\frac{dp_0}{(2\pi)}$$

$$\varphi_n(\Omega;0) = \begin{cases}\left(-\frac{1}{2}\right)^{n/2}(n-1)!! & (n \text{ par }) \\ 0 & (n \text{ impar })\end{cases}$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \frac{1}{(4\pi\tau)^{(d+1)/2}}\sum_n a_n^T(x)\tau^n$$

$$a_0^T(x) = \varphi_0(\Omega(x);\tau/\beta^2),$$

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = \frac{1}{\beta}\sum_{p_0} \langle x_0,\vec{x}|e^{-\tau(M-Q^2-D_i^2)}|0,\vec{x}\rangle, Q = ip_0 + D_0$$

$$\mathcal{K} = \mathcal{Y} - D_i^2, \mathcal{Y} = M - Q^2,$$

$$\langle x_0,\vec{x}|e^{-\tau(\mathcal{Y}-D_i^2)}|0,\vec{x}\rangle = \frac{1}{(4\pi\tau)^{d/2}}\sum_{n=0}^{\infty} a_n(\widehat{D}_i,\mathcal{Y})\tau^n$$

$$\langle x_0,\vec{x}|e^{-\tau(M-Q^2-D_i^2)}|0,\vec{x}\rangle = \frac{1}{(4\pi\tau)^{d/2}}\sum_{n=0}^{\infty} e^{\tau Q^2}\tilde{a}_n(Q^2,M,\widehat{D}_i)\tau^n$$

$$\sum_{n=0}^{\infty} a_n(\widehat{D}_i,\mathcal{Y})\tau^n = e^{\tau Q^2}\sum_{n=0}^{\infty}\tilde{a}_n(Q^2,M,\widehat{D}_i)\tau^n$$

$$\tilde{a}_0 = 1$$
$$\tilde{a}_1 = -M$$
$$\tilde{a}_2 = \frac{1}{2}M^2 - \frac{1}{6}M_{ii} + \frac{1}{12}F_{ij}^2 + \frac{1}{2}[Q^2,M] + \frac{1}{6}(Q^2)_{ii}.$$

$$[Q^2, X] = Q[Q, X] + [Q, X]Q = 2Q[Q, X] - [Q, [Q, X]] = 2QX_0 - X_{00}$$

$$\tilde{a}_2 = \frac{1}{2}M^2 - \frac{1}{6}M_{ii} + \frac{1}{12}F_{ij}^2 - \frac{1}{2}M_{00} + \frac{1}{3}E_i^2 + \frac{1}{6}E_{0ii} + Q\left(M_0 - \frac{1}{3}E_{ii}\right).$$

$$\tilde{a}_2 \to \varphi_0(\Omega)\left(\frac{1}{2}M^2 - \frac{1}{6}M_{ii} + \frac{1}{12}F_{ij}^2 - \frac{1}{2}M_{00} + \frac{1}{3}E_i^2 + \frac{1}{6}E_{0ii}\right)\tau^2 + \varphi_1(\Omega)\left(M_0 - \frac{1}{3}E_{ii}\right)\tau^{3/2}$$

$$\begin{aligned}
&a_0 \sim \tilde{a}_0 \sim \varphi_0 a_0^T\\
&a_1 \sim \tilde{a}_1 \sim \varphi_0 a_1^T\\
&a_2 \sim \tilde{a}_2 \sim \varphi_0 a_2^T + \varphi_1 a_{3/2}^T\\
&a_3 \sim \tilde{a}_3 \sim \varphi_0 a_3^T + \varphi_1 a_{5/2}^T + \varphi_2 a_2^T\\
&a_4 \sim \tilde{a}_4 \sim \varphi_0 a_4^T + \varphi_1 a_{7/2}^T + \varphi_2 a_3^T + \varphi_3 a_{5/2}^T\\
&a_5 \sim \tilde{a}_5 \sim \varphi_0 a_5^T + \varphi_1 a_{9/2}^T + \varphi_2 a_4^T + \varphi_3 a_{7/2}^T + \varphi_4 a_3^T\\
&\quad \cdots\\
&a_k \sim \tilde{a}_k \sim \varphi_0 a_k^T + \varphi_1 a_{(2k-1)/2}^T + \cdots + \varphi_{k-1} a_{(k+1)/2}^T
\end{aligned}$$

$$\begin{aligned}
a_0^T &= \varphi_0,\\
a_{1/2}^T &= 0,\\
a_1^T &= -\varphi_0 M,\\
a_{3/2}^T &= \varphi_1\left(M_0 - \frac{1}{3}E_{ii}\right),\\
a_2^T &= \varphi_0 a_2^{T=0} + \frac{1}{6}\bar{\varphi}_2(E_i^2 + E_{0ii} - 2M_{00}),\\
a_{5/2}^T &= \frac{1}{3}(2\varphi_1 + \varphi_3)M_{000} + \frac{1}{6}\varphi_1 M_{0ii} - \frac{1}{3}\varphi_1(2M_0M + MM_0)\\
&\quad + \frac{1}{6}\varphi_1(\{M_i, E_i\} + \{M, E_{ii}\}) - \left(\frac{1}{3}\varphi_1 + \frac{1}{5}\varphi_3\right)E_{00ii} - \frac{1}{30}\varphi_1 E_{iijj}
\end{aligned}$$

$$\begin{aligned}
&-\left(\frac{5}{6}\varphi_1 + \frac{2}{5}\varphi_3\right)E_{0i}E_i - \left(\frac{1}{2}\varphi_1 + \frac{4}{15}\varphi_3\right)E_iE_{0i} + \frac{1}{30}\varphi_1[E_j, F_{iij}]\\
&-\varphi_1\left(\frac{1}{10}F_{0ij}F_{ij} + \frac{1}{15}F_{ij}F_{0ij}\right),
\end{aligned}$$

$$\begin{aligned}
a_3^T =&\varphi_0 a_3^{T=0} - \left(\frac{1}{4}\bar{\varphi}_2 - \frac{1}{10}\bar{\varphi}_4\right)M_{0000} - \frac{1}{60}\bar{\varphi}_2(3M_{00ii} - 15M_{00}M - 5MM_{00} - 15M_0^2\\
&+4\{M, E_i^2\} + 2E_iME_i + 4ME_{0ii} + 6E_{0ii}M + 4M_iE_{0i} + 6E_{0i}M_i\\
&+7M_0E_{ii} + 3E_{ii}M_0 + 6M_{0i}E_i + 4E_iM_{0i})\\
&+\left(\frac{3}{20}\bar{\varphi}_2 - \frac{1}{15}\bar{\varphi}_4\right)E_{000ii} + \frac{1}{60}\bar{\varphi}_2 E_{0iijj} + \left(\frac{1}{2}\bar{\varphi}_2 - \frac{1}{5}\bar{\varphi}_4\right)E_{00i}E_i\\
&+\left(\frac{7}{30}\bar{\varphi}_2 - \frac{1}{10}\bar{\varphi}_4\right)E_iE_{00i} + \left(\frac{19}{30}\bar{\varphi}_2 - \frac{4}{15}\bar{\varphi}_4\right)E_{0i}^2\\
&+\frac{1}{180}\bar{\varphi}_2\big(2\{E_i, E_{jij}\} + 4\{E_i, E_{ijj}\} + 5E_{ii}^2 + 4E_{ij}^2 + 4F_{0iij}E_j - 2E_jF_{0iij} - 2E_{0ij}F_{ij}\\
&-[E_{ij}, F_{0ij}] - 4E_{0i}F_{jji} + 2F_{jji}E_{0i} + 2E_iF_{ij}E_j + 2\{E_iE_j, F_{ij}\} + 7F_{00ij}F_{ij}\\
&+3F_{ij}F_{00ij} + 8F_{0ij}^2\big).
\end{aligned}$$

$$\bar{\varphi}_2 = \varphi_0 + 2\varphi_2, \bar{\varphi}_4 = \varphi_0 - \frac{4}{3}\varphi_4, \ldots\ldots, \bar{\varphi}_{2n} = \varphi_0 - \frac{(-2)^n}{(2n-1)!!}\varphi_{2n}$$

$$\mathrm{Tr}\left(e^{-\tau(M-D_\mu^2)}\right) = \frac{1}{(4\pi\tau)^{(d+1)/2}}\sum_{n=0}^{\infty}\int_0^\beta dx_0 \int d^dx\,\mathrm{tr}(b_n^T(x))\tau^n$$

$$[X, f(Y)] = -f'(Y)[Y, X] + \frac{1}{2} f''(Y)[Y, [Y, X]] - \frac{1}{3!} f^{(3)}(Y)[Y, [Y, [Y, X]]] + \cdots$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(Y) D_Y^n(X)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n e^{\lambda Y} D_Y^n(X) = e^{\lambda Y}\left(e^{-\lambda D_Y} - 1\right)X = e^{\lambda Y}\left(e^{-\lambda Y} X e^{\lambda Y} - X\right) = \left[X, e^{\lambda Y}\right]$$

$$[X, f] = -f' X_0 + \frac{1}{2} f'' X_{00} - \frac{1}{3!} f^{(3)} X_{000} + \cdots$$

$$\hat{D}_0 f \ = 0$$
$$\hat{D}_i f \ = -f' E_i + \frac{1}{2} f'' E_{0i} - \frac{1}{3!} f^{(3)} E_{00i} + \cdots$$

$$\varphi_n' = \sqrt{\tau}(n\varphi_{n-1} + 2\varphi_{n+1})$$

$$\begin{aligned}
b_0^T &= \varphi_0, \\
b_{1/2}^T &= 0, \\
b_1^T &= -\varphi_0 M, \\
b_{3/2}^T &= 0, \\
b_2^T &= \varphi_0 b_2^{T=0} - \frac{1}{6} \bar{\varphi}_2 E_i^2, \\
b_{5/2}^T &= -\frac{1}{6} \varphi_1 \{M_i, E_i\}, \\
b_3^T &= \varphi_0 b_3^{T=0} + \frac{1}{6} \bar{\varphi}_2 \left(\frac{1}{2} M_0^2 + E_i M E_i + \frac{1}{10} E_{ii}^2 + \frac{1}{10} F_{0ij}^2 - \frac{1}{5} E_i F_{ij} E_j\right) \\
&\quad - \left(\frac{1}{6} \bar{\varphi}_2 - \frac{1}{10} \bar{\varphi}_4\right) E_{0i}^2.
\end{aligned}$$

$$\mathrm{Tr}\left(e^{-\tau(M - D_\mu^2)}\right) = \frac{1}{(4\pi\tau)^{(d+1)/2}} \sum_{n=0}^{\infty} B_n^T \tau^n, B_n^T = \mathrm{Tr} b_n^T(x)$$

$$a_n^T(x) \simeq -\frac{\delta}{\delta M(x)} \sum_{1 \le k \le n+1} B_k^T \tau^{k-n-1}$$

$$H = -\frac{1}{g^2} \int d^3 x \mathrm{tr}[(\partial_0 A_i)^2 + B_i^2]$$

$$\langle A_i'(\vec{x}) | e^{-\beta H} | A_i''(\vec{x}) \rangle = \int \mathcal{D}A_i(x_0, \vec{x}) \exp\left\{\frac{1}{g^2} \int_0^\beta dx_0 \int d^3 x \mathrm{tr}[(\partial_0 A_i)^2 + B_i^2]\right\}$$

$$Z_{\mathrm{YM}} = \mathrm{Tr}\left(e^{-\beta H}\right) = \int \mathcal{D}A_i(\vec{x}) \langle A_i(\vec{x}) | e^{-\beta H} | A_i(\vec{x}) \rangle$$
$$= \int \mathcal{D}A_i^{(0)}(\vec{x}) \int_{A_i(0,\vec{x}) = A_i^{(0)}}^{A_i(\beta,\vec{x}) = A_i^{(0)}} \mathcal{D}A_i(x_0, \vec{x}) \exp\left\{\frac{1}{g^2} \int_0^\beta dx_0 \int d^3 x \mathrm{tr}[(\partial_0 A_i)^2 + B_i^2]\right\}$$

$$\vec{D} \cdot \vec{E}(\vec{x}) | \psi_{\mathrm{fis}} \rangle = 0 \ \forall \vec{x},$$

$$\exp\left(\int d^3 x \mathrm{tr}[\vec{D}\Lambda(\vec{x}) \cdot \vec{E}(\vec{x})]\right) | \psi_{\mathrm{fis}} \rangle = | \psi_{\mathrm{fis}} \rangle$$

$$Z_{\text{YM}} = \text{Tr}\left(Pe^{-\beta H}\right) = \int_{\Lambda(\infty)=0} \mathcal{D}\Lambda(\vec{x})\mathcal{D}A_i(\vec{x})\langle A_i^U(\vec{x})|e^{-\beta H}|A_i(\vec{x})\rangle$$

$$= \int_{\Lambda(\infty)=0} \mathcal{D}\Lambda(\vec{x}) \int_{A_i(\beta,\vec{x})=A_i^U(0,\vec{x})} \mathcal{D}A_i(x_0,\vec{x})\exp\left\{\frac{1}{g^2}\int_0^\beta dx_0 \int d^3x\text{tr}[(\partial_0 A_i)^2 + B_i^2]\right\}$$

$$Z_{\text{YM}} = \lim_{N\to\infty} \text{Tr}(Pe^{-\varepsilon H})^N$$

$$= \int \mathcal{D}\Lambda(x_0,\vec{x})\mathcal{D}A_i(x_0,\vec{x})\exp\left\{\frac{1}{g^2}\int_0^\beta dx_0 \int d^3x\text{tr}\left[\left(\partial_0 A_i - \widehat{D}_i\Lambda\right)^2 + B_i^2\right]\right\}$$

$$Z_{\text{YM}} = \int_{A_\mu(\beta,\vec{x})=A_\mu(0,\vec{x})} \mathcal{D}A_\mu(x_0,\vec{x})\exp\left\{\frac{1}{2g^2}\int_0^\beta dx_0 \int d^3x\text{tr}\left(F_{\mu\nu}^2\right)\right\} =: \int \mathcal{D}A_\mu(x)e^{-S_{\text{YM}}^E}$$

$$\widehat{D}_\mu F_{\mu\nu} = 0, \widehat{D}_\lambda F_{\mu\nu} + \widehat{D}_\mu F_{\nu\lambda} + \widehat{D}_\nu F_{\lambda\mu} = 0$$

$$Z_{\text{QCD}} = \int_{A_\mu(\beta,\vec{x})=A_\mu(0,\vec{x})} \mathcal{D}A_\mu(x_0,\vec{x}) \int_{q(\beta,\vec{x})=-q(0,\vec{x})} \prod_{\alpha=1}^{N_f} \mathcal{D}\bar{q}_\alpha(x_0,\vec{x})\mathcal{D}q_\alpha(x_0,\vec{x})\exp\left(-S_E\right)$$

$$S_E = -\frac{1}{2g^2}\int_0^\beta dx_0 \int d^3x\text{tr}\left(F_{\mu\nu}^2\right) + \int_0^\beta dx_0 \int d^3x \sum_{\alpha=1}^{N_f} \bar{q}_\alpha(\not{D} + m_\alpha)q_\alpha$$

$$Z_{\text{QCD}} = Z_q Z_{\text{YM}}$$

$$Z_q[A] = \int_{q(\beta,\vec{x})=-q(0,\vec{x})} \prod_{\alpha=1}^{N_f} \mathcal{D}\bar{q}_\alpha(x_0,\vec{x})\mathcal{D}q_\alpha(x_0,\vec{x})\exp\left(-S_q^E\right)$$

$$S_q^E = \int_0^\beta dx_0 \int d^3x \sum_{\alpha=1}^{N_f} \bar{q}_\alpha \not{D} q_\alpha$$

$$Z_q[A] = \text{Det}(\not{D})^{N_f},$$

$$\Gamma_q^{\text{desn}}[A] = -N_f \log\ \text{Det}(\not{D}) = -N_f \text{Trlog}\,(\not{D})$$

$$\Gamma_q[A] = -\frac{N_f}{2}\text{Trlog}\left(\not{D}^2\right) = \frac{N_f}{2}\int_0^\infty \frac{d\tau}{\tau}\text{Tr}e^{\tau\not{D}^2} =: \int_0^\beta dx_0 \int d^3x\mathcal{L}_q(x)$$

$$\mathcal{L}_q(x) = \frac{N_f}{2}\int_0^\infty \frac{d\tau}{\tau}\frac{\mu^{2\epsilon}}{(4\pi\tau)^{D/2}}\sum_n \tau^n \text{tr}\left(b_{n,q}^T(x)\right)$$

$$\gamma_\mu = \gamma_\mu^\dagger, \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \text{tr}_{\text{Dirac}}\,(\mathbf{1}) = 4$$

$$-\not{D}^2 = -D_\mu^2 - \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$$

$$\text{tr}_{\text{Dirac}}\left(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{2n+1}}\right) = 0$$
$$\text{tr}_{\text{Dirac}}\left(\gamma_\mu\gamma_\nu\right) = 4\delta_{\mu\nu}$$
$$\text{tr}_{\text{Dirac}}\left(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\right) = 4\left(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}\right)$$

$$\text{tr}_{\text{Dirac}}\left(\gamma_\mu\gamma_\nu\gamma_\alpha\cdots\right) = \text{tr}_{\text{Dirac}}\left(\cdots\gamma_\alpha\gamma_\nu\gamma_\mu\right)$$

$$b_{0,q}^{T} = 4\varphi_0,$$
$$b_{2,q}^{T} = -\frac{2}{3}\left(\varphi_0 F_{\mu\nu}^2 + \bar{\varphi}_2 E_i^2\right),$$
$$b_{3,q}^{T} = \varphi_0\left(\frac{32}{45}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu} + \frac{1}{6}F_{\lambda\mu\nu}^2 - \frac{1}{15}F_{\mu\mu\nu}^2\right) + \bar{\varphi}_2\left(\frac{1}{15}E_{ii}^2 - \frac{1}{10}F_{0ij}^2 - \frac{2}{15}E_iF_{ij}E_j\right) + \left(\frac{2}{5}\bar{\varphi}_4 - \bar{\varphi}_2\right)E_{0i}^2.$$

$$I_{\ell,n}^{\pm}(\Omega) := \int_0^\infty \frac{d\tau}{\tau}(4\pi\mu^2\tau)^\epsilon \tau^\ell \varphi_n^{\pm}(\Omega), |\Omega| = 1$$

$$I_{\ell,2n}^{-}(e^{i2\pi\nu}) = (-1)^n(4\pi)^\epsilon\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma\left(\ell+n+\epsilon+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times\left[\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}+\nu\right)+\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}-\nu\right)\right]$$
$$-\frac{1}{2}<\nu<\frac{1}{2}$$

$$I_{-2,0}^{-} = -\frac{2}{3}\left(\frac{2\pi}{\beta}\right)^4 B_4\left(\frac{1}{2}+\nu\right) + \mathcal{O}(\epsilon)$$

$$\mathcal{L}_{0,q}(x) = \frac{\pi^2 N_f}{\beta^4}\left(\frac{2N_c}{45} - \frac{1}{12}\mathrm{tr}[(1-4\nu^2)^2]\right), \Omega(x) = e^{i2\pi\nu}, -\frac{1}{2}<\nu<\frac{1}{2}$$

$$I_{0,0}^{-} = \frac{1}{\epsilon} + \log(4\pi) - \gamma_E + 2\log(\mu\beta/4\pi) - \psi\left(\frac{1}{2}+\nu\right) - \psi\left(\frac{1}{2}-\nu\right) + \mathcal{O}(\epsilon),$$
$$I_{0,\bar{2}}^{-} := I_{0,0}^{-} + 2I_{0,2}^{-} = -2 + \mathcal{O}(\epsilon).$$

$$\zeta(1+z,q) = \frac{1}{z} - \psi(q) + \mathcal{O}(z)$$

$$\mathcal{L}_{2,q}(x) = -\frac{1}{3}\frac{1}{(4\pi)^2}N_f\mathrm{tr}\left[\left(2\log(\mu\beta/4\pi) - \psi\left(\frac{1}{2}+\nu\right) - \psi\left(\frac{1}{2}-\nu\right)\right)F_{\mu\nu}^2 - 2E_i^2\right]$$

$$I_{1,0}^{-} = -\left(\frac{\beta}{4\pi}\right)^2\left(\psi''\left(\frac{1}{2}+\nu\right) + \psi''\left(\frac{1}{2}-\nu\right)\right) + \mathcal{O}(\epsilon)$$
$$I_{1,\bar{2}}^{-} = -2I_{1,0}^{-} + \mathcal{O}(\epsilon), I_{1,\bar{4}}^{-} = -4I_{1,0}^{-} + \mathcal{O}(\epsilon)$$

$$\mathcal{L}_{3,q}(x) = -\frac{2}{(4\pi)^4}N_f\beta^2\mathrm{tr}\left[\left(\psi''\left(\frac{1}{2}+\nu\right)+\psi''\left(\frac{1}{2}-\nu\right)\right)\right.$$
$$\left.\times\left(\frac{8}{45}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu} + \frac{1}{24}F_{\lambda\mu\nu}^2 - \frac{1}{60}F_{\mu\mu\nu}^2 + \frac{1}{20}F_{0\mu\nu}^2 - \frac{1}{30}E_{ii}^2 + \frac{1}{15}E_iF_{ij}E_j\right)\right]$$

$$Z_g = \int_{A_\mu(\beta,\vec{x})=A_\mu(0,\vec{x})} \mathcal{D}A_\mu(x_0,\vec{x})\exp(-S_{\mathrm{YM}}^E)$$

$$S_{\mathrm{YM}}^E = -\frac{1}{2g^2}\int_0^\beta dx_0\int d^3x\,\mathrm{tr}\left(F_{\mu\nu}^2\right)$$

$$\bar{A}_\mu(x) = A_\mu(x) + a_\mu(x)$$

$$F_{\mu\nu}[\bar{A}] = F_{\mu\nu}[A] + \widehat{D}_\mu a_\nu - \widehat{D}_\nu a_\mu + [a_\mu, a_\nu]$$

$$\delta A_\mu = 0, \delta a_\mu = \widehat{D}_\mu\Lambda + [a_\mu, \Lambda]$$

$$\delta A_\mu = \widehat{D}_\mu \Lambda, \delta a_\mu = [a_\mu, \Lambda]$$

$$S_{\text{fix}} = \frac{1}{\alpha}\int_0^\beta dx_0 \int d^3x \text{tr}(G^2)$$

$$S_{\text{FP}} = 2\int_0^\beta dx_0 \int d^3x \text{tr}\left(C^* \frac{\delta G}{\delta \Lambda} C\right)$$

$$Z_g[A, J, \eta^*, \eta] = N \int \mathcal{D}a\mathcal{D}C^*\mathcal{D}C \exp\left(-S_{\text{tot}} + J \cdot a + \eta^* \cdot C + C^* \cdot \eta\right)$$

$$W_g[A, J, \eta^*, \eta] = \log Z_g[A, J, \eta^*, \eta]$$

$$\tilde{a} = \frac{\delta W_g}{\delta J}, \tilde{C} = \frac{\delta W_g}{\delta \eta^*}, \tilde{C}^* = \frac{\delta W_g}{\delta \eta}$$

$$\Gamma_g[A, \tilde{a}, \tilde{C}^*, \tilde{C}] = J \cdot \tilde{a} + \eta \cdot \tilde{C}^* + \tilde{C} \cdot \eta^* - W_g[A, J, \eta^*, \eta]$$

$$G = \widehat{D}_\mu a_\mu + \lambda a_\mu^2$$

$$S_{\text{FP}} = 2\int_0^\beta dx_0 \int d^3x \text{tr}\left(C^* \widehat{D}_\mu\left(\widehat{D}_\mu C + [a_\mu, C]\right)\right)$$

$$J_\mu = \mathcal{J}_\mu + j_\mu$$

$$j_\mu = \left.\frac{\delta S_{\text{tot}}[A, a]}{\delta a_\mu}\right|_{a=0}$$

$$\begin{aligned} S_{\text{tot}}[A, a] - J \cdot a =& S_{\text{YM}}^E[A] + \frac{1}{2}\int_0^\beta dx_0 \int d^3x \text{tr}\left(a_\mu \Delta_{\mu\nu}[A] a_\nu\right) \\ & - \int_0^\beta dx_0 \int d^3x \text{tr}(C^* \Delta[A] C) - S_{\text{int}}[A, a] - \mathcal{J} \cdot a \end{aligned}$$

$$\Delta_{\mu\nu}[A] = -\left(\delta_{\mu\nu}\widehat{D}_\lambda^2 + 2\widehat{F}_{\mu\nu}\right), \Delta[A] = -\widehat{D}_\mu^2$$

$$S_{\text{int}}[A, a] = \int_0^\beta dx_0 \int d^3x \text{tr}\left(\left(\widehat{D}_\mu a_\nu\right)[a_\mu, a_\nu] + \frac{1}{4}[a_\mu, a_\nu]^2 + C^* \widehat{D}_\mu[a_\mu, C]\right)$$

$$\begin{aligned} & Z_g[A, \mathcal{J}, \eta^*, \eta] \\ = & \int \mathcal{D}a\mathcal{D}C^*\mathcal{D}C \exp\left[-S_{\text{YM}}^E[A] - \frac{1}{2}\int_0^\beta dx_0 \int d^3x a_\mu \Delta_{\mu\nu} a_\nu + \int_0^\beta dx_0 \int d^3x C^* \Delta C \right. \\ & \left. -S_{\text{int}}[A, a] + \mathcal{J} \cdot a + \eta^* \cdot C + C^* \cdot \eta\right] \end{aligned}$$

$$\Gamma_g[A, 0,0,0] = -W_g[A, \mathcal{J}, \eta^*, \eta]\Big|_{\delta W_g/\delta \mathcal{J} = \delta W_g/\delta \eta^* = \delta W_g/\delta \eta = 0}$$

$$\Gamma_g[A, 0,0,0] = S_{\text{YM}}^E[A] + \frac{1}{2}\text{Tr}\log\left(-\delta_{\mu\nu}\widehat{D}_\lambda^2 - 2\widehat{F}_{\mu\nu}\right) - \text{Tr}\log\left(-\widehat{D}_\mu^2\right) + \mathcal{O}(\hbar^2)$$

$$\Gamma[A] = -\frac{\mu^{-2\epsilon}}{2g_0^2}\int d^4x \text{tr}\left(F_{\mu\nu}^2\right) + \Gamma_q[A] + \Gamma_g[A]$$

$$\Gamma_g[A] = \frac{1}{2}\text{Tr}\log\left(-\delta_{\mu\nu}\widehat{D}_\lambda^2 - 2\widehat{F}_{\mu\nu}\right) - \text{Tr}\log\left(-\widehat{D}_\mu^2\right) =: \int_0^\beta dx_0 \int d^3x \mathcal{L}_g(x)$$

$$\mathcal{L}_g(x) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\frac{\mu^{2\epsilon}}{(4\pi\tau)^{D/2}}\sum_n \tau^n \widehat{\mathrm{tr}}\left(b_{n,g}^T(x)\right)$$

$$\begin{aligned}
b_{0,g}^T &= (D-2)\varphi_0(\widehat{\Omega}) \\
b_{2,g}^T &= \left(-2+\frac{D-2}{12}\right)\varphi_0(\widehat{\Omega})\hat{F}_{\mu\nu}^2 - \frac{D-2}{6}\bar{\varphi}_2(\widehat{\Omega})\hat{E}_i^2 \\
b_{3,g}^T &= \varphi_0(\widehat{\Omega})\left(\left(\frac{4}{3}+\frac{D-2}{90}\right)\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu} + \frac{1}{3}\hat{F}_{\lambda\mu\nu}^2 - \frac{D-2}{60}\hat{F}_{\mu\mu\nu}^2\right) \\
&\quad + \frac{1}{6}\bar{\varphi}_2(\widehat{\Omega})\left(-2\hat{F}_{0\mu\nu}^2 + \frac{D-2}{10}\left(\hat{E}_{ii}^2 + \hat{F}_{0ij}^2 - 2\hat{E}_i\hat{F}_{ij}\hat{E}_j\right)\right) \\
&\quad + (D-2)\left(\frac{1}{10}\bar{\varphi}_4(\widehat{\Omega}) - \frac{1}{6}\bar{\varphi}_2(\widehat{\Omega})\right)\hat{E}_{0i}^2.
\end{aligned}$$

$$\begin{aligned}
I_{\ell,2n}^{+}\left(e^{i2\pi\hat{v}}\right) &= (-1)^n(4\pi)^\epsilon\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma\left(\ell+n+\epsilon+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\
&\quad \times[\zeta(1+2\ell+2\epsilon,\hat{v}) + \zeta(1+2\ell+2\epsilon,1-\hat{v})] \\
&\quad 0<\hat{v}<1
\end{aligned}$$

$$I_{-2,0}^{+} = -\frac{1}{3}\left(\frac{2\pi}{\beta}\right)^4 (B_4(\hat{v}) + B_4(1-\hat{v})) + \mathcal{O}(\epsilon).$$

$$\begin{aligned}
\mathcal{L}_{0,g}(x) &= \frac{\pi^2}{3\beta^4}\widehat{\mathrm{tr}}(B_4(\hat{v}) + B_4(1-\hat{v})) \\
&= -\frac{\pi^2}{45\beta^4}\hat{N}_c + \frac{2\pi^2}{3\beta^4}\widehat{\mathrm{tr}}[\hat{v}^2(1-\hat{v})^2], \widehat{\Omega}(x) = e^{i2\pi\hat{v}}, 0<\hat{v}<1,
\end{aligned}$$

$$\begin{aligned}
I_{0,0}^{+} &= \frac{1}{\epsilon} + \log(4\pi) - \gamma_E + 2\log(\mu\beta/4\pi) - \psi(\hat{v}) - \psi(1-\hat{v}) + \mathcal{O}(\epsilon) \\
I_{0,\bar{2}}^{+} &:= I_{0,0}^{+} + 2I_{0,2}^{+} = -2 + \mathcal{O}(\epsilon)
\end{aligned}$$

$$\mathcal{L}_{2,g}(x) = \frac{1}{(4\pi)^2}\widehat{\mathrm{tr}}\left[\frac{11}{12}\left(2\log(\mu\beta/4\pi) + \frac{1}{11} - \psi(\hat{v}) - \psi(1-\hat{v})\right)\hat{F}_{\mu\nu}^2 - \frac{1}{3}\hat{E}_i^2\right]$$

$$\begin{aligned}
I_{1,0}^{+} &= -\left(\frac{\beta}{4\pi}\right)^2(\psi''(\hat{v}) + \psi''(1-\hat{v})) + \mathcal{O}(\epsilon) \\
I_{1,\bar{2}}^{+} &= -2I_{1,0}^{+} + \mathcal{O}(\epsilon), I_{1,\bar{4}}^{+} = -4I_{1,0}^{+} + \mathcal{O}(\epsilon)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{3,g}(x) &= \frac{1}{2}\frac{\beta^2}{(4\pi)^4}\widehat{\mathrm{tr}}[(\psi''(\hat{v}) + \psi''(1-\hat{v})) \\
&\times\left(\frac{61}{45}\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu} + \frac{1}{3}\hat{F}_{\lambda\mu\nu}^2 - \frac{1}{30}\hat{F}_{\mu\mu\nu}^2 + \frac{3}{5}\hat{F}_{0\mu\nu}^2 - \frac{1}{15}\hat{E}_{ii}^2 + \frac{2}{15}\hat{E}_i\hat{F}_{ij}\hat{E}_j\right)]
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}_{\text{árbol}}(x) + \mathcal{L}_q^{\text{div}}(x) + \mathcal{L}_g^{\text{div}}(x) = \\
&= -\frac{1}{2g_0^2}\mathrm{tr}\left(F_{\mu\nu}^2\right) + \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log(4\pi) - \gamma_E\right)\left(\frac{11}{12}\widehat{\mathrm{tr}}\left(\hat{F}_{\mu\nu}^2\right) - \frac{N_f}{3}\mathrm{tr}\left(F_{\mu\nu}^2\right)\right) \\
&= -\frac{1}{2g^2(\mu)}\mathrm{tr}\left(F_{\mu\nu}^2\right)
\end{aligned}$$

$$\widehat{\mathrm{tr}}\left(\hat{F}_{\mu\nu}^2\right) = 2\mathrm{tr}(\mathbf{1})\mathrm{tr}\left(F_{\mu\nu}^2\right) - 2\left(\mathrm{tr}\left(F_{\mu\nu}\right)\right)^2 = 2N_c\mathrm{tr}\left(F_{\mu\nu}^2\right),$$

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \beta_0\left(\frac{1}{\epsilon} + \log(4\pi) - \gamma_E\right), \beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right),$$

$$\begin{aligned}\mathcal{L}_{2,\mathrm{QCD}}(x) = &\left(-\frac{1}{2g^2(\mu)} + \beta_0 \log(\mu\beta/4\pi) + \frac{1}{6}\frac{1}{(4\pi)^2}N_c\right)\mathrm{tr}\left(F_{\mu\nu}^2\right)\\ &-\frac{11}{12}\frac{1}{(4\pi)^2}\widehat{\mathrm{tr}}\left[(\psi(\hat{\nu}) + \psi(1-\hat{\nu}))\hat{F}_{\mu\nu}^2\right]\\ &+\frac{1}{3}\frac{1}{(4\pi)^2}N_f \mathrm{tr}\left[\left(\psi\left(\frac{1}{2}+\nu\right) + \psi\left(\frac{1}{2}-\nu\right)\right)F_{\mu\nu}^2\right]\\ &-\frac{2}{3}\left(N_c - N_f\right)\frac{1}{(4\pi)^2}\mathrm{tr}[E_i^2], -\frac{1}{2} < \nu < \frac{1}{2}, 0 < \hat{\nu} < 1\end{aligned}$$

$$Z[A]|_{\mathrm{reg}} = \frac{Z[A]}{Z'[A, M^2]}$$

$$\mathrm{Det}(\mathbf{K})|_{\mathrm{reg}} = \frac{\mathrm{Det}(\mathbf{K})}{\mathrm{Det}(\mathbf{K} + M^2)} = \exp\left[-\int_0^\infty \frac{d\tau}{\tau}\left(1 - e^{-\tau M^2}\right)\mathrm{Tr}e^{-\tau\mathbf{K}}\right]$$

$$I_{0,0}^{+,\mathrm{PV}} = 2\log(M/\mu) + 2\log(\mu\beta/4\pi) - \psi(\hat{\nu}) - \psi(1-\hat{\nu}) + \mathcal{O}(M^{-1}), 0 < \hat{\nu} < 1,$$

$$I_{0,0}^{-,\mathrm{PV}} = 2\log(M/\mu) + 2\log(\mu\beta/4\pi) - \psi\left(\frac{1}{2}+\nu\right) - \psi\left(\frac{1}{2}-\nu\right) + \mathcal{O}(M^{-1}), -\frac{1}{2} < \nu < \frac{1}{2}$$

$$\frac{1}{g_0^2(M)} = \frac{2}{(4\pi)^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\log\frac{M}{\mu}.$$

$$\begin{aligned}\mathcal{L}_{\text{árbol}}(x) + \mathcal{L}_q^{\mathrm{div}}(x) + \mathcal{L}_g^{\mathrm{div}}(x) &= -\frac{1}{2g_0^2(M)}\mathrm{tr}\left(F_{\mu\nu}^2\right) + \frac{1}{(4\pi)^2}\log(M)\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\mathrm{tr}\left(F_{\mu\nu}^2\right)\\ &= -\frac{1}{2g^2(\mu)}\mathrm{tr}\left(F_{\mu\nu}^2\right)\end{aligned}$$

$$\log\left(\Lambda_{\mathrm{PV}}^2/\Lambda_{\mathrm{MS}}^2\right) = \frac{1}{11 - \frac{2N_f}{N_c}}$$

$$\begin{aligned}&\psi(\hat{\nu}) + \psi(1-\hat{\nu})|_{\hat{\nu}=0} \to \psi(1+\hat{\nu}) + \psi(1-\hat{\nu})|_{\hat{\nu}=0} = -2\gamma_E,\\ &\psi''(\hat{\nu}) + \psi''(1-\hat{\nu})|_{\hat{\nu}=0} \to \psi''(1+\hat{\nu}) + \psi''(1-\hat{\nu})|_{\hat{\nu}=0} = -4\zeta(3),\end{aligned}$$

$$I_{\ell,0}^{+}(1)\Big|_{p_0=0} = \frac{\sqrt{4\pi}\,\Gamma\left(\frac{1}{2}+\ell\right)}{\beta m^{2\ell+1}}$$

$$\begin{aligned}\mathcal{L}_{2,\mathrm{IR}} = &\frac{1}{48\pi}\frac{T}{m}\mathrm{tr}\left[11F_{\mu\nu\perp}^2 + 2E_{i\perp}^2\right]\\ \mathcal{L}_{3,\mathrm{IR}} = &\frac{1}{240\pi}\frac{T}{m^3}\mathrm{tr}\left[-\frac{61}{3}F_{\mu\nu\perp}F_{\nu\alpha}F_{\alpha\mu} + E_{i\perp}F_{ij}E_{j\perp} + E_iF_{ij\parallel}E_j\right.\\ &\left.-5F_{\mu\nu\lambda\perp}^2 + \frac{1}{2}F_{\mu\mu\nu\perp}^2 + \frac{9}{2}F_{0\mu\nu\perp}^2 + 3E_{0i\perp}^2 - \frac{1}{2}E_{ii\perp}^2\right]\end{aligned}$$

$$A_\mu(x_0, \vec{x}) = \sum_{n=-\infty}^{\infty} A_\mu(\omega_n, \vec{x})e^{i\omega_n x_0}, \omega_n = \frac{2\pi n}{\beta}$$

$$\int d^4x\mathcal{L}_{\mathrm{QCD}}(x) \longrightarrow \int d^3x\mathcal{L}'(\vec{x})$$

$$\mathcal{L}'_{\text{árbol}}(\vec{x}) = \beta\mathcal{L}_{\text{árbol}}(\vec{x}).$$

$$\mathcal{L}'_{0,g}(\vec{x}) = \frac{2\pi^2}{3}T^3\widehat{\mathrm{tr}}[\hat{v}^2(1+\hat{v}^2)], \hat{v} = \log(\widehat{\Omega})/(2\pi i), -1 \le \hat{v} \le 1$$

$$\mathcal{L}'_{2,g}(\vec{x}) = \frac{1}{(4\pi)^2 T}\mathrm{tr}\left[\frac{11}{12}\left(2\log(\mu/4\pi T) + \frac{1}{11} - \psi(\hat{v}) - \psi(-\hat{v})\right)\hat{F}^2_{\mu\nu} - \frac{1}{3}\hat{E}^2_i\right]$$

$$\mathcal{L}'_{3,g}(\vec{x}) = \frac{1}{2}\frac{1}{(4\pi)^4}\frac{1}{T^3}\mathrm{tr}[(\psi''(\hat{v}) + \psi''(-\hat{v}))$$

$$\times\left(\frac{61}{45}\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu} + \frac{1}{3}\hat{F}^2_{\lambda\mu\nu} - \frac{1}{30}\hat{F}^2_{\mu\mu\nu} + \frac{3}{5}\hat{F}^2_{0\mu\nu} - \frac{1}{15}\hat{E}^2_{ii} + \frac{2}{15}\hat{E}_i\hat{F}_{ij}\hat{E}_j\right)\Big]$$

$$\mathcal{L}'_0(\vec{x}) = -\left(\frac{N_c}{3} + \frac{N_f}{6}\right)T\langle A_0^2\rangle + \frac{1}{4\pi^2 T}\langle A_0^2\rangle^2 + \frac{1}{12\pi^2 T}(N_c - N_f)\langle A_0^4\rangle.$$

$$Z_M = Z_g = 1 + 2g^2\beta_0(\log(\mu/4\pi T) + \gamma_E) - \frac{g^2}{3(4\pi)^2}(-N_c + 8N_f\log 2)$$

$$Z_E = Z_M - \frac{2g^2}{3(4\pi)^2}(N_c - N_f)$$

$$\mathcal{L}'_{(4)}(\vec{x}) = -\frac{1}{Tg_E^2(T)}\langle E_i^2\rangle - \frac{1}{Tg_M^2(T)}\langle B_i^2\rangle$$

$$\frac{1}{g_E^2(T)} = \frac{1}{g^2(\mu)} - 2\beta_0(\log(\mu/4\pi T) + \gamma_E) + \frac{1}{3(4\pi)^2}\left(N_c + 8N_f\left(\log 2 - \frac{1}{4}\right)\right)$$

$$\frac{1}{g_M^2(T)} = \frac{1}{g^2(\mu)} - 2\beta_0(\log(\mu/4\pi T) + \gamma_E) + \frac{1}{3(4\pi)^2}(-N_c + 8N_f\log 2)$$

$$\frac{1}{g_{E,M}^2(T)} = 2\beta_0\log\left(T/\Lambda^T_{E,M}\right)$$

$$\log\left(\Lambda_E^T/\Lambda_{\overline{\mathrm{MS}}}\right) = \gamma_E - \log(4\pi) - \frac{N_c + 8N_f(\log 2 - 1/4)}{22N_c - 4N_f}$$

$$\log\left(\Lambda_M^T/\Lambda_{\overline{\mathrm{MS}}}\right) = \gamma_E - \log(4\pi) + \frac{N_c - 8N_f\log 2}{22N_c - 4N_f}$$

$$\widehat{\mathrm{tr}}(\hat{X}^2) = 2N_c\langle X^2\rangle, X \in \mathrm{su}(N_c)$$

$$\widehat{\mathrm{tr}}(\hat{X}^2\hat{Y}^2) = 2N_c\langle X^2Y^2\rangle + 2\langle X^2\rangle\langle Y^2\rangle + 4\langle XY\rangle^2, X, Y \in \mathrm{su}(N_c)$$

$$\mathcal{L}'_{(6)}(\vec{x}) = -\frac{2}{15}\frac{\zeta(3)}{(4\pi)^4 T^3}\Big[\left(\frac{2}{3}N_c - \frac{14}{3}N_f\right)\langle F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}\rangle + (19N_c - 28N_f)\langle F^2_{\mu\mu\nu}\rangle$$

$$+(18N_c - 21N_f)\langle F^2_{0\mu\nu}\rangle + (110N_c - 140N_f)\langle A_0^2F^2_{\mu\nu}\rangle - (2N_c - 14N_f)\langle E^2_{ii}\rangle$$

$$+(4N_c - 28N_f)\langle E_iF_{ij}E_j\rangle + 110\langle A_0^2\rangle\langle F^2_{\mu\nu}\rangle + 220\langle A_0F_{\mu\nu}\rangle^2\Big]$$

$$A_0 = A_0^a t_a = -\frac{i}{2}\vec{\sigma}\cdot\vec{A}_0, F_{\mu\nu} = F^a_{\mu\nu}t_a = -\frac{i}{2}\vec{\sigma}\cdot\vec{F}_{\mu\nu}, \ldots$$

$$[t_a, t_b] = \epsilon_{abc}t_c, \mathrm{tr}(t_at_b) = -\frac{1}{2}\delta_{ab}$$

$$A_0 = -\frac{i}{2}\sigma_3\phi, \phi = \sqrt{A_0^aA_0^a}$$

$$\hat{A}_0 = A_0^aT_a, (T^a)_{bc} = f_{bac} = -\epsilon_{abc}$$

$$\mathcal{L}_{0,q}(x) = \frac{2\pi^2}{3}T^4 N_f\left(\frac{2}{15} - \frac{1}{4}(1-4\bar{v}^2)^2\right)$$

$$\mathcal{L}_{2,q}(x) = \frac{N_f}{48\pi^2}\left(2\log\left(\frac{\mu}{4\pi T}\right) - \psi\left(\frac{1}{2}+\bar{v}\right) - \psi\left(\frac{1}{2}-\bar{v}\right) - 1\right)\vec{E}_i^2 + \frac{N_f}{48\pi^2}\left(2\log\left(\frac{\mu}{4\pi T}\right) - \psi\left(\frac{1}{2}+\bar{v}\right) - \psi\left(\frac{1}{2}-\bar{v}\right)\right)\vec{B}_i^2$$

$$\bar{v} = \left(\frac{\beta\phi}{4\pi} + \frac{1}{2}\right)(\mathrm{mod}1) - \frac{1}{2}$$

$$\mathcal{L}_{0,g}(x) = \frac{\pi^2}{3}T^4\left(-\frac{1}{5} + 4\hat{v}^2(1-\hat{v})^2\right)$$

$$\mathcal{L}_{2,g}(x) = -\frac{11}{48\pi^2}\left(2\log\left(\frac{\mu}{4\pi T}\right) - \frac{1}{11} - \psi(\hat{v}) - \psi(1-\hat{v})\right)\vec{E}_{i\parallel}^2 - \frac{11}{48\pi^2}\left(\frac{12}{11}\frac{\pi T}{m} + 2\log\left(\frac{\mu}{4\pi T}\right) - \frac{1}{11} + \gamma_E - \frac{1}{2}\psi(\hat{v}) - \frac{1}{2}\psi(1-\hat{v})\right)\vec{E}_{i\perp}^2 - \frac{11}{48\pi^2}\left(2\log\left(\frac{\mu}{4\pi T}\right) + \frac{1}{11} - \psi(\hat{v}) - \psi(1-\hat{v})\right)\vec{B}_{i\parallel}^2 - \frac{11}{48\pi^2}\left(\frac{\pi T}{m} + 2\log\left(\frac{\mu}{4\pi T}\right) + \frac{1}{11} + \gamma_E - \frac{1}{2}\psi(\hat{v}) - \frac{1}{2}\psi(1-\hat{v})\right)\vec{B}_{i\perp}^2$$

$$\hat{v} = \frac{\beta\phi}{2\pi}(\mathrm{mod}1)$$

$$\vec{E}_i = \vec{E}_{i\parallel} + \vec{E}_{i\perp}, \vec{B}_i = \vec{B}_{i\parallel} + \vec{B}_{i\perp}$$

$$\mathcal{L}_{2,q}(x) = -f_{1,q}(\bar{v})\vec{E}_{i\perp}^2 - f_{2,q}(\bar{v})\vec{E}_{i\parallel}^2 - h_{1,q}(\bar{v})\vec{B}_{i\perp}^2 - h_{2,q}(\bar{v})\vec{B}_{i\parallel}^2$$
$$\mathcal{L}_{2,g}(x) = -f_{1,g}(\hat{v})\vec{E}_{i\perp}^2 - f_{2,g}(\hat{v})\vec{E}_{i\parallel}^2 - h_{1,g}(\hat{v})\vec{B}_{i\perp}^2 - h_{2,g}(\hat{v})\vec{B}_{i\parallel}^2$$

$f_{1,q} = f_{2,q} \equiv f_q, h_{1,q} = h_{2,q} \equiv h_q.$

$$A_\mu \to U^{-1}\partial_\mu U + U^{-1}A_\mu U, U(x_0,\vec{x}) = \exp\left[-i\frac{\sigma_3}{2}(\alpha(\vec{x}) + x_0 2\pi n/\beta)\right]$$

$$A_0'^3(\vec{x}) = A_0^3(\vec{x}) + 2\pi n/\beta$$
$$A_i'^1(x_0,\vec{x}) = A_i^1\cos\chi + A_i^2\sin\chi$$
$$A_i'^2(x_0,\vec{x}) = -A_i^1\sin\chi + A_i^2\cos\chi$$
$$A_i'^3(x_0,\vec{x}) = A_i^3 + \partial_i\alpha(\vec{x})$$

$$U(x_0,\vec{x}) = \exp\left[-i\frac{\sigma_3}{2}(\alpha(\vec{x}) + x_0 4\pi n/\beta)\right]$$

$$A_0 = A_0^a t_a = -\frac{i}{2}\lambda_a A_0^a, F_{\mu\nu} = F_{\mu\nu}^a t_a = -\frac{i}{2}\lambda_a F_{\mu\nu}^a, \ldots$$

$$[t_a, t_b] = f_{abc}t_c, \mathrm{tr}(t_a t_b) = -\frac{1}{2}\delta_{ab}$$

$$A_0 = -i\frac{\lambda_3}{2}\phi_3 - i\frac{\sqrt{3}}{2}\lambda_8\phi_8$$

$$\omega_1 = \exp\left(i\frac{\beta}{2}(\phi_3+\phi_8)\right), \omega_2 = \exp\left(i\frac{\beta}{2}(-\phi_3+\phi_8)\right), \omega_3 = \exp(-i\beta\phi_8)$$

$$\nu_1 = \frac{\beta}{4\pi}(\phi_3+\phi_8), \nu_2 = \frac{\beta}{4\pi}(-\phi_3+\phi_8), \nu_3 = -\frac{\beta}{2\pi}\phi_8$$

$$\mathcal{L}_{0,q}(x) = -\frac{\pi^2 T^4 N_f}{12}\left(-\frac{8}{5} + (1-4\bar{v}_1^2)^2 + (1-4\bar{v}_2^2)^2 + (1-4\bar{v}_3^2)^2\right)$$

$$\begin{aligned}\mathcal{L}_{2,q}(x) =& \frac{N_f}{24\pi^2}\left[\log\left(\frac{\mu}{4\pi T}\right) - \frac{1}{2}\right]\vec{E}_i^2 + \frac{N_f}{24\pi^2}\log\left(\frac{\mu}{4\pi T}\right)\vec{B}_i^2 \\ &- \frac{N_f}{12(4\pi)^2}\left(f^-(\nu_1) + f^-(\nu_2)\right)\left(\left(F_{\mu\nu}^1\right)^2 + \left(F_{\mu\nu}^2\right)^2 + \left(F_{\mu\nu}^3\right)^2\right) \\ &- \frac{N_f}{12(4\pi)^2}\left(f^-(\nu_1) + f^-(\nu_3)\right)\left(\left(F_{\mu\nu}^4\right)^2 + \left(F_{\mu\nu}^5\right)^2\right) \\ &- \frac{N_f}{12(4\pi)^2}\left(f^-(\nu_2) + f^-(\nu_3)\right)\left(\left(F_{\mu\nu}^6\right)^2 + \left(F_{\mu\nu}^7\right)^2\right) \\ &- \frac{N_f}{36(4\pi)^2}\left(f^-(\nu_1) + f^-(\nu_2) + 4f^-(\nu_3)\right)\left(F_{\mu\nu}^8\right)^2 \\ &- \frac{N_f}{6\sqrt{3}(4\pi)^2}\left(f^-(\nu_1) - f^-(\nu_2)\right)F_{\mu\nu}^3 F_{\mu\nu}^8\end{aligned}$$

$$f^-(\nu) = \psi\left(\frac{1}{2} + \bar{\nu}\right) + \psi\left(\frac{1}{2} - \bar{\nu}\right), \bar{\nu} = \left(\nu + \frac{1}{2}\right)(\mathrm{mod}1) - \frac{1}{2}.$$

$$\left(\hat{A}_0\right)_{ab} = (A_0^c T_c)_{ab} = -f_{abc}A_0^c = -f_{ab3}\phi_3 - f_{ab8}\sqrt{3}\phi_8$$

$$1, 1, \exp\left(\pm i\beta\phi_3\right), \exp\left(\pm i\frac{\beta}{2}(\phi_3 + 3\phi_8)\right), \exp\left(\pm i\frac{\beta}{2}(\phi_3 - 3\phi_8)\right)$$

$$\nu_{12} = \frac{\beta}{2\pi}\phi_3, \nu_{31} = \frac{\beta}{4\pi}(\phi_3 + 3\phi_8), \nu_{23} = \frac{\beta}{4\pi}(\phi_3 - 3\phi_8).$$

$$\mathcal{L}_{0,g}(x) = \frac{4}{3}\pi^2 T^4\left(-\frac{2}{15} + \hat{\nu}_{12}^2(1-\hat{\nu}_{12})^2 + \hat{\nu}_{31}^2(1-\hat{\nu}_{31})^2 + \hat{\nu}_{23}^2(1-\hat{\nu}_{23})^2\right),$$

$$\begin{aligned}\mathcal{L}_{2,g}(x) =& -\frac{1}{(4\pi)^2}\left(11\log\left(\frac{\mu}{4\pi T}\right) - \frac{1}{2}\right)\vec{E}_i^2 - \frac{1}{(4\pi)^2}\left(11\log\left(\frac{\mu}{4\pi T}\right) + \frac{1}{2}\right)\vec{B}_i^2 \\ &- \frac{T}{4\pi m}\left(\vec{E}_{i\perp}^2 + \frac{11}{12}\vec{B}_{i\perp}^2\right) \\ &+ \frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0) + f^+(\nu_{12}) + \frac{1}{2}f^+(\nu_{31}) + \frac{1}{2}f^+(\nu_{23})\right)\left(\left(F_{\mu\nu}^1\right)^2 + \left(F_{\mu\nu}^2\right)^2\right) \\ &+ \frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0) + \frac{1}{2}f^+(\nu_{12}) + f^+(\nu_{31}) + \frac{1}{2}f^+(\nu_{23})\right)\left(\left(F_{\mu\nu}^4\right)^2 + \left(F_{\mu\nu}^5\right)^2\right) \\ &+ \frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0) + \frac{1}{2}f^+(\nu_{12}) + \frac{1}{2}f^+(\nu_{31}) + f^+(\nu_{23})\right)\left(\left(F_{\mu\nu}^6\right)^2 + \left(F_{\mu\nu}^7\right)^2\right) \\ &+ \frac{1}{(4\pi)^2}\frac{11}{12}\left(2f^+(\nu_{12}) + \frac{1}{2}f^+(\nu_{31}) + \frac{1}{2}f^+(\nu_{23})\right)\left(F_{\mu\nu}^3\right)^2 \\ &+ \frac{1}{(4\pi)^2}\frac{11}{8}\left(f^+(\nu_{31}) + f^+(\nu_{23})\right)\left(F_{\mu\nu}^8\right)^2 \\ &+ \frac{1}{(4\pi)^2}\frac{11}{4\sqrt{3}}\left(f^+(\nu_{31}) - f^+(\nu_{23})\right)F_{\mu\nu}^3 F_{\mu\nu}^8\end{aligned}$$

$$f^+(\nu) = \psi(\hat{\nu}) + \psi(1-\hat{\nu})\ (\nu \notin \mathbb{Z}), \hat{\nu} = \nu(\mathrm{mod}1)$$

$$\mathcal{L}_{\text{árbol}}(x) = \frac{1}{4g^2(\mu)}\vec{F}_{\mu\nu}^2$$

$$\mathcal{L}_{2,q}(x)+\mathcal{L}_{2,g}(x)=f_{12}(\phi_3,\phi_8)((E_i^1)^2+(E_i^2)^2)+f_{45}(\phi_3,\phi_8)\left((E_i^4)^2+(E_i^5)^2\right)$$
$$+f_{67}(\phi_3,\phi_8)((E_i^6)^2+(E_i^7)^2)+f_{33}(\phi_3,\phi_8)(E_i^3)^2+f_{88}(\phi_3,\phi_8)(E_i^8)^2$$
$$+f_{38}(\phi_3,\phi_8)(E_i^3E_i^8)$$
$$+\text{(misma estructura para } B_iB_i\text{)}.$$

$$\theta=\frac{\beta}{2\pi}\phi_3,\rho=\frac{\beta}{2\pi}\phi_8$$

$$L(T)=\left\langle\frac{1}{N_c}\mathrm{tr}_c\mathcal{T}\left(e^{ig\int_0^{1/T}dx_0A_0(\vec{x},x_0)}\right)\right\rangle$$

$$L(T)=1+\frac{1}{16\pi}\frac{N_c^2-1}{N_c}g^2\frac{m_D}{T}+\frac{N_c^2-1}{32\pi^2}g^4\left(\log\frac{m_D}{2T}+\frac{3}{4}\right)+\mathcal{O}(g^5)$$

$$m_D=gT\left(N_c/3+N_f/6\right)^{1/2}$$

$$T\mathcal{L}_3(\vec{x})=m_D^2\mathrm{tr}(A_0^2)+\frac{g^4(\mu)}{4\pi^2}\left(\mathrm{tr}(A_0^2)\right)^2+\frac{g^4(\mu)}{12\pi^2}\left(N_c-N_f\right)\mathrm{tr}(A_0^4)$$
$$+\frac{g^2(\mu)}{g_E^2(T)}\mathrm{tr}([D_i,A_0]^2)+\frac{g^2(\mu)}{g_M^2(T)}\frac{1}{2}\mathrm{tr}\left(F_{ij}^2\right)+T\delta\mathcal{L}_3.$$

$$\frac{1}{g^2(\mu)}=2\beta_0\log\left(\mu/\Lambda_{\overline{\mathrm{MS}}}\right),\beta_0=\left(11N_c/3-2N_f/3\right)/(4\pi)^2$$

$$A_0(\vec{x})=\frac{g(\mu)}{g_E(T)}A_0^{\overline{\mathrm{MS}}}(\vec{x})$$

$$\mathcal{L}_3(\vec{x})=\frac{m_D^2}{T}\mathrm{tr}(A_0^2)+\frac{1}{T}\mathrm{tr}([D_i,A_0]^2)+\cdots$$
$$\frac{1}{g^2(T)}=2\beta_0\log\left(T/\Lambda_E\right)$$

$$\Lambda_E=\frac{\Lambda_{\overline{\mathrm{MS}}}}{4\pi}\exp\left(\gamma_E-\frac{N_c+8N_f(\log 2-1/4)}{22N_c-4N_f}\right)$$

$$L(T)=\frac{1}{N_c}\left\langle\mathrm{tr}e^{igA_0(\vec{x})/T}\right\rangle$$

$$L(T)=1-\frac{g^2}{2T^2}\frac{1}{N_c}\langle\mathrm{tr}(A_0^2)\rangle+\frac{g^4}{24T^4}\frac{1}{N_c}\langle\mathrm{tr}(A_0^4)\rangle+\cdots.$$

$$\left\langle A_{0,a}^2\right\rangle=(N_c^2-1)T\int\frac{d^3k}{(2\pi)^3}D_{00}(\vec{k})$$

$$D_{00}^{\mathrm{Pert}}(\vec{k})=\frac{1}{\vec{k}^2+m_D^2}$$

$$\left\langle A_{0,a}^2\right\rangle^{\mathrm{Pert}}=-(N_c^2-1)\frac{Tm_D}{4\pi}$$

$$\mathcal{L}_3^{\mathrm{ren}}(\vec{x})=\frac{1}{2}\mathrm{tr}\left(F_{ij}^2\right)+\mathrm{tr}([D_i,\mathcal{A}_0]^2)+m_3^2\mathrm{tr}(\mathcal{A}_0^2)+\lambda_1\left(\mathrm{tr}(\mathcal{A}_0^2)\right)^2+\lambda_2\mathrm{tr}(\mathcal{A}_0^4)$$
$$D_i=\partial_i-ig_3\mathcal{A}_i$$

$$\epsilon(g_3, m_3, \lambda_1) = \sum_{\ell\geq 1}\sum_{k=0}^{\ell-1} f_{\ell k} m_3^{4-\ell} g_3^{2k} \lambda_1^{\ell-k-1}$$

$$\frac{g^2}{T^2}\langle \mathrm{tr}(A_0^2)\rangle \sim \frac{g^2}{T}\frac{\partial\epsilon(g_3, m_3, \lambda_1)}{\partial m_3^2} \sim \sum_{\ell\geq 1}\sum_{n=\ell+2}^{3\ell} g^n,$$
$$\frac{g^4}{T^4}\langle \mathrm{tr}(A_0^4)\rangle \sim \frac{g^4}{T^2}\frac{\partial\epsilon(g_3, m_3, \lambda_1)}{\partial \lambda_1} \sim \sum_{\ell\geq 2}\sum_{n=\ell+4}^{3\ell} g^n.$$

$$\delta\mathcal{L}_3 = \frac{g^2}{T^2}\mathrm{tr}\left(\left[D_i, F_{\mu\nu}\right]^2\right) + \frac{g^3}{T^{3/2}}\mathrm{tr}\left(F_{\mu\nu}^3\right) + \frac{g^4}{T}\mathrm{tr}\left(A_0^2 F_{\mu\nu}^2\right)$$

$$\mathcal{O}(g^5) = \frac{(N_c^2-1)g^4 T m_D}{384\pi^3}\left[-\frac{m_D^2}{(gT)^2}\left(9Q + 3c_m + 4N_f + 2N_c(6\zeta - 7)\right) + \frac{N_c^2}{4}(89 + 4\pi^2 - 44\log\, 2)\right]$$

$$Q = \frac{22}{3}N_c \log\frac{\mu}{\mu_T} - \frac{4}{3}N_f \log\frac{4\mu}{\mu_T}, c_m = \frac{10N_c^2 + 2N_f^2 + 9N_f/N_c}{6N_c + 3N_f}$$

$$L = \exp\left[-\frac{g^2\langle A_{0,a}^2\rangle}{4N_c T^2}\right]$$

$$\langle A_{0,a}^2\rangle^{\mathrm{Pert}} = -\frac{N_c^2-1}{4\pi}m_D T - \frac{N_c(N_c^2-1)}{8\pi^2}g^2T^2\left(\log\frac{m_D}{2T} + \frac{3}{4}\right) + \mathcal{O}(g^3)$$

$$D_{00}(\vec{k}) = D_{00}^{\mathrm{Pert}}(\vec{k}) + D_{00}^{\mathrm{No\ Pert}}(\vec{k})$$

$$D_{00}^{\mathrm{No\ Pert}}(\vec{k}) = \frac{m_G^2}{\left(\vec{k}^2 + m_D^2\right)^2}$$

$$\langle A_{0,a}^2\rangle^{\mathrm{No\ Pert}} = \frac{(N_c^2-1)T m_G^2}{8\pi m_D}$$

$$D_{00}^{\mathrm{No\ Pert}}(\vec{k}) = \frac{8\pi}{N_c^2-1}\frac{m_D}{T}\frac{\langle A_{0,a}^2\rangle^{\mathrm{No\ Pert}}}{\left(\vec{k}^2 + m_D^2\right)^2}$$

$$-2\log\, L = \frac{g^2\langle A_{0,a}^2\rangle^{\mathrm{Pert}}}{2N_c T^2} + \frac{g^2\langle A_{0,a}^2\rangle^{\mathrm{No\ Pert}}}{2N_c T^2}$$

$$-2\log\, L = a + b\left(\frac{T_c}{T}\right)^2$$

$$e^{-F_1(\vec{x},T)/T + C(T)} = \frac{1}{N_c}\langle \mathrm{Tr}\Omega^{\mathrm{desn}}(\vec{x})\Omega^{\dagger\mathrm{desn}}(0)\rangle,$$

$$e^{-F_8(\vec{x},T)/T + C(T)} = \frac{1}{N_c^2-1}\langle \mathrm{Tr}\Omega^{\mathrm{desn}}(\vec{x})\mathrm{Tr}\Omega^{\dagger\,\mathrm{desn}}(0)\rangle - \frac{1}{N_c(N_c^2-1)}\langle \mathrm{Tr}\Omega^{\mathrm{desn}}(\vec{x})\Omega^{\dagger\,\mathrm{desn}}(0)\rangle$$

$$\frac{1}{N_c}\langle \mathrm{Tr}\Omega_{\mathcal{R}}(\vec{x})\Omega_{\mathcal{R}}^{\dagger}(0)\rangle = \frac{1}{N_c}e^{-C(T)}\langle \mathrm{Tr}\Omega^{\mathrm{desn}}(\vec{x})\Omega^{\dagger\mathrm{desn}}(0)\rangle = e^{-F_1(r,T)/T} \underset{r\to\infty}{\longrightarrow} L^2(T)$$

$$a = -\frac{1}{8\pi}\frac{N_c^2-1}{N_c}g^2\frac{m_D}{T} - \frac{N_c^2-1}{16\pi^2}g^4\left(\log\frac{m_D}{2T}+\frac{3}{4}\right)+\mathcal{O}(g^5),$$

$$g^2\langle A_{0,a}^2\rangle^{\text{No Pert}} = 2N_cT_c^2b.$$

$$-2\log L = a^{\text{NLO}} + b\left(\frac{T_c}{T}\right)^2$$

N_τ	b	$g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}$ (GeV)^2	χ^2/ DOF
4	2.20(6)	$(0{,}98(2))^2$	0.75
8	2.14(4)	$(0{,}97(1))^2$	1.43

N_τ	a	b	$g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}$ (GeV)^2	χ^2/DOF
4	−0.27(5)	1.81(13)	$(0{,}89(3))^2$	1.07
8	−0.23(1)	1.72(5)	$(0{,}87(2))^2$	0.45

$$a^{\text{NLO}} = -0{,}22(1)\ (T = 6T_c)$$

N_τ	a	b	c	χ^2/DOF
8	a^{NLO}	2.18(20)	-0.04 ± 0.24	1.89
8	−0.22(2)	1.61(24)	0.13 ± 0.28	0.42

N_τ	a	b	$g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}$ (GeV)^2	χ^2/DOF
4	a^{NLO}	2.99(12)	$(0{,}86(2))^2$	1.87
4	−0.31(6)	2.19(13)	$(0{,}73(3))^2$	0.25

$$a^{\text{NLO}} = -0{,}35(2)\ (T = 6T_c)$$

N_τ	a	b	c	χ^2/DOF

4	a^{NLO}	$2.44(21)$	$1.07(19)$	12.8

$$\langle \mathcal{P}_{\mathcal{R}}(\vec{x})\rangle = \frac{1}{\mathcal{Z}_{\mathcal{R}}}\langle \mathcal{P}^{\text{desn}}(\vec{x})\rangle, \mathcal{Z}_{\mathcal{R}} = \exp\left(-\frac{m_{\mathcal{R}}^{\text{div}}}{T}\right),$$

$$m_{\mathcal{R}}^{\text{div}} \sim \frac{1}{a}$$

$$-\log L^{\text{desn}}(T) = f^{\text{div}} N_\tau + f^{\text{ren}} + f^{\text{lat}} N_\tau^{-1}$$

$$L^{\text{desn}}(T) = \frac{1}{N_c}\langle \text{Tr}\Omega^{\text{desn}}(\vec{x})\rangle, L(T) = \frac{1}{N_c}\langle \text{Tr}\Omega_{\mathcal{R}}(\vec{x})\rangle = e^{-f^{\text{ren}}}$$

$$-2\log L = a + b\left(\frac{T_c}{T}\right)^2 + \delta a_{-1}\frac{T_c}{T} + \delta a + \delta a_1 \frac{T}{T_c}$$

a	δa	$a + \delta a$	b	δa_{-1}	δa_1	χ^2/DOF
a^{NLO}	1.8 ± 1.8	-	1.4 ± 2.6	-1.0 ± 3.8	-0.29 ± 0.26	0.0349
-	-	1.6 ± 1.8	1.3 ± 2.6	-1.4 ± 3.8	-0.28 ± 0.26	0.0350

Referencia	$g^2\langle A_{\mu,a}^2\rangle(\text{GeV})^2$
gluón	$(2{,}4 \pm 0{,}6)^2$
vértice simétrico	$(3{,}6 \pm 1{,}2)^2$
propagador del quark	$(2{,}1 \pm 0{,}1)^2$
cola del propagador del quark	$(3{,}0 - 3{,}4)^2$

$$g^2\langle A_{0,a}^2\rangle^{\text{No Pert}} = (0{,}93(7)\text{GeV})^2$$

$$F_1(\vec{x},T) = -\frac{N_c^2 - 1}{2N_c} g^2 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} D_{00}(\vec{k})$$

$$F_1(r,T) = -\frac{N_c^2-1}{2N_c}\left(\frac{g^2}{4\pi r} + \frac{1}{N_c^2-1}\frac{g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}}{T}\right)e^{-m_D r} - \frac{N_c^2-1}{2N_c}\frac{g^2 m_D}{4\pi} + \frac{g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}}{2N_c T}.$$

$$F_\infty(T) \equiv F_1(r \to \infty, T) = -2T\log L(T) = -\frac{N_c^2-1}{2N_c}\frac{g^2 m_D}{4\pi} + \frac{g^2\langle A_{0,a}^2\rangle^{\text{No Pert}}}{2N_c T} + \mathcal{O}(g^4)$$

$$F_1(r,T) \overset{T\to 0}{\sim} -\frac{N_c^2-1}{2N_c}\frac{g^2}{4\pi r} + \sigma r \equiv V_{\bar{q}q}(r)$$

$$\sigma = \left(\frac{N_c}{3}+\frac{N_f}{6}\right)^{1/2} \frac{g^3\langle A_{0,a}^2\rangle_{T=0}}{2N_c}$$

$$\bar{\alpha}_s(r,T) = \lambda\alpha_s^{\text{Pert}}(r,T), \lambda > 1$$

$$F_{\text{fit}}(r,T) = -\frac{4\alpha(T)}{3r}\exp\left(-\sqrt{4\pi\bar{\alpha}(T)}rT\right) + b(T)$$

$$F_\infty(T) = V_{\bar{q}q}(r)\big|_{r=1/m_D}$$

$$\alpha_s(r) \equiv \alpha_s(r, T=0), \alpha_s(T) \equiv \alpha_s(r\to\infty, T)$$

$$\rho F_\infty(T)/T = rV_{\bar{q}q}(r)\big|_{r=\gamma/T}, \rho = \gamma B$$

$$BF_\infty(T) = V_{\bar{q}q}(r)\big|_{r=\gamma/T}$$

$$B = \left(\frac{\alpha_s(r)}{\alpha_s(T)}\right)^{3/4}, \gamma = \frac{1}{\sqrt{4\pi\left(N_c/3+N_f/6\right)}}\left(\frac{\alpha_s(r)}{\alpha_s(T)^3}\right)^{1/4}$$

$$\alpha_s(T)\langle A_{0,a}^2\rangle_T^{\text{No Pert}} = \alpha_s(r)\langle A_{0,a}^2\rangle_{T=0}.$$

$$g(x_0,\vec{x}) = g(x_0+\beta,\vec{x})$$

$$g(x_0) = e^{i2\pi x_0\Lambda/\beta}$$

$$A_0 \to A_0 + \frac{2\pi}{\beta}\Lambda$$

$$\Omega(x) \to g^{-1}(x)\Omega(x)g(x)$$

$$g(x_0+\beta,\vec{x}) = zg(x_0,\vec{x}), z^{N_c} = 1$$

$$g(x_0) = e^{i2\pi x_0\Lambda/N_c\beta}$$

$$A_0 \to A_0 + \frac{2\pi}{N_c\beta}\Lambda, \Omega \to z\Omega$$

$$e^{-\beta F_q(x)} = \frac{1}{N_c}\langle \text{tr}_c\Omega(x)\rangle$$

$$\langle \text{tr}_c\Omega(x)\rangle = z\langle \text{tr}_c\Omega(x)\rangle$$

$$\langle \text{tr}_c\Omega^n(x)\rangle = 0 \ \text{ para } \ n \neq mN_c, m \in \mathbb{Z}$$

$$q(\beta,\vec{x}) \to g(\beta,\vec{x})q(\beta,\vec{x}) = -zg(0,\vec{x})q(0,\vec{x})$$

$$\langle \bar{q}(n\beta)q(0)\rangle = 0 \ \text{ para } \ n \neq mN_c, m \in \mathbb{Z}$$

$$\mathcal{L}_{\text{NJL}} = \bar{q}(\partial\!\!\!/ + \hat{m}_0)q + \frac{1}{2a_s^2}\sum_{a=0}^{N_f^2-1}\left((\bar{q}\lambda_a q)^2 + (\bar{q}\lambda_a i\gamma_5 q)^2\right) + \frac{1}{2a_v^2}\sum_{a=0}^{N_f^2-1}\left(\left(\bar{q}\lambda_a\gamma_\mu q\right)^2 + \left(\bar{q}\lambda_a\gamma_\mu\gamma_5 q\right)^2\right)$$

$$Z_{\mathrm{NJL}}[s,p,v,a,\eta,\bar{\eta}] = \int \mathcal{D}\bar{q}\mathcal{D}q\exp\left[-\int d^4x\left(\mathcal{L}_{\mathrm{NJL}} + \bar{q}(\not{v} + \not{a}\gamma_5 + s + i\gamma_5 p)q + \bar{\eta}q + \bar{q}\eta\right)\right]$$

$$s = \sum_{a=0}^{N_f^2-1} s_a \frac{\lambda_a}{2}, \ldots$$

$$Z_{\mathrm{NJL}}[s,p,v,a,\eta,\bar{\eta}] = \int \mathcal{D}\bar{q}\mathcal{D} \quad q\mathcal{D}S\mathcal{D}P\mathcal{D}V\mathcal{D}A\exp\left[-\int d^4x(\bar{q}(\not{\partial} + \mathcal{V} + \mathcal{A}\gamma_5 + \mathcal{S} + i\gamma_5\mathcal{P})q + \frac{a_s^2}{4}\mathrm{tr}((S-\hat{m}_0)^2 + P^2) - \frac{a_v^2}{4}\mathrm{tr}\left(V_\mu^2 + A_\mu^2\right) + \bar{\eta}q + \bar{q}\eta\right)\right]$$

$$Z_{\mathrm{NJL}}[s,p,v,a,\eta,\bar{\eta}] = \int \mathcal{D}S\mathcal{D}P\mathcal{D}V\mathcal{D}A\mathrm{Det}(\mathbf{D})^{N_c}\exp\left(\langle\bar{\eta}|\mathbf{D}^{-1}|\eta\rangle\right)$$
$$\exp\left[-\int d^4x\left(\frac{a_s^2}{4}\mathrm{tr}((S-\hat{m}_0)^2 + P^2) - \frac{a_v^2}{4}\mathrm{tr}\left(V_\mu^2 + A_\mu^2\right)\right)\right]$$

$$\mathbf{D} = \not{\partial} + \mathcal{Y} + \mathcal{A}\gamma_5 + \mathcal{S} + i\gamma_5\mathcal{P}$$

$$\mathbf{D} = \not{P}_{\mathcal{V}} + \mathcal{A}\gamma_5 + MU^{\gamma_5}$$

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

$$\Gamma_{\mathrm{NJL}} = \Gamma_q[\mathbf{D}] + \Gamma_m$$

$$\Gamma_q[\mathbf{D}] = -N_c\mathrm{Tr}\log(\mathbf{D})$$

$$\Gamma_m = \int d^4x\left\{\frac{a_s^2}{4}\mathrm{tr}(S^2 + P^2) - \frac{a_s^2}{2}\mathrm{tr}(\hat{m}_0 S) + \frac{a_s^2}{4}\mathrm{tr}(\hat{m}_0^2) - \frac{a_v^2}{4}\mathrm{tr}\left(V_\mu^2 + A_\mu^2\right)\right\}$$

$$\mathbf{D}_5[\mathcal{S},\mathcal{P},\mathcal{V},\mathcal{A}] = \gamma_5\mathbf{D}[\mathcal{S},-\mathcal{P},\mathcal{V},-\mathcal{A}]\gamma_5$$

$$\Gamma_q^+[\mathbf{D}] = -\frac{N_c}{2}\mathrm{Tr}\sum_i c_i\log\left(\mathbf{D}_5\mathbf{D} + \Lambda_i^2\right)$$

$$\Gamma_q^+[\mathbf{D}] = \frac{N_c}{2}\int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\mathrm{Tr}e^{-\tau\mathbf{D}_5\mathbf{D}}$$

$$\phi(\tau) = \sum_i c_i e^{-\tau\Lambda_i^2}$$

$$\left.\frac{\delta\Gamma_{\mathrm{NJL}}^+[S]}{\delta S(x)}\right|_{S(x)=\Pi} = \frac{a_s^2}{2}\mathrm{tr}(\Pi - \hat{m}_0) - \frac{N_c}{2}\mathrm{Tr}\left((\mathbf{D}_5\mathbf{D})^{-1}\frac{\delta(\mathbf{D}_5\mathbf{D})}{\delta S(x)}\right)_{S(x)=\Pi} = 0$$

$$a_s^2(\Pi - \hat{m}_0) - 8N_c\Pi g(\Pi) = 0$$

$$g(\Pi) = \int \frac{d^4p}{(2\pi)^4}\int_0^\infty d\tau\phi(\tau)e^{-\tau(p^2+\Pi^2)}$$

$$\langle\bar{q}q\rangle = -\frac{a_s^2}{2}\mathrm{tr}(\Pi - \hat{m}_0)$$

$$S(p) = \int d^4 x e^{-px} \langle 0|T\{q(x)\bar{q}(0)\}|0\rangle$$

$$S(p) = \int_{\mathcal{C}} d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

$$S(p) = A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)}$$

$$A(p) = \int_{\mathcal{C}} d\omega \frac{\rho(\omega)}{p^2 - \omega^2}, B(p) = \int_{\mathcal{C}} d\omega \frac{\rho(\omega)\omega}{p^2 - \omega^2}$$

$$M(p) = \frac{B(p)}{A(p)}, Z(p) = (p^2 - M^2(p))A(p)$$

$$\rho_n = \int_{\mathcal{C}} d\omega \omega^n \rho(\omega), \rho_n' = \int_{\mathcal{C}} d\omega \log(\omega^2/\mu^2)\omega^n \rho(\omega), n \in \mathbb{Z}$$

$$\langle \bar{q}q \rangle = -N_c \int_{\mathcal{C}} d\omega \rho(\omega) \int \frac{d^4 p}{(2\pi)^4} \mathrm{tr}_{\mathrm{Dirac}} \frac{1}{p - \omega}$$

$$\int d^4 p \longrightarrow 4\pi \int dp_0 \int_0^{\Lambda} dp p^2, p = |\vec{p}|$$

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2} \int_{\mathcal{C}} d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{4\Lambda^2}\right) + \omega^2\right]$$

$$\Gamma_{\mathrm{SQM}} = -N_c \int d^4 x \int_{\mathcal{C}} d\omega \rho(\omega) \mathrm{tr} \log(\not{D}_V + A\gamma_5 + \omega U^{\gamma_5})$$

$$\int \frac{dk_0}{2\pi} F(k_0, \vec{k}) \to iT \sum_{n=-\infty}^{\infty} F(iw_n, \vec{k})$$

$$\langle \bar{q}q \rangle = -iN_c \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{tr}_{\mathrm{Dirac}} S(x)\Big|_{x_0 = in\beta} = 4MT\mathrm{tr}_c \sum_{\omega_n} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{k}^2 + M^2}$$

$$\langle \bar{q}q \rangle_T = \langle \bar{q}q \rangle_{T=0} - 2\frac{N_c M^2 T}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} K_1(nM/T)$$

$$\overset{\text{T pequeño}}{\sim} \langle \bar{q}q \rangle_{T=0} - \frac{N_c}{2} \sum_{n=1}^{\infty} (-1)^n \left(\frac{2MT}{n\pi}\right)^{3/2} e^{-nM/T},$$

$$S(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{\not{k} - M} = (i\not{\partial} + M) \frac{M^2}{4\pi^2 i} \frac{K_1(\sqrt{-M^2 x^2})}{\sqrt{-M^2 x^2}}$$

$$S(\vec{x}, i\beta) \overset{\text{T}}{\sim} \text{pequeño } e^{-M/T}$$

$$\langle \bar{q}q \rangle_T = \sum_{n=-\infty}^{\infty} (-1)^n \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = in\beta}$$

$$\langle \bar{q}q \rangle_T \to \sum_{n=-\infty}^{\infty} (-z)^n \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = in\beta}$$

$$\langle \bar{q}q \rangle_T |_{\text{singlete}} = \sum_{n=-\infty}^{\infty} (-1)^n \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = iN_c n\beta}$$

$$\langle \bar{q}q \rangle_T |_{\text{TQP}} = \langle \bar{q}q \rangle_{T=0} \left(1 - \frac{T^2}{8 f_\pi^2} - \frac{T^4}{384 f_\pi^4} + \cdots \right)$$

$$\mathbf{D} = \mathbb{P}_{\mathcal{V}} + \mathcal{A}^f \gamma_5 + M U^{\gamma_5},$$

$$\Omega_f(x_0, \vec{x}) = \mathcal{T}\exp\left(- \int_{x_0}^{x_0+\beta} dx_0' \left(\mathcal{V}_0^f(x_0', \vec{x}) + \gamma_5 \mathcal{A}_0^f(x_0', \vec{x}) \right) \right)$$

$$\Omega_{R,L}(x_0, \vec{x}) = \mathcal{T}\exp\left(- \int_{x_0}^{x_0+\beta} dx_0' \left(\mathcal{V}_0^f(x_0', \vec{x}) \pm \mathcal{A}_0^f(x_0', \vec{x}) \right) \right)$$

$$\Omega_c(x_0, \vec{x}) = \mathcal{T}\exp\left(-g \int_{x_0}^{x_0+\beta} dx_0' \mathcal{V}_0^c(x_0', \vec{x}) \right)$$

$$\mathcal{L}(x) = \sum_n \mathrm{tr}[f_n(\Omega(x)) \mathcal{O}_n(x)]$$

$$\widehat{\omega}_n = 2\pi T(n + 1/2 + \hat{\nu}), \hat{\nu} = (2\pi i)^{-1} \log\, \Omega$$

$$\tilde{F}(x; x) \to \sum_{n=-\infty}^{\infty} (-\Omega(\vec{x}))^n \tilde{F}(\vec{x}, x_0 + in\beta; \vec{x}, x_0)$$

$$\partial_0 \to \partial_0 + g\mathcal{V}_0^c$$

$$Z = \int \mathcal{D}U \mathcal{D}\Omega e^{-\Gamma_G[\Omega]} e^{-\Gamma_Q[U,\Omega]}$$

$$\left\langle \frac{1}{N_c} \mathrm{tr}_c f(\Omega) \right\rangle = \int_{\mathrm{SU}(N_c)} \mathcal{D}\Omega \rho(\Omega) \frac{1}{N_c} \sum_{j=1}^{N_c} f\left(e^{i\phi_j}\right) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \hat{\rho}(\phi) f\left(e^{i\phi}\right)$$

$$\hat{\rho}(\phi) := \int_{\mathrm{SU}(N_c)} \mathcal{D}\Omega \rho(\Omega) \frac{1}{N_c} \sum_{j=1}^{N_c} 2\pi\delta\left(\phi - \phi_j\right)$$

$$\hat{\rho}(\phi) = 1 - \frac{2(-1)^{N_c}}{N_c} \cos\,(N_c \phi)$$

$$\langle \mathrm{tr}_c(-\Omega)^n \rangle_{\mathrm{SU}(N_c)} = \begin{cases} N_c, & n = 0 \\ -1, & n = \pm N_c \\ 0, & \text{otro caso} \end{cases}$$

$$\langle \bar{q}q \rangle_T = \sum_{n=-\infty}^{\infty} \frac{1}{N_c} \langle \mathrm{tr}_c(-\Omega)^n \rangle \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = in\beta}$$

$$\langle\bar{q}q\rangle_T = \langle\bar{q}q\rangle_{T=0} + \frac{2M^2T}{\pi^2N_c}K_1(N_cM/T) + \cdots \overset{\text{T pequeño}}{\sim} \langle\bar{q}q\rangle_{T=0} + 4\left(\frac{MT}{2\pi N_c}\right)^{3/2} e^{-N_cM/T}$$

$$\mathcal{L}_q^{*(0)} = \frac{2N_f}{(4\pi)^2}\langle \mathrm{tr}_c\mathcal{J}_{-2}(\Lambda, M, \hat{v})\rangle$$

$$\mathcal{L}_q^{*(2)} = \frac{f_\pi^{*2}}{4}\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}_\mu U - (\chi^\dagger U + \chi U^\dagger)\right)$$

$$\begin{aligned}
\mathcal{L}_q^{*(4)} = & -L_1^*\left(\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}^\mu U\right)\right)^2 - L_2^*\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}_\nu U\right)\mathrm{tr}_f\left(\hat{D}^\mu U^\dagger \hat{D}^\nu U\right) \\
& -L_3^*\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}^\mu U \hat{D}_\nu U^\dagger \hat{D}^\nu U\right) - \bar{L}_3^*\mathrm{tr}_f\left(\hat{D}_0 U^\dagger \hat{D}^0 U \hat{D}_\mu U^\dagger \hat{D}^\mu U\right) \\
& +L_4^*\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}^\mu U\right)\mathrm{tr}_f(\chi^\dagger U + \chi U^\dagger) \\
& +L_5^*\mathrm{tr}_f\left(\hat{D}_\mu U^\dagger \hat{D}^\mu U(\chi^\dagger U + U^\dagger\chi)\right) + \bar{L}_5^*\mathrm{tr}_f\left(\hat{D}_0 U^\dagger \hat{D}^0 U(\chi^\dagger U + U^\dagger\chi)\right) \\
& +\bar{L}_5^{*\prime}\mathrm{tr}_f\left(\hat{D}_0\hat{D}^0 U^\dagger\chi + \hat{D}_0\hat{D}^0 U\chi^\dagger\right) - L_6^*\left(\mathrm{tr}_f(\chi^\dagger U + \chi U^\dagger)\right)^2 - L_7^*\left(\mathrm{tr}_f(\chi^\dagger U - \chi U^\dagger)\right)^2 \\
& +\bar{L}^{*\prime}\mathrm{tr}_f\left(U^\dagger \hat{D}_0\hat{D}^0 U - U\hat{D}_0\hat{D}^0 U^\dagger\right)\mathrm{tr}_f(\chi^\dagger U - \chi U^\dagger) \\
& -L_8^*\mathrm{tr}_f(\chi^\dagger U\chi^\dagger U + \chi U^\dagger \bar{\chi} U^\dagger) \\
& -L_9^*\mathrm{tr}_f\left(F_{\mu\nu}^R \hat{D}^\mu U^\dagger \hat{D}^\nu U + F_{\mu\nu}^L \hat{D}^\mu U \hat{D}^\nu U^\dagger\right) \\
& -\bar{L}_9^*\mathrm{tr}_f\left(E_i^R\left(\hat{D}^0 U^\dagger \hat{D}^i U - \hat{D}^i U^\dagger \hat{D}^0 U\right) + E_i^L\left(\hat{D}^0 U \hat{D}^i U^\dagger - \hat{D}^i U \hat{D}^0 U^\dagger\right)\right) \\
& -\bar{L}_9^{*\prime}\mathrm{tr}_f\left(\hat{D}_0 E_i^R U^\dagger \hat{D}^i U + \hat{D}_0 E_i^L U \hat{D}^i U^\dagger\right) \\
& +L_{10}^*\mathrm{tr}_f\left(U^\dagger F_{\mu\nu}^L U F^{\mu\nu R}\right) \\
& +H_1^*\mathrm{tr}_f\left(\left(F_{\mu\nu}^R\right)^2 + \left(F_{\mu\nu}^L\right)^2\right) + \bar{H}_1^*\mathrm{tr}_f((E_i^R)^2 + (E_i^L)^2) - H_2^*\mathrm{tr}_f(\chi^\dagger\chi)
\end{aligned}$$

$$\begin{aligned}
\hat{D}_\mu U &= D_\mu^L U - U D_\mu^R = \partial_\mu U + l_\mu U - U r_\mu \\
F_{\mu\nu}^R &= \left[D_\mu^R, D_\nu^R\right] = \partial_\mu r_\nu - \partial_\nu r_\mu + \left[r_\mu, r_\nu\right] \\
F_{\mu\nu}^L &= \left[D_\mu^L, D_\nu^L\right] = \partial_\mu l_\nu - \partial_\nu l_\mu + \left[l_\mu, l_\nu\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_q^{*(0)} &= \frac{2N_f}{(4\pi)^2}\langle \mathrm{tr}_c\mathcal{J}_{-2}\rangle, f_\pi^{*2} = \frac{M^2}{4\pi^2}\langle \mathrm{tr}_c\mathcal{J}_0\rangle, f_\pi^{*2}B_0^* = \frac{M}{4\pi^2}\langle \mathrm{tr}_c\mathcal{J}_{-1}\rangle \\
L_1^* &= \frac{M^4}{24(4\pi)^2}\langle \mathrm{tr}_c\mathcal{J}_2\rangle, L_2^* = 2L_1^*, L_3^* = -8L_1^* + \frac{1}{2}L_9^* \\
\overline{L_3^*} &= -\frac{M^2}{6(4\pi)^2}\langle \mathrm{tr}_c\overline{\mathcal{J}}_1\rangle, L_4^* = 0, L_5^* = \frac{M}{2B_0^*}\left(\frac{f_\pi^{*2}}{4M^2} - 3L_9^*\right) \\
\overline{L_5^*} &= \frac{1}{2}\overline{L_3^*}, \overline{L_5^{*\prime}} = \frac{1}{2}\overline{L_3^*}, L_6^* = 0, L_7^* = \frac{1}{8N_f}\left(-\frac{f_\pi^{*2}}{2B_0^*M} + L_9^*\right) \\
\overline{L^{*\prime}} &= -\frac{1}{4N_f}\bar{L}_3^*, L_8^* = \frac{1}{16B_0^*}\left(\frac{1}{M} - \frac{1}{B_0^*}\right)f_\pi^{*2} - \frac{1}{8}L_9^* \\
L_9^* &= \frac{M^2}{3(4\pi)^2}\langle \mathrm{tr}_c\mathcal{J}_1\rangle, \overline{L_9^*} = -\bar{L}_3^*, \overline{L_9^{*\prime}} = -\bar{L}_3^*, L_{10}^* = -\frac{1}{2}L_9^* \\
H_1^* &= -\frac{f_\pi^{*2}}{24M^2} + \frac{1}{4}L_9^*, \overline{H_1^*} = -\frac{1}{6(4\pi)^2}\langle \mathrm{tr}_c\overline{\mathcal{J}}_0\rangle, H_2^* = -\frac{f_\pi^{*2}}{8B_0^{*2}} + \frac{1}{4}L_9^*
\end{aligned}$$

$$\mathcal{L}_q^{*(0)} = \frac{2N_f}{(4\pi)^2}\langle \text{tr}_c\mathcal{J}_{-2}\rangle, f_\pi^{*2} = \frac{1}{4\pi^2}\langle \omega^2\text{tr}_c\mathcal{J}_0\rangle, f_\pi^{*2}B_0^* = \frac{1}{4\pi^2}\langle \omega\text{tr}_c\mathcal{J}_{-1}\rangle$$

$$L_1^* = \frac{1}{24(4\pi)^2}\langle \omega^4\text{tr}_c\mathcal{J}_2\rangle, L_9^* = \frac{1}{3(4\pi)^2}\langle \omega^2\text{tr}_c\mathcal{J}_1\rangle, \overline{L_3^*} = -\frac{1}{6(4\pi)^2}\langle \omega^2\text{tr}_c\overline{\mathcal{J}}_1\rangle$$

$$L_5^* = \frac{1}{2(4\pi)^2B_0^*}(\langle \omega\text{tr}_c\mathcal{J}_0\rangle - \langle \omega^3\text{tr}_c\mathcal{J}_1\rangle)$$

$$L_7^* = \frac{1}{2(4\pi)^2N_f}\left(-\frac{1}{2B_0^*}\langle \omega\text{tr}_c\mathcal{J}_0\rangle + 4\pi^2L_9^*\right)$$

$$L_8^* = \frac{1}{4(4\pi)^2B_0^*}\langle \omega\text{tr}_c\mathcal{J}_0\rangle - \frac{f_\pi^{*2}}{16B_0^{*2}} - \frac{1}{8}L_9^*$$

$$H_1^* = -\frac{1}{6(4\pi)^2}\langle \text{tr}_c\mathcal{J}_0\rangle + \frac{1}{4}L_9^*$$

$$\bar{H}_1^* = -\frac{1}{6(4\pi)^2}\langle \text{tr}_c\overline{\mathcal{J}}_0\rangle$$

$$H_2^* = \frac{1}{2(4\pi)^2B_0^*}\left(\frac{1}{B_0^*}\langle \text{tr}_c\mathcal{J}_{-1}\rangle - \langle \omega\text{tr}_c\mathcal{J}_0\rangle\right) - \frac{f_\pi^{*2}}{8B_0^{*2}} + \frac{1}{4}L_9^*$$

$$L_7^* = -\frac{1}{N_f}\left(\frac{f_\pi^{*2}}{16B_0^{*2}} + L_8^*\right)$$

$$\langle \text{tr}_c\mathcal{J}_{-2}\rangle = -\frac{N_c}{2}\rho_4' - \frac{2M_V^4}{3x_V^4}(48 + 24x_V + 6x_V^2 + x_V^3)e^{-x_V/2}$$

$$\langle \text{tr}_c\mathcal{J}_{-1}\rangle = N_c\rho_2' - \frac{2M_V^2}{3x_V^2}(12 + 6x_V + x_V^2)e^{-x_V/2}$$

$$\langle \omega\text{tr}_c\mathcal{J}_{-1}\rangle = \rho_3'\left(N_c - 2e^{-x_S/2}\right)$$

$$\langle \text{tr}_c\mathcal{J}_0\rangle = -N_c(\rho_0 + \rho_0') + 2\gamma_E - 4\log(4) + 4\log(x_V) - 2\psi(5/2) - \frac{x_V^5}{1800}\,{}_1F_2\left[\left\{\frac{5}{2}\right\},\left\{\frac{7}{2},\frac{7}{2}\right\},\left(\frac{x_V}{4}\right)^2\right] - \frac{x_V^2}{12}\,{}_2F_3\left[\{1,1\},\left\{-\frac{1}{2},2,2\right\},\left(\frac{x_V}{4}\right)^2\right]$$

$$\langle \omega\text{tr}_c\mathcal{J}_0\rangle = -N_c\rho_1' - \frac{2\rho_3'}{M_S^2}(2 + x_S)e^{-x_S/2}$$

$$\langle \omega^2\text{tr}_c\mathcal{J}_0\rangle = -N_c\rho_2' - \frac{M_V^2}{6}(2 + x_V)e^{-x_V/2}$$

$$\langle \omega^2\text{tr}_c\mathcal{J}_1\rangle = N_c\rho_0 - \frac{1}{6}(12 + 6x_V + x_V^2)e^{-x_V/2}$$

$$\langle \omega^3\text{tr}_c\mathcal{J}_1\rangle = -\frac{\rho_3'x_S^2}{2M_S^2}e^{-x_S/2}$$

$$\langle \omega^4\text{tr}_c\mathcal{J}_2\rangle = N_c\rho_0 - \frac{1}{24}(48 + 24x_V + 6x_V^2 + x_V^3)e^{-x_V/2}$$

$$\langle \text{tr}_c\overline{\mathcal{J}}_0\rangle = -\frac{1}{3}(12 + 6x_V + x_V^2)e^{-x_V/2}$$

$$\langle \omega^2\text{tr}_c\overline{\mathcal{J}}_1\rangle = -\frac{x_V^2}{12}(2 + x_V)e^{-x_V/2}$$

$$x_V := N_c\beta M_V, x_S := N_c\beta M_S,$$

$$ב = \sum_{w^{(1)},w^{(2)},w^{(3)}} S(w^{(1)}) \otimes S(w^{(1)}) \otimes S(w^{(2)}) \otimes S(w^{(3)}) \otimes S(w^{(1)} + w^{(3)} - w^{(2)})$$

$$\aleph = \sum_{n_1,n_2,n_3} \langle \Omega^{n_1+n_2+n_3} \rangle \int_{-\infty}^{\infty} d\tau_1 d\tau_3 S(\tau_1) \otimes S(-\tau_1 - \tau_3 + n_1\beta + n_3\beta) \otimes$$
$$\otimes S(-\tau_3 + n_2\beta + n_3\beta) \otimes S(\tau_3) \otimes S(\tau_3 - n_3\beta)$$
$$\overset{\beta\to\infty}{\sim} \sum_{n_1,n_2,n_3} \langle \Omega^{n_1+n_2+n_3} \rangle e^{-\beta M(|n_1|+|n_2|+|n_3|)}$$

$$\gimel = \sum_{n,m=-\infty}^{\infty} \int_0^{\beta} dx_4 F(x_4 + n\beta + m\beta) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx_4 F(x_4 + n\beta)$$

$$\daleth = \prod_{i=0}^{L} \int d^4 z_i G^{2L} \sum_{n_1,\dots,n_L} \prod_{i=1}^{L} (-\Omega)^{n_i} S(\vec{x}_i, t_i + \mathrm{i}n_i\beta)$$

$$\mathcal{O}^T = \sum_L \sum_{n_1,\dots,n_L} \mathcal{O}_{n_1\dots n_L} \langle \Omega^{n_1+\dots n_L} \rangle e^{-M\beta(|n_1|+\dots+|n_L|)}$$

$$n_1 + \dots + n_L = kN_c$$

$$Z_{\bar{q}q} \sim \frac{1}{N_c} e^{-2M/T},$$
$$Z_{qqq} \sim e^{-N_c M/T},$$
$$Z_{qqq\bar{q}q} \sim \frac{1}{N_c} e^{-(2+N_c)M/T},$$
$$\dots$$
$$Z_{(\bar{q}q)^{N_M}(qqq)^{N_B}} \sim \frac{1}{N_c^{N_M}} e^{-(2N_M + N_B N_c)M/T}.$$

$$\mathcal{O}^T = \mathcal{O}^{T=0} + \sum_m \mathcal{O}_m \frac{1}{N_c} e^{-m/T} + \sum_B \mathcal{O}_B e^{-M_B/T} + \cdots$$

$$\sum_L \sum_{n_1,\dots,n_L} \mathcal{O}_{n_1\dots n_L} \langle \Omega^{1+n_1+\dots+n_L} \rangle e^{-M\beta(|n_1|+\dots+|n_L|)}$$

$$1 + n_1 + \dots + n_L = kN_c$$

$$\langle |\mathrm{tr}_c \Omega|^2 \rangle = 1$$

$$\Gamma_G[\Omega] = V_{\text{glue}}[\Omega] \cdot a^3/T = -2(d-1)\mathrm{e}^{-\sigma a/T} |\mathrm{tr}_c \Omega|^2$$

$$e^{-\Gamma_G[\Omega]} = 1 - \Gamma_G[\Omega] + \frac{1}{2}\Gamma_G[\Omega]^2 + \cdots,$$

$$\mathcal{M} = nN_c M_q + mM_{\bar{q}q} + lm_G$$

$$V_{\text{glue}}[\Omega] = T \int \frac{d^3k}{(2\pi)^3} \widehat{\mathrm{tr}}_c \ln\left[1 - e^{-\beta\omega_k} \widehat{\Omega}\right]$$

$$V_{\text{glue}}[\Omega] = -T \sum_{n=1}^{\infty} \frac{1}{n} (|\mathrm{tr}_c \Omega^n|^2 - 1) \int \frac{d^3k}{(2\pi)^3} e^{-n\beta\omega_k}$$

$$\widehat{\mathrm{tr}}_c \widehat{\Omega}^n = |\mathrm{tr}_c \Omega^n|^2 - 1.$$

$$\langle \mathrm{tr}_c \Omega(\vec{x}_1, \beta) \mathrm{tr}_c \Omega^{-1}(\vec{x}_2, \beta) \rangle \simeq e^{-\beta\sigma|\vec{x}_1 - \vec{x}_2|}.$$

$$V=\frac{1}{T}\int d^3x\longrightarrow\frac{1}{T}\int d^3xe^{-\sigma r/T}=\frac{8\pi T^2}{\sigma^3}$$

$$Z_Q[U,\Omega]:=e^{-\Gamma_Q[U,\Omega]}=\mathrm{Det}(\mathbf{D})\exp\left(-\frac{a_s^2}{4}\mathrm{tr}_f\int d^4x(M-\hat{m}_0)^2\right)$$

$$\mathrm{Det}(\mathbf{D})=e^{-\int d^4x\mathcal{L}_q^*(x)}=\exp\left(-\int d^4x\left(\mathcal{L}_q^{*(0)}(x)+\mathcal{L}_q^{*(2)}(x)+\mathcal{L}_q^{*(4)}(x)+\cdots\right)\right)$$

$$\begin{aligned}\mathcal{L}_q^{*(0)}(x)=&-\frac{N_cN_f}{(4\pi)^2}\sum_i c_i(\Lambda_i^2+M^2)^2\log(\Lambda_i^2+M^2)\\&+\frac{N_f}{\pi^2}(MT)^2\sum_{n=1}^{\infty}(-1)^n\frac{K_2(nM/T)}{n^2}(\mathrm{tr}_c\Omega^n(x)+\mathrm{tr}_c\Omega^{-n}(x))\\&=\mathcal{L}_q^{(0)}(T=0)+\mathcal{L}_q^{(0)}(\Omega(x),T),\end{aligned}$$

$$Z=\int\mathcal{D}\Omega e^{-\Gamma_G[\Omega]}\exp\left(-\int d^4x\left\{\frac{a_s^2}{4}\mathrm{tr}_f(M-\hat{m}_0)^2+\mathcal{L}_q^{(0)}(T=0)+\mathcal{L}_q^{(0)}(\Omega(x),T)\right\}\right)$$

$$L=\frac{1}{N_c}\langle\mathrm{tr}_c\Omega\rangle=\frac{1}{N_cZ}\int\mathcal{D}\Omega e^{-\Gamma_G[\Omega]}e^{-\Gamma_Q[\Omega]}\mathrm{tr}_c\Omega(x)$$

$$\int\mathcal{D}\Omega\Omega_{ij}\Omega_{kl}^*=\frac{1}{N_c}\delta_{ik}\delta_{jl}$$

$$\int\mathcal{D}\Omega\mathrm{tr}_c\Omega\mathrm{tr}_c\Omega^{-1}=1$$

$$\int\mathcal{D}\Omega\mathrm{tr}_c\Omega(\vec{x})\mathrm{tr}_c\Omega^{-1}(\vec{y})=e^{-\sigma|\vec{x}-\vec{y}|/T}$$

$$\int d^4x\int\mathcal{D}\Omega\mathrm{tr}_c\Omega(\vec{x})\mathrm{tr}_c\Omega(\vec{y})=0$$

$$\int d^4x\int\mathcal{D}\Omega\mathrm{tr}_c\Omega(\vec{x})\mathrm{tr}_c\Omega^{-1}(\vec{y})=\int d^4xe^{-\sigma|\vec{x}-\vec{y}|/T}=\frac{8\pi T^2}{\sigma^3}$$

$$L(T)\overset{\text{T pequeño}}{\sim}\frac{4N_f}{N_c\sigma^3}\sqrt{\frac{2M^3T^9}{\pi}}e^{-M/T}$$

$$\langle\bar{q}q\rangle_T=-f_\pi^{*2}B_0^*=-\frac{M}{4\pi^2}\langle\mathrm{tr}_c\mathcal{J}_{-1}\rangle$$

$$\langle\bar{q}q\rangle_T=-\frac{M}{4\pi^2}\frac{1}{Z}\int\mathcal{D}\Omega e^{-\Gamma_G[\Omega]}e^{-\Gamma_Q[\Omega]}\mathrm{tr}_c\mathcal{J}_{-1}(M,\Omega)$$

$$\langle\bar{q}q\rangle_T\overset{\text{T pequeño}}{\sim}\langle\bar{q}q\rangle_{T=0}+\frac{8N_f}{\pi^2}\frac{M^3T^6}{\sigma^3}e^{-2M/T}$$

$$\Omega=\mathrm{diag}\left(e^{i\phi_1},e^{i\phi_2},e^{-i(\phi_1+\phi_2)}\right)$$

$$Z=\int\mathcal{D}\Omega e^{-\Gamma_G[\Omega]}e^{-\Gamma_Q[\Omega]}=\int_{-\pi}^{\pi}\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\rho_G(\phi_1,\phi_2)\rho_Q(\phi_1,\phi_2)$$

$$\mathcal{D}\Omega e^{-\Gamma_G[\Omega]}=\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\rho_G(\phi_1,\phi_2),e^{-\Gamma_Q[\Omega]}=\rho_Q(\phi_1,\phi_2)$$

$$\langle \mathrm{tr}_c f(\Omega)\rangle = \frac{1}{Z}\int_{-\pi}^{\pi}\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\rho_G(\phi_1,\phi_2)\rho_Q(\phi_1,\phi_2)\left(f\left(e^{i\phi_1}\right)+f\left(e^{i\phi_2}\right)+f\left(e^{-i(\phi_1+\phi_2)}\right)\right)$$
$$= \frac{1}{Z}\int_{-\pi}^{\pi}\frac{d\phi_1}{2\pi}\hat{\rho}(\phi_1)f\left(e^{i\phi_1}\right)$$

$$\hat{\rho}(\phi_1) = 3\int_{-\pi}^{\pi}\frac{d\phi_2}{2\pi}\rho_G(\phi_1,\phi_2)\rho_Q(\phi_1,\phi_2)$$

$$\langle \bar{q}q\rangle_T = -\frac{a_s^2}{2}\mathrm{tr}_f(M(T)-\hat{m}_0)$$

$$\delta \equiv \frac{1}{N_c}\sqrt{\langle \mathrm{tr}_c\Omega \mathrm{tr}_c\Omega^{-1}\rangle - \langle \mathrm{tr}_c\Omega\rangle^2} = \frac{1}{N_c}\sqrt{1+\langle \widehat{\mathrm{tr}}_c\widehat{\Omega}\rangle - \langle \mathrm{tr}_c\Omega\rangle^2}$$

$$\frac{1}{2}\theta^{\mu\nu}(x) = \left.\frac{\delta S}{\delta g_{\mu\nu}(x)}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

$$S = \int d^4x\sqrt{-g}\mathcal{L}(x)$$

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)} + \theta_{\mu\nu}^{(2)} + \theta_{\mu\nu}^{(4)} + \cdots$$

$$\theta_{\mu\nu}^{(0)} = -\eta_{\mu\nu}\mathcal{L}^{(0)}$$
$$\theta_{\mu\nu}^{(2)} = \frac{f_\pi^2}{2}\langle D_\mu U^\dagger D_\nu U\rangle - \eta_{\mu\nu}\mathcal{L}^{(2)}$$
$$\begin{aligned}\theta_{\mu\nu}^{(4)} &= -\eta_{\mu\nu}\mathcal{L}^{(4)} + 2L_4\langle D_\mu U^\dagger D_\nu U\rangle\langle \chi^\dagger U + U^\dagger\chi\rangle \\ &+L_5\langle D_\mu U^\dagger D_\nu U + D_\nu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + U^\dagger\chi\rangle \\ &-2L_{11}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle D_\alpha U^\dagger D^\alpha U\rangle \\ &-2L_{13}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle \chi^\dagger U + U^\dagger\chi\rangle \\ &-L_{12}\left(\eta_{\mu\alpha}\eta_{\nu\beta}\partial^2 + \eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\mu\alpha}\partial_\nu\partial_\beta - \eta_{\nu\alpha}\partial_\mu\partial_\beta\right)\langle D^\alpha U^\dagger D^\beta U\rangle\end{aligned}$$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2,g)} + \mathcal{L}^{(2,R)} + \mathcal{L}^{(4,g)} + \mathcal{L}^{(4,R)} + \cdots$$

$$g^{\mu\nu}(x) = e_A^\mu(x)e_B^\nu(x)\eta^{AB}$$

$$\delta_\nu^\mu = \eta^{AB}e_A^\mu e_{\nu B} = e_A^\mu e_\nu^A, \delta_B^A = g^{\mu\nu}e_\mu^A e_{\nu B} = e_\mu^A e_B^\mu$$

$$e_\mu^A \to \frac{\partial x^\nu}{\partial x'^\mu}e_\nu^A, e_\mu^A \to \Lambda_B^A(x)e_\mu^B$$

$$T^{AB} = e_\mu^A e_\nu^B T^{\mu\nu}$$

$$T_{\nu A}^\alpha \to \frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\alpha}{\partial x^\beta}\Lambda_A^B(x)T_{\mu B}^\beta$$

$$d_\mu T_{\nu A}^\alpha = \partial_\mu T_{\nu A}^\alpha - \Gamma_{\nu\mu}^\lambda T_{\lambda A}^\alpha + \Gamma_{\mu\lambda}^\alpha T_{\nu A}^\lambda + \omega_{AB\mu}T_\nu^{\alpha B},$$

$$\Gamma_{\lambda\mu}^\sigma = \frac{1}{2}g^{\nu\sigma}\{\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\}$$

$$d_\mu e_{\nu A} = \partial_\mu e_{\nu A} - \Gamma_{\nu\mu}^\lambda e_{\lambda A} + \omega_{AB\mu}e_\nu^B = 0$$

$$d_\mu \eta_{AB} = \omega_{AB\mu} + \omega_{BA\mu} = 0$$

$$\omega_{AB\mu} = e_A^{\gamma}\left[\partial_\mu e_{\nu B} - \Gamma_{\nu\mu}^{\lambda} e_{\lambda B}\right]$$

$$\begin{aligned}
U(x) &\to U(x)\\
\Psi(x) &\to S(\Lambda(x))\Psi(x)\\
A_\mu(x) &\to \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)\\
\Psi_\mu(x) &\to \frac{\partial x^\nu}{\partial x'^\mu} S(\Lambda(x))\Psi_\nu(x)
\end{aligned}$$

$$d_\mu U = \partial_\mu U$$

$$A_{\nu;\mu} := d_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\nu\mu}^{\lambda} A_\lambda$$

$$-R_{\sigma\mu\nu}^{\lambda} = \partial_\mu \Gamma_{\nu\sigma}^{\lambda} - \partial_\nu \Gamma_{\mu\sigma}^{\lambda} + \Gamma_{\mu\alpha}^{\lambda}\Gamma_{\nu\sigma}^{\alpha} - \Gamma_{\nu\alpha}^{\lambda}\Gamma_{\mu\sigma}^{\alpha}$$

$$R_{\mu\nu} = R_{\mu\lambda\nu}^{\lambda}, R = g^{\mu\nu} R_{\mu\nu}$$

$$\left[d_\mu, d_\nu\right] A_\alpha = R_{\alpha\mu\nu}^{\lambda} A_\lambda$$

$$d_\mu \Psi = \partial_\mu \Psi(x) - i\omega_\mu \Psi(x)$$

$$\omega_\mu = \frac{1}{4}\sigma^{AB}\omega_{AB\mu}$$

$$\gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB}$$

$$\gamma_\mu(x) = \gamma_A e_\mu^A(x)$$

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x)$$

$$d_\mu \gamma_A = \partial_\mu \gamma_A - i\left[\omega_\mu, \gamma_A\right] + \omega_{AB\mu}\gamma^B = 0$$

$$d_\mu \gamma_\nu(x) = 0$$

$$\Psi_{\nu;\mu} := d_\mu \Psi_\nu = \partial_\mu \Psi_\nu - \Gamma_{\mu\nu}^{\lambda}\Psi_\lambda - i\omega_\mu \Psi_\nu$$

$$\left[d_\mu, d_\nu\right]\Psi = \frac{i}{4}\sigma^{\alpha\beta} R_{\alpha\beta\mu\nu}\Psi$$

$$d^\mu d_\mu \Psi = \frac{1}{\sqrt{-g}}\left\{\left(\partial_\mu - i\omega_\mu\right)\left[\sqrt{-g}\, g^{\mu\nu}\left(\partial_\nu - i\omega_\nu\right)\right]\Psi\right\}$$

$$\nabla_\mu \Psi = \left(d_\mu - iV_\mu\right)\Psi$$

$$\gamma_5(x) = \frac{1}{4!\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}\gamma_\mu(x)\gamma_\nu(x)\gamma_\alpha(x)\gamma_\beta(x) = \frac{1}{4!}\epsilon^{ABCD}\gamma_A\gamma_B\gamma_C\gamma_D = \gamma_5$$

$$i\mathbf{D} = i\not{d} - MU^5 - \hat{m}_0 + (\not{v} + \not{a}\gamma_5 - s - i\gamma_5 p)$$

$$Y = \gamma^\mu(x)V_\mu(x)$$

$$\begin{aligned}
\nabla_\mu U &= \widehat{D}_\mu U = \partial_\mu U - i\left[v_\mu, U\right] - i\left\{a_\mu, U\right\},\\
\nabla_\mu \Psi &= \widehat{\mathcal{D}}_\mu \Psi = \partial_\mu \Psi - i\left(\omega_\mu + v_\mu + \gamma_5 a_\mu\right)\Psi,\\
\nabla_\mu \Psi_\nu &= \partial_\mu \Psi_\nu - i\left(\omega_\mu + v_\mu + \gamma_5 a_\mu\right)\Psi_\nu - \Gamma_{\mu\nu}^{\lambda}\Psi_\lambda,
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}_5\mathbf{D} = {} & [D_L^2 + i\mathcal{M}^\dagger \not{D}_L - i\not{D}_R \mathcal{M}^\dagger + \mathcal{M}^\dagger\mathcal{M}]P_R\\
& + \left[\not{D}_R^2 + i\mathcal{M} D_L - i\not{D}_R \mathcal{M} + \mathcal{M}\mathcal{M}^\dagger\right]P_L,
\end{aligned}$$

$$\mathbf{D}_5[s,p,v,a,U] = \gamma_5 \mathbf{D}[s,-p,v,-a,U^\dagger]\gamma_5$$

$$\begin{aligned} D_\mu &= \partial_\mu - i(v_\mu + \gamma_5 a_\mu) = D_\mu^R P_R + D_\mu^L P_L, \\ D_\mu^R &= \partial_\mu - i(v_\mu + a_\mu), \\ D_\mu^L &= \partial_\mu - i(v_\mu - a_\mu), \end{aligned}$$

$$\mathcal{M} = MU^5 + (s + i\gamma_5 p) + \hat{m}_0$$

$$\hat{\mathcal{D}}_\mu = \partial_\mu - i(\omega_\mu + v_\mu + \gamma_5 a_\mu)$$

$$\hat{\mathcal{D}}_\mu \Psi = \nabla_\mu \Psi$$

$$\hat{\mathcal{D}}_\mu \nabla\Psi = \nabla_\mu \nabla\Psi$$

$$\begin{aligned} \mathrm{D}_5 \mathrm{D}\Psi = {} & [\nabla_L^2 + i\mathcal{M}\nabla_L - i\nabla_R\mathcal{M} + \mathcal{M}^\dagger\mathcal{M}]P_R\Psi \\ & +[\nabla_R^2 + i\mathcal{M}^\dagger\nabla_L - i\nabla_R\mathcal{M}^\dagger + \mathcal{M}\mathcal{M}^\dagger]P_L\Psi. \end{aligned}$$

$$\mathbb{D}^2\Psi = \nabla^2\Psi = \left[\nabla^\mu\nabla_\mu - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} + \frac{1}{4}R\right]\Psi$$

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\Psi &= [\mathcal{D}_\mu, \mathcal{D}_\nu]\Psi \\ &= [D_\mu, D_\nu]\Psi + \frac{i}{4}\sigma^{\alpha\beta}R_{\alpha\beta\mu\nu}\Psi \end{aligned}$$

$$\nabla^\mu\nabla_\mu\Psi = \frac{1}{\sqrt{-g}}\mathcal{D}_\mu\left(\sqrt{-g}g^{\mu\nu}\mathcal{D}_\nu\Psi\right)$$

$$\mathbf{D}_5\mathbf{D} = \frac{1}{\sqrt{-g}}\left[\mathcal{D}_\mu\left(\sqrt{-g}g^{\mu\nu}\mathcal{D}_\nu\right)\right] + \mathcal{V},$$

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_R P_R + \mathcal{V}_L P_L \\ \mathcal{V}_R &= -\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}^R + \frac{1}{4}R - i\gamma^\mu\nabla_\mu\mathcal{M} + \mathcal{M}^\dagger\mathcal{M} \\ \mathcal{V}_L &= -\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}^L + \frac{1}{4}R - i\gamma^\mu\nabla_\mu\mathcal{M}^\dagger + \mathcal{M}\mathcal{M}^\dagger \end{aligned}$$

$$S_{\mathrm{NJL}} = \int d^4x\sqrt{-g}\mathcal{L}_{\mathrm{NJL}}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{NJL}} = {} & \bar{q}(i\partial\!\!\!/ + \psi - \hat{m}_0)q + \frac{1}{2a_s^2}\sum_{a=0}^{N_f^2-1}((\bar{q}\lambda_a q)^2 + (\bar{q}\lambda_a i\gamma_5 q)^2) \\ & -\frac{1}{2a_v^2}\sum_{a=0}^{N_f^2-1}\left((\bar{q}\lambda_a\gamma_\mu q)^2 + (\bar{q}\lambda_a\gamma_\mu\gamma_5 q)^2\right) \end{aligned}$$

$$\omega_\mu(x) = \frac{i}{8}[\gamma^\nu(x), \gamma_{\nu;\mu}(x)]$$

$$Z_{\mathrm{NJL}}[g;s,p,v,a] = \int \mathcal{D}S\mathcal{D}P\mathcal{D}V\mathcal{D}A e^{i\Gamma_{\mathrm{NJL}}[g;\bar{S},\bar{P},\bar{V},\bar{A}]}$$

$$\Gamma_{\mathrm{NJL}}[g;\bar{S},\bar{P},\bar{V},\bar{A}] = \Gamma_q[\mathbf{D}] + \Gamma_m[g;S,P,V,A]$$

$$\Gamma_q[\mathbf{D}] = -iN_c \mathrm{Trlog}\,(i\mathbf{D})$$

$$\Gamma_m[g;S,P,V,A] = \int d^4x\sqrt{-g}\left\{-\frac{a_s^2}{4}\mathrm{tr}(S^2+P^2)+\frac{a_v^2}{4}\mathrm{tr}\left(V_\mu^2+A_\mu^2\right)\right\}$$

$$i\mathbf{D} = i\partial\!\!\!/ + \psi\!\!\!/ - \hat{m}_0 + (\bar{Y} + \bar{A}\gamma_5 - \bar{S} - i\gamma_5\bar{P})$$

$$\Gamma_q^+[\mathbf{D}] = -i\frac{N_c}{2}\mathrm{Tr}\sum c_i \log\,(\mathbf{D}_5\mathbf{D} + \Lambda_i^2 + i\epsilon)$$

$$\mathcal{L}_{\mathrm{GM}} = \bar{q}\left(i\partial\!\!\!/ + \psi\!\!\!/ - MU^5 - \hat{m}_0 + \frac{1}{2}(1-g_A)U^5 i\partial\!\!\!/ U^5\right)q =: \bar{q}i\mathbf{D}q$$

$$\Gamma_{\mathrm{GM}} = -iN_c\mathrm{Trlog}\,(i\mathbf{D}),$$

$$V_\mu = \frac{1}{4}(1-g_A)\left[U^\dagger\partial_\mu U - \partial_\mu U^\dagger U\right]$$

$$A_\mu = \frac{1}{4}(1-g_A)\left[U^\dagger\partial_\mu U + \partial_\mu U^\dagger U\right]$$

$$\sum_i c_i \mathrm{Trlog}\,(\mathbf{D}_5\mathbf{D} + \Lambda_i^2) = -\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau\mathbf{D}_5\mathbf{D}}\phi(\tau)$$

$$\langle x|e^{-i\tau\mathbf{D}_5\mathbf{D}}|x\rangle = e^{-i\tau M^2}\langle x|e^{-i\tau(\mathbf{D}_5\mathbf{D}-M^2)}|x\rangle = \frac{i}{(4\pi i\tau)^2}e^{-i\tau M^2}\sum_{n=0}^\infty a_n(x)(i\tau)^n$$

$$a_0 = 1$$

$$a_1 = M^2 - \overline{\mathcal{V}} + \frac{1}{6}R$$

$$a_2 = \frac{1}{180}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{12}\overline{\mathcal{F}}^{\mu\nu}\overline{\mathcal{F}}_{\mu\nu} + \frac{1}{30}\bar{\nabla}^2 R - \frac{1}{6}\bar{\nabla}^2\overline{\mathcal{V}} + \frac{1}{2}\left[M^2 - \overline{\mathcal{V}} + \frac{1}{6}R\right]^2$$

$$a_3 = \frac{1}{6}\left[M^2 - \overline{\mathcal{V}} + \frac{1}{6}R\right]^3 - \frac{1}{12}\bar{\nabla}^\mu\overline{\mathcal{V}}\bar{\nabla}_\mu\overline{\mathcal{V}} + \mathcal{O}(p^6),$$

$$a_4 = \frac{1}{24}\left[\overline{\mathcal{V}} - M^2\right]^4 + \mathcal{O}(p^6).$$

$$\overline{\mathcal{D}}_\mu = \partial_\mu - i(\overline{V_\mu} + \gamma_5\bar{A}_\mu),$$

$$\bar{\nabla}_\mu = d_\mu - i(\overline{V_\mu} + \gamma_5\bar{A}_\mu),$$

$$\mathcal{I}_{2l} := M^{2l}\int_0^\infty \frac{d\tau}{\tau}\phi(\tau)(i\tau)^l e^{-i\tau M^2}$$

$$M^4\mathcal{I}_{-4} = -\frac{1}{2}\sum_i c_i(\Lambda_i^2 + M^2)^2\log\,(\Lambda_i^2 + M^2)$$

$$M^2\mathcal{I}_{-2} = \sum_i c_i(\Lambda_i^2 + M^2)\log\,(\Lambda_i^2 + M^2)$$

$$\mathcal{I}_0 = -\sum_i c_i\log\,(\Lambda_i^2 + M^2)$$

$$\mathcal{I}_{2n} = \Gamma(n)\sum_i c_i\left(\frac{M^2}{\Lambda_i^2 + M^2}\right)^n, \mathrm{Re}(n) > 0$$

$$\mathcal{L}_q^{(2)} = \frac{N_c}{(4\pi)^2}\left\{M^2\mathcal{I}_0\langle\bar{\nabla}_\mu U^\dagger\bar{\nabla}^\mu U\rangle + 2M^3\mathcal{I}_{-2}\langle\bar{m}^\dagger U + U^\dagger\bar{m}\rangle + \frac{M^2}{6}\mathcal{I}_{-2}\langle R\rangle\right\}$$

$$\begin{aligned}\mathcal{L}_q^{(4)} = \frac{N_c}{(4\pi)^2}\Big\{ & -\frac{1}{6}\mathcal{I}_0\left\langle\left(\bar{F}_{\mu\nu}^R\right)^2 + \left(\bar{F}_{\mu\nu}^L\right)^2\right\rangle + \mathcal{I}_0\left\langle\frac{7}{720}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{144}R^2 + \frac{1}{90}R_{\mu\nu}R^{\mu\nu}\right\rangle \\ & -\frac{i}{2}\mathcal{I}_2\langle\bar{F}_{\mu\nu}^R\bar{\nabla}^\mu U^\dagger\bar{\nabla}^\nu U + \bar{F}_{\mu\nu}^L\bar{\nabla}^\mu U\bar{\nabla}^\nu U^\dagger\rangle \\ & +\frac{1}{12}\mathcal{I}_4\left\langle\left(\bar{\nabla}_\mu U\bar{\nabla}_\nu U^\dagger\right)^2\right\rangle - \frac{1}{6}\mathcal{I}_4\left\langle\left(\bar{\nabla}_\mu U\bar{\nabla}^\mu U^\dagger\right)^2\right\rangle \\ & +\frac{1}{6}\mathcal{I}_2\langle\bar{\nabla}_\mu\bar{\nabla}_\mu U\bar{\nabla}^\nu\bar{\nabla}^\nu U^\dagger\rangle \\ & +2M^2\mathcal{I}_{-2}\langle\bar{m}^\dagger\bar{m}\rangle - M^2\mathcal{I}_0\langle(\bar{m}^\dagger U + U^\dagger\bar{m})^2\rangle \\ & -M\mathcal{I}_2\langle\bar{\nabla}_\mu U^\dagger\bar{\nabla}^\mu U(\bar{m}^\dagger U + U^\dagger\bar{m})\rangle \\ & +M\mathcal{I}_0\langle\bar{\nabla}_\mu U^\dagger\bar{\nabla}^\mu\bar{m} + \bar{\nabla}_\mu\bar{m}^\dagger\bar{\nabla}^\mu U\rangle \\ & -\frac{M}{6}\mathcal{I}_0 R\langle U^\dagger\bar{m} + \bar{m}^\dagger U\rangle - \frac{1}{12}\mathcal{I}_2 R\langle\bar{\nabla}_\mu U^\dagger\bar{\nabla}^\mu U\rangle\Big\}\end{aligned}$$

$$\begin{aligned}\bar{\nabla}_\mu U &= \nabla_\mu U - iV_\mu^L U + iUV_\mu^R \\ \bar{F}_{\mu\nu}^r &= \partial_\mu\bar{V}_\nu^r - \partial_\nu\bar{V}_\mu^r - i\left[\bar{V}_\mu^r, \bar{V}_\nu^r\right]\end{aligned}$$

$$\bar{m} = (S + iP - MU) + \frac{1}{2B_0}\chi, \chi = 2B_0(s + ip)$$

$$\mathcal{L}_{A,V}^{(2)} = \frac{N_c}{(4\pi)^2}M^2\mathcal{I}_0\langle\bar{\nabla}_\mu U^\dagger\bar{\nabla}^\mu U\rangle + \frac{a_v^2}{4}\langle V_\mu V^\mu + A_\mu A^\mu\rangle$$

$$\bar{V}_\mu^R = v_\mu^R + \frac{i}{2}(1 - g_A)U^\dagger\nabla_\mu U, \bar{V}_\mu^L = v_\mu^L + \frac{i}{2}(1 - g_A)U\nabla_\mu U^\dagger,$$

$$\begin{aligned}\bar{F}_{\mu\nu}^R =& \frac{1}{2}(1 + g_A)F_{\mu\nu}^R + \frac{1}{2}(1 - g_A)U^\dagger F_{\mu\nu}^L U \\ & -\frac{i}{4}(1 - g_A^2)\left(\nabla_\mu U^\dagger\nabla_\nu U - \nabla_\nu U^\dagger\nabla_\mu U\right),\end{aligned}$$

$$\begin{aligned}\bar{F}_{\mu\nu}^L =& \frac{1}{2}(1 - g_A)UF_{\mu\nu}^R U^\dagger + \frac{1}{2}(1 + g_A)F_{\mu\nu}^L \\ & -\frac{i}{4}(1 - g_A^2)\left(\nabla_\mu U\nabla_\nu U^\dagger - \nabla_\nu U\nabla_\mu U^\dagger\right), \\ \bar{\nabla}_\mu U =& g_A\nabla_\mu U, \\ \bar{\nabla}^2 U =& g_A\nabla^2 U + ig_A(1 - g_A)U\nabla_\mu U^\dagger\nabla^\mu U.\end{aligned}$$

$$S + iP = \sqrt{U}\Sigma\sqrt{U}$$

$$\bar{m} = \sqrt{U}\Phi\sqrt{U} + \frac{1}{2B_0}\chi$$

$$\mathcal{L}_m = -\frac{a_s^2}{4}\langle M^2 + 2M\Phi + \Phi^2\rangle$$

$$\mathcal{L}_\Phi(x) = -\frac{N_c}{(4\pi)^2}\Big\langle 4M^2\mathcal{J}_0\Phi^2 + \frac{1}{3}M\mathcal{J}_0 R\Phi$$
$$+M\mathcal{J}_0\sqrt{U}\Phi\sqrt{U^\dagger}(U\bar{\nabla}^2U^\dagger + \bar{\nabla}^2UU^\dagger)$$
$$+\frac{M^2}{B_0}(2\mathcal{J}_0 - \mathcal{J}_{-2})\sqrt{U}\Phi\sqrt{U^\dagger}(U\chi^\dagger + U^\dagger\chi)$$
$$+\frac{M}{B_0}\mathcal{J}_2\sqrt{U}\Phi\sqrt{U^\dagger}\bar{\nabla}_\mu U\bar{\nabla}^\mu U^\dagger\Big\rangle.$$

$$\sqrt{U}\Phi\sqrt{U^\dagger} = -\frac{1}{24M}R + \frac{1}{4M}\left(1-\frac{\mathcal{J}_2}{\mathcal{J}_0}\right)\bar{\nabla}_\mu U\bar{\nabla}^\mu U^\dagger$$
$$-\frac{1}{4B_0}\left(1-\frac{\mathcal{J}_{-2}}{2\mathcal{J}_0}\right)(U\chi^\dagger + \chi U^\dagger).$$

$$\langle\nabla^2U^\dagger\nabla^2U\rangle = \left\langle\left(\nabla_\mu U^\dagger\nabla^\mu U\right)^2\right\rangle - \frac{1}{4}\langle(\chi^\dagger U - U^\dagger\chi)^2\rangle + \frac{1}{12}\langle\chi^\dagger U - U^\dagger\chi\rangle^2$$

$$\langle\chi^\dagger\nabla^2U + \nabla^2U^\dagger\chi\rangle = 2\langle\chi^\dagger\chi\rangle - \frac{1}{2}\langle(\chi^\dagger U + U^\dagger\chi)^2\rangle - \langle(\chi^\dagger U + U^\dagger\chi)\nabla^\mu U^\dagger\nabla_\mu U\rangle$$
$$+\frac{1}{6}\langle\chi^\dagger U + U^\dagger\chi\rangle^2$$

$$\int d^4x\sqrt{-g}\langle\nabla_\mu\nabla_\nu U^\dagger\nabla^\mu\nabla^\nu U\rangle = \int d^4x\sqrt{-g}\big[\langle\nabla^2U^\dagger\nabla^2U\rangle + i\langle F^R_{\mu\nu}\nabla^\mu U^\dagger\nabla^\nu U + F^L_{\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger\rangle$$
$$-\langle F^L_{\mu\nu}UF^{\mu\nu}U^\dagger\rangle + \frac{1}{2}\left\langle\left(F^R_{\mu\nu}\right)^2 + \left(F^L_{\mu\nu}\right)^2\right\rangle + R^{\mu\nu}\langle\nabla_\mu U^\dagger\nabla_\nu U\rangle\big]$$

$$\left\langle\left(\nabla_\mu U^\dagger\nabla_\nu U\right)^2\right\rangle = -2\left\langle\left(\nabla_\mu U^\dagger\nabla^\mu U\right)^2\right\rangle + \langle\nabla_\mu U^\dagger\nabla_\nu U\rangle^2 + \frac{1}{2}\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle^2$$

$$\mathcal{L}^{(2,g)} = \frac{f_\pi^2}{4}\langle\nabla_\mu U^\dagger\nabla^\mu U + (\chi^\dagger U + U^\dagger\chi)\rangle,$$

$$\mathcal{L}^{(4,g)} = L_1\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle^2 + L_2\langle\nabla_\mu U^\dagger\nabla_\nu U\rangle^2 + L_3\left\langle\left(\nabla_\mu U^\dagger\nabla^\mu U\right)^2\right\rangle$$
$$+L_4\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle\langle\chi^\dagger U + U^\dagger\chi\rangle + L_5\langle\nabla_\mu U^\dagger\nabla^\mu U(\chi^\dagger U + U^\dagger\chi)\rangle$$
$$+L_6\langle\chi^\dagger U + U^\dagger\chi\rangle^2 + L_7\langle\chi^\dagger U - U^\dagger\chi\rangle^2 + L_8\langle(\chi^\dagger U)^2 + (U^\dagger\chi)^2\rangle$$
$$-iL_9\langle F^L_{\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger + F^R_{\mu\nu}\nabla^\mu U^\dagger\nabla^\nu U\rangle + L_{10}\langle F^L_{\mu\nu}UF^{\mu\nu}U^\dagger\rangle$$
$$+H_1\left\langle\left(F^R_{\mu\nu}\right)^2 + \left(F^L_{\mu\nu}\right)^2\right\rangle + H_2\langle\chi^\dagger\chi\rangle.$$

$$\mathcal{L}^{(2,R)} = -H_0R,$$

$$\mathcal{L}^{(4,R)} = -L_{11}R\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle - L_{12}R^{\mu\nu}\langle\nabla_\mu U^\dagger\nabla_\nu U\rangle - L_{13}R\langle\chi^\dagger U + U^\dagger\chi\rangle$$
$$+H_3R^2 + H_4R_{\mu\nu}R^{\mu\nu} + H_5R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

$$f_\pi^2 = \frac{N_c}{4\pi^2}g_A^2M^2\mathcal{J}_0$$

$$B_0 = \frac{M}{g_A^2}\frac{\mathcal{J}_{-2}}{\mathcal{J}_0}$$

$$\rho \equiv \frac{M}{B_0} = M\frac{f_\pi^2}{|\langle\bar{q}q\rangle|} = g_A^2\frac{\mathcal{J}_0}{\mathcal{J}_{-2}}$$

$$L_1=\frac{N_c}{48(4\pi)^2}[(1-g_A^2)^2\mathcal{J}_0+4g_A^2(1-g_A^2)\mathcal{J}_2+2g_A^4\mathcal{J}_4],L_2=2L_1,$$
$$L_3=-\frac{N_c}{24(4\pi)^2}[3(1-g_A^2)^2\mathcal{J}_0+8g_A^4\mathcal{J}_4+4g_A^2(3-4g_A^2)\mathcal{J}_2],L_4=0,$$

$$L_5=\frac{N_c}{2(4\pi)^2}\rho g_A^2[\mathcal{J}_0-\mathcal{J}_2],L_6=0,L_7=-\frac{N_c}{24(4\pi)^2N_f}g_A[6\rho\mathcal{J}_0-g_A\mathcal{J}_2]$$
$$L_8=-\frac{N_c}{24(4\pi)^2}[6\rho(\rho-g_A)\mathcal{J}_0+g_A^2\mathcal{J}_2],L_9=\frac{N_c}{6(4\pi)^2}[(1-g_A^2)\mathcal{J}_0+2g_A^2\mathcal{J}_2]$$
$$L_{10}=-\frac{N_c}{6(4\pi)^2}[(1-g_A^2)\mathcal{J}_0+g_A^2\mathcal{J}_2],L_{11}=\frac{N_c}{12(4\pi)^2}g_A^2\mathcal{J}_2$$
$$L_{12}=-\frac{N_c}{6(4\pi)^2}g_A^2\mathcal{J}_2,L_{13}=\frac{N_c}{12(4\pi)^2}\rho\mathcal{J}_0=\frac{\rho}{48M^2}\frac{f_\pi^2}{g_A^2}$$
$$H_0=-\frac{N_cN_f}{6(4\pi)^2}M^2\mathcal{J}_{-2}=-\frac{N_f}{24}\frac{f_\pi^2}{\rho},H_1=\frac{N_c}{12(4\pi)^2}[-(1+g_A^2)\mathcal{J}_0+g_A^2\mathcal{J}_2]$$
$$H_2=\frac{N_c}{12(4\pi)^2}[6\rho^2\mathcal{J}_{-2}-6\rho(\rho+g_A)\mathcal{J}_0+g_A^2\mathcal{J}_2]$$
$$H_3=-\frac{N_cN_f}{144(4\pi)^2}\mathcal{J}_0=-\frac{N_f}{576M^2}\frac{f_\pi^2}{g_A^2},H_4=\frac{N_cN_f}{90(4\pi)^2}\mathcal{J}_0,H_5=\frac{7N_cN_f}{720(4\pi)^2}\mathcal{J}_0$$

$$\Lambda=1470\text{MeV},B_0=4913\text{MeV}$$
$$\mathcal{J}_{-2}=20{,}8,\mathcal{J}_0=2{,}26,\mathcal{J}_2=0{,}922,\mathcal{J}_4=0{,}995$$

$$\Lambda=828\text{MeV},B_0=1299\text{MeV}$$
$$\mathcal{J}_{-2}=5{,}50,\mathcal{J}_0=1{,}27,\mathcal{J}_2=0{,}781,\mathcal{J}_4=0{,}963$$

$$f_\pi^2=\frac{N_c}{4\pi^2}g_AM^2\mathcal{J}_0$$

$$B_0=\frac{a_s^2M}{2f_\pi^2}=\frac{M}{g_A}\frac{\mathcal{J}_{-2}}{\mathcal{J}_0},g_A=1-2\frac{f_\pi^2}{a_v^2}$$

$$\rho\equiv\frac{M}{B_0}=g_A\frac{\mathcal{J}_0}{\mathcal{J}_{-2}}$$

$$L_3^S=\frac{N_c}{4(4\pi)^2}\frac{g_A^4}{\mathcal{J}_0}[\mathcal{J}_0-\mathcal{J}_2]^2,L_5^S=\frac{N_c}{4(4\pi)^2}g_A^2(g_A-2\rho)[\mathcal{J}_0-\mathcal{J}_2]$$
$$L_8^S=\frac{N_c}{16(4\pi)^2}(g_A-2\rho)^2\mathcal{J}_0,$$
$$L_{11}^S=\frac{N_c}{12(4\pi)^2}g_A^2[\mathcal{J}_0-\mathcal{J}_2],L_{13}^S=\frac{N_c}{24(4\pi)^2}(g_A-2\rho)\mathcal{J}_0$$
$$H_2^S=2L_8^S,H_3^S=\frac{N_cN_f}{144(4\pi)^2}\mathcal{J}_0=\frac{N_f}{576M^2}\frac{f_\pi^2}{g_A}$$

$$L_3=-\frac{N_c}{24(4\pi)^2}\left[3(1-2g_A^2-g_A^4)\mathcal{J}_0+8g_A^4\mathcal{J}_4+2g_A^2\left(2(3-g_A^2)-3g_A^2\frac{\mathcal{J}_2}{\mathcal{J}_0}\right)\mathcal{J}_2\right]$$
$$L_5=\frac{N_c}{4(4\pi)^2}g_A^3[\mathcal{J}_0-\mathcal{J}_2],L_8=\frac{N_c}{48(4\pi)^2}g_A^2[3\mathcal{J}_0-2\mathcal{J}_2]$$
$$L_{11}=\frac{N_c}{12(4\pi)^2}g_A^2\mathcal{J}_0=\frac{g_Af_\pi^2}{48M^2},L_{13}=\frac{N_c}{24(4\pi)^2}g_A\mathcal{J}_0=\frac{f_\pi^2}{96M^2}$$
$$H_2=\frac{N_c}{24(4\pi)^2}[12\rho^2\mathcal{J}_{-2}+3g_A(g_A-8\rho)\mathcal{J}_0+2g_A^2\mathcal{J}_2],H_3=0$$

$\Lambda = 1344\text{MeV}, B_0 = 4015\text{MeV}$
$\mathcal{I}_{-2} = 17{,}0, \mathcal{I}_0 = 2{,}10, \mathcal{I}_2 = 0{,}907, \mathcal{I}_4 = 0{,}993$

$L_{11} = 1{,}58 \cdot 10^{-3}, L_{12} = -3{,}2 \cdot 10^{-3}, L_{13} = 0{,}3 \cdot 10^{-3}$

	TQP [1]	NJL	NJL (g_A 1)	QC	GM	SQM[2] (MDM)	Large N_c	Dual [2] Large N_c
L_1	0.53 ± 0.25	0.77	0.76	0.76	0.78	0.79	0.9	0.79
L_2	0.71 ± 0.27	1.54	1.52	1.52	1.56	1.58	1.8	1.58
L_3	-2.72 ± 1.12	-4.02	-2.73	-3.62	-4.25	-3.17	-4.3	-3.17
L_4	0	0	0	0	0	0	0	0
L_5	0.91 ± 0.15	1.26	2.32	1.08	0.44	2.0 ± 0.1	2.1	3.17
L_6	0	0	0	0	0	0	0	0
L_7	-0.32 ± 0.15	-0.06	-0.26	-0.26	-0.03	-0.07 ± 0.01	-0.3	
L_8	0.62 ± 0.20	0.65	0.89	0.46	0.04	0.08 ± 0.04	0.8	1.18
L_9	5.93 ± 0.43	6.31	4.95	4.95	6.41	6.33	7.1	6.33

L_{10}	-4.40 ± 0.70[4]	-5.25	-2.47	-2.47	-4.77	-3.17	-5.4	-4.75
L_{11}	1.85 ± 0.90[5]	1.22	2.01	1.24	0.82	1.58	1.6 [5]	
L_{12}	-2.7[5]	-1.06	-2.47	-2.47	-1.64	-3,17	-2.7[5]	
L_{13}	1.7 ± 0.80[5]	1.01	1.01	0.47	0.22	0.33 ± 0.01	1.1[5]	
H_0		-14.6	-4.67	-4.67	-17.7	1.09		
H_1		-4.01	-2.78	-2.78	-4.76			
H_2		1.46	1.45	0.59	0.49	-1.0 ± 0.2		
H_3		0	0	-0.50	-0.89			
H_4		1.33	0.80	0.80	1.43			
H_5		1.16	0.70	0.70	1.25			

$$\mathrm{Tr}A = \int d^4x\sqrt{-g}\,\mathrm{tr}\langle A(x,x)\rangle$$

$$\Gamma_{\mathrm{SQM}}[U,s,p,v,a,g] = -iN_c\int_{\mathcal{C}} d\omega\rho(\omega)\mathrm{Trlog}\,(i\mathbf{D})$$

$$i\mathbf{D} = i\not{d} - \omega U^5 - \hat{m}_0 + (\not{v} + \not{a}\gamma_5 - s - i\gamma_5 p) = iD - \omega U^5$$

$$\mathcal{L}_A = -\frac{f_\pi^2}{4}m_{\eta_1}^2\left\{\theta - \frac{i}{2}[\log\,\det\bar{U} - \log\,\det\bar{U}^\dagger]\right\}^2$$

$$-i\mathbf{D}_5 = \gamma_5(i\not{d} - \omega U^{5\dagger} - \hat{m}_0 + \not{v} - \gamma_5\not{a} - s + i\gamma_5 p)\gamma_5.$$

$$\Gamma_{\mathrm{SQM}}^{+} = -\frac{i}{2}N_c\int_{\mathcal{C}} d\omega\rho(\omega)\mathrm{Trlog}\,(\mathbf{D}_5\mathbf{D})$$

$$\mathbf{D} \to e^{+i\epsilon_V(x) - i\epsilon_A(x)\gamma_5}\mathbf{D}e^{-i\epsilon_V(x) - i\epsilon_A(x)\gamma_5},$$

$$\epsilon_V(x) = \sum_a \epsilon_V^a(x)\lambda_a, \epsilon_A(x) = \sum_a \epsilon_A^a(x)\lambda_a$$

$$\delta\mathbf{D} = i[\epsilon_V, \mathbf{D}] - i\{\epsilon_A\gamma_5, \mathbf{D}\}$$

$$\delta S = -iN_c\mathrm{Tr}\int_{\mathcal{C}} d\omega\rho(\omega)[\delta\mathbf{D}\mathbf{D}^{-1}]$$

$$\delta_A S \equiv \mathcal{A}_A = \int d^4x\mathrm{tr}\int_{\mathcal{C}} d\omega\rho(\omega)\langle 2i\epsilon_A\gamma_5\rangle = \rho_0\int d^4x\mathrm{tr}\langle 2i\epsilon_A\gamma_5\rangle$$

$$\delta_A S \equiv \mathcal{A}_A = \mathrm{Tr}\int_{\mathcal{C}} d\omega\rho(\omega)(2i\epsilon_A\gamma_5[i\mathbf{D}]^0)$$
$$= \int d^4x\mathrm{tr}\int_{\mathcal{C}} d\omega\rho(\omega)\left\langle 2i\epsilon_A(x)\gamma_5\langle x|\mathbf{D}^0 \mid x\rangle\right\rangle$$

$$\langle x|\mathbf{D}^0|x\rangle = \frac{1}{(4\pi)^2}\left\{\frac{1}{2}\mathbf{D}^4 + \frac{1}{3}\left(\mathbf{D}^2\Gamma_\mu^2+\Gamma_\mu\mathbf{D}^2\Gamma_\mu + \Gamma_\mu^2\mathbf{D}^2\right)\right.$$
$$\left.+\frac{1}{6}\left(\Gamma_\mu^2\Gamma_\nu^2 + \left(\Gamma_\mu\Gamma_\nu\right)^2 + \Gamma_\mu\Gamma_\nu^2\Gamma_\mu\right)\right\}$$

$$\mathcal{A}_A = \int_{\mathcal{C}} d\omega\rho(\omega)(\mathcal{A}_A[s,p,v,a] + \mathcal{A}_A[s,p,v,a,\omega,U]) = \rho_0\mathcal{A}_A[s,p,v,a]$$

$$U_t^5 = e^{it\sqrt{2}\gamma_5\Phi/f_\pi}$$

$$\Gamma_{\mathrm{SQM}}[U,s,p,v,a] - \Gamma_{\mathrm{SQM}}[\mathbf{1},s,p,v,a] = -iN_c\int_0^1 dt\frac{d}{dt}\int_{\mathcal{C}} d\omega\rho(\omega)\mathrm{Trlog}\,(iD-\omega U_t^5)$$
$$= iN_c\int_0^1 dt\int_{\mathcal{C}} d\omega\rho(\omega)\mathrm{Tr}\left[\omega\frac{dU_t^5}{dt}\frac{1}{iD-\omega U_t^5}\right].$$

$$\Gamma_{\mathrm{SQM}}^{-(4)} = -iN_c\int_0^1 dt\int_{\mathcal{C}} d\omega\rho(\omega)\int d^4x\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-\omega^2]^5}$$
$$\times\mathrm{Tr}\left\{-\omega\gamma_5 U_t^\dagger\frac{dU_t}{dt}\omega\left[\omega U_t^\dagger i\,\partial\!\!\!/ U_t\right]^4\right\}$$

$$\Gamma_{\mathrm{SQM}}^{-(4)} = \rho_0\frac{N_c}{48\pi^2}\int_0^1 dt\int d^4x\epsilon_{\mu\nu\alpha\beta}\left\langle U_t^\dagger\frac{dU_t}{dt}U_t^\dagger\partial^\mu U_t U_t^\dagger\partial^\nu U_t U_t^\dagger\partial^\alpha U_t U_t^\dagger\partial^\beta U_t\right\rangle$$

$$-\frac{i}{2}\mathrm{Trlog}\;\mathbf{D}_5\mathbf{D} = -\frac{1}{2}\frac{N_c}{(4\pi^2)}\int d^4x\sqrt{-g}\int_{\mathcal{C}} d\omega\rho(\omega)$$
$$\times\mathrm{tr}\left\langle -\frac{1}{2}\omega^4\log\,\omega^2 a_0 + \omega^2\log\,\omega^2 a_1 - \log\,(\omega^2/\mu^2)a_2 + \frac{1}{\omega^2}a_3 + \frac{1}{\omega^4}a_4 + \cdots\right\rangle$$
$$= \int d^4x\sqrt{-g}\left(\mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \cdots\right)$$

$$\mathcal{L}^{(2)} = \frac{N_c}{(4\pi)^2}\int_{\mathcal{C}}\rho(\omega)\left\{-\omega^2\log\,\omega^2\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle\right.$$
$$\left.+2\omega^3\log\,\omega^2\langle m^\dagger U + U^\dagger m\rangle + \omega^2\log\,\omega^2\frac{1}{12}\langle R\rangle\right\},$$

$$
\begin{aligned}
\mathcal{L}^{(4)} = \frac{N_c}{(4\pi)^2}\int_{\mathcal{C}} \rho(\omega)\{ \\
&+\frac{1}{6}\log\ \omega^2 \left\langle \left(F_{\mu\nu}^{R}\right)^2 + \left(F_{\mu\nu}^{L}\right)^2 \right\rangle - \log\ \omega^2 \left\langle \frac{7}{720} R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - \frac{1}{144} R^2 + \frac{1}{90} R^{\mu\nu} R_{\mu\nu} \right\rangle \\
&-\frac{i}{3}\langle F_{\mu\nu}^{R} \nabla_\mu U^\dagger \nabla_\nu U + F_{\mu\nu}^{L} \nabla_\mu U \nabla_\nu U^\dagger \rangle + \frac{1}{12}\left\langle \left(\nabla_\mu U \nabla_\nu U^\dagger\right)^2 \right\rangle - \frac{1}{6}\left\langle \left(\nabla_\mu U \nabla^\mu U^\dagger\right)^2 \right\rangle \\
&+\frac{1}{6}\langle \nabla^\mu \nabla^\nu U \nabla_\mu \nabla_\nu U^\dagger \rangle - \frac{1}{6}\langle F_{\mu\nu}^{L} U F_{\mu\nu}^{R} U^\dagger \rangle \\
&+\log\ \omega^2 \omega^2 (2\langle m^\dagger m\rangle + \langle (m^\dagger U + U^\dagger m)^2 \rangle) \\
&-\frac{1}{2}\omega \langle \nabla_\mu U^\dagger \nabla^\mu U (m^\dagger U + U^\dagger m) \rangle - \log\ \omega^2 \omega \langle \nabla_\mu U^\dagger \nabla^\mu m + \nabla_\mu m^\dagger \nabla^\mu U \rangle \\
&-\omega \log\ \omega^2 \frac{1}{6} R \langle U^\dagger m + m^\dagger U \rangle + \frac{1}{12} R \nabla_\mu U^\dagger \nabla^\mu U \rangle \}
\end{aligned}
$$

$$
\begin{aligned}
f_\pi^2 &= -\frac{4N_c}{(4\pi)^2}\rho_2' \\
f_\pi^2 B_0 &= -\langle \bar{q} q \rangle = \frac{4N_c}{(4\pi)^2}\rho_3'
\end{aligned}
$$

$$
\begin{aligned}
&L_3 = -2L_2 = -4L_1 = -\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}, L_4 = L_6 = 0, \\
&L_5 = -\frac{N_c}{(4\pi)^2}\frac{\rho_1'}{2B_0}, L_7 = \frac{N_c}{(4\pi)^2}\frac{1}{2N_f}\left(\frac{\rho_1'}{2B_0} + \frac{\rho_0}{12}\right), \\
&L_8 = \frac{N_c}{(4\pi)^2}\left[\frac{\rho_2'}{4B_0^2} - \frac{\rho_1'}{4B_0} - \frac{\rho_0}{24}\right], L_9 = -2L_{10} = \frac{N_c}{(4\pi)^2}\frac{\rho_0}{3}, \\
&L_{12} = -2L_{11} = -\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}, L_{13} = -\frac{N_c}{(4\pi)^2}\frac{\rho_1'}{12B_0} = \frac{1}{6}L_5, \\
&H_0 = -\frac{f_\pi^2}{4}\frac{N_f}{6}, H_1 = \frac{N_c}{(4\pi)^2}\frac{\rho_0'}{6}, H_2 = \frac{N_c}{(4\pi)^2}\left(\frac{\rho_2'}{B_0^2} + \frac{\rho_1'}{2B_0} + \frac{\rho_0}{12}\right) \\
&H_3 = \frac{N_c}{(4\pi)^2}N_f\frac{\rho_0'}{144}, H_4 = -\frac{N_c}{(4\pi)^2}N_f\frac{\rho_0'}{90}, H_5 = -\frac{N_c}{(4\pi)^2}N_f\frac{7\rho_0'}{720}.
\end{aligned}
$$

$$
\begin{aligned}
L_7 &= -\frac{L_5}{2N_f} + \frac{N_c}{384\pi^2 N_f} \simeq -0{,}09 \cdot 10^{-3}, \\
L_8 &= \frac{L_5}{2} - \frac{N_c}{384\pi^2} - \frac{f_\pi^2}{16B_0^2} \simeq 0{,}13 \cdot 10^{-3}, \\
H_2 &= -L_5 + \frac{N_c}{192\pi^2} - \frac{f_\pi^2}{4B_0^2} \simeq -1{,}02 \cdot 10^{-3}.
\end{aligned}
$$

$$
F_V(t) = \frac{M_V^2}{M_V^2 + t}
$$

$$
\rho_{2-2n} = \frac{2^{2n+3}\pi^{3/2} f_\pi^2}{N_c M_V^{2n}} \frac{n\Gamma(n+3/2)}{\Gamma(n+1)}, n = 1,2,3, \ldots
$$

$$
f_\pi^2 = \frac{N_c M_V^2}{24\pi^2}
$$

$$
\rho_n' = \int_{\mathcal{C}} d\omega \log(\omega^2)\omega^n \rho(\omega) = 2\frac{d}{dz}\int_{\mathcal{C}} d\omega \omega^z \rho(\omega)\Big|_{z=n} = 2\frac{d}{dz}\rho_z\Big|_{z=n}
$$

$$
\rho_{2n}' = \left(-\frac{M_V^2}{4}\right)^n \frac{\Gamma(n)\Gamma\left(\frac{5}{2} - n\right)}{\Gamma(5/2)}, n = 1,2,3, \ldots
$$

$$\rho_V(\omega) = \frac{1}{2\pi i}\frac{1}{\omega}\frac{1}{(1-4\omega^2/M_V^2)^{d_V}}$$

$$\rho_S(\omega) = \frac{1}{2\pi i}\frac{16(d_S-1)(d_S-2)\rho_3'}{M_S^4(1-4\omega^2/M_S^2)^{d_S}}$$

$$S(p) = \int_C d\omega \frac{\rho_V(\omega)\not{p} + \rho_S(\omega)\omega}{p^2-\omega^2} = \frac{Z(p^2)}{\not{p} - M(p^2)}$$

$$\rho_1'^{\mathrm{MD}} = \frac{8\pi^2\langle\bar{q}q\rangle}{N_c M_S^2} = -\frac{5M_Q M_S^2}{6M_V^2},$$
$$\rho_2'^{\mathrm{MD}} = -\frac{4\pi^2 f_\pi^2}{N_c} = -\frac{M_V^2}{6},$$
$$\rho_3'^{\mathrm{MD}} = -\frac{4\pi^2\langle\bar{q}q\rangle}{N_c} = \frac{5M_Q M_S^4}{12M_V^2},$$

$$M_Q \equiv M(0) = -\frac{48M_V^2\pi^2\langle\bar{q}q\rangle}{5N_c M_S^4}$$

$$L_5 = \frac{N_c}{96\pi^2}\frac{M_V^2}{M_S^2}$$
$$L_7 = \frac{N_c}{32\pi^2 N_f}\left(\frac{1}{12} - \frac{M_V^2}{6M_S^2}\right)$$
$$L_8 = \frac{N_c}{16\pi^2}\left(-\frac{M_V^{10}}{150M_Q^2 M_S^8} + \frac{M_V^2}{12M_S^2} - \frac{1}{24}\right)$$

$$\bar{l}_1 = -\bar{l}_2 = -\frac{1}{2}\bar{l}_5 = -\frac{1}{4}\bar{l}_6 = -N_c$$
$$\bar{l}_3 = \frac{4N_c}{3} + \frac{16N_c M_V^{10}}{75M_Q^2 M_S^8}$$
$$\bar{l}_4 = \frac{2N_c M_V^2}{3M_S^2}$$

$$\langle r^2\rangle_V = \frac{1}{16\pi^2 f_\pi^2}\bar{l}_6 = \frac{6}{M_V^2}, \langle r^2\rangle_S = \frac{3}{8\pi^2 f_\pi^2}\bar{l}_4 = \frac{6}{M_S^2}$$

$$\langle r^2\rangle_{G,0} = \langle r^2\rangle_{G,2} = \frac{N_c}{48\pi^2 f_\pi^2}$$

$$M_{f_0} = M_{f_2} = 4\pi f_\pi\sqrt{3/N_c} = 1105 - 1168\mathrm{MeV}$$

$$2L_1^{\mathrm{SRA}} = L_2^{\mathrm{SRA}} = \frac{1}{4}L_9^{\mathrm{SRA}} = -\frac{1}{3}L_{10}^{\mathrm{SRA}} = \frac{f_\pi^2}{8M_V^2}$$
$$L_5^{\mathrm{SRA}} = \frac{8}{3}L_8^{\mathrm{SRA}} = \frac{f_\pi^2}{4M_S^2}$$
$$L_3^{\mathrm{SRA}} = -3L_2^{\mathrm{SRA}} + \frac{1}{2}L_5^{\mathrm{SRA}}$$
$$2L_{13}^{\mathrm{SRA}} = 3L_{11}^{\mathrm{SRA}} + L_{12}^{\mathrm{SRA}} = \frac{f_\pi^2}{4M_{f_0}^2}$$
$$L_{12}^{\mathrm{SRA}} = -\frac{f_\pi^2}{2M_{f_2}^2}$$

$$M_V = c_V f_\pi/\sqrt{N_c}, M_S = c_S f_\pi/\sqrt{N_c}$$

$$\rho_1'\text{SRA} = \frac{8\pi^2\langle\bar{q}q\rangle}{N_c M_S^2},$$
$$\rho_2'\text{SRA} = -\frac{4\pi^2 f_\pi^2}{N_c} = -\frac{M_V^2}{6},$$

$$2L_1 = L_2 = -\frac{1}{2}L_3 = \frac{1}{2}L_5 = \frac{2}{3}L_8 = \frac{1}{4}L_9 = -\frac{1}{3}L_{10} = \frac{N_c}{192\pi^2}$$

$$M_A = M_P = \sqrt{2}M_V = \sqrt{2}M_S = 4\pi\sqrt{\frac{3}{N_c}}f_\pi$$

$$\langle r^2\rangle_S^{1/2} = \langle r^2\rangle_V^{1/2} = \frac{\sqrt{N_c}}{2\pi f_\pi}$$

$$\langle r^2\rangle_S^{1/2} = \langle r^2\rangle_V^{1/2} = 0{,}58 - 0{,}62\text{fm}$$

$$-\bar{l}_1 = \bar{l}_2 = \frac{3}{2}\bar{l}_3 = \frac{3}{2}\bar{l}_4 = \frac{1}{3}\bar{l}_5 = \frac{1}{4}\bar{l}_6 = N_c$$

$$\begin{aligned} \bar{l}_1 &= -0{,}4 \pm 0{,}6, & \bar{l}_2 &= 6{,}0 \pm 1{,}3, \\ \bar{l}_4 &= 4{,}4 \pm 0{,}2, & \bar{l}_3 &= 2{,}9 \pm 2{,}4 \\ &= 13{,}0 \pm 1{,}0, & \bar{l}_6 &= 16{,}0 \pm 1{,}0 \end{aligned}$$

$$\begin{aligned} \bar{l}_2 - \bar{l}_1 &= 2N_c \text{ (Exp. } 6{,}4 \pm 1{,}4), \\ \bar{l}_3 - \bar{l}_1 &= \frac{5N_c}{3} \text{ (Exp. } 3{,}3 \pm 2{,}5), \\ \bar{l}_4 - \bar{l}_1 &= \frac{5N_c}{3} \text{ (Exp. } 4{,}8 \pm 0{,}6), \\ \bar{l}_5 - \bar{l}_1 &= 4N_c \text{ (Exp. } 13{,}4 \pm 1{,}2), \\ \bar{l}_6 - \bar{l}_1 &= 5N_c \text{ (Exp. } 16{,}4 \pm 1{,}2). \end{aligned}$$

$$\bar{l}_2 - \bar{l}_1 = 8{,}3, \bar{l}_3 - \bar{l}_1 = 6{,}2, \bar{l}_4 - \bar{l}_1 = 6{,}2, \bar{l}_5 - \bar{l}_1 = 15{,}2, \bar{l}_6 - \bar{l}_1 = 18{,}7$$

$$\begin{aligned} M^U(x) &= U^{-1}(x)M(x)U(x) \\ A_\mu^U(x) &= U^{-1}(x)\partial_\mu U(x) + U^{-1}(x)A_\mu(x)U(x) \end{aligned}$$

$$f(M^U, D_\mu^U) = U^{-1}f(M, D_\mu)U$$

$$B_0(\vec{x}) = U^{-1}(x)\partial_0 U(x) + U^{-1}(x)A_0(\vec{x})U(x).$$

$$B_0(\vec{x}) = V^{-1}(x)\partial_0 V(x).$$

$$V(x) = U_0(\vec{x})e^{x_0 B_0(\vec{x})}.$$

$$B_0(\vec{x}) = U_0^{-1}(\vec{x})(A_0(\vec{x}) + \Lambda(\vec{x}))U_0(\vec{x}),$$

$$U(x) = e^{-x_0 A_0(\vec{x})}e^{x_0(A_0(\vec{x})+\Lambda(\vec{x}))}U_0(\vec{x}).$$

$$U(x_0 + \beta, \vec{x}) = e^{i\alpha}U(x_0, \vec{x}).$$

$$e^{\beta(A_0(\vec{x})+\Lambda(\vec{x}))} = e^{i\alpha}e^{\beta A_0(\vec{x})},$$

$$e^{\beta\Lambda(\vec{x})} = e^{i\alpha}, [A_0(\vec{x}), \Lambda(\vec{x})] = 0$$

$$U(x) = e^{x_0 \Lambda(\vec{x})} U_0(\vec{x})$$

$$U(x) = \exp\,(x_0 \lambda_a \Lambda^a) U_0(\vec{x})$$

$$\lambda_{N_c^2-1} = \mathrm{diag}(1,1,\cdots,1-N_c)\rho$$

$$U(x_0+\beta,\vec{x}) = zU(x_0,\vec{x}), z = e^{i2\pi n/N_c}$$

$$\mathcal{L}_{0,q}(x) = -\frac{(2\pi)^2}{3\beta^4} N_f \mathrm{tr} B_4\left(\frac{1}{2}+\bar{\nu}\right), \Omega(x) = e^{i2\pi\bar{\nu}}, -\frac{1}{2} < \bar{\nu} < \frac{1}{2}$$

$$\mathcal{L}_{0,g}(x) = \frac{2\pi^2}{3\beta^4} \widehat{\mathrm{tr}} B_4(\hat{\nu}), \widehat{\Omega}(x) = e^{i2\pi\hat{\nu}}, 0 < \hat{\nu} < 1$$

$$B_{2\ell}(x) = \frac{(-1)^{\ell-1} 2(2\ell)!}{(2\pi)^{2\ell}} \sum_{n=1}^{\infty} \frac{\cos\,(2\pi n x)}{n^{2\ell}}, 0 \le x \le 1, n = 1,2,\ldots$$

$$\mathcal{L}_{0,q}(x) = \frac{4N_f}{\pi^2\beta^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \{(N_c-1)\cos\,(2\pi n\rho) + \cos\,((N_c-1)2\pi n\rho)\}$$

$$\mathcal{L}_{0,g}(x) = -\frac{2}{\pi^2\beta^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \{2(N_c-1)\cos\,(2\pi n N_c \rho) + (N_c-1)^2\}$$

$$I_{\ell,n}^{\pm}(\nu) = \int_0^{\infty} \frac{d\tau}{\tau} (4\pi\mu^2\tau)^{\epsilon} \tau^{\ell} \varphi_n^{\pm}(e^{i2\pi\nu}), \nu, \ell, \epsilon \in \mathbb{R}, n = 0,1,2,\ldots$$

$$\varphi_n^{\pm}(\Omega;\tau/\beta^2) = \frac{(4\pi\tau)^{1/2}}{\beta} \sum_{p_0^{\pm}} \tau^{n/2} Q^n e^{\tau Q^2}, Q = ip_0^{\pm} - \frac{1}{\beta}\log\,(\Omega)$$

$$I_{\ell,n}^{+}(\nu) = (4\pi\mu^2)^{\epsilon} \frac{\sqrt{4\pi}}{\beta} \left(\frac{2\pi i}{\beta}\right)^n \sum_{k\in\mathbb{Z}} (k-\nu)^n \int_0^{\infty} d\tau \tau^{\ell+\epsilon+(n-1)/2} e^{-\left(\frac{2\pi}{\beta}\right)^2 (k-\nu)^2 \tau}, \nu \notin \mathbb{Z}$$

$$I_{\ell,n}^{+}(\nu) = i^n (4\pi\mu^2)^{\epsilon} \left(\frac{\beta}{2\pi}\right)^{2(\ell+\epsilon)} \frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)} \sum_{k\in\mathbb{Z}} \frac{(k-\nu)^n}{|k-\nu|^n} \frac{1}{|k-\nu|^{2(\ell+\epsilon)+1}}$$

$$I_{\ell,n}^{+}(\nu) = i^n (4\pi\mu^2)^{\epsilon} \left(\frac{\beta}{2\pi}\right)^{2(\ell+\epsilon)} \frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times \left(\sum_{k\le k_0} \frac{(-1)^n}{(k_0+\hat{\nu}-k)^{2(\ell+\epsilon)+1}} + \sum_{k>k_0} \frac{1}{(k-k_0-\hat{\nu})^{2(\ell+\epsilon)+1}} \right)$$

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}\ [\mathrm{Re} z > 1, q \ne 0,-1,-2,\ldots]$$

$$I_{\ell,n}^{+}(\nu) = (4\pi)^{\epsilon} \left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon} \left(\frac{\beta}{2\pi}\right)^{2\ell} \frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times [(-i)^n \zeta(1+2\ell+2\epsilon,\hat{\nu}) + i^n \zeta(1+2\ell+2\epsilon, 1-\hat{\nu})]$$

$$I_{\ell,n}^{-}(\nu) = (4\pi)^{\epsilon}\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times\left[(-i)^{n}\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}+\bar{\nu}\right)+i^{n}\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}-\bar{\nu}\right)\right]$$

$$I_{\ell,2n}^{\pm}(\nu) = (-1)^{n}\frac{\Gamma\left(\ell+\epsilon+n+\frac{1}{2}\right)}{\Gamma\left(\ell+\epsilon+\frac{1}{2}\right)}I_{\ell,0}^{\pm}(\nu), I_{\ell,2n+1}^{\pm}(\nu) = (-1)^{n}\frac{\Gamma(\ell+\epsilon+n+1)}{\Gamma(\ell+\epsilon+1)}I_{\ell,1}^{\pm}(\nu)$$

$$I_{\ell,n}^{\pm}(\nu) = (-1)^{n}I_{\ell,n}^{\pm}(-\nu)$$

$$I_{\ell,n}^{\prime +}(\nu) = (4\pi)^{\epsilon}\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times\begin{cases}(-i)^{n}\zeta(1+2\ell+2\epsilon,1+\hat{\nu})+i^{n}\zeta(1+2\ell+2\epsilon,1-\hat{\nu}), & 0\leq\hat{\nu}<\frac{1}{2}\\ (-i)^{n}\zeta(1+2\ell+2\epsilon,\hat{\nu})+i^{n}\zeta(1+2\ell+2\epsilon,2-\hat{\nu}), & \frac{1}{2}<\hat{\nu}\leq 1\end{cases}$$

$$I_{\ell,n}^{+}(\nu) = I_{\ell,n}^{\prime +} = (4\pi)^{\epsilon}\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma(\ell+\epsilon+(n+1)/2)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times\begin{cases}2(-1)^{n/2}\zeta(1+2\ell+2\epsilon), & (n\text{ par})\\ 0, & (n\text{ impar})\end{cases}\nu\in\mathbb{Z}.$$

$$\mathcal{L}_{\text{árbol}}(x) = \frac{1}{4g^{2}(\mu)}\vec{F}_{\mu\nu}^{2}$$
$$\mathcal{L}_{0,g}(x) = \frac{\pi^{2}T^{4}}{3}\left(-\frac{1}{5}+4\hat{\nu}^{2}(1-\hat{\nu})^{2}\right)$$
$$\mathcal{L}_{2,g}(x) = -\frac{11}{96\pi^{2}}\left[\frac{1}{11}+2\log\left(\frac{\mu}{4\pi T}\right)-\psi(\hat{\nu})-\psi(1-\hat{\nu})\right]\vec{F}_{\mu\nu\parallel}^{2}$$
$$-\frac{11}{96\pi^{2}}\left[\frac{\pi T}{m}+\frac{1}{11}+2\log\left(\frac{\mu}{4\pi T}\right)+\gamma_{E}-\frac{1}{2}\psi(\hat{\nu})-\frac{1}{2}\psi(1-\hat{\nu})\right]\vec{F}_{\mu\nu\perp}^{2}$$
$$+\frac{1}{24\pi^{2}}\vec{E}_{i}^{2}-\frac{1}{48\pi^{2}}\left(\frac{\pi T}{m}\right)\vec{E}_{i\perp}^{2}$$
$$\mathcal{L}_{3,g}(x) = \frac{61}{2160\pi^{2}}\left(\frac{1}{4\pi T}\right)^{2}\left[8\left(\frac{\pi T}{m}\right)^{3}+2\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right](\vec{F}_{\mu\nu}\times\vec{F}_{\nu\alpha})\cdot\vec{F}_{\alpha\mu}$$
$$-\frac{1}{48\pi^{2}}\left(\frac{1}{4\pi T}\right)^{2}[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}_{\lambda\mu\nu\parallel}^{2}$$
$$+\frac{1}{96\pi^{2}}\left(\frac{1}{4\pi T}\right)^{2}\left[16\left(\frac{\pi T}{m}\right)^{3}+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}_{\lambda\mu\nu\perp}^{2}$$

$$
\begin{aligned}
&+\frac{1}{480\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}^2_{\mu\mu\nu\parallel}\\
&-\frac{1}{960\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[16\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}^2_{\mu\mu\nu\perp}\\
&-\frac{3}{80\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}^2_{0\mu\nu\parallel}\\
&+\frac{3}{160\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[-8\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}^2_{0\mu\nu\perp}\\
&-\frac{1}{10\pi^2}\left(\frac{1}{4\pi T}\right)^2\left(\frac{\pi T}{m}\right)^3\vec{E}^2_{0i\perp}\\
&+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{E}^2_{ii\parallel}\\
&-\frac{1}{480\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[-8\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{E}^2_{ii\perp}\\
&+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\epsilon_{ijk}\left(\vec{E}_i\times\vec{E}_j\right)\cdot\vec{B}_k\\
&+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[8\left(\frac{\pi T}{m}\right)^3-4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\epsilon_{ijk}\left(\vec{E}_{i\perp}\times\vec{E}_{j\perp}\right)\cdot\vec{B}_{k\parallel},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{0,q}(x) =&\frac{2}{3}\pi^2T^4N_f\left(\frac{2}{15}-\frac{1}{4}(1-4\bar{\nu}^2)^2\right)\\
\mathcal{L}_{2,q}(x) =&\frac{N_f}{96\pi^2}\left[2\log\left(\frac{\mu}{4\pi T}\right)-\psi\left(\frac{1}{2}+\bar{\nu}\right)-\psi\left(\frac{1}{2}-\bar{\nu}\right)\right]\vec{F}^2_{\mu\nu}-\frac{N_f}{48\pi^2}\vec{E}_i^2\\
\mathcal{L}_{3,q}(x) =&\frac{N_f}{960\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[\psi''\left(\frac{1}{2}+\bar{\nu}\right)+\psi''\left(\frac{1}{2}-\bar{\nu}\right)\right]\\
&\times\left(\frac{16}{3}\left(\vec{F}_{\mu\nu}\times\vec{F}_{\nu\alpha}\right)\cdot\vec{F}_{\alpha\mu}+\frac{5}{2}\vec{F}^2_{\lambda\mu\nu}-\vec{F}^2_{\mu\mu\nu}-2\epsilon_{ijk}\left(\vec{E}_i\times\vec{E}_j\right)\cdot\vec{B}_k+3\vec{F}^2_{0\mu\nu}-2\vec{E}^2_{ii}\right)
\end{aligned}
$$

$$(\vec{a}\times\vec{b})_i=\epsilon_{ijk}a_jb_k$$

$$\mathbf{D}=\not{D}+h, h=m+z,$$

$$
\begin{aligned}
h_{LR}&=MU+\frac{1}{2B_0^*}\chi,\\
h_{RL}&=MU^\dagger+\frac{1}{2B_0^*}\chi^\dagger.
\end{aligned}
$$

$$\Gamma_q^+[v,h]=-\frac{1}{2}\mathrm{Trlog}\left(\mathbf{D}^\dagger\mathbf{D}\right)=:\int_0^\beta dx_0\int d^3x\mathcal{L}_q^*(x)$$

$$
\begin{aligned}
\mathbf{D}^\dagger\mathbf{D}&=-D_\mu^2-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}-\gamma_\mu\widehat{D}_\mu h+m^2+\bar{h}^2\\
\bar{h}^2&=h^2-m^2=\{m,z\}+z^2
\end{aligned}
$$

$$\mathcal{L}_q^*=\frac{1}{2}\int_0^\infty\frac{d\tau}{\tau}\phi(\tau)\mathrm{Tr}e^{-\tau D^\dagger D}=\frac{1}{2}\int_0^\infty\frac{d\tau}{\tau}\phi(\tau)\frac{e^{-\tau M^2}}{(4\pi\tau)^2}\sum_n\tau^n\mathrm{tr}b_n^T$$

$$\phi(\tau)=\sum_i c_ie^{-\tau\Lambda_i^2}$$

$$
\begin{aligned}
b_0^T &= 4\varphi_0(\Omega),\\
b_{1/2}^T &= 0,\\
b_1^T &= -4\varphi_0(\Omega)\bar{h}^2 = -4\varphi_0(\Omega)(\{m,z\}+z^2),\\
b_{3/2}^T &= 0,\\
b_2^T &= 2\varphi_0(\Omega)\left(\left(h_\mu\right)^2+\bar{h}^4-\frac{1}{3}F_{\mu\nu}^2\right)-\frac{2}{3}\bar{\varphi}_2E_i^2\\
&= 2\varphi_0(\Omega)\left(\left(m_\mu\right)^2+\{m_\mu,z_\mu\}+\{m,z\}\{m,z\}-\frac{1}{3}F_{\mu\nu}^2\right)-\frac{2}{3}\bar{\varphi}_2(\Omega)E_i^2+\mathcal{O}(p^6),\\
b_{5/2}^T &= -\frac{2}{3}\varphi_1\left\{E_i,\left(\bar{h}^2\right)_i\right\} = -\frac{2}{3}\varphi_1\{E_i,\widehat{D}_i\{m,z\}\} = \mathcal{O}(p^5),\\
b_3^T &= -\frac{2}{3}\varphi_0(\Omega)\big(m_\mu\{m_\mu,\{m,z\}\}+\{m,z\}m_\mu m_\mu+\{F_{\mu\nu},m_\mu m_\nu\}-m_\mu F_{\mu\nu}m_\nu\\
&\quad+\frac{1}{2}\left(m_{\mu\nu}\right)^2\big)+\frac{1}{3}\bar{\varphi}_2\left(m_{0\mu}\right)^2+\mathcal{O}(p^5),
\end{aligned}
$$

$$
\begin{aligned}
b_{7/2}^T &= \mathcal{O}(p^5)\\
b_4^T &= \frac{1}{6}\varphi_0(\Omega)(m_\mu m_\mu m_\nu m_\nu + m_\mu m_\nu m_\nu m_\mu - m_\mu m_\nu m_\mu m_\nu)+\mathcal{O}(p^5)
\end{aligned}
$$

$$
\mathrm{tr}(ABAB) = -2\mathrm{tr}(A^2B^2)+\frac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}(B^2)+(\mathrm{tr}(AB))^2
$$

$$
\begin{aligned}
\mathrm{tr}_f\left(m_\mu m_\nu m_\mu m_\nu\right) &= -2\mathrm{tr}_f\left(\left(m_\mu\right)^2(m_\nu)^2\right)+\frac{1}{2}\mathrm{tr}_f\left(\left(m_\mu\right)^2\right)\mathrm{tr}_f((m_\nu)^2)+\left(\mathrm{tr}_f\left(m_\mu m_\nu\right)\right)^2\\
\mathrm{tr}_f\left(m_0 m_\mu m_0 m_\mu\right) &= -2\mathrm{tr}_f\left((m_0)^2\left(m_\mu\right)^2\right)+\frac{1}{2}\mathrm{tr}_f((m_0)^2)\mathrm{tr}_f\left(\left(m_\mu\right)^2\right)+\left(\mathrm{tr}_f\left(m_0 m_\mu\right)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}_f\left(\left(m_{\mu\nu}\right)^2\right) &= \mathrm{tr}_f\left(\left(m_{\mu\nu}\right)^2\right)-2\mathrm{tr}_f\left(F_{\mu\nu}m_\mu m_\nu\right)+\mathrm{tr}_f\left(mF_{\mu\nu}mF_{\mu\nu}\right)-M^2\mathrm{tr}_f\left(F_{\mu\nu}^2\right)\\
\mathrm{tr}_f\left(\left(m_{0\mu}\right)^2\right) &= \mathrm{tr}_f\left(m_{00}m_{\mu\mu}\right)-2\mathrm{tr}_f(E_i[m_0,m_i])-2\mathrm{tr}_f(E_{0i}mm_i)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}_f\left(m_\mu z_\mu\right) &= \frac{1}{2B_0^*M^2}\mathrm{tr}_f\left(m_\mu m_\mu mx\right)-\frac{1}{4B_0^*M}\mathrm{tr}_f(mxmx)+\frac{M}{4B_0^*}\mathrm{tr}_f(x^2)\\
&\quad+\frac{1}{8MN_fB_0^*}\mathrm{tr}_f([m,x])\mathrm{tr}_f([m,x])\\
\mathrm{tr}_f\left(m_{\mu\mu}m_{\nu\nu}\right) &= \frac{1}{M^2}\mathrm{tr}_f\left(m_\mu m_\mu m_\nu m_\nu\right)-\frac{1}{2}\mathrm{tr}_f(mxmx)+\frac{M^2}{2}\mathrm{tr}_f(x^2)\\
&\quad+\frac{1}{4N_f}\mathrm{tr}_f([m,x])\mathrm{tr}_f([m,x])\\
\mathrm{tr}_f\left(m_{00}m_{\mu\mu}\right) &= \frac{1}{M^2}\mathrm{tr}_f\left(m_0 m_0 m_\mu m_\mu\right)-M\mathrm{tr}_f(m_{00}x)-\frac{1}{M}\mathrm{tr}_f(m_0 m_0 mx)\\
&\quad+\frac{1}{2MN_f}\mathrm{tr}_f(m_{00}m)\mathrm{tr}_f([m,x])
\end{aligned}
$$

$$
x_{LR}=\chi, x_{RL}=\chi^\dagger
$$

$$
\begin{aligned}
\mathrm{tr}_f b_0^T &= 4N_f \varphi_0(\Omega)\\
\mathrm{tr}_f b_1^T &= -\varphi_0(\Omega)\left(\frac{4}{B_0^*}\mathrm{tr}_f(mx) + \frac{1}{B_0^{*2}}\mathrm{tr}_f(x^2)\right)\\
\mathrm{tr}_f b_2^T &= 2\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu) + \frac{2}{B_0^* M^2}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu mx)\\
&\quad + \frac{1}{B_0^*}\left(\frac{1}{B_0^*} - \frac{1}{M}\right)\varphi_0(\Omega)\mathrm{tr}_f(mxmx) + \frac{M}{B_0^*}\left(\frac{M}{B_0^*}+1\right)\varphi_0(\Omega)\mathrm{tr}_f(x^2)\\
&\quad - \frac{2}{3}\varphi_0(\Omega)\mathrm{tr}_f(F_{\mu\nu}^2) - \frac{2}{3}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(E_i^2) + \frac{1}{2MN_f B_0^*}\varphi_0(\Omega)\mathrm{tr}_f([m,x])\mathrm{tr}_f([m,x]),\\
\mathrm{tr}_f b_3^T &= -\frac{4}{3}\varphi_0(\Omega)\mathrm{tr}_f(F_{\mu\nu}m_\mu m_\nu) - \frac{1}{3}\varphi_0(\Omega)\mathrm{tr}_f(mF_{\mu\nu}mF_{\mu\nu}) + \frac{1}{3}M^2\varphi_0(\Omega)\mathrm{tr}_f(F_{\mu\nu})\\
&\quad - \frac{1}{6}M^2\varphi_0(\Omega)\mathrm{tr}_f(x^2) + \frac{1}{6}\varphi_0(\Omega)\mathrm{tr}_f(mxmx) - \frac{2}{B_0^*}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu mx)\\
&\quad - \frac{1}{3M}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(m_0 m_0 mx) - \frac{M}{3}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(m_{00}x) - \frac{2}{3}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(E_i[m_0,m_i])\\
&\quad - \frac{2}{3}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(E_{0i}mm_i) - \frac{1}{3M^2}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu m_\nu m_\nu)\\
&\quad + \frac{1}{3M^2}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(m_0 m_0 m_\mu m_\mu) - \frac{1}{12N_f}\varphi_0(\Omega)\mathrm{tr}_f([m,x])\mathrm{tr}_f([m,x])\\
&\quad + \frac{1}{6MN_f}\bar{\varphi}_2(\Omega)\mathrm{tr}_f(m_{00}m)\mathrm{tr}_f([m,x])\Big),\\
\mathrm{tr}_f b_4^T &= -\frac{1}{12}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu)\mathrm{tr}_f(m_\nu m_\nu) - \frac{1}{6}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\nu)\mathrm{tr}_f(m_\mu m_\nu)\\
&\quad + \frac{2}{3}\varphi_0(\Omega)\mathrm{tr}_f(m_\mu m_\mu m_\nu m_\nu).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_l(\Lambda, M, \nu) &:= \int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\tau^l e^{-\tau M^2}\varphi_0(\Omega)\\
\bar{\mathcal{J}}_l(\Lambda, M, \nu) &:= \int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\tau^l e^{-\tau M^2}\bar{\varphi}_2(\Omega)
\end{aligned}
$$

$$
\varphi_0(\Omega) = \sum_{n\in\mathbb{Z}} e^{-\frac{n^2\beta^2}{4\tau}}(-\Omega)^n,
$$

$$
\bar{\varphi}_2(\Omega) = \frac{\beta^2}{2\tau}\sum_{n\in\mathbb{Z}} n^2 e^{-\frac{n^2\beta^2}{4\tau}}(-\Omega)^n
$$

$$\mathcal{J}_l(\Lambda, M, \nu) = \mathbf{1}_{N_c\times N_c}\Gamma(l)\sum_i c_i(\Lambda_i^2 + M^2)^{-l}$$

$$+2\left(\frac{\beta}{2M}\right)\sum_{n=1}^{l} n^l K_l(n\beta M)((-\Omega)^n + (-\Omega)^{-n}), \mathrm{Re}(l) > 0$$

$$\mathcal{J}_0(\Lambda, M, \nu) = -\mathbf{1}_{N_c\times N_c}\sum_i c_i \log(\Lambda_i^2 + M^2)$$

$$+2\sum_{n=1}^{\infty} K_0(n\beta M)((-\Omega)^n + (-\Omega)^{-n})$$

$$\mathcal{J}_{-1}(\Lambda, M, \nu) = \mathbf{1}_{N_c\times N_c}\sum_i c_i(\Lambda_i^2 + M^2)\log(\Lambda_i^2 + M^2)$$

$$+\frac{4M}{\beta}\sum_{n=1}^{\infty}\frac{K_1(n\beta M)}{n}((-\Omega)^n + (-\Omega)^{-n})$$

$$\mathcal{J}_{-2}(\Lambda, M, \nu) = -\mathbf{1}_{N_c\times N_c}\frac{1}{2}\sum_i c_i(\Lambda_i^2 + M^2)^2\log(\Lambda_i^2 + M^2)$$

$$+8\left(\frac{M}{\beta}\right)^2\sum_{n=1}^{\infty}\frac{K_2(n\beta M)}{n^2}((-\Omega)^n + (-\Omega)^{-n}),$$

$$\overline{\mathcal{J}}_l(\Lambda, M, \nu) = \frac{\beta^{l+1}}{(2M)^{l-1}}\sum_{n=1}^{\infty} n^{l+1}K_{l-1}(n\beta M)((-\Omega)^n + (-\Omega)^{-n}), l \in \mathbb{R}.$$

$$\mathcal{L}_q^{*(2)} = \int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\frac{e^{-\tau M^2}}{(4\pi)^2}\mathrm{tr}_c\varphi_0(\Omega)\left[\mathrm{tr}_f\left(m_\mu m_\mu\right) - \frac{4}{\tau}\mathrm{tr}_f(mz)\right]$$
$$= \frac{1}{(4\pi)^2}\left(\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)\mathrm{tr}_f\left(m_\mu m_\mu\right) - 4\mathrm{tr}_c\mathcal{J}_{-1}(\Lambda, M, \nu)\mathrm{tr}_f(mz)\right)$$

$$= \frac{M^2}{(4\pi)^2}\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)\left(\mathrm{tr}_f\left(\widehat{D}_\mu U^\dagger \widehat{D}_\mu U\right) - \frac{2}{M}\frac{\mathrm{tr}_c\mathcal{J}_{-1}(\Lambda, M, \nu)}{\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)}\mathrm{tr}_f(z_{RL}U + z_{LR}U^\dagger)\right)$$
$$= \frac{M^2}{(4\pi)^2}\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)\mathrm{tr}_f\left(\widehat{D}_\mu U^\dagger \widehat{D}_\mu U - (\chi^\dagger U + \chi U^\dagger)\right)$$

$$\chi = 2B_0^* z_{LR}, \chi^\dagger = 2B_0^* z_{RL}, B_0^* = \frac{1}{M}\frac{\mathrm{tr}_c\mathcal{J}_{-1}(\Lambda, M, \nu)}{\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)}.$$

$$\frac{f_\pi^{*2}}{4} = \frac{M^2}{(4\pi)^2}\mathrm{tr}_c\mathcal{J}_0(\Lambda, M, \nu)$$

$$m_{\mu\mu}m + m_\mu m_\mu - \frac{M}{2}[m, x] + \frac{M}{2N_f}\mathrm{tr}_f([m, x]) = 0$$

$$\mathcal{L}_q^* = \mathcal{L}_q^{*(0)} + \mathcal{L}_q^{*(2)} + \mathcal{L}_q^{*(4)} + \cdots$$

$$\mathcal{L}_q^{*(0)} = \frac{2N_f}{(4\pi)^2} \mathrm{tr}_c \mathcal{J}_{-2}(\Lambda, M, \nu)$$

$$\mathcal{L}_q^{*(2)} = \frac{f_\pi^{*2}}{4} \mathrm{tr}_f \left(\widehat{D}_\mu U^\dagger \widehat{D}_\mu U - (\chi^\dagger U + \chi U^\dagger) \right)$$

$$\begin{aligned} \mathcal{L}_q^{*(4)} = & -L_1^* \mathrm{tr}_f(u_\mu u_\mu) \mathrm{tr}_f(u_\nu u_\nu) - L_2^* \mathrm{tr}_f(u_\mu u_\nu) \mathrm{tr}_f(u_\mu u_\nu) - L_3^* \mathrm{tr}_f(u_\mu u_\mu u_\nu u_\nu) \\ & -\bar{L}_3^* \mathrm{tr}_f(u_0 u_0 u_\mu u_\mu) + 2L_4^* \mathrm{tr}_f(u_\mu u_\mu) \mathrm{tr}_f(xu) + 2L_5^* \mathrm{tr}_f(u_\mu u_\mu u x) \\ & +2\bar{L}_5^* \mathrm{tr}_f(u_0 u_0 u x) + 2\bar{L}_5^{*\prime} \mathrm{tr}_f(u_{00} x) - 2(L_6^* + L_7^*) \mathrm{tr}_f(ux) \mathrm{tr}_f(ux) \\ & -2(L_6^* - L_7^*) \mathrm{tr}_f(ux) \mathrm{tr}_f(xu) + 2\bar{L}^{*\prime} \mathrm{tr}_f(u_{00} u) \mathrm{tr}_f([u, x]) - 2L_8^* \mathrm{tr}_f(uxux) \\ & -2L_9^* \mathrm{tr}_f(F_{\mu\nu} u_\mu u_\nu) - 2\bar{L}_9^* \mathrm{tr}_f(E_i [u_0, u_i]) - 2\bar{L}_9^{*\prime} \mathrm{tr}_f(E_{0i} u u_i) \\ & +L_{10}^* \mathrm{tr}_f(u F_{\mu\nu} u F_{\mu\nu}) + 2H_1^* \mathrm{tr}_f(F_{\mu\nu}^2) + 2\bar{H}_1^* \mathrm{tr}_f(E_i^2) - H_2^* \mathrm{tr}_f(x^2) \end{aligned}$$

2. Cosmología cuántica en espacios cuánticos relativistas. Formalización matemática.

$$q_{ab} = e_a^i e_b^j \delta_{ij}$$

$$e_i^a e_a^j = \delta_i^j, e_i^a e_b^i = \delta_b^a,$$

$$E_i^a = \sqrt{q} e_i^a$$

$$K_a^i = K_{ab} e_j^b \delta^{ij}$$

$$\nabla_b E_i^a + \epsilon_{ijk} \Gamma_b^j E^{ak} = 0$$

$$\Gamma_a^i = -\frac{1}{2} \epsilon^{ijk} E_{jb} \left(\partial_a E_k^b + \Gamma_{ca}^b E_k^c \right)$$

$$A_a^i = \Gamma_a^i + \gamma K_a^i$$

$$\left\{ A_i^a(x), E_b^j(y) \right\} = 8\pi G \gamma \delta_b^a \delta_i^j \delta(x - y)$$

$$\mathcal{G}_i = \partial_a E_i^a + \epsilon_{ijk} \Gamma_a^j E^{ak} = 0$$

$$\mathcal{C}_a = F_{ab}^i E_i^b = 0$$

$$\mathcal{C} = \frac{1}{\sqrt{|\det(E)|}} \epsilon_{ijk} \left[F_{ab}^i - (1 + \gamma^2) \epsilon_{mn}^i K_a^m K_b^n \right] E^{aj} E^{bk} = 0$$

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{ijk} A_a^j A_b^k$$

$$C_{\text{grav}} = \int_\Sigma d^3x \mathcal{C} = -\frac{1}{\gamma^2} \int_\Sigma d^3x \frac{\epsilon_{ijk} F_{ab}^i E^{aj} E^{bk}}{\sqrt{|\det(E)|}}$$

$$A_a^i = c V_o^{-1/3 o} e_a^i, E_i^a = p V_o^{-2/3} \sqrt{{}^o q}\, {}^o e_i^a$$

$$\{c, p\} = \frac{8\pi G \gamma}{3}$$

$$\left\{ \phi, P_\phi \right\} = 1$$

$$C = C_{\text{grav}} + C_{\text{mat}} = -\frac{6}{\gamma^2} c^2 \sqrt{|p|} + 8\pi G \frac{P_\phi^2}{V} = 0$$

$$h_e(A) = \mathcal{P} e^{\int_e dx^a A_a^i(x) \tau_i},$$

$$E(S,f)=\int_S f^i E_i^a \epsilon_{abc} dx^b dx^c$$

$$\{E(S,f),h_e(A)\}=2\pi G\gamma\epsilon(e,S)f^i\tau_i h_e(A)$$

$$h_i^\mu(c)=e^{\mu c\tau_i}=\cos\left(\frac{\mu c}{2}\right)\mathbb{I}+2\sin\left(\frac{\mu c}{2}\right)\tau_i$$

$$\mathcal{N}_\mu(c)=e^{\frac{i}{2}\mu c}$$

$$E(S,f)=pV_o^{-2/3}A_{S,f}$$

$$\{\mathcal{N}_\mu(c),p\}=i\frac{4\pi G\gamma}{3}\mu\mathcal{N}_\mu(c)$$

$$\langle\mu\mid\mu'\rangle=\delta_{\mu\mu'}$$

$$\hat{\mathcal{N}}_{\mu'}|\mu\rangle=|\mu+\mu'\rangle$$

$$\hat{p}|\mu\rangle=p(\mu)|\mu\rangle, p(\mu)=\frac{4\pi l_{\mathrm{Pl}}^2\gamma}{3}\mu$$

$$h_{\square_{ij}}^\mu=h_i^\mu h_j^\mu\left(h_i^\mu\right)^{-1}\left(h_j^\mu\right)^{-1},$$

$$F_{ab}^i=-2\lim_{A_\square\to 0}\operatorname{tr}\left(\frac{h_{\square_{jk}}^\mu-\delta_{jk}}{A_\square}\tau^i\right)\,{}^oe_a^{jo}e_b^k,$$

$$F_{ab}^i=-2\operatorname{tr}\left(\frac{h_{\square_{jk}}^{\bar\mu}-\delta_{jk}}{\bar\mu^2V_o^{2/3}}\tau^i\right)\,{}^oe_a^{jo}e_b^k$$

$$\frac{1}{\bar\mu}=\sqrt{\frac{|p|}{\Delta}}$$

$$\hat{\mathcal{N}}_{\bar\mu}=e^{i\widehat{\frac{\bar\mu c}{2}}}$$

$$\hat{\mathcal{N}}_{\bar\mu}|v\rangle=|v+1\rangle.$$

$$\hat{p}|v\rangle=\left(2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}\right)^{2/3}\operatorname{sgn}(v)|v|^{2/3}|v\rangle.$$

$$\hat{V}=\widehat{|p|}^{3/2}, \hat{V}|v\rangle=2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}|v||v\rangle$$

$$\frac{1}{V}=\frac{\sqrt{{}^oq}}{\sqrt{|\det(E)|}V_o}$$

$$\frac{\epsilon_{ijk}E^{aj}E^{bk}}{\sqrt{|\det(E)|}}=\sum_{k=1}^{3}\frac{\operatorname{sgn}(p)}{2\pi\gamma GV_o^{1/3}}\frac{1}{l}\,{}^oe_c^k\,{}^o\epsilon^{abc}\operatorname{tr}\left(h_k^l(c)\left\{\left[h_k^l(c)\right]^{-1},V\right\}\tau_i\right)$$

$$\{,\}\to-\frac{i}{\hbar}[\hat{},\hat{}]$$

$$\hat{C}_{\text{grav}} = i\frac{\widehat{\text{sgn}(p)}}{2\pi\gamma^3 l_{\text{Pl}}^2 \Delta^{3/2}}\hat{V}\sum_{ijk}\epsilon^{ijk}\text{tr}\left(\hat{h}_i^{\bar{\mu}}\hat{h}_j^{\bar{\mu}}\left(\hat{h}_i^{\bar{\mu}}\right)^{-1}\left(\hat{h}_j^{\bar{\mu}}\right)^{-1}\hat{h}_k^{\bar{\mu}}\left[\left(\hat{h}_k^{\bar{\mu}}\right)^{-1},\hat{V}\right]\right),$$

$$\left(\hat{h}_i^{\bar{\mu}}\right)^{\pm 1} = \cos\widehat{\left(\frac{\bar{\mu}c}{2}\right)}\mathbb{I} \pm 2\sin\widehat{\left(\frac{\bar{\mu}c}{2}\right)}\tau_i$$

$$\cos\widehat{\left(\frac{\bar{\mu}c}{2}\right)} = \frac{\hat{\mathcal{N}}_{\bar{\mu}} + \hat{\mathcal{N}}_{-\bar{\mu}}}{2}, \sin\widehat{\left(\frac{\bar{\mu}c}{2}\right)} = \frac{\hat{\mathcal{N}}_{\bar{\mu}} - \hat{\mathcal{N}}_{-\bar{\mu}}}{2i}$$

$$\hat{C}_{\text{grav}} = \sin\widehat{(\bar{\mu}c)}\left[i\frac{3\widehat{\text{sgn}(p)}}{2\pi\gamma^3 l_{\text{Pl}}^2\Delta^{3/2}}\hat{V}\left(\hat{\mathcal{N}}_{\bar{\mu}}\hat{V}\hat{\mathcal{N}}_{-\bar{\mu}} - \hat{\mathcal{N}}_{-\bar{\mu}}\hat{V}\hat{\mathcal{N}}_{\bar{\mu}}\right)\right]\sin\widehat{(\bar{\mu}c)}$$

$$\hat{C}_{\text{grav}}|v\rangle = \tilde{f}_+(v)|v+4\rangle + \tilde{f}_o(v)|v\rangle + \tilde{f}_-(v)|v-4\rangle$$

$$\begin{aligned}\tilde{f}_+(v) &= \frac{3\pi l_{\text{Pl}}^2}{2\gamma\sqrt{\Delta}}|v+2||\,|v+1| - |v+3|\,|,\\ \tilde{f}_-(v) &= \tilde{f}_+(v+4),\\ \tilde{f}_o(v) &= -\tilde{f}_+(v) - \tilde{f}_-(v).\end{aligned}$$

$$\frac{\text{sgn}(p)}{|p|^{1-a}} = \frac{1}{a4\pi\gamma G}\frac{1}{l}\text{tr}\left(\sum_i \tau^i h_i^l(c)\left\{\left[h_i^l(c)\right]^{-1},|p|^a\right\}\right)$$

$$\begin{aligned}\widehat{\left[\frac{1}{\sqrt{|p|}}\right]} &= -\frac{i}{2\pi\gamma l_{\text{Pl}}^2\sqrt{\Delta}}\widehat{\text{sgn}(p)}\widehat{\sqrt{|p|}}\text{tr}\left(\sum_i \tau^i\hat{h}_i^{\bar{\mu}}\left[\widehat{\left(h_i^{\bar{\mu}}\right)^{-1}},\widehat{\sqrt{|p|}}\right]\right)\\ &= \frac{3}{4\pi\gamma l_{\text{Pl}}^2\sqrt{\Delta}}\widehat{\text{sgn}(p)}\widehat{\sqrt{|p|}}\left(\hat{\mathcal{N}}_{-\bar{\mu}}\widehat{\sqrt{|p|}}\hat{\mathcal{N}}_{\bar{\mu}} - \hat{\mathcal{N}}_{\bar{\mu}}\widehat{\sqrt{|p|}}\hat{\mathcal{N}}_{-\bar{\mu}}\right).\end{aligned}$$

$$\widehat{\left[\frac{1}{\sqrt{|p|}}\right]}|v\rangle = b(v)|v\rangle,\ b(v) = \frac{3}{2}\frac{1}{\left(2\pi\gamma l_{\text{Pl}}^2\sqrt{\Delta}\right)^{1/3}}|v|^{1/3}\left||v+1|^{1/3} - |v-1|^{1/3}\right|$$

$$\hat{C}_{\text{mat}} = -8\pi l_{\text{Pl1}}^2\hbar\left[\widehat{\left[\frac{1}{\sqrt{|p|}}\right]}\right]^3\partial_\phi^2$$

$$\text{Cil}_S \otimes \mathcal{S}(\mathbb{R}) \subset \mathcal{H}_{\text{cin}} \subset (\text{Cil}_S \otimes \mathcal{S}(\mathbb{R}))^*.$$

$$(\psi \mid = \int_{\mathbb{R}} d\phi \sum_v \psi(v,\phi)\langle\phi| \otimes \langle v|$$

$$(\psi \mid \hat{C}^\dagger = 0 \leftrightarrow (\psi \mid \hat{C}_{\text{mat}}^\dagger = -(\psi \mid \hat{C}_{\text{grav}}^\dagger.$$

$$8\pi l_{\text{Pl}}^2\hbar[b(v)]^3\partial_\phi^2\psi(v,\phi) = -\left[\tilde{f}_+(v)\psi(v+4,\phi) + \tilde{f}_o(v)\psi(v,\phi) + \tilde{f}_-(v)\psi(v-4,\phi)\right]$$

$$\partial_\phi^2\psi(v,\phi) = -\widehat{\Theta}\psi(v,\phi)$$

$$\widehat{\Theta} = [B(v)]^{-1}\hat{C}_{\text{grav}}, B(v) = 8\pi l_{\text{Pl}}^2\hbar[b(v)]^3,$$

$$\langle v \mid v'\rangle = B(v)\delta_{vv'}$$

$$\mathcal{L}_{\pm|\epsilon|} := \{v = \pm|\epsilon| + 4n, n \in \mathbb{Z}\}, |\epsilon| \in [0,2]$$

$$e_{-|k|}^{\pm}(v) \overset{v\gg 1}{\rightarrow} \frac{1}{\sqrt{2\pi}}e^{-ik\ln|v|}, e_{-|k|}^{\pm}(v) \overset{v\ll -1}{\rightarrow} \frac{A}{\sqrt{2\pi}}e^{-ik\ln|v|} + \frac{B}{\sqrt{2\pi}}e^{ik\ln|v|},$$

$$\psi(v,\phi)=\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dve_k^{(s)}(v)\left[\tilde{\psi}_+(k)e^{i\nu(k)\phi}+\tilde{\psi}_-(k)e^{-i\nu(k)\phi}\right]$$

$$\mathcal{H}_{\text{fis}}^{\epsilon}=L^2(\mathbb{R},dk)$$

$$\langle\psi_1\mid\psi_2\rangle_\epsilon=\sum_{v\in\mathcal{L}_{+|\epsilon|}\cup\mathcal{L}_{-|\epsilon|}}B(v)\psi_1^*(v,\phi)\psi(\,_2v,\phi),$$

$$\psi_\pm(v,\phi)=\int_{-\infty}^{\infty}dk\tilde{\psi}_\pm(k)e_k^{(s)}(v)e^{\pm i\nu(k)\phi}$$

$$\psi_\pm(v,\phi)=U_\pm(\phi-\phi_0)\psi_\pm(v,\phi_0),U_\pm(\phi-\phi_0)=e^{\pm i\sqrt{\widehat{\Theta}}(\phi-\phi_0)}$$

$$\widehat{|v|}\phi_0\psi(v,\phi)=U_+(\phi-\phi_0)|v|\psi_+(v,\phi_0)+U_-(\phi-\phi_0)|v|\psi_-(v,\phi_0).$$

$$\tilde{\psi}_+(k)=e^{-\frac{(k-k^*)^2}{2\sigma^2}}e^{-i\nu(k)\phi^*}$$

$$\psi_+(v,\phi)=\int_{-\infty}^{0}dke^{-\frac{(k-k^*)^2}{2\sigma^2}}e_{-|k|}^{(s)}(v)e^{i\nu(k)(\phi-\phi^*)}$$

$$\phi(\theta)=\sum_{m\in\mathbb{Z}}\frac{1}{\sqrt{2\pi}}\phi_m e^{im\theta},\phi_m=\frac{1}{\sqrt{2\pi}}\oint d\theta\phi(\theta)e^{-im\theta}$$

$$ds^2=q_{\theta\theta}\left[-\tau^2N^2dt^2+\left(d\theta+N^\theta dt\right)^2\right]+q_{\sigma\sigma}d\sigma^2+q_{\delta\delta}d\delta^2,$$
$$q_{\theta\theta}=\frac{4G}{\pi}e^{\bar{\gamma}-\frac{\xi}{\sqrt{\tau}}-\frac{\xi^2}{4\tau}},q_{\sigma\sigma}=\frac{\pi}{4G}\tau^2e^{-\frac{\xi}{\sqrt{\tau}}},q_{\delta\delta}=\frac{4G}{\pi}e^{\frac{\xi}{\sqrt{\tau}}}.$$

$$S=\int_{t_i}^{t_f}dt\oint d\theta\left[P_\tau\dot{\tau}+P_{\bar{\gamma}}\dot{\bar{\gamma}}+P_\xi\dot{\xi}-\left(N\tilde{\mathcal{C}}+N^\theta\mathcal{C}_\theta\right)\right]$$
$$\mathcal{C}_\theta=P_\tau\tau'+P_{\bar{\gamma}}\bar{\gamma}'+P_\xi\xi'-2P_{\bar{\gamma}}'=0$$

$$\tilde{\mathcal{C}}=\frac{4G}{\pi}\left[\frac{\tau}{2}P_\xi^2+\frac{\xi^2}{8\tau}P_{\bar{\gamma}}^2-\tau P_\tau P_{\bar{\gamma}}\right]+\frac{\pi}{4G}\frac{\tau}{2}\left[4\tau''-2\bar{\gamma}'\tau'-\left(\frac{\xi\tau'}{2\tau}\right)^2+(\xi')^2\right]=0$$

$$g_1\equiv P_{\bar{\gamma}}-\frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}}=0$$
$$g_2\equiv\tau+t\frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}}=0$$

$$P_{\bar{\gamma}_0}=\frac{1}{\sqrt{2\pi}}\oint d\theta P_{\bar{\gamma}}(\theta)$$

$$\left\{g_1(\theta),\oint d\bar{\theta}\left[G(\bar{\theta})\mathcal{C}_\theta(\bar{\theta})+\underset{\sim}{F}(\bar{\theta})\tilde{\mathcal{C}}(\bar{\theta})\right]\right\}=\frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}}G'(\theta)$$
$$\left\{g_2(\theta),\oint d\bar{\theta}\left[G(\bar{\theta})\mathcal{C}_\theta(\bar{\theta})+\underset{\sim}{F}(\bar{\theta})\tilde{\mathcal{C}}(\bar{\theta})\right]\right\}=t\left(\frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}}\right)^2\underset{\sim}{F}(\theta)$$

$$\dot{g}_i=\partial_t g_i+\left\{g_i,\oint d\theta\left[N^\theta(\theta)\mathcal{C}_\theta(\theta)+\underset{\sim}{N}(\theta)\tilde{\mathcal{C}}(\theta)\right]\right\}=0$$

$$\underset{\sim}{N}=-\frac{\sqrt{2\pi}}{tP_{\bar{\gamma}_0}}$$

$$-\frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}}\bar{\gamma}' = P_\xi \xi'$$

$$\bar{\gamma}(\theta) = \frac{\bar{\gamma}_0}{\sqrt{2\pi}} + \sum_{m\neq 0} \frac{i}{\sqrt{2\pi} m P_{\bar{\gamma}_0}} \oint d\bar{\theta} e^{im(\theta-\bar{\theta})} P_\xi(\bar{\theta})\xi'(\bar{\theta})$$

$$S = \int_{t_i}^{t_f} dt \left[P_{\bar{\gamma}_0}\dot{\bar{\gamma}}_0 + \oint d\theta P_\xi \dot{\xi} - \left(N^\theta C_\theta + \bar{H}_r\right) \right]$$

$$C_\theta = \oint d\theta P_\xi \xi' = 0$$

$$\bar{H}_r = \frac{1}{2}\oint d\theta \left[P_\xi^2 + (\xi')^2 + \frac{\xi^2}{4t^2} \right]$$

$$\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0$$

$$a_0 = \sqrt{\frac{\pi}{8G}}\left(\xi_0 + i\frac{4G}{\pi}P_{\xi_0}\right), a_0^* = \sqrt{\frac{\pi}{8G}}\left(\xi_0 - i\frac{4G}{\pi}P_{\xi_0}\right),$$

$$a_m = \sqrt{\frac{\pi}{8G|m|}}\left(|m|\xi_m + i\frac{4G}{\pi}P_{\xi_m}\right), a_{-m}^* = \sqrt{\frac{\pi}{8G|m|}}\left(|m|\xi_m - i\frac{4G}{\pi}P_{\xi_m}\right),$$

$$C_\theta = \sum_{m=1}^{\infty} m(a_m^* a_m - a_{-m}^* a_{-m}) = 0$$

$$\hat{a}_m|0\rangle = 0, \forall m \in \mathbb{Z}$$

$$\hat{C}_\theta = \hbar \sum_{m>0}^{\infty} m\left(\hat{a}_m^\dagger \hat{a}_m - \hat{a}_{-m}^\dagger \hat{a}_{-m}\right)$$

$$\hat{C} := \widehat{\left[\frac{1}{V}\right]}^{1/2}\left(-\frac{6}{\gamma^2}\widehat{\Omega}^2 + 8\pi G \hat{P}_\phi^2\right)\widehat{\left[\frac{1}{V}\right]}^{1/2}$$

$$\widehat{\left[\frac{1}{V}\right]} = \widehat{\left[\frac{1}{\sqrt{|p|}}\right]}^3$$

$$\widehat{\Omega} = \frac{1}{4i\sqrt{\Delta}}\widehat{\sqrt{|p|}}\widehat{\left[\frac{1}{\sqrt{|p|}}\right]}^{-1/2}\left[\left(\hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}}\right)\widehat{\mathrm{sgn}(p)} + \widehat{\mathrm{sgn}(p)}\left(\hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}}\right)\right]$$
$$\times \widehat{\left[\frac{1}{\sqrt{|p|}}\right]}^{-1/2}\widehat{\sqrt{|p|}},$$

$$\sin(\bar{\mu}c)\mathrm{sgn}(p) \to \frac{1}{2}\left[\widehat{\sin(\bar{\mu}c)}\widehat{\mathrm{sgn}(p)} + \widehat{\mathrm{sgn}(p)}\widehat{\sin(\bar{\mu}c)}\right]$$

$$\widetilde{\mathrm{Cyl}}_S = \mathrm{lin}\{|v\rangle; v \in \mathbb{R} - \{0\}\},$$

$$\left(\tilde{\psi}\right| \to \left(\psi\right| = \left(\tilde{\psi}\right|\widehat{\left[\frac{1}{V}\right]}^{1/2}$$

$$\left(\psi \mid \hat{\mathcal{C}}^\dagger = 0, \hat{\mathcal{C}} = \widehat{\left[\frac{1}{V}\right]}^{-1/2} \hat{C} \widehat{\left[\frac{1}{V}\right]}^{-1/2}\right.$$

$$\hat{C} = -\frac{6}{\gamma^2}\widehat{\Omega}^2 + 8\pi G \hat{P}_\phi^2$$

$$\tilde{C} = \int_{\mathcal{V}} d^3x \tilde{\mathcal{C}} = VC$$

$$\widehat{\Omega}^2|v\rangle = -f_+(v)f_+(v+2)|v+4\rangle + [f_+^2(v) + f_-^2(v)]|v\rangle - f_-(v)f_-(v-2)|v-4\rangle,$$

$$f_\pm(v) = \frac{\pi\gamma l_{\mathrm{Pl}}^2}{3} g(v\pm 2)s_\pm(v)g(v)$$
$$s_\pm(v) = \mathrm{sgn}(v\pm 2) + \mathrm{sgn}(v)$$
$$g(v) = \begin{cases} \left|\left|1+\frac{1}{v}\right|^{\frac{1}{3}} - \left|1-\frac{1}{v}\right|^{\frac{1}{3}}\right|^{-\frac{1}{2}} & \text{si } v \neq 0 \\ 0 & \text{si } v = 0 \end{cases}$$

$f_-(v)f_-(v-2) = 0,$ si $v \in (0,4],$

$f_+(v)f_+(v+2) = 0,$ si $v \in [-4,0).$

$$\mathcal{L}_{\tilde{\varepsilon}}^{\pm} = \{v = \pm(\tilde{\varepsilon} + 4n), n \in \mathbb{N}\}, \tilde{\varepsilon} \in (0,4]$$

$$\mathrm{Cil}_{\tilde{\varepsilon}}^{\pm} = \mathrm{lin}\{|v\rangle, v \in \mathcal{L}_{\tilde{\varepsilon}}^{\pm}\}$$

$$\widetilde{\mathcal{H}}_{\mathrm{grav}} = \oplus_{\tilde{\varepsilon}} (\mathcal{H}_{\tilde{\varepsilon}}^{+} \oplus \mathcal{H}_{\tilde{\varepsilon}}^{-})$$

$$\hat{H}'_{\mathrm{APS}} = -\frac{4\pi G}{3}\left[\hat{\mathcal{N}}_{2\bar{\mu}}(\hat{v}^2 - 2 - a)\hat{\mathcal{N}}_{2\bar{\mu}} + \hat{\mathcal{N}}_{-2\bar{\mu}}(\hat{v}^2 - 2 - a)\hat{\mathcal{N}}_{-2\bar{\mu}} - 2\hat{v}^2 + 2a\right]$$

$$\widehat{\Omega}^2 = -\hat{\mathcal{N}}_{2\bar{\mu}} f_-(\hat{v}) f_+(\hat{v}) \hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}} f_+(\hat{v}) f_-(\hat{v}) \hat{\mathcal{N}}_{-2\bar{\mu}} + \{[f_+(\hat{v})]^2 + [f_-(\hat{v})]^2\}$$

$$\widehat{\Omega}^2 - \frac{4}{3\pi G}\hat{H}'_{\mathrm{APS}} = -\hat{\mathcal{N}}_{2\bar{\mu}} h_1(\hat{v})\hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}} h_1(\hat{v})\hat{\mathcal{N}}_{-2\bar{\mu}} + h_2(\hat{v})$$

$$h_1(\hat{v}) = f_-(\hat{v})f_+(\hat{v}) - (\hat{v}^2 - 2 - a)$$
$$h_2(\hat{v}) = \{[f_+(\hat{v})]^2 + [f_-(\hat{v})]^2\} - 2\hat{v}^2 + 2a$$

$$e_\lambda^{\tilde{\varepsilon}}(v+4) = \left[\frac{f_-(v+2)}{f_+(v+2)} + \frac{f_-(v)}{f_+(v)}\frac{f_+(v-2)}{f_+(v+2)} - \frac{\lambda}{f_+(v)f_+(v+2)}\right] e_\lambda^{\tilde{\varepsilon}}(v) - \frac{f_-(v)}{f_+(v)}\frac{f_-(v-2)}{f_+(v+2)} e_\lambda^{\tilde{\varepsilon}}(v-4).$$

$$e_\lambda^{\tilde{\varepsilon}}(\tilde{\varepsilon}+4n) = \left[\mathcal{S}_{\tilde{\varepsilon}}(0,2n) + \frac{F(\tilde{\varepsilon})}{G_\lambda(\tilde{\varepsilon}-2)}\mathcal{S}_{\tilde{\varepsilon}}(1,2n)\right] e_\lambda^{\tilde{\varepsilon}}(\tilde{\varepsilon}), n \in \mathbb{N}^+,$$

$$F(v) = \frac{f_-(v)}{f_+(v)}, G_\lambda(v) = -\frac{i\sqrt{\lambda}}{f_+(v)}$$
$$\mathcal{S}_{\tilde{\varepsilon}}(a,b) = \sum_{O(a\to b)}\left[\prod_{\{r_p\}} F(\tilde{\varepsilon} + 2r_p + 2)\prod_{\{s_q\}} G_\lambda(\tilde{\varepsilon} + 2s_q)\right]$$

$$1 \to 2 \to 3 \to 4{:}\{r_p\} = \emptyset, \{s_q\} = \{1,2,3\};$$
$$1 \to 2 \to 4{:}\{r_p\} = \{2\}, \{s_q\} = \{1\};$$
$$1 \to 3 \to 4{:}\{r_p\} = \{1\}, \{s_q\} = \{3\}.$$

$$\langle e^{\tilde{\varepsilon}}_{\lambda} \mid e^{\tilde{\varepsilon}}_{\lambda'} \rangle = \delta(\lambda - \lambda')$$

$$\mathbb{I} = \int_{\mathbb{R}^+} d\lambda \left| e^{\tilde{\varepsilon}}_{\lambda} \right\rangle \left\langle e^{\tilde{\varepsilon}}_{\lambda} \right|$$

$$\Psi(v,\phi) = [\mathcal{P}\psi](v,\phi) = \int_{\mathbb{R}} dt e^{it\hat{\mathcal{C}}} \psi(v,\phi)$$

$$\psi(v,\phi) = \int_0^{\infty} d\lambda \int_{-\infty}^{\infty} dv e^{\tilde{\varepsilon}}_{\lambda}(v) \left[\tilde{\psi}_+(\lambda) e^{iv\phi} + \tilde{\psi}_-(\lambda) e^{-iv\phi} \right]$$

$$\Psi(v,\phi) = \int_0^{\infty} \frac{d\lambda}{\nu(\lambda)} e^{\tilde{\varepsilon}}_{\lambda}(v) \left[\tilde{\psi}_+(\lambda) e^{i\nu(\lambda)\phi} + \tilde{\psi}_-(\lambda) e^{-i\nu(\lambda)\phi} \right]$$

$$\nu(\lambda) = \sqrt{\frac{3\lambda}{4\pi l_{\mathrm{Pl}}^2 \hbar \gamma^2}}$$

$$\langle \Psi_1 \mid \Psi_2 \rangle_{\mathrm{fis}} = \langle \mathcal{P}\psi_1 \mid \psi_2 \rangle_{\mathrm{cin}} = \int_0^{\infty} \frac{d\lambda}{\nu(\lambda)} \left[\tilde{\psi}^*_{1+}(\lambda) \tilde{\psi}_{2+}(\lambda) + \tilde{\psi}^*_{1-}(\lambda) \tilde{\psi}_{2-}(\lambda) \right]$$

$$\mathcal{H}^{\tilde{\varepsilon}}_{\mathrm{fis}} = L^2 \left(\mathbb{R}^+, \frac{d\lambda}{\nu(\lambda)} \right)$$

$$\Psi_{\pm}(v,\phi) = U_{\pm}(\phi - \phi_0) \Psi_{\pm}(v,\phi_0), U_{\pm}(\phi - \phi_0) = \exp\left[\pm i \sqrt{\frac{3}{4\pi l_{\mathrm{Pl}}^2 \hbar \gamma^2}} \widehat{\Omega}^2 (\phi - \phi_0) \right]$$

$$p = \mathrm{sgn}(v) \left(2\pi\gamma l_{\mathrm{Pl}}^2 \sqrt{\Delta} |v| \right)^{2/3}$$

$$\hat{c} = i2(2\pi\gamma l_{\mathrm{Pl}}^2/\Delta)^{1/3} |v|^{1/6} \partial_v |v|^{1/6}$$

$$\begin{aligned} \underline{\widehat{\Omega}}^2 &= -\frac{\alpha^2}{4} \sqrt{|v|} [\mathrm{sgn}(v)\partial_v + \partial_v \mathrm{sgn}(v)] |v| [\mathrm{sgn}(v)\partial_v + \partial_v \mathrm{sgn}(v)] \sqrt{|v|} \\ &= -\frac{\alpha^2}{4} [1 + 4v\partial_v + 4(v\partial_v)^2], \end{aligned}$$

$$\underline{e}_{\omega}(v) = \frac{1}{\sqrt{2\pi\alpha|v|}} \exp\left(-i\omega \frac{\ln |v|}{\alpha} \right)$$

$$\langle \underline{e}_{\omega} \mid \underline{e}_{\omega'} \rangle = \delta(\omega - \omega')$$

$$\vec{e}^{\tilde{\varepsilon}}_{\lambda}(v) = \begin{pmatrix} e^{\tilde{\varepsilon}}_{\lambda}(v) \\ e^{\tilde{\varepsilon}}_{\lambda}(v-4) \end{pmatrix}$$

$$\begin{matrix} \underline{e}_{\omega}(v) & \underline{e}_{-\omega}(v) \\ \underline{e}_{\omega}(v-4) & \underline{e}_{-\omega}(v-4) \end{matrix}$$

$$\boldsymbol{M}(v) = \mathbb{I} + \mathcal{O}(v^{-3})$$

$$\vec{\psi} = \lim_{v \to \infty} \vec{\psi}(v)$$

$$\vec{\psi}(v) = \vec{\psi} + \mathcal{O}(v^{-2})$$

$$e^{\tilde{\varepsilon}}_{\lambda}(v) \overset{v \gg 1}{\rightarrow} r \left\{ \exp\left[i\phi_{\tilde{\varepsilon}}(\omega) \right] \underline{e}_{\omega}(v) + \exp\left[-i\phi_{\tilde{\varepsilon}}(\omega) \right] \underline{e}_{-\omega}(v) \right\},$$

$$\phi_{\tilde{\varepsilon}}(\omega) = T(|\omega|) + c_{\tilde{\varepsilon}} + R_{\tilde{\varepsilon}}(|\omega|),$$

$$A_a^i = \frac{c^i}{2\pi}\delta_a^i, E_i^a = \frac{p_i}{4\pi^2}\delta_i^a\sqrt{{}^\sigma q}$$

$$\{c^i, p_j\} = 8\pi G\gamma\delta_j^i$$

$$ds^2 = -N^2dt^2 + \frac{|p_\theta p_\sigma p_\delta|}{4\pi^2}\left(\frac{d\theta^2}{p_\theta^2} + \frac{d\sigma^2}{p_\sigma^2} + \frac{d\delta^2}{p_\delta^2}\right),$$

$$C_{\mathrm{BI}} = -\frac{2}{\gamma^2}\frac{c^\theta p_\theta c^\sigma p_\sigma + c^\theta p_\theta c^\delta p_\delta + c^\sigma p_\sigma c^\delta p_\delta}{V} = 0$$

$$h_i^{\mu_i}(c^i) = e^{\mu_i c^i \tau_i}$$

$$E(A_\square^i, f = 1) = \frac{p_i}{4\pi^2}A_\square^i$$

$$\mathrm{Cil}_\mathrm{S} = \otimes_i \mathrm{Cil}_\mathrm{S}^i = \mathrm{lin}\{|\mu_\theta, \mu_\sigma, \mu_\delta\rangle\}$$

$$\mathcal{H}_{\mathrm{grav}} = \otimes_i \mathcal{H}_{\mathrm{grav}}^i$$

$$\langle \mu_i \mid \mu_i' \rangle = \delta_{\mu_i \mu_i'}$$

$$\hat{p}_i|\mu_i\rangle = p_i(\mu_i)|\mu_i\rangle, p_i(\mu_i) = 4\pi\gamma l_{\mathrm{Pl}}^2\mu_i$$
$$\hat{\mathcal{N}}_{\mu_i'}|\mu_i\rangle = |\mu_i + \mu_i'\rangle$$

$$\frac{1}{\bar{\mu}_i} = \frac{\sqrt{|p_i|}}{\sqrt{\Delta}}.$$

$$\frac{1}{\bar{\mu}_i} = \frac{1}{\sqrt{\Delta}}\sqrt{\left|\frac{p_j p_k}{p_i}\right|}$$

$$v = \mathrm{sgn}(p_\theta p_\sigma p_\delta)\frac{\sqrt{|p_\theta p_\sigma p_\delta|}}{2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}},$$

$$F_{ab}^i = -2\sum_{j,k}\mathrm{tr}\left(\frac{h_{\square_{jk}}^{\bar{\mu}} - \delta_{jk}}{4\pi^2\bar{\mu}_j\bar{\mu}_k}\tau^i\right)\delta_a^j\delta_b^k,$$

$$h_{\square_{jk}}^{\bar{\mu}} = h_j^{\bar{\mu}_j}h_k^{\bar{\mu}_k}\left(h_j^{\bar{\mu}_j}\right)^{-1}\left(h_k^{\bar{\mu}_k}\right)^{-1}.$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_i}|v_i\rangle = |v_i \pm 1\rangle$$
$$\hat{p}_i|v_i\rangle = 3^{2/3}\left(2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}\right)^{2/3}\mathrm{sgn}(v_i)|v_i|^{2/3}|v_i\rangle$$

$$\bar{\mu}_i\partial_{\mu_i} = 4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}\sqrt{\left|\frac{p_i}{p_j p_k}\right|}\partial_{p_i}$$

$$\partial_{\lambda_i} = 2\sqrt{|p_i|}\partial_{p_i}$$

$$\bar{\mu}_i\partial_{\mu_i} = \frac{1}{2|\lambda_j\lambda_k|}\partial_{\lambda_i}$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\theta}|\lambda_\theta,\lambda_\sigma,\lambda_\delta\rangle = \left|\lambda_\theta \pm \frac{1}{2|\lambda_\sigma\lambda_\delta|},\lambda_\sigma,\lambda_\delta\right\rangle$$

$$\hat{p}_i|\lambda_\theta,\lambda_\sigma,\lambda_\delta\rangle = \left(4\pi\gamma l_{\mathrm{P}1}^2\sqrt{\Delta}\right)^{2/3}\mathrm{sgn}(\lambda_i)\lambda_i^2|\lambda_\theta,\lambda_\sigma.\lambda_\delta\rangle.$$

$$v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\theta}|v,\lambda_\sigma,\lambda_\delta\rangle = |v \pm \mathrm{sgn}(\lambda_\sigma\lambda_\delta),\lambda_\sigma,\lambda_\delta\rangle$$

$$\hat{p}_\theta|v,\lambda_\sigma,\lambda_\delta\rangle \ = \left(4\pi\gamma l_{\mathrm{P}1}^2\sqrt{\Delta}\right)^{\frac{2}{3}}\mathrm{sgn}\left(\frac{v}{\lambda_\sigma\lambda_\delta}\right)\frac{v^2}{4\lambda_\sigma^2\lambda_\delta^2}|v,\lambda_\sigma,\lambda_\delta\rangle,$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\sigma}|v,\lambda_\sigma,\lambda_\delta\rangle \ = \left|v \pm \mathrm{sgn}(\lambda_\sigma v),\left(\frac{v \pm \mathrm{sgn}(v\lambda_\sigma)}{v}\right)\lambda_\sigma,\lambda_\delta\right\rangle,$$

$$\hat{p}_\sigma|v,\lambda_\sigma,\lambda_\delta\rangle \ = \left(4\pi\gamma l_{\mathrm{P}1}^2\sqrt{\Delta}\right)^{\frac{2}{3}}\mathrm{sgn}(\lambda_\sigma)\lambda_\sigma^2|v,\lambda_\sigma,\lambda_\delta\rangle,$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\delta}|v,\lambda_\sigma,\lambda_\delta\rangle \ = \left|v \pm \mathrm{sgn}(\lambda_\delta v),\lambda_\sigma,\left(\frac{v \pm \mathrm{sgn}(v\lambda_\delta)}{v}\right)\lambda_\delta\right\rangle,$$

$$\hat{p}_\delta|v,\lambda_\sigma,\lambda_\delta\rangle \ = \left(4\pi\gamma l_{\mathrm{P}1}^2\sqrt{\Delta}\right)^{\frac{2}{3}}\mathrm{sgn}(\lambda_\delta)\lambda_\delta^2|v,\lambda_\sigma,\lambda_\delta\rangle.$$

$$\hat{V} = \sqrt{|p_\theta p_\sigma p_\delta|},\hat{V}|v,\lambda_\sigma,\lambda_\delta\rangle = 2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}|v||v,\lambda_\sigma,\lambda_\delta\rangle.$$

$$C_{\mathrm{BI}} = \frac{2}{\gamma^2}\frac{1}{V}\sum_{i,j,k}\epsilon^{ijk}p_jp_k\frac{\mathrm{tr}\left(\tau_i h_{\square_{jk}}^{\bar{\mu}}\right)}{\bar{\mu}_j\bar{\mu}_k}$$

$$C_{\mathrm{BI}}^{\mathrm{A}} = \frac{2}{\gamma^2\Delta}\sum_{i,j,k}\epsilon^{ijk}\frac{1}{\sqrt{|p_i|}}|p_j||p_k|\mathrm{sgn}(p_j)\mathrm{sgn}(p_k)\mathrm{tr}\left(\tau_i h_{\square_{jk}}^{\bar{\mu}}\right).$$

$$\hat{C}_{\mathrm{BI}}^{\mathrm{A}} = -\frac{2}{\gamma^2}\left\{\widehat{\Lambda}_\theta\widehat{\Lambda}_\sigma\widehat{\left[\frac{1}{\sqrt{|p_\delta|}}\right]} + \widehat{\Lambda}_\theta\widehat{\Lambda}_\delta\widehat{\left[\frac{1}{\sqrt{|p_\sigma|}}\right]} + \widehat{\Lambda}_\sigma\widehat{\Lambda}_\delta\widehat{\left[\frac{1}{\sqrt{|p_\theta|}}\right]}\right\}$$

$$\widehat{\Lambda}_i \ = \frac{1}{4i\sqrt{\Delta}}\widehat{\sqrt{|p_\iota|}}\left[\left(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}\right)\widehat{\mathrm{sgn}(p_\iota)} + \widehat{\mathrm{sgn}(p_\iota)}\left(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}\right)\right]\widehat{\sqrt{|p_\iota|}}$$

$$\widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]} \ = \frac{1}{4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}}\widehat{\mathrm{sgn}(p_\iota)}\widehat{\sqrt{|p_\iota|}}\left(\hat{\mathcal{N}}_{-\bar{\mu}_i}\widehat{\sqrt{|p_\iota|}}\hat{\mathcal{N}}_{\bar{\mu}_i} - \hat{\mathcal{N}}_{\bar{\mu}_i}\widehat{\sqrt{|p_\iota|}}\hat{\mathcal{N}}_{-\bar{\mu}_i}\right)$$

$$\widehat{\left[\frac{1}{V}\right]} = \otimes_i\widehat{\left[\frac{1}{\sqrt{|p_\iota|}}\right]}.$$

$$C_{\mathrm{BI}}^{\mathrm{B}} = \frac{2}{\gamma^2\Delta}\frac{1}{V}V^2\sum_{i,j,k}\epsilon^{ijk}\mathrm{sgn}(p_j)\mathrm{sgn}(p_k)\mathrm{tr}\left(\tau_i h_{\square_{jk}}^{\bar{\mu}}\right).$$

$$\hat{C}_{\mathrm{BI}}^{\mathrm{B}} = \widehat{\left[\frac{1}{V}\right]}^{1/2}\left[\hat{\mathcal{C}}^{(\theta)} + \hat{\mathcal{C}}^{(\sigma)} + \hat{\mathcal{C}}^{(\delta)}\right]\widehat{\left[\frac{1}{V}\right]}^{1/2}$$

$$\hat{\mathcal{C}}^{(i)} \ = -\frac{1}{4\gamma^2\Delta}\widehat{\sqrt{V}}\left[\hat{F}\hat{F}_j\hat{V}\hat{F}_k + \hat{F}_k\hat{V}\hat{F}_j\right]\widehat{\sqrt{V}}$$

$$\hat{F}_i = \frac{\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}}{2i}\widehat{\mathrm{sgn}(p_\iota)} + \widehat{\mathrm{sgn}(p_\iota)}\frac{\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}}{2i}$$

$$\widehat{\left[\frac{1}{|p_\theta|^{\frac{1}{4}}}\right]} = \frac{\widehat{\mathrm{sgn}(p_\theta)}}{2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}}\sqrt{|p_\sigma p_\delta|}\left[\hat{\mathcal{N}}_{-\bar{\mu}_\theta}|\hat{p}_\theta|^{\frac{1}{4}}\hat{\mathcal{N}}_{\bar{\mu}_\theta} - \hat{\mathcal{N}}_{\bar{\mu}_\theta}|\hat{p}_\theta|^{\frac{1}{4}}\hat{\mathcal{N}}_{-\bar{\mu}_\theta}\right]$$

$$\widehat{\left[\frac{1}{|p_i|^{\frac{1}{4}}}\right]}|v,\lambda_\sigma,\lambda_\delta\rangle = \frac{b_i^\star(v,\lambda_\sigma,\lambda_\delta)}{\left(4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}\right)^{\frac{1}{6}}}|v,\lambda_\sigma,\lambda_\delta\rangle,$$

$$b_\theta^\star(v,\lambda_\sigma,\lambda_\delta) = \sqrt{2|\lambda_\sigma\lambda_\delta|}\left|\sqrt{|v+1|} - \sqrt{|v-1|}\right|$$

$$b_a^\star(v,\lambda_\sigma,\lambda_\delta) = \sqrt{\left|\frac{v}{\lambda_a}\right|}\left|\sqrt{|v+1|} - \sqrt{|v-1|}\right|, a = \sigma,\delta$$

$$\widehat{\left[\frac{1}{V}\right]} = \otimes_i \widehat{\left[\frac{1}{|p_i|^{\frac{1}{4}}}\right]}^2$$

$$\hat{F}_\theta|v,\lambda_\sigma,\lambda_\delta\rangle = \frac{\mathrm{sgn}(\lambda_\sigma\lambda_\delta)}{2i}\{[\mathrm{sgn}(v - 2\mathrm{sgn}(\lambda_\sigma\lambda_\delta)) + \mathrm{sgn}(v)]|v - 2\mathrm{sgn}(\lambda_\sigma\lambda_\delta),\lambda_\sigma,\lambda_\delta\rangle$$

$$-[\mathrm{sgn}(v) + \mathrm{sgn}(v + 2\mathrm{sgn}(\lambda_\sigma\lambda_\delta))]|v + 2\mathrm{sgn}(\lambda_\sigma\lambda_\delta),\lambda_\sigma,\lambda_\delta\rangle\}$$

$$\hat{F}_\sigma|v,\lambda_\sigma,\lambda_\delta\rangle = \frac{\mathrm{sgn}(\lambda_\sigma)}{2i}$$

$$\times\left\{[1 + \mathrm{sgn}(|v| - 2\mathrm{sgn}(\lambda_\sigma))]\left|v - 2\mathrm{sgn}(v\lambda_\sigma), \frac{v - 2\mathrm{sgn}(v\lambda_\sigma)}{v}\lambda_\sigma,\lambda_\delta\right\rangle\right.$$

$$\left.-[1 + \mathrm{sgn}(|v| + 2\mathrm{sgn}(\lambda_\sigma))]\left|v + 2\mathrm{sgn}(v\lambda_\sigma), \frac{v + 2\mathrm{sgn}(v\lambda_\sigma)}{v}\lambda_\sigma,\lambda_\delta\right\rangle\right\}$$

$$\widetilde{\mathrm{Cil}}_{\mathrm{S}} = \otimes_i \widetilde{\mathrm{Cil}}_{\mathrm{S}}^i = \mathrm{lin}\{|v_\theta,v_\sigma,v_\delta\rangle; v_\theta v_\sigma v_\delta \neq 0\}$$
$$= \mathrm{lin}\{|v,\lambda_\sigma,\lambda_\delta\rangle; v\lambda_\sigma\lambda_\delta \neq 0\}$$

$$\hat{C}_{\mathrm{BI}} = \widehat{\left[\frac{1}{V}\right]}^{-1/2}\hat{C}_{\mathrm{BI}}\widehat{\left[\frac{1}{V}\right]}^{-1/2}$$

$$\hat{C}_{\mathrm{BI}}^{\mathrm{A}} = -\frac{2}{\gamma^2}(\hat{\Omega}_\theta\hat{\Omega}_\sigma + \hat{\Omega}_\theta\hat{\Omega}_\delta + \hat{\Omega}_\sigma\hat{\Omega}_\delta).$$

$$\hat{\Omega}_i = \widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}^{-\frac{1}{2}}\hat{\Lambda}_i\widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}^{-\frac{1}{2}}$$

$$\hat{\Omega}_i = \frac{1}{4i\sqrt{\Delta}}\widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}^{-\frac{1}{2}}\widehat{\sqrt{|p_i|}}\left[\left(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}\right)\widehat{\mathrm{sgn}(p_i)} + \widehat{\mathrm{sgn}(p_i)}\left(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}\right)\right]$$
$$\times\widehat{\sqrt{|p_i|}}\widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}^{-\frac{1}{2}}.$$

$$\hat{\Omega}_i|v_i\rangle = -i3[f_+(v_i)|v_i + 2\rangle - f_-(v_i)|v_i - 2\rangle],$$

$$\mathcal{L}_{\varepsilon_i}^{\pm} = \{\pm(\varepsilon_i + 2k), k \in \mathbb{N}\}, \varepsilon_i \in (0,2],$$

$$\mathrm{Cil}_{\varepsilon_i}^{\pm} = \mathrm{lin}\{|v_i\rangle; v_i \in \mathcal{L}_{\varepsilon_i}^{\pm}\}$$

$$\mathbb{I} = \int_{\mathbb{R}^+} d\lambda_i \left| {}^{(4)} e_{\lambda_i}^{\tilde{\varepsilon}_i} \right\rangle \left\langle {}^{(4)} e_{\lambda_i}^{\tilde{\varepsilon}_i} \right|$$

$$\mathcal{H}_{\varepsilon_i}^+ = {}^{(4)}\mathcal{H}_{\tilde{\varepsilon}_i=\varepsilon_i}^+ \oplus {}^{(4)}\mathcal{H}_{\tilde{\varepsilon}_i=\varepsilon_i+2}^+$$

$$\left| e_{\pm|\omega_i|}^{\varepsilon_i} \right\rangle = |X_{\pm|\omega_i|}| \left\{ \Box \left| {}^{(4)} e_{\lambda_i}^{\varepsilon_i} \right\rangle \mp i |Y_{\pm|\omega_i|}| \left| {}^{(4)} e_{\lambda_i}^{\varepsilon_i+2} \right\rangle \right\}$$

$$\mathbb{I} = \int_{\mathbb{R}} d\omega_i |e_{\omega_i}^{\varepsilon_i}\rangle\langle e_{\omega_i}^{\varepsilon_i}|.$$

$$\mathbb{I} = \int_{\mathbb{R}^+} d\lambda_i \left| {}^{(4)} e_{\lambda_i}^{\varepsilon_i} \right\rangle \left\langle {}^{(4)} e_{\lambda_i}^{\varepsilon_i} \right| + \int_{\mathbb{R}^+} d\lambda_i \left| {}^{(4)} e_{\lambda_i}^{\varepsilon_i+2} \right\rangle \left\langle {}^{(4)} e_{\lambda_i}^{\varepsilon_i+2} \right|$$

$$|Y_{-|\omega_i|}| = |Y_{+|\omega_i|}| = 1, |X_{+|\omega_i|}|^2 + |X_{-|\omega_i|}|^2 = 2|\omega_i|.$$

$$\left\langle e_{\omega_i}^{\varepsilon_i} \mid e_{\omega_i'}^{\varepsilon_i'} \right\rangle = \delta(\omega_i - \omega_i'), \left\langle {}^{(4)} e_{\lambda_i}^{\tilde{\varepsilon}_i} \middle| {}^{(4)} e_{\lambda_i'}^{\tilde{\varepsilon}_i} \right\rangle = \delta(\lambda_i - \lambda_i')$$

$$|X_{\omega_i}| \left| X_{\omega_i'} \right| \left(1 + |Y_{\omega_i}| \left| Y_{\omega_i'} \right|\right) = 2|\omega_i|$$

$$|e_{\omega_i}^{\varepsilon_i}\rangle = \sqrt{|\omega_i|} \left[\left| {}^{(4)} e_{\omega_i^2}^{\varepsilon_i} \right\rangle - i\mathrm{sgn}(\omega_i) \left| {}^{(4)} e_{\omega_i^2}^{2+\varepsilon_i} \right\rangle \right]$$

$$|e_0^{\varepsilon_i}\rangle = |{}^{(4)} e_0^{\varepsilon_i}\rangle$$

$$|e_{\omega_i}^{\varepsilon_i}\rangle = \sum_{v_i \in \mathcal{L}_{\varepsilon_i}^+} e_{\omega_i}^{\varepsilon_i}(\varepsilon_i) |v_i\rangle$$

$$e_{\omega_i}^{\varepsilon_i}(\varepsilon_i + 2n) = \sum_{O(0 \to n)} \left[\prod_{\{r_p\}} F(\varepsilon_i + 2r_p + 2) \prod_{\{s_q\}} G_{\omega_i}(\varepsilon_i + 2s_q) \right] e_{\omega_i}^{\varepsilon_i}(\varepsilon_i)$$

$$F(v_i) = \frac{f_-(v_i)}{f_+(v_i)}, G_{\omega_i}(v_i) = \frac{-i\omega_i}{3f_+(v_i)}$$

$$\hat{\mathcal{C}}_{\mathrm{BI}}^{\mathrm{B}} = \hat{\mathcal{C}}^{(\theta)} + \hat{\mathcal{C}}^{(\sigma)} + \hat{\mathcal{C}}^{(\delta)}$$

$$\mathrm{Cil}_{\mathrm{S}}^+ = \mathrm{lin}\{|v, \lambda_\sigma, \lambda_\delta\rangle; v, \lambda_\sigma, \lambda_\delta > 0\}$$

$$\hat{\mathcal{C}}_{\mathrm{BI}}^{\mathrm{B}} |v, \lambda_\sigma, \lambda_\delta\rangle = \frac{(\pi l_{\mathrm{Pl}}^2)^2}{4} [x_-(v)|v-4, \lambda_\sigma, \lambda_\delta\rangle_- - x_0^-(v)|v, \lambda_\sigma, \lambda_\delta\rangle_{0^-}$$
$$- x_0^+(v)|v, \lambda_\sigma, \lambda_\delta\rangle_{0^+} + x_+(v)|v+4, \lambda_\sigma, \lambda_\delta\rangle_+]$$

$$x_-(v) = 2\sqrt{v}(v-2)\sqrt{v-4}[1 + \mathrm{sgn}(v-4)], \quad x_+(v) = x_-(v+4)$$
$$x_0^-(v) = 2(v-2)v[1 + \mathrm{sgn}(v-2)], \quad x_0^+(v) = x_0^-(v+2)$$

$$|v \pm 4, \lambda_\sigma, \lambda_\delta\rangle_\pm = \left| v \pm 4, \lambda_\sigma, \frac{v \pm 4}{v \pm 2}\lambda_\delta \right\rangle + \left| v \pm 4, \frac{v \pm 4}{v \pm 2}\lambda_\sigma, \lambda_\delta \right\rangle$$
$$+ \left| v \pm 4, \frac{v \pm 2}{v}\lambda_\sigma, \lambda_\delta \right\rangle + \left| v \pm 4, \lambda_\sigma, \frac{v \pm 2}{v}\lambda_\delta \right\rangle$$
$$+ \left| v \pm 4, \frac{v \pm 2}{v}\lambda_\sigma, \frac{v \pm 4}{v \pm 2}\lambda_\delta \right\rangle + \left| v \pm 4, \frac{v \pm 4}{v \pm 2}\lambda_\sigma, \frac{v \pm 2}{v}\lambda_\delta \right\rangle$$

$$|v,\lambda_\sigma,\lambda_\delta\rangle_{0\pm} = \left|v,\lambda_\sigma,\frac{v}{v\pm 2}\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm 2}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v,\lambda_\sigma,\frac{v\pm 2}{v}\lambda_\delta\right\rangle$$
$$+\left|v,\frac{v\pm 2}{v}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v,\frac{v\pm 2}{v}\lambda_\sigma,\frac{v}{v\pm 2}\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm 2}\lambda_\sigma,\frac{v\pm 2}{v}\lambda_\delta\right\rangle.$$

$$\mathcal{L}^+_{\tilde{\varepsilon}} = \{\tilde{\varepsilon}+4k, k=0,1,2\ldots\}, \tilde{\varepsilon}\in(0,4]$$

$$\left(\lambda_a^\star,\frac{v-4}{v-2}\lambda_b^\star\right),\left(\lambda_a^\star,\frac{v-2}{v}\lambda_b^\star\right),\left(\frac{v-2}{v}\lambda_a^\star,\frac{v-4}{v-2}\lambda_b^\star\right),$$

$$\left(\lambda_a^\star,\frac{v}{v+2}\lambda_b^\star\right),\left(\lambda_a^\star,\frac{v+2}{v}\lambda_b^\star\right),\left(\frac{v+2}{v}\lambda_a^\star,\frac{v}{v+2}\lambda_b^\star\right),$$

$$\left(\lambda_a^\star,\frac{v-2}{v}\lambda_b^\star\right),\left(\lambda_a^\star,\frac{v}{v-2}\lambda_b^\star\right),\left(\frac{v-2}{v}\lambda_a^\star,\frac{v}{v-2}\lambda_b^\star\right),$$

$$\left(\lambda_a^\star,\frac{v+4}{v+2}\lambda_b^\star\right),\left(\lambda_a^\star,\frac{v+2}{v}\lambda_b^\star\right),\left(\frac{v+2}{v}\lambda_a^\star,\frac{v+4}{v+2}\lambda_b^\star\right).$$

$$\mathcal{W}_{\tilde{\varepsilon}} = \left\{\left(\frac{\tilde{\varepsilon}-2}{\tilde{\varepsilon}}\right)^z \prod_{m,n\in\mathbb{N}} \left(\frac{\tilde{\varepsilon}+2m}{\tilde{\varepsilon}+2n}\right)^{k_n^m};\ k_n^m\in\mathbb{N}, z\in\mathbb{Z} \text{ si } \tilde{\varepsilon}>2, z=0 \text{ si } \tilde{\varepsilon}<2\right\}$$

$$|v,\lambda_\sigma,\lambda_\delta\rangle \quad \text{con} \quad \left|v+4,\lambda_\sigma,\frac{v+4}{v+2}\lambda_\delta\right\rangle,$$
$$|v,\lambda_\sigma,\lambda_\delta\rangle \quad \text{con} \quad \left|v,\lambda_\sigma,\frac{v-2}{v}\lambda_\delta\right\rangle,$$
$$\mid siv > 2.$$

$$\left|v+4,\lambda_\sigma,\frac{v+4}{v+2}\lambda_\delta\right\rangle \ \text{con}\ \left|v+4,\lambda_\sigma,\frac{v+2}{v+4}\frac{v+4}{v+2}\lambda_\delta\right\rangle = |v+4,\lambda_\sigma,\lambda_\delta\rangle,$$

$$|v,\lambda_\sigma^\star,\lambda_\delta^\star\rangle \quad \text{con}$$
$$|v,\lambda_\sigma^\star,\lambda_\delta^\star\rangle \quad \text{con}$$
$$\left|v,\lambda_\sigma^\star,\frac{v+2}{v}\lambda_\delta^\star\right\rangle,$$
$$\left|v,\frac{v+2}{v}\lambda_\sigma^\star,\lambda_\delta^\star\right\rangle.$$

$$\left|v,\frac{v+2}{v}\lambda_\sigma^\star,\lambda_\delta^\star\right\rangle \ \text{con}\ \left|v,\frac{v+2}{v}\lambda_\sigma^\star,\frac{v+2}{v}\lambda_\delta^\star\right\rangle,$$

$$\frac{\lambda_\sigma}{\lambda_\sigma^\star} = \frac{\lambda_\delta}{\lambda_\delta^\star} = \frac{v+2}{v}$$

$$U_{\tilde{\varepsilon}} = \left\{\frac{\tilde{\varepsilon}+4m}{\tilde{\varepsilon}+4n}; m,n\in\mathbb{N}\right\} \subset \mathcal{W}_{\tilde{\varepsilon}}, V_{\tilde{\varepsilon}} = \left\{\frac{\tilde{\varepsilon}}{4}+n; n\in\mathbb{N}\right\}.$$

$$sc+1 < sd$$

$$sc < \frac{\tilde{\varepsilon}}{4}+m \le sc+1$$

$$c < \frac{\tilde{\varepsilon}+4m}{\tilde{\varepsilon}+4n} < d$$

$$\mathrm{Cil}^+_{\tilde{\varepsilon},\lambda_\sigma^\star,\lambda_\delta^\star} = \mathrm{lin}\{|v,\lambda_\sigma,\lambda_\delta\rangle; v\in\mathcal{L}^+_{\tilde{\varepsilon}}, \lambda_a = \omega_{\tilde{\varepsilon}}\lambda_a^\star, \omega_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}, \lambda_a^\star\in\mathbb{R}^+\},$$

$$\langle v,\lambda_\sigma,\lambda_\delta \mid v',\lambda'_\sigma,\lambda'_\delta\rangle = \delta_{vv'}\delta_{\lambda_\sigma\lambda'_\sigma}\delta_{\lambda_\delta\lambda'_\delta}$$

$$\phi(\vec{v}) = \int_{\mathbb{R}^3} d\vec{\omega}\tilde{\phi}(\vec{\omega}) e^{\varepsilon_\theta}_{\omega_\theta}(v_\theta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\Phi(\vec{v}) = [\mathcal{P}\phi](\vec{v}) = \int_{\mathbb{R}} dt e^{it\frac{\gamma^2}{2}\hat{C}^{\mathrm{A}}_{\mathrm{BI}}} \phi(\vec{v})$$

$$= \int_{\mathbb{R}^3} d\vec{\omega} \delta(\omega_\theta\omega_\sigma + \omega_\theta\omega_\delta + \omega_\sigma\omega_\delta)\tilde{\phi}(\vec{\omega}) e^{\varepsilon_\theta}_{\omega_\theta}(v_\theta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\sum_i \frac{1}{\omega_i} = 0.$$

$$\omega_\theta(\omega_\sigma, \omega_\delta) = -\frac{\omega_\sigma\omega_\delta}{\omega_\sigma + \omega_\delta}$$

$$\Phi(\vec{v}) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\Phi}(\omega_\sigma, \omega_\delta) e^{\varepsilon_\theta}_{\omega_\theta(\omega_\sigma,\omega_\delta)}(v_\theta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\langle \Phi_1 \mid \Phi_2 \rangle_{\mathrm{fis}} = \langle \mathcal{P}\phi_1 \mid \phi_2 \rangle_{\mathrm{cin}} = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta |\omega_\sigma + \omega_\delta| \tilde{\Phi}_1^*(\omega_\sigma, \omega_\delta) \tilde{\Phi}_2(\omega_\sigma, \omega_\delta)$$

$$\mathcal{H}^{\vec{\varepsilon}}_{\mathrm{fis}} = L^2(\mathbb{R}^2, |\omega_\sigma + \omega_\delta| d\omega_\sigma d\omega_\delta)$$

$$\Big(\psi \mid = \sum_{v\in\mathcal{L}^+_{\tilde{\varepsilon}}} \sum_{\omega_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}} \sum_{\bar{\omega}_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}} \psi(v, \omega_{\tilde{\varepsilon}}\lambda^\star_\sigma, \bar{\omega}_{\tilde{\varepsilon}}\lambda^\star_\delta) \langle v, \omega_{\tilde{\varepsilon}}\lambda^\star_\sigma, \bar{\omega}_{\tilde{\varepsilon}}\lambda^\star_\delta|.$$

$$\psi_+(v+4, \lambda_\sigma, \lambda_\delta) = \frac{1}{x_+(v)}[x_0^-(v)\psi_{0^-}(v, \lambda_\sigma, \lambda_\delta) + x_0^+(v)\psi_{0^+}(v, \lambda_\sigma, \lambda_\delta) \\ - x_-(v)\psi_-(v-4, \lambda_\sigma, \lambda_\delta)].$$

$$\psi_\pm(v \pm 4, \lambda_\sigma, \lambda_\delta) = \big(\psi|v \pm 4, \lambda_\sigma, \lambda_\delta\rangle_\pm, \psi_{0^\pm}(v, \lambda_\sigma, \lambda_\delta) = \big(\psi|v, \lambda_\sigma, \lambda_\delta\rangle_{0^\pm}$$

$$\psi_+(\tilde{\varepsilon}+4, \lambda_\sigma, \lambda_\delta) = \frac{1}{x_+(\tilde{\varepsilon})}[x_0^-(\tilde{\varepsilon})\psi_{0^-}(\tilde{\varepsilon}, \lambda_\sigma, \lambda_\delta) + x_0^+(\tilde{\varepsilon})\psi_{0^+}(\tilde{\varepsilon}, \lambda_\sigma, \lambda_\delta)]$$

$$\psi_+(\tilde{\varepsilon}+4, \lambda_\sigma, \lambda_\delta) = \psi\left(\tilde{\varepsilon}+4, \lambda_\sigma, \frac{\tilde{\varepsilon}+4}{\tilde{\varepsilon}+2}\lambda_\delta\right) + \psi\left(\tilde{\varepsilon}+4, \frac{\tilde{\varepsilon}+4}{\tilde{\varepsilon}+2}\lambda_\sigma, \lambda_\delta\right) \\ +\psi\left(\tilde{\varepsilon}+4, \lambda_\sigma, \frac{\tilde{\varepsilon}+2}{\tilde{\varepsilon}}\lambda_\delta\right) + \psi\left(\tilde{\varepsilon}+4, \frac{\tilde{\varepsilon}+2}{\tilde{\varepsilon}}\lambda_\sigma, \lambda_\delta\right) \\ +\psi\left(\tilde{\varepsilon}+4, \frac{\tilde{\varepsilon}+2}{\tilde{\varepsilon}}\lambda_\sigma, \frac{\tilde{\varepsilon}+4}{\tilde{\varepsilon}+2}\lambda_\delta\right) + \psi\left(\tilde{\varepsilon}+4, \frac{\tilde{\varepsilon}+4}{\tilde{\varepsilon}+2}\lambda_\sigma, \frac{\tilde{\varepsilon}+2}{\tilde{\varepsilon}}\lambda_\delta\right)$$

$$\{\psi_+(v+4, \lambda_\sigma, \lambda_\delta); \lambda_\sigma = \omega_{\tilde{\varepsilon}}\lambda^\star_\sigma, \lambda_\delta = \bar{\omega}_{\tilde{\varepsilon}}\lambda^\star_\delta, \omega_{\tilde{\varepsilon}}, \bar{\omega}_{\tilde{\varepsilon}} \in \mathcal{W}_{\tilde{\varepsilon}}\}$$

$$\{\psi(v+4, \lambda_\sigma, \lambda_\delta); \lambda_\sigma = \omega_{\tilde{\varepsilon}}\lambda^\star_\sigma, \lambda_\delta = \bar{\omega}_{\tilde{\varepsilon}}\lambda^\star_\delta, \omega_{\tilde{\varepsilon}}, \bar{\omega}_{\tilde{\varepsilon}} \in \mathcal{W}_{\tilde{\varepsilon}}\}$$

$$\psi_+(v+4, \lambda_\sigma, \lambda_\delta) = \hat{U}_6(v+4)\psi(v+4, \lambda_\sigma, \lambda_\delta),$$

$$\mathrm{Cil}_{\lambda^\star_\sigma,\lambda^\star_\delta} = \mathrm{Cil}_{\lambda^\star_\sigma} \otimes \mathrm{Cil}_{\lambda^\star_\delta} = \mathrm{lin}\{|\lambda_\sigma, \lambda_\delta\rangle; \lambda_\sigma = \omega_{\tilde{\varepsilon}}\lambda^\star_\sigma, \lambda_\delta = \bar{\omega}_{\tilde{\varepsilon}}\lambda^\star_\delta, \omega_{\tilde{\varepsilon}}, \bar{\omega}_{\tilde{\varepsilon}} \in \mathcal{W}_{\tilde{\varepsilon}}\}$$

$$\hat{U}_6(v+4)|\lambda_\sigma, \lambda_\delta\rangle = \left|\frac{v}{v+2}\lambda_\sigma, \lambda_\delta\right\rangle + \left|\lambda_\sigma, \frac{v}{v+2}\lambda_\delta\right\rangle + \left|\frac{v+2}{v+4}\lambda_\sigma, \lambda_\delta\right\rangle + \left|\lambda_\sigma, \frac{v+2}{v+4}\lambda_\delta\right\rangle \\ + \left|\frac{v}{v+2}\lambda_\sigma, \frac{v+2}{v+4}\lambda_\delta\right\rangle + \left|\frac{v+2}{v+4}\lambda_\sigma, \frac{v}{v+2}\lambda_\delta\right\rangle$$

$$\hat{U}^{(\varpi_\sigma,\varpi_\delta)}\psi(v, x_\sigma, x_\delta) = \psi(v, x_\sigma + \varpi_\sigma, x_\delta + \varpi_\delta)$$

$$\rho_a(\varpi_a)\rho_a(\bar{\varpi}_a) = \rho_a(\varpi_a + \bar{\varpi}_a).$$

$$\omega_6(\rho_\sigma, \rho_\delta) = \sum_{a=\sigma,\delta} \left\{\rho_a\left[\ln\left(\frac{v}{v-2}\right)\right] + \rho_a\left[\ln\left(\frac{v-2}{v-4}\right)\right]\right\} + \sum_{a,b=\sigma,\delta;a\neq b} \left\{\rho_a\left[\ln\left(\frac{v}{v-2}\right)\right]\rho_b\left[\ln\left(\frac{v-2}{v-4}\right)\right]\right\}.$$

$$\{\psi(\tilde{\varepsilon}, x_\sigma, x_\delta) = \psi(\tilde{\varepsilon}, \ln(\lambda_\sigma^\star) + \varpi_\sigma, \ln(\lambda_\delta^\star) + \bar{\varpi}_\delta)\}$$

$$\widehat{e^{ix_\sigma}}\tilde{\psi}(x_\sigma, x_\delta) = e^{ix_\sigma}\tilde{\psi}(x_\sigma, x_\delta)$$
$$\hat{U}_\sigma^{\varpi_\sigma}\tilde{\psi}(x_\sigma, x_\delta) = \tilde{\psi}(x_\sigma + \varpi_\sigma, x_\delta)$$

$$E(A^i = 4\pi^2\bar{\mu}_j\bar{\mu}_k, f = 1) = p_i\bar{\mu}_j\bar{\mu}_k = \frac{\Delta p_i}{\sqrt{p_j p_k}}$$

$$\underline{\mathcal{H}}_{\text{grav}} = \otimes_i \underline{\mathcal{H}}_{\text{grav}}^i, \underline{\mathcal{H}}_{\text{grav}}^i = L^2(\mathbb{R}, dv_i)$$

$$p_i = \left(6\pi\gamma l_{\text{Pl}}^2\sqrt{\Delta}\right)^{2/3}\text{sgn}(v_i)|v_i|^{2/3}$$

$$\hat{c}^i = i2(6\pi\gamma l_{\text{Pl}}^2/\Delta)^{1/3}|v_i|^{1/6}\partial_{v_i}|v_i|^{1/6}$$

$$\hat{\Omega}_i = -i\bar{\alpha}\sqrt{|v_i|}[\text{sgn}(v_i)\partial_{v_i} + \partial_{v_i}\text{sgn}(v_i)]\sqrt{|v_i|},$$

$$\hat{\Omega}_i = -i\bar{\alpha}(2v_i\partial_{v_i} + 1),$$

$$\hat{C}_{\text{BI}} = -\frac{2}{\gamma^2}\left(\hat{\hat{\Omega}}_\theta\hat{\Omega}_\sigma + \hat{\hat{\Omega}}_\theta\hat{\Omega}_\delta + \hat{\hat{\Omega}}_\sigma\hat{\Omega}_\delta\right)$$

$$\underline{e}_{\omega_i}(v_i) = \frac{1}{\sqrt{2\pi\bar{\alpha}|v_i|}}\exp\left(-i\omega_i\frac{\ln|v_i|}{\bar{\alpha}}\right)$$

$$\left\langle \underline{e}_{\omega_i} \mid \underline{e}_{\omega_i'}\right\rangle = \delta(\omega_i - \omega_i')$$

$$\underline{\mathcal{H}}_{\text{fis}} = L^2(\mathbb{R}^2, |\omega_\sigma + \omega_\delta| d\omega_\sigma d\omega_\delta)$$

$$\underline{\Phi}(\vec{v}) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta)\underline{e}_{\omega_\theta(\omega_\sigma,\omega_\delta)}(v_\theta)\underline{e}_{\omega_\sigma}(v_\sigma)\underline{e}_{\omega_\delta}(v_\delta)$$

$$\underline{\hat{P}}_T : \underline{\mathcal{H}}_{\text{fis}} \to \underline{\mathcal{H}}_T,$$

$$\underline{\hat{P}}_{v_\theta} : \underline{\mathcal{H}}_{\text{fis}} \to \underline{\mathcal{H}}_{v_\theta} = L^2(\mathbb{R}^2, |\omega_\sigma + \omega_\delta| d\omega_\sigma d\omega_\delta)$$
$$\underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) \mapsto \underline{\tilde{\Phi}}_{v_\theta}(\omega_\sigma, \omega_\delta) = \underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta)\sqrt{2\pi\bar{\alpha}v_\theta}\underline{e}_{\omega_\theta}(v_\theta)$$
$$\underline{\Phi}_{v_\theta}(v_\sigma, v_\delta) = \int_{\mathbb{R}^2} \frac{d\omega_\sigma d\omega_\delta}{\sqrt{2\pi\bar{\alpha}v_\theta}}\tilde{\Phi}_{v_\theta}(\omega_\sigma, \omega_\delta)\underline{e}_{\omega_\sigma}(v_\sigma)\underline{e}_{\omega_\delta}(v_\delta)$$

$$\underline{\mathcal{H}}(T) \otimes \underline{\mathcal{H}}',$$

$$\hat{O}' : \underline{\mathcal{H}}' \to \underline{\mathcal{H}}'.$$

$$\hat{O}_T : \underline{\mathcal{H}}_{\text{fis}} \to \underline{\mathcal{H}}_{\text{fis}}$$

$$\underline{\hat{R}}_T : \underline{\mathcal{H}}_{\text{fis}} \to \underline{\mathcal{H}}_T \to \underline{\mathcal{H}}'$$

$$\underline{\mathcal{H}}' = \underline{\mathcal{H}}_{\text{grav}}^{\sigma,+} \otimes \underline{\mathcal{H}}_{\text{grav}}^{\delta,+} = L^2((\mathbb{R}^+)^2, dv_\sigma dv_\delta),$$

$$\underline{\boldsymbol{\chi}}(v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \underline{\tilde{\boldsymbol{\chi}}}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_\sigma}(v_\sigma) \underline{e}_{\omega_\delta}(v_\delta)$$

$$\underline{\mathcal{H}}' = \underline{\mathcal{H}}_{\text{grav}}^{\sigma,+} \otimes \underline{\mathcal{H}}_{\text{grav}}^{\delta,+} = L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta).$$

$$\underline{\tilde{\Phi}}_{v_\theta}(\omega_\sigma, \omega_\delta) \mapsto \underline{\tilde{\chi}}_{v_\theta}(\omega_\sigma, \omega_\delta) = |\omega_\sigma + \omega_\delta|^{\frac{1}{2}} \underline{\tilde{\Phi}}_{v_\theta}(\omega_\sigma, \omega_\delta)$$

$$\underline{\Phi}_{v_\theta}(v_\sigma, v_\delta) \to \underline{\boldsymbol{\chi}}_{v_\theta}(v_\sigma, v_\delta)$$

$$\underline{\boldsymbol{\chi}}_{v_\theta}(v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \underline{\tilde{\boldsymbol{\chi}}}_{v_\theta}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_\sigma}(v_\sigma) \underline{e}_{\omega_\delta}(v_\delta)$$

$$\underline{\chi}_{v_\theta}(v_\sigma, v_\delta) = \sqrt{2\pi\bar{\alpha}v_\theta} \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta |\omega_\sigma + \omega_\delta|^{\frac{1}{2}} \underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_\theta}(v_\theta) \underline{e}_{\omega_\sigma}(v_\sigma) \underline{e}_{\omega_\delta}(v_\delta)$$

$$\ln(v_a) \underline{e}_{\omega_a}(v_a) = i\bar{\alpha} \partial_{\omega_a} \underline{e}_{\omega_a}(v_a)$$

$$\ln(\hat{v}_\sigma) \underline{\chi}_{v_\theta}(v_\sigma, v_\delta) = i\bar{\alpha}\sqrt{2\pi\bar{\alpha}v_\theta} \times \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta |\omega_\sigma + \omega_\delta|^{\frac{1}{2}} \underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_\theta}(v_\theta) [\partial_{\omega_\sigma} \underline{e}_{\omega_\sigma}(v_\sigma)] \underline{e}_{\omega_\delta}(v_\delta),$$

$$\ln(\hat{v}_a)_{v_\theta} : \mathcal{S}(\mathbb{R}^2) \subset \underline{\mathcal{H}}_{\text{fis}} \to \underline{\mathcal{H}}_{\text{fis}}$$

$$[\ln(\hat{v}_a)_{v_\theta} \underline{\tilde{\Phi}}](\omega_\sigma, \omega_\delta) = \frac{-i\bar{\alpha}}{\underline{e}_{\omega_\theta}(v_\theta)} |\omega_\sigma + \omega_\delta|^{-\frac{1}{2}} \partial_{\omega_a} \left[|\omega_\sigma + \omega_\delta|^{\frac{1}{2}} \underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_\theta}(v_\theta) \right]$$

$$\left[\underline{\hat{Q}}_{v_\theta, v_\theta^\star} \tilde{\Phi} \right](\omega_\sigma, \omega_\delta) = \sqrt{\frac{v_\theta}{v_\theta^\star} \frac{e_{\omega_\theta}(v_\theta)}{e_{\omega_\theta}(v_\theta^\star)}} \tilde{\Phi}(\omega_\sigma, \omega_\delta)$$

$$\ln(\hat{v}_a)_{v_\theta^\star} = \underline{\hat{Q}}_{v_\theta, v_\theta^\star} \ln(\hat{v}_a)_{v_\theta} \underline{\hat{Q}}_{v_\theta^\star, v_\theta}$$

$$\underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) = \frac{K}{\sqrt{|\omega_\sigma + \omega_\delta|}} \prod_{a=\sigma}^{\delta} e^{-\frac{(\omega_a - \omega_a^\star)^2}{2\sigma_a^a}} e^{iv^a \omega_a}$$

$$\partial_{\omega_a} e_{\omega_\theta}(v_\theta) = \frac{i}{\bar{\alpha}} \ln(v_\theta) \left[\frac{\omega_\theta(\omega_\sigma, \omega_\delta)}{\omega_a} \right]^2 \underline{e}_{\omega_\theta}(v_\theta),$$

$$\langle \underline{\tilde{\Phi}} \mid \ln(\hat{v}_a)_{v_\theta} \tilde{\Phi} \rangle = A_a \ln v_\theta + B_a$$

$$A_a = \left\| \omega_\theta(\omega_\sigma, \omega_\delta) \omega_a^{-1} \underline{\tilde{\Phi}} \right\|^2$$

$$B_a = \bar{\alpha} \left\langle \underline{\tilde{\Phi}} \middle\| \omega_\sigma + \omega_\delta|^{-\frac{1}{2}} (-i\partial_{\omega_a}) \middle| \omega_\sigma + \omega_\delta|^{\frac{1}{2}} \underline{\tilde{\Phi}} \right\rangle$$

$$A_a \simeq \left[\frac{\omega_\theta(\omega_\sigma^\star, \omega_\delta^\star)}{\omega_a^\star} \right]^2 \|\underline{\tilde{\Phi}}\|^2 = \left[\frac{\omega_\theta(\omega_\sigma^\star, \omega_\delta^\star)}{\omega_a^\star} \right]^2$$

$$B_a = -\frac{i\bar{\alpha}}{2} \langle \tilde{\Phi} || \omega_\sigma + \omega_\delta|^{-1} \underline{\tilde{\Phi}} \rangle - i\bar{\alpha} \langle \tilde{\Phi} \mid \partial_{\omega_a} \underline{\tilde{\Phi}} \rangle$$

$$\langle \underline{\tilde{\Phi}} \mid \ln(\hat{v}_a)_{v_\theta} \underline{\tilde{\Phi}} \rangle \simeq \left[\frac{\omega_\theta(\omega_\sigma^\star, \omega_\delta^\star)}{\omega_a^\star}\right]^2 \ln\ v_\theta + \bar{\alpha} v^a$$

$$\left\langle \underline{\tilde{\Phi}} \mid \left[\ln(\hat{v}_a)_{v_\theta}\right]^2 \underline{\tilde{\Phi}} \right\rangle = W_a[\ln(v_\theta)]^2 + Y_a \ln\ v_\theta + X_a$$

$$W_a = \left\| \omega_\theta^2(\omega_\sigma, \omega_\delta)\omega_a^{-2} \underline{\tilde{\Phi}} \right\|^2$$

$$Y_a = -2i\bar{\alpha} \left\langle \underline{\tilde{\Phi}} \| \omega_\sigma + \omega_\delta |^{-\frac{1}{2}} \omega_\theta(\omega_\sigma, \omega_\delta)\omega_a^{-1}\left(\partial_{\omega_a}\right) \right| \omega_\sigma + \omega_\delta |^{\frac{1}{2}} \omega_\theta(\omega_\sigma, \omega_\delta)\omega_a^{-1} \underline{\tilde{\Phi}} \right\rangle$$

$$X_a = -\bar{\alpha}^2 \left\langle \underline{\tilde{\Phi}} \| \omega_\sigma + \omega_\delta |^{-\frac{1}{2}} \partial_{\omega_a}^2 \right| \omega_\sigma + \omega_\delta |^{\frac{1}{2}} \underline{\tilde{\Phi}} \right\rangle$$

$$\left\langle \Delta \ln(\hat{v}_a)_{v_\theta} \right\rangle^2 = \left\langle \left[\ln(\hat{v}_a)_{v_\theta}\right]^2 \right\rangle - \left\langle \ln(\hat{v}_a)_{v_\theta} \right\rangle^2$$

$$\lim_{v_\theta \to \infty} \frac{\left\langle \Delta \ln(\hat{v}_a)_{v_\theta} \right\rangle}{\left\langle \ln(\hat{v}_a)_{v_\theta} \right\rangle} = \frac{\left\langle \Delta[\omega_\theta^2(\omega_\sigma, \omega_\delta)\omega_a^{-2}] \right\rangle}{\left\langle \omega_\theta^2(\omega_\sigma, \omega_\delta)\omega_a^{-2} \right\rangle}$$

$$\beta_i = \hbar\sqrt{\Delta} c^i |p_i|^{-\frac{1}{2}}$$

$$\{\beta_i, v_j\} = 2\delta_{ij}$$

$$[\underline{\mathcal{F}}\psi](\beta_i) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} dv_i \psi(v_i) e^{-\frac{i}{2} v_i \beta_i}.$$

$$\underline{\mathcal{F}}: \underline{\mathcal{H}_{\text{grav}}^i} = L^2(\mathbb{R}, dv_i) \to \underline{\mathcal{H}_{\text{grav}}^i}^{\,i} = L^2(\mathbb{R}, d\beta_i).$$

$$\hat{v}_i \to 2i\partial_{\beta_i}, \partial_{v_i} \to i\hat{\beta}_i/2$$

$$\underline{\mathcal{F}}\left(\underline{\widehat{\Omega}}_i\right) = i\bar{\alpha}\left(1 + 2\beta_i \partial_{\beta_i}\right)$$

$$|\underline{\Phi}\rangle_{\text{fin}} = \hat{\rho}_s |\underline{\Phi}\rangle_{\text{in}}$$

$$\left\langle \underline{e}_{\omega_\sigma}, \underline{e}_{\omega_\delta} \middle| \hat{\rho}_s \middle| \underline{e}_{\omega'_\sigma}, \underline{e}_{\omega'_\delta} \right\rangle = \hat{\rho}_\theta\left(\omega_\theta(\omega_\sigma, \omega_\delta), \omega_\theta(\omega'_\sigma, \omega'_\delta)\right) \hat{\rho}_\sigma(\omega_\sigma, \omega'_\sigma) \hat{\rho}_\delta(\omega_\delta, \omega'_\delta)$$

$$\hat{\rho}_i(\omega_i, \omega'_i) := \left\langle \underline{e}_{\omega_i} \middle| \hat{\rho}_i \middle| \underline{e}_{\omega'_i} \right\rangle$$

$$\widehat{\Omega}_i^2 e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) = \omega_i^2 e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i), e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) := e_{\omega_i}^{\varepsilon_i}(v_i)\Big|_{v_i \in {}^{(4)}\mathcal{L}_{\tilde{\varepsilon}_i}^+}$$

$$e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) \to \underline{e}_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) = r(\omega_i)\left[e^{i\phi(\omega_i)} \underline{e}_{\omega_i}(v_i) + e^{-i\phi(\omega_i)} \underline{e}_{-\omega_i}(v_i)\right],$$

$$\phi(\omega_i) = T(|\omega_i|) + c_{\tilde{\varepsilon}_i} + R_{\tilde{\varepsilon}_i}(|\omega_i|).$$

$$T(|\omega_i|) = (\ln|\omega_i| + a)(|\omega_i| + b),$$

$$\hat{\rho}_i(\omega_i, \omega'_i) = e^{-2i\phi(\omega_i)} \delta(\omega_i + \omega'_i)$$

$$e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) = \frac{1}{2}\left[e_{\omega_i}^{\varepsilon_i}(v_i) \pm e_{-\omega_i}^{\varepsilon_i}(v_i)\right]$$

$$\left\langle e_{\omega_i}^{\tilde{\varepsilon}_i} \mid e_{\omega'_i}^{\tilde{\varepsilon}_i} \right\rangle = \frac{1}{2}\delta(\omega_i - \omega'_i)$$

$$\left\| e_{\omega_i}^{\tilde{\varepsilon}_i} \right\|^2 = \sum_{v_i \in \mathcal{L}_{\tilde{\varepsilon}_i}^+ = {}^{(4)}\mathcal{L}_{\tilde{\varepsilon}_i}^+} \left[e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i)\right]^* e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i)$$

$$\left\| \underline{e}_{\omega_i}^{\tilde{\varepsilon}_i} \right\|_{\mathrm{WDW}}^2 = \int_{\mathbb{R}^+} dv_i \left[\underline{e}_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) \right]^* \underline{e}_{\omega_i}^{\tilde{\varepsilon}_i}(v_i)$$

$$\left\| e_{\omega_i}^{\tilde{\varepsilon}_i} \right\|^2 = \frac{1}{8} \left\| e_{\omega_i}^{\tilde{\varepsilon}_i} \right\|_{\mathrm{WDW}}^2$$

$$\left\langle e_{\omega_i}^{\tilde{\varepsilon}_i} \mid e_{\omega_i'}^{\tilde{\varepsilon}_i} \right\rangle = O_1(\omega_i, \omega_i') + \sum_{v_i^\star < v_i \epsilon^{(4)} \mathcal{L}_{\tilde{\varepsilon}_i}^+} \left[e_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) \right]^* \underline{e}_{\omega_i'}^{\tilde{\varepsilon}_i}(v_i),$$

$$O_3(\omega_i, \omega_i') = \frac{1}{4} \int_{v_i > v_i^\star} dv_i \left[\underline{e}_{\omega_i}^{\tilde{\varepsilon}_i}(v_i) \right]^* \underline{e}_{\omega_i'}^{\tilde{\varepsilon}_i}(v_i)$$

$$\int_{\mathbb{R}^+} dx e^{i(\omega_i - \omega_i')x} = \frac{N}{2} \delta(\omega_i - \omega_i') + f(\omega_i, \omega_i'),$$

$$\left\langle e_{\omega_i}^{\tilde{\varepsilon}_i} \mid e_{\omega_i'}^{\tilde{\varepsilon}_i} \right\rangle = O_1(\omega_i, \omega_i') + O_2(\omega_i, \omega_i') + \frac{1}{8} \left\langle e_{\omega_i}^{\tilde{\varepsilon}_i} \mid e_{\omega_i'}^{\tilde{\varepsilon}_i} \right\rangle \mathrm{WDW} + O_4(\omega_i, \omega_i'),$$

$$r(\omega_i) = \sqrt{2} z_i$$

$$\underline{\Phi}(\vec{v}) = \sum_{s_\sigma, s_\delta = \pm 1} \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\Phi}_{\vec{s}}(\omega_\sigma, \omega_\delta) \underline{e}_{\omega_{\vec{s}}(\omega_\sigma, \omega_\delta)}(v_\theta) \underline{e}_{\omega_\sigma}(v_\sigma) \underline{e}_{\omega_\delta}(v_\delta)$$

$$\omega_{\vec{s}}(\omega_\sigma, \omega_\delta) = \omega_\theta(s_\sigma \omega_\sigma, s_\delta \omega_\delta) = -\frac{s_\sigma s_\delta \omega_\sigma \omega_\delta}{s_\sigma \omega_\sigma + s_\delta \omega_\delta}$$

$$\tilde{\Phi}_{\vec{s}}(\omega_\sigma, \omega_\delta) = 2\sqrt{2} \sum_{s_\theta = \pm 1} \tilde{\Phi}(s_\theta s_\sigma \omega_\sigma, s_\theta s_\delta \omega_\delta) e^{i s_\theta \phi\left(\omega_{s_\theta \vec{s}}\right)} s_\theta^{(3 - z_\theta^2 - z_\sigma^2 - z_\delta^2)/2} z_\theta$$
$$\times \prod_{a=\sigma}^{\delta} s_a^{(1 - z_a^2)/2} z_a e^{i s_1 s_a \phi(s_\theta s_a \omega_a)}$$

$$A_{a,\vec{s}} := s_a \left\| \omega_{\vec{s}}(\omega_\sigma, \omega_\delta) \omega_a^{-1} \underline{\tilde{\Phi}} \right\|^2$$

$$\underline{\tilde{\Phi}}(\omega_\sigma, \omega_\delta) = 2\sqrt{2} \sum_{s = \pm 1} s^{\frac{3 - z_\theta^2 - z_\sigma^2 - z_\delta^2}{2}} \tilde{\Phi}(s\omega_\sigma, s\omega_\delta) \prod_{i = \theta, \sigma, \delta} z_i e^{i s \phi(s\omega_i)}$$

$$\Phi_{v_\theta}(v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \left| e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta) \right| \tilde{\Phi}'_{v_\theta}(\omega_\sigma, \omega_\delta) e_{\omega_\sigma}^{\varepsilon_\sigma}(v_\sigma) e_{\omega_\delta}^{\varepsilon_\delta}(v_\delta)$$

$$\hat{P}'_{v_\theta} : \mathcal{H}_{\mathrm{fis}}^{\vec{\varepsilon}} \to \mathcal{H}'_{v_\theta} = L^2(\mathbb{R}^2, |\omega_\sigma + \omega_\delta| d\omega_\sigma d\omega_\delta)$$
$$\tilde{\Phi}(\omega_\sigma, \omega_\delta) \mapsto \tilde{\Phi}'_{v_\theta}(\omega_\sigma, \omega_\delta) = \tilde{\Phi}(\omega_\sigma, \omega_\delta) \frac{e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta)}{\left| e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta) \right|}$$

$$\hat{P}_{v_\theta} : \mathcal{H}_{\mathrm{fis}}^{\vec{s}} \to \mathcal{H}_{v_\theta} = L^2\left(\mathbb{R}^2, |\omega_\sigma + \omega_\delta| \left| e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta) \right|^{-2} d\omega_\sigma d\omega_\delta\right),$$
$$\tilde{\Phi}(\omega_\sigma, \omega_\delta) \mapsto \tilde{\Phi}_{v_\theta}(\omega_\sigma, \omega_\delta) = \tilde{\Phi}(\omega_\sigma, \omega_\delta) e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta),$$

$$\Phi_{v_\theta}(v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\Phi}_{v_\theta}(\omega_\sigma, \omega_\delta) e_{\omega_\sigma}^{\varepsilon_\sigma}(v_\sigma) e_{\omega_\delta}^{\varepsilon_\delta}(v_\delta)$$

$$\tilde{\Phi}(\omega_\sigma, \omega_\delta) \mapsto \chi_{v_\theta}(\omega_\sigma, \omega_\delta) = |\omega_\sigma + \omega_\delta|^{\frac{1}{2}} \frac{e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta)}{\left| e_{\omega_\theta}^{\varepsilon_\theta}(v_\theta) \right|} \tilde{\Phi}(\omega_\sigma, \omega_\delta)$$

$$[\ln(\hat{v}_a)\Phi_{v_\theta}](v_\sigma, v_\delta) = \ln(v_a)\Phi_{v_\theta}(v_\sigma, v_\delta)$$

$$[\ln(\hat{v}_\sigma)_{v_\theta}\tilde{\Phi}](\omega_\sigma,\omega_\delta) = \frac{1}{e^{\varepsilon_\theta}_{\omega_\theta}(v_\theta)}\int_{\mathbb{R}} d\omega'_\sigma \left\langle e^{\varepsilon_\sigma}_{\omega_\sigma} \mid \ln(\hat{v}_\sigma) e^{\varepsilon_\sigma}_{\omega'_\sigma}\right\rangle_{\mathcal{H}^+_{\varepsilon_\sigma}} e^{\varepsilon_\theta}_{\omega_\theta(\omega'_\sigma,\omega_\delta)}(v_\theta)\tilde{\Phi}(\omega'_\sigma,\omega_\delta)$$

$$\left[\hat{Q}_{v_\theta,v^\star_\theta}\tilde{\Phi}\right](\omega_\sigma,\omega_\delta) = \frac{e^{\varepsilon_\theta}_{\omega_\theta}(v_\theta)}{e^{\varepsilon_\theta}_{\omega_\theta}(v^\star_\theta)}\tilde{\Phi}(\omega_\sigma,\omega_\delta)$$

$$\ln(\hat{v}_a)_{v^\star_\theta} = \hat{Q}_{v_\theta,v^\star_\theta}\ln(\hat{v}_a)_{v_\theta}\hat{Q}_{v^\star_\theta,v_\theta}$$

$$e^{\varepsilon_\theta}_{\omega_\theta} \to e^{\varepsilon_\theta s}_{\omega_\theta} = \mathcal{F}^{-1}\left\{\theta\left[s\left(\beta_\theta - \frac{\pi}{2}\right)\right]\mathcal{F}e^{\varepsilon_\theta}_{\omega_\theta}\right\}$$

$$[\mathcal{F}f](\beta_\theta) = \sum_{v_\theta\in(4)\mathcal{L}^+_{\varepsilon_\theta}} f(v_\theta)v_\theta^{-\frac{1}{2}}e^{-\frac{i}{2}v_\theta\beta_\theta}, \beta_\theta\in[0,\pi]$$

$$e^{\varepsilon_\theta}_{\omega_\theta}(v_\theta) \mapsto e^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta) = |\omega_\sigma+\omega_\delta|^{\frac{1}{2}}\frac{e^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta)}{\left|e^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta)\right|}$$

$$\begin{aligned}&\hat{R}^s_{v_\theta}: \mathcal{H}^{\vec{\varepsilon}}_{\text{fis}} \to \mathcal{H}'^s = L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta)\\ &\tilde{\Phi}(\omega_\sigma,\omega_\delta) \mapsto \tilde{\chi}^s_{v_\theta}(\omega_\sigma,\omega_\delta) = \tilde{\Phi}(\omega_\sigma,\omega_\delta)e'^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta)\end{aligned}$$

$$[\ln(\hat{v}_\sigma)^s_{v_\theta}\tilde{\Phi}](\omega_\sigma,\omega_\delta) = \frac{1}{e^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta)}\int_{\mathbb{R}} d\omega'_\sigma \left\langle e^{\varepsilon_\sigma}_{\omega_\sigma} \mid \ln(\hat{v}_\sigma) e^{\varepsilon_\sigma}_{\omega'_\sigma}\right\rangle_{\mathcal{H}^+_{\varepsilon_\sigma}} e^{\varepsilon_\theta s}_{\omega_\theta(\omega'_\sigma,\omega_\delta)}(v_\theta)\tilde{\Phi}(\omega'_\sigma,\omega_\delta)$$

$$\left[\hat{Q}^s_{v_\theta,v^\star_\theta}\tilde{\Phi}\right](\omega_\sigma,\omega_\delta) = \frac{e'^{\varepsilon_\theta s}_{\omega_\theta}(v_\theta)}{e^{\varepsilon_\theta s}_{\omega_\theta}(v^\star_\theta)}\tilde{\Phi}(\omega_\sigma,\omega_\delta)$$

$$\ln(\hat{v}_a)^s_{v^\star_\theta} = \hat{Q}^s_{v_\theta,v^\star_\theta}\ln(\hat{v}_a)^s_{v_\theta}\hat{Q}^s_{v^\star_\theta,v_\theta}$$

$$\tilde{\Phi}(\omega_\sigma,\omega_\delta) = \frac{K}{\sqrt{|\omega_\sigma+\omega_\delta|}}\prod_{a=\sigma}^{\delta} e^{-\frac{(\omega_a-\omega^\star_a)^2}{2\sigma^a_a}} e^{iv^a\omega_a}$$

$$\langle \underline{\Phi}_\pm \mid \ln(\hat{v}_a)_{v_\theta}\underline{\Phi}_\pm\rangle =: \langle \ln(\hat{v}_a)_{v_\theta}\rangle_\pm$$

$$\begin{aligned}\phi(\omega_i) &\approx D(\omega^\star_i)(\omega_i - \omega^\star_i) + E(\omega^\star_i)\\ D(\omega^\star_i) &= \operatorname{sgn}(\omega^\star_i)(1 + a + \ln|\omega^\star_i|)\end{aligned}$$

$$\langle \ln(\hat{v}_a)_{v_\theta}\rangle_\pm = \left[\frac{\omega_\theta(\omega^\star_\sigma,\omega^\star_\delta)}{\omega^\star_a}\right]^2 [\ln\, v_\theta - \alpha D(\omega^\star_\theta)] + \alpha[D(\omega^\star_a) \pm v^a]$$

$$U = \prod_{i=\theta,\sigma,\delta} e^{2i\phi(\omega_i)}$$

$$\lim_{v_\theta\to\infty}\frac{\langle \Delta\ln(\hat{v}_a)_{v_\theta}\rangle_-}{\langle \ln(\hat{v}_a)_{v_\theta}\rangle_-} = \frac{\langle \Delta[\omega^2_\theta(\omega_\sigma,\omega_\delta)\omega_a^{-2}]\rangle}{\langle \omega^2_\theta(\omega_\sigma,\omega_\delta)\omega_a^{-2}\rangle}$$

$$\lim_{v_\theta\to\infty}\frac{\langle \Delta\ln(\hat{v})_a\rangle_+}{\langle \ln(\hat{v})_a\rangle_+} = \lim_{v_\theta\to\infty}\frac{\langle \Delta\ln(\hat{v})_a\rangle_-}{\langle \ln(\hat{v})_a\rangle_-}$$

$$w_a := \omega_\theta(\omega_\sigma,\omega_\delta)\omega_a^{-1}$$
$$\Upsilon_a^{(n)} := w_a^n \ln|\omega_\theta(\omega_\sigma,\omega_\delta)|$$
$$\Sigma_a^{(n)} := w_a^n \ln|\omega_a|$$

$$\boldsymbol{\chi}^s_{v_\theta}(v_\sigma,v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\boldsymbol{\chi}}^s_{v_\theta}(\omega_\sigma,\omega_\delta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\langle \boldsymbol{\chi}^s_{v_\theta} \mid \boldsymbol{\chi}'^s_{v_\theta}\rangle = \sum_{\mathcal{L}^+_{\varepsilon_\sigma}\times\mathcal{L}^+_{\varepsilon_\delta}} \left[\boldsymbol{\chi}^s_{v_\theta}(v_\sigma,v_\delta)\right]^* \boldsymbol{\chi}'^s_{v_\theta}(v_\sigma,v_\delta).$$

$$\left[\ln(\hat{v}_a)^s_{v_\theta}\boldsymbol{\chi}^s_{v_\theta}\right](v_\sigma,v_\delta) = \ln(v_a)\boldsymbol{\chi}^s_{v_\theta}(v_\sigma,v_\delta)$$

$$\langle \tilde{\Phi} \mid \ln(\hat{v}_a)^s_{v_\theta}\tilde{\Phi}\rangle = \left\|\boldsymbol{\chi}^s_{v_\theta}\right\|^{-2} \sum_{\mathcal{L}^+_{\sigma_\sigma}\times\mathcal{L}^+_{\varepsilon_\delta}} \ln(v_a)\left|\boldsymbol{\chi}^s_{v_\theta}(v_\sigma,v_\delta)\right|^2$$

$$\langle \Delta\ln(\hat{v}_a)^s_{v_\theta}\rangle^2 = \left\langle\left(\ln(\hat{v}_a)^s_{v_\theta}\right)^2\right\rangle - \langle\ln(\hat{v}_a)^s_{v_\theta}\rangle^2$$

$$[\mathcal{F}\psi](\beta_i) = \sum_{\mathcal{L}^+_{\varepsilon_i}} \psi(v_i) v_i^{-\frac{1}{2}} e^{-\frac{i}{2}v_i\beta_i}$$

$$\hat{v}_i \to 2i\partial_{\beta_i}$$
$$\hat{v}_i^{1/2}\left(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i}\right)\hat{v}_i^{-1/2} \to 2i\sin\left(\hat{\beta}_i\right)$$

$$[\mathcal{F}_1\psi](\beta_i) = \sum_{v_i\in{}^{(4)}\mathcal{L}^+_{\varepsilon_i}} \psi(v_i) v_i^{-\frac{1}{2}} e^{-\frac{i}{2}v_i\beta_i}$$

$$\Phi(\beta_\theta,v_\sigma,v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\Phi}(\omega_\sigma,\omega_\delta)\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta) = \begin{cases}\left[\mathcal{F}_1 e^{\varepsilon_\theta}_{\omega_\theta}\right](\beta_\theta), & \beta_\theta \in [0,\pi)\\ \left[\mathcal{F}_2 e^{\varepsilon_\theta}_{\omega_\theta}\right](\beta_\theta - \pi), & \beta_\theta \in [\pi, 2\pi)\end{cases}$$

$$\hat{P}_{\beta_\theta}: \mathcal{H}^{\vec{\varepsilon}}_{\text{fis}} \to \mathcal{H}_{\beta_\theta} = L^2\left(\mathbb{R}^2, |\omega_\sigma+\omega_\delta|\left|\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)\right|^{-2} d\omega_\sigma d\omega_\delta\right),$$
$$\tilde{\Phi}(\omega_\sigma,\omega_\delta) \mapsto \tilde{\Phi}_{\beta_\theta}(\omega_\sigma,\omega_\delta) = \tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)\tilde{\Phi}(\omega_\sigma,\omega_\delta).$$

$$\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta) \to \tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta) = |\omega_\sigma+\omega_\delta|^{\frac{1}{2}} \frac{\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)}{\left|\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)\right|}$$

$$\hat{R}_{\beta_\theta}: \mathcal{H}^{\vec{\varepsilon}}_{\text{fis}} \to \mathcal{H}'_{\beta_\theta} = L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta)$$
$$\tilde{\Phi}(\omega_\sigma,\omega_\delta) \mapsto \tilde{\boldsymbol{\chi}}_{\beta_\theta}(\omega_\sigma,\omega_\delta) = \tilde{\Phi}(\omega_\sigma,\omega_\delta)\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)$$

$$\boldsymbol{\chi}_{\beta_\theta}(v_\sigma,v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\boldsymbol{\chi}}_{\beta_\theta}(\omega_\sigma,\omega_\delta) e^{\varepsilon_\sigma}_{\omega_\sigma}(v_\sigma) e^{\varepsilon_\delta}_{\omega_\delta}(v_\delta)$$

$$\left[\ln(\hat{v}_\sigma)_{\beta_\theta}\tilde{\Phi}\right](\omega_\sigma,\omega_\delta) = \frac{1}{\tilde{e}^{\varepsilon_\theta}_{\omega_\theta}(\beta_\theta)} \int_{\mathbb{R}} d\omega'_\sigma \left\langle e^{\varepsilon_\sigma}_{\omega_\sigma} \mid \ln(\hat{v}_\sigma) e^{\varepsilon_\sigma}_{\omega'_\sigma}\right\rangle_{\mathcal{H}^+_{\varepsilon_\sigma}} \tilde{e}^{\varepsilon_\theta}_{\omega_\theta(\omega'_\sigma,\omega_\delta)}(\beta_\theta)\tilde{\Phi}(\omega'_\sigma,\omega_\delta)$$

$$\ln(v_\theta) = \ln(v_\theta^o) - \sum_{n=1}^{\infty} \frac{1}{n(v_\theta^o)^n}(v_\theta^o - v_\theta)^n$$

$$\ln(\hat{v}_\theta) = \ln(v_\theta^o) - \sum_{n=1}^{\infty} \frac{1}{n(v_\theta^o)^n}\left(v_\theta^o - 2i\partial_{\beta_\theta}\right)^n$$

$$[\ln(\hat{v}_\theta)\tilde{\boldsymbol{\chi}}_{\beta_\theta}](\omega_\sigma, \omega_\delta) = \tilde{\Phi}(\omega_\sigma, \omega_\delta)[\ln(\hat{v}_\theta)\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}](\beta_\theta)$$

$$[\ln(\hat{v}_\theta)_{\beta_\theta}\tilde{\Phi}](\omega_\sigma, \omega_\delta) = \frac{[\ln(\hat{v}_\theta)\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}](\beta_\theta)}{\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta)}\tilde{\Phi}(\omega_\sigma, \omega_\delta)$$

$$\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta) \mapsto \tilde{f}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta) := \frac{[\ln(\hat{v}_\theta)\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}](\beta_\theta)}{\left|\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta)\right|}$$

$$\tilde{e}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta) \sim |\omega_\sigma + \omega_\delta|^{\frac{1}{2}}[e^{i\omega_\theta x(\beta_\theta)} + O(\omega_\theta^{-2})],$$
$$x(\beta) = \ln\left[\tan\left(\frac{\beta}{2}\right)\right].$$

$$[\ln(\hat{v}_\theta)\boldsymbol{\chi}_{\beta_\theta}](v_\sigma, v_\delta) = \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \tilde{\Phi}(\omega_\sigma, \omega_\delta)\tilde{f}_{\omega_\theta}^{\varepsilon_\theta}(\beta_\theta)e_{\omega_\sigma}^{\varepsilon_\sigma}(v_\sigma)e_{\omega_\delta}^{\varepsilon_\delta}(v_\delta)$$

$$\langle\Phi \mid \ln(\hat{v}_\theta)_{\beta_\theta}\Phi\rangle = \|\boldsymbol{\chi}_{\beta_\theta}\|^{-2}\sum_{\bar{\mathcal{L}}^2} \overline{\boldsymbol{\chi}}_{\beta_\theta}(v_\sigma, v_\delta)[\ln(\hat{v}_\theta)\boldsymbol{\chi}_{\beta_\theta}](v_\sigma, v_\delta)$$

$$g_1 \equiv P_{\bar{\gamma}} - \frac{P_{\bar{\gamma}_0}}{\sqrt{2\pi}} = 0$$

$$g_2' \equiv \tau - \frac{\tau_0}{\sqrt{2\pi}} = 0$$

$$S = \int_{t_i}^{t_f} dt\left[P_{\tau_0}\dot{\tau}_0 + P_{\bar{\gamma}_0}\dot{\bar{\gamma}}_0 + \oint d\theta P_\xi \dot{\xi} - \left(N\tilde{C} + N^\theta C_\theta\right)\right]$$

$$C_\theta = \oint d\theta P_\xi \xi' = 0$$

$$\tilde{C} = \frac{1}{\sqrt{2\pi}}\frac{4G}{\pi}\left\{-\tau_0 P_{\tau_0}P_{\bar{\gamma}_0} + \frac{P_{\bar{\gamma}_0}^2}{8\tau_0}\oint d\theta\xi^2 + \frac{\tau_0}{2}\oint d\theta\left[P_\xi^2 + \left(\frac{\pi}{4G}\right)^2(\xi')^2\right]\right\} = 0$$

$$q_{ii} = \frac{|p_\theta p_\sigma p_\delta|}{(2\pi p_i)^2}$$

$$q_{\theta\theta} = \frac{4G}{\pi}\exp\left[\frac{1}{\sqrt{2\pi}}\left(\bar{\gamma}_0 - (2\pi)^{1/4}\frac{\xi_0}{\sqrt{\tau_0}} - \frac{\xi_0^2}{4\tau_0}\right)\right],$$
$$q_{\sigma\sigma} = \frac{\pi}{4G}\frac{\tau_0^2}{2\pi}\exp\left(-\frac{1}{(2\pi)^{1/4}}\frac{\xi_0}{\sqrt{\tau_0}}\right),$$
$$q_{\delta\delta} = \frac{4G}{\pi}\exp\left(\frac{1}{(2\pi)^{1/4}}\frac{\xi_0}{\sqrt{\tau_0}}\right).$$

$$\xi_0 = \frac{1}{\sqrt{2\pi}}|p_\theta|^{1/2}\ln\left(\frac{1}{16\pi G}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\right)$$
$$\tau_0 = \frac{\sqrt{2\pi}}{4\pi^2}|p_\theta|$$
$$\bar{\gamma}_0 = \sqrt{2\pi}\left\{2\ln\left(\frac{|p_\sigma|}{16\pi G}\right) + \frac{1}{4}\left[\ln\left(\frac{1}{16\pi G}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\right)\right]^2\right\}$$
$$P_{\xi_0} = \frac{1}{8\pi G\gamma}\sqrt{2\pi}\frac{1}{|p_\theta|^{1/2}}\left[c^\delta p_\delta + \frac{1}{4}(c^\sigma p_\sigma + c^\delta p_\delta)\ln\left(\frac{1}{16\pi G}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\right)\right]$$
$$P_{\tau_0} = \frac{1}{8\pi G\gamma}\frac{4\pi^2}{\sqrt{2\pi}}\frac{1}{|p_\theta|}\left\{-c^\theta p_\theta - c^\delta p_\delta - \frac{1}{8}(c^\sigma p_\sigma + c^\delta p_\delta)\left[\ln\left(\frac{1}{16\pi G}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\right)\right]^2 - \frac{c^\delta p_\delta}{2}\ln\left(\frac{1}{16\pi G}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\right)\right\}$$
$$P_{\bar{\gamma}_0} = -\frac{1}{8\pi G\gamma}\frac{1}{2\sqrt{2\pi}}(c^\sigma p_\sigma + c^\delta p_\delta)$$

$$q_{\theta\theta} = \frac{1}{4\pi^2}\left|\frac{p_\sigma p_\delta}{p_\theta}\right|\exp\left\{\frac{2\pi}{\sqrt{|p_\theta|}}\frac{c^\delta p_\delta - c^\sigma p_\sigma}{c^\sigma p_\sigma + c^\delta p_\delta}\tilde{\xi}(\theta) - \frac{\pi^2}{|p_\theta|}[\tilde{\xi}(\theta)]^2 - \frac{8\pi G\gamma}{c^\sigma p_\sigma + c^\delta p_\delta}\zeta(\theta)\right\}$$
$$q_{\sigma\sigma} = \frac{1}{4\pi^2}\left|\frac{p_\theta p_\delta}{p_\sigma}\right|\exp\left\{-\frac{2\pi}{\sqrt{|p_\theta|}}\tilde{\xi}(\theta)\right\}$$
$$q_{\delta\delta} = \frac{1}{4\pi^2}\left|\frac{p_\theta p_\sigma}{p_\delta}\right|\exp\left\{\frac{2\pi}{\sqrt{|p_\theta|}}\tilde{\xi}(\theta)\right\}$$

$$\tilde{\xi}(\theta) = \frac{1}{\pi}\sum_{m\neq 0}\sqrt{\frac{G}{|m|}}(a_m + a^*_{-m})e^{im\theta}$$
$$\zeta(\theta) = i\sum_{m\neq 0}\sum_{\tilde{m}\neq 0}\operatorname{sgn}(m+\tilde{m})\frac{\sqrt{|m+\tilde{m}||\tilde{m}|}}{|m|}(a_{-\tilde{m}} - a^*_{\tilde{m}})(a_{m+\tilde{m}} + a^*_{-(m+\tilde{m})})e^{im\theta}$$

$$d\theta + dtN^\theta(t) \to d\theta$$

$$ds^2 = -q_{\theta\theta}\left(\frac{|p_\theta|}{4\pi^2}\right)^2 N^2dt^2 + q_{\theta\theta}d\theta^2 + q_{\sigma\sigma}d\sigma^2 + q_{\delta\delta}d\delta^2$$

$$S = \int_{t_i}^{t_f} dt\left[\frac{1}{8\pi G\gamma}(p_\theta\dot{c}^\theta + p_\sigma\dot{c}^\sigma + p_\delta\dot{c}^\delta) + i\sum_{m\neq 0} a^*_m\dot{a}_m - \left(N^\theta C_\theta + \frac{1}{16\pi G}\frac{N_0}{(2\pi)^3}C_{\mathrm{G}}\right)\right]$$
$$C_\theta = \sum_{m=1}^{\infty} m(a^*_m a_m - a^*_{-m}a_{-m}) = 0$$

$$C_{\mathrm{G}} = C_{\mathrm{BI}} + C_\xi = 0, C_\xi = \frac{G}{V}\left[\frac{(c^\sigma p_\sigma + c^\delta p_\delta)^2}{\gamma^2|p_\theta|}H^\xi_{\mathrm{int}} + 32\pi^2|p_\theta|H^\xi_0\right].$$

$$H^\xi_{\mathrm{int}} = \sum_{m\neq 0}\frac{1}{2|m|}[2a^*_m a_m + a_m a_{-m} + a^*_m a^*_{-m}], H^\xi_0 = \sum_{m\neq 0}|m|a^*_m a_m,$$

$$|\mathfrak{n}\rangle := |\ldots, n_{-2}, n_{-1}, n_1, n_2, \ldots\rangle,$$

$$\hat{a}_m|\ldots, n_m, \ldots\rangle = \sqrt{n_m}|\ldots, n_m - 1, \ldots\rangle;\ \hat{a}^\dagger_m|\ldots, n_m, \ldots\rangle = \sqrt{n_m + 1}|\ldots, n_m + 1, \ldots\rangle.$$

$$\hat{C}_\theta = \hbar\sum_{m>0}^{\infty} m\hat{X}_m, \hat{X}_m = \hat{a}^\dagger_m\hat{a}_m - \hat{a}^\dagger_{-m}\hat{a}_{-m}$$

$$\sum_{m>0}^{\infty} mX_m = 0, X_m = n_m - n_{-m}$$

$$\hat{C}_\xi = \widehat{\left[\frac{1}{V}\right]}^{\frac{1}{2}} \hat{C}_\xi \left[\frac{1}{V}\right]^{\frac{1}{2}}$$

$$\hat{C}_{\mathrm{G}} = \widehat{\left[\frac{1}{V}\right]}^{-\frac{1}{2}} \hat{C}_{\mathrm{G}} \widehat{\left[\frac{1}{V}\right]}^{-\frac{1}{2}} = \hat{C}_{\mathrm{BI}} + \hat{C}_\xi$$

$$\hat{H}_0^\xi = \sum_{m>0}^{\infty} m\hat{N}_m, \hat{N}_m = \hat{a}_m^\dagger \hat{a}_m + \hat{a}_{-m}^\dagger \hat{a}_{-m}$$

$$\hat{H}_{\mathrm{int}}^\xi = \sum_{m>0}^{\infty} \frac{\hat{N}_m + \hat{Y}_m}{m}, \hat{Y}_m = \hat{a}_m \hat{a}_{-m} + \hat{a}_m^\dagger \hat{a}_{-m}^\dagger$$

$$|X_1, X_2, \dots; N_1, N_2, \dots\rangle := |\mathfrak{X}; \mathfrak{N}\rangle.$$

$$\begin{aligned} \hat{X}_m|\mathfrak{X};\mathfrak{N}\rangle &= X_m|\mathfrak{X};\mathfrak{N}\rangle \\ \hat{N}_m|\mathfrak{X};\mathfrak{N}\rangle &= N_m|\mathfrak{X};\mathfrak{N}\rangle \\ \hat{Y}_m|\mathfrak{X};\mathfrak{N}\rangle &= \frac{\sqrt{N_m^2 - X_m^2}}{2}|\mathfrak{X}; \dots, N_m - 2, \dots\rangle + \frac{\sqrt{(N_m+2)^2 - X_m^2}}{2}|\mathfrak{X}; \dots, N_m + 2, \dots\rangle \end{aligned}$$

$$\mathcal{F} = \bigoplus_{\mathfrak{X}} \mathcal{F}_{\mathfrak{X}}.$$

$$\|\hat{H}_{\mathrm{int}}^\xi|\mathfrak{N}\rangle\|^2 = \left(\sum_{m>0}^{\infty} \frac{N_m}{|m|}\right)^2 + \sum_{m>0}^{\infty} \frac{N_m^2 - X_m^2 + 2N_m}{2m^2} + \sum_{m>0}^{\infty} \frac{1}{m^2}$$

$$\hat{C}_{\mathrm{G}}^s = \hat{C}_{\mathrm{BI}}^{\mathrm{s}} + \hat{C}_\xi^{\mathrm{s}}, s = \mathrm{A}, \mathrm{B}$$

$$\hat{C}_\xi^{\mathrm{A}} = l_{\mathrm{Pl}}^2 \left\{ \frac{(\hat{\Omega}_\sigma + \hat{\Omega}_\delta)^2}{\gamma^2} \left[\frac{\hat{1}}{\sqrt{|p_\theta|}}\right]^2 \hat{H}_{\mathrm{int}}^\xi + 32\pi^2 \middle| \widehat{p_\theta} \,|\, \hat{H}_0^\xi \right\}$$

$$\hat{G} = \frac{1}{4\gamma^2\Delta}\widehat{\sqrt{V}}[\hat{F}_\sigma \hat{V} \hat{F}_\sigma + \hat{F}_\delta \hat{V}' \hat{F}_\delta]\widehat{\sqrt{V}} - \hat{C}^{(\theta)}$$

$$\hat{C}_\xi^{\mathrm{B}} = l_{\mathrm{Pl}}^2 \left\{ \widehat{\left[\frac{1}{|p_\theta|^{\frac{1}{4}}}\right]}^2 \hat{G}^2 \widehat{\left[\frac{1}{|p_\theta|^{\frac{1}{4}}}\right]}^2 \hat{H}_{\mathrm{int}}^\xi + 32\pi^2 \middle| \widehat{p_\theta} \,|\, \hat{H}_0^\xi \right\}$$

$$\mathcal{H}_{\vec{\varepsilon}}^+ \otimes \mathcal{F}, \text{ con } \mathcal{H}_{\vec{\varepsilon}}^+ = \bigotimes_i \mathcal{H}_{\varepsilon_i}^+$$

$$\mathcal{H}_{\tilde{\varepsilon}, \lambda_\sigma^\star, \lambda_\delta^\star} \otimes \mathcal{F}$$

$$(\psi \,|\, \hat{C}_{\mathrm{G}}^{\mathrm{A}\dagger} = 0.$$

$$\Big(\psi \,| = \sum_{v_\theta \in \mathcal{L}_{\varepsilon_\theta}^+} \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \langle v_\theta| \otimes \langle e_{\omega_\sigma}^{\varepsilon_\sigma}| \otimes \langle e_{\omega_\delta}^{\varepsilon_\delta}| \otimes (\psi_{\omega_\sigma,\omega_\delta}(v_\theta) \,|$$

$$\begin{aligned} (\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta + 2k + 2) \,| = &\left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta + 2k) \,|\, \hat{H}_{\omega_\sigma,\omega_\delta}^\xi(\varepsilon_\theta + 2k)\right. \\ &+ \left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta + 2k - 2) \,|\, F(\varepsilon_\theta + 2k),\right. \end{aligned}$$

$$\widehat{H}^{\xi}_{\omega_\sigma,\omega_\delta}(v_\theta) = \frac{i}{6\pi(\omega_\sigma+\omega_\delta)f_+(v_\theta)}\Bigg[2\omega_\sigma\omega_\delta - \frac{(\omega_\sigma+\omega_\delta)^2}{9\gamma}b^2(v_\theta)\widehat{H}^{\xi}_{\text{int}} \\ -32\pi^2\left(6\pi\gamma l_{\text{Pl}}^2\sqrt{\Delta}\right)^{2/3}|v_\theta|^{2/3}\widehat{H}^{\xi}_0\Bigg].$$

$$\left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta+2)\right| = \left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)\right| \widehat{H}^{\xi}_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta).$$

$$\left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta+2k)\right| = \left(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)\right| \sum_{O(0\to k)}\left[\prod_{\{r_p\}}F\left(\varepsilon_\theta+2r_p+2\right)\right]\mathcal{P}\left[\prod_{\{s_q\}}\widehat{H}^{\xi}_{\omega_\sigma,\omega_\delta}\left(\varepsilon_\theta+2s_q\right)\right],$$

$$\left\{\widehat{\Omega}_a, -i|\omega_\sigma+\omega_\delta|^{-1/2}\partial_{\omega_a}|\omega_\sigma+\omega_\delta|^{1/2}, a=\sigma,\delta\right\},$$

$$\left\{\left(\hat{a}_m+\hat{a}_m^\dagger\right)\pm\left(\hat{a}_{-m}+\hat{a}_{-m}^\dagger\right), i\left[\left(\hat{a}_m-\hat{a}_m^\dagger\right)\pm\left(\hat{a}_{-m}-\hat{a}_{-m}^\dagger\right)\right];\ m\in\mathbb{N}^+\right\}$$

$$L^2(\mathbb{R}^2, |\omega_\sigma+\omega_\delta|d\omega_\sigma d\omega_\delta)\otimes\mathcal{F}$$

$$\mathcal{H}^{\text{A}}_{\text{fis}} = L^2(\mathbb{R}^2, |\omega_\sigma+\omega_\delta|d\omega_\sigma d\omega_\delta)\otimes\mathcal{F}_f$$

$$\left(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)\right|\hat{P}_M^\perp = 0$$
$$\left(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)\right|\hat{P}_M\hat{\mathcal{C}}_{\text{G}}^\dagger\hat{P}_M = 0$$
$$\left(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)\right|\hat{P}_M\hat{\mathcal{C}}_{\theta}^\dagger\hat{P}_M = 0$$

$$L^2(\mathbb{R}^2, |\omega_\sigma+\omega_\delta|d\omega_\sigma d\omega_\delta)\otimes\left(\mathcal{F}_f\right)_M$$

$$(\psi|\,\hat{\mathcal{C}}_{\text{G}}^{\text{B}\dagger} = 0.$$

$$\left(\psi\right| = \sum_{v\in\mathcal{L}_{\tilde{\varepsilon}}}\ \sum_{\omega_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}}\ \sum_{\bar{\omega}_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}}\langle v,\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star|\otimes(\psi(v,\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star)|,$$

$$(\psi(v,\lambda_\sigma,\lambda_\delta)| = (\psi(v,\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star)|$$

$$(\psi_\pm(v\pm4,\lambda_\sigma,\lambda_\delta)| = (\psi|v\pm4,\lambda_\sigma,\lambda_\delta\rangle_\pm, (\psi_{0^\pm}(v,\lambda_\sigma,\lambda_\delta)| = (\psi|v,\lambda_\sigma,\lambda_\delta\rangle_{0^\pm}$$

$$|v\pm4,\lambda_\sigma,\lambda_\delta\rangle'_\pm = \left|v\pm4,\lambda_\sigma,\frac{v\pm4}{v}\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm4}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle \\ +\left|v\pm4,\frac{v\pm4}{v}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm2}{v}\lambda_\sigma,\frac{v\pm4}{v\pm2}\lambda_\delta\right\rangle$$
$$|v,\lambda_\sigma,\lambda_\delta\rangle'_{0^\pm} = 2|v,\lambda_\sigma,\lambda_\delta\rangle + \left|v,\frac{v\pm2}{v}\lambda_\sigma,\frac{v}{v\pm2}\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle,$$

$$(\psi'_\pm(v\pm4,\lambda_\sigma,\lambda_\delta)| = (\psi|v\pm4,\lambda_\sigma,\lambda_\delta\rangle'_\pm, (\psi'_{0^\pm}(v,\lambda_\sigma,\lambda_\delta)| = (\psi|v,\lambda_\sigma,\lambda_\delta\rangle'_{0^\pm}$$

$$(\psi_+(v+4,\lambda_\sigma,\lambda_\delta)\Big| -\eta[b_\theta^\star(v,\lambda_\sigma,\lambda_\delta)b_\theta^\star(v+4,\lambda_\sigma,\lambda_\delta)]^2\frac{v+4}{v}\left(\psi'_+(v+4,\lambda_\sigma,\lambda_\delta)\right|\widehat{H}^{\xi}_{\text{int}} \\ = -\frac{1}{\eta}\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2x_+(v)}\left(\psi(v,\lambda_\sigma,\lambda_\delta)\right|\widehat{H}^{\xi}_0 + \frac{x_0^-(v)}{x_+(v)}\left(\psi_{0^-}(v,\lambda_\sigma,\lambda_\delta)\right| + \frac{x_0^+(v)}{x_+(v)}(\psi_{0^+}(v,\lambda_\sigma,\lambda_\delta)| \\ -\frac{x_-(v)}{x_+(v)}\left(\psi_-(v-4,\lambda_\sigma,\lambda_\delta)\right| + \eta[b_\theta^\star(v,\lambda_\sigma,\lambda_\delta)]^4\left\{\left[\frac{b_\theta^\star(v-4,\lambda_\sigma,\lambda_\delta)}{b_\theta^\star(v,\lambda_\sigma,\lambda_\delta)}\right]^2\frac{v-4}{v}\frac{x_-(v)}{x_+(v)} \\ \times\left(\psi'_-(v-4,\lambda_\sigma,\lambda_\delta)\right| - \left[\frac{x_0^-(v)}{x_+(v)}\left(\psi'_{0^-}(v,\lambda_\sigma,\lambda_\delta)\right| + \frac{x_0^+(v)}{x_+(v)}(\psi'_{0^+}(v,\lambda_\sigma,\lambda_\delta)|\right]\right\}\widehat{H}^{\xi}_{\text{int}}$$

$$\eta = \left(\frac{l_{\mathrm{Pl}}}{4\pi\gamma\sqrt{\Delta}}\right)^{2/3}$$

$$\Big(\psi(\tilde{\varepsilon}+4k,\lambda_\sigma,\lambda_\delta)\,| = \sum_{n\in\mathbb{N}} \eta^{n-k}(\,{}^{n}\psi(\tilde{\varepsilon}+4k,\lambda_\sigma,\lambda_\delta)\,|, \forall k\in\mathbb{N}^{+}$$

$$\left(\,{}^{0}\psi_{+}(v+4)\right| = -\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}\left(\,{}^{0}\psi(v)\,|\,\hat{H}_0^{\xi}\right.$$

$$(\,{}^{1}\psi_{+}(v+4)\,| = -\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}\left(\,{}^{1}\psi(v)\right|\hat{H}_0^{\xi} + \frac{x_0^-(v)}{x_+(v)}\left(\,{}^{0}\psi_{0^-}(v)\right| + \frac{x_0^+(v)}{x_+(v)}(\,{}^{0}\psi_{0^+}(v)\,|$$
$$+[b_\theta^\star(v)b_\theta^\star(v+4)]^2\frac{v+4}{v}(\,{}^{0}\psi_+'(v+4)\,|\,\hat{H}_{\mathrm{int}}^{\xi}$$

$$(\,{}^{2}\psi_{+}(v+4)\,| = -\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}\left(\,{}^{2}\psi(v)\right|\hat{H}_0^{\xi} + \frac{x_0^-(v)}{x_+(v)}\left(\,{}^{1}\psi_{0^-}(v)\right| + \frac{x_0^+(v)}{x_+(v)}(\,{}^{1}\psi_{0^+}(v)\,|$$
$$-\frac{x_-(v)}{x_+(v)}\left(\,{}^{0}\psi_-(v-4)\right| - [b_\theta^\star(v)]^4\left\{\left[\frac{x_0^-(v)}{x_+(v)}(\,{}^{0}\psi_{0^-}'(v)\,|\right.\right.$$
$$\left.+\frac{x_0^+(v)}{x_+(v)}(\,{}^{0}\psi_{0^+}'(v)\,|\right] - \left[\frac{b_\theta^\star(v+4)}{b_\theta^\star(v)}\right]^2\frac{v+4}{v}(\,{}^{1}\psi_+'(v+4)\,|\Big]\Big\}\hat{H}_{\mathrm{int}}^{\xi},$$

$$(\,{}^{n}\psi_{+}(v+4)\,| = -\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}\left(\,{}^{n}\psi(v)\right|\hat{H}_0^{\xi} + \frac{x_0^-(v)}{x_+(v)}\left(\,{}^{n-1}\psi_{0^-}(v)\right| + \frac{x_0^+(v)}{x_+(v)}(\,{}^{n-1}\psi_{0^+}(v)\,|$$
$$-\frac{x_-(v)}{x_+(v)}\left(\,{}^{n-2}\psi_-(v-4)\right| - [b_\theta^\star(v)]^4\left\{\frac{x_0^-(v)}{x_+(v)}(\,{}^{n-2}\psi_{0^-}'(v)\,|\right.$$
$$+\frac{x_0^+(v)}{x_+(v)}(\,{}^{n-2}\psi_{0^+}'(v)\,|] - \left[\frac{b_\theta^\star(v+4)}{b_\theta^\star(v)}\right]^2\frac{v+4}{v}(\,{}^{n-1}\psi_+'(v+4)\,|$$
$$\left.-\left[\frac{b_\theta^\star(v-4)}{b_\theta^\star(v)}\right]^2\frac{v-4}{v}(\,{}^{n-3}\psi_-'(v-4)\,|\right\}\hat{H}_{\mathrm{int}}^{\xi}.$$

$${}^{n}\psi_{+}(v+4,\lambda_\sigma,\lambda_\delta)\,| = \left(\,{}^{n}\psi\left(v+4,\lambda_\sigma,\frac{v+4}{v+2}\lambda_\delta\right)\right| + \left(\,{}^{n}\psi\left(v+4,\lambda_\sigma,\frac{v+2}{v}\lambda_\delta\right)\right|$$
$$+\left(\,{}^{n}\psi\left(v+4,\frac{v+4}{v+2}\lambda_\sigma,\frac{v+2}{v}\lambda_\delta\right)\right| + \left(\,{}^{n}\psi\left(v+4,\frac{v+4}{v+2}\lambda_\sigma,\lambda_\delta\right)\right|$$
$$+\left(\,{}^{n}\psi\left(v+4,\frac{v+2}{v}\lambda_\sigma,\frac{v+4}{v+2}\lambda_\delta\right)\right| + \left(\,{}^{n}\psi\left(v+4,\frac{v+2}{v}\lambda_\sigma,\lambda_\delta\right)\right|.$$

$$\{(\,{}^{n}\psi_{+}(v+4,\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star)\,|;\omega_{\tilde{\varepsilon}},\bar{\omega}_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}},\}$$

$$\{(\,{}^{n}\psi(v+4,\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star)\,|;\omega_{\tilde{\varepsilon}},\bar{\omega}_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}\}$$

$$\{(\psi(\tilde{\varepsilon},\omega_{\tilde{\varepsilon}}\lambda_\sigma^\star,\bar{\omega}_{\tilde{\varepsilon}}\lambda_\delta^\star)\,|;\omega_{\tilde{\varepsilon}},\bar{\omega}_{\tilde{\varepsilon}}\in\mathcal{W}_{\tilde{\varepsilon}}\}$$

$$\mathcal{H}_{\mathrm{fis}}^{\mathrm{B}} = \mathcal{H}_{\lambda_\sigma^\star,\lambda_\delta^\star}\otimes\mathcal{F}_f.$$

$$\Gamma(A) = \{(\phi,\psi)\in\mathcal{H}\times\mathcal{H};\phi\in\mathcal{D}(A),\psi=A\phi\}.$$

$$\langle\phi\,|\,A\psi\rangle = \langle A^\dagger\phi\,|\,\psi\rangle,$$

$$\mathbb{I} = \int_{\mathbb{R}} dE_\lambda^A$$

$$A=\int_{\mathbb{R}} \lambda d E_\lambda^A$$

$$\langle\psi_1 \mid A \psi_2\rangle=\int_{\mathbb{R}} \lambda d\langle\psi_1 \mid E_\lambda^A \psi_2\rangle$$

$$f(A)=\int_{\mathbb{R}} f(\lambda) d E_\lambda^A$$

$$\mathcal{H}=\mathcal{H}_{p.p} \oplus \mathcal{H}_{a.c} \oplus \mathcal{H}_{s.c}.$$

$$A_{p.p}=A \upharpoonright \mathcal{H}_{p.p}, A_{a.c}=A \upharpoonright \mathcal{H}_{a.c}, A_{s.c}=A \upharpoonright \mathcal{H}_{s.c},$$

$$\sigma(A)=\sigma_{\text{disc}}(A) \cup \sigma_{\text{es}}(A)$$

$$\|V \psi\| \leq a\|A \psi\|+b\|\psi\|, \text{ para todo } \psi \in \mathcal{D}(A)$$

$$\sum_v \langle v \mid (A^{\dagger} A)^{1/2} v\rangle<\infty$$

$$f(A, A^{\dagger})=\int_{\sigma(N)} \int_{\sigma(M)} f(z, z^*) d E^A(x, y)$$

$$B_a=\langle\mathcal{D}_a\rangle, Y_a=\langle\mathcal{D}_a'\rangle, X_a-B_a^2=\langle\Delta \mathcal{D}_a\rangle^2=: \sigma_{\mathcal{D}_a}^2$$

$$\mathcal{D}_a=-i\alpha|\omega_\sigma+\omega_\delta|^{-\frac{1}{2}}(\partial_{\omega_a})|\omega_\sigma+\omega_\delta|^{\frac{1}{2}}$$
$$\mathcal{D}_a'=2 \omega_\theta(\omega_\sigma, \omega_\delta) \omega_a^{-1} \mathcal{D}_a \omega_\theta(\omega_\sigma, \omega_\delta) \omega_a^{-1}$$

$$\langle\mathcal{D}_a\rangle_+=-\langle\mathcal{D}_a\rangle_- -2 \alpha \sum_{i=1}^3 \langle[\partial_{\omega_a} \phi(-\omega_i)]\rangle_-$$

$$\langle\mathcal{D}_a'\rangle_+=-\langle\mathcal{D}_a'\rangle_- -4 \alpha \sum_{i=1}^3 \left\langle\frac{\omega_\theta^2}{\omega_a^2}[\partial_{\omega_a} \phi(-\omega_i)]\right\rangle_-$$

$$\sigma_{\mathcal{D}_a+} \leq \sigma_{\mathcal{D}_a-}+2 \alpha \sum_{i=1}^3 \langle\Delta[\partial_{\omega_a} \phi(-\omega_i)]\rangle_-$$

$$|\partial_\omega \phi(\omega)| \leq C_\theta|\ln |\omega||+C_0$$
$$|\omega \partial_\omega^2 \phi(\omega)| \leq C_\sigma$$

$$\left|\sum_{i=1}^3 \langle[\partial_{\omega_a} \phi(-\omega_i)]\rangle_-\right| \leq C_\theta[\langle|\ln |\omega_a||\rangle_-+\langle|\Upsilon_a^{(2)}|\rangle_-]+C_0(1+\langle w_a^2\rangle),$$

$$\left|\sum_{i=1}^3 \left\langle\frac{\omega_\theta^2}{\omega_a^2}[\partial_{\omega_a} \phi(-\omega_i)]\right\rangle_-\right| \leq C_\theta[\langle|\Sigma_a^{(2)}|\rangle_-+\langle|\Upsilon_a^{(4)}|\rangle_-]+C_0[\langle w_a^2\rangle+\langle w_a^4\rangle]$$

$$\sum_{i=1}^3 \langle\Delta[\partial_{\omega_a} \phi(-\omega_i)]\rangle_- \leq C_\sigma\left[\langle\Delta \ln |\omega_a|\rangle_-+\left\langle\Delta \Upsilon_a^{(2)}\right\rangle_-\right]$$

$$\tilde{\Phi} \rightarrow \tilde{\Phi}': \tilde{\Phi}'(\omega_\sigma, \omega_\delta)=\tilde{\Phi}(\omega_\theta(\omega_\sigma, \omega_\delta), \omega_\delta)$$

$$\tilde{\Phi}(\omega_\sigma, \omega_\delta) \mapsto \frac{\omega_\theta^2(\omega_\sigma, \omega_\delta)}{\omega_\sigma^2} \tilde{\Phi}(\omega_\theta(\omega_\sigma, \omega_\delta), \omega_\delta)$$

$$H^{\mathrm{ef}} = -\frac{2}{\gamma^2}\left[\Omega_\theta^{\mathrm{ef}}\Omega_\sigma^{\mathrm{ef}} + \Omega_\theta^{\mathrm{ef}}\Omega_\delta^{\mathrm{ef}} + \Omega_\sigma^{\mathrm{ef}}\Omega_\delta^{\mathrm{ef}}\right],$$
$$\Omega_i^{\mathrm{ef}} := 6\pi\gamma G v_i \sin(\beta_i),$$

$$(\partial_\tau v_i)^2 = 9(8\pi G)^4(\mathcal{K} - \mathcal{K}_i)^2\left[v_i^2 - \left(\frac{4\mathcal{K}_i}{3}\right)^2\right]$$
$$\partial_\tau b_i = 3(8\pi G)^2(\mathcal{K}_i - \mathcal{K})\sin(\beta_i)$$

$$v_i(\tau) = \frac{4}{3}\mathcal{K}_i \cosh\left[3(8\pi G)^2(\mathcal{K} - \mathcal{K}_i)(\tau - \tau_o)\right]$$

3. Modelo TOV para partículas estrella o supermasivas en campos cuánticos relativistas o curvos.

$$s^2 \equiv c^2\frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}P}{\mathrm{d}\varepsilon}$$

$$\xi \equiv \frac{GM_{\mathrm{NS}}}{Rc^2} = \frac{M_{\mathrm{NS}}}{R}$$

$$\phi \equiv P/\varepsilon$$

$$\gamma \equiv \frac{\mathrm{d}\ln P}{\mathrm{d}\ln \varepsilon} = \frac{s^2}{P/\varepsilon} = \frac{s^2}{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}\hat{r}}\hat{P} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2}\frac{(1+\hat{P}/\hat{\varepsilon})\left(1+\hat{r}^3\hat{P}/\widehat{M}\right)}{1-2\widehat{M}/\hat{r}};\ \frac{\mathrm{d}}{\mathrm{d}\hat{r}}\widehat{M} = \hat{\varepsilon}^2\, E(\rho,\delta) \approx E_0(\rho) + E_{\mathrm{sym}}(\rho)\delta^2$$

$$\frac{M_{\mathrm{NS}}}{R} \leftrightarrow \frac{P}{\varepsilon} \leftrightarrow \frac{\mathrm{d}P}{\mathrm{d}\varepsilon} \leftrightarrow \frac{\mathrm{d}\ln P}{\mathrm{d}\ln \varepsilon}$$
$$\hat{P} = P/\varepsilon_{\mathrm{c}} \approx \mathrm{X} + b_2\hat{r}^2 + b_4\hat{r}^4 + \cdots$$
$$\hat{\varepsilon} = \varepsilon/\varepsilon_{\mathrm{c}} \approx 1 + a_2\hat{r}^2 + a_4\hat{r}^4 + \cdots$$
$$\widehat{M} \approx \frac{1}{3}\hat{r}^3 + \frac{1}{5}a_2\hat{r}^5 + \frac{1}{7}a_4\hat{r}^7 + \cdots$$
$$\mathrm{X} \equiv P_{\mathrm{c}}/\varepsilon_{\mathrm{c}}, \mu \equiv \hat{\varepsilon} - 1$$
$$\mathcal{U}/\mathcal{U}_{\mathrm{c}} \approx 1 + \sum_{i+j\geq 1} u_{ij}\mathrm{X}^i\mu^j$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\varepsilon}{r^2}\left(1+\frac{p}{\varepsilon}\right)\left(1+\frac{4\pi r^3 P}{M}\right)\left(1-\frac{2GM}{r}\right)^{-1}, \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\varepsilon$$

$$W = \frac{1}{G}\frac{1}{\sqrt{4\pi G\varepsilon_{\mathrm{c}}}} = \frac{1}{\sqrt{4\pi\varepsilon_{\mathrm{c}}}}, Q = \frac{1}{\sqrt{4\pi G\varepsilon_{\mathrm{c}}}} = \frac{1}{\sqrt{4\pi\varepsilon_{\mathrm{c}}}}$$

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2}\frac{(1+\hat{P}/\hat{\varepsilon})\left(1+\hat{r}^3\hat{P}/\widehat{M}\right)}{1-2\widehat{M}/\hat{r}}, \frac{\mathrm{d}\widehat{M}}{\mathrm{d}\hat{r}} = \hat{r}^2\hat{\varepsilon}$$

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} = -\frac{\widehat{M}\hat{\varepsilon}}{\hat{r}^2}, \frac{\mathrm{d}\widehat{M}}{\mathrm{d}\hat{r}} = \hat{r}^2\hat{\varepsilon}$$

$$P(R) = 0 \leftrightarrow \hat{P}(\hat{R}) = 0$$

$$M_{\rm NS} = \hat{M}_{\rm NS} W, \text{ with } \hat{M}_{\rm NS} \equiv \hat{M}(\hat{R}) = \int_0^{\hat{R}} \mathrm{d}\hat{r}\hat{r}^2 \hat{\varepsilon}(\hat{r})$$

$$\hat{P}(\hat{r}) = \frac{1}{\hat{r}^2}\frac{2\zeta^2}{1+6\zeta+\zeta^2}, \hat{\varepsilon}(\hat{r}) = \frac{1}{\hat{r}^2}\frac{2\zeta}{1+6\zeta+\zeta^2}, \hat{M}(\hat{r}) = \frac{2\zeta\hat{r}}{1+6\zeta+\zeta^2}$$

$$P = \varepsilon/3.$$

$$\hat{M}_{\rm NS} = 3\hat{R}/14 \leftrightarrow M_{\rm NS} = 3R/14G$$

$$R/\mathrm{nm} \approx 6.9 M_{\rm NS}/M_{\odot}$$

$$\hat{P} = \zeta\hat{\varepsilon} + \hat{\Phi}, \text{ donde} \hat{\Phi} = \hat{P}_{\rm c} - \zeta\hat{\varepsilon}_{\rm c} = \hat{P}_{\rm c} - \zeta$$

$$\begin{aligned}
\hat{P}(\hat{r}) &\approx \frac{1}{\hat{r}^2}\frac{2\zeta^2}{1+6\zeta+\zeta^2} + \frac{2(1+2\zeta)\hat{\Phi}}{(1+3\zeta)(2+\zeta)} \\
\hat{\varepsilon}(\hat{r}) &\approx \frac{1}{\hat{r}^2}\frac{2\zeta}{1+6\zeta+\zeta^2} - \frac{3(1+\zeta)\hat{\Phi}}{(1+3\zeta)(2+\zeta)} \\
\hat{M}(\hat{r}) &\approx \frac{2\zeta\hat{r}}{1+6\zeta+\zeta^2} - \frac{(1+\zeta)\hat{\Phi}\hat{r}^3}{(1+3\zeta)(2+\zeta)}
\end{aligned}$$

$$\text{EOS} \hat{P} = \hat{\varepsilon} + \hat{\Phi}: \hat{P}(\hat{r}) = \frac{1}{4\hat{r}^2} + \frac{\hat{\Phi}}{2}, \hat{\varepsilon}(\hat{r}) = \frac{1}{4\hat{r}^2} - \frac{\hat{\Phi}}{2}, \hat{M}(\hat{r}) = \frac{\hat{r}}{4} - \frac{\hat{\Phi}\hat{r}^3}{6}$$

$$\xi = \hat{M}/\hat{R} \approx 4^{-1}\left(1 - 32\hat{M}^2\hat{\Phi}/3\right) \approx 4^{-1}\left(1 - 2\hat{R}^2\hat{\Phi}/3\right)$$

$$\hat{M}(\hat{r}) = \int_0^{\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(x)$$

$$\hat{M}(\hat{r}) \to -\int_0^{-\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(-x)$$

$$\hat{M}(-\hat{r}) = \int_0^{-\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(x)$$

$$\begin{aligned}
\hat{P}(\hat{r}) &= -\int_0^{-\hat{r}} \mathrm{d}x \frac{\hat{\varepsilon}(x)\hat{M}(x)}{x^2}\frac{[1+\hat{P}(-x)/\hat{\varepsilon}(x)]\left[1+x^3\hat{P}(-x)/\hat{M}(x)\right]}{1-2\hat{M}(x)/x} \\
\hat{P}(-\hat{r}) &= -\int_0^{-\hat{r}} \mathrm{d}x \frac{\hat{\varepsilon}(x)\hat{M}(x)}{x^2}\frac{[1+\hat{P}(x)/\hat{\varepsilon}(x)]\left[1+x^3\hat{P}(x)/\hat{M}(x)\right]}{1-2\hat{M}(x)/x}
\end{aligned}$$

$$\begin{aligned}
\hat{\varepsilon}(\hat{r}) &\approx 1 + a_2\hat{r}^2 + a_4\hat{r}^4 + a_6\hat{r}^6 + \cdots, \\
\hat{P}(\hat{r}) &\approx \mathrm{X} + b_2\hat{r}^2 + b_4\hat{r}^4 + b_6\hat{r}^6 + \cdots, \\
\hat{M}(\hat{r}) &\approx \frac{1}{3}\hat{r}^3 + \frac{1}{5}a_2\hat{r}^5 + \frac{1}{7}a_4\hat{r}^7 + \frac{1}{9}a_6\hat{r}^9 + \cdots,
\end{aligned}$$

$$\mathrm{X} \equiv \hat{P}_{\rm c} \equiv P_{\rm c}/\varepsilon_{\rm c}.$$

$$\hat{M}_{\rm NS} = M_{\rm NS}/W \approx 0.18, \hat{R} = R/Q \approx 1.1$$

$$\mu \equiv \hat{\varepsilon} - \hat{\varepsilon}_{\rm c} = \hat{\varepsilon} - 1 < 0$$

$$\mathcal{U}/\mathcal{U}_{\rm c} \approx 1 + \sum_{i+j\geq 1} u_{ij} X^i \mu^j$$

$$\xi = \tau(\mathrm{X}) \approx \tau_1\mathrm{X} + \tau_2\mathrm{X}^2 + \cdots$$

$$b_2 = -\frac{1}{6}(1 + 4X + 3X^2)$$

$$b_4 = -\frac{2a_2}{15} + \left(\frac{1}{12} - \frac{3}{10}a_2\right)X + \frac{1}{3}X^2 + \frac{1}{4}X^3$$

$$b_6 = -\frac{1}{216} - \frac{a_2^2}{30} - \frac{a_2}{54} - \frac{5a_4}{63} + \left(\frac{a_2}{45} - \frac{4a_4}{21} - \frac{1}{54}\right)X + \left(\frac{2a_2}{15} - \frac{1}{18}\right)X^2 - \frac{1}{6}X^3 - \frac{1}{8}X^4$$

$$s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}} = \frac{b_2 + 2b_4\hat{r}^2 + \cdots}{a_2 + 2a_4\hat{r}^2 + \cdots}$$

$$a_2 = b_2/s_c^2$$

$$b_4 = -\frac{1}{2}b_2\left(\mathrm{X} + \frac{4 + 9\mathrm{X}}{15s_c^2}\right)$$

$$\hat{P}/\hat{\varepsilon} \approx X - \frac{1}{6}\frac{1+\Psi}{4+\Psi}\left[1 + \frac{7+\Psi}{4+\Psi}\cdot 4X + \frac{\Psi^2 + 14\Psi + 88}{(4+\Psi)^2}\cdot 3X^2\right]\hat{r}^2$$

$$\hat{r}^3\hat{P}/\hat{M} \approx 3X - \frac{1}{10}\frac{11+5\Psi}{4+\Psi}\left[1 + \frac{5\Psi^2 + 40\Psi + 53}{5\Psi^2 + 31\Psi + 44}\cdot 4X + \frac{5\Psi^3 + 69\Psi^2 + 402\Psi + 392}{(11+5\Psi)(4+\Psi)^2}\cdot 3X^2\right]\hat{r}^2$$

$$2\hat{M}/\hat{r} \approx \frac{2}{3}\hat{r}^2\left[1 - \frac{3}{10X}\frac{1}{4+\Psi}\left[1 + \frac{3}{4+\Psi}\cdot 4X - \frac{\Psi^2 + 18\Psi + 8}{(4+\Psi)^2}\cdot 3X^2\right]\hat{r}^2\right]$$

$$\hat{P}/\hat{\varepsilon} \approx \mathrm{X} - \frac{1}{24}\left(1 + 7\mathrm{X} + \frac{33}{2}\mathrm{X}^2\right)\hat{r}^2$$

$$\hat{r}^3\hat{P}/\hat{M} \approx 3\mathrm{X} - \frac{11}{40}\left(1 + \frac{53}{11}\mathrm{X} + \frac{147}{22}\mathrm{X}^2\right)\hat{r}^2$$

$$2\hat{M}/\hat{r} \approx \frac{2}{3}\hat{r}^2\left[1 - \frac{3}{40\mathrm{X}}\left(1 + 3\mathrm{X} - \frac{3}{2}\mathrm{X}^2\right)\hat{r}^2\right]$$

$$\frac{1}{\omega^2}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}\theta}{\mathrm{d}\omega}\right) + \theta^n = 0$$

$$\omega = \frac{\hat{r}}{\sqrt{(n+1)X}}$$

$$P = K\varepsilon^{1+1/n} = K\varepsilon_c^{1+1/n}\theta^{n+1} = P_c\theta^{n+1}$$

$$\theta(\omega) = 1 - \frac{1}{6}\omega^2 + \frac{n}{120}\omega^4 + \cdots$$

$$P/P_c = \theta^{n+1}(\omega) \approx 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{n}{n+1}\frac{1}{45\mathrm{X}^2}\hat{r}^4 + \cdots.$$

$$P/P_c \approx 1 + \frac{b_2}{\mathrm{X}}\hat{\boldsymbol{r}}^2 + \frac{b_4}{\mathrm{X}}\hat{\boldsymbol{r}}^4 + \cdots \approx 1 - \frac{1}{6\mathrm{X}}\hat{\boldsymbol{r}}^2 + \frac{1}{45\mathrm{X}s_c^2}\hat{\boldsymbol{r}}^4$$

$$P/P_c \approx 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{3}{4+\Psi}\frac{1}{45\mathrm{X}^2}\hat{r}^4 \overset{\Psi=0}{\rightarrow} 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{1}{60\mathrm{X}^2}\hat{r}^4$$

$$n = \frac{3}{1+\Psi};$$

$$\hat{P} = \sum_{k=1} d_k\hat{\varepsilon}^k \approx d_1\hat{\varepsilon} + d_2\hat{\varepsilon}^2 + d_3\hat{\varepsilon}^3 + \cdots$$

$$\mathrm{X} = \sum_{k=1} d_k, s_c^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}}\bigg|_{\hat{r}=0\leftrightarrow\hat{\varepsilon}_c=1} = \sum_{k=1} k d_k$$

$$E_0(\rho) \approx E_0(\rho_0) + \frac{1}{2}K_0\chi^2 + \frac{1}{6}J_0\chi^3 + \mathcal{O}(\chi^4), \chi \equiv \frac{\rho - \rho_0}{3\rho_0}$$

$$E_{\mathrm{sym}}(\rho) \equiv \frac{1}{2}\frac{\partial^2 E(\rho,\delta)}{\partial\delta^2}\bigg|_{\delta=0} \approx S + L\chi + \frac{1}{2}K_{\mathrm{sym}}\chi^2 + \frac{1}{6}J_{\mathrm{sym}}\chi^3 + \mathcal{O}(\chi^4)$$

$$P(\rho,\delta) = \rho^2\frac{\partial E(\rho,\delta)}{\partial\rho}$$

$$r \sim (P/\varepsilon)^{1/2}/\sqrt{G\varepsilon} \sim 1/\sqrt{G\varepsilon}$$

$$R \sim P^{\sigma}$$

$$R \sim \left(\mathrm{X}^{1/2}/\sqrt{\varepsilon_c}\right)\cdot\vartheta(\mathrm{X})$$

$$\hat{R} = \left(\frac{6X}{1+3X^2+4X}\right)^{1/2} \sim \left(\frac{X}{1+3X^2+4X}\right)^{1/2}, \hat{P}(\hat{R}) = 0$$

$$\vartheta(X) = \left(\frac{1}{1+3X^2+4X}\right)^{1/2}$$

$$R = \hat{R}Q = \left(\frac{3}{2\pi G}\right)^{1/2} v_c \sim v_c,\ \text{con}\ v_c \equiv \frac{X^{1/2}}{\sqrt{\varepsilon_c}}\left(\frac{1}{1+3X^2+4X}\right)^{1/2},$$

$$M_{\mathrm{NS}} \approx \frac{1}{3}\hat{R}^3 W = \left(\frac{6}{\pi G^3}\right)^{1/2}\Gamma_c \sim \Gamma_c,\ \text{con}\ \Gamma_c \equiv \frac{\mathrm{X}^{3/2}}{\sqrt{\varepsilon_c}}\left(\frac{1}{1+3\mathrm{X}^2+4\mathrm{X}}\right)^{3/2},$$

$$\xi = \frac{M_{\mathrm{NS}}}{R} \approx \frac{2\Pi_c}{G} \sim \Pi_c,\ \text{con}\ \Pi_c = \frac{\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}.$$

partícula - estrella: $\vartheta \approx 1, M_{\mathrm{NS}} \sim \frac{\mathrm{X}^{3/2}}{\sqrt{\varepsilon_c}} \sim P_c^{3/2}\varepsilon_c^{-2}, R \sim \frac{\mathrm{X}^{1/2}}{\sqrt{\varepsilon_c}} \sim P_c^{1/2}\varepsilon_c^{-1}, \xi \approx 2\mathrm{X}.$

$$P_c \sim M_{\mathrm{NS}}^2/R^4$$

$$I = -\frac{2}{3G}\int_0^R \mathrm{d}r r^3\omega(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}j(r)\right) = \frac{8\pi}{3}\int_0^R \mathrm{d}r r^4[\varepsilon(r) + P(r)]\exp[\lambda(r)]j(r)\omega(r),$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}}{\mathrm{d}r}\omega(r)\right) + 4r^3\omega(r)\frac{\mathrm{d}}{\mathrm{d}r}j(r) = 0$$

$$M_{\mathrm{NS}}^{\max} \sim D_{\mathrm{M}}\varepsilon_c^{-1/2}$$

$$\frac{\mathrm{d}M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_c}\bigg|_{M_{\mathrm{NS}}=M_{\mathrm{NS}}^{\max}=M_{\mathrm{TOV}}} = 0,\ \frac{\mathrm{d}^2 M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_c^2}\bigg|_{M_{\mathrm{NS}}=M_{\mathrm{NS}}^{\max}=M_{\mathrm{TOV}}} < 0$$

estabilización NSs a lo largo de la curva M-R: $s_c^2 = X\left(1+\frac{1+\Psi}{3}\frac{1+3X^2+4X}{1-3X^2}\right)$

$$\Psi = \frac{2\varepsilon_c}{M_{NS}}\frac{dM_{NS}}{d\varepsilon_c} = 2\frac{d\ln\ M_{NS}}{d\ln\ \varepsilon_c} \geq 0$$

configuración TOV para la estabilización NSs: $s_c^2 = X\left(1+\frac{1}{3}\frac{1+3X^2+4X}{1-3X^2}\right)$

$$s_c^2 \leq 1 \leftrightarrow X \lesssim 0.374 \equiv X_+$$

$$\frac{dR}{d\varepsilon_c} = \frac{dR}{dM_{NS}}\frac{dM_{NS}}{d\varepsilon_c} = \frac{d}{d\varepsilon_c}\left[\left(\frac{3}{2\pi G}\right)^{1/2} v_c\right] = \left(\frac{R}{\varepsilon_c}\right)\cdot\left(\frac{\Psi}{6}-\frac{1}{3}\right).$$

$$R \sim \varepsilon_c^{\Psi/6-1/3}$$

$$\begin{aligned} M_{NS} &\sim \frac{1}{\sqrt{\varepsilon_c}}\left(\frac{X(s_c^2,\Psi)}{1+3X^2(s_c^2,\Psi)+4X(s_c^2,\Psi)}\right)^{3/2} \equiv \frac{\mathcal{M}(s_c^2,\Psi)}{\sqrt{\varepsilon_c}} \\ &= \frac{1}{\sqrt{\varepsilon_c}}\cdot\frac{3\sqrt{3}s_c^3}{(4+\Psi)^{3/2}}\left[1-18\frac{5+2\Psi}{(4+\Psi)^2}s_c^2+\frac{81}{2}\frac{148+126\Psi+29\Psi^2}{(4+\Psi)^4}s_c^4+\cdots\right] \\ &\to \frac{1}{\sqrt{\varepsilon_c}}\frac{3\sqrt{3}s_c^3}{8}\left(1-\frac{45s_c^2}{8}+\frac{2997s_c^4}{128}+\cdots\right) \end{aligned}$$

$$X \approx X_+(\Psi) + L_1\varphi + L_2\varphi^2 + \mathcal{O}(\varphi^3),$$

$$L_1 = \frac{[1-3X_+^2(\Psi)]X_+(\Psi)[1+X_+(\Psi)][1+3X_+(\Psi)]}{1+8X_+(\Psi)+8X_+^2(\Psi)-12X_+^3(\Psi)-21X_+^4(\Psi)},$$

$$L_2 = -\frac{2[1-3X_+^2(\Psi)]X_+^2(\Psi)[1-X_+(\Psi)][1+X_+(\Psi)]^2[1+3X_+(\Psi)]^2[2+9X_+(\Psi)+18X_+^2(\Psi)+9X_+^3(\Psi)]}{[1+8X_+(\Psi)+8X_+^2(\Psi)-12X_+^3(\Psi)-21X_+^4(\Psi)]^3}$$

$$\mathcal{M}(s_c^2,\Psi)/\mathcal{M}(1,\Psi) \approx 1+T_1\varphi+T_2\varphi^2+\mathcal{O}(\varphi^3), \mathcal{M}(1,\Psi) = \left(\frac{X_+(\Psi)}{1+3X_+^2(\Psi)+4X_+(\Psi)}\right)^{3/2}$$

$$T_1 = \frac{3}{2}\frac{[1-3X_+^2(\Psi)]^2}{1+8X_+(\Psi)+8X_+^2(\Psi)-12X_+^3(\Psi)-21X_+^4(\Psi)}$$

$$\begin{aligned} T_2 = \frac{3}{8}\frac{[1-3X_+^2(\Psi)]^2}{[1+8X_+(\Psi)+8X_+^2(\Psi)-12X_+^3(\Psi)-21X_+^4(\Psi)]^3} &\times [1-24X_+(\Psi)-282X_+^2(\Psi)-820X_+^3(\Psi) \\ &-504X_+^4(\Psi)+1344X_+^5(\Psi)+1746X_+^6(\Psi)-324X_+^7(\Psi)-945X_+^8(\Psi)] \end{aligned}$$

$$\frac{d^2M_{NS}}{d\varepsilon_c^2} \sim \left(1-\frac{s_c^2}{X}\right)\left[\left(\frac{s_c^2}{X}-\frac{ds_c^2}{dX}\right)+X\left(1-\frac{s_c^2}{X}\right)\frac{12X^2+12X+4}{9X^4+12X^3-4X-1}\right]$$

$$\left.\frac{ds_c^2}{dX}\right|_{M_{NS}^{max}} < \sigma_c^2 \equiv \frac{d}{dX}\left[X\left(1+\frac{1}{3}\frac{1+3X^2+4X}{1-3X^2}\right)\right] = \frac{2}{3}\frac{9X^4-3X^2+4X+2}{(3X^2-1)^2}$$

$$R_{max}/\text{nm} \approx A_R^{max}v_c + B_R^{max} \approx 1.05_{-0.03}^{+0.03}\times 10^3\left(\frac{v_c}{\text{fm}^{3/2}/\text{MeV}^{1/2}}\right)+0.64_{-0.25}^{+0.25},$$

$$M_{NS}^{max}/M_\odot \approx A_M^{max}\Gamma_c + B_M^{max} \approx 1.73_{-0.03}^{+0.03}\times 10^3\left(\frac{\Gamma_c}{\text{fm}^{3/2}/\text{Me}^{1/2}}\right)-0.106_{-0.035}^{+0.035}$$

$$\frac{M_{\rm NS}^{\rm max}}{M_\odot} \approx \frac{1.65\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}\left(\frac{R_{\rm max}}{\rm nm} - 0.64\right) - 0.106.$$

radio	$10^3 v_c$	$\varepsilon_{\rm c}$	$P_{\rm c}$	$s_{\rm c}^2$
$12.39^{+1.30}_{-0.98}$	$11.2^{+1.2}_{-0.9}$	901^{+214}_{-287}	218^{+93}_{-125}	$0.45^{+0.14}_{-0.18}$
$13.7^{+2.6}_{-1.5}$	$12.4^{+2.5}_{-1.4}$	656^{+187}_{-339}	124^{+53}_{-99}	$0.32^{+0.08}_{-0.14}$
$12.90^{+1.25}_{-0.97}$	$11.7^{+1.2}_{-0.9}$	794^{+181}_{-235}	173^{+69}_{-89}	$0.39^{+0.09}_{-0.13}$
$12.49^{+1.28}_{-0.88}$	$11.3^{+1.3}_{-0.9}$	879^{+208}_{-312}	208^{+94}_{-140}	$0.44^{+0.14}_{-0.21}$
$12.76^{+1.49}_{-1.02}$	$11.5^{+1.8}_{-1.2}$	822^{+255}_{-383}	184^{+105}_{-157}	$0.40^{+0.15}_{-0.22}$

$$P_{\rm c}(\varepsilon_{\rm c}) \approx f_{\rm M}^{2/3}\varepsilon_{\rm c}^{4/3}\cdot\left(1+4f_{\rm M}^{2/3}\varepsilon_{\rm c}^{1/3}+19f_{\rm M}^{4/3}\varepsilon_{\rm c}^{2/3}+100f_{\rm M}^2\varepsilon_{\rm c}+\cdots\right)$$

$$P_{\rm c}(\varepsilon_{\rm c}) \approx f_{\rm R}^2\varepsilon_{\rm c}^2\cdot\left(1+4f_{\rm R}^2\varepsilon_{\rm c}+19f_{\rm R}^4\varepsilon_{\rm c}^2+100f_{\rm R}^6\varepsilon_{\rm c}^3+\cdots\right)$$

PSR J0740+6620: $\mathrm{X} \approx 0.24^{+0.05}_{-0.07}$, $R_{\rm max} \approx 12.39^{+1.30}_{-0.98}$ nm.

$$\gamma_{\rm c}^{(\rm M)} = \frac{s_{\rm c}^2}{\mathrm{X}} \approx \frac{4}{3}\left(1+f_{\rm M}^{2/3}\varepsilon_{\rm c}^{1/3}+\frac{11}{2}f_{\rm M}^{4/3}\varepsilon_{\rm c}^{3/3}+34f_{\rm M}^2\varepsilon_{\rm c}+\cdots\right), f_{\rm M} \approx \left(\frac{M_{\rm NS}^{\rm max}+0.106}{1730}\right)\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}$$

$$\gamma_{\rm c}^{(\rm R)} \approx 2\left(1+2f_{\rm R}^2\varepsilon_{\rm c}+11f_{\rm R}^4\varepsilon_{\rm c}^2+68f_{\rm R}^6\varepsilon_{\rm c}^3+\cdots\right), f_{\rm R} \approx \left(\frac{M_{\rm NS}^{\rm max}-0.64}{1050}\right)\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}$$

PSR J0740+6620: $P_{\rm c}(\varepsilon_{\rm c}) \approx 0.012\varepsilon_{\rm c}^{4/3}\cdot\left(1+0.047\varepsilon_{\rm c}^{1/3}+0.0026\varepsilon_{\rm c}^{2/3}+0.00016\varepsilon_{\rm c}+\cdots\right)$

$$\delta P = P_{\rm t} - P = \frac{\lambda_{\rm BL}}{3}\frac{(\varepsilon+3P)(\varepsilon+P)r^3}{r-2M}$$

$$M_{\rm NS}^{\rm max} \sim \alpha \equiv \Gamma_{\rm c}\left(1-\frac{\lambda_{\rm BL}}{2\pi}\right)^{-3/2}, R \sim \beta \equiv v_{\rm c}\left(1-\frac{\lambda_{\rm BL}}{2\pi}\right)^{-1/2}$$

$$M_{\rm NS}^{\rm max} \approx \frac{a_{\rm M}\varepsilon_{\rm c}^{-1/2}\mathrm{X}^{3/2}}{b_{\rm M}+c_{\rm M}\mathrm{X}^{p_{\rm M}}\varepsilon_{\rm c}^{q_{\rm M}}}, R_{\rm max} \approx \frac{a_{\rm R}\varepsilon_{\rm c}^{-1/2}\mathrm{X}^{1/2}}{b_{\rm R}+c_{\rm R}\mathrm{X}^{p_{\rm R}}\varepsilon_{\rm c}^{q_{\rm R}}}, \mathrm{X} = P_{\rm c}/\varepsilon_{\rm c}$$

Partícula supermasiva: $s^2 = \mathrm{d}\hat{P}/\mathrm{d}\hat{\varepsilon} = b_2/a_2 = \mathrm{X}$.

$$s_{\rm c}^2 = \frac{b_2}{a_2} = \frac{\mathrm{X}+b_4\hat{R}^4}{1+a_4\hat{R}^4}, s_{\rm c}^2 - \mathrm{X} = \frac{(b_4-a_4\mathrm{X})\hat{R}^4}{1+a_4\hat{R}^4}$$

$$s_{\rm surf}^2 = \frac{b_2+2b_4\hat{R}^2}{a_2+2a_4\hat{R}^2} = \frac{X-b_4\hat{R}^4}{1-a_4\hat{R}^4}$$

$$s_{\rm c}^2 - s_{\rm surf}^2 = \frac{\mathrm{X}+b_4\hat{R}^4}{1+a_4\hat{R}^4} - \frac{\mathrm{X}-b_4\hat{R}^4}{1-a_4\hat{R}^4} = \frac{2(b_4-a_4\mathrm{X})\hat{R}^4}{(1-a_4\hat{R}^4)(1+a_4\hat{R}^4)} = \frac{2(s_{\rm c}^2-\mathrm{X})}{1-a_4\hat{R}^4}$$

$$\mathcal{O} = a[\mathcal{O}]\left(\frac{M_{\rm NS}^{\max}}{M_\odot}\right)^{b[\mathcal{O}]}\left(\frac{R_{\max}}{10\ \rm nm}\right)^{c[\mathcal{O}]}$$

$$s_{\rm c}^2 = \mathrm{X}\left(1+\frac{1}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right) \sim \mathrm{X}^k,\ \text{with } k \approx 2$$

$$P_{\rm c} \sim (M_{\rm NS}^{\max})^{1+q^{-1}} R_{\max}^{-3-q^{-1}} \sim M_{\rm NS}^{\max,3}/R_{\max}^5,\ \text{with } q \approx 1/2$$

$$s_{\rm c} \sim \mathrm{X}^{k/2} \sim (M_{\rm NS}^{\max}/R_{\max})^{k/2q} \approx (M_{\rm NS}^{\max}/R_{\max})^2, \text{with } q \approx 1/2, k \approx 2.$$

$$\frac{\rho_{\rm c}}{\rho_{\rm sat}} \approx \frac{7.35\times 10^3\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}\left(\frac{R_{\max}}{\rm nm}-0.64\right)^{-2}$$

$$\frac{\rho_{\rm c}}{\rho_{\rm sat}} \approx \bar{d}_0\left[1-\left(\frac{R_{\max}}{10\ \rm nm}\right)\right]+\bar{d}_1\left(\frac{R_{\max}}{10\ \rm nm}\right)^2$$

$$\rho_{\rm c}/\rho_{\rm sat} \sim R_{\max}^{-2}\cdot[1+\text{correcciones para } R_{\max}^{-1}]$$

$$\rho_{\rm c}/\rho_{\rm sat} \approx 2\times 10^4\left(\frac{\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}\right)^3\left(\frac{M_{\rm NS}^{\max}}{M_\odot}+0.106\right)^{-2} \sim M_{\rm NS}^{\max,-2}\cdot\left[1+\ \text{correcciones de } M_{\rm NS}^{\max,-1}\right]$$

$$\hat{\rho} \equiv \rho/\rho_{\rm c} \approx 1+\left(\frac{b_2/s_{\rm c}^2}{1+\mathrm{X}}\right)\hat{r}^2+\frac{1}{1+\mathrm{X}}\left(a_4-\frac{b_2^2/2s_{\rm c}^2}{1+\mathrm{X}}\right)\hat{r}^4$$

$$\rho/\rho_{\rm c} \approx \hat{\varepsilon}-\mu\left(1+\frac{4}{3}\mu\right)\mathrm{X}(1-\mathrm{X})$$

$$\hat{P}/X \approx 1+\frac{b_2}{X}\hat{r}^2, \hat{\varepsilon}/\hat{\varepsilon}_{\rm c} \approx 1+\frac{b_2}{s_{\rm c}^2}\hat{r}^2$$

$$\phi/\mathrm{X} \approx 1+b_2\left(\frac{1}{\mathrm{X}}-\frac{1}{s_{\rm c}^2}\right)\hat{r}^2 = 1+\frac{b_2}{\mathrm{X}}\left[1-\left(1+\frac{1}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right)^{-1}\right]\hat{r}^2$$

$$a_4 \leq -\frac{b_2}{2s_{\rm c}^2}\frac{1}{\hat{R}^2},\ \text{y}\ \ a_4 \leq \frac{b_2}{2s_{\rm c}^2}\left(\frac{b_2}{1+\mathrm{X}}-\frac{1}{\hat{R}^2}\right)$$

$$\begin{aligned}\hat{p}/\mathrm{X} = p/p_{\rm c} \approx & 1+\frac{4}{3}\mu+\frac{16}{15}\mu^2+\frac{4}{15}\mu^3+\left(\frac{4}{3}-\frac{4}{5}\mu-\frac{268}{135}\mu^2\right)\mu\mathrm{X}\\ &+\left[2+\left(\frac{28}{3}-\frac{256a_4}{3}\right)\mu+\left(\frac{370}{27}-\frac{30208}{315}a_4\right)\mu^2\right]\mu\mathrm{X}^2\\ &+\left[4+\left(\frac{1280a_4}{3}-\frac{262}{15}\right)\mu+\left(\frac{68608}{105}a_4+\frac{2048}{3}a_6-\frac{1496}{27}\right)\mu^2\right]\mu\mathrm{X}^3+\mathcal{O}(\mathrm{X}^4,\mu^4),\end{aligned}$$

$$\gamma_{\rm c} \equiv \left.\frac{\mathrm{dln}\ P}{\mathrm{dln}\ \varepsilon}\right|_{\rm centro} = s_{\rm c}^2/\mathrm{X} = 1$$

$$\phi \approx \frac{\mathrm{X}}{1+\mu}\left[1+\frac{4\mu}{3}+\frac{16\mu^2}{15}+\left(\frac{4}{3}-\frac{4\mu}{5}\right)\mu\mathrm{X}\right] \approx \mathrm{X}\left[1+\frac{\mu}{3}(4\mathrm{X}+1)+\frac{\mu^2}{15}(11-32\mathrm{X})\right]$$

$$\mu \approx \frac{3}{4X+1}\left(\frac{\phi}{X}-1\right)+\frac{9}{5}\frac{32X-11}{4X+1}\left(\frac{\phi}{X}-1\right)^2$$

$$s^2/\phi_{\rm c} \approx \frac{4}{3}+\frac{32}{5}\left(1-\frac{19}{4}\mathrm{X}\right)\left(\frac{\phi}{\phi_{\rm c}}-1\right)-\frac{876}{25}\left(1-\frac{3439}{219}\mathrm{X}\right)\left(\frac{\phi}{\phi_{\rm c}}-1\right)^2, \phi_{\rm c} \equiv \mathrm{X}$$

$$\varepsilon(\rho,\delta) = [E(\rho,\delta) + M_N]\rho + \varepsilon_\ell(\rho,\delta)$$

$$\mu_{\mathrm{n}} - \mu_{\mathrm{p}} = \mu_{\mathrm{e}} \approx \mu_\mu \approx 4\delta E_{\mathrm{sym}}(\rho)$$

$$\xi \ \approx A_\xi \Pi_{\mathrm{c}} + B_\xi \approx 2.31^{+0.03}_{-0.03}\Pi_{\mathrm{c}} - 0.032^{+0.003}_{-0.003}$$

$$M_{\mathrm{NS}}/M_\odot \ \approx A_{\mathrm{M}} + B_{\mathrm{M}} \approx 1242^{+15}_{-15}\left(\frac{\Gamma_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) - 0.08^{+0.02}_{-0.02}$$

$$R/\mathrm{nm} \ \approx A_{\mathrm{R}}\nu_{\mathrm{c}} + B_{\mathrm{R}} \approx 572^{+25}_{-25}\left(\frac{\nu_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) + 4.22^{+0.35}_{-0.35}$$

$\xi \lesssim 0.264^{+0.005}_{-0.005} \equiv \xi_{\mathrm{GR}}$, para estabilización general NSs a lo largo de la curva M-R.

$$M_{\mathrm{NS}} \sim \frac{1}{\sqrt{\varepsilon_{\mathrm{c}}}}\left(\frac{\mathrm{X}}{1 + 3\mathrm{X}^2 + 4\mathrm{X}}\right)^{3/2} \cdot (1 + \kappa_1\mathrm{X} + \kappa_2\mathrm{X}^2 + \cdots)$$

$$\xi \sim \frac{X}{1 + 3X^2 + 4X} \cdot \left(1 + \frac{18}{25}X\right)$$

		X	ε_{c}	p_{c}	Masa y radio.
$M_{\mathrm{NS}}^{\mathrm{max}}/M_\odot$-análisis.	PSR J0740+6620 [a]	$0.24^{+0.05}_{-0.07}$	901^{+214}_{-287}	218^{+93}_{-125}	$2.08^{+0.08}_{-0.07}M_\odot, 12.39^{+1.30}_{-0.98}$ nm
(escalares)	PSR J0030+0451 [a]	$0.11^{+0.03}_{-0.02}$	550^{+186}_{-178}	58^{+33}_{-31}	$1.34^{+0.15}_{-0.16}M_\odot, 12.71^{+1.14}_{-1.19}$ nm
	PSR J0437-4715	$0.14^{+0.02}_{-0.02}$	828^{+166}_{-250}	115^{+37}_{-55}	$1.418^{+0.037}_{-0.037}M_\odot, 11.36^{+0.95}_{-0.63}$ nı
	canonical NS	$0.12^{+0.02}_{-0.02}$	687^{+197}_{-197}	85^{+38}_{-38}	$1.4M_\odot, 12^{+1}_{-1}$ nm
$M_{\mathrm{NS}}/M_\odot$-análisis	PSR J0740+6620 [a]	$0.30^{+0.14}_{-0.18}$	597^{+235}_{-312}	181^{+154}_{-204}	
(escalares)	PSR J0030+0451 [a]	$0.12^{+0.03}_{-0.03}$	350^{+166}_{-160}	40^{+31}_{-29}	
	PSR J0437-4715	$0.17^{+0.04}_{-0.04}$	620^{+192}_{-288}	106^{+54}_{-82}	
	canonical NS	$0.14^{+0.04}_{-0.04}$	474-204	68^{+46}_{-46}	

$M_{\mathrm{NS}}/M_{\odot}$-análisis	PSR J0740+6620 [a]	$0.30^{+0.08}_{-0.10}$	588^{+139}_{-184}	175^{+85}_{-113}
(escalar $+\xi$)	PSR J0030+0451 [a]	$0.13^{+0.03}_{-0.03}$	$411\,^{+186}_{-178}$	52^{+34}_{-32}
	PSR J0437-4715	$0.16^{+0.02}_{-0.02}$	565^{+94}_{-140}	91^{+24}_{-36}
	canonical NS	$0.15^{+0.02}_{-0.02}$	$487\,^{+108}_{-108}$	71^{+25}_{-25}

$$\varepsilon_{\mathrm{c}} = \left(\frac{\xi - B_{\xi}}{A_{\xi}}\right)^{3}\left(\frac{A_{\mathrm{M}}}{M_{\mathrm{NS}}/M_{\odot} - B_{\mathrm{M}}}\right)^{2} = \underbrace{\left(\frac{\mathrm{X}}{1 + 3\mathrm{X}^{2} + 4\mathrm{X}}\right)^{3}}_{\text{delimitada superior}}\left(\frac{A_{\mathrm{M}}}{M_{\mathrm{NS}}/M_{\odot} - B_{\mathrm{M}}}\right)^{2}$$

$$\Psi \approx a_{\Psi}\left(\frac{M_{\mathrm{NS}}}{M_{\odot}}\right) + b_{\Psi} \approx -1.62^{+0.13}_{-0.13}\left(\frac{M_{\mathrm{NS}}}{M_{\odot}}\right) + 5.12^{+0.22}_{-0.22}$$

$$M_{\mathrm{NS}}/M_{\odot} \sim \varepsilon_{\mathrm{c}}^{\Psi/2}.$$

$$\text{at } M_{\mathrm{NS}} \approx 1.4M_{\odot}\text{: } M_{\mathrm{NS}}/M_{\odot} \sim \varepsilon_{\mathrm{c}}^{1.43\pm0.14}$$

$$\text{at } M_{\mathrm{NS}} \approx 1.4M_{\odot}\text{: } R \sim \varepsilon_{\mathrm{c}}^{0.14\pm0.05}$$

$$\frac{\mathrm{d}R}{\mathrm{d}M_{\mathrm{NS}}} = \text{ const. } \times\left(1 - \frac{2}{\Psi}\right)\cdot \varepsilon_{\mathrm{c}}^{-3^{-1}(1+\Psi)} = \text{ const. } \times\left(1 - \frac{2}{\Psi}\right)\cdot\left(\frac{M_{\mathrm{NS}}}{M_{\odot}}\right)^{-(2/3)(1+\Psi^{-1})}$$

$$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \sim (1 - 2\Psi^{-1})\cdot R^{-2(1+\Psi^{-1})/(1-2\Psi^{-1})}$$

$$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \approx 3^{-1}\xi^{-1}(1 - 2\Psi^{-1})$$

$$\mathrm{Y} \equiv \frac{\varepsilon_{\mathrm{c}}}{\varepsilon_{0}} \lesssim \frac{21.71}{\left(M_{\mathrm{NS}}/M_{\odot} + 0.08\right)^{2}} \equiv \mathrm{Y}_{+} \sim M_{\mathrm{NS}}^{-2}, \mathrm{X} \lesssim 0.374 \leftrightarrow \Delta_{\mathrm{c}} \gtrsim \Delta_{\mathrm{GR}} \approx -0.041,$$

$$\mathrm{Y} \lesssim 51\left(M_{\mathrm{NS}}/M_{\odot}\right)^{-2}$$

$$R/\mathrm{nm} = \frac{1.477M_{\mathrm{NS}}/M_{\odot}}{A_{\xi}\Pi_{\mathrm{c}} + B_{\xi}} \gtrsim 5.58M_{\mathrm{NS}}/M_{\odot}$$

$$Y_{+} \leftrightarrow R_{-}$$

$\mathrm{d}M_{\mathrm{NS}}/\mathrm{d}\varepsilon_{\mathrm{c}} > 0 \leftrightarrow \mathrm{dY}/\mathrm{d}\left(M_{\mathrm{NS}}/M_{\odot}\right) > 0$, para estabilización general NSs a lo largo de la curva M-R,

$$R/\mathrm{nm} = \frac{\sum\ M_{\mathrm{NS}}/M_{\odot}}{A_{\xi}\left(\frac{M_{\mathrm{NS}}/M_{\odot} - B_{\mathrm{M}}}{A_{\mathrm{M}}}\sqrt{Y\varepsilon_{0}}\right)^{2/3} + B_{\xi}} \approx \frac{1.477M_{\mathrm{NS}}/M_{\odot}}{0.106\left(M_{\mathrm{NS}}/M_{\odot} + 0.08\right)^{2/3}Y^{1/3} - 0.032},$$

$$Y_{-} \leftrightarrow R_{+}$$

$$X(s_c^2,\Psi) \approx \frac{3s_c^2}{4+\Psi}\left[1-\frac{12(1+\Psi)}{(4+\Psi)^2}s_c^2+\frac{18(1+\Psi)(4+13\Psi)}{(4+\Psi)^4}s_c^4+\cdots\right]$$
$$\to \frac{3s_c^2}{4}\left(1-\frac{3}{4}s_c^2+\frac{9}{32}s_c^4+\cdots\right)$$

$$\Pi_c(s_c^2,\Psi) \approx \frac{3s_c^2}{4+\Psi}\left[1-\frac{12(5+2\Psi)}{(4+\Psi)^2}s_c^2+\frac{9(344+298\Psi+71\Psi^2)}{(4+\Psi)^4}s_c^4+\cdots\right]$$
$$\to \frac{3s_c^2}{4}\left(1-\frac{15}{4}s_c^2+\frac{387}{32}s_c^4+\cdots\right).$$

$$R/\text{nm} \gtrsim 3.59 M_{NS}/M_\odot + 4.51$$

$$R/\text{nm} = \frac{(A_M B_R \Pi_c - A_R B_M)\Sigma}{(A_M\Sigma - A_\xi A_R)\Pi_c - A_R B_\xi}$$

$$\frac{R\Sigma^{-1}}{\text{nm}} \approx \underbrace{\frac{A_M B_R}{A_M\Sigma - A_\xi A_R}}_{\text{no }\Pi_c\text{ factor}} \times \left[1+\underbrace{\overbrace{\left(\frac{A_R B_\xi}{A_M\Sigma - A_\xi A_R}-\frac{A_R B_M}{A_M B_R}\right)}^{\text{negativo: }-0.027\pm0.007}\frac{1}{\Pi_c}}_{\text{delimitada superior}}+\mathcal{O}\left(\frac{1}{\Pi_c^2}\right)\right].$$

$$\left(\frac{A_R B_\xi}{A_M\Sigma - A_\xi A_R}-\frac{A_R B_M}{A_M B_R}\right) < 0$$

$$\frac{R\Sigma^{-1}}{\text{nm}} \approx \frac{A_M B_R}{A_M\Sigma - A_\xi A_R}$$

$$\left(\frac{M_{NS}}{M_\odot}\right) = \frac{(A_\xi\Pi_c + B_\xi)(A_M B_R \Pi_c - A_R B_M)}{(A_M\Sigma - A_\xi A_R)\Pi_c - A_R B_\xi} \approx \frac{A_\xi A_M B_R}{A_M\Sigma - A_\xi A_R}\Pi_c \approx \left(\frac{R\Sigma^{-1}}{\text{nm}}\right)A_\xi\Pi_c,$$

$$M_{NS}/M_\odot \lesssim 2.26 \pm 0.28, R/\text{nm} \lesssim 12.62 \pm 1.51$$

$$\xi_{max} \equiv \xi_{TOV} \approx A_\xi^{max}\Pi_c + B_\xi^{max} \approx 2.59\Pi_c - 0.05$$

$$M_{TOV}/M_\odot \approx 2.28 \pm 0.28, R_{TOV}/\text{nm} \approx 11.91 \pm 1.51$$

$$s^2 \equiv \mathrm{d}P/\mathrm{d}\varepsilon = \mathrm{d}\hat{P}/\mathrm{d}\hat{\varepsilon} = \phi f(\phi), \phi = P/\varepsilon$$

$$f \approx f_0 + f_1\phi + f_2\phi^2 + \cdots$$

$$P(\rho_0) \approx P_0(\rho_0) + P_{sym}(\rho_0)\delta^2 \approx 3^{-1}L\rho_0\delta^2 \lesssim 3\text{MeV/fm}^3$$

$$f_0 \approx s^2/\phi \gtrsim 1 \sim 2$$

$$\psi(\hat{\varepsilon}) \approx C_1 + C_0\frac{f_1}{f_0-1}\hat{\varepsilon}^{f_0-1}$$

$$\hat{\varepsilon}_* \approx \left(-\frac{2C_0 f_1}{f_0-1}\right)^{-\frac{1}{f_0-1}}, s^2(\hat{\varepsilon}_*) \approx C_0 f_0 \hat{\varepsilon}_*^{f_0-1}\left(1+\frac{C_0 f_1}{f_0-1}\hat{\varepsilon}_*^{f_0-1}\right) = \frac{f_0(1-f_0)}{4f_1}, f_0 > 1,$$

$$\left[\frac{\mathrm{d}^2 s^2}{\mathrm{d}\hat{\varepsilon}^2}\right]_{\hat{\varepsilon}_*} \approx C_0 f_0 \hat{\varepsilon}_*^{f_0-3}\left[f_0^2 - 3f_0 + 2 + 4C_0\left(f_0-\frac{3}{2}\right)f_1\hat{\varepsilon}_*^{f_0-1}\right] = -C_0 f_0 (f_0-1)^2\left(-\frac{2C_0 f_1}{f_0-1}\right)^{-\frac{f_0-3}{f_0-1}} < 0$$

$$-\left(\frac{1}{2}\right)^{\frac{f_0-3}{1-f_0}}(C_0 f_0)^{\frac{2}{f_0-1}}(1-f_0)^2 \lesssim \left[\frac{\mathrm{d}^2 s^2}{\mathrm{d}\hat{\varepsilon}^2}\right]_{\hat{\varepsilon}_*} \lesssim -C_0 f_0 (1-f_0)^2 \left(\frac{1}{2}\frac{1}{f_0-1}\right)^{\frac{f_0-3}{1-f_0}}$$

$$\text{GR}: s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2\,\mathrm{d}\hat{\varepsilon}/\mathrm{d}\hat{r}}\frac{(1+\hat{P}/\hat{\varepsilon})(1+\hat{r}^3\hat{P}/\widehat{M})}{1-2\widehat{M}/\hat{r}}$$

$$\text{einsteniano}: s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2\,\mathrm{d}\hat{\varepsilon}/\mathrm{d}\hat{r}}$$

$$s^2 \approx s_c^2 + l_2\hat{r}^2, l_2 = \frac{2s_c^2}{b_2}(b_4 - s_c^2 a_4)$$

$$a_4 > \frac{1}{12}\frac{1+3X^2+4X}{s_c^2}\left(X + \frac{4+9X}{15s_c^2}\right) \approx \frac{1}{80X^2}\left(1 + \frac{17}{4}X + \frac{9}{2}X^2 - \frac{13}{4}X^3 - \frac{49}{2}X^4 + \cdots\right).$$

$$\frac{1}{12}\frac{1+3X^2+4X}{s_c^2}\left(X + \frac{4+9X}{15s_c^2}\right) \lesssim a_4 \lesssim \frac{1}{\hat{R}^4} \text{ con } \hat{R} \sim \mathcal{O}(1)$$

$$s^2 \approx \Delta\hat{P}/\Delta\hat{\varepsilon}$$

$$s^2 = s_{\text{deriv}}^2 + s_{\text{no-deriv}}^2 = -\bar{\varepsilon}\frac{\mathrm{d}\Delta}{\mathrm{d}\bar{\varepsilon}} + \frac{1}{3} - \Delta, \bar{\varepsilon} \equiv \varepsilon/\varepsilon_0$$

$$\Delta' \approx \Delta_p' + 2^{-1}\Delta_p'''(\bar{\varepsilon} - \bar{\varepsilon}_p)^2, \Delta_p' < 0, \Delta_p''' > 0$$

$$\mathrm{d}s^2/\mathrm{d}\bar{\varepsilon} = -2\Delta_p' + (3\bar{\varepsilon}_p\bar{\varepsilon} - 2\bar{\varepsilon}^2 - \bar{\varepsilon}_p^2)\Delta_p'''$$

$$\bar{\varepsilon}_p^* = \frac{3\bar{\varepsilon}_p\Delta_p''' + \sqrt{\bar{\varepsilon}_p^2\Delta_p'''^2 - 16\Delta_p'\Delta_p'''}}{4\Delta_p'''} \approx \bar{\varepsilon}_p\left(1 - \frac{2}{\bar{\varepsilon}_p^2}\frac{\Delta_p'}{\Delta_p'''}\right) > \bar{\varepsilon}_p, \therefore \Delta_p'$$

$$s^2(\bar{\varepsilon}_p^*) \approx \frac{1}{3} - \Delta_p - \bar{\varepsilon}_p\Delta_p', \left.\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}_p^*} \approx \Delta_p'''\bar{\varepsilon}_p\left(\frac{8}{\bar{\varepsilon}_p^2}\frac{\Delta_p'}{\Delta_p'''} - 1\right) < 0$$

$$s_{\text{deriv}}^2(\bar{\varepsilon}_p^*) \approx -\bar{\varepsilon}_p\Delta_p', s_{\text{no-deriv}}^2(\bar{\varepsilon}_p^*) \approx \frac{1}{3} - \Delta_p, \therefore \frac{s^2(\bar{\varepsilon}_p^*)}{s_{\text{deriv}}^2(\bar{\varepsilon}_p^*)} = 1 - \frac{3^{-1} - \Delta_p}{\bar{\varepsilon}_p\Delta_p'} > 1.$$

$$\left|\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}_q^*} \approx \Delta_q'''\bar{\varepsilon}_q\left(\frac{8}{\bar{\varepsilon}_q^2}\frac{\Delta_q'}{\Delta_q'''} - 1\right) > 0, \text{ at } \bar{\varepsilon}_q^* \approx \bar{\varepsilon}_q\left(1 - \frac{2}{\bar{\varepsilon}_q^2}\frac{\Delta_q'}{\Delta_q'''}\right) > \bar{\varepsilon}_q,$$

$$s^2(\phi) \approx 3\phi - \frac{a}{2b}\left(a + \sqrt{a^2 - 4b\phi}\right) \approx 2\phi\left(1 - \frac{b}{2a^2}\phi\right) + \mathcal{O}(\phi^3) \text{ y } s^2/\phi \approx 2\left(1 - \frac{b}{2a^2}\phi\right)$$

Δ (en construcción)	0.19	0.18	0.17	0.02	0.01	0.00
$\mathrm{Y} = \varepsilon_c/\varepsilon_0$ (en construcción)	3.0	3.3	3.6	3.7	4.0	4.3
$M_{\mathrm{NS}}/M_{\odot}$ (usando escalares)	1.44	1.45	1.47	2.19	2.13	2.05
R/nm (usando escalares)	12.5	12.0	11.7	12.9	12.4	11.9

$$s^2 \approx s_c^2 + l_2^N \hat{r}^2 + l_4^N \hat{r}^4 \approx s_c^2 + \left(12a_4 s_c^4 - \frac{4}{15}\right)\hat{r}^2 + \left(144a_4^2 s_c^6 + 18a_6 s_c^4 - \frac{62}{35}a_4 s_c^2 + \frac{1}{60s_c^2}\right)\hat{r}^4$$

$$a_2 = b_2/s_c^2 \approx -1/6s_c^2 \lesssim \mathcal{O}(10^{k-1}), 12a_4 s_c^4 \lesssim \mathcal{O}(10^{2-k})$$

$$144a_4^2 s_c^6 \sim 18a_6 s_c^4 \lesssim \mathcal{O}(10^{5-k}), (62/35)a_4 s_c^2 \lesssim \mathcal{O}(10^1), 1/60s_c^2 \lesssim \mathcal{O}(10^{k-2})$$

$$s^2 \approx s_c^2 - (4/15)\hat{r}^2 + (60s_c^2)^{-1}\hat{r}^4$$

$$\begin{aligned} s^2 \approx & s_c^2 + \left(12a_4 s_c^4 - \frac{4}{15}\right)\hat{r}^2 + \left(144a_4^2 s_c^6 + 18a_6 s_c^4 - \frac{62}{35}a_4 s_c^2 + \frac{1}{60s_c^2}\right)\hat{r}^4 \\ & + \left[1728a_4^3 s_c^8 + 432a_4 a_6 s_c^6 + \left(24a_8 - \frac{74_4}{35}a_4^2\right)s_c^4 - \frac{52}{15}a_6 s_c^2 + \frac{1}{35}a_4\right]\hat{r}^6 \\ \approx & s_c^2 - (4/15)\hat{r}^2 + (60s_c^2)^{-1}\hat{r}^4 + (a_4/35)\hat{r}^6, \end{aligned}$$

$$\begin{aligned} \hat{r}_{\min}^2 &\approx 8s_c^2(1 + 285a_4 s_c^4/7 + 1416a_6 s_c^6) \\ s_{\min}^2 &\approx -\frac{s_c^2}{15}(1 + 288a_4 s_c^4/7 + 9344a_6 s_c^6) \\ \hat{R}^2 &\approx 6s_c^2(1 + 144a_4 s_c^4/7 + 1620a_6 s_c^6) \end{aligned}$$

$$\left(\frac{ds^2}{d\hat{\varepsilon}}\right)_N = -\frac{3\hat{\varepsilon}}{\hat{r}^3}\left(\frac{d\hat{\varepsilon}}{d\hat{r}}\right)^2\left(\frac{\hat{r}^3\hat{\varepsilon}}{3} - \widehat{M}\right) + \frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2}\left(\frac{d\hat{\varepsilon}}{d\hat{r}}\right)^{-3}\left[\frac{d^2\hat{\varepsilon}}{d\hat{r}^2} - \frac{1}{\hat{r}}\frac{d\hat{\varepsilon}}{d\hat{r}}\left(1 + \frac{\hat{r}}{\hat{\varepsilon}}\frac{d\hat{\varepsilon}}{d\hat{r}}\right)\right]$$

PSR J0740+6620: $s_c^2 \approx 0.45_{-0.18}^{+0.14}$

$s_c^2(X \leq 1/3) \leq 7/9 \approx 0.778$, configuración general TOV.

$$s_c^2 \approx \frac{4}{3}X\left(1 + X + \frac{3}{2}X^2 + 3X^3\right) + \mathcal{O}(X^5) \approx \frac{4}{3}H\left(1 + 5H + \frac{57}{2}H^2 + 175H^3\right) + \mathcal{O}(H^5)$$

$$X = -\frac{4\xi - \tau + \sqrt{4\xi^2 - 8\tau\xi + \tau^2}}{6\xi} \approx \frac{\xi}{\tau} + 4\left(\frac{\xi}{\tau}\right)^2 + 19\left(\frac{\xi}{\tau}\right)^3 + 100\left(\frac{\xi}{\tau}\right)^4 + \cdots$$

fluido incompresible: $X \approx \frac{\xi}{2} + \xi^2$;

fluido Tolman VII: $X \approx \frac{\xi}{2} + \frac{133\xi^2}{120}$;

fluido Buchdahl: $X \approx \frac{\xi}{2} + \frac{5\xi^2}{4}$.

$$s_c^2 \approx \frac{4+\Psi}{3\tau}\xi + \frac{4}{3}\frac{5+2\Psi}{\tau^2}\xi^2 + \frac{38+19\Psi}{\tau^3}\xi^3 + \frac{100}{3}\frac{7+4\Psi}{\tau^4}\xi^4 + \cdots$$

compactación $\xi \leftrightarrow$ presión central sobre sobre densidad de energía con radio X (promedio SSS) $\leftrightarrow$ rigidez s_c^2.

$$\xi\text{-escalar} \underset{\xi \approx A_\xi \Pi_c + B_\xi}{M_{NS}/R} \; X \underset{s_c^2 = X\left(1 + \frac{1+\Psi}{3}\frac{1+3X^2+4X^2}{1-3X^2}\right)}{\overset{M_{NS}/M_\odot \to \Psi(\text{ FIG. } 32)}{\longrightarrow}} s_c^2$$

$s_c^2 \approx 0.47_{-0.09}^{+0.09}$, para un canonical NS con $R \approx 12_{-1}^{+1}$ nm

canonical NS	s_c^2	radius
א	0.49 ± 0.18	

ב	0.47 ± 0.17	$12.2^{+0.9}_{-1.0}$ nm
Escalares de masa y radio para X	0.46 ± 0.16	12^{+1}_{-1} nm
ξ-escalar para X	0.47 ± 0.09	
PSR J0030+0451 (ξ- escalar para X)	0.40 ± 0.11	$12.71^{+1.14}_{-1.19}$ nm
PSR J0437-4715 (ξ- escalar para X)	0.55 ± 0.10	$11.36^{+0.95}_{-0.63}$ nm

$$a_4 = s_c^{-2}\left(b_4 - a_2^2 \sum_{k=1}^{K} 2^{-1} k(k-1) d_k\right)$$

$$s^2(\hat{r}) \approx s_c^2[1 + (2/b_2)(b_4 - s_c^2 a_4)\hat{r}^2] + \mathcal{O}(\hat{r}^4) \approx s_c^2 + 2a_2 D\hat{r}^2 + \mathcal{O}(\hat{r}^4), D = \sum_{k=1}^{K} 2^{-1} k(k-1) d_k$$

$$s^2(\hat{\varepsilon}) \approx s_c^2 + 2D(\hat{\varepsilon} - 1) = s_c^2 + 2D\mu$$

$$D < 0 \leftrightarrow D\mu > 0 \leftrightarrow s_c^2 < s^2(\hat{\varepsilon}) \leftrightarrow \text{"reducción de } s^2 \text{ hacia los centros NS".}$$

$$0 \leq d_1 + 2d_2\hat{\varepsilon} + 3d_3\hat{\varepsilon}^2 + \cdots \leq 1$$

$$\left.\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}\hat{\varepsilon}^2}\right|_{\hat{\varepsilon}=\hat{\varepsilon}_c=1} = \sum_{k=1}^{K} k(k-1) d_k = 2D < \sigma_c^2 s_c^2$$

$$d_2 = -\mathrm{X} + s_c^2 - \sum_{k=3}^{K} (k-1) d_k$$

$$d_3 = -2\mathrm{X} + s_c^2 - \sum_{k=4}^{K} (k-2) d_k$$

$$D = \sum_{k=1}^{K} \frac{k(k-1)}{2} d_k = d_2 + 3d_3 + \sum_{k=4}^{K} \frac{k(k-1)}{2} d_k = 2s_c^2 - 3X + \sum_{k=4}^{K} \frac{(k-2)(k-3)}{2} d_k.$$

$$\mathrm{prob}(D < 0) \approx \frac{\#[0 \leq s^2 \leq 1 \text{ and } 2D < \sigma_c^2 s_c^2 \text{ and } D < 0]}{\#[0 \leq s^2 \leq 1 \text{ and } 2D < \sigma_c^2 s_c^2]}$$

$$d_4^{(1)}(\hat{\varepsilon}) = \frac{\hat{\varepsilon}(\mathrm{X} - s_c^2) - \mathrm{X} + \hat{\varepsilon}^{-1}}{4\hat{\varepsilon}^2 - 3\hat{\varepsilon} - 1}, d_4^{(\mathrm{u})}(\hat{\varepsilon}) = \frac{\hat{\varepsilon}(\mathrm{X} - s_c^2) - \mathrm{X}}{4\hat{\varepsilon}^2 - 3\hat{\varepsilon} - 1}$$

$$u_c = 2s_c^2 - 3\mathrm{X} \approx -\frac{1 - 2\Psi}{3}\mathrm{X}, \Psi > 0$$

$$\langle 2a_2 D/s_c^2 \rangle = (2a_2/s_c^2) \sum_{k=\pm} \mathrm{prob}(D_k) D_k$$

$$s^2/s_c^2 \approx 1 + \frac{9.4\mathrm{X}\langle a_2 D/s_c^2 \rangle}{1 + 3\mathrm{X}^2 + 4\mathrm{X}}\left(\frac{r}{R_{\max}}\right)^2$$

$$s^2/s_c^2 \approx 1 + \frac{2}{b_2}(b_4 - s_c^2 a_4)\hat{r}^2 + \underbrace{\frac{3}{b_2}\left[(b_6 - s_c^2 a_6) - \frac{4}{3}\frac{a_4}{a_2}(b_4 - s_c^2 a_4)\right]\hat{r}^4}_{\text{término relevante para estimar el pico}}.$$

$$J = \sum_{k=1}^{K} \frac{k(k-1)(k-2)}{6} d_k = d_3 + 4d_4 + 10d_5 + \cdots$$

$$s^2(\hat{r}) \approx s_c^2 + 2a_2 D\hat{r}^2 + (3a_2^2 J + 2a_4 D)\hat{r}^4$$

$$\hat{r}_{\mathrm{pk}} = \left(-\frac{a_2 D}{3a_2^2 J + 2a_4 D}\right)^{1/2}$$

$$s^2(\mu) \approx s_c^2 + 2D\mu + 3J\mu^2 - \frac{2}{a_2^2}\left(3a_4 J + \frac{a_6}{a_2} D\right)\mu^3$$

$$\hat{P}(\mu) \approx \mathrm{X} + s_c^2\mu + \frac{1}{2}\frac{\mathrm{d}s^2}{\mathrm{d}\mu}\bigg|_{\mu=0}\mu^2 + \frac{1}{6}\frac{\mathrm{d}^2 s^2}{\mathrm{d}\mu^2}\bigg|_{\mu=0}\mu^3 \approx \mathrm{X} + s_c^2\mu + D\mu^2 + J\mu^3$$

$$\mu_{\mathrm{pk}} \approx -D/3J$$

$$\hat{r}_{\mathrm{pk}} \approx \sqrt{-\frac{1}{3a_2}\frac{D}{J}} \cdot \left(1 - \frac{a_4}{3a_2^{3/2}}\frac{D}{J}\right)$$

$$-t(\hat{\varepsilon}) = -t(\mu) \approx -t_c + (2D + t_c)\mu + (3J - D - t_c)\mu^2$$

$$\mu_{\mathrm{pk}}^{(-t)} = \hat{\varepsilon}_{\mathrm{pk}}^{(-t)} - 1 = \frac{1}{2}\frac{2D + t_c}{D + t_c - 3J}$$

$$-t_{\mathrm{pk}} \equiv -t\left(\mu_{\mathrm{pk}}^{(-t)}\right) = \frac{4D^2 + 12Jt_c - 3t_c^2}{4D - 12J + 4t_c}$$

$$\mu_{\mathrm{pk}} - \mu_{\mathrm{pk}}^{(-t)} = \frac{2}{9}\left(\frac{D}{J}\right)^2 \cdot \frac{1 + t_c/D + 3Jt_c/2D^2}{1 - D/3J - t_c/3J} > 0$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}\hat{\varepsilon}} = \frac{Y}{1 - 2\hat{M}/\hat{r}}\left\{\left(\frac{\mathrm{d}s^2}{\mathrm{d}\hat{\varepsilon}}\right)_{\mathrm{N}} - \frac{\hat{\varepsilon}\hat{M}}{1 - 2\hat{M}/\hat{r}}\frac{2}{\hat{r}^4}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)^2(\hat{r}^3\hat{\varepsilon} - \hat{M}) - \frac{\hat{\varepsilon}}{\hat{M}}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1}\left[\hat{r}\hat{M}s^2 + \hat{P}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}(3\hat{M} - \hat{r}^3\hat{\varepsilon})\right] - \frac{\hat{M}}{\hat{r}^2}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{p}}{\hat{\varepsilon}}\right)^{-1}\left(s^2 - \frac{\hat{p}}{\hat{\varepsilon}}\right)\right\},$$

$$\hat{r}^3\hat{\varepsilon} - \hat{M} \approx \frac{2\hat{r}^3}{3}\left(1 + \frac{6}{5}a_2\hat{r}^2\right) > 0$$

$$-\frac{Y}{1 - 2\hat{M}/\hat{r}}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)^2\left[\frac{\hat{\varepsilon}\hat{M}}{1 - 2\hat{M}/\hat{r}}\frac{2}{\hat{r}^4}(\hat{r}^3\hat{\varepsilon} - \hat{M}) + \frac{\hat{p}\hat{\varepsilon}}{\hat{M}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1}(3\hat{M} - \hat{r}^3\hat{\varepsilon})\right] < 0$$

$$-\frac{Y}{1 - 2\hat{M}/\hat{r}}\left[\frac{\hat{\varepsilon}}{\hat{M}}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1}\hat{r}\hat{M}s^2 + \frac{\hat{M}}{\hat{r}^2}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{p}}{\hat{\varepsilon}}\right)^{-1}s^2\right] = -\frac{\hat{\varepsilon}\hat{r}s^2}{1 - 2\hat{M}/\hat{r}}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)\left(1 + \frac{2\hat{p}}{\hat{\varepsilon}} + \frac{\hat{M}}{\hat{\varepsilon}\hat{r}^3}\right) > 0.$$

$$s^2 \approx s_c^2 + \left[\left(12a_4 s_c^4 - \frac{4}{15}\right) - \left(48a_4 s_c^4 + s_c^2 + \frac{3}{5}\right)\mathrm{X}\right]\hat{r}^2 + \left[\left(144a_4^2 s_c^6 + 18a_6 s_c^4 - \frac{62}{35}a_4 s_c^2 + \frac{1}{60s_c^2} + \frac{1}{12}s_c^2 - \frac{1}{18}\right) + \left(\frac{1}{15s_c^2} + \frac{1}{15} - 12a_4 s_c^4 - 72a_6 s_c^4 - 1152a_4^2 s_c^6 + \frac{116}{35}a_4 s_c^2\right)\mathrm{X}\right]\hat{r}^4.$$

$$s^2 \approx \frac{4}{3}X + \frac{4}{3}X^2 + \left[-\frac{4}{15} - \frac{3}{5}X + \left(\frac{64a_4}{3} - \frac{4}{3}\right)X^2\right]\hat{r}^2 + \left[\frac{1}{80X} - \frac{13}{720} + \left(\frac{229}{1440} - \frac{248a_4}{105}\right)X + \left(\frac{157}{360} + \frac{72a_4}{35} + 32a_6\right)X^2\right]\hat{r}^4.$$

un pico s^2: $l_2 > 0$ necesita $a_4 > 0$ así como $\mathrm{X} \gtrsim \mathcal{O}(0.1)$

$$a_6 < \frac{b_6}{s_c^2} + \frac{4}{3}\frac{a_4}{b_2}(s_c^2 a_4 - b_4) \equiv a_6^{(\mathrm{up})}$$

$$\hat{r}_{\mathrm{pk}} = \sqrt{-l_2/2l_4}.$$

$$s_{\max}^2 \equiv s^2(\hat{r}_{\mathrm{pk}}) = s_c^2 - l_2^2/4l_4$$

$$\Delta s^2 \equiv s_{\max}^2/s_c^2 - 1 = -l_2^2/4l_4 s_c^2$$

$$l_2 + 2l_4\hat{r}^2 + 3l_6\hat{r}^4 = 0$$

$$l_6 = \frac{4}{b_2}\left[(b_8 - s_c^2 a_8) - \frac{3}{2}\frac{a_4}{a_2}(b_6 - s_c^2 a_6) - \left(\frac{3}{2}\frac{a_6}{a_2} - 2\left(\frac{a_4}{a_2}\right)^2\right)(b_4 - s_c^2 a_4)\right]$$

$$b_8 = -\frac{1}{648}(1 + 3X^2 + 4X)\left(1 - 3X - \frac{27}{2}X^3\right) - \left(\frac{19}{1620} + \frac{X}{54} + \frac{X^2}{90} + \frac{7X^3}{120}\right)a_2 - \left(\frac{4}{225} + \frac{X}{150}\right)a_2^2 - \left(\frac{11}{756} - \frac{X}{252} - \frac{X^2}{12}\right)a_4 - \frac{3a_2a_4}{70} - \left(\frac{1}{18} + \frac{5X}{36}\right)a_6,$$

$$\hat{\varepsilon} \approx (\rho_c/\varepsilon_c)\left[M_{\mathrm{N}} + E_0(\rho_0) + 2^{-1}K_0\chi^2 + 6^{-1}J_0\chi^3 + \left(S + L\chi + 2^{-1}K_{\mathrm{sym}}\chi^2 + 6^{-1}J_{\mathrm{sym}}\chi^3\right)\delta^2\right].$$

$$a_4 \approx -\frac{\beta_3}{54(\beta_3\beta_2 - 1)^2\hat{\rho}_0^3}\Bigg\{+2\left((\beta_3\beta_2 - 1)\beta_1^2 + \frac{2\beta_2\beta_3}{3}\right)\left[\left(\frac{J_0}{\bar{M}_{\mathrm{N}}}\right) + \left(\frac{J_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right)\delta^2\right] -3\left((\beta_3\beta_2 - 1)\beta_1^2 + \beta_2\beta_3\right)\left[\left(\frac{J_0}{\bar{M}_{\mathrm{N}}}\right) - 3\left(\frac{K_0}{\bar{M}_{\mathrm{N}}}\right) + \left(\left(\frac{J_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right) - 3\left(\frac{K_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right)\right)\delta^2\right]\hat{\rho}_0 +\left((\beta_3\beta_2 - 1)\beta_1^2 + 2\beta_2\beta_3\right)\left[\left(\frac{J_0}{\bar{M}_{\mathrm{N}}}\right) - 6\left(\frac{K_0}{\bar{M}_{\mathrm{N}}}\right) + \left(\left(\frac{J_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right) - 6\left(\frac{K_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right) + 18\left(\frac{L}{\bar{M}_{\mathrm{N}}}\right)\right)\delta^2\right]\hat{\rho}_0^2 -5t_4\beta_2\beta_3\left[(\beta_3\beta_2 - 1) + \frac{1}{162}\left(\frac{J_0}{\bar{M}_{\mathrm{N}}}\right) - \frac{1}{18}\left(\frac{K_0}{\bar{M}_{\mathrm{N}}}\right) + \left(\frac{1}{162}\left(\frac{J_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right) - \frac{1}{18}\left(\frac{K_{\mathrm{sym}}}{\bar{M}_{\mathrm{N}}}\right) + \frac{1}{3}\left(\frac{L}{\bar{M}_{\mathrm{N}}}\right) - \left(\frac{S}{\bar{M}_{\mathrm{N}}}\right)\right)\delta^2\right]\hat{\rho}_0^3\Bigg\}$$

$$\Delta(\bar{\varepsilon}) \approx \Delta_\ell + \frac{1}{2}\Delta_\ell''(\bar{\varepsilon} - \bar{\varepsilon}_\ell)^2, \Delta_\ell \equiv \Delta(\bar{\varepsilon}_\ell) > 0, \Delta_\ell'' \equiv \Delta''(\bar{\varepsilon}_\ell) < 0$$

$$\bar{\varepsilon}_\ell^* \approx 2\bar{\varepsilon}_\ell/3 \leftrightarrow \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 2/3.$$

$$\mathrm{H}(k) \equiv \left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right) = \frac{3}{4}\left(1 - \frac{1}{k}\right) + \frac{\sqrt{k^2 - 2k + 9}}{4k},$$

$$k \equiv \frac{\bar{\varepsilon}_\ell \Delta_\ell'''}{\Delta_\ell''} = \frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell} = \left[2 - 3\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right)\right]\left[1 - 3\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right) + 2\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right)^2\right]^{-1},$$

$$\left|\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}=\bar{\varepsilon}_\ell^*} = -\Delta_\ell''\sqrt{k^2 - 2k + 9} > 0, \text{ desde } \Delta_\ell'' < 0$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 2^{-1}(1 - k^{-1} - 2k^{-2} - 2k^{-3} + 2k^{-4} + 10k^{-5})\ k$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 1 - \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} - \frac{1}{k^4} - \frac{5}{k^5},\ k;$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \frac{2}{3}\left(1 + \frac{1}{18}k + \frac{1}{162}k^2 - \frac{1}{1458}k^3 - \frac{5}{13122}k^4 - \frac{1}{39366}k^5\right)\ k \approx 0$$

$$\frac{\bar{\varepsilon}_\ell^* \leftrightarrow \text{ posición curva en SSS } s^2}{\bar{\varepsilon}_\ell \leftrightarrow \text{ anomalía por curvatura } \Delta} : \begin{cases} 1/2 \le \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \le 2/3 & \Delta_\ell''' \ge 0; \\ 2/3 \le \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \le 1 & \Delta_\ell''' \le 0, \end{cases}$$

$$s^2(\bar{\varepsilon}_\ell^*) = \frac{1}{3} - \Delta_\ell - \frac{1}{96}\frac{\Delta_\ell''\bar{\varepsilon}_\ell^2}{k^2}\left(3 + k - \sqrt{k^2 - 2k + 9}\right)^2\left(6 - k + \sqrt{k^2 - 2k + 9}\right)\ k = \frac{\bar{\varepsilon}_\ell \Delta_\ell'''}{\Delta_\ell''}.$$

$$s^2(\bar{\varepsilon}_\ell^*) \approx \frac{1}{3} - \Delta_\ell + \bar{\varepsilon}_\ell^2\Delta_\ell''\left(\frac{k}{12} - \frac{1}{8}\right) = \frac{1}{3} - \Delta_\ell + \frac{\bar{\varepsilon}_\ell^3\Delta_\ell'''}{12}\left(1 - \frac{3}{2k}\right) \approx \frac{1}{3} - \Delta_\ell + \frac{\bar{\varepsilon}_\ell^3\Delta_\ell'''}{12}\ k$$

$$s^2(\bar{\varepsilon}_\ell^*) \approx \frac{1}{3} - \Delta_\ell - \frac{1}{96}\frac{\Delta_\ell''\bar{\varepsilon}_\ell^2}{k^2}\left(80 - \frac{96}{k}\right) \approx \frac{1}{3} - \Delta_\ell - \frac{5\Delta_\ell''}{6}\left(\frac{\Delta_\ell''}{\Delta_\ell'''}\right)^2\ k,$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \mathrm{H}(k)\left(1 + \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{6\Delta_\ell''}\frac{(2 - 5\mathrm{H}(k))(1 - \mathrm{H}(k))^2}{\mathrm{H}(k)\sqrt{k^2 - 2k + 9}}\right)$$

$$\begin{aligned}
\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell &\approx \mathrm{H}(k)\left[1 + \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\frac{1}{k}\left(1 - \frac{1}{k} - \frac{20}{k^2} + \cdots\right)\right] \\
&\approx \frac{1}{2}\left[1 - \frac{1}{k}\left(1 - \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\right) - \frac{2}{k^2}\left(1 + \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\right) - \frac{2}{k^3}\left(1 + \frac{7\bar{\varepsilon}_\ell^2\Delta_\ell''''}{16\Delta_\ell''}\right) + \cdots\right] \\
&\approx \frac{1}{2}\left[1 - \frac{1}{k} - \frac{2}{k^2} - \frac{2}{k^3} + \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\frac{1}{k}\left(1 - \frac{2}{k} - \frac{21}{k^2}\right) + \cdots\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{k} - \frac{2}{k^2} - \frac{2}{k^3} + \frac{1}{24}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(1 - \frac{2}{k} - \frac{21}{k^2}\right) + \cdots\right]
\end{aligned}$$

$$\begin{aligned}
\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell &\approx 1 - \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} - \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{2\Delta_\ell''}\frac{1}{k^3}\left(1 - \frac{8}{3k} - \frac{8}{3k^2}\right) + \cdots \\
&\approx 1 - \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} - \frac{1}{2}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(\frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)^{-2}\left(1 - \frac{8}{3k} - \frac{8}{3k^2}\right) + \cdots
\end{aligned}$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \frac{2}{3}\left[1 + \frac{1}{18}k + \frac{1}{162}k^2 - \frac{1}{1458}k^3 - \frac{1}{81}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(\frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(1 + \frac{1}{36}k - \frac{2}{27}k^2 - \frac{4}{243}k^3\right)\right].$$

causalidad: $R/\mathrm{nm} \gtrsim$ valor mínimo de M_{NS}.

M-R curve: $\phi = P/\varepsilon = \hat{P}/\hat{\varepsilon} \le \mathrm{X} \le 0.374$.

$$P^{(\omega)} \approx \varepsilon^{(\omega)} \approx \frac{1}{2}g_\omega^2\omega^2 \approx \frac{1}{2}\left(\frac{g_\omega}{m_\omega}\right)^2\rho^2$$

$$\Delta \ge \Delta_{\mathrm{GR}} \approx -0.04$$

$\Delta \to 1/3$ y $s^2 \to 1/3$, que equivale a $\gamma \to 1$

Cuantificación	Rango	$X \approx 0.18$	0.24	1/3	0.374
$\Delta_c = 1/3 - X$	$-0.041 \lesssim \Delta_c \leq 1/3$	0.15	0.09	0	-0.041
$s_c^2 = dP_c/d\varepsilon_c$	$0 \leq s_c^2 \leq 1$	0.31	0.45	7/9	1
$\gamma_c = s_c^2/X$	$4/3 \leq \gamma_c \lesssim 2.67$	1.68	1.86	7/3	2.67
$-t_c = s_c^2 - X$	$0.63 \leqslant t_c \leq 0$	0.13	0.21	4/9	0.63
$\Theta_c = (\Delta_c^2 + t_c^2)^{1/2}$	$0.19 \lesssim \Theta_c \lesssim 0.63$	0.19	0.22	4/9	0.63

$$\gamma_c = \frac{2}{3}\frac{(6\Lambda-1)\sqrt{1-8\Lambda+4\Lambda^2}+1-10\Lambda+6\Lambda^2}{(4\Lambda-1)\sqrt{1-8\Lambda+4\Lambda^2}+1-8\Lambda+4\Lambda^2} \approx \frac{4}{3}\left(1+\Lambda+\frac{11}{2}\Lambda^2+34\Lambda^3\right)+\cdots,$$

$$\Lambda \approx \frac{105}{173}\frac{\xi_{max}/\Sigma + 0.106/(R_{max}/\mathrm{nm})}{1-0.64/(R_{max}/\mathrm{nm})} \lesssim \frac{105}{173}\frac{\xi_{max}}{\Sigma}$$

$$\gamma_c = 1 + \frac{1+\Psi}{3}\frac{1+3X^2+4X}{1-3X^2} \geq \frac{4+\Psi}{3} \geq 4/3$$

$$\Theta \equiv \sqrt{\Delta^2 + (P/\varepsilon - s^2)^2} = \sqrt{(1/3-\phi)^2+(\phi-s^2)^2}$$

$$\Theta_c \approx 1/3 - \eta + 2\eta^2/3 + 13\eta^3/6 + 41\eta^4/8 + \cdots$$

$\Theta_c \approx 0.38 \pm 0.05$, para un NS canonical con $R \approx 12^{+1}_{-1}$ nm,

$$\gamma/\gamma_c \approx 1 + \frac{b_2}{s_c^2}\left(1+\frac{2D}{s_c^2}-\frac{s_c^2}{X}\right)\hat{r}^2 \approx 1 - \frac{3D}{16X^2}\hat{r}^2$$

$$\Theta/\Theta_c \approx 1 + \frac{b_2}{s_c^2}\frac{3t_c(1+3s_c^2-6X-6D)}{1+9s_c^2-6X(1+3s_c^2)+18X^2}\hat{r}^2 \approx 1 + \frac{1-6D}{8}\hat{r}^2$$

$$\gamma/\gamma_c \approx 1 + \left(\frac{t_c}{X} + \frac{2D}{s_c^2}\right)\mu \approx 1 + \frac{3D}{2X}\mu$$

$$\Theta/\Theta_c \approx 1 + \frac{3t_c + 9s_c^2(t_c + 2D) - 18X(t_c + D)}{1 + 9s_c^2 - 6X(1 + 3s_c^2) + 18X^2}\mu \approx 1 + (6D - 1)X\mu$$

$$\frac{\Delta\Theta}{\Delta\gamma} \approx \left(4 - \frac{2D}{3}\right)X^2 \approx \left(4 - \frac{2D}{3}\right)X^2,$$

NS	$X \equiv P_c/\varepsilon_c$	$Y \equiv \varepsilon_c/\varepsilon_0$
PSR J0030+0451 [a]	$0.126^{+0.026}_{-0.024}$	$2.73^{+1.25}_{-1.21}$
PSR J0030+0451 [a]	$0.134^{+0.025}_{-0.029}$	$2.69^{+1.07}_{-1.12}$
PSR J0437+4715	$0.161^{+0.017}_{-0.026}$	$3.76^{+0.92}_{-1.05}$
PSR J0740+6620 [a]	$0.297^{+0.077}_{-0.103}$	$3.92^{+0.90}_{-1.19}$
PSR J0740+6620 [b]	$0.231^{+0.058}_{-0.102}$	$3.00^{+0.93}_{-1.61}$
PSR J0740+6620 [c]	$0.267^{+0.057}_{-0.075}$	$3.52^{+0.84}_{-1.08}$
GW 170817$(M_{NS}^{(1)})$	$0.159^{+0.036}_{-0.036}$	$3.42^{+1.37}_{-1.37}$
GW 170817$(M_{NS}^{(2)})$	$0.128^{+0.025}_{-0.025}$	$3.12^{+1.21}_{-1.21}$
GW 190425 ($M_{NS}^{(1)}$)	$0.176^{+0.055}_{-0.055}$	$2.89^{+1.44}_{-1.44}$
GW 190425$(M_{NS}^{(2)})$	$0.152^{+0.037}_{-0.037}$	$2.72^{+1.15}_{-1.15}$
canonical NS	$0.146^{+0.020}_{-0.020}$	$3.25^{+0.79}_{-0.79}$
GS 1826-24 (12 nm)	$0.224^{+0.082}_{-0.082}$	$3.84^{+2.16}_{-2.16}$
GS 1826-24 (13 nm)	$0.224^{+0.082}_{-0.082}$	$3.29^{+1.82}_{-1.82}$
GS 1826-24 (14 nm)	$0.224^{+0.082}_{-0.082}$	$2.85^{+1.61}_{-1.61}$
PSR J2215+5135 (12 nm)	$0.374^{+0.080}_{-0.080}$	$4.37^{+0.71}_{-0.71}$
PSR J2215+5135 (13 nm)	$0.283^{+0.039}_{-0.039}$	$3.53^{+0.57}_{-0.57}$
PSR J2215+5135 (14nm)	$0.237^{+0.026}_{-0.026}$	$2.91^{+0.47}_{-0.47}$

$$\Delta \approx \frac{1}{3}(1 - ft\bar{\varepsilon}^{a})\exp\left(-t\bar{\varepsilon}^{a}\right)$$

a	t	f	$\bar{\varepsilon}_{\mathrm{GR}}$	$\bar{\varepsilon}_{\mathrm{pk}}$	s_{pk}^{2}	$\bar{\varepsilon}_{\mathrm{vl}}$	s_{vl}^{2}	$\bar{\varepsilon}_{\mathrm{deriv,\,pk}}$	$s_{\mathrm{deriv,\,pk}}^{2}$	$\bar{\varepsilon}_{\mathrm{deriv,\,vl}}$	$s_{\mathrm{deriv,\,vl}}^{2}$
2.2	0.0	0.953	6.5	4.	0.6	8.	0.2	3.7	0.34	8.2	-0.11

	$\bar{\varepsilon}_{\mathrm{GR}}$	$\bar{\varepsilon}_{\mathrm{pk}}$	s_{pk}^{2}	$\bar{\varepsilon}_{\mathrm{vl}}$	s_{vl}^{2}
con PSR J04317-4715	6.5	4.3	0.62	8.4	0.25
sin PSR J04317-4715	5.8	4.0	0.67	7.2	0.23

$$\Delta \approx \frac{1}{3}\exp\left(-k_{1}\bar{\varepsilon}^{2}\right) - k_{2}\bar{\varepsilon}$$

$$\mathrm{X} = \frac{P_{\mathrm{c}}}{\varepsilon_{\mathrm{c}}} = \frac{1 - \sqrt{1 - 2\xi}}{3\sqrt{1 - 2\xi} - 1} \approx \frac{\xi}{2}\left(1 + 2\xi + \frac{17}{4}\xi^{2} + \frac{37}{4}\xi^{3} + \cdots\right), \xi = M_{\mathrm{NS}}/R$$

causalidad Buchdahl: $R_{\mathrm{Buch}}/\mathrm{nm} \gtrsim 3.32 M_{\mathrm{NS}}/M_{\odot}$.

causalidad Buchdahl EDC: $R_{\mathrm{Buch}}^{\mathrm{EDC}}/\mathrm{nm} \gtrsim 3.94 M_{\mathrm{NS}}/M_{\odot}$.

causalidad Buchdahl para X_{+}:

$$R_{\mathrm{Buch}}^{(+)} \gtrsim \frac{(1 + 3\mathrm{X}_{+})^{2}}{2\mathrm{X}_{+}(1 + 2\mathrm{X}_{+})} M_{\mathrm{NS}} \leftrightarrow \frac{R_{\mathrm{Buch}}^{(+)}}{\mathrm{nm}} \gtrsim 1.477 \frac{(1 + 3\mathrm{X}_{+})^{2}}{2\mathrm{X}_{+}(1 + 2\mathrm{X}_{+})}\left(\frac{M_{\mathrm{NS}}}{M_{\odot}}\right),$$

causalidad Buchdahl para $X_{+} \approx 0.374$: $R_{\mathrm{Buch}}^{X_{+}\approx 0.374}/\mathrm{nm} \gtrsim 5.01 M_{\mathrm{NS}}/M_{\odot}$.

$$\mathrm{X} = \frac{\xi}{2 - 5\xi} \approx \frac{\xi}{2}\left(1 + \frac{5}{2}\xi + \frac{25}{4}\xi^{2} + \frac{125}{8}\xi^{3} + \cdots\right), \text{ or } \frac{R_{\mathrm{BF}}^{(+)}}{\mathrm{nm}} \gtrsim 1.477\left(\frac{5}{2} + \frac{1}{2\mathrm{X}_{+}}\right)\left(\frac{M_{\mathrm{NS}}}{M_{\odot}}\right)$$

$$R_{\mathrm{BF}}/\mathrm{nm} \gtrsim 3.69 M_{\mathrm{NS}}/M_{\odot};\ R_{\mathrm{BF}}^{\mathrm{EDC}}/\mathrm{nm} \gtrsim 4.43 M_{\mathrm{NS}}/M_{\odot};\ R_{\mathrm{BF}}^{\mathrm{X}_{+}\approx 0.374}/\mathrm{nm} \gtrsim 5.67 M_{\mathrm{NS}}/M_{\odot};$$

א: $R/\mathrm{nm} \gtrsim 4.18 M_{\mathrm{NS}}/M_{\odot}$.

ב: $R/\mathrm{nm} \gtrsim 4.51 M_{\mathrm{NS}}/M_{\odot}$.

ג: $R/\mathrm{nm} \gtrsim 4.34 M_{\mathrm{NS}}/M_{\odot}$,

ד: $R/\mathrm{nm} \gtrsim 3.6 + 3.9 M_{\mathrm{NS}}/M_{\odot}$.

$$\xi_{\max} \equiv \frac{M_{\mathrm{NS}}^{\max}}{R_{\max}} = \frac{M_{\mathrm{NS}}^{\max}/M_{\odot}}{R_{\max}/\mathrm{nm}}\left(\frac{M_{\odot}}{\mathrm{nm}}\right) < \frac{1.73 \times 10^{3}\Gamma_{\mathrm{c}}}{1.05 \times 10^{3} v_{\mathrm{c}}}\left(\frac{M_{\odot}}{\mathrm{nm}}\right) \approx \frac{2.44\mathrm{X}}{1 + 3\mathrm{X}^{2} + 4\mathrm{X}} \equiv \xi_{\max}^{(\mathrm{up})}$$

$$\xi_{\max} \lesssim \Pi_c(0.374)\left(\frac{M_\odot}{\mathrm{nm}}\right)\frac{A_{\mathrm{M}}^{\max}}{A_{\mathrm{R}}^{\max}}\left[1+\frac{\mathrm{nm}}{R_{\max}}\left(\frac{B_{\mathrm{M}}^{\max}A_{\mathrm{R}}^{\max}}{A_{\mathrm{M}}\Pi_c(0.374)}-B_{\mathrm{R}}^{\max}\right)\right] \approx 0.313_{-0.01}^{+0.01}\cdot\left(1-\frac{1.14_{-0.3}^{+0.3}\ \mathrm{nm}}{R_{\max}}\right)$$

$$\xi_{\max} \equiv \xi_{\mathrm{TOV}} \lesssim 0.283_{-0.014}^{+0.014},\ \text{para NSs por configuración TOV.}$$

$$1+z=\left(1-\frac{2M_{\mathrm{NS}}}{R_{\mathrm{GR}}}\right)^{-1}=\left(1-\frac{2\xi}{\sqrt{1+z}}\right)^{-1},R_{\mathrm{GR}}=\sqrt{1+z}R,\xi=M_{\mathrm{NS}}/R$$

$$\frac{\Delta I}{I}\approx\frac{28\pi P_{\mathrm{t}}R^3}{3M_{\mathrm{NS}}}\frac{1.67\xi-0.6\xi^2}{\xi}\left[1+\frac{2P_{\mathrm{t}}(1+5\xi-14\xi^2)}{\rho_{\mathrm{t}}M_{\mathrm{N}}\xi^2}\right]^{-1}$$

$$R_{\max}/\mathrm{nm} \gtrsim 4.73M_{\mathrm{NS}}^{\max}/M_\odot+1.14.$$

$$R_{\max}/\mathrm{nm} \gtrsim 4.83M_{\mathrm{NS}}^{\max}/M_\odot+0.04.$$

$$R_{\max}/\mathrm{nm} \gtrsim 3.75M_{\mathrm{NS}}^{\max}/M_\odot+2.27$$

$$R_{\max} \gtrsim 12.25\ \mathrm{nm},\ \text{para PSR J0952-0607.}$$

$$R_{\max} \gtrsim 10.6\ \mathrm{nm},\ \text{for } M_{\mathrm{NS}}^{\max}/M_\odot \gtrsim 2,$$

$$X=\frac{2}{15}\left\{\sqrt{\frac{3}{\xi}}\tan\left[\arctan\sqrt{\frac{1}{3}\frac{\xi}{1-2\xi}}+\frac{1}{2}\ln\left(\frac{1}{6}+\sqrt{\frac{1-2\xi}{3\xi}}\right)-\frac{1}{2}\ln\left(\frac{1}{\sqrt{3\xi}}-\frac{5}{6}\right)\right]-\frac{5}{2}\right\}$$
$$\approx\frac{\xi}{2}\left(1+\frac{133}{60}\xi+\frac{599}{112}\xi^2+\frac{17915}{134t_4}\xi^3+\cdots\right).$$

$$\varepsilon_c \leq \varepsilon_{\mathrm{ult}} \equiv 6.32\left(\frac{M_{\mathrm{NS}}^{\max}}{M_\odot}+0.106\right)^{-2}\ \mathrm{GeV/fm^3}$$

$$P_c \leq P_{\mathrm{ult}} \equiv 2.36\left(\frac{M_{\mathrm{NS}}^{\max}}{M_\odot}+0.106\right)^{-2}\ \mathrm{GeV/fm^3}$$

$$\varepsilon_{\mathrm{ult}} \approx 7.62\left(\frac{M_\odot}{M_{\mathrm{NS}}^{\max}}\right)^2\mathrm{GeV/fm^3}, P_{\mathrm{ult}} \approx 5.12\left(\frac{M_\odot}{M_{\mathrm{NS}}^{\max}}\right)^2\mathrm{GeV/fm^3},$$

$M_{\mathrm{NS}}^{\max}/M_\odot$	$\varepsilon_{\mathrm{ult}}$ (GeV/fm^3)	P_{ult} (GeV/fm^3)
$\gtrsim 1.97$	1	570_{-320}^{+320}
$\gtrsim 2.0$	1.46	1
$\gtrsim 2.01$	$1.18_{-0.17}^{+0.17}$	/
$\gtrsim 2.2$	$1.16_{-0.10}^{+0.11}$	/

$$\langle s^2(\varepsilon)\rangle \equiv \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}\varepsilon' s^2(\varepsilon')=\frac{p}{\varepsilon}=\phi.$$

$$\langle s_{\rm rf}^2\rangle \equiv \frac{1}{\varepsilon_{\rm rf}}\int_0^{\varepsilon_{\rm rf}} \mathrm{d}\varepsilon s^2(\varepsilon) = \frac{1}{3} \leftrightarrow \int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\rm low}}^{\varepsilon_{\rm rf}} \mathrm{d}\varepsilon s^2(\varepsilon) = \frac{1}{3}\varepsilon_{\rm low} + \frac{1}{3}(\varepsilon_{\rm rf} - \varepsilon_{\rm low})$$

$$\langle s_{\rm low}^2\rangle \equiv \frac{1}{\varepsilon_{\rm low}}\int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3} \leftrightarrow \int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3}\varepsilon_{\rm low}$$

$$\alpha \equiv \langle s^2(\varepsilon_{\rm low} \to \varepsilon_{\rm rf})\rangle \equiv \frac{1}{\varepsilon_{\rm rf} - \varepsilon_{\rm low}}\int_{\varepsilon_{\rm low}}^{\varepsilon_{\rm rf}} \mathrm{d}\varepsilon s^2(\varepsilon) \geq \frac{1}{3}$$

$$\alpha = \frac{1}{3} + \frac{3^{-1} - \langle s_{\rm low}^2\rangle}{1 - \varepsilon_{\rm low}/\varepsilon_{\rm rf}}\frac{\varepsilon_{\rm low}}{\varepsilon_{\rm rf}} > \frac{1}{3}$$

$$\langle s_{\rm high}^2\rangle \equiv \frac{1}{\varepsilon_{\rm high}}\int_0^{\varepsilon_{\rm high}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3}$$

$$\leftrightarrow \frac{1}{\varepsilon_{\rm high}}\left[\int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\rm low}}^{\varepsilon_{\rm rft}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\rm rf}}^{\varepsilon_{\rm hhgh}} \mathrm{d}\varepsilon s^2(\varepsilon)\right] \leq \frac{1}{3}$$

$$\leftrightarrow \int_{\varepsilon_{\rm rf}}^{\varepsilon_{\rm hlow}} \mathrm{d}\varepsilon s^2(\varepsilon) < \frac{1}{3}\varepsilon_{\rm low} - \int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon) + \frac{1}{3}(\varepsilon_{\rm high} - \varepsilon_{\rm rf}) \int_{\varepsilon_{\rm low}}^{\varepsilon_{\rm rf}} \mathrm{d}\varepsilon s^2(\varepsilon) > \frac{1}{3}(\varepsilon_{\rm rf} - \varepsilon_{\rm low})$$

$$\leftrightarrow \langle s^2(\varepsilon_{\rm rf} \to \varepsilon_{\rm high})\rangle \equiv \frac{1}{\varepsilon_{\rm high} - \varepsilon_{\rm rf}}\int_{\varepsilon_{\rm rf}}^{\varepsilon_{\rm hyh}} \mathrm{d}\varepsilon s^2(\varepsilon) < \frac{1}{3} + \frac{3^{-1}\varepsilon_{\rm low} - \int_0^{\varepsilon_{\rm low}} \mathrm{d}\varepsilon s^2(\varepsilon)}{\varepsilon_{\rm high} - \varepsilon_{\rm rf}}$$

$$\langle s^2(\varepsilon_{\rm rf} \to \varepsilon_{\rm high})\rangle < \frac{1}{3}$$

$$\langle s^2(\varepsilon_{\rm low} \to \varepsilon_{\rm rf})\rangle \gtrsim \mathrm{X}_{\rm rf} + \frac{\mathrm{X}_{\rm rf} - \mathrm{X}_{\rm low}}{\varepsilon_{\rm rf}/\varepsilon_{\rm low} - 1} \approx \frac{(\varepsilon_{\rm rf}/\varepsilon_{\rm low})\mathrm{X}_{\rm rf} - \mathrm{X}_{\rm low}}{\varepsilon_{\rm rf}/\varepsilon_{\rm low} - 1}$$

$$s^2 = \frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho}\right) = \frac{2\rho}{\mu}\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho} + \frac{\rho^2}{\mu}\frac{\mathrm{d}^2(\varepsilon/\rho)}{\mathrm{d}\rho^2}$$

$$\ell_{\rm sp} = \frac{2\rho}{\mu}\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho} = 2\left(1 - \frac{\varepsilon}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\varepsilon}\right) = 2\left(1 - \frac{\varepsilon}{\rho}\frac{\rho}{P + \varepsilon}\right) = \frac{2P}{P + \varepsilon} = \frac{2}{1 + \phi^{-1}} = 2\frac{\Delta - 1/3}{\Delta - 4/3}$$

$$\beta = s^2 - \ell_{\rm sp} = -\varepsilon\mathrm{d}\Delta/\mathrm{d}\varepsilon + (3^{-1} - \Delta)[1 + 2/(\Delta - 4/3)]$$

$$K_{\rm NM}(\rho) \equiv 9\rho^2\frac{\mathrm{d}^2(\varepsilon/\rho)}{\mathrm{d}\rho^2} = 9\mu\left(s^2 - \frac{2P}{P + \varepsilon}\right) = 9\mu\left(s^2 - \frac{2}{1 + \phi^{-1}}\right), \phi = P/\varepsilon$$

$$\beta(\varepsilon) \approx \beta'(\varepsilon - \varepsilon_\beta) + \frac{1}{2}\beta''(\varepsilon - \varepsilon_\beta)^2,\ \beta' \equiv \left.\frac{\mathrm{d}\beta}{\mathrm{d}\varepsilon}\right|_{\varepsilon=\varepsilon_\beta} < 0,\ \beta'' \equiv \left.\frac{\mathrm{d}^2\beta}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=\varepsilon_\beta}$$

$$\Delta(\varepsilon) \approx \Delta_\beta + \Delta'_\beta(\varepsilon - \varepsilon_\beta) + 2^{-1}\Delta''_\beta(\varepsilon - \varepsilon_\beta)^2.$$

$$\begin{aligned}\beta(\varepsilon) \approx & -\varepsilon_\beta\Delta'_\beta - \frac{1}{3}\frac{2 - 3\Delta_\beta - 9\Delta_\beta^2}{4 - 3\Delta_\beta} \\ & - \frac{1}{(4 - 3\Delta_\beta)^2}\left[2(7 - 24\Delta_\beta + 9\Delta_\beta^2)\Delta'_\beta + (16 - 24\Delta_\beta + 9\Delta_\beta^2)\varepsilon_\beta\Delta''_\beta\right](\varepsilon - \varepsilon_\beta) \\ & + \frac{3}{2}\frac{1}{(4 - 3\Delta_\beta)^3}\left[36\Delta'^2_\beta - (40 - 126\Delta_\beta + 108\Delta_\beta^2 - 27\Delta_\beta^3)\Delta''_\beta\right](\varepsilon - \varepsilon_\beta)^2\end{aligned}$$

$$\Delta'_\beta = -\frac{1}{3\varepsilon_\beta}\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)}{4-3\Delta_\beta}$$

$$\Delta''_\beta = -\frac{\beta'}{\varepsilon_\beta} + \frac{2}{3\varepsilon_\beta^2}\frac{(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2}{(4-3\Delta_\beta)^3}$$

$$\Delta(\varepsilon) \approx \Delta_\beta - \frac{\delta\varepsilon}{3\varepsilon_\beta}\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)}{4-3\Delta_\beta} - \frac{\delta\varepsilon^2}{2\varepsilon_\beta^2}\left[\varepsilon\beta' - \frac{2}{3}\frac{(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2}{(4-3\Delta_\beta)^3}\right].$$

$$\varepsilon_\Delta/\varepsilon_\beta = 1 - \frac{\Delta'_\beta}{\Delta''_\beta\varepsilon_\beta} = \frac{3(2+3\Delta_\beta)(1-3\Delta_\beta)(10-24\Delta_\beta+9\Delta_\beta^2) - 3\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}{2(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2 - 3\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}$$

$$\Delta_{\min} = \Delta_\beta - \frac{1}{2}\frac{\Delta_\beta'^2}{\Delta''_\beta} = -\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)^2(8-78\Delta_\beta+27\Delta_\beta^2) + 18\varepsilon_\beta\beta'\Delta_\beta(4-3\Delta_\beta)^3}{12(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2 - 18\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}$$

$$\Delta(\varepsilon) \approx -\frac{1}{6}\frac{\delta\varepsilon}{\varepsilon_\beta} + \frac{1}{2}\left(\frac{7}{48} - \varepsilon_\beta\beta'\right)\frac{\delta\varepsilon^2}{\varepsilon_\beta^2}, \varepsilon_\Delta/\varepsilon_\beta \approx \frac{15-48\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}, \Delta_{\min} \approx \frac{2}{3}\frac{1}{48\varepsilon_\beta\beta'-7}$$

$$\Delta_\beta \approx \frac{2}{3}\frac{29-240\varepsilon_\beta\beta'+128\varepsilon_\beta^2\beta''}{159-744\varepsilon_\beta\beta'+320\varepsilon_\beta^2\beta''}$$

$$\frac{\varepsilon_{\text{pk}}}{\varepsilon_\beta} \approx \frac{10-32\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}\left[1+12\Delta_\beta\cdot\frac{3+8\varepsilon_\beta\beta'}{(5-16\varepsilon_\beta\beta')(7-48\varepsilon_\beta\beta')}\right]$$

$$s_{\text{pk}}^2 \equiv s^2(\varepsilon_{\text{pk}}) \approx \frac{1}{32}\frac{121-672\varepsilon_\beta\beta'+256\varepsilon_\beta^2\beta'^2}{7-48\varepsilon_\beta\beta'}\left[1-\frac{3}{4}\frac{2001-34464\varepsilon_\beta\beta'+108800\varepsilon_\beta^2\beta'^2}{(7-48\varepsilon_\beta\beta')(121-672\varepsilon_\beta\beta'+256\varepsilon_\beta^2\beta'^2)}\cdot\Delta_\beta\right]$$

$$\left.\frac{\mathrm{d}^2s^2}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=\varepsilon_{\text{pk}}} \approx -\frac{28-192\varepsilon_\beta\beta'-75\Delta_\beta}{64\varepsilon_\beta^2} < 0$$

$$\frac{\varepsilon_{\text{deriv,pk}}}{\varepsilon_\beta} \approx \frac{3}{4}\frac{10-32\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}\left[1+12\Delta_\beta\cdot\frac{3+8\varepsilon_\beta\beta'}{(5-16\varepsilon_\beta\beta')(7-48\varepsilon_\beta\beta')}\right]$$

$$\frac{\varepsilon_{\text{pk}}-\varepsilon_\beta^*}{\varepsilon_\beta} \approx \frac{3+16\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'} + \frac{\beta'}{\varepsilon_\beta\beta''}$$

$$\Delta_\beta = \frac{4}{3}\frac{s_\beta^2-1/2}{s_\beta^2-2} = \frac{4}{3}\frac{\gamma_\beta-3/2}{\gamma_\beta},\ \text{ or }\ s_\beta^2 = \frac{2-6\Delta_\beta}{4-3\Delta_\beta}, \gamma_\beta = \frac{6}{4-3\Delta_\beta}$$

$$\gamma(\varepsilon) \approx \frac{6}{4-3\Delta_\beta}\left[1-\frac{\delta\varepsilon}{\varepsilon_\beta}\frac{(2-9\Delta_\beta)-8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(1-3\Delta_\beta)(4-3\Delta_\beta)^2}\right], \beta' < 0$$

$$s^2/s_\beta^2 \approx 1 + \frac{2}{s_\beta^2}\frac{\delta\varepsilon}{\varepsilon_\beta}\frac{3(2-3\Delta_\beta)+8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(4-3\Delta_\beta)^3}, \beta' < 0$$

$$\Delta(\varepsilon) \approx \Delta_\beta - \frac{\delta\varepsilon}{3\varepsilon_\beta}\frac{2+12B-(3+9B)\Delta_\beta-9\Delta_\beta^2}{4-3\Delta_\beta}$$
$$-\frac{\delta\varepsilon^2}{2\varepsilon_\beta^2}\left[\varepsilon_\beta\beta' - \frac{2}{3}\frac{(7-3\Delta_\beta)(1-3\Delta_\beta)[2+12B-(3+9B)\Delta_\beta-9\Delta_\beta^2]}{(4-3\Delta_\beta)^3}\right]$$

$$\gamma(\varepsilon) \approx \frac{6}{4-3\Delta_\beta}\left[1-\frac{2(1+6B)-9(1+5B)\Delta_\beta-8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(1-3\Delta_\beta)(4-3\Delta_\beta)^2}\frac{\delta\varepsilon}{\varepsilon_\beta}\right]$$

$$s^2/s_\beta^2 \approx 1-\frac{1}{s_\beta^2}\frac{18(1+3B)\Delta_\beta-12(1+6B)-16\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(4-3\Delta_\beta)^3}\frac{\delta\varepsilon}{\varepsilon_\beta}, s_\beta^2 = B+\frac{2-6\Delta_\beta}{4-3\Delta_\beta}$$

$$\frac{6\Delta_\beta-2}{4-3\Delta_\beta} \leq B \leq \frac{2+3\Delta_\beta}{4-3\Delta_\beta}$$

$$2(1+6B)-9(1+5B)\Delta_\beta-8\varepsilon_\beta\beta'(4-9\Delta_\beta)<0$$

$$0<B<\min\left[\frac{8\varepsilon_\beta\beta'(4-9\Delta_\beta)-(2-9\Delta_\beta)}{12-45\Delta_\beta},\frac{2+3\Delta_\beta}{4-3\Delta_\beta}\right],\ \text{donde}\ \ \varepsilon_\beta\beta'>0$$

$$s^2/s_\beta^2 \gtrsim 1+\frac{\delta\varepsilon}{\varepsilon_\beta}\frac{1}{s_\beta^2}\frac{16(1+6B)-36(1+4B)\Delta_\beta}{(4-3\Delta_\beta)^3}$$

NS compacto $\xi = M_{\mathrm{NS}}/R$ ↔ radio central $\mathrm{X} = \phi_{\mathrm{c}} = \hat{p}_{\mathrm{c}} = p_{\mathrm{c}}/\varepsilon_{\mathrm{c}}$

↔ promedio SSS para una superficie $\mathrm{X}= \langle s_{\mathrm{c}}^2\rangle = \frac{1}{\varepsilon_{\mathrm{c}}}\int_0^{\varepsilon_{\mathrm{c}}} \mathrm{d}\varepsilon' s^2(\varepsilon')$

↔ rigidez central $s_{\mathrm{c}}^2 = \mathrm{X}\left(1+\frac{1+\Psi}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right)$.

$$\text{observaciones NS}: \begin{bmatrix}(M_{\mathrm{NS}},R) \leftrightarrow \text{NICER}\\ (\Lambda, M_{\mathrm{NS}},R) \leftrightarrow \text{GW}\\ \xi = M_{\mathrm{NS}}/R \leftrightarrow \text{ redshift}\end{bmatrix}$$

$$\mathrm{EOS}P(\varepsilon): \begin{bmatrix}\phi = P/\varepsilon = \langle s^2\rangle\\ s^2 = \mathrm{d}P/\mathrm{d}\varepsilon = \phi + \overbrace{\varepsilon \mathrm{d}\phi/\mathrm{d}\varepsilon}^{\text{pico}}\\ \gamma = s^2\phi^{-1} = 1+\varepsilon\phi^{-1}\,\mathrm{d}\phi/\mathrm{d}\varepsilon\end{bmatrix}$$

escalares:

$$M_{\mathrm{NS}} \sim \frac{\Pi_{\mathrm{c}}^{3/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}, R \sim \frac{\Pi_{\mathrm{c}}^{1/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}, \xi \sim \Pi_{\mathrm{c}}$$
$$\Pi_{\mathrm{c}} = \mathrm{X}/[1+3\mathrm{X}^2+4\mathrm{X}]$$

Compacto NS:

$$\xi = M_{\mathrm{NS}}/R \leftrightarrow \mathrm{X} = \langle s_{\mathrm{c}}^2\rangle \leftrightarrow s_{\mathrm{c}}^2$$

Central SSS:

$$s_{\mathrm{c}}^2 = \mathrm{X}\left[1+\frac{1+\Psi}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right]$$
$$\Psi = 2\ \mathrm{dln}\ M_{\mathrm{NS}}/\mathrm{dln}\varepsilon_{\mathrm{c}}$$

Métrica GR Δ :

$$\Delta \equiv 1/3 - P/\varepsilon \gtrsim -0.04$$

$$\phi = \frac{P}{\varepsilon}\frac{\text{densidad y rigidez}}{\text{1 st-derivada}} s^2 = \frac{\mathrm{d}P}{\mathrm{d}\varepsilon}$$

$$\xi = \frac{M_{\mathrm{NS}}}{R}\frac{\text{Compacto NS para una curva M-R}}{\text{1 st-derivada}}\frac{\mathrm{d}M_{\mathrm{NS}}}{\mathrm{d}R} \approx \frac{3M_{\mathrm{NS}}/R}{1-2/\Psi}$$

$$k_2 = \frac{8}{5}\xi^5(1-2\xi)^5[2-y_R+2\xi(y_R-1)] \times \{6\xi[2-y_R+\xi(5y_R-8)]$$
$$+4\xi^3[13-11y_R+\xi(3y_R-2)+2\xi^2(1+y_R)]$$
$$+3(1-2\xi)^2\ln(1-2\xi)[2-y_R+2\xi(y_R-1)]\}$$

$$vy' + y^2 + y\mathrm{e}^{\lambda}[1+4\pi r^2(P-\varepsilon)] + r^2Q = 0, Q = 4\pi\mathrm{e}^{\lambda}\left(5\varepsilon + 9P + \frac{\varepsilon+P}{\mathrm{d}P/\mathrm{d}\varepsilon}\right) - \frac{6\mathrm{e}^{\lambda}}{r^2} - v'^2$$

$$(M_{\mathrm{NS}}, R); (M_{\mathrm{NS}}^{\max}, R_{\max})\Gamma_{\mathrm{c}}, v_{\mathrm{c}}\text{EOS}: E(\rho,\delta)\rho_0, E_0(\rho_0), K_0, J_0, S, L, K_{\mathrm{sym}}, J_{\mathrm{sym}}, \cdots \varepsilon = [E(\rho,\delta)+M_{\mathrm{N}}]\rho, P$$
$$= \rho^2\partial E(\rho,\delta)/\partial\rho$$

$$-rp'(r-2m) = (p+\rho)(m+4\pi r^3 p)$$

$$\begin{cases} 4\pi r x(\ln r) = m(r) \\ 4\pi y(\ln r) = m'(r) = r^2\rho(r) \end{cases}$$

$$\begin{cases} x'(s) = -x(s) + y(s) \\ y'(s) = 2y(s) - \dfrac{8\pi y(s)(x(s)+y(s))}{1-8\pi x(s)} \end{cases}$$

$$L(x,y) = 2 + 16\pi(y-3x) - \log(128\pi y(1-8\pi x)^3)$$

$$\frac{d}{dt}L(x(t),y(t)) = x'(t)(x(t)-2) + y'(t) - 2(d-2)y'(t)/y(t) = -(x(t)-2)^2 \le 0$$

$$L(x,y) \sim \frac{1}{2}\left(x - \frac{1}{16\pi}\right)^2 + \left(y - \frac{1}{16\pi}\right)^2$$

$$\lim_{s\to-\infty} x(s)e^{s} = 0$$

$$\lim_{s\to-\infty} x(s)e^{-2s} < \infty$$

$$\rho_0 = \rho(0) = |\rho|_\infty = \lim_{s\to-\infty} y(s)e^{-2s} < \infty$$

$$\lim_{s\to-\infty} x(s)e^{-2s} = \lim_{s\to-\infty} Q'(s)e^{(1-d)s} = \lim_{r\to 0^+} r^{d-1}\rho(r)r^{1-d} = \rho(0)$$

$$\lim_{s\to-\infty}\frac{x(s)}{y(s)} = \frac{1}{3}$$

$$N = \lim_{s\to-\infty}\frac{x(s)}{y(s)} = \lim_{s\to-\infty}\frac{x'(s)}{y'(s)} = \lim_{s\to-\infty}\frac{1-\dfrac{x(s)}{y(s)}}{2-\dfrac{8\pi y(s)(x(s)/y(s)+1)}{1-8\pi x(s)}} = \frac{1-N}{2}$$

$$\begin{cases} x' = y - x \\ y' = 2y \end{cases}$$

$$x'\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)=(y-x)\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)$$

$$x'\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)+y'=-x\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)$$

$$y'(y-C)/y=\left(2-\frac{8\pi(x+y)}{1-8\pi x}\right)(y-C)$$

$$3x'\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)+4y'-Cy'/y=(C-y-3x)\left(-2+\frac{8\pi(x+y)}{1-8\pi x}\right)$$

$$3x'\left(-2+\frac{8\pi(2x+x')}{1-8\pi x}\right)+4y'-Cy'/y=\frac{(C-y-3x)(-2+24\pi x+8\pi y)}{1-8\pi x}$$

$$-6x'+48\pi xx'/(1-8\pi x)+4y'-\frac{1}{4\pi}y'/y=\frac{-(1-12\pi x-4\pi y)^2}{2\pi(1-8\pi x)}\leq 0$$

$$(-48\pi x-3\log(1-8\pi x)+16\pi y-\log y)'\leq 0$$

$$L(x,y)=2+16\pi(y-3x)-\log(128\pi y(1-8\pi x)^3)$$

$$L(1/(16\pi),1/(16\pi))=0$$

$$m/(4\pi r)=x<9/(96\pi)=(3/4)\cdot(1/8\pi)=(3/8)\cdot(1/(16\pi))$$

$$\frac{m}{r}\leq\frac{3}{8}$$

$$\frac{GM}{Rc^2}<3/8<4/9<1/2$$

$$\frac{2GM}{Rc^2}<3/4<8/9<1$$

$$\frac{2GM}{Rc^2}<0.55<0.64<0.75<0.97$$

$$\frac{Rc^2}{GM}>2\frac{2}{3}>2\frac{1}{4}>2$$

$$L(x,y)=1-\log(2)>0$$

$$y'/x'=3(1-4w/(1-w))\in[-3,3]$$

$$2-32\pi x-\log(128\pi x(1-8\pi x)^3)=1-\log(2)$$

$$p=\kappa\rho$$

$$\begin{cases} x'(s) &= -x(s)+y(s) \\ y'(s) &= 2y(s)-\dfrac{1+\kappa}{2\kappa}\cdot\dfrac{y(s)(x(s)+\kappa y(s))}{1-x(s)} \end{cases}$$

$$(x_\kappa,y_\kappa)=\left(\frac{4\kappa}{(1+\kappa)^2+4\kappa},\frac{4\kappa}{(1+\kappa)^2+4\kappa}\right)$$

$$V=2y-(5+1/\kappa)x-2x_k\log\left(y(1-x)^{\delta_\kappa}\right)+C_\kappa$$

$$8\kappa^2\delta_\kappa=(5\kappa+1)(\kappa+1)^2$$

$$C_\kappa = (3 + 1/\kappa)x_\kappa + 2x_\kappa \log\left(x_\kappa(1 - x_\kappa)^{\delta_\kappa}\right).$$

$$V = C + 2y - \gamma x - \beta \log\left(y(1 - x)^{\delta}\right)$$

$$V' = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y'\ (1 - x)\ -(1 + \kappa)y^2 + y(\delta\beta - \gamma + 4 + \beta(1 + \kappa)/2 - \gamma x^2 + x(\gamma - \delta\beta + 2\beta + \beta(1 + \kappa)/(2\kappa) - 2\beta.$$

$$(1 - x)V' = -(5 + 1/\kappa)(x - x_\kappa)^2 - (1 + \kappa)(y - y_\kappa)^2$$

$$\begin{cases} v' = -v + w \\ w' = -\dfrac{(1+\kappa)y}{\kappa(1 - x)^2}v + \left(2 - \dfrac{1+\kappa}{2\kappa} \cdot \dfrac{x + 2\kappa y}{1 - x}\right)w \end{cases}$$

$$p = p(4\pi\rho) = p(r^{-2}y)$$

$$\begin{cases} x'(s) = -x(s) + y(s) \\ y'(s) = 2y(s) - \dfrac{4\pi\left(y + r^2 p(yr^{-2})\right)\left(x + r^2 p(yr^{-2})\right)}{(1 - 8\pi x)p'(r^{-2}y)} \end{cases}$$

$$p = \rho$$

$$\rho = Cp^{1/\Gamma} + \frac{p}{1 - \Gamma}$$

$$\rho = Cp^{3/4} + 3p$$

$$\rho = Cp^{5/7} + 5p/2.$$

$$p \leq \rho$$

$$\rho_0 = \lim_{s \to -\infty} y(s)e^{-2s}$$

$$\lim_{s \to -\infty} \frac{x(s)}{y(s)} = \frac{1}{3}$$

$$N = \lim_{s \to -\infty} \frac{x(s)}{y(s)} = \lim_{s \to -\infty} \frac{x'(s)}{y'(s)} = \lim_{s \to -\infty} \frac{1 - \dfrac{x(s)}{y(s)}}{2 - \dfrac{4\pi y(s)\left(1 + \dfrac{p(y(s)r - 2)}{y(s)r - 2}\right)\left(\dfrac{x(s)}{y(s)} + \dfrac{p(y(s)r - 2)}{y(s)r - 2}\right)}{(1 - 8\pi x(s))p'(r^{-2}y(s))}}$$

$$N = \frac{1 - N}{2}$$

$$\rho = \frac{2m}{h^3}\int_0^\infty f_c(\hat{p})\left(1 + \frac{\varepsilon(\hat{p})}{mc^2}\right)d^3\hat{p}$$

$$f_c(\hat{p}) = \left(\frac{1 - \exp\left((\varepsilon(\hat{p}) - \varepsilon_c)/kT\right)}{\exp\left((\varepsilon(\hat{p}) - \mu)/kT\right) + 1}\right)_+$$

$$p = \frac{4}{3h^3}\int_0^\infty f_c(\hat{p})\varepsilon(\hat{p})\frac{1 + \varepsilon(\hat{p})/2mc^2}{1 + \varepsilon(\hat{p})/mc^2}d^3\hat{p}$$

$$\varepsilon(\hat{p}) = \sqrt{c^2\hat{p}^2 + m^2c^4} - mc^2$$

$$p(\rho) = \left((p_\infty \rho)^{-1} + (p_0 \rho)^{-7/5}\right)^{-1}$$

$$(\varepsilon + mc^2)/T \equiv \text{const} = mc^2/T_R$$

$$(\varepsilon_c/(mc^2) + 1)T_R/T = 1$$

$$\kappa = kT_R/(mc^2)$$

$$y = \varepsilon_c/(kT)$$

$$1 - y\kappa = T_R/T$$

$$\hat{p} = mv/\sqrt{1 - |v|^2/c^2}$$

$$xkT = \varepsilon(\hat{p})$$

$$1 + \varepsilon(\hat{p})/(mc^2) = 1 + x\kappa T/T_R = 1 + x\kappa/(1 - y\kappa)$$

$$\hat{p}^2 = m^2c^2 \frac{x\kappa}{1 - \kappa y}\left(\frac{x\kappa}{1 - y\kappa} + 2\right)$$

$$\frac{\kappa m^2 c^2}{1 - y\kappa}(x\kappa/(1 - y\kappa) + 1)dx = |\hat{p}|d|\hat{p}|$$

$$\rho = \frac{8\pi c^3 \kappa m^4}{h^3(1 - \kappa y)} \int_0^y \left(\frac{\kappa x}{1 - \kappa y} + 1\right)^2 \left(\left(\frac{\kappa x}{1 - \kappa y} + 1\right)^2 - 1\right)^{1/2} \frac{1 - \exp(x - y)}{1 + \chi \exp(x - y)} dx$$

$$\rho = \frac{8\sqrt{2}\pi c^3 \kappa^{3/2} m^4}{h^3(1 - \kappa y)^{3/2}} \int_0^y x^{1/2} \left(\frac{\kappa x}{1 - \kappa y} + 1\right)^2 \left(\frac{\kappa x/2}{1 - \kappa y} + 1\right)^{1/2} \frac{1 - \exp(x - y)}{1 + \chi \exp(x - y)} dx$$

$$\rho(y) \sim \frac{8\sqrt{2}\pi m \kappa^{3/2}}{1 + \chi}\left(\frac{mc}{h}\right)^3 \int_0^y \sqrt{x}(y - x)dx = \rho_0 y^{5/2}$$

$$\rho_0 = \frac{32\sqrt{2}\pi m \kappa^{3/2}}{15(1 + \chi)}\left(\frac{mc}{h}\right)^3$$

$$\rho(y) \sim \frac{\rho_\infty}{(1 - \kappa y)^\gamma}$$

$$\rho_\infty = 8\pi c^3 m^4 h^{-3} \int_0^1 w^3 \frac{1 - \exp((w - 1)/\kappa)}{1 + \chi \exp((w - 1)/\kappa)} dw$$

$$p = \frac{\eta 2\sqrt{2}\kappa}{1 - \kappa y} \int_0^y \left(\frac{\kappa x}{1 - \kappa y}\right)^{3/2} \left(\frac{\kappa x/2}{1 - \kappa y} + 1\right)^{3/2} \frac{1 - \exp(x - y)}{1 + \chi \exp(x - y)} dx$$

$$p = \eta \left(\frac{\kappa}{1 - \kappa y}\right)^{5/2} \int_0^y x^{3/2} \left(\frac{\kappa x}{1 - \kappa y} + 2\right)^{3/2} \frac{1 - \exp(x - y)}{1 + \chi \exp(x - y)} dx$$

$$\eta = \frac{8\pi m^4 c^5}{3h^3}$$

$$p(y) \sim p_0 y^{7/2}$$

$$p_0 = \frac{64\sqrt{2}\pi m^4 c^5 \kappa^{5/2}}{105(1 + \chi)h^3}$$

$$p(\rho) \sim p_0(\rho/\rho_0)^{7/5}$$

$$p(y) \sim \frac{c^2\rho_\infty}{3(1-\kappa y)^4}$$

$$\rho_\infty = 8\pi c^3\kappa^4 m^4 h^{-3}\int_0^{1/\kappa} x^3 \frac{1-\exp{(x-1/\kappa)}}{1+\chi\exp{(x-1/\kappa)}}dx$$

$$p(\rho) \sim c^2\rho/3$$

$$p(\rho) = \rho^{7/5}/\left(a+\rho^{2/5}/3\right)$$

$$\rho = \frac{1}{\pi^2}\int_0^\infty \frac{y^2\sqrt{1+y^2}}{1+e^{-\alpha}e^{x\sqrt{1+y^2}}}dy$$

$$p = \frac{1}{3\pi^2}\int_0^\infty \frac{y^4}{\sqrt{1+y^2}\left(1+e^{-\alpha}e^{x\sqrt{1+y^2}}\right)}dy$$

$$z = x\sqrt{1+y^2}$$

$$p(x) \sim \frac{1}{3\pi x^4}\int_0^\infty z^3(1+e^{-\alpha}e^z)^{-1}dz$$

$$\rho(x) \sim \frac{1}{\pi x^4}\int_0^\infty z^3(1+e^{-\alpha}e^z)^{-1}dz$$

$$p(\rho) \sim \rho/3$$

$$\pi^2 p(x) \sim \sqrt{2}x^{-3/2}e^{-x+\alpha}\int_0^\infty w^{1/2}e^{-w}dw$$

$$\pi^2 \rho(x) \sim 2\sqrt{2}x^{-5/2}e^{-x+\alpha}\int_0^\infty w^{1/2}e^{-w}dw$$

$$p(\rho) \sim -\rho/\log{(\rho)}$$

$$p(\rho) = \rho/(3-\log{(\rho)}(1+\rho)^{-1})$$

$$\rho = \kappa\left(\sqrt{1+x^2}(x+2x^3) - \mathrm{arcsinh}(x)\right)$$

$$p = \frac{\kappa}{3}\left(x\sqrt{1+x^2}(2x^2-3) + 3\mathrm{arcsinh}(x)\right)$$

$$\rho \sim 2\kappa x^4$$

$$3p \sim 2\kappa x^4$$

$$3p \sim \rho$$

$$3p \sim \kappa\chi x^5$$

$$3\rho \sim 8x^3$$

$$p \sim p_\infty \rho^{5/3}$$

$$\kappa^{-1}\rho' = 8x^2\sqrt{x^2+1}$$

$$3\kappa^{-1}\sqrt{x^2+1}p' = 8x^4$$

$$3\kappa^{-1}\sqrt{(x^2+1)^3}p'' = 8x^3(3x^2+4)$$

$$\frac{d}{d\rho}(p(x(\rho))) = \frac{p'(x(\rho)}{\rho'(x(\rho))} = \frac{1}{1+x^{-2}(\rho)}$$

$$3p/\rho = 1 - \frac{4}{2x^2+1} \leq 1$$

$$S = \int d^4xe\left[\frac{f(T)}{2\kappa^2} + \mathcal{L}_m\right]$$

$$S_\mu{}^{\nu\rho}\partial_\rho T f_{TT} + \left[e^{-1}e^i{}_\mu\partial_\rho\left(ee_i{}^\alpha S_\alpha{}^{\nu\lambda}\right) + T^\alpha{}_{\lambda\mu}S_\alpha{}^{\nu\lambda}\right]f_T + \frac{1}{4}\delta_\mu^\nu f = \frac{\kappa^2}{2}\mathcal{T}_\mu^\nu$$

$$(\rho+p_t)u_\mu u^\nu - p_t\delta_\mu^\nu + (p_r - p_t)v_\mu v^\nu$$

$$ds^2 = e^{\alpha(r)}dt^2 - e^{\beta(r)}dr^2 - r^2(d\theta^2 + \sin^2\,\theta d\phi^2)$$

$$\frac{f}{4} - \left[T - \frac{1}{r^2} - \frac{e^{-\beta}}{r}(\alpha' + \beta')\right]\frac{f_T}{2} = 4\pi\rho$$

$$\left(T - \frac{1}{r^2}\right)\frac{f_T}{2} - \frac{f}{4} = 4\pi p_r$$

$$\frac{T}{2} + e^{-\beta}\left(\frac{\alpha''}{2} + \left(\frac{\alpha'}{4} + \frac{1}{2r}\right)(\alpha' - \beta')\right)\Bigg]\frac{f_T}{2} - \frac{f}{4} = 4\pi p_t$$

$$\frac{\cot\,\theta}{2r^2}T' f_{TT} = 0$$

$$T = 2\frac{e^{-\beta}}{r^2}(1 + r\alpha')$$

$$\frac{dp_r}{dr} = -\frac{1}{2}(\rho + p_r)\frac{d\alpha}{dr} + \frac{2}{r}(p_t - p_r)$$

$$\frac{e^{-\beta}}{4r^2}\left[4\left(e^\beta - 3\right) + 4r(\beta' - 3\alpha') + r^2(\alpha'\beta' - 2\alpha'' - \alpha'^2)\right]f_T + f = 4\pi(\rho - p_r - 2p_t)$$

$$e^{-\beta} = 1 - \frac{2GM}{r}$$

$$e^{-\beta} = 1 - \frac{2m}{r} \Longrightarrow \beta' = \frac{2m}{r^2}\frac{1 - \frac{rm'}{r}}{\frac{2m}{r} - 1}$$

$$\lim_{r\to\infty} T(r) = 0, \lim_{r\to\infty} m(r) = cste$$

$$\left[\frac{2}{r(\rho+p)}\left(1-\frac{2m}{r}\right)\frac{dp}{dr}+\frac{1}{r^2}\frac{dm}{dr}-\frac{1}{2r^2}+\frac{m}{r^3}\right]f_T+\frac{1}{4}f-4\pi\rho=0$$

$$\left[-\frac{\alpha'}{r}+\frac{2\alpha' m}{r^2}-\frac{1}{2r^2}+\frac{2m}{r^3}\right]f_T+\frac{1}{4}f+4\pi p=0$$

$$\left[-\frac{2}{\rho+p}\left(\frac{1}{r}\frac{dm}{dr}-\frac{6}{r}+\frac{11m}{r^2}\right)\frac{dp}{dr}+\frac{4}{(\rho+p)^2}\left(\frac{2m}{r}-1\right)\left(\frac{dp}{dr}\right)^2\right.$$
$$\left.-2\left(\frac{2m}{r}-1\right)\frac{d}{dr}\left(\frac{1}{\rho+p}\frac{dp}{dr}\right)-\frac{2}{r^2}\frac{dm}{dr}-\frac{2}{r^2}+\frac{4m}{r^3}\right]f_T+f-4\pi(\rho-3p)=0$$

$$\{e^i_\mu\}=\begin{pmatrix} e^\alpha & 0 & 0 & 0\\ 0 & e^\beta\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ 0 & e^\beta\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ 0 & e^\beta\cos\theta & -r\sin\theta & 0\end{pmatrix}$$

$$-\frac{1}{r^2}\sqrt{r(r-2m)}\left(\sqrt{\frac{r-2m}{r}}-1\right)T' f_{TT}$$
$$+\frac{1}{2r^3}\sqrt{\frac{r}{r-2m}}\left[(\alpha' r^2-2m' r-2\alpha' mr+2r-2m)\sqrt{\frac{r-2m}{r}}\right.$$
$$\left.-\alpha' r^2+2\alpha' mr-2r+4m\right]f_T+\frac{1}{4}f=4\pi\rho$$

$$\frac{1}{2r^3}\sqrt{r(r-2m)}\left[2(\alpha' r+1)\sqrt{\frac{r-2m}{r}}-\alpha' r-2\right]f_T-\frac{1}{4}f=4\pi p_r$$

$$\frac{1}{4r^2}\sqrt{r(r-2m)}\left[(\alpha' r+2)\sqrt{\frac{r-2m}{r}}-2\right]T' f_{TT}-\frac{1}{8r^3}\left[4r(\alpha' r+2)\sqrt{\frac{r-2m}{r}}\right.$$
$$-2\alpha'' r^3-\alpha'^2 r^3+2\alpha' m' r^2+4\alpha'' mr^2+2\alpha'^2 mr^2$$
$$\left.-6\alpha' r^2+4m' r+10\alpha' mr-8r+4m\right]f_T-\frac{1}{4}f=4\pi p_t$$

$$\left[\frac{1}{r}\left(2\sqrt{1-\frac{2m}{r}}+\alpha' m-2\right)+\frac{4m}{r^2}-\frac{\alpha'}{2}\right]T' f_{TT}+\left(\frac{\alpha' m'}{2r}+\frac{\alpha'' m}{r}+\frac{\alpha'^2}{2r}-\frac{3\alpha'}{r}\right.$$
$$\left.\frac{2m'}{r^2}+\frac{11\alpha' m}{2r^2}-\frac{4}{r^2}+\frac{4m}{r^3}-\frac{\alpha''}{2}-\frac{\alpha'^2}{4}\right)f_T+f=4\pi(\rho-p_r-2p_t)$$

$$\zeta=2\xi+5$$

$$\xi=\log(\rho/\mathrm{g\ cm^{-3}}),\zeta=\log(p/\mathrm{dyn}\ cm^{-2})$$

$$\zeta=\frac{a_1+a_2\xi+a_3\xi^3}{1+a_4\xi}f_0a_5(\xi-a_6)+f_0(a_7+a_8\xi)[a_9(a_{10}-\xi)]$$

$$\zeta=\frac{a_1+a_2\xi+a_3\xi^3}{1+a_4\xi}f_0a_5(\xi-a_6)+f_0(a_7+a_8\xi)[a_9(a_{10}-\xi)]$$
$$+f_0(a_{11}+a_{12}\xi)[a_{13}(a_{14}-\xi)]+f_0(a_{15}+a_{16}\xi)[a_{17}(a_{18}-\xi)].$$

$$f_0(x)=\frac{1}{e^x+1}$$

i	a_i (SLy)	i	a_i (SLy)
1	6.22	10	11.4950
2	6.121	11	-22.775
3	0.005925	12	1.5707
4	0.16326	13	4.3
5	6.48	14	14.08
6	11.4971	15	27.80
7	19.105	16	-1.653
8	0.8938	17	1.50
9	6.54	18	14.67

$$p = \omega(\rho - 4\gamma)$$

$$S = S_g + S_v + S_s + S_m,$$

$$S_g = (16\pi G)^{-1} \int g^{\alpha\beta} R_{\alpha\beta} (-g)^{1/2} d^4x,$$
$$S_s = -\frac{1}{2} \int \left[\sigma^2 h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \frac{1}{2} G \ell^{-2} \sigma^4 F(kG\sigma^2) \right] (-g)^{1/2} d^4x,$$
$$S_v = -\frac{K}{32\pi G} \int \left[g^{\alpha\beta} g^{\mu\nu} \mathfrak{U}_{[\alpha,\mu]} \mathfrak{U}_{[\beta,\nu]} - 2(\tilde{\lambda}/K)\left(g^{\mu\nu} \mathfrak{U}_\mu \mathfrak{U}_\nu + 1\right) \right] (-g)^{1/2} d^4x,$$
$$S_m = \int \mathcal{L}\left(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha{}_{|\mu}, \cdots\right) (-\tilde{g})^{1/2} d^4x.$$

$$G_{\alpha\beta} = 8\pi G\left[\tilde{T}_{\alpha\beta} + \left(1 - e^{-4\phi}\right) \mathfrak{U}^\mu \tilde{T}_{\mu(\alpha} \mathfrak{U}_{\beta)} + \tau_{\alpha\beta}\right] + \Theta_{\alpha\beta}$$

$$K \mathfrak{U}^{[\alpha;\beta]}{}_{;\beta} + \tilde{\lambda} \mathfrak{U}^\alpha + 8\pi G \sigma^2 \mathfrak{U}^\beta \phi_{,\beta} g^{\alpha\gamma} \phi_{,\gamma} = 8\pi G \left(1 - e^{-4\phi}\right) g^{\alpha\mu} \mathfrak{U}^\beta \tilde{T}_{\mu\beta},$$

$$\left[\mu h^{\alpha\beta} \phi_{,\alpha}\right]_{;\beta} = kG\left[g^{\alpha\beta} + \left(1 + e^{-4\phi}\right) \mathfrak{U}^\alpha \mathfrak{U}^\beta\right] \tilde{T}_{\alpha\beta}$$

$$\tau_{\alpha\beta} \equiv \sigma^2 \left[\phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} g_{\alpha\beta} - \mathfrak{U}^\mu \phi_{,\mu} \left(\mathfrak{U}_{(\alpha} \phi_{,\beta)} - \frac{1}{2} \mathfrak{U}^\nu \phi_{,\nu} g_{\alpha\beta} \right) \right] - \frac{1}{4} G \ell^{-2} \sigma^4 F(\mu) g_{\alpha\beta},$$
$$\Theta_{\alpha\beta} \equiv K \left(g^{\mu\nu} \mathfrak{U}_{[\mu,\alpha]} \mathfrak{U}_{[\nu,\beta]} - \frac{1}{4} g^{\sigma\tau} g^{\mu\nu} \mathfrak{U}_{[\sigma,\mu]} \mathfrak{U}_{[\tau,\nu]} g_{\alpha\beta} \right) - \tilde{\lambda} \mathfrak{U}_\alpha \mathfrak{U}_\beta,$$

$$\tilde{T}_{\alpha\beta} = \tilde{\rho} \tilde{u}_\alpha \tilde{u}_\beta + \tilde{p}\left(\tilde{g}_{\alpha\beta} + \tilde{u}_\alpha \tilde{u}_\beta\right),$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$

$$\mathfrak{U}^\alpha = \left\{e^{-\nu/2}, 0,0,0\right\}$$

$$\frac{e^{-\lambda}}{r^2}\left(-1+e^{\lambda}+r\lambda'\right)=8\pi G\left[\tilde{p}e^{2\phi}+\frac{1}{kG}\frac{e^{-\lambda}}{2}(\phi')^2\right]+Ke^{-\lambda}\left[\frac{(\nu')^2}{8}+\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'}{r}\right]$$

$$\frac{e^{-\lambda}}{r^2}\left(1-e^{\lambda}+r\nu'\right)=8\pi G\left[\tilde{p}e^{-2\phi}+\frac{1}{kG}\frac{e^{-\lambda}}{2}(\phi')^2\right]+Ke^{-\lambda}\left[-\frac{(\nu')^2}{8}\right]$$

$$\frac{e^{-\lambda}}{2}\left[\frac{(\nu')^2}{2}+\nu''-\frac{\nu'\lambda'}{2}+\frac{\nu'-\lambda'}{r}\right]=8\pi G\left[\tilde{p}e^{-2\phi}-\frac{1}{kG}\frac{e^{-\lambda}}{2}(\phi')^2\right]+Ke^{-\lambda}\left[\frac{(\nu')^2}{8}\right].$$

$$\left[\frac{\phi' r^2}{e^{(\lambda-\nu)/2}}\right]'=kG(\tilde{\rho}+3\tilde{p})e^{(\nu+\lambda)/2}e^{-2\phi}r^2$$

$$m_s=m_s(R)=4\pi\int_0^R(\tilde{\rho}+3\tilde{p})e^{(\nu+\lambda)/2}e^{-2\phi}r^2dr$$

$$\phi'=k\frac{Gm_s}{4\pi}\frac{e^{(\lambda-\nu)/2}}{r^2}$$

$$\frac{e^{-\lambda}}{r^2}\left(-1+e^{\lambda}+r\lambda'\right)=k\frac{(Gm_s)^2}{4\pi}\frac{e^{-\nu}}{r^4}+Ke^{-\lambda}\left[\frac{(\nu')^2}{8}+\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'}{r}\right]$$

$$\frac{e^{-\lambda}}{r^2}\left(1-e^{\lambda}+r\nu'\right)=k\frac{(Gm_s)^2}{4\pi}\frac{e^{-\nu}}{r^4}+Ke^{-\lambda}\left[-\frac{(\nu')^2}{8}\right].$$

$$e^{\nu}=\alpha_0\left[1-\frac{r_e}{r}+\alpha_2\left(\frac{r_e}{r}\right)^2+\alpha_3\left(\frac{r_e}{r}\right)^3+\cdots\right],$$
$$e^{\lambda}=\beta_0\left[1+\beta_1\left(\frac{r_e}{r}\right)+\beta_2\left(\frac{r_e}{r}\right)^2+\beta_3\left(\frac{r_e}{r}\right)^3+\cdots\right],$$

$$\begin{aligned}\alpha_2&=0\\ \alpha_3&=-\frac{K}{48}+\frac{k}{a_0}\frac{(Gm_s)^2}{4\pi}\frac{1}{6r_e^2},\ldots\\ \beta_0&=1\\ \beta_1&=1\\ \beta_2&=1+\frac{K}{8}-\frac{k}{a_0}\frac{(Gm_s)^2}{4\pi}\frac{1}{r_e^2}\\ \beta_3&=1+\frac{5K}{16}-\frac{k}{a_0}\frac{(Gm_s)^2}{4\pi}\frac{5}{2r_e^2},\ldots\end{aligned}$$

$$\phi=\phi_c-\frac{kGm_s}{4\pi}\frac{1}{r}-\frac{kGm_sr_e}{8\pi}\frac{1}{r^2}+\mathcal{O}(r^{-3}),$$

$$\begin{aligned}\tilde{g}_{tt}&=-1+\frac{2G_Nm}{r}-\frac{2G_Nm\left(G_Nm-\frac{r_e}{2}\right)}{r^2}+\mathcal{O}(r^{-3})\\ \tilde{g}_{rr}&=1+\frac{2G_Nm}{r}+\frac{2G_Nm\left(G_Nm+\frac{r_e}{2}\right)-k\frac{(Gm_s)^2}{4\pi}+\frac{K}{8}r_e^2}{r^2}+\mathcal{O}(r^{-3})\\ \tilde{g}_{\theta\theta}&=r^2+(2G_Nm-r_e)r+2G_Nm\left(G_Nm-\frac{r_e}{2}\right)+\mathcal{O}(r^{-1})\\ \tilde{g}_{\varphi\varphi}&=\sin^2\theta\,\tilde{g}_{\theta\theta}\end{aligned}$$

$$\frac{d\left(\tilde{p}e^{-2\phi}\right)}{dr}=-\frac{\left(\tilde{p}e^{-2\phi}+\rho e^{2\phi}\right)}{2}\frac{d\nu}{dr}.$$

$$\frac{e^{-\lambda}}{r^2}\left(-1+e^{\lambda}+r\lambda'\right)=8\pi G\bar{\rho},$$

$$\frac{e^{-\lambda}}{r^2}\left(1-e^{\lambda}+r\nu'\right)=8\pi G\bar{p}$$

$$\bar{\rho}=\tilde{\rho}e^{2\phi}+\frac{1}{kG}\frac{e^{-\lambda}}{2}(\phi')^2+\frac{K}{8\pi G}e^{-\lambda}\left[\frac{(\nu')^2}{8}+\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'}{r}\right]\bar{p}=\left[\tilde{p}e^{-2\phi}+\frac{1}{kG}\frac{e^{-\lambda}}{2}(\phi')^2\right]+\frac{K}{8\pi G}e^{-\lambda}\left[-\frac{(\nu')^2}{8}\right].$$

$$m(r)=\frac{1}{2G}\left(r-re^{-\lambda}\right),$$

$$e^{-\lambda}=1-\frac{2Gm(r)}{r}.$$

$$\frac{dm(r)}{dr}=4\pi r^2\bar{\rho},$$

$$m(r)=4\pi\int_0^r\bar{\rho}(r')r^2dr'$$

$$e^{-\lambda}=1-\frac{2G\left[4\pi\int_0^R\bar{\rho}(r')r'^2dr'+4\pi\int_R^r\bar{\rho}(r')r'^2dr'\right]}{r}=1-\frac{2Gm_g(R)}{r}-\frac{8\pi G\int_R^r\bar{\rho}(r')r'^2dr'}{r}$$

$$r_e-\frac{Kr_e^2}{8R}+\frac{kG^2m_s^2}{4\pi R}+\mathcal{O}\left(\frac{r_e^3}{R^2}\right)=2Gm_g$$

$$\frac{d\nu}{dr}=\frac{2Gm(r)+8\pi G\bar{p}r^3}{r[r-2Gm(r)]}$$

$$\frac{d\left(\tilde{p}e^{-2\phi}\right)}{dr}=-\left(\tilde{p}e^{-2\phi}+\tilde{\rho}e^{2\phi}\right)\frac{Gm(r)+4\pi G\bar{p}r^3}{r[r-2Gm(r)]}$$

$$e^{\nu}=a_0\left[1+\left(\frac{r}{r_i}\right)^2+a_2\left(\frac{r}{r_i}\right)^4+a_3\left(\frac{r}{r_i}\right)^6+\cdots\right]$$

$$e^{\lambda}=b_0\left[1+b_1\left(\frac{r}{r_i}\right)^2+b_2\left(\frac{r}{r_i}\right)^4+b_3\left(\frac{r}{r_i}\right)^6+\cdots\right]$$

$$\tilde{p}=c_0\left[1+c_1\left(\frac{r}{r_i}\right)^2+c_2\left(\frac{r}{r_i}\right)^4+c_3\left(\frac{r}{r_i}\right)^6+\cdots\right]$$

$$\phi=d_1\left(\frac{r}{r_i}\right)^2+d_2\left(\frac{r}{r_i}\right)^4+d_3\left(\frac{r}{r_i}\right)^6+\cdots$$

$$a_2=\frac{27k(2-K)}{80\pi}+\frac{8\pi G\tilde{\rho}_0(5-4K)r_i^2}{30(2-K)}+\frac{5-4K^2+6kG\tilde{\rho}_0r_i^2+K(8-3kG\tilde{\rho}_0r_i^2)}{10(2-K)},\ldots$$

$$b_0=1$$

$$b_1=K+\frac{8\pi G\tilde{\rho}_0}{3}r_i^2$$

$$b_2=\frac{K^3(16\pi-9k)}{16\pi(-2+K)}+\frac{K^2\left(-348\pi+243k+640\pi^2Gr_i^2\tilde{\rho}_0\right)}{120\pi(-2+K)}$$

$$-\frac{K\left[-243k-1632\pi^2Gr_i^2\tilde{\rho}_0+1280\pi^3G^2r_i^4\tilde{\rho}_0^2+72\pi\left(2+3kGr_i^2\tilde{\rho}_0\right)\right]}{180\pi(-2+K)}-\frac{1280\pi^3G^2\tilde{\rho}_0^2+27k\left(3+8\pi Gr_i^2\tilde{\rho}_0\right)}{90\pi(-2+K)},\ldots$$

$$c_0=\frac{\tilde{\rho}_0}{3}+\frac{2-K}{8\pi Gr_i^2}$$

$$c_1=\frac{1}{3}+\frac{kG(2-K)}{16\pi G}+\frac{(10-5K)[8\pi G-3kG(2-K)]}{48\pi G(-2+K+8\pi G\tilde{\rho}_0 r_i^2)},$$

$$c_2=\frac{27k^2(-2+K)^3}{128\pi^2(-6+3K+8\pi Gr_i^2\tilde{\rho}_0)}+\frac{32(-5+4K)\pi^2G^2r_i^4\tilde{\rho}_0^2}{15(-2+K)(-6+3K+8\pi Gr_i^2\tilde{\rho}_0)}$$
$$-\frac{4\pi Gr_i^2\tilde{\rho}_0[-15+48k\pi Gr_i^2\tilde{\rho}_0+K(27-44kGr_i^2\tilde{\rho}_0)+2K^2(-3+5kGr_i^2\tilde{\rho}_0)]}{15(-2+K)(-6+3K+8\pi Gr_i^2\tilde{\rho}_0)}$$
$$-\frac{3k(-2+K)^2\left(5K-2(9+5kGr_i^2\tilde{\rho}_0)\right)}{40\pi(-6+3K+8\pi Gr_i^2\tilde{\rho}_0)}$$
$$-\frac{60-492kGr_i^2\tilde{\rho}_0+40k^2G^2r_i^4\tilde{\rho}_0^2+8K^2(3+10kGr_i^2\tilde{\rho}_0)+K(-93+86kGr_i^2\tilde{\rho}_0-20k^2G^2r_i^4\tilde{\rho}_0^2)}{40(-6+3K+8\pi Gr_i^2\tilde{\rho}_0)},\ldots$$

$$d_1=\frac{3k(2-K)}{8\pi}$$
$$d_2=\frac{k}{192\pi}[-12K^2+6(-3+2(4\pi-k)Gr_i^2\tilde{\rho}_0)+K(33+2(-16\pi+3k)Gr_i^2\tilde{\rho}_0)],\ldots$$

$$\left(K+\frac{8}{3}\pi G\tilde{\rho}_0 r_i^2\right)\frac{R^3}{r_i^2}$$
$$-\left\{\frac{45K^3k}{80\pi(-2+K)}+\frac{18K^2(4\pi-9k)}{80\pi(-2+K)}-\frac{4K\left[-27k+32\pi^2G\tilde{\rho}_0r_i^2+8\pi\left(2+3kG\tilde{\rho}_0r_i^2\right)\right]}{80\pi(-2+K)}+\frac{24k\left(3+8\pi G\tilde{\rho}_0r_i^2\right)}{80\pi(-2+K)}\right\}\frac{R^5}{r_i^4}$$
$$+O\left(\frac{R^7}{r_i^6}\right)=2Gm_g.$$

$$\tilde{g}_{tt}=-e^{2\phi+\nu},\tilde{g}_{rr}=e^{-2\phi+\lambda}$$

$$\tilde{g}_{\theta\theta}=\tilde{g}_{\varphi\varphi}/\sin^2\ \theta=r^2e^{-2\phi}$$

$$\tilde{g}_{tt}\ =-\left(\frac{5}{2}-\frac{9Gm_g}{2R}-\frac{3}{2}\sqrt{1-\frac{2Gm_g}{R}}\right)$$
$$-\left(\frac{5}{2}-\frac{9Gm_g}{2R}-\frac{3}{2}\sqrt{1-\frac{2Gm_g}{R}}\right)\left[1-\frac{3k(-2+K)}{8\pi}\right]\left(\frac{r}{r_i}\right)^2+O\left[\left(\frac{r}{r_i}\right)^4\right],$$
$$\tilde{g}_{rr}\ =1+\left[K+\frac{3k(-2+K)}{8\pi}+\frac{8}{3}\pi G\tilde{\rho}_0r_i^2\right]\left(\frac{r}{r_i}\right)^2+O\left[\left(\frac{r}{r_i}\right)^4\right],$$
$$\tilde{g}_{\theta\theta}\ =r^2\left\{\left[1+\frac{3k(-2+K)}{8\pi}\right]\left(\frac{r}{r_i}\right)^2+O\left[\left(\frac{r}{r_i}\right)^4\right]\right\},$$
$$\tilde{g}_{\varphi\varphi}\ =\sin^2\ \theta\tilde{g}_{\theta\theta}.$$

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta=-e^{\nu(\rho)}dt^2+e^{\varsigma(\rho)}[dr^2+\rho^2(d\theta^2+\sin^2\ \theta d\varphi^2)]$$

$$e^{\nu(\rho)}\ =e^{\nu(r)}$$
$$e^{\varsigma(\rho)}d\rho^2\ =e^{\lambda(r)}dr^2$$
$$e^{\varsigma(\rho)}\rho^2\ =r^2$$

$$\phi(\rho)=\phi_c+\frac{kGm_s}{8\pi\rho_c}\ln\left(\frac{\rho-\rho_c}{\rho+\rho_c}\right)$$
$$e^\nu\ =\left(\frac{\rho-\rho_c}{\rho+\rho_c}\right)^{\rho_g/2\rho_c}$$
$$e^\varsigma\ =\frac{(\rho^2-\rho_c^2)^2}{\rho^4}\left(\frac{\rho-\rho_c}{\rho+\rho_c}\right)^{-\rho_g/2\rho_c}$$

$$\rho_c = \frac{\rho_g}{4}\sqrt{1 + \frac{k}{\pi}\left(\frac{Gm_s}{\rho_g}\right)^2} - \frac{K}{2}\rho_g$$

$$\rho_g - \frac{3K\rho_g^2}{8R} - \frac{kG^2m_s^2}{4\pi R} + O(\rho_g^3/R^2) = 2Gm_g$$

$$\tilde{g}_{tt} = -\left(\frac{\rho - \rho_c}{\rho + \rho_c}\right)^a$$

$$\tilde{g}_{\rho\rho} = \frac{\tilde{g}_{\theta\theta}}{\rho^2} = \frac{\tilde{g}_{\varphi\varphi}}{\rho^2 \sin^2\ \theta} = \frac{(\rho^2 - \rho_c^2)^2}{\rho^4}\left(\frac{\rho - \rho_c}{\rho + \rho_c}\right)^{-a}$$

$$a \equiv \frac{\rho_g}{2\rho_c} + \frac{kGm_s}{4\pi\rho_c}$$

$$\tilde{g}^{\alpha\beta}p_\alpha p_\beta + m^2 = 0$$

$$\tilde{g}^{tt}\left(\frac{\partial S}{\partial t}\right)^2 + \tilde{g}^{\rho\rho}\left(\frac{\partial S}{\partial \hat{\rho}}\right)^2 + \tilde{g}^{\rho\rho}\rho^{-2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \tilde{g}^{\rho\rho}\rho^{-2}\sin^{-2}\ \theta\left(\frac{\partial S}{\partial \varphi}\right)^2 - m^2 = 0$$

$$\hat{\rho} = \sqrt{g_{rr}}r$$

$$S = -\tilde{E}t + S_1(\hat{\rho}) + S_2(\theta) + S_3(\varphi)$$

$$\tilde{g}^{tt}\tilde{E}^2 + \tilde{g}^{\rho\rho}\left(\frac{\partial S_1}{\partial \hat{\rho}}\right)^2 + \tilde{g}^{\rho\rho}\rho^{-2}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + \tilde{g}^{\rho\rho}\rho^{-2}\sin^{-2}\ \theta\left(\frac{\partial S_3}{\partial \varphi}\right)^2 + m^2 = 0$$

$$S = -\tilde{E}t + \int^\rho \sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}^2 - \frac{m^2}{\tilde{g}^{\rho\rho}} - \frac{\tilde{L}^2}{\rho^2}}\sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}^2 - \frac{m^2}{\tilde{g}^{\rho\rho}} - \frac{\tilde{L}^2}{\rho^2}}\,d\hat{\rho} + \int^\theta \sqrt{\tilde{L}^2 - \frac{p_\varphi}{\sin^2\ \theta}}\sqrt{\tilde{L}^2 - \frac{p_\varphi}{\sin^2\ \theta}}\,d\theta + p_\varphi\varphi$$

$$\frac{\partial S}{\partial \tilde{E}} = -t + \int^\rho \frac{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}}{\sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}^2 - \frac{m^2}{\tilde{g}^{\rho\rho}} - \frac{\tilde{L}^2}{\rho^2}}}\frac{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}}{\sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}^2 - \frac{m^2}{\tilde{g}^{\rho\rho}} - \frac{\tilde{L}^2}{\rho^2}}}\,d\hat{\rho}$$

$$\frac{d\hat{\rho}}{dt} = \frac{\sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}\rho\rho}\tilde{E}^2 - \frac{m^2}{\tilde{g}^{\rho\rho}} - \frac{\tilde{L}^2}{\rho^2}}}{-\frac{\tilde{g}^{tt}}{\tilde{g}^{\rho\rho}}\tilde{E}}$$

$$\tilde{U} = \left[-\frac{\tilde{g}^{\rho\rho}}{\tilde{g}^{tt}}\left(\frac{m^2}{\tilde{g}^{\rho\rho}} + \frac{\tilde{L}^2}{\rho^2}\right)\right]^{1/2}$$

$$U = \left(\frac{\rho - \rho_c}{\rho + \rho_c}\right)^{a/2}\left[1 + \frac{\tilde{L}^2}{m^2}\frac{\rho^2}{(\rho^2 - \rho_c^2)^2}\left(\frac{\rho - \rho_c}{\rho + \rho_c}\right)^a\right]^{1/2}$$

$$U = \left(\frac{\mathfrak{r} - \mathfrak{r}_c}{\mathfrak{r} + \mathfrak{r}_c}\right)^{a/2}\left[1 + L^2\frac{\mathfrak{r}^2}{(\mathfrak{r}^2 - \mathfrak{r}_c)^2}\left(\frac{\mathfrak{r} - \mathfrak{r}_c}{\mathfrak{r} + \mathfrak{r}_c}\right)^a\right]^{1/2},$$

$$a = \frac{1}{2\mathfrak{r}_c}\left(1 + \frac{k}{2\pi}\frac{Gm_s}{\rho g}\right) \text{ and } \mathfrak{r}_c = \frac{1}{4}\sqrt{1 + \frac{k}{\pi}\left(\frac{Gm_s}{\rho g}\right)^2 - \frac{K}{2}}$$

$$\Delta L_{\text{lowest}} = \frac{L_{\text{lowest}}(K) - L_{\text{lowest}}(K=0)}{L_{\text{lowest}}(K=0)}$$

$$\Delta L_{\text{crit}} = \frac{L_{\text{crit}}(K) - L_{\text{crit}}(K=0)}{L_{\text{crit}}(K=0)}$$

K	L_{lowest}	ΔL_{lowest}	L_{crit}	ΔL_{crit}
0	1.73205	0	2	0
0.01	1.73221	0.009013%	2.0002	0.00992684%
0.02	1.73236	0.018022%	2.0004	0.019847%
0.03	1.73252	0.027025%	2.0006	0.0297605%
0.04	1.73267	0.036023%	2.00079	0.0396673%
0.05	1.73283	0.045016%	2.00099	0.0495674%
0.06	1.73299	0.054003%	2.00119	0.059461%
0.07	1.73314	0.062985%	2.00139	0.0693478%
0.08	1.73330	0.071961%	2.00158	0.0792281%
0.09	1.73345	0.080932%	2.00178	0.0891018%
0.1	1.73361	0.089899%	2.00198	0.0989688%
0.2	1.73516	0.179271%	2.00395	0.19728%
0.3	1.73669	0.268126%	2.0059	0.292944%
0.4	1.73822	0.356470%	2.00784	0.391974%
0.5	1.73975	0.444313%	2.00977	0.488381%
0.6	1.74126	0.531661%	2.01168	0.584175%
0.7	1.74276	0.618522%	2.01359	0.679367%
0.8	1.74426	0.704904%	2.01548	0.773968%

0.9	1.74575	0.790813%	2.01736	0.867986%
1.0	1.74723	0.876256%	2.01923	0.961433%
1.1	1.74870	0.961241%	2.02109	1.05432%
1.2	1.75016	1.045774%	2.02293	1.14665%
1.3	1.75162	1.129861%	2.02477	1.23843%
1.4	1.75307	1.213508%	2.02659	1.32968%
1.5	1.75451	1.296722%	2.02841	1.4204%
1.6	1.75594	1.379509%	2.03021	1.5106%
1.7	1.75737	1.461874%	2.03201	1.6003%
1.8	1.75879	1.543824%	2.03379	1.68948%
1.9	1.76020	1.625363%	2.03556	1.77817%

$$\frac{dp}{dr}\left[\frac{1}{\kappa_G}-\frac{M(r)c^2}{4\pi r}\right]=-\frac{p+\rho c^2}{2}\left[pr+\frac{M(r)c^2}{4\pi r^2}\right].$$

$$\kappa_G=8\pi G/c^4$$

$$H_{jkl}=\kappa_G T_{jkl}$$

$$\begin{aligned}H_{jkl}=&\nabla_j R_{kl}+\nabla_k R_{lj}+\nabla_l R_{jk}\\&-\frac{1}{3}\left(g_{kl}\nabla_j R+g_{lj}\nabla_k R+g_{jk}\nabla_l R\right)\\T_{jkl}=&\nabla_j T_{kl}+\nabla_k T_{lj}+\nabla_l T_{jk}\\&-\frac{1}{6}\left(g_{kl}\nabla_j T+g_{lj}\nabla_k T+g_{jk}\nabla_l T\right)\end{aligned}$$

$$\begin{gathered}R_{kl}-\frac{1}{2}Rg_{kl}=\kappa_G(T_{kl}+K_{kl})\\\nabla_j K_{kl}+\nabla_k K_{jl}+\nabla_l K_{jk}\\=\frac{1}{6}\left(g_{kl}\nabla_j K+g_{jl}\nabla_k K+g_{jk}\nabla_l K\right)\end{gathered}$$

$$ds^2=-y(r)dt^2+\frac{h(r)}{y(r)}dr^2+r^2d\Omega_2^2$$

$$\nabla_i u_j=-u_i\dot{u}_j,\nabla_i\dot{u}_j=\nabla_j\dot{u}_i$$

$$\begin{aligned}R_{kl}=&\frac{R+4\nabla_p\dot{u}^p}{3}u_k u_l+\frac{R+\nabla_p\dot{u}^p}{3}g_{kl}\\&+\Sigma(r)\left[\chi_k\chi_l-\frac{u_k u_l+g_{kl}}{3}\right].\end{aligned}$$

$$R = R^\star - 2\nabla_p \dot{u}^p$$
$$\frac{R^\star}{2} = \frac{1}{r^2} + \frac{y}{r}\frac{h'}{h^2} - \frac{y'}{rh} - \frac{y}{r^2 h}$$
$$\nabla_p \dot{u}^p = \frac{y''}{2h} - \frac{y'}{4}\frac{h'}{h^2} + \frac{y'}{rh}$$
$$\Sigma = -\frac{y''}{2h} + \frac{y'}{4}\frac{h'}{h^2} + \frac{y}{r^2 h} + \frac{y}{2r}\frac{h'}{h^2} - \frac{1}{r^2}$$

$$\frac{R^\star}{2} + \nabla_p \dot{u}^p + \Sigma = \frac{3y}{2r}\frac{h'}{h^2}$$

$$K_{kl} = \mathrm{A}(r) u_k u_l + \mathrm{B}(r) g_{kl} + \mathrm{C}(r) \chi_k \chi_l$$

$$\mathrm{A}(r) = \kappa_2 r^2 - 2\kappa_3 y(r)$$
$$\mathrm{B}(r) = \kappa_1 + 2\kappa_2 r^2 + \kappa_3 y(r)$$
$$\mathrm{C}(r) = -\kappa_2 r^2$$

$$T_{kl} = (\mu_m + P_m) u_k u_l + P_m g_{kl} + (p_{mr} - p_{m\perp}) \left[\chi_k \chi_l - \frac{g_{kl} + u_k u_l}{3}\right]$$

$$K_{kl} = (\mu_d + P_d) u_k u_l + P_d g_{kl} + (p_{dr} - p_{d\perp}) \left[\chi_k \chi_l - \frac{g_{kl} + u_k u_l}{3}\right]$$

$$\mu_d = \mathrm{A} - \mathrm{B} = -\kappa_1 - \kappa_2 r^2 - 3\kappa_3 y(r)$$
$$p_{dr} = \mathrm{B} + \mathrm{C} = \kappa_1 + \kappa_2 r^2 + \kappa_3 y(r)$$
$$p_{d\perp} = \mathrm{B} = \kappa_1 + 2\kappa_2 r^2 + \kappa_3 y(r)$$

$$\mu_d(r) + 5p_{dr}(r) - 2p_{d\perp}(r) = 2\kappa_1$$

$\frac{1}{2}R^\star + \nabla_p \dot{u}^p + \Sigma = \frac{3}{2}(\mu_m + \mu_d + p_{mr} + p_{dr})$, $\mu_d + p_{dr} = -2\kappa_3 r$

$$\frac{1}{2}R^\star = \mu_m + \mu_d$$
$$\nabla_p \dot{u}^p = \frac{3}{2}(P_m + P_d) + \frac{1}{2}(\mu_m + \mu_d)$$
$$\Sigma = (p_{mr} - p_{m\perp}) + (p_{dr} - p_{d\perp})$$

$$\mu_m + p_{mr} = \frac{y(r)}{r}\frac{d}{dr}\left[\kappa_3 r^2 - \frac{1}{h(r)}\right]$$

$$\frac{dp_{mr}}{dr} = -\frac{y'}{2y}(\mu_m + p_{mr}) - \frac{2}{r}(p_{mr} - p_{m\perp})$$

$p_{mr} = p_{m\perp} \equiv p_m$ 。

$$\frac{1}{2}R^\star = -\kappa_1 - \kappa_2 r^2 - 3\kappa_3 y(r)$$
$$\nabla_p \dot{u}^p = \kappa_1 + 2\kappa_2 r^2$$
$$\Sigma = -\kappa_2 r^2$$

$$h(r) = \frac{1}{\kappa_3 r^2 + \kappa_4}$$

$$\frac{y'}{r}(\kappa_3 r^2 + \kappa_4) + \frac{y}{r^2}\kappa_4 = \kappa_1 + \kappa_2 r^2 + \frac{1}{r^2}$$
$$y''(\kappa_3 r^2 + \kappa_4) + \frac{y'}{r}(3\kappa_3 r^2 + 2\kappa_4) = 4\kappa_2 r^2 + 2\kappa_1$$
$$y''(\kappa_3 r^2 + \kappa_4) + y'\kappa_3 r - 2\frac{y}{r^2}\kappa_4 = 2\kappa_2 r^2 - \frac{2}{r^2}$$

$\kappa_4 = 1, \kappa_3 = 0.$

$\frac{y'}{r} + \frac{y}{r^2} = \kappa_1 + \kappa_2 r^2 + \frac{1}{r^2}.\ C = -2\bar{M}, \kappa_1 = -\Lambda, \kappa_2 = -\lambda$)

$$y(r) = 1 - \frac{2\bar{M}}{r} - \frac{\Lambda}{3}r^2 - \frac{\lambda}{5}r^4$$

$$y' = \frac{\kappa_1}{\kappa_3}\frac{1}{r} + \frac{\kappa_2}{\kappa_3}r + \frac{1}{\kappa_3}\frac{1}{r^3}$$

$$y(r) = \frac{\kappa_1}{\kappa_3}\log\left(r\sqrt{\kappa_1}\right) + \frac{\kappa_2}{2\kappa_3}r^2 - \frac{1}{2\kappa_3}\frac{1}{r^2} + \text{ const.}$$

$$y(r) = C\frac{\sqrt{\kappa_3 r^2 + \kappa_4}}{r} + \frac{1}{\kappa_4} + \frac{\kappa_2}{2\kappa_3}r^2$$
$$+\left[\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right]\left[\frac{\sqrt{\kappa_3 r^2 + \kappa_4}}{r\sqrt{|\kappa_3|}}F\left(r\sqrt{\frac{|\kappa_3|}{|\kappa_4|}}\right) - 1\right]$$
$$F(x) = \begin{cases} \text{ArcSinh}x & \kappa_3 > 0, \kappa_4 > 0 \\ \text{ArcCosh}x & \kappa_3 > 0, \kappa_4 < 0 \\ \text{ArcSin}x & \kappa_3 < 0, \kappa_4 > 0 \end{cases}$$

$r^2 \gg \frac{|\kappa_4|}{\kappa_3}$

$$y(r) \approx \frac{\kappa_2}{2\kappa_3}r^2 + \left[\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right]\log\left(2r\sqrt{\frac{\kappa_3}{|\kappa_4|}}\right)$$
$$+\left[\frac{1}{\kappa_4} - \frac{\kappa_1}{\kappa_3} + \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right] + \cdots$$

$$ds^2 = -b^2(r)dt^2 + \left[1 - \frac{2M(r)}{r}\right]^{-1}dr^2 + r^2 d\Omega_2^2$$

$$R^\star = \frac{4M'}{r^2}$$
$$\Sigma = -\left[\frac{b''}{b} - \frac{b'}{br}\right]\left[1 - \frac{2M(r)}{r}\right] + \frac{b'}{b}\left[\frac{M'r - M}{r^2}\right]$$
$$-\frac{3M - rM'}{r^3}$$
$$\nabla_p \dot{u}^p = \left[\frac{b''}{b} + \frac{2b'}{br}\right]\left[1 - \frac{2M(r)}{r}\right] - \frac{b'}{b}\left[\frac{M'r - M}{r^2}\right]$$

$$M(r) = \frac{1}{2}\int_0^r dr' r^2 [\mu_m(r') + \mu_d(r')]$$

$$\left[p'_{mr} + 2\frac{p_{mr} - p_{m\perp}}{r}\right]\left[1 - \frac{2M(r)}{r}\right] = -\frac{\mu_m + p_{mr}}{2}\left[r(p_m + p_{dr}) + \frac{2M(r)}{r^2}\right]$$

$$\nabla_p \dot{u}^p + \Sigma - \frac{R^\star}{4} = \frac{3b'}{br}\left[1 - \frac{2M(r)}{r}\right] - \frac{3M(r)}{r^3}$$

$$ds_-^2 = -b_-^2(r)dt^2 + \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\Omega_2^2 \quad r < R$$
$$ds_+^2 = -b_+^2(r)dt^2 + f_{1+}^2(r)dr^2 + r^2 d\Omega_2^2 \quad r > R$$

$$b_-^2(R) = b_+^2(R)$$
$$1 - \frac{2M(R)}{R} = \frac{1}{f_{1+}^2(R)}$$
$$p_{mr}(R) = p_{dr}^+(R) - p_{dr}^-(R)$$

$$K_{kl}^\pm(r) = \mathrm{A}^\pm(r)u_k^\pm(r)u_l^\pm(r) + \mathrm{B}^\pm(r)g_{kl}^\pm(r) + \mathrm{C}^\pm(r)\chi_k^\pm(r)\chi_l^\pm(r)$$

$$\delta K_{kl}(R) = K_{kl}^+(R) - K_{kl}^-(R) = \delta\mathrm{A}(R)u_k(R)u_l(R) + \delta\mathrm{B}(R)g_{kl}(R) + \delta\mathrm{C}(R)\chi_k(R)\chi_l(R)$$

$$\delta\mathrm{A}(R) = (\kappa_2^+ - \kappa_2^-)R^2 - 2(\kappa_3^+ - \kappa_3^-)b^2(R)$$
$$\delta\mathrm{B}(R) = (\kappa_1^+ - \kappa_1^-) + 2(\kappa_2^+ - \kappa_2^-)R^2 + (\kappa_3^+ - \kappa_3^-)b^2(R)$$
$$\delta\mathrm{C}(R) = -(\kappa_2^+ - \kappa_2^-)R^2$$

$$\nabla_k^- \chi_l^-(R) = \nabla_k^+ \chi_l^+(R)$$

$$\delta R_{jklm} = \chi_j\chi_l B_{km} - \chi_j\chi_m B_{kl} + \chi_k\chi_m B_{jl} - \chi_k\chi_l B_{jm}$$

$$0 = \delta G_{kl}(R)\chi^k = G_{kl}^+(R)\chi^k - G_{kl}^-(R)\chi^k$$

$$p_{mr}(R) = \delta\mathrm{B}(R) + \delta\mathrm{C}(R) = (\kappa_1^+ - \kappa_1^-) + (\kappa_2^+ - \kappa_2^-)R^2 + (\kappa_3^+ - \kappa_3^-)b^2(R)$$

$$\kappa_1^+ = \kappa_1^-, \kappa_2^+ = \kappa_2^- \; \kappa_3^+ = \kappa_3^-$$
$$p_{mr}(R) = 0$$

$$1 - \frac{2M(R)}{R} = 1 - \frac{2\bar{M}}{R} - \frac{\Lambda}{3}R^2 - \frac{\lambda}{5}R^4$$

$$\frac{1}{2}\int_0^R dr' [\mu_m(r') + \mu_d(r')]r'^2 = \bar{M} + \frac{\Lambda}{6}R^3 + \frac{\lambda}{10}R^5$$

$$\kappa_1 = -\Lambda, \kappa_2 = -\lambda, \kappa_3 = 0$$

$$\mu_d(r) = \lambda r^2 + \Lambda$$
$$p_{dr}(r) = -\lambda r^2 - \Lambda$$
$$p_{d\perp}(r) = -2\lambda r^2 - \Lambda$$

$$\frac{1}{2}\int_0^R drr^2\mu_m(r) = \bar{M}$$

$$\left[1-\frac{2M(r)}{r}\right]b_-'' - \left[\frac{1}{r}+\frac{rM'(r)-3M(r)}{r^2}\right]b_-' + \left[\lambda r^2 - \frac{rM'(r)-3M(r)}{r^3}\right]b_- = 0$$

$$\left[1-\frac{2M(r)}{r}\right]b_-'' + \left[\frac{2}{r}-\frac{rM'(r)+3M(r)}{r^2}\right]b_-' - \left[\frac{3}{2}p_m+\frac{1}{2}\mu_m - 2\lambda r^2 - \Lambda\right]b_- = 0$$

$$\frac{p_m'}{p_m+\mu_m} = -\frac{b'}{b}$$

$$M(r) = \frac{\mu_m+\Lambda}{6}r^3 + \frac{\lambda}{10}r^5$$

$$b_-(r) = \gamma\frac{\mu_m}{p_m(r)+\mu_m}$$

$$\gamma = \sqrt{1-\frac{2M(R)}{R}} = \sqrt{1-\frac{\mu_m+\Lambda}{3}R^2-\frac{\lambda}{5}R^4}$$

$$y'\left[1-\frac{\mu_m+\Lambda}{3}r^2-\frac{\lambda}{5}r^4\right]+\frac{y^2}{2}r - y\left[\frac{\mu_m+\Lambda}{3}r+2\frac{\lambda}{5}r^3\right]$$

$$\frac{1}{p_m(r)+\mu_m} = \frac{1}{4}\frac{\frac{\mu_m+\Lambda}{6}+\frac{\lambda}{5}r^2}{\left(\frac{\mu_m+\Lambda}{6}\right)^2+\frac{\lambda}{5}} + \sqrt{\frac{1-\frac{2M(r)}{r}}{1-\frac{2M(R)}{R}}}\left[\frac{1}{\mu_m}-\frac{1}{4}\frac{\frac{\mu_m+\Lambda}{6}+\frac{\lambda}{5}R^2}{\left(\frac{\mu_m+\Lambda}{6}\right)^2+\frac{\lambda}{5}}\right]$$

$$p_m(r) = \mu_m\frac{\sqrt{1-2M(R)/R}-\sqrt{1-2M(r)/r}}{\sqrt{1-2M(r)/r}-3\sqrt{1-2M(R)/R}}$$

$$b_-(r) = \sqrt{1-\frac{2M(r)}{r}}\left[1-\frac{\mu_m}{4}\frac{\frac{\mu_m+\Lambda}{6}+\frac{\lambda}{5}R^2}{\left(\frac{\mu_m+\Lambda}{6}\right)^2+\frac{\lambda}{5}}\right] + \frac{\mu_m}{4}\frac{\frac{\mu_m+\Lambda}{6}+\frac{\lambda}{5}r^2}{\left(\frac{\mu_m+\Lambda}{6}\right)^2+\frac{\lambda}{5}}\sqrt{1-\frac{2M(R)}{R}}$$

$$\left[1-\frac{\Lambda+\mu_m}{3}r^2-\frac{\lambda}{5}r^4\right]b_-'' - \left[\frac{1}{r}+\frac{\lambda}{5}r^3\right]b_-' + \frac{4\lambda}{5}r^2b_-$$

$$\left[1-\frac{2M(r)}{r}\right]b_-' + 2\left[\frac{\mu_m+\Lambda}{6}r+\frac{\lambda}{5}r^2\right]b_- = \frac{r}{2}\gamma\mu_m$$

$$\bar{M} = \frac{1}{2}\int_0^R drr^2\mu_m = \frac{1}{6}\mu_m R^3$$

$$0=\frac{1}{\mu_m}\left[\frac{\mu_m^2}{36}+\frac{\lambda}{5}\right]-\frac{1}{4}\left[\frac{\mu_m}{6}+\frac{\lambda}{5}R^2\right]+\frac{\mu_m}{24}\sqrt{1-\frac{2M(R)}{R}}$$

$$X=\frac{1}{3}\mu_m R^2, Y=\frac{\lambda}{5}R^4$$

$$0=-\frac{1}{6}+\frac{4}{3}\frac{Y}{X^2}-\frac{Y}{X}+\frac{1}{2}\sqrt{1-X-Y}$$

$$Y=\frac{\frac{4}{9}-\frac{X}{3}-\frac{X^2}{4}\pm\left(\frac{X^2}{4}-\frac{5X}{3}+\frac{4}{3}\right)}{2\left(\frac{4}{3X}-1\right)^2}$$

$\left(\frac{4}{3}-X\right)^2 Y=X^2\left(\frac{8}{9}-X\right)$

$$\frac{\lambda}{5}(\mu_m R^2)^2-\left(\frac{8}{5}\lambda-\frac{\mu_m^2}{3}\right)(\mu_m R^2)+\frac{16}{5}\lambda-\frac{8}{9}\mu_m^2=0$$

$$\mu_m R^2=4-\frac{5}{6}\frac{\mu_m^2}{\lambda}\left[1-\sqrt{1-\frac{16}{5}\frac{\lambda}{\mu_m^2}}\right]$$

$$y_0(r)=\frac{\sqrt{\kappa_3 r^2+\kappa_4}}{r}, v(r)=\int dr\frac{1+\kappa_1 r^2+\kappa_2 r^4}{(\kappa_3 r^2+\kappa_4)^{3/2}}$$

$$\kappa_3>0, \kappa_4>0$$

$$r=\sqrt{\frac{\kappa_4}{\kappa_3}}\mathrm{sh}\theta, y_0=\sqrt{\kappa_3}\frac{\mathrm{ch}\theta}{\mathrm{sh}\theta}$$

$$v(r)=\frac{1}{\sqrt{\kappa_3}}\int d\theta\frac{\frac{1}{\kappa_4}+\frac{\kappa_1}{\kappa_3}\mathrm{sh}^2\theta+\frac{\kappa_2\kappa_4}{\kappa_3^2}\mathrm{sh}^4\theta}{\mathrm{ch}^2\theta}$$
$$=\frac{1}{\sqrt{\kappa_3}}\left[\left(\frac{\kappa_1}{\kappa_3}-\frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\theta+\left(\frac{1}{\kappa_4}-\frac{\kappa_1}{\kappa_3}+\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\frac{\mathrm{sh}\theta}{\mathrm{ch}\theta}\right.$$
$$\left.+\frac{\kappa_2\kappa_4}{4\kappa_3^2}\mathrm{sh}(2\theta)\right]$$
$$y_P(r)=\left[\frac{\kappa_1}{\kappa_3}-\frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right]\theta\frac{\mathrm{ch}\theta}{\mathrm{sh}\theta}+\left[\frac{1}{\kappa_4}-\frac{\kappa_1}{\kappa_3}+\frac{\kappa_2\kappa_4}{\kappa_3^2}\right]$$
$$+\frac{\kappa_2\kappa_4}{2\kappa_3^2}(\mathrm{sh}^2\theta+1)$$
$$=\left[\frac{\kappa_1}{\kappa_3}-\frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right]\left[\frac{\sqrt{\kappa_3 r^2+\kappa_4}}{r\sqrt{\kappa_3}}\mathrm{Arsh}\sqrt{\frac{\kappa_3}{\kappa_4}}r-1\right]$$
$$+\frac{1}{\kappa_4}+\frac{\kappa_2}{2\kappa_3}r^2$$

$$r=\sqrt{\frac{|\kappa_4|}{\kappa_3}}\mathrm{ch}\theta, y_0=\sqrt{\kappa_3}\frac{\mathrm{sh}\theta}{\mathrm{ch}\theta}$$

$$v(r) = \frac{1}{\sqrt{\kappa_3}} \int d\theta \frac{\frac{1}{|\kappa_4|} + \frac{\kappa_1}{\kappa_3}\mathrm{ch}^2\theta + \frac{\kappa_2|\kappa_4|}{\kappa_3^2}\mathrm{ch}^4\theta}{\mathrm{sh}^2\theta}$$

$$= \frac{1}{\sqrt{\kappa_3}}\left[\left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\theta + \left(\frac{1}{\kappa_4} - \frac{\kappa_1}{\kappa_3} + \frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\frac{\mathrm{ch}\theta}{\mathrm{sh}\theta}\right.$$

$$\left. - \frac{\kappa_2\kappa_4}{4\kappa_3^2}\mathrm{sh}(2\theta)\right]$$

$$y_P(r) = \left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\theta\frac{\mathrm{sh}\theta}{\mathrm{ch}\theta} + \left(\frac{1}{\kappa_4} - \frac{\kappa_1}{\kappa_3} + \frac{\kappa_2\kappa_4}{\kappa_3^2}\right)$$

$$- \frac{\kappa_2\kappa_4}{2\kappa_3^2}(\mathrm{ch}^2\theta - 1)$$

$$= \left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\left[\frac{\sqrt{\kappa_3 r^2 + \kappa_4}}{r\sqrt{\kappa_3}}\mathrm{Arch}\sqrt{\frac{\kappa_3}{|\kappa_4|}}r - 1\right]$$

$$+ \frac{1}{\kappa_4} + \frac{\kappa_2}{2\kappa_3}r^2$$

$$\kappa_4 > 0 (r^2 \leq \kappa_4/|\kappa_3|)$$

$$r = \sqrt{\frac{\kappa_4}{|\kappa_3|}}\sin\theta, y_0 = \sqrt{|\kappa_3|}\frac{\cos\theta}{\sin\theta}$$

$$v(r) = \frac{1}{\sqrt{|\kappa_3|}} \int d\theta \frac{\frac{1}{\kappa_4} + \frac{\kappa_1}{|\kappa_3|}\sin^2\theta + \frac{\kappa_2\kappa_4}{\kappa_3^2}\sin^4\theta}{\cos^2\theta}$$

$$= \frac{1}{\sqrt{\kappa_3}}\left[\left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\theta + \left(\frac{1}{\kappa_4} - \frac{\kappa_1}{\kappa_3} + \frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\frac{\sin\theta}{\cos\theta}\right.$$

$$\left. + \frac{\kappa_2\kappa_4}{4\kappa_3^2}\sin(2\theta)\right]$$

$$y_P(r) = \left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\theta\frac{\cos\theta}{\sin\theta} + \left(\frac{1}{\kappa_4} - \frac{\kappa_1}{\kappa_3} + \frac{\kappa_2\kappa_4}{\kappa_3^2}\right)$$

$$+ \frac{\kappa_2\kappa_4}{2\kappa_3^2}(1 - \sin^2\theta)$$

$$= \left(\frac{\kappa_1}{\kappa_3} - \frac{3}{2}\frac{\kappa_2\kappa_4}{\kappa_3^2}\right)\left[\frac{\sqrt{\kappa_3 r^2 + \kappa_4}}{r\sqrt{|\kappa_3|}}\mathrm{ArcSin}\sqrt{\frac{|\kappa_3|}{\kappa_4}}r - 1\right]$$

$$+ \frac{1}{\kappa_4} + \frac{\kappa_2}{2\kappa_3}r^2$$

$$\frac{d}{dr}\left(\frac{1}{y\sqrt{F}}\right) = \frac{1}{2}\frac{r}{F^{3/2}}$$

$$\frac{1}{y(r)} = \sqrt{F(r)}\left[C - \frac{1}{2}\int_r^R \frac{r'dr'}{F(r')^{3/2}}\right]$$

$$= \sqrt{F(r)}\left[C - \frac{1}{2(a^2+4b)}\left(\frac{a+2bR^2}{\sqrt{F(R)}} - \frac{a+2br^2}{\sqrt{F(r)}}\right)\right]$$

$$\frac{1}{y(r)} = \sqrt{\frac{F(r)}{F(R)}}\left[\frac{1}{\mu_m} - \frac{a+2bR^2}{2(a^2+4b)}\right] + \frac{a+2br^2}{2(a^2+4b)}$$

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + f(G)\right] + S_m$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\left[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho}\right.$$
$$\left. + \frac{R}{2}\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho}\right)\right]\nabla^\rho\nabla^\sigma f_G + (Gf_G - f)g_{\mu\nu} = \kappa^2 T_{\mu\nu},$$

$$ds^2 = c^2e^{2\phi}dt^2 - e^{2\lambda}dr^2 - r^2(d\theta^2 + \sin^2\,\theta d\varphi^2)$$

$$-\frac{1}{r^2}\left(2r\lambda' + e^{2\lambda} - 1\right) + 8e^{-2\lambda}(f_{GG}(G'' - 2\lambda' G') + f_{GGG}(G')^2)\left(\frac{1 - e^{2\lambda}}{r^2} - 2(\phi'' + \phi'^2(y)\right)$$
$$+(Gf_G - f)e^{2\lambda} = \kappa^2\rho c^2 e^{2\lambda}$$
$$-\frac{1}{r^2}\left(2r\phi' - e^{2\lambda} + 1\right) - (Gf_G - f)e^{2\lambda} = \kappa^2 p e^{2\lambda}$$

$$R + 8G_{\rho\sigma}\nabla^\rho\nabla^\sigma f_G - 4(Gf_G - f) = -\kappa^2(\rho c^2 - p)$$

$$2\left(\phi'' + \phi'^2 - \phi'\lambda' + \frac{2}{r}(\phi' - \lambda') + \frac{1 - e^{2\lambda}}{r^2}\right)$$
$$+8e^{-2\lambda}\left(\frac{2\phi'}{r} + \frac{1 - e^{2\lambda}}{r^2}\right)(f_{GG}(G'' - 2\lambda' G') + f_{GGG}(G')^2) + 4(Gf_G - f)e^{2\lambda} = \kappa^2 e^{2\lambda}(\rho c^2 - 3p)$$

$$\frac{dp}{dr} = -(p + \rho c^2)\phi'$$

$$e^{-2\lambda} = 1 - \frac{2GM}{c^2 r} \Longrightarrow \frac{GdM}{c^2 dr} = \frac{1}{2}\left[1 - e^{-2\lambda}(1 - 2r\lambda')\right]$$

$$M \to mM_\star, r \to r_g r, \rho \to \frac{\rho M_\star}{r_g^3}, p \to \frac{pM_\star c^2}{r_g^3}, G \to \frac{G}{r_g^4}$$

$$\frac{dp}{dr} = -(p + \rho)\phi'$$

$$\frac{d\lambda}{dr} = \frac{m}{r^3}\frac{1 - \frac{r}{m}\frac{dm}{dr}}{\frac{2m}{r} - 1}$$

$$\frac{2}{r}\frac{1 - \frac{2m}{r}}{p + \rho c^2} + \frac{2m}{r^3} - r_g^2(Gf_G - f) = 8\pi p$$

$$-\frac{2}{r^2}\frac{dm}{dr} + 8r_g^2\left(1 - \frac{2m}{r}\right)^2\left[f_{GG}\left(r_g^2 G'' - r_g^3\frac{2m}{r^2}\frac{1 - r\frac{dm}{dr}}{\frac{2m}{r} - 1}G'\right) + r_g^2 f_{GGG}G^2\right]$$
$$\times\left[-\frac{2m/r^3}{1 - \frac{2m}{r}} + 2\frac{d}{dr}\left(\frac{\frac{dp}{dr}}{p + \rho}\right) - 2\left(\frac{\frac{dp}{dr}}{p + \rho}\right)^2\right] + (Gf_G - f) = 8\pi\rho$$

$$2\left(1-\frac{2m}{r}\right)\left(-\frac{d}{dr}\left(\frac{\frac{dp}{dr}}{p+\rho}\right)+\left(\frac{\frac{dp}{dr}}{p+\rho}\right)^2+\frac{m}{r^3}\frac{1-\frac{r}{m}\frac{dm}{dr}}{\frac{2m}{r}-1}\left(\frac{\frac{dp}{dr}}{p+\rho}\right)-\frac{2m/r^3}{1-\frac{2m}{r}}\right)$$

$$+8\left[-\frac{2\frac{dp}{dr}}{r(p+\rho)}-\frac{2m/r^3}{1-\frac{2m}{r}}\right]\left[f_{GG}\left(r_g^2G''-r_g^3\frac{2m}{r^2}\frac{1-r\frac{dm}{dr}}{\frac{2m}{r}-1}G'\right)+r_g^2f_{GGG}G^2\right]$$

$$4(Gf_G-f)=8\pi(\rho-3p)$$

$$\Gamma_{12}^1=\phi',\Gamma_{11}^2=\phi'e^{2\phi-2\lambda},\Gamma_{22}^2=\lambda',\Gamma_{23}^2=-re^{-2\lambda},$$
$$\Gamma_{44}^2=-r\sin^2\ \theta e^{-2\lambda},\Gamma_{23}^3=\frac{1}{r},\Gamma_{44}^3=-\sin\ \theta\cos\ \theta,\Gamma_{24}^4=\frac{1}{r},\Gamma_{34}^4=\cot\ \theta.$$

$$G_{11}=-e^{2\phi-2\lambda}\left(\frac{2\lambda'}{r}+\frac{e^{2\lambda}-1}{r^2}\right)$$
$$G_{22}=-\frac{2\phi'}{r}+\frac{e^{2\lambda}-1}{r^2}$$
$$G_{33}=-re^{-2\lambda}(\phi'-\lambda'+r(\phi''+\phi'^2)-r\phi'\lambda')$$
$$G_{44}=-\sin^2\ \theta G_{33}$$

$$R_{1212}=e^{2\phi}(\phi'\lambda'-\phi''-\phi'^2),R_{1313}=-re^{2\phi-2\lambda}\phi',R_{1414}=\sin^2\ \theta R_{1313}$$
$$R_{2323}=-r\lambda',R_{2424}=\sin^2\ \theta R_{2323},R_{3434}=\sin^2\ \theta r^2e^{-2\lambda}\left(1-e^{2\lambda}\right)$$

$$R=-2e^{-2\lambda}\left(\phi''+\phi^2-\phi'\lambda'+2\frac{\phi'-\lambda'}{r}+\frac{1-e^{2\lambda}}{r^2}\right)$$

$$\frac{-e^{4\lambda}G}{8}=\frac{1}{r^2}\left[(\phi''+\phi'^2-\lambda'\phi')\left(e^{2\lambda}-1\right)+2\phi'\lambda'\right]$$

$$\nabla_\mu T_2^\mu=\partial_\mu T_2^\mu+\Gamma_{\mu\sigma}^\mu T_2^\sigma-\Gamma_{\mu2}^\sigma T_\sigma^\mu=0,\partial_\mu(-p\delta^{\mu,2})-p\Gamma_{\mu2}^\mu+p\Gamma_{a2}^a-\rho c^2\Gamma_{12}^1=0$$
$$-\frac{dp}{dr}-(p+\rho c^2)\Gamma_{12}^1=0\Longrightarrow\frac{dp}{dr}=-(p+\rho c^2)\phi'$$

4. Modelo Klein – Gordon para espacios cuánticos relativistas.

$$\psi_{tt}-\psi_{xx}+\mathrm{m}\psi+a^{(2)}(\omega t,x)\psi_{xx}+a^{(1)}(\omega t,x)\psi_x+a^{(0)}(\omega t,x)\psi=0,$$

$$a^{(i)}:\mathbb{T}_\varphi^\nu\times\mathbb{T}_x\to\mathbb{R},(\varphi,x)\mapsto a^{(i)}(\varphi,x),i=0,1,2$$

$$a^{(i)}(\varphi,-x)=a^{(i)}(\varphi,x),i=0,2,\qquad a^{(1)}(\varphi,-x)=-a^{(1)}(\varphi,x),$$
$$a^{(i)}(-\varphi,x)=a^{(i)}(\varphi,x),i=0,1,2,$$

$$\mathfrak{d}:=\max_{i=0,1,2}\left\|a^{(i)}\right\|_{H^{\bar{s}}(\mathbb{T}^{\nu+1},\mathbb{R})}\leq\mathfrak{d}_0(s)$$

$$\begin{cases}\psi_{tt}-\psi_{xx}+m\psi+a^{(2)}(\omega t,x)\psi_{xx}+a^{(1)}(\omega t,x)\psi_x+a^{(0)}(\omega t,x)\psi=0\\ \psi(0,x)=\psi_0(x)\\ \partial_t\psi(0,x)=v_0(x)\end{cases}$$

$$\sup_{t\in\mathbb{R}}\left(\|\psi(t,\cdot)\|_{H^{s+1}(\mathbb{T},\mathbb{R})}+\|(\partial_t\psi)(t,\cdot)\|_{H^s(\mathbb{T},\mathbb{R})}\right)\leq C(s)\left(\|\psi_0\|_{H^{s+1}(\mathbb{T},\mathbb{R})}+\|v_0\|_{H^s(\mathbb{T},\mathbb{R})}\right)$$

$$\partial_t \begin{bmatrix}\psi\\ v\end{bmatrix} = X(\omega t)\begin{bmatrix}\psi\\ v\end{bmatrix}$$

$$X(\omega t) := \begin{pmatrix} 0 & 1 \\ -\mathrm{D}_{\mathrm{m}}^2 - a^{(2)}(\omega t,x)\partial_{xx} - a^{(1)}(\omega t,x)\partial_x - a^{(0)}(\omega t,x) & 0\end{pmatrix}$$

$$\mathrm{D}_{\mathrm{m}} e^{\mathrm{i}j\cdot x} = \mathrm{D}_{\mathrm{m}}(j)e^{\mathrm{i}j\cdot x}, \mathrm{D}_{\mathrm{m}}(j) := \sqrt{j^2+\mathrm{m}}, \forall j \in \mathbb{Z}$$

$$\left(\mathcal{P}\begin{bmatrix}\psi\\ v\end{bmatrix}\right)(x) := \begin{bmatrix}\psi(-x)\\ v(-x)\end{bmatrix}, \mathcal{P}^2 = \mathbb{I} := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$X(\omega t)\circ\mathcal{P} = \mathcal{P}\circ X(\omega t), \forall t \in \mathbb{R}$$

$$E \circ X(\omega t) = -X(-\omega t)\circ E$$

$$E\begin{bmatrix}\psi\\ v\end{bmatrix} = \begin{bmatrix}\psi\\ -v\end{bmatrix}, E := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, E^2 = \mathbb{I}$$

$$\mathcal{X}_{\mathbb{R}}^s := H^{s+1}(\mathbb{T},\mathbb{R})\times H^s(\mathbb{T},\mathbb{R})$$

$$\mathcal{X}_{odd,\mathbb{R}}^s := \mathcal{X}_{\mathbb{R}}^s \cap \left\{\begin{bmatrix}\psi(-x)\\ v(-x)\end{bmatrix} = -\begin{bmatrix}\psi(x)\\ v(x)\end{bmatrix}\right\}, \mathcal{X}_{\mathrm{even},\mathbb{R}}^s := \mathcal{X}_{\mathbb{R}}^s \cap \left\{\begin{bmatrix}\psi(-x)\\ v(-x)\end{bmatrix} = \begin{bmatrix}\psi(x)\\ v(x)\end{bmatrix}\right\}$$

$$\begin{bmatrix}u\\ \bar{u}\end{bmatrix} := \mathcal{C}\begin{bmatrix}\psi\\ v\end{bmatrix} \Leftrightarrow \begin{bmatrix}\psi\\ v\end{bmatrix} = \mathcal{C}^{-1}\begin{bmatrix}u\\ \bar{u}\end{bmatrix}$$

$$\mathcal{C} := \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{D}_{\mathrm{m}} & -\mathrm{i}\\ \mathrm{D}_{\mathrm{m}} & \mathrm{i}\end{pmatrix}, \mathcal{C}^{-1} := \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{D}_{\mathrm{m}}^{-1} & \mathrm{D}_{\mathrm{m}}^{-1}\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}$$

$$\partial_t U = \mathrm{i}E\mathfrak{D}(\omega t)U, U := \begin{bmatrix}u\\ \bar{u}\end{bmatrix}$$

$$\mathfrak{D}(\omega t) := \mathrm{D}_{\mathrm{m}}\mathbb{I} + 1\left(\frac{1}{2}a^{(2)}(\omega t,x)\partial_{xx} + \frac{1}{2}a^{(1)}(\omega t,x)\partial_x + \frac{1}{2}a^{(0)}(\omega t,x)\right)\mathrm{D}_{\mathrm{m}}^{-1}$$

$$\mathbf{1} := \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

$$\mathcal{C}: \mathcal{X}_{\mathbb{R}}^s \to \mathcal{H}^s, \mathcal{C}: \mathcal{X}_{p,\mathbb{R}}^s \to \mathcal{H}_p^s, p \in \{\text{ odd , even }\}$$

$$\mathcal{H}^s := \left\{U = \begin{bmatrix}u^+\\ u^-\end{bmatrix} \in H^s(\mathbb{T};\mathbb{C}^2): \overline{u^+} = u^-\right\}$$
$$\mathcal{H}_{odd}^s := \mathcal{H}^s \cap \{U(-x) = -U(x)\}, \mathcal{H}_{\mathrm{even}}^s := \mathcal{H}^s \cap \{U(-x) = U(x)\}.$$

$$S := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathcal{C}\circ E\circ \mathcal{C}^{-1}, S\begin{bmatrix}u\\ \bar{u}\end{bmatrix} = \begin{bmatrix}\bar{u}\\ u\end{bmatrix}$$

$$\gamma^{-7/2}\left\|a^{(i)}\right\|_{H^{s_0+\mu}(\mathbb{T}^{\nu+1},\mathbb{R})} \leq \delta_0(s_1), i = 0,1,2$$

$$|\Lambda \setminus \mathcal{G}_\infty(\gamma)| \to 0 \text{ as } \gamma \to 0$$

$$\|\mathfrak{F}(\varphi)h\|_{H^s(\mathbb{T},\mathbb{C}^2)} \leq C(s)\|h\|_{H^s(\mathbb{T},\mathbb{C}^2)} + \frac{C(s)}{\gamma^{7/2}}\sup_{i=0,1,2}\left\|a^{(i)}\right\|_{H^{s+\mu}(\mathbb{T}^{\nu+1},\mathbb{R})}\|h\|_{H^{s_0}(\mathbb{T},\mathbb{C}^2)}$$

$$\|\mathfrak{F}(\varphi)w\|_{H^s(\mathbb{T}^{\nu+1},\mathbb{C}^2)} \leq C(s)\|w\|_{H^s(\mathbb{T}^{\nu+1},\mathbb{C}^2)} + \frac{C(s)}{\gamma^{7/2}}\sup_{i=0,1,2}\left\|a^{(i)}\right\|_{H^{s+\mu}(\mathbb{T}^{\nu+1},\mathbb{R})}\|w\|_{H^{s_0}(\mathbb{T}^{\nu+1},\mathbb{C}^2)}$$

$$\mathfrak{r}_j^{\sigma j} = \mathfrak{r}_{-j}^{-\sigma j}, \forall j \in \mathbb{N}_0, \sigma = \pm$$

$$\sup_{\omega\in\Lambda}|\mathfrak{c}(\omega)|+\gamma\sup_{\omega_1\neq\omega_2}\frac{|\mathfrak{c}(\omega_1)-\mathfrak{c}(\omega_2)|}{|\omega_1-\omega_2|}\leq C\sup_{i=0,1,2}\left\|a^{(i)}\right\|_{H^{s_0+\mu}(\mathbb{T}^{\nu+1},\mathbb{R})}$$

$$\sup_{j\in\mathbb{N}_0,\sigma=\pm}\left(\sup_{\omega\in\Lambda}\left|\mathfrak{r}_j^{\sigma j}(\omega)\right|+\gamma^{3/2}\sup_{\omega_1\neq\omega_2}\frac{\left|\mathfrak{r}_j^{\sigma j}(\omega_1)-\mathfrak{r}_j^{\sigma j}(\omega_2)\right|}{|\omega_1-\omega_2|}\right)\leq\frac{C}{\gamma^2}\sup_{i=0,1,2}\left\|a^{(i)}\right\|_{H^{s_0+\mu}(\mathbb{T}^{\nu+1},\mathbb{R})}$$

$$\partial_t Z=\begin{pmatrix}\mathrm{i}\mathfrak{D}_\infty^+ z\\ -\mathrm{i}\mathfrak{D}_\infty^+\bar{z}\end{pmatrix}$$

$$\mathfrak{D}_\infty^+ z:=\left((1+\mathfrak{c})\sqrt{\mathrm{m}}+\mathfrak{r}_0^0\right)z_0+\sum_{j\in\mathbb{Z}\setminus\{0\}}e^{\mathrm{i}jx}\left(\left((1+\mathfrak{c})\mathrm{D}_\mathrm{m}(j)+\frac{\mathfrak{r}_j^j}{\langle j\rangle}\right)z_j+\frac{\mathfrak{r}_j^{-j}}{\langle j\rangle}z_{-j}\right).$$

$$\lambda_{0,\pm}^{(\infty)}:=\lambda_{0,\pm}^{(\infty)}(\omega)=(1+\mathfrak{c})\sqrt{\mathrm{m}}+\mathfrak{r}_0^0,$$

$$\lambda_{j,\pm}^{(\infty)}:=\lambda_{j,\pm}^{(\infty)}(\omega)=(1+\mathfrak{c})\mathrm{D}_\mathrm{m}(j)+\frac{\mathfrak{r}_j^j\pm\mathfrak{r}_j^{-j}}{\langle j\rangle},\forall j\in\mathbb{N},$$

$$\Lambda_0:=\{\omega\in\Lambda:|\omega\cdot\ell|\geq 2\gamma|\ell|^{-\nu},\ell\in\mathbb{Z}^\nu\setminus\{0\}\}$$
$$\Lambda_1:=\{\omega\in\Lambda:|\omega\cdot\ell-(1+\mathfrak{c})j|\geq 2\gamma\langle\ell\rangle^{-\tau},j\in\mathbb{N},\ell\in\mathbb{Z}^\nu\}$$
$$\Lambda_2^+:=\left\{\omega\in\Lambda:\left|\omega\cdot\ell+\lambda_{j,\eta}^{(\infty)}+\lambda_{k,\eta}^{(\infty)}\right|\geq\frac{2\gamma}{\langle\ell\rangle^\tau},j,k\in\mathbb{N}_0,\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\right\}$$
$$\Lambda_2^-:=\left\{\omega\in\Lambda:\left|\omega\cdot\ell+\lambda_{j,\eta}^{(\infty)}-\lambda_{k,\eta}^{(\infty)}\right|\geq\frac{2\gamma^{3/2}}{\langle\ell\rangle^\tau},\eta\in\{\pm\}\right.$$
$$\left.j,k\in\mathbb{N}_0,\ell\in\mathbb{Z}^\nu,(\ell,j,k)\neq(0,j,j)\right\}$$

$$\Lambda_0\cap\Lambda_1\cap\Lambda_2^+\cap\Lambda_2^-\subseteq\mathcal{G}_\infty(\gamma)\ \text{ and }\ |(\Lambda_0\cap\Lambda_1\cap\Lambda_2^+\cap\Lambda_2^-)^c|\leq C\gamma$$

$$z(t)=z_0(0)e^{\mathrm{i}\lambda_{0,+}^{(\infty)}t}+\sum_{j\in\mathbb{N}}w_j^{(\mathrm{ev})}(0)e^{\mathrm{i}\lambda_{j,+}^{(\infty)}t}\cos(jx)+w_j^{(\mathrm{odd})}(0)e^{\mathrm{i}\lambda_{j,-}^{(\infty)}t}\sin(jx)$$

$$\mathfrak{F}(\omega t)\circ\mathrm{i}E\mathfrak{D}(\omega t)\circ\mathfrak{F}^{-1}(\omega t)+(\partial_t\mathfrak{F}(\omega t))\circ\mathfrak{F}^{-1}(\omega t)=\mathrm{i}E\begin{pmatrix}\mathfrak{D}_\infty^+&0\\0&\mathfrak{D}_\infty^+\end{pmatrix}$$

$$(\mathfrak{A}U)(\varphi,x):=\mathfrak{A}(\varphi)U(\varphi,x),\forall U(\varphi,x)\in H^s(\mathbb{T}^{\nu+1},\mathbb{C}^2)$$

$$\mathcal{L}:=\omega\cdot\partial_\varphi-\mathrm{i}E\mathfrak{D}$$

$$\mathfrak{F}\circ\mathcal{L}\circ\mathfrak{F}^{-1}-\omega\cdot\partial_\varphi=\mathfrak{F}\circ\mathrm{i}E\mathfrak{D}\circ\mathfrak{F}^{-1}+\left(\omega\cdot\partial_\varphi\mathfrak{F}\right)\circ\mathfrak{F}^{-1}$$

$$-\mathrm{i}E\begin{pmatrix}\mathfrak{D}_\infty^+&0\\0&\mathfrak{D}_\infty^+\end{pmatrix}$$

$$\mathcal{L}=\omega\cdot\partial_\varphi-\mathrm{i}E\begin{pmatrix}1+b_1&b_1\\b_1&1+b_1\end{pmatrix}\mathrm{D}_\mathrm{m}+\mathbf{O}(\mathfrak{d}\partial_x^0)$$

$$\mathcal{L}_3=\omega\cdot\partial_\varphi-\mathrm{i}\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}\mathrm{D}_\mathrm{m}+\begin{pmatrix}O(\mathfrak{d}\partial_x^0)&0\\0&O(\mathfrak{d}\partial_x^0)\end{pmatrix}+\mathbf{O}\left(\mathfrak{d}\partial_x^{-\rho}\right),\rho\gg 1,$$

$$\mathrm{i}\mathrm{D}_\mathrm{m}=\mathrm{i}|\partial_x|+O(\partial_x^{-1})=\Pi_+\partial_x-\Pi_-\partial_x+O(\partial_x^{-1})$$

$$\omega\cdot\partial_\varphi-\lambda(\Pi_+\partial_x-\Pi_-\partial_x)=\left(\omega\cdot\partial_\varphi-\lambda\partial_x\right)\Pi_++\left(\omega\cdot\partial_\varphi+\lambda\partial_x\right)\Pi_-$$

$$L:=\mathcal{C}_{\alpha_+}\Pi_++\mathcal{C}_{\alpha_-}\Pi_-,$$

$$\omega\cdot\partial_\varphi-\mathrm{i}(1+\mathfrak{c})|\partial_x|+O\left(\mathfrak{d}\partial_x^{-\rho}\right)$$

$$\|\mathrm{A}\|_s := \sup_{s_* \le p \le s_1} |M|_{p,p} + |R|_{s_*,s}$$

$$\mathcal{L}_4 := \Theta_1 \mathcal{L}_3 \Theta_1^{-1} = \omega \cdot \partial_\varphi - \mathrm{i}\begin{pmatrix} 1+\mathfrak{c} & 0 \\ 0 & -(1+\mathfrak{c}) \end{pmatrix} \mathrm{D_m} + \begin{pmatrix} O(\mathfrak{d}\partial_x^0) & 0 \\ 0 & O(\mathfrak{d}\partial_x^0) \end{pmatrix} + \mathbf{O}(\mathfrak{d}\partial_x^{-1})$$

$$\mathbb{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathfrak{U} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\nu \in \mathbb{N}, s_* > (\nu+5)/2, s_0 := s_* + 1, \tau > 2\nu + 4.$$

$$s_1 > s_0, \gamma \in \left(0, \frac{1}{2}\right)$$

$$u(\varphi, x) = \sum_{j \in \mathbb{Z}} u_j(\varphi) e^{\mathrm{i}jx} = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} u_{\ell,j} e^{\mathrm{i}(\ell \cdot \varphi + jx)}$$

$$u_{\ell,j} := u_{\ell,j} := \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} u(\varphi, x) e^{-\mathrm{i}(\ell\cdot\varphi + jx)} d\varphi dx$$

$$H^s := H^s(\mathbb{T}^{\nu+1}) := H^s(\mathbb{T}^{\nu+1}, \mathbb{C})$$

$$:= \left\{ u(\varphi, x) \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}) : \|u\|_s^2 := \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} \langle \ell, j \rangle^{2s} |u_{\ell,j}|^2 < \infty \right\}$$

$$\|\underline{u}\|_s = \|u\|_s$$

$$\|u\|_s^{\gamma,\mathcal{O}} := \|u\|_s^{\sup} + \gamma \|u\|_{s-1}^{\mathrm{lip}} := \sup_{\omega \in \mathcal{O}} \|u(\omega)\|_s + \gamma \sup_{\substack{\omega_1 \neq \omega_2 \\ \omega_1, \omega_2 \in \mathcal{O}}} \frac{\|u(\omega_1) - u(\omega_2)\|_{s-1}}{|\omega_1 - \omega_2|}$$

$$\|uv\|_s^{\gamma,\mathcal{O}} \lesssim_s \|u\|_s^{\gamma,\mathcal{O}} \|v\|_{s_0}^{\gamma,\mathcal{O}} + \|u\|_{s_0}^{\gamma,\mathcal{O}} \|v\|_s^{\gamma,\mathcal{O}}$$

$$\|u\|_{a_0+p}^{\gamma,\mathcal{O}} \|v\|_{b_0+q}^{\gamma,\mathcal{O}} \le \|u\|_{a_0+p+q}^{\gamma,\mathcal{O}} \|v\|_{b_0}^{\gamma,\mathcal{O}} + \|u\|_{a_0}^{\gamma,\mathcal{O}} \|v\|_{b_0+p+q}^{\gamma,\mathcal{O}}$$

$$\mathcal{A} := \left(\mathcal{A}_{j,\ell}^{j',\ell'}\right)_{\substack{j,j' \in \mathbb{Z} \\ \ell,\ell' \in \mathbb{Z}^\nu}}, \mathcal{A}_{j,\ell}^{j',\ell'} := \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} \mathcal{A}\left[e^{\mathrm{i}(\ell'\cdot\varphi + j'x)}\right] \cdot e^{-\mathrm{i}(\ell\cdot\varphi + jx)} d\varphi dx$$

$$\mathcal{A}_{(j,\ell)}^{(j',\ell')} = A_j^{j'}(\ell - \ell')$$

$$A : \mathbb{T}^\nu \to \mathcal{L}(L^2(\mathbb{T}, \mathbb{C})), \varphi \mapsto A(\varphi),$$

$$(Au)(\varphi, x) = (A(\varphi)u(\varphi, \cdot))(x) = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} \left(\sum_{\ell' \in \mathbb{Z}^\nu, j' \in \mathbb{Z}} A_j^{j'}(\ell - \ell') u_{\ell',j'} \right) e^{\mathrm{i}(\ell\cdot\varphi + jx)}$$

$$\mathcal{T} := \left(\mathcal{T}_\sigma^{\sigma'}\right)_{\sigma,\sigma'} := \begin{pmatrix} \mathcal{T}_+^+ & \mathcal{T}_+^- \\ \mathcal{T}_-^+ & \mathcal{T}_-^- \end{pmatrix}, \sigma, \sigma' \in \{\pm\}$$

$$u^\sigma(\varphi, x) = \sum_{j \in \mathbb{Z}} u_j^\sigma(\varphi) e^{\sigma \mathrm{i} jx} = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} u_{\ell,j}^\sigma e^{\mathrm{i}(\ell\cdot\varphi + \sigma jx)}, \sigma \in \{\pm\},$$

$$\mathcal{T}\begin{bmatrix} u^+ \\ u^- \end{bmatrix} = \begin{pmatrix} \mathcal{T}_+^+ u^+ + \mathcal{T}_+^- u^- \\ \mathcal{T}_-^+ u^+ + \mathcal{T}_-^- u^- \end{pmatrix}$$

$$\left(\mathcal{T}_{\sigma,j}^{\sigma',j'}(\ell)\right)_{\ell \in \mathbb{Z}^\nu, j,j' \in \mathbb{Z}, \sigma,\sigma' = \pm}$$

$$\mathcal{T}_{\sigma,j}^{\sigma',j'}(\ell) := \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} \mathcal{T}_{\sigma}^{\sigma'}\left[e^{\mathrm{i}\sigma' j' x}\right] e^{-\mathrm{i}(\sigma j x + \ell\cdot\varphi)} d\varphi dx$$

$$\|A\|_{s,s'} := \sup_{\|u\|_s \leq 1} \|Au\|_{s'} = \sup_{\|u\|_s \leq 1} \left(\sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s'} \left| \sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} A_j^{j'}(\ell-\ell') u_{\ell',j'} \right|^2 \right)^{1/2}$$

$$\underline{A} := \left(\underline{A}_j^{j'}(\ell)\right)_{\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{Z}}, \underline{A}_j^{j'}(\ell) := \left|A_j^{j'}(\ell)\right|, \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{Z}$$

$$|A|_{s,s'} := \|\underline{A}\|_{s,s'}$$

$$\mathcal{M}^{\mathrm{T}}\left(H^s, H^{s'}\right) := \left\{A \in \mathcal{L}^{\mathrm{T}}\left(H^s, H^{s'}\right) \text{ s.t. } |A|_{s,s'} < \infty\right\}$$

$$A \preceq B \;\Leftrightarrow\; \underline{A} \leq \underline{B} \;\Leftrightarrow\; \left|A_j^{j'}(\ell)\right| \leq \left|B_j^{j'}(\ell)\right|, \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{Z}$$
$$\text{so that } A \preceq B \implies |A|_{s,s'} \leq |B|_{s,s'}$$

$$\|AB\|_{s,s'} \leq \|A\|_{s_1,s'} \|B\|_{s,s_1}, |AB|_{s,s'} \leq |A|_{s_1,s'} |B|_{s,s_1}$$

$$\Delta_{12} A := \frac{A(\omega_1) - A(\omega_2)}{|\omega_1 - \omega_2|}$$

$$|A|_{s,s'}^{\gamma,\mathcal{O}} := \sup_{\omega\in\mathcal{O}} |A(\omega)|_{s,s'} + \gamma \sup_{\substack{\omega_1\neq\omega_2 \\ \omega_1,\omega_2\in\mathcal{O}}} |\Delta_{12} A|_{s,s'}$$
$$\|A\|_{s,s'}^{\gamma,\mathcal{O}} := \sup_{\omega\in\mathcal{O}} \|A(\omega)\|_{s,s'} + \gamma \sup_{\substack{\omega_1\neq\omega_2 \\ \omega_1,\omega_2\in\mathcal{O}}} \|\Delta_{12} A\|_{s,s'}$$

$$|T|_{s,s'}^{\gamma,\mathcal{O}} := \max_{\sigma,\sigma'\in\{\pm\}} \left\{ \left|T_\sigma^{\sigma'}\right|_{s,s'}^{\gamma,\mathcal{O}} \right\} \left(\text{resp. } \|T\|_{s,s'}^{\gamma,\mathcal{O}} := \max_{\sigma,\sigma'\in\{\pm\}} \left\{ \left\|T_\sigma^{\sigma'}\right\|_{s,s'}^{\gamma,\mathcal{O}} \right\} \right).$$

$$\|AB\|_{s,s'}^{\gamma,\mathcal{O}} \leq \|A\|_{s_1,s'}^{\gamma,\mathcal{O}} \|B\|_{s,s_1}^{\gamma,\mathcal{O}}, \|Au\|_{s'}^{\gamma,\mathcal{O}} \leq \left(\|A\|_{s,s'}^{\gamma,\mathcal{O}} + \|A\|_{s-1,s'-1}^{\gamma,\mathcal{O}} \right) \|u\|_s^{\gamma,\mathcal{O}}$$

$$\left(\mathrm{d}_{\varphi_h} A\right)_j^{j'}(\ell) := \mathrm{i}\ell_h A_j^{j'}(\ell), \left(\mathrm{d}_x A\right)_j^{j'}(\ell) := \mathrm{i}(j - j') A_j^{j'}(\ell)$$

$$\left(\langle \mathrm{d}_\varphi \rangle A\right)_j^{j'}(\ell) := \langle \ell \rangle A_j^{j'}(\ell) \;\text{ and }\; \mathrm{d}_\varphi^p A := \prod_{h=1}^{\nu} \mathrm{d}_{\varphi_h}^{p_h} A, \forall p \in \mathbb{N}_0^\nu$$

$$(\Pi_N T)_{\sigma,j}^{\sigma',k}(\ell) := \begin{cases} T_{\sigma,j}^{\sigma',k}(\ell) & \text{if } |\ell| \leq N \\ 0 & \text{otherwise} \end{cases} \quad \Pi_N^\perp := \mathrm{Id} - \Pi_N$$

$$\mathrm{ad}_A[B] := [A, B] \equiv A \circ B - B \circ A$$

$$\underline{M} \preceq \underline{\langle \mathrm{d}_{\varphi_h} \rangle M}, \underline{\langle \mathrm{d}_\varphi \rangle M} \preceq \underline{M} + \sum_{h=1}^{\nu} \underline{\mathrm{d}_{\varphi_h} M}$$
$$\underline{\langle \mathrm{d}_\varphi \rangle^{\mathrm{b}} M} \preceq_{\mathrm{b},\nu} \underline{M} + \sup_{\substack{\beta\in\mathbb{N}_0^\nu \\ |\beta|\leq \mathrm{b}}} \prod_{h=1}^{\nu} \mathrm{d}_{\varphi_h}^{\beta_h} M \preceq_{\mathrm{b},\nu} \underline{M} + \sum_{h=1}^{\nu} \underline{\mathrm{d}_{\varphi_h}^{\mathrm{b}} M}$$

$$|A|_{s,s}^{\gamma,\mathcal{O}} \lesssim_s \|A\|_{s,s}^{\gamma,\mathcal{O}} + \|\mathrm{d}_x A\|_{s,s}^{\gamma,\mathcal{O}} + \max_{1\leq h\leq \nu} \left\{ \left\|\mathrm{d}_{\varphi_h}^{\beta} A\right\|_{s,s}^{\gamma,\mathcal{O}}, \left\|\mathrm{d}_x\, \mathrm{d}_{\varphi_h}^{\beta} A\right\|_{s,s}^{\gamma,\mathcal{O}} \right\}$$

$$
\begin{aligned}
\|\underline{A}u\|_s^2 &\le \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s} \left(\sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} \left|A_j^{j'}(\ell-\ell')\right| \left|u_{\ell',j'}\right| \right)^2 \\
&\lesssim \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s} \left(\sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} \langle \ell-\ell'\rangle^{2\beta}\langle j-j'\rangle^{2} \left|A_j^{j'}(\ell-\ell')\right|^2 \left|u_{\ell',j'}\right|^2 \right) \\
&= \sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} \left|u_{\ell',j'}\right|^2 \left(\sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s} \langle \ell-\ell'\rangle^{2\beta}\langle j-j'\rangle^{2} \left|A_j^{j'}(\ell-\ell')\right|^2 \right).
\end{aligned}
$$

$$
\langle \ell-\ell'\rangle^{2\beta}\langle j-j'\rangle^{2} \lesssim_\beta 1 + |j-j'|^2 + \sum_{h=1}^{\nu} |\ell_h-\ell'_h|^{2\beta} + |j-j'|^2 \sum_{h=1}^{\nu} |\ell_h-\ell'_h|^{2\beta}
$$

$$
\sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s} \left|A_j^{j'}(\ell-\ell')\right|^2 \le \|A\|_{s,s}^2 \langle \ell', j'\rangle^{2s}
$$

$$
\sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell, j\rangle^{2s} \langle \ell-\ell'\rangle^{2\beta}\langle j-j'\rangle^{2} \left|A_j^{j'}(\ell-\ell')\right|^2 \lesssim \beta
$$

$$
\left(\|A\|_{s,s}^2 + \|\mathrm{d}_x A\|_{s,s}^2 + \max_{h=1,\dots,\nu}\left\{\left\| \mathrm{d}_{\varphi_h}^{\beta} A\right\|^2, \left\| \mathrm{d}_x\, \mathrm{d}_{\varphi_h}^{\beta} A\right\|^2\right\}\right) \langle \ell', j'\rangle^{2s}
$$

$$
\|\underline{A}u\|_s^2 \lesssim_\beta \left(\|A\|_{s,s}^2 + \|\mathrm{d}_x A\|_{s,s}^2 + \max_{h=1,\dots,\nu}\left\{\left\| \mathrm{d}_{\varphi_h}^{\beta} A\right\|_{s,s}^2, \left\| \mathrm{d}_{\varphi_h}^{\beta} \mathrm{d}_x A\right\|_{s,s}^2\right\}\right) \|u\|_s^2
$$

$$
\begin{gathered}
\mathcal{U} := \{(u^+, u^-) \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}^2) : \overline{u^-} = u^+\} \\
\mathbf{X} := (X \times X) \cap \mathcal{U}, X := \left\{u \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}) : u(\varphi, x) = \overline{u(-\varphi, x)}\right\} \\
\mathbf{Y} := (Y \times Y) \cap \mathcal{U}, Y := \left\{u \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}) : u(\varphi, x) = \overline{u(-\varphi, x)}\right\} \\
O := \{u \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}) : u(\varphi, x) = -u(\varphi, -x)\} \\
P := \{u \in L^2(\mathbb{T}^{\nu+1}, \mathbb{C}) : u(\varphi, x) = u(\varphi, -x)\}
\end{gathered}
$$

$$
A := \left(A_\sigma^{\sigma'}\right)_{\sigma,\sigma'\in\{\pm\}} := \begin{pmatrix} A_+^+ & A_+^- \\ A_-^+ & A_-^- \end{pmatrix} \in \mathcal{L}^{\mathrm{T}}\left(H^s, H^{s'}\right) \otimes \mathcal{M}_2(\mathbb{C})
$$

$$
\bar{B}[h] := \overline{B[\bar{h}]}
$$

$$
(\bar{B})_j^{j'}(\ell) = \overline{B_{-j}^{-j'}(-\ell)}, \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{Z}
$$

$$
\begin{aligned}
&\text{reversible} \quad \Leftrightarrow B(-\varphi) = -\bar{B}(\varphi), \forall \varphi \in \mathbb{T}^\nu \\
&\text{reversibility preserving} \quad \Leftrightarrow B(-\varphi) = \bar{B}(\varphi), \forall \varphi \in \mathbb{T}^\nu \\
&\text{parity preserving} \quad \Leftrightarrow B_j^k(\varphi) = B_{-j}^{-k}(\varphi), \forall \varphi \in \mathbb{T}^\nu, j, k \in \mathbb{Z}
\end{aligned}
$$

$$
A = A(\varphi) = \left(A_\sigma^{\sigma'}\right)_{\sigma,\sigma'\in\{\pm\}} \in \mathcal{L}^{\mathrm{T}}\left(H^s, H^{s'}\right) \otimes \mathcal{M}_2(\mathbb{C})
$$

$$
\begin{aligned}
&\text{real to real} \quad \Leftrightarrow \bar{A}(\varphi) S = S A(\varphi), S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \forall \varphi \in \mathbb{T}^\nu \\
&\text{reversible} \quad \Leftrightarrow A(-\varphi) = -S A(\varphi) S \overset{\sqrt{-g}}{=} -\bar{A}(\varphi), \forall \varphi \in \mathbb{T}^\nu \\
&\text{reversibility pres.} \quad \Leftrightarrow A(-\varphi) = S A(\varphi) S \overset{\sqrt{-g}}{=} \bar{A}(\varphi) \\
&\text{parity pres.} \quad \Leftrightarrow A_{\sigma,j}^{\sigma',k}(\varphi) = A_{\sigma,-j}^{\sigma',-k}(\varphi), \forall \varphi \in \mathbb{T}^\nu, j, k \in \mathbb{Z}, \sigma, \sigma' \in \{\pm\}
\end{aligned}
$$

$$\text{real - to - real} \;\Leftrightarrow\; A_{\sigma,j}^{\sigma',k}(\ell) = \overline{A_{-\sigma,-j}^{-\sigma',-k}(-\ell)}$$

$$\text{reversible} \;\Leftrightarrow\; A_{\sigma,j}^{\sigma',k}(\ell) = -A_{-\sigma,j}^{-\sigma',k}(-\ell) \overset{\sqrt{-g}}{=} -\overline{A_{\sigma,-j}^{\sigma',-k}(\ell)}$$

$$\text{revers.pres.} \;\Leftrightarrow\; A_{\sigma,j}^{\sigma',k}(\ell) = A_{-\sigma,j}^{-\sigma',k}(-\ell) \overset{\sqrt{-g}}{=} \overline{A_{\sigma,-j}^{\sigma',-k}(\ell)}$$

$$A = A(\varphi) = \left(A_{\sigma}^{\sigma'}(\varphi)\right)_{\sigma,\sigma'\in\{\pm\}} \in \mathcal{L}^{\mathrm{T}}\left(H^{s}, H^{s'}\right) \otimes \mathcal{M}_2(\mathbb{C})$$

$$\begin{aligned} A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) &:= \begin{pmatrix} A_{\sigma,j}^{\sigma',k}(\ell) & A_{\sigma,j}^{\sigma',-k}(\ell) \\ A_{\sigma,-j}^{\sigma',k}(\ell) & A_{\sigma,-j}^{\sigma',-k}(\ell) \end{pmatrix}, j,k \in \mathbb{N}, \\ A_{\sigma,\vec{0}}^{\sigma',\vec{k}}(\ell) &:= \left(A_{\sigma,0}^{\sigma',k}(\ell) \quad A_{\sigma,0}^{\sigma',-k}(\ell)\right), j = 0, k \in \mathbb{N}, \\ A_{\sigma,\vec{j}}^{\sigma',\vec{0}}(\ell) &:= \left(A_{\sigma,j}^{\sigma',0}(\ell) \quad A_{\sigma,-j}^{\sigma',0}(\ell)\right)^{T}, j \in \mathbb{N}, k = 0, \\ A_{\sigma,\vec{0}}^{\sigma',\vec{0}}(\ell) &:= \left(A_{\sigma,0}^{\sigma',0}(\ell)\right), j = k = 0. \end{aligned}$$

$$A(\varphi) = \left(A_{\sigma,j}^{\sigma',k}(\ell)\right)_{\sigma,\sigma'\in\{\pm\},\ell\in\mathbb{Z}^{\nu},j,k\in\mathbb{Z}} \overset{\sqrt{2.53}}{=} \left(A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\right)_{\substack{\sigma,\sigma'\in\{\pm\} \\ \ell\in\mathbb{Z}^{\nu},j,k\in\mathbb{N}_0}}$$

$$\frac{1}{4}|\bar{A}_{s,s'} \le |A|_{s,s'} \le |\bar{A}_{s,s'}$$

$$\bar{A}_{\sigma,\vec{j}}^{\sigma'\vec{k}}(\ell) := \left\|A_{\sigma,\vec{j}}^{\sigma'\vec{k}}(\ell)\right\|_{\infty} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, j,k \in \mathbb{N}, \bar{A}_{\sigma,\vec{0}}^{\sigma'\vec{k}}(\ell) := \left\|A_{\sigma,\vec{0}}^{\sigma'\vec{k}}(\ell)\right\|_{\infty} (1 \quad 1), j = 0, k \in \mathbb{N},$$

$$\left|\overline{A}\,\right|_{s,s'} = \sup_{\substack{\|u\|_s \le 1 \\ u_j^{\sigma} = u_{-j}^{\sigma} \ge 0 \forall j\in\mathbb{N}_0, \sigma\in\{\pm\}}} \|\underline{\tilde{A}u}\|_{s'} \le 4 \sup_{\substack{\|u\|_s \le 1 \\ u_j^{\sigma} = u_{-j}^{\sigma} \ge 0 \forall j\in\mathbb{N}_0, \sigma\in\{\pm\}}} \|\underline{A}u\|_{s'} \le 4 \left|\left| A \right|\right|_{s,s'}$$

$$\begin{aligned} (\underline{\tilde{A}u})_j^{\sigma} + (\underline{\tilde{A}u})_{-j}^{\sigma} &= 2 \sum_{j'\in\mathbb{N},\sigma'\in\{\pm\}} \left\|A_{\sigma,\vec{j}}^{\sigma',\vec{j}'}\right\|_{\infty} \left(\left|u_{j'}^{\sigma'}\right| + \left|u_{-j'}^{\sigma'}\right|\right) + 2 \sum_{\sigma'\in\{\pm\}} \left\|A_{\sigma,\vec{j}}^{\sigma',\vec{0}'}\right\|_{\infty} \left|u_0^{\sigma'}\right| \\ &= 4 \sum_{j'\in\mathbb{N},\sigma'\in\{\pm\}} \left\|A_{\sigma,\vec{j}}^{\sigma',\vec{j}'}\right\|_{\infty} z_{j'}^{\sigma'} + 2 \sum_{\sigma'\in\{\pm\}} \left\|A_{\sigma,\vec{j}}^{\sigma',\vec{0}'}\right\|_{\infty} \left|u_0^{\sigma'}\right| \\ &= (\underline{\tilde{A}}z)_j^{\sigma} + (\underline{\tilde{A}}z)_{-j}^{\sigma} \end{aligned}$$

$$\begin{aligned} (\underline{A}u)_j^{\sigma} + (\underline{A}u)_{-j}^{\sigma} = &\sum_{j'\in\mathbb{N},\sigma'\in\{\pm\}} \left(\left|A_{\sigma,j}^{\sigma',j'}\right| + \left|A_{\sigma,-j}^{\sigma',j'}\right|\right) u_{j'}^{\sigma'} + \left(\left|A_{\sigma,j}^{\sigma',-j'}\right| + \left|A_{\sigma,-j}^{\sigma',-j'}\right|\right) u_{-j'}^{\sigma'} \\ &+ \sum_{\sigma'\in\{\pm\}} \left(\left|A_{\sigma,j}^{\sigma',0}\right| + \left|A_{\sigma,-j}^{\sigma',0}\right|\right) u_0^{\sigma'} \\ &= \frac{1}{2} \sum_{j'\in\mathbb{N},\sigma'\in\{\pm\}} \sum_{\eta,\eta'\in\{\pm\}} \left|A_{\sigma,\eta j}^{\sigma',\eta' j'}\right| \left(u_{j'}^{\sigma'} + u_{-j'}^{\sigma'}\right) + \sum_{\sigma'\in\{\pm\}} \left(\left|A_{\sigma,j}^{\sigma',0}\right| + \left|A_{\sigma,-j}^{\sigma',0}\right|\right) u_0^{\sigma'} \\ &\ge \frac{1}{2} \sum_{j'\in\mathbb{N}_0,\sigma'\in\{\pm\}} \left\|A_{\sigma,\vec{j}}^{\sigma',\vec{j}'}\right\|_{\infty} \left(u_{j'}^{\sigma'} + u_{-j'}^{\sigma'}\right) = \frac{1}{4}\left((\underline{\hat{A}}u)_j^{\sigma} + (\underline{\hat{A}}u)_{-j}^{\sigma}\right) \end{aligned}$$

$$\text{real - to - real} \;\Leftrightarrow\; \varphi \;\Leftrightarrow\; A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) = S A_{-\sigma,\vec{j}}^{-\sigma',\vec{k}}(-\ell) S$$

$$\text{reversible} \;\Leftrightarrow\; \zeta \;\Leftrightarrow\; A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) = -A_{-\sigma,\vec{j}}^{-\sigma',\vec{k}}(-\ell) \overset{\xi}{=} -\overline{S A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) S},$$

$$\text{revers. pres.} \;\Leftrightarrow\; \delta \;\Leftrightarrow\; A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) = A_{-\sigma,\vec{j}}^{-\sigma',\vec{k}}(-\ell) \overset{\Pi}{=} S A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) S,$$

$$\text{parity pres.} \;\Leftrightarrow\; \gamma \;\Leftrightarrow\; A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) = S A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) S.$$

$$A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) = A_{\sigma,j}^{\sigma',k}(\ell)\mathbb{I} + A_{\sigma,j}^{\sigma',-k}(\ell)S, \quad j,k \in \mathbb{N}$$

$$A_{\sigma,\vec{0}}^{\sigma',\vec{k}}(\ell) = A_{\sigma,0}^{\sigma',k}(\ell)(11), k \neq 0, A_{\sigma,\vec{j}}^{\sigma',\vec{0}}(\ell) = A_{\sigma,j}^{\sigma',0}(\ell)(11)^T, j \neq 0$$

$$\begin{aligned}\tilde{A}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) &:= \mathfrak{U}^{-1} A_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\mathfrak{U} \\ &= \begin{pmatrix} A_{\sigma,j}^{\sigma',k}(\ell) + A_{\sigma,j}^{\sigma',-k}(\ell) & 0 \\ 0 & A_{\sigma,j}^{\sigma',k}(\ell) - A_{\sigma,j}^{\sigma',-k}(\ell) \end{pmatrix} = A_{\sigma,j}^{\sigma',k}(\ell)\mathbb{I} + A_{\sigma,j}^{\sigma',-k}(\ell)E\end{aligned}$$

$$\tilde{A}_{\sigma,\vec{j}}^{\sigma',\vec{0}}(\ell) := \mathfrak{U}^{-1} A_{\sigma,\vec{j}}^{\sigma',\vec{0}}(\ell), \tilde{A}_{\sigma,\vec{0}}^{\sigma',\vec{k}}(\ell) := A_{\sigma,\vec{0}}^{\sigma',\vec{k}}(\ell)\mathfrak{U}, \text{ are proportional to (10).}$$

$$\begin{aligned} A \text{ is real} - \text{to} - \text{real} &\Leftrightarrow \tilde{A}_{\sigma,\eta j}^{\sigma',\eta k}(\ell) = \overline{\tilde{A}_{-\sigma,\eta j}^{-\sigma',\eta k}(-\ell)} \\ A \text{ is reversible} &\Leftrightarrow \tilde{A}_{\sigma,\eta j}^{\sigma',\eta k}(\ell) = -\overline{\tilde{A}_{\sigma,\eta j}^{\sigma',\eta k}(\ell)} \overset{\overline{\text{o}}}{\equiv} -\tilde{A}_{-\sigma,\eta j}^{-\sigma',\eta k}(-\ell) \\ A \text{ is reversibility preserving} &\Leftrightarrow \tilde{A}_{\sigma,\eta j}^{\sigma',\eta k}(\ell) = \overline{\tilde{A}_{\sigma,\eta j}^{\sigma',\eta k}(\ell)} \overset{\Omega}{=} \tilde{A}_{-\sigma,\eta j}^{-\sigma',\eta k}(-\ell) \end{aligned}$$

$$\mathfrak{U}^{-1}S\mathfrak{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: E, (1 \quad 1)\mathfrak{U} = (2 \quad 0), \mathfrak{U}^{-1}(1 \quad 1)^T = (1 \quad 0)^T.$$

$$[A]_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) := \begin{cases} A_{\sigma,\vec{j}}^{\sigma,\vec{j}}(0), & \ell = 0, k = j, \sigma = \sigma' \\ 0 & \text{otherwise} \end{cases}$$

$$[A] := \begin{pmatrix} [A]_+^+(0) & 0 \\ 0 & [A]_-^-(0) \end{pmatrix}, [A]_-^-(0) = [A]_+^+(0) = \operatorname{diag}_{j\in\mathbb{N}_0} A_{+,\vec{j}}^{+,\vec{j}}(0)$$

$$A_{+,\vec{j}}^{+,\vec{j}}(0) \overset{\text{告.48}}{=} \begin{pmatrix} A_{+,j}^{+,j}(0) & A_{+,j}^{+,-j}(0) \\ A_{+,j}^{+,-j}(0) & A_{+,j}^{+,j}(0) \end{pmatrix} = A_{+,j}^{+,j}(0)\mathbb{I} + A_{+,j}^{+,-j}(0)S, j \in \mathbb{N},$$

$$A_{+,\vec{0}}^{+,\vec{0}}(0) = \left(A_{+,0}^{+,0}(0)\right), A_{+,j}^{+,j}(0), A_{+,j}^{+,-j}(0) \in \mathbb{R}, \forall j \in \mathbb{N}_0.$$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\beta}, \forall (x,\xi) \in \mathbb{T} \times \mathbb{R}$$

$$\operatorname{Op}(a(x,\xi))u := a(x,D)u := \sum_{j\in\mathbb{Z}} a(x,j)u_j e^{\mathrm{i}jx}$$

$$|D|^m := \operatorname{Op}(\chi(\xi)|\xi|^m)$$

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq \dfrac{1}{2} \\ 1 & \text{if } |\xi| \geq \dfrac{2}{3} \end{cases}, \partial_\xi \chi(\xi) > 0 \ \forall \xi \in \left(\frac{1}{2}, \frac{2}{3}\right)$$

$$\langle D\rangle := \operatorname{Op}(\langle\xi\rangle), \langle\xi\rangle := \sqrt{|\xi|^2 + 1}, \xi \in \mathbb{R}$$

$$\overline{Op(a(x,\xi))} = Op\left(\overline{a(x,-\xi)}\right)$$

$$a(\varphi,x,\xi) = \sum_{j\in\mathbb{Z}} \hat{a}(\varphi,j,\xi)e^{\mathrm{i}jx} = \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \hat{\hat{a}}(\ell,j,\xi)e^{\mathrm{i}(\ell\cdot\varphi + jx)}$$

$$A_j^k(\varphi) := \hat{a}(\varphi, j-k, k)$$

$$\|a\|_{m,s,p}^{\gamma,\mathcal{O}} := \sup_{\omega\in\mathcal{O}} \|a(\omega)\|_{m,s,p} + \gamma \sup_{\substack{\omega_1\neq\omega_2 \\ \omega_1,\omega_2\in\mathcal{O}}} \frac{\|a(\omega_1;\varphi,x,\xi) - a(\omega_2;\varphi,x,\xi)\|_{m,s-1,p}}{|\omega_1 - \omega_2|}$$

$$\|a(\omega;\cdot)\|_{m,s,p} := \max_{0\le\beta\le p}\sup_{\xi\in\mathbb{R}} \left\|\partial_\xi^\beta a(\omega;\cdot,\cdot,\xi)\right\|_s \langle\xi\rangle^{-m+\beta}$$

$$\forall m \le m' \ \Rightarrow \|\cdot\|_{m',s,p}^{\gamma,\mathcal{O}} \le \|\cdot\|_{m,s,p}^{\gamma,\mathcal{O}}$$
$$\forall s \le s', p \le p' \ \Rightarrow \|\cdot\|_{m,s,p}^{\gamma,\mathcal{O}} \le \|\cdot\|_{m,s',p'}^{\gamma,\mathcal{O}}, \|\cdot\|_{m,s,p}^{\gamma,\mathcal{O}} \le \|\cdot\|_{m,s,p'}^{\gamma,\mathcal{O}}$$

$$A := A(\omega;\varphi,x,\xi) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S^m \otimes \mathcal{M}_2(\mathbb{C})$$

$$\|A\|_{m,s,p}^{\gamma,\mathcal{O}} := \max\left\{\|f\|_{m,s,p}^{\gamma,\mathcal{O}}, f = a,b,c,d\right\}$$

$$\mathrm{Op}(A) := \begin{pmatrix} \mathrm{Op}(a) & \mathrm{Op}(b) \\ \mathrm{Op}(c) & \mathrm{Op}(d) \end{pmatrix}$$

$$\|\underline{\mathrm{Op}(a)}u\|_s^{\gamma,\mathcal{O}} \le C(s_0)\|a\|_{m,s_0,0}^{\gamma,\mathcal{O}}\|u\|_{s+m}^{\gamma,\mathcal{O}} + C(s)\|a\|_{m,s,0}^{\gamma,\mathcal{O}}\|u\|_{s_0+m}^{\gamma,\mathcal{O}}$$

$$a \triangle b(\omega;\varphi,x,\xi) := \sum_{j\in\mathbb{Z}} a(\omega;\varphi,x,\xi+j)\hat{b}(\omega;\varphi,j,\xi)e^{\mathrm{i}jx}$$
$$= \sum_{j,j'\in\mathbb{Z}} \hat{a}(\omega;\varphi,j'-j,\xi+j)\hat{b}(\omega;\varphi,j,\xi)e^{\mathrm{i}j'x}$$

$$(a \circledcirc b)(x,\xi) = \sum_{n=0}^{N-1} \frac{1}{n!\,i^n}\partial_\xi^n a(x,\xi)\partial_x^n b(x,\xi) + r_N(x,\xi)$$

$$r_N(x,\xi) := \frac{1}{(N-1)!\,\mathrm{i}^N}\int_0^1 (1-\tau)^N \sum_{j\in\mathbb{Z}} \left(\partial_\xi^N a\right)(x,\xi+\tau j)\widehat{\partial_x^N} b(j,\xi)e^{\mathrm{i}jx}d\tau$$

$$a\#_n b := \frac{1}{n!\,i^n}\left(\partial_\xi^n a\right)\left(\partial_x^n b\right) \in S^{m+m'-n}$$
$$a\#_{<N} b := \sum_{n=0}^{N-1} a\#_n b \in S^{m+m'}, a\#_{\ge N} b := r_N := r_{N,ab} \in S^{m+m'-N}$$

$$\begin{pmatrix} A_+^+ & A_+^- \\ A_-^+ & A_-^- \end{pmatrix} \qquad \begin{pmatrix} B_+^+ & B_+^- \\ B_-^+ & B_-^- \end{pmatrix} = \begin{pmatrix} A_+^+\#B_+^+ + A_+^-\#B_-^+ & A_+^+\#B_+^- + A_+^-\#B_-^- \\ A_-^+\#B_+^+ + A_-^-\#B_-^+ & A_-^+\#B_+^- + A_-^-\#B_-^- \end{pmatrix}$$

$$\|a \triangleq b\|_{m+m',s,p}^{\gamma,\mathcal{O}} \lesssim_{m,s,p} \|a\|_{m,s,p}^{\gamma,\mathcal{O}}\|b\|_{m',s_0+p+|m|,p}^{\gamma,\mathcal{O}} + \|a\|_{m,s_0,p}^{\gamma,\mathcal{O}}\|b\|_{m',s+p+|m|,p}^{\gamma,\mathcal{O}}$$

$$\|a\#_n b\|_{m+m'-n,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p \sum_{\substack{\beta_1,\beta_2\in\mathbb{N}_0 \\ \beta_1+\beta_2=p}} \|a\|_{m,s,\beta_1+n}^{\gamma,\mathcal{O}}\|b\|_{m',s_0+n,\beta_2}^{\gamma,\mathcal{O}} + \|a\|_{m,s_0,\beta_1+n}^{\gamma,\mathcal{O}}\|b\|_{m',s+n,\beta_2}^{\gamma,\mathcal{O}}$$

$$\|r_N\|_{m+m'-N,s,p}^{\gamma,\mathcal{O}} \lesssim m,N,s,p$$
$$\|a\|_{m,s,N+p}^{\gamma,\mathcal{O}}\|b\|_{m',s_0+2N+p+|m|,p}^{\gamma,\mathcal{O}} + \|a\|_{m,s_0,N+p}^{\gamma,\mathcal{O}}\|b\|_{m',s+2N+p+|m|,p}^{\gamma,\mathcal{O}}$$

$$[\mathrm{Op}(a),\mathrm{Op}(b)] = \mathrm{ad}_{\mathrm{Op}(a)}[\mathrm{Op}(b)] = \mathrm{Op}(a \star b)$$

$$a \star b = a \ddagger b - b \diamond a = -\mathrm{i}\{a,b\} + \sum_{\beta=2}^{N-1}\left(a\#_\beta b - b\#_\beta a\right) + \mathrm{r}_N$$

$$\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b, \mathrm{r}_N := r_{N,ab} - r_{N,ba}$$

$$\|a \star b\|^{\gamma,\mathcal{O}}_{m+m'-1,s,p} \lesssim m,m',s,p\ \|a\|^{\gamma,\mathcal{O}}_{m,s+2+|m'|+p,p+1}\|b\|^{\gamma,\mathcal{O}}_{m',s_0+2+|m|+p,p+1}$$
$$+\|a\|^{\gamma,\mathcal{O}}_{m,s_0+2+|m'|+p,p+1}\|b\|^{\gamma,\mathcal{O}}_{m',s+2+|m|+p,p+1}$$

$$\|\{a,b\}\|^{\gamma,\mathcal{O}}_{m+m'-1,s,p} \lesssim_{s,p} \|a\|^{\gamma,\mathcal{O}}_{m,s+1,p+1}\|b\|^{\gamma,\mathcal{O}}_{m',s_0+1,p+1} + \|a\|^{\gamma,\mathcal{O}}_{m,s_0+1,p+1}\|b\|^{\gamma,\mathcal{O}}_{m',s+1,p+1}$$

$$\|\mathrm{r}_N\|^{\gamma,\mathcal{O}}_{m+m'-N,s,p} \lesssim m,m',s,N,p\|a\|^{\gamma,\mathcal{O}}_{m,s+2N+|m'|+p,p+N}\|b\|^{\gamma,\mathcal{O}}_{m',s_0+2N+|m|+p,p+N}$$
$$+\|a\|^{\gamma,\mathcal{O}}_{m,s_0+2N+|m'|+p,p+N}\|b\|^{\gamma,\mathcal{O}}_{m',s+2N+|m|+p,p+N}$$

$$a^{\#0}:=1, a^{\#1}:=a, a^{\#k}:=a \circledast a^{\#k-1}, \forall k \geq 1$$

$$\|a^{\#k}\|^{\gamma,\mathcal{O}}_{m,s,p} \leq \left(\mathrm{C}(m,s,p)\|a\|^{\gamma,\mathcal{O}}_{m,s_0+p,p}\right)^{k-1}\|a\|^{\gamma,\mathcal{O}}_{m,s+p,p}$$

$$\|a^{\#k+1}\|^{\gamma,\mathcal{O}}_{0,s_0,p} \leq \left(\mathrm{C}(s_0,p)\right)^{k}\left(\|a\|^{\gamma,\mathcal{O}}_{0,s_0+p,p}\right)^{k+1}$$
$$\|a^{\#k+1}\|^{\gamma,\mathcal{O}}_{0,s,p} \leq \left(\mathrm{C}(s,p)\|a\|^{\gamma,\mathcal{O}}_{0,s_0+p,p}\right)^{k}\|a\|^{\gamma,\mathcal{O}}_{0,s+p,p}$$

$$\|a^{\#k}\#a\|^{\gamma,\mathcal{O}}_{m,s,p} \overset{\square}{\lesssim}$$
$$\overset{\square\,\square}{\lesssim}, \lesssim \left(\mathrm{C}(s,p)\|a\|^{\gamma k}_{0,s,p}\|\ ^{\gamma,\mathcal{O}}_{0,s,p}\|a\|\ ^{\gamma,\mathcal{O}}_{m,s_0+p,p} + \|a^{\#k}\|\ ^{\gamma,\mathcal{O}}_{0,s_0,p}\|a\|^{\gamma,\mathcal{O}}_{m,s+p,p}\right.$$
$$<m,s,p$$
$$k-1\|a\|^{\gamma,\mathcal{O}}_{0,s+p,p}\|a\|^{\gamma,\mathcal{O}}_{m,s_0+p,p} + \left(\mathrm{C}(s_0,p)\right)^{k-1}\left(\|a\|^{\gamma,\mathcal{O}}_{0,s_0+p,p}\right)^{k}\|a\|^{\gamma,\mathcal{O}}_{m,s+p,p}$$

$$\|\Phi\|^{\gamma,\mathcal{O}}_{0,s,p} \lesssim_{s,p} \|a\|^{\gamma,\mathcal{O}}_{0,s+p,p}$$

$$\|a\|^{\gamma,\mathcal{O}}_{m,s,p} \lesssim m,s,p\|f\|^{\gamma,\mathcal{O}}_{s+p}$$

$$\|f\|^{\gamma,\mathcal{O}}_{s_0+\tilde{\sigma}} \leq \delta$$

$$(\mathbb{I}-\mathrm{Op}(a))^{-1} = \sum_{k\geq 0}(\mathrm{Op}(a))^{k} = \mathbb{I} + \mathrm{Op}(a) + \mathrm{Op}\left(g_{<\rho}\right) + \mathrm{Op}\left(g_{\geq\rho}\right)$$

$$\left\|g_{<\rho}\right\|^{\gamma,\mathcal{O}}_{2m,s,p} \lesssim m,s,p\|f\|^{\gamma,\mathcal{O}}_{s+\tilde{\sigma}+p}, \forall p \geq 0$$
$$\left\|g_{\geq\rho}\right\|^{\gamma,\mathcal{O}}_{-\rho,s,p} \lesssim m,s,\rho,p_*\|f\|^{\gamma,\mathcal{O}}_{s+\tilde{\sigma}}, \forall 0 \leq p \leq p_*$$

$$a^{\#k} = a^{(k)}_{<\rho} + a^{(k)}_{\geq\rho}, a^{(1)}_{<\rho}:=a, a^{(1)}_{\geq\rho}:=0$$

$$\|a^{(k)}_{<\rho}\|^{\gamma,\mathcal{O}}_{m,s,p} \leq \mathrm{C}_1\sum_{\sum_{i=1}^{k}\beta_i=p}\left(\prod_{i=1}^{k-1}\|a\|^{\gamma,\mathcal{O}}_{m,s_0+\sigma_k,\beta_i+\sigma_k}\right)\|a\|^{\gamma,\mathcal{O}}_{m,s+\sigma_k,\beta_k+\sigma_k}$$
$$\|a^{(k)}_{\geq\rho}\|^{\gamma,\mathcal{O}}_{-\rho,s,p} \leq \mathrm{C}_2\left(\|a\|^{\gamma,\mathcal{O}}_{0,s_0+\sigma_k+p,p+\sigma_k}\right)^{k-1}\|a\|^{\gamma,\mathcal{O}}_{0,s+\sigma_k+p,p+\sigma_k}$$

$$a^{\#k+1} = a^{\#k} \otimes a = a^{(k)}_{<\rho} \bowtie a + a^{(k)}_{\geq\rho} \star a \overset{\sqrt{-g}}{=} \underbrace{a^{(k)}_{<\rho}\#_{<\rho}a}_{=:a^{(k+1)}_{<\rho}} + \underbrace{a^{(k)}_{<\rho}\#_{\geq\rho}a + a^{(k)}_{\geq\rho}\#a}_{=:a^{(k+1)}_{\geq\rho}}$$

$$\left\|a_{<\rho}^{(k)}\#_{<\rho}a\right\|_{m,s,p}^{\gamma,\mathcal{O}} \leq \left\|a_{<\rho}^{(k)}\#_{<\rho}a\right\|_{2m,s,p}^{\gamma,\mathcal{O}}$$

$$\lesssim_{m,s,p,\rho} \sum_{\substack{\beta_1,\beta_2\in\mathbb{N}_0\\ \beta_1+\beta_2=p}} \left\|a_{<\rho}^{(k)}\right\|_{m,s,\beta_1+\rho}^{\gamma,\mathcal{O}}\|a\|_{m,s_0+\rho,\beta_2}^{\gamma,\mathcal{O}} + \left\|a_{<\rho}^{(k)}\right\|_{m,s_0,\beta_1+\rho}^{\gamma,\mathcal{O}}\|a\|_{m,s+\rho,\beta_2}^{\gamma,\mathcal{O}}$$

$$\lesssim_{m,s,p,\rho,k} \sum_{\beta_1+\beta_2=p}\ \sum_{\sum_{i=1}^{k}\beta_i'=\beta_1+\rho} \left[\left(\prod_{i=1}^{k-1}\|a\|_{m,s_0+\sigma_k,\beta_i'+\sigma_k}^{\gamma,\mathcal{O}}\right)\|a\|_{m,s+\sigma_k,\beta_k'+\sigma_k}^{\gamma,\mathcal{O}}\|a\|_{m,s_0+\rho,\beta_2}^{\gamma,\mathcal{O}} + \left(\prod_{i=1}^{k}\|a\|_{m,s_0+\sigma_k,\beta_i'+\sigma_k}^{\gamma,\mathcal{O}}\right)\|a\|_{m,s+\rho,\beta_2}^{\gamma,\mathcal{O}}\right]$$

$$\leq \mathrm{C}_1(k+1)\sum_{\sum_{i=1}^{k+1}\beta_i=p}\left(\prod_{i=1}^{k}\|a\|_{m,s_0+\sigma_k+\rho,\beta_i+\sigma_k+\rho}^{\gamma,\mathcal{O}}\right)\|a\|_{m,s+\sigma_k+\rho,\beta_{k+1}+\sigma_k+\rho}^{\gamma,\mathcal{O}}$$

$$\left\|a_{<\rho}^{(k)}\#_{\geq\rho}a\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \lesssim s,\rho,p\left\|a_{<\rho}^{(k)}\right\|_{0,s,\rho+p}^{\gamma,\mathcal{O}}\|a\|_{0,s_0+2\rho+p,p}^{\gamma,\mathcal{O}} + \left\|a_{<\rho}^{(k)}\right\|_{0,s_0,\rho+p}^{\gamma,\mathcal{O}}\|a\|_{0,s+2\rho+p,p}^{\gamma,\mathcal{O}}$$

$$\overset{3.37}{\underset{s,\rho,p,k}{=}} \sum_{\sum_{i=1}^{k}\beta_i=p+\rho}\left[\left(\prod_{i=1}^{k-1}\|a\|_{0,s_0+\sigma_k,\beta_i+\sigma_k}^{\gamma,\mathcal{O}}\right)\|a\|_{0,s+\sigma_k,\beta_k+\sigma_k}^{\gamma,\mathcal{O}}\|a\|_{0,s_0+2\rho+p,p}^{\gamma,\mathcal{O}} + \left(\prod_{i=1}^{k}\|a\|_{0,s_0+\sigma_k,\beta_i+\sigma_k}^{\gamma,\mathcal{O}}\right)\|a\|_{0,s+2\rho+p,p}^{\gamma,\mathcal{O}}\right]$$

$$\lesssim_{s,\rho,p,k}\left(\|a\|_{0,s_0+\sigma_k+p+\rho,p+\sigma_k+\rho}^{\gamma,\mathcal{O}}\right)^{k}\|a\|_{m,s+\sigma_k+p+\rho,\sigma_k+\rho}^{\gamma,\mathcal{O}}$$

$$\left\|a_{\geq\rho}^{(k)}\#a\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \lesssim \rho,s,p\left\|a_{\geq\rho}^{(k)}\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}}\|a\|_{0,s_0+p+\rho,p}^{\gamma,\mathcal{O}} + \left\|a_{\geq\rho}^{(k)}\right\|_{-\rho,s_0,p}^{\gamma,\mathcal{O}}\|a\|_{0,s+p+\rho,p}^{\gamma,\mathcal{O}}$$

$$\lesssim_{s,\rho,p,k}^{\alpha\beta\rho\sigma}\left[\left(\|a\|_{0,s_0+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}\right)^{k-1}\|a\|_{0,s+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}\|a\|_{0,s_0+p+\rho,p}^{\gamma,\mathcal{O}} + \left(\|a\|_{0,s_0+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}\right)^{k}\|a\|_{0,s+p+\rho,p}^{\gamma,\mathcal{O}}\right] \lesssim_{s,\rho,p,k}\left(\|a\|_{0,s_0+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}\right)^{k}\|a\|_{0,s+p+\sigma_k,p+\sigma_k}^{\gamma,\mathcal{O}}$$

$$g_{<\rho} := 0,\ g_{\geq\rho} := a\#a\#b = a^{\#2}\#b$$

$$(\mathbb{I}-\mathrm{Op}(a))^{-1} = \mathbb{I} + \mathrm{Op}\left(\sum_{k=1}^{\rho-1}a^{\#k} + a^{\#\rho}\#b\right)$$

$$\overset{\beth}{=} \mathbb{I} + \mathrm{Op}(a) + \mathrm{Op}\Big(\underbrace{\sum_{k=2}^{\rho-1}a_{<\rho}^{(k)}}_{=:g_{<\rho}} + \underbrace{\sum_{k=2}^{\rho-1}a_{\geq\rho}^{(k)} + a^{\#\rho}\#b}_{=:g_{\geq\rho}}\Big)$$

$$\|g_{<\rho}\|_{2m,s,p}^{\gamma,\mathcal{O}} \leq \sum_{k=2}^{\rho-1}\mathrm{C}_1(k)\sum_{\sum_{i=1}^{k}\beta_i=p}\left(\prod_{i=1}^{k-1}\|a\|_{m,s_0+\sigma,\beta_i+\sigma}^{\gamma,\mathcal{O}}\right)\|a\|_{m,s+\sigma,\beta_k+\sigma}^{\gamma,\mathcal{O}}$$

$$\|g_{<\rho}\|_{2m,s,p}^{\gamma,\mathcal{O}} \lesssim \underset{m,s,\rho,p}{\beth}\sum_{k=2}^{\rho-1}\sum_{\sum_{i=1}^{k}\beta_i=p}\left(\prod_{i=1}^{k-1}\|f\|_{s_0+2\sigma+\beta_i}^{\gamma,\mathcal{O}}\right)\|f\|_{s+2\sigma+\beta_k}^{\gamma,\mathcal{O}}$$

$$\|g_{<\rho}\|_{2m,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,\rho,p\sum_{k=2}^{\rho-1}\left(\|f\|_{s_0+2\sigma}^{\gamma,\mathcal{O}}\right)^{k-1}\|f\|_{s+2\sigma+p}^{\gamma,\mathcal{O}} \lesssim_{m,s,\rho,p}\|f\|_{s+2\sigma+p}^{\gamma,\mathcal{O}}$$

$$\left\|\sum_{k=2}^{\rho-1} a_{\geq\rho}^{(k)}\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \overset{\sqrt{\aleph}}{\leq} \sum_{k=2}^{\rho-1} C_2(k)\left(\|a\|_{0,s_0+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}\right)^{k-1}\|a\|_{0,s+\sigma_k+p,p+\sigma_k}^{\gamma,\mathcal{O}}$$

$$\|f\|_{s+2p_*+2\sigma}^{\gamma,\mathcal{O}}$$

$$\overset{\sqrt{\mathfrak{H}}}{\lesssim} s,\rho,p \sum_{k=2}^{\rho-1} \left(\|f\|_{s_0+2\sigma_k+2p}^{\gamma,\mathcal{O}}\right)^{k-1}\|f\|_{s+2\sigma_k+2p}^{\gamma,\mathcal{O}} \lesssim s,\rho,p_*$$

$$\begin{aligned}\|a^{\#\rho}\|_{-\rho,s,p}^{\gamma,\mathcal{O}} &= \|a^{\#\rho-1}\#a\|_{-\rho,s,p}^{\gamma,\mathcal{O}}\\ &\lesssim_{s,p,\rho} \|a^{\#\rho-1}\|_{-\rho+1,s,p}^{\gamma,\mathcal{O}}\|a\|_{-1,s_0+p+\rho,p}^{\gamma,\mathcal{O}} + \|a^{\#\rho-1}\|_{-\rho+1,s_0,p}^{\gamma,\mathcal{O}}\|a\|_{-1,s+p+\rho,p}^{\gamma,\mathcal{O}}\\ &\lesssim_{s,p,\rho} \|a\|_{-1,s+p+\rho,p}^{\gamma,\mathcal{O}}\left(\|a\|_{-1,s_0+p+\rho,p}^{\gamma,\mathcal{O}}\right)^{\rho-1}\\ &\overset{\zeta}{\lesssim}_{m,s,\rho,p} \|f\|_{s+2p+\rho}^{\gamma,\mathcal{O}}\left(\|f\|_{s_0+2p+\rho}^{\gamma,\mathcal{O}}\right)^{\rho-1} \lesssim_{s,\rho,p} \|f\|_{s+2p+\rho}^{\gamma,\mathcal{O}}\end{aligned}$$

$$\begin{aligned}\|b\|_{0,s,p}^{\gamma,\mathcal{O}} &\leq \sum_{k\geq 0} \left(C(s,p)\|a\|_{0,s_0+p,p}^{\gamma,\mathcal{O}}\right)^k\|a\|_{0,s+p,p}^{\gamma,\mathcal{O}}\\ &\overset{\mathfrak{I}}{\lesssim} s,\rho,p \sum_{k\geq 0} \left(C(s,p)\|f\|_{s_0+2p}^{\gamma,\mathcal{O}}\right)^k\|f\|_{s+2p}^{\gamma,\mathcal{O}} \overset{\xi}{\lesssim} \|f,p\|f\|_{s+2p}^{\gamma,\mathcal{O}},\end{aligned}$$

$$\begin{aligned}\|a^{\#\rho}\#b\|_{-\rho,s,p}^{\gamma,\mathcal{O}} &\lesssim_{s,p,\rho} \|a^{\#\rho}\|_{-\rho,s,p}^{\gamma,\mathcal{O}}\|b\|_{0,s_0+p+\rho,p}^{\gamma,\mathcal{O}} + \|a^{\#\rho}\|_{-\rho,s_0,p}^{\gamma,\mathcal{O}}\|b\|_{0,s+p+\rho,p}^{\gamma,\mathcal{O}}\\ &\lesssim_{s,p,\rho} \|f\|_{s+2p+\rho}^{\gamma,\mathcal{O}}\|f\|_{s_0+3p+\rho}^{\gamma,\mathcal{O}} + \|f\|_{s_0+2p+\rho}^{\gamma,\mathcal{O}}\|f\|_{s+3p+\rho}^{\gamma,\mathcal{O}}\end{aligned}$$

$$\left\|g_{\geq\rho}\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \lesssim_{s,\rho,p_*} \|f\|_{s+3\sigma+3p_*}^{\gamma,\mathcal{O}}, \forall 0 \leq p \leq p_*$$

$$a(-\varphi,x,\xi) = -\overline{a(\varphi,x,-\xi)}$$

$$a(-\varphi,x,\xi) = \overline{a(\varphi,x,-\xi)}$$

$$a(\varphi,-x,-\xi) = a(\varphi,x,\xi)$$

$$A = A(\varphi,x,\xi) = \begin{pmatrix} a(\varphi,x,\xi) & b(\varphi,x,\xi)\\ \overline{b(\varphi,x,-\xi)} & \overline{a(\varphi,x,-\xi)}\end{pmatrix}$$

$$(\partial_\xi^n a)(-\varphi,x,\xi) = -(-1)^n\overline{(\partial_\xi^n a)(\varphi,x,-\xi)}, (\partial_x^n a)(-\varphi,x,\xi) = -\overline{(\partial_x^n a)(\varphi,x,-\xi)}$$
$$(\partial_\xi^n b)(-\varphi,x,\xi) = (-1)^n\overline{(\partial_\xi^n b)(\varphi,x,-\xi)}, (\partial_x^n b)(-\varphi,x,\xi) = \overline{(\partial_x^n b)(\varphi,x,-\xi)}$$

$$\overline{(a\#_n b)(\varphi,x,-\xi)} = \overline{\frac{1}{n!\,i^n}(\partial_\xi^n a)(\varphi,x,-\xi)(\partial_x^n b)(\varphi,x,-\xi)} = -(a\#_n b)(-\varphi,x,\xi)$$

$\mathcal{M}_s^{\mathrm{T}} := \bigcap_{s_*\leq p\leq s} \mathcal{M}^{\mathrm{T}}(H^p,H^p)$ endowed with the norm $|\cdot|_{\mathcal{M}_s^{\mathrm{T}}} := \sup_{s_*\leq p\leq s} |\cdot|_{p,p}$

$$|AB|_{\mathcal{M}_s^{\mathrm{T}}} \leq |A|_{\mathcal{M}_s^{\mathrm{T}}}|B|_{\mathcal{M}_s^{\mathrm{T}}}$$

$$\mathrm{A} = (M,R), M \in \mathcal{M}_{s_1}^{\mathrm{T}}, R \in \mathcal{M}^{\mathrm{T}}(H^{s_*},H^s)$$

$$(M_1,R_1) + (M_2,R_2) := (M_1+M_2, R_1+R_2), k(M,R) := (kM,kR)$$

$$\|\mathrm{A}\|_s = \|(M,R)\|_s := \|(M,R)\|_{s,s_*,s_1} := |M|_{\mathcal{M}_{s_1}^{\mathrm{T}}} + |R|_{s_*,s} := \sup_{s_*\leq p\leq s_1} |M|_{p,p} + |R|_{s_*,s}$$

$$\|A\|_s^{\gamma,\mathcal{O}} := \sup_{\omega}\|A\|_s + \gamma \sup_{\omega_1\neq\omega_2}\frac{\| A(\omega_1)-A(\omega_2)\|_s}{|\omega_1-\omega_2|}$$

$$A_1 \circ A_2 = (M_1,R_1)\circ(M_2,R_2) := (M_1M_2, M_1R_2 + R_1M_2 + R_1R_2)$$

$$\mathfrak{S}: E_s \to \mathcal{M}_s^{\mathrm{T}}, (M,R) \mapsto \mathfrak{S}(M,R) := M + R$$

$$|\mathfrak{S}(M,R)|_{\mathcal{M}_s^{\mathrm{T}}} = |M+R|_{\mathcal{M}_s^{\mathrm{T}}} = \sup_{s_*\leq p\leq s}|M+R|_{p,p} \leq \|(M,R)\|_{s,s_*,s_1}$$

$$s_* \leq s_*', s_1' \leq s_1, s' \leq s \implies E_{s,s_*,s_1} \subseteq E_{s',s_*',s_1'}, \|\cdot\|_{s',s_*',s_1}^{\gamma,\mathcal{O}} \leq \|\cdot\|_{s,s_*,s_1}^{\gamma,\mathcal{O}}$$

$$(M_1,R_1) \preceq (M_2,R_2) \Leftrightarrow M_1 \preceq M_2, R_1 \preceq R_2.$$

$$A = \mathfrak{S}(A') \text{ and } \frac{1}{c}\| A\|_s^{\gamma,\mathcal{O}} \leq \|A'\|_s^{\gamma,\mathcal{O}} \leq c\| A\|_s^{\gamma,\mathcal{O}}$$

$$M_{\sigma,j}^{\sigma',k}(\ell) - M_{\sigma,-j}^{\sigma',-k}(\ell) = R_{\sigma,-j}^{\sigma',-k}(\ell) - R_{\sigma,j}^{\sigma',k}(\ell), j,k\in\mathbb{Z}, \ell\in\mathbb{Z}^\nu, \sigma,\sigma'\in\{\pm\}$$

$$(M_1')_{\sigma,j}^{\sigma',k}(\ell) := \frac{1}{2}\left(M_{\sigma,j}^{\sigma',k}(\ell) + M_{\sigma,-j}^{\sigma',-k}(\ell)\right), (R_1')_{\sigma,j}^{\sigma',k}(\ell) := \frac{1}{2}\left(R_{\sigma,j}^{\sigma',k}(\ell) + R_{\sigma,-j}^{\sigma',-k}(\ell)\right)$$

$$\|Au\|_s^{\gamma,\mathcal{O}} \leq \|\underline{A}u\|_s^{\gamma,\mathcal{O}} \leq \|A\|_{s_*}^{\gamma,\mathcal{O}}\|u\|_s^{\gamma,\mathcal{O}} + \|A\|_s^{\gamma,\mathcal{O}}\|u\|_{s_0}^{\gamma,\mathcal{O}}$$

$$\|\underline{A}u\|_s \leq |M|_{s,s}\|u\|_s + |R|_{s_*,s}\|u\|_{s_*} \leq \sup_{s_*\leq p\leq s_1}|M|_{p,p}\|u\|_s + \|A\|_s\|u\|_{s_*} \leq \|A\|_{s_*}\|u\|_s + \|A\|_s\|u\|_{s_*}.$$

$$\begin{aligned}\left\|\Delta_{12}(\underline{A}u)\right\|_{s-1} &\leq \left\|(\Delta_{12}\underline{A})u\right\|_{s-1} + \left\|\underline{A}\Delta_{12}u\right\|_{s-1}\\ &\leq \|(\Delta_{12}A)u\|_s + \|A\|_{s_*}\|\Delta_{12}u\|_{s-1} + \|A\|_s\|\Delta_{12}u\|_{s_*}\\ &\leq \|\Delta_{12} A\|_{s_*}\|u\|_s + \|\Delta_{12} A\|_s\|u\|_{s_*} + \|A\|_{s_*}\|\Delta_{12}u\|_{s-1} + \|A\|_s\|\Delta_{12}u\|_{s_*}\\ &\leq \|\Delta_{12} A\|_{s_*}\|u\|_s + \|\Delta_{12} A\|_s\|u\|_{s_0} + \|A\|_{s_*}\|\Delta_{12}u\|_{s-1} + \|A\|_s\|\Delta_{12}u\|_{s_0-1}\end{aligned}$$

$$\|A_1 A_2\|_s^{\gamma,\mathcal{O}} \leq \|A_1\|_s^{\gamma,\mathcal{O}}\|A_2\|_s^{\gamma,\mathcal{O}}$$
$$\|A_1 A_2\|_s^{\gamma,\mathcal{O}} \leq \|A_1\|_{s_*}^{\gamma,\mathcal{O}}\|A_2\|_s^{\gamma,\mathcal{O}} + \|A_1\|_s^{\gamma,\mathcal{O}}\|A_2\|_{s_*}^{\gamma,\mathcal{O}}$$

$$\begin{aligned}&|(M_1,R_1)\circ(M_2,R_2)|_s = \sup_{s_*\leq p\leq s_1}|M_1M_2|_{p,p} + |M_1R_2 + R_1M_2 + R_1R_2|_{s_*,s}\\ &\leq \sup_{s_*\leq p\leq s_1}|M_1|_{p,p}|M_2|_{p,p} + |M_1|_{s,s}|R_2|_{s_*,s} + |R_1|_{s_*,s}|M_2|_{s_*,s_*} + |R_1|_{s_*,s}|R_2|_{s_*,s_*}\\ &\leq \sup_{s_*\leq p\leq s_1}|M_1|_{p,p}\left(\sup_{s_*\leq p\leq s_1}|M_2|_{p,p} + |R_2|_{s_*,s}\right) + |R_1|_{s_*,s}\left(|M_2|_{s_*,s_*} + |R_2|_{s_*,s_*}\right)\\ &\leq \sup_{s_*\leq p\leq s_1}|M_1|_{p,p}| A_2|_s + |R_1|_{s_*,s}| A_2|_{s_*} \leq |A_1|_{s_*}| A_2|_s + |A_1|_s| A_2|_{s_*}\end{aligned}$$

$$\begin{aligned}&\|\Delta_{12}(A_1 \circ A_2)\|_s \leq \|(\Delta_{12} A_1)\circ A_2\|_s + \|A_1\circ(\Delta_{12} A_2)\|_s\\ &\leq \|\Delta_{12} A_1\|_{s*}\| A_2\|_s + \|\Delta_{12} A_1\|_{sl}\| A_2\|_{s_*} + \|A_1\|_{s_*}\|\Delta_{12} A_2\|_s + \|A_1\|_s\|\Delta_{12} A_2\|_{s_*}\end{aligned}$$

$$\langle \mathrm{d}_\varphi\rangle A := (\langle \mathrm{d}_\varphi\rangle M, \langle \mathrm{d}_\varphi\rangle R)$$

$$\|A(\varphi)u\|_{H_x^s} \leq \left\|\langle \mathrm{d}_\varphi\rangle^b A\right\|_{s_*}\|u\|_{H_x^s} + \left\|\langle \mathrm{d}_\varphi\rangle^b A\right\|_s\|u\|_{H_x^{s_*}}$$

$$\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}(\mathrm{AB}) \preceq_{\mathrm{b}} \left(\left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}} A\right)B + A\left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}} B\right)\right)$$

$$\langle \ell-\ell'\rangle^{\mathrm{b}} \leq (\langle \ell-\ell_1\rangle + \langle \ell_1-\ell'\rangle)^{\mathrm{b}} \lesssim_{\mathrm{b}} \langle \ell-\ell_1\rangle^{\mathrm{b}} + \langle \ell_1-\ell'\rangle^{\mathrm{b}}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}(\mathrm{AB})\right\|_s^{\gamma,\mathcal{O}} \lesssim_{\mathrm{b}} \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\right\|_s^{\gamma,\mathcal{O}} \|\mathrm{B}\|_{s_*}^{\gamma,\mathcal{O}} + \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\right\|_{s_*}^{\gamma,\mathcal{O}} \|\mathrm{B}\|_s^{\gamma,\mathcal{O}} + \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{B}\right\|_s^{\gamma,\mathcal{O}} \|\mathrm{A}\|_{s_*}^{\gamma,\mathcal{O}} +$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{B}\right\|_{s_*}^{\gamma,\mathcal{O}} \|\mathrm{A}\|_s^{\gamma,\mathcal{O}}.$$

$$\|\mathrm{A}^k\|_{s_*}^{\gamma,\mathcal{O}} \leq \left(\|\mathrm{A}\|_{s_*}^{\gamma,\mathcal{O}}\right)^k, \|\mathrm{A}^k\|_s^{\gamma,\mathcal{O}} \leq 2^{k-2}k\left(\|\mathrm{A}\|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1}\|\mathrm{A}\|_s^{\gamma,\mathcal{O}}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}^k\right\|_s^{\gamma,\mathcal{O}} \lesssim_{\mathrm{b}} 2^{k-1}k\left(\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\right\|_s^{\gamma,\mathcal{O}}\left(\|\mathrm{A}\|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1} + \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\right\|_{s_*}^{\gamma,\mathcal{O}}\|A\|_s^{\gamma,\mathcal{O}}\left(\|\mathrm{A}\|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-2}\right)$$

$$c(\mathrm{b})\|\mathrm{Q}\|_{s_*}^{\gamma,\mathcal{O}} < 1$$

$$\|\mathrm{A}^{-1} - \mathrm{Id}\|_s^{\gamma,\mathcal{O}} \leq \|\mathrm{Q}\|_s^{\gamma,\mathcal{O}}\left(1 + \|\mathrm{Q}\|_{s_*}^{\gamma,\mathcal{O}}\right)$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}^{-1} - \mathrm{Id}\right\|_s^{\gamma,\mathcal{O}} \lesssim \mathrm{b}\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{Q}\right\|_{s_*}^{\gamma,\mathcal{O}}\|\mathrm{Q}\|_s^{\gamma,\mathcal{O}}\left(1 + \|\mathrm{Q}\|_{s_*}^{\gamma,\mathcal{O}}\right) + \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{Q}\right\|_s^{\gamma,\mathcal{O}}\|\mathrm{Q}\|_{s_*}^{\gamma,\mathcal{O}}\left(1 + \|\mathrm{Q}\|_{s_*}^{\gamma,\mathcal{O}}\right)$$

$$\langle D\rangle^{n_1}\mathrm{A}\langle D\rangle^{n_2} := \left(\langle D\rangle^{n_1}M\langle D\rangle^{n_2}, \langle D\rangle^{n_1}R\langle D\rangle^{n_2}\right)$$

$$\Pi_N\,\mathrm{A} := (\Pi_N M, \Pi_N R), \Pi_N^{\perp} := \mathrm{Id} - \Pi_N$$

$$\|\Pi_N^{\perp}\mathrm{A}\langle D\rangle^m\|_s^{\gamma,\mathcal{O}} \leq N^{-\mathrm{b}}\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\langle D\rangle^m\right\|_s^{\gamma,\mathcal{O}}, \mathrm{b}, m \in \mathbb{N}_0$$

$$\mathrm{ad}_{\mathrm{A}}[\mathrm{B}] := [\mathrm{A}, \mathrm{B}] := \mathrm{A} \circ \mathrm{B} - \mathrm{B} \circ \mathrm{A},$$

$$\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}] := \mathrm{A} \circ \mathrm{ad}_{\mathrm{A}}^{k-1}[\mathrm{B}] - \mathrm{ad}_{\mathrm{A}}^{k-1}[\mathrm{B}] \circ \mathrm{A}, \mathrm{ad}_{\mathrm{A}}^0[\mathrm{B}] := \mathrm{B}$$

$$\mathrm{l}\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}]\langle D\rangle^m \boldsymbol{l}_s^{\gamma,\mathcal{O}} \leq$$
$$2^k\left(\left(\mathrm{l}\,\mathrm{A}\langle D\rangle^m\boldsymbol{\Gamma}_{s_*}^{\gamma,\mathcal{O}}\right)^k\Big|\,\mathrm{B}\langle D\rangle^m\boldsymbol{\Gamma}_s^{\gamma,\mathcal{O}} + k\left(\mathrm{l}\,\mathrm{A}\langle D\rangle^m\boldsymbol{\Gamma}_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1}\Big|\mathrm{A}\langle D\rangle^m\boldsymbol{\Gamma}_s^{\gamma,\mathcal{O}}\,\mathrm{l}\,\mathrm{B}\langle D\rangle^m\boldsymbol{\Gamma}_{s_*}^{\gamma,\mathcal{O}}\right)$$

$$\begin{aligned}\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}]\langle D\rangle^m\right|_s^{\gamma,\mathcal{O}} &\leq 2^{k(\mathrm{b}+1)}k|\,\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\left(|\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1}|\,\mathrm{B}\langle D\rangle^m|_s \\ &+2^{k(\mathrm{b}+1)}k\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\langle D\rangle^m\right|_s^{\gamma,\mathcal{O}}\left(|\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1}|\,\mathrm{B}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}} \\ &+2^{k(\mathrm{b}+1)}k|\,\mathrm{A}\langle D\rangle^m|_s^{\gamma,\mathcal{O}}\left(|\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-1}\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{B}\langle D\rangle^m\right|_{s_*}^{\gamma,\mathcal{O}} \\ &+2^{k(\mathrm{b}+1)}k(k-1)|\mathrm{A}\langle D\rangle^m|_s^{\gamma,\mathcal{O}}\left(|\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\right)^{k-2}\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\langle D\rangle^m\right|_{s_*}^{\gamma,\mathcal{O}}|\mathrm{B}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}} \\ &+2^{k(\mathrm{b}+1)}\left(|\mathrm{A}\langle D\rangle^m|_{s_*}^{\gamma,\mathcal{O}}\right)^k\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{B}\langle D\rangle^m\right|_s^{\gamma,\mathcal{O}}\end{aligned}$$

$\mathrm{ad}_A^k[B] = \mathfrak{S}\left(\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}]\right), k \geq 0.$

$$\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}]\langle D\rangle^m = \mathrm{ad}_{\mathrm{A}}(\mathrm{C}_k)\langle D\rangle^m \preceq \mathrm{A}\langle D\rangle^m\mathrm{C}_k\langle D\rangle^m + \mathrm{C}_k\langle D\rangle^m\,\mathrm{A}\langle D\rangle^m,$$

$$\begin{aligned}\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{ad}_{\mathrm{A}}^k[\mathrm{B}]\langle D\rangle = \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{ad}_{\mathrm{A}}(\mathrm{C}_k)\langle D\rangle &\preceq \left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\langle D\rangle\right)\mathrm{C}_k\langle D\rangle + \mathrm{A}\langle D\rangle\left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{C}_k\langle D\rangle\right) \\ &+ \left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{C}_k\langle D\rangle\right)\mathrm{A}\langle D\rangle + \mathrm{C}_k\langle D\rangle\left(\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{A}\langle D\rangle\right)\end{aligned}$$

$$\mathfrak{S}(\mathrm{L}) = L, L := \langle D\rangle^{-n_1}\mathrm{Op}(a)\langle D\rangle^{-n_2}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{L}\right\|_{s,s_*,s_1}^{\gamma,\mathcal{O}} \lesssim s, s_1, n_1, \mathrm{b}\|a\|_{m,s+s_*+|n_1|+\mathrm{b}+1,0}^{\gamma,\mathcal{O}}$$

$$|A|_s^{\mathrm{dec}} := \left(\sum_{p\in\mathbb{Z}^\nu, h\in\mathbb{Z}} \langle p,h\rangle^{2s} \sup_{\substack{j-j'=h \\ \ell-\ell'=p}} \left| A_j^{j'}(\ell-\ell') \right|^2 \right)^{1/2}$$

$$|A|_s^{\mathrm{dec},\gamma,\mathcal{O}} := \sup_{\omega\in\mathcal{O}} |A(\omega)|_s^{\mathrm{dec}} + \gamma \sup_{\substack{\omega_1\neq\omega_2 \\ \omega_1,\omega_2\in\mathcal{O}}} \frac{|A(\omega_1)-A(\omega_2)|_s^{\mathrm{dec}}}{|\omega_1-\omega_2|}$$

$$|A|_{s,s}^{\gamma,\mathcal{O}} \leq C_2(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}}$$

$$\|\underline{A}u\|_s^2 \leq \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell,j\rangle^{2s} \left(\sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} |\mathrm{a}_{j-j'}(\ell-\ell')||u_{\ell',j'}| \right)^2$$
$$= \|\mathrm{a}\cdot\underline{u}\|_s^2 \overset{\Upsilon}{\lesssim} \|\mathrm{a}\|_s^2\|\underline{u}\|_s^2 \lesssim_s \left(|A|_s^{\mathrm{dec}}\right)^2\|u\|_s^2$$

$$(L^B)_j^{j'}(\ell-\ell') := \begin{cases} L_j^{j'}(\ell-\ell') & \text{if } |\ell-\ell'|+|j-j'| < \frac{1}{2}(|\ell|+|j|) \\ 0 & \text{otherwise} \end{cases}$$

$$|L^B|_{s,s}^{\gamma,\mathcal{O}} \leq 3^{s-s_*}|L|_{s_*,s_*}^{\gamma,\mathcal{O}}, \qquad |L^U|_{s_*,s}^{\gamma,\mathcal{O}} \leq 4^s C(s_*)|L|_s^{\mathrm{dec},\gamma,\mathcal{O}}$$

$$\left\|\underline{L^B}u\right\|_s^2 \leq \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell,j\rangle^{2s} \left(\sum_{|\ell-\ell'|+|j-j'|<\frac{1}{2}(|\ell|+|j|)} \left| L_j^{j'}(\ell-\ell')\|u_{\ell',j'} \right. \right)^2$$
$$\leq 3^{2(s-s_*)} \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell,j\rangle^{2s_*} \left(\sum_{|\ell-\ell'|+|j-j'|<\frac{1}{2}(|\ell|+|j|)} \left|L_j^{j'}(\ell-\ell')\right| \langle \ell',j'\rangle^{s-s_*}|u_{\ell',j'}| \right)^2$$
$$\leq 3^{2(s-s_*)} \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell,j\rangle^{2s_*} \left(\sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} \left|L_j^{j'}(\ell-\ell')\right| \langle \ell',j'\rangle^{s-s_*}|u_{\ell',j'}| \right)^2$$
$$= 3^{2(s-s_*)}\left\|\underline{L}\left(\langle \mathbf{D}\rangle^{s-s_*}\underline{u}\right)\right\|_{s_*}^2 \leq 3^{2(s-s_*)}|L|_{s_*,s_*}^2\left\|\langle \mathbf{D}\rangle^{s-s_*}\underline{u}\right\|_{s_*}^2 = 3^{2(s-s_*)}|L|_{s_*,s_*}^2\|u\|_s^2$$

$$\left\|\underline{L^U}u\right\|_s^2 \leq \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \langle \ell,j\rangle^{2s} \left(\sum_{|\ell-\ell'|+|j-j'|\geq\frac{1}{2}(|\ell|+|j|)} \left| L_j^{j'}(\ell-\ell')\|u_{\ell',j'} \right. \right)^2$$
$$\leq 4^{2s} \sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \left(\sum_{\ell'\in\mathbb{Z}^\nu, j'\in\mathbb{Z}} \langle \ell-\ell',j-j'\rangle^{s}\left|L_j^{j'}(\ell-\ell')\right|\langle \ell',j'\rangle^{s_*}|u_{\ell',j'}|\frac{1}{\langle \ell',j'\rangle^{s_*}} \right)^2$$
$$\leq 4^{2s}C(s_*)^2\sum_{\ell',j'}\langle \ell',j'\rangle^{2s_*}|u_{\ell',j'}|^2\sum_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}}\langle \ell-\ell',j-j'\rangle^{2s}\left|L_j^{j'}(\ell-\ell')\right|^2$$
$$\leq 4^{2s}C(s_*)^2\sum_{\ell',j'}\langle \ell',j'\rangle^{2s_*}|u_{\ell',j'}|^2\left(|L|_s^{\mathrm{dec}}\right)^2 \leq 4^{2s}C(s_*)^2\|u\|_{s_*}^2\left(|L|_s^{\mathrm{dec}}\right)^2$$

$$\|\mathrm{L}\|_{s,s_*,s_1}^{\gamma,\mathcal{O}} \leq c(s,s_*,s_1)|L|_s^{\mathrm{dec},\gamma,\mathcal{O}}$$

$$\left|\langle D\rangle^{-n_1}\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{Op}(a)\langle D\rangle^{-n_2}\right|_s^{\mathrm{dec},\gamma,\mathcal{O}} \lesssim_{s,n_1,\mathrm{b}} \|a\|_{m,s+s_*+|n_1|+\mathrm{b}+1,0}^{\gamma,\mathcal{O}}$$

$$\langle \ell,h\rangle^{s+s_*}|\hat{\hat{a}}(\ell,h,\xi)| \lesssim_s \langle\xi\rangle^m\|a\|_{m,s+s_*,0}$$

$$P_j^{j'}(\ell-\ell') = \langle \ell-\ell\rangle^{\mathrm{b}} \hat{\hat{a}}(\ell-\ell', j-j', j')\langle j\rangle^{-n_1}\langle j'\rangle^{-n_2}$$

$$\left|P_j^{j'}(\ell-\ell')\right| \lesssim \langle \ell-\ell'\rangle^{\mathrm{b}}\left|\hat{\hat{a}}(\ell-\ell', j-j', j')\right|\langle j'\rangle^{-m}\frac{\langle j'\rangle^{n_1}}{\langle j\rangle^{n_1}}$$
$$\lesssim \langle \ell-\ell', j-j'\rangle^{\mathrm{b}+|n_1|}\left|\hat{\hat{a}}(\ell-\ell', j-j', j')\right|\langle j'\rangle^{-m}$$

$$\langle \ell-\ell', j-j'\rangle^{s}\left|P_j^{j'}(\ell-\ell')\right| \lesssim_{s,n_1,\mathrm{b}} \|a\|_{m,s+s_*+\mathrm{b}+|n_1|,0}\langle \ell-\ell', j-j'\rangle^{-s_*}$$

$$x\mapsto y = x+\tau\alpha(\varphi,x), x\in\mathbb{T}, \varphi\in\mathbb{T}^\nu, \tau\in[0,1],$$

$$y\mapsto x = y+\breve{\alpha}(\tau;\varphi,y)$$

$$\mathcal{C}_\alpha^\tau h(\varphi,x) := h(\varphi, x+\tau\alpha(\varphi,x)), (\mathcal{C}_\alpha^\tau)^{-1}h(\varphi,y) := h(\varphi, y+\breve{\alpha}(\tau;\varphi,y))$$

$$\|\alpha\|_{s_0+\mu}^{\gamma,\mathcal{O}} \leq \delta$$

$$\mathfrak{S}(\mathrm{P}^\tau) = P^\tau, P^\tau := \langle D\rangle^{-N_1}\circ(\mathcal{C}_\alpha^\tau - \mathrm{Id})\circ\langle D\rangle^{-N_2}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^q \mathrm{P}^\tau\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,q,N_1,N_2} \|\alpha\|_{s+\mu}^{\gamma,\mathcal{O}}, q = 0, \mathrm{b}$$

$$\sup_{\tau\in[0,1]} \|\mathcal{C}_\alpha^\tau u\|_s^{\gamma,\mathcal{O}} \lesssim_s \|u\|_s^{\gamma,\mathcal{O}} + \|\alpha\|_{s+\sigma}^{\gamma,\mathcal{O}}\|u\|_{s_0}^{\gamma,\mathcal{O}}$$

$$\sup_{\tau\in[0,1]} \|\langle D\rangle^{-m_1}(\mathcal{C}_\alpha^\tau - \mathbb{I})\langle D\rangle^{-m_2}u\|_s^{\gamma,\mathcal{O}} \lesssim s, m_1, m_2\|\alpha\|_{s_0+\sigma}^{\gamma,\mathcal{O}}\|u\|_s^{\gamma,\mathcal{O}} + \|\alpha\|_{s+\sigma}^{\gamma,\mathcal{O}}\|u\|_{s_0}^{\gamma,\mathcal{O}}$$

$$\sup_{\tau\in[0,1]} \left\|\langle D\rangle^{-m_1}\,\mathrm{d}_\varphi^q\,\mathrm{d}_x^i\mathcal{C}_\alpha^\tau\langle D\rangle^{-m_2}w\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,|q|,m_1,m_2} \|w\|_s\|\alpha\|_{s_0+\mu}^{\gamma,\mathcal{O}} + \|\alpha\|_{s+\mu}^{\gamma,\mathcal{O}}\|w\|_{s_0}$$

$$\sup_{\tau\in[0,1]} \left\|\langle D\rangle^{-m_1}\,\mathrm{d}_\varphi^q\,\mathrm{d}_x^i\mathcal{C}_\alpha^\tau\langle D\rangle^{-m_2}\right\|_{s,s}^{\gamma,\mathcal{O}} \lesssim_{s,|q|,m_1,m_2} \|\alpha\|_{s+\mu}^{\gamma,\mathcal{O}}$$

$$P := P^1 := \langle D\rangle^{-N_1}\circ(\mathcal{C}_\alpha - \mathbb{I})\circ\langle D\rangle^{-N_2}, N_1+N_2 = N,$$

$$\mathcal{C}_\alpha u = \sum_{j'\in\mathbb{Z}} \mathrm{t}_\alpha(\varphi,x,j')u_{j'}e^{\mathrm{i}j'x} = \sum_{j\in\mathbb{Z}}\left(\sum_{j'\in\mathbb{Z}}\hat{\mathrm{t}}_\alpha(\varphi,j-j',j')u_{j'}\right)e^{\mathrm{i}jx}$$

$$\mathrm{t}_\alpha(\varphi,x,\xi) = \sum_{k\in\mathbb{Z}}\hat{\mathrm{t}}_\alpha(\varphi,k,\xi)e^{\mathrm{i}kx}, \hat{\mathrm{t}}_\alpha(\varphi,k,\xi) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{\mathrm{i}\xi\alpha(\varphi,x)}e^{-\mathrm{i}kx}dx$$

$$\left|\widehat{\widehat{f\mathrm{t}}}_\alpha(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} \lesssim s\frac{1}{\langle \ell,k\rangle^s}\left(\|\alpha\|_{s+\eta}^{\gamma,\mathcal{O}}\|f\|_{s_0}^{\gamma,\mathcal{O}} + \|f\|_{s+1}^{\gamma,\mathcal{O}}\right)$$
$$\left|(\widehat{\mathrm{t}_\alpha - 1})(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} \lesssim_s \frac{1}{\langle \ell,k\rangle^s}\|\alpha\|_{s+\eta}^{\gamma,\mathcal{O}}$$

$$|\,\widehat{\widehat{\mathrm{t}_\alpha - 1}}\big)(\ell,k,\xi)|^{\gamma,\mathcal{O}} \lesssim s\frac{1}{\langle \ell,k\rangle^s}\|\alpha\|_{s+\eta}^{\gamma,\mathcal{O}}$$

$$\widehat{\widehat{f\mathrm{t}}}_\alpha(\ell,k,\xi) = \frac{1}{(2\pi)^{\nu+1}}\int_{\mathbb{T}^{\nu+1}} f(\varphi,x)e^{\mathrm{i}\xi\alpha(\varphi,x)-\mathrm{i}kx}e^{-\mathrm{i}\ell\cdot\varphi}d\varphi dx$$

$$\widehat{\widehat{f\mathrm{t}_\alpha}}(\ell,k,\xi)=\frac{1}{(2\pi)^{\nu+1}}\int_{\mathbb{T}^{\nu+1}} f(\varphi,x)e^{-\mathrm{i}k(x-\eta\alpha(\varphi,x))}e^{-\mathrm{i}\ell\cdot\varphi}d\varphi dx$$

$$=\frac{1}{(2\pi)^{\nu+1}}\int_{\mathbb{T}^{\nu+1}} f(\varphi,y+\breve{\alpha}(\varphi,y,\eta))e^{-\mathrm{i}ky}e^{-\mathrm{i}\ell\cdot\varphi}\Big(1+\breve{\alpha}_y(\varphi,y,\eta)\Big)\Big)d\varphi dy$$

$$\|\breve{\alpha}\|_s^{\gamma,\mathcal{O}}\lesssim_s\|\alpha\|_{s+s_0}^{\gamma,\mathcal{O}}$$

$$|\widehat{g}(\ell,k)|^{\gamma,\mathcal{O}}\leq\frac{1}{\langle\ell,k\rangle^s}\|g\|_{s+1}^{\gamma,\mathcal{O}}$$

$$\left|\widehat{\widehat{f\mathrm{t}_\alpha}}(\ell,0,\xi)\right|^{\gamma,\mathcal{O}}\lesssim\frac{1}{\langle\ell,0\rangle^s}\|fe^{\mathrm{i}\xi\alpha}\|_{s+1}^{\gamma,\mathcal{O}}\lesssim s\frac{1}{\langle\ell,0\rangle^s}\left(\|f\|_{s+1}^{\gamma,\mathcal{O}}+\|f\|_{s_0}^{\gamma,\mathcal{O}}\|\alpha\|_{s+1}^{\gamma,\mathcal{O}}\right)$$

$$(\widehat{\widehat{\mathrm{t}_\alpha-1}})(\ell,k,\xi)=\frac{\mathrm{i}}{(2\pi)^{\nu+1}}\int_0^1 d\tau\int_{\mathbb{T}^{\nu+1}}\alpha(\varphi,x)\xi e^{\mathrm{i}\tau\xi\alpha(\varphi,x)}e^{-\mathrm{i}kx}e^{-\mathrm{i}\ell\cdot\varphi}d\varphi dx$$

$$(\widehat{\widehat{\mathrm{t}_\alpha-1}})(\ell,k,\xi)=\frac{\mathrm{i}\xi}{(2\pi)^{\nu+1}}\int_0^1 d\tau\int_{\mathbb{T}^{\nu+1}}\alpha(\varphi,x)e^{-\mathrm{i}k(x-\eta\alpha(\varphi,x))}e^{-\mathrm{i}\ell\cdot\varphi}d\varphi dx$$

$$\left(\widehat{\widehat{\mathrm{t}_\alpha-1}}\right)(\ell,k,\xi)=\frac{\mathrm{i}\xi}{(2\pi)^{\nu+1}}\int_0^1 d\tau\int_{\mathbb{T}^{\nu+1}}\alpha(\varphi,x)e^{-\mathrm{i}\ell_1(\varphi_1-\eta\alpha(\varphi,x))}e^{-\mathrm{i}\sum_{i=2}^n\ell_i\varphi_i}e^{-\mathrm{i}kx}d\varphi dx$$

$$\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{P}\right|_s^{\gamma,\mathcal{O}}=\sup_{s_*\leq p\leq s_1}\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}P^B\right|_{p,p}^{\gamma,\mathcal{O}}+\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}P^U\right|_{s_*,s}^{\gamma,\mathcal{O}}$$
$$\overset{\Xi}{\leq}\sup_{s_*\leq p\leq s_1}3^{p-s_*}\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}P\right|_{s_*,s_*}^{\gamma,\mathcal{O}}+\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}P^U\right|_{s_*,s}^{\gamma,\mathcal{O}}.$$

$$\left|\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}P\right|_{s_*,s_*}^{\gamma,\mathcal{O}}\overset{\Lambda}{\lesssim}|P|_{s_*,s_*}^{\gamma,\mathcal{O}}+\sum_{h=1}^{\nu}\left|\mathrm{d}_{\varphi_h}^{\mathrm{b}}P\right|_{s_*,s_*}^{\gamma,\mathcal{O}}$$
$$\overset{\Gamma}{\lesssim}\sum_{q=0,\,\mathrm{b}}\sum_{h=1}^{\nu}\left(\left\|\mathrm{d}_{\varphi_h}^{q}P\right\|_{s_*,s_*}^{\gamma,\mathcal{O}}+\left\|\mathrm{d}_x\,\mathrm{d}_{\varphi_h}^{q}P\right\|_{s_*,s_*}^{\gamma,\mathcal{O}}+\left\|\mathrm{d}_{\varphi_h}^{q+\beta}P\right\|_{s_*,s_*}^{\gamma,\mathcal{O}}+\left\|\mathrm{d}_x\,\mathrm{d}_{\varphi_h}^{q+\beta}P\right\|_{s_*,s_*}^{\gamma,\mathcal{O}}\right)$$
$$\overset{\mathcal{H},\wp}{\lesssim}\|\alpha\|_{s_*+\sigma}^{\gamma,\mathcal{O}}$$

$$(\langle D\rangle^{-N_1}(\mathcal{C}_\alpha-\mathbb{I})\langle D\rangle^{-N_2})_j^{j'}(\ell-\ell')=\left(\widehat{\widehat{\mathrm{t}_\alpha-1}}\right)(\ell-\ell',j-j',j')\langle j\rangle^{-N_1}\langle j'\rangle^{-N_2}$$

$$|\ell-\ell'|+|j-j'|\geq\max\{|j'|,|j|\}/3$$

$$\left|\left(\langle\mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{-N_1}(\mathcal{C}_\alpha-\mathbb{I})\langle D\rangle^{-N_2}\right)_j^{j'}(\ell-\ell')\right|^{\gamma,\mathcal{O}}\overset{<s}{\lesssim}\|\alpha\|_{s+s_*+\sigma+\mathrm{b}}^{\gamma,\mathcal{O}}\frac{\langle j\rangle^{-N_1}\langle j'\rangle^{-N_2}}{\langle\ell-\ell',j-j'\rangle^{s+|N_1|+|N_2|+s_*}}$$
$$\overset{\psi}{\lesssim}\|\alpha\|_{s+s_*+\sigma+\mathrm{b}}^{\gamma,\mathcal{O}}\frac{1}{\langle\ell-\ell',j-j'\rangle^{s+s_*}}$$

$$\|\alpha\|_{s_0+\sigma}^{\gamma,\mathcal{O}}<\delta$$

$$\mathcal{C}_\alpha^\tau\circ\mathrm{Op}(w)\circ(\mathcal{C}_\alpha^\tau)^{-1}=\mathrm{Op}(q(\tau))+R^\tau$$

$$q(\tau;\varphi,x,\xi)=q_m(\tau;\varphi,x,\xi)+\tilde{q}_{m-1}(\tau;\varphi,x,\xi)$$

$$
\begin{aligned}
q_m(1;\varphi,x,\xi) &= w\left(\varphi, x+\alpha(\varphi,x), \frac{\xi}{1+\alpha_x(\varphi,x)}\right) \\
&= w\left(\varphi, x+\alpha(\varphi,x), \xi\left(1+\partial_y \breve{\alpha}(\varphi,y)\right)\right)_{|y=x+\alpha(\varphi,x)}
\end{aligned}
$$

$$
\|q_m\|_{m,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p,M,\mathrm{b} \|w\|_{m,s,p}^{\gamma,\mathcal{O}} + \sum_{s}^{*} \|w\|_{m,k_1,p+k_2}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+s_0+2}^{\gamma,\mathcal{O}}
$$

$$
\|\tilde{q}_{m-1}\|_{m-1,s,p}^{\gamma,\mathcal{O}} \lesssim_{m,s,p,M,\mathrm{b}} \sum_{s}^{*} \|w\|_{m,k_1,p+k_2+\sigma}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma}^{\gamma,\mathcal{O}}
$$

$$
\sup_{\tau\in[0,1]} \left\| \langle \mathrm{d}_\varphi \rangle^j \langle D \rangle^{m_1} \mathrm{R}^\tau \langle D \rangle^{m_2} \right\|_s^{\gamma,\mathcal{O}} \lesssim s, s_1, m, M, \mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1+\sigma,k_2+\sigma}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma}^{\gamma,\mathcal{O}}
$$

$$
Aw := w\left(\varphi, x+\alpha(\varphi,x), \frac{\xi}{1+\alpha_x(\varphi,x)}\right)
$$

$$
\|Aw\|_{m,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p \|w\|_{m,s,p}^{\gamma,\mathcal{O}} + \sum_{s}^{*} \|w\|_{m,k_1,p+k_2}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+s_0+2}^{\gamma,\mathcal{O}}
$$

$$
\partial_\tau \mathcal{C}_\alpha^\tau = X^\tau \mathcal{C}_\alpha^\tau, \mathcal{C}_\alpha^0 = \mathrm{Id}
$$

$$
X^\tau := \mathcal{A}(\tau;\varphi,x)\partial_x = \mathrm{Op}(\chi), \chi := \chi(\tau;x,\xi) := \chi(\tau;\varphi,x,\xi) := \mathrm{i}\mathcal{A}(\tau;\varphi,x)\xi
$$

$$
\mathcal{A}(\tau;\varphi,x) = \frac{\alpha(\varphi,x)}{1+\tau\alpha_x(\varphi,x)}
$$

$$
\|\chi\|_{1,s,p}^{\gamma,\mathcal{O}} \lesssim_s \|\mathcal{A}\|_s^{\gamma,\mathcal{O}} \lesssim_s \|\alpha\|_{s+1}^{\gamma,\mathcal{O}}, \forall p \geq 0
$$

$$
\partial_\tau P^\tau = [X^\tau, P^\tau], P^0 = \mathrm{Op}(w)
$$

$$
\rho := M + 4(\lfloor \nu/2 \rfloor + 4 + \mathrm{b}) + 1
$$

$$
Q^\tau := \mathrm{Op}(q(\tau;x,\xi)), q = q(\tau;x,\xi) = \sum_{k=0}^{m+\rho-1} q_{m-k}(\tau;x,\xi)
$$

$$
\partial_\tau Q^\tau = [X^\tau, Q^\tau] + \mathcal{M}^\tau, Q^0 = \mathrm{Op}(w)
$$

$$
\begin{cases}
\partial_\tau q(\tau;x,\xi) = \chi(\tau;x,\xi) \star q(\tau;x,\xi) + \mathrm{r}_{-\rho}(\tau;x,\xi) \\
q(0;x,\xi) = w(x,\xi)
\end{cases}
$$

$$
\begin{aligned}
\chi \star q &= \chi \circledast q - q \diamond \chi = \chi \wr q + \frac{1}{\mathrm{i}}(\partial_\xi \chi)(\partial_x q) - q \uplus \chi \\
&\overset{\widehat{\sqrt{-g}}}{=} \chi q + \sum_{k=0}^{m+\rho-1} \frac{1}{\mathrm{i}}(\partial_\xi \chi)(\partial_x q_{m-k}) - \left(\sum_{k=0}^{m+\rho-1} q_{m-k}(\tau;x,\xi) \right) \doteq \chi \\
&\overset{\hbar}{=}, \overset{\mathfrak{K}}{=}, \overset{\mathbb{R}}{=} \sum_{k=0}^{m+\rho-1} \frac{1}{\mathrm{i}}\{\chi, q_{m-k}\} - \sum_{k=0}^{m+\rho-1} \sum_{n=2}^{m-k+\rho} q_{m-k} \setminus \eta_\dagger \chi - \sum_{k=0}^{m+\rho-1} q_{m-k} \square \geq_{\mathrm{m}-k+\rho+1} \chi
\end{aligned}
$$

$$
\chi \star q = \underbrace{-\mathrm{i}\{\chi, q_m\}}_{\in S^m} + \overbrace{\sum_{k=1}^{m+\rho-1} \underbrace{(-\imath\{\chi, q_{m-k}\} + r_{m-k})}_{\in S^{m-k}}}^{\text{orders from } -\rho+1 \text{ to } m-1} - \underbrace{r_{-\rho}}_{\in S^{-\rho}}
$$

$$r_{m-k} := -\sum_{h=0}^{k-1} q_{m-h} \#_{\mathrm{w}} \chi$$

$$\overset{3.18\sqrt{\equiv}}{=} -\sum_{h=0}^{k-1} \frac{1}{\mathrm{w}!\,\mathrm{i}^{\mathrm{w}}} \left(\partial_\xi^{\mathrm{w}} q_{m-h}\right)(\partial_x^{\mathrm{w}} \chi) \in S^{(m-h)+1-(k-h+1)} \equiv S^{m-k}$$

$$\mathrm{r}_{-\rho} := \sum_{k=0}^{m+\rho-1} q_{m-k} \#_{\geq \rho+m-k+1} \chi \overset{\widehat{\sqrt{-g}}}{\epsilon} S^{m-k+1-(m-k+1+\rho)} \equiv S^{-\rho}$$

$$\begin{cases} \partial_\tau q_m(\tau; x, \xi) = \{\mathcal{A}(\tau; x)\xi, q_m(\tau; x, \xi)\} \\ q_m(0; x, \xi) = w(x, \xi) \end{cases}$$

$$\begin{cases} \partial_\tau q_{m-k}(\tau; x, \xi) = \{\mathcal{A}(\tau; x)\xi, q_{m-k}(\tau; x, \xi)\} + r_{m-k}(\tau; x, \xi) \\ q_{m-k}(0; x, \xi) = 0 \end{cases}$$

$$\begin{cases} \dfrac{d}{ds} x(s) = -\mathcal{A}(s, x(s)) \\ \dfrac{d}{ds} \xi(s) = \mathcal{A}_x(s, x(s))\xi(s) \end{cases} \quad (x(0), \xi(0)) = (x_0, \xi_0) \in \mathbb{T} \times \mathbb{R}$$

$$q_m(\tau; x, \xi) = w(\gamma^{\tau,0}(x, \xi))$$

$$\gamma^{\tau,0}(x, \xi) = (f(\tau; x), \xi g(\tau; x)), f(\tau; x) := x + \tau\alpha(x), g(\tau; x) := \frac{1}{1 + \tau\alpha_x(x)}$$

$$\|q_{m-h}\|_{m-h,s,p}^{\gamma,\mathcal{O}} \lesssim m, s, p, M, \mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\sigma_h+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma_h}^{\gamma,\mathcal{O}}, 1 \leq h < k$$

$$\frac{d}{d\tau} f_{m-k}(\tau) = r_{m-k}(\tau; x(\tau), \xi(\tau)) \;\Rightarrow\; f_{m-k}(\tau) = \int_0^\tau r_{m-k}(\sigma; x(\sigma), \xi(\sigma)) d\sigma$$

$$q_{m-k}(\tau; x, \xi) = \int_0^\tau r_{m-k}(\gamma^{0,\sigma} \gamma^{\tau,0}(x, \xi)) d\sigma$$

$$\tilde{f}(\sigma, \tau, x) := x + \tau\alpha(x) + \breve{\alpha}(\sigma, x + \tau\alpha(x)), \tilde{g}(\sigma, \tau, x) := \frac{1}{\tilde{f}_x(\sigma, \tau, x)}$$

$$\|r_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} \lesssim \sum_{h=0}^{k-1} \frac{1}{\mathrm{w}!} \left\|\left(\partial_\xi^{\mathrm{w}} q_{m-h}\right)(\partial_x^{\mathrm{w}} \chi)\right\|_{m-k,s,p}^{\gamma,\mathcal{O}}$$

$$\overset{\mathfrak{I}}{\lesssim} m, s, p \|q_m\|_{m,s,p+k+1}^{\gamma,\mathcal{O}} \|\alpha\|_{s_0+k+2}^{\gamma,\mathcal{O}} + \|q_m\|_{m,s_0,p+k+1}^{\gamma,\mathcal{O}} \|\alpha\|_{s+k+2}^{\gamma,\mathcal{O}}$$

$$+ \sum_{h=1}^{k-1} \|q_{m-h}\|_{m-h,s,p+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{s_0+1+\mathrm{w}}^{\gamma,\mathcal{O}} + \|q_{m-h}\|_{m-h,s_0,p+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{s+1+\mathrm{w}}^{\gamma,\mathcal{O}}$$

$$\underline{\mathcal{K}} \overset{\sharp}{\lesssim}$$

$$+ \sum_{h,s,p,\rho} \sum_{h=1}^{k-1} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\sigma_h+p+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma_h+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{s_0+1+\mathrm{w}}^{\gamma,\mathcal{O}}$$

$$\lesssim m,s,p,\rho \sum_{h=1}^{k-1} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\sigma_h+p+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma_h+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{s+1+\mathrm{w}}^{\gamma,\mathcal{O}}$$

$$+ \sum_{h=1}^{k-1} \sum_{1 \leq k_1+k_2 \leq s_0}^{\gamma,\mathcal{O}} \|w\|_{m,k_1,k_2+\sigma_h+p+\mathrm{w}}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma_h+\mathrm{w}}^{\gamma,\mathcal{O}}$$

$$\lesssim m,s,p,M,\mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\hat{\sigma}_{k-1}+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}}$$

$$\|r_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p,M,\mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\hat{\sigma}_{k-1}+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}}$$

$$\|q_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} \lesssim s,p \|r_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} + \sum_{s}^{*} \|r_{m-k}\|_{m-k,k_1,p+k_2}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+2s_0+2}^{\gamma,\mathcal{O}}$$

$$\|q_{m-k}\|_{m-k,s_0,p}^{\gamma,\mathcal{O}} \lesssim \|r_{m-k}\|_{m-k,s_0,p+s_0}^{\gamma,\mathcal{O}}$$

$$\|q_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p,M,\mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\hat{\sigma}_{k-1}+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}}$$

$$+ \sum_{s}^{*} \left(\sum_{k_1}^{*} \|w\|_{m,k_1',k_2'+\hat{\sigma}_{k-1}+p+k_2}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3'+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}} \right) \|\alpha\|_{k_3+2s_0+2}^{\gamma,\mathcal{O}}$$

$$\|\alpha\|_{k_3'+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+2s_0+2}^{\gamma,\mathcal{O}} \lesssim \|\alpha\|_{k_3'+k_3+\hat{\sigma}_{k-1}+2s_0+2}^{\gamma,\mathcal{O}}$$

$$\|q_{m-k}\|_{m-k,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p,M,\mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\hat{\sigma}_{k-1}+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}}$$

$$+ \sum_{s}^{*} \sum_{k_1}^{*} \|w\|_{m,k_1',k_2'+\hat{\sigma}_{k-1}+p+k_2}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3'+k_3+2s_0+2+\hat{\sigma}_{k-1}}^{\gamma,\mathcal{O}}$$

$$\lesssim m,s,p,M,\mathrm{b} \sum_{s}^{*} \|w\|_{m,k_1,k_2+\sigma_k+p}^{\gamma,\mathcal{O}} \|\alpha\|_{k_3+\sigma_k}^{\gamma,\mathcal{O}}$$

$q_{m-k} \in S^{m-k}, b \rightsquigarrow \chi \in S^1, N \rightsquigarrow m-k+1+\rho$ (note that $2 < N \leq 2|m| + 2\rho + 1$),

$$\|r_{-\rho}\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \lesssim \sum_{k=0}^{m+\rho-1} \|q_{m-k}\# \geq \rho + m - k + 1\chi\|_{-\rho,s,p}^{\gamma,\mathcal{O}}$$

$$\lesssim_{m,s,p,M,\mathrm{b}} \sum_{k=0}^{4} \|q_{m-k}\|_{m-k,s,2|m|+2\rho+1+p}^{\gamma,\mathcal{O}} \|\chi\|_{1,s_0+6|m|+3+6\rho+p,p}^{\gamma,\mathcal{O}}$$

$$+ \|q_{m-k}\|_{m-k,s_0,2|m|+2\rho+1+p}^{\gamma,\mathcal{O}} \|\chi\|_{1,s+6|m|+3+6\rho+p,p}^{\gamma,\mathcal{O}}$$

$$\lesssim_{m,s,p,M,\mathrm{b}} \sum_{k=0}^{(6,11)} \|q_{m-k}\|_{m-k,s,2|m|+2\rho+1+p}^{\gamma,\mathcal{O}} \|\alpha\|_{s_0+6|m|+4+6\rho+p}^{\gamma,\mathcal{O}}$$

$$+ \|q_{m-k}\|_{m-k,s_0,2|m|+2\rho+1+p}^{\gamma,\mathcal{O}} \|\alpha\|_{s+6|m|+4+6\rho+p}^{\gamma,\mathcal{O}}$$

$$\left\|r_{-\rho}\right\|_{-\rho,s,p}^{\gamma,\mathcal{O}} \lesssim m,s,p,M,\mathrm{b} \sum_{k=0}^{m+\rho-1}\left(\sum_{s}^{*}\|w\|_{m,k_1,k_2+\sigma_k+2|m|+1+2\rho+p}^{\gamma,\mathcal{O}}\|\alpha\|_{k_3+\sigma_k}^{\gamma,\mathcal{O}}\right)\|\alpha\|_{s_0+6|m|+4+6\rho+p}^{\gamma,\mathcal{O}}$$

$$+\sum_{k=0}^{m+\rho-1}\left(\sum_{s_0}^{*}\|w\|_{m,k_1,k_2+\sigma_k+2|m|+1+2\rho+p}^{\gamma,\mathcal{O}}\|\alpha\|_{k_3+\sigma_k}^{\gamma,\mathcal{O}}\right)\|\alpha\|_{s+6|m|+4+6\rho+p}^{\gamma,\mathcal{O}}$$

$$\lesssim m,s,p,M,\mathrm{b}\sum_{s}^{\widehat{\epsilon,\hbar}}\|w\|_{m,k_1,k_2+\hat{\sigma}+p}^{\gamma,\mathcal{O}}\|\alpha\|_{k_3+\hat{\sigma}+p}^{\gamma,\mathcal{O}}$$

$$\partial_\tau Q^\tau=[X^\tau,Q^\tau]+\mathcal{M}^\tau,\mathcal{M}^\tau:=\mathrm{Op}\left(\mathbf{r}_{-\rho}\right),Q^0=\mathrm{Op}(w)$$

$$\partial_\tau R^\tau=[X^\tau,R^\tau]-\mathcal{M}^\tau,R^0=0$$

$$\partial_\tau V^\tau=\partial_\tau((\mathcal{C}_\alpha^\tau)^{-1})\circ R^\tau\circ\mathcal{C}_\alpha^\tau+(\mathcal{C}_\alpha^\tau)^{-1}\circ(\partial_\tau R^\tau)\circ\mathcal{C}_\alpha^\tau+(\mathcal{C}_\alpha^\tau)^{-1}\circ R^\tau\circ(\partial_\tau\mathcal{C}_\alpha^\tau)$$

$$\overset{\sqrt{-g},\sqrt{\mathrm{d}^4\partial\chi}}{=}-(\mathcal{C}_\alpha^\tau)^{-1}\circ X^\tau\circ R^\tau\circ\mathcal{C}_\alpha^\tau$$

$$+(\mathcal{C}_\alpha^\tau)^{-1}\circ([X^\tau,R^\tau]-\mathcal{M}^\tau)\circ\mathcal{C}_\alpha^\tau+(\mathcal{C}_\alpha^\tau)^{-1}\circ R^\tau\circ X^\tau\circ\mathcal{C}_\alpha^\tau=-(\mathcal{C}_\alpha^\tau)^{-1}\circ\mathcal{M}^\tau\circ\mathcal{C}_\alpha^\tau$$

$$R^\tau=-\int_0^\tau\mathcal{C}_\alpha^\tau\circ(\mathcal{C}_\alpha^t)^{-1}\circ\mathcal{M}^t\circ\mathcal{C}_\alpha^t\circ(\mathcal{C}_\alpha^\tau)^{-1}dt$$

$$\langle D\rangle^{m_1}R^\tau\langle D\rangle^{m_2}=-\int_0^\tau\underbrace{\langle D\rangle^{m_1}\mathcal{C}_\alpha^\tau\langle D\rangle^{-N-m_1}}_{=:A_1}\underbrace{\langle D\rangle^{N+m_1}(\mathcal{C}_\alpha^t)^{-1}\langle D\rangle^{-2N-m_1}}_{=:A_2}$$

$$\times\underbrace{\langle D\rangle^{m_1+2N}\mathcal{M}^t\langle D\rangle^{m_2+2N}}_{=:B}\underbrace{\langle D\rangle^{-2N-m_2}\mathcal{C}_\alpha^t\langle D\rangle^{m_2+N}}_{=:A_3}\underbrace{\langle D\rangle^{-N-m_2}(\mathcal{C}_\alpha^\tau)^{-1}\langle D\rangle^{m_2}}_{=:A_4}d\tau$$

$$N_1:=-m_1,N_2:=N+m_1,N_1+N_2=N:=\lfloor\nu/2\rfloor+4+\mathrm{b}>\lfloor\nu/2\rfloor+3+\mathrm{b}.$$

$$\left\|\langle\mathrm{d}_\varphi\rangle^q\,\mathrm{A}_1\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,q}1+\|\alpha\|_{s+\sigma}^{\gamma,\mathcal{O}},q=0,\mathrm{b}$$

$$\overline{\overline{\left\|\langle\mathrm{d}_\varphi\rangle^q\,\mathrm{B}\right\|_s^{\gamma,\mathcal{O}}}}\lesssim_{s,s_1,n_1,q}\left\|r_{-\rho}\right\|_{-\rho+1,s+s_*+|m_1+2N|+q+1,0}^{\gamma,\mathcal{O}}$$

$$\overset{\partial\dagger/\partial t\text{ with }p=0}{\lesssim}{}^{*}_{s,s_1,M,q}\sum_{s}^{*}\|w\|_{m,k_1+\hat{\sigma},k_2+\hat{\sigma}}^{\gamma,\mathcal{O}}\|\alpha\|_{k_3+\hat{\sigma}}^{\gamma,\mathcal{O}}$$

$$\Pi_\pm:=\chi_\pm(D):=\mathrm{Op}(\chi_\pm(\xi))$$

$$\chi_+(\xi):=\begin{cases}0 & \text{if}\quad \xi\le-\frac{1}{2}\\ \frac{1}{2} & \text{if}\quad \xi=0\\ 1 & \text{if}\quad \xi\ge\frac{1}{2}\end{cases}\quad\partial_\xi\chi_+(\xi)\ge0\;\forall\xi\in\mathbb{R},\chi_+(\xi)+\chi_-(\xi)=1$$

$$C(s,\mathrm{b})\gamma^{-1}\|a\|_{s_0+\sigma}^{\gamma,\mathcal{O}}\le1$$

$$|\mathfrak{a}_+|^{\gamma,\Lambda}\le2\|a\|_{s_0+\sigma}^{\gamma,\mathcal{O}}$$

$$\Omega_1:=\{\omega\in\mathcal{O}:|\omega\cdot\ell+(1+\mathfrak{a}_+(\omega))j|>2\gamma\langle\ell\rangle^{-\tau},\forall(\ell,j)\in\mathbb{Z}^{\nu+1}\setminus\{0\}\}$$

$$\alpha_\sigma(\varphi,x)=-\alpha_\sigma(-\varphi,-x),\alpha_-(\varphi,x):=-\alpha_+(\varphi,-x)$$

$$\|\alpha_\sigma\|_s^{\gamma,\Omega_1}\lesssim_s\gamma^{-1}\|a\|_{s+\sigma}^{\gamma,\mathcal{O}},\forall s\ge s_0$$

$$L := \mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-$$

$$L \circ \left(\omega \cdot \partial_\varphi - \mathrm{i}(1+a)|D|\right) \circ L^{-1} = \omega \cdot \partial_\varphi - \mathrm{i}(1+\mathfrak{a}_+)|D| + R$$

$$\left\| \langle \mathrm{d}_\varphi \rangle^q \langle D \rangle^{m_1} \mathrm{R} \langle D \rangle^{m_2} \right\|_s^{\gamma,\Omega_1} \lesssim_{s,s_1,q,M} \gamma^{-1} \|a\|_{s+\sigma}^{\gamma,\mathcal{O}}, q = 0, \mathrm{b}$$

$$[\Pi_\sigma, \mathcal{C}_\alpha] = \mathrm{Op}(g_\sigma), \sigma \in \{\pm\}, [\mathrm{Op}(\chi), \mathcal{C}_\alpha] = \mathrm{Op}(h),$$

$$\|g_\sigma\|_{-N,s,p}^{\gamma,\mathcal{O}} \lesssim N, s, p \|\alpha\|_{s+\mu}^{\gamma,\mathcal{O}}, \|h\|_{-N,s,p}^{\gamma,\mathcal{O}} \lesssim_{N,s,p} \|\alpha\|_{s+\mu}^{\gamma,\mathcal{O}}$$

$$\|f_\sigma\|_{-N,s,p}^{\gamma,\mathcal{O}} \lesssim N, s, p \|r\|_{m,s+\mu,p+\mu}^{\gamma,\mathcal{O}}$$

$$\begin{aligned} g_\sigma(\omega;\varphi,x,\xi) \equiv g_\sigma(\varphi,x,\xi) &:= \sum_{k\in\mathbb{Z}} (\chi_\sigma(\xi+k) - \chi_\sigma(\xi))\hat{\mathfrak{t}}_\alpha(\varphi,k,\xi)e^{\mathrm{i}kx} \\ &= \sum_{k\in\mathbb{Z}} (\chi_\sigma(\xi+k) - \chi_\sigma(\xi))(\hat{\mathfrak{t}}_\alpha(\varphi,k,\xi) - \delta(k,0))e^{\mathrm{i}kx} \end{aligned}$$

$$\left|\partial_\xi^p \hat{g}_\sigma(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} \lesssim_{s,N,p} \frac{1}{\langle \ell,k\rangle^s} \|\alpha\|_{s+N+p+\eta}^{\gamma,\mathcal{O}} \langle\xi\rangle^{-N-p},$$

$$\chi_\sigma(k+\xi) - \chi_\sigma(\xi) \neq 0 \Rightarrow |k| > |\xi| - \frac{1}{2}$$

$$\begin{aligned} \chi_\sigma(k+\xi) - \chi_\sigma(\xi) &= \chi_\sigma(k+\xi)(\chi_+(\xi) + \chi_-(\xi)) - \chi_\sigma(\xi)(\chi_+(k+\xi) + \chi_-(k+\xi)) \\ &= -\sigma\chi_+(\xi)\chi_-(k+\xi) + \sigma\chi_-(\xi)\chi_+(k+\xi) \end{aligned}$$

$$\begin{aligned} &\left|\partial_\xi^p \hat{g}_\sigma(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} = \\ &= \frac{1}{(2\pi)^\nu} \left| \int_{\mathbb{T}^\nu} d\varphi e^{-\mathrm{i}\ell\cdot\varphi} \sum_{p_1+p_2=p} \binom{p}{p_1} \partial_\xi^{p_1}(\chi_\sigma(\xi+k) - \chi_\sigma(\xi)) \partial_\xi^{p_2}(\hat{\mathfrak{t}}_\alpha(\varphi,k,\xi) - \delta(k,0)) \right|^{\gamma,\mathcal{O}} \\ &\lesssim p \sum_{p_1+p_2=p} \left|\partial_\xi^{p_1}(\chi_\sigma(\xi+k) - \chi_\sigma(\xi))\right| \left| \int_{\mathbb{T}^\nu} d\varphi e^{-\mathrm{i}\ell\cdot\varphi} \partial_\xi^{p_2(\hat{\mathfrak{t}}_\alpha(\varphi,k,\xi) - \delta(k,0))} \right|^{\gamma,\mathcal{O}} \\ &\lesssim p \sum_{p_1+p_2=p} \left| \int_{\mathbb{T}^{\nu+1}} d\varphi dx e^{-\mathrm{i}\ell\cdot\varphi - \mathrm{i}kx} \partial_\xi^{p_2}\left(e^{\mathrm{i}\alpha(\varphi,x)\xi} - 1\right)\right|^{\gamma,\mathcal{O}} = C_p \sum_{p_1+p_2=p} \left|\partial_\xi^{p_2} \widehat{\left(\overline{\mathfrak{t}_\alpha} - 1\right)}(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} \end{aligned}$$

$$\begin{aligned} \left|\widehat{(\mathfrak{t}_\alpha - 1)}(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} &\lesssim_{s,N,p} \frac{\langle\xi\rangle^{N+p}}{\langle \ell,k\rangle^{s+N+p}} \|\alpha\|_{s+N+p+\eta}^{\gamma,\mathcal{O}} \langle\xi\rangle^{-N-p} \\ &\lesssim_{s,N,p} \frac{1}{\langle \ell,k\rangle^s} \|\alpha\|_{s+N+p+\eta}^{\gamma,\mathcal{O}} \langle\xi\rangle^{-N-p} \end{aligned}$$

$$\left|\widehat{\left(\alpha^{p_2}\mathfrak{t}_\alpha\right)}(\ell,k,\xi)\right|^{\gamma,\mathcal{O}} \lesssim_{s,N} \frac{1}{\langle \ell,k\rangle^s} \|\alpha\|_{s+N+p+\eta}^{\gamma,\mathcal{O}} \langle\xi\rangle^{-N-p},$$

$$C(s_1, N, \mathrm{b})\mathfrak{d}(s_0 + \sigma) < 1$$

$$L^{-1} = \left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right) \circ (\mathrm{Id} + \tilde{R}), (\bar{L})^{-1} = \overline{L^{-1}}$$

$$\left\| \langle \mathrm{d}_\varphi \rangle^q \langle D \rangle^{m_1} \tilde{\mathrm{R}} \langle D \rangle^{m_2} \right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,\mathrm{b},N} \mathfrak{d}(s+\sigma), q = 0, \mathrm{b}$$

$$\begin{aligned} L\circ\Gamma &= \left(\mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-\right)\circ\left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right)\\ &= \mathcal{C}_{\alpha_+}\Pi_+\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_+}\Pi_+\mathcal{C}_{\alpha_-}^{-1}\Pi_- + \mathcal{C}_{\alpha_-}\Pi_-\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-\mathcal{C}_{\alpha_-}^{-1}\Pi_-\\ &= \Pi_+^2 + \Pi_-^2 + 2\Pi_+\Pi_-\\ &+\left(\mathcal{C}_{\alpha_+}\mathcal{C}_{\alpha_-}^{-1} - \mathrm{Id}\right)\Pi_+\Pi_- + \left(\mathcal{C}_{\alpha_-}\mathcal{C}_{\alpha_+}^{-1} - \mathrm{Id}\right)\Pi_-\Pi_+\\ &+\mathcal{C}_{\alpha_+}\left[\Pi_+,\mathcal{C}_{\alpha_+}^{-1}\right]\Pi_+ + \mathcal{C}_{\alpha_+}\left[\Pi_+,\mathcal{C}_{\alpha_-}^{-1}\right]\Pi_- + \mathcal{C}_{\alpha_-}\left[\Pi_-,\mathcal{C}_{\alpha_+}^{-1}\right]\Pi_+ + \mathcal{C}_{\alpha_-}\left[\Pi_-,\mathcal{C}_{\alpha_-}^{-1}\right]\Pi_- \end{aligned}$$

$$L\circ\Gamma = \mathrm{Id} + Q,\, Q := ד + ב.$$

$$\begin{aligned} \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{m_1}B\langle D\rangle^{m_2} &= \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{m_1}\mathcal{C}_{\alpha_+}\left[\Pi_+,\mathcal{C}_{\alpha_+}^{-1}\right]\langle D\rangle^{m_2}\\ &= \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{m_1}\mathcal{C}_{\alpha_+}\langle D\rangle^{-m_1-N'}\langle D\rangle^{m_1+N'}\left[\Pi_+,\mathcal{C}_{\alpha_+}^{-1}\right]\langle D\rangle^{m_2} \end{aligned}$$

$$\mathfrak{S}(\mathrm{R}_1) = \langle D\rangle^{m_1}\left(\mathcal{C}_{\alpha_+} - \mathrm{Id}\right)\langle D\rangle^{-m_1-N'} \quad \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_1\right\|_s^{\gamma,\mathcal{O}} \lesssim s, s_1 N, \mathrm{b}\mathfrak{d}(s+\sigma),$$

$$\mathfrak{S}(\mathrm{R}_2) = \langle D\rangle^{m_1+N'}\left[\Pi_+,\mathcal{C}_{\alpha_+}^{-1}\right]\langle D\rangle^{m_2}, \left\|\langle\, \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_2\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,N,\,\mathrm{b}} \mathfrak{d}(s+\sigma)$$

$$\|\beta_+\|_s^{\gamma,\Omega_1} \lesssim_s \gamma^{-1}\|a\|_{s+\sigma}^{\gamma,\mathcal{O}}, \forall s \geq s_0$$

$$\omega\cdot\partial_\varphi\beta_+ - (1+a)(1+\partial_x\beta_+) = -(1+\mathfrak{a}_+)$$

$$\beta_+(\varphi,x) = -\beta_+(-\varphi,-x)$$

$$\omega\cdot\partial_\varphi\beta_- + (1+a)(1+\partial_x\beta_-) = 1+\mathfrak{a}_+$$

$$x\mapsto y = x+\alpha_\sigma(\varphi,x) \Leftrightarrow y\mapsto x = y+\beta_\sigma(\varphi,y), \sigma\in\{\pm\}.$$

$$x\mapsto y = x+\tau\alpha_\sigma(\varphi,x) \;\Leftrightarrow\; y\mapsto x = y+\breve{\alpha}_\sigma(\tau;\varphi,y), \breve{\alpha}_\sigma(1;\varphi,x) = \beta_\sigma(\varphi,x).$$

$$A := \omega\cdot\partial_\varphi - \mathrm{i}(1+a)|D| = A_+\Pi_+ + A_-\Pi_- \text{with } A_\sigma := \omega\cdot\partial_\varphi - \sigma(1+a)\partial_x \mathrm{Op}(\chi)$$

$$L\circ A\circ L^{-1} = \mathcal{C}_{\alpha_+}\circ A_+\circ\mathcal{C}_{\alpha_+}^{-1}\circ\Pi_+ + \mathcal{C}_{\alpha_-}\circ A_-\circ\mathcal{C}_{\alpha_-}^{-1}\circ\Pi_- + Q$$

$$\begin{aligned} \left[L,\omega\cdot\partial_\varphi\right]L^{-1} = \left[\mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-,\omega\cdot\partial_\varphi\right]L^{-1} &= \sum_{\sigma\in\{\pm\}}\left[\mathcal{C}_{\alpha_\sigma},\omega\cdot\partial_\varphi\right]\Pi_\sigma L^{-1}\\ &= -\sum_{\sigma\in\{\pm\}}\left(\omega\cdot\partial_\varphi\alpha_\sigma\right)\mathcal{C}_{\alpha_\sigma}\partial_x\Pi_\sigma\left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right) + Q_1 \end{aligned}$$

$$Q_1 := -\sum_{\sigma\in\{\pm\}}\left(\omega\cdot\partial_\varphi\alpha_\sigma\right)\mathcal{C}_{\alpha_\sigma}\partial_x\Pi_\sigma\left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right)\tilde{R}$$

$$\begin{aligned} \left[L,\omega\cdot\partial_\varphi\right]L^{-1} &= -\sum_{\sigma,\sigma'\in\{\pm\}}\left(\omega\cdot\partial_\varphi\alpha_\sigma\right)\mathcal{C}_{\alpha_\sigma}\partial_x\Pi_\sigma\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'} + Q_1\\ &= -\sum_{\sigma\in\{\pm\}}\left(\omega\cdot\partial_\varphi\alpha_\sigma\right)\mathcal{C}_{\alpha_\sigma}\partial_x\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma + Q_1 + Q_2\\ &= \sum_{\sigma\in\{\pm\}}\left[\mathcal{C}_{\alpha_\sigma},\omega\cdot\partial_\varphi\right]\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma + Q_1 + Q_2 \end{aligned}$$

$$Q_2 := -\sum_{\sigma\in\{\pm\}}\left(\omega\cdot\partial_\varphi\alpha_\sigma\right)\mathcal{C}_{\alpha_\sigma}\partial_x\left(\left(\mathcal{C}_{\alpha_{-\sigma}}^{-1} - \mathcal{C}_{\alpha_\sigma}^{-1}\right)\Pi_\sigma\Pi_{-\sigma} + \left[\Pi_\sigma,\mathcal{C}_{\alpha_{-\sigma}}^{-1} - \mathcal{C}_{\alpha_\sigma}^{-1}\right]\Pi_{-\sigma} + \left[\Pi_\sigma,\mathcal{C}_{\alpha_\sigma}^{-1}\right]\right)$$

$$L\circ\omega\cdot\partial_\varphi\circ L^{-1}=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}\circ\omega\cdot\partial_\varphi\circ\mathcal{C}_{\alpha_\sigma}^{-1}\circ\Pi_\sigma+Q_1+Q_2$$

$$L\circ B\circ L^{-1}=\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}\Pi_\sigma B\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}+Q_3$$

$$Q_3:=(\mathcal{C}_{\alpha_+}\Pi_++\mathcal{C}_{\alpha_-}\Pi_-)B(\mathcal{C}_{\alpha_+}^{-1}\Pi_++\mathcal{C}_{\alpha_-}^{-1}\Pi_-)\tilde{R}$$

$$\begin{aligned}L\circ B\circ L^{-1}&=\\&=\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_\sigma\Pi_{\sigma'}+\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B\left[\Pi_\sigma,\mathcal{C}_{\alpha_{\sigma'}}^{-1}\right]\Pi_{\sigma'}+\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}[\Pi_\sigma,B]\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}+Q_3\\&=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma+P_1+P_2+Q_3\end{aligned}$$

$$P_1:=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B\left(\mathcal{C}_{\alpha_{-\sigma}}^{-1}-\mathcal{C}_{\alpha_\sigma}^{-1}\right)\Pi_\sigma\Pi_{-\sigma}$$

$$P_2:=\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B\left[\Pi_\sigma,\mathcal{C}_{\alpha_{\sigma'}}^{-1}\right]\Pi_{\sigma'}+\sum_{\sigma,\sigma'\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}[\Pi_\sigma,B]\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}$$

$$B=B_+\Pi_++B_-\Pi_-\quad\text{where}\quad B_+:=-(1+a)\partial_x\mathrm{Op}(\chi),B_-:=(1+a)\partial_x\mathrm{Op}(\chi)$$

$$\begin{aligned}L\circ B\circ L^{-1}&=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_\sigma\Pi_\sigma\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma+\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_{-\sigma}\Pi_{-\sigma}\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma+P_1+P_2+Q_3\\&=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}\circ B_\sigma\circ\mathcal{C}_{\alpha_\sigma}^{-1}\circ\Pi_\sigma+P_3+P_4+P_1+P_2+Q_3\end{aligned}$$

$$\begin{aligned}P_3&:=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_\sigma[\Pi_\sigma,\mathcal{C}_{\alpha_\sigma}^{-1}]\Pi_\sigma+\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_{-\sigma}[\Pi_{-\sigma},\mathcal{C}_{\alpha_\sigma}^{-1}]\Pi_\sigma\\P_4&:=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_{-\sigma}\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_{-\sigma}\Pi_\sigma+\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}B_\sigma\mathcal{C}_{\alpha_\sigma}^{-1}(\Pi_\sigma^2-\Pi_\sigma)\\&=\sum_{\sigma\in\{\pm\}}\mathcal{C}_{\alpha_\sigma}(B_{-\sigma}-B_\sigma)\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_{-\sigma}\Pi_\sigma\\&=\left(\mathcal{C}_{\alpha_+}(B_--B_+)\left(\mathcal{C}_{\alpha_+}^{-1}-\mathcal{C}_{\alpha_-}^{-1}\right)-\left(\mathcal{C}_{\alpha_-}-\mathcal{C}_{\alpha_+}\right)(B_--B_+)\mathcal{C}_{\alpha_-}^{-1}\right)\Pi_-\Pi_+\end{aligned}$$

$$\begin{aligned}&\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{m_1}\mathcal{M}\mathcal{C}_{\alpha_\sigma}\Pi_\sigma\mathcal{Q}\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}\tilde{R}\langle D\rangle^{m_2}\\&\quad=\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\langle D\rangle^{m_1}\mathcal{M}\langle D\rangle^{-m_1}\circ\langle D\rangle^{m_1}\mathcal{C}_{\alpha_\sigma}\langle D\rangle^{-m_1-N'}\circ\langle D\rangle^{m_1+N'}\Pi_\sigma\mathcal{Q}\langle D\rangle^{-m_1-N'-1}\\&\quad\circ\langle D\rangle^{m_1+N'+1}\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}\langle D\rangle^{-m_1-2N'-1}\circ\langle D\rangle^{m_1+2N'+1}\tilde{R}\langle D\rangle^{m_2}\end{aligned}$$

$$\mathfrak{S}(\mathrm{R}_0)=\langle D\rangle^{m_1}\mathcal{M}\langle D\rangle^{-m_1},\ \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_0\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,\mathrm{b}}1+\gamma^{-1}\|a\|_{s_0+\mu}^{\gamma,\mathcal{O}}$$

$$\mathfrak{S}(\mathrm{R}_1)=\langle D\rangle^{m_1}\left(\mathcal{C}_{\alpha_\sigma}-\mathrm{Id}\right)\langle D\rangle^{-m_1-N'},\ \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_1\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,\mathrm{b}}\gamma^{-1}\|a\|_{s_0+\mu}^{\gamma,\mathcal{O}}$$

$$\mathfrak{S}(\mathrm{R}_2)=\langle D\rangle^{m_1+N'}\Pi_\sigma\mathcal{Q}\langle D\rangle^{-m_1-N'-1},\ \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_2\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,\mathrm{b}}1+\gamma^{-1}\|a\|_{s_0+\mu}^{\gamma,\mathcal{O}}$$

$$\mathfrak{S}(\mathrm{R}_3)=\langle D\rangle^{m_1+N'+1}\left(\mathcal{C}_{\alpha_{\sigma'}}^{-1}-\mathrm{Id}\right)\langle D\rangle^{-m_1-2N'-1},\ \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_3\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,\mathrm{b}}\gamma^{-1}\|a\|_{s_0+\mu}^{\gamma,\mathcal{O}}$$

$$\mathfrak{S}(\mathrm{R}_4)=\langle D\rangle^{m_1+2N'+1}\tilde{R}\langle D\rangle^{m_2},\ \left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_4\right\|_s^{\gamma,\mathcal{O}}\lesssim_{s,s_1,M,\mathrm{b}}\gamma^{-1}\|a\|_{s_0+\mu}^{\gamma,\mathcal{O}}$$

$$\mathfrak{S}(\mathrm{R}_5) = \langle D\rangle^{m_1}\Pi_\sigma\Pi_{-\sigma}\langle D\rangle^{m_2}, \left\|\left\langle \mathrm{d}_\varphi\right\rangle^{\mathrm{b}}\mathrm{R}_5\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,M,\mathrm{b}} 1$$

$$\begin{aligned}
\mathcal{C}_{\alpha_\sigma} \circ A_\sigma \circ \mathcal{C}_{\alpha_\sigma}^{-1} &= \mathcal{C}_{\alpha_\sigma} \circ \left(\omega\cdot\partial_\varphi - \sigma(1+a)\partial_x \mathrm{Op}(\chi)\right) \circ \mathcal{C}_{\alpha_\sigma}^{-1} \\
&= \omega\cdot\partial_\varphi + \mathcal{C}_{\alpha_\sigma}(\omega\cdot\partial_\varphi\breve{\alpha}_\sigma)\partial_x - \sigma\mathcal{C}_{\alpha_\sigma}\left((1+a)(1+\partial_x\breve{\alpha}_\sigma)\right)\partial_x \circ \mathcal{C}_{\alpha_\sigma}\mathrm{Op}(\chi)\mathcal{C}_{\alpha_\sigma}^{-1} \\
&= \omega\cdot\partial_\varphi + \left(\mathcal{C}_{\alpha_\sigma}(\omega\cdot\partial_\varphi\breve{\alpha}_\sigma) - \sigma\mathcal{C}_{\alpha_\sigma}\left((1+a)(1+\partial_x\breve{\alpha}_\sigma)\right)\mathrm{Op}(\chi)\right)\partial_x + Q_{4,\sigma} \\
&\overset{\alpha_\sigma}{\sigma}{}^{\beta_\sigma}\omega\cdot\partial_\varphi + \mathcal{C}_{\alpha_\sigma}\left(\omega\cdot\partial_\varphi\beta_\sigma - \sigma(1+a)(1+\partial_x\beta_\sigma)\right)\mathrm{Op}(\chi)\partial_x + Q_{4,\sigma} + Q_{5,\sigma} \\
&\partial_\dagger^\star/d^4\chi,\sqrt{-g}\,\omega\cdot\partial_\varphi - \sigma(1+\mathfrak{a}_+)\mathrm{Op}(\chi)\partial_x + Q_{4,\sigma} + Q_{5,\sigma}
\end{aligned}$$

$$R := Q + \left(Q_{4,+} + Q_{5,+}\right)\Pi_+ + \left(Q_{4,-} + Q_{5,-}\right)\Pi_-$$

$$\begin{aligned}
L \circ A \circ L^{-1} &= \left(\omega\cdot\partial_\varphi - (1+\mathfrak{a}_+)\mathrm{Op}(\chi)\partial_x\right)\Pi_+ + \left(\omega\cdot\partial_\varphi + (1+\mathfrak{a}_+)\mathrm{Op}(\chi)\partial_x\right)\Pi_- + R \\
&= \left(\omega\cdot\partial_\varphi - \mathrm{i}(1+\mathfrak{a}_+)\mathrm{Op}(\chi(\xi)|\xi|)\right)(\Pi_+ + \Pi_-) + R \\
&= \omega\cdot\partial_\varphi - \mathrm{i}(1+\mathfrak{a}_+)|D| + R
\end{aligned}$$

$$\mathcal{L} = \omega\cdot\partial_\varphi - \mathrm{i}E\mathcal{O}\mathrm{p}\left((\mathbb{I} + b_1(\varphi,x)\mathbf{1})\mathrm{D_m}(\xi) + \mathrm{i}b_0(\varphi,x)\mathbf{1}\xi\mathrm{D_m^{-1}}(\xi) + b_{-1}(\varphi,x)\mathbf{1}\mathrm{D_m^{-1}}(\xi)\right)$$

$$b_1 := -\frac{1}{2}a^{(2)}, b_0 := \frac{1}{2}a^{(1)}, b_{-1} := \frac{1}{2}\left(\mathrm{m}a^{(2)} + a^{(0)}\right).$$

$$\epsilon(s) := \|b_1\|_s + \|b_0\|_s + \|b_{-1}\|_s$$

$$\gamma^{-7/2}\epsilon(s_0 + \sigma_*) \le \delta_*$$

$$\begin{aligned}
\mathcal{L}_2 &:= \mathcal{U}^{-1}\mathcal{L}\mathcal{U} \\
&= \omega\cdot\partial_\varphi - \mathrm{i}E\mathrm{Op}\left(\lambda(\varphi,x)\mathbb{I}\mathrm{D_m}(\xi) + \mathrm{i}b_0(\varphi,x)\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi) + A_0^{(2)}(\varphi,x,\xi) + A_{-1}^{(2)}(\varphi,x,\xi)\right)
\end{aligned}$$

$$\lambda(\varphi,x) := \sqrt{1 + 2b_1(\varphi,x)}, A_0^{(2)}(\varphi,x,\xi) := \left(\frac{0}{c^{(2)}(\varphi,x,-\xi)} c^{(2)}(\varphi,x,\xi)\right), c^{(2)} \in S^0,$$

$$\|A_0^{(2)}\|_{0,s,p}^{\gamma,\Lambda}, \|A_{-1}^{(2)}\|_{-1,s,p}^{\gamma,\Lambda} \lesssim_{s,p} \epsilon(s+p+5), \|\lambda - 1\|_{0,s,p}^{\gamma,\Lambda}, \|U - \mathbb{I}\|_{0,s,p}^{\gamma,\Lambda} \lesssim_s \epsilon(s)$$

$$f := f(\varphi,x) := \frac{1 + b_1 + \lambda}{\sqrt{(1 + b_1 + \lambda)^2 - b_1^2}}, g := g(\varphi,x) := \frac{-b_1}{\sqrt{(1 + b_1 + \lambda)^2 - b_1^2}}$$

$$\|\lambda - 1\|_s^{\gamma,\Lambda} + \|f - 1\|_s^{\gamma,\Lambda} + \|g\|_s^{\gamma,\Lambda} \lesssim_s \|b_1\|_s^{\gamma,\Lambda}$$

$$\det(U) = f^2 - g^2 = 1, U^{-1} = \begin{pmatrix} f & -g \\ -g & f \end{pmatrix}, U^{-1}E(\mathbb{I} + b_1\mathbf{1})U = E\lambda\mathbb{I}$$

$$\mathcal{L}_2 = \mathcal{U}^{-1}\omega\cdot\partial_\varphi\mathcal{U} - \mathrm{i}\mathrm{Op}\left(U^{-1}E\left((\mathbb{I} + b_1\mathbf{1})\mathrm{D_m}(\xi) + \mathrm{i}b_0\mathbf{1}\xi\mathrm{D_m^{-1}}(\xi) + b_{-1}\mathbf{1}\mathrm{D_m^{-1}}(\xi)\right)\#U\right)$$

$$\mathcal{U}^{-1}\omega\cdot\partial_\varphi\mathcal{U} = \omega\cdot\partial_\varphi - \mathrm{i}E\begin{pmatrix} 0 & \mathrm{i}\left(f(\omega\cdot\partial_\varphi g) - g(\omega\cdot\partial_\varphi f)\right) \\ -\mathrm{i}\left(f(\omega\cdot\partial_\varphi g) - g(\omega\cdot\partial_\varphi f)\right) & 0 \end{pmatrix}$$

$$U^{-1}E(\mathbb{I} + b_1\mathbf{1})\mathrm{D_m}(\xi) \qquad U = E\lambda\mathbb{I}U^{-1}\mathrm{D_m}(\xi)U = E\lambda\mathbb{I}\mathrm{D_m}(\xi) + E\lambda U^{-1}(\mathrm{D_m}(\xi)\mathbb{I} \star U)$$

$$E\lambda U^{-1}(\mathrm{D_m}(\xi)\mathbb{I} \star U) = E\begin{pmatrix} e_1 & h_1 \\ h_1 & e_1 \end{pmatrix}, \quad \begin{aligned} e_1 &:= \lambda[f(\mathrm{D_m}(\xi) \star f) - g(\mathrm{D_m}(\xi) \star g)], \\ h_1 &:= \lambda[f(\mathrm{D_m}(\xi) \star g) - g(\mathrm{D_m}(\xi) \star f)], \end{aligned}$$

$$f(D_m(\xi) \star f) = -\frac{\mathrm{i}}{2}\{D_m(\xi), f^2\} + fD_m(\xi)\#_{\ge 2}f$$

$$e_1 = \lambda[f\mathrm{D_m}(\xi)\#_{\geq 2}(f-1) - g\mathrm{D_m}(\xi)\#_{\geq 2}g] \in S^{-1}$$

$$U^{-1}\mathrm{i}b_0 E\mathbf{1}\xi\mathrm{D_m^{-1}}(\xi) \qquad U = E\big(\mathrm{i}b_0\xi\mathrm{D_m^{-1}}(\xi)\mathbf{1} + \mathrm{i}EU^{-1}b_0(\xi\mathrm{D_m^{-1}}(\xi)E\mathbf{1}\star U)\big)$$

$$A_{-1}^{(2)} := \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} + \mathrm{i}EU^{-1}b_0(\xi\mathrm{D_m^{-1}}(\xi)E\mathbf{1}\star U) + Ub_{-1}\mathbf{1}\mathbf{D}_\mathrm{m}^{-1}(\xi)\#U$$

$$c^{(2)} := \mathrm{i}\left(f\big(\omega\cdot\partial_\varphi g\big) - g\big(\omega\cdot\partial_\varphi f\big)\right) + h_1 + \mathrm{i}b_0\xi\mathrm{D_m^{-1}}(\xi)$$

$$\left\|f\big(\omega\cdot\partial_\varphi g\big) - g\big(\omega\cdot\partial_\varphi f\big)\right\|_s^{\gamma,\Lambda} \lesssim_s \|b_1\|_{s+1}^{\gamma,\Lambda} + \|b_1\|_s^{\gamma,\Lambda}\|b_1\|_{s_0+1}^{\gamma,\Lambda} \lesssim_s \epsilon(s+1)$$

$$\begin{aligned}\|\lambda g(\mathrm{D_m}(\xi)\star f)\|_{0,s,p}^{\gamma,\Lambda} &\lesssim_{s,p} \|\lambda g\|_s^{\gamma,\Lambda}\|\mathrm{D_m}(\xi)\star f\|_{0,s_0,p}^{\gamma,\Lambda} + \|\lambda g\|_{s_0}^{\gamma,\Lambda}\|\mathrm{D_m}(\xi)\star f\|_{0,s,p}^{\gamma,\Lambda} \\ &\lesssim_{s,p} \|b_1\|_s^{\gamma,\Lambda}\|b_1\|_{s_0+p+3}^{\gamma,\Lambda} + \|b_1\|_{s_0}^{\gamma,\Lambda}\|b_1\|_{s+p+3}^{\gamma,\Lambda} \lesssim_{s,p} \epsilon(s+p+3)\end{aligned}$$

$$\begin{aligned}\|\lambda g(\mathrm{D_m}(\xi)\star f)\|_{0,s,p}^{\gamma,\Lambda} &\lesssim s,p\|\lambda g\|_s^{\gamma,\Lambda}\|\mathrm{D_m}(\xi)\star f\|_{0,s_0,p}^{\gamma,\Lambda} + \|\lambda g\|_{s_0}^{\gamma,\Lambda}\|\mathrm{D_m}(\xi)\star f\|_{0,s,p}^{\gamma,\Lambda} \\ &\lesssim s,p\|\lambda g\|_s^{\gamma,\Lambda}\|f-1\|_{s_0+p+3}^{\gamma,\Lambda} + \|\lambda g\|_{s_0}^{\gamma,\Lambda}\|f-1\|_{s+p+3}^{\gamma,\Lambda} \\ &\lesssim s,p\|b_1\|_s^{\gamma,\Lambda}\|b_1\|_{s_0+p+3}^{\gamma,\Lambda} + \|b_1\|_{s_0}^{\gamma,\Lambda}\|b_1\|_{s+p+3}^{\gamma,\Lambda} \lesssim_{s,p} \epsilon(s+p+3).\end{aligned}$$

$$\begin{aligned}\|\lambda f\mathrm{D_m}(\xi)\#_{\geq 2}(f-1)\|_{-1,s,p}^{\gamma,\Lambda} &\lesssim_{s,p} \|\lambda f\|_s^{\gamma,\Lambda}\|f-1\|_{s_0+p+5}^{\gamma,\Lambda} + \|\lambda f\|_{s_0}^{\gamma,\Lambda}\|f-1\|_{s+p+5}^{\gamma,\Lambda} \\ &\lesssim_{s,p} \epsilon(s+p+5)\end{aligned}$$

$$\|U^{-1}b_0(\xi\mathrm{D_m^{-1}}(\xi)E\mathbf{1}\star U)\|_{-1,s,p}^{\gamma,\Lambda}, \|U^{-1}b_{-1}\mathbf{1}\mathbf{D}_\mathrm{m}^{-1}(\xi)\#U\|_{-1,s,p}^{\gamma,\Lambda} \lesssim_{s,p} \epsilon(s+p+3)$$

$$\begin{aligned}\mathcal{L}_3 &:= \mathbf{\Psi}\mathcal{L}_2\mathbf{\Psi}^{-1} \\ &= \omega\cdot\partial_\varphi - \mathrm{i}E\mathrm{Op}\big(\lambda(\varphi,x)\mathbb{I}\mathrm{D_m}(\xi) + \mathrm{i}b_0(\varphi,x)\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi) + A^{(3)}(\varphi,x,\xi) + R_{-\rho}^{(3)}(\varphi,x,\xi)\big)\end{aligned}$$

$$A^{(3)}(\varphi,x,\xi) := \begin{pmatrix} c^{(3)}(\varphi,x,\xi) & 0 \\ 0 & \dfrac{c^{(3)}(\varphi,x,-\xi)}{c^{(3)} \in S^{-1}} \end{pmatrix},$$

$$\left\|c^{(3)}\right\|_{-1,s,p}^{\gamma,\Lambda} \lesssim s,p\epsilon(s+p+\mu), \forall p \geq 0$$

$$\left\|R_{-\rho}^{(3)}\right\|_{-\rho,s,p}^{\gamma,\Lambda} \lesssim_{s,p,\rho} \epsilon(s+\mu), 0 \leq p \leq p_*$$

$$\|(\Psi^{\pm 1} - \mathbb{I})h\|_s^{\gamma,\Lambda} \lesssim_{s,\rho} \epsilon(s_0+\mu)\|h\|_s^{\gamma,\Lambda} + \epsilon(s+\mu)\|h\|_{s_0}^{\gamma,\Lambda}$$

$$\mathcal{Y}^{(0)} := \mathcal{L}_2 =: = \omega\cdot\partial_\varphi - \mathrm{i}E\mathrm{Op}(d(\varphi,x,\xi) + Q_0)$$
$$d(\varphi,x,\xi) := \lambda(\varphi,x)\mathbb{I}\mathbb{D}_\mathrm{m}(\xi) + \mathrm{i}b_0(\varphi,x)\mathbb{I}\xi\mathbb{D}_\mathrm{m}^{-1}(\xi)$$

$$Q_0 := A_0^{(2)} + A_{-1}^{(2)} := \begin{pmatrix} \dfrac{r_0(\varphi,x,\xi)}{q_0(\varphi,x,-\xi)} & \dfrac{q_0(\varphi,x,\xi)}{r_0(\varphi,x,-\xi)} \end{pmatrix}, r_0 \in S^{-1}, q_0 \in S^0$$

$$\mathcal{Y}^{(j)} := \omega\cdot\partial_\varphi - \mathrm{i}E\mathrm{Op}\big(d(\varphi,x,\xi) + Q_j + R_j\big)$$

$$Q_j = \begin{pmatrix} \dfrac{r_j(\varphi,x,\xi)}{q_j(\varphi,x,-\xi)} & \dfrac{q_j(\varphi,x,\xi)}{r_j(\varphi,x,-\xi)} \end{pmatrix}, r_j \in S^{-1}, q_j \in S^{-j}, R_j \in S^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$$

$$\left\|r_j\right\|_{-1,s,p}^{\gamma,\Lambda}, \left\|q_j\right\|_{-j,s,p}^{\gamma,\Lambda} \lesssim_{s,p,\rho,j} \epsilon(s+\mu_j+p), \forall p \geq 0$$
$$\left\|R_j\right\|_{-\rho,s,p}^{\gamma,\Lambda} \lesssim_{s,p_*,\rho,j} \epsilon\big(s+\mu_j\big), \forall 0 \leq p \leq p_*$$

$$\mathbf{\Psi}_j := \mathbb{I} + \mathrm{Op}\big(M_j(\varphi, x, \xi)\big)$$

$$M_j(\varphi, x, \xi) := \begin{pmatrix} 0 & m_j(\varphi, x, \xi) \\ \overline{m_j(\varphi, x, -\xi)} & 0 \end{pmatrix} \; m_j = \frac{-q_j(\varphi, x, \xi)}{2\lambda(\varphi, x)\mathrm{D_m}(\xi)} \in S^{-(j+1)}$$

$$\mathcal{Y}^{(j+1)} = \mathbf{\Psi}_j^{-1}\mathcal{Y}^{(j)}\mathbf{\Psi}_j$$

$$\|m_j\|_{-j-1,s,p}^{\gamma,\Lambda} \lesssim s, p, \rho, j\epsilon\big(s + \mu_j + p\big), \forall p \geq 0$$

$$\mathbf{\Psi}_j^{-1} - \mathbb{I} = \sum_{p=1}^{\infty} \Big(\mathrm{Op}\big(-M_j\big)\Big)^p = \mathrm{Op}\big(\widetilde{M}_j\big), \widetilde{M}_j := -M_j + M_{j,<\rho} + M_{j,\geq\rho}$$

$$\|M_{j,<\rho}\|_{-j-2,s,p}^{\gamma,\Lambda} \lesssim j, s, \rho, p$$

$$\|M_{j,\geq\rho}\|_{-\rho,s,p}^{\gamma,\Lambda} \lesssim \big(s + \hat{\mu}_j + p\big) \; \forall p \geq 0$$

$$j, s, \rho, p_*\epsilon\big(s + \hat{\mu}_j\big), 0 \leq p \leq p_*$$

$$\mathbf{\Psi}_j^{-1}\mathcal{Y}^{(j)}\mathbf{\Psi}_j = \mathbf{\Psi}_j^{-1}\big(\omega \cdot \partial_\varphi \mathbf{\Psi}_j\big) - \mathrm{i}\mathbf{\Psi}_j^{-1}\mathrm{E}\mathrm{Op}\big(d(\varphi, x, \xi) + Q_j + R_j\big)\mathbf{\Psi}_j$$

$$\mathbf{\Psi}_j^{-1}\big(\omega \cdot \partial_\varphi \mathbf{\Psi}_j\big) = \omega \cdot \partial_\varphi + \Big(\mathbb{I} + \mathrm{Op}\big(\widetilde{M}_j\big)\Big) \circ \mathrm{Op}\big(\omega \cdot \partial_\varphi M_j\big)$$

$$= \omega \cdot \partial_\varphi + \mathrm{Op}\Big(\omega \cdot \partial_\varphi M_j + \widetilde{M}_j \# \big(\omega \cdot \partial_\varphi M_j\big)\Big) = \omega \cdot \partial_\varphi - \mathrm{i}E\mathrm{Op}\big(Q_{j+1}^{(1)} + R_{j+1}^{(1)}\big)$$

$$Q_{j+1}^{(1)} := \mathrm{i}E\Big(\omega \cdot \partial_\varphi M_j - M_j \#_{<\rho}\big(\omega \cdot \partial_\varphi M_j\big) + M_{j,<\rho}\#_{<\rho}\big(\omega \cdot \partial_\varphi M_j\big)\Big)$$

$$R_{j+1}^{(1)} := \mathrm{i}E\Big(-M_j \#_{\geq\rho}\big(\omega \cdot \partial_\varphi M_j\big) + M_{j,<\rho}\#_{\geq\rho}\big(\omega \cdot \partial_\varphi M_j\big) + M_{j,\geq\rho}\#\big(\omega \cdot \partial_\varphi M_j\big)\Big)$$

$$\|Q_{j+1}^{(1)}\|_{-j-1,s,p}^{\gamma,\Lambda} \lesssim j, s, \rho, p$$

$$\|R_{j+1}^{(1)}\|_{-\rho,s,p}^{\gamma,\Lambda} \lesssim \big(s + \hat{\mu}_j + p\big) \; \forall p \geq 0$$

$$j, s, \rho, p_*\epsilon\big(s + \hat{\mu}_j\big), 0 \leq p \leq p_*$$

$$-\mathrm{i}\mathbf{\Psi}_j^{-1}\mathrm{Op}\big(ER_j\big)\mathbf{\Psi}_j = -\mathrm{i}E\mathrm{Op}\big(R_{j+1}^{(2)}\big)$$

$$ER_{j+1}^{(2)} := ER_j + \widetilde{M}_j \# ER_j + ER_j \# M_j + \widetilde{M}_j \# ER_j \# M_j$$

$$-\mathrm{i}\mathbf{\Psi}_j^{-1}\mathrm{Op}\big(EQ_j\big)\mathbf{\Psi}_j = -\mathrm{i}\mathrm{Op}\big(EQ_j + \widetilde{M}_j \# EQ_j + EQ_j \# M_j + \widetilde{M}_j \# EQ_j \# M_j\big)$$

$$= -\mathrm{i}E\mathrm{Op}\big(Q_j + Q_{j+1}^{(3)} + R_{j+1}^{(3)}\big)$$

$$EQ_{j+1}^{(3)} := \big(-M_j + M_{j,<\rho}\big)\#_{<\rho} EQ_j \#_{<\rho}\big(\mathbb{I} + M_j\big) + EQ_j \#_{<\rho} M_j$$

$$\begin{aligned}
-\mathrm{i}\mathbf{\Psi}_j^{-1}E\mathrm{Op}(d(\varphi, x, \xi))\mathbf{\Psi}_j &\overset{\Theta}{=} -\mathrm{i}\mathbf{\Psi}_j^{-1}E\mathrm{Op}(\lambda\mathbb{I}\mathrm{D_m}(\xi) + \mathrm{i}b_0\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi))\mathbf{\Psi}_j \\
&\overset{\phi}{=} -\mathrm{i}E\mathrm{Op}(\lambda\mathbb{I}\mathrm{D_m}(\xi) + \mathrm{i}b_0\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi)) \\
&\quad -\mathrm{i}E\mathrm{Op}\Big((E\lambda\mathbb{I}\mathrm{D_m}(\xi)) \star \big(EM_j\big)\Big) \\
&\quad -\mathrm{i}E\mathrm{Op}\big(E\big(M_{j,<\rho} + M_{j,\geq\rho}\big)\# E\lambda\mathbb{I}_{\mathrm{m}}(\xi)\big) \\
&\quad -\mathrm{i}E\mathrm{Op}\big(E\widetilde{M}_j \# E\mathrm{i}b_0\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi) + \mathrm{i}b_0\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi)\# M_j\big) \\
&\quad -\mathrm{i}E\mathrm{Op}\big(E\widetilde{M}_j \# E(\lambda\mathbb{I}\mathrm{D_m}(\xi) + \mathrm{i}b_0\mathbb{I}\xi\mathrm{D_m^{-1}}(\xi))\# M_j\big)
\end{aligned}$$

$$\mathrm{r}(\varphi, x, \xi) := \lambda\mathrm{D_m}(\xi) \diamond m_j + m_j \wr \lambda\mathrm{D_m}(\xi) = 2m_j\lambda\mathrm{D_m}(\xi) + \mathrm{q}_{j,<\rho} + \mathrm{q}_{j,\geq\rho}$$

$$\mathrm{q}_{j,<\rho} := \sum_{k=1}^{\rho-1} \lambda \mathrm{D_m}(\xi) \#_k m_j + m_j \#_k \lambda \mathrm{D_m}(\xi) \in S^{-(j+1)}$$
$$\mathrm{q}_{j,\geq\rho} := \lambda \mathrm{D_m}(\xi) \#_{\geq\rho} m_j + m_j \#_{\geq\rho} \lambda \mathrm{D_m}(\xi) \in S^{-\rho}$$

$$\delta_\dagger^{*} + \delta_\dagger^{\star} + \delta_\dagger^{\circ} = -\mathrm{i}EOp\left(Q_{j+1}^{(4)} + R_{j+1}^{(4)}\right)$$

$$\begin{aligned} Q_{j+1}^{(4)} := {} & EM_{j,<\rho} \#_{<\rho} E\lambda \mathbb{I}\mathbb{D}_\mathrm{m}(\xi) \\ & + E\left(-M_j + M_{j,<\rho}\right) \#_{<\rho} E \mathrm{i} b_0 \mathbb{I} \xi \mathrm{D}_\mathrm{m}^{-1}(\xi) + \mathrm{i} b_0 \mathbb{I} \xi \mathrm{D}_\mathrm{m}^{-1}(\xi) \#_{<\rho} M_j \\ & + E\left(-M_j + M_{j,<\rho}\right) \#_{<\rho} E\left(\lambda \mathbb{I}\mathbb{D}_\mathrm{m}(\xi) + \mathrm{i} b_0 \mathbb{I} \xi \mathrm{D}_\mathrm{m}^{-1}(\xi)\right) \#_{<\rho} M_j \end{aligned}$$

$$\begin{aligned} \Psi_j^{-1} \mathcal{Y}^{(j)} \Psi_j = {} & -\mathrm{i}E\mathrm{Op}\left(\lambda \mathbb{I}\mathbb{D}_\mathrm{m}(\xi) + \mathrm{i} b_0 \mathbb{I} \xi \mathrm{D}_\mathrm{m}^{-1}(\xi)\right) \\ & -\mathrm{i}E\mathrm{Op}\left(Q_{j+1}^{(1)} + Q_{j+1}^{(3)} + Q_{j+1}^{(4)} + R_{j+1}^{(1)} + R_{j+1}^{(2)} + R_{j+1}^{(3)} + R_{j+1}^{(4)}\right) \\ & -\mathrm{i}E\mathrm{Op}\left(Q_j + \left(E\lambda \mathbb{I}\mathbb{D}_\mathrm{m}(\xi)\right) \star \left(EM_j\right)\right). \end{aligned}$$

$$Q_j + \left(E\lambda \mathbb{I}\mathbb{D}_\mathrm{m}(\xi)\right) \star (EMj) = \begin{pmatrix} r_j(\varphi,x,\xi) & \tilde{q}_j(\varphi,x,\xi) \\ \tilde{q}_j(\varphi,x,-\xi) & r_j(\varphi,x,-\xi) \end{pmatrix}$$

$$\tilde{q}_j := q_j + 2m_j \lambda \mathrm{D_m}(\xi) + \mathrm{q}_{j,<\rho} + \mathrm{q}_{j,\geq\rho} = \mathrm{q}_{j,<\rho} + \mathrm{q}_{j,\geq\rho} \in S^{-(j+1)}$$

$$Q_{j+1} := Q_{j+1}^{(1)} + Q_{j+1}^{(3)} + Q_{j+1}^{(4)} + \begin{pmatrix} r_j(\varphi,x,\xi) & \mathrm{q}_{j,<\rho}(\varphi,x,\xi) \\ \mathrm{q}_{j,<\rho}(\varphi,x,-\xi) & r_j(\varphi,x,-\xi) \end{pmatrix}$$
$$R_{j+1} := R_{j+1}^{(1)} + R_{j+1}^{(2)} + R_{j+1}^{(3)} + R_{j+1}^{(4)} + \begin{pmatrix} 0 & \mathrm{q}_{j,\geq\rho}(\varphi,x,\xi) \\ \mathrm{q}_{j,\geq\rho}(\varphi,x,-\xi) & 0 \end{pmatrix}$$

$$|\mathrm{c}|^{\gamma,\Lambda} \lesssim \epsilon(s_0 + \mu)$$

$$\Omega_1 := \{\omega \in \Lambda : |\omega \cdot \ell - (1+\mathrm{c})j| \geq 2\gamma \langle \ell \rangle^{-\tau}, j \in \mathbb{Z}, \ell \in \mathbb{Z}^\nu, (\ell, j) \neq (0,0)\}$$

$$\mathcal{L}_4 := \Theta_1 \mathcal{L}_3 \Theta_1^{-1} = \omega \cdot \partial_\varphi - \mathrm{i}E\mathrm{Op}\left((1+\mathrm{c})\mathrm{D_m}(\xi)\mathbb{I} + \begin{pmatrix} \mathrm{i}^{(4)}(\varphi,x,\xi) & 0 \\ 0 & -\mathrm{i}c^{(4)}(\varphi,x,-\xi) \end{pmatrix}\right) - \mathrm{i}E\mathcal{R}^{(4)}$$

$$c^{(4)}(\varphi,x,\xi) := c_+^{(4)}(\varphi,x)\chi_+(\xi) - c_-^{(4)}(\varphi,x)\chi_-(\xi)$$

$$c_-^{(4)}(\varphi,x) = -c_+^{(4)}(\varphi,-x), c_\sigma^{(4)}(\varphi,x) = -c_\sigma^{(4)}(-\varphi,-x), \sigma \in \{\pm\}$$

$$\left\| c^{(4)} \right\|_{0,s,p}^{\gamma,\Omega_1} \lesssim s, \mathrm{b}, p\gamma^{-1}\epsilon(s+\mu), \forall p \geq 0$$

$$\left\| \langle \mathrm{d}_\varphi \rangle^q \mathrm{R}^{(4)} \langle D \rangle \right\|_s^{\gamma,\Omega_1} \lesssim s, s_1, \mathrm{b}\gamma^{-1}\epsilon(s+\mu), q = 0, \mathrm{b}$$

$$\|(\Theta_1)^{\pm} h\|_s^{\gamma,\Omega_1} \lesssim_s \|h\|_s^{\gamma,\Omega_1} + \gamma^{-1}\epsilon(s+\mu)\|h\|_{s_0}^{\gamma,\Omega_1}$$

$$\alpha_-(\varphi,x) := -\alpha_+(\varphi,-x), \alpha_+(\varphi,x) = -\alpha_+(-\varphi,-x), \forall (\varphi,x) \in \mathbb{T}^{\nu+1}$$
$$y = x + \alpha_\sigma(\varphi,x) \Leftrightarrow x = y + \breve{\alpha}_\sigma(\varphi,y), \sigma \in \{\pm\}, \forall x, y \in \mathbb{T}, \varphi \in \mathbb{T}^\nu$$

$$\|\breve{\alpha}_\sigma\|_s^{\gamma,\Omega_1}, \|\alpha_\sigma\|_s^{\gamma,\Omega_1} \lesssim s\gamma^{-1}\epsilon(s+\mu), \sigma \in \{\pm\}$$

$$\Theta_1 := \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix}, \quad \begin{aligned} L &:= \mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_- \\ \bar{L} &:= \mathcal{C}_{\alpha_+}\Pi_- + \mathcal{C}_{\alpha_-}\Pi_+ \end{aligned}$$

$$\Theta_1^{-1} = \begin{pmatrix} L^{-1} & 0 \\ 0 & \overline{L^{-1}} \end{pmatrix}, L^{-1} = \left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right) \circ (\mathrm{Id} + \tilde{R})$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^q \langle D\rangle^{m_1} \widetilde{\mathrm{R}} \langle D\rangle^{m_2}\right\|_s^{\gamma,\Omega_1} \lesssim_{s,s_1,\mathrm{b}} \gamma^{-1}\epsilon(s+\mu), q=0,\mathrm{b}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^q \langle D\rangle^{-N_1}(\mathrm{L}^{\pm}-\mathrm{Id})\langle D\rangle^{-N_2}\right\|_s^{\gamma,\Omega_1} \lesssim s, s_1, \mathrm{b}\gamma^{-1}\epsilon(s+\mu), q=0,\mathrm{b}$$

$$\mathcal{L}_4 = \Theta_1 \mathcal{L}_3 \Theta_1^{-1} = \begin{pmatrix} B_1 & 0 \\ 0 & \bar{B}_1 \end{pmatrix} - \mathrm{i}E\begin{pmatrix} B_2 & 0 \\ 0 & \bar{B}_2 \end{pmatrix} - \mathrm{i}EF$$

$$\begin{aligned} B_1 &:= L\left(\omega\cdot\partial_\varphi - \mathrm{iOp}(\lambda \mathrm{D_m}(\xi))\right)L^{-1} \\ B_2 &:= L\mathrm{Op}\left(\mathrm{i}_0\xi \mathrm{D_m^{-1}}(\xi) + c^{(3)}(\varphi,x,\xi)\right)L^{-1} \\ F &:= \Theta_1 \mathrm{Op}\left(R_{-\rho}^{(3)}(\varphi,x,\xi)\right)\Theta_1^{-1} \end{aligned}$$

$$B_1 = \omega\cdot\partial_\varphi - \mathrm{iOp}((1+\mathrm{c})\mathrm{D_m}(\xi)) + Q$$

$$\mathrm{D_m}(\xi) = |\xi|\chi(\xi) + \widetilde{\mathrm{D}}_\mathrm{m}(\xi), \widetilde{\mathrm{D}}_\mathrm{m}(\xi) := \sqrt{|\xi|^2+\mathrm{m}} - |\xi|\chi(\xi) \in S^{-1}$$

$$\begin{aligned} B_1 &= L(\omega\cdot\partial_\varphi - \mathrm{i}\lambda|D|)L^{-1} \\ &-\mathrm{i}L\mathrm{Op}\left(\lambda\widetilde{\mathrm{D}}_\mathrm{m}(\xi)\right)L^{-1} \end{aligned}$$

$$\eth = \omega\cdot\partial_\varphi - \mathrm{i}(1+\mathrm{c})|D| + Q_1$$

$$L\circ A\circ L^{-1} = \left(\mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-\right)\circ A\circ\left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right) + Q_2$$

$$Q_2 := \left(\mathcal{C}_{\alpha_+}\Pi_+ + \mathcal{C}_{\alpha_-}\Pi_-\right)\circ A\circ\left(\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}^{-1}\Pi_-\right)\circ\tilde{R}.$$

$$\begin{aligned} \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}} \mathcal{C}_{\alpha_\sigma}\Pi_\sigma A\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}\tilde{R}\langle D\rangle &= \langle \mathrm{d}_\varphi\rangle^{\mathrm{b}} \mathcal{C}_{\alpha_\sigma}\langle D\rangle^{-N'}\circ\langle D\rangle^{N'}\Pi_\sigma A\langle D\rangle^{-N'+1} \\ &\circ\langle D\rangle^{N'-1}\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'}\langle D\rangle^{-2N'+1}\circ\langle D\rangle^{2N'-1}\tilde{R}\langle D\rangle \end{aligned}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_1\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,\mathrm{b}} \gamma^{-1}\epsilon(s+\mu), \mathfrak{S}(\mathrm{R}_1) = \left(\mathcal{C}_{\alpha_\sigma} - \mathrm{Id}\right)\langle D\rangle^{-N'}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_2\right\|_s^{\gamma,\mathcal{O}} \lesssim_{s,s_1,\mathrm{b}} 1+\gamma^{-1}\epsilon(s+\mu), \mathfrak{S}(\mathrm{R}_2) = \langle D\rangle^{N'}\Pi_\sigma A\langle D\rangle^{-N'+1}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_3\right\|_s^{\gamma,\mathcal{O}} \lesssim s, s_1, \mathrm{b}\gamma^{-1}\epsilon(s+\mu), \mathfrak{S}(\mathrm{R}_3) = \langle D\rangle^{N'-1}\left(\mathcal{C}_{\alpha_{\sigma'}}^{-1} - \mathrm{Id}\right)\langle D\rangle^{-2N'+1}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}_4\right\|_s^{\gamma,\mathcal{O}} \lesssim s, s_1, \mathrm{b}\gamma^{-1}\epsilon(s+\mu), \mathfrak{S}(\mathrm{R}_4) = \langle D\rangle^{2N'-1}\tilde{R}\langle D\rangle$$

$$\begin{aligned} L\circ A\circ L^{-1} &= \sum_{\sigma,\sigma'\in\{\pm\}} \mathcal{C}_{\alpha_\sigma}\Pi_\sigma A\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'} + Q_2 \\ &= \sum_{\sigma\in\{\pm\}} \mathcal{C}_{\alpha_\sigma} A\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma \\ &+ \sum_{\sigma\in\{\pm\}} \mathcal{C}_{\alpha_\sigma} A\mathcal{C}_{\alpha_\sigma}^{-1}(\Pi_\sigma^2 - \Pi_\sigma) + \sum_{\sigma\in\{\pm\}} \mathcal{C}_{\alpha_\sigma} A\mathcal{C}_{\alpha_{-\sigma}}^{-1}\Pi_\sigma\Pi_{-\sigma} \\ &+ \sum_{\sigma,\sigma'\in\{\pm\}} \mathcal{C}_{\alpha_\sigma} A\left[\Pi_\sigma, \mathcal{C}_{\alpha_{\sigma'}}^{-1}\right]\Pi_{\sigma'} + \sum_{\sigma,\sigma'\in\{\pm\}} \mathcal{C}_{\alpha_\sigma}[\Pi_\sigma, A]\mathcal{C}_{\alpha_{\sigma'}}^{-1}\Pi_{\sigma'} + Q_2 \end{aligned}$$

$$L\circ A\circ L^{-1} = \mathcal{C}_{\alpha_+}A\mathcal{C}_{\alpha_+}^{-1}\Pi_+ + \mathcal{C}_{\alpha_-}A\mathcal{C}_{\alpha_-}^{-1}\Pi_- + Q_3$$

$$\aleph = \sum_{\sigma\in\{\pm\}} \mathcal{C}_{\alpha_\sigma} A\left(\mathrm{Id} - \mathcal{C}_{\alpha_\sigma}^{-1} + \mathcal{C}_{\alpha_{-\sigma}}^{-1} - \mathrm{Id}\right)\Pi_\sigma\Pi_{-\sigma}$$

$$\mathcal{C}_{\alpha_\sigma} A \mathcal{C}_{\alpha_\sigma}^{-1} = \mathrm{Op}\big((1+\mathrm{c})\widetilde{\mathrm{D}}_{\mathrm{m}}(\xi)\big) + Q_{5,\sigma}$$

$$\mathcal{C}_{\alpha_\sigma}\mathrm{Op}\big(\lambda\widetilde{\mathrm{D}}_{\mathrm{m}}(\xi)\big)\mathcal{C}_{\alpha_\sigma}^{-1} = \mathrm{Op}\big(\mathrm{r}_\sigma^{(1)}(\varphi,x,\xi) + \mathrm{r}_\sigma^{(2)}(\varphi,x,\xi)\big) + \tilde{Q}_{4,\sigma}$$

$$\mathrm{r}_\sigma^{(1)}(\varphi,x,\xi) := \lambda(\varphi,x+\alpha_\sigma(\varphi,x))\widetilde{\mathrm{D}}_{\mathrm{m}}\left(\xi\big(1+\partial_y\breve{\alpha}_\sigma(\varphi,y)\big)\right)_{|y=x+\alpha_\sigma(\varphi,x)}$$

$$\left\|\mathrm{r}_\sigma^{(2)}\right\|_{-2,s,p}^{\gamma,\mathcal{O}} \lesssim s,p\gamma^{-1}\epsilon(s+\mu+p)$$

$$\|\lambda(\varphi,x+\alpha_\sigma(\varphi,x)) - (1+\mathrm{c})\|_s^{\gamma,\Omega_1} \lesssim_s \gamma^{-1}\epsilon(s+\mu)$$

$$\begin{aligned}
&\widetilde{\mathrm{D}}_{\mathrm{m}}\left(\xi\big(1+\partial_y\breve{\alpha}_\sigma\big)\right) - \big(1+\partial_y\breve{\alpha}_\sigma\big)\widetilde{\mathrm{D}}_{\mathrm{m}}(\xi)\\
&= \big(1+\partial_y\breve{\alpha}_\sigma\big)[\underbrace{\left(\sqrt{\xi^2 + \frac{\mathrm{m}}{\big(1+\partial_y\breve{\alpha}_\sigma\big)^2}} - \sqrt{\xi^2+\mathrm{m}}\right)}_{=:(1)} + \underbrace{|\xi|\left(\chi(\xi) - \chi\left(\xi\big(1+\partial_y\breve{\alpha}_\sigma\big)\right)\right)}_{=:(2)}]
\end{aligned}$$

$$\left\|\widetilde{D}_{\mathrm{m}}\left(\xi\big(1+\partial_y\breve{\alpha}_\sigma\big)\right) - \big(1+\partial_y\breve{\alpha}_\sigma\big)\widetilde{D}_{\mathrm{m}}(\xi)\right\|_{-1,s,p}^{\gamma,\Omega_1} \lesssim_{s,p} \gamma^{-1}\epsilon(s+\mu)$$

$$\mathbf{r}_\sigma^{(1)}(\varphi,x,\xi) = (1+\mathrm{c})\widetilde{D}_{\mathrm{m}}(\xi) + q_\sigma$$

$$\begin{aligned}
B_1 &= \omega\cdot\partial_\varphi - \mathrm{i}(1+\mathrm{c})\mathrm{Op}(|\xi|\chi(\xi)) - \mathrm{i}\sum_{\sigma\in\{\pm\}}\big((1+\mathrm{c})\mathrm{Op}\big(\widetilde{\mathrm{D}}_{\mathrm{m}}(\xi)\big) + Q_{5,\sigma}\big)\Pi_\sigma + Q_1 - \mathrm{i}Q_3\\
&\triangleq \omega\cdot\partial_\varphi - \mathrm{i}\mathrm{Op}((1+\mathrm{c})\mathrm{D}_{\mathrm{m}}(\xi)) + Q
\end{aligned}$$

$$B_2 = \mathrm{Op}\big(\mathrm{i}c^{(4)}(\varphi,x,\xi)\big) + R$$

$$\Re = \sum_{\sigma\in\{\pm\}} \mathcal{C}_{\alpha_\sigma}\mathrm{Op}\big(\mathrm{i}b_0\xi\mathrm{D}_{\mathrm{m}}^{-1}(\xi) + c^{(3)}\big)\mathcal{C}_{\alpha_\sigma}^{-1}\Pi_\sigma + R_1$$

$$\mathcal{C}_{\alpha_\sigma}\mathrm{Op}\big(\mathrm{i}b_0\xi\mathrm{D}_{\mathrm{m}}^{-1}(\xi) + c^{(3)}\big)\mathcal{C}_{\alpha_\sigma}^{-1} = \mathrm{Op}(\mathrm{g}_\sigma) + R_{2,\sigma},$$

$$\mathrm{g}_\sigma(\varphi,x,\xi) = \mathrm{i}b_0(\varphi,x+\alpha_\sigma(\varphi,x))\frac{\xi\big(1+\partial_y\breve{\alpha}_\sigma(\varphi,y)\big)}{\sqrt{\left|\xi\big(1+\partial_y\breve{\alpha}_\sigma(\varphi,y)\big)\right|^2 + \mathrm{m}}_{|y=x+\alpha_\sigma(\varphi,x)}}. \square$$

$$\begin{aligned}
\mathrm{g}_\sigma(\varphi,x,\xi) &= \mathrm{i}c_\sigma^{(4)}(\varphi,x)\frac{\xi}{|\xi|}\chi(\xi) + \tilde{\mathrm{g}}_\sigma(\varphi,x,\xi)\\
\tilde{\mathrm{g}}_\sigma(\varphi,x,\xi) &:= (1-\chi(\xi))\mathrm{g}_\sigma(\varphi,x,\xi) + \mathrm{i}b_0(\varphi,x+\alpha_\sigma)\xi\mathrm{f}_\sigma(\varphi,x+\alpha_\sigma,\xi)\\
\mathrm{f}_\sigma(\varphi,y,\xi) &:= \chi(\xi)\left(\frac{1+\partial_y\breve{\alpha}_\sigma(\varphi,y)}{\sqrt{\left|\xi\big(1+\partial_y\breve{\alpha}_\sigma(\varphi,y)\big)\right|^2+\mathrm{m}}} - \frac{1}{|\xi|}\right).
\end{aligned}$$

$$\left\|c_\sigma^{(4)}\right\|_s^{\gamma,\Omega_1} \lesssim s\gamma^{-1}\epsilon(s+\mu).$$

$$\|\tilde{g}_\sigma\|_{-2,s,p}^{\gamma,\Omega_1} \lesssim_{s,p} \epsilon(s+\mu), \sigma\in\{\pm\}$$

$$\mathrm{f}_\sigma(\varphi,y,\xi)=\chi(\xi)\frac{|\xi|(1+\partial_y\breve{\alpha}_\sigma)-\sqrt{|\xi(1+\partial_y\breve{\alpha}_\sigma)|^2+\mathrm{m}}}{|\xi|\sqrt{|\xi(1+\partial_y\breve{\alpha}_\sigma)|^2+\mathrm{m}}}$$
$$=\frac{-\mathrm{m}\chi(\xi)}{\left(|\xi|(1+\partial_y\breve{\alpha}_\sigma)+\sqrt{|\xi(1+\partial_y\breve{\alpha}_\sigma)|^2+\mathrm{m}}\right)|\xi|\sqrt{|\xi(1+\partial_y\breve{\alpha}_\sigma)|^2+\mathrm{m}}},$$

$$\|\mathrm{f}_\sigma(\varphi,x+\alpha_\sigma(\varphi,x),\xi)\|_{-3,s,p}^{\gamma,\Omega_1}\lesssim s,p1+\gamma^{-1}\epsilon(s+\mu)$$

$$\underline{9.17}=\sum_{\sigma\in\{\pm\}}\mathrm{Op}\left(\mathrm{i}c_\sigma^{(4)}\frac{\xi}{|\xi|}\chi(\xi)\chi_\sigma(\xi)\right)+R_4=\mathrm{Op}\big(\mathrm{i}c^{(4)}(\varphi,x,\xi)\big)+\mathrm{Op}(\mathrm{r})+R_4$$

$$R_4:=R_1+\sum_{\sigma\in\{\pm\}}\left(R_{2,\sigma}+\mathrm{Op}(\tilde{\mathrm{g}}_\sigma)\right)\Pi_\sigma$$
$$\mathrm{r}:=\sum_{\sigma\in\{\pm\}}\mathrm{i}_\sigma^{(4)}(\xi|\xi|^{-1}\chi(\xi)\chi_\sigma(\xi)-\sigma\chi_\sigma(\xi))$$

$$R_{-\rho}^{(3)}=\begin{pmatrix}\frac{r_{-\rho}(\varphi,x,\xi)}{q_{-\rho}(\varphi,x,-\xi)} & \frac{q_{-\rho}(\varphi,x,\xi)}{r_{-\rho}(\varphi,x,-\xi)}\end{pmatrix},r_{-\rho},q_{-\rho}\in S^{-\rho}$$

$$\underline{\mathcal{H}}=\begin{pmatrix}\frac{L\mathrm{Op}(r_{-\rho})L^{-1}}{L\mathrm{Op}(q_{-\rho})\bar{L}^{-1}} & \frac{L\mathrm{Op}(q_{-\rho})\bar{L}^{-1}}{L\mathrm{Op}(r_{-\rho})L^{-1}}\end{pmatrix}$$

$$LOp(r_{-\rho})L^{-1}=\sum_{\sigma\in\{\pm\}}\mathrm{Op}\left(r_{-\rho}\left(\varphi,y,\xi(1+\partial_y\breve{\alpha}_\sigma)\right)_{|y=x+\alpha_\sigma}\chi_\sigma(\xi)\right)+T_1$$

$$\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}L\mathrm{Op}(q_{-\rho})\bar{L}^{-1}\langle D\rangle=\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}L\langle D\rangle^{-N'}\langle D\rangle^{N'}\mathrm{Op}(q_{-\rho})\langle D\rangle^{N'+1}\langle D\rangle^{-N'-1}\bar{L}^{-1}\langle D\rangle$$

$$\mathcal{L}_5:=\Theta_2\mathcal{L}_4\Theta_2^{-1}=\omega\cdot\partial_\varphi-\mathrm{i}(1+\mathrm{c})E\mathrm{D_m}-\mathrm{i}E\mathcal{R}^{(5)}$$

$$\left\|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}}\mathrm{R}^{(5)}\langle D\rangle\right\|_s^{\gamma,\Omega_1}\lesssim s,s_1,\mathrm{b}\gamma^{-2}\epsilon(s+\mu)$$

$$\left\|\Theta_2^{\pm}h\right\|_s^{\gamma,\Omega_1}\lesssim_s\|h\|_s^{\gamma,\Omega_1}+\gamma^{-2}\epsilon(s+\mu)\|h\|_{s_0}^{\gamma,\Omega_1}$$

$$\|d\|_{0,s,p}^{\gamma,\Omega_1},\|r_{-2}\|_{-2,s,p}^{\gamma,\Omega_1}\lesssim_{s,p}\gamma^{-2}\epsilon(s+\mu),$$

$$\omega\cdot\partial_\varphi d(\varphi,x,\xi)-(1+\mathrm{c})\partial_x d(\varphi,x,\xi)\xi\mathrm{D_m^{-1}}(\xi)=c^{(4)}(\varphi,x,\xi)+r_{-2}(\varphi,x,\xi)$$

$$d(\varphi,x,\xi):=d_+(\varphi,x)\chi_+(\xi)+d_-(\varphi,x)\chi_-(\xi),d_-(\varphi,x):=d_+(\varphi,-x).$$

$$\left(\omega\cdot\partial_\varphi-(1+\mathrm{c})\partial_x\right)d_+(\varphi,x)=c_+^{(4)}(\varphi,x)$$

$$\mathrm{i}(\omega\cdot\ell-(1+\mathrm{c})j)(d_+)_{\ell,j}=\left(c_+^{(4)}\right)_{\ell,j},\forall(\ell,j)\in\mathbb{Z}^{\nu+1}.$$

$$(d_+)_{\ell,j}=\frac{\left(c_+^{(4)}\right)_{\ell,j}}{\mathrm{i}(\omega\cdot\ell-(1+\mathrm{c})j)},\forall(\ell,j)\in\mathbb{Z}^{\nu+1}\setminus\{(0,0)\},(d_+)_{0,0}:=0$$

$$d_+(-\varphi,-x)=d_+(\varphi,x).$$

$$(\omega\cdot\partial_\varphi+(1+\mathrm{c})\partial_x)d_-(\varphi,x)=-c_-^{(4)}(\varphi,x)$$

$$\chi_\sigma(\xi)\xi \mathrm{D}_{\mathrm{m}}^{-1}(\xi) = \sigma\chi_\sigma(\xi) + \mathrm{r}_\sigma(\xi), \sigma \in \{\pm\}$$

$$\mathrm{r}_\sigma(\xi) := \chi_\sigma(\xi)(\xi \mathrm{D}_{\mathrm{m}}^{-1}(\xi) - \sigma 1) = -\mathrm{m}\chi_\sigma(\xi)\sigma \mathrm{D}_{\mathrm{m}}^{-1}(\xi)\left(\sigma\xi + \sqrt{\xi^2 + \mathrm{m}}\right)^{-1}$$

$$\begin{aligned}
&\omega \cdot \partial_\varphi d(\varphi, x, \xi) - (1 + \mathfrak{c})\partial_x d(\varphi, x, \xi)\xi \mathrm{D}_{\mathrm{m}}^{-1}(\xi) - c^{(4)}(\varphi, x, \xi) \\
&= \left(\omega \cdot \partial_\varphi d_+(\varphi, x) - (1 + \mathfrak{c})\partial_x d_+(\varphi, x) - c_+^{(4)}(\varphi, x)\right)\chi_+(\xi) \\
&+ \left(\omega \cdot \partial_\varphi d_-(\varphi, x) + (1 + \mathfrak{c})\partial_x d_-(\varphi, x) + c_-^{(4)}(\varphi, x)\right)\chi_-(\xi) \\
&-(1 + \mathfrak{c})\partial_x d_+(\varphi, x) r_+(\xi) - (1 + \mathfrak{c})\partial_x d_-(\varphi, x) r_-(\xi) \\
&\underline{\xi, \sqrt{\delta_\varsigma^\zeta}} - (1 + \mathfrak{c})\partial_x d_+(\varphi, x) r_+(\xi) - (1 + \mathfrak{c})\partial_x d_-(\varphi, x) r_-(\xi) =: r_{-2}(\varphi, x, \xi)
\end{aligned}$$

$$\Theta_2 := \begin{pmatrix} \Psi_2 & 0 \\ 0 & \Psi_2 \end{pmatrix} \text{ where } \Psi_2 := \Psi_2^1, \Psi_2^\tau := \exp\{\tau \mathrm{Op}(d(\varphi, x, \xi))\}$$

$$\|\Psi_2^\tau - \mathrm{Id}\|_{0,s,0}^{\gamma,\mathcal{O}} \lesssim_s \|d\|_{0,s,0}^{\gamma,\mathcal{O}} \lesssim_s \gamma^{-2}\epsilon(s + \mu)$$

$$\left\|\langle \mathrm{d}_\varphi \rangle^q \langle D \rangle^{-m_1}(\mathrm{B}^\pm - \mathrm{Id})\langle D \rangle^{-m_2}\right\|_s \lesssim s, s_1, \mathrm{b}\gamma^{-2}\epsilon(s + \mu), q = 0, \mathrm{b}$$

$$\begin{aligned}
\Theta_2\mathcal{L}_4\Theta_2^{-1} &= \Theta_2\omega \cdot \partial_\varphi\Theta_2^{-1} \\
&- \mathrm{i}\Theta_2 E \mathrm{Op}\left((1 + \mathfrak{c})\mathrm{D}_{\mathrm{m}}(\xi) + \begin{pmatrix} \mathrm{i}c^{(4)}(\varphi, x, \xi) & 0 \\ 0 & \overline{\mathrm{i}c^{(4)}(\varphi, x, -\xi)} \end{pmatrix}\right)\Theta_2^{-1} - \Theta_2 \mathrm{i} E \mathcal{R}^{(4)}\Theta_2^{-1} \\
&= \begin{pmatrix} F & 0 \\ 0 & \overline{F} \end{pmatrix} - \mathrm{i}E\begin{pmatrix} G_1 & 0 \\ 0 & \overline{G_1} \end{pmatrix} - \mathrm{i}E\begin{pmatrix} G_2 & 0 \\ & \overline{G_2} \end{pmatrix} - \mathrm{i}E\Theta_2\mathcal{R}^{(4)}\Theta_2^{-1}
\end{aligned}$$

$$F := \Psi_2\omega \cdot \partial_\varphi\Psi_2^{-1}, G_1 := \Psi_2 \mathrm{X}\Psi_2^{-1}, \mathrm{X} := \mathrm{Op}((1 + \mathfrak{c})\mathrm{D}_{\mathrm{m}}(\xi)), G_2 := \Psi_2\mathrm{Y}\Psi_2^{-1}, \mathrm{Y} := \mathrm{Op}\left(\mathrm{i}c^{(4)}\right)$$

$$F = \Psi_2\omega \cdot \partial_\varphi\Psi_2^{-1} = \omega \cdot \partial_\varphi - \mathrm{Op}\left(\omega \cdot \partial_\varphi d\right) + \mathcal{Q}_1$$

$$\mathcal{Q}_1 := -\int_0^1 (1 - \tau)\Psi_2^\tau \mathrm{ad}_{\mathrm{Op}(d)}\left[\mathrm{Op}\left(\omega \cdot \partial_\varphi d\right)\right]\Psi_2^{-\tau} d\tau$$

$$G_1 = \Psi_2 \mathrm{X}\Psi_2^{-1} = \mathrm{X} + [\mathrm{Op}(d), \mathrm{X}] + \mathcal{Q}_2, G_2 = \Psi_2\mathrm{Y}\Psi_2^{-1} = \mathrm{Y} + \mathcal{Q}_3$$

$$\mathcal{Q}_2 := \int_0^1 (1 - \tau)\Psi_2^\tau \mathrm{ad}_{\mathrm{Op}(d)}^2[\mathrm{X}]\Psi_2^{-\tau} d\tau, \mathcal{Q}_3 := \int_0^1 \Psi_2^\tau \mathrm{ad}_{\mathrm{Op}(d)}[\mathrm{Y}]\Psi_2^{-\tau} d\tau$$

$$[\mathrm{Op}(d), \mathrm{X}] = \mathrm{Op}\left(d \star ((1 + \mathfrak{c})\mathrm{D}_{\mathrm{m}}(\xi))\right) = \mathrm{Op}(\mathrm{i}(1 + \mathfrak{c})(\partial_x d)(\varphi, x, \xi)\xi \mathrm{D}_{\mathrm{m}}^{-1}(\xi) + \mathrm{r}_2)$$

$$F - \mathrm{i}(G_1 + G_2) = \omega \cdot \partial_\varphi - \mathrm{i}\mathrm{Op}((1 + \mathfrak{c})\mathrm{D}_{\mathrm{m}}(\xi)) + \mathrm{Op}(r) + \mathcal{Q}_4$$

$$\mathcal{M}_0 := \omega \cdot \partial_\varphi - \mathrm{i}E\mathcal{D}_0 - \mathrm{i}E\mathcal{P}_0$$

$$\mathrm{b} := 6\tau + 6$$

$$\mathcal{L}^{\mathrm{T}}(H^s, H^{s-1}) \otimes \mathcal{M}_2(\mathbb{C})$$

$$\begin{aligned}
&(\mathcal{D}_0)_-^- = (\mathcal{D}_0)_+^+ := \mathrm{diag}_{j \in \mathbb{N}_0} d_j^{(0)}, \quad (\mathcal{D}_0)_+^- = (\mathcal{D}_0)_-^+ \equiv 0 \\
&d_j^{(0)} := (1 + \mathfrak{c})\mathbb{ID}_{\mathrm{m}}(j), j \in \mathbb{N}, \qquad d_{\vec{0}}^{(0)} := (1 + \mathfrak{c})\sqrt{\mathrm{m}}
\end{aligned}$$

$$\gamma^{-1}|\mathfrak{c}|^{\gamma,\Lambda} \leq v_0$$

$$\varepsilon_0(s) := \gamma^{-3/2}\|\mathrm{P}_0\langle D \rangle\|_s^{\gamma^{3/2},\hat{\mathcal{O}}} < +\infty, \varepsilon_0(s, \mathrm{b}) := \gamma^{-3/2}\left\|\langle \mathrm{d}_\varphi \rangle^{\mathrm{b}}\mathrm{P}_0\langle D \rangle\right\|_s^{\gamma^{3/2},\hat{\mathcal{O}}} < +\infty.$$

$$C_0\varepsilon_0(s_*,\mathrm{b})\leq v_0$$

$$(\mathcal{D}_\infty)_-^- = (\mathcal{D}_\infty)_+^+ := \mathrm{diag}_{j\in\mathbb{N}_0} d_{\vec{j}}^{(\infty)}, (\mathcal{D}_\infty)_+^- = (\mathcal{D}_\infty)_-^+ \equiv 0$$
$$d_{\vec{j}}^{(\infty)} := d_{\vec{j}}^{(0)} + r_{\vec{j}}^{(\infty)}, r_{\vec{j}}^{(\infty)} := (r_\infty)_j^j \mathbb{I} + (r_\infty)_j^{-j} S, j\in\mathbb{N}, r_{\vec{0}}^{(\infty)} := (r_\infty)_0^0$$

$$(\mathrm{r}_\infty)_j^{\sigma j}:\Lambda\to\mathbb{R}, \sup_{j\in\mathbb{N}_0,\sigma\in\{\pm\}}\langle j\rangle\left|(\mathrm{r}_\infty)_j^{\sigma j}\right|^{\gamma^{3/2},\Lambda}\lesssim\gamma^{3/2}\varepsilon_0(s_*)$$

$$\lambda_{j,\pm}^{(\infty)} := (1+\mathrm{c})\mathrm{D_m}(j) + (\mathrm{r}_\infty)_j^j \pm (\mathrm{r}_\infty)_j^{-j}, j\in\mathbb{N}, \lambda_{0,\pm}^{(\infty)} := (1+\mathrm{c})\sqrt{\mathrm{m}} + (\mathrm{r}_\infty)_0^0$$

$$\mathcal{O}_\infty := \Omega_\infty^+ \cap \Omega_\infty^-$$
$$\Omega_\infty^+ := \left\{\omega\in\hat{\mathcal{O}}: \left|\omega\cdot\ell + \lambda_{j,\eta}^{(\infty)} + \lambda_{k,\eta}^{(\infty)}\right| \geq \frac{2\gamma}{\langle\ell\rangle^\tau}, j,k\in\mathbb{N}_0, \ell\in\mathbb{Z}^\nu, \eta\in\{\pm\}\right\}$$
$$\Omega_\infty^- := \left\{\omega\in\hat{\mathcal{O}}: \left|\omega\cdot\ell + \lambda_{j,\eta}^{(\infty)} - \lambda_{k,\eta}^{(\infty)}\right| \geq \frac{2\gamma^{3/2}}{\langle\ell\rangle^\tau}, \eta\in\{\pm\}\right.$$
$$\left. j,k\in\mathbb{N}_0, \ell\in\mathbb{Z}^\nu, (\ell,j,k)\neq(0,j,j)\right\}$$

$$\mathcal{M}_\infty(\omega) := \Phi_\infty(\omega)\circ\mathcal{M}_0\circ\Phi_\infty^{-1}(\omega) = \omega\cdot\partial_\varphi - \mathrm{i}E\mathcal{D}_\infty$$

$$\|\Phi_\infty^{\pm1} - \mathrm{Id}\|_s^{\gamma^{3/2},\mathcal{O}_\infty} \lesssim \varepsilon_0(s,\mathrm{b}), \forall s_*\leq s\leq s_1$$

$$\|(\Phi_\infty^{\pm1}(\varphi)-\mathrm{Id})u\|_{H_x^s} \lesssim \varepsilon_0(s_0,\mathrm{b})\|u\|_{H_x^s} + \varepsilon_0(s,\mathrm{b})\|u\|_{H_x^{s_0}}$$

$$\mathcal{M} = \omega\cdot\partial_\varphi - \mathrm{i}E\mathcal{D} - \mathrm{i}E\mathcal{P}$$

$$(\mathcal{D})_{+,\vec{j}}^{+,\vec{j}}(0) = d_{\vec{j}}^{(0)} + \mathrm{r}_{\vec{j}}, \mathrm{r}_{\vec{j}} := \mathrm{r}_j^j\mathbb{I} + \mathrm{r}_j^{-j}S, j\in\mathbb{N}, \mathrm{r}_{\vec{0}} := (\mathrm{r}_\infty)_0^0$$

$$\sup_{j\in\mathbb{N}_0,\eta\in\{\pm\}}\langle j\rangle\left|r_j^{\eta j}\right|^{\gamma^{3/2},\Lambda}\lesssim\gamma^{\frac{3}{2}}\varepsilon_0(s_*)$$

$$\varepsilon(s) := \gamma^{-3/2}\|\mathrm{P}\langle D\rangle\|_s^{\gamma^{3/2},\mathcal{O}} < +\infty, \varepsilon(s,\mathrm{b}) := \gamma^{-3/2}\left\|\left\langle \mathrm{d}_\varphi\right\rangle^{\mathrm{b}}\mathrm{P}\langle D\rangle\right\|_s^{\gamma^{3/2},\mathcal{O}} < +\infty$$

$$\mathcal{O}_+ := \Omega_+^+ \cap \Omega_+^-,$$
$$\Omega_+^+ := \left\{\omega\in\mathcal{O}: \left|\omega\cdot\ell + \lambda_{j,\eta} + \lambda_{k,\eta}\right| \geq \frac{\gamma}{\langle\ell\rangle^\tau}, \eta\in\{\pm\}, |\ell|\leq N, j,k\in\mathbb{N}_0\right\}$$
$$\Omega_+^- := \left\{\omega\in\mathcal{O}: \left|\omega\cdot\ell + \lambda_{j,\eta} - \lambda_{k,\eta}\right| \geq \frac{\gamma^{3/2}}{\langle\ell\rangle^\tau}, \eta\in\{\pm\},\right.$$
$$\left. |\ell|\leq N, j,k\in\mathbb{N}_0, \ell\in\mathbb{Z}^\nu, (\ell,j,k)\neq(0,j,j)\right\}$$

$$\lambda_{j,\pm} := (1+\mathrm{c})\mathrm{D_m}(j) + \mathrm{r}_j^j \pm \mathrm{r}_j^{-j}, \forall j\in\mathbb{N}, \lambda_{0,\pm} := (1+\mathrm{c})\sqrt{\mathrm{m}} + \mathrm{r}_0^0$$

$$\mathcal{S} = \left(\mathcal{S}_\sigma^{\sigma'}(\varphi)\right)_{\sigma,\sigma'\in\{\pm\}} \in \mathcal{L}^{\mathrm{T}}(H^s,H^s)\otimes\mathcal{M}_2(\mathbb{C}), \forall s_*\leq s\leq s_1$$

$$-\omega\cdot\partial_\varphi\mathcal{S}_\sigma^{\sigma'}(\varphi) + \mathrm{i}\sigma\mathcal{D}_\sigma^\sigma\mathcal{S}_\sigma^{\sigma'}(\varphi) - \mathrm{i}\sigma'\mathcal{S}_\sigma^{\sigma'}(\varphi)\mathcal{D}_{\sigma'}^{\sigma'} = \mathrm{i}\sigma\left((\Pi_N\mathcal{P})_\sigma^{\sigma'}(\varphi) - [\mathcal{P}]_\sigma^{\sigma'}\right)$$

$$\|\mathrm{S}\langle D\rangle\|_s^{\gamma^{3/2},\mathcal{O}_+} \lesssim N^{2\tau+1}\varepsilon(s), \left\|\left\langle\mathrm{d}_\varphi\right\rangle^{\mathrm{b}}\mathrm{S}\langle D\rangle\right\|_s^{\gamma^{3/2},\mathcal{O}_+} \lesssim N^{2\tau+1}\varepsilon(s,\mathrm{b})$$

$$-\mathrm{i}\omega\cdot\ell\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) + \mathrm{i}\sigma\mathcal{D}_{\sigma,\vec{j}}^{\sigma,\vec{j}}\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell) - \mathrm{i}\sigma'\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\mathcal{D}_{\sigma',\vec{k}}^{\sigma',\vec{k}} = \mathrm{i}\sigma\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)$$

$$\widetilde{\mathcal{D}}_{-,\vec{j}}^{-,\vec{j}}(0) := \mathfrak{U}^{-1}\mathcal{D}_{-,\vec{j}}^{-,\vec{j}}(0)\mathfrak{U} = \widetilde{\mathcal{D}}_{+,\vec{j}}^{+,\vec{j}}(0) = \begin{pmatrix}\lambda_{j,+} & 0\\ 0 & \lambda_{j,-}\end{pmatrix}, j\in\mathbb{N}, \widetilde{\mathcal{D}}_{-,\vec{j}}^{-,\vec{j}}(0) := \mathcal{D}_{-,\vec{j}}^{-,\vec{j}}(0) = \lambda_{0,\pm}$$

$$-\mathrm{i}\omega\cdot\ell\tilde{\mathcal{S}}_{\sigma,\eta j}^{\sigma',\eta' k}(\ell)+\left(\mathrm{i}\sigma\lambda_{j,\eta}-\mathrm{i}\sigma'\lambda_{k,\eta'}\right)\tilde{\mathcal{S}}_{\sigma,\eta j}^{\sigma',\eta' k}(\ell)=\mathrm{i}\sigma\tilde{\mathcal{P}}_{\sigma,\eta j}^{\sigma',\eta' k}(\ell),\forall\eta,\eta'\in\{\pm\}$$

$$\tilde{\mathcal{S}}_{\sigma,\eta j}^{\sigma',\eta k}(\ell)=\frac{\sigma\tilde{\mathcal{P}}_{\sigma,\eta j}^{\sigma',\eta k}(\ell)}{-\omega\cdot\ell+\sigma\lambda_{j,\eta}-\sigma'\lambda_{k,\eta}},\tilde{\mathcal{S}}_{\sigma,\eta j}^{\sigma',-\eta k}(\ell)=0$$

$$\left\|\tilde{\mathcal{P}}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\right\|_{\infty}\sim\left\|\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\right\|_{\infty}$$

$$\psi_{\ell,j,k}^{\sigma,\sigma',\eta}:=\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega):=-\omega\cdot\ell+\sigma\lambda_{j,\eta}(\omega)-\sigma'\lambda_{k,\eta}(\omega)$$

$$\tilde{\mathcal{S}}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)=\tilde{\mathcal{S}}_{-\sigma,\vec{j}}^{-\sigma',\vec{k}}(-\ell)$$

$$\left\|\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\right\|_{\infty}\lesssim\gamma^{-3/2}N^{\tau}\left\|\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\ell)\right\|_{\infty}$$

$$\begin{aligned}\left|\frac{1}{\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_1)}-\frac{1}{\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_2)}\right|&=\left|\frac{\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_1)-\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_2)}{\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_1)\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_2)}\right|\\&\lesssim\frac{|\omega_1-\omega_2|}{\left|\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_1)\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_2)\right|}\Bigg(|\ell|+\sup_{\omega_1\neq\omega_2}\frac{|\mathfrak{c}(\omega_1)-\mathfrak{c}(\omega_2)|}{|\omega_1-\omega_2|}|\sigma\mathrm{D_m}(j)-\sigma'\mathrm{D_m}(k)|\\&\quad+\sup_{\substack{j\in\mathbb{N}_0,\eta\in\{\pm\}\\\omega_1\neq\omega_2}}\frac{\left|r_j^{\eta j}(\omega_1)-r_j^{\eta j}(\omega_2)\right|}{|\omega_1-\omega_2|}\Bigg)\\&\overset{\text{ג,ד}}{\lesssim}|\omega_1-\omega_2|\frac{|\ell|+\langle\sigma|j|-\sigma'|k|\rangle}{\left|\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_1)\psi_{\ell,j,k}^{\sigma,\sigma',\eta}(\omega_2)\right|}\lesssim\gamma^{-2-\frac{\sigma+\sigma'}{2}}N^{2\tau+1}|\omega_1-\omega_2|.\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\omega_1;\ell)-\mathcal{S}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\omega_2;\ell)\right\|_{\infty}&\lesssim\gamma^{-\frac{3}{2}}N^{\tau}\left\|\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\omega_1;\ell)-\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\omega_2;\ell)\right\|_{\infty}\\&+\gamma^{-3}N^{2\tau+1}\left\|\mathcal{P}_{\sigma,\vec{j}}^{\sigma',\vec{k}}(\omega_1;\ell)\right\|_{\infty}|\omega_1-\omega_2|\end{aligned}$$

$$\check{\mathcal{S}}(\omega)\leq\gamma^{3/2}CN^{\tau}\check{\mathcal{P}}(\omega),\overline{\Delta_{12}\mathcal{S}}\leq\gamma^{3/2}CN^{\tau}\left(\overline{\Delta_{12}\mathcal{P}}\right)+\gamma^{-3}CN^{2\tau+1}\check{\mathcal{P}}(\omega_1),$$

$$\left(\tilde{M}_{\mathrm{S}}\right)_{\sigma}^{\sigma'}(\omega)\leq\gamma^{3/2}CN^{\tau}\left(\tilde{M}_{\mathrm{P}}\right)_{\sigma}^{\sigma'}(\omega),\left(\tilde{R}_{\mathrm{S}}\right)_{\sigma}^{\sigma'}(\omega)\leq\gamma^{3/2}CN^{\tau}\left(\tilde{R}_{\mathrm{P}}\right)_{\sigma}^{\sigma'}(\omega)$$

$$CN^{2\tau+1}\varepsilon(s_*)\leq 1$$

$$\begin{aligned}|\Phi^{\pm1}-\mathrm{Id}|_{s}^{\gamma^{3/2},\mathcal{O}_+}&\lesssim N^{2\tau+1}\varepsilon(s),\forall s_*\leq s\leq s_1,\\\left|\langle\mathrm{d}_{\varphi}\rangle^{b}\Phi^{\pm1}-\mathrm{Id}\right|_{s}^{\gamma^{3/2},\mathcal{O}_+}&\lesssim N^{2\tau+1+b}\varepsilon(s),\forall b>\nu/2,\end{aligned}$$

$$\mathcal{M}_+:=\Phi\circ\mathcal{M}\circ\Phi^{-1}=\omega\cdot\partial_{\varphi}-\mathrm{i}E\mathcal{D}_+-\mathrm{i}E\mathcal{P}_+$$

$$(\mathcal{D}_+)_{+,\vec{j}}^{+,\vec{j}}=d_j^{(0)}+\mathrm{r}_j^+\ \mathrm{r}_j^+:=(\mathrm{r}_+)_j^j\mathbb{I}+(\mathrm{r}_+)_j^{-j}S,j\in\mathbb{N},\mathrm{r}_0^+:=(\mathrm{r}_+)_0^0$$

$$\sup_{j\in\mathbb{Z},\eta\in\{\pm\}}\langle j\rangle\left|(r_+)_j^{\eta j}-r_j^{\eta j}\right|^{\gamma^{3/2},\Lambda}\lesssim\gamma^{3/2}\varepsilon(s_*)$$

$$\varepsilon_+(s):=\gamma^{-3/2}\|\mathrm{P}_+\langle D\rangle\|_{s}^{\gamma^{3/2},\mathcal{O}_+},\varepsilon_+(s,\mathrm{b}):=\gamma^{-3/2}\left\|\langle\mathrm{d}_{\varphi}\rangle^{\mathrm{b}}\mathrm{P}_+\langle D\rangle\right\|_{s}^{\gamma^{3/2},\mathcal{O}_+}$$

$$\begin{aligned}\varepsilon_+(s)&\leq\mathrm{c}N^{-\mathrm{b}}\varepsilon(s,\mathrm{b})+\mathrm{c}N^{2\tau+1}\varepsilon(s)\varepsilon(s_*)\\\varepsilon_+(s,\mathrm{b})&\leq\varepsilon(s,\mathrm{b})\left(1+\mathrm{c}N^{2\tau+1}\varepsilon(s_*)\right)+\mathrm{c}N^{2\tau+1}\varepsilon(s_*,\mathrm{b})\varepsilon(s)\end{aligned}$$

$$\Phi \circ \mathcal{M} \circ \Phi^{-1} = \omega \cdot \partial_\varphi - \mathrm{i}E\mathcal{D}$$
$$-\omega \cdot \partial_\varphi \mathcal{S} + [\mathcal{S}, -\mathrm{i}E\mathcal{D}] - \mathrm{i}E\mathcal{P}$$
$$+[\mathcal{S}, -\mathrm{i}E\mathcal{P}] + \sum_{k\geq 2} \frac{1}{k!} \mathrm{ad}_{\mathcal{S}}^{k}(-\mathrm{i}E\mathcal{D} - \mathrm{i}E\mathcal{P}) - \sum_{k\geq 2} \frac{1}{k!} \mathrm{ad}_{\mathcal{S}}^{k-1}(\omega \cdot \partial_\varphi \mathcal{S})$$

$$\mathcal{P}_{+,\vec{j}}^{+,\vec{j}}(0) = \mathcal{P}_{-,\vec{j}}^{-,\vec{j}}(0) = \mathcal{P}_{+,j}^{+,j}(0)\mathbb{I} + \mathcal{P}_{+,j}^{+,-j}(0)S, \mathcal{P}_{+,j}^{+,\eta j}(0) \in \mathbb{R}$$

$$\left|\breve{\mathcal{P}}_{+,j}^{+,\eta j}(0)\right|^{\gamma,\Lambda} \leq \left|\mathcal{P}_{+,j}^{+,\eta j}(0)\right|^{\gamma,\mathcal{O}}, \forall j \in \mathbb{N}_0, \eta \in \{\pm\}$$

$$\mathcal{D}_+ := \mathcal{D} + \mathrm{diag}_{\sigma\in\{\pm\}, j\in\mathbb{N}_0} \breve{\mathcal{P}}_{\sigma,\vec{j}}^{\sigma,\vec{j}}(0), \breve{\mathcal{P}}_{\sigma,\vec{j}}^{\sigma,\vec{j}}(0) = \breve{\mathcal{P}}_{+,j}^{+,j}(0)\mathbb{I} + \breve{\mathcal{P}}_{+,,j}^{+,-j}(0)S, \sigma \in \{\pm\}$$

$$-\mathrm{i}E\mathrm{P}_+ := -\mathrm{i}E\Pi_N^{\perp}\mathrm{P} + \sum_{k\geq 1} \frac{1}{k!} \mathrm{ad}_{\mathrm{S}}^{k}(-\mathrm{i}E\mathrm{P}) + \sum_{k\geq 2} \frac{1}{k!} \mathrm{ad}_{\mathrm{S}}^{k-1}\left(\mathrm{i}E(\Pi_N \mathrm{P} - [\mathrm{P}])\right)$$

$$\mathrm{a} := 6\tau + 4$$

$$N_0^{2\tau+2}\varepsilon_0(s_*, \mathrm{b}) \leq 1$$

$$\mathcal{M}_n = \omega \cdot \partial_\varphi - \mathrm{i}E\mathcal{D}_n - \mathrm{i}E\mathcal{P}_n$$

$$(\mathcal{D}_n)_{-,\vec{j}}^{-,\vec{j}} = (\mathcal{D}_n)_{+,\vec{j}}^{+,\vec{j}} = d_j^{(0)} + \mathrm{r}_j^{(n)}, j \in \mathbb{N}_0,$$
$$\mathrm{r}_j^{(n)} := (\mathrm{r}_n)_j^j\mathbb{I} + (\mathrm{r}_n)_j^{-j}S, j \in \mathbb{N}, \mathrm{r}_{\vec{0}}^{(n)} := (\mathrm{r}_n)_0^0,$$

$$(\mathrm{r}_n)_j^{\eta j} = (\mathrm{r}_n)_{-j}^{-\eta j}$$

$$\sup_{j\in\mathbb{N}_0, \eta\in\{\pm\}} \langle j\rangle\left|(\mathrm{r}_n)_j^{\eta j} - (\mathrm{r}_{n-1})_j^{\eta j}\right|^{\gamma^{3/2},\Lambda} \leq \gamma^{3/2}\varepsilon_0(s_*, \mathrm{b})N_{n-2}^{-\mathrm{a}}$$

$$\Omega_n^+ := \Big\{\omega \in \mathcal{O}_{n-1} : \left|\omega \cdot \ell + \lambda_{j,\eta}^{(n-1)} + \lambda_{k,\eta}^{(n-1)}\right| \geq \frac{\gamma}{\langle \ell\rangle^\tau}, \eta \in \{\pm\},$$
$$|\ell| \leq N_{n-1}, j, k \in \mathbb{N}_0\Big\}$$
$$\Omega_n^- := \Big\{\omega \in \mathcal{O}_{n-1} : \left|\omega \cdot \ell + \lambda_{j,\eta}^{(n-1)} - \lambda_{k,\eta}^{(n-1)}\right| \geq \frac{\gamma^{3/2}}{\langle \ell\rangle^\tau}, \eta \in \{\pm\},$$
$$|\ell| \leq N_{n-1}, j, k \in \mathbb{N}_0, \ell \in \mathbb{Z}^\nu, (\ell, j, k) \neq (0, j, j)\Big\}$$

$$\lambda_{j,\pm}^{(n)} := (1 + \mathrm{c})\mathrm{D_m}(j) + (\mathrm{r}_n)_j^j \pm (\mathrm{r}_n)_j^{-j}, j \in \mathbb{N}, \lambda_{0,\pm}^{(n)} := (1 + \mathrm{c})\sqrt{\mathrm{m}} + (\mathrm{r}_n)_0^0$$

$$\varepsilon_n(s) := \gamma^{-3/2}|\mathrm{P}_n\langle D\rangle|_s^{3/2}, \mathcal{O}_n \leq \varepsilon_0(s, \mathrm{b})N_{n-1}^{-\mathrm{a}},$$
$$\varepsilon_n(s, \mathrm{b}) := \gamma^{-3/2}\left|\langle \mathrm{d}_\varphi\rangle^{\mathrm{b}} \mathrm{P}_n\langle D\rangle\right|_s^{3/2}, \mathcal{O}_n \leq \varepsilon_0(s, \mathrm{b})N_{n-1}.$$

$$\mathcal{M}_n = \Phi_{n-1} \circ \mathcal{M}_{n-1} \circ \Phi_{n-1}^{-1}$$

$$\left\|\Phi_{n-1}^{\pm 1} - \mathrm{Id}\right\|_s^{\gamma^{3/2},\mathcal{O}_n} \leq N_{n-1}^{2\tau+2}N_{n-2}^{-\mathrm{a}}\varepsilon_0(s, \mathrm{b})$$
$$\left\|\langle \mathrm{d}_\varphi\rangle^b \Phi_{n-1}^{\pm 1} - \mathrm{Id}\right\|_s \leq N_{n-1}^{2\tau+2+b}N_{n-2}^{-\mathrm{a}}\varepsilon_0(s, \mathrm{b}), \forall b > \frac{\nu}{2}$$

$$U_n := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{n-1}$$

$$\|\mathrm{U}_n - \mathrm{U}_{n-1}\|_s^{\gamma^{3/2},\mathcal{O}_n}, \|\mathrm{U}_n^{-1} - \mathrm{U}_{n-1}^{-1}\|_s^{\gamma^{3/2},\mathcal{O}_n} \leq 2^{-n}\varepsilon_0(s, \mathrm{b})$$

$$N_n^{2\tau+1}\varepsilon_n(s_*) \leq N_n^{2\tau+1}\varepsilon_0(s_*, \mathrm{b})N_{n-1}^{-\mathrm{a}} = \varepsilon_0(s_*, \mathrm{b})N_n^{2\tau+1-\frac{2}{3}\mathrm{a}} \leq \varepsilon_0(s_*, \mathrm{b})N_0^{-2\tau-\frac{5}{3}}$$

$$\|\mathrm{S}_n\langle D\rangle\|_s^{\gamma^{3/2},\mathcal{O}_{n+1}} \leq \mathrm{C}N_n^{2\tau+1}N_{n-1}^{-\mathrm{a}}\varepsilon_0(s,\mathrm{b})$$

$$\begin{aligned}\varepsilon_{n+1}(s) &\leq \mathrm{c}N_n^{-\mathrm{b}}\varepsilon_n(s,\mathrm{b}) + \mathrm{c}N_n^{2\tau+1}\varepsilon_n(s)\varepsilon_n(s_*)\\ &\leq \mathrm{c}N_n^{-\mathrm{b}}N_{n-1}\varepsilon_0(s,\mathrm{b}) + \mathrm{c}N_n^{2\tau+1}N_{n-1}^{-2\mathrm{a}}\varepsilon_0(s,\mathrm{b})\varepsilon_0(s_*,\mathrm{b}) \leq \varepsilon_0(s,\mathrm{b})N_n^{-\mathrm{a}}\end{aligned}$$

$$\begin{aligned}\varepsilon_{n+1}(s,\mathrm{b}) &\leq \varepsilon_n(s,\mathrm{b})\left(1+\mathrm{c}N_n^{2\tau+1}\varepsilon_n(s_*)\right) + \mathrm{c}N_n^{2\tau+1}\varepsilon_n(s_*,\mathrm{b})\varepsilon_n(s)\\ &\leq \varepsilon_0(s,\mathrm{b})N_{n-1}\left(1+\mathrm{c}N_n^{2\tau+1}N_{n-1}^{-\mathrm{a}}\varepsilon_0(s_*,\mathrm{b})\right)\\ &\quad+\varepsilon_0(s,\mathrm{b})\varepsilon_0(s_*,\mathrm{b})\mathrm{c}N_n^{2\tau+1}N_{n-1}^{1-\mathrm{a}} \leq \varepsilon_0(s,\mathrm{b})N_n\end{aligned}$$

$$(\mathrm{r}_\infty)_j^{\eta j} := \lim_{n\to\infty}(\mathrm{r}_n)_j^{\eta j}, \forall\eta\in\{\pm\}, j\in\mathbb{N}_0$$

$$\mathcal{O}_\infty \subseteq \cap_{n\geq 0}\mathcal{O}_n$$

$$\Phi_\infty := \lim_{n\to\infty}U_n \stackrel{\Lambda}{=} \lim_{n\to\infty}\Phi_0\circ\Phi_1\circ\cdots\circ\Phi_{n-1}$$

$$\begin{aligned}\omega\cdot\ell + \lambda_{j,\eta}^{(n-1)} - \lambda_{k,\eta}^{(n-1)} &= \omega\cdot\ell + \lambda_{j,\eta}^{(\infty)} - \lambda_{k,\eta}^{(\infty)}\\ &+(\mathrm{r}_{n-1})_j^j - (\mathrm{r}_\infty)_j^j + \hat{\eta}(j)\left((\mathrm{r}_{n-1})_j^{-j} - (\mathrm{r}_\infty)_j^{-j}\right)\\ &-(\mathrm{r}_{n-1})_k^k + (\mathrm{r}_\infty)_k^k - \hat{\eta}(k)\left((\mathrm{r}_{n-1})_k^{-k} + (\mathrm{r}_\infty)_k^{-k}\right)\end{aligned}$$

$$\sup_{j\in\mathbb{N}_0,\sigma\in\{\pm\}}\langle j\rangle\left|(\mathrm{r}_\infty)_j^{\sigma j} - (\mathrm{r}_{n-1})_j^{\sigma j}\right|^{\gamma^{3/2},\Lambda} \leq \mathrm{C}\varepsilon(s_*,\mathrm{b})N_{n-2}^{-\mathrm{a}}$$

$$\left|\omega\cdot\ell + \lambda_{j,\eta}^{(n-1)} - \lambda_{k,\eta}^{(n-1)}\right| \stackrel{10.10}{\geq} 2\gamma^{3/2}\langle\ell\rangle^{-\tau} - 4\mathrm{C}\varepsilon(s_*,\mathrm{b})N_{n-2}^{-\mathrm{a}} \geq \gamma^{3/2}\langle\ell\rangle^{-\tau}$$

$$\varepsilon_0(s) \leq \varepsilon_0(s,\mathrm{b}) = \gamma^{-3/2}\left\|\left\langle \mathrm{d}_\varphi\right\rangle^{\mathrm{b}}\mathrm{P}_0\langle D\rangle\right\|_s^{\gamma,\Omega_1} \lesssim s_1\gamma^{-7/2}\epsilon(s+\mu)$$

$$\mathcal{O}_\infty \equiv \Lambda_0\cap\Lambda_1\cap\Lambda_2^+\cap\Lambda_2^-$$

$$\mathfrak{F} := \Phi_\infty\circ\Theta_2\circ\Theta_1\circ\boldsymbol{\Psi}\circ\mathcal{U}^{-1}$$

$$|\Lambda\setminus\mathcal{O}_\infty| \leq C\gamma$$

$$\begin{aligned}Q_\ell^{(0)} &:= Q_\ell^{(0)}(\gamma,\nu) := \{\omega\in\Lambda : |\omega\cdot\ell| < 2\gamma\langle\ell\rangle^{-\nu}\},\\ Q_{\ell j}^{(1)} &:= Q_{\ell j}^{(1)}(\gamma,\tau) := \{\omega\in\Lambda : |\omega\cdot\ell + (1+\mathrm{c})j| < 2\gamma\langle\ell\rangle^{-\tau}\},\\ R_{\ell jk\eta}^{(+)} &:= R_{\ell jk\eta}^{(+)}(\gamma,\tau) := \left\{\omega\in\Lambda : \left|\omega\cdot\ell + \lambda_{j,\eta}^{(\infty)} + \lambda_{k,\eta}^{(\infty)}\right| < 2\gamma\langle\ell\rangle^{-\tau}\right\},\\ R_{\ell jk\eta}^{(-)} &:= R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau) := \left\{\omega\in\Lambda : \left|\omega\cdot\ell + \lambda_{j,\eta}^{(\infty)} - \lambda_{k,\eta}^{(\infty)}\right| < 2\gamma^{3/2}\langle\ell\rangle^{-\tau}\right\},\end{aligned}$$

$$\begin{aligned}\Lambda\setminus\Lambda_0 &= \bigcup_{\ell\in\mathbb{Z}^\nu\setminus\{0\}} Q_\ell^{(0)}, \qquad \Lambda\setminus\Lambda_1 = \bigcup_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{N}} Q_{\ell j}^{(1)},\\ \Lambda\setminus\Lambda_2^+ &= \bigcup_{\substack{\ell\in\mathbb{Z}^\nu, j,k\in\mathbb{N}_0\\ \eta\in\{\pm\}}} R_{\ell jk\eta}^{(+)}, \quad \Lambda\setminus\Lambda_2^- = \bigcup_{\substack{\ell\in\mathbb{Z}^\nu, j,k\in\mathbb{N}_0,\eta\in\{\pm\}\\ (\ell,j,k)\neq(0,j,j)}} R_{\ell jk\eta}^{(-)}.\end{aligned}$$

$$\mathrm{D_m}(j) = \sqrt{j^2+\mathrm{m}} = j + \frac{\mathrm{m}}{2j} + \mathfrak{r}(j), |\mathfrak{r}(j)| \leq \frac{\mathrm{m}^2}{8j^3}$$

$$\left|\lambda_{j,\eta}^{(\infty)}(\omega) - \lambda_{k,\eta}^{(\infty)}(\omega)\right| < 2\gamma^{3/2}\langle\ell\rangle^{-\tau} + |\omega\cdot\ell|$$

$$\left|\lambda_{j,\eta}^{(\infty)}(\omega)-\lambda_{k,\eta}^{(\infty)}(\omega)\right| \geq |1+\mathrm{c}||\mathrm{D_m}(j)-\mathrm{D_m}(k)| - 2\langle j\rangle^{-1}\sup_{\sigma\in\{\pm\}}\left|\mathfrak{r}_j^{\sigma j}\right| - 2\langle k\rangle^{-1}\sup_{\sigma\in\{\pm\}}\left|\mathfrak{r}_k^{\sigma k}\right|$$
$$\geq \frac{1}{3}|\mathrm{D_m}(j)-\mathrm{D_m}(k)|$$

$$\phi_{R^-}(\omega) := \omega\cdot\ell + \lambda_{j,\eta}^{(\infty)}(\omega) - \lambda_{k,\eta}^{(\infty)}(\omega)$$
$$\stackrel{\wp}{=} \omega\cdot\ell + (1+\mathrm{c})(\mathrm{D_m}(j)-\mathrm{D_m}(k)) + \frac{\mathfrak{r}_j^{j}+\hat{\eta}(j)\mathfrak{r}_j^{-j}}{\langle j\rangle} - \frac{\mathfrak{r}_k^{k}+\hat{\eta}(k)\mathfrak{r}_k^{-k}}{\langle k\rangle}.$$

$$|\Psi_R(s_1)-\Psi_R(s_2)| \geq |s_1-s_2|\left(|\ell| - C|\ell||\mathrm{c}|^{\mathrm{lip}} - 4\sup_{j\in\mathbb{Z},\sigma\in\{\pm\}}\left|\mathfrak{r}_j^{\sigma j}\right|^{\mathrm{lip}}\right) \geq \frac{|\ell|}{2}|s_1-s_2|$$

$$R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau) \subseteq Q_{\ell,j-k}^{(1)}(\gamma,\tau_1),\ \text{for } j,k \geq \mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}$$

$$\left|\omega\cdot\ell+\lambda_{j,\eta}^{(\infty)}-\lambda_{k,\eta}^{(\infty)}\right| \geq |\omega\cdot\ell+(1+\mathrm{c})(j-k)| - \frac{\mathrm{m}}{2}\left|\frac{1}{j}-\frac{1}{k}\right|$$
$$-\frac{\mathrm{m}^2}{8\min\{j,k\}^3} - \frac{4}{\min\{j,k\}}\sup_{j\in\mathbb{Z},\sigma\in\{\pm\}}\left|\mathfrak{r}_j^{\sigma j}\right|$$
$$\stackrel{\Lambda,\nabla}{\geq} \frac{2\gamma}{\langle\ell\rangle^{\tau_1}} - |j-k|\frac{\mathrm{m}\gamma}{2\mathrm{C}^2\langle\ell\rangle^{2\tau_1}} - \frac{\mathrm{m}^2\gamma^{3/2}}{8\mathrm{C}^3\langle\ell\rangle^{3\tau_1}} - \frac{C\gamma^2\delta_0}{\mathrm{C}\langle\ell\rangle^{\tau_1}}.$$

$$\left|\omega\cdot\ell+\lambda_{j,\eta}^{(\infty)}-\lambda_{k,\eta}^{(\infty)}\right| \geq \frac{2\gamma}{\langle\ell\rangle^{\tau_1}} - c_1\left(\frac{\gamma}{\langle\ell\rangle^{2\tau_1-1}}+\frac{\gamma^{3/2}}{\langle\ell\rangle^{3\tau_1}}+\frac{\gamma^2}{\langle\ell\rangle^{\tau_1}}\right) \geq \frac{2\gamma^{3/2}}{\langle\ell\rangle^{\tau}}$$

$$\mathcal{A}(\ell) := \left\{(j,k)\in\mathbb{N}^2 : j>k\geq \mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}\right\}$$
$$\mathcal{B}(\ell) := \left\{(j,k)\in\mathbb{N}_0^2\setminus\mathcal{A}(\ell) : k<j\leq 2\mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}\right\}$$
$$\mathcal{C}(\ell) := \left\{(j,k)\in\mathbb{N}_0^2\setminus\mathcal{A}(\ell) : j\geq 2\mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}\right\}$$

$$\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,j,k\in\mathbb{N}_0,n\in\{\pm\}\\(\ell,j,k)\neq(0,j,j)}} R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau) \leq \sum_{\ell\in\mathbb{Z}^\nu\setminus\{0\}}\left|Q_\ell^{(0)}\right| + \left|\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{A}(\ell)}} R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right|$$
$$+\sum_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{B}(\ell)}}\left|R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right| + \sum_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{C}(\ell)}}\left|R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right|.$$

$$\left|\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{A}(\ell)}} R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right| \lesssim \sum_{\ell\in\mathbb{Z}^\nu,|h|\lesssim\langle\ell\rangle}\left|Q_{\ell h}^{(1)}(\gamma,\tau_1)\right| \lesssim \sum_{\ell\in\mathbb{Z}^\nu,|h|\lesssim\langle\ell\rangle}\gamma\langle\ell\rangle^{-\tau_1-1} \lesssim \gamma\sum_{\ell\in\mathbb{Z}^\nu}\langle\ell\rangle^{-\tau_1} \lesssim \gamma.$$

$$\sum_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{B}(\ell)}}\left|R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right| \lesssim \sum_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{B}(\ell)\\2\,\mathrm{m}<j-k<2C\langle\ell\rangle}}\left|R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right| + \sum_{\substack{\ell\in\mathbb{Z}^\nu,\eta\in\{\pm\}\\(j,k)\in\mathcal{B}(\ell)\\j-k\leq 2\,\mathrm{m}}}\left|R_{\ell jk\eta}^{(-)}(\gamma^{3/2},\tau)\right|$$
$$\lesssim \gamma^{3/2}\sum_{\ell\in\mathbb{Z}^\nu}\frac{\langle\ell\rangle\langle\ell\rangle^{\tau_1}}{\sqrt{\gamma}\langle\ell\rangle^{\tau+1}} + \gamma^{3/2}\sum_{\ell\in\mathbb{Z}^\nu}\frac{\langle\ell\rangle^{\tau_1}}{\sqrt{\gamma}\langle\ell\rangle^{\tau+1}} \lesssim \gamma$$

$$C\langle\ell\rangle \geq \mathrm{D_m}(j)-\mathrm{D_m}(k) > j-k-\mathrm{m} \geq \mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}-\mathrm{m} \geq \frac{\mathrm{C}\langle\ell\rangle^{\tau_1}\gamma^{-1/2}}{2}$$

4.1. Agujeros negros cuánticos y gravedad bajo las Ecuaciones de Klein – Gordon para campos cuánticos relativistas.

$$ds^2 = \frac{\Delta_r^{KN}}{\Xi^2\rho^2}(\,dt - a\sin^2\,\theta\, d\phi)^2 - \frac{\rho^2}{\Delta_r^{KN}}\, dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2$$
$$-\frac{\Delta_\theta \sin^2\,\theta}{\Xi^2\rho^2}(a\, dt - (r^2 + a^2) d\phi)^2$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\,\theta, \Xi := 1 + \frac{a^2\Lambda}{3},$$
$$\Delta_r^{KN} := \left(1 - \frac{\Lambda}{3} r^2\right)(r^2 + a^2) - 2Mr + e^2,$$
$$\rho^2 = r^2 + a^2\cos^2\,\theta,$$

$$A = -\frac{er}{\Xi(r^2 + a^2\cos^2\,\theta)}(dt - a\sin^2\,\theta\, d\phi)$$

$$M \geq \left[\left(\frac{J}{M}\right)^2 + e^2\right]^{1/2} \Leftrightarrow M^2 \geq a^2 + e^2$$

$$\Box\,\Phi + \mu^2\Phi = 0$$

$$\Box\,\Phi = \frac{1}{\sqrt{-g}} D_\nu\left(\sqrt{-g}\, g^{\mu\nu} D_\mu \Phi\right)$$

$$D_\mu = \partial_\mu - iqA_\mu$$

$$\frac{\Xi^2}{\rho^2}\left[\frac{(r^2 + a^2)^2}{\Delta_r^{KN}} - \frac{a^2\sin^2\,\theta}{\Delta_\theta}\right]\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\Delta_r^{KN}\frac{\partial\Phi}{\partial r}\right) - \frac{1}{\rho^2}\frac{1}{\sin\,\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\Delta_\theta\frac{\partial\Phi}{\partial\theta}\right)$$
$$+2\frac{a\Xi^2}{\rho^2}\left\{-\frac{1}{\Delta_\theta} + \frac{r^2 + a^2}{\Delta_r^{KN}}\right\}\frac{\partial^2\Phi}{\partial t\partial\phi} - \frac{\Xi^2}{\rho^2\sin^2\,\theta}\left\{\frac{1}{\Delta_\theta} - \frac{a^2\sin^2\,\theta}{\Delta_r^{KN}}\right\}\frac{\partial^2\Phi}{\partial\phi^2} + \mu^2\Phi = 0.$$

$$\Phi = \Phi(\vec{r}, t) = R(r)S(\theta)e^{im\varphi}e^{-i\omega t}$$

$$\frac{1}{R(r)}\frac{d}{dr}\left(\Delta_r^{KN}\frac{dR}{dr}\right) - \Xi^2\left[\frac{(r^2 + a^2)^2}{\Delta_r^{KN}} - \frac{a^2\sin^2\,\theta}{\Delta_\theta}\right](-\omega^2)$$
$$+\frac{1}{S(\theta)}\frac{1}{\sin\,\theta}\frac{d}{d\theta}\left(\sin\,\theta\Delta_\theta\frac{dS(\theta)}{d\theta}\right) + \frac{\Xi^2}{\sin^2\,\theta}\left\{\frac{1}{\Delta_\theta} - \frac{a^2\sin^2\,\theta}{\Delta_r^{KN}}\right\}(-m^2)$$
$$-2a\Xi^2\left\{-\frac{1}{\Delta_\theta} + \frac{r^2 + a^2}{\Delta_r^{KN}}\right\}m\omega - \rho^2\mu^2 = 0,$$

$$\frac{1}{\sin\,\theta}\frac{d}{d\theta}\left(\sin\,\theta\Delta_\theta\frac{dS(\theta)}{d\theta}\right)$$
$$+S(\theta)\left[-\frac{m^2\Xi^2}{\sin^2\,\theta}\frac{1}{\Delta_\theta} + \frac{2a\Xi^2}{\Delta_\theta}m\omega - \frac{\Xi^2 a^2\sin^2\,\theta\omega^2}{\Delta_\theta} - \mu^2 a^2\cos^2\,\theta + K_{lm}\right] = 0$$
$$\frac{d}{dr}\left(\Delta_r^{KN}\frac{dR}{dr}\right) + \frac{R(r)}{\Delta_r^{KN}}[\Xi^2K^2 - r^2\mu^2\Delta_r^{KN} - K_{lm}\Delta_r^{KN}] = 0$$

$$K(r) := \omega(r^2 + a^2) - am$$

$$A^\rho A_\rho = g^{00}A_0A_0 + g^{03}A_3A_0 + g^{30}A_3A_0 + g^{33}A_3A_3 = -\frac{q^2e^2r^2}{\rho^2\Delta_r^{KN}},$$
$$-2iqA^\mu\partial_\mu = -2iqA^0\partial_0 - 2iqA^3\partial_3 = \frac{2iq\Xi}{\rho^2\Delta_r^{KN}}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\phi}\right]$$

$$\frac{d}{dr}\left(\Delta_r^{KN}\frac{dR}{dr}\right) + \frac{R(r)}{\Delta_r^{KN}}\left[\Xi^2\left(K - \frac{eqr}{\Xi}\right)^2 - r^2\mu^2\Delta_r^{KN} - K_{lm}\Delta_r^{KN}\right] = 0$$

$$\left[\left(1+\frac{a^2\Lambda}{3}x^2\right)(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\frac{a^2\Lambda}{3}x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}-2\left(1+\frac{a^2\Lambda}{3}x^2\right)x\frac{\mathrm{d}}{\mathrm{d}x}\right]S$$
$$+\left[-\frac{\Xi^2a^2\omega^2(1-x^2)}{1+\frac{a^2\Lambda}{3}x^2}+\frac{2a\omega m\Xi^2}{1+\frac{a^2\Lambda}{3}x^2}-\frac{m^2\Xi^2}{\left(1+\frac{a^2\Lambda}{3}x^2\right)(1-x^2)}\right]S$$
$$+\left[-2\frac{a^2\Lambda}{3}x^2+K_{lm}\right]S=0$$

$$z=\frac{a_2-a_4}{a_2-a_1}\frac{x-a_1}{x-a_4}=\frac{1-\frac{i}{\sqrt{\alpha_\lambda}}}{2}\frac{x+1}{x-\frac{i}{\sqrt{\alpha_\Lambda}}},\alpha_\Lambda:=\frac{a^2\Lambda}{3},$$

$$(1+\alpha_\Lambda x^2)(1-x^2)=\frac{\alpha_\Lambda 16i\Xi^2}{\sqrt{\alpha_\Lambda}}\frac{z(z-1)(z-z_3)}{[2z\sqrt{\alpha_\Lambda}-\sqrt{\alpha_\Lambda}+i]^4}$$

$$z_3=-\frac{1}{2}\left(-1+\frac{\alpha_\Lambda-1}{2i\sqrt{\alpha_\Lambda}}\right)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}\right.$$
$$-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}+$$
$$\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}+\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}+\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}\right.$$
$$-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}+$$
$$\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}+\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}+\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]$$
$$\left.+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\right\}S(z)=0$$

$$S(z)=z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}(z-z_\infty)^{\alpha_4}\bar{S}(z)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{2\alpha_1+1}{z}+\frac{2\alpha_2+1}{z-1}+\frac{2\alpha_3+1}{z-z_3}\right]\frac{\mathrm{d}}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-z_3)}\right\}\bar{S}(z)=0$$

$$q=\frac{i}{4\sqrt{\alpha_\Lambda}}\Big\{-\left(1+i\sqrt{\alpha_\Lambda}\right)^2[2\alpha_1\alpha_2+\alpha_2+\alpha_1]-4\sqrt{\alpha_\Lambda}i[2\alpha_1\alpha_3+\alpha_3+\alpha_1]$$
$$-\frac{m^2}{2}\left(\left(1+i\sqrt{\alpha_\Lambda}\right)^2+4\alpha_\Lambda\right)+K_{lm}-2i\sqrt{\alpha_\Lambda}+2\Xi m\xi\Big\}$$

$$\alpha\beta=q-(z_3-1)\times\mathcal{B}$$
$$=\frac{i}{4\sqrt{\alpha_\Lambda}}\Big\{-\left(1+i\sqrt{\alpha_\Lambda}\right)^2[2\alpha_1\alpha_2+\alpha_2+\alpha_1]-4\sqrt{\alpha_\Lambda}i[2\alpha_1\alpha_3+\alpha_3+\alpha_1]$$
$$-\frac{m^2}{2}\left(\left(1+i\sqrt{\alpha_\Lambda}\right)^2+4\alpha_\Lambda\right)+K_{lm}-2i\sqrt{\alpha_\Lambda}+2\Xi m\xi\Big\}$$
$$+\frac{i}{4\sqrt{\alpha_\Lambda}}\Big\{\frac{m^2}{2}\left(\left(1-i\sqrt{\alpha_\Lambda}\right)^2+4\alpha_\Lambda\right)-2m\xi\Xi-K_{lm}-2\sqrt{\alpha_\Lambda}i$$
$$+\left(1-i\sqrt{\alpha_\Lambda}\right)^2[2\alpha_1\alpha_2+\alpha_2+\alpha_1]+i4\sqrt{\alpha_\Lambda}(-2\alpha_2\alpha_3-\alpha_3-\alpha_2)\Big\}$$

$$u(z)=\sum_{\nu=0}^{\infty}c_\nu y_\nu(z)$$

$$y_\nu(z)=F(-\nu,\nu+\omega,\gamma,z)=\frac{\nu!\,\Gamma(\gamma)}{\Gamma(\nu+\gamma)}P_\nu^{(\gamma-1,\omega-\gamma)}(1-2z)$$

$$y_\nu''(z)+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}\right]y_\nu'(z)-\frac{\nu(\nu+\omega)}{z(z-1)}y_\nu(z)=0$$

$$zy_\nu(z)=P_\nu y_{\nu+1}(z)+Q_\nu y_\nu(z)+R_\nu y_{\nu-1}(z)$$
$$z(z-1)\frac{\mathrm{d}}{\mathrm{dz}}y_\nu(z)=P_\nu' y_{\nu+1}(z)+Q_\nu' y_\nu(z)+R_\nu' y_{\nu-1}(z)$$

$$\begin{cases}P_\nu=-\dfrac{(\nu+\omega)(\nu+\gamma)}{(2\nu+\omega)(2\nu+\omega+1)}\\ Q_\nu=\dfrac{(\omega-1)(\gamma-\delta)}{2(2\nu+\omega+1)(2\nu+\omega-1)}+\dfrac{1}{2}\\ R_\nu=-\dfrac{\nu(\nu+\delta-1)}{(2\nu+\omega)(2\nu+\omega-1)}\end{cases}\begin{cases}P_\nu'=-\dfrac{\nu(\nu+\omega)(\nu+\gamma)}{(2\nu+\omega)(2\nu+\omega+1)}\\ Q_\nu'=\dfrac{\nu(\nu+\omega)(\gamma-\delta)}{(2\nu+\omega+1)(2\nu+\omega-1)}\\ R_\nu'=\dfrac{\nu(\nu+\omega)(\nu+\delta-1)}{(2\nu+\omega)(2\nu+\omega-1)}\end{cases}$$

$$\begin{cases}P_0=-\dfrac{\gamma}{\omega+1}\\ Q_0=\dfrac{\gamma}{\omega+1}\\ R_0=0\end{cases}\begin{cases}P_0'=0\\ Q_0'=0\\ R_0'=0\end{cases}$$

$$c_{\nu+1}=E_\nu c_\nu+F_\nu c_{\nu-1}$$

$$\begin{cases}E_\nu=-\dfrac{[\nu(\nu+\omega)+\alpha\beta]Q_\nu+\varepsilon Q_\nu'-a\nu(\nu+\omega)-q}{\varepsilon R_{\nu+1}'+[(\nu+1)(\nu+1+\omega)+\alpha\beta]R_{\nu+1}}\\ F_\nu=-\dfrac{\varepsilon P_{\nu-1}'+((\nu-1)(\nu-1+\omega)+\alpha\beta)P_{\nu-1}}{\varepsilon R_{\nu+1}'+[(\nu+1)(\nu+1+\omega)+\alpha\beta]R_{\nu+1}}\end{cases}$$

$$c_1=-\frac{c_0(\alpha\beta\gamma-q(\omega+1))(2+\omega)}{\delta((1+\omega)(\varepsilon-1)-\alpha\beta)}$$

$$\begin{cases} P_\nu = -\frac{1}{4} + \frac{1-2\gamma}{8\nu} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \\ Q_\nu = \frac{1}{2} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \quad \text{as } \nu \to \infty \\ R_\nu = -\frac{1}{4} - \frac{1-2\gamma}{8\nu} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \end{cases}$$

$$\lim_{\nu\to\infty} \frac{y_{\nu+1}(z)}{y_\nu(z)} = s(z) \equiv s_2(z) = \frac{(1-z^{-1})^{1/2}+1}{(1-z^{-1})^{1/2}-1} \equiv \frac{Z+1}{Z-1}$$

$$\lim_{\nu\to\infty} \frac{c_{\nu+1}}{c_\nu} = t_2 = \frac{(1-a^{-1})^{1/2}+1}{(1-a^{-1})^{1/2}-1} \equiv \frac{A+1}{A-1},$$

$$\lim_{\nu\to\infty} \frac{c_{\nu+1}}{c_\nu} = t_1 = \frac{(1-a^{-1})^{1/2}-1}{(1-a^{-1})^{1/2}+1} \equiv \frac{A-1}{A+1}$$

$$\lim_{\nu\to\infty} \left|\frac{c_{\nu+1}y_{\nu+1}(z)}{c_\nu y_\nu(z)}\right| = |t_n s_2(z)| < 1, n = 1{,}2$$

$$\left|\frac{Z+1}{Z-1}\right| = |s_2(z)| = \frac{1}{|t_2|} = \left|\frac{A-1}{A+1}\right| < 1$$

$$\left|\frac{Z+1}{Z-1}\right| = |s_2(z)| < \frac{1}{|t_1|} = \left|\frac{A+1}{A-1}\right|$$

$$\lim_{\nu\to\infty} \nu^{\Re}\left(\frac{c_{\nu+1}y_{\nu+1}(z)}{c_\nu y_\nu(z)} - 1\right) = -1 - c$$

$$\lim_{\nu\to\infty} \nu\Re\left(\frac{c_{\nu+1}y_{\nu+1}(z)}{c_\nu y_\nu(z)} - 1\right) = \Re\varepsilon - 2.$$

$$\mathcal{D}_\nu c_{\nu+1} + \mathcal{E}_\nu c_\nu + \mathcal{F}_\nu c_{\nu-1} = 0,$$
$$\mathcal{F}_\nu = -\frac{(\nu-1+\omega)(\nu-1+\gamma)(\nu-1+\alpha)(\nu-1+\beta)}{(2\nu+\omega-2)(2\nu+\omega-1)}$$
$$\mathcal{D}_\nu = -\frac{(\nu+\delta)(\nu+1)(\nu+1+\omega-\alpha)(\nu+1+\omega-\beta)}{(2\nu+\omega+2)(2\nu+\omega+1)}$$
$$\mathcal{E}_\nu = \frac{J_\nu}{(2\nu+\omega+1)(2\nu+\omega-1)} - z_3\nu(\nu+\omega) - q,$$
$$J_\nu = [\nu(\nu+\omega)+\alpha\beta][2\nu(\nu+\omega)+\gamma(\omega-1)] + \varepsilon\nu(\nu+\omega)(\gamma-\delta)$$

$$v_{\nu-1} = \frac{-\mathcal{F}_\nu}{\mathcal{D}_\nu v_\nu + \mathcal{E}_\nu}$$

$$v_{\nu-1} = \frac{-\mathcal{F}_\nu}{Q_\nu - q -} \frac{\mathcal{D}_\nu \mathcal{F}_{\nu+1}}{Q_{\nu+1} - q + \mathcal{D}_{\nu+1} v_{\nu+1}}$$

$$\mathcal{E}_0 = \frac{\mathcal{D}_0\mathcal{F}_1}{Q_1 - q -} \frac{\mathcal{D}_1\mathcal{F}_2}{Q_2 - q -} \cdots$$

$$R_\nu L_{\nu-1} = 1$$

$$R_\nu = \frac{c_\nu}{c_{\nu-1}}, L_\nu = \frac{c_\nu}{c_{\nu+1}}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) - E\right) f(x) = 0$$

$$w=\frac{e_1-e_3}{\wp(x)-e_3}, t=\frac{e_1-e_3}{e_2-e_3},$$
$$\tilde{\Phi}(w)=w^{\frac{l_0+1}{2}}(w-1)^{\frac{l_1+1}{2}}(w-t)^{\frac{l_2+1}{2}}$$

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^2+\left(\frac{l_0+\frac{3}{2}}{w}+\frac{l_1+\frac{3}{2}}{w-1}+\frac{l_2+\frac{3}{2}}{w-t}\right)\frac{\mathrm{d}}{\mathrm{d}w}+\frac{\left(\frac{\sum_{i=0}^{3} l_i+4}{2}\right)\left(\frac{3+\sum_{i=0}^{2} l_i-l_3}{2}\right)w-q}{w(w-1)(w-t)}\right)\tilde{f}(w)$$

$$q=-\frac{t}{4}\left(\frac{E}{e_1-e_3}+\left(\frac{t+1}{3t}\right)\sum_{i=0}^{3} l_i(l_i+1)-\frac{1}{t}(l_0+l_2+2)^2-(l_0+l_1+2)^2\right)$$

$$f(x)=\tilde{f}\left(\frac{e_1-e_3}{\wp(x)-e_3}\right)\tilde{\Phi}\left(\frac{e_1-e_3}{\wp(x)-e_3}\right)=\tilde{f}(w)\tilde{\Phi}(w)$$

$$\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2=4(e_2-e_3)w(w-1)(w-t)$$
$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\left[\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right], \frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{(e_1-e_3)\wp'(x)}{(\wp(x)-e_3)^2}$$

$$\wp(x+\omega_1)=\frac{(e_2-e_1)w}{w-1}+e_1, \wp(x+\omega_2)=-\frac{(e_2-e_1)w}{w-t}+e_2$$
$$\wp(x+\omega_3)=e_3-w(e_3-e_2)$$

$$\gamma+\delta+\varepsilon=l_0+l_1+l_2+\frac{9}{2}=\alpha+\beta+1$$

$$\omega_1=\int_{e_1}^{\infty}\frac{\mathrm{d}x}{\sqrt{X}}=\int_{e_3}^{e_2}\frac{\mathrm{d}x}{\sqrt{X}}=\frac{K}{\sqrt{e_1-e_3}}$$

$$\omega_3=\int_{e_2}^{e_1}\frac{\mathrm{d}x}{\sqrt{X}}=\int_{-\infty}^{e_3}\frac{\mathrm{d}x}{\sqrt{X}}=\frac{iK'}{\sqrt{e_1-e_3}}$$

$$\frac{\mathrm{d}^2\bar{S}}{\mathrm{d}u^2}+\left[(4\alpha_1+1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u}-(4\alpha_2+1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u}-k^2(4\alpha_3+1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}\bar{S}}{\mathrm{d}u}$$

$$\frac{\mathrm{d}^2\bar{S}}{\mathrm{d}u^2}+\left[(4\alpha_1+1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u}-(4\alpha_2+1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u}-k^2(4\alpha_3+1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}\bar{S}}{\mathrm{d}u}$$
$$+(4\alpha\beta k^2\mathrm{sn}^2u-4k^2q)\bar{S}=0$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}=f(z)\frac{\mathrm{d}Y}{\mathrm{d}z}+g(z)Y$$

$$f(z)=\frac{1}{z-a_j}+f_0+O(z-a_j), g(z)=\frac{g_{-1}}{z-a_j}+g_0+O(z-a_j)$$

$$Y(z)=\sum_{m=0}^{\infty} c_m(z-a_j)^m=c_0+c_1(z-a_j)+c_2(z-a_j)^2+O\left((z-a_j)^3\right)$$

$$g_{-1}f_0-g_0+(g_{-1})^2=0$$

$$L=x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+xp(x)\frac{\mathrm{d}}{\mathrm{d}x}+q(x)$$

$$p(x)=\sum_{j=0}^{\infty} p_j x^j, q(x)=\sum_{j=0}^{\infty} q_j x^j$$

$$y(x)=x^r\sum_{m=0}^{\infty} c_m x^m, (c_0\neq 0, c_0=1)$$

$$Ly=\sum_{m\geq 0}\{((m+r)(m+r-1)+(m+r)p_0+q_0)c_m+R_m\}x^{m+r}$$

$$R_0=0, R_m=\sum_{i+j=m, i\neq m}\{(i+r)c_ip_j+c_iq_j\}=\sum_{k=0}^{m-1}[(k+r)p_{m-k}+q_{m-k}]c_k, m>0$$

$$F(r)=r(r-1)+rp_0+q_0$$

$$F(r+m)c_m+R_m=0, m=0,1,2,\ldots$$

$$\begin{aligned}R_2&=[-rp_2+q_2]c_0+[(1+r)p_1+q_1]c_1=q_2c_0+(p_1+q_1)c_1\\&=-g_0c_0+(-f_0-g_{-1})(-g_{-1}c_0)=-g_0c_0+f_0g_{-1}c_0+g_{-1}^2c_0,\end{aligned}$$

$$\begin{aligned}&\left\{\frac{d^2}{dz^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{d}{dz}\right.\\&-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{a^2\mu^2}{\alpha_\Lambda(z-z_\infty)^2}+\\&\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{a^2\mu^2}{(-i+\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]\\&+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}-\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{a^2\mu^2}{(i+\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]\\&+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{4ia^2\mu^2}{\sqrt{\alpha_\Lambda}\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\end{aligned}$$

$$\begin{aligned}&\left\{\frac{d^2}{dz^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{d}{dz}\right.\\&-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{a^2\mu^2}{\alpha_\Lambda(z-z_\infty)^2}+\\&\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{a^2\mu^2}{(-i+\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]\\&+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}-\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{a^2\mu^2}{(i+\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]\\&+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{4ia^2\mu^2}{\sqrt{\alpha_\Lambda}\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\\&\left.+\frac{1}{z-z_\infty}\frac{-4ia^2\mu^2}{\sqrt{\alpha_\Lambda}\Xi}\right\}S(z)=0\end{aligned}$$

$$F(r)=r(r-1)+p_0r+q_0=0$$

$$g_{-1} = \frac{-i\sqrt{\alpha_\Lambda}}{\Xi}$$

$$f_0 = \frac{2\alpha_1 + 1}{z_\infty} + \frac{2\alpha_2 + 1}{z_\infty - 1} + \frac{2\alpha_3 + 1}{z_\infty - z_3}$$

$$g_0 = \left[\frac{m^2(1 + 2i\sqrt{\alpha_\Lambda} + 3\alpha_\Lambda)}{2(-i + \sqrt{\alpha_\Lambda})^2} + \frac{2m\xi\Xi}{(1 + i\sqrt{\alpha_\Lambda})^2} + \frac{-2\alpha_\Lambda}{(1 + i\sqrt{\alpha_\Lambda})^2} + \frac{K_{lm}}{(1 + i\sqrt{\alpha_\Lambda})^2}\right]\frac{1}{z_\infty}$$
$$+ \left[\frac{m^2}{2}\left(1 + \frac{4\alpha_\Lambda}{(1 - i\sqrt{\alpha_\Lambda})^2} - \right)\frac{2m\xi\Xi}{(1 - i\sqrt{\alpha_\Lambda})^2} + \frac{2\alpha_\Lambda}{(1 - i\sqrt{\alpha_\Lambda})^2} - \frac{K_{lm}}{(1 - i\sqrt{\alpha_\Lambda})^2}\right]\frac{1}{z_\infty - 1}$$
$$+ \left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2} + \frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi} + \frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2} + \frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\frac{1}{z_\infty - z_3}$$

$$\Delta_r^{KN} = -\frac{\Lambda}{3}(r - r_+)(r - r_-)(r - r_\Lambda^+)(r - r_\Lambda^-)$$

$$z = \left(\frac{r_+ - r_\Lambda^-}{r_+ - r_-}\right)\left(\frac{r - r_-}{r - r_\Lambda^-}\right)$$

$$\Delta_r^{KN} = -\frac{\Lambda}{3}\frac{Hz_\infty^3 z(z-1)(z - z_r)}{(z_\infty - z)^4}$$

$$r = \frac{r_- z_\infty - r_\Lambda^- z}{z_\infty - z}, z_\infty := \frac{r_+ - r_\Lambda^-}{r_+ - r_-}, z_r := z_\infty\left(\frac{r_\Lambda^+ - r_-}{r_\Lambda^+ - r_\Lambda^-}\right)$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{z_\infty(r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^2} = \frac{1}{z_\infty}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2 = \frac{r_+ - r_-}{r_+ - r_\Lambda^-}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = \frac{-2z_\infty(r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^3}, \frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} = \frac{-2}{z_\infty - z}$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \frac{1}{\left(\frac{\mathrm{d}z}{\mathrm{d}z}\right)^2}\frac{1}{\Delta_r^{KN}}\frac{\mathrm{d}\Delta_r^{KN}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)}\frac{\mathrm{d}R}{\mathrm{d}z} + \frac{\Xi^2\left(K(r) - \frac{eqr}{\Xi}\right)^2}{(\Delta_r^{KN})^2\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} + \frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}} - \frac{K_{lm}R}{\Delta_r^{KN}\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}$$
$$= \frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z - z_r} - \frac{2}{z - z_\infty}\right\}\frac{\mathrm{d}R}{\mathrm{d}z} + \left[\frac{A'}{z^2} + \frac{B'}{z} + \frac{C'}{(z-1)^2} + \frac{D'}{z-1} + \frac{E'}{(z - z_r)^2} + \frac{H'}{z - z_r}\right]R$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \frac{1}{\left(\frac{\mathrm{d}z}{\mathrm{d}z}\right)^2}\frac{1}{\Delta_r^{KN}}\frac{\mathrm{d}\Delta_r^{KN}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)}\frac{\mathrm{d}R}{\mathrm{d}z} + \frac{\Xi^2\left(K(r) - \frac{eqr}{\Xi}\right)^2}{(\Delta_r^{KN})^2\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} + \frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}} - \frac{K_{lm}R}{\Delta_r^{KN}\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}$$
$$= \frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z - z_r} - \frac{2}{z - z_\infty}\right\}\frac{\mathrm{d}R}{\mathrm{d}z} + \left[\frac{A'}{z^2} + \frac{B'}{z} + \frac{C'}{(z-1)^2} + \frac{D'}{z-1} + \frac{E'}{(z - z_r)^2} + \frac{H'}{z - z_r}\right]R$$
$$+ \left[\frac{A}{(z_\infty - z)^2} + \frac{B}{z_\infty - z} + \frac{C}{z} + \frac{D}{z-1} + \frac{F}{z - z_r}\right]R + \left[\frac{\mathcal{B}_{K_{lm}}}{z} + \frac{\mathcal{D}_{K_{lm}}}{z-1} + + \frac{\mathcal{H}_{K_{lm}}}{z - z_r}\right]R = 0.$$

$$A = \frac{3\mu^2}{\Lambda}$$

$$B = \frac{3\mu^2}{\Lambda}\frac{1}{r_- - r_\Lambda^-}\left[\frac{(r_\Lambda^- + r_-)z_r - 2r_- z_\infty - 2r_- z_r z_\infty - (r_\Lambda^- - 3r_-)z_\infty^2}{(1 - z_\infty)(z_r - z_\infty)z_\infty}\right]$$

$$C = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{r_-^2}{z_\infty}$$

$$D = -\frac{3\mu^2}{\Lambda}\frac{z_r}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{[r_\Lambda^- - r_- z_\infty]^2}{(z_r - 1)(z_\infty - 1)}$$

$$F = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{(r_\Lambda^- z_r - r_- z_\infty)^2}{(z_r - 1)(z_r - z_\infty)^2}$$

$$A' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_-) - eqr_-]^2}{(r_- - r_\Lambda^-)^2(r_+ - r_-)^2(r_\Lambda^+ - r_-)^2}$$

$$C' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_+) - eqr_+]^2}{(r_+ - r_\Lambda^-)^2(r_+ - r_\Lambda^+)^2(r_+ - r_-)^2}$$

$$E' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_\Lambda^+) - eqr_\Lambda^+]^2}{(r_\Lambda^+ - r_-)^2(r_\Lambda^+ - r_\Lambda^-)^2(r_+ - r_\Lambda^+)^2}$$

$$F(r) = r(r-1) - 2r + \frac{3\mu^2}{\Lambda} = 0$$

$$r_{\mu z_\infty}^{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - \frac{12\mu^2}{\Lambda}}$$

$$F(r) = r(r-1) + r + \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_+) - eqr_+]^2}{(r_+ - r_\Lambda^-)^2(r_+ - r_\Lambda^+)^2(r_+ - r_-)^2} = 0$$

$$r_{z=1}^{1,2} \equiv \mu_2 = \pm\frac{ia^2}{\alpha_\Lambda}\frac{\Xi K(r_+) - eqr_+}{(r_\Lambda^- - r_+)(r_- - r_+)(r_\Lambda^+ - r_+)}$$

$$r_{z=0}^{1,2} \equiv \mu_1 = \pm\frac{ia^2}{\alpha_\Lambda}\frac{\Xi K(r_-) - eqr_-}{(r_- - r_\Lambda^-)(r_+ - r_-)(r_\lambda^+ - r_-)}$$

$$r_{z=z_r}^{1,2} \equiv \mu_3 = \pm\frac{ia^2}{\alpha_\Lambda}\frac{[\Xi K(r_\Lambda^+) - eqr_\Lambda^+]}{(r_\Lambda^- - r_\Lambda^+)(r_+ - r_\Lambda^+)(r_- - r_\Lambda^+)}$$

$$R(z) = z^{\mu_1}(z-1)^{\mu_2}(z - z_r)^{\mu_3}(z - z_\infty)^{r_{z_\infty}^2}\bar{R}(z)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{2\mu_1 + 1}{z} + \frac{2\mu_2 + 1}{z - 1} + \frac{2\mu_3 + 1}{z - z_r}\right]\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z - z_r)}\right\}\bar{R}(z) = 0$$

$$\frac{1}{z - z_\infty}\left(\frac{1}{z_\infty} - \frac{1}{1 - z_\infty} - \frac{1}{z_r - z_\infty}\right) - \frac{B}{z - z_\infty}$$

$$\frac{1}{z - z_\infty}\left(\frac{1}{z_\infty} - \frac{1}{1 - z_\infty} - \frac{1}{z_r - z_\infty}\right) - \frac{B}{z - z_\infty} = \frac{1}{z - z_\infty}\frac{(r_- - r_+)(r_\Lambda^- + r_\Lambda^+ + r_- + r_+)}{(r_\Lambda^- - r_-)(r_\Lambda^- - r_+)} = 0,$$

$$\begin{aligned}\alpha &= \frac{1}{2}\frac{\wp'(-x_1/2+\omega)-\wp'(x_1)}{\wp(-x_1/2+\omega)-\wp(x_1)}\\ \beta &= \frac{1}{2}\frac{\wp'(-x_1/2+\omega+\omega')-\wp'(x_1)}{\wp(-x_1/2+\omega+\omega')-\wp(x_1)}\\ \gamma &= \frac{1}{2}\frac{\wp'(-x_1/2+\omega')-\wp'(x_1)}{\wp(-x_1/2+\omega')-\wp(x_1)}\\ \delta &= \frac{1}{2}\frac{\wp'(-x_1/2)-\wp'(x_1)}{\wp(-x_1/2)-\wp(x_1)}\end{aligned}$$

$$-6\wp(x_1) = -\frac{3}{\Lambda}+a^2$$

$$4\wp'(x_1) = \frac{6}{\Lambda}, -3\wp^2(x_1)+g_2 = -\frac{3}{\Lambda}(a^2+e^2)$$

$$g_2 = \frac{1}{12}\left(-\frac{3}{\Lambda}+a^2\right)^2 - \frac{3}{\Lambda}(a^2+e^2)$$

$$g_3 = -\frac{1}{216}\left(-\frac{3}{\Lambda}+a^2\right)^3 - \frac{3}{\Lambda}\frac{1}{6}(a^2+e^2)\left(-\frac{3}{\Lambda}+a^2\right) - \frac{9}{4\Lambda^2}$$

$$\frac{\mathrm{d}^2\bar{R}}{\mathrm{d}u^2} + \left[(4\mu_1+1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u} - (4\mu_2+1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u} - k^2(4\mu_3+1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}\bar{R}}{\mathrm{d}u}$$
$$+(4\alpha\beta k^2\mathrm{sn}^2u - 4k^2q)\bar{R} = 0$$

$$\frac{\mathrm{d}^2\bar{R}}{\mathrm{d}u^2} + \left[(4\mu_1+1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u} - (4\mu_2+1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u} - k^2(4\mu_3+1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}\bar{R}}{\mathrm{d}u}$$

$$k^{-2} = z_r = \frac{r_+ - r_\Lambda^-}{r_+ - r_-}\frac{r_\Lambda^+ - r_-}{r_\Lambda^+ - r_\Lambda^-}$$

$$\begin{aligned}\bar{R}_\nu(z) &= \sum_{\varrho=-\infty}^{+\infty} c_\varrho^\nu u_{\nu+\varrho}(z)\\ u_\nu(z) &= F(-\nu, \nu+2(\mu_1+\mu_2)+1, 2\mu_1+1, z)\end{aligned}$$

$$D_\varrho^\nu c_{\varrho+1}^\nu + E_\varrho^\nu c_\varrho^\nu + F_\varrho^\nu c_{\varrho-1}^\nu = 0$$

$$\begin{aligned}D_\varrho^\nu &= -\frac{(\nu+\varrho+\delta)(\nu+\varrho+1)(\nu+\varrho+1+\omega-\alpha)(\nu+\varrho+1+\omega-\beta)}{(2\nu+2\varrho+\omega+2)(2\nu+2\varrho+\omega+1)},\\ F_\varrho^\nu &= -\frac{(\nu+\varrho-1+\omega)(\nu+\varrho-1+\gamma)(\nu+\varrho-1+\alpha)(\nu+\varrho-1+\beta)}{(2\nu+2\varrho+\omega-2)(2\nu+2\varrho+\omega-1)},\\ E_\varrho^\nu &= \frac{J_\varrho^\nu}{(2\nu+2\varrho+\omega+1)(2\nu+2\varrho+\omega-1)} - z_r(\nu+\varrho)(\nu+\varrho+\omega) - q,\\ J_\varrho^\nu &= [(\nu+\varrho)(\nu+\varrho+\omega)+\alpha\beta](2(\nu+\varrho)(\nu+\varrho+\omega)+\gamma(\omega-1))\\ &\quad +\varepsilon(\nu+\varrho)(\nu+\varrho+\omega)(\gamma-\delta)\end{aligned}$$

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + (\tau^2-\mu^2a^2)x^2 + \frac{-m^2}{1-x^2} + E\right]S = 0$$

$$S(z) = z^{\frac{m}{2}}(1-z)^{\frac{m}{2}}\exp(2\tau' z)w(z)$$

$$w''(z) + \left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right)w'(z) + \frac{4p\alpha z - \sigma}{z(z-1)}w(z) = 0$$

$$\sigma := E + \tau^2 - \mu^2 a^2 - m(m+1) + 2\tau'(m+1), p = \tau'$$
$$\alpha = m+1,$$
$$\gamma = m+1, \delta = m+1$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{a\mu\sin\theta}{\lambda + a\mu\cos\theta}\frac{d}{d\theta} + \left(\frac{1}{2} + a\omega\cos\theta\right)^2 - \left(\frac{m - \frac{1}{2}\cos\theta}{\sin\theta}\right)^2 - \frac{3}{4}\right.$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{a\mu\sin\theta}{\lambda + a\mu\cos\theta}\frac{d}{d\theta} + \left(\frac{1}{2} + a\omega\cos\theta\right)^2 - \left(\frac{m - \frac{1}{2}\cos\theta}{\sin\theta}\right)^2 - \frac{3}{4}\right.$$
$$\left. + 2a\omega m - a^2\omega^2 + \frac{a\mu\left(\frac{1}{2}\cos\theta + a\omega\sin^2\theta - m\right)}{\lambda + a\mu\cos\theta} - a^2\mu^2\cos^2\theta + \lambda^2\right] S^{(-)}(\theta) = 0.$$

$$\left\{(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{a\mu(1-x^2)}{\lambda + a\mu x}\frac{d}{dx}\right.$$
$$+ \frac{a\mu\left(\frac{x}{2} + a\omega(1-x^2) - m\right)}{\lambda + a\mu x} + a^2(\omega^2 - \mu^2)x^2 + a\omega x - \frac{1}{4}$$
$$\left. + \lambda^2 + 2am\omega - a^2\omega^2 + \frac{-m^2 + mx - \frac{1}{4}}{1-x^2}\right\} S^- = 0.$$

$$\left\{(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{a\mu(1-x^2)}{\lambda + a\mu x}\frac{d}{dx}\right.$$
$$+ \frac{a\mu\left(\frac{x}{2} + a\omega(1-x^2) - m\right)}{\lambda + a\mu x} + a^2(\omega^2 - \mu^2)x^2 + a\omega x - \frac{1}{4}$$

$$z = \frac{x - a_1}{a_2 - a_1} = \frac{x+1}{2}$$

$$z_3 = \frac{a_3 - a_1}{a_2 - a_1} = \frac{-\lambda/a\mu + 1}{2}$$

$$\left\{\frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} - \frac{1}{z-z_3}\right]\frac{d}{dz} + 4a^2(\mu^2 - \omega^2) + \frac{a^2(\mu^2-\omega^2)}{z-1} - \frac{a^2(\mu^2-\omega^2)}{z}\right.$$
$$+ \frac{1}{16}\frac{-4m^2 - 4m - 1}{z^2} + \frac{1}{8}\frac{4m^2+1}{z-1} + \frac{1}{16}\frac{-4m^2+4m-1}{(z-1)^2} + \frac{1}{8}\frac{-4m^2-1}{z}$$
$$+ \frac{1}{4}\frac{8a\omega z_3^2 - 8a\omega z_3 + 2m - 2z_3 + 1}{z_3(z_3-1)(z-z_3)} + \frac{-2m+1}{(z-1)(-4+4z_3)} + \frac{1}{4}\frac{2m+1}{z_3 z}$$

$$\left\{\frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} - \frac{1}{z-z_3}\right]\frac{d}{dz} + 4a^2(\mu^2 - \omega^2) + \frac{a^2(\mu^2-\omega^2)}{z-1} - \frac{a^2(\mu^2-\omega^2)}{z}\right.$$
$$+ \frac{1}{16}\frac{-4m^2 - 4m - 1}{z^2} + \frac{1}{8}\frac{4m^2+1}{z-1} + \frac{1}{16}\frac{-4m^2+4m-1}{(z-1)^2} + \frac{1}{8}\frac{-4m^2-1}{z}$$
$$+ \frac{1}{4}\frac{8a\omega z_3^2 - 8a\omega z_3 + 2m - 2z_3 + 1}{z_3(z_3-1)(z-z_3)} + \frac{-2m+1}{(z-1)(-4+4z_3)} + \frac{1}{4}\frac{2m+1}{z_3 z}$$
$$\left. + \frac{1}{4}\frac{4a^2\omega^2 - 8am\omega - 4a\omega - 4\lambda^2 + 1}{z-1} + \frac{1}{4}\frac{-4a^2\omega^2 + 8am\omega - 4a\omega + 4\lambda^2 - 1}{z}\right\} S(z) = 0.$$

$$F(r) = r(r-1) + r - \frac{1}{4}\left(m + \frac{1}{2}\right)^2 = 0$$

$$S(z) = z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}\bar{S}(z)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{2\alpha_1+1}{z} + \frac{2\alpha_2+1}{z-1} + \frac{-1}{z-z_3}\right]\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-z_3)}\right\}\bar{S}(z) = 0$$

$$q = -z_3\left\{-a^2(\mu^2-\omega^2) + \frac{1}{8}(-4m^2-1) + \frac{2m+1}{4z_3}\right.$$
$$\left.+\frac{-4a^2\omega^2 + 8ma\omega - 4a\omega + 4\lambda^2 - 1}{4} - \alpha_2 - \frac{\alpha_3}{z_3} - \alpha_1 + \frac{\alpha_1}{z_3} - 2\alpha_1\alpha_2 - \frac{2\alpha_1\alpha_3}{z_3}\right\}$$

$$w = \sum_\mu a_\mu w_\mu, w_\mu = F(\alpha_\mu, \gamma_\mu, s_0 z)$$

$$w'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)w' + \frac{\alpha z - q}{z(z-1)}w = 0$$

$$t\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\gamma - t)\frac{\mathrm{d}y}{\mathrm{d}t} - \alpha y = 0$$

$$w_\mu'' + \left(\frac{\gamma_\mu}{z} - s_0\right)w_\mu' - \frac{\alpha_\mu s_0}{z}w_\mu = 0$$

$$\alpha_\mu = \alpha_0 + \mu, \gamma_\mu = \gamma_0 = \text{ constant }, \mu \in \mathbb{Z}$$

$$\alpha_\mu w_{\mu+1} = \alpha_\mu\left[1 + \frac{(\alpha_0+\mu+1)zs_0}{\gamma_0 1!} + \frac{(\alpha_0+\mu+1)(\alpha_0+\mu+2)z^2 s_0^2}{(\gamma_0+1)\gamma_0 2!} + \cdots\right]$$
$$(\alpha_\mu - \gamma_0)w_{\mu-1} = \alpha_\mu\left[1 + \frac{\alpha_0+\mu-1}{\gamma_0}zs_0 + \frac{(\alpha_0+\mu-1)(\alpha_0+\mu)z^2 s_0^2}{\gamma_0(\gamma_0+1)2!} + \cdots\right]$$
$$(\gamma_0 - 2\alpha_\mu)w_\mu = \left[1 + \frac{(\alpha_0+\mu)zs_0}{\gamma_0 1!} + \frac{(\alpha_0+\mu)(\alpha_0+\mu+1)z^2 s_0^2}{\gamma_0(\gamma_0+1)2!} + \cdots\right]$$

$$\alpha_\mu w_{\mu+1} = \alpha_\mu\left[1 + \frac{(\alpha_0+\mu+1)zs_0}{\gamma_0 1!} + \frac{(\alpha_0+\mu+1)(\alpha_0+\mu+2)z^2 s_0^2}{(\gamma_0+1)\gamma_0 2!} + \cdots\right]$$
$$(\alpha_\mu - \gamma_0)w_{\mu-1} = \alpha_\mu\left[1 + \frac{\alpha_0+\mu-1}{\gamma_0}zs_0 + \frac{(\alpha_0+\mu-1)(\alpha_0+\mu)z^2 s_0^2}{\gamma_0(\gamma_0+1)2!} + \cdots\right]$$

$$s_0 z w_\mu = (\alpha_\mu - \gamma_0)w_{\mu-1} + (\gamma_0 - 2\alpha_\mu)w_\mu + \alpha_\mu w_{\mu+1},$$
$$zw_\mu' = \alpha_\mu(w_{\mu+1} - w_\mu)$$

$$s_0 z^2 w_\mu' = \alpha_\mu[(\alpha_\mu+1)w_{\mu+2} + w_{\mu+1}(\gamma_0 - 3\alpha_\mu - 2)$$
$$+w_\mu(3\alpha_\mu + 1 - 2\gamma_0) + (\gamma_0 - \alpha_\mu)w_{\mu-1}].$$

$$\sum_\mu a_\mu\{[z^2(\varepsilon + s_0) + z(-\varepsilon - s_0 + \gamma + \delta - \gamma_0) + \gamma_0 - \gamma]w_\mu'$$
$$+[(\alpha_\mu s_0 + \alpha)z - (q + \alpha_\mu s_0)]w_\mu\} = 0.$$

$$\sum_\mu a_\mu\{z\varepsilon\alpha_\mu(w_{\mu+1} - w_\mu) + \alpha_\mu(\alpha_\mu+1)w_{\mu+2}$$
$$-\alpha_\mu(2\alpha_\mu + 2 - \gamma_0)w_{\mu+1} + \alpha_\mu(\alpha_\mu - \gamma_0 + 1)w_\mu$$
$$+(\delta - \epsilon)\alpha_\mu(w_{\mu+1} - w_\mu) - s_0\alpha_\mu(w_{\mu+1} - w_\mu) + \alpha z w_\mu - q w_\mu - \alpha_\mu s_0 w_\mu\} = 0$$

$$R_\nu a_\nu + Q_{\nu-1}a_{\nu-1} + P_{\nu-2}a_{\nu-2} + S_{\nu-3}a_{\nu-3} = 0$$

$$R_\nu a_\nu + Q_{\nu-1}a_{\nu-1} + P_{\nu-2}a_{\nu-2} = 0$$

$$\begin{aligned} R_\nu &= (\alpha_\nu - \gamma)\left(\alpha_\nu - \frac{\alpha}{\varepsilon}\right) \\ Q_\nu &= \left(\alpha_\nu - \frac{\alpha}{\varepsilon}\right)(\gamma - 2\alpha_\nu) + \alpha_\nu(\varepsilon - \delta) - q \\ P_\nu &= \alpha_\nu\left[(\alpha_\nu + \delta) - \frac{\alpha}{\varepsilon}\right] \end{aligned}$$

$$w = \sum_{\nu=0}^{\infty} a_\nu F(\alpha_0 + \nu, \gamma, -\varepsilon z)$$

$$\begin{aligned} R_\mu &= (\alpha_0 + \mu - \gamma)(\alpha_0 + \mu - \alpha/\varepsilon) \\ Q_\mu &= (\alpha_0 + \mu - \alpha/\varepsilon)(\gamma - 2\alpha_0 - 2\mu) + (\alpha_0 + \mu)(\varepsilon - \delta) - q \\ P_\mu &= (\alpha_0 + \mu)\big((\alpha_0 + \mu + \delta) - \alpha/\varepsilon\big) \end{aligned}$$

$$\alpha_0 = -N \Rightarrow \alpha/\varepsilon = -N, \text{ or } \delta = -N$$

$$\gamma + \delta + \left(-\frac{\alpha}{\varepsilon}\right) = -N$$

$$R_1 a_1 + Q_0 a_0 = 0 \Rightarrow a_1 = \frac{-Q_0 a_0}{R_1}$$

$$Q_1 a_1 + P_0 a_0 = 0, (a_2 = 0)$$

$$\left[\left(\alpha_1 - \frac{\alpha}{\varepsilon}\right)(\gamma - 2\alpha_1) + \alpha_1(\varepsilon - \delta) - q\right]\left[-\frac{(\alpha_0(\varepsilon - \delta) - q)a_0}{1 + \alpha_0 - \gamma}\right] + \alpha_0 \delta a_0 = 0$$

$$Q_2 a_2 + P_1 a_1 = 0, (a_3 = 0)$$

$$\begin{aligned} R_2 a_2 + Q_1 a_1 + P_0 a_0 = 0 \Rightarrow a_2 &= \frac{-Q_1 a_1 - P_0 a_0}{R_2} \\ &= -\frac{\left[\left(\alpha_1 - \frac{\alpha}{\varepsilon}\right)(\gamma - 2\alpha_1) + \alpha_1(\varepsilon - \delta) - q\right]\left[-\frac{(\alpha_0(\varepsilon - \delta) - q)a_0}{1 + \alpha_0 - \gamma}\right] - P_0 a_0}{R_2} \end{aligned}$$

$$Q_2 = 2(\gamma - 2\alpha_0 - 4) + (\alpha_0 + 2)(\varepsilon - \delta) - q$$

$$\Delta^{KN}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right) + [\omega^2(r^2 + a^2)^2 - 4Ma\omega mr + 2e^2 a\omega m - \mu^2 r^2 \Delta^{KN}$$

$$\begin{aligned} &\Delta^{KN}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right) + [\omega^2(r^2 + a^2)^2 - 4Ma\omega mr + 2e^2 a\omega m - \mu^2 r^2 \Delta^{KN} \\ &+ m^2 a^2 - (\omega^2 a^2 + \mathcal{K}_{lm})\Delta^{KN} - 2eqr[(r^2 + a^2)\omega - am] + e^2 q^2 r^2]R = 0 \end{aligned}$$

$$M\chi = r - r_+, r_\pm = M \pm Md$$

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\chi(\chi + 2d)\frac{\mathrm{d}R}{\mathrm{d}\chi}\right] + \left[\frac{\omega^2}{M^2\chi(\chi + 2d)}\{M^2[(\chi + d + 1)^2 - (d^2 - 1)] - e^2\}^2 + \frac{2e^2 a\omega m}{M^2\chi(\chi + 2d)}\right.$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\chi(\chi + 2d)\frac{\mathrm{d}R}{\mathrm{d}\chi}\right] + \left[\frac{\omega^2}{M^2\chi(\chi + 2d)}\{M^2[(\chi + d + 1)^2 - (d^2 - 1)] - e^2\}^2 + \frac{2e^2 a\omega m}{M^2\chi(\chi + 2d)}\right. \\ &\left. - \frac{4a\omega m(\chi + d + 1)}{\chi(\chi + 2d)} - \mu^2 M^2(\chi + d + 1)^2 + \frac{m^2 a^2}{M^2\chi(\chi + 2d)} - (\omega^2 a^2 + \mathcal{K}_{lm})\right]R = 0, \end{aligned}$$

$$\Delta^{KN} = M^2\chi(\chi + 2d), \Delta^{KN} + 2Mr = M^2(\chi + d + 1)^2 - M^2(d^2 - 1)$$

$$R(\chi) = Z(\chi)(\chi(\chi + 2d))^{-1/2}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + Z\Big\{(\omega^2 - \mu^2)M^2 + \frac{1}{M^2\chi^2(\chi + 2d)^2}\big((\omega^2[M^4 4(\chi + d + 1)^2 + 4M^4(\chi + d + 1)\chi(\chi + 2d)$$
$$-2e^2M^2[\chi(\chi + 2d) + 2(\chi + d + 1)] + e^4]$$
$$-4a\omega m M^2(\chi + d + 1) + 2e^2 a\omega m - \mu^2 M^4[2\chi + (d + 1)^2]\chi(\chi + 2d)$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + Z\Big\{(\omega^2 - \mu^2)M^2 + \frac{1}{M^2\chi^2(\chi + 2d)^2}\big((\omega^2[M^4 4(\chi + d + 1)^2 + 4M^4(\chi + d + 1)\chi(\chi + 2d)$$
$$-2e^2M^2[\chi(\chi + 2d) + 2(\chi + d + 1)] + e^4]$$
$$-4a\omega m M^2(\chi + d + 1) + 2e^2 a\omega m - \mu^2 M^4[2\chi + (d + 1)^2]\chi(\chi + 2d)$$
$$+m^2a^2 - (\omega^2 a^2 + \mathcal{K}_{lm})M^2\chi(\chi + 2d) + d^2M^2)\big)\Big\} = 0$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + \left[M^2(\omega^2 - \mu^2) + \frac{1}{M^2}\left\{\frac{A}{\chi^2} + \frac{B}{\chi} + \frac{C}{(\chi + 2d)^2} + \frac{D}{\chi + 2d}\right\}\right] Z = 0$$

$$A = \frac{d^2M^2 + (am + (-2(1 + d)M^2 + e^2)\omega)^2}{4d^2}$$

$$B = \frac{1}{4d^3}(-a^2m^2 + d^2M^2(-1 - 2\mathcal{K}_{lm} - 2(1 + d)^2M^2\mu^2) + 2am(2M^2 - e^2)\omega$$
$$-(2a^2d^2M^2 - 4(1 + d)^2(-1 + 2d)M^4 + 4(-1 + d^2)M^2e^2 + e^4)\omega^2)$$

$$C = \frac{d^2M^2 + (am + (2(-1 + d)M^2 + e^2)\omega)^2}{4d^2}$$

$$D = \frac{1}{4d^3}(d^2M^2(1 + 2\mathcal{K}_{lm} + 2(-1 + d)^2M^2\mu^2) + 2am(-2M^2 + e^2)\omega$$
$$+(4(-1 + d)^2(1 + 2d)M^4 + 4(-1 + d^2)M^2e^2 + e^4)\omega^2 + a^2(m^2 + 2d^2M^2\omega^2))$$

$$\zeta = -\frac{\chi}{2d},$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\zeta^2} + \left[4d^2M^2(\omega^2 - \mu^2) + \frac{1}{M^2}\left(\frac{A}{\zeta^2} + \frac{-2dB}{\zeta} + \frac{C}{(\zeta - 1)^2} + \frac{-2dD}{\zeta - 1}\right)\right] Z = 0$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} + \frac{\mathrm{d}w}{\mathrm{d}\zeta}\left[\frac{1}{\zeta} + \frac{1}{\zeta - 1}\right] + w(\zeta)\left\{\left(\frac{A}{M^2} - \frac{1}{4}\right)\frac{1}{\zeta^2} + \left(\frac{C}{M^2} - \frac{1}{4}\right)\frac{1}{(\zeta - 1)^2}\right.$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} + \frac{\mathrm{d}w}{\mathrm{d}\zeta}\left[\frac{1}{\zeta} + \frac{1}{\zeta - 1}\right] + w(\zeta)\left\{\left(\frac{A}{M^2} - \frac{1}{4}\right)\frac{1}{\zeta^2} + \left(\frac{C}{M^2} - \frac{1}{4}\right)\frac{1}{(\zeta - 1)^2}\right.$$
$$\left. + \left(\frac{-2dB}{M^2} - \frac{1}{2}\right)\frac{1}{\zeta} + \left(\frac{-2dD}{M^2} + \frac{1}{2}\right)\frac{1}{\zeta - 1} + 4d^2M^2(\omega^2 - \mu^2)\right\} = 0$$

$$r(r - 1) + r + B_i' = 0, i = 1,2$$

$$2\mu_1^{(1,2)} = \pm 2i\sqrt{B_1'} = \pm\frac{i}{M}\sqrt{4A - M^2},$$
$$2\mu_2^{(1,2)} = \pm 2i\sqrt{B_2'} = \pm\frac{i}{M}\sqrt{4C - M^2}$$

$$w(\zeta) = e^{\nu\zeta}\prod_{i=1}^{2}(\zeta - \zeta_i)^{\mu_i}Y(\zeta) = e^{\pm 2i\zeta dM\sqrt{\omega^2 - \mu^2}}\zeta^{\mu_1}(\zeta - 1)^{\mu_2}Y(\zeta)$$

$$w(\zeta) = e^{\nu\zeta}\prod_{i=1}^{2}(\zeta - \zeta_i)^{\mu_i}Y(\zeta) = e^{\pm 2i\zeta dM\sqrt{\omega^2 - \mu^2}}\zeta^{\mu_1}(\zeta - 1)^{\mu_2}Y(\zeta)$$
$$= e^{\pm 2i\zeta dM\sqrt{\omega^2 - \mu^2}}\zeta^{\frac{\pm i}{2M}\sqrt{4A - M^2}}(\zeta - 1)^{\frac{\pm i}{2M}\sqrt{4C - M^2}}Y(\zeta)$$

$$Y''(\zeta)+\left(\alpha+\frac{\gamma}{\zeta}+\frac{\delta}{\zeta-1}\right)Y'(\zeta)+\frac{w\zeta-\sigma}{\zeta(\zeta-1)}Y(\zeta)=0$$

$$\alpha_{\pm}\ =2\nu_{\pm}=\pm 4idM\sqrt{\omega^2-\mu^2},\gamma_{\pm}=1\pm\frac{i}{M}\sqrt{4A-M^2},\delta_{\pm}\ =1\pm\frac{i}{M}\sqrt{4C-M^2}$$

$$\sigma_{\pm}\ =\left(\frac{-2dB}{M^2}-\frac{1}{2}\right)+\frac{1}{2}\pm\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left(1\pm\frac{i}{M}\sqrt{4A-M^2}\right)$$
$$-\frac{1}{2}\left(1\pm\frac{i}{M}\sqrt{4A-M^2}\right)\left(1\pm\frac{i}{M}\sqrt{4C-M^2}\right)$$

$$w_{\pm}\ =\frac{-2d}{M^2}(B+D)\pm 4idM\sqrt{\omega^2-\mu^2}\pm\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\pm\frac{i}{M}\sqrt{4A-M^2}\pm\frac{i}{M}\sqrt{4C-M^2}\right]$$

$$R(\zeta)=\frac{M}{\sqrt{\Delta^{KN}}}e^{-2idM\sqrt{\omega^2-\mu^2}\zeta}\zeta^{\frac{1}{2}-\frac{i}{2M}\sqrt{4A-M^2}}(\zeta-1)^{\frac{1}{2}-\frac{i}{2M}\sqrt{4C-M^2}}H_c(\alpha_-,w_-,\gamma_-,\delta_-,\sigma_-,\zeta)$$

$$\delta_{\pm}=-N,\frac{w_{\pm}}{\alpha_{\pm}}=-N$$

$$\gamma_{\pm}+\delta_{\pm}+\left(-\frac{w_{\pm}}{\alpha_{\pm}}\right)=-N$$

$$\frac{C}{(\chi+2d)^2}=C\left[\frac{1}{\chi^2}-\frac{4d}{\chi^3}+\cdots\right]$$
$$\frac{D}{\chi+2d}=D\left[\frac{1}{\chi}-\frac{2d}{\chi^2}+\cdots\right]$$
$$\frac{\mathrm{d}^2Z}{\mathrm{d}\chi^2}=\left[M^2(\mu^2-\omega^2)-\frac{1}{M^2}\left(\frac{A+C-2dD}{\chi^2}+\frac{B+D}{\chi}\right)+\mathcal{O}\left(\frac{1}{\chi^3}\right)\right]Z$$

$$\frac{\mathrm{d}^2Z}{\mathrm{d}\xi^2}=\left(\frac{1}{4}-\frac{k}{\xi}+\frac{m^2-\frac{1}{4}}{\xi^2}\right)Z$$

$$k=\frac{B+D}{2M^3\sqrt{\mu^2-\omega^2}},\frac{1}{4}-m^2=\frac{1}{M^2}[A+C-2dD]$$

$$M_{k,m}(\xi)\ =e^{-\xi/2}\xi^{m+1/2}M\left(m-k+\frac{1}{2},2m+1,\xi\right)$$
$$W_{k,m}(\xi)\ =e^{-\xi/2}\xi^{m+1/2}U\left(m-k+\frac{1}{2},2m+1,\xi\right)$$

$$W_{k,m}(\xi)\sim e^{-\xi/2}\xi^{k}\ \left(\xi\to\infty,|\mathrm{Arg}(\xi)|\leq\frac{3}{2}\pi-\delta_1\right)$$

$$R(r)\sim\frac{M}{\sqrt{\Delta^{KN}}}e^{-\xi/2}\xi^{k}\Rightarrow$$
$$R(r)\sim\frac{M}{\sqrt{\Delta^{KN}}}e^{-\sqrt{\mu^2-\omega^2}(r-r_+)}[2(\mu^2-\omega^2)^{1/2}(r-r_+)]\frac{B+D}{2^3\sqrt{\mu^2-\omega^2}}.$$

$$\frac{\mathrm{d}^2Z}{\mathrm{d}\eta^2}=\left(\frac{1}{4}-\frac{k_h}{\xi}+\frac{m_h^2-\frac{1}{4}}{\eta^2}\right)Z$$

$$k_h = \frac{B}{2M^2 \underbrace{\sqrt{M^2(\mu^2-\omega^2) - \frac{C}{4M^2d^2} - \frac{D}{2M^2d}}}_{=:\sqrt{\mathcal{F}}}}, m_h^2 = \frac{1}{4} - \frac{A}{M^2}$$

$$M_{k_h,m_h}(\eta) = e^{-\frac{1}{2}\eta}\eta^{\frac{1}{2}+m_h}F\left(m_h + \frac{1}{2} - k_h, 2m_h + 1, \eta\right)$$

$$M_{k_h,m_h}(\eta) \sim \eta^{\frac{1}{2}+m_h}, \eta \to 0$$

$$R(r) \sim \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mathcal{F}}\frac{(r-r_+)}{M}} \left(2\sqrt{\mathcal{F}}\frac{(r-r_+)}{M}\right)^{\frac{1}{2}+m_h}, r \to r_+$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + Z\Big\{(\omega^2-\mu^2)M^2 + \frac{1}{M^2\chi^2(\chi+2d)^2}\big((\omega^2[M^4 4(\chi+d+1)^2 + 4M^4(\chi+d+1)\chi(\chi+2d)$$
$$-2e^2M^2[\chi(\chi+2d) + 2(\chi+d+1)] + e^4]$$
$$-4a\omega mM^2(\chi+d+1) + 2e^2a\omega m - \mu^2M^4[2\chi + (d+1)^2]\chi(\chi+2d)$$
$$+m^2a^2 - (\omega^2a^2 + \mathcal{K}_{lm})M^2\chi(\chi+2d) + d^2M^2$$
$$-2eqM^3\chi(\chi+2d)(\chi+d+1)\omega - 4eqM^3(\chi+d+1)^2\omega + 2e^3qM(\chi+d+1)\omega$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + Z\Big\{(\omega^2-\mu^2)M^2 + \frac{1}{M^2\chi^2(\chi+2d)^2}\big((\omega^2[M^4 4(\chi+d+1)^2 + 4M^4(\chi+d+1)\chi(\chi+2d)$$
$$-2e^2M^2[\chi(\chi+2d) + 2(\chi+d+1)] + e^4]$$
$$-4a\omega mM^2(\chi+d+1) + 2e^2a\omega m - \mu^2M^4[2\chi + (d+1)^2]\chi(\chi+2d)$$
$$+m^2a^2 - (\omega^2a^2 + \mathcal{K}_{lm})M^2\chi(\chi+2d) + d^2M^2$$
$$-2eqM^3\chi(\chi+2d)(\chi+d+1)\omega - 4eqM^3(\chi+d+1)^2\omega + 2e^3qM(\chi+d+1)\omega +$$
$$+2eqM(\chi+d+1)am + e^2q^2M^2[\chi(\chi+2d) + 2\chi + (d+1)^2]\big)\Big\} = 0$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} + \left[M^2(\omega^2-\mu^2) + \frac{1}{M^2}\left\{\frac{A'}{\chi^2} + \frac{B'}{\chi} + \frac{C'}{(\chi+2d)^2} + \frac{D'}{\chi+2d}\right\}\right]Z = 0,$$
$$A' = A - \frac{1}{4d^2}(-e^2q^2M^2(1+d)^2 + 4eM^3q\omega(1+d)^2 - 2e^3qM\omega(d+1))$$
$$B' = B - \frac{1}{4d^3}(2aemM + e^2M^2q^2(1-d^2) + 2d^2M^4\mu^2(1+d)^2 + 4eM^3q\omega(d^3+2d^2-1) + 2e^3qM\omega)$$
$$C' = C - \frac{1}{4d^2}(2aeqmM(d-1) - e^2q^2M^2(1-d)^2 + 4eqM^3\omega(d-1)^2 + 2e^3qM\omega(d-1))$$
$$D' = D - \frac{1}{4d^3}(-2aeqmM - e^2q^2M^2(1-d^2) - 2d^2M^4\mu^2(d-1)^2 + 4eq\omega M^3(1-2d^2+d^3) - 2e^3qM\omega)$$

$$R(\zeta) = \frac{M}{\sqrt{\Delta^{KN}}} e^{-2idM\sqrt{\omega^2-\mu^2}\zeta}\zeta^{\frac{1}{2}-\frac{i}{2M}\sqrt{4A'-M^2}}(\zeta-1)^{\frac{1}{2}-\frac{i}{2M}\sqrt{4C'-M^2}}H_c(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta)$$

$$\alpha'_\pm = \pm 4idM\sqrt{\omega^2-\mu^2}, \gamma'_\pm = 1 \pm \frac{i}{M}\sqrt{4A'-M^2}, \delta'_\pm = 1 \pm \frac{i}{M}\sqrt{4C'-M^2}$$
$$\sigma'_\pm = \left(\frac{-2dB'}{M^2} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\pm 4idM\sqrt{\omega^2-\mu^2}}{2}\left(1 \pm \frac{i}{M}\sqrt{4A'-M^2}\right)$$
$$-\frac{1}{2}\left(1 \pm \frac{i}{M}\sqrt{4A'-M^2}\right)\left(1 \pm \frac{i}{M}\sqrt{4C'-M^2}\right)$$
$$w'_\pm = \frac{-2d}{M^2}(B'+D') \pm 4idM\sqrt{\omega^2-\mu^2} \pm \frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\pm\frac{i}{M}\sqrt{4A'-M^2} \pm \frac{i}{M}\sqrt{4C'-M^2}\right]$$

$$Hc(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta) \rightarrow \mathrm{HeunC}(\alpha_\mathrm{M}, \beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta)$$
$$\alpha'_- = \alpha_\mathrm{M}, \gamma'_- = 1 + \beta_\mathrm{M}, \delta'_- = 1 + \gamma_\mathrm{M}, \delta_\mathrm{M} = -\frac{2d}{M^2}(B' + D'), \eta_\mathrm{M} = \frac{1}{2} + \frac{2dB'}{M^2}$$
$$\sigma'_- = -\eta_\mathrm{M} + \frac{1}{2} + \frac{\alpha_\mathrm{M}}{2}(1 + \beta_\mathrm{M}) - \frac{1}{2}(1 + \beta_\mathrm{M})(1 + \gamma_\mathrm{M})$$

$$R(\zeta) = \frac{M}{\sqrt{\Delta^{KN}}} e^{\frac{1}{2}\alpha_\mathrm{M}\zeta} \zeta^{\frac{1}{2}(1+\beta_\mathrm{M})} (\zeta - 1)^{\frac{1}{2}(1+\gamma_\mathrm{M})} \{c_1 \mathrm{HeunC}(\alpha_\mathrm{M}, \beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta)$$

$$Hc(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta) \rightarrow \mathrm{HeunC}(\alpha_\mathrm{M}, \beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta)$$
$$\alpha'_- = \alpha_\mathrm{M}, \gamma'_- = 1 + \beta_\mathrm{M}, \delta'_- = 1 + \gamma_\mathrm{M}, \delta_\mathrm{M} = -\frac{2d}{M^2}(B' + D'), \eta_\mathrm{M} = \frac{1}{2} + \frac{2dB'}{M^2}$$
$$\sigma'_- = -\eta_\mathrm{M} + \frac{1}{2} + \frac{\alpha_\mathrm{M}}{2}(1 + \beta_\mathrm{M}) - \frac{1}{2}(1 + \beta_\mathrm{M})(1 + \gamma_\mathrm{M})$$
$$w'_- = \delta_\mathrm{M} + \alpha_\mathrm{M} + \frac{\alpha_\mathrm{M}}{2}(\beta_\mathrm{M} + \gamma_\mathrm{M})$$

$$R(\zeta) = \frac{M}{\sqrt{\Delta^{KN}}} e^{\frac{1}{2}\alpha_\mathrm{M}\zeta} \zeta^{\frac{1}{2}(1+\beta_\mathrm{M})} (\zeta - 1)^{\frac{1}{2}(1+\gamma_\mathrm{M})} \{c_1 \mathrm{HeunC}(\alpha_\mathrm{M}, \beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta)$$
$$+ c_2 \zeta^{-\beta_\mathrm{M}} \mathrm{HeunC}(\alpha_\mathrm{M}, -\beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta)\}$$

$$\delta'_\pm = 1 \pm \frac{i}{M}\sqrt{4C' - M^2} = -N$$

$$\frac{w'_\pm}{\alpha'_\pm} = \frac{\frac{-2d}{M^2}(B' + D') \pm 4idM\sqrt{\omega^2 - \mu^2} \pm \frac{4idM\sqrt{\omega^2 - \mu^2}}{2}\left[\pm \frac{i}{M}\sqrt{4A' - M^2} \pm \frac{i}{M}\sqrt{4C' - M^2}\right]}{\pm 4idM\sqrt{\omega^2 - \mu^2}} = -N$$

$$\gamma'_\pm + \delta'_\pm + \left(-\frac{w'_\pm}{\alpha'_\pm}\right) = -N$$

$$U(a, c, \xi) \sim \xi^{-a}\left[1 - \frac{ab}{\xi} + \frac{a(a+1)b(b+1)}{2!\xi^2} - \cdots\right] = \frac{1}{\xi^a}\sum_{\nu=0} \frac{(a)_\nu (b)_\nu}{(1)_\nu}\left(\frac{-1}{\xi}\right)^\nu \xi$$

$$W_{k,m}(\xi) \sim e^{-\xi/2}\xi^k\left[1 + \frac{m^2 - \left(k - \frac{1}{2}\right)^2}{\xi} + \frac{\left(m^2 - \left(k - \frac{1}{2}\right)^2\right)\left(m^2 - \left(k - \frac{3}{2}\right)^2\right)}{2!\,\xi^2} + \cdots\right]$$

$$R(r) \sim \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mu^2 - \omega^2}(r - r_+)} \left[2(\mu^2 - \omega^2)^{1/2}(r - r_+)\right]^{\frac{B' + D'}{2M^3\sqrt{\mu^2 - \omega^2}}}$$

$$R(r) \sim \frac{M}{\sqrt{\Delta^{KN}}} e^{+\sqrt{\mu^2 - \omega^2}(r - r_+)} \left[-2(\mu^2 - \omega^2)^{1/2}(r - r_+)\right]^{-\frac{B' + D'}{2M^3\sqrt{\mu^2 - \omega^2}}}$$

$$Y = e^{\lambda\zeta}\zeta^\mu \sum_{s=0}^{\infty} \frac{a_s}{\zeta^s}$$

$$\lambda_1 = 0, \mu_1 = -\frac{w'_-}{\alpha'_-} = -\left[\frac{\delta_\mathrm{M}}{\alpha_\mathrm{M}} + \frac{1}{2}(2 + \beta_\mathrm{M} + \gamma_\mathrm{M})\right]$$
$$\lambda_2 = -\alpha'_- = -\alpha_\mathrm{M}, \mu_2 = \left[-(\gamma'_- + \delta'_-) + \frac{w'_-}{\alpha'_-}\right] = \frac{\delta_\mathrm{M}}{\alpha_\mathrm{M}} - \frac{1}{2}(2 + \beta_\mathrm{M} + \gamma_\mathrm{M})$$

$$H_c(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta) \sim \begin{cases} \zeta^{-\frac{w'_-}{\alpha'_-}} \\ e^{-\alpha'_-\zeta}\zeta^{-(\gamma'_-+\delta'_-)+\frac{w'_-}{\alpha'_-}} \end{cases} \Leftrightarrow$$

$$\mathrm{HeunC}(\alpha_\mathrm{M}, \beta_\mathrm{M}, \gamma_\mathrm{M}, \delta_\mathrm{M}, \eta_\mathrm{M}, \zeta) \sim \begin{cases} \zeta^{-\left[\frac{\delta_\mathrm{M}}{\alpha_\mathrm{M}}+\frac{1}{2}(2+\beta_\mathrm{M}+\gamma_\mathrm{M})\right]} \\ e^{-\alpha_\mathrm{M}\zeta}\zeta^{\frac{\delta_\mathrm{M}}{\alpha_\mathrm{M}}-\frac{1}{2}(2+\beta_\mathrm{M}+\gamma_\mathrm{M})} \end{cases}$$

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mu^2-\omega^2}(r-r_+)} \left(-\frac{r-r_+}{2Md}\right)^{\frac{(B'+D')}{2M^3\sqrt{\mu^2-\omega^2}}} \\ \frac{M}{\sqrt{\Delta^{KN}}} e^{+\sqrt{\mu^2-\omega^2}(r-r_+)} \left(-\frac{r-r_+}{2Md}\right)^{-\frac{(B'+D')}{2M^3\sqrt{\mu^2-\omega^2}}} \end{cases}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\eta'^2} = \left[\frac{1}{4} - \frac{1}{M^2}\left(\frac{A'}{\eta'^2} + \frac{B'}{2\sqrt{\mathcal{F}'}}\frac{1}{\eta'}\right)\right] Z$$

$$\mathcal{F}' = M^2(\mu^2 - \omega^2) - \frac{C'}{4M^2d^2} - \frac{D'}{2M^2d}, \eta' := 2\sqrt{\mathcal{F}'}\chi$$

$$M_{k',m'_h}(\eta') = e^{-\frac{\eta'}{2}}\eta'^{\frac{1}{2}+m'_h} F\left(m'_h + \frac{1}{2} - k', 2m'_h + 1, \eta'\right) = e^{-\frac{\eta'}{2}}\eta'^{\frac{1}{2}+m'_h} \sum_{\nu=0}^{\infty} \frac{\left(m'_h + \frac{1}{2} - k'\right)_\nu}{(2m'_h+1)_\nu} \frac{\eta'^\nu}{\nu!}$$

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}} \left(2\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}\right)^{\frac{1}{2}+m'_h} & , r \to r_+ \\ \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}} \left(2\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}\right)^{\frac{1}{2}-m'_h} & , r \to r_+ \end{cases}$$

$$H_c(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta) = \sum_{k=0}^{\infty} c_k\zeta^k = 1 + \frac{\sigma'_-}{-\gamma'_-}\zeta + \frac{-(-\alpha'_- + \gamma'_- + \delta'_-)\sigma'_- + \sigma'^2_- + w'_-\gamma'_-}{2\gamma'_-(1+\gamma'_-)}\zeta^2 + \cdots$$

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}}\left[-\frac{(r-r_+)}{2dM}\right]^{\frac{1}{2}-\frac{i}{2M}\sqrt{4A'-M^2}} \\ \frac{M}{\sqrt{\Delta^{KN}}}\left[-\frac{(r-r_+)}{2dM}\right]^{\frac{1}{2}+\frac{i}{2M}\sqrt{4A'-M^2}} \end{cases}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

$$\gamma + \delta + \varepsilon = \alpha + \beta + 1$$

$$\mathrm{Hl}(a, q; \alpha, \beta, \gamma, \delta; z)$$

$$z(z-1)\left(\frac{z}{a} - 1\right)y''(z) + \left[\gamma(z-1)\left(\frac{z}{a} - 1\right) + \delta z\left(\frac{z}{a} - 1\right) + \frac{\varepsilon}{a}z(z-1)\right]y'(z)$$

$$z(z-1)\left(\frac{z}{a} - 1\right)y''(z) + \left[\gamma(z-1)\left(\frac{z}{a} - 1\right) + \delta z\left(\frac{z}{a} - 1\right) + \frac{\varepsilon}{a}z(z-1)\right]y'(z) + \left(\alpha\frac{\beta}{a}z - \frac{q}{a}\right)y(z) = 0$$

$$\frac{\beta}{a} \to \frac{\varepsilon}{a} \to -\nu, \frac{q}{a} \to -\sigma$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\nu\right]\frac{\mathrm{d}y}{\mathrm{d}z}+\left[\frac{\alpha\nu z-\sigma}{z(z-1)}\right]y(z)=0$$

$$z=\mathrm{sn}^2(u,k),z_3=a=k^{-2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2}+\left[(2\gamma-1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u}-(2\delta-1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u}-k^2(2\varepsilon-1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}y}{\mathrm{d}u}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2}+\left[(2\gamma-1)\frac{\mathrm{cn}u\mathrm{dn}u}{\mathrm{sn}u}-(2\delta-1)\frac{\mathrm{sn}u\mathrm{dn}u}{\mathrm{cn}u}-k^2(2\varepsilon-1)\frac{\mathrm{sn}u\mathrm{cn}u}{\mathrm{dn}u}\right]\frac{\mathrm{d}y}{\mathrm{d}u}$$
$$+(4\alpha\beta k^2\mathrm{sn}^2u-4k^2q)y=0$$

$$\mathrm{sn}(u+4mK+4niK')=\mathrm{sn}u$$
$$\mathrm{cn}(u+4mK+4niK')=\mathrm{cn}u$$
$$\mathrm{dn}(u+4mK+4niK')=\mathrm{dn}u$$

$$K=\int_0^1\frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}},K'=\int_1^{1/k}\frac{\mathrm{d}x}{\sqrt{(x^2-1)(1-k^2x^2)}}=\int_0^{\cos^{-1}k}\frac{\mathrm{d}\theta}{\sqrt{\cos^2\theta-k^2}}$$

$$\frac{\mathrm{d}z}{\mathrm{d}u}=\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn}^2(u,k)=2\mathrm{sn}(u,k)\frac{\mathrm{dsn}u}{\mathrm{d}u}=2\mathrm{sn}(u,k)\mathrm{cn}(u,k)\mathrm{dn}(u,k)$$
$$\frac{\mathrm{d}y}{\mathrm{d}z}=\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z}=\frac{1}{2\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}\frac{\mathrm{d}y}{\mathrm{d}u}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}=\frac{1}{2\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}\frac{\mathrm{d}u}{\mathrm{d}z}\frac{\mathrm{d}^2y}{\mathrm{d}u^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}=\frac{1}{2\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}\frac{\mathrm{d}u}{\mathrm{d}z}\frac{\mathrm{d}^2y}{\mathrm{d}u^2}$$
$$-\frac{1}{2\mathrm{sn}^2u\mathrm{cn}^2u\mathrm{dn}^2u}(\mathrm{cn}^2u\mathrm{dn}^2u-\mathrm{sn}^2u\mathrm{dn}^2u-k^2\mathrm{sn}^2u\mathrm{cn}^2u)\frac{\mathrm{d}u}{\mathrm{d}z}\frac{\mathrm{d}y}{\mathrm{d}u},$$

$$4z(1-z)(1-k^2z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2}=\frac{\mathrm{d}^2y}{\mathrm{d}u^2}-\frac{(\mathrm{cn}^2u\mathrm{dn}^2u-\mathrm{sn}^2u\mathrm{dn}^2u-k^2\mathrm{sn}^2u\mathrm{cn}^2u)}{\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}\frac{\mathrm{d}y}{\mathrm{d}u}$$

$$u=\int_x^\infty\frac{\mathrm{d}x}{\sqrt{4x^3-g_2x-g_3}}=\int_x^\infty\frac{\mathrm{d}x}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}=\wp^{-1}x$$
$$=\frac{1}{\sqrt{e_1-e_3}}\mathrm{sn}^{-1}\sqrt{\frac{e_1-e_3}{x-e_3}}$$
$$=\frac{1}{\sqrt{e_1-e_3}}\mathrm{cn}^{-1}\sqrt{\frac{x-e_1}{x-e_3}}=\frac{1}{\sqrt{e_1-e_3}}\mathrm{dn}^{-1}\sqrt{\frac{x-e_2}{x-e_3}}$$

$$u=\int_x^\infty\frac{\mathrm{d}x}{\sqrt{4x^3-g_2x-g_3}}=\int_x^\infty\frac{\mathrm{d}x}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}=\wp^{-1}x$$
$$=\frac{1}{\sqrt{e_1-e_3}}\mathrm{sn}^{-1}\sqrt{\frac{e_1-e_3}{x-e_3}}$$

$$\frac{e_1-e_3}{\wp(u)-e_3}=\mathrm{sn}^2\left(u\sqrt{e_1-e_3}\right),\frac{\wp(u)-e_2}{\wp(u)-e_3}=\mathrm{dn}^2\left(u\sqrt{e_1-e_3}\right),\frac{\wp(u)-e_1}{\wp(u)-e_3}=\mathrm{cn}^2\left(u\sqrt{e_1-e_3}\right)$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2}+\left(\frac{\gamma}{\zeta}+\frac{\delta}{\zeta-1}+\frac{-1}{\zeta-a}\right)\frac{\mathrm{d}Y}{\mathrm{d}\zeta}+\frac{(\alpha\beta\zeta-q)Y)}{\zeta(\zeta-1)(\zeta-a)}=0$$

$$Y(\zeta) = (1-a)(\gamma-1)F(\alpha,\beta,\gamma-1,\zeta) + (q - a(1+\alpha+\beta+\alpha\beta-\gamma))F(\alpha,\beta,\gamma,\zeta)$$

$$\zeta(\zeta-1)(\zeta-a)\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} + \{\gamma(\zeta-1)(\zeta-a) + \delta\zeta(\zeta-a) - \zeta(\zeta-1)\}\frac{\mathrm{d}Y}{\mathrm{d}\zeta} + (\alpha\beta\zeta - q)Y = 0$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\zeta} = (1-a)\alpha\beta F(\alpha+1,\beta+1,\gamma,\zeta) + (q - a(1+\alpha+\beta+\alpha\beta-\gamma))\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,\zeta)$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} = (1-a)\alpha\beta\frac{(\alpha+1)(\beta+1)}{\gamma}F(\alpha+2,\beta+2,\gamma+1,\zeta)$$
$$+(q - a(1+\alpha+\beta+\alpha\beta-\gamma))\frac{\alpha\beta}{\gamma}\frac{(\alpha+1)(\beta+1)}{\gamma+1}F(\alpha+2,\beta+2,\gamma+2,\zeta)$$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\alpha+m,\beta+m,\gamma+m,x)$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} = \frac{(1-a)\alpha\beta(\alpha+1)(\beta+1)}{\gamma}\left\{\frac{-\gamma(\gamma-1)}{(\alpha+1)(\beta+1)\zeta}\left[F(\alpha+1,\beta+1,\gamma,\zeta)\right.\right.$$
$$-\frac{(\alpha+1-\gamma)(\beta+1-\gamma)\zeta}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma+1,\zeta)$$
$$\left.\left.+\frac{\gamma(1-\gamma-(\alpha+\beta+3-2\gamma)\zeta)}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma,\zeta)\right]\right\}$$
$$+(q - a(1+\alpha+\beta+\alpha\beta-\gamma))\frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma(\gamma+1)}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} = \frac{(1-a)\alpha\beta(\alpha+1)(\beta+1)}{\gamma}\left\{\frac{-\gamma(\gamma-1)}{(\alpha+1)(\beta+1)\zeta}\left[F(\alpha+1,\beta+1,\gamma,\zeta)\right.\right.$$
$$-\frac{(\alpha+1-\gamma)(\beta+1-\gamma)\zeta}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma+1,\zeta)$$
$$\left.\left.+\frac{\gamma(1-\gamma-(\alpha+\beta+3-2\gamma)\zeta)}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma,\zeta)\right]\right\}$$
$$+(q - a(1+\alpha+\beta+\alpha\beta-\gamma))\frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma(\gamma+1)}$$
$$\times\left\{\frac{-\gamma(\gamma+1)}{(\alpha+1)(\beta+1)}\left[\frac{F(\alpha+1,\beta+1,\gamma+1,\zeta) - F(\alpha+1,\beta+1,\gamma,\zeta)}{\zeta}\right]\right\}$$

$$F(\alpha+2,\beta+2,\gamma+1,\zeta) = \frac{-\gamma(\gamma-1)}{(\alpha+1)(\beta+1)\zeta}[F(\alpha+1,\beta+1,\gamma,\zeta) - F(\alpha+1,\beta+1,\gamma-1,\zeta)]$$

$$-F(\alpha+1,\beta+1,\gamma-1,\zeta) = \frac{-(\alpha+1-\gamma)(\beta+1-\gamma)}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma+1,\zeta)\zeta$$

$$-F(\alpha+1,\beta+1,\gamma-1,\zeta) = \frac{-(\alpha+1-\gamma)(\beta+1-\gamma)}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma+1,\zeta)\zeta$$
$$+\frac{\gamma(1-\gamma-(\alpha+\beta+3-2\gamma)\zeta)}{\gamma(\gamma-1)(1-\zeta)}F(\alpha+1,\beta+1,\gamma,\zeta)$$

$$(\gamma(\zeta-1)(\zeta-a)+\delta\zeta(\zeta-a)-\zeta(\zeta-1))\frac{\mathrm{d}Y}{\mathrm{d}\zeta}$$
$$=[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\alpha\beta F(\alpha,\beta,\gamma,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)(-\alpha(\gamma-1))F(\alpha,\beta,\gamma,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\alpha(\gamma-1)F(\alpha,\beta,\gamma-1,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\frac{\alpha\beta(\beta+1)\zeta}{\gamma}F(\alpha+1,\beta+2,\gamma+1,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)][q-a(1+\alpha+\beta+\alpha\beta-\gamma)]\frac{1-\gamma}{\zeta}F(\alpha,\beta,\gamma,\zeta)$$

$$(\gamma(\zeta-1)(\zeta-a)+\delta\zeta(\zeta-a)-\zeta(\zeta-1))\frac{\mathrm{d}Y}{\mathrm{d}\zeta}$$
$$=[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\alpha\beta F(\alpha,\beta,\gamma,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)(-\alpha(\gamma-1))F(\alpha,\beta,\gamma,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\alpha(\gamma-1)F(\alpha,\beta,\gamma-1,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)](1-a)\frac{\alpha\beta(\beta+1)\zeta}{\gamma}F(\alpha+1,\beta+2,\gamma+1,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)][q-a(1+\alpha+\beta+\alpha\beta-\gamma)]\frac{1-\gamma}{\zeta}F(\alpha,\beta,\gamma,\zeta)$$
$$+[\gamma(\zeta-1)(\zeta-a)+[2-\gamma+\beta+\alpha](\zeta-a)\zeta-\zeta(\zeta-1)][q-a(1+\alpha+\beta+\alpha\beta-\gamma)]\frac{\gamma-1}{\zeta}F(\alpha,\beta,\gamma-1,\zeta)$$

$$F(\alpha+1,\beta+1,\gamma,\zeta)=F(\alpha,\beta,\gamma,\zeta)+\frac{\zeta\alpha}{\gamma}F(\alpha+1,\beta+1,\gamma+1,\zeta)+\frac{(\beta+1)\zeta}{\gamma}F(\alpha+1,\beta+2,\gamma+1,\zeta)$$

$$F(\alpha+1,\beta+2,\gamma+1,\zeta)=\frac{\gamma-\beta-1-\alpha\zeta}{(\beta+1)(\zeta-1)}\left(\frac{-\gamma(\gamma-1)}{\alpha\beta\zeta}\right)[F(\alpha,\beta,\gamma,\zeta)-F(\alpha,\beta,\gamma-1,\zeta)]$$

$$F(\alpha+1,\beta+2,\gamma+1,\zeta)=\frac{\gamma-\beta-1-\alpha\zeta}{(\beta+1)(\zeta-1)}\left(\frac{-\gamma(\gamma-1)}{\alpha\beta\zeta}\right)[F(\alpha,\beta,\gamma,\zeta)-F(\alpha,\beta,\gamma-1,\zeta)]$$
$$+\frac{1}{\alpha(\beta+1)(\zeta-1)}(\gamma(\beta-\gamma)+(\gamma-\alpha-\beta)\gamma)F(\alpha,\beta,\gamma,\zeta)$$

$$\{-q^2+q[a(\alpha+\beta+2\alpha\beta)+1-\gamma]-a\alpha\beta[a(1+\alpha+\beta+\alpha\beta)-\gamma]\}F(\alpha,\beta,\gamma,\zeta)=0$$

$$v=z^{\gamma}(z-1)^{\delta}(z-a)^{\varepsilon}\frac{\mathrm{d}y}{\mathrm{d}z}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}z^2}+\left(\frac{1-\gamma}{z}+\frac{1-\delta}{z-1}+\frac{1-\varepsilon}{z-a}-\frac{\alpha\beta}{\alpha\beta z-q}\right)\frac{\mathrm{d}v}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}v=0$$

$$\frac{\mathrm{d}v}{\mathrm{d}z}=-(\alpha\beta z-q)z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\varepsilon-1}y$$
$$\frac{\mathrm{d}^2v}{\mathrm{d}z^2}=-\frac{(\alpha\beta z-q)}{z(z-1)(z-a)}v-\alpha\beta z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\varepsilon-1}y-(\gamma-1)(\alpha\beta z-q)z^{\gamma-2}(z-1)^{\delta-1}(z-a)^{\varepsilon-1}y$$
$$-(\delta-1)(\alpha\beta z-q)z^{\gamma-1}(z-1)^{\delta-2}(z-a)^{\varepsilon-1}y-(\varepsilon-1)(\alpha\beta z-q)z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\varepsilon-2}y$$

$$v=(z-z_0)^{\mu}\sum_{\nu=0}^{\infty}a_{\nu}(z-z_0)^{\nu}$$

$$y=C_0+\sum_{\nu=0}^{\infty}a_{\nu}\left(\int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon}(z-z_0)^{\mu+\nu}\mathrm{d}z\right)$$

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\mathrm{d}u=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y)$$

$$v = z^{\mu}\sum_{\nu=0}^{+\infty} a_{\nu}^{(1)} z^{\nu}, \mu = 0, \gamma$$

$$u = C_0 + \sum_{\nu=0}^{\infty}\left(\int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon} z^{\mu+\nu}\mathrm{dz}\right)$$

$$\begin{aligned} y_{\nu} &= \int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon} z^{\mu+\nu}\mathrm{dz} \\ &= \frac{(-1)^{(-\delta)}}{(-a)^{\varepsilon}} z^{1-\gamma+\mu+\nu}\int_0^1 u^{-\gamma+\mu+\nu}(1-uz)^{-\delta}\left(1-u\frac{z}{a}\right)^{-\varepsilon}\mathrm{d}u \\ &= \frac{(-1)^{(-\delta)}}{(-a)^{\varepsilon}} z^{\gamma_0+\nu}\frac{\Gamma(\gamma_0+\nu)}{\Gamma(\gamma_0+\nu+1)} F_1\left(\gamma_0+\nu, \delta, \varepsilon, 1+\gamma_0+\nu, z, \frac{z}{a}\right) \end{aligned}$$

$$\begin{aligned} y_{\nu} &= \int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon} z^{\mu+\nu}\mathrm{dz} \\ &= \frac{(-1)^{(-\delta)}}{(-a)^{\varepsilon}} z^{1-\gamma+\mu+\nu}\int_0^1 u^{-\gamma+\mu+\nu}(1-uz)^{-\delta}\left(1-u\frac{z}{a}\right)^{-\varepsilon}\mathrm{d}u \end{aligned}$$

$$y = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(1)} y_{\nu} = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(1)}\frac{(-1)^{(-\delta)}}{(-a)^{\varepsilon}} z^{\gamma_0+\nu}\frac{1}{\gamma_0+\nu} F_1\left(\gamma_0+\nu, \delta, \varepsilon, 1+\gamma_0+\nu, z, \frac{z}{a}\right) \Rightarrow$$
$$Hl(a, q; \alpha, \beta, \gamma, \delta; z) = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(1)}\frac{(-1)^{-\delta}}{(-a)^{\varepsilon}}\frac{z^{\gamma_0+\nu}}{\gamma_0+\nu} F_1\left(\gamma_0+\nu, \delta, \varepsilon, 1+\gamma_0+\nu, z, \frac{z}{a}\right), \mu = 0, \gamma$$

$$y = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(1)} y_{\nu} = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(1)}\frac{(-1)^{(-\delta)}}{(-a)^{\varepsilon}} z^{\gamma_0+\nu}\frac{1}{\gamma_0+\nu} F_1\left(\gamma_0+\nu, \delta, \varepsilon, 1+\gamma_0+\nu, z, \frac{z}{a}\right)$$

$$y = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(2)}\left(\int z^{-\gamma}(z-1)^{-\delta+\mu+\nu}(z-a)^{-\varepsilon}\mathrm{dz}\right) \Rightarrow$$
$$y = C_0 + \frac{z^{1-\gamma}(-1)^{-\delta+\mu+\nu}}{(-a)^{\varepsilon}}\sum_{\nu=0}^{\infty} a_{\nu}^{(2)}\frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)} F_1\left(1-\gamma, \delta-\mu-\nu, \varepsilon, 2-\gamma, z, \frac{z}{a}\right), \mu = 0, \delta$$

$$y = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(2)}\left(\int z^{-\gamma}(z-1)^{-\delta+\mu+\nu}(z-a)^{-\varepsilon}\mathrm{dz}\right)$$

$$\begin{aligned} y &= C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(3)}\left(\int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon+\mu+\mu}\mathrm{dz}\right) \\ &= C_0 + \frac{(-1)^{-\delta}}{(-a)^{\varepsilon-\mu-\nu}} z^{1-\gamma}\sum_{\nu=0}^{\infty} a_{\nu}^{(3)}\frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)} F_1\left(1-\gamma, \delta, \varepsilon-\mu-\nu, 2-\gamma, z, \frac{z}{a}\right), \mu = 0, \varepsilon \end{aligned}$$

$$y = C_0 + \sum_{\nu=0}^{\infty} a_{\nu}^{(3)}\left(\int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon+\mu+\mu}\mathrm{dz}\right)$$

$$y_\nu = \int z^{-\gamma}(z-1)^{-\delta}(z-a)^{-\varepsilon}(z-z_0)^{2+\nu}\mathrm{d}z \Rightarrow$$

$$y_\nu = \int_0^1 u^{-\gamma} z^{-\gamma}(-1)^{-\delta}(1-uz)^{-\delta}(-a)^{-\varepsilon}\left(1-\frac{zu}{a}\right)^{-\varepsilon}(-z_0)^{2+\nu}\left[1-\frac{uz}{z_0}\right]^{2+\nu} z\,\mathrm{d}u$$
$$= \frac{(-1)^{-\delta}}{(-a)^{\varepsilon}}(-z_0)^{2+\nu} z^{1-\gamma}\int_0^1 u^{-\gamma}(1-uz)^{-\delta}\left(1-\frac{uz}{a}\right)^{-\varepsilon}\left(1-\frac{zu}{z_0}\right)^{2+\nu}\mathrm{d}u$$
$$= \frac{(-1)^{-\delta}}{(-a)^{\varepsilon}}(-z_0)^{2+\nu} z^{1-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(2-\gamma)}F_D\left(1-\gamma,\delta,\varepsilon,-2-\nu,2-\gamma,z,\frac{z}{a},\frac{z}{z_0}\right)$$

$$y_\nu = \frac{(-1)^{-\delta}}{(-a)^{\varepsilon}}\frac{z^{1-\gamma}}{(-z_0)^{-2-\nu}}\frac{1}{1-\gamma}F_1\left(1-\gamma,\delta,-2-\nu,2-\gamma,z,\frac{z}{z_0}\right)$$

$$Y_{j,1}(\zeta) = (\zeta-a_j)^{\alpha_j} y_1(\zeta), Y_{j,2}(\zeta) = (\zeta-a_j)^{\beta_j} y_2(\zeta)$$
$$Y_{j,1}(\zeta) = (\zeta-a_j)^{\alpha_j} y_1(\zeta) + \log(\zeta-a_j) Y_{j,2}(\zeta), Y_{j,2}(\zeta) = (\zeta-a_j)^{\beta_j} y_2(\zeta)$$

$$Y_{j,1}(\zeta) = \zeta^{-\alpha_j} y_1(1/\zeta), Y_{j,2}(\zeta) = \zeta^{-\beta_j} y_2(1/\zeta)$$
$$Y_{j,1}(\zeta) = \zeta^{-\alpha_j} y_1(1/\zeta) + \log(\zeta) Y_{j,2}(\zeta), Y_{j,2} = \zeta^{-\beta_j} y_2(1/\zeta)$$

$$Y_1(\zeta) = e^{\lambda_1\zeta}\zeta^{-\mu_1}\sum_{\nu=0}^{\infty} c_\nu \zeta^{-\nu}, Y_2(\zeta) = e^{\lambda_2\zeta}\zeta^{-\mu_2}\sum_{\nu=0}^{\infty} d_\nu \zeta^{-\nu}$$

$$\mathrm{U} \overset{\varphi}{\to} \mathrm{V}$$

$$D(\zeta) = \begin{vmatrix} \varphi(U_1) & \psi(U_1) \\ \varphi(U_2) & \psi(U_2) \end{vmatrix}$$

$$U = R_1\varphi(U) + R_2\psi(U)$$

$$D(\zeta) = \begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix}$$

$$U''(\zeta) = f(\zeta)U'(\zeta) + g(\zeta)U(\zeta)$$

$$U_1'' = f(\zeta)U_1' + g(\zeta)U_1(\zeta),$$
$$U_2'' = f(\zeta)U_2' + g(\zeta)U_2(\zeta),$$

$$f(\zeta) = \frac{\begin{vmatrix} U_2'' & U_2 \\ U_1'' & U_1 \end{vmatrix}}{\begin{vmatrix} U_2' & U_2 \\ U_1' & U_1 \end{vmatrix}} = \frac{(-)\begin{vmatrix} U_1'' & U_1 \\ U_2'' & U_2 \end{vmatrix}}{(-)\begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix}} = \frac{\begin{vmatrix} U_1'' & U_1 \\ U_2'' & U_2 \end{vmatrix}}{D(\zeta)}$$
$$g(\zeta) = \frac{\begin{vmatrix} U_1' & U_1'' \\ U_2' & U_2'' \end{vmatrix}}{D(\zeta)}$$

$$D(\zeta) = \begin{vmatrix} U_1' & U_1 \\ U_2' & U_2 \end{vmatrix}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\zeta^2} + p(\zeta)\frac{\mathrm{d}u}{\mathrm{d}\zeta} + q(\zeta)u = 0$$

$$V(\zeta) = J(\zeta)U(\zeta) + H(\zeta)U'(\zeta)$$

$$V''(\zeta) = f^*(\zeta)V' + g^*(\zeta)V(\zeta)$$

$$V'(\zeta) = I(\zeta)U(\zeta) + Z(\zeta)U'(\zeta)$$

$$I(\zeta) := J'(\zeta) + H(\zeta)g(\zeta), Z(\zeta) := J(\zeta) + H(\zeta)f(\zeta) + H'(\zeta)$$

$$V''(\zeta) = [I'(\zeta) + Z(\zeta)g(\zeta)]U(\zeta) + [I(\zeta) + Z'(\zeta) + Z(\zeta)f(\zeta)]U'(\zeta)$$

$$f^*(\zeta) = \frac{\begin{vmatrix} I'(\zeta) + Z(\zeta)g(\zeta) & J(\zeta) \\ I(\zeta) + Z'(\zeta) + Z(\zeta)f(\zeta) & H(\zeta) \end{vmatrix}}{\begin{vmatrix} I(\zeta) & J(\zeta) \\ Z(\zeta) & H(\zeta) \end{vmatrix}}$$

$$g^*(\zeta) = \frac{\begin{vmatrix} I(\zeta) & I'(\zeta) + Z(\zeta)g(\zeta) \\ Z(\zeta) & I(\zeta) + Z'(\zeta) + Z(\zeta)f(\zeta) \end{vmatrix}}{-J(\zeta)Z(\zeta) + I(\zeta)H(\zeta)}$$

$$\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} J & H \\ I & Z \end{bmatrix} \begin{bmatrix} U \\ U' \end{bmatrix}$$

$$\begin{bmatrix} U \\ U' \end{bmatrix} = \frac{1}{JZ - IH} \begin{bmatrix} Z & -H \\ -I & J \end{bmatrix} \begin{bmatrix} V \\ V' \end{bmatrix}$$

$$D(\zeta) = P_\epsilon(\zeta) \prod_{i=1}^{r} (\zeta - a_1)^{\alpha_i + \beta_i - 1}, \epsilon = -\sum_{i=1}^{r} (\alpha_i + \beta_i - 1) - 2$$

4.2. Curvatura de Fock - Klein – Gordon.

$$K := \square_A + Y = |g|^{-\frac{1}{2}}(D_\mu - A_\mu)|g|^{\frac{1}{2}} g^{\mu v}(D_v - A_v) + Y,$$

$$KG = 0 \text{ and } GK = 0.$$

$$KG = \mathbb{1} \text{ and } GK = \mathbb{1}.$$

$$\int_{\mathrm{iR}}' f(t)\mathrm{d}t = \lim_{R\to\infty} \int_{-\mathrm{i}R}^{\mathrm{i}R} f(t)\mathrm{d}t.$$

$$(u \mid v)_\gamma := \int_M \bar{u} v \gamma, u, v \in C_c^\infty(M)$$

$$(\tilde{u} \mid \tilde{v}) = \int_M \bar{\tilde{u}} \tilde{v}, \tilde{u}, \tilde{v} \in C_c^\infty\left(\Omega^{\frac{1}{2}} M\right)$$

$$L^2(M, \gamma) \ni u \mapsto \tilde{u} := u\gamma^{\frac{1}{2}} \in L^2\left(\Omega^{\frac{1}{2}} M\right)$$

$$K_{\frac{1}{2}} := |g|^{\frac{1}{4}} K |g|^{-\frac{1}{4}} = |g|^{-\frac{1}{4}}(D_\mu - A_\mu)|g|^{\frac{1}{2}} g^{\mu v}(D_v - A_v)|g|^{-\frac{1}{4}} + Y.$$

$$\partial_t := \frac{\mathrm{d}}{\mathrm{d}t} \epsilon_t.$$

$$\frac{1}{\alpha^2} := -g^{-1}(\mathrm{d}t, \mathrm{d}t) > 0$$

$$g_\Sigma(\beta, \beta) < \alpha^2.$$

$$\beta := \partial_t + \alpha^2 g^{-1}(\,\mathrm{d}t,\cdot).$$

$$g^{-1} = -\frac{1}{\alpha^2}(\partial_t - \beta) \otimes (\partial_t - \beta) + g_\Sigma^{-1}$$

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\alpha^2\,\mathrm{d}t^2 + g_{\Sigma,ij}(\,\mathrm{d}x^i + \beta^i\,\mathrm{d}t)(\mathrm{d}x^j + \beta^j\,\mathrm{d}t)$$
$$g^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{\alpha^2}(\partial_t - \beta^i\partial_i)^2 + g_\Sigma^{ij}\partial_i\partial_j.$$

$$\tilde{K} := \alpha K \alpha.$$

$$\begin{aligned}\tilde{K} &= -\gamma^{-\frac{1}{2}}(D_t - D_i\beta^i + V)\gamma\big(D_t - \beta^j D_j + V\big)\gamma^{-\frac{1}{2}}\\ &\quad+\gamma^{-\frac{1}{2}}(D_i - A_i)\alpha^2\gamma g_\Sigma^{ij}\big(D_j - A_j\big)\gamma^{-\frac{1}{2}} + \alpha^2 Y\\ &= -(D_t + W^*)(D_t + W) + L\end{aligned}$$

$$\begin{aligned}\gamma &:= \alpha^{-2}|g|^{\frac{1}{2}} = \alpha^{-1}|g_\Sigma|^{\frac{1}{2}}\\ V &:= -A_0 + A_i\beta^i\\ W &:= \beta^i D_i + V - \frac{1}{2}\gamma^{-1}(D_t\gamma - \beta^i D_i\gamma)\\ L &:= D_i^{A,\gamma^*}\tilde{g}_\Sigma^{ij} D_j^{A,\gamma} + \tilde{Y}\end{aligned}$$

$$\begin{aligned}\tilde{g}_\Sigma^{ij}(t) &:= \alpha(t)^2 g_\Sigma^{ij}(t)\\ \tilde{Y}(t) &:= \alpha(t)^2 Y(t)\\ D^{A,\gamma}(t) &:= \gamma(t)^{\frac{1}{2}}(D - A(t))\gamma(t)^{-\frac{1}{2}}\end{aligned}$$

$$B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & W(t)^* \end{pmatrix}$$

$$(\partial_t + \mathrm{i}B(t))\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = 0$$

$$\left\|L(t)^{-\frac{1}{2}}(L(t) - L(s))L(t)^{-\frac{1}{2}}\right\| + 2\left\|(W(t) - W(s))L(t)^{-\frac{1}{2}}\right\| \leq \left|\int_s^t C(r)\mathrm{d}r\right|$$

$$(u \mid L(t)v) = \int_\Sigma \Big(\big(\overline{D_i^{A,\gamma}(t)u}\big)\tilde{g}_\Sigma^{ij}(t)\big(D_j^{A,\gamma}(t)v\big) + \bar{u}\tilde{Y}(t)v\Big)$$

$$\left|\beta^k(x)p_k\big(\tilde{g}_\Sigma^{ij}(x)p_i p_j\big)^{-\frac{1}{2}}\right| < 1$$

$$\tilde{g}_{\Sigma,ij}\beta^i\beta^j < 1$$

$$L(t) \leq c(t,s)L(s)$$

$$\mathcal{K}^\delta := L(t)^{-\delta/2}L^2\left(\Omega^{\frac{1}{2}}\Sigma\right)$$

$$\left\|L(t)^{-\frac{1}{2}}(L(t) - L(s))L(t)^{-\frac{1}{2}}\right\| + 2\left\|(W(t) - W(s))L(t)^{-\frac{1}{2}}\right\| \leq \left|\int_s^t C(r)\mathrm{d}r\right|$$

$$\mathcal{H} := L^2\left(\Omega^{\frac{1}{2}}\Sigma\right) \oplus L^2\left(\Omega^{\frac{1}{2}}\Sigma\right) = \mathcal{K}^0 \oplus \mathcal{K}^0$$

$$\mathcal{H}_\lambda := \mathcal{K}^{(\lambda+1)/2} \oplus \mathcal{K}^{(\lambda-1)/2}$$

$$\mathcal{H}_\lambda = (L(t) \oplus L(t))^{-\lambda/4}\mathcal{H}_0, \lambda \in [-1,1].$$

$$\mathcal{H}_{\text{en}} := \mathcal{H}_1 = \left(L(t)^{-\frac{1}{2}} \oplus \mathbb{1}\right)\mathcal{H} = H_0(t)^{-\frac{1}{2}}\mathcal{H}$$

$$\mathcal{H}_{\text{dyn}} := \mathcal{H}_0 = \left(L(t)^{-\frac{1}{4}} \oplus L(t)^{\frac{1}{4}}\right)\mathcal{H}$$

$$\mathcal{H}_{\text{en}}^* := \mathcal{H}_{-1} = \left(\mathbb{1} \oplus L(t)^{\frac{1}{2}}\right)\mathcal{H} = (QH_0(t)Q)^{\frac{1}{2}}\mathcal{H}$$

$$H_0(t) := L(t) \oplus \mathbb{1} = \begin{pmatrix} L(t) & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$(u \mid Qv) := (u_1 \mid v_2) + (u_2 \mid v_1), Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\operatorname{Im}(u \mid Qv) = \frac{1}{2\mathrm{i}}((u \mid Qv) - (v \mid Qu))$$

$$H(t) = QB(t) = B(t)^*Q$$

$$H(t) := \begin{pmatrix} L(t) & W(t)^* \\ W(t) & \mathbb{1} \end{pmatrix}$$

$$(1 - a(t))H_0(t) \le H(t) \le (1 + a(t))H_0(t)$$

$$\begin{aligned}
(u \mid H(t)u) &\le \left\|L(t)^{\frac{1}{2}}u_1\right\|^2 + \|u_2\|^2 + 2\|W(t)u_1\|\|u_2\| \\
&\le \left\|L(t)^{\frac{1}{2}}u_1\right\|^2 + \|u_2\|^2 + 2a(t)\left\|L(t)^{\frac{1}{2}}u_1\right\|\|u_2\| \\
&\le (1 + a(t))\left(\left\|L(t)^{\frac{1}{2}}u_1\right\|^2 + \|u_2\|^2\right) \\
&= (1 + a(t))(u \mid H_0(t)u)
\end{aligned}$$

$$(u \mid v)_{\text{en},t} := (u \mid H(t)v)$$

$$(1 + a(t))^{-1}QH_0(t)^{-1}Q \le QH(t)^{-1}Q \le (1 - a(t))^{-1}QH_0(t)^{-1}Q.$$

$$(u \mid v)_{\text{en}^*,t} := (u \mid QH(t)^{-1}Qv)$$

$$B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & W(t)^* \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} W & \mathbb{1} \\ L & W^* \end{pmatrix}\begin{pmatrix} L^{-\frac{1}{2}} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} WL^{-\frac{1}{2}} & \mathbb{1} \\ \mathbb{1} & L^{-\frac{1}{2}}W^* \end{pmatrix}$$

$$B = \begin{pmatrix} \mathbb{1} & 0 \\ W^* & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ L - W^*W & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ W & \mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} L^{\frac{1}{2}} & 0 \\ 0 & \mathbb{1} \end{pmatrix}B^{-1}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ -L^{-\frac{1}{2}}W^* & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \left(\mathbb{1} - L^{-\frac{1}{2}}W^*WL^{-\frac{1}{2}}\right)^{-1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -WL^{-\frac{1}{2}} & \mathbb{1} \end{pmatrix}$$

$$\mathbb{1} - L^{-\frac{1}{2}}W^*WL^{-\frac{1}{2}}$$

$$(QHQ)^{-1}B^{-1} = (BQHQ)^{-1} = (QHQHQ)^{-1} = (QHQB^*)^{-1} = B^{*-1}(QHQ)^{-1}.$$

$$\mathcal{H}_{\lambda,t} := |B(t)|^{-(1+\lambda)/2}\mathcal{H}_{\text{en},t}^*, \lambda \in \mathbb{R}$$

$$(u \mid v)_{\lambda,t} := \left(u | |B(t)|^{1+\lambda} v\right)_{\mathrm{en}^*,t}, \; u, v \in \mathcal{H}_{\lambda,t}$$

$$|B(t)| = \sqrt{B(t)^2} = \sqrt{QH(t)QH(t)}$$

$$\mathcal{H}_{\mathrm{dyn},t} := \mathcal{H}_{0,t},$$

$$\mathcal{H}_{\mathrm{en},t} \subset \mathcal{H}_{\mathrm{dyn},t} \subset \mathcal{H}^*_{\mathrm{en},t}$$

$$\mathcal{H}_{\lambda,t} = \mathcal{H}_{\lambda}, \lambda \in [-1,1],$$

$$\mathcal{H}_{\mathrm{en},t} = \mathcal{H}_{\mathrm{en}}, \mathcal{H}_{\mathrm{dyn},t} = \mathcal{H}_{\mathrm{dyn}}, \mathcal{H}^*_{\mathrm{en},t} = \mathcal{H}^*_{\mathrm{en}}$$

$$c^{-1} \left\| (L(t) \oplus L(t))^{\frac{1}{2}} u \right\|_{\mathrm{en}*} \leq \| |B(t)| u \|_{\mathrm{en}*} \leq c \left\| (L(t) \oplus L(t))^{\frac{1}{2}} u \right\|_{\mathrm{en}*}$$

$$c^{-\delta} \left\| (L(t) \oplus L(t))^{\delta/2} u \right\|_{\mathrm{en}*} \leq \left\| |B(t)|^{\delta} u \right\|_{\mathrm{en}*} \leq c^{\delta} \left\| (L(t) \oplus L(t))^{\delta/2} u \right\|_{\mathrm{en}*}$$

$$\partial_t u(t) + \mathrm{i} B(t) u(t) = 0$$

$$\mathcal{X}_t = \mathcal{H}^*_{\mathrm{en},t} \text{ and } \mathcal{Y}_t = \mathcal{H}_{\mathrm{en},t}$$

$$c_{s,t} := \sup_{\tau \in [s,t]} (1 - a(\tau))^{-1}$$

$$\|u\|_{\lambda,t} \exp\left(-c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right) \leq \|u\|_{\lambda,s} \leq \|u\|_{\lambda,t} \exp\left(c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right)$$

$$\begin{aligned} &\left\| \left(L(t)^{-\frac{1}{2}} \oplus \mathbb{1}\right) (H(s) - H(t)) \left(L(t)^{-\frac{1}{2}} \oplus \mathbb{1}\right) \right\| \\ &\leq \left\| L(t)^{-\frac{1}{2}} (L(s) - L(t)) L(t)^{-\frac{1}{2}} \right\| + 2 \left\| (W(s) - W(t)) L(t)^{-\frac{1}{2}} \right\| \\ &\leq \int_s^t C(\tau) \mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} &\left\| \left(L(t)^{-\frac{1}{2}} \oplus \mathbb{1}\right) (H(s) - H(t)) \left(L(t)^{-\frac{1}{2}} \oplus \mathbb{1}\right) \right\| \\ &\leq \left\| L(t)^{-\frac{1}{2}} (L(s) - L(t)) L(t)^{-\frac{1}{2}} \right\| + 2 \left\| (W(s) - W(t)) L(t)^{-\frac{1}{2}} \right\| \end{aligned}$$

$$\left\| H(t)^{-\frac{1}{2}} (L(t) \oplus \mathbb{1}) H(t)^{-\frac{1}{2}} \right\| \leq c_{s,t}$$

$$\left\| H(t)^{-\frac{1}{2}} (H(s) - H(t)) H(t)^{-\frac{1}{2}} \right\| \leq c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau$$

$$\left| \|u\|^2_{\mathrm{en},s} - \|u\|^2_{\mathrm{en},t} \right| \leq \|u\|^2_{\mathrm{en},t} \left(c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right)$$

$$\begin{aligned} \|u\|^2_{\mathrm{en},s} &\leq \|u\|^2_{\mathrm{en},t} \left(1 + c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right) \\ &\leq \|u\|^2_{\mathrm{en},t} \exp\left(c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right) \end{aligned}$$

$$\|u\|^2_{\mathrm{en},s} \geq \|u\|^2_{\mathrm{en},t} \exp\left(-c_{s,t} \int_s^t C(\tau) \mathrm{d}\tau\right)$$

$$\|(B(s)-B(t))u\|_{\mathrm{en}^*,t} \leq \|u\|_{\mathrm{en},t} \left| c_{s,t} \int_s^t C(\tau)\mathrm{d}\tau \right|$$

$$\begin{aligned} &\|(\mathbb{1}\oplus L(t)^{-\frac{1}{2}})(B(s)-B(t))\left(L(t)^{-\frac{1}{2}}\oplus \mathbb{1}\right)\| \\ &\quad = \left\|Q\left(L(t)^{-\frac{1}{2}}\oplus \mathbb{1}\right)Q(B(s)-B(t))\left(L(t)^{-\frac{1}{2}}\oplus \mathbb{1}\right)\right\| \\ &\quad \leq \left\|\left(L(t)^{-\frac{1}{2}}\oplus \mathbb{1}\right)(H(s)-H(t))\left(L(t)^{-\frac{1}{2}}\oplus \mathbb{1}\right)\right\| \\ &\quad \leq \left|\int_s^t C(\tau)\mathrm{d}\tau\right| \end{aligned}$$

$$U(t,t)=\mathbb{1}, U(t,r)U(r,s)=U(t,s).$$

$$\begin{aligned} \|U(t,r)\|_{\lambda,s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau\right) \\ \|U(r,t)\|_{\lambda,s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau\right) \end{aligned}$$

$$\begin{aligned} \mathrm{i}\partial_t U(t,s)u &= B(t)U(t,s)u \\ -\mathrm{i}\partial_s U(t,s)u &= U(t,s)B(s)u \end{aligned}$$

$$\begin{aligned} \|U(t,r)\|_{\mathrm{en},s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau\right) \\ \|U(t,r)\|_{\mathrm{en}^*,s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau\right) \end{aligned}$$

$$\|U(t,s)\|_{\lambda,r} \leq \exp\left(\int_{\mathbb{R}} C_1(\tau)\mathrm{d}\tau\right)$$

$$\begin{aligned} \|U(t,r)\|_{\lambda,s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau + (t-r)b\left\|(L(s)+b)^{-\frac{1}{2}}\right\|\right) \\ \|U(r,t)\|_{\lambda,s} &\leq \exp\left(2c_{r,t}\int_r^t C(\tau)\mathrm{d}\tau + (t-r)b\left\|(L(s)+b)^{-\frac{1}{2}}\right\|\right) \end{aligned}$$

$$B_b(t) = B(t) + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L^{-\frac{1}{2}}b & 0 \end{pmatrix}$$

$$\pi_2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := u_2$$

$$\iota_2 u := \begin{pmatrix} 0 \\ u \end{pmatrix}, \rho u := \begin{pmatrix} u \\ -(D_t+W)u \end{pmatrix}.$$

$$\tilde{K} = -\mathrm{i}\pi_2(\partial_t + \mathrm{i}B)\rho \ \text{ and } \ K = -\mathrm{i}\alpha^{-1}\pi_2(\partial_t+\mathrm{i}B)\rho\alpha^{-1}.$$

$$-\mathrm{i}(\partial_t + \mathrm{i}B)\rho\tilde{u} = \iota_2\tilde{f}.$$

$$\begin{aligned} \pi_2 &: \mathcal{H}_\lambda \to \mathcal{K}^{(\lambda-1)/2}, \\ \pi_2 Q &: \mathcal{H}_\lambda \to \mathcal{K}^{(\lambda+1)/2}, \\ \iota_2 &: \mathcal{K}^{(\lambda-1)/2} \to \mathcal{H}_\lambda. \end{aligned}$$

$$\begin{pmatrix}\tilde{u}_1(s)\\ \tilde{u}_2(s)\end{pmatrix} = \alpha(s)^{-1}\begin{pmatrix}u_1(s)\\ u_2(s)\end{pmatrix} \text{ and } \tilde{f} = \alpha f$$

$$\tilde{u}(t) = \pi_2 Q U(t,s)\begin{pmatrix}\tilde{u}_1(s)\\ \tilde{u}_2(s)\end{pmatrix} + \mathrm{i}\int_s^t \pi_2 Q U(t,r)\iota_2 \tilde{f}(r)\mathrm{d}r$$

$$u \in C(\mathbb{R};\mathcal{K}^1) \cap C^1(\mathbb{R};\mathcal{K}^0) \text{ and } \rho\tilde{u}(s) = \begin{pmatrix}\tilde{u}_1(s)\\ \tilde{u}_2(s)\end{pmatrix}.$$

$$\begin{gathered}\iota_2 : \mathcal{K}^0 \to \mathcal{H}_{\mathrm{en}},\\ \pi_2 Q : \mathcal{H}_{\mathrm{en}} \to \mathcal{K}^1,\\ \pi_2 Q : \mathcal{H}_{\mathrm{en}}^* \to \mathcal{K}^0.\end{gathered}$$

$$\begin{pmatrix}\tilde{u}_1(t)\\ \tilde{u}_2(t)\end{pmatrix} = U(t,s)\begin{pmatrix}\tilde{u}_1(s)\\ \tilde{u}_2(s)\end{pmatrix} + \mathrm{i}\int_s^t U(t,r)\iota_2 \tilde{f}(r)\mathrm{d}r.$$

$$\mathrm{i}\partial_t\begin{pmatrix}\tilde{u}_1(t)\\ \tilde{u}_2(t)\end{pmatrix} = B(t)\begin{pmatrix}\tilde{u}_1(t)\\ \tilde{u}_2(t)\end{pmatrix} - \iota_2 \tilde{f}(t)$$

$$\rho\tilde{u}(t) = \begin{pmatrix}\tilde{u}_1(t)\\ \tilde{u}_2(t)\end{pmatrix}$$

$$\tilde{f}(t) = -\mathrm{i}\pi_2(\partial_t + \mathrm{i}B)\begin{pmatrix}\tilde{u}_1(t)\\ \tilde{u}_2(t)\end{pmatrix} = -\mathrm{i}\pi_2(\partial_t + \mathrm{i}B)\rho\tilde{u}(t) = \tilde{K}\tilde{u}(t)$$

$$Ku = Ku' = f \text{ and } \rho\tilde{u}(s) = \rho\tilde{u}'(s) = \begin{pmatrix}\tilde{u}_1(s)\\ \tilde{u}_2(s)\end{pmatrix}$$

$$\begin{gathered}E^{\mathrm{PJ}}(t,s) := U(t,s)\\ E^{\vee}(t,s) := \theta(t-s)U(t,s)\\ E^{\wedge}(t,s) := -\theta(s-t)U(t,s)\end{gathered}$$

$$(E^{\blacksquare} f)(t) = \int_{\mathbb{R}} E^{\cdot}(t,s)f(s)\mathrm{d}s$$

$$\begin{gathered}E^{\blacksquare}: L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_\lambda) \to C(\mathbb{R};\mathcal{H}_\lambda)\\ E^{\blacksquare}: L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \to C^1(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\end{gathered}$$

$$\begin{gathered}E^{\vee/\wedge}: L^1_{\mathrm{loc}}(I;\mathcal{H}_\lambda) \to C(I;\mathcal{H}_\lambda)\\ E^{\vee/\wedge}: L^1_{\mathrm{loc}}(I;\mathcal{H}_{\mathrm{en}}) \to C^1(I;\mathcal{H}_{\mathrm{en}}^*)\end{gathered}$$

$$\begin{gathered}E^{\blacksquare}: L^1(\mathbb{R};\mathcal{H}_\lambda) \to C_{\mathrm{b}}(\mathbb{R};\mathcal{H}_\lambda)\\ E^{\blacksquare}: L^1(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \to C^1_{\mathrm{b}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\end{gathered}$$

$$\begin{aligned}&(\partial_t + \mathrm{i}B)E^{\mathrm{PJ}} f = 0, \quad f \in L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}})\\ &E^{\mathrm{PJ}}(\partial_t + \mathrm{i}B) f = 0, \quad f \in L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \cap AC_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\end{aligned}$$

$$\begin{aligned}&(\partial_t + \mathrm{i}B)E^{\vee/\wedge} f = f, \quad f \in L^1_{\mathrm{loc}}(I,\mathcal{H}_{\mathrm{en}})\\ &E^{\vee/\wedge}(\partial_t + \mathrm{i}B) f = f, \quad f \in L^1_{\mathrm{loc}}(I;\mathcal{H}_{\mathrm{en}}) \cap AC(I,\mathcal{H}_{\mathrm{en}}^*)\end{aligned}$$

$$\begin{gathered}E^{\blacksquare}: L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \to C(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \cap C^1(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\\ (\partial_t + \mathrm{i}B): C(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \cap C^1(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*) \to C(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\end{gathered}$$

$$\begin{gathered}(\partial_t + \mathrm{i}B) : L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}) \cap AC_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*) \to L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\\ E^{\blacksquare} : L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*) \to C(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)\end{gathered}$$

$$E^{\blacksquare}\colon \langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{H}_\lambda)\to\langle t\rangle^{s}L^2(\mathbb{R};\mathcal{H}_\lambda)$$
$$E^{\blacksquare}\colon \langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{H}_{\mathrm{en}})\to\langle t\rangle^{s}\langle\partial_t\rangle^{-1}L^2(\mathbb{R};\mathcal{H}_{\mathrm{en}}^*)$$

$$\langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{X})\subset L^1(\mathbb{R};\mathcal{X})\ \text{ and }\ \langle t\rangle^{s}L^2(\mathbb{R};\mathcal{X})\supset C_{\mathrm{b}}(\mathbb{R};\mathcal{X})$$

$$G^{\blacksquare}=\mathrm{i}\alpha\pi_2QE^{\blacksquare}\iota_2\alpha$$

$$G^{\blacksquare}\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-\delta})\to C(\mathbb{R};\mathcal{K}^{1-\delta})$$
$$G^{\blacksquare}\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{0})\to C^1(\mathbb{R};\mathcal{K}^{0})$$

$$G^{\vee/\wedge}\colon L^1_{\mathrm{loc}}(I;\mathcal{K}^{-\delta})\to C(I;\mathcal{K}^{1-\delta})$$
$$G^{\vee/\wedge}\colon L^1_{\mathrm{loc}}(I;\mathcal{K}^{0})\to C^1(I;\mathcal{K}^{0})$$

$$G^{\cdot}\colon L^1(\mathbb{R};\mathcal{K}^{-\delta})\to C_{\mathrm{b}}(\mathbb{R};\mathcal{K}^{1-\delta})$$
$$G^{\cdot}\colon L^1(\mathbb{R};\mathcal{K}^{0})\to C^1_{\mathrm{b}}(\mathbb{R};\mathcal{K}^{0})$$

$$KG^{\mathrm{PJ}}f=0,\quad f\in L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)$$
$$G^{\mathrm{PJ}}Kf=0,\quad f\in L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^1)\cap AC_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)\cap AC^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-1})$$

$$KG^{\vee/\wedge}f=f,\quad f\in L^1_{\mathrm{loc}}(I;\mathcal{K}^0)$$
$$G^{\vee/\wedge}Kf=f,\quad f\in L^1_{\mathrm{loc}}(I;\mathcal{K}^1)\cap AC(\mathbb{R};\mathcal{K}^0)\cap AC^1(I;\mathcal{K}^{-1})$$

$$\rho\colon C(\mathbb{R};\mathcal{K}^1)\cap C^1(\mathbb{R};\mathcal{K}^0)\to C(\mathbb{R};\mathcal{H}_{\mathrm{en}}),$$
$$\rho\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^1)\cap AC_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)\to L^1_{\mathrm{c}}(\mathbb{R};\mathcal{H}_{\mathrm{en}}),$$
$$\rho\colon AC_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)\cap AC^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-1})\to AC_{\mathrm{c}}(\mathbb{R};\mathcal{H}^*_{\mathrm{en}}).$$

$$G^{\blacksquare}\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)\to C(\mathbb{R};\mathcal{K}^1)\cap C^1(\mathbb{R};\mathcal{K}^0)$$
$$K\colon C(\mathbb{R};\mathcal{K}^1)\cap C^1(\mathbb{R};\mathcal{K}^0)\to C^{-1}(\mathbb{R};\mathcal{K}^0)\cap C(\mathbb{R};\mathcal{K}^{-1})$$

$$K\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^1)\cap AC_{\mathrm{c}}(\mathbb{R};\mathcal{K}^0)\cap AC^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-1})\to L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-1})$$
$$G^{\blacksquare}\colon L^1_{\mathrm{c}}(\mathbb{R};\mathcal{K}^{-1})\to C(\mathbb{R};\mathcal{K}^0)$$

$$G^{\blacksquare}\colon\langle t\rangle^{-s}L^2\left(\Omega^{\frac12}M\right)\to\langle t\rangle^{s}L(t)^{-\frac12}L^2\left(\Omega^{\frac12}M\right)$$
$$G^{\blacksquare}\colon\langle t\rangle^{-s}L^2\left(\Omega^{\frac12}M\right)\to\langle t\rangle^{s}\langle\partial_t\rangle^{-1}L^2\left(\Omega^{\frac12}M\right)$$

$$G^{\blacksquare}\colon\langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{K}^{-\delta})\to\langle t\rangle^{s}L^2(\mathbb{R};\mathcal{K}^{1-\delta})$$

$$G^{\blacksquare}\colon\langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{K}^{0})\to\langle t\rangle^{s}L(t)^{-\frac12}L^2(\mathbb{R};\mathcal{K}^{0})$$

$$G^{\blacksquare}\colon\langle t\rangle^{-s}L^2(\mathbb{R};\mathcal{K}^{0})\to\langle t\rangle^{s}\langle\partial_t\rangle^{-1}L^2(\mathbb{R};\mathcal{K}^{0})$$

$$0=Ku=K\chi u-K(\chi-1)u$$

$$G^{\mathrm{PI}}K\chi u=G^{\vee}K\chi u-G^{\wedge}K(\chi-1)u=u.$$

$$E^{\blacksquare}=-\mathrm{i}\begin{pmatrix}-\alpha^{-1}G\cdot\alpha^{-1}(D_t+W^*) & \alpha^{-1}G\cdot\alpha^{-1}\\ \mathbb{1}+(D_t+W)\alpha^{-1}G\cdot\alpha^{-1}(D_t+W^*) & -(D_t+W)\alpha^{-1}G\cdot\alpha^{-1}\end{pmatrix}$$

$$E^{\blacksquare}=-\mathrm{i}\begin{pmatrix}-\alpha^{-1}G^{\cdot}\alpha^{-1}(D_t+W^*) & \alpha^{-1}G^{\cdot}\alpha^{-1}\\ (D_t+W)\alpha^{-1}G^{\cdot}\alpha^{-1}(D_t+W^*) & -(D_t+W)\alpha^{-1}G^{\cdot}\alpha^{-1}\end{pmatrix}$$

$$\Pi^{(\pm)}_\tau:=\mathbb{1}_{[0,\infty[}(\pm B(\tau)).$$

$$\pm\left(u\mid Q\Pi^{(\pm)}_\tau u\right)=\pm\left(\Pi^{(\pm)}_\tau u\mid Qu\right)=\pm\left(\Pi^{(\pm)}_\tau u\mid Q\Pi^{(\pm)}_\tau u\right)\geq 0$$

$$E_\tau^{(\pm)}(t,s) := \pm U(t,\tau)\Pi_\tau^{(\pm)} U(\tau,s).$$

$$E_\tau^{\mathrm{F}}(t,s) := \theta(t-s)E_\tau^{(+)}(t,s) + \theta(s-t)E_\tau^{(-)}(t,s)$$
$$E_\tau^{\overline{\mathrm{F}}}(t,s) := -\theta(t-s)E_\tau^{(-)}(t,s) - \theta(s-t)E_\tau^{(+)}(t,s)$$

$$E_\tau^{\mathrm{F}}(t,s) = E^\wedge(t,s) + E_\tau^{(+)}(t,s) = E^\vee(t,s) + E_\tau^{(-)}(t,s)$$
$$E_\tau^{\overline{\mathrm{F}}}(t,s) = E^\vee(t,s) - E_\tau^{(+)}(t,s) = E^\wedge(t,s) - E_\tau^{(-)}(t,s)$$

$$E_\tau^\dagger : L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\lambda) \to C(\mathbb{R};\mathcal{H}_\lambda)$$
$$E_\tau^\dagger : L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\mathrm{en}) \to C^1(\mathbb{R};\mathcal{H}_\mathrm{en}^*)$$

$$E_\tau^\circ : L^1(\mathbb{R};\mathcal{H}_\lambda) \to C_\mathrm{b}(\mathbb{R};\mathcal{H}_\lambda),$$
$$E_\tau^\circ : L^1(\mathbb{R};\mathcal{H}_\mathrm{en}) \to C_\mathrm{b}^1(\mathbb{R};\mathcal{H}_\mathrm{en}^*).$$

$$(\partial_t + \mathrm{i}B)E_\tau^{(\pm)} f = 0, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\mathrm{en})$$
$$E_\tau^{(\pm)}(\partial_t + \mathrm{i}B) f = 0, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\mathrm{en}) \cap AC_\mathrm{c}(\mathbb{R};\mathcal{H}_\mathrm{en}^*)$$

$$(\partial_t + \mathrm{i}B)E_\tau^{\mathrm{F}/\overline{\mathrm{F}}} f = f, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\mathrm{en})$$
$$E_\tau^{\mathrm{F}/\overline{\mathrm{F}}}(\partial_t + \mathrm{i}B) f = f, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{H}_\mathrm{en}) \cap AC_\mathrm{c}(\mathbb{R};\mathcal{H}_\mathrm{en}^*)$$

$$E_\tau^{\mathrm{F}} = E^\wedge + E_\tau^{(+)} = E^\vee + E_\tau^{(-)}, \quad E_\tau^{\mathrm{F}} + E_\tau^{\overline{\mathrm{F}}} = E^\vee + E^\wedge, \qquad E_\tau^{(+)} - E_\tau^{(-)} = E^{\mathrm{PJ}}$$
$$E_\tau^{\overline{\mathrm{F}}} = E^\vee - E_\tau^{(+)} = E^\wedge - E_\tau^{(-)}, \quad E_\tau^{\mathrm{F}} - E_\tau^{\overline{\mathrm{F}}} = E_\tau^{(+)} + E_\tau^{(-)}$$

$$E_\tau^\circ : \langle t\rangle^{-s} L^2(\mathbb{R};\mathcal{H}_\lambda) \to \langle t\rangle^{s} L^2(\mathbb{R};\mathcal{H}_\lambda)$$
$$E_\tau^\circ : \langle t\rangle^{-s} L^2(\mathbb{R};\mathcal{H}_\mathrm{en}) \to \langle t\rangle^{s}\langle \partial_t\rangle^{-1} L^2(\mathbb{R};\mathcal{H}_\mathrm{en}^*).$$

$$G_\tau^{(\pm)} := \alpha\pi_2 Q E_\tau^{(\pm)}\iota_2\alpha, G_\tau^{\mathrm{F}/\overline{\mathrm{F}}} := \mathrm{i}\alpha\pi_2 Q E_\tau^{\mathrm{F}/\overline{\mathrm{F}}}\iota_2\alpha.$$

$$G_\tau^\circ : L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^{-\delta}) \to C(\mathbb{R};\mathcal{K}^{1-\delta})$$
$$G_\tau^\circ : L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^0) \to C^1(\mathbb{R};\mathcal{K}^0)$$

$$G_\tau^\circ : L^1(\mathbb{R};\mathcal{K}^{-\delta}) \to C_\mathrm{b}(\mathbb{R};\mathcal{K}^{1-\delta})$$
$$G_\tau^\circ : L^1(\mathbb{R};\mathcal{K}^0) \to C_\mathrm{b}^1(\mathbb{R};\mathcal{K}^0)$$

$$K G_\tau^{(\pm)} f = 0, f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^0)$$
$$G_\tau^{(\pm)} K f = 0, f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^1) \cap AC_\mathrm{c}(\mathbb{R};\mathcal{K}^0) \cap AC_\mathrm{c}^1(\mathbb{R};\mathcal{K}^{-1})$$

$$K G_\tau^{\mathrm{F}/\overline{\mathrm{F}}} f = f, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^0)$$
$$G_\tau^{\mathrm{F}/\overline{\mathrm{F}}} K f = f, \quad f \in L_\mathrm{c}^1(\mathbb{R};\mathcal{K}^1) \cap AC_\mathrm{c}(\mathbb{R};\mathcal{K}^0) \cap AC_\mathrm{c}^1(\mathbb{R};\mathcal{K}^{-1})$$

$$G_\tau^{\mathrm{F}} = G^\wedge + \mathrm{i}G_\tau^{(+)} = G^\vee + \mathrm{i}G_\tau^{(-)}, \quad G_\tau^{\mathrm{F}} + G_\tau^{\overline{\mathrm{F}}} = G^\vee + G^\wedge, G_\tau^{(+)} - G_\tau^{(-)} = -\mathrm{i}G^{\mathrm{PJ}}$$
$$G_\tau^{\overline{\mathrm{F}}} = G^\vee - \mathrm{i}G_\tau^{(+)} = G^\wedge - \mathrm{i}G_\tau^{(-)}, \quad G_\tau^{\mathrm{F}} - G_\tau^{\overline{\mathrm{F}}} = \mathrm{i}G_\tau^{(+)} + \mathrm{i}G_\tau^{(-)}$$

$$\left(f \mid G_\tau^{(\pm)} f\right) = \int_M \bar{f} G_\tau^{(\pm)} f \geq 0$$

$$\left(f \mid G_\tau^{(\pm)} f\right) = \iint \left(\iota_2 \tilde{f}(t) \mid Q E_\tau^{(\pm)}(t,s)\iota_2 \tilde{f}(s)\right) \mathrm{d}s\, \mathrm{d}t$$
$$= \left(\tilde{u}(\tau) \mid Q \Pi_\tau^{(\pm)} \tilde{u}(\tau)\right) \geq 0$$

$$G_\tau^\dagger : \langle t\rangle^{-s} L^2\left(\Omega^{\frac{1}{2}} M\right) \to \langle t\rangle^{s} L(t)^{-\frac{1}{2}} L^2\left(\Omega^{\frac{1}{2}} M\right)$$
$$G_\tau^\dagger : \langle t\rangle^{-s} L^2\left(\Omega^{\frac{1}{2}} M\right) \to \langle t\rangle^{s} \langle \partial_t\rangle^{-1} L^2\left(\Omega^{\frac{1}{2}} M\right)$$

$$\Pi_+^{(\pm)} := \mathbb{1}_{[0,\infty[}(\pm B(+\infty))$$
$$\Pi_-^{(\pm)} := \mathbb{1}_{[0,\infty[}(\pm B(-\infty))$$

$$\Pi_+^{(\pm)}(t) := \mathrm{s} - \lim_{\tau\to+\infty} U(t,\tau)\Pi_+^{(\pm)} U(\tau,t),$$
$$\Pi_-^{(\pm)}(t) := \mathrm{s} - \lim_{\tau\to-\infty} U(t,\tau)\Pi_-^{(\pm)} U(\tau,t)$$

$$U(s,t)\Pi_+^{(\pm)}(t)U(t,s) = \Pi_+^{(\pm)}(s)$$
$$U(s,t)\Pi_-^{(\pm)}(t)U(t,s) = \Pi_-^{(\pm)}(s)$$

$$U(t,r)\Pi_+^{(\pm)} U(r,t) = U(t,r)\mathrm{e}^{\mathrm{i}(t-r)B(+\infty)}\Pi_+^{(\pm)}\mathrm{e}^{\mathrm{i}(r-t)B(+\infty)}U(r,t).$$

$$\begin{aligned} U(t,r)\mathrm{e}^{\mathrm{i}(t-r)B(+\infty)}u &= u + \int_t^r \partial_s\left(U(t,s)\mathrm{e}^{\mathrm{i}(t-s)B(+\infty)}\right)u\,\mathrm{d}s \\ &= u - \mathrm{i}\int_t^r U(t,s)(B(s) - B(+\infty))\mathrm{e}^{\mathrm{i}(t-s)B(+\infty)}u\,\mathrm{d}s \end{aligned}$$

$$\begin{aligned} &\left\| U(t,r)\mathrm{e}^{\mathrm{i}(t-r)B(+\infty)}u - u\right\|_{\mathrm{en}^*,\tau} \\ &\le C\|u\|_{\mathrm{en},\tau}\int_t^r \left\|\left(\mathbb{1}\oplus L(\tau)^{-\frac{1}{2}}\right)(B(s) - B(+\infty))\left(L(\tau)^{-\frac{1}{2}}\oplus\mathbb{1}\right)\right\|\mathrm{d}s \end{aligned}$$

$$\left\|\left(\mathbb{1}\oplus L(\tau)^{-\frac{1}{2}}\right)(B(s) - B(+\infty))\left(L(\tau)^{-\frac{1}{2}}\oplus\mathbb{1}\right)\right\|$$

$$\left\| U(t,r)\mathrm{e}^{\mathrm{i}(t-r)B(+\infty)}u - u\right\|_{\mathrm{en}^*,\tau} \to 0$$

$$E_+^{(\pm)}(t,s) := \pm U(t,\tau)\Pi_+^{(\pm)}(\tau)U(\tau,s)$$
$$E_-^{(\pm)}(t,s) := \pm U(t,\tau)\Pi_-^{(\pm)}(\tau)U(\tau,s)$$

$$E_\pm^{\mathrm{F}} = E^\wedge + E_\pm^{(+)} = E^\vee + E_\pm^{(-)},$$
$$E_\pm^{\overline{\mathrm{F}}} = E^\vee - E_\pm^{(+)} = E^\wedge - E_\pm^{(-)}.$$

$$L = D_i g^{ij}(x) D_j - A^i(x)D_i - D_i A^i(x) + Y_0(x)$$

$$L = (D_i - A_i)g^{ij}\left(D_j - A_j\right) + Y_1$$

$$A_i := g_{ij}A^j, Y_1 := Y_0 - A^i g_{ij} A^j$$

$$L = \gamma^{-\frac{1}{2}}(D_i - A_i)\gamma g^{ij}\left(D_j - A_j\right)\gamma^{-\frac{1}{2}} + Y_\gamma$$

$$Y_\gamma := Y - \frac{1}{2}\left(D_i g^{ij}\gamma^{-1}\left(D_j\gamma\right)\right) - \frac{1}{4}g^{ij}\gamma^{-2}(D_i\gamma)\left(D_j\gamma\right).$$

$$L = |g|^{-\frac{1}{4}}(D_i - A_i)|g|^{\frac{1}{2}}g^{ij}\left(D_j - A_j\right)|g|^{-\frac{1}{4}} + Y.$$

$$L = g^{ij}(\mathrm{i}\nabla_i + A_i)\left(\mathrm{i}\nabla_j + A_j\right) + Y.$$

$$D^{A,\gamma} := \gamma^{\frac{1}{2}}(D - A)\gamma^{-\frac{1}{2}},$$

$$(u \mid Lv) = \int_\Sigma \left(\overline{\left(D_i^{A,\gamma} u\right)}\, g^{ij} \left(D_j^{A,\gamma} v\right) + \bar{u} Y_\gamma v\right)$$

$$l_{\mathrm{mx}}[u,v] = \int_\Sigma \left(\overline{\left(D_i^{A,\gamma} u\right)}\, g^{ij} \left(D_j^{A,\gamma} v\right) + \bar{u} Y_\gamma v\right)$$

$$\mathrm{dom} l_{\mathrm{mx}} = \left\{ u \in L^2\left(\Omega^{\frac{1}{2}}\Sigma\right) \middle| D^{A,\gamma} u \in L^2\left(\Omega^{\frac{1}{2}} T^*\Sigma, g\right), Y_\gamma^{\frac{1}{2}} u \in L^2\left(\Omega^{\frac{1}{2}}\Sigma\right) \right\}$$

$$u \mapsto \left(\int_\Sigma \bar{u}_i g^{ij} u_j\right)^{\frac{1}{2}}$$

$$\mathrm{Dom} L_{\mathrm{mx}} = \left\{ v \in \mathrm{dom} l_{\mathrm{mx}} \middle| | l_{\mathrm{mx}}[u,v] | \leq C_v \|u\| \text{ for all } u \in L^2\left(\Omega^{\frac{1}{2}}\Sigma\right) \right\}$$

$$(u \mid L_{\mathrm{mx}} v) = l_{\mathrm{mx}}[u,v]$$

$$\mathrm{dom} l_{\mathrm{mx}} \ni u \mapsto \left(l_{\mathrm{mx}}[u,u] + (1-C)\|u\|^2\right)^{\frac{1}{2}}$$

$$u_n \to u, Y_\gamma^{\frac{1}{2}} u_n \to v \text{ in } L^2\left(\Omega^{\frac{1}{2}}\Sigma\right)$$

$$D^{A,\gamma} u_n \to w \text{ in } L^2\left(\Omega^{\frac{1}{2}} T^*\Sigma, g\right)$$

$$W(t) := \beta(t)^i D_i + V(t) - \frac{1}{2}\gamma(t)^{-1}\left(D_t \gamma(t) - \beta(t)^i D_i \gamma(t)\right)$$

$$W(t) := \beta(t)^i D_i + V(t) - \frac{1}{2}\gamma(t)^{-1}\left(D_t \gamma(t) - \beta(t)^i D_i \gamma(t)\right),$$
$$(u \mid L(t) v) := \int_\Sigma \left(\overline{\left(D_i^{A,\gamma}(t) u\right)} \tilde{g}^{ij}(t) \left(D_j^{A,\gamma}(t) v\right) + \bar{u} \tilde{Y}(t) v\right),$$

$$\left\| L(t)^{-\frac{1}{2}} (L(t) - L(s)) L(t)^{-\frac{1}{2}} \right\| + 2 \left\| (W(t) - W(s)) L(t)^{-\frac{1}{2}} \right\| \leq \left| \int_s^t C(r) \mathrm{d}r \right|$$

$$\|X\|_t = \left(\int_\Sigma \tilde{g}^{ij}(t) \bar{X}_i X_j\right)^{\frac{1}{2}}$$

$$\left\| L(t)^{-\frac{1}{2}} \partial_s \tilde{Y}(s) L(t)^{-\frac{1}{2}} \right\| \leq C_Y(s)$$
$$\left\| \partial_s W(s) L(t)^{-\frac{1}{2}} \right\| \leq C_W(s)$$
$$\left\| \partial_s A(s) L(t)^{-\frac{1}{2}} \right\|_t \leq C_A(s)$$
$$\left\| \partial_s \gamma(s)^{-1} \mathrm{d}\gamma(s) L(t)^{-\frac{1}{2}} \right\|_t \leq C_\gamma(s)$$
$$\left| \partial_s \tilde{g}^{ij}(s) X_i X_j \right| \leq C_g(s) \tilde{g}^{ij}(t) X_i X_j, X \in C(T^*\Sigma)$$

$$\left\| L(t)^{-\frac{1}{2}}(\tilde{Y}(t) - \tilde{Y}(s))L(t)^{-\frac{1}{2}} \right\| \le \left| \int_s^t C_Y(r)\mathrm{d}r \right|$$

$$\left\| (W(t) - W(s))L(t)^{-\frac{1}{2}} \right\| \le \left| \int_s^t C_W(r)\mathrm{d}r \right|$$

$$\left\| (A(t) - A(s))L(t)^{-\frac{1}{2}} \right\|_t \le \left| \int_s^t C_A(r)\mathrm{d}r \right|$$

$$\left\| (\gamma(t)^{-1}\,\mathrm{d}\gamma(t) - \gamma(s)^{-1}\,\mathrm{d}\gamma(s))L(t)^{-\frac{1}{2}} \right\|_t \le \left| \int_s^t C_\gamma(r)\mathrm{d}r \right|$$

$$\left| \tilde{g}^{ij}(t)X_iX_j - \tilde{g}^{ij}(s)X_iX_j \right| \le \left| \int_s^t C_g(r)\mathrm{d}r \right| \tilde{g}^{ij}(t)X_iX_j$$

$$\begin{aligned}(u \mid (L(t) - L(s))u) &= \int_\Sigma \tilde{g}^{ij}(t)\Big(\big(\overline{D_i(t)u}\big)\big(D_j(t)u - D_j(s)u\big) + \big(\overline{D_i(t)u - D_i(s)u}\big)\big(D_j(t)u\big) \\ &\quad -\big(\overline{D_i(t)u - D_i(s)u}\big)\big(D_j(t)u - D_j(s)u\big)\Big) \\ &\quad + \int_\Sigma (\tilde{g}^{ij}(t) - \tilde{g}^{ij}(s))\Big(\big(\overline{D_i(t)u}\big)\big(D_j(t)u\big) - \big(\overline{D_i(t)u}\big)\big(D_j(t)u - D_j(s)u\big) \\ &\quad -\big(\overline{D_i(t)u - D_i(s)u}\big)\big(D_j(t)u\big) + \big(\overline{D_i(t)u - D_i(s)u}\big)\big(D_j(t)u - D_j(s)u\big)\Big) \\ &\quad + \int_\Sigma (\tilde{Y}(t) - \tilde{Y}(s))|u|^2\end{aligned}$$

$$D_i(t) - D_i(s) = -A_i(t) + A_i(s) + \frac{\mathrm{i}}{2}\gamma(t)^{-1}\partial_i\gamma(t) - \frac{\mathrm{i}}{2}\gamma(s)^{-1}\partial_i\gamma(s).$$

$$|(u \mid (L(t) - L(s))u)| \le \tilde{C}(t,s)(u \mid L(t)u)$$

$$\begin{aligned}\tilde{C}(t,s) &= 2\left| \int_s^t C_D(r)\mathrm{d}r \right| + \left| \int_s^t C_D(r)\mathrm{d}r \right|^2 \\ &\quad + \left| \int_s^t C_g(r)\mathrm{d}r \right| \left(1 + \left| \int_s^t C_D(r)\mathrm{d}r \right|\right)^2 + \left| \int_s^t C_Y(r)\mathrm{d}r \right|\end{aligned}$$

$$\left| \int_s^t C_D(r)\mathrm{d}r \right|^2 \le |t - s| \left| \int_s^t C_D(r)^2\,\mathrm{d}r \right| \le \left| \int_s^t C_D(r)^2\,\mathrm{d}r \right|$$

$$\tilde{C}(t,s) \le \left| \int_s^t \big(c(t)(2C_D + C_D^2) + C_\gamma + C_g\big)\mathrm{d}r \right|$$

$$(u \mid L_0 v) := \int_\Sigma \Big(\big(\overline{D_i^{\gamma_0}u}\big)\, g_0^{ij}(t)\big(D_j^{\gamma_0}v\big) + \bar{u}v\Big).$$

$$\tilde{g}^{ij}(t)X_iX_j \ge C_g(t)g_0^{ij}X_iX_j$$

$$\varepsilon_0(t)\gamma_0^2\gamma(t)^{-2}(\partial_i\gamma_0^{-1}\gamma(t))\tilde{g}^{ij}(t)\big(\partial_j\gamma_0^{-1}\gamma(t)\big) + \tilde{Y}(t) \ge C_0(t)$$

$$\left\| L(t)^{\frac{1}{2}}u \right\| \ge C(t)\left\| L_0^{\frac{1}{2}}|u| \right\|, u \in \mathrm{Dom}L(t)^{\frac{1}{2}}$$

$$
\begin{aligned}
(u \mid L(t)u) &\geq \int_{\Sigma} \left(-\left(D_i^{\gamma}(t)|u|\right)\tilde{g}^{ij}(t)\left(D_j^{\gamma}(t)|u|\right) + \tilde{Y}(t)|u|^2\right) \\
&\geq \int_{\Sigma} (\varepsilon(t) - 1)\left(D_i^{\gamma_0}|u|\right)\tilde{g}^{ij}(t)\left(D_j^{\gamma_0}|u|\right) \\
&\quad + \int_{\Sigma} \left(\varepsilon_0(t)\gamma_0^2\gamma(t)^{-2}(\partial_i\gamma_0^{-1}\gamma(t))\tilde{g}^{ij}(t)\left(\partial_j\gamma_0^{-1}\gamma(t)\right) + \tilde{Y}(t)\right)|u|^2 \\
&\geq \min\left(C_g(t)(1-\varepsilon(t)), C_0(t)\right)(|u||L_0|u \mid)
\end{aligned}
$$

$$
|(\partial_x - \mathrm{i}V(x))f(x)| \geq |\partial_x|f(x)||
$$

$$
\|X\| = \left(\int_{\Sigma} g_0^{ij}\bar{X}_i X_j\right)^{\frac{1}{2}}
$$

$$
\tilde{g}^{ij}(t)X_iX_j \leq C_g(t)g_0^{ij}X_iX_j, X \in C(T^*\Sigma)
$$

$$
\left\| L_0^{-\frac{1}{2}} |\partial_t \tilde{Y}(t)| L_0^{-\frac{1}{2}} \right\| \leq C_{Y,0}(t)
$$

$$
\left\| \partial_t W(t) L_0^{-\frac{1}{2}} \right\| \leq C_{W,0}(t)
$$

$$
\left\| \partial_t A(t) L_0^{-\frac{1}{2}} \right\| \leq C_{A,0}(t)
$$

$$
\left\| \partial_t \gamma(t)^{-1}\, \mathrm{d}\gamma(t) L_0^{-\frac{1}{2}} \right\| \leq C_{\gamma,0}(t)
$$

$$
\left|\partial_t \tilde{g}^{ij}(t)X_iX_j\right| \leq C_{g,0}(t)g_0^{ij}X_iX_j, X \in C(T^*\Sigma)
$$

$$
\|(A-\lambda)^{-n}\| \leq M(\lambda-\beta)^{-n}, \lambda > \beta, n = 1,2, \dots .
$$

$$
\|(A-\lambda)^{-1}\| \leq (\lambda-\beta)^{-1}, \lambda > \beta
$$

$$
\mathrm{Dom}(\tilde{A}) := \{y \in \mathrm{Dom}(A) \cap \mathcal{Y} \mid Ay \in \mathcal{Y}\}
$$

$$
\left\| \prod_{j=1}^{k} \left(A(t_j) - \lambda\right)^{-1} \right\| \leq M(\lambda-\beta)^{-k}, \lambda > \beta
$$

$$
\left\| \prod_{j=1}^{k} \mathrm{e}^{\mu_j A(t_j)} \right\| \leq M\mathrm{e}^{\beta(\mu_1+\dots+\mu_k)}, \mu_j \geq 0
$$

$$
\|u\|_s \leq \|u\|_t \exp\left|\int_s^t C(r)\mathrm{d}r\right|, u \in \mathcal{X}, s, t \in [0,T]
$$

$$
\|(A(t)-\lambda)^{-1}\|_t \leq (\lambda-\beta)^{-1}, \lambda > \beta
$$

$$
\left\| \prod_{j=1}^{k} \left(A(t_j) - \lambda\right)^{-1} \right\|_s \leq (\lambda-\beta)^{-k} \exp\left(\int_0^T 2C(r)\mathrm{d}r\right), t_1 \leq s \leq t_k
$$

$$\left\|\prod_{j=1}^{k}\left(A(t_j)-\lambda\right)^{-1}u\right\|_{t_k} \leq (\lambda-\beta)^{-1}\left\|\prod_{j=1}^{k-1}\left(A(t_j)-\lambda\right)^{-1}u\right\|_{t_k}$$
$$\leq (\lambda-\beta)^{-1}\exp\left(\int_{t_{k-1}}^{t_k} C(r)\mathrm{d}r\right)\left\|\prod_{j=1}^{k-1}\left(A(t_j)-\lambda\right)^{-1}u\right\|_{t_{k-1}}$$
$$\leq \cdots$$
$$\leq (\lambda-\beta)^{-k}\exp\left(\int_{t_1}^{t_k} C(r)\mathrm{d}r\right)\|u\|_{t_1}$$

$$U(t,t)=\mathbb{1}, U(t,s)U(s,r)=U(t,r)$$

$$\partial_t^+ U(t,s)y|_{t=s} = A(s)y$$
$$-\partial_s U(t,s)y = U(t,s)A(s)y$$

$$A_n(t)=A(T\lfloor \mathrm{tn}/T\rfloor/n)$$

$$\|A_n(t)-A(t)\|_{B(\mathcal{Y},\mathcal{X})} \to 0 \text{ as } n\to\infty$$

$$U_n(t,s)=\mathrm{e}^{(t-s)A_n(s)}$$

$$U_n(t,s)=U_n(t,r)U_n(r,s)$$

$$\|U_n(t,s)\|_{\mathcal{X}} \leq M\mathrm{e}^{\beta(t-s)}, \|U_n(t,s)\|_{\mathcal{Y}} \leq \tilde{M}\mathrm{e}^{\tilde{\beta}(t-s)}$$

$$\partial_t U_n(t,s)y = A_n(t)U_n(t,s)y$$
$$\partial_s U_n(t,s)y = -U_n(t,s)A_n(s)y$$

$$U_n(t,r)y-U_m(t,r)y=\int_r^t U_n(t,s)(A_n(s)-A_m(s))U_m(s,r)y\,\mathrm{d}s$$

$$\|U_n(t,r)y-U_m(t,r)y\|_{\mathcal{X}} \leq M\tilde{M}\mathrm{e}^{\gamma(t-r)}\|y\|_{\mathcal{Y}}\int_r^t \|A_n(s)-A_m(s)\|_{B(\mathcal{Y},\mathcal{X})}\mathrm{d}s$$

$$U(t,s)=s-\lim_{n\to\infty}U_n(t,s)$$

$$U_n(t,s)y-V(t,s)y=\int_s^t U_n(t,r)(A_n(r)-A'(r))V(r,s)y\,\mathrm{d}r$$

$$\|U_n(t,s)y-V(t,s)y\|_{\mathcal{X}} \leq M\tilde{M}\mathrm{e}^{\gamma(t-s)}\|y\|_{\mathcal{Y}}\int_s^t \|A_n(r)-A'(r)\|_{B(\mathcal{Y},\mathcal{X})}\mathrm{d}r$$

$$(t-s)^{-1}\left\|U(t,s)y-\mathrm{e}^{(t-s)A(\tau)}y\right\|_{\mathcal{X}} \leq (t-s)^{-1}M\tilde{M}\mathrm{e}^{\gamma(t-s)}\|y\|_{\mathcal{Y}}\int_s^t \|A_n(r)-A(\tau)\|_{B(\mathcal{Y},\mathcal{X})}\mathrm{d}r$$

$$\partial_s^- U(t,s)y|_{s=t} = -A(t)y$$

$$\partial_s^+ U(t,s)y = \mathrm{s}-\lim_{h\searrow 0} h^{-1}(U(t,s+h)y-U(t,s)y)$$
$$\partial_s^- U(t,s)y = \mathrm{s}-\lim_{h\searrow 0} h^{-1}(U(t,s)y-U(t,s-h)y)$$

$$\partial_s^+ U(t,s)y = \mathrm{s}-\lim_{h\searrow 0} h^{-1}(U(t,s+h)y - U(t,s)y)$$
$$= U(t,s+h)\ \mathrm{s}-\lim_{h\searrow 0} h^{-1}(y - U(s+h,s)y) = -U(t,s)A(s)y,$$

$$\partial_s^- U(t,s)y = \mathrm{s}-\lim_{h\searrow 0} h^{-1}(U(t,s)y - U(t,s-h)y)$$
$$= U(t,s)\ \mathrm{s}-\lim_{h\searrow 0} h^{-1}(y - U(s,s-h)y) = -U(t,s)A(s)y.$$

$$\|U(t,r)\|_{\mathcal{Y}} \leq \tilde{M}\mathrm{e}^{\tilde{\beta}(t-s)}, 0 \leq r \leq s \leq t \leq T$$

$$\|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta$$

$$\|y\|_{\mathcal{Y},s} \leq \|y\|_{\mathcal{Y},t} \exp\left|\int_s^t C(r)\mathrm{d}r\right|, s,t \in [0,T]$$

$$\left\|(\tilde{A}(t)-\lambda)^{-1}\right\|_{\mathcal{Y},t} \leq (\lambda-\tilde{\beta})^{-1}, \lambda > \tilde{\beta}$$

$$\|U(t,r)\|_{\mathcal{Y},s} \leq \exp\left(\int_r^t (\tilde{\beta}+2C(\tau))\mathrm{d}\tau\right), 0 \leq r \leq s \leq t \leq T$$

$$\|y\| \leq \liminf_{r,t\to s} \|U(t,r)y\|_{\mathcal{Y},s} \leq \limsup_{r,t\to s} \|U(t,r)y\|_{\mathcal{Y},s}$$
$$\leq \limsup_{r,t\to s} \exp\left(\int_r^t (\tilde{\beta}+2C(\tau))\mathrm{d}\tau\right)\|y\| = \|y\|$$

$$\lim_{r,t\to s} U(t,r)y = y$$

$$\|U(t,s')y - U(t,s)y\|_{\mathcal{Y}} \leq \|U(t,s')\|_{\mathcal{Y}} \|y - U(s',s)y\|_{\mathcal{Y}} \to 0$$

$$\|U(t',s)y - U(t,s)y\|_{\mathcal{Y}} \leq \|(U(t',t)-\mathbb{1})U(t,s)y\|_{\mathcal{Y}} \to 0$$

$$U(t,t) = \mathbb{1}, U(t,s)U(s,r) = U(t,r).$$

$$\partial_t U(t,s)y = A(t)U(t,s)y,$$
$$-\partial_s U(t,s)y = U(t,s)A(s)y,$$

$$U(s,t) = V(T-s, T-t).$$

$$U(t,s)U(s,t) = \mathbb{1}$$

$$\partial_t U(t,s)y|_{t=s} = A(s)y$$
$$-\partial_s U(t,s)y = U(t,s)A(s)y$$

$$\partial_t U(t,s)y = A(t)U(t,s)y.$$

$$\|x\|_{\mathcal{X},s} \leq \|x\|_{\mathcal{X},t} \exp\left|\int_s^t C(r)\mathrm{d}r\right|,$$
$$\|y\|_{\mathcal{Y},s} \leq \|y\|_{\mathcal{Y},t} \exp\left|\int_s^t C(r)\mathrm{d}r\right|.$$

$$U(t,t) = \mathbb{1}, U(t,s)U(s,r) = U(t,r)$$

$$\|U(t,s)\|_{\mathcal{X},s} \leq \exp\left|\int_s^t 2C(r)\mathrm{d}r\right|, s,t \in I$$

$$i\partial_t U(t,s)y = A(t)U(t,s)y$$
$$-i\partial_s U(t,s)y = U(t,s)A(s)y$$

$$\|U(t,s)\|_{y,s} \leq \exp\left|\int_s^t 2C(r)\mathrm{d}r\right|, s,t \in I$$

$$V = U + U * B * U + U * B * U * B * U + \cdots,$$

$$(U * B * U)(t,r) = \int_r^t U(t,s)B(s)U(s,r)\mathrm{d}s$$

$$\|Tx\| \leq C_0\|x\|, \|BTx\| \leq C_1\|Ax\|$$

$$\|B^\lambda Tx\| \leq C_0^\lambda C_1^{1-\lambda}\|A^\lambda x\|, \lambda \in [0,1].$$

$$Ku := -g^{\mu\nu}(\nabla_\mu - iA_\mu)(\nabla_\nu - iA_\nu)u + Yu$$

$$|\dot{g}^{\mu\nu}X_\mu X_\nu| \leq C_g|g^{\mu\nu}X_\mu X_\nu|$$

$$\mathrm{supp}u \subset J(\mathrm{supp}Ku \cup \{t\} \times (\mathrm{supp}u(t) \cup \mathrm{supp}\dot{u}(t)))$$

$$\mathcal{L}[u] := -|g|^{\frac{1}{2}}\left(\left((\partial_\mu + iA_\mu)\bar{u}\right) g^{\mu\nu}\left((\partial_\nu - iA_\nu)u\right) + Y|u|^2\right).$$

$$\mathcal{P}^\mu[u] := -\delta_0^\mu \mathcal{L}[u] + \frac{\partial\mathcal{L}[u]}{\partial(\partial_\mu \bar{u})}\partial_t\bar{u} + \frac{\partial\mathcal{L}[u]}{\partial(\partial_\mu u)}\partial_t u.$$

$$\tilde{\mathcal{L}}[u] := -|g|^{\frac{1}{2}}\left(\left((\partial_\mu + iA_\mu)\bar{u}\right) g^{\mu\nu}\left((\partial_\nu - iA_\nu)u\right) - (1 + \alpha^{-2}A_0^2)|u|^2\right),$$

$$\tilde{\mathcal{E}}[u] := \tilde{\mathcal{P}}^0[u] = |g|^{\frac{1}{2}}\left(\alpha^{-2}|\dot{u}|^2 + ((\partial_i + iA_i)\bar{u})g_\Sigma^{ij}\left((\partial_j - iA_j)u\right) + |u|^2\right)$$

$$\tilde{\mathcal{P}}^i[u] = \mathcal{P}^i[u] = -|g|^{\frac{1}{2}}\left(\dot{\bar{u}}g_\Sigma^{ij}\left((\partial_j - iA_j)u\right) + \dot{u}g_\Sigma^{ij}\left((\partial_j + iA_j)\bar{u}\right)\right)$$

$$\hat{g} > g.$$

$$e^{C(s-t)}\int_{K_t}\tilde{\mathcal{E}}[u](t) \leq \int_{K_s}\tilde{\mathcal{E}}[u](s) + \int_\Omega |g|^{\frac{1}{2}}|Ku|^2.$$

$$\begin{aligned}
\partial_\mu\tilde{\mathcal{P}}^\mu[u] &= -\partial_t\tilde{\mathcal{L}}[u] + \left(\partial_\mu\frac{\partial\tilde{\mathcal{L}}[u]}{\partial(\partial_\mu\bar{u})}\right)\dot{\bar{u}} + \frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\bar{u})}\partial_\mu\partial_t\bar{u} + \left(\partial_\mu\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu u)}\right)\dot{u} + \frac{\partial\tilde{\mathcal{L}}[u]}{\partial(\partial_\mu u)}\partial_\mu\partial_t u \\
&= -\partial_t\tilde{\mathcal{L}}[u] + \left(|g|^{\frac{1}{2}}\tilde{K}u + \frac{\partial\tilde{\mathcal{L}}[u]}{\partial\bar{u}}\right)\dot{\bar{u}} + \frac{\partial\tilde{\mathcal{L}}[u]}{\partial(\partial_\mu\bar{u})}\partial_t\partial_\mu\bar{u} + \left(|g|^{\frac{1}{2}}\bar{\tilde{K}}u + \frac{\partial\tilde{\mathcal{L}}[u]}{\partial u}\right)\dot{u} \\
&\quad + \frac{\partial\tilde{\mathcal{L}}[u]}{\partial(\partial_\mu u)}\partial_t\partial_\mu u \\
&= -2|g|^{\frac{1}{2}}\mathrm{Re}(\dot{\bar{u}}\tilde{K}u) - \frac{\partial\tilde{\mathcal{L}}[u]}{\partial g^{\mu\nu}}\dot{g}^{\mu\nu} - \frac{\partial\tilde{\mathcal{L}}[u]}{\partial A_\mu}\dot{A}_\mu - \frac{\partial\tilde{\mathcal{L}}[u]}{\partial|g|}\partial_t|g| \\
&= |g|^{\frac{1}{2}}\Big(2\mathrm{Re}(\dot{\bar{u}}\tilde{K}u) + \left((\partial_\mu + iA_\mu)\bar{u}\right)\dot{g}^{\mu\nu}\left((\partial_\nu - iA_\nu)u\right) - 2\alpha^{-3}\dot{\alpha}A_0^2|u|^2 \\
&\quad -2\mathrm{Im}\left(\bar{u}\dot{A}_\mu g^{\mu\nu}(\partial_\nu - iA_\nu)u\right) + 2\alpha^{-2}A_0\dot{A}_0|u|^2 - \frac{1}{2}|g|^{-1}(\partial_t|g|)\tilde{\mathcal{L}}[u]\Big).
\end{aligned}$$

$$\tilde{K} = K - Y + 1 + \alpha^{-2}A_0^2$$

$$\partial_\mu \tilde{\mathcal{P}}^\mu[u] \leq |g|^{\frac{1}{2}}\left(|Ku|^2 + C_1\alpha^{-2}|\dot{u}|^2 + C_2\big((\partial_i + \mathrm{i}A_i)\bar{u}\big)g_\Sigma^{ij}\left((\partial_j + \mathrm{i}A_j)u\right) + C_3|u|^2\right)$$

$$\int_\Omega \partial_\mu \tilde{\mathcal{P}}^\mu[u] \leq \int_\Omega \left(|g|^{\frac{1}{2}}|Ku|^2 + C\tilde{\mathcal{E}}[u]\right)$$

$$\int_\Omega \partial_\mu \tilde{\mathcal{P}}^\mu[u] = \int_{\partial\Omega} n_\mu \tilde{\mathcal{P}}^\mu[u] = \int_{K_t} \tilde{\mathcal{E}}[u](t) - \int_{K_s} \tilde{\mathcal{E}}[u](s) + \int_\Lambda n_\mu \tilde{\mathcal{P}}^\mu[u],$$

$$\begin{aligned}\xi_\mu \tilde{\mathcal{P}}^\mu[u] &= \xi_0\tilde{\mathcal{E}}[u] - 2|g|^{\frac{1}{2}}\mathrm{Re}\big(\xi_i\dot{\bar{u}}g_\Sigma^{ij}(\partial_j - \mathrm{i}A_j)u\big)\\ &\geq \xi_0\tilde{\mathcal{E}}[u] - |g|^{\frac{1}{2}}\alpha|\vec{\xi}|\left(\alpha^{-2}|\dot{u}|^2 + \big((\partial_i + \mathrm{i}A_i u)\big)g_\Sigma^{ij}\left((\partial_j - \mathrm{i}A_j)u\right)\right)\\ &\geq \big(\xi_0 - \alpha|\vec{\xi}|\big)\tilde{\mathcal{E}}[u] \geq 0\end{aligned}$$

$$\int_{K_t} \tilde{\mathcal{E}}[u](t) - \int_{K_s} \tilde{\mathcal{E}}[u](s) \leq \int_s^t \left(\int_{K_r} \left(|g|^{\frac{1}{2}}|Ku(r)|^2 + C\tilde{\mathcal{E}}[u](r)\right)\right)\mathrm{d}r$$

$$\begin{aligned}\mathrm{supp}u \cap M_\pm &\subset J_g^\pm\left((\mathrm{supp}Ku \cap M_\pm) \cup \{t\} \times (\mathrm{supp}u(t) \cup \mathrm{supp}\dot{u}(t))\right),\\ \mathrm{supp}u &\subset J_g(\mathrm{supp}Ku \cup \{t\} \times (\mathrm{supp}u(t) \cup \mathrm{supp}\dot{u}(t))),\end{aligned}$$

$$x \in M \setminus J_{\hat{g}}^+\left((\mathrm{supp}Ku \cap M_+) \cup \{t\} \times \mathrm{supp}\tilde{\mathcal{E}}[u](t)\right)$$

$$\mathrm{supp}u \cap M_\pm \subset J_{\hat{g}}^\pm\left((\mathrm{supp}Ku \cap M_\pm) \cup \{t\} \times (\mathrm{supp}u(t) \cup \mathrm{supp}\dot{u}(t))\right)$$

$$J_g^\pm(\Omega) = \bigcap_{\hat{g} \succ g} J_{\hat{g}}^\pm(\Omega), \Omega \subset M$$

4.3. Dunkl -Klein – Gordon en altas dimensiones, para campos cuánticos relativistas.

$$D_x = \frac{\partial}{\partial x} + \frac{\mu}{x}(\mathbf{1} - R)$$

$$Rf(x) = f(-x)$$

$$D_Y = \frac{\partial}{\partial x} - \frac{\mu}{x}R$$

$$\left[E^2 + D_j^2 - m^2\right]\Psi(\mathbf{x}) = 0$$

$$\begin{aligned}x_1 &= r\cos\,\theta_1\sin\,\theta_2\sin\,\theta_3 \cdots \sin\,\theta_{d-1}\\ x_2 &= r\sin\,\theta_1\sin\,\theta_2\sin\,\theta_3 \cdots \sin\,\theta_{d-1}\\ x_3 &= r\cos\,\theta_2\sin\,\theta_3\sin\,\theta_4 \cdots \sin\,\theta_{d-1}\\ x_j &= r\cos\,\theta_{j-1}\sin\,\theta_j\sin\,\theta_{j+1} \cdots \sin\,\theta_{d-1}\\ &\vdots\\ x_d &= r\cos\,\theta_{d-1}\end{aligned}$$

$$\begin{gathered}
x_1 = r\cos\ \theta_1 \sin\ \theta_2 \sin\ \theta_3 \cdots \sin\ \theta_{d-1} \\
x_2 = r\sin\ \theta_1 \sin\ \theta_2 \sin\ \theta_3 \cdots \sin\ \theta_{d-1} \\
x_3 = r\cos\ \theta_2 \sin\ \theta_3 \sin\ \theta_4 \cdots \sin\ \theta_{d-1} \\
\vdots \\
x_j = r\cos\ \theta_{j-1} \sin\ \theta_j \sin\ \theta_{j+1} \cdots \sin\ \theta_{d-1} \\
\vdots \\
x_d = r\cos\ \theta_{d-1}
\end{gathered}$$

$$\sum_{j=1}^{d} x_j^2 = r^2$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{j=1}^{d-2} \frac{1}{\sin^2\ \theta_{j+1} \sin^2\ \theta_{j+2} \cdots \sin^2\ \theta_{d-1}} \left\{ \frac{\partial^2}{\partial \theta_j^2} + (j-1)\tan\ \theta_j \frac{\partial}{\partial \theta_j} \right\}$$

$$\begin{aligned}
\Delta = &\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{j=1}^{d-2} \frac{1}{\sin^2\ \theta_{j+1} \sin^2\ \theta_{j+2} \cdots \sin^2\ \theta_{d-1}} \left\{ \frac{\partial^2}{\partial \theta_j^2} + (j-1)\tan\ \theta_j \frac{\partial}{\partial \theta_j} \right\} \\
&+ \frac{1}{r^2}\left\{ \frac{1}{\sin^{d-2}\ \theta_{d-1}} \frac{\partial}{\partial \theta_{d-1}} \sin^{d-2}\ \theta_{d-1} \frac{\partial}{\partial \theta_{d-1}} \right\}
\end{aligned}$$

$$\prod_{j=1}^{d} dx_j = r^{d-1} dr \prod_{j=1}^{d-1} \left(\sin\ \theta_j\right)^{j-1} d\theta_j$$

$$\left[\mathcal{A}_r + \frac{\mathcal{J}_{\theta_1}}{r^2 \sin^2\ \theta_2 \sin^2\ \theta_3 \cdots \sin^2\ \theta_{d-1}} + \frac{\mathcal{J}_{\theta_2}}{\sin^2\ \theta_3 \cdots \sin^2\ \theta_{d-1}} + \cdots + \frac{1}{r^2}\mathcal{J}_{\theta_{d-1}} \right] \Psi(\mathbf{x}) = 0$$

$$\mathcal{A}_r = \frac{\partial^2}{\partial r^2} + \frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{r}\frac{\partial}{\partial r} + E^2 - m^2$$

$$\begin{aligned}
\mathcal{J}_{\theta_1} &= -\frac{\partial^2}{\partial \theta_1^2} + \frac{2(\mu_1 \tan\ \theta_1 - \mu_2 \cot\ \theta_1)}{r^2}\frac{\partial}{\partial \theta_1} + \frac{\mu_1}{\cos^2\ \theta_1}(1-R_1) + \frac{\mu_2(1-R_2)}{\sin^2\ \theta_1} \\
\mathcal{J}_{\theta_2} &= -\frac{\partial^2}{\partial \theta_2^2} - \left[\left(1+2(\mu_1+\mu_2)\right)\cot\ \theta_2 - 2\mu_3 \tan\ \theta_2\right]\frac{\partial}{\partial \theta_2} + \frac{\mu_3(1-R_3)}{\cos^2\ \theta_2} \\
\mathcal{J}_{\theta_3} &= -\frac{\partial^2}{\partial \theta_3^2} - \left[\left(2+2(\mu_1+\mu_2+\mu_3)\right)\cot\ \theta_3 - 2\mu_4 \tan\ \theta_3\right]\frac{\partial}{\partial \theta_3} + \frac{\mu_4(1-R_4)}{\cos^2\ \theta_3} \\
&\vdots \\
\mathcal{J}_{\theta_{d-1}} &= \frac{\partial^2}{\partial \theta_{d-1}^2} + \left[\left((d-2)+2(\mu_1+\mu_2+\cdots+\mu_{d-1})\right)\cot\ \theta_{d-1} - 2\mu_d \tan\ \theta_{d-1}\right]\frac{\partial}{\partial \theta_{d-1}} + \frac{\mu_d(1-R_d)}{\cos^2\ \theta_{d-1}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{\theta_1} &= -\frac{\partial^2}{\partial \theta_1^2} + \frac{2(\mu_1 \tan\ \theta_1 - \mu_2 \cot\ \theta_1)}{r^2}\frac{\partial}{\partial \theta_1} + \frac{\mu_1}{\cos^2\ \theta_1}(1-R_1) + \frac{\mu_2(1-R_2)}{\sin^2\ \theta_1} \\
\mathcal{J}_{\theta_2} &= -\frac{\partial^2}{\partial \theta_2^2} - \left[\left(1+2(\mu_1+\mu_2)\right)\cot\ \theta_2 - 2\mu_3 \tan\ \theta_2\right]\frac{\partial}{\partial \theta_2} + \frac{\mu_3(1-R_3)}{\cos^2\ \theta_2} \\
\mathcal{J}_{\theta_3} &= -\frac{\partial^2}{\partial \theta_3^2} - \left[\left(2+2(\mu_1+\mu_2+\mu_3)\right)\cot\ \theta_3 - 2\mu_4 \tan\ \theta_3\right]\frac{\partial}{\partial \theta_3} + \frac{\mu_4(1-R_4)}{\cos^2\ \theta_3} \\
&\vdots \\
\mathcal{J}_{\theta_{d-2}} &= -\frac{\partial^2}{\partial \theta_{d-2}^2} - \left[\left((d-3)+2(\mu_1+\mu_2+\cdots+\mu_{d-2})\right)\cot\ \theta_{d-2} - 2\mu_{d-1} \tan\ \theta_{d-2}\right]\frac{\partial}{\partial \theta_{d-2}} + \frac{\mu_{d-1}(1-R_{d-1})}{\cos^2\ \theta_{d-2}} \\
\mathcal{J}_{\theta_{d-1}} &= \frac{\partial^2}{\partial \theta_{d-1}^2} + \left[\left((d-2)+2(\mu_1+\mu_2+\cdots+\mu_{d-1})\right)\cot\ \theta_{d-1} - 2\mu_d \tan\ \theta_{d-1}\right]\frac{\partial}{\partial \theta_{d-1}} + \frac{\mu_d(1-R_d)}{\cos^2\ \theta_{d-1}}
\end{aligned}$$

$$R_1 f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,\pi-\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1})$$
$$R_2 f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,-\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1})$$
$$\vdots$$
$$\vdots$$
$$R_d f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\pi-\theta_{d-1})$$

$$R_1 f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,\pi-\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1})$$
$$R_2 f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,-\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1})$$
$$\vdots$$
$$R_j f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,\theta_1,\theta_2,\cdots,\pi-\theta_j,\cdots,\theta_{d-1})$$
$$\vdots$$
$$R_d f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\theta_{d-1}) = f(r,\theta_1,\theta_2,\cdots,\theta_j,\cdots,\pi-\theta_{d-1})$$

$$\psi(r,\theta_1,\cdots,\theta_{d-1}) = \mathcal{R}(r)\Theta_1(\theta_1)\Theta_2(\theta_2)\cdots\Theta_{d-1}(\theta_{d-1})$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{r}\frac{\partial}{\partial r} + E^2 - m^2 - \frac{\varpi^2}{r^2}\right]\mathcal{R}(r) = E\mathcal{R}(r)$$

$$(\mathcal{J}_{\theta_1} + \lambda_1^2)\Theta_1(\theta_1) = 0$$
$$\left(\mathcal{J}_{\theta_2} + \frac{\lambda_1^2}{\sin^2\ \theta_2} + \lambda_2^2\right)\Theta_2(\theta_2) = 0$$
$$\vdots$$
$$\left(\mathcal{J}_{\theta_{d-1}} + \frac{\lambda_{d-2}^2}{\sin^2\ \theta_{d-1}} + \varpi^2\right)\Theta_{d-1}(\theta_{d-1}) = 0$$

$$\Theta_1^{s_1,s_2}(\theta_1) = i_{\ell_1}\cos^{e_1}\ \theta_1 \sin^{e_2}\ \theta_1 \mathbf{P}_{\ell_1-(e_1+e_2)/2}^{(\mu_2+e_2-1/2;\mu_1+e_1-1/2)}(\cos\ 2\theta_1),$$

$$e_j = \begin{cases} 1 & \text{if } s_j = -1 \\ 0 & \text{if } s_j = +1 \end{cases}$$

$$\Theta_2^{s_3}(\theta_2) = i_{\ell_2}\cos^{e_3}\ \theta_1 \sin^{2\ell_1}\ \theta_2 \mathbf{P}_{\ell_2-\frac{e_3}{2}}^{(2\ell_1+\mu_1+\mu_2;\mu_3+e-1/2)}(\cos\ 2\theta_2)$$

$$\lambda_2^2 = 4(\ell_2+\ell_1)(\ell_2+\ell_1+\mu_1+\mu_2+\mu_3+1/2)$$

j	$\Theta_j^{s_{j+1}}(\theta_j)$
3	$\cos^{e_4}\ \theta_3 \sin^{2(\ell_2+\ell_1)}\ \theta_3 \mathbf{P}_{\ell_3-\frac{e_4}{2}}^{(1/2+2(\ell_2+\ell_1)+\mu_1+\mu_2+\mu_3,\mu_4+e_4-1/2)}(\cos\ 2\theta_3)$
4	$\cos^{e_5}\ \theta_2 \sin^{2(\ell_2+\ell_1+\ell_3)}\ \theta_2 \mathbf{P}_{\ell_4-\frac{e_5}{2}}^{(1+2(\ell_1+\ell_2+\ell_3)+\mu_1+\cdots+\mu_4,\mu_5+e_5-1/2)}(\cos\ 2\theta_4)$
5	$\cos^{e_6}\ \theta_5 \sin^{2(\ell_1+\cdots+\ell_4)}\ \theta_5 \mathbf{P}_{\ell_5-\frac{e_6}{2}}^{(3/2+2(\ell_1+\cdots+\ell_4)+\mu_1+\cdots+\mu_5,\mu_6+e_6-1/2)}(\cos\ 2\theta_5)$
6	$\cos^{e_7}\ \theta_6 \sin^{2(\ell_1+\cdots+\ell_5)}\ \theta_6 \mathbf{P}_{\ell_6-\frac{e_7}{2}}^{(2+2(\ell_1+\cdots+\ell_5)+\mu_1+\cdots+\mu_5,\mu_7+e_7-1/2)}(\cos\ 2\theta_6)$
$\vdots$	$\vdots$

k	$\cos^{e_{k+1}}\theta_k \sin^{2(\ell_1+\cdots+\ell_{k-1})}\theta_k \mathbf{P}_{\ell_k-\frac{e_{k+1}}{2}}^{\left(\frac{k-2}{t_1}+2(\ell_1+..+\ell_{k-1})+\mu_1+\cdots+\mu_k,\mu_{k+1}+e_{k+1}-1/2\right)}(\cos\ 2\theta_k)$

j	λ_j^2
3	$4(\ell_1+\ell_2+\ell_3)(\ell_1+\ell_2+\ell_3+\mu_1+\cdots+\mu_4+1)$
4	$4(\ell_1+\ell_2+\ell_3+\ell_4)(\ell_1+\cdots+\ell_4+\mu_1+\cdots+\mu_5+3/2)$
5	$4(\ell_1+\cdots+\ell_5)(\ell_1+\cdots+\ell_5+\mu_1+\cdots+\mu_6+2)$
6	$4(\ell_1+\ell_2+\cdots+\ell_6)(\ell_1+\ell_2+\cdots+\ell_6+\mu_1+\cdots+\mu_7 + 5/2)$
⋮	⋮
k	$4(\ell_1+\ell_2+\cdots+\ell_k)\left(\ell_1+\ell_2+\cdots+\ell_k+\mu_1+\cdots+\mu_{k+1} + \frac{k-1}{2}\right)$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{r}\frac{\partial}{\partial r}+E^2-m^2-\frac{\varpi^2}{r^2}\right]\mathcal{R}(r)=0$$

$$\varpi^2=4(\ell_1+\ell_2+\cdots+\ell_{d-1})\left(\ell_1+\ell_2+\cdots+\ell_{d-1}+\mu_1+\cdots+\mu_d+\frac{d-2}{2}\right).$$

$$\left[E^2-\left(\frac{1}{i}D_j+im\omega x_j\right)\left(\frac{1}{i}D_j-im\omega x_j\right)-m^2\right]\Psi(\mathbf{x})=0$$

$$\left[(D_1^2+\cdots+D_d^2)+2m\omega\left(\mu_1R_1+\cdots+\mu_dR_d+\frac{d}{2}\right)-m^2\omega^2(x_1^2+\cdots+x_d^2)+E^2-m^2\right]\Psi(\mathbf{x})$$

$$\left[\frac{d^2}{dr^2}+\frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{r}\frac{d}{dr}+2m\omega\left(\mu_1s_1+\cdots+\mu_ds_d+\frac{d}{2}\right) -m^2\omega^2r^2+E^2-m^2-\frac{\varpi^2}{r^2}\right]\mathcal{R}(r)$$

$$\rho=m\omega r^2$$

$$\left[\rho\frac{d^2}{d\rho^2}+\left(\frac{d}{2}+\mu_1+\mu_2+\cdots+\mu_d\right)\frac{d}{d\rho}-\frac{\rho}{4}-\frac{\varpi^2}{4\rho}+\frac{1}{2}\left(\mu_1s_1+\cdots+\mu_ds_d+\frac{d}{2}\right)+\frac{E^2-m^2}{4m\omega}\right]\mathcal{R}(\rho)$$

$$\mathcal{R}(\rho)=e^{-\frac{\rho}{2}}\rho^{\ell_1+\ell_2+\cdots+\ell_{d-1}}\digamma(\rho)$$

$$\left[\rho\frac{d^2}{d\rho^2}+\left(2(\ell_1+\ell_2+\cdots+\ell_{d-1})+\frac{d}{2}+(\mu_1+\mu_2+\cdots+\mu_d)-\rho\right)\frac{d}{d\rho}-(\ell_1+\ell_2+\cdots+\ell_{d-1})\right.$$
$$\left.+\frac{E^2-m^2}{4m\omega}-\frac{1}{2}\big((\mu_1+\mu_2+\cdots+\mu_d)-(\mu_1 s_1+\mu_2 s_2+\cdots+\mu_d s_d)\big)\right]\digamma(\rho)=0$$

$$\mathcal{R}(\rho)=C\,\rho^{\ell_1+\ell_2+\cdots+\ell_{d-1}}e^{-\frac{\rho}{2}}\mathbf{F}(a,b;\rho)$$

$$\begin{cases} a=(\ell_1+\ell_2+\cdots+\ell_{d-1})+\dfrac{1}{2}[(\mu_1+\mu_2+\cdots+\mu_d)-(\mu_1 s_1+\mu_2 s_2+\cdots+\mu_d s_d)]-\dfrac{E^2-m^2}{4m\omega} \\ b=2(\ell_1+\ell_2+\cdots+\ell_{d-1})+\dfrac{d}{2}+(\mu_1+\cdots+\mu_d) \end{cases}$$

$$\ell_1+\ell_2+\cdots+\ell_{d-1}+\frac{\mu_1+\mu_2+\cdots+\mu_d-(\mu_1 s_1+\cdots+\mu_d s_d)}{2}-\frac{E^2-m^2}{4m\omega}=-n$$

$$E_{n,s_1,\cdots,s_d}=\pm\sqrt{2m\omega[2(n+\ell_1+\ell_2+\cdots+\ell_{d-1})+\mu_1+\mu_2+\cdots+\mu_d-(\mu_1 s_1+\cdots+\mu_d s_d)]+m^2}.$$

$$E_{nr}=\omega[2(n+\ell_1+\ell_2+\cdots+\ell_{d-1})+(\mu_1+\mu_2+\cdots+\mu_d)-(\mu_1 s_1+\cdots+\mu_d s_d)]$$

$$V=\frac{-Ze^2}{r}$$

$$\left[\frac{d^2}{dr^2}+\frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{r}\frac{d}{dr}+\left(E+\frac{Ze^2}{r}\right)^2-m^2-\frac{\varpi^2}{r^2}\right]\mathcal{R}(r)$$

$$\mathcal{R}=\eta^{\delta}e^{-\frac{\eta}{2}}\Xi(\eta)$$

$$\eta=2\varkappa r\ \ \text{and}\ \ \varkappa=\sqrt{m^2-E^2}$$

$$\delta=1-\frac{d+2(\mu_1+\cdots+\mu_d)}{2}+\sqrt{\varpi^2+(\mu_1+\cdots+\mu_d)(\mu_1+\cdots+\mu_d+d-2)+\left(\frac{d}{2}-1\right)^2-Z^2e^4}$$

$$\left[\eta\frac{d^2}{d\eta^2}+(2\delta+d-1+2(\mu_1+\cdots+\mu_d)-\eta)\frac{d}{d\eta}+\frac{EZe^2}{\varkappa}-\frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{2}-\delta\right]\Xi(\eta)$$

$$\Xi(\eta)=\mathbf{F}(-n,2\delta+d-1+2(\mu_1+\cdots+\mu_d);\eta)$$

$$\frac{E_n}{m}=\left\{1+\frac{Z^2e^4}{\left(n-\frac{1}{2}-\sqrt{\varpi^2+(\mu_1+\cdots+\mu_d)(\mu_1+\cdots+\mu_d+d-2)+\left(\frac{d}{2}-1\right)^2-Z^2e^4}\right)^2}\right\}^{-1/2}$$

$$\varpi^2+(\mu_1+\cdots+\mu_d)(\mu_1+\cdots+\mu_d+d-2)+\left(\frac{d}{2}-1\right)^2-Z^2e^4\geq 0$$

$$\zeta=-2i\kappa r$$

$$\left[\frac{\partial^2}{\partial\zeta^2}+\frac{d-1+2(\mu_1+\mu_2+\mu_3+\cdots+\mu_d)}{\zeta}\frac{\partial}{\partial\zeta}-\frac{i}{\kappa}\frac{EZe^2}{\zeta}+\frac{Z^2e^4-\varpi^2}{\zeta^2}-\frac{1}{4}\right]\mathcal{R}(r)=0$$

$$\mathcal{R}(\zeta)=\zeta^{\vartheta}\Phi(\zeta)$$

$$\left[\frac{d^2}{d\zeta^2}-\frac{i}{\kappa}\frac{EZe^2}{\zeta}-\frac{1}{4}+\frac{(Ze^2)^2-\vartheta(\vartheta+1)-\varpi^2}{\zeta^2}\right]\Phi(\zeta)=0$$

$$2\vartheta=-[d-1+2(\mu_1+\cdots+\mu_d)].$$

$$\left[\frac{d^2}{d\zeta^2}-\frac{1}{4}\right]\Phi(\zeta)=0$$

$$\Phi(\zeta)\simeq e^{\frac{-\zeta}{2}}$$

$$\left[\frac{d^2}{d\zeta^2}-\frac{\varpi^2+\vartheta(\vartheta+1)-(Ze^2)^2}{\zeta^2}\right]\Phi(\zeta)=0$$

$$\Phi(\zeta)\simeq\zeta^{\frac{1}{2}\pm i\sqrt{(Ze^2)^2-\varpi^2-\vartheta(\vartheta+1)-\frac{1}{4}}}$$

$$\Phi(\zeta)=C_1W_{\alpha,\beta}(\zeta)+C_2M_{\alpha,\beta}(\zeta)$$

$$\alpha=-i\frac{EZe^2}{\kappa};\ \beta=\pm i\sqrt{(Ze^2)^2-\varpi^2-\vartheta(\vartheta+1)-\frac{1}{4}}$$

$$\Phi_{in}^{+}=A\Phi_{out}^{+}+B\Phi_{out}^{-}$$

$$|A|^2-|B|^2=1$$

$$\lim_{\zeta\to\infty}W_{\alpha,\beta}(\zeta)\simeq\zeta^{\alpha}e^{\frac{-\zeta}{2}}$$

$$\Phi_{out}^{-}(\zeta)=W_{\alpha,\beta}(\zeta)$$
$$\Phi_{out}^{+}(\zeta)=\left(W_{\alpha,\beta}(\zeta)\right)^{*}=W_{-\alpha,\beta}(-\zeta)$$

$$\lim_{\zeta\to 0}M_{\alpha,\beta}(\zeta)=\zeta^{1/2+\beta}e^{\frac{-\zeta}{2}}$$

$$\Phi_{in}^{+}(\zeta)=M_{\alpha,\beta}(\zeta)$$

$$M_{\alpha,\beta}(\zeta)=\frac{\Gamma(1+2\beta)}{\Gamma(1/2+\beta-\alpha)}e^{i\pi\alpha}W_{-\alpha,\beta}(-\zeta)+\frac{\Gamma(1+2\beta)e^{i\pi(\alpha-\beta-1/2)}}{\Gamma(1/2+\beta+\alpha)}W_{\alpha,\beta}(\zeta)$$

$$\Phi_{in}^{+}(\zeta)=\frac{\Gamma(1+2\beta)}{\Gamma(1/2+\beta-\alpha)}e^{i\pi\alpha}\Phi_{out}^{+}(\zeta)+\frac{\Gamma(1+2\beta)e^{i\pi(\alpha-\beta-1/2)}}{\Gamma(1/2+\beta+\alpha)}\Phi_{out}^{-}(\zeta)$$

$$A=\frac{\Gamma(1+2\beta)}{\Gamma(1/2+\beta-\alpha)}e^{i\pi\alpha}$$
$$B=\frac{\Gamma(1+2\beta)e^{i\pi(\alpha-\beta-1/2)}}{\Gamma(1/2+\beta+\alpha)}$$

$$\mathcal{P}=\left|\frac{B}{A}\right|^2$$

$$\left|\Gamma\left(\frac{1}{2}+ix\right)\right|^2=\frac{\pi}{\cosh\pi x}$$

$$\mathcal{P} = \frac{\cosh\ \pi\left(\tilde{\beta} + \frac{EZe^2}{\kappa}\right)}{\cosh\ \pi\left(\tilde{\beta} - \frac{EZe^2}{\kappa}\right)} e^{-2\pi\tilde{\beta}},$$

$$\tilde{\beta} = \sqrt{(Ze^2)^2 - \varpi^2 - \vartheta(\vartheta+1) - \frac{1}{4}},\ \text{and}\ (Ze^2)^2 - \varpi^2 - \vartheta(\vartheta+1) - \frac{1}{4} \geq 0$$

$$\mathcal{N} = \left(\left|\frac{B}{A}\right|^{-2} - 1\right)^{-1}$$

$$\mathcal{N} = \frac{\cosh\ \pi\left(\tilde{\beta} + \frac{EZe^2}{\kappa}\right)}{e^{2\pi\tilde{\beta}}\cosh\ \pi\left(\tilde{\beta} - \frac{EZe^2}{\kappa}\right) - \cosh\ \pi\left(\tilde{\beta} + \frac{EZe^2}{\kappa}\right)}$$

$$Z^2e^4 \geq \varpi^2 + \vartheta(\vartheta+1) + \frac{1}{4}$$

$$Z \geq \frac{1}{e^2}\left(\ell + \frac{d}{2} - 1\right)$$

$$Z \geq \frac{1}{e^2}\left(\ell + \frac{1}{2}\right)$$

d	ℓ_i	$Z(\mu_i = +0.4)$	$Z(\mu_i = 0)$	$Z(\mu_i = -0.4)$
3	1	5.7×137	$\frac{3}{2} \times 137$	3.3×137
4		8.6×137	2×137	5.4×137
5		11.5×137	$\frac{5}{2} \times 137$	7.5×137
6		14.4×137	3×137	9.6×137
3	2	9.7×137	$\frac{5}{2} \times 137$	7.3×137
4		14.6×137	3×137	11.4×137
5		19.5×137	$\frac{7}{2} \times 137$	15.5×137
6		24.4×137	4×137	19.6×137
3	3	13.7×137	$\frac{7}{2} \times 137$	11.3×137

4	20.6×137	4×137	17.4×137
5	27.5×137	$\frac{9}{2} \times 137$	23.5×137
6	34.4×137	5×137	23.6×137

5. Niveles y despliegue de energía en partículas – estrella en relación a campos cuánticos relativistas.

$$\hat{\rho} = \sum_n p_n |E_n\rangle\langle E_n|$$

$$C_{\mathrm{AA}'} - \mathrm{Tr}_{\mathrm{A}'}[d_{\mathrm{A}}|\Phi\rangle\langle\Phi|C_{\mathrm{AA}'}] \otimes \mathbb{1}_{\mathrm{A}'} \geq 0$$

$$p_* \leq \left(1 + \frac{E_1\left(q_{0,\min}^{\mathrm{AB}}\right)^2}{\max\limits_{i\geq 1}\left[E_i\left(q_{i,\max}^{\mathrm{AB}}\right)^2\right]}\right)^{-1}$$

$$\max_{\hat{M},\hat{N}} |\mathrm{Tr}[\hat{M} \otimes \hat{N}\hat{\rho}] - \mathrm{Tr}[\hat{M}\hat{\rho}]\mathrm{Tr}[\hat{N}\hat{\rho}]| \leq \|\hat{M}\|\|\hat{N}\|\epsilon(l),$$

$$\hat{H}_{\mathrm{AB}} = \hat{H}_{\mathrm{AB}_1} + \hat{V}_{\mathrm{B}_1\,\mathrm{B}_2} + \hat{H}_{\mathrm{B}_2}$$

$$\hat{H}_{\mathrm{AB}_1} = \hat{H}_{\mathrm{A}} + \hat{V}_{\mathrm{AB}_1} + \hat{H}_{\mathrm{B}_1},$$

$$E_1^{\mathrm{AB}_1}\left(q_{0,\min}^{\mathrm{AB}_1}\right)^2 > \lambda(l)$$

$$\lambda(l) = K d_{\mathrm{A}}^2 \left\|\hat{H}_{\mathrm{A}}\right\| \|\partial B_2 \mid (\epsilon(l/2) + c_1 e^{-c_2 l})$$

$$\mathrm{Tr}\left[e^{-\beta_* \hat{H}_{\mathrm{AB}_1}}\right]^{-1} \leq \left(1 + \frac{\lambda(l)}{\max\limits_{i\geq 1}\left[E_i^{\mathrm{AB}_1}\left(q_{i,\max}^{\mathrm{AB}_1}\right)^2\right]}\right)$$

$$\mathrm{Tr}\left[e^{-\beta_* \hat{H}_{\mathrm{AB}_1}}\right]^{-1} \leq \left(1 + \frac{\lambda(l)}{\max\limits_{i\geq 1}\left[E_i^{\mathrm{AB1}}\left(q_{i,\max}^{\mathrm{AB1}}\right)^2\right]}\right) \times \left(1 + \frac{E_1^{\mathrm{AB1}}\left(q_{0,\min}^{\mathrm{AB1}}\right)^2}{\max\limits_{i\geq 1}\left[E_i^{\mathrm{AB1}}\left(q_{i,\max}^{\mathrm{AB1}}\right)^2\right]}\right)^{-1}$$

$$\hat{H} = \hat{H}_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} + \mathbb{1}_{\mathrm{A}} \otimes \hat{H}_{\mathrm{B}} + \hat{V}_{\mathrm{AB}}$$

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{V}_{\mathrm{AB}}$$

$$\hat{H}_i = h\hat{\sigma}_z^i + f(h,k)\mathbb{1}$$

$$\hat{V}_{\mathrm{AB}} = 2\left(k\hat{\sigma}_x^{\mathrm{A}}\hat{\sigma}_x^{\mathrm{B}} + \frac{k^2}{h^2} f(h,k)\mathbb{1}\right)$$

$$\langle g|\hat{H}_{\mathrm{A}}|g\rangle = \langle g|\hat{H}_{\mathrm{B}}|g\rangle = \langle g|\hat{V}_{\mathrm{AB}}|g\rangle = \langle g|\hat{H}|g\rangle = 0$$

$$f(h,k) = \frac{h^2}{\sqrt{h^2 + k^2}}$$

$$|g\rangle = \frac{1}{\sqrt{2}}(C^{-}|1_{\mathrm{A}}1_{\mathrm{B}}\rangle - C^{+}|0_{\mathrm{A}}0_{\mathrm{B}}\rangle)$$

$$C^{\pm} = \sqrt{1 \pm \frac{f(h,k)}{h}}$$

$$\hat{P}_{\mathrm{A}}(\alpha) = \frac{1}{2}(\mathbb{1} + \alpha\hat{\sigma}_x^{\mathrm{A}})$$

$$\hat{\rho}_1 = \sum_{\alpha=\pm 1} \hat{P}_{\mathrm{A}}(\alpha)|g\rangle\langle g|\hat{P}_{\mathrm{A}}(\alpha)$$

$$\begin{aligned} E_{P_{\mathrm{A}}} &= \mathrm{Tr}(\hat{\rho}_1\hat{H}) \\ &= \sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{A}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle \\ &+ \sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{B}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle \end{aligned}$$

$$\sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{B}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$

$$\begin{aligned} E_{P_{\mathrm{A}}} &= \mathrm{Tr}(\hat{\rho}_1\hat{H}) \\ &= \sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{A}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle \\ &+ \sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{B}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle \\ &+ \sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{V}_{\mathrm{AB}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle. \end{aligned}$$

$$\sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{B}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle = \frac{1}{2}\sum_{\alpha=\pm 1} \alpha h\langle g|\hat{\sigma}_z^{\mathrm{B}}\hat{\sigma}_x^{\mathrm{A}}|g\rangle + \alpha f(h,k)\langle g|\hat{\sigma}_x^{\mathrm{A}}|g\rangle = 0,$$

$$\sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{V}_{\mathrm{AB}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle = \sum_{\alpha=\pm 1} \alpha k\langle g|\hat{\sigma}_x^{\mathrm{B}}|g\rangle + \alpha\frac{k^2}{h^2}f(h,k)\langle g|\hat{\sigma}_x^{\mathrm{A}}|g\rangle = 0$$

$$\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{A}}\hat{P}_{\mathrm{A}}(\alpha) = \frac{f(h,k)}{4}((\alpha+2)\mathbb{1} + \alpha\hat{\sigma}_x^{\mathrm{A}}).$$

$$\sum_{\alpha=\pm 1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{A}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle = \frac{1}{2}\sum_{\alpha=\pm 1} f(h,k)\langle g|\mathbb{1}|g\rangle + \frac{1}{4}\sum_{\alpha=\pm 1} \alpha f(h,k)\langle g|\mathbb{1}|g\rangle$$

$$\sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{H}_{\mathrm{A}}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle =$$
$$\frac{1}{2}\sum_{\alpha=\pm1} f(h,k)\langle g|\mathbb{1}|g\rangle + \frac{1}{4}\sum_{\alpha=\pm1} \alpha f(h,k)\langle g|\mathbb{1}|g\rangle$$
$$+\frac{1}{4}\sum_{\alpha=\pm1} \alpha f(h,k)\langle g|\hat{\sigma}_x^{\mathrm{A}}|g\rangle = f(h,k) > 0$$

$$\hat{U}_{\mathrm{B}}(\alpha) = \cos(\theta)\mathbb{1} - \mathrm{i}\alpha\sin(\theta)\hat{\sigma}_y^{\mathrm{B}}$$

$$\cos(2\theta) = \frac{h^2 + 2k^2}{\sqrt{(h^2+2k^2)^2 + h^2k^2}},$$
$$\sin(2\theta) = \frac{hk}{\sqrt{(h^2+2k^2)^2 + h^2k^2}}.$$

$$\hat{\rho}_2 = \sum_{\alpha=\pm1} \hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle\langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)$$

$$E_{U_{\mathrm{B}}} := \mathrm{Tr}(\hat{\rho}_2\hat{H}) - \mathrm{Tr}(\hat{\rho}_1\hat{H})$$

$$\mathrm{Tr}(\hat{\rho}_2\hat{H}) = \sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{H}_{\mathrm{A}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$+\sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{H}_{\mathrm{B}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$+\sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{V}_{\mathrm{AB}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$

$$\mathrm{Tr}(\hat{\rho}_2\hat{H}) = \sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{H}_{\mathrm{A}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$+\sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{H}_{\mathrm{B}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$+\sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)\hat{V}_{\mathrm{AB}}\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$= E_{P_{\mathrm{A}}} + \sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)\hat{U}_{\mathrm{B}}^{\dagger}(\alpha)(\hat{H}_{\mathrm{B}} + \hat{V}_{\mathrm{AB}})\hat{U}_{\mathrm{B}}(\alpha)\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$

$$E_{U_{\mathrm{B}}} = \frac{-1}{\sqrt{h^2+k^2}}(hk\sin(2\theta) - (h^2+2k^2)(1-\cos(2\theta))$$

$$E_{U_{\mathrm{B}}} \approx \frac{-2hk\theta}{\sqrt{h^2+k^2}} < 0$$

$$\langle\hat{H}_{\mathrm{B}}(t)\rangle = \sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)e^{\mathrm{i}\hat{H}t}\hat{H}_{\mathrm{B}}e^{-\mathrm{i}\hat{H}t}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$

$$\langle\hat{H}_{\mathrm{B}}(t)\rangle = \sum_{\alpha=\pm1} \langle g|\hat{P}_{\mathrm{A}}(\alpha)e^{\mathrm{i}\hat{H}t}\hat{H}_{\mathrm{B}}e^{-\mathrm{i}\hat{H}t}\hat{P}_{\mathrm{A}}(\alpha)|g\rangle$$
$$= \frac{1}{2}f(h,k)(1-\cos(4kt))$$

$$\hat{\rho}_2^{W} = \hat{W}_{\mathrm{B}}\hat{\rho}_1\hat{W}_{\mathrm{B}}^{\dagger}$$

$$E_W = \mathrm{Tr}(\hat{\rho}_2^W \hat{H}) - \mathrm{Tr}(\hat{\rho}_1 \hat{H})$$

$$\mathrm{Tr}(\hat{\rho}_2^W \hat{H}) = E_{P_A} + \sum_{\alpha=\pm 1} \langle g|\hat{P}_A(\alpha)\hat{W}_B^\dagger(\hat{H}_B + \hat{V}_{AB})\hat{W}_B\hat{P}_A(\alpha)|g\rangle.$$

$$E_W = \langle g|\hat{W}_B^\dagger[\hat{H}_B + \hat{V}_{AB}]\hat{W}_B|g\rangle$$

$$\langle g|\hat{W}_B^\dagger \hat{H}_A \hat{W}_B|g\rangle = \langle g|\hat{H}_A|g\rangle = 0$$

$$E_W = \langle g|\hat{W}_B^\dagger[\hat{H}_A + \hat{H}_B + \hat{V}_{AB}]\hat{W}_B|g\rangle = \langle g|\hat{H}|g\rangle$$

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{V}_{AB}$$

$$\hat{V}_{AB} = 2k\hat{\sigma}_x^A\hat{\sigma}_x^B + \frac{4k^2}{h_A + h_B} f(h_A, h_B, k)\mathbb{1}$$

$$f(h_A, h_B, k) = \sqrt{\frac{(h_A + h_B)^2 + 1}{4k^2}}.$$

$$|g\rangle = \frac{1}{\sqrt{2}}(F_+|0_A 0_B\rangle - F_-|1_A 1_B\rangle),$$

$$|g\rangle = \frac{1}{\sqrt{2}}(F_+|0_A 0_B\rangle - F_-|1_A 1_B\rangle) \otimes |0_{A_n}\rangle.$$

$$\hat{U}_{A_n A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{U}_{A_n A}|0_{A_n}0_A\rangle = \frac{1}{\sqrt{2}}(|0_{A_n}0_A\rangle - |1_{A_n}1_A\rangle) = |\Phi^-_{A_n A}\rangle$$
$$\hat{U}_{A_n A}|1_{A_n}1_A\rangle = \frac{1}{\sqrt{2}}(|0_{A_n}0_A\rangle + |1_{A_n}1_A\rangle) = |\Phi^+_{A_n A}\rangle$$
$$\hat{U}_{A_n A}|0_{A_n}1_A\rangle = \frac{1}{\sqrt{2}}(|0_{A_n}1_A\rangle - |1_{A_n}0_A\rangle) = |\Psi^-_{A_n A}\rangle$$
$$\hat{U}_{A_n A}|1_{A_n}0_A\rangle = \frac{1}{\sqrt{2}}(|0_{A_n}1_A\rangle + |1_{A_n}0_A\rangle) = |\Psi^+_{A_n A}\rangle$$

$$|\Psi_{BA_n A}\rangle = \frac{1}{\sqrt{2}}(F_+|0_B\rangle|\Phi^-_{A_n A}\rangle - F_-|1_B\rangle|\Psi^-_{A_n A}\rangle).$$

$$\begin{aligned}(Z \otimes \mathbb{1})C_{\mathrm{NOT}}(H \otimes \mathbb{1})C_{\mathrm{NOT}}|0_{A_n}0_A\rangle \\ &= (Z \otimes \mathbb{1})C_{\mathrm{NOT}}(H \otimes \mathbb{1})|0_{A_n}0_A\rangle \\ &= \frac{1}{\sqrt{2}}(Z \otimes \mathbb{1})C_{\mathrm{NOT}}(|0_{A_n}0_A\rangle + |1_{A_n}0_A\rangle)\end{aligned}$$

$$\begin{aligned}(Z \otimes \mathbb{1})C_{\mathrm{NOT}}(H \otimes \mathbb{1})C_{\mathrm{NOT}}|0_{A_n}1_A\rangle \\ &= (Z \otimes \mathbb{1})C_{\mathrm{NOT}}(H \otimes \mathbb{1})|0_{A_n}1_A\rangle \\ &= \frac{1}{\sqrt{2}}(Z \otimes \mathbb{1})C_{\mathrm{NOT}}(|0_{A_n}1_A\rangle + |1_{A_n}1_A\rangle)\end{aligned}$$

$$
\begin{aligned}
&(Z\otimes\mathbb{1})C_{\mathrm{NOT}}(H\otimes\mathbb{1})C_{\mathrm{NOT}}|0_{\mathrm{A}_n}0_{\mathrm{A}}\rangle\\
&\qquad=(Z\otimes\mathbb{1})C_{\mathrm{NOT}}(H\otimes\mathbb{1})|0_{\mathrm{A}_n}0_{\mathrm{A}}\rangle\\
&\qquad=\frac{1}{\sqrt{2}}(Z\otimes\mathbb{1})C_{\mathrm{NOT}}\left(|0_{\mathrm{A}_n}0_{\mathrm{A}}\rangle+|1_{\mathrm{A}_n}0_{\mathrm{A}}\rangle\right)\\
&\qquad=\frac{1}{\sqrt{2}}\left(|0_{\mathrm{A}_n}0_{\mathrm{A}}\rangle-|1_{\mathrm{A}_n}1_{\mathrm{A}}\rangle\right)=|\Phi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&(Z\otimes\mathbb{1})C_{\mathrm{NOT}}(H\otimes\mathbb{1})C_{\mathrm{NOT}}|0_{\mathrm{A}_n}1_{\mathrm{A}}\rangle\\
&\qquad=(Z\otimes\mathbb{1})C_{\mathrm{NOT}}(H\otimes\mathbb{1})|0_{\mathrm{A}_n}1_{\mathrm{A}}\rangle\\
&\qquad=\frac{1}{\sqrt{2}}(Z\otimes\mathbb{1})C_{\mathrm{NOT}}\left(|0_{\mathrm{A}_n}1_{\mathrm{A}}\rangle+|1_{\mathrm{A}_n}1_{\mathrm{A}}\rangle\right)\\
&\qquad=\frac{1}{\sqrt{2}}\left(|0_{\mathrm{A}_n}1_{\mathrm{A}}\rangle-|1_{\mathrm{A}_n}0_{\mathrm{A}}\rangle\right)=|\Psi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle.
\end{aligned}
$$

$$\hat{U}_{\mathrm{BA}_n}=\hat{U}_{\mathrm{B}}(0)\otimes|0_{\mathrm{A}_n}\rangle\langle 0_{\mathrm{A}_n}|+\hat{U}_{\mathrm{B}}(1)\otimes|1_{\mathrm{A}_n}\rangle\langle 1_{\mathrm{A}_n}|.$$

$$\hat{\rho}_{\mathrm{B}}=\mathrm{Tr}_{\mathrm{AA}_n}\left(\hat{U}_{\mathrm{BA}_n}|\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle\langle\Psi_{\mathrm{BA}_n\,\mathrm{A}}|\hat{U}^{\dagger}_{\mathrm{BA}_n}\right)$$

$$\langle 0_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle=\frac{1}{\sqrt{2}}\left(F_{+}|0_{\mathrm{B}}\rangle\langle 0_{\mathrm{A}_n}\mid\Phi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle-F_{-}|1_{\mathrm{B}}\rangle\langle 0_{\mathrm{A}_n}\mid\Psi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle\right)$$

$$
\begin{aligned}
\langle 1_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle&=\frac{1}{\sqrt{2}}\left(F_{+}|0_{\mathrm{B}}\rangle\langle 1_{\mathrm{A}_n}\mid\Phi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle-F_{-}|1_{\mathrm{B}}\rangle\langle 1_{\mathrm{A}_n}\mid\Psi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle\right)\\
&=-\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(F_{+}|1_{\mathrm{A}}0_{\mathrm{B}}\rangle-F_{-}|0_{\mathrm{A}}1_{\mathrm{B}}\rangle)\right]\\
&=-\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(F_{+}\hat{\sigma}^{\mathrm{A}}_{x}|0_{\mathrm{A}}0_{\mathrm{B}}\rangle-F_{-}\hat{\sigma}^{\mathrm{A}}_{x}|1_{\mathrm{A}}1_{\mathrm{B}}\rangle)\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_{\mathrm{B}}&=\mathrm{Tr}_{\mathrm{AA}_n}\left(\hat{U}_{\mathrm{BA}_n}|\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle\langle\Psi_{\mathrm{BA}_n\,\mathrm{A}}|\hat{U}^{\dagger}_{\mathrm{BA}_n}\right)\\
&=\mathrm{Tr}_{\mathrm{A}}\left[\sum_{\alpha}\hat{U}_{\mathrm{B}}(\alpha)\langle\alpha_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle\langle\Psi_{\mathrm{BA}_n\,\mathrm{A}}\mid\alpha_{\mathrm{A}_n}\rangle\hat{U}^{\dagger}_{\mathrm{B}}(\alpha)\right],
\end{aligned}
$$

$$
\begin{aligned}
\langle 0_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle&=\frac{1}{\sqrt{2}}\left(F_{+}|0_{\mathrm{B}}\rangle\langle 0_{\mathrm{A}_n}\mid\Phi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle-F_{-}|1_{\mathrm{B}}\rangle\langle 0_{\mathrm{A}_n}\mid\Psi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle\right)\\
&=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(F_{+}|0_{\mathrm{A}}0_{\mathrm{B}}\rangle-F_{-}|1_{\mathrm{A}}1_{\mathrm{B}}\rangle)\right]=\frac{1}{\sqrt{2}}|g\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\langle 1_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle&=\frac{1}{\sqrt{2}}\left(F_{+}|0_{\mathrm{B}}\rangle\langle 1_{\mathrm{A}_n}\mid\Phi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle-F_{-}|1_{\mathrm{B}}\rangle\langle 1_{\mathrm{A}_n}\mid\Psi^{-}_{\mathrm{A}_n\,\mathrm{A}}\rangle\right)\\
&=-\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(F_{+}|1_{\mathrm{A}}0_{\mathrm{B}}\rangle-F_{-}|0_{\mathrm{A}}1_{\mathrm{B}}\rangle)\right]\\
&=-\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(F_{+}\hat{\sigma}^{\mathrm{A}}_{x}|0_{\mathrm{A}}0_{\mathrm{B}}\rangle-F_{-}\hat{\sigma}^{\mathrm{A}}_{x}|1_{\mathrm{A}}1_{\mathrm{B}}\rangle)\right]\\
&=-\frac{1}{\sqrt{2}}\hat{\sigma}^{\mathrm{A}}_{x}|g\rangle
\end{aligned}
$$

$$\langle\alpha_{\mathrm{A}_n}\mid\Psi_{\mathrm{BA}_n\,\mathrm{A}}\rangle=\frac{1}{\sqrt{2}}(\mathbb{1}-\alpha\hat{\sigma}^{\mathrm{A}}_{x})|g\rangle,$$

$$\hat{\rho}_{\mathrm{B}}=\mathrm{Tr}_{\mathrm{A}}\left[\hat{U}_{\mathrm{B}}(\alpha)\left(\frac{\mathbb{1}-\alpha\hat{\sigma}^{\mathrm{A}}_{x}}{\sqrt{2}}\right)|g\rangle\langle g|\left(\frac{\mathbb{1}-\alpha\hat{\sigma}^{\mathrm{A}}_{x}}{\sqrt{2}}\right)\hat{U}^{\dagger}_{\mathrm{B}}(\alpha)\right]$$

$$E_{U_{\mathrm{B}}}=\mathrm{Tr}\left[(\hat{H}_{\mathrm{B}}+\hat{V}_{\mathrm{AB}})\hat{\rho}_{\mathrm{B}}\right]\leq 0.$$

$$E_0^{\mathrm{AB}} \leq E_{U_\mathrm{B}} \leq 0,$$

$$E_{U_\mathrm{B}} \leq \sqrt{h_\mathrm{B}^2 + 4k^2} - \frac{h_\mathrm{B}(h_\mathrm{A} + h_\mathrm{B}) + 4k^2}{\sqrt{(h_\mathrm{A} + h_\mathrm{B})^2 + 4k^2}}$$

$$t_{\mathrm{QET}} \ll t_c = \frac{1}{1.16}\ \mathrm{s} \approx 862\ \mathrm{ms}$$

$$\hat{Y}(\theta) = \cos(\theta)\mathbb{1} - \mathrm{i}\sin(\theta)\hat{\sigma}_y^{\mathrm{B}},$$

$$\cos(\theta) = \frac{F_+}{\sqrt{2}}, \sin(\theta) = \frac{F_-}{\sqrt{2}},$$

$$F_\pm = \sqrt{1 \pm \frac{h_\mathrm{A} + h_\mathrm{B}}{\sqrt{4k^2 + (h_\mathrm{A} + h_\mathrm{B})^2}}}$$

$$\hat{U}_\mathrm{p}|0_\mathrm{A}0_{\mathrm{A}_n}0_\mathrm{B}\rangle = C_{\mathrm{NOT}} \circ Y(\theta)|0_\mathrm{A}0_{\mathrm{A}_n}0_\mathrm{B}\rangle$$

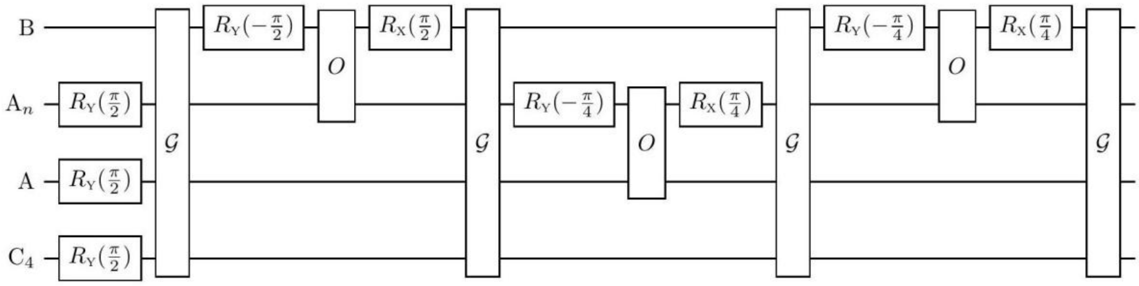
$$\begin{aligned}\hat{U}_\mathrm{p}|0_\mathrm{A}0_{\mathrm{A}_n}0_\mathrm{B}\rangle &= C_{\mathrm{NOT}} \circ Y(\theta)|0_\mathrm{A}0_{\mathrm{A}_n}0_\mathrm{B}\rangle \\ &= -\frac{1}{\sqrt{2}}(F_+|0_\mathrm{A}0_\mathrm{B}\rangle - F_-|1_\mathrm{A}1_\mathrm{B}\rangle) \otimes |0_{\mathrm{A}_n}\rangle.\end{aligned}$$

$$\hat{H} = \hat{H}_\mathrm{z} + \hat{H}_\mathrm{J}$$

$$\hat{H}_\mathrm{Z} = \pi \sum_j \omega_j \hat{\sigma}_z^j$$

$$\hat{H}_\mathrm{J} = \frac{\pi}{2} \sum_{j,k}^{j<k} J_{jk} \hat{\boldsymbol{\sigma}}^j \cdot \hat{\boldsymbol{\sigma}}^k$$

$$\hat{\rho}_T = \frac{1}{2^n}\exp(-\beta\hat{H}) \approx \frac{1}{2^n}(\mathbb{1} + \xi\bar{\rho}_T)$$



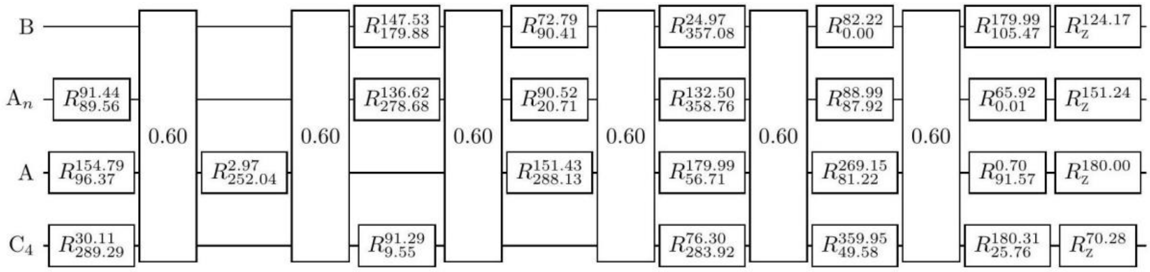
$$\hat{O} = e^{-\mathrm{i}\frac{\pi}{4}\hat{\sigma}_z^j\hat{\sigma}_z^k}$$

$$\hat{R}_\phi^\theta = \exp\left(-\mathrm{i}\theta\left(\cos(\phi)\hat{\sigma}_x + \sin(\phi)\hat{\sigma}_y\right)/2\right)$$
$$\hat{R}_\mathrm{z}^\theta = \exp(-\mathrm{i}\theta\hat{\sigma}_z/2)$$

$$\hat{U}_{\mathrm{BA}_n} = \hat{U}_{\mathrm{rot}}\hat{U}_{\mathrm{diag}}$$

$$\hat{U}_{\mathrm{rot}} = \frac{1}{\sqrt{2}}\begin{pmatrix} F_{2+} & F_{2-} & 0 & 0 \\ 0 & 0 & -F_{2+} & F_{2-} \\ 0 & 0 & F_{2-} & F_{2+} \\ -F_{2-} & F_{2+} & 0 & 0 \end{pmatrix},$$

$$\hat{U}_{\mathrm{diag}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & F_{+} & F_{-} & 0 \\ F_{-} & 0 & 0 & -F_{+} \\ F_{+} & 0 & 0 & F_{-} \\ 0 & -F_{-} & F_{+} & 0 \end{pmatrix},$$



$$t_{\mathrm{tot}} = t_{\mathrm{A}_n\,\mathrm{A}} + t_{\mathrm{A}_n\,\mathrm{B}} + t_p \approx 37.6 \text{ ms}$$

$$t_{\mathrm{QET}} < t_{\mathrm{tot}} \approx 37.6 \text{ ms} \ll t_{\mathrm{AB}} \approx 862 \text{ ms}$$

$$\begin{aligned} |g\rangle &= C_{\mathrm{NOT}}\big(\hat{R}_{\mathrm{Y,\,A}}(2\theta) \otimes \mathbb{1}_{\mathrm{B}}\big)|0_{\mathrm{A}}0_{\mathrm{B}}\rangle \\ &= \cos(\theta)|0_{\mathrm{A}}0_{\mathrm{B}}\rangle + \sin(\theta)|1_{\mathrm{A}}1_{\mathrm{B}}\rangle \end{aligned}$$

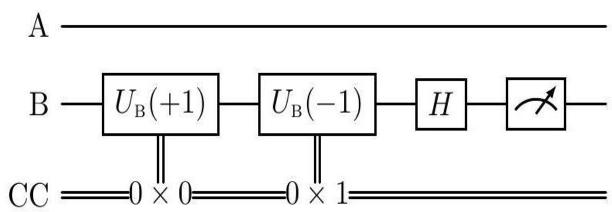
$$\begin{aligned} |g\rangle &= C_{\mathrm{NOT}}\big(\hat{R}_{\mathrm{Y,\,A}}(2\theta) \otimes \mathbb{1}_{\mathrm{B}}\big)|0_{\mathrm{A}}0_{\mathrm{B}}\rangle \\ &= \cos(\theta)|0_{\mathrm{A}}0_{\mathrm{B}}\rangle + \sin(\theta)|1_{\mathrm{A}}1_{\mathrm{B}}\rangle \\ &= \frac{1}{\sqrt{2}}\Big(\sqrt{1-g(h,k)}|0_{\mathrm{A}}0_{\mathrm{B}}\rangle - \sqrt{1+g(h,k)}|1_{\mathrm{A}}1_{\mathrm{B}}\rangle\Big) \end{aligned}$$

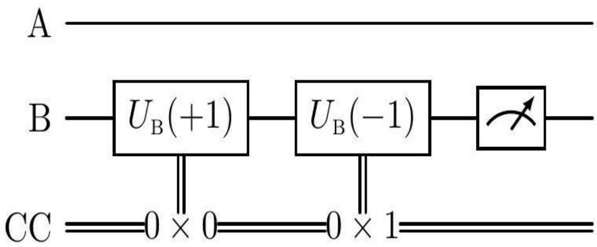
$$\theta = -\arccos\left(\frac{1}{\sqrt{2}}\sqrt{1-g(h,k)}\right)$$

$$g(h,k) = \frac{h}{\sqrt{h^2+k^2}}$$

$$\hat{U}_{\mathrm{B}}(\alpha) = \cos(\phi)\mathbb{1} - \mathrm{i}\alpha\sin(\phi)\hat{\sigma}_y^{\mathrm{B}} = \hat{R}_{\mathrm{Y,\,B}}(2\alpha\phi),$$

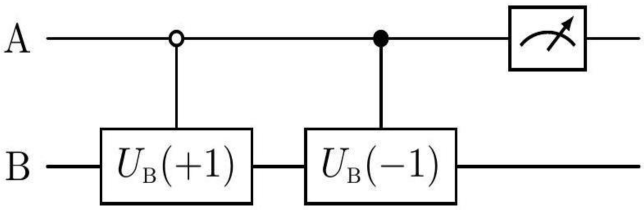
$$\begin{aligned} \cos(2\phi) &= \frac{h^2+2k^2}{\sqrt{(h^2+2k^2)^2+h^2k^2}}, \\ \sin(2\phi) &= \frac{hk}{\sqrt{(h^2+2k^2)^2+h^2k^2}}. \end{aligned}$$

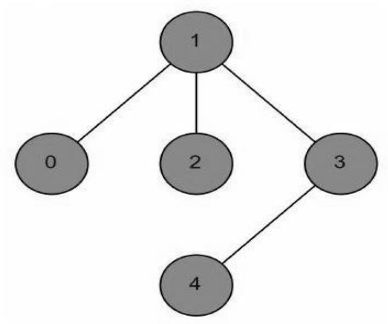


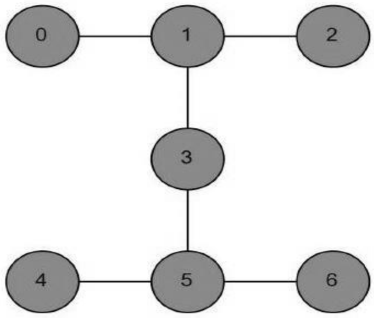


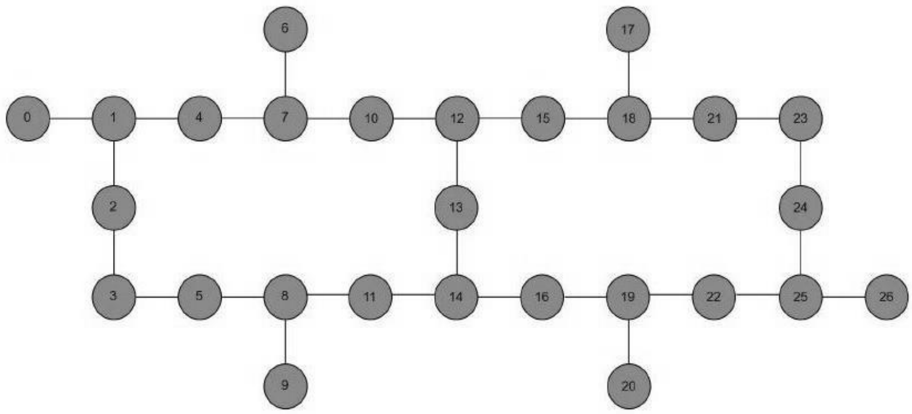
$$\hat{\Lambda}(U) = |0_A\rangle\langle 0_A| \otimes \mathbb{1}_B + |1_A\rangle\langle 1_A| \otimes \hat{U}_B$$

$$\hat{\bar{\Lambda}}(U) = (\hat{\sigma}_x^A \otimes \mathbb{1}_B)\hat{\Lambda}(U)(\hat{\sigma}_x^A \otimes \mathbb{1}_B)$$









		(h, k)			
Backend	Mode				
		(1,0.2)	(1,0.5)	(1,1)	(1.5,1)

Analytical value		E_{P_A}	0.9806	0.8944	0.7071	1.2481
qasm			0.9827 $\pm$ 0.0031	0.8941 $\pm$ 0.0001	0.7088 $\pm$ 0.0001	1.2437 $\pm$ 0.0047
ibmq	error mitigated		0.9423 $\pm$ 0.0032	0.8169 $\pm$ 0.0032	0.6560 $\pm$ 0.0031	1.2480 $\pm$ 0.0047
	unmitigated		0.9049 $\pm$ 0.0017	0.8550 $\pm$ 0.0032	0.6874 $\pm$ 0.0031	1.4066 $\pm$ 0.0047
ibmq	error mitigated		0.9299 $\pm$ 0.0056	0.8888 $\pm$ 0.0056	0.7039 $\pm$ 0.0056	1.2318 $\pm$ 0.0084
	unmitigated		0.9542 $\pm$ 0.0056	0.9089 $\pm$ 0.0056	0.7232 $\pm$ 0.0056	1.2624 $\pm$ 0.0083
ibm	error mitigated		0.9571 $\pm$ 0.0032	0.8626 $\pm$ 0.0031	0.7277 $\pm$ 0.0031	1.2072 $\pm$ 0.0047
	unmitigated		0.9578 $\pm$ 0.0031	0.8735 $\pm$ 0.0031	0.7362 $\pm$ 0.0031	1.2236 $\pm$ 0.0047
Analytical value		$\langle \hat{H}_B \rangle$	0.0521	0.1873	0.2598	0.3480
qasm			0.0547 $\pm$ 0.0012	0.1857 $\pm$ 0.0022	0.2550 $\pm$ 0.0028	0.3487 $\pm$ 0.0038
ibmq	error mitigated		0.0733 $\pm$ 0.0032	0.1934 $\pm$ 0.0032	0.2526 $\pm$ 0.0032	0.3590 $\pm$ 0.0047

	unmitigated	0.1295 ± 0.0053	0.2422 ± 0.0024	0.2949 ± 0.0028	0.4302 ± 0.0039
ibmq	error mitigated	0.0736 ± 0.0055	0.2018 ± 0.0056	0.2491 ± 0.0056	0.3390 ± 0.0084
	unmitigated	0.0852 ± 0.0022	0.2975 ± 0.0045	0.3365 ± 0.0052	0.4871 ± 0.0073
ibm	error mitigated	0.0674 ± 0.0032	0.1653 ± 0.0031	0.2579 ± 0.0031	0.3559 ± 0.0047
	unmitigated	0.0905 ± 0.0014	0.1825 ± 0.0022	0.2630 ± 0.0027	0.3737 ± 0.0037
Analytical value	$\langle \hat{V}_{\mathrm{AB}} \rangle$	-0.0701	-0.2598	-0.3746	-0.4905
qasm		−0.0708 ± 0.0012	−0.2608 ± 0.0032	−0.3729 ± 0.0063	−0.4921 ± 0.0038
ibmq	error mitigated	−0.0655 ± 0.0012	−0.2041 ± 0.0031	−0.2744 ± 0.0063	−0.4091 ± 0.0063
	unmitigated	−0.0538 ± 0.0011	−0.1471 ± 0.0025	−0.1233 ± 0.0041	−0.2737 ± 0.0046
ibmq	error mitigated	0.0515 ± 0.0022	−0.2348 ± 0.0056	−0.3255 ± 0.0112	−0.4469 ± 0.0112
	unmitigated	−0.0338 ± 0.0021	−0.1371 ± 0.0046	−0.0750 ± 0.0075	−0.2229 ± 0.0083
ibm	error mitigated	0.0497 ± 0.0013	−0.1968 ± 0.0031	−0.2569 ± 0.0063	−0.3804 ± 0.0063

	unmitigated	0.0471 ± 0.0012	−0.1682 ± 0.0026	−0.1733 ± 0.0038	−0.3089 ± 0.0045
Analytical value	$E_{U_{\mathrm{B}}}$	-0.0180	-0.0726	-0.1147	-0.1425
qasm_		−0.0161 ± 0.0017	−0.0751 ± 0.0040	−0.1179 ± 0.0069	0.1433 ± 0.0054
	error mitigated	0.0078 ± 0.0034	−0.0107 ± 0.0045	0.0217 ± 0.0071	0.0501 ± 0.0079
ibmq_	unmitigated	0.0757 ± 0.0054	0.0950 ± 0.0035	0.1715 ± 0.0050	0.1565 ± 0.0060
	error mitigated	0.0221 ± 0.0059	−0.0330 ± 0.0079	−0.0764 ± 0.0125	0.1079 ± 0.0140
ibmq	unmitigated	0.0514 ± 0.0030	0.1604 ± 0.0064	0.2615 ± 0.0091	0.2642 ± 0.0011
	error mitigated	0.0177 ± 0.0035	−0.0315 ± 0.0044	0.0010 ± 0.0070	0.0245 ± 0.0079
ibm	unmitigated	0.0433 ± 0.0018	0.0143 ± 0.0034	0.0897 ± 0.0047	0.0648 ± 0.0058

$$\hat{M}_{\mathrm{A}}(\alpha) = e^{\mathrm{i}\delta_\alpha}(m_\alpha \mathbb{1} + e^{\mathrm{i}\gamma_\alpha} l_\alpha \hat{\sigma}_x^{\mathrm{A}})$$

$$\sum_\alpha (m_\alpha^2 + l_\alpha^2) = 1,$$

$$\sum_\alpha m_\alpha l_\alpha \cos(\alpha) = 0.$$

$$\hat{U}_{\mathrm{B}}(\alpha) = \cos(\Omega_\alpha)\mathbb{1} + \mathrm{i}\sin(\Omega_\alpha)\hat{\sigma}_y^{\mathrm{B}}$$

$$\cos(2\Omega_\alpha) = \frac{(h^2+2k^2)p_A(\alpha)}{\sqrt{(h^2+2k^2)^2 p_A^2(\alpha) + h^2k^2 q_A^2(\alpha)}},$$
$$\sin(2\Omega_\alpha) = -\frac{hkq_A(\alpha)}{\sqrt{(h^2+2k^2)^2 p_A^2(\alpha) + h^2k^2 q_A^2(\alpha)}},$$

$$\mathcal{P}_0^{\mathrm{B}} = \mathrm{Tr}(\hat{\rho}_{\mathrm{B}}^2),$$

$$\hat{\rho}_{\mathrm{B}} = \frac{1}{2}(C_-^2|1_{\mathrm{B}}\rangle\langle 1_{\mathrm{B}}| + C_+^2|0_{\mathrm{A}}\rangle\langle 0_{\mathrm{B}}|)$$

$$\mathcal{P}_0^{\mathrm{B}} = \frac{2h^2+k^2}{2(h^2+k^2)}$$

$$\hat{\rho}_f = \sum_{\alpha=\pm1} \hat{U}_{\mathrm{B}}(\alpha)\hat{M}_{\mathrm{A}}(\alpha)|g\rangle\langle g|\hat{M}_{\mathrm{A}}^\dagger(\alpha)\hat{U}_{\mathrm{B}}^\dagger(\alpha)$$

$$\mathcal{P}_f^{\mathrm{B}} = \frac{2}{h^2+k^2}\left(\frac{h^2}{2}+\frac{k^2}{4} - hkl_1m_1\sin\left(2(\Omega_0-\Omega_1)\right)\right.$$
$$\left. + \left(4k^2l_1^2m_1^2 + h^2(l_1^2+m_1^2-1)(l_1^2+m_1^2)\right)\sin^2(\Omega_0-\Omega_1)\right)$$

$$\hat{\rho}_\beta = \frac{1}{\mathrm{Tr}\left(e^{-\beta\hat{H}}\right)} e^{-\beta\hat{H}}$$

$$\hat{U}_{\mathrm{A}} = e^{\mathrm{i}\hat{\sigma}_y^A\hat{\sigma}_y^{An}}$$

$$\hat{U}_{\mathrm{B}} = e^{\mathrm{i}\hat{\sigma}_x^{\mathrm{B}}\hat{\sigma}_z^{An}}$$

$$\hat{H}_{\mathrm{A}_n} = h_{\mathrm{A}_n}\hat{\sigma}_z^{\mathrm{A}n}$$

$$\mathcal{P}_f^{\mathrm{B}} = \frac{1}{2} + \frac{h_- S_+^2\left[(h_{\mathrm{A}}^2+h_{\mathrm{B}}^2) + k^2\sin^4(2)\tanh^2\left(\beta h_{\mathrm{A}_n}\right)\right]}{2h_-h_+(C_-+C_+)^2}$$
$$+ \frac{S_-^2\left[h_+\left[(h_{\mathrm{A}}-h_{\mathrm{B}})^2 + k^2\sin^4(2)\tanh^2\left(\beta h_{\mathrm{A}_n}\right)\right] + 2h_{\mathrm{B}}^2h_r\right]}{2h_-h_+(C_-+C_+)^2}$$

$$\mathcal{P}_f^{\mathrm{B}} = \frac{1}{2} + \frac{h_- S_+^2\left[(h_{\mathrm{A}}^2+h_{\mathrm{B}}^2) + k^2\sin^4(2)\tanh^2\left(\beta h_{\mathrm{A}_n}\right)\right]}{2h_-h_+(C_-+C_+)^2}$$
$$+ \frac{S_-^2\left[h_+\left[(h_{\mathrm{A}}-h_{\mathrm{B}})^2 + k^2\sin^4(2)\tanh^2\left(\beta h_{\mathrm{A}_n}\right)\right] + 2h_{\mathrm{B}}^2h_r\right]}{2h_-h_+(C_-+C_+)^2}$$
$$- \frac{2h_rS_+S_-\left[h_{\mathrm{A}}^2 + k^2\sin^4(2)\tanh^2\left(\beta h_{\mathrm{A}_n}\right)\right]}{2h_-h_+(C_-+C_+)^2}$$

$$h_\pm := (h_{\mathrm{A}} \pm h_{\mathrm{B}})^2 + k^2, \quad h_r := \sqrt{\frac{1}{2}(h_-^2+h_+^2) - 8h_{\mathrm{A}}^2h_{\mathrm{B}}^2}$$
$$S_\pm := \sinh\left(\sqrt{h_\pm\beta}\right), \qquad C_\pm := \cosh\left(\sqrt{h_\pm\beta}\right)$$

$$\hat{U}_{\mathrm{A}} = e^{\mathrm{i}\hat{H}_{\mathrm{probe}}^{\mathrm{A}}}, \qquad \hat{U}_{\mathrm{B}} = e^{\mathrm{i}\hat{H}_{\mathrm{probe}}^{\mathrm{B}}},$$
$$H_{\mathrm{probe}}^{\mathrm{A}} = \sum_{i,j} \hat{\sigma}_i^{\mathrm{A}}\mathcal{J}^{ij}\hat{\sigma}_j^{\mathrm{A}n}, \quad H_{\mathrm{probe}}^{\mathrm{B}} = \sum_{i,j} \hat{\sigma}_i^{\mathrm{B}}\mathcal{K}^{ij}\hat{\sigma}_j^{\mathrm{A}n},$$

$$\hat{H}_{\mathrm{I}} = \delta(t-t_0)\hat{\sigma}_x^{\mathrm{A}} \int \mathrm{d}x\lambda(x)\hat{\pi}(\mathrm{x})$$

$$\hat{H}_{\mathrm{I}} = \delta(t-T-t_0)\hat{\sigma}_z^{\mathrm{B}} \int \mathrm{d}x\mu(x)\hat{\phi}(\mathrm{x})$$

$$\hat{T}_{\mu\nu}(\mathrm{x}) = \partial_\mu\hat{\phi}(\mathrm{x})\partial_\nu\hat{\phi}(\mathrm{x}) - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\rho\hat{\phi}(\mathrm{x})\partial^\rho\hat{\phi}(\mathrm{x})\right)$$

$$|\psi(0)\rangle = |0\rangle \otimes |A_0\rangle$$

$$|\psi(t)\rangle = e^{-\mathrm{i}\hat{\sigma}_x^A \int \mathrm{d}x\lambda(x)\hat{\pi}(\mathrm{x})}|0\rangle \otimes |A_0\rangle$$

$$\begin{aligned} |\psi(t)\rangle &= e^{-\mathrm{i}\hat{\sigma}_x^A \int \mathrm{d}x\lambda(x)\hat{\pi}(\mathrm{x})}|0\rangle \otimes |A_0\rangle \\ &= |\boldsymbol{\alpha}(t)\rangle|+\rangle\langle+| A_0\rangle + |-\boldsymbol{\alpha}(t)\rangle|-\rangle\langle-| A_0\rangle \end{aligned}$$

$$\begin{aligned} \alpha_k(t) &= e^{-\mathrm{i}|k|t}\sqrt{\frac{|k|}{4\pi}}\int \mathrm{d}x\lambda(x)e^{\mathrm{i}kx} \\ |\boldsymbol{\alpha}(t)\rangle &= \bigotimes_{k\in\mathbb{R}} |\alpha_k(t)\rangle \end{aligned}$$

$$\langle\psi(t)|{:}\hat{T}_{00}(\mathrm{x}){:}|\psi(t)\rangle = \frac{1}{4}(\lambda'(x-t))^2 + \frac{1}{4}(\lambda'(x+t))^2$$

$$\langle\psi(t)|{:}\hat{T}_{00}(\mathrm{x}){:}|\psi(t)\rangle := \langle\psi(t)|\hat{T}_{00}(\mathrm{x})|\psi(t)\rangle - \langle 0|\hat{T}_{00}(\mathrm{x})|0\rangle.$$

$$\begin{aligned} &|\psi(t)\rangle = \\ &\hat{D}(\boldsymbol{\beta}(t))\left(\frac{\langle+| A_0\rangle}{\sqrt{2}}|\boldsymbol{\alpha}(t)\rangle|e\rangle + \frac{\langle-| A_0\rangle}{\sqrt{2}}|-\boldsymbol{\alpha}(t)\rangle|e\rangle\right) \\ &+\hat{D}(-\boldsymbol{\beta}(t))\left(\frac{\langle+| A_0\rangle}{\sqrt{2}}|\boldsymbol{\alpha}(t)\rangle|g\rangle - \frac{\langle-| A_0\rangle}{\sqrt{2}}|-\boldsymbol{\alpha}(t)\rangle|g\rangle\right) \end{aligned}$$

$$\beta_k(t) = -\frac{\mathrm{i}e^{-\mathrm{i}|k|(t-T)}}{\sqrt{4\pi|k|}}\int \mathrm{d}x\mu(x)e^{\mathrm{i}kx}$$

$$\begin{aligned} \langle{:}\hat{T}_{00}(x,t){:}\rangle &= \frac{(\lambda'(x-t))^2}{4} + \frac{(\lambda'(x+t))^2}{4} + \frac{(\mu(x-(t-T)))^2}{4} + \frac{(\mu(x+(t-T)))^2}{4} \\ &+ \underbrace{\frac{e^{-2\|\alpha\|}\left(A_0\middle|\hat{\sigma}_y\middle|A_0\right)}{2\pi}\mu(x-(t-T))\int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x+t}}_{\text{Right moving QET term}} + \underbrace{\frac{e^{-2\|\alpha\|}\langle A_0|\hat{\sigma}_y|A_0\rangle}{2\pi}\mu(x+(t-T))\int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x-t}}_{\text{Left moving QET term}}. \end{aligned}$$

$$\int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x+t} = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{x-t-\epsilon} + \int_{x-t+\epsilon}^{\infty}\right]\mathrm{d}y\frac{\lambda'(y)}{y-x+t}$$

$$\left[\int_a^b + \int_c^d\right]\mathrm{d}xf(x) := \int_a^b \mathrm{d}xf(x) + \int_c^d \mathrm{d}xf(x)$$

$$\begin{aligned} \int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x+t} &= \left[\int_{-\infty}^{x-t-a} + \int_{x-t+a}^{\infty}\right]\mathrm{d}y\frac{\lambda'(y)}{y-x+t} \\ &+\lim_{\epsilon\to 0}\left[\int_{x-t-a}^{x-t-\epsilon} + \int_{x-t+\epsilon}^{x-t+a}\right]\mathrm{d}y\frac{\lambda'(y)}{y-x+t} \end{aligned}$$

$$\int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x+t} = \left[\int_{-\infty}^{x-t-a} + \int_{x-t+a}^{\infty}\right]\mathrm{d}y\frac{\lambda'(y)}{y-x+t}$$

$$\int \mathrm{d}y\lambda'(y)\frac{\mathrm{P.P}}{y-x+t} = \left[\int_{-\infty}^{x-t-a} + \int_{x-t+a}^{\infty}\right]\mathrm{d}y\frac{\lambda'(y)}{y-x+t} + 2a\lambda''(x-t) + \frac{a^3}{9}\mathcal{O}\left(\lambda^{(4)}(\xi)\right)$$

$$f(z,\sigma,\delta) = \begin{cases} S\left(\frac{\sigma/2+\pi\delta+z}{\delta}\right) & -\pi\delta < z+\sigma/2 < 0 \\ 1 & -\sigma/2 \le z \le \sigma/2 \\ S\left(\frac{\sigma/2+\pi\delta-z}{\delta}\right) & 0 < z-\sigma/2 < \pi\delta \\ 0 & \text{otherwise} \end{cases}$$

$$g(z,\delta) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2\delta^2}}$$

$$h(z,\delta) = \frac{1}{\pi}\frac{1}{1+\left(\frac{z}{\delta}\right)^2}$$

$$\lambda(x) = \lambda_0 f(x,\sigma_{\mathrm{A}},\delta_{\mathrm{A}}), \mu(x) = \mu_0 f(x-x_{\mathrm{B}},\sigma_{\mathrm{B}},\delta_{\mathrm{B}}),$$
$$\lambda(x) = \lambda_0 g(x,\delta_{\mathrm{A}}), \mu(x) = \mu_0 g(x-x_{\mathrm{B}},\delta_{\mathrm{B}}),$$
$$\lambda(x) = \lambda_0 h(x,\delta_{\mathrm{A}}), \mu(x) = \mu_0 h(x-x_{\mathrm{B}},\delta_{\mathrm{B}}),$$

$$\langle\psi(T+\Delta T)|{:}\hat{T}_{\mu\nu}{:}(x)|\psi(T+\Delta T)\rangle = \frac{1}{4^4\pi^6}[\underbrace{\left(I_\mu^1 I_\nu^1 - \eta_{\mu\nu}\frac{I_\lambda^1 I^{1,\lambda}}{2}\right)}_{\text{QET positive contribution}} - \underbrace{\left(I_\mu^2 I_\nu^2 - \eta_{\mu\nu}\frac{I_\lambda^2 I^{2,\lambda}}{2}\right)}_{\text{QET neutral contribution}}$$
$$-e^{-2\|\alpha\|}\langle A_0|\hat{\sigma}_y|A_0\rangle \underbrace{\left(\left(I_\mu^1 I_\nu^3 - \eta_{\mu\nu}\frac{I_\lambda^1 I^{3,\lambda}}{2}\right) + \left(I_\mu^3 I_\nu^1 - \eta_{\mu\nu}\frac{I_\lambda^1 I^{3,\lambda}}{2}\right)\right)}_{\text{QET negative contribution}}]$$

$$I_\mu^1 = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{k}\tilde{k}_\mu\left(e^{|\boldsymbol{k}|(-2\varepsilon+\mathrm{i}\Delta T)+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})} + e^{|\boldsymbol{k}|(-2\varepsilon-\mathrm{i}\Delta T)-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})}\right)\mu(\boldsymbol{r})$$
$$I_\mu^2 = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{k}\tilde{k}_\mu\left(e^{|\boldsymbol{k}|(-2\varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})} - e^{|\boldsymbol{k}|(-2\varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})}\right)|\boldsymbol{k}|\lambda(\boldsymbol{r})$$
$$I_\mu^3 = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{k}\tilde{k}_\mu\left(e^{|\boldsymbol{k}|(-2\varepsilon-\mathrm{i}(\Delta T+T))-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})} + e^{|\boldsymbol{k}|(-2\varepsilon+\mathrm{i}(\Delta T+T))+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{x})}\right)|\boldsymbol{k}|\lambda(\boldsymbol{r})$$

$$\|\alpha\|_\Upsilon = \frac{1}{2(2\pi)^{n-1}}\int \mathrm{d}^{n-1}\boldsymbol{x}\int \mathrm{d}^{n-1}\boldsymbol{y}\int \mathrm{d}^{n-1}\boldsymbol{k} \times \lambda(\Upsilon\boldsymbol{x})\lambda(\Upsilon\boldsymbol{y})|\boldsymbol{k}|e^{-2\epsilon|\boldsymbol{k}|-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

$$\|\alpha\|_\Upsilon = \frac{1}{2(2\pi)^{n-1}}\int \frac{\mathrm{d}^{n-1}\tilde{\boldsymbol{x}}}{\Upsilon^{n-1}}\int \frac{\mathrm{d}^{n-1}\tilde{\boldsymbol{y}}}{\Upsilon^{n-1}}\int \mathrm{d}^{n-1}\tilde{\boldsymbol{k}}\Upsilon^{n-1} \times \lambda(\tilde{\boldsymbol{x}})\lambda(\tilde{\boldsymbol{y}})|\Upsilon\tilde{\boldsymbol{k}}|e^{-2\epsilon|\Upsilon\boldsymbol{k}|-\mathrm{i}\tilde{\boldsymbol{k}}\cdot(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{y}})}$$

$$\|\alpha\|_\Upsilon = \frac{1}{2(2\pi)^{n-1}}\int \frac{\mathrm{d}^{n-1}\tilde{\boldsymbol{x}}}{\Upsilon^{n-1}}\int \frac{\mathrm{d}^{n-1}\tilde{\boldsymbol{y}}}{\Upsilon^{n-1}}\int \mathrm{d}^{n-1}\tilde{\boldsymbol{k}}\Upsilon^{n-1} \times \lambda(\tilde{\boldsymbol{x}})\lambda(\tilde{\boldsymbol{y}})|\Upsilon\tilde{\boldsymbol{k}}|e^{-2\epsilon|\Upsilon\boldsymbol{k}|-\mathrm{i}\tilde{\boldsymbol{k}}\cdot(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{y}})} = \frac{1}{\Upsilon^{n-2}}\|\alpha\|_{\Upsilon=1}$$

$$\left(I_\mu^1(\Upsilon\boldsymbol{x},\Upsilon\Delta T)_\Upsilon = \Upsilon^{2\xi}\left(I_\mu^1(\boldsymbol{x},\Delta T)\right)_{\Upsilon=1}$$

$$\langle\psi(\Upsilon t)|:\hat{T}_{\mu\nu}:(\Upsilon\boldsymbol{x})|\psi(\Upsilon t)\rangle_{\Upsilon}=[\Upsilon^{2\xi}\underbrace{\left(I_{\mu}^{1}I_{\nu}^{1}-\eta_{\mu\nu}\frac{I_{\lambda}^{1}I^{1,\lambda}}{2}\right)}_{\text{QET positive contribution}}-\Upsilon^{n}\underbrace{\left(I_{\mu}^{2}I_{\nu}^{2}-\eta_{\mu\nu}\frac{I_{\lambda}^{2}I^{2,\lambda}}{2}\right)}_{\text{QET neutral contribution}}$$

$$-\Upsilon^{\frac{n}{2}+\xi}\langle A_0|\hat{\sigma}_y|A_0|e\rangle^{-2\|\alpha\|}\underbrace{\left(\left(I_{\mu}^{1}I_{\nu}^{3}-\eta_{\mu\nu}\frac{I_{\lambda}^{1}I^{3,\lambda}}{2}\right)+\left(I_{\mu}^{3}I_{\nu}^{1}-\eta_{\mu\nu}\frac{I_{\lambda}^{1}I^{3,\lambda}}{2}\right)\right)}_{\text{QET negative contribution}}].$$

$$\lambda(\boldsymbol{x})\to\Upsilon^{\frac{n-2}{2}}\lambda(\Upsilon\boldsymbol{x})$$
$$\mu(\boldsymbol{x})\to\Upsilon^{\frac{n}{2}}\mu(\Upsilon\boldsymbol{x})$$

$$\langle\psi(\Upsilon t)|:\hat{T}_{\mu\nu}(\Upsilon\boldsymbol{x}):|\psi(\Upsilon t)\rangle=$$
$$\Upsilon^{n}\langle\psi(t)|:\hat{T}_{\mu\nu}(\boldsymbol{x}):|\psi(t)\rangle_{\Upsilon=1}.$$

6. Factores gravitacionales de las superpartículas o partículas – estrella en campos cuánticos relativistas.

$$\mathcal{L}=\sum_{q}\bar{\psi}_q(i\not{D}+m_q)\psi_q-\frac{1}{4}F^2,$$

$$T_q^{\mu\nu}=\bar{\psi}_q\gamma^{\mu}iD^{\nu}\psi_q,$$
$$T_G^{\mu\nu}=-F^{c\mu\lambda}F_{\lambda}^{c\nu}+\frac{1}{4}g^{\mu\nu}F^2$$

$$g_{\mu\nu}T^{\mu\nu}=\sum_{q}(1+\gamma_m)m_q\bar{\psi}_q\psi_q+\frac{\beta(g)}{2g}F^2$$

$$\langle p',\vec{s}'|T_a^{\mu\nu}|p,\vec{s}\rangle=\bar{u}(p',\vec{s}')\left[A_a(t)\frac{P^{\mu}P^{\nu}}{M_N}\right.$$
$$\left.+J_a(t)\frac{P^{\{\mu}i\sigma^{\nu\}\lambda}\Delta_{\lambda}}{M_N}-S_a(t)\frac{P^{[\mu}i\sigma^{\nu]\lambda}\Delta_{\lambda}}{M_N}\right]u(p,\vec{s})$$

$$\langle p',\vec{s}'|T_a^{\mu\nu}|p,\vec{s}\rangle=\bar{u}(p',\vec{s}')\left[A_a(t)\frac{P^{\mu}P^{\nu}}{M_N}\right.$$
$$+D_a(t)\frac{\Delta^{\mu}\Delta^{\nu}-g^{\mu\nu}\Delta^2}{4M_N}+\bar{C}_a(t)M_Ng^{\mu\nu}$$
$$\left.+J_a(t)\frac{P^{\{\mu}i\sigma^{\nu\}\lambda}\Delta_{\lambda}}{M_N}-S_a(t)\frac{P^{[\mu}i\sigma^{\nu]\lambda}\Delta_{\lambda}}{M_N}\right]u(p,\vec{s})$$

$$A(0)=\sum_{q}A_q(0)+A_G(0)=1$$
$$J(0)=\sum_{q}J_q(0)+J_G(0)=\frac{1}{2}$$
$$\frac{1}{2}\Delta\Sigma=\sum_{q}S_q(0)$$
$$\bar{C}(t)=\sum_{q}\bar{C}_q(t)+\bar{C}_G(t)=0$$

$$D \equiv D(0) = \sum_q D_q(0) + D_G(0)$$

$$\mathcal{T}_a^{\mu\nu}(\vec{r}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3 2E} e^{-i\vec{\Delta}\cdot\vec{r}} \langle p'|T_a^{\mu\nu}|p\rangle$$

$$\int \mathrm{d}^3 r \mathcal{T}_a^{\mu\nu}(\vec{r}) = \left.\frac{\langle p|T_a^{\mu\nu}|p\rangle}{2M_N}\right|_{\vec{p}=\vec{0}}$$

$$\int \mathrm{d}^3 r \mathcal{T}_a^{\mu\nu}(\vec{r}) = \begin{pmatrix} U_a & 0 & 0 & 0 \\ 0 & W_a & 0 & 0 \\ 0 & 0 & W_a & 0 \\ 0 & 0 & 0 & W_a \end{pmatrix}$$

$$\sum_a U_a = M_N$$

$$\sum_a W_a = 0$$

$$M_m = \sum_q \sigma_q \equiv \left.\frac{\langle p|\sum_q m_q \bar{\psi}_q \psi_q|p\rangle}{2M_N}\right|_{\vec{p}=\vec{0}}$$

$$M_N = \sum_q M_q + M_m + M_G$$

$$M_G = \bar{M}_G + \frac{1}{4} M_A$$

$$M_A = \left.\frac{\langle p|\sum_q \gamma_m m_q \bar{\psi}_q \psi_q + \frac{\beta(g)}{2g} F^2|p\rangle}{2M_N}\right|_{\vec{p}=\vec{0}}$$

$$\langle p|g_{\mu\nu} T^{\mu\nu}|p\rangle = 2p^2 = 2M_N^2$$

$$M_N = M_m + M_A$$

$$M_N = \sum_a \int \mathrm{d}^3 r g_{\mu\nu} \mathcal{T}_a^{\mu\nu}(\vec{r}) = \sum_a (U_a - 3W_a)$$

$$\mathcal{J}^i = \int \mathrm{d}^3 r \epsilon^{ijk} r^j T_{\mathrm{Bel}}^{0k}$$

$$J_a^z = J_a(0)$$

$$\mathcal{J}_q^i = \int \mathrm{d}^3 r \epsilon^{ijk} r^j T_q^{0k} + \int \mathrm{d}^3 r \frac{1}{2} \bar{\psi}_q \gamma^i \gamma_5 \psi_q$$

$$L_q^z = J_q(0) - S_q(0)$$

$$\sum_q S_q^z = \frac{1}{2}\Delta\Sigma$$

$$\int_{-1}^{1} \mathrm{d}x x H_q(x,\xi,t) = A_q(t) + \xi^2 D_q(t)$$
$$\int_{-1}^{1} \mathrm{d}x x E_q(x,\xi,t) = B_q(t) - \xi^2 D_q(t)$$

$$\mathrm{ReH}(\xi,t) + i\mathrm{Im}\mathcal{H}(\xi,t) = \sum_q e_q^2 \int_{-1}^{1} dx \left[\frac{1}{\xi - x - i\epsilon} - \frac{1}{\xi + x - i\epsilon}\right] H_q(x,\xi,t),$$

$$\mathrm{Re}\mathcal{H}(\xi,t) = \mathcal{C}_{\mathcal{H}}(t) + \frac{1}{\pi}\,\mathrm{P.V.} \int_0^1 \mathrm{d}\xi' \left[\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'}\right] \mathrm{Im}\mathcal{H}(\xi',t),$$

$$\mathcal{C}_{\mathcal{H}}(t) = 2\sum_q e_q^2 \int_{-1}^{1} \mathrm{d}z \frac{D_{\mathrm{term}}^q(z,t)}{1-z}$$

$$D_{\mathrm{term}}^q(z,t) = (1-z^2) \sum_{\mathrm{odd}\,n} d_n^q(t) C_n^{3/2}(z)$$

$$D_q(t) = \frac{4}{5} d_1^q(t) = \int_{-1}^{1} \mathrm{d}z z D_{\mathrm{term}}^q(z,t)$$

$$A_C = \frac{\sigma^{e^-} - \sigma^{e^+}}{\sigma^{e^-} + \sigma^{e^+}}$$

C. $\gamma\gamma^* \to \pi^0\pi^0$

$$\lim_{m_\pi \to 0} D_\pi = -1$$

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t)\Big|_{t=0} = -\frac{g_A^2 M_N}{40\pi f_\pi^2 m_\pi} + \cdots$$

$$A_{LU}(\xi,t) = \frac{N^+(\xi,t) - N^-(\xi,t)}{N^+(\xi,t) + N^-(\xi,t)}$$

$$\mathrm{Im}\mathcal{H}(\xi,t) = \frac{\mathcal{N}}{1+\xi} \frac{\left(\frac{2\xi}{1+\xi}\right)^{-\alpha(t)} \left(\frac{1-\xi}{1+\xi}\right)^{b}}{\left(1 - \frac{1-\xi}{1+\xi}\frac{t}{M^2}\right)^{p}}$$

$$\mathcal{C}_{\mathcal{H}}(t) = \mathcal{C}_{\mathcal{H}}(0)\left[1 + \frac{(-t)}{M^2}\right]^{-\lambda}$$

$$\mathcal{C}_{\mathcal{H}}(0) = -2.27 \pm 0.16 \pm 0.36,$$
$$M^2 = 1.02 \pm 0.13 \pm 0.21\,\mathrm{GeV}^2,$$
$$\lambda = 2.76 \pm 0.23 \pm 0.48.$$

$$S = \frac{\mathcal{C}_{\mathcal{H}}(0)}{\sigma_{\mathcal{C}_{\mathcal{H}}(0)}} \approx 5.8$$

$$\mathcal{C}_{\mathcal{H}}(t) \approx \frac{10}{9} d_1^{u+d}(t) = \frac{25}{18} D_{u+d}(t)$$

$$\mathcal{T}^{ij}(\vec{r}) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right) s(r) + \delta^{ij} p(r)$$

$$D(0) = -\frac{4}{15} M_N \int \ \mathrm{d}^3 r r^2 s(r) = M_N \int \ \mathrm{d}^3 r r^2 p(r)$$

$$s(r) = -\frac{1}{4M_N} r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \widetilde{D}(r)$$

$$p(r) = \frac{1}{6M_N} \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}}{\mathrm{d}r} \widetilde{D}(r)$$

$$p_n(r) = \frac{2}{3} s(r) + p(r) > 0$$

$$D(0) < 0$$

$$r_{\mathrm{mech}}^2 = \frac{\int \ \mathrm{d}^3 r r^2 p_n(r)}{\int \ \mathrm{d}^3 r p_n(r)} = \frac{6D(0)}{\int_{-\infty}^{0} \ \mathrm{d}t D(t)}$$

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