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MONOTONICITY AND CONVERGENCE IN THE COLLATZ CONJECTURE: A NEW PERSPECTIVE

MONOTONICIDAD Y CONVERGENCIA EN LA CONJETURA DE COLLATZ: UNA NUEVA PERSPECTIVA

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Monotonicity and Convergence in the Collatz Conjecture: A New Perspective

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ABSTRACT

The Collatz conjecture declares that every positive integer will eventually reach 1 when subjected to a simple iterative process: if the number is even, it is divided by 2, and if it is odd, it is multiplied by 3 and then increased by 1. Despite the straightforward nature of these rules, a general proof of the conjecture remains elusive. For the above, this study introduces an alternative interpretation of the conjecture. This approach involves multiplying an odd integer N_1 by 3 and subsequently adding the largest power-of-2 factor within N_1. Repeated iterations of this alternative process show that any initial odd integer N_1 will eventually convert into a power of 2, leading the sequence towards convergence. The behavior of the sequence was studied by representing the resulting integers as a power of 2 multiplied by an odd component. Using this representation under the modified rules, we developed a structured proof framework that demonstrates the consistent reduction of the odd component's relative value after each iteration, the accelerated increase of the power-of-2 factor's relative value, and the absence of any divergent cycles or alternative behaviors. This analysis provides insights into the mechanics of convergence in the Collatz sequence and proposes a new perspective for understanding the conjecture's underlying dynamics.

Keywords: Collatz conjecture, convergence, iterative sequences, number theory

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RESUMEN

La conjetura de Collatz establece que todo entero positivo llegará a ser 1 si es sometido a un proceso iterativo simple: si el número es par, se divide entre 2, y si es impar, se multiplica por 3 y luego se suma 1. A pesar de la naturaleza sencilla de estas reglas, no se ha podido demostrar de manera general la conjetura. Por lo anterior, este estudio presenta una interpretación alternativa de la conjetura. Este enfoque implica multiplicar un entero impar N_1 por 3 y, posteriormente, sumar el factor-potencia de 2 más grande de N_1. Al repetir este proceso iterativo, cualquier entero impar inicial N_1 se convertirá eventualmente en una potencia de 2, lo que lleva la secuencia hacia la convergencia. Se estudió el comportamiento de la secuencia representando los números enteros resultantes como una potencia de 2 multiplicada por un componente impar. Utilizando esta representación bajo las reglas modificadas, desarrollamos un procedimiento que demuestra la reducción consistente del valor relativo del componente impar después de cada iteración, el aumento acelerado del valor relativo del factor-potencia de 2 y la ausencia de ciclos divergentes o comportamientos alternativos. Este análisis proporciona información sobre la mecánica de la convergencia en la secuencia de Collatz y propone una nueva perspectiva para comprender el comportamiento subyacente de la conjetura.

Palabras clave: conjetura de Collatz, convergencia, secuencias iterativas, teoría de números

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INTRODUCTION

The Collatz conjecture explores the behavior of a sequence initiated by any positive integer, where each subsequent term is generated according to the rules given in Eq. (1). The conjecture posits that this series always reaches 1 regardless of the chosen initial integer [7, 15].

$$n_{t+1} \begin{cases} 3 \cdot n_t + 1, & \text{if } n_t \text{ is odd} \\ \frac{n_t}{2}, & \text{if } n_t \text{ is even} \end{cases} \quad (1)$$

For any positive integer n_t, the Collatz sequence proceeds as follows: if n_t is even, the next term is obtained by dividing it by 2; if n_t is odd, it is transformed by multiplying it by 3 and adding 1 [14]. The conjecture also implies that the followed sequence will always arrive to the trivial cycle "1-4-2-1" [2]. Originally proposed by Lothar Collatz in 1937, the Collatz conjecture is also known by several other names, including the $3n + 1$ conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, the Syracuse problem or the hailstone sequence [1]. Although a formal proof of the conjecture remains elusive, extensive experimental evidence and heuristic arguments suggest that it is valid [5], and the conjecture has been verified for values up to $2^{1000000} - 1$ [6]. In addition, one of the most notable recent advances was made by Terence Tao, who proved that most orbits of the Collatz map attain almost bounded values [8, 13]. Finally, proving the Collatz conjecture is equivalent to demonstrating the absence of cycles other than "1-4-2-1" and of divergent orbits [10].

Bottom-up Approach to the Collatz Conjecture

A bottom-up representation of the Collatz conjecture suggests that any positive integer can be reached by applying an inverse form of the rules described in Eq. (1), these being $2 \cdot n_t$ and $\frac{n_t - 1}{3}$ [12]. Starting from $n_1 = 1$, both rules can be applied as long as only integers are produced.

Table 1 illustrates the steps required to reach the first 10 integers, showing a complex behavior [3]. However, to prove the validity of the Collatz conjecture it suffices to prove that it holds true for every positive odd integer [9]. This occurs because each odd number n_t can generate multiples of itself in the form $n_t \cdot 2^x$, demonstrating that every even number originates from an odd number. Accordingly, the formulas in this study are designed to generate only positive odd numbers.

Table 1 Steps required to generate the first 10 positive integers n_t following the rules of the Collatz conjecture.

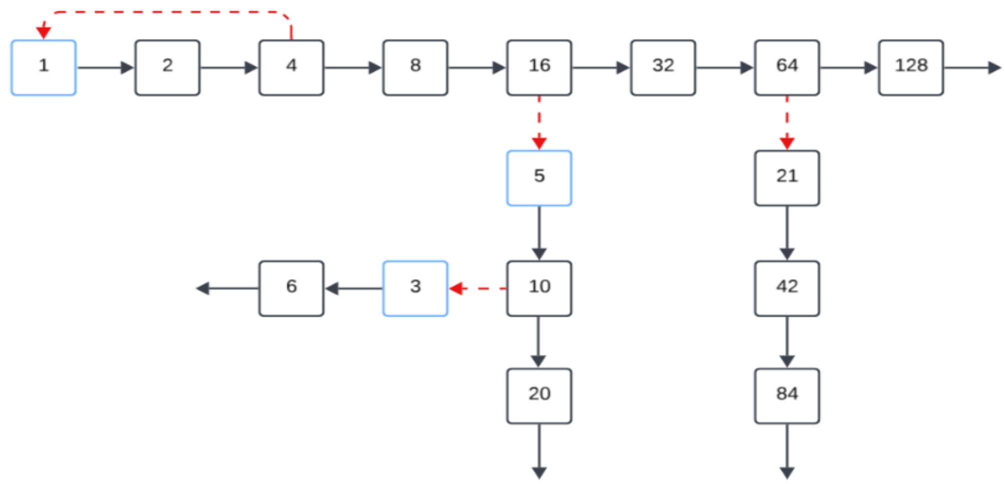
n_t	Steps to Reach n_t
1	1
2	1, 2
3	1, 2, 4, 8, 16, 5, 10, 3
4	1, 2, 4
5	1, 2, 4, 8, 16, 5
6	1, 2, 4, 8, 16, 5, 10, 3, 6
7	1, 2, 4, 8, 16, 5, 10, 20, 40, 13, 26, 52, 17, 34, 11, 22, 7
8	1, 2, 4, 8
9	1, 2, 4, 8, 16, 5, 10, 20, 40, 13, 26, 52, 17, 34, 11, 22, 7, 14, 28, 9
10	1, 2, 4, 8, 16, 5, 10

Progression of the Bottom-Up Collatz Conjecture: Examples and Diagram

Fig. 1 illustrates the initial integers that can be generated by starting from $n_1 = 1$ and applying the rules of the bottom-up Collatz conjecture. The diagram indicates that the $\frac{n_t-1}{3}$ formula cannot be applied to all numbers or "branches", but it consistently generates odd numbers when used. As an example, the equations in the figure detail the steps taken to reach the numbers 5 and 3.

Figura 1. a) Representation of the first integers that can be generated by starting from n1 = 1 and following both rules of the bottom-up Collatz conjecture. Black arrows indicate the doubling of the current number (2·n_t), while red arrows represent the use of the (n_t-1)/3 formula. b) Algebraic expressions that indicate the steps required to reach number 5. c) Algebraic expressions that indicate the steps needed to reach number 3.

a)



b) $1 \rightarrow 2^4 \rightarrow \frac{2^4-1}{3^1} = 5 \quad \left(n_t = \frac{2^\alpha-1}{3^1}\right)$

c) $1 \rightarrow 2^4 \rightarrow \frac{2^4-1}{3^1} \rightarrow \frac{2^5-2^1}{3^1} \rightarrow \frac{2^5-2^1-3^1}{3^2} = 3 \quad \left(n_t = \frac{2^\alpha-2^x-3^1}{3^2}\right)$

General Formula to Represent the Steps Required to Generate Any Odd Number n_t

The reverse Collatz iterations used to reach an odd integer can be represented as functions [11]. As illustrated in Fig. 1, the steps needed to reach any odd number n_t starting from $n_1 = 1$ can be represented by the following general formula:

$$n_t = \frac{2^{\alpha} - 2^{x} - 2^{y}.3^{1} - 2^{z}.3^{2} - \ldots - 2^{w}.3^{\beta-2} - 3^{\beta-1}}{3^{\beta}}, \qquad (2)$$

where: n_t, α, β, x, y, z, ..., w ∈ Z^+; α > x > y > z > ... > w. Here, α is the total number of times the transformation $2 \cdot n_t$ was applied, while β is the total number of times the $\frac{n_t - 1}{3}$ formula was applied. Eq. (2) captures the iterative process of the bottom-up Collatz sequence to generate any odd integer n_t. The exponents α, x, y, z, ..., w denote the cumulative number of times that n_t was multiplied by 2 between successive applications of the $\frac{n_t - 1}{3}$ formula. For example, the first step to generate 17 from $n_1 = 1$ involves doubling four times before applying $\frac{n_t - 1}{3}$ for the first time, which corresponds to α − x = 4. Then, between the first and second applications of $\frac{n_t - 1}{3}$, n_t is doubled three times, so $x - y = 3$, and so on.

Alternative Interpretation of the Collatz Conjecture

The terms of Eq. (2) can be rearranged to yield Eq. (3), in which N_1 is an initial odd positive integer.

$$N_1 \cdot 3^{\beta} + 2^{0} \cdot 3^{\beta-1} + 2^{w} \cdot 3^{\beta-2} + \cdots + 2^{z} \cdot 3^{2} + 2^{y} \cdot 3^{1} + 2^{x} \cdot 3^{0} = 2^{\alpha}, \qquad (3)$$

where: N_1, α, β, x, y, z, ..., w ∈ Z^+; α > x > y > z > ... > w. Here, α represents the total number of times the transformation $2 \cdot n_t$ was applied, while β is the total number of times the $\frac{n_t - 1}{3}$ formula was applied.

Transformation of Odd Positive Integers N_1 Into Powers of 2 Via Iteratively Multiplying by 3 and Adding a Power-of-2 Factor

Based on Eq. (3), an odd positive integer N_1 can be transformed into a power of 2 by applying the following iterative steps:

- Multiply N_1 by 3 and add 1 (2^0) to obtain an even integer N_2.
- For each subsequent iteration, multiply N_t by 3 and add the largest power of 2 that is a factor of N_t.

- Repeat this process until N_t transforms into a power of 2, denoted 2^α.

The iterative process is represented by Eq. (4):

$$N_{t+1} = 3 \cdot N_t + 2^{k_t}, \tag{4}$$

where 2^{k_t} is the biggest power of 2 that divides N_t.

As an example, Table 2 summarizes the steps that must be followed to convert odd positive integers N_1, from 3 to 13, into powers of 2. Each positive integer N_t is expressed as a power of 2 multiplied by an odd number.

Table 2. Summary of the transformation of positive odd integers N_1 from 3 to 13 by multiplying each by 3 and adding the largest power of 2 that divides it (including 2^0). This cycle is repeated until the integer becomes a power of 2. Each resulting integer is represented as a power of 2 multiplied by an odd factor.

N_1	$3 \cdot N_1 + 1$		$3 \cdot N_2 + 2^a$		$3 \cdot N_3 + 2^b$		$3 \cdot N_4 + 2^c$
3	$1 \cdot 9+1 = \mathbf{2^1 \cdot 5}$	→	$2^1 \cdot 15+2^1 = \mathbf{2^5}$				
5	$1 \cdot 15+1 = \mathbf{2^4}$						
7	$1 \cdot 21+1 = \mathbf{2^1 \cdot 11}$	→	$2^1 \cdot 33+2^1 = \mathbf{2^2 \cdot 17}$	→	$2^2 \cdot 51+2^2 = \mathbf{2^4 \cdot 13}$	→	$2^4 \cdot 39+2^4 = \mathbf{2^7 \cdot 5} \to \mathbf{2^{11}}$
9	$1 \cdot 27+1 = \mathbf{2^2 \cdot 7}$	→	$2^2 \cdot 21+2^2 = \mathbf{2^3 \cdot 11}$	→	$2^3 \cdot 33+2^3 = \mathbf{2^4 \cdot 17}$	→	$2^4 \cdot 51+2^4 = \mathbf{2^6 \cdot 13} \to \mathbf{2^9 \cdot 5} \to \mathbf{2^{13}}$
11	$1 \cdot 33+1 = \mathbf{2^1 \cdot 17}$	→	$2^1 \cdot 51+2^1 = \mathbf{2^3 \cdot 13}$	→	$2^3 \cdot 39+2^3 = \mathbf{2^6 \cdot 5}$	→	$2^6 \cdot 15+2^6 = \mathbf{2^{10}}$
13	$1 \cdot 39+1 = \mathbf{2^3 \cdot 5}$	→	$2^3 \cdot 15+2^3 = \mathbf{2^7}$				

Note: For $N_1 = 7$ and $N_1 = 9$, the final steps were simplified.

Monotonic Reduction and Convergence of the Alternative Interpretation of the Collatz Conjecture

Eq. (4) presents a unique scenario that cannot be replicated by iteratively multiplying by larger odd factors at the start of each cycle (e.g., $5 \cdot N_t + 2^{k_t}$, $7 \cdot N_t + 2^{k_t}$, etc.). This behavior arises because the rules of the Collatz conjecture allow the power-of-2 factor of N_t to increase with each cycle, including a growth in the power-of-2 factor's relative value.

The following sections outline the key properties that explain why any initial positive odd integer N_1 can ultimately be transformed into a power of 2 (2^α) by applying the steps of this alternative interpretation of the Collatz conjecture.

Lemma 4.1 Iteratively adding the largest power-of-2 factor of an odd positive integer N_1 will ultimately transform N_1 into the closest power of 2 that is greater than N_1.

Proof Every positive integer N_t can be uniquely factorized into a power of 2 multiplied by an odd component:

$$N_t = 2^{k_t} \cdot O_t,$$

where O_t is the odd component of N_t, and 2^{k_t} is the largest power-of-2 factor of N_t.

Similarly, every positive integer N_t has a unique binary representation [4]:

$$N_t = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_n},$$

where $0 \le a_1 < a_2 < \ldots < a_n$.

The smallest term of the binary representation, 2^{a_1}, is equal to the biggest power-of-2 factor, 2^{k_t}. This can be proven if the previous formula is rewritten to resemble a power-of-2 factor multiplied by an odd component:

$$N_t = 2^{a_1} \cdot (1 + 2^{a_2 - a_1} + \cdots + 2^{a_n - a_1}),$$

where the term in parentheses is equal to the odd component O_t.

After one iteration, upon summing 2^{a_1} (or equivalently 2^{k_t}) to N_t, 2^{a_1} doubles:

$$N_{t+1} = 2^{a_1+1} + 2^{a_2} + \cdots + 2^{a_n}$$

After sufficient iterations, the initial 2^{a_1} becomes equal to 2^{a_2}, and their sum produces 2^{a_2+1}. This process continues, systematically combining and doubling terms until all binary terms coalesce into 2^{a_n+1}, the closest power of 2 that is greater than the positive odd integer N_1. For a detailed example of this iterative process, refer to Table 3.

Table 3. Test to analyze the effect over positive odd integers N_1 from 3 to 15 (decomposed into series of sums of non-repeating powers of 2), after adding the smallest power of 2 found in their respective series. The cycle is repeated until the number becomes a power of 2.

N_1	$N_1 + 1$	$N_2 + 2^a$	$N_3 + 2^b$
3 (1+2)	1+1+2 = **4**		
5 (1+4)	1+1+4 = **2+4** →	2+2+4 = **8**	
7 (1+2+4)	1+1+2+4 = **8**		
9 (1+8)	1+1+8 **= 2+8** →	2+2+8 = **4+8** →	4+4+8 = **16**
11 (1+2+8)	1+1+2+8 = **4+8** →	4+4+8 = **16**	
13 (1+4+8)	1+1+4+8 = **2+4+8** →	2+2+4+8 = **16**	
15 (1+2+4+8)	1+1+2+4+8 = **16**		

Conclusion

The iterative summation of the largest power-of-2 factor of the odd positive integer N_1 successively eliminates all smaller binary terms of N_1 by combining and doubling them, ultimately transforming the integer into the nearest power of 2 larger than N_1.

Lemma 4.2 Let N_1 denote an odd positive integer undergoing the iterative transformation $N_{t+1} = 3 \cdot N_t + 2^{k_t}$, with 2^{k_t} being the largest power-of-2 factor in N_t. N_t's odd factor divided by its largest power-of-2 factor, undergoes a strict monotonic reduction in each cycle.

Proof Representing each number N_t as a power of 2 multiplied by an odd component:

$$N_t = 2^{k_t} \cdot O_t,$$

where O_t is the odd component of N_t and 2^{k_t} is the biggest power-of-2 factor of N_t. We seek to show that the value $\frac{O_t}{2^{k_t}}$ decreases in every iteration, demonstrating a monotonic reduction of the relative value of O_t in the sequence.

Worst-Case Scenario Let N_{1*} be a hypothetical positive odd integer that undergoes the iterative transformation $N_{t*+1} = 3 \cdot N_{t*} + 2^{k_{t*}}$. Assume that in this process only consecutive powers of 2 (i.e. 1, 2, 4, 8, …) are added in each iteration. Under these conditions, this sequence represents a worst-case scenario in which the sequence progresses at its slowest rate. Even so, it can be shown that this sequence progresses monotonically, with the odd component decreasing relative to its corresponding power of 2 at each step.

$$1 \cdot O_{1*}, \quad 2^1 \cdot O_{2*}, \quad 2^2 \cdot O_{3*}, \quad 2^3 \cdot O_{4*}, \ \ldots$$

$$1 \cdot N_{1*}, \ 2^1 \cdot (\frac{3 \cdot N_{1*} + 1}{2^1}), \ 2^2 \cdot (\frac{9 \cdot N_{1*} + 5}{2^2}), \ 2^3 \cdot (\frac{27 \cdot N_{1*} + 19}{2^3}), \ \ldots$$

Dividing the odd components by their powers of 2, we obtain:

$$\frac{N_{1*}}{1}, \ \frac{3 \cdot N_{1*}+1}{2^2}, \ \frac{9 \cdot N_{1*}+5}{2^4}, \ \frac{27 \cdot N_{1*}+19}{2^6}, \ \ldots$$

When $N_1 > 1$, the terms of the series always decrease more rapidly after every new iteration.

Non-Worst-Case Scenario For any iteration of the process $N_{t*+1} = 3 \cdot N_{t*} + 2^{k_{t*}}$ applied to the hypothetical positive odd integer N_{1*}, if in any step the added power-of-2 factor ($2^{k_{t*}}$) does not follow the minimal consecutive sequence, the ratio of the odd component O_{t*} to the power of 2 will decrease even more rapidly than in the worst-case scenario:

$$\frac{O_{t*}}{2^{k_{t*}+j}}$$

where j is a positive integer.

CONCLUSION

The relative value of the odd component O_t decreases with each iteration, as shown by the construction of the worst-case scenario sequence. Even in this scenario, the sequence progresses monotonically, with the odd component's influence diminishing over time as the relative value of the power of 2 factor increases. Also, any variation in the sum of powers of 2 that includes major factors accelerates the process of turning N_t to a power of 2. These cases eliminate the possibility of strange cycles or sequence divergence, supporting the hypothesis that any positive odd integer N_1 approaches a power of 2 through repeated iterations under the Collatz process.

Author Contributions

Conceptualization: Wells Abascal G.; Methodology: Wells Abascal G.; Formal analysis and investigation: Wells Abascal G.; Writing - original draft preparation: Wells Abascal G.; Writing - review and editing: Wells Abascal G., Zavala Raus A.

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Declarations

Conflict of interest We have nothing to declare and there is no conflict of interest.

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