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CAMPOS CUÁNTICOS RELATIVISTAS: APROXIMACIONES TEÓRICO – MATEMÁTICAS RELATIVAS A LOS ESPACIOS CUÁNTICOS GEOMÉTRICAMENTE DEFORMADOS O PERFORADOS POR PARTÍCULAS Y ANTIPARTÍCULAS SUPERMASIVAS Y MASIVAS E HIPERPARTÍCULAS Y SUPRAPARTÍCULAS

RELATIVISTIC QUANTUM FIELDS: THEORETICAL-MATHEMATICAL APPROACHES RELATIVE TO GEOMETRICALLY DEFORMED OR PERFORATED QUANTUM SPACES BY SUPERMASSIVE AND MASSIVE PARTICLES AND ANTIPARTICLES AND HYPERPARTICLES AND SUPRAPARTICLES

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Campos Cuánticos Relativistas: Aproximaciones Teórico – Matemáticas Relativas a los Espacios Cuánticos Geométricamente Deformados o Perforados por Partículas y Antipartículas Supermasivas y Masivas E Hiperpartículas y Suprapartículas

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RESUMEN

En anteriores trabajos, este investigador, ha desarrollado planteamientos o alternativas de solución relativas al Problema del Milenio de Yang – Mills. Sin embargo, ha sido indispensable ampliar el espectro, pues, la teoría de Yang – Mills, alcanza diversos escenarios de la física moderna, y ciertamente, es a mi criterio, un puente crucial para conciliar la relatividad einsteniana y la mecánica cuántica. Tal es así, que, desde mi perspectiva, es la teoría cuántica de campos curvos o lo que he denominado como "campos cuánticos relativistas", la que se constituye como una alternativa legítima para unificar las áreas de la física antes mencionadas. Mi investigación ha arrojado resultados significativos, sin embargo, resta mucho por desarrollar e investigar en la medida de alcanzar la mentada unificación. En este artículo, cumplo con fortalecer la teoría cuántica de campos relativistas, desde las bases formales de la relatividad general y la relatividad especial respectivamente, hasta la teoría cuántica de campos, con especial énfasis, en los campos de Yang – Mills – Einstein.

Palabras clave: mecánica cuántica, relatividad general, relatividad especial, teoría cuántica de campos curvos

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Relativistic Quantum Fields: Theoretical-Mathematical Approaches Relative to Geometrically Deformed or Perforated Quantum Spaces by Supermassive and Massive Particles and Antiparticles and Hyperparticles and Supraparticles

ABSTRACT

In previous works, this researcher has developed approaches or alternative solutions related to the Yang-Mills Millennium Problem. However, it has been essential to broaden the spectrum, since the Yang-Mills theory reaches various scenarios of modern physics, and certainly, in my opinion, it is a crucial bridge to reconcile Einsteinian relativity and quantum mechanics. So much so, that, from my perspective, it is the quantum theory of curved fields or what I have called "relativistic quantum fields", which constitutes a legitimate alternative to unify the areas of physics mentioned above. My research has yielded significant results, however, there is still much to be developed and researched to the extent of achieving the aforementioned unification. In this article, I strengthen the quantum theory of relativistic fields, from the formal bases of general relativity and special relativity respectively, to quantum field theory, with special emphasis on the Yang – Mills – Einstein fields.

Keywords: quantum mechanics, general relativity, special relativity, quantum theory of curved fields

INTRODUCCIÓN

Como se ha mencionado anteriormente, a lo largo de mis investigaciones, el propósito esencial de mi trabajo ha sido aproximar una alternativa de solución al Problema del Milenio de Yang – Mills. Sin embargo, en el cumplimiento de ese objetivo, he arribado a una conclusión ineludible, la teoría cuántica de campo de Yang – Mills, es por mucho, la conexión necesaria para unificar la física relativista y la mecánica cuántica. Parece improbable, pero no imposible. Si bien ambos campos de la física contemporánea, son esencialmente distintos, en la medida en que el primero es determinista y el segundo, puramente probabilista, no es menos cierto que existe un componente unificador que permite fundir ambas teorías, y es la curvatura. Y es que, la teoría cuántica de campos curvos, de cardinal bagaje, ofrece esta posibilidad. Ha sido ampliamente desarrollado, el principio de dualidad holográfica, y no es para menos, pues, lo macroscópico y lo microscópico o subatómico, convergen y coinciden, sobre todo, cuando existe una curvatura. Al igual que sucede con el espacio – tiempo a nivel cósmico, esto es, que sufre de curvatura ante un objeto masivo o supermasivo, según sea el caso, deformando o perforando (agujeros negros) el tejido circundante, pues, ocurre lo mismo, con los espacios cuánticos en los que interactúan las distintas fuerzas fundamentales, y es precisamente esto, lo que propugna la teoría cuántica de campos relativistas.

Para estos efectos, se han deducido las siguientes conjeturas: 1. Que, los espacios cuánticos, son susceptibles de deformación o perforación, a propósito de la interacción de las partículas y antipartículas que les sean congénitas. 2. Que, un campo cuántico, es susceptible de deformación geométrica, es decir, padece de curvatura, cuando una partícula o antipartícula, a propósito de su masa superior, modifica espacialmente su posición, repercutiendo en sus perímetros aproximados. 3. Que, un campo cuántico, es susceptible de deformación geométrica, es decir, padece de curvatura, cuando una partícula o antipartícula, se aproxima, alcanza o supera la velocidad de la luz, lo que explica además la brecha de masa que no es arbitraria y es habitualmente superior a cero. 4. Que, un campo cuántico, es susceptible de perforación, es decir, adherido a la existencia de microagujeros cuánticos, cuando una partícula o antipartícula, a propósito de su masa superlativa, modifica espacialmente su posición, repercutiendo en sus perímetros aproximados. 5. Las partículas y antipartículas mencionadas en el numeral 2, se denominarán para efectos de entender la presente teoría, como partículas masivas o antipartículas masivas, según sea el caso, es decir, aquellas que, por su interacción temporal – espacial, deforman geométricamente el espacio cuántico en el que interactúan. 6. Las partículas y antipartículas mencionadas en el numeral 3, se denominarán para efectos de entender la presente teoría, como hiperpartículas, indistintamente si se trata de una partícula o una antipartícula, esto, cuando pese a no tener masa, se aproximan, alcanzan o superan la velocidad de la luz. Considérese también en este escenario, la existencia de suprapartículas, es decir, aquellas que a más de ser masivas o supermasivas, según sea el caso, a razón de su masa, son capaces de aproximarse, alcanzar o superar la velocidad de la luz, en cuyo caso, el surgimiento de microagujeros cuánticos es inevitable. 7. Que, la deformación de un espacio cuántico o en su defecto, la generación de agujeros negros cuánticos, ocurren por la existencia de fenómenos cuánticos propios e inherentes al sistema de partículas y antipartículas propuesto, esto es, fenómenos tales como la superposición, el entrelazamiento o la colisión, lo que, para todos los casos, aplica el puente Einstein – Rosen (Paradoja EPR). 8. En mérito a la deformación o perforación de los espacios cuánticos, se supone la existencia de ondas cuánticas, al igual que las ondas gravitacionales en cuanto a su fenomenología. 9. La perforación de un campo cuántico específico, no solamente supone la existencia de un agujero negro cuántico, el cual, incluso puede ser supermasivo, sino que además, supone la existencia de pluridimensiones en las que ocurre un fenómeno de dualidad divergente.

Para sustentar las hipótesis antes referidas, se ha desplegado, no solamente en este trabajo, sino también en investigaciones a priori, modelos matemáticos que comprenden desde el cálculo tensorial hasta bases propias de la electrodinámica y cromodinámica cuánticas, además de la formalización matemática que exige la mecánica cuántica pero incorporando magnitudes relativistas esenciales, como el tensor y flujo de Ricci o símbolos de Christoffel, las identidades de Bianchi o la misma métrica de Einstein, para explicar la curvatura, o verbigracia, las ecuaciones de Tolman-Oppenheimer-Volkoff, para explicar la morfología y la sistematicidad de los objetos subatómicos antes referidos sin perder de vista sus características específicas inherentes, etc.

METODOLOGÍA

La formalización matemática formulada en el presente trabajo, comporta la aplicación de una metodología de investigación integral, esto es, bajo un enfoque híbrido, es decir, tanto en lo cualitativo como en lo cuantitativo. El tipo de investigación aplicado, es preeminentente predictivo, lo que es propio en la física teórica. El estado del arte, es diferencialmente constructivista. Cabe precisar, que dada la naturaleza de esta investigación, no existe población de estudio. El componente bibliográfico ha sido un elemento indispensable para la elaboración de este trabajo. La técnica metodológica, dada la temática, es deductiva, pues la teorización ha sido desarrollada en fundamento a teorías físicas validadas y aceptadas por la comunidad científica internacional. Finalmente, las ecuaciones contenidas en esta investigación, se proyectan esencialmente a demostrar las hipótesis explicadas en la parte introductoria.

RESULTADOS Y DISCUSIÓN

Para efectos de demostrar los postulados contenidos en la parte introductoria de este trabajo, así como en los manuscritos anteriores, cada ecuación ha sido clasificada según la naturaleza y complejidad de cada elemento teórico estudiado, verbigracia la geometría riemanniana para explicar la curvatura de un campo cuántico específico, etc. En consecuencia, para la aplicación de las fórmulas en relación, se sugiere ajustarse a los distintos escenarios de suposición planteados por este investigador, es decir y en su núcleo duro, el comportamiento, morfología y sistematicidad de los objetos subatómicos en espacios cuánticos curvos o relativistas, incluyendo verbigracia, la existencia irrefutable de agujeros negros cuánticos u ondas cuánticas, etc.

Las ecuaciones por componente, pasan a exponerse a detalle:

Geometría de Lorentz en espacios cuánticos relativistas

Cuestiones preliminares

$$\frac{\partial f_{\phi}}{\partial \chi^{i}} = \frac{\partial f_{\phi'}}{\partial \chi^{j'}} \frac{\partial \chi^{j'}}{\partial \chi^{i}}$$

$$\upsilon^{i}_{\phi_{\mathcal{J}}} = \upsilon^{j}_{\phi_{\mathcal{J}}} \frac{\partial \chi^{i}_{\mathcal{J}}}{\partial \chi^{j}_{\mathcal{J}}}$$

$$\omega_{i} = \omega_{j} \frac{\partial \chi^{i}}{\partial \chi'^{i}}$$

$$\theta^{(j)} e_{i} = \delta^{j}_{i}$$

$$\upsilon(f) \coloneqq \upsilon^{i}_{\phi} \frac{\partial f_{\phi}}{\partial \chi^{i}_{\phi}}$$

$$\upsilon(f+g) = \upsilon(f) + \upsilon(g)$$

$$\upsilon(fg) = f\upsilon(g) + g\upsilon(f)$$

$$\upsilon(f) \coloneqq e_{\phi,(i)}(f) = \frac{\partial f(\phi^{-1})}{\partial \chi_{\phi}^{i}}$$

Campos tensoriales y tensores

$$\mathcal{T}_{i'j'} = \frac{\partial \chi^{\hbar}}{\partial \chi^{i'}} \frac{\partial \chi^{\kappa}}{\partial \chi^{i'}} \mathcal{T}_{\hbar\kappa}$$

$$\mathcal{T} = \mathcal{T}_{ij} d\chi^{i} \bigotimes d\chi^{j}$$

$$\left(d\chi^{i} \bigotimes d\chi^{j}\right)(\upsilon, m) = \upsilon^{i}\omega^{j}$$

$$\left(\omega \bigotimes \mathcal{T}\right)_{i}^{j\kappa} = \omega_{i}\mathcal{T}^{j\kappa}$$

$$(f^{*}\omega)_{i}(\chi) = \frac{\partial \gamma^{\alpha}}{\partial \chi^{i}} \omega_{\alpha}(\gamma(\chi))$$

$$(\mathcal{L} \times \mathcal{T})(\chi) \coloneqq \lim_{t=0} \left[(f_{t}^{-1})' \mathcal{T}\left(f_{t}(\chi) - \mathcal{T}(\chi)\right)\right]$$

$$(\mathcal{L} \times \mathcal{T})^{j\kappa} = \chi^{i} \frac{\partial \mathcal{T}_{j\kappa}}{\partial \chi^{i}} - \mathcal{T}^{i\kappa} \frac{\partial \chi^{j}}{\partial \chi^{i}} - \mathcal{T}^{ji} \frac{\partial \chi^{\kappa}}{\partial \chi^{i}}$$

$$(\mathcal{L} \times \mathcal{T})_{j\kappa} = \chi^{i} \frac{\partial \mathcal{T}_{j\kappa}}{\partial \chi^{i}} + \mathcal{T}_{i\kappa} \frac{\partial \chi^{i}}{\partial \chi^{j}} + \mathcal{T}_{ji} \frac{\partial \chi^{i}}{\partial \chi^{\kappa}}$$

$$\omega = \frac{1}{\wp!} \omega_{i_{1}\cdots i_{\wp}} d\chi^{i_{1}} \wedge \cdots \wedge d\chi^{i_{\wp}}$$

$$d\chi^{i} \wedge d\chi^{j} = -d\chi^{j} \wedge d\chi^{i} \coloneqq \frac{1}{2}\left(d\chi^{i} \bigotimes d\chi^{j} - d\chi^{j} \bigotimes d\chi^{i}\right)$$

$$d\omega = \frac{1}{\wp!} d\omega_{i_{1}\cdots i_{\wp}} d\chi^{i_{1}} \wedge \cdots \wedge d\chi^{i_{\wp}}$$

Métrica de Riemann

$$\det(g) = \det(g') \left(\frac{\mathcal{D}(\chi')}{\mathcal{D}(\chi)}\right)^{2}$$

$$\mathcal{T}_{ij} = g_{i\hbar} g_{j\kappa} \mathcal{T}^{\hbar\kappa}$$

$$g = g_{ij}\, d\chi^{i} d\chi^{j}$$

$$\mu_{g} = |\det g|^{1/2}\, d\chi^{1} \wedge \cdots \wedge d\chi^{\eta}$$

$$\mu_{g} = |\det g|^{1/2}\, d\chi^{1} \cdots d\chi^{\eta}$$

$$|g_\chi(v,v)|^{1/2} \coloneqq |g_{ij}(\chi)v^i v^j|^{1/2}$$

$$\ell = \int_{\tau_1}^{\tau_2} \left| g_{ij}(\chi(\tau)) \frac{d\chi^i}{d\tau} \frac{d\chi^j}{d\tau} \right|^{1/2} d\tau$$

$$\left| g_{ij}(\chi(s)) \frac{d\chi^i}{ds} \frac{d\chi^j}{ds} \right| = 1$$

$$g \equiv g_{\alpha\beta} d\chi^\alpha d\chi^\beta$$

$$g \equiv -(\theta^0)^2 + \sum_{i=1\cdots\eta} (\theta^i)^2$$

$$\theta^\alpha = a^\alpha_\beta d\chi^\beta$$

$$-\mathcal{N}^2 dt^2 + g_{ij}\, d\chi^i d\chi^j$$

$$g'_{i0} = \frac{\partial \chi^j}{\partial \chi'^i} \left(g_{j0} + g_{j\hbar} \frac{\partial \chi^\hbar}{\partial \chi'^0} \right)$$

Conexión de Riemann

$$\nabla(v + \omega) = \nabla v + \nabla \omega$$

$$\nabla(fv) = f\nabla v + df \bigotimes v$$

$$\nabla v = v^i \nabla e_i + dv^i \bigotimes e_i$$

$$\nabla e_i = \omega^\hbar_{ji} \theta^j \bigotimes e_\hbar$$

$$\omega^\hbar_{ji} = \mathcal{A}^\hbar_{\hbar'} \partial_j \mathcal{A}^{\hbar'}_i + \mathcal{A}^{\hbar'}_\hbar \mathcal{A}^j_{j'} \mathcal{A}^i_{i'} \omega^{\hbar'}_{j'i'}$$

$$\nabla v = (\nabla_j v^i)\theta^j \bigotimes e_\hbar$$

$$\nabla_j v^i = \partial_j v^i + \omega^\hbar_{ji} v^\hbar$$

$$\nabla_j \mu_i = \partial_j \mu_i + \omega^\hbar_{ji} \mu_\hbar$$

$$\nabla_i \mathcal{T}^{\hbar\kappa}_{jl} = \partial_i \mathcal{T}^{\hbar\kappa}_{jl} - \omega^m_{ij} \mathcal{T}^{\hbar\kappa}_{ml} + \omega^\hbar_{im} \mathcal{T}^{m\kappa}_{jl} + \omega^\kappa_{im} \mathcal{T}^{\hbar m}_{jl} - \omega^m_{il} \mathcal{T}^{\hbar\kappa}_{jm}$$

Métrica de Christoffel

$$\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$$

$$\partial_\lambda g_{\alpha\beta} - \Gamma^\mu_{\lambda\alpha} g_{\mu\beta} - \Gamma^\mu_{\lambda\beta} g_{\alpha\mu}$$

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$$

$$\left(\mathcal{L}_g \chi\right)_{\alpha\beta} \equiv \nabla_\alpha \chi_\beta + \nabla_\alpha \chi_\beta$$

$$(\circledast\, \omega)_{\alpha_{\wp+1}\cdots\alpha_\eta} := \frac{1}{\wp!} \mu_{\alpha_0\alpha_1\cdots\alpha_\eta} \omega^{\alpha_0\alpha_1\cdots\alpha_\wp}$$

$$(\delta\mathfrak{F})^\beta \equiv \nabla_\alpha \mathfrak{F}^{\alpha\beta}$$

Métrica Geodésica

$$\mu^\alpha \nabla_\alpha \upsilon^\beta = 0, \mu^\alpha = \frac{d\chi^\alpha}{d\lambda}$$

$$\mu^\alpha \nabla_\alpha \mu^\beta \frac{d\chi^\alpha}{d\lambda} \left\{ \frac{\partial \mu^\beta}{\partial \chi^\alpha} + \Gamma^\beta_{\alpha\lambda} \mu^\beta \right\} \frac{d^2\chi^\beta}{d\lambda^2} + \Gamma^\beta_{\alpha\gamma} \frac{d\chi^\alpha}{d\lambda} \frac{d\chi^\gamma}{d\lambda} - g_{\alpha\beta}(\chi) \frac{d\chi^\alpha}{d\lambda} \frac{d\chi^\beta}{d\lambda} = \ell_\sigma^2$$

$$\delta\ell_\sigma \equiv -\frac{\ell_\sigma^{-1}}{2} \int_0^1 \left\{ \frac{\partial g_{\alpha\beta}}{\partial \chi^\lambda} \delta\chi^\lambda \frac{d\chi^\alpha}{d\lambda} \frac{d\chi^\beta}{d\lambda} + 2 g_{\alpha\beta} \frac{d\delta\chi^\alpha}{d\lambda} \frac{d\chi^\beta}{d\lambda} \right\} d\lambda$$

$$\delta\ell \equiv -\frac{\ell_\sigma^{-1}}{2} \int_0^1 \delta\chi^\alpha \left\{ \frac{d}{d\lambda} \left(2 g_{\alpha\beta} \frac{d\chi^\beta}{d\lambda} - \frac{\partial g_{\lambda\beta}}{\partial \chi^\alpha} \frac{d\chi^\lambda}{d\lambda} \frac{d\chi^\beta}{d\lambda} \right) \right\} d\lambda$$

$$\delta\ell_{\sigma_0} \equiv \ell_\sigma^{-1} \int_0^1 \delta\chi^\alpha \left\{ \mu^\lambda \partial_\lambda \mu_\alpha - \Gamma^\beta_{\lambda\alpha} \mu^\lambda \mu_\beta \right\} d\lambda$$

$$\upsilon^\alpha \nabla_\alpha \chi^\beta - \chi^\alpha \nabla_\alpha \upsilon^\beta$$

$$\upsilon^\lambda \upsilon^\alpha \nabla_\lambda \nabla_\alpha \chi^\beta - \upsilon^\lambda \nabla_\lambda \chi^\alpha \nabla_\alpha \upsilon^\beta - \chi^\alpha \upsilon^\lambda \nabla_\lambda \nabla_\alpha \upsilon^\beta$$

$$\nabla^2_{\upsilon^2} \chi^\beta \equiv \frac{\mathcal{D}^2}{\mathcal{D}s^2} \chi^\beta = \chi^\alpha \upsilon^\lambda \upsilon^\mu \mathcal{R}^{\beta\mu}_{\lambda\alpha}$$

Curvatura

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) \upsilon^\lambda \equiv \mathcal{R}^{\lambda\mu}_{\alpha\beta} \upsilon^\mu$$

$$\nabla_\alpha \upsilon^\beta = \frac{\partial}{\partial \chi^\alpha} \upsilon^\beta + \Gamma^\beta_{\alpha\lambda} \upsilon^\lambda$$

$$\mathcal{R}^{\lambda\mu}_{\alpha\beta} = \partial_\alpha \Gamma^\lambda_{\beta\mu} - \partial_\beta \Gamma^\lambda_{\alpha\mu} + \Gamma^\lambda_{\alpha\wp} \Gamma^\wp_{\beta\mu} - \Gamma^\lambda_{\beta\wp} \Gamma^\wp_{\alpha\mu}$$

$$\mathcal{R}^{\lambda\mu}_{\alpha\beta} + \mathcal{R}^{\lambda\beta}_{\mu\alpha} + \mathcal{R}^{\lambda\alpha}_{\beta\mu} - \mathcal{R}_{\alpha\beta,\lambda\mu} \equiv \mathcal{R}_{\lambda\mu,\alpha\beta}$$

$$\left\{ \frac{d^2\ell_\sigma}{d\sigma^2} \right\}_{\sigma=0} \frac{d\ell_\sigma}{d\sigma} \equiv \ell_\sigma^{-1} \int_0^1 \left(\mu_\sigma^\lambda \nabla_\lambda \mu_\sigma^\alpha \right) g_{\alpha\beta}(\chi) \hbar_\sigma^\beta \, d\lambda$$

$$\frac{d^2\ell_\sigma}{d\sigma^2} = \frac{d}{d\sigma}\left\{\ell_\sigma^{-1}\int_0^1 \mu_\sigma^\alpha \nabla_\alpha \mu_{\beta,\sigma} \hbar_\sigma^\beta d\lambda\right\} \mu_\sigma^\lambda \nabla_\lambda \mu_{\beta,\sigma}$$

$$\left\{\frac{d^2\ell_\sigma}{d\sigma^2}\right\}_{\sigma=0} \equiv \ell_\sigma^{-1}\int_0^1 \left\{\frac{\partial}{\partial\sigma}\left\{\left(\mu_\sigma^\lambda \nabla_\lambda \mu_\sigma^\alpha\right) g_{\alpha\beta}(\psi) \hbar_\sigma^\beta\right\} d\lambda\right\}_{\sigma=0}$$

$$\left\{\frac{d^2\ell_\sigma}{d\sigma^2}\right\}_{\sigma=0} \equiv \ell_\sigma^{-1}\int_0^1 \left\{\hbar^\mu \hbar_\alpha \nabla_\mu\left(\upsilon^\lambda \nabla_\lambda \upsilon^\alpha\right)\right\}_{\sigma=0} ds$$

$$\hbar^\mu \nabla_\mu\left(\upsilon^\lambda \nabla_\lambda \upsilon^\alpha\right) \equiv \hbar^\mu \nabla_\mu \upsilon^\lambda \nabla_\lambda \upsilon^\alpha + \hbar^\mu \upsilon^\lambda \nabla_\mu \nabla_\lambda \upsilon^\alpha$$

$$\hbar^\mu \nabla_\mu\left(\upsilon^\lambda \nabla_\lambda \upsilon^\alpha\right) \equiv \hbar^\mu \nabla_\mu \upsilon^\lambda \nabla_\lambda \upsilon^\alpha + \hbar^\mu \upsilon^\lambda\left(\nabla_\mu \nabla_\lambda \upsilon^\alpha + \mathcal{R}_{\mu\lambda\cdots\beta}^{\cdots\alpha} \upsilon^\beta\right)$$

$$\hbar^\mu \nabla_\mu \upsilon^\lambda = \upsilon^\mu \nabla_\mu \hbar^\lambda$$

$$\hbar^\mu \nabla_\mu\left(\upsilon^\lambda \nabla_\lambda \upsilon^\alpha\right) \equiv \upsilon^\mu \nabla_\mu\left(\hbar^\lambda \nabla_\lambda \upsilon^\alpha\right) + \hbar^\mu \upsilon^\lambda \mathcal{R}_{\mu\lambda}^{\alpha\beta} \upsilon^\beta$$

$$\mathcal{J}^\alpha(\hbar) \coloneqq \left\{\hbar^\mu \nabla_\mu\left(\upsilon^\lambda \nabla_\lambda \upsilon^\alpha\right)\right\}_{\alpha=0} = \upsilon^\mu \upsilon^\lambda \nabla_\mu \nabla_\lambda \hbar^\alpha + \hbar^\mu \upsilon^\lambda \upsilon^\beta \mathcal{R}_{\mu\lambda}^{\alpha\beta}$$

$$\left\{\frac{d^2\ell(\mathfrak{C}_\alpha)}{d\alpha^2}\right\}_{\alpha=0} = \ell^{-1}\int_0^1 \mathcal{J}^\alpha(\hbar)\hbar_\alpha ds$$

$$\mathcal{J}^\alpha(\hbar) \equiv \frac{\mathcal{D}^2\hbar^\alpha}{\mathcal{D}\lambda^2} - \hbar^\mu \upsilon^\lambda \upsilon^\beta \mathcal{R}_{\mu\lambda}^{\alpha\beta}$$

Identidades de Bianchi

$$\nabla_\gamma \mathcal{R}_{\alpha\beta}^{\lambda\mu} + \nabla_\beta \mathcal{R}_{\gamma\alpha}^{\lambda\mu} + \nabla_\alpha \mathcal{R}_{\beta\gamma}^{\lambda\mu}$$

$$\nabla_\gamma \mathcal{R}_{\beta\mu} - \nabla_\beta \mathcal{R}_{\gamma\mu} + \nabla_\alpha \mathcal{R}_{\beta\gamma}^{\lambda\mu}$$

$$\nabla_\alpha \delta_\mu^\beta \equiv \delta_{\alpha\beta}\mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{R}$$

$$\delta(g) \coloneqq Ricc\,(g) - \frac{1}{2} g\mathcal{R}$$

Tensor de Ricci

$$\mathcal{R}_{\alpha\beta} \coloneqq \mathcal{R}_{\lambda\alpha}^{\lambda\beta} \equiv \partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda\mu}^\mu - \Gamma_{\alpha\mu}^\lambda \Gamma_{\beta\gamma}^\mu$$

$$\mathcal{R} \coloneqq g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$$

$$\upsilon^\lambda \upsilon^\alpha \nabla_\lambda \nabla_\alpha \chi^\beta - \upsilon^\lambda \nabla_\lambda \chi^\alpha \nabla_\alpha \upsilon^\beta - \chi^\alpha \upsilon^\lambda\left(\nabla_\alpha \nabla_\lambda \upsilon^\beta + \mathcal{R}_{\lambda\alpha}^{\beta\mu} \upsilon^\mu\right)$$

$$\upsilon^\lambda \upsilon^\alpha \nabla_\lambda \nabla_\alpha \chi^\beta - \upsilon^\lambda \nabla_\lambda \chi^\alpha \nabla_\alpha \upsilon^\beta - \chi^\alpha \nabla_\alpha \upsilon^\lambda \nabla_\lambda \upsilon^\beta - \chi^\alpha \upsilon^\lambda \mathcal{R}_{\lambda\alpha}^{\beta\mu} \upsilon^\mu$$

$$\upsilon^{\lambda}\upsilon^{\alpha}\nabla_{\lambda}\nabla_{\alpha}\chi^{\beta} = \chi^{\alpha}\upsilon^{\lambda}\upsilon^{\mu}\mathcal{R}_{\lambda\alpha}^{\beta\mu}$$

1.10. Tensores de Ricci y Einstein

$$\delta\mathcal{P} \coloneqq \mathcal{P}'_{\mu}\delta\mu, \mathcal{P}(\mu+\delta\mu) - \mathcal{P}(\mu) = \mathcal{P}'_{\mu}(\mu)\delta\mu + o(|\delta\mu|)g^{\alpha\lambda}g_{\alpha\beta} = \delta_{\beta}^{\lambda}$$

$$\delta g^{\alpha\beta} = -g^{\alpha\lambda}g^{\beta\mu}\hbar_{\lambda\mu} \coloneqq \delta g_{\lambda\mu}$$

$$\delta\Gamma_{\alpha\beta}^{\lambda} \equiv \frac{1}{2}\{\nabla_{\alpha}\hbar_{\beta}^{\lambda} + \nabla_{\beta}\hbar_{\alpha}^{\lambda} - \nabla^{\gamma}\hbar_{\alpha\beta}\}$$

$$\delta\mathcal{R}_{\alpha\beta} \equiv -\frac{1}{2}\nabla^{\lambda}\nabla_{\lambda}\hbar_{\alpha\beta} + \frac{1}{2}\{\nabla_{\lambda}\nabla_{\alpha}\hbar_{\beta}^{\lambda} + \nabla_{\lambda}\nabla_{\beta}\hbar_{\alpha}^{\lambda} - \nabla_{\alpha}\nabla_{\beta}\hbar_{\lambda}^{\lambda}\}$$

$$\delta\mathcal{R} \equiv g^{\alpha\beta}\delta\mathcal{R}_{\alpha\beta} + \mathcal{R}_{\alpha\beta}\delta g^{\alpha\beta}$$

$$g^{\alpha\beta}\delta\mathcal{R}_{\alpha\beta} \equiv -\nabla_{\lambda}\{\nabla^{\lambda}\hbar_{\alpha}^{\alpha} - \nabla_{\alpha}\hbar^{\lambda\alpha}\}$$

$$\delta^{2}\mathcal{P} \coloneqq \mathcal{P}''_{\mu^{2}}(\delta\mu,\delta\mu)\mathcal{P}(\mu+\delta\mu) - \mathcal{P}(\mu) = \mathcal{P}'_{\mu}(\mu)\delta\mu + \frac{1}{2}\mathcal{P}''_{\mu^{2}}(\mu)(\delta\mu,\delta\mu) + o(|\delta\mu|^{2})$$

$$\delta^{2}\mathcal{R}_{\alpha\beta} \coloneqq \mathcal{R}''_{\alpha\beta}(g)(\hbar,\hbar)$$

$$\delta^{2}\mathcal{R}_{\alpha\beta} \equiv -\hbar^{\lambda\mu}\{\nabla_{\lambda}(\nabla_{\alpha}\hbar_{\beta\mu} + \nabla_{\beta}\hbar_{\alpha\mu} - \nabla_{\mu}\hbar_{\alpha\beta}) - \nabla_{\alpha}\nabla_{\beta}\hbar_{\lambda\mu}\} - \nabla_{\lambda}\hbar^{\lambda\mu}(\nabla_{\alpha}\hbar_{\beta\mu} + \nabla_{\beta}\hbar_{\alpha\mu} - \nabla_{\mu}\hbar_{\alpha\beta})$$
$$+ \frac{1}{2}\nabla_{\beta}\hbar^{\lambda\mu}\nabla_{\alpha}\hbar_{\lambda\mu} + \frac{1}{2}\nabla^{\lambda}\hbar_{\wp}^{\wp}(\nabla_{\alpha}\hbar_{\beta\lambda} + \nabla_{\beta}\hbar_{\alpha\lambda} - \nabla_{\lambda}\hbar_{\alpha\beta}) + \nabla_{\lambda}\hbar_{\alpha}^{\mu}\nabla^{\lambda}\hbar_{\beta\mu}$$
$$- \nabla_{\lambda}\hbar_{\alpha}^{\mu}\hbar_{\beta\mu}\nabla_{\mu}\hbar_{\beta}^{\lambda}$$

Relatividad especial en espacios cuánticos curvos

Tensor de Maxwell

$$\tau_{\alpha\beta} \coloneqq \mathfrak{F}_{\alpha}^{\lambda}\mathfrak{F}_{\beta\lambda} - \frac{1}{4}\eta_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\mathfrak{F}_{\lambda\mu}$$

$$\tau_{00} = \frac{1}{2}(\xi^{2} + \mathcal{H}^{2}), \tau_{0i} = -\varepsilon_{ijl}\xi^{j}\mathcal{H}^{l}$$

$$\nabla_{\alpha}\tau^{\alpha\beta} = \mathcal{J}^{\lambda}\mathfrak{F}_{\beta\lambda}$$

$$\nabla_{\alpha}\tau_{\beta}^{\alpha} \equiv (\nabla_{\alpha}\mathfrak{F}^{\alpha\lambda})\mathfrak{F}_{\beta\lambda} + \mathfrak{F}^{\alpha\lambda}\nabla_{\alpha}\mathfrak{F}_{\beta\lambda} - \frac{1}{2}\mathfrak{F}^{\lambda\mu}\nabla_{\beta}\mathfrak{F}_{\lambda\mu}$$

$$\nabla_{\alpha}\tau_{\beta}^{\alpha} = (\nabla_{\alpha}\mathfrak{F}^{\alpha\lambda})\mathfrak{F}_{\beta\lambda} + \frac{1}{2}\mathfrak{F}^{\alpha\lambda}(\nabla_{\alpha}\mathfrak{F}_{\beta\lambda} + \nabla_{\lambda}\mathfrak{F}_{\alpha\beta} + \nabla_{\beta}\mathfrak{F}_{\lambda\alpha})$$

Grupo de Poincaré

$$\chi^{\alpha} = f^{\alpha}(\chi'^{1} \cdots \chi'^{\eta}) - (d\chi^{0})^{2} + \sum_{i=1\cdots\eta} \left(d\chi^{i}\right)^{2} = -(d\chi'^{0})^{2} + \sum_{i=1\cdots\eta} \left(d\chi'^{i}\right)^{2} - (\upsilon^{0})^{2} + \sum_{i=1\cdots\eta} \left(\upsilon^{i}\right)^{2}$$

$$- \left(\mathcal{L}^{0}_{\alpha'}\upsilon^{\alpha'}\right)^{2} + \sum_{i=1\cdots\eta} \left(\mathcal{L}^{i}_{\alpha'}\upsilon^{\alpha'}\right)^{2} \equiv -(\upsilon'^{0})^{2} + \sum_{i=1\cdots\eta} \left(\upsilon'^{i}\right)^{2}$$

Grupo de Lorentz

$$\left(\mathcal{L}^{0}_{0'}\right)^{2} = 1 + \sum_{i=1\cdots\eta} \left(\mathcal{L}^{i}_{0'}\right)^{2} \geq 1 \det(\mathcal{L}) = 1$$

$$\mathcal{L} = \mathcal{R}_{1}\mathcal{L}_{\delta}\mathcal{R}_{2}$$

$$\left(\mathcal{L}^{0}_{0'}\right)^{2} - \left(\mathcal{L}^{1}_{0'}\right)^{2} = 1$$

$$\left(\mathcal{L}^{1}_{1'}\right)^{2} - \left(\mathcal{L}^{0}_{1'}\right)^{2} = 1$$

$$\mathcal{L}^{0}_{0'}\mathcal{L}^{0}_{1'} - \mathcal{L}^{1}_{1'}\mathcal{L}^{1}_{0'} = 0$$

$$\mathcal{L}^{0}_{0'} = \mathcal{L}^{1}_{1'} = \cosh\varphi$$

$$\mathcal{L}^{1}_{0'} = \mathcal{L}^{0}_{1'} = \sinh\varphi$$

$$t - \tau = \frac{t' - \tau' + \mathcal{V}(\chi'^{1} - \xi'^{1})}{\sqrt{1 - \mathcal{V}^{2}}}, \chi^{1} - \xi^{1} = \frac{\chi'^{1} - \xi'^{1} + \mathcal{V}(t' - \tau')}{\sqrt{1 - \mathcal{V}^{2}}}$$

$$\left.\frac{d\chi^{1}}{dt'}\right|_{\chi'^{1}=const} = \frac{\mathcal{V}}{\sqrt{1 - \mathcal{V}^{2}}}$$

$$\left.\frac{d\chi'^{1}}{dt}\right|_{\chi^{1}=const} = \frac{-\mathcal{V}}{\sqrt{1 - \mathcal{V}^{2}}}$$

$$\mathfrak{F}_{\alpha\beta} \equiv \frac{\partial\chi^{\alpha'}}{\partial\chi^{\alpha}}\frac{\partial\chi^{\beta'}}{\partial\chi^{\beta}}\mathfrak{F}_{\alpha'\beta'} = \mathcal{L}^{\alpha'}_{\alpha}\mathcal{L}^{\beta'}_{\beta}\mathfrak{F}_{\alpha'\beta'}$$

$$\xi = \xi' + \upsilon \wedge \mathcal{H}, \mathcal{H} = \mathcal{H}' - \upsilon \wedge \xi$$

Contracción y dilatación de Lorentz

$$\chi^{1} - \xi^{1} = \frac{\chi'^{1} - \xi'^{1}}{\sqrt{1 - \mathcal{V}^{2}}}$$

$$t - \tau \geq t' - \tau'$$

Operador temporal

$$\int_{t_1}^{t_2} \left\{-\eta\left(\frac{d\mathfrak{C}}{dt}, \frac{d\mathfrak{C}}{dt}\right)\right\}^{1/2} dt$$

$$\eta(\mu, \mu) = -1$$

$$\mathcal{V}^i = \frac{\upsilon^i}{\upsilon^0}$$

$$|\mathcal{V}| \equiv \left\{\sum_i (\mathcal{V}^i)^2\right\}^{1/2} \leq 1$$

$$\mathcal{V} = \frac{\tilde{\mu}^1}{\tilde{\mu}^0} = \frac{\mathcal{L}^1_{0'}}{\mathcal{L}^0_{0'}}$$

$$\frac{d\chi^1}{dt} = \frac{d\chi'^1 + \mathcal{V}dt'}{dt' + \mathcal{V}d\chi'^1}$$

$$\mathcal{U} = \frac{\mathcal{U}' + \mathcal{V}}{1 + \mathcal{U}'\mathcal{V}}$$

Dinámica relativista de masa

$$\frac{d(m\upsilon)}{dt} = f$$

$$\frac{d}{dt}\left(\frac{1}{2}m\upsilon^2\right) = f \bigotimes \upsilon$$

$$\mu^\alpha = \frac{d\chi^\alpha}{ds}, \mu^\alpha\mu_\alpha = -1$$

$$\mu^\alpha\nabla_\alpha\left(m_0\mu^\beta\right) = \mathfrak{F}^\beta - \mu^\alpha\partial_\alpha m_0 = \mu_\beta\mathfrak{F}^\beta - \eta(\wp, \wp) \equiv -\wp_\alpha\wp^\alpha = (m_0)^2$$

$$\mu^\alpha\nabla_\alpha\wp^\beta = \mathfrak{F}^\beta$$

$$\wp^0 = m_0$$

$$m_0\left\{\frac{d^2\chi^\beta}{ds^2} + \Gamma^\beta_{\alpha\lambda}\frac{d\chi^\alpha}{ds}\frac{d\chi^\lambda}{ds}\right\} = \mathfrak{F}^\beta$$

Equivalencia de masa y energía

$$\mathcal{U}^i \coloneqq \frac{d\chi^i}{dt} = \frac{d\chi^i}{ds}\frac{ds}{dt} = \frac{\mu^i}{\mu^0}$$

$$\mu^i = \frac{\mathcal{U}^i}{\sqrt{1 - |\mathcal{U}|^2}}$$

$$\mu^0 = \frac{\mathcal{U}^0}{\sqrt{1-|\mathcal{U}|^2}}$$

$$|\mathcal{U}|^2 = \sum_i (\mathcal{U}^i)^2$$

$$\frac{d}{dt}\frac{m_0\mathcal{U}^i}{\sqrt{1-|\mathcal{U}|^2}} = \mathfrak{F}^i\sqrt{1-|\mathcal{U}|^2}$$

$$m = \frac{m_0}{\sqrt{1-|\mathcal{U}|^2}}\mathfrak{F}^i\sqrt{1-|\mathcal{U}|^2} = f^i$$

$$\frac{dm}{dt} = \mathfrak{F}^0\sqrt{1-|\mathcal{U}|^2} = f\otimes\mathcal{U}$$

$$m \cong m_0\left(1+\frac{1}{2}|\mathcal{U}|^2\right)$$

Materia continua

$$\frac{\partial\wp}{\partial t} + \partial_i(\rho\upsilon^i)\frac{d(\rho\upsilon^i)}{dt}\partial_\kappa t^{i\kappa}, \chi^i = t^{ij}\eta_j$$

$$\nabla_\alpha(r\mu^\alpha) = 0$$

$$\mu^\alpha\nabla_\alpha\mu^\beta = 0$$

$$\mathcal{T}^{\alpha\beta} = r\mu^\alpha\mu^\beta$$

$$\nabla_\alpha\mathcal{T}^{\alpha\beta} = 0$$

$$\mathcal{T}^{00} = r, \mathcal{T}^{0i} = \mathcal{T}^{i0} = 0, \mathcal{T}^{ij} = 0$$

$$\mathcal{T}^{ij} = \wp e^{ij}$$

$$\mathcal{T}^{\alpha\beta} = u\mu^\alpha\mu^\beta + \wp\left(\eta^{\alpha\beta} + \mu^\alpha\mu^\beta\right)$$

Relatividad general en espacios cuánticos curvos

Cuestiones preliminares

$$\mathfrak{F}_{grav} = m\gamma_{grav}$$

$$\frac{d^2\chi^i}{dt^2} = \gamma^i_{grav} = \frac{\partial\mathcal{U}}{\partial\chi^i}$$

$$\Delta\mathcal{U} = -4\pi\kappa\rho$$

$$g = -\mathcal{N}^2(\chi^1)dt^2 + g_{11}(\chi^1)(d\chi^1)^2 + g_{\alpha\beta}d\chi^\alpha d\chi^\beta$$

$$\frac{d\chi^1}{dt} = \frac{\sqrt{g_{11}}}{\mathcal{N}}$$

$$t(\hbar) = \int_0^\hbar \frac{\mathcal{N}}{\sqrt{g_{11}}}(\chi^1)(d\chi^1) + t(0)$$

$$t_2(0) - t_1(0) = \mathcal{N}^{-1}(0)\mathcal{T}(0) = t_2(\hbar) - t_1(\hbar) = \mathcal{N}^{-1}(\hbar)\mathcal{T}(\hbar)$$

$$\mathcal{T}(\hbar) = \frac{\mathcal{N}(\hbar)}{\mathcal{N}(0)}\mathcal{T}$$

$$\mathcal{N}^2 \cong 1 - 2\frac{m}{\chi^1}$$

$$\mathcal{R}icci(g) = 0$$

$$\mathcal{R}_{\alpha\beta} \equiv \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda}$$

$$\nabla_\alpha \delta^{\alpha\beta} \equiv \delta^{\alpha\beta} \equiv \mathcal{R}^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\mathcal{R} \equiv g^{\lambda\mu}\mathcal{R}_{\lambda\mu}$$

$$\zeta_{einstein}(g) + \Lambda g \equiv \mathcal{R}icci(g) - \frac{1}{2}g\mathcal{R}(g) + \Lambda g = \mathcal{T}$$

$$\delta_{\alpha\beta} + \Lambda g_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R} + \Lambda g_{\alpha\beta} = \mathcal{T}_{\alpha\beta}$$

$$\mathcal{R}_{\alpha\beta} = \rho_{\alpha\beta} \equiv \mathcal{T}_{\alpha\beta} + \left(\frac{d}{d-2}\Lambda - \frac{\mathcal{T}^\lambda_\lambda}{d-2}\right)g_{\alpha\beta}$$

$$\nabla_\alpha \mathcal{T}^{\alpha\beta} = 0$$

$$\mathfrak{F} \equiv \frac{1}{2}\mathfrak{F}_{\alpha\beta}d\chi^\alpha \wedge d\chi^\beta$$

$$\nabla_\alpha \mathfrak{F}_{\beta\gamma} + \nabla_\gamma \mathfrak{F}_{\alpha\beta} + \nabla_\beta \mathfrak{F}_{\gamma\alpha}$$

$$\tau_{\alpha\beta} \coloneqq \mathfrak{F}^\lambda_\alpha \mathfrak{F}_{\beta\lambda} - \frac{1}{4}g_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\mathfrak{F}_{\lambda\mu}$$

$$\nabla_\alpha \tau^{\alpha\beta} = \mathcal{J}^\lambda \mathfrak{F}^\beta_\lambda$$

$$\delta_{\alpha\beta} = \tau_{\alpha\beta}$$

$$\xi_\alpha = \mathfrak{F}_{\beta\alpha}\mu^\beta$$

$$({}^*\mathcal{F})_{\alpha\beta} = \frac{1}{2}\mu_{\alpha\beta\lambda\mu}\mathfrak{F}_{\lambda\mu}$$

$$\mathcal{H}_\alpha \coloneqq ({}^*\mathcal{F})_{\alpha\beta}\mu^\beta$$

$$\mathcal{J}^{\alpha} \equiv q\mu^{\alpha} + \sigma\xi^{\alpha}$$

$$\mathcal{J}^{\alpha} \coloneqq \tau\mathcal{P}_{\alpha}$$

$$\mathcal{P}_{\alpha} \coloneqq \mu_{\alpha\beta\lambda\mu}\mu^{\beta}\xi^{\lambda}\mathcal{H}^{\mu}$$

$$\mathfrak{F}_{\alpha\beta} = \partial_{\alpha}\mathcal{A}_{\beta} - \partial_{\beta}\mathcal{A}_{\alpha}$$

$$\nabla_{\alpha}\left(\nabla^{\alpha}\mathcal{A}^{\beta} - \nabla^{\beta}\mathcal{A}^{\alpha}\right) = \mathcal{J}^{\beta}$$

$$\nabla_{\alpha}\nabla^{\alpha}\mathcal{A}^{\beta} - \mathfrak{R}_{\lambda}^{\beta}\mathcal{A}^{\lambda} = \mathcal{J}^{\beta}$$

Campos de Yang – Mills

$$\mathfrak{F} = d\mathcal{A} + [\mathcal{A},\mathcal{A}], \mathfrak{F}_{\alpha\beta} = \partial_{\alpha}\mathcal{A}_{\beta} - \partial_{\beta}\mathcal{A}_{\alpha} + \left[\mathcal{A}_{\alpha},\mathcal{A}_{\beta}\right]\hat{d}\mathfrak{F}\langle\hat{\nabla}_{\alpha}\mathfrak{F}_{\beta\gamma} + \hat{\nabla}_{\gamma}\mathfrak{F}_{\alpha\beta}\rangle$$

$$\hat{\nabla}_{\alpha}\mathfrak{F}_{\beta\gamma} \coloneqq \nabla_{\alpha}\mathfrak{F}_{\beta\gamma} + \left[\mathcal{A}_{\alpha},\mathfrak{F}_{\beta\gamma}\right]$$

$$\hat{\nabla}_{\alpha}\mathfrak{F}^{\alpha\beta} \coloneqq \nabla_{\alpha}\mathfrak{F}^{\alpha\beta}\left[\mathcal{A}_{\alpha},\mathfrak{F}^{\alpha\beta}\right]$$

$$\tau_{\alpha\beta} \coloneqq \mathfrak{F}_{\alpha}^{\lambda}\otimes\mathfrak{F}_{\beta\lambda} - \frac{1}{4}g_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\otimes\mathfrak{F}_{\lambda\mu}$$

$$\mathcal{R}_{\alpha\beta} = \rho_{\alpha\beta} \equiv \mathfrak{F}_{\alpha}^{\lambda}\otimes\mathfrak{F}_{\beta\lambda} - \frac{1}{2(d-2)}g_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\otimes\mathfrak{F}_{\lambda\mu}$$

Campos escalares

$$\mathcal{T}_{\alpha\beta} \equiv \partial_{\alpha}\psi\partial_{\beta}\psi - \frac{1}{2}g_{\alpha\beta}\partial^{\lambda}\psi\partial_{\lambda}\psi - g_{\alpha\beta}\mathcal{U}(\psi)$$

$$\nabla_{\alpha}\mathcal{T}^{\alpha\beta} \equiv \{\nabla^{\alpha}\partial_{\alpha}\psi - \mathcal{U}'(\psi)\}\partial^{\beta}\psi\frac{d\mathcal{U}}{d\psi}$$

$$\nabla^{\alpha}\partial_{\alpha}\psi - \mathcal{U}'(\psi) = 0$$

$$\mathcal{T}_{\lambda}^{\lambda} \equiv \partial^{\lambda}\psi\partial_{\lambda}\psi\left(\frac{2-d}{2}\right) - d\mathcal{U}(\psi)$$

$$\mathcal{R}_{\alpha\beta} = \partial_{\alpha}\psi\partial_{\beta}\psi + \frac{d-3}{2}g_{\alpha\beta}\partial^{\lambda}\psi\partial_{\lambda}\psi + \frac{d}{d-2}g_{\alpha\beta}\mathcal{U}(\psi)$$

$$\mathcal{T}_{\alpha\beta} = \frac{1}{2}\left\{\mathfrak{D}_{\alpha}\psi\left(\mathfrak{D}_{\beta}\psi\right)^{*} + (\mathfrak{D}_{\alpha}\psi)^{*}\mathfrak{D}_{\beta}\psi - g_{\alpha\beta}\mathfrak{D}^{\lambda}\psi\left(\mathfrak{D}^{\lambda}\psi\right)^{\dagger}\right\}$$

$$\mathfrak{D}_{\alpha}\psi = \partial_{\alpha}\psi + i\mathcal{A}_{\alpha}\psi$$

Mapa de ondas cuánticas.

$$\chi^{\alpha} \mapsto \gamma^{\mathrm{A}} = \mu^{\mathrm{A}}(\chi^{\alpha})$$

$$\partial\mu(\chi) \in \mathcal{T}_{\chi}^{\dagger}\mathcal{V}\otimes\mathcal{T}_{\mu(\chi)}\mathcal{M}$$

$$\widehat{\nabla}_{\alpha} f_{\beta}^{A}(\chi) \equiv \partial_{\alpha} f_{\beta}^{A}(\chi) - \Gamma_{\alpha\beta}^{\mu}(\chi) f_{\mu}^{A}(\chi) + \partial_{\alpha}\mu^{\beta}(\chi) \Gamma_{\mathcal{BC}}^{\mathcal{A}}(\mu(\chi)) f_{\beta}^{\mathcal{C}}(\chi)$$

$$(\widehat{\nabla}_{\alpha}\widehat{\nabla}_{\beta} - \widehat{\nabla}_{\beta}\widehat{\nabla}_{\alpha}) f_{\lambda}^{A} = \mathcal{R}_{\alpha\beta\lambda^{\mu}} f_{\mu}^{A} + \partial_{\alpha}\mu^{\mathcal{C}} \partial_{\beta}\mu^{B} \mathcal{R}_{\mathcal{BC}}^{\mathcal{AD}} f_{\mu}^{\mathcal{D}}$$

$$g^{\alpha\beta}\widehat{\nabla}_{\alpha}\partial_{\beta}\mu^{A} \equiv g^{\alpha\beta}\left\{\partial_{\alpha}\partial_{\beta}\mu^{A} - \Gamma_{\alpha\beta}^{\mu}(\chi)\partial_{\mu}\mu^{A} + \Gamma_{\mathcal{BC}}^{\mathcal{A}}\partial_{\alpha}\mu^{\beta}\partial_{\beta}\mu^{\mathcal{C}}\right\}$$

$$\mathcal{T}_{\alpha\beta} = \partial_{\alpha}\mu \otimes \partial_{\beta}\mu - \frac{1}{2} g_{\alpha\beta}\partial^{\lambda}\mu \otimes \partial_{\lambda}\mu \int_{\mathcal{V}} g(\partial\mu, \partial\mu)\mu_{g}$$

Condiciones de energía

$$\mathcal{T}_{\alpha\beta}\chi^{\alpha}\chi^{\beta} \geq 0$$

$$\rho_{\alpha\beta}\chi^{\alpha}\chi^{\beta} \geq 0$$

$$\rho_{\alpha\beta} \coloneqq \mathcal{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{T}_{\lambda}^{\lambda}$$

Lagrangianos

$$\mathcal{L}_{grav}(g) \coloneqq \int \mathcal{R}(g)\mu_{g}$$

$$\delta\mathcal{L}_{grav}(g) = \int \delta\mathcal{R}(g)\mu_{g} + \int \mathcal{R}(g)\delta\mu_{g}$$

$$\int \delta\mathcal{R}(g)\mu_{g} = \int \mathcal{R}_{\alpha\beta}\, \delta g^{\alpha\beta}\mu_{g}$$

$$\delta\mu_{g} = \frac{1}{2} g^{\alpha\beta}\delta g_{\alpha\beta}\mu_{g} = \frac{1}{2} g_{\alpha\beta}\delta g^{\alpha\beta}\mu_{g}$$

$$\delta\mathcal{L}_{grav}(g) \equiv \int \left(\mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{R}\right)\delta g^{\alpha\beta}\mu_{g} = -\int \delta^{\alpha\beta}\delta g_{\alpha\beta}\mu_{g}$$

$$\mathcal{L}_{e\otimes m}(\mathcal{F}, g) \coloneqq \int \mathfrak{F}^{\alpha\beta}\mathfrak{F}_{\alpha\beta}\mu_{g}$$

$$\mathcal{L}_{scal}(\varphi, g) \coloneqq \int g^{\alpha\beta}\partial_{\alpha}\varphi\partial_{\beta}\varphi\mu_{g}$$

$$\mathcal{L}_{sour}(g, \psi) \coloneqq \int \ell(g, \psi)\mu_{g}$$

$$\delta\mathcal{L}_{sour} \equiv \int \{\psi(g, \psi) \otimes \delta\psi + \mathcal{T}^{\alpha\beta}(g, \psi)\delta g_{\alpha\beta}\}\mu_{g}$$

$$\delta g^{\lambda\mu} = (\mathcal{L} \otimes g)^{\lambda\mu} \equiv \nabla^{\lambda}\chi^{\mu} + \nabla^{\mu}\chi^{\lambda}$$

$$\delta(\mathcal{L}_{grav} + \mathcal{L}_{sour}) = \iint (-\delta^{\alpha\beta} + \mathcal{T}^{\alpha\beta})\delta g_{\alpha\beta}\mu_{g}$$

Fluidos relativistas en espacios cuánticos curvos

$$\mathcal{T}^{\alpha\beta} := u\mu^{\alpha}\mu^{\beta} + \wp\left(g^{\alpha\beta} + \mu^{\alpha}\mu^{\beta}\right)$$

$$\nabla_{\alpha}\mathcal{T}^{\alpha\beta} \equiv \mu^{\beta}\nabla_{\alpha}[(\mu + \wp)\mu^{\alpha}] + (\mu + \wp)\mu^{\alpha}\nabla_{\alpha}\mu^{\beta} + \partial^{\beta}\wp$$

$$\nabla_{\alpha}[(\mu + \wp)\mu^{\alpha}] - \mu^{\beta}\partial_{\beta}\wp(\mu + \wp)\mu^{\alpha}\nabla_{\alpha}\mu^{\beta} + \left(g^{\alpha\beta} + \mu^{\alpha}\mu^{\beta}\right)\partial_{\alpha}\wp$$

Espacio – tiempo cuántico einsteniano

$$\zeta_{einstein}(g) = \mathcal{T}, \delta_{\alpha\beta} = \mathcal{T}_{\alpha\beta}$$

$$\nabla_{\alpha}\tau^{\alpha\beta} = 0$$

$$\Gamma(g) \in \mathcal{H}_{s-1}^{loc}(\mathcal{V}) \otimes \mathcal{R}ieman(g) \otimes \mathcal{R}icci(g) \in \mathcal{H}_{s-2}^{loc}(\mathcal{V}) - (d\chi^{0})^{2} + \sum_{i=1}^{3}\left(d\chi^{i}\right)^{2}$$

$$g_{00} = -(1 + f_{00}), g_{0i} = f_{0i}, g_{ij} = \delta_{ij} + f_{ij}$$

$$\mathcal{R}_{00} \cong \partial_{i}\Gamma_{00}^{i} \cong \frac{1}{2}\Delta f_{00}$$

$$\Delta\mathcal{U} = -4\pi\kappa\rho$$

$$\mathcal{T}_{\alpha\beta} = u\mu^{\alpha}\mu^{\beta}$$

$$\rho_{00} \cong \mathcal{T}_{00} - \frac{1}{2}g_{00}\mathcal{T} \cong \frac{1}{2}\mu$$

$$\Delta f_{00} \cong \mu$$

$$\delta_{\alpha\beta} = 8\pi\kappa\mathcal{T}_{\alpha\beta}$$

$$\delta_{\alpha\beta} = \mathcal{T}_{\alpha\beta}$$

$$\mathcal{T}_{\alpha\beta} \equiv u\mu^{\alpha}\mu^{\beta}$$

$$\frac{d^{2}\chi^{\alpha}}{ds^{2}} + \Gamma_{\mu\lambda}^{\alpha}\frac{d\chi^{\lambda}}{ds}\frac{d\chi^{\mu}}{ds}\frac{d^{2}\chi^{i}}{(d\chi^{0})^{2}} \sim -\Gamma_{00}^{i} \sim -\frac{1}{2}\partial_{i}f_{00}$$

Ondas relativistas en espacios cuánticos curvos

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \hbar_{\alpha\beta}$$

$$\mathcal{R}_{\alpha\beta} \cong -\frac{1}{2}\eta^{\lambda\mu}\partial_{\lambda\mu}^{2}\hbar_{\alpha\beta} + \frac{1}{2}\partial_{\alpha}f_{\beta} + \frac{1}{2}\partial_{\beta}f_{\alpha}$$

$$f_{\alpha} := \partial_{\lambda}\hbar_{\alpha}^{\lambda} - \frac{1}{2}\partial_{\alpha}\hbar_{\lambda}^{\lambda}$$

$$\eta^{\lambda\mu}\partial_{\lambda\mu}^{2}\hbar_{\alpha\beta} = 0$$

$$\mathcal{R}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta}^{(\hbar)} + \mathcal{L}_{\alpha\beta} \equiv \frac{1}{2}\{g_{\alpha\lambda}\partial_\beta \mathcal{F}^\lambda + g_{\beta\lambda}\partial_\alpha \mathcal{F}^\lambda\}, \mathcal{F}^\lambda \coloneqq g^{\alpha\beta}\Gamma_{\alpha\beta}^\lambda$$

$$\mathcal{R}_{\alpha\beta}^{(\hbar)} \equiv -\frac{1}{2} g^{\lambda\mu}\partial_{\lambda\mu}^2 g_{\alpha\beta} + \mathcal{H}_{\alpha\beta}^{\rho\sigma\gamma\delta\lambda\mu}(g)\partial_\wp g_{\gamma\delta}\partial_\sigma g_{\lambda\mu}$$

$$\Box_g \mu \coloneqq \nabla^\lambda \partial_\lambda \mu \equiv g^{\lambda\mu}(\partial_{\lambda\mu}^2 \mu - \Gamma_{\lambda\mu}^\alpha \partial_\alpha \mu)$$

$$\mu(\chi) = \alpha(\chi) e^{i\omega\phi(\chi)}$$

$$g_{\alpha\beta}(\chi, \omega\phi(\chi)) \equiv \underline{g}_{\alpha\beta}(\chi) + \omega^{-2} v_{\alpha\beta}(\chi, \omega\phi(\chi))$$

$$\mathcal{R}_{\alpha\beta} = \omega^{-2}\mathfrak{R}_{\alpha\beta}(\chi, \omega)$$

$$\frac{\partial}{\partial \chi^\alpha} f(\chi, \omega\phi(\chi)) \equiv \partial_\alpha f(\chi, \xi) + \omega\phi_\alpha f'(\chi, \xi)|_{\xi=\omega\phi(\chi)}$$

$$\partial_\alpha f \coloneqq \frac{\partial}{\partial \chi^\alpha} f(\chi, \xi), \phi_\alpha \coloneqq \frac{\partial \phi}{\partial \chi^\alpha}, f' \coloneqq \frac{\partial}{\partial \xi} f(\chi, \xi) \lim_{\mathcal{T}=0} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f'(\chi, \xi) d\xi$$

$$g^{\alpha\beta}(\chi, \omega\phi) = \underline{g}^{\alpha\beta} - \omega^{-2} v^{\alpha\beta}(\chi, \omega\phi) + \omega^{-3} g^{(3)\alpha\beta}(\chi, \omega)$$

$$\Gamma_{\alpha\beta}^\lambda = \underline{\Gamma}_{\alpha\beta}^\lambda + \omega^{-1}\Gamma_{\alpha\beta}^{\lambda(1)} + \omega^{-2}\Gamma_{\alpha\beta}^{\lambda(2)} + \mathcal{O}(\omega^{-3})$$

$$\Gamma_{\alpha\beta}^{\lambda(1)} \coloneqq \gamma_{\alpha\beta}^\lambda \equiv \frac{1}{2}(\phi_\alpha v_\beta'^\lambda + \phi_\beta v_\alpha'^\lambda - \phi^\lambda v_{\alpha\beta}'), \gamma_{\alpha\beta}^\lambda \coloneqq \frac{1}{2}\phi_\alpha v_\lambda'^\lambda$$

$$\Gamma_{\alpha\beta}^{\lambda(2)} \coloneqq \varphi_{\alpha\beta}^\lambda \equiv \frac{1}{2}(\underline{\nabla}_\alpha v_\beta^\lambda + \underline{\nabla}_\beta v_\alpha^\lambda - \underline{\nabla}^\lambda v_{\alpha\beta}), \varphi_{\alpha\lambda}^\lambda : \equiv \frac{1}{2}\underline{\nabla}_\alpha(v_\lambda^\lambda)$$

$$(\underline{\nabla}_\alpha \mu^\beta)(\chi, \omega\phi) = \left\{\frac{\partial}{\partial \chi^\alpha}\mu^\beta(\chi, \xi) + \underline{\Gamma}_{\alpha\lambda}^\lambda(\chi)\mu^\lambda(\chi, \xi)\right\}_{\xi=\omega\phi}$$

$$\mathcal{R}_{\alpha\beta} \equiv \partial_\lambda \Gamma_{\alpha\beta}^\lambda - \partial_\alpha \Gamma_{\beta\lambda}^\lambda + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda\mu}^\mu - \Gamma_{\alpha\mu}^\lambda \Gamma_{\beta\lambda}^\mu$$

$$\mathcal{R}_{\alpha\beta}(\chi, \omega\phi) = \mathcal{R}_{\alpha\beta}^{(0)}(\chi, \omega\phi(\chi)) + \omega^{-1}\mathcal{R}_{\alpha\beta}^{(1)}(\chi, \omega\phi(\chi)) + \omega^{-2}\mathfrak{R}_{\alpha\beta}(\chi, \omega)$$

$$\mathcal{R}_{\alpha\beta}^{(0)}(\chi, \omega\phi(\chi)) \equiv \underline{\mathcal{R}}_{\alpha\beta} - \frac{1}{2}\left\{\underline{g}^{\lambda\mu}\phi_\lambda\phi_\mu v_{\alpha\beta}'' - \phi_\beta \mathcal{P}_\alpha'' - \phi_\alpha \mathcal{P}_\beta''\right\}(\chi, \omega\phi(\chi))$$

$$\mathcal{P}_\alpha'' \coloneqq \phi^\mu \left(v_{\mu\alpha}'' - \frac{1}{2}\underline{g}_{\mu\alpha} v_\rho''^\rho\right)$$

$$\mathcal{P}_\alpha \equiv \phi^\mu \left(v_{\mu\alpha} - \frac{1}{2}\underline{g}_{\mu\alpha} v_\rho^\rho\right) \langle v^{\lambda\mu} | \phi_\lambda | \phi_\mu \rangle$$

$$v_{\alpha\beta} = \phi_\alpha f_\beta + \phi_\beta f_\alpha$$

$$\chi^\alpha = \tilde{\chi}^\alpha + \omega^{-3}\hbar^\alpha(\tilde{\chi}, \omega\phi(\tilde{\chi}))$$

$$\tilde{g}_{\lambda\mu} = \left(\underline{g}_{\alpha\beta} + \omega^{-2}\upsilon_{\alpha\beta}\right)\frac{\partial\chi^{\alpha}}{\partial\tilde{\chi}^{\lambda}}\frac{\partial\chi^{\beta}}{\partial\tilde{\chi}^{\mu}}$$

$$\frac{\partial\chi^{\alpha}}{\partial\tilde{\chi}^{\lambda}} = \delta^{\alpha}_{\lambda} + \omega^{-2}\phi_{\lambda}\hbar'^{\alpha} + \omega^{-3}\tilde{\partial}_{\lambda}\hbar^{\alpha}$$

$$\tilde{g}_{\lambda\mu} = \underline{g}_{\lambda\mu} + \omega^{-2}\left(\upsilon_{\lambda\mu} + \underline{g}_{\lambda\beta}\phi_{\mu}\hbar'^{\beta} + \underline{g}_{\mu\alpha}\phi_{\lambda}\hbar'^{\alpha}\right) + \omega^{-3}\mathcal{M}_{\lambda\mu}$$

$$\mathcal{R}^{(1)}_{\alpha\beta} \equiv \underline{\nabla}_{\lambda}\gamma^{\lambda}_{\alpha\beta} - \underline{\nabla}_{\alpha}\gamma^{\lambda}_{\beta\lambda} + \phi_{\lambda}\varphi'^{\lambda}_{\alpha\beta} - \phi_{\alpha}\varphi'^{\lambda}_{\beta\lambda}$$

$$\mathcal{R}^{(1)}_{\alpha\beta} \equiv \frac{1}{2}\{\underline{\nabla}_{\lambda}(\phi_{\alpha}\upsilon'^{\lambda}_{\beta} + \phi_{\beta}\upsilon'^{\lambda}_{\alpha} - \phi^{\lambda}\upsilon'_{\alpha\beta}) - \underline{\nabla}_{\alpha}(\phi_{\beta}\upsilon'^{\lambda}_{\lambda})\}$$

$$+\frac{1}{2}\{\phi_{\lambda}(\underline{\nabla}_{\alpha}\upsilon'^{\lambda}_{\beta} + \underline{\nabla}_{\beta}\upsilon'^{\lambda}_{\alpha} - \underline{\nabla}^{\lambda}\upsilon'_{\alpha\beta}) - \phi_{\alpha}(\underline{\nabla}_{\beta}\upsilon'^{\lambda}_{\lambda})\}$$

$$\phi_{\lambda}(\underline{\nabla}_{\alpha}\upsilon'^{\lambda}_{\beta}) \equiv \underline{\nabla}_{\alpha}(\phi_{\lambda}\upsilon'^{\lambda}_{\beta}) - \upsilon'^{\lambda}_{\beta}\underline{\nabla}_{\alpha}\phi_{\lambda}\underline{\nabla}_{\lambda}\phi_{\alpha} - \mathcal{P}_{\alpha\beta}$$

$$+\frac{1}{2}\left\{\phi_{\beta}\left(\underline{\nabla}_{\mu}\upsilon'^{\mu}_{\alpha} - \frac{1}{2}\partial_{\alpha}\upsilon'^{\mu}_{\mu}\right) + \phi_{\alpha}\left(\underline{\nabla}_{\mu}\upsilon'^{\mu}_{\beta} - \frac{1}{2}\partial_{\beta}\upsilon'^{\mu}_{\mu}\right)\right\}$$

$$\mathcal{P}_{\alpha\beta} \coloneqq \phi^{\lambda}\underline{\nabla}_{\lambda}\upsilon'_{\alpha\beta} + \frac{1}{2}\underline{\nabla}_{\lambda}\phi^{\lambda}\upsilon'_{\alpha\beta}$$

$$\mathcal{P}_{ij} \equiv \phi^{\lambda}\underline{\nabla}_{\lambda}\upsilon'_{ij} + \frac{1}{2}\underline{\nabla}_{\lambda}\phi^{\lambda}\upsilon'_{ij}$$

$$\phi^{\lambda}\underline{\nabla}_{\lambda}(\phi^{i}\upsilon'_{ij}) + \frac{1}{2}\underline{\nabla}_{\lambda}\phi^{\lambda}(\phi^{i}\upsilon'_{ij}) - (\phi^{\lambda}\underline{\nabla}_{\lambda}\phi^{i})\upsilon'_{ij}$$

$$\mathfrak{F}_{\alpha\beta}(\chi) \coloneqq \underline{\mathfrak{F}}_{\alpha\beta}(\chi) + \omega^{-1}\mathfrak{H}_{\alpha\beta}(\chi, \omega\phi(\chi))$$

$$\tau_{\alpha\beta} = \underline{\tau}_{\alpha\beta} + \omega^{-1}\left\{\underline{\mathfrak{F}}^{\lambda}_{\alpha}\mathfrak{H}_{\beta\lambda} + \underline{\mathfrak{F}}_{\beta\lambda}\mathfrak{H}^{\lambda}_{\alpha}\right\} + \mathcal{O}(\omega^{-2})$$

$$\tau_{\alpha\beta} \equiv \mathfrak{F}^{\lambda}_{\alpha}\mathfrak{F}_{\beta\lambda} - \frac{1}{4}g_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\mathfrak{F}_{\lambda\mu}(\mathcal{R}_{\alpha\beta} - \tau_{\alpha\beta})(\chi, \omega\phi(\chi)) = \omega^{-2}\mathcal{M}(\chi,\omega)\mathcal{R}^{(0)}_{\alpha\beta} - \underline{\tau}_{\alpha\beta}\langle\underline{\mathcal{R}}_{\alpha\beta} + \underline{\tau}_{\alpha\beta}\rangle$$

$$(d\mathfrak{F})^{(0)}_{\alpha\beta\gamma} = \left(d\underline{\mathfrak{F}}\right)_{\alpha\beta\gamma} + \phi_{\alpha}\mathfrak{H}'_{\beta\gamma} + \phi_{\gamma}\mathfrak{H}'_{\alpha\beta} + \phi_{\beta}\mathfrak{H}'_{\gamma\alpha}$$

$$\left(\nabla_{\alpha}\mathfrak{F}^{\alpha}_{\beta}\right)^{(0)} \equiv \underline{\nabla}_{\alpha}\underline{\mathfrak{F}}^{\alpha}_{\beta} + \phi_{\alpha}\mathfrak{H}'_{\beta\gamma} + \phi_{\gamma}\mathfrak{H}'_{\alpha\beta} + \phi_{\beta}\mathfrak{H}'_{\gamma\alpha}\|\phi^{\alpha}\mathfrak{H}'_{\alpha\beta}\|\langle\mathfrak{H}'_{ij}|\phi^{i}|\mathfrak{H}'_{i0}\rangle$$

$$\mathfrak{H}_{\alpha\beta} = a_{\alpha}\phi_{\beta} - a_{\beta}\phi_{\alpha}|\phi^{\alpha}a_{\alpha}|$$

$$\mathcal{R}^{(1)}_{\alpha\beta} - \tau^{(1)}_{\alpha\beta} = 0$$

$$\phi^{\lambda}\underline{\nabla}_{\lambda}\upsilon'_{ij} + \frac{1}{2}\underline{\nabla}_{\lambda}\phi^{\lambda}\upsilon'_{ij} + \underline{g}_{ij}\underline{\mathfrak{F}}^{0\hbar}\alpha_{\hbar} - \underline{\mathfrak{F}}^{0}_{i}\alpha_{j} - \underline{\mathfrak{F}}^{0}_{j}\alpha_{i}\left\langle\phi_{\lambda}\middle|\underline{\mathfrak{F}}^{\lambda\mu}\right\rangle\langle\kappa\phi^{\mu}\rangle$$

$$(d\mathfrak{F})^{(1)}_{\alpha\beta\gamma} \equiv \left(d\underline{\mathfrak{H}}\right)_{\alpha\beta\gamma} \equiv \partial_{\alpha}\mathfrak{H}_{\beta\gamma} + \partial_{\gamma}\mathfrak{H}_{\alpha\beta} + \partial_{\beta}\mathfrak{H}_{\gamma\alpha}$$

$$\left(\nabla_\alpha \mathfrak{F}_\beta^\alpha\right)^{(1)} \equiv \underline{\nabla}_\alpha \underline{\mathfrak{H}}_\beta^\alpha + \Gamma_{\alpha\lambda}^{\alpha(1)} \underline{\mathfrak{F}}_\beta^\lambda - \Gamma_{\alpha\beta}^{\lambda(1)} \underline{\mathfrak{F}}_\lambda^\alpha$$

$$\underline{\nabla}_\alpha \underline{\mathfrak{H}}_\beta^\alpha \equiv \phi_\beta \underline{\nabla}^\alpha a_\alpha + a_\alpha \underline{\nabla}^\alpha \phi_\beta - \phi_\alpha \underline{\nabla}^\alpha a_\beta - a_\beta \underline{\nabla}^\alpha \phi_\alpha$$

$$a_\alpha \underline{\nabla}^\alpha \phi_\beta \equiv a_\alpha \underline{\nabla}_\beta \phi^\alpha \equiv \underline{\nabla}_\beta (a_\alpha \phi^\alpha) - \phi^\alpha \underline{\nabla}_\beta a_\alpha = -\phi^\alpha \underline{\nabla}_\beta a_\alpha \langle \partial_i \alpha_j - \partial_j \alpha_i \rangle$$

$$\underline{\nabla}_\alpha \mathfrak{H}_i^\alpha = -2\phi^\alpha \underline{\nabla}_\alpha \alpha_i - \alpha_i \underline{\nabla}_\alpha \phi^\alpha$$

$$\Gamma_{\alpha i}^{\lambda(1)} \underline{\mathfrak{F}}_\lambda^\alpha = \underline{\mathfrak{F}}^{0j} \upsilon'_{ij} \left\langle 2\phi^\alpha \underline{\nabla}_\alpha \alpha_i \middle| \alpha_i \underline{\nabla}_\alpha \phi^\alpha \middle| \underline{\mathfrak{F}}^{0j} \upsilon'_{ij} \right\rangle$$

Condiciones de gauge

$$g_{\alpha\beta} = \underline{g}_{\alpha\beta} + \omega^{-2} \upsilon_{\alpha\beta} + \omega^{-3} \omega_{\alpha\beta}$$

$$\phi_\beta \left(\phi^\mu \omega''_{\mu\alpha} - \frac{1}{2} \phi_\alpha \omega''^{\rho}_{\rho} \right) + \phi_\alpha \left(\phi^\mu \omega''_{\mu\beta} - \frac{1}{2} \phi_\beta \omega''^{\rho}_{\rho} \right)$$

$$\underline{\nabla}_\mu \upsilon'^{\mu}_{\alpha} - \frac{1}{2} \partial_\alpha \upsilon'^{\mu}_{\mu} + \phi^\mu \omega''_{\mu\alpha} - \frac{1}{2} \phi_\alpha \omega''^{\rho}_{\rho}$$

Conservación de la energía

$$\underline{\nabla}_\lambda \left(\phi^\lambda \overline{\upsilon}'^{ij} \upsilon'_{ij} \right)$$

$$\overline{\upsilon}'^{ij} := \underline{g}^{i\hbar} \underline{g}^{j\kappa} \upsilon'_{\hbar\kappa}$$

$$\mathfrak{E} := \frac{1}{4} \overline{\upsilon}'^{ij} \upsilon'_{ij}, \overline{\upsilon}'^{ij} := \underline{g}^{i\hbar} \underline{g}^{j\kappa} \upsilon'_{\hbar\kappa}$$

$$\mathfrak{E} = \frac{1}{4} \left\{ \upsilon'^{\alpha\beta} \upsilon'_{\alpha\beta} - \frac{1}{2} (\upsilon'^{\alpha}_{\alpha})^2 \right\} \underline{\nabla}_\lambda (\phi^\lambda \mathfrak{E})$$

$$\underline{\nabla}_\alpha \left\{ \phi^\alpha \left(\alpha^i \alpha_i + \frac{1}{4} \upsilon'^{ij} \upsilon'_{ij} \right) \right\} \alpha^i \alpha_i + \frac{1}{4} \upsilon'^{ij} \upsilon'_{ij} \underline{\nabla}_\alpha (\phi^\alpha \xi)$$

$$\xi = \xi_{grav} + \xi_{e \otimes m} := \frac{1}{4} \left\{ \upsilon'^{\alpha\beta} \upsilon'_{\alpha\beta} - \frac{1}{2} (\upsilon'^{\alpha}_{\alpha})^2 \right\} + \frac{1}{2} \mathfrak{H}^{\alpha\beta} \mathfrak{H}_{\alpha\beta}$$

Agujeros negros cuánticos

Métrica de Schwarzschild

$$e^{\hbar(\rho)} d\rho^2 + f^2(\rho)(d\theta^2 + sin^2\theta d\varphi^2)$$

$$\bar{g}_t = e^{\lambda(r,t)} dr^2 + r^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

$$g = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

$$g = -\alpha^2(r,t) dt^2 + 2\beta(r,t) dt dr + e^{\lambda(r,t)} dr^2 + r^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

$$e^{-2\nu} \omega \equiv e^{-2\nu}(t,r)(\alpha dt - \beta dr) \equiv d\tau$$

$$g_{\text{Schwarzschild}} = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

$$\Gamma_{00}^0 = \frac{1}{2}\partial_t\nu, \Gamma_{00}^1 = \frac{1}{2}e^{\nu-\lambda}\nu', \Gamma_{01}^0 = \frac{\nu'}{2}, \Gamma_{11}^0 = \frac{\partial_t\lambda}{2}e^{\lambda-\nu}, \Gamma_{01}^1 = \frac{\partial_t\lambda}{2}$$

$$\Gamma_{11}^1 = \frac{\lambda'}{2}, \Gamma_{22}^1 = -re^{-\lambda}, \Gamma_{12}^2 = \Gamma_{13}^3 = r^{-1}, \Gamma_{33}^1 = -rsin^2\theta e^{-\lambda}, \Gamma_{33}^2 = -\sin\theta\cos\theta, \Gamma_{23}^3 = \cot\theta$$

$$\mathcal{R}_{10} \equiv r^{-1}\partial_t\lambda, \mathcal{R}_{22} \equiv -e^{-\lambda}\left(1+\frac{r}{2}(\nu'-\lambda')\right)+1, \mathcal{R}_{33} \equiv \sin^2\theta\mathcal{R}_{22}$$

$$\nu(t,r) = \nu(r) + f(t) set\ \tau \coloneqq \int e^{\frac{1}{2}f(t)^2}dt$$

$$\mathcal{R}_{00} \equiv e^{\nu-\lambda}\left\{\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{\nu'}{r}\right\}$$

$$\mathcal{R}_{11} \equiv -\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\lambda'\nu'}{4}+\frac{\lambda'}{r}$$

$$r\left(e^{\lambda-\nu}\mathcal{R}_{00}+\mathcal{R}_{11}\right) \equiv \nu'+\lambda'-e^{-\lambda}(1-r\lambda')+1\left(e^{-\lambda}\right)'+\frac{e^{-\lambda}}{r} = \frac{1}{r}$$

$$e^{-\lambda} = 1+\frac{\mathrm{A}}{r}, e^{\nu} = \mathrm{B}\left(1+\frac{\mathrm{A}}{r}\right)$$

Coordenadas isotrópicas

$$r \coloneqq \mathcal{R}\left(1+\frac{m}{2\mathcal{R}}\right)^2$$

$$\chi \coloneqq \Re r^{-1}\chi, \gamma \coloneqq \Re r^{-1}\gamma, \mathcal{Z} \coloneqq \Re r^{-1}\mathcal{Z}$$

$$g_{\text{Schwarzschild}} = -\left(\frac{2\mathcal{R}-m}{2\mathcal{R}+m}\right)^2 dt^2 + \left(1+\frac{m}{2\mathcal{R}}\right)^4 (d\chi^2 + d\gamma^2 + dz^2)$$

Coordenadas de ondas cuánticas

$$-\frac{\bar{r}-m}{\bar{r}+m}dt^2 + \frac{\bar{r}+m}{\bar{r}-m}d\bar{r}^2 + (\bar{r}+m)^2(d\theta^2 + sin^2\theta d\phi^2) - \mathrm{A}^2dt^2 + \mathrm{B}^2d\bar{r}^2$$

$$+ r^2(d\theta^2 + sin^2\theta d\phi^2) - \frac{1}{\mathrm{A}^2}\frac{\partial^2\psi}{\partial t} + \frac{1}{r^2}\left\{\frac{1}{\mathrm{AB}}\frac{\partial}{\partial\bar{r}}\left(\mathrm{AB}^{-1}r^2\frac{\partial\psi}{\partial\bar{r}}\right) + \Delta^{\dagger}\psi\right\}$$

$$\frac{\partial\chi^i}{\partial\bar{r}} = \frac{\chi^i}{\bar{r}}, \frac{\partial^2\chi^i}{\partial\bar{r}^2}\Delta^{\dagger}\chi^i = -2\chi^i\frac{1}{\mathrm{AB}}\frac{d}{d\bar{r}}(\mathrm{AB}^{-1}r^2) - 2\bar{r}$$

$$\mathrm{A} \equiv \sqrt{1-\frac{2m}{r}}, \mathrm{B} \equiv \mathrm{A}^{-1}\frac{dr}{d\bar{r}}$$

$$\frac{d}{dr}\left(\frac{dr}{d\bar{r}}\mathrm{A}^2 r^2\right) - 2\bar{r} \equiv \frac{d}{dr}\left(\frac{d\bar{r}}{dr}(r^2 - 2mr)\right) - 2\bar{r}$$

$$r = m(1+z)$$

$$\frac{d}{dz}\left[(z^2-1)\frac{d\bar{r}}{dz}\right] - 2z|\mathfrak{C}_1 z + \mathfrak{C}_2|\left(\frac{z}{2}\ln\frac{z+1}{z-1} - 1\right)$$

$$\bar{r} = mz, r = \bar{r} + m$$

Coordenadas de Painlevé–Gullstrand

$$-\left(1-\frac{2m}{r}\right)dt^2 + d\chi^2 + d\gamma^2 + dz^2 + \frac{2}{r}\sqrt{\frac{2m}{r}}(\chi d\chi + \gamma d\gamma + z dz)dt$$

Coordenadas de Regge–Wheeler

$$\rho = r + 2m\log(r-2m)$$

$$g_{\text{Schwarzschild}} = \left(1-\frac{2m}{r}\right)(-dt^2 + d\rho^2) + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Espacio – tiempo cuántico de Schwarzschild

$$ds^2 \equiv -g_{\alpha\beta}d\chi^\alpha d\chi^\beta \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 \frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} r^2 \frac{d\varphi}{ds} = \ell$$

$$\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds}, \nu \equiv \log\left(1-\frac{2m}{r}\right)\frac{dt}{ds} = \mathfrak{E}$$

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{ds}\right)^2 = e^{-\lambda}\left(\frac{d\varphi}{ds}\right)^2 + \frac{e^{\nu-\lambda}}{2}\frac{d\nu}{ds}\left(\frac{dt}{ds}\right)^2 \frac{d^2\mu}{d\varphi^2} + \mu\frac{m}{\ell^2} + 3m\mu^2, \mu \equiv \frac{1}{r}$$

$$\mu = \mu_{newton} + \upsilon\frac{d^2\upsilon}{d\varphi^2} + \upsilon - 3m\mu^2$$

$$\frac{1}{r} = m\ell^2(1 + e\cos\varphi) \coloneqq \mu_{newton}$$

$$\frac{d^2\upsilon}{d\varphi^2} + \upsilon - 3m\mu_{newton}^2 \equiv 3m^3\mu_{newton}^2$$

$$\equiv 3m^3\ell^{-4}(1 + 2e\cos\varphi + e^2\cos^2\varphi)3m^3\ell^{-4}\left(1 + e\varphi\sin\varphi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\varphi\right)\right)$$

$$\mu_{einstein} \sim \mu_{newton} + 3m^2\ell^{-4}e\varphi\sin\varphi = m\ell^{-2}(1 + e\cos\varphi + e3m^2\ell^{-2}\varphi\sin\varphi)$$

$$\mu_{einstein} \sim m\ell^{-2}\big(1 + e\cos\big((1 - 3m^2\ell^{-2})\varphi\big)\big)\left|\frac{6\pi m}{\alpha(1-e^2)}\right|$$

$$\left(1-\frac{2m}{r}\right)\dot{t} = \xi = \left(1-\frac{2m}{r_0}\right)\dot{t}_0$$

$$1 = \left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2$$

$$\dot{r}^2 = \xi^2 - 1 + \frac{2m}{r}$$

$$r_{\mathcal{M}} = \frac{2m}{1-\xi^2}, \dot{r}_0^2 = \frac{2m}{r_0}$$

$$\theta^0 = \left(1-\frac{2m}{r_0}\right)^{1/2} dt, \theta^1 = \left(1-\frac{2m}{r_0}\right)^{-1/2} dr$$

$$\mathcal{V}^1 = \left(1-\frac{2m}{r_0}\right)^{-1/2} \dot{r}_0, \mathcal{V}^0 = \left(1-\frac{2m}{r_0}\right)^{1/2} \dot{t}_0$$

$$\mathcal{V} := \frac{\mathcal{V}^1}{\mathcal{V}^0} = \sqrt{\frac{2m}{r_0}}$$

$$g_{\text{Schwarzschild}} = -\left(1-\frac{2m}{r^{\eta-2}}\right)dt^2 + \left(1-\frac{2m}{r^{\eta-2}}\right)^{-1} dr^2 + r^2 d\omega^2$$

$$\frac{d}{dr}\left[\frac{d\bar{r}}{dr} r^{\eta-1}(1-2mr^{2-\eta})\right] - (\eta-1)\bar{r}$$

$$\frac{d}{ds}\left[\delta^{3-\eta}(1-2m\delta^{\eta-2})\frac{d\bar{r}}{ds}\right] = (\eta-1)\delta^{1-\eta}\bar{r}$$

$$\bar{r} = r + \frac{m}{(\eta-2)r^{\eta-3}} + \begin{cases} \frac{m^2}{4} r^{-3}\ln r + \mathcal{O}(r^{-5}\ln r) & \eta = 4 \\ \mathcal{O}(r^{5-2\eta}) & \eta \geq 5 \end{cases}$$

$$\left(g_{\text{Schwarzschild,m}}\right)_{\mu\nu} = \eta_{\mu\nu} + (f_m)_{\mu\nu}$$

$$f_{m,\mu\nu} = \frac{1}{r^{\eta-2}} \hbar_{\mu\nu}\left(m, \frac{1}{r}, \frac{\bar{\chi}}{r}\right) \bar{\chi} := (\chi^i)$$

Dinámica orbital en espacios cuánticos curvos

$$g_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta = -1$$

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{\ell^2}{r^2}+1\right) = \frac{1}{2}\xi^2$$

$$\mathcal{V}(r) \equiv \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{\ell^2}{r^2}+1\right)\ddot{r} + \frac{d\mathcal{V}}{dr} \equiv r^{-4}[mr^2 - \ell^2 r + 3m\ell^2]$$

$$\mathcal{R}_{\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 - 12\ell^2 m^2}}{2mr^2}$$

Desviación de la luz

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\frac{\ell^2}{r^2} = \frac{1}{2}\xi^2$$

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right) + \frac{1}{r^2} - \frac{2m}{r^3} = \kappa^2$$

$$\left(\frac{d\mu}{d\phi}\right)^2 + \mu^2 - 2m\mu^3 = \kappa^2$$

$$\frac{d^2\mu}{d\varphi^2} + \mu = 3m\mu^2$$

$$\frac{1}{r} \equiv \mu_{\delta tr} = \frac{1}{r_0}\cos(\phi - \phi_0)$$

$$\frac{d^2\mu}{d\varphi^2} + \mu = 3m\mu_{\delta tr}^2$$

$$\frac{1}{r} = \frac{1}{r_0}\cos\phi + \frac{m}{r_0^2}(1 + \sin^2\phi) - \sin\alpha + \frac{m}{r_0}(1 + \cos^2\alpha)$$

$$\alpha \cong \frac{2}{r_0}, \delta \cong \frac{4m}{r_0}$$

Principio de Fermat

$$ds^2 = g_{00}dt^2 + g_{ij}d\chi^i d\chi^j$$

$$d\sigma^2 = \frac{g_{ij}d\chi^i d\chi^j}{g_{00}}$$

$$\ell = \int_{t_2}^{t_1} \sqrt{\frac{g_{ij}d\chi^i d\chi^j}{g_{00}}}$$

Dilatación del tiempo

$$\mathcal{T}_{\mathcal{O}} = \left(1 - \frac{2m}{r_{\mathcal{O}}}\right)\left(1 - \frac{2m}{r_{\mathrm{A}}}\right)^{-1}\mathcal{T}_{\mathrm{A}}$$

$$\frac{\mathcal{T}_{\mathrm{A}}}{\mathcal{T}_{\mathcal{O}}} \equiv \frac{\nu_{\mathcal{O}}}{\nu_{\mathrm{A}}} \cong 1 - 2m\left(\frac{1}{r_{\mathrm{A}}} - \frac{1}{r_{\mathcal{O}}}\right)$$

$$\delta_{\mathrm{A}} = \int_0^{t_{\mathcal{M}}} \frac{ds}{dt}dt = \int_0^{t_{\mathcal{M}}} \xi^{-1}\left(1 - \frac{2m}{r}\right)dt = \int_0^{t_{\mathcal{M}}} \frac{1 - \frac{2m}{r}\frac{ds}{}}{1 - \frac{2m}{r_{\mathcal{O}}}dt}(0)dt$$

$$\left(\frac{ds}{dt}(0)\right)^2 = \left(1-\frac{2m}{r_0}\right)-\left(1-\frac{2m}{r_0}\right)^{-1}\left(\frac{dr}{dt}(0)\right)^2$$

$$\delta_{\mathrm{A}} \cong \int_0^{t_{\mathcal{M}}} \left(1-\frac{2m}{r}+\frac{m}{r_0}+\frac{mv^2}{r_0}\right)dt$$

$$\delta_{\mathcal{O}} = \int_0^{t_{\mathcal{M}}} \sqrt{1-\frac{2m}{r_0}}\,dt \cong \int_0^{t_{\mathcal{M}}} \left(1-\frac{m}{r_0}\right)dt$$

$$\delta_{\mathrm{A}}-\delta_{\mathcal{O}} \cong \int_0^{t_{\mathcal{M}}} \left(-\frac{2m}{r}+2\frac{m}{r_0}+\frac{mv^2}{r_0}\right)dt > 0$$

Simetría de masa - Ecuación Tolman–Oppenheimer–Volkov

$$\mathcal{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\mathcal{R} = \mathcal{T}_{\alpha\beta}$$

$$g = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

$$\mathcal{T}_{\alpha\beta} \equiv (\mu+\rho)\mu_\alpha\mu_\beta + \wp g_{\alpha\beta}$$

$$\mathcal{R}_{\alpha\beta} = \rho_{\alpha\beta} \equiv (\mu+\wp)\mu_\alpha\mu_\beta + \frac{1}{2}(\mu-\wp)g_{\alpha\beta}$$

$$\mathcal{T}_{00} \equiv \mu e^{\nu}, \mathcal{T}_{0i} = 0, \mathcal{T}_{ij} \equiv \wp g_{ij}$$

$$\mathcal{R}_{22} + \frac{1}{\sin^2\theta}\mathcal{R}_{33} \equiv 2\left\{-e^{-\lambda}\left(1+\frac{r}{2}(\nu'-\lambda')\right)+1\right\}r\left(e^{\lambda-\nu}\mathcal{R}_{00}+\mathcal{R}_{11}\right) \equiv \nu'+\lambda'$$

$$\mathcal{R} \equiv -e^{-\nu}\mathcal{R}_{00} + e^{-\lambda}\mathcal{R}_{11} + r^{-2}(\mathcal{R}_{22} + sin^{-2}\theta\mathcal{R}_{33}) \equiv -e^{-\lambda}\left(\nu''+\frac{\nu'^2}{2}-\frac{\nu'\lambda'}{2}+\frac{\nu'-\lambda'}{r}\right)$$

$$\delta_{00} \equiv \mathcal{R}_{00} - \frac{1}{2}g_{00}\mathcal{R} = \mathcal{T}_{00} \equiv \mu e^{\nu}$$

$$e^{-\lambda}(r\lambda'-1)+1 = 2r^2\mu$$

$$\frac{d}{dr}\left[r - re^{-\lambda}\right] = 2r^2\mu$$

$$e^{-\lambda} = 1-\frac{2\mathcal{M}(r)}{r}, \mathcal{M}(r) \equiv \int_0^{\tau} \rho^2\mu(\rho)d\rho$$

$$\mathcal{M}(\alpha) \equiv \int_0^{\alpha} \rho^2\mu(\rho)d\rho \leq \frac{\alpha}{2}$$

$$\mathcal{M}_\alpha := \int_0^\alpha \rho^2\mu(\rho)\left(1-\frac{2\mathcal{M}(\rho)}{\rho}\right)^{-1/2} d\rho$$

$$\nu' + \lambda' \equiv r\left(e^{\lambda-\nu}\mathcal{R}_{00} + \mathcal{R}_{11}\right) = r\left(e^{\lambda-\nu}\rho_{00} + \rho_{11}\right) \equiv re^{\lambda}(\mu + \wp)$$

$$\nu' = \left(e^{\lambda} - 1\right)r^{-1} + re(\wp - \mu) = \frac{2\mathcal{M}(r) + r^3(\wp - \mu)}{r[r - 2\mathcal{M}(r)]}$$

$$(\mu + \wp)\mu^{\alpha}\nabla_{\alpha}\mu_1 + \partial_1\wp \equiv (\mu + \wp)\mu^0\mu_0\Gamma^1_{01} + \partial_1\wp$$

$$\wp' = -\frac{1}{2}(\wp + \mu)\nu'$$

$$\wp' = -\frac{1}{2}(\wp + \mu)\frac{2\mathcal{M}(r) + r^3(\wp - \mu)}{r[r - 2\mathcal{M}(r)]}$$

$$\mathcal{M}(r) \equiv \mu_0 \int_0^\tau \rho^2 d\rho = \frac{1}{3}\mu_0 r^3$$

$$\wp(0) = \mu_0 \frac{1 - \left(1 - 2\alpha^{-1}\mathcal{M}(\alpha)\right)^{1/2}}{3[1 - 2\alpha^{-1}\mathcal{M}(\alpha)]^{1/2} - 1}, \mathcal{M}(\alpha) \geq \frac{4\alpha}{9}$$

Modelo einsteniano

$$g = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

$$g_m = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

$$m = \mathcal{M}\left(r_{particle}\right) \equiv \int_0^{r_{particle}} r^2\mu(r)dr$$

$$\mu(r) = 2r^{-3}\mathcal{M}(r) \equiv 2r^{-3}\int_0^\tau \rho^2\mu(\rho)d\rho$$

Agujero negro cuántico de Schwarzschild

Horizonte de eventos

$$ds^2 = (1 - 2mr^{-1})dt^2 - (1 - 2mr^{-1})^{-1}dr^2$$

$$\frac{dr}{dt} = \pm(1 - 2mr^{-1})$$

$$t - t_0 = \pm\left(r - r_0 + 2m\log\left[\left(\frac{r}{2m} - 1\right)\right]\left[\left(\frac{r_0}{2m} - 1\right)\right]\right)^{-1}$$

$$v = t + r + 2m\ln\left(\frac{r}{2m} - 1\right)$$

$$-\left(\frac{r}{2m}-1\right)dv^2+2drdv+r^2(\sin^2\theta\, d\varphi^2+d\theta^2)$$

$$v=2r+4m\log|r-2m|+constant$$

$$\dot{r}^2=\xi^2-(1-2mr^{-1})\int_0^{r_0}\frac{r^{\frac{1}{2}}dr}{\sqrt{2m-(1-\xi^2)r}}-\left(1-\frac{2m}{r}\right)dv^2+2drdv+r^2(\sin^2\theta\, d\varphi^2+d\theta^2)$$

Coordenadas de Kruskal–Szekeres

$$\frac{32m^3}{r}e^{-r/2m}[dz^2-d\omega^2]+r^2(d\theta^2+sin^2\theta d\phi^2)$$

$$z^2-\omega^2\geq -1$$

$$z^2-\omega^2=\frac{1}{2m}(r-2m)e^{r/2m}$$

$$\mu=t-r-2m\log\left(\frac{r}{2m}-1\right)$$

Métrica de Tolman

$$-dt^2+e^{2\omega}dr^2+\mathcal{R}^2(d\theta^2+sin^2\theta d\phi^2)$$

$$\mathcal{T}_{\alpha\beta}=u\mu_\alpha\mu_\beta,\dot{\mathcal{R}}'-\dot{\omega}\mathcal{R}'e^{-\omega}f(r)$$

$$\nabla_\alpha\tau^{\alpha 0}\equiv\dot{\mu}+\Gamma^\alpha_{\alpha 0}\mu\equiv\dot{\mu}+\mu\left(\dot{\omega}+2\mathcal{R}^{-1}\dot{\mathcal{R}}\right)$$

$$\mu(t,r)=\frac{e^{-\omega}}{\mathcal{R}^2}\phi(r)$$

$$\mu(t,r)=\frac{r^2\mu_0}{\mathcal{R}^2\mathcal{R}'}\frac{1}{2}\dot{\mathcal{R}}^2-\frac{\mathcal{M}(r)}{\mathcal{R}}=\frac{1}{2}[f^2(r)-1]$$

$$\mathcal{M}(r)=\int_0^r f(\rho)\mu(t,\rho)\mathcal{R}^2(t,\rho)e^\omega d\rho$$

$$\mathcal{M}(r)=\int_0^r \mu_0(\rho)\rho^2 d\rho$$

$$\dot{\mathcal{R}}(0,r)=2\frac{\mathcal{M}(r)}{r}+f^2(r)-1$$

$$\mathcal{R}^{1/2}\dot{\mathcal{R}}=\pm\sqrt{2\mathcal{M}(r)}$$

$$\mathcal{R}(r,t)^{3/2}=\phi(r)-\frac{3}{2}\sqrt{2\mathcal{M}(r)}t$$

$$\frac{3}{2}\sqrt{2\mathcal{M}(r)}\coloneqq\hbar^{1/2}(r)$$

$$\mathcal{R}(r,t)=\left\{r^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(r)t\right\}^{\frac{2}{3}}$$

$$e^{\omega}=\mathcal{R}'=\left\{r^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(r)t\right\}^{-\frac{1}{3}}\left\{r^{\frac{1}{2}}-\frac{1}{3}\hbar^{-\frac{1}{2}}\hbar'(r)t\right\}$$

$$\hbar'(r)=\frac{9}{2}r^{2}\mu_{0}(r)$$

$$\mu(t,r)=\frac{r^{2}\mu_{0}}{\left\{r^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(r)t\right\}\left\{r^{\frac{1}{2}}-\frac{3}{2}\hbar^{-\frac{1}{2}}\hbar'(r)t\right\}}$$

$$d\mathcal{R}=e^{\omega}dr+\dot{\mathcal{R}}dt=e^{\omega}dr-\mathcal{R}^{1/2}\sqrt{2\mathcal{M}(r)}dt$$

$$-\left(1-\frac{\sqrt{2\mathcal{M}(r)}}{\mathcal{R}}\right)dt^{2}+2\mathcal{R}^{\frac{1}{2}}\sqrt{2\mathcal{M}(r)}d\mathcal{R}dt+d\mathcal{R}^{2}+\mathcal{R}^{2}(d\theta^{2}+sin^{2}\theta d\phi^{2})$$

Densidad decreciente

$$t_{1}(r)=\hbar^{-\frac{1}{2}}(r)r^{\frac{3}{2}}$$

$$\frac{dt_{1}(r)}{dr}=\frac{1}{2}r^{\frac{1}{2}}\hbar^{-\frac{3}{2}}\{3\hbar(r)-r\hbar'(r)\}=\frac{9}{4}r^{\frac{1}{2}}\hbar^{-\frac{3}{2}}\{3\mathcal{M}(r)-r\mathcal{M}'(r)\}$$

$$3\mathcal{M}(r)-r\mathcal{M}'(r)\equiv 3\int_{0}^{r}\mu_{0}(\rho)\rho^{2}d\rho-\mu_{0}(r)r^{3}$$

$$\frac{3}{2}\mathcal{R}^{3/2}\mathcal{R}'=\left\{\frac{3}{2}r^{\frac{1}{2}}-\frac{1}{2}\hbar^{-\frac{1}{2}}(r)\hbar'(r)t\right\}$$

$$t_{2}=\frac{3\hbar^{\frac{1}{2}}(r)r^{\frac{1}{2}}}{\hbar'(r)}=\frac{2\hbar^{\frac{1}{2}}(r)}{3r^{\frac{3}{2}}\mu_{0}(r)}\equiv\frac{\sqrt{2\mathcal{M}(r)}}{r^{\frac{3}{2}}\mu_{0}(r)},\frac{t_{1}}{t_{2}}=\frac{\hbar'(r)r}{3\hbar(r)}$$

$$t_{0}=\lim_{r=0}\left[\frac{r^{3}}{\hbar(r)}\right]^{1/2}=\frac{1}{\sqrt{\frac{3}{2}\mu_{0}(0)}}$$

$$0\leq\mathcal{R}\leq\left\{\alpha^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(\alpha)t\right\}^{\frac{2}{3}}$$

$$\mathcal{M}(r)=\mathcal{M}_{\alpha}:=\int_{0}^{\alpha}\mu_{0}(\rho)\rho^{2}d\rho$$

$$-\left(1-\frac{2\mathcal{M}_{\alpha}}{\mathcal{R}}\right)dt^{2}+2\sqrt{\frac{2\mathcal{M}_{\alpha}}{\mathcal{R}}}d\mathcal{R}dt+d\mathcal{R}^{2}+\mathcal{R}^{2}(d\theta^{2}+sin^{2}\theta d\phi^{2})$$

$$-\left(1-\frac{2\mathcal{M}_\alpha}{\mathcal{R}}\right)dt^2+\left(1-\frac{2\mathcal{M}_\alpha}{\mathcal{R}}\right)^{-1}d\mathcal{R}^2+\mathcal{R}^2(d\theta^2+sin^2\theta d\phi^2)$$

$$d\tau = dt-\left(1-\frac{2\mathcal{M}_\alpha}{\mathcal{R}}\right)^{-1}\sqrt{\frac{2\mathcal{M}_\alpha}{\mathcal{R}}}d\mathcal{R}$$

$$g = g_{dust}, 0 \leq \mathcal{R} \leq \left\{\alpha^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(\alpha)t\right\}^{\frac{2}{3}}$$

$$g = g_{ext}, \mathcal{R} \geq \left\{\alpha^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(\alpha)t\right\}^{\frac{2}{3}}$$

$$\left\{\alpha^{\frac{3}{2}}-\hbar^{\frac{1}{2}}(\alpha)t\right\}^{\frac{2}{3}} < 2\mathcal{M}_\alpha$$

$$t_3 = \hbar^{\frac{1}{2}}(\alpha)\left[\alpha^{\frac{3}{2}}-(2\mathcal{M}_\alpha)^{3/2}\right]$$

Solución de Reissner–Nordström

$$-\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right)dt^2+\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right)^{-1}dr^2+r^2(d\theta^2+sin^2\theta d\varphi^2)$$

$$r_\pm = m \pm \sqrt{m^2-Q^2}$$

Bases cosmológicas de la teoría cuántica de campos curvos

Isotropía

$$\mathcal{K}(\mathcal{P}) := \frac{\mathcal{R}iemann(\chi,\gamma;\chi,\gamma)}{g(\chi,\chi)g(\gamma,\gamma)-g(\chi,\gamma)^2}$$

$$\mathcal{K}(\mathcal{P}) := \frac{\mathcal{R}_{ij,\hbar\kappa}\chi^i,\gamma^j;\chi^\hbar,\gamma^\kappa}{\chi^i\chi_i\gamma^j\gamma_j-(\chi^i\gamma_i)^2}$$

$$\mathcal{R}_{ij,\hbar\kappa}(\chi) = \kappa(\chi)\big(g_{i\hbar}g_{j\kappa}-g_{j\hbar}g_{i\kappa}\big)(\chi)$$

Homogeneidad

$$\mathcal{R}_{i\hbar} = (\eta-1)\kappa g_{i\hbar}; \mathcal{R} = \eta(\eta-1)\kappa$$

$$0 \equiv \nabla_\hbar\left(\mathcal{R}_i^\hbar-\frac{1}{2}\delta_i^\hbar\mathcal{R}\right) = -\frac{1}{2}(\eta-1)(\eta-2)\partial_i\kappa$$

$$g = \frac{(d\chi^1)^2+\cdots+(d\chi^\eta)^2}{1+\frac{\kappa}{4}[(\chi^1)^2+\cdots+(\chi^\eta)^2]^2}$$

$$g = \sum_{i=1\cdots\eta} \frac{(d\chi^i)^2}{f^2(\chi^1 \cdots \chi^\eta)} \frac{\partial^2 f}{\partial \chi^i \partial \chi^j} f \left[\frac{\partial^2 f}{(\partial \chi^i)^2} + \frac{\partial^2 f}{(\partial \chi^j)^2} \right]$$

$$= \kappa + \sum_{\hbar} \left(\frac{\partial f}{\partial \chi^\hbar} \right)^2 \frac{\partial^2 f}{(\partial \chi^j)^2} \frac{\partial^2 f}{(\partial \chi^\kappa)^2} \frac{d^2 \chi_j}{(d\chi^j)^2} \frac{d^2 \chi_\kappa}{(d\chi^\kappa)^2}$$

$$\frac{d^2 \chi_j}{(d\chi^j)^2} = c, \frac{d\chi_j}{d\chi^j} = c\chi^i + \lambda_j$$

$$\chi_j = \frac{c}{2}(\chi^j)^2 + \mu_j, f = \frac{c}{2} \sum_i (\chi^i)^2 + \sum_i \mu_i$$

$$f = 1 + \frac{c}{2} \sum_i (\chi^i)^2 \sum_{\mathrm{A}=1\cdots\eta+1} (\chi^{\mathrm{A}})^2 = \kappa^{-1}$$

$$\sum_{i=1\cdots\eta} (\chi^i)^2 - (\chi^{\eta+1})^2 = |\kappa|^{-1}$$

Espacio – tiempo cuántico de Robertson–Walker

$$^{(3)}g \equiv e^\mu + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

$$^{(3)}\mathcal{R}_{i\kappa} \equiv 0, i \neq \kappa, {}^{(3)}\mathcal{R}_{11} \equiv r^{-1}\mu' = 2\kappa e^\mu, \sin^{-2}\theta\, {}^{(3)}\mathcal{R}_{33} \equiv {}^{(3)}\mathcal{R}_{22} \equiv -e^{-\mu} + 1 + \frac{r}{2} e^{-\mu}\mu' = 2\kappa r^2|$$

$$e^{-\mu} = 1 - \kappa r^2$$

$$^{(3)}g \equiv |\kappa|^{-1}\gamma_\varepsilon, \gamma_\varepsilon \equiv \frac{dr^2}{1 - \varepsilon r^2} + r^2(sin^2\theta d\phi^2 + d\theta^2)$$

$$\gamma_+ = d\alpha^2 + \sin^2\alpha\, (sin^2\theta d\phi^2 + d\theta^2)$$

$$\gamma_- = d\chi^2 + \sin\hbar^2 \chi\, (sin^2\theta d\phi^2 + d\theta^2)$$

$$-dt^2 + \mathcal{R}^2(t)\gamma_\varepsilon$$

$$\mathcal{R}_{\alpha\beta} = \Lambda g_{\alpha\beta} + \rho_{\alpha\beta} \equiv \mathcal{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{T}$$

$$\Gamma^0_{ij} \equiv \mathcal{R}\dot{\mathcal{R}}\gamma_{ij}, \Gamma^j_{0i} \equiv \mathcal{R}^{-1}\dot{\mathcal{R}}\delta^j_i, \Gamma^i_{j\hbar} = \gamma^i_{j\hbar}, \dot{\mathcal{R}} = \frac{d\mathcal{R}}{dt}$$

$$\mathcal{R}_{00} \equiv 3\mathcal{R}^{-1}\ddot{\mathcal{R}}, \mathcal{R}_{ij} \equiv \{2\varepsilon + (\mathcal{R}\ddot{\mathcal{R}} + 2\dot{\mathcal{R}}^2)\}\gamma_{ij}$$

$$\rho_{\alpha\beta} \equiv \mathcal{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{T} = (\mu + \wp)\mu_\alpha\mu_\beta + \frac{1}{2} g_{\alpha\beta}\wp$$

$$\rho_{00} \equiv \frac{1}{2}(\mu + 3\wp), \rho_{ij} \equiv \frac{1}{2}\mathcal{R}^2(\mu - \wp)\gamma_{ij}, \gamma^{ij}\rho_{ij} \equiv \frac{3}{2}\mathcal{R}^2(\mu - \wp)$$

$$\mathcal{R}_{00} = -\Lambda + \rho_{00}, \mathcal{R}_{ij} = \Lambda g_{ij} + \rho_{ij} - 3\mathcal{R}^{-1}\ddot{\mathcal{R}} = \frac{1}{2}(\mu + 3\wp) - \Lambda + 2\mathcal{R}^{-2}\epsilon + \left(\mathcal{R}^{-1}\ddot{\mathcal{R}} + 2\mathcal{R}^{-2}\dot{\mathcal{R}}^2\right)$$

$$= \frac{1}{2}(\mu - \wp) + \Lambda$$

$$\mu \equiv \frac{1}{2}\left\{\mathcal{R}^{-2}\gamma^{ij}\rho_{ij} + \rho_{00}\right\} = 3\mathcal{R}^{-2}\left(\dot{\mathcal{R}}^2 + \epsilon\right) - \Lambda$$

$$\wp = -2\mathcal{R}^{-1}\ddot{\mathcal{R}} - 2\mathcal{R}^{-2}\epsilon - \mathcal{R}^{-2}\dot{\mathcal{R}}^2 + \Lambda$$

$$\dot{\mu} = -3(\mu + \wp)\mathcal{R}^{-1}\dot{\mathcal{R}}$$

$$\Re\frac{d\mu}{d\mathcal{R}} + 3\mu = -3\wp, \frac{d}{d\mathcal{R}}(\mathcal{R}^3\mu) = -3\wp - dt^2 + \mathcal{R}_0^2\gamma_+, \mathcal{R}_{0\alpha} = 0, \mathcal{R}_{ij} = 2\gamma_{ij}, \mu_0 = 3\mathcal{R}_0^{-2} - \Lambda, \rho_0$$

$$= -2\mathcal{R}_0^{-2} + \Lambda$$

$$z \equiv \frac{\nu_\delta - \nu_0}{\nu_0}$$

$$1 + z \equiv \frac{\nu_\delta}{\nu_0} = \frac{\mathcal{R}(t_0)}{\mathcal{R}(t_\delta)} - dt^2 + d\alpha^2 + f^2\alpha(d\theta^2 + sin^2\theta d\phi^2)$$

$$\alpha_\delta = \int_{t_\delta}^{t_\mathcal{O}} \frac{dt}{\mathcal{R}(t)} = \int_{t_\delta + \mathcal{T}_\delta}^{t_\mathcal{O} + \mathcal{T}_\mathcal{O}} \frac{dt}{\mathcal{R}(t)} \int_{t_\delta}^{t_\delta + \mathcal{T}_\delta} \frac{dt}{\mathcal{R}(t)} = \int_{t_\mathcal{O}}^{t_\mathcal{O} + \mathcal{T}_\mathcal{O}} \frac{dt}{\mathcal{R}(t)} \frac{\mathcal{T}_\delta}{\mathcal{T}_\mathcal{O}} = \frac{\mathcal{R}(t_\mathcal{O})}{\mathcal{R}(t_\delta)}$$

$$\frac{dt}{d\alpha} = \pm\mathcal{R}(t)$$

$$\alpha_\delta = \int_{t_\delta}^{t_0} \frac{dt}{\mathcal{R}(t)}$$

$$\alpha_\delta \cong (t_0 - t_\delta)\mathcal{R}^{-1}(t_\delta)$$

$$\mathcal{R}(t_0) \cong \mathcal{R}(t_\delta) + (t_\mathcal{O} - t_\delta)\dot{\mathcal{R}}(t_0)$$

$$z \cong d\mathfrak{H}(t_\mathcal{O}), d \coloneqq \alpha_\delta\mathcal{R}(t_0)$$

$$q = \frac{\mathcal{R}\ddot{\mathcal{R}}}{\dot{\mathcal{R}}^2}$$

$$\mathcal{R}(t_\delta) = \mathcal{R}(t_0)\left[1 + (t_\delta - t_0)\mathcal{H}(t_0) - \frac{1}{2}q\mathcal{H}^2(t_\delta - t_0)^2\right]$$

$$\alpha_\delta = \mathcal{R}^{-1}(t_0)\left[t_0 - t_\delta + \frac{\mathcal{H}}{2}(t_\mathcal{O} - t_\delta)^2\right]$$

$$z = \mathcal{H}d + \frac{1}{2}(q + 1)\mathcal{H}^2d^2$$

Espacio – tiempo cuántico de Sitter

$$\mathcal{R}_{00} = -\Lambda, \mathcal{R}_{ij} = \Lambda g_{ij}$$

$$\ddot{\mathcal{R}} - \kappa^2\mathcal{R} = 0, \kappa^2 := \frac{\Lambda}{3}$$

$$\dot{\mathcal{R}}^2 - \kappa^2\mathcal{R}^2 = -\varepsilon$$

$$\mathcal{R} = \mathrm{A}e^{\kappa t} + \mathrm{B}e^{-\kappa t} - dt^2 + e^{\kappa t}\{d\chi^2 + d\gamma^2 + dz^2\}$$

$$g_{d\delta i} := -dt^2 + \kappa^{-2}\cosh^2(\kappa t)\,\{d\alpha^2 + \sin^2\alpha\,(d\theta^2 + sin^2\theta d\phi^2)\}$$

$$-d\mathcal{T}^2 + \sum_{\mathcal{A}=1\cdots4}\left(d\chi^{\mathrm{A}}\right)^2\mathcal{T}^2 - \sum_{\mathcal{A}=1\cdots4}\left(\chi^{\mathrm{A}}\right)^2 = \kappa^{-2}$$

$$\mathcal{T} = \kappa^{-1}\delta\hbar(\kappa t), \chi^{\mathrm{A}} = \kappa^{-1}c\hbar(\kappa t)\mu^{\mathrm{A}}$$

$$\sum_{\mathcal{A}=1\cdots4}\left(\mu^{\mathrm{A}}\right)^2 = 1, \sum_{\mathcal{A}=1\cdots4}\mu^{\mathrm{A}}d\mu^{\mathrm{A}} = 0$$

$$d\mathcal{T} = \mathcal{T}^{-1}\sum_{\mathcal{A}=1\cdots4}\chi^{\mathrm{A}}d\chi^{\mathrm{A}} = \cosh(\kappa t)dt$$

$$\left[-d\mathcal{T}^2 + \sum_{\mathcal{A}=1\cdots4}\left(d\chi^{\mathrm{A}}\right)^2\right]_{\mathfrak{H}} = -dt^2 + \kappa^{-2}\cosh^2(\kappa t)(d\delta^3)^2$$

$$\tau := \kappa^{-1}\log[\kappa(\mathcal{T} + \chi^4)]\,, \xi^i = \frac{\kappa^{-1}\chi^i}{\mathcal{T} + \chi^4} - d\tau^2 + e^{2\kappa\tau}\sum_{i=1,2,3}\left(d\xi^i\right)^2$$

$$t' = 2\arctan(\exp\kappa t)$$

$$g_{d\delta i} = \kappa^{-2}(\cos\chi)^2 g_{\xi i}$$

Espacio – tiempo cuántico Anti de Sitter

$$\mathcal{R} = \mathrm{A}\cos\omega t + \mathrm{B}\sin\omega t - dt^2 + \cos^2 t\gamma\,,\gamma_{-}$$

$$:= d\chi^2 + \sinh^2\chi(sin^2\theta d\phi^2 + d\theta^2) - dt^2(\cosh\chi)^2 + d\chi^2$$

$$+ \sinh^2\chi(sin^2\theta d\phi^2 + d\theta^2)$$

$$dt = \frac{d\chi}{\cosh\chi}, t = t^0 + \int_{\chi_0}^{\chi}\frac{d\chi}{\cosh\chi}$$

$$t - t_0 \leq \int_0^{\chi}\frac{d\chi}{\cosh\chi} \leq 2\int_0^{\infty}\frac{d\chi}{e^{\chi}} := \mathcal{T}$$

Modelos Friedmann–Lemaître

$$\wp = (\gamma - 1)\mu$$

$$\dot{\mu} = -3\gamma\mu\mathcal{R}^{-1}\dot{\mathcal{R}}$$

$$\mu\mathcal{R}^{3\gamma} = \mathcal{M}$$

$$\dot{\mathcal{R}}^2 = \frac{1}{3}\mathcal{M}\mathcal{R}^{2-3\gamma} - \epsilon$$

$$\dot{\mathcal{R}}(t_0) = \frac{\mathcal{R}(t_0)}{t_0 - \mathcal{T}}$$

$$t_0 = \mathcal{H}(t_0)^{-1} + \mathcal{T} < \mathcal{H}(t_0)^{-1}$$

$$\dot{\mathcal{R}}^2 = \frac{1}{3}\mathcal{M}\mathcal{R}^{-1} - \epsilon$$

$$\mathcal{R}(t) = ct^{2/3}, c \equiv \left(\frac{3\mathcal{M}}{4}\right)^{\frac{1}{3}} - dt^2 + t^{4/3}\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\}$$

$$d\mathcal{T} = \frac{dt}{\mathcal{R}(t)}$$

$$\mathcal{R} = \frac{\mathcal{M}}{6}(1 - \cos\mathcal{T}), t = \frac{\mathcal{M}}{6}(1 - \sin\mathcal{T})$$

$$\mathcal{R} = \frac{\mathcal{M}}{6}(\cosh\mathcal{T} - 1)$$

$$\mathcal{R}(t) \cong \left(\frac{3\mathcal{M}}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}}$$

$$\dot{\mathcal{R}}\mathcal{R}^{\frac{3}{2}\gamma - 1} = \mathfrak{C}$$

$$\mathcal{R}(t)^{\frac{3}{2}\gamma} = \mathfrak{C}t + \mathfrak{C}_0 - dt^2 + t^{\frac{4}{3}\gamma}\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} - dt^2$$
$$+ t^2\{dr^2 + \sinh^2(r)(d\theta^2 + sin^2(\theta)d\phi^2)\}$$

$$\Omega \equiv \frac{\mu}{3\mathcal{H}^2}, \Omega \equiv 1 + \frac{\varepsilon}{\mathcal{H}^2\mathcal{R}^2}$$

Cosmologías cuánticas no isotrópicas

$$[\chi, \gamma] = -[\gamma, \chi] \coloneqq \mathcal{L}_\chi\gamma - \mathcal{L}_\gamma\chi$$

$$\left[[\chi, \gamma], \mathcal{Z}\right] + \left[[\mathcal{Z}, \chi], \gamma\right] + \left[[\gamma, \mathcal{Z}], \chi\right]\left[\chi_{(j)}, \chi_{(\hbar)}\right] = c^i_{j\hbar}\chi_{(i)}$$

$$\mathrm{A}_i \coloneqq c^j_{ji}, \mathcal{Q}^{ij} \coloneqq \frac{1}{2}\varepsilon^{i\hbar\kappa}\left(c^j_{\hbar\kappa} - \delta^j_\hbar \mathrm{A}_\kappa\right)$$

$$Q^{ij}\mathrm{A}_i = 0$$

$$c^{j}_{lm} = \varepsilon_{ilm}Q^{ij} + \frac{1}{2}\left(\delta^{j}_{l}\mathrm{A}_m - \delta^{j}_{m}\mathrm{A}_l\right)$$

Espacio – tiempo cuántico de Bianchi

$$g \equiv \sum_{i=1,2,3} \alpha_i(t)\left(d\chi^i\right)^2$$

$$^{(4)}\mathcal{R}_{00} \equiv -\frac{1}{4}(\partial_0 \log \alpha_i)^2 - \frac{1}{2}\partial_0\partial_0 \log \alpha_i$$

$$^{(4)}\mathcal{R}_{ij} \equiv 0$$

$$^{(4)}\mathcal{R}_{ii} \equiv -\left\{\frac{1}{2}(\partial_0 \log \alpha_i)\partial_0\alpha_i \sum_{\wp=1,2,3}\left(\partial_0 \log \alpha_{\wp}\right) - \frac{1}{2}\partial^2_{00}\alpha_i\right\}$$

$$\frac{1}{2}\partial_0 \log \alpha_i = \beta_i, {}^{(4)}\mathcal{R}_{00} \equiv \sum_{i=1,2,3}\{\beta_i^2 + \partial_0\beta_i\}\frac{1}{2}\alpha_i^{-1}\partial^2_{00}\alpha_i \equiv \partial_0\left(\frac{1}{2}\alpha_i^{-1}\partial_0\alpha_i\right) + \frac{1}{2}\alpha_i^{-2}(\partial_0\alpha_i)^2$$

$$\alpha_i^{-1}\,{}^{(4)}\mathcal{R}_{ii} \equiv \beta_i\left(\sum_{j=1,2,3}\beta_j\right) + \partial_0\beta_i$$

$$g^{ij}\,{}^{(4)}\mathcal{R}_{ij} \equiv \left(\sum_{i=1,2,3}\beta_i\right) + \sum(\partial_0\beta_i)$$

$$^{(4)}\delta_{00} \equiv \frac{1}{2}\left({}^{(4)}\mathcal{R}_{00} + g^{ij}\,{}^{(4)}\mathcal{R}_{ij}\right) \equiv -\left(\sum_{i=1,2,3}\beta_i\right)^2 + \sum_{i=1,2,3}\beta_i^2$$

$$\kappa \equiv \sum_{i=1,2,3}\beta_i \equiv -\chi^{-1}\partial_0\chi$$

$$\chi \coloneqq (\det g)^{1/2} \equiv (\alpha_1\alpha_2\alpha_3)^{1/2}$$

$$^{(4)}\mathcal{R}_{ii}\alpha_i^{-1} \equiv \frac{\beta_i}{t} + \partial_0\beta_i\left(\sum_{i=1,2,3}\wp_i\right)^2 \sum_{i=1,2,3}\wp_i^2 - dt^2 + t^{2\wp_1}(d\chi^1)^2 + t^{2\wp_2}(d\chi^2)^2 + t^{2\wp_2}(d\chi^3)^2$$

$$t = (t'^2 - \chi'^2)^{\frac{1}{2}}, \chi' = \tanh^{-1}\left(\frac{\chi'}{t'}\right)$$

Modelos de cosmología cuántica con materia

$$\rho_{00} = \frac{1}{2}\mu, \rho_{ij} = \frac{1}{2}\mu g_{ij}$$

$$^{(4)}\delta_{00} \equiv -\left(\sum_{i=1,2,3} \beta_i\right)^2 + \sum_{i=1,2,3} \beta_i^2 = \mu$$

$$g^{ij\,(4)}\mathcal{R}_{ij} \equiv (\partial_0\gamma)^2 + \partial_{00}^2\gamma = \frac{9}{2}me^{-\gamma}$$

$$e^{\gamma} \equiv \chi = 9\frac{m}{2}t(t+\kappa)$$

$$\alpha_i^{-1\,(4)}\mathcal{R}_{ii} \equiv \beta_i\frac{\partial_0\chi}{\chi} + \partial_0\beta_i = \frac{1}{2}\mu = \frac{3m}{4\chi}\partial_0(\beta_i\chi) = 3\frac{m}{4}$$

$$^{(4)}\delta_{00} \equiv -\left(\sum_{i=1,2,3} \beta_i\right)^2 + \sum_{i=1,2,3} \beta_i^2 = \frac{3}{2}\frac{m}{\chi} - dt^2 + \sum_i \alpha_i(t)\left(\theta^i\right)^2$$

$$^{(4)}\mathcal{R}_{00} \equiv \sum_i\{\beta_i^2 + \partial_0\beta_i\} \equiv \sum_i \frac{\partial_{00}^2\alpha_i^{1/2}}{\alpha_i^{1/2}}$$

$$\alpha_i^{-1\,(4)}\mathcal{R}_{ii} \equiv \beta_i\chi^{-1}\partial_0\chi + \partial_0\beta_i + \alpha_i^{-1}\mathcal{R}_{ii}\chi^{-1}\partial_0\left(\alpha_2^{\frac{1}{2}}\alpha_3^{\frac{1}{2}}\partial_0\alpha_1^{\frac{1}{2}}\right) = \alpha_{11}^{-1}\mathcal{R}_{11}$$

$$\equiv \frac{1}{2\chi^2}\{(\alpha_1)^2 - (\alpha_2 - \alpha_3)^2\}$$

Modelo Kantowski–Sachs

$$-dt^2\mathrm{A}^2(t)d\chi^2 + \mathrm{B}^2(t)g^2(\chi)(d\theta^2 + f^2(\theta)d\phi^2)$$

Espacio – tiempo cuántico de Taub

$$\phi^2(d\psi + \alpha)^2 + {}^{(3)}g$$

$$-\mathcal{U}^{-1}dt^2 + (2\ell)^2\mathcal{U}(d\psi + \cos\theta d\phi)^2 + (t^2 + \ell^2)(d\theta^2 + sin^2\theta d\phi^2)$$

$$\mathcal{U} \equiv -1 + \frac{2(mt + \ell^2)}{t^2 + \ell^2}$$

$$t\pm = m \pm (m^2 + \ell^2)^{1/2}$$

$$\psi' = \psi + \frac{1}{2\ell}\int_0^t \frac{d\tau}{\mathcal{U}(\tau)} 4\ell^2 d\psi'^2 - 4\ell d\psi' dt + 4\ell^2\mathcal{U}\cos\theta d\phi d\psi' - 2\ell\cos\theta dt d\phi$$

$$+ (t^2 + \ell^2)(d\theta^2 + sin^2\theta d\phi^2)$$

$$\psi'' = \psi - \frac{1}{2\ell}\int_0^t \frac{d\tau}{\mathcal{U}(\tau)}\nabla_i v_j + \nabla_j v_i$$

$$\nabla^i\left(\nabla_i \upsilon_j + \nabla_j \upsilon_i\right) = \nabla^i \nabla_i \upsilon_j + \mathcal{R}^i_j \upsilon_i \iiint\limits_{\mathcal{M}} \left\{-|\nabla \upsilon|^2 + \mathcal{R}_{ij} \upsilon^i \upsilon^j\right\} d\mu_g$$

Problema de Cauchy a escala cuántica.

Cuestiones preliminares

$$d\theta^\alpha \equiv -\frac{1}{2} \mathfrak{C}^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma$$

$$\left[e_\alpha, e_\beta\right] = \frac{1}{2} \mathfrak{C}^\gamma_{\alpha\beta} e^\gamma$$

$$df \equiv \frac{\partial f}{\partial \chi^\alpha} d\chi^\alpha \equiv \partial_\alpha f \theta^\alpha$$

$$\partial_\alpha f \equiv \mathrm{A}^\beta_\alpha \frac{\partial f}{\partial \chi^\beta}$$

$$d^2 f \equiv \frac{1}{2}\left\{\partial_\beta \partial_\gamma f - \partial_\gamma \partial_\beta f - \mathfrak{C}^\alpha_{\beta\gamma} \partial_\alpha f\right\} \theta^\beta \wedge \theta^\gamma \left(\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha\right) f \equiv \mathfrak{C}^\gamma_{\alpha\beta} \partial_\gamma f$$

$$g \equiv g_{\alpha\beta} \theta^\alpha \theta^\beta$$

$$\omega^\beta_\gamma \coloneqq \omega^\beta_{\alpha\gamma} \theta^\alpha$$

$$\nabla_\alpha \upsilon^\beta \coloneqq \partial_\alpha \upsilon^\beta + \omega^\beta_{\alpha\gamma} \upsilon^\gamma$$

Conexión de Riemann

$$d\theta^\gamma + \omega^\gamma_{\alpha\beta} \theta^\alpha \wedge \theta^\beta$$

$$\omega^\alpha_{\beta\gamma} - \omega^\alpha_{\gamma\beta} = \mathfrak{C}^\alpha_{\beta\gamma} \langle \nabla_\alpha \partial_\beta f - \nabla_\beta \partial_\alpha f \rangle \langle \partial_\alpha g_{\beta\gamma} - \omega^\lambda_{\alpha\gamma} g_{\beta\lambda} - \omega^\lambda_{\alpha\beta} g_{\lambda\gamma} \rangle$$

$$\omega^\beta_{\alpha\gamma} \equiv \Gamma^\beta_{\alpha\gamma} + g^{\beta\mu} \widetilde{\omega}_{\alpha\gamma,\mu}$$

$$\widetilde{\omega}_{\alpha\gamma,\mu} \equiv \frac{1}{2}\left(g_{\mu\lambda} \mathfrak{C}^\lambda_{\alpha\gamma} - g_{\lambda\gamma} \mathfrak{C}^\lambda_{\alpha\mu} - g_{\alpha\lambda} \mathfrak{C}^\lambda_{\gamma\mu}\right)$$

$$\Gamma^\beta_{\alpha\gamma} \equiv \frac{1}{2} g^{\beta\mu}\left(\partial_\alpha g_{\gamma\mu} + \partial_\gamma g_{\alpha\mu} - \partial_\mu g_{\alpha\gamma}\right)$$

Curvatura de Riemann

$$\left(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda\right) \upsilon^\alpha \equiv \mathcal{R}^{\alpha\beta}_{\lambda\mu} \upsilon^\beta$$

$$\mathcal{R}^{\alpha\beta}_{\lambda\mu} \equiv \partial_\lambda \omega^\alpha_{\mu\beta} - \partial_\mu \omega^\alpha_{\lambda\beta} + \omega^\alpha_{\lambda\rho} \omega^\rho_{\mu\beta} - \omega^\alpha_{\mu\rho} \omega^\rho_{\lambda\beta} - \omega^\alpha_{\rho\beta} \mathfrak{C}^\rho_{\lambda\mu}$$

$$\langle \mathcal{R}_{\alpha\beta,\lambda\mu} + \mathcal{R}_{\mu\alpha,\beta\lambda} + \mathcal{R}_{\lambda\mu,\alpha\beta} \rangle$$

Identidades de Bianchi

$$\nabla_{\alpha}\mathcal{R}_{\beta\gamma,\lambda\mu}+\nabla_{\beta}\mathcal{R}_{\gamma\alpha,\lambda\mu}+\nabla_{\gamma}\mathcal{R}_{\alpha\beta,\lambda\mu}$$

Tensor de Ricci

$$\mathcal{R}_{\alpha\beta}\coloneqq\mathcal{R}^{\lambda\beta}_{\lambda\alpha}$$

Curvatura escalar

$$\mathcal{R}\coloneqq g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$$

Tensor de Einstein

$$\delta_{\alpha\beta}\coloneqq\mathcal{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\mathcal{R}$$

Identidad de Conservación

$$\nabla_{\alpha}\mathcal{R}^{\alpha\mu}_{\beta\gamma}-\nabla_{\beta}\mathcal{R}_{\gamma\mu}+\nabla_{\gamma}\mathcal{R}_{\beta\mu}\langle\nabla_{\alpha}\delta^{\alpha\beta}\nabla_{\alpha}\mathcal{T}^{\alpha\beta}\delta_{\alpha\beta}|\mathcal{T}_{\alpha\beta}|\rangle$$

Sistema de Cauchy

$$\partial_i=\frac{\partial}{\partial\chi^i}$$

$$\theta^i=d\chi^i+\beta^i dt$$

$$\theta^0=dt$$

$$\partial_0\equiv\partial_t-\beta^j\partial_j,\partial_t\coloneqq\frac{\partial}{\partial t}$$

$$ds^2=-\mathcal{N}^2(\theta^0)^2+g_{ij}\theta^i\theta^j$$

Coeficientes de estructura

$$\mathfrak{C}^i_{0j}=-\mathfrak{C}^i_{j0}=\partial_j\beta^i$$

$$\omega^i_{j\kappa}\equiv\Gamma^i_{j\kappa}\equiv\bar{\Gamma}^i_{j\kappa}$$

$$\omega^i_{00}\equiv\mathcal{N}g^{ij}\partial_j\mathcal{N},\omega^0_{0i}\equiv\omega^0_{i0}\equiv\mathcal{N}^{-1}\partial_i\mathcal{N},\omega^0_{00}\equiv\mathcal{N}^{-1}\partial_0\mathcal{N}$$

$$\omega^0_{ij}\equiv\frac{1}{2}\mathcal{N}^{-2}\{\partial_0 g_{ij}+g_{\hbar j}\mathfrak{C}^{\hbar}_{i0}+g_{i\hbar}\mathfrak{C}^{\hbar}_{j0}\}$$

$$\omega^0_{ij}\equiv\frac{1}{2}\mathcal{N}^{-2}\bar{\partial}_0 g_{ij}$$

$$\bar{\partial}_0\coloneqq\frac{\partial}{\partial t}-\bar{\mathcal{L}}_{\beta}$$

$$\kappa_{ij} \equiv -\omega_{ij}^{0}\eta_0 \equiv -\mathcal{N}\omega_{ij}^{0} \equiv -\frac{1}{2}\mathcal{N}^{-1}\bar{\partial}_0 g_{ij}$$

$$\omega_{0j}^{i} \equiv -\mathcal{N}\kappa_j^i + \partial_j\beta^i, \omega_{j0}^{i} \equiv -\mathcal{N}\kappa_j^i$$

$$\tau \equiv tr_{\bar{g}}\kappa \equiv \bar{g}^{ij}\kappa_{ij}$$

Curvatura extrínseca

$$\nabla_i\eta_j = \omega_{ij}^{0}\eta_0 = -\omega_{ij}^{0}\mathcal{N} \equiv \kappa_{ij}$$

$$\nabla_\alpha\eta^\alpha = \nabla_i\eta^i = \bar{g}^{ij}\kappa_{ij} \coloneqq \tau$$

Tensor de Riemann

$$\mathcal{R}_{ij,\kappa l} \equiv \bar{\mathcal{R}}_{ij,\kappa l} + \kappa_{i\kappa}\kappa_{jl} - \kappa_{il}\kappa_{\kappa j}$$

$$\mathcal{R}_{0i,j\kappa} \equiv \mathcal{N}\left(\bar{\nabla}_j\kappa_{\kappa i} - \bar{\nabla}_\kappa\kappa_{ji}\right)$$

$$\mathcal{R}_{0i,0j} \equiv \mathcal{N}\left(\bar{\partial}_0\kappa_{ij} + \mathcal{N}\kappa_{i\kappa}\kappa_j^\kappa + \bar{\nabla}_i\partial_j\mathcal{N}\right)$$

Curvatura de Ricci

$$\mathcal{N}\mathcal{R}_{ij} \equiv \mathcal{N}\bar{\mathcal{R}}_{ij} - \bar{\partial}_0\kappa_{ij} + \mathcal{N}\kappa_{ij}\kappa_\hbar^\hbar - 2\mathcal{N}\kappa_{i\kappa}\kappa_j^\kappa - \bar{\nabla}_i\partial_j\mathcal{N}$$

$$\mathcal{N}^{-1}\mathcal{R}_{0j} \equiv \partial_j\kappa_\hbar^\hbar - \bar{\nabla}_\hbar\kappa_j^\hbar, \mathcal{R}_{00} \equiv \mathcal{N}\left(\bar{\partial}_0\kappa_\hbar^\hbar - \mathcal{N}\kappa_{ij}\kappa^{ij} + \bar{\Delta}\mathcal{N}\right)$$

$$\bar{\mathcal{R}} \coloneqq g^{ij}\bar{\mathcal{R}}_{ij}, g^{ij}\mathcal{R}_{ij} = \bar{\mathcal{R}} - \mathcal{N}^{-1}\bar{\partial}_0\kappa_\hbar^\hbar + \left(\kappa_\hbar^\hbar\right)^2 - \mathcal{N}^{-1}\bar{\Delta}\mathcal{N}$$

$$\mathcal{R} \equiv -\mathcal{N}^{-2}\mathcal{R}_{00} + g^{ij}\mathcal{R}_{ij} = \bar{\mathcal{R}} + \kappa_{ij}\kappa^{ij} + \left(\kappa_\hbar^\hbar\right)^2 - 2\mathcal{N}^{-1}\bar{\partial}_0\kappa_\hbar^\hbar - 2\mathcal{N}^{-1}\bar{\Delta}\mathcal{N}$$

$$\delta_{00} \equiv \mathcal{R}_{00} - \frac{1}{2}g_{00}\mathcal{R} \equiv \frac{1}{2}\left(\mathcal{R}_{00} + g^{ij}\mathcal{R}_{ij}\right)$$

$$2\mathcal{N}^{-2}\delta_{00} \equiv -2\delta_0^0 \equiv \bar{\mathcal{R}} - \kappa_{ij}\kappa^{ij} + \left(\kappa_\hbar^\hbar\right)^2$$

$$\mathcal{N}\mathcal{R}_j^i = \mathcal{N}\bar{\mathcal{R}}_j^i - \bar{\partial}_0\kappa_j^i + \mathcal{N}\kappa_j^i\kappa_\hbar^\hbar - \bar{\nabla}_i\partial^j\mathcal{N}$$

$$\bar{\partial}_0 g^{ij} = 2\mathcal{N}\kappa^{ij}$$

Restricciones y evolución

$$\delta_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R} = \mathcal{T}_{\alpha\beta} \equiv \rho_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\rho$$

$$\mathfrak{C}_i \equiv \frac{1}{\mathcal{N}}\left(\mathcal{R}_{0i} - \rho_{0i}\right) \equiv -\bar{\nabla}_\hbar\kappa_i^\hbar + \bar{\nabla}_i\kappa_\hbar^\hbar - \mathcal{N}^{-1}\rho_{0i}$$

$$\mathfrak{C}_0 \equiv \frac{2}{\mathcal{N}^2}\left(\delta_{00} - \mathcal{T}_{00}\right) \equiv \bar{\mathcal{R}} - \kappa_j^i\kappa_i^j + \left(\kappa_\hbar^\hbar\right)^2 + 2\mathcal{T}_0^0$$

$$\mathcal{R}_{ij} \equiv \bar{\mathcal{R}}_{ij} - \frac{\bar{\partial}_0 \kappa_{ij}}{\mathcal{N}} - 2\kappa_{j\hbar}\kappa_i^{\hbar} + \kappa_{ij}\kappa_{\hbar}^{\hbar} - \frac{\bar{\nabla}_j \partial_i \mathcal{N}}{\mathcal{N}} = \rho_{ij}$$

$$\mathcal{R} - \rho = -\mathcal{N}^2(\mathcal{R}^{00} - \rho^{00})$$

$$\delta_{00} - \mathcal{T}_{00} = \frac{1}{2}(\mathcal{R}^{00} - \rho^{00}), \mathcal{R} - \rho = -2\mathcal{N}^2(\delta^{00} - \mathcal{T}^{00}) = 2(\delta_0^0 - \mathcal{T}_0^0)$$

$$\delta^{ij} - \mathcal{T}^{ij} = \frac{1}{2}\bar{g}^{ij}(\mathcal{R} - \rho) = -\bar{g}^{ij}(\delta_0^0 - \mathcal{T}_0^0)$$

$$\mathcal{N}^{-2}\partial_0\Sigma_0^i + \bar{g}^{ij}\partial_j\Sigma_0^0, \partial_0\Sigma_0^0 + \partial_i\Sigma_0^i$$

Lagrangiano y hamiltoniano

$$\mathcal{L}(g) := \int \mathcal{R}\mu_g \int_{t_1}^{t_2} \{\wp\dot{q} - \mathcal{H}(\wp, q)\}\, dt$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial \wp}, \dot{\wp} = -\frac{\partial \mathcal{H}}{\partial q}$$

$$\mathcal{L}(g) := \int_{t_1}^{t_2} \int_{\mathcal{M}} \mathcal{R}\mathcal{N}\mu_{\bar{g}} dt \equiv \int_{t_1}^{t_2} \int_{\mathcal{M}} \{\mathcal{N}(\bar{\mathcal{R}} + |\kappa|^2 + \tau^2) - 2\partial_0\tau - 2\bar{\Delta}\mathcal{N}\}\mu_{\bar{g}} dt$$

$$\mathcal{H} := \bar{\mathcal{R}} - |\kappa|^2 + \tau^2$$

$$\mathcal{L}(g) := \int_{t_1}^{t_2} \int_{\mathcal{M}} \{2\mathcal{N}|\kappa|^2 + \mathcal{N}\mathcal{H} - 2\partial_0\tau - 2\bar{\Delta}\mathcal{N}\}\mu_{\bar{g}} dt$$

$$\int_{t_1}^{t_2} \int_{\mathcal{M}} \partial_0\tau\mu_{\bar{g}} dt \equiv \int_{t_1}^{t_2} \int_{\mathcal{M}} \left\{\frac{\partial\tau}{\partial t} - \beta^i \frac{\partial\tau}{\partial\chi^i}\right\} (\det \bar{g})^{\frac{1}{2}} d\chi^1 \cdots d\chi^\eta dt$$

$$\int_{t_1}^{t_2} \int_{\mathcal{M}} \left\{\frac{\partial}{\partial t}\left[\tau(\det \bar{g})^{\frac{1}{2}}\right] + \left[-\tau(\det \bar{g})^{-\frac{1}{2}} \frac{\partial}{\partial t}(\det \bar{g})^{\frac{1}{2}} - \nabla_i(\beta^i\tau) + \tau\bar{\nabla}_i\beta^i\right](\det \bar{g})^{\frac{1}{2}}\right\} d\chi^1 \cdots d\chi^\eta dt$$

$$(\det \bar{g})^{-\frac{1}{2}} \frac{\partial}{\partial t}(\det \bar{g})^{\frac{1}{2}} - \bar{\nabla}_i\beta^i \equiv \frac{1}{2} g^{ij}\bar{\partial}_0 g_{ij} \iiint \partial_0\tau\mu_{\bar{g}} dt \cong -\frac{1}{2}\iint \tau g^{ij}\bar{\partial}_0 g_{ij}\mu_{\bar{g}} dt$$

$$\mathcal{L}(g) \cong \int_{t_1}^{t_2} \int_{\mathcal{M}} \{2\mathcal{N}\kappa^{ij}\kappa_{ij} + \mathcal{N}\mathcal{H} + \tau g^{ij}\bar{\partial}_0 g_{ij}\}\mu_{\bar{g}} dt$$

$$\mathcal{L}(g) \cong \int_{t_1}^{t_2} \int_{\mathcal{M}} \{\mathcal{P}^{ij}\bar{\partial}_0 g_{ij} + \mathcal{N}\mathcal{H}\}\mu_{\bar{g}} dt \equiv \int_{t_1}^{t_2} \int_{\mathcal{M}} \left\{\mathcal{P}^{ij}\left(\frac{\partial}{\partial t} g_{ij} - 2\bar{\nabla}_j\beta_i\right) + \mathcal{N}\mathcal{H}\right\}\mu_{\bar{g}} dt$$

$$\mathcal{L}(g) \cong \int_{t_1}^{t_2} \int_{\mathcal{M}} \left\{\mathcal{P}^{ij} \frac{\partial}{\partial t} g_{ij} - 2\beta_i \mathcal{M}^i + \mathcal{N}\mathcal{H}\right\} \mu_{\bar{g}} dt$$

$$\mathcal{L}(\bar{g}, \mathcal{P}, \mathcal{N}, \beta) := \int_{t_1}^{t_2} \int_{\mathcal{M}} \left\{\mathcal{P}^{ij} \frac{\partial}{\partial t} g_{ij} - 2\beta_i \mathcal{M}^i + \mathcal{N}\mathcal{H}\right\} d\mu_{\bar{g}} dt$$

$$\mathcal{F} := \iiint_{\mathcal{M}} \mathcal{H}_{tot} \mu_{\bar{g}}$$

$$\mathcal{H}_{tot} := 2\beta_i \mathcal{M}^i - \mathcal{N}\mathcal{H}$$

$$\mathcal{H} \equiv \bar{\mathcal{R}} - \mathcal{P}_{ij}\mathcal{P}^{ij} + \frac{1}{\eta - 1}\left(\mathcal{P}_{\hbar}^{\hbar}\right)^2$$

$$\delta\mathcal{L} \equiv \iiint_{\mathcal{M}} \left\{\frac{\partial \mathcal{L}}{\partial g_{ij}} \delta g_{ij} + \frac{\partial \mathcal{L}}{\partial \mathcal{P}^{ij}} \delta \mathcal{P}^{ij} + \frac{\partial \mathcal{L}}{\partial \beta^i} \delta \beta^i + \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \delta \mathcal{N}\right\} \mu_{\bar{g}} dt$$

$$\delta_{\mathcal{P}}\mathcal{L} \equiv \iiint_{\mathcal{M}} \left\{\frac{\partial g_{ij}}{\partial t} - \frac{\partial \mathcal{H}_{tot}}{\partial \mathcal{P}^{ij}}\right\} \delta \mathcal{P}^{ij} \mu_{\bar{g}}$$

$$\iiint_{\mathcal{M}} \frac{\partial \mathcal{H}_{tot}}{\partial \mathcal{P}^{ij}} \delta \mathcal{P}^{ij} \mu_{\bar{g}} = \iiint_{\mathcal{M}} \{-2\beta_i \bar{\nabla}_j \delta \mathcal{P}^{ij} - \mathcal{N} \delta_{\mathcal{P}} \mathcal{H}\} \mu_{\bar{g}}$$

$$\delta_{\mathcal{P}}\mathcal{H} \equiv \left(\mathcal{P}_{ij} - \frac{1}{\eta - 1} g_{ij} \mathcal{P}_{\hbar}^{\hbar}\right) \delta \mathcal{P}^{ij} \equiv 2\kappa_{ij} \delta \mathcal{P}^{ij}$$

$$\frac{\partial \mathcal{H}_{tot}}{\partial \mathcal{P}^{ij}} = \bar{\nabla}_j \beta_i + \bar{\nabla}_i \beta_j - 2\mathcal{N}\kappa_{ij}$$

$$\delta_{\bar{g},\mathcal{P}}\mathcal{F} \equiv \iiint_{\mathcal{M}} \left\{\frac{\partial \mathcal{H}_{tot}}{\partial g_{ij}} \delta g_{ij} + \frac{\partial \mathcal{H}_{tot}}{\partial \mathcal{P}^{ij}} \delta \mathcal{P}^{ij}\right\} d\mu_{\bar{g}} dt \equiv \iiint_{\mathcal{M}} \{2\beta_i \delta_{\bar{g},\mathcal{P}} \mathcal{M}^i - \mathcal{N} \delta_{\bar{g},\mathcal{P}} \mathcal{H}\} d\mu_{\bar{g}} dt$$

$$\delta_{\bar{g},\mathcal{P}}\mathcal{F} \equiv \langle(\beta, \mathcal{N}), \left(\phi' \otimes \left(\delta_{\bar{g}}, \delta\mathcal{P}\right)\right)\rangle$$

$$\delta_{\bar{g},\mathcal{P}}\mathcal{F} \equiv \langle(\beta, \mathcal{N}), \phi' \otimes \left(\delta_{\bar{g}}, \delta\mathcal{P}\right)\rangle \equiv \langle \phi'^{\dagger} \otimes (\beta, \mathcal{N}), \left(\delta_{\bar{g}}, \delta\mathcal{P}\right)\rangle$$

$$\langle \phi'^{\dagger} \otimes (\beta, \mathcal{N}), \left(\delta_{\bar{g}}, \delta\mathcal{P}\right)\rangle = \langle\left(-\frac{\partial \mathcal{P}}{\partial t}, \frac{\partial \bar{g}}{\partial t}\right), \left(\delta_{\bar{g}}, \delta\mathcal{P}\right)\rangle$$

$$\left(\frac{\partial \bar{g}}{\partial t}, \frac{\partial \mathcal{P}}{\partial t}\right) = \mathcal{J}\phi'^{\dagger} \otimes (\beta, \mathcal{N})$$

Ecuaciones ADM

$$\partial_t g_{ij} = -2\mathcal{N}\kappa_{ij} + \overline{\nabla}_i\beta_j + \overline{\nabla}_j\beta_i$$

$$\partial_t \kappa_{ij} = \mathcal{N}\{\bar{\mathcal{R}}_{ij} - 2\kappa_{i\hbar}\kappa_j^{\hbar} + \kappa_{ij}\kappa_{\hbar}^{\hbar}\} - \overline{\nabla}_i\partial_j\mathcal{N} + \beta^{\hbar}\overline{\nabla}\kappa_{ij} + \kappa_{i\hbar}\overline{\nabla}_j\beta^{\hbar} + \kappa_{\hbar j}\overline{\nabla}_i\beta^{\hbar} - \mathcal{N}\rho_{ij}$$

Coordenadas de ondas cuánticas

$$g^{\lambda\mu}\nabla_\lambda\partial_\mu\mu \equiv g^{\lambda\mu}\left(\partial^2_{\lambda\mu}\mu - \Gamma^\rho_{\lambda\mu}\partial_\rho\mu\right) \equiv -g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu}$$

$$\mathcal{R}^{(\hbar)}_{\alpha\beta} \equiv -\frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g_{\alpha\beta} + \mathcal{P}_{\alpha\beta}(g,\partial_g)$$

$$\mathcal{R}_{\alpha\beta} \equiv \mathcal{R}^{(\hbar)}_{\alpha\beta} + \mathcal{L}_{\alpha\beta}$$

$$\mathcal{L}_{\alpha\beta} \equiv \frac{1}{2}\{g_{\alpha\lambda}\partial_\beta\mathcal{F}^\lambda + g_{\beta\lambda}\partial_\alpha\mathcal{F}^\lambda\}$$

$$\mathcal{R}^{(\hbar)}_{\alpha\beta} \equiv -\frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g_{\alpha\beta} + \mathcal{P}^{\rho\sigma\gamma\delta\lambda\mu}_{\alpha\beta}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu}$$

$$\mathcal{P}^{\rho\sigma\gamma\delta\lambda\mu}_{\alpha\beta}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu} \equiv -\frac{1}{2}\left(\partial_\beta g^{\lambda\mu}\partial_\lambda g_{\alpha\mu} + \partial_\alpha g^{\lambda\mu}\partial_\lambda g_{\beta\mu}\right) - \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\mu}$$

$$\mathcal{R}^{\alpha\beta} \equiv \mathcal{R}^{\alpha\beta}_{(\hbar)} + \mathcal{M}^{\alpha\beta}$$

$$\mathcal{M}^{\alpha\beta} \equiv \frac{1}{2}\{g^{\alpha\lambda}\partial_\lambda\mathcal{F}^\beta + g^{\beta\lambda}\partial_\lambda\mathcal{F}^\alpha\}$$

$$\mathcal{R}^{\alpha\beta}_{(\hbar)} \equiv \frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g^{\alpha\beta} + \mathcal{Q}^{\rho\sigma\gamma\delta\lambda\mu}_{\alpha\beta}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu}$$

$$\partial_\lambda g^{\alpha\beta} \equiv -g^{\alpha\sigma}g^{\beta\rho}\partial_\lambda g_{\sigma\rho}$$

$$\mathcal{F}^\alpha \equiv -\nabla_\lambda g^{\lambda(\alpha)}$$

$$\nabla_\lambda\nabla^\alpha g^{\lambda(\beta)} \equiv \mathcal{R}^{\alpha,\lambda}_{\lambda\mu}g^{\mu(\beta)} \equiv \mathcal{R}^{\alpha\beta}$$

$$\mathcal{R}^{\alpha\beta}_{(\hbar)} \equiv \frac{1}{2}\{\nabla_\lambda\nabla^\alpha g^{\lambda(\beta)} + \nabla_\lambda\nabla^\beta g^{\lambda(\alpha)}\}$$

$$\nabla^\alpha g^{\lambda(\beta)} = -g^{\alpha\rho}g^{\lambda\mu}\Gamma^{(\beta)}_{\rho\mu} = -g^{\alpha\rho}g^{\lambda\mu}g^{\gamma(\beta)}[\gamma,\rho\mu]$$

$$\mathcal{R}^{\alpha\beta}_{(\hbar)} \equiv \frac{1}{4}g^{\lambda\mu}\nabla_\lambda\{(g^{\alpha\rho}g^{(\beta)\gamma} + g^{\beta\rho}g^{(\alpha)\gamma})\partial_\mu g_{\gamma\rho}\} \equiv \frac{1}{4}g^{\lambda\mu}\nabla_\lambda\{\partial_\mu g^{(\alpha)\beta} + \partial_\mu g^{\beta(\alpha)}\}$$

$$\mathcal{R}^{\alpha\beta}_{(\hbar)} \equiv \frac{1}{2}g^{\lambda\mu}\partial^2_{\lambda\mu}g^{\alpha\beta} + \mathcal{Q}^{\alpha\beta,\rho\sigma\gamma\delta\lambda\mu}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu}$$

$$\mathcal{Q}^{\alpha\beta,\rho\sigma\gamma\delta\lambda\mu}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu} \equiv g^{\lambda\mu}\left\{-2\Gamma^\rho_{\lambda\mu}\partial_\rho g^{\alpha\beta} + \Gamma^\alpha_{\lambda\rho}\partial_\mu g^{\rho\beta} + \Gamma^\beta_{\rho\lambda}\partial_\mu g^{\alpha\rho}\right\}$$

$$\delta^{\alpha\beta} := \mathcal{R}^{\alpha\beta} - \frac{1}{2}\mathcal{R}g^{\alpha\beta} \equiv \delta^{\alpha\beta}_{(\hbar)} + \frac{1}{2}\{g^{\alpha\lambda}\partial_\lambda\mathcal{F}^\beta + g^{\beta\lambda}\partial_\lambda\mathcal{F}^\alpha - g^{\alpha\beta}\partial_\lambda\mathcal{F}^\lambda\}$$

$$\mathcal{G}^{\alpha\beta} \equiv g^{\alpha\beta}(\det g)^{\frac{1}{2}}$$

$$\delta^{\alpha\beta}_{(\hbar)} \equiv \mathcal{R}^{\alpha\beta}_{(\hbar)} - \frac{1}{2} g^{\alpha\beta}\mathcal{R}_{(\hbar)}$$

$$\mathcal{R}_{(\hbar)} \equiv g^{\lambda\mu} g_{\alpha\beta}\partial^2_{\lambda\mu} g^{\alpha\beta} + g_{\alpha\beta}\mathcal{Q}^{\alpha\beta,\rho\sigma\gamma\delta\lambda\mu}(g)\partial_\rho g_{\gamma\delta}\partial_\sigma g_{\lambda\mu}$$

$$g_{\alpha\beta}\partial^2_{\lambda\mu} g^{\alpha\beta} \equiv \partial_\lambda\left(g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\right) - \partial_\lambda g_{\alpha\beta}\partial_\mu g^{\alpha\beta}$$

$$g_{\alpha\beta}\partial^2_{\lambda\mu} g^{\alpha\beta} \equiv -\partial_\lambda\left(g^{\alpha\beta}\partial_\mu g_{\alpha\beta}\right) - \partial_\lambda g_{\alpha\beta}\partial_\mu g^{\alpha\beta}$$

$$g_{\alpha\beta}\partial^2_{\lambda\mu} g^{\alpha\beta} \equiv -\partial_\lambda\left(|\det g|^{-1}\partial_\mu |\det g|\right) - \partial_\lambda g_{\alpha\beta}\partial_\mu g^{\alpha\beta}$$

$$\mathcal{R}_{(\hbar)} \sim -\frac{1}{2} g^{\alpha\beta} g^{\lambda\mu}\partial^2_{\lambda\mu} g_{\alpha\beta} \sim -\frac{g^{\lambda\mu}}{|\det g|^{\frac{1}{2}}}\partial^2_{\lambda\mu}|\det g|^{\frac{1}{2}}$$

$$\delta^{\alpha\beta}_{(\hbar)} \sim \frac{1}{2}|\det g|^{-\frac{1}{2}} g^{\lambda\mu}\partial^2_{\lambda\mu}\mathcal{G}^{\alpha\beta}$$

$$\mathcal{R}_{\alpha\beta} \equiv \mathcal{R}^{(\hbar)}_{\alpha\beta} + \frac{1}{2}\left\{g_{\alpha\lambda}\partial_\beta\left(\mathcal{F}^\lambda_{\mathcal{H}} + \mathcal{H}^\lambda\right) + g_{\beta\lambda}\partial_\alpha\left(\mathcal{F}^\lambda_{\mathcal{H}} + \mathcal{H}^\lambda\right)\right\}$$

$$\mathcal{R}^{(\hbar,\mathcal{H})}_{\alpha\beta} \equiv \mathcal{R}^{(\hbar)}_{\alpha\beta} + \frac{1}{2}\left\{g_{\alpha\lambda}\partial_\beta\mathcal{H}^\lambda + g_{\beta\lambda}\partial_\alpha\mathcal{H}^\lambda\right\}$$

$$\mathcal{R}^{(\hbar,\mathcal{H},\gamma_0)}_{\alpha\beta} \equiv \mathcal{R}^{(\hbar)}_{\alpha\beta} + \frac{1}{2}\left\{g_{\alpha\lambda}\partial_\beta\mathcal{H}^\lambda + g_{\beta\lambda}\partial_\alpha\mathcal{H}^\lambda\right\} + \frac{1}{2}\gamma_0\mathcal{F}^\lambda_{\mathcal{H}}\left(g_{\beta\lambda}\eta_\alpha + g_{\alpha\lambda}\eta_\beta - g_{\alpha\beta}\eta_\lambda\right)$$

$$g^{\alpha\beta}\left(\partial^2_{\alpha\beta} f^{\mathrm{A}} - \Gamma^\lambda_{\alpha\beta}\partial_\lambda f^{\mathrm{A}} - \Gamma^{\mathcal{A}}_{\mathcal{BC}}(f)\partial_\alpha f^{\mathrm{B}}\partial_\beta f^{\mathfrak{C}}\right)$$

$$\hat{\mathcal{F}}^\lambda \equiv g^{\alpha\beta}\left(\Gamma^\lambda_{\alpha\beta} - \hat{\Gamma}^\lambda_{\alpha\beta}\right)$$

$$\mathcal{R}^{(\hat{e})}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2}\left\{g_{\alpha\lambda}\mathfrak{D}_\beta\hat{\mathcal{F}}^\lambda + g_{\beta\lambda}\mathfrak{D}_\alpha\hat{\mathcal{F}}^\lambda\right\}$$

$$\mathcal{R}^{(\hat{e})}_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu}\mathfrak{D}_\lambda\mathfrak{D}_\mu g_{\alpha\beta} + f_{\alpha\beta}$$

$$\mathfrak{D}_\lambda\mathfrak{D}_\mu g_{\alpha\beta} = \partial^2_{\lambda\mu} g_{\alpha\beta} - g_{\rho\beta}\partial_\lambda\hat{\Gamma}^\rho_{\alpha\mu} - g_{\alpha\rho}\partial_\lambda\hat{\Gamma}^\rho_{\mu\beta}$$

$$\hat{\mathcal{F}}^\beta \equiv \mathcal{F}^\beta - g^{\lambda\mu}\hat{\Gamma}^\beta_{\lambda\mu}$$

$$\hat{\mathcal{R}}^\lambda_{\mu\alpha\beta} = \partial_\alpha\hat{\Gamma}^\lambda_{\mu\beta} - \partial_\beta\hat{\Gamma}^\lambda_{\mu\alpha}$$

$$f_{\alpha\beta} \equiv \mathcal{P}^{\rho\sigma\gamma\delta\lambda\mu}_{\alpha\beta}\mathfrak{D}_\rho g_{\gamma\delta}\mathfrak{D}_\sigma g_{\lambda\mu} + \frac{1}{2} g^{\lambda\mu}\left\{g_{\alpha\rho}\hat{\mathcal{R}}^\rho_{\lambda\beta\mu} + g_{\beta\rho}\hat{\mathcal{R}}^\rho_{\lambda\alpha\mu}\right\}$$

$$\mathfrak{D}_\lambda\mathfrak{D}_\mu g^{\alpha\beta} = \partial^2_{\lambda\mu} g^{\alpha\beta} + g^{\rho\beta}\partial_\lambda\hat{\Gamma}^\alpha_{\mu\rho} + g^{\alpha\rho}\partial_\lambda\hat{\Gamma}^\beta_{\mu\rho}$$

$$\mathcal{R}^{\alpha\beta} \equiv \mathcal{R}^{\alpha\beta}_{(\hat{e})} + \frac{1}{2}\left\{g^{\alpha\lambda}\mathfrak{D}_\lambda\hat{\mathcal{F}}^\beta + g^{\beta\lambda}\mathfrak{D}_\lambda\hat{\mathcal{F}}^\alpha\right\}$$

$$\mathcal{R}^{\alpha\beta}_{(\hat{e})} \equiv \frac{1}{2} g^{\lambda\mu}\mathfrak{D}_\lambda\mathfrak{D}_\mu g^{\alpha\beta} + \mathcal{Q}^{\alpha\beta\rho\sigma}_{\gamma\delta\lambda\mu}(g)\mathfrak{D}_\rho g^{\gamma\delta}\mathfrak{D}_\sigma g^{\lambda\mu} - \frac{1}{2} g^{\lambda\mu}\left\{g^{\alpha\rho}\hat{\mathfrak{R}}^{\beta}_{\mu\lambda\rho} + g^{\beta\rho}\hat{\mathfrak{R}}^{\alpha}_{\mu\lambda\rho}\right\}$$

$$Ricc(g)^{(\hat{e})} \equiv \frac{1}{2} g \otimes \mathfrak{D}^2 g + \mathcal{P}(g)(\mathfrak{D}g, \mathfrak{D}g) - g \otimes \hat{\mathfrak{R}}iemann$$

$$-\bar{\mathcal{N}}^{-3}\left(\overline{\partial_0\mathcal{N}} + \bar{\mathcal{N}}\bar{g}^{ij}\bar{\kappa}_{ij}\right) = g^{\lambda\mu}\hat{\Gamma}^{0}_{\lambda\mu}$$

$$\bar{\mathcal{N}}^{-1}\bar{g}^{i\hbar}\partial_\hbar\bar{\mathcal{N}} - \bar{\mathcal{N}}^{-2}\bar{g}^{i\hbar}\partial_t g_{0\hbar} + \bar{g}^{j\hbar}\hat{\Gamma}^{i}_{j\hbar} = g^{\lambda\mu}\hat{\Gamma}^{i}_{\lambda\mu}$$

Preservación de las ondas cuánticas

$$\nabla_\alpha \delta^{\alpha\beta} \equiv 0$$

$$\mathcal{R}^{\alpha\beta}_{(\hbar)} = \rho^{\alpha\beta}$$

$$\delta_{\alpha\beta} - \mathcal{T}_{\alpha\beta} = -\frac{1}{2}\left(g^{\alpha\lambda}\partial_\lambda\hat{\mathcal{F}}^{\beta} + g^{\beta\lambda}\partial_\lambda\hat{\mathcal{F}}^{\alpha} - g^{\alpha\beta}\partial_\lambda\hat{\mathcal{F}}^{\lambda}\right)$$

$$g^{\alpha\lambda}\partial^2_{\alpha\lambda}\hat{\mathcal{F}}^{\beta} + \mathrm{A}^{\beta\lambda}_{\alpha}\partial_\lambda\hat{\mathcal{F}}^{\alpha}$$

$$\mathcal{R}^{(\hbar)}_{\alpha\beta} + \frac{1}{2}\{g_{\alpha\lambda}\partial_\beta\mathcal{H}^{\lambda} + g_{\beta\lambda}\partial_\alpha\mathcal{H}^{\lambda}\} = \rho_{\alpha\beta}$$

$$g^{\alpha\lambda}\partial^2_{\alpha\lambda}\hat{\mathcal{F}}^{\beta}_{\mathcal{H}} + \mathrm{A}^{\beta\lambda}_{\alpha}\partial_\lambda\hat{\mathcal{F}}^{\alpha}_{\mathcal{H}}$$

$$\delta_{\alpha\beta} - \mathcal{T}_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\lambda}\mathfrak{D}_\beta\hat{\mathcal{F}}^{\lambda} + g_{\beta\lambda}\mathfrak{D}_\alpha\hat{\mathcal{F}}^{\lambda} - g_{\alpha\beta}\mathfrak{D}_\lambda\hat{\mathcal{F}}^{\lambda}\right)$$

$$\delta^{\alpha 0} \equiv \delta^{\alpha 0}_{(\hbar)} + \frac{1}{2}\left(g^{\alpha\lambda}\partial_\lambda\hat{\mathcal{F}}^{0} + g^{0\lambda}\partial_\lambda\hat{\mathcal{F}}^{\alpha} - g^{\alpha 0}\partial_\lambda\hat{\mathcal{F}}^{\lambda}\right)$$

$$\delta^{\alpha 0} \equiv \delta^{\alpha 0}_{(\hbar)} + \frac{1}{2} g^{00}\partial_0\mathcal{F}^{\alpha}$$

$$g_{\lambda\mu} = \frac{\partial f^\alpha}{\partial\chi^\lambda}\frac{\partial f^\beta}{\partial\chi^\mu}\underline{g}_{\alpha\beta}$$

$$\Gamma^{0}_{ij} = \frac{\partial f^0}{\partial\chi^\lambda}\frac{\partial\mathcal{F}^\alpha}{\partial\underline{\chi}^i}\frac{\partial\mathcal{F}^\beta}{\partial\underline{\chi}^j}\Gamma^{\lambda}_{\alpha\beta} + \frac{\partial f^0}{\partial\chi^\alpha}\frac{\partial}{\partial\underline{\chi}^i}\frac{\partial\mathcal{F}^\alpha}{\partial\underline{\chi}^j}$$

$$\mathfrak{C}^{\alpha} \equiv \frac{1}{2} g^{ij}\partial^2_{ij}\mathcal{G}^{\alpha 0} - \frac{1}{2} g^{00}\partial^2_{it}\mathcal{G}^{\alpha i} + g^{i0}\partial^2_{it}\mathcal{G}^{\alpha 0} + \mathcal{K}^{\alpha 0} - \mathcal{T}^{\alpha 0}$$

$$\mathfrak{C}^{0} \equiv \frac{1}{2} g^{ij}\partial^2_{ij}\mathcal{G}^{00} - \mathcal{A}$$

$$\mathcal{A} \coloneqq \frac{1}{2} g^{00}\partial^2_{it}\mathcal{G}^{\alpha i} + g^{i0}\partial^2_{it}\mathcal{G}^{\alpha 0} + \mathcal{K}^{\alpha 0} - \mathcal{T}^{\alpha 0}$$

$$\mathfrak{C}^{\hbar} \equiv -\frac{1}{2} g^{00}\partial_i\left(\partial_t\mathcal{G}^{\hbar i}\right) + \mathfrak{B}^{\hbar}$$

$$\mathfrak{B}^{\hbar} \coloneqq \frac{1}{2} g^{ij} \partial_{ij}^{2} \mathcal{G}^{\hbar 0} + g^{i0} \partial_{it}^{2} \mathcal{G}^{\hbar 0} + \mathcal{K}^{\alpha 0} - \mathcal{T}^{\alpha 0}$$

$$\mathfrak{C}^{0} \equiv \frac{1}{2} g^{ij} \partial_{ij}^{2} \mathcal{G}^{00} - g^{0i} \partial_{ij}^{2} \mathcal{G}^{j0} - f^{0}$$

$$\mathfrak{C}^{\hbar} \equiv \frac{1}{2} g^{ij} \partial_{ij}^{2} \mathcal{G}^{0\hbar} - f^{\hbar}$$

$$\delta_{\alpha\beta} = \tau_{\alpha\beta} \equiv \mathcal{F}_{\alpha}^{\lambda} \mathcal{F}_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} \mathcal{F}^{\lambda\mu} \mathcal{F}_{\lambda\mu}$$

$$\mathcal{R}_{\alpha\beta} = \mathcal{F}_{\alpha}^{\lambda} \mathcal{F}_{\beta\lambda} - \frac{1}{2(\eta - 1)} g_{\alpha\beta} \mathcal{F}^{\lambda\mu} \mathcal{F}_{\lambda\mu}$$

$$\nabla^{\alpha} \mathcal{F}_{\alpha\beta} = g^{\alpha\lambda} \left(\partial_{\lambda} \mathcal{F}_{\alpha\beta} - \Gamma_{\lambda\beta}^{\mu} \mathcal{F}_{\alpha\mu} \right) - g^{\alpha\lambda} \Gamma_{\lambda\alpha}^{\mu} \mathcal{F}_{\mu\beta}$$

$$\Gamma^{\mu} \coloneqq g^{\alpha\lambda} \Gamma_{\lambda\alpha}^{\mu}$$

$$g^{\alpha\lambda} \left\{ \partial_{\lambda} \partial_{\alpha} \mathrm{A}_{\beta} - \partial_{\lambda} \partial_{\beta} \mathrm{A}_{\alpha} - \Gamma_{\lambda\beta}^{\mu} \left(\partial_{\alpha} \mathrm{A}_{\mu} - \partial_{\mu} \mathrm{A}_{\alpha} \right) \right\}$$

$$g^{\alpha\lambda} \partial_{\lambda} \partial_{\alpha} \mathrm{A}_{\beta} - \partial_{\beta} \left(g^{\alpha\lambda} \partial_{\lambda} \mathrm{A}_{\alpha} \right) + \left(\partial_{\beta} g^{\alpha\lambda} \right) \partial_{\lambda} \mathrm{A}_{\alpha} - \Gamma_{\lambda\beta}^{\mu} \left(\partial_{\alpha} \mathrm{A}_{\mu} - \partial_{\mu} \mathrm{A}_{\alpha} \right)$$

$$g^{\alpha\lambda} \partial_{\lambda} \partial_{\alpha} \mathrm{A}_{\beta} = f_{\beta} \left(g, \partial_{g}, \partial \mathrm{A} \right)$$

$$g^{\alpha\lambda} \partial_{\lambda} \mathrm{A}_{\alpha} \equiv g^{\alpha\lambda} \left(\nabla_{\lambda} \mathrm{A}_{\alpha} + \Gamma_{\alpha\lambda}^{\mu} \mathrm{A}_{\mu} \right)$$

Restricciones

Restricción hamiltoniana

$$\mathcal{H}(\bar{g}, \kappa) \equiv \mathcal{R}(\bar{g}) - |\kappa|_{\bar{g}}^{2} + \left(tr_{\bar{g}} \kappa \right)^{2} - 2\rho$$

$$\bar{\mathcal{R}} \equiv e^{-2\lambda} \{ \mathcal{R}(\gamma) - 2(\eta - 1) \Delta_{\gamma} \lambda - (\eta - 1)(\eta - 2) \bar{g}^{ij} \partial_{i} \lambda \partial_{j} \lambda \}$$

$$\mathcal{R}(\bar{g}) \equiv \varphi^{-\frac{(\eta+2)}{(\eta-2)}} \left(\varphi \mathcal{R}(\gamma) - \frac{4(\eta - 1)}{\eta - 2} \Delta_{\gamma} \varphi \right), \bar{g} \equiv \varphi^{\frac{4}{(\eta-2)}} \gamma^{ij}$$

$$\Delta_{\gamma} \varphi - \kappa_{\eta} \mathcal{R}(\gamma) \varphi + \kappa_{\eta} \left(|\kappa|_{\bar{g}}^{2} - \tau^{2} + 2\rho \right) \varphi^{\frac{(\eta+2)}{(\eta-2)}}$$

$$\bar{\nabla}_{i} \mathcal{P}^{ij} \equiv \varphi^{-2\frac{(\eta+2)}{(\eta-2)}} \mathcal{D}_{i} \left\{ \varphi^{-2\frac{(\eta+2)}{(\eta-2)}} \mathcal{P}^{ij} \right\} - \frac{2}{\eta - 2} \varphi^{-1} \gamma^{ij} \partial_{i} \varphi tr_{\gamma} \mathcal{P}$$

$$\bar{\Gamma}_{j\hbar}^{i} = \mathfrak{C}_{j\hbar}^{i} + \frac{2}{\eta - 2} \varphi^{-1} \{ \delta_{j}^{i} \partial_{\hbar} \varphi + \delta_{\hbar}^{i} \partial_{j} \varphi - \gamma^{i\kappa} \gamma_{j\hbar} \partial_{\kappa} \varphi \}$$

$$\mathfrak{K}^{ij} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}} \tilde{\mathfrak{K}}^{ij} + \frac{1}{\eta} \bar{g}^{ij} \tau$$

$$\mathfrak{K}_{ij} = \varphi^{-2}\tilde{\mathfrak{K}}_{ij} + \frac{1}{\eta}\bar{g}^{ij}\tau, \bar{g}_{ij} = \varphi^{\frac{4}{\eta-2}}\gamma_{ij}$$

$$tr_\gamma\tilde{\mathfrak{K}} \equiv \gamma^{ij}\tilde{\mathfrak{K}}_{ij} = \varphi^{\frac{2\eta}{\eta-2}}\bar{g}^{ij}\left(\mathfrak{K}_{ij} - \frac{1}{\eta}\bar{g}^{ij}\tau\right)$$

$$\mathcal{D}_i\tilde{\mathfrak{K}}^{ij} = \frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\gamma^{ij}\partial_i\tau + \varphi^{2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{J}$$

$$|\mathfrak{K}|_{\bar{g}}^2 \coloneqq \bar{g}_{i\hbar}\bar{g}_{j\kappa}\mathfrak{K}^{ij}\mathfrak{K}^{\hbar\kappa} = \varphi^{\frac{(-3\eta+2)}{(\eta-2)}}\bar{\gamma}_{i\hbar}\bar{\gamma}_{j\kappa}\tilde{\mathfrak{K}}^{ij}\tilde{\mathfrak{K}}^{\hbar\kappa} + \frac{1}{\eta}\tau^2 \equiv \varphi^{\frac{(-3\eta+2)}{(\eta-2)}}\left|\tilde{\mathfrak{K}}\right|_\gamma^2 + \frac{1}{\eta}\tau^2$$

$$\Delta_\gamma\varphi - \kappa_\eta\mathcal{R}(\gamma)\varphi + \kappa_\eta\varphi^{\frac{(-3\eta+2)}{(\eta-2)}}\left|\tilde{\mathfrak{K}}\right|_\gamma^2 - \frac{\eta-2}{4\eta}\varphi^{\frac{(\eta+2)}{(\eta-2)}}\tau^2 + \frac{\eta-2}{2(\eta-1)}\rho\varphi^{\frac{(\eta+2)}{(\eta-2)}}$$

$$\mathcal{H} \equiv \Delta_\gamma\varphi - f(\varphi)$$

Restricción de momentum

$$\mathcal{M}(\bar{g},\kappa) \equiv \bar{\nabla}\otimes\kappa - \bar{\nabla}tr\kappa - \mathfrak{J}$$

$$\mathcal{M}^i \equiv \mathcal{D}_j\tilde{\mathfrak{K}}^{ij} - \mathfrak{F}^i$$

$$\mathfrak{F}^i \coloneqq \frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\gamma^{ij}\partial_j\tau + \bar{\mathcal{J}}^i\varphi^{2\frac{(\eta+2)}{(\eta-2)}}$$

$$f(\varphi) \equiv \mathfrak{K}_\eta\mathcal{R}(\gamma) - \mathfrak{K}_\eta\left|\tilde{\mathfrak{K}}\right|_\gamma^2\varphi^{\frac{(-3\eta+2)}{(\eta-2)}} + \frac{\eta-2}{4\eta}\tau^2\varphi^{\frac{(\eta+2)}{(\eta-2)}}$$

$$f(\varphi) \equiv r\varphi - \alpha\varphi^{-\frac{3\eta+2}{\eta-2}} - q_1\varphi^{-\frac{\eta}{\eta-2}} - q_2\varphi^{\frac{-6+\eta}{\eta-2}} + \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$r \equiv \mathfrak{K}_\eta\left(\mathcal{R}(\gamma) - \left|\mathcal{D}\bar{\psi}\right|_\gamma^2\right), \alpha \equiv \mathfrak{K}_\eta\left(\left|\tilde{\mathfrak{K}}\right|_\gamma^2 + |\tilde{\pi}|^2\right), \beta \equiv \frac{\eta-2}{4\eta}\tau^2 - q_0$$

$$q_1 = \mathfrak{K}_\eta\left(\gamma_{ij}\bar{\mathfrak{E}}^i\bar{\mathfrak{E}}^j + 2\tilde{\rho}\right), q_2 = \mathfrak{K}_\eta\gamma^{i\hbar}\gamma^{j\kappa}\bar{\mathfrak{F}}_{ij}\bar{\mathfrak{F}}_{\hbar\kappa}$$

$$q_0 = 2\mathfrak{K}_\eta(\mathcal{V}(\bar{\psi}) + \bar{\rho}_0)$$

$$\bar{\mathcal{J}}^i\varphi^{2\frac{(\eta+2)}{(\eta-2)}} \equiv \tilde{\mathcal{J}}_1^i + \varphi^{2\frac{(\eta+1)}{(\eta-2)}}\tilde{\mathcal{J}}_0^i$$

$$\tilde{\mathcal{J}}_1^i \equiv -\gamma^{ij}\left(\partial_j\bar{\psi}\tilde{\pi} + \tilde{\eta}_{j\kappa l}\bar{\mathfrak{E}}^\kappa\bar{\mathfrak{H}}^l\right) + \left(1 + \gamma_{\hbar j}\tilde{\mu}^\hbar\tilde{\mu}^j\right)^{\frac{1}{2}}\tilde{\mu}^i(\tilde{\mu} + \tilde{\wp})$$

$$\tilde{\mathcal{J}}_0^i \equiv \left(1 + \gamma_{\hbar j}\tilde{\mu}^\hbar\tilde{\mu}^j\right)^{1/2}\tilde{\mu}^i(\tilde{\mu} + \tilde{\wp})$$

$$\left(\Delta_{\gamma,conf}\chi\right)^i \equiv \mathfrak{D}_j(\mathfrak{L}\chi)^{ij} = \mathfrak{F}^i$$

$$\mathfrak{F}^i \equiv \frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\gamma^{ij}\partial_j\tau + \varphi^{\frac{4}{(\eta-2)}}\tilde{\mathcal{J}}_0^i + \tilde{\mathcal{J}}_1^i - \mathfrak{D}_j\mathfrak{U}^{ij}$$

$$\|\chi\|_{\mathcal{W}_2^{\wp}} \leq \mathfrak{C}_\gamma \|\mathfrak{F}\|_{\mathfrak{L}^{\wp}}$$

$$\iiint\limits_{\mathcal{M}} \chi_j \left(\Delta_{\gamma,conf}\chi - \kappa\chi\right)^j \mu_\gamma = -\iiint\limits_{\mathcal{M}} \left\{\left|\mathcal{L}_{\gamma,conf}\chi\right|_\gamma^2 + \mathscr{k}|\chi|_\gamma^2\right\}\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}} \chi_j \left(\widetilde{\Delta}_{\gamma,conf}\chi - \kappa\chi\right)^j \mu_\gamma \equiv \iiint\limits_{\mathcal{M}} \chi_j \mathfrak{D}_i \left\{\widetilde{\mathcal{N}}^{-1}\left(\mathfrak{D}^i\chi^j + \mathfrak{D}^j\chi^i - \frac{2}{\eta}\gamma^{ij}\right)\mathfrak{D}_\kappa \chi^\kappa\right\}\mu_\gamma$$

$$= -\iiint\limits_{\mathcal{M}} \widetilde{\mathcal{N}}^{-1}\left(\mathfrak{D}^i\chi^j + \mathfrak{D}^j\chi^i - \frac{2}{\eta}\gamma^{ij}\mathfrak{D}_\kappa\chi^\kappa\right) \bigotimes \left(\mathfrak{D}_i\chi_j + \mathfrak{D}_j\chi_i - \frac{2}{\eta}\gamma_{ij}\mathfrak{D}_l\chi^l\right)\mu_\gamma$$

$$\mathfrak{D}^i\mathfrak{D}_i\chi_j + \left(1 - \frac{2}{\eta}\right)\mathfrak{D}_j\mathfrak{D}^i\chi_i + \rho_{jl}\chi^l$$

$$\iiint\limits_{\mathcal{M}} \left\{-\mathfrak{D}^i\chi^j\mathfrak{D}_i\chi_j - \left(1 - \frac{2}{\eta}\right)\mathfrak{D}_j\chi^j\mathfrak{D}^i\chi_i + \rho_{jl}\chi^l\chi^j\right\}\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}} \mathfrak{D}_i\mathfrak{U}^{ij}\chi_j\mu_\gamma = -\iiint\limits_{\mathcal{M}} \mathfrak{U}^{ij}\left(\mathcal{L}_{\gamma,conf}\chi\right)_{ij}\mu_\gamma$$

Restricciones de map

$$\mathcal{H}(\bar{g},\kappa) \equiv \bar{\mathcal{R}} - |\kappa|_{\bar{g}}^2 + \left(tr_{\bar{g}}\kappa\right)^2, \bar{\mathcal{R}} \coloneqq \mathcal{R}(\bar{g}), |\kappa|_{\bar{g}}^2 \coloneqq \kappa_j^i\kappa_i^j, tr_{\bar{g}}\kappa \coloneqq \kappa_i^i$$

$$\mathcal{M}(\bar{g},\kappa) \equiv \overline{\nabla}\kappa - \partial tr\kappa, \mathcal{M}_i(\bar{g},\kappa) \equiv \overline{\nabla}_j\kappa_j^i - \partial_i(tr\kappa)$$

$$\phi' \coloneqq (\hbar,\kappa) \mapsto \delta\phi \equiv (\delta\mathcal{H}, 2\delta\mathcal{M})$$

$$\delta\mathcal{H} = \delta\bar{\mathcal{R}} - 2\kappa\otimes\kappa + 2\left(tr_{\bar{g}}\kappa\right)\left(tr_{\bar{g}}\mathfrak{K}\right)$$

$$\delta\bar{\mathcal{R}} = -\Delta_{\bar{g}}tr_{\bar{g}}\hbar + div_{\bar{g}}\left(div_{\bar{g}}\hbar\right) - \hbar\otimes Ricci(\bar{g})$$

$$div_{\bar{g}}\left(div_{\bar{g}}\hbar\right) \equiv \overline{\nabla}^i\overline{\nabla}^j\hbar_{ij}$$

$$\delta\mathcal{M}_i = \overline{\nabla}_j\kappa_i^j - \partial_i(tr\kappa) + \frac{1}{2}\kappa_i^l\partial_l tr_{\bar{g}}\hbar - \frac{1}{2}\kappa^{jl}\overline{\nabla}_i\hbar_{jl}$$

$$\phi'^\dagger \coloneqq (f,\chi) \mapsto \phi'^\star \coloneqq (f,\chi) \equiv \left(\phi_{\bar{g}}'^\dagger(f,\chi), \phi_{\mathcal{P}}'^\star(f,\chi)\right)$$

$$\langle(\hbar,\kappa), \phi'^\dagger(f,\chi)\rangle = \langle\phi'^\star(\hbar,\kappa),(f,\chi)\rangle$$

$$\iiint\left\{\hbar\otimes\phi_{\bar{g}}'^\dagger(f,\chi) + \kappa\otimes\phi_\kappa'^\star(f,\chi)\right\}d\mu_{\bar{g}} = \iiint\left\{f\delta\mathcal{H} + 2\chi^i\delta\mathcal{M}_i\right\}\mu_{\bar{g}}$$

$$\phi_{\bar{g}}'^\dagger(f,\chi)^{ij} = -g^{ij}\Delta_{\bar{g}}f + \overline{\nabla}^i\overline{\nabla}^j f - \bar{\mathcal{R}}^{ij}f - g^{ij}\overline{\nabla}_\hbar\left(\chi^\kappa\kappa_\kappa^\hbar\right) + \overline{\nabla}_\hbar\left(\chi^\hbar\kappa_j^i\right)$$

$$\phi_{\kappa}^{\prime\star}(f,\chi)_j^i \equiv -2f\left(\kappa_j^i - \delta_j^i\kappa_\hbar^\hbar\right) - 2\overline{\nabla}_j\chi^i + 2\delta_j^i\overline{\nabla}_\hbar\chi^\hbar$$

$$\begin{pmatrix} -g^{ij}\Delta_{\bar{g}}f + \overline{\nabla}^i\overline{\nabla}^j f & 0 \\ 0 & -2\overline{\nabla}_j\chi^i + 2\delta_j^i\overline{\nabla}_\hbar\chi^\hbar \end{pmatrix}$$

$$\mathrm{A}^* \equiv \begin{pmatrix} \left(-g^{ij}\xi^\hbar\xi_\hbar + \xi^i\xi^j\right)f & 0 \\ 0 & -2\xi_j\chi^i + 2\delta_j^i\xi_\hbar\chi^\hbar \end{pmatrix}$$

$$\mathcal{W}_\delta^{\wp} = range\phi' + ker\phi'^*$$

$$\mathcal{W}_\delta^{\wp} = \mathfrak{E}_1 + \mathfrak{E}_2, \mathfrak{E}_1 \coloneqq ker\phi', \mathfrak{E}_2 \coloneqq range\phi'^*$$

$$Einstein\ (g) \coloneqq \delta(g)$$

$$\delta_g'\otimes\chi \iiint_{\mathcal{M}_t} \chi\otimes\delta_{g,g}''\otimes(\chi,\chi)\otimes\eta d\mu_{\bar{g}}$$

Conservación de Taub

$$\frac{d^2}{d\lambda^2}\phi\left(\mu(\lambda)\right) \equiv \phi'(\mu)\frac{d^2\mu}{(d\lambda)^2} + \phi''(\mu)\left(\frac{d\mu}{d\lambda},\frac{d\mu}{d\lambda}\right)$$

Criterio de Fisher–Marsden

$$\delta\mathcal{H} \equiv -\Delta_{\bar{g}}tr_{\bar{g}}\hbar + \overline{\nabla}^i\overline{\nabla}^j\hbar_{ij} - \bar{\mathcal{R}}^{ij}\hbar_{ij} - 2\kappa_i^j\mathcal{K}_j^i + 2(tr\kappa)(tr\mathcal{K}) = f_0$$

$$\delta\mathcal{M}_i \equiv \overline{\nabla}_j\kappa_i^j - \partial_i(tr\kappa) + \frac{1}{2}\kappa_i^l\partial_l tr_{\bar{g}}\hbar - \frac{1}{2}\kappa^{jl}\overline{\nabla}_i\hbar_{jl} = f_i$$

$$\hbar_{ij} = \frac{1}{\eta}\lambda\bar{g}_{ij}, \kappa_i^j = \frac{1}{2}\left\{-\lambda\mathcal{K}_i^j + \left(\mathcal{L}_{conf,\bar{g}}\chi\right)_i^j\right\}$$

$$\delta\mathcal{H} = \left(\frac{1}{\eta} - 1\right)\Delta_{\bar{g}}\lambda - \lambda\left(\frac{1}{\eta}\bar{\mathcal{R}} - |\kappa|^2\right) = f_0$$

$$\left(\frac{1}{\eta} - 1\right)\left(\Delta_{\bar{g}}\lambda - \lambda|\kappa|^2\right) - \kappa_j^i\left(\mathcal{L}_{conf,\bar{g}}\chi\right)_i^j = f_0$$

$$\delta\mathcal{M}_i = \left(\overline{\Delta}_{\bar{g},conf}\chi\right) \coloneqq \overline{\nabla}_j\left(\mathcal{L}_{conf,\bar{g}}\chi\right)_i^j = f_i$$

$$\left(\mathcal{L}_{\gamma,conf}\gamma\right)^{ij} \equiv \mathcal{D}^i\gamma^j + \mathcal{D}^j\gamma^i - \frac{2}{\eta}\gamma^{ij}\mathfrak{D}_\kappa\gamma^\kappa$$

$$\Delta_{\gamma,conf}\gamma \coloneqq \mathfrak{D}\otimes\left(\mathcal{L}_{\gamma,conf}\gamma\right) = \mathfrak{F}$$

$$\Delta_{\gamma,conf}\mathcal{Z} \equiv \mathfrak{D}\otimes\left(\mathcal{L}_{\gamma,conf}\mathcal{Z}\right) = -\mathfrak{D}\otimes\mathfrak{U}$$

$$\mathfrak{K}^{ij} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\left[\left(\mathcal{L}_{\gamma,conf}\chi\right)^{ij} + \mathfrak{U}^{ij}\right] + \frac{1}{\eta}\bar{g}^{ij}\tau$$

$$\Delta_{\gamma,conf}\chi = \mathfrak{F}$$

$$\mathfrak{F} \coloneqq -\mathfrak{D}\otimes\mathfrak{U} + \frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\partial\tau + \varphi^{2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{J}$$

$$\varphi^{2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{J} = \tilde{\mathcal{J}}_1 + \tilde{\mathcal{J}}_0\varphi^{\frac{4}{(\eta-2)}}$$

$$\mathfrak{U}'^{ij} \equiv \theta^{-2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{U}^{ij}$$

$$\left(\mathcal{L}_{\gamma',conf}\chi\right)^{ij} \equiv \theta^{\frac{-4}{(\eta-2)}}\left(\mathcal{L}_{\gamma,conf}\chi\right)^{ij}$$

$$\tilde{\mathcal{L}}_{\gamma,conf}\chi \coloneqq \widetilde{\mathcal{N}}^{-1}\mathcal{L}_{\gamma,conf}\chi$$

$$\left(\widetilde{\Delta}_{\gamma,conf}\chi\right)^{j} \coloneqq \mathcal{D}_i\left(\tilde{\mathcal{L}}_{\gamma,conf}\chi\right)^{ij}$$

$$\widetilde{\mathfrak{K}}^{ij} \equiv \left(\tilde{\mathcal{L}}_{\gamma,conf}\chi\right)^{ij} + \mathfrak{U}^{ij}$$

$$\left(\tilde{\mathcal{L}}_{\gamma',conf}\chi\right)^{ij} = \theta^{-2\frac{(\eta+2)}{(\eta-2)}}\left(\tilde{\mathcal{L}}_{\gamma,conf}\chi\right)^{ij}$$

$$\mathcal{M}^{i} \equiv \left(\widetilde{\Delta}_{\gamma,conf}\chi\right)^{j} + \mathcal{D}_i\mathfrak{U}^{ij} - \left\{\frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\gamma^{ij}\partial\tau + \tilde{\mathcal{J}}_1 + \tilde{\mathcal{J}}_0\varphi^{\frac{4}{(\eta-2)}}\right\}$$

$$\widetilde{\mathfrak{K}}^{ij} \equiv \left(\tilde{\mathcal{L}}_{\gamma,conf}\chi\right)^{ij} + \mathfrak{U}^{ij}$$

$$\widetilde{\mathfrak{K}}'^{ij} \equiv \left(\tilde{\mathcal{L}}_{\gamma',conf}\chi\right)^{ij} + \mathfrak{U}'^{ij}, \mathfrak{U}'^{ij} = \theta^{-2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{U}^{ij}, \widetilde{\mathcal{N}}' = \theta^{\frac{2\eta}{\eta-2}}\widetilde{\mathcal{N}}$$

$$\mathfrak{K}_{\mathcal{T}}^{ij} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\left\{\widetilde{\mathcal{N}}^{-1}\left(\mathcal{L}_{\gamma,conf}\chi\right)^{ij} + \mathfrak{U}^{ij}\right\}$$

$$\mathfrak{K}^{ij} = \frac{1}{2}\bar{\mathcal{N}}^{-1}\left(\overline{\partial_t g^{ij}} + \left(\bar{\mathcal{L}}_\beta\bar{g}\right)^{ij}\right), \left(\bar{\mathcal{L}}_\beta\bar{g}\right)^{ij} \equiv \bar{\nabla}^i\beta^j + \bar{\nabla}^j\beta^i$$

$$\tau \coloneqq \bar{g}_{ij}\mathfrak{K}^{ij} = -\frac{1}{2}\bar{\mathcal{N}}^{-1}\left(\overline{\partial_t \log(\det\bar{g})} + 2\bar{\nabla}_i\beta^i\right)$$

$$\mathfrak{K}_{\mathcal{T}}^{ij} = \frac{1}{2}\bar{\mathcal{N}}^{-1}\left\{\left(\mathcal{L}_{\bar{g},conf}\beta\right)^{ij} + (\det\bar{g})^{-1/\eta}\overline{\partial_t\left(\bar{g}^{ij}(\det\bar{g})^{-1/\eta}\right)}\right\}$$

$$\frac{1}{2}\bar{\mathcal{N}}^{-1}(\det\bar{g})^{-1/\eta}\overline{\partial_t\left(\bar{g}^{ij}(\det\bar{g})^{-1/\eta}\right)} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{U}^{ij}$$

$$\overline{\partial_t\left(\bar{g}^{ij}(\det\bar{g})^{-1/\eta}\right)} = 2\widetilde{\mathcal{N}}(\det\gamma)^{1/\eta}\mathfrak{U}^{ij}$$

$$\bar{g} = \exp(2\lambda)\gamma, \mathfrak{K}^{ij} = \exp(4\lambda)\widetilde{\mathfrak{K}}^{ij} + \frac{1}{2}\bar{g}^{ij}\tau$$

$$\bar{\nabla}^i\mathfrak{K}^{ij} \equiv \exp(4\lambda)\mathcal{D}_i\widetilde{\mathfrak{K}}^{ij} + \frac{1}{2}\bar{g}^{ij}\partial_i\tau$$

$$\overline{\nabla}^i \mathfrak{K}^{ij} - \bar{g}^{ij}\partial_i \tau \equiv \exp(4\lambda)\mathcal{D}_i \tilde{\mathfrak{K}}^{ij} - \frac{1}{2}\bar{\gamma}^{ij}\exp(-2\lambda)\partial_i \tau$$

Tensor de energía

$$\mathcal{T}_{\alpha\beta} \equiv \mathfrak{F}_{\alpha}^{\lambda}\mathfrak{F}_{\beta\gamma} - \frac{1}{4} g_{\alpha\beta}\mathfrak{F}^{\lambda\mu}\mathfrak{F}_{\lambda\mu}$$

$$\mathcal{J}^i = -\mathcal{N}^{-1}\mathcal{T}_0^i = -\mathcal{N}^{-1}\mathfrak{F}_0^j\mathfrak{F}_j^i$$

$$\overline{\mathfrak{E}}^i := \bar{\mathcal{N}}^{-1}\overline{\mathfrak{F}}_0^i \equiv \varphi^{\frac{2\eta}{\eta-2}}\tilde{\mathfrak{E}}^i, \tilde{\mathfrak{F}}_{ij} = \overline{\mathfrak{F}}_{ij}$$

$$\overline{\mathfrak{J}}^i = \varphi^{\frac{-2(\eta+2)}{\eta-2}}\tilde{\mathfrak{J}}^i, \tilde{\mathfrak{J}}^i \equiv \tilde{\mathfrak{E}}^i\gamma^{i\kappa}\tilde{\mathfrak{F}}_{\kappa j}$$

$$\bar{\rho} \equiv \bar{\mathcal{N}}^{-2}\bar{\mathcal{T}}_{00} = \frac{1}{2}\bar{\mathcal{N}}^{-2}\overline{\mathfrak{F}}_{0i}\overline{\mathfrak{F}}_0^i + \frac{1}{4}\overline{\mathfrak{F}}^{ij}\overline{\mathfrak{F}}_{ij}$$

$$\bar{\mathcal{N}}^{-1}\overline{\mathfrak{F}}_{0i} \equiv \bar{\mathcal{N}}^{-1}\bar{g}_{ij}\overline{\mathfrak{F}}_0^i = \varphi^{-2}\gamma_{ij}\tilde{\mathfrak{E}}^i$$

$$\bar{\mathcal{N}}^{-2}\overline{\mathfrak{F}}_{0i}\overline{\mathfrak{F}}_0^i = \varphi^{\frac{2(\eta+1)}{\eta-2}}\gamma_{ij}\tilde{\mathfrak{E}}^i\tilde{\mathfrak{E}}^j$$

$$\overline{\mathfrak{F}}^{ij}\overline{\mathfrak{F}}_{ij} = \varphi^{-\frac{8}{\eta-2}}\gamma^{i\hbar}\gamma^{j\kappa}\tilde{\mathfrak{F}}_{ij}\tilde{\mathfrak{F}}_{\hbar\kappa}$$

$$\overline{\nabla_\alpha \mathfrak{F}_0^\alpha} \equiv \overline{\partial_\alpha \mathfrak{F}_0^\alpha - \omega_{\alpha 0}^{\beta}\mathfrak{F}_\beta^\alpha + \omega_{\alpha\beta}^{\alpha}\mathfrak{F}_0^\beta}$$

$$\overline{\nabla_\alpha \mathfrak{F}_0^\alpha} \equiv \bar{\mathcal{N}}\overline{\nabla}\overline{\mathfrak{E}}^i$$

$$\overline{\nabla}_i\overline{\mathfrak{E}}^i \equiv \frac{1}{\sqrt{\det \bar{g}}}\partial_i\left(\sqrt{\det \bar{g}}\,\bar{g}^{ij}\partial_j\overline{\mathfrak{E}}^i\right) = \varphi^{\frac{-2\eta}{\eta-2}}\tilde{\nabla}_i\tilde{\mathfrak{E}}^i$$

$$\overline{\mathfrak{H}}^i = \frac{1}{2}\bar{\eta}^{ij}\overline{\mathfrak{F}}_{j\kappa} = \frac{1}{2}\varphi^{-6}\tilde{\eta}^{ij\kappa}\tilde{\mathfrak{F}}_{j\kappa} \equiv \varphi^{-6}\tilde{\mathfrak{H}}^i$$

$$\bar{\rho} = \frac{1}{2}\bar{g}_{ij}\left(\overline{\mathfrak{E}}^i\overline{\mathfrak{E}}^j + \overline{\mathfrak{H}}^i\overline{\mathfrak{H}}^j\right) \equiv \varphi^{-8}\tilde{\rho}$$

$$\tilde{\rho} \equiv \frac{1}{2}\gamma_{ij}\left(\tilde{\mathfrak{E}}^i\tilde{\mathfrak{E}}^j + \tilde{\mathfrak{H}}^i\tilde{\mathfrak{H}}^j\right)$$

$$\bar{\rho} = \frac{1}{2}\left(\bar{\pi}^2 + \bar{g}^{ij}\partial_i\bar{\psi}\partial_j\bar{\psi}\right) + \mathcal{V}(\bar{\psi}), \bar{\pi} := \bar{\mathcal{N}}^{-1}\overline{\partial_0\psi}$$

$$\bar{\pi} \equiv \bar{\mathcal{N}}^{-1}\overline{\partial_0\psi} = \varphi^{\frac{-2\eta}{(\eta-2)}}\tilde{\pi}, \tilde{\pi} := \tilde{\mathcal{N}}^{-1}\overline{\partial_0\psi}$$

$$\bar{\rho} = \frac{1}{2}\left(\varphi^{\frac{-4\eta}{\eta-2}}|\tilde{\pi}|^2 + \varphi^{\frac{-4}{\eta-2}}\gamma^{ij}\partial_i\bar{\psi}\partial_j\bar{\psi}\right) + \mathcal{V}(\bar{\psi})$$

$$\bar{\mathcal{J}}^i = -\bar{\mathcal{N}}^{-1}\bar{g}^{ij}\partial_j\bar{\psi}\overline{\partial_0\psi} = -\varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\gamma^{ij}\partial_j\bar{\psi}\tilde{\pi} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\tilde{\mathcal{J}}^i$$

Tensor de materia

$$\mathcal{T}_{\alpha\beta} \equiv (\mu + \wp)\mu_\alpha\mu_\beta + \wp g_{\alpha\beta}$$

$$\bar{\mathcal{N}}^2(\bar{\mu}^0)^2 - \bar{g}_{ij}\bar{\mu}^i\bar{\mu}^j$$

$$\bar{\rho} \coloneqq (\bar{\mathcal{N}}\bar{\mu}^0)^2(\bar{\mu} + \bar{\wp}) - \bar{\wp}, \bar{\mathcal{J}}^i \coloneqq \bar{\mathcal{N}}(\bar{\mu} + \bar{\wp})\bar{\mu}^0\bar{\mu}^i$$

$$\tilde{\mathcal{N}}^2(\tilde{\mu}^0)^2 - \gamma_{ij}\tilde{\mu}^i\tilde{\mu}^j$$

$$\tilde{\mu}^i = \varphi^{\frac{2}{\eta-2}}\bar{\mu}^i$$

$$\bar{\mathcal{N}}\bar{\mu}^0 = \tilde{\mathcal{N}}\tilde{\mu}^0 = \left(\gamma_{ij}\tilde{\mu}^i\tilde{\mu}^j + 1\right)^{\frac{1}{2}}$$

$$\bar{\mathcal{J}}^i \coloneqq \bar{\mathcal{N}}\bar{\mu}^0(\bar{\mu} + \bar{\wp})\bar{\mu}^i = \varphi^{-\frac{2}{\eta-2}}\tilde{\mathcal{N}}\tilde{\mu}^0\tilde{\mu}^i(\bar{\mu} + \bar{\wp})$$

$$\bar{\mu} = \varphi^{\frac{-2(\eta+1)}{\eta-2}}\tilde{\mu}, \bar{\wp} = \varphi^{\frac{-2(\eta+1)}{\eta-2}}\tilde{\wp}, \tilde{\mathcal{J}}^i \equiv \left(\gamma_{ij}\tilde{\mu}^i\tilde{\mu}^j + 1\right)^{\frac{1}{2}}\tilde{\mu}^i(\tilde{\mu} + \tilde{\wp})$$

$$\bar{\rho} = \varphi^{\frac{-2(\eta+1)}{\eta-2}}\left[(\tilde{\mu} + \tilde{\wp})\left(\tilde{\mathcal{N}}\tilde{\mu}^0\right)^2 - \tilde{\wp}\right] \equiv \varphi^{\frac{-2(\eta+1)}{\eta-2}}\tilde{\rho}$$

$$\bar{\mu}\bar{\mathcal{N}}(\det\bar{g})^{1/2} = \tilde{\mu}\tilde{\mathcal{N}}(\det\gamma)^{1/2}$$

$$\bar{\mathcal{J}}^i = \bar{\mathcal{N}}\bar{\mu}^0\bar{\rho}\varphi^{-\frac{2}{\eta-2}}\tilde{\mu}^i \coloneqq \varphi^{-\frac{2}{\eta-2}}\tilde{\mathcal{J}}^i$$

Restricciones CF

$$\gamma'_{ij} \equiv \theta^{\frac{4}{(\eta-2)}}\gamma_{ij}$$

$$\left|\mathcal{D}\bar{\psi}\right|_\gamma^2 \equiv \theta^{\frac{4}{(\eta-2)}}\left|\mathcal{D}\bar{\psi}\right|_{\gamma'}^2$$

$$\Delta_\gamma\varphi - \mathfrak{K}_\eta\mathcal{R}(\gamma)\varphi \equiv \theta^{\frac{(\eta+2)}{(\eta-2)}}\{\Delta_{\gamma'}\varphi'\} - \mathfrak{K}_\eta\mathcal{R}(\gamma')\varphi'$$

$$\Delta_\gamma\varphi - r\varphi \equiv \theta^{\frac{(\eta+2)}{(\eta-2)}}\{\Delta_{\gamma'}\varphi' - r'\varphi'\}$$

$$\mathcal{D}'_i\tilde{\mathfrak{K}}'^{ij} \equiv \theta^{-2\frac{(\eta+2)}{(\eta-2)}}\mathfrak{D}_i\tilde{\mathfrak{K}}^{ij}$$

$$q'_2 = \theta^{\frac{-8}{\eta-2}}q_2$$

$$\tilde{\pi}' \equiv \theta^{\frac{-2\eta}{(\eta-2)}}\pi$$

$$\alpha' \equiv \mathfrak{K}_\eta\left(\tilde{\mathfrak{K}}'\ '\tilde{\mathfrak{K}}' + (\tilde{\pi}')^2\right) \equiv \theta^{\frac{-4\eta}{(\eta-2)}}\alpha$$

$$\tilde{\rho}' = \theta^{\frac{-4}{(\eta-2)}}\tilde{\rho}$$

$$q_1' = \theta^{\frac{-4}{(\eta-2)}}q_1$$

$$\gamma'^{ij}\partial_i\tau = \theta^{\frac{-4}{(\eta-2)}}\gamma^{ij}\partial_i\tau$$

$$\mathfrak{H}'(\tilde{\mathfrak{K}}', \varphi') \equiv \theta^{-\frac{(\eta+2)}{(\eta-2)}}\mathfrak{H}(\tilde{\mathfrak{K}}, \varphi)$$

$$\mathcal{M}'^{i}(\tilde{\mathfrak{K}}', \varphi') \equiv \theta^{-2\frac{(\eta+2)}{(\eta-2)}}\mathcal{M}^{i}(\tilde{\mathfrak{K}}, \varphi)$$

Ecuación de Lichnerowicz

$$\mathcal{H} \equiv \Delta_\gamma\varphi - f(\varphi) \equiv r\varphi - \sum_{\mathbb{I}} \alpha_{\mathcal{J}}\varphi^{\mathcal{J}}$$

$$r \equiv \mathfrak{K}_\eta\left[\mathcal{R}(\gamma) - \left|\mathcal{D}\bar{\psi}\right|_\gamma^2\right], \mathfrak{K}_\eta = \frac{\eta-2}{4(\eta-1)}$$

$$\sum_{\mathbb{I}} \alpha_{\mathcal{J}}\varphi^{\mathcal{J}} \equiv \mathfrak{K}_\eta\left\{\left(\left|\tilde{\mathfrak{K}}\right|_\gamma^2 + |\tilde{\pi}|^2\right)\varphi^{-\frac{3\eta-2}{\eta-2}} - \left[\frac{\eta-2}{4\eta}\tau^2 - \mathcal{V}(\bar{\psi})\right]\varphi^{\frac{\eta+2}{\eta-2}}\right\}$$

$$\mathcal{J}_{\gamma,q}(\varphi) \equiv \frac{\iiint_{\mathcal{M}}\left(\mathfrak{K}_\eta^{-1}|\nabla\varphi|_\gamma^2 + \mathcal{R}(\gamma)\varphi^2\right)\mu_\gamma}{\left(\iiint_{\mathcal{M}}\varphi^{2q}\mu_\gamma\right)^{1/q}}$$

$$\iiint_{\mathcal{M}} \varphi^{\frac{2\eta}{(\eta-2)}}\mu_\gamma = \iiint_{\mathcal{M}} \varphi^{\frac{\prime 2\eta}{(\eta-2)}}\mu_{\gamma'}$$

$$\iiint_{\mathcal{M}} \left(\mathfrak{K}_\eta^{-1}|\nabla\varphi|_\gamma^2 + \mathcal{R}(\gamma)\varphi^2\right)\mu_\gamma = \iiint_{\mathcal{M}} \left(\mathfrak{K}_\eta^{-1}|\nabla\varphi|_{\gamma'}^2 + \mathcal{R}(\gamma')\varphi^2\right)\mu_{\gamma'}$$

$$\iiint_{\mathcal{M}} \left(\mathfrak{K}_\eta^{-1}|\nabla\varphi|_\gamma^2 + \mathcal{R}(\gamma)\varphi^2\right)\mu_\gamma \geq -\left|\iiint_{\mathcal{M}} \mathcal{R}(\gamma)\varphi^2\mu_\gamma\right| \geq -\|\mathcal{R}(\gamma)\|_{\mathcal{L}^{q/(q-1)}}\|\varphi^2\|_{\mathcal{L}^q}$$

$$\mu \equiv \operatorname*{Inf}_{\varphi\in\mathcal{W}_2^{\wp}, \varphi\not\equiv 0} \mathcal{J}_\gamma(\varphi)$$

Ecuación de Yamabe

$$\mathfrak{K}_\eta^{-1}\Delta_\gamma\varphi_{m,q} - \mathcal{R}(\gamma)\varphi_{m,q} + \mu_q\varphi_{m,q}^{2q-1}$$

$$r = \mathfrak{K}_\eta\left[\mathcal{R}(\gamma) - \left|\mathcal{D}\bar{\psi}\right|_\gamma^2\right]$$

$$\mathcal{J}_{\gamma,\bar{\psi}}(\varphi) \equiv \frac{\iiint_{\mathcal{M}} \left\{ \mathfrak{K}_\eta^{-1} |\nabla\varphi|_\gamma^2 + \left[\mathcal{R}(\gamma) - \left|\mathcal{D}\bar{\psi}\right|_\gamma^2 \right] \varphi^2 \right\} \mu_\gamma}{\left(\iiint_{\mathcal{M}} \varphi^{\frac{2\eta}{(\eta-2)}} \mu_\gamma \right)^{\frac{(\eta-2)}{\eta}}}$$

$$\left|\mathcal{D}\bar{\psi}\right|_\gamma^2 \equiv \gamma^{ij}\partial_i\bar{\psi}\partial_j\bar{\psi} = \theta^{\frac{4}{\eta-2}}\gamma'^{ij}\partial_i\bar{\psi}\partial_j\bar{\psi} \equiv \theta^{\frac{4}{\eta-2}}\left|\mathcal{D}\bar{\psi}\right|_{\gamma'}^2,$$

$$\Delta_{\varphi_2^{2q}\gamma}(\varphi_1\varphi_2^{-1}) - r(\varphi_2^{2q}\gamma)(\varphi_1\varphi_2^{-1}) \equiv -(\varphi_1\varphi_2^{-1})^{(\eta+2)/(\eta-2)} r(\varphi_1^{2q}\gamma)$$

$$r(\varphi_1^{2q}\gamma) \equiv -\varphi_1^{-\frac{(\eta+2)}{(\eta-2)}}\{\Delta_\gamma\varphi_1 - \varphi_1 r(\gamma)\} = \varphi_1^{-\frac{(\eta+2)}{(\eta-2)}}\alpha_{\mathcal{J}}\varphi_1^{\mathcal{P}_{\mathcal{J}}}$$

$$\Delta_{\varphi_2^{2q}\gamma}(\varphi_1\varphi_2^{-1} - 1) - \lambda\{(\varphi_1\varphi_2^{-1} - 1)\}$$

$$\lambda \equiv \alpha_{\mathcal{J}}\varphi_1^{\mathcal{P}_{\mathcal{J}}}\varphi_2^{-\frac{(\eta+2)}{(\eta-2)}}\mu, \mu := \frac{(\varphi_1\varphi_2^{-1})^{-\mathcal{P}_{\mathcal{J}}}}{\varphi_1\varphi_2^{-1} - 1}$$

$$\Delta_\gamma\varphi - r\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} + q\varphi^{\frac{-\eta}{\eta-2}} - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\gamma' = \theta^{\frac{4}{\eta-2}}\gamma$$

$$r(\gamma') = \theta^{-\frac{\eta+2}{\eta-2}}\left(-\Delta_\gamma\theta + r(\gamma)\theta\right)$$

$$\Delta_\gamma\theta - \mathscr{k}\theta = -(\alpha + q)$$

$$r(\gamma') = \theta^{-\frac{\eta+2}{\eta-2}}[-\mathscr{k}\theta + r(\gamma)\theta + \alpha + q]$$

$$\theta^{-\frac{\eta+2}{\eta-2}}\Delta_{\gamma'}\varphi' - [-\mathscr{k}\theta + r(\gamma)\theta + \alpha + q]\varphi' + \alpha'\varphi'^{\frac{-3\eta+2}{\eta-2}} + q'\varphi'^{\frac{-\eta}{\eta-2}} - \beta'\varphi'^{\frac{\eta+2}{\eta-2}}$$

$$[-\mathscr{k}\theta + r(\gamma)\theta - \alpha - q]\gamma + \alpha'\gamma'^{\frac{-3\eta+2}{\eta-2}} + q'\gamma'^{\frac{-\eta}{\eta-2}} - \beta'\gamma'^{\frac{\eta+2}{\eta-2}}$$

$$m \geq Max\left\{Sup\theta^{\frac{-3\eta+2}{\eta-2}}, Sup\theta^{\frac{\eta}{\eta-2}}, Sup\lambda\theta^{-1}\right\}$$

$$0 < \ell \leq Min\left\{Inf\theta^{\frac{-3\eta+2}{\eta-2}}, Inf\theta^{\frac{\eta}{\eta-2}}, Inf\lambda\theta^{-1}\right\}$$

$$\underline{f} \equiv \frac{1}{Vol\,(\mathcal{M},\gamma)}\iiint_{\mathcal{M}} f\mu_\gamma$$

$$f(\gamma) \equiv \gamma^{\frac{-3\eta+2}{\eta-2}}\hbar(\gamma)$$

$$\hbar(\gamma) \equiv \beta\gamma^{\frac{4\eta}{\eta-2}} + r\gamma^{\frac{4(\eta-1)}{\eta-2}} - \alpha$$

$$\hbar(\ell) \equiv \beta \ell^{\frac{4\eta}{\eta-2}} + r\ell^{\frac{4(\eta-1)}{\eta-2}} - \alpha \leq 0$$

$$\hbar(m) \equiv \beta m^{\frac{4\eta}{\eta-2}} + rm^{\frac{4(\eta-1)}{\eta-2}} - \alpha \geq 0$$

$$m_{+}^{\frac{2(\eta-1)}{(\eta-2)}} Max\left\{1, \frac{\underset{\mathcal{M}_{+}}{\text{Supa}}}{\underset{\mathcal{M}_{+}}{\text{Inf}}(\beta + r)}\right\}, 0 < \ell_{+} \leq Max\left\{1, \frac{\underset{\mathcal{M}_{+}}{\text{Infa}}}{\underset{\mathcal{M}_{+}}{\text{Sup}}(\beta + r)}\right\}$$

$$[\gamma_{Max}(\chi)]^{\frac{4}{\eta-2}} = \frac{(\eta-1)r(\chi)}{\eta|\beta(\chi)|}$$

$$\hbar(\gamma_{Max}) \equiv \frac{1}{\eta}\left[\frac{(\eta-1)}{\eta}\right]^{\eta-1} \frac{r^{\eta}}{|\beta|^{\eta-1}} - \alpha$$

$$\eta((\eta-1)^2|\beta|^{\eta-1}\alpha \leq r^{\eta})$$

$$\left[\underset{\mathcal{M}_{-}}{\text{Supz}_1}(\chi)\right] \leq \underset{\mathcal{M}_{-}}{\text{Infz}_2}(\chi)$$

Sistema de restricciones

$$\bar{g}_{ij} = \varphi^{\frac{4\eta}{\eta-2}}\gamma_{ij}, \mathfrak{K}^{ij} = \varphi^{-2\frac{(\eta+2)}{(\eta-2)}}\left[\left(\mathcal{L}_{\gamma,conf}\chi\right)^{ij} + \mathfrak{U}^{ij}\right] + \frac{1}{\eta}\bar{g}^{ij}\tau$$

$$\phi: (\chi, \gamma) \mapsto \phi(\chi, \gamma) \equiv \big(\mathcal{H}(\chi, \gamma), \mathcal{M}(\chi, \gamma)\big), \chi \equiv (\gamma, \tau, \mathcal{U}, \bar{\psi}, \tilde{\pi}, q_1, q_2, q_0, \mathcal{J}_1, \mathcal{J}_0), \gamma \equiv (\varphi, \chi)$$

$$\mathcal{H}(\chi, \gamma) \equiv \Delta_{\gamma}\varphi - r(\gamma, \bar{\psi})\varphi + \alpha(\mathcal{U}, \tilde{\pi}, \chi)\varphi^{\frac{-3\eta+2}{\eta-2}} + q_1\varphi^{-\frac{\eta}{\eta-2}} + q_2\varphi^{-\frac{6-\eta}{\eta-2}} - \left(q_0 - \frac{\eta-2}{4\eta}\tau^2\varphi^{\frac{\eta+1}{\eta-2}}\right)$$

$$-\left\{\frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{(\eta-2)}}\partial^i\tau + \mathcal{J}_1^i + \mathcal{J}_0^i\varphi^{2\frac{(\eta+2)}{(\eta-2)}} - \mathcal{D}_j\mathfrak{B}^{ij}\right\}$$

$$\Omega_1 \equiv \mathfrak{B}_1 \cap \{\gamma \in \mathcal{M}_2^{\wp}\}, \Omega_2 \equiv \mathfrak{B}_2 \cap \{\varphi > 0\}$$

$$\mathfrak{B}_1 \equiv \left(\mathcal{W}_2^{\wp} \bigotimes \mathcal{W}_1^{\wp} \bigotimes ({}^2\otimes\mathcal{W}_1^{\wp}) \bigotimes \mathcal{L}^{\wp} \bigotimes \mathcal{L}^{\wp} \otimes {}^1 \bigotimes \mathcal{L}^{\wp} \otimes {}^1 \bigotimes \mathcal{L}^{\wp}\right)$$

$$\mathfrak{B}_2 \equiv \mathcal{W}_2^{\wp} \bigotimes ({}^1\otimes\mathcal{W}_2^{\wp})$$

$$\delta\phi_{\chi,\gamma}: (\delta\chi, \delta\gamma) \mapsto \delta\phi_{\chi,\gamma}(\delta\chi, \delta\gamma) \equiv \left(\delta\mathcal{H}_{\chi,\gamma}(\delta\chi, \delta\gamma), \delta\mathcal{M}_{\chi,\gamma}(\delta\chi, \delta\gamma)\right)$$

$$\delta\phi_{\gamma}: \delta\gamma \mapsto \delta\phi_{\gamma}(\delta\gamma) \equiv \left(\delta\mathcal{H}_{\gamma}(\delta\gamma), \delta\mathcal{M}_{\gamma}(\delta\gamma)\right)$$

$$\delta\mathcal{H}_{\gamma}(\delta\gamma) \equiv \Delta_{\gamma}\delta\varphi$$

$$-\left\{r + \frac{3\eta-2}{\eta-2}\alpha\varphi^{-\frac{2\eta}{\eta-2}} + \frac{\eta}{\eta-2}q_1\varphi^{-\frac{2(\eta-1)}{\eta-2}} + \frac{6-\eta}{\eta-2}q_2\varphi^{-\frac{4}{\eta-2}} + \frac{\eta+2}{\eta-2}(\bar{\beta})\varphi^{\frac{4}{\eta-2}}\right\}\delta\varphi$$

$$+\alpha'_{\chi}\varphi^{\frac{-3\eta+2}{\eta-2}}\delta\chi$$

$$\delta\mathcal{M}_{\chi,\gamma}(\delta\chi) \equiv \left(\Delta_{\gamma,conf}\delta\chi\right)^{i} - \left(\frac{2(\eta-1)}{\eta-2}\varphi^{\frac{\eta+2}{\eta-2}}\partial^{i}\tau + \frac{2(\eta+2)}{\eta-2}\varphi^{\frac{\eta+6}{\eta-2}}\mathcal{J}_0^{i}\right)\delta\varphi$$

$$\left(\Delta_{\bar{g},conf}\delta\chi\right)^{i} = \gamma$$

$$\Delta_{\bar{g}}\delta\varphi - \left\{\bar{r} + \frac{3\eta-2}{\eta-2}\bar{\alpha} + \frac{\eta}{\eta-2}\bar{q}_1 + \frac{6-\eta}{\eta-2}\bar{q}_2 + \frac{\eta+2}{\eta-2}\bar{\beta}\right\}\delta\varphi + \bar{\alpha}'_{\chi}\delta\chi = \psi$$

$$\Delta_{\bar{g}}\delta\varphi - \left\{2\bar{\alpha} + \frac{2(\eta-1)}{\eta-2}\bar{q}_1 + \frac{4}{\eta-2}\bar{q}_2 + \frac{4}{\eta-2}\bar{\beta}\right\}\delta\varphi + \bar{\alpha}'_{\chi}\delta\chi = \psi$$

Soluciones euclídeas

$$\|\mu\|_{\mathcal{W}^{\wp}_{s,\delta}} = \left\{\sum_{0\le m\le s}\iiint_{\mathcal{V}}|\partial^{m}\mu|^{\wp}(1+d^{2})^{\frac{1}{2}\wp(\delta+m)}d\mu\right\}^{1/\wp}$$

$$\|\mu\|_{\mathfrak{C}^{m}_{\beta}} \equiv \sum_{0\le m\le s}\sup_{\mathcal{M}}\left(\left|\partial^{\ell}\mu\right|(1+d^{2})^{\frac{1}{2}(\beta+\ell)}\right)$$

$$\mathcal{D}_j(\mathcal{L}\chi)^{ij} \equiv \left(\Delta_{\gamma,conf}\chi\right)^{i} = \mathfrak{F}^{i}(\varphi)$$

$$\mathfrak{F}^{i}(\varphi) \equiv \mathcal{D}_j\mathcal{U}^{ij} + \frac{\eta-1}{\eta}\varphi^{\frac{2\eta}{\eta-2}}\gamma^{ij}\partial_j\tau + \varphi^{2\frac{(\eta+2)}{(\eta-2)}}\mathcal{J}_0^{i} + \mathcal{J}_1^{i}$$

Solución de Lichnerowicz

$$\mathcal{H}(\chi,\varphi) \equiv \Delta_{\gamma}\varphi - f(\chi,\varphi)$$

$$f(\chi,\varphi) \equiv r\varphi - \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} - q_1\varphi^{-\frac{\eta}{\eta-2}} - q_2\varphi^{-\frac{6-\eta}{\eta-2}} + \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\iiint_{\mathcal{M}}\{|\mathfrak{D}f|^{2} + r(\gamma,\bar{\psi})f^{2}\}\mu_{\gamma}$$

$$\Delta_{\gamma}\varphi - r(\gamma,\bar{\psi})\varphi$$

$$\Delta_{\gamma}\mu - r(\gamma,\bar{\psi})\mu = r(\gamma,\bar{\psi})$$

$$\Delta_{\gamma}\varphi - \mathfrak{K}r(\gamma,\bar{\psi})\varphi$$

$$\Delta_{\gamma}\mu - \mathfrak{K}r(\gamma,\bar{\psi})\mu = \mathfrak{K}r(\gamma,\bar{\psi})$$

$$\left\|\varphi_{\mathscr{k}'}\right\|_{\mathcal{L}^q(\mathfrak{B}_{2\mathfrak{R}})} \leq \mathfrak{C} Inf_{\mathfrak{B}_{\mathfrak{R}}} \varphi_{\mathscr{k}'}$$

$$Inf_{f\in\mathfrak{D},f\not\equiv 0}\left(\iiint\limits_{\mathcal{M}}\{|\mathfrak{D}f|^2+\mathfrak{K}_\eta\mathcal{R}(\gamma)f^2\}\mu_\gamma\right)\setminus\|f\|^2_{\mathcal{L}^{2\eta/(\eta-2)}}$$

$$|\mathfrak{D}f|^2 = |\mathfrak{D}\theta|^2\varphi^2 + \varphi\mathfrak{D}\varphi\otimes\mathfrak{D}(\theta^2) + \theta^2|\mathfrak{D}\varphi|^2$$

$$\iiint\limits_{\mathcal{M}} \varphi\mathfrak{D}\varphi\otimes\mathfrak{D}(\theta^2)\mu_\gamma = \iiint\limits_{\mathcal{M}} -\theta^2\mathfrak{D}(\varphi\mathfrak{D}\varphi)\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}} \varphi\mathfrak{D}\varphi\otimes\mathfrak{D}(\theta^2)\mu_\gamma = \iiint\limits_{\mathcal{M}} -\theta^2\mathfrak{D}\left(\varphi\Delta_\gamma\varphi + |\mathfrak{D}\varphi|^2\right)\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}} |\mathfrak{D}\varphi|^2\mu_\gamma = \iiint\limits_{\mathcal{M}}\left\{|\mathfrak{D}\theta|^2\varphi^2 - \theta^2\varphi\Delta_\gamma\varphi\right\}\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}}\{|\mathfrak{D}f|^2+\alpha_0 f^2\}d\mu_\gamma$$

$$\iiint\limits_{\mathcal{M}}\{|\mathfrak{D}f|^2_\gamma + r(\gamma,\bar{\psi})f^2\}\mu_\gamma$$

$$\Delta_\gamma\varphi - r\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} + q\varphi^{-\frac{\eta}{\eta-2}} - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\Delta_\gamma\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} + q\varphi^{-\frac{\eta}{\eta-2}} - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\Delta_\gamma\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} + q\varphi^{-\frac{\eta}{\eta-2}}$$

$$\Delta_\gamma\mu_+ = -(\alpha+q)$$

$$\Delta_\gamma\varphi_+ = -(\alpha+q) \leq -\left(\alpha{\varphi_+}^{\frac{-3\eta+2}{\eta-2}} + q{\varphi_+}^{-\frac{\eta}{\eta-2}}\right)$$

$$\Delta_\gamma\varphi - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\Delta_\gamma\varphi - \mathscr{k}\beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$f(\gamma) \equiv d\gamma^{-\frac{\eta}{\eta-2}} - r\gamma^{-\frac{\eta}{\eta-2}} + \alpha$$

$$\alpha d^{\eta-1} \leq \left[\frac{(\eta-1)^{\eta-1}}{\eta^\eta}\right] r^\eta$$

$$\Delta_\gamma \varphi - r\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\operatorname*{Inf}_{\chi\in\mathcal{M}} \gamma_1(\chi) > 0,\ \operatorname*{Inf}_{\chi\in\mathcal{M}} \gamma_2(\chi) \geq \max\left\{1, \sup_{\chi\in\mathcal{M}} \gamma_1(\chi)\right\}$$

$$f_\chi(\gamma) \equiv -\beta(\chi)\gamma^{\frac{\eta}{\eta-2}} - r(\chi)\gamma^{\frac{\eta-1}{\eta-2}} + \alpha(\chi)\langle f_\chi(\ell^4)|f_\chi(m^4)\rangle$$

$$\ell = \min\left\{1, \inf_{\chi\in\mathcal{M}} \gamma_1(\chi)\right\}, m = \max\left\{1, \inf_{\chi\in\mathcal{M}} \gamma_2(\chi)\right\}$$

$$\Delta_\gamma \varphi - r\varphi + \alpha\varphi^{\frac{-3\eta+2}{\eta-2}} + q\varphi^{-\frac{\eta}{\eta-2}} - \beta\varphi^{\frac{\eta+2}{\eta-2}}$$

$$\Delta_\gamma \mu_\eta = -r\varphi_{\eta-1} + \alpha\varphi_{\eta-1}^{\frac{-3\eta+2}{\eta-2}} + q\varphi_{\eta-1}^{-\frac{\eta}{\eta-2}} - \beta\varphi_{\eta-1}^{\frac{\eta+2}{\eta-2}}$$

$$\|\mu_\eta\|_{\mathcal{W}^{\wp}_{2,\tilde{\delta}}} \leq \mathfrak{C}_{\mathfrak{E}}\left\{\mathcal{A} + \mathcal{R}(1+\mathcal{M}) + \mathcal{B}(1+\mathcal{M})^{\frac{\eta+2}{\eta-2}}\right\}$$

Estándar de Schauder

$$\|\mathcal{N}\|_{\mathcal{H}^2(\Omega)} \leq \mathfrak{C}\left(\left\|\mathfrak{D}\mathfrak{R}^\dagger\mathcal{N}\right\|_{\mathcal{L}^2(\Omega)} + \|\mathcal{N}\|_{\mathcal{L}^2(\Omega)}\right)$$

$$\|\mathcal{N}\|^2_{\mathcal{H}^2_\rho(\Omega)} \coloneqq \iiint_\Omega \rho(\mathcal{N}^2 + |\nabla^2\mathcal{N}|^2)dv_g \leq \mathfrak{C}\iiint_\Omega \rho\left(\left\|\mathfrak{D}\mathfrak{R}^\dagger\mathcal{N}\right\|^2 + \mathcal{N}^2\right)dv_g$$

Restricción de Schwarzschild

$$g_{\text{Schwarzschild}} = \left(1 + \frac{m}{2|x-c|}\right)^4 \delta_{ij}$$

Singularidades en espacios cuánticos relativistas

Cuestiones preliminares

$$\ell(\mathfrak{C}) \equiv \int_{t_1}^{+\infty}\left(\mathcal{N}^2 - g_{ij}\left(\frac{d\mathfrak{C}^i}{dt} + \beta^i\right)\left(\frac{d\mathfrak{C}^j}{dt} + \beta^j\right)\right)^{\frac{1}{2}} dt$$

$$\mathcal{N}^2 - g_{ij}\left(\frac{d\mathfrak{C}^i}{dt} + \beta^i\right)\left(\frac{d\mathfrak{C}^j}{dt} + \beta^j\right) \geq \kappa^2$$

$$\mu^\alpha\nabla_\alpha\mu^\lambda \equiv \mu^\alpha\partial_\alpha\mu^\lambda + \omega^\lambda_{\alpha\gamma}\mu^\alpha\mu^\gamma$$

$$\mu^\alpha\partial_\alpha\mu^\lambda \equiv \frac{dt}{ds}\left[\left(\frac{\partial}{\partial t} - \beta^i\frac{\partial}{\partial\chi^i}\right)\mu^\lambda\right] + \left(\frac{d\chi^i}{ds} + \beta^i\frac{dt}{ds}\right)\frac{\partial}{\partial\chi^i}\mu^\lambda$$

$$\equiv \frac{d\mu^\lambda}{ds}\frac{d}{dt}\left(\frac{dt}{ds}\right) + \frac{dt}{ds}\left(\omega^0_{00} + 2\omega^0_{0i}v^i + \omega^0_{ij}v^iv^j\right)$$

$$\log\frac{\gamma(t)}{\gamma(t_1)} = -\int_{t_1}^{t}\left(\omega_{00}^{0} + 2\omega_{0i}^{0}v^i + \omega_{ij}^{0}v^i v^j\right)dt$$

$$\log\frac{\gamma(t)}{\gamma(t_1)} = \int_{t_1}^{t}\mathcal{N}^{-1}\left(-\partial_0\mathcal{N} - 2\partial_i\mathcal{N}v^i + \kappa_{ij}v^i v^j\right)dt$$

$$\log\frac{\gamma(t)}{\gamma(t_1)} \leq 2\log\mathcal{N}_m^{-1} + \mathcal{N}_m^{-1}\int_{t_1}^{t}\left(|\nabla\mathcal{N}|_{gt}\mathcal{N}_{\mathcal{M}} + |\kappa|_{gt}\mathcal{N}_{\mathcal{M}}^2\right)dt$$

$$\kappa_{ij}v^i v^j \equiv \mathcal{P}_{ij}v^i v^j + \frac{1}{\eta}\tau g_{ij}v^i v^j \leq \mathcal{P}_{ij}v^i v^j$$

$$\log\frac{z(t)}{z(t_0)} = \int_{t_0}^{t}\mathcal{N}^{-1}\left(\partial_0\mathcal{N} + 2\partial_i\mathcal{N}v^i - \kappa_{ij}v^i v^j\right)dt$$

$$z(t) = z(t_1)\exp\int_{t_0}^{t}\mathcal{N}^{-1}\left(\partial_0\mathcal{N} + 2\partial_i\mathcal{N}v^i - \kappa_{ij}v^i v^j\right)dt$$

$$z(t) = z(t_0)\frac{\mathcal{N}(t)}{\mathcal{N}(t_0)}\exp\int_{t_0}^{t}\left(\mathcal{N}^{-1}\partial_i\mathcal{N}v^i - \kappa_{ij}v^i v^j\right)dt$$

$$\left|\mathcal{N}^{-1}\partial_i\mathcal{N}v^i\right| \leq |\nabla\mathcal{N}|_{\bar{g}}|\mathcal{N}^{-1}v|_{\bar{g}} \leq |\nabla\mathcal{N}|_{\bar{g}}\int_{t_0}^{t}|\nabla\mathcal{N}|_{\bar{g}}\,dt = c$$

$$z(t) \leq z(t_0)\frac{\mathcal{N}_{\mathcal{M}}}{\mathcal{N}(t_0)}(\exp c)\exp[-\kappa(t-t_0)]$$

$$\left|\mathcal{P}_{ij}v^i v^j\right| \leq |\mathcal{P}|_{\bar{g}}|v|_{\bar{g}}^2 \leq \mathcal{N}^2|\mathcal{P}|_{\bar{g}}$$

Singularidad de Hawking–Penrose.

$$\frac{d\chi^\alpha}{d\lambda} = \mu^\alpha(\chi)$$

$$\Omega_{\alpha\beta} \coloneqq \nabla_\alpha\mu_\beta - \nabla_\beta\mu_\alpha$$

$$\mu^\lambda\nabla_\lambda\nabla_\alpha\mu_\beta \equiv \mu^\lambda\nabla_\alpha\nabla_\lambda\mu_\beta + \mu^\lambda\mathcal{R}_{\lambda\alpha\beta\gamma}\mu^\gamma$$

$$\mu^\lambda\nabla_\lambda\nabla_\alpha\mu_\beta = -\nabla_\alpha\mu^\lambda\nabla_\lambda\mu_\beta + \mu^\lambda\mathcal{R}_{\lambda\alpha\beta\gamma}\mu^\gamma$$

$$\mu^\lambda\nabla_\lambda\Omega_{\alpha\beta} = \mu^\lambda\nabla_\lambda\{\nabla_\alpha\mu_\beta - \nabla_\beta\mu_\alpha\}$$

$$\mu^\lambda\nabla_\lambda\Omega_{\alpha\beta} = -\nabla_\alpha\mu^\lambda\nabla_\lambda\mu_\beta + \nabla_\beta\mu^\lambda\nabla_\lambda\mu_\alpha = (\nabla_\lambda\mu_\alpha - \nabla_\alpha\mu_\lambda)\nabla_\beta\mu^\lambda + \nabla_\alpha\mu^\lambda\left(\nabla_\beta\mu_\lambda - \nabla_\lambda\mu_\beta\right)$$

$$\mu^{\lambda}\nabla_{\lambda}\Omega_{\alpha\beta} = \nabla_{\alpha}\mu^{\lambda}\Omega_{\beta\lambda} + \nabla_{\beta}\mu^{\lambda}\Omega_{\lambda\alpha}$$

$$\mu^{\lambda}\nabla_{\lambda}\hbar^{i} = \mu^{\lambda}\,\partial_{\lambda}\hbar^{i} = \frac{d\hbar^{i}}{ds}\left\{\frac{d}{ds}\hbar^{i}_{(j)}\right\}\delta^{i}_{j}$$

$$\frac{\mathfrak{D}\hbar^{i}_{(j)}}{\mathfrak{D}s} \equiv \mu^{\lambda}\nabla_{\lambda}\hbar^{i}_{(j)} = \hbar^{\mathscr{k}}_{(j)}\nabla_{\mathscr{k}}\mu^{i}$$

$$\frac{1}{\det\mathcal{A}}\frac{d(\det\mathcal{A})}{ds} = \frac{d(\log\det\mathcal{A})}{ds} = trace\mathcal{B}$$

$$\theta \coloneqq \nabla_{\alpha}\mu^{\alpha} = \omega^{\alpha}_{\alpha 0} = \omega^{i}_{i0} = \nabla_{i}\mu^{i} \coloneqq tr\mathfrak{B}$$

$$\hbar^{i} = \mathcal{A}^{i}_{j}\omega^{j}$$

$$\ell''(\hbar,\hbar) = -\int_{\delta_{\alpha}}^{\delta_{\beta}}\sum_{i=1}^{\eta}\left(\mathrm{A}_{ij}\frac{d\omega^{j}}{ds}\right)^{2} ds$$

Tensores espaciales

$$\nabla_{\alpha}\mu_{\beta} \equiv \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{\eta}\theta\pi_{\alpha\beta} - (\mu^{\lambda}\nabla_{\lambda}\mu^{\alpha})\mu_{\beta}$$

$$\omega_{\alpha\beta} \coloneqq \frac{1}{2}\pi^{\rho}_{\alpha}\pi^{\sigma}_{\beta}\{\nabla_{\rho}\mu_{\sigma} - \nabla_{\rho}\mu_{\sigma}\}$$

$$\sigma_{\alpha\beta} \coloneqq \frac{1}{2}\pi^{\rho}_{\alpha}\pi^{\sigma}_{\beta}\{\nabla_{\rho}\mu_{\sigma} - \nabla_{\rho}\mu_{\sigma}\} - \frac{1}{\eta}\theta\pi_{\alpha\beta}$$

$$\omega_{\alpha\beta} \coloneqq \frac{1}{2}\{\nabla_{\alpha}\mu_{\beta} - \nabla_{\beta}\mu_{\alpha} + (\mu^{\lambda}\nabla_{\lambda}\mu_{\alpha})\mu_{\beta} - (\mu^{\lambda}\nabla_{\lambda}\mu_{\beta})\mu_{\alpha}\}$$

$$\sigma_{\alpha\beta} \coloneqq \frac{1}{2}\{\nabla_{\alpha}\mu_{\beta} - \nabla_{\beta}\mu_{\alpha} + (\mu^{\lambda}\nabla_{\lambda}\mu^{\alpha})\mu_{\beta} + (\mu^{\lambda}\nabla_{\lambda}\mu^{\beta})\mu_{\alpha}\} - \frac{1}{\eta}\theta\pi_{\alpha\beta}$$

$$\omega_{\alpha\beta} = \frac{1}{2}\{\nabla_{\alpha}\mu_{\beta} - \nabla_{\beta}\mu_{\alpha}\}$$

$$\sigma_{\alpha\beta} = \frac{1}{2}\{\nabla_{\alpha}\mu_{\beta} - \nabla_{\beta}\mu_{\alpha}\} - \frac{1}{\eta}\theta\pi_{\alpha\beta}$$

Derivación geodésica

$$\langle\mu^{\alpha}\nabla_{\alpha}\hbar^{\beta} - \hbar^{\alpha}\nabla_{\alpha}\mu^{\beta}\rangle$$

$$\langle\mu^{\lambda}\mu^{\alpha}\nabla_{\lambda}\nabla_{\alpha}\hbar^{\beta} - \mu^{\lambda}\nabla_{\lambda}\hbar^{\alpha}\nabla_{\alpha}\mu^{\beta} - \hbar^{\alpha}\mu^{\lambda}\nabla_{\lambda}\nabla_{\alpha}\mu^{\beta}\rangle$$

$$\mu^{\lambda}\mu^{\alpha}\nabla_{\alpha}\hbar^{\beta} - \mu^{\lambda}\nabla_{\lambda}\hbar^{\alpha}\nabla_{\alpha}\mu^{\beta} - \hbar^{\alpha}\mu^{\lambda}\left(\nabla_{\alpha}\nabla_{\lambda}\mu^{\beta} + \mathcal{R}^{\beta\mu}_{\lambda\alpha}\mu^{\mu}\right)$$

$$\hbar^{\alpha}\mu^{\lambda}\nabla_{\alpha}\nabla_{\lambda}\mu^{\beta} = -\hbar^{\alpha}\nabla_{\alpha}\mu^{\lambda}\nabla_{\lambda}\mu^{\beta} = -\mu^{\alpha}\nabla_{\alpha}\hbar^{\lambda}\nabla_{\lambda}\mu^{\beta}$$

$$\mu^{\lambda}\mu^{\alpha}\nabla_{\lambda}\nabla_{\alpha}\hbar^{\beta} = \hbar^{\alpha}\mu^{\lambda}\mu^{\mu}\mathcal{R}^{\beta\mu}_{\lambda\alpha}$$

$$\nabla^2_{\mu^2}\hbar^{\beta} := \frac{\mathfrak{D}^2}{\mathfrak{D}s^2}\hbar^{\beta} = \hbar^{\alpha}\mu^{\lambda}\mu^{\mu}\mathcal{R}^{\beta\mu}_{\lambda\alpha}$$

Ecuación de Raychauduri

$$\frac{d\theta}{ds} := \mu^{\lambda}\nabla_{\lambda}\theta = -\nabla_{\alpha}\mu^{\lambda}\nabla_{\lambda}\mu^{\alpha} - \mathcal{R}_{\lambda\gamma}\mu^{\lambda}\mu^{\gamma}$$

$$\frac{d\theta}{ds} = -\frac{\theta^2}{\eta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - \mathcal{R}_{\lambda\gamma}\mu^{\lambda}\mu^{\gamma}$$

$$\frac{d\theta}{ds} \leq -\frac{1}{\eta}\theta^2, \frac{d\tilde{\theta}}{ds} \geq \frac{1}{\eta}\tilde{\theta}^2$$

$$\delta(\tilde{\theta}) \leq \delta(\tilde{\theta}_0) + \int_{\tilde{\theta}_0}^{\tilde{\theta}} \frac{\eta}{\tilde{\theta}^2} d\tilde{\theta} \leq \delta_{\infty} = \delta_0 + \frac{\eta}{\tilde{\theta}_0}$$

Espacio – tiempo cuántico Eddington–Finkelstein

$$(\nabla_{\mu}\upsilon)^{\ddagger} = \eta_{\alpha}(\nabla_{\mu}\upsilon)^{\alpha}$$

$$(\nabla_{\mu}\upsilon)^{j} = \mu^{i}\nabla_{i}\upsilon^{j} = \mu^{i}(\partial_{i}\upsilon^{j} + \Gamma^{j}_{i\kappa}\upsilon^{\kappa}) \equiv \mu^{i}\overline{\nabla}_{i}\upsilon^{j}$$

$$\mathfrak{K}(\mu,\upsilon) := \eta_{\alpha}(\nabla_{\mu}\upsilon)^{\alpha} = -\mathcal{N}(\nabla_{\mu}\upsilon)^{0} = -\mathcal{N}\mu^{i}\nabla_{i}\upsilon^{0} = -\mathcal{N}\mu^{i}\omega^{0}_{i\mathfrak{k}}\upsilon^{\kappa}$$

$$\mathfrak{K}^{+}(\mu,\upsilon) := g(\ell^{+},\nabla_{\mu}\upsilon) \equiv \ell^{+}_{\alpha}(\nabla_{\mu}\upsilon)^{\alpha}, \mathfrak{K}^{-}(\mu,\upsilon) := \ell^{-}_{\alpha}(\nabla_{\mu}\upsilon)^{\alpha}$$

$$\mathfrak{K}^{*}_{\alpha\beta} = g(\nabla_{e_{\alpha}}e_{\beta},\ell^{+}) = g(e_{\beta},\nabla_{e_{\alpha}}\ell^{+}) \equiv \nabla_{\alpha}\ell^{+}_{\beta}$$

$$\chi^{+} = \hbar^{\alpha\beta}\nabla_{\alpha}\ell^{+}_{\beta}, \chi^{-} = \hbar^{\alpha\beta}\nabla_{\alpha}\ell^{-}_{\beta}$$

$$\hbar := g + \eta\otimes\eta - \upsilon\otimes\nu \equiv g + \frac{1}{2}(\ell^{+}\otimes\ell^{-} + \ell^{-}\otimes\ell^{+})$$

$$\hbar^{\alpha\beta} := g^{\alpha\beta} + \eta^{\alpha}\eta^{\beta} - \nu^{\alpha}\nu^{\beta} = g^{\alpha\beta} - \frac{1}{2}\left(\ell^{\alpha}_{+}\ell^{\beta}_{-} + \ell^{\alpha}_{-}\ell^{\beta}_{+}\right)$$

$$\chi^{+} = g^{\alpha\beta}\nabla_{\alpha}\ell^{+}_{\beta} \equiv \nabla_{\alpha}\ell^{\alpha}_{+}$$

$$\chi^{+} = -tr_{\bar{g}}\mathcal{K} + \mathcal{K}(\nu,\nu) + tr_{\hbar}\mathfrak{K}$$

$$g^{\alpha\beta}\nabla_{\alpha}\ell^{+}_{\beta} = (g^{\alpha\beta} + \eta^{\alpha}\eta^{\beta} - \nu^{\alpha}\nu^{\beta})\nabla_{\alpha}(\eta_{\beta} + \nu_{\beta})$$

$$(g^{\alpha\beta} + \eta^{\alpha}\eta^{\beta})\nabla_{\alpha}\eta_{\beta} = -tr_{\bar{g}}\mathcal{K}$$

$$\nu^{\alpha}\nu^{\beta}\nabla_{\alpha}\eta_{\beta} = -\nu^{i}\nu^{j}\mathfrak{K}_{ij}$$

$$\left(g^{\alpha\beta}+\eta^{\alpha}\eta^{\beta}-\nu^{\alpha}\nu^{\beta}\right)\nabla_{\alpha}\nu_{\beta}=\hbar^{\alpha\beta}\overline{\nabla}_{\alpha}\nu_{\beta}=tr_{\hbar}\mathfrak{K}$$

$$\chi^{+}(\Sigma)\coloneqq -tr_{\bar{g}}\mathcal{K}+\mathcal{K}(\nu,\nu)+tr_{\hbar}\mathfrak{K}$$

Espacio – tiempo cuántico de Schwarzschild (métrica)

$$ds^{2}=-(1-2mr^{-1})dt^{2}+(1-2mr^{-1})^{-1}dr^{2}+r^{2}(\sin^{2}\theta\, d\phi^{2}+d\theta^{2})$$

Singularidad en agujeros negros cuánticos.

$$g_{ij}=\mu^{4}(\chi)\delta_{ij}$$

$$\mu(\chi)=\alpha+\frac{\beta}{|\chi|}+\mathcal{O}\left(\frac{1}{|\chi|^{2}}\right)$$

$$m\geq\sqrt{\frac{\mathrm{A}_{0}}{16\pi}}$$

Espacio – tiempo cuántico simétrico

$$g=-g\Sigma+r^{2}(d\theta^{2}+\sin^{2}\theta\, d\varphi^{2})$$

$$g_{\Sigma}=-e^{2\nu}d\mu^{2}-2e^{\nu+\lambda}d\mu dr$$

$$g_{\Sigma}=-\left(e^{\nu}d\mu+e^{\lambda}dr\right)^{2}+e^{2\lambda}dr^{2}$$

$$\delta_{\alpha\beta}\coloneqq\mathcal{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\mathcal{R}=8\pi\mathcal{T}_{\alpha\beta}\equiv 8\pi\partial_{\alpha}\phi\partial_{\beta}\phi$$

$$\ell^{+}\coloneqq e^{-\lambda}\frac{\partial}{\partial r},\ell^{-}\coloneqq e^{-\nu}\frac{\partial}{\partial\mu}-\frac{1}{2}e^{-\lambda}\frac{\partial}{\partial r}$$

$$\theta^{+}=\frac{1}{2}e^{\nu}d\mu+e^{\lambda}dr,\theta^{-}=e^{\nu}d\mu,\theta^{3}=d\theta,\theta^{4}=\sin\theta\, d\phi$$

$$g=-2\theta^{+}\theta^{-}+r^{2}(d\theta^{2}+\sin^{2}\theta\, d\varphi^{2})$$

$$\delta_{--}\equiv\mathcal{R}_{--}\equiv\frac{1}{r}\left\{-2e^{-(\nu+\lambda)}\frac{\partial\lambda}{\partial r}+\frac{1}{2}e^{-2\lambda}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\nu}{\partial r}\right)\right\}$$

$$\delta_{++}\equiv\mathcal{R}_{++}\equiv 2\frac{e^{-2\lambda}}{r}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\nu}{\partial r}\right)$$

$$\delta_{+-}\equiv\mathcal{R}_{33}\equiv\mathcal{R}_{44}\equiv\frac{e^{-2\lambda}}{r}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\nu}{\partial r}\right)+\frac{1}{r^{2}(1-e^{-2\lambda})}$$

$$\delta_{33}\equiv\delta_{44}\equiv\mathcal{R}_{+-}\equiv -e^{-(\nu+\lambda)}\frac{\partial^{2}(\nu+\lambda)}{\partial\mu\partial r}+e^{-2\lambda}\left\{\frac{\partial^{2}\nu}{\partial r^{2}}+\left(\frac{\partial\nu}{\partial r}-\frac{\partial\lambda}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial\nu}{\partial r}\right)\right\}$$

$$\mathcal{T}_{++}=e^{-2\lambda}\left(\frac{\partial\phi}{\partial r}\right)^{2},\mathcal{T}_{--}=\left\{e^{-\nu}\frac{\partial\phi}{\partial\mu}-\frac{1}{2}e^{-\lambda}\frac{\partial\phi}{\partial r}\right\}^{2}$$

$$\mathcal{T}_{33} = \mathcal{T}_{44} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

$$\Sigma_{\alpha\beta} \coloneqq \delta_{\alpha\beta} - 8\pi \mathcal{T}_{\alpha\beta} \frac{\partial \Sigma_{--}}{\partial r} + 2\left(\frac{\partial \nu}{\partial r} + \frac{1}{r}\right)\Sigma_{-}$$

$$\frac{\partial(\nu + \lambda)}{\partial r} = 4\pi r \left(\frac{\partial \phi}{\partial r}\right)^2 \frac{\partial}{\partial r}\left(r e^{\nu-\lambda} - \frac{1}{r} e^{\nu+\lambda}\right)$$

$$\lim_{r=\infty} \nu = 0, \lim_{r=\infty} \lambda = 0$$

$$\nu + \lambda = -4\pi \iiint_r^\infty r \left(\frac{\partial \phi}{\partial r}\right)^2 dr$$

$$e^{\nu-\lambda} = \frac{1}{r} \iiint_0^r e^{\nu+\lambda} dr$$

$$\hbar \coloneqq \frac{\partial (r\phi)}{\partial r}, \alpha \coloneqq e^{\nu+\lambda}$$

$$\bar{f}(\mu, r) \coloneqq \frac{1}{r} \iiint_0^r f(\mu, \rho) d\rho$$

$$\phi = \bar{\hbar}, \alpha = \exp\left\{-4\pi \iiint_r^\infty \frac{1}{r}\left(\hbar - \bar{\hbar}\right)^2 dr\right\}, e^{\nu-\lambda} = \bar{\alpha}$$

$$\mathfrak{D}\hbar = \frac{1}{2r}(\alpha + \bar{\alpha})\left(\hbar - \bar{\hbar}\right)$$

$$\mathfrak{D} \coloneqq e^{\nu} \ell_{-} \equiv \frac{\partial}{\partial \mu} - \frac{1}{2}\bar{\alpha}\frac{\partial}{\partial r}$$

Masa de Bondi

$$m(\mu, r) = \frac{r}{2}\left(1 - e^{-2\lambda}\right) \equiv \frac{r}{2}\left(1 - \frac{\bar{\alpha}}{\alpha}\right)$$

$$\frac{\partial m}{\partial r} = 2\pi \frac{\bar{\alpha}}{\alpha}\left(\hbar - \bar{\hbar}\right)^2$$

$$m(\mu, r) = 2\pi \iiint_0^r \frac{\bar{\alpha}}{\alpha}\left(\hbar - \bar{\hbar}\right)^2 (\mu, \rho) d\rho$$

$$\mathfrak{D}m = -4\pi \frac{r^2}{\alpha}\left(\mathfrak{D}\bar{\hbar}\right)^2$$

$$\lim_{r=\infty} m(0, r) \coloneqq \mathcal{M}_0$$

$$\lim_{r=\infty} m(\mu, r) \coloneqq \mathcal{M}(\mu)\frac{d\mathcal{M}(\mu)}{d\mu}$$

$$\|\hbar_0\| \coloneqq \inf_{\alpha>0} \sup_{\alpha>0}\left\{\left[1+\frac{r}{\alpha}\right]^3 |\hbar_0(r)| + \left[1+\frac{r}{\alpha}\right]^4 \left|\frac{d\hbar_0(r)}{dr}\right|\right\}$$

$$\mathcal{M}_1 \coloneqq \lim_{\mu\mapsto\infty} \mathcal{M}(\mu)$$

$$e^{\nu(\mu,r_0)} \geq \bar{\alpha}(\mu, r_0) \geq 1 - \frac{2\mathcal{M}(\mu)}{r_0} \geq \frac{1}{2}\left\{1 - \frac{2\mathcal{M}_1}{r_0}\right\}$$

$$\iiint_{\mu_1}^{\mu_2} e^{\nu(\mu,r_0)} d\mu \geq \frac{1}{2}\left\{1 - \frac{2\mathcal{M}_1}{r_0}\right\}(\mu_2 - \mu_1)$$

$$\mathfrak{D}\hbar = \frac{1}{2r}(\alpha + \bar{\alpha})(\hbar - \bar{\hbar}), \mathfrak{D}m = -\frac{1}{2}r^2\frac{|\mathfrak{D}\bar{\hbar}|^2}{\alpha}, m \coloneqq \frac{r}{2}\left(1 - \frac{\bar{\alpha}}{\alpha}\right)$$

$$\xi \coloneqq \lim_{\delta\geq 0} \iiint_{\delta}^{r} \bar{\alpha}\,(\hbar - \bar{\hbar})r^{-1} dr$$

$$\mathfrak{D}\bar{\hbar} = \frac{\xi}{2r}, \mathfrak{D}m = -\pi\alpha^{-1}\xi^2 \iiint_0^{\chi(0,r_1)} \frac{\alpha}{\bar{\alpha}}(0, r) dr$$

$$= \iiint_0^{r_1} \frac{\alpha}{\bar{\alpha}}(\mu_1, r) dr + \frac{1}{2}\iiint_0^{\mu_1} \alpha(\mu, 0) d\mu + 2\pi \iiint_0^{\mu_1} \iiint_0^{\chi(\mu_1,r_1)} \frac{\alpha\xi^2}{\bar{\alpha}^2 r} dr d\mu$$

$$\mathfrak{D}_\epsilon \hbar_\epsilon = \frac{1}{2(r+\epsilon)}(\alpha_\epsilon + \bar{\alpha}_\epsilon)(\hbar_\epsilon - \bar{\hbar}_\epsilon)$$

$$\bar{f}_\epsilon(\mu, r) \coloneqq \frac{1}{r+\epsilon} \iiint_0^{r} f_\epsilon\,(\mu, \rho) d\rho \frac{\partial \bar{f}_\epsilon}{\partial r}\frac{f_\epsilon - \bar{f}_\epsilon}{\partial r}$$

$$\alpha_\epsilon = \exp\left\{-4\pi \iiint_r^{\infty} \frac{1}{r+\epsilon}(\hbar_\epsilon - \bar{\hbar}_\epsilon)^2 dr\right\}$$

$$\mathfrak{D}_\epsilon \hbar_\epsilon = \frac{1}{2(r+\epsilon)}(\alpha_\epsilon - \bar{\alpha}_\epsilon)(\hbar_\epsilon + \bar{\hbar}_\epsilon)$$

$$\mathfrak{D}_\epsilon \coloneqq \frac{\partial}{\partial\mu} - \frac{1}{2}\bar{\alpha}_\epsilon \frac{\partial}{\partial r}$$

$$m_\epsilon \coloneqq \frac{1+\epsilon}{2}\left(1 - \frac{\bar{\alpha}_\epsilon}{\alpha_\epsilon}\right)$$

$$\mathfrak{D}_\epsilon m_\epsilon = -\frac{\pi}{\alpha_\epsilon}\xi_\epsilon^2, \xi_\epsilon \coloneqq \iiint_0^r \bar{\alpha}_\epsilon\left(\hbar_\epsilon + \bar{\hbar}_\epsilon\right)\frac{dr}{r+\epsilon}$$

$$\iiint_0^{\chi_\epsilon(0,r_1)} \frac{\alpha_\epsilon}{\bar{\alpha}_\epsilon}(0,r)dr = \iiint_0^{r_1} \frac{\alpha_\epsilon}{\bar{\alpha}_\epsilon}(\mu_1,r)dr + \frac{1}{2}\iiint_0^{\mu_1} \alpha_\epsilon(\mu,0)d\mu + 2\pi\iiint_0^{\mu_1}\iiint_0^{\chi_\epsilon(\mu_1,r_1)} \frac{\alpha_\epsilon\xi^2}{\bar{\alpha}^2(r+\epsilon)}drd\mu$$

$$g_\epsilon \coloneqq -\alpha_\epsilon\bar{\alpha}_\epsilon d\mu^2 - 2\alpha_\epsilon d\mu dr + (r+\epsilon)^2(d\theta^2 + \sin^2\theta)$$

$$\iiint_0^{r_0} \frac{\alpha\xi^2}{\bar{\alpha}^2 r}(\mu_1,r)dr$$

$$\mathcal{M}(\mu) = \mathcal{M}(0) - \pi\iiint_0^{\mu} \Xi^2(\upsilon)d\upsilon$$

$$\mathcal{M}_1 \coloneqq \lim_{\mu=\infty}\lim_{r=\infty}\frac{1}{2}r\left(1-e^{-2\lambda(\mu,r)}\right) = 2\pi\lim_{\mu=\infty}\iiint_0^{\infty}\frac{\bar{\alpha}}{\alpha}\left(\hbar-\bar{\hbar}\right)^2(\mu,\rho)d\rho$$

$$g_Q = -\Omega^2 d\mu d\upsilon$$

Singularidades en cosmología cuántica

$$g_{ij}(t,\chi) = \alpha^2\ell_i\ell_j + \beta^2 m_i m_j + c^2\eta_i\eta_j$$

$$|dt| = \alpha\beta c d\tau$$

$$ds^2 = -\left(\tilde{\mathcal{N}}\sqrt{g}d\tau\right)^2 + g_{ij}\omega^i\omega^j$$

$$g_{ij} = \sum_{\alpha=1}^{d} e^{-2\beta^\alpha}\mathfrak{N}_i^\alpha\mathfrak{N}_j^\alpha$$

$$\mathfrak{H}^{asymp}(\beta,\pi) = \frac{1}{4}\mathfrak{G}^{\alpha\beta}\pi_\alpha\pi_\beta + \mathfrak{V}_{\mathfrak{S}}^{asymp} + \mathfrak{V}_{\mathfrak{G}}^{asymp}$$

$$\mathfrak{G}^{\alpha\beta}\pi_\alpha\pi_\beta \equiv \sum_{\alpha=1}^{d}\pi_\alpha^2 - \frac{1}{d-1}\left(\sum_{\alpha=1}^{d}\pi_\alpha\right)^2$$

$$\mathfrak{G}_{\alpha\beta}\beta^\alpha\beta^{\mathfrak{B}} = \sum_{\alpha=1}^{d}(\beta^\alpha)^2 - \left(\sum_{\alpha=1}^{d}\beta^\alpha\right)^2$$

$$\mathfrak{V}_{\mathfrak{S}}^{asymp} = \frac{1}{2}\sum_{\alpha=1}^{d-1} e^{-2\left(\beta^{\alpha+1}-\beta^\alpha\right)}\left(\mathfrak{P}_{(0)\alpha}^{i}\mathfrak{N}_{(0)i}^{\alpha+1}\right)^2$$

$$\mathfrak{V}_{\mathfrak{G}}^{asymp} = \frac{1}{2} e^{-2\alpha_{1d-d}(\beta)} \left(\mathfrak{C}_{(0)d-1d}^{1}\right)^{2}$$

$$\partial_\tau \beta_{(0)}^{\alpha} = \frac{1}{2} \mathfrak{G}^{\alpha\beta} \pi_{\beta}^{(0)}, \partial_\tau \pi_{\alpha}^{(0)}$$

$$= -\frac{\partial}{\partial \beta_{(0)}^{\alpha}} \{\mathfrak{V}_{\mathfrak{S}}^{asymp}(\beta_{(0)}; \mathfrak{P}_{(0)}; \mathfrak{N}_{(0)})$$

$$+ \mathfrak{V}_{\mathfrak{G}}^{asymp}(\beta_{(0)}; \mathfrak{P}_{(0)}; \mathfrak{N}_{(0)}; \partial_\chi \mathfrak{N}_{(0)})\} \langle \partial_\tau \mathfrak{N}_{(0)i}^{\alpha} | \partial_\tau \mathfrak{P}_{(0)\alpha}^{i} \rangle$$

$$\mathfrak{H}^{asymp}(\beta_{(0)}; \pi_{(0)}; \mathfrak{N}_{(0)}; \partial_\chi \mathfrak{N}_{(0)}; \mathfrak{P}_{(0)};)$$

$$\mathfrak{H}_{\alpha}^{asymp}({}_{(0)}\partial_{\chi(0),(0)})$$

Espacio – tiempo cuántico AVTD

$$\partial_\tau \mu - \mathrm{A}(\chi)\mu = e^{-t_\mu} f(t, \chi, \mu, \mathfrak{D}_\chi \mu)$$

$$f(t, \chi, \mu, \mathfrak{D}_\chi \mu) \equiv f_0(t, \chi, \mu) + f_1(t, \chi, \mu) \mathfrak{D}_\chi \mu$$

$$Sup_{z \in \widehat{\mathcal{M}}} \left| \sigma^{\mu^{-1} \mathrm{A}(z)} \right| \sigma^{\alpha} \leq \Sigma$$

$$\hat{g} \equiv e^{-2\phi\,(3)} g + e^{2\phi}(d\theta + \alpha)^2, {}^{(3)}g \equiv {}^{(3)}g_{\alpha\beta} d\chi^{\alpha} d\chi^{\beta}$$

$$\mathfrak{G} \equiv 2(d\phi)^2 + \frac{1}{2} e^{-4\gamma}(d\omega)^2 e^{4\phi} \otimes \mathfrak{F}$$

$$\mathfrak{F}_{\alpha\beta} \equiv \frac{1}{2} e^{-4\phi} \eta_{\alpha\beta\lambda} \partial^{\lambda} \omega$$

$$g^{\alpha\beta} \left({}^{(3)}\nabla_\alpha \partial_\beta \phi + \frac{1}{2} e^{-4\phi} \partial_\alpha \omega \partial_\beta \omega\right) - g^{\alpha\beta} \left({}^{(3)}\nabla_\alpha \partial_\beta \omega - 4 \partial_\alpha \omega \partial_\beta \omega\right) \left\langle g^{\alpha\beta} \right| {}^{(3)} |\nabla_\alpha| \left| \partial_\beta \phi \right\rangle$$

$${}^{(3)}\mathcal{R}_{\alpha\beta} = \partial_\alpha \phi \otimes \partial_\beta \phi \coloneqq 2 \partial_\alpha \phi \otimes \partial_\beta \phi + \frac{1}{2} e^{-4\gamma} \partial_\alpha \omega \partial_\beta \omega$$

$${}^{(3)}g \equiv -\mathfrak{N}^2 dt^2 + g_{\alpha\beta} d\chi^{\alpha} d\chi^{\beta} \langle e^{\lambda} | \sigma_{\alpha\beta} \rangle$$

$$\mathfrak{K}_{\alpha\beta} \coloneqq -\frac{1}{2\mathfrak{N}} \partial_\tau g_{\alpha\beta} \equiv -\frac{1}{2}(\sigma_{\alpha\beta} \partial_\tau \lambda + \partial_\tau \sigma_{\alpha\beta})$$

$$\tau \coloneqq g^{\alpha\beta} \mathfrak{K}_{\alpha\beta} \equiv -e^{-\lambda} \left(\partial_\tau \lambda + \frac{1}{2} \psi\right)$$

$$\psi \coloneqq \sigma^{\alpha\beta} \partial_\tau \sigma_{\alpha\beta}$$

$${}^{(3)}\Gamma_{\alpha\beta}^{c} = \Gamma_{\alpha\beta}^{c}(g) = \Gamma_{\alpha\beta}^{c}(\sigma) + \frac{1}{2}(\delta_{\beta}^{c} \partial_\alpha \lambda + \delta_{\alpha}^{c} \partial_\beta \lambda - \sigma^{cd} \sigma_{\alpha\beta} \partial_d \lambda)$$

$${}^{(3)}\Gamma_{00}^{0} = \partial_\tau \lambda, {}^{(3)}\Gamma_{0\alpha}^{0} = \partial_\alpha \lambda, {}^{(3)}\Gamma_{00}^{\alpha} = \sigma^{\alpha\beta} e^{\lambda} \partial_\alpha \lambda$$

$$ {}^{(3)}\Gamma^{0}_{\alpha\beta} = -e^{-\lambda}\mathfrak{K}_{\alpha\beta}, {}^{(3)}\Gamma^{\beta}_{\alpha 0} = -e^{\lambda}\mathfrak{K}^{\beta}_{\alpha} $$

$$ {}^{(3)}g^{\alpha\beta\,(3)}\Gamma^{0}_{\alpha\beta} = \frac{1}{2}\psi e^{-2\lambda} $$

$$ \mathfrak{C}_0 \equiv \Sigma^0_0 \equiv -\frac{1}{2}\{\mathcal{R}(g) - \mathfrak{K}\otimes\mathfrak{K} + \tau^2 - e^{-2\lambda}\partial_\tau\phi\otimes\partial_\tau\phi - g^{\alpha\beta}\partial_\alpha\phi\otimes\partial_\beta\phi\} $$

$$ \mathfrak{C}_\alpha \equiv e^{\lambda}\Sigma^0_\alpha \equiv -\{\nabla_\beta\mathfrak{K}^{\alpha}_{\beta} - \partial_\alpha\tau + e^{-\lambda}\partial_\alpha\phi\otimes\partial_\beta\phi\} $$

$$ \mathcal{N}\left({}^{(3)}\mathcal{R}^{\beta}_{\alpha} - \rho^{\beta}_{\alpha}\right) \equiv -\partial_\tau\mathfrak{K}^{\beta}_{\alpha} + \mathcal{N}\tau\mathfrak{K}^{\beta}_{\alpha} - \nabla^{\beta}\partial_\alpha\mathfrak{N} + \mathfrak{N}\mathfrak{R}^{\beta}_{\alpha} - \mathcal{N}\partial_\alpha\phi\otimes\partial^{\beta}\phi $$

$$ \partial_t\phi = \phi_t, \partial_t\phi_\alpha = \partial_\alpha\phi_t, \partial_t\sigma^{\alpha\beta}_c = \widetilde{\nabla}_c\partial_t\sigma^{\alpha\beta} $$

$$ \partial_t\sigma^{\alpha\beta} = 2e^{2\lambda}\mathfrak{K}^{\alpha\beta} + \sigma^{\alpha\beta}\partial_t\lambda $$

$$ \hat{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta}, \hat{g}_{\alpha\beta} = e^{\tilde{\lambda}}\tilde{\sigma}_{\alpha\beta}, \hat{\psi} = 0, \hat{g}_{\alpha\beta} = e^{-\tilde{\lambda}}\tilde{\sigma}_{\alpha\beta}\partial_t\hat{g}_{\alpha\beta} = e^{\tilde{\lambda}}\tilde{\sigma}_{\alpha\beta}\partial_t\hat{\lambda} $$

$$ \widehat{\mathfrak{K}}_{\alpha\beta} = -\frac{1}{2}\tilde{\sigma}_{\alpha\beta}\partial_t\hat{\lambda}, \widehat{\mathfrak{K}}^{\beta}_{\alpha} = -\frac{1}{2}e^{-\hat{\lambda}}\delta^{\beta}_{\alpha}\partial_t\hat{\lambda}, \hat{\tau} \coloneqq \widehat{\mathfrak{K}}^{\alpha}_{\alpha} = -e^{-\hat{\lambda}}\partial_t\hat{\lambda} $$

$$ \partial_t\widehat{\mathfrak{K}}^{\beta}_{\alpha} = \widehat{\mathcal{N}}\hat{\tau}\widehat{\mathfrak{K}}^{\beta}_{\alpha} $$

$$ \widehat{\mathfrak{K}}_{\alpha\beta} = \frac{1}{2}\tilde{\upsilon}\tilde{\sigma}_{\alpha\beta}, \widehat{\mathfrak{K}}^{\beta}_{\alpha} = \frac{1}{2}\tilde{\upsilon}e^{-\hat{\lambda}}\delta^{\beta}_{\alpha}, \hat{\tau} = e^{-\hat{\lambda}}\tilde{\upsilon} $$

$$ \widehat{\mathfrak{C}}_0 \equiv \hat{\Sigma}^0_0 \equiv -\frac{1}{2}\left\{-\widehat{\mathfrak{K}}\otimes\widehat{\mathfrak{K}} + \hat{\tau}^2 - e^{-2\hat{\lambda}}\partial_\tau\hat{\phi}\otimes\partial_\tau\hat{\phi}\right\}\left\langle\partial^2_{tt}\hat{\phi}\middle|\partial_\tau\hat{\phi} = -\frac{1}{2}\widetilde{\omega}\right\rangle $$

$$ -\widehat{\mathfrak{K}}\otimes\widehat{\mathfrak{K}} + \hat{\tau}^2 - 2e^{-2\hat{\lambda}}\partial_\tau\hat{\phi}\otimes\partial_\tau\hat{\phi} = \frac{e^{-2\lambda}}{2}(\tilde{\upsilon}^2 - \widetilde{\omega}^2), \hat{\phi} = \tilde{\phi} - \frac{1}{2}\tilde{\upsilon}t $$

$$ \sigma^{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} + e^{-\varepsilon_\sigma\tau}\mu^{\alpha\beta}_{\sigma} $$

$$ \mathfrak{K}^{\beta}_{\alpha} = e^{-\lambda}\left(\frac{1}{2}\tilde{\upsilon}\delta^{\beta}_{\alpha} + e^{-\varepsilon_\kappa\tau}\mu^{\beta}_{\kappa,\alpha}\right), \tau \coloneqq \mathfrak{K}^{\alpha}_{\alpha} \equiv e^{-\lambda}\left(\tilde{\upsilon} + e^{-\varepsilon_\kappa\tau}\mu^{\alpha}_{\kappa,\alpha} + \frac{1}{2}\psi\right) $$

$$ \phi = \hat{\phi} + e^{-\varepsilon_\phi\tau}\mu_\phi \equiv \tilde{\phi} - \frac{1}{2}\tilde{\upsilon}t + e^{-\varepsilon_\phi\tau}\mu_\phi $$

$$ \partial_\tau\phi = \phi_\tau, \partial_\tau\phi_\alpha = \partial_\alpha\phi_\tau, \partial_\tau\sigma^{\alpha\beta}_c = \widetilde{\nabla}_c\partial_\tau\sigma^{\alpha\beta} $$

$$ \phi_\tau \equiv -\frac{1}{2}\tilde{\upsilon} + e^{-\varepsilon_\phi\tau}\mu_{\phi_\tau}, \phi_\alpha \equiv \partial_\alpha\tilde{\phi} - \frac{1}{2}t\partial_\alpha\tilde{\upsilon} + e^{-\varepsilon_{\phi_\alpha}\tau}\mu_{\phi_\alpha} $$

$$ \sigma^{\alpha\beta}_c \equiv e^{-\varepsilon_{\sigma'}\tau}\mu^{\alpha\beta}_{\sigma' c} $$

$$ \partial_\tau\mu^{\alpha\beta}_{\sigma} - \varepsilon_\sigma\mu^{\alpha\beta}_{\sigma} = 2e^{(\varepsilon_\sigma+\varepsilon_\kappa)^\tau}\sigma^{\alpha c}\mu^{\beta}_{\kappa,c} $$

$$ \partial_\tau\mu^{\alpha\beta}_{\sigma' c} - \varepsilon_{\sigma'}\mu^{\alpha\beta}_{\sigma' c} = e^{\left(\varepsilon_{\sigma'}+\varepsilon_\sigma\right)^\tau}\widetilde{\nabla}_c\mu^{\alpha\beta}_{\sigma} $$

$$\partial_\tau \mu^\beta_{\kappa,\alpha} - \varepsilon_\kappa \mu^\beta_{\kappa,\alpha} - \frac{1}{2}\tilde{\upsilon}\delta^\beta_\alpha \mu^c_{\kappa,c} = e^{-\varepsilon_\kappa \tau}\mu^c_{\kappa,c}\mu^\beta_{\kappa,\alpha} + e^{\lambda+\varepsilon_\kappa \tau}\mathfrak{F}^\beta_\alpha(\tau,\chi,\mu,\mu_\chi)$$

$$\partial_\tau \mu^\alpha_{\kappa,\alpha} - \varepsilon_\kappa \mu^\alpha_{\kappa,\alpha} - \tilde{\upsilon}\mu^\alpha_{\kappa,\alpha} = e^{-\varepsilon_\kappa \tau}\mu^c_{\kappa,c}\mu^\alpha_{\kappa,\alpha} + e^{\lambda+\varepsilon_\kappa \tau}\mathfrak{F}^\alpha_\alpha(\tau,\chi,\mu,\mathfrak{D}_\chi \mu)$$

$$\partial^{\mathfrak{T}}_\tau \mu^\beta_{\kappa,\alpha} - \varepsilon^{\mathfrak{T}}_\tau \mu^\beta_{\kappa,\alpha} = e^{-\varepsilon_\kappa \tau}\mu^{c\mathfrak{T}}_{\kappa,c}\mu^\beta_{\kappa,\alpha} + e^{\lambda+\varepsilon_\kappa \tau \mathfrak{T}}\mathfrak{F}^\beta_\alpha(\tau,\chi,\mu,\mu_\chi)$$

$$\partial_\tau \mu_{\phi_\tau} - \varepsilon_{\phi_\tau}\mu_{\phi_\tau} = e^{(\varepsilon_{\phi_\tau}-\varepsilon_\kappa)^\tau}\mathfrak{F} + e^{(\lambda+\varepsilon_{\phi_\tau}\mathfrak{T})}\mathfrak{F}$$

$$\partial_\tau \mu - \mathrm{A}\mu = e^{-\mu\tau} f(\tau,\chi,\mu,\mathfrak{D}_\chi \mu)$$

$$\partial_\tau(\phi_\alpha - \partial_\alpha \phi) = \partial_\alpha \phi_\tau - \partial_\alpha \partial_\tau \phi$$

$$\phi_\alpha - \partial_\alpha \phi = e^{-\varepsilon_{\phi_\alpha}\mathfrak{T}}\mu_{\phi_\alpha} - e^{-\varepsilon_\phi \tau}(\partial_\alpha \mu_\phi - \varepsilon_\phi \mu_\phi)$$

$$\mathcal{G} \equiv \frac{1}{2}\left\{\frac{d\omega^2 + d\gamma^2}{\gamma^2}\right\} = \gamma e^{2\phi}\omega'' - 2\gamma^{-1}\omega'\gamma', \gamma'' + \gamma^{-1}\omega'\chi', \widehat{\omega} = \mathfrak{B} + \mathrm{A}\cos\theta, \hat{\gamma} = \mathrm{A}\sin\theta, \frac{\theta''}{\theta'}$$

$$= \frac{\cos\theta}{\sin\theta}\theta' - \widetilde{\omega}\sin\theta, tg\frac{\theta}{2} = \widetilde{\Theta}e^{-i\widetilde{\omega}t}$$

$$\hat{\phi} = \tilde{\phi} + \frac{1}{2}\log(\sin\theta), \widehat{\omega} = \widetilde{\omega} + e^{2\widetilde{\phi}}\cos\theta$$

$$\widetilde{\omega} + e^{2\widetilde{\phi}} = constant$$

Restricciones

$$\mathfrak{C}_0 \equiv \hat{\Sigma}^0_0 \equiv -\frac{1}{2}\{\mathcal{R}(g) - \mathfrak{K}\otimes\mathfrak{K} + \tau^2 - e^{-2\lambda}2|\partial_\tau \phi|^2\}$$

$$\mathfrak{C}_\alpha \equiv e^\lambda \Sigma^0_\alpha \equiv -\left\{\nabla_\beta \mathfrak{K}^\beta_\alpha - \partial_\alpha \tau + e^{-\lambda}2\partial_\tau \phi \partial_\alpha \phi\right\}$$

Espacios cuánticos no estacionarios y agujeros negros cuánticos

Cuestiones preliminares

$$\hat{g} \equiv -\psi^2(dt+\alpha)^2 + g$$

$$\alpha \equiv \alpha_i d\chi^i, g \equiv g_{ij}d\chi^i d\chi^j$$

$$t' = t + f$$

$$\hat{g} \equiv -\psi^2 dt'^2 + g_{ij}d\chi^i d\chi^j$$

$$\hat{g} \equiv \pi^\dagger g^{\dagger(\star)} + (\pi^* \phi^{\star(*)})\theta^2$$

$$\pi^\dagger g^{\dagger(\star)} = g^\diamond = g_{\alpha\beta}d\chi^\alpha d\chi^\beta$$

$$\theta = d\chi^m + \mathrm{A}_\alpha d\chi^\alpha$$

$$\xi \equiv e_m \equiv \frac{\partial}{\partial \chi^m}$$

$$\mathrm{A} \mapsto \mathrm{A} + \hbar^{-1} d\hbar$$

$$f = d\mathrm{A}, f_{\alpha\beta} = \nabla_\alpha \mathrm{A}_\beta - \nabla_\beta \mathrm{A}_\alpha$$

$$\hat{c}^m_{\alpha\beta} = -f_{\alpha\beta}$$

$$\widehat{\omega}^\alpha_{\beta c} = \Gamma^\alpha_{\beta c}$$

$$\widehat{\omega}^m_{mm} = 0, \widehat{\omega}^m_{\alpha m} = \frac{1}{2}\phi^{-1}\partial_\alpha \phi, \widehat{\omega}^\alpha_{mm} = \frac{1}{2} g^{\alpha\beta}\partial_\beta \phi$$

$$\widehat{\omega}^m_{\alpha\beta} = -\frac{1}{2} f^m_{\alpha\beta}, \widehat{\omega}^\alpha_{m\beta} = \widehat{\omega}^\alpha_{\beta m} = -\frac{1}{2} f^\alpha_{\beta,m}$$

$$f^m_{\alpha\beta} \coloneqq f_{\alpha\beta}, f_{\alpha\beta,m} \coloneqq \phi f_{\alpha\beta}$$

Tensor de curvatura

$$\hat{\mathcal{R}}^{\alpha\beta}_{cd} \equiv \mathcal{R}^{\alpha\beta}_{cd} + \frac{1}{2} f^\alpha_{\beta,m} f^m_{cd} + \frac{1}{4} f^\alpha_{m,c} f^m_{\beta d} - \frac{1}{4} f^\alpha_{m,d} f^m_{\beta c}$$

$$\hat{\mathcal{R}}^{\alpha m}_{cm} \equiv \nabla_c \widehat{\omega}^\alpha_{mm} - \widehat{\omega}^\alpha_{md}\widehat{\omega}^d_{cm}$$

$$\hat{\mathcal{R}}^{\alpha m}_{cm} \equiv -\frac{1}{2}\nabla_c \nabla^\alpha \phi + \frac{1}{4} f^\alpha_{m,d} f^d_{m,c} - \frac{1}{4}\phi^{-1}\partial^\alpha \phi \mathfrak{D}_c \phi$$

$$\hat{\mathcal{R}}^{\alpha m}_{cd} \equiv \frac{1}{2}\nabla_{[cfm,d]^\alpha} + \frac{1}{2} g^{\alpha\beta}\partial_\beta \phi f^m_{cd} + \frac{1}{4} f_{m[c}{}^\alpha \phi^{-1}\partial_{d]}\phi$$

$$\hat{\mathcal{R}}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2} f_{mb}{}^c f^m_{\alpha c} - \frac{1}{2}\phi^{-1}\nabla_\alpha \partial_\beta \phi + \frac{1}{4}\phi^{-2}\partial_\alpha \phi \partial_\beta \phi$$

$$\hat{\mathcal{R}}_{\alpha m} \equiv \frac{1}{2}\nabla_\alpha f^\beta_{m,\alpha} + \frac{1}{4} f_{m,\alpha}{}^\beta \phi^{-1}\partial_\beta \phi$$

$$\hat{\mathcal{R}}_{mm} \equiv \frac{1}{4} f_{m,\alpha\beta} f^{\alpha\beta}_m + \phi^{-1}\partial_\alpha \phi \partial^\alpha \phi - 2\nabla_\alpha \nabla^\alpha \phi$$

$$\hat{\mathcal{R}}_{\alpha m} \equiv \frac{1}{2\phi^{\frac{1}{2}}}\nabla_\beta \left(f_{m,\alpha}{}^\beta \phi^{\frac{1}{2}}\right) \equiv \frac{1}{2}\nabla_\beta \left(f_\alpha{}^\beta \psi^{\frac{3}{2}}\right), \psi \coloneqq \phi^{\frac{1}{2}}$$

$$\hat{\mathcal{R}}_{mm} \equiv \frac{1}{4}\psi^4 f^\beta_\alpha f^\alpha_\beta - \psi \nabla^\alpha \partial_\alpha \psi$$

Espacio – tiempo cuántico en sentido estricto

$$\hat{g} \coloneqq -\psi^2 \left(dt + \alpha_i d\chi^i\right)^2 + g_{ij} d\chi^i d\chi^j$$

$$\hat{\mathcal{R}}_{ij} \equiv \mathcal{R}_{ij} + \frac{\psi^2}{2} f_i{}^\hbar f_{j\hbar} - \psi^{-1}\nabla_i \partial_j \psi = \rho_{ij}$$

$$\hat{\mathcal{R}}_{i0} \equiv -\frac{1}{2\psi}\nabla_j\left(f_i{}^j\psi^3\right) = \rho_{i0}$$

$$\hat{\mathcal{R}}_{i0} \equiv \frac{1}{4}\psi^4 f_i^j f_j^i + \psi\Delta_g\psi = \rho_{00}$$

$$\hat{g} \coloneqq -\psi^2 dt^2 + g_{ij}d\chi^i d\chi^j$$

$$\hat{\mathcal{R}}_{ij} \equiv \mathcal{R}_{ij} - \psi^{-1}\nabla_i\partial_j\psi = \rho_{ij}$$

$$\hat{\mathcal{R}}_{i0} \equiv 0 = \rho_{0i}$$

$$\psi^{-1}\hat{\mathcal{R}}_{00} \equiv \Delta_g\psi = \psi^{-1}\rho_{00}$$

$$\nabla_j\left(\alpha^i f_j^i\psi^3\right) = \frac{1}{2}\psi^3 f_i^j f_j^i$$

$$\iiint_{\partial\mathfrak{B}_\wp} \alpha^i f_i^j \psi^3 \eta_j \mu_{\partial\mathfrak{B}_\wp} = \frac{1}{2}\iiint_{\mathfrak{B}_\wp} \psi^3 f_i^j f_j^i \mu_{\mathfrak{B}_\wp}$$

$$\hat{g} \coloneqq -\psi^2 dt^2 - 2\psi\alpha_i d\chi^i dt + \left(g_{ij} - \alpha_i\alpha_j\right)d\chi^i d\chi^j, \xi_0 = -\psi^2, \xi_i = -\alpha_i\psi$$

$$d\xi \equiv 2\psi\partial_i\psi d\chi^i \wedge dt - \frac{1}{2}\psi f_{ij}d\chi^i \wedge d\chi^j$$

$$\psi \cong 1 - \frac{m}{r} + o\left(\frac{1}{r}\right), \partial_i\psi = \frac{m}{r^2}\frac{\chi^i}{r} + o\left(\frac{1}{r^2}\right)$$

$$\circledast\, d\xi|_r \cong \frac{\partial\psi}{\partial r}\omega_{\delta^2}$$

$$\mathcal{R}icci(\hat{g})(\xi,\xi) \geq \mathcal{M}$$

$$d \circledast d\xi|_{\mathcal{M}} = 4\hat{\mathcal{R}}_{00}\omega_{\mathcal{M}} \equiv 4\hat{\mathcal{R}}icci(\xi,\xi)\omega_{\mathcal{M}}$$

$$\iiint_{\mathcal{M}} \Delta_g\psi\omega_{\mathcal{M}} = lim_{r\mapsto\infty} \iiint_{\delta^2\otimes\{r\}} \eta^i\partial_i\psi r^2\omega_{\delta^2}$$

Ecuación de divergencia.

$$\psi\alpha^i\hat{\mathcal{R}}_{i0} \equiv -\frac{1}{2}\nabla_j\left(\alpha^i f_i^j\psi^3\right) + \frac{1}{4}\psi^3 f_i^j f_j^i$$

$$\mathcal{R}_t^t \equiv \mathcal{R}_t^t \equiv -\psi^{-1}\nabla^i\left\{\partial_i\psi + \frac{1}{2}\left(\alpha^j f_{ji}\psi^3\right)\right\}$$

$$\alpha^i\nabla_j\left(f_i^j\psi^3\right) \equiv \nabla_j\left(\alpha^i f_i^j\psi^3\right) - f_i^j\psi^3\nabla_j\alpha^i$$

$$\theta^0 = dt + \alpha_i d\chi^i, \theta^i = d\chi^i$$

$$\mathcal{R}_t^t \equiv \hat{\mathcal{R}}_0^0 + \alpha_i \hat{\mathcal{R}}_0^i$$

$$\hat{\mathcal{R}}_0^0 \equiv -\psi^{-2}\hat{\mathcal{R}}_{00}, \hat{\mathcal{R}}_0^i \equiv g^{ij}\hat{\mathcal{R}}_{0j}$$

$$\mathcal{R}_t^t \equiv -\left\{\frac{1}{4}\psi^2 f_i^{\ j} f_j^{\ i} + \psi^{-1}\Delta_g\psi + \frac{1}{2}\psi^{-1}\alpha^i\nabla_j\left(f_i^{\ j}\psi^3\right)\right\}$$

$$\mathcal{R}_t^t \equiv -\psi^{-1}\left\{\Delta_g\psi + \frac{1}{2}\nabla_j\left(\alpha^i f_i^{\ j}\psi^3\right)\right\}$$

Solitones cuánticos

$$\iiint_{\mathcal{M}} \psi^{-1}\rho_{00}\mu_g\, \nabla_j\left(f_i^{\ j}\psi^3\right)$$

$$\Delta_g\psi = -\frac{1}{4}\psi^3 f_i^{\ j} f_j^{\ i}, \mathcal{R}_{ij} = \psi^{-1}\nabla_i\partial_j\psi - \frac{\psi^2}{2} f_i^{\ \hbar} f_{j\hbar}$$

$$\iiint_{\partial\mathfrak{B}_{\wp}} \alpha^i f_i^{\ j}\psi^3\eta_j\mu_{\partial\mathfrak{B}_{\wp}} = \iiint_{\mathfrak{B}_{\wp}} \frac{1}{2}\psi^3 f_i^{\ j} f_j^{\ i}\,\mu_{\wp}$$

Masa de Komar y masa ADM

$$d\xi \equiv \frac{1}{2}\left(\hat{\nabla}^\lambda\xi^\mu - \hat{\nabla}^\mu\xi^\lambda\right) d\chi^\lambda \wedge d\chi^\mu$$

$$\circledast\, d\xi \coloneqq \hat{\eta}_{\alpha\beta\lambda\mu}\hat{\nabla}^\lambda\xi^\mu d\chi^\alpha \wedge d\chi^\beta$$

$$m_{Komar} \coloneqq lim_{r\mapsto\infty}\frac{1}{4\pi}\iiint_{\delta^2\otimes\{r\}} \circledast\, d\xi$$

$$d \circledast d\xi \equiv \hat{\eta}_{\alpha\beta\lambda\mu}\hat{\nabla}_\gamma\hat{\nabla}^\lambda\xi^\mu d\chi^\alpha \wedge d\chi^\beta$$

$$\circledast\, d \circledast d\xi \equiv \hat{\eta}_{\rho\gamma\alpha\beta}\hat{\eta}^{\alpha\beta\lambda\mu}\hat{\nabla}^\gamma\hat{\nabla}_\lambda\xi_\mu d\chi^\rho = 4\left(\hat{\nabla}^\gamma\hat{\nabla}_\rho\xi_\gamma - \hat{\nabla}_\rho\hat{\nabla}^\gamma\xi_\gamma\right)d\chi^\rho = 4\hat{\mathcal{R}}_{\rho\mu}\xi^\mu d\chi^\rho$$

$$m_{Komar} = \frac{1}{4\pi}\iiint_{\partial\mathcal{M}} \circledast\, d\xi$$

$$m_{Komar} = \frac{1}{4\pi}\iiint_{\mathcal{M}} d \circledast d\xi$$

$$m_{\mathfrak{ADM}}(g) \coloneqq \frac{1}{16\pi}\lim_{r=\infty}\iiint_{\delta^2\otimes\{r\}} \left(\partial^j g_{ij} - \partial_i g^{ij}\right)\eta_i r^2\omega_{\delta^2}$$

$$m_{\mathfrak{ADM}}(\gamma) = \frac{1}{16\pi} \lim_{r\mapsto\infty} \iiint_{\delta^2 \otimes \{r\}} \delta_{ij}\eta^i\chi^j r^2 \omega_{\delta^2}$$

$$g = \psi^{-2}\tilde{g}, \partial_i g_{\hbar\Bbbk} \equiv \psi^{-2}(\partial_i \tilde{g}_{\hbar\Bbbk} - 2\tilde{g}_{\hbar\Bbbk}\psi^{-1}\partial_i\psi)$$

$$\partial_i\psi = 0\left(\frac{1}{r^2}\right), g_{ij} = \delta_{ij} + 0\left(\frac{1}{r}\right)$$

$$m_{\mathfrak{ADM}}(g) = m_{\mathfrak{ADM}}(\tilde{g}) - \lim_{r=\infty} \iiint_{\delta^2 \otimes \{r\}} (\partial_i\psi - 3\partial_i\psi)\eta^i r^2 \omega_{\delta^2} = m_{\mathfrak{ADM}}(g)$$

$$= m_{\mathfrak{ADM}}(\tilde{g}) + 2\lim_{r=\infty} \iiint_{\delta^2 \otimes \{r\}} \partial_i\psi\eta^i r^2 \omega_{\delta^2}$$

$$m(g)_{\mathfrak{ADM}} = m_{\mathfrak{ADM}}(\tilde{g}) + m_{Komar}$$

Tensor de Landau–Lifshitz

$$lim_{r\mapsto\infty} \iiint_{\delta^2 \otimes \{r\}} t_{ij}\eta^i\chi^j r^2 \omega_{\delta^2}$$

$$\delta_{ij} + t_{ij} \equiv (\det\gamma)^{-1}\partial_{\Bbbk}\hbar^{ij\Bbbk}$$

$$\hbar^{ij\Bbbk} := \partial_{\hbar}\left[\det\gamma\left(\gamma^{ij}\gamma^{\Bbbk\ell} - \gamma^{j\Bbbk}\gamma^{i\ell}\right)\right]$$

$$\partial_{\Bbbk}\partial_j\hbar^{ij\Bbbk} \equiv \iiint_{\mathbb{R}^4} \chi^i\partial_{\Bbbk}\partial_j\hbar^{ij\Bbbk} d\chi^1 d\chi^2 d\chi^3 \lim_{r\mapsto\infty} \iiint_{\delta^2 \otimes \{r\}} \delta_{ij}\eta^i\chi^j r^2 \omega_{\delta^2}$$

$$- \iiint_{\mathbb{R}^4} \partial_{\Bbbk}\hbar^{ij\Bbbk} d\chi^1 d\chi^2 d\chi^3 \lim_{r=\infty} \iiint_{\delta^2 \otimes \{r\}} (\partial^j\gamma_{ij} - \partial_i\gamma^{ij})\eta_i r^2 \omega_{\delta^2}$$

Solitones cuánticos electrovac

$$\mathfrak{F}_{i0} = \partial_i \mathrm{A}_0$$

$$\hat{\nabla}^{\alpha}\mathfrak{F}^{\alpha 0} \equiv -\psi^{-1}\nabla^i(\psi^{-1}\partial_i \mathrm{A}_0) \equiv -\psi^{-Z}\left(\Delta_g \mathrm{A}_0 - \psi^{-1}\partial_i\psi\partial_i \mathrm{A}_0\right)\left\langle\hat{\nabla}_i\mathfrak{F}^{ij}\middle|\nabla_i\mathfrak{F}^{ij}\right\rangle$$

$$\mathrm{A}_j\nabla_i\mathfrak{F}^{ij} \equiv \nabla_i\left(\mathrm{A}_j\mathfrak{F}^{ij}\right) - \left(\nabla_i\mathrm{A}_j\right)\mathfrak{F}^{ij}, \nabla_i\left(\mathrm{A}_j\mathfrak{F}^{ij}\right) = \frac{1}{2}\mathfrak{F}_{ij}\mathfrak{F}^{ij}$$

Fórmula Kaluza – Klein

$$\hat{g} = g_{\delta} + \phi_{m\eta}(\, d\chi^m + \mathrm{A}_{\alpha}^m\, d\chi^{\alpha})(\, d\chi^m + \mathrm{A}_{\alpha}^m\, d\chi^{\alpha})$$

$$g_{\delta} \equiv g_{\alpha\beta}d\chi^{\alpha}d\chi^{\beta}, \phi \equiv \phi_{m\eta}(\, d\chi^m + \mathrm{A}_{\alpha}^m\, d\chi^{\alpha})(\, d\chi^m + \mathrm{A}_{\alpha}^m\, d\chi^{\alpha})$$

$$\phi \equiv -\mathcal{N}^2 dt^2 + 2\alpha dt d\phi + \beta^2 d\phi^2$$

Métrica de Kerr

$$g_{\mathfrak{BL}} = \mathrm{A}\left(\frac{dr^2}{\mathfrak{B}} + d\theta^2\right) - \left(1 - \frac{2mr}{\mathrm{A}}\right) dt^2 - \frac{4m\alpha r \sin^2\theta}{\mathrm{A}} d\phi dt + \frac{2m\alpha^2 \sin^4\theta}{\mathrm{A}} d\phi^2$$

$$\mathrm{A} \coloneqq r^2 + \alpha^2 \cos^2\theta \, , \mathfrak{B} \coloneqq r^2 - 2mr + \alpha^2$$

$$dv = dt + \frac{\alpha^2 + r^2}{\mathfrak{B}}$$

$$d\Phi = d\phi + \frac{\alpha}{\mathfrak{B}} dr$$

$$g_{\kappa\delta} \equiv \left(1 - \frac{2mr}{\mathrm{A}}\right) dv^2 + 2drdv + \mathrm{A} d\theta^2 + \frac{\Sigma^2}{\mathrm{A}} \sin^2\theta \, d\phi^2 + 4\alpha mr \frac{\sin^2\theta}{\mathrm{A}} dv d\Phi + 2\alpha \sin^2\theta \, dr d\phi$$

$$\Sigma^2 \equiv (r^2 + \alpha^2)^2 - \mathfrak{B}\alpha^2 \sin^2\theta$$

$$\chi = (r^2 + \alpha^2)^{1/2} \sin\phi \sin\theta \, , \gamma = (r^2 + \alpha^2)^{1/2} \cos\phi \sin\theta \, , z = r\cos\theta$$

Horizonte de eventos

$$r_{\pm} = m \pm \sqrt{m^2 - \alpha^2} \left| \frac{2mr}{\mathrm{A}} \right\rangle$$

$$r_{stat} = m + \sqrt{m^2 - \alpha^2 \cos^2\theta} \geq r_{\pm}$$

Proceso de Penrose

$$g_{\alpha\beta}\chi^{\alpha}\chi^{\beta} \equiv -\left(1 - \frac{2mr}{\mathrm{A}}\right)$$

Solución de Majumdar–Papapetrou

$$g = -\mu^{-2} dt^2 + \mu^2 (d\chi^2 + d\gamma^2 + dz^2), \mathrm{A} = \mu^{-1} dt, \mu = 1 + \sum_{i=1}^{\mathfrak{I}} \frac{\mu_i}{\vec{\chi} - \vec{\alpha}_i}$$

$$d\big(g(\chi,\chi)\big) = -2\kappa\chi^{\flat}$$

$$\langle \mathcal{M}_{ext} \rangle = \mathfrak{I}^{+}\big(\cup_t \phi_t(\Sigma_{ext})\big) \cap \mathfrak{I}^{-}\big(\cup_t \phi_t(\Sigma_{ext})\big)$$

$$\partial\bar{\Sigma} \subset \xi^{\circ} \coloneqq \partial(\langle \mathcal{M}_{ext} \rangle) \cap \mathfrak{I}^{+}(\Sigma_{ext})$$

Agujeros negros cuánticos completos

$$g=-\frac{\mathfrak{F}(\chi)}{\mathfrak{F}(\gamma)}\left(dt+\sqrt{\frac{\nu}{\xi_{\mathfrak{F}}}}\frac{\xi_1-\gamma}{\mathrm{A}}d\psi\right)^2$$

$$+\frac{\mathfrak{F}(\gamma)}{\mathrm{A}^2(\chi-\gamma)^2}\left[-\mathfrak{F}(\chi)\left(\frac{d\gamma^2}{\mathfrak{G}(\gamma)}+\frac{\mathfrak{G}(\gamma)}{\mathfrak{F}(\gamma)}d\psi^2\right)+\mathfrak{F}(\gamma)\left(\frac{d\chi^2}{\mathfrak{G}(\chi)}+\frac{\mathfrak{G}(\chi)}{\mathfrak{F}(\chi)}d\varphi^2\right)\right]$$

$$\mathfrak{F}(\xi)=1\frac{\xi}{\xi_{\mathfrak{F}}},\mathfrak{G}(\xi)=\nu\xi^3-\xi^2+1=\nu(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)$$

$$\xi_{\mathfrak{F}}=\frac{\xi_1\xi_2-\xi_3^2}{\xi_1-2\xi_3+\xi_2}$$

$$\tilde{\rho}=2\sqrt{\chi-\xi_1},\varphi=\lambda\tilde{\varphi}$$

$$\lambda=\frac{2\sqrt{\xi_{\mathfrak{F}}-\xi_1}}{\nu\sqrt{\xi_{\mathfrak{F}}}(\xi_2-\xi_1)(\xi_3-\xi_1)}$$

$$\underbrace{\mathcal{R}}_{t}\bigotimes\left\{\left(\underbrace{\mathcal{R}^2}_{\gamma,\psi\Leftrightarrow\hat{\rho},\hat{\varphi}}\bigotimes\underbrace{\delta^2}_{\chi,\psi\Leftrightarrow\tilde{\rho},\tilde{\varphi}}\right)\setminus\left(\{0\}\bigotimes\{\mathcal{N}\}\right)\right\}$$

$$\tilde{r}=\frac{\tilde{\rho}}{\mathfrak{B}(\tilde{\rho}^2+\hat{\rho}^2)},\hat{r}=\frac{\hat{\rho}}{\mathfrak{B}(\tilde{\rho}^2+\hat{\rho}^2)}$$

$$(4\mathfrak{B})^2\mathfrak{F}^2(\xi_1)=\nu\mathrm{A}^2(\xi_1-\xi_2)(\xi_1-\xi_3)$$

$$g=\{\eta_{\mu\nu}+\mathcal{O}(r^{-2})\}\,d\chi^\mu\wedge d\chi^\nu$$

$$g=-\mathfrak{F}(\chi)\left[\frac{dt^2}{\mathfrak{F}(\gamma)}+2\sqrt{\frac{\nu}{\xi_{\mathfrak{F}}}}\frac{\xi_1-\gamma}{\mathrm{A}\mathfrak{F}(\gamma)}dtd\psi+\frac{1}{\mathrm{A}^2}\left\{\frac{\nu(\xi_1-\gamma)^2}{\xi_{\mathfrak{F}}-\gamma}+\frac{\mathfrak{G}(\gamma)}{(\chi-\gamma)^2}\right\}d\psi^2+\frac{\mathfrak{F}(\gamma)\gamma^4}{\mathrm{A}^2(\chi-\gamma)^2\mathfrak{G}(\gamma)}d\gamma^2\right]$$

$$+\frac{\mathfrak{F}^2(\gamma)}{\mathrm{A}^2(\chi-\gamma)^2}\left(\frac{d\chi^2}{\mathfrak{G}(\chi)}+\frac{\mathfrak{G}(\chi)}{\mathfrak{F}(\chi)}d\varphi^2\right)$$

$$-\mathfrak{F}(\chi)\left[2\frac{\sqrt{\nu\xi_{\mathfrak{F}}}}{\mathrm{A}}dtd\psi+\frac{2\xi_1+\acute{e}\chi-1-\nu\xi_{\mathfrak{F}}}{\mathrm{A}^2}d\psi^2+\frac{1}{\mathrm{A}^2\nu\xi_{\mathfrak{F}}}d\gamma^2\right]+\frac{1}{\mathrm{A}^2\xi_{\mathfrak{F}}^2}\left(\frac{d\chi^2}{\mathfrak{G}(\chi)}+\frac{\mathfrak{G}(\chi)}{\mathfrak{F}(\chi)}d\varphi^2\right)$$

$$g=(\partial_\tau,\partial_\tau)=g_{\tau\tau}=-\frac{\mathfrak{F}(\chi)}{\mathfrak{F}(\gamma)}=-\frac{\xi_{\mathfrak{F}}-\chi}{\xi_{\mathfrak{F}}-\gamma}=-\frac{(\xi_{\mathfrak{F}}-\chi)\gamma}{\gamma\xi_{\mathfrak{F}}+1}$$

$$d\chi=d\psi+\frac{\sqrt{-\mathfrak{F}(\mathcal{Z})}}{\mathfrak{G}(\mathcal{Z})}dz,dv=dt+\sqrt{\frac{\nu}{\xi_{\mathfrak{F}}}}(z-\xi_1)\frac{\sqrt{-\mathfrak{F}(\mathcal{Z})}}{\mathrm{A}\mathfrak{G}(\mathcal{Z})}dz$$

$$ds^2 = -\frac{\mathfrak{F}(\chi)}{\mathfrak{F}(\gamma)}\left(dv - \sqrt{\frac{\nu}{\xi_{\mathfrak{F}}}}\frac{z-\xi_1}{\mathrm{A}}d\chi\right)^2$$

$$+\frac{1}{\mathrm{A}^2(\chi - z)^2}\left[\mathfrak{F}(\chi)\left(-\mathfrak{G}(\mathcal{Z})d\chi^2 + 2\sqrt{-\mathfrak{F}(\mathcal{Z})}d\chi dz\right)\right.$$

$$\left.+\mathfrak{F}(\mathcal{Z})^2\left(\frac{d\chi^2}{\mathfrak{G}(\chi)} + \frac{\mathfrak{G}(\chi)}{\mathfrak{F}(\chi)}d\phi^2\right)\right]$$

Cinética cuántica relativista

Cuestiones preliminares

$$f\colon \mathcal{P}_{\mathcal{V}} \mapsto \mathcal{R}, (\chi, \wp) \mapsto f(\chi, \wp), \chi \in \mathcal{V}, \wp \in \mathcal{P}_\chi \subset \mathcal{T}_\chi\mathcal{V}$$

$$\mathcal{P}_{m\chi} \equiv \mathcal{T}_\chi\mathcal{V} \cap \{g(\wp, \wp) = -m^2, \wp^0\}$$

$$\theta = \omega_g \wedge \omega_\wp$$

$$\omega_g = (\det g)^{1/2} d\chi^0 \wedge d\chi^1 \wedge \cdots \wedge d\chi^\eta$$

$$\omega_\wp = (\det \wp)^{1/2} d\wp^0 \wedge d\wp^1 \wedge \cdots \wedge d\wp^\eta$$

$$d\left[\frac{1}{2}\left(g_{\alpha\beta} - m^2\right)\right] \wedge \omega_{m,\wp} = \omega_\wp$$

$$\omega_{m,\wp} = \frac{(\det g)^{1/2} d^\eta \wp}{\wp_0}, d^\eta\wp \coloneqq d\wp^1 \wedge \cdots \wedge d\wp^\eta$$

Momento de distribución

$$r_0(\chi) \coloneqq \iiint\limits_{\mathcal{P}_\chi} f\omega_\wp = \iiint\limits_{\mathcal{P}_\chi} f\mu_\wp$$

$$\mathcal{N}_\Sigma = \iiint\limits_{\Sigma} f \ \ {}^{i}\chi^\theta$$

$${}^{i}\chi^\theta \equiv i_\chi\left(\omega_g \wedge \omega_\wp\right) = i_\chi\omega_g \wedge \omega_\wp + (-1)^{\eta+1}\omega_g \wedge i_\chi\omega_\wp$$

$$\left(\ {}^{i}\chi^\theta\right)_\Sigma = \left(i_\chi\omega_g\right)_\delta \wedge \omega_\wp = \wp^\alpha\eta_\alpha\overline{\omega}_{g,\delta} \wedge \omega_\wp$$

$$r_\delta(\chi) = \iiint\limits_{\mathcal{P}_\chi} f\wp^\alpha\eta_\alpha\,\omega_\wp$$

$$\mathcal{P}^{\alpha}(\chi) := \iiint_{\mathcal{P}_{\chi}} \wp^{\alpha} f(\chi, \wp) \mu_{\wp}$$

$$r^{2} := -\mathcal{P}^{\alpha}\mathcal{P}_{\alpha}, \mu^{\alpha} := r^{-1}\mathcal{P}^{\alpha}$$

$$\mathcal{T}^{\alpha\beta}(\chi) := \iiint_{\mathcal{P}_{\chi}} f(\chi, \wp) \wp^{\alpha} \wp^{\beta} \mu_{\wp}$$

$$f(\chi, \wp) \equiv \mathcal{F}(\chi, \mathcal{V}_{\alpha}(\chi) \wp^{\alpha})$$

$$\mathcal{P}^{\alpha}(\chi) \equiv \iiint_{\mathcal{P}_{\chi}} \wp^{\alpha} \mathcal{F}(\chi, \mathcal{V}_{\alpha}(\chi) \wp^{\alpha}) \mu_{\wp}$$

$$\mathcal{P}^{i}(\chi) = \iiint_{\mathcal{P}_{\chi}} \wp^{i} \mathcal{F}\,(\chi, -\lambda(\chi) \wp^{0}) \mu_{\wp}$$

$$\mathcal{P}^{0}(\chi) = \iiint_{\mathcal{P}_{\chi}} \wp^{0} \mathcal{F}\,(\chi, -\lambda(\chi) \wp^{0}) \mu_{\wp}$$

$$\mathcal{T}^{ii} = \iiint_{\mathcal{P}_{\chi}} (\wp^{i})^{2} \mathcal{F}\,(\chi, -\lambda(\chi) \wp^{0}) \mu_{\wp}$$

$$\mathcal{T}^{00} = \iiint_{\mathcal{P}_{\chi}} (\wp^{0})^{2} \mathcal{F}\,(\chi, -\lambda(\chi) \wp^{0}) \mu_{\wp}$$

$$\mathcal{M}^{\alpha_{1} \cdots \alpha_{\wp}} := \iiint_{\mathcal{P}_{\chi}} f(\chi, \wp) \wp^{\alpha_{1}} \cdots \wp^{\alpha_{\wp}} \omega$$

$$\mathcal{T}^{\alpha\beta}(\chi) := \sum_{\alpha} \iiint_{\mathcal{P}_{m_{\alpha}, \chi}} f_{\alpha}(\chi, \wp) \wp^{\alpha} \wp^{\beta} \omega_{m_{\alpha}, \wp}$$

Ecuaciones de Vlasov

$$\wp^{\alpha} := \frac{d\chi^{\alpha}}{d\lambda}, \frac{d\wp^{\alpha}}{d\lambda} = Q^{\alpha} := -\Gamma^{\alpha}_{\lambda\mu} \wp^{\lambda} \wp^{m}$$

$$\frac{\partial \wp^{\alpha'}}{\partial \wp^{\alpha}} = \mathrm{A}^{\alpha'}_{\alpha}, \frac{\partial \wp^{\alpha'}}{\partial \chi^{\beta}} = \partial_{\beta} \mathrm{A}^{\alpha'}_{\alpha}$$

$$\wp'^{\alpha'} = \mathrm{A}^{\alpha'}_{\alpha} \wp^{\alpha}, Q'^{\alpha'} = \wp^{\beta} \partial_{\beta} \mathrm{A}^{\alpha'}_{\alpha} - \mathrm{A}^{\alpha'}_{\alpha} \Gamma^{\alpha}_{\lambda\mu} \wp^{\lambda} \wp^{m}$$

$$\Gamma'^{\alpha'}_{\beta'\gamma'} = \mathrm{A}^{\alpha'}_{\alpha}\partial_{\beta'}\mathrm{A}^{\alpha}_{\gamma'} + \mathrm{A}^{\alpha'}_{\alpha}\mathrm{A}^{\beta}_{\beta'}\mathrm{A}^{\gamma}_{\gamma'}\Gamma^{\alpha}_{\beta\gamma}$$

$$\mathcal{Q}'^{\alpha'} \equiv -\Gamma'^{\alpha'}_{\beta'\gamma'}\wp^{\beta'}\wp^{\gamma'}$$

$$\left(\wp^{\alpha}\frac{\partial}{\partial\chi^{\alpha}} - \Gamma^{\alpha}_{\lambda\mu}\wp^{\lambda}\wp^{\mu}\frac{\partial}{\partial\wp^{\alpha}}\right)g_{\sigma\rho}\wp^{\sigma}\wp^{\rho}$$

$$(\mathcal{L}\otimes\theta)_{01\cdots\eta\eta+1\cdots2(\eta+1)}$$

$$= \wp^{\alpha}\frac{\partial}{\partial\chi^{\alpha}}\theta_{01\cdots\eta\eta+1\cdots2(\eta+1)} + \theta_{\mathrm{A}1\cdots2(\eta+1)}\frac{\partial\chi^{\mathrm{A}}}{\partial\chi^{0}} + \theta_{0\mathrm{A}1\cdots2\eta+1\mathrm{A}}\frac{\partial\chi^{\mathrm{A}}}{\partial\chi^{1}} + \cdot\cdot$$

$$\cdot + \theta_{01\cdots2\eta+1\mathrm{A}}\frac{\partial\chi^{\mathrm{A}}}{\partial\chi^{2(\eta+1)}}$$

$$\frac{\partial}{\partial\chi^{\alpha}}\theta_{01\cdots\eta\eta+1\cdots2(\eta+1)} = g^{\lambda\mu}\left(\frac{\partial}{\partial\chi^{\alpha}}g_{\lambda\mu}\right)\theta_{01\cdots\eta\eta+1\cdots2(\eta+1)}$$

$$\theta_{01\cdots2(\eta+1)}\frac{\partial\chi^{\mathrm{A}}}{\partial\chi^{\mathrm{A}}} = \theta_{01\cdots2(\eta+1)}\frac{\partial\mathcal{Q}^{\alpha}}{\partial\wp^{\alpha}} = -2\Gamma^{\lambda}_{\lambda\alpha}\wp^{\alpha}$$

$$\mathcal{L}_{\chi} \equiv di_{\chi} + i_{\chi}d, \mathcal{L}_{\alpha\chi}i_{\chi}\theta \equiv d\left(i_{\alpha\chi}i_{\chi}\theta\right)$$

$$\iiint_{\gamma} i_{\chi}\tau d\lambda, i_{\chi}\tau \equiv \chi^{\mathrm{A}}\tau_{\mathrm{A}} \equiv \wp^{\alpha}\tau_{\alpha} + \mathcal{Q}^{\bar{\alpha}}\tau_{\bar{\alpha}}$$

$$\tau \wedge i_{\chi}\theta = \tau_{\mathrm{A}}d\chi^{\mathrm{A}} \wedge (-1)^{\mathrm{A}}\chi^{\mathrm{A}}\theta_{0\cdots\mathrm{A}\cdots2(\eta+1)}d\chi^{0} \wedge\cdots\wedge d\hat{\chi}^{\mathrm{A}} \wedge\cdots d\chi^{2(\eta+1)} = \theta$$

$$\mathcal{L}_{\chi}f \equiv \wp^{\alpha}\frac{\partial f}{\partial\chi^{\alpha}} + \mathcal{Q}^{\alpha}\frac{\partial f}{\partial\wp^{\alpha}}$$

$$\left(\chi^{\mathrm{A}}\right) = \left(\wp^{\alpha}, \mathcal{Q}^{\beta}, +\phi^{\beta}\right), \phi^{\alpha} \coloneqq e\mathcal{F}^{\alpha\beta}\wp_{\beta}\phi^{\alpha}\frac{\partial g(\wp,\wp)}{\partial\wp^{\alpha}} \equiv 2e\mathcal{F}^{\alpha\beta}\wp_{\beta}\wp_{\alpha}$$

$$i_{(0,\phi)}\theta = \omega_{\chi} \wedge i_{\phi}\omega_{\wp}, di_{\phi}\omega_{\wp} \equiv \frac{\partial\phi^{\alpha}}{\partial\wp^{\alpha}}\omega_{\wp}$$

Ecuaciones de Yang – Mills – Vlasov

$$\frac{d\chi^{\alpha}}{ds} = \wp^{\alpha}, \frac{d\wp^{\alpha}}{ds} = \mathcal{Q}^{\alpha} + q\otimes\mathcal{F}^{\alpha\beta}\wp_{\beta}, \frac{dq^{\alpha}}{ds} = \hat{\mathcal{Q}}^{\alpha} \coloneqq -\wp^{\alpha}[\mathrm{A}_{\alpha}, q]^{\alpha}$$

$$\mathcal{L}_{\chi}f \equiv \wp^{\alpha}\frac{\partial f}{\partial\chi^{\alpha}} + \left(\mathcal{Q}^{\alpha} + q\otimes\mathcal{F}^{\alpha\beta}\wp_{\beta}\right)\frac{\partial f}{\partial\wp^{\alpha}} + \mathcal{Q}^{\alpha}\frac{\partial f}{\partial q^{\alpha}}$$

$$\mathcal{L}_{\chi_{\mathbb{I}}}f_{\mathbb{I}} = 0, \chi_{\mathbb{I}} \coloneqq (\wp, \mathcal{Q} + \phi_{\mathbb{I}}), \phi^{\alpha}_{\mathbb{I}} \coloneqq e_{\mathbb{I}}\mathcal{F}^{\alpha\beta}\wp_{\alpha}$$

$$g_{\alpha\beta}\wp^{\alpha}\wp^{\beta} = -m^{2}_{\mathbb{I}}$$

$$f_{m_{\mathbb{I}}}(\chi^\alpha,\wp^i)=f_{\mathbb{I}}\left(\chi^\alpha,\wp^i,\wp^0(\wp^i)\right)\frac{\partial\wp^0}{\partial\wp^i}=-\frac{\wp^i}{\wp^0},\frac{\partial\wp^0}{\partial\chi^\alpha}=-\frac{\wp^\lambda\wp^\mu}{2\wp^0}\frac{\partial g_{\lambda\mu}}{\partial\chi^\alpha}$$

$$\wp^\alpha\frac{\partial f_{m_{\mathbb{I}}}}{\partial\chi^\alpha}+\psi^i\frac{\partial f_{m_{\mathbb{I}}}}{\partial\wp^i}=\wp^\alpha\left(\frac{\partial f_{\mathbb{I}}}{\partial\chi^\alpha}-\frac{\wp^\lambda\wp^\mu}{2\wp^0}\frac{\partial g_{\lambda\mu}}{\partial\chi^\alpha}\frac{\partial f_{\mathbb{I}}}{\partial\wp^0}\right)+\psi^i\left(\frac{\partial f_{\mathbb{I}}}{d\wp^i}-\frac{\wp^i}{\wp^0}\frac{\partial f_{\mathbb{I}}}{\partial\wp^0}\right)$$

$$\mathcal{Q}^\alpha\wp_\alpha+\frac{\wp^\alpha\wp^\lambda\wp^\mu}{2\wp^0}\frac{\partial g_{\lambda\mu}}{\partial\chi^\alpha}\phi^\alpha\wp_\alpha$$

$$\wp^0\frac{\partial f_{m_{\mathbb{I}}}}{\partial\chi^0}+\wp^i\frac{\partial f_{m_{\mathbb{I}}}}{\partial\chi^i}+\psi^i\frac{\partial f_{m_{\mathbb{I}}}}{\partial\wp^i}\equiv\wp^\alpha\frac{\partial f_{\mathbb{I}}}{\partial\chi^\alpha}+\mathcal{Q}^\alpha\frac{\partial f_{\mathbb{I}}}{\partial\wp^i}+\phi^\alpha\frac{\partial f_{\mathbb{I}}}{d\wp^\alpha}\equiv\mathcal{L}_\chi f_{\mathbb{I}}$$

$$\frac{\partial f_{m_{\mathbb{I}}}}{\partial\chi^0}+\frac{\wp^i}{\wp^0}\frac{\partial f_{m_{\mathbb{I}}}}{\partial\chi^i}+\frac{\psi^i}{\wp^0}\frac{\partial f_{m_{\mathbb{I}}}}{\partial\wp^i},\psi^i\equiv-\Gamma^i_{\lambda\mu}\wp^\lambda\wp^\mu+\phi^i_{\mathbb{I}}$$

$$\nabla_{\alpha_1}\mathcal{M}^{\alpha_1\alpha_2\cdots\alpha_\wp}=\iiint_{\mathcal{P}_\chi}(\mathcal{L}_\chi f)(\chi,\wp)\wp^{\alpha_2}\cdots\wp^{\alpha_\wp}\mu_\wp$$

Ecuación de Liouville–Vlasov

$$\frac{df(\chi(\lambda),\wp(\lambda))}{d\lambda},\frac{d\chi^\alpha}{d\lambda}=\wp^\alpha,\frac{d\wp^\alpha}{d\lambda}=\mathcal{Q}^\alpha\equiv\Gamma^\alpha_{\lambda\mu}\wp^\lambda\wp^\mu$$

$$\chi^\alpha(\lambda,\xi^i,\pi^\alpha),\wp^\alpha(\lambda,\xi^i,\pi^\alpha)$$

$$\xi^0=\chi^0(0,\xi^i,\pi^\alpha),\xi^i=\chi^i(0,\xi^i,\pi^\alpha),\pi^\alpha=\wp^\alpha(0,\xi^i,\pi^\alpha)$$

$$f(\chi,\wp)\equiv\bar{f}(\xi(\chi,\wp),\pi(\chi,\wp))$$

Espacio – tiempo cuántico de Robertson–Walker

$$g\equiv-dt^2+\mathcal{R}^2(t)\sigma^2\equiv\gamma_{ij}\,d\chi^i d\chi^j$$

$$\mathcal{L}_\chi f\equiv\wp^\alpha\frac{\partial f}{\partial\chi^\alpha}-\mathcal{R}\mathcal{R}'\gamma_{ij}\wp^i\wp^j\frac{\partial f}{\partial\wp^0}-2\mathcal{R}^{-1}\mathcal{R}'\wp^0\wp^i\frac{\partial f}{\partial\wp^i}\mathcal{R}^2\gamma_{ij}\wp^i\wp^j=(\wp^0)^2-m^2$$

$$\wp^0\frac{\partial f_m}{\partial t}-\mathcal{R}^{-1}\mathcal{R}'\{(\wp^0)^2-m^2\}\frac{\partial f_m}{\partial\wp^0}$$

$$\mathcal{R}\frac{\partial f_m}{\partial\mathcal{R}}-\{(\wp^0)^2-m^2\}\frac{1}{\wp^0}\frac{\partial f_m}{\partial\wp^0},\frac{d}{d\mathcal{R}}=-\frac{\wp^0 d\wp^0}{(\wp^0)^2-1}=d\lambda\log\mathcal{R}+\frac{1}{2}\log((\wp^0)^2-m^2)$$

$$=constant,\mathcal{R}^2\{(\wp^0)^2-m^2\}=constant$$

$$(\wp^0)^2\geq m^2+\mathfrak{K}\mathcal{R}^{-2}(t_0),(\wp^0)^2\geq m^2+\mathfrak{K}\mathcal{R}^{-2}(t)$$

Estimaciones de energía

$$g\equiv-\mathcal{N}^2dt^2+g_{ij}\theta^i\theta^j,\theta^i\equiv\ d\chi^i+\beta^i dt$$

$$\hat{e} = e_{ij}d\chi^i d\chi^j + dt^2 + e_{ij}d\wp^i d\wp^j + (d\wp^0)^2$$

$$\mathcal{P}_{\chi,t} \equiv \mathcal{T}_{\chi,t}\mathcal{V} \cap \{0 < m^2 \leq \mathcal{N}^2(\wp^0)^2 - g_{ij}\wp^i\wp^j \leq \mathcal{M}^2, \wp^0\}$$

$$m^2 = \mathcal{N}^2(\wp^0)^2 - g_{ij}\wp^i\wp^j, \mathcal{N}^2(\wp^0)^2 - g_{ij}\wp^i\wp^j = \mathcal{M}^2$$

$$\mathcal{L}_\chi(\hbar f^2\theta) \equiv di_\chi(\hbar f^2\theta) \iiint\limits_{\hat{\mathcal{V}}_\mathcal{T}} \mathcal{L}_\chi(\hbar f^2\theta) = \iiint\limits_{\partial\hat{\mathcal{V}}_\mathcal{T}} i_\chi(\hbar f^2\theta) \equiv \iiint\limits_{\partial\hat{\mathcal{V}}_\mathcal{T}} \hbar f^2 i_\chi\theta$$

$$\mathcal{L}_\chi(\hbar f^2\theta) = (\mathcal{L}_\chi\hbar)f^2\theta, i_\chi\theta = \wp^0\mathcal{N}_t^2\omega_{\bar{g}_t} \wedge \omega_{\wp,\bar{g}_t}|\widehat{\mathcal{M}}_t\rangle$$

$$2g^{\alpha\beta}\wp^\beta d\wp^\alpha + dg_{\alpha\beta}\wp^{\alpha\beta}$$

$$\mathfrak{E}_\hbar(t) \coloneqq \iiint\limits_{\widehat{\mathcal{M}}_t} \hbar f^2 i_\chi\theta \equiv \iiint\limits_{\widehat{\mathcal{M}}_t} \hbar f^2\wp^0\mathcal{N}_t\mu_{g_t}\mu_\wp$$

$$\mu_{g_t} = (\det\bar{\bar{g}}_t)^{1/2}d\chi^1 \cdots d\chi^\eta, \mu_\wp = (\det g)^{1/2}d\wp^0 d\wp^1 \cdots d\wp^\eta$$

$$\mathfrak{E}_\hbar(\mathcal{T}) - \mathfrak{E}_\hbar(0) = \iiint\limits_0^\mathcal{T} \iiint\limits_{\widehat{\mathcal{M}}_t} (\mathcal{L}_\chi\hbar)f^2\mathcal{N}_t\mu_{\bar{\bar{g}}_t}\mu_\wp dt$$

$$\mathcal{L}_\chi\hbar = \mathcal{P}(v_\alpha\wp^\alpha)^{\mathcal{P}-1}\chi^\mathrm{A}\partial_\mathrm{A}(v_\lambda\wp^\lambda)$$

$$\chi^\mathrm{A}\partial_\mathrm{A}(v_\lambda\wp^\lambda) = \wp^\mu\wp^\lambda\left(\frac{\partial}{\partial\chi^\mu}v_\lambda - v_\alpha\Gamma^\alpha_{\lambda\mu}\right) \equiv \wp^\mu\wp^\lambda\nabla_\mu v_\lambda$$

$$(\mathcal{L}_v g)_{\mu\lambda} \equiv \nabla_\mu v_\lambda + \nabla_\lambda v_\mu$$

$$\mathcal{L}_\chi\hbar = \frac{1}{2}\mathcal{P}(v_\alpha\wp^\alpha)^{\mathcal{P}-1}\wp^\mu\wp^\lambda(\mathcal{L}_v g)_{\mu\lambda}$$

$$\mathcal{L}_\chi\hbar = -\mathcal{P}(\wp^0)^{\mathcal{P}-1}\wp^\mu\wp^\lambda\Gamma^0_{\lambda\mu}$$

$$\mathfrak{C} = \sup_{\mathcal{V}_\mathcal{T}}\left|\mathcal{P}\frac{\wp^\lambda\wp^\mu}{(\wp^0)^2}\Gamma^0_{\lambda\mu}\right|, \mathfrak{E}_\hbar(t) \leq \mathfrak{E}_\hbar(0)e^{\mathfrak{C}t}$$

$$\mathfrak{E}_\hbar(t) \leq \mathfrak{E}_\hbar(0) + \mathfrak{C}\iiint\limits_0^t \mathfrak{E}_\hbar(\tau)d\tau$$

$$|\partial_\chi f|^2 \coloneqq e^{\alpha\beta}\frac{\partial f}{\partial\chi^\alpha}\frac{\partial f}{\partial\chi^\beta}, |\partial_\wp f|^2 \coloneqq e^{\alpha\beta}\frac{\partial f}{\partial\wp^\alpha}\frac{\partial f}{\partial\wp^\beta}$$

$$\mathcal{L}_\chi|\partial_\chi f|^2 = q_\chi(\partial f, \partial f), \mathcal{L}_\chi|\partial_\wp f|^2 = q_\wp(\partial f, \partial f)$$

$$|q_\chi(\partial f, \partial f)| \leq \mathfrak{C}\{(\wp^0)^2|\partial_\chi f||\partial_\wp f|\}, |q_\wp(\partial f, \partial f)| \leq \mathfrak{C}\left\{|\partial_\chi f||\partial_\wp f| + \wp^0|\partial_\wp f|^2\right\}$$

$$\mathcal{L}_\chi|\partial_\chi f|^2 \equiv 2e^{\alpha\beta}\frac{\partial f}{\partial\chi^\alpha}\chi^{\mathrm{A}}\frac{\partial f}{\partial\chi^{\mathrm{A}}}\frac{\partial f}{\partial\chi^\beta}f$$

$$\mathcal{L}_\chi|\bar{\bar{\partial}}f|^2 \equiv 2e^{\alpha\beta}\frac{\partial f}{\partial\chi^\alpha}\left\{\frac{\partial}{\partial\chi^\beta}\left(\chi^{\mathrm{A}}\frac{\partial f}{\partial\chi^{\mathrm{A}}}\right)-\frac{\partial\chi^{\mathrm{A}}}{\partial\chi^\beta}\frac{\partial f}{\partial\chi^{\mathrm{A}}}\right\}$$

$$\mathcal{L}_\chi|\partial_\chi f|^2 = q_\chi(\partial f,\partial f) \equiv -2e^{\alpha\beta}\frac{\partial Q^\beta}{\partial\chi^\alpha}\frac{\partial f}{\partial\chi^\beta}\frac{\partial f}{\partial\wp^\lambda}$$

$$\mathcal{L}_\chi|\partial_\wp f|^2 \equiv 2e^{\alpha\beta}\frac{\partial f}{\partial\wp^\alpha}\left\{\frac{\partial}{\partial\wp^\beta}\left(\chi^{\mathrm{A}}\frac{\partial f}{\partial\chi^{\mathrm{A}}}\right)-\frac{\partial\chi^{\mathrm{A}}}{\partial\wp^\beta}\frac{\partial f}{\partial\chi^{\mathrm{A}}}\right\}$$

$$\mathcal{L}_\chi|\partial_\wp f|^2 = q_\wp(\partial f,\partial f) \coloneqq -2e^{\alpha\beta}\frac{\partial f}{\partial\wp^\alpha}\left\{\frac{\partial f}{\partial\chi^\beta}+\frac{\partial Q^\gamma}{\partial\wp^\beta}\frac{\partial f}{\partial\wp^\gamma}\right\}$$

$$Q^\alpha = -\Gamma^\alpha_{\lambda\mu}\wp^\lambda\wp^\mu, \frac{\partial Q^\alpha}{\partial\chi^i} = -\wp^\lambda\wp^\mu\frac{\partial\Gamma^\alpha_{\lambda\mu}}{\partial\chi^j}, \frac{\partial Q^\alpha}{\partial\wp^\gamma} = -2\wp^\mu\Gamma^\alpha_{\lambda\mu}$$

$$\mathfrak{E}^{(1)}_{\mathcal{P}}(t) \coloneqq \iiint\limits_{\widehat{\mathcal{M}}_t}\left\{(\wp^0)^{\mathcal{P}}|\partial_\chi f|^2+(\wp^0)^{\mathcal{P}+2}|\partial_\wp f|^2\right\}\mathcal{N}\mu_{\bar{g}}\mu_\wp$$

$$\mathfrak{E}^{(1)}_{\mathcal{P}}(t) \le \mathfrak{E}^{(1)}_{\mathcal{P}}(0)e^{\mathfrak{C}t}, (\wp^0)^{\mathcal{P}-1}|\partial_\chi f|^2 i_\chi\theta + (\wp^0)^{\mathcal{P}'-1}|\partial_\wp f|^2 i_\chi\theta$$

$$\mathfrak{E}^{(1)}_{\mathcal{P}}(t)-\mathfrak{E}^{(1)}_{\mathcal{P}}(0) = \iiint\limits_0^{\mathcal{T}}\iiint\limits_{\widehat{\mathcal{M}}_t}\mathcal{L}_\chi\left\{(\wp^0)^{\mathcal{P}-1}|\partial_\chi f|^2+(\wp^0)^{\mathcal{P}'-1}|\partial_\wp f|^2\right\}\theta$$

$$\mathcal{L}_\chi(\hbar\mathcal{F}\theta) = \{(\mathcal{L}_\chi\hbar)\mathcal{F}+\hbar(\mathcal{L}_\chi\mathcal{F})\}\theta$$

$$\mathcal{L}_\chi\left\{(\wp^0)^{\mathcal{P}-1}|\partial_\chi f|^2\right\} = |\partial_\chi f|^2\mathcal{L}_\chi(\wp^0)^{\mathcal{P}-1}+(\wp^0)^{\mathcal{P}-1}\mathcal{L}_\chi|\partial_\chi f|^2$$

$$\mathcal{L}_\chi(\wp^0)^{\mathcal{P}-1} = (\mathcal{P}-1)(\wp^0)^{\mathcal{P}-2}\mathcal{L}_\chi\wp^0 \le \mathfrak{C}(\wp^0)^{\mathcal{P}}$$

$$(\wp^0)^{\mathcal{P}-1}\left|\mathcal{L}_\chi|\partial_\chi f|^2\right| \le \mathfrak{C}(\wp^0)^{\mathcal{P}/2}|\partial_\chi f|(\wp^0)^{\mathcal{P}/2+1}|\partial_\wp f|$$

$$\mathcal{L}_\chi\left\{(\wp^0)^{\mathcal{P}'-1}|\partial_\wp f|^2\right\} \le \mathfrak{C}(\wp^0)^{\mathcal{P}'}|\partial_\wp f|^2+(\wp^0)^{\mathcal{P}'-1}|\partial_\wp f|\{|\wp^0||\partial_\wp f|+|\partial_\chi f|\}$$

$$\mathcal{L}_\chi\left\{(\wp^0)^{\mathcal{P}'-1}|\partial_\wp f|^2\right\} \le \mathfrak{C}(\wp^0)^{\mathcal{P}'}|\partial_\wp f|^2+(\wp^0)^{\mathcal{P}'/2}|\partial_\wp f|(\wp^0)^{\mathcal{P}/2}|\partial_\chi f|$$

$$\mathfrak{E}^{(1)}_{\mathcal{P}}(t)-\mathfrak{E}^{(1)}_{\mathcal{P}}(0) \le \mathfrak{C}\iiint\limits_0^{\mathcal{T}}\mathfrak{E}^{(1)}_{\mathcal{P}}(\tau)d\tau$$

$$\mathcal{L}_\chi|\partial^2 f|^2 = e^{\mathfrak{B}\mathfrak{C}}e^{\mathfrak{D}\mathfrak{E}}\partial_{\mathfrak{B}\mathfrak{D}}f\chi^{\mathrm{A}}\partial_{\mathrm{A}}\partial_{\mathfrak{C}\mathfrak{D}}f = e^{\mathfrak{B}\mathfrak{C}}e^{\mathfrak{D}\mathfrak{E}}\partial_{\mathfrak{B}\mathfrak{D}}f\left[\partial_{\mathfrak{C}\mathfrak{D}}(\chi^{\mathrm{A}}\partial_{\mathrm{A}}f)-\partial_{\mathfrak{C}}(\partial_{\mathfrak{D}}\chi^{\mathrm{A}})\partial_{\mathfrak{A}}f-\partial_{\mathfrak{C}}\chi^{\mathrm{A}}\partial_{\mathfrak{A}\mathfrak{D}}f\right]$$

$$\mathcal{L}_\chi\frac{\partial\mathcal{F}}{\partial\chi^{\mathrm{A}}} = \frac{\partial}{\partial\chi^{\mathrm{A}}}\mathcal{L}_\chi\mathcal{F}-\frac{\partial\mathcal{F}}{\partial\chi^{\mathfrak{B}}}\frac{\partial\chi^{\mathfrak{B}}}{\partial\chi^{\mathrm{A}}}$$

$$\mathcal{L}_\chi \frac{\partial^{\mathfrak{K}}\mathcal{F}}{\partial\chi^{A_1}\cdots\partial\chi^{A_k}} \equiv \frac{\partial^{\mathfrak{K}}(\mathcal{L}_\chi\mathcal{F})}{\partial\chi^{A_1}\cdots\partial\chi^{A_k}} + \sum_{\substack{0\le \ell \le k\\ 0<i<j}} \mathfrak{C}_\ell \frac{\partial^{\mathfrak{K}-\ell}\mathcal{F}}{\partial\chi^{A_1}\cdots\partial\chi^{A_{k-\ell}}\partial\chi^{\mathfrak{B}}} \frac{\partial^{\ell}\chi^{\mathfrak{B}}}{\partial\chi^{A_{k-\ell+1}}\cdots\partial\chi^{A_k}}$$

$$\left|\frac{\partial^{\ell}\chi^{\mathfrak{B}}_{\wp}}{\partial\chi^{A_{k-\ell+1}}\cdots\partial\chi^{A_k}}\right| \le \mathfrak{C}|\wp|^2, \left|\frac{\partial^{\ell}\chi^{\mathfrak{B}}_{\wp}}{\partial\chi^{A_{k-\ell+1}}\cdots\partial\chi^{A_k}}\right| \le \mathfrak{C}|\wp|, \left|\frac{\partial^{\ell}\chi^{\mathfrak{B}}_{\wp}}{\partial\chi^{A_{k-\ell+1}}\cdots\partial\chi^{A_k}}\right| \le \mathfrak{C}$$

$$\mathfrak{E}^{(\delta)}_{\mathcal{P}}(t) := \sum_{|k|=1\cdots\delta} \iiint_{\widehat{\mathcal{M}}_t} (\wp^0)^{\mathcal{P}+k_\wp}\left|\partial^{\mathfrak{K}} f\right|^2 \mathcal{N}\mu_{\bar{g}}\mu_\wp$$

$$\mathfrak{E}^{(\delta)}_{\mathcal{P}}(t) \le \mathfrak{E}^{(\delta)}_{\mathcal{P}}(0)e^{\mathfrak{C}t}$$

$$\sum_{|k|=1\cdots\delta} (\wp^0)^{\mathcal{P}+k_\wp}\left|\partial^{\mathfrak{K}} f\right|^2 i_\chi\theta, \mathfrak{E}^{(\delta)}_{\mathcal{P}}(t) - \mathfrak{E}^{(\delta)}_{\mathcal{P}}(0) = \iiint_0^t \iiint_{\widehat{\mathcal{M}}_t} \mathcal{L}_\chi \left\{\sum_{|k|=1\cdots\delta} (\wp^0)^{\mathcal{P}+k_\wp}\left|\partial^{\mathfrak{K}} f\right|^2\right\}\theta$$

$$\mathfrak{E}^{(\delta)}_{\mathcal{P}}(t) \le \mathfrak{E}^{(\delta)}_{\mathcal{P}}(0) \le \mathfrak{C}\iiint_0^{\mathcal{T}} \mathfrak{E}^{(\delta)}_{\mathcal{P}}(\tau)d\tau\Big|$$

$$|\mathcal{T}(\chi,t)|^2 := \left|\iiint_{\mathcal{P}_{\chi,t}} f(\chi,t,\wp)\wp\otimes\wp\mu_\wp\right|^2 \le A\iiint_{\mathcal{P}_{\chi,t}} f^2(\chi,t,\wp)|\wp|^{\mathcal{P}}\mu_\wp$$

$$A := \iiint_{\mathcal{P}_{\chi,t}} |\wp|^{4-\mathcal{P}}\mu_\wp, \wp^0 \ge \left\{\mathcal{N}^{-2}g_{ij}\wp^i\wp^j + m^2\right\}^{1/2} \ge m$$

$$\|\mathcal{T}\|^2_{\mathcal{L}^2(\mathcal{M}_t)} \le A\mathfrak{C}^{-\mathcal{P}} \iiint_{\mathcal{M}_t}\iiint_{\mathcal{P}_{\chi,t}} f^2(\chi,t,\wp)|\wp^0|^{\mathcal{P}}\mathcal{N}\mu_{\bar{g}}\mu_\wp \equiv A\mathfrak{C}^{-\mathcal{P}}\mathfrak{E}_{\mathcal{P}}(t)$$

$$\left|\bar{\partial}^{\mathfrak{K}}\mathcal{T}(\chi,t)\right|^2 = \left|\iiint_{\mathcal{P}_{\chi,t}} \bar{\partial}^{\mathfrak{K}} f(\chi,t,\wp)\wp\otimes\wp\mu_\wp\right|^2$$

$$\mathcal{R}_{\alpha\beta} = \rho_{\alpha\beta}, \rho_{\alpha\beta} \equiv \mathcal{T}_{\alpha\beta} - \frac{1}{1-\eta}g_{\alpha\beta}\mathcal{T}^{\lambda}_{\lambda}$$

$$\mathcal{T}_{\alpha\beta}(\chi) := \iiint_{\mathcal{P}_{\chi,t}} f(\chi,\wp)\wp_\alpha\wp_\beta\mu_\wp$$

$$\mathcal{L}_\chi f \equiv \wp^\alpha\frac{\partial f}{\partial\wp^\alpha} + \mathcal{Q}^\alpha\frac{\partial f}{\partial\wp^\alpha}, \mathcal{Q}^\alpha := -\Gamma^\alpha_{\lambda\mu}\wp^\lambda\wp^\mu$$

$$\mathcal{R}(g) - |\mathcal{K}|^2_{\bar{g}} + \left(tr_{\bar{g}}\mathcal{K}\right)^2 = 2\mathcal{N}^2\mathcal{T}^{00}\bar{\bar{\nabla}}\otimes\mathcal{K} - \bar{\bar{\nabla}}tr\mathcal{K} = \mathcal{N}\mathcal{T}^{0i}$$

$$f\left(\chi, \wp^0, \wp^i\right) = f(\chi, \wp^0, -\widehat{\wp}), \widehat{\wp} = \left(\wp^{\mathfrak{I}}\right)$$

$$\mathcal{T}^{0i} := \iiint\limits_{\mathcal{P}_\chi} f(\wp^0, -\widehat{\wp})\wp^0\wp^i\omega$$

$$\mathcal{R}_{\alpha\beta}^{(\hbar)} \equiv -\frac{1}{2}g^{\lambda\mu}\partial_{\lambda\mu}^2 g_{\alpha\beta} + \mathfrak{H}_{\alpha\beta}(g)(\partial g, \partial g) = \rho_{\alpha\beta}$$

$$\mathcal{R}_{\alpha\beta}^{(\hbar)}(g_2) = \rho_{\alpha\beta}(f_1), \mathcal{L}_{\chi_1}f_2$$

Sistema Einstein–Maxwell–Vlasov

$$\mathcal{L}_\chi f := \wp^\alpha \frac{\partial f}{\partial \chi^\alpha} + (\mathcal{Q}^\alpha + \phi^\alpha)\frac{\partial f}{\partial \wp^\alpha}$$

$$\mathfrak{J}^\alpha = e \iiint\limits_{\mathcal{P}_\chi} f(\chi, \wp)\wp^\alpha \omega_\chi$$

$$\mathfrak{J}^\alpha = \sum_{\mathfrak{J}} e_{\mathfrak{J}} \iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} f_{m_{\mathfrak{J}}}(\chi, \wp)\wp^\alpha \omega_{m_{\mathfrak{J}},\wp}$$

$$\delta^{\alpha\beta} = \mathcal{T}^{\alpha\beta} \equiv \mathcal{F}^{\alpha}{}_{\lambda}\mathcal{F}^{\beta\gamma} - \frac{1}{4}g^{\alpha\beta}\mathcal{F}_{\lambda\mu}\mathcal{F}^{\lambda\mu} + \iiint\limits_{\mathcal{P}_\chi} f(\chi, \wp)\wp^\alpha\wp^\beta\omega_\wp$$

$$\frac{\partial \mathfrak{J}^\alpha}{\partial \chi^\alpha} = \sum_{\mathfrak{J}} e_{\mathfrak{J}} \iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} \phi^\alpha \frac{\partial f_{m_{\mathbb{I}}}(\chi, \wp)}{\partial \wp^\alpha}\omega_{m_{\mathfrak{J}},\wp}$$

$$\iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} \phi^\alpha \frac{\partial f_{m_{\mathbb{I}}}(\chi, \wp)}{\partial \wp^\alpha}\omega_{m_{\mathfrak{J}},\wp} = -\iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} f_{m_{\mathbb{I}}}(\chi, \wp)\frac{\partial \phi^\alpha}{\partial \wp^\alpha}\omega_{m_{\mathfrak{J}},\wp}$$

$$\nabla^\alpha \tau^{\alpha\beta} = \mathfrak{J}^\lambda \mathcal{F}_\lambda^\beta$$

$$\nabla_\alpha \iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} f(\chi, \wp)\wp^\alpha\wp^\beta\omega_{m_{\mathfrak{J}},\wp} = \iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} \phi^\alpha \frac{\partial f(\chi, \wp)}{\partial \chi^\alpha}\omega_{m_{\mathfrak{J}},\wp}$$

$$\nabla_\alpha \iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} f_{m_{\mathbb{I}}}(\chi, \wp)\wp^\alpha\wp^\beta\omega_{m_{\mathfrak{J}},\wp} = -\iiint\limits_{\mathcal{P}_{m_{\mathfrak{J}},\chi}} \phi^\alpha \frac{\partial f_{m_{\mathbb{I}}}(\chi, \wp)}{\partial \wp^\alpha}\omega_{m_{\mathfrak{J}},\wp}$$

$$\phi^\alpha \frac{\partial f_{m_{\mathbb{I}}}(\chi, \wp)}{\partial \wp^\alpha}\wp^\beta \equiv \frac{\partial}{\partial \wp^\alpha}\left[\phi^\alpha f_{m_{\mathbb{I}}}(\chi, \wp)\wp^\beta\right] - \frac{\partial \phi^\alpha}{\partial \wp^\alpha}f_{m_{\mathbb{I}}}\wp^\beta - \phi^\beta f_{m_{\mathbb{I}}}$$

$$\nabla_\alpha \sum_{\mathfrak{I}} e_{\mathfrak{I}} \iiint_{\mathcal{P}_{m_{\mathfrak{I}},\chi}} f_{m_{\mathbb{I}}}(\chi,\wp)\wp^\alpha \wp^\beta \omega_{m_{\mathfrak{I}},\wp} = \sum_{\mathfrak{I}} e_{\mathfrak{I}} \iiint_{\mathcal{P}_{m_{\mathfrak{I}},\chi}} \phi^\beta f_{m_{\mathbb{I}}} \omega_{m_{\mathfrak{I}},\wp}$$

Ecuaciones de Boltzmann

$$\mathcal{L}_\chi f = \mathfrak{I} f$$

$$\wp' + q' = \wp + q$$

$$\xi' \wedge \left(\bigwedge_\alpha \left(d(\wp'^\alpha + q'^\alpha) \right) \right) = \omega_{\wp'} \wedge \omega_{q'}$$

$$(\mathfrak{I} f)(\chi,\wp) \equiv \iiint_{\mathcal{P}_\chi(q)} \iiint_{\Sigma_{\wp q}} [f(\chi,\wp')f(\chi,q') - f(\chi,\wp)f(\chi,q)] \mathrm{A}(\chi,\wp,q,\wp',q')\xi' \wedge \omega_{q'}$$

$$\mathrm{A}(\chi,\wp,q,\wp',q') = \mathrm{A}(\chi,\wp',q',\wp,q)$$

$$\mathrm{A}(\chi,\wp,q,\wp',q')\xi' = \mathcal{S}(\chi,\wp,q,\theta,\varphi) \sin\theta \, d\theta \wedge d\varphi$$

$$\wp^\alpha q_\alpha = \wp'^\alpha q'_\alpha = m^2 - 2\lambda^2 \leq -m^2$$

$$\alpha = \frac{1}{2}(-\wp^\alpha q_\alpha - m^2)^{1/2} = \frac{1}{2} g(\wp - q, \wp - q)^{1/2}$$

$$\xi' = (2\lambda)^{-1} \alpha \sin\theta \, d\theta \wedge d\varphi$$

$$\mathcal{S}(\wp,q,\theta,\varphi) = \mathrm{A}(\chi,\wp,q,\wp',q')(2\lambda)^{-1}\alpha$$

$$\wp'_{\Sigma_{\wp q}} = q'_{\Sigma_{\wp q}} = \left\{ -\frac{1}{2} g(\wp + q)^{\frac{1}{2}} + \frac{1}{2} g(\wp - q, \wp - q)^{1/2} (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \right\}$$

Leyes de conservación de la energía

$$\mathcal{T}^{\alpha_1 \cdots \alpha_\eta}(\chi) := \iiint_{\mathcal{P}_\chi} \wp^{\alpha_1} \cdots \wp^{\alpha_\eta} f(\chi,\wp)\omega_\wp$$

$$\nabla^\alpha \mathcal{T}^{\alpha_2 \cdots \alpha_\eta} \equiv \iiint_{\mathcal{P}_\chi} \wp^{\alpha_2} \cdots \wp^{\alpha_\eta} \mathcal{L}_\chi f(\chi,\wp)\omega_\wp$$

$$\mathcal{P}^\alpha := \iiint_{\mathcal{P}_\chi} \wp^\alpha f(\chi,\wp)\omega_\wp$$

$$\nabla_\alpha \mathcal{P}^\alpha \equiv \iiint_{\mathcal{P}_\chi} \mathcal{L}_\chi f(\chi,\wp)\omega_\wp = \iiint_{\mathcal{P}_\chi} (\mathfrak{I} f)(\chi,\wp)\omega_\wp$$

$$\mathcal{T}^{\alpha\beta} := \iiint\limits_{\mathcal{P}_\chi} \wp^\alpha \wp^\beta f(\chi, \wp)\omega_\wp$$

$$\nabla_\alpha \mathcal{T}^{\alpha\beta} \equiv \iiint\limits_{\mathcal{P}_\chi} \wp^\beta (\mathfrak{T} f)(\chi, \wp)\omega_\wp$$

Cuestiones termodinámicas relativas a agujeros negros cuánticos

$$\mathcal{H}^\alpha(\chi) := \mathcal{k} \iiint\limits_{\mathcal{P}_\chi} \wp^\alpha (f \log f)(\chi, \wp)\omega_\wp$$

$$(\nabla_\alpha \mathcal{H}^\alpha)(\chi) \equiv -\mathfrak{K} \iiint\limits_{\mathcal{P}_\chi} \big(\mathcal{L}_\chi [f \log f]\big)(\chi, \wp)\omega_\wp \equiv -\mathfrak{K} \iiint\limits_{\mathcal{P}_\chi} \big(\mathcal{L}_\chi f [\log f + 1]\big)(\chi, \wp)\omega_\wp$$

$$\iiint\limits_{\mathcal{P}_\chi(q)} \iiint\limits_{\Sigma_{\wp q}} [f(\chi, \wp')f(\chi, q') - f(\chi, \wp)f(\chi, q)] \mathrm{A}(\chi, \wp, q, \wp', q')\xi' \wedge \omega_q \wedge \omega_\wp$$

$$(\nabla_\alpha \mathcal{H}^\alpha)(\chi) \equiv -\mathfrak{K} \iiint\limits_{\mathcal{P}_\chi(q)} \iiint\limits_{\Sigma_{\wp q}} [f(\chi, \wp')f(\chi, q') - f(\chi, \wp)f(\chi, q)](\log f)(\chi, \wp)\, \mathrm{A}(\chi, \wp, q, \wp', q')\xi'$$

$$\wedge\, \omega_q \wedge \omega_\wp$$

$$\mathrm{A}(\chi, \wp, q, \wp', q') = \mathrm{A}(\chi, \wp', q', \wp, q)$$

$$\nabla_\alpha \mathcal{H}^\alpha = -\frac{\mathfrak{K}}{4} \iiint\limits_{\mathcal{P}_\chi(q)} \iiint\limits_{\Sigma_{\wp q}} [f(\wp')f(q') - f(\wp)f(q)] \log \frac{f(\wp)f(q)}{f(\wp')f(q')} \mathrm{A}(\wp, q, \wp', q')\xi' \wedge \omega_q \wedge \omega_\wp$$

$$\iiint\limits_{\mathcal{M}_\mathcal{T}} \mathcal{H}^0 \mathcal{N} \mu_{\bar{g}} \geq \iiint\limits_{\mathcal{M}_0} \mathcal{H}^0 \mathcal{N} \mu_{\bar{g}}$$

$$\frac{d(\mathcal{R}^3 \Sigma)}{dt}, f(\chi, \wp) = \alpha(\chi) \exp(\beta_\alpha(\chi)\wp^\alpha)$$

$$\mathcal{L}_\chi f \equiv e^{\beta_\lambda \wp^\lambda} \left\{ \wp^\alpha \frac{\partial \alpha}{\partial \chi^\alpha} + \frac{1}{2} \wp^\alpha \wp^\beta \big(\nabla_\alpha \beta_\lambda + \nabla_\beta \beta_\alpha\big) \right\}$$

$$\lambda = (\mathfrak{K}\mathfrak{T})^{-1}, \nabla_\alpha \mathfrak{T}^{\alpha\beta\mu} \equiv \left\langle \mathfrak{J}^{\beta\mu} \middle| \mathfrak{J}^\alpha_\alpha \middle| \mathfrak{T}^{\alpha\beta}_\beta \right\rangle = m^4 c^4$$

$$\mathfrak{T}^{\alpha\beta\mu} \equiv \hat{\mathfrak{T}}^{\alpha\beta\mu}(\mathfrak{T}^\nu, \mathfrak{T}^{\gamma\rho}), \mathfrak{J}^{\alpha\beta} \equiv \hat{\mathfrak{J}}^{\alpha\beta}(\mathfrak{T}^\nu, \mathfrak{T}^{\gamma\rho})$$

$$\mathcal{H}^\alpha \equiv \hat{\mathcal{H}}^\alpha(\mathfrak{T}^\nu, \mathfrak{T}^{\gamma\rho})$$

$$\mathfrak{T}^{\alpha} = r\mu^{\alpha}, \mathfrak{T}^{\alpha\beta} = t^{\langle\alpha\beta\rangle} + (\wp + \pi)\hbar^{\alpha\beta} + \frac{1}{c^4}\left(\mu^{\alpha}q^{\beta} + \mu^{\beta}q^{\alpha}\right) + \frac{e}{c^4}\mu^{\alpha}\mu^{\beta}, \mathfrak{T}^{\alpha\beta\gamma}$$

$$= (\mathfrak{C}_1 + \mathfrak{C}_2\pi)\mu^{\alpha}\mu^{\beta}\mu^{\gamma} + \frac{c^4}{6}(r - \mathfrak{C}_1 + \mathfrak{C}_2\pi)\left(g^{\alpha\beta}\mu^{\gamma} + g^{\beta\gamma}\mu^{\alpha} + g^{\alpha\gamma}\mu^{\beta}\right)$$

$$+ \mathfrak{C}_3\left(g^{\alpha\beta}q^{\gamma} + g^{\beta\gamma}q^{\alpha} + g^{\alpha\gamma}q^{\beta}\right) + \mathfrak{C}_4\left(t^{\langle\alpha\beta\rangle}\mu^{\gamma} + t^{\langle\beta\gamma\rangle}\mu^{\alpha} + t^{\langle\alpha\gamma\rangle}\mu^{\beta}\right)$$

$$- \frac{6}{c^4}\mathfrak{C}_5\left(\mu^{\alpha}\mu^{\beta}q^{\gamma} + \mu^{\beta}\mu^{\gamma}q^{\alpha} + \mu^{\alpha}\mu^{\gamma}q^{\beta}\right)$$

$$\mathfrak{J}^{\alpha\beta} = \mathfrak{B}_1\pi\left(g^{\alpha\beta} - \frac{4}{c^2}\mu^{\alpha}\mu^{\beta}\right)\mathfrak{B}_2 t^{\langle\alpha\beta\rangle} + \frac{1}{c^2}\mathfrak{B}_3\left(q^{\alpha}\mu^{\beta} + q^{\beta}\mu^{\alpha}\right)$$

$$\mathfrak{T}^{\alpha\mathrm{A}}(\chi) = \iiint \wp^{\alpha}\wp^{\mathrm{A}} f\omega_{\wp}, \pi^{\mathrm{A}}(\chi) = \iiint \mathcal{Q}\wp^{\mathrm{A}} f\omega_{\wp}$$

$$\wp^{\mathrm{A}} = \begin{cases} 1 \\ \wp^{\alpha_1}\wp^{\alpha_2}\dots\wp^{\alpha_{\mathrm{A}}} \end{cases}, \mathcal{F}^{\alpha\mathrm{A}} = \begin{cases} \mathcal{F}^{\alpha} \\ \mathcal{F}^{\alpha\alpha_1\cdots\alpha_{\mathrm{A}}} \end{cases}, \pi^{\mathrm{A}} = \begin{cases} 0 \\ \pi^{\alpha_1\cdots\alpha_{\mathrm{A}}} \end{cases}$$

$$f_{\eta} = e^{\chi/\hbar}, \chi = \sum_{\mathrm{A}=0}^{\eta}\mu'_{\mathrm{A}}(\chi)\wp^{\mathrm{A}}, \sum_{\mathfrak{B}=0}^{\eta}\mathcal{H}^{\alpha\mathfrak{A}\mathfrak{B}}(\mu'_{\mathfrak{C}})\partial_{\alpha}\mu'_{\mathfrak{B}} = \pi^{\mathrm{A}}(\mu'_{\mathfrak{C}})$$

$$\mathcal{H}^{\alpha\mathfrak{A}\mathfrak{B}} = \iiint \frac{1}{\mathfrak{K}}e^{(\chi/\hbar)}\wp^{\alpha}\wp^{\mathrm{A}}\wp^{\mathfrak{B}}\omega_{\wp}\det\left(\mathcal{H}^{\alpha\mathfrak{A}\mathfrak{B}}\partial_{\alpha}\phi\right)$$

$$\det\left(\mathcal{H}^{i\mathfrak{A}\mathfrak{B}}\eta_i - \lambda\mathcal{H}^{0\mathfrak{A}\mathfrak{B}}\right)\frac{2\eta - 1}{\gamma}\frac{\mathfrak{K}_{\eta+1}(\gamma)}{\mathfrak{K}_{\eta+2}(\gamma)} \leq \frac{\lambda_{max}^2}{c^4} \geq 1$$

Supersimetría de Yang – Mills, supermembranas y multidimensiones.

Cuestiones preliminares – espacio – tiempo cuántico multidimensional.

$$\chi(\xi): \xi^{\mu}(0 \leq \mu \leq \wp) \mapsto \chi^{\mathcal{M}}(0 \leq \mathcal{M} \leq \mathfrak{D} - 1)$$

$$\delta_{p-brane} = \iiint d^{\wp+1}\xi\mathcal{L}_{p-brane}, \mathcal{L}_{p-brane} = \mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}} + \mathcal{L}_{\mathfrak{C}_{\wp+1}}$$

$$\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}} = -\mathfrak{T}\sqrt{-\det\mathcal{G}_{\mu\nu}}, \mathcal{L}_{\mathfrak{C}_{\wp+1}} = \frac{1}{(\wp+1)!}\epsilon^{\mu_1\mu_2\cdots\mu_{\wp+1}}\mathfrak{C}_{\mu_1\mu_2\cdots\mu_{\wp+1}}$$

$$\mathcal{G}_{\mu\nu}(\xi) = \partial_{\mu}\chi^{\mathcal{M}}\partial_{\nu}\chi^{\mathcal{N}}\mathcal{G}_{\mathcal{M}\mathcal{N}}(\chi), \mathfrak{C}_{\mu_1\mu_2\cdots\mu_{\wp+1}}(\xi) = \partial_{\mu_1}\chi^{\mathcal{M}_1}\partial_{\mu_2}\chi^{\mathcal{M}_2}\cdots\partial_{\mu_{\wp+1}}\chi^{\mathcal{M}_{\wp+1}}\mathfrak{C}_{\mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_{\wp+1}}(\chi)$$

$$\mathcal{L}_{Poly} = -\frac{1}{2}\mathfrak{T}\sqrt{-\hbar}\left[\hbar^{-1\mu\nu}\partial_{\mu}\chi^{\mathcal{L}}\partial_{\nu}\chi^{\mathcal{M}}\mathcal{G}_{\mathcal{L}\mathcal{M}}(\chi) + 1 - \wp\right]$$

$$\hbar_{\mu\nu} = \partial_{\mu}\chi^{\mathcal{L}}\partial_{\nu}\chi^{\mathcal{M}}\mathcal{G}_{\mathcal{L}\mathcal{M}}(\chi)$$

$$\hbar_{\mu\nu} \propto \partial_{\mu}\chi^{\mathcal{L}}\partial_{\nu}\chi^{\mathcal{M}}\mathcal{G}_{\mathcal{L}\mathcal{M}}(\chi)$$

$$\chi^{\pm} = \frac{1}{\sqrt{2}}\left(\pm\chi^0 + \chi^{\mathfrak{D}-1}\right)\frac{\partial\mathcal{G}_{\mathcal{L}\mathcal{M}}}{\partial\chi^-}\frac{\partial\mathfrak{F}_{(\wp+2)}}{\partial\chi^-}$$

$$\mathcal{G}_{--} = 0, \mathcal{G}_{-\alpha} = 0, \mathfrak{F}_{\mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_{\wp+1}} = 0, \mathfrak{F}_{\alpha_1\alpha_2\cdots\alpha_{\wp+2}}$$

$$ds^2 = \mathrm{A}(\gamma,\chi^+)\left[2d\chi^+d\chi^- - 2\mathcal{V}(\gamma,\chi^+)d\chi^+d\chi^+ + 2\mathfrak{J}_\alpha(\gamma,\chi^+)d\chi^+d\gamma^\alpha + g_{\alpha\beta}(\gamma,\chi^+)d\gamma^\alpha d\gamma^\beta\right]$$

$$\mathfrak{F}_{(\wp+2)} = \frac{1}{(\wp+1)!}\mathfrak{F}_{+\alpha_1\alpha_2\cdots\alpha_{\wp+1}}(\gamma,\chi^+)\, d\chi^+ \wedge d\gamma^{\alpha_1} \wedge\cdots\wedge d\gamma^{\alpha_{\wp+1}}$$

$$\mathfrak{F}_{+\alpha_1\alpha_2\cdots\alpha_{\wp+1}} = \partial_{\alpha_1}\mathcal{V}_{\alpha_2\cdots\alpha_{\wp+2}} + (-1)^{\wp}\partial_{\alpha_2}\mathcal{V}_{\alpha_3\cdots\alpha_{\wp+2}\alpha_1} + \cdots + (-1)^{\wp}\partial_{\alpha_{\wp+1}}\mathcal{V}_{\alpha_1\cdots\alpha_{\wp}}$$

Morfología y sistematicidad a las partículas y antipartículas supermasivas y masivas e hiperpartículas.

$$\tau = \xi^0 \equiv \chi^+$$

$$\delta_{particle} = \iiint d\tau\left(\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}} - \mathfrak{C}_+(\chi) - \ddot{\chi}^-\mathfrak{C}_-(\chi) - \hat{\gamma}^\alpha\mathfrak{C}_\alpha(\chi)\right)$$

$$\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}} = -m\sqrt{-\mathrm{A}(\gamma,\tau)\left(2\grave{\chi}^- - 2\mathcal{V}(\gamma,\tau) + 2\mathfrak{J}_\alpha(\gamma,\tau)\bar{\bar{\gamma}}^\alpha + g_{\alpha\beta}(\gamma,\tau)\bar{\bar{\gamma}}^\alpha\bar{\bar{\gamma}}^\beta\right)}$$

$$\wp_- = \frac{m\mathrm{A}}{\sqrt{-\mathrm{A}\left(2\grave{\chi}^- - 2\mathcal{V} + 2\mathfrak{J}_\alpha\ddot{\bar{\bar{\gamma}}}^\alpha + g_{\alpha\beta}\ddot{\bar{\bar{\gamma}}}^\alpha\ddot{\bar{\bar{\gamma}}}^\beta\right)}} - \mathfrak{C}_- = -\frac{m^4\mathrm{A}}{\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}} - \mathfrak{C}_-$$

$$\wp_\alpha = \frac{m\mathrm{A}\left(g_{\alpha\beta}\bar{\bar{\gamma}}^\beta + \mathfrak{J}_\alpha\right)}{\sqrt{-\mathrm{A}(2\grave{\chi}^- - 2\mathcal{V} + 2\mathfrak{J}_c\bar{\bar{\gamma}}^c + g_{cd}\bar{\bar{\gamma}}^c\bar{\bar{\gamma}}^d)}} - \mathfrak{C}_\alpha = -(\wp_- + \mathfrak{C}_-)\left(g_{\alpha\beta}\bar{\bar{\gamma}}^\beta + \mathfrak{J}_\alpha\right)$$

$$\ddot{\bar{\bar{\gamma}}}^\alpha = \frac{\bar{g}^{\alpha\beta}\mathcal{P}_\beta}{\mathcal{P}_-} - \bar{\bar{\mathfrak{J}}}_\alpha, \grave{\chi}^- = \hat{\bar{\mathcal{V}}} + \frac{1}{2}\tilde{\mathfrak{J}}_\alpha\bar{\bar{\mathfrak{J}}}^\alpha - \frac{\bar{g}^{\alpha\beta}\mathcal{P}_\alpha\mathcal{P}_\beta + m^4\mathrm{A}}{2\mathcal{P}_-^2}$$

$$\mathcal{P}_-(\wp,\chi) \coloneqq \wp_- + \mathfrak{C}_-(\chi^-,\gamma,\tau), \mathcal{P}_\alpha(\wp,\chi) \coloneqq \wp_\alpha + \mathfrak{C}_\alpha(\chi^-,\gamma,\tau)$$

$$\bar{g}^{\alpha\beta}g_{\beta c} = \delta_c^\alpha, \bar{\mathfrak{J}}^\alpha(\gamma,\tau) \coloneqq \bar{g}^{\alpha\beta}(\gamma,\tau)\mathfrak{J}_\beta(\gamma,\tau), \mathfrak{J}^2(\gamma,\tau) \coloneqq \mathfrak{J}_\alpha(\gamma\tau)\bar{\mathfrak{J}}^\alpha(\gamma,\tau)$$

$$\mathfrak{H} = \frac{\overline{\bar{g}^{\alpha\beta}(\gamma,\tau)\mathcal{P}_\alpha(\wp,\chi)\mathcal{P}_\beta(\wp,\chi) + m^4\mathrm{A}(\gamma,\tau)}}{2\mathcal{P}_-(\wp,\chi)} + \mathfrak{C}_+(\chi^-,\gamma,\tau) - \mathcal{P}_\alpha(\wp,\chi)\bar{\mathfrak{J}}^\alpha(\gamma,\tau)$$

$$+ \mathcal{P}_-(\wp,\chi)\left(\mathcal{V}(\gamma,\tau) + \frac{1}{2}\mathfrak{J}^2(\gamma,\tau)\right)$$

$$\frac{d\mathcal{P}_-}{d\tau} = \mathfrak{F}_{+-} + \mathfrak{F}_{\alpha-}\frac{\partial\mathcal{H}}{\partial\wp_\alpha}, \frac{d\mathcal{P}_\alpha}{d\tau} = \mathfrak{F}_{+\alpha} + \mathfrak{F}_{-\alpha}\frac{\partial\mathcal{H}}{\partial\wp_-} + \mathcal{F}_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\wp_\beta} - \frac{\hat{\partial}}{\hat{\partial}\gamma^\alpha}(\mathcal{H} - \mathfrak{C}_+)\left\langle\frac{d\mathcal{P}_-}{d\tau}\right|$$

$$\mathfrak{H}^-\left(\mathcal{P}_\alpha,\gamma^\beta,\tau\right) = \frac{\bar{g}^{\alpha\beta}(\gamma,\tau)\mathcal{P}_\alpha\mathcal{P}_\beta + \mathrm{m}^4\mathrm{A}(\gamma,\tau)}{2\mathcal{P}_-} - \mathcal{P}_\alpha\bar{\mathfrak{J}}^\alpha(\gamma,\tau) + \mathcal{P}_-\mathcal{V}(\gamma,\tau) + \frac{1}{2}\mathcal{P}_-\mathfrak{J}^2(\gamma,\tau) - \mathcal{V}(\gamma,\tau)$$

$$\frac{d\gamma^\alpha}{d\tau} = \frac{\hat{\partial}\mathcal{H}^-}{\hat{\partial}\mathcal{P}_\alpha}, \frac{d\mathcal{P}_\alpha}{d\tau} = -\frac{\hat{\partial}\mathcal{H}^-}{\hat{\partial}\gamma^\alpha}$$

$$\mathcal{L}_{-}(\gamma,\tau)=\mathcal{P}_{-}\left[\frac{1}{2}g_{\alpha\beta}(\gamma,\tau)\dddot{\bar{\gamma}}^{\alpha}\dddot{\bar{\gamma}}^{\beta}+\mathfrak{J}_{\alpha}(\gamma,\tau)\dddot{\bar{\gamma}}^{\alpha}-\mathcal{V}(\gamma,\tau)-\frac{1}{2}\widehat{m}^{4}\mathrm{A}(\gamma,\tau)\right]$$

$$\mathcal{L}_{\mathcal{YM}}^{-}=\mathcal{P}_{-}tr\left[\frac{1}{2}\mathfrak{D}_{t}\chi^{\alpha}\mathfrak{D}_{t}\chi_{\alpha}+\mathfrak{J}_{\alpha}(\chi,\tau)\mathfrak{D}_{t}\chi^{\alpha}-\mathcal{V}(\chi,\tau)-\frac{1}{2}\widehat{m}^{4}\mathrm{A}(\chi,\tau)\right]+tr[\mathcal{V}(\chi,\tau)],\mathfrak{D}_{t}\chi$$

$$=\underline{\dot{\chi}}-i[\mathrm{A}_{0},\chi]$$

$$\chi\longrightarrow\mathcal{U}^{-1}\chi\mathcal{U},\mathrm{A}_{0}\longrightarrow\mathcal{U}^{-1}\mathrm{A}_{0}\mathcal{U}+i\mathcal{U}^{-1}\partial_{t}\mathcal{U},\mathcal{U}\in\mathcal{U}(\mathcal{N})$$

Morfología y sistematicidad de la supermembrana

$$\chi^{\mathcal{M}}(\xi)\longrightarrow\chi'^{\mathcal{M}}(\xi)=\chi^{\mathcal{M}}(\xi')$$

$$\mathcal{G}_{\mu\nu}(\xi)\longrightarrow\mathcal{G}'_{\mu\nu}(\xi)=\frac{\partial\xi'^{\kappa}}{\partial\xi^{\mu}}\frac{\partial\xi'^{\lambda}}{\partial\xi^{\nu}}\mathcal{G}_{\kappa\lambda}(\xi'),\xi^{0}=\tau,\xi^{i}=\sigma^{i}(1\leq i\leq\wp),\tau\equiv\chi^{+},\tau\longrightarrow\tau'=\tau,\sigma^{i}\longrightarrow\sigma'^{i}$$

$$=f^{i}(\tau,\sigma)$$

$$\mathcal{G}_{\tau i}(\xi)\longrightarrow\mathcal{G}'_{\tau i}(\xi)=\frac{\partial f^{i}}{\partial\sigma^{i}}\mathcal{G}_{j\kappa}(\xi')\underline{\left(\partial_{\tau}f^{\kappa}(\tau,\sigma)+\bar{\mathcal{G}}^{\kappa l}(\xi')\mathcal{G}_{\tau i}(\xi)\right)}$$

$$\partial_{\tau}f^{\kappa}(\tau,\sigma)=-\bar{\mathcal{G}}^{\kappa l}\big(\tau,f(\tau,\sigma)\big)\mathcal{G}_{\tau l}\big(\tau,f(\tau,\sigma)\big)$$

$$\tau\longrightarrow\tau'=\tau,\sigma^{i}\longrightarrow\sigma'^{i}=f^{i}(\sigma)$$

$$\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}=-\mathcal{T}\mathrm{A}^{\frac{\wp+1}{2}}\sqrt{\left(-2\underline{\chi}^{-}+2\mathcal{V}-2\mathfrak{J}_{\alpha}\dddot{\bar{\gamma}}^{\alpha}+g_{\alpha\beta}\dddot{\bar{\gamma}}^{\alpha}\dddot{\bar{\gamma}}^{\beta}\right)\det\left(\partial_{i}\gamma^{\alpha}\partial_{j}\gamma^{\beta}g_{\alpha\beta}\right)}$$

$$\mathcal{L}_{\mathfrak{C}_{\wp+1}}=-\frac{1}{\wp!}\epsilon^{\tau j_{1}\cdots j_{\wp}}\partial_{j_{i}}\chi^{\mathcal{M}_{1}}\cdots\partial_{j_{\wp}}\chi^{\mathcal{M}_{\wp}}\left(\mathfrak{C}_{+\mathcal{M}_{1}\cdots\mathcal{M}_{\wp}}(\chi)+\underline{\dot{\chi}}^{-}\mathfrak{C}_{-\mathcal{M}_{1}\cdots\mathcal{M}_{\wp}}(\chi)+\dddot{\bar{\gamma}}^{\alpha}\mathfrak{C}_{\alpha\mathcal{M}_{1}\cdots\mathcal{M}_{\wp}}(\chi)\right)$$

$$\mathcal{P}_{-}:=\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\underline{\dot{\chi}}^{-}}=\mathcal{T}\mathrm{A}^{\frac{\wp+1}{2}}\sqrt{\frac{\det\left(\partial_{i}\gamma^{\alpha}\partial_{j}\gamma^{\beta}g_{\alpha\beta}(\gamma,\tau)\right)}{-2\underline{\chi}^{-}+2\mathcal{V}-2\mathfrak{J}_{\alpha}\dddot{\bar{\gamma}}^{\alpha}+g_{\alpha\beta}\dddot{\bar{\gamma}}^{\alpha}\dddot{\bar{\gamma}}^{\beta}}}$$

$$\mathcal{P}_{-}(\tau,\sigma)\longrightarrow\mathcal{P}'_{-}(\tau,\sigma)=\left|\det\left(\frac{\partial\sigma'}{\partial\sigma}\right)\right|\mathcal{P}_{-}(\tau,\sigma)\frac{\partial\mathcal{P}_{-}(0,\sigma)}{\partial\sigma^{i}}$$

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\partial_{\mu}\chi^{m}}\right)-\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\chi^{m}}+\partial_{\mu}\left(\frac{\partial\mathcal{L}_{\mathfrak{C}_{\wp+1}}}{\partial\partial_{\mu}\chi^{m}}\right)-\frac{\partial\mathcal{L}_{\mathfrak{C}_{\wp+1}}}{\partial\chi^{m}}$$

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\partial_{\mu}\chi^{m}}\right)-\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\chi^{m}}=\frac{1}{(\wp+1)!}\epsilon^{\mu_{1}\mu_{2}\cdots\mu_{\wp+1}}\partial_{\mu_{1}}\chi^{\mathcal{M}_{1}}\cdots\partial_{\mu_{\wp+1}}\chi^{\mathcal{M}_{\wp+1}}\mathfrak{F}_{m\mathcal{M}_{1}\cdots\mathcal{M}_{\wp+1}}$$

$$\frac{\partial\mathcal{P}_{-}}{\partial\tau}=\frac{1}{\wp!}\epsilon^{\tau j_{1}\cdots j_{\wp}}\partial_{j_{1}}\chi^{\mathcal{M}_{1}}\cdots\partial_{j_{\wp}}\chi^{\mathcal{M}_{\wp}}\left(\mathfrak{F}_{+-\mathcal{M}_{1}\cdots\mathcal{M}_{\wp+1}}+\dot{\hat{\gamma}}^{\alpha}\mathfrak{F}_{\alpha-\mathcal{M}_{1}\cdots\mathcal{M}_{\wp+1}}\right)\left\langle\frac{\partial\mathcal{P}_{-}}{\partial\xi^{\mu}}\right|$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\dot{\hat{\gamma}}^{\alpha}}\right)+\frac{\partial}{\partial\sigma^{i}}\left(\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\partial_{i}\hat{\gamma}^{\alpha}}\right)-\frac{\partial\mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial\dot{\hat{\gamma}}^{\alpha}}-\frac{1}{\wp!}\epsilon^{\tau j_{1}\cdots j_{\wp}}\partial_{j_{1}}\chi^{\mathcal{M}_{1}}\cdots\partial_{j_{\wp}}\chi^{\mathcal{M}_{\wp}}\mathfrak{F}_{+\alpha\beta_{1}\cdots\beta_{\wp}}$$

$$\frac{\partial \mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial \dot{\hat{\gamma}}^{\alpha}} = \mathcal{P}_{-}\left(g_{\alpha\beta}\dot{\hat{\gamma}}^{\beta} + \mathfrak{J}_{\alpha}\right), \frac{\partial \mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial \partial_i \dot{\hat{\gamma}}^{\alpha}} = -\left(\frac{\mathcal{T}^2}{2\mathcal{P}_{-}}\right) \mathrm{A}^{\wp+1} \frac{\partial}{\partial \partial_i \dot{\hat{\gamma}}^{\alpha}} \det\left(\partial_j \gamma^{\alpha} \partial_{\Bbbk} \gamma^{\beta} g_{\alpha\beta}(\gamma,\tau)\right)$$

$$\frac{\partial \mathcal{L}_{\mathcal{N}\otimes\mathfrak{G}}}{\partial \dot{\hat{\gamma}}^{\alpha}} = \frac{\partial}{\partial \dot{\hat{\gamma}}^{\alpha}}\left[\mathcal{P}_{-}\left(\frac{1}{2} g_{\beta c} \ddot{\bar{\gamma}}^{\beta} \ddot{\bar{\gamma}}^{c} + \mathfrak{J}_{\beta}\dot{\hat{\gamma}}^{\beta} - \mathcal{V}\right) - \left(\frac{\mathcal{T}^2}{2\mathcal{P}_{-}}\right) \mathrm{A}^{\wp+1} \det\left(\partial_j \gamma^{\alpha} \partial_{\Bbbk} \gamma^{\beta} g_{\alpha\beta}\right)\right]$$

$$\mathcal{L}_{\mathcal{V}} \coloneqq \frac{1}{\wp!} \epsilon^{\tau j_1 j_2 \cdots j_{\wp}} \partial_{j_1}\gamma^{\alpha_1} \partial_{j_2}\gamma^{\alpha_2} \cdots \partial_{j_{\wp}}\gamma^{\alpha_{\wp}} \mathcal{V}_{\alpha_1\alpha_2\cdots\alpha_{\wp}}(\gamma,\tau)$$

$$\partial_i\left(\frac{\partial \mathcal{L}_{\mathcal{V}}}{\partial \partial_i \gamma^{\alpha}}\right) - \frac{\partial \mathcal{L}_{\mathcal{V}}}{\partial \gamma^{i}} = -\frac{1}{\wp!} \epsilon^{\tau j_1 \cdots j_{\wp}} \partial_{j_1}\gamma^{\beta_1} \cdots \partial_{j_{\wp}}\gamma^{\beta_{\wp}} \mathfrak{F}_{+\alpha\beta_1\cdots\beta_{\wp}}$$

$$\mathcal{L}^{-} = \mathcal{P}_{-}\left[\frac{1}{2} g_{\alpha\beta}(\gamma,\tau)\ddot{\bar{\gamma}}^{\alpha}\ddot{\bar{\gamma}}^{\beta} + \mathfrak{J}_{\alpha}(\gamma,\tau)\dot{\hat{\gamma}}^{\alpha} - \mathcal{V}(\gamma,\tau) - \frac{1}{2}\mathcal{T}_{-}^{2}\mathrm{A}(\gamma,\tau)^{\wp+1} \det\left(\partial_i\gamma^{\alpha}\partial_j\gamma^{\beta} g_{\alpha\beta}(\gamma,\tau)\right)\right]$$

$$+ \frac{1}{\wp!} \epsilon^{\tau j_1 j_2 \cdots j_{\wp}} \partial_{j_1}\gamma^{\alpha_1} \partial_{j_2}\gamma^{\alpha_2} \cdots \partial_{j_{\wp}}\gamma^{\alpha_{\wp}} \mathcal{V}_{\alpha_1\alpha_2\cdots\alpha_{\wp}}(\gamma,\tau)$$

$$\mathcal{L}_{string} = \frac{1}{2}\left(\partial_{\tau}\gamma^{\alpha}\partial_{\tau}\gamma^{\beta} - \mathrm{A}(\gamma,\tau)^{2}\partial_{\sigma}\gamma^{\alpha}\partial_{\sigma}\gamma^{\beta}\right) g_{\alpha\beta}(\gamma,\tau) + \mathfrak{J}_{\alpha}(\gamma,\tau)\dot{\hat{\gamma}}^{\alpha} - \mathcal{V}(\gamma,\tau) + \mathcal{T}^{-1}\partial_{\sigma}\gamma^{\alpha}\mathcal{V}_{\alpha}$$

$$\{\gamma^{\alpha_1}, \gamma^{\alpha_2} \cdots \gamma^{\alpha_{\wp}}\}_{\mathcal{N}\otimes\mathfrak{B}} \coloneqq \epsilon^{j_1 j_2 \cdots j_{\wp}} \frac{\partial \gamma^{\alpha_1}}{\partial \sigma^{j_1}} \frac{\partial \gamma^{\alpha_2}}{\partial \sigma^{j_2}} \cdots \frac{\partial \gamma^{\alpha_{\wp}}}{\partial \sigma^{j_{\wp}}} \det\left(\partial_i\gamma^{\alpha}\partial_j\gamma^{\beta} g_{\alpha\beta}\right)$$

$$= \frac{1}{\wp!}\{\gamma^{\alpha_1}, \gamma^{\alpha_2} \cdots \gamma^{\alpha_{\wp}}\}_{\mathcal{N}\otimes\mathfrak{B}}\{\gamma^{\beta_1}, \gamma^{\beta_2} \cdots \gamma^{\beta_{\wp}}\}_{\mathcal{N}\otimes\mathfrak{B}} g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} \cdots g_{\alpha_{\wp}\beta_{\wp}}$$

$$\{\gamma^{\alpha_1}, \gamma^{\alpha_2} \cdots \gamma^{\alpha_{\wp}}\}_{\mathcal{N}\otimes\mathfrak{B}} \Leftrightarrow \left(\sqrt{-1}\right)^{\frac{1}{2}\wp(\wp-1)}\{\chi^{\alpha_1}, \chi^{\alpha_2} \cdots \chi^{\alpha_{\wp}}\}$$

$$[\mathcal{M}^{\alpha_1}, \mathcal{M}^{\alpha_2} \cdots \mathcal{M}^{\alpha_{\wp}}] \coloneqq \epsilon^{j_1 j_2 \cdots j_{\wp}} \mathcal{M}_{j_1}\mathcal{M}_{j_2} \cdots \mathcal{M}_{j_{\wp}}$$

$$\left\{\{f_1 f_2 \cdots f_{\wp}\}_{\mathcal{N}\otimes\mathfrak{B}} g_2 \cdots g_{\wp}\right\}_{\mathcal{N}\otimes\mathfrak{B}} = \sum_{j=1}^{\wp}\left\{f_1 \cdots f_{j-1}\{f_j, g_2 \cdots g_{\wp}\}_{\mathcal{N}\otimes\mathfrak{B}}, f_{j+1} \cdots f_{\wp}\right\}_{\mathcal{N}\otimes\mathfrak{B}}$$

$$\delta_{\mathcal{M}\otimes\mathfrak{M}} = \iiint d\tau \mathcal{P}_{-}\mathcal{L}_{\mathcal{M}\otimes\mathfrak{M}}^{-}$$

$$\mathcal{L}_{\mathcal{M}\otimes\mathfrak{M}}^{-} = tr\left(\frac{1}{2}\mathfrak{D}_t\chi^{\alpha}\mathfrak{D}_t\chi_{\alpha} + \mathfrak{J}_{\alpha}(\chi,\tau)\mathfrak{D}_t\chi^{\alpha} - \mathcal{V}(\chi,\tau)\right)$$

$$+ tr\left(-\frac{\kappa_{\wp}^{2}}{2\wp!}(-1)^{\frac{1}{2}\wp(\wp-1)}\mathrm{A}(\chi,\tau)^{\wp+1}[\chi^{\alpha_1}, \chi^{\alpha_2} \cdots \chi^{\alpha_{\wp}}]^{2}\right)$$

$$+ tr\left(\frac{\lambda_{\wp}}{\wp!}\left(\sqrt{-1}\right)^{\frac{1}{2}\wp(\wp-1)}[\chi^{\beta_1}, \chi^{\beta_2} \cdots \chi^{\beta_{\wp}}]\mathcal{V}_{\beta_1\beta_2\cdots\beta_{\wp}}(\chi,\tau)\right)$$

$$f(\sigma) \star g(\sigma) = f(\sigma) e^{i\frac{\theta}{2}\overleftarrow{\partial}_i \epsilon^{ij} \overrightarrow{\partial}_j} g(\sigma) \Longrightarrow \sigma^1 \star \sigma^2 - \sigma^2 \star \sigma^1 = i\theta$$

$$\mathcal{O}(f)\mathcal{O}(g) = \mathcal{O}(f \star g)$$

$$\mathcal{O}\left(\sigma^{j_1}\sigma^{j_2}\cdots\sigma^{j_\eta}\right) := \sum_{\mathcal{P}=1}^{\eta!}\frac{1}{\eta!}\hat{\sigma}^{\mathcal{P}_1}\hat{\sigma}^{\mathcal{P}_2}\cdots\hat{\sigma}^{\mathcal{P}_\eta}$$

$$[f,g]_\star = i\theta\{f,g\}_{\mathcal{P}\otimes\mathfrak{B}} + \mathfrak{O}(\theta^2)$$

Matrices

$$ds^2 = 2d\chi^+d\chi^- - \frac{1}{\zeta}\mu^2\left(\chi_1^2\chi_2^2\cdots\chi_\eta^2\right)d\chi^+d\chi^+ + \sum_{\alpha=1}^{\zeta_{multidimensional}}\iiint_{\zeta_{multidimensional}}^{\langle\alpha\beta\gamma\delta\epsilon\zeta\eta\lambda\mu\rho\sigma\rangle'} d\chi^\alpha d\chi^\beta$$

$$\mathcal{L}_{\mathfrak{BMN}}^{\mathcal{N}=multidimensional}$$

$$= tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}\left[\chi^\alpha,\chi^\beta\right]^2 + i\frac{1}{2}\psi^\dagger\mathfrak{D}_\tau\psi - \frac{1}{2}\psi^\dagger\Gamma^\alpha[\chi^\alpha,\psi]\right)$$

$$+ i\mu\left(\chi_1^2\chi_2^2\cdots\chi_\eta^2\right)\sum_{\alpha=1}^{\zeta_{multidimensional}}\iiint_{\zeta_{multidimensional}}^{\langle\alpha\beta\gamma\delta\epsilon\zeta\eta\lambda\mu\rho\sigma\rangle'} d\chi^\alpha d\chi^\beta$$

$$(\Gamma^\alpha)^{\mathfrak{T}} = (\Gamma^\alpha)^{\circledast} = \mathfrak{C}^{-1}\Gamma^\alpha\mathfrak{C}, \mathfrak{C} = \mathfrak{C}^{\mathfrak{T}} = \left(\mathfrak{C}^\dagger\right)^{-1}$$

$$\delta\mathrm{A}_0 = i\psi^\dagger\varepsilon(\tau), \delta\chi^\alpha = i\psi^\dagger\Gamma^\alpha\varepsilon(\tau), \delta\psi$$

$$= {}_{\hat{\mathcal{G}}^{\alpha\beta}}^{\mathcal{G}''(\tau)}\left(\mathfrak{D}_\tau\chi^\alpha\Gamma_\alpha - i\frac{1}{2}\left[\chi^\alpha,\chi^\beta\right]^2\Gamma_{\alpha\beta}\overbrace{\overbrace{\overbrace{\langle\mu|\chi|\Gamma\rangle_{cd}^{\alpha\beta}}^{\langle\Gamma_{\lambda\mu}^\alpha\widehat{\nabla_\beta^\alpha\Delta_{\rho\sigma}^{\lambda\mu}}\rangle\zeta}}{\langle\xi|\gamma|\Gamma\rangle_{cd}^{\alpha\beta}}}{\langle\epsilon|\zeta|\Gamma\rangle_{cd}^{\alpha\beta}}\right)_{\widehat{\mathfrak{R}}_{\langle\alpha}\hat{\mathcal{T}}_{|\beta|}\hat{\delta}_{c\rangle}}^{\mathcal{R}''(g)} \underline{|\varepsilon(t)|^{\delta_j^i}}$$

$$\varepsilon_{\wp qr}(\tau) = e^{\frac{1}{2}\tau\mu\Gamma^{\lambda\mu}} - \frac{1}{4}\underbrace{\left\|\mu_\beta^\alpha\bigotimes\Gamma_\beta^\alpha\right\|}_{\mu_{\langle\beta|}^{|\alpha\rangle}\Gamma_{|\beta\rangle}^{\langle\alpha|}}\Delta_g\varphi$$

$$\mathcal{L}_{\mathcal{SYM}} = tr\left(-\frac{1}{4}\mathfrak{F}_{\mathcal{LM}}\mathfrak{F}^{\mathcal{LM}} - i\frac{1}{2}\bar{\bar{\hat{\psi}}}^i\Gamma^{\mathcal{L}}\mathfrak{D}_{\mathfrak{L}}\underline{\bar{\bar{\hat{\psi}}}_i}\right)\otimes\left\|\widehat{\mathfrak{R}}_{\langle\alpha}\hat{\mathcal{T}}_{|\beta|}\hat{\delta}_{c\rangle}\right\|$$

$$\mathfrak{D}_{\mathfrak{L}}\underline{\bar{\bar{\hat{\psi}}}_i} = \partial_{\mathfrak{L}}\underline{\bar{\bar{\hat{\psi}}}_i} - i\left[\mathrm{A}_{\mathfrak{L}},\underline{\psi_i}\right], \mathfrak{F}_{\mathcal{LM}} = \partial_{\mathfrak{L}}\mathrm{A}_{\mathcal{M}} - \partial_{\mathcal{M}}\mathrm{A}_{\mathfrak{L}} - i[\mathrm{A}_{\mathfrak{L}},\mathrm{A}_{\mathcal{M}}]$$

$$\delta\mathrm{A}_{\mathcal{M}} = +i\bar{\bar{\varepsilon}}^i\Gamma_{\mathcal{M}}\underline{\bar{\bar{\hat{\psi}}}_i} = i\bar{\bar{\hat{\psi}}}_i\Gamma_{\mathcal{M}}\epsilon_i, \delta\underline{\bar{\bar{\hat{\psi}}}_i} = -\frac{1}{2}\mathfrak{F}_{\mathcal{MN}}\Gamma^{\mathcal{MN}}\epsilon_i$$

$$\left(\Gamma^{\mathfrak{L}}\mathcal{P}\right)_{\alpha\beta}(\Gamma_{\mathcal{L}}\mathcal{P})_{\gamma\delta} + \left(\Gamma^{\mathfrak{L}}\mathcal{P}\right)_{\gamma\beta}(\Gamma_{\mathcal{L}}\mathcal{P})_{\alpha\delta}tr\left(\underline{\overline{\psi^i}}\Gamma^{\mathfrak{L}}\left[i\bar{\bar{\varepsilon}}^j\Gamma_{\mathcal{L}}\underline{\bar{\bar{\hat{\psi}}}_j},\underline{\bar{\bar{\hat{\psi}}}_i}\right]\right)$$

$$\mathcal{L}_0^{\mathcal{N}=multidimensional}$$

$$= tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}[\chi^\alpha,\chi^\beta]^2 + i\frac{1}{2}\hat{\hat{\bar{\psi}}}^i\Gamma^t\mathfrak{D}_\tau\underline{\hat{\hat{\psi}}_i}\right.$$

$$\left.-\frac{1}{2}\hat{\hat{\bar{\psi}}}^i\Gamma^\alpha\left[\chi_\alpha,\underline{\hat{\hat{\psi}}_i}\right]\right) tr\left[\bar{\psi}\left(\mathcal{M}_{\mathfrak{L}}\Gamma^{\mathfrak{L}} - i\frac{1}{3!}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}\right)\psi\right]$$

$$\mathcal{L}_{type\ I}^{\mathcal{N}=multidimensional}$$

$$= tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}[\chi^\alpha,\chi^\beta]^2 - i\frac{1}{2}\hat{\hat{\bar{\psi}}}^i\Gamma^\tau\mathfrak{D}_\tau\underline{\hat{\hat{\psi}}_i} - \frac{1}{2}\hat{\hat{\bar{\psi}}}^i\Gamma^\alpha\left[\chi_\alpha,\underline{\hat{\hat{\psi}}_i}\right] - i\mu\right)\otimes(\chi_1^2\chi_2^2$$

$$\cdots\chi_\eta^2)\sum_{\alpha=1}^{\zeta_{multidimensional}}\iiint_{\zeta_{multidimensional}}^{\langle\alpha\beta\gamma\delta\epsilon\zeta\eta\lambda\mu\rho\sigma\rangle'} d\chi^\alpha d\chi^\beta$$

$$\delta\mathrm{A}_0 = \hat{\hat{\bar{\psi}}}^i\Gamma_0\epsilon_i(\tau), \delta\chi_\alpha = i\hat{\hat{\bar{\psi}}}^i\Gamma_\alpha\epsilon_i(\tau)$$

$$\delta\psi_i = \left(\Gamma^{t\alpha}\mathfrak{D}_\tau\chi_\alpha - i\frac{1}{2}[\chi_\alpha,\chi_\beta]\Gamma^{\alpha\beta} - \frac{1}{2}\mu\chi\Gamma^{\lambda\mu} - \frac{1}{4}\mu\Gamma^{\lambda\mu}\chi\right)$$

$$\chi = \Gamma^\alpha\chi_\alpha, \epsilon_i(\tau) = e^{i\frac{1}{2}\tau'\mu''\Gamma''\tau''^{(1)}}\epsilon_i(0)$$

$$tr\left(-i\hat{\hat{\bar{\psi}}}^i\Gamma^\tau\delta\underline{\hat{\hat{\psi}}_i}\right) \coloneqq i\bar{\mathcal{Q}}^i\epsilon_i(\tau), \bar{\mathcal{Q}}^i = (\mathcal{Q}_i)^\dagger\mathrm{A} = \epsilon^{ij}(\mathcal{Q}_j)^{\mathfrak{T}}\mathfrak{C}$$

$$[\mathcal{H},\mathcal{Q}_i] = i\frac{1}{2}\mu\Gamma_i\mathcal{Q}_i, [\mathcal{H},\bar{\mathcal{Q}}_i] = -i\frac{1}{2}\mu\bar{\mathcal{Q}}_i\Gamma_i$$

$$\{\mathcal{Q}_j\bar{\mathcal{Q}}^j\} = 2\delta_j^i\left(\mathrm{A}\left(\mathfrak{H} - \frac{1}{6}\mu\mathcal{M}_\wp\right) - \frac{1}{6}\mu\epsilon_{\wp qr}\Gamma^{\wp q}\Gamma_{\wp q}\mathcal{M}^{qr}\mathcal{M}_{\wp q}\right)\mathcal{P}_+ + i\frac{2}{3}\mu\mathcal{T}_j{}^i\Gamma_\zeta\mathcal{P}_+$$

$$[\mathcal{M}_{\wp q},\mathcal{M}_{rs}] = i(\delta_{\wp r}\mathcal{M}_{qs} - \delta_{\wp s}\mathcal{M}_{qr} - \delta_{qr}\mathcal{M}_{\wp s} + \delta_{qs}\mathcal{M}_{\wp r}), [\mathcal{T}_j{}^i,\mathcal{Q}_k] = \delta^j{}_k\mathcal{Q}_i - \frac{1}{2}\delta_i^j\mathcal{Q}_k, [\mathcal{T}_j{}^i,\mathcal{T}_k^\ell]$$

$$= \delta_k^j\mathcal{T}_i^\ell - \delta_i^\ell\mathcal{T}_k^j[\![\mathcal{H},\mathcal{M}_{\wp q}]\!][\mathcal{H},\mathcal{T}_j{}^i]\left\langle\mathcal{M}_{\alpha\beta}\middle|\mathcal{T}_i{}^j\right\rangle$$

$$\chi_1 = \mathcal{R}\cos\left(\frac{1}{6}\tau\mu\right), \chi_2 = \mathcal{R}\sin\left(\frac{1}{6}\tau\mu\right), \chi_\wp = \frac{1}{3}\mu\mathfrak{I}_\wp$$

Deformaciones

$$\psi = \psi_1, \hat{\bar{\psi}} = \psi^\dagger\mathrm{A} = \hat{\bar{\psi}}^1$$

$\mathcal{L}_{type\ II}^{\mathcal{N}=multidimensional}$

$$= tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}\left[\chi^\alpha,\chi^\beta\right]^2 - i\hat{\bar{\psi}}\Gamma^\tau\mathfrak{D}_\tau\psi - \hat{\bar{\psi}}\Gamma^\alpha[\chi_\alpha,\psi]\right)$$

$$+ tr\left(\frac{1}{4}\mu''\hat{\bar{\psi}}''\Gamma'^{\lambda'\mu'}[\![\psi\mu]\!]^2\right)\otimes\left(\chi_1^2\chi_2^2\cdots\chi_\eta^2\right)\sum_{\alpha=1}^{\zeta_{multidimensional}}\iiint_{\zeta_{multidimensional}}^{\langle\alpha\beta\gamma\delta\epsilon\zeta\eta\lambda\mu\rho\sigma\rangle'} d\chi^\alpha d\chi^\beta$$

$$\delta A_0 = \hat{\bar{\psi}}\Gamma_0\epsilon(\tau) + \bar{\epsilon}(\tau)\Gamma_0\psi, \delta\chi_\alpha = \hat{\bar{\psi}}\Gamma_\alpha\epsilon(\tau) + \bar{\epsilon}(\tau)\Gamma_\alpha\psi$$

$$\delta\psi = \left(-i\mathfrak{D}_\tau\chi_\alpha\Gamma^{\tau\alpha} - \frac{1}{2}[\chi_\alpha,\chi_\beta]\Gamma^{\alpha\beta} - \frac{1}{4}\mu\Gamma^{\lambda\mu(1)}\chi - \frac{1}{2}\mu''\Gamma''^{\lambda\mu(1)}\chi\right)\epsilon(\tau)$$

$$\chi = \Gamma^\alpha\chi_\alpha, \epsilon(\tau) = e^{-i\frac{1}{2}\tau'\mu''\Gamma''\tau''^{(1)}}\epsilon(0)$$

$$\{\mathcal{Q},\bar{\mathcal{Q}}\} = 2\left(A\mathcal{H} + i\frac{1}{2}\mu''\mathcal{M}_{m\eta}\Gamma^{m\eta} - \frac{1}{2}\mu\mathcal{T}'\Gamma'\right)\mathcal{P}_+\{\mathcal{Q},\mathcal{Q}\}[\mathcal{H},\mathcal{T}],[\mathcal{H},\mathcal{Q}] = \frac{1}{2}\mu\Gamma^\tau,[\mathcal{H},\bar{\mathcal{Q}}]$$

$$= \frac{1}{2}\mu\bar{\mathcal{Q}}\Gamma^\tau,[\mathcal{T},\mathcal{Q}] = \mathcal{Q},[\mathcal{T},\bar{\mathcal{Q}}] = -\bar{\mathcal{Q}}$$

Análisis en cuatro dimensiones $\mathbb{R}^4$.

$$\Gamma^{\mu\dagger} = \Gamma_\mu = -A\Gamma^\mu A^\dagger, A = \Gamma^\tau = -A^\dagger$$

$$\Gamma^{\mu\star} = \mathfrak{B}\Gamma^\mu\mathfrak{B}^\dagger, \mathfrak{B}^\tau = \mathfrak{B}, \mathfrak{B}^\dagger = \mathfrak{B}^{-1}$$

$$\Gamma^{\mu\mathcal{T}\circledast} = -\mathfrak{C}\Gamma^\mu\mathfrak{C}^\dagger, \mathfrak{C} = -\mathfrak{C}^{\mathcal{T}} = \mathfrak{B}\Gamma^\tau, \mathfrak{C}^\dagger = \mathfrak{C}^{-1}$$

$$\bar{\psi} = \psi^\dagger\Gamma^\tau = \psi^{\mathcal{T}}\mathfrak{C} \Leftrightarrow \psi^* = \mathfrak{B}\psi$$

$$\mathcal{L}_{4D\,\mathcal{SYM}} = tr\left(-\frac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} - i\frac{1}{2}\hat{\hat{\bar{\psi}}}^i\Gamma^\mu\mathfrak{D}_\mu\underline{\psi}\right)$$

$$\delta A_\mu = i\bar{\varepsilon}\Gamma_\mu\psi = -i\bar{\psi}\Gamma_\mu\varepsilon, \delta\psi = -\frac{1}{2}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}\varepsilon$$

$$(\mathfrak{C}\Gamma^\mu)_{\alpha\beta}(\mathfrak{C}\Gamma_\mu)_{\gamma\delta} + (\mathfrak{C}\Gamma^\mu)_{\beta\gamma}(\mathfrak{C}\Gamma_\mu)_{\alpha\delta} + (\mathfrak{C}\Gamma^\mu)_{\gamma\alpha}(\mathfrak{C}\Gamma_\mu)_{\beta\delta}$$

$$\mathcal{L}_0^{\mathcal{N}=4} = tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}\left[\chi^\alpha,\chi^\beta\right]^2 + i\frac{1}{2}\hat{\hat{\bar{\psi}}}\Gamma^\tau\mathfrak{D}_\tau\psi - \frac{1}{2}\hat{\hat{\bar{\psi}}}\Gamma^\alpha[\chi_\alpha,\psi]\right)$$

$$\mathcal{L}_{type\ I}^{\mathcal{N}=4} = tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}\left[\chi^\alpha,\chi^\beta\right]^2 - i\frac{1}{2}\hat{\bar{\psi}}\Gamma^\tau\mathfrak{D}_\tau\psi - \hat{\bar{\psi}}\Gamma^\alpha[\chi_\alpha,\psi]\right)$$

$$+ tr\left(i\frac{1}{4}\mu_1\hat{\bar{\psi}}''\psi' + \frac{1}{4}\mu_2\hat{\bar{\psi}}''\Gamma'^{\lambda'\mu'}\psi' - i\mu_2[\![\chi_1,\chi_2]\!]\chi_3 - \frac{1}{2}(\mu_1^2+\mu_2^2)\chi^\alpha\chi_\alpha\right)$$

$$\delta A_0 = -i\hat{\bar{\psi}}\Gamma_\tau\epsilon(\tau), \delta\chi_\alpha = -i\hat{\bar{\psi}}\Gamma_\alpha\epsilon(\tau)$$

$$\delta\psi = \left(-\Gamma^{t\alpha}\mathfrak{D}_\tau\chi_\alpha + i\frac{1}{2}[\chi_\alpha,\chi_\beta]\Gamma^{\alpha\beta} - \frac{1}{3}\mu_1\Gamma^{\lambda\mu(1)}\chi + \frac{1}{3}\mu_2\chi\Gamma^{\lambda\mu(2)}\right)\epsilon(\tau)$$

$$\epsilon(\tau) = e^{\frac{1}{6}\tau(\mu_1\Gamma^{\tau(1)}-\mu_2\Gamma^{\tau(2)})}\epsilon(0)$$

$$[\mathcal{H},\mathcal{Q}] = -i\frac{1}{6}(\mu_1\Gamma^{\tau(1)} - \mu_2\Gamma^{\tau(2)})\mathcal{Q}, \{\mathcal{Q},\bar{\mathcal{Q}}\} = 2\left(\Gamma^\tau\mathcal{H} + \frac{1}{6}\mu_1\Gamma^{\alpha\beta}\mathcal{M}_{\alpha\beta} - \frac{1}{6}\mu_2\epsilon_{\alpha\beta c}\Gamma^\alpha\mathcal{M}^{\beta c}\right)$$

$$\mathfrak{D}_\tau\chi_\alpha = 0, [\chi_\alpha,\chi_\beta] = i\frac{1}{3}\mu_2\epsilon_{\alpha\beta c}\chi^c$$

$$\mathcal{L}_{type\ I}^{\mathcal{N}=4} = tr\left[\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}[\chi^\alpha,\chi^\beta]^2 - i\frac{1}{3}\mu_2\epsilon_{\alpha\beta c}\chi^c\right)^2 - \frac{1}{9}\mu_1^2\chi^\alpha\chi_\alpha\right]$$

$$+ tr\left(-i\frac{1}{2}\hat{\bar{\psi}}\Gamma^t\mathfrak{D}_\tau\psi - \frac{1}{2}\hat{\bar{\psi}}\Gamma^\alpha[\chi_\alpha,\psi] + i\frac{1}{4}\hat{\bar{\psi}}(\mu_1 + \mu_2\Gamma'^{\lambda'\mu'})\psi\right)$$

$$\psi \longrightarrow e^{\frac{1}{2}\tau(\mu_3-\mu_2)\Gamma'^{\lambda'\mu'}}\psi$$

$$\mathcal{L}_{type\ I}^{\mathcal{N}=4}\Big|_{\mu_1=0} = tr\left[\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}\left([\chi^\alpha,\chi^\beta] - i\frac{1}{3}\mu_2\epsilon_{\alpha\beta c}\chi^c\right)^2\right]$$

$$+ tr\left[-i\frac{1}{2}\hat{\bar{\psi}}\Gamma^\tau\mathfrak{D}_\tau\psi - \frac{1}{2}\hat{\bar{\psi}}\Gamma^\alpha[\chi_\alpha,\psi] + i\frac{1}{4}\mu_3\hat{\bar{\psi}}\Gamma'^{\lambda'\mu'}\psi\right]$$

$$\delta\mathrm{A}_0 = -i\hat{\bar{\psi}}\Gamma_\tau\epsilon(\tau), \delta\chi_\alpha = -i\hat{\bar{\psi}}\Gamma_\alpha\epsilon(\tau)$$

$$\delta\psi = \left(-\Gamma^{\tau\alpha}\mathfrak{D}_\tau\chi_\alpha + i\frac{1}{2}[\chi_\alpha,\chi_\beta]\Gamma^{\alpha\beta} + \frac{1}{3}\mu_2\chi\Gamma^{\lambda\mu(2)}\right)\epsilon(\tau)$$

$$\epsilon(\tau) = e^{\frac{1}{6}\tau(2\mu_2-3\mu_3)\Gamma^{\tau(2)(3)}}\epsilon(0)$$

$$[\mathcal{H},\mathcal{Q}] = i\frac{1}{6}(2\mu_2 - 3\mu_3)\Gamma^{\tau(2)(3)}\mathcal{Q}, [\mathcal{R},\mathcal{Q}] = i\Gamma^{\tau(1)(2)(3)}\mathcal{Q}, [\mathcal{H},\mathcal{R}] = 0, \{\mathcal{Q},\bar{\mathcal{Q}}\}$$

$$= 2\left(\Gamma^\tau\left(\mathcal{H} + \frac{1}{2}(\mu_3 - \mu_2)\mathfrak{R}\right) - \frac{1}{6}\mu_2\epsilon_{\alpha\beta c}\Gamma^\alpha\mathcal{M}^{\beta c}\right)$$

Deformaciones tipo 2

$$\mathcal{L}_{type\ I}^{\mathcal{N}=4} = tr\left(\frac{1}{2}\mathfrak{D}_\tau\chi^\alpha\mathfrak{D}_\tau\chi_\alpha + \frac{1}{4}[\chi^\alpha,\chi^\beta]^2 - i\frac{1}{2}\hat{\bar{\psi}}\Gamma^\tau\mathfrak{D}_\tau\psi - \frac{1}{2}\hat{\bar{\psi}}\Gamma^\alpha[\chi_\alpha,\psi]\right)$$

$$+ tr\left(i\frac{1}{2}\mu\hat{\bar{\psi}}\Gamma^\tau\psi - \frac{1}{2}\mu^2(\chi_1^2 + \chi_2^2 + 4\chi_3^2)\right)$$

$$\delta\mathrm{A}_0 = -i\hat{\bar{\psi}}\Gamma_\tau\epsilon(\tau), \delta\chi_\alpha = -i\hat{\bar{\psi}}\Gamma_\alpha\epsilon(\tau)$$

$$\delta\psi = \left(-\Gamma^{\tau\alpha}\mathfrak{D}_\tau\chi_\alpha + i\frac{1}{2}[\chi_\alpha,\chi_\beta]\Gamma^{\alpha\beta} + \frac{1}{4}\mu\Gamma^\tau\chi\right)\epsilon(\tau)$$

$$\epsilon(\tau) = e^{-\frac{1}{2}\tau\mu\Gamma^{\tau}}\epsilon(0)$$

$$[\mathcal{H}, \mathcal{Q}] = i\frac{1}{2}\mu\Gamma_t\mathcal{Q}, \{\mathcal{Q}, \bar{\mathcal{Q}}\} = 2\Gamma^{\tau}\left(\mathcal{H} + \frac{1}{6}\mu\mathcal{M}_{\tau}\mathfrak{R}\right)$$

$$\chi_1 = \mathcal{R}\cos\left(\frac{1}{6}\tau\mu\right), \chi_2 = \mathcal{R}\sin\left(\frac{1}{6}\tau\mu\right), \chi_{\wp} = \frac{1}{3}\mu\mathfrak{J}_{\wp}$$

$$\mathcal{L}_0^{\mathcal{N}=2} = tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + \frac{1}{4}\left[\chi^{\alpha}, \chi^{\beta}\right]^2 - i\frac{1}{2}\hat{\bar{\psi}}\Gamma^{\tau}\mathfrak{D}_{\tau}\psi - \frac{1}{2}\hat{\bar{\psi}}\Gamma^{\alpha}[\chi_{\alpha}, \psi]\right) \bigotimes i\frac{1}{4}tr\left|\hat{\bar{\psi}}\psi\right|$$

$$\mathcal{L}_{Massive/Supermassive}^{\mathcal{N}=2}$$

$$= tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + \frac{1}{2}\left[\chi^{\alpha}, \chi^{\beta}\right]^2 - i\frac{1}{2}\hat{\bar{\psi}}\Gamma^{\tau}\mathfrak{D}_{\tau}\psi - \frac{1}{2}\hat{\bar{\psi}}\Gamma^{\alpha}[\chi_{\alpha}, \psi]\right)$$

$$+ tr\left(i\frac{1}{4}tr\left|\hat{\bar{\psi}}\psi\right| - \frac{1}{6}\mu^2(\chi_1^2 + \chi_2^2)\right)$$

$$\delta A_0 = -i\hat{\bar{\psi}}\Gamma_{\tau}\epsilon(\tau), \delta\chi_{\alpha} = -i\hat{\bar{\psi}}\Gamma_{\alpha}\epsilon(\tau)$$

$$\delta\psi = \left(-\Gamma^{\tau\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + i[\chi_1, \chi_2]\Gamma^{\alpha\beta} - \frac{1}{6}\mu\Gamma^{\alpha}\chi_{\alpha}\right)\epsilon(\tau)$$

$$\epsilon(\tau) = e^{\frac{1}{2}\tau\mu\Gamma^{t}}\epsilon(0)$$

$$[\mathcal{H}, \mathcal{Q}] = -i\frac{1}{2}\mu\Gamma^{\tau}\mathcal{Q}, \{\mathcal{Q}, \bar{\mathcal{Q}}\} = 2\Gamma^{\tau}\left(\mathcal{H} - \frac{1}{6}\mu\mathcal{M}_{\tau}\mathfrak{R}\right)$$

$$\chi_1 = \mathcal{R}\cos\left(\frac{1}{6}\tau\mu\right), \chi_2 = \mathcal{R}\sin\left(\frac{1}{6}\tau\mu\right)$$

$$\mathcal{L}_{Massive/Supermassive}^{\mathcal{N}=1+1} = tr\left[\frac{1}{2}(\mathfrak{D}_{\tau}\chi)^2 + i\frac{1}{2}\psi\mathfrak{D}_{\tau}\psi - \chi\psi\psi + \frac{1}{2}\Lambda\left(\sqrt{-g}\right)\chi^2 + \rho(\tau)\chi\right]$$

$$\delta_{\pm}A_0 = \delta_{\pm}\chi = if_{\pm}(\tau)\psi\varepsilon_{\pm}$$

$$\delta_{\pm}\psi = \left(f_{\pm}(\tau)\mathfrak{D}_{\tau}\chi - \ddot{f}_{\pm}(\tau)\chi - \kappa_{\pm}(\tau)\right)\varepsilon_{\pm}$$

$$\ddot{f}_{\pm}(\tau) = f_{\pm}(\tau)\Lambda\left(\sqrt{-g}\right)$$

$$\kappa_{\pm}(\tau) \coloneqq \iiint_{\tau_0}^{\tau} dt'\rho(\tau')f_{\pm}(\tau')$$

$$\delta_{++}A_0 = \delta_{++}\chi = f_{+}\left(f_{+}\mathfrak{D}_{\tau}\chi - \dot{f}_{+}\chi - \kappa_{+}1\right), \delta_{++}\psi = 0$$

$$\delta_{--}A_0 = \delta_{--}\chi = f_{-}\left(f_{-}\mathfrak{D}_{\tau}\chi - \dot{f}_{-}\chi - \kappa_{-}1\right), \delta_{--}\psi = 0$$

$$\delta_{\{+,-\}}A_0 = \delta_{\{+,-\}}\chi = 2f_{+}f_{-}\mathfrak{D}_{\tau}\chi - \left(\dot{f}_{+}f_{-} + f_{-}\dot{f}_{+}\right)\chi - (f_{+}\kappa_{-} + f_{-}\kappa_{+})1, \delta_{\{+,-\}}\psi = 0$$

$$f_{\pm}(\tau)\mathfrak{D}_{\tau}\chi = \dot{f}_{+}(\tau)\chi + \kappa_{\pm}(\tau)1$$

$$\chi(\tau) = f_{+}(\tau)\chi + \hbar_{+}(\tau)1$$

$$\chi(\tau) = f_{-}(\tau)\chi + \hbar_{-}(\tau)1$$

Cuántica supermasiva de Yang – Mills

$$ds^2 = 2d\chi^{+}d\chi^{-} - \frac{1}{72}\mu^2(\chi_1^2 + \chi_2^2 + 4\chi_3^2)d\chi^{+}d\chi^{+} + \sum_{\alpha=1}^{4} d\chi^{\alpha}d\chi^{\alpha}$$

Deformaciones de masa

$$\mathcal{L}_0^{\mathcal{N}=multidimensional} = tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + \frac{1}{4}\left[\chi^{\alpha},\chi^{\beta}\right]^2 - i\frac{1}{2}\hat{\hat{\bar{\psi}}}^{i}\Gamma^{\tau}\mathfrak{D}_{\tau}\underline{\hat{\hat{\bar{\psi}}}_{i}} - \frac{1}{2}\hat{\hat{\bar{\psi}}}^{i}\Gamma^{\alpha}\left[\chi_{\alpha},\underline{\hat{\hat{\bar{\psi}}}_{i}}\right]\right)$$

$$\mathcal{L}_0^{\mathcal{N}=multidimensional} = tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + \frac{1}{4}\left[\chi^{\alpha},\chi^{\beta}\right]^2 - i\hat{\bar{\psi}}\Gamma^{\tau}\mathfrak{D}_{\tau}\hat{\bar{\psi}} - \hat{\bar{\psi}}\Gamma^{\alpha}\left[\chi_{\alpha},\hat{\bar{\psi}}\right]\right)$$

$$\mathcal{L}_1^{\mathcal{N}=multidimensional} = tr\left(\bar{\psi}\mathcal{M}\psi + \frac{1}{3!}\delta_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c} + \mathfrak{J}_{\alpha\beta}\chi^{\alpha}\mathfrak{D}_{\tau}\chi^{\beta}\right)$$

$$\mathcal{M} = \mathcal{M}_{\tau}\Gamma^{\tau} + \mathcal{M}_{\alpha}\Gamma^{\alpha} - i\frac{1}{3!}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}$$

$$\mathcal{L}_2^{\mathcal{N}=multidimensional} = -tr\left(\frac{1}{2}\delta_{(\alpha\beta)}\chi^{\alpha}\chi^{\beta}\right)$$

$$tr\left(\mathfrak{J}_{\alpha\beta}\chi^{\alpha}\mathfrak{D}_{\tau}\chi^{\beta}\right) = tr\left(\mathfrak{J}_{[\alpha\beta]}\chi^{\alpha}\mathfrak{D}_{\tau}\chi^{\beta}\right) + \frac{d}{d\tau}tr\left(\frac{1}{2}\mathfrak{J}_{(\alpha\beta)}\chi^{\alpha}\chi^{\beta}\right) - tr\left(\frac{1}{2}\ddot{\mathfrak{J}}_{(\alpha\beta)}\chi^{\alpha}\chi^{\beta}\right)$$

$$\mathcal{L}^{\mathcal{T}} = \mathcal{L}^{-1}, \hat{\mathcal{L}}^{\dagger} = \hat{\mathcal{L}}^{-1}, \hat{\mathcal{L}}\Gamma_{\alpha}\hat{\mathcal{L}}^{-1} = \Gamma_{\alpha}\mathcal{L}_{\alpha}^{\beta}, \hat{\mathcal{L}}\Gamma_{\tau} = \Gamma_{\tau}\hat{\mathcal{L}}$$

$$\mathfrak{J}_{\alpha\beta} \longrightarrow \mathfrak{J}_{\alpha\beta} - 2\left(\mathcal{L}^{\mathcal{T}}\ddot{\mathcal{L}}\right)_{\alpha\beta}$$

$$\mathcal{M} = \mathcal{M}_{\alpha}\Gamma^{\alpha} - i\frac{1}{6}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}$$

$$\Gamma^{\tau}\mathcal{M}\Gamma^{\tau} = \mathcal{M}, \Gamma^{\alpha}\mathcal{M}\Gamma^{\beta} - \Gamma^{\beta}\mathcal{M}\Gamma^{\alpha} + \mathcal{M}\Gamma^{\alpha\beta} = 4i\mathcal{M}^{\alpha\beta c}\Gamma_{c} - i\Gamma^{\alpha\beta}\mathcal{M}$$

$$\left[\mathcal{M}, \Gamma^{\int_{\infty}^{\infty}\eta}\right] = 0, \mathcal{M}^2 = \mathcal{M}_{\alpha}\mathcal{M}^{\alpha} + \frac{1}{6}\mathcal{M}_{\alpha\beta c}\mathcal{M}^{\alpha\beta c} - i\mathcal{M}_{\alpha\beta c}\mathcal{M}^{c}\Gamma^{\alpha\beta} - \frac{1}{4}\mathcal{M}_{\alpha\beta e}\mathcal{M}_{cd}{}^{e}\Gamma^{\alpha\beta cd}$$

$$\delta A_0 = \bar{\psi}\Gamma_0\varepsilon(\tau)\Gamma_0\psi, \delta\chi_{\alpha} = \bar{\psi}\Gamma_{\alpha}\varepsilon(\tau) + \bar{\varepsilon}(\tau)\Gamma_{\alpha}\psi$$

$$\delta\psi = \left(-i\mathfrak{D}_{\tau}\chi_{\alpha}\Gamma^{\tau\alpha} - \frac{1}{2}\left[\chi_{\alpha},\chi_{\beta}\right]\Gamma^{\alpha\beta} + \mu\Delta\right)\varepsilon(\tau)$$

$$\varepsilon \coloneqq \mathfrak{G}(\tau)\hat{\varepsilon}, \partial_{\tau}\varepsilon(\tau) = \mu\prod(\tau)\varepsilon(\tau), \mu\prod(\tau) \coloneqq \partial_{\tau}\mathfrak{G}(\tau)\mathfrak{G}(\tau)^{-1}$$

$$\delta\mathcal{L}_0^{\mathcal{N}=multidimensional}$$

$$\cong \mu tr\left[\bar{\psi}\Gamma^{\tau}\left(-i\mathfrak{D}_{\tau}\Delta+\Gamma^{\tau\alpha}[\chi_{\alpha},\Delta]-\mathfrak{D}_{\tau}\chi_{\alpha}\Gamma^{\tau\alpha}\prod+i\frac{1}{2}[\chi_{\alpha},\chi_{\beta}]\Gamma^{\alpha\beta}\prod-i\mu\Delta\prod\right)\varepsilon\right]$$

$$+c\otimes c$$

$$\delta tr[\bar{\psi}\mathcal{M}\psi]=tr\left[\bar{\psi}\mathcal{M}\left(-i\mathfrak{D}_{\tau}\chi_{\alpha}\Gamma^{\tau\alpha}-\frac{1}{2}[\chi_{\alpha},\chi_{\beta}]\Gamma^{\alpha\beta}+\mu\Delta\right)\varepsilon\right]+c\otimes c$$

$$\delta tr\left[\frac{1}{3}\delta_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c}\right]=tr\left[\frac{1}{2}\delta_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\Gamma^{c}\varepsilon\right]+c\otimes c$$

$$\delta\mathcal{L}_{Massive/Supermassive}^{\mathcal{N}=multidimensional}\Rightarrow\mu tr[\bar{\psi}\mathfrak{D}_{\tau}(-i\Gamma^{\tau}\Delta+\chi_{\alpha}\Gamma^{\alpha}\prod-i\mathcal{M}\Gamma^{\tau}\chi_{\alpha}\Gamma^{\alpha})\varepsilon]+c\otimes c$$

$$\Delta\left(1+\Gamma^{(\lambda\mu)}\right)=(\mathcal{M}\chi-i\chi\Gamma^{\tau}\prod)\left(1+\Gamma^{(\lambda\mu)}\right)$$

$$\Delta=\mathcal{M}\chi-i\chi\Gamma^{\tau}\prod$$

$$\delta\left[\mathcal{L}_0^{\mathcal{N}=multidimensional}+\mu tr\left|\bar{\psi}\left(\mathcal{M}_{\alpha}\Gamma^{\alpha}-i\frac{1}{3!}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}\right)\psi+i\frac{2}{3}\mathcal{M}_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c}\right|\right]$$

$$=\frac{1}{2}\mu tr[\bar{\psi}[\chi_{\alpha},\chi_{\beta}]\Gamma^{\alpha\beta}(\mathcal{M}+3i\Gamma^{\tau}\prod)\varepsilon]$$

$$+\mu tr[\bar{\psi}(\mu\mathcal{M}\Delta-i\mu\Gamma^{\tau}\Delta\prod-\chi\widehat{\prod}+i\widetilde{\mathcal{M}}\Gamma^{\tau}\chi)\epsilon]+c\otimes c$$

$$\delta\left[\mathcal{L}_0^{\mathcal{N}=multidimensional}+\mu tr\left|\bar{\psi}\left(\mathcal{M}_{\alpha}\Gamma^{\alpha}-i\frac{1}{3!}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}\right)\psi+i\frac{2}{3}\mathcal{M}_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c}\right|\right]$$

$$=\mu^2 tr\left[\bar{\psi}\left(\mathcal{M}^2\chi+\frac{2}{3}\mathcal{M}\chi\mathcal{M}+\frac{1}{9}\chi\mathcal{M}^2-i\mu^{-1}\Gamma^{\tau}\left(\dot{\mathcal{M}}\chi+\frac{1}{3}\chi\dot{\mathcal{M}}\right)\right)\varepsilon\right]+c\otimes c$$

$$\mathcal{L}_0^{\mathcal{N}=4}=tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha}+\frac{1}{4}[\chi^{\alpha},\chi^{\beta}]^2-i\frac{1}{2}\hat{\bar{\psi}}\Gamma^{\tau}\mathfrak{D}_{\tau}\psi-\frac{1}{2}\hat{\bar{\psi}}\Gamma^{\alpha}[\chi_{\alpha},\psi]\right)$$

$$\mu\mathcal{L}_{\psi}^{\mathcal{N}=4}=i\mu tr\left[\hat{\bar{\psi}}(c\Gamma^{\lambda\mu}+\Gamma^{\tau}\mathfrak{H}+r\cos\theta+r\sin\theta\,\Gamma^{\tau})\psi\right]$$

$$(\psi,\bar{\psi})\longrightarrow\left(e^{\phi\Gamma^{\tau}}\psi,\bar{\psi}e^{\phi\Gamma^{\tau}}\right)$$

$$u\mathcal{L}_{Myers}=\mu tr(4i[\chi_1,\chi_2]\chi_3)$$

$$\delta A_0=-i\hat{\bar{\psi}}\Gamma_{\tau}\varepsilon(\tau),\delta\chi_{\alpha}=-i\hat{\bar{\psi}}\Gamma_{\alpha}\varepsilon(\tau)$$

$$\delta\psi=\left(-\Gamma^{\tau\alpha}\mathfrak{D}_{\tau}\chi_{\alpha}+i\frac{1}{2}[\chi_{\alpha},\chi_{\beta}]\Gamma^{\alpha\beta}+\mu\Delta\right)\varepsilon(\tau)$$

$$\partial_{\tau}\varepsilon(\tau)=\mu\prod\varepsilon(\tau)$$

$$-\mu tr\left[\hat{\bar{\psi}}(\mathfrak{D}_{\tau}\chi\prod+\Gamma^{\tau}\mathfrak{D}_{\tau}\Delta-2(c\Gamma^{\lambda\mu}+\Gamma^{\tau}\mathfrak{H}+r)\Gamma^{\tau}\mathfrak{D}_{\tau}\chi)\varepsilon\right]$$

$$\Delta=2\left(r+\Gamma^{\tau}\mathfrak{H}-c\Gamma^{\lambda\mu}\right)\chi+\Gamma^{\tau}\chi\prod\langle e^2+r^2\rangle$$

$$\mu\mathcal{L}_{\psi}^{multidimensional} = -i\frac{1}{2}\mu tr(\hat{\psi}\psi)$$

$$-i\mu tr\left[\hat{\hat{\psi}}(\mathfrak{D}_{\tau}\chi\prod + \Gamma^{\tau}\mathfrak{D}_{\tau}\Delta - \Gamma^{\tau}\mathfrak{D}_{\tau}\chi)\varepsilon\right]$$

$$\Delta \coloneqq \chi + \Gamma^{\tau}\chi\prod$$

$$-i\mu tr\left[\hat{\hat{\psi}}\mathcal{F}(1 - 3\Gamma^{\tau}\prod)\varepsilon\right]$$

$$\mathcal{L}_0 = tr\left[\frac{1}{2}\mathfrak{D}_{\tau}\chi\mathfrak{D}_{\tau}\chi + i\frac{1}{2}\psi\mathfrak{D}_{\tau}\psi + \chi\psi\psi\right]$$

$$\delta_{\mathcal{YM}}\mathrm{A}_0 = \delta_{\mathcal{YM}}\chi = i\psi\epsilon, \delta_{\mathcal{YM}}\psi = \mathfrak{D}_{\tau}\chi\epsilon$$

$$tr\left[i\frac{1}{2}\psi\mathfrak{D}_{\tau}\psi + \chi\psi\psi\right] = tr\left[i\frac{1}{2}\psi\partial_{\tau}\psi + (\chi - \mathrm{A}_0)\psi\psi\right]$$

$$\delta\mathrm{A}_0 = \delta\chi = if(\tau)\psi\epsilon, \delta\psi = (f(\tau)\mathfrak{D}_{\tau}\chi + \Delta)\epsilon$$

$$\delta\mathcal{L}_0 = tr\left[i\psi\epsilon\left(\mathfrak{D}_{\tau}(\hat{f}\chi + \triangle) - \tilde{f}\chi + i[\chi, \Delta]\right)\right] + \partial_{\tau}\mathfrak{K}$$

$$\mathfrak{K} = tr\left(\mathfrak{D}_{\tau}\chi\delta\chi - i\frac{1}{2}\psi\delta\psi\right)$$

$$\Delta \coloneqq -\tilde{f}\chi - \kappa_1$$

$$\delta\left[\mathcal{L}_0 + tr\left(\frac{1}{2}\left(\frac{\hat{f}}{f}\right)\chi^2 + \left(\frac{\bar{\kappa}}{f}\right)\chi\right)\right]$$

$$\mathcal{L}_0^{\mathcal{N}=multidimensional} = tr\left(\frac{1}{2}\mathfrak{D}_{\tau}\chi^{\alpha}\mathfrak{D}_{\tau}\chi_{\alpha} + \frac{1}{4}[\chi^{\alpha}, \chi^{\beta}]^2 + i\frac{1}{2}\Psi^{\mathfrak{T}}\mathfrak{D}_{\tau}\Psi - \frac{1}{2}\Psi^{\mathfrak{T}}\Gamma^{\alpha}[\chi_{\alpha}, \Psi]\right)$$

$$\mathcal{L}_{Massive/Supermassive}^{\mathcal{N}=multidimensional} = tr\left(\frac{1}{2}i\Psi^{\mathfrak{T}}\mathcal{M}\Psi + \frac{1}{3!}\delta_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c} + \mathfrak{I}_{\alpha\beta}\chi^{\alpha}\mathfrak{D}_{\tau}\chi^{\beta}\right)$$

$$\mathcal{M} = \frac{1}{2}\mathcal{M}_{\alpha\beta}\Gamma^{\alpha\beta} + \frac{1}{3!}\mathcal{M}_{\alpha\beta c}\Gamma^{\alpha\beta c}$$

$$\delta\mathrm{A}_0 = i\Psi^{\mathfrak{T}}\epsilon(\tau), \delta\chi^{\alpha} = i\Psi^{\mathfrak{T}}\Gamma^{\alpha}\epsilon(\tau), \delta\Psi = (\mathfrak{D}_{\tau}\chi + \mathfrak{F} + \mu\Delta)\epsilon(\tau)$$

$$\delta\mathcal{L}_0^{\mathcal{N}=multidimensional} = i\mu tr\left[\Psi^{\mathfrak{T}}(\mathfrak{D}_{\tau}\Delta + \mathfrak{D}_{\tau}\chi\prod + \mathfrak{F}\prod + i\Gamma^{\alpha}[\chi_{\alpha}, \Delta] + \mu\Delta\prod)\right]\epsilon(\tau)$$

$$\delta\mathcal{L}_1^{\mathcal{N}=multidimensional} = itr\left[\Psi^{\mathfrak{T}}\left(\mathcal{M}(\mathfrak{D}_{\tau}\chi + \mathfrak{F} + \mu\Delta) + \frac{1}{2}\delta_{\alpha\beta c}\chi^{\alpha}\chi^{\beta}\chi^{c}\right)\right]\epsilon(\tau)$$

$$\Delta \coloneqq -\chi\prod - \mathcal{M}\chi$$

$$\delta\left(\mathcal{L}_0^{\mathcal{N}=multidimensional} + \mu\mathcal{L}_1^{\mathcal{N}=multidimensional}\right) = i\mu tr\left[\Psi^{\mathfrak{T}}\mathfrak{F}_{(\alpha\beta}\mathcal{M}_{cd)}\Gamma^{\alpha\beta cd}\right]\epsilon(\tau) + \mathcal{O}(\mu^2)$$

$$\delta\left(\mathcal{L}_0^{\mathcal{N}=multidimensional} + \mu\mathcal{L}_1^{\mathcal{N}=multidimensional}\right) = -i\mu^2 tr\left[\Psi^{\mathfrak{T}}\chi_{\alpha}\mathfrak{M}^{\alpha}\epsilon(\tau)\right]$$

$$\mathfrak{M}^\alpha \equiv \mu^{-1}\left(\dddot{\mathcal{M}}\Gamma^\alpha + \frac{1}{3}\Gamma^\alpha \dddot{\mathcal{M}}\right) + \mathcal{M}^4\Gamma^\alpha + \frac{2}{3}\mathcal{M}\Gamma^\alpha\mathcal{M} + \frac{1}{2}\Gamma^\alpha\mathcal{M}^4$$

$$= \frac{1}{4}\mathcal{M}_{\beta cd}\mathcal{M}^{\beta cd}\Gamma^\alpha - \frac{2}{3}\mathcal{M}^{\alpha\beta c}\mathcal{M}_{\beta cd}\Gamma^d + \frac{1}{3}\mu^{-1}\dddot{\mathcal{M}}^\alpha{}_{\beta c}\Gamma^{\beta c} - \frac{1}{4}\mathcal{M}^\alpha{}_{\beta c}\mathcal{M}_{de}{}^c\Gamma^{\beta de}$$

$$-\frac{1}{2}\mu^{-1}\dddot{\mathcal{M}}_{\beta cd}\Gamma^{\alpha\beta cd} + \frac{1}{3}\mathcal{M}_{\beta cd}\mathcal{M}_{ef}{}^d\Gamma^{\alpha\beta cef} + \frac{1}{3}\mathcal{M}^\alpha{}_{\beta c}\mathcal{M}_{def}\Gamma^{\beta cdef}$$

Modelo de Cuantización de Dirac para campos cuánticos relativistas.

Cálculos preliminares

$$\mathcal{Q}|\psi\rangle = 0, \delta|\psi\rangle = \mathcal{Q}|\Lambda\rangle$$

$$\mathcal{Q}_{\mathrm{A}} = \mathcal{Q} + \mathcal{V}(\mathrm{A}) + \frac{1}{2}\mathcal{V}_2(\mathrm{A},\mathrm{A})$$

$$\delta_{\mathcal{YM}} = \frac{1}{2}\langle\mathcal{V}(\mathrm{A})\mathcal{Q}\mathcal{V}(\mathrm{A})\rangle + \frac{1}{3}\langle\mathcal{V}^3(\mathrm{A})\rangle + \frac{1}{4}\langle\mathcal{V}(\mathrm{A})\{\mathcal{V}_2(\mathrm{A},\mathrm{A}),\mathcal{V}(\mathrm{A})\}\rangle$$

Comportamientos endógeno y exógeno de las partículas y antipartículas supermasivas y masivas e hiperpartículas

$$\delta_{symp} = \iiint d\tau\left[\wp_\mu\dddot{\chi}^\mu - i\bar{\bar{\alpha}}^\mu\hat{\hat{\alpha}}_\mu\right]$$

$$\mathfrak{H} \coloneqq \frac{1}{2}\wp^2, \mathcal{L} \coloneqq \alpha^\mu\wp_\mu, \tilde{\mathcal{L}} \coloneqq \tilde{\alpha}^\mu\wp_\mu$$

$$\delta = \iiint d\tau\left[\wp_\mu\dddot{\chi}^\mu - i\bar{\bar{\alpha}}^\mu\hat{\hat{\alpha}}_\mu - e\mathcal{H} - \bar{\mu}\mathcal{L} - \mu\bar{\mathcal{L}}\right]$$

$$\delta\chi^\mu = \epsilon\wp^\mu + \xi\bar{\bar{\alpha}}^\mu + \bar{\bar{\xi}}\alpha^\mu, \delta\wp_\mu$$

$$\delta\alpha^\mu = i\xi\wp^\mu, \delta\bar{\bar{\alpha}}^\mu = -i\bar{\xi}\wp^\mu$$

$$\delta\mu = \hat{\xi}, \delta\bar{\mu} = \hat{\hat{\xi}}, \delta\epsilon = \ddot{\epsilon} + 2i\mu\bar{\xi} - 2i\bar{\mu}\xi$$

$$[\chi^\mu, \wp_\nu] = i\delta^\mu_\nu, [\bar{\alpha}^\mu, \alpha^\nu] = \eta^{\mu\nu}$$

$$\left[\tilde{\mathcal{L}}, \mathcal{L}\right] = 2\mathfrak{H}, [\mathcal{H}, \mathfrak{L}] = 0, [\mathcal{H}, \bar{\mathfrak{L}}]$$

$$|\varphi\rangle = \sum_{\delta=0}^{\infty}|\varphi_\delta\rangle, |\varphi_\delta\rangle = \frac{1}{\delta!}\varphi_{\mu_1\cdots\mu_\delta}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_\delta}|0\rangle$$

$$\wp_\mu = -i\partial_\mu, \bar{\alpha}^\mu = \eta^{\mu\nu}\frac{\partial}{\partial\alpha^\nu}, \mathfrak{H} = -\frac{1}{2}\Box, \mathfrak{L} = -i\alpha^\mu\partial_\mu, \bar{\mathfrak{L}} = -i\frac{\partial}{\partial\alpha^\mu}\partial^\mu$$

$$i\mathfrak{L}|\varphi_\delta\rangle = \frac{1}{\delta!}\partial_{\left(\mu_1\varphi_{\mu_2\cdots\mu_{\delta+1}}\right)}\alpha^{\mu_1}\cdots\alpha^{\mu_{\delta+1}}|0\rangle, i\bar{\mathfrak{L}}|\varphi_\delta\rangle = \frac{1}{(\delta-1)!}\partial^\nu_{\varphi_{\nu\mu_2\cdots\mu_{\delta+1}}}\alpha^{\mu_2}\cdots\alpha^{\mu_\delta}|0\rangle$$

$$\langle\varphi_\delta| = \frac{1}{\delta!}\varphi^{\circledast}_{\mu_1\cdots\mu_\delta}(\chi)\langle 0|\hat{\alpha}^{\mu_1}\cdots\hat{\alpha}^{\mu_\delta}$$

$$\langle\chi'_\delta|\varphi_\delta\rangle = \frac{1}{\delta!\,\delta'!}\iiint d^{\mathfrak{D}}x\chi^{\circledast}_{\mu_1\cdots\mu_{\delta'}}\varphi^{\diamond}_{\nu_1\cdots\nu_\delta}\langle 0|\hat{\alpha}^{\mu_1}\cdots\hat{\alpha}^{\mu_{\delta'}}\hat{\alpha}^{\nu_1}\cdots\hat{\alpha}^{\nu_{\delta'}}|0\rangle$$

$$= \delta_{\mathscr{s}\mathscr{s}'}\iiint d^{\mathfrak{D}}x\chi^{\circledast}_{\mu_1\cdots\mu_{\delta'}}\varphi^{\nu_1\cdots\nu_\delta}$$

$$\langle\chi|\mathfrak{H},\mathfrak{L},\bar{\mathfrak{L}}|\psi\rangle = \forall\chi,\psi\in\mathfrak{H}_{phys}$$

$$|\psi\rangle\in\mathfrak{H}_{phys}\Leftrightarrow\mathfrak{H}|\varphi\rangle = 0,\bar{\mathfrak{L}}|\varphi\rangle$$

$$\Box\varphi_{\mu_1\cdots\mu_\delta} = 0,\partial^{\nu}_{\varphi_{\nu\mu_1\cdots\mu_{\delta-1}}}$$

$$\delta\varphi_{\mu_1\cdots\mu_\delta} = \delta\partial_{(\mu_1}\xi_{\mu_2\cdots\mu_\delta)}$$

$$\Box\mathrm{A}_\mu = 0,\partial^\mu\mathrm{A}_\mu$$

$$|\varphi_{null}\rangle = \mathcal{L}|\xi\rangle,\mathfrak{H}|\xi\rangle = \bar{\mathcal{L}}|\xi\rangle$$

Cuantización BRST

$$[\mathfrak{G}_i,\mathfrak{G}_j] = f_{ij}^{\mathscr{k}}\mathfrak{G}_{\mathscr{k}}$$

$$\mathfrak{G}_i\mapsto(\beta_i,c^i),\{\beta_i,c^i\} = \delta_i^j$$

$$\mathcal{Q} \coloneqq c^i\mathfrak{G}_i - \frac{1}{2}f_{ij}^{\mathscr{k}}c^ic^j\beta_{\mathscr{k}}$$

$$\mathfrak{H}\mapsto(\beta,c),\{\beta,c\},\mathfrak{L}\mapsto(\mathfrak{B},\bar{\mathfrak{C}}),\{\mathfrak{B},\bar{\mathfrak{C}}\},\bar{\mathfrak{L}}\mapsto(\bar{\mathfrak{B}},\mathfrak{C}),\{\bar{\mathfrak{B}},\mathfrak{C}\}$$

$$\mathcal{Q} \coloneqq c\Box + (\bar{\mathfrak{C}}\alpha^\mu + \mathfrak{C}\bar{\alpha}^\mu)\partial_\mu - \mathfrak{C}\bar{\mathfrak{C}}\beta,\mathcal{Q}^2$$

$$(\bar{\alpha}^\mu,\beta,\bar{\mathfrak{B}},\bar{\mathfrak{C}})|0\rangle$$

$$|\psi\rangle = \sum_{\delta=0}^{\infty}\sum_{\wp,q,r=0}^{4}c^{\wp}\,\mathfrak{C}^q\mathfrak{B}^r|\psi_{\delta,\wp,q,r}\rangle,|\psi_{\delta,\wp,q,r}\rangle = \frac{1}{\delta!}\psi^{(\wp,q,r)}_{\mu_1\cdots\mu_\delta}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_\delta}|0\rangle$$

$$\bar{\alpha}^\mu = \frac{\partial}{\partial\alpha^\mu},\beta = \frac{\partial}{\partial c},\bar{\mathfrak{B}} = \frac{\partial}{\partial\mathfrak{C}},\bar{\mathfrak{C}} = \frac{\partial}{\partial\mathfrak{B}}$$

$$c^\dagger = c,\beta^\dagger = \beta,\mathfrak{C}^\dagger = -\bar{\mathfrak{C}},\mathfrak{B}^\dagger = -\bar{\mathfrak{B}}$$

$$\mathfrak{G} \coloneqq c\beta + \mathfrak{C}\bar{\mathfrak{B}} - \mathfrak{B}\bar{\mathfrak{C}} = \mathfrak{N}_c + \mathfrak{N}_{\mathfrak{C}} - \mathfrak{N}_{\mathfrak{B}}$$

$$\mathfrak{H} = \bigoplus{}_{\delta}\overset{\infty}{=}{}_0\bigoplus{}_{\mathscr{k}}\overset{2}{=}{}_{-1}\mathfrak{H}_{\mathscr{s},\mathscr{k}}$$

$$|\varphi_\delta\rangle = |\varphi_\delta\rangle + c\mathfrak{B}|f_{\delta-1}\rangle, |\varphi_\delta\rangle = \frac{1}{\delta!}\varphi_{\mu_1\cdots\mu_\delta}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_\delta}|0\rangle, |f_{\delta-1}\rangle$$

$$= \frac{1}{(\delta-1)!}f_{\mu_1\cdots\mu_{\delta-1}}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_{\delta-1}}|0\rangle, |\chi_{\delta-2}\rangle$$

$$= \frac{1}{(\delta-2)!}x_{\mu_1\cdots\mu_{\delta-2}}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_{\delta-2}}|0\rangle$$

$$\Box\varphi_{\mu_1\cdots\mu_\delta} - \delta\partial_{(\mu_1}f_{\mu_2\cdots\mu_\delta)}, \Box\chi_{\mu_1\cdots\mu_\delta} - \partial^{\wp}f_{\wp\mu_1\mu_2\cdots\mu_{\delta-2}}, \partial^{\wp}\varphi_{\wp\mu_1\mu_2\cdots\mu_{\delta-1}} - (\delta-1)\partial_{(\mu_1}\chi_{\mu_2\cdots\mu_{\delta-1})}$$
$$- f_{\mu_1\cdots\mu_{\delta-1}}$$

$$|\Lambda_\delta\rangle = \mathfrak{B}|\xi_{\delta-1}\rangle, |\xi_{\delta-1}\rangle = \frac{1}{(\delta-1)!}\xi_{\mu_1\cdots\mu_{\delta-1}}(\chi)\alpha^{\mu_1}\cdots\alpha^{\mu_{\delta-1}}|0\rangle$$

$$\delta\varphi_{\mu_1\cdots\mu_\delta} = \delta\partial_{(\mu_1}\xi_{\mu_2\cdots\mu_\delta)}, \delta\chi_{\mu_1\cdots\mu_{\delta-2}} = \partial^{\wp}\xi_{\wp\mu_1\mu_2\cdots\mu_{\delta-1}}, \delta f_{\mu_1\cdots\mu_{\delta-1}} = \Box\xi_{\mu_1\cdots\mu_{\delta-1}}$$

$$\delta_{sft} = \frac{1}{2}\langle\varphi_\delta|\mathcal{Q}|\varphi_\delta\rangle$$

$$= \frac{1}{2}\iiint d^{\mathfrak{D}}x\left[\frac{1}{\delta!}\varphi^{\mu_1\cdots\mu_\delta}\Box\varphi_{\mu_1\cdots\mu_\delta} - \frac{1}{(\delta-1)!}f^{\mu_1\cdots\mu_{\delta-1}}\Box f_{\mu_1\cdots\mu_{\delta-1}}\right.$$
$$+ \frac{2}{(\delta-1)!}f^{\mu_1\cdots\mu_{\delta-1}}\left(\partial\bigotimes\varphi_{\mu_1\cdots\mu_{\delta-1}} - (\delta-1)\partial_{\mu_1}\chi_{\mu_2\cdots\mu_{\delta-1}}\right)$$
$$\left.- \frac{1}{(\delta-2)!}\chi^{\mu_1\cdots\mu_{\delta-2}}\Box\chi_{\mu_1\cdots\mu_{\delta-2}}\right]$$

$$\delta = \iiint d^{\mathfrak{D}}x\left[\frac{1}{2}\mathrm{A}^\mu\Box\mathrm{A}_\mu - \frac{1}{2}f^2 + f\partial\bigotimes\mathrm{A}\right]$$

$$\delta = \iiint d^{\mathfrak{D}}x\left[\frac{1}{2}\mathrm{A}^\mu\Box\mathrm{A}_\mu - \frac{1}{2}\left(\partial\bigotimes\mathrm{A}\right)^2\right] = -\frac{1}{4}\iiint d^{\mathfrak{D}}x\mathfrak{F}^{\mu\nu}\Box\mathfrak{F}_{\mu\nu}$$

$$\langle\varphi_\delta|\mathfrak{E}_\delta\rangle = \langle\varphi_\delta|\mathfrak{E}_\delta\rangle$$

$$= \iiint d^{\mathfrak{D}}x\left[\frac{1}{\delta!}\varphi^{\mu_1\cdots\mu_\delta}\Box\mathfrak{E}_{\mu_1\cdots\mu_\delta} + \frac{1}{(\delta-1)!}f^{\mu_1\cdots\mu_{\delta-1}}\Box\mathfrak{E}_{\mu_1\cdots\mu_{\delta-1}}\right.$$
$$\left.- \frac{1}{(\delta-2)!}\chi^{\mu_1\cdots\mu_{\delta-2}}\Box\mathfrak{E}_{\mu_1\cdots\mu_{\delta-2}}\right]$$

$$\langle\Lambda_\delta|\mathfrak{N}_\delta\rangle = \langle\mathfrak{N}_\delta|\Lambda_\delta\rangle = \frac{1}{(\delta-1)!}\iiint d^{\mathfrak{D}}x\xi^{\mu_1\cdots\mu_\delta}\Box\mathfrak{N}_{\mu_1\cdots\mu_\delta}$$

Cálculos cromodinámicos

$$\delta_{pigment} = \iiint d\tau[-i\widehat{\omega}^\alpha\widetilde{\omega}_\alpha]$$

$$|\mathcal{V}\rangle_{pigment} = \sum_{r=0}^{\infty}\frac{1}{r!}\mathcal{V}^{\alpha_1\cdots\alpha_r}\omega_{\alpha_1\cdots\alpha_r}|0\rangle_{pigment}$$

$$\langle\mathcal{U}|\mathcal{V}\rangle = \mathcal{U}^{\alpha}\mathcal{V}^{\beta}\langle\widetilde{\omega}_{\alpha}|\omega_{\beta}\rangle = \delta_{\alpha\beta}\mathcal{U}^{\alpha}\mathcal{V}^{\beta}$$

$$tr(\mathfrak{T}_{\alpha}\mathfrak{T}_{\beta}) = \underline{\langle\overline{\omega}^{c}|\mathfrak{T}_{\alpha}\mathfrak{T}_{\beta}|\omega_{c}\rangle} = f_{\alpha c}{}^{d}f_{\beta d}^{c} = \delta_{\alpha\beta}$$

$$\chi \coloneqq \mathfrak{H}_1 \bigotimes \mathfrak{H}_{pigment}^{1}$$

$$|\psi\rangle = \left(\alpha_{\mu}^{\alpha}(\chi)\alpha^{\mu}|0\rangle + f^{\alpha}(\chi)c\mathfrak{B}|0\rangle\right)\bigotimes|\omega_{\alpha}\rangle$$

$$\mathcal{Q} = c\Box + \delta^{\mu}\partial_{\mu} - \mathcal{M}\beta, \delta^{\mu} \coloneqq \overline{\mathfrak{C}}\alpha^{\mu} + \mathfrak{C}\bar{\alpha}^{\mu}, \mathcal{M} \coloneqq \mathfrak{C}\overline{\mathfrak{C}}$$

$$\delta^{\mu\nu} \coloneqq \bar{\alpha}^{\mu}\alpha^{\nu} - \bar{\alpha}^{\nu}\alpha^{\mu}, [\delta^{\mu\nu}, \delta^{\rho}] = 2\eta^{[\nu}\delta^{\mu]}, [\delta^{\mu\nu}, \delta^{\rho\sigma}] = 4\eta^{[\rho[\nu}\delta^{\mu]\sigma]}$$

$$\delta^{\mu}\delta^{\nu} = \mathfrak{M}(\eta^{\mu\nu} - \delta^{\mu\nu}), \delta^{\mu}\mathfrak{M} = \mathfrak{M}\delta^{\mu}, [\delta^{\mu\nu}, \mathfrak{Q}] = 2\delta^{[\mu}\partial^{\nu]}$$

$$\mathrm{A}_{\mu} \coloneqq \mathrm{A}_{\mu}^{\alpha}\tau_{\alpha} = \mathrm{A}_{\mu}^{\alpha}(\chi)f_{\alpha\beta}^{c}\omega_{c}\overline{\omega}^{\beta}, \mathfrak{D}_{\mu} \coloneqq \partial_{\mu} + \mathrm{A}_{\mu}, [\mathfrak{D}_{\mu}, \mathfrak{D}_{\nu}] = \mathfrak{F}_{\mu\nu}, \mathfrak{F}_{\mu\nu}$$

$$\coloneqq \partial_{\mu}\mathrm{A}_{\nu} - \partial_{\nu}\mathrm{A}_{\mu} + [\mathrm{A}_{\mu}, \mathrm{A}_{\nu}], \mathfrak{F}_{\mu\nu} \coloneqq \mathfrak{F}_{\mu\nu}^{\alpha}\tau_{\alpha}, \mathfrak{F}_{\mu\nu}^{\alpha} = \partial_{\mu}\mathrm{A}_{\nu}^{\alpha} - \partial_{\nu}\mathrm{A}_{\mu}^{\alpha} + f_{\beta c}^{\alpha}\mathrm{A}_{\mu}^{\beta}\mathrm{A}_{\nu}^{c}$$

$$\mathcal{Q}_{\mathrm{A}} \coloneqq c\Delta + \delta^{\mu}\mathfrak{D}_{\mu} - \mathfrak{M}\beta, \Delta \coloneqq \mathfrak{D}^{\mu}\mathfrak{D}_{\mu} + \mathfrak{F}_{\mu\nu}\delta^{\mu\nu}$$

$$\mathcal{Q}_{\mathrm{A}}^{2} \coloneqq -\frac{3}{2}\mathfrak{M}\delta^{\mu\nu}\mathfrak{F}_{\mu\nu} - c\left(\delta^{\mu}\mathfrak{D}^{\nu}\mathfrak{F}_{\mu\nu} + \mathfrak{D}_{\mu}\mathfrak{F}_{\nu\rho}\delta^{\mu}\delta^{\nu\rho}\right)$$

$$\mathfrak{D}_{\mu}\mathfrak{F}_{\nu\rho} \coloneqq [\mathfrak{D}_{\mu}, \mathfrak{F}_{\nu\rho}] = \mathfrak{D}_{\mu}\mathfrak{F}_{\nu\rho}^{\alpha}\mathfrak{T}_{\alpha}$$

$$\delta^{\mu}\delta^{\nu\rho}|_{\chi} = 2\left(\overline{\mathfrak{C}}\alpha^{\mu} + \mathfrak{C}\bar{\alpha}^{\mu}\right)\alpha^{|\nu}\alpha^{\rho|}\Big|_{\chi} = 2\alpha^{\mu}\alpha^{|\nu}\alpha^{\rho|}\overline{\mathfrak{C}} + \mathfrak{C}\alpha^{|\nu}\alpha^{\rho|}\bar{\alpha}^{\mu} + \mathfrak{C}\eta^{\mu|\nu}\bar{\alpha}^{\rho|}\Big|_{\chi} = 2\mathfrak{C}\eta^{\mu|\nu}\bar{\alpha}^{\rho|}$$

$$\mathcal{Q}_{\mathrm{A}}^{2}\Big|_{\chi} = c\left(\overline{\mathfrak{C}}\alpha^{\mu} + \mathfrak{C}\bar{\alpha}^{\mu}\right)\mathfrak{D}^{\rho}\mathfrak{F}_{\rho\mu}$$

Espacio – tiempo cuántico relativista. Cálculos formales

$$\delta_{sft,\mathrm{A}}[\psi] = \frac{1}{2}\langle\psi|\mathcal{Q}_{\mathrm{A}}|\psi\rangle = \iiint d^{\mathfrak{D}}x\mathfrak{D}^{\mu}\alpha_{\alpha}^{\nu}\mathfrak{D}_{\mu}\alpha_{\nu}^{\alpha} - \frac{1}{2}f_{\alpha}f^{\alpha} + f_{\alpha}\mathfrak{D}^{\mu}\alpha_{\mu}^{\alpha} - f_{\beta c}^{\alpha}\mathfrak{F}_{\alpha}^{\mu\nu}\alpha_{\mu}^{\beta}\alpha_{\nu}^{c}$$

$$\delta_{sft,\mathrm{A}}[\alpha] = \iiint d^{\mathfrak{D}}x\left[-\frac{1}{4}\left(\mathfrak{D}^{\mu}\alpha_{\alpha}^{\nu} - \mathfrak{D}^{\nu}\alpha_{\alpha}^{\mu}\right)\left(\mathfrak{D}_{\mu}\alpha_{\nu}^{\alpha} - \mathfrak{D}_{\nu}\alpha_{\mu}^{\alpha}\right) - \frac{1}{2}f_{\beta c}^{\alpha}\mathfrak{F}_{\alpha}^{\mu\nu}\alpha_{\mu}^{\beta}\alpha_{\nu}^{c}\right]$$

$$\delta_{\mathcal{YM}}[\mathrm{A}] = \delta_{\mathcal{YM}}[\mathrm{A}] + \delta_{1}[\mathrm{A}, \alpha] + \delta_{2}[\mathrm{A}; \alpha] + \mathfrak{O}(\alpha^{3}), \delta_{1}[\mathrm{A}, \alpha]$$

$$= \iiint d^{\mathfrak{D}}x\alpha_{\mu}^{\alpha}\frac{\delta\delta_{\mathcal{YM}}}{\delta\mathrm{A}_{\mu}^{\alpha}}\Bigg|_{\mathrm{A}=\mathfrak{A}} = \iiint d^{\mathfrak{D}}x\left(\mathfrak{D}^{\mu}\mathfrak{F}_{\mu\nu}^{\alpha}\alpha_{\alpha}^{\nu}\right), \delta_{2}[\mathrm{A}; \alpha] = \delta_{sft,\mathrm{A}}[\alpha]$$

$$\delta\alpha_{\mu}^{\alpha} = \mathfrak{D}_{\mu}\lambda^{\alpha} + f_{\beta c}^{\alpha}\alpha_{\mu}^{\beta}\lambda^{c} = \delta_{0}\alpha_{\mu}^{\alpha} + \delta_{1}\alpha_{\mu}^{\alpha}$$

$$\delta_{0}\delta_{2}[\mathrm{A}; \alpha] + \delta_{1}\delta_{1}[\mathrm{A}; \alpha]$$

$$|1\rangle := \mathfrak{B}|0\rangle, \langle 1| := \langle 0|\widehat{\mathfrak{B}} = -(|1\rangle)^{\dagger}$$

$$\langle \mathfrak{O}(\chi)\rangle := \iiint d^{\mathfrak{D}}x tr\langle 1|\mathfrak{O}(\chi)|1\rangle = \iiint d^{\mathfrak{D}}x \langle \widehat{\omega}^{\alpha}|\otimes\langle 1|\mathfrak{O}(\chi)|1\rangle\otimes|\omega_{\alpha}\rangle$$

$$\mathcal{V}(\lambda)|1\rangle = \lambda^{\alpha}(\chi)\mathfrak{B}|0\rangle\otimes\tau_{\alpha}, \mathcal{V}(\mathfrak{A})|1\rangle = \left(\mathrm{A}_{\mu}^{\alpha}(\chi)\alpha^{\mu} + \partial^{\mu}\mathrm{A}_{\mu}^{\alpha}(\chi)c\mathfrak{B}\right)|0\rangle\otimes\tau_{\alpha}, \mathcal{V}(\mathfrak{E})|1\rangle$$

$$= \mathfrak{E}_{\mu}^{\alpha}(\chi)c\alpha^{\mu}|0\rangle\otimes\tau_{\alpha}, \mathcal{V}(\mathfrak{N})|1\rangle = \mathfrak{N}^{\alpha}(\chi)c\mathfrak{C}|0\rangle\otimes\tau_{\alpha}$$

$$\langle \mathcal{V}(\mathfrak{A}), \mathcal{V}(\mathfrak{E})\rangle = \iiint d^{\mathfrak{D}}x\mathrm{A}_{\mu}^{\alpha}(\chi)\mathfrak{E}_{\alpha}^{\mu}(\chi)$$

$$\delta_{\mathcal{YM}}[\mathrm{A}] = \iiint d^{\mathfrak{D}}x\mathrm{A}_{\mu}^{\alpha}\left[\frac{1}{2}\mathfrak{B}_1(\mathrm{A}) + \frac{1}{3!}\mathfrak{B}_2(\mathrm{A},\mathrm{A}) + \frac{1}{4!}\mathfrak{B}_3(\mathrm{A},\mathrm{A},\mathrm{A})\right]_{\alpha}^{\mu}$$

$$\delta_{\mathcal{YM}}[\mathrm{A}] = \frac{1}{2}\langle \mathcal{V}(\mathfrak{A})\mathcal{Q}\mathcal{V}(\mathfrak{A})\rangle + \frac{1}{3}\langle \mathcal{V}^3(\mathfrak{A})\{\mathcal{V}_2(\mathrm{A},\mathrm{A}), \mathcal{V}(\mathfrak{A})\}\rangle \equiv -\frac{1}{4}\iiint d^{\mathfrak{D}}x\mathfrak{F}_{\alpha}^{\mu\nu}\Box\mathfrak{F}_{\mu\nu}^{\alpha}$$

Ecuación de Maurer-Cartan

$$\mathcal{Q}_{\mathrm{A}} = c\left(\mathfrak{D}^{\mu}\mathfrak{D}_{\mu} + \mathfrak{F}_{\mu\nu}\delta^{\mu\nu}\right) + \delta^{\mu}\mathfrak{D}_{\mu} - \mathfrak{M}\beta, \mathcal{Q}_{\mathrm{A}} = \mathcal{Q} + \mathcal{V}(\mathfrak{A}) + \frac{1}{2}\mathcal{V}_2(\mathrm{A},\mathrm{A}), \mathcal{V}(\mathfrak{A})$$

$$:= \delta^{\mu}\mathrm{A}_{\mu} + c\left(2\mathrm{A}^{\mu}\partial_{\mu} + \left(\mathrm{A}^{\mu}\partial_{\mu}\right) + 2\left(\partial_{\mu}\mathrm{A}_{\nu}\right)\delta^{\mu\nu}\right), \mathcal{V}_2(\mathrm{A},\mathrm{A}) := 2c\left(\mathrm{A}^2 + [\mathrm{A}^{\mu},\mathrm{A}^{\nu}]\delta_{\mu\nu}\right)$$

$$\mathcal{Q}_{\mathrm{A}}^2 := \{\mathcal{Q}, \mathcal{V}(\mathfrak{A})\} + \frac{1}{2}\{\mathcal{V}(\mathfrak{A}), \mathcal{V}(\mathfrak{A})\} + \frac{1}{2}\{\mathcal{Q}, \mathcal{V}_2(\mathrm{A},\mathrm{A})\} + \frac{1}{2}\{\mathcal{V}(\mathfrak{A}), \mathcal{V}_2(\mathrm{A},\mathrm{A})\}$$

$$\mathcal{V}(\mathfrak{E}) := c\tilde{\delta}^{\mu}\mathfrak{E}_{\mu}, \tilde{\delta}^{\mu} := \overline{\mathfrak{C}}\alpha^{\mu} - \mathfrak{C}\bar{\alpha}^{\mu}, \mathfrak{E}_{\mu} = \mathfrak{E}_{\mu}^{\alpha}\tau_{\alpha}$$

$$\delta\mathcal{Q}_{\mathrm{A}} = [\mathcal{Q}_{\mathrm{A}}, \mathcal{V}(\lambda)] \mapsto \delta\left(\mathcal{Q}_{\mathrm{A}}^2\right) = \left[\mathcal{Q}_{\mathrm{A}}^2, \mathcal{V}(\lambda)\right]$$

$$[\mathcal{Q}_{\mathrm{A}}, \mathcal{V}(\lambda)] = c\left(2(\mathfrak{D}^{\mu}\Lambda)\mathfrak{D}_{\mu} + (\mathfrak{D}^2\Lambda) + \left[\mathfrak{F}_{\mu\nu}, \Lambda\right]\delta^{\mu\nu}\right) + \delta^{\mu}\left(\mathfrak{D}_{\mu}\Lambda\right)$$

$$[\mathcal{Q}_{\mathrm{A}}, \mathcal{V}(\lambda)] = \mathcal{Q}_{\mathrm{A}+\delta_{\lambda}\mathrm{A}} - \mathcal{Q}_{\mathrm{A}}$$

$$[\mathcal{Q}_{\mathrm{A}}, \mathcal{V}(\mathfrak{E})] = -c\mathfrak{M}\mathfrak{D}^{\mu}\mathfrak{E}_{\mu}^{\alpha}\tau_{\alpha}, \mathcal{V}(\mathfrak{N}) := c\mathfrak{M}\mathfrak{N}$$

CONCLUSIONES.

En mérito a la formalización matemática contenida en el apartado de Resultados y Discusión, queda demostrado lo que sigue:

1. Que, los espacios cuánticos, son susceptibles de deformación o perforación, a propósito de la interacción de las partículas y antipartículas que les sean congénitas.

2. Que, un campo cuántico, es susceptible de deformación geométrica, es decir, padece de curvatura, cuando una partícula o antipartícula, a propósito de su masa superior, modifica espacialmente su posición, repercutiendo en sus perímetros aproximados.

3. Que, un campo cuántico, es susceptible de deformación geométrica, es decir, padece de curvatura, cuando una partícula o antipartícula, se aproxima, alcanza o supera la velocidad de la luz, lo que explica además la brecha de masa que no es arbitraria y es habitualmente superior a cero.

4. Que, un campo cuántico, es susceptible de perforación, es decir, adherido a la existencia de microagujeros cuánticos, cuando una partícula o antipartícula, a propósito de su masa superlativa, modifica espacialmente su posición, repercutiendo en sus perímetros aproximados.

5. Las partículas y antipartículas mencionadas en el numeral 2, se denominan partículas masivas o antipartículas masivas, según sea el caso, es decir, aquellas que, por su interacción temporal – espacial, deforman geométricamente el espacio cuántico en el que interactúan, provocando la curvatura en sentido estricto.

6. Las partículas y antipartículas mencionadas en el numeral 3, se denominan hiperpartículas, indistintamente si se trata de una partícula o una antipartícula, esto, cuando pese a no tener masa, se aproximan, alcanzan o superan la velocidad de la luz. Considérese también en este escenario, la existencia de suprapartículas, es decir, aquellas que a más de ser masivas o supermasivas, según sea el caso, a razón de su masa, son capaces de aproximarse, alcanzar o superar la velocidad de la luz, en cuyo caso, el surgimiento de microagujeros cuánticos es inevitable.

7. Que, la deformación de un espacio cuántico o en su defecto, la generación de agujeros negros cuánticos, ocurren por la existencia de fenómenos cuánticos propios e inherentes al sistema de partículas y antipartículas propuesto, esto es, fenómenos tales como la superposición, el entrelazamiento o la colisión, lo que, para todos los casos, aplica el puente Einstein – Rosen (Paradoja EPR).

8. En mérito a la deformación o perforación de los espacios cuánticos, se supone la existencia de ondas cuánticas, al igual que las ondas gravitacionales en cuanto a su fenomenología.

9. La perforación de un campo cuántico específico, no solamente supone la existencia de un agujero negro cuántico, el cual, incluso puede ser supermasivo, sino que, además, supone la existencia de pluridimensiones en las que ocurre un fenómeno de dualidad divergente.

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APÉNDICE A

Cuestiones secundarias relativas a la teoría cuántica de campos relativistas.

Métrica de Levi-Civita.

$$\widehat{\mathfrak{G}}_{\mu\nu}\left[\mathcal{V}^{(\lambda)}\right]^{\nu} = \lambda\mathcal{V}_{\mu}^{(\lambda)}\widehat{\mathfrak{G}}_{\mu\sigma}g^{\mu\sigma}\left[\mathcal{V}^{(\lambda)}\right]_{\nu} = \lambda\mathcal{V}_{\mu}^{(\lambda)}\left[\ln\widehat{\mathfrak{G}}\right]_{\mu\nu} = \mathcal{V}_{\mu}^{(\lambda)}\mathcal{V}_{\nu}^{(\lambda)}\ln(\lambda), \left[\widehat{\mathfrak{G}}^{-1}\right]^{\mu\nu} = \left[\mathcal{V}^{(\lambda)}\right]^{\mu}\left[\mathcal{V}^{(\lambda)}\right]^{\nu}\lambda^{-1}\left[\ln\left(\widehat{\mathfrak{G}}^{-1}\right)\right]^{\mu\nu} = -\left[\mathcal{V}^{(\lambda)}\right]^{\mu}\left[\mathcal{V}^{(\lambda)}\right]^{\nu}\ln(\lambda)$$

$$Tr\widehat{\mathfrak{G}} = \sum_{\lambda}\lambda\ Tr_{\mathcal{M}}\widehat{\mathfrak{G}}g^{-1} = \widehat{\mathfrak{G}}_{\mu\nu}g^{\mu\nu}, \mathfrak{H} = Tr\widehat{\mathfrak{G}}\ln\widehat{\mathfrak{G}}^{-1} = \widehat{\mathfrak{G}}_{\mu\nu}\left[\ln\left(\widehat{\mathfrak{G}}^{-1}\right)\right]^{\nu\mu} = -\sum\lambda\ln\lambda$$

$$\mathfrak{H} = Trg\ln g^{-1}$$

$$\mathcal{L} = -Trg\ln g^{-1} + Trg\ln\mathfrak{G}^{-1}, \mathcal{L} \equiv -Tr_{\mathcal{M}}\mathfrak{G}g^{-1} = \sum_{\lambda'}\ln(\lambda')$$

$$\delta = \frac{1}{\ell_{\mathcal{P}}^{d}}\iiint\sqrt{|-g|}\,\mathcal{L}dr, \tilde{\delta} = \frac{1}{\ell_{\mathcal{P}}^{d}}\iiint\sqrt{|-g|}\,\tilde{\mathcal{L}}dr, \tilde{\delta} = \beta\delta_{\mathfrak{G}} + \alpha\delta_{\mathcal{M}}, \delta_{\mathfrak{G}} = \frac{1}{\ell_{\mathcal{P}}^{d}}\iiint\sqrt{|-g|}\mathcal{L}_{\mathfrak{G}}dr, \delta_{\mathcal{M}} = \frac{1}{\ell_{\mathcal{P}}^{d}}\iiint\sqrt{|-g|}\mathcal{L}_{\mathcal{M}}dr, \mathcal{L}_{\mathfrak{G}} = \left(\mathfrak{R}_{\mathcal{G}} - 2\Lambda_{\mathcal{G}}\right), \mathcal{L}_{\mathcal{M}} = -\mathcal{M}_{\mathcal{G}}, \mathfrak{R}_{\mathcal{G}} = Tr_{\mathfrak{F}}\tilde{g}_{\mathcal{G}}^{-1}\widetilde{\mathfrak{R}}, \mathcal{M}_{\mathcal{G}} = Tr_{\mathfrak{F}}\tilde{g}_{\mathcal{G}}^{-1}\widetilde{\mathcal{M}}, \Lambda_{\mathcal{G}} = \frac{1}{2\beta}Tr_{\mathfrak{F}}\left(\tilde{\mathcal{G}} - \tilde{\mathfrak{F}} - \ln\tilde{\mathcal{G}}\right) - \frac{1}{\sqrt{|-g|}}\frac{\mathcal{S}\delta_{\mathcal{M}}}{dg_{\mu\nu}} = \mathfrak{T}_{\mu\nu}, \delta = \frac{1}{\ell_{\mathcal{P}}}\iiint\sqrt{|-g|}\,\mathcal{L}dr$$

Campos cuánticos relativistas entrópicos.

$$\mathfrak{G} = g + \alpha\mathcal{M}, \mathcal{M}_{\mu\nu} = \left[\!\left[\nabla^{\mu}\bar{\phi}\right]\!\right]\left[\!\left[\nabla^{\mu}\phi\right]\!\right], |\nabla\phi|^2 = \nabla_{\mu}\bar{\phi}g^{\mu\nu}\nabla_{\nu}\phi, [\mathfrak{G}^{-1}]^{\mu\nu} = g^{\mu\nu} - \alpha\frac{\mathcal{M}^{\mu\nu}}{1+\alpha|\nabla\phi|^2}[\ln\mathfrak{G}]_{\mu\nu} = f(|\nabla\phi|^2)\mathcal{M}_{\mu\nu}, [\ln\mathfrak{G}^{-1}]^{\mu\nu} = -f(|\nabla\phi|^2)\mathcal{M}^{\mu\nu}, f(\omega) = \frac{\ln(1+\alpha\omega)}{\omega}$$

$$\mathcal{L} = -\ln(1+\alpha|\nabla\phi|^2)\,\nabla_{\mu}\hbar(|\nabla\phi|^2)g^{\mu\nu}\nabla_{\nu}\phi, \hbar(\omega) = \frac{\alpha}{1+\alpha\omega}$$

$$\delta\mathfrak{S} = -\hbar(|\nabla\phi|^2)\mathcal{M}^{\mu\nu} - \frac{1}{2}\mathcal{L}g^{\mu\nu}, |\phi\rangle = \phi\bigoplus\omega_{\mu}d\chi^{\mu}\bigoplus\zeta_{\mu\nu}d\chi^{\mu}\wedge d\chi^{\nu}, \langle\phi| = \bar{\phi}\bigoplus\bar{\omega}_{\mu}d\chi^{\mu}\bigoplus\bar{\zeta}_{\mu\nu}d\chi^{\mu}\wedge d\chi^{\nu}$$

$$g_{(2)} = g_{\mu\rho}g_{\nu\sigma}(d\chi^{\mu}\wedge d\chi^{\nu})\bigotimes(d\chi^{\rho}\wedge d\chi^{\sigma})\left[\!\left[g_{(2)}\right]\!\right]_{\mu\nu\rho\sigma}(d\chi^{\mu}\wedge d\chi^{\nu})\bigotimes(d\chi^{\rho}\wedge d\chi^{\sigma})\left[\!\left[g_{(2)}\right]\!\right]_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

$$\widetilde{g} = 1 \bigoplus g_{\mu\nu}\, d\chi^{\mu} \bigotimes d\chi^{\nu} \bigoplus [\![g_{(2)}]\!]_{\mu\nu\rho\sigma}\, (d\chi^{\mu} \wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma}), \langle\phi|\phi\rangle$$
$$= |\phi|^2 + \bar{\varpi}^{\mu}\omega_{\mu} + \bar{\zeta}^{\mu\nu}\zeta_{\mu\nu}, |\phi\rangle\langle\phi|$$
$$= \bar{\phi}\phi \bigoplus \left(\bar{\varpi}_{\mu}\omega_{\nu} d\chi^{\mu} \bigotimes d\chi^{\nu}\right) \bigoplus \bar{\zeta}_{\mu\nu}\zeta_{\rho\sigma}\, (d\chi^{\mu}$$
$$\wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma}), \mathfrak{D}|\phi\rangle$$
$$= -\nabla^{\mu}\omega_{\mu} \bigoplus (\nabla_{\mu}\phi - \nabla^{\rho}\zeta_{\rho\mu}) d\chi^{\mu} \bigoplus \nabla_{\mu}\omega_{\mu} d\chi^{\mu} \wedge d\chi^{\nu}$$

$$\widetilde{\mathfrak{G}} = \mathfrak{G}_{(0)} \bigoplus [\mathfrak{G}_{(1)}]_{\mu\nu} d\chi^{\mu} \bigotimes d\chi^{\nu} \bigoplus [\mathfrak{G}_{(1)}]_{\mu\nu\rho\sigma}\, (d\chi^{\mu} \wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma})\, \widetilde{g}$$
$$+ \alpha(\mathfrak{D}|\phi\rangle\langle\phi|\mathfrak{D})$$

$$\widetilde{\mathfrak{R}} = \mathcal{R} \bigoplus \left(\mathcal{R}_{\mu\nu} d\chi^{\mu} \bigotimes d\chi^{\nu}\right) \bigoplus \mathcal{R}_{\mu\nu\rho\sigma}\, (d\chi^{\mu} \wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma})$$

$$\widetilde{\mathcal{M}} = \mathfrak{D}|\phi\rangle\langle\phi|\mathfrak{D} + (m^2 + \xi\mathfrak{R})|\phi\rangle\langle\phi|$$

$$\widetilde{\mathfrak{G}} = \widetilde{g} + \alpha\widetilde{\mathcal{M}} - \beta\widetilde{\mathfrak{R}}$$

$$\widetilde{\mathfrak{H}} = Tr\widetilde{g} \ln \widetilde{g}^{-1} = 1 \ln 1 + Tr g \ln g^{-1} + Tr g_{(2)} \ln g_{(2)}^{-1}, \mathcal{L} \coloneqq -Tr\widetilde{g} \ln \widetilde{g}^{-1} + Tr\widetilde{g} \ln \widetilde{\mathfrak{G}}^{-1}, \mathcal{L}$$
$$\coloneqq -Tr\widetilde{g} \ln \widetilde{\mathfrak{G}}^{-1} = -Tr_{\mathfrak{F}} \ln \widetilde{\mathfrak{G}}\, \widetilde{g}^{-1}, \mathcal{L}$$
$$\coloneqq \ln[\![\mathfrak{G}_{(0)}]\!]^{-1} + Tr g \ln[\![\mathfrak{G}_{(1)}]\!]^{-1} + Tr g_{(2)} \ln[\![\mathfrak{G}_{(2)}]\!]^{-1}, \mathcal{L} \coloneqq -Tr_{\mathfrak{F}} \ln \widetilde{\mathfrak{G}}\, \widetilde{g}^{-1}$$
$$= -\ln[\![\mathfrak{G}_{(0)}]\!] - Tr_{\mathfrak{F}} \ln[\![\mathfrak{G}_{(1)}]\!]\, g^{-1} - Tr_{\mathfrak{F}} \ln[\![\mathfrak{G}_{(2)}]\!]^{-1} [\![g_{(2)}]\!]^{-1}, \mathcal{L}$$
$$= 3\beta\mathfrak{R} - \alpha\langle\phi|\mathfrak{D}\widetilde{g}^{-1}\mathfrak{D}|\phi\rangle - \alpha(m^2 + \xi\mathfrak{R})\left(|\phi|^2 + \bar{\varpi}^{\mu}\omega_{\mu} + \bar{\zeta}^{\mu\nu}\zeta_{\mu\nu}\right), \langle\phi|\mathfrak{D}\widetilde{g}^{-1}\mathfrak{D}|\phi\rangle$$
$$= |\nabla\phi|^2 + \left|\nabla^{\mu}\omega_{\mu}\right|^2 + \left|\nabla^{\rho}\zeta_{\rho\mu}\right|^2 + \left|\epsilon^{\mu\nu\rho}\nabla_{\mu}\omega_{\mu\nu}\right|^2, \mathcal{L}$$
$$= 3\beta\mathfrak{R} - \alpha|\nabla\phi|^2 - \alpha(m^2 + \xi\mathfrak{R})|\phi|^2, \widetilde{\mathfrak{G}}\widetilde{g}^{-1} = \widetilde{\Theta}, \tilde{\mathcal{L}}$$
$$= -Tr_{\mathfrak{F}} \ln \widetilde{\Theta} - Tr_{\mathfrak{F}}\tilde{\mathcal{G}}\left(\widetilde{\mathfrak{G}}\widetilde{g}^{-1} - \widetilde{\Theta}\right)$$

$$\widetilde{\Theta} = \Theta_{(0)} \bigoplus \Theta_{(1)} \bigoplus \Theta_{(2)}, \tilde{\mathcal{G}} = \mathcal{G}_{(0)} \bigoplus \mathcal{G}_{(1)} \bigoplus \mathcal{G}_{(2)}, \mathcal{W} = Tr_{\mathfrak{F}}\tilde{\mathcal{G}}\left(\widetilde{\mathfrak{G}}\widetilde{g}^{-1} - \widetilde{\Theta}\right) = \sum_{\eta=0}^{4} \mathcal{W}_{\eta}, \mathcal{W}_0$$
$$= \mathcal{G}_{(0)}\left(\mathfrak{G}_{(0)} - \Theta_{(0)}\right), \mathcal{W}_1 = [\![\mathcal{G}_{(1)}]\!]_{\rho}^{\mu} \left(\|\mathfrak{G}_{(1)}\|_{\mu\nu} g^{\nu\rho} - [\![\Theta_{(1)}]\!]_{\mu}^{\rho}\right), \mathcal{W}_2$$
$$= [\![\mathcal{G}_{(2)}]\!]_{\eta} \theta^{\mu\nu} \left(\|\mathfrak{G}_{(2)}\|_{\mu\nu\rho\sigma} \|g_{(2)}\|^{\rho\sigma\eta\theta} - [\![\Theta_{(2)}]\!]_{\mu\nu}^{\eta\theta}\right)$$

Ecuación de movimiento.

$$\mathfrak{D}\tilde{g}_{\mathcal{G}^{-1}}\mathfrak{D}|\phi\rangle + \widetilde{g}_{\mathcal{G}^{-1}}(m^2 + \xi\mathfrak{R})|\phi\rangle\widetilde{g}_{\mathcal{G}} = \tilde{\mathcal{G}}^{-1} g$$

Gravedad cuántica modificada.

$$\widetilde{\Theta} = \widetilde{\mathfrak{I}} + \alpha\widetilde{\mathcal{M}}\widetilde{g}^{-1} - \beta\widetilde{\mathfrak{R}}\widetilde{g}^{-1}, \Theta_{(0)} = 1 + \alpha\mathcal{M}_{(0)} - \beta\mathcal{R}, [\![\Theta_{(1)}]\!]_{\mu}^{\nu}$$
$$= \delta_{\mu}^{\nu} + \alpha\|\mathcal{M}_{(1)}\|_{\mu\rho} g^{\rho\nu} - \beta\mathcal{R}_{\mu\rho} g^{\rho\nu}, [\![\Theta_{(2)}]\!]_{\vec{\mu}}^{\vec{\nu}}$$
$$= \delta_{\vec{\mu}}^{\vec{\rho}} + \alpha\|\mathcal{M}_{(2)}\|_{\overrightarrow{\mu\rho}} [\![g_{(2)}]\!]^{\overrightarrow{\rho\nu}} - \beta\mathcal{R}_{\overrightarrow{\mu\rho}} [\![g_{(2)}]\!]^{\overrightarrow{\rho\nu}} \binom{d}{\eta} - Tr_{\mathfrak{F}}\Theta_{(\eta)} + \alpha Tr_{\mathfrak{F}}\mathcal{M}_{(\eta)} g_{(\eta)}^{-1}$$
$$= \beta\mathfrak{R}, \widetilde{\Theta}^{-1} = \tilde{\mathcal{G}}, \tilde{\mathcal{G}}^{-1} = \widetilde{\mathfrak{I}} + \alpha\widetilde{\mathcal{M}}\widetilde{g}^{-1} - \beta\widetilde{\mathfrak{R}}\widetilde{g}^{-1}$$

Ecuación de Einstein – Hilbert.

$$\mathfrak{R}^{\mathcal{G}}_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu}\left(\mathfrak{R}_{\mathcal{G}} - 2\Lambda_{\mathcal{G}}\right) + \mathfrak{D}_{(\mu\nu)} = \kappa \mathfrak{T}_{\mu\nu}$$

$$\mathfrak{R}^{\mathcal{G}}_{\mu\nu} = \mathcal{G}_{(0)}\mathcal{R}_{\mu\nu} + [\![\mathcal{G}_{(1)}]\!]^{\rho}_{\mu}\mathcal{R}_{\rho\nu} - [\![\mathcal{G}_{(2)}]\!]_{\rho_1\rho_2\mu\eta}\mathfrak{R}^{\eta\rho_1\rho_2}_{\nu} + 2[\![\mathcal{G}_{(2)}]\!]^{\eta\rho_1\rho_2}_{\mu}\mathcal{R}_{\rho_1\rho_2\nu\eta}$$

$$\mathfrak{D}_{\mu\nu} = \left(\nabla^{\rho}\nabla_{\rho} g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\right)\mathcal{G}_{(0)} - \nabla^{\rho}\nabla_{\nu}\left[\mathcal{G}_{(1)}\right]_{(\rho\mu)} + \frac{1}{2}\nabla^{\rho}\nabla_{\rho}\left[\mathcal{G}_{(1)}\right]_{\mu\nu} + \frac{1}{2}\nabla^{\rho}\nabla^{\eta}\left[\mathcal{G}_{(1)}\right]_{\rho\eta} g_{\mu\nu}$$
$$+ \nabla^{\eta}\nabla^{\rho}\left[\mathcal{G}_{(2)}\right]_{\mu\rho\nu\eta} + \nabla^{\rho}\nabla^{\eta}\left[\mathcal{G}_{(2)}\right]_{\eta\mu\rho\nu} + \frac{1}{2}[\nabla^{\rho},\nabla^{\eta}]\left[\mathcal{G}_{(2)}\right]_{\rho\eta\mu\nu} + \left|\Box^{2}\right\rangle\left\langle\Box^{2}\right|$$
$$+ \left|\Box^{\dagger}\right\rangle\left\langle\Box^{\star}\right|$$

Campo abeliano.

$$\widetilde{\mathfrak{F}} = \zeta \bigoplus \zeta^{\mu}\, d\chi_{\mu} \bigoplus \mathfrak{F}_{\mu\nu}\, \mathfrak{F}_{\rho\sigma}(d\chi^{\mu} \wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma})$$

$$\widetilde{\mathfrak{G}} = \widetilde{g} + \alpha\left(\widetilde{\mathcal{M}} + \widetilde{\mathfrak{F}}\right) - \beta\widetilde{\mathfrak{R}}$$

$$\nabla_{\mu} \mapsto \nabla^{(\mathrm{A})}_{\mu} = \nabla_{\mu} - ie\mathrm{A}_{\mu}$$

Operadores cuánticos.

$$\Big\langle \omega = \omega_{\rho\sigma} d\chi^{\rho\sigma}, \mathfrak{G}_{(1)} = \langle\mathfrak{G}_{(1)}\rangle_{\mu\nu} d\chi^{\mu} \bigotimes d\chi^{\nu}, \mathfrak{G}_{(1)} \times \omega = \widehat{\omega} = \widehat{\omega}_{\mu\nu} d\chi^{\mu} \bigotimes d\chi^{\nu}, \widehat{\omega}_{\mu\nu}$$
$$\coloneqq \langle\mathfrak{G}_{(1)}\rangle_{\mu\nu}\omega_{\rho\sigma}\langle d\chi^{\sigma}|d\chi^{\rho}\rangle = g^{\sigma\rho}, \widehat{\omega}_{\mu\nu} = \langle\mathfrak{G}_{(1)}\rangle_{\mu\nu} g^{\sigma\rho}\omega_{\rho\sigma}, \mathfrak{G}_{(1)} \times \omega$$
$$= \langle\mathfrak{G}_{(1)}\rangle_{\mu\nu} g^{\sigma\rho}\omega_{\rho\sigma} d\chi^{\mu} \bigotimes d\chi^{\nu}, \mathfrak{G}_{(1)} \times \omega \equiv \lambda\omega, \langle\mathfrak{G}_{(1)}\rangle_{\mu\nu} g^{\sigma\rho}\omega_{\rho\sigma} d\chi^{\mu} \bigotimes d\chi^{\nu}$$
$$= \lambda\omega_{\mu\nu}\Big|$$

$$\Big|\zeta = \zeta_{\eta\theta} d\chi^{\eta} \bigotimes d\chi^{\theta}, \mathfrak{G}_{(2)} = \langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}(d\chi^{\mu} \wedge d\chi^{\nu}) \bigotimes (d\chi^{\rho} \wedge d\chi^{\sigma}), \mathfrak{G}_{(2)} \times \zeta$$
$$= \hat{\zeta}_{\mu\nu} d\chi^{\mu} \wedge d\chi^{\nu}, \hat{\zeta}_{\mu\nu} \coloneqq \frac{1}{2}\langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\zeta_{\eta\theta}\left\langle d\chi^{\rho} \wedge d\chi^{\sigma}\middle| d\chi^{\eta} \wedge d\chi^{\theta}\right\rangle, \hat{\zeta}_{\mu\nu}$$
$$\coloneqq \frac{1}{2}\langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\zeta_{\eta\theta}\left\| g^{\rho\eta} g^{\sigma\theta} - g^{\rho\theta} g^{\sigma\eta}\right\| = \langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\langle g_{(2)}\rangle^{\rho\sigma\eta\theta}\zeta_{\eta\theta}, \langle g_{(2)}\rangle_{\mu\nu\rho\sigma}$$
$$= \frac{1}{2}\left\| g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right\|, \mathfrak{G}_{(2)} \times \zeta = \langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\langle g_{(2)}\rangle^{\rho\sigma\eta\theta}\zeta_{\eta\theta} d\chi^{\mu} \wedge d\chi^{\nu}, \mathfrak{G}_{(2)} \times \zeta$$
$$\equiv \lambda\zeta, \langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\langle g_{(2)}\rangle^{\rho\sigma\eta\theta}\zeta_{\eta\theta}$$
$$= \lambda\zeta_{\mu\nu} - \frac{1}{4}\left\| g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right\|\left\| g^{\rho\eta} g^{\sigma\theta} - g^{\rho\theta} g^{\sigma\eta}\right\|\zeta_{\eta\theta} = \lambda\zeta_{\mu\nu}\Big\rangle$$

$$\zeta = \zeta_{\nu_1\nu_2\cdots\nu_\eta} d\chi^{\nu_1} \wedge d\chi^{\nu_2} \wedge \cdots d\chi^{\nu_\eta}, \mathfrak{G}_{(\eta)}$$
$$= \langle\mathfrak{G}_{(\eta)}\rangle_{\mu_1\mu_2\cdots\mu_\eta\nu_1\nu_2\cdots\nu_\eta}|d\chi^{\mu_1} \wedge d\chi^{\mu_2} \wedge \cdots d\chi^{\mu_\eta}| \bigotimes |d\chi^{\nu_1} \wedge d\chi^{\nu_2} \wedge \cdots d\chi^{\nu_\eta}|, g_{(\eta)}$$
$$= \prod_{i=1}^{\eta} g_{\mu_i\rho_i}|d\chi^{\mu_1} \wedge d\chi^{\mu_2} \wedge \cdots d\chi^{\mu_\eta}| \bigotimes |d\chi^{\nu_1} \wedge d\chi^{\nu_2} \wedge \cdots d\chi^{\nu_\eta}|, g_{(\eta)}$$
$$= \langle g_{(\eta)}\rangle_{\mu_1\mu_2\cdots\mu_\eta\nu_1\nu_2\cdots\nu_\eta}|d\chi^{\mu_1} \wedge d\chi^{\mu_2} \wedge \cdot\cdot$$
$$\cdot\, d\chi^{\mu_\eta}| \bigotimes |d\chi^{\nu_1} \wedge d\chi^{\nu_2} \wedge \cdots d\chi^{\nu_\eta}|, \langle g_{(\eta)}\rangle_{\mu_1\mu_2\cdots\mu_\eta\nu_1\nu_2\cdots\nu_\eta}$$
$$= \frac{1}{\eta!}\delta^{\rho_1\rho_2\cdots\rho_\eta}_{\sigma_1\sigma_2\cdots\sigma_\eta}\prod_{i=1}^{\eta} g_{\mu_i\rho_i}, \mathfrak{G}_{(\eta)} \cdot \zeta = \hat{\zeta}$$

$$\hat{\zeta}_{\overrightarrow{\mu\nu}} \coloneqq \frac{1}{\eta!}\langle\mathfrak{G}_{(\eta)}\rangle_{\overrightarrow{\mu\nu}}\hat{\zeta}_{\overrightarrow{\rho\sigma}}|d\chi^{\mu_1}\wedge d\chi^{\mu_2}\wedge\cdots d\chi^{\mu_\eta}, d\chi^{\nu_1}\wedge d\chi^{\nu_2}\wedge\cdots$$
$$\cdot\, d\chi^{\nu_\eta}|\bigotimes|d\chi^{\rho_1}\wedge d\chi^{\rho_2}\wedge\cdots d\chi^{\rho_\eta}, d\chi^{\sigma_1}\wedge d\chi^{\sigma_2}\wedge\cdots d\chi^{\sigma_\eta}|, \langle\mathfrak{G}_{(\eta)}\rangle_{\overrightarrow{\mu\nu}}\langle g_{(\eta)}\rangle^{\overrightarrow{\rho\sigma}}\hat{\zeta}_{\overrightarrow{\rho\sigma}}$$
$$= \lambda\hat{\zeta}_{\overrightarrow{\mu\nu}}$$

$$\langle\mathcal{N}_{(1)}\rangle_{\mu}^{\nu} \coloneqq \langle\mathfrak{G}_{(\eta)}\rangle_{\mu\nu}g^{\rho\sigma}, \langle\mathfrak{G}_{(2)}^{\mathfrak{F}}\rangle_{\mu\nu,\rho\sigma} = 2\langle\mathfrak{G}_{(2)}\rangle_{\mu\nu,\rho\sigma}, \langle g_{(2)}^{\mathfrak{F}}\rangle_{\mu\nu,\rho\sigma} = 2\langle g_{(2)}\rangle_{\mu\nu,\rho\sigma}, \langle\mathcal{N}_{(2)}\rangle_{\mu\nu}^{\eta\theta}$$
$$\coloneqq \langle\mathfrak{G}_{(2)}^{\mathfrak{F}}\rangle_{\mu\nu\rho\sigma}\langle g_{(2)}^{\mathfrak{F}}\rangle^{\rho\sigma\eta\theta}, \langle\mathfrak{G}_{(2)}^{\mathfrak{F}}\rangle_{\vec{\mu};\vec{\nu}} = \eta!\,\langle\mathfrak{G}_{(2)}\rangle_{\vec{\mu};\vec{\nu}}, \langle g_{(2)}^{\mathfrak{F}}\rangle_{\vec{\mu};\vec{\nu}} = \eta!\,\langle g_{(2)}\rangle_{\vec{\mu};\vec{\nu}}, \langle\mathcal{N}_{(\eta)}\rangle_{\vec{\mu}}^{\vec{\eta}}$$
$$\coloneqq \langle\mathfrak{G}_{(\eta)}^{\mathfrak{F}}\rangle_{\overrightarrow{\mu\rho}}\langle g_{(\eta)}^{\mathfrak{F}}\rangle^{\overrightarrow{\rho\eta}}$$

$$g_{(2)}^{\mathfrak{F}} = \left\langle\begin{matrix} g_{00}g_{11} - g_{01}g_{10} & g_{00}g_{12} - g_{02}g_{10} & g_{01}g_{12} - g_{02}g_{11} \\ g_{00}g_{21} - g_{01}g_{20} & g_{00}g_{22} - g_{02}g_{20} & g_{01}g_{22} - g_{02}g_{21} \\ g_{10}g_{21} - g_{11}g_{20} & g_{10}g_{22} - g_{12}g_{20} & g_{11}g_{22} - g_{12}g_{21}\end{matrix}\right\rangle$$

Tensores métricos.

$$Tr\mathfrak{G}_{(\eta)} = Tr_{\mathcal{M}}\mathcal{N}_{(\eta)}, Tr_{\mathfrak{F}}\mathfrak{G}_{(\eta)}g_{(\eta)}^{-1} = Tr\mathfrak{G}_{(\eta)} = Tr_{\mathcal{M}}\mathcal{N}_{(\eta)}, Tr_{\mathfrak{F}}\mathfrak{G}_{(\eta)}g_{(\eta)}^{-1} = \langle\mathfrak{G}_{(\eta)}\rangle_{\overrightarrow{\mu\rho}}\langle g_{(\eta)}\rangle^{\overrightarrow{\rho\mu}}, Tr\mathfrak{G}_{(1)}$$
$$= \langle\mathfrak{G}_{(1)}\rangle_{\mu\rho}g^{\rho\mu}, Tr\mathfrak{G}_{(2)} = \langle\mathfrak{G}_{(2)}\rangle_{\mu\nu\rho\sigma}\langle g_{(2)}\rangle^{\rho\sigma\mu\nu}, Tr g_{(\eta)} = \binom{d}{\eta}, \mathfrak{H} = Tr\mathfrak{G}_{(\eta)}\ln\mathfrak{G}_{(\eta)}^{-1}$$
$$\coloneqq -\sum_{\lambda}\lambda\ln\lambda\,, \mathfrak{H}_{(\eta)} = Tr g_{(\eta)}\ln g_{(\eta)}^{-1}\,, Tr g_{(\eta)}\ln\mathfrak{G}_{(\eta)}^{-1} \coloneqq -Tr_{\mathfrak{F}}\ln\mathfrak{G}_{(\eta)}g_{(\eta)}^{-1}$$
$$\coloneqq -Tr_{\mathcal{M}}\mathcal{N}_{(\eta)} = -\sum_{\mu}\ln(\lambda')$$

Operadores cuánticos topológicos.

$$|\psi\rangle = \psi\bigoplus\omega_{\mu}d\chi^{\mu}\bigoplus\zeta_{\mu\nu}d\chi^{\mu}\wedge d\chi^{\nu}, |\phi\rangle = \hat{\phi}\bigoplus\hat{\omega}_{\mu}d\chi^{\mu}\bigoplus\hat{\zeta}_{\mu\nu}d\chi^{\mu}\wedge d\chi^{\nu}$$

$$\langle\langle\psi,\phi\rangle\rangle = \iiint\sqrt{|-g|}\left(\bar{\phi}\hat{\phi} + \bar{\omega}_{\mu}\hat{\omega}^{\mu} + \bar{\zeta}_{\mu\nu}\hat{\zeta}^{\mu\nu}\right)dr\,, \tilde{g}^{-1}$$
$$= 1\bigoplus g^{\mu\nu}d\chi_{\mu}\bigotimes d\chi_{\nu}\bigoplus\langle g_{(2)}\rangle^{\mu\nu\rho\sigma}(d\chi_{\mu}$$
$$\wedge\, d\chi_{\nu})\bigotimes(d\chi_{\rho}\wedge d\chi_{\sigma}), \langle\langle\psi, c_1\phi_1 + c_2\phi_2\rangle\rangle$$
$$= c_1\langle\langle\psi,\phi_1\rangle\rangle + c_2\langle\langle\psi,\phi_2\rangle\rangle, \langle\langle c_1\psi_1 + c_2\psi_2\phi\rangle\rangle = \bar{c}_1\langle\langle\psi_1|\phi\rangle\rangle + \bar{c}_2\langle\langle\psi_2|\phi\rangle\rangle, \langle\langle\psi,\phi\rangle\rangle$$
$$\in\mathbb{R}^4, \langle\langle\phi|\phi\rangle\rangle \geq \infty$$

$$\tilde{\mathfrak{G}} = [\![\mathfrak{G}_{(0)}]\!]\bigoplus[\![\mathfrak{G}_{(1)}]\!]_{\mu\nu}d\chi^{\mu}\bigotimes d\chi^{\nu}\bigoplus[\![\mathfrak{G}_{(2)}]\!]_{\mu\nu\rho\sigma}(d\chi^{\mu}\wedge d\chi^{\nu})\bigotimes(d\chi^{\rho}\wedge d\chi^{\sigma}), \tilde{\mathfrak{G}}\times|\phi\rangle$$
$$= \phi\bigoplus[\![\mathfrak{G}_{(1)}]\!]_{\mu\nu}\omega^{\nu}d\chi^{\mu}\bigoplus[\![\mathfrak{G}_{(2)}]\!]_{\mu\nu\rho\sigma}\zeta^{\rho\sigma}d\chi^{\mu}\wedge d\chi^{\nu}$$

$$|\psi^{\star}\rangle = \psi\bigoplus\omega^{\mu}d\chi_{\mu}\bigoplus\zeta^{\mu\nu}d\chi_{\mu}\wedge d\chi_{\nu}, |\phi^{\star}\rangle = \hat{\phi}\bigoplus\hat{\omega}^{\mu}d\chi_{\mu}\bigoplus\hat{\zeta}^{\mu\nu}d\chi_{\mu}\wedge d\chi_{\nu}$$

$$\langle\langle\psi,\phi\rangle\rangle = \langle\langle\psi^{\star},\phi^{\star}\rangle\rangle_{\star}$$

$$\tilde{\mathfrak{G}}^{\star} = \mathfrak{G}_{(0)}^{\star}\bigoplus[\![\mathfrak{G}_{(1)}^{\star}]\!]^{\mu\nu}d\chi_{\mu}\bigotimes d\chi_{\nu}\bigoplus[\![\mathfrak{G}_{(2)}^{\star}]\!]^{\mu\nu\rho\sigma}(d\chi_{\mu}\wedge d\chi_{\nu})\bigotimes(d\chi_{\rho}\wedge d\chi_{\sigma})$$

$$\langle\langle\psi,\widetilde{\mathfrak{G}}\times\phi\rangle\rangle=\langle\langle\widetilde{\mathfrak{G}}^{\star}\times\psi^{\star},\phi^{\star}\rangle\rangle_{\star}\,\widetilde{\mathfrak{G}}^{\star}\times|\phi^{\star}\rangle$$
$$=[\![\mathfrak{G}^{\star}_{(0)}]\!]\phi\bigoplus[\![\mathfrak{G}^{\star}_{(1)}]\!]^{\mu\nu}\,\omega_{\nu}d\chi_{\mu}\bigoplus[\![\mathfrak{G}^{\star}_{(2)}]\!]^{\mu\nu\rho\sigma}\,\zeta_{\rho\sigma}d\chi_{\mu}\wedge d\chi_{\nu},\mathfrak{G}^{\star}_{(0)}$$
$$=\mathfrak{G}_{(0)},[\![\mathfrak{G}^{\star}_{(1)}]\!]^{\mu\nu}=[\![\mathfrak{G}_{(1)}]\!]^{\mu\nu},[\![\mathfrak{G}^{\star}_{(2)}]\!]^{\mu\nu\rho\sigma}=[\![\mathfrak{G}_{(2)}]\!]^{\mu\nu\rho\sigma},[\![\mathfrak{G}_{(1)}]\!]^{\mu\nu}$$
$$=\mathscr{g}^{\mu\rho}[\![\mathfrak{G}_{(1)}]\!]_{\rho\sigma}\mathscr{g}^{\nu\sigma},[\![\mathfrak{G}_{(2)}]\!]^{\mu\nu\rho\sigma}=[\![\mathscr{g}_{(2)}]\!]^{\mu\nu\eta_1\eta_2}[\![\mathfrak{G}_{(2)}]\!]_{\eta_1\eta_2\theta_1\theta_2}[\![\mathscr{g}_{(2)}]\!]^{\theta_1\theta_2\rho\sigma},\widetilde{\mathfrak{G}}^{\star}$$
$$=\widetilde{\mathscr{g}}^{-1}\widetilde{\mathfrak{G}}\widetilde{\mathscr{g}}^{-1},\widetilde{\mathscr{g}}^{\star}=\widetilde{\mathscr{g}}^{-1},\widetilde{\mathfrak{G}}=\widetilde{\mathfrak{G}}^{\star\star}=\widetilde{\mathscr{g}}\widetilde{\mathfrak{G}}^{\star}\widetilde{\mathscr{g}},\widetilde{\mathfrak{G}}_{\eta}$$
$$=\mathfrak{G}_{(0),\eta}\bigoplus[\![\mathfrak{G}_{(1),\eta}]\!]_{\mu\nu}\,d\chi^{\mu}\bigotimes d\chi^{\nu}\bigoplus[\![\mathfrak{G}_{(2),\eta}]\!]_{\mu\nu\rho\sigma}\,(d\chi^{\mu}$$
$$\wedge\,d\chi^{\nu})\bigotimes(d\chi^{\rho}\wedge d\chi^{\sigma})$$

$$\widetilde{\mathfrak{G}}_1\widetilde{\mathscr{g}}^{-1}\widetilde{\mathfrak{G}}_2=[\![\mathfrak{G}_{(0),12}]\!]\bigoplus[\![\mathfrak{G}_{(1),12}]\!]_{\mu\nu}\,d\chi^{\mu}\bigotimes d\chi^{\nu}\bigoplus[\![\mathfrak{G}_{(2),12}]\!]_{\mu\nu\rho\sigma}\,(d\chi^{\mu}$$
$$\wedge\,d\chi^{\nu})\bigotimes(d\chi^{\rho}\wedge d\chi^{\sigma}),[\![\mathfrak{G}_{(0),12}]\!]=[\![\mathfrak{G}_{(0),1}]\!][\![\mathfrak{G}_{(0),2}]\!],[\![\mathfrak{G}_{(1),12}]\!]_{\mu\nu}$$
$$=[\![\mathfrak{G}_{(1),1}]\!]_{\mu\eta_1}\mathscr{g}^{\eta_1\eta_2}[\![\mathfrak{G}_{(1),2}]\!]_{\eta_2\nu},[\![\mathfrak{G}_{(2),12}]\!]_{\vec{\mu}\vec{\rho}}=[\![\mathfrak{G}_{(1),1}]\!]_{\vec{\mu}\vec{\eta}_1}[\![\mathscr{g}_{(2)}]\!]^{\vec{\eta}_1,\vec{\eta}_2}[\![\mathfrak{G}_{(2),2}]\!]_{\vec{\eta}_2\vec{\rho}}$$

$$\widetilde{\mathfrak{G}}_1^{\star}\widetilde{\mathscr{g}}\widetilde{\mathfrak{G}}_2^{\star}=[\![\mathfrak{G}^{\star}_{(0),12}]\!]\bigoplus[\![\mathfrak{G}^{\star}_{(1),12}]\!]^{\mu\nu}\,d\chi_{\mu}\bigotimes d\chi_{\nu}\bigoplus[\![\mathfrak{G}^{\star}_{(2),12}]\!]^{\mu\nu\rho\sigma}\,(d\chi_{\mu}$$
$$\wedge\,d\chi_{\nu})\bigotimes(d\chi_{\rho}\wedge d\chi_{\sigma}),[\![\mathfrak{G}^{\star}_{(0),12}]\!]=[\![\mathfrak{G}^{\star}_{(0),1}]\!][\![\mathfrak{G}^{\star}_{(0),2}]\!],[\![\mathfrak{G}^{\star}_{(1),12}]\!]^{\mu\nu}$$
$$=[\![\mathfrak{G}^{\star}_{(1),1}]\!]^{\mu\eta_1}\mathscr{g}_{\eta_1\eta_2}[\![\mathfrak{G}^{\star}_{(1),2}]\!]^{\eta_2\nu},[\![\mathfrak{G}^{\star}_{(2),12}]\!]^{\vec{\mu}\vec{\rho}}=[\![\mathfrak{G}^{\star}_{(2),1}]\!]_{\vec{\mu}\vec{\eta}_1}[\![\mathscr{g}_{(2)}]\!]_{\vec{\eta}_1,\vec{\eta}_2}[\![\mathfrak{G}^{\star}_{(2),1}]\!]^{\vec{\eta}_2\vec{\rho}}$$

$$(\widetilde{\mathfrak{G}}^{\star})^{\star}=\widetilde{\mathfrak{G}}^{\star\star}=\widetilde{\mathfrak{G}},(\widetilde{\mathfrak{G}}_1\widetilde{\mathscr{g}}^{-1}\widetilde{\mathfrak{G}}_2)^{\star}=\widetilde{\mathfrak{G}}_1^{\star}\widetilde{\mathscr{g}}\widetilde{\mathfrak{G}}_2^{\star},(\widetilde{\mathfrak{G}}_1^{\star}\widetilde{\mathscr{g}}\widetilde{\mathfrak{G}}_2^{\star})^{\star}=\widetilde{\mathfrak{G}}_1\widetilde{\mathscr{g}}^{-1}\widetilde{\mathfrak{G}}_2(c_1\widetilde{\mathfrak{G}}_1+c_2\widetilde{\mathfrak{G}}_2)^{\star}$$
$$=\bar{c}_1\widetilde{\mathfrak{G}}_1^{\star}+\bar{c}_2\widetilde{\mathfrak{G}}_2^{\star},(c_1\widetilde{\mathfrak{G}}_1^{\star}+c_2\widetilde{\mathfrak{G}}_2^{\star})^{\star}=\bar{c}_1\widetilde{\mathfrak{G}}_1+\bar{c}_2\widetilde{\mathfrak{G}}_2$$

$$\|\widetilde{\mathfrak{G}}\|=\|\widetilde{\mathfrak{G}}^{\star}\|=\iiint\sqrt{|-\mathscr{g}|}Tr_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}}\widetilde{\mathfrak{G}}^{\star})dr$$

$$Tr_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}}\widetilde{\mathfrak{G}}^{\star})=\mathfrak{G}_{(0)}^{2}+[\![\mathfrak{G}_{(1)}]\!]_{\mu\nu}[\![\mathfrak{G}_{(1)}]\!]^{\nu\mu}+[\![\mathfrak{G}_{(2)}]\!]_{\mu\nu\rho\sigma}[\![\mathfrak{G}_{(2)}]\!]^{\rho\sigma\mu\nu}$$

$$\Delta^{1/2}_{\widetilde{\mathfrak{G}},\mathscr{g}}=\sqrt{\widetilde{\mathfrak{G}}\widetilde{\mathfrak{G}}^{\star}}=\widetilde{\mathfrak{G}}\widetilde{\mathscr{g}}^{-1},\Delta^{1/2}_{\widetilde{\mathfrak{G}},\mathscr{g}}|\phi\rangle$$
$$=\mathfrak{G}_{(0)}\phi\bigoplus[\![\mathfrak{G}_{(1)}]\!]_{\mu\rho}\,\mathscr{g}^{\rho\nu}\omega_{\nu}\,d\chi^{\mu}\bigoplus[\![\mathfrak{G}_{(2)}]\!]_{\mu\nu\eta_1\eta_2}\,\mathscr{g}_{(2)}^{\eta_1\eta_2\rho\sigma}\zeta_{\rho\sigma}(d\chi^{\mu}\wedge d\chi^{\nu}),\mathcal{L}$$
$$=-Tr_{\widetilde{\mathfrak{F}}}\ln\Delta^{\frac{1}{2}}_{\widetilde{\mathfrak{G}},\mathscr{g}}=-Tr_{\widetilde{\mathfrak{F}}}\ln\widetilde{\mathfrak{G}}\widetilde{\mathscr{g}}^{-1}$$

Morfología y sistematicidad de una partícula o antipartícula relativista en materia oscura.

$$\frac{d\gamma(\chi)}{d\chi} = \frac{1}{\chi^2}\frac{\delta(m)}{\mathfrak{H}(m)}\langle\sigma\upsilon\rangle\gamma_{eq}(\chi)^2, \gamma(\chi_0 \preccurlyeq 1) = \gamma_0 \preccurlyeq 1, \mathfrak{H}(\tau) = \sqrt{\frac{4\pi^3}{90}}g_*^{\frac{1}{2}}\frac{\tau^2}{m_{\wp\ell}}, \delta(\tau)$$

$$= \frac{2\pi^2}{90}g_{*\mathcal{S}}\tau^3, \mathfrak{H}(\chi) \mapsto \mathfrak{H}(\chi)\cdot\mathcal{F}(\chi,\chi_\tau,\gamma), \mathcal{F}(\chi,\chi_\tau,\gamma) = \begin{cases}\left(\frac{\chi_\tau}{\chi}\right)^\gamma & \chi \le \chi_\tau \\ 1 & \chi \ge \chi_\tau\end{cases}$$

$$\gamma_{eq}(\chi) = \frac{90}{4\pi^4}\frac{g}{g_{*\mathcal{S}}}\chi^2\mathfrak{K}_2(\chi)\underset{\chi\gg 4}{\longrightarrow}\frac{90}{\sqrt{64\pi^9}}\frac{g}{g_{*\mathcal{S}}}\chi^{3/2}\exp(-\chi), \Omega\hbar^2 = \frac{\delta_0}{\rho_{crit}^0/\hbar^2}m\gamma(\chi \mapsto \infty), \gamma_{Std}(\chi)$$

$$= \frac{2710^{3/2}}{256\varpi^{13/4}}\frac{g^2 g_{*\mathcal{S}}}{g_*^{5/2}}m_{\wp\ell}m\langle\sigma\upsilon\rangle[1-\exp(-2\chi)(1-2\chi)], \Omega\hbar_{Std}^2$$

$$= \frac{\delta_0}{\rho_{crit}^0/\hbar^2}\frac{2710^{3/2}}{256\varpi^{13/4}}\frac{g^2 g_{*\mathcal{S}}}{g_*^{5/2}}m_{\wp\ell}m^2\langle\sigma\upsilon\rangle, \gamma_{(\chi_\tau,\gamma)}(\chi)$$

$$= \frac{2710^{3/2}}{256\varpi^{13/4}}\frac{g^2 g_{*\mathcal{S}}}{g_*^{5/2}}m_{\wp\ell}m^2\langle\sigma\upsilon\rangle\frac{[\Gamma(2+\gamma)-\Gamma(2+\gamma,2\chi)]}{(2\chi_\tau)^\gamma}$$

$$\Gamma(\alpha,\beta) = \iiint_\beta^\infty d\gamma\gamma^{\alpha-1}\exp(-\gamma), \alpha,\beta \in \mathfrak{C}, \Omega\hbar_{(\chi_\tau,\gamma)}^2 = \Omega\hbar_{Std}^2\frac{\Gamma(2+\gamma)}{(2\chi_\tau)^\gamma}$$

$$z \equiv \ln\chi, \mathcal{W}(z) \equiv \ln\mathcal{Y}(z), \mathcal{W}_\eta(z) = -\frac{\mathcal{W}_\eta(z)}{\mathcal{W}_0}\exp[\mathfrak{C}(m\langle\sigma\upsilon\rangle)] \equiv \frac{\langle\sigma\upsilon\rangle\delta(m)}{\mathfrak{H}(m)}$$

$$= \sqrt{\frac{\pi}{90}}\frac{g_{*\mathcal{S}}}{g_*^{\frac{1}{2}}}m_{\wp\ell}m\langle\sigma\upsilon\rangle, \mathcal{F}(z,z_\tau,\gamma) = \exp[\gamma\,\mathrm{ReLU}(z_\tau - z)], \mathcal{W}_{eq}(z)$$

$$= \ln\left(\frac{90}{\sqrt{64\varpi^9}}\frac{g}{g_{*\mathcal{S}}}\right) + \frac{3}{2}z - \exp(z)\,\xi\left[z,\mathcal{W},\frac{d\mathcal{W}}{dz};\mathfrak{C},z_\tau,\gamma\right]$$

$$\equiv \frac{d\mathcal{W}}{dz} - \exp\left(\mathfrak{C} - \gamma\,\mathrm{ReLU}(z_\tau - z) - z + 2\mathcal{W}_{eq} - \mathcal{W}\right)$$

$$\mathcal{L}_{Fwd} = \lambda_{\mathfrak{BE}}\mathcal{L}_{\mathfrak{BE}} + \lambda_{\mathfrak{IC}}\mathcal{L}_{\mathfrak{IC}}$$

$$\mathcal{L}_{\mathfrak{BE}} = \frac{1}{\mathcal{N}_z}\sum_{j=1}^{\mathcal{N}_z}\left|\xi\left(z_j,\mathcal{W}(z_j),\frac{d\mathcal{W}}{dz}(z_j);\mathfrak{C},z_\tau,\gamma\right)\right|, \mathcal{L}_{\mathfrak{IC}} = |\mathcal{W}(z_0) - \mathcal{W}_0|, \mathcal{L}'_{Fwd} = \mathcal{L}_{Fwd} + \lambda_+\mathcal{L}_+, \mathcal{L}_+$$

$$= \frac{1}{\mathcal{N}_z}\sum_{j=1}^{\mathcal{N}_z}\mathrm{ReLU}\left(-\frac{d\mathcal{W}}{dz}(z_j)\right)\mathcal{L}_{Inv} = \mathcal{L}_{Fwd} + \lambda_{\Omega\hbar^2}\mathcal{L}_{\Omega\hbar^2}, \mathcal{L}_{\Omega\hbar^2}$$

$$= \left|\mathcal{W}_\eta(z_f) - \mathcal{W}_{\Omega\hbar^2}\right|\,\mathrm{Softplus}_\beta(\chi) := \frac{1}{\beta}\ln\left(1 + e^{\beta\chi}\right)\lim_{\beta\mapsto\infty}\mathrm{Softplus}_\beta(\chi) = \mathrm{ReLU}(\chi)$$

$$= \max(0,\chi), \mathrm{Softplus}_\beta(\chi) \sim \chi, \chi \mapsto \infty, \forall\beta \ge 0, \mathrm{Softplus}_\beta(\chi) \sim 0, \chi \mapsto -\infty, \forall\beta$$

$$\ge 0, \left.\frac{\partial\mathrm{Softplus}_\beta(\chi)}{\partial\chi}\right|_{\chi=0}\forall\beta$$

$$\mathcal{G}(\chi,\chi_\tau,\gamma;\beta)=\exp[\gamma\mathrm{Softplus}_\beta(\chi)(z_\tau-z)]=\left[1+\left(\frac{\chi_\tau}{\chi}\right)^\beta\right]^{\frac{\gamma}{\beta}}\lim_{\beta\mapsto\infty}\mathcal{G}(\chi,\chi_\tau,\gamma;\beta)$$
$$=\mathcal{F}(\chi,\chi_\tau,\gamma),\xi\left[z,\mathcal{W},\frac{d\mathcal{W}}{dz};\mathfrak{C},z_\tau,\gamma;\beta\right]$$
$$\equiv\frac{d\mathcal{W}}{dz}-\exp(\mathfrak{C}-\gamma\ \mathrm{Softplus}_\beta(\chi)(z_\tau-z)-z+2\mathcal{W}_{eq}-\mathcal{W})$$
$$\mathfrak{C}_{sample}=\mu_{\mathfrak{C}}+\sigma_{\mathfrak{C}}\times\epsilon,\mathcal{L}_{Bayesian}=\mathcal{L}_{Inv}+\lambda_{\mathfrak{KL}}\mathfrak{D}_{\mathfrak{KL}}\langle\wp(\mathfrak{C})||q(\mathfrak{C})\rangle$$

Cálculos cromodinámicos.

$$\mathcal{W}^{\rho,\mu\nu}_{\delta,\delta'}=(\wp,\wp')=\langle\wp',\delta'|\bar{q}(0)\gamma^\rho ig\mathfrak{G}^{\mu\nu}(0)q(0)|\wp,\delta\rangle,\mathcal{W}^{+,+i}(\wp,\wp')$$
$$=\bar{\mu}(\wp',\delta')\Big\{(\mathcal{P}^+\Delta^i_\perp-\mathcal{P}^i\Delta^+)\gamma^+\phi_1(\tau)+\mathcal{P}^+m_{\mathcal{N}}i\sigma^{+i}\phi_2(\tau)$$
$$+\frac{1}{m_{\mathcal{N}}}i\sigma^{+\nu}\Delta_\nu[\mathcal{P}^+\Delta^i_\perp\phi_3(\tau)-\mathcal{P}^i\Delta^+\phi_4(\tau)]+\frac{\mathcal{P}^+\Delta^+}{m_{\mathcal{N}}}i\sigma^\nu\Delta_\nu\phi_5(\tau)\Big\}\mu(\wp,\delta)$$
$$\mathcal{O}^{[5]}_{[i\{j]4\}}=-\frac{1}{4}\bar{q}(0)\gamma_{[i}\gamma_5\overleftrightarrow{\mathfrak{D}}_{\{j]}\overleftrightarrow{\mathfrak{D}}_{4\}}q(0),\phi_i(\tau)=\frac{\phi_i(0)+\beta_i\alpha}{\left[1-\tau\left(\frac{1}{\Lambda_i^2}+c_i\alpha\right)\right]^2}$$
$$\mathfrak{F}^j_{\delta'\delta}(\beta)=\iiint\frac{d^2\Delta_\perp}{(4\pi)^2}e^{-i\beta\otimes\Delta_\perp}\mathfrak{F}^j_{\delta'\delta}(\Delta_\perp),\mathfrak{F}^j_{\delta'\delta}(\Delta_\perp)$$
$$=\frac{i}{\sqrt{2}\mathcal{P}^+}\bigotimes\left\langle\wp^+,\frac{\Delta_\perp}{2},\delta'\middle|\bar{q}(0)\gamma^+ig\mathfrak{G}^{+j}(0)q(0)\middle|\wp^+,-\frac{\Delta_\perp}{2},\delta\right\rangle,\mathfrak{F}^j_{\delta'\delta}(\beta)$$
$$=\frac{i}{\sqrt{2}\mathcal{P}^+}\iiint\frac{d^2\Delta_\perp}{(4\pi)^2}e^{-i\beta\otimes\Delta_\perp}\bigotimes\bar{\mu}(\wp',\delta')\Big[\mathcal{P}^+\Delta^j\gamma^+\phi_1(\tau)+\mathcal{P}^+m_{\mathcal{N}}i\sigma^{+j}\phi_2(\tau)$$
$$+\frac{\mathcal{P}^+\Delta^j}{m_{\mathcal{N}}}i\sigma^{+\nu}\Delta_\nu\phi_3(\tau)\Big]\mu(\wp,\delta)$$

Partículas y antipartículas relativistas - QCD.

$$\mathcal{L}_{QCD}=g_{\alpha\gamma\gamma}\xi\otimes\beta\alpha\left|m_\alpha^2-m_\gamma^2\right|\le\frac{8\hslash_\gamma}{\mathcal{L}_{eff}},m_\alpha=\sqrt{m_\gamma^2+2q_\tau\hslash_\gamma\cos(\theta_\beta)\Delta\theta},\mathcal{P}(\alpha\leftrightarrow\gamma)$$
$$=\left(\frac{1}{4}g_{\alpha\gamma\gamma}\xi_{eff}\mathcal{L}_{eff}\cos\theta_\beta\right)^2,\mathcal{L}_{eff}=2\mathcal{L}^{\mathrm{B}}_{att}\left(1-e^{\frac{\mathcal{L}_\chi}{2\mathcal{L}^{\mathrm{B}}_{att}}}\right)\Delta\theta_{RC}\Delta_\tau\cong\frac{\lambda_\chi\tan\theta_\beta}{c},c\Delta_\tau$$
$$=2\ell\tan\theta_\beta\sin\theta_\beta,\mathcal{P}(\alpha\leftrightarrow\gamma)\cong\left(\frac{1}{4}g_{\alpha\gamma\gamma}\xi_{eff}\mathcal{L}_{eff}\xi_{\mathrm{B}}\cos\theta_\beta\right)^2,g_{\alpha\gamma\gamma}$$
$$\le\left(\frac{1}{4}\xi_{eff}\mathcal{L}_{eff}\xi_{\mathrm{B}}\cos\theta_\beta\right)^{-1}\mathcal{P}(\alpha\leftrightarrow\gamma)^{\frac{1}{2}}$$
$$\eta_i=\frac{1}{\mathfrak{T}^2_{Ge}}\frac{\mathfrak{E}_i^{JF,ch}}{\mathfrak{E}_{i\backslash\varrho}^{in,ch}}\eta\mathcal{N}_{in}=\sum_i\eta_i\frac{\mathfrak{E}_i^{in,aq}}{\hslash_\gamma}$$

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APÉNDICE B.

1. Campos de Yang – Mills y gravedad cuántica.

1.1. Transformaciones de Gauge.

1.1.1. Campos de Yang – Mills.

$$\hat{g}^{\alpha} \equiv (g^{-1})^{\alpha} g_{\star}\hat{g}^{\alpha} = \mathfrak{I}^{\alpha}, (g_{\star}\hbar)_{\star}\phi^{\alpha} = g_{\star}(\hbar_{\star}\phi), (g_{\star}\hbar)_{\star}\phi^{\alpha} \equiv g_{\star}\hbar_{\star}\phi^{\alpha} \equiv g_{\star}(\hbar_{\star}\phi)$$

$$\mathfrak{G}(g) = e^{i\mathfrak{g}^{i}t_{i}} trt_{i}t_{j} = \frac{1}{2}\delta_{ij}[t_{i}, t_{j}] = if^{ij\kappa}t_{\kappa}\mathfrak{G}(g)\mathfrak{G}(\hbar) = \mathfrak{G}(g_{\star}\hbar), ig_{\star}\hbar^{i}t_{i} = \log\left(e^{i\mathfrak{g}^{i}t_{i}}e^{i\hbar^{i}t_{i}}\right), g_{\star}\hbar^{i}$$
$$= -2itrt_{i}\log\left(e^{i\mathfrak{g}^{j}t_{j}}e^{i\hbar^{\kappa}t_{\kappa}}\right), e^{i\mathfrak{g}^{j}t_{j}}t_{i}e^{-ig^{j}t_{j}} = \left(e^{\mathfrak{F}}(\mathfrak{g})\right)^{i\kappa} t_{\kappa}\left(e^{-\mathfrak{F}(\mathfrak{g})}\right)^{\kappa i}, \left(\mathfrak{F}(\mathfrak{g})\right)^{i}_{\kappa}$$
$$\coloneqq f^{ij\kappa}g^{j}, \mathcal{O}(\mathfrak{g}) = e^{\mathfrak{F}(g)}, g_{\star}\alpha_{\mu} = \mathfrak{G}(g)\alpha_{\mu}\mathcal{G}^{-1}(g) + i\mathfrak{G}(g)\partial_{\mu}\mathcal{G}^{-1}(g), g_{\star}\alpha^{\kappa}_{\mu}$$
$$= \alpha^{i}_{\mu}\left(e^{\mathfrak{F}(\mathfrak{g})}\right)^{i\kappa} + \partial_{\mu}\mathfrak{g}^{i}\left(e^{\mathfrak{F}(\mathfrak{g})}\right)^{i\kappa}$$

1.1.2. Gravedad cuántica.

$$\hbar_{\star}\mathfrak{g}^{\mu}(\chi) = \hbar^{\mu}(\mathfrak{g}(\chi)), \hat{\mathfrak{g}}^{\mu}(\mathfrak{g}(\chi)) = \chi^{\mu}, \mathfrak{g}^{\mu}(\hat{\mathfrak{g}}(\chi)) = \chi^{\mu}, \hbar^{\star}\mathfrak{g}^{\mu}(\chi) = \mathfrak{g}^{\mu}(\hbar(\chi))$$

$$\mathfrak{g}_{\star}g^{\mu\nu}(\chi) = \frac{\partial\hat{\mathfrak{g}}^{\rho}(\chi)}{\partial\chi^{\mu}}\frac{\partial\hat{\mathfrak{g}}^{\lambda}(\chi)}{\partial\chi^{\nu}}g_{\rho\lambda}(\hat{\mathfrak{g}}(\chi))$$

1.1.3. Parametrizaciones estándar.

$$g_{\star}\phi^{\alpha} = \mathfrak{T}^{\alpha}{}_{\beta}[\mathfrak{g}]\phi^{\beta} + \mathfrak{T}^{\alpha}[\mathfrak{g}], \mathfrak{T}^{\alpha}{}_{\beta}[\hat{\mathfrak{g}}]\mathfrak{T}^{\beta}{}_{c}[\mathfrak{g}] = \delta^{\alpha}_{c}, \mathfrak{T}^{\alpha}{}_{\beta}[\mathfrak{g}]\mathfrak{T}^{\beta}{}_{c}[\hat{\mathfrak{g}}] = \delta^{\alpha}_{c}, \mathfrak{T}^{\alpha}[\hat{\mathfrak{g}}] = -\mathfrak{T}^{\alpha}{}_{\beta}[\hat{\mathfrak{g}}]\mathfrak{T}^{\beta}[\mathfrak{g}]$$

$$\hat{\phi}^{\alpha}[\phi] = \hat{\mathfrak{G}}[\phi]_{\star}\phi^{\alpha} = \hat{\mathfrak{G}}[g_{\star}\phi] = \hat{\mathfrak{G}}[\phi]_{\star}\hat{\mathfrak{g}}, \hat{\phi}^{\alpha}[g_{\star}\phi] = \hat{\mathfrak{G}}[g_{\star}\phi]_{\star}g_{\star}\phi^{\alpha} = \hat{\mathfrak{G}}[\phi]_{\star}\hat{g}_{\star}g_{\star}\phi^{\alpha} = \hat{\mathfrak{G}}[\phi]_{\star}\phi^{\alpha}$$
$$= \hat{\phi}^{\alpha}[\phi]$$

$$\mathfrak{G}[\phi]_{\star}\hat{\phi}^{\alpha}[\phi] = \phi^{\alpha}, \mathfrak{G}^{\alpha}[g_{\star}\phi] = g_{\star}\mathfrak{G}^{\alpha}[\phi]$$

$$\mathfrak{E}[g_{\star}\phi] = g_{\star}\mathfrak{E}[\phi]_{\star}\hat{\mathfrak{g}}, \hat{\mathfrak{E}}[g_{\star}\phi] = g_{\star}\hat{\mathfrak{E}}[\phi]_{\star}\hat{\mathfrak{g}}, \mathfrak{E}[g_{\star}\phi]_{\star}g_{\star}\phi = g_{\star}\mathfrak{E}[\phi]_{\star}\phi, \phi^{\alpha} \longrightarrow \mathfrak{E}^{\alpha}[\phi] \equiv \hat{\mathfrak{E}}[\phi]_{\star}\phi^{\alpha}$$

$$\hat{\varphi}^{\alpha} = \mathfrak{E}^{\alpha}[\hat{\phi}] = \hat{\mathfrak{E}}[\hat{\phi}]_{\star}\hat{\phi}^{\alpha}, \hat{\varphi}^{\alpha}[\phi] = \hat{\mathfrak{G}}[\phi]_{\star}\hat{\mathfrak{E}}[\phi]_{\star}\phi, \hat{\varphi}^{\alpha}[\phi] = \hat{\hbar}[\phi]_{\star}\phi, \hat{\hbar}[\phi] = \hat{\mathfrak{G}}^{\alpha}[\mathfrak{E}[\phi]_{\star}\phi]$$
$$= \hat{\mathfrak{G}}[\phi]_{\star}\hat{\mathfrak{E}}[\phi]$$

$$\hat{\hbar}^{\alpha}[g_{\star}\phi] = \hat{\mathfrak{G}}^{\alpha}[g_{\star}\phi]_{\star}\hat{\mathfrak{E}}[g_{\star}\phi] = \hat{\mathfrak{G}}^{\alpha}[\phi]_{\star}\hat{g}_{\star}g_{\star}\hat{\mathfrak{E}}[\phi]_{\star}\hat{\mathfrak{g}} = \hat{\mathfrak{G}}^{\alpha}[\phi]_{\star}\hat{\mathfrak{E}}[\phi]_{\star}\hat{\mathfrak{g}} = \hat{\hbar}^{\alpha}[\phi]_{\star}\hat{\mathfrak{g}}, \mathfrak{E}[\phi]$$
$$= \hbar[\phi]_{\star}\hat{\mathfrak{G}}^{\alpha}[\phi]$$

1.2. Transformaciones dinámicas en campos de Yang – Mills y gravedad cuántica.

$$\mathfrak{D}[\phi] = \mathcal{G}_{\hat{\mathfrak{E}}[\phi]}, \mathfrak{D}[g_{\star}\phi] = \mathfrak{D}[\phi]\mathcal{G}_{\hat{\mathfrak{g}}}$$

$$\hat{\mathfrak{G}}^{\hat{\mu}}(\chi)\big|_{\phi\mapsto g_{\star}\phi} = \hat{\mathfrak{G}}^{\hat{\mu}}(\hat{g}(\chi)), \mathfrak{G}^{\mu}(\hat{\chi})\big|_{\phi\mapsto g_{\star}\phi} = g^{\mu}(\mathfrak{G}^{\mu}(\hat{\chi})), \hat{\mathfrak{G}}^{\mu}(\chi)\big|_{\phi\mapsto g_{\star}\phi} = g\left(\hat{\mathfrak{E}}(\hat{g}(\chi))\right), \hat{\mathfrak{G}}^{\hat{\alpha}}[\phi]$$
$$= \mathfrak{D}^{\hat{\alpha}}[\phi] = \mathfrak{D}^{\hat{\mu}}(\chi)[\phi]$$

1.3. Derivadas funcionales.

$$\frac{\delta g^{\alpha}}{\delta g^{\beta}} = \delta^{\alpha}_{\beta} = \delta(\chi-\gamma)\delta^{i}_{j}, \delta^{\alpha}_{\beta} = \delta(\chi-\gamma)\delta^{\mu}_{\nu}, \mathfrak{T}^{\mathfrak{a}}_{\alpha}[g,\phi] \coloneqq \frac{\delta(g_{\star}\phi^{\alpha})}{\delta g^{\alpha}}, \mathfrak{T}^{\mathfrak{a}}_{\alpha}[\phi] \coloneqq \mathfrak{T}^{\mathfrak{a}}_{\alpha}[1,\phi], \mathfrak{T}^{\mathfrak{a}}_{\alpha}[\phi]\epsilon^{\alpha}$$

$$= -\mathcal{L}_{\epsilon} g_{\mu\nu} = -\nabla^{(g)}_{\mu}\epsilon_{\mu}, \mathfrak{T}^{\mathfrak{a}}_{\alpha}[\phi]\epsilon^{\alpha} = \partial_{\mu}\epsilon^{i} + f^{ijk}a^{j}_{\mu}\epsilon^{\kappa} \equiv \nabla^{(A)}_{\mu}\epsilon^{i}, \hbar_{\star}\phi^{\alpha}$$

$$= \hbar_{\star}\widehat{g}_{\star}g_{\star}\phi^{\alpha}, \mathfrak{T}^{\mathfrak{a}}_{\alpha}[g,\phi] = \mathcal{U}_{\alpha}{}^{\beta}[\widehat{g}_{\star}]\mathfrak{T}^{\alpha}_{\beta}[g_{\star}\phi], \mathcal{U}_{\gamma}{}^{\alpha}[\widehat{g}_{\star}] \coloneqq \left.\frac{\delta\hbar_{\star}\widehat{g}^{\alpha}}{\delta\hbar^{\gamma}}\right|_{\mathfrak{h}=\mathfrak{g}}, \mathfrak{T}^{\mathfrak{a}}_{\alpha}[g,\widehat{g}_{\star}\phi]$$

$$= \mathcal{U}_{\alpha}{}^{\beta}[\widehat{g}]\mathfrak{T}^{\alpha}_{\beta}[\phi], \mathfrak{g}[\phi]_{\star}\hat{\mathfrak{g}}[\phi]_{\star}\phi^{\alpha} = \phi^{\alpha}\mathcal{T}^{\alpha}{}_{c}[\mathfrak{g}]\frac{\delta\widehat{g}_{\star}\phi^{c}}{\delta\phi^{\beta}} + \frac{\delta g^{\alpha}}{\delta\phi^{\beta}}\mathcal{U}_{\alpha}{}^{\beta}[\widehat{g}]\mathfrak{T}^{\alpha}_{\beta}[\phi]$$

$$= \delta^{\alpha}_{\beta}, \frac{\delta g_{\star}\phi^{\alpha}}{\delta\phi^{\beta}} = \mathcal{T}^{\alpha}{}_{c}[\hat{\mathfrak{g}}]\left(\delta^{c}_{\beta} - \frac{\delta g^{\alpha}}{\delta\phi^{\beta}}\mathcal{U}_{\alpha}{}^{\beta}[\widehat{g}]\mathfrak{T}^{c}_{\beta}[\phi]\right), \frac{\delta\widehat{g}_{\star}\phi^{\alpha}}{\delta\phi^{\beta}}$$

$$= \mathcal{T}^{\alpha}{}_{c}[g]\left(\delta^{c}_{\beta} - \frac{\delta\widehat{g}^{\alpha}}{\delta\phi^{\beta}}\mathcal{U}_{\alpha}{}^{\beta}[g]\mathfrak{T}^{c}_{\beta}[\phi]\right)$$

1.4. Integrales funcionales.

$$\mathcal{Z} = \iiint d\phi\tilde{\mu}[\phi]\delta\left(1-\widehat{\mathfrak{G}}[\phi]\right)e^{-\delta[\phi]}, \mathcal{Q}^{\alpha}{}_{\beta}[\phi] \coloneqq \left.\frac{\delta}{\delta g^{\beta}}\left(\mathfrak{G}[\phi]_{\star}\hat{\mathfrak{g}}\right)^{\alpha}\right|_{\mathfrak{g}=1} = \left.\frac{\delta}{\delta g^{\beta}}\left(\mathfrak{G}[\phi]_{\star}\mathfrak{g}\right)^{\alpha}\right|_{\mathfrak{g}=1}$$

$$\mathcal{Z} = \iiint d\phi\tilde{\mu}[\phi]\delta\left(1-\widehat{\mathfrak{H}}[\phi]\right)e^{-\delta[\phi]}\, d(\mathfrak{E}[\phi]_{\star}\phi)\tilde{\mu}[\mathfrak{E}[\phi]_{\star}\phi] = d\phi\tilde{\mu}[\phi], d(\mathfrak{E}[\phi]_{\star}\phi)$$

$$= d\phi\,\det\mathcal{T}^{\alpha}{}_{\beta}[\mathfrak{E}] \bigotimes \det\left(\delta^{\alpha}_{\beta} - \frac{\delta\widehat{\mathfrak{E}}^{\alpha}}{\delta g^{\beta}}\mathcal{U}_{\alpha}{}^{\beta}[\mathfrak{E}]\mathfrak{T}^{\alpha}_{\beta}[\phi]\right), d(\mathfrak{E}[\phi]_{\star}\phi)$$

$$= d\phi\,\det\mathcal{T}^{\alpha}{}_{\beta}[\mathfrak{E}]\,\det\mathcal{T}^{\alpha}{}_{\beta}[\widehat{\mathfrak{E}}]$$

$$\mathcal{T}^{\alpha}{}_{\beta}[\mathfrak{g}] \coloneqq \left.\frac{\delta g_{\star}\hbar_{\star}\widehat{g}^{\alpha}}{\delta\mathfrak{h}^{\beta}}\right|_{\mathfrak{h}=1} \mathcal{T}^{\alpha}{}_{\beta}[\mathfrak{g}]\mathcal{T}^{\beta}{}_{\gamma}[\hat{\mathfrak{g}}] = \delta^{\alpha}_{\gamma}, \gamma_{\alpha\beta}[\phi] = \gamma_{\alpha\beta}[g_{\star}\phi]$$

$$= \mathcal{T}^{c}{}_{\alpha}[\hat{\mathfrak{g}}]\gamma_{cd}[\phi]\mathcal{T}^{d}{}_{\beta}[\hat{\mathfrak{g}}], \eta_{\alpha\beta}[g_{\star}\phi] = \mathcal{T}^{\gamma}{}_{\alpha}[\hat{\mathfrak{g}}]\eta_{\gamma\delta}[\phi]\mathcal{T}^{\delta}{}_{\beta}[\hat{\mathfrak{g}}]$$

$$\tilde{\mu}[\phi] = \frac{\sqrt{\det\gamma_{\alpha\beta}[\phi]}}{\sqrt{\det\eta_{\alpha\beta}[\phi]}}\mathcal{T}^{\alpha}{}_{\beta}v^{\beta} = \left(e^{-\mathcal{F}(\mathfrak{g}(\chi))}\right)^{ij}v^{j}(\chi), \mathcal{T}^{\alpha}{}_{\beta}v^{\beta} = \frac{\partial\mathfrak{g}^{\mu}(\chi)}{\partial\chi^{\nu}}v^{\nu}\left(\hat{\mathfrak{g}}(\chi)\right), \gamma^{\mu,\nu}_{i,j}(\chi,\gamma)$$

$$= \frac{\mu^{2}}{g}\delta^{\mu\nu}\delta_{ij}\delta(\chi-\gamma), \eta_{i,j}(\chi,\gamma) = \frac{\mu^{4}}{g}\delta_{ij}\delta(\chi-\gamma)$$

$$\gamma^{\mu\nu,\rho\sigma}(\chi,\gamma) = \frac{\mu^{2}}{64\pi\mathcal{G}_{\mathcal{N}}}\sqrt{g}\left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}\right)\delta(\chi-\gamma), \eta_{\mu,\nu}(\chi,\gamma)$$

$$= \frac{\mu^{4}}{32\pi\mathcal{G}}\sqrt{g}g_{\mu\nu}\delta(\chi-\gamma)$$

1.5. Métrica de Faddeev-Popov.

$$\chi^{\alpha}[g_{\star}\phi] = 0 \Longrightarrow g = \widehat{\mathfrak{G}}[\phi], \delta\left(\mathfrak{g}-\widehat{\mathfrak{G}}[\phi]\right) = \delta(\chi[g_{\star}\phi])\left|\det\frac{\delta\chi^{\alpha}[g_{\star}\phi]}{\delta g^{\beta}}\right|, \delta\left(\mathfrak{g}-\widehat{\mathfrak{G}}[\phi]\right)$$

$$= \delta\left(1-\widehat{\mathfrak{G}}[\phi]\right) = \delta(\chi[\phi])\left|\det\frac{\delta\chi^{\alpha}[g_{\star}\phi]}{\delta g^{\beta}}\right|_{\mathfrak{g}=1}$$

1.6. Campos armónicos de Landau.

$$\mathfrak{I}(\alpha,\hat{\mathfrak{g}}) = \iiint_{\chi} tr(\hat{\mathcal{g}}_{\star}\alpha_{\mu})(\hat{\mathcal{g}}_{\star}\alpha_{\mu}), \mathfrak{D}^{\dagger}\nabla_{\mu}\nabla^{\mu}\mathfrak{D} + (\nabla_{\mu}\mathfrak{D})^{\dagger}(\nabla_{\mu}\mathfrak{D}), \mathfrak{I}[g,\hat{\mathfrak{g}}]$$

$$= \iiint_{\chi} \mathcal{g}_{\star}(\sqrt{g}g^{\mu\nu})\delta_{\mu\nu}, \mathfrak{I}[g,\hat{\mathfrak{g}}] = \iiint_{\chi} \sqrt{g}g^{\rho\lambda}\delta_{\mu\nu}\,\partial_{\rho}\widehat{\mathcal{g}^{\mu}}\partial_{\lambda}\widehat{\mathcal{g}^{\nu}}\delta_{\widehat{\mu\nu}}$$

1.7. Acción BRST.

$$\mathcal{Z} = \iiint d\phi\tilde{\mu}[\phi]\delta(\mathfrak{G}[\phi])e^{-\delta[\phi]}\lim_{\epsilon\mapsto 0}\delta_{\epsilon}(\mathcal{M}-1) = \delta(\mathcal{M}-1), \delta(\mathcal{M}-1)\mathcal{F}(\mathcal{M})$$
$$= \delta(\mathcal{M}-1)\mathcal{F}(1), \delta_{\epsilon}(\mathcal{M}-1) = \iiint d\mathcal{B}e^{-\iiint_{\chi}\frac{1}{2}[\epsilon tr\bar{\mathcal{B}}\mathcal{B}+tr\bar{\mathcal{B}}(\mathcal{M}-1)+tr(\bar{\mathcal{M}}-1)\mathcal{B}]}$$

$$\mathcal{B}_{\mathfrak{I}\mathfrak{I}} = \mathfrak{B}^{\mathfrak{R}}_{\mathfrak{I}\mathfrak{I}} + i\mathfrak{B}^{\mathfrak{I}}_{\mathfrak{I}\mathfrak{I}}, \bar{\mathcal{B}}_{\mathfrak{I}\mathfrak{I}} = \mathfrak{B}^{\mathfrak{R}}_{\mathfrak{I}\mathfrak{I}} - i\mathfrak{B}^{\mathfrak{I}}_{\mathfrak{I}\mathfrak{I}}, \mathfrak{B}^{\mathfrak{R}}_{\mathfrak{I}\mathfrak{I}} = \frac{1}{2}(\mathcal{B}_{\mathfrak{I}\mathfrak{I}} + \bar{\mathcal{B}}_{\mathfrak{I}\mathfrak{I}}), \mathfrak{B}^{\mathfrak{I}}_{\mathfrak{I}\mathfrak{I}} = -\frac{1}{2}(\mathcal{B}_{\mathfrak{I}\mathfrak{I}} + \bar{\mathcal{B}}_{\mathfrak{I}\mathfrak{I}}), \delta(\mathcal{g})$$
$$= \lim_{\epsilon\mapsto 0}\mathcal{N}_{\epsilon}\delta_{\epsilon}(\mathcal{G}[\mathfrak{g}]-1), \mathcal{N}_{\epsilon} = \frac{1}{\iiint d\mathfrak{h}\delta_{\epsilon}(\mathcal{G}[\mathfrak{h}]-1)}\mathcal{G}[\mathfrak{g}] = 1 \Longrightarrow \mathfrak{g}^{i}$$
$$= 0, \iiint d\mathfrak{g}\lim_{\epsilon\mapsto 0}\mathcal{N}_{\epsilon}\mathfrak{F}[\mathfrak{g}]\delta_{\epsilon}(\mathcal{G}[\mathfrak{g}]-1)$$

$$\delta\hat{\mathfrak{G}}[\phi] = \lim_{\epsilon\mapsto 0}\mathcal{N}_{\epsilon}\iiint d\mathcal{B}e^{-\iiint_{\chi}\frac{1}{2}[\epsilon tr\bar{\mathcal{B}}\mathcal{B}+tr\bar{\mathcal{B}}(\mathfrak{D}[\phi]-1)+tr(\bar{\mathfrak{D}}[\phi]-1)\mathcal{B}]}$$

$$\mathcal{N}_{\epsilon} \sim \iiint d\mathfrak{C}d\mathcal{A}e^{-\frac{1}{2}\iiint_{\chi} tr[\epsilon^{2}\bar{\mathcal{A}}\mathcal{A}+i\bar{\mathcal{A}}t_{i}\mathfrak{C}^{i}-i\mathcal{A}\bar{t}_{i}\mathfrak{C}^{i}]}, \mathcal{Q}_{i}(\chi,\gamma) \coloneqq -\frac{\delta}{\delta\mathfrak{g}^{i}(\gamma)}\mathfrak{D}[\mathcal{g}_{\star}\phi](\chi)|_{\mathcal{g}=1}, \mathcal{Q}_{i}(\chi,\gamma)$$
$$= iDt_{i}\delta(\chi-\gamma)$$

$$\mathcal{Z} = \iiint d\phi dAdBd\mathcal{C}\tilde{\mu}e^{-\mathcal{S}+\frac{1}{2}\iiint_{\chi}(tr\mathcal{B}+tr\bar{\mathcal{B}})}, \mathcal{S} = \mathfrak{S} + \frac{1}{2}\iiint_{\chi} tr[\bar{\mathcal{B}}D + \bar{D}\mathcal{B} + (\bar{\mathcal{A}}Q_{i} + \mathcal{A}\bar{Q}_{i})\mathfrak{C}^{i}], \mathcal{D}^{\hat{\alpha}}$$

$$= (\mathcal{D})^{\mathcal{I}\mathcal{J}}(\chi), \mathcal{S} = \mathfrak{S} + \iiint_{\chi}\frac{1}{2}[\mathcal{B}^{*}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}} + \mathcal{B}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}} + (\mathcal{A}^{*}_{\hat{\alpha}}\mathcal{Q}^{\hat{\alpha}}_{\alpha} + \mathrm{A}_{\hat{\alpha}}\bar{\mathcal{Q}}^{\hat{\alpha}}_{\alpha})\mathfrak{C}^{\alpha}]$$

$$\mathcal{Z} = \iiint d\phi d\mathcal{B}\tilde{\mu}[\phi]e^{-\delta[\phi]-\mathcal{B}_{\hat{\alpha}}(\mathcal{D}^{\hat{\alpha}}[\phi]-1^{\hat{\alpha}})}\,\bar{\mathcal{Q}}^{\hat{\mu}}_{\mu}(\chi,\gamma) \coloneqq -\frac{\delta}{\delta\mathfrak{g}^{\mu}(\gamma)}\mathcal{D}^{\hat{\mu}}[\mathcal{g}_{\star}\phi](\chi)\big|_{\mathcal{g}=1}, \bar{\mathcal{Q}}^{\hat{\mu}}_{\mu}(\chi,\gamma)$$
$$= \partial_{\mu}\mathcal{D}^{\hat{\mu}}\delta(\chi-\gamma)\iiint d\mathrm{A}d\mathcal{C}e^{-\mathrm{A}_{\hat{\alpha}}\mathcal{Q}^{\hat{\alpha}}{}_{\beta}\mathcal{C}^{\beta}}$$

$$\mathcal{S} = \mathfrak{S} + \iiint_{\chi}(\mathrm{A}_{\hat{\mu}}(\chi)\partial_{\mu}\mathcal{D}^{\hat{\alpha}}\mathfrak{C}^{\mu} + \mathrm{B}_{\hat{\mu}}\mathcal{D}^{\hat{\mu}}), \mathcal{S} = \mathfrak{S} + \mathrm{A}_{\hat{\alpha}}\mathcal{Q}^{\hat{\alpha}}{}_{\beta}\mathcal{C}^{\beta} + \mathrm{B}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}}, \mathrm{B}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}}$$

$$= \iiint_{\chi}\mathrm{B}_{\hat{\mu}}\mathcal{D}^{\hat{\mu}}, \mathrm{A}_{\hat{\alpha}}\mathcal{Q}^{\hat{\alpha}}{}_{\beta}\mathcal{C}^{\beta} = \iiint_{\chi}\mathrm{A}_{\hat{\mu}}(\chi)\partial_{\mu}\mathcal{D}^{\hat{\mu}}\mathfrak{C}^{\mu}, 1^{\hat{\alpha}}\mathrm{B}_{\hat{\alpha}} = \iiint_{\chi}\chi^{\hat{\mu}}\mathrm{B}_{\hat{\mu}}(\chi)$$

$$\mathcal{Z} = \iiint d\phi d\mathcal{B}d\mathcal{C}\tilde{\mu}[\phi]e^{-\delta[\phi,\mathcal{A},\mathcal{B},\mathcal{C}]+1^{\hat{\alpha}}\mathrm{B}_{\hat{\alpha}}}$$

1.8. Simetrías.

$$[\vec{\mathcal{T}}_{\alpha},\vec{\mathcal{T}}_{\beta}] = f^{\gamma}{}_{\alpha\beta}\vec{\mathcal{T}}_{\gamma}, f^{\gamma}{}_{\alpha\beta}\xi^{\alpha}\epsilon^{\beta} = f^{\kappa}_{ij}\xi^{i}\epsilon^{j}, f^{\gamma}{}_{\alpha\beta}\xi^{\alpha}\epsilon^{\beta} = \mathcal{L}_{\xi}\epsilon^{\mu} = \xi^{\nu}\partial_{\nu}\epsilon^{\mu} - \partial_{\nu}\xi^{\mu}\epsilon^{\nu}, \mathcal{Q}^{\hat{\alpha}}_{\alpha} = -\frac{\delta\mathcal{D}^{\hat{\alpha}}}{\delta\phi^{\alpha}}\mathfrak{T}^{\mathfrak{a}}_{\alpha}$$

$$\delta_\theta \phi^\alpha = \mathfrak{T}^{\mathfrak{a}}_\alpha[\phi]\mathfrak{C}^\alpha\theta, \delta_\theta\mathfrak{C}^\alpha = \frac{1}{2}\mathfrak{C}^\beta f^\alpha_{\beta\gamma}\mathfrak{C}^\gamma\theta, \delta_\theta \mathrm{A}_{\widehat{\alpha}} = -\mathcal{B}_{\widehat{\alpha}}\theta, \delta_\theta\mathcal{B}_{\widehat{\alpha}} = 1$$

$$\mathcal{S} = \mathfrak{S} + \frac{1}{2}\iiint_\chi tr\left[\bar{\mathcal{B}}\mathcal{D} + \bar{\mathcal{D}}\mathcal{B} + i\left(\bar{\mathcal{A}}\mathfrak{D}\mathfrak{C} - \mathcal{A}\overline{\mathfrak{C}\mathfrak{D}}\right)\mathfrak{C}^i\right], \mathfrak{C} \equiv \mathfrak{C}^i t_i, \bar{\mathfrak{C}} \equiv \mathfrak{C}^i \bar{t}_i, \delta_\theta a_\mu = \nabla^\alpha_\mu\mathfrak{C}\theta, \delta\theta\mathfrak{C}$$
$$= -i\mathfrak{C}^2\theta, \delta_\theta g_{\mu\nu} = -\left(\nabla_\mu\mathfrak{C}_\nu + \nabla_\nu\mathfrak{C}_\mu\right)\theta, \delta\theta\mathfrak{C}^\mu = \mathfrak{C}^\mu\nabla^g_\nu\mathfrak{C}^\nu\theta = \mathfrak{C}^\mu\partial_\nu\mathfrak{C}^\nu\theta$$

$$\mathscr{g}_\star\mathcal{B} = \mathcal{B}\bar{\mathcal{G}}[\mathscr{g}], \mathscr{g}_\star\mathcal{A} = \mathcal{A}\bar{\mathcal{G}}[\mathscr{g}], \mathscr{g}_\star\mathfrak{C} = \mathfrak{G}[\mathscr{g}]\mathfrak{C}\mathfrak{G}^{-1}[\mathscr{g}], \mathscr{g}_\star\bar{\mathcal{B}} = \mathfrak{G}[\mathscr{g}]\mathcal{B}, \mathscr{g}_\star\bar{\mathcal{A}} = \mathfrak{G}[\mathscr{g}]\mathcal{A}, \mathscr{g}_\star\bar{\mathcal{C}}$$
$$= \bar{\mathfrak{G}}^{-1}[\mathscr{g}]\bar{\mathcal{C}}\bar{\mathfrak{G}}[\mathscr{g}], \mathscr{g}_\star\mathfrak{C}^i = \left(e^{\mathcal{F}[\mathscr{g}]}\right)^i{}_j\mathcal{C}^j$$

$$\mathscr{g}_\star\mathfrak{C}^\mu(\chi) = \frac{\partial\mathscr{g}^\mu(\chi)}{\partial\chi^\nu}\mathfrak{C}^\nu\left(\widehat{\mathscr{g}}(\chi)\right), \mathscr{g}_\star\mathrm{A}_{\widehat{\mu}}(\chi) = \det\left|\frac{\partial\widehat{\mathscr{g}}(\chi)}{\partial\chi}\right|\mathrm{A}_{\widehat{\mu}}\left(\widehat{\mathscr{g}}(\chi)\right), \mathscr{g}_\star\mathrm{B}_{\widehat{\mu}}(\chi)$$
$$= \det\left|\frac{\partial\widehat{\mathscr{g}}(\chi)}{\partial\chi}\right|\mathrm{B}_{\widehat{\mu}}\left(\widehat{\mathscr{g}}(\chi)\right)$$

$$d[\mathscr{g}_\star\phi]\sqrt{\det\gamma[\mathscr{g}_\star\phi]} = d\phi\sqrt{\det\varphi[\phi]}, d[\mathscr{g}_\star\mathfrak{C}]\frac{1}{\sqrt{\det\eta[\mathscr{g}_\star\phi]}} = d\mathfrak{C}\frac{1}{\sqrt{\det\eta[\phi]}}$$

$$d\mathcal{A}d\mathcal{B} = d[\mathscr{g}_\star\mathcal{A}]d[\mathscr{g}_\star\mathcal{B}]$$

1.9. Acción efectiva.

$$\frac{1}{2}\iiint_\chi tr(\mathcal{B} + \bar{\mathcal{B}}) \mapsto \iiint_\chi tr\left(\bar{\mathfrak{K}}\mathfrak{B} + tr\mathfrak{K}\bar{\mathfrak{B}}\right) = 2\iiint_\chi\left(\mathfrak{B}^{\mathfrak{R}}_{\mathfrak{I}\mathfrak{J}}\mathfrak{K}^{\mathfrak{I}\mathfrak{J}}_{\mathfrak{R}} + \mathfrak{B}^{\mathfrak{I}}_{\mathfrak{I}\mathfrak{J}}\mathfrak{K}^{\mathfrak{I}\mathfrak{J}}_{\mathfrak{I}}\right)\iiint_\chi \chi^{\widehat{\mu}}\mathcal{B}_{\widehat{\mu}} \mapsto \iiint_\chi \mathcal{K}^{\widehat{\mu}}\mathcal{B}_{\widehat{\mu}}$$

$$e^{\mathcal{W}[\mathfrak{J}]} = \iiint d\phi\tilde{\mu}[\phi]e^{-\mathcal{S}[\phi]+\mathfrak{J}_{\mathfrak{A}}\phi^{\mathcal{A}}}, \Gamma[\phi] = \sup_{\mathfrak{J}}\mathfrak{J}_{\mathfrak{A}}\phi^{\mathcal{A}} - \mathcal{W}[\mathfrak{J}], \phi^{\mathcal{A}} = \frac{\delta\mathcal{W}[\mathfrak{J}]}{\delta\mathfrak{J}_{\mathfrak{A}}}$$
$$\equiv \mathcal{U}^{\mathcal{A}}[\mathfrak{J}], \Gamma[\phi]\mathcal{V}_{\mathfrak{A}}[\phi]\phi^{\mathcal{A}} - \mathcal{W}[\mathcal{V}[\phi]], \mathfrak{J}_{\mathfrak{A}} = \mathcal{V}_{\mathfrak{A}}[\phi] = \Gamma[\phi]\frac{\overleftarrow{\delta}}{\delta\phi^{\mathcal{A}}}, \mathfrak{J}_{\mathfrak{A}} = \mathcal{V}_{\mathfrak{A}}[\mathcal{U}[\mathcal{J}]]$$
$$= \mathfrak{J}_{\mathfrak{A}}, \mathcal{U}^{\mathcal{A}}[\mathcal{V}[\phi]] = \phi^{\mathcal{A}}, \mathcal{P}^{\mathcal{A}\mathcal{B}}[\phi]$$
$$= \langle\phi^{\mathcal{A}}\phi^{\mathcal{B}}\rangle_{\mathcal{J}}$$
$$- \langle\phi^{\mathcal{A}}\rangle\langle\phi^{\mathcal{B}}\rangle_{\mathcal{J}}\left\langle\frac{\delta\Gamma}{\delta\phi^\alpha}\middle|\frac{\delta\Gamma}{\delta\mathrm{A}_{\widehat{\alpha}}}\middle|\frac{\delta\Gamma}{\delta\mathfrak{C}^\alpha}\right\rangle\left\langle\frac{\delta\Gamma}{\mathfrak{B}^{\mathfrak{R}}_{\mathfrak{I}\mathfrak{J}}(\chi)}\middle|\frac{\delta\Gamma}{\delta\mathfrak{B}^{\mathfrak{I}}_{\mathfrak{I}\mathfrak{J}}(\chi)}\middle|\frac{\delta\Gamma}{\delta\mathrm{B}_{\widehat{\mu}}(\chi)}\right\rangle\langle\delta_{\mathfrak{I}\mathfrak{J}}|\chi^{\widehat{\mu}}\rangle, \bar{\phi}^\alpha$$
$$= \langle\widehat{\phi}^\alpha\rangle, \mathcal{P}^{\alpha\beta}[\bar{\phi}] = \langle\widehat{\phi}^\alpha\widehat{\phi}^\beta\rangle - \langle\widehat{\phi}^\alpha\rangle\langle\widehat{\phi}^\beta\rangle$$

1.10. Difeomorfismo e invariancias.

$$\mathscr{g}_\star\phi^{\mathcal{A}} = \mathcal{T}^{\mathrm{A}}{}_{\mathrm{B}}[\mathfrak{g}]\phi^{\mathcal{B}} + \mathcal{T}^{\mathrm{A}}[\mathfrak{g}], \mathscr{g}_\star\mathfrak{J}_{\mathfrak{A}} = \mathfrak{J}_{\mathfrak{B}}\mathcal{T}^{\mathfrak{B}}{}_{\mathfrak{A}}[\widehat{\mathfrak{g}}], \mathcal{W}[\mathscr{g}_\star\mathcal{J}] = \mathcal{W}[\mathcal{J}] - \mathcal{T}^{\mathrm{A}}[\widehat{\mathfrak{g}}]\mathfrak{J}_{\mathfrak{A}}, \mathscr{g}_\star\mathcal{U}^{\mathcal{A}}[\mathcal{J}]$$
$$= \mathcal{U}^{\mathcal{A}}[\mathscr{g}_\star\mathcal{J}], \mathcal{V}_{\mathfrak{A}}[\mathscr{g}_\star\mathcal{U}[\mathcal{J}]] = \mathcal{V}_{\mathfrak{A}}[\mathcal{U}[\mathscr{g}_\star\mathcal{J}]] = \mathscr{g}_\star\mathcal{J}, \mathcal{V}_{\mathfrak{A}}[\mathscr{g}_\star\phi] = \mathscr{g}_\star\mathcal{V}_{\mathfrak{A}}[\phi], \Gamma[\mathscr{g}_\star\phi]$$
$$= \Gamma[\phi]$$

1.11. Análisis SU(4) en Yang – Mills.

$$\mathcal{D}^{\widehat{\alpha}} = \bar{\mathcal{D}}^{\mathfrak{I}\mathfrak{J}} = (\mathfrak{D}^{-1})^{\mathfrak{I}\mathfrak{J}}, \mathcal{Y}^{\mathcal{I}\mathcal{J}}_i(\chi,\gamma) = it^{\mathcal{I}\mathcal{J}}_i\delta(\chi-\gamma), \mathcal{Y}^i_{\mathcal{I}\mathcal{J}}(\chi,\gamma) = -2it^{\mathfrak{I}\mathfrak{J}}_i\delta(\chi-\gamma), \mathcal{Y}^{\widehat{\alpha}}_\alpha\mathcal{Y}^\beta_{\widehat{\alpha}}$$
$$= \delta^\beta_\alpha, \mathcal{Y}^{\widehat{\alpha}}_\alpha\mathcal{Y}^{\widehat{\beta}}_\alpha = \Xi^{\widehat{\beta}}_{\perp\widehat{\alpha}}, \Xi^{\mathfrak{I}\mathfrak{J}}_{\perp}{}_{\mathfrak{K}\mathfrak{L}}(\chi,\gamma) = 2t^{\mathcal{J}\mathfrak{I}}_i t^i_{\mathfrak{L}\mathfrak{K}}\delta(\chi-\gamma)$$

$$\sigma^{\widehat{\alpha\beta}} = \sigma^{\mathfrak{I}\mathfrak{J},\mathfrak{K}\mathfrak{L}}(\chi,\gamma) = -\frac{g}{2\mu^4}\delta^{\mathfrak{I}\mathfrak{L}}\delta^{\mathfrak{J}\mathfrak{K}}\delta(\chi-\gamma), \sigma^{\widehat{\alpha\beta}}\mathcal{M}_{\hat{\beta}} = -\frac{g}{2\mu^4}\left(\mathcal{M}^{\mathcal{T}}\right)^{\hat{\alpha}}, \sigma_{\widehat{\beta\alpha}}\mathcal{Y}^{\hat{\alpha}}_{\hat{\alpha}} = \mathcal{Y}_{\widehat{\alpha\beta}}$$

$$= \eta_{\alpha\beta}\mathcal{Y}^{\beta}_{\hat{\beta}}, \mathfrak{T}^{\mathfrak{a}}_{\hat{\alpha}} \coloneqq \mathcal{Y}^{\alpha}_{\hat{\alpha}}\mathfrak{T}^{\mathfrak{a}}_{\alpha}, \mathcal{D}^{\hat{\alpha}}_{\alpha} = \frac{\delta D^{\hat{\alpha}}}{\delta\phi^{\alpha}}, \bar{\mathcal{D}}^{\hat{\alpha}}_{\alpha} = \frac{\delta D^{\hat{\alpha}}}{\delta\phi^{\alpha}}, \bar{\mathcal{D}}^{\hat{\alpha}}_{\alpha} = -\mathcal{D}^{\hat{\alpha}}_{\alpha}, \mathcal{D}^{\hat{\alpha}}_{\alpha}\mathfrak{T}^{\mathfrak{a}}_{\alpha} = -\mathcal{Y}^{\hat{\alpha}}_{\alpha}, \mathcal{D}^{\hat{\alpha}}_{\alpha}\mathfrak{T}^{\alpha}_{\hat{\beta}}$$

$$= -\Xi^{\hat{\alpha}}_{\perp\hat{\beta}}\frac{\delta\mathfrak{S}}{\delta\phi^{\alpha}}\mathfrak{D}[\phi]$$

$$\mathcal{S}^{(2)}_{phy} = \begin{vmatrix} \mathcal{S}_{\alpha\beta} & \frac{1}{2}\mathcal{D}^{\hat{\beta}}_{\alpha} & -\frac{1}{2}\mathcal{D}^{\hat{\beta}}_{\alpha} \\ \frac{1}{2}\mathcal{D}^{\hat{\alpha}}_{\beta} & 0 & -\epsilon\sigma^{\widehat{\alpha\beta}} \\ -\frac{1}{2}\mathcal{D}^{\hat{\alpha}}_{\beta} & -\epsilon\sigma^{\widehat{\alpha\beta}} & 0 \end{vmatrix}$$

$$\mathcal{S}_{\alpha\beta} \equiv \frac{\delta^2\mathfrak{S}}{\delta\phi^{\alpha}\delta\phi^{\beta}}$$

$$\mathfrak{P}_{phy} = \begin{vmatrix} \mathcal{P}^{\mathrm{bc}} & \mathcal{P}^{\beta}_{\hat{\gamma}} & \bar{\mathcal{P}}^{\beta}_{\hat{\gamma}} \\ \mathcal{P}^{c}_{\hat{\beta}} & \mathcal{P}^{\widehat{\alpha\beta}} & \mathcal{R}^{\widehat{\alpha\beta}} \\ \bar{\mathcal{P}}^{c}_{\hat{\beta}} & \mathcal{R}^{\widehat{\alpha\beta}} & \bar{\mathcal{P}}^{\widehat{\alpha\beta}} \end{vmatrix}$$

$$\left\|\begin{vmatrix} \mathcal{S}_{\alpha\beta} & \frac{1}{2}\mathcal{D}^{\hat{\beta}}_{\alpha} & -\frac{1}{2}\mathcal{D}^{\hat{\beta}}_{\alpha} \\ \frac{1}{2}\mathcal{D}^{\hat{\alpha}}_{\beta} & 0 & -\epsilon\sigma^{\widehat{\alpha\beta}} \\ -\frac{1}{2}\mathcal{D}^{\hat{\alpha}}_{\beta} & -\epsilon\sigma^{\widehat{\alpha\beta}} & 0 \end{vmatrix}\begin{vmatrix} \mathcal{P}^{\mathrm{bc}} & \mathcal{P}^{\beta}_{\hat{\gamma}} & \bar{\mathcal{P}}^{\beta}_{\hat{\gamma}} \\ \mathcal{P}^{c}_{\hat{\beta}} & \mathcal{P}^{\widehat{\alpha\beta}} & \mathcal{R}^{\widehat{\alpha\beta}} \\ \bar{\mathcal{P}}^{c}_{\hat{\beta}} & \mathcal{R}^{\widehat{\alpha\beta}} & \bar{\mathcal{P}}^{\widehat{\alpha\beta}} \end{vmatrix}\begin{vmatrix} \delta^{\alpha}_{c} & 0 & 0 \\ 0 & \delta^{\hat{\alpha}}_{\hat{\gamma}} & 0 \\ 0 & 0 & \delta^{\hat{\alpha}}_{\hat{\gamma}} \end{vmatrix}\right\|$$

$$\left\langle \mathcal{S}_{\alpha\beta}\middle|\mathfrak{T}^{\beta}_{\alpha}\middle|\mathcal{P}^{\alpha}_{\hat{\alpha}}\right\rangle = -\bar{\mathcal{P}}^{\alpha}_{\hat{\alpha}} = -\mathfrak{T}^{\mathfrak{a}}_{\hat{\alpha}}, \mathcal{P}_{\widehat{\alpha\beta}} = \bar{\mathcal{P}}_{\widehat{\alpha\beta}} = \frac{1}{2\epsilon}\Xi^{\perp}_{\widehat{\alpha\beta}}, \mathcal{R}_{\widehat{\alpha\beta}} = -\frac{1}{\epsilon}\sigma_{\widehat{\alpha\beta}} + \frac{1}{2\epsilon}\Xi^{\perp}_{\widehat{\alpha\beta}}, \mathcal{S}_{\alpha c}\mathcal{P}^{c\beta} = \prod\nolimits^{\beta}_{\perp\alpha}$$

$$\equiv \delta^{\alpha}_{\beta} + \mathfrak{T}^{\mathfrak{a}}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}}_{\beta}\mathcal{P}^{\beta c} = \epsilon\mathcal{P}^{c\hat{\alpha}} = -\epsilon\mathfrak{T}^{c\hat{\alpha}}, \Box_{\alpha\beta} = \mathcal{S}_{\alpha\beta} + \mathcal{D}^{\hat{\alpha}}_{\alpha}\mathcal{D}_{\widehat{\alpha\beta}}, \Box_{\alpha\beta}\mathfrak{T}^{\beta}_{\hat{\alpha}}$$

$$= -\mathfrak{D}_{\alpha\hat{\alpha}}, \wp^{\alpha\beta}\mathcal{D}^{\hat{\alpha}}_{\beta} = -\mathfrak{T}^{\alpha\hat{\alpha}}$$

$$\mathcal{P}^{\alpha\beta}_{\perp} = \wp^{\alpha\beta} - \mathfrak{T}^{\mathfrak{a}}_{\hat{\alpha}}\sigma^{\widehat{\alpha\beta}}\mathfrak{T}^{\beta}_{\hat{\beta}} = \wp^{\alpha\beta} - \mathfrak{T}^{\mathfrak{a}}_{\alpha}\mathfrak{T}^{\alpha}_{\beta}$$

$$\mathfrak{S}^{(2)}_{gh} = \begin{vmatrix} 0 & \frac{1}{2}\mathcal{Y}^{\alpha}_{\hat{\alpha}} & -\frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} \\ \frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} & 0 & -\epsilon\sigma^{\widehat{\alpha\beta}} \\ -\frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} & -\epsilon\sigma^{\widehat{\alpha\beta}} & 0 \end{vmatrix}$$

$$\mathfrak{P}_{gh} = \begin{vmatrix} 0 & \mathcal{Y}^{\alpha}_{\hat{\alpha}} & -\mathcal{Y}^{\alpha}_{\hat{\alpha}} \\ \mathcal{Y}^{\alpha}_{\hat{\alpha}} & \mathcal{P}_{\widehat{\alpha\beta}} & \mathcal{R}_{\widehat{\alpha\beta}} \\ -\mathcal{Y}^{\alpha}_{\hat{\alpha}} & \mathcal{R}_{\widehat{\alpha\beta}} & \mathcal{P}_{\widehat{\alpha\beta}} \end{vmatrix}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}}\frac{\sqrt{\det\gamma_{\alpha\beta}}}{\sqrt{\det\eta_{\alpha\beta}}}\frac{\sqrt{\det\mathfrak{S}^{(2)}_{gh}}}{\sqrt{\det\mathcal{S}^{(2)}_{phy}}}$$

$$\det\begin{vmatrix} \delta^{\alpha}_{\beta} & \frac{1}{2}\mathcal{D}^{\beta\hat{\alpha}} & -\frac{1}{2}\mathcal{D}^{\beta\hat{\alpha}} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\beta}}_{\hat{\alpha}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} = \det\begin{vmatrix} \wp^{\alpha}_{\beta} & 0 & 0 \\ 0 & \delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \\ 0 & 0 & \delta^{\hat{\alpha}}_{\hat{\beta}} \end{vmatrix} \det\begin{vmatrix} \delta^{\alpha}_{\beta} & \frac{1}{2}\mathcal{D}^{\beta\hat{\alpha}} & -\frac{1}{2}\mathcal{D}^{\beta\hat{\alpha}} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\beta}}_{\hat{\alpha}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \det\Box^{\alpha}_{\beta}$$

$$= \det\begin{vmatrix} \Pi^{\alpha}_{\perp\beta} & -\frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} & \frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \det\Box^{\alpha}_{\beta}$$

$$\det\begin{vmatrix} \Pi^{\alpha}_{\perp\beta} & -\frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} & \frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \sim \det\begin{vmatrix} 0 & \frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} & -\frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} \\ \frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & 0 & -\epsilon\delta^{\widehat{\alpha\beta}} \\ -\frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & -\epsilon\delta^{\widehat{\alpha\beta}} & 0 \end{vmatrix}$$

$$\begin{vmatrix} \Pi^{\alpha}_{\perp\beta} & -\frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} & \frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \begin{vmatrix} \sqrt{2}\mathfrak{T}^{\beta}_{\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\beta\hat{\alpha}}\upsilon^{\alpha} \\ \mathcal{Y}_{\beta\hat{\alpha}}\upsilon^{\alpha} \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2}\mathfrak{T}^{\alpha}_{\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \\ \mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \end{vmatrix}$$

$$\begin{vmatrix} \Pi^{\alpha}_{\perp\beta} & -\frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} & \frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \begin{vmatrix} \sqrt{2}\mathfrak{T}^{\beta}_{\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\beta\hat{\alpha}}\upsilon^{\alpha} \\ \mathcal{Y}_{\beta\hat{\alpha}}\upsilon^{\alpha} \end{vmatrix} = -\frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2}\mathfrak{T}^{\alpha}_{\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \\ \mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \end{vmatrix}$$

$$\begin{vmatrix} \Pi^{\alpha}_{\perp\beta} & -\frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} & \frac{1}{2}\mathfrak{T}^{\hat{\beta}\alpha} \\ \frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & 0 & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} \\ -\frac{1}{2}\mathcal{D}_{\hat{\alpha}\beta} & -\epsilon\delta^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{vmatrix} \begin{vmatrix} \upsilon^{\beta}_{\perp} \\ 0 \\ 0 \end{vmatrix} \begin{vmatrix} \upsilon^{\alpha}_{\perp} \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & \frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} & -\frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} \\ \frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & 0 & 0 \\ -\frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & 0 & 0 \end{vmatrix} \begin{vmatrix} \sqrt{2}\upsilon^{\beta} \\ \mathcal{Y}_{\hat{\beta}\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\beta}\alpha}\upsilon^{\alpha} \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2}\upsilon^{\alpha} \\ \mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \end{vmatrix}$$

$$\begin{vmatrix} 0 & \frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} & -\frac{1}{2}\mathcal{Y}^{\hat{\alpha}}_{\alpha} \\ \frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & 0 & 0 \\ -\frac{1}{2}\mathcal{Y}_{\alpha\hat{\alpha}} & 0 & 0 \end{vmatrix} \begin{vmatrix} \sqrt{2}\upsilon^{\beta} \\ \mathcal{Y}_{\hat{\beta}\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\beta}\alpha}\upsilon^{\alpha} \end{vmatrix} = -\frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2}\upsilon^{\alpha} \\ \mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \\ -\mathcal{Y}_{\hat{\alpha}\alpha}\upsilon^{\alpha} \end{vmatrix}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}} \frac{1}{\sqrt{\det \Box^{\alpha}_{\beta}}} = e^{-\mathfrak{S}} \sqrt{\det \wp^{\alpha}_{\beta}}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}} = \sqrt{\frac{\det \mathfrak{T}^{\alpha}_{\beta} \mathfrak{T}^{\mathfrak{b}}_{\beta}}{\det_{\perp} \mathfrak{S}^{\alpha}_{\beta}}}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}} = \frac{\det \mathfrak{T}^{\mathfrak{b}}_{\beta} \mathfrak{T}^{\beta}_{\gamma}}{\sqrt{\det \mathfrak{S}^{\alpha}_{\beta} + \mathfrak{T}^{\alpha}_{\alpha} \mathfrak{T}^{\alpha}_{\beta}}}$$

1.12. Cálculos gravitatorios a escala cuántica.

$$\frac{\delta \mathfrak{S}}{\delta \phi^{\alpha}} \mathcal{D}^{\hat{\mu}}(\chi) = \chi^{\hat{\mu}}$$

$$\begin{vmatrix} \delta_{\alpha\beta} & \mathcal{D}^{\hat{\beta}}_{\alpha} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} \begin{vmatrix} \mathcal{P}^{\beta c} & \mathcal{P}^{\beta}_{\hat{\gamma}} \\ \mathcal{P}^{c}_{\hat{\beta}} & \mathcal{P}_{\widehat{\beta\gamma}} \end{vmatrix} = \begin{vmatrix} \delta_{\alpha\beta}\mathcal{P}^{\beta c} + \mathcal{D}^{\hat{\beta}}_{\alpha}\mathcal{P}^{c}_{\hat{\beta}} & \delta_{\alpha\beta}\mathcal{P}^{\beta}_{\hat{\gamma}} + \mathcal{D}^{\hat{\beta}}_{\alpha}\mathcal{P}_{\widehat{\beta\gamma}} \\ \mathcal{D}^{\hat{\alpha}}_{\beta}\mathcal{P}^{\beta c} & \mathcal{D}^{\hat{\alpha}}_{\beta}\mathcal{P}^{\beta}_{\hat{\gamma}} \end{vmatrix} = \begin{vmatrix} \delta^{\alpha}_{c} & 0 \\ 0 & \delta^{\hat{\alpha}}_{\hat{\gamma}} \end{vmatrix}$$

$$\mathcal{P}^{\alpha}_{\hat{\alpha}} = \mathfrak{T}^{\alpha}_{\hat{\alpha}}, \mathcal{P}^{\alpha}_{\hat{\beta}}\mathcal{D}^{\hat{\beta}}_{\beta} = \prod\nolimits^{\alpha}_{\parallel\beta} \equiv -\mathfrak{T}^{\alpha}_{\hat{\alpha}}\mathcal{D}^{\hat{\alpha}}_{\beta}, \delta_{\alpha\beta}\mathcal{P}^{\beta c} = \prod\nolimits^{\alpha}_{\perp c} \equiv \delta^{\alpha}_{\beta} - \prod\nolimits^{\alpha}_{\parallel\beta}$$

$$\begin{aligned} \Box_{\alpha\beta} &\equiv \delta_{\alpha\beta} + \mathcal{D}^{\hat{\alpha}}_{\alpha}\eta_{\widehat{\alpha\beta}}\mathcal{D}^{\hat{\beta}}_{\beta}, \Box_{\alpha\beta}\mathcal{P}^{\beta c} = \delta^{c}_{\alpha}, \wp^{\alpha\beta} = \mathcal{P}^{\alpha\beta} + \mathfrak{T}^{\alpha}_{\alpha}\eta^{\alpha\beta}\mathfrak{T}^{\beta}_{\beta}, \mathcal{P}^{\alpha\beta} \\ &= \wp^{\alpha\beta} - \mathfrak{T}^{\alpha}_{\alpha}\eta^{\alpha\beta}\mathfrak{T}^{\beta}_{\beta}, \Box_{\alpha c}\mathcal{P}^{\beta c} = \prod\nolimits^{\beta}_{\perp\alpha}, \mathcal{P}^{\alpha\beta} = \prod\nolimits^{\alpha}_{\perp c}\wp^{c\beta}, \mathcal{P}^{\alpha\beta} \\ &= \prod\nolimits^{\alpha}_{\perp c}\wp^{cd}\prod\nolimits^{\beta}_{\perp d}, \wp^{\alpha\beta}\mathcal{D}^{\alpha}_{\beta} = -\mathfrak{T}^{\alpha}_{\beta}\eta^{\beta\alpha}, \Box_{\alpha\beta}\mathfrak{T}^{\beta}_{\hat{\alpha}} = \mathcal{D}^{\hat{\beta}}_{\alpha}\eta_{\widehat{\beta\alpha}} \end{aligned}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}} \frac{\sqrt{\det \gamma_{\alpha\beta}}}{\sqrt{\det \begin{vmatrix} \delta_{\alpha\beta} & \mathcal{D}^{\hat{\beta}}_{\alpha} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} \sqrt{\det \eta_{\alpha\beta}}}}$$

$$\det \begin{vmatrix} \delta_{\alpha\beta} & \mathcal{D}^{\hat{\beta}}_{\alpha} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} \det \gamma^{\alpha\beta} \det \eta_{\alpha\beta} = \det \begin{vmatrix} \delta^{\alpha}_{\beta} & \mathcal{D}^{\hat{\alpha}}_{\beta} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix}$$

$$\det \begin{vmatrix} \delta^{\alpha}_{\beta} & \mathcal{D}^{\hat{\alpha}}_{\beta} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} = \det \begin{vmatrix} \delta^{\alpha}_{\beta} & \mathcal{D}^{\hat{\alpha}}_{\beta} \\ \mathcal{D}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} = \det \begin{vmatrix} \wp^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\alpha}_{\beta} \end{vmatrix} \det \Box^{\alpha}_{\beta} = \det \begin{vmatrix} \prod^{\alpha}_{\perp\beta} & \mathcal{D}^{\alpha}_{\beta} \\ -\mathfrak{T}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} \det \Box^{\alpha}_{\beta}$$

$$\det \begin{vmatrix} \prod^{\alpha}_{\perp\beta} & \mathcal{D}^{\alpha}_{\beta} \\ -\mathfrak{T}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix} = 1$$

$$\begin{vmatrix} \prod^{\alpha}_{\perp\beta} & \mathcal{D}^{\alpha}_{\beta} \\ -\mathfrak{T}^{\hat{\alpha}}_{\beta} & 0 \end{vmatrix}^{2} = \begin{vmatrix} \delta^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\hat{\alpha}}_{\beta} \end{vmatrix}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}} \frac{1}{\sqrt{\det \Box^{\alpha}_{\beta}}} = e^{-\mathfrak{S}} \sqrt{\det \wp^{\alpha}_{\beta}}$$

$$e^{-\Gamma} = e^{-\mathfrak{S}}\sqrt{\frac{\det \mathfrak{T}^{\ell}_{\beta}\,\mathfrak{T}^{\beta}_{\gamma}}{\det_{\perp} \mathfrak{S}^{\alpha}_{\beta}}}$$

1.13. Gauge independiente.

$$\delta_\theta \phi^{\mathcal{A}} = \delta^{\mathrm{A}}\theta, s = \delta^{\mathrm{A}}\frac{\delta}{\delta\phi^{\mathcal{A}}} = \mathfrak{T}^{\alpha}[\phi]_{\alpha}\mathfrak{C}^{\alpha}\frac{\delta}{\delta\phi^{\alpha}} + \frac{1}{2}f^{\alpha}_{\beta\gamma}\mathfrak{C}^{\beta}\mathfrak{C}^{\gamma}\frac{\delta}{\delta\mathfrak{C}^{\alpha}} - \mathcal{B}_{\widehat{\alpha}}\frac{\delta}{\delta \mathrm{A}_{\widehat{\alpha}}} - \overline{\mathcal{B}}_{\widehat{\alpha}}\frac{\delta}{\delta\overline{\mathrm{A}}_{\widehat{\alpha}}}$$

$$\tilde{\rho} \equiv \tilde{\mu}e^{-\mathfrak{S}}\frac{\delta}{\delta\phi^{\mathcal{A}}}\left(\delta^{\mathrm{A}}\tilde{\rho}\right)$$

$$\frac{\partial}{\partial\zeta}\mathcal{Z} = \iiint d\phi\tilde{\rho}\delta^{\mathrm{A}}\frac{\delta}{\delta\phi^{\mathcal{A}}}\Upsilon = \iiint d\phi\frac{\delta}{\delta\phi^{\mathcal{A}}}\left(\tilde{\rho}\delta^{\mathrm{A}}\right)\Upsilon$$

1.14. Ecuación fundamental.

$$\delta[\phi] \mapsto \delta[\phi] - \delta^{\mathrm{A}}[\phi]\phi^{\star}_{\mathrm{A}}, \tilde{\rho}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}} = \delta^{\mathrm{A}}\tilde{\rho}, \frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}\tilde{\rho}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\mathcal{J}_{\mathcal{A}}\frac{\delta\mathcal{W}}{\delta\phi^{\star}_{\mathrm{A}}}\Gamma\frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\Gamma[\mathscr{g}_{\star}\phi, \mathscr{g}_{\star}\phi^{\star}]$$

$$\delta^{\mathrm{A}}\phi^{\star}_{\mathrm{A}} = \iiint_{\chi}\left(\nabla_{\mu}\mathfrak{C}^{i}a^{\mu}_{\star i} + \frac{1}{2}f^{k}_{ij}\mathfrak{C}^{i}\mathfrak{C}^{j}\mathfrak{C}^{\star}_{\kappa}\right) + \iiint_{\chi} tr\left(\mathrm{BA}^{\mathfrak{T}}_{\star} + \overline{\mathrm{BA}^{\mathfrak{T}}_{\star}}\right)$$

$$\nabla_{\mu}\mathfrak{C}^{i}a^{\mu}_{\star i} = \frac{1}{2}tr\left(\partial_{\mu}\mathfrak{C} - i\left[\alpha_{\mu}, \mathfrak{C}\right]\right)a^{\mu}_{\star}, \mathscr{g}_{\star}a^{\mu}_{\star} = \mathfrak{G}[\mathscr{g}]\alpha_{\star}\mathfrak{G}^{-1}[\mathscr{g}], \frac{1}{2}f^{k}_{ij}\mathfrak{C}^{i}\mathfrak{C}^{j}\mathfrak{C}^{\star}_{\kappa} = -i2tr\mathfrak{C}^{2}\mathfrak{C}^{\star}_{\kappa}, \mathscr{g}_{\star}\mathfrak{C}_{\star}$$
$$= \mathfrak{G}[\mathscr{g}]\mathfrak{C}_{\star}\mathfrak{G}^{-1}[\mathscr{g}], \mathscr{g}_{\star}\mathrm{A}_{\star} = \mathrm{A}_{\star}\overline{\mathfrak{G}}^{-1}[\mathscr{g}]$$

1.15. Cálculos referenciales relativos a BRST.

$$r = \Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\mathcal{A}}}\frac{\vec{\delta}}{\delta\phi^{\star}_{\mathrm{A}}} - \Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}, \frac{\partial}{\partial\zeta}\Gamma = \psi^{\mathrm{A}}\frac{\delta\Gamma}{\delta\phi^{\mathcal{A}}}, r\Upsilon = \psi^{\mathrm{A}}\frac{\delta\Gamma}{\delta\phi^{\mathcal{A}}}, r\Upsilon$$
$$= \frac{\overleftarrow{\delta}}{\delta\phi^{\mathcal{A}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\Upsilon - \Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\mathcal{A}}}\Upsilon - \Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{A}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\mathcal{A}}}\Upsilon = \frac{\delta\Gamma}{\delta\phi^{\mathcal{A}}}\Theta^{\mathrm{A}}$$

$$\frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}\Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\mathrm{B}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{B}}}\Gamma + \frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}\Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{B}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\mathrm{B}}}\Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathfrak{C}}}\Gamma + \mathcal{P}^{\mathcal{C}\mathcal{B}}\frac{\vec{\delta}}{\delta\phi^{\mathcal{A}}}\frac{\overleftarrow{\delta}}{\delta\phi^{\star}_{\mathrm{B}}}\frac{\vec{\delta}}{\delta\phi^{\mathrm{B}}}\Gamma$$

$$\frac{\partial}{\partial\zeta}\Gamma = \mathrm{r}\Upsilon\Lambda\frac{\partial}{\partial\Lambda}\Gamma = \mathrm{r}\Upsilon$$

1.16. Cálculos suplementarios relativos a la teoría de Yang – Mills.

$$[\mathcal{M}, \mathcal{N}]_{\eta} = \left[[\mathcal{M}, \mathcal{N}]_{\eta-1}, \mathcal{N}\right], e^{\mathcal{N}}\mathcal{M}e^{-\mathcal{N}} = \sum_{\eta=0}^{\infty}(-1)^{\eta}\frac{1}{\eta!}[\mathcal{M}, \mathcal{N}]_{\eta}, \left[t_{i}, i\mathrm{g}^{j}t_{j}\right]_{\eta}$$
$$= (-1)^{\eta}\left(\mathcal{F}^{\eta}(\mathrm{g})\right)^{i\hbar}t_{\hbar}, e^{i\mathrm{g}^{j}t_{j}}t_{i}e^{-i\mathrm{g}^{j}t_{j}} = \sum_{\eta=0}^{\infty}\frac{1}{\eta!}\left(\mathcal{F}^{\eta}(\mathrm{g})\right)^{i\hbar}t_{\hbar} = \left(e^{\mathcal{F}}(\mathrm{g})\right)^{i\hbar}t_{\hbar}$$
$$= t_{\hbar}\left(e^{-\mathcal{F}(\mathrm{g})}\right)^{\hbar i}, e^{-i\mathrm{g}^{j}t_{j}}t_{i} = t_{\hbar}e^{-i\mathrm{g}^{j}t_{j}}\left(e^{-\mathcal{F}(\mathrm{g})}\right)^{ki}$$

$$i\frac{\partial}{\partial\mathfrak{h}^i}(\mathcal{g}_\star\mathfrak{h}_\star\hat{\mathfrak{g}})^j t_j\Big|_{\mathfrak{h}=1} = \frac{\partial}{\partial\mathfrak{h}^i}e^{i\mathcal{g}_\star\mathfrak{h}_\star\hat{\mathfrak{g}}^j t_j}\Big|_{\mathfrak{h}=1} i\frac{\partial}{\partial\mathfrak{h}^i}(\mathcal{g}_\star\mathfrak{h}_\star\hat{\mathfrak{g}})^j t_j\Big|_{\mathfrak{h}=1} = \frac{\partial}{\partial\mathfrak{h}^i}e^{i\mathfrak{g}^j t_j}e^{i\mathfrak{h}^j t_j}e^{-i\mathfrak{g}^j t_j}\Big|_{\mathfrak{h}=1}$$

$$= e^{i\mathfrak{g}^j t_j} i t_i e^{-i\mathfrak{g}^j t_j}, \frac{\partial}{\partial\mathfrak{h}^i}(\mathcal{g}_\star\mathfrak{h}_\star\hat{\mathfrak{g}})^{k}\Big|_{\mathfrak{h}=1} = \left(e^{\mathcal{F}}\right)^{ik} = \left(e^{-\mathcal{F}}\right)^{ki}$$

$$\mathcal{g}_\star a_\mu^i t_i = e^{i\mathfrak{g}^j t_j} a_\mu^i t_i e^{-i\mathfrak{g}^j t_j} + e^{i\mathfrak{g}^j t_j}\partial_\mu \mathfrak{g}^i t_i e^{-i\mathfrak{g}^j t_j} = a_\mu^i\left(e^{\mathcal{F}}\right)^{ik} t_k + \partial_\mu\mathfrak{g}^i\left(e^{\mathcal{F}}\right)^{ik} t_k$$

$$\mathcal{g}_\star a_\mu^{k} = a_\mu^i\left(e^{\mathcal{F}}\right)^{ik} + \partial_\mu\mathfrak{g}^i\left(e^{\mathcal{F}}\right)^{ik}, \mathcal{g}_\star a_\mu^{k} = \left(e^{-\mathcal{F}}\right)^{ki} a_\mu^i + \left(e^{-\mathcal{F}}\right)^{ki}\partial_\mu\mathfrak{g}^i$$

$$\mathfrak{T}_\alpha^\alpha[\phi] = \mathfrak{T}_{\mu j}^i\epsilon^j = \partial_\mu\epsilon^i + f^{ijk}\mathrm{A}_\mu^j\epsilon^{k}$$

1.17. Transformaciones de Yang – Mills.

$$\det\delta_\beta^\alpha - \frac{\delta\hat{\mathfrak{E}}^\alpha}{\delta\phi^\beta}\mathcal{U}_\alpha{}^\beta[\mathfrak{E}]\mathfrak{T}_\beta^\alpha[\phi] = \det\delta_\beta^\alpha - \mathfrak{T}_\beta^\alpha[\phi]\frac{\delta\hat{\mathfrak{E}}^\gamma}{\delta\phi^\alpha}\mathcal{U}_\gamma{}^\alpha[\mathfrak{E}], \mathfrak{T}_\beta^\alpha[\phi]\frac{\delta\hat{\mathfrak{E}}^\gamma}{\delta\phi^\alpha}$$

$$= \frac{\delta}{\delta\mathcal{g}^\beta}\hat{\mathfrak{E}}^\gamma[\mathcal{g}_\star\phi]\Big|_{\mathcal{g}=1}\mathcal{U}_\gamma{}^\alpha[\mathfrak{E}] = \frac{\delta\mathfrak{h}_\star\mathfrak{E}^\alpha}{\delta\mathfrak{h}^\gamma}\Big|_{\mathfrak{h}=\hat{\mathfrak{E}}}, \mathfrak{T}_\beta^\alpha[\phi]\frac{\delta\hat{\mathfrak{E}}^\gamma}{\delta\phi^\alpha}\mathcal{U}_\gamma{}^\alpha[\mathfrak{E}]$$

$$= \frac{\delta}{\delta\mathcal{g}^\beta}\hat{\mathfrak{E}}^\gamma[\mathcal{g}_\star\phi]\Big|_{\mathcal{g}=1}\frac{\delta\mathfrak{h}_\star\mathfrak{E}^\alpha}{\delta\mathfrak{h}^\gamma}\Big|_{\mathfrak{h}=\hat{\mathfrak{E}}}$$

$$\mathfrak{T}_\beta^\alpha[\phi]\frac{\delta\hat{\mathfrak{E}}^\gamma}{\delta\phi^\alpha}\mathcal{U}_\gamma{}^\alpha[\mathfrak{E}] = \frac{\delta\hat{\mathfrak{E}}[\mathcal{g}_\star\phi]_\star\mathfrak{E}^\alpha[\phi]}{\delta\mathcal{g}^\beta}\Big|_{\mathcal{g}=1}, \mathfrak{T}_\beta^\alpha[\phi]\frac{\delta\hat{\mathfrak{E}}}{\delta\phi^\alpha}\mathcal{U}_\gamma{}^\alpha[\mathfrak{E}]$$

$$= \frac{\delta\mathcal{g}_\star\hat{\mathfrak{E}}^\gamma[\phi]_\star\mathfrak{E}^\alpha[\phi]}{\delta\mathcal{g}^\beta}\Big|_{\mathcal{g}=1}, \frac{\delta\mathcal{g}_\star\hat{\mathfrak{E}}[\phi]_\star\mathfrak{E}^\alpha[\phi]}{\delta\mathcal{g}^\beta}\Big|_{\mathcal{g}=1} = \delta_\beta^\alpha + \frac{\delta\hat{\mathfrak{E}}[\phi]_\star\hat{\mathcal{g}}_\star\mathfrak{E}^\alpha[\phi]}{\delta\mathcal{g}^\beta}\Big|_{\mathcal{g}=1}$$

$$= \delta_\beta^\alpha - \frac{\delta\hat{\mathfrak{E}}[\phi]_\star\mathcal{g}_\star\mathfrak{E}^\alpha[\phi]}{\delta\mathcal{g}^\beta}\Big|_{\mathcal{g}=1}$$

2. Supersimetría de Yang – Mills.

2.1. Correlaciones de Wilson.

$$(\alpha;q)_0 \coloneqq 1, (\alpha;q)_\eta \coloneqq \prod_{\kappa=0}^{\eta-1}(1-\alpha q^\kappa), (q)_\eta \coloneqq \prod_{\kappa=1}^{\eta}(1-q^\kappa), (\alpha;q)_\infty \coloneqq \prod_{\kappa=0}^{\infty}(1-\alpha q^\kappa), (q)_\infty$$

$$\coloneqq \prod_{\kappa=1}^{\infty}(1-q^\kappa)$$

$$(\alpha\chi^\pm;q)_\eta \coloneqq (\alpha\chi;q)_\eta(\alpha\chi^{-1};q)_\eta$$

$$\langle\mathcal{W}_{\mathcal{R}_1}\cdots\mathcal{W}_{\mathcal{R}_\kappa}\rangle^{\mathcal{G}}(\tau;q)$$

$$= \frac{1}{|\mathrm{Weyl}(\mathfrak{G})|}\frac{(q)_\infty^{2rank(\mathfrak{G})}}{\left(q^{\frac{1}{2}}\tau^{\pm2};q\right)_\infty^{rank(\mathfrak{G})}}\oiiint\prod_{\alpha\in root(\mathfrak{G})}ds\frac{(\delta^\alpha;q)_\infty(q\delta^\alpha;q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta^\alpha;q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta^\alpha;q\right)_\infty}\prod_{i=1}^{\kappa}\chi_{\mathcal{R}_i}^{\mathfrak{g}}$$

$$\mathfrak{T}^{\mathfrak{G}}(\tau;q) \coloneqq Tr(-1)^{\mathcal{F}}q^{\mathfrak{J}+\frac{\mathfrak{H}+\mathfrak{C}}{4}}\tau^{\mathfrak{H}-\mathfrak{C}}$$

$$\langle\mathcal{W}_{\mathcal{R}_1}\cdots\mathcal{W}_{\mathcal{R}_\kappa}\rangle^{\mathcal{G}}(\tau;q) = \frac{\langle\mathcal{W}_{\mathcal{R}_1}\cdots\mathcal{W}_{\mathcal{R}_\kappa}\rangle^{\mathcal{G}}(\tau;q)}{\mathfrak{T}^{\mathfrak{G}}(\tau;q)}$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{G}}_{\frac{1}{2}\mathcal{BPS}}(q) = \frac{1}{|\mathrm{Weyl}(\mathfrak{G})|} \frac{1}{(1-q^2)^{rank(\mathfrak{G})}} \oiiint \prod_{\alpha \in root(\mathfrak{G})} ds \frac{(1-\delta^\alpha)}{(1-q^2\delta^\alpha)} \prod_{i=1}^{\kappa} \chi^{\mathfrak{g}}_{\mathcal{R}_i}$$

$$\frac{1}{\mathcal{N}!} \oiiint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i s_i} \frac{\prod_{i\neq j} 1 - \frac{\delta_i}{\delta_j}}{\prod_{i=j} 1 - \tau \frac{\delta_i}{\delta_j}} \mathcal{P}_\mu(\delta,\tau)\mathcal{P}_\lambda(\delta^{-1},\tau) = \frac{\delta_{\mu\lambda}}{(\tau;\tau)_{\mathcal{N}-\ell(\mu)} \prod_{j\geq 1}(\tau;\tau)_{m_j(\mu)}}$$

$$\frac{1}{|\mathrm{Weyl}(\mathfrak{G})|} \oiiint \prod_{\alpha \in root(\mathfrak{G})} ds(1-\delta^\alpha) \prod_{i=1}^{\kappa} \chi^{\mathfrak{g}}_{\mathcal{R}_i}$$

2.2. Correaciones de 't Hooft.

$$\langle \mathfrak{T}_{\mathfrak{B}} \mathfrak{T}_{\mathfrak{B}} \rangle^{\mathcal{G}}(\tau;q) = \sum_{v \in \mathrm{Rep}(\mathfrak{B})} \frac{1}{|\mathrm{Weyl}(\mathfrak{G})|} \frac{(q)_\infty^{2rank(\mathfrak{G})}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{rank(\mathfrak{G})}} \oiiint \prod_{\alpha \in root(\mathfrak{G})} ds \bigotimes \frac{\left(\delta^{\frac{|\alpha(\mathfrak{B})|}{2}}\delta^\alpha;q\right)_\infty \left(q^{1+\frac{|\alpha(\mathfrak{B})|}{2}}\delta^\alpha;q\right)_\infty}{\left(q^{\frac{1+|\alpha(\mathfrak{B})|}{2}}\tau^2\delta^\alpha;q\right)_\infty \left(q^{\frac{1+|\alpha(\mathfrak{B})|}{2}}\tau^{-2}\delta^\alpha;q\right)_\infty} \mathcal{Z}^{(\mathfrak{B},v)}_{bubb}(t,\delta;q)$$

2.3. Teoría SYM en dimensión $\mathbb{R}^4$.

$$(z_e, z_m) \in \mathbb{Z}_2 \otimes \mathbb{Z}_2$$

$$\chi^{\delta o(2\mathcal{N}+1)}_{\delta\wp} = \prod_{i=1}^{\mathcal{N}} \left(\delta_i^{\frac{1}{2}} + \delta_i^{-\frac{1}{2}}\right)$$

$$\chi^{\delta o(2\mathcal{N}+1)}_{\square} = 1 + \prod_{i=1}^{\mathcal{N}} (\delta_i + \delta_i^{-1})$$

$$\chi^{\delta o(2\mathcal{N}+1)}_{\lambda} = \frac{\det\left(\delta_j^{\lambda_i+\mathcal{N}-i+1/2} - \delta_j^{-(\lambda_i+\mathcal{N}-i+1/2)}\right)}{\det\left(\delta_j^{\mathcal{N}-i+1/2} - \delta_j^{-(\mathcal{N}-i+1/2)}\right)}$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{SO}(2\mathcal{N}+1)} = \iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)} \exp\left[\sum_{\eta=1}^{\infty} \frac{1}{\eta} f_\eta(q,\tau) \frac{\bar{\mathcal{P}}_\eta(\delta)^2 - \bar{\mathcal{P}}_{2\eta}(\delta)}{2}\right] \prod_{i=1}^{\kappa} \chi^{\mathcal{SO}(2\mathcal{N}+1)}_{\mathcal{R}_i}(\delta)$$

$$d\mu^{\mathcal{SO}(2\mathcal{N}+1)} = \frac{1}{2^{\mathcal{N}}\mathcal{N}!} \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i\delta_i} (1-\delta_i)(1-\delta_i^{-1}) \bigotimes \prod_{1\leq i\leq j\leq \mathcal{N}} (1-\delta_i\delta_j)(1-\delta_i^{-1}\delta_j^{-1})(1 - \delta_i\delta_j^{-1})(1-\delta_j\delta_i^{-1})$$

$$f_\eta(q,\tau) = \frac{q^{\eta/2}(\tau^{2\eta}+\tau^{-2\eta}) - 2q^\eta}{1-q^\eta}$$

$$\mathcal{P}_m(\delta) := \sum_{i=1}^{\mathcal{N}} (\delta_i^m + \delta_i^{-m})$$

$$\bar{\mathcal{P}}_m(\delta) := 1 + \mathcal{P}_m(\delta) = 1 + \sum_{i=1}^{\mathcal{N}} (\delta_i^m + \delta_i^{-m})$$

$$\overline{\mathcal{M}}_\eta(\delta) = \frac{\overline{\mathcal{P}}_\eta(\delta)^2 - \bar{\mathcal{P}}_{2\eta}(\delta)}{2} = \mathcal{P}_\eta(\delta) + \frac{\mathcal{P}_\eta(\delta)^2 - \mathcal{P}_{2\eta}(\delta)}{2} \exp\left(\sum_{\eta=1}^{\infty} \frac{1}{\eta} f_\eta(q,\tau)\overline{\mathcal{M}}_\eta(\delta)\right)$$
$$= \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau)\overline{\mathcal{M}}_\lambda(\delta)$$

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i} m_i!\,, f_\lambda(q,\tau) = \prod_{i=1}^{\ell(\lambda)} f_{\lambda_i}(q,\tau)\,, \overline{\mathcal{M}}_\lambda(\delta) = \prod_{i=1}^{\ell(\lambda)} \overline{\mathcal{M}}_{\lambda_i}(\delta)$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa}\rangle^{\mathcal{SO}(2\mathcal{N}+1)} = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau) \iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)}\, \overline{\mathcal{M}}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{\mathcal{SO}(2\mathcal{N}+1)}(\delta)$$

$$\iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)}\, \overline{\mathcal{M}}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{\mathcal{SO}(2\mathcal{N}+1)}(\delta)$$

$$\iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)}\, \overline{\mathcal{P}}_\mu(\delta) = \sum_{\nu\in\mathfrak{R}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu) + \sum_{\nu\in\mathcal{W}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu)$$

$$\mathfrak{R}_\eta(\wp) = \{\lambda \vdash \wp \,|\, \ell(\lambda) \le \eta \forall \lambda_i\}$$

$$\mathcal{W}_\eta(\wp) = \{\lambda \vdash \wp \,|\, \ell(\lambda) = \eta \forall \lambda_i\}$$

$$\delta_\lambda = \sum_{\mu\vdash\lambda} \frac{\chi_\lambda^\delta(\mu)}{z_\mu} \wp_\mu$$

$$\overline{\mathcal{M}}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{\mathcal{SO}(2\mathcal{N}+1)}(\delta) = \sum_\mu \alpha_{\lambda,\mathcal{R}}^\mu\, \overline{\mathcal{P}}_\mu(\delta)$$

$$\iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)}\, \overline{\mathcal{M}}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{\mathcal{SO}(2\mathcal{N}+1)}(\delta) = \sum_\mu \alpha_{\lambda,\mathcal{R}}^\mu \iiint d\mu^{\mathcal{SO}(2\mathcal{N}+1)}\, \overline{\mathcal{P}}_\mu(\delta)$$
$$= \sum_\mu \alpha_{\lambda,\mathcal{R}}^\mu \left(\sum_{\nu\in\mathfrak{R}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu) + \sum_{\nu\in\mathcal{W}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu) \right)$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa}\rangle^{\mathcal{SO}(2\mathcal{N}+1)} = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau) \sum_\mu \alpha_{\lambda,\mathcal{R}}^\mu \left(\sum_{\nu\in\mathfrak{R}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu) + \sum_{\nu\in\mathcal{W}_{2\mathcal{N}+1}(|\mu|)} \chi_\nu^\delta(\mu) \right)$$

$$\left(\chi_{\delta\wp}^{\mathcal{SO}(2\mathcal{N}+1)}\right)^2 = \sum_{i=1}^{\mathcal{N}} (1+\delta_i)(1+\delta_i^{-1})$$

$$d\mu^{\mathcal{SO}(2\mathcal{N}+1)} \left(\chi_{\delta\wp}^{\mathcal{SO}(2\mathcal{N}+1)}\right)^2 = d\mu^{\mathcal{U}\delta\wp(2\mathcal{N})}$$

$$d\mu^{\mathcal{U}\delta\wp(2\mathcal{N})} = \frac{1}{2^{\mathcal{N}}\mathcal{N}!} \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i \delta_i} (1-\delta_i^2)(1-\delta_i^{-2}) \bigotimes \prod_{1\le i\le j\le\mathcal{N}} (1-\delta_i\delta_j)(1-\delta_i^{-1}\delta_j^{-1})(1$$
$$- \delta_i\delta_j^{-1})(1-\delta_j\delta_i^{-1})$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\mathcal{N}+1)} = \iiint d\mu^{\mathcal{U}\delta\wp(2\mathcal{N})} \exp\left(\sum_{\eta=1}^{\infty}\frac{1}{\eta}f_\eta(q,\tau)\overline{\mathcal{M}}_\eta(\delta)\right)$$

$$= \iiint d\mu^{\mathcal{U}\delta\wp(2\mathcal{N})} \exp\left[\sum_{\eta=1}^{\infty}\frac{1}{\eta}f_\eta(q,\tau)\left(\mathcal{P}_\eta(\delta)+\frac{\mathcal{P}_\eta(\delta)^2-\mathcal{P}_{2\eta}(\delta)}{2}\right)\right]$$

$$\mathfrak{T}^{\mathcal{SO}(3)}(\tau,q) = \mathfrak{T}^{\mathcal{U}\delta\wp(2)}(\tau;q) = -\frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}{(q;q)_\infty^2}\sum_{\substack{\wp_1,\wp_2\in\mathbb{Z}\\ \wp_1\le\wp_2}}\frac{\left(q^{\frac{1}{2}}\tau^{-2}\right)^{\wp_1+\wp_2-2}}{\left(1-q^{\wp_1-\frac{1}{2}}\tau^2\right)\left(1-q^{\wp_2-\frac{1}{2}}\tau^2\right)}$$

2.4. Spin de Wilson.

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}(\tau;q) = \frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}\oiiint\frac{ds}{2\pi i\delta}\frac{\left(\delta^{\pm};q\right)_\infty\left(q\delta^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta^{\pm};q\right)_\infty}\left(\delta^{1/2}+\delta^{-1/2}\right)^2$$

$$\langle \mathfrak{T}_{\left(\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(2)/\mathbb{Z}_2}(\tau;q)$$

$$= \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}\oiiint\frac{ds}{2\pi i\delta}\frac{\left(q^{\frac{1}{2}}\delta^{\pm 2};q\right)_\infty\left(q^{\frac{3}{2}}\delta^{\pm 2};q\right)_\infty}{(q\tau^2\delta^{\pm 2};q)_\infty(q\tau^{-2}\delta^{\pm 2};q)_\infty}\left(\delta^{1/2}+\delta^{-1/2}\right)^2$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}(\tau;q) = \langle \mathfrak{T}_{\left(\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(2)/\mathbb{Z}_2}(\tau;q)$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}(\tau;q) = \langle \mathfrak{T}_{\left(\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(2)/\mathbb{Z}_2}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}(\tau;q)$$

$$= \frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}{(q\tau^{\pm 4};q)_\infty}\sum_{m\in\mathbb{Z}\setminus\{0,\eta\}}\frac{\tau^{2m}-\tau^{-2m}}{\tau^2-\tau^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(2)/\mathbb{Z}_2}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2}{1-q^4} = \frac{1}{1-q^2}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}}_{2\kappa}(\tau;q) = \underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}}_{2\kappa}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}}^{2\kappa}(\tau;q) = \underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}}^{2\kappa}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)\sum_{i=0}^{\kappa}\alpha_\kappa^{\delta o(3)}{}_{\delta\wp}(i)q^{2i} = \frac{1}{1-q^4}\sum_{i=0}^{\kappa}\alpha_\kappa^{\delta o(3)}{}_{\delta\wp}(i)q^{2i}$$

$$\alpha_\kappa^{\delta o(3)}{}_{\delta\wp}(i) = (2i+1)\frac{(2\kappa)!}{(\kappa-i)!\,(\kappa+i+1)!} = \mathfrak{C}_{\kappa+i+1,2i+1}$$

$$\mathfrak{C}_{m,\eta} = \frac{m}{\eta}\binom{2\eta - m - 1}{\eta - 1}$$

$$\mathfrak{C}_{\kappa} = \frac{1}{\kappa+1}\binom{2\kappa}{\kappa} = \prod_{1\le i\le j\le \kappa-1} \frac{i+j+2}{i+j}$$

$$\frac{1-\sqrt{1-4\chi}}{2\chi} = \sum_{\kappa=0}^{\infty} \alpha_{\kappa}^{\delta o(3)}{}_{\delta\wp}(0)\chi^{\kappa}$$

$$\frac{1}{\chi^{i+1}}\left(\frac{1-\sqrt{1-4\chi}}{2\chi}\right)^{2i+1} = \sum_{\kappa=0}^{\infty} \alpha_{\kappa}^{\delta o(3)}{}_{\delta\wp}(i)\chi^{\kappa}$$

$$\sum_{\kappa=0}^{\infty} \chi^{\kappa} \underbrace{\langle \mathcal{W}_{\delta\wp} \cdots \mathcal{W}_{\delta\wp}\rangle^{Spin(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{1}{1-q^4} \bigotimes \frac{2\left(1-\sqrt{1-4\chi}\right)}{4\chi - \left(1-\sqrt{1-4\chi}\right)^2 q^2}$$

$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}(\tau;q)$

$$= \frac{1}{8}\frac{(q)_{\infty}^4}{\left(q^{\frac{1}{2}}\tau^{\pm};q\right)_{\infty}^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i}\frac{(\delta_i^{\pm};q)_{\infty}(q\delta_i^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_{\infty}} \bigotimes \frac{(\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}(q\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{2}\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}}$$

$\langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(4)/\mathbb{Z}_2}(\tau;q)$

$$= \frac{1}{2}\frac{(q)_{\infty}^4}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i}\frac{\left(q^{\frac{1}{2}}\delta_i^{\pm 2};q\right)_{\infty}\left(q^{\frac{3}{2}}\delta_i^{\pm 2};q\right)_{\infty}}{(q\tau^{2}\delta_i^{\pm 2};q)_{\infty}(q\tau^{-2}\delta_i^{\pm 2};q)_{\infty}} \bigotimes \frac{(\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}\left(q^{\frac{1}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}(q\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}\left(q^{\frac{3}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}(q\tau^{2}\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}(q\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}(\tau;q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(4)/\mathbb{Z}_2}(\tau;q)$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{U}\delta\wp(4)/\mathbb{Z}_2}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6}{(1-q^4)(1-q^8)} = \frac{1}{(1-q^2)(1-q^4)}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{4}(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}{(1-q^4)(1-q^8)}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{6}(q) = \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{8}(q) = \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp} \cdots \mathcal{W}_{\delta\wp} \rangle^{Spin(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{\sum_{i=0}^{3\kappa} \alpha_{\kappa}^{\delta o(5)}{}_{\delta\wp}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\alpha_{\kappa}^{\delta o(5)}{}_{\delta\wp}(0) = \mathfrak{C}_{\kappa}\mathfrak{C}_{\kappa+2} - \mathfrak{C}_{\kappa+1}^2 = \frac{48(2\kappa+1)!\,(2\kappa-1)!}{(\kappa-1)!\,\kappa!\,(\kappa+2)!\,(\kappa+3)!} = \prod_{1 \le i \le j \le \kappa-1} \frac{i+j+4}{i+j}$$

$${}_3\mathfrak{F}_2\left(1, \frac{1}{2}, \frac{3}{2}, 6{,}8; 32\chi\right) = \sum_{\kappa=0}^{\infty} \alpha_{\kappa}^{\delta o(5)}{}_{\delta\wp}(0)\chi^{\kappa}$$

$${}_{\wp}\mathfrak{F}_q = (\alpha_1 \cdots \alpha_{\wp}; \beta_1 \cdots \beta_{\wp}; \zeta) = \sum_{\kappa=0}^{\infty} \frac{(\alpha_1)_{\kappa}(\alpha_2)_{\kappa} \cdots (\alpha_{\wp})_{\kappa}}{(\beta_1)_{\kappa}(\beta_2)_{\kappa} \cdots (\beta_q)_{\kappa}} \frac{\zeta^{\kappa}}{\kappa!}$$

$$\alpha_{\kappa}^{\delta o(5)}{}_{\delta\wp}(1) = \frac{120(2\kappa)!\,(2\kappa+2)!}{(\kappa-1)!\,\kappa!\,(\kappa+3)!\,(\kappa+4)!}$$

$$\underbrace{\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \frac{\sum_{i=0}^{2\kappa} \alpha_{\kappa}^{\mathcal{SO}(5)}{}^{\square}_{\square}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4}{(1-q^4)(1-q^8)}, \langle \mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6+q^8}{(1-q^4)(1-q^8)}$$
$$= \frac{1-q^{10}}{(1-q^2)(1-q^4)(1-q^8)}, \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$
$$= \frac{1+q^2+q^4+q^6+q^8+q^{10}}{(1-q^4)(1-q^8)}, \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$
$$= \frac{1}{(1-q^4)(1-q^8)}(1+q^2+q^4+q^6+q^8$$
$$+ q^{10}), \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$
$$= \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\alpha_{\kappa}^{\mathcal{SO}(5)}{}^{\square}_{\square}(i) = \sum_{i=0}^{\left\lfloor \frac{\kappa}{2} \right\rfloor} \mathfrak{C}_i \mathfrak{C}_{i+1} \binom{\kappa}{2i} - \sum_{i=0}^{\left\lfloor \frac{\kappa+1}{2} \right\rfloor} \mathfrak{C}_i^2 \binom{\kappa}{2i-1} = \kappa\,{}_3\mathfrak{F}_2\left(\frac{3}{2}, \frac{1}{2} - \frac{\kappa}{2}\right) + {}_3\mathfrak{F}_2\left(1, \frac{1}{2}, \frac{3}{2} + \frac{\kappa}{2}\right)$$

$$\underbrace{\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \frac{\sum_{i=0}^{3\kappa} \alpha_{\kappa}^{\mathcal{SO}(5)}{}^{\square}_{\square\square}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2+q^6}{(1-q^4)(1-q^8)} = \frac{q^2}{(1-q^4)^2}, \langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}{(1-q^4)(1-q^8)}$$

$$= \frac{(1-q^6)(1-q^8)}{(1-q^2)(1-q^4)^3}, \langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1-q^4)(1-q^8)}(1+q^2+q^4+q^6+q^8$$

$$+ q^{10}), \langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{(2)} \cdots \mathcal{W}_{(2)} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \frac{\sum_{i=0}^{4\kappa} \alpha_{\kappa}^{\mathcal{SO}(5)\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8}{(1-q^4)(1-q^8)} = \frac{q^4}{(1-q^4)^2}, \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+q^{14}+q^{16}}{(1-q^4)(1-q^8)}, \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{(\kappa)} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{4\kappa-4}+q^{4\kappa}}{(1-q^4)(1-q^8)} = \frac{q^{4\kappa-4}}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{(\kappa)} \mathcal{W}_{(\kappa)} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{8\kappa} \alpha_{2}^{\mathcal{SO}(5)\square_{\kappa}}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square\square} \mathcal{W}_{\square\square\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\square\square\square\square} \mathcal{W}_{\square\square\square\square} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{(\infty)} \mathcal{W}_{(\infty)} \rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1-q^{24}}{(1-q^2)(1-q^4)^2(1-q^6)(1-q^8)^2(1-q^{12})}$$

$$\langle \mathcal{W}_{\delta_{\wp}} \mathcal{W}_{\delta_{\wp}} \rangle^{Spin(7)}(\tau; q)$$

$$= \frac{1}{96} \frac{(q)_{\infty}^{6}}{\left(q^{\frac{1}{2}}\tau^{\pm}; q\right)_{\infty}^{3}} \oiiint \prod_{i=1}^{6} \frac{ds_i}{2\pi i \delta_i} \frac{(\delta_i^{\pm}; q)_{\infty} (q\delta_i^{\pm}; q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}; q\right)_{\infty} \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}; q\right)_{\infty}} \bigotimes \prod_{i<j} \frac{(\delta_i^{\pm}\delta_j^{\mp}; q)_{\infty} (\delta_i^{\pm}\delta_j^{\pm}; q)_{\infty} (q\delta_i^{\pm}\delta_j^{\mp}; q)_{\infty} (q\delta_i^{\pm}\delta_j^{\pm}; q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\mp}; q\right)_{\infty} \left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\pm}; q\right)_{\infty} \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp}; q\right)_{\infty} \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm}; q\right)_{\infty}}$$

$$\prod_{i=1}^{6} \left(\delta_i^{\frac{1}{2}} - \delta_i^{-\frac{1}{2}}\right)^2$$

$$\langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \rangle^{\mathcal{U}\delta\wp(6)/\mathbb{Z}_2} (\tau; q)$$

$$= \frac{1}{6} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}\tau^{\pm}; q\right)_\infty^3} \oiiint \prod_{i=1}^{6} \frac{ds_i}{2\pi i \delta_i} \frac{(\delta_i^{\pm}; q)_\infty (q\delta_i^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}}\tau^2 \delta_i^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}; q\right)_\infty} \bigotimes \prod_{i<j} \frac{(\delta_i^{\pm}\delta_j^{\mp}; q)_\infty (\delta_i^{\pm}\delta_j^{\pm}; q)_\infty (q\delta_i^{\pm}\delta_j^{\mp}; q)_\infty (q\delta_i^{\pm}\delta_j^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty}$$

$$\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(7)} (q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{U}\delta\wp(6)/\mathbb{Z}_2} (q) = \frac{1 + q^2 + q^4 + q^6 + q^8 + q^{10} + q^{12}}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$= \frac{1}{(1 - q^2)(1 - q^4)(1 - q^6)}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp} \cdots \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(7)}}_{2\kappa} (q) = \frac{\sum_{i=0}^{6\kappa} \alpha_\kappa^{\mathcal{SO}(7)\,\square}{}_{\delta\wp}(i) q^{2i}}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(7)} (q)$$

$$= \frac{1}{(1 - q^4)(1 - q^8)(1 - q^{12})} \bigotimes (1 + q^2 + q^4 + q^6 + q^8 + q^{10} + \cdots + q^\eta)$$

$$\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(7)} (q)$$

$$= \frac{1}{(1 - q^4)(1 - q^8)(1 - q^{12})} \bigotimes (1 + q^2 + q^4 + q^6 + q^8 + q^{10} + \cdots + q^\eta)$$

$$\alpha_\kappa^{\delta o(7)}{}_{\delta\wp}(0) = \prod_{1 \le i \le j \le \kappa - 1} \frac{i + j + 6}{i + j}$$

$${}_4\mathfrak{F}_3 \left(1, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} 8, 10; 12; 64\chi\right) = \sum_{\kappa=0}^{\infty} \alpha_\kappa^{\delta o(7)}{}_{\delta\wp}(0) \chi^\kappa$$

$$\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(7)} (q) = \frac{\sum_{i=0}^{3\kappa} \alpha_\kappa^{\mathcal{SO}(7)\,\square\!\square}(i) q^{2i}}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$\langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(7)} (q) = \frac{q^6}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(7)} (q) = \frac{1 + q^2 + q^4 + q^6 + q^8 + q^{10} + q^{12}}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$= \frac{1 - q^{14}}{(1 - q^2)(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(7)} (q) = \frac{1 + q^2 + q^4 + q^6 + q^8 + q^{10} + \cdots + q^\eta}{(1 - q^4)(1 - q^8)(1 - q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cdots \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{5\kappa} \alpha_{\kappa}^{SO(7)\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(i) q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \cdots \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{6\kappa} \alpha_{\kappa}^{SO(7)\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}(i) q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2+q^6+q^{10}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8+q^{12}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle^{Spin(7)}_{\frac{1}{2}\mathfrak{BPS}}(q)}_{\kappa} = \frac{\sum_{i=0}^{3\ell\kappa} \alpha_{\kappa}^{SO(7)\underset{(\ell)}{\square}}(i) q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\square\square} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8+q^{12}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{SO(7)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

2.5. Operador fundamental de Wilson.

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle^{SO(3)}(\tau; q) = \frac{1}{2} \frac{(q)_{\infty}^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2}; q\right)_{\infty}} \oiiint \frac{ds}{2\pi i \delta} \frac{(\delta^{\pm}; q)_{\infty} (q\delta^{\pm}; q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta^{\pm}; q\right)_{\infty} \left(q^{\frac{1}{2}}\tau^{-2}\delta^{\pm}; q\right)_{\infty}} (1+\delta+\delta^{-1})^2$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}(\tau;q) = \langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{\mathcal{SU}(2)}(\tau;q)$$

$$= \frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}{(q\tau^{\pm 4};q)_\infty}\left[\frac{3}{2}\sum_{m\in\mathbb{Z}\backslash\{0\}}\left(\frac{\tau^{2m}-\tau^{-2m}}{\tau^2-\tau^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}\right)-\frac{2}{1-q}-\frac{q^{\frac{1}{2}}(\tau^2+\tau^{-2})}{1-q^2}\right]$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{\mathcal{SU}(2)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+q^2+q^4}{1-q^4} = \frac{1-q^6}{(1-q^2)(1-q^4)}$$

$$\underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}}_{\kappa}(\tau;q) = \sum_{i=0}^{\kappa}\binom{\kappa}{i}(-1)^i\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}}_{2(\kappa-i)}(\tau;q)$$

$$\langle \mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}(\tau;q) - \mathfrak{T}^{\mathcal{SO}(3)}(\tau,q)$$

$$= -\frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}{(q;q)_\infty^2}\sum_{\substack{\wp_1,\wp_2\in\mathbb{Z}\\ \wp_1\le\wp_2}}\frac{\left(q^{\frac{1}{2}}\tau^{-2}\right)^{\wp_1+\wp_2-2}}{\left(1-q^{\wp_1-\frac{1}{2}}\tau^2\right)\left(1-q^{\wp_2-\frac{1}{2}}\tau^2\right)}$$

$$\langle \mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) - \mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{q^2}{1-q^4}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q})$$

$$= \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) - 3\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q})$$

$$+ 3\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) - \mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6}{1-\mathfrak{q}^4}$$

$$\underbrace{\langle \mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(\mathfrak{q}) = \frac{\sum_{i=0}^{\kappa}\alpha_\kappa^{\delta o(3)\frac{\square}{\square}}(i)\mathfrak{q}^{2i}}{1-\mathfrak{q}^4}$$

$$\alpha_\kappa^{\delta o(3)\frac{\square}{\square}}(i) = c_\kappa^{(i)} - c_\kappa^{(i+1)}$$

$$(1+\chi+\chi^2)^\eta = \sum_{i=-\kappa}^{\kappa} c_\kappa^{(i)}\chi^{\kappa+i}$$

2.6. Números de Riordan.

$$\mathcal{R}_\eta = \sum_{i=0}^{\eta}(-1)^{\eta-1}\binom{\eta}{i}\mathfrak{C}_i$$

$$\sum_{\eta=1}^{\infty}\mathcal{R}_\eta\,\chi^\eta = \frac{1}{2\chi}\left(1-\frac{\sqrt{1-3\chi}}{\sqrt{1+\chi}}\right)$$

2.7. Simetría de Wilson.

$$
\begin{aligned}
&\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}(\tau;q)\\
&\quad=\frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}}{(q\tau^{\pm 4};q)_{\infty}}\left[\frac{2\kappa+1}{2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\left(\frac{\tau^{2m}-\tau^{-2m}}{\tau^{2}-\tau^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^{m}}\right)\right.\\
&\quad\left.-\sum_{m=1}^{2\kappa}(2\kappa+m+1)\left(\frac{\tau^{2m}-\tau^{-2m}}{\tau^{2}-\tau^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^{m}}\right)\right]
\end{aligned}
$$

$$
\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)=\frac{1+q^{2}\cdots q^{4}}{1-q^{4}}=\frac{1-q^{4\kappa+2}}{(1-q^{2})(1-q^{4})}
$$

$$
\langle \mathcal{W}_{(\infty)}\mathcal{W}_{(\infty)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)=\frac{1}{(1-q^{2})(1-q^{4})}
$$

$$
\underbrace{\langle \mathcal{W}_{\square\square}\cdots\mathcal{W}_{\square\square}\rangle^{\mathcal{SO}(3)}}_{\kappa}(\tau;q)=\sum_{\kappa_1+\kappa_2+\kappa_3=\kappa}\binom{\kappa}{\kappa_1,\kappa_2,\kappa_3}(-3)^{\kappa_2}\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}}_{4\kappa_1+2\kappa_2}(\tau;q)
$$

$$
\begin{aligned}
\langle \mathcal{W}_{\square\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)&=\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)-6\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)+\mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)\\
&=\frac{q^{4}}{1-q^{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)\\
&\quad=\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{12}(q)-18\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{10}(q)\\
&\quad+60\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{8}(q)-90\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{6}(q)\\
&\quad+60\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)}_{4}-18\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)+\mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)\\
&\quad=\frac{1+q^{2}+q^{4}+q^{6}+q^{8}+q^{10}+q^{12}}{1-q^{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\langle \mathcal{W}_{\square\square\square}\cdots\mathcal{W}_{\square\square\square}\rangle^{\mathcal{SO}(3)}(\tau;q)\\
&\quad=\sum_{\kappa_1+\kappa_2+\kappa_3+\kappa_4=\kappa}\binom{\kappa}{\kappa_1,\kappa_2,\kappa_3,\kappa_4}(-1)^{\kappa_2+\kappa_4}\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}}_{6\kappa_2+4\kappa_1+2\kappa_2}(\tau;q)
\end{aligned}
$$

$$
\begin{aligned}
&\langle \mathcal{W}_{\square\square\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)\\
&\quad=\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{6}(q)-10\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{4}(q)\\
&\quad+12\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)-\mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q)=\frac{q^{6}}{1-q^{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\langle \mathcal{W}_{\square\square\square}\mathcal{W}_{\square\square\square}\mathcal{W}_{\square\square\square}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) \\
&= \underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{18}(q) - 30\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{16}(q) \\
&+ 186\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{14}(q) - 616\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{12}(q) \\
&+ 1176\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{10}(q) - 1302\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{8}(q) \\
&+ 798\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{6}(q) - 246\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}}_{4}(q) \\
&+ 36\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) - \mathfrak{T}^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) \\
&= \frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+q^{14}+q^{16}+q^{18}}{1-q^4}
\end{aligned}
$$

$$
\chi^{\mathcal{SO}(3)}_{(\kappa)} = \sum_{\eta=0}^{\kappa}(-1)^{\eta}\binom{2\kappa-\eta}{\eta}\chi^{\mathcal{SO}(3)^{2\kappa-2\eta}}_{\delta\wp}
$$

$$
\langle \mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{2\kappa}}{1-q^4}
$$

$$
\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\ell)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{2(\ell-\kappa)}(1-q^{4\kappa+2})}{(1-q^2)(1-q^4)}
$$

$$
\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2-3q^{2\kappa+2}+q^{6\kappa+4}}{(1-q^2)^2(1-q^4)}
$$

$$
\langle \mathcal{W}_{(\infty)}\mathcal{W}_{(\infty)}\mathcal{W}_{(\infty)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^2)^3}
$$

$$
\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{2\kappa+1-3q^2-(2\kappa+1)q^4+4q^{4\kappa+4}-q^{8\kappa+6}}{(1-q^2)^3(1-q^4)}
$$

$$
\begin{aligned}
&\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\mathcal{N}+1)}(\tau;q) = \frac{1}{2^{\mathcal{N}}\mathcal{N}!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm};q\right)_\infty^{\mathcal{N}}}\oiiint\prod_{i=1}^{\mathcal{N}}\frac{ds_i}{2\pi i\delta_i}\frac{(\delta_i^{\pm};q)_\infty(q\delta_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty} \\
&\bigotimes\prod_{i<j}\frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}-\delta_i^{-\frac{1}{2}}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\langle \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)} \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)} \rangle^{\mathcal{U}\delta\wp(2\mathcal{N})/\mathbb{Z}_2}(\tau;q) \\
&= \frac{1}{\mathcal{N}!} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}\tau^{\pm};q\right)_\infty^3} \oiiint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i \delta_i} \frac{\left(q^{\frac{1}{2}}\delta_i^{\pm 2};q\right)_\infty \left(q^{\frac{3}{2}}\delta_i^{\pm 2};q\right)_\infty}{\left(q\tau^2\delta_i^{\pm 2};q\right)_\infty \left(q\tau^{-2}\delta_i^{\pm 2};q\right)_\infty} \\
&\quad \bigotimes \prod_{i<j} \frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty \left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{\frac{3}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}
\end{aligned}
$$

$$
\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(2\mathcal{N}+1)}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)} \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{U}\delta\wp(2\mathcal{N})/\mathbb{Z}_2}(q) = \prod_{i=1}^{\mathcal{N}} \frac{1}{(1-q^{2\eta})}
$$

$$
\mathfrak{T}_{\frac{1}{2}\mathfrak{BPS}}^{Spin(2\mathcal{N}+1)}(q) = \prod_{i=1}^{\mathcal{N}} \frac{1}{1-q^{4\eta}}
$$

$$
\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(2\mathcal{N}+1)}(q) = \prod_{i=1}^{\mathcal{N}} (1-q^{2\eta})
$$

$$
\langle \underbrace{\mathcal{W}_{\delta\wp} \cdots \mathcal{W}_{\delta\wp}}_{2\kappa} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{Spin(2\mathcal{N}+1)}(q) = \frac{\sum_{i=0}^{\frac{\mathcal{N}(\mathcal{N}+1)\kappa}{2}} \alpha_{\kappa}^{\delta o(2\mathcal{N}+1)\square}{}_{\delta\wp}(i)\, q^{2i}}{\prod_{i=1}^{\mathcal{N}}(1-q^{4\eta})}
$$

$$
\alpha_{\kappa}^{\delta o(2\mathcal{N}+1)\square}{}_{\delta\wp}(0) = \det\left(\mathfrak{C}_{2\mathcal{N}-i-j+\kappa}\right) = \prod_{1\le i\le j\le \kappa-1} \frac{i+j+2\mathcal{N}}{i+j}
$$

$$
\langle \underbrace{\mathcal{W}_{\square} \cdots \mathcal{W}_{\square}}_{2\kappa} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = \frac{\sum_{i=0}^{\mathcal{N}\kappa} \alpha_{\kappa}^{\delta o(2\mathcal{N}+1)\begin{smallmatrix}\square\\\square\end{smallmatrix}}(i)\, q^{2i}}{\prod_{i=1}^{\mathcal{N}}(1-q^{4\eta})}
$$

$$
\begin{aligned}
\langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) &= \frac{q^{2\mathcal{N}}}{\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})}, \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = \frac{1+q^2+q^4\cdots+q^{4\mathcal{N}}}{\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})} \\
&= \frac{1-q^{4\mathcal{N}+2}}{(1-q^2)\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})}
\end{aligned}
$$

$$
\langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = q^{2\mathcal{N}}, \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = \frac{1-q^{4\mathcal{N}+2}}{1-q^2}
$$

$$
\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS},c}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) - \langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q)^2 = \frac{1-q^{4\mathcal{N}}}{1-q^2}
$$

$$
\langle \underbrace{\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cdots \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{2\kappa} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(2\mathcal{N}+1)}(q) = \frac{\sum_{i=0}^{(2\mathcal{N}-1)\kappa} \alpha_{\kappa}^{\delta o(2\mathcal{N}+1)\begin{smallmatrix}\square\\\square\end{smallmatrix}}(i)\, q^{2i}}{\prod_{i=1}^{\mathcal{N}}(1-q^{4\eta})}
$$

$$\langle \mathcal{W}_{\substack{\square\\\square}} \rangle^{\mathcal{SO}(2\mathcal{N}+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2 + q^6 + \cdots + q^{4\mathcal{N}-2}}{\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})} = \frac{q^2(1-q^{4\mathcal{N}})}{(1-q^4)\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})}$$

$$\langle \mathcal{W}_{\substack{\square\\\square}} \rangle^{\mathcal{SO}(2\mathcal{N}+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2(1-q^{4\mathcal{N}})}{\prod_{\eta=1}^{\mathcal{N}} 1-q^4}$$

$$\underbrace{\langle \mathcal{W}_\ell \cdots \mathcal{W}_\ell \rangle}_{\kappa}{}^{\mathcal{SO}(2\mathcal{N}+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{\mathcal{N}\ell\kappa} \alpha_\kappa^{\mathcal{So}(2\mathcal{N}+1)\square_{(\ell)}}(i) q^{2i}}{\prod_{i=1}^{\mathcal{N}}(1-q^{4\eta})}$$

$$\chi_{\square}^{usp(2\mathcal{N})} = \sum_{i=1}^{\mathcal{N}} (\delta_i + \delta_i^{-1})$$

$$\chi_{\square}^{usp(2\mathcal{N})} = \frac{\det\left(\delta_j^{\lambda_i+\mathcal{N}-i+1} - \delta_j^{-\lambda_i-\mathcal{N}+i-1}\right)}{\det(\delta_j^{\mathcal{N}-i+1} - \delta_j^{-\mathcal{N}+i-1})}$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{USp}(2\mathcal{N})} = \iiint d\mu^{\mathcal{USp}(2\mathcal{N})} \exp\left(\sum_{\eta=1}^{\infty} \frac{1}{\eta} f_\eta(q,\tau) \mathcal{L}_\eta(\delta)\right) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{usp(2\mathcal{N})}(\delta)$$

$$\mathcal{L}_\eta(\delta) = \frac{\mathcal{P}_\eta(\delta)^2 + \mathcal{P}_{2\eta}(\delta)}{2}$$

$$\exp\left(\sum_{\eta=1}^{\infty} \frac{1}{\eta} f_\eta(q,\tau) \mathcal{L}_\eta(\delta)\right) = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau) \mathcal{L}_\lambda(\delta)$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{USp}(2\mathcal{N})} = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau) \iiint d\mu^{\mathcal{USp}(2\mathcal{N})} \mathcal{L}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{usp(2\mathcal{N})}(\delta)$$

$$\mathcal{L}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{usp(2\mathcal{N})}(\delta) = \sum_\mu \beta_{\lambda,\mathcal{R}}^\mu \mathcal{P}_\mu(\delta)$$

$$\iiint d\mu^{\mathcal{USp}(2\mathcal{N})} \mathcal{P}_\mu(\delta) = \sum_{\nu \in \mathcal{R}_{2\mathcal{N}+1}^c(|\mu|)} \chi_\nu^\delta(\mu)$$

$$\mathcal{R}_\eta^c(\wp) = \{\lambda \dashv \wp | \ell(\lambda) \leq \eta \forall \lambda_i'\}$$

$$\iiint d\mu^{\mathcal{USp}(2\mathcal{N})} \mathcal{L}_\lambda(\delta) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{usp(2\mathcal{N})}(\delta) = \sum_\mu \beta_{\lambda,\mathcal{R}}^\mu \iiint d\mu^{\mathcal{USp}(2\mathcal{N})} \mathcal{P}_\mu(\delta) = \sum_\mu \beta_{\lambda,\mathcal{R}}^\mu \sum_{\nu \in \mathcal{R}_{2\mathcal{N}}^c(|\mu|)} \chi_\nu^\delta(\mu)$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{USp}(2\mathcal{N})} = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(q,\tau) \sum_\mu \beta_{\lambda,\mathcal{R}}^\mu \sum_{\nu \in \mathcal{R}_{2\mathcal{N}}^c(|\mu|)} \chi_\nu^\delta(\mu)$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle^{\mathcal{USp}(2\mathcal{N})}(\tau;q) = \frac{1}{2} \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty} \oiiint \frac{ds}{2\pi i \delta} \frac{\left(\delta^{\pm 2};q\right)_\infty \left(q\delta^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta^{\pm 2};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta^{\pm 2};q\right)_\infty} \left(\delta + \delta^{-1}\right)^2$$

$$\langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(3)}(\tau;q) = \frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty} \oiiint \frac{ds}{2\pi i\delta} \frac{\left(q^{\frac{1}{2}}\delta^{\pm};q\right)_\infty \left(q^{\frac{3}{2}}\delta^{\pm 2};q\right)_\infty}{\left(q\tau^2\delta^{\pm};q\right)_\infty \left(q\tau^{-2}\delta^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(2\mathcal{N})}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(3)}(\tau;q) = \langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(3)}(\tau;q)$$
$$= \langle \mathfrak{T}_{(1/2)}\mathfrak{T}_{(1/2)}\rangle^{\mathcal{USp}(2)/\mathbb{Z}_2}(\tau;q)$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(3)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^2}{1-\mathfrak{q}^4} = \frac{1}{1-\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{\mathcal{USp}(2\mathcal{N})}(\tau;q)$$
$$= \frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty} \oiiint \frac{ds}{2\pi i\delta} \frac{\left(\delta^{\pm 2};q\right)_\infty \left(q\delta^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta^{\pm 2};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta^{\pm 2};q\right)_\infty} \left(1+\delta^2+\delta^{-2}\right)^2$$

$$\langle \mathcal{W}_{(2\kappa)}\rangle^{\mathcal{USp}(2\mathcal{N})}(\tau;q) = \langle \mathcal{W}_{(\kappa)}\rangle^{\mathcal{SO}(3)}(\tau;q)$$

$$\langle \mathcal{W}_{(2\kappa)}\rangle^{\mathcal{USp}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2\kappa}}{(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{(\kappa)}\mathcal{W}_{(\kappa)}\rangle^{\mathcal{USp}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{2\kappa+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{(\infty)}\mathcal{W}_{(\infty)}\rangle^{\mathcal{USp}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(4)}(\tau;q)$$
$$= \frac{1}{8}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i} \frac{\left(\delta_i^{\pm 2};q\right)_\infty \left(q\delta_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm 2};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm 2};q\right)_\infty} \bigotimes \frac{\left(\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty \left(q\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty} \left[\sum_{i=1}^{2}(\delta_i$$
$$+ \delta_i^{-1})\Big]^2$$

$$\langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)}\rangle^{\mathcal{SO}(5)}(\tau;q) = \frac{1}{2}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i} \frac{\left(q^{\frac{1}{2}}\delta_1^{\pm};q\right)_\infty \left(\delta_2^{\pm};q\right)_\infty \left(q^{\frac{3}{2}}\delta_1^{\pm};q\right)_\infty \left(q\delta_2^{\pm};q\right)_\infty}{\left(q\tau^2\delta_1^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}\tau^2\delta_2^{\pm};q\right)_\infty \left(q\tau^{-2}\delta_1^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_2^{\pm};q\right)_\infty} \bigotimes \frac{\left(q^{\frac{1}{2}}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty \left(q^{\frac{3}{2}}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q^{\frac{3}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}{\left(q\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty \left(q\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty \left(q\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(4)}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(5)}(\tau;q) = \langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)}\rangle^{\mathcal{SO}(5)}(\tau;q)$$
$$= \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{USp}(4)/\mathbb{Z}_2}(\tau;q)$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)}\rangle^{\mathcal{SO}(5)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{\substack{\square\\\square}}\rangle^{\mathcal{USp}(4)}(\tau;q) = \langle \mathcal{W}_{\square}\rangle^{\mathcal{SO}(5)}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\substack{\square\\\square}} \cdots \mathcal{W}_{\substack{\square\\\square}} \rangle}_{\kappa}^{\mathcal{USp}(4)} (\tau; q) = \underbrace{\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle}_{\kappa}^{\mathcal{SO}(5)} (\tau; q)$$

$$\underbrace{\langle \mathcal{W}_{\square\square} \cdots \mathcal{W}_{\square\square} \rangle}_{\kappa}^{\mathcal{USp}(4)} (\tau; q) = \underbrace{\langle \mathcal{W}_{\substack{\square\\\square}} \cdots \mathcal{W}_{\substack{\square\\\square}} \rangle}_{\kappa}^{\mathcal{SO}(5)} (\tau; q)$$

$$\underbrace{\langle \mathcal{W}_{(2\ell)} \cdots \mathcal{W}_{(2\ell)} \rangle}_{\kappa}^{\mathcal{USp}(4)} (\tau; q) = \underbrace{\langle \mathcal{W}_{(\ell^2)} \cdots \mathcal{W}_{(\ell^2)} \rangle}_{\kappa}^{\mathcal{SO}(5)} (\tau; q)$$

$$\underbrace{\langle \mathcal{W}_{(2\ell)} \cdots \mathcal{W}_{(2\ell)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{USp}(4)}}_{\kappa} (q) = \frac{\sum_{i=0}^{3\ell\kappa} \alpha_{\kappa}^{usp(4)} \,{}_{(2\ell)}^{\square}(i) q^{2i}}{(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{(\infty)} \cdots \mathcal{W}_{(\infty)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{USp}(4)} (q) = \langle \mathcal{W}_{(\infty^2)} \cdots \mathcal{W}_{(\infty^2)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(5)} (q)$$

$$= \frac{1-q^{16}}{(1-q^2)(1-q^4)^3(1-q^6)(1-q^8)^2}$$

$$\underbrace{\langle \mathcal{W}_{(\ell^2)} \cdots \mathcal{W}_{(\ell^2)} \rangle}_{\kappa}^{\mathcal{USp}(4)} (\tau; q) = \underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{\kappa}^{\mathcal{SO}(5)} (\tau; q)$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle^{\mathcal{USp}(6)} (\tau; q)$$

$$= \frac{1}{96} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}\tau^{\pm 2}; q\right)_\infty^3} \oiiint \prod_{i=1}^{6} \frac{ds_i}{2\pi i \delta_i} \frac{\left(\delta_i^{\pm 2}; q\right)_\infty \left(q\delta_i^{\pm 2}; q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm 2}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm 2}; q\right)_\infty} \bigotimes \prod_{i<j} \frac{\left(\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty \left(q\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty} \left[\sum_{i=1}^{4} (\delta_i + \delta_i^{-1})\right]^2$$

$$\langle \mathfrak{T}_{(1,0,0)} \mathfrak{T}_{(1,0,0)} \rangle^{\mathcal{SO}(7)} (\tau; q)$$

$$= \frac{1}{8} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}\tau^{\pm 2}; q\right)_\infty^3} \oiiint \prod_{i=1}^{6} \frac{ds_i}{2\pi i \delta_i} \frac{\left(q^{\frac{1}{2}\delta_{i,1}}; \delta_i^{\pm}, q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i,1}}\delta_i^{\pm 2}; q\right)_\infty}{\left(q^{\frac{1+\delta_{i,1}}{2}}\tau^2\delta_i^{\pm}; q\right)_\infty \left(q^{\frac{1+\delta_{i,1}}{2}}\tau^{-2}\delta_i^{\pm}; q\right)_\infty} \bigotimes \prod_{i<j} \square$$

$$\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp}; q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm}; q\right)_\infty}$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{USp}(6)} (q) = \langle \mathfrak{T}_{(1,0,0)} \mathfrak{T}_{(1,0,0)} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{SO}(7)} = \frac{1+q^2+q^4+q^6+q^8+q^{10}}{(1-q^4)\left(1-q^8(1-q^{12})\right)}$$

$$= \frac{1}{(1-q^2)(1-q^4)(1-q^8)}$$

$$\underbrace{\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathfrak{BPS}}^{\mathcal{USp}(6)}}_{2\kappa} (q) = \frac{\sum_{i=0}^{5\kappa} \alpha_{\kappa}^{usp(6)} \substack{\square\\\square}(i) q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\det\begin{pmatrix} \mathfrak{F}_0 & \mathfrak{F}_1 & \mathfrak{F}_2 \\ \mathfrak{F}_1 & \mathfrak{F}_2 & \mathfrak{F}_3 \\ \mathfrak{F}_2 & \mathfrak{F}_3 & \mathfrak{F}_4 \end{pmatrix} = \sum_{\kappa=0}^{\infty} \frac{\alpha_{\kappa}^{usp(6)\begin{smallmatrix}\square\\\square\end{smallmatrix}}(0)}{(2\kappa)!}\chi^{2\kappa}$$

$$\mathfrak{F}_m(\chi) := \sum_{j=0}^{m} \binom{m}{j} \left(\mathfrak{I}_{2j-m}(2\chi) - \mathfrak{I}_{2j-m+2}(2\chi)\right)$$

$$\mathfrak{I}_{\kappa}(2\chi) := \sum_{\kappa=0}^{\infty} \frac{\chi^{2\eta+\kappa}}{\eta!\,(\eta+\kappa)!}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\mathcal{W}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\underbrace{\langle \mathcal{W}_{\square\square}\cdots\mathcal{W}_{\square\square}\rangle}_{\kappa}{}^{\mathcal{USp}(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\cdots\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle}_{\kappa}{}^{\mathcal{SO}(7)}(\tau;q)$$

$$\langle \mathcal{W}_{\square\square}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2+q^6+q^{10}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{(2\ell)}\cdots\mathcal{W}_{(2\ell)}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \frac{\sum_{i=0}^{5\ell\kappa}\alpha_{\kappa}^{usp(6)\,\underset{(2\ell)}{\square}}(i)\,q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\square\square\square\square}\rangle^{\mathcal{USp}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8+q^{12}+q^{16}+q^{20}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\Box}\mathcal{W}_{\Box}\rangle^{\mathcal{US}\wp(2\mathcal{N})}(\tau;q)$$

$$= \frac{1}{2^{\mathcal{N}}\mathcal{N}!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm};q\right)_\infty^{\mathcal{N}}}\oiiint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i\delta_i}\frac{\left(\delta_i^{\pm 2};q\right)_\infty\left(q\delta_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm 2};q\right)_\infty} \bigotimes \prod_{i<j}\frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}\left[\sum_{i=1}^{\mathcal{N}}\left(\delta_i+\delta_i^{-1}\right)\right]^2$$

$$\langle \mathfrak{T}_{(1,0^{\mathcal{N}-1})}\mathfrak{T}_{(1,0^{\mathcal{N}-1})}\rangle^{\mathcal{SO}(2\mathcal{N}+1)}(\tau;q)$$

$$= \frac{1}{2^{\mathcal{N}-1}(\mathcal{N}-1)!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm};q\right)_\infty^{\mathcal{N}}}\oiiint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i\delta_i}\frac{\left(q^{\frac{1}{2}\delta_{i,1}};\delta_i^{\pm},q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}\delta_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1+\delta_{i,1}}{2}}\tau^{2}\delta_i^{\pm};q\right)_\infty\left(q^{\frac{1+\delta_{i,1}}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty} \bigotimes \prod_{i<j}\Box \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1+\delta_{i+j,1}}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\Box}\mathcal{W}_{\Box}\rangle^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{(1,0^{\mathcal{N}-1})}\mathfrak{T}_{(1,0^{\mathcal{N}-1})}\rangle^{\mathcal{SO}(2\mathcal{N}+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^2)\prod_{\eta=1}^{\mathcal{N}-1}(1-q^{4\eta})}$$

$$\mathfrak{T}^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \prod_{\eta=1}^{\mathcal{N}}\frac{1}{1-q^{4\eta}}$$

$$\langle \mathcal{W}_{\Box}\mathcal{W}_{\Box}\rangle^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1-q^{4\mathcal{N}}}{1-q^2}$$

$$\underbrace{\langle \mathcal{W}_{\Box}\cdots\mathcal{W}_{\Box}\rangle}_{\kappa}{}^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{(2\mathcal{N}-1)\kappa}\alpha_{\kappa}^{usp(2\mathcal{N})\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}(i)\,q^{2i}}{\prod_{\eta=1}^{\mathcal{N}-1}(1-q^{4\eta})}$$

$$\det\left(\mathfrak{F}_{i+j-2}(\chi)\right) = \sum_{\kappa=0}^{\infty}\frac{\alpha_{\kappa}^{usp(2\mathcal{N})\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}(0)}{(2\kappa)!}\chi^{2\kappa}$$

$$\underbrace{\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\cdots\mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\rangle}_{\kappa}{}^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{(2\mathcal{N}-1)\kappa}\alpha_{\kappa}^{usp(2\mathcal{N})\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}}(i)\,q^{2i}}{\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})}$$

$$\langle \mathcal{W}_{\Box\Box}\cdots\mathcal{W}_{\Box\Box}\rangle^{\mathcal{US}\wp(2\mathcal{N})}(\tau;q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\cdots\mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\rangle^{\mathcal{SO}(2\mathcal{N}+1)}(\tau;q)$$

$$\langle \mathcal{W}_{\Box\Box}\rangle^{\mathcal{US}\wp(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2+q^6+\cdots+q^{4\mathcal{N}}}{\prod_{\eta=1}^{\mathcal{N}-1}(1-q^{4\eta})} = \frac{q^2(1-q^{4\mathcal{N}})}{(1-q^4)\prod_{\eta=1}^{\mathcal{N}-1}(1-q^{4\eta})}$$

$$\underbrace{\langle \mathcal{W}_{(2\ell)} \cdots \mathcal{W}_{(2\ell)} \rangle^{\mathcal{USp}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa} (q) = \frac{\sum_{i=0}^{(2\mathcal{N}-1)\kappa} \alpha_{\kappa}^{usp(2\mathcal{N})} \square_{(2\ell)}(i) q^{2i}}{\prod_{\eta=1}^{\mathcal{N}} (1-q^{4\eta})}$$

$$\chi_{\delta\wp}^{\delta o(2\mathcal{N})} = \frac{1}{2}\left[\prod_{\eta=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}} - \delta_i^{-\frac{1}{2}}\right) + \prod_{\eta=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}} - \delta_i^{-\frac{1}{2}}\right)\right]$$

$$\chi_{\delta\wp}^{\delta o(2\mathcal{N})} = \frac{1}{2}\left[\prod_{\eta=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}} - \delta_i^{-\frac{1}{2}}\right) - \prod_{\eta=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}} - \delta_i^{-\frac{1}{2}}\right)\right]$$

$$\chi_{\square}^{\delta o(2\mathcal{N})} = \sum_{\eta=1}^{\mathcal{N}} (\delta_i + \delta_i^{-1})$$

$$\chi_{\lambda}^{\delta o(2\mathcal{N})} = \frac{\det\left(\delta_j^{\lambda_i+\mathcal{N}-i} + \delta_j^{-\lambda_i-\mathcal{N}+i}\right) + \det\left(\delta_j^{\lambda_i+\mathcal{N}-i} - \delta_j^{-\lambda_i-\mathcal{N}+i}\right)}{\det\left(\delta_j^{\mathcal{N}-i} - \delta_j^{-\mathcal{N}+i}\right)}$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_\kappa} \rangle^{\mathcal{SO}(2\mathcal{N})} = \iiint d\mu^{\mathcal{SO}(2\mathcal{N})} \exp\left(\sum_{\eta=1}^{\infty} \frac{1}{\eta} f_\eta(q,\tau) \mathcal{M}_\eta(\delta)\right) \prod_{i=1}^{\kappa} \chi_{\mathcal{R}_i}^{\delta o(2\mathcal{N})}(\delta)$$

$$d\mu^{\mathcal{SO}(2\mathcal{N})} = \frac{1}{2^{\mathcal{N}-1}\mathcal{N}!} \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i \delta_i} \prod_{1\le i\le j\le \mathcal{N}} (1-\delta_i\delta_j)(1-\delta_i^{-1}\delta_j^{-1})(1-\delta_i\delta_j^{-1})(1-\delta_j\delta_i^{-1})$$

$$\mathcal{M}_\eta(\delta) = \frac{\mathcal{P}_\eta(\delta)^2 + \mathcal{P}_{2\eta}(\delta)}{2}$$

$$\mathfrak{T}^{\mathcal{SO}(4)}(\tau;q) = \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q) \circledast \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q)$$

$$= \frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^2}{(q,q)_\infty^4} \left(\sum_{\substack{\wp_1,\wp_2\in\mathbb{Z} \\ \wp_1\le\wp_2}} \frac{\left(q^{\frac{1}{2}}\tau^{-2}\right)^{\wp_1+\wp_2-2}}{\left(1-q^{\wp_1-\frac{1}{2}}\tau^2\right)\left(1-q^{\wp_2-\frac{1}{2}}\tau^2\right)}\right)^2$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp} \rangle^{Spin(4)}(\tau;q)$$

$$= \frac{1}{4} \frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i} \otimes \frac{(\delta_1^{\pm}\delta_2^{\mp};q)_\infty(\delta_1^{\pm}\delta_2^{\pm};q)_\infty(q\delta_1^{\pm}\delta_2^{\mp};q)_\infty(q\delta_1^{\pm}\delta_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty} \bigotimes \left(\delta_1^{1/2}\delta_2^{1/2} + \delta_1^{-1/2}\delta_2^{-1/2}\right)$$

$$\langle \mathfrak{T}_{\left(\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2}\right)} \rangle^{\mathcal{SO}(4)/\mathbb{Z}_2}(\tau;q)$$

$$= \frac{1}{2} \frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^2} \oiiint \prod_{i=1}^{4} \frac{ds_i}{2\pi i\delta_i} \otimes \frac{(q^{1/2}\delta_1^{\pm}\delta_2^{\mp};q)_\infty(\delta_1^{\pm}\delta_2^{\pm};q)_\infty(q^{3/2}\delta_1^{\pm}\delta_2^{\mp};q)_\infty(q\delta_1^{\pm}\delta_2^{\pm};q)_\infty}{(q\tau^2\delta_1^{\pm}\delta_2^{\mp};q)_\infty(q^{1/2}\tau^2\delta_1^{\pm}\delta_2^{\pm};q)_\infty(q\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q)_\infty(q^{1/2}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q)_\infty} \bigotimes \square$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}(\tau;q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\frac{\mathcal{SO}(4)}{\mathbb{Z}_2}}(\tau;q) = \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q)\,\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2}\right)}\rangle^{\frac{\mathcal{SO}(4)}{\mathbb{Z}_2}}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2}{(1-q^4)^2} = \frac{1}{(1-q^2)(1-q^4)}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle}_{2\kappa}^{Spin(4)}(\tau;q) = \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q)\underbrace{\langle \mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle}_{2\kappa}^{\mathcal{SU}(2)}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\cdots\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \mathfrak{T}^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\sum_{i=0}^{\kappa}\alpha_{\kappa}^{\delta o(4)\square}{}_{\delta\wp}(i)q^{2i} = \frac{1}{(1-q^4)^2}\sum_{i=0}^{\kappa}\alpha_{\kappa}^{\delta o(4)\square}{}_{\delta\wp}(i)q^{2i}$$

$$\alpha_{\kappa}^{\delta o(4)\square}{}_{\delta\wp}(i) = \frac{(2i+1)(2\kappa)!}{(\kappa-i)!\,(\kappa+i+1)!}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4\cdots q^{\eta}}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4\cdots q^{\eta}}{(1-q^4)^2}$$

$$\begin{aligned}\langle \mathcal{W}^{2\kappa}_{\delta\wp}\mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) &= \left(\mathfrak{T}^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\right)^{-1}\langle \mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\langle \mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\\ &= (1-q^4)^2\langle \mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\langle \mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)\end{aligned}$$

$$\langle \mathcal{W}^{2\kappa}_{\delta\wp}\mathcal{W}^{2\kappa}_{\delta\wp}\rangle^{Spin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)^2}\left(\sum_{i=0}^{\kappa}\alpha_{\kappa}^{\delta o(4)\square}{}_{\delta\wp}(i)q^{2i}\right)\left(\sum_{j=0}^{m}\alpha_{m}^{\delta o(4)\square}{}_{\delta\wp}(j)q^{2j}\right)$$

$$\begin{aligned}&\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}(\tau;q)\\ &= \frac{1}{4}\frac{(q)^4_{\infty}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)^2_{\infty}}\oiiint\prod_{i=1}^{4}\frac{ds_i}{2\pi i\delta_i}\otimes\frac{(\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}(q\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_{\infty}}\\ &\qquad\bigotimes\left(\delta_1+\delta_2+\delta_1^{-1}+\delta_2^{-1}\right)\end{aligned}$$

$$\begin{aligned}&\langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)}\rangle^{\mathcal{SO}(4)}(\tau;q)\\ &= \frac{(q)^4_{\infty}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)^2_{\infty}}\oiiint\prod_{i=1}^{4}\frac{ds_i}{2\pi i\delta_i}\otimes\frac{(q^{1/2}\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q^{1/2}\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}(q^{3/2}\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q^{3/2}\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}}{(q\tau^2\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q\tau^2\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}(q\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q)_{\infty}(q\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q)_{\infty}}\\ &\qquad\bigotimes\square\end{aligned}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}(\tau;q)^2$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^2)^2}$$

$$\underbrace{\langle \mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle}_{2\kappa}^{\mathcal{SO}(4)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle}_{\kappa}^{\mathcal{SU}(2)}(\tau;q)^2$$

$$\langle \mathcal{W}_{\square}\cdots\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{2\kappa}\alpha_{\kappa}^{\mathcal{so}(4)\frac{\square}{\square}}(i)q^{2i}}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4\cdots q^{\eta}}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{S}pin(4)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$
$$= \frac{1+q^2+q^4\cdots q^{\eta}}{(1-q^4)^2}\bigotimes(1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\alpha_{\kappa}^{\mathcal{so}(4)\frac{\square}{\square}}(0) = \mathfrak{C}_{\kappa}^2$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)^-}(\tau;q) = \frac{1}{2}\frac{(q)_{\infty}^2(-q;q)_{\infty}^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}}$$

$$\oiiint\prod_{i=1}^{4}\frac{ds}{2\pi i\delta}\otimes\frac{(\delta^{\pm};q)_{\infty}(-\delta;q)_{\infty}(q\delta^{\pm};q)_{\infty}(-q\delta^{\pm};q)_{\infty}}{\left(q^{\frac{1}{2}}\tau^2\delta^{\pm};q\right)_{\infty}\left(-q^{\frac{1}{2}}\tau^2\delta^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta^{\pm};q\right)_{\infty}\left(-q^{\frac{1}{2}}\tau^{-2}\delta^{\pm};q\right)_{\infty}}$$
$$\bigotimes\left(\delta+\delta^{-1}\right)^2$$

$$\langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(4)^-}(\tau;q) = \frac{(q)_{\infty}^2(-q;q)_{\infty}^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}}$$

$$\oiiint\prod_{i=1}^{4}\frac{ds}{2\pi i\delta}\otimes\frac{\left(q^{\frac{1}{2}}\delta^{\pm};q\right)_{\infty}\left(-q^{\frac{1}{2}}\delta;q\right)_{\infty}\left(q^{\frac{3}{2}}\delta^{\pm};q\right)_{\infty}\left(-q^{\frac{3}{2}}\delta^{\pm};q\right)_{\infty}}{(q\tau^2\delta^{\pm};q)_{\infty}(-q\tau^2\delta^{\pm};q)_{\infty}(q\tau^{-2}\delta^{\pm};q)_{\infty}(-q\tau^{-2}\delta^{\pm};q)_{\infty}}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)^-}(\tau;q) = \langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(4)^-}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SU}(2)}(\tau;q)^2$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{(1)}\mathfrak{T}_{(1)}\rangle^{\mathcal{SO}(4)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{1-q^4}$$

$$\underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(4)^-}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{\sum_{i=0}^{\kappa}\alpha_{\kappa}^{\mathcal{so}(4)^-\frac{\square}{\square}}(i)q^{4i}}{1-q^8}$$

$$\alpha_{\kappa}^{\mathcal{so}(4)^-\frac{\square}{\square}}(i) = (2i+1)\frac{(2\kappa)!}{(\kappa-i)!\,(\kappa+i+1)!}$$

$$\langle \mathcal{W}_{\lambda_1} \cdots \mathcal{W}_{\lambda_\kappa} \rangle^{O(4)^-}(\tau;q) = \frac{1}{2} \langle \mathcal{W}_{\lambda_1} \cdots \mathcal{W}_{\lambda_\kappa} \rangle^{SO(4)}(\tau;q) + \langle \mathcal{W}_{\lambda_1} \cdots \mathcal{W}_{\lambda_\kappa} \rangle^{SO(4)^-}(\tau;q)$$

$$\langle \mathcal{W}_{\Box} \mathcal{W}_{\Box} \rangle^{O(4)^+}(\tau;q) = \langle \mathfrak{T}_{(1)} \mathfrak{T}_{(1)} \rangle^{O(4)^+}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\Box} \mathcal{W}_{\Box} \rangle^{O(4)^+}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(\mathfrak{q}) = \frac{\sum_{i=0}^{2\kappa+1} \alpha_\kappa^{o(4)\,\substack{\Box\\\Box}}(i) \mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \mathcal{W}_{\Box} \mathcal{W}_{\Box} \rangle^{O(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4 \cdots \mathfrak{q}^\eta}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)} = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \rangle^{O(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4 \cdots \mathfrak{q}^\eta}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \mathcal{W}_{\Box} \rangle^{O(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q})$$
$$= \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}+\cdots+\mathfrak{q}^\eta)$$

$$\alpha_\kappa^{o(4)\,\substack{\Box\\\Box}}(0) = \frac{1}{2}(\mathfrak{C}_\kappa^2 + \mathfrak{C}_{\mathfrak{k}})$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \cdots \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}(\tau;q) = \langle \mathcal{W}_{\substack{\Box\\\Box}} \cdots \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}(\tau;q)$$
$$= \mathfrak{T}^{SU(2)}(\tau;q) \langle \mathcal{W}_{\Box\Box} \cdots \mathcal{W}_{\Box\Box} \rangle^{SU(2)}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\substack{\Box\\\Box}} \cdots \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(\mathfrak{q}) = \frac{\sum_{i=0}^{\kappa} \alpha_\kappa^{\mathcal{S}o(4)\,\substack{\Box\\\Box\\\Box}}(i) \mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \mathcal{W}_{\substack{\Box\\\Box}} \rangle^{SO(4)}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)^2}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1 + q^2 + q^4 + q^6 + q^8 + q^{10} + \cdots + q^{\eta}}{(1 - q^4)^2}$$

$$\underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{\kappa}{}^{\mathcal{SO}(4)}(\tau; q) = \underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{\kappa}{}^{\mathcal{SU}(2)}(\tau; q)^2$$

$$\underbrace{\langle \mathcal{W}_{\square\square} \cdots \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}}_{\kappa}(q) = \frac{\sum_{i=0}^{2\kappa} \alpha_{\kappa}^{\mathcal{So}(4)^-, \begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}}(i)\, q^{2i}}{(1 - q^4)^2}$$

$$\langle \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4}{(1 - q^4)^2}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{(1 + q^2 + q^4)^2}{(1 - q^4)^2}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{(1 + q^2 + q^4 + \cdots + q^{\eta})^2}{(1 - q^4)^2}$$

$$\alpha_{\kappa}^{\mathcal{So}(4)^-, \begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}}(0) = \mathfrak{R}_{\kappa}^2$$

$$\alpha_{\kappa}^{\mathcal{So}(4)^-, \begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}}(1) = 2\mathfrak{R}_{\kappa}\mathfrak{R}_{\kappa+1}$$

$$\langle \left(\mathcal{W}_{\square\square}\right)^{\kappa} \rangle^{\mathcal{SO}(4)}(\tau; q) = \langle \left(\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right)^{\kappa} \left(\mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right)^{\kappa} \rangle^{\mathcal{SO}(4)}$$

$$\langle \mathcal{W}_{(2\kappa)} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{4\kappa}}{(1 - q^4)^2}$$

$$\langle \mathcal{W}_{(\kappa)} \mathcal{W}_{(\kappa)} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{(1 - q^{2\kappa+2})^2}{(1 - q^2)^2 (1 - q^4)^2}$$

$$\langle \mathcal{W}_{(\infty)} \mathcal{W}_{(\infty)} \rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1 - q^2)^2 (1 - q^4)^2}$$

$$\underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{\kappa}{}^{\mathcal{SO}(4)^-}(\tau; q) = \underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{\kappa}{}^{\mathcal{SU}(2)}(\tau; q)^2$$

$$\langle \mathcal{W}_{(2\ell)} \rangle^{\mathcal{SO}(4)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1 - q^{4\ell+4}}{(1 - q^4)(1 - q^8)}$$

$$\langle \mathcal{W}_{(2\ell)} \rangle^{\mathcal{SO}(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{4\ell}}{(1 - q^4)(1 - q^8)}$$

$$\langle \mathcal{W}_{(\ell)} \mathcal{W}_{(\ell)} \rangle^{\mathcal{O}(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1 - q^2 + q^4 - q^{2\ell+2} - q^{2\ell+6} + q^{4\ell+6}}{(1 - q^2)^2 (1 - q^4)(1 - q^8)}$$

$$\langle \mathcal{W}_{(\infty)}\mathcal{W}_{(\infty)}\rangle^{\mathcal{O}(4)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1-q^2+q^4}{(1-q^2)^2(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{(\ell,\ell)}\cdots\mathcal{W}_{(\ell,\ell)}\rangle^{\mathcal{SO}(4)}(\tau;q) = \underbrace{\langle \mathcal{W}_{(\ell,\ell)}\cdots\mathcal{W}_{(\ell,\ell)}\rangle}_{\kappa}^{\mathcal{SO}(4)}(\tau;q)$$
$$= \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q)\underbrace{\langle \mathcal{W}_{(2\ell)}\cdots\mathcal{W}_{(2\ell)}\rangle}_{\kappa}^{\mathcal{SU}(2)}(\tau;q)$$

$$\langle \mathcal{W}_{(\ell,\ell)}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{(\ell,-\ell)}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{2\ell}}{(1-q^4)^2}$$

$$\langle \mathcal{W}_{(\ell,\ell)}\mathcal{W}_{(\ell,\ell)}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{(\ell,-\ell)}\mathcal{W}_{(\ell,-\ell)}\rangle^{\mathcal{SO}(4)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1-q^{4\ell+2}}{(1-q^2)(1-q^4)^2}$$

$$\langle \mathcal{W}^{\kappa}_{(\ell,\ell)}\mathcal{W}^{\kappa}_{(\ell,-\ell)}\rangle^{\mathcal{SO}(4)}(q) = \langle \mathcal{W}^{\kappa}_{(2\ell)}\rangle^{\mathcal{SO}(4)}$$

$$\mathfrak{T}^{\mathcal{SO}(6)}(\tau;q) = \mathfrak{T}^{\mathcal{SU}(2)}(\tau;q)$$
$$= -\frac{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}{(q,q)^2_\infty}\sum_{\substack{\wp_1,\wp_2,\wp_3,\wp_4\in\mathbb{Z}\\ \wp_1\le\wp_2\le\wp_3\le\wp_4}}\frac{\left(q^{\frac{1}{2}}\tau^{-2}\right)^{\wp_1+\wp_2+\wp_3+\wp_4-16}}{(1-q^{\wp_1-1}\tau^4)(1-q^{\wp_2-1}\tau^4)(1-q^{\wp_3-1}\tau^4)(1-q^{\wp_4-1}\tau^4)}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}(\tau;q) = \frac{1}{48}\frac{(q)^6_\infty}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)^3_\infty}$$

$$\oiiint \prod_{i=1}^{6}\frac{ds_i}{2\pi i\delta_i}\otimes\prod_{i\neq j}\frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$
$$\bigotimes(1+\delta_1\delta_2+\delta_1\delta_3+\delta_2\delta_3)\,(1+\delta_1^{-1}\delta_2^{-1}+\delta_1^{-1}\delta_3^{-1}+\delta_2^{-1}\delta_3^{-1})$$

$$\langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{SO}(6)/\mathbb{Z}_2}(\tau;q) = \frac{1}{6}\frac{(q)^6_\infty}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)^3_\infty}$$

$$\oiiint \prod_{i=1}^{6}\frac{ds_i}{2\pi i\delta_i}\otimes\prod_{i\le j}\frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}(\tau;q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\rangle^{\mathcal{SO}(6)/\mathbb{Z}_2}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\overline{\square}}\rangle^{\mathcal{SU}(4)}(\tau;q)$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\mathfrak{T}_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\rangle^{\frac{\mathcal{SO}(6)}{\mathbb{Z}_2}}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^2)(1-q^4)(1-q^6)}$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\cdots\mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle}_{2\kappa}^{Spin(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\overline{\square}}\cdots\mathcal{W}_{\square}\mathcal{W}_{\overline{\square}}\rangle}_{2\kappa}^{\mathcal{SU}(4)}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\cdots\mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{\sum_{i=0}^{3\kappa}\alpha_{\kappa}^{\delta o(6)\square}{}_{\delta\wp}(i)q^{2i}}{(1-q^4)(1-q^6)(1-q^8)}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+\cdots+q^{\eta}}{(1-q^4)(1-q^6)(1-q^8)}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^6)(1-q^8)}\bigotimes(1+q^2+q^4+\cdots+q^{\eta})$$

$$\det\begin{pmatrix}\mathfrak{I}_0(2\chi) & \mathfrak{I}_1(2\chi) & \mathfrak{I}_2(2\chi)\mathfrak{I}_3(2\chi)\\ \mathfrak{I}_1(2\chi) & \mathfrak{I}_0(2\chi) & \mathfrak{I}_1(2\chi)\mathfrak{I}_2(2\chi)\\ \mathfrak{I}_2(2\chi) & \mathfrak{I}_2(2\chi) & \mathfrak{I}_0(2\chi)\mathfrak{I}_1(2\chi)\end{pmatrix} = \sum_{\kappa=0}^{\infty}\frac{\alpha_{\kappa}^{\mathcal{SO}(6)\square}{}_{\mathcal{SP}}(0)}{(\kappa!)^2}\chi^{2\kappa}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)}(\tau;q) = \frac{1}{48}\frac{(q)_{\infty}^{6}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}^{3}}\oiiint\prod_{i=1}^{6}\frac{ds_i}{2\pi i\delta_i}$$

$$\otimes\prod_{i\neq j}\frac{\left(\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}\left(q\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}{\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{2}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}\bigotimes\left(\delta_1+\delta_2+\delta_3+\delta_1^{-1}+\delta_2^{-1}+\delta_3^{-1}\right)^2$$

$$\langle \mathfrak{T}_{(1,0,0)}\mathfrak{T}_{(1,0,0)}\rangle^{\mathcal{SO}(6)}(\tau;q) = \frac{1}{4}\frac{(q)_{\infty}^{6}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_{\infty}^{3}}\oiiint\prod_{i=1}^{6}\frac{ds_i}{2\pi i\delta_i}$$

$$\otimes\prod_{i\leq j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{2}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{2}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}$$

$$\bigotimes\frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_{\infty}\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_{\infty}}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)}(\tau;q) = \langle \mathfrak{T}_{(1,0,0)}\mathfrak{T}_{(1,0,0)}\rangle^{\mathcal{SO}(6)}(\tau;q) = \langle \mathcal{W}_{\substack{\square\\\square}}\mathcal{W}_{\substack{\square\\\square}}\rangle^{\mathcal{SU}(4)}(\tau,q)$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{(1,0,0)}\mathfrak{T}_{(1,0,0)}\rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^8}{(1-q^4)(1-q^6)(1-q^8)} = \frac{1}{(1-q^2)(1-q^4)^2}$$

$$\underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle}_{2\kappa}{}^{\mathcal{SO}(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\substack{\square\\\square}}\mathcal{W}_{\substack{\square\\\square}}\rangle}_{2\kappa}{}^{\mathcal{SU}(4)}(\tau;q)$$

$$\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{\sum_{i=0}^{4\kappa} \alpha_{\kappa}^{\mathcal{so}(6)\frac{\square}{\square}}(i) q^{2i}}{(1-q^4)(1-q^6)(1-q^8)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^6)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^6)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)^-}(\tau;q) = \frac{1}{8}\frac{(q)^4_\infty(-q;q)^4_\infty}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}$$

$$\otimes \oiiint \prod_{i=1}^{4}\frac{ds_i}{2\pi i\delta_i}\frac{(\delta_i^{\pm};q)_\infty(-\delta_i^{\pm};q)_\infty(q\delta_i^{\pm};q)_\infty(-q\delta_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^2\delta_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty}$$

$$\otimes \frac{\left(\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty\left(q\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty} \bigotimes \left(\delta_1+\delta_2+\delta_1^{-1}+\delta_2^{-1}\right)^2$$

$$\langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)} \rangle^{\mathcal{SO}(6)^-}(\tau;q) = \frac{1}{2}\frac{(q)^4_\infty(-q;q)^2_\infty}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty}$$

$$\oiiint \prod_{i=1}^{4}\frac{ds_i}{2\pi i\delta_i} \otimes \frac{\left(q^{\frac{1}{2}}\delta_1^{\pm};q\right)_\infty(\delta_2^{\pm};q)_\infty\left(-q^{\frac{1}{2}}\delta_1^{\pm};q\right)_\infty(-\delta_2^{\pm};q)_\infty}{(q\tau^2\delta_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_2^{\pm};q\right)_\infty(-q\tau^2\delta_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}\tau^2\delta_2^{\pm};q\right)_\infty}$$

$$\otimes \frac{\left(q^{\frac{3}{2}}\delta_1^{\pm};q\right)_\infty(q\delta_2^{\pm};q)_\infty\left(-q^{\frac{3}{2}}\delta_1^{\pm};q\right)_\infty(-q\delta_2^{\pm};q)_\infty}{(q\tau^{-2}\delta_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_2^{\pm};q\right)_\infty(-q\tau^{-2}\delta_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}\tau^{-2}\delta_2^{\pm};q\right)_\infty}$$

$$\otimes \frac{\left(q^{\frac{1}{2}}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}{\left(q\tau^2\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q\tau^2\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty\left(q\tau^{-2}\delta_1^{\pm}\delta_2^{\mp};q\right)_\infty\left(q\tau^{-2}\delta_1^{\pm}\delta_2^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{(1,0)}\mathfrak{T}_{(1,0)} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^6+q^8}{(1-q^6)(1-q^4)(1-q^8)} = \frac{1}{(1-q^2)(1-q^8)}$$

$$\underbrace{\langle \mathcal{W}_{\square} \cdots \mathcal{W}_{\square} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{\sum_{i=0}^{4\kappa} \alpha_{\kappa}^{\mathcal{so}(6)^-\frac{\square}{\square}}(i) q^{2i}}{(1-q^6)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+\cdots+q^{\eta}}{(1-q^6)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^6)(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\alpha_{\kappa}^{\mathcal{SO}(6)^-}(0) = \mathfrak{C}_{\kappa}\mathfrak{C}_{\kappa+2} - \mathfrak{C}_{\kappa+1}^2$$

$$\langle \mathcal{W}_{\lambda_1}\cdots\mathcal{W}_{\lambda_\kappa}\rangle^{\mathcal{O}(6)^+}(\tau;q) = \frac{1}{2}\left[\langle \mathcal{W}_{\lambda_1}\cdots\mathcal{W}_{\lambda_\kappa}\rangle^{\mathcal{SO}(6)^+}(\tau;q) + \langle \mathcal{W}_{\lambda_1}\cdots\mathcal{W}_{\lambda_\kappa}\rangle^{\mathcal{SO}(6)^-}(\tau;q)\right]$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+\cdots+q^{\eta}}{(1-q^4)(1-q^8)(1-q^{12})} = \frac{1}{(1-q^2)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^2)(1-q^4)(1-q^8)} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\underbrace{\langle \mathcal{W}_{\substack{\square\\\square}}\mathcal{W}_{\substack{\square\\\square}}\rangle}_{\kappa}{}^{\mathcal{SO}(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\substack{\square\square\\\square\\\square}}\mathcal{W}_{\substack{\square\square\\\square\\\square}}\rangle}_{\kappa}{}^{\mathcal{SU}(4)}(\tau;q)$$

$$\underbrace{\langle \mathcal{W}_{\substack{\square\\\square\\\square}}\mathcal{W}_{\substack{\square\\\square\\\square}}\rangle}_{2\kappa}{}^{\mathcal{SO}(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{\square\square}\mathcal{W}_{\square\square}\rangle}_{2\kappa}{}^{\mathcal{SU}(4)}(\tau;q)$$

$$\langle \mathcal{W}_{\substack{\square\\\square}}\rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2}{(1-q^2)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\substack{\square\\\square}}\mathcal{W}_{\substack{\square\\\square}}\rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\substack{\square\\\square\\\square}}\mathcal{W}_{\substack{\square\\\square\\\square}}\rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^{\eta})$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2}{(1-q^2)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+\cdots+q^\eta}{(1-q^2)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2}{(1-q^4)^2(1-q^8)}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+\cdots+q^\eta}{(1-q^4)^2(1-q^8)}$$

$$\underbrace{\langle \mathcal{W}_{(\ell)} \cdots \mathcal{W}_{(\ell)} \rangle}_{2\kappa}{}^{\mathcal{SO}(6)}(\tau;q) = \underbrace{\langle \mathcal{W}_{(\ell^2)} \cdots \mathcal{W}_{\overline{(\ell^2)}} \rangle}_{\kappa}{}^{\mathcal{SU}(4)}(\tau;q)$$

$$\langle \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8}{(1-q^4)(1-q^6)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{(1-q^4)(1-q^8)(1-q^{12})} \bigotimes (1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^\eta)$$

$$\langle \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8}{(1-q^6)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^\eta}{(1-q^6)(1-q^2)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^4+q^8}{(1-q^6)(1-q^4)(1-q^8)}$$

$$\langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{\mathcal{SO}(6)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}+\cdots+q^\eta}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle \mathcal{W}_{\mathcal{S}\wp} \mathcal{W}_{\mathcal{S}\wp} \rangle^{Spin(2\mathcal{N})}(\tau;q) = \frac{1}{2^{\mathcal{N}-1}\mathcal{N}!} \frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}}} \oiiint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i \delta_i}$$

$$\bigotimes \prod_{i\neq j} \frac{(\delta_i^{\pm}\delta_j^{\mp};q)_\infty (\delta_i^{\pm}\delta_j^{\pm};q)_\infty (q\delta_i^{\pm}\delta_j^{\mp};q)_\infty (q\delta_i^{\pm}\delta_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty} \bigotimes \frac{1}{4}\left[\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}+\delta_i^{-\frac{1}{2}}\right) + \prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}-\delta_i^{-\frac{1}{2}}\right)\right]^2$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(2\mathcal{N})}(\tau;q) = \frac{1}{2^{\mathcal{N}-1}\mathcal{N}!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}}}\oiiint \prod_{i=1}^{\mathcal{N}}\frac{ds_i}{2\pi i\delta_i}$$

$$\bigotimes \prod_{i\neq j}\frac{(\delta_i^{\pm}\delta_j^{\mp};q)_\infty(\delta_i^{\pm}\delta_j^{\pm};q)_\infty(q\delta_i^{\pm}\delta_j^{\mp};q)_\infty(q\delta_i^{\pm}\delta_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}\bigotimes\frac{1}{4}\left[\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}+\delta_i^{-\frac{1}{2}}\right)\right.$$

$$\left.+\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}-\delta_i^{-\frac{1}{2}}\right)\right]\left[\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}+\delta_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{\mathcal{N}}\left(\delta_i^{\frac{1}{2}}-\delta_i^{-\frac{1}{2}}\right)\right]$$

$$\langle \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\rangle^{\mathcal{SO}(2\mathcal{N})/\mathbb{Z}_2}(\tau;q) = \frac{1}{\mathcal{N}!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}}}\oiiint \prod_{i=1}^{\mathcal{N}}\frac{ds_i}{2\pi i\delta_i}$$

$$\bigotimes \prod_{i\leq j}\frac{(\delta_i^{\pm}\delta_j^{\mp};q)_\infty\left(q^{\frac{1}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty(q\delta_i^{\pm}\delta_j^{\mp};q)_\infty\left(q^{\frac{3}{2}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty(q\tau^2\delta_i^{\pm}\delta_j^{\pm};q)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty(q\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q)_\infty}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\rangle^{\mathcal{SO}(2\mathcal{N})/\mathbb{Z}_2}_{\frac{1}{2}\mathfrak{BPS}} = \prod_{\eta=1}^{\mathcal{N}}\frac{1}{1-q^{2\eta}}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\mathfrak{T}_{\left(\frac{1}{2},\mathcal{N}\right)}\rangle^{\mathcal{SO}(2\mathcal{N})/\mathbb{Z}_2}_{\frac{1}{2}\mathfrak{BPS}} = \prod_{\eta=1}^{\mathcal{N}}\frac{1}{1-q^{2\eta}}$$

$$\mathfrak{T}^{Spin(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{1-q^{2\mathcal{N}}}\prod_{\eta=1}^{\mathcal{N}-1}\frac{1}{1-q^{4\eta}}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(q) = \prod_{\eta=1}^{\mathcal{N}-1}(1+q^{2\eta})$$

$$\underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\mathcal{N}=4\eta)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \underbrace{\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(2\mathcal{N}=4\eta+2)}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(q) = \frac{\sum_{i=0}^{\frac{\mathcal{N}(\mathcal{N}+1)\kappa}{2}}\alpha_\kappa^{\delta o(2\mathcal{N})\square}{}_{\delta\wp}(i)q^{2i}}{1-q^{2\mathcal{N}}\prod_{\eta=1}^{\mathcal{N}-1}1-q^{4\eta}}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\mathcal{N})}(\tau;q) = \frac{1}{2^{\mathcal{N}-1}\mathcal{N}!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}}}\oiiint \prod_{i=1}^{\mathcal{N}}\frac{ds_i}{2\pi i\delta_i}$$

$$\bigotimes \prod_{i\neq j}\frac{(\delta_i^{\pm}\delta_j^{\mp};q)_\infty(\delta_i^{\pm}\delta_j^{\pm};q)_\infty(q\delta_i^{\pm}\delta_j^{\mp};q)_\infty(q\delta_i^{\pm}\delta_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}\bigotimes\left[\sum_{i=1}^{\mathcal{N}}(\delta_i+\delta_i^{-1})\right]^2$$

$$\langle \mathfrak{T}_{(1,0,\mathcal{N}-1)}\mathfrak{T}_{(1,0,\mathcal{N}-1)}\rangle^{\mathcal{SO}(2\mathcal{N})}(\tau;q) = \frac{1}{2^{\mathcal{N}-2}(\mathcal{N}-1)!}\frac{(q)_\infty^{2\mathcal{N}}}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}}} \otimes \oint\!\!\oint\!\!\oint \prod_{i=1}^{\mathcal{N}} \frac{ds_i}{2\pi i\delta_i}$$

$$\prod_{i\leq j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\bigotimes\frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}+\cdots+\mathfrak{q}^{\eta}}{(1-\mathfrak{q}^{2\mathcal{N}})\prod_{\eta=1}^{\mathcal{N}-1}(1-\mathfrak{q}^{4\eta})}$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^{2(\mathcal{N}-1)})\prod_{\eta=1}^{\mathcal{N}-2}(1-\mathfrak{q}^{4\eta})}$$

$$\underbrace{\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\mathcal{N})}_{\frac{1}{2}\mathfrak{BPS}}}_{2\kappa}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2\mathcal{N}-2)\kappa}\alpha_\kappa^{\mathcal{S}o(2\mathcal{N})\frac{\square}{\square}}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^{2\mathcal{N}})\prod_{\eta=1}^{\mathcal{N}-1}(1-\mathfrak{q}^{4\eta})}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\mathcal{N})^-}(\tau;q) = \frac{1}{2^{\mathcal{N}-1}(\mathcal{N}-1)!}\frac{(q)_\infty^{2\mathcal{N}-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}-1}\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty} \otimes \oint\!\!\oint\!\!\oint \prod_{i=1}^{\mathcal{N}-1}\frac{ds_i}{2\pi i\delta_i}$$

$$\frac{(\delta_i^{\pm};q)_\infty(-\delta_i^{\pm};q)_\infty(q\delta_i^{\pm};q)_\infty(-q\delta_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^2\delta_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm};q\right)_\infty}$$

$$\bigotimes\prod_{i\leq j}\frac{(\delta_i^{\pm}\delta_j^{\mp};q)_\infty(\delta_i^{\pm}\delta_j^{\pm};q)_\infty(q\delta_i^{\pm}\delta_j^{\mp};q)_\infty(q\delta_i^{\pm}\delta_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}\bigotimes\left[\sum_{i=1}^{\mathcal{N}-1}(\delta_i+\delta_i^{-1})\right]^2$$

$$\langle \mathfrak{T}_{(1,0,\mathcal{N}-2)}\mathfrak{T}_{(1,0,\mathcal{N}-2)}\rangle^{\mathcal{SO}(2\mathcal{N})^-}(\tau;q)$$
$$= \frac{1}{2^{\mathcal{N}-2}(\mathcal{N}-2)!}\frac{(q)_\infty^{2\mathcal{N}-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty^{\mathcal{N}-1}\left(-q^{\frac{1}{2}}\tau^{\pm 2};q\right)_\infty} \otimes \oint\!\!\oint\!\!\oint \prod_{i=1}^{\mathcal{N}-1}\frac{ds_i}{2\pi i\delta_i}$$

$$\bigotimes\frac{\left(q^{\frac{1}{2}\delta_{i,1}}\delta_i^{\pm};q\right)_\infty\left(q^{-\frac{1}{2}\delta_{i,1}}\delta_i^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}\delta_i^{\pm};q\right)_\infty\left(-q^{1+\frac{1}{2}\delta_{i,1}}\delta_i^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i,1})}\tau^2\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}\tau^2\delta_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i,1})}\tau^{-2}\delta_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}\tau^{-2}\delta_i^{\pm};q\right)_\infty}$$

$$\prod_{i\leq j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^2\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^2\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\otimes \frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})}\tau^{-2}\delta_i^{\pm}\delta_j^{\pm};q\right)_\infty}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\mathcal{N})^-}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathfrak{T}_{(1,0,\mathcal{N}^{-2})}\mathfrak{T}_{(1,0,\mathcal{N}^{-2})}\rangle^{\mathcal{SO}(2\mathcal{N})^-}_{\frac{1}{2}\mathfrak{BPS}}(q) =$$

$$\frac{1+q^2+\cdots+q^{2\mathcal{N}-4}}{\prod_{\eta=1}^{\mathcal{N}-1}(1-q^{4\eta})} = \frac{1-q^{2(\mathcal{N}-1)}}{1-q^2}\prod_{\eta=1}^{\mathcal{N}-1}\frac{1}{1-q^{4\eta}}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{O}(2\mathcal{N})^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1+q^2+\cdots+q^{2\mathcal{N}-4}}{\prod_{\eta=1}^{\mathcal{N}}(1-q^{4\eta})} == \frac{1}{1-q^2}\prod_{\eta=1}^{\mathcal{N}-1}\frac{1}{1-q^{4\eta}}$$

$$\mathfrak{T}^{\mathcal{O}(2\mathcal{N})^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \prod_{\eta=1}^{\mathcal{N}-1}\frac{1}{1-q^{4\eta}}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{O}(2\mathcal{N})^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1-q^{4\mathcal{N}}}{1-q^2}$$

$$\mathfrak{T}^{\mathcal{SO}(2\infty+1)}(\tau;q) = \mathfrak{T}^{\mathcal{USp}(2\infty)}(\tau;q) = \mathfrak{T}^{\mathcal{SO}(2\infty)}(\tau;q) = \mathfrak{T}^{\mathcal{O}(2\infty)^+}(\tau;q)$$

$$= \prod_{\eta,m,\ell=0}\frac{\left(1-q^{\eta+m+\ell+\frac{3}{2}}\tau^{-4m+4\ell\pm2}\right)^2}{(1-q^{\eta+m+\ell+1}\tau^{-4m+4\ell\pm4})(1-q^{\eta+m+1}\tau^{-4m+4\ell})(1-q^{\eta+m+\ell+3}\tau^{-4m+4\ell})}$$

$$i^{\mathcal{AdS}_5\times\mathbb{RP}^5}(\tau;q) = \frac{q^{\frac{1}{2}}(\tau^2+\tau^{-2})-q-q^2}{(1-q\tau^4)(1-q\tau^{-4})} - \frac{q^{\frac{1}{2}}(\tau^2+\tau^{-2})}{\left(1+q^{\frac{1}{2}}\tau^2\right)\left(1+q^{\frac{1}{2}}\tau^{-2}\right)(1-q)}$$

$$i^{\chi}(\tau;q) = Tr(-1)^{\mathcal{F}}q^{\frac{\hbar+j}{2}}\tau^{2(q_2-q_3)}$$

$$i^{\chi}_{\frac{1}{2}\mathfrak{BPS}}(q) = Tr(-1)^{\mathcal{F}}q^{2(q_2-q_3)}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\infty)}_{\frac{1}{2}\mathfrak{BPS},c}(q)$$

$$= \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{O}(2\infty)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{1}{1-q^2}$$

$$\langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\infty+1)}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{USp}(2\infty)}(\tau;q) = \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{SO}(2\infty)}(\tau;q)$$

$$= \langle \mathcal{W}_{\square}\mathcal{W}_{\square}\rangle^{\mathcal{O}(2\infty)^+}(\tau;q) = \frac{1-q}{\left(1-q^{\frac{1}{2}}\tau^2\right)\left(1-q^{\frac{1}{2}}\tau^{-2}\right)}$$

$$i^{string}(\tau;q) = -q + q^{\frac{1}{2}}\tau^2 - q^{\frac{1}{2}}\tau^{-2}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\square\square}\rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle^{\mathcal{O}(2\infty)^+}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2}{(1-q^4)} = q^2+\cdots+q^{\eta}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\Box\Box} \mathcal{W}_{\Box\Box} \rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{O}(2\infty)^{+}}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1 + q^2 + q^4}{(1 - q^4)^2} = q^2 + \cdots + q^{\eta}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\Box\Box} \mathcal{W}_{\Box\Box} \rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{O}(2\infty)^{+}}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1 - q^2)(1 - q^4)} = q^2 + \cdots + q^{\eta}$$

$$\langle \mathcal{W}_{\lambda} \mathcal{W}_{\lambda} \rangle^{\mathcal{G}}_{\frac{1}{2}\mathfrak{BPS},c}(q) := \langle \mathcal{W}_{\lambda} \mathcal{W}_{\lambda} \rangle^{\mathcal{G}}_{\frac{1}{2}\mathfrak{BPS}} - \langle \mathcal{W}_{\lambda} \rangle^{\mathcal{G}}_{\frac{1}{2}\mathfrak{BPS},c}(q)^2$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\Box\Box} \rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{O}(2\infty)^{+}}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^{\frac{1}{2}}(\tau^2 + \tau^{-2}) - q - q^2}{(1 - q\tau^4)(1 - q\tau^{-4})}$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\Box\Box} \mathcal{W}_{\Box\Box} \rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{O}(2\infty)^{+}}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{1}{(1 - q\tau^4)(1 - q\tau^{-4})}\Big(1 + (\tau^2 + \tau^{-2})q^{\frac{1}{2}} + (3 + \tau^4 + \tau^{-4})q$$
$$- 3(\tau^2 + \tau^{-2})q^{\frac{3}{2}} - (\tau^2 + \tau^{-2})q^2 - 3(\tau^2 + \tau^{-2})q^{\frac{5}{2}} + (3 + \tau^4 + \tau^{-4})q^3$$
$$+ (\tau^2 + \tau^{-2})q^{\frac{7}{2}} + q^4\Big)$$

$$\langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{SO}(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\Box\Box} \mathcal{W}_{\Box\Box} \rangle^{\mathcal{USp}(2\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \rangle^{\mathcal{O}(2\infty)^{+}}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \frac{(1 - q)\left(1 + q - q^{\frac{3}{2}}(\tau^2 + \tau^{-2})\right)}{\left(1 - q^{\frac{1}{2}}\tau^2\right)\left(1 - q^{\frac{1}{2}}\tau^{-2}\right)(1 - q\tau^4)(1 - q\tau^{-4})}$$

$$\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle^{Spin(2\infty+1)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle^{Spin(4\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\overline{\delta\wp}} \rangle^{Spin(4\infty+2)}_{\frac{1}{2}\mathfrak{BPS}}(q)$$

$$= \prod_{\eta=1}^{\infty} \frac{1}{1 - q^{4\eta-2}}$$

$$\langle \mathcal{W}_{\delta\wp} \mathcal{W}_{\delta\wp} \rangle^{Spin(\infty)}_{\frac{1}{2}\mathfrak{BPS}}(q) = \sum_{\eta\geq 0} d^{(\mathfrak{H})}_{\{\delta\wp,\delta\wp\}}(\eta) q^{2\eta}$$

$$i^{fat\ string}_{\frac{1}{2}\mathfrak{BPS}}(q) = \frac{q^2}{1 - q^4} = q^2 + \cdots + q^{\eta}$$

$$\delta_{\mathfrak{D}5} = \mathcal{T}_5 \iiint d^6\sigma \sqrt{\det(g + 2\pi\alpha'\mathcal{F})} - i\mathcal{T}_5 \iiint 2\pi\alpha'\mathcal{F} \wedge \mathcal{C}_{(4)}$$

$$\underset{\widetilde{\mathfrak{D}5}}{\delta_{\mathcal{A}d\mathcal{S}_2\times\mathbb{RP}^4}} = \mathcal{T}_5 \iiint d^6\sigma\sqrt{\det g} = \mathcal{T}_5 \operatorname{Vol}(\mathcal{A}d\mathcal{S}_2)\operatorname{Vol}(\mathbb{RP}^4)$$

$$ds^2_{\mathcal{A}d\mathcal{S}_2} = \frac{1}{r^2}(-d\tau^2 + dr^2), ds_4^2 = g_{ij}d\sigma^i d\sigma^j$$

$$\delta = \mathcal{T}_5 \iiint d^6\sigma\sqrt{g^{(4)}}\frac{1}{2}\frac{1}{r^2}\left[r^2(\partial_\tau\phi)^2 - r^2(\partial_r\phi)^2 + \left(\nabla_i\phi\nabla^i\phi - 4\phi^2\right)\right]$$

$$\phi(\tau, r, \Theta) = \sum_\omega \phi_\omega(\tau, r)\,\Upsilon^\omega(\Theta)$$

$$\delta = \mathcal{T}_5\sum_\omega \frac{3}{2}\pi^2 \iiint d^6\sigma\frac{1}{r^2}\left[r^2(\partial_\tau\phi_\omega)^2 - r^2(\partial_r\phi_\omega)^2 - \omega(\omega+1)\phi_\omega^2\right]$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\infty+1)}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4\infty)}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(4\infty+2)}(\tau;q)$$
$$= \prod_{\eta=0}^{\infty}\prod_{m=0}^{\infty}\frac{(1-q^{1+\eta+m}\tau^{4\eta-4m})(1-q^{2+\eta+m}\tau^{4\eta-4m})}{\left(1-q^{\frac{1}{2}+\eta+m}\tau^{2+4\eta-4m}\right)\left(1-q^{\frac{1}{2}+\eta+m}\tau^{-2+4\eta+4m}\right)}$$

$$\langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(2\infty+1)}(\tau;q) = \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\delta\wp}\rangle^{Spin(4\infty)}(\tau;q)$$
$$= \langle \mathcal{W}_{\delta\wp}\mathcal{W}_{\overline{\delta\wp}}\rangle^{Spin(4\infty+2)}(\tau;q)\prod_{\eta=1}^{\infty}\frac{(1-q^\eta)^{2\eta-1}}{(1-q^{\eta-1/2})^{2\eta}} = q^2 + \cdots + q^\eta$$

$$i^{fat\,string}(q) = \frac{q^{\frac{1}{2}}(\tau^2+\tau^{-2}) - q - q^2}{(1-q\tau^4)(1-q\tau^{-4})}$$

3. Modelo Yang – Mills – Higgs para espacios cuánticos relativistas.

3.1. Cálculos estocásticos.

$$\partial_t \mathrm{A}_i = \Delta\mathrm{A}_i + \left[\mathrm{A}_j, 2\partial_j\mathrm{A}_i - \partial_i\mathrm{A}_j + \left[\mathrm{A}_j; \mathrm{A}_i\right]\right] + (\mathfrak{C}_\mathrm{A}^\varepsilon\mathrm{A})_i$$
$$+ \xi_i^\varepsilon \lim_{\varepsilon\downarrow 0}\left|\mathfrak{C}_\mathrm{A}^\varepsilon - \overline{\mathfrak{C}}_\mathrm{A}^\varepsilon\right|\left|\mathfrak{E}\mathcal{W}_\ell[\mathcal{F}_s(\mathrm{A}^\alpha)] - \mathfrak{E}\mathcal{W}_\ell\left[\mathcal{F}_s\left(\mathrm{A}^\beta\right)\right]\right|$$

$$\chi^\varepsilon(t,\chi) = \varepsilon^{-5}\chi(\varepsilon^{-2}t, \varepsilon^{-1}\chi), \xi = g^3 \oplus \mathcal{V}$$

$$\partial_t\mathrm{A}_i = \Delta\mathrm{A}_i + \left[\mathrm{A}_j, 2\partial_j\mathrm{A}_i - \partial_i\mathrm{A}_j + \left[\mathrm{A}_j;\mathrm{A}_i\right]\right] - \mathfrak{B}\left((\partial_i\phi + \mathrm{A}_i\phi)\otimes\phi\right) + (\mathfrak{C}_\mathrm{A}^\varepsilon\mathrm{A})_i + \xi_i^\varepsilon, \partial_t\phi$$
$$= \Delta\phi + 2\mathrm{A}_j\partial_j\phi + \mathrm{A}_j^2\phi - |\phi|^2\phi + \mathfrak{C}_\Phi^\varepsilon\Phi + \xi_\mathfrak{H}^\varepsilon\left(\mathrm{A}(0), \phi(0)\right) = (\alpha, \varphi) \in \mathfrak{C}^\infty$$

$$\langle\mathfrak{B}(\mu\otimes\nu), \hbar\rangle_g = \langle\mu, \hbar\nu\rangle_\mathcal{V}$$

$$\partial_t\chi = \Delta\chi + \chi\partial\chi + \chi^3 + \mathfrak{C}^\varepsilon\chi + \chi^\varepsilon \circledast \xi, \chi \coloneqq (\mathrm{A}, \phi): [0, \mathcal{T}] \times \mathcal{T}^3 \mapsto \xi$$

$$\mathfrak{C} = \{\mathfrak{C}^\varepsilon\}_{\varepsilon\in(0,1)} = \left\{\mathfrak{C}_\mathrm{A}^\varepsilon, \mathfrak{C}_\Phi^\varepsilon\right\}_{\varepsilon\in(0,1)}, \mathfrak{G}^\varrho \stackrel{\text{def}}{=} \mathfrak{G}^\varrho(\mathcal{T}^3, \mathfrak{G}), g\cdot\mathrm{A} \triangleq \mathrm{Ad}_g(\mathrm{A}) - (dg)g^{-1}, g\cdot\Phi \triangleq g\phi$$

$$g\cdot\mathrm{SYMH}\left(\mathfrak{C}, (\alpha,\varphi)\right)\underline{\underline{\mathrm{law}}}\,\mathrm{SYMH}\left(\mathfrak{C}, g(0)\cdot(\alpha,\varphi)\right)$$

$$g^{-1}(\partial_t g) = \partial_j\left(g^{-1}\partial_j g\right) + \left[\mathrm{A}_j, g^{-1}\partial_j g\right], \breve{\mathfrak{C}} \in \mathcal{L}_\mathcal{G}(g)$$

$$\left[\mathrm{SYMH}\left(\mathfrak{C},(\alpha,\varphi)\right)\right]\underline{\underline{\mathrm{law}}}\left[\mathrm{SYMH}\left(\mathfrak{C},g(0)\cdot(\alpha,\varphi)\right)\right]$$

$$\partial_s \mathrm{A}_i = \Delta \mathrm{A}_i + \left[\mathrm{A}_j, 2\partial_j \mathrm{A}_i - \partial_i \mathrm{A}_j + \left[\mathrm{A}_j; \mathrm{A}_i\right]\right] \mathrm{A}(0) = \alpha$$

$$d\mathcal{Y}_t = \mathcal{Y}_t d\ell_{\mathrm{A}}, \ell_{\mathrm{A}}(t) = \iiint_0^t \langle \mathrm{A}(\ell_s), \dot{\ell}_s \rangle ds$$

$$\mathcal{W}_\ell(\mathrm{A}) = Tr\, \mathrm{hol}(\mathrm{A}, \ell)$$

$$\mathring{\tilde{\mathfrak{C}}}_{\mathrm{A}} = \tilde{\mathfrak{C}} + c \in \mathcal{L}_{\mathcal{G}}(\mathcal{g}^3), \mathfrak{C}^\varepsilon = \left(\mathfrak{C}^\varepsilon_{\mathcal{YM}} + \mathring{\tilde{\mathfrak{C}}}_{\mathrm{A}}, \mathfrak{C}^\varepsilon_\Phi\right) \in \mathcal{L}_{\mathcal{G}}(\mathcal{g}^3) \oplus \mathcal{L}_{\mathcal{G}}(\mathcal{V})$$

$$\chi = (\alpha, \varphi) \in \mathfrak{G}^\infty(\mathcal{T}^3, \xi), g \in \mathcal{G}^\infty$$

$$\mathrm{SYMH}(\mathfrak{C}, \chi) = \left(\mathrm{A}^{(1)}, \phi^{(1)}\right), \mathrm{SYMH}(\mathfrak{C}, g\cdot\chi) = \left(\mathrm{A}^{(2)}, \phi^{(2)}\right)$$

$$\left|\mathfrak{E}\mathcal{W}_\ell\left[\mathcal{F}_s\left(\mathrm{A}_t^{(1)}\right)\right] - \mathfrak{E}\mathcal{W}_\ell\left[\mathcal{F}_s\left(\mathrm{A}_t^{(2)}\right)\right]\right| \geq \sigma t^{1+r}$$

$$\left|\mathfrak{E}\mathcal{W}_\ell[\mathcal{F}_s(\mathrm{A}_t)] - \mathfrak{E}\mathcal{W}_\ell[\mathcal{F}_s(\tilde{\mathrm{A}}_t)]\right| \geq \sigma t^{1+r}$$

$$\partial_t \tilde{\chi} = \Delta\tilde{\chi} + \tilde{\chi}\partial\tilde{\chi} + \tilde{\chi}^3 + \chi^\varepsilon \ast \xi + \mathfrak{C}^\varepsilon \tilde{\chi} + (c d\tilde{g}\tilde{g}^{-1}, 0)$$

$$\partial_t \tilde{g} = \Delta\tilde{g} - (\partial_j \tilde{g})\tilde{g}^{-1}(\partial_j \tilde{g}) + [\tilde{\mathrm{A}}_j, (\partial_j \tilde{g})\tilde{g}^{-1}]\tilde{g}$$

$$(\partial_t g)g^{-1} = \partial_j\left((\partial_j g)g^{-1}\right) + [\mathfrak{B}_j, (\partial_j g)g^{-1}]$$

$$\partial_t \gamma = \Delta\gamma + \gamma\partial\gamma + \gamma^3 + \mathfrak{C}^\varepsilon\gamma + (\mathfrak{C}^\varepsilon d g g^{-1}, 0) + \mathrm{Ad}_{\mathcal{g}}(\chi^\varepsilon \ast \xi)$$

$$\partial_t \bar{\chi} = \Delta\bar{\chi} + \bar{\chi}\partial\bar{\chi} + \bar{\chi}^3 + \mathfrak{C}^\varepsilon\bar{\chi} + (c d\bar{g}\bar{g}^{-1}, 0) + \chi^\varepsilon \ast \left(\mathrm{Ad}_{\mathcal{g}}\xi\right)$$

$$\mathfrak{E}\mathcal{W}_\ell\left[\mathcal{F}_s\left(\mathrm{A}_t^{(1)}\right)\right] = \mathfrak{E}\mathcal{W}_\ell[\mathcal{F}_s(\tilde{\mathrm{A}}_t)] + \mathcal{O}(t^{\mathcal{M}})$$

$$\left|\mathfrak{E}\mathcal{W}_\ell[\mathcal{F}_s(\tilde{\mathrm{A}}_t)] - \mathfrak{E}\mathcal{W}_\ell\left[\mathcal{F}_s\left(\mathrm{A}_t^{(2)}\right)\right]\right| \geq \sigma t^{1+r}$$

$$\mathcal{P}_t \star f = \iiint_0^t \mathcal{P}_{t-s} f_s ds$$

$$|f|_{\mathfrak{C}^{\mathfrak{B}}} = \sup_{s\in(0,1)} s^{-\beta/2} |\mathcal{P}_s f|_\infty |f|_{\mathfrak{C}^{\mathfrak{B}}(\mathbb{R}^4)} \overset{\mathrm{def}}{=} \max_{|\kappa|\leq\lfloor\beta\rfloor} |\partial^\kappa f|_{\mathfrak{C}^{\mathfrak{B}-\lfloor\beta\rfloor}} \leq \infty$$

$$|f|_{\mathfrak{C}^\eta} = \sup_{\chi\neq\gamma} |\chi-\gamma|^{-\eta}|f(\chi) - f(\gamma)|$$

$$\mathcal{O} = [-1,2] \times \mathcal{T}^3$$

$$|\xi|_{\mathfrak{C}^{\mathfrak{B}}(\mathcal{O})} = \sup_{z\in\mathcal{O}} \sup_{\varphi\in\mathfrak{B}^r} \sup_{\lambda\in\mathfrak{B}^r(0,1]} \lambda^{-\beta}\left|\langle\xi, \varphi_z^\lambda\rangle\right|$$

$$\varphi^\lambda_{(s,y)}(t,\chi) = \lambda^{-5}\varphi\left((t-s)\lambda^{-2}, (\chi-\gamma)\lambda^{-1}\right)$$

$$\partial_t \chi = \Delta\chi + \chi\partial\chi + \chi^3 + \chi^\varepsilon \circledast \xi + \mathfrak{C}^\varepsilon \chi + (c\hbar, 0), \partial_t \hbar_i$$
$$= \Delta\hbar_i - [\hbar_j, \partial_j \hbar_i] + \big[[\mathrm{A}_j, \hbar_j], \hbar_i\big] + \partial_i [\mathrm{A}_j, \hbar_j]$$

$$|f_1 f_2|_{\mathfrak{D}_\alpha^{\gamma,\eta}} \lesssim |f_1|_{\mathfrak{D}_{\alpha_1}^{\gamma_1,\eta_1}} |f_2|_{\mathfrak{D}_{\alpha_2}^{\gamma_2,\eta_2}} |\partial f|_{\mathfrak{D}_{\alpha-1}^{\gamma-1,\eta-1}} |f|_{\mathfrak{D}_\alpha^{\gamma,\eta}}$$

$$\chi = \mathcal{P}\chi(0) + \psi + \mathcal{P}1_+\left\{\chi\partial\chi + \chi^3 + \mathring{\tilde{\mathfrak{C}}}\chi + c\mathcal{H}\right\} \stackrel{\text{def}}{=} \mathcal{P}\chi(0) + \psi + \mathcal{P}1_+\{\mathcal{Q}^{\mathcal{YMH}}(\chi) + c\mathcal{H}\}, \mathfrak{H}$$
$$= \mathcal{P}\hbar(0) + \mathcal{P}1_+(\mathfrak{H}\partial\mathfrak{H} + \chi\mathfrak{H}^2) + \mathcal{P}'1_+(\chi\mathfrak{H})$$

$$\psi = \mathcal{P}^{1+\xi}1_+\Xi \in \mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}$$

$$\gamma = \mathcal{P}\chi(0) + \mathcal{P}1_+\left\{\chi^3 + c\mathcal{H} + \mathring{\tilde{\mathfrak{C}}}\chi\right\} + \mathcal{P}^{\psi\partial\psi}(\psi\partial\psi) + \mathcal{P}1_+(\gamma\partial\psi + \psi\partial\gamma + \gamma\partial\gamma)$$
$$= \mathcal{P}\chi(0) + \mathcal{P}1_+\{\tilde{\mathcal{Q}}^{\mathcal{YMH}}(\gamma) + c\mathcal{H}\} + \mathcal{P}^{\psi\partial\psi}(\psi\partial\psi)$$

$$\tilde{\mathcal{Q}}^{\mathcal{YMH}}(\gamma) = \chi^3 + \mathring{\tilde{\mathfrak{C}}}\chi + \gamma\partial\psi + \psi\partial\gamma + \gamma\partial\gamma, \tilde{\mathcal{Q}}^{\mathcal{YMH}} : \mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}$$
$$\nrightarrow \mathfrak{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}, \mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega} \otimes \mathfrak{D}_{-\frac{3}{2}-2\kappa}^{\frac{1}{2}+2\kappa,-\frac{3}{2}-\kappa} \nleftarrow \mathfrak{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}$$

$$(\gamma, \mathfrak{H}) \in \mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega} \otimes \mathfrak{D}_0^{1+2\kappa,0}$$

$$|\mathcal{P}1_+ f|_{\mathfrak{D}^{\gamma+2,\bar{\eta}}} \cong t^{\frac{\theta}{2}} |f|_{\mathfrak{D}^{\gamma,\eta}}, |\mathcal{P}'1_+ f|_{\mathfrak{D}^{\gamma+1,\bar{\eta}-1}} \cong t^{\frac{\theta}{2}} |f|_{\mathfrak{D}^{\gamma,\eta}}$$

$$|\mathcal{P}^\omega 1_+ f|_{\mathfrak{D}^{\gamma+2,\bar{\eta}}} \cong t^{\frac{\theta}{2}} \left(|f|_{\mathfrak{D}^{\gamma,\eta}} + |\omega|_{\mathfrak{C}^{\eta\wedge\alpha}(\mathcal{O})}\right)$$

$$\tau^{-1/q} \cong 2 + |||\mathcal{Z}|||_{\frac{3}{2}+2\kappa;\mathcal{O}} + |\chi(0)|_{\mathfrak{C}^3} + |\hbar(0)|_{\mathfrak{C}^3} + |\mathcal{P} \star 1_+\xi|_{\mathfrak{C}\left([-1,3],\mathfrak{C}^{-\frac{1}{2}-\kappa}\right)}$$
$$+ |\mathcal{P} \star (\psi\partial\psi)|_{\mathfrak{C}([-1,3],\mathfrak{C}^{-2\kappa})}$$

$$|\mathcal{P}'1_+(\chi\mathfrak{H})|_{\mathfrak{D}_0^{1+2\kappa-\kappa}} \lesssim t^{\frac{1}{4}} |\chi\mathfrak{H}|_{\mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{1}{2}+\kappa,-\frac{1}{2}-\kappa}} \lesssim t^{\frac{1}{4}} |\chi|_{\mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}} |\mathfrak{H}|_{\mathfrak{D}_0^{1+2\kappa,0}} \lesssim t^{\frac{1}{4}}$$

$$|\mathcal{P}1_+(\mathcal{H}\partial\mathcal{H})|_{\mathfrak{D}_0^{1+2\kappa-\kappa}} \lesssim t, |\mathcal{P}1_+(\chi\mathcal{H}^2)|_{\mathfrak{D}_0^{\frac{3}{2}+\kappa,-\kappa}} \lesssim t^{\frac{3}{4}}$$

$$\gamma = \mathcal{P}\chi(0) + \mathcal{O}_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\left(t^{-\frac{\omega}{2}-\frac{\kappa}{2}}\right), \left|\mathcal{P}1_+\left(\tilde{\mathcal{Q}}^{\mathcal{YMH}}(\gamma)\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{4}-\frac{\kappa}{2}} \left|\tilde{\mathcal{Q}}^{\mathcal{YMH}}(\gamma)\right|_{\mathfrak{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}}$$
$$\lesssim t^{\frac{1}{4}-\frac{\kappa}{2}}$$

$$\left|\mathcal{P}^{\psi\partial\psi}(\psi\partial\psi)\right|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{-\frac{(\kappa+\omega)}{2}}$$

$$|\mathcal{PH}|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq |\mathfrak{P}\mathcal{P}\hbar(0)|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} + \left|\mathcal{P}\mathcal{O}_{\mathfrak{D}_0^{1+2\kappa-\kappa}}\left(t^{\frac{1}{4}}\right)\right|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq t^{-\frac{\omega}{2}}|\mathcal{P}\hbar(0)|_{\mathfrak{D}_0^{0+,0}} + t^{\frac{1}{4}+1-\frac{\kappa}{2}-\frac{\omega}{2}} \preccurlyeq t^{1-\frac{\omega}{2}}$$

$$\mathfrak{B}_0 = \mathcal{P}\chi(0) + \psi, \hbar_0 = \mathcal{P}\hbar(0)$$

$$\mathfrak{B}_\eta = \mathcal{P}\left(\mathring{\tilde{\mathfrak{C}}}\mathfrak{B}_{\eta-4}1_{\eta\geq 4}\right) + \sum_{\kappa_1+\kappa_2=\eta-1}\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\partial\mathfrak{B}_{\kappa_2}\right) + \sum_{\kappa_1+\kappa_2+\kappa_3=\eta-2}\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\mathfrak{B}_{\kappa_2}\mathfrak{B}_{\kappa_3}\right)$$

$$\chi = \sum_{i=1}^{\eta}\mathfrak{B}_i + c1_{\eta=5}\mathcal{P}\hbar_0 + r_\eta, \mathcal{H} = \hbar_0 + q_0, r_0 = \mathcal{O}_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\left(t^{-\frac{\omega}{2}-\frac{\kappa}{2}}\right), q_0 = \mathcal{O}_{\mathfrak{D}_0^{1+2\kappa,-\kappa}}\left(t^{\frac{1}{4}}\right)$$

$$\eta(0) = -\frac{1}{2} - \kappa, \eta(n) = -\frac{1}{2} + 2\kappa(1 \leq \eta \leq 5), \mathscr{b}(0) = 0, \mathscr{b}(\eta) = \left(\frac{1}{4} - \frac{\kappa}{2}\right)\eta - \frac{3\kappa}{2}(1 \leq \eta \leq 5)$$

$$\left|\mathfrak{B}_\eta\right|_{\mathfrak{D}_{\alpha(n)}^{\frac{3}{2}+2\kappa,\eta(n)}} \preccurlyeq t^{\mathscr{b}(\eta)}\forall 0 \leq \eta \leq 5, r_0 = \mathcal{O}_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\left(t^{-\frac{\omega}{2}-\frac{\kappa}{2}}\right), r_\eta = \mathcal{O}_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}}\left(t^{\frac{(\eta+1)}{4}-\kappa_\eta}\right)\forall 1 \leq \eta \leq 5, \kappa_\eta \stackrel{\text{def}}{=} \frac{1}{2}\left(\omega + \frac{1}{2}\right) + \left(1 + \frac{\eta}{2}\right)\kappa$$

$$\frac{1}{2}\eta(n) + \mathscr{b}(\eta) = -\frac{1}{4} - \frac{\kappa}{2} + \left(\frac{1}{4} - \frac{\kappa}{2}\right)\eta$$

$$\left|\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\partial\mathfrak{B}_{\kappa_2}\right)\right|_{\mathfrak{D}_{\alpha(\eta)}^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}} \preccurlyeq t^{\frac{1}{2}\left(\eta(\kappa_1)+\eta(\kappa_2)+\frac{3}{2}-2\kappa\right)}\left|\mathfrak{B}_{\kappa_1}\partial\mathfrak{B}_{\kappa_2}\right|_{\mathfrak{D}_{\alpha(\kappa_1)+\alpha(\kappa_2)-1}^{\kappa,\eta(\kappa_1)+\eta(\kappa_2)-1}}$$
$$\preccurlyeq t^{\frac{1}{2}\left(\eta(\kappa_1)+\eta(\kappa_2)+\frac{3}{2}-2\kappa\right)}\left|\mathfrak{B}_{\kappa_1}\right|_{\mathfrak{D}_{\alpha(\kappa_1)}^{\frac{3}{2}+2\kappa,\eta(\kappa_1)}}\left|\partial\mathfrak{B}_{\kappa_2}\right|_{\mathfrak{D}_{\alpha(\kappa_2)-1}^{\frac{1}{2}+2\kappa,\eta(\kappa_2)-1}}$$
$$\preccurlyeq t^{\frac{1}{2}\left(\eta(\kappa_1)+\eta(\kappa_2)+\frac{3}{2}-2\kappa\right)} \circledast t^{\mathscr{b}(\kappa_1)} \circledast t^{\mathscr{b}(\kappa_2)} = t^{\mathscr{b}(\eta)}$$

$$\left|\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\mathfrak{B}_{\kappa_2}\mathfrak{B}_{\kappa_3}\right)\right|_{\mathfrak{D}_{\alpha(\eta)}^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}} \preccurlyeq t^{\frac{1}{2}\left(\sum_i\eta(\kappa_i)+\frac{5}{2}-2\kappa\right)}\prod_{i=1}^{3}\left|\mathfrak{B}_{\kappa_i}\right|_{\mathfrak{D}_{\alpha(\kappa_i)}^{\frac{3}{2}+2\kappa,\eta(\kappa_i)}}$$
$$\preccurlyeq t^{\frac{1}{2}\left(\sum_i\eta(\kappa_i)+\frac{5}{2}-2\kappa\right)}\odot t^{\sum_i\mathscr{b}(\kappa_i)} = t^{\mathscr{b}(\eta)}$$

$$\left|\mathcal{P}\left(\mathfrak{B}_{\eta-4}\right)\right|_{\mathfrak{D}_{\alpha(\eta)}^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}} \preccurlyeq t^{\frac{1}{2}\left(\eta(\eta-4)+\frac{5}{2}-2\kappa\right)}\left|\mathfrak{B}_{\eta-4}\right|_{\mathfrak{D}_{\alpha(\eta-4)}^{\frac{3}{2}+2\kappa,\eta(\eta-4)}} \preccurlyeq t^{\left(\frac{1}{4}-\frac{\kappa}{2}\right)\eta+\frac{\kappa}{2}}\odot t^{\mathscr{b}(\eta)}$$

$$\sum_{i=1}^{\eta}\mathfrak{B}_i + c1_{\eta=5}\mathcal{P}\hbar_0 + r_\eta = \mathfrak{B}_0 + c1_+\{\mathcal{Q}^{\mathcal{YMH}}(\chi) + c\mathcal{H}\}$$

$$r_\eta = \mathcal{P}1_+\left\{\mathcal{Q}^{\mathcal{YMH}}\left(\sum_{i=0}^{\eta-1}\mathfrak{B}_i + r_{\eta-1}\right) + c\left(\hbar_0 + q_0 - 1_{\eta=5}\hbar_0\right)\right\} - \sum_{i=1}^{\eta}\mathfrak{B}_i$$

$$
\begin{aligned}
r_\eta = \mathcal{P}1_+ \Bigg(& \sum_{\kappa_1+\kappa_2\geq\eta} \mathfrak{B}_{\kappa_1}\partial\mathfrak{B}_{\kappa_2} + \sum_{\kappa_1+\kappa_2+\kappa_3\geq\eta-1} \mathfrak{B}_{\kappa_1}\mathfrak{B}_{\kappa_2}\mathfrak{B}_{\kappa_3} + r_{\eta-1}\partial r_{\eta-1} + \left(\sum_{i=0}^{\eta-1}\mathfrak{B}_i\right)\partial r_{\eta-1} \\
& + r_{\eta-1}\partial\left(\sum_{i=0}^{\eta-1}\mathfrak{B}_i\right) + r_{\eta-1}^3 + 3r_{\eta-1}^2\left(\sum_{i=0}^{\eta-1}\mathfrak{B}_i\right) + 3r_{\eta-1}\left(\sum_{i=0}^{\eta-1}\mathfrak{B}_i\right)^2 \\
& + c\left(\hbar_0 + q_0 - 1_{\eta=5}\hbar_0\right) + \mathring{\tilde{\mathfrak{C}}}\left(\sum_{i=(\eta-3)\vee 0}^{\eta-1}\mathfrak{B}_i + r_{\eta-1}\right)\Bigg)
\end{aligned}
$$

$$
\left|\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\partial\mathfrak{B}_{\kappa_2}\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq t^{\frac{1}{2}(\eta(\kappa_1)+\eta(\kappa_2)+1-\omega)} \circledast t^{\mathscr{b}(\kappa_1)} \circledast t^{\mathscr{b}(\kappa_2)} \preccurlyeq t^{-\frac{\omega}{2}-\kappa+1\left(\frac{1}{4}-\frac{\kappa}{2}\right)\eta} = t^{\frac{(\eta+1)}{4-\kappa_\eta}}
$$

$$
\left|\mathcal{P}\left(\mathfrak{B}_{\kappa_1}\mathfrak{B}_{\kappa_2}\mathfrak{B}_{\kappa_3}\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq t^{\frac{1}{2}(\sum_i\eta(\kappa_i)+2-\omega)} \circledast t^{\sum_i\mathscr{b}(\kappa_i)} \preccurlyeq t^{\frac{1}{4}-\frac{\omega}{2}-\frac{3}{2}\kappa+\left(\frac{1}{4}-\frac{\kappa}{2}\right)(\eta-1)} = t^{\frac{(\eta+1)}{4-\kappa_\eta}}
$$

$$
\left|\mathcal{P}(\mathfrak{B}_i)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq t^{\frac{1}{2}(\eta(i)+2-\omega)}\left|\mathfrak{B}_i\right|_{\mathfrak{D}_{\alpha(i)}^{\frac{3}{2}+2\kappa,\eta(i)}} \preccurlyeq t^{(\eta(i)+2-\omega)} \boxtimes t^{\mathscr{b}(i)} = t^{\frac{3}{4}-\frac{\omega}{2}-\frac{\kappa}{2}+\left(\frac{1}{4}-\frac{\kappa}{2}\right)i} \leq t^{\frac{(\eta+1)}{4-\kappa_\eta}}
$$

$$
\begin{aligned}
\left|\mathcal{P}\left(\mathfrak{B}_0\partial r_{\eta-1}\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} & \preccurlyeq t^{\frac{1}{4}-\frac{\kappa}{2}}\left|\mathfrak{B}_0\partial r_{\eta-1}\right|_{\mathfrak{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}} \preccurlyeq t^{\frac{1}{4}-\frac{\kappa}{2}}\left|\mathfrak{B}_0\right|_{\mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}\left|\partial r_{\eta-1}\right|_{\mathfrak{D}_{-1-\kappa}^{\frac{1}{2}+2\kappa,\omega-1}} \\
& \preccurlyeq t^{\frac{1}{4}-\frac{\kappa}{2}} t^{\frac{\eta}{4-\kappa_{\eta-1}}} = t^{\frac{(\eta+1)}{4-\kappa_\eta}}
\end{aligned}
$$

$$
\left|\mathcal{P}\left(r_{\eta-1}\partial\mathfrak{B}_0\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} \preccurlyeq t^{\frac{(\eta+1)}{4-\kappa_\eta}}
$$

$$
\begin{aligned}
\left|\mathcal{P}\left(\mathfrak{B}_0^2 r_{\eta-1}\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} & \preccurlyeq t^{\frac{1}{2}-\kappa}\left|\mathfrak{B}_0^2 r_{\eta-1}\right|_{\mathfrak{D}_{-1-3\kappa}^{\frac{1}{2},\omega-1-2\kappa}} \preccurlyeq t^{\frac{1}{2}-\kappa}\left|\mathfrak{B}_0\right|^2_{\mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}\left|r_{\eta-1}\right|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} \\
& \preccurlyeq t^{\frac{1}{4}-\kappa} t^{\frac{\eta}{4-\kappa_{\eta-1}}} = t^{\frac{1}{4}-\frac{\kappa}{2}} t^{\frac{(\eta+1)}{4-\kappa_\eta}} \preccurlyeq t^{\frac{(\eta+1)}{4-\kappa_\eta}}
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathcal{P}\left(\mathfrak{B}_0 r_{\eta-1}^2\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} & \preccurlyeq t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}\left|\mathfrak{B}_0 r_{\eta-1}^2\right|_{\mathfrak{D}_{-\frac{1}{2}-3\kappa}^{1,2\omega-\frac{1}{2}-\kappa}} \preccurlyeq t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}\left|\mathfrak{B}_0\right|_{\mathfrak{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}\left|r_{\eta-1}\right|^2_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}} \\
& \preccurlyeq t^{\frac{1}{4}-\kappa} t^{\frac{\eta}{4-\kappa_{\eta-1}}} \preccurlyeq t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}\left(t^{\frac{\eta}{4-\kappa_{\eta-1}}}\right)^2 \preccurlyeq t^{\frac{(\eta+1)}{4-\kappa_\eta}}
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathcal{P}\left(r_{\eta-1}\partial r_{\eta-1}\right)\right|_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}} & \preccurlyeq t^{(1+\omega)/2}\left|r_{\eta-1}\partial r_{\eta-1}\right|_{\mathfrak{D}_{-1-2\kappa}^{\frac{1}{2}+\kappa,2\omega-1}} \\
& \preccurlyeq t^{(1+\omega)/2}\left|r_{\eta-1}\right|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\left|\partial r_{\eta-1}\right|_{\mathfrak{D}_{-1-\kappa}^{\frac{1}{2}+2\kappa,\omega-1}} \preccurlyeq t^{(1+\omega)/2}\left(t^{\frac{\eta}{4-\kappa_{\eta-1}}}\right)^2 \preccurlyeq t^{\frac{(\eta+1)}{4-\kappa_\eta}}
\end{aligned}
$$

$$\left|\mathcal{P}\left(r_{\eta-1}^{3}\right)\right|_{\mathfrak{D}_{0}^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{1+\omega}\left|r_{\eta-1}^{3}\right|_{\mathfrak{D}_{-3\kappa}^{\frac{3}{2},3\omega}} \lesssim t^{1+\omega}\left|r_{\eta-1}^{3}\right|_{\mathfrak{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}^{3} \lesssim t^{1+\omega}\left(t^{\frac{\eta}{4-\kappa_{\eta-1}}}\right)^{3} \lesssim t^{\frac{(\eta+1)}{4-\kappa_{\eta}}}$$

$$|\mathcal{P}(q_0)|_{\mathfrak{D}_{0}^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(2-\kappa-\omega)}|q_0|_{\mathfrak{D}_{0}^{\frac{1}{2}+2\kappa,-\kappa}} \lesssim t^{\frac{1}{2}(2-\kappa-\omega)}\otimes t^{\frac{1}{4}} \lesssim t^{\frac{(\eta+1)}{4-\kappa_{\eta}}}$$

$$|\mathcal{P}(\hbar_0)|_{\mathfrak{D}_{0}^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(2-\omega)}|\hbar_0|_{\mathfrak{D}_{0}^{\frac{1}{2}+2\kappa,0}} \lesssim t^{\frac{(\eta+1)}{4-\kappa_{\eta}}}$$

$$\psi = \mathcal{R}\psi = \mathcal{P} \star 1_{+}\xi, (\mathcal{R}\chi)(t) = \chi(0) + \psi_t + \mathcal{P} \star (\psi\partial\psi) + \mathcal{O}_{\mathcal{L}^\infty}\left(t^{\frac{1}{4}-\frac{3\kappa}{2}}\right)$$

$$\mathcal{R}\mathfrak{B}_1 = \mathcal{P} \star (\mathcal{P}\chi(0)\mathcal{P}'\chi(0) + \mathcal{P}\chi(0)\partial\psi + \mathcal{P}'\chi(0)\psi + \psi\partial\psi)$$

$$|\psi|_{\mathcal{L}^\infty} \lesssim t^{-\frac{3}{4}-\frac{\kappa}{2}}, |r_1|_{\mathfrak{D}_{0}^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}-\kappa_1}, |(\mathcal{R}r_1)(t)|_{\mathcal{L}^\infty} \lesssim t^{\frac{1}{2}-\kappa_1+\frac{\omega}{2}} = t^{\frac{1}{4}-\frac{3\kappa}{2}}$$

$$\mathcal{N}(\mathcal{A}): (0,\infty) \mapsto \mathfrak{C}^\infty(\mathfrak{T}^3, \mathcal{g}^3 \otimes (\mathcal{g}^3)^3), \mathcal{N}_s(\mathcal{A}) \stackrel{\text{def}}{=} \mathcal{P}_s\mathcal{A} \otimes \nabla\mathcal{P}_s\mathcal{A}$$

$$\mathcal{L} = \mathfrak{T}^3 \times \left\{v \in \mathbb{R}^4 : \leq \frac{1}{4}\right\}$$

$$|f|_{\gamma-gr} = \sup_{\ell\in\mathcal{L}} \left|\iiint_{\ell} f\right| / |\ell|^\gamma$$

$$\iiint_{\ell} f = \iiint_{0}^{1} |v| f(\gamma + tv) dt \in \mathcal{F}$$

$$[\![\mathrm{A};\mathrm{B}]\!]_{\gamma,\delta} \triangleq \sup_{s\in(0,1)} s^\delta |\mathcal{N}_s\mathrm{A} - \mathcal{N}_s\mathrm{B}|_{\gamma-gr}, [\![\mathrm{A}]\!]_{\gamma,\delta} = [\![\mathrm{A};0]\!]_{\gamma,\delta}$$

$$[\![\mathrm{A};\mathrm{B}]\!]_{\beta,\delta} \stackrel{\text{m}}{=} \sup_{s\in(0,1)} s^\delta |\mathcal{N}_s\mathrm{A} - \mathcal{N}_s\mathrm{B}|_{\mathfrak{C}^\beta}, |||\mathrm{A}|||_{\alpha,\theta} \triangleq \sup_{s\in(0,1)} |\mathcal{P}_s\mathcal{A}|_{\alpha-gr;\leq s^\theta}, |\mathrm{A}|_{\alpha-gr;\leq r}$$

$$\stackrel{\text{def}}{=} \sup_{\ell\in\mathcal{L},|\ell|\leq r} \frac{(\mathrm{A}(\ell))}{|\ell|^\alpha} |\mathrm{A}|_{\mathfrak{C}^\eta} \lesssim |||\mathrm{A}|||_{\alpha,\theta}, \sum(\mathrm{A}) \stackrel{\text{def}}{=} |||\mathrm{A}-\mathrm{B}|||_{\alpha,\theta} + [\![\mathrm{A};\mathrm{B}]\!]_{\gamma,\delta}$$

$$\lesssim \infty \sum(\mathrm{A},\mathrm{B}) \stackrel{\text{def}}{=} |||\mathrm{A}-\mathrm{B}|||_{\alpha,\theta} + [\![\mathrm{A};\mathrm{B}]\!]_{\gamma,\delta} \lesssim \infty$$

$$\alpha \in \left(0,\frac{1}{2}\right), \theta \geq 0, \gamma \in \left(\frac{1}{2},1\right], \delta \in (0,1), \eta \stackrel{\text{def}}{=} (1+2\theta)(\alpha-1) \geq -\frac{2}{3}, \mu$$

$$\stackrel{\text{def}}{=} \gamma - 1 + 2(1-\delta) \in \left(-\frac{1}{2},0\right) \eta + \mu \geq -1$$

$$|\mathcal{F}_s\mathrm{A}|_\infty \lesssim s^{\frac{\eta}{2}} \sum(\mathrm{A}), |\partial\mathcal{F}_s\mathrm{A}|_\infty \lesssim s^{\frac{\eta}{2}-\frac{1}{2}} \sum(\mathrm{A})$$

$$|\mathcal{P}_s\mathrm{A}|_\infty \lesssim s^{\frac{\eta}{2}} |\mathrm{A}|_{\mathfrak{C}^\eta}, |\partial\mathcal{P}_s\mathrm{A}|_\infty \lesssim s^{\frac{\eta}{2}-\frac{1}{2}} |\mathrm{A}|_{\mathfrak{C}^\eta}$$

$$|\mathcal{P}_s\mathrm{A}|_{\gamma-gr} \lesssim s^\lambda |||\mathrm{A}|||_{\alpha,\theta}^{\zeta} |\mathrm{A}|_{\mathfrak{C}^\eta}^{1-\zeta}$$

$$\lambda \stackrel{\text{def}}{=} (1-\zeta)\eta/2 - \theta(1-\alpha)\zeta$$

$$|\mathcal{P}_s \mathrm{A}|_{\gamma-gr} \lesssim s^{\lambda} |||\mathrm{A}|||_{\alpha,\theta}$$

$$0 \leq \nu \leq \min\left\{\frac{\eta}{2} + \frac{\mu}{2} + \frac{1}{2}, 1 + \frac{3\eta}{2}, \mu + \frac{1}{2}\right\}$$

$$|\mathcal{P}_s \star \mathcal{N}\mathcal{A}|_{\gamma-gr} \lesssim \iiint_0^s |\mathcal{N}_r \mathrm{A}|_{\gamma-gr} dr \lesssim \iiint_0^s r^{-\delta} [\![\mathrm{A}]\!]_{\gamma,\delta} dr \lesssim s^{1-\delta} [\![\mathrm{A}]\!]_{\gamma,\delta}$$

$$\mathcal{F}_s \mathrm{A} = \mathcal{P}_s \mathrm{A} + \mathcal{P}_s \star \mathcal{N}\mathcal{A} + \mathcal{R}_s \mathrm{A}, |\mathcal{R}_s \mathrm{A}|_{\infty} \lesssim s^{\nu} \left(\sum(\mathrm{A}) + \sum(\mathrm{A})^3\right)$$

$$\mathcal{R}_s = \iiint_0^s \mathcal{P}_{s-r} \{(\mathcal{P}_r \mathrm{A} + \mathcal{P}_r \star \mathcal{N}\mathcal{A} + \mathcal{R}_r) \partial (\mathcal{P}_r \mathrm{A} + \mathcal{P}_r \star \mathcal{N}\mathcal{A} + \mathcal{R}_r) + (\mathcal{F}_r \mathrm{A})^3\} dr - \mathcal{P}_s \star \mathcal{N}\mathcal{A}$$

$$|\mathcal{P}_s \star \mathcal{N}\mathcal{A}|_{\infty} \lesssim \iiint_0^s \mathcal{P}_{s-r} |\mathcal{N}_r \mathrm{A}|_{\infty} dr \lesssim \iiint_0^s (s-r)^{\frac{\gamma-1}{2}} |\mathcal{N}_r \mathrm{A}|_{\mathfrak{C}^{\gamma-1}} ds$$

$$\lesssim \iiint_0^s (s-r)^{\frac{\gamma-1}{2}} |\mathcal{N}_r \mathrm{A}|_{\gamma-gr} ds \lesssim \iiint_0^s (s-r)^{\frac{\gamma-1}{2}} r^{-\delta} [\![\mathrm{A}]\!]_{\gamma,\delta} ds \asymp s^{\frac{\mu}{2}} [\![\mathrm{A}]\!]_{\gamma,\delta}$$

$$|\partial \mathcal{P}_s \star \mathcal{N}\mathcal{A}|_{\infty} \lesssim s^{\frac{\mu}{2}-\frac{1}{2}} [\![\mathrm{A}]\!]_{\gamma,\delta} \iiint_0^s \left\{r^{\frac{\eta}{2}+\frac{\mu}{2}-\frac{1}{2}} + r^{\mu-\frac{1}{2}} + r^{\frac{3\eta}{2}}\right\} dr \asymp s^{\frac{\eta}{2}+\frac{\mu}{2}+\frac{1}{2}} + s^{\mu+\frac{1}{2}} + r^{1+\frac{3\eta}{2}}$$

$$|\mathcal{R}|_{\mathcal{B}} \overset{\text{def}}{=} \sup_{s \in (0,\mathcal{T})} \left\{s^{-\nu} |\mathcal{R}_s|_{\infty} + s^{-\nu+\frac{1}{2}} |\partial \mathcal{R}_s|_{\infty}\right\}$$

$$s^{-\nu} |\mathcal{R}|_{\infty} \lesssim \sum(\mathrm{A}) + \sum(\mathrm{A})^3 + s^{\kappa'} |\mathcal{R}|_{\mathcal{B}} \left(\sum(\mathrm{A}) + \sum(\mathrm{A})^2\right) + s^{\kappa''} |\mathcal{R}|_{\mathcal{B}}^2 \left(1 + \sum(\mathrm{A})\right) + s^{1+2\nu} |\mathcal{R}|_{\mathcal{B}}^3$$

$$\kappa' = \min\left\{\frac{\eta}{2} + \frac{1}{2}, 1 + \eta\right\} = \frac{\eta}{2} + \frac{1}{2}$$

$$\kappa'' = \min\left\{\frac{1}{2} + \nu, 1 + \frac{\eta}{2} + \nu\right\} = \frac{1}{2} + \nu$$

$$|\mathcal{R}|_{\mathcal{B}} \lesssim \sum(\mathrm{A}) + \sum(\mathrm{A})^3 + \mathcal{T}^{\kappa} (|\mathcal{R}|_{\mathcal{B}} + |\mathcal{R}|_{\mathcal{B}}^3) \left(1 + \sum(\mathrm{A})^2\right)$$

$$\mathcal{F}_s \mathrm{A} = \mathcal{P}_s \mathrm{A} + \mathcal{O}_{\Omega_{\gamma-gr}\{\mathscr{g},\mathscr{g}\}} \left(s^{\nu} \left(\sum(\mathrm{A}) + \sum(\mathrm{A})^3\right)\right), \mathcal{P}_s \mathrm{A} = \mathcal{O}_{\Omega_{\gamma-gr}} (s^{\lambda})$$

$$\mathcal{F}_s \widetilde{\mathrm{A}} = \mathcal{F}_s \mathrm{A} + \mathcal{P}_s r + \mathcal{O}_{\mathcal{L}^{\infty}\{\mathscr{g},\mathscr{g}\}} \left(s^{\frac{\eta}{2}+\frac{1}{2}} |r|_{\infty}\right)$$

$$\mathcal{F}_s \mathrm{A} + \mathcal{Q}_s = \mathcal{P}_s (\mathcal{A} + r) \iiint_0^s \mathcal{P}_{s-\mu} \left\{(\mathcal{F}_{\mu} \mathrm{A} + \mathcal{Q}_{\mu}) \partial (\mathcal{F}_{\mu} \mathrm{A} + \mathcal{Q}_{\mu}) + (\mathcal{F}_{\mu} \mathrm{A} + \mathcal{Q}_{\mu})^3\right\} d\mu$$

$$\mathcal{Q}_s = \mathcal{P}_s r + \iiint_0^s \mathcal{P}_{s-\mu}\left(\mathcal{Q}_\mu \partial \mathcal{F}_\mu \mathrm{A} + (\mathcal{F}_\mu \mathrm{A})\partial \mathcal{Q}_\mu + \mathcal{Q}_\mu \partial \mathcal{Q}_\mu + (\mathcal{F}_\mu \mathrm{A})^2 \mathcal{Q}_\mu + (\mathcal{F}_\mu \mathrm{A})\mathcal{Q}_\mu^2 + \mathcal{Q}_\mu^3\right) d\mu$$

$$\iiint_0^s \left\{|\mathcal{Q}_\mu|_\infty \mu^{\frac{\eta}{2}-\frac{1}{2}} + \mu^{\frac{\eta}{2}}|\mathcal{Q}_\mu|_{\mathfrak{C}^1} + |\mathcal{Q}_\mu|_\infty |\mathcal{Q}_\mu|_{\mathfrak{C}^1} + \mu^\eta |\mathcal{Q}_\mu|_\infty + \mu^{\frac{\eta}{2}}|\mathcal{Q}_\mu|_\infty^2 + |\mathcal{Q}_\mu|_\infty^3\right\} d\mu$$

$$\lesssim |\mathcal{Q}_\mu|_\infty s^{\frac{\eta}{2}-\frac{1}{2}} + s^{\frac{\eta}{2}+\frac{1}{2}}|\mathcal{Q}_\mu|_{\mathcal{L}_{\frac{1}{2}}^\infty \mathfrak{C}^1} + s^{\frac{1}{2}}|\mathcal{Q}_\mu|_\infty |\mathcal{Q}_\mu|_{\mathcal{L}_{\frac{1}{2}}^\infty \mathfrak{C}^1} + s^{\eta+1}|\mathcal{Q}_\mu|_\infty + s^{\frac{\eta}{2}+1}|\mathcal{Q}|_\infty^2 + s|\mathcal{Q}|_\infty^3$$

$$|\mathcal{Q}|_\infty + |\mathcal{Q}|_{\mathcal{L}_{\frac{1}{2}}^\infty \mathfrak{C}^1} \lesssim |r|_\infty$$

$$\mathcal{Q}_s = \mathcal{P}_s r + \mathcal{O}_{\mathcal{L}^\infty}\left(s^{\frac{\eta}{2}+\frac{1}{2}}|r|_\infty\right)$$

$$\tilde{\chi} = \mathrm{B} + c\bar{\hbar} + \mathcal{O}_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}}\left(t^{\frac{3}{2}-\kappa_5}\right), \tilde{\chi} = \mathrm{B} + c\mathfrak{P}\mathcal{P}\hbar(0) + \mathcal{O}_{\mathfrak{D}_0^{\frac{3}{2}+2\kappa,\omega}}\left(t^{\frac{3}{2}-\kappa_5}\right)$$

$$\tilde{\mathcal{A}} = \mathcal{A} + ct\mathcal{P}_t\hbar(0) + \mathcal{O}_{\mathcal{L}^\infty}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}}\right)$$

$$\mathcal{F}_s\tilde{\mathrm{A}} = \mathcal{F}_s\mathrm{A} + \mathcal{P}_s\left(ct\mathcal{P}_t\hbar(0) + \mathcal{O}_{\mathcal{L}^\infty}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}}\right)\right) + \mathcal{O}_{\mathcal{L}^\infty\{\mathscr{g},\mathscr{g}\}}\left(s^{\frac{\eta}{2}+\frac{1}{2}}t\right)$$

$$|f|_{\wp-var} \stackrel{\text{def}}{=} \sup_{\mathcal{P}\subset[0,1]}\left(\sum_{[s,t]\in\mathcal{P}}|f(t)+f(s)|^{\wp}\right)^{1/\wp}$$

$$d\mathfrak{J}^\gamma(\chi) = \mathfrak{J}^\gamma(\chi)d\gamma(\chi), \mathfrak{J}^\gamma(0) = id$$

$$\mathfrak{J}^{\gamma+\zeta}(1) = \mathfrak{J}^\gamma(1)\iiint_0^1 d\zeta(\chi)\iiint_0^1\iiint_0^\chi\{d\zeta(\chi)d\gamma(\mathscr{y}) + d\gamma(\chi)d\zeta(\mathscr{y})\} + \mathcal{O}\{\upsilon(\omega^2+\omega^{\mathcal{L}-1}) + \omega^{\mathcal{L}} + \omega^{\mathcal{L}+1} + \upsilon^{\mathcal{L}+1} + \upsilon^2(1+\omega+\upsilon+\omega^{\mathcal{L}-3})\}$$

$$\mathfrak{J}^\gamma(1) = id + \mathfrak{J}^\gamma + \iiint_0^1 \cdots \iiint_0^{\chi_{\mathcal{L}-1}} \mathfrak{J}^\gamma(\chi_{\mathcal{L}})\, d\gamma(\chi_{\mathcal{L}})d\gamma(\chi_{\mathcal{L}-1})\cdots d\gamma(\chi_1)$$

$$\mathfrak{J}^{\gamma+\zeta} = \mathfrak{J}^\gamma + \iiint_0^1 d\zeta(\chi)\iiint_0^1\iiint_0^\chi\{d\zeta(\chi)d\gamma(\mathscr{y}) + d\gamma(\chi)d\zeta(\mathscr{y})\} + \mathcal{O}\{\upsilon^2 + \upsilon(\omega^2+\omega^{\mathcal{L}-1})\} + \mathcal{O}\{\upsilon^2(\omega+\upsilon+\omega^{\mathcal{L}-3})\}$$

$$|\ell_\alpha|_{\frac{1}{\gamma}-var} \leq |\ell_\alpha|_{\gamma-\mathcal{H}\ddot{o}l} \leq |\alpha|_{\gamma-gr}$$

$$\mathcal{W}_\ell \mathcal{F}_s \widetilde{\mathrm{A}} = \mathcal{W}_\ell(\mathcal{F}_s\mathrm{A}) + tTr\iiint_\ell c\hbar(0) + tTr \iiint_{[0,1]^2} d\ell_{\mathcal{A}(0)}(\chi_1) d\ell_{c\hbar(0)}(\chi_2)$$
$$+ \mathcal{O}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}} + ts + ts^\lambda \left\|\left|\Psi_t^{\mathcal{YM}}\right|\right\|_{\alpha,\theta} + ts^\lambda \left\|\left|\mathcal{P}_t \star (\psi\partial\psi)^{\mathcal{YM}}\right|\right\|_{\alpha,\theta} + |\mathcal{A}(0)|_{\mathcal{L}^\infty} s^{\frac{\eta}{2}+\frac{1}{2}} t + s^\nu t\left(\sum(\mathrm{A}) + \sum(\mathrm{A})^3\right) + t(\mu^2 + \mu^{\mathcal{L}-1}) + s^{\nu\mathcal{L}} + \mu^{\mathcal{L}} + t^2\mu\right)$$

$$\gamma(\chi) = \iiint_0^\chi (\mathcal{F}_s\mathrm{A})_1(\gamma, 0,0) d\gamma = \ell_{\mathcal{F}_s\mathrm{A}}(\chi)$$

$$\mathfrak{D}_\hbar \overset{\text{def}}{\equiv} ct\mathcal{P}_{t+s}\hbar(0) = ct\hbar(0) + \mathcal{O}_{\mathcal{L}^\infty}\big(t(t+s)\big)$$

$$\mathfrak{D}_{err} \overset{\text{def}}{\equiv} \mathcal{O}_{\mathcal{L}^\infty\{\mathcal{g},\mathcal{g}\}}\left(s^{\frac{\eta}{2}+\frac{1}{2}} t\right) + \mathcal{O}_{\mathcal{L}^\infty}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}}\right)$$

$$\mathcal{W}_\ell(\mathcal{F}_s\mathrm{A}) = Tr\mathfrak{J}^\gamma(1), \mathcal{W}_\ell\big(\mathcal{F}_s\widetilde{\mathrm{A}}\big) = Tr\mathfrak{J}^{\gamma+\zeta}(1), \mathcal{W}_\ell\big(\mathcal{F}_s\widetilde{\mathrm{A}}\big)$$
$$= \mathcal{W}_\ell(\mathcal{F}_s\mathrm{A}) + Tr\left(\iiint_0^1 d\zeta(\chi)\right) + Tr\iiint_0^1 \iiint_0^\chi \{d\zeta(\chi)d\gamma(\mathcal{y}) + d\gamma(\chi)d\zeta(\mathcal{y})\}$$
$$+ \mathcal{O}\{\upsilon(\omega^2 + \omega^{\mathcal{L}-1}) + \omega^{\mathcal{L}} + \omega^{\mathcal{L}+1} + \upsilon^{\mathcal{L}+1} + \upsilon^2(1 + \omega + \upsilon + \omega^{\mathcal{L}-3})\}$$

$$Tr\left(\iiint_0^1 d\zeta(\chi)\right) = tTr\iiint_\ell c\hbar(0) + tTr + \mathcal{O}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}} + ts\right)$$

$$Tr\iiint_0^1 \iiint_0^\chi \{d\zeta(\chi)d\gamma(\mathcal{y}) + d\gamma(\chi)d\zeta(\mathcal{y})\} = \iiint_{[0,1]^2} \{d\zeta(\chi)d\gamma(\mathcal{y})\}$$

$$\mathcal{F}_s\mathrm{A} = \mathcal{P}_s\mathrm{A} + \mathcal{O}_{\gamma-gr}\left(s^\nu\left(\sum(\mathrm{A}) + \sum(\mathrm{A})^3\right)\right)$$

$$\mathcal{A} = \mathcal{A}(0) + \Psi_t^{\mathcal{YM}} + \mathcal{P}_t \star (\psi\partial\psi)^{\mathcal{YM}} + \mathcal{O}_{\mathcal{L}^\infty}\left(t^{\frac{1}{4}-\frac{3\kappa}{2}}\right)$$

$$\iiint_0^1 d\gamma(\chi) = \iiint_{[0,1]^2} d\ell_{\mathcal{A}(0)}(\chi) + \mathcal{O}\left(s^\lambda \left\|\left|\Psi_t^{\mathcal{YM}}\right|\right\|_{\alpha,\theta} + s^\lambda \left\|\left|\mathcal{P}_t \star (\psi\partial\psi)^{\mathcal{YM}}\right|\right\|_{\alpha,\theta}\right)$$
$$+ \mathcal{O}\left(t^{\frac{1}{4}-\frac{3\kappa}{2}} + s^\nu\left(\sum(\mathrm{A}) + \sum(\mathrm{A})^3\right)\right)$$

$$\iiint_0^1 d\zeta(\mathcal{y}) = t\iiint_{[0,1]^2} d\ell_{c\hbar(0)}(\mathcal{y}) + \mathcal{O}\left(t^{\frac{5}{4}-\frac{7\kappa}{2}} + s^{\frac{\eta}{2}+\frac{1}{2}} t\right)$$

$$Tr \iiint_{[0,1]^2} \{d\zeta(\chi)d\gamma(\psi)\}$$

$$= tTr\left(\iiint_{[0,1]^2} d\ell_{\mathcal{A}(0)}(\chi_1)d\ell_{c\hbar(0)}(\chi_2)\right)$$
$$+ \mathcal{O}\left(ts^{\lambda}\left\|\left|\Psi_t^{\mathcal{YM}}\right\|\right|_{\alpha,\theta} + ts^{\lambda}\left\|\left|\mathcal{P}_t \star (\psi\partial\psi)^{\mathcal{YM}}\right\|\right|_{\alpha,\theta} + t^{\frac{5}{4}-\frac{7\kappa}{2}}\right)$$
$$+ \mathcal{O}\left(ts^{\nu}\sum(\mathrm{A}) + \sum(\mathrm{A})^3 + s^{\frac{\eta}{2}+\frac{1}{2}}t|\mathcal{A}(0)|_{\mathcal{L}^\infty}\right)$$

$$\omega = |\gamma|_{\frac{1}{\gamma}-var} \leq |\mathcal{F}_s\mathrm{A}|_{\gamma-gr} = |\mathcal{P}_s\mathrm{A}|_{\gamma-gr} + \mathcal{O}(s^{\nu}) = \mathcal{O}(s^{\nu})\||\mathcal{A}(0)\||_{\alpha,\theta} + \mathcal{O}(s^{\nu})$$

$$\upsilon = |\zeta|_{\frac{1}{\gamma}-var} \leq |\mathfrak{D}_{\hbar} + \mathfrak{D}_{err}|_{\gamma-gr} \lesssim s^{\frac{\eta}{2}+\frac{1}{2}}t + t = \mathcal{O}(t)$$

$$[\![\mathrm{A}+\mathrm{B}]\!]_{\gamma,\delta} \lesssim [\![\mathrm{A}]\!]_{\gamma,\delta} + |\mathrm{B}|_{\mathfrak{C}^{\bar{\eta}}}\left(|\mathrm{A}|_{\mathfrak{C}^{\eta}} + |\mathrm{B}|_{\mathfrak{C}^{\bar{\eta}}}\right)$$

$$\mathcal{N}_s(\mathrm{A}+\mathrm{B}) = \mathcal{N}_s\mathrm{A} + \mathcal{N}_s\mathrm{B} + \mathcal{P}_s\mathrm{A}\otimes\nabla\mathcal{P}_s\mathrm{B} + \mathcal{P}_s\mathrm{B}\otimes\nabla\mathcal{P}_s\mathrm{A}$$

$$|\mathcal{P}_s\mathrm{A}\otimes\nabla\mathcal{P}_s\mathrm{B}|_\infty + |\mathcal{P}_s\mathrm{B}\otimes\nabla\mathcal{P}_s\mathrm{A}|_\infty \lesssim s^{(\eta+\bar{\eta}-1)/2}|\mathrm{A}|_{\mathfrak{C}^{\eta}}|\mathrm{B}|_{\mathfrak{C}^{\bar{\eta}}} \lesssim s^{-\delta}|\mathrm{A}|_{\mathfrak{C}^{\eta}}|\mathrm{B}|_{\mathfrak{C}^{\bar{\eta}}}$$

$$\mathcal{Z}_{s,t} = s^{\delta}\mathcal{N}_s\psi_t$$

$$\mathfrak{E}\left[\left|\sup_{(s,t)\neq(\bar{s},\bar{t})} \frac{|\mathcal{Z}_{s,t} - \mathcal{Z}_{\bar{s},\bar{t}}|_{\gamma-gr}}{(|t-\bar{t}| + |s-\bar{s}|)^{\bar{\kappa}}}\right|^{\wp}\right]^{1/\wp}$$

$$\left(\mathfrak{E}|\mathcal{Z}_{s,t} - \mathcal{Z}_{\bar{s},\bar{t}}|_{\gamma-gr}^{\wp}\right)^{1/\wp} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa/2}$$

$$\mathfrak{E}\left|\sup_{\mathcal{P}\subset[0,1]} t^{\bar{\kappa}}[\![\psi_t]\!]_{\gamma,\delta}\right|^{\wp} \left|\mathfrak{C}_{s,\bar{s};t,\bar{t}}(\chi)\right| \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa}|\chi|^{4\delta-4-2\kappa}\left|\nabla\mathfrak{C}_{s,\bar{s};t,\bar{t}}(\chi)\right|$$
$$\lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa}|\chi-\gamma|^{4\delta-4-2\kappa}$$

$$|\nabla\mathfrak{C}_{r,s}(\chi)| \lesssim |\chi|^{-2}, \left|\nabla\left(\mathfrak{C}_{r,r} - \mathfrak{C}_{r,s}(\chi)\right)\right| \lesssim |r-s|^{\kappa}|\chi|^{-2-2\kappa}$$

$$d\big((\chi,\upsilon),(\overline{\chi,\upsilon})\big) \stackrel{\text{def}}{=} |\chi-\bar{\chi}| \vee |\chi+\upsilon-(\bar{\chi}+\bar{\upsilon})|$$

$$\left(\mathfrak{E}\left|\iiint_{\ell}(\mathcal{Z}_{s,t} - \mathcal{Z}_{\bar{s},\bar{t}})\right|^{\wp}\right)^{1/\wp} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa}|\ell|^{2\delta-1-\kappa}$$

$$\left(\mathfrak{E}\left|\left(\iiint_{\ell} - \iiint_{\bar{\ell}}\right)(\mathcal{Z}_{s,t} - \mathcal{Z}_{\bar{s},\bar{t}})\right|^{\wp}\right)^{1/\wp} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa/2}d|\ell,\bar{\ell}|^{2\delta-3/2-\kappa}$$

$$\mathfrak{E}\left|\iiint_{\ell}(\mathcal{Z}_{s,t}-\mathcal{Z}_{\bar{s},\bar{t}})\right|^2 = |\ell|^2 \iiint_{[0,1]^2}\mathfrak{C}\big((r-\bar{r})v\big)drd\bar{r}$$

$$\lesssim (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{2\delta-2-2\kappa}\iiint_{[0,1]^2}|r-\bar{r}|^{4\delta-4+2\kappa}drd\bar{r}$$

$$\lesssim (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}$$

$$\mathfrak{E}\left|\left(\iiint_{\ell}-\iiint_{\ell}\quad\right)(\mathcal{Z}_{s,t}-\mathcal{Z}_{\bar{s},\bar{t}})\right|^2$$

$$=|\ell|^2\iiint_{[0,1]^2}\{\mathfrak{C}\big((r-\bar{r})v\big)-\mathfrak{C}(rv-\overline{rv})-\mathfrak{C}(rv-\overline{rv})+\mathfrak{C}\big((r-\bar{r})\bar{v}\big)\}drd\bar{r}$$

$$=|\ell|^2\iiint_{[0,1]^2}\{2\mathfrak{C}\big((r-\bar{r})v\big)-2\mathfrak{C}(rv-\overline{rv})\}drd\bar{r}$$

$$(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}\iiint_{0}^{\hbar}r^{4\delta-4-2\kappa}\,dr \asymp (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}\hbar^{4\delta-3-2\kappa}$$

$$=(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}|v-\bar{v}|\iiint_{0}^{1}r^{4\delta-5-2\kappa}\,dr$$

$$\asymp(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-3-2\kappa}|v-\bar{v}|\hbar^{4\delta-4-2\kappa}$$

$$=(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$\mathfrak{E}\left|\left(\iiint_{\ell}-\iiint_{\ell}\quad\right)(\mathcal{Z}_{s,t}-\mathcal{Z}_{\bar{s},\bar{t}})\right|^2\lesssim(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$\left(\mathfrak{E}\left|\iiint_{\ell}\mathrm{A}\right|^{\wp}\right)^{1/\wp}\leq\mathcal{M}_{\wp}|\ell|^{\alpha}$$

$$\left(\mathfrak{E}\left|\left(\iiint_{\ell}-\iiint_{\ell}\quad\right)\mathrm{A}\right|^{\wp}\right)^{1/\wp}\leq\mathcal{M}_{\wp}d|\ell,\bar{\ell}|^{\beta}$$

$$\mathfrak{E}|\mathrm{A}|^{\wp}_{\gamma-gr}\leq\lambda\mathcal{M}_{\wp}$$

$$\sup_{\ell\in\mathcal{D}}\frac{\left|\iiint_{\ell}\mathcal{A}\right|}{|\ell|^{\gamma}}\lesssim\sup_{\alpha\in\mathcal{D}_{\mathrm{N}}}\frac{\left|\iiint_{\alpha}\mathcal{A}\right|}{2^{-\gamma\mathrm{N}/\omega}}+\sup_{\mathrm{N}\geq1}\sup_{\substack{\alpha,\mathscr{b}\in\mathcal{D}_{\mathrm{N}}\\ d(\alpha,\mathscr{b})\leq\mathcal{K}2^{-\mathcal{N}}}}\frac{\left|\left(\iiint_{\alpha}\iiint_{\mathscr{b}}\quad\right)\mathcal{A}\right|}{2^{-\gamma\mathrm{N}/\omega}}$$

$$\mathfrak{E}\left|\sup_{\ell\in\mathcal{D}}\frac{\left|\iiint_{\ell}\ \mathcal{A}\right|}{|\ell|^{\gamma}}\right|\lesssim\mathcal{M}_{\wp}^{\wp}\sum_{\mathcal{N}\geq 1}\left\{2^{\mathcal{N}(6-\wp(\alpha-\gamma)/\omega)}+2^{\mathcal{N}\left(12-\wp(\beta-\gamma/\omega)\right)}\right\}$$

$$|||f|||_{\alpha,\theta}\lesssim|f|_{\mathfrak{C}^{\bar{\eta}}},\mathcal{P}\star(\psi\partial\psi)=\lim_{\varepsilon\downarrow 0}\mathcal{P}\star(\psi_{\varepsilon}\partial\psi_{\varepsilon}),\mathfrak{E}|||\psi_t|||_{\alpha,\theta}^{\wp}+\mathfrak{E}|\mathcal{P}\star(\psi\partial\psi)|_{\mathfrak{C}^{\bar{\eta}}}^{\wp}+\mathfrak{E}[\![\psi]\!]_{\gamma,\delta}^{\wp} = \mathcal{O}(t^{\wp\varepsilon})$$

$$\mathcal{Q}_t=\left\{\|\mathcal{Z}\|_{\frac{3}{2}+2\kappa;\mathcal{O}}+|\mathcal{P}\star 1_{+}\xi|_{\mathfrak{C}([-1,3],\mathfrak{C}^{-1/2-\kappa})}+|\mathcal{P}\star(\psi\partial\psi)|_{\mathfrak{C}([-1,3],\mathfrak{C}^{2-\kappa})}+t^{-\varepsilon}|||\psi_t|||_{\alpha,\theta} + t^{-\varepsilon}|\mathcal{P}\star(\psi\partial\psi)|_{\mathfrak{C}^{\bar{\eta}}}+t^{-\varepsilon}[\![\psi]\!]_{\gamma,\delta}\right\}$$

$$\left\{\|\mathcal{Z}\|_{\frac{3}{2}+2\kappa;\mathcal{O}}+|\mathcal{P}\star 1_{+}\xi|_{\mathfrak{C}([-1,3],\mathfrak{C}^{-1/2-\kappa})}+|\mathcal{P}\star(\psi\partial\psi)|_{\mathfrak{C}([-1,3],\mathfrak{C}^{2-\kappa})}\right\}\supset\mathcal{Q}_t$$

$$\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\widetilde{\mathrm{A}})1_{\mathcal{Q}_t}-\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\mathrm{A})1_{\mathcal{Q}_t} = \mathcal{P}(\mathcal{Q}_t)\left\{tTr\left(\iiint_{\ell} c\hbar(0)\right)+tTr\left(\iiint_{[0,1]^2} d\ell_{\mathcal{A}(0)}(\chi_1)d\ell_{c\hbar(0)}(\chi_2)\right) + \mathcal{O}\left(t^{1+r+\beta/6}+t^{1+3r/2}+t^{1+r+\nu\beta}\right)\right\}$$

$$|||\mathrm{A}|||_{\alpha,\theta}\leq|\mathrm{A}(0)|_{\infty}+|||\psi_t|||_{\alpha,\theta}+|||\mathcal{P}_t\star(\psi\partial\psi)|||_{\alpha,\theta}+\mathcal{O}\left(t^{\frac{1}{4}-\frac{3\kappa}{2}}\right)\lesssim t^{r},\beta=-\frac{r}{4\lambda},\mu = s^{\lambda}|||\mathrm{A}|||_{\alpha,\theta}\lesssim s^{\lambda}t^{r}=t^{\frac{3r}{4}}$$

$$[\![\mathrm{A}]\!]_{\gamma,\delta}\lesssim[\![\psi_t]\!]_{\gamma,\delta}+|\psi|_{\mathfrak{C}^{\eta}}\left\{|\mathrm{A}(0)|_{\mathcal{L}^{\infty}}+|\mathcal{P}_t\star(\psi\partial\psi)|_{\mathfrak{C}^{\bar{\eta}}}+\mathcal{O}\left(t^{\frac{1}{4}-\frac{3\kappa}{2}}\right)\right\}\lesssim t^{r}$$

$$t^{\frac{5}{4}-\frac{7\kappa}{2}}+ts\lesssim t^{1+\frac{3r}{2}},ts^{\lambda}\left(|||\psi_t|||_{\alpha,\theta}+|||\mathcal{P}_t\star(\psi\partial\psi)|||_{\alpha,\theta}\right)\lesssim s^{\lambda}t^{1+\varepsilon}\lesssim t^{1+\frac{3r}{2}},|\mathrm{A}(0)|_{\mathcal{L}^{\infty}}s^{\frac{\eta}{2}+\frac{1}{2}}t \lesssim t^{r+1}t^{\left(\frac{\eta}{2}+\frac{1}{2}\right)\beta}\lesssim t^{1+r+\beta/6},s^{\nu}t\left(\sum(\mathrm{A})+\sum(\mathrm{A})^{3}\right)\lesssim t^{1+r+\nu\beta},t(\mu^{2}+\mu^{\mathcal{L}-1}) \lesssim t^{1+3r/2}$$

$$\left|\mathfrak{E}\{\mathcal{W}_{\ell}(\mathcal{F}_s\widetilde{\mathrm{A}})-\mathcal{W}_{\ell}(\mathcal{F}_s\mathrm{A})\}1_{\mathcal{Q}_t}\right|\gtrsim t^{1+r},\mathrm{A}(0)=t^{r}c\hbar(0)\iiint_{\ell} c\hbar(0)$$

$$=\iiint_{\ell} c_1^{(1)}\hbar_1(0)=c_1^{(1)}\zeta(1)$$

$$\left|\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\widetilde{\mathrm{A}})1_{\mathcal{Q}_t}-\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\mathrm{A})1_{\mathcal{Q}_t}\right|\gtrsim t$$

$$\left|\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\widetilde{\mathrm{A}})1_{\mathcal{Q}_t}-\mathfrak{E}\mathcal{W}_{\ell}(\mathcal{F}_s\mathrm{A})1_{\mathcal{Q}_t}\right|\gtrsim t^{1+r}\left|Tr\left(\left\{c_1^{(1)}\zeta(1)\right\}^{2}\right)\right|-o(t^{1+r})\gtrsim t^{1+r}$$

$$\mu(\chi,\gamma,z)=\begin{cases}e^{\psi(\gamma)\chi} & ::\gamma\in[0,1/4]\\ 1 & ::\gamma\in[1/4,3/4]\\ e^{\psi(\gamma-1)\chi} & ::\gamma\in[3/4,1]\end{cases}$$

$$\hbar_2(\chi,0{,}0) \stackrel{\text{def}}{=} (\partial_2\mu)\mu^{-1}(\chi,0{,}0) = \chi$$

$$\iiint_{\ell} c\hbar \iiint_{\ell} c_1^{(2)}\hbar_2(\chi,0{,}0) = c_1^{(2)}\chi$$

4. Mecanismo de Brout-Englert-Higgs para espacios cuánticos relativistas.

4.1. Cálculos de simetría en lagrangiano.

$$\mathcal{L}_{\hbar}(\chi) = \left|\left(i\partial_\mu + g\mathfrak{B}_\mu\right)\hbar\right|^2 - \mu^2|\hbar|^2 - \lambda|\hbar|^4 - \frac{m_f}{v_\hbar}\hbar\bar{\varphi}\varphi$$

$$\mathcal{L}_0(\chi) = -\frac{1}{4}\mathfrak{F}^{\mu\nu}\mathfrak{F}_{\mu\nu} + \bar{\varphi}\left(i\partial_\mu + g\mathfrak{B}_\mu\right)\gamma^\mu\varphi$$

$$\varphi(\chi) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\wp,s}\left[\alpha^{s}(\wp)\mu^{s}(\wp)e^{i\wp\chi} + \beta^{s\dagger}(\wp)\nu^{s}(\wp)e^{i\wp\chi}\right]$$

4.2. Transformaciones de Lorentz.

$$t_2' - t_1' = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left[t_2 - t_1 - \frac{v}{c^2}(\chi_2-\chi_1)\right], \tilde{\alpha}^{s}(\wp)$$
$$= \cos\theta_\rho\alpha^{s}(\wp) + \sin\theta_\rho\beta^{s\dagger}(-\wp), \tilde{\beta}^{s}(-\wp)$$
$$= \cos\theta_\rho\beta^{s}(-\wp) + \sin\theta_\rho\alpha^{s\dagger}(\wp)$$

$$|\tilde{0}\rangle = \prod_{\wp,s}\left[\cos\theta_\rho + \sin\theta_\rho e^{i\alpha(\chi)}\beta^{s\dagger}(-\wp)\alpha^{s\dagger}(\wp)\right]|0\rangle$$

4.3. Cálculos de simetría parametrizados.

$$\bar{\varphi}(\chi)(i\partial + \mathcal{U}_0)\varphi(\chi)\cos^2\theta_\rho = \frac{1}{2}\left(1 + \frac{\epsilon_\rho}{\sqrt{\epsilon_\rho^2 + \mathcal{U}_0^2}}\right), \sin^2\theta_\rho = \frac{1}{2}\left(1 - \frac{\epsilon_\rho}{\sqrt{\epsilon_\rho^2 + \mathcal{U}_0^2}}\right)$$

$$\left[\mathcal{P}_{k}, \mathcal{P}_{k'}^{\dagger}\right] = 0, k \neq k', \left[\mathcal{P}_{k}, \mathcal{P}_{k}^{\dagger}\right] = 1 - \left(\eta_{k,\uparrow} + \eta_{-k,\downarrow}\right), \mathcal{P}_{k}^2 = \mathcal{P}_{k}^{\dagger 2}$$

$$f(\chi_1,\chi_2\cdots) = \frac{1}{\sqrt[3]{\mathcal{V}}}\sum_{k,s}\left(\mathcal{P}_{k}\bar{v}^{s}(-k)\mu^{s}(k) + \mathcal{P}_{-k}^{\dagger}\bar{\mu}^{s}(-k)\nu^{s}(k)\right)\exp\left(i\sum_{j}^{\infty}k\chi_j\right)$$

$$\mathfrak{H}_{ef} = \iiint\sum_{i}^{\infty}\left|\frac{\partial}{\partial\chi_i}f(\chi_1,\chi_2\cdots)\right|^2\prod_{i}^{\infty}d^3\chi_i$$
$$= -\iiint f^{\dagger}(\chi_1,\chi_2\cdots)\sum_{i}^{\infty}\Delta_i f(\chi_1,\chi_2\cdots)\prod_{i}^{\infty}d^3\chi_i$$

$$g_{rr}(r) = \frac{d_m^2}{r^2}, \left(d_m \lesssim r \gtreqless \ell_c\right), g_{rr}(r) = 1, \left(0 \lesssim r \gtreqless d_m, \ell_c \ggg r\right)$$

$$\mathfrak{H}_{ef} = \iiint g_{\mu\nu} \frac{\partial \widehat{f^\dagger}}{\partial \chi_i^\mu} \frac{\partial \hat{f}}{\partial \chi_i^\nu} \prod_i^\infty d^3\chi_i + \iiint \sum_i^\infty \mathcal{W}(\chi_i) \widehat{f^\dagger}(\chi_1, \chi_2 \cdots) \hat{f}(\chi_1, \chi_2 \cdots) \prod_i^\infty d^3\chi$$

$$\mathcal{W}(\chi_i) = \frac{1}{4} \frac{\partial}{\partial \chi_i^\mu} \left(g_{\mu\nu} \frac{\partial \ln g}{\partial \chi_i^\nu} \right) + \frac{1}{32} g_{\mu\nu} \left(\frac{\partial \ln g}{\partial \chi_i^\mu} \right) \left(\frac{\partial \ln g}{\partial \chi_i^\nu} \right)$$

$$\mathcal{W}(r) = \frac{3}{2} \frac{d_m^2}{r^4} + \frac{1}{4} \frac{d_m^2}{r^4}$$

$$\epsilon_0^2 = \frac{1}{d_m} \iiint_{d_m}^{\ell_c} \mathcal{W}(r) dr = \frac{1}{d_m^2} \left[1 - \left(\frac{d_m}{\ell_c} \right)^3 \right]$$

$$\begin{aligned} &\langle \tilde{0} | \iiint d^4\chi_1 \mathcal{L}_0^{min}(\chi_1) \exp\left(i \iiint \mathfrak{H}_{\mathfrak{I}}(\chi_2) d^4\chi_2 \right) | \tilde{0} \rangle \\ &\quad = \langle \tilde{0} | \iiint d^4\chi_1 \bar{\varphi}(\chi_1) \gamma^\mu [i\partial_\mu + g\mathfrak{B}_\mu(\chi_1)] \bar{\varphi}(\chi_1) | \tilde{0} \rangle \\ &\quad + \langle \tilde{0} | \iiint d^4\chi_1 \bar{\varphi}(\chi_1) \gamma^\mu [i\partial_\mu + g\mathfrak{B}_\mu(\chi_1)] \bar{\varphi}(\chi_1) ig \iiint d^4\chi_2 \mathfrak{j}^\nu(\chi_2) \mathfrak{B}_\nu(\chi_2) | \tilde{0} \rangle \end{aligned}$$

$$\langle \tilde{0} | \iiint d^4\chi_1 \mathfrak{H}_{\mathfrak{I}}(\chi_1) \iiint d^4\chi_2 \mathfrak{j}^\nu \mathfrak{H}_{\mathfrak{I}}(\chi_2) | \tilde{0} \rangle$$

$$\begin{aligned} &g^2 \iiint \langle \tilde{0} | \iiint \mathfrak{j}_\mu(\chi_1) d^2\chi_1 \iiint \mathfrak{j}^\mu(\chi_2) d^2\chi_2 | \tilde{0} \rangle \mathfrak{B}^\mu(\chi_2) d^2\chi_1 d^2\chi_2 \\ &\quad = g^2 \iiint \langle \tilde{0} | \iiint \mathfrak{j}_\mu(\mathcal{Y}) \mathfrak{j}^\mu(0) d^4\mathcal{Y} | \tilde{0} \rangle \bigotimes \mathfrak{B}^\mu(\mathcal{X}) \mathfrak{B}_\mu(\mathcal{X}) d^4\mathcal{X} \\ &\quad = \mathcal{M}^2 \iiint \mathfrak{B}^\mu(\mathcal{X}) \mathfrak{B}_\mu(\mathcal{X}) d^4\mathcal{X} = g^2 \langle \tilde{0} | \iiint \mathfrak{j}_\mu(\mathcal{Y}) \mathfrak{j}^\mu(0) d^4\mathcal{Y} | \tilde{0} \rangle \equiv \mathcal{M}^2 \end{aligned}$$

$$\begin{aligned} &\langle \tilde{0} | \mathfrak{j}_\mu(\mathcal{Y}) \mathfrak{j}^\mu(0) | \tilde{0} \rangle \\ &\quad = \langle \tilde{0} | \left(\mathfrak{j}_\mu(0) + \left[\frac{\partial}{\partial \mathcal{Y}_\mu} [\bar{\varphi}(\mathcal{Y}) \gamma_\mu \varphi(\mathcal{Y})] \right]_{\mathcal{Y}_\mu = 0} \mathcal{Y}^\mu \right. \\ &\quad \left. + \cdots \right) \mathfrak{j}^\mu(0) | \tilde{0} \rangle \binom{\circlearrowright}{\circlearrowleft} \langle \tilde{0} | \mathfrak{j}_\mu(0) \mathfrak{j}^\mu(0) | \tilde{0} \rangle \end{aligned}$$

$$m_{\mathfrak{B}}^2 = 2g^2 \langle \tilde{0} | \iiint_{\mathcal{Y} \in 2\mathcal{Z}_c} [\varphi^\dagger(0) \varphi(0)]^2 d^4\mathcal{Y} | \tilde{0} \rangle$$

$$\begin{aligned} &\langle \tilde{0} | [\varphi^\dagger(0) \varphi(0)]^2 | \tilde{0} \rangle \\ &\quad = \langle \tilde{0} | \frac{1}{d_m^6} \sum_{\mathscr{p}, s} ([\alpha^{s\dagger}(\wp) \mu^{s\dagger}(\mathscr{p}) + \beta^s(-\wp) \nu^{s\dagger}(-\mathscr{p})][\alpha^s(\wp) \mu^s(\mathscr{p}) \\ &\quad + \beta^{s\dagger}(-\wp) \nu^s(-\mathscr{p})])^2 | \tilde{0} \rangle = \frac{1}{d_m^6} \prod_{\mathscr{p}, s} \left(\sin^2\theta_\rho + \cos^2\theta_\rho \right)^2 = \frac{1}{d_m^6} \end{aligned}$$

$$m_{\mathfrak{B}}^2 = 2g^2 \frac{1}{d_m^6} \iiint_{\mathcal{Y}\in 2\mathcal{Z}_c} d^4\mathcal{Y} = g^2 \left(\frac{2\ell_c^2}{d_m^3}\right)^2$$

$$g\langle\tilde{0}| \iiint d^4\chi_1 \mathfrak{j}_\mu(\chi_1) \iiint d^4\chi_2\, \mathfrak{j}^\nu(\chi_2)\mathfrak{B}_\nu(\chi_2)\partial^\mu|\tilde{0}\rangle$$
$$+ g\partial^\mu\langle\tilde{0}| \iiint d^4\chi_1 \mathfrak{j}_\mu(\chi_1) \iiint d^4\chi_2\, \mathfrak{j}^\nu(\chi_2)\mathfrak{B}_\nu(\chi_2)\partial^\mu|\tilde{0}\rangle$$

$$\frac{2i}{g} m_{\mathfrak{B}}^2 \iiint \mathfrak{B}_\mu(\mathcal{X})\partial^\mu\alpha(\mathcal{X})d^4\mathcal{X}$$
$$\equiv m_{\mathcal{B}} \iiint \mathfrak{B}_\mu(\mathcal{X})\partial^\mu\mathcal{G}(\mathcal{X})d^4\mathcal{X} \iiint \frac{d\mathcal{X}^4}{(2\pi)^4}\langle\tilde{0}|\mathcal{T}[\mathcal{G}(\mathcal{X})\mathcal{G}(0)]|\tilde{0}\rangle e^{iq\mathcal{X}} = \frac{i}{q^2}$$

$$\mathfrak{B}^\mu(q)\left[im_{\mathfrak{B}}^2 g^{\mu\nu} - m_{\mathcal{B}}q^\mu \frac{i}{q^2} m_{\mathcal{B}}q^\nu\right]\mathfrak{B}_\nu(q) = im_{\mathfrak{B}}^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\mathfrak{B}^\mu(q)\mathfrak{B}^\nu(q)$$

$$\mathfrak{D}^{\mu\nu}(q) = \frac{-i}{q^2 - m_{\mathfrak{B}}^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \equiv i\mathfrak{D}(q^2)\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

$$i\langle\tilde{0}| \iiint d^4\chi_1 \mathfrak{j}^\mu(\chi_1)\,\partial_\mu|\tilde{0}\rangle + i\,\partial_\mu\langle\tilde{0}| \iiint d^4\chi_1 \mathfrak{j}^\mu(\chi_1)|\tilde{0}\rangle$$

$$\frac{g}{m_{\mathcal{B}}} = \langle\tilde{0}| \iiint d^4\chi_1 \bar{\varphi}(\chi_1)\gamma^\mu\varphi(\chi_1)\,\partial_\mu\mathcal{G}(\chi_1)|\tilde{0}\rangle$$

4.4. Superpartículas. Entiéndase por superpartículas, como aquellas partículas o antipartículas, según sea el caso, cuya masa superior o superlativa o cuya energía (carga) superior o superlativa o cuando pese a no tener carga ni masa, despliegan una cantidad de movimiento que se aproxima, igual o supera la velocidad de la luz, las mismas que, al comportar la morfología y sistematicidad antes referidas, deforman el espacio – tiempo cuántico o en su defecto, crean agujeros negros cuánticos. En consecuencia, el modelo matemático propuesto por este autor, en trabajos anteriores, aplica en sentido general, a las superpartículas, siguiendo la definición antes esgrimida. Precísese adicionalmente, que cuando una superpartícula deforma o perfora el espacio – tiempo cuántico, según sea el caso, desarrolla una brecha de masa superior a cero.

$$\bar{\varphi}(\chi)[i\partial + \mathcal{U}_0 + \mathcal{U}_1(\chi)]\varphi(\chi), \bar{\varphi}(\chi)[i\partial + \mathcal{U}_0 + \hat{g}\mathfrak{H}(\chi)]\varphi(\chi)$$

$$\hat{g} = \frac{m_f}{m_{\mathcal{B}}}g$$

$$iq^2\chi(q^2) = (-i\hat{g}^2)^2(-1)\iiint_0^\Lambda \frac{d^4\rho}{(2\pi)^4} tr\left[\frac{i}{\rho - m_f}\frac{i}{\rho + \varrho - m_f}\right]$$

$$q^2\chi(q^2) = -4\hat{g}^4 \iiint_0^1 d\chi \iiint \frac{d\Omega_4}{(2\pi)^4} \iiint_{\sqrt{\chi^2\rho^2}}^{\sqrt{\Lambda^2+\chi^2\varrho^2}} \ell_\xi^3 d\ell_\xi \left[\frac{-\ell_\xi^2}{\left(\ell_\xi^2 + \Delta\right)^2} + \frac{\Delta}{\left(\ell_\xi^2 + \Delta\right)^2}\right]$$

$$\mathfrak{I}(m,\eta) \equiv \iiint \ell_\xi^m\left(\ell_\xi^2+\Delta\right)^\eta d\ell_\xi\, \mathfrak{I}(5,-2) - \Delta\otimes\mathfrak{I}(3,-2)$$
$$= \mathfrak{I}(1,0) - 3\Delta\otimes\mathfrak{I}(1,-1) + 2\Delta^2\otimes\mathfrak{I}(1,-2), \mathfrak{I}(1,0) = \frac{1}{2}\ell_\xi^2, \mathfrak{I}(1,-1)$$
$$= \frac{1}{2}\ln\left|\ell_\xi^2+\Delta\right|, \mathfrak{I}(1,-2) = -\frac{1}{2\left(\ell_\xi^2+\Delta\right)}$$

$$q^2\chi(q^2) = \frac{\hat{g}^4}{4\pi^2}\Lambda^2 - \frac{\hat{g}^4}{2\pi^2}\iiint_0^1 d\chi\Delta^2\left(\frac{1}{\Lambda^2+\chi^2\varrho^2+\Delta} - \frac{1}{\chi^2\varrho^2+\Delta}\right)$$
$$-\frac{\hat{g}^4}{2\pi^2}\iiint_0^1 d\chi\frac{3}{2}\Delta\ln\left|1+\frac{\Lambda^2}{\chi^2\varrho^2+\Delta}\right| \iiint \frac{d^4\chi}{(2\pi)^4}\langle\tilde{0}|\mathcal{T}[\mathfrak{H}(\chi)\mathfrak{H}(0)]|\tilde{0}\rangle e^{i\varrho\chi}$$
$$= \frac{1}{\varrho^2[1-\chi(\varrho^2)]}$$

$$\iiint_0^1 d\chi\frac{1}{\Lambda^2+\chi^2\varrho^2+\Delta} \leftrightsquigarrow \frac{3}{4\left(\Lambda^2-m_f^2\right)}\iiint_0^1 d\chi\ln\left|1+\frac{\Lambda^2}{\chi^2\varrho^2+\Delta}\right| \leftrightarrows\rightleftarrows \ln\left|\frac{\lambda^2+m_f^2}{m_f^2}\right|$$

$$m_\mathfrak{h}^2 = \frac{\hat{g}^4}{4\pi^2}\left[\Lambda^2+\frac{3}{2}m_f^2\left(1-\frac{m_f^2}{\Lambda^2+m_f^2}\right)3m_f^2\ln\left|\frac{\Lambda^2+m_f^2}{m_f^2}\right|\right]\left(\partial_\mu\mathfrak{H}\right)^2 - m_\mathfrak{h}^2\mathfrak{H}^2 + \frac{m_f}{m_\mathcal{B}}g\bar{\varphi}\varphi\mathfrak{H}$$

$$\tilde{\mathcal{L}}(\chi) = -\frac{1}{4}\mathfrak{F}^{\mu\nu}\mathfrak{F}_{\mu\nu} + m_\mathfrak{B}^2\mathcal{B}^\mu\mathcal{B}_\mu + \bar{\psi}\left(i\partial_\mu+g\mathfrak{B}_\mu\right)\gamma^\mu\psi - m_f\bar{\psi}\psi + \left(\partial_\mu\mathfrak{G}\right)^2 + m_\mathcal{B}\mathcal{B}_\mu\partial^\mu\mathcal{G}$$
$$+\frac{g}{m_\mathcal{B}}\bar{\psi}\gamma^\mu\psi\partial_\mu\mathfrak{G} + \left(\partial_\mu\mathfrak{H}\right)^2 - m_\mathfrak{h}^2\mathfrak{H}^2 + \frac{m_f}{m_\mathcal{B}}g\bar{\varphi}\varphi\mathfrak{H}$$

$$\frac{\mu^2}{2\lambda} = \left(\frac{2\ell_c^2}{d_m^3}\right), (\ddagger\, v_\mathfrak{h}^2)2\mu^2 = \frac{\hat{g}^4}{4\pi^2}\left[\Lambda^2+\frac{3}{2}\mathcal{U}_0^2\left(1-\frac{\mathcal{U}_0^2}{\Lambda^2+\mathcal{U}_0^2}\right) - 3\mathcal{U}_0^2\ln\left(1+\frac{\Lambda^2}{\mathcal{U}_0^2}\right)\right](\ddagger\, m_\mathfrak{h}^2)$$

$$g^2v_\mathfrak{h}^2\mathcal{B}^\mu\mathcal{B}_\mu\left(1+\frac{\mathfrak{h}_1}{v_\mathfrak{h}}\right)^2 + g^2\mathcal{B}^\mu\mathcal{B}_\mu\mathfrak{h}_2^2 + 2g\mathcal{B}^\mu\left(\mathfrak{h}_1\partial_\mu\mathfrak{h}_2+\mathfrak{h}_2\partial_\mu\mathfrak{h}_1\right) + (c\odot c)$$

$$\sigma\left(e^+e^- \looparrowright \zeta_{supermassive\ particle}\right)$$
$$= \frac{4\pi\alpha^2}{\delta}\left[\operatorname{Im}c^1(\varrho^2) + \operatorname{Im}c^{\bar{\varphi}\varphi}\langle 0|m\bar{\varphi}\varphi|0\rangle + \operatorname{Im}c^{\mathfrak{F}^2}(\varrho^2)\langle 0\|\mathfrak{F}_{\alpha\beta}^a\|0\rangle\right]$$

$$e^{-i\mathcal{K}}\mathcal{F}e^{i\mathcal{K}} = \mathcal{F} + [-i\mathcal{K},\mathcal{F}] + \frac{1}{2!}[-i\mathcal{K},[-i\mathcal{K},\mathcal{F}]]$$

$$\mathcal{K} = i\sum_{\rho,\delta}\theta_\rho[\beta^{s\dagger}(-\wp)\alpha^{s\dagger}(\wp) - \alpha^s(\wp)\beta^s(-\wp)], \tilde{\alpha}^s(\wp) = e^{-i\mathcal{K}}\alpha^s(\wp)e^{i\mathcal{K}}, \tilde{\beta}^s(-\wp)$$
$$= e^{-i\mathcal{K}}\beta^s(-\wp)e^{i\mathcal{K}}$$

$$|\tilde{0}\rangle = \exp\left(\sum_{\rho,\delta}\theta_\rho[\beta^{\mathscr{s}\dagger}(-\wp)\alpha^{\mathscr{s}\dagger}(\wp) - \alpha^{\mathscr{s}}(\wp)\beta^{\mathscr{s}}(-\wp)]\right)|0\rangle$$

$$= \prod_{\rho,\delta}\left[\sum_\eta \frac{1}{\eta!}\theta_\rho^\eta[\beta^{\mathscr{s}\dagger}(-\wp)\alpha^{\mathscr{s}\dagger}(\wp) - \alpha^{\mathscr{s}}(\wp)\beta^{\mathscr{s}}(-\wp)]^\eta\right]|0\rangle$$

$$\sum_\eta \frac{\theta^\eta}{\eta!}(\beta^\dagger\alpha^\dagger - \alpha\beta)^\eta|0\rangle = |0\rangle + \theta\beta^\dagger\alpha^\dagger|0\rangle - \frac{\theta^2}{2!}\alpha\beta\beta^\dagger\alpha^\dagger|0\rangle - \frac{\theta^3}{3!}\beta^\dagger\alpha^\dagger\alpha\beta\beta^\dagger\alpha^\dagger|0\rangle$$

$$+ \frac{\theta^4}{4!}\alpha\beta\beta^\dagger\alpha^\dagger\alpha\beta\beta^\dagger\alpha^\dagger|0\rangle + \cdots$$

$$\mathfrak{H}_0 = -\iiint d^3\chi f^\dagger(\chi)\Delta f(\chi)\,\langle f(\chi)|f(\chi)\rangle = \iiint \sqrt{g(\chi)}d^3\chi f^\dagger(\chi)f(\chi)\frac{\mathfrak{D}\mathfrak{A}^\mu}{d\chi^\mu}$$

$$= \frac{d\mathfrak{A}^\mu}{d\chi^\mu} + \Gamma^\mu_{\nu\mu}\mathrm{A}^\nu$$

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2g}\frac{\partial g}{\partial\chi^\nu}\frac{\mathfrak{D}\mathfrak{A}^\mu}{d\chi^\mu} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\mathrm{A}^\mu\right)}{\partial\chi^\mu}$$

$$\langle f(\chi)|\Delta|f(\chi)\rangle = \iiint \sqrt{g}d^3\chi f^\dagger\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi^\mu}\left(\sqrt{g}g_{\mu\nu}\frac{\partial f}{\partial\chi^\nu}\right)$$

$$= \iiint \sqrt{g(\chi)}g_{\mu\nu}d^3\chi\frac{\partial f^\dagger}{\partial\chi^\mu}\frac{\partial f}{\partial\chi^\nu}\iiint \sqrt{g}d^3\chi f^\dagger(\chi)f(\chi) = \iiint d^3\chi\widehat{f^\dagger}(\chi)\hat{f}(\chi)$$

$$\frac{\partial f}{\partial\chi} = g^{-1/4}\left(\frac{\partial}{\partial\chi} - \frac{1}{4}\frac{\partial\ln g}{\partial\chi}\right)\hat{f}(\chi)$$

$$\langle f(\chi)|\Delta|f(\chi)\rangle = \iiint d^3\chi g_{\mu\nu}\left(\frac{\partial}{\partial\chi^\mu} - \frac{1}{4}\frac{\partial\ln g}{\partial\chi^\mu}\right)\widehat{f^\dagger}(\chi)\left(\frac{\partial}{\partial\chi^\nu} - \frac{1}{4}\frac{\partial\ln g}{\partial\chi^\nu}\right)\hat{f}(\chi)$$

$$\langle f(\chi)|\Delta|f(\chi)\rangle = \iiint d^3\chi g_{\mu\nu}\frac{\partial\widehat{f^\dagger}}{\partial\chi^\mu}\frac{\partial\hat{f}}{\partial\chi^\nu} + \iiint \mathcal{W}(\chi)\widehat{f^\dagger}(\chi)\hat{f}(\chi)d^3\chi$$

$$\mathcal{W}(\chi) = \frac{1}{4}\frac{\partial}{\partial\chi^\mu}\left(g_{\mu\nu}\frac{\partial\ln g}{\partial\chi^\nu}\right) + \frac{1}{32}g_{\mu\nu}\left(\frac{\partial\ln g}{\partial\chi^\mu}\right)\left(\frac{\partial\ln g}{\partial\chi^\nu}\right)$$

$$\langle f(\chi_1,\cdots)|f(\chi_1,\cdots)\rangle = \iiint\prod_{i=1}^{\infty}\sqrt{g(\chi_i)}d\mathcal{V}_i f^\dagger(\chi_1,\cdots)f(\chi_1,\cdots)$$

$$\langle f(\chi_1,\cdots)|\textstyle\sum_{i=1}^{\infty}\Delta_i|f(\chi_1,\cdots)\rangle$$

$$= \iiint f^\dagger(\chi_1,\cdots)\sum_{i=1}^{\infty}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi^\mu}\left(\sqrt{g}g_{\mu\nu}\frac{\partial f(\chi_1,\cdots)}{\partial\chi^\nu}\right)\prod_{i=1}^{\infty}\sqrt{g(\chi_i)}d\mathcal{V}_i$$

$$= \iiint g_{\mu\nu}\sum_{i=1}^{s}\frac{\partial f^\dagger(\chi_1,\cdots)}{\partial\chi_i^\mu}\frac{\partial f(\chi_1,\cdots)}{\partial\chi_i^\nu}\prod_{i=1}^{\infty}\sqrt{g(\chi_i)}d\mathcal{V}_i$$

$$\hat{f}(\chi_1,\cdots) = \prod_{i=1}^{\infty} g(\chi_i)^{1/4} f(\chi_1,\cdots) \iiint \prod_{i=1}^{\infty} \sqrt{g(\chi_i)} d\mathcal{V}_i f^{\dagger}(\chi_1,\cdots) f(\chi_1,\cdots)$$

$$= \prod_{i=1}^{\infty} d\mathcal{V}_i \widehat{f^{\dagger}}(\chi_1,\cdots) \hat{f}(\chi_1,\cdots)$$

$$\frac{\partial f(\chi_1,\cdots)}{\partial \chi_i} = g^{-1/4}\left(\frac{\partial}{\partial \chi_i} - \frac{1}{4}\frac{\partial \ln g}{\partial \chi_i}\right)\widehat{f^{\dagger}}(\chi_1,\cdots)$$

$$\langle f(\chi_1,\cdots)|\,\textstyle\sum_{i=1}^{\infty}\Delta_i\,|f(\chi_1,\cdots)\rangle$$
$$= \iiint \sum_{i=1}^{\infty} g_{\mu\nu}\left(\frac{\partial}{\partial \chi^{\mu}} - \frac{1}{4}\frac{\partial \ln g}{\partial \chi^{\mu}}\right)_i \widehat{f^{\dagger}}(\chi_1,\cdots)\left(\frac{\partial}{\partial \chi^{\nu}}\right.$$
$$\left. - \frac{1}{4}\frac{\partial \ln g}{\partial \chi^{\nu}}\right)_i \hat{f}(\chi_1,\cdots) \prod_{i=1}^{\infty} d\mathcal{V}_i$$

$$\langle f(\chi_1,\cdots)|\,\textstyle\sum_{i=1}^{\infty}\Delta_i\,|f(\chi_1,\cdots)\rangle$$
$$= \iiint \sum_{i=1}^{\infty} g_{\mu\nu}\frac{\partial f^{\dagger}(\chi_1,\cdots)}{\partial \chi_i^{\mu}}\frac{\partial f(\chi_1,\cdots)}{\partial \chi_i^{\nu}} \prod_{i=1}^{\infty} d\mathcal{V}_i$$
$$+ \iiint \sum_{i=1}^{\infty} \mathcal{W}(\chi_i)\,\widehat{f^{\dagger}}(\chi_1,\cdots)\hat{f}(\chi_1,\cdots) \prod_{i=1}^{\infty} d\mathcal{V}_i$$

$$i\psi(\chi) = \iiint \frac{d^2\rho}{(2\pi)^3}\sum_{s}\frac{1}{\sqrt{2\xi_{\rho}}}\left[\hat{\alpha}^{s}(\wp)\hat{\mu}^{s}(\wp)e^{-i\wp\chi} + \hat{\beta}^{s\dagger}(\wp)\hat{\nu}^{s}(\wp)e^{i\wp\chi}\right]$$

5. Modelo de Dirac para espacios cuánticos relativistas.

$$z = e^{\frac{r}{\hbar}+i\theta}, \bar{z} = e^{\frac{r}{\hbar}-i\theta}, \partial_z = e^{-\left(\frac{r}{\hbar}+i\theta\right)}\left(\frac{\hbar}{2}\partial_r - \frac{i}{2}\partial_\theta\right), \partial_{\bar{z}} = e^{-\left(\frac{r}{\hbar}-i\theta\right)}\left(\frac{\hbar}{2}\partial_r + \frac{i}{2}\partial_\theta\right), dz$$
$$= e^{\frac{r}{\hbar}+i\theta}\left(\frac{1}{\hbar}dr + id\theta\right), d\bar{z} = e^{\frac{r}{\hbar}-i\theta}\left(\frac{1}{\hbar}dr - id\theta\right), z = e^{\hbar' r+i\theta}, \bar{z} = e^{\hbar' r-i\theta},$$

$$\widehat{ds}^2 = g_{z\bar{z}}dzd\bar{z}, \widehat{ds}^2 = g_{z\bar{z}}e^{\frac{2r}{\hbar}}\left(\frac{1}{\hbar^2}dr^2 + d\theta^2\right), ds^2 = dr^2, g \doteq \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}, \hbar \doteq \begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}, \hbar g$$
$$= g\hbar \doteq \begin{vmatrix}0 & 0\\ 0 & 0\end{vmatrix}, \varepsilon = \begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix} \tilde{r} = \tilde{r}(r), \tilde{\theta} = \tilde{\theta}_0(r) + \theta\partial_r\tilde{r}(r), \delta r = f(r), \delta\theta$$
$$= \mathcal{F}(r) + \theta\partial_r f(r)$$

$$\hat{d}f = \frac{1}{\hbar}\partial_r f dr + \partial_\theta f d\theta, df = \partial_r f(r,\theta)dr$$

$$\langle f_1|f_2\rangle = \iiint_{\hat{\Sigma}} d^2\sigma\sqrt{\hat{g}}f_1(\sigma)f_2(\sigma)$$

$$\langle \omega_1|\omega_2\rangle = \iiint_{\hat{\Sigma}} d^2\sigma\sqrt{\hat{g}\hat{g}^{\alpha\beta}}(\sigma)\omega_\alpha(\sigma)\omega_\beta(\sigma)$$

$$\widehat{\Delta} = \widehat{d^\dagger}\hat{d} + \hat{d}\widehat{d^\dagger}, \widehat{\Delta}f = \frac{1}{\sqrt{\hat{g}}}\partial_i\left(\sqrt{\hat{g}\hat{g}^{ij}}\partial_i f\right), \widehat{\Delta}f = 4\partial_z\partial_{\bar{z}}f, \widehat{\Delta}f = e^{-\frac{2r}{\hbar}}\left|\hbar^2\partial_r^2 f + \partial_\theta^2 f\right|, \Delta f = e^{-\frac{2r}{\hbar}}\partial_\theta^2 f, \Delta = \lim_{\hbar \rightsquigarrow 0} e^{\frac{2r}{\hbar}}\widehat{\Delta} = \partial_\theta^2$$

$$\mathfrak{D} = \gamma^r\partial_r + \gamma^\theta\partial_\theta, f(r,\theta) = f_0(r) + \theta f_1(r), f = \exp\left(\frac{\mathcal{F}(r,\theta)}{\hbar} + i\Theta(r,\theta)\right)$$

$$\frac{\widehat{\Delta}f}{f} = e^{-\frac{2r}{\hbar}}\left[(\partial_r\mathcal{F})^2 - (\partial_\theta\Theta)^2 + \frac{1}{\hbar^2}(\partial_\theta\mathcal{F})^2 - \hbar^2(\partial_r\Theta)^2 + \hbar\partial_r^2\mathcal{F} + \frac{1}{\hbar}\partial_\theta^2\mathcal{F}\right] + ie^{-\frac{2r}{\hbar}}\left[2\hbar\partial_r\mathcal{F}\partial_r\Theta + \frac{2\partial_\theta\mathcal{F}\partial_\theta\Theta}{\hbar} + \hbar^2\partial_r^2\Theta + \partial_\theta^2\Theta\right]$$

$$\widehat{\Delta}f = \lim_{\hbar \rightsquigarrow 0} e^{\frac{2r}{\hbar}}\frac{\widehat{\Delta}f}{f} = (\partial_\theta\mathcal{F})^2 + i(2\partial_\theta\mathcal{F}\partial_\theta\Theta)$$

$$\frac{\widehat{\Delta}f}{f} = e^{-\frac{2r}{\hbar}}[(\partial_r\mathcal{F})^2 - (\partial_\theta\Theta)^2 - \hbar^2(\partial_r\Theta)^2 + \hbar\partial_r^2\mathcal{F}] + ie^{-\frac{2r}{\hbar}}[2\hbar\partial_r\mathcal{F}\partial_r\Theta + \hbar^2\partial_r^2\Theta + \partial_\theta^2\Theta]$$

$$\widehat{\Delta}f = \lim_{\hbar \rightsquigarrow 0} e^{\frac{2r}{\hbar}}\frac{\widehat{\Delta}f}{f} = [(\partial_\theta\mathcal{F})^2 - (\partial_\theta\Theta)^2] + i[\partial_\theta^2\Theta]\otimes(\partial_r\mathcal{F})^2 - (\partial_\theta\Theta)^2\otimes\langle\partial_\theta^2\Theta\rangle$$

$$\Theta(r,\theta) = \Theta_0(r) + \theta\Theta_1(r), \mathcal{F}(r) = \mathcal{F}_0 + \iiint_0^r d\tilde{r}\Theta_1(\tilde{r}), \Theta(r,\theta) = \Theta_0(r) + \eta\theta, \mathcal{F}(r) = \mathcal{F}_0 + \eta r$$

$$\widehat{ds}^2 = \left(\frac{dr^2}{d\hbar^2} + d\theta^2 + d\tau^2\right), \widehat{\Delta} = \left(\hbar^2\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\tau^2}\right), \Delta = \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\tau^2} \triangleq \hbar \doteq \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} g \doteq \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\mathcal{D} = \alpha\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial\theta} + c\frac{\partial}{\partial\tau}, \{\gamma^{\mathfrak{I}}, \gamma^{\mathfrak{I}}\} = \hbar^{\mathfrak{I}\mathfrak{I}}$$

$$\mathfrak{A} = \mathfrak{B}\otimes\alpha\mathcal{B}, \alpha = \begin{vmatrix} 0_2 & \mathfrak{E} \\ -\mathfrak{E} & 0_2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \mathcal{E} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \beta = \begin{vmatrix} \mathfrak{I}_2 & 0 \\ 0 & \mathfrak{I}_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, c = \begin{vmatrix} 0 & \mathfrak{I}_2 \\ \mathfrak{I}_2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \mathfrak{D}$$

$$= \begin{vmatrix} \partial_\theta & 0 & \partial_\tau & \partial_r \\ 0 & \partial_\theta & 0 & \partial_\tau \\ \partial_\tau & -\partial_r & -\partial_\theta & 0 \\ 0 & \partial_\tau & 0 & -\partial_\theta \end{vmatrix}, \mathfrak{D}^2$$

$$= \begin{vmatrix} \partial_\theta^2 + \partial_\tau^2 & 0 & 0 & 0 & 0 \\ 0 & \partial_\theta^2 + \partial_\tau^2 & 0 & 0 & 0 \\ 0 & -\partial_r & \partial_\theta^2 + \partial_\tau^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_\theta^2 + \partial_\tau^2 \end{vmatrix}, \mathfrak{D}$$

$$= \begin{vmatrix} \epsilon\partial_r & \partial_\theta - i\partial_\tau \\ \partial_\theta + i\partial_\tau & -\epsilon\partial_r \end{vmatrix}, \mathfrak{D}^2 = \begin{vmatrix} \partial_\theta^2 + \partial_\tau^2 & 0 \\ 0 & \partial_\theta^2 + \partial_\tau^2 \end{vmatrix}$$

$$\beta = \frac{1}{\sqrt{2}}(\beta + ic), \beta^\dagger = \frac{1}{\sqrt{2}}(\beta - ic), \beta^2 = (\beta^\dagger)^2 = \{\beta, \beta^\dagger\} = \beta\beta^\dagger + \beta^\dagger\beta = \beta^2 + c^2, \beta$$

$$= \begin{Vmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{Vmatrix}, \beta^\dagger = \begin{Vmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{Vmatrix}, \beta = \frac{1}{\sqrt{2}}(\beta + \beta^\dagger) = \frac{1}{\sqrt{2}}\begin{Vmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{Vmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \frac{-i}{\sqrt{2}}(\beta - \beta^\dagger) = \frac{-i}{\sqrt{2}}\begin{Vmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{Vmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \alpha = \epsilon\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\overset{\star}{\tilde{\Delta}}\phi = \frac{\partial^2\phi}{\partial r^2}\{\gamma^{\mathfrak{I}}, \gamma^{\mathfrak{I}}\} = \overset{\star\mathfrak{I}\mathfrak{I}}{\tilde{\hbar}} \doteq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, c$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\overset{\star}{\tilde{\mathfrak{D}}} = \begin{bmatrix} \partial_r & 0 & 0 & 0 \\ \partial_\theta & -\partial_r & 0 & 0 \\ \partial_\tau & 0 & -\partial_r & 0 \\ 0 & -\partial_\tau & \partial_\theta & \partial_r \end{bmatrix}, \overset{\star}{\tilde{\mathfrak{D}}}^2 = \begin{bmatrix} \partial_r^2 & 0 & 0 & 0 \\ 0 & \partial_r^2 & 0 & 0 \\ 0 & 0 & \partial_r^2 & 0 \\ 0 & 0 & 0 & \partial_r^2 \end{bmatrix}$$

$$\alpha = \left\langle\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\rangle, \beta = \left\langle\begin{matrix} 0 & \epsilon \\ 0 & 0 \end{matrix}\right\rangle, c = \left\langle\begin{matrix} 0 & 0 \\ \epsilon & 0 \end{matrix}\right\rangle$$

$$\overset{\star}{\tilde{\mathfrak{D}}} = \begin{bmatrix} \partial_r & \epsilon\partial_\theta \\ \epsilon\partial_\tau & -\partial_r \end{bmatrix}$$

$$\mathcal{D}\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \epsilon\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$(\partial_\theta - i\partial_\tau)\psi_2 + \epsilon(\partial_r\psi_1 + (\partial_\theta - i\partial_\tau)\phi_2), (\partial_\theta + i\partial_\tau)\psi_1 + \epsilon((\partial_\theta - i\partial_\tau)\phi_1 - \partial_r\psi_2)$$

$$(\partial_\theta + i\partial_\tau)\psi_1(\partial_\theta - i\partial_\tau)\psi_2\partial_r\psi_1 + (\partial_\theta - i\partial_\tau)\phi_2(\partial_\theta + i\partial_\tau)\phi_1, (\partial_\theta - i\partial_\tau)\phi_2 = -\partial_r\psi_1, (\partial_\theta + i\partial_\tau)\phi_1 = \partial_r\psi_2$$

$$\psi^\dagger = (\psi_1^\dagger - \epsilon\phi_1^\dagger, \psi_2^\dagger - \epsilon\phi_2^\dagger), \|q^2\| = (\chi - \epsilon\gamma)(\chi + \epsilon\gamma) = \chi^2$$

$$\phi' = e^{\mho_\epsilon}\phi = (1 + \mho\epsilon)\phi = \phi + \mho\epsilon\phi$$

$$\|\phi'\|^2 = (\phi - \mho\epsilon\phi)(\phi + \mho\epsilon\phi) = \|\phi\|^2$$

$$\mathcal{L} = \psi^\dagger\mathfrak{D}\psi$$

$$\mathcal{L} = \psi_1^\dagger\big((\partial_\theta - i\partial_\tau)\psi_2 + \epsilon(\partial_r\psi_1 + (\partial_\theta - i\partial_\tau)\phi_2)\big) - \epsilon\phi_1^\dagger(\partial_\theta - i\partial_\tau)\psi_2 + \psi_2^\dagger\Big((\partial_\theta + i\partial_\tau)\psi_1 + \epsilon\big((\partial_\theta + i\partial_\tau)\phi_1 - \partial_r\psi_2\big)\Big) - \epsilon\phi_2^\dagger(\partial_\theta + i\partial_\tau)\psi_1$$

$$\prod_{\psi_1} = i\psi_2^\dagger - i\epsilon\phi_2^\dagger, \prod_{\phi_1} = i\epsilon\psi_2^\dagger, \prod_{\psi_2} = -i\psi_1^\dagger + i\epsilon\phi_1^\dagger, \prod_{\phi_2} = -i\epsilon\psi_1^\dagger\{\psi_1, \prod_{\psi_1}\} = \{\psi_2, \prod_{\psi_2}\} = i$$

$$\delta\phi_1 = \alpha_{\phi_1}, \delta\phi_2 = \alpha_{\phi_2}, \delta\phi_1^\dagger = \alpha_{\phi_1^\dagger}, \delta\phi_2^\dagger = \alpha_{\phi_2^\dagger}$$

$$\chi_1 = \prod_{\psi_1} - i\psi_2^\dagger, \chi_2 = \prod_{\psi_2} - i\psi_1^\dagger, \chi_3 = \prod_{\psi_1^\dagger}, \chi_4 = \prod_{\psi_2^\dagger}$$

$$\{\psi_i, \psi_j^\dagger\}_{\mathcal{D}} = \{\psi_i, \psi_j^\dagger\} - \{\psi_i, \chi_{k}\}\mathfrak{C}^{k\ell}\{\chi_\ell, \psi_j^\dagger\}, \{\psi_1, \psi_1^\dagger\}_{\mathcal{D}} = -\{\psi_2, \psi_2^\dagger\}_{\mathcal{D}} = 1$$

$$\psi'(r, \theta, \tau) = e^{i\alpha(r,\theta,\tau)}\psi(r, \theta, \tau)$$

$$\mathcal{D}^{\mathcal{A}} = \gamma^{\mathfrak{I}}\mathfrak{D}_{\mathfrak{I}}^{\mathfrak{A}} = \gamma^{\mathfrak{I}}(\partial_{\mathfrak{I}} + \mathfrak{A}_{\mathfrak{I}}) = \alpha(\partial_r + \mathfrak{A}_r) + \gamma^i(\partial_i + \mathfrak{A}_i)$$

$$\mathcal{D}^{\mathcal{A}} = \begin{Vmatrix} \epsilon(\partial_r + \mathfrak{A}_r) & (\partial_\theta - i\partial_\tau) + (\mathrm{A}_\theta - i\mathrm{A}_\tau) \\ (\partial_\theta + i\partial_\tau) + (\mathrm{A}_\theta + i\mathrm{A}_\tau) & -\epsilon(\partial_r + \mathfrak{A}_r) \end{Vmatrix}$$

$$(\mathcal{D}^{\mathcal{A}})^2 = \gamma^{\mathfrak{I}}\gamma^{\mathfrak{I}}(\partial_{\mathfrak{I}} + \mathfrak{A}_{\mathfrak{I}})(\partial_{\mathfrak{I}} + \mathfrak{A}_{\mathfrak{I}}), (\mathcal{D}^{\mathcal{A}})^2 = \mathfrak{h}^{\mathfrak{I}\mathfrak{I}}\mathfrak{D}_{\mathfrak{I}}^{\mathfrak{A}}\mathfrak{D}_{\mathfrak{I}}^{\mathfrak{A}} + \frac{1}{2}[\gamma^{\mathfrak{I}}, \gamma^{\mathfrak{I}}](\partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}} - \partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}}) = \mathfrak{h}^{ij}\mathfrak{D}_i^{\mathfrak{A}}\mathfrak{D}_j^{\mathfrak{A}} + \frac{1}{2}[\gamma^{\mathfrak{I}}, \gamma^{\mathfrak{I}}](\partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}} - \partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}})$$

$$\mathcal{F}_{\mathfrak{I}\mathfrak{I}} = \partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}} - \partial_{\mathfrak{I}}\mathfrak{A}_{\mathfrak{I}}$$

$$[\alpha, \beta] = \alpha\beta - \beta\alpha = 2i\epsilon\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, [\alpha, c] = \alpha c - c\alpha = -2i\epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, [\beta, c] = \beta c - c\beta = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\mathcal{D}^{\mathcal{A}})^2 = \begin{Vmatrix} (\partial_i + \mathfrak{A}_i)^2 + i\mathfrak{F}_{\theta\tau} & \epsilon\mathfrak{F}_{r\theta} - i\epsilon\mathfrak{F}_{r\tau} \\ -\epsilon\mathfrak{F}_{r\theta} - i\epsilon\mathfrak{F}_{r\tau} & (\partial_i + \mathfrak{A}_i)^2 - i\mathfrak{F}_{\theta\tau} \end{Vmatrix}$$

$$\mathcal{L} = \psi^\dagger\mathcal{D}^{\mathcal{A}}\psi$$

$$\mathcal{L}_0 = \psi_1^\dagger\big((\partial_\theta - i\partial_\tau)\psi_2 + (\mathcal{A}_\theta - i\mathcal{A}_\tau)\big)\psi_2 + \psi_2^\dagger\big((\partial_\theta + i\partial_\tau)\psi_1 + (\mathcal{A}_\theta + i\mathcal{A}_\tau)\big)\psi_1$$

$$\mathcal{L}_1 = \epsilon\big[\psi_1^\dagger(\partial_r + \mathrm{A}_r)\psi_1 - \psi_2^\dagger(\partial_r - \mathrm{A}_r)\psi_2 - \phi_2^\dagger\big((\partial_\theta + i\partial_\tau) + (\mathrm{A}_\theta + i\mathrm{A}_\tau)\big)\psi_1 + \psi_1^\dagger\big((\partial_\theta - i\partial_\tau) + (\mathrm{A}_\theta - i\mathrm{A}_\tau)\big)\phi_2 + \psi_2^\dagger\big((\partial_\theta + i\partial_\tau) + (\mathrm{A}_\theta + i\mathrm{A}_\tau)\big)\phi_1 - \phi_1^\dagger\big((\partial_\theta + i\partial_\tau) + (\mathrm{A}_\theta + i\mathrm{A}_\tau)\big)\psi_2\big]$$

$$\iiint d\epsilon e^{-\delta_0-\epsilon\delta_1} = \iiint d\epsilon e^{-\delta_0}(1-\epsilon\delta_1) = -e^{-\delta_0}\delta_1$$

$$\eta_{\mathrm{A}}(\mathscr{s}) = \sum_{\lambda \nparallel 0} \frac{\mathrm{sign}(\lambda)}{|\lambda|^{\delta}} = \frac{1}{\Gamma\left(\frac{\delta+1}{2}\right)} \iiint_0^{\infty} t^{\frac{\delta-1}{2}} Tr\left(\mathrm{A}e^{-t\mathfrak{A}^2}\right)dt$$

6. Modelo de Englert-Brout-Higgs para espacios cuánticos relativistas. Cálculos secundarios en relación al rompimiento de simetrías de gauge.

$$\mathfrak{L} = \partial_\alpha\phi_1^\star\partial^\alpha\phi_1 + \partial_\alpha\phi_2^\star\partial^\alpha\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(\phi_1^\star\phi_2 - \phi_2^\star\phi_1)$$

$$\mathcal{M}_\pm^2 = \frac{1}{2}(m_1^2+m_2^2) \pm \frac{1}{2}\sqrt{(m_1^2+m_2^2)^2 - 4\mu^4}$$

$$\eta \equiv \frac{2\mu^2}{|m_1^2+m_2^2|} \lesssim 1$$

$$\mathcal{PT} \doteqdot \phi \nVdash \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \leftrightsquigarrow \begin{pmatrix}\phi_1^\star\\ \phi_2^\star\end{pmatrix}$$

$$\mathrm{Im}\,\mathfrak{S} = i\mu^2 \iiint d^4\chi(\phi_1^\star\phi_2 - \phi_2^\star\phi_1)$$

$$\frac{\delta\mathfrak{S}}{\delta\phi^\dagger} \equiv \frac{\partial\mathfrak{L}}{\partial\phi^\dagger} - \partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi^\dagger)} \gnapprox \frac{\delta\mathfrak{S}}{\delta\phi} \equiv \frac{\partial\mathfrak{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi)}$$

$$\mathfrak{L} = \phi^\ddagger\begin{pmatrix}-\Box - m_1^2 & -\mu^2\\ -\mu^2 & \Box + m_2^2\end{pmatrix}\Phi$$

$$\frac{\delta\mathfrak{S}}{\delta\phi^\ddagger} \equiv \frac{\partial\mathfrak{L}}{\partial\phi^\ddagger} - \partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi^\ddagger)} \lnsim \left(\frac{\delta\mathfrak{S}}{\delta\phi}\right)^\ddagger \equiv \left(\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi)}\right)^\ddagger$$

$$\frac{\delta\mathfrak{S}}{\delta\phi} \equiv \frac{\partial\mathfrak{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi)} \nvDash \frac{\delta\mathfrak{S}}{\delta\phi^\star}\frac{\delta\mathfrak{S}^\star}{\delta\phi} \nprec \frac{\delta\mathfrak{S}}{\delta\phi}\left(\frac{\delta\mathfrak{S}}{\delta\phi}\right)^\star_\circ$$

$$\partial_t\Pi^\dagger = -\frac{\partial\mathcal{H}}{\partial\phi^\dagger}, \partial_t\Pi = -\frac{\partial\mathcal{H}}{\partial\phi} \gnapprox (\partial_t\Pi^\dagger)^\dagger$$

$$\Xi \equiv \mathfrak{R}\phi, \bar{\Xi} \equiv \mathfrak{R}^{-1}$$

$$\mathcal{R} = \mathcal{N}\begin{vmatrix}\eta & 1-\sqrt{1-\eta^2}\\ 1-\sqrt{1-\eta^2} & \eta\end{vmatrix}$$

$$\mathcal{N}^{-1} \cong \sqrt{2\eta^2 - 2 + 2\sqrt{1-\eta^2}}$$

$$\mathfrak{L} = \bar{\Xi}\begin{pmatrix}-\Box - \mathcal{M}_+^2 & 0\\ 0 & -\Box - \mathcal{M}_-^2\end{pmatrix}\left\langle\frac{\delta\mathfrak{S}}{\delta\bar{\Xi}}\middle|\frac{\delta\mathfrak{S}}{\delta\Xi}\right\rangle$$

$$j_+^\alpha \equiv i(\phi_1^\star\partial^\alpha\phi_1 - \phi_1\partial^\alpha\phi_1^\star) + i(\phi_2^\star\partial^\alpha\phi_2 - \phi_2\partial^\alpha\phi_2^\star)$$

$$\phi \upharpoonright e^{-i\theta\mathfrak{P}}\phi = \begin{Vmatrix} e^{-i\theta}\phi_1 \\ e^{i\theta}\phi_2 \end{Vmatrix}$$

$$j_-^\alpha = i(\phi_1^\star\partial^\alpha\phi_1 - \phi_1\partial^\alpha\phi_1^\star) - i(\phi_2^\star\partial^\alpha\phi_2 - \phi_2\partial^\alpha\phi_2^\star)$$

$$\mathfrak{L}_\theta = \partial_\alpha\phi_1^\star\partial^\alpha\phi_1 + \partial_\alpha\phi_2^\star\partial^\alpha\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(e^{2i\theta}\phi_1^\star\phi_2 - e^{-2i\theta}\phi_2^\star\phi_1\right)$$

$$\mathfrak{L} = \partial_\alpha\phi_1^\star\partial^\alpha\phi_1 + \partial_\alpha\phi_2^\star\partial^\alpha\phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(\phi_1^\star\phi_2 - \phi_2^\star\phi_1) - \frac{g}{4}\sqrt{\frac{\langle|\phi_1|^4|\phi_2|^4\rangle}{\langle|\phi_1^\star|^4|\phi_2^\star|^4\rangle}}$$

$$\zeta = (g|\phi_1|^2 - 2m_1^2)\phi_1 + 2\mu^2\phi_2 = m_2^2\phi_2 - \mu^2\phi_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{2\frac{m_1^2m_2^2 - \mu^4}{gm_1^2m_2^2}}\begin{pmatrix} 1 \\ \frac{\mu^4}{m_2^2} \end{pmatrix}$$

$$\mathcal{U}(\hat{\phi}_1, \hat{\phi}_2) = -\frac{2\mu^4}{m_2^2}v_1\hat{\phi}_1 + 2m_2^2v_2\hat{\phi}_2 + \widetilde{m}_1^2|\hat{\phi}_1|^2 + \frac{g}{4}v_1^2\left(\hat{\phi}_1^2 + (\hat{\phi}_1^\star)^2\right) + m_2^2|\hat{\phi}_2|^2 + \mu^2(\hat{\phi}_1^\star\hat{\phi}_2 - \hat{\phi}_2^\star\hat{\phi}_1)^2 + \frac{g}{2}v_1(\hat{\phi}_1 + \hat{\phi}_1^\star)|\phi_1|^2 + \frac{g}{4}|\hat{\phi}_1|^4$$

$$(-\square - \widetilde{m}_1^2)\hat{\phi}_1 = \mu^2\hat{\phi}_2 + \frac{g}{2}v_1^2\hat{\phi}_1^\star + \frac{g}{2}\left(v_1\hat{\phi}_1^2 + 2v_1|\hat{\phi}_1|^2 + |\hat{\phi}_1|^2\hat{\phi}_1\right)(-\square - m_2^2)\hat{\phi}_2 = -\mu^2\hat{\phi}_1$$

$$\mathcal{M}^2 = \begin{vmatrix} \widetilde{m}_1^2 & \frac{g}{2}v_1^2 & \mu^2 & 0 \\ \frac{g}{2}v_1^2 & \widetilde{m}_1^2 & 0 & \mu^2 \\ & & & 0 \\ -\mu^2 & 0 & m_2^2 & \\ 0 & -\mu^2 & 0 & m_2^2 \end{vmatrix}$$

$$\mathcal{G}_1 = \sqrt{\frac{2m_2^4}{m_2^4 - \mu^4}}\left[\mathrm{Im}(\hat{\phi}_1) - \frac{\mu^2}{m_2^2}\mathrm{Im}(\hat{\phi}_2)\right]$$

$$\lambda_2 = m_2^2 - \frac{\mu^4}{m_2^2}, \lambda_\pm = \frac{1}{2m_2^2}\left(2m_1^2m_2^2 - 3\mu^4 + m_2^4 \pm \sqrt{(2m_1^2m_2^2 - 3\mu^4 + m_2^4) - 4\mu^4m_2^4}\right)$$

$$\mathcal{G}_2 = \sqrt{\frac{2m_2^4}{m_2^4 - \mu^4}}\left[\mathrm{Im}(\hat{\phi}_2) - \frac{\mu^2}{m_2^2}\mathrm{Im}(\hat{\phi}_1)\right]$$

$$\mathfrak{G}_\pm = \frac{\sqrt{2}}{\sqrt{\left|\lambda_\pm - m_2^2\right|^2 - \mu^4}}\left[\left|\lambda_\pm - m_2^2\right|\mathrm{Re}(\hat{\phi}_1) + \mu^2\,\mathrm{Re}(\hat{\phi}_2)\right]$$

$$\mathfrak{L} = [\mathfrak{D}_\alpha^+\phi_1]^\star\mathfrak{D}_+^\alpha\phi_1 + [\mathfrak{D}_\alpha^-\phi_2]^\star\mathfrak{D}_-^\alpha\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(\phi_1^\star\phi_2 - \phi_2^\star\phi_1) - \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$$

$$\jmath_{A,-}^{\alpha} = i q (\phi_1^\star \mathfrak{D}_+^\alpha \phi_1 - \phi_1 [\mathfrak{D}_+^\alpha \phi_1]^\star) - i q (\phi_2^\star \mathfrak{D}_-^\alpha \phi_2 - \phi_2 [\mathfrak{D}_-^\alpha \phi_2]^\star)$$

$$\phi_1(\chi) \rightleftharpoons \phi_1(\chi) e^{-iq f(\chi)}, \phi_2(\chi) \leftrightharpoons \phi_2(\chi) e^{iq f(\chi)}, \mathrm{A}^\alpha(\chi) \binom{\circlearrowright}{\circlearrowleft} \mathrm{A}^\alpha(\chi) + \partial^\alpha f(\chi)$$

$$\mathcal{M}^2(\chi) = \begin{Vmatrix} m_1^2 & \mu^2 e^{iq f(\chi)} \\ -\mu^2 e^{-iq f(\chi)} & m_2^2 \end{Vmatrix} \cong \begin{Vmatrix} m_1^2 & \tilde{\mu}^2(\chi) \\ [-\tilde{\mu}^2(\chi)]^\star & m_2^2 \end{Vmatrix}$$

$$\mathfrak{R}(\chi) = \mathcal{N} \begin{pmatrix} \eta e^{-2iq f(\chi)} & 1 - \sqrt{1-\eta^2} \\ 1 - \sqrt{1-\eta^2} & \eta e^{2iq f(\chi)} \end{pmatrix}$$

$$\kappa_\alpha \prod^{\alpha\beta}(\kappa^2) = \frac{q^2}{16\varpi^2} \frac{\kappa^\beta \mu^4}{[\![\mathcal{M}_+^2 - \mathcal{M}_-^2]\!]^3} \langle \mathcal{M}_+^4 - \mathcal{M}_-^4 + 2\mathcal{M}_+^2 \mathcal{M}_-^2 \ln \left| \frac{\mathcal{M}_-^2}{\mathcal{M}_+^2} \right| \rangle$$

$$\begin{aligned} \mathfrak{L}_{\mathfrak{W}} = {} & [\mathfrak{D}_\alpha^+ \phi_1]^\star \mathfrak{D}_+^\alpha \phi_1 + [\mathfrak{D}_\alpha^- \phi_2]^\star \mathfrak{D}_-^\alpha \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 \\ & - \mu^2 (\mathfrak{W}^{\star 2}(\chi) \phi_1^\star \phi_2 - \mathfrak{W}^2(\chi) \phi_2^\star \phi_1) - \frac{1}{4} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} \end{aligned}$$

$$\mathfrak{W}(\chi) = \exp \left[iq \iiint^{\chi} \mathfrak{A}_\alpha d\gamma^\alpha \right] \mathfrak{W}(\chi) e^{iq f(\chi)}$$

$$\jmath_{A,+}^{\alpha} = i q (\phi_1^\star \mathfrak{D}^\alpha \phi_1 - \phi_1 [\mathfrak{D}^\alpha \phi_1]^\star) + i q (\phi_2^\star \mathfrak{D}^\alpha \phi_2 - \phi_2 [\mathfrak{D}^\alpha \phi_2]^\star)$$

$$\partial_\alpha \jmath_{A,+}^{\alpha} = 2 i q \mu^2 (\phi_2^\star \phi_1 - \phi_1^\star \phi_2) - \frac{1}{2\xi} (\partial_\alpha \mathrm{A}^\alpha)^2$$

$$\Box \mathrm{A}^\alpha - \left(1 - \frac{1}{\xi} \partial^\alpha \partial_\beta \mathrm{A}^\beta\right) = \jmath_{A,+}^{\alpha} \frac{1}{\xi} \Box \partial_\alpha \mathrm{A}^\alpha = 2 i q \mu^2 (\phi_2^\star \phi_1 - \phi_1^\star \phi_2)$$

$$\Box \pi_0 = 2 i q \mu^2 (\phi_1^\star \phi_2 - \phi_2^\star \phi_1)$$

$$\begin{aligned} \mathfrak{L}_\rho = {} & [\mathcal{D}_\alpha \phi_1]^\star \mathfrak{D}^\alpha \phi_1 + [\mathcal{D}_\alpha \phi_2]^\star \mathfrak{D}^\alpha \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 (\phi_1^\star \phi_2 - \phi_2^\star \phi_1) \\ & - \frac{1}{4} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} + \frac{1}{2} (m_0 \mathrm{A}_\alpha - \partial_\alpha \rho)(m_0 \mathrm{A}^\alpha - \partial^\alpha \rho) - \frac{1}{2\xi} (\partial_\alpha \mathrm{A}^\alpha + \xi m_0 \rho)^2 \end{aligned}$$

$$\phi_{1,2}(\chi) \rightleftharpoons \phi_{1,2}(\chi) e^{-iq f(\chi)}, \mathrm{A}^\alpha(\chi) \binom{\circlearrowright}{\circlearrowleft} \mathrm{A}^\alpha(\chi) + \partial^\alpha f(\chi), \rho(\chi) \binom{\circlearrowleft}{\circlearrowright} \rho(\chi) + m_0 f(\chi)$$

$$\mathrm{A}_\beta^\alpha = \frac{1}{\mathcal{Z}} \frac{\delta \mathcal{Z}}{\delta \mathfrak{J}_\alpha}$$

$$\mathcal{Z} = \iiint \mathfrak{D}[\mathrm{A}_\alpha, \phi, \phi^\ddagger] \exp \left(-\delta_\epsilon + \iiint d^4 \chi \left(\mathfrak{J}_\alpha \mathrm{A}^\alpha + \chi_1^{\mathcal{PT}} \phi_1 + \phi_1^{\mathcal{PT}} \chi_1 + \chi_2^{\mathcal{PT}} \phi_2 + \phi_2^{\mathcal{PT}} \chi_2 \right) \right)$$

$$\mathcal{Z} = \iiint \mathfrak{D}[\mathrm{A}_\alpha, \phi, \phi^\ddagger] \exp \left(-\delta_\epsilon + \iiint d^4 \chi \left(\mathfrak{J}_\alpha \mathrm{A}^\alpha + \chi_1^{\mathcal{PT}} \phi_1 + \phi_1^\star \chi_1 + \chi_2^{\mathcal{PT}} \phi_2 + \phi_2^\star \chi_2 \right) \right)$$

$$\begin{aligned} \mathcal{Z}^\star = \iiint \mathfrak{D}[\mathrm{A}_\alpha, \phi, \phi^\ddagger] \exp \Big(& -\delta_\epsilon^\star \\ & + \iiint d^4 \chi \left(\mathfrak{J}_\alpha \mathrm{A}^\alpha + \left(\chi_1^{\mathcal{PT}}\right)^\star \phi_1^\star + \phi_1 \chi_1^\star + \left(\chi_2^{\mathcal{PT}}\right)^\star \phi_2^\star + \phi_2 \chi_2^\star \right) \Big) \end{aligned}$$

$$\mathcal{Z}^{\star} = \iiint \mathfrak{D}[\mathrm{A}_{\alpha}, \phi, \phi^{\ddagger}] \exp\Big(-\delta_{\epsilon}^{\star} + \iiint d^4\chi\Big(\mathfrak{J}_{\alpha}\mathrm{A}^{\alpha} + \left(\chi_1^{\mathcal{PT}}\right)^{\star}\phi_1^{\star} + \phi_1\chi_1^{\star} + \left(\chi_2^{\mathcal{PT}}\right)^{\star}\phi_2^{\star} + \phi_2\chi_2^{\star}\Big)\Big)$$

$$\phi_2^{\beta} = \frac{1}{\mathcal{Z}}\frac{\delta\mathcal{Z}}{\delta\chi_2^{\mathcal{PT}}}, \left(\phi_2^{\beta}\right)^{\mathcal{PT}} = \frac{1}{\mathcal{Z}}\frac{\delta\mathcal{Z}}{\delta\chi_2} = -\left(\phi_2^{\beta}\right)^{\star}$$

$$\mathfrak{L} = [\mathcal{D}_{\alpha}\phi_1]^{\star}\mathfrak{D}^{\alpha}\phi_1 + [\mathcal{D}_{\alpha}\phi_2]^{\star}\mathfrak{D}^{\alpha}\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(\phi_1^{\star}\phi_2 - \phi_2^{\star}\phi_1) - \frac{g}{4}|\phi_1|^4 - \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{2\xi}(\partial_{\alpha}\mathrm{A}^{\alpha})^2$$

$$\mathfrak{L} = \partial_{\alpha}\hat{\phi}_1^{\star}\partial^{\alpha}\hat{\phi}_1 + \partial_{\alpha}\hat{\phi}_2^{\star}\partial^{\alpha}\hat{\phi}_2 - \mathcal{U}(\hat{\phi}_1, \hat{\phi}_2) - -\frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{2\xi}(\partial_{\alpha}\mathrm{A}^{\alpha})^2 + q^2\mathrm{A}_{\alpha}\mathrm{A}^{\alpha}\left(\left|v_1 + \hat{\phi}_1\right|^2 + \left|v_2 + \hat{\phi}_2\right|^2\right) - \mathrm{A}_{\alpha}j_{+}^{\alpha}$$

$$(-\mathcal{D}^2 - \widetilde{m}_1^2)\hat{\phi}_1 = \mu^2\hat{\phi}_2 - q^2v_1\mathcal{A}^2 + iqv_1\partial_{\alpha}\mathrm{A}^{\alpha} + \frac{g}{2}v_1^2\hat{\phi}_1^{\star} + \frac{g}{2}\left(v_1\hat{\phi}_1^2 + 2v_1\left|\hat{\phi}_1\right|^2 + \left|\hat{\phi}_1\right|^2\hat{\phi}_1\right)(-\mathcal{D}^2 - m_2^2)\hat{\phi}_2$$
$$= -\mu^2\hat{\phi}_1 - q^2v_2\mathrm{A}^2 + iqv_2\partial_{\alpha}\mathrm{A}^{\alpha}(-\Box - \mathcal{M}_{\mathcal{A}}^2)\mathrm{A}^{\alpha} + \left(1 - \frac{1}{\xi}\right)\partial^{\alpha}\partial_{\beta}\mathrm{A}^{\beta}$$
$$= 2q^2(v_1^{\star}\hat{\phi}_1 + v_1\hat{\phi}_1^{\star} + v_2^{\star}\hat{\phi}_2 + v_2\hat{\phi}_2^{\star})\mathrm{A}^{\alpha} + 2q^2\left(\left|\hat{\phi}_1\right|^2 + \left|\hat{\phi}_2\right|^2\right)\mathrm{A}^{\alpha} - j_{+}^{\alpha}$$

$$\mathcal{M}_{\mathcal{A}}^2 = (|v_1|^2 + |v_2|^2)$$

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APÉNDICE C.

1. Campos cuánticos relativistas, supergravedad cuántica y supermembranas.

1.1. Métrica de Green-Schwarz.

$$\mathfrak{S} = \iiint_{\Sigma} d^2\sigma\sqrt{-g} + \iiint_{\Sigma} \mathcal{L}_{WZ} + \iiint_{\partial\Sigma} \mathcal{A}$$
$$\mathcal{L}_{WZ} = -i\bar{\theta}\Gamma_\mu\Gamma_{11}d\theta\left(d\chi^\mu - \frac{i}{2}\bar{\theta}\Gamma^\mu d\theta\right)$$
$$g_{rs} = \prod\nolimits_r^\mu \prod\nolimits_s^\nu \eta_{\mu\nu}(r,s=\tau,\sigma), \prod\nolimits_r^\mu = \partial_r\chi^\mu - i\bar{\theta}\Gamma^\mu\partial_r\theta$$
$$\delta_\epsilon\chi^\mu = -i\bar{\theta}\Gamma^\mu\epsilon, \delta_\epsilon\theta = \epsilon, \delta_\kappa\chi^\mu = -i\bar{\theta}\Gamma^\mu(1+\Gamma)\kappa, \delta_\kappa\theta = (1+\Gamma)\kappa$$
$$\Gamma = \frac{\epsilon^{rs}}{2\sqrt{-g}}\prod\nolimits_r^\mu \prod\nolimits_s^\nu \Gamma_{\mu\nu}\Gamma_{11}, (\Gamma)^2 = \mathfrak{I}_{32}, \mathfrak{C}^{-1}\Gamma^{\mathfrak{T}}\mathfrak{C} = \Gamma$$

$$\delta_\kappa\mathcal{S} = \iiint_{\partial\Sigma}\left[\frac{1}{2}\left(\bar{\theta}\Gamma_\mu\Gamma_{11}d\theta\bar{\theta}\Gamma^\mu\delta_\kappa\theta + \bar{\theta}\Gamma_\mu\Gamma_{11}\delta_\kappa\theta\bar{\theta}\Gamma^\mu d\theta\right) + i\bar{\theta}\Gamma_\mu\Gamma_{11}\delta_\kappa\theta d\chi^\mu\right] + \iiint_{\partial\Sigma} d\chi^\mu\delta_\kappa\,\chi^\nu\mathfrak{F}_{\nu\mu}$$
$$d\chi^{\bar{\alpha}} = \bar{\theta}\Gamma^{\bar{\alpha}}d\theta = \bar{\theta}\Gamma^{\bar{\alpha}}(1+\Gamma)\kappa, \bar{\theta}\Gamma_{\underline{\mu}}\Gamma_{11}(1+\Gamma)\kappa = \mathfrak{F}_{\underline{\mu\nu}}\bar{\theta}\Gamma^{\underline{\nu}}(1+\Gamma)\kappa$$

$$\delta_\epsilon\mathcal{S} = \iiint_{\partial\Sigma}\left[-\frac{1}{6}\left(\bar{\theta}\Gamma_\mu\Gamma_{11}\epsilon\bar{\theta}\Gamma^\mu d\theta + \bar{\theta}\Gamma^\mu\epsilon\bar{\theta}\Gamma_\mu\Gamma_{11}d\theta\right) - i\bar{\theta}\Gamma_\mu\Gamma_{11}\epsilon d\chi^\mu\right] + \iiint_{\partial\Sigma} d\chi^\mu(-i\bar{\theta}\Gamma^\nu\epsilon)\mathfrak{F}_{\nu\mu}$$
$$d\chi^{\bar{\alpha}} = \bar{\theta}\Gamma^{\bar{\alpha}}d\theta = \bar{\theta}\Gamma^{\bar{\alpha}}\epsilon, \bar{\theta}\Gamma_{\underline{\mu}}\Gamma_{11}\epsilon = \mathfrak{F}_{\underline{\mu\nu}}\bar{\theta}\Gamma^{\underline{\nu}}\epsilon$$
$$\delta\chi^{\bar{\alpha}} = \bar{\theta}\Gamma^{\bar{\alpha}}\delta\theta = 0\ (\bar{a} = p+1,\ldots,9), \bar{\theta}\Gamma_{\underline{\mu}}\Gamma_{11}\delta\theta = F_{\underline{\mu\nu}}\bar{\theta}\Gamma^{\underline{\nu}}\delta\theta\ (\underline{\mu} = 0,1,\cdots,\wp)$$
$$\chi^{\bar{\alpha}} = \text{ const}, \theta = e^{\frac{1}{2}\gamma_{\underline{\mu\nu}}\Gamma^{\underline{\mu\nu}}\Gamma_{11}}(\Gamma_{11})^{\frac{\wp-2}{2}}\Gamma_{(\wp)}\theta, = e^{\frac{1}{4}\gamma_{\underline{\mu\nu}}\Gamma^{\underline{\mu\nu}}\Gamma_{11}}(\Gamma_{11})^{\frac{\wp-2}{2}}\Gamma_{(\wp)}e^{-\frac{1}{4}\gamma_{\underline{\mu\nu}}\Gamma^{\underline{\mu\nu}}\Gamma_{11}}\theta$$
$$\theta^{(+)} \equiv \frac{1}{2}\left(1 + e^{\frac{1}{2}\gamma_{\underline{\mu\nu}}\Gamma^{\underline{\mu\nu}}\Gamma_{11}}(\Gamma_{11})^{\frac{\wp-2}{2}}\Gamma_{(\wp))}\right)\theta$$
$$\phi^{\underline{\mu}} \equiv \sqrt{-g}g^{\sigma s}\Pi_s^{\underline{\mu}} - \mathfrak{F}^{\underline{\mu}}{}_{\underline{\nu}}\Pi_\tau^{\underline{\nu}}, \Pi_r^\mu\Gamma_\mu\left(\sqrt{-g}g^{rs}\partial_s + \epsilon^{rs}\Gamma_{11}\partial_s\right)\theta\Pi_\sigma^{\underline{\mu}}, \partial_\sigma\chi^{\underline{\mu}} = \partial_\sigma\theta^{(+)} = \partial\Sigma$$
$$\chi^+ = \tau, \Gamma^+\theta = 0, g_{rs} \propto \delta_{rs}, \partial_\sigma\chi^{\underline{\alpha}} = \mathfrak{F}_{\underline{\alpha\beta}}\partial_\tau\chi^{\underline{b}} + F_{\underline{a}+}, \left[1 + e^{\frac{1}{2}}Y_{ab}\underline{a^{\underline{ab}}\Gamma_{11}}(\Gamma_{11})^{\frac{p-2}{2}}\Gamma_{(p))}\right]\partial_\sigma\theta = \partial\Sigma$$
$$S = -\int_\Sigma d^3\xi\sqrt{-g} + \int_\Sigma \mathcal{L}_{WZ} + \int_{\partial\Sigma} B$$
$$\mathcal{L}_{WZ} = \frac{i}{2}\bar{\theta}\Gamma_{\mu\nu}d\theta\left[(dX^\mu - i\bar{\theta}\Gamma^\mu d\theta)dX^\nu - \frac{1}{3}\bar{\theta}\Gamma^\mu d\theta\bar{\theta}\Gamma^\nu d\theta\right]$$
$$g_{ij} = \Pi_i^\mu\Pi_j^\nu\eta_{\mu\nu}, (i,j = 0,1,2)$$
$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\theta}\Gamma^\mu\partial_i\theta$$
$$\delta_\epsilon X^\mu = -i\bar{\theta}\Gamma^\mu\epsilon, \delta_\epsilon\theta = \epsilon$$
$$\delta_\kappa X^\mu = i\bar{\theta}\Gamma^\mu(1+\Gamma)\kappa, \delta_\kappa\theta = (1+\Gamma)\kappa$$
$$\Gamma = \frac{\epsilon^{ijk}}{3!\sqrt{-g}}\Pi_i^\mu\Pi_j^\nu\Pi_k^\rho\Gamma_{\mu\nu\rho},$$
$$(\Gamma)^2 = I_{32}, \mathcal{C}^{-1}\Gamma^T\mathcal{C} = \Gamma, \Pi_i^\mu\Gamma\Gamma_\mu = \Pi_i^\mu\Gamma_\mu\Gamma = \frac{g_{im}}{2\sqrt{-g}}\epsilon^{mkl}\Pi_k^\nu\Pi_l^\rho\Gamma_{\nu\rho}$$

$$\delta_\kappa S = \int_{\partial\Sigma} \left[\frac{i}{2}\bar\theta\Gamma_{\mu\nu}d\theta\left(idX^\mu\bar\theta\Gamma^\nu\delta_\kappa\theta + \frac{1}{3}\bar\theta\Gamma^\mu d\theta\bar\theta\Gamma^\nu\delta_\kappa\theta\right)\right.$$
$$\left.+\frac{i}{2}\bar\theta\Gamma_{\mu\nu}\delta_\kappa\theta\left(dX^\mu dX^\nu - i\bar\theta\Gamma^\mu d\theta dX^\nu - \frac{1}{3}\bar\theta\Gamma^\mu d\theta\bar\theta\Gamma^\nu d\theta\right)\right]$$
$$+\int_{\partial\Sigma}\left(-\frac{i}{2}dX^\mu dX^\nu H_{\mu\nu\rho}\bar\theta\Gamma^\rho\delta_\kappa\theta\right), \qquad + \int_{\partial\Sigma}\frac{i}{2}dX^\mu dX^\nu H_{\mu\nu\rho}\bar\theta\Gamma^\rho\epsilon$$
$$\delta_\epsilon S = \int_{\partial\Sigma}\left[-\frac{i}{2}\bar\theta\Gamma_{\mu\nu}\epsilon\left(dX^\mu dX^\nu - \frac{i}{3}\bar\theta\Gamma^\mu d\theta dX^\nu - \frac{1}{15}\bar\theta\Gamma^\mu d\theta\bar\theta\Gamma^\nu d\theta\right)\right.$$
$$\left.+\frac{1}{6}\bar\theta\Gamma^\nu\epsilon\bar\theta\Gamma_{\mu\nu}d\theta\left(dX^\mu - \frac{i}{5}\bar\theta\Gamma^\mu d\theta\right)\right]$$
$$\delta X^{\bar a} = \bar\theta\Gamma^{\bar a}\delta\theta$$
$$\bar\theta\Gamma_{\underline{\mu\nu}}\delta\theta = H_{\underline{\mu\nu\rho}}\bar\theta\Gamma^{\underline{\rho}}\delta\theta$$
$$\theta = F\left(\Gamma^{\underline{\mu}}; H_{\underline{\mu\nu\rho}}\right)\Gamma_{(p)}\theta, F(\Gamma^{\underline{\mu}}; 0) = I_{32}$$
$$I_{32} = \left(F\left(\Gamma^{\underline{\mu}}; H_{\underline{\mu\nu\rho}}\right)\Gamma_{(p)}\right)^2$$
$$\theta = \exp\left(\frac{-1}{3}h_{\underline{\mu\nu\rho}}\Gamma^{\underline{\mu\nu\rho}}\right)\Gamma_{(5)}\theta, h_{\underline{\mu\nu\rho}} = \frac{1}{3!}\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}h^{\underline{\sigma\kappa\lambda}}(= \text{ const. })\ldots$$
$$H_{\underline{\mu\nu\rho}} = 4h_{\underline{\mu\nu\sigma}}(1-2k)^{-1\sigma}_{\underline{\rho}},$$
$$\Phi^\mu \equiv \sqrt{-g}g^{1k}\Pi^{\underline{\mu}}_k - H^{\underline{\mu}}{}_{\underline{\nu\rho}}\Pi^{\underline{\nu}}_2\Pi^{\underline{\rho}}_\tau$$
$$\partial_1 X^{\underline{\mu}} = 0$$
$$(1+\Gamma_{(p)})\partial_1\theta = 0$$
$$\delta_v X^\mu = v^i\partial_i X^\mu, \delta_v(\partial_i X^\mu) = v^j\partial_j(\partial_i X^\mu) + \partial_i v^j\partial_j X^\mu$$
$$\delta_v S = \int_\Sigma d^3\xi\partial_i(v^i\mathcal{L}) = -\int_{\partial\Sigma}d\tau d\sigma^2 v^1\mathcal{L}.$$
$$\begin{aligned} v^1 &= 0 \\ \partial_1 v^0 = \partial_1 v^2 &= 0 \end{aligned}$$

1.2. Formulación de gauge.

$$X^+ = \tau$$
$$\Gamma^+\theta = 0$$
$$g_{\tau r} = \partial_r X^- - i\bar\theta\Gamma^-\partial_r\theta + \partial_\tau X^a\partial_r X^a,$$
$$g_{\tau\tau} = 2(\partial_\tau X^- - i\bar\theta\Gamma^-\partial_\tau\theta) + \partial_\tau X^a\partial_\tau X^a = -\frac{1}{\left(P_0^+\sqrt{w}\right)^2}\det(g_{rs}) = -\frac{1}{2(P_0)^2}\left(\{X^a, X^b\}\right)^2$$
$$\{A, B\} = \frac{\epsilon^{rs}}{\sqrt{w}}\partial_r A\partial_s B$$
$$S = \int d\tau\int_{\Sigma^{(2)}}d^2\sigma\sqrt{w}\left[\frac{1}{2}P_0^+(\partial_\tau X^a)^2 - iP_0^+\bar\theta\Gamma^-\partial_\tau\theta - \frac{1}{4P_0^+}\left(\{X^a, X^b\}\right)^2 + i\bar\theta\Gamma^-\Gamma^a\{X^a, \theta\}\right]$$
$$S = \int d\tau\int_{\Sigma^2}d^2\sigma\sqrt{w}\left[\frac{P_0^+}{2}(\partial_\tau X^a)^2 + \frac{i}{2}\theta^T\partial_\tau\theta - \frac{1}{4P_0^+}\left(\{X^a, X^b\}\right)^2 + \frac{i}{2P_0^+}\theta^T\gamma^a\{X^a, \theta\}\right]$$
$$\delta X^{\bar a} = (1-\Gamma_{(5)})\theta$$
$$\partial_1 X^{\underline{\mu}} = (1+\Gamma_{(5)})\partial_1\theta = \partial\Sigma$$
$$w(\sigma) = \det(w_{rs}(\sigma)),\ w_{12}|_{\partial\Sigma^{(2)}} = 0,\ \partial_r\left(\sqrt{w}w^{r1}\right)\big|_{\partial\Sigma^{(2)}}$$
$$X^{\bar a}(\sigma) = \sum_A Y_A^{(D)}(\sigma)X^{\bar a A}, \quad \theta^{(+)}(\sigma) = \sum_A Y_A^{(D)}(\sigma)\theta^{(+)A},$$
$$X^{\underline{a}}(\sigma) = \sum_A Y_A^{(N)}(\sigma)X^{\underline{a}A}, \quad \theta^{(-)}(\sigma) = \sum_A Y_A^{(N)}(\sigma)\theta^{(-)A},$$

$$\Delta Y_A^{(D,N)} \equiv \frac{1}{\sqrt{w}}\partial_r\left(\sqrt{w}w^{rs}\partial_s Y_A^{(D,N)}\right) = -\omega_A^{(D,N)}Y_A^{(D,N)}$$
$$\int_{\Sigma^{(2)}} d^2\sigma\sqrt{w}Y_A^{(D)}\left(Y_B^{(D)}\right)^* = \delta_A^B$$
$$\int_{\Sigma^{(2)}} d^2\sigma\sqrt{w}Y_A^{(N)}\left(Y_B^{(N)}\right)^* = \delta_A^B$$
$$\theta|_{SO(10,1)} = \frac{1}{2^{3/4}\sqrt{P_0^+}}\begin{pmatrix}0\\ \theta|_{SO(9)}\end{pmatrix}$$
$$\left(X^{\bar{a}}(\sigma), P^{\bar{b}}(\sigma')\right)_{DB} = \delta^{\bar{a}\bar{b}}\delta^{(D)}(\sigma,\sigma'),$$
$$\left(\theta_\alpha^{(+)}(\sigma), \theta_\beta^{(+)}(\sigma')\right)_{DB} = \frac{-i}{\sqrt{w(\sigma)}}\left(\frac{1+\gamma_{(4)}}{2}\right)_{\alpha\beta}\delta^{(D)}(\sigma,\sigma'),$$
$$\left(X^{\underline{a}}(\sigma), P^{\underline{b}}(\sigma')\right)_{DB} = \delta^{\underline{ab}}\delta^{(N)}(\sigma,\sigma')$$
$$\left(\theta_\alpha^{(-)}(\sigma), \theta_\beta^{(-)}(\sigma')\right)_{DB} = \frac{-i}{\sqrt{w(\sigma)}}\left(\frac{1-\gamma_{(4)}}{2}\right)_{\alpha\beta}\delta^{(N)}(\sigma,\sigma')$$
$$\delta^{(D,N)}(\sigma,\sigma') \equiv \sqrt{w(\sigma)}\sum_A Y_A^{(D,N)}(\sigma)\left(Y_A^{(D,N)}(\sigma')\right)^*$$
$$H = \int_{\Sigma(2)} d^2\sigma\frac{\sqrt{w}}{P_0^+}\left[\frac{(P^a)^2}{2w} + \frac{1}{4}\left(\{X^a,X^b\}\right)^2 - \frac{i}{2}\theta^T\gamma^a\{X^a,\theta\}\right] = \frac{\left(P_0^{\underline{a}}\right)^2 + \mathcal{M}^2}{2P_0^+}$$
$$\varphi(\sigma) = -\left\{\frac{P^a}{\sqrt{w}}, X^a\right\} - \frac{i}{2}\{\theta^T,\theta\}$$
$$\varphi_\lambda = \int d^2\sigma\phi^{(\lambda)^r}\left(P^a\partial_r X^a + \frac{i}{2}\sqrt{w}\theta^T\partial_r\theta\right)$$
$$\delta_\zeta X^a = \{\zeta, X^a\}$$
$$\delta_{(\lambda)}X^a = \phi^{(\lambda)r}\partial_r X^a$$
$$\zeta(\sigma) = \partial\Sigma$$
$$S = \int dt\int_{\Sigma^{(2)}} d^2\sigma\sqrt{w}\left[\frac{1}{2}(D_tX^a)^2 + \frac{i}{2}\theta^T D_t\theta - \frac{1}{4}\left(\{X^a,X^b\}\right)^2 + \frac{i}{2}\theta^T\gamma^a\{X^a,\theta\}\right],$$
$$\delta_\zeta\omega = D_t\zeta = \partial_t\zeta - \{\omega,\zeta\}$$
$$\omega(t,\sigma) = \partial\Sigma$$

1.3. Regularizaciones.

$$(w_{rs}) = \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}.$$
$$\text{Dirichlet (DD sector)}: Y_A^{(D)}(\sigma^1 = 0,\sigma^2) = Y_A^{(D)}(\sigma^1 = 1/2,\sigma^2)$$
$$\text{Neumann (NN sector)}: \partial_1 Y_A^{(N)}(\sigma^1 = 0,\sigma^2) = \partial_1 Y_A^{(N)}(\sigma^1 = 1/2,\sigma^2)$$
$$Y_A^{(D)}(\sigma) = \sqrt{2}e^{2\pi i A_2\sigma^2}\sin(2\pi A_1\sigma^1)$$
$$Y_A^{(N)}(\sigma) = \sqrt{2}e^{2\pi i A_2\sigma^2}\cos(2\pi A_1\sigma^1)\ \text{para}\ A_1 \neq 0$$
$$Y_{(0,A_2)}^{(N)}(\sigma) = e^{2\pi i A_2\sigma^2}$$
$$\int_{\Sigma^{(2)}} d^2\sigma\sqrt{w}[(\text{ Dirichlet }) \times (\text{ Dirichlet }) + (\text{ Neumann }) \times (\text{ Neumann })]$$
$$S = \int dt\mathrm{Tr}\left(\frac{1}{2}(D_tX^a)^2 + \frac{i}{2}\theta^T D_t\theta + \frac{1}{4}\left([X^a,X^b]\right)^2 - \frac{1}{2}\theta^T\gamma^a[X^a,\theta]\right),$$
$$\varphi = -i[P^a,X^a] + \frac{1}{2}[\theta^\alpha,\theta^\alpha]_+$$

$$\varphi_2 = \int d^2\sigma\sqrt{w}\left[P_0^+\partial_2 X^- + \frac{P^a}{\sqrt{w}}\partial_2 X^a + \frac{i}{2}\theta^T\partial_2\theta\right] \approx 0.$$

$$\int_0^{1/2} d\sigma e^{2\pi i m\sigma} = \begin{cases} 1/2 & m = 0 \\ i\dfrac{1-(-1)^m}{2\pi m} & m \neq 0 \end{cases}$$

$$\mathrm{Tr}'(AB) = \mathrm{Tr}'(BA)$$

$$\mathrm{Tr}'A = \mathrm{Tr}\mathcal{P}A, \mathcal{P} = \begin{pmatrix} I_M & 0_M \\ 0_M & 0_M \end{pmatrix}$$

$$\frac{1}{N}\mathrm{Tr}'U^mV^n = \begin{cases} \frac{1}{2}\delta_{n0} & m = 0 \\ \delta_{n0}\dfrac{1-(-1)^m}{N(1-\omega^m)} & m \neq 0 \end{cases}$$

$$U' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega' & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & (\omega')^{N-1} \end{pmatrix} \omega' = e^{\pi i/N}$$

1.4. Algebra de gauge.

$$Q^- = \sqrt{P_0^+}\int d^2\sigma\sqrt{w}\theta$$

$$Q^+ = \frac{1}{\sqrt{P_0^+}}\int d^2\sigma\left(P^a\gamma_a + \frac{\sqrt{w}}{2}\{X^a, X^b\}\gamma_{ab}\right)\theta$$

$$\delta_- X^a = 0, \delta_-\theta = \epsilon'$$

$$\delta_+ X^a = \epsilon\gamma^a\theta, \delta_+\theta = +i\left(\frac{P^a}{\sqrt{w}}\gamma_a - \frac{1}{2}\{X^a, X^b\}\gamma_{ab}\right)\epsilon$$

$$Q_{(-)}^- \equiv \frac{1-\gamma_{(4)}}{2}Q^-$$

$$Q_{(+)}^+ \equiv \frac{1+\gamma_{(4)}}{2}Q^+$$

$$i\left(Q_{(-)\alpha}^-, Q_{(-)\beta}^-\right)_{DB} = \left(\mathcal{P}^{(-)}\right)_{\alpha\beta}P_0^+$$

$$i\left(Q_{(-)\alpha}^-, Q_{(+)\beta}^+\right)_{DB} = \left(\mathcal{P}^{(-)}\gamma_{\underline{a}}\right)_{\alpha\beta}P_0^{\underline{a}} + \left(\mathcal{P}^{(-)}\gamma_{\bar{a}\underline{b}}\right)_{\alpha\beta}Z^{\bar{a}\underline{b}},$$

$$i\left(Q_{(+)\alpha}^+, Q_{(+)\beta}^+\right)_{DB} = 2H\left(\mathcal{P}^{(+)}\right)_{\alpha\beta} + 2(\mathcal{P}^+\gamma_{\bar{a}})_{\alpha\beta}Z^{\bar{a}}$$

$$Z^{ab} = -\int d^2\sigma\sqrt{w}\{X^a, X^b\},$$

$$Z^a = \frac{1}{P_0^+}\int d^2\sigma\sqrt{w}\left(\{X^a, X^b\}\frac{P^b}{\sqrt{w}} + \frac{i}{2}\theta^T\{X^a, \theta\}\right)$$

$$= -\int d^2\sigma\sqrt{w}\{X^a, X^-\}$$

$$Q \equiv \begin{pmatrix} \sqrt{2}Q^- \\ Q^+ \end{pmatrix}$$

$$\tilde{Q} \equiv \begin{pmatrix} \sqrt{2}Q_{(-)}^- \\ Q_{(+)}^+ \end{pmatrix} = \frac{1-\Gamma_{(5)}}{2}Q$$

$$i\left(\tilde{Q},\tilde{Q}^T\right)_{DB} = \begin{pmatrix} 2P_0^+\mathcal{P}^{(-)} & \sqrt{2}\mathcal{P}^{(-)}\left(\not{P}+\not{Z}_{(2)}\right) \\ \sqrt{2}\mathcal{P}^{(+)}\left(\not{P}-\not{Z}_{(2)}\right) & 2\mathcal{P}^{(+)}\left(H\cdot I_{16}+\not{Z}_{(1)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}P_0^+\mathcal{P}^{(-)} & 0 \\ \mathcal{P}^{(+)}\left(\not{P}-\not{Z}_{(2)}\right) & \mathcal{P}^{(+)} \end{pmatrix} \begin{pmatrix} \frac{1}{P_0^+}\mathcal{P}^{(-)} & 0 \\ 0 & \frac{1}{P_0^+}\mathbf{m} \end{pmatrix} \begin{pmatrix} \sqrt{2}P_0^+\mathcal{P}^{(-)} & \mathcal{P}^{(-)}\left(\not{P}-\not{Z}_{(2)}\right) \\ 0 & \mathcal{P}^{(+)} \end{pmatrix}$$

$$\mathbf{m} \equiv \mathcal{P}^{(+)}\left[2P_0^+\left(H\cdot I_{16}+\not{Z}_{(1)}\right)-\left(P-\not{Z}_{(2)}\right)\left(P+\not{Z}_{(2)}\right)\right]$$

$$= P^{(+)}\left[\left(M^2-Z^{\bar{a}\underline{b}}-Z^{\bar{a}\underline{b}}-\right)I_{16}+2\left(Z^{\bar{a}}P_0^+ + Z^{\bar{a}\underline{c}}-P_0^{\underline{c}}\right)\gamma_{\bar{a}}-Z^{\bar{a}\underline{b}}-Z^{\underline{c}\underline{d}}-\gamma_{\bar{a}\bar{c}\underline{b}\underline{d}}\right]$$

$$\int d^2\sigma\sqrt{w}\{X^{[a},X^b\}\{X^c,X^{d]}\}$$

1.5. Configuraciones BPS.

$$X^{\bar{a}}\big|_{\sigma^1=0}=0,\ X^{\bar{a}}\big|_{\sigma^1=1/2}=b\delta_{10}^{\bar{a}},\ \partial_1 X^{\underline{a}}|_{\sigma^1=0}=\partial_1 X^{\underline{a}}|_{\sigma^1=1/2}=0$$

$$X^{\underline{a}}\sim X^{\underline{a}}+2\pi R^{\underline{a}}, X^-\sim X^-+2\pi R$$

$$X^{\underline{a}}=\frac{Rm^{\underline{a}}}{R^{\underline{a}}m}\tau+2\pi R^{\underline{a}}n^{\underline{a}}\sigma^2+X_0^{\underline{a}}+\hat{X}^{\underline{a}}(\tau,\sigma),$$

$$X^{\bar{a}}=2b\delta_{10}^{\bar{a}}\sigma^1+\hat{X}^{\bar{a}}(\tau,\sigma),$$

$$X^-=-\frac{R}{m}H\tau+2\pi Rn\sigma^2+\hat{X}^-(\tau,\sigma)$$

$$\theta^{(-)}=\theta_0^{(-)}+\hat{\theta}^{(-)}(\tau,\sigma),\theta^{(+)}=\hat{\theta}^{(+)}(\tau,\sigma)$$

$$0\approx\varphi(\sigma)=\nabla^a\left(\frac{\hat{P}^a}{\sqrt{w}}\right)+\left\{\hat{X}^a,\frac{\hat{P}^a}{\sqrt{w}}\right\}-\frac{i}{2}\{\hat{\theta}^T,\hat{\theta}\}$$

$$0\approx\varphi_2=2\pi(nm+n^{\underline{a}}m^{\underline{a}})+\int d^2\sigma\left(\hat{P}^a\partial_2\hat{X}^a+\frac{i}{2}\sqrt{w}\hat{\theta}^T\partial_2\hat{\theta}\right)$$

$$Z^{\bar{a}\bar{b}}=Z^{\underline{ab}}=Z^{\underline{a}}=0,$$

$$Z^{\bar{a}\underline{b}}=-\left(2\pi bR^{\underline{b}}n^{\underline{b}}\right)\delta_{10}^{\bar{a}}$$

$$Z^{\bar{a}}=-(2\pi bRn)\delta_{10}^{\bar{a}}$$

$$0=\mathcal{M}^2-Z^{\bar{a}b}-Z^{\bar{a}\underline{b}}$$

$$=\int d^2\sigma\sqrt{w}\left[\left(\hat{P}^a/\sqrt{w}\right)^2+\frac{1}{2}\left(\{X^a,X^b\}+Z^{ab}\right)^2-i\hat{\theta}^T\gamma^a\{X^a,\hat{\theta}\}\right]$$

$$0=Z^{\bar{a}}P_0^+ + Z^{\bar{a}\underline{c}}P_0^{\underline{c}}=-2\pi b\delta_{10}^{\bar{a}}(nm+n^{\underline{c}}m^{\underline{c}})$$

$$\approx b\delta_{10}^{\bar{a}}\int d^2\sigma\left(\hat{P}^a\partial_2\hat{X}^a+\frac{i}{2}\sqrt{w}\hat{\theta}^T\partial_2\hat{\theta}\right)$$

$$X^{\underline{a}}=\frac{Rm^{\underline{a}}}{R^{\underline{a}}m}\tau+2\pi R^{\underline{a}}n^{\underline{a}}\sigma^2,X^{\bar{a}}=2b\delta_{10}^{\bar{a}}\sigma^1,$$

$$X^-=-\frac{R}{m}H\tau+2\pi Rn\sigma^2\ (mn+n^{\underline{a}}m^{\underline{a}})$$

$$\theta^{(-)}=\theta_0^{(-)},\theta^{(+)}=0$$

$$\mathcal{M}^2-Z^{\bar{a}\underline{b}}Z^{\bar{a}\underline{b}}\mp 2\left(Z^{10}P_0^+ + Z^{10\underline{c}}P_0^{\underline{c}}\right)$$

$$\hat{P}^{10}=0$$

$$\frac{\hat{P}^i}{\sqrt{w}}=\pm\left(\{X^{10},X^i\}+Z^{10i}\right)$$

$$0=\{X^i,X^j\}+Z^{ij}$$

$$\tilde{Q}^{(\mp)}\equiv\mathcal{P}^{(+)}\frac{1\mp\gamma_{10}}{2}\int d^2\sigma\left[\hat{P}^a\gamma_a+\frac{\sqrt{w}}{2}\left(\{X^a,X^b\}+Z^{ab}\right)\gamma_{ab}\right]\hat{\theta}$$

$$X^{\bar{a}} = 2b\delta_{10}^{\bar{a}}\sigma^1,$$
$$X^{\underline{a}} = \frac{Rm^{\underline{a}}}{R^{\underline{a}}m}\tau + 2\pi R^{\underline{a}}n^{\underline{a}}\sigma^2 + \hat{X}^{\underline{a}}(\tau,\sigma^2)$$
$$\int d^2\sigma\sqrt{w}\epsilon^{\underline{abcd}}\{X^{\underline{a}},X^{\underline{b}}\}\{X^{\underline{c}},X^{\underline{d}}\}$$
$$X^- = -\frac{R}{m}H\tau + 2\pi Rn\sigma^2 + \hat{X}^-(\tau,\sigma^2)$$
$$\theta^{(+)} = 0$$
$$\theta^{(-)} = \theta_0^{(-)} + \hat{\theta}^{(-)}(\tau,\sigma^2)$$
$$0 \approx \varphi_2 = 2\pi(nm + n^{\underline{a}}m^{\underline{a}}) + \oiiint d\sigma^2\left(\frac{\hat{P}^{\underline{a}}}{\sqrt{w}}\partial_2\hat{X}^{\underline{a}} + \frac{i}{2}\hat{\theta}^{(-)T}\partial_2\hat{\theta}^{(-)}\right)$$
$$\partial_\tau\hat{X}^{\underline{a}} = \pm\frac{Rb}{m}\partial_2\hat{X}^{\underline{a}}$$
$$\hat{\theta}^{(-)} = \mp\gamma_{10}\hat{\theta}^{(-)}$$

1.6. Métrica de Clifford.

$$\Gamma^\mu\Gamma^\nu = \eta^{\mu\nu}I_{32} + \Gamma^{\mu\nu}$$
$$\Gamma^{\mu_1\cdots\mu_n} \equiv \Gamma^{[\mu_1}\dots\Gamma^{\mu_n]}.$$
$$\mathcal{C}^{-1}(\Gamma^\mu)^T\mathcal{C} = -\Gamma^\mu$$
$$\mathcal{C}^{-1}(\Gamma^{\mu_1\cdots\mu_n})^T\mathcal{C} = (-)^{\frac{n(n+1)}{2}}\Gamma^{\mu_1\cdots\mu_n}$$
$$\left(\mathcal{C}\Gamma_{\mu\nu}\right)_{(\alpha\beta}(\mathcal{C}\Gamma^\nu)_{\gamma\delta)}$$
$$\mathcal{C}^{-1}\left(\Gamma_{(p)}\right)^T\mathcal{C} = (-)^p\left(\Gamma_{(p)}\right)^{-1}$$
$$\Gamma^0 = \mathcal{C} = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix}, \Gamma^1 = \begin{pmatrix} 0 & -I_{16} \\ -I_{16} & 0 \end{pmatrix}, \Gamma^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix}$$
$$\Gamma_{(5)} = \Gamma_{01\cdots5} = \frac{\epsilon^{\underline{\mu\nu\rho\sigma\kappa\lambda}}}{6!}\Gamma_{\underline{\mu\nu\rho\sigma\kappa\lambda}}$$
$$\left(\Gamma_{(5)}\right)^2 = I_{32}, \Gamma_{(5)}\Gamma^{\underline{\mu}} + \Gamma^{\underline{\mu}}\Gamma_{(5)}$$
$$\Gamma_{\underline{\mu\nu\rho\sigma\kappa\lambda}} = -\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}\Gamma_{(5)},$$
$$\Gamma_{\underline{\mu\nu\rho\sigma\kappa}} = -\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}\Gamma_{(5)}\Gamma^{\underline{\lambda}},$$
$$\Gamma_{\underline{\mu\nu\rho\sigma}} = \frac{1}{2}\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}\Gamma_{(5)}\Gamma^{\underline{\kappa\lambda}},$$
$$\Gamma_{\underline{\mu\nu\rho}} = \frac{1}{3!}\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}\Gamma_{(5)}\Gamma^{\underline{\sigma\kappa\lambda}}$$
$$h_{\underline{\mu\nu\rho}} = \frac{1}{3!}\epsilon_{\underline{\mu\nu\rho\sigma\kappa\lambda}}h^{\underline{\sigma\kappa\lambda}}$$
$$k_{\underline{\mu}}^{\underline{\mu}} = \mathrm{h}^{\underline{\mu\nu\rho}}h_{\underline{\mu\nu\rho}}, \mathrm{h}^{\underline{\mu\nu\kappa}}h_{\underline{\rho\sigma\kappa}} = \delta_{[\underline{\rho}}^{[\underline{\mu}}\kappa_{\underline{\sigma}]}^{\underline{\nu}]}, k_{\underline{\sigma}}^{\underline{\mu}}\mathrm{h}^{\underline{\sigma\nu\rho}} = k_{\underline{\sigma}}^{\underline{\nu}}\mathrm{h}^{\underline{\mu\sigma\rho}}, h^{\mu[\underline{\nu}}{}_{\underline{\kappa}}h^{\underline{\rho\sigma}]\underline{\kappa}}, k_{\underline{\mu}}^{\underline{\rho}}k_{\underline{\rho}}^{\underline{\nu}}$$
$$= \frac{1}{6}\delta_{\underline{\mu}}^{\underline{\nu}}\left(k_{\underline{\sigma}}^{\underline{\rho}}k_{\underline{\rho}}^{\underline{\sigma}}\right), k_{\underline{\mu}}^{\underline{\sigma}}k_{\underline{\nu}}^{\underline{\kappa}}h_{\underline{\sigma\kappa\rho}} = \frac{1}{6}\left(k_{\underline{\kappa}}^{\underline{\lambda}}k_{\underline{\lambda}}^{\underline{\kappa}}\right)h_{\underline{\mu\nu\rho}}$$
$$\theta' \equiv \exp\left(\frac{1}{3!}h_{\underline{\mu\nu\rho}}\Gamma^{\underline{\mu\nu\rho}}\right)\theta = \Gamma_{(5)}\theta'$$
$$0 = \bar{\theta}'\Gamma_{\underline{\mu\nu}}\delta\theta'$$
$$= \bar{\theta}\exp\left(\frac{1}{3!}h_{\underline{\rho\sigma\kappa}}\Gamma^{\underline{\rho\sigma\kappa}}\right)\Gamma_{\underline{\mu\nu}}\exp\left(\frac{1}{3!}h_{\underline{\rho\sigma\kappa}}\Gamma^{\underline{\rho\sigma\kappa}}\right)\delta\theta$$
$$= \bar{\theta}\Gamma_{\underline{\mu\nu}}\delta\theta - 2h_{\underline{\mu\nu\rho}}\bar{\theta}\Gamma^{\underline{\rho}}\left(1 + \Gamma_{(5)}\right)\delta\theta$$
$$\Gamma_{(5)}\theta = \left(1 + \frac{1}{3}h_{\underline{\mu\nu\rho}}\Gamma^{\underline{\mu\nu\rho}}\right)\theta$$

$$(1+2k)\underline{{}^{\mu}_{\nu}}\Gamma^{\underline{\nu}}\Gamma_{(5)}\theta=(1-2k)\underline{{}^{\mu}_{\nu}}\Gamma^{\underline{\nu}}\theta+2\mathrm{h}^{\underline{\mu\nu\rho}}\Gamma_{\underline{\nu\rho}}\theta$$
$$\bar{\theta}\Gamma_{\underline{\mu\nu}}\delta\theta-k^{\rho}_{\underline{\mu}}\bar{\theta}\Gamma_{\underline{\rho\nu}}\delta\theta-k^{\rho}_{\underline{\nu}}\Gamma_{\underline{\mu\rho}}\delta\theta=4h_{\underline{\mu\nu\rho}}\bar{\theta}\Gamma^{\underline{\rho}}\delta\theta$$
$$\bar{\theta}\Gamma_{\underline{\mu\nu}}\delta\theta=4\left(1-\frac{2}{3}k^{\rho}_{\underline{\sigma}}k^{\sigma}_{\underline{\rho}}\right)^{-1}\left(h_{\underline{\mu\nu\kappa}}+k^{\lambda}_{\underline{\mu}}h_{\underline{\lambda\nu\kappa}}+k^{\lambda}_{\underline{\nu}}h_{\underline{\mu\lambda\kappa}}\right)\bar{\theta}\Gamma^{\underline{\kappa}}\delta\theta$$

1.7. Aproximaciones matriciales.

$$Y_A(\sigma)\equiv e^{2\pi i(A_1\sigma^1+A_2\sigma^2)}\overset{N\to\infty}{\longleftarrow}T_A\equiv\frac{1}{\sqrt{N}}e^{-\frac{\pi i}{N}A_1A_2}V^{A_1}U^{A_2}$$
$$U^N=V^N=1,VU=e^{\frac{2\pi}{N}i}UV,U^{\dagger}=U^{-1},V^{\dagger}=V^{-1}$$
$$(T_A)^{\dagger}=T_{-A}$$
$$U=\begin{pmatrix}1&&&&0\\&e^{\frac{2\pi}{N}i}&&&\\&&\ddots&&\\&&&\ddots&\\0&&&&e^{\frac{2\pi i}{N}(N-1)}\end{pmatrix},V=\begin{pmatrix}0&1&&&0\\&0&1&&\\&&\ddots&\ddots&\\&&&\ddots&1\\1&&&&0\end{pmatrix}$$
$$T_{(-A_1,A_2)}=\left(T_{(A_1,A_2)}\right)^T$$
$$Y_A^{(D)}=\frac{-i}{\sqrt{2}}\left(Y_{(A_1,A_2)}-Y_{(-A_1,A_2)}\right)\overset{N\to\infty}{\longleftarrow}T_A^{(D)}\equiv\frac{-i}{\sqrt{2}}(T_A-(T_A)^T),$$
$$Y_A^{(N)}=\frac{1}{\sqrt{2}}\left(Y_{(A_1,A_2)}+Y_{(-A_1,A_2)}\right)\overset{N\to\infty}{\longleftarrow}T_A^{(N)}\equiv\frac{1}{\sqrt{2}}(T_A-(T_A)^T)$$
$$T_{(-A_1,A_2)}=\left(T_{(A_1,A_2)}\right)^*$$
$$\int_0^{1/2}d\sigma^1\int_0^1 d\sigma^2 2F(\sigma)=\int_0^1 d\sigma^1\int_0^1 d\sigma^2F(\sigma)$$
$$\int_{\Sigma^{(2)}}d^2\sigma\sqrt{w}A\{B,C\}=\int_{\Sigma^{(2)}}d^2\sigma\sqrt{w}B\{C,A\}$$
$$\text{L.H.S.}=\int d^2\sigma A\epsilon^{rs}\partial_rB\partial_sC$$
$$=-\int d^2\sigma B\epsilon^{rs}\partial_rA\partial_sC+\int d\sigma^2(AB\partial_2C)||_{\sigma^1=0}^{\sigma^1=1/2}$$
$$\text{L.H.S.}=\int d^2\sigma\sqrt{w}B\{C,A\}+\int d\sigma^2(AB\partial_2C)\Big|_{\sigma^1=0}^{\sigma^1=1/2}=R.H.S.$$

1.8. Soluciones BPS.

$$X^-=-Ht+2\pi Rn_r\sigma^r+\hat{X}^-(\sigma^1,\sigma^2,t)$$
$$X^a=\frac{m^a}{R^a}t+2\pi R^an_r^a\sigma^r+\hat{X}^a(\sigma^1,\sigma^2,t)$$
$$P^a\equiv\partial_tX^a=\frac{m^a}{R^a}+\hat{P}^a(\sigma^1,\sigma^2,t)$$
$$V^9=e_a^{(9)}V^a,V^i=e_a^{(i)}V^a$$
$$\hat{P}^9=0$$
$$\hat{P}^i=\pm\left(\left\{X^9,X^i\right\}+z^{9i}\right)$$
$$0=\left\{X^i,X^j\right\}+z^{ij}$$
$$0=\varphi(\sigma)=\nabla^a\hat{P}^a+\left\{\hat{X}^a,\hat{P}^a\right\}$$
$$0=\varphi_r=mn_r+m^an_r^a+\frac{1}{2\pi}\int d^2\sigma\hat{P}^a\partial_r\hat{X}^a$$
$$0=P_0^+z^i-P_0^cz^{ci}$$

$$P_0^+ z^i - P_0^c z^{ci} = \int d^2\sigma \hat{P}^c \nabla^i \hat{X}^c$$
$$= \int d^2\sigma \hat{P}^c \left(\nabla^i \hat{X}^c - \nabla^c \hat{X}^i + \{\hat{X}^i, \hat{X}^c\}\right)$$
$$= \int d^2\sigma \hat{P}^j \left(\{X^i, X^j\} + z^{ij}\right)$$
$$X^9 = P_0^9 t + 2\pi \tilde{R}_r^9 \sigma^r + \xi(\sigma^1, \sigma^2)$$
$$X^9 = P_0^9 t + 2\pi \tilde{R}_r^9 \sigma^r + \xi\left(\tilde{R}_r^9 \sigma^r\right)$$
$$\nabla^i \hat{X}^j - \nabla^j \hat{X}^i = 0, \nabla^i \equiv e_a^{(i)} \nabla^a = 2\pi\left(\tilde{R}_1^i \partial_2 - \tilde{R}_2^i \partial_1\right)$$
$$\hat{X}^i = \nabla^i \epsilon_{ij}(\sigma^1, \sigma^2, t) + \eta^i(\sigma^1, \sigma^2, t)$$
$$\hat{X}^j = \nabla^j \epsilon_{ij}(\sigma^1, \sigma^2, t) + \eta^i(\sigma^1, \sigma^2, t)$$
$$\nabla^j \eta^i = \nabla^i \eta^j$$
$$X^i = P_0^i t + 2\pi k^i \tilde{R}_r \sigma^r + \nabla^i \epsilon(\sigma^1, \sigma^2, t) + \eta^i\left(\tilde{R}_r \sigma^r, t\right)$$
$$X^i = P_0^i t + 2\pi \tilde{R}_r^i \sigma^r + \nabla^i \epsilon(\sigma^1, \sigma^2, t)$$
$$\nabla^9 \epsilon(\sigma^1, \sigma^2) - \xi\left(\tilde{R}_r^9 \sigma^r\right)$$
$$\xi = 0, \epsilon = \epsilon\left(\tilde{R}_r^9 \sigma^r\right)$$
$$X^9 = P_0^9 t + 2\pi \tilde{R}_r^9 \sigma^r, X^i = P_0^i t + 2\pi \tilde{R}_r^i \sigma^r$$
$$(\partial_t \mp \nabla^9)\eta^i\left(\tilde{R}_r \sigma^r, t\right) = \pm k^i \nabla \left(\nabla^9 \epsilon(\sigma^1, \sigma^2) - \xi\left(\tilde{R}_r^9 \sigma^r\right)\right)$$
$$\left(\frac{m}{R} \partial_t \mp \nabla^9\right) \eta^i = 0$$
$$\epsilon(\sigma^1, \sigma^2) = \epsilon^{(1)}\left(\tilde{R}_r \sigma^r\right) + \epsilon^{(2)}\left(\tilde{R}_r^9 \sigma^r\right)$$
$$\xi\left(\tilde{R}_r^9 \sigma^r\right) = 0$$
$$X^9 = P_0^9 t + 2\pi \tilde{R}_r^9 \sigma^r$$
$$X^i = P_0^i t + 2\pi k^i \tilde{R}_r \sigma^r + \eta^i\left(\tilde{R}_r \sigma^r\right)$$
$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$$
$$X^9 = P_0^9 t + 2\pi \tilde{R}_r'^9 \sigma^r$$
$$X^i = P_0^i t + 2\pi k^i \tilde{R}' \sigma^1 + \hat{X}^i(\sigma^1, t)$$
$$\partial_t \hat{X}^i = \mp 2\pi \tilde{R}_2^9 \partial_1 \hat{X}^i$$

2. Modelo de Supermembrana de Yang – Mills.

2.1. Cálculos preliminares.

$$\tau_2^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z\big|_{m=0}$$
$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}$$
$$\langle S(\vec{x}_1, Y_1) \cdots S(\vec{x}_4, Y_4) \rangle = \frac{1}{\vec{x}_{12}^4 \vec{x}_{34}^4} \left[\vec{\mathcal{S}}_{\text{free}} + \mathcal{T}(U,V)\vec{\Theta}\right] \cdot \vec{\mathcal{B}}$$
$$\vec{\mathcal{S}}_{\text{free}} \equiv \begin{pmatrix} 1 & U^2 & \frac{U^2}{V^2} & \frac{1}{c}\frac{U^2}{V} & \frac{1}{c}\frac{U}{V} & \frac{1}{c}U \end{pmatrix}$$
$$\mathcal{B} \equiv (Y_{12}^2 Y_{34}^2 \quad Y_{13}^2 Y_{24}^2 \quad Y_{14}^2 Y_{23}^2 \quad Y_{13}Y_{14}Y_{23}Y_{24} \quad Y_{12}Y_{14}Y_{23}Y_{34} \quad Y_{12}Y_{13}Y_{24}Y_{34}).$$
$$\vec{\Theta} \equiv (V \quad UV \quad U \quad U(U-V-1) \quad 1-U-V \; V(V-U-1))$$
$$\mathcal{T}(U,V) = \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{u}{2}-2} \Gamma\left[2 - \frac{s}{2}\right]^2 \Gamma\left[2 - \frac{t}{2}\right]^2 \Gamma\left[2 - \frac{u}{2}\right]^2 \mathcal{M}(s,t)$$
$$\mathcal{M}(s,t) = \frac{\alpha}{(s-2)(t-2)(u-2)}\frac{1}{c} + \frac{\beta}{c^{7/4}} + \frac{\mathcal{M}_{\text{1-loop}}(s,t)}{c^2} + \frac{\gamma_1(s^2+t^2+u^2)+\gamma_2}{c^{\frac{9}{4}}} + \cdots$$
$$\mathcal{A}(\mathbf{s},\mathbf{t}) = \mathcal{A}_{\text{SG tree}}(\mathbf{s},\mathbf{t}) f(\mathbf{s},\mathbf{t})$$

$$f(\mathbf{s},\mathbf{t}) \equiv 1 + f_{R^4}(\mathbf{s},\mathbf{t})\ell_s^6 + f_{1-\text{loop}}(\mathbf{s},\mathbf{t})\ell_s^8 + f_{D^4R^4}(\mathbf{s},\mathbf{t})\ell_s^{10} + \cdots$$
$$f_{R^4} = \frac{\mathbf{stu}}{64} g_s^{\frac{3}{2}} E\left(\frac{3}{2},\tau_s,\bar{\tau}_s\right)$$
$$f_{D^4R^4} = \frac{\mathbf{stu}(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2)}{2^{11}} g_s^{\frac{5}{2}} E\left(\frac{5}{2},\tau_s,\bar{\tau}_s\right)$$
$$E(r,\tau_s,\bar{\tau}_s) = \frac{2\zeta(2r)}{g_s^r} + 2\sqrt{\pi} g_s^{r-1} \frac{\Gamma\left(r-\frac{1}{2}\right)}{\Gamma(r)}\zeta(2r-1)$$
$$+ \frac{2\pi^r}{\Gamma(r)\sqrt{g_s}} \sum_{k\neq 0} |k|^{r-\frac{1}{2}} \sigma_{1-2r}(|k|) K_{r-\frac{1}{2}}(2\pi g_s^{-1}|k|) e^{2\pi i k \chi_s}$$
$$f(\mathbf{s},\mathbf{t}) = \frac{\mathbf{stu}}{2^{11}\pi^2 g_s^2 \ell_s^8} \lim_{L/\ell_s\to\infty} L^{14} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{d\alpha}{2\pi i} e^{\alpha}\alpha^{-6} \mathcal{M}\left(\frac{L^2}{2\alpha}\mathbf{s}, \frac{L^2}{2\alpha}\mathbf{t}\right)$$
$$\tau_s = \tau, \frac{L^4}{\ell_s^4} = \lambda = g_{\text{YM}}^2\sqrt{4c+1}$$
$$\beta(\tau,\bar{\tau}) = \frac{15}{4\sqrt{2\pi^3}} E\left(\frac{3}{2},\tau,\bar{\tau}\right), \gamma_1(\tau,\bar{\tau}) = \frac{315}{128\sqrt{2\pi^5}} E\left(\frac{5}{2},\tau,\bar{\tau}\right)$$
$$\left.\frac{\partial_\tau\partial_{\bar{\tau}}\partial_m^2 \log Z}{\partial_\tau\partial_{\bar{\tau}} \log Z}\right|_{m=0} = 2 - \frac{\beta(\tau,\bar{\tau})}{5c^{3/4}} + \frac{C_{1\text{-loop}}}{c} - \frac{16\gamma_1(\tau,\bar{\tau}) + 7\gamma_2(\tau,\bar{\tau})}{35c^{5/4}} + \cdots$$
$$\partial_\tau\partial_{\bar{\tau}} \log Z|_{m=0} = \frac{c}{2(\text{Im}\tau)^2}$$
$$\partial_m^2 \log Z|_{m=0} = -(4c+1)\log \text{Im}\tau - \frac{\sqrt{2}}{\pi^{3/2}} E\left(\frac{3}{2},\tau,\bar{\tau}\right)c^{1/4} + \frac{3}{16\sqrt{2\pi^5}} E\left(\frac{5}{2},\tau,\bar{\tau}\right)\frac{1}{c^{1/4}} + \cdots$$
$$\gamma_2 = -3\gamma_1$$

2.2. Series de Eisenstein.

$$Z(m,\tau,\bar{\tau}) = \int d^{N-1}a \frac{\prod_{i<j} a_{ij}^2 H^2(a_{ij})}{H(m)^{N-1}\prod_{i\neq j} H(a_{ij}+m)} e^{-\frac{8\pi^2}{g_{\text{YM}}}\sum_i a_i^2} \left|Z_{\text{inst}}(m,\tau,a_{ij})\right|^2$$
$$Z_{\text{inst}}(m,\tau,a_{ij}) = \sum_{k=0}^{\infty} e^{2\pi i k\tau} Z_{\text{inst}}^{(k)}(m,a_{ij})$$
$$\partial_m^2 \log Z|_{m=0} = \partial_m^2 \log Z|_{m=0}^{\text{pert}} + \partial_m^2 \log Z|_{m=0}^{\text{inst}}$$
$$\partial_m^2 \log Z|_{m=0}^{\text{pert}} \equiv \left.\left\langle \partial_m^2 \prod_{i<j} \frac{H^2(a_{ij})}{H(a_{ij}-m)H(a_{ij}+m)}\right\rangle\right|_{m=0},$$
$$\partial_m^2 \log Z|_{m=0}^{\text{inst}} \equiv \sum_{k=1}^{\infty} (e^{ik\theta}+e^{-ik\theta}) e^{-\frac{8\pi^2 k}{g_{\text{YM}}^2}} \left.\left\langle \partial_m^2 Z_{\text{inst}}^{(k)}(m,a_{ij})\right\rangle\right|_{m=0}$$
$$\partial_m^2 \log Z|_{m=0}^{\text{pert}} = 2N^2 \log g_{\text{YM}} + \sqrt{N}\left[\frac{16\zeta(3)}{g_{\text{YM}}^3} + \frac{g_{\text{YM}}}{3}\right] - \frac{1}{\sqrt{N}}\left[\frac{12\zeta(5)}{g_{\text{YM}}^5} + \frac{g_{\text{YM}}^3}{1440}\right] + \cdots$$
$$e^{-\frac{8\pi^2 k}{g_{\text{YM}}^2}} \left.\left\langle \partial_m^2 Z_{\text{inst}}^{(k)}(m,a_{ij})\right\rangle\right|_{m=0} = -\frac{16\sqrt{N}}{g_{\text{YM}}} k\sigma_{-2}(k) K_1(8\pi^2 k/g_{\text{YM}}^2)$$
$$+ \frac{2}{g_{\text{YM}}\sqrt{N}} k^2 \sigma_{-4}(k) K_2(8\pi^2 k/g_{\text{YM}}^2) + \cdots$$
$$Z^{SU(N)}\Big|_{m=0} = \int d^N a\,\delta\left(\sum_i a_i\right) e^{-\frac{8\pi^2}{g_{\text{YM}}}\sum_i a_i^2} \prod_{i<j} a_{ij}^2,\ Z^{U(N)}\Big|_{m=0} = \int d^N a\, e^{-\frac{8\pi^2}{g_{\text{YM}}}\sum_i a_i^2} \prod_{i<j} a_{ij}^2$$

$$\int d^N a\delta\left(\sum_i a_i\right)e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\sum_i a_i^2}F(a_{ij}) = \sqrt{\frac{8\pi}{g_{\mathrm{YM}}^2 N}}\int d^N a e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\sum_i a_i^2}F(a_{ij})$$

2.3. Sector de Instantones.

$$Z_{\mathrm{inst}}^{(1)}(m,a_{ij}) = -m^2\sum_{l=1}^{N}\prod_{j\neq l}\frac{(a_l-a_j+i)^2-m^2}{(a_l-a_j)(a_l-a_j+2i)}$$

$$\partial_m^2 Z_{\mathrm{inst}}^{(1)}\Big|_{m=0} = I_1, I_1 \equiv -2\sum_{l=1}^{N}\prod_{j\neq l}\frac{(a_l-a_j+i)^2}{(a_l-a_j)(a_l-a_j+2i)}$$

$$I_1 = 4\int\frac{dz}{2\pi}\left[\prod_j\frac{(z-a_j)^2}{(z-a_j)^2+1}-1\right] = 4\int\frac{dz}{2\pi}\left[\exp\left(\sum_j\log\frac{(z-a_j)^2}{(z-a_j)^2+1}\right)-1\right]$$

$$\langle I_1\rangle \approx 4\int_{-\infty}^{\infty}\frac{dz}{2\pi}\left[\exp\left\langle\sum_j\log\frac{(z-a_j)^2}{(z-a_j)^2+1}\right\rangle-1\right]$$

$$\rho(b) = \frac{2}{\pi}\sqrt{1-b^2}, b\in[-1,1]$$

$$\langle I_1\rangle \approx 2\sqrt{N}\frac{g_{\mathrm{YM}}}{\pi}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\left[\exp\left(-N\int_{-1}^{1}db\rho(b)\log\left(1+\frac{4\pi^2}{Ng_{\mathrm{YM}}^2(x-b)^2}\right)\right)-1\right]$$

$$\langle I_1\rangle \approx 2\sqrt{N}\frac{g_{\mathrm{YM}}}{\pi}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\left[\exp\left(-\frac{8\pi}{g_{\mathrm{YM}}^2}\int_{-1}^{1}db\frac{\sqrt{1-b^2}}{(x-b)^2}\right)-1\right]$$

$$\int_{-1}^{1}db\frac{\sqrt{1-b^2}}{(x-b)^2} = \begin{cases}\pi\left(-1+\frac{1}{\sqrt{1-x^{-2}}}\right), & \text{Si } |x|>1\\ \infty, & \text{Si } |x|<1\end{cases}$$

$$\langle I_1\rangle \approx 2\sqrt{N}\frac{g_{\mathrm{YM}}}{\pi}e^{\frac{8\pi^2}{g_{\mathrm{YM}}}}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\left[e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\frac{1}{\sqrt{1-x^{-2}}}}\theta(|x|-1)-e^{-\frac{8\pi^2}{g_{\mathrm{YM}}}}\right]$$

$$t = \frac{x}{\sqrt{x^2-1}} \Leftrightarrow x = \frac{t}{\sqrt{t^2-1}}$$

$$\langle I_1\rangle \approx 2\sqrt{N}\frac{g_{\mathrm{YM}}}{\pi^2}e^{\frac{8\pi^2}{g_{\mathrm{YM}}^2}}\left[-e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}}+\int_1^{\infty}\frac{dt}{(t^2-1)^{3/2}}\left(e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}t}-e^{-\frac{8\pi^2}{g_{\mathrm{YM}}}}\right)\right]$$

$$\int_1^{\infty}dt\frac{e^{-at}-e^{-a}}{(t^2-1)^{3/2}} = e^{-a}-aK_1(a)$$

$$\langle I_1\rangle|_{\sqrt{N}} = -\sqrt{N}\frac{16}{g_{\mathrm{YM}}}e^{\frac{8\pi^2}{g_{\mathrm{YM}}}}K_1(8\pi^2/g_{\mathrm{YM}}^2)$$

$$e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}}\left\langle\partial_m^2 Z_{\mathrm{inst}}^{(1)}(m,a_{ij})\right\rangle\Big|_{m=0} \approx -\sqrt{N}\frac{16K_1(8\pi^2/g_{\mathrm{YM}}^2)}{g_{\mathrm{YM}}}$$

$$N\log\left(1+\frac{4\pi^2}{(x-b)^2g_{\mathrm{YM}}^2N}\right) = \frac{4\pi^2}{(x-b)^2g_{\mathrm{YM}}^2}-\frac{8\pi^4}{(x-b)^4g_{\mathrm{YM}}^4N}+\cdots$$

$$\langle I_1\rangle|_{1/\sqrt{N}} = \frac{1}{\sqrt{N}}\frac{32\pi^2}{g_{\mathrm{YM}}^3}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\left(\int_{-1}^{1}db\frac{\sqrt{1-b^2}}{(x-b)^4}\right)\exp\left(-\frac{8\pi}{g_{\mathrm{YM}}^2}\int_{-1}^{1}db\frac{\sqrt{1-b^2}}{(x-b)^2}\right)$$

$$\langle I_1\rangle|_{1/\sqrt{N}} = \frac{1}{\sqrt{N}}\frac{16\pi^2}{g_{\mathrm{YM}}^3}\int_1^{\infty}dtt\sqrt{t^2-1}e^{-\frac{8\pi^2}{g_{\mathrm{YM}}}(t-1)} = \frac{1}{\sqrt{N}}e^{\frac{8\pi^2}{g_{\mathrm{YM}}}}\frac{2K_2(8\pi^2/g_{\mathrm{YM}}^2)}{g_{\mathrm{YM}}}$$

$$e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}}\left\langle\partial_m^2 Z_{\mathrm{inst}}^{(1)}(m,a_{ij})\right\rangle\Big|_{m=0} = -\sqrt{N}\frac{16K_1(8\pi^2/g_{\mathrm{YM}}^2)}{g_{\mathrm{YM}}}+\frac{1}{\sqrt{N}}\frac{2K_2(8\pi^2/g_{\mathrm{YM}}^2)}{g_{\mathrm{YM}}}+\mathcal{O}(N^{-1})$$

$$Z_{\text{inst}}^{(k)}(m,a_{ij}) = \sum_{|\vec{Y}|=k} Z_{k,\vec{Y}}(m,a_{ij})$$

$$Z_{k,\vec{Y}}(m,a_{ij}) = \frac{1}{k!}\left(\frac{\epsilon_+(m^2+\epsilon_-^2/4)}{\epsilon_1\epsilon_2(m^2+\epsilon_+^2/4)}\right)^k \oiiint \prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\prod_{i=1}^{N}\frac{\left(\phi_I-a_j\right)^2-m^2}{\left(\phi_I-a_j\right)^2+\epsilon_+^2/4}$$

$$\times\prod_{I<J}^{k}\frac{\phi_{IJ}^2[\phi_{IJ}^2+\epsilon_+^2][\phi_{IJ}^2+(im-\epsilon_-/2)^2][\phi_{IJ}^2+(im+\epsilon_-/2)^2]}{[\epsilon_1^2][\phi_{IJ}^2+\epsilon_2^2][\phi_{IJ}^2+(im+\epsilon_+/2)^2][\phi_{IJ}^2+(im-\epsilon_+/2)^2]}$$

$$\{\phi_I \mid 1\le I\le k\} = \{a_j + i\epsilon_+/2 + (\alpha-1)i\epsilon_1 + (\beta-1)i\epsilon_2 \mid (\alpha,\beta)\in Y_i, 1\le i\le N\}$$

$$Z_{k,\vec{Y}}(m,a_{ij}) = \frac{1}{k!}\left(\frac{2m^2}{m^2+1}\right)^k \oiiint \prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\prod_{i=1}^{N}\frac{(\phi_I-a_i)^2-m^2}{(\phi_I-a_i)^2+1}$$

$$\times\prod_{I<J}^{k}\frac{\phi_{IJ}^2(\phi_{IJ}^2+4)\left(\phi_{IJ}^2-m^2\right)^2}{\left(\phi_{IJ}^2+1\right)^2\left[\left(\phi_{IJ}-m\right)^2+1\right]\left[\left(\phi_{IJ}+m\right)^2+1\right]}$$

$$\{\phi_I \mid 1\le I\le k\} = \{a_j + (\alpha+\beta-1)i \mid (\alpha,\beta)\in Y_i, 1\le i\le N\}$$

$$\vec{Y} = \{\dots,\square\square,\dots\},\ \vec{Y} = \{\dots,\begin{smallmatrix}\square\\\square\end{smallmatrix},\dots\},\ \vec{Y} = \{\dots,\square,\dots,\square,\dots\}$$

2.4. Diagramas de Young.

$$I_k \equiv \partial_m^2 Z_{\text{inst}}^{(k)}(m,a_{ij})\Big|_{m=0}$$

$$Y_i = \emptyset \ \text{ Si } \ i\neq\hat{\imath}$$

$$Y_{\hat{\imath}} = Y_{p\times q} \quad \text{for } \ p\le q\in\mathbb{Z}_+, pq=k$$

$$\vec{Y} = \{\dots,\square\square\square\square,\dots\},\ \vec{Y} = \{\dots,\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix},\dots\}$$

$$\vec{Y} = \{\dots,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\dots\}$$

$$h(\alpha,\beta)\equiv\lambda_\beta^T-\alpha, v(\alpha,\beta)\equiv\lambda_\alpha-\beta$$

$$I_{k,\vec{Y}} \equiv \partial_m^2 Z_{k,\vec{Y}}(m,a_{ij})\Big|_{m=0} \sim (\epsilon_-)^{-2+\sum_{i=1}^{N}\mu(Y_i)}$$

$$\mu(Y) = 2n_0(Y) - n_{-1}(Y) - n_1(Y)$$

$$\mu\left(Y_{p\times q}\right) = \begin{cases}1, & \text{Si } \ p\neq q\\ 2, & \text{Si } \ p=q\end{cases}$$

$$\vec{Y} = \{\dots,\begin{smallmatrix}\square\\\square\end{smallmatrix},\dots\},\ \vec{Y} = \{\dots,\square\square,\dots\}$$

$$R_1 = \text{Res}_{\phi_1=a_j+i\epsilon_+/2}\text{Res}_{\phi_2=\phi_1+i\epsilon_1}, R_2 = \text{Res}_{\phi_2=a_j+i\epsilon_+/2}\text{Res}_{\phi_1=\phi_2+i\epsilon_1}$$

$$I_{1\times 2} = \oiiint \frac{dz}{2\pi}\prod_{k_a}\prod_{j=1}^{N}\frac{\left(z-a_j+k_a i\right)^2}{\left(z-a_j+k_a i\right)^2+1}$$

$$\times\left[5+\sum_{j=1}^{N}\frac{3i}{\left(z-a_j+2i\right)\left(z-a_j+i\right)\left(z-a_j\right)}\right]$$

$$\vec{Y} = \{\dots,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\dots\},\ \vec{Y} = \{\dots,\square\square\square,\dots\}$$

$$\vec{Y}=\{\dots,\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix},\dots\},\vec{Y}=\{\dots,\begin{smallmatrix}\square\\\square\\\square\square\end{smallmatrix},\dots\},\vec{Y}=\{\dots,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}\}$$

$$I_{1\times 4}=\oint\!\!\oint\!\!\oint\frac{dz}{2\pi}\prod_{k_a}\prod_{j=1}^{N}\frac{\left(z-a_j+k_a i\right)^2}{\left(z-a_j+k_a i\right)^2+1}$$

$$\times\left[\frac{17}{4}+\sum_{j=1}^{N}\frac{45i}{2(z-a_j+4i)(z-a_j+3i)(z-a_j)}\right]$$

$$I_{2\times 2}=\oint\!\!\oint\!\!\oint\frac{dz}{2\pi}\prod_{k_a}\prod_{j=1}^{N}\frac{\left(z-a_j+k_a i\right)^2}{\left(z-a_j+k_a i\right)^2+1}$$

$$I_{p\times q}=\oint\!\!\oint\!\!\oint\frac{dz}{2\pi}\prod_{k_a}\prod_{j=1}^{N}\frac{\left(z-a_j+k_a i\right)^2}{\left(z-a_j+k_a i\right)^2+1}\times\left[\frac{4}{1+\delta_{pq}}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)\right.$$

$$\left.+\sum_{j=1}^{N}\frac{if(p,q)}{\left(z-a_j+(p+q-1)i\right)\left(z-a_j+(q-1)i\right)\left(z-a_j+(p-1)i\right)}\right]$$

$$k_a=\{0,1,\cdots,p-1;1,2,\cdots,p;\cdots;q-1,q,\cdots,p+q-2\}.$$

$$f(p,q)=\frac{2(q+p)(q-p)^2}{pq}$$

$$\langle I_{p\times q}\rangle\approx\frac{\sqrt{N}g_{\mathrm{YM}}}{2\pi}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\left(\exp\left[-N\int_{-1}^{1}db\rho(b)\sum_{k_a}\log\left(1+\frac{4\pi^2}{Ng_{\mathrm{YM}}^2\left(x-b+\frac{2\pi i}{\sqrt{N}g_{\mathrm{YM}}}k_a\right)^2}\right)\right]\right.$$

$$\left.\times\left(\frac{4}{1+\delta_{pq}}\frac{p^2+q^2}{p^2q^2}+i\frac{\left(\frac{2\pi}{g_{\mathrm{YM}}}\right)^3}{\sqrt{N}}\int_{-1}^{1}db\rho(b)\frac{f(p,q)}{g(x,b)}\right)-\frac{4}{1+\delta_{pq}}\frac{p^2+q^2}{p^2q^2}\right)$$

$$g(x,b)=\left[x-b+\frac{2\pi i}{\sqrt{N}g_{\mathrm{YM}}}(p+q-1)\right]\left[x-b+\frac{2\pi i}{\sqrt{N}g_{\mathrm{YM}}}(q-1)\right]\left[x-b+\frac{2\pi i}{\sqrt{N}g_{\mathrm{YM}}}(p-1)\right]$$

$$\langle I_{p\times q}\rangle\big|_{\sqrt{N}}=\frac{\sqrt{N}g_{\mathrm{YM}}}{2\pi^2}\left(-\frac{4}{1+\delta_{pq}}\frac{p^2+q^2}{p^2q^2}\right)+\frac{\sqrt{N}g_{\mathrm{YM}}}{\pi}\exp\left[k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right]\left(\frac{4}{1+\delta_{pq}}\frac{p^2+q^2}{p^2q^2}\right)$$

$$\times\int_{1}^{\infty}\frac{dx}{2\pi}\left(\exp\left[-\frac{8k\pi^2x}{g_{\mathrm{YM}}^2(x^2-1)^{\frac{1}{2}}}\right]-\exp\left[-k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right]\right)$$

$$\langle I_{p\times q}\rangle\big|_{\sqrt{N}}=-\frac{k}{1+\delta_{pq}}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)\frac{16\sqrt{N}}{g_{\mathrm{YM}}}\exp\left[k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right]K_1\left(k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right)$$

$$\sum_{pq=k,0<p\leq q}\langle I_{p\times q}\rangle\Big|_{\sqrt{N}}=-\frac{16\sqrt{N}}{g_{\mathrm{YM}}}k\sigma_{-2}(k)\exp\left[k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right]K_1\left(k\frac{8\pi^2}{g_{\mathrm{YM}}^2}\right)$$

$$\sum_{pq=k,0<p\leq q}\frac{1}{1+\delta_{pq}}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)=\sigma_{-2}(k)$$

$$\langle I_{p\times q}\rangle\big|_{1/\sqrt{N}} = \frac{1}{1+} + \frac{16\pi_{pq}^2}{\sqrt{N}g_{\rm YM}^5 k^2}\exp\left[k\frac{8\pi^2}{g_{\rm YM}^2}\right]\int_1^\infty dx\left(\exp\left[-\frac{8k\pi^2 x}{g_{\rm YM}^2(x^2-1)^{\frac{1}{2}}}\right]\right.$$

$$\left.\times\left[\frac{c_1 g_{\rm YM}^2 x}{(x^2-1)^{\frac{5}{2}}} - \frac{c_2\pi^2}{(x^2-1)^3}\right]\right)$$

$$c_1 = k(p-q)^2(p+q)(2p+2q-3) + \left(k+6\sum_a k_a^2\right)(p^2+q^2),$$

$$c_2 = 8k(p-q)^2(p+q)\sum k_a + 16(p^2+q^2)\left(\sum_a k_a\right)^2$$

$$\langle I_{p\times q}\rangle\big|_{1/\sqrt{N}} = \frac{1}{1+\delta_{pq}}\frac{16\pi^2}{\sqrt{N}g_{\rm YM}^5 k^2}\exp\left[k\frac{8\pi^2}{g_{\rm YM}^2}\right]\left[\frac{c_1 g_{\rm YM}^2}{\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right)} - \frac{3c_2\pi^2}{\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right)^2}\right]K_2\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right)$$

$$= \frac{k^2}{1+\delta_{pq}}\left(\frac{1}{p^4}+\frac{1}{q^4}\right)\frac{2}{\sqrt{N}g_{\rm YM}}\exp\left[k\frac{8\pi^2}{g_{\rm YM}^2}\right]K_2\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right).$$

$$\sum_{pq=k,0<p\leq q}\frac{1}{1+\delta_{pq}}\left(\frac{1}{p^4}+\frac{1}{q^4}\right) = \sigma_{-4}(k)$$

$$-\frac{16\sqrt{N}}{g_{\rm YM}}k\sigma_{-2}(k)\exp\left[k\frac{8\pi^2}{g_{\rm YM}^2}\right]K_1\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right) + \frac{2}{\sqrt{N}g_{\rm YM}}k^2\sigma_{-4}(k)\exp\left[k\frac{8\pi^2}{g_{\rm YM}^2}\right]K_2\left(k\frac{8\pi^2}{g_{\rm YM}^2}\right)$$

2.5. Series de Eisenstein. Cálculos complementarios.

$$\partial_m^2 \log Z\big|_{m=0} = 2N^2\log g_{\rm YM} - \frac{\sqrt{N}}{\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) + \frac{3}{16\sqrt{N}\pi^{\frac{5}{2}}}E\left(\frac{5}{2},\tau,\bar\tau\right)$$

$$+\frac{1}{N^{\frac{3}{2}}}\left[-\frac{13}{2^9\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) + \frac{135}{2^{11}\pi^{\frac{7}{2}}}E\left(\frac{7}{2},\tau,\bar\tau\right)\right]$$

$$+\frac{1}{N^{\frac{7}{2}}}\left[\frac{1533}{2^{18}\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) - \frac{80325}{2^{21}\pi^{\frac{7}{2}}}E\left(\frac{7}{2},\tau,\bar\tau\right) + \frac{2480625}{2^{23}\pi^{\frac{11}{2}}}E\left(\frac{11}{2},\tau,\bar\tau\right)\right]$$

$$+O\left(N^{-\frac{9}{2}}\right) + \text{(anti)ambiguedad holomórfica}$$

$$\tau_2^2\partial_\tau\partial_{\bar\tau}\partial_m^2\log Z\big|_{m=0} = \frac{N^2}{4} - \frac{3\sqrt{N}}{2^4\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) + \frac{45}{2^8\sqrt{N}\pi^{\frac{5}{2}}}E\left(\frac{5}{2},\tau,\bar\tau\right)$$

$$+\frac{1}{N^{\frac{3}{2}}}\left[-\frac{39}{2^{13}\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) + \frac{4725}{2^{15}\pi^{\frac{7}{2}}}E\left(\frac{7}{2},\tau,\bar\tau\right)\right]$$

$$+\frac{1}{N^{\frac{5}{2}}}\left[-\frac{1125}{2^{16}\pi^{\frac{5}{2}}}E\left(\frac{5}{2},\tau,\bar\tau\right) + \frac{99225}{2^{18}\pi^{\frac{9}{2}}}E\left(\frac{9}{2},\tau,\bar\tau\right)\right]$$

$$+\frac{1}{N^{\frac{7}{2}}}\left[\frac{4599}{2^{22}\pi^{\frac{3}{2}}}E\left(\frac{3}{2},\tau,\bar\tau\right) - \frac{2811375}{2^{25}\pi^{\frac{7}{2}}}E\left(\frac{7}{2},\tau,\bar\tau\right) + \frac{245581875}{2^{27}\pi^{\frac{11}{2}}}E\left(\frac{11}{2},\tau,\bar\tau\right)\right]$$

$$+O\left(N^{-\frac{9}{2}}\right)$$

$$\partial_m^2 \log Z|_{m=0}^{\text{pert}} = 2N^2 \log g_{\text{YM}} + \sqrt{N}\left[-\frac{16\zeta(3)}{g_{\text{YM}}^3} - \frac{g_{\text{YM}}}{3}\right] + \frac{1}{\sqrt{N}}\left[\frac{12\zeta(5)}{g_{\text{YM}}^5} + \frac{g_{\text{YM}}^3}{1440}\right]$$
$$+\frac{1}{N^{\frac{3}{2}}}\left[\frac{135\zeta(7)}{8g_{\text{YM}}^7} + \frac{g_{\text{YM}}^5}{215040} - \frac{13\zeta(3)}{32g_{\text{YM}}^3} - \frac{13g_{\text{YM}}}{1536}\right]$$
$$+\frac{1}{N^{\frac{7}{2}}}\left[\frac{2480625\zeta(11)}{2048g_{\text{YM}}^{11}} + \frac{25g_{\text{YM}}^9}{2491416576} - \frac{80325\zeta(7)}{8192g_{\text{YM}}^7} - \frac{17g_{\text{YM}}^5}{6291456}\right.$$
$$\left. + \frac{1533\zeta(3)}{16384g_{\text{YM}}^3} + \frac{511g_{\text{YM}}}{262144}\right] + O\left(N^{-\frac{9}{2}}\right)$$

$$W^n(y_1, \dots, y_n) \equiv N^{n-2}\left\langle \sum_{i_1}\frac{1}{y_1 - a_{i_1}} \cdots \sum_{i_n}\frac{1}{y_n - a_{i_n}}\right\rangle_{\text{conn.}}$$

$$W^n(y_1, \dots, y_n) \equiv \sum_{m=0}^{\infty}\frac{1}{N^{2m}}W_m^n(y_1, \dots, y_n)$$

$$I_1 = -\frac{4\Gamma\left(N+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(N)} - \frac{3\Gamma\left(N-\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(N+2)}C_2 + \frac{315\Gamma\left(N-\frac{3}{2}\right)}{32\sqrt{\pi}\Gamma(N+4)}C_2^2 - \frac{15(3-N+4N^2)\Gamma\left(N-\frac{3}{2}\right)}{16\sqrt{\pi}\Gamma(N+4)}C_4 + \cdots$$

$$C_p = \sum_{j,k}(a_j - a_k)^p$$

$$\langle C_2\rangle = \lambda\left[\frac{N^2}{8\pi^2} - \frac{1}{8\pi^2}\right], \langle C_2^2\rangle = \lambda^2\left[\frac{N^4}{64\pi^4} - \frac{1}{64\pi^4}\right], \langle C_4\rangle = 5\lambda^2\left[\frac{N^2}{128\pi^4} - \frac{1}{128\pi^4}\right]$$

$$\langle I_1\rangle = \sqrt{N}\left[-\frac{4}{\sqrt{\pi}} - \frac{3g_{\text{YM}}^2}{16\pi^{5/2}} + \frac{15g_{\text{YM}}^4}{2048\pi^{9/2}} + O(g_{\text{YM}}^6)\right]$$
$$+\frac{1}{\sqrt{N}}\left[\frac{1}{2\sqrt{\pi}} + \frac{15g_{\text{YM}}^2}{128\pi^{5/2}} + \frac{105g_{\text{YM}}^4}{16384\pi^{9/2}} + O(g_{\text{YM}}^6)\right]$$
$$+\frac{1}{N^{\frac{5}{2}}}\left[-\frac{5}{256\sqrt{\pi}} + \frac{285g_{\text{YM}}^2}{16384\pi^{5/2}} + \frac{24675g_{\text{YM}}^4}{2097152\pi^{9/2}} + O(g_{\text{YM}}^6)\right]$$
$$+\frac{1}{N^{\frac{7}{2}}}\left[\frac{21}{8192\sqrt{\pi}} + \frac{5103g_{\text{YM}}^2}{524288\pi^{5/2}} + \frac{1158885g_{\text{YM}}^4}{67108864\pi^{9/2}} + O(g_{\text{YM}}^6)\right] + O\left(N^{-\frac{9}{2}}\right)$$

$$\left\langle \partial_m^2 Z_{\text{inst}}^{(1)}(m, a_{ij})\right\rangle\Big|_{m=0} = e^{\frac{8\pi^2}{g_{\text{YM}}}}\left[-\sqrt{N}\frac{16K_1(8\pi^2/g_{\text{YM}}^2)}{g_{\text{YM}}} + \frac{2K_2(8\pi^2/g_{\text{YM}}^2)}{\sqrt{N}g_{\text{YM}}}\right.$$
$$+\frac{1}{32g_{\text{YM}}N^{\frac{3}{2}}}\left[-13K_1(8\pi^2/g_{\text{YM}}^2) + 9K_3(8\pi^2/g_{\text{YM}}^2)\right]$$
$$\left.+\frac{1}{g_{\text{YM}}N^{\frac{7}{2}}}\left[\frac{1533K_1\left(\frac{8\pi^2}{g_{\text{YM}}^2}\right)}{16384} - \frac{5355K_3\left(\frac{8\pi^2}{g_{\text{YM}}^2}\right)}{32768} + \frac{2625K_5\left(\frac{8\pi^2}{g_{\text{YM}}^2}\right)}{32768}\right] + O\left(N^{-\frac{9}{2}}\right)\right]$$

2.6. Series no holomórficas de Eisenstein.

$$\mathcal{A}(\mathbf{s},\mathbf{t}) = \frac{R^4}{\ell_s^8 g_s^2}\left(\frac{1}{\mathbf{stu}} + \frac{\ell_s^6 g_s^{\frac{3}{2}}}{2^6} E\left(\frac{3}{2},\tau,\bar{\tau}\right) + \frac{\ell_s^{10} g_s^{\frac{5}{2}}}{2^{11}}(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2) E\left(\frac{5}{2},\tau,\bar{\tau}\right)\right.$$

$$\left. + \frac{\ell_s^{12} g_s^3}{2^{12}}(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3)\mathcal{E}(\tau,\bar{\tau}) + \cdots\right)$$

$$(\Delta_\tau - r(r-1))E(r,\tau,\bar{\tau})$$

$$E(r,\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^r}{|m\tau+n|^{2r}}$$

$$E(r,\tau,\bar{\tau}) \to E(r,\tau',\bar{\tau}'),\ \tau\to\tau' = \frac{a\tau+b}{c\tau+d}$$

$$E(r,\tau,\bar{\tau}) = \sum_{k\in\mathbb{Z}} \mathcal{F}_k(r,\tau_2) e^{2\pi i k \tau_1}$$

$$\mathcal{F}_0(r,\tau_2) = 2\zeta(2r)\tau_2^r + \frac{2\sqrt{\pi}\,\Gamma\left(r-\frac{1}{2}\right)\zeta(2r-1)}{\Gamma(r)}\tau_2^{1-r}$$

$$\mathcal{F}_k(r,\tau_2) = \frac{2\pi^r}{\Gamma(r)}|k|^{r-\frac{1}{2}}\sigma_{1-2r}(|k|)\sqrt{\tau_2}K_{r-\frac{1}{2}}(2\pi|k|\tau_2), k\neq 0$$

$$\sigma_p(k) = \sum_{d>0, d|k} d^p$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}\left(1 + O\left(\frac{1}{z}\right)\right)$$

$$(\Delta_\tau - 12)\mathcal{E}(\tau,\bar{\tau}) = -E^2\left(\frac{3}{2},\tau,\bar{\tau}\right)$$

$$\mathcal{F}_{0,\mathcal{E}}(\tau_2) := \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \mathcal{E}(\tau,\bar{\tau}) = \frac{2\zeta(3)^2}{3}\tau_2^3 + \frac{4\zeta(2)\zeta(3)}{3}\tau_2 + \frac{4\zeta(4)}{\tau_2} + O(e^{-4\pi\tau_2})$$

$$Z_{\text{inst}}^{(k)}(m, a_{ij}, \epsilon_{1,2}) = \sum_{|\vec{Y}|=k} Z_{\vec{Y}}$$

$$Z_{\vec{Y}} \equiv \frac{\prod_{i,j=1}^N \prod_{s\in Y_i}\left(E(a_{ij},Y_i,Y_j,s) - im - \epsilon_+/2\right)\prod_{t\in Y_j}\left(-E(a_{ji},Y_j,Y_i,t) - im + \epsilon_+/2\right)}{\prod_{s\in Y_i} E(a_{ij},Y_i,Y_j,s)\prod_{t\in Y_j}\left(\epsilon_+ - E(a_{ji},Y_j,Y_i,t)\right)}$$

$$E(a_{ij},Y_i,Y_j,s) \equiv i a_{ji} - \epsilon_1 h_j(s) + \epsilon_2(v_i(s)+1)$$

$$h(s) = \lambda_\beta^T - \alpha, v(s) = \lambda_\alpha - \beta$$

$$I_k(a_{ij}) \equiv \partial_m^2\left(\lim_{\epsilon_{1,2}\to 1} Z_{\text{inst}}^{(k)}(m, a_{ij}, \epsilon_{1,2})\right)\Big|_{m=0}$$

$$Y_i = \emptyset \text{ Si } i \neq \hat{\imath}$$

$$Y_{\hat{\imath}} = \underbrace{[p,p,\ldots,p]}_{q}\ pq = k \text{ for } p,q\in\mathbb{Z}_+$$

$$\text{PT}_\alpha(Y) = [\lambda_1,\lambda_2,\ldots,\lambda_{\alpha-1},\lambda'_1,\ldots,\lambda'_{\lambda_\alpha}]$$

$$\text{PT}_\alpha(Y) = (Y\setminus P)\sqcup P^T$$

$$\max[\Delta(Y_{p\times q})] = p+q$$

$$\Delta_B(Y) \equiv \{\alpha+\lambda_\alpha \mid 1\leq\alpha\leq M\}$$

$$(\alpha,\beta) = (\alpha,\lambda_\alpha+1) = (\lambda_\beta^T+1,\beta)$$

$$\mu(Y) = 2n_0(Y) - (n_1(Y) + n_{-1}(Y))$$

$$\{\rho_\beta^T + \beta + (\hat{\alpha} - \lambda_\alpha - \alpha) \mid 1\leq\beta\leq\lambda_\alpha\}$$

$$\{\rho_\beta + \beta + (\hat{\alpha} - \lambda_\alpha - \alpha) \mid 1\leq\beta\leq\lambda_\alpha\}$$

$$n_0(Y) = \min(p,q), n_1(Y) = \min(p-1,q), n_{-1}(Y) = \min(p,q-1)$$

$$\mu(Y_{p\times q}) = \begin{cases} 2 & \text{Si } p = q \\ 1 & \text{Si } p \neq q \end{cases}$$

$$Y_{\min} = Y_{p\times q}$$

$$\mu(Y) \geq c$$

$$G_{Y_1,Y_2}(a) \equiv \prod_{s\in Y_1} \left(a - \left(h_{Y_1}(s) - v_{Y_2}(s)\right)\right) \prod_{t\in Y_2} \left(-a - \left(h_{Y_2}(t) - v_{Y_1}(t)\right)\right)$$

$$G_{Y_1,Y_2}(a) = G_{Y_1,Y_2^T}(a)$$

$$G_{Y_1,Y_2}(a) = \frac{\prod_{(\alpha,\beta)\in Y_1} (a+1+M-\alpha-\beta) \prod_{(\alpha,\beta)\in Y_2} (-a+1+M-\alpha-\beta)}{\prod_{\alpha=\beta=1}^{M} (a+\alpha-\beta)}$$

$$\times \prod_{\alpha=\beta=1}^{M} \left(a - \left(h_{Y_1}((\alpha,\beta))\right) - v_{Y_2}((\alpha,\beta))\right)$$

$$\prod_{\alpha=1}^{M} (\alpha+\rho_\alpha) = \prod_{\alpha=1}^{M} (\alpha+\rho_\alpha^T)$$

$$Z_{\vec{Y}} = F_1(Y)F_2(Y)$$

$$F_1(Y) = \prod_{s\in Y} \frac{(E(0,Y,Y,s) - im - \epsilon_+/2)(-E(0,Y,Y,s) - im + \epsilon_+/2)}{E(0,Y,Y,s)(\epsilon_+ - E(0,Y,Y,s))},$$

$$F_2(Y) = \prod_{j=1,j\neq\hat{\imath}}^{N} \prod_{s\in Y} \frac{\left(E\left(a_{\hat{\imath}j},Y,\emptyset,s\right) - im - \epsilon_+/2\right)}{E\left(a_{\hat{\imath}j},Y,\emptyset,s\right)} \frac{\left(E\left(a_{\hat{\imath}j},Y,\emptyset,s\right) + im - \epsilon_+/2\right)}{E\left(a_{\hat{\imath}j},Y,\emptyset,s\right) - \epsilon_+}.$$

$$F_1(Y) = F_1^0(Y)F_1^+(Y)F_1^-(Y)F_1^r(Y),$$

$$Y_0 = \{s \in Y \mid h(s) - v(s) = 0\},$$

$$Y_\pm = \{s \in Y \mid h(s) - v(s) = \pm 1\},$$

$$Y_r = Y \setminus (Y_0 \sqcup Y_+ \sqcup Y_-)$$

$$F_1^0 = \prod_{s\in Y_0} \frac{\left(h\epsilon_- + \frac{\epsilon_-}{2} + m\right)\left(h\epsilon_- + \frac{\epsilon_-}{2} - m\right)}{(h\epsilon_- - \epsilon_2)(h\epsilon_- + \epsilon_1)},$$

$$F_1^+ = \prod_{s\in Y_+} \frac{\left(h\epsilon_- + \frac{\epsilon_+}{2} + m\right)\left(h\epsilon_- + \frac{\epsilon_+}{2} - m\right)}{h\epsilon_-(h\epsilon_- + \epsilon_+)},$$

$$F_1^- = \prod_{s\in Y_-} \frac{\left(h\epsilon_- + \frac{\epsilon_+}{2} + m - 2\epsilon_2\right)\left(h\epsilon_- + \frac{\epsilon_+}{2} - m - 2\epsilon_2\right)}{(h\epsilon_- - 2\epsilon_2)(h+1)\epsilon_-}$$

$$F_1^0 = \prod_{s\in Y_0} (-m^2 - (h+1/2)^2\epsilon_-^2)\left(1 + \mathcal{O}(\epsilon_-^2)\right),$$

$$F_1^+ = \prod_{s\in Y_+} \frac{1}{2h\epsilon_-}\left(1 + \frac{3}{2}h\epsilon_- + \mathcal{O}(\epsilon_-^2)\right),$$

$$F_1^- = \prod_{s\in Y_-} \frac{1}{-2(h+1)\epsilon_-}\left(1 - \frac{3}{2}(h+1)\epsilon_- + \mathcal{O}(\epsilon_-^2)\right)$$

$$F_1^0 \simeq \prod_{s\in Y_0} (-m^2 - (h+1/2)^2\epsilon_-^2),$$

$$F_1^+ \simeq \prod_{s\in Y_+} \frac{1}{2h\epsilon_-}\left(1 + \frac{3}{2}h\epsilon_-\right),$$

$$F_1^- \simeq \prod_{s\in Y_-} \frac{1}{-2(h+1)\epsilon_-}\left(1 - \frac{3}{2}(h+1)\epsilon_-\right)$$

$$F_1^r \simeq F_1^r|_{m=0}$$

$$F_1(Y) \simeq m^2 \epsilon_-^{\mu(Y)-2} \left(F_1^{(0)} + \epsilon_- F_1^{(1)}\right) F_1^r$$

$$F_1^{(0)}(Y) = \frac{1}{\prod_{s\in Y_+} (2h) \prod_{s\in Y_-} (-2v)} \prod_{s\in Y_0} \left(h + \frac{1}{2}\right)^2 \sum_{s\in Y_0} \frac{1}{\left(h+\frac{1}{2}\right)^2}$$

$$F_1^{(1)}(Y) = \frac{3}{2} F_1^{(0)}(Y) \left(\sum_{s\in Y_+} h - \sum_{s\in Y_-} v \right)$$

$$F_1^r = F_1^{r(0)} + \epsilon_- F_1^{r(1)}$$

$$F_1^{r(0)}(Y) = \prod_{s\in Y} \frac{(h-v)^2}{(h-v)^2 - 1}$$

$$F_2(Y) \simeq F_2^{(0)}(Y) + \epsilon_- F_2^{(1)}(Y)$$

$$F_2^{(0)}(Y) = \prod_{j=1, j\neq \hat{\imath}}^{N} \prod_{s\in Y} \frac{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2}{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2 - 1}$$

$$F_2^{(1)}(Y)$$
$$= F_2^{(0)}(Y) \sum_{j=1, j\neq \hat{\imath}}^{N} \sum_{s\in Y} \frac{h_\emptyset(s) + v_Y(s) + 1}{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))) \left(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s) + 1)\right) \left(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s) - 1)\right)}$$

$$I_{k,\vec{Y}} \sim \lim_{\epsilon_- \to 0} \epsilon_-^{\mu(Y)-2}$$

$$F_1(Y) \simeq m^2 F_1^{(0)}(Y), F_2(Y) \simeq F_2^{(0)}(Y)$$

$$I_{k,\vec{Y}} = 2F_1^{(0)}(Y) F_2^{(0)}(Y).$$

$$q_2 \geq p_1 + p_2 + 1, q_1 > p_2$$

$$I_{k,\vec{Y}} + I_{k,\vec{Y}'} = 0$$

$$F_2^{(0)}(Y) = \prod_{j=1, j\neq \hat{\imath}}^{N} \prod_{s\in Y\backslash P} \frac{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2}{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2 - 1} \prod_{s\in P} \frac{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2}{(ia_{j\hat{\imath}} - (h_\emptyset(s) - v_Y(s))^2 - 1}$$

$$h_\emptyset(s) - v_Y(s) = -\alpha - (\lambda_\alpha - \beta)$$

$$h_\emptyset(\alpha,\beta) - v_Y(\alpha,\beta) = -\alpha - (l-m) - \rho_\alpha + \beta$$

$$h_\emptyset(\beta,\alpha) - v_{\mathrm{PT}(Y)}(\beta,\alpha) = -\beta - (l-m) - \rho_\beta^T + \alpha$$

$$\alpha \to 1 + \rho_\beta^T - \alpha, \beta \to 1 + \rho_\alpha - \beta$$

$$F_2^{(0)}(Y) = F_2^{(0)}(\mathrm{PT}(Y))$$

$$F_1^{(0)}(Y) = f_1(Y) f_2(Y) f_3(Y)$$

$$f_1 = \frac{1}{\prod_{s\in Y_+} (2h) \prod_{s\in Y_-} (-2v)}$$

$$f_2 = \prod_{s\in Y_0} \left(h + \frac{1}{2}\right)^2 \sum_{s\in Y_0} \frac{1}{\left(h + \frac{1}{2}\right)^2}$$

$$f_3 = \prod_{s\in Y_r} \frac{(h-v)^2}{(h-v)^2 - 1}$$

$$f_i(Y) = \prod_{s\in Y\backslash W^Y} \cdots \prod_{s\in W^Y} \cdots \equiv f_i^1(Y) f_i^2(Y)$$

$$f_1^2(Y) = -f_1^2(\iota_Y(Y)), f_2^2(Y) = f_2^2(\iota_Y(Y)), f_3^2(Y) = f_3^2(\iota_Y(Y))$$

$$r - \alpha + (q + p + t) - \lambda_\alpha \leq -2$$

$$(p,q,t,r) \to (q,p,r-x+1,t+x-1)$$

$$t + x - 1 - \alpha + (q + p + t - x + 1) - \lambda_\alpha \leq -2$$

$$F_1^{(0)}(Y) = -F_1^{(0)}(\iota_Y(Y))$$

$$\vec{Y} = \{Y_1, Y_2, \emptyset, \dots, \emptyset\}$$
$$\vec{Y}' = \{Y_1, Y_2^T, \emptyset, \dots, \emptyset\}$$
$$Z_{\vec{Y}} = Z_{Y_1} Z_{Y_2} Z_{Y_1,Y_2}$$
$$Z_{Y_1,Y_2} \simeq \prod_{s\in Y_1} \frac{\left(ia_{21} - \left(h_{Y_1}(s) - v_{Y_2}(s)\right)^2\right.}{\left(ia_{21} - \left(h_{Y_1}(s) - v_{Y_2}(s)\right)^2 - 1\right.} \prod_{t\in Y_2} \frac{\left(ia_{12} - \left(h_{Y_2}(t) - v_{Y_1}(t)\right)^2\right.}{\left(ia_{12} - \left(h_{Y_2}(t) - v_{Y_1}(t)\right)^2 - 1\right.}$$
$$Z_{Y_1,Y_2} = Z_{Y_1,Y_2^T}$$

2.7. Relaciones de recursividad.

$$Z_{k,\vec{Y}}(m, a_{ij}, \epsilon_i) = \frac{1}{k!}\left(\frac{\epsilon_+(m^2 + \epsilon_-^2/4)}{\epsilon_1\epsilon_2(m^2 + \epsilon_+^2/4)}\right)^k \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi} \prod_{j=1}^{N} \frac{\left(\phi_I - a_j\right)^2 - m^2}{\left(\phi_I - a_j\right)^2 + \epsilon_+^2/4}$$
$$\times \prod_{I<J}^{k} \frac{\phi_{IJ}^2[\phi_{IJ}^2 + \epsilon_+^2][\phi_{IJ}^2 + (im - \epsilon_-/2)^2][\phi_{IJ}^2 + (im + \epsilon_-/2)^2]}{[\phi_{IJ}^2 + \epsilon_1^2][\phi_{IJ}^2 + \epsilon_2^2][\phi_{IJ}^2 + (im + \epsilon_+/2)^2][\phi_{IJ}^2 + (im - \epsilon_+/2)^2]}$$
$$Z_{k,\vec{Y}}(m, a_{ij}) = \frac{1}{k!}\left(\frac{2m^2}{m^2 + 1}\right)^k \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi} \prod_{i=1}^{N} \frac{(\phi_I - a_i)^2 - m^2}{(\phi_I - a_i)^2 + 1}$$
$$\times \prod_{I<J}^{k} \frac{\phi_{IJ}^2(\phi_{IJ}^2 + 4)(\phi_{IJ}^2 - m^2)^2}{(\phi_{IJ}^2 + 1)^2\left[(\phi_{IJ} - m)^2 + 1\right]\left[(\phi_{IJ} + m)^2 + 1\right]}$$

$$Z_{k+1,\hat{Y}_+}(m, a_{ij}, \epsilon_i) = Z_{k,\vec{Y}}(m, a_{ij}, \epsilon_i) \frac{1}{k+1} \frac{\epsilon_+(m^2 + \epsilon_-^2/4)}{\epsilon_1\epsilon_2(m^2 + \epsilon_+^2/4)} \oint \frac{d\phi}{2\pi} \prod_{j=1}^{N} \frac{\left(\phi - a_j\right)^2 - m^2}{\left(\phi - a_j\right)^2 + \epsilon_+^2/4}$$
$$\times \prod_{J=1}^{k} \frac{\left(\phi - \hat{\phi}_J\right)^2\left[\left(\phi - \hat{\phi}_J\right)^2 + \epsilon_+^2\right]\left[\left(\phi - \hat{\phi}_J\right)^2 + (im - \epsilon_-/2)^2\right]\left[\left(\phi - \hat{\phi}_J\right)^2 + (im + \epsilon_-/2)^2\right]}{\left[\left(\phi - \hat{\phi}_J\right)^2 + \epsilon_1^2\right]\left[\left(\phi - \hat{\phi}_J\right)^2 + \epsilon_2^2\right]\left[\left(\phi - \hat{\phi}_J\right)^2 + (im + \epsilon_+/2)^2\right]\left[\left(\phi - \hat{\phi}_J\right)^2 + (im - \epsilon_+/2)^2\right]}$$

$$I_{1\times k}(\epsilon_1, \epsilon_2) = \partial_m^2 Z_{k,(\dots,Y_{1\times k},\dots)}(m, a_{ij}, \epsilon_i)\big|_{m=0}$$

$$I_{1\times 2}(\epsilon_1, \epsilon_2) = \oiiint \frac{dz}{2\pi}\left[\frac{3(z - a_1)(z - a_1 + i\epsilon_1)}{4\epsilon_1^2(z - a_1 - i\epsilon_1)(z - a_1 + 2i\epsilon_1)} \frac{1}{\epsilon_1 - \epsilon_2} + \frac{3\epsilon_1^4 + (z - a_1)(z - a_1 + i\epsilon_1)\left(22\epsilon_1^2 + 8(z - a_1)(z - a_1 + i\epsilon_1)\right)}{4\epsilon_1^3(z - a_1 - i\epsilon_1)^2(z - a_1 + 2i\epsilon_1)^2}\right]$$

$$I_{1\times k}(\epsilon_1, \epsilon_2) = \oiiint \frac{dz}{2\pi}\left[\frac{(2k^2 - 2)(z - a_1)\left((z - a_1) + i(k-1)\epsilon_1\right)}{\epsilon_1^2 k^2 k!\left((z - a_1) - i\epsilon_1\right)\left((z - a_1) + ik\epsilon_1\right)} \frac{1}{\epsilon_1 - \epsilon_2} + \frac{(k-1)^2(k+1)\epsilon_1^4 + (z - a_1)(z - a_1 + i(k-1)\epsilon_1)\left((6k^2 - 2)\epsilon_1^2 + 4k(z - a_1)(z - a_1 + i(k-1)\epsilon_1)\right)}{kk!\,\epsilon_1^3(z - a_1 - i\epsilon_1)^2(z - a_1 + ik\epsilon_1)^2}\right]$$

$$I_{k\times 1}(\epsilon_1, \epsilon_2) = I_{1\times k}(\epsilon_2, \epsilon_1)$$
$$I_{1\times k} = [I_{1\times k}(\epsilon_1, \epsilon_2) + I_{k\times 1}(\epsilon_1, \epsilon_2)]\big|_{\epsilon_1=\epsilon_2=1}$$
$$= \oiiint \frac{dz}{2\pi} \prod_{k_a=0}^{k-1} \frac{(z - a_1 + k_a i)^2}{(z - a_1 + k_a i)^2 + 1} \times \frac{1}{k!}\left[\frac{4}{1 + \delta_{1k}}\left(1 + \frac{1}{k^2}\right) + \frac{2i(k+1)(k-1)^2}{k(z - a_1 + ki)(z - a_1 + (k-1)i)(z - a_1)}\right]$$

$$I_{1\times k} = \oiiint \frac{dz}{2\pi} \prod_{k_a=0}^{k-1} \prod_{j=1}^{N} \frac{\left(z-a_j+k_a i\right)^2}{\left(z-a_j+k_a i\right)^2+1} \times \frac{1}{k!}\left[\frac{4}{1+\delta_{1k}}\left(1+\frac{1}{k^2}\right)\right.$$

$$\left. + \sum_{j=1}^{N} \frac{2i(k+1)(k-1)^2}{k\left(z-a_j+ki\right)\left(z-a_j+(k-1)i\right)\left(z-a_j\right)}\right]$$

2.8. Instantones de alto orden.

$$I_{p\times q}\left(N,a_{ij}\right) = I_{p\times q}^{(0)}(N) + I_{p\times q}^{(2)}(N) C_2\left(a_{ij}\right) + \cdots$$

$$(-1600N^2 - 7310N - 8256) I_{2\times1}^{(0)}(N+2) + (1440N^2 + 5859N + 5958) I_{2\times1}^{(0)}(N+3)$$

$$+(N+2)(160N+491) I_{2\times1}^{(0)}(N+1) = 0, I_{2\times1}^{(0)}(2) = -\frac{134}{27}, I_{2\times1}^{(0)}(3) = -\frac{517}{81}$$

$$(-320N^2 - 734N - 456) I_{2\times1}^{(2)}(N+2) + (288N^2 + 1323N + 684) I_{2\times1}^{(2)}(N+3)$$

$$+N(32N+51) I_{2\times1}^{(2)}(N+1) = 0, I_{2\times1}^{(2)}(2) = -\frac{10}{243}, I_{2\times1}^{(2)}(3) = -\frac{20}{729}$$

$$I_{2\times1}^{(0)} = -5\sqrt{\frac{2}{\pi}}\sqrt{N} + \frac{17\sqrt{\frac{1}{N}}}{8\sqrt{2\pi}} + \frac{325\left(\frac{1}{N}\right)^{3/2}}{1024\sqrt{2\pi}} + \frac{2155\left(\frac{1}{N}\right)^{5/2}}{8192\sqrt{2\pi}} + \frac{1543605\left(\frac{1}{N}\right)^{7/2}}{4194304\sqrt{2\pi}} + O\left(N^{-\frac{9}{2}}\right),$$

$$I_{2\times1}^{(2)} = -\frac{15}{8\sqrt{2\pi}N^{5/2}} + \frac{255}{128\sqrt{2\pi}N^{7/2}} - \frac{11025}{16384\sqrt{2\pi}N^{9/2}} + \frac{478485}{131072\sqrt{2\pi}N^{11/2}} + O\left(N^{-\frac{13}{2}}\right)$$

$$\left\langle \partial_m^2 Z_{\text{inst}}^{p\times q}\left(m,a_{ij}\right)\right\rangle\Big|_{m=0} = \frac{e^{\frac{8pq\pi^2}{g_{\text{YM}}^2}}}{1+\delta_{p,q}}\left[-\sqrt{N}\frac{16K_1\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)}{g_{\text{YM}}}\left(\frac{p}{q}+\frac{q}{p}\right) + \frac{2K_2\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)}{g_{\text{YM}}\sqrt{N}}\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}\right)\right.$$

$$+\frac{1}{32g_{\text{YM}}N^{\frac{3}{2}}}\left[-13K_1\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)\left(\frac{p}{q}+\frac{q}{p}\right) + 9K_3\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)\left(\frac{p^3}{q^3}+\frac{q^3}{p^3}\right)\right]$$

$$+\frac{1}{128g_{\text{YM}}N^{\frac{5}{2}}}\left[-25K_2\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}\right) + 15K_4\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)\left(\frac{p^4}{q^4}+\frac{q^4}{p^4}\right)\right]$$

$$+\frac{1}{g_{\text{YM}}N^{\frac{7}{2}}}\left[\frac{1533K_1\left(\frac{8pp\pi^2}{g_{\text{YM}}^2}\right)}{16384}\left(\frac{p}{q}+\frac{q}{p}\right) - \frac{5355K_3\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)}{32768}\left(\frac{p^3}{q^3}+\frac{q^3}{p^3}\right) + \frac{2625K_5\left(\frac{8pq\pi^2}{g_{\text{YM}}^2}\right)}{32768}\left(\frac{p^5}{q^5}+\frac{q^5}{p^5}\right)\right]$$

$$\left.+O\left(N^{-\frac{9}{2}}\right)\right]$$

$$Z^{SU(N)}\Big|_{m=0} = \int d^N a\,\delta\left(\sum_i a_i\right) e^{-\frac{8\pi^2}{g_{\text{YM}}}\sum_i a_i^2} \prod_{i<j} a_{ij}^2,\ Z^{U(N)}\Big|_{m=0} = \int d^N a e^{-\frac{8\pi^2}{g_{\text{YM}}}\sum_i a_i^2} \prod_{i<j} a_{ij}^2$$

$$\int d^N a\,\delta\left(\sum_i a_i\right) e^{-\frac{8\pi^2}{g_{\text{Y}}^2}\sum_i a_i^2} F\left(a_{ij}\right) = \sqrt{\frac{8\pi}{g_{\text{YM}}^2 N}} \int d^N a e^{-\frac{8\pi^2}{g_{\text{YM}}^2}\sum_i a_i^2} F\left(a_{ij}\right)$$

$$\hat{E}(r,\tau,\bar{\tau}) = \pi^{-r}\Gamma(r)\zeta(2r)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \text{Im}(\gamma(\tau))^r$$

2.9. Correlaciones, localización y acoplamiento.

$$\mathcal{M}_p = c^{-1}\Big(B_1^1\mathcal{M}_p^1 + \lambda^{-\frac{3}{2}}[B_4^4\mathcal{M}_p^4 + B_1^4\mathcal{M}_p^1] + \lambda^{-2}[B_5^5\mathcal{M}_p^5 + B_4^5\mathcal{M}_p^4 + B_1^5\mathcal{M}_p^1]$$
$$+\lambda^{-\frac{5}{2}}[B_{6,1}^6\mathcal{M}_p^{6,1} + B_{6,2}^6\mathcal{M}_p^{6,2} + B_5^6\mathcal{M}_p^5 + B_4^6\mathcal{M}_p^4 + B_1^6\mathcal{M}_p^1] + O(\lambda^{-3})\Big) + O(c^{-2})$$

$$\mathcal{A} = \mathcal{A}_0 f(s,t)\, f(s,t) \equiv -\frac{stu\ell_s^6}{64}\frac{\Gamma\left(-\frac{\ell_s^2 s}{4}\right)\Gamma\left(-\frac{\ell_s^2 t}{4}\right)\Gamma\left(-\frac{\ell_s^2 u}{4}\right)}{\Gamma\left(1+\frac{\ell_s^2 s}{4}\right)\Gamma\left(1+\frac{\ell_s^2 t}{4}\right)\Gamma\left(1+\frac{\ell_s^2 u}{4}\right)} + O(g_s^2)$$

$$f(s,t) = [1 + \ell_s^6 f_{R^4}(s,t) + \ell_s^{10} f_{D^4R^4}(s,t) + \ell_s^{12} f_{D^6R^4}(s,t) + \cdots] + O(g_s^2)$$

$$f_{R^4}(s,t) = \frac{\zeta(3)}{32} stu, f_{D^4R^4}(s,t) = \frac{\zeta(5)}{2^{10}} stu(s^2+t^2+u^2), f_{D^6R^4}(s,t) = \frac{\zeta(3)^2(stu)^2}{2^{11}}$$

$$f(s,t) = 1 + \ell_s^6 \tilde{f}_{R^4}(s,t) + \ell_s^8 \tilde{f}_{1-\text{loop}}(s,t) + \ell_s^{10}\tilde{f}_{D^4R^4}(s,t) + O(\ell_s^{12})$$

$$c = \frac{N^2-1}{4}$$

$$S_{I_1\ldots I_p}(\vec{x}) = N_p\left[\text{tr}\left(\phi_{I_1}\cdots\phi_{I_p}\right) - SO(6)\text{ rastros}\right]$$

$$S_p(\vec{x},Y) \equiv S_{I_1\ldots I_p}(\vec{x})Y^{I_1}\ldots Y^{I_p}$$

$$\langle S_p(\vec{x}_1,Y_1)S_p(\vec{x}_2,Y_2)\rangle = \frac{Y_{12}^p}{|\vec{x}_{12}|^{2p}}, Y_{12} \equiv Y_1\cdot Y_2, \vec{x}_{12} \equiv \vec{x}_1 - \vec{x}_2$$

$$N_p = \frac{(2\pi)^p}{\sqrt{p}(4c)^{p/4}}$$

$$\langle S_2(\vec{x}_1,Y_1)S_2(\vec{x}_2,Y_2)S_p(\vec{x}_3,Y_3)S_p(\vec{x}_4,Y_4)\rangle =$$
$$\frac{Y_{34}^{p-2}}{\vec{x}_{12}^4\vec{x}_{34}^{2p}}[\mathcal{S}_p^1(U,V)Y_{12}^2Y_{34}^2 + \mathcal{S}_p^2(U,V)Y_{13}^2Y_{24}^2 + \mathcal{S}_p^3(U,V)Y_{14}^2Y_{23}^2$$
$$+\mathcal{S}_p^4(U,V)Y_{13}Y_{14}Y_{23}Y_{24} + \mathcal{S}_p^5(U,V)Y_{12}Y_{14}Y_{23}Y_{34} + \mathcal{S}_p^6(U,V)Y_{12}Y_{13}Y_{24}Y_{34}]$$

$$U \equiv \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, V \equiv \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

$$\mathcal{S}_p^i(U,V) = \Theta^i(U,V)\mathcal{T}_p(U,V) + \mathcal{S}_{p,\text{ free}}^i(U,V),$$
$$\Theta^i(U,V) \equiv \begin{pmatrix} V & UV & U & U(U-V-1) & 1-U-V & V(V-U-1)\end{pmatrix}$$

$$\mathcal{S}_{2,\text{ free}}^i(U,V) = \begin{pmatrix} 1 & U^2 & \frac{U^2}{V^2} & \frac{1}{c}\frac{U^2}{V} & \frac{1}{c}\frac{U}{V} & \frac{1}{c}U\end{pmatrix}$$

$$\mathcal{S}_{p,\text{free}}^i(U,V) = \begin{pmatrix} 1 & 0 & 0 & \frac{p(p-1)}{2c}\frac{U^2}{V} & \frac{p}{2c}\frac{U}{V} & \frac{p}{2c}U\end{pmatrix} + O(1/c^2)$$

$$g_s = \frac{g_{\text{YM}}^2}{4\pi}$$

$$\mathcal{T}_p(U,V) = \int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{u-p-2}{2}}\Gamma\left[2-\frac{s}{2}\right]\Gamma\left[p-\frac{s}{2}\right]$$
$$\times\Gamma^2\left[\frac{2+p}{2}-\frac{t}{2}\right]\Gamma^2\left[\frac{2+p}{2}-\frac{u}{2}\right]\mathcal{M}_p(s,t)$$

$$\mathcal{M}_p(s,t) = \mathcal{M}_p(s,u), \mathcal{M}_2(s,t) = \mathcal{M}_2(u,t)$$

$$\mathcal{M}_p = c^{-1}\Big(B_1^1\mathcal{M}_p^1 + \lambda^{-\frac{3}{2}}[B_4^4\mathcal{M}_p^4 + B_1^4\mathcal{M}_p^1] + \lambda^{-2}[B_5^5\mathcal{M}_p^5 + B_4^5\mathcal{M}_p^4 + B_1^5\mathcal{M}_p^1]$$
$$+\lambda^{-\frac{5}{2}}[B_{6,1}^6\mathcal{M}_p^{6,1} + B_{6,2}^6\mathcal{M}_p^{6,2} + B_5^6\mathcal{M}_p^5 + B_4^6\mathcal{M}_p^4 + B_1^6\mathcal{M}_p^1] + \cdots\Big) + O(c^{-2})$$

$$\mathcal{M}_p^1 = \frac{1}{(s-2)(t-p)(u-p)},$$
$$\mathcal{M}_p^4 = 1, \qquad \mathcal{M}_p^5 = s$$
$$\mathcal{M}_p^{6,1} = s^2 + t^2 + u^2, \qquad \mathcal{M}_p^{6,2} = s^2$$
$$\mathcal{T}_p(U,V) = c^{-1}\Big[B_1^1\mathcal{T}_p^1 + \lambda^{-\frac{3}{2}}\big(B_4^4\mathcal{T}_p^4 + B_1^4\mathcal{T}_p^1\big) + \lambda^{-2}\big(B_5^5\mathcal{T}_p^5 + B_4^5\mathcal{T}_p^4 + B_1^5\mathcal{T}_p^1\big)$$
$$+\lambda^{-\frac{5}{2}}\big(B_{6,1}^6\mathcal{T}_p^{6,1} + B_{6,2}^6\mathcal{T}_p^{6,2} + B_5^6\mathcal{T}_p^5 + B_4^6\mathcal{T}_p^4 + B_1^6\mathcal{T}_p^1\big) + \cdots\Big] + O(c^{-2})$$
$$\mathcal{T}_p^1 = -\frac{1}{8}U^p\bar{D}_{p,p+2,2,2}(U,V)$$
$$\mathcal{T}_p^4 = U^p\bar{D}_{p+2,p+2,4,4}(U,V)$$
$$\mathcal{T}_p^5 = 2U^p\big(2\bar{D}_{p+2,p+2,4,4}(U,V) - \bar{D}_{p+2,p+2,5,5}(U,V)\big),$$
$$\mathcal{T}_p^{6,1} = 2U^p\big(2(1+U+V)\bar{D}_{p+3,p+3,5,5} - (4+4p-p^2)\bar{D}_{p+2,p+2,4,4}(U,V)\big),$$
$$\mathcal{T}_p^{6,2} = 4U^p\big(\bar{D}_{p+2,p+2,6,6}(U,V) - 5\bar{D}_{p+2,p+2,5,5}(U,V) + 4\bar{D}_{p+2,p+2,4,4}(U,V)\big)$$
$$B_1^1(p) = \frac{4p}{\Gamma(p-1)}, B_1^n(p)\ n > 1$$
$$\mathbf{20'} \to (\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{2},\mathbf{2})_{\pm\mathbf{1}}$$
$$S_2 \to \{\mathcal{A}_2, \overline{\mathcal{A}}_2\} \oplus \mathcal{B}_{(ab)}^{(\alpha\beta)} \oplus \mathcal{C} \oplus \{\mathcal{D}_a^\alpha, \overline{\mathcal{D}}_a^\alpha\}$$
$$[0p0] \to (\mathbf{1},\mathbf{1})_{\pm\mathbf{p}} \oplus \ldots$$
$$S_p \to \{\mathcal{A}_p, \overline{\mathcal{A}}_p\} \oplus \ldots$$
$$m_{\alpha\beta}\int d^4x\tilde{Q}^2\mathcal{B}^{(\alpha\beta)}(x) + \text{ c.c.}$$
$$\tau_p\int d^4x\tilde{Q}^4\mathcal{A}_p(x) + \text{ c.c}$$
$$S_m = \int d^4x\sqrt{g}\left(m\left[\frac{i}{r}J + K\right] + m^2L\right)$$
$$J \equiv \frac{1}{2}\sum_{i=1}^{2}\operatorname{tr}[(Z_i)^2 + (\bar{Z}_i)^2], K \equiv -\frac{1}{2}\sum_{i=1}^{2}\operatorname{tr}(\chi_i\sigma_2\chi_i + \tilde{\chi}_i\sigma_2\tilde{\chi}_i), L \equiv \sum_{i=1}^{2}\operatorname{tr}|Z_i|^2$$
$$\mathcal{C} \equiv \frac{1}{3}\operatorname{tr}[|Z_1|^2 + |Z_2|^2 - 2|Z_3|^2], \mathcal{K} \equiv \frac{2}{3}\sum_{i=1}^{3}\operatorname{tr}|Z_i|^2$$
$$J = N_J[S_{11} + S_{22} - S_{44} - S_{55}],$$
$$\mathcal{C} = N_{\mathcal{C}}[S_{11} + S_{22} + S_{33} + S_{44} - 2S_{55} - 2S_{66}]$$
$$K = N_K[P_{11} + P_{22} + \bar{P}^{11} + \bar{P}^{22}]$$
$$N_K^2 = 8N_J^2 = 36N_{\mathcal{C}}^2 = \frac{N^2-1}{4\pi^4}$$
$$S_{\tau_p} = \tau_p\int d^4x\sqrt{g}\left(M_p(x) - i\frac{p-2}{r}(\sigma_2)_{ab}N_p^{(ab)}(x) + \frac{2(p-2)(p-3)}{r^2}\mathcal{A}_p(x)\right)$$
$$S_{\tau_p} = \tau_p\mathcal{A}_p(N)$$
$$S_{\bar{\tau}_p} = \bar{\tau}_p\overline{\mathcal{A}}_p(S)$$
$$\mathcal{A}_p \propto S_p(Y_0), \overline{\mathcal{A}}_p \propto S_p(\bar{Y}_0)$$
$$l_p \equiv \frac{\partial_m^2\partial_{\tau_p}\partial_{\bar{\tau}_p}\log Z}{\partial_{\tau_p}\partial_{\bar{\tau}_p}\log Z} = l_p^{(4)} + l_p^{(3)}$$
$$l_p^{(4)} = \frac{\int d^4\vec{x}_1 d^4\vec{x}_2\sqrt{g(\vec{x}_1)}\sqrt{g(\vec{x}_2)}\big\langle\big(iJ(\vec{x}_1) + K(\vec{x}_1)\big)\big(iJ(\vec{x}_2) + K(\vec{x}_2)\big)\mathcal{A}_p(N)\overline{\mathcal{A}}_p(S)\big\rangle}{\big\langle\mathcal{A}_p(N)\overline{\mathcal{A}}_p(S)\big\rangle}$$

$$l_p^{(3)} = \frac{2\int \ d^4\vec{x}\sqrt{g(\vec{x})}\langle \mathcal{C}(\vec{x})\mathcal{A}_p(N)\overline{\mathcal{A}}_p(S)\rangle}{\langle \mathcal{A}_p(N)\overline{\mathcal{A}}_p(S)\rangle}$$

$$\frac{\langle \mathcal{C}(\vec{x}_1)\mathcal{A}_p(\vec{x}_2)\overline{\mathcal{A}}_p(\vec{x}_3)\rangle}{\langle \mathcal{A}_p(\vec{x}_2)\overline{\mathcal{A}}_p(\vec{x}_3)\rangle} = -\frac{p}{6\pi^2}\frac{\vec{x}_{23}^2}{\vec{x}_{12}^2\vec{x}_{13}^2} + O(1/c)$$

$$\vec{x}_{ij} \to \Omega(\vec{x}_i)^{-1/2}\Omega(\vec{x}_j)^{-1/2}\vec{x}_{ij}$$

$$l_p^{(3)} = -\frac{p}{3\pi^2}\int \ d^4\vec{x}\frac{1}{\vec{x}^2\left(1+\frac{\vec{x}^2}{4}\right)^2} + O(1/c) = -\frac{4p}{3} + O(1/c)$$

$$\frac{\langle (iJ(\vec{x}_1)+K(\vec{x}_1))(iJ(\vec{x}_2)+K(\vec{x}_2))\mathcal{A}_p(\vec{x}_3)\overline{\mathcal{A}}_p(\vec{x}_4)\rangle}{\langle \mathcal{A}_p(\vec{x}_3)\overline{\mathcal{A}}_p(\vec{x}_4)\rangle} = 4N_J^2\left[-\frac{\mathcal{S}_p^1}{\vec{x}_{12}^4} + 8\frac{\mathcal{R}_p^1 - \mathcal{R}_p^2 + \mathcal{R}_p^3}{\vec{x}_{12}^6}\right]$$

$$l_p^{(4)} = 4N_J^2\left[-\tilde{I}_2[\mathcal{S}_p^1] + 8\tilde{I}_3[\mathcal{R}_p^1 - \mathcal{R}_p^2 + \mathcal{R}_p^3]\right]$$

$$\tilde{I}_\Delta[\mathcal{G}] = \lim_{\substack{|\vec{x}_3|\to 0 \\ |x_4|\to\infty}} \int \ d^4\vec{x}_1 d^4\vec{x}_2 \frac{\left(1+\frac{x_1^2}{4}\right)^{\Delta-4}\left(1+\frac{x_2^2}{4}\right)^{\Delta-4}}{\vec{x}_{12}^{2\Delta}}\mathcal{G}(U,V)$$

$$\tilde{I}_\Delta[\mathcal{G}]$$
$$= \mathrm{Vol}(S^3)\mathrm{Vol}(S^2)\int dr_1 dr_2 d\theta r_1^3 r_2^3 \sin^2\theta \frac{\left(1+\frac{r_1^2}{4}\right)^{\Delta-4}\left(1+\frac{r_2^2}{4}\right)^{\Delta-4}}{(r_1^2+r_2^2-2r_1r_2\cos\,\theta)^\Delta}\mathcal{G}\left(\frac{r_1^2+r_2^2-2r_1r_2\cos\,\theta}{r_1^2}, \frac{r_2^2}{r_1^2}\right)$$

$$\tilde{I}_\Delta[\mathcal{G}] = 2^{11-2\Delta}\pi^3\int \ drd\rho d\theta r^3\rho^7\sin^2\ \theta\frac{(1+\rho^2)^{\Delta-4}(1+\rho^2r^2)^{\Delta-4}}{\rho^{2\Delta}(1+r^2-2r\cos\ \theta)^\Delta}\mathcal{G}(1+r^2-2r\cos\ \theta, r^2)$$

$$\tilde{I}_2[\mathcal{G}] = 128\pi^3\int \ drd\theta r^3\sin^2\ \theta\frac{1-r^2+(1+r^2)\log\ r\mathcal{G}(1+r^2-2r\cos\ \theta, r^2)}{(r^2-1)^3}$$

$$\tilde{I}_3[\mathcal{G}] = 32\pi^3\int \ drd\theta r^3\sin^2\ \theta\frac{\log\ r\cos\ \theta)^2}{r^2-1}\frac{\mathcal{G}(1+r^2-2r\cos\ \theta, r^2)}{(1+r^2-2r\cos\ \theta)^3}$$

$$-\tilde{I}_2[\mathcal{S}_p^1] + 8\tilde{I}_3[\mathcal{R}_p^1 - \mathcal{R}_p^2 + \mathcal{R}_p^3] = 16\pi^4 I[\mathcal{T}_p] - \frac{2\pi^4}{c}l_p^{(3)}$$

$$I[\mathcal{G}] \equiv \frac{4}{\pi}\int \ drd\theta r^3\sin^2\ \theta\frac{r^2-1-2r^2\log\ r}{(r^2-1)^2}\frac{\mathcal{G}(1+r^2-2r\cos\ \theta, r^2)}{(1+r^2-2r\cos\ \theta)^2}$$

$$l_p = l_p^{(3)} + l_p^{(4)} = 8cI[\mathcal{T}_p]$$

$$Z(m,\lambda) = \int \ d^{N-1}a\left(\prod_{i<j}\frac{(a_i-a_j)^2H^2(a_i-a_j)}{H(a_i-a_j-m)H(a_i-a_j+m)}\right)e^{-\frac{8\pi^2N}{\lambda}\sum_i a_i^2}|Z_{\mathrm{inst}}|^2$$

$$Z(m,\lambda,\tau_p',\bar{\tau}_p') = \int \ d^{N-1}a\left(\prod_{i<j}\frac{(a_i-a_j)^2H^2(a_i-a_j)}{H(a_i-a_j-m)H(a_i-a_j+m)}\right)e^{-\frac{8\pi^2N}{\lambda}\sum_i a_i^2}$$
$$\times\, e^{i\sum_p \pi^{p/2}(\tau_p'-\tau_p')\sum_i a_i^p}|Z_{\mathrm{inst}}|^2$$

$$\mathbf{A}_{pq} \equiv \left.\frac{\partial^2\log\ Z}{\partial\tau_p'\partial\bar{\tau}_q'}\right|_{m=\tau_p'=\bar{\tau}_p'=0}$$

$$l_n = \frac{\mathbf{B}_{pq}v_n^q v_n^p}{\mathbf{A}_{pq}v_n^p v_n^q}$$

$$\mathbf{B}_{pq} \equiv \left.\frac{\partial^4\log\ Z}{\partial m^2\partial\tau_p'\partial\bar{\tau}_q'}\right|_{m=\tau_p'=\bar{\tau}_p'=0}$$

$$Z(m,\lambda,\tau_p,\bar{\tau}_p) = \int d^N a e^{-N^2 F(a)}$$

$$F \approx \frac{1}{2}\int dxdy\rho(x)\rho(y)\log\frac{H(x-y+m)H(x-y-m)}{H^2(x-y)|x-y|^2} + \int dx\rho(x)V(x),$$

$$V(x) \equiv \frac{8\pi^2}{\lambda}x^2 + \sum_{p=2}^{\infty}\frac{8\pi^{p/2+1}}{\lambda_p}x^p$$

$$\int dx\rho(x) = 1$$

$$\int dy\rho(y)\left(\frac{1}{x-y} - K(x-y) + \frac{1}{2}K(x-y+m) + \frac{1}{2}K(x-y-m)\right) = \frac{8\pi^2}{\lambda}x + \sum_{p=2}^{\infty}\frac{4p\pi^{p/2+1}}{\lambda_p}x^{p-1}$$

$$\rho_0(x) = \frac{Q(x)}{2\pi}\sqrt{b^2-x^2}$$

$$\frac{V'(x)}{\sqrt{x^2-b^2}} - Q(x) \to \frac{2}{x^2}, x \to \infty$$

$$\log Z(0,\lambda,\tau_p',\bar{\tau}_p') \approx -N^2 4\pi^2 \int dx\rho_0(x)\left[\frac{x^2}{\lambda} - \frac{\log x}{4\pi^2} + \sum_p \frac{\pi^{p/2-1}x^p}{\lambda_p}\right]$$

$$\mathbf{A}_{pq} = \frac{2\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\pi(p+q)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{\lambda}{4\pi}\right)^{\frac{p+q}{2}}$$

$$v_2^p = (1 \quad 0 \quad 0 \quad 0 \quad \cdots)$$

$$v_4^p = \left(-\frac{\lambda}{4\pi} \quad 1 \quad 0 \quad 0 \quad \cdots\right),$$

$$v_6^p = \left(\frac{9\lambda^2}{256\pi^2} \quad -\frac{3\lambda}{8\pi} \quad 1 \quad 0 \quad \cdots\right)$$

$$v_n^p = (-1)^{n/2}2^{-n}n\pi^{\frac{-1}{2}}\left(\frac{\lambda}{4\pi}\right)^{\frac{n-p}{2}}\frac{\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{p+n}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{n-p+2}{2}\right)}$$

$$v_n^p\mathbf{A}_{pq}v_n^q = n\left(\frac{\lambda}{16\pi}\right)^n$$

$$\frac{\partial^2\log Z}{\partial m^2} \approx N^2\int dxdy\rho_0(x)\rho_0(y)K'(x-y)$$

$$K'(x) = -\int_0^\infty d\omega\frac{2\omega[\cos(2\omega x)-1]}{\sinh^2\omega}$$

$$\frac{\partial^2\log Z}{\partial m^2} \approx -\frac{N^2b^4}{2\pi^2}\int_{-1}^{1} d\xi d\eta Q(b\xi)Q(b\eta)\sqrt{1-\xi^2}\sqrt{1-\eta^2}\int_0^\infty \frac{\omega[\cos[2b\omega(\xi-\eta)]-1]}{\sinh^2\omega}$$

$$l_p = 4p\int_0^\infty d\omega\omega\frac{J_1\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2 - J_p\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2}{\sinh^2\omega}$$

$$l_p = 2(p-1) - \frac{4p(p^2-1)\zeta(3)}{\lambda^{3/2}} - \frac{3p(p^2-1)(3-2p^2)\zeta(5)}{\lambda^{5/2}} - \frac{15p(p^2-1)(135-124p^2+16p^4)\zeta(7)}{32\lambda^{7/2}} - \frac{35p(p^2-1)(1575-1654p^2+320p^4-16p^6)\zeta(9)}{64\lambda^{9/2}} + \cdots$$

$$\mathcal{M}_p(s,t) = \frac{4p}{\Gamma(p-1)}\frac{1}{c}\Bigg[\frac{1}{(s-2)(t-p)(u-p)} + \frac{(p+1)_3}{4}\zeta(3)\frac{1}{\lambda^{3/2}}$$

$$+\frac{(p+1)_5}{32}\zeta(5)\left[s^2+t^2+u^2+\frac{2p(p-2)}{p+5}s+\left(-2p^2+\frac{50+20p(p+2)}{(p+4)(p+5)}\right)\right]\frac{1}{\lambda^{5/2}}$$

$$+\cdots\Bigg] + O(c^{-2})$$

$$\mathcal{M}_2(s,t) = \frac{8}{c}\left[\frac{1}{(s-2)(t-2)(u-2)} + \frac{15\zeta(3)}{\lambda^{3/2}} + \frac{315\zeta(5)}{4\lambda^{5/2}}(s^2+t^2+u^2-3)+\cdots\right] + O(c^{-2})$$

$$I\left[\mathcal{T}_p^1\right] = \frac{\Gamma(p)}{16p}, I\left[\mathcal{T}_p^4\right] = -\frac{\Gamma(p+2)}{2(p)_4}, I\left[\mathcal{T}_p^5\right] = -2p\frac{\Gamma(p+2)}{(p)_5}$$

$$I\left[\mathcal{T}_p^{6,1}\right] = (16+20p-12p^2-5p^3-p^4)\frac{\Gamma(p+2)}{(p)_6},$$

$$I\left[\mathcal{T}_p^{6,2}\right] = -8p(p-1)\frac{\Gamma(p+2)}{(p)_6}$$

$$l_p = 8\Bigg[B_1^1(p)\frac{\Gamma(p)}{16p} - B_4^4(p)\frac{\Gamma(p+2)}{2(p)_4}\frac{1}{\lambda^{\frac{3}{2}}} - \frac{\left[4pB_5^5(p)+(p+4)B_4^5(p)\right]\Gamma(p+2)}{2(p)}\frac{1}{\lambda^2}$$

$$+\left[\frac{2B_{6,1}^6(p)(16+20p-12p^2-5p^3-p^4)-16p(p-1)B_{6,2}^6(p)-B_5^6(p)4p(p+5)-B_4^6(p)(p+4)(p+5)}{2(p)_6}\right]$$

$$\times\Gamma(p+2)\frac{1}{\lambda^{\frac{5}{2}}}+\cdots\Bigg]$$

$$R^4{:}B_4^4(p) = \zeta(3)\frac{(p)_4}{\Gamma(p-1)},$$

$$D^2R^4{:}4pB_5^5(p) + (4+p)B_4^5(p) = 0,$$

$$D^4R^4{:}2B_{6,1}^6(p)(16+20p-12p^2-5p^3-p^4) - 16p(p-1)B_{6,2}^6(p)$$

$$-B_5^6(p)4p(p+5) - B_4^6(p)(p+4)(p+5) = \frac{3(2p^2-3)(p)_6}{4\Gamma(p-1)}\zeta(5)$$

$$\mathcal{A}(\eta_i,s,t) \ = \Gamma\left(\frac{1}{2}\Delta_\Sigma - 2\right)\left[\int_{S^5} d^5x\sqrt{g}\prod_{i=1}^{4}\Psi_{\eta_i}^{\mathcal{O}_i}(\vec{n})\right]$$

$$\times\lim_{L\to\infty} L^6\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{d\alpha}{2\pi i}e^{\alpha}\alpha^{2-\frac{1}{2}\Delta_\Sigma}M^{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}\left(\frac{L^2}{2\alpha}s,\frac{L^2}{2\alpha}t\right)$$

$$\frac{L^4}{\ell_s^4} = \lambda = g_{\mathrm{YM}}^2N, g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}$$

$$M_p^i(s,t) \approx \frac{1}{16}(t^2u^2 \quad s^2u^2 \quad s^2t^2 \quad 2s^2tu \quad 2st^2u \quad 2stu^2)\mathcal{M}_p(s,t)$$

$$f(s,t) = \frac{1}{\mathcal{N}}\lim_{L\to\infty}L^{14}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{d\alpha}{2\pi i}e^{\alpha}\alpha^{-4-p}\mathcal{M}_p\left(\frac{L^2}{2\alpha}s,\frac{L^2}{2\alpha}t\right)$$

$$\mathcal{N} = (4\pi)^2B_1^1(p)\frac{32g_s^2\ell_s^8}{stu}\int\frac{d\alpha}{2\pi i}e^{\alpha}\alpha^{-1-p} = \frac{2048\pi^2g_s^2\ell_s^8}{stu}\frac{p}{\Gamma(p-1)\Gamma(p+1)}$$

$$f(s,t) = \frac{stu}{B_1^1(p)}\Bigg[\frac{B_1^1(p)}{stu} + \frac{B_4^4(p)}{2^3(p+1)_3}\ell_s^6 + \frac{B_5^5(p)s}{2^4(p+1)_4}\ell_s^8$$

$$+\frac{B_{6,1}^6(p)(s^2+t^2+u^2)+B_{6,2}^6(p)s^2}{2^5(p+1)_5}\ell_s^{10}+\cdots\Bigg] + O(g_s^2)$$

$$R^4 : B_4^4(p) = \zeta(3)\frac{(p)_4}{\Gamma(p-1)}$$
$$D^2R^4 : B_5^5(p) = 0$$
$$D^4R^4 : B_{6,1}^6(p) = \zeta(5)\frac{(p)_6}{8\Gamma(p-1)}, B_{6,2}^6(p) = 0$$
$$B_{6,1}^6(2) = -\frac{B_4^6(2)}{3} = 630\zeta(5), B_{6,2}^6(2) = 0$$
$$B_k^n(p) = C_k^n(p)\frac{(p)_k}{\Gamma(p-1)}$$
$$B_5^6(p) = (p-2)(a_1 p + a_0)\frac{\zeta(5)(p)_5}{\Gamma(p-1)}, B_4^6(p) = (b_4 p^4 + b_3 p^3 + b_2 p^2 + b_1 p + b_0)\frac{\zeta(5)(p)_4}{\Gamma(p-1)}$$
$$b_0 = \frac{25}{4}, b_1 = 8a_0 + 5, b_2 = 8a_1 - 4a_0 - \frac{9}{2}$$
$$b_3 = -4a_1 - \frac{5}{4}, b_4 = -\frac{1}{4}$$
$$B_5^6(p) = \frac{1}{4}p(p-2)\frac{\zeta(5)(p)_5}{\Gamma(p-1)}, B_4^6(p) = -\frac{1}{4}(p^4 + 9p^3 + c_2 p^2 - 20p + c_1)\frac{\zeta(5)(p)_4}{\Gamma(p-1)}$$
$$a_1 = -\frac{1}{4}, a_0 = 1, c_1 = -25, c_2 = 10$$
$$\gamma_j\big|_{c^{-1}} = -\frac{24}{(j+1)(j+6)} - \frac{1}{\lambda^{\frac{3}{2}}}\frac{4320\zeta(3)}{7}\delta_{j,0} - \frac{\zeta(5)}{\lambda^{\frac{5}{2}}}\left[30600\delta_{j,0} + \frac{201600}{11}\delta_{j,2}\right] + O(\lambda^{-3})$$
$$\gamma_j\big|_{c^{-2}\lambda^{-\frac{3}{2}}} = -103680\frac{(j+2)_4(j^2+7j+16)(j^2+7j+54)}{(j-4)_6(j+6)_6}\zeta(3),\ \text{for}\ \ j>4,$$
$$\gamma_j\big|_{c^{-2}\lambda^{-\frac{5}{2}}} = -(77j^4(j+7)^4 + 15452j^3(j+7)^3 + 1610364j^2(j+7)^2 + 48199536j(j+7) + 401725440)$$
$$\times\frac{1036800(j+2)_4}{(j-6)_8(j+6)_8}\frac{\zeta(5)}{8}, \text{para}\ \ j>6$$
$$\mathcal{T}(U,V) = \tilde{B}_1^1(\tau)\mathcal{T}^1(U,V)\frac{1}{c} + \tilde{B}_4^4(\tau)\mathcal{T}^4(U,V)\frac{1}{c^{\frac{7}{4}}} + \mathcal{T}^{1-\text{loop}}(U,V)\frac{1}{c^2} +$$
$$+\left(\tilde{B}_6^6(\tau)\mathcal{T}^6(U,V) + \tilde{B}_4^6(\tau)\mathcal{T}^4(U,V)\right)\frac{1}{c^{\frac{9}{4}}} + \cdots$$
$$\tilde{B}_1^1 = B_1^1(2) = 8$$
$$f(s,t) = 1 + \ell_s^6\tilde{f}_{R^4}(s,t) + \ell_s^8\tilde{f}_{1-\text{loop}}(s,t) + \ell_s^{10}\tilde{f}_{D^4R^4}(s,t) + O(\ell_s^{12})$$
$$\tilde{f}_{R^4}(s,t) = \frac{stu}{64}g_s^{\frac{3}{2}}\mathcal{E}_{3/2}(\tau_s,\bar{\tau}_s) = \frac{stu}{64}\left[2\zeta(3) + \frac{2\pi^2}{3}g_s^2 + O\left(e^{-1/g_s}\right)\right]$$
$$\tilde{f}_{D^4R^4}(s,t) = g_s^{\frac{5}{2}}\mathcal{E}_{5/2}(\tau_s,\bar{\tau}_s)\frac{stu}{2^{11}}(s^2+t^2+u^2) = \frac{stu}{2^{10}}\left(\zeta(5) + \frac{2\pi^4}{135}g_s^4 + O\left(e^{-1/g_s}\right)\right)(s^2+t^2+u^2)$$
$$\mathcal{E}_r(\tau_s,\bar{\tau}_s) = \sum_{(m,n)\neq(0,0)}\frac{g_s^{-r}}{|m+n\tau_s|^{2r}}$$
$$= 2\zeta(2r)g_s^{-r} + 2\sqrt{\pi}g_s^{r-1}\frac{\Gamma(r-1/2)}{\Gamma(r)}\zeta(2r-1) + \frac{2\pi^r}{\Gamma(r)\sqrt{g_s}}\sum_{m,n\neq 0}\left|\frac{m}{n}\right|^{r-1/2}K_{r-1/2}(2\pi g_s^{-1}|mn|)e^{2\pi i m n\chi_s}$$
$$\tilde{B}_4^4 = 60g_s^{3/2}\mathcal{E}_{3/2}(\tau,\bar{\tau})$$
$$\frac{\partial^4\log Z}{\partial\tau\partial\bar{\tau}\partial m^2}\bigg|_{m=0} = \frac{cg_{\text{YM}}^4}{16\pi^2} - \frac{3c^{\frac{1}{4}}}{128\sqrt{2}\pi^{7/2}}g_{\text{YM}}^4\mathcal{E}_{3/2}(\tau,\bar{\tau}) + O(c^0)$$
$$4g_s^{-2}\partial_{\tau_s}\partial_{\bar{\tau}_s}\mathcal{E}_r = r(r-1)\mathcal{E}_r$$

$$\left.\frac{\partial^2 \log Z}{\partial m^2}\right|_{m=0} = \left[8c\log g_{\rm YM} - \frac{\sqrt{2}c^{\frac{1}{4}}}{\pi^{3/2}}\mathcal{E}_{3/2}(\tau,\bar{\tau}) + O(c^0)\right] + \kappa_1(\tau) + \kappa_2(\bar{\tau})$$

$$\gamma_j = -\frac{1}{c}\frac{24}{(j+1)(j+6)} - \frac{1}{c^{\frac{7}{4}}}\frac{135\mathcal{E}_{3/2}(\tau_1,\tau_2)}{7\sqrt{2}\pi^{\frac{3}{2}}}\delta_{j,0} + O(c^{-2})$$

$$\lambda \to g_{\rm YM}^2\sqrt{4c+1}, \zeta(3) \to \frac{g_{\rm YM}^3}{16\pi^{\frac{3}{2}}}\mathcal{E}_{3/2}(\tau,\bar{\tau}), \zeta(5) \to \frac{g_{\rm YM}^5}{64\pi^{\frac{5}{2}}}\mathcal{E}_{5/2}(\tau,\bar{\tau})$$

$$P(\vec{x},\bar{X}) \equiv P_{AB}(\vec{x})\bar{X}^A\bar{X}^B, \bar{P}(\vec{x},X) \equiv \bar{P}^{AB}(\vec{x})X_AX_B$$

$$C_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \quad C_6 = -i\begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, l$$

$$C_4 = -i\begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \quad C_5 = -i\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

$$X \wedge X \to Y^I := X_A C^{IAB} X_B, \bar{X} \wedge \bar{X} \to Y^I := \bar{X}^A \bar{C}^I_{AB} \bar{X}^B$$

$$Y \wedge X \to \bar{X}^A := Y_I C^{IAB} X_B, Y \wedge \bar{X} \to X_A := Y_I \bar{C}^I_{AB} \bar{X}^B$$

$$\chi^\alpha(\vec{x},\bar{X},Y) \equiv \chi^\alpha{}_{IA}(\vec{x})Y^I\bar{X}^A, \bar{\chi}^{\dot{\alpha}}(\vec{x},X,Y) \equiv \bar{\chi}_I^{\dot{\alpha}A}(\vec{x})Y^IX_A$$

$$j^\mu(\vec{x},Y_1,Y_2) \equiv j^\mu_{[I_1I_2]}(\vec{x})Y_1^{I_1}Y_2^{I_2}, F^{\alpha\beta}(\vec{x},Y) \equiv F_I^{\alpha\beta}(\vec{x})Y^I, \bar{F}^{\dot{\alpha}\dot{\beta}}(\vec{x},Y) \equiv \bar{F}_I^{\dot{\alpha}\dot{\beta}}(\vec{x})Y^I$$

$$\left\langle \chi^\alpha(x_1,\bar{X}_1,Y_1)\bar{\chi}^{\dot{\alpha}}(x_2,X_2,Y_2)\right\rangle = \frac{Y_{12}(\bar{X}_1\cdot X_2)ix_{12}^\mu\sigma_\mu^{\alpha\dot{\alpha}}}{x_{12}^6},$$

$$\langle j^\mu(\vec{x}_1,Y_1,Y_{1'})j^\nu(\vec{x}_2,Y_2,Y_{2'})\rangle = \frac{4[Y_{12}Y_{1'2'} - Y_{12'}Y_{1'2}]}{x_{12}^6}\left(\delta^{\mu\nu} - 2\frac{x_{12}^\mu x_{12}^\nu}{x_{12}^2}\right)$$

$$\left\langle F^{\alpha\beta}(x_1,Y_1)\bar{F}^{\dot{\alpha}\dot{\beta}}(x_2,Y_2)\right\rangle = \frac{Y_{12}x_{12}^\mu x_{12}^\nu\sigma_\mu^{(\alpha\dot{\alpha}}\sigma_\nu^{\beta)\dot{\beta}}}{x_{12}^8},$$

$$\langle P(\vec{x}_1,\bar{X}_1)\bar{P}(\vec{x}_2,X_2)\rangle = \frac{(\bar{X}_1\cdot X_2)^2}{x_{12}^6}$$

$$\left\langle P(\vec{x}_1,\bar{X}_1)\bar{P}(\vec{x}_2,X_2)S_p(\vec{x}_3,Y_3)S_p(\vec{x}_4,Y_4)\right\rangle = \frac{Y_{34}^{p-2}}{x_{12}^6x_{34}^{2p}}\left[\mathcal{R}_{1,p}(U,V)Y_{34}^2(\bar{X}_1\cdot X_2)^2\right.$$

$$\left.+\mathcal{R}_{2,p}(U,V)Y_{34}(\bar{X}_1\cdot X_2)\{X_2,\bar{X}_1,Y_3,Y_4\} + \mathcal{R}_{3,p}(U,V)\{X_2,\bar{X}_1,Y_3,Y_4\}^2\right]$$

$$\{X,\bar{X}',Y,Y'\} = \frac{1}{2}X_A\bar{X}'^BY_IY_J'\left(\bar{C}^I_{BC}C^{JCA} - \bar{C}^J_{BC}C^{ICA}\right)$$

$$\left\langle S_2(\vec{x}_1,Y_1)S_2(\vec{x}_2,Y_2)\chi^\alpha(\vec{x}_3,\bar{X}_3,Y_3)\bar{\chi}^{\dot{\beta}}(\vec{x}_4,X_4,Y_4)\right\rangle$$

$$= \frac{ix_{34}^\mu\sigma_\mu^{\alpha\dot{\beta}}}{x_{12}^4x_{34}^6}[Y_{12}(\bar{X}_3\cdot X_4)(Y_{12}Y_{34}\mathcal{A}_{11} + Y_{13}Y_{24}\mathcal{A}_{12} + Y_{14}Y_{23}\mathcal{A}_{13})$$

$$+\{X_4,\bar{X}_3,Y_1,Y_2\}(Y_{12}Y_{34}\mathcal{A}_{14} + Y_{13}Y_{24}\mathcal{A}_{15} + Y_{14}Y_{23}\mathcal{A}_{16})]$$

$$+\frac{i\sigma_\mu^{\alpha\dot{\beta}}}{2x_{12}^6x_{34}^6}\left(x_{24}^2x_{31}^\mu - x_{14}^2x_{32}^\mu + x_{23}^2x_{41}^\mu - x_{13}^2x_{42}^\mu - x_{34}^2x_{21}^\mu - x_{12}^2x_{43}^\mu - 2\varepsilon^{\mu\nu\rho\sigma}x_{\nu42}x_{\rho13}x_{\sigma12}\right)$$

$$[Y_{12}(\bar{X}_3\cdot X_4)(Y_{12}Y_{34}\mathcal{A}_{21} + Y_{13}Y_{24}\mathcal{A}_{22} + Y_{14}Y_{23}\mathcal{A}_{23})$$

$$+\{X_4,\bar{X}_3,Y_1,Y_2\}(Y_{12}Y_{34}\mathcal{A}_{24} + Y_{13}Y_{24}\mathcal{A}_{25} + Y_{14}Y_{23}\mathcal{A}_{26})]$$

$$\langle S_2(\vec{x}_1,Y_1)S_2(\vec{x}_2,Y_2)S_2(\vec{x}_3,Y_3)j^\mu(\vec{x}_4,Y_4,Y_5)\rangle$$
$$= \frac{1}{x_{12}^4 x_{34}^4}\left(\frac{x_{24}^\mu}{x_{24}^2}-\frac{x_{34}^\mu}{x_{34}^2}\right)[(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{13}Y_{23}\mathcal{W}_{11}$$
$$+(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{13}Y_{12}\mathcal{W}_{12}+(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{12}Y_{23}\mathcal{W}_{13}]$$
$$+\frac{1}{x_{12}^4 x_{34}^4}\left(\frac{x_{24}^\mu}{x_{24}^2}-\frac{x_{14}^\mu}{x_{14}^2}\right)[(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{13}Y_{23}\mathcal{W}_{21}$$
$$+(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{13}Y_{12}\mathcal{W}_{22}+(Y_{14}Y_{25}-Y_{24}Y_{15})Y_{12}Y_{23}\mathcal{W}_{23}]$$

$$\left\langle S_2(\vec{x}_1,Y_1)P(\vec{x}_2,\bar{X}_2)\bar{\chi}^{\dot{\alpha}}(\vec{x}_3,X_3,Y_3)\bar{\chi}^{\dot{\beta}}(\vec{x}_4,X_4,Y_4)\right\rangle$$
$$=\frac{x_{14}^2}{x_{12}^6 x_{34}^6 x_{24}^2}\left((x_{32}^2+x_{42}^2-x_{34}^2)\varepsilon^{\dot{\alpha}\dot{\beta}}-4x_{23}^\mu x_{24}^\nu \bar{\sigma}_{\mu\nu}^{\dot{\alpha}\dot{\beta}}\right)$$
$$[Y_{13}Y_{14}(\bar{X}_2\cdot X_3)(\bar{X}_2\cdot X_4)\mathcal{B}_{11}+\{X_3,\bar{X}_2,Y_1,Y_4\}\{X_4,\bar{X}_2,Y_1,Y_3\}\mathcal{B}_{12}$$
$$+Y_{14}(\bar{X}_2\cdot X_3)\{X_4,\bar{X}_2,Y_1,Y_3\}\mathcal{B}_{13}+Y_{13}(\bar{X}_2\cdot X_4)\{X_3,\bar{X}_2,Y_1,Y_4\}\mathcal{B}_{14}]$$
$$+\frac{1}{x_{12}^6 x_{34}^6}\left((x_{31}^2+x_{41}^2-x_{34}^2)\varepsilon^{\dot{\alpha}\dot{\beta}}-4x_{13}^\mu x_{14}^\nu \bar{\sigma}_{\mu\nu}^{\dot{\alpha}\dot{\beta}}\right)$$
$$[Y_{13}Y_{14}(\bar{X}_2\cdot X_3)(\bar{X}_2\cdot X_4)\mathcal{B}_{21}+\{X_3,\bar{X}_2,Y_1,Y_4\}\{X_4,\bar{X}_2,Y_1,Y_3\}\mathcal{B}_{22}$$
$$+Y_{14}(\bar{X}_2\cdot X_3)\{X_4,\bar{X}_2,Y_1,Y_3\}\mathcal{B}_{23}+Y_{13}(\bar{X}_2\cdot X_4)\{X_3,\bar{X}_2,Y_1,Y_4\}\mathcal{B}_{24}]$$

$$\left\langle S_2(\vec{x}_1,Y_1)S_2(\vec{x}_2,Y_2)P(\vec{x}_3,\bar{X}_3)\bar{F}^{\dot{\alpha}\dot{\beta}}(\vec{x}_4,Y_4)\right\rangle$$
$$=\frac{x_{14}^2 x_{23}^\mu x_{43}^\nu + x_{24}^2 x_{31}^\mu x_{41}^\nu + x_{34}^2 x_{12}^\mu x_{42}^\nu}{x_{12}^6 x_{34}^8}\bar{\sigma}_{\mu\nu}^{\dot{\alpha}\dot{\beta}} Y_{12}\{(Y_1\wedge\bar{X}_3),\bar{X}_3,Y_2,Y_4\}\mathcal{C}_{11}$$

2.10. Identidades de Ward.

$$\bar{\delta}^{\dot{\alpha}}(\bar{X})S_2(\vec{x},Y)=\bar{\chi}^{\dot{\alpha}}(\vec{x},\bar{X}\wedge Y,Y),$$
$$\bar{\delta}^{\dot{\alpha}}(\bar{X})\chi^{\beta}(\vec{x},\bar{X}',Y)=\frac{1}{4}\sigma_\mu^{\dot{\alpha}\beta}j^\mu(\vec{x},\bar{X}\wedge\bar{X}',Y)+2\sigma_\mu^{\dot{\alpha}\beta}\partial^\mu S(\vec{x},\bar{X}\wedge\bar{X}',Y),$$
$$\bar{\delta}^{\dot{\alpha}}(\bar{X})\bar{\chi}^{\dot{\beta}}(\vec{x},X',Y)=\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{P}(\vec{x},X',\bar{X}\wedge Y)+(\bar{X}\cdot X')\bar{F}^{\dot{\alpha}\dot{\beta}}(\vec{x},Y),$$
$$\bar{\delta}^{\dot{\alpha}}(\bar{X})P(\vec{x},\bar{X}')=\frac{1}{4}\sigma_\mu^{\dot{\alpha}\beta}\partial^\mu\chi_\beta(\vec{x},\bar{X}',\bar{X}\wedge\bar{X}'),$$
$$0=\bar{\delta}\langle S_2S_2S_2\chi\rangle=\langle\bar{\chi}S_2S_2\chi\rangle+\langle S_2\bar{\chi}S_2\chi\rangle+\langle S_2S_2\bar{\chi}\chi\rangle+\langle S_2S_2S_2j\rangle+\langle S_2S_2S_2\partial S_2\rangle$$
$$\partial_U\mathcal{S}_2^4(U,V)=\frac{2}{U}\mathcal{S}_2^4(U,V)+\left(\frac{2}{U}-\partial_U-\partial_V\right)\mathcal{S}_2^2(U,V)+\left(\frac{2}{U}+(U-1)\partial_U+V\partial_V\right)\mathcal{S}_2^3(U,V)$$
$$\partial_V\mathcal{S}_2^4(U,V)=-\frac{1}{V}\mathcal{S}_2^4(U,V)-\frac{1}{V}(2-U\partial_U+(1-U)\partial_V)\mathcal{S}_2^2(U,V)-(\partial_U+\partial_V)\mathcal{S}_2^3(U,V)$$
$$\mathcal{S}_2^3(U,V)=\mathcal{S}_2^2\left(\frac{U}{V},\frac{1}{V}\right),\qquad \mathcal{S}_2^6(U,V)=\mathcal{S}_2^4\left(\frac{U}{V},\frac{1}{V}\right)$$
$$\mathcal{S}_2^2(U,V)=U^2\mathcal{S}_2^1\left(\frac{1}{U},\frac{V}{U}\right),\quad \mathcal{S}_2^5(U,V)=U^2\mathcal{S}_2^4\left(\frac{1}{U},\frac{V}{U}\right)$$
$$0=\bar{\delta}\langle S_2S_2P\bar{\chi}\rangle=\langle\bar{\chi}S_2P\bar{\chi}\rangle+\langle S_2\bar{\chi}P\bar{\chi}\rangle+\langle S_2S_2\partial\chi\bar{\chi}\rangle+\langle S_2S_2P\bar{P}\rangle+\langle S_2S_2P\bar{F}\rangle$$

$$\mathcal{R}_{1,p}(U,V)=\frac{1}{8}\big[2U(U-V-3)\partial_U\mathcal{S}_{1,p}(U,V)+UV(2-U+2V)\partial_V^2\mathcal{S}_{1,p}(U,V)$$
$$+U^2(U-2-2V)\partial_U^2\mathcal{S}_{1,p}(U,V)-(4V^2-4+U[1+U-5V])\partial_V\mathcal{S}_{1,p}(U,V)$$
$$-U(U-2-2V)(U+V-1)\partial_V\partial_U\mathcal{S}_{1,p}(U,V)+8\mathcal{S}_{1,p}(U,V)\big]$$
$$\mathcal{R}_{2,p}(U,V)=\frac{1}{4}\big[(4V+2UV-2-2V^2)\partial_V\mathcal{S}_{1,p}(U,V)+UV(V-1)\partial_V^2\mathcal{S}_{1,p}(U,V)$$
$$+U(1+U-V)\partial_U\mathcal{S}_{1,p}(U,V)+U(V-1)(U+V-1)\partial_V\partial_U\mathcal{S}_{1,p}(U,V)$$
$$+U^2(V-1)\partial_U^2\mathcal{S}_{1,p}(U,V)\big]$$
$$\mathcal{R}_{3,p}(U,V)=\frac{1}{8}\big[U(1+U-V)\partial_V\mathcal{S}_{1,p}(U,V)+U^2V\partial_V^2\mathcal{S}_{1,p}$$
$$+U^2(U+V-1)\partial_V\partial_U\mathcal{S}_{1,p}(U,V)+U^3\partial_U^2\mathcal{S}_{1,p}(U,V)\big]$$

2.11. Amplitudes de Mellin.

$$\mathcal{S}^i_{p,\mathrm{conn}}(U,V)\equiv\mathcal{S}^i_p(U,V)-\mathcal{S}^i_{p,\mathrm{disc}}(U,V)$$
$$\mathcal{S}^i_{2,\text{ disc}}(U,V)=\begin{pmatrix}1 & U^2 & \frac{U^2}{V^2} & 0 & 0 & 0\end{pmatrix}$$
$$\mathcal{S}^i_{p,\text{ disc}}(U,V)=(1\quad 0\quad 0\quad 0\quad 0\quad 0)\text{ for }p>2.$$
$$\mathcal{S}^i_{p,\text{ conn}}(U,V)=\int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{u-p-2}{2}}\Gamma\left[2-\frac{s}{2}\right]\Gamma\left[p-\frac{s}{2}\right]$$
$$\times\Gamma^2\left[\frac{2+p}{2}-\frac{t}{2}\right]\Gamma^2\left[\frac{2+p}{2}-\frac{u}{2}\right]M^i_p(s,t)$$
$$\mathcal{S}^i_p(U,V)=\Theta^i(U,V)\mathcal{T}_p(U,V)+\mathcal{S}^i_{p,\text{ free}}(U,V)$$
$$\Theta^i(U,V)\equiv\big(V\quad UV\quad U\quad U(U-V-1)\quad 1-U-V\;V(V-U-1)\big)$$
$$M^i_p(s,t)=\widehat{\Theta}_i(U,V)\circ\mathcal{M}_p$$

$$\widehat{U^mV^n}\circ\mathcal{M}_p(s,t)=\mathcal{M}_p(s-2m,t-2n)\left(\frac{4-s}{2}\right)^2_m\left(\frac{4-t}{2}\right)^2_{2-m-n}\left(\frac{4-u}{2}\right)^2_n$$
$$\widehat{U^mV^n}\circ\mathcal{M}_p(s,t)\underset{s,t\to\infty}{\longrightarrow}\frac{1}{16}s^{2m}t^{4-2m-2n}u^{2n}\mathcal{M}_p(s,t)$$

2.12. Expansión supersimétrica usando las representaciones de Mellin-Barnes.

$$l_p=4p\int_0^\infty d\omega\omega\frac{J_1\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2-J_p\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2}{\sinh^2(\omega)}$$

$$I_p(x)=\int_0^\infty d\omega\frac{\omega J_p(x\omega)^2}{\sinh^2\omega}$$

$$J_\mu(x)J_\nu(x)=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Gamma(-s)\Gamma(2s+\mu+\nu+1)\left(\frac{1}{2}x\right)^{\mu+\nu+2s}}{\Gamma(s+\mu+1)\Gamma(s+\nu+1)\Gamma(s+\mu+\nu+1)}$$

$$I_p(x)=\int_0^\infty d\omega\int_{-i\infty}^{\infty}ds\frac{\Gamma(-s)\Gamma(2s+2p+1)x^{2p+2s}}{2^{2p+2s}\Gamma(s+p+1)^2\Gamma(s+2p+1)}\frac{\omega^{2p+2s+1}}{\sinh^2\omega}$$

$$\int_0^\infty d\omega \frac{\omega^{2p+2s+1}}{\sinh^2\ \omega} = \frac{1}{2^{2p+2s}}\Gamma(2p+2s+2)\zeta(2p+2s+1)$$

$$I_p(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(-s)\Gamma(2s+2p+1)\Gamma(2p+2s+2)\zeta(2p+2s+1)}{\Gamma(s+p+1)^2\Gamma(s+2p+1)}\left(\frac{x}{4}\right)^{2p+2s}$$

$$I_p(x)\ \sim \frac{1}{2p} - \frac{1}{\pi}x^{-1} - \sum_{n=1}^{\infty}\frac{2(-1)^n\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+p+\frac{1}{2}\right)\zeta(2n+1)}{\pi^{\frac{3}{2}+2n}x^{2n+1}\Gamma(n)\Gamma\left(p-n+\frac{1}{2}\right)}$$

$$= \frac{1}{2p} - \frac{1}{\pi}x^{-1} + \frac{4p^2-1}{4\pi^3}\zeta(3)x^{-3} - \frac{3(9-40p^2+16p^4)}{32\pi^5}\zeta(5)x^{-5} + \cdots$$

2.13. Amplitudes de Yang – Mills. Acción Pura de Spin en superpartículas.

$$A = \sum_{i\in\Gamma_i}\frac{c_i n_i}{D_i}$$

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0$$

$$S = \int\ d\tau\left[P_m\partial_\tau X^m + p_\alpha\partial_\tau\theta^\alpha + w_\alpha\partial_\tau\lambda^\alpha - \frac{1}{2}P^2\right]$$

$$Q = \lambda^\alpha d_\alpha$$

$$\Psi(x,\theta,\lambda) = \Psi^{(0)}(x,\theta,\lambda) + \Psi^{(1)}(x,\theta,\lambda) + \Psi^{(2)}(x,\theta,\lambda) + \Psi^{(3)}(x,\theta,\lambda)$$

$$Q\Psi^{(1)} = 0, \delta\Psi^{(1)} = Q\Lambda$$

$$(\gamma^{mnpqr})^{\alpha\beta}D_\alpha A_\beta = 0, \delta A_\alpha = D_\alpha\Lambda$$

$$A_\alpha(x,\theta) = \frac{1}{2}(\gamma^m\theta)_\alpha a_m(x) - \frac{1}{3}(\gamma^m\theta)_\alpha(\theta\gamma_m\chi(x)) - \frac{1}{16}\left(\gamma_p\theta\right)_\alpha(\theta\gamma^{mnp}\theta)\partial_m a_n(x)$$
$$+\frac{1}{60}\left(\gamma_p\theta\right)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma_m\partial_n\chi(x)) + \cdots$$

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)\left(\theta\gamma_{mnp}\theta\right)\rangle = 1$$

$$S = \int\ d\tau\left[P_m\partial_\tau X^m + p_\alpha\partial_\tau\theta^\alpha + w_\alpha\partial_\tau\lambda^\alpha + \bar{w}^\alpha\partial_\tau\bar{\lambda}_\alpha + s^\alpha\partial_\tau r_\alpha - \frac{1}{2}P^2\right]$$

$$Q = Q_0 + r_\alpha\bar{w}^\alpha$$

$$[d\lambda]\lambda^\beta\lambda^\delta\lambda^\gamma\ = (\epsilon T^{-1})^{\beta\delta\gamma}_{\alpha_1\ldots\alpha_{11}}d\lambda^{\alpha_1}\ldots d\lambda^{\alpha_{11}}$$

$$[d\bar{\lambda}]\bar{\lambda}_\beta\bar{\lambda}_\delta\bar{\lambda}_\gamma\ = (\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\delta\gamma}d\bar{\lambda}_{\alpha_1}\ldots d\bar{\lambda}_{\alpha_{11}}$$

$$[dr]\ = (\epsilon T^{-1})^{\beta\delta\gamma}\bar{\lambda}_\beta\bar{\lambda}_\delta\bar{\lambda}_\gamma\left(\frac{\partial}{\partial r_{\alpha_1}}\right)\ldots\left(\frac{\partial}{\partial r_{\alpha_{11}}}\right)$$

$$(\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\gamma\delta}\ = \epsilon^{\alpha_1\ldots\alpha_{16}}(\gamma^m)_{\alpha_{12}\eta}(\gamma^n)_{\alpha_{13}\epsilon}(\gamma^p)_{\alpha_{14}\kappa}\left(\gamma_{mnp}\right)_{\alpha_{15}\alpha_{16}}\left[\delta^\eta_{(\beta}\delta^\epsilon_\gamma\delta^\kappa_{\delta)} - \frac{1}{40}(\gamma^q)_{(\beta\gamma}\delta^\eta_{\delta)}\left(\gamma_q\right)^{\epsilon\kappa}\right]$$

$$S = \int\ d^{10}x d^{16}\theta[dZ]\mathcal{N}\left(\frac{1}{2}\Psi Q\Psi\right)$$

$$(A,B) = \int\ \frac{\delta_R A}{\delta\Psi(Z)}[dZ]\frac{\delta_L B}{\delta\Psi(Z)}$$

$$(S,S) = 0$$

$$S = \int\ d^{10}x d^{16}\theta[dZ]\mathcal{N}\mathrm{Tr}\left(\frac{1}{2}\Psi Q\Psi + \frac{g}{3}\Psi\Psi\Psi\right)$$

$$Q\Psi + g\Psi\Psi = 0, \delta\Psi = Q\Lambda + g[\Psi,\Lambda]$$

$$(\gamma^{mnpqr})^{\alpha\beta}\left(D_\alpha A_\beta + gA_\alpha A_\beta\right) = 0, \delta A_\alpha = D_\alpha\Lambda + g[A_\alpha,\Lambda]$$

$$\{Q,b\} = \frac{P^2}{2}$$

$$b = \frac{(\bar{\lambda}\gamma^m d)}{2(\lambda\bar{\lambda})}P_m + \frac{(\bar{\lambda}\gamma^{mnp}r)[-(d\gamma_{mnp}d)+24N_{mn}P_p]}{192(\lambda\bar{\lambda})^2} - \frac{(r\gamma^{mnp}r)(\bar{\lambda}\gamma_m d)N_{np}}{16(\lambda\bar{\lambda})^3}$$
$$-\frac{(r\gamma^{mnp}r)(\bar{\lambda}\gamma^{pqr}r)N_{mn}N_{qr}}{128(\lambda\bar{\lambda})^4}$$

$$[Q,\mathbf{A}_\alpha] = -d_\alpha + (\gamma^m\lambda)_\alpha \mathbf{A}_m$$
$$\{Q,\mathbf{A}_m\} = P_m + (\lambda\gamma_m\mathbf{W})$$
$$[Q,\mathbf{W}^\alpha] = \frac{1}{4}(\gamma^{mn})_\alpha^\beta \lambda^\alpha \mathbf{F}_{mn}$$
$$\{Q,\mathbf{F}_{mn}\} = -2\big(\lambda\gamma_{[m}\partial_{n]}\mathbf{W}\big)$$
$$D_\alpha A_\beta + D_\beta A_\alpha = (\gamma^m)_{\alpha\beta} A_m, D_\alpha A_m = \partial_m A_\alpha + (\gamma_m W)_\alpha,$$
$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha^\beta F_{mn}, D_\alpha F_{mn} = -2\big(\gamma_{[m}\partial_{n]}W\big)_\alpha$$
$$\mathbf{A}_\alpha = \frac{1}{4(\lambda\bar{\lambda})}\left[N^{mn}\big(\gamma_{mn}\bar{\lambda}\big)_\alpha + J\bar{\lambda}_\alpha\right]$$
$$\mathbf{A}_m = \frac{(\bar{\lambda}\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(\bar{\lambda}\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np}$$
$$\mathbf{W}^\alpha = \frac{(\gamma^m\bar{\lambda})^\alpha}{2(\lambda\bar{\lambda})}\mathbf{\Delta_m}$$
$$\mathbf{F}_{mn} = -\frac{(r\gamma_{mn}\mathbf{W})}{2(\lambda\bar{\lambda})} = \frac{(\bar{\lambda}\gamma_{mn}^p r)}{4(\lambda\bar{\lambda})}\mathbf{\Delta_p}$$
$$\mathbf{\Delta_m} = -P_m + \frac{(r\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(r\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np}$$
$$\hat{\mathbf{A}}_\alpha \Psi^{(1)} = A_\alpha + (\lambda\gamma_m)_\alpha \sigma_m$$
$$\hat{\mathbf{A}}_m \Psi^{(1)} = A_m - (\lambda\gamma_m\rho) + Q\sigma_m$$
$$\hat{\mathbf{W}}^\alpha \Psi^{(1)} = W^\alpha - Q\rho^\alpha + (\gamma^{mn}\lambda)^\alpha s_{mn} + \lambda^\alpha s$$
$$\hat{\mathbf{F}}_{mn}\Psi^{(1)} = F_{mn} - 4Qs_{mn} + \big(\lambda\gamma_{[m}g_{n]}\big) + (\lambda\gamma_{mn}g)$$
$$\sigma_m = -\frac{(\bar{\lambda}\gamma_m A)}{2(\lambda\bar{\lambda})}, \rho^\alpha = \frac{(\gamma^p\bar{\lambda})^\alpha}{2(\lambda\bar{\lambda})}\big(A_p + Q\sigma_p\big), \xi^\alpha = W^\alpha - Q\rho^\alpha$$
$$s_{mn} = \frac{(\bar{\lambda}\gamma_{mn}\xi)}{8(\lambda\bar{\lambda})}, s = \frac{(\bar{\lambda}\xi)}{4(\lambda\bar{\lambda})}, r_{mn} = -F_{mn} + 4Qs_{mn},$$
$$g_\alpha = \frac{(\gamma^{mn}\bar{\lambda})_\alpha}{8(\lambda\bar{\lambda})} r_{mn} - \frac{\bar{\lambda}_\alpha}{2(\lambda\bar{\lambda})} Qs, g_m^\alpha = \frac{(\gamma^n\bar{\lambda})^\alpha}{(\lambda\bar{\lambda})} r_{nm}$$
$$P_m \to \partial_m, d_\alpha \to D_\alpha, w_\alpha \to -\partial_{\lambda^\alpha}$$
$$b = \frac{1}{2}\left[P^m\mathbf{A}_m - d_\alpha\mathbf{W}^\alpha - \frac{1}{2}N^{mn}\mathbf{F}_{mn}\right]$$
$$P^m\hat{\mathbf{A}}_m - D_\alpha\hat{\mathbf{W}}_\alpha - \frac{1}{2}N^{mn}\hat{\mathbf{F}}_{mn} = \partial^m\mathbf{A}_m - d_\alpha\mathbf{W}^\alpha + \frac{1}{4}(\lambda\gamma^{mn}\partial_\lambda)\mathbf{F}_{mn}$$
$$\{b,\Psi^{(1)}\} = P^m A_m - D_\alpha W^\alpha - \frac{1}{2}N^{mn}F_{mn} + Q\Sigma,$$
$$\Sigma = P^m\sigma_m - d_\alpha\rho^\alpha + (\lambda\gamma^{mn}w)s_{mn} - (\lambda w)s$$

2.14. Reglas de Feynman.

$$S = \int d^{10}x d^{16}\theta [dZ]\mathcal{N}\mathrm{Tr}\left[\frac{1}{2}\Psi Q\Psi + \frac{g}{3}\Psi\Psi\Psi + e(b_0\Psi - Q\Xi)\right]$$
$$b_0\Psi^{(1)} = Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn} + 7(\lambda\partial_\lambda)s]$$
$$S = \int d^{10}x d^{16}\theta [dZ]\mathcal{N}\left[\frac{1}{2}\Psi^a Q\Psi^a + \frac{g}{6}f^{abc}\Psi^a\Psi^b\Psi^c + e^a(b_0\Psi^a - Q\Xi^a)\right]$$
$$\mathcal{G}^{ab}(Z,Z') = \delta^{ab}\frac{2b_0}{\Box}\delta(Z-Z')$$
$$\mathcal{V}^{abc} = gf^{abc}\int d^{10}x d^{16}\theta [dZ]\mathcal{N}$$
$$\mathcal{A}_n = \sum_{i\in\Gamma_n}\frac{c_i n_i}{D_i}$$
$$\mathcal{A}_n = \sum_{\sigma\in S_{n-1}} A_{\sigma(1,\ldots,n-1),n}\mathrm{Tr}(T^{\sigma_1}\ldots T^{\sigma_{n-1}}T^n)$$
$$\mathcal{G}(Z,Z') = \frac{2b_0}{\Box}\delta(Z-Z')$$
$$\mathcal{V} = g\int d^{10}x d^{16}\theta [dZ]\mathcal{N}$$

2.15. Reglas de Berends-Giele.

$$Q\Psi + \Psi\Psi - g b_0(e) = 0$$

$$b_0\Psi - Q\Xi = 0$$

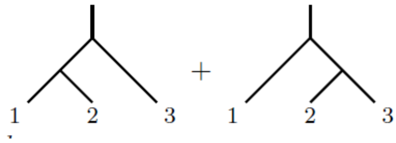
$$\frac{1}{2}\Box\Psi + b_0(\Psi\Psi) = 0,$$

$$\Psi = \sum_P \Psi_P T^P e^{k_P\cdot x} = \sum_i \Psi_i T^{a_i} e^{k_i\cdot x} + \sum_{i,j} \Psi_{ij} T^{a_i} T^{a_j} e^{k_{ij}\cdot x} + \cdots$$

$$QV_i = 0, \Psi_P = -\frac{1}{s_P}\sum_{QR=P} b_0\big(\Psi_Q\Psi_R\big), b_0 V_i = Q\Xi_i$$

$$A_{1\ldots n} = (-1)^n \sum_{PQ=(1\ldots n-1)} \langle\Psi_P\Psi_Q V_n\rangle$$

$$A_{123} = \langle V_1 V_2 V_3\rangle$$

$\Psi_{123} =$ 

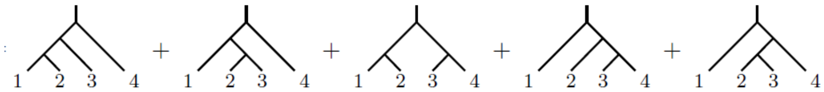
$$= -\frac{b_0}{s_{123}}(\Psi_{12}V_3 + V_1\Psi_{23})$$

$$= \frac{b_0}{s_{123}}\left(\frac{b_0}{s_{12}}(V_1V_2)V_3 + V_1\frac{b_0}{s_{23}}(V_2V_3)\right)$$

$$A_{1234} = \frac{1}{s_{12}}\langle b_0(V_1V_2)V_3V_4\rangle + \frac{1}{s_{23}}\langle V_1 b_0(V_2V_3)V_4\rangle$$

$$b_0(V_1V_2) = V_{12} + Q\Lambda_{12}$$

$$A_{1234} = \frac{1}{s_{12}}\langle V_{12}V_3V_4\rangle + \frac{1}{s_{23}}\langle V_1V_{23}V_4\rangle$$

$\Psi_{1234} =$ 

$$= -\frac{b_0}{s_{1234}}\left[\frac{b_0(b_0(V_1V_2)V_3)V_4}{s_{123}s_{12}} + \frac{b_0\big(V_1b_0(V_2V_3)\big)V_4}{s_{123}s_{23}} + \frac{b_0(V_1V_2)b_0(V_3V_4)}{s_{12}s_{34}} + \frac{V_1b_0\big(V_2b_0(V_3V_4)\big)}{s_{234}s_{34}} + \frac{V_1b_0(b_0(V_2V_3)V_4)}{s_{234}s_{23}}\right]$$

$$A_{12345} = -\frac{\langle V_1b_0\big(V_2\big(b_0(V_3V_4)\big)V_5\big\rangle}{s_{234}s_{34}} - \frac{\langle V_1b_0(b_0(V_2V_3)V_4)V_5\rangle}{s_{234}s_{23}} - \frac{\langle b_0(V_1V_2)b_0(V_3V_4)V_5\rangle}{s_{12}s_{34}} - \frac{\langle b_0\big(V_1\big(b_0(V_2V_3)\big)V_4V_5\big\rangle}{s_{123}s_{23}} - \frac{\langle b_0(b_0(V_1V_2)V_3)V_4V_5\rangle}{s_{123}s_{12}}$$

$$b_0\big(b_0\big(V_iV_j\big)V_k\big) = V_{ijk} + s_{ij}T_{ijk} + s_{ijk}\Lambda_{ij}V_k + Q\big(\Lambda_{[ij]k}\big)$$

$$T_{ijk} = \Lambda_iV_jV_k - \Lambda_{ij}V_k + \text{cyclic}(ijk)$$

$$A_{12345} = \frac{\langle[V_{342}+s_{34}T_{342}+s_{342}\Lambda_{34}V_2]V_1V_5\rangle}{s_{234}s_{34}} - \frac{\langle[V_{234}+s_{23}T_{234}+s_{123}\Lambda_{23}V_4]V_1V_5\rangle}{s_{234}s_{23}} + \frac{\langle(V_{12}+Q\Lambda_{12})(V_{34}+Q\Lambda_{34})V_5\rangle}{s_{12}s_{34}} - \frac{\langle[V_{231}+s_{23}T_{231}+s_{231}\Lambda_{23}V_1]V_4V_5\rangle}{s_{123}s_{23}} + \frac{\langle[V_{123}+s_{12}T_{123}+s_{123}\Lambda_{12}V_3]V_4V_5\rangle}{s_{123}s_{12}}$$

$$A_{12345}\big|_{s_{123}^{-1}} = \langle T_{123}V_4V_5\rangle - \langle T_{231}V_4V_5\rangle$$

$$A_{12345}\big|_{s_{34}^{-1}} = \langle\Lambda_{34}V_2V_1V_5\rangle + \frac{\langle(V_{12}+Q\Lambda_{12})Q\Lambda_{34}V_5\rangle}{s_{12}}$$

$$= \langle\Lambda_{34}V_2V_1V_5\rangle + \frac{\langle\Lambda_{34}Q(V_{12}+Q\Lambda_{12})V_5\rangle}{s_{12}}$$

$$= \langle\Lambda_{34}V_2V_1V_5\rangle + \langle\Lambda_{34}V_1V_2V_5\rangle$$

$$A_{12345} = \langle V_1M_{234}V_5\rangle + \langle M_{12}M_{34}V_5\rangle + \langle M_{123}V_4V_5\rangle$$

$$M_{ij} = \frac{V_{ij}}{s_{ij}}, \text{ and } M_{ijk} = \frac{1}{s_{ijk}}\left(\frac{V_{ijk}}{s_{ij}} - \frac{V_{jki}}{s_{jk}}\right)$$

2.16. Invariancia de gauge y bloques BRST.

$$Qb_0\big(b_0\big(\Psi_\alpha\Psi_\beta\big)\Psi_\gamma\big) = s_{\alpha\beta\gamma}b_0\big(\Psi_\alpha\Psi_\beta\big)\Psi_\gamma - s_{\alpha\beta}b_0\big(\Psi_\alpha\Psi_\beta\Psi_\gamma\big) + \cdots$$

$$Qb_0(b_0(b_0(V_1V_2)V_3)V_4) = s_{1234}b_0(b_0(V_1V_2)V_3)V_4 - s_{123}b_0(b_0(V_1V_2)V_3V_4) + s_{12}b_0(b_0(V_1V_2V_3)V_4)$$

$$Q\Psi_Z = -\sum_{XY=Z} \Psi_X\Psi_Y$$

$$\begin{aligned}\delta A_{12\ldots n} &= \sum_{XY=(1\ldots n-1)} \langle \Psi_X\Psi_Y Q\omega\rangle \\ &= \sum_{XY=(1\ldots n-1)} \langle Q(\Psi_X\Psi_Y)\omega\rangle \\ &= \sum_{XY=(1\ldots n-1)} \Big(-\sum_{AB=X} \langle \Psi_A\Psi_B\Psi_Y\omega\rangle + \sum_{AB=Y} \langle \Psi_X\Psi_A\Psi_B\omega\rangle\Big)\end{aligned}$$

$$b_0(b_0(b_0(Q\omega V_2)V_3)V_4) = s_{1234}b_0(b_0(\omega V_2)V_3)V_4 - s_{123}b_0(b_0(\omega V_2)V_3V_4) + s_{12}b_0(b_0(\omega V_2V_3)V_4)$$

$$b_0(b_0(\ldots)V_n)|_{V_i\to Q\omega} = \pm[Qb_0(b_0(\ldots)V_n)]|_{V_i\to\omega}$$

$$QV_{1\ldots n} = \sum_{j=2}^{n} s_{1\ldots j}\Big(\sum_{\alpha\in P(\beta_j)} V_{1\ldots(j-1)\alpha}V_{j(\beta_j-\alpha)} + \sum_{\tilde\alpha\in P(\beta_{(j+1)})} V_{1\ldots(j)\tilde\alpha}V_{j+1(\beta_{(j-1)}-\tilde\alpha)}\Big),$$

$$QV_{1234} = s_{1234}V_{123}V_4 - s_{123}(V_{123}V_4 - V_{12}V_{34} - V_{124}V_3) + s_{12}(V_1V_{234} - V_{12}V_{34} + V_{13}V_{24} + V_{14}V_{23} - V_{124}V_3 + V_{134}V_2)$$

$$b_0(b_0(\ldots)V_n) = V_{12\ldots n} + \mathcal{C}_n$$

$$b_0(V_{12\ldots n}V_{n+1}) = V_{12\ldots n+1} + \mathcal{C}$$

2.17. Dualidad Color – Kinemática.

$$b = \left[P^m + \frac{(\lambda\gamma^{mn}r)}{4(\lambda\bar\lambda)}\mathbf{A}_n\right]\mathbf{A}_m$$

$$b = -\Delta^m\mathbf{A}_m$$

$$b_0(V_1V_2) = (b_0V_1)V_2 + \hat\Delta^m V_1\hat{\mathbf{A}}_m V_2 - \hat{\mathbf{A}}^m V_1\hat\Delta_m V_2 - V_1(b_0V_2)$$

$$b_0(V_1V_2V_3) = b_0(V_1V_2)V_3 + b_0(V_2V_3)V_1 + b_0(V_3V_1)V_2 - (b_0V_1)V_2V_3 + V_1(b_0V_2)V_3 - V_1V_2(b_0V_3)$$

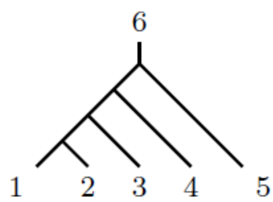
$$b_0(\mathcal{V}_1\mathcal{V}_2) = \hat\Delta^m\mathcal{V}_1\hat{\mathbf{A}}_m\mathcal{V}_2 - \hat{\mathbf{A}}^m\mathcal{V}_1\hat\Delta_m\mathcal{V}_2$$

$$b_0(\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3) = b_0(\mathcal{V}_1\mathcal{V}_2)\mathcal{V}_3 + b_0(\mathcal{V}_2\mathcal{V}_3)\mathcal{V}_1 + b_0(\mathcal{V}_3\mathcal{V}_1)\mathcal{V}_2$$

$$\{X, Y\} = \hat\Delta^m X\hat{\mathbf{A}}_m Y - \hat{\mathbf{A}}^m X\hat\Delta_m Y$$

$$b_0(\mathcal{V}_1\mathcal{V}_2) = \{\mathcal{V}_1, \mathcal{V}_2\}$$

$$b_0(b_0(\mathcal{V}_1\mathcal{V}_2)\mathcal{V}_3) + b_0(b_0(\mathcal{V}_2\mathcal{V}_3)\mathcal{V}_1) + b_0(b_0(\mathcal{V}_3\mathcal{V}_1)\mathcal{V}_2)$$



$$n_{123456} = \\ = \langle b_0(b_0(b_0(\mathcal{V}_1\mathcal{V}_2)\mathcal{V}_3)\mathcal{V}_4)\mathcal{V}_5\mathcal{V}_6\rangle$$

$$\{\{\mathcal{V}_1, \mathcal{V}_2\}, \mathcal{V}_3\} + \{\{\mathcal{V}_2, \mathcal{V}_3\}, \mathcal{V}_1\} + \{\{\mathcal{V}_3, \mathcal{V}_1\}, \mathcal{V}_2\}$$

$$L_\psi := \hat\Delta^m(\psi)\hat{\mathbf{A}}_m - \hat{\mathbf{A}}_m(\psi)\hat\Delta^m$$

$$[L_\psi, L_\phi] = L_{b_0(\psi\phi)}$$

2.18. Operador fantasma.

$$b_0V = \left(\partial^m \hat{\mathbf{A}}_m - D_\alpha \hat{\mathbf{W}}^\alpha + \frac{1}{4}(\lambda\gamma^{mn}\partial_\lambda)\hat{\mathbf{F}}_{mn}\right)V$$

$$b_0V = Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn}] - (\lambda D)s + \frac{9}{4}(\lambda\gamma^n g_n) + \frac{1}{4}\lambda^\alpha(\lambda\gamma^m)_\beta\partial_{\lambda^\alpha}g_m^\beta$$
$$-\frac{45}{2}(\lambda g) - \frac{5}{2}\lambda^\alpha\lambda^\beta\partial_{\lambda^\alpha}g_\beta$$
$$= Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn}] - (\lambda D)s + 2(\lambda\gamma^n g_n) - 20(\lambda g) + \frac{5}{4}(\lambda D)s - \frac{5}{4}Qs$$
$$-\frac{(\lambda\gamma^{mn}\bar\lambda)}{4(\lambda\bar\lambda)}(\lambda D)s_{mn} + \frac{(\lambda\gamma^{mn}\bar\lambda)}{4(\lambda\bar\lambda)}Qs_{mn}$$

$$\partial_{\lambda^\alpha}g_m^\beta = -\frac{\bar\lambda_\alpha}{(\lambda\bar\lambda)}g_m^\beta + \frac{4(\gamma^n\bar\lambda)^\beta}{(\lambda\bar\lambda)}\left[D_\alpha s_{mn} - Q\left(\frac{\bar\lambda_\alpha}{(\lambda\bar\lambda)}s_{mn}\right)\right]$$

$$\partial_{\lambda^\alpha}g_\beta = -\frac{\bar\lambda_\alpha}{(\lambda\bar\lambda)}g_\beta - \frac{\bar\lambda_\alpha}{2(\lambda\bar\lambda)}\left[D_\beta s - Q\left(\frac{\bar\lambda_\beta}{(\lambda\bar\lambda)}s\right)\right]$$
$$+\frac{(\gamma^{mn}\bar\lambda)^\beta}{2(\lambda\bar\lambda)}\left[D_\beta s_{mn} - Q\left(\frac{\bar\lambda_\beta}{(\lambda\bar\lambda)}s_{mn}\right)\right]$$

$$2(\lambda\gamma^n g_n) - 20(\lambda g) = \frac{(\lambda\gamma^{mn}\bar\lambda)}{2(\lambda\bar\lambda)}F_{mn} - 2\frac{(\lambda\gamma^{mn}\bar\lambda)}{(\lambda\bar\lambda)}Qs_{mn} + 10Qs$$

$$b_0V = Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn}] + \frac{1}{4}(\lambda D)s - \frac{5}{4}Qs + \frac{(\lambda\gamma^{mn}\bar\lambda)}{2(\lambda\bar\lambda)}F_{mn}$$
$$-\frac{(\lambda\gamma^{mn}\bar\lambda)}{4(\lambda\bar\lambda)}(\lambda D)s_{mn} + \frac{(\lambda\gamma^{mn}\bar\lambda)}{4(\lambda\bar\lambda)}Qs_{mn} - 2\frac{(\lambda\gamma^{mn}\bar\lambda)}{(\lambda\bar\lambda)}Qs_{mn} + 10Qs$$

$$\frac{(\lambda\gamma^{mn}\bar\lambda)}{(\lambda\bar\lambda)}Qs_{mn} = \frac{\bar\lambda Q\xi}{(\lambda\bar\lambda)} + Qs$$

$$b_0V = Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn}] + \frac{1}{4}(\lambda D)s - \frac{5}{4}Qs + 8(\lambda D)s - \frac{5}{4}(\lambda D)s$$
$$-7(\lambda D)s - \frac{7}{4}Qs + 10Qs$$
$$= Q[\partial^m\sigma_m - D_\alpha\rho^\alpha - (\lambda\gamma^{mn}\partial_\lambda)s_{mn} + 7s]$$

$$\Lambda = -\frac{\partial^m(\bar\lambda\gamma_m A)}{(\lambda\bar\lambda)} + 2\frac{(\bar\lambda W)}{(\lambda\bar\lambda)} + \frac{(\bar\lambda\gamma^m D)}{4(\lambda\bar\lambda)^2}(r\gamma_m A)$$

$$b_0(V_1V_2) = V_{12} + Q\Lambda_{12}$$

$$A_{12\alpha} = \frac{1}{2}[(k_2\cdot A_1)A_{2\alpha} - (k_1\cdot A_2)A_{1\alpha} + (\gamma^p W_1)_\alpha A_{2p} - (\gamma^p W_2)_\alpha A_{1p}]$$

$$\Lambda_{12} = -\frac{b_0(V_{12})}{s_{12}}$$

$$D_\alpha A_{12\beta} + D_\beta A_{12\alpha} = (\gamma^m)_{\alpha\beta}A_{12m} + (k_1\cdot k_2)(A_{1\alpha}A_{2\beta} + A_{1\beta}A_{2\alpha})$$
$$D_\alpha A_{12m} = (\gamma_m W_{12})_\alpha + k_{12m}A_{12\alpha} + (k_1\cdot k_2)(A_{1\alpha}A_{2m} - A_{2\alpha}A_{1m})$$
$$D_\alpha W_{12}^\beta = \frac{1}{4}(\gamma^{mn})_\alpha^\beta F_{12mn} + (k_1\cdot k_2)\left(A_{1\alpha}W_2^\beta - A_{2\alpha}W_1^\beta\right)$$
$$D_\alpha F_{12mn} = k_{12m}(\gamma_n W_{12})_\alpha - k_{12n}(\gamma_m W_{12})_\alpha + (k_1\cdot k_2)[A_{1\alpha}F_{2mn} - A_{2\alpha}F_{1mn}$$
$$+A_{1n}(\gamma_m W_2)_\alpha - A_{2n}(\gamma_m W_1)_\alpha - A_{1m}(\gamma_n W_2)_\alpha + A_{2m}(\gamma_n W_1)_\alpha]$$
$$A_{12m} = \frac{1}{2}[A_{2m}(k_2\cdot A_1) - A_{1m}(k_1\cdot A_2) + (k_{2m} - k_{1m})(A_1\cdot A_2) + 2(W_1\gamma_m W_2)]$$
$$W_{12}^\alpha = \frac{1}{4}(\gamma^{mn}W_2)^\alpha F_{1mn} + W_2^\alpha(k_2\cdot A_1) - \frac{1}{4}(\gamma^{mn}W_1)^\alpha F_{2mn} - W_1^\alpha(k_1\cdot A_2)$$
$$F_{12mn} = k_{12m}A_{12n} - k_{12n}A_{12m} - (k_1\cdot k_2)(A_{1m}A_{2n} - A_{1n}A_{2m})$$

$$\hat{\mathbf{A}}_\alpha V_{12} = A_{12\alpha} + (\lambda\gamma^m)_\alpha\sigma_{12m}$$

$$\sigma_{12m} = -\frac{(\bar\lambda\gamma_m A_{12})}{2(\lambda\bar\lambda)}$$

$$\hat{\mathbf{A}}_m V_{12} = \frac{(\gamma^m\bar\lambda)^\alpha\lambda^\beta}{2(\lambda\bar\lambda)}[D_\alpha A_{12\beta} + D_\beta A_{12\alpha}] + \frac{(\bar\lambda\gamma^m\gamma^p\lambda)}{2(\lambda\bar\lambda)}Q\left[-\frac{(\bar\lambda\gamma_p A_{12})}{2(\lambda\bar\lambda)}\right]$$

$$\hat{\mathbf{A}}_m V_{12} = A_{12m} - (\lambda\gamma_m\rho_{12}) + Q(\sigma_{12m}) + (k_1\cdot k_2)\left[\frac{1}{2(\lambda\bar\lambda)}\Big(V_1(\bar\lambda\gamma_m A_2) - V_2(\bar\lambda\gamma_m A_1)\Big)\right]$$

$$\rho_{12}^\alpha = \frac{(\gamma^p\bar\lambda)^\alpha}{2(\lambda\bar\lambda)}[A_{12p} + Q(\sigma_{12p})]$$

$$\hat{\mathbf{W}}^{\alpha}V_{12} = \frac{(\gamma^m\bar{\lambda})^{\alpha}}{2(\lambda\bar{\lambda})}[-\partial_m V_{12} + QA_{12m} - (\lambda\gamma_m Q\rho_{12})$$

$$\hat{\mathbf{W}}^{\alpha}V_{12} = \frac{(\gamma^m\bar{\lambda})^{\alpha}}{2(\lambda\bar{\lambda})}[-\partial_m V_{12} + QA_{12m} - (\lambda\gamma_m Q\rho_{12})$$

$$+(k_1\cdot k_2)Q\left[\frac{1}{2(\lambda\bar{\lambda})}\left(V_1(\bar{\lambda}\gamma_m A_2) - V_2(\bar{\lambda}\gamma_m A_1)\right)\right]\Bigg]$$

$$\hat{\mathbf{W}}^{\alpha}V_{12} = \xi_{12}^{\alpha} + (\gamma^{mn}\lambda)^{\alpha}s_{12mn} + \lambda^{\alpha}s_{12} + (k_1\cdot k_2)\left[\frac{(\gamma^m\bar{\lambda})^{\alpha}}{2(\lambda\bar{\lambda})}(V_1A_{2m} - V_2A_{1m})\right.$$

$$\left.+\frac{(\gamma^m\bar{\lambda})^{\alpha}}{4(\lambda\bar{\lambda})^2}\left(-V_1(r\gamma_m A_2) + V_2(r\gamma_m A_1)\right)\right]$$

$$\xi_{12}^{\alpha} = W_{12}^{\alpha} - Q\rho_{12}^{\alpha}\; s_{12mn} = \frac{(\bar{\lambda}\gamma_{mn}\xi_{12})}{8(\lambda\bar{\lambda})}, s_{12} = \frac{(\bar{\lambda}\xi_{12})}{4(\lambda\bar{\lambda})}$$

$$\hat{\mathbf{F}}_{mn}V_{12} = F_{12mn} - 4Qs_{12mn} - (\lambda\gamma_{[m}g_{12n]}) + (\lambda\gamma_{mn}g_{12}) + \frac{(k_1\cdot k_2)}{2(\lambda\bar{\lambda})}\left[V_1(\bar{\lambda}\gamma_{mn}W_2)\right.$$

$$-V_2(\bar{\lambda}\gamma_{mn}W_1) + \frac{(\bar{\lambda}\gamma^{mnp}r)}{2(\lambda\bar{\lambda})}(V_1A_{2m} - V_2A_{1m})$$

$$\left.+\frac{(\bar{\lambda}\gamma^{mnp}r)}{4(\lambda\bar{\lambda})^2}\left(-V_1(r\gamma_p A_2)\} + V_2(r\gamma_m A_1)\right)\right]$$

$$r_{12mn} = -F_{12mn} + 4Qs_{12mn}, g_{12\alpha} = \frac{(\gamma^{mn}\bar{\lambda})_{\alpha}}{8(\lambda\bar{\lambda})}r_{12mn} - \frac{\bar{\lambda}_{\alpha}}{2(\lambda\bar{\lambda})}Qs_{12}$$

$$g_{12m}^{\alpha} = \frac{(\gamma^n\bar{\lambda})^{\alpha}}{(\lambda\bar{\lambda})}r_{12mn}$$

$$-\Lambda_{12} = \frac{b_0(V_{12})}{s_{12}}$$

$$= -(A_1W^2 - A_2W^1) - \frac{2}{(\lambda\bar{\lambda})}[V_1(\bar{\lambda}W_2) - V_2(\bar{\lambda}W_1)] + \frac{k_{12}^m}{2(\lambda\bar{\lambda})}[V_1(\bar{\lambda}\gamma_m A_2) - V_2(\bar{\lambda}\gamma_m A_1)]$$

$$-\frac{(\bar{\lambda}\gamma^m D)}{2(\lambda\bar{\lambda})}[V_1A_{2m} - V_2A_{1m}] - \frac{(\bar{\lambda}\gamma^m D)}{4(\lambda\bar{\lambda})^2}[-V_1(r\gamma_m A_2) + V_2(r\gamma_m A_1)]$$

$$+\frac{1}{4}(\lambda\gamma^{mn}\partial_\lambda)\left[\frac{1}{2(\lambda\bar{\lambda})}\left(V_1(\bar{\lambda}\gamma_{mn}W_2) - V_2(\bar{\lambda}\gamma_{mn}W_1)\right) + \frac{(\bar{\lambda}\gamma^{mnp}r)}{4(\lambda\bar{\lambda})^2}(V_1A_{2p} - V_2A_{1p})\right.$$

$$\left.+\frac{(\bar{\lambda}\gamma_{mnp}r)}{8(\lambda\bar{\lambda})^3}\left(-V_1(r\gamma^p A_2) + V_2(r\gamma^p A_1)\right)\right]$$

$$-\Lambda_{12} = k_{12m}V_1(\bar{\lambda}\gamma^m A_2) - k_{12m}V_2(\bar{\lambda}\gamma^m A_1) + 2V_1(\bar{\lambda}W_2) - 2V_2(\bar{\lambda}W_1)$$

$$-\frac{1}{(\lambda\bar{\lambda})}(\lambda\gamma^{mp}\bar{\lambda})A_{1m}A_{2p} + Q_0\left[\frac{1}{2(\lambda\bar{\lambda})}[(\bar{\lambda}\gamma^m A_1)A_{2m} - (\bar{\lambda}\gamma^m A_2)A_{1m}]\right] + O(r)$$

$$-\frac{1}{2(\lambda\bar{\lambda})}(\bar{\lambda}\gamma^m D)[V_1A_{2m} - V_2A_{1m}] = Q_0\left[\frac{1}{2(\lambda\bar{\lambda})}[(\bar{\lambda}\gamma^m A_1)A_{2m} - (\bar{\lambda}\gamma^m A_2)A_{1m}]\right]$$

$$+\frac{k_{12}m}{2(\bar{\lambda})}(\bar{\lambda}\gamma^m A_1)V_2 - \frac{k_{12m}}{2(\bar{\lambda})}(\bar{\lambda}\gamma^m A_2)V_1$$

$$+\frac{(\lambda\gamma^{mp}\bar{\lambda})}{(\lambda\bar{\lambda})}A_{1p}A_{2m} + 5V_1(\bar{\lambda}W_2) - 5V_2(\bar{\lambda}W_1)$$

$$+\frac{(\bar{\lambda}\gamma^m A_1)}{2(\lambda\bar{\lambda})}(\lambda\gamma_m W_2) - \frac{(\bar{\lambda}\gamma^m A_2)}{2(\lambda\bar{\lambda})}(\lambda\gamma_m W_1)$$

$$\frac{1}{4}(\lambda\gamma^{mn}\partial_\lambda)\left[\frac{1}{2(\lambda\bar{\lambda})}\left(V_1(\bar{\lambda}\gamma_{mn}W_2) - V_2(\bar{\lambda}\gamma_{mn}W_1)\right)\right] = \frac{5}{4(\lambda\bar{\lambda})^2}[V_2(\bar{\lambda}W_1) - V_1(\bar{\lambda}W_2)]$$

$$+\frac{1}{2(\lambda\bar{\lambda})}[(\lambda\gamma^m W_2)(\bar{\lambda}\gamma_m A_1) - (\lambda\gamma^m W_1)(\bar{\lambda}\gamma_m A_2)]$$

$$+\frac{1}{4(\lambda\bar{\lambda})}[V_1(\bar{\lambda}W_2) - V_2(\bar{\lambda}W_1)] + A_1W_2 - A_2W_1$$

$$\frac{(\bar{\lambda}\gamma^{mnp}r)}{16(\lambda\bar{\lambda})^2}[(\lambda\gamma_{mn}A_1)A_{2p}-(\lambda\gamma_{mn}A_2)A_{1p}]=\frac{1}{2(\lambda\bar{\lambda})}(r\gamma^pA_1)A_{2p}-\frac{(\lambda r)}{2(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^pA_1)A_{2p}$$
$$-\frac{1}{4(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^mA_1)(\lambda\gamma^p\gamma_mr)A_{2p}$$
$$-\frac{1}{2(\lambda\bar{\lambda})}(r\gamma^pA_2)A_{1p}+\frac{(\lambda r)}{2(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^pA_2)A_{1p}$$
$$+\frac{1}{4(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^mA_2)(\lambda\gamma^p\gamma_mr)A_{1p}$$
$$\frac{(\bar{\lambda}\gamma^{mnp}r)}{32(\lambda\bar{\lambda})^2}[-(\lambda\gamma_{mn}A_1)(r\gamma^pA_2)+(\lambda\gamma_{mn}A_2)(r\gamma^pA_1)]=-\frac{(r\gamma^pA_1)}{4(\lambda\bar{\lambda})^2}(r\gamma_pA_2)$$
$$+\frac{(\lambda r)}{2(\lambda\bar{\lambda})^3}(\bar{\lambda}\gamma^pA_1)(r\gamma_pA_2)-\frac{(\bar{\lambda}\gamma^mA_1)}{8(\lambda\bar{\lambda})^3}(\lambda\gamma_m\gamma^pr)(r\gamma_pA_2)$$
$$+\frac{(r\gamma^pA_2)}{4(\lambda\bar{\lambda})^2}(r\gamma_pA_1)-\frac{(\lambda r)}{2(\lambda\bar{\lambda})^3}(\bar{\lambda}\gamma^pA_2)(r\gamma_pA_1)$$
$$+\frac{(\bar{\lambda}\gamma^mA_2)}{8(\lambda\bar{\lambda})^3}(\lambda\gamma_m\gamma^pr)(r\gamma_pA_1)$$
$$\frac{1}{4(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^mD)[V_1(r\gamma_mA_2)-V_2(r\gamma_mA_1)]=Q_0\left[-\frac{(\bar{\lambda}\gamma^mA_1)}{4(\lambda\bar{\lambda})^2}(r\gamma_mA_2)+\frac{(\bar{\lambda}\gamma^mA_2)}{4(\lambda\bar{\lambda})^2}(r\gamma_mA_1)\right]$$
$$+\frac{(r\gamma^mD)}{4(\bar{\lambda})^2}[V_2(\bar{\lambda}\gamma_mA_1)-V_1(\bar{\lambda}\gamma_mA_2)]$$
$$+\frac{1}{4(\lambda\bar{\lambda})^2}[(\bar{\lambda}\gamma^mA_1)(r\gamma_m\gamma^s\lambda)A_{2s}-(\bar{\lambda}\gamma^mA_2)(r\gamma_m\gamma^s\lambda)A_{1s}$$
$$+(\bar{\lambda}\gamma^m\gamma^s\lambda)A_{1s}(r\gamma_mA_2)-(\bar{\lambda}\gamma^m\gamma^s\lambda)A_{2s}(r\gamma_mA_1)]$$
$$\Lambda_{12}=k_{12m}V_1(\bar{\lambda}\gamma^mA_2)-k_{12}V_2(\bar{\lambda}\gamma^mA_1)+2V_1(\bar{\lambda}W_2)-2V_2(\bar{\lambda}W_1)-\frac{(\lambda\gamma^{mp}\bar{\lambda})}{(\lambda\bar{\lambda})}A_{1m}A_{2p}$$
$$+\frac{(\lambda\gamma^p\gamma_mr)}{4(\bar{\lambda})^2}(\bar{\lambda}\gamma^mA_1)A_{2p}-\frac{(\lambda\gamma^p\gamma_mr)}{4(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma^mA_2)A_{1p}+\frac{(\lambda\gamma_m\gamma^pr)}{8(\bar{\lambda})^3}(\bar{\lambda}\gamma^mA_1)(r\gamma_pA_2)$$
$$-\frac{(\lambda\gamma_m\gamma^pr)}{8(\lambda\bar{\lambda})^3}(\bar{\lambda}\gamma^mA_2)(r\gamma_pA_1)+\frac{1}{4(\lambda\bar{\lambda})^2}(r\gamma^mD)[V_2(\bar{\lambda}\gamma_mA_1)-V_1(\bar{\lambda}\gamma_mA_2)]$$
$$+Q\left[\frac{1}{2(\lambda\bar{\lambda})}[(\bar{\lambda}\gamma^mA_1)A_{2m}-(\bar{\lambda}\gamma^mA_2)A_{1m}]\right]$$
$$b_0(b_0(V_1V_2)V_3)$$
$$D_\alpha A_{123\beta}+D_\beta A_{123\alpha}=(\gamma^m)_{\alpha\beta}A_{123m}+(k_1\cdot k_2)[A_{1\alpha}A_{23\beta}+A_{2\alpha}A_{31\beta}+(\alpha\leftrightarrow\beta)]$$
$$+(k_{12}\cdot k_3)[A_{12\alpha}A_{3\beta}-(12\leftrightarrow 3)]$$
$$A_{123\alpha}=\frac{1}{2}[(k_3\cdot A_{12})A_{3\alpha}-(k_{12}\cdot A_3)A_{12\alpha}+(\gamma^pW_{12})_\alpha A_{3p}-(\gamma^pW_3)_\alpha A_{12p}]$$
$$QV_{123}=s_{123}V_{12}V_3+s_{12}[V_1V_{23}+V_2V_{31}+V_3V_{12}]$$
$$s_{123}V_{123}-Q(b_0(V_{123}))=s_{123}b_0(V_{12}V_3)+s_{12}b_0[V_1V_{23}+V_2V_{31}+V_3V_{12}]$$
$$s_{123}V_{123}-Q(b_0(V_{123}))=s_{123}[b_0(b_0(V_1V_2)V_3)+s_{12}(V_1\Lambda_{23}+V_2\Lambda_{31}+V_3\Lambda_{12})-s_{123}\Lambda_{12}V_3]$$
$$+s_{12}b_0(V_1b_0(V_2V_3))+s_{12}b_0(V_2b_0(V_3V_1))+s_{12}b_0(V_3b_0(V_1V_2))$$
$$+Q[s_{123}b_0(\Lambda_{12}V_3)-s_{12}(b_0(V_1\Lambda_{23})+b_0(V_2\Lambda_{31})+b_0(V_3\Lambda_{12}))]$$
$$V_{123}=[b_0(b_0(V_1V_2)V_3)+s_{12}(V_1\Lambda_{23}+V_2\Lambda_{31}+V_3\Lambda_{12})-s_{123}\Lambda_{12}V_3]$$
$$-s_{12}(\Lambda_1V_2V_3+V_1\Lambda_2V_3+V_1V_2\Lambda_3)+Q\left[b_0(\Lambda_{12}V_3)-\frac{s_{12}}{s_{123}}[b_0(V_1\Lambda_{23}\right.$$
$$\left.+V_2\Lambda_{31}+V_3\Lambda_{12}-\Lambda_1V_2V_3-V_1\Lambda_2V_3-V_1V_2\Lambda_3)]+\frac{b_0(V_{123})}{s_{123}}\right]$$
$$b_0(b_0(V_1V_2)V_3)=V_{123}+s_{12}T_{123}+s_{123}\Lambda_{12}V_3+Q\Lambda_{[12]3}$$
$$\Lambda_{[12]3}=-b_0\left[\frac{s_{12}}{s_{123}}T_{123}+\frac{V_{123}}{s_{123}}+\Lambda_{12}V_3\right]$$

2.19. Teorías de Super Yang-Mills y deformación de masa.

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left[-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} - D^{\alpha}\phi_a D_{\alpha}\phi_a + \frac{g^2}{2}[\phi_a,\phi_b]^2 + i\bar{\psi}_p\gamma^{\alpha}D_{\alpha}\psi_p - g\left(\bar{\psi}_p\left(\Gamma_a^{pq}P_+ + \bar{\Gamma}_a^{pq}P_-\right)[\phi_a,\psi_q]\right)\right]$$

$$\delta_{\epsilon}A_{\alpha} = i\bar{\psi}_p\gamma_{\alpha}\epsilon_p, \delta_{\epsilon}\phi_a = -i\bar{\psi}_p\left(\Gamma_a^{pq}P_+ + \bar{\Gamma}_a^{pq}P_-\right)\epsilon_q,$$

$$\delta_{\epsilon}\psi_p = iF_{\alpha\beta}\Sigma^{\alpha\beta}\epsilon_p + \gamma^{\alpha}D_{\alpha}\phi_a\left(\Gamma_a^{pq}P_+ + \bar{\Gamma}_a^{pq}P_-\right)\epsilon_q - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{pq}P_+ + \bar{\Gamma}_{ab}^{pq}P_-\right)\epsilon_q$$

$$\Gamma_{ab}^{pq} = \frac{i}{4}\left(\bar{\Gamma}_a^{pr}\Gamma_b^{rq} - \bar{\Gamma}_b^{pr}\Gamma_a^{rq}\right), \bar{\Gamma}_{ab}^{pq} = \frac{i}{4}\left(\Gamma_a^{pr}\bar{\Gamma}_b^{rq} - \Gamma_b^{pr}\bar{\Gamma}_a^{rq}\right)$$

$$\Gamma_1^{pq} = i\left(\delta_{p1}\delta_{q4} - \delta_{p4}\delta_{q1} + \delta_{p2}\delta_{q3} - \delta_{p3}\delta_{q2}\right), \quad \Gamma_2^{pq} = i\left(\delta_{p1}\delta_{q2} - \delta_{p2}\delta_{q1} + \delta_{p3}\delta_{q4} - \delta_{p4}\delta_{q3}\right),$$

$$\Gamma_3^{pq} = i\left(\delta_{p1}\delta_{q3} - \delta_{p3}\delta_{q1} - \delta_{p2}\delta_{q4} + \delta_{p4}\delta_{q2}\right), \quad \Gamma_4^{pq} = -\left(\delta_{p1}\delta_{q4} - \delta_{p4}\delta_{q1} - \delta_{p2}\delta_{q3} + \delta_{p3}\delta_{q2}\right),$$

$$\Gamma_5^{pq} = \left(\delta_{p1}\delta_{q2} - \delta_{p2}\delta_{q1} - \delta_{p3}\delta_{q4} + \delta_{p4}\delta_{q3}\right), \quad \Gamma_6^{pq} = -\left(\delta_{p1}\delta_{q3} - \delta_{p3}\delta_{q1} + \delta_{p2}\delta_{q4} - \delta_{p4}\delta_{q2}\right),$$

$$\epsilon_p = \delta_{p4}\epsilon$$

$$\delta_{\epsilon}A_{\alpha} = i\bar{\psi}_4\gamma_{\alpha}\epsilon, \delta_{\epsilon}\phi_a = -i\bar{\psi}_p\left(\Gamma_a^{p4}P_+ + \bar{\Gamma}_a^{p4}P_-\right)\epsilon,$$

$$\delta_{\epsilon}\psi_p = iF_{\alpha\beta}\Sigma^{\alpha\beta}\delta_{p4}\epsilon + \gamma^{\alpha}D_{\alpha}\phi_a\left(\Gamma_a^{p4}P_+ + \bar{\Gamma}_a^{p4}P_-\right)\epsilon - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right)\epsilon$$

$$\delta'_{\epsilon}\psi_p = \mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\epsilon$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left(2i\mu_{pq}\bar{\psi}_p(\delta_{\epsilon} + \delta'_{\epsilon})\psi_q + 2M_{ab}\phi_a\delta_{\epsilon}\phi_b - 3igT_{abc}[\phi_b,\phi_c]\delta_{\epsilon}\phi_a\right)$$

$$T_{234} = \frac{1}{3}(\mu_1 - \mu_2 - \mu_3), \quad T_{126} = \frac{1}{3}(\mu_1 - \mu_2 + \mu_3),$$

$$T_{135} = \frac{1}{3}(\mu_1 + \mu_2 - \mu_3), \quad T_{456} = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$$

$$\mathcal{L}_{\mu} = \mathrm{tr}\left(-i\mu_{pq}\bar{\psi}_p\psi_q - M_{ab}\phi_a\phi_b + igT_{abc}\phi_a[\phi_b,\phi_c]\right)$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right)$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right) = \delta'_{\epsilon}\mathcal{L}_{\mathrm{SYM}} + \delta_{\epsilon}\mathcal{L}_{\mu} + \delta'_{\epsilon}\mathcal{L}_{\mu}$$

$$\delta'_{\epsilon}\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}[2i(\partial_{\alpha}\mu_m)\bar{\psi}_m\gamma^{\alpha}\phi_a(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)\epsilon + 2i\mu_m\bar{\psi}_m\gamma^{\alpha}D_{\alpha}\phi_a(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)\epsilon$$

$$\delta'_{\epsilon}\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}[2i(\partial_{\alpha}\mu_m)\bar{\psi}_m\gamma^{\alpha}\phi_a(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)\epsilon + 2i\mu_m\bar{\psi}_m\gamma^{\alpha}D_{\alpha}\phi_a(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)\epsilon$$
$$-2g\mu_m\bar{\psi}_p[\phi_a,\phi_b]\left(\Gamma_a^{pm}\Gamma_b^{m4}P_+ + \bar{\Gamma}_a^{pm}\bar{\Gamma}_b^{m4}P_-\right)\epsilon]$$

$$\delta_{\epsilon}\mathcal{L}_{\mu} = \mathrm{tr}\left[-2i\mu_m D_{\alpha}\phi_a\bar{\psi}_m\gamma^{\alpha}(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)\epsilon - g\mu_m[\phi_a,\phi_b]\bar{\psi}_m\left(\bar{\Gamma}_a^{mp}\Gamma_b^{p4}P_+ + \Gamma_a^{mp}\bar{\Gamma}_b^{p4}P_-\right)\epsilon\right.$$
$$\left.+2iM_{ab}\phi_a\bar{\psi}_p\left(\Gamma_b^{p4}P_+ + \bar{\Gamma}_b^{p4}P_-\right)\epsilon + 3gT_{abc}[\phi_b,\phi_c]\right)\bar{\psi}_p\left(\Gamma_a^{p4}P_+ + \bar{\Gamma}_a^{p4}P_-\right)\epsilon]$$

$$\delta'_{\epsilon}\mathcal{L}_{\mu} = \mathrm{tr}\left[-2i\mu_{pr}\mu_{rq}\phi_a\bar{\psi}_p\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\epsilon\right]$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right) = 2i\left(\partial_{\alpha}\mu_m\right)\mathrm{tr}[(-\textstyle\sum_{a=1}^{3} + \sum_{a=4}^{6})\phi_a\bar{\psi}_m(\Gamma_a^{m4}P_+ + \bar{\Gamma}_a^{m4}P_-)]\gamma^{\alpha}\epsilon$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right) = 2i\left(\partial_{\alpha}\mu_p\right)\mathrm{tr}\left[(-\textstyle\sum_{a=1}^{3} + \sum_{a=4}^{6})\phi_a\bar{\psi}_p\left(\Gamma_a^{p4}P_+ + \bar{\Gamma}_a^{p4}P_-\right)\right]\gamma^{\alpha}\epsilon$$
$$= \mathrm{tr}\left[-2i(\partial_{\alpha}J_{ab})\phi_a\bar{\psi}_p\left(\Gamma_b^{p4}P_+ + \bar{\Gamma}_b^{p4}P_-\right)\gamma^{\alpha}\epsilon\right]$$

$$J_{ab} = \mathrm{diag}(\mu_1,\mu_3,\mu_2,-\mu_1,-\mu_3,-\mu_2)$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right) = \mathrm{tr}\left[-2iJ'_{ab}\phi_a\bar{\psi}_p\left(\Gamma_b^{p4}P_+ + \bar{\Gamma}_b^{p4}P_-\right)\gamma^1\epsilon\right]$$

$$\gamma^1\epsilon = \epsilon$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\left(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mu}\right) = \mathrm{tr}\left[2J'_{ab}\phi_a\left(-i\bar{\psi}_p\left(\Gamma_b^{p4}P_+ + \bar{\Gamma}_b^{p4}P_-\right)\epsilon\right)\right] = \mathrm{tr}[2J'_{ab}\phi_a\delta\phi_b]$$

$$\mathcal{L}_J = -\mathrm{tr}(J'_{ab}\phi_a\phi_b)$$

$$T_{\mu\nu} = \mathrm{tr}\left[2\partial_{\mu}\phi_a\partial_{\nu}\phi_a + g_{\mu\nu}\left(-\partial^{\alpha}\phi_a\partial_{\alpha}\phi_a + \frac{g^2}{2}[\phi_a,\phi_b]^2 - (M_{ab} + J'_{ab})\phi_a\phi_b + igT_{abc}\phi_a[\phi_b,\phi_c]\right)\right]$$

$$E_0 = \int\ d^3xT_{00} = \int\ d^3x\mathrm{tr}\left[\phi'_a\phi'_a - \frac{g^2}{2}[\phi_a,\phi_b]^2 + (M_{ab} + J'_{ab})\phi_a\phi_b - igT_{abc}\phi_a[\phi_b,\phi_c]\right]$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\psi_p = iF_{\alpha\beta}\Sigma^{\alpha\beta}\delta_{p4}\epsilon + D_{\alpha}\phi_a\left(\Gamma_a^{p4}P_- + \bar{\Gamma}_a^{p4}P_+\right)\gamma^{\alpha}\epsilon - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right)\epsilon$$
$$+\mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\epsilon$$

$$(\delta_{\epsilon} + \delta'_{\epsilon})\psi_p = \left[\phi'_a\left(\Gamma_a^{p4}P_- + \bar{\Gamma}_a^{p4}P_+\right) - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right) + \mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\right]\epsilon$$

$$\mathrm{tr}\left[\left|\phi_a'\left(\Gamma_a^{p4}P_- + \bar{\Gamma}_a^{p4}P_+\right) - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right) + \mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\right|^2\right]$$

$$= \mathrm{tr}\left[\phi_a'\phi_a' - \frac{ig}{3}\left(\tilde{T}_{abc}\phi_a[\phi_b,\phi_c]\right)' - J_{ab}(\phi_a\phi_b)' - \frac{g^2}{2}[\phi_a,\phi_b]^2 - igT_{abc}\phi_a[\phi_b,\phi_c] + M_{ab}\phi_a\phi_b\right]$$

$$\tilde{T}_{126} = \tilde{T}_{135} = -\tilde{T}_{234} = -\tilde{T}_{456} = -1$$

$$E_0 = \int d^3x\mathrm{tr}\left[\left|\phi_a'\left(\Gamma_a^{p4}P_- + \bar{\Gamma}_a^{p4}P_+\right) - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right) + \mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)\right|^2\right] + \int\ d^3x\mathcal{K}'$$

$$\mathcal{K} = \mathrm{tr}\left(J_{ab}\phi_a\phi_b + \frac{ig}{3}\tilde{T}_{abc}\phi_a[\phi_b,\phi_c]\right)$$

$$\phi_a'\left(\Gamma_a^{p4}P_- + \bar{\Gamma}_a^{p4}P_+\right) - g[\phi_a,\phi_b]\left(\Gamma_{ab}^{p4}P_+ + \bar{\Gamma}_{ab}^{p4}P_-\right) + \mu_{pq}\phi_a\left(\Gamma_a^{q4}P_+ + \bar{\Gamma}_a^{q4}P_-\right)$$

$$\phi_a'\bar{\Gamma}_a^{p4} - g[\phi_a,\phi_b]\Gamma_{ab}^{p4} + \mu_{pq}\phi_a\Gamma_a^{q4}, \phi_a'\Gamma_a^{p4} - g[\phi_a,\phi_b]\bar{\Gamma}_{ab}^{p4} + \mu_{pq}\phi_a\bar{\Gamma}_a^{q4}$$

$$\phi_a'\bar{\Gamma}_a^{p4} - \frac{ig}{2}[\phi_a,\phi_b]\bar{\Gamma}_a^{pr}\Gamma_b^{r4} + \mu_{pq}\phi_a\Gamma_a^{q4}$$

$$\int_{x_L}^{x_R} dx\mathcal{K}' = 0 \iff \mathcal{K}|_{x\to x_L} = \mathcal{K}|_{x\to x_R}$$

$$i\phi_1' + \phi_4' - g\big(i([\phi_2,\phi_3] + [\phi_5,\phi_6]) + ([\phi_2,\phi_6] + [\phi_3,\phi_5])\big) - \mu_1(i\phi_1 - \phi_4),$$
$$i\phi_3' - \phi_6' + g\big(-i([\phi_1,\phi_2] - [\phi_4,\phi_5]) + ([\phi_1,\phi_5] - [\phi_2,\phi_4])\big) - \mu_2(i\phi_3 + \phi_6),$$
$$i\phi_2' + \phi_5' + g\big(i([\phi_1,\phi_3] + [\phi_4,\phi_6]) + ([\phi_1,\phi_6] + [\phi_3,\phi_4])\big) - \mu_3(i\phi_2 - \phi_5),$$
$$[\phi_1,\phi_4] + [\phi_2,\phi_5] - [\phi_3,\phi_6]$$

$$\Phi_1 = g(\phi_1 + i\phi_4), \Phi_3 = g(\phi_2 + i\phi_5), \Phi_2 = g(\phi_3 - i\phi_6)$$

$$\Phi_i^{\dagger'} + \frac{1}{2}\sum_{j,k=1}^3\ \epsilon_{ijk}\left[\Phi_j,\Phi_k\right] - \mu_i\Phi_i\ \sum_{i=1}^3\ \left[\Phi_i,\Phi_i^\dagger\right]$$

$$\lim_{x_L\to-\infty}\mu_i(x_L) = \mu_{Li}, \lim_{x_R\to\infty}\mu_i(x_R) = \mu_{Ri}, (i = 1,2,3)$$

$$\left[\Phi_i,\Phi_j\right] - \epsilon_{ijk}(\mu_{0k}\Phi_k)\ \sum_{i=1}^3\ \left[\Phi_i^\dagger,\Phi_i\right]$$

$$\Phi_1 = -i\sqrt{\mu_{02}\mu_{03}}T_1, \Phi_2 = -i\sqrt{\mu_{01}\mu_{03}}T_2, \Phi_3 = -i\sqrt{\mu_{01}\mu_{02}}T_3$$

$$T_i = \begin{pmatrix} T_i^{(n_1)} & & \\ & \ddots & \\ & & T_i^{(n_l)} \end{pmatrix}$$

$$\sum_{k=1}^l\ n_k = N$$

$$\sum_{n=1}^\infty\ nN_n = N$$

$$\phi_1 = \phi_2 = \phi_3 = 0, \phi_4 = -\frac{\sqrt{\mu_{02}\mu_{03}}}{g}T_1, \phi_5 = -\frac{\sqrt{\mu_{01}\mu_{02}}}{g}T_3, \phi_6 = \frac{\sqrt{\mu_{01}\mu_{03}}}{g}T_2$$

$$\mathcal{K}|_{\mathrm{vac}} = -\frac{1}{3g^2}\mu_{01}\mu_{02}\mu_{03}\mathrm{tr}(T_1^2 + T_2^2 + T_3^2)$$
$$= -\frac{1}{12g^2}\mu_{01}\mu_{02}\mu_{03}\sum_{n=1}^\infty\ n(n^2 - 1)N_n$$

$$\mu_{L1}\mu_{L2}\mu_{L3}\sum_{n=1}^\infty\ n(n^2 - 1)N_n^{(L)} = \mu_{R1}\mu_{R2}\mu_{R3}\sum_{n=1}^\infty\ n(n^2 - 1)N_n^{(R)}$$

$$R_k^2 \sim \frac{\mu_{01}\mu_{02}\mu_{03}}{4N}n_k\left(n_k^2 - 1\right)$$

$$\mu_i(x) = m_{i0} + m_i(x)\ \int_0^\tau\ m_i(x)dx$$

$$\Phi_i(x) = e^{K_i(x)}\tilde{\Phi}_i(x),$$
$$K_i(x) = m_{i0}(\xi_i - x) - \Lambda_i(x),$$
$$\Lambda_i = \int^x m_i\, m_i(x')dx'$$

$$\frac{d\xi_i}{dx} = e^{K_i - \sum_{i'\neq i}K_{i'}}$$

$$e^{\sum_{i'\neq i}K_{i'}}\left(\frac{d\tilde{\Phi}_i^\dagger}{d\xi_i} + \frac{1}{2}\sum_{j,k=1}^3\ \epsilon_{ijk}\left[\tilde{\Phi}_j,\tilde{\Phi}_k\right] + m_{i0}\tilde{\Phi}_i^\dagger\right) = e^{\sum_{i'\neq i}K_{i'}}\mu_i\left(\tilde{\Phi}_i^\dagger + \tilde{\Phi}_i\right)$$

$$\sum_{i=1}^3\ e^{2K_i}\left[\tilde{\Phi}_i,\tilde{\Phi}_i^\dagger\right]$$

$$\frac{d\tilde{\Phi}_i}{d\xi_i} - \frac{1}{2}\sum_{j,k=1}^3\ \epsilon_{ijk}\left[\tilde{\Phi}_j,\tilde{\Phi}_k\right] + m_{i0}\tilde{\Phi}_i$$

$$\Phi_i(x) = e^{K(x)}\tilde{\Phi}_i(x)$$
$$K(x) = m_0(\xi - x) - \Lambda(x)\ \Lambda = \int^x m\, m(x')dx',$$
$$\left(\frac{d\xi}{dx}\right)(x) = e^{-K(x)} = e^{-m_0(\xi - x)+\Lambda(x)}$$
$$\left(\frac{dx}{d\xi}\right)(x) = m_0 \int_{-\infty}^{x} e^{m_0(x'-x)+(\Lambda(x')-\Lambda(x))}dx'$$
$$\left(\frac{dx}{d\xi}\right)(x+\tau) = m_0 \int_{-\infty}^{x+\tau} e^{m_0(x'-x-\tau)+(\Lambda(x')-\Lambda(x))}dx'$$
$$= m_0 \int_{-\infty}^{x} e^{m_0(x''-x)+(\Lambda(x'')-\Lambda(x))}dx'' = \left(\frac{dx}{d\xi}\right)(x)$$
$$\frac{d\tilde{\Phi}_i}{d\xi_i} - \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\left[\tilde{\Phi}_j, \tilde{\Phi}_k\right]$$
$$\tilde{\Phi}_D^i = \mathrm{diag}\left(a_1^i, a_2^i, \cdots, a_N^i\right)$$
$$\mu(x) = m_1 \sin\, qx$$
$$\Phi_i(x) = e^{\frac{m_1}{q}\cos\, qx}\tilde{\Phi}_D^i$$
$$\Phi_i(x) = \left(\frac{dx}{d\xi}\right)\tilde{\Phi}_i(x)$$
$$\left(\frac{dx}{d\xi}\right)_x = m_0 \int_{-\infty}^{x} e^{m_0(x'-x)-\frac{m_1}{q}(\cos\, qx' - \cos\, qx)}dx'$$
$$\tilde{\Phi}_i(x) = -im_0T_i$$

2.20. Supersimetrías.

$$(\delta_\epsilon + \delta'_\epsilon)(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_\mu) = 2i(\partial_\alpha\mu)\mathrm{tr}\left[\left(-\sum_{a=1}^{3} + \sum_{a=4}^{6}\right)\phi_a\bar{\psi}_m\left(\Gamma_a^{mi}P_+ + \bar{\Gamma}_a^{mi}P_-\right)\gamma^\alpha\epsilon_i\right]$$
$$(\delta_\epsilon + \delta'_\epsilon)(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_\mu) = 2i(\partial_\alpha\mu)\mathrm{tr}\left[\left(-\sum_{a=1,3} + \sum_{a=4,6}\right)\phi_a\bar{\psi}_p\left(\Gamma_a^{pi}P_+ + \bar{\Gamma}_a^{pi}P_-\right)\gamma^\alpha\epsilon_i\right]$$
$$= \mathrm{tr}\left[-2i\partial_\alpha J_{ab}\phi_a\bar{\psi}_p\left(\Gamma_b^{pi}P_+ + \bar{\Gamma}_b^{pi}P_-\right)\gamma^\alpha\epsilon_i\right]$$
$$(\delta_\epsilon + \delta'_\epsilon)(\mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_\mu) = \mathrm{tr}\left[2J'_{ab}\phi_a\left(-i\bar{\psi}_p\left(\Gamma_b^{pi}P_+ + \bar{\Gamma}_b^{pi}P_-\right)\epsilon_i\right)\right] = \mathrm{tr}(2J'_{ab}\phi_a\delta\phi_b)$$
$$\mathcal{L}_J = -\mathrm{tr}(J'_{ab}\phi_a\phi_b)$$
$$\phi'_a\bar{\Gamma}_a^{pi} - \frac{ig}{2}[\phi_a, \phi_b]\bar{\Gamma}_a^{pr}\Gamma_b^{ri} + \mu_{pr}\phi_a\Gamma_a^{ri}\ \left(J_{ab}\phi_a\phi_b + \frac{ig}{3}\tilde{T}_{abc}\phi_a[\phi_b, \phi_c]\right)\Big|_{\mathrm{boundary}}$$
$$\begin{aligned}
&\phi'_3 - ig[\phi_1, \phi_5] - \mu\phi_3 = 0, && [\phi_1, \phi_2] = 0,\\
&\phi'_6 - ig[\phi_4, \phi_5] + \mu\phi_6 = 0, && [\phi_2, \phi_4] = 0,\\
&\phi'_4 - ig[\phi_5, \phi_6] + \mu\phi_4 = 0, && [\phi_2, \phi_6] = 0,\\
&\phi'_1 + ig[\phi_3, \phi_5] - \mu\phi_1 = 0, && [\phi_2, \phi_3] = 0,\\
&[\phi_1, \phi_4] - [\phi_3, \phi_6] = 0, && [\phi_2, \phi_5] = 0,\\
&\phi'_5 + ig([\phi_1, \phi_3] + [\phi_4, \phi_6]) = 0, && \phi'_2 = 0,\\
&[\phi_1, \phi_6] + [\phi_3, \phi_4] = 0
\end{aligned}$$

2.21. Superespacios y Superamplitudes.

$$\Phi\left(p^{a\dot{a}},\eta^{A}\right)=g^{+}(p)+\eta^{A}\tilde{g}_{A}(p)+\frac{1}{2}\eta^{A}\eta^{B}S_{AB}(p)+\frac{1}{6}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\overline{\tilde{g}}^{D}(p)+\frac{1}{24}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}g^{-}(p)$$

$$p^{a\dot{a}}=\lambda^{a}\tilde{\lambda}^{\dot{a}}$$

$$\{\eta^{A},\eta^{B}\}=0$$

$$\delta^{4}\left(\sum_{i=1}^{n}\lambda_{i}^{a}\tilde{\lambda}_{i}^{\dot{a}}\right)\delta^{8}\left(\sum_{i=1}^{n}\lambda_{i}^{a}\eta_{i}^{A}\right)$$

$$\mathcal{A}_{n}\left(\lambda_{i}^{a},\tilde{\lambda}_{i}^{\dot{a}},\eta_{i}^{A}\right)=\sum_{k=0}^{n-4}\mathcal{A}_{n}^{\mathrm{N}^{k}\mathrm{MHV}}\left(\lambda_{i}^{a},\tilde{\lambda}_{i}^{\dot{a}},\eta_{i}^{A}\right)$$

$$x_{i}^{a\dot{a}}-x_{i+1}^{a\dot{a}}=p_{i}^{a\dot{a}}$$

$$\theta_{i}^{aA}-\theta_{i+1}^{aA}=\lambda_{i}^{a}\eta_{i}^{A}$$

$$\delta^{4}(p)\delta^{8}(q)=\delta^{4}(x_{n+1}-x_{1})\delta^{8}(\theta_{n+1}-\theta_{1})$$

$$\log\frac{\mathcal{A}_{n}^{\mathrm{MHV}}(p_{1},\ldots,p_{n})}{\mathcal{A}_{n}^{\mathrm{MHV}}(p_{1},\ldots,p_{n})|_{\mathrm{tree-level}}}=\log\langle W(x_{1},\ldots,x_{n})\rangle$$

$$\log\langle W(x_{1},\ldots,x_{n})\rangle=\sum_{i=1}^{n}\mathrm{Div}\left(x_{i-1,i+1}^{2};\epsilon\right)+\mathrm{Fin}_{n}\left(x_{ij}^{2}\right)$$

$$\mathrm{Div}(x^{2};\epsilon)=-\frac{1}{4}\sum_{L=1}^{\infty}g^{2L}(-x^{2}\mu^{2})^{L\epsilon}\left[\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{(L\epsilon)^{2}}+\frac{\Gamma_{\mathrm{collinear}}^{(L)}}{L\epsilon}\right]$$

$$g^{2}\equiv\frac{g_{\mathrm{YM}}^{2}N}{16\pi^{2}}=\frac{\lambda}{16\pi^{2}}$$

$$K^{\mu}\mathrm{Fin}_{n}\left(x_{ij}^{2}\right)=\frac{1}{2}\Gamma_{\mathrm{cusp}}(g^{2})\sum_{i=1}^{n}x_{i,i+1}^{\mu}\log\left(x_{i,i+2}^{2}/x_{i-1,i+1}^{2}\right)$$

$$K^{\mu}=\sum_{i=1}^{n}\left[2x_{i}^{\mu}x_{i}^{\nu}\frac{\partial}{\partial x_{i\nu}}-x_{i}^{2}\frac{\partial}{\partial x_{i\mu}}\right]$$

$$u=\frac{x_{13}^{2}x_{46}^{2}}{x_{14}^{2}x_{36}^{2}},\ v=\frac{x_{24}^{2}x_{51}^{2}}{x_{25}^{2}x_{41}^{2}},\ w=\frac{x_{35}^{2}x_{62}^{2}}{x_{36}^{2}x_{52}^{2}}$$

$$Z_{i}^{I}=\left(\lambda_{i}^{a},x_{i}^{b\dot{a}}\lambda_{ib}\right)$$

$$Z\in\mathrm{Gr}(4,n)/\mathrm{GL}(1)^{n-1}$$

$$x_{ij}^{2}\to\det\left(Z_{i-1}Z_{i}Z_{j-1}Z_{j}\right)$$

$$\Omega_{n,k}^{\ell-\mathrm{loop}}=d\mu_{1}d\mu_{2}\ldots d\mu_{\ell}\mathcal{I}_{n,k}^{\ell-\mathrm{loop}}$$

$$\Omega_{n,k}^{\ell-\mathrm{loop}}=\omega_{n,k}^{\ell-\mathrm{loop}}(dZ_{i}\to\eta_{i})$$

$$\mathcal{S}_{6}=\{u,v,w,1-u,1-v,1-w,y_{u},y_{v},y_{w}\}$$

$$y_{u}=\frac{u-z_{+}}{u-z_{-}},\ y_{v}=\frac{v-z_{+}}{v-z_{-}},\ y_{w}=\frac{w-z_{+}}{w-z_{-}}$$

$$z_{\pm}=\frac{1}{2}\left[-1+u+v+w\pm\sqrt{\Delta}\right],\ \Delta=(1-u-v-w)^{2}-4uvw$$

2.22. Correspondencias.

$$R_{6}=\frac{g_{YM}^{2}}{8\pi^{2}}$$

$$F=-\frac{\left(9/4+m^{2}\right)^{2}}{96\pi}\lambda N^{2}$$

$$I_{AdS}=-\frac{5\pi R_{6}}{12r}N^{3}$$

$$\langle W\rangle\sim\exp\left(\frac{\lambda}{8\pi}\right)$$

$$\langle W\rangle\sim\exp\left((9/4+m^{2})\frac{\lambda}{8\pi}\right)$$

$$\langle W\rangle_{AdS}\sim\exp\left(\frac{2\pi NR_{6}}{r}\right)$$

$$R_{6}=\frac{5g_{YM}^{2}}{32\pi^{2}}$$

$$L_{\mathrm{vector}}=\frac{1}{g_{YM}^{2}}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn}-D_{m}\sigma D^{m}\sigma-\frac{1}{2}D_{IJ}D^{IJ}+\frac{2}{r}\sigma t^{IJ}D_{IJ}-\frac{10}{r^{2}}t^{IJ}t_{IJ}\sigma^{2}+i\lambda_{I}\Gamma^{m}D_{m}\lambda^{I}-\lambda_{I}[\sigma,\lambda^{I}]-\frac{i}{r}t^{IJ}\lambda_{I}\lambda_{J}\right]$$

$$L_{vector} = \frac{1}{g_{YM}^2}\left[\frac{1}{2}F_{mn}F^{mn} - D_m\sigma D^m\sigma - \frac{4}{r^2}\sigma^2 + \cdots\right]$$

$$L_{\text{escalar}} = D_m\phi D^m\phi + \frac{d-2}{4(d-1)}\mathcal{R}\phi^2$$

$$L_{\text{escalar}} = D_m\phi D^m\phi + \frac{15}{4r^2}\phi^2$$

$$L_{\text{materia}} = \epsilon^{IJ}D_m\bar{q}_I D^m q_J - \epsilon^{IJ}\bar{q}_I\sigma^2 q_J + \frac{15}{4r^2}\epsilon^{IJ}\bar{q}_I q_J - 2i\bar{\psi}\not{D}\psi - 2\bar{\psi}\sigma\psi$$
$$-4\epsilon^{IJ}\bar{\psi}\lambda_I q_J - iq_I D^{IJ}q_J,$$

$$L_{masa} = -M^2\epsilon^{IJ}\bar{q}_I q_J + \frac{2i}{r}Mt^{IJ}\bar{q}_I q_J - 2M\bar{\psi}\psi$$
$$-\frac{4}{r^2}\sigma^2 + \left(\frac{15}{4r^2} - M^2\right)\epsilon^{IJ}\bar{q}_I q_J + \frac{2i}{r}Mt^{IJ}\bar{q}_I q_J$$
$$-\frac{4}{r^2}\sigma^2 + \frac{3}{r^2}\bar{q}_1 q^1 + \frac{4}{r^2}\bar{q}_2 q^2$$

$$Z = \int_{\text{Cartan}} [d\phi] e^{-\frac{8\pi^3 r}{g_{YM}^2}\text{Tr}(\phi^2) - \frac{\pi k}{3}\text{Tr}(\phi^3)} Z_{1-\text{loop}}^{\text{vect}}(\phi) Z_{1-\text{loop}}^{\text{hyper}}(\phi) + \mathcal{O}\left(e^{-\frac{16\pi^3 r}{g_{YM}^2}}\right)$$

$$Z_{1-\text{loop}}^{\text{vect}}(\phi) = \prod_\beta \prod_{t\neq 0} (t - \langle\beta, i\phi\rangle)^{\left(1+\frac{3}{2}t+\frac{1}{2}t^2\right)}$$

$$Z_{1-\text{loop}}^{\text{hyper}}(\phi) = \prod_\mu \prod_t \left(t - \langle i\phi, \mu\rangle + \frac{3}{2}\right)^{-\left(1+\frac{3}{2}t+\frac{1}{2}t^2\right)}$$

$$S = \frac{1}{g_{YM}^2}\int_{S^5} \text{Tr}(F \wedge * F) + \cdots + \frac{ik}{24\pi^2}\int_{S^5} \text{Tr}(A \wedge dA \wedge dA) + \cdots$$

$$\mathcal{P} = x\prod_{t=1}^{\infty} (t + x)^{\left(1+\frac{3}{2}t+\frac{1}{2}t^2\right)}(t - x)^{\left(1-\frac{3}{2}t+\frac{1}{2}t^2\right)}$$

$$\log \mathcal{P} = \sum_{t=1}^{\infty} \left(3x - \frac{x^2}{2}\right) + \text{ parte convergente}$$

$$S_3(x) = 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4} - \frac{3}{2}x} \prod_{t=1}^{\infty} \left(\left(1 + \frac{x}{t}\right)^{\left(1+\frac{3}{2}t+\frac{1}{2}t^2\right)} \left(1 - \frac{x}{t}\right)^{\left(1-\frac{3}{2}t+\frac{1}{2}t^2\right)} e^{\frac{x^2}{2} - 3x}\right)$$

$$\log\left(Z_{1-\text{loop}}^{\text{vect}}(\phi) Z_{1-\text{loop}}^{\text{hyper}}(\phi)\right) = -\frac{\pi\Lambda r}{2}\sum_\beta (\langle\beta, i\phi\rangle)^2 + \frac{\pi\Lambda r}{2}\sum_\mu (\langle i\phi, \mu\rangle)^2 + \text{ parte convergente}$$

$$= \pi\Lambda r(C_2(\text{adj}) - C_2(R))\text{Tr}(\phi^2) + \text{parte convergente}$$

$$\frac{1}{g_{eff}^2} = \frac{1}{g_{YM}^2} - \frac{\Lambda}{8\pi^2}(C_2(\text{adj}) - C_2(R))$$

$$Z = \int d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2}\text{Tr}(\phi^2) - \frac{\pi k}{3}\text{Tr}(\phi^3)} \det_{Ad}(S_3(i\phi)) \det_R^{-1}\left(S_3\left(i\phi + \frac{3}{2}\right)\right)$$

$$S_3(-x) = S_3(x + 3), S_3\left(x + \frac{3}{2}\right) = S_3\left(-x + \frac{3}{2}\right)$$

$$\det_R\left(S_3\left(i\phi + \frac{3}{2}\right)\right) = \det_R\left(S_3\left(-i\phi + \frac{3}{2}\right)\right) = \det_{\bar{R}}\left(S_3\left(i\phi + \frac{3}{2}\right)\right)$$

$$Z = \int d\phi e^{-\frac{8\pi^3 r}{g_Y^2}\text{Tr}(\phi^2) - \frac{\pi k}{3}\text{Tr}(\phi^3)} \det_{Ad}(S_3(i\phi)) \det_R^{-1}\left(S_3\left(i\phi + im + \frac{3}{2}\right)\right)$$

$$\det_R\left(S_3\left(i\phi + im + \frac{3}{2}\right)\right) = \det_{\bar{R}}\left(S_3\left(i\phi - im + \frac{3}{2}\right)\right)$$

$$\int d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2}\text{Tr}(\phi^2) - \frac{\pi k}{3}\text{Tr}(\phi^3)} \det_{Ad}(S_3(i\phi))$$
$$\times \det_R^{-1/2}\left(S_3\left(i\phi + im + \frac{3}{2}\right)\right) \det_{\bar{R}}^{-1/2}\left(S_3\left(i\phi - im + \frac{3}{2}\right)\right)$$
$$\int d\phi e^{-\mathcal{F}}$$

$$\mathcal{F} = \frac{8\pi^3 r}{g_{YM}^2}\text{Tr}(\phi^2) + \frac{\pi k}{3}\text{Tr}(\phi^3) - \sum_\beta \log S_3(\langle i\phi, \beta\rangle) + \sum_\mu \log S_3\left(\langle i\phi, \mu\rangle + im + \frac{3}{2}\right)$$

$$\log S_3(z) \sim -\text{sgn}(\text{Im}z)\pi i\left(\frac{1}{6}z^3 - \frac{3}{4}z^2 + z + \cdots\right)$$

$$\frac{1}{2\pi r^3}\mathcal{F} = \frac{4\pi^2}{g_{YM}^2}\text{Tr}(\phi^2) + \frac{k}{6}\text{Tr}(\phi^3) + \frac{1}{12}\left(\sum_\beta |\langle\phi, \beta\rangle|^3 - \sum_\mu |\langle\phi, \mu\rangle + m|^3\right) + O(r^{-2})$$

$$\mathcal{F} = \frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2) + \frac{\pi k}{3}\mathrm{Tr}(\phi^3) + \frac{\pi}{6}\left(\sum_\beta \ |\langle\phi,\beta\rangle|^3 - \sum_\mu \ |\langle\phi,\mu\rangle|^3\right)$$

$$-\frac{\pi}{2}m\sum_\mu \ \mathrm{sgn}(\langle\phi,\mu\rangle)(\langle\phi,\mu\rangle)^2 - \pi\sum_\beta \ |\langle\phi,\beta\rangle| - \frac{\pi}{2}\left(m^2 + \frac{1}{4}\right)\sum_\mu \ |\langle\phi,\mu\rangle| + \cdots,$$

$$\mathcal{F} = \frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2) + \frac{\pi k}{3}\mathrm{Tr}(\phi^3) - \sum_\beta \ \log \ S_3(\langle i\phi,\beta\rangle) - \mathrm{sgn}(m)\frac{\pi}{2}\sum_\mu \ \left(\frac{1}{3}(\langle\phi,\mu\rangle)^3 + m(\langle\phi,\mu\rangle)^2\right)$$

$$\mathrm{Tr}(T_A T_B T_C + T_A T_C T_B) = C_3(R) d_{ABC}$$

$$\sum_\mu \ (\langle\phi,\mu\rangle)^3 = C_3(R)\mathrm{Tr}(\phi^3)$$

$$k_{eff} = k - \mathrm{sgn}(m)\frac{C_3(R)}{2}$$

$$\frac{r}{g_{eff}^2} = \frac{r}{g_{YM}^2} - \frac{|m|}{8\pi^2}C_2(R)$$

$$S_3(z) = 2e^{-\zeta'(-2)}\sin(\pi z)e^{\frac{1}{2}f(z)}e^{\frac{3}{2}l(z)}$$

$$l(z) = -z\log\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12}$$

$$f(z) = \frac{i\pi z^3}{3} + z^2\log\left(1 - e^{-2\pi i z}\right) + \frac{iz}{\pi}\mathrm{Li}_2\left(e^{-2\pi i z}\right) + \frac{1}{2\pi^2}\mathrm{Li}_3\left(e^{-2\pi i z}\right) - \frac{\zeta(3)}{2\pi^2}$$

$$Z = \int_{\mathrm{Cartan}} \ [d\phi]e^{-\frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2)}\prod_\beta \left(\sin(\pi\langle\beta,i\phi\rangle)e^{-\frac{1}{4}l\left(\frac{1}{2}-im-\langle\beta,i\phi\rangle\right)-\frac{1}{4}l\left(\frac{1}{2}-im+\langle\beta,i\phi\rangle\right)}\right.$$

$$\left.\times\, e^{\frac{1}{2}f(\langle\beta,i\phi\rangle)-\frac{1}{4}f\left(\frac{1}{2}-im-\langle\beta,i\phi\rangle\right)-\frac{1}{4}f\left(\frac{1}{2}-im+\langle\beta,i\phi\rangle\right)}\right) + \cdots$$

$$\lambda = \frac{g_{YM}^2 N}{r}$$

$$Z \sim \int \ \prod_{i=1}^N \ d\phi_i \exp\left(-\frac{8\pi^3 N}{\lambda}\sum_i \ \phi_i^2 + \sum_{j\neq i} \ \sum_i \ \left[\log\left[\sinh\left(\pi(\phi_i - \phi_j)\right)\right]\right.\right.$$

$$-\frac{1}{4}l\left(\frac{1}{2} - im + i(\phi_i - \phi_j)\right) - \frac{1}{4}l\left(\frac{1}{2} - im - i(\phi_i - \phi_j)\right) + \frac{1}{2}f\left(i(\phi_i - \phi_j)\right) -$$

$$\left.\left.-\frac{1}{4}f\left(\frac{1}{2} - im + i(\phi_i - \phi_j)\right) - \frac{1}{4}f\left(\frac{1}{2} - im - i(\phi_i - \phi_j)\right)\right]\right).$$

$$\frac{df(z)}{dz} = \pi z^2\cot(\pi z);\quad \frac{dl(z)}{dz} = -\pi z\cot(\pi z)$$

$$\lim_{|x|\to\infty}\mathrm{Re}f\left(\frac{1}{2} + ix\right) = -\frac{\pi}{3}|x|^3 + \frac{\pi}{4}|x|;\qquad \lim_{x\to\infty}\mathrm{Im}f\left(\frac{1}{2} \pm ix\right) = \pm\frac{\pi}{2}x^2;$$

$$\lim_{|x|\to\infty}\mathrm{Re}l\left(\frac{1}{2} + ix\right) = -\frac{\pi}{2}|x|;\qquad \lim_{x\to\infty}\mathrm{Im}l\left(\frac{1}{2} \pm ix\right) = \mp\frac{\pi}{2}x^2;$$

$$\lim_{|x|\to\infty}\mathrm{Re}f(ix) = -\frac{\pi}{3}|x|^3;\qquad \mathrm{Im}f(ix) = 0$$

$$\frac{16\pi^3 N}{\lambda}\phi_i = \pi\sum_{j\neq i} \ \left[\left(2 - (\phi_i - \phi_j)^2\right)\coth\left(\pi(\phi_i - \phi_j)\right)\right.$$

$$+\frac{1}{2}\left(\frac{1}{4} + (\phi_i - \phi_j - m)^2\right)\tanh\left(\pi(\phi_i - \phi_j - m)\right)$$

$$\left.+\frac{1}{2}\left(\frac{1}{4} + (\phi_i - \phi_j + m)^2\right)\tanh\left(\pi(\phi_i - \phi_j + m)\right)\right]$$

$$\frac{16\pi^3 N}{\lambda}\phi_i \approx 2\sum_{j\neq i} \ \frac{1}{\phi_i - \phi_j}$$

$$\rho(\phi) \equiv \frac{1}{N}\frac{dn}{d\phi} = \frac{2}{\pi\phi_0^2}\sqrt{\phi_0^2 - \phi^2}\ \phi_0 = \sqrt{\frac{\lambda}{4\pi^3}}$$

$$\int \rho(\phi)d\phi = 1$$

$$F = -\log Z \approx -N^2\log\sqrt{\lambda}$$

$$\frac{16\pi^3 N}{\lambda}\phi_i = \pi\left(\frac{9}{4} + m^2\right)\sum_{j\neq i} \ \mathrm{sign}(\phi_i - \phi_j)$$

$$\phi_i = \frac{(9+4m^2)\lambda}{64\pi^2 N}(2i - N)$$

$$\rho(\phi) \ = \frac{32\pi^2}{(9+4m^2)\lambda} \qquad |\phi| \leq \phi_m, \phi_m = \frac{(9+4m^2)\lambda}{64\pi^2}$$

$$= \qquad 0 \qquad |\phi| > \phi_m$$

$$Z \sim \int \ \prod_i \ d\phi_i e^{-\frac{8\pi^3 N}{\lambda}\sum_i \phi_i^2 + \frac{\pi}{2}\left(\frac{9}{4}+m^2\right)\sum_{j\neq i} \sum_i |\phi_i - \phi_j|}$$

$$F \equiv -\log Z \approx -\frac{g_{YM}^2 N^3}{96\pi r}\left(\frac{9}{4}+m^2\right)^2$$
$$\sum_{i=1}^{N}(2i-N)^2 \approx \frac{1}{3}N^3, \sum_{j\neq i}\sum_{i=1}^{N}|i-j| \approx \frac{1}{3}N^3$$
$$F = -\frac{25 g_{YM}^2 N^3}{384\pi r}$$
$$\frac{16\pi^3 N}{\lambda}\phi_i = \pi\sum_{j\neq i}\left[\left(2-(\phi_i-\phi_j)^2\right)\coth\left(\pi(\phi_i-\phi_j)\right)+2m(\phi_i-\phi_j)\right]$$
$$= 2\pi m N\phi_i + \pi\sum_{j\neq i}\left(2-(\phi_i-\phi_j)^2\right)\coth\left(\pi(\phi_i-\phi_j)\right)$$
$$\frac{16\pi^3 N}{\lambda_{eff}}\phi_i = \pi\sum_{j\neq i}\left(2-(\phi_i-\phi_j)^2\right)\coth\left(\pi(\phi_i-\phi_j)\right)$$
$$\frac{1}{\lambda_{eff}} = \frac{1}{\lambda}-\frac{m}{8\pi^2}$$

2.23. Supersimetrías y Loops de Wilson.

$$\langle W\rangle = \frac{1}{N}\langle \mathrm{Tr} e^{2\pi\phi_i}\rangle$$
$$\langle W\rangle \sim \frac{1}{N}\int \prod_i d\phi_i \sum_i e^{2\pi\phi_i} e^{-\frac{8\pi^3 N}{\lambda}\sum_i \phi_i^2+\frac{\pi}{2}\left(\frac{9}{4}+m^2\right)\sum_{j\neq i}\sum_i|\phi_i-\phi_j|}$$
$$\langle W\rangle = \int d\phi\rho(\phi)e^{2\pi\phi}$$
$$\langle W\rangle \approx \int d\phi\rho(\phi)(1+2\pi^2\phi^2) = 1+\frac{\lambda}{8\pi} \approx \exp\left(\frac{\lambda}{8\pi}\right)$$
$$\langle W\rangle \approx \frac{32\pi^2}{(9+4m^2)\lambda}\int_{-\phi_m}^{\phi_m} e^{2\pi\phi}d\phi \sim \exp\left(\frac{\lambda}{8\pi}\left(\frac{9}{4}+m^2\right)\right)$$
$$\frac{16\pi^3 N}{\lambda}\psi_i^{(r)} = \pi\left[\sum_{j\neq i}\left(2-\left(\psi_i^{(r)}-\psi_j^{(r)}\right)^2\right)\coth\left(\pi\left(\psi_i^{(r)}-\psi_j^{(r)}\right)\right)\right.$$
$$+\left(\sum_j\left[\frac{1}{4}\left(\frac{1}{4}+\left(\psi_i^{(r)}-\psi_j^{(r+1)}-m\right)^2\right)\tanh\left(\pi\left(\psi_i^{(r)}-\psi_j^{(r+1)}-m\right)\right)\right.\right.$$
$$\left.\left.+\frac{1}{4}\left(\frac{1}{4}+\left(\psi_i^{(r)}-\psi_j^{(r-1)}-m\right)^2\right)\tanh\left(\pi\left(\psi_i^{(r)}-\psi_j^{(r-1)}-m\right)\right)\right]\right]$$
$$+(m\to -m)).$$
$$\psi_i^{(r)} = \frac{(9+4m^2)\lambda}{64\pi^2 N}(2i-N/k)$$
$$F \approx -k\frac{g_{YM}^2 N^3}{96\pi r k^3}\left(\frac{9}{4}+m^2\right)^2 = -\frac{g_{YM}^2 N^3}{96\pi r k^2}\left(\frac{9}{4}+m^2\right)^2$$
$$\langle W\rangle \approx \exp\left(\frac{\lambda}{8\pi k}\right).$$
$$\langle W\rangle \sim \exp\left(\frac{\lambda}{8\pi k}\left(\frac{9}{4}+m^2\right)\right)$$
$$\tau\frac{d\phi_i}{dt} = -\frac{\partial\mathcal{F}}{\partial\phi_i}$$
$$a_3 = c_1+c_2 m+c_3 m^2+c_4 m^3+\cdots$$

2.24. Supergravedad.

$$ds^2 = \ell^2\left(\cosh^2\rho d\tau^2+d\rho^2+\sinh^2\rho d\Omega_5^2\right)$$
$$I_{AdS} = I_{\text{bulk}}+I_{\text{surface}}+I_{\text{ct}}$$
$$I_{\text{bulk}} = -\frac{1}{16\pi G_N}\mathrm{Vol}(S^4)\int d^7x\sqrt{g}(R-2\Lambda)$$
$$R-2\Lambda = -\frac{12}{\ell^2}$$
$$I_{\text{bulk}} = -\frac{1}{256\pi^8\ell_{pl}^9}\left(\frac{\pi^2\ell^4}{6}\right)\frac{2\pi R_6}{r}\pi^3(-12\ell^5)\int_0^{\rho_0}\cosh\rho\sinh^5\rho d\rho = \frac{4\pi R_6}{3r}N^3\sinh^6\rho_0$$
$$\sinh^6\rho_0 = \frac{1}{64}\epsilon^{-6}-\frac{3}{32}\epsilon^{-4}+\frac{15}{64}\epsilon^{-2}-\frac{5}{16}+O(\epsilon^2)$$
$$I_{AdS} = -\frac{5\pi R_6}{12r}N^3$$
$$\langle W\rangle \sim e^{-T^{(2)}\int dV}$$
$$\int dV = l^3\int_0^{\frac{2\pi R_6}{r}} d\tau\int_0^{2\pi} d\phi\int_0^{\rho_0} d\rho\sinh(\rho)\cosh(\rho)$$

$$T^{(2)}\int\ dV=\frac{\pi NR_6}{r}\left(\frac{1}{\epsilon}-2+\epsilon\right)$$
$$\langle W\rangle\sim\exp\left(\frac{2\pi NR_6}{r}\right)$$
$$R_6=\frac{g_{YM}^2}{16\pi^2}\left(\frac{9}{4}+m^2\right)$$
$$R_6=\frac{g_{YM}^2}{16\pi^2}\frac{5}{2}$$
$$I_{AdS}=-\frac{5\pi R_6}{12rk}N^3$$
$$R_6=\frac{g_{YM}^2}{16\pi^2k}\left(\frac{9}{4}+m^2\right)$$

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Apéndice D.

1. Supermembranas y supergravedad cuántica. Ecuaciones de movimiento en supercampos cuánticos relativistas y superespacios cuánticos con curvatura.

$$\{Z^M\} \equiv \{x^\mu, \theta^{\underline{\alpha}}\}, \mu = 0,1,2,3, \underline{\alpha} = 1,2,3,4$$
$$E^a = dZ^M E_M^a(Z), E^\alpha = dZ^M E_M^\alpha(Z), \bar{E}^{\dot{\alpha}} = dZ^M \bar{E}_M^{\dot{\alpha}}(Z)$$
$$a = 0,1,2,3, \alpha = 1,2, \dot{\alpha} = 1,2$$
$$E^A = (E^a, E^{\underline{\alpha}}) = \left(E^a, E^\alpha, \bar{E}^{\dot{\alpha}}\right) = dZ^M E_M^A(Z)$$
$$T^a := \mathcal{D}E^a = dE^a - E^b \wedge w_b{}^a = \frac{1}{2} E^B \wedge E^C T_{CB}{}^a$$
$$T^\alpha := \mathcal{D}E^\alpha = dE^\alpha - E^\beta \wedge w_\beta{}^\alpha = \frac{1}{2} E^B \wedge E^C T_{CB}{}^\alpha, w_\beta{}^\alpha := \frac{1}{4} w^{ab} \sigma_{ab\beta}{}^\alpha$$
$$T^{\dot{\alpha}} := \mathcal{D}E^{\dot{\alpha}} = dE^{\dot{\alpha}} - E^{\dot{\beta}} \wedge w_{\dot{\beta}}^{\dot{\alpha}} = \frac{1}{2} E^B \wedge E^C T_{CB}{}^{\dot{\alpha}}, w_{\dot{\beta}}{}^{\dot{\alpha}} := \frac{1}{4} w^{ab} \tilde{\sigma}_{ab}{}^{\dot{\alpha}}{}_{\dot{\beta}}$$
$$\sigma^a \tilde{\sigma}^b = \eta^{ab} + \frac{i}{2} \epsilon^{abcd} \sigma_c \tilde{\sigma}_d, \tilde{\sigma}^a \sigma^b = \eta^{ab} - \frac{i}{2} \epsilon^{abcd} \tilde{\sigma}_c \sigma_d$$
$$\mathcal{D}T^a + E^b \wedge R_b{}^a = 0, \mathcal{D}T^\alpha + E^\beta \wedge R_\beta{}^\alpha = 0, \mathcal{D}T^{\dot{\alpha}} + E^{\dot{\beta}} \wedge R_{\dot{\beta}}^{\dot{\alpha}} = 0$$
$$R^{ab} = (dw - w \wedge w)^{ab} = \frac{1}{2} E^B \wedge E^C R_{CB}{}^{ab}$$
$$S_{SG} = \int d^8 Z E := \int d^4 x \tilde{d}^4 \theta \operatorname{sdet}(E_M^A)$$
$$T^a = -2i \sigma_{\alpha\dot{\alpha}}^a E^\alpha \wedge \bar{E}^{\dot{\alpha}} - \frac{1}{8} E^b \wedge E^c \varepsilon^a{}_{bcd} G^d$$
$$T^\alpha := \left(T^{\dot{\alpha}}\right)^* = \frac{i}{8} E^c \wedge E^\beta (\sigma_c \tilde{\sigma}_d)_\beta^\alpha G^d - \frac{i}{8} E^c \wedge \bar{E}^{\dot{\beta}} \epsilon^{\alpha\beta} \sigma_{c\beta\dot{\beta}} R + \frac{1}{2} E^c \wedge E^b T_{bc}{}^\alpha$$
$$\mathcal{D}_\alpha \bar{R} = 0, \qquad \overline{\mathcal{D}}_{\dot{\alpha}} R = 0,$$
$$\overline{\mathcal{D}}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} = -\mathcal{D}_\alpha R, \quad \mathcal{D}^\alpha G_{\alpha\dot{\alpha}} = -\overline{\mathcal{D}}_{\dot{\alpha}} \bar{R}$$
$$T_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma} \equiv \sigma_{\alpha\dot{\alpha}}^a \sigma_{\beta\dot{\beta}}^b \epsilon_{\gamma\delta} T_{ab}{}^\delta = -\frac{1}{8} \epsilon_{\alpha\beta} \overline{\mathcal{D}}_{(\dot{\alpha}|} G_{\gamma|\dot{\beta})} - \frac{1}{8} \epsilon_{\dot{\alpha}\dot{\beta}} \left[W_{\alpha\beta\gamma} - 2\epsilon_{\gamma(\alpha} \mathcal{D}_{\beta)} R\right]$$
$$\overline{\mathcal{D}}_{\dot{\alpha}} W^{\alpha\beta\gamma} = 0, \mathcal{D}_\alpha \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} = 0$$
$$\mathcal{D}_\gamma W^{\alpha\beta\gamma} = \overline{\mathcal{D}}_{\dot{\gamma}} \mathcal{D}^{(\alpha} G^{\beta)\dot{\gamma}}$$
$$\epsilon^{abcd} T_{bc}{}^\alpha \sigma_{d\alpha\dot{\alpha}} = \frac{i}{8} \tilde{\sigma}^{a\dot{\beta}\beta} \overline{\mathcal{D}}_{(\dot{\beta}|} G_{\beta|\dot{\alpha})} + \frac{3i}{8} \sigma_{\beta\dot{\alpha}}^a \mathcal{D}^\beta R$$
$$R_{bc}{}^{ac} = \frac{1}{32} \left(\mathcal{D}^\beta \overline{\mathcal{D}}^{\dot{\alpha}|} G^{\alpha|\dot{\beta})} - \overline{\mathcal{D}}^{\dot{\beta}} \mathcal{D}^{(\beta} G^{\alpha)\dot{\alpha}}\right) \sigma_{\alpha\dot{\alpha}}^a \sigma_{b\beta\dot{\beta}} - \frac{3}{64} (\overline{\mathcal{D}\mathcal{D}} \bar{R} + \mathcal{D}\mathcal{D}R - 4R\bar{R}) \delta_b^a$$
$$G_a = 0,$$
$$R = 0, \quad \bar{R} = 0$$
$$\delta S_{SG} = \int d^8 Z E \left[\frac{1}{6} G_a \delta H^a - 2R \delta \overline{\mathcal{U}} - 2\bar{R} \delta \mathcal{U}\right]$$
$$S_{p=2} = \frac{1}{2} \int d^3 \xi \sqrt{g} - \int_{W^3} \hat{C}_3 = -\frac{1}{6} \int_{W^3} * \hat{E}_a \wedge \hat{E}^a - \int_{W^3} \hat{C}_3$$
$$g_{mn} = \hat{E}_m^a \eta_{ab} \hat{E}_n^b, \hat{E}_m^a := \partial_m \hat{Z}^M(\xi) E_M^a(\hat{Z})$$
$$W^3 \subset \Sigma^{(4|4)}: Z^M = \hat{Z}^M(\xi) = \left(\hat{x}^\mu(\xi), \hat{\theta}^{\check{\alpha}}(\xi)\right)$$
$$\hat{E}^a = d\xi^m \hat{E}_m^a = d\hat{Z}^M(\xi) E_M^a(\hat{Z})$$
$$* \hat{E}^a := \frac{1}{2} d\xi^m \wedge d\xi^n \sqrt{g} \epsilon_{mnk} g^{kl} \hat{E}_l^a$$
$$C_3 = \frac{1}{3!} dZ^M \wedge dZ^N \wedge dZ^K C_{KNM}(Z) = \frac{1}{3!} E^C \wedge E^B \wedge E^A C_{ABC}(Z)$$

$$\hat{C}_3 = \frac{1}{3!} d\hat{Z}^M \wedge d\hat{Z}^N \wedge d\hat{Z}^K C_{KNM}(\hat{Z}(\xi)) = \frac{1}{3!} \hat{E}^C \wedge \hat{E}^B \wedge \hat{E}^A C_{ABC}(\hat{Z}) =$$

$$= \frac{1}{3!} d\xi^m \wedge d\xi^n \wedge d\xi^k \hat{C}_{knm} = d^3\xi \epsilon^{mnk} \hat{C}_{knm}$$

$$H_{4L} = -\frac{i}{4} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta \sigma_{ab\alpha\beta} - \frac{1}{128} E^d \wedge E^c \wedge E^b \wedge E^a \epsilon_{abcd} R, dH_{4L} = 0$$

$$H_4 := dC_3 = \frac{1}{4!} E^{A_4} \wedge \ldots \wedge E^{A_1} H_{A_1 \ldots A_4}(Z) = H_{4L} + H_{4R}$$

$$\int_{W^3} C_3 = \int_{W^4 : \partial W^4 = W^3} H_4$$

$$\delta S_{p=2} = \int_{W^3} \left(\frac{1}{2} \mathcal{M}_{3a} E_M^a(\hat{Z}) + i\Psi_{3\alpha} E_M^\alpha(\hat{Z}) + i\Psi_{3\dot{\alpha}} E_M^{\dot{\alpha}}(\hat{Z}) \right) \delta\hat{Z}^M(\xi)$$

$$\mathcal{M}_{3a} := \mathcal{D} * \hat{E}_a + i\hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^\beta \sigma_{ab\beta\alpha} - i\hat{E}^b \wedge \hat{\bar{E}}^{\dot{\alpha}} \wedge \hat{\bar{E}}^{\dot{\beta}} \tilde{\sigma}_{ab\dot{\beta}\dot{\alpha}} -$$

$$-\frac{1}{8} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{abcd} (R + \bar{R}) = 0$$

$$\bar{\Psi}_{3\dot{\alpha}} := * \hat{E}_a \wedge \left(\hat{E}^\alpha \sigma^a_{\alpha\dot{\alpha}} - \left(\bar{\gamma}\sigma^a\right)_{\dot{\alpha}\dot{\beta}} \hat{\bar{E}}^{\dot{\beta}} \right) = 0$$

$$\Psi_{3\alpha} := * \hat{E}_a \wedge \left(\sigma^a_{\alpha\dot{\alpha}} \hat{\bar{E}}^{\dot{\alpha}} + \hat{E}^\beta (\sigma^a \tilde{\bar{\gamma}})_{\alpha\beta} \right) = 0$$

$$\bar{\gamma}_{\beta\dot{\alpha}} = \epsilon_{\beta\alpha} \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\bar{\gamma}}^{\dot{\beta}\alpha} = \frac{i}{3! \sqrt{g}} \sigma^a_{\beta\dot{\alpha}} \epsilon_{abcd} \epsilon^{mnk} \hat{E}_m^b \hat{E}_n^c \hat{E}_k^d = -\left(\bar{\gamma}_{\alpha\dot{\beta}}\right)^*$$

$$\bar{\gamma}_{\beta\dot{\alpha}} \tilde{\bar{\gamma}}^{\dot{\alpha}\alpha} = \delta_\beta{}^\alpha, \tilde{\bar{\gamma}}^{\dot{\alpha}\alpha} \bar{\gamma}_{\alpha\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}}$$

$$\bar{\gamma}\tilde{\sigma}^a = -\sigma^a \tilde{\bar{\gamma}} + \frac{i}{3! \sqrt{g}} \epsilon_{abcd} \epsilon^{mnk} \hat{E}_m^b \hat{E}_n^c \hat{E}_k^d$$

$$* \hat{E}_a \bar{\gamma} \tilde{\sigma}^a \bar{\gamma} = * \hat{E}_a \sigma^a, * \hat{E}_a \bar{\gamma} \tilde{\sigma}^a = - * \hat{E}_a \sigma^a \tilde{\bar{\gamma}}$$

$$\frac{1}{2} \hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^\beta \sigma_{ab\beta\alpha} = * \hat{E}_a \wedge \hat{E}^\beta (\sigma^a \tilde{\bar{\gamma}})_{\beta\alpha}$$

$$\frac{1}{2} \hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^{\dot{\beta}} \tilde{\sigma}_{ab\dot{\beta}\dot{\alpha}} = - * \hat{E}_a \wedge \hat{\bar{E}}^{\dot{\beta}} (\tilde{\sigma}^a \bar{\gamma})_{\dot{\beta}\dot{\alpha}}$$

$$\delta_\kappa \hat{Z}^M = \kappa^\alpha(\xi) \left(E_\alpha^M(\hat{Z}) + \bar{\gamma}_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} E_{\dot{\beta}}^M(\hat{Z}) \right)$$

$$\kappa^\alpha(\xi) = -\bar{\kappa}_{\dot{\alpha}}(\xi) \tilde{\bar{\gamma}}^{\dot{\alpha}\alpha} \Leftrightarrow \bar{\kappa}_{\dot{\alpha}}(\xi) = -\kappa^\alpha(\xi) \bar{\gamma}_{\alpha\dot{\alpha}}$$

$$\delta C_3 = \frac{1}{3!} E^C \wedge E^B \wedge E^A \beta_{ABC}(\delta)$$

$$\delta \mathcal{U} = \frac{i}{12} \delta V, \delta \overline{\mathcal{U}} = -\frac{i}{12} \delta V$$

$$\delta S_{SG} = \frac{1}{6} \int d^8 Z E [G_a \delta H^a + (R - \bar{R}) i \delta V]$$

$$G_a = 0$$

$$R - \bar{R} = 0$$

$$R = 4c, \bar{R} = 4c, c = \text{ const } = c^*$$

$$R_{bc}{}^{ac} = 3c^2 \delta_b{}^a$$

$$S = S_{SG} + T_2 S_{p=2} = \int d^8 Z E(Z) + \frac{T_2}{2} \int d^3\xi \sqrt{g} - T_2 \int_{W^3} \hat{C}_3$$

$$G_a = T_2 J_a$$

$$R - \bar{R} = -iT_2 \mathcal{X}$$

$$J_a = \int_{W^3} \frac{3}{\hat{E}} \hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^\beta \sigma_{ab\alpha\beta} \delta^8(Z-\hat{Z}) -$$
$$-\int_{W^3} \frac{3i}{\hat{E}} \left(* \hat{E}_a \wedge \hat{E}^\alpha + \frac{i}{2} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}_{\dot{\beta}} \epsilon_{abcd} \tilde{\sigma}^{d\dot{\beta}\alpha} \right) \mathcal{D}_\alpha \delta^8(Z-\hat{Z}) + c.c -$$
$$-\int_{W^3} \frac{i}{8\hat{E}} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{abcd} \left(\mathcal{D}\mathcal{D} - \frac{1}{2}\bar{R} \right) \delta^8(Z-\hat{Z}) + c.c. +$$
$$+\int_{W^3} \frac{1}{4\hat{E}} * \hat{E}_b \wedge \hat{E}^b G_a \delta^8(Z-\hat{Z}) -$$
$$-\int_{W^3} \frac{1}{4\hat{E}} * \hat{E}_c \wedge \hat{E}^b \tilde{\sigma}^{d\dot{\alpha}\alpha} (3\delta_a^c \delta_b^d - \delta_a^d \delta_b^c) [\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}] \delta^8(Z-\hat{Z})$$

$$\mathcal{X} = \frac{6i}{E} \int_{W^3} \hat{E}^a \wedge \hat{E}^\alpha \wedge \hat{E}^{\dot{\alpha}} \sigma^a_{\alpha\dot{\alpha}} \delta^8(Z-\hat{Z}) -$$
$$-\frac{3}{2} \int_{W^3} \frac{\hat{E}^b \wedge \hat{E}^a \wedge \hat{E}^\alpha}{\hat{E}} \sigma_{ab\alpha}^{\ \ \ \beta} \mathcal{D}_\beta \delta^8(Z-\hat{Z}) + c.c +$$
$$+\int_{W^3} \frac{\hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d}{8\hat{E}} \epsilon_{abcd} \tilde{\sigma}^{a\dot{\alpha}\alpha} [\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}] \delta^8(Z-\hat{Z}) +$$
$$+i \int_{W^3} \frac{* \hat{E}_a \wedge \hat{E}^a}{4\hat{E}} (\mathcal{D}\mathcal{D} - \bar{R}) \delta^8(Z-\hat{Z}) + c.c. +$$
$$+\int_{W^3} \frac{1}{4\hat{E}} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{abcd} G^a \delta^8(Z-\hat{Z}).$$

$$\overline{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = i\mathcal{D}_\alpha \mathcal{X}, \mathcal{D}^\alpha J_{\alpha\dot{\alpha}} = -i\overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{X}$$

$$i_{\underline{\theta}} E^\alpha := \theta^{\underline{\alpha}} E_{\underline{\alpha}}{}^{\alpha} = \theta^\alpha, i_{\underline{\theta}} E^{\dot{\alpha}} := \theta^{\underline{\alpha}} E_{\underline{\alpha}}^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}}$$

$$\theta^\alpha := \theta \underline{\underline{B}} \delta^\alpha_{\underline{\beta}}, \bar{\theta}^{\dot{\alpha}} := \theta \underline{\underline{\beta}} \delta^{\dot{\alpha}}_{\underline{\beta}}$$

$$i_{\underline{\theta}} E^{\underline{a}} := \theta ⅄ E^{\underline{a}}_{\underline{\alpha}} = 0$$

$$i_\theta w^{\underline{ab}} := \theta \underline{\underline{\beta}} w^{\underline{ab}}_{\underline{\beta}} = 0$$

$$\hat{\theta}^\alpha(\xi) = 0 \Leftrightarrow \hat{\theta}^\alpha(\xi) = 0, \hat{\bar{\theta}}^{\dot{\alpha}}(\xi) = 0$$

$$E_N^A \Big|_{\theta=0} = \begin{pmatrix} e_\nu^a(x) & \psi \frac{\alpha}{\nu}(x) \\ 0 & \delta_{\breve{\beta}}\underline{\alpha} \end{pmatrix} \Rightarrow E_A^N \Bigg|_{\theta=0} = \begin{pmatrix} e_a^\nu(x) & -\psi_a^{\breve{\beta}}(x) \\ 0 & \delta^{\breve{\beta}}_{\underline{\alpha}} \end{pmatrix}$$

$$T^\alpha_{ab}|_{\theta=0} = 2e_a^\mu e_b^\nu \mathcal{D}_{[\mu} \psi^\alpha_{\nu]}(x) - \frac{i}{4} \left(\psi_{[a} \sigma_{b]} \right)_{\dot{\beta}} G^{\alpha\dot{\beta}} \Big|_{\theta=0} - \frac{i}{4} \left(\bar{\psi}_{[a} \tilde{\sigma}_{b]} \right)^\alpha R \Big|_{\theta=0}$$

$$\hat{\epsilon}^\alpha = \hat{\bar{\epsilon}}_{\dot{\alpha}} \tilde{\bar{\gamma}}^{\dot{\alpha}\alpha}$$

$$\hat{E}^a = \hat{e}^a = d\hat{x}^\mu e_\mu^a(\hat{x}), \hat{E}^\alpha = \hat{\psi}^\alpha = d\hat{x}^\mu \psi_\mu^\alpha(\hat{x})$$

$$\mathcal{D}_\alpha \delta^8(Z-\hat{Z}) = \frac{1}{8} \theta_\alpha \bar{\theta}\bar{\theta} \delta^4(x-\hat{x}) + \propto \underline{\theta}^{\wedge 4}, \overline{\mathcal{D}}_{\dot{\alpha}} \delta^8(Z-\hat{Z}) = -\frac{1}{8} \bar{\theta}_{\dot{\alpha}} \theta\theta \delta^4(x-\hat{x}) + \underline{\theta}^{\wedge 4}$$

$$\mathcal{D}^\alpha \mathcal{D}_\alpha \delta^8(Z-\hat{Z}) = -\frac{1}{4} \bar{\theta}\bar{\theta} \delta^4(x-\hat{x}) + \propto \theta\bar{\theta}\bar{\theta}$$

$$\overline{\mathcal{D}}_{\dot{\alpha}} \overline{\mathcal{D}}^{\dot{\alpha}} \delta^8(Z-\hat{Z}) = -\frac{1}{4} \theta\theta \delta^4(x-\hat{x}) + \propto \theta\theta\bar{\theta}$$

$$[\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}] \delta^8(Z-\hat{Z}) = -\frac{1}{2} \theta_\alpha \bar{\theta}_{\dot{\alpha}} \delta^4(x-\hat{x}) + \propto \underline{\theta}^{\wedge 3}$$

$$\mathcal{P}_a{}^b(x) := \int_{W^3} \frac{1}{\hat{e}} * \hat{e}_a \wedge \hat{e}^b \delta^4(x-\hat{x})$$

$$\mathcal{P}_a(x) := \int_{W^3} \frac{1}{\hat{e}} \epsilon_{abcd} \hat{e}^b \wedge \hat{e}^c \wedge \hat{e}^d \delta^4(x-\hat{x}) =$$

$$= e_a^\mu(x) \int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(x-\hat{x})$$

$$J_{\alpha\dot{\alpha}}|_{\hat{\theta}=0} = \frac{\theta_\beta \bar{\theta}_{\dot{\beta}}}{8}\left(3\mathcal{P}_a{}^b(x)\sigma^a_{\alpha\dot{\alpha}}\tilde{\sigma}_b^{\beta\dot{\beta}} - 2\delta_\alpha{}^\beta \delta^{\dot{\beta}}_{\dot{\alpha}} \mathcal{P}_b{}^b(x)\right) - i\frac{(\theta\theta - \bar{\theta}\bar{\theta})}{32}\sigma^a_{\alpha\dot{\alpha}}\mathcal{P}_a(x) + \propto \underline{\theta}^{\wedge 3}$$

$$\mathcal{X}|_{\hat{\theta}=0} = -\frac{\theta\sigma^a\bar{\theta}}{16}\mathcal{P}_a + i\frac{(\theta\theta - \bar{\theta}\bar{\theta})}{16}\mathcal{P}_a{}^a(x) + \propto \underline{\theta}^{\wedge 3}$$

$$\epsilon^{\mu\nu\rho\sigma} e^a_\nu(x) \mathcal{D}_\rho \psi^\alpha_\sigma(x) \sigma_{a\alpha\dot{\alpha}} = 0$$

$$R^{ac}_{bc}|_{\theta=0} = \frac{1}{32}\left(\mathcal{D}^\beta \overline{\mathcal{D}}^{(\dot{\alpha}|} J^{\alpha|\dot{\beta})} - \overline{\mathcal{D}}^{\dot{\beta}} \mathcal{D}^{(\beta} J^{\alpha)\dot{\alpha}}\right)\Big|_{\theta=0} \sigma^a_{\alpha\dot{\alpha}}\sigma_{b\beta\dot{\beta}} - \frac{3i}{64}(\overline{\mathcal{D}\mathcal{D}}\mathcal{X} - \mathcal{D}\mathcal{D}\mathcal{X})\Big|_{\theta=0} \delta^a_b +$$

$$+ \frac{3}{16}(R\bar{R})\Big|_{\theta=0} \delta^a_b$$

$$R_{bc}{}^{ac}|_{\theta=0,\hat{\theta}=0} = -\frac{3}{32}T_2\left(\mathcal{P}_b{}^a(x) - \frac{1}{2}\delta^a_b \mathcal{P}_c{}^c(x)\right) + \frac{3}{64}(R+\bar{R})^2\Big|_{\theta=0} \delta^a_b$$

$$\partial_\mu(R+\bar{R})|_{\theta=0} = \frac{T_2}{16}\int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(x-\hat{x})$$

$$R(x) + \bar{R}(x) = 8c + \frac{T_2}{16}\int_{x_0}^x d\tilde{x}^\mu \int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(\tilde{x}-\hat{x})$$

$$\Theta(x, x_0 \mid \hat{x}) := \int_{x_0}^x d\tilde{x}^\mu \int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(\tilde{x}-\hat{x})$$

$$\partial_\mu \Theta(x, x_0 \mid \hat{x}) = \int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(x-\hat{x})$$

$$R(x) + \bar{R}(x) = 8c + \frac{T_2}{16}\Theta(x, x_0 \mid \hat{x})$$

$$R_{bc}{}^{ac}(x) = -\frac{3T_2}{32}\left(\mathcal{P}_b{}^a(x) - \frac{1}{2}\delta^a_b \mathcal{P}_c{}^c(x)\right) + 3\delta^a_b\left(c^2 + \left(\left(\frac{T_2}{128} + c\right)^2 - c^2\right)\Theta(x, x_0 \mid \hat{x})\right)$$

$$R^c_{acb}(x) = \eta_{ab} 3c^2 + T_2\left(\mathcal{T}^{sing}_{ab}(x) + \mathcal{T}^{reg}_{ab}(x)\right)$$

$$\mathcal{T}^{\text{sing}}_{ab}(x) = -T_2\frac{3}{32}\left(\mathcal{P}_{ba}(x) - \frac{1}{2}\eta_{ba}\mathcal{P}^c_c(x)\right) =$$

$$= -\frac{3T_2}{32}\int_{W^3} \frac{1}{\hat{e}} * \hat{e}_a \wedge \hat{e}_b \delta^4(x-\hat{x}) + \frac{3T_2}{64}\eta_{ba}\int_{W^3} \frac{1}{\hat{e}} * \hat{e}_c \wedge \hat{e}^c \delta^4(x-\hat{x})$$

$$\mathcal{T}^{reg}_{ab}(x) = \eta_{ab}\mathcal{T}^{reg}(x), \mathcal{T}^{reg}(x) = +\frac{3T_2}{64}\left(\frac{T_2}{256} + c\right)\Theta(x, x_0 \mid \hat{x})$$

$$M^4_+: \quad R_{acb}{}^c(x) = 3\eta_{ab}\left(\frac{T_2}{128} + c\right)^2$$

$$M^4_-: \quad R^c_{acb}(x) = 3\eta_{ab}c^2$$

$$D\epsilon^\alpha + \frac{i}{8}e^c(\epsilon\sigma_c\tilde{\sigma}_d)^\alpha_\beta G^d\Big|_{\theta=0} + \frac{i}{8}e^c(\bar{\epsilon}\tilde{\sigma}_c)^\alpha R\Big|_{\theta=0} = 0$$

$$D\epsilon^\alpha + \frac{i}{2}e^a(\bar{\epsilon}\tilde{\sigma}_a)^\alpha\left(c + \frac{T_2}{128}\Theta(x, x_0 \mid \hat{x})\right) = 0$$

$$M^4_-: \quad D\epsilon^\alpha + \frac{i}{2}e^a(\bar{\epsilon}\tilde{\sigma}_a)^\alpha c = 0$$

$$M^4_+: \quad D\epsilon^\alpha + \frac{i}{2}e^a(\bar{\epsilon}\tilde{\sigma}_a)^\alpha\left(c + \frac{T_2}{128}\right) = 0$$

$$W^3 = \pm\partial M^4_\pm: \hat{\epsilon}^\alpha = \hat{\bar{\epsilon}}_{\dot{\alpha}}\tilde{\hat{\gamma}}^{\dot{\alpha}\alpha}, \hat{\epsilon}^\alpha := \epsilon^\alpha(\hat{x}(\xi)), \hat{\bar{\epsilon}}_{\dot{\alpha}} := \bar{\epsilon}_{\dot{\alpha}}(\hat{x}(\xi))$$

$$M_{-}^{4}: \quad R^{ab}\epsilon^{\beta}\sigma_{ab\beta}{}^{\alpha} = \frac{1}{4}|c|^2 e^d \wedge e^c \epsilon^{\beta}\sigma_{cd\beta}{}^{\alpha}$$

$$M_{+}^{4}: \quad R^{ab}\epsilon^{\beta}\sigma_{ab\beta}{}^{\alpha} = \frac{1}{4}\left|c + \frac{T_2}{128}\right|^2 e^d \wedge e^c \epsilon^{\beta}\sigma_{cd\beta}{}^{\alpha}$$

$$M_{-}^{4}: \quad R_{cd}{}^{ab} = \frac{1}{2}|c|^2 \delta_{[c}{}^{a}\delta_{d]}^{b}$$

$$M_{+}^{4}: \quad R_{cd}{}^{ab} = \frac{1}{2}\left|c + \frac{T_2}{128}\right|^2 \delta_{[c}{}^{a}\delta_{d]}^{b}$$

$$\delta E^a = E^a(\Lambda(\delta) + \bar{\Lambda}(\delta)) - \frac{1}{4}E^b \tilde{\sigma}_b^{\dot{\alpha}\alpha}[\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}]\delta H^a + iE^\alpha \mathcal{D}_\alpha \delta H^a - i\bar{E}^{\dot{\alpha}}\overline{\mathcal{D}}_{\dot{\alpha}}\delta H^a$$

$$\delta E^\alpha = E^a \Xi_a^\alpha(\delta) + E^\alpha \Lambda(\delta) + \frac{1}{8}\bar{E}^{\dot{\alpha}} R \sigma_{a\dot{\alpha}}^{\alpha}\delta H^a$$

$$2\Lambda(\delta) + \bar{\Lambda}(\delta) = \frac{1}{4}\tilde{\sigma}_a^{\dot{\alpha}\alpha}\mathcal{D}_\alpha \overline{\mathcal{D}}_{\dot{\alpha}}\delta H^a + \frac{1}{8}G_a \delta H^a + 3(\mathcal{D}\mathcal{D} - \bar{R})\delta\mathcal{U}$$

$$\delta H_4 = \frac{1}{2}E^b \wedge E^\alpha \wedge E^\beta \wedge E^\gamma \sigma_{ab(\alpha\beta}D_{\gamma)}\delta H^a - \frac{1}{2}E^b \wedge E^\alpha \wedge E^\beta \wedge \bar{E}^{\dot{\gamma}}\sigma_{ab\alpha\beta}\bar{D}_{\dot{\gamma}}\delta H^a + c.c. -$$

$$-\frac{i}{2}E^b \wedge E^a \wedge E^\alpha \wedge E^\beta\left(\sigma_{ab\alpha\beta}(2\Lambda(\delta) + \bar{\Lambda}(\delta)) + \frac{1}{4}\sigma_{c[a|\alpha\beta}\tilde{\sigma}_{|b]}{}^{\dot{\gamma}\gamma}[D_\gamma, \bar{D}_{\dot{j}}]\delta H^c\right) + c.c. +$$

$$+\frac{i}{16}E^b \wedge E^a \wedge E^\alpha \wedge \bar{E}^{\dot{\beta}}(R\sigma_{ab}\tilde{\sigma}_c - \bar{R}\sigma_c\tilde{\sigma}_{ab})_{\alpha\dot{\beta}}\delta H^c + \propto E^c \wedge E^b \wedge E^a$$

$$\delta H_4 = d(\delta C_3)$$

$$(\mathcal{D}\mathcal{D} - \bar{R})\delta\mathcal{U} = \frac{1}{12}(\mathcal{D}\mathcal{D} - \bar{R})\left(i\delta V + \frac{1}{2}\overline{\mathcal{D}}_{\dot{\alpha}}\delta\bar{\kappa}^{\dot{\alpha}}\right)$$

$$\delta\mathcal{U} = \frac{i}{12}\delta V + \frac{1}{24}\overline{\mathcal{D}}_{\dot{\alpha}}\delta\bar{\kappa}^{\dot{\alpha}} + \frac{i}{24}\mathcal{D}_\alpha \delta\nu^\alpha$$

$$\beta_{\alpha\beta\gamma}(\delta) = 0 = \beta_{\alpha\beta\dot{\gamma}}(\delta), \beta_{\alpha\dot{\beta}a}(\delta) = i\sigma_{a\alpha\dot{\beta}}\delta V$$

$$\beta_{\alpha\beta a}(\delta) = -\sigma_{ab\alpha\beta}\left(\delta H^b + \tilde{\sigma}^{b\gamma\dot{\gamma}}D_\gamma \delta\bar{\kappa}_{\dot{\gamma}}\right)$$

$$\beta_{\alpha ab}(\delta) = \frac{1}{2}\epsilon_{abcd}\sigma_{\alpha\dot{\alpha}}^{c}\bar{D}^{\dot{\alpha}}\delta H^d + \frac{1}{2}\sigma_{ab\alpha}{}^{\beta}D_\beta \delta V - \frac{i}{4}\tilde{\sigma}_{ab}{}^{\dot{\beta}}{}_{\dot{\gamma}}\bar{D}_{\dot{\beta}}D_\alpha \bar{\kappa}^{\dot{\gamma}} + \frac{i}{4}\sigma_{ab\alpha}^{\beta}\bar{D}_{\dot{\beta}}D_\beta \bar{\kappa}^{\dot{\beta}}$$

$$\beta_{abc}(\delta) = \frac{i}{8}\epsilon_{abcd}\left(\left(\overline{\mathcal{D}\mathcal{D}} - \frac{1}{2}R\right)\delta H^d - c.c.\right) +$$

$$+\frac{1}{4}\epsilon_{abcd}G^d \delta V + \frac{1}{8}\epsilon_{abcd}\tilde{\sigma}^{d\dot{\gamma}\gamma}[\mathcal{D}_\gamma, \overline{\mathcal{D}}_{\dot{\gamma}}]\delta V - \frac{i}{16}\epsilon_{abcd}\tilde{\sigma}^{d\dot{\gamma}\gamma}\left(\left(\mathcal{D}\mathcal{D} + \frac{5}{2}\bar{R}\right)\overline{\mathcal{D}}_{\dot{\gamma}}\kappa_\gamma - c.c.\right)$$

$$\delta S_{SG} = \frac{1}{6}\int d^8 Z E[G_a \delta H^a + (R - \bar{R})i\delta V] -$$

$$-\frac{1}{12}\int d^8 Z E\left(R\mathcal{D}_\alpha \delta\kappa^\alpha + \bar{R}\overline{\mathcal{D}}_{\dot{\alpha}}\delta\bar{\kappa}^{\dot{\alpha}}\right)$$

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Apéndice E.

Teoría de Yang-Mills-Utiyama para campos cuánticos relativistas.

$$d_{\mathrm{G}}:\mathrm{End}(\Gamma):\mathfrak{a}\mapsto d_{\mathrm{G}}\mathfrak{a}:=d\mathfrak{a}-ic_{\mathrm{G}}[\mathfrak{A}_{\mathrm{G}},\mathfrak{a}]_{\wedge},[\mathfrak{A}_{\mathrm{G}},\mathfrak{a}]_{\wedge}:=\mathfrak{A}_{\mathrm{G}}\wedge\mathfrak{a}-(-1)^{p}\mathfrak{a}\wedge\mathfrak{A}_{\mathrm{G}}$$

$$\mathfrak{F}_{\mathrm{G}}:=d\mathfrak{A}_{\mathrm{G}}-ic_{\mathrm{G}}\mathfrak{A}_{\mathrm{G}}\wedge\mathfrak{A}_{\mathrm{G}}=\sum_{I}\left(d\mathfrak{A}_{\mathrm{G}}^{I}+\frac{c_{\mathrm{G}}}{2}\sum_{J,K}f_{JK}^{I}\mathfrak{A}_{\mathrm{G}}^{J}\wedge\mathfrak{A}_{\mathrm{G}}^{K}\right)\tau_{I}$$

$$[\tau_{J},\tau_{K}]=\tau_{J}\tau_{K}-\tau_{K}\tau_{J}=:i\sum_{I}f_{JK}^{I}\tau_{I}$$

$$d_{\mathrm{G}}\boldsymbol{\phi}=d\boldsymbol{\phi}-ic_{\mathrm{G}}[\mathfrak{A}_{\mathrm{G}},\boldsymbol{\phi}]_{\wedge}=\sum_{I}\left(d\phi^{I}+c_{G}\sum_{J,K}f_{JK}^{I}\mathfrak{A}_{\mathrm{G}}^{J}\phi^{K}\right)\tau_{I}\in\Gamma(M,V(E),G)$$

$$\Gamma_{\mu\nu}^{\lambda}:=\frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma}+\partial_{\nu}g_{\mu\sigma}-\partial_{\sigma}g_{\mu\nu}\right)$$

$$(\mathcal{M}_{\mathrm{G}}\otimes\mathcal{M}_{\mathrm{L}},\pi_{\mathrm{I}},\mathcal{M}_{\mathrm{G}},SO(1,3))$$

$$\pi_{\mathrm{I}}:\mathcal{M}_{\mathrm{G}}\otimes\mathcal{M}_{\mathrm{L}}\to\mathcal{M}_{\mathrm{G}}:\mathcal{M}_{\mathrm{L}}(p)\mapsto p$$

$$\pi_{I}^{\#}:\Omega^{1}(T^{*}\mathcal{M}_{\mathrm{G}})\to\Gamma(\mathcal{M}_{\mathrm{L}},T^{*}\mathcal{M}_{\mathrm{L}},SO(1,3)):dx^{\mu}\mapsto\mathrm{e}^{a}:=\mathcal{E}_{\mu}^{a}(p)dx^{\mu}$$

$$\xi^{a}(x,x)=0,\ \mathcal{E}_{\mu}^{a}(x):=\left.\frac{\partial\xi^{a}(y,x)}{\partial y^{\mu}}\right|_{y=x}$$

$$\mathfrak{v}:=\frac{1}{4!}\epsilon_{\circ0\circ\circ}\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}=\det[\mathcal{E}]dx^{0}\wedge dx^{1}\wedge dx^{2}\wedge dx^{3},\det[\mathcal{E}]>0$$

$$g_{\mu\nu}=[\mathcal{E}^{t}\boldsymbol{\eta}_{\mathrm{L}}\mathcal{E}]_{\mu\nu}=\eta_{\mathrm{L}\circ\circ}\mathcal{E}_{\mu}^{\circ}\mathcal{E}_{\nu}^{\circ},\det[\boldsymbol{g}]=\det[\boldsymbol{\eta}_{\mathrm{L}}]\det[\mathcal{E}]^{2}=-\det[\mathcal{E}]^{2}<0$$

$$\mathfrak{v}=\sqrt{-\det[\boldsymbol{g}]}dx^{0}\wedge dx^{1}\wedge dx^{2}\wedge dx^{3}$$

$$\mathfrak{S}_{ab}:=\frac{1}{2}\epsilon_{ab\circ\circ}\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ},\mathfrak{V}_{a}:=\frac{1}{3!}\epsilon_{a\circ\circ\circ}\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}$$

$$\mathfrak{w}^{ab}=\omega_{\mu}^{ab}dx^{\mu}\in V^{2}(T\mathcal{M}_{\mathrm{L}})\otimes\Omega^{1}(T^{*}\mathcal{M}_{\mathrm{L}})\otimes\mathrm{Ad}\left(\mathfrak{g}_{SO(1,3)}\right)$$

$$d_{\mathfrak{w}}\mathfrak{a}^{a}:=d\mathfrak{a}^{a}+c_{gr}\eta_{\mathrm{L}\circ\circ}\mathfrak{w}^{a\circ}\wedge\mathfrak{a}^{\circ},\mathfrak{a}\in\Omega^{1}(T^{*}\mathcal{M}_{\mathrm{L}})$$

$$\mathfrak{T}^{a}:=d_{\mathfrak{w}}\mathrm{e}^{a}\in V^{1}(T\mathcal{M}_{\mathrm{L}})\otimes\Omega^{2}(T^{*}\mathcal{M}_{\mathrm{L}})$$

$$\mathfrak{a}=a^{i_{1}\cdots i_{p}}{}_{j_{1}\cdots j_{q}}\left(\partial_{i_{1}}\otimes\cdots\otimes\partial_{i_{p}}\right)\left(\mathrm{e}^{j_{1}}\wedge\cdots\wedge\mathrm{e}^{j_{q}}\right)$$

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_{\mathrm{L}}=\boldsymbol{u}^{t}\cdot\boldsymbol{\eta}_{\mathrm{L}}\cdot\boldsymbol{v}=\eta_{\mathrm{L}\circ\circ}u^{\circ}v^{\circ}$$

$$\boldsymbol{u}'=\boldsymbol{g}_{\mathrm{L}}\cdot\boldsymbol{u}$$

$$G_{SO}:\mathrm{e}\mapsto G_{SO}(\mathrm{e})=\mathrm{e}'=\boldsymbol{g}_{\mathrm{L}}\cdot\mathrm{e}$$

$$G_{SO}:\mathfrak{w}\mapsto G_{SO}(\mathfrak{w})=\mathfrak{w}'=\boldsymbol{g}_{\mathrm{L}}\cdot\mathfrak{w}\cdot\boldsymbol{g}_{\mathrm{L}}^{-1}+c_{gr}^{-1}\boldsymbol{g}_{\mathrm{L}}\cdot d\boldsymbol{g}_{\mathrm{L}}^{-1}$$

$$G_{SO}(d_{\mathfrak{w}}\mathrm{e})=d_{\mathfrak{w}'}\cdot\mathrm{e}'$$

$$SO^{\uparrow}(1,3):=\{\boldsymbol{g}_{\mathrm{L}}\in SO(1,3)\mid[\boldsymbol{g}_{\mathrm{L}}]_{0}^{0}>0\}\ \ SO^{\downarrow}(1,3):=\{\boldsymbol{g}_{\mathrm{L}}\in SO(1,3)\mid[\boldsymbol{g}_{\mathrm{L}}]_{0}^{0}<0\}$$

$$\boldsymbol{O}^{\mathrm{T}}:=\mathrm{diag}[-1,1,1,1],\boldsymbol{O}^{\mathrm{P}}:=\mathrm{diag}[1,-1,-1,-1]$$

$$\mathfrak{R}^{ab}:=d\mathfrak{w}^{ab}+c_{gr}\mathfrak{w}^{a}{}_{\circ}\wedge\mathfrak{w}^{\circ b}\in V^{2}(T\mathcal{M}_{\mathrm{L}})\otimes\Omega^{2}(T^{*}\mathcal{M}_{\mathrm{L}})\otimes Ad\left(\mathfrak{g}_{SO(1,3)}\right)$$

$$\mathfrak{R}^{ab}=\sum_{c<d}R_{cd}^{ab}\mathrm{e}^{c}\wedge\mathrm{e}^{d}=\frac{1}{2}R_{\circ_{\circ}}{}^{ab}\mathrm{e}^{\circ}\wedge\mathrm{e}^{\circ}$$

$$R_{ab}:=\eta_{\mathrm{L}a\star}R_{\circ\star}^{\circ\star},R:=R_{\circ\star}^{\circ\star}$$

$$d_{\mathfrak{w}}(d_{\mathfrak{w}}\mathrm{e}^{a})=c_{gr}\eta_{\mathrm{L}\circ\circ}\mathfrak{R}^{a\circ}\wedge\mathrm{e}^{\circ},d_{\mathfrak{w}}\mathfrak{R}^{ab}=0$$

$$\boldsymbol{v},\boldsymbol{u}\in V(T\mathcal{M}_{\mathrm{E}}),\langle\boldsymbol{u},\boldsymbol{v}\rangle_{\mathrm{E}}=\boldsymbol{u}^{t}\cdot\boldsymbol{\eta}_{\mathrm{E}}\cdot\boldsymbol{v}=\eta_{\mathrm{E}\circ\circ}u^{\circ}v^{\circ}$$

$$\left|\left.\frac{\partial\Gamma}{\partial\zeta}\right|_{\mathrm{L}\to\mathrm{E}}\Longrightarrow\mathfrak{v}_{\mathrm{E}}:=\det[E_{\mathrm{E}}]dx^{0}\wedge dx^{1}\wedge dx^{2}\wedge dx^{3}\ \det[\boldsymbol{g}_{\mathrm{E}}]=\det[\boldsymbol{\eta}_{\mathrm{E}}]\det[E]^{2}=\det[\mathcal{E}]^{2}>0;$$

$$\mathfrak{v}_{\mathrm{E}}=\sqrt{\det[\boldsymbol{g}_{\mathrm{E}}]}dx^{0}\wedge dx^{1}\wedge dx^{2}\wedge dx^{3}$$

$$\mathrm{ISO}(1,3)=SO(1,3)\ltimes T^{4}$$

$$[P_a, P_b] = 0$$
$$[J_{ab}, P_c] = -\eta_{ac}P_b + \eta_{bc}P_a$$
$$[J_{ab}, J_{cd}] = -\eta_{ac}J_{bd} + \eta_{bc}J_{ad} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}$$
$$(\mathcal{M}_{\mathrm{G}} \otimes \mathcal{M}_{\mathrm{L}}, \pi_{\mathrm{I}}, \mathcal{M}_{\mathrm{G}}, G_{\mathrm{cP}})$$
$$[\Theta_I]_{ab} := \begin{cases} P_{ab}, & I = 1 \\ J_{ab}, & I = 2 \end{cases}$$
$$[P_{ab}, P_{cd}] = 0, [J_{ab}, P_{cd}] = -\eta_{ac}P_{bd} + \eta_{bc}P_{ad},\ [J_{ab}, J_{cd}] = \left| \frac{\partial\Gamma}{\partial\zeta} \right|_{\mathrm{L}\to\mathrm{E}}$$
$$[\Theta_I, \Theta_J] := \mathcal{F}_{IJ}^{K}\Theta_K$$
$$\begin{cases} \mathcal{F}_{11}^{1} = \mathcal{F}^{2}{}_{11} = \mathcal{F}_{12}^{2} = \mathcal{F}^{2}{}_{21} = \mathcal{F}^{1}{}_{22} = 0, \\ [\mathcal{F}^{1}{}_{12}]_{ab;cd}^{ef} = -[\mathcal{F}_{21}^{1}{}_{21}]_{cd;ab} = \eta_{ac}\delta_b^e\delta_d^f - \eta_{bc}\delta_a^e\delta_d^f, \\ [\mathcal{F}_{22}^{2}]_{ab;cd}^{ef} = -\eta_{ac}\delta_b^e\delta_d^f + \eta_{bc}\delta_a^e\delta_d^f - \eta_{bd}\delta_a^e\delta_c^f + \eta_{ad}\delta_b^e\delta_c^f \end{cases}$$
$$\mathfrak{A}_{\mathrm{cP}}^{I} = \begin{cases} J_{\circ\circ} \otimes \mathfrak{w}^{\circ\circ}, & I = 1, \\ P_{\circ\circ} \otimes \mathfrak{S}^{\circ\circ}, & I = 2, \end{cases} \in \Omega^1(T^*\mathcal{M}) \otimes Ad(\mathfrak{g}_{\mathrm{cP}})$$
$$\mathfrak{F}_{\mathrm{cP}}^{I} = \begin{cases} J_{\circ\circ} \otimes \mathfrak{R}^{\circ\circ}, & I = 1, \\ P_{\circ\circ} \otimes d_{\mathfrak{w}}\mathfrak{S}^{\circ\circ}, & I = 2, \end{cases} \in \Omega^2(T^*\mathcal{M}) \otimes \mathrm{Ad}(\mathfrak{g}_{\mathrm{cP}})$$
$$\mathfrak{a} := \frac{1}{p!} a_{i_1\cdots i_p} \mathrm{e}^{i_1} \wedge \cdots \wedge \mathrm{e}^{i_p} \in \Omega^p(T^*\mathcal{M}_\blacksquare), \mathfrak{b} := \frac{1}{p!} b_{i_1\cdots i_p} \mathrm{e}^{i_1} \wedge \cdots \wedge \mathrm{e}^{i_p} \in \Omega^p(T^*\mathcal{M}_\blacksquare)$$
$$\langle \mathfrak{a}, \mathfrak{b} \rangle_\blacksquare := \frac{1}{p!} \eta_\blacksquare^{i_1 j_1} \cdots \eta_\blacksquare^{i_p j_p} a_{i_1\cdots i_p} b_{j_1\cdots j_p}$$
$$\|\mathfrak{a}\|_\blacksquare^2 := \langle \mathfrak{a}, \mathfrak{a} \rangle_\blacksquare$$
$$\mathfrak{a} \wedge \widehat{\mathrm{H}}_\blacksquare(\mathfrak{b}) := \det[\boldsymbol{\eta}_\blacksquare]^{1/2} \langle \mathfrak{a}, \mathfrak{b} \rangle_\blacksquare \mathfrak{v}_\blacksquare$$
$$\widehat{\mathrm{H}}_\blacksquare : \Omega^p \to \Omega^{n-p} : \mathfrak{b} \mapsto \hat{\mathfrak{b}} := \widehat{\mathrm{H}}(\mathfrak{b}) = \frac{\det[\boldsymbol{\eta}_\blacksquare]^{1/2}}{p!\,(n-p)!} b_{i_1\cdots i_p} [\epsilon]^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n} \mathrm{e}^{i_{p+1}} \wedge \cdots \wedge \mathrm{e}^{i_n}$$
$$[\epsilon]^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n} := \frac{1}{p!} \eta_\blacksquare^{i_1 j_1} \cdots \eta_\blacksquare^{i_p j_p} \epsilon_{j_1\cdots j_p i_{p+1}\cdots i_n}$$
$$\widehat{\mathrm{H}}_\blacksquare : b_{i_1\cdots i_p} \mapsto \hat{b}_{i_{p+1}\cdots i_n} := \frac{\det[\boldsymbol{\eta}_\blacksquare]^{1/2}}{p!} b_{i_1\cdots i_p} [\boldsymbol{\epsilon}]^{i_1\cdots i_p}_{i_{p+1}\cdots i_n}$$
$$\hat{\mathfrak{b}} = \frac{1}{(n-p)!} \hat{b}_{i_1\cdots i_{n-p}} \mathrm{e}^{i_1} \wedge \cdots \wedge \mathrm{e}^{i_{n-p}}$$
$$\widehat{\mathrm{H}}_\blacksquare \circ \widehat{\mathrm{H}}_\blacksquare(\mathfrak{a}) = \det[\boldsymbol{\eta}_\blacksquare](-1)^{p(n-p)}\mathfrak{a} \Longrightarrow \widehat{\mathrm{H}}_\blacksquare^{-1}(\mathfrak{a}) = \det[\boldsymbol{\eta}_\blacksquare]^{-1}(-1)^{p(n-p)}\widehat{\mathrm{H}}_\blacksquare(\mathfrak{a})$$
$$\widehat{\mathrm{H}}_\blacksquare(1) = \det[\boldsymbol{\eta}_\blacksquare]^{1/2}\mathrm{e}^0 \wedge \cdots \wedge \mathrm{e}^{n-1} = \det[\boldsymbol{\eta}_\blacksquare]^{1/2}\mathfrak{v} = \det[\boldsymbol{\eta}_\blacksquare]\det[\mathcal{E}]dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$
$$\begin{aligned} \hat{d} : \Omega^p(T^*\mathcal{M}_\blacksquare) \to \Omega^{p-1}(T^*\mathcal{M}_\blacksquare) : \mathfrak{a} \mapsto \hat{d}\mathfrak{a} &:= (-1)^p \widehat{\mathrm{H}}_\blacksquare^{-1}\big(d\widehat{\mathrm{H}}_\blacksquare(\mathfrak{a})\big) \\ &= \det[\boldsymbol{\eta}_\blacksquare]^{-1}(-1)^{p(n-p+1)}\widehat{\mathrm{H}}_\blacksquare\big(d\widehat{\mathrm{H}}_\blacksquare(\mathfrak{a})\big) \end{aligned}$$
$$\begin{aligned} \int \langle d\mathfrak{a}, \mathfrak{b} \rangle_\blacksquare \mathfrak{V}_\blacksquare &= \det[\boldsymbol{\eta}_\blacksquare]^{-1/2} \int d\mathfrak{a} \wedge \widehat{\mathrm{H}}_\blacksquare(\mathfrak{b}) = \det[\boldsymbol{\eta}_\blacksquare]^{-1/2}(-1)^p \int \mathfrak{a} \wedge d\widehat{\mathrm{H}}_\blacksquare(\mathfrak{b}) \\ &= \det[\boldsymbol{\eta}_\blacksquare]^{-1/2} \int \mathfrak{a} \wedge \widehat{\mathrm{H}}_\blacksquare\Big((-1)^p \widehat{\mathrm{H}}_\blacksquare^{-1}\big(d\widehat{\mathrm{H}}_\blacksquare(\mathfrak{b})\big)\Big) = \int \langle \mathfrak{a}, \hat{d}\hat{\mathfrak{b}} \rangle_\blacksquare \mathfrak{v}_\blacksquare \end{aligned}$$
$$\Delta_\blacksquare := d\hat{d} + \hat{d}d$$
$$\langle \Delta_\blacksquare \mathfrak{a}, \mathfrak{b} \rangle_\blacksquare = \langle d\hat{d}\mathfrak{a}, \mathfrak{b} \rangle_\blacksquare + \langle \hat{d}d\mathfrak{a}, \mathfrak{b} \rangle_\blacksquare = \langle \mathfrak{a}, d\hat{d}\mathfrak{b} \rangle_\blacksquare + \langle \mathfrak{a}, \hat{d}d\mathfrak{b} \rangle_\blacksquare = \langle \mathfrak{a}, \Delta\Delta_\blacksquare \mathfrak{b} \rangle_\blacksquare$$
$$\left| \frac{\partial\Gamma}{\partial\zeta} \right|_{\mathrm{L}\to\mathrm{E}} \Longrightarrow \widehat{\mathrm{H}}_\blacksquare \circ \widehat{\mathrm{H}}_\blacksquare(\mathfrak{a}) = (-1)^{n^2/4}\det[\boldsymbol{\eta}_\blacksquare]\mathfrak{a} \in \Omega^p(T^*\mathcal{M})$$
$$\tilde{\mathfrak{S}}^I := \mathrm{e}^{i_{p(1}} \wedge \cdots \wedge \mathrm{e}^{i_{n/2)}}\ I = 1, \cdots, m := \frac{n!}{((n/2)!)^2}$$
$$\widehat{\mathrm{H}}_\blacksquare(\tilde{\mathfrak{S}}^I) = (-1)^{n/2}\tilde{\mathfrak{S}}^J, \widehat{\mathrm{H}}_\blacksquare(\tilde{\mathfrak{S}}^J) = \tilde{\mathfrak{S}}^I$$
$$\tilde{\mathfrak{S}}^{\pm} := \tilde{\mathfrak{S}}^I \pm (-1)^{n/2}\tilde{\mathfrak{S}}^J$$
$$\widehat{\mathrm{H}}_\blacksquare(\tilde{\mathfrak{S}}^+) = (-1)^{n/2}(\tilde{\mathfrak{S}}^J + \tilde{\mathfrak{S}}^I) = +(-1)^{n/2}\tilde{\mathfrak{S}}^+ \in \mathbb{V}_{\widehat{\mathrm{H}}}^+(\Omega^{n/2})$$
$$\widehat{\mathrm{H}}_\blacksquare(\tilde{\mathfrak{S}}^-) = (-1)^{n/2}(\tilde{\mathfrak{S}}^J - \tilde{\mathfrak{S}}^I) = -(-1)^{n/2}\tilde{\mathfrak{S}}^- \in \mathbb{V}_{\widehat{\mathrm{H}}}^-(\Omega^{n/2})$$
$$\mathbb{V}(\Omega^{n/2}) = \mathbb{V}^+(\Omega^{n/2}) \oplus \mathbb{V}^-(\Omega^{n/2})$$

$$P_{\hat{\mathfrak{H}}}^{\pm} := \frac{1}{2}\left(1 \pm \hat{\mathfrak{H}}_{\blacksquare}\right)$$

$$P_{\hat{\mathfrak{H}}}^{\pm}\mathfrak{a} = \mathfrak{a}^{\pm}\ \hat{\mathfrak{H}}_{\blacksquare}(\mathfrak{a}^{\pm}) = \pm\mathfrak{a}^{\pm} \in \mathbb{V}_{\hat{\mathfrak{H}}}^{\pm}\left(\Omega^2(T^*\mathcal{M}_{\blacksquare})\right)$$

$$\hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^1\right) = \mathrm{e}_{\mathrm{E}}^2 \wedge \mathrm{e}_{\mathrm{E}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^2\right) = -\mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^3\right) = \mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^2,$$

$$\hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^2\right) = \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^3\right) = -\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^2,\quad \hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathrm{e}_{\mathrm{E}}^2 \wedge \mathrm{e}_{\mathrm{E}}^3\right) = \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^1$$

$$\mathfrak{S}_{\mathrm{E}}^{+a} := \left\{\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^1 + \mathrm{e}_{\mathrm{E}}^2 \wedge \mathrm{e}_{\mathrm{E}}^3, \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^2 - \mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^3, \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^3 + \mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^2\right\}$$

$$\mathfrak{S}_{\mathrm{E}}^{-a} := \left\{\mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^1 - \mathrm{e}_{\mathrm{E}}^2 \wedge \mathrm{e}_{\mathrm{E}}^3, \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^2 + \mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^3, \mathrm{e}_{\mathrm{E}}^0 \wedge \mathrm{e}_{\mathrm{E}}^3 - \mathrm{e}_{\mathrm{E}}^1 \wedge \mathrm{e}_{\mathrm{E}}^2\right\}$$

$$\hat{\mathfrak{H}}_{\mathrm{E}}\left(\mathfrak{S}_{\mathrm{E}}^{\pm}\right) = \pm\mathfrak{S}_{\mathrm{E}}^{\pm} \in \mathbb{V}_{\hat{\mathfrak{H}}}^{\pm}, P_{\hat{\mathfrak{H}}}^{\pm}\left(\mathfrak{S}_{\mathrm{E}}^{\pm}\right) = \mathfrak{S}_{\mathrm{E}}^{\pm},\ P_{\hat{\mathfrak{H}}}^{\pm}\left(\mathfrak{S}_{\mathrm{E}}^{\mp}\right) = 0$$

$$\hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^1\right) = i\mathrm{e}_{\mathrm{L}}^2 \wedge \mathrm{e}_{\mathrm{L}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^2\right) = -i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^3\right) = i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^2$$

$$\hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^2\right) = -i\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^3,\quad \hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^3\right) = i\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^2,\quad \hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathrm{e}_{\mathrm{L}}^2 \wedge \mathrm{e}_{\mathrm{L}}^3\right) = -i\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^1$$

$$\mathfrak{S}_{\mathrm{L}}^{+a} := \left\{\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^1 + i\mathrm{e}_{\mathrm{L}}^2 \wedge \mathrm{e}_{\mathrm{L}}^3, \mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^2 - i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^3, \mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^3 + i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^2\right\}$$

$$\mathfrak{S}_{\mathrm{L}}^{-a} := \left\{\mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^1 - i\mathrm{e}_{\mathrm{L}}^2 \wedge \mathrm{e}_{\mathrm{L}}^3, \mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^2 + i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^3, \mathrm{e}_{\mathrm{L}}^0 \wedge \mathrm{e}_{\mathrm{L}}^3 - i\mathrm{e}_{\mathrm{L}}^1 \wedge \mathrm{e}_{\mathrm{L}}^2\right\}$$

$$\mathfrak{F}_{\mathrm{L}}^{\pm} := P_{\hat{\mathfrak{H}}}^{\pm}\mathfrak{F}_{\mathrm{L}},\ \hat{\mathfrak{H}}_{\mathrm{L}}\left(\mathfrak{F}_{\mathrm{L}}^{\pm}\right) = \pm\mathfrak{F}_{\mathrm{L}}^{\pm} \in \mathbb{V}_{\hat{\mathfrak{H}}}^{\pm}\left(\Omega_{\mathrm{L}}^2\right),$$

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

$$\mathfrak{so}(1{,}3) \otimes \mathbb{C} = \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$$

$$\mathfrak{so}(3) = \mathfrak{su}(2)$$

$$\mathfrak{so}(3) \otimes \mathbb{C} = \mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$$

$$\mathrm{Spin}(4) = SU(2) \times SU(2),\ \mathrm{Spin}(1{,}3) = SL(2,\mathbb{C})$$

$$\mathrm{Spin}(4) = SO(4) \times_{\mathbb{Z}_2} U(1),\ \mathrm{Spin}(1{,}3) = SO(1{,}3) \times_{\mathbb{Z}_2} U(1)$$

$$S^3 := \{z_1, z_2 \in \mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}$$

$$\boldsymbol{M} := \begin{pmatrix} z_1 & -z_2 \\ z_2^* & z_1^* \end{pmatrix} \det[\boldsymbol{M}] = 1, \boldsymbol{M}^{\dagger} \cdot \boldsymbol{M} = \mathbf{1}_2$$

$$\mathbb{V}_s^{\mathrm{E}} \ni \lambda_s(z_1, z_2) = \sum_{0 \le k \le 2s} c_k z_1^k z_2^{2s-k},\ c_k \in \mathbb{R}, \{z_1, z_2\} \in S^3$$

$$(\pi_s(\boldsymbol{g}_{\mathrm{E}}) \circ \lambda_s)(z_1, z_2) = \lambda_s(z_1', z_2')\ \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \boldsymbol{g}_{\mathrm{E}}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\dot{\pi}_s(\mathfrak{g}_{\mathrm{E}}) \circ \lambda_s := \frac{d}{dt}\lambda_s\left(e^{t\mathfrak{g}}z_1, e^{t\mathfrak{g}}z_2\right)\Big|_{t=0}$$

$$\langle\lambda_s', \lambda_s\rangle := \frac{1}{\pi^2}\int_{\mathbb{C}^2} \lambda_s'^* \lambda_s e^{-(|z_1|^2 + |z_2|^2)} dx_1 dy_1 dx_2 dy_2, z_a := x_a + iy_a < \infty$$

$$\|\lambda_s\| := \langle\lambda_s, \lambda_s\rangle^{1/2}$$

$$c_k = (k!\,(2s-k)!)^{-1/2}$$

$$\lambda_{1/2}(z_1, z_2) = \tilde{\boldsymbol{c}}^t \cdot \boldsymbol{z} = (\tilde{c}_1, \tilde{c}_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tilde{c}_1 z_1 + \tilde{c}_2 z_2$$

$$\left(\pi_{1/2}(\boldsymbol{g}_{\mathrm{E}}) \circ \lambda_{1/2}\right)(z_1, z_2) = \lambda_{1/2}(z_1', z_2') = \tilde{\boldsymbol{c}}^t \cdot \left(\boldsymbol{g}_{\mathrm{E}}^{\dagger} \cdot \boldsymbol{z}\right) = (\boldsymbol{g}_{\mathrm{E}}\tilde{\boldsymbol{c}})^{\dagger} \cdot \boldsymbol{z}$$

$$\pi_{s_1 \otimes s_2}^{\mathrm{E}} := \left(\pi_{s_1}(\boldsymbol{g}_{\mathrm{E}}) \otimes \pi_{s_2}(\boldsymbol{g}_{\mathrm{E}}), \mathbb{V}_{s_1}^{\mathrm{E}} \otimes \mathbb{V}_{s_2}^{\mathrm{E}}\right)\ ,\ s_1, s_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

$$\pi_{1/2}(\boldsymbol{g}_{\mathrm{L}}) = \boldsymbol{g}_{\mathrm{L}} = (\boldsymbol{g}_{\mathrm{L}}^t)^{-1}\ ,\ \pi_{\overline{1/2}}(\boldsymbol{g}_{\mathrm{L}}) = \boldsymbol{g}_{\mathrm{L}}^* = \left(\boldsymbol{g}_{\mathrm{L}}^{\dagger}\right)^{-1}$$

$$\dot{\pi}_{s_1 \otimes \bar{s}_2}^{\mathrm{L}} := \dot{\pi}_{s_1}(\mathfrak{g}_{\mathrm{L}}) \otimes \dot{\pi}_{s_2}(\mathfrak{g}_{\mathrm{L}}) \implies \pi_{s_1 \otimes \bar{s}_2}^{\mathrm{L}} := \left(\pi_{s_1}(\boldsymbol{g}_{\mathrm{L}}) \otimes \pi_{\bar{s}_2}(\boldsymbol{g}_{\mathrm{L}}), \mathbb{V}_{s_1}^{\mathrm{L}} \otimes \mathbb{V}_{\bar{s}_2}^{\mathrm{L}}\right)$$

$$\mathbb{V}_{1/2,1/2}^{\mathrm{L}} := \mathbb{V}_{1/2}^{\mathrm{L}} \otimes \mathbb{V}_{1/2}^{\mathrm{L}}$$

$$\zeta, \chi \in \mathcal{C}\ell(V_n), \zeta\chi + \chi\zeta = 2\langle\zeta, \chi\rangle_{\mathcal{C}\ell}$$

$$\langle\zeta, \chi\rangle_{\mathcal{C}\ell} := \boldsymbol{\zeta}^t \cdot \boldsymbol{\eta} \cdot \boldsymbol{\chi} = \eta_{\circ\circ}\zeta^{\circ}\chi^{\circ}$$

$$\gamma: V_n \to \mathcal{C}\ell(V_n): e^a \mapsto \gamma^a, a = 0, 1, \cdots, n-1$$

$$\mathrm{Spin}(V_n) := \left\{\gamma \in \mathcal{C}\ell_0(V_n) \mid {}_{\gamma}\gamma = \pm\mathbf{1}_{sp}, \gamma\boldsymbol{v}\overset{*}{\gamma} \in V_n, \forall\boldsymbol{v} \in V_n\right\}$$

$$\tau(\gamma)(\boldsymbol{v}) := \gamma\boldsymbol{v}\overset{*}{\gamma}$$

$$\langle\tau(\gamma)(\boldsymbol{v}), \tau(\gamma)(\boldsymbol{u})\rangle = \left\langle\gamma\boldsymbol{v}^{\,*}, \gamma\boldsymbol{u}^{\,*}_{\,\gamma}\right\rangle = \left(\mathrm{sign}[\gamma\overset{*}{\gamma}]\right)^2\langle\boldsymbol{v}, \boldsymbol{u}\rangle = \langle\boldsymbol{v}, \boldsymbol{u}\rangle.$$

$$\mathrm{Spin}(V_n) \ni \Gamma_s := \sqrt{\frac{1}{\det[\boldsymbol{\eta}]}}\gamma^0\gamma^1\gamma^2\gamma^3 \ \det[\boldsymbol{\eta}] = \begin{cases} +1 & \boldsymbol{\eta} = \boldsymbol{\eta}_{\mathrm{E}} \\ -1 & \boldsymbol{\eta} = \boldsymbol{\eta}_{\mathrm{L}} \end{cases}$$

$$|\frac{\partial \Gamma}{\partial \zeta}\bigg|_{\mathrm{L}\to\mathrm{E}} \Longrightarrow \Gamma_s\gamma^a = -\gamma^a\Gamma_s, \ {}^{\forall}a \in \{0,1,2,3\}$$

$$P_s^{\pm} := \frac{1}{2}\left(\mathbf{1}_{sp} \pm \Gamma_s\right) \in (\mathbb{V}_s), P_s^{\pm}P_s^{\pm} = P_s^{\pm}, P_s^{\pm}P_s^{\mp} = 0$$

$$\mathbb{V}_s = \mathbb{V}_s^+ \oplus \mathbb{V}_s^-$$

$$\xi^{\pm} := P_s^{\pm}\xi \in \mathbb{V}_s^{\pm}, \mathbb{V}_s^{\pm} := \mathrm{Im}\left(P_s^{\pm}\right) = \mathrm{Ker}\left(P_s^{\mp}\right)$$

$$\Gamma_s : \mathrm{End}\left(\mathbb{V}_s^{\pm}\right) : \xi^{\pm} \mapsto \Gamma_s\xi^{\pm} = \pm\xi^{\pm}$$

$$\gamma_{\mathrm{SE}} = \left\{\gamma_{\mathrm{SE}}^0, \gamma_{\mathrm{SE}}^1, \gamma_{\mathrm{SE}}^2, \gamma_{\mathrm{SE}}^3\right\}^t = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\right\}^t$$

$$i^2 = j^2 = k^2 = -1, ij + ji = jk + kj = ki + ik = 0, \ , \ ij = k$$

$$\Gamma_{\mathrm{SE}} = \sqrt{\frac{1}{\det[\boldsymbol{\eta}_{\mathrm{E}}]}}\gamma_{\mathrm{SE}}^0\gamma_{\mathrm{SE}}^1\gamma_{\mathrm{SE}}^2\gamma_{\mathrm{SE}}^3 = \gamma_{\mathrm{SE}}^0\gamma_{\mathrm{SE}}^1\gamma_{\mathrm{SE}}^2\gamma_{\mathrm{SE}}^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_{\mathrm{SE}}^+ := \frac{1 + \Gamma_{\mathrm{SE}}}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, P_{\mathrm{SE}}^- := \frac{1 - \Gamma_{\mathrm{SE}}}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Sigma_{\mathrm{E}} : Sp(1) \to SU(2) : \{1, i, j, k\} \mapsto \boldsymbol{\sigma}_{\mathrm{E}} := \{\mathbf{1}_2, -i\boldsymbol{\sigma}\}$$

$$\Sigma_{\mathrm{E}} : \boldsymbol{\gamma}_{\mathrm{SE}} \mapsto \boldsymbol{\gamma}_{\mathrm{VE}} = \left(\gamma_{\mathrm{VE}}^0, \gamma_{\mathrm{VE}}^1, \gamma_{\mathrm{VE}}^2, \gamma_{\mathrm{VE}}^3\right)^t$$

$$\gamma_{\mathrm{VE}}^a\gamma_{\mathrm{VE}}^{a\dagger} = \mathbf{1}_2, \det[\gamma_{\mathrm{VE}}^a] = 1 \Longrightarrow \gamma_{\mathrm{VE}}^a \in SU(2) \text{ for } a = 1, \cdots, 4$$

$$\Sigma_{\mathrm{E}} : P_{\mathrm{SE}}^{\pm} \mapsto P_{\mathrm{VE}}^{\pm} = P_{\mathrm{SE}}^{\pm}\big|_{0\to\mathbf{0}_2, 1\to\mathbf{1}_2}$$

$$\Sigma_{\mathrm{L}}^{\pm} : Sp(1) \to \mathcal{H}(2) \cap SL(2, \mathbb{C}) : \{1, i, j, k\} \mapsto \boldsymbol{\sigma}_{\mathrm{L}}^{\pm} := \{\mathbf{1}_2, \pm\boldsymbol{\sigma}\}$$

$$\boldsymbol{\gamma}_{\mathrm{VL}} = \left(\gamma_{\mathrm{VL}}^0, \gamma_{\mathrm{VL}}^1, \gamma_{\mathrm{VL}}^2, \gamma_{\mathrm{VL}}^3\right)^t = \left(\begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}^1 \\ -\boldsymbol{\sigma}^1 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}^2 \\ -\boldsymbol{\sigma}^2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}^3 \\ -\boldsymbol{\sigma}^3 & \mathbf{0}_2 \end{pmatrix}\right)^t$$

$$\gamma_{\mathrm{VE}}^a = \gamma_{\mathrm{VE}}^{a\dagger}, \det[\gamma_{\mathrm{VL}}^a] = 1 \Longrightarrow \gamma_{\mathrm{VL}}^a \in \mathcal{H}(2) \cap SL(2, \mathbb{C}), a = 1, \cdots, 4$$

$$\Gamma_{\mathrm{VL}} = \sqrt{\frac{1}{\det[\boldsymbol{\eta}_{\mathrm{L}}]}}\gamma_{\mathrm{VL}}^0\gamma_{\mathrm{VL}}^1\gamma_{\mathrm{VL}}^2\gamma_{\mathrm{VL}}^3 = i\gamma_{\mathrm{VL}}^0\gamma_{\mathrm{VL}}^1\gamma_{\mathrm{VL}}^2\gamma_{\mathrm{VL}}^3 = \begin{pmatrix} -\mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}$$

$$\Sigma_{\mathrm{L}}^{\pm} : P_{\mathrm{SE}}^{\pm} \mapsto P_{\mathrm{VE}}^{\pm} = P_{\mathrm{SE}}^{\pm}\big|_{0\to\mathbf{0}_2, 1\to\mathbf{1}_2}$$

$$P_{\mathrm{VL}}^{\pm} = P_{\mathrm{VE}}^{\pm}$$

$$\alpha_i \in \mathbb{C}, \beta_i \in \mathbb{H} \longrightarrow (\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = \alpha_1\alpha_2 \otimes \beta_1\beta_2$$

$$\alpha \in \mathbb{C}, \beta \in \mathbb{H}(2) \longrightarrow (\alpha \otimes \beta)^{\dagger} = \alpha^* \otimes \beta^{\dagger}$$

$$(\sqrt{-1} \otimes 1) \times (1 \otimes i) = \sqrt{-1} \otimes i \neq -1 \overset{s_{\blacksquare}h_{\blacksquare}n_{\blacksquare}}{\rightarrow} \sqrt{-1}i \neq -1$$

$$\Gamma_{\mathrm{SL}} = \sqrt{-1}\gamma_{\mathrm{SL}}^0\gamma_{\mathrm{SL}}^1\gamma_{\mathrm{SL}}^2\gamma_{\mathrm{SL}}^3 = \Gamma_{\mathrm{SE}} = -\bar{\Gamma}_{\mathrm{SL}}$$

$$P_{\mathrm{SL}}^+ = \bar{P}_{\mathrm{SL}}^- = P_{\mathrm{SE}}^+ \ , \ P_{\mathrm{SL}}^- = \bar{P}_{\mathrm{SL}}^+ = P_{\mathrm{SE}}^-$$

$$(\gamma_{\blacksquare}{}_{\mathrm{L}}^{a})^{\dagger} = \gamma_{\blacksquare}{}_{\mathrm{L}}^{0}\gamma_{\blacksquare}{}_{\mathrm{L}}^{a}\gamma_{\blacksquare}{}_{\mathrm{L}}^{0}, a \in \{0,1,2,3\}$$

$$\lambda_{1/2}(\phi_{\mathrm{U}}, \phi_{\mathrm{D}}) := \phi_{\mathrm{W}} = \begin{pmatrix} \phi_{\mathrm{U}} \\ \phi_{\mathrm{D}} \end{pmatrix} \in \mathbb{V}_{1/2}^{\mathrm{E}}, \phi_{\mathrm{U}}, \phi_{\mathrm{D}} \in \mathbb{C}$$

$$\Gamma_{\mathrm{SE}}\boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{U}} = -\boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{U}} \in \mathbb{V}_{1/2}^{\mathrm{EU}} \Longrightarrow P_{\mathrm{SE}}^-\boldsymbol{\phi}_{\mathrm{w}} = \boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{U}} = \begin{pmatrix} \phi_{\mathrm{U}} \\ 0 \end{pmatrix}$$

$$\Gamma_{\mathrm{SE}}\boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{D}} = +\boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{D}} \in \mathbb{V}_{1/2}^{\mathrm{ED}} \Longrightarrow P_{\mathrm{SE}}^+\boldsymbol{\phi}_{\mathrm{W}} = \boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{D}} = \begin{pmatrix} 0 \\ \phi_{\mathrm{D}} \end{pmatrix}$$

$$\boldsymbol{\phi}_{\mathrm{H}} \in \left\{\boldsymbol{\phi}_{\mathrm{W}}^{\mathrm{U}}, \boldsymbol{\phi}_{\mathrm{w}}^{\mathrm{D}}\right\} \subset \mathbb{V}_{1/2}^{\mathrm{E}} = \mathbb{V}_{1/2}^{\mathrm{EU}} \oplus \mathbb{V}_{1/2}^{\mathrm{ED}}$$

$$\hat{\phi}_{\mathrm{W}}^{\mathrm{U}} := \phi_{\mathrm{W}}^{\mathrm{U}}/\phi_{\mathrm{U}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \hat{\phi}_{\mathrm{w}}^{\mathrm{D}} := \phi_{\mathrm{w}}^{\mathrm{D}}/\phi_{\mathrm{D}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\phi'_A = \left[\boldsymbol{g}_{\mathrm{E}}^{\dagger}\right]_A^B\phi_B$$

$$\pi_{1/2}(\boldsymbol{g}_r) = \boldsymbol{g}_r^a(\varphi) := e^{-i(\varphi/2)\sigma^a} = \begin{cases} \begin{pmatrix} \cos\varphi/2 & -i\sin\varphi/2 \\ -i\sin\varphi/2 & \cos\varphi/2 \end{pmatrix} & a=1 \\ \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix} & a=2 \\ \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} & a=3 \end{cases} \in S(2),$$

$$\langle \phi_{\mathrm{w}}, \varphi_{\mathrm{w}} \rangle := \phi_A^* \delta^{AB} \varphi_B = (\phi^A)^* \varphi_A = \phi_{\mathrm{w}}^\dagger \cdot \boldsymbol{\varphi}_{\mathrm{w}},\ \delta^{AB} = [\mathbf{1}_2]_B^A$$

$$Spin(4) = SU_R(2) \times SU_L(2) = SO(4) \otimes \{R, L\}$$

$$\lambda_{1/2}(\xi_{\mathrm{U}}, \xi_{\mathrm{D}}) := \boldsymbol{\xi} = \begin{pmatrix} \xi_{\mathrm{U}} \\ \xi_{\mathrm{D}} \end{pmatrix} \in \mathbb{V}_{1/2}^{\mathrm{L}}, \xi_{\mathrm{U}}, \xi_{\mathrm{D}} \in \mathbb{C}$$

$$\lambda_{1/2}(\dot{\xi}_{\dot{\mathrm{U}}}, \dot{\xi}_{\dot{\mathrm{D}}}) := \dot{\boldsymbol{\xi}} = \begin{pmatrix} \dot{\xi}_{\dot{\mathrm{U}}} \\ \dot{\xi}_{\dot{\mathrm{D}}} \end{pmatrix} \in \mathbb{V}\frac{\mathrm{L}}{1/2}, \dot{\xi}_{\dot{\mathrm{U}}}, \dot{\xi}_{\dot{\mathrm{D}}} \in \mathbb{C}$$

$$\boldsymbol{\xi}' = \pi_{1/2}^{\mathrm{L}}(\boldsymbol{g}_{\mathrm{L}})\boldsymbol{\xi} = \boldsymbol{g}_{\mathrm{L}}^\dagger(\varphi)\boldsymbol{\xi}, [\boldsymbol{\xi}']_A = [\boldsymbol{g}_{\mathrm{L}}(\varphi)]_A^B [\boldsymbol{\xi}]_B$$

$$\dot{\boldsymbol{\xi}}' = \pi\frac{\mathrm{L}}{1/2}(\boldsymbol{g}_{\mathrm{L}})\dot{\boldsymbol{\xi}} = \boldsymbol{g}_{\mathrm{L}}^*(\varphi)\dot{\boldsymbol{\xi}}, [\dot{\boldsymbol{\xi}}']_{\dot{A}} = [\boldsymbol{g}_{\mathrm{L}}^*(\varphi)]_{\dot{A}}^{\dot{B}} [\dot{\boldsymbol{\xi}}]_{\dot{B}}$$

$$\xi^A = \epsilon_2{}^{AB} \xi_B, \dot{\xi}^{\dot{A}} = \epsilon_2^{\dot{A}\dot{B}} \dot{\xi}_{\dot{B}}$$

$$[\epsilon_2]^{AB} = [\epsilon_2]^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -[\epsilon_2]_{AB} = -[\epsilon_2]_{\dot{A}\dot{B}}$$

$$\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle := \xi_A \epsilon_2^{AB} \zeta_B = \xi_{\mathrm{U}}\zeta_{\mathrm{D}} - \xi_{\mathrm{D}}\zeta_{\mathrm{U}}, \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\zeta}} \rangle := \dot{\xi}_{\dot{A}} \epsilon_2^{\dot{A}\dot{B}} \dot{\zeta}_{\dot{B}} = \dot{\xi}_{\dot{\mathrm{U}}}\dot{\zeta}_{\dot{\mathrm{D}}} - \dot{\xi}_{\dot{\mathrm{D}}}\dot{\zeta}_{\dot{\mathrm{U}}}$$

$$\langle \boldsymbol{\xi}', \boldsymbol{\zeta}' \rangle = \langle \boldsymbol{g}_{\mathrm{L}}\boldsymbol{\xi}, \boldsymbol{g}_{\mathrm{L}}\boldsymbol{\zeta} \rangle = [\boldsymbol{g}_{\mathrm{L}}\boldsymbol{\xi}]_A \epsilon_2{}^{AB} [\boldsymbol{g}_{\mathrm{L}}\boldsymbol{\zeta}]_B = \xi_A [\boldsymbol{g}_{\mathrm{L}}^t]_C^A \epsilon_2^{CD} [\boldsymbol{g}_{\mathrm{L}}]_D^B \zeta_B = \xi^A \zeta_A = \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle$$

$$[\boldsymbol{g}_{\mathrm{L}}^t]_C^A \epsilon_2^{CD} [\boldsymbol{g}_{\mathrm{L}}]_D^B = \epsilon_2{}^{AB}$$

$$[\boldsymbol{\xi}^T]^B := \xi_A \epsilon_2^{AB} = \xi^B, [\dot{\boldsymbol{\xi}}^T]^{\dot{B}} := \dot{\xi}_{\dot{A}} \epsilon_2{}^{\dot{A}\dot{B}} = \dot{\xi}^{\dot{B}} \Longrightarrow \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle = \boldsymbol{\xi}^T \cdot \boldsymbol{\zeta}, \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\zeta}} \rangle = \dot{\boldsymbol{\xi}}^T \cdot \dot{\boldsymbol{\zeta}}$$

$$\pi_{1/2}(\boldsymbol{g}_b) = \boldsymbol{g}_b^a(\chi) := e^{(\chi/2)\sigma^a} = \begin{cases} \begin{pmatrix} \cosh\chi/2 & \sinh\chi/2 \\ \sinh\chi/2 & \cosh\chi/2 \end{pmatrix} & a=1 \\ \begin{pmatrix} \cosh\chi/2 & -i\sinh\chi/2 \\ i\sinh\chi/2 & \cosh\chi/2 \end{pmatrix} & a=2 \\ \begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix} & a=3 \end{cases} \in \mathcal{H}(2)$$

$$\mathbb{V}_{1/2}^{\mathrm{L}} = \mathbb{V}_{1/2}^{\mathrm{UL}} \oplus \mathbb{V}_{1/2}^{\mathrm{LD}}$$

$$\Gamma_{\mathrm{SL}}\boldsymbol{\xi}_{\mathrm{U}} = -\boldsymbol{\xi}_{\mathrm{U}} \in \mathbb{V}_{1/2}^{\mathrm{LR}} \Longrightarrow P_{\mathrm{SL}}^- \boldsymbol{\xi} = \boldsymbol{\xi}_{\mathrm{U}} = \begin{pmatrix} \xi_{\mathrm{U}} \\ 0 \end{pmatrix}$$

$$\Gamma_{\mathrm{SL}}\boldsymbol{\xi}_{\mathrm{D}} = +\boldsymbol{\xi}_{\mathrm{D}} \in \mathbb{V}_{1/2}^{\mathrm{LD}} \Longrightarrow P_{\mathrm{SL}}^+ \boldsymbol{\xi} = \boldsymbol{\xi}_{\mathrm{D}} = \begin{pmatrix} 0 \\ \xi_{\mathrm{D}} \end{pmatrix}$$

$$\boldsymbol{\xi} \in \{\boldsymbol{\xi}_{\mathrm{U}}, \boldsymbol{\xi}_{\mathrm{D}}\} \subset \mathbb{V}_{1/2}^{\mathrm{L}} = \mathbb{V}_{1/2}^{\mathrm{LU}} \oplus \mathbb{V}_{1/2}^{\mathrm{LD}}$$

$$\bar{\Gamma}_{\mathrm{SL}}\dot{\boldsymbol{\xi}}_{\mathrm{U}} = +\dot{\boldsymbol{\xi}}_{\mathrm{U}} \in \mathbb{V}\frac{\mathrm{LU}}{1/2} \Longrightarrow \bar{P}_{\mathrm{SL}}^+ \dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}_{\mathrm{U}} = \begin{pmatrix} \dot{\xi}_{\dot{\mathrm{U}}} \\ 0 \end{pmatrix}$$

$$\bar{\Gamma}_{\mathrm{SL}}\dot{\boldsymbol{\xi}}_{\mathrm{D}} = -\dot{\boldsymbol{\xi}}_{\mathrm{D}} \in \mathbb{V}\frac{\mathrm{LD}}{1/2} \Longrightarrow \bar{P}_{\mathrm{SL}}^- \dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}_{\mathrm{D}} = \begin{pmatrix} 0 \\ \dot{\xi}_{\dot{\mathrm{D}}} \end{pmatrix}$$

$$\dot{\boldsymbol{\xi}} \in \{\dot{\boldsymbol{\xi}}_{\mathrm{U}}, \dot{\boldsymbol{\xi}}_{\mathrm{D}}\} \subset \mathbb{V}_{\frac{\mathrm{L}}{1/2}} = \mathbb{V}\frac{\mathrm{LU}}{1/2} \oplus \mathbb{V}_{1/2}^{\mathrm{LD}}$$

$$\mathfrak{g}_{Spin(1,3)} = \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} = \mathfrak{so}(1,3) \otimes \{R, L\}$$

$$\psi := \begin{pmatrix} \dot{\boldsymbol{\xi}} \\ \boldsymbol{\xi} \end{pmatrix} \in \mathbb{V}_{1/2 \otimes \overline{1/2}}^{\mathrm{L}} := \mathbb{V}_{1/2}^{\mathrm{L}} \otimes \mathbb{V}_{1/2}^{\mathrm{L}},\ \boldsymbol{\xi} \in \mathbb{V}_{1/2}^{\mathrm{L}}, \dot{\boldsymbol{\xi}} \in \mathbb{V}_{1/2}^{\mathrm{L}}$$

$$\psi^L := P_{\mathrm{VL}}^- \psi = \begin{pmatrix} \dot{\boldsymbol{\xi}} \\ \mathbf{0}_{\mathrm{SL}} \end{pmatrix}, \Gamma_{\mathrm{VL}}\psi^L = -\psi^L$$

$$\psi^R := P_{\mathrm{VL}}^+ \psi = \begin{pmatrix} \dot{\mathbf{0}}_{\mathrm{SL}} \\ \boldsymbol{\xi} \end{pmatrix}, \Gamma_{\mathrm{VL}}\psi^R = +\psi^R$$

$$\Sigma_{\mathrm{E}}: S^3 \to SU(2): \hat{\boldsymbol{v}}_{\mathrm{VE}} \mapsto \hat{\boldsymbol{v}}_{\mathrm{SE}} := \boldsymbol{\sigma}_{\mathrm{E}}^t \cdot \hat{\boldsymbol{v}}_{\mathrm{VE}} = \begin{pmatrix} \hat{v}^0 - i\hat{v}^3 & -i\hat{v}^1 - \hat{v}^2 \\ -i\hat{v}^1 + \hat{v}^2 & \hat{v}^0 + i\hat{v}^3 \end{pmatrix}$$

$$\mathbb{V}^{\mathrm{E}}_{1/2,1/2} \ni \begin{pmatrix}\phi_{\mathrm{U}}\\ \phi_{\mathrm{D}}\end{pmatrix} \otimes \begin{pmatrix}\phi_{\mathrm{U}}\\ \phi_{\mathrm{D}}\end{pmatrix}^{*} \mapsto \begin{pmatrix}\phi_{\mathrm{U}} & -\phi_{\mathrm{D}}^{*}\\ \phi_{\mathrm{D}} & \phi_{\mathrm{U}}^{*}\end{pmatrix}$$

$$\langle \hat{\boldsymbol{u}}_{\mathrm{SE}}, \hat{\boldsymbol{v}}_{\mathrm{SE}}\rangle := \frac{1}{2}\left(\mathrm{Tr}[\hat{\boldsymbol{u}}_{\mathrm{SE}}]\mathrm{Tr}[\hat{\boldsymbol{v}}_{\mathrm{SE}}] - \mathrm{Tr}[\hat{\boldsymbol{u}}_{\mathrm{SE}}\hat{\boldsymbol{v}}_{\mathrm{SE}}]\right) = \hat{u}^0\hat{v}^0 + \hat{u}^1\hat{v}^1 + \hat{u}^2\hat{v}^2 + \hat{u}^3\hat{v}^3$$

$$\|\hat{\boldsymbol{u}}_{\mathrm{SE}}\|^2 ::= \langle \hat{\boldsymbol{u}}_{\mathrm{SE}}, \hat{\boldsymbol{u}}_{\mathrm{SE}}\rangle = \det[\hat{\boldsymbol{u}}_{\mathrm{SE}}] = 1$$

$$\boldsymbol{u}_{\mathrm{SE}} = \|\boldsymbol{u}_{\mathrm{SE}}\|\hat{\boldsymbol{v}}_{\mathrm{SE}},\ , \|\boldsymbol{u}_{\mathrm{SE}}\|^2 > 0$$

$$\Sigma_{\mathrm{E}}: O(4) \to SU(2) \otimes \mathbb{R}: \boldsymbol{v}_{\mathrm{VE}} \mapsto \boldsymbol{v}_{\mathrm{SE}}$$

$$\hat{\boldsymbol{v}}'_{\mathrm{SE}} = \tau(\boldsymbol{g}_r^a(\varphi))(\hat{\boldsymbol{v}}_{\mathrm{SE}}) = \boldsymbol{\sigma}_{\mathrm{E}}^{t} \cdot \begin{cases} \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1 \\ (\hat{v}^2\cos\,\varphi - \hat{v}^3\sin\,\varphi \\ \hat{v}^2\sin\,\varphi + \hat{v}^3\cos\,\varphi \end{pmatrix} & a = 1 \\ \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1\cos\,\varphi + \hat{v}^3\sin\,\varphi \\ \hat{v}^2 \\ -\hat{v}^1\sin\,\varphi + \hat{v}^3\cos\,\varphi \end{pmatrix} & a = 2 \\ \begin{pmatrix} \hat{v}^0 \\ \hat{v}^1\cos\,\varphi - \hat{v}^2\sin\,\varphi \\ \hat{v}^1\sin\,\varphi + \hat{v}^2\cos\,\varphi \\ \hat{v}^3 \end{pmatrix} & a = 3 \end{cases}$$

$$\tau(\boldsymbol{g}_r)(\boldsymbol{v}_{\mathrm{SE}}) := \boldsymbol{g}_r\boldsymbol{v}_{\mathrm{SE}}\boldsymbol{g}_r^{\dagger}$$

$$g: \mathbb{R} \to G: \varphi \mapsto g(\varphi), g(\vartheta + \varphi) = g(\vartheta)g(\varphi)\,, g(0) = \mathbf{1}_{\mathrm{id}}$$

$$\Sigma_{\mathrm{L}}^{\pm}: \mathrm{Hyp}^3 \to \mathcal{H}(2) \cap (SL(2,\mathbb{C}) \otimes \{-1,1\}): \hat{\boldsymbol{u}}_{\mathrm{VL}} \mapsto \hat{\boldsymbol{u}}_{\mathrm{SL}}^{\pm} := \left(\boldsymbol{\sigma}_{\mathrm{L}}^{\pm}\right)^{t} \cdot \hat{\boldsymbol{u}}_{\mathrm{VL}} = \begin{pmatrix} \hat{u}^0 \pm \hat{u}^3 & \pm\hat{u}^1 \mp i\hat{u}^2 \\ \pm\hat{u}^1 \pm i\hat{u}^2 & \hat{u}^0 \mp \hat{u}^3 \end{pmatrix}$$

$$\mathbb{V}^{\mathrm{L}}_{1/2,\overline{1/2}} \ni \begin{pmatrix}\xi_{\mathrm{U}}\\ \xi_{\mathrm{D}}\end{pmatrix} \otimes \begin{pmatrix}\xi_{\mathrm{U}}\\ \xi_{\mathrm{D}}\end{pmatrix}^{*} \mapsto \begin{pmatrix}(\xi_{\mathrm{U}} + \xi_{\mathrm{U}}^{*})/2 & \xi_{\mathrm{D}}^{*}\\ \xi_{\mathrm{D}} & i(\xi_{\mathrm{U}}^{*} - \xi_{\mathrm{U}})/2\end{pmatrix}.$$

$$\langle \hat{\boldsymbol{u}}_{\mathrm{SL}}, \hat{\boldsymbol{v}}_{\mathrm{SL}}\rangle := \frac{1}{2}\left(\mathrm{Tr}[\hat{\boldsymbol{u}}_{\mathrm{SL}}]\mathrm{Tr}[\hat{\boldsymbol{v}}_{\mathrm{SL}}] - \mathrm{Tr}[\hat{\boldsymbol{u}}_{\mathrm{SL}}\hat{\boldsymbol{v}}_{\mathrm{SL}}]\right) = \hat{v}^0\hat{v}^0 - \hat{u}^1\hat{v}^1 - \hat{u}^2\hat{v}^2 - \hat{u}^3\hat{v}^3$$

$$\|\hat{\boldsymbol{u}}_{\mathrm{SL}}\|^2 := \langle \hat{\boldsymbol{u}}_{\mathrm{SL}}, \hat{\boldsymbol{u}}_{\mathrm{SL}}\rangle = (\hat{u}^0)^2 - (\hat{u}^1)^2 - (\hat{u}^2)^2 - (\hat{u}^3)^2 = \det[\hat{\boldsymbol{u}}_{\mathrm{SL}}]$$

$$\Sigma_{\mathrm{L}}^{\pm}: O(1,3) \otimes \mathbb{R} \to \mathcal{H}(2) \otimes \mathbb{R}: \boldsymbol{u}_{\mathrm{VL}} \mapsto \boldsymbol{u}_{\mathrm{SL}}$$

$$\boldsymbol{u}_{\mathrm{SL}}^{\pm\prime} = \begin{cases} \tau(\boldsymbol{g}_b^a(\chi))(\boldsymbol{u}_{\mathrm{L}}^{+}) \\ \tau(\boldsymbol{g}_b^{a**}(\chi))(\boldsymbol{u}_{\mathrm{SL}})' \end{cases} \boldsymbol{\sigma}_{\mathrm{L}}^{\pm t} \cdot \begin{cases} \begin{pmatrix} u^0\cosh\,\chi + u^1\sinh\,\chi \\ u^0\sinh\,\chi + u^1\cosh\,\chi \\ u^2 \\ u^3 \end{pmatrix} & a = 1 \\ \begin{pmatrix} u^0\cosh\,\chi + u^2\sinh\,\chi \\ u^1 \\ u^0\sinh\,\chi + u^2\cosh\,\chi \\ u^3 \end{pmatrix} & a = 2 \\ \begin{pmatrix} u^0\cosh\,\chi + u^3\sinh\,\chi \\ u^1 \\ u^2 \\ u^0\sinh\,\chi + u^3\cosh\,\chi \end{pmatrix} & a = 3 \end{cases}$$

$$\tau(\boldsymbol{g}_b)(\boldsymbol{v}_{\mathrm{SL}}) := \boldsymbol{g}_b\boldsymbol{v}_{\mathrm{SL}}\boldsymbol{g}_b^{\dagger}$$

$$\boldsymbol{g}_L^{+}(\varphi,\chi) := \boldsymbol{g}_r(\varphi) \otimes \boldsymbol{g}_b(\chi)\,, \boldsymbol{g}_L^{-}(\varphi,\chi) := \boldsymbol{g}_r(\varphi) \otimes \boldsymbol{g}_b^{*}(\chi)$$

$$\boldsymbol{g}_r(\varphi) \circ \boldsymbol{g}_b^{(*)}(\chi) = \boldsymbol{g}_b^{(*)}(\chi) \circ \boldsymbol{g}_r(\varphi)$$

$$\Omega^2(T^*\mathcal{M}) \otimes V^2(T\mathcal{M}) \ni \mathfrak{R} = \frac{1}{2}\mathfrak{R}^{ab}(\partial_a \times \partial_b) = \frac{1}{4}R^{ab}_{cd}\left(\mathrm{e}^c \wedge \mathrm{e}^d\right)(\partial_a \times \partial_b)$$

$$R^{ab}{}_{cd} = \mathcal{W}^{ab}_{bd} + \frac{2}{3}\delta^{[a}_{[c}\eta^{b]\circ}\hat{R}_{d]\circ} + \frac{1}{2}\left(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c\right)R$$

$$\hat{R}_{ab} = R_{ab} - \frac{1}{4}\eta_{ab}R$$

$$\Omega^1(T^*\mathcal{M}_{\mathrm{G}}) \otimes V^1(T\mathcal{M}_{\mathrm{L}}) \ni \mathbb{e} = \mathbb{e}^a\partial_a = \mathcal{E}_\mu{}^a(x \in \mathcal{M}_{\mathrm{G}})dx^\mu\partial_a$$

$$g_{\mu\nu} = \eta_{\circ\circ}\mathcal{E}_\mu^\circ\mathcal{E}_\nu^\circ \Rightarrow \boxed{\mu} \otimes \boxed{\nu} = \boxed{}\boxed{} \oplus \begin{array}{c}\boxed{}\\\boxed{}\end{array} = \widehat{\boxed{}\boxed{}} \oplus \blacksquare \oplus \begin{array}{c}\boxed{}\\\boxed{}\end{array}$$

$$g_{\mu\nu}(x \in \mathcal{M}_{\mathrm{G}}) = \eta_{\mu\nu} + h_{\mu\nu}(x \in \mathcal{M}_{\mathrm{G}}),\ h_{\mu\nu}({}^\forall x \in \mathcal{M}_{\mathrm{G}}) \ll 1$$

$$\gamma_{\mu\nu}(x) := h_{\mu\nu}(x) - \frac{1}{\boldsymbol{\eta}\cdot\boldsymbol{\eta}}\eta_{\mu\nu}\left(h_{\rho\sigma}(x)\eta^{\rho\sigma}\right) \Longrightarrow \eta^{\mu\nu}\gamma_{\mu\nu}(x) = 0$$

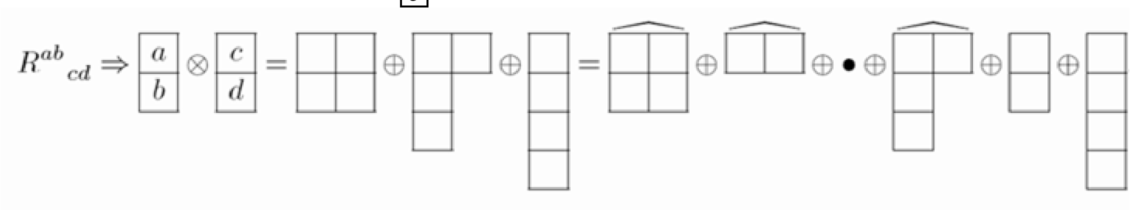
$$\Omega^1(T^*\mathcal{M}) \otimes V^2(T\mathcal{M}) \ni \mathfrak{w} = \frac{1}{2}\mathfrak{w}^{ab}(\partial_a \times \partial_b) = \frac{1}{2}\omega_c^{ab}\mathbb{e}^c(\partial_a \times \partial_b)$$

$$\mathbb{e}^a\partial_b = \mathcal{E}_\mu^a\mathcal{E}_b^\nu dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta_b^a$$

$$\boxed{a} \Rightarrow \omega^a := \omega_\circ^{a\circ},$$

$$\begin{array}{c}\boxed{\hat{b}}\\\boxed{a}\end{array}\boxed{\hat{b}} \Rightarrow \omega^{a\hat{b}} := \omega_{\hat{b}}{}^{a\hat{b}} - \frac{1}{3}\omega^a$$

$$\begin{array}{c}\boxed{a}\\\boxed{b}\\\boxed{c}\end{array} \Rightarrow \bar{\omega}_c^{ab} = 0$$



$$\left(\omega^{a\hat{b}} = 0\right) \wedge (\omega^a \neq 0) \wedge (\partial_\circ\omega^\circ = 0)$$

$$\not\partial : \mathbb{V}^\pm \to \mathbb{V}^\mp : \xi^\pm \mapsto \not\partial\xi^\pm \in \mathbb{V}^\mp$$

$$\not\partial := \iota_\gamma d : \Gamma(\mathcal{M},\Omega^0) \xrightarrow{d} \Gamma(\mathcal{M},\Omega^1) \xrightarrow{\iota_\gamma} \Gamma(\mathcal{M},\Omega^0)$$

$$\boldsymbol{\xi} \in \mathbb{V}_{1/2}^{\mathrm{L}} \to \not\boldsymbol{\xi} \in \mathbb{V}_{1/2}^{\mathrm{L}},\ \dot{\boldsymbol{\xi}} \in \mathbb{V}_{1/2}^{\mathrm{L}} \nrightarrow \dot{\boldsymbol{\xi}} \in \mathbb{V}_{1/2}^{\mathrm{L}}$$

$$\iota_\gamma \not\partial\phi_{\mathrm{SE}} = \not\partial_{\mathrm{SE}} := \gamma_{\mathrm{SE}}^\circ\partial_\circ = \begin{pmatrix} 0 & h_{\mathrm{E}}^\circ\partial_\circ \\ \bar{h}_{\mathrm{E}}^\circ\partial_\circ & 0 \end{pmatrix}$$

$$h_{\mathrm{E}}^a := (1,i,j,k),\ \bar{h}_{\mathrm{E}}^a := (1,-i,-j,-k)$$

$$P_s^\pm \not\partial_{\mathrm{SE}} P_s^\pm = 0,\ P_s^\mp \not\partial_{\mathrm{SE}} P_s^\pm = \not\partial_{\mathrm{SE}} P_s^\pm$$

$$d_{\mathrm{SL}}\boldsymbol{\xi} := \iota_\gamma d_{\mathrm{SL}}\boldsymbol{\xi} = \gamma_{\mathrm{SL}}^\circ(\partial_\circ\boldsymbol{\xi}) =: \not\partial_{\mathrm{SL}},\ \bar{\not\phi}_{\mathrm{SL}}\dot{\boldsymbol{\xi}} := \iota_{\bar{\gamma}}\bar{d}_{\mathrm{SL}}\dot{\boldsymbol{\xi}} = \bar{\gamma}_{\mathrm{SL}}^\circ\left(\partial_\circ\dot{\boldsymbol{\xi}}\right) =: \bar{\not\phi}_{\mathrm{SL}}\boldsymbol{\xi}$$

$$\boldsymbol{\partial}_{\mathrm{SL}} := \boldsymbol{\sigma}_{\mathrm{E}}{}^t \cdot \boldsymbol{\partial} = \mathbf{1}_2\partial_0 - i\boldsymbol{\sigma}^1\partial_1 - i\boldsymbol{\sigma}^2\partial_2 - i\sigma^3\boldsymbol{\partial}_3 = \begin{pmatrix} \partial_0 - i\partial_3 & -i\partial_1 - \partial_2 \\ -i\partial_1 + \partial_2 & \partial_0 + i\partial_3 \end{pmatrix}$$

$$\boldsymbol{\partial}_{\mathrm{SL}} := \boldsymbol{\sigma}_{\mathrm{L}}^{+t} \cdot \boldsymbol{\partial} = \mathbf{1}_2\partial_0 + \boldsymbol{\sigma}^1\partial_1 + \boldsymbol{\sigma}^2\partial_2 + \boldsymbol{\sigma}^3\partial_3 = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix}$$

$$\langle \boldsymbol{\partial}_{\mathrm{s}\cdot}, \boldsymbol{v}_{\mathrm{s}\cdot} \rangle = \partial_0 v^0 \pm \partial_1 v^1 \pm \partial_2 v^2 \pm \partial_3 v^3$$

$$\not d_{\mathrm{VL}}\psi = \iota_\gamma d\psi = \iota_\gamma\left(\left(\partial_j\psi\right)\mathbb{e}^j\right) = \gamma_{\mathrm{VL}}^\circ\partial_\circ\psi =: \not\partial_{\mathrm{VL}}\psi$$

$$\psi^\pm \in \mathbb{V}_{1/2\otimes\overline{1/2}}^{\mathrm{L}^\pm} \Longrightarrow \not\partial_{\mathrm{VL}}\psi^\pm = \not\partial_{\mathrm{VL}}\left(\frac{\mathbf{1}_{sp} \pm \Gamma_{\mathrm{VL}}}{2}\psi\right) = \frac{\mathbf{1}_{sp} \mp \Gamma_{\mathrm{VL}}}{2}(\not\partial_{\mathrm{VL}}\psi) \in \mathbb{V}_{1/2\otimes\overline{1/2}}^{\mathrm{L}^\mp}$$

$$\not\partial_{\mathrm{VL}}^2 = ((\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2)\mathbf{1}_{sp} =: \Delta_{\mathrm{VL}}$$

$$\not d_{\mathrm{VE}} = \iota_\gamma d = \iota_\gamma\left(\left(\partial_j\right)\mathbb{e}^j\right) = \gamma_{\mathrm{VE}}^\circ\partial_\circ =: \not\partial_{\mathrm{VE}}$$

$$\not\partial_{\mathrm{VE}}^2 = ((\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2)\mathbf{1}_{sp} =: \Delta_{\mathrm{VE}}$$

$$\phi_{\mathrm{SL}}\boldsymbol{\phi}_{\mathrm{H}} = \ddot{\partial}_{\mathrm{SL}}\boldsymbol{\phi}_{\mathrm{H}},\ \not\partial_{\mathrm{SL}}\left(P_{\mathrm{SE}}^\pm\boldsymbol{\phi}_{\mathrm{H}}\right) = P_{\mathrm{SE}}^\mp(\partial_{\mathrm{SL}}\boldsymbol{\phi}_{\mathrm{H}})$$

$$i\phi_{\mathrm{SL}} := i\phi_{\mathrm{SL}}^\dagger\langle i\phi_{\mathrm{SL}}\phi_{\mathrm{H}}, \phi_{\mathrm{H}}\rangle = \langle \phi_{\mathrm{H}}, i\phi_{\mathrm{SL}}\phi_{\mathrm{H}}\rangle$$

$$\int_{\mathcal{M}_{\mathrm{L}}} \langle i\delta_{\mathrm{SL}}\phi_{\mathrm{H}}, \phi_{\mathrm{H}}\rangle = -i\int_{\mathcal{M}_{\mathrm{L}}} \left(\gamma_{\mathrm{SL}}^{\circ}{}^{\dagger}\partial_{\circ}\phi_{\mathrm{H}}\right)^{\dagger}\phi_{\mathrm{H}} = -i\int_{\mathcal{M}_{\mathrm{L}}} (\partial_{\circ}\phi_{\mathrm{H}}^{\dagger})\gamma_{\mathrm{SL}}^{\circ}\phi_{\mathrm{H}}$$

$$= -i\int_{\mathcal{M}_{\mathrm{L}}} \partial_{\circ}\left(\phi_{\mathrm{H}}^{\dagger}\gamma_{\mathrm{SL}}^{\circ}\phi_{\mathrm{H}}\right) + i\int_{\mathcal{M}_{\mathrm{L}}} \phi_{\mathrm{H}}^{\dagger}\gamma_{\mathrm{SL}}^{\circ}(\partial_{\circ}\phi_{\mathrm{H}}),$$

$$= \int_{\mathcal{M}_{\mathrm{L}}} \langle \phi_{\mathrm{H}}, i\,\partial\!\!\!/_{\mathrm{SL}}\phi_{\mathrm{H}}\rangle$$

$$\phi_{\mathrm{SL}}\langle\phi_{\mathrm{H}}, \phi_{\mathrm{H}}\rangle := \langle\phi_{\mathrm{SL}}\phi_{\mathrm{H}}, \phi_{\mathrm{H}}\rangle + \langle\phi_{\mathrm{H}}, \phi_{\mathrm{SL}}\phi_{\mathrm{H}}\rangle = 2\langle\phi_{\mathrm{H}}, \phi_{\mathrm{SL}}\phi_{\mathrm{H}}\rangle$$

$$\left(\mathcal{M}\otimes\left(\mathbb{V}_{1/2}^{\mathrm{L}}\oplus\frac{\mathbb{V}_{1/2}^{\mathrm{L}}}{\mathrm{L}}\right), \pi_{\mathrm{sp}}, \mathcal{M}, G_{\mathrm{cP}}\otimes \mathrm{Spin}(1,3)\right)$$

$$\tau_{\mathrm{cov}} : S\mathrm{Spin}(1,3) \to SO(1,3)\otimes\{R,L\}$$

$$\pi_{\mathrm{sp}} := \tau_{cov}/\{R,L\} : \mathcal{M}\otimes\left(\mathbb{V}_{1/2}\oplus\mathbb{V}_{\overline{1/2}}\right) \to \mathcal{M}/\{R,L\} : \boldsymbol{\xi}|_p \mapsto p\in\mathcal{M}$$

$$\overline{\mathfrak{A}}_{sp} := \mathrm{Tr}_{\mathcal{C}}[\mathfrak{w}\bar{S}] = \frac{i}{2}\mathfrak{w}^{\circ\circ}\bar{S}_{\circ\circ}, \mathfrak{A}_{sp} := \mathrm{Tr}_{\mathcal{C}}[\mathfrak{w}S] = \frac{i}{2}\mathfrak{w}^{\circ\circ}S_{\circ\circ}$$

$$\bar{S}^{ab} := i\left[\frac{\bar{\gamma}^a}{2}, \frac{\bar{\gamma}^b}{2}\right], S^{ab} := i\left[\frac{\gamma^a}{2}, \frac{\gamma^b}{2}\right]$$

$$\bar{d}_{sp}\dot{\boldsymbol{\xi}} := d\dot{\boldsymbol{\xi}} - ic_{gr}\overline{\mathfrak{A}}_{sp}\dot{\boldsymbol{\xi}}, d_{sp}\boldsymbol{\xi} := d\boldsymbol{\xi} - ic_{gr}\mathfrak{A}_{sp}\boldsymbol{\xi}$$

$$\overline{\mathfrak{F}}_{sp} := d\overline{\mathfrak{A}}_{sp} + ic_{gr}\overline{\mathfrak{A}}_{sp}\wedge\overline{\mathfrak{A}}_{sp} = \frac{1}{2}\bar{S}_{\circ\circ}\mathfrak{R}^{\circ\circ}, \mathfrak{F}_{sp} := d\mathfrak{A}_{sp} + ic_{gr}\mathfrak{A}_{sp}\wedge\mathfrak{A}_{sp} = \frac{1}{2}S_{\circ\circ}\mathfrak{R}^{\circ\circ}$$

$$\bar{\phi}_{sp} := \iota_{\gamma}\bar{d}_{sp} = \left(\bar{\Lambda} - ic_{gr}\overline{\mathfrak{A}}_{sp}\right), \phi_{sp} := \iota_{\gamma}d_{sp} = \left(\partial\!\!\!/ - ic_{gr}\mathfrak{A}_{sp}\right)$$

$$\left(\mathcal{M}\otimes\mathbb{V}_{1/2}^{\mathrm{E}}, \pi_{SU}, \mathcal{M}, SU_{\mathrm{w}}(2)\right)$$

$$G_{SU} : \mathrm{End}(\Gamma_{\mathrm{w}}) : \boldsymbol{\phi}_{\mathrm{w}} \mapsto G_{SU}(\phi_{\mathrm{w}}) := \boldsymbol{g}_{SU}\boldsymbol{\phi}_{\mathrm{w}}$$

$$[G_{SU}(\boldsymbol{\phi}_{\mathrm{w}})]_I = [\boldsymbol{g}_{SU}]_I^J[\boldsymbol{\phi}_{\mathrm{w}}]_J, I, J \in \{\mathrm{U}, \mathrm{D}\}$$

$$G_{SU}(\mathfrak{A}_{SU}) = \boldsymbol{g}_{SU}^{-1}\mathfrak{A}_{SU}\boldsymbol{g}_{SU} + ic_{SU}^{-1}\boldsymbol{g}_{SU}^{-1}d\boldsymbol{g}_{SU} = \boldsymbol{g}_{SU}^{-1}\mathfrak{A}_{SU}\boldsymbol{g}_{SU}$$

$$G_{SO} : \mathrm{End}\left(\Omega^1(T^*\mathcal{M})\right) : \mathfrak{A}_{SU} \mapsto G_{SO}(\mathfrak{A}_{SU}) = \boldsymbol{\Lambda}\mathfrak{A}_{SU}$$

$$\mathfrak{F}_{SU} = \mathfrak{F}_{SU}^I\tau_I := d\mathfrak{A}_{SU} - ic_{SU}\mathfrak{A}_{SU}\wedge\mathfrak{A}_{SU}$$

$$= \left(d\mathfrak{A}_{SU}^I + \frac{c_{SU}}{2}f_{JK}^I\mathfrak{A}_{SU}^J\wedge\mathfrak{A}_{SU}^K\right)\tau_I \in \Omega^2(T^*\mathcal{M})\otimes\mathrm{Ad}(\mathfrak{su}(2)).$$

$$d_g\mathfrak{a} := \mathbf{1}_{SU}d\mathfrak{a} - \frac{i}{2}c_{SU}[\mathfrak{A}_{SU}, \mathfrak{a}]_{\wedge} = \mathbf{1}_{SU}d\mathfrak{a} - ic_{SU}\mathfrak{A}_{SU}\wedge\mathfrak{a}$$

$$\left(d_g\wedge d_g\right)\mathfrak{a}^a = -ic_{SU}\mathfrak{F}_{SU}\wedge\mathfrak{a}^a, d_g\mathfrak{F}_{SU} = 0$$

$$\mathfrak{A}_{SU} = \mathfrak{A}_{SU}^I\tau_I =: \mathcal{A}_a^I\mathrm{e}^a\tau_I,\ \mathfrak{F}_{SU} = \mathfrak{F}_{SU}^I\tau_I =: \frac{1}{2}\mathcal{F}_{ab}^I\mathrm{e}^a\wedge\mathrm{e}^b\tau_I$$

$$\Longrightarrow \mathcal{F}_{ab}^I = \partial_a\mathcal{A}_b^I - \partial_b\mathcal{A}_a^I + c_{SU}f^I{}_{JK}\mathcal{A}_a^J\mathcal{A}_b^K$$

$$\phi_g\phi_{\mathrm{w}} := (\mathbf{1}_{SU}\,\partial\!\!\!/ - ic_{SU}\mathfrak{A}_{SU})\phi_{\mathrm{w}},\ \mathfrak{A}_{SU} = \gamma^{\circ}\mathcal{A}_{\circ}^I\tau_I$$

$$[\tau_I\boldsymbol{\phi}_{\mathrm{H}}]_A := [\tau_I]_A^B\phi_B, A, B \in \{\mathrm{U}, \mathrm{D}\}$$

$$\left(\mathcal{M}\otimes\left(\mathbb{V}_{1/2}^{\mathrm{L}}\oplus\mathbb{V}_{1/2}^{\mathrm{L}}\right)\otimes\mathbb{V}_{1/2}^{\mathrm{E}}, \pi_{sp}\oplus\pi_{SU}, \mathcal{M}, G_{\mathrm{cp}}\otimes\mathrm{Spin}(1,3)\otimes SU_{\mathrm{w}}(2)\right)$$

$$\mathfrak{A}_{sg} = \mathfrak{A}_{SU}\otimes\mathbf{1}_{sp} + \mathbf{1}_{SU}\otimes\mathfrak{A}_{sp},\ \mathfrak{F}_{sg} = \mathfrak{F}_{SU}\otimes\mathbf{1}_{sp} + \mathbf{1}_{SU}\otimes\mathfrak{F}_{sp}$$

$$d_{sg} := \left(\mathbf{1}_{SU}\otimes\mathbf{1}_{sp}\right)d - ic_{SU}\left(\mathfrak{A}_{SU}\otimes\mathbf{1}_{sp}\right) - ic_{gr}\left(\mathbf{1}_{SU}\otimes\mathfrak{A}_{sp}\right)$$

$$\Xi^1 = \boldsymbol{\xi}^1\otimes\boldsymbol{\phi}_{\mathrm{w}}{}^1, \Xi^2 = \boldsymbol{\xi}^2\otimes\boldsymbol{\phi}_{\mathrm{w}}{}^2 \Longrightarrow \begin{cases} \langle\Xi^1, \Xi^2\rangle & = \langle\boldsymbol{\xi}^1, \boldsymbol{\xi}^2\rangle\langle\boldsymbol{\phi}_{\mathrm{w}}{}^1, \boldsymbol{\phi}_{\mathrm{w}}{}^2\rangle \\ \langle\boldsymbol{\xi}^1, \Xi^2\rangle & = \langle\boldsymbol{\xi}^1, \boldsymbol{\xi}^2\rangle\boldsymbol{\phi}_{\mathrm{w}}^2 \\ \langle\Xi^1, \boldsymbol{\xi}^2\rangle & = \langle\boldsymbol{\xi}^1, \boldsymbol{\xi}^2\rangle\boldsymbol{\phi}_{\mathrm{w}}{}^2 \\ \left\langle\Xi^1, \boldsymbol{\phi}_{\mathrm{w}}{}^2\right\rangle & = \left\langle\langle\boldsymbol{\phi}_{\mathrm{w}}^1, \boldsymbol{\phi}_{\mathrm{w}}{}^2\right\rangle\boldsymbol{\xi}^1 \\ \langle\boldsymbol{\phi}_{\mathrm{w}}^1, \Xi^2\rangle & = \langle\boldsymbol{\phi}_{\mathrm{w}}^1, \boldsymbol{\phi}_{\mathrm{w}}^2\rangle\boldsymbol{\xi}^2 \end{cases}$$

$$\boldsymbol{\xi}\otimes\boldsymbol{\phi}_{\mathrm{w}}{}^1 + \boldsymbol{\xi}\otimes\boldsymbol{\phi}_{\mathrm{w}}{}^2 = \boldsymbol{\xi}\otimes(\boldsymbol{\phi}_{\mathrm{w}}{}^1 + \boldsymbol{\phi}_{\mathrm{w}}{}^2), \boldsymbol{\xi}^1\otimes\boldsymbol{\phi}_{\mathrm{w}} + \boldsymbol{\xi}^2\otimes\boldsymbol{\phi}_{\mathrm{w}} = (\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2)\otimes\boldsymbol{\phi}_{\mathrm{w}}$$

$$\Xi_{\mathrm{w}} := \boldsymbol{\xi}^1\otimes\hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{U}} + \boldsymbol{\xi}^2\otimes\hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{D}} \in \Gamma\left(T\mathcal{M}_{\mathrm{L}}, \mathbb{V}_{1/2}^{\mathrm{L}}\otimes\mathbb{V}_{1/2}^{\mathrm{E}}, \mathrm{Spin}\ (1,3)\otimes SU_{\mathrm{w}}(2)\right)$$

$$\dot{\Xi}_{\mathrm{w}} := \dot{\boldsymbol{\xi}}^1\otimes\hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{U}} + \dot{\boldsymbol{\xi}}^2\otimes\hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{D}} \in \Gamma\left(T\mathcal{M}_{\mathrm{L}}, \mathbb{V}_{1/2}^{\mathrm{L}}\otimes\mathbb{V}_{1/2}^{\mathrm{E}}, \mathrm{Spin}(1,3)\otimes SU_{\mathrm{w}}(2)\right)$$

$$\left\langle\dot{\Xi}_{\mathrm{w}}, \dot{\Xi}_{\mathrm{w}}\right\rangle = \left\langle\dot{\boldsymbol{\xi}}^1, \boldsymbol{\xi}^1\right\rangle\left\langle\hat{\phi}_{\mathrm{w}}^{\mathrm{U}}, \hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{U}}\right\rangle + \left\langle\dot{\boldsymbol{\xi}}^2, \boldsymbol{\xi}^1\right\rangle\left\langle\hat{\phi}_{\mathrm{w}}^{\mathrm{D}}, \hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{U}}\right\rangle + \left\langle\dot{\boldsymbol{\xi}}^1, \boldsymbol{\xi}^2\right\rangle\left\langle\hat{\phi}_{\mathrm{w}}^{\mathrm{U}}, \hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{D}}\right\rangle + \left\langle\dot{\boldsymbol{\xi}}^2, \boldsymbol{\xi}^2\right\rangle\left\langle\hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{D}}, \hat{\boldsymbol{\phi}}_{\mathrm{w}}^{\mathrm{D}}\right\rangle$$

$$= \left\langle\dot{\boldsymbol{\xi}}^1, \boldsymbol{\xi}^1\right\rangle + \left\langle\dot{\boldsymbol{\xi}}^2, \boldsymbol{\xi}^2\right\rangle$$

$$\pi_{1/2}(\boldsymbol{g}_{\mathrm{E}})\dot{\Xi}_{\mathrm{W}} = \dot{\boldsymbol{\xi}}^1 \otimes (\boldsymbol{g}_{\mathrm{E}}^{\dagger}\hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{U}}) + \dot{\boldsymbol{\xi}}^2 \otimes (\boldsymbol{g}_{\mathrm{E}}^{\dagger}\hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{D}}),$$

$$= [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_1{}^1\dot{\xi}^1 \otimes \begin{pmatrix}1\\0\end{pmatrix} + [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_2{}^1\dot{\boldsymbol{\xi}}^1 \otimes \begin{pmatrix}0\\1\end{pmatrix} + [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_1{}^2\dot{\boldsymbol{\xi}}^2 \otimes \begin{pmatrix}1\\0\end{pmatrix} + [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_2{}^2\dot{\boldsymbol{\xi}}^2 \otimes \begin{pmatrix}0\\1\end{pmatrix},$$

$$= \left([\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_1{}^1\dot{\boldsymbol{\xi}}^1 + [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_1{}^2\dot{\boldsymbol{\xi}}^2\right) \otimes \begin{pmatrix}1\\0\end{pmatrix} + \left([\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_2{}^1\dot{\boldsymbol{\xi}}^1 + [\boldsymbol{g}_{\mathrm{E}}^{\dagger}]_2^2\dot{\boldsymbol{\xi}}^2\right) \otimes \begin{pmatrix}0\\1\end{pmatrix},$$

$$= [\boldsymbol{g}_{\mathrm{E}}^{*}]^1{}_A[\dot{\boldsymbol{\xi}}^A]_1 \otimes \hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{U}} + [\boldsymbol{g}_{\mathrm{E}}^{*}]^2{}_A[\dot{\boldsymbol{\xi}}^A]_2 \otimes \hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{D}}$$

$$\langle\dot{\Xi}_{\mathrm{W}}', \dot{\Xi}_{\mathrm{w}}'\rangle = \langle\dot{\Xi}_{\mathrm{W}}, \dot{\Xi}_{\mathrm{w}}\rangle,\ \dot{\Xi}_{\mathrm{w}}' = \pi_{1/2}(\boldsymbol{g}_{\mathrm{E}})\dot{\Xi}_{\mathrm{w}}$$

$$\boldsymbol{\tau}_I \cdot \dot{\Xi}_{\mathrm{w}} := \dot{\boldsymbol{\xi}}^1 \otimes \boldsymbol{\tau}_I \cdot \hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{U}} + \dot{\boldsymbol{\xi}}^2 \otimes \boldsymbol{\tau}_I \cdot \hat{\boldsymbol{\phi}}_{\mathrm{W}}^{\mathrm{D}} = \boldsymbol{\tau}_I$$

$$\gamma^a\boldsymbol{\Xi} = (\gamma_{\mathrm{SL}}^a\boldsymbol{\xi}) \otimes \boldsymbol{\phi}_{\mathrm{W}} \Longrightarrow \langle\dot{\Xi}_{\mathrm{W}}^*, \gamma_{\mathrm{SL}}^a\Xi_{\mathrm{W}}\rangle = \langle\dot{\boldsymbol{\xi}}^{1*}, \gamma_{\mathrm{SL}}^a\boldsymbol{\xi}^1\rangle + \langle\dot{\boldsymbol{\xi}}^{2*}, \gamma_{\mathrm{SL}}^a\boldsymbol{\xi}^2\rangle. = \dot{\boldsymbol{\xi}}^{1\dagger\epsilon}\gamma_{\mathrm{SL}}^a\boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}^{2\dagger\epsilon}\gamma_{\mathrm{SL}}^a\boldsymbol{\xi}^2$$

$$\bar{\phi}_{sg}: \mathrm{End}\left(\mathbb{V}_{\frac{\mathrm{L}}{\mathrm{L}}}^{1/2} \otimes \mathbb{V}_{1/2}^{\mathrm{E}}\right): \dot{\Xi}_{\mathrm{w}} \mapsto \bar{\phi}_{sg}\dot{\Xi}_{\mathrm{w}} := (\iota\bar{\gamma}d_{sg})\dot{\Xi}_{\mathrm{W}}$$

$$= \partial\dot{\Xi}_{\mathrm{W}} - ic_{gr}\left((\overline{\mathfrak{A}}_{sp}\dot{\xi}^1) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{U}} + (\overline{\mathfrak{A}}_{sp}\dot{\xi}^2) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{D}}\right) - ic_{SU}\left(\dot{\xi}^1 \otimes (\mathfrak{A}_{SU}\hat{\phi}_{\mathrm{W}}^{\mathrm{U}}) + \dot{\xi}^2 \otimes (\mathfrak{A}_{SU}\hat{\phi}_{\mathrm{W}}^{\mathrm{D}})\right)$$

$$= (\partial\dot{\xi}^1) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{U}} + (\partial\dot{\xi}^2) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{D}}$$

$$-ic_{gr}\left((\overline{\mathfrak{A}}_{sp}\dot{\xi}^1) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{U}} + (\overline{\mathfrak{A}}_{sp}\dot{\xi}^2) \otimes \hat{\phi}_{\mathrm{W}}^{\mathrm{D}}\right) - ic_{SU}\mathcal{A}_{\circ}^I\bar{\gamma}^{\circ}\left(\dot{\xi}^1 \otimes (\tau_I \cdot \hat{\phi}_{\mathrm{W}}^{\mathrm{U}}) + \dot{\xi}^2 \otimes (\tau_I \cdot \hat{\phi}_{\mathrm{W}}^{\mathrm{D}})\right)$$

$$\mathfrak{L}_{\mathrm{YMU}} = \mathfrak{L}_{\mathrm{YM}} + \mathfrak{L}_{\mathrm{GR}} + \mathfrak{L}_{\mathrm{F_M}} + \mathfrak{L}_{\mathrm{H}} + \mathfrak{L}_{\mathrm{F_M-H}}$$

$$\mathcal{I}_{\blacksquare} := \int_{\mathfrak{T}*\mathcal{M}_{\mathrm{L}}} \mathfrak{L}_{\blacksquare}, \cdot \in \{\mathrm{YM}, \mathrm{GR}, \mathrm{F_M}, \mathrm{H}, \mathrm{M-H}\}$$

$$\mathfrak{L}_G(\mathfrak{A}_G) := C_{\mathrm{DIM}}\mathrm{Tr}_G[\mathfrak{F}_G \wedge \hat{\mathfrak{F}}_G] = iC_{\mathrm{DIM}}\mathrm{Tr}_G[\|\mathfrak{F}_G\|^2]\mathfrak{v}$$

$$\frac{\delta S_G(\mathfrak{A}_G)}{\delta\mathfrak{A}_G} = 0 \Longrightarrow \hat{d}_G\mathfrak{F}_G := \hat{d}\mathfrak{F}_G - ic_G[\hat{\mathfrak{A}}_G, \mathfrak{F}_G]_{\wedge} = 0$$

$$\hat{d}\mathfrak{a} = \hat{\mathfrak{A}}_G \wedge \mathfrak{a} := \det[\boldsymbol{\eta}]^{-1}(-1)^{p(n-p+1)}\widehat{\mathrm{H}}(\mathfrak{A}_G \wedge \widehat{\mathrm{H}}(\mathfrak{a}))$$

$$\Longrightarrow \int \langle d_G\mathfrak{a}, \mathfrak{b}\rangle\mathfrak{v} = \int \langle\mathfrak{a}, \hat{d}_G\mathfrak{b}\rangle\mathfrak{v}$$

$$\frac{1}{iC_{\mathrm{DIM}}}\frac{\delta\mathcal{I}_G(\mathfrak{A}_G)}{\delta\mathfrak{A}_G} := \frac{1}{iC_G}\lim_{t\to 0}\frac{d}{dt}\mathcal{I}_G(\mathfrak{A}_G + t\mathfrak{a}) = \lim_{t\to 0}\frac{d}{dt}\int \|\mathfrak{F}_G(\mathfrak{A}_G + t\mathfrak{a})\|^2\mathfrak{v}$$

$$= 2\lim_{t\to 0}\int \left\langle\frac{d}{dt}\mathfrak{F}_G(\mathfrak{A}_G + t\mathfrak{a}), \mathfrak{F}_G(\mathfrak{A}_G + t\mathfrak{a})\right\rangle\mathfrak{v} = 2\lim_{t\to 0}\int \langle d_G\mathfrak{a}, \mathfrak{F}_G(\mathfrak{A}_G)\rangle\mathfrak{v}$$

$$= 2\lim_{t\to 0}\int \langle\mathfrak{a}, \hat{d}_G\mathfrak{F}_G(\mathfrak{A}_G)\rangle\mathfrak{v} = 0 \Longrightarrow \delta$$

$$\mathfrak{L}_{\mathrm{YM}}(\mathfrak{A}_{SU}) := i\mathrm{Tr}_{SU}\left[\|\mathfrak{F}_{sg}(\mathfrak{A}_{SU})\|^2\right]\mathfrak{v} = \mathrm{Tr}_{SU}[\mathfrak{F}_{sg}(\mathfrak{A}_{SU}) \wedge \hat{\mathfrak{F}}_{SU}(\hat{\mathfrak{A}}_{SU})]$$

$$\frac{\delta\mathcal{F}_{\mathrm{YM}}(\mathfrak{A}_{SU})}{\delta\mathfrak{A}_{SU}} = \dot{0} \Longrightarrow \hat{d}_{SU}\mathfrak{F}_{SU} = \dot{0}$$

$$\partial\Lambda = \frac{1}{4}\mathcal{F}_{\circ}^I\mathcal{F}_I^{\circ\circ}\mathfrak{v}, \mathcal{F}_{ab}^I = \delta\epsilon$$

$$\eta_{\mathrm{L}}{}^{\circ\circ}(\partial_{\circ}\mathcal{F}^{\mathrm{o}a\,I} + c_{SU}f^I{}_{JK}\mathcal{A}_0^J\mathcal{F}_{\circ a}^K)\tau_I = \dot{0}$$

$$[\partial] = [\mathcal{A}] = [\mathcal{T}] = L^{-1}, [\mathfrak{v}] = L^4 \Longrightarrow [\mathfrak{L}_{\mathrm{YM}}(\mathfrak{A}_{SU})] = 1$$

$$\mathfrak{L}_{\mathrm{GR}}(\mathfrak{w}, \mathrm{e}) := -\frac{i}{\hbar\kappa_{\mathrm{E}}}\mathrm{Tr}_{\mathrm{CP}}[\|\mathfrak{F}_{\mathrm{CP}}\|^2]\mathfrak{v} = -\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathrm{Tr}_{\mathrm{CP}}[\mathfrak{F}_{\mathrm{CP}} \wedge \hat{\mathfrak{F}}_{\mathrm{CP}}]$$

$$\hbar\kappa_{\mathrm{E}}\mathfrak{L}_{\mathrm{GR}} = -\mathrm{Tr}_{\mathrm{CP}}[\mathfrak{F}_{\mathrm{CP}} \wedge \hat{\mathfrak{F}}_{\mathrm{CP}}] = -\mathcal{F}_{IJ}^K\mathrm{Tr}_{SO}[\mathfrak{F}_{\mathrm{CP}}^I \wedge \hat{\mathfrak{F}}_{\mathrm{CP}}^J]\Theta_K$$

$$= -\mathrm{Tr}_{SO}[\mathfrak{R}(\mathfrak{w}) \wedge \mathfrak{S}(\mathrm{e})] = \frac{1}{2}\mathfrak{R}^{\circ\circ}(\mathfrak{w}) \wedge \mathfrak{S}_{\circ\circ}(\mathrm{e})$$

$$\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathrm{Tr}_{SO}[\mathfrak{R}(\mathfrak{w}) \wedge \mathfrak{S}(\mathrm{e})] = \frac{1}{\hbar\kappa_{\mathrm{E}}}R\mathfrak{v}$$

$$[R] = [\hbar\kappa_{\mathrm{E}}]^{-1} = L^{-2}, [\mathfrak{v}] = L^4 \Longrightarrow [\mathfrak{L}_{\mathrm{GR}}] = 1$$

$$\frac{\delta\mathfrak{I}_{\mathrm{GR}}(\mathfrak{w},\mathfrak{e})}{\delta\mathfrak{e}}=\dot{0}\Longrightarrow\frac{1}{\kappa_{\mathrm{E}}}\mathfrak{G}_a=\dot{0},\mathfrak{G}_a:=\frac{1}{2}\epsilon_{a\circ 0\circ}\mathfrak{R}^{\circ\circ}\wedge\mathfrak{e}^{\circ}$$
$$\frac{\delta\mathcal{G}_{\mathrm{GR}}(\mathfrak{w},\mathfrak{e})}{\delta\mathfrak{w}}=\dot{0}\Longrightarrow\frac{1}{\hbar\kappa_{\mathrm{E}}}d_{\mathfrak{w}}\mathfrak{e}^a=\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathfrak{T}^a=\dot{0}$$
$$\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathfrak{G}_a=\dot{0}\Longrightarrow\frac{1}{\hbar\kappa_{\mathrm{E}}}G_{\mathrm{E}}^{ab}=\dot{0},\ G_{\mathrm{E}}^{ab}:=R^{ab}-\frac{1}{2}\eta^{ab}R$$
$$\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathfrak{T}^a=\dot{0}\Longrightarrow\frac{1}{2\hbar\kappa_{\mathrm{E}}}\left(\partial_\mu\mathcal{E}_\nu^a+c_{gr}\omega_{\mu\circ}{}^a\mathcal{E}_\nu^{\circ}-(\mu\leftrightarrow\nu)\right)=\dot{0}$$
$$\mathfrak{L}_{\mathrm{F_N}}:=\mathfrak{L}_{\mathrm{F_u}}^R+\mathfrak{L}_{\mathrm{F_u}}^L=\left(\mathcal{L}_{\mathrm{F_u}}^R+\mathcal{L}_{\mathrm{F_N}}^L\right)\mathfrak{v}$$
$$\mathcal{L}_{\mathrm{F_M}}^R\left(\boldsymbol{\xi}^e,\boldsymbol{\xi}^{e\dagger_\epsilon}\right):=\left\langle\boldsymbol{\xi}^{e*},id_{sp}\boldsymbol{\xi}^e\right\rangle=i\boldsymbol{\xi}^{e\dagger_\epsilon}\left(d_{sp}\boldsymbol{\xi}^e\right)$$
$$\frac{\delta\mathfrak{I}_{\mathrm{F_u}}^R\left(\boldsymbol{\xi}^e,\boldsymbol{\xi}^{\dagger_\epsilon}\right)}{\delta\boldsymbol{\xi}^{\ominus\dagger_\epsilon}}=\dot{0}\Longrightarrow i\phi_{sp}\boldsymbol{\xi}^e=\dot{0}\Longrightarrow\gamma^{\circ}\left(i\partial_{\circ}-\frac{1}{2}c_{gr}\omega_{\circ}^{\star\star}S_{\star\star}\right)\boldsymbol{\xi}^e=\dot{0}$$
$$\mathcal{L}_{\mathrm{F_N}}^L\left(\dot{\Xi}_{\mathrm{W}},\dot{\Xi}_{\mathrm{W}}^{\dagger_e}\right):=\left\langle\dot{\Xi}_{\mathrm{W}}^*,i\bar{\phi}_{sg}\dot{\Xi}_{\mathrm{w}}\right\rangle=i\dot{\Xi}_{\mathrm{W}}^{\dagger_e}\left(\bar{\phi}_{sg}\dot{\Xi}_{\mathrm{w}}\right)$$
$$\frac{\delta\mathfrak{I}_{\mathrm{F_N}}^L\left(\dot{\Xi}_{\mathrm{W}},\dot{\Xi}_{\mathrm{W}}^{\dagger_e}\right)}{\delta\dot{\Xi}_{\mathrm{W}}^{\dagger_e}}=\dot{0}\Longrightarrow i\bar{\phi}_{sg}\dot{\Xi}_{\mathrm{W}}=\bar{\gamma}^{\circ}\left(\mathbf{1}_{SU}\left(i\partial_{\circ}-\frac{1}{2}c_{gr}\omega_{\circ}^{\star\star}\bar{S}_{\star\star}\right)+c_{SU}\mathcal{A}_{\circ}^I\boldsymbol{\tau}_I\right)\dot{\Xi}_{\mathrm{w}}=\dot{0}$$
$$\frac{\delta\mathfrak{I}_{\mathrm{F_N}}^L(\mathfrak{A}_{SU})}{\delta\mathfrak{A}_{SU}}=\dot{0}\Longrightarrow-c_{SU}\dot{\Xi}_{\mathrm{W}}^{\dagger_\epsilon}\gamma^{\circ}\dot{\Xi}_{\mathrm{w}}\mathfrak{V}_{\circ}\frac{\nabla}{}-c_{SU}\left(\dot{\boldsymbol{\xi}}^{\nu\dagger_\epsilon}\gamma^{\circ}\dot{\boldsymbol{\xi}}^{\nu}+\dot{\boldsymbol{\xi}}^{e\dagger_\epsilon}\gamma^{\circ}\dot{\boldsymbol{\xi}}^{e}\right)\mathfrak{V}_{\circ}=\dot{0}$$
$$\frac{\delta\mathfrak{I}_{\mathrm{F_N}}^L(\mathfrak{A}_{SU})}{\delta\mathfrak{A}_{SU}}\Longrightarrow\hat{d}_{sg}\mathfrak{F}_{sg}=c_{SU}\dot{\Xi}_{\mathrm{W}}^{\dagger_\epsilon}\gamma^{\circ}\dot{\Xi}_{\mathrm{w}}\mathfrak{V}_{\circ},\Longrightarrow\eta^{a\circ}[\hbar\daleth]\Big]_{\circ}=c_{SU}\dot{\Xi}_{\mathrm{W}}^{\dagger_\epsilon}\gamma^a\dot{\Xi}_{\mathrm{w}}$$
$$\mathfrak{L}_{\mathrm{H}}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right):=\mathfrak{L}_{\mathrm{H}}^{kin}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)-V\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)\mathfrak{v}$$
$$\mathfrak{L}_{\mathrm{H}}^{kin}(\phi_{\mathrm{H}},\phi_{\mathrm{H}}^{\dagger}):=\frac{1}{4}(i\phi_{SU})\circ(i\phi_{SU})\langle\phi_{\mathrm{H}}^*,\phi_{\mathrm{H}}\rangle\mathfrak{v}=\frac{1}{2}(i\phi_{SU})\langle\phi_{\mathrm{H}}^*,i\phi_{SU}\boldsymbol{\phi}_{\mathrm{H}}\rangle\mathfrak{v},$$
$$=\langle\phi_{\mathrm{H}}^*,(i\phi_{SU})^2\boldsymbol{\phi}_{\mathrm{H}}\rangle\mathfrak{v}=\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\cdot((i\phi_{SU})^2\boldsymbol{\phi}_{\mathrm{H}})\mathfrak{v}$$
$$\frac{\delta\mathfrak{I}_{\mathrm{H}}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)}{\delta\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}}=\dot{0}\Longrightarrow\left((i\phi_{SU})^2-\frac{\delta V\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)}{\delta\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}}\right)\boldsymbol{\phi}_{\mathrm{H}}=\dot{0}$$
$$-(id_{SU})^2=\mathbf{1}_{SU}\Delta+\widehat{\mathcal{X}}_{SU}$$
$$\hat{\mathfrak{x}}_{SU}:=\iota_\gamma\iota_\gamma(d_{SU}\wedge d_{SU})$$
$$\frac{\delta\mathfrak{I}_{\mathrm{H}}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)}{\delta\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}}=\iota_\gamma\iota_\gamma\left((\mathbf{1}_{SU}d-ic_{SU}\mathfrak{A}_{SU})\wedge(\mathbf{1}_{SU}d-ic_{SU}\mathfrak{A}_{SU})\right)$$
$$=-ic_{SU}(\mathfrak{A}_{SU}\,\partial\!\!\!/+(\partial\!\!\!/\mathfrak{A}_{SU})-ic_{SU}\mathfrak{A}_{SU}\mathfrak{A}_{SU})$$
$$=-(i\mathbf{1}_{SU}\,\partial\!\!\!/+c_{SU}\mathfrak{A}_{SU})(i\mathbf{1}_{SU}\,\partial\!\!\!/+c_{SU}\mathfrak{A}_{SU})-\mathbf{1}_{SU}\Delta=-(id_{SU})^2-\mathbf{1}_{SU}\Delta$$
$$\Longrightarrow\frac{\delta\mathfrak{I}_{\mathrm{F_N}}^L\left(\dot{\Xi}_{\mathrm{W}},\dot{\Xi}_{\mathrm{W}}^{\dagger_e}\right)}{\delta\dot{\Xi}_{\mathrm{W}}^{\dagger_e}}$$
$$\mathbf{1}_{SU}\Delta=\left(\iota_\gamma d_{SU}\right)\left(\iota_\gamma d_{SU}\right)-\iota_\gamma\iota_\gamma(d_{SU}\wedge d_{SU})$$
$$(\mathfrak{A}_{SU}\wedge d+d\mathfrak{A}_{SU})\phi_{\mathrm{H}}=\mathfrak{A}_{SU}\wedge d\boldsymbol{\phi}_{\mathrm{H}}+d(\mathfrak{A}_{SU}\boldsymbol{\phi}_{\mathrm{H}})=\mathfrak{A}_{SU}\wedge d\boldsymbol{\phi}_{\mathrm{H}}+d(\mathfrak{A}_{SU})\boldsymbol{\phi}_{\mathrm{H}}-\mathfrak{A}_{SU}\wedge d\boldsymbol{\phi}_{\mathrm{H}}$$
$$=(d\mathfrak{A}_{SU})\boldsymbol{\phi}_{\mathrm{H}}$$
$$\widehat{\mathfrak{F}}_{SU}=-ic_{SU}\iota_\gamma\iota_\gamma\left((d\mathfrak{A}_{SU})-ic_{SU}\mathfrak{A}_{SU}\wedge\mathfrak{A}_{SU}\right)=-ic_{SU}\mathfrak{B}_{SU}=-ic_{SU}\left(\frac{1}{2}\mathcal{F}_{\circ\circ}^I\gamma^{\circ}\gamma^{\circ}\tau_I\right)$$
$$\frac{\delta\mathfrak{I}_{\mathrm{H}}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)}{\delta\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}}\Longrightarrow\mathfrak{L}_{\mathrm{H}}\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)=-\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}(\mathbf{1}_{SU}\Delta+ic_{SU}\mathcal{X}_{SU})\boldsymbol{\phi}_{\mathrm{H}}-V(\boldsymbol{\phi}_{\mathrm{H}})$$
$$\frac{\delta\mathfrak{I}_{\mathrm{F_N}}^L\left(\dot{\Xi}_{\mathrm{W}},\dot{\Xi}_{\mathrm{W}}^{\dagger_e}\right)}{\delta\dot{\Xi}_{\mathrm{W}}^{\dagger_e}}\Longrightarrow\left(\mathbf{1}_{SU}\Delta+ic_{SU}\mathcal{B}_{SU}+\frac{\delta V\left(\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}\right)}{\delta\boldsymbol{\phi}_{\mathrm{H}}^{\dagger}}\right)\boldsymbol{\phi}_{\mathrm{H}}=\dot{0}$$
$$\mathcal{L}_{\mathrm{F_M}}^{m_e}=m_e\left\langle\dot{\boldsymbol{\xi}}^{e*},\boldsymbol{\xi}^e\right\rangle=m_e\left(\dot{\boldsymbol{\xi}}^e\right)^{\dagger_\epsilon}\cdot\boldsymbol{\xi}^e=m_e\left[\dot{\boldsymbol{\xi}}^{e^*}\right]_A\epsilon_2^{AB}[\boldsymbol{\xi}^e]_B$$
$$\boldsymbol{\xi}^e=\boldsymbol{\xi}_{\mathrm{U}}^e+\boldsymbol{\xi}_{\mathrm{D}}^e,\dot{\boldsymbol{\xi}}^e=\dot{\boldsymbol{\xi}}_{\mathrm{U}}^e+\dot{\boldsymbol{\xi}}_{\mathrm{D}}^e$$
$$\left\langle\dot{\boldsymbol{\xi}}_{\mathrm{U}}^{e*},\boldsymbol{\xi}_{\mathrm{U}}^e\right\rangle=\left\langle\dot{\boldsymbol{\xi}}_{\mathrm{D}}^{e*},\boldsymbol{\xi}_{\mathrm{D}}^e\right\rangle=0\Longrightarrow\mathcal{L}_{\mathrm{F_T}}^{m_e}=-m_e\left(\left\langle\dot{\boldsymbol{\xi}}_{\mathrm{U}}^{e*},\boldsymbol{\xi}_{\mathrm{D}}^e\right\rangle-\left\langle\dot{\boldsymbol{\xi}}_{\mathrm{D}}^{e*},\boldsymbol{\xi}_{\mathrm{U}}^e\right\rangle\right)$$

$$\langle \dot{\Xi}_{\rm w}, \boldsymbol{\phi}_{\rm w} \rangle \xrightarrow{\sqrt{-g}} \langle \boldsymbol{\phi}^{\rm U}_{\rm w}, \hat{\boldsymbol{\phi}}^{\rm U}_{\rm w} \rangle \dot{\boldsymbol{\xi}}^{\nu} + \langle \boldsymbol{\phi}^{\rm D}_{\rm w}, \hat{\boldsymbol{\phi}}^{\rm D}_{\rm w} \rangle \dot{\boldsymbol{\xi}}^{e} = \phi_{\rm U} \dot{\boldsymbol{\xi}}^{\nu} + \phi_{\rm D} \dot{\boldsymbol{\xi}}^{e}$$

$$\mathcal{L}^{m_e}_{\rm F_M} = \lambda \left\langle \langle \dot{\Xi}^*_{\rm W}, \boldsymbol{\phi}_{\rm W} \rangle, \Xi_{\rm W} \right\rangle = \lambda \phi_{\rm U} \langle \dot{\boldsymbol{\xi}}^{\nu*}, \boldsymbol{\xi}^{\nu} \rangle + \lambda \phi_{\rm D} \langle \dot{\boldsymbol{\xi}}^{e*}, \boldsymbol{\xi}^{e} \rangle$$

$$= -\lambda \phi_{\rm D} \left(\langle \dot{\boldsymbol{\xi}}^{e*}_{\rm U}, \boldsymbol{\xi}^{e}_{\rm D} \rangle - \langle \dot{\boldsymbol{\xi}}^{e*}_{\rm D}, \boldsymbol{\xi}^{e}_{\rm U} \rangle \right) - \lambda \phi_{\rm U} \left(\langle \dot{\boldsymbol{\xi}}^{\nu*}_{\rm U}, \boldsymbol{\xi}^{\nu}_{\rm D} \rangle - \langle \dot{\boldsymbol{\xi}}^{**}_{\rm D}, \boldsymbol{\xi}^{\nu}_{\rm U} \rangle \right)$$

$$\left(\phi_{\rm U} = 0, \phi_{\rm D} = v_0/\sqrt{2} \in \mathbb{R}, m_e = \lambda v_0/\sqrt{2} \right)$$

$$\frac{1}{2} \epsilon_{ab\circ\circ} \mathfrak{T}^{\circ}_{\blacksquare} \wedge \mathrm{e}^{\circ} := \frac{\delta \mathfrak{L}_{\blacksquare}}{\delta \mathfrak{w}^{ab}} - d \left(\frac{\delta \mathfrak{L}_{\blacksquare}}{\delta (d\mathfrak{w}^{ab})} \right), \ \mathfrak{E}^{\blacksquare}_{a} := \hbar \left(\frac{\delta \mathfrak{L}_{\blacksquare}}{\delta \mathrm{e}^{a}} - d \left(\frac{\delta \mathfrak{L}_{\blacksquare}}{\delta (d\mathrm{e}^{a})} \right) \right)$$

$$[\mathfrak{T}_{\blacksquare}] = L^{-1} \ , \ [\mathfrak{E}^{\blacksquare}] = E$$

$$\mathfrak{T}^{a}_{\blacksquare} =: \frac{1}{2} T^{a}_{\blacksquare\ \circ\circ} \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ}, \ \mathfrak{E}^{\blacksquare}_{a} =: E^{\blacksquare} \mathfrak{V}_{a}$$

$$[T_{\blacksquare}] = L^{-1}/L^{2} \ , \ [E^{\blacksquare}] = E/L^{3}$$

$$\frac{1}{2} \epsilon_{ab\circ\circ} \mathfrak{T}^{\circ}_{\rm F_M} \wedge \mathrm{e}^{\circ} = \frac{\delta \mathfrak{L}_{\rm F_M}}{\delta \mathfrak{w}^{ab}} = c_{gr} K^{\circ}_{ab} \mathfrak{V}_{\circ}$$

$$c_{gr} \left(\boldsymbol{\xi}^{\dagger\epsilon} \gamma^{\circ} S_{ab} \boldsymbol{\xi} \right) \frac{1}{3!} \epsilon_{0\star\star\star} \mathrm{e}^{\star} \wedge \mathrm{e}^{\star} \wedge \mathrm{e}^{\star} = c_{gr} \left(\boldsymbol{\xi}^{\dagger\epsilon} \gamma^{\circ} S_{ab} \boldsymbol{\xi} \right) \mathfrak{V}_{\circ}$$

$$K^{c}_{ab} := \boldsymbol{\xi}^{\dagger\epsilon} \gamma^{c} S_{ab} \boldsymbol{\xi}$$

$$\epsilon_{ab\circ\circ} \mathfrak{T}^{\circ}_{\rm F_M} \wedge \mathrm{e}^{\circ} = c_{gr} K^{\circ}_{ab} \mathfrak{V}_{\circ} \Longrightarrow T^{a}_{{\rm F_M}bc} = \boldsymbol{\xi}^{\dagger} \left(\gamma^{a} S_{bc} + 2 \delta^{a}_{[b} \eta_{c]\circ} \gamma^{\circ} \right) \boldsymbol{\xi} \Longrightarrow \mathfrak{T}^{a}_{\rm F_M} = \frac{1}{2} T^{a}_{\rm F_M \circ\circ} \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ}$$

$$\mathfrak{L}_{\blacksquare}(\mathfrak{v}) = \Phi_{\blacksquare}(f) \mathfrak{v} \text{ as } \mathfrak{E}^{\blacksquare}_{a} = \hbar \frac{\delta \mathfrak{L}_{\blacksquare}(\mathfrak{v})}{\delta \mathrm{e}^{a}} = \hbar \Phi_{\blacksquare}(f) \frac{\delta}{\delta \mathrm{e}^{a}} \left(\frac{1}{4!} \epsilon_{\circ\circ\circ\circ} \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ} \right)$$

$$= \hbar \Phi_{\blacksquare}(f) \frac{\hbar}{3!} \epsilon_{a\circ\circ\circ} \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ} \wedge \mathrm{e}^{\circ} = \hbar \Phi_{\blacksquare}(f) \mathfrak{V}_{a}$$

$$\Longrightarrow E^{\blacksquare} = \hbar \Phi_{\blacksquare}(f)$$

$$\Longrightarrow E^{\rm YM} = -\frac{\hbar}{4} \mathcal{F}^{I}_{\circ\circ} \mathcal{F}^{\circ\circ}_{I}$$

$$\Longrightarrow E^{\rm F_M} = \hbar \boldsymbol{\xi}^{\dagger\epsilon} \gamma^{\circ} \left(i \partial_{\circ} - \frac{1}{2} c_{gr} \omega^{\star\star}_{\circ} S_{\star\star} + c_{SU} \mathcal{A}^{I}_{\circ} \tau_{I} \right) \boldsymbol{\xi}$$

$$\Longrightarrow E^{\rm H}_{a} = \hbar \left(\eta^{\circ\circ} (\partial_{\circ} \boldsymbol{\phi}^{\dagger}_{\rm H})(\partial_{\circ} \boldsymbol{\phi}_{\rm H}) - i c_{SU} \frac{1}{2} \mathcal{F}^{I}_{\circ\circ} \gamma^{\circ} \gamma^{\circ} \tau_{I} (\boldsymbol{\phi}^{\dagger}_{\rm H} \boldsymbol{\phi}_{\rm H}) - V(\boldsymbol{\phi}_{\rm H}, \boldsymbol{\phi}^{\dagger}_{\rm H}) \right)$$

$$\Longrightarrow E^{m_e} = -\hbar m_e \left(\langle \dot{\boldsymbol{\xi}}^{e*}_{\rm U}, \boldsymbol{\xi}^{e}_{\rm D} \rangle - \langle \dot{\boldsymbol{\xi}}^{e*}_{\rm D}, \boldsymbol{\xi}^{e}_{\rm U} \rangle \right)$$

$$\mathfrak{G}^{\rm GR}_{a} = \mathfrak{E}^{\rm YM}_{a} + \mathfrak{E}^{\rm F_M}_{a} + \mathfrak{E}^{\rm H}_{a} + \mathfrak{E}^{m_e}_{a}, \ \mathfrak{T}^{a}_{\rm GR} = \mathfrak{T}^{a}_{\rm F_N}$$

$$\mathfrak{G}^{\rm GR}_{a} := \frac{1}{\kappa_{\rm E}} \mathfrak{G}_{a} = \frac{1}{2\kappa_{\rm E}} \epsilon_{a\circ\circ\circ} \mathfrak{R}^{\circ\circ} \wedge \mathrm{e}^{\circ}, \ \mathfrak{T}^{a}_{\rm GR} := \frac{1}{\hbar\kappa_{\rm E}} \mathfrak{T}^{a} = \frac{1}{\hbar\kappa_{\rm E}} d_{\mathfrak{w}} \mathrm{e}^{a}$$

$$\Longrightarrow \Sigma_{\theta}: Sp(1) \to GL(2,\mathbb{C}): \{1, i, j, k\}^{t} \mapsto \boldsymbol{\sigma}_{\theta} := \{\mathbf{1}_{2}, \kappa(\theta) \boldsymbol{\sigma}\}^{t}$$

$$\kappa(\theta) := e^{i\pi(\theta-1)/2} = \begin{cases} +1 & \theta = +1 \\ -i & \theta = 0 \\ -1 & \theta = -1 \end{cases}$$

$$\Sigma_{\theta}: \mathbb{R}^{4} \to Ampho(2) \subset GL(2,\mathbb{C}): \boldsymbol{v}_{\rm v\theta} \mapsto \boldsymbol{v}_{\rm s\theta} := \boldsymbol{\sigma}^{t}_{\theta} \cdot \boldsymbol{v}_{\rm v\theta} = \begin{pmatrix} v^{0} + \kappa(\theta) v^{3} & \kappa(\theta)(v^{1} - i v^{2}) \\ \kappa(\theta)(v^{1} + i v^{2}) & v^{0} - \kappa(\theta) v^{3} \end{pmatrix}$$

$$\langle \boldsymbol{u}_{\rm S\theta}, \boldsymbol{v}_{\rm S\theta} \rangle := \frac{1}{2} \left(\mathrm{Tr}[\boldsymbol{u}_{s\theta}] \mathrm{Tr}[\boldsymbol{v}_{s\theta}] - \mathrm{Tr}[\boldsymbol{u}_{s\theta} \boldsymbol{v}_{s\theta}] \right) = \boldsymbol{u}^{t}_{\rm S\theta} \boldsymbol{\eta}_{\theta} \boldsymbol{v}_{s\theta} = u^{0} v^{0} + e^{i\pi\theta} (u^{1} v^{1} + u^{2} v^{2} + u^{3} v^{3})$$

$$\Longrightarrow \boldsymbol{\eta}_{\theta} = \mathrm{diag}\left[1, e^{i\pi\theta}, e^{i\pi\theta}, e^{i\pi\theta}\right], \boldsymbol{\eta}_{\theta} = \begin{cases} \boldsymbol{\eta}_{\rm E}, & \theta = 0 \\ \boldsymbol{\eta}_{\rm L}, & \theta = \pm 1 \end{cases}$$

$$\|\boldsymbol{v}_{s\theta}\|^{2} := \langle \boldsymbol{v}_{s\theta}, \boldsymbol{v}_{s\theta} \rangle = \det[\boldsymbol{v}_{s\theta}] = (v^{0})^{2} + e^{i\pi\theta} \left((v^{1})^{2} + (v^{2})^{2} + (v^{3})^{2} \right)$$

$$\boldsymbol{g}_{c}(\theta = \pm 1) \in SO^{\uparrow}(1,3) \subset SO(1,3) \ , \boldsymbol{g}_{c}(\theta = 0) \in SO^{\perp}(3) \subset SO(4)$$

$$\boldsymbol{g}_{c}: \mathrm{End}(\mathrm{Caus}(2)): \boldsymbol{v}_{s\theta} \mapsto \boldsymbol{v}'_{s\theta} := \tau(\boldsymbol{g}_{c})(\boldsymbol{v}_{s\theta})$$

$$\boldsymbol{g}^{a}_{c}(\theta; \varphi, \chi) := \boldsymbol{g}^{a}_{r}(\varphi) \circ \boldsymbol{g}^{a}_{b}(\chi)|_{\boldsymbol{\sigma} \to (\chi\theta - i\varphi)\boldsymbol{\sigma}} = \exp\left(-i\sigma^{a} \frac{\varphi}{2} \right) \exp\left(\sigma^{a} \frac{\chi\theta}{2} \right)$$

$$\boldsymbol{v}'_{s\theta} = \tau(\boldsymbol{g}_c^a(\theta;\varphi,\chi))(\boldsymbol{v}_{s\theta}) = \Sigma_\theta \begin{cases} \begin{pmatrix} v^0\cosh(\chi\theta) + v_\theta^1\sinh(\chi\theta) \\ v^0\sinh(\chi\theta) + v_\theta^1\cosh(\chi\theta) \\ v_\theta^2\cos(\varphi) - v_\theta^3\sin(\varphi) \\ v_\theta^2\sin(\varphi) + v_\theta^3\cos(\varphi) \end{pmatrix} & a = 1, \\ \begin{pmatrix} v^0\cosh(\chi\theta) + v_\theta^2\sinh(\chi\theta) \\ v_\theta^1\cos(\varphi) + v_\theta^3\sin(\varphi) \\ v^0\sinh(\chi\theta) + v_\theta^2\cosh(\chi\theta) \\ -v_\theta^1\sin(\varphi) + v_\theta^3\cos(\varphi) \end{pmatrix} & a = 2, \\ \begin{pmatrix} v^0\cosh(\chi\theta) + v_\theta^3\sinh(\chi\theta) \\ v_\theta^1\cos(\varphi) - v_\theta^2\sin(\varphi) \\ v_\theta^1\sin(\varphi) + v_\theta^2\cos(\varphi) \\ v^0\sinh(\chi\theta) + v_\theta^3\cosh(\chi\theta) \end{pmatrix} & a = 3, \end{cases}$$

$$(v^0, v_\theta^1, v_\theta^2, v_\theta^3) := (v^0, \kappa(\theta)v^1, \kappa(\theta)v^2, \kappa(\theta)v^3)$$

$$\det[\boldsymbol{g}_c^a(\theta;\varphi,\chi)] = \det\left[\exp\left(-i\sigma^a\frac{\varphi}{2}\right)\right]\det\left[\exp\left(\sigma^a\frac{\chi\theta}{2}\right)\right] = \exp\left(\mathrm{Tr}\left[-i\sigma^a\frac{\varphi}{2}\right]\right)\exp\left(\mathrm{Tr}\left[\sigma^a\frac{\chi\theta}{2}\right]\right)$$
$$= 1,$$

$$\boldsymbol{g}_c^a(\theta;\varphi,\chi)^* = \boldsymbol{g}_r^{a=2}(\pi)\boldsymbol{g}_c^a(-\theta;\varphi,\chi)\boldsymbol{g}_r^{a=2}(\pi)^\dagger \text{ for } a = 1,2,3$$

$$\boldsymbol{g}_c = \boldsymbol{g}_c^{a_1}(\theta;\varphi_1,\chi_1)\cdots\boldsymbol{g}_c^{a_n}(\theta;\varphi_n,\chi_n)$$

$$\begin{array}{ccccc} \theta = -1 & \leftrightarrow & \theta = 0 & \leftrightarrow & \theta = 1 \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{v}_{\mathrm{SL}}^- & \leftrightarrow & \boldsymbol{v}_{\mathrm{SE}} & \leftrightarrow & \boldsymbol{v}_{\mathrm{SL}}^+ \\ \downarrow & & \downarrow & & \downarrow \\ \tau(\boldsymbol{g}_{\mathrm{L}}^*)(\boldsymbol{v}_{\mathrm{SL}}^-) & \leftrightarrow & \tau(\boldsymbol{g}_{\mathrm{E}})^-(\boldsymbol{v}_{\mathrm{SE}}) & \leftrightarrow & \tau(\boldsymbol{g}_{\mathrm{L}})(\boldsymbol{v}_{\mathrm{SE}}^+) \end{array}$$

$$z_1 = \mathrm{Re}[\tilde{z}_1] + \kappa(\theta)\mathrm{Im}[\tilde{z}_1], z_2 = -i\kappa(\theta)(\mathrm{Re}[\tilde{z}_2] + i\mathrm{Im}[\tilde{z}_2]),\ |\tilde{z}_1|^2 + |\tilde{z}_2|^2 = 1$$

$$\mathrm{Amp}: SU(2) \otimes [-1,1] \to \mathrm{Caus}(2): (\tilde{z}_1,\tilde{z}_2) \otimes \theta \mapsto \mathrm{Amp}(\theta)(\tilde{z}_1,\tilde{z}_2)) \in \mathrm{Caus}(2) \subset \mathrm{Ampho}(2)$$

$$\mathrm{Amp}(\theta)\left(\sqrt{\partial\Gamma}\left|d^4\xi\right|_{z_i\to\tilde{z}_i}\right) \Longrightarrow (\pi_s^\theta(\boldsymbol{g}_\theta) \circ \lambda_s^\theta)(z_1,z_2) = \lambda_s^\theta(z'_1,z'_2),\ \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \boldsymbol{g}_\theta^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\pi_s^\theta(\boldsymbol{g}_c) = \boldsymbol{g}_c, \ ,\ \dot{\pi}_s^\theta(\mathfrak{g}_c) = \mathfrak{g}_c \in \mathfrak{C}_\theta$$

$$\pi_s^\theta(\boldsymbol{g}_c)\big|_{\theta=0} = \boldsymbol{g}_{\mathrm{E}},\ \pi_s^\theta(\boldsymbol{g}_c)\big|_{\theta=1} = \boldsymbol{g}_{\mathrm{L}},\ \pi_s^\theta(\boldsymbol{g}_c)\big|_{\theta=-1} = \pi_{\bar{s}}(\boldsymbol{g}_{\mathrm{L}}) = \boldsymbol{g}_{\mathrm{L}}^*$$

$$\begin{array}{cccccc} \longrightarrow & \mathfrak{so}(4) & = & \mathfrak{su}(2) & \oplus & \mathfrak{su}(2) \\ & & & \cup & & \cup \\ & & & \mathfrak{C}_\theta|_{\theta=0} & \oplus & \mathfrak{C}_\theta|_{\theta=0} \\ & & & \updownarrow & & \updownarrow \\ & \mathfrak{C}_\theta \otimes [-1,1] & = & \mathfrak{C}_\theta \otimes [0,1] & \oplus & \mathfrak{C}_\theta \otimes [-1,0] \\ & & & \updownarrow & & \updownarrow \\ & & & \mathfrak{C}_\theta|_{\theta=1} & \oplus & \mathfrak{C}_\theta|_{\theta=-1} \\ & & & \| & & \| \\ \longrightarrow & \mathfrak{so}(1,3) \otimes \mathbb{C} & = & \mathfrak{sl}(2,\mathbb{C}) & \oplus & \overline{\mathfrak{sl}(2,\mathbb{C})} \end{array}$$

$$\mathbb{V}_{\mathrm{S}}(\theta_1,\theta_2;G) := \mathbb{V}_{1/2}^{\theta_1}[\boldsymbol{g}_c \in \mathrm{Caus}(2)] \otimes_G \mathbb{V}_{1/2}^{\theta_2}[\boldsymbol{g}_c \in \mathrm{Caus}(2)]$$

$$\mathbb{V}_{1/2}^\theta[\boldsymbol{g}_\theta \in \mathrm{Caus}(2)]\big|_{\theta=0} = \mathbb{V}_{1/2}^{\mathrm{E}},\ \mathbb{V}_{1/2}^\theta[\boldsymbol{g}_\theta \in \mathrm{Caus}(2)]\big|_{\theta=1} = \mathbb{V}_{1/2}^{\mathrm{L}},\ ,\ \mathbb{V}_{1/2}^\theta[\boldsymbol{g}_\theta \in \mathrm{Caus}(2)]\big|_{\theta=-1}$$
$$= \mathbb{V}_{1/2}^{\mathrm{L}}$$

$$\langle\boldsymbol{\xi}(\theta),\boldsymbol{\zeta}(\theta)\rangle = [\xi(-\theta)]_A^*[\boldsymbol{\epsilon}(\theta)]^{AB}[\boldsymbol{\zeta}(\theta)]_B, \boldsymbol{\epsilon}(\theta) := \begin{pmatrix} \cos(\pi\theta/2) & \sin(\pi\theta/2) \\ -\sin(\pi\theta/2) & \cos(\pi\theta/2) \end{pmatrix}$$

$$\boldsymbol{\epsilon}(0) = \mathbf{1}_2 \ \Longrightarrow \langle\boldsymbol{\xi}(0),\boldsymbol{\zeta}(0)\rangle = \langle\boldsymbol{\phi}_{\mathrm{H}},\boldsymbol{\varphi}_{\mathrm{H}}\rangle$$

$$\boldsymbol{\xi}(\theta)|_{\theta=+1} \in \mathbb{V}_{1/2}^{\mathrm{L}} \Leftrightarrow \boldsymbol{\xi}^*(-\theta)|_{\theta=+1} \in \mathbb{V}_{1/2}^{\mathrm{L}}$$

$$\boldsymbol{\xi}(\theta)|_{\theta=-1} \in \mathbb{V}_{1/2}^{\mathrm{L}} \Leftrightarrow \boldsymbol{\xi}^*(-\theta)|_{\theta=-1} \in \mathbb{V}_{\overline{1/2}}^{\overline{\mathrm{L}}}$$

$$\boldsymbol{\xi}^*(-\theta)=\boldsymbol{\xi}(\theta)\Longrightarrow\boldsymbol{\xi}^*(-1)=\boldsymbol{\xi}(1)=\boldsymbol{\xi}_{\mathrm{L}},\boldsymbol{\xi}^*(1)=\boldsymbol{\xi}(-1)=\dot{\boldsymbol{\xi}}_{\mathrm{L}}$$
$$\boldsymbol{\epsilon}(\pm1)=\pm\boldsymbol{\epsilon}_2\Longrightarrow\langle\boldsymbol{\xi}(1),\boldsymbol{\zeta}(1)\rangle=\langle\boldsymbol{\xi}_{\mathrm{L}},\boldsymbol{\zeta}_{\mathrm{L}}\rangle,\langle\boldsymbol{\xi}(-1),\boldsymbol{\zeta}(-1)\rangle=-\langle\dot{\boldsymbol{\xi}}_{\mathrm{L}},\dot{\boldsymbol{\zeta}}_{\mathrm{L}}\rangle$$
$$\left[\boldsymbol{\xi}^{\dagger_\epsilon}(\theta)\right]^A:=[\boldsymbol{\xi}(\theta)]^*_A[\boldsymbol{\epsilon}(\theta)]^{AB}\Longrightarrow\langle\boldsymbol{\xi}(\theta),\boldsymbol{\zeta}(\theta)\rangle=\boldsymbol{\xi}^{\dagger_\epsilon}(-\theta)\boldsymbol{\zeta}(\theta)=\boldsymbol{\xi}^t(\theta)\cdot\boldsymbol{\epsilon}(\theta)\cdot\boldsymbol{\zeta}(\theta)$$
$$\begin{array}{ccccc}
\theta=-1 & \leftrightarrow & \theta=0 & \leftrightarrow & \theta=1\\
\downarrow & & \downarrow & & \downarrow\\
\boldsymbol{\xi}(\theta)=\boldsymbol{\xi}(0) & \leftrightarrow & & & \boldsymbol{\xi}(1)\\
\| & \leftrightarrow & \| & & \|\\
\dot{\boldsymbol{\xi}}^{\xi} & \leftrightarrow & \boldsymbol{\phi}_{\mathrm{H}} & \leftrightarrow & \boldsymbol{\xi}
\end{array}$$
$$\begin{array}{ccccccccc}
\theta=-1 & \longleftrightarrow & -1<\theta<0 & \longleftrightarrow & \theta=0 & \longleftrightarrow & 0<\theta<1 & \longleftrightarrow & \theta=1\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\overline{\mathfrak{sl}(2.\mathbb{C})} & \longleftrightarrow & \mathfrak{Caus}(2) & \longleftrightarrow & \mathfrak{su}(2) & \longleftrightarrow & \mathfrak{Caus}(2) & \longleftrightarrow & \mathfrak{sl}(2.\mathbb{C})\\
\Cup & & \Cup & & \Cup & & \Cup & & \Cup\\
\boldsymbol{g}^a_r(\varphi;-1) & \longleftrightarrow & \boldsymbol{g}^a_r(\varphi;\theta) & \longleftrightarrow & \boldsymbol{g}^a_r(\varphi;0) & \longleftrightarrow & \boldsymbol{g}^a_r(\varphi;\theta) & \longleftrightarrow & \boldsymbol{g}^a_r(\varphi;1)\\
\| & & & & \| & & & & \|\\
\boldsymbol{g}^a_r(\varphi;1)^* & & & & \boldsymbol{g}^a_r(\varphi) & & & & \pi_{1/2}(\boldsymbol{g}_r)\\
\| & & & & & & & & \\
\pi_{\overline{1/2}}(\boldsymbol{g}_r) & & & & & & & &
\end{array}$$
$$\to\mathfrak{a}_\theta\wedge\widehat{\mathrm{H}}_\theta(\mathfrak{b}_\theta):=\det[\boldsymbol{\eta}_\theta]^{1/2}\langle\mathfrak{a}_\theta,\mathfrak{b}_\theta\rangle\mathfrak{v}_\theta$$
$$\det[\boldsymbol{\eta}_\theta]^{1/2}\Rightarrow e^{3i\pi\theta/2}$$
$$\langle\mathfrak{a}_\blacksquare,\mathfrak{b}_\blacksquare\rangle\overset{\eta_\blacksquare\mapsto\eta_\theta}{\to}\langle\mathfrak{a}_\theta,\mathfrak{b}_\theta\rangle,\|\mathfrak{a}_\blacksquare\|\mathfrak{v}_\blacksquare\overset{\eta_\blacksquare\mapsto\eta_\theta}{\to}\|\mathfrak{a}_\theta\|\mathfrak{v}_\theta\ \cdot\in\{\mathrm{E},\mathrm{L}\}$$
$$\mathfrak{v}_\theta:=\frac{1}{4!}\epsilon_{\circ\circ\circ\circ}\mathfrak{e}^\circ_\theta\wedge\mathfrak{e}^\circ_\theta\wedge\mathfrak{e}^\circ_\theta\wedge\mathfrak{e}^\circ_\theta,\ \mathfrak{a}_\blacksquare=\frac{1}{p!}a_{i_1\cdots i_p}\mathfrak{e}^{i_1}_\blacksquare\wedge\cdots\wedge\mathfrak{e}^{i_p}_\blacksquare\to\mathfrak{a}_\theta=\frac{1}{p!}a_{i_1\cdots i_p}\mathfrak{e}^{i_1}_\theta\wedge\cdots\wedge\mathfrak{e}^{i_p}_\theta$$
$$\mathfrak{e}^a_\theta|_{\theta=0}=\mathfrak{e}^a_{\mathrm{E}},\ \mathfrak{e}^a_\theta|_{\theta=\pm1}=\mathfrak{e}^a_{\mathrm{L}}$$
$$\mathfrak{e}^a_\theta=\left(\mathfrak{e}^0_\theta,\mathfrak{e}^1_\theta,\mathfrak{e}^2_\theta,\mathfrak{e}^3_\theta\right):=(\mathfrak{e}^0,i\kappa(\theta)\mathfrak{e}^1,i\kappa(\theta)\mathfrak{e}^2,i\kappa(\theta)\mathfrak{e}^3)$$
$$\Delta_\theta=d_\theta\hat{d}_\theta+\hat{d}_\theta d_\theta=\partial_0+e^{i\pi\theta}(\partial_1+\partial_2+\partial_3)$$
$$\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^0_\theta\wedge\mathfrak{e}^1_\theta\right)=\mathfrak{e}^2_\theta\wedge\mathfrak{e}^3_\theta,\quad\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^0_\theta\wedge\mathfrak{e}^2_\theta\right)=\mathfrak{e}^3_\theta\wedge\mathfrak{e}^1_\theta,\quad\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^0_\theta\wedge\mathfrak{e}^3_\theta\right)=\mathfrak{e}^1_\theta\wedge\mathfrak{e}^2_\theta,$$
$$\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^1_\theta\wedge\mathfrak{e}^2_\theta\right)=\mathfrak{e}^0_\theta\wedge\mathfrak{e}^3_\theta,\quad\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^1_\theta\wedge\mathfrak{e}^3_\theta\right)=\mathfrak{e}^2_\theta\wedge\mathfrak{e}^0_\theta,\quad\widehat{\mathrm{H}}_\theta\left(\mathfrak{e}^2_\theta\wedge\mathfrak{e}^3_\theta\right)=\mathfrak{e}^0_\theta\wedge\mathfrak{e}^1_\theta,$$
$$\mathfrak{S}^{+a}_\theta:=\left\{\mathfrak{e}^0_\theta\wedge\mathfrak{e}^1_\theta+\mathfrak{e}^2_\theta\wedge\mathfrak{e}^3_\theta,\mathfrak{e}^0_\theta\wedge\mathfrak{e}^2_\theta-\mathfrak{e}^1_\theta\wedge\mathfrak{e}^3_\theta,\mathfrak{e}^0_\theta\wedge\mathfrak{e}^3_\theta+\mathfrak{e}^1_\theta\wedge\mathfrak{e}^2_\theta\right\}$$
$$\mathfrak{S}^{-a}_\theta:=\left\{\mathfrak{e}^0_\theta\wedge\mathfrak{e}^1_\theta-\mathfrak{e}^2_\theta\wedge\mathfrak{e}^3_\theta,\mathfrak{e}^0_\theta\wedge\mathfrak{e}^2_\theta+\mathfrak{e}^1_\theta\wedge\mathfrak{e}^3_\theta,\mathfrak{e}^0_\theta\wedge\mathfrak{e}^3_\theta-\mathfrak{e}^1_\theta\wedge\mathfrak{e}^2_\theta\right\}$$
$$P^{\pm}_{\widehat{\mathrm{H}}}\mathfrak{S}^{\pm a}_\theta=\pm\mathfrak{S}^{\pm a}_\theta$$
$$\mathfrak{S}^{\pm a}_\theta\wedge\mathfrak{S}^{\pm b}_\theta=\begin{cases}0 & a\neq b\\ \pm2e^{3i\pi\theta/2}\mathfrak{v}_\theta & a=b\end{cases},\quad\mathfrak{S}^{\pm a}_\theta\wedge\mathfrak{S}^{\mp b}_\theta=0$$
$$\mathfrak{S}^{\pm a}_1:=\mathfrak{S}^{\pm a}_\theta\Big|_{\theta=1}=\mathfrak{S}^{\pm a}_{\mathrm{L}},\mathfrak{S}^{\pm a}_{-1}:=\mathfrak{S}^{\pm a}_\theta\Big|_{\theta=-1}=\mathfrak{S}^{\mp a}_{\mathrm{L}},\mathfrak{S}^{\pm a}_0:=\mathfrak{S}^{\pm a}_\theta\Big|_{\theta=0}=\mathfrak{S}^{\pm a}_{\mathrm{E}}$$
$$\Longrightarrow P^{\pm}_0:=P^{\pm}_{\widehat{\mathrm{H}}}\Big|_{\theta=0}=P^{\pm}_{\mathrm{E}},P^{\pm}_1:=P^{\pm}_{\widehat{\mathrm{H}}}\Big|_{\theta=1}=P^{\pm}_{\mathrm{L}}=P^{\mp}_{-1}$$
$$P^{\pm}_{-1}\mathfrak{S}^{\pm}_{-1}=P^{\mp}_{\mathrm{L}}\mathfrak{S}^{\mp}_{\mathrm{L}}=\mathfrak{S}^{\mp}_{\mathrm{L}}$$
$$\mathfrak{F}^{\pm}_\theta:=P^{\pm}_{\widehat{\mathrm{H}}}\mathfrak{F}_\theta\widehat{\mathrm{H}}_\theta\left(\mathfrak{F}^{\pm}_\theta\right)=\pm\mathfrak{F}^{\pm}_\theta\in\mathbb{V}^{\pm}_{\widehat{\mathrm{H}}}\left(\Omega^2_\theta(T^*\mathcal{M}_\theta)\right)\otimes\mathrm{Ad}(\mathfrak{g})$$
$$\mathfrak{e}^a_{ch}:=(\mathfrak{e}^0-\kappa(\theta)\mathfrak{e}^3,\mathfrak{e}^0+\kappa(\theta)\mathfrak{e}^3,\mathfrak{e}^1+i\mathfrak{e}^2,\mathfrak{e}^1-i\mathfrak{e}^2)$$
$$\mathfrak{e}^1_{ch}\wedge\mathfrak{e}^3_{ch}=[\mathfrak{S}^+_\theta]^1+i[\mathfrak{S}^+_\theta]^2\in\mathbb{V}^+_{\hat{H}}\left(\Omega^2_\theta\right),\quad\mathfrak{e}^1_{ch}\wedge\mathfrak{e}^4_{ch}=[\mathfrak{S}^-_\theta]^1-i[\mathfrak{S}^-_\theta]^2\in\mathbb{V}^-_{\hat{H}}\left(\Omega^2_\theta\right)$$
$$\mathfrak{e}^2_{ch}\wedge\mathfrak{e}^3_{ch}=[\mathfrak{S}^-_\theta]^1+i[\mathfrak{S}^-_\theta]^2\in\mathbb{V}^-_{\widehat{\mathrm{H}}}\left(\Omega^2_\theta\right),\quad\mathfrak{e}^2_{ch}\wedge\mathfrak{e}^4_{ch}=[\mathfrak{S}^+_\theta]^1-i[\mathfrak{S}^+_\theta]^2\in\mathbb{V}^+_{\widehat{\mathrm{H}}}\left(\Omega^2_\theta\right)$$
$$\longrightarrow\begin{cases}\mathfrak{e}^3_{ch}\in\text{ polarización derecha} & \overset{\eta_\blacksquare\mapsto\eta_\theta}{\to}\quad\left\{\mathfrak{e}^1_{ch},\mathfrak{e}^3_{ch}\right\}\in\text{ quiral-derecha}\quad=\mathrm{SD}\\ \mathfrak{e}^4_{ch}\in\text{ polarización izquierda} & \overset{\eta_\blacksquare\mapsto\eta_\theta}{\to}\quad\left\{\mathfrak{e}^2_{ch},\mathfrak{e}^4_{ch}\right\}\in\text{ quiral-izquierda}\quad=\mathrm{ASD}\end{cases}$$
$$\left\{\gamma^a_\theta,\gamma^b_\theta\right\}=2\eta^{ab}_\theta\mathbf{1}_{sp}$$
$$\gamma^0_\theta:=\gamma^0_{\mathrm{E}},\gamma^a_\theta:=e^{i\pi\theta/2}\gamma_{\mathrm{E}}{}^a\text{ for }a=1{,}2{,}3$$

$$\gamma_\theta = \begin{cases} \gamma_{\mathrm{L}}, & \theta = +1 \\ \gamma_{\mathrm{E}}, & \theta = 0 \\ \bar{\gamma}_{\mathrm{L}}, & \theta = -1 \end{cases}$$

$$S_\theta^{ab} := i\left[\frac{\gamma_\theta{}^a}{2}, \frac{\gamma_\theta{}^b}{2}\right]$$

$$\Gamma_\theta = \sqrt{\frac{1}{\det[\boldsymbol{\eta}_\theta]}}\gamma_\theta^0\gamma_\theta^1\gamma_\theta^2\gamma_\theta^3 = f_\Gamma(\theta)\gamma_{\mathrm{E}}^0\gamma_{\mathrm{E}}^1\gamma_{\mathrm{E}}^2\gamma_{\mathrm{E}}^3,\ f_\Gamma(\theta) := \frac{e^{3i\pi\theta/2}}{\sqrt{e^{3i\pi\theta}}}$$

$$f_\Gamma(\theta) = \begin{cases} +1, & \theta \to 1 + 0, \\ -1, & \theta \to -1 - 0, \quad \mathrm{Re}[\sqrt{\blacksquare}] \geq 0 \\ -1 - 0, & \theta \to -1 + 0 \end{cases}$$

$$\sqrt{e^{3i\pi\theta}} = e^{3i\pi\theta/2} \Longrightarrow f_\Gamma(\theta) = 1$$

$$\Lambda = \gamma_{\mathrm{E}}^0\gamma_{\mathrm{E}}^1\gamma_{\mathrm{E}}^2\gamma_{\mathrm{E}}^3 = \begin{cases} \begin{pmatrix} -\mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}, & \dim - 4 \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \dim - 2 \end{cases}$$

$$P_\theta^\pm := \frac{1}{2}\left(\mathbf{1}_{sp} \pm \Gamma_\theta\right) = P_\theta^\pm \circ P_\theta^\pm = P_\theta^\pm, P_\theta^\pm P_\theta^\mp = 0$$

$$\not{d}_\theta := \iota_\gamma d_\theta = \not{\partial}_\theta = \gamma_\theta^a\partial_a,\ \not{\partial}_\theta P_\theta^\pm = P_\theta^\mp \not{\partial}_\theta$$

$$(\not{\partial}\theta)^2 = \Delta_\theta \mathbf{1}_{sp}, \Delta_\theta := (\partial_0)^2 + e^{i\pi\theta}((\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2)$$

$$\Gamma_\theta \boldsymbol{\xi}_{\mathrm{U}}(\theta) = -\boldsymbol{\xi}_{\mathrm{U}}(\theta) \Longrightarrow P_\theta^- \boldsymbol{\xi}(\theta) = \boldsymbol{\xi}_{\mathrm{U}}(\theta) \in \mathbb{V}_{1/2}^{\theta\mathrm{U}}$$

$$\Gamma_\theta \boldsymbol{\xi}_{\mathrm{D}}(\theta) = +\boldsymbol{\xi}_{\mathrm{D}}(\theta) \Longrightarrow P_\theta^+ \boldsymbol{\xi}(\theta) = \boldsymbol{\xi}_{\mathrm{D}}(\theta) \in \mathbb{V}_{1/2}^{\theta\mathrm{D}}$$

$$\begin{array}{ccccc} \theta = -1 & \leftrightarrow & \theta = 0 & \leftrightarrow & \theta = 1 \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{\xi}_{\mathrm{U}}(\theta) = \boldsymbol{\xi}_{\mathrm{U}}(-1) = \dot{\boldsymbol{\xi}}_{\mathrm{U}} & \leftrightarrow & \boldsymbol{\xi}_{\mathrm{U}}(0) = \boldsymbol{\phi}_{\mathrm{U}} & \leftrightarrow & \boldsymbol{\xi}_{\mathrm{U}}(1) = \boldsymbol{\xi}_{\mathrm{U}} \\ \boldsymbol{\xi}_{\mathrm{D}}(\theta) = \boldsymbol{\xi}_{\mathrm{D}}(-1) = \dot{\boldsymbol{\xi}}_{\mathrm{D}} & \leftrightarrow & \boldsymbol{\xi}_{\mathrm{D}}(0) = \boldsymbol{\phi}_{\mathrm{D}} & \leftrightarrow & \boldsymbol{\xi}_{\mathrm{D}}(1) = \boldsymbol{\xi}_{\mathrm{D}} \end{array}$$

$$\Longrightarrow \mathfrak{L}_{\mathrm{YM}}(\mathfrak{A}_\theta) := \det[\boldsymbol{\eta}_\theta]^{1/2}\mathrm{Tr}[\|\mathfrak{F}_\theta(\mathfrak{A}_\theta)\|^2]\mathfrak{v}_\theta = \mathrm{Tr}\left[\mathfrak{F}_\theta(\mathfrak{A}_\theta) \wedge \hat{\mathfrak{F}}_\theta(\hat{\mathfrak{A}}_\theta)\right]$$

$$\Lambda = -\frac{1}{4}\eta_\theta{}^{\circ\star}\eta_\theta{}^{\circ\star}\mathcal{F}_{\theta\circ\circ}\mathcal{F}_{\theta\star\star}\mathfrak{v}_\theta, \mathcal{F}_{\theta ab} := \mathcal{F}_{\theta ab}^I\tau_I$$

$$\hat{d}_{\mathfrak{A}}\mathfrak{F}_\theta = \hat{d}\mathfrak{F}_\theta + \frac{c_{SU}}{2}\left[\hat{\mathfrak{A}}_\theta, \mathfrak{F}_\theta\right]_\wedge = \eta_\theta{}^{\circ\circ}\left(\partial_\circ\mathcal{F}_{\theta\circ\star}^I + c_{SU}f^I{}_{JK}\mathcal{A}_{\theta\circ}^J\mathcal{F}_{\theta\circ\star}^K\right)\mathfrak{V}_\theta{}^\star$$

$$\mathfrak{F}_\theta = \mathfrak{F}_\theta^+ + \mathfrak{F}_\theta^-,\ \hat{\mathfrak{F}}_\theta = \mathfrak{F}_\theta^+ - \mathfrak{F}_\theta^-$$

$$(\|\mathfrak{F}_\theta^+\|^2 + \|\mathfrak{F}_\theta^-\|^2)\mathfrak{v}_\theta$$

$$\mathcal{I}_{\mathrm{YM}} = \det[\boldsymbol{\eta}_\theta]^{1/2}\int_{T^*\mathcal{M}_\theta}\|\mathfrak{F}_\theta\|^2\mathfrak{v}_\theta = \det[\boldsymbol{\eta}_\theta]^{1/2}\int_{T^*\mathcal{M}_\theta}(\|\mathfrak{F}_\theta^+\|^2 + \|\mathfrak{F}_\theta^-\|^2)\mathfrak{v}_\theta$$

$$c_2(\mathfrak{F}_\theta) = \frac{1}{8\pi^2}\int_{T^*\mathcal{M}_\theta}(\mathrm{Tr}[\mathfrak{F}_\theta \wedge \mathfrak{F}_\theta] - \mathrm{Tr}[\mathfrak{F}_\theta]^2)\mathfrak{v}_\theta = \frac{\det[\boldsymbol{\eta}_\theta]^{1/2}}{8\pi^2}\int_{T^*\mathcal{M}_\theta}(\|\mathfrak{F}_\theta^+\|^2 - \|\mathfrak{F}_\theta^-\|^2)\mathfrak{v}_\theta$$

$$\mathcal{I}_{\mathrm{YM}} \geq 8\pi^2|c_2(\mathfrak{F}_{\mathrm{E}})|$$

$$\mathcal{W}^+ := \mathcal{M}_\theta \otimes \theta \in [0,1], \mathcal{W}^- := \mathcal{M}_\theta \otimes \theta \in [-1,0],\ \mathcal{W} := \mathcal{W}^+ \cup \mathcal{W}^- = \mathcal{M}_\theta \otimes \theta \in [-1,1]$$

$$\mathfrak{F}_\theta = \sum_i \alpha_i\left(\left\|\boldsymbol{r}_\theta^i\right\|^2 + \lambda\right)^{-m} + \beta$$

$$\hat{d}\mathfrak{F}_0^+ = 0, \mathfrak{F}_0^+ = d\mathfrak{A}_0^+ - ic_{SU}\mathfrak{A}_0^+ \wedge \mathfrak{A}_0^+$$

$$T^*\mathcal{W}_\theta^+ := T^*\mathcal{M}_\theta \otimes \theta \in [0,1], \overline{T^*\mathcal{M}_{\mathrm{L}}} := T^*\mathcal{M}_{\mathrm{L}} \setminus \left\{\boldsymbol{r}_{\mathrm{L}}^i(t) \mid \left\|\boldsymbol{r}_{\mathrm{L}}^i(t)\right\|^2 + \lambda = 0\right\}$$

$$\mathfrak{a} \in \Omega_{\mathrm{SD}}^2(T^*\mathcal{W}_\theta) \Longrightarrow \widehat{\mathrm{H}}_\theta(\mathfrak{a}) = +\mathfrak{a}$$

$$\{\mathfrak{A}_{\mathrm{L}}^+, \mathfrak{F}_{\mathrm{L}}^+\} \in \Omega_{\mathrm{SD}}^2\left(\overline{T^*\mathcal{M}_{\mathrm{L}}}\right) \otimes \mathrm{Ad}(\mathfrak{sl}(2,\mathbb{C}))$$

$$\infty > \left|\lim_{\theta \to \pm 1}\int_{\overline{T^*\mathcal{M}_\theta}}\|\mathfrak{F}_\theta^+\|^2\mathfrak{v}_\theta\right| =: |c_2(\mathfrak{F}_{\mathrm{L}})| = \left|\mathcal{F}_{\mathrm{YM}}^{\mathrm{L}}\right|$$

$$\mathfrak{F}_{\mathrm{L}} = \alpha\mathfrak{F}_{\mathrm{L}}^+ + \beta\mathfrak{F}_{\mathrm{L}}^-\ \alpha, \beta \in \mathbb{C}$$

$$\mathfrak{S}_{\mathrm{L}}^{\pm a} \to \mathfrak{S}_\theta^{\pm a}, \boldsymbol{\eta}_{\mathrm{L}} \to \boldsymbol{\eta}_\theta \text{ in } \theta \in [-1,1]$$

$$\Longrightarrow \mathfrak{L}_{\mathrm{GR}}(\mathfrak{w}_\theta,\mathfrak{e}_\theta) := -\frac{\det[\boldsymbol{\eta}_\theta]^{1/2}}{\hbar\kappa_{\mathrm{E}}}\mathrm{Tr}_{\mathrm{CP}}[\|\mathfrak{F}_{\mathrm{CP}}(\mathfrak{A}_\theta)\|_\theta^2]\mathfrak{v}_\theta = -\frac{1}{\hbar\kappa_{\mathrm{E}}}\mathrm{Tr}_{\mathrm{CP}}[\mathfrak{F}_{\mathrm{CP}}(\mathfrak{A}_\theta)\wedge\hat{\mathfrak{F}}_{\mathrm{CP}}(\hat{\mathfrak{A}}_\theta)]$$

$$\mathrm{Tr}_{SO}[\mathfrak{R}_\theta(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta(\mathfrak{e}_\theta)] = \mathrm{Tr}_{SO}[\mathfrak{R}_\theta^+(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^+(\mathfrak{e}_\theta)] + \mathrm{Tr}_{SO}[\mathfrak{R}_\theta^-(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^-(\mathfrak{e}_\theta)]$$

$$\mathfrak{R}_\theta^\pm(\mathfrak{w}_\theta) := P_{\hat{\mathrm{H}}}^\pm\mathfrak{R}_\theta(\mathfrak{w}_\theta)\,,\ \mathfrak{S}_\theta^\pm(\mathfrak{e}_\theta) := P_{\hat{\mathrm{H}}}^\pm\mathfrak{S}(\mathfrak{e}_\theta)$$

$$-\mathrm{Tr}_{SO}[\mathfrak{R}_\theta(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta(\mathfrak{e}_\theta)] = R(\mathfrak{w}_\theta)\mathfrak{v}_\theta\ ,\quad -\mathrm{Tr}_{SO}\left[\mathfrak{R}_\theta^\pm(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^\pm(\mathfrak{e}_\theta)\right] = \frac{1}{2}\left(R(\mathfrak{w}_\theta)\pm(R^0+R_0)\right)\mathfrak{v}_\theta$$

$$R^0 := i\frac{\kappa(-\theta)}{2}\epsilon_{\theta 0\star\star}R^{0\circ}_{\star\star}(\mathfrak{w}_\theta) = i\kappa(-\theta)(R^{01}_{23}-R^{02}_{13}+R^{03}_{12})(\mathfrak{w}_\theta)$$

$$R_0 := i\frac{\kappa(\theta)}{2}\epsilon_{\theta\circ\circ}{}^{0\star}R^{\circ\circ}_{0\star}(\mathfrak{w}_\theta) = i\kappa(\theta)(R^{12}_{03}-R^{02}_{13}+R^{23}_{01})(\mathfrak{w}_\theta)$$

$$0 = \mathfrak{w}_\theta{}^a{}_\circ\wedge\mathfrak{T}^\circ_\theta = \mathfrak{w}_\theta{}^a{}_\circ\wedge d\mathfrak{e}^\circ_\theta + \mathfrak{w}_\theta{}^a{}_\circ\wedge\mathfrak{w}_\theta{}_{\star\star}\wedge\mathfrak{e}_\theta{}^\star = \eta_{\theta\circ\circ}\mathfrak{R}^{a\circ}(\mathfrak{w}_\theta)\wedge\mathfrak{e}^\circ - d(\mathfrak{w}_\theta{}^a{}_\circ\wedge\mathfrak{e}^\circ_\theta)$$

$$\mathfrak{T}^a(\mathfrak{e}_\theta) = 0 \Longrightarrow \eta_{\theta\circ\circ}\mathfrak{R}^{a\circ}(\mathfrak{w}_\theta)\wedge\mathfrak{e}^\circ = 0$$

$$\eta_{\theta a\circ}\eta_{\theta b\circ}R^{\circ\circ}_{cd}(\mathfrak{w}_\theta) = \eta_{\theta c\circ}\eta_{\theta d\circ}R^{\circ\circ}_{ab}(\mathfrak{w}_\theta)$$

$$R_{abcd} = R_{cdab}$$

$$R^0 = -R_0$$

$$\mathrm{Tr}_{SO}[\mathfrak{R}_\theta^+(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^+(\mathfrak{e}_\theta)] = \mathrm{Tr}_{SO}[\mathfrak{R}_\theta^-(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^-(\mathfrak{e}_\theta)] = -\frac{1}{2}R(\mathfrak{w}_\theta)\mathfrak{v}_\theta$$

$$\mathfrak{L}_{\mathrm{GR}}(\mathfrak{w}_\theta) = -2\frac{\det[\boldsymbol{\eta}_\theta]^{1/2}}{\hbar\kappa}\mathrm{Tr}_{SO}[\mathfrak{R}_\theta^+(\mathfrak{w}_\theta)\wedge\mathfrak{S}_\theta^+(\mathfrak{e}_\theta)],\ \mathfrak{T}^a(\mathfrak{e}_\theta)$$

$$\frac{1}{3!}\mathfrak{v} = -f(\theta)\mathrm{Tr}_{SO}[\mathfrak{S}_\theta^+(\mathfrak{e}_\theta)\wedge\mathfrak{S}_\theta^+(\mathfrak{e}_\theta)] = f(\theta)\mathrm{Tr}_{SO}[\mathfrak{S}_\theta^-(\mathfrak{e}_\theta)\wedge\mathfrak{S}_\theta^-(\mathfrak{e}_\theta)]$$

$$f(\theta) := \frac{i}{4}(3\kappa(-\theta)+\kappa(\theta)) = \begin{cases} 1, & \theta = 0 \\ -i/2, & \theta = \pm 1 \end{cases}$$

$$\Xi_{\mathrm{w}}(\theta_\nu,\theta_e) := \boldsymbol{\xi}^\nu(\theta_\nu)\otimes\hat{\boldsymbol{\phi}}^{\mathrm{U}}_{\mathrm{w}} + \boldsymbol{\xi}^e(\theta_e)\otimes\hat{\boldsymbol{\phi}}^{\mathrm{D}}_{\mathrm{w}} \longrightarrow \begin{cases} & \boldsymbol{\xi}^\nu\otimes\hat{\boldsymbol{\phi}}^{\mathrm{U}}_{\mathrm{W}} + \boldsymbol{\xi}^e\otimes\hat{\boldsymbol{\phi}}^{\mathrm{D}}_{\mathrm{w}} \\ \dot{\boldsymbol{\xi}}^\nu\otimes\hat{\boldsymbol{\phi}}^{\mathrm{U}}_{\mathrm{W}} + \dot{\boldsymbol{\xi}}^e\otimes\hat{\boldsymbol{\phi}}^{\mathrm{D}}_{\mathrm{W}}, & \theta_\nu = \theta_e = +1 \\ & \theta_e = -1 \end{cases}$$

$$\mathcal{L}^{\theta L}_{\mathrm{FM}}(\Xi_{\mathrm{W}},\Xi^{\dagger_\epsilon}_{\mathrm{W}}) := -\langle\Xi^*_{\mathrm{W}}(\theta_\nu,\theta_e), i\phi^{sg}_\Theta\Xi_{\mathrm{W}}(\theta_\nu,\theta_e)\rangle$$

$$i\phi^{sg}_\Theta := i\phi^{sg}_{\theta_\nu}\hat{\phi}^{\mathrm{U}}_{\mathrm{W}} + i\phi^{sg}_{\theta_e}\hat{\phi}^{\mathrm{D}}_{\mathrm{W}},\ ,\ i\phi^{sg}_\theta := \gamma^\circ_\theta\left(\mathbf{1}_{SU}\otimes\left(i\partial_\circ - \frac{1}{2}c_{gr}\omega^{\star\star}_{\theta\circ}S_{\theta\star\star}\right) + c_{SU}\mathcal{A}^I_\circ\tau_I\right)$$

$$\overset{\theta_\nu=\theta_e=-1}{\rightarrow}\ \mathcal{L}^L_{\mathrm{F}M}$$

$$\hat{d}_{\mathfrak{A}_\theta}\mathfrak{F}_\theta = c_{SU}\Xi^{\dagger_\epsilon}(\theta_\nu,\theta_e)\gamma^\circ_{S\theta}\Xi(\theta_\nu,\theta_e)\mathfrak{V}_{\theta\circ}\ \overset{\theta_\nu=\theta_e=-1}{\rightarrow}$$

$$\mathcal{L}^{\theta R}_{\mathrm{FM}}(\boldsymbol{\xi}^e,\boldsymbol{\xi}^{e\dagger_\epsilon}) = \langle\boldsymbol{\xi}^{e*}(\theta), i\chi^{sp}_\theta\boldsymbol{\xi}^e(\theta)\rangle\ \overset{\theta=1}{\rightarrow}\ \mathcal{L}^R_{\mathrm{FM}}$$

$$i\phi^{sp}_\theta := \gamma^\circ_\theta\left(i\partial_\circ - \frac{1}{2}c_{gr}\omega^{\star\star}_{\theta\circ}S_{\theta\star\star}\right)$$

$$\begin{array}{ccccccccc}
\theta=-1 & \longleftrightarrow & & & \theta=0 & & \longleftrightarrow & \theta=+1 \\
\downarrow & & \swarrow & & & \searrow & & \downarrow \\
(\dot{\boldsymbol{\xi}}^\nu & \otimes & \hat{\phi}^{\mathrm{U}}_{\mathrm{W}}) & & & (\hat{\phi}'^{\mathrm{U}}_{\mathrm{W}} & \otimes & \boldsymbol{\xi}^\nu) \\
 & + & & & & & + & \\
(\dot{\boldsymbol{\xi}}^e & \otimes & \hat{\phi}^{\mathrm{D}}_{\mathrm{W}}) & & & (\hat{\phi}'^{\mathrm{D}}_{\mathrm{W}} & \otimes & \boldsymbol{\xi}^e) \\
\Cup & \| & \Cup & & & \Cup & \| & \Cup \\
\underline{\mathfrak{sl}(2,\mathbb{C})} & \dot{\Xi}_{\mathrm{W}} & (\mathfrak{su}_L(2) & & \oplus & \mathfrak{su}_R(2)) & \Xi_{\mathrm{W}} & \mathfrak{sl}(2,\mathbb{C}) \\
 & & & & \| & & & \\
 & & & & \mathfrak{so}(4) & & &
\end{array}\ .$$

$$\rightarrow \mathfrak{L}_{\mathrm{H}}(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)) := \mathfrak{L}^{kin}_{\mathrm{H}}(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)) - V(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta))\mathfrak{v}_\theta$$

$$\mathfrak{L}^{kin}_{\mathrm{H}}(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)) := \frac{1}{4}(i\boldsymbol{d}_\theta)\circ(id_\theta)\langle\boldsymbol{\xi}(\theta),\boldsymbol{\xi}(\theta)\rangle\mathfrak{v}_\theta$$

$$\frac{\delta \mathcal{I}_{\mathrm{H}}\left(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)\right)}{\delta \boldsymbol{\xi}^{\dagger\theta}(\theta)} = 0 \Longrightarrow \left(\Delta_\theta + ic_{SU}\mathcal{X}_\theta + \frac{\delta V\left(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)\right)}{\delta \boldsymbol{\xi}^{\dagger}(\theta)}\right)\boldsymbol{\xi}(\theta) = \mathcal{F}_\theta := \mathcal{F}_{SU}|_{\gamma_{\mathrm{SL}}\to\gamma_\theta}$$

$$\to -(id_\theta)^2 = \Delta_\theta + \widehat{\mathfrak{F}}_\theta, \hat{\mathcal{F}}_\theta := \iota_{\gamma_\theta}\iota_{\gamma_\theta}\left(d_{sg}\wedge d_{sg}\right)$$

$$\widehat{\mathfrak{B}}_\theta = \iota_{\gamma_\theta}\iota_{\gamma_\theta}\left(d - ic_{SU}\left(\mathfrak{A}_{SU}\otimes \mathbf{1}_{sp}\right) - ic_{gr}\left(\mathbf{1}_{SU}\otimes\mathfrak{A}_{sp}\right)\right)^{2\wedge}$$

$$= -i\iota_{\gamma_\theta\iota_{\gamma_\theta}}\left(c_{SU}(d\mathfrak{A}_{SU} - ic_{SU}\mathfrak{A}_{SU}\wedge\mathfrak{A}_{SU}) + c_{gr}(d\mathfrak{A}_{sp} - ic_{gr}\mathfrak{A}_{sp}\wedge\mathfrak{A}_{sp})\right)$$

$$\mathfrak{A}_{sp} = \frac{i}{2}\left(\iota_{\gamma_\theta}\mathfrak{w}_\theta^{\circ\circ}\right)S_{\mathrm{SL}\circ\circ}\Big|_{\gamma_s\to\gamma_\theta} = -\frac{1}{8}\omega_\circ^{\circ\star}\gamma_\theta^\circ\eta_{\theta\circ\circ}\eta_{\theta\star\star}[\gamma_\theta^\circ,\gamma_\theta^\star] = -\frac{1}{4}\omega_\circ^{\circ\star}\gamma_\theta^\circ\eta_{\theta\circ\circ}\eta_{\theta\star\star}\gamma_\theta^\circ\gamma_\theta^\star$$

$$\widehat{\mathfrak{R}} = -c_{gr}\iota_{\gamma_\theta}\iota_{\gamma_\theta}\mathfrak{R} = -\frac{c_{gr}}{8}R_\theta^{\circ\circ}{}_{\star\star}\gamma_{\theta\circ}\gamma_{\theta\circ}(\eta_\theta{}^{\star\circ}\gamma_{\theta\circ})(\eta_\theta{}^{\star\circ}\gamma_{\theta\circ})$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} \otimes R^a_{bcd} := \eta_{\mathrm{L}b\circ}R^{a\circ}{}_{cd}$$

$$\sum_{c\neq b\neq d} R^a_b{}_{cd}\gamma^b_{\mathrm{SL}}\gamma^c_{\mathrm{SL}}\gamma^d_{\mathrm{SL}}$$

$$R_{abcd}\gamma^a_{\mathrm{SL}}\gamma^b_{\mathrm{SL}}\gamma^c_{\mathrm{SL}}\gamma^d_{\mathrm{SL}} \triangleq \eta_{\mathrm{L}}^{\circ\circ}(-R_{\circ ba\circ} + R_{\circ b\circ a})\gamma^a_{\mathrm{SL}}\gamma^b_{\mathrm{SL}} = 2\eta_{\mathrm{L}}^{\circ\circ}R_{b\circ a\circ}\gamma^a_{\mathrm{SL}}\gamma^b_{\mathrm{SL}},$$

$$\triangleq 2R^{\circ\star}_{\circ\star} \overset{\theta=1}{\to} -c_{gr}\frac{R}{4}$$

$$\to -(id_\theta)^2 = \Delta(\theta) - ic_{SU}\mathcal{Z}_\theta - c_{gr}\frac{R_\theta}{4}$$

$$\Longrightarrow \left(\Delta_\theta + ic_{SU}\mathcal{B}_\theta + c_{gr}\frac{R_\theta}{4} + \frac{\delta V\left(\boldsymbol{\xi}(\theta),\boldsymbol{\xi}^{\dagger\theta}(\theta)\right)}{\delta\boldsymbol{\xi}_\theta(\theta)}\right)\boldsymbol{\xi}(\theta)$$

$$\Longrightarrow \left\langle \Xi_{\mathrm{W}}(\theta_\nu,\theta_e), \boldsymbol{\xi}^{\mathrm{H}}(\theta_{\mathrm{H}} = 0)\right\rangle$$

$$\xi^{\mathrm{H}}(0) = \phi_{\mathrm{W}}$$

$$\mathcal{L}_{\mathrm{FM}}^{\theta m_e} = \lambda\left\langle\langle \Xi^*_{\mathrm{W}}(-\theta_\nu,-\theta_e),\phi_{\mathrm{W}}\rangle, \Xi_{\mathrm{W}}(\theta_\nu,\theta_e)\right\rangle = -\lambda\phi_{\mathrm{D}}(\langle\boldsymbol{\xi}^{e*}_{\mathrm{U}}(-\theta_e),\boldsymbol{\xi}^e_{\mathrm{D}}(\theta_e)\rangle - \langle\boldsymbol{\xi}^{e*}_{\mathrm{D}}(-\theta_e),\boldsymbol{\xi}^e_{\mathrm{U}}(\theta_e)\rangle)$$

$$-\lambda\phi_{\mathrm{U}}(\langle\boldsymbol{\xi}^{\nu*}_{\mathrm{U}}(-\theta_\nu),\boldsymbol{\xi}^\nu_{\mathrm{D}}(\theta_\nu)\rangle - \langle\boldsymbol{\xi}^{\nu*}_{\mathrm{D}}(-\theta_\nu),\boldsymbol{\xi}^\nu_{\mathrm{U}}(\theta_\nu)\rangle)$$

$$\Lambda|_{\theta^\nu=\theta^e=1} \Longrightarrow \delta$$

$$\begin{array}{ccccc}
\mathfrak{R}^+_{\mathrm{L}} & \in \Omega^2_{\mathrm{SD}}(T^*\mathcal{M}_{\mathrm{L}}) & \oplus & \mathrm{Ad}(\mathfrak{C}_1 = \mathfrak{sl}(2,\mathbb{C})) \\
\uparrow & & & \uparrow\theta\to 1 \\
\uparrow & \theta\to 1\uparrow & & \\
\mathfrak{F}^+_\theta & \in \Omega^2_{\mathrm{SD}}(T^*\mathcal{W}^+_\theta) & \oplus & \mathrm{Ad}(\mathfrak{C}_\theta) \\
\downarrow & \theta\to 1\downarrow & & \downarrow\theta\to 0 \\
\mathfrak{F}^+_{SU} & \in \Omega^2_{\mathrm{SD}}(T^*\mathcal{M}_{\mathrm{L}}) & \oplus & \mathrm{Ad}(\mathfrak{C}_0 = \mathfrak{su}_R(2))
\end{array}$$

$$\begin{array}{ccccccccc}
\Xi(\theta_1,\theta_2) & = & \boldsymbol{\xi}(\theta_1) & \otimes & & & \boldsymbol{\xi}(\theta_2) & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\Xi(-1,0) & = & \dot{\boldsymbol{\xi}} & \otimes & (\phi^{\mathrm{U}}_{\mathrm{W}} & & \oplus & & \phi^{\mathrm{D}}_{\mathrm{W}}) \\
 & & \Cup & & \Cup & & & & \Cup \\
 & & \Gamma_{\mathrm{W}}\left(\mathcal{M}_{\mathrm{L}},\mathbb{V}^{\mathrm{L}}_{1/2}\right) & \otimes & \left(\Gamma_{\mathrm{W}}\left(\mathcal{M}_{\mathrm{L}},\mathbb{V}^{\mathrm{EU}}_{1/2}\right)\right. & & \oplus & & \left.\Gamma_{\mathrm{W}}\left(\mathcal{M}_{\mathrm{L}},\mathbb{V}^{\mathrm{ED}}_{1/2}\right)\right)
\end{array}$$

$$\xrightarrow{\{P^-_{\mathrm{SE}},P^+_{\mathrm{SE}}\}} \qquad \left\{\dot{\Xi}^{\mathrm{U}}_{\mathrm{W}} = \dot{\boldsymbol{\xi}}\oplus\phi^{\mathrm{U}}_{\mathrm{W}}, \dot{\Xi}^{\mathrm{D}}_{\mathrm{W}} = \dot{\boldsymbol{\xi}}\oplus\phi^{\mathrm{D}}_{\mathrm{W}}\right\}.$$

$$\Longrightarrow -\frac{1}{2}c_{gr}\omega_0^{\star\star}(\phi^\dagger_{\mathrm{H}}\nu^\circ S_{\star\star}\boldsymbol{\phi}_{\mathrm{H}})$$

$$ds^2_{\mathrm{Schw}} = f^2_{\mathrm{Schw}}(r)dt^2 - f^{-2}_{\mathrm{Schw}}(r)dr^2 - r^2(d\vartheta^2 + \sin^2\ \varphi d\varphi^2),\ f^2_{\mathrm{Schw}}(r) = 1 - 16\pi\kappa_{\mathrm{E}}m_e/r$$

$$\mathrm{e}^a_{\mathrm{Schw}} = \left(f_{\mathrm{Schw}}\,dt, f^{-1}_{\mathrm{Schw}}\,dr, rd\vartheta, r\sin\ \vartheta d\varphi\right)$$

$$\mathfrak{w}_{\mathrm{Schw}} = \begin{pmatrix} 0 & -8\pi\kappa_{\mathrm{E}}m_e/r^2dt & 0 & 0 \\ & 0 & f_{\mathrm{Schw}}d\vartheta & f_{\mathrm{Schw}}\sin\ \vartheta d\varphi \\ & & 0 & \cos\ \vartheta d\varphi \\ & & & 0 \end{pmatrix}$$

$$\Lambda \approx -\frac{1}{2}c_{gr}\sum_{p_i\in\{t,r,\vartheta,\varphi\}}\eta_{p_1p_2}\eta_{p_3p_4}\omega_t^{p_1p_3}\gamma^t S^{p_2p_4}$$

$$\gamma^t = \gamma^0_{\mathrm{VL}},$$
$$\gamma^r = \sin\vartheta\cos\varphi\gamma^1_{\mathrm{VL}} + \sin\vartheta\sin\varphi\gamma^2_{\mathrm{VL}} + \cos\vartheta\gamma^3_{\mathrm{VL}},$$
$$\gamma^\vartheta = \cos\vartheta\cos\varphi\gamma^1_{\mathrm{VL}} + \cos\vartheta\sin\varphi\gamma^2_{\mathrm{VL}} + \sin\vartheta\gamma^3_{\mathrm{VL}},$$
$$\gamma^\varphi = -\sin\varphi\gamma^2_{\mathrm{VL}} + \cos\varphi\gamma^3_{\mathrm{VL}},$$
$$\eta_{tt} = 1, \eta_{rr} = \eta_{\vartheta\vartheta} = \eta_{\varphi\varphi} = -1,\ \eta_{ab} = 0,$$
$$\Longrightarrow \{\gamma^{p_1}, \gamma^{p_2}\} = 2\eta^{p_1p_2}_{\mathrm{L}}\mathbf{1}_{sp}\ \text{ for } p_i \in \{t, r, \vartheta, \varphi\}$$
$$\Delta = 4\pi i c_{gr}\frac{\kappa_{\mathrm{E}} m_e}{r^2}\langle\boldsymbol{\phi}_{\mathrm{H}}, \gamma^1_{\mathrm{VL}}\boldsymbol{\phi}_{\mathrm{H}}\rangle.$$
$$K := \begin{cases} SO(2) & (\mathbb{F} = \mathbb{R}) \\ SU(2) & (\mathbb{F} = \mathbb{C}) \end{cases}$$
$$T := \left\{ t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle|\ a \in \mathbb{F} \setminus \{0\} \right\}$$
$$A := \left\{ a(\varphi) = \begin{pmatrix} e^\varphi & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \middle|\ \varphi \in \mathbb{R} \right\}$$
$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle|\ b \in \mathbb{F} \right\}$$
$$B := TN^t = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \middle|\ a \in \mathbb{F} \setminus \{0\}, b \in \mathbb{F} \right\}$$
$$M := K \cap T = \begin{cases} \pm\mathbf{1}_2 & (\mathbb{F} = \mathbb{R}) \\ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} & (\mathbb{F} = \mathbb{C}) \end{cases}$$
$$T = MA, G = KTN = KAN,\ ,\ G = KTN^t = KAN^t$$
$$G = KAN = KAN^t$$
$$K := \left\{ k(\vartheta) = \begin{pmatrix} \cos(\vartheta/2) & i\sin(\vartheta/2) \\ i\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \middle|\ 0 \le \vartheta < 4\pi \right\}$$
$$M := \left\{ m(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \middle|\ 0 \le \varphi < 4\pi \right\}$$
$$m(\varphi)k(\vartheta)m(\psi) = \begin{pmatrix} \cos(\vartheta/2)e^{i(\varphi+\psi)/2} & i\sin(\vartheta/2)e^{i(\varphi-\psi)/2} \\ i\sin(\vartheta/2)e^{-i(\varphi-\psi)/2} & \cos(\vartheta/2)e^{-i(\varphi+\psi)/2} \end{pmatrix}$$
$$0 \le \psi < 4\pi, 0 \le \vartheta < \pi,\ ,\ 0 \le \varphi < 2\pi$$
$$d\psi d\phi = \hat{\boldsymbol{v}}_{\mathrm{SE}} \Longrightarrow \hat{\boldsymbol{v}}_{\mathrm{VE}} = \Sigma_{\mathrm{E}}^{-1}\hat{\boldsymbol{v}}_{\mathrm{SE}} = \begin{pmatrix} \cos(\vartheta/2)\cos((\varphi+\psi)/2) \\ \sin(\vartheta/2)\cos((\varphi-\psi)/2) \\ -\sin(\vartheta/2)\sin((\varphi-\psi)/2) \\ \cos(\vartheta/2)\sin((\varphi+\psi)/2) \end{pmatrix}$$
$$\overline{\mathfrak{V}}^0_{\mathrm{E}} = \frac{1}{16\pi^2}\sin\vartheta\sin\varphi d\vartheta d\varphi d\psi$$
$$\hat{\boldsymbol{v}}_{\mathrm{VE}} = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi\cos\psi \\ \sin\vartheta\sin\varphi\sin\psi \end{pmatrix},\ 0 \le \psi \le 2\pi, 0 \le \vartheta < \pi,\ ,0 \le \varphi < \pi$$
$$\overline{\mathfrak{V}}^0_{\mathrm{E}} = \frac{1}{2\pi^2}\sin^2\vartheta\sin\varphi d\vartheta d\varphi d\psi$$
$$\Delta_\theta u(x,\theta) = f(x,\theta)$$
$$u(x,\theta = 0) \in W^2_2$$
$$u(x,\theta) := \|\mathfrak{u},\mathfrak{u}\|_\theta^{\frac{1}{2}}\ ,\ f(x,\theta) := \|\mathfrak{f},\mathfrak{f}\|_\theta^{\frac{1}{2}}$$
$$\|u\|_{L_2} \le N\|\Delta_0 u\|_{L_2} = N\|f\|_{L_2}$$
$$\|v(\theta)\|_{L_2} := \left(\int_{D(\Delta_0)} |v(x,\theta)|^2 \mathfrak{v}_{\mathrm{E}}\right)^{\frac{1}{2}} \ge 0$$
$$\Delta_{\theta_0+\delta_\theta} u(x,\theta_0+\delta_\theta) = f(x,\theta_0+\delta_\theta)$$
$$|\delta_\theta| < \left|\Delta_{\theta_0}\partial_\theta u(x,\theta)\right|_{\theta\to\theta_0}$$

$$\Delta_{\theta_0+\delta_\theta} u(x,\theta_0+\delta_\theta)$$
$$=\left(\Delta_{\theta_0} - i\pi e^{-i\pi\theta}\delta_\theta(\partial_\theta)^2\big|_{\theta\to\theta_0}\right)\left(u(x,\theta)+\delta_\theta[\partial_\theta u(x,\theta)]_{\theta\to\theta_0}\right),$$
$$\simeq f(x,\theta_0)+\delta_\theta\left[\left(\Delta_\theta - i\pi e^{-i\pi\theta}(\partial_\theta)^2\right)(\partial_\theta u(x,\theta))\right]_{\theta\to\theta_0}$$
$$=f(x,\theta_0)+\delta_\theta[\partial_\theta(\Delta_\theta u(x,\theta))]_{\theta\to\theta_0}$$
$$=f(x,\theta_0)+\delta_\theta[\partial_\theta f(x,\theta)]_{\theta\to\theta_0}$$
$$=f(x,\theta_0+\delta_\theta)+\mathcal{O}\left(\delta_\theta^2\right)$$
$$d(x,\theta)=\sum_i\prod_j\left(e^{i\pi\theta}d_0^{ij}(x)+d_1^{ij}(x)\right)^{\alpha_{ij}}$$
$$\Omega_C^n := \{n\text{-dimensional}\}/\sim$$
$$(\partial W,\partial\omega_W)=(M,-\omega_M)\amalg(N,\omega_N)$$
$$f(p_1,p_2,\cdots)[M]=\int_M f(p(M))=\int_{\partial M}\iota^\#(f(p(W)))=\int_W df(p(W))$$
$$\mathcal{M}_{\mathrm{E}}\hookrightarrow\mathcal{W}_\theta:\mathcal{M}_{\mathrm{E}}=\mathcal{W}_0,\ \mathcal{M}_{\mathrm{L}}\hookrightarrow\mathcal{W}_\theta:\mathcal{M}_{\mathrm{L}}=\mathcal{W}_1$$
$$\langle v\mid u\rangle_\theta := \sum_{a,b=1,n}[\boldsymbol{\eta}_\theta]_{ab}v^a u^b\in\mathbb{C},\eta_\theta := \mathrm{diag}\left(1,e^{i\pi\theta},e^{i\pi\theta},\cdots,e^{i\pi\theta}\right)$$
$$\pi_\theta:\mathcal{M}_{\mathrm{E}}\to\mathcal{M}_{\mathrm{L}}:(x^1,\cdots,x^n,0)\in\mathcal{W}_0\mapsto(x^1,\cdots,x^n,1)\in\mathcal{W}_1$$
$$\mathcal{A}_\theta(\rho)=i\kappa(\theta)\frac{\rho^2}{\rho^2+\lambda^2}\boldsymbol{g}(\rho)d\boldsymbol{g}^{-1}(\rho)\Big)\ 0<\lambda\in\mathbb{R}$$
$$\boldsymbol{g}(\boldsymbol{\rho}) := \frac{1}{\sqrt{\boldsymbol{\rho}^2}}\boldsymbol{\sigma}_\theta\boldsymbol{\rho},\ \rho^2 := \eta_{\theta\circ\circ}\rho^\circ\rho^\circ$$
$$\mathfrak{A}_\theta^I=\left[\mathcal{A}_\theta^I\right]_\circ\mathfrak{e}_\theta^\circ=\frac{1}{\rho^2+\lambda^2}\times\begin{cases}x_\theta dt-tdx_\theta+z_\theta dy_\theta-y_\theta dz_\theta, & (I=1)\\ y_\theta dt-z_\theta dx_\theta-tdy_\theta+x_\theta dz_\theta, & (I=2)\\ z_\theta dt+y_\theta dx_\theta-x_\theta dy_\theta-tdz_\theta, & (I=3)\end{cases}$$
$$\mathfrak{F}_\theta^I=\frac{1}{2}\left[\mathcal{F}_\theta^I\right]_{\circ\circ}\mathfrak{e}_\theta^\circ\wedge\mathfrak{e}_\theta^\circ=-\frac{\lambda^2}{\rho^2+\lambda^2}\times\begin{cases}dt\wedge dx_\theta+dy_\theta\wedge dz_\theta=\mathfrak{S}_\theta^{+1} & (I=1)\\ dt\wedge dy_\theta-dx_\theta\wedge dz_\theta=\mathfrak{S}_\theta^{+2} & (I=2)\\ dt\wedge dz_\theta+dx_\theta\wedge dy_\theta=\mathfrak{S}_\theta^{+3} & (I=3)\end{cases}$$
$$\mathfrak{F}_\theta|_{\theta=0}=\mathfrak{F}_{\mathrm{E}}^+,\ \mathfrak{F}_\theta|_{\theta=1}=\mathfrak{F}_{\mathrm{L}}^+.$$
$$\boldsymbol{\rho}_{\mathrm{E}}^t := (r_{\mathrm{E}}\cos\vartheta_1,r_{\mathrm{E}}\sin\vartheta_1\cos\vartheta_2,r_{\mathrm{E}}\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3,r_{\mathrm{E}}\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3)$$
$$\int\mathfrak{v}_{\mathrm{E}}=\int_0^\infty dr_{\mathrm{E}}\int_0^\pi d\vartheta_1\int_0^\pi d\vartheta_2\int_0^{2\pi}d\vartheta_3 r_{\mathrm{E}}^3\sin^2\vartheta_1\sin\vartheta_2$$
$$r_{\mathrm{E}}\mapsto r'_{\mathrm{E}}=\begin{cases}r_{\mathrm{E}}/\lambda, & 0\le r_{\mathrm{E}}\le\lambda\\ \lambda/r_{\mathrm{E}}, & \lambda<r_{\mathrm{E}}<\infty\end{cases}$$
$$\overline{\mathcal{M}_{\mathrm{E}}}=D_{\mathrm{in}}^4\cup D_{\mathrm{out}}^4,\ D_{\mathrm{in}}^4\cap D_{\mathrm{out}}^4=\partial D_{\mathrm{in}}^4=-\partial D_{\mathrm{out}}^4=S^3$$
$$c_2(\mathfrak{F}_\theta)=-\frac{1}{8\pi^2}\int\mathrm{Tr}_{SU}[\mathfrak{F}_\theta\wedge\mathfrak{F}_\theta]=e^{3i\pi\theta/2}\frac{6}{\pi^2}\int\left(\frac{\lambda}{\boldsymbol{\rho}_\theta^2+\lambda^2}\right)^4\mathfrak{v}_\theta$$
$$\int_{D_{\mathrm{in}}^4}c_2(\mathfrak{F}_{\mathrm{E}})=\int_{D_{\mathrm{out}}^4}c_2(\mathfrak{F}_{\mathrm{E}})=\frac{6}{\pi^2}\int_0^1\frac{r_{\mathrm{E}}'^3}{(r_{\mathrm{E}}'^2+1)^4}dr'_{\mathrm{E}}\int_{S^3}d\Omega_4=\frac{1}{2}$$
$$\mathrm{ch}\left(\overline{\mathcal{M}_{\mathrm{E}}}\right)=\int_{D_{\mathrm{in}}^4}c_2(\mathfrak{F}_{\mathrm{E}})+\int_{D_{\mathrm{out}}^4}c_2(\mathfrak{F}_{\mathrm{E}})=1.$$
$$\partial D_{\mathrm{in}}\ni\vartheta_i^{\mathrm{in}}=\vartheta_i^{\mathrm{out}}\in-\partial D_{\mathrm{out}}$$

$$\mathrm{ch}(\overline{\mathcal{M}_\theta}) = e^{i\pi\theta/2}\frac{\Lambda^4}{\pi^2}\int_0^1 dr_\theta \int_0^\pi d\vartheta_1 \int_0^\pi d\vartheta_2 \int_0^{2\pi} d\vartheta_3 r_\theta^3 \sin^2\,\vartheta_1 \sin\,\vartheta_2$$

$$\times\left[\left(2 + r_\theta^2\left(1 + \cos\,2\vartheta_1 + 2e^{i\pi\theta}\sin^2\,\vartheta_1\right)\right)^{-4} + \left(1 + 2r_\theta^2 + \cos\,2\vartheta_1 + 2e^{i\pi\theta}\sin^2\,\vartheta_1\right)^{-4}\right]$$

$$= e^{i\pi\theta/2}\frac{2^3}{\pi}\int_0^\pi d\vartheta_1 \frac{\sin^2\,\vartheta_1}{(1 + \cos\,2\vartheta_1 + 2e^{i\pi\theta}\sin^2\,\vartheta_1)^2}$$

$$= \begin{cases} +1 & 4n - 1 < \theta < 4n + 1 \\ -1 & 4n + 1 < \theta < 4n + 3, \quad n \in \mathbb{Z} \\ \text{indefinido} & \theta = 2n + 1 \end{cases}$$

$$\mathrm{ch}(\mathcal{M}_\mathrm{L}) := \mathrm{ch}(\mathcal{M}_{\theta=1-\epsilon})|_{\epsilon\to+0} = 1$$

$$\mathfrak{F}_\mathrm{L}^I = -\frac{\lambda^2}{\rho^2 + \lambda^2}\mathfrak{S}_\mathrm{L}^{+a=I}$$

$$\mathcal{A}_\theta(\rho) = \Gamma|_{\lambda\to\lambda_\theta},\ \lambda_\theta := i\kappa(\theta)\lambda$$

$$\Gamma \implies \mathfrak{F}_\theta^I = \begin{cases} -\lambda^2/(\rho^2 + \lambda^2)\mathfrak{S}_\mathrm{E}^{+a=I}, & \theta = 0 \\ +\lambda^2/(\rho^2 - \lambda^2)\mathfrak{S}_\mathrm{L}^{+a=I}, & \theta = \pm 1 \end{cases}$$

$$1 = \mathrm{ch}(\mathcal{M}_\theta) = \mathrm{ch}(\mathcal{M}_\mathrm{E}) = \mathrm{ch}(\mathcal{M}_{\theta=1-\epsilon})|_{\epsilon\to+0} =: \mathrm{ch}(\mathcal{M}_\mathrm{L})$$

$$\mathfrak{A}_\mathrm{L}^I = \left[\mathcal{A}_\mathrm{L}^I\right]_\mathrm{o}\mathfrak{e}_\mathrm{L}^\circ = \frac{1}{\rho^2 - \lambda^2}\times\begin{cases} xd\tilde{t} - \tilde{t}dx - zdy + ydz, & (I = 1) \\ yd\tilde{t} + zdx - \tilde{t}dy - xdz, & (I = 2), \\ zd\tilde{t} - ydx + xdy - \tilde{t}dz, & (I = 3) \end{cases}, \rho^2 = t^2 - x^2 - y^2 - z^2$$

REFERENCIAS BIBLIOGRÁFICAS ADICIONALES.

Yoshimasa Kurihara, Yang–Mills–Utiyama Theory and Graviweak Correspondence, arXiv:2501.04738v1 [gr-qc] 8 Jan 2025.

Apéndice F.

1. Supergravedad cuántica, agujeros negros cuánticos, supersimetrías de gauge y supermembranas para espacios cuánticos curvos o relativistas.

$$\Gamma^{012}\varepsilon = -\varepsilon, \Gamma^{013456}\varepsilon = \varepsilon$$

$$\delta\psi_\mu \equiv \nabla_\mu\epsilon + \frac{1}{288}\left(\Gamma_\mu^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu\Gamma^{\rho\lambda\sigma}\right)F_{\nu\rho\lambda\sigma}\epsilon$$

$$\Gamma^{012345678910} = \mathbb{1}$$

$$\Gamma^{0178910}\varepsilon = -\varepsilon$$

$$ds_{11}^2 = e^{2A_0}\Big[-dt^2 + dy^2 + e^{-3A_0}(-\partial_z w)^{-\frac{1}{2}}d\vec{u}\cdot d\vec{u} + e^{-3A_0}(-\partial_z w)^{\frac{1}{2}}d\vec{v}\cdot d\vec{v} \\ +(-\partial_z w)\left(dz + (\partial_z w)^{-1}\left(\vec{\nabla}_{\vec{u}}w\right)\cdot d\vec{u}\right)^2\Big]$$

$$e^0 = e^{A_0}dt, e^1 = e^{A_0}dy, \qquad e^2 = e^{A_0}(-\partial_z w)^{\frac{1}{2}}\left(dz + (\partial_z w)^{-1}\left(\vec{\nabla}_{\vec{u}}w\right)\cdot d\vec{u}\right), \\ e^{i+2} = e^{-\frac{1}{2}A_0}(-\partial_z w)^{-\frac{1}{4}}du_i, \qquad e^{i+6} = e^{-\frac{1}{2}A_0}(-\partial_z w)^{\frac{1}{4}}dv_i, i = 1,2,3,4$$

$$C^{(3)} = -e^0\wedge e^1\wedge e^2 + \frac{1}{3!}\epsilon_{ijk\ell}\left((\partial_z w)^{-1}\left(\partial_{u_\ell}w\right)du^i\wedge du^j\wedge du^k - \left(\partial_{v_\ell}w\right)dv^i\wedge dv^j\wedge dv^k\right)$$

$$\mathcal{L}_u \equiv \nabla_{\vec{u}}\cdot\nabla_{\vec{u}}, \mathcal{L}_v \equiv \nabla_{\vec{v}}\cdot\nabla_{\vec{v}}$$

$$\mathcal{L}_v G_0 = (\mathcal{L}_u G_0)(\partial_z\partial_z G_0) - (\nabla_{\vec{u}}\partial_z G_0)\cdot(\nabla_{\vec{u}}\partial_z G_0)$$

$$w = \partial_z G_0, e^{-3A_0}(-\partial_z w)^{\frac{1}{2}} = \mathcal{L}_v G_0$$

$$e^{-3A_0}(\partial_z w)^{-\frac{1}{2}} - (\partial_z w)^{-1}(\nabla_{\vec{u}}w)\cdot(\nabla_{\vec{u}}w) = -\mathcal{L}_u G_0$$

$$ds_{11}^2 = e^{2A_0}[-dt^2 + dy^2 + (-\partial_z w)(dz + (\partial_z w)^{-1}(\partial_u w)du)^2 \\ +e^{-3A_0}(-\partial_z w)^{-\frac{1}{2}}(du^2 + u^2 d\Omega_3^2) + e^{-3A_0}(-\partial_z w)^{\frac{1}{2}}\left(dv^2 + v^2 d\Omega'^2_3\right)\Big]$$

$$e^0 = e^{A_0}dt, \qquad e^1 = e^{A_0}dy, e^2 = e^{A_0}(-\partial_z w)^{\frac{1}{2}}(dz + (\partial_z w)^{-1}(\partial_u w)du), \\ e^3 = e^{-\frac{1}{2}A_0}(-\partial_z w)^{-\frac{1}{4}}du, \qquad e^4 = e^{-\frac{1}{2}A_0}(-\partial_z w)^{\frac{1}{4}}dv \\ e^{i+4} = e^{-\frac{1}{2}A_0}(-\partial_z w)^{-\frac{1}{4}}\sigma_i, \qquad e^{i+7} = e^{-\frac{1}{2}A_0}(-\partial_z w)^{\frac{1}{4}}\tilde{\sigma}_i, i = 1,2,3$$

$$C^{(3)} = -e^0\wedge e^1\wedge e^2 + (\partial_z w)^{-1}(u^3\partial_u w)\mathrm{Vol}(S^3) + (v^3\partial_v w)\mathrm{Vol}(S'^3)$$

$$G_0 = -\frac{1}{2}z^2\hat{g}_2(u,v) + z\hat{g}_1(u,v) + \hat{g}_0(u,v)$$

$$\mathcal{L}_{\vec{v}}\hat{g}_2 + \hat{g}_2\mathcal{L}_{\vec{u}}\hat{g}_2 - 2\left(\vec{\nabla}_{\vec{u}}\hat{g}_2\right)^2 = 0$$
$$\mathcal{L}_{\vec{v}}\hat{g}_1 + \hat{g}_2\mathcal{L}_{\vec{u}}\hat{g}_1 - 2\left(\vec{\nabla}_{\vec{u}}\hat{g}_1\right)\cdot\left(\vec{\nabla}_{\vec{u}}\hat{g}_2\right) = 0$$
$$\mathcal{L}_{\vec{v}}\hat{g}_0 + \hat{g}_2\mathcal{L}_{\vec{v}}\hat{g}_0 - \left(\vec{\nabla}_{\vec{u}}\hat{g}_1\right)^2 = 0$$

$$\mathcal{L}_{\vec{v}}\hat{g}_2 - \hat{g}_2^3\mathcal{L}_{\vec{u}}(\hat{g}_2^{-1}) = 0$$

$$\hat{g}_2 = \frac{h_2(\vec{v})}{h_1(\vec{u})}$$

$$G_0 = -\frac{1}{2}z^2\frac{h_2(\vec{v})}{h_1(\vec{u})} + \hat{g}_0(u, v)$$

$$\frac{1}{h_1(\vec{u})}\mathcal{L}_{\vec{u}}\hat{g}_0 + \frac{1}{h_2(\vec{v})}\mathcal{L}_{\vec{v}}\hat{g}_0 = 0$$

$$w = \partial_z G_0 = -z\frac{h_2(\vec{v})}{h_1(\vec{u})}$$

$$e^{-A_0}e^2 = (-\partial_z w)^{\frac{1}{2}}\left(dz + (\partial_z w)^{-1}\left(\vec{\nabla}_{\vec{u}}w\right)\cdot d\vec{u}\right) = \left(\frac{h_2(\vec{v})}{h_1(\vec{u})}\right)^{\frac{1}{2}}\left[dz - \frac{z}{h_1(\vec{u})}\left(\vec{\nabla}_{\vec{u}}h_1(\vec{u})\right)\cdot d\vec{u}\right]$$
$$= (h_1(\vec{u})h_2(\vec{v}))^{\frac{1}{2}}\left[\frac{dz}{h_1(\vec{u})} - \frac{z}{(h_1(\vec{u}))^2}\left(\vec{\nabla}_{\vec{u}}h_1(\vec{u})\right)\cdot d\vec{u}\right] = (h_1(\vec{u})h_2(\vec{v}))^{\frac{1}{2}}d\hat{z}$$

$$\hat{z} \equiv \frac{z}{h_1(\vec{u})}$$

$$\mathcal{L}_{\vec{u}}\hat{g}_0 = -h_0(\vec{u},\vec{v})h_1(\vec{u}), \mathcal{L}_{\vec{v}}\hat{g}_0 = h_0(\vec{u},\vec{v})h_2(\vec{v})$$

$$\hat{g}_0 = f_2(\vec{v})h_1(\vec{u}) - f_1(\vec{u})h_2(\vec{v}),\ \text{donde}\ \mathcal{L}_{\vec{u}}f_1 = h_1^2, \mathcal{L}_{\vec{v}}f_2 = h_2^2$$

$$G_0 = -\frac{1}{2}z^2\frac{h_2(\vec{v})}{h_1(\vec{u})} + f_2(\vec{v})h_1(\vec{u}) - f_1(\vec{u})h_2(\vec{v})$$

$$w = -z\frac{h_2(\vec{v})}{h_1(\vec{u})}, e^{-3A_0}\left(\frac{h_2(\vec{v})}{h_1(\vec{u})}\right)^{\frac{1}{2}} = h_1(\vec{u})h_2^2(\vec{v}) \ \Rightarrow\ e^{-2A_0} = h_1(\vec{u})h_2(\vec{v})$$

$$ds_{11}^2 = (h_1(\vec{u})h_2(\vec{v}))^{-1}(-dt^2 + dy^2) + d\hat{z}^2 + h_1(\vec{u})d\vec{u}\cdot d\vec{u} + h_2(\vec{v})d\vec{v}\cdot d\vec{v}$$

$$ds_{11}^2 = e^{2A}\left(\hat{f}_1^2 ds_{AdS_3}^2 + \hat{f}_2^2 ds_{S^3}^2 + \hat{f}_3^2 ds_{S'}^2 + h_{ij}d\sigma^i d\sigma^j\right)$$
$$C^{(3)} = b_1\hat{e}^{012} + b_2\hat{e}^{345} + b_3\hat{e}^{678}$$

$$h_{ij}d\sigma^i d\sigma^j = \frac{\partial_w h\partial_{\bar{w}}h}{h^2}|dw|^2$$

$$\partial_w\partial_{\bar{w}}h = 0$$

$$w = \xi + i\rho \;\Rightarrow\; \partial_w = \frac{1}{2}(\partial_\xi - i\partial_\rho), \partial_{\bar{w}} = \frac{1}{2}(\partial_\xi + i\partial_\rho)$$

$$\partial_{\bar{w}}(-\tilde{h} + ih) = 0$$

$$-\tilde{h} + ih = \beta w = \beta(\xi + i\rho)$$

$$h_{ij} d\sigma^i d\sigma^j = \frac{d\xi^2 + d\rho^2}{4\rho^2}$$

$$\partial_w G = \frac{1}{2}(G + \bar{G})\partial_w \log(h)$$

$$\partial_\xi g_1 + \partial_\rho g_2 = 0, \partial_\xi g_2 - \partial_\rho g_1 = -\frac{1}{\rho} g_1$$

$$\partial_w \Phi = \bar{G}\partial_w h \;\Leftrightarrow\; \partial_\xi \Phi = -\beta g_2, \partial_\rho \Phi = \beta g_1$$

$$\left(\partial_\xi^2 + \partial_\rho^2 - \frac{1}{\rho}\partial_\rho\right)\Phi = 0$$

$$\partial_\xi \tilde{\Phi} = -\frac{\beta}{\rho} g_1 = -\frac{1}{\rho}\partial_\rho \Phi, \partial_\rho \tilde{\Phi} = -\frac{\beta}{\rho} g_2 = \frac{1}{\rho}\partial_\xi \Phi$$

$$\partial_\xi^2 \tilde{\Phi} + \frac{1}{\rho}\partial_\rho\left(\rho \partial_\rho \tilde{\Phi}\right) = 0$$

$$W_\pm \equiv |G \pm i|^2 + \gamma^{\pm 1}(G\bar{G} - 1)$$

$$\gamma(G\bar{G} - 1) \geq 0$$

$$\gamma > 0, |G| \geq 1$$

$$c_1 = \gamma^{1/2} + \gamma^{-1/2} > 0, c_2 = -\gamma^{1/2} < 0, c_3 = -\gamma^{-1/2} < 0, \sigma = +1$$

$$\hat{f}_1^{-2} = \gamma^{-1}(\gamma + 1)^2(G\bar{G} - 1), \hat{f}_2^{-2} = W_+, \hat{f}_3^{-2} = W_-$$

$$e^{6A} = h^2(G\bar{G} - 1)W_+ W_- = \gamma(\gamma + 1)^{-2} h^2 \hat{f}_1^{-2}\hat{f}_2^{-2}\hat{f}_3^{-2}$$

$$b_1 = \frac{\nu_1}{c_1^3}\left[\frac{h(G + \bar{G})}{(G\bar{G} - 1)} + \gamma^{-1}(\gamma + 1)^2\Phi - (\gamma - \gamma^{-1})\tilde{h}\right],$$
$$b_2 = \frac{\nu_2}{c_2^3}\left[-\frac{h(G + \bar{G})}{W_+} + (\Phi - \tilde{h})\right], b_3 = \frac{\nu_3}{c_3^3}\left[\frac{h(G + \bar{G})}{W_-} - (\Phi + \tilde{h})\right]$$

$$ds_{AdS_3}^2 = \frac{d\mu^2}{\mu^2} + \mu^2(-dt^2 + dy^2)$$

$$\mu \to \lambda\mu, (t, y) \to \lambda^{-1}(t, y)$$

$$(u, v) \to \sqrt{\lambda}(u, v), z \to \lambda^{-1} z$$

$$e^{A_0} \to \lambda e^{A_0}, w \to \lambda^{-1} w$$

$$u = \sqrt{\mu} m_1(\rho, \xi), \qquad v = \sqrt{\mu} m_2(\rho, \xi), z = \mu^{-1} m_3(\rho, \xi)$$
$$w = \mu^{-1} m_4(\rho, \xi), \qquad e^{A_0} = \mu m_5(\rho, \xi),$$

$$e^{2A} \hat{f}_1^2 \mu^2 = e^{2A_0}, e^{2A} \hat{f}_2^2 = e^{-A_0} (-\partial_z w)^{-\frac{1}{2}} u^2, e^{2A} \hat{f}_3^2 = e^{-A_0} (-\partial_z w)^{\frac{1}{2}} v^2$$

$$e^{2A} \left(\hat{f}_1^2 \frac{d\mu^2}{\mu^2} + \frac{d\xi^2 + d\rho^2}{4\rho^2} \right) = e^{-A_0} \left((-\partial_z w)^{-\frac{1}{2}} du^2 + (-\partial_z w)^{\frac{1}{2}} dv^2 \right)$$
$$+ e^{2A_0} (-\partial_z w)(dz + (-\partial_z w)^{-1} (\partial_u w) du)^2$$

$$\frac{\gamma}{(1+\gamma)^2} \frac{1}{(G\bar{G} - 1)} \frac{d\mu^2}{\mu^2} + \frac{d\xi^2 + d\rho^2}{4\rho^2}$$
$$= \frac{1}{W_+} \frac{du^2}{u^2} + \frac{1}{W_-} \frac{dv^2}{v^2} + \frac{1}{\beta^2 \rho^2 (G\bar{G} - 1)} \frac{W_+}{W_-} \left(u^2 dz + (\partial_z w)^{-1} (u^3 \partial_u w) \frac{du}{u} \right)^2$$

$$u^2 v^2 = \frac{\beta^2 \gamma}{(\gamma + 1)^2} \mu^2 \rho^2, (-\partial_z w) \frac{v^2}{u^2} = \frac{W_+}{W_-}, e^{A_0} = \frac{\beta \sqrt{\gamma} \mu \rho}{(\gamma + 1)} e^{-2A} (W_+ W_-)^{\frac{1}{2}}$$

$$u = \sqrt{a \mu \rho} e^{\alpha(\rho, \xi)}, v = \sqrt{a \mu \rho} e^{-\alpha(\rho, \xi)}, z = \mu^{-1} e^{-2\alpha(\rho, \xi)} p(\rho, \xi)$$

$$a \equiv \frac{\beta \sqrt{\gamma}}{(\gamma + 1)}$$

$$(\partial_z w)^{-1} (u^3 \partial_u w) = b_2, (v^3 \partial_v w) = b_3$$

$$\frac{\gamma}{(1+\gamma)^2} \frac{1}{(G\bar{G} - 1)} \frac{d\mu^2}{\mu^2} + \frac{d\xi^2 + d\rho^2}{4\rho^2}$$
$$= \frac{1}{W_+} \frac{du^2}{u^2} + \frac{1}{W_-} \frac{dv^2}{v^2} + \frac{1}{\beta^2 \rho^2 (G\bar{G} - 1)} \frac{W_+}{W_-} \left(u^2 dz + b_2 \frac{du}{u} \right)^2$$

$$\partial_\xi \alpha = -\frac{\varepsilon_1}{2\rho} g_1, \partial_\rho \alpha = \frac{1}{2\rho} g_2, b_2 = 2a\rho p + \frac{\varepsilon_2 \beta \rho g_1}{g_1^2 + g_2^2 + g_2}$$
$$\partial_\xi p = -\frac{\varepsilon_1 \varepsilon_2 \beta}{2a\rho} (g_2 - 1), \partial_\rho p = -\frac{1}{\rho} \left(p + \frac{\varepsilon_2 \beta}{2a} g_1 \right)$$

$$\alpha = -\frac{1}{2\beta} \tilde{\Phi}$$

$$p = -\frac{\varepsilon_2}{2a\rho} (\Phi + \beta\xi)$$

$$b_2 = \varepsilon_2 \left(\frac{\beta \rho g_1}{g_1^2 + g_2^2 + g_2} - (\Phi + \beta\xi) \right) = \varepsilon_2 \left(\frac{h(G + \bar{G})}{W_+} - (\Phi - \tilde{h}) \right)$$

$$\gamma = 1, u = \left(\frac{1}{2} \beta \mu \rho \right)^{\frac{1}{2}} e^{-\frac{1}{2\beta} \tilde{\Phi}}, v = \left(\frac{1}{2} \beta \mu \rho \right)^{\frac{1}{2}} e^{+\frac{1}{2\beta} \tilde{\Phi}}, z = -\frac{\varepsilon_2}{\beta \rho \mu} e^{\frac{1}{\beta} \tilde{\Phi}} (\Phi + \beta\xi)$$

$$\partial_z w = -\frac{g_1^2 + g_2^2 + g_2}{g_1^2 + g_2^2 - g_2} e^{-\frac{2}{\beta}\tilde{\Phi}}, (\partial_z w)^{-1}(u^3 \partial_u w) = b_2, (v^3 \partial_v w) = b_3$$

$$dw = (\partial_z w)dz + (\partial_u w)du + (\partial_v w)dv = d\left[\frac{\varepsilon_2}{\beta\rho\mu} e^{-\frac{1}{\beta}\tilde{\Phi}}(\Phi - \beta\xi)\right]$$

$$w = \frac{\varepsilon_2}{\beta\rho\mu} e^{-\frac{1}{\beta}\tilde{\Phi}}(\Phi - \beta\xi)$$

$$b_3 = \varepsilon_2\left(\frac{\beta\rho g_1}{g_1^2 + g_2^2 - g_2} - (\Phi - \beta\xi)\right) = \varepsilon_2\left(\frac{h(G + \bar{G})}{W_-} - (\Phi + \tilde{h})\right)$$

$$u^2 z = -\frac{1}{2}\varepsilon_2(\Phi + \beta\xi), v^2 w = \frac{1}{2}\varepsilon_2(\Phi - \beta\xi)$$

$$\Phi \to -\Phi, \tilde{\Phi} \to -\tilde{\Phi} \;\Rightarrow\; u \leftrightarrow v, z \leftrightarrow w$$

$$\omega \equiv e^{3A_0}(-\partial_z w)^{\frac{1}{2}}(dz + (\partial_z w)^{-1}(\partial_u w)du)$$

$$\begin{aligned}
\omega &= \frac{W_+\mu^2}{4(G\bar{G} - 1)}\left(u^2 dz + (\partial_z w)^{-1}(u^3 \partial_u w)\frac{du}{u}\right) \\
&= \frac{\varepsilon_2}{4}\left[\left(\frac{\beta\rho g_1}{g_1^2 + g_2^2 - 1} + 2\Phi\right)\mu d\mu - d(\mu^2\Phi)\right] \\
&= \frac{\varepsilon_2}{8}\left[\left(\frac{h(G + \bar{G})}{(G\bar{G} - 1)} + 4\Phi\right)\mu d\mu - d(2\mu^2\Phi)\right] = \frac{\varepsilon_2}{\nu_1} b_1 \mu d\mu - \frac{\varepsilon_2}{4} d(\mu^2\Phi)
\end{aligned}$$

$$b_1 = \frac{\nu_1}{4}\left(\frac{\beta\rho g_1}{g_1^2 + g_2^2 - 1} + 2\Phi\right)$$

$$C^{(3)}_{tyz} = -e^0 \wedge e^1 \wedge e^2 = -dt \wedge dy \wedge \omega = -\frac{\varepsilon_2}{\nu_1} b_1 \mu dt \wedge dy \wedge d\mu + \frac{\varepsilon_2}{4} d(\mu^2\Phi dt \wedge dy)$$

$$C^{(3)}_{tyz} = b_1 \mu dt \wedge dy \wedge d\mu$$

$$\begin{aligned}
e^0 &= \frac{\mu e^A}{2\sqrt{G\bar{G} - 1}} dt, e^1 = \frac{\mu e^A}{2\sqrt{G\bar{G} - 1}} dy, \\
e^2 &= \frac{\varepsilon_2 e^A}{\rho\sqrt{(G\bar{G} - 1)W_+ W_-}}\left(\rho g_1 \frac{d\mu}{\mu} + (G\bar{G} - 1)(g_2 d\xi - g_1 d\rho)\right) \\
e^3 &= \frac{e^A}{2\sqrt{W_+}}\left(\frac{d\mu}{\mu} + \frac{d\rho}{\rho} + \frac{1}{\rho}(g_1 d\xi + g_2 d\rho)\right), \\
e^4 &= \frac{e^A}{2\sqrt{W_-}}\left(\frac{d\mu}{\mu} + \frac{d\rho}{\rho} - \frac{1}{\rho}(g_1 d\xi + g_2 d\rho)\right), \\
e^{i+4} &= \frac{e^A}{2\sqrt{W_+}}\sigma_i, e^{i+7} = \frac{e^A}{2\sqrt{W_-}}\tilde{\sigma}_i, i = 1,2,3
\end{aligned}$$

$$\partial_\xi b_1 = \varepsilon_2 \partial_\xi \left[-\frac{\beta\rho g_1}{4(G\bar{G}-1)} \right] + \frac{1}{2}\varepsilon_2 \beta g_2, \quad \partial_\rho b_1 = \varepsilon_2 \partial_\rho \left[-\frac{\beta\rho g_1}{4(G\bar{G}-1)} \right] - \frac{1}{2}\varepsilon_2 \beta g_1,$$
$$\partial_\xi b_2 = \varepsilon_2 \partial_\xi \left[-\frac{2\beta\rho g_1}{W_+} + \beta\xi \right] - \varepsilon_2 \beta g_2, \quad \partial_\rho b_2 = \varepsilon_2 \partial_\rho \left[-\frac{2\beta\rho g_1}{W_+} + \varepsilon_2 \beta\xi \right] + \varepsilon_2 \beta g_1,$$
$$\partial_\xi b_3 = \varepsilon_2 \partial_\xi \left[-\frac{2\beta\rho g_1}{W_-} - \beta\xi \right] - \varepsilon_2 \beta g_2, \quad \partial_\rho b_3 = \varepsilon_2 \partial_\rho \left[-\frac{2\beta\rho g_1}{W_-} - \beta\xi \right] + \varepsilon_2 \beta g_1$$

$$b_1 = -\varepsilon_2 \left(\frac{\beta\rho g_1}{4(G\bar{G}-1)} + \frac{1}{2}\Phi \right), b_2 = -\varepsilon_2 \left(\frac{2\beta\rho g_1}{W_+} - (\Phi + \beta\xi) \right), b_3 = -\varepsilon_2 \left(\frac{2\beta\rho g_1}{W_-} - (\Phi - \beta\xi) \right)$$

$$\Gamma^{012}\varepsilon = \eta_1 \varepsilon, \Gamma^{013567}\varepsilon = \eta_2 \varepsilon, \Gamma^{0148910}\varepsilon = -\eta_1\eta_2 \varepsilon$$

$$b_1 = \varepsilon_2 \eta_1 \left(\frac{\beta\rho g_1}{4(G\bar{G}-1)} + \frac{1}{2}\Phi \right), b_2 = \varepsilon_2 \eta_1 \eta_2 \left(\frac{2\beta\rho g_1}{W_+} - (\Phi + \beta\xi) \right)$$
$$b_3 = -\varepsilon_2 \eta_2 \left(\frac{2\beta\rho g_1}{W_-} - (\Phi - \beta\xi) \right)$$

$$\eta_1 = -1, \eta_2 = +1, \varepsilon_2 = +1$$

$$ds^2_{\mathrm{IIB}} = \sqrt{h_{11}} \left[-e^{3A} dt^2 + e^{3A} h_{ab} dr^a dr^b + \frac{e^{-3A}}{\det h} dw_2^2 + d\boldsymbol{y}_6^2 \right]$$
$$e^{2\phi} = \frac{h_{11}^2}{\det h}, C_0 = -\frac{h_{12}}{h_{11}}, B_2 = e^{3A} h_{1a} dt \wedge dr^a, C_2 = e^{3A} h_{2a} dt \wedge dr^a$$

$$h_{ab} = \frac{1}{2} \partial_a \partial_b K(r^1, r^2, \boldsymbol{y})$$

$$\Delta_y K + 2e^{-3A} = 0$$

$$r^1 \to z, r^2 \to u_1, w_2 \to u_2, y_1 \to u_3, y_2 \to y, y_{3,4,5,6} \to v_{3,4,5,6}$$

$$ds^2 = \frac{1}{\sqrt{\det h}}(-dt^2 + dy^2) + \frac{\sqrt{\det h}}{h_{11}}(du_2^2 + du_3^2) + \sqrt{\det h}\left(e^{3A} h_{ab} dr^a dr^b + ds^2_{\mathbb{R}^4}\right)$$
$$e^{2\phi} = \frac{\sqrt{\det h}}{h_{11}}, B_2 = \frac{h_{12}}{h_{11}} du_2 \wedge du_3$$

$$C_3 = e^{3A} h_{1a} dt \wedge dr^a \wedge dy - \frac{v^3}{2} \partial_v \partial_z K d\Omega_3'$$
$$C_5 = \frac{1}{h_{11}} dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy + \frac{v^3}{2} \left(\frac{h_{12}}{h_{11}} \partial_v \partial_z K - \partial_v \partial_{u_1} K \right) du_2 \wedge du_3 \wedge d\Omega_3'$$

$$ds^2_{11} = e^{\frac{-2\phi}{3}} ds^2_{10} + e^{\frac{4\phi}{3}} (dx + C_1)^2$$
$$C_3' = C_3 + B_2 \wedge dx$$

$$ds^2_{11} = \frac{h_{11}^{1/3}}{(\det h)^{2/3}}(-dt^2 + dy^2) + \frac{(\det h)^{1/3}}{h_{11}^{2/3}}(du_2^2 + du_3^2 + du_4^2)$$
$$+ (\det h)^{1/3} h_{11}^{1/3} \left(e^{3A} h_{ab} dr^a dr^b + ds^2_{\mathbb{R}^4} \right)$$

$$C_3 = \frac{h_{11}}{\det h} dt \wedge dz \wedge dy + \frac{h_{12}}{\det h} dt \wedge du_1 \wedge dy - \frac{h_{12}}{h_{11}} du_2 \wedge du_3 \wedge du_4 - \frac{v^3}{2} \partial_v \partial_z K d\Omega_3'$$

$$ds_{11}^2 = \frac{h_{11}^{1/3}}{(\det h)^{2/3}}(-dt^2 + dy^2) + \frac{(\det h)^{1/3}}{h_{11}^{2/3}}(du_1^2 + du_2^2 + du_3^2 + du_4^2)$$
$$+ \frac{h_{11}^{4/3}}{(\det h)^{2/3}}\left(dz + \frac{h_{12}}{h_{11}} du_1\right)^2 + (\det h)^{1/3} h_{11}^{1/3}(dv^2 + v^2 d\Omega_3'^2)$$

$$ds_{11}^2 = e^{2A_0}(-dt^2 + dy^2) + e^{-A_0}(-\partial_z w)^{-\frac{1}{2}}(du_1^2 + du_2^2 + du_3^2 + du_4^2)$$
$$+ e^{2A_0}(-\partial_z w)\left(dz + (\partial_z w)^{-1}(\partial_{u_1} w) du_1\right)^2 + e^{-A_0}(-\partial_z w)^{\frac{1}{2}}(dv^2 + v^2 d\Omega_3^2)$$
$$C^{(3)} = -e^{3A_0}(-\partial_z w)^{\frac{1}{2}} dt \wedge dy \wedge dz + e^{3A_0}(-\partial_z w)^{-\frac{1}{2}}(\partial_{x_1} w) dt \wedge dy \wedge dx_1$$
$$+ (-\partial_z w)^{-1}(\partial_{u_1} w) du_2 \wedge du_3 \wedge du_4 + (v^3 \partial_v w) d\Omega_3'$$

$$e^{2A_0} = \frac{h_{11}^{1/3}}{(\det h)^{2/3}}, h_{11} = -\partial_z w, h_{12} = -\partial_{u_1} w$$

$$t = \eta_0, y = \eta_1, z = \eta_2, \vec{u}, \vec{v} \text{ constante}$$

$$C_{tyz}^{(3)} = -e^0 \wedge e^1 \wedge e^2$$

$$\rho = k_1 \mu^{-1}, \tilde{\Phi}(\xi, \rho) = k_2$$

$$t = \eta_0, y = \eta_1, \mu = e^{\eta_2}, \xi = \sigma_1(\eta_2), \rho = \sigma_2(\eta_2)$$

$$d\hat{s}_3^2 = e^{2A}\left[\hat{f}_1^2\left(d\eta_2^2 + e^{2\eta_2}(-d\eta_0^2 + d\eta_1^2)\right) + \frac{(\sigma_1')^2 + (\sigma_2')^2}{4\sigma_2^2} d\eta_2^2\right]$$

$$\mathcal{L}_{\text{DBI}} = e^{3A}\hat{f}_1^2 e^{2\eta_2}\left(\hat{f}_1^2 + \frac{(\sigma_1')^2 + (\sigma_2')^2}{4\sigma_2^2}\right)^{\frac{1}{2}}$$
$$= h\hat{f}_1^2 e^{2\eta_2}\left[(G\bar{G} - 1)W_+ W_-\left(\hat{f}_1^2 + \frac{(\sigma_1')^2 + (\sigma_2')^2}{4\sigma_2^2}\right)\right]^{\frac{1}{2}}$$

$$\frac{(\sigma_1')^2 + (\sigma_2')^2}{\sigma_2^2} = \frac{g_1^2 + g_2^2}{g_1^2}$$

$$\left[(G\bar{G} - 1)W_+ W_-\left(\hat{f}_1^2 + \frac{(\sigma_1')^2 + (\sigma_2')^2}{4\sigma_2^2}\right)\right] = \left(\frac{W_+ W_-}{4g_1}\right)^2$$

$$\mathcal{L}_{\text{DBI}} = e^{2\eta_2}\frac{\beta\sigma_2((g_1^2 + g_2^2)^2 - g_2^2)}{4g_1(g_1^2 + g_2^2 - 1)}$$

$$\hat{C}^{(3)} = b_1 e^{2\eta_2} d\eta_0 \wedge d\eta_1 \wedge d\eta_2$$

$$\tilde{C}^{(3)} = e^{2\eta_2}\left(b_1 + 2\Lambda + (\partial_\xi \Lambda)\sigma_1' + (\partial_\rho \Lambda)\sigma_2'\right) d\eta_0 \wedge d\eta_1 \wedge d\eta_2$$

$$\tilde{C}^{(3)} = e^{2\eta_2}\frac{\nu_1}{c_1^3}\left[\frac{h(G+\bar{G})}{(G\bar{G}-1)} - 2(\partial_\xi\Phi)\sigma_1' - 2(\partial_\rho\Phi)\sigma_2'\right] d\eta_0 \wedge d\eta_1 \wedge d\eta_2$$
$$= \nu_1 e^{2\eta_2}\frac{\beta\sigma_2}{4}\left[\frac{g_1}{(g_1^2+g_2^2-1)} + g_2\frac{\sigma_1'}{\sigma_2} - g_1\frac{\sigma_2'}{\sigma_2}\right] d\eta_0 \wedge d\eta_1 \wedge d\eta_2$$

$$\sigma_2 = k_1 e^{-\eta_2}, \frac{\sigma_1'}{\sigma_2} = \frac{g_2}{g_1}$$

$$g_1\sigma_1' + g_2\sigma_2' = 0 \;\Leftrightarrow\; \partial_\xi\tilde{\Phi}\sigma_1' + \partial_\rho\tilde{\Phi}\sigma_2' = 0$$

$$\eta_0 = t, \eta_1 = -u_1, \eta_2 = u_2, \eta_3 = u_3, \eta_4 = y$$

$$d\tilde{s}_5^2 = \frac{1}{\sqrt{\mathrm{deth}}}(-dt^2 + dy^2) + \frac{h_{22}}{\sqrt{\mathrm{deth}}}du_1^2 + \frac{\sqrt{\mathrm{deth}}}{h_{11}}(du_2^2 + du_3^2)$$

$$\tilde{B}_2 = \frac{h_{12}}{h_{11}}du_2 \wedge du_3,$$
$$\tilde{C}_3 = -\frac{h_{12}}{\mathrm{deth}}dt \wedge du_1 \wedge dy,$$
$$\tilde{C}_5 = -\frac{1}{h_{11}}dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy$$

$$S_{\mathrm{DBI}} = -T_4\int d^5\sigma e^{-\phi}\sqrt{-\det(\tilde{G}_{\alpha\beta} + F_{\alpha\beta} + \tilde{B}_{\alpha\beta})} = -T_4\int d^5\sigma\frac{h_{22}}{\mathrm{deth}}$$
$$S_{WZ} = -T_4\int e^{\tilde{B}_2+\tilde{F}_2}\wedge\bigoplus_n \tilde{C}_n = T_4\int d^5\sigma\frac{h_{22}}{\mathrm{deth}}$$

$$h = -iw + i\bar{w}, G = \pm\left[i + \sum_{a=1}^{n+1}\frac{\zeta_a \mathrm{Im}(w)}{(\bar{w}-\xi_a)|w-\xi_a|}\right]$$

$$g_1 = \pm\sum_{a=1}^{n+1}\frac{\zeta_a\rho(\xi-\xi_a)}{((\xi-\xi_a)^2+\rho^2)^{\frac{3}{2}}}, g_2 = \pm\left[1 + \sum_{a=1}^{n+1}\frac{\zeta_a\rho^2}{((\xi-\xi_a)^2+\rho^2)^{\frac{3}{2}}}\right]$$

$$\tilde{\Phi} = \pm 2\left[-\log\rho + \sum_{a=1}^{n+1}\frac{\zeta_a}{\sqrt{(\xi-\xi_a)^2+\rho^2}}\right], \Phi = \mp 2\left[\xi + \sum_{a=1}^{n+1}\frac{\zeta_a(\xi-\xi_a)}{\sqrt{(\xi-\xi_a)^2+\rho^2}}\right]$$

$$ds_{11} = -e^{2A_0}dt^2 + e^{2A_1}(dy - Pdt)^2 + e^{2A_2}du^2 + e^{2A_3}dv^2 + u^2e^{2A_4}d\Omega_3^2 + v^2e^{2A_5}d\Omega_3'^2 + e^{2A_6}(dz + B_1du)^2$$

$$e^0 = e^{A_0}dt, e^1 = e^{A_1}(dy - Pdt), e^2 = e^{A_6}(dz + B_1du)$$
$$e^3 = e^{A_2}du, e^4 = e^{A_3}dv, e^{i+4} = ue^{A_4}\sigma_i, e^{i+7} = ve^{A_5}\tilde{\sigma}_i, i = 1,2,3$$

$$\Gamma^{01}\varepsilon = -\varepsilon, \Gamma^{012}\varepsilon = -\varepsilon, \Gamma^{013456}\varepsilon = \varepsilon$$

$$\Gamma^{0178910}\varepsilon = -\varepsilon$$

$$\delta\psi_\mu \equiv \nabla_\mu\epsilon + \frac{1}{288}\left(\Gamma_\mu^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu\Gamma^{\rho\lambda\sigma}\right)F_{\nu\rho\lambda\sigma}\epsilon = 0$$

$$P \equiv 1 - e^{A_0 - A_1}$$

$$\hat{A}_0 \equiv \frac{1}{2}(A_0 + A_1), \hat{A}_1 \equiv \frac{1}{2}(A_0 - A_1)$$

$$ds_{11} = e^{2\hat{A}_0}\Big[-e^{2\hat{A}_1}dt^2 + e^{-2\hat{A}_1}\left(dy + \left(e^{2\hat{A}_1} - 1\right)dt\right)^2 + (-\partial_z w)(dz + (\partial_z w)^{-1}(\partial_u w)du)^2 \\ +e^{-3\hat{A}_0}(-\partial_z w)^{-\frac{1}{2}}(du^2 + u^2 d\Omega_3^2) + e^{-3\hat{A}_0}(-\partial_z w)^{\frac{1}{2}}\left(dv^2 + v^2 d\Omega'^2_3\right)\Big]$$

$$e^0 = e^{\hat{A}_0 + \hat{A}_1}dt, e^1 = e^{\hat{A}_0 - \hat{A}_1}\left(dy + \left(e^{2\hat{A}_1} - 1\right)dt\right)$$
$$e^2 = e^{\hat{A}_0}(-\partial_z w)^{\frac{1}{2}}(dz + (\partial_z w)^{-1}(\partial_u w)du)$$
$$e^3 = e^{-\frac{1}{2}\hat{A}_0}(-\partial_z w)^{-\frac{1}{4}}du, e^4 = e^{-\frac{1}{2}\hat{A}_0}(-\partial_z w)^{\frac{1}{4}}dv,$$
$$e^{i+4} = \frac{1}{2}ue^{-\frac{1}{2}\hat{A}_0}(-\partial_z w)^{-\frac{1}{4}}\sigma_i, e^{i+7} = \frac{1}{2}ve^{-\frac{1}{2}\hat{A}_0}(-\partial_z w)^{\frac{1}{4}}\tilde{\sigma}_i, i = 1,2,3$$

$$C^{(3)} = -e^0 \wedge e^1 \wedge e^2 + (\partial_z w)^{-1}(u^3 \partial_u w)\mathrm{Vol}(S^3) + (v^3 \partial_v w)\mathrm{Vol}\left(S'^3\right)$$

$$F_1 \equiv (-\partial_z w)^{\frac{1}{2}}e^{-3\hat{A}_0}, F_2 \equiv (-\partial_z w)^{-\frac{1}{2}}e^{-3\hat{A}_0} + (-\partial_z w)^{-1}(\partial_u w)^2$$

$$\mathcal{L}(H) = e^{2\hat{A}_0}(-\partial_z w)^{-\frac{1}{2}}\Big[(-\partial_z w)\frac{1}{u^3}\partial_u(u^3 \partial_u H) + \frac{1}{v^3}\partial_v(v^3 \partial_v H) + 2(\partial_u w)\partial_u \partial_z H \\ + \left((-\partial_z w)^{-\frac{1}{2}}e^{-3\hat{A}_0} + (-\partial_z w)^{-1}(\partial_u w)^2\right)\partial_z^2 H\Big]$$

$$\mathcal{L}\left(e^{-2\hat{A}_1}\right) = 0$$

$$ds_{11}^2 = e^{2A}\Big[\hat{f}_1^2\left(\frac{d\mu^2}{\mu^2} + \mu^2\left(-e^{2\hat{A}_1}dt^2 + e^{-2\hat{A}_1}\left(dy + \left(e^{2\hat{A}_1} - 1\right)dt\right)^2\right)\right) \\ + \hat{f}_2^2 ds_{S^3}^2 + \hat{f}_3^2 ds_{S'^3}^2 + \frac{d\xi^2 + d\rho^2}{4\rho^2}\Big]$$

$$\mathcal{L}(H) = 4e^{-A}\left[(G\bar{G} - 1)\frac{1}{\mu}\partial_\mu\left(\mu^3 \partial_\mu H\right) + \frac{1}{\rho}\partial_\rho\left(\rho^3 \partial_\rho H\right) + \rho^2 \partial_\xi^2 H\right]$$

$$\frac{1}{\rho}\partial_\rho\left(\rho^3 \partial_\rho K\right) + \rho^2 \partial_\xi^2 K + p(p+2)(G\bar{G} - 1)K = 0$$

$$K = \frac{c_1}{u^2} + \frac{c_2}{v^2} = \frac{2}{\beta\rho}\left(c_1 e^{\frac{1}{\beta}\tilde{\Phi}} + c_2 e^{-\frac{1}{\beta}\tilde{\Phi}}\right)$$

$$(u^2 + v^2)^{-3} \sim \mu^{-3}\rho^{-3}$$

$$\frac{1}{\rho^3}\partial_\rho\left(\rho^3 \partial_\rho K\right) + \partial_\xi^2 K + \frac{p(p+2)}{\rho^2}(G\bar{G} - 1)K = 0$$

$$\mathcal{L}_4(K) \equiv \frac{1}{\rho^3}\partial_\rho\left(\rho^3 \partial_\rho K\right) + \partial_\xi^2 K$$

$$ds_5^2 \equiv d\rho^2 + \rho^2 d\Omega_3^2 + d\xi^2$$

$$\mathcal{L}_4\left(\frac{1}{(\rho^2+\xi^2)^{\frac{3}{2}}}\right)$$

$$\frac{p(p+2)}{\rho^2}(G\bar{G}-1)\sim\frac{c_0}{(\rho^2+\xi^2)^{\frac{3}{2}}}$$

$$K=\frac{Q}{(\rho^2+\xi^2)^{\frac{3}{2}}}\left(1-\frac{c_0}{4}\frac{1}{\sqrt{\rho^2+\xi^2}}+\cdots\right)$$

$$e^{-2\hat{A}_1}=V_0+V_1\mu^{-2}$$

$$V_0=1+\sum_{a=1}^{m}\frac{k_a}{\left(\left(\xi-\tilde{\xi}_a\right)^2+\rho^2\right)^{\frac{3}{2}}},V_1=q_0+\sum_{a=1}^{m'}\frac{q_a}{\left(\left(\xi-\hat{\xi}_a\right)^2+\rho^2\right)^{\frac{3}{2}}}$$

$$e^{-2\hat{A}_1}=1+\alpha+Q\mu^{-2}$$

$$\begin{aligned}&\frac{d\mu^2}{\mu^2}+\mu^2\left(-e^{2\hat{A}_1}dt^2+e^{-2\hat{A}_1}\left(dy+\left(e^{2\hat{A}_1}-1\right)dt\right)^2\right)\\&=\frac{d\mu^2}{\mu^2}-\frac{\mu^4}{Q+\mu^2}dt^2+(Q+\mu^2)\left(dy-\frac{Q}{Q+\mu^2}dt\right)^2\\&=\frac{d\mu^2}{\mu^2}+\mu^2(-dt^2+dy^2)+Q(dy-dt)^2\end{aligned}$$

$$ds_{11}^2=e^{2\alpha_0}(-dt^2+dy^2)+e^{2\alpha_1}d\Omega_3^2+e^{2\alpha_2}d\Omega_3'^2+g_{ij}dz^idz^j$$

$$ds_3^2=g_{ij}dz^idz^j=e^{2\alpha_3}dz^2+e^{2\alpha_4}du^2+e^{2\alpha_5}dv^2$$

$$\begin{aligned}e^0&=e^{\alpha_0}dt, && e^1=e^{\alpha_0}dy, && e^2=e^{\alpha_1}dz\; e^3=e^{\alpha_2}du, e^4=e^{\alpha_3}dv,\\ e^{i+4}&=e^{\alpha_4}\sigma_i, && e^{i+7}=e^{\alpha_5}\tilde{\sigma}_i, && i=1,2,3\end{aligned}$$

$$\Gamma^{012}\varepsilon=-\varepsilon,\Gamma^{013567}\varepsilon=\varepsilon,\Gamma^{0148910}\varepsilon=-\varepsilon$$

$$\begin{aligned}&e^0=e^{A_0}dt,\quad e^1=e^{A_0}dy,\quad e^2=e^{A_1}(dz+B_1du+B_2dv)\\&e^3=e^{A_2}du,\quad e^4=e^{A_3}dv,\quad e^{i+4}=ue^{A_4}\sigma_i,e^{i+7}=ve^{A_5}\tilde{\sigma}_i,i=1,2,3\end{aligned}$$

$$B_2\equiv 0$$

$$\begin{aligned}&e^0=e^{A_0}dt,\quad e^1=e^{A_0}dy,\quad e^2=e^{A_1}(dz+B_1du)\\&e^3=e^{A_2}du,\quad e^4=e^{A_3}dv,\quad e^{i+4}=ue^{A_4}\sigma_i,e^{i+7}=ve^{A_5}\tilde{\sigma}_i,i=1,2,3\end{aligned}$$

$$\begin{aligned}ds_{11}^2=e^{2A_0}(-dt^2+dy^2)&+e^{2A_2}du^2+e^{2A_3}dv^2+u^2e^{2A_4}d\Omega_3^2+v^2e^{2A_5}d\Omega_3'^2\\&+e^{2A_1}(dz+B_1du)^2\end{aligned}$$

$$\begin{aligned}F^{(4)}=&e^0\wedge e^1\wedge(b_1e^2\wedge e^3+b_2e^2\wedge e^4+b_3e^3\wedge e^4)\\&+(b_4e^2+b_5e^3+b_6e^4)\wedge e^5\wedge e^6\wedge e^7+(b_7e^2+b_8e^3+b_9e^4)\wedge e^8\wedge e^9\wedge e^{10}\end{aligned}$$

$$\epsilon = e^{\frac{1}{2}A_0}\epsilon_0$$

$$\partial_u(A_5 - A_3) = \partial_z(A_5 - A_3) = 0, \partial_v(A_4 - A_2) = \partial_z(A_4 - A_2) = 0$$

$$A_4 = A_2, A_5 = A_3$$

$$ds_{11}^2 = e^{2A_0}(-dt^2 + dy^2) + e^{2A_2}(du^2 + u^2 d\Omega_3^2) + e^{2A_3}(dv^2 + v^2 d\Omega_3'^2) \\ + e^{2A_1}(dz + B_1 du)^2$$

$$A_3 = -(A_0 + A_2)$$

$$A_1 = -2A_2$$

$$ds_{11}^2 = e^{2A_0}[(-dt^2 + dy^2) + e^{2(A_2 - A_0)}(du^2 + u^2 d\Omega_3^2) + e^{-2(A_2 + 2A_0)}\left(dv^2 + v^2 d\Omega'^2_{3\,3}\right) \\ + e^{-2(A_0 + 2A_2)}(dz + B_1 du)^2]$$

$$\partial_z\left(B_1 e^{-2(A_0 + 2A_2)}\right) = \partial_u\left(e^{-2(A_0 + 2A_2)}\right)$$

$$B_1 e^{-2(A_0 + 2A_2)} = -\partial_u w, e^{-2(A_0 + 2A_2)} = -\partial_z w$$

$$B_1 = (\partial_z w)^{-1}\partial_u w, e^{-2(A_0 + 2A_2)} = -\partial_z w$$

$$F_1 \equiv (-\partial_z w)^{\frac{1}{2}} e^{-3A_0}, F_2 \equiv (-\partial_z w)^{-\frac{1}{2}} e^{-3A_0} + (-\partial_z w)^{-1}(\partial_u w)^2$$

$$H_1 \equiv \mathcal{L}_v w - \partial_z F_1, H_2 \equiv \mathcal{L}_u w + \partial_z F_2$$

$$\partial_z H_1 = \partial_u H_1 = \partial_z H_2 = \partial_v H_2 = 0$$

$$w = \partial_z G_0, F_1 = \mathcal{L}_v G_0, F_2 = -\mathcal{L}_u G_0$$

$$\mathcal{L}_v G_0 = (\partial_z^2 G_0)(\mathcal{L}_u G_0) - (\partial_u \partial_z G_0)^2$$

$$\left(\frac{\partial F}{\partial \eta}\right)_{\zeta,\xi}$$

$$dw = \left(\frac{\partial w}{\partial z}\right)_{\vec{u},\vec{v}} dz + \left(\frac{\partial w}{\partial u_i}\right)_{z,\vec{v}} du_i + \left(\frac{\partial w}{\partial v_i}\right)_{z,\vec{u}} dv_i$$

$$\left(\frac{\partial z}{\partial u_i}\right)_{w,\vec{v}} = -\left(\left(\frac{\partial w}{\partial z}\right)_{\vec{u},\vec{v}}\right)^{-1}\left(\frac{\partial w}{\partial u_i}\right)_{z,\vec{v}}, \left(\frac{\partial z}{\partial v_i}\right)_{w,\vec{u}} = -\left(\left(\frac{\partial w}{\partial z}\right)_{\vec{u},\vec{v}}\right)^{-1}\left(\frac{\partial w}{\partial v_i}\right)_{z,\vec{u}}$$

$$\left(\frac{\partial z}{\partial w}\right)_{\vec{u},\vec{v}} = \left(\left(\frac{\partial w}{\partial z}\right)_{\vec{u},\vec{v}}\right)^{-1}$$

$$
\begin{aligned}
e^2 &= (-\partial_z w)^{\frac{1}{2}}\left(dz + (\partial_z w)^{-1}\left(\vec{\nabla}_{\vec{u}} w\right)\cdot d\vec{u}\right)\\
&= -(-\partial_z w)^{-\frac{1}{2}}\left((\partial_z w)dz + \left(\vec{\nabla}_{\vec{u}} w\right)\cdot d\vec{u}\right) = -(-\partial_z w)^{-\frac{1}{2}}\left(dw - \left(\vec{\nabla}_{\vec{v}} w\right)\cdot d\vec{v}\right)\\
&= -\left((-\partial_w z)_{\vec{u},\vec{v}}\right)^{\frac{1}{2}}\left(dw + \left(\left(\frac{\partial z}{\partial w}\right)_{\vec{v},\vec{u}}\right)^{-1}\left(\frac{\partial z}{\partial v_i}\right)_{w,\vec{v}} dv_i\right)\\
&= -(-\partial_w z)^{\frac{1}{2}}\left(dw + (\partial_w z)^{-1}\left(\vec{\nabla}_{\vec{v}} z\right)\cdot d\vec{v}\right)
\end{aligned}
$$

$$
C^{(3)} = -e^0\wedge e^1\wedge e^2 + \frac{1}{3!}\epsilon_{ijk\ell}\left(-\left(\partial_{u_\ell} z\right)du^i\wedge du^j\wedge du^k + (\partial_w z)^{-1}\left(\partial_{v_\ell} z\right)dv^i\wedge dv^j\wedge dv^k\right)
$$

$$
\begin{aligned}
ds^2 &= G_{xx}\left(dx + A_\mu dx^\mu\right)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu,\\
B_2 &= B_{\mu x}dx^\mu\wedge\left(dx + A_\mu dx^\mu\right) + \hat{B}_2,\\
C_p &= C_{(p-1)x}\wedge\left(dx + A_\mu dx^\mu\right) + \hat{C}_p
\end{aligned}
$$

$$
\begin{aligned}
d\tilde{s}^2 &= G_{xx}^{-1}\left(dx + B_{\mu x}dx^\mu\right)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu,\\
e^{2\tilde{\phi}} &= G_{xx}^{-1}e^{2\phi},\\
\tilde{B}_2 &= A_\mu dx^\mu\wedge dx + \hat{B}_2,\\
\tilde{C}_p &= \hat{C}_{p-1}\wedge\left(dx + B_{\mu x}dx^\mu\right) + C_{(p)x}
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{g}_{\mu\nu} = \sqrt{C_0^2 + e^{-2\phi}}\,g_{\mu\nu}, e^{-\tilde{\phi}} = \frac{e^{-\phi}}{C_0^2 + e^{-2\phi}}, \tilde{C}_0 = -\frac{C_0}{C_0^2 + e^{-2\phi}}\\
&\tilde{B}_2 = -C_2, \tilde{C}_2 = B_2, \tilde{C}_4 = C_4 + B_2\wedge C_2
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= \frac{\sqrt{\det h}}{h_{11}}\left(du_2^2 + du_3^2\right) + \frac{1}{\sqrt{\det h}}\left(-dt^2 + dy^2\right) + \sqrt{\det h}\left(e^{3A}h_{ab}dr^a dr^b + ds_{\mathbb{R}^4}^2\right)\\
e^{2\phi} &= \frac{\sqrt{\det h}}{h_{11}}, B_2 = \frac{h_{12}}{h_{11}}du_2\wedge du_3\\
C_3 &= e^{3A}h_{1a}dt\wedge dr^a\wedge dy, C_5 = \frac{1}{h_{11}}dt\wedge du_1\wedge du_2\wedge du_3\wedge dy
\end{aligned}
$$

$$
\begin{aligned}
F_p &= dC_{p-1} && \text{para } p < 3,\\
F_p &= dC_{p-1} + H_3\wedge C_{p-3} && \text{para } p \geq 3,\\
F_6 &= \star F_4, F_8 = \star F_2.
\end{aligned}
$$

$$
\begin{aligned}
v_3 &= v\cos\,\phi_1\\
v_4 &= v\sin\,\phi_1\cos\,\phi_2\\
v_5 &= v\sin\,\phi_1\sin\,\phi_2\cos\,\phi_3\\
v_6 &= v\sin\,\phi_1\sin\,\phi_2\sin\,\phi_3\\
ds_{\mathbb{R}^4}^2 &= dv^2 + v^2\left(d\phi_1^2 + \sin^2\,\phi_1\left(d\phi_2^2 + \sin^2\,\phi_2 d\phi_3^2\right)\right)
\end{aligned}
$$

$$dC_5^e = -\frac{1}{h_{11}^2}(\partial_z h_{11} dz + \partial_v h_{11} dv) \wedge dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy$$

$$\begin{aligned} H_3 \wedge C_3^e = & \left[\partial_z \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{12} - \partial_{u_1} \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{11}\right] dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dz \wedge dy \\ & -\partial_v \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{11} dz \wedge dt \wedge dv \wedge du_2 \wedge du_3 \wedge dy \\ & -\partial_v \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{12} du_1 \wedge dt \wedge dv \wedge du_2 \wedge du_3 \wedge dy \end{aligned}$$

$$\begin{aligned} F_6^e = & f_1 dt \wedge dz \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy \\ & + f_2 dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy \wedge dv \\ & - f_3 dt \wedge dz \wedge du_2 \wedge du_3 \wedge dy \wedge dv \end{aligned}$$

$$\begin{aligned} f_1 &= \frac{1}{h_{11}^2} \partial_z h_{11} - \partial_z \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{12} + \partial_{u_1} \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{11} \\ f_2 &= \frac{1}{h_{11}^2} \partial_v h_{11} - \partial_v \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{12}, \\ f_3 &= \partial_v \left(\frac{h_{12}}{h_{11}}\right) e^{3A} h_{11}. \end{aligned}$$

$$\begin{aligned} F_4^m = & -v^3 h_{11}(f_2 h_{11} + f_3 h_{12}) dz \wedge d\Omega_3' - v^3 h_{11}(f_2 h_{12} + f_3 h_{22}) du_1 \wedge d\Omega_3' \\ & + r^3 f_1 h_{11} \mathrm{det} h dr \wedge d\Omega_3' \end{aligned}$$

$$\begin{aligned} F_4^m = & -(v^3 \partial_v h_{11} dz + v^3 \partial_v h_{12} du_1) \wedge d\Omega_3' \\ & + v^3 \left(h_{22} \partial_z h_{11} - h_{12} \partial_z h_{12} - h_{12} \partial_{u_1} h_{11} + h_{11} \partial_{u_1} h_{12}\right) dv \wedge d\Omega_3' \end{aligned}$$

$$\begin{aligned} & -\frac{1}{2} v^3 \partial_v \partial_z^2 K dz - \frac{1}{2} v^3 \partial_v \partial_z \partial_{u_1} K du_1 = -\frac{1}{2} d(v^3 \partial_v \partial_z K) + \frac{1}{2} \partial_v (v^3 \partial_v \partial_z K) dv \\ & = -\frac{1}{2} d(v^3 \partial_v \partial_z K) + \frac{v^3}{2} \partial_z \left(\frac{1}{v^3} \partial_v (v^3 \partial_v K)\right) dv = -\frac{1}{2} d(v^3 \partial_v \partial_z K) + \frac{v^3}{2} \partial_z \Delta_y K dv \end{aligned}$$

$$\frac{v^3}{4}\left(\partial_{u_1}^2 K \partial_z^3 K - 2 \partial_z \partial_{u_1} K \partial_z^2 \partial_{u_1} K + \partial_z^2 K \partial_z \partial_{u_1}^2 K\right) dv = v^3 \partial_z (\mathrm{det} h)$$

$$F_4^m = -\frac{1}{2} d(v^3 \partial_v \partial_z K) \wedge d\Omega_3' + \frac{v^3}{2} \partial_z \left(\Delta_y K + 2 \mathrm{det} h\right) dv \wedge d\Omega_3'$$

$$C_3^m = -\frac{1}{2} v^3 \partial_v \partial_z K d\Omega_3'$$

$$\begin{aligned} dC_3^e = & -\left[\partial_{u_1}(e^{3A} h_{11}) - \partial_z (e^{3A} h_{12})\right] dt \wedge du_1 \wedge dz \wedge dy \\ & -[\partial_v (e^{3A} h_{11}) dz + \partial_v (e^{3A} h_{12}) du_1] \wedge dt \wedge dv \wedge dy \end{aligned}$$

$$F_6^m = \frac{v^3 \det h}{h_{11}} [h_{12} \partial_v (e^{3A} h_{11}) - h_{11} \partial_v (e^{3A} h_{12})] dz \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$+ \frac{v^3 \det h}{h_{11}} [h_{22} \partial_v (e^{3A} h_{11}) - h_{12} \partial_v (e^{3A} h_{12})] du_1 \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$- \frac{v^3 (\det h)^2}{h_{11}} \left[\partial_{u_1} (e^{3A} h_{11}) - \partial_z (e^{3A} h_{12})\right] du_2 \wedge du_3 \wedge dv \wedge d\Omega_3',$$

$$F_6^m = v^3 \left(\frac{h_{12}}{h_{11}} \partial_v h_{11} - \partial_v h_{12}\right) dz \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$+ v^3 \left(\frac{h_{12}}{h_{11}} \partial_v h_{12} - \partial_v h_{22}\right) du_1 \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$+ v^3 \left(\partial_{u_1} (\det h) - \frac{h_{12}}{h_{11}} \partial_z (\det h)\right) dv \wedge dx_2 \wedge du_3 \wedge d\Omega_3'$$

$$\frac{1}{2} \frac{h_{12}}{h_{11}} d_\Sigma (v^3 \partial_v \partial_z K) \wedge du_2 \wedge du_3 \wedge d\Omega_3' - \frac{1}{2} d_\Sigma (v^3 \partial_v \partial_1 K) \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$

$$F_6^m - H \wedge C_3^m = \left[\frac{h_{12}}{h_{11}} d_\Sigma \left(\frac{v^3}{2} \partial_v \partial_z K\right) + d\left(\frac{h_{12}}{h_{11}}\right) \frac{v^3}{2} \partial_v \partial_z K\right] \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$- d_\Sigma \left(\frac{v^3}{2} \partial_v \partial_{u_1} K\right) \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$+ v^3 \left(\partial_{u_1} (\det h) - \frac{h_{12}}{h_{11}} \partial_z (\det h)\right) dv \wedge dx_2 \wedge dx_3 \wedge d\Omega_3'$$

$$d\left(\frac{v^3}{2} \frac{h_{12}}{h_{11}} \partial_v \partial_z K\right) \wedge du_2 \wedge du_3 \wedge d\Omega_3' - \frac{h_{12}}{h_{11}} \partial_v \left(\frac{v^3}{2} \partial_v \partial_z K\right) dv \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$
$$- d\left(\frac{v^3}{2} \partial_v \partial_{u_1} K\right) \wedge du_2 \wedge du_3 \wedge d\Omega_3' + \partial_v \left(\frac{v^3}{2} \partial_v \partial_{u_1} K\right) dv \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$

$$\frac{v^3}{2} \partial_{u_1} \left[2 \det h + \frac{1}{v^3} \partial_v (v^3 \partial_v K)\right] - \frac{v^3}{2} \frac{h_{12}}{h_{11}} \partial_z \left[2 \det h + \frac{1}{v^3} \partial_v (v^3 \partial_v K)\right]$$

$$F_6^m - H \wedge C_3^m = d\left(\frac{v^3}{2} \frac{h_{12}}{h_{11}} \partial_v \partial_z K - \frac{v^3}{2} \partial_v \partial_{u_1} K\right) \wedge du_2 \wedge du_3 \wedge d\Omega_3'$$

$$C_5^m = \frac{v^3}{2} \left(\frac{h_{12}}{h_{11}} \partial_v \partial_z K - \partial_v \partial_{u_1} K\right) du_2 \wedge du_3 \wedge d\Omega_3'$$

$$ds^2 = \frac{1}{\sqrt{\det h}} (-dt^2 + dy^2) + \frac{\sqrt{\det h}}{h_{11}} (du_2^2 + du_3^2) + \sqrt{\det h} \left(e^{3A} h_{ab} dr^a dr^b + ds_{\mathbb{R}^4}^2\right)$$
$$e^{2\phi} = \frac{\sqrt{\det h}}{h_{11}}, B_2 = \frac{h_{12}}{h_{11}} du_2 \wedge du_3$$
$$C_3 = e^{3A} h_{1a} dt \wedge dr^a \wedge dy - \frac{v^3}{2} \partial_v \partial_z K d\Omega_3'$$
$$C_5 = \frac{1}{h_{11}} dt \wedge du_1 \wedge du_2 \wedge du_3 \wedge dy + \frac{v^3}{2} \left(\frac{h_{12}}{h_{11}} \partial_v \partial_z K - \partial_v \partial_{u_1} K\right) du_2 \wedge du_3 \wedge d\Omega_3'$$

$$ds^2 = Z^{-1/2}(-dt^2 + dx_1^2 + dx_2^2) + Z^{1/2}(dx_3^2 + \cdots + dx_9^2)$$
$$e^{\Phi} = Z^{1/4}$$
$$C_3 = Z^{-1} dt \wedge dx_1 \wedge dx_2$$

$$Z = 1 + \frac{Q}{r^2}, r^2 \equiv x_6^2 + \cdots + x_9^2$$

$$x_2 = x_2' c + x_3' s$$
$$x_3 = -x_2' s + x_3' c$$

$$W \equiv c^2 Z + s^2$$

$$\begin{aligned} ds^2 =& Z^{-1/2}(-dt^2 + dx_1^2) + Z^{1/2}(dx_4^2 + \cdots + dx_9^2) \\ & + Z^{-1/2} W (dx_3 - cs(Z-1)W^{-1} dx_2)^2 + Z^{1/2} W^{-1} dx_2^2 \\ e^{2\Phi} =& Z^{1/2} \\ C_3 =& Z^{-1} s dt \wedge dx_1 \wedge (dx_3 - cs(Z-1)W^{-1} dx_2) \\ & + W^{-1} c dt \wedge dx_1 \wedge dx_2 \end{aligned}$$

$$\begin{aligned} ds^2 &= Z^{-1/2}(-dt^2 + dx_1^2) + Z^{1/2}(dx_4^2 + \cdots + dx_9^2) + Z^{1/2} W^{-1}(dx_2^2 + dx_3^2) \\ e^{2\Phi} &= W^{-1} Z, B_2 = -cs(Z-1)W^{-1} dx_2 \wedge dx_3 \\ C_2 &= Z^{-1} s dt \wedge dx_1, C_4 = W^{-1} c dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

$$ds^2 = Z^{-1/2}(-dt^2 + dx_1^2 + dx_4^2) + Z^{1/2}\left(dx_5^2 + \cdots + dx_9^2\right) + Z^{1/2} W^{-1}(dx_2^2 + dx_3^2)$$
$$e^{2\Phi} = Z^{1/2} W^{-1}$$
$$C_3 = Z^{-1} s dt \wedge dx_1 \wedge dx_4, C_5 = W^{-1} c dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

$$h_{11} = c^2 Z + s^2, h_{22} = s^2 Z + c^2$$
$$h_{12} = cs(Z-1), \det h = Z$$

$$\begin{aligned} dB_2 \wedge C_3 &= [-cs\partial_l((Z-1)W^{-1}) dx_l \wedge dx_2 \wedge dx_3] \wedge [Z^{-1} s dt \wedge dx_1 \wedge dx_4] \\ &= cs^2 W^{-2} Z^{-1} (\partial_l Z) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_l \end{aligned}$$

$$\begin{aligned} dC_5 &= c\partial_l(W^{-1}) dx_l \wedge dt \wedge dx_1 \wedge \cdots \wedge dx_4 \\ &= c^3 W^{-2} (\partial_l Z) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_l \end{aligned}$$

$$\begin{aligned} F_6 &= c W^{-1} Z^{-1} (\partial_l Z) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_l \\ &= c Z^{-1} (\partial_l Z) e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^l \end{aligned}$$

$$\begin{aligned} F_4^{(m)} &= c \frac{\partial_l Z}{Z} \frac{\epsilon_{(l-4),abcd}}{4!} e^{4+a} \wedge e^{4+b} \wedge e^{4+c} \wedge e^{4+d} \\ &= c(\partial_l Z) \frac{\epsilon_{(l-4),abcd}}{4!} dx^{4+a} \wedge dx^{4+b} \wedge dx^{4+c} \wedge dx^{4+d} \end{aligned}$$

$$F_4^{(m)} = -c\frac{x_l}{r}(\partial_r Z)\frac{\epsilon_{(l-5),abc}}{3!} dx^5 \wedge dx^{5+a} \wedge dx^{5+b} \wedge dx^{5+c}$$

$$\begin{aligned} C_3^{(m)} &= -c\frac{x_5 x_l}{r}(\partial_r Z)\frac{\epsilon_{(l-5),abc}}{3!} dx^{5+a} \wedge dx^{5+b} \wedge dx^{5+c} \\ &= -c x_5 r^3 (\partial_r Z) d\Omega_3' \end{aligned}$$

$$
\begin{aligned}
ds^2 =& Z^{-1/6}W^{1/3}\left[Z^{-1/2}(-dt^2+dx_1^2+dx_4^2)+Z^{1/2}\left(dx_5^2+\cdots+dx_9^2\right)\right]\\
&+Z^{1/3}W^{-2/3}(dx_2^2+dx_3^2+dx_{11}^2),\\
C_3 =& Z^{-1}sdt\wedge dx_1\wedge dx_4 - cs(Z-1)W^{-1}dx_2\wedge dx_3\wedge dx_{11} - cx_5r^3(\partial_r Z)d\Omega_3'
\end{aligned}
$$

$$
x_4 \to cu_1+sz, x_5 \to -su_1+cz
$$
$$
x_1 \to y, x_2 \to u_2, x_3 \to u_3, x_{11} \to -u_4, x_{6,7,8,9} \to v_{1,2,3,4}
$$

$$
\begin{aligned}
ds^2 =& W^{1/3}Z^{-2/3}(-dt^2+dy^2)+W^{4/3}Z^{-2/3}(dz-cs(Z-1)W^{-1}du_1)^2\\
&+W^{1/3}Z^{1/3}(dv_1^2+\cdots+dv_4^2)+W^{-2/3}Z^{1/3}(du_1^2+du_2^2+du_3^2+du_4^2)\\
C_3 =& Z^{-1}sdt\wedge dy\wedge(sdz+cdu_1)+cs(Z-1)W^{-1}du_2\wedge du_3\wedge du_4 + c(su_1-cz)v^3(\partial_v Z)d\Omega_3'
\end{aligned}
$$

$$
e^{A_0} = W^{1/6}Z^{-1/3}, (-\partial_z w) = W
$$
$$
\left(\partial_{u_1} w\right) = cs(Z-1)
$$

$$
\left(\partial_{v_l} w\right) = c(su_1-cz)v_l\frac{\partial_v Z}{v}
$$

$$
\delta C_3 = -c^2 dt\wedge dy\wedge dz + csdt\wedge dy\wedge du_1
$$

$$
w \equiv -zW + cs(Z-1)u_1
$$

$$
G_0 = -\frac{1}{2}Z(cz-su_1)^2 - \frac{1}{2}(sz+cu_1)^2 + f(v)
$$

Entiéndase que la supergravedad cuántica, para efectos de este trabajo, comporta la simetría entre dos partículas o antipartículas, según sea el caso, de las cuales, una de ellas, es una superpartícula (Véase la definición proporcionada por este autor respecto de las superpartículas en sentido lato), a propósito de la deformación o perforación del espacio cuántico de que se trate, combinando en consecuencia, relatividad general y supersimetría. Entiéndase por supersimetría, para efectos de este trabajo, comporta la interacción de dos partículas o antipartículas, según sea el caso, de las cuales, una de ellas, es una superpartícula (Véase la definición proporcionada por este autor), a propósito de la deformación o perforación del espacio cuántico de que se trate, por acción de las superpartículas. Entiéndase que las supermembranas, para efectos de este trabajo, comporta la existencia de infinitas dimensiones a propósito de la deformación o perforación del espacio cuántico de que se trate, por acción de las superpartículas. Finalmente, entiéndase por superespacio, para efectos de este trabajo, como la existencia de un espacio cuántico relativista, el mismo que posee dimensiones ordinarias y anticonmutativas, a propósito de la deformación o perforación del espacio cuántico de que se trate, por acción de las superpartículas.

REFERENCIAS BIBLIOGRÁFICAS ADICIONALES.

Iosif Bena, Anthony Houppe, Dimitrios Toulikas *y* Nicholas P. Warner, Maze topiary in supergravity, JHEP03(2025)120.

Apéndice G.

1. Modelo Diósi – Penrose para campos cuánticos relativistas, a propósito de la existencia de superpartículas por gravitación.

$$\tau_{\mathrm{DP}} = \frac{\hbar}{\Delta E_{\mathrm{DP}}}$$

$$\Delta E_{\mathrm{DP}}(\boldsymbol{d}) = -8\pi G \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{\mu(\boldsymbol{r})[\mu(\boldsymbol{r}' + \boldsymbol{d}) - \mu(\boldsymbol{r}')]}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] - \frac{4\pi G}{\hbar} \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} [\hat{M}(\boldsymbol{y}), [\hat{M}(\boldsymbol{x}), \rho(t)]]$$

$$\frac{d\Gamma_t}{d\omega} = \frac{2}{3} \frac{G e^2 N^2 N_a}{\pi^{3/2} \varepsilon_0 c^3 R_0^3 \omega}$$

$$z_s(R_0) = \sum_i \int_{\Delta E} \left. \frac{d\Gamma_t}{dE} \right|_i T\epsilon_i(E) dE = \frac{a}{R_0^3}$$

$$\frac{d}{d\omega}\Gamma_t = \frac{k^2}{c} \sum_\mu \int d\Omega_k \frac{d}{dt} \langle a_{\mathbf{k}\mu}^\dagger a_{\mathbf{k}\mu} \rangle_t$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \int d\boldsymbol{Q} \sum_{n,n'} \tilde{\Gamma}_{n,n'}(\boldsymbol{Q}) \left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_n} \rho(t) e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{n'}} - \frac{1}{2}\left\{ e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{n'}} e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_n}, \rho(t) \right\} \right)$$

$$\tilde{\Gamma}_{n,n'}(\boldsymbol{Q}) = \frac{4G}{\pi\hbar^2} \frac{\tilde{\mu}_n(\boldsymbol{Q}) \tilde{\mu}_{n'}^*(\boldsymbol{Q})}{Q^2}$$

$$\tilde{\mu}(\boldsymbol{Q}) = \frac{1}{2\pi\hbar^3} \int d\boldsymbol{y} \mu(\boldsymbol{y}) e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{y}}$$

$$\frac{d}{dt}O(t) = \frac{i}{\hbar}[H, O(t)] + \int d\boldsymbol{Q} \sum_{k,k'} \tilde{\Gamma}_{k,k'}(\boldsymbol{Q}) \left(e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{k'}} O(t) e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_k} - \frac{1}{2}\left\{ O(t), e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{k'}} e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_k} \right\} \right)$$

$$H = H_{\mathrm{s}} + H_{\mathrm{R}} + H_{\mathrm{INT}}$$

$$H_{\mathrm{s}} = \sum_j \left(\frac{\boldsymbol{p}_j^2}{2m_j} + V(\boldsymbol{x}_j) + \sum_{i<j} U(\boldsymbol{x}_j - \boldsymbol{x}_i) \right)$$

$$H_{\mathrm{R}} = \sum_\mu \int d\boldsymbol{k} \hbar\omega_k \left(\frac{1}{2} + a_{\boldsymbol{k},\mu}^\dagger a_{\boldsymbol{k},\mu} \right)$$

$$H_{\mathrm{INT}} = \sum_j \left(-\frac{e_j}{m_j}\right) \boldsymbol{A}(\boldsymbol{x}_j) \cdot \boldsymbol{p}_j + \sum_j \frac{e_j^2}{2m_j} \boldsymbol{A}^2(\boldsymbol{x}_j)$$

$$\boldsymbol{A}(\boldsymbol{x}) = \int d\boldsymbol{k} \sum_\mu \alpha_k \left[\vec{\epsilon}_{\boldsymbol{k},\mu} a_{\boldsymbol{k},\mu} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \vec{\epsilon}_{\boldsymbol{k},\mu} a^\dagger_{\boldsymbol{k},\mu} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$

$$\int_{\Delta E} \left.\frac{d\Gamma_t}{dE}\right|_i T\epsilon_i(E) dE$$

$$p(z_c \mid \Lambda_c) = \frac{\Lambda_c^{z_c} e^{-\Lambda_c}}{z_c!}$$

$$\Lambda_c(R_0) = \Lambda_b + \Lambda_s(R_0) = z_b + z_s(R_0)$$

$$\tilde{p}\left(\Lambda_c(R_0) \mid p\left(z_c \mid \Lambda_c(R_0)\right)\right) = \frac{p\left(z_c \mid \Lambda_c(R_0)\right) \cdot \tilde{p}_0\left(\Lambda_c(R_0)\right)}{\int_D p\left(z_c \mid \Lambda_c(R_0)\right) \cdot \tilde{p}_0\left(\Lambda_c(R_0)\right) d[\Lambda_c(R_0)]}$$

$$\tilde{p}_0\left(\Lambda_c(R_0)\right) = \theta\left(\Lambda_c^{\max} - \Lambda_c(R_0)\right)$$

$$\tilde{p}\left(\Lambda_c(R_0)\right) = \frac{\Lambda_c^{z_c} e^{-\Lambda_c} \theta(\Lambda_c^{\max} - \Lambda_c)}{\int_0^{\Lambda_c^{\max}} \Lambda_c^{z_c} e^{-\Lambda_c} d\Lambda_c}$$

$$\tilde{P}(\bar{\Lambda}_c) = \frac{\gamma(z_c + 1, \bar{\Lambda}_c)}{\gamma(z_c + 1, \Lambda_c^{\max})}$$

$$\Lambda_c(R_0) = \Lambda_s(R_0) + \Lambda_b < 617 \Rightarrow \frac{a}{R_0^3} + \Lambda_b + 1 < 617 \Rightarrow R_0 > \sqrt[3]{\frac{a}{616 - \Lambda_b}}$$

$$\sum_i \int_{E_2}^{E_3} \left.\frac{d\Gamma_t}{dE}\right|_i T dE = \sum_i \int_{E_2}^{E_3} N_i^2 N_{ai} \beta T \frac{1}{R_0^3 E} dE = \frac{b}{R_0^3}$$

$$\beta = \frac{2}{3} \frac{Ge^2}{\pi^{3/2} \varepsilon_0 c^3}$$

$$z_s(R_0) = \frac{a}{R_0^3} + \frac{fb}{R_0^3} > \frac{a}{R_0^3}$$

$$z_s(R_0) = \sum_i \int_{E_1}^{E_2} \left.\frac{d\Gamma}{dE}\right|_i T\epsilon_i(E) dE + \sum_i \epsilon_i^{\max} \int_{E_2}^{E_3} \left.\frac{d\Gamma}{dE}\right|_i T dE =$$
$$((1.756 + 5.712) \times 10^{-29}) \frac{\mathrm{m}^3}{R_0^3} = \frac{a + fb}{R_0^3}$$

$$p(z_c \mid \Lambda_c) = \frac{\Lambda_c^{z_c} e^{-\Lambda_c}}{z_c!}$$

$$\epsilon_i(E) = \sum_{j=0}^{c_i} \xi_{ij} E^j$$

$$z_s(R_0) = \sum_i \int_{E_1}^{E_2} \frac{d\Gamma_t}{dE}\bigg|_i T\epsilon_i(E)dE =$$

$$= \sum_i \int_{E_1}^{E_2} N_i^2 N_{ai} \beta T \frac{1}{R_0^3 E} \sum_{j=0}^{ci} \xi_{ij} E^j dE$$

$$= 1.756 \times 10^{-29} \frac{\mathrm{m}^3}{R_0^3} = \frac{a}{R_0^3}$$

$$\beta = \frac{2}{3} \frac{Ge^2}{\pi^{3/2} \varepsilon_0 c^3}$$

$$z_{b,ij} = \frac{m_i A_{ij} T N_{rec,ij}}{N_{ij}}$$

$$z_b = \sum_{i,j} z_{b,ij}$$

$$\Lambda_c(R_0) = \Lambda_s + \Lambda_b = \frac{a}{R_0^3}$$

$$\tilde{p}\big(\Lambda_c \mid p(z_c \mid \Lambda_c)\big) = \frac{p(z_c \mid \Lambda_c) \cdot \tilde{p}_0(\Lambda_c)}{\int_D p(z_c \mid \Lambda_c) \cdot \tilde{p}_0(\Lambda_c) d\Lambda_c}$$

$$\tilde{p}\big(\Lambda_c \mid p(z_c \mid \Lambda_c)\big) = \frac{\Lambda_c^{z_c} e^{-\Lambda_c}}{\Gamma(z_c+1)}$$

$$\tilde{P}(\bar{\Lambda}_c) = \frac{\int_0^{\bar{\Lambda}_c} \Lambda_c^{z_c} e^{-\Lambda_c} d\Lambda_c}{\Gamma(z_c+1)} = \frac{\gamma(z_c+1, \bar{\Lambda}_c)}{\Gamma(z_c+1)}$$

$$\tilde{p}_0(\Lambda_c) = \theta(\Lambda_c^{\max} - \Lambda_c)$$

$$\Lambda_c^{\max} = \frac{a}{\left[R_0^{\min}\right]^3}$$

$$\tilde{p}\big(\Lambda_c \mid p(z_c \mid \Lambda_c)\big) = \frac{\Lambda_c^{z_c} e^{-\Lambda_c} \theta(\Lambda_c^{\max} - \Lambda_c)}{\int_0^{\Lambda_c^{\max}} \Lambda_c^{z_c} e^{-\Lambda_c} d\Lambda_c}$$

$$\tilde{P}(\bar{\Lambda}_c) = \frac{\gamma(z_c+1, \bar{\Lambda}_c)}{\gamma(z_c+1, \Lambda_c^{\max})}$$

$$\Lambda_s(R_0) + \Lambda_b < 617 \Rightarrow \frac{a}{R_0^3} + \Lambda_b + 1 < 617 \Rightarrow R_0 > \sqrt[3]{\frac{a}{616 - \Lambda_b}}$$

$$\Delta E_{\mathrm{DP}} = \frac{1}{G}\int d\boldsymbol{r}(g_a(\boldsymbol{r}) - g_b(\boldsymbol{r}))^2$$

$$\Delta E_{\mathrm{DP}} = 4\pi G \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{[\mu_a(\boldsymbol{r}) - \mu_b(\boldsymbol{r})][\mu_a(\boldsymbol{r}') - \mu_b(\boldsymbol{r}')]}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$\mu(\boldsymbol{r}) = \sum_{i=1}^{N} \mu_{R_0}(\boldsymbol{r} - \boldsymbol{x}_i)$$

$$\mu_{R_0}(\boldsymbol{r}) = \frac{3m}{4\pi R_0^3}\theta(R_0 - r), r = |\boldsymbol{r}|$$

$$\begin{aligned}\Delta E_{\mathrm{DP}}(\boldsymbol{d}) &= -8\pi G \int d\boldsymbol{r}'\left[\mu_{R_0}(\boldsymbol{r}' + \boldsymbol{d}) - \mu_{R_0}(\boldsymbol{r}')\right] I_{R_0}(\boldsymbol{r}') \\ &= -8\pi G \int d\boldsymbol{r}' \mu_{R_0}(\boldsymbol{r}')\left[I_{R_0}(\boldsymbol{r}' - \boldsymbol{d}) - I_{R_0}(\boldsymbol{r}')\right]\end{aligned}$$

$$I_{R_0}(\boldsymbol{r}') := \int d\boldsymbol{r} \frac{\mu_{R_0}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} = \begin{cases} \frac{3m}{2R_0^3}\left(R_0^2 - \frac{r'^2}{3}\right) & \text{si } r' \leq R_0 \\ \frac{m}{r'} & \text{si } r' \geq R_0 \end{cases}$$

$$I_{R_0}(\boldsymbol{r}' - \boldsymbol{d}) - I_{R_0}(\boldsymbol{r}') = \begin{cases} \frac{m}{R_0^3}\left(\boldsymbol{r}' \cdot \boldsymbol{d} - \frac{d^2}{2}\right) & \text{si } |\boldsymbol{r}' - \boldsymbol{d}| \leq R_0 \\ \frac{m}{|\boldsymbol{r}' - \boldsymbol{d}|} - \frac{3m}{2R_0^3}\left(R_0^2 - \frac{r'^2}{3}\right) & \text{si } |\boldsymbol{r}' - \boldsymbol{d}| \geq R_0 \end{cases}$$

$$\Delta E_{\mathrm{DP}}(\boldsymbol{d}) \simeq -\frac{8\pi G m}{R_0^3}\int d\boldsymbol{r}' \mu_{R_0}(\boldsymbol{r}')\left(\boldsymbol{r}' \cdot \boldsymbol{d} - \frac{d^2}{2}\right) = \frac{4\pi G m^2 d^2}{R_0^3}$$

$$\Delta E_{\mathrm{DP}}(\boldsymbol{d}) = -8\pi G \int d\boldsymbol{r}' \mu_{R_0}(\boldsymbol{r}')\left[\frac{m}{|\boldsymbol{r}' - \boldsymbol{d}|} - \frac{3m}{2R_0^3}\left(R_0^2 - \frac{r'^2}{3}\right)\right] = \frac{8\pi G m^2}{R_0}\left(\frac{6}{5} - \frac{R_0}{d}\right)$$

$$\Delta E_{\mathrm{DP}}(\boldsymbol{d}) = -8\pi G \sum_{i,j=1}^{N} \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{\mu_{R_0}(\boldsymbol{r} - \boldsymbol{x}_i)\left[\mu_{R_0}(\boldsymbol{r}' + \boldsymbol{d} - \boldsymbol{x}_j) - \mu_{R_0}(\boldsymbol{r}' - \boldsymbol{x}_j)\right]}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$\Delta E_{\mathrm{Dp}}(\boldsymbol{d}) = N\frac{8\pi G m^2}{R_0}\left(\frac{6}{5} - \frac{R_0}{d}\right)$$

$$\Delta E_{\mathrm{DP}}(\boldsymbol{d}) \simeq 4.61 \times 10^{-32}\ \mathrm{J}$$

$$\tau_{\mathrm{DP}} = \frac{\hbar}{\Delta E_{\mathrm{DP}}(\boldsymbol{d})} \simeq 0.0023\ \mathrm{s}$$

$$\tau_{\mathrm{DP}} = \frac{\hbar}{\Delta E_{\mathrm{DP}}(\boldsymbol{d})} \simeq 0.013\ \mathrm{s}$$

$$D := -8\pi G I$$

$$I := \sum_{i\neq j}^{N} \int d\boldsymbol{y} \int d\boldsymbol{y}' \frac{\mu_{R_0}(\boldsymbol{y})[\mu_{R_0}(\boldsymbol{y}'+\boldsymbol{d}) - \mu_{R_0}(\boldsymbol{y}')]}{|\boldsymbol{y}-\boldsymbol{y}'+\boldsymbol{x}_i-\boldsymbol{x}_j|}$$

$$I = \sum_{i\neq j}^{N} \int d\boldsymbol{y} \int d\boldsymbol{y}' \frac{\mu_{R_0}(\boldsymbol{y})[\mu_{R_0}(\boldsymbol{y}'+\boldsymbol{d}) - \mu_{R_0}(\boldsymbol{y}')]}{|\boldsymbol{y}-\boldsymbol{y}'+\boldsymbol{x}_i-\boldsymbol{x}_j|} \simeq \sum_{i\neq j}^{N} \left(\frac{m^2}{|\boldsymbol{d}+\boldsymbol{x}_{ij}|} - \frac{m^2}{|x_{ij}|}\right)$$

$$= \sum_{i\neq j}^{N} \frac{m^2}{x_{ij}} \left(\frac{1}{\sqrt{\left(\frac{d}{x_{ij}}\right)^2+1}} - 1 \right) \simeq \sum_{i\neq j}^{N} \frac{m^2}{x_{ij}} \left(-\frac{1}{2}\left(\frac{d}{x_{ij}}\right)^2 \right) = -\frac{d^2 m^2}{2} \sum_{i\neq j}^{N} \frac{1}{x_{ij}^3}$$

$$\boldsymbol{x}_i = i_1 a\hat{\boldsymbol{x}} + i_2 a\hat{\boldsymbol{y}} + i_3 a\hat{\boldsymbol{z}}$$

$$x_{ij} = a\sqrt{(i_1-j_1)^2+(i_2-j_2)^2+(i_3-j_3)^2}$$

$$I \simeq -\frac{d^2 m^2}{2a^3} S$$

$$S := \sum_{i=1}^{N} \sum_{j=1,(j\neq i)}^{N} \frac{1}{[(i_1-j_1)^2+(i_2-j_2)^2+(i_3-j_3)^2]^{\frac{3}{2}}}$$

$$S \ll N \sum_{\substack{j_1=-n/2 \\ (j_1,j_2,j_3)\neq(0,0,0)}}^{n/2} \sum_{j_2=-n/2}^{n/2} \sum_{j_3=-n/2}^{n/2} \frac{1}{(j_1^2+j_2^2+j_3^2)^{\frac{3}{2}}}$$

$$S \ll N \sum_{\substack{j_1=-n/2 \\ (j_1,j_2,j_3)\neq(0,0,0)}}^{n/2} \sum_{j_2=-n/2}^{n/2} \sum_{j_3=-n/2}^{n/2} \frac{1}{(j_1^2+j_2^2+j_3^2)^{\frac{3}{2}}}$$

$$S \ll N2^3 \sum_{\substack{j_1=0 \\ (j_1,j_2,j_3)\neq(0,0,0)}}^{n/2} \sum_{j_2=0}^{n/2} \sum_{j_3=0}^{n/2} \frac{1}{(j_1^2+j_2^2+j_3^2)^{\frac{3}{2}}}$$

$$= N2^3 \left(\sum_{\substack{j_1=0 \\ (j_1,j_2,j_3)\neq(0,0,0)}}^{1} \sum_{j_2=0}^{1} \sum_{j_3=0}^{1} + 3 \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} \sum_{j_3=2}^{n/2} + 3 \sum_{j_1=0}^{1} \sum_{j_2=2}^{n/2} \sum_{j_3=2}^{n/2} + \sum_{j_1=2}^{n/2} \sum_{j_2=2}^{n/2} \sum_{j_3=2}^{n/2} \right) \frac{1}{(j_1^2+j_2^2+j_3^2)^{\frac{3}{2}}}$$

$$\sum_{j=k}^{n} \frac{1}{(a+j^2)^{\frac{3}{2}}} \leq \int_{k-1}^{n} dx \frac{1}{(a+x^2)^{\frac{3}{2}}}$$

$$S \ll N2^3\left[4.25+3\int_1^{n/2}dx\left(\frac{1}{x^3}+\frac{2}{(1+x^2)^{\frac{3}{2}}}+\frac{1}{(2+x^2)^{\frac{3}{2}}}\right)+\right.$$

$$\left.3\frac{\pi}{2}\int_1^{n/\sqrt{2}}dr\left(\frac{1}{r^2}+\frac{r}{(1+r^2)^{\frac{3}{2}}}\right)+\frac{\pi}{2}\int_1^{\sqrt{3}n/2}dr\frac{1}{r}\right]$$

$$3\int_1^{n/2}dx\left(\frac{1}{x^3}+\frac{2}{(1+x^2)^{\frac{3}{2}}}+\frac{1}{(2+x^2)^{\frac{3}{2}}}\right)=\frac{6n}{\sqrt{n^2+4}}+\frac{3n}{2\sqrt{n^2+8}}-\frac{6}{n^2}-3\sqrt{2}-\frac{\sqrt{3}}{2}+\frac{3}{2}\simeq$$

$$\simeq 9-3\sqrt{2}-\frac{\sqrt{3}}{2}\simeq 3.89$$

$$3\frac{\pi}{2}\int_1^{n/\sqrt{2}}dr\left(\frac{1}{r^2}+\frac{r}{(1+r^2)^{\frac{3}{2}}}\right)=\frac{3}{4}\pi\left(-\frac{2\sqrt{2}}{\sqrt{n^2+2}}-\frac{2\sqrt{2}}{n}+\sqrt{2}+2\right)\simeq\frac{3}{4}\pi(\sqrt{2}+2)\simeq 8.04$$

$$\frac{\pi}{2}\int_1^{\sqrt{3}n/2}dr\frac{1}{r}=\frac{\pi}{2}\ln\left(\frac{\sqrt{3}n}{2}\right)$$

$$S \ll N2^3\left(17+\frac{\pi}{2}\ln\left[\frac{\sqrt{3}}{2}(\sqrt[3]{N}-1)\right]\right)\sim\frac{4\pi}{3}N\ln(N)\sim 10^{16}$$

$$|I|=\frac{d^2m^2}{2a^3}|S|\ll\frac{10^{-26}(5\times10^{-26})^2}{2\times10^{-30}}10^{16}\sim 10^{-31}\mathrm{Kg}^2\ \mathrm{m}^{-1},$$

$$|D|=8\pi G|I|\ll 1.7\times10^{-40}\ \mathrm{J}$$

$$\rho(t+dt)=(1-\lambda dt)\left[\rho(t)-\frac{i}{\hbar}[H,\rho(t)]dt\right]+\lambda dt\mathcal{G}[\rho(t)]$$

$$\frac{d\rho(t)}{dt}=-\frac{i}{\hbar}[H,\rho(t)]+\lambda(\mathcal{G}[\rho(t)]-\rho(t))$$

$$\mathcal{L}[\rho(t)]=\int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})\sum_{j=1}^{\infty}\left[\left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}L_j(\boldsymbol{Q},\hat{\boldsymbol{p}})\rho(t)L_j^\dagger(\boldsymbol{Q},\hat{\boldsymbol{p}})e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}-\frac{1}{2}\{L_j^\dagger(\boldsymbol{Q},\hat{\boldsymbol{p}})L_j(\boldsymbol{Q},\hat{\boldsymbol{p}}),\rho(t)\}\right)\right]$$

$$\mathcal{L}[\rho(t)]=\int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})\left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}\rho(t)e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}-\rho(t)\right)$$

$$\frac{d\rho(t)}{dt}=-\frac{i}{\hbar}[H,\rho(t)]+\int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})\left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}\rho(t)e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}-\rho(t)\right)$$

$$\langle\boldsymbol{a}|\mathcal{L}[\rho(t)]|\boldsymbol{b}\rangle=\left[\frac{8\pi G}{\hbar}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{\mu(\boldsymbol{r}-\boldsymbol{a})\mu(\boldsymbol{r}'-\boldsymbol{b})-\mu(\boldsymbol{r})\mu(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\langle\boldsymbol{a}|\hat{\rho}_t|\boldsymbol{b}\rangle$$

$$\frac{8\pi G}{\hbar}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{\mu(\boldsymbol{r}-\boldsymbol{a})\mu(\boldsymbol{r}'-\boldsymbol{b})-\mu(\boldsymbol{r})\mu(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}=\int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})\left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot(\boldsymbol{a}-\boldsymbol{b})}-1\right)$$

$$\Gamma(\boldsymbol{y})=\frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{y}}$$

$$\Gamma(\boldsymbol{d}) = \frac{8\pi G}{\hbar(2\pi\hbar)^3}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{\mu(\boldsymbol{r})\mu(\boldsymbol{r}'+\boldsymbol{d})}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\int d\boldsymbol{s}\frac{e^{-\frac{i}{\hbar}Q\cdot\boldsymbol{s}}}{|\boldsymbol{s}|} = \frac{4\pi\hbar^2}{Q^2}$$

$$\tilde{\Gamma}(\boldsymbol{Q}) = \frac{8\pi G}{\hbar(2\pi\hbar)^3}\tilde{\mu}(\boldsymbol{Q})\frac{4\pi\hbar^2}{Q^2}\tilde{\mu}(-\boldsymbol{Q})$$

$$\tilde{\Gamma}(\boldsymbol{Q}) = \frac{4G}{\pi\hbar^2}\frac{|\tilde{\mu}(\boldsymbol{Q})|^2}{Q^2}$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H,\rho(t)] + \int d\boldsymbol{Q}\tilde{\Gamma}(\boldsymbol{Q})\left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}}\rho(t)e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}} - \rho(t)\right)$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H,\rho(t)] - \frac{4\pi G}{\hbar}\int d\boldsymbol{x}\int d\boldsymbol{y}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}[\mu(\boldsymbol{y},\hat{\boldsymbol{x}}),[\mu(\boldsymbol{x},\hat{\boldsymbol{x}}),\rho(t)]]$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H,\rho(t)] - \frac{4\pi G}{\hbar}\int d\boldsymbol{x}\int d\boldsymbol{y}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}[\hat{M}(\boldsymbol{y}),[\hat{M}(\boldsymbol{x}),\rho(t)]]$$

$$\frac{d}{d\omega_k}\Gamma_t = \frac{k^2}{c}\sum_{\nu}\int d\Omega_k\frac{d}{dt}\langle a^{\dagger}_{\mathbf{k}\nu}a_{\mathbf{k}\nu}\rangle_t$$

$$\frac{d}{dt}O(t) = \frac{i}{\hbar}[H,O(t)] + \int d\boldsymbol{Q}\sum_{k,k'}\tilde{\Gamma}_{k,k'}(\boldsymbol{Q})\left(e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{k'}}O(t)e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_k} - \frac{1}{2}\left\{O(t),e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{k'}}e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_k}\right\}\right)$$

$$\frac{d}{dt}O^I(t) = \mathcal{C}^I_t[O^I(t)] + \mathcal{L}^I_t[O^I(t)]$$

$$\mathcal{C}^I_t[O^I(t)] := \frac{i}{\hbar}\left[H^I_{\text{INT}}(t),O^I(t)\right]$$

$$\mathcal{L}^I_t[O^I(t)] := \int d\boldsymbol{Q}\sum_{n,n'}\tilde{\Gamma}_{n,n'}(\boldsymbol{Q})\left(e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}^I_{n'}(t)}O^I(t)e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}^I_n(t)} - \frac{1}{2}\left\{O^I(t),e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\left(\boldsymbol{x}^I_{n'}(t)-\boldsymbol{x}^I_n(t)\right)}\right\}\right)$$

$$O(t) = O(0) + \int_0^t dt_1\mathcal{C}_{t_1}O(t_1) + \int_0^t dt_1\mathcal{L}_{t_1}O(t_1)$$

$$O(t) = O(0) + \int_0^t dt_1\mathcal{C}_{t_1}O(0) + \int_0^t dt_1\mathcal{L}_{t_1}O(0) + \int_0^t dt_1\int_0^{t_1} dt_2\mathcal{C}_{t_1}\mathcal{C}_{t_2}O(t_2)$$
$$+ \int_0^t dt_1\int_0^{t_1} dt_2\mathcal{C}_{t_1}\mathcal{L}_{t_2}O(t_2) + \int_0^t dt_1\int_0^{t_1} dt_2\mathcal{L}_{t_1}\mathcal{C}_{t_2}O(t_2) + \int_0^t dt_1\int_0^{t_1} dt_2\mathcal{L}_{t_1}\mathcal{L}_{t_2}O(t_2)$$

$$O(t) = O + \int_0^t dt_1 \mathcal{C}_{t_1} O + \int_0^t dt_1 \mathcal{L}_{t_1} O + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{C}_{t_1} \mathcal{C}_{t_2} O + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{C}_{t_1} \mathcal{L}_{t_2} O$$
$$+ \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{L}_{t_1} \mathcal{C}_{t_2} O + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathcal{C}_{t_1} \mathcal{C}_{t_2} \mathcal{L}_{t_3} O$$
$$+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathcal{C}_{t_1} \mathcal{L}_{t_2} \mathcal{C}_{t_3} O + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathcal{L}_{t_1} \mathcal{C}_{t_2} \mathcal{C}_{t_3} O$$

$$\langle a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}\nu\nu} \rangle_t = \mathrm{Tr}[(a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}\nu})(t)\rho]$$

$$\langle a^\dagger_{\mathbf{k}\mu} a_{\mathbf{k}\mu} \rangle_t = A_1 + A_2 + A_3$$

$$A_1 := \int_0^t dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}[\rho_i \mathcal{L}_{t_1} \mathcal{C}_{t_2} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}\nu}]$$
$$A_2 := \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathrm{Tr}[\rho_i \mathcal{C}_{t_1} \mathcal{L}_{t_2} \mathcal{C}_{t_3} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}\nu}]$$
$$A_3 := \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathrm{Tr}[\rho_i \mathcal{L}_{t_1} \mathcal{C}_{t_2} \mathcal{C}_{t_3} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}\nu}]$$

$$\frac{d}{dt} A_3 = \frac{1}{\hbar^2 \omega_k^2} \frac{4\pi G}{\hbar} \left(\frac{\alpha_k^2 \hbar^2}{(2\pi)^3} \right) \left[\sum_n \frac{e_n^2}{R_{0n}^{10}} \left(2 \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \left(e^{-\frac{y^2 + x^2}{2R_{0n}^2}} (\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{y})(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{x}) \right) \right) \right]$$

$$\frac{d}{d\omega_k} \Gamma_t = \frac{k^2}{c} \sum_\nu \int d\Omega_k \frac{d}{dt} A_3$$

$$\int d\Omega_k \sum_\nu \epsilon^i_{\mathbf{k}\nu} \epsilon^j_{\mathbf{k}\nu} = \frac{8}{3} \pi \delta^{ij}$$

$$\frac{d}{d\omega_k} \Gamma_t = \frac{k^2}{c} \frac{1}{\hbar^2 \omega_k^2} \frac{4\pi G}{\hbar} \left(\frac{\alpha_k^2 \hbar^2}{(2\pi)^3} \right) 2 \left(\frac{8}{3} \pi \right) \left[\sum_n \frac{e_n^2}{R_{0n}^{10}} \left(\int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} e^{-\frac{y^2 + x^2}{2R_{0n}^2}} \boldsymbol{y} \cdot \boldsymbol{x} \right) \right].$$

$$\int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} e^{-\frac{y^2 + x^2}{2R_{0n}^2}} \boldsymbol{y} \cdot \boldsymbol{x} = 4\pi^{5/2} R_{0n}^7$$

$$\frac{d}{d\omega_k} \Gamma_t = \frac{2G}{3\pi^{3/2} \varepsilon_0 c^3 \omega_k} \left[\sum_n \frac{e_n^2}{R_{0n}^3} \right]$$

$$\frac{d}{d\omega_k} \Gamma_t = \frac{2Ge^2}{3\pi^{3/2} \varepsilon_0 c^3 \omega_k} \left[\frac{N}{R_{0e}^3} + \frac{N^2}{R_0^3} \right]$$

$$\frac{d}{d\omega_k} \Gamma_t = \frac{2Ge^2 N^2 N_A}{3\pi^{3/2} \varepsilon_0 c^3 R_0^3 \omega_k}$$

$$H_{\mathrm{INT}}(t) = H_1(t) + H_2(t)$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \int d\boldsymbol{k} \sum_{\nu} \alpha_k \left[\vec{\epsilon}_{\boldsymbol{k},\nu} a_{\boldsymbol{k},\nu} e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_k t} + \vec{\epsilon}_{\boldsymbol{k},\nu} a^{\dagger}_{\boldsymbol{k},\nu} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega_k t}\right]$$

$$H_1(t) := \sum_j \left(-\frac{e_j}{m_j}\right) \boldsymbol{A}\big(\boldsymbol{x}_j(t),t\big) \cdot \boldsymbol{p}_j(t) = \int d\boldsymbol{k} \sum_{\nu} \left(R_{\boldsymbol{k},\nu}(t) a_{\boldsymbol{k},\nu} + R^{\dagger}_{\boldsymbol{k},\nu}(t) a^{\dagger}_{\boldsymbol{k},\nu}\right)$$

$$R_{\boldsymbol{k},\nu}(t) := -\alpha_k e^{+i\omega_k t} \sum_{j=1}^{N_p} \frac{e_j}{m_j} \vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j(t)}$$

$$H_2(t) = \sum_j \frac{e_j^2}{2m_j} \boldsymbol{A}^2\big(\boldsymbol{x}_j(t),t\big)$$

$$\mathcal{C}_t a^{\dagger}_{\mathbf{k}\nu} a_{\mathbf{k}\nu} = \frac{i}{\hbar}\left[H_{\mathrm{INT}}(t), a^{\dagger}_{\mathbf{k}\nu} a_{\mathbf{k}\nu}\right] = a^{\dagger}_{\mathbf{k}\nu} \mathcal{C}_t a_{\mathbf{k}\nu} + \mathcal{C}_t\left[a^{\dagger}_{\mathbf{k}\nu}\right] a_{\mathbf{k}\nu}$$

$$\mathcal{C}_t a_{\mathbf{k}\nu} = \frac{i}{\hbar}\left[H_1(t), a_{\mathbf{k}\nu}\right] = -\frac{i}{\hbar} R^{\dagger}_{\mathbf{k},\nu}(t)$$

$$\mathcal{C}_t\left[a^{\dagger}_{\mathbf{k}\nu}\right] = \left(\mathcal{C}_t a_{\mathbf{k}\nu}\right)^{\dagger} = \frac{i}{\hbar} R_{\mathbf{k},\nu}(t)$$

$$\mathcal{C}_t a^{\dagger}_{\mathbf{k}\nu} a_{\mathbf{k}\nu} = -\frac{i}{\hbar} a^{\dagger}_{\mathbf{k}\nu} R^{\dagger}_{\boldsymbol{k},\nu}(t) + \frac{i}{\hbar} R_{\boldsymbol{k},\nu}(t) a_{\mathbf{k}\nu}$$

$$\mathcal{C}_{t_2}\mathcal{C}_{t_3} a^{\dagger}_{\mathbf{k}\nu} a_{\mathbf{k}\nu} = \mathcal{C}_{t_2}\left(a^{\dagger}_{\mathbf{k}\nu}\mathcal{C}_{t_3} a_{\mathbf{k}\nu} + \mathcal{C}_{t_3}\left[a^{\dagger}_{\mathbf{k}\nu}\right] a_{\mathbf{k}\nu}\right)$$
$$= \mathcal{C}_{t_2}\left[a^{\dagger}_{\mathbf{k}\nu}\right]\mathcal{C}_{t_3}\left[a_{\mathbf{k}\nu}\right] + \mathcal{C}_{t_3}\left[a^{\dagger}_{\mathbf{k}\nu}\right]\mathcal{C}_{t_2}\left[a_{\mathbf{k}\nu}\right] = \frac{1}{\hbar^2}\left(R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\mathbf{k},\nu}(t_3) + R_{\boldsymbol{k},\nu}(t_3) R^{\dagger}_{\mathbf{k},\nu}(t_2)\right)$$

$$\int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \mathcal{C}_{t_2}\mathcal{C}_{t_3} a^{\dagger}_{\mathbf{k}\nu} a_{\mathbf{k}\nu} = \frac{1}{\hbar^2}\int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left(R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\boldsymbol{k},\nu}(t_3) + R_{\boldsymbol{k},\nu}(t_3) R^{\dagger}_{\boldsymbol{k},\nu}(t_2)\right) =$$
$$\frac{1}{\hbar^2}\int_0^{t_1} dt_2 \int_0^{t_2} dt_3 R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\boldsymbol{k},\nu}(t_3) + \frac{1}{\hbar^2}\int_0^{t_1} dt_3 \int_{t_3}^{t_1} dt_2 R_{\boldsymbol{k},\nu}(t_3) R^{\dagger}_{\boldsymbol{k},\nu}(t_2) =$$
$$= \frac{1}{\hbar^2}\int_0^{t_1} dt_2 \int_0^{t_1} dt_3 R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\boldsymbol{k},\nu}(t_3)$$

$$A_3 = \int_0^{t} dt_1 \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \mathrm{Tr}\left[\rho_g\left(\frac{1}{\hbar^2}\int d\boldsymbol{Q} \sum_{n,n'} \tilde{\Gamma}_{n,n'}(\boldsymbol{Q}) \left[e^{-\frac{i}{\hbar}Q\cdot x_{n'}(t_1)} R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\boldsymbol{k},\nu}(t_3) e^{\frac{i}{\hbar}Q\cdot\boldsymbol{x}_n(t_1)}\right.\right.\right.$$
$$\left.\left.\left. -\frac{1}{2}\left\{R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{k,\nu}(t_3), e^{-\frac{i}{\hbar}Q\cdot\left(x_{n'}(t_1)-x_n(t_1)\right)}\right\}\right]\right)\right]$$

$$R_{\boldsymbol{k},\nu}(t_2) R^{\dagger}_{\boldsymbol{k},\nu}(t_3)$$
$$= \alpha_k^2 e^{+i\omega_k(t_2-t_3)} \sum_{j,j'} \frac{e_j e_{j'}}{m_j m_{j'}} \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j(t_2) e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j(t_2)}\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu}\right.$$
$$\left.\cdot \boldsymbol{p}_{j'}(t_3) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{j'}(t_3)}\right)$$

$$T_1 := \int_0^{t_1} dt_2 \int_0^{t_1} dt_3 \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}(t_1)} R_{k,\nu}(t_2) R_{k,\nu}^{\dagger}(t_3) e^{\frac{i}{\hbar} Q\cdot x_n(t_1)} |g\rangle =$$
$$= \sum_f \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}(t_1)} \left(\int_0^{t_1} dt_2 R_{k,\nu}(t_2) \right) |f\rangle\langle f| \left(\int_0^{t_1} dt_3 R_{k,\nu}^{\dagger}(t_3) \right) e^{\frac{i}{\hbar} Q\cdot x_n(t_1)} |g\rangle$$

$$\langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}(t_1)} \left(\int_0^{t_1} dt_2 R_{k,\nu}(t_2) \right) |f\rangle = \int_0^{t_1} dt_2 \sum_E \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}(t_1)} |E\rangle\langle E| R_{k,\nu}(t_2) |f\rangle =$$
$$= \sum_E e^{-\frac{i}{\hbar}(E_g - E)t_1} \left(\int_0^{t_1} dt_2 e^{-\frac{i}{\hbar}(E - E_f - \hbar\omega_k)t_2} \right) \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} |E\rangle\langle E| R_{k,\nu}(0) |f\rangle =$$
$$= \sum_E \left(\frac{e^{-\frac{i}{\hbar}(E_g - E_f - \hbar\omega_k)t_1} - e^{-\frac{i}{\hbar}(E_g - E)t_1}}{-\frac{i}{\hbar}(E - E_f - \hbar\omega_k)} \right) \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} |E\rangle\langle E| R_{k,\nu}(0) |f\rangle$$

$$\int_0^{t_1} dt_2 e^{-\frac{i}{\hbar}(E - E_f - \hbar\omega_k)t_2} \longrightarrow \int_{-\infty}^{t_1} dt_2 e^{-\frac{i}{\hbar}(E - E_f - \hbar\omega_k)t_2 + \epsilon t_2} = \frac{e^{\left[-\frac{i}{\hbar}(E - E_f - \hbar\omega_k) + \epsilon\right]t_1}}{\epsilon - \frac{i}{\hbar}(E - E_f - \hbar\omega_k)}$$

$$\langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}(t_1)} \int_0^{t_1} dt_2 R_{k,\nu}(t_2) |f\rangle = \sum_E \left(\frac{e^{\left[-\frac{i}{\hbar}(E_g - E_f - \hbar\omega_k) + \epsilon\right]t_1}}{\epsilon - \frac{i}{\hbar}(E - E_f - \hbar\omega_k)} \right) \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} |E\rangle\langle E| R_{k,\nu}(0) |f\rangle$$
$$\simeq \frac{e^{\left[-\frac{i}{\hbar}(E_g - E_f - \hbar\omega_k) + \epsilon\right]t_1}}{\epsilon + i\omega_k} \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} R_{k,\nu}(0) |f\rangle$$

$$T_1 = \sum_f \frac{e^{2\epsilon t_1}}{\epsilon^2 + \omega_k^2} \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} R_{\boldsymbol{k},\nu}(0) |f\rangle\langle f| R_{\boldsymbol{k},\nu}^{\dagger}(0) e^{\frac{i}{\hbar} \boldsymbol{Q}\cdot x_n(t_1)} |g\rangle$$
$$= \frac{e^{2\epsilon\epsilon_1}}{\epsilon^2 + \omega_k^2} \langle g| e^{-\frac{i}{\hbar} Q\cdot x_{n'}} R_{\boldsymbol{k},\nu}(0) R_{\boldsymbol{k},\nu}^{\dagger}(0) e^{\frac{i}{\hbar} Q\cdot x_n(t_1)} |g\rangle.$$

$$\frac{d}{dt} A_3 = \lim_{\epsilon\to 0} \frac{d}{dt} \int_{-\infty}^{t} dt_1 \frac{e^{3\epsilon t_1}}{\epsilon^2 + \omega_k^2} \frac{1}{\hbar^2} \mathrm{Tr}\left[\rho_g X\right] = \lim_{\epsilon\to 0} \frac{e^{3\epsilon t}}{\epsilon^2 + \omega_k^2} \frac{1}{\hbar^2} \mathrm{Tr}\left[\rho_g X\right] = \frac{1}{\hbar^2 \omega_k^2} \mathrm{Tr}\left[\rho_g X\right]$$

$$X := \int d\boldsymbol{Q} \sum_{n,n'} \tilde{\Gamma}_{n,n'}(\boldsymbol{Q}) \left[e^{-\frac{i}{\hbar} \boldsymbol{Q}\cdot \boldsymbol{x}_{n'}} R_{\boldsymbol{k},\nu}(0) R_{\boldsymbol{k},\nu}^{\dagger}(0) e^{\frac{i}{\hbar} \boldsymbol{Q}\cdot \boldsymbol{x}_n} - \frac{1}{2} \left\{ R_{\boldsymbol{k},\nu}(0) R_{\boldsymbol{k},\nu}^{\dagger}(0), e^{-\frac{i}{\hbar} \boldsymbol{Q}\cdot(\boldsymbol{x}_{n'} - \boldsymbol{x}_n)} \right\} \right]$$

$$X = \frac{8\pi G}{\hbar} \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \left(M(\boldsymbol{x}) R_{\boldsymbol{k},\nu}(0) R_{\boldsymbol{k},\nu}^{\dagger}(0) M(\boldsymbol{y}) - \frac{1}{2} \left\{ M(\boldsymbol{x}) M(\boldsymbol{y}), R_{\boldsymbol{k},\nu}(0) R_{\boldsymbol{k},\nu}^{\dagger}(0) \right\} \right)$$
$$= -\frac{4\pi G}{\hbar} \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \left[M(\boldsymbol{y}), \left[M(\boldsymbol{x}), R_{\boldsymbol{k},\nu}(0) R_{k,\nu}^{\dagger}(0) \right] \right]$$

$$M(\boldsymbol{x}) = \sum_n \mu_n(\boldsymbol{x}, \boldsymbol{x}_n) = \sum_n m_n \frac{1}{(2\pi R_{0n}^2)^{\frac{3}{2}}} e^{-\frac{(\boldsymbol{x} - \boldsymbol{x}_n)^2}{2R_{0n}^2}} := \sum_n m_n g_n(\boldsymbol{x})$$

$$R_{k,\nu}(0)R^{\dagger}_{k,\nu}(0) = \alpha_k^2 \sum_{j,j'=1}^{N_p} \frac{e_j e_{j'}}{m_j m_{j'}} \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j e^{i\boldsymbol{k}\cdot x_j}\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'} e^{-i\boldsymbol{k}\cdot x_{j'}}\right)$$

$$= \alpha_k^2 \sum_{j,j'=1}^{N_p} \frac{e_j e_{j'}}{m_j m_{j'}} e^{i\boldsymbol{k}\cdot\left(x_j - x_{j'}\right)}\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right),$$

$$X = -\frac{4\pi G\alpha_k^2}{\hbar} \sum_{n,n',j,j'} \frac{m_{n'} m_n e_j e_{j'}}{m_j m_{j'}} \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{\left[g_{n'}(\boldsymbol{y}), \left[g_n(\boldsymbol{x}), e^{i\boldsymbol{k}\cdot\left(x_j - x_{j'}\right)}\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right)\right]\right]}{|\boldsymbol{x} - \boldsymbol{y}|}$$

$$= -\frac{4\pi G\alpha_k^2}{\hbar} \sum_{n,n',j,j'} \frac{m_{n'} m_n e_j e_{j'}}{m_j m_{j'}} e^{i\boldsymbol{k}\cdot\left(x_j - x_{j'}\right)} \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{\left[g_{n'}(\boldsymbol{y}), \left[g_n(\boldsymbol{x}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right)\right]\right]}{|\boldsymbol{x} - \boldsymbol{y}|}$$

$$C := \left[g_{n'}(\boldsymbol{y}), \left[g_n(\boldsymbol{x}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right)\right]\right]$$

$$= \left[g_{n'}(\boldsymbol{y}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\right]\left[g_n(\boldsymbol{x}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right)\right] + \left[g_n(\boldsymbol{x}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\right]\left[g_{n'}(\boldsymbol{y}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_{j'}\right)\right]$$

$$\left[g_n(\boldsymbol{x}), \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{p}_j\right)\right] = \frac{\vec{\epsilon}_{\boldsymbol{k},\nu}}{\left(2\pi R_{0n}^2\right)^{\frac{3}{2}}} \cdot \left[e^{-\frac{(x - x_n)^2}{2R_{0n}^2}}, \boldsymbol{p}_j\right] = \frac{\vec{\epsilon}_{\boldsymbol{k},\nu}}{\left(2\pi R_{0n}^2\right)^{\frac{3}{2}}} \cdot \left(i\hbar\nabla_j e^{-\frac{(x - x_n)^2}{2R_{0n}^2}}\right)$$

$$= \delta_{jn} i\hbar \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot (\boldsymbol{x} - \boldsymbol{x}_n)\right) \frac{e^{-\frac{(x - x_n)^2}{2R_{0n}^2}}}{(2\pi)^{3/2} R_{0n}^5}$$

$$C = -\frac{\hbar^2\left(\delta_{jn'}\delta_{j'n} + \delta_{jn}\delta_{j'n'}\right)}{(2\pi)^3 R_{0n'}^5 R_{0n}^5} \left(e^{-\frac{\left(y - x_{n'}\right)^2}{2R_{0n'}} - \frac{(x - x_n)^2}{2R_{0n}^2}} \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot (\boldsymbol{x} - \boldsymbol{x}_n)\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}_{n'})\right)\right)$$

$$X = \frac{4\pi G\alpha_k^2}{\hbar}\left(\frac{\hbar^2}{(2\pi)^3}\right) \sum_{n,n'} \frac{e_{n'} e_n}{R_{0n'}^5 R_{0n}^5} \{2\cos\left[\boldsymbol{k} \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n'})\right] \times$$

$$\times \int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\left[e^{-\frac{\left(y - x'\right)^2}{2R_{0n'}^2} - \frac{(x - x_n)^2}{2R_{0n}^2}} \left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot (\boldsymbol{x} - \boldsymbol{x}_n)\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}_{n'})\right)\right]\bigg\}.$$

$$\mathrm{Tr}\left[\rho_g X\right] \simeq \frac{4\pi G}{\hbar}\left(\frac{\alpha_k^2\hbar^2}{(2\pi)^3}\right)\left[\sum_n \frac{e_n^2}{R_{0n}^{10}}\left(2\int d\boldsymbol{x} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\left(e^{-\frac{y^2 + x^2}{2R_{0n}^2}}\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{y}\right)\left(\vec{\epsilon}_{\boldsymbol{k},\nu} \cdot \boldsymbol{x}\right)\right)\right)\right].$$

$$\frac{d}{dt}A_3 = \frac{1}{\hbar^2\omega_k^2}\frac{4\pi G}{\hbar}\left(\frac{\alpha_k^2\hbar^2}{(2\pi)^3}\right)\left[\sum_n \frac{e_n^2}{R_{0n}^{10}}\left(2\int d\boldsymbol{x}\int d\boldsymbol{y}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\left(e^{-\frac{y^2+x^2}{2R_{0n}^2}}(\vec{\epsilon}_{\boldsymbol{k},\nu}\cdot\boldsymbol{y})(\vec{\epsilon}_{\boldsymbol{k},\nu}\cdot\boldsymbol{x})\right)\right)\right]$$

$$A_2 := \int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 \mathrm{Tr}\left[\rho_i \mathcal{C}_{t_1}\mathcal{L}_{t_2}\mathcal{C}_{t_3}a_{\mathbf{k}\nu}^\dagger a_{\mathbf{k}\nu}\right]$$

$$\mathcal{C}_{t_1}\mathcal{L}_{t_2}\mathcal{C}_{t_3}a_{\mathbf{k}\nu}^\dagger a_{\mathbf{k}\nu} = -\frac{i}{\hbar}\mathcal{C}_{t_1}\left(a_{\mathbf{k}\nu}^\dagger\mathcal{L}_{t_2}[R_{\boldsymbol{k},\nu}^\dagger(t_3)]\right) + \frac{i}{\hbar}\mathcal{C}_{t_1}\left(\mathcal{L}_{t_2}[R_{\mathbf{k},\nu}(t_3)]a_{\mathbf{k}\nu\nu}\right)$$

$$\mathcal{L}_t O^\dagger = [\mathcal{L}_t O]^\dagger$$

$$\mathcal{C}_t O^\dagger = \frac{i}{\hbar}[H_1(t), O^\dagger] = \left(-\frac{i}{\hbar}[O, H_1(t)]\right)^\dagger = (\mathcal{C}_t O)^\dagger$$

$$\mathcal{C}_{t_1}\mathcal{L}_{t_2}\mathcal{C}_{t_3}a_{\mathbf{k}\nu}^\dagger a_{\mathbf{k}\nu} = \frac{i}{\hbar}\mathcal{C}_{t_1}\left(\mathcal{L}_{t_2}[R_{\mathbf{k},\nu}(t_3)]a_{\mathbf{k}\nu}\right) + \text{ H.c}$$

$$\mathcal{C}_{t_1}\left(\mathcal{L}_{t_2}[R_{\boldsymbol{k},\nu}(t_3)]a_{\mathbf{k}\nu}\right) = \mathcal{C}_{t_1}\left(\mathcal{L}_{t_2}[R_{\boldsymbol{k},\nu}(t_3)]\right)a_{\mathbf{k}\nu} + \mathcal{L}_{t_2}[R_{\boldsymbol{k},\nu}(t_3)]\left(-\frac{i}{\hbar}R_{k,\nu}^\dagger(t_1)\right)$$

$$\mathcal{C}_{t_1}\mathcal{L}_{t_2}\mathcal{C}_{t_3}a_{\mathbf{k}\nu}^\dagger a_{\mathbf{k}\nu} = \frac{1}{\hbar^2}\int d\boldsymbol{Q}\sum_{n,n'}\tilde{\Gamma}_{n,n'}(\boldsymbol{Q})\left(e^{-\frac{i}{\hbar}Q\cdot\boldsymbol{x}_{n'}(t_2)}R_{\boldsymbol{k},\nu}(t_3)e^{\frac{i}{\hbar}Q\cdot x_n(t_2)}\right.$$
$$\left. -\frac{1}{2}\left\{R_{k,\nu}(t_3), e^{-\frac{i}{\hbar}Q\cdot\left(x_{n'}(t_2)-x_n(t_2)\right)}\right\}\right)\left(R_{\boldsymbol{k},\nu}^\dagger(t_1)\right) + H.c$$

$$\frac{d}{dt}A_2 := \frac{1}{\hbar^2}\int d\boldsymbol{Q}\sum_{n,n'}\tilde{\Gamma}_{n,n'}(\boldsymbol{Q})\int_0^t dt_2\int_0^{t_2} dt_3\langle g|\left(e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{n'}(t_2)}R_{k,\nu}(t_3)e^{\frac{i}{\hbar}Q\cdot x_n(t_2)}\right.$$
$$\left.-\frac{1}{2}\left\{R_{\boldsymbol{k},\nu}(t_3), e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\left(\boldsymbol{x}_{n'}(t_2)-\boldsymbol{x}_n(t_2)\right)}\right\}\right)\left(R_{k,\nu}^\dagger(t)\right)|g\rangle + H.c$$

$$\int_{-\infty}^t dt_2 e^{\epsilon t_2}\int_{-\infty}^{t_2} dt_3 e^{\epsilon t_3}\sum_{f,E,E'}\langle g|e^{-\frac{i}{\hbar}Q\cdot x_{n'}(t_2)}|E'\rangle\langle E'|R_{k,\nu}(t_3)|E\rangle\langle E|e^{\frac{i}{\hbar}Q\cdot x_n(t_2)}|f\rangle\langle f|R_{k,\nu}^\dagger(t)|g\rangle =$$
$$= \sum_{f,E,E'}\int_{-\infty}^t dt_2 e^{\left[-\frac{i}{\hbar}(E_g-E'+E-E_f)t_2+\epsilon t_2\right]}\int_{-\infty}^{t_2} dt_3 e^{\left[-\frac{i}{\hbar}(E'-E-\hbar\omega_k)t_3+\epsilon t_3\right]}e^{-\frac{i}{\hbar}(E_f+\hbar\omega_k-E_g)t}\times$$
$$\times\langle g|e^{-\frac{i}{\hbar}Q\cdot x_{n'}}|E'\rangle\langle E'|R_{k,\nu}(0)|E\rangle\langle E|e^{\frac{i}{\hbar}Q\cdot x_n}|f\rangle\langle f|R_{k,\nu}^\dagger(0)|g\rangle=$$
$$= -\sum_{f,E,E'}\frac{\hbar^2 e^{2t\epsilon}}{[E-E'+\hbar(\omega_k-i\epsilon)][E_f-E_g+\hbar(\omega_k-2i\epsilon)]}\times$$
$$\times\langle g|e^{-\frac{i}{\hbar}Q\cdot x_{n'}}|E'\rangle\langle E'|R_{k,\nu}(0)|E\rangle\langle E|e^{\frac{i}{\hbar}Q\cdot x_n}|f\rangle\langle f|R_{k,\nu}^\dagger(0)|g\rangle\simeq$$
$$\simeq -\frac{e^{2t\epsilon}}{[(\omega_k-i\epsilon)][(\omega_k-2i\epsilon)]}\langle g|e^{-\frac{i}{\hbar}Q\cdot x_{n'}}R_{k,\nu}(0)e^{\frac{i}{\hbar}Q\cdot x_n}R_{k,\nu}^\dagger(0)|g\rangle\rightarrow$$
$$\overrightarrow{\epsilon\rightarrow 0}\longrightarrow -\frac{1}{\omega_k^2}\langle g|e^{-\frac{i}{\hbar}Q\cdot x_{n'}}R_{k,\nu}(0)e^{\frac{i}{\hbar}Q\cdot x_n}R_{k,\nu}^\dagger(0)|g\rangle.$$

$$\frac{d}{dt}A_2 := -\frac{1}{\hbar^2\omega_k^2}\int \; d\boldsymbol{Q}\sum_{n,n'}\tilde{\Gamma}_{n,n'}(\boldsymbol{Q})$$

$$\times \langle g|\left(e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_{n'}}R_{\boldsymbol{k},\nu}(0)e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}_n} - \frac{1}{2}\left\{R_{\boldsymbol{k},\nu}(0), e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot(x_{n'}-x_n)}\right\}\right)R_{\boldsymbol{k},\nu}^{\dagger}(0)|g\rangle + H.c$$

$$\int \; d\boldsymbol{Q}\sum_{n,n'}\tilde{\Gamma}_{n,n'}(\boldsymbol{Q})\left(e^{-\frac{i}{\hbar}Q\cdot\boldsymbol{x}_{n'}}R_{\boldsymbol{k},\nu}(0)e^{\frac{i}{\hbar}Q\cdot\boldsymbol{x}_n} - \frac{1}{2}\left\{R_{\boldsymbol{k},\nu}(0), e^{-\frac{i}{\hbar}Q\cdot(x_{n'}-x_n)}\right\}\right)$$

$$= -\frac{4\pi G}{\hbar}\int \; d\boldsymbol{x}\int \; d\boldsymbol{y}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\left[M(\boldsymbol{y}),\left[M(\boldsymbol{x}),R_{k,\nu}(0)\right]\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -\frac{i}{\hbar}\left[\hat{H}_N,\hat{\rho}(t)\right] + \mathcal{D}[\hat{\rho}(t)]$$

$$\mathcal{D}[\hat{\rho}(t)] = -\frac{4\pi G}{\hbar}\int \; \mathrm{d}\mathbf{r}\int \; \mathrm{d}\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}[\hat{\mu}(\mathbf{r}'),[\hat{\mu}(\mathbf{r}),\hat{\rho}(t)]]$$

$$\hat{\mu}(\mathbf{r}) = \sum_{i=1}^{N}\frac{m_i}{\left(2\pi R_{\mathrm{eff},i}^2\right)^{3/2}}e^{-\frac{(\mathbf{r}-\hat{x}_i)^2}{2R_{\mathrm{eff},i}^2}}$$

$$\langle\mathbf{x}|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{y}\rangle \simeq \langle\mathbf{x}|\hat{\rho}_{\mathrm{CM}}(0)|\mathbf{y}\rangle\exp\left[-t/\tau(\mathbf{x}-\mathbf{y})\right]$$

$$\Delta E(\mathbf{d}) = -8\pi G\int \; \mathrm{d}\mathbf{r}\int \; \mathrm{d}\mathbf{r}'\frac{\mu(\mathbf{r})[\mu(\mathbf{r}'+\mathbf{d})-\mu(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Delta E(\mathbf{d}) = 8\pi G m^2\sum_{i=1}^{N}\sum_{j=1}^{N} f\left(\mathbf{r}_{ij},R_0,\mathbf{d}\right)$$

$$f\left(\mathbf{r}_{ij},R_0,\mathbf{d}\right) = \frac{\mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right)}{r_{ij}} - \frac{\mathrm{erf}\left(\frac{\left|\mathbf{d}-\mathbf{r}_{ij}\right|}{2R_{\mathrm{eff}}}\right)}{\left|\mathbf{d}-\mathbf{r}_{ij}\right|}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f\left(\mathbf{r}_{ij},R_0,\mathbf{d}\right) = \sum_{\mathbf{r}\in\mathcal{D}}\omega(\mathbf{r})f(\mathbf{r},R_0,\mathbf{d})$$

$$\mathcal{D} = \{|n_1\mathbf{a}_1 + n_2\mathbf{a}_2|, \text{donde } n_i \in [-(N_i-1),(N_i-1)]\}$$

$$\omega(n_1,n_2) = (N_1-|n_1|)(N_2-|n_2|)$$

$$\sum_{n_1=-(N_1-1)}^{N_1-1}\;\sum_{n_2=-(N_2-1)}^{N_2-1}\omega(n_1,n_2)f(n_1\mathbf{a}_1+n_2\mathbf{a}_2,R_0,\mathbf{d})$$

$$\mathcal{D}[\hat{\rho}_{\mathrm{CM}}(t)] = -\frac{8\pi G}{\hbar}\int_0^t \mathrm{d}s\int \; \mathrm{d}\mathbf{r}\int \; \mathrm{d}\mathbf{r}'\frac{f(t-s)}{|\mathbf{r}-\mathbf{r}'|}$$

$$\times\left[\hat{\mu}(\mathbf{r}'),\left[e^{\frac{i}{\hbar}\hat{H}_N(s-t)}\hat{\mu}(\mathbf{r})e^{-\frac{i}{\hbar}\hat{H}_N(s-t)},\hat{\rho}_{\mathrm{CM}}(t)\right]\right]$$

$$\tau(\mathbf{d},t)=\frac{\hbar}{\Delta E(\mathbf{d})}\frac{t}{g(t)}$$

$$g(t>0)=t\left[1-\frac{1}{\Omega_C t}\left(1-e^{-\Omega_C t}\right)\right]$$

$$\hat{\mathbf{x}}_i=\hat{\mathbf{x}}+\mathbf{x}_i^{(0)}$$

$$\mathcal{D}[\hat{\rho}(t)]=-\frac{4\pi G}{\hbar}\int\ \mathrm{d}\mathbf{r}\int\ \mathrm{d}\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left[\hat{\mu}_{\mathrm{CM}}(\mathbf{r}'),\left[\hat{\mu}_{\mathrm{CM}}(\mathbf{r}),\hat{\rho}_{\mathrm{CM}}\right]\right]$$

$$\hat{\mu}_{\mathrm{CM}}(\mathbf{r})=\sum_{i=1}^{N}\frac{m_i}{\left(2\pi R_{\mathrm{eff}}^2\right)^{3/2}}\exp\left[-\frac{\left(\hat{\mathbf{x}}+\mathbf{x}_i^{(0)}-\mathbf{r}\right)^2}{2R_{\mathrm{eff}}^2}\right]$$

$$\hat{\mu}_{\mathrm{CM}}(\mathbf{r})=\sum_{i=1}^{N}\frac{m_i}{(2\pi)^3}\int\ \mathrm{d}\mathbf{q}e^{-i\mathbf{q}\cdot\left(\hat{\mathbf{x}}+\mathbf{x}_i^{(0)}-\mathbf{r}\right)}e^{-R_{\mathrm{eff}}^2q^2/2}$$

$$\mathcal{D}[\hat{\rho}_{\mathrm{CM}}(t)]=$$
$$-\frac{4\pi G}{\hbar}\int\ \mathrm{d}\mathbf{r}\int\ \mathrm{d}\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\sum_{i,j=1}^{N}\frac{m_im_j}{(2\pi)^6}\int\ \mathrm{d}\mathbf{q}\int\ \mathrm{d}\mathbf{k}e^{-R_{\mathrm{eff}}^2(\mathbf{q}^2+\mathbf{k}^2)/2}e^{-i\mathbf{q}\cdot\left(\mathbf{x}_i^{(0)}-\mathbf{r}'\right)}e^{-i\mathbf{k}\cdot\left(\mathbf{x}_j^{(0)}-\mathbf{r}\right)}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{x}}},\left[e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}},\hat{\rho}(t)\right]\right]$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{2\pi^2}\int\ \mathrm{d}\mathbf{p}\frac{e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{p^2}$$

$$\frac{\mathrm{d}\hat{\rho}_{\mathrm{CM}}(t)}{\mathrm{d}t}=-\frac{i}{\hbar}\left[\hat{H}_{\mathrm{CM}},\hat{\rho}_{\mathrm{CM}}(t)\right]+\int\ \mathrm{d}\mathbf{p}F(\mathbf{p})\left(e^{i\mathbf{p}\cdot\hat{\mathbf{x}}}\hat{\rho}_{\mathrm{CM}}(t)e^{-i\mathbf{p}\cdot\hat{\mathbf{x}}}-\hat{\rho}_{\mathrm{CM}}(t)\right)$$

$$F(\mathbf{p})=\frac{4G}{\pi\hbar}\sum_{i,j=1}^{N}m_im_j\frac{e^{-R_{\mathrm{eff}}^2p^2}}{p^2}e^{i\mathbf{p}\cdot\left(\mathbf{x}_i^{(0)}-\mathbf{x}_j^{(0)}\right)}$$

$$\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)=\mathrm{Tr}\left[\hat{\rho}_{\mathrm{CM}}(t)e^{\frac{i}{\hbar}(\boldsymbol{\nu}\cdot\hat{\mathbf{x}}+\boldsymbol{\mu}\cdot\hat{\mathbf{p}})}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)=\frac{1}{M}\boldsymbol{\nu}\cdot\nabla_{\boldsymbol{\mu}}\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)+\frac{1}{\hbar}(E(\boldsymbol{\mu})-E(\mathbf{0}))\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)$$

$$E(\boldsymbol{\mu})=\hbar\int\ \mathrm{d}\mathbf{p}F(\mathbf{p})e^{i\boldsymbol{\mu}\cdot\mathbf{p}}$$

$$\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)=\chi_0\left(\boldsymbol{\nu},\boldsymbol{\mu}+\boldsymbol{\nu}\frac{t}{M},t\right)\exp\left[\frac{1}{\hbar}\int_0^t\ \mathrm{d}\tau\left(E\left(\boldsymbol{\mu}+\boldsymbol{\nu}\frac{\tau}{M}\right)-E(\mathbf{0})\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_0(\boldsymbol{\nu},\boldsymbol{\mu},t)=\frac{1}{M}\boldsymbol{\nu}\cdot\nabla_{\boldsymbol{\mu}}\chi_0(\boldsymbol{\nu},\boldsymbol{\mu},t)$$

$$\langle\mathbf{x}|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{y}\rangle=\frac{1}{(2\pi\hbar)^3}\int\ \mathrm{d}\boldsymbol{\nu}e^{-\frac{i}{2\hbar}\boldsymbol{\nu}\cdot(\mathbf{x}+\mathbf{y})}\chi(\boldsymbol{\nu},\mathbf{x}-\mathbf{y},t)$$

$$\langle \mathbf{x}|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{y}\rangle = \frac{1}{(2\pi\hbar)^3}\int \mathrm{d}\boldsymbol{\nu}\int \mathrm{d}\mathbf{z}e^{\frac{i}{\hbar}\boldsymbol{\nu}\cdot\mathbf{z}}\langle \mathbf{z}+\mathbf{x}|\hat{\rho}_{\mathrm{CM}}^{\mathrm{QM}}(t)|\mathbf{z}+\mathbf{y}\rangle \exp\left[\frac{1}{\hbar}\int_0^t \mathrm{d}\tau\left(E\left(\boldsymbol{\nu}\frac{\tau}{M}+\mathbf{x}-\mathbf{y}\right)-E(\mathbf{0})\right)\right]$$

$$E(\boldsymbol{\mu}) = 8\pi G\sum_{i,j=1}^{N} m_i m_j \frac{\mathrm{erf}\left(\frac{\left|\mathbf{x}_i^{(0)}-\mathbf{x}_j^{(0)}+\boldsymbol{\mu}\right|}{2R_{\mathrm{eff}}}\right)}{\left|\mathbf{x}_i^{(0)}-\mathbf{x}_j^{(0)}+\boldsymbol{\mu}\right|}$$

$$\Delta E(\mathbf{d}) = 8\pi G m^2\sum_{i,j=1}^{N}\left[\frac{\mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right)}{r_{ij}}-\frac{\mathrm{erf}\left(\frac{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}{2R_{\mathrm{eff}}}\right)}{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}\right]$$

$$\psi(\mathbf{x},0) = \frac{1}{\mathcal{N}}\left[e^{-\frac{1}{2\sigma^2}(\mathbf{x}-\mathbf{d}/2)^2}+e^{-\frac{1}{2\sigma^2}(\mathbf{x}+\mathbf{d}/2)^2}\right]$$

$$\mathcal{N} = \left[2(\sqrt{\pi}\sigma)^3\left(1+e^{-\frac{d^2}{4\sigma^2}}\right)\right]^{1/2}$$

$$\psi(\mathbf{x},t) = \frac{1}{\mathcal{N}}\left(\frac{\sigma}{\sqrt{\sigma^2+\frac{i\hbar t}{M}}}\right)^3\left(e^{-\frac{(\mathbf{x}+\mathrm{d}/2)^2}{2\left(\sigma^2+\frac{i\hbar t}{M}\right)}}+e^{-\frac{(\mathbf{x}-\mathrm{d}/2)^2}{2\left(\sigma^2+\frac{i\hbar t}{M}\right)}}\right)$$

$$\langle \mathbf{x}|\hat{\rho}_{\mathrm{CM}}^{\mathrm{QM}}(t)|\mathbf{y}\rangle = \psi^*(\mathbf{x},t)\psi(\mathbf{y},t) = \frac{1}{\mathcal{N}^2}\frac{\sigma^6\left(e^{-\frac{(\mathbf{x}+\mathrm{d}/2)^2}{2\left(\sigma^2-\frac{i\hbar t}{M}\right)}}+e^{-\frac{(\mathbf{x}-\mathrm{d}/2)^2}{2\left(\sigma^2-\frac{i\hbar t}{M}\right)}}\right)\left(e^{-\frac{(\mathbf{y}+\mathrm{d}/2)^2}{2\left(\sigma^2+\frac{\hbar t}{M}\right)}}+e^{-\frac{(\mathbf{y}-\mathrm{d}/2)^2}{2\left(\sigma^2+\frac{i\hbar t}{M}\right)}}\right)}{\left(\sigma^4+\left(\frac{\hbar t}{M}\right)^2\right)^{3/2}}$$

$$\langle -\mathbf{d}/2|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{d}/2\rangle = \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3,\ \text{donde}\ \mathcal{K}_j = \int \mathrm{d}\boldsymbol{\nu}e^{-\boldsymbol{\nu}^2}\mathcal{F}(\mathbf{d}/2,\boldsymbol{\nu})\mathcal{G}_j(\mathbf{d}/2,\boldsymbol{\nu})$$

$$\mathcal{K}_1 = \frac{1}{\mathcal{N}^2}\left(\frac{1}{1+\left(\frac{\hbar t}{M\sigma^2}\right)^2}\right)^{3/2}\exp\left[(f(\mathbf{d})-f(0))t\right],$$
$$\mathcal{K}_2 = \exp\left[-\frac{d^2}{\sigma^2}\left(1-\frac{t^2\hbar^2}{M^2\sigma^4+t^2\hbar^2}\right)\right]\mathcal{K}_1,$$
$$\mathcal{K}_3 = 2\exp\left(-\frac{d^2}{4\sigma^2}\frac{2M^2\sigma^4}{M^2\sigma^4+t^2\hbar^2}\right)\cos\left(\frac{M\sigma^2 t\hbar}{M^2\sigma^4+t^2\hbar^2}\frac{d^2}{2\sigma^2}\right)\mathcal{K}_1.$$

$$\mathcal{K}_1 = \frac{1}{\mathcal{N}^2}\exp\left[f(\mathbf{d})-f(\mathbf{0})\right)t\Big]$$
$$\mathcal{K}_2 = e^{-\frac{d^2}{\sigma^2}}\mathcal{K}_1$$
$$\mathcal{K}_3 = 2e^{-\frac{d^2}{2\sigma^2}}\mathcal{K}_1$$

$$\langle -\mathbf{d}/2|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{d}/2\rangle = \frac{1}{\mathcal{N}^2}\left(1 + e^{-\frac{d^2}{2\sigma^2}}\right)^2 \exp\left[(f(\mathbf{d}) - f(\mathbf{0}))t\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(\boldsymbol{\nu},\boldsymbol{\mu},t) = (f(\boldsymbol{\mu}) - f(\mathbf{0}))\chi(\boldsymbol{\nu},\boldsymbol{\mu},t)$$

$$\chi(\boldsymbol{\nu},\boldsymbol{\mu},t) = \chi(\boldsymbol{\nu},\boldsymbol{\mu},0)\exp\left[(f(\mathbf{d}) - f(\mathbf{0}))t\right]$$

$$\langle \mathbf{x}|\hat{\rho}_{\mathrm{CM}}(t)|\mathbf{y}\rangle = \langle \mathbf{x}|\hat{\rho}_{\mathrm{CM}}^{\mathrm{QM}}(0)|\mathbf{y}\rangle\exp\left[(f(\mathbf{d}) - f(\mathbf{0}))t\right]$$

$$\langle \mathbf{x}|\hat{\rho}_{\mathrm{CM}}^{\mathrm{QM}}(0)|\mathbf{y}\rangle = \psi^*(\mathbf{x},0)\psi(\mathbf{y},0)$$

$$S_1 = \sum_{i,j=1}^{N} \frac{\mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right)}{r_{ij}},$$

$$S_2 = \sum_{i,j=1}^{N} \frac{\mathrm{erf}\left(\frac{|\mathrm{d} + \mathbf{r}_{ij}|}{2R_{\mathrm{eff}}}\right)}{|\mathbf{d} + \mathbf{r}_{ij}|},$$

$$\sum_{i=1}^{N} \frac{\mathrm{erf}\left(\frac{r_{ii}}{2R_{\mathrm{eff}}}\right)}{r_{ii}} \underset{r_{ii}\to 0}{\longrightarrow} \sum_{i=1}^{N} \frac{1}{\sqrt{\pi}R_{\mathrm{eff}}}$$

$$S_1 = \frac{N}{\sqrt{\pi}R_{\mathrm{eff}}} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right)}{r_{ij}}$$

$$\begin{cases} \text{para } 1 \leq \frac{r_{ij}}{R_{\mathrm{eff}}} \leq 4, & \frac{1}{2} < \mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right) < 1 \\ \text{para } 4 \leq \frac{r_{ij}}{R_{\mathrm{eff}}}, & \mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right) \sim 1 \end{cases}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{2}\frac{1}{r_{ij}} \leq \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\mathrm{erf}\left(\frac{r_{ij}}{2R_{\mathrm{eff}}}\right)}{r_{ij}} \leq \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{a\sqrt{\left(n_x^i - n_x^j\right)^2 + \left(n_y^i - n_y^j\right)^2}}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} \leq \sum_{i=1}^{N} \sum_{\substack{n_x^j,n_y^j=-\sqrt{N}/2\\ \left(n_x^j,n_y^j\right)\neq(0,0)}}^{\sqrt{N}/2} \frac{1}{a\sqrt{\left(n_x^j\right)^2 + \left(n_y^j\right)^2}} = \frac{N}{a} \sum_{\substack{n_x^j,n_y^j=-\sqrt{N}/2\\ \left(n_x^j,n_y^j\right)\neq(0,0)}}^{\sqrt{N}/2} \frac{1}{\sqrt{\left(n_x^j\right)^2 + \left(n_y^j\right)^2}}$$

$$\frac{N}{a}\int_{-\sqrt{N}/2}^{\sqrt{N}/2} \mathrm{d}x \int_{-\sqrt{N}/2}^{\sqrt{N}/2} \mathrm{d}y \frac{1}{\sqrt{x^2 + y^2}} = \frac{\eta_+ N\sqrt{N}}{a}$$

$$\mathbf{r}_{ij} = a\left(-\sqrt{N}/2 - n_x^j, -\sqrt{N}/2 - n_y^j\right) \qquad r_{ij} = a\sqrt{\left(\sqrt{N}/2 + n_x^j\right)^2 + \left(\sqrt{N}/2 + n_y^j\right)^2}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} \geq \sum_{i=1}^{N} \sum_{\substack{n_x^j, n_y^j = -\sqrt{N}/2\\ \left(n_x^j, n_y^j\right)\neq(0,0)}}^{\sqrt{N}/2} \frac{1}{a\sqrt{\left(\sqrt{N}/2+n_x^j\right)^2+\left(\sqrt{N}/2+n_y^j\right)^2}}$$

$$\frac{N}{a}\int_{-\sqrt{N}/2}^{\sqrt{N}/2} \mathrm{d}x \int_{-\sqrt{N}/2}^{\sqrt{N}/2} \mathrm{d}y \frac{1}{\sqrt{(\sqrt{N}/2 + x)^2 + (\sqrt{N}/2 + y)^2}} = \frac{2\eta_- N\sqrt{N}}{a}$$

$$\frac{N}{\sqrt{\pi}R_{\text{eff}}} + \frac{\eta_- N\sqrt{N}}{a} \leq S_1 \leq \frac{N}{\sqrt{\pi}R_{\text{eff}}} + \frac{\eta_+ N\sqrt{N}}{a}$$

$$R_{\text{eff}} \lesssim \frac{a}{\eta_\pm \sqrt{\pi N}}$$

$$\operatorname{erf}\left(\frac{x}{2}\right) \simeq \begin{cases} \frac{x}{\sqrt{\pi}}, & x \leq 2 \\ 1, & x \geq 2 \end{cases}$$

$$\frac{\operatorname{erf}\left(\frac{r_{ij}}{2R_{\text{eff}}}\right)}{r_{ij}} \simeq \frac{1}{\sqrt{\pi}R_{\text{eff}}}$$

$$\sum_{\substack{i,j=1\\ i\neq j, r_{ij}\leq 2R_{\text{eff}}}}^{N} \frac{\operatorname{erf}\left(\frac{r_{ij}}{2R_{\text{eff}}}\right)}{r_{ij}} \simeq N \times \left(\frac{2R_{\text{eff}}}{a}\right)^2 \frac{1}{\sqrt{\pi}R_{\text{eff}}} = 4N\frac{R_{\text{eff}}}{\sqrt{\pi}a^2}$$

$$\frac{\operatorname{erf}\left(\frac{r_{ij}}{2R_{\text{eff}}}\right)}{r_{ij}} \simeq \frac{1}{r_{ij}}$$

$$\sum_{\substack{i,j=1\\ i\neq j, r_{ij}\geq 2R_{\text{eff}}}}^{N} \frac{\operatorname{erf}\left(\frac{r_{ij}}{2R_{\text{eff}}}\right)}{r_{ij}} \simeq \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} - \sum_{\substack{i,j=1\\ i\neq j, r_{ij}\leq 2R_{\text{eff}}}}^{N} \frac{1}{r_{ij}}$$

$$\sum_{\substack{i,j=1\\ i\neq j, r_{ij}\leq 2R_{\text{eff}}}}^{N} \frac{1}{r_{ij}} \geq \sum_{\substack{i,j=1\\ i\neq j, r_{ij}\leq 2R_{\text{eff}}}}^{N} \frac{1}{2R_{\text{eff}}} = N \times \left(\frac{2R_{\text{eff}}}{a}\right)^2 \frac{1}{2R_{\text{eff}}} = \frac{4NR_{\text{eff}}}{a^2}$$

$$\sum_{\substack{i,j=1\\ i\neq j, r_{ij}\leq 2R_{\text{eff}}}}^{N} \frac{1}{r_{ij}} \leq \frac{2\pi N}{a}\mathcal{F}_G \int_1^{\frac{2R_{\text{eff}}}{a}} \mathrm{d}r = \frac{2\pi N}{a}\mathcal{F}_G\left(\frac{2R_{\text{eff}}}{a} - 1\right)$$

$$\mathcal{F}_G = \frac{L^2}{L^2 + 4LR_{\text{eff}} + \pi R_{\text{eff}}^2} \frac{4R_{\text{eff}}^2}{\pi\left(4R_{\text{eff}}^2 - a^2\right)}$$

$$\frac{N}{\sqrt{\pi}R_{\text{eff}}}+\frac{\eta_{-}N\sqrt{N}}{a}+\frac{4NR_{\text{eff}}}{\sqrt{\pi}a^{2}}-\frac{2\pi N}{a}\mathcal{F}_{G}\left(\frac{2R_{\text{eff}}}{a}-1\right)\leq S_{1}\leq\frac{N}{\sqrt{\pi}R_{\text{eff}}}+\frac{\eta_{+}N\sqrt{N}}{a}+\frac{4NR_{\text{eff}}}{a^{2}}\left(\frac{1}{\sqrt{\pi}}-1\right)$$

$$R_{\text{eff}}\gtrsim\frac{\eta_{\pm}N\sqrt{N\pi}a}{4}$$

c. tercer intervalo: $L\leq R_{eff}\leq d$

$$S_{1}\simeq\frac{N}{\sqrt{\pi}R_{\text{eff}}}+\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{\sqrt{\pi}R_{\text{eff}}}$$

$$S_{1}\simeq\frac{N^{2}}{\sqrt{\pi}R_{\text{eff}}}$$

d. cuarto intervalo: $d\leq R_{\text{eff}}$

$$S_{1}\simeq\frac{N^{2}}{\sqrt{\pi}R_{\text{eff}}}$$

$$d-L\leq\left|\mathbf{d}+\mathbf{r}_{ij}\right|\leq\sqrt{(d+L)^{2}+L^{2}}\leq d+2L$$

$$S_{2}\simeq\sum_{i,j=1}^{N}\frac{1}{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}$$

$$\frac{N^{2}}{d+2L}\leq S_{2}\leq\frac{N^{2}}{d-L}$$

b. segundo intervalo: $(d-L)/2\leq R_{\text{eff}}\leq(d+2L)/2$

$$\frac{d-L}{d+2L}\leq\frac{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}{2R_{\text{eff}}}\leq\frac{d+2L}{d-L}$$

$$\frac{1}{2R_{\text{eff}}}\frac{\operatorname{erf}\left(\frac{d+2L}{d-L}\right)}{\frac{d+2L}{d-L}}\leq\frac{\operatorname{erf}\left(\frac{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}{2R_{\text{eff}}}\right)}{\left|\mathbf{d}+\mathbf{r}_{ij}\right|}\leq\frac{1}{2R_{\text{eff}}}\frac{\operatorname{erf}\left(\frac{d-L}{d+2L}\right)}{\frac{d-L}{d+2L}}$$

$$\frac{\epsilon_{-}N^{2}}{2R_{\text{eff}}}\leq S_{2}\leq\frac{\epsilon_{+}N^{2}}{2R_{\text{eff}}}$$

$$\epsilon_{\pm}=\frac{\operatorname{erf}\left[\left(\frac{d-L}{d+2L}\right)^{\pm1}\right]}{\left(\frac{d-L}{d+2L}\right)^{\pm1}}$$

c. primer intervalo: $(d+2L)/2\leq R_{\text{eff}}$

$$S_2 \simeq \frac{N^2}{\sqrt{\pi} R_{\text{eff}}}$$

$$\Delta E(\mathbf{d}) \simeq 8\pi G m^2 \sum_{i,j=1}^{N} \left(-\frac{r_{ij}^2}{12\sqrt{\pi} R_{\text{eff}}^3} + \frac{\left|\mathbf{d} + \mathbf{r}_{ij}\right|^2}{12\sqrt{\pi} R_{\text{eff}}^3} \right) = \frac{2}{3}\sqrt{\pi} G m^2 \sum_{i,j=1}^{N} \frac{d^2 + 2\mathbf{r}_{ij} \cdot \mathbf{d}}{R_{\text{eff}}^3}$$

Intervalos para R_{eff}	Valores o límites en $\Delta E(\mathrm{d})/(8\pi G m^2) = S_1 - S_2 = \Delta S$
$R_{\text{eff}} \leq a$	$\frac{N}{\sqrt{\pi} R_{\text{eff}}} + \frac{\eta_- N\sqrt{N}}{a} - \frac{N^2}{d-L} \leq \Delta S \leq \frac{N}{\sqrt{\pi} R_{\text{eff}}} + \frac{\eta_+ N\sqrt{N}}{a} - \frac{N^2}{d+L}$
$a \leq R_{\text{eff}} \leq L$	$\frac{N}{\sqrt{\pi} R_{\text{eff}}} + \frac{\eta_- N\sqrt{N}}{a} - \frac{N^2}{d-L} + \frac{4N R_{\text{eff}}}{\sqrt{\pi} a^2} - \frac{2\pi N}{a} \mathcal{F}_G\left(\frac{2R_{\text{eff}}}{a} - 1\right) \leq \Delta S \leq \frac{N}{\sqrt{\pi} R_{\text{eff}}} + \frac{\eta_+ N\sqrt{N}}{a} - \frac{N^2}{d+L} + \frac{4N R_{\text{eff}}}{a^2}\left(\frac{1}{\sqrt{\pi}} - 1\right)$
$L \leq R_{\text{eff}} \leq \frac{(d-L)}{2}$	$\frac{N^2}{\sqrt{\pi} R_{\text{eff}}} - \frac{N^2}{d-L} \leq \Delta S \leq \frac{N^2}{\sqrt{\pi} R_{\text{eff}}} - \frac{N^2}{d+2L}$
$\frac{(d-L)}{2} \leq R_{\text{eff}} \leq \frac{(d+2L)}{2}$	$\frac{N^2}{\sqrt{\pi} R_{\text{eff}}} - \frac{\epsilon_+ N^2}{2R_{\text{eff}}} \leq \Delta S \leq \frac{N^2}{\sqrt{\pi} R_{\text{eff}}} - \frac{\epsilon_- N^2}{2R_{\text{eff}}}$
$\frac{(d+2L)}{2} \leq R_{\text{eff}} \leq d$	$\Delta S = \frac{N^2 d^2}{12\sqrt{\pi} R_{\text{eff}}^3}$
$d \leq R_{\text{eff}}$	$\Delta S = \frac{N^2 d^2}{12\sqrt{\pi} R_{\text{eff}}^3}$

$$\Delta E(\mathbf{d}) \simeq \frac{2}{3}\sqrt{\pi} G m^2 N^2 \frac{d^2}{R_{\text{eff}}^3}$$

$$S_1 = \frac{N}{\sqrt{\pi} a} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\operatorname{erf}\left(\frac{r_{ij}}{2a}\right)}{r_{ij}}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{a\left(\sum_{k=1}^{D} \left(n_k^i - n_k^j\right)^2\right)^{\frac{1}{2}}}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{r_{ij}} \lesssim \frac{N}{a} \sum_{\left\{n_k^j\right\}_{k=1}^{D} \in S_D} \frac{1}{\left(\sum_{k=1}^{D} \left(n_k^j\right)^2\right)^{\frac{1}{2}}}$$

$$\frac{N}{a}\Omega_D \int_0^{N_D/2} \mathrm{d}x x^{D-1} \frac{1}{x} = \frac{N}{a} \frac{2\pi^{D/2}}{\Gamma(D/2)} \frac{N^{\frac{D-1}{D}}}{2^{D-1}(D-1)}$$

$$\Delta E(\mathbf{d}) \simeq \frac{8\pi G m^2}{a} \frac{\pi^{D/2}}{\Gamma(D/2)} \frac{N^{\frac{2D-1}{D}}}{2^{D-2}(D-1)}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} f\left(\mathbf{r}_{ij}, R_0, \mathbf{d}\right)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} f\left(\mathbf{r}_{ij}, R_0, \mathbf{d}\right) = \sum_{\mathbf{r}\in\mathcal{D}} \omega(\mathbf{r}) f(\mathbf{r}, R_0, \mathbf{d})$$

$$\mathbf{r}_{ij\alpha\beta} = \mathbf{x}_{i,\alpha} - \mathbf{x}_{j,\beta} = n_1^{ij}\mathbf{a}_1 + n_2^{ij}\mathbf{a}_2 + \mathbf{b}_{\alpha\beta}, \text{ con } n_1^{ij}, n_2^{ij} \in \mathbb{Z}$$

$$\mathbf{r}(n_1, n_2, \gamma) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + \mathbf{c}_\gamma$$

$$\mathbf{c}_\gamma \in \mathcal{D}_b = \left\{\mathbf{b}_\alpha - \mathbf{b}_\beta \mid \alpha, \beta = 1, \dots, K\right\}$$

$$\mathcal{D} = \left\{\mathbf{r}(n_1, n_2, \gamma) \mid n_i \in [-N_i + 1, N_i - 1], \mathbf{c}_\gamma \in \mathcal{D}_b\right\}$$

$$\omega(n_1, n_2, \gamma) = \omega_\gamma (N_1 - |n_1|)(N_2 - |n_2|)$$

$$\begin{aligned}\mathcal{D}_b &= \{\mathbf{c}_1 = \mathbf{b}_1 - \mathbf{b}_1, \mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_2, \mathbf{c}_3 = \mathbf{b}_2 - \mathbf{b}_1, \mathbf{c}_4 = \mathbf{b}_2 - \mathbf{b}_2\},\\ &= \left\{\mathbf{c}_1 = (0{,}0), \mathbf{c}_2 = -a\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \mathbf{c}_3 = a\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \mathbf{c}_4 = (0{,}0)\right\}\end{aligned}$$

$$\sum_{\mathbf{r}\in\mathcal{D}} \omega(\mathbf{r}) f(\mathbf{r}, R_0, \mathbf{d}) = \sum_{n_1=-N_1+1}^{N_1-1} \sum_{n_2=-N_2+1}^{N_2-1} \sum_{\mathbf{c}_\gamma\in\mathcal{D}_b} \omega_\gamma (N_1 - |n_1|)(N_2 - |n_2|) f(\mathbf{r}(n_1, n_2, \gamma), R_0, \mathbf{d})$$

$$(2N_1 - 1)(2N_2 - 1)[1 + K(K-1)] \simeq 4N_1N_2[1 + K(K-1)] = 4\frac{[1 + K(K-1)]}{K} N$$

$$\mathrm{d}|\psi_t\rangle = \Big[-\frac{i}{\hbar}\hat{H}\,\mathrm{d}t + \int \mathrm{d}^3\mathbf{x}\big(\hat{M}(\mathbf{x}) - \langle\hat{M}(\mathbf{x})\rangle_t\big)\mathrm{d}W_t(\mathbf{x})$$

$$-\frac{1}{2}\iint \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}\,\mathcal{G}(\mathbf{x}-\mathbf{y})\big(\hat{M}(\mathbf{x}) - \langle\hat{M}(\mathbf{x})\rangle_t\big)\big(\hat{M}(\mathbf{y}) - \langle\hat{M}(\mathbf{y})\rangle_t\big)\mathrm{d}t\Big]|\psi_t\rangle$$

$$\hat{M}(\mathbf{x}) = \sum_{j=1}^{N} m_j\delta\big(\mathbf{x} - \hat{\mathbf{r}}_j\big)$$

$$\mathbb{E}(w(t,\mathbf{x})) = 0, \mathbb{E}\big(w(t_1,\mathbf{x})w(t_2,\mathbf{y})\big) = \delta(t_1 - t_2)\mathcal{G}(\mathbf{x}-\mathbf{y})$$

$$\mathcal{G}(\mathrm{x}) = \frac{G}{\hbar}\frac{1}{|\mathrm{x}|}$$

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] + \mathcal{L}[\hat{\rho}(t)]$$

$$\mathcal{L}[\hat{\rho}(t)] = \frac{G}{\hbar}\iint \frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\Big(\hat{M}(\mathbf{x})\hat{\rho}(t)\hat{M}(\mathbf{y}) - \frac{1}{2}\{\hat{M}(\mathbf{x})\hat{M}(\mathbf{y}),\hat{\rho}(t)\}\Big)$$

$$\mathcal{L}[\hat{\rho}(t)] = \frac{G}{2\pi^2\hbar^2}\sum_{j,l=1}^{N} m_j m_l \int \frac{\mathrm{d}^3\mathbf{Q}}{Q^2}\Big(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l} - \frac{1}{2}\Big\{e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l},\hat{\rho}(t)\Big\}\Big)$$

$$= -\Lambda\hat{\rho}(t) + \frac{G}{2\pi^2\hbar^2}\sum_{j=1}^{N} m_j^2 \int \frac{\mathrm{d}^3\mathbf{Q}}{Q^2} e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}$$

$$+\frac{G}{2\pi^2\hbar^2}\sum_{j\neq l=1}^{N} m_j m_l \int \frac{\mathrm{d}^3\mathbf{Q}}{Q^2}\Big(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l} - \Big\{e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l},\hat{\rho}(t)\Big\}\Big)$$

$$\Lambda = \frac{G}{2\pi^2\hbar^2}\sum_{j,l=1}^{N} m_j m_l \int \frac{\mathrm{d}^3\mathbf{Q}}{Q^2} = \frac{2GM^2}{\pi\hbar^2}\int_0^\infty \mathrm{d}Q$$

$$\hat{M}'(\mathbf{x}) = \frac{3}{4\pi R_0^3}\int \mathrm{d}^3\mathbf{y}\,\theta(R_0 - |\mathbf{x}-\mathbf{y}|)\hat{M}(\mathbf{y})$$

$$\hat{M}'(\mathbf{x}) = (2\pi R_0^2)^{-3/2}\int \mathrm{d}^3\mathbf{y}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{2R_0^2}\right)\hat{M}(\mathbf{y})$$

$$\mathcal{L}[\hat{\rho}(t)] = \frac{G}{2\pi^2\hbar^2}\sum_{j,l=1}^{N} m_j m_l \int \frac{\mathrm{d}^3\mathbf{Q}}{Q^2} f(Q) \times$$
$$\Big(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l} - \frac{1}{2}\Big\{e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_j}e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}_l},\hat{\rho}(t)\Big\}\Big)$$

$$f(Q) = \frac{9\hbar^6}{R_0^6 Q^6}\left(\sin\left(\frac{QR_0}{\hbar}\right) - \frac{QR_0}{\hbar}\cos\left(\frac{QR_0}{\hbar}\right)\right)^2$$

$$f(Q) = \exp\left(-\frac{Q^2R_0^2}{\hbar^2}\right)$$

$$\mathcal{L}[\hat{\rho}(t)] = \frac{G}{2\pi^2\hbar^2}\sum_{j,l=}^{N} m_j m_l \int_0^{Q_{\max}} \mathrm{d}Q \iint \mathrm{d}^2\tilde{\mathbf{n}} \times$$
$$\left(e^{\frac{i}{\hbar}Q\tilde{\mathbf{n}}\cdot\hat{\mathbf{r}}_j}\hat{\rho}(t)e^{-\frac{i}{\hbar}Q\tilde{\mathbf{n}}\cdot\hat{\mathbf{r}}_l} - \frac{1}{2}\left\{e^{\frac{i}{\hbar}Q\tilde{\mathbf{n}}\cdot\hat{\mathbf{r}}_j}e^{-\frac{i}{\hbar}Q\tilde{\mathbf{n}}\cdot\hat{\mathbf{r}}_l}, \hat{\rho}(t)\right\}\right)$$

$$\mathcal{L}[\hat{\rho}(t)] = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\mathrm{DP}}(\mathbf{Q})\left(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}} - \hat{\rho}(t)\right)$$

$$\Gamma_{\mathrm{DP}}(\mathbf{Q}) = \frac{Gm^2}{2\pi^2\hbar^2}\frac{1}{Q^2}\exp\left(-\frac{Q^2R_0^2}{\hbar^2}\right)$$

$$\Lambda_{\mathrm{DP}} = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\mathrm{DP}}(\mathbf{Q}) = \frac{Gm^2}{\sqrt{\pi}\hbar R_0}, \text{ en la dimensión } [\Lambda_{\mathrm{DP}}] = \mathrm{s}^{-1}$$

$$\rho(\mathbf{x},\mathbf{x}',t) = \iint \mathrm{d}^3\mathbf{y}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\rho_0(\mathbf{x}+\mathbf{y},\mathbf{x}'+\mathbf{y},t) \times$$
$$\exp\left(-\frac{i}{\hbar}\mathbf{y}\cdot\mathbf{p} - \frac{1}{\hbar}\int_0^t d\tau\left(U\left(-\frac{\mathbf{p}\tau}{m}+\mathbf{x}-\mathbf{x}'\right) - U(0)\right)\right)$$

$$U(\mathbf{x}) = -G\iint \frac{\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'M'(\mathbf{r})M'(\mathbf{r}')}{|\mathbf{x}+\mathbf{r}-\mathbf{r}'|} = -Gm^2\frac{\mathrm{Erf}(|\mathbf{x}|/2R_0)}{|\mathbf{x}|}$$

$$\rho(\mathbf{x},\mathbf{x}',t) = \exp\left(-\frac{t}{\tau(\mathbf{x},\mathbf{x}')}\right)\rho(\mathbf{x},\mathbf{x}',0)$$

$$\tau(\mathbf{x},\mathbf{x}') = \frac{\hbar}{U(\mathbf{x}-\mathbf{x}') - U(0)}$$

$$\frac{\partial}{\partial t}|\psi_t\rangle = \left[-\frac{i}{\hbar}\hat{H} + \frac{\sqrt{\gamma}}{m_0}\int \mathrm{d}^3\mathbf{x}\left(\hat{M}(\mathbf{x}) - \langle\hat{M}(\mathbf{x})\rangle_t\right)\mathrm{d}W_t(\mathbf{x})\right.$$
$$\left.-\frac{\gamma}{2m_0^2}\iint \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}\mathcal{G}(\mathbf{x}-\mathbf{y})\left(\hat{M}(\mathbf{x}) - \langle\hat{M}(\mathbf{x})\rangle_t\right)\left(\hat{M}(\mathbf{y}) - \langle\hat{M}(\mathbf{y})\rangle_t\right)\right]|\psi_t\rangle$$

$$\mathcal{G}(\mathbf{x}) = \frac{1}{(4\pi r_c)^{3/2}}\exp\left(-\frac{\mathbf{x}^2}{4r_c^2}\right)$$

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] + \int \mathrm{d}^3\mathbf{Q}\Gamma_{\mathrm{CSL}}(\mathbf{Q})\left(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}}\hat{\rho}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}} - \hat{\rho}(t)\right)$$

$$\Gamma_{\mathrm{CSL}}(\mathbf{Q}) = \frac{\gamma}{(2\pi\hbar)^3}\frac{m^2}{m_0^2}\exp\left(-\frac{Q^2r_c^2}{\hbar^2}\right);\ m_0 = 1\mathrm{amu}$$

$$\Phi(\mathbf{x}) = \int \mathrm{d}^3\mathbf{Q}e^{\frac{i}{\hbar}\mathbf{Q}\cdot\mathbf{x}}\Gamma_{\mathrm{CSL}}(\mathbf{Q}) = \frac{m^2}{m_0^2}\frac{\gamma}{(4\pi r_c)^{3/2}}\exp\left(-\frac{x^2}{4r_c^2}\right)$$

$$\tau_{\mathrm{CSL}}(\mathbf{x},\mathbf{x}') = \frac{1}{\Phi(0) - \Phi(\mathbf{x}-\mathbf{x}')}$$

$$\Lambda_{\rm CSL} = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\rm CSL}(\mathbf{Q}) = \frac{m^2}{m_0^2}\frac{\gamma}{(4\pi r_c^2)^{3/2}}$$

$$\mathcal{L}[\hat{\rho}_{\rm M}(t)] = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\rm M}(\mathbf{Q})\left(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{R}}}\hat{\rho}_{\rm M}(t)e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{R}}} - \hat{\rho}_{\rm M}(t)\right)$$

$$\Gamma_{\rm DP}^{\rm M}(\mathbf{Q}) = \frac{G}{2\pi^2\hbar^2 Q^2}|\tilde{\varrho}_{\rm rel}(\mathbf{Q})|^2 \exp\left(-\frac{Q^2R_0^2}{\hbar^2}\right)$$

$$\Gamma_{\rm DP}^{\rm M}(Q) \approx \frac{GM^2}{2\pi^2\hbar^2}\frac{1}{Q^2}\exp\left(-\frac{Q^2(R^2+R_0^2)}{\hbar^2}\right)$$

$$\Lambda_{\rm DP}^{\rm M} = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\rm M}(\mathbf{Q}) \approx \frac{GM^2}{\hbar\sqrt{\pi(R^2+R_0^2)}}$$

$$\Gamma_{\rm CSL}^{\rm M}(Q) \approx \frac{\gamma}{(2\pi\hbar)^3}\frac{M^2}{m_0^2}\exp\left(-\frac{Q^2(R^2+r_C^2)}{\hbar^2}\right)$$

$$\Lambda_{\rm CSL}^{\rm M} = \int \mathrm{d}^3\mathbf{Q}\Gamma_{\rm M}(\mathbf{Q}) \approx \frac{\gamma M^2}{8\pi^{3/2}m_0^2(R^2+r_C^2)^{3/2}}$$

$$\frac{\mathrm{d}E_{\rm DP}(t)}{\mathrm{d}t} = \frac{2\pi}{m}\int_0^\infty \mathrm{d}Q\Gamma_{\rm DP}(Q)Q^4 = \frac{mG\hbar}{4\sqrt{\pi}R_0^3}$$

$$\mathcal{L}[\hat{\rho}(t)] = \int \mathrm{d}^3\mathbf{Q}\left(e^{\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}}\hat{L}(\mathbf{Q},\hat{\mathbf{p}})\hat{\rho}(t)\hat{L}^\dagger(\mathbf{Q},\hat{\mathbf{p}})e^{-\frac{i}{\hbar}\mathbf{Q}\cdot\hat{\mathbf{r}}} - \frac{1}{2}\{\hat{L}^\dagger(\mathbf{Q},\hat{\mathbf{p}})\hat{L}(\mathbf{Q},\hat{\mathbf{p}}),\hat{\rho}(t)\}\right)$$

$$\hat{L}(\mathbf{Q},\hat{\mathbf{p}}) \equiv \frac{m\sqrt{G}}{\pi\sqrt{2}\hbar}\frac{1}{Q}\exp\left[-\frac{R_0^2}{2\hbar^2}\left((1+k)Q + 2k\frac{\hat{\mathbf{p}}\cdot\mathbf{Q}}{Q}\right)^2\right]$$

$$k = \frac{m_r}{m}$$

$$\hat{L}_{\rm DEC}(\mathbf{Q},\hat{\mathbf{p}}) \propto \exp\left[-\frac{1}{16m_{\rm en}k_BT}\left(\left(1+\frac{m_{\rm en}}{m}\right)Q + 2\frac{m_{\rm en}}{m}\frac{\hat{\mathbf{p}}\cdot\mathbf{Q}}{Q}\right)^2\right]$$

$$m_r \leftrightarrow m_{\rm en}\,;\quad R_0 \leftrightarrow \frac{\hbar}{\sqrt{8m_{\rm en}k_BT}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\rm DP}(t) = \frac{1}{2m}\int \mathrm{d}^3\mathbf{Q}\,\mathrm{Tr}\left(\hat{\rho}(t)\hat{L}^\dagger(\mathbf{Q},\hat{\mathbf{p}})\hat{L}(\mathbf{Q},\hat{\mathbf{p}})(Q^2+2\hat{\mathbf{p}}\cdot\mathbf{Q})\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\rm DP}(t) = \gamma_{\rm DP} - \xi_{\rm DP}E_{\rm DP}(t)$$

$$\gamma_{\rm DP} = \frac{mG\hbar}{4\sqrt{\pi}(1+k)^3R_0^3}\,;\quad \xi_{\rm DP} = \frac{4m^2Gk}{3\sqrt{\pi}(1+k)^3\hbar R_0}$$

$$E_{\mathrm{DP}}(t) = E_{\mathrm{DP}}(0)e^{-\xi_{\mathrm{DP}}t} + \frac{\gamma_{\mathrm{DP}}}{\xi_{\mathrm{DP}}}\left(1 - e^{-\xi_{\mathrm{DP}}t}\right)$$

$$T = \frac{2\gamma_{\mathrm{DP}}}{3k_B\xi_{\mathrm{DP}}} = \frac{\hbar^2}{8k_B}\frac{1}{m_r R_0^2}$$

$$\nu_{eq}(\hat{\mathbf{p}}) = (2\pi M k_B T)^{-3/2}\exp\left(-\frac{\hat{\mathbf{p}}^2}{2Mk_BT}\right)$$

$$\hat{\hat{L}}(\mathbf{Q},\hat{\mathbf{p}}) \equiv \frac{m\sqrt{G}}{\pi\sqrt{2}\hbar}\frac{1}{Q}\exp\left[-\frac{R_0^2}{2\hbar^2}((1+k)\mathbf{Q} + 2k\hat{\mathbf{p}}\cdot\mathbf{Q})^2\right]$$

$$T = \frac{10^{-19}}{m_r R_0^2},\ \text{en las dimensiones } [T] = \text{ Kelvin, } [m_r] = \text{ amu, y } [R_0] = \mathrm{m}$$

$$T \sim 1\ \mathrm{K} \implies m_r R_0^2 \sim 10^{-19},\ \text{con } [m_r] = \text{amu, y } [R_0] = \mathrm{m}$$

$$P(\mathbf{Q}) = \frac{C}{Q^2}\exp\left(-\frac{k^2R_0^2}{\hbar^2}(Q + 2\mathbf{p}\cdot\mathbf{Q}/Q)^2\right)$$

$$|\psi_1\rangle := \sum_{j=1}^{2} \alpha_j|\mathbf{a}^j\rangle, |\psi_2\rangle := \sum_{j=1}^{2} \beta_j|\mathbf{b}^j\rangle$$

$$\rho_0 = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|$$

$$\sum_i p_i\rho_i^1 \otimes \rho_i^2$$

$$H_G = \sum_{i=1}^{2}\frac{\mathbf{P}_i^2}{2m_i} - \frac{Gm_1m_2}{\|\mathbf{X}_1 - \mathbf{X}_2\|}$$

$$|\psi(t)\rangle \approx \sum_{j,k=1}^{2} \alpha_j\beta_k e^{-i\frac{Gm_1m_2t}{\hbar\|\mathbf{a}^j-\mathbf{b}^k\|}}|\mathbf{a}^j\rangle|\mathbf{b}^k\rangle$$

$$C = \sum_{i,j,k,l} C_{i,j,k,l}|i\rangle\langle j| \otimes |k\rangle\langle l|$$

$$C^{T_1} := \sum_{i,j,k,l} C_{i,j,k,l}|j\rangle\langle i| \otimes |k\rangle\langle l|$$

$$\frac{Gm_1m_2t}{2\pi\hbar}\left(\frac{1}{d} + \frac{1}{d+2L} - \frac{2}{d+L}\right) \notin \mathbb{Z}$$

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} = -\frac{i}{\hbar}[H,\rho] + \frac{G}{2\hbar}A(\rho)$$

$$A(\rho) = -\int \mathrm{drds}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}[\mu_\sigma(\mathrm{r}),[\mu_\sigma(\mathbf{s}),\rho]]$$
$$+i\int \mathrm{drds}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}[\mu_\sigma(\mathbf{r})\mu_\sigma(\mathbf{s}),\rho]$$

$$\mu_\sigma(\mathbf{x})\mathrm{d}\mathbf{x} := \frac{1}{(2\pi)^{3/2}\sigma^3}\sum_{j=1}^{n} m_j e^{-\frac{\|\mathbf{x}-\mathbf{x}_j\|^2}{2\sigma^2}}\mathrm{d}\mathbf{x}$$

$$V_G := -\int \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{s}\frac{Gm(\mathbf{s})\mu_\sigma(\mathbf{r})}{\|\mathbf{r}-\mathbf{s}\|}$$

$$\langle \mathbf{x}|A(\rho)|\mathbf{y}\rangle = g(\mathbf{x},\mathbf{y})\langle \mathbf{x}|\rho|\mathbf{y}\rangle$$

$$g(\mathrm{x},\mathrm{y}) := \sum_{j,k=1}^{n} m_j m_k\big[(i-1)f(\mathrm{x}_j,\mathrm{x}_k)$$
$$+(-i-1)f(\mathbf{y}_i,\mathbf{y}_k)+2f(\mathrm{x}_j,\mathbf{y}_k)\big]$$

$$\tilde{f}(z) := \begin{cases} \operatorname{erf}\left(\frac{z}{2\sigma}\right)/z & \text{si } z \neq 0 \\ \frac{1}{\sigma\sqrt{\pi}} & \text{si } z = 0 \end{cases}$$

$$\tilde{f}(z) = \frac{1}{\sigma\sqrt{\pi}} + \frac{z^2}{12\sqrt{\pi}\sigma^3} + O(z^3/\sigma^4)$$

$$|\bar{\psi}_1\rangle := \frac{1}{\sqrt{2}}(|\mathbf{a}^1\rangle + |\mathbf{a}^2\rangle),$$
$$|\bar{\psi}_2\rangle := \frac{1}{\sqrt{2}}(|\mathbf{b}^1\rangle + |\mathbf{b}^2\rangle).$$

$$|0\rangle_1 := |\mathbf{a}^1\rangle, |1\rangle_1 := |\mathbf{a}^2\rangle, |0\rangle_2 := |\mathbf{b}^1\rangle, |1\rangle_2 := |\mathbf{b}^2\rangle$$

$$(\mathbb{I} - |+\rangle\langle + \quad |^{\otimes 2})\dot{\rho}(0)^{T_1}(\mathbb{I} - |+\rangle\langle + \quad |^{\otimes 2})$$

$$\langle\phi|\rho(0)^{T_1}|\phi\rangle = \left|\langle\phi \mid +\rangle^{\otimes 2}\right|^2 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi|\rho(t)^{T_1}|\phi\rangle\Big|_{t=0} = \lambda$$

$$E_- := \frac{Gm^2}{2\hbar}\Bigg(\frac{1}{\sqrt{\pi}\sigma} - \tilde{f}(L)$$
$$-\frac{1}{\sqrt{2}}|\tilde{f}(d+2L)+\tilde{f}(d)-2\tilde{f}(L+d)|\Bigg)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(t) = \frac{i}{\hbar}[H,\tilde{\rho}(t)] - \frac{i}{\hbar}[H,\tilde{\rho}(t)] + \frac{G}{2\hbar}e^{iHt}A(\rho_H(t))e^{-iHt}$$
$$= \frac{G}{2\hbar}e^{iHt}A(\rho_H(t))e^{-iHt}$$

$$\lim_{L\to 0}\frac{E_-}{L^2}=\frac{Gm^2}{2\hbar}\left(-\frac{1}{2}\ddot{\tilde{f}}(0)-\frac{1}{\sqrt{2}}|\ddot{\tilde{f}}(d)|\right)=:\gamma(d)$$

$$\frac{1}{\sqrt{\pi}\sigma}-\tilde{f}(L)-\sqrt{2}\left(\tilde{f}(d)-\tilde{f}\left(\sqrt{L^2+d^2}\right)\right)<0$$

$$\langle \mathbf{x}|\rho(t)|\mathbf{y}\rangle:=e^{\frac{G}{2\hbar}g(\mathbf{x},\mathbf{y})t}\langle \mathbf{x}|\rho(0)|\mathbf{y}\rangle$$

$$\int \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{s}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\mu_\sigma(\mathbf{r})\mu_\sigma(\mathbf{s})\rho$$

$$\sum_{i,j=1}^{n}\frac{m_i m_j}{(2\pi)^3\sigma^6}\int \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{s}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}e^{-\frac{\|\mathbf{s}-\mathbf{x}_i\|^2+\|\mathbf{r}-\mathbf{x}_j\|^2}{2\sigma^2}}\langle \mathbf{x}|\rho|\mathbf{y}\rangle$$

$$I:=\int \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{s}\frac{1}{\|\mathbf{r}-\mathbf{s}\|}e^{-\|\mathbf{s}-\mathbf{x}\|^2/2\sigma^2}e^{-\|\mathbf{r}-\mathbf{y}\|^2/2\sigma^2}$$

$$\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{s}\end{pmatrix}$$

$$I=\frac{e^{-(\mathbf{x}^2+\mathbf{y}^2)/2\sigma^2}}{2^3}J_1J_2$$

$$J_1:=\int \mathrm{d}\mathbf{v}e^{-\mathbf{v}^2/4\sigma^2}e^{\mathbf{v}\cdot(\mathbf{x}+\mathbf{y})/2\sigma^2}$$

$$J_2:=\int \mathrm{d}\mathbf{u}\frac{1}{\|\mathbf{u}\|}e^{-\mathbf{u}^2/4\sigma^2}e^{\mathbf{u}\cdot(\mathbf{x}-\mathbf{y})/2\sigma^2}$$

$$J_1=\left(\sqrt{4\pi\sigma^2}\right)^3e^{(\mathbf{x}+\mathbf{y})^2/4\sigma^2}$$

$$\begin{aligned}J_2&=2\pi\int_0^\infty \mathrm{d}\lambda\lambda e^{-\lambda^2/4\sigma^2}\int_0^\pi \mathrm{d}\theta\sin(\theta)e^{\lambda\|\mathbf{x}-\mathbf{y}\|\cos(\theta)/2\sigma^2}\\ &=8\pi\sigma^2\int_0^\infty \mathrm{d}\lambda\frac{1}{\|\mathbf{x}-\mathbf{y}\|}e^{-\lambda^2/4\sigma^2}\sinh(\lambda\|\mathbf{x}-\mathbf{y}\|/2\sigma^2)\\ &=\frac{8\sigma^3\sqrt{\pi}}{\|\mathbf{x}-\mathbf{y}\|}e^{\|\mathbf{x}-\mathbf{y}\|^2/4\sigma^2}\mathrm{erf}(\|\mathbf{x}-\mathbf{y}\|/2\sigma)\end{aligned}$$

$$I=\frac{(2\pi\sigma^2)^3}{\|\mathbf{x}-\mathbf{y}\|}\mathrm{erf}(\|\mathbf{x}-\mathbf{y}\|/2\sigma)$$

$$\rho\to\big(r(\mathbf{X}_1)\otimes s(\mathbf{X}_2)\big)\rho\big(r(\mathbf{X}_1)\otimes s(\mathbf{X}_2)\big)^\dagger$$

$$\rho(t;\psi_1,\psi_2)=(C_1\otimes C_2)\rho(t;\bar{\psi}_1,\bar{\psi}_2)(C_1\otimes C_2)^\dagger$$

$$\langle\varphi_i(0)\mid\varphi_i(t)\rangle=1, i=0,\ldots,3$$

$$(\rho^{T_1}(t)-\lambda_i(t))|\varphi_i(t)\rangle=0$$

$$(\rho^{T_1}(0)-\lambda_i(0))|\varphi_i(0)\rangle = 0,$$
$$(\rho^{T_1}(0)-\lambda_i(0))|\dot\varphi_i(0)\rangle + \left(\dot\rho^{T_1}(0)-\dot\lambda_i(0)\right)|\varphi_i(0)\rangle = 0,$$
$$\frac{1}{2}(\rho^{T_1}(0)-\lambda_i(0))|\ddot\varphi_i(0)\rangle + \left(\dot\rho^{T_1}(0)-\dot\lambda_i(0)\right)|\dot\varphi_i(0)\rangle$$
$$+\frac{1}{2}\left(\ddot\rho^{T_1}(0)-\ddot\lambda_i(0)\right)|\varphi_i(0)\rangle = 0$$

$$\langle\varphi_j(0)|\dot\rho^{T_1}(0)|\varphi_i(0)\rangle = \dot\lambda_i(0)\delta_{ij}$$

$$P\dot\rho^{T_1}(0)P|\varphi_i(0)\rangle = \dot\lambda_i(0)|\varphi_i(0)\rangle$$

$$E_\pm := \frac{Gm^2}{2\hbar}\left(\frac{1}{\sqrt{\pi}\sigma} - \tilde f(L)\right.$$
$$\left. \pm\frac{1}{\sqrt{2}}|\tilde f(d+2L)+\tilde f(d)-2\tilde f(L+d)|\right)$$

$$\frac{1}{2}\left(\langle\varphi_3(0)|\ddot\rho^{T_1}(0)|\varphi_3(0)\rangle - \ddot\lambda_3(0)\right)$$
$$+\langle\varphi_3(0)|\dot\rho^{T_1}(0)|\dot\varphi_3(0)\rangle = 0$$

$$\langle\varphi_0(0)\mid\dot\varphi_3(0)\rangle + \langle\varphi_0(0)|\dot\rho(0)^{T_1}|\varphi_3(0)\rangle = 0$$

$$\langle\varphi_3(0)|\dot\rho^{T_1}(0)|\dot\varphi_3(0)\rangle$$
$$= \langle\varphi_3(0)|\dot\rho^{T_1}(0)P|\dot\varphi_3(0)\rangle$$
$$+\langle\varphi_3(0)|\dot\rho^{T_1}(0)|\varphi_0(0)\rangle\langle\varphi_0(0)\mid\dot\varphi_3(0)\rangle$$
$$= -|\langle\varphi_3(0)|\dot\rho^{T_1}(0)|\varphi_0(0)\rangle\quad|^2,$$

$$\ddot\lambda_3(0) = \langle - \quad|^{\otimes 2}\ddot\rho^{T_1}(0)\mid -\rangle^{\otimes 2}$$
$$-2\left|\langle - \quad|^{\otimes 2}\dot\rho^{T_1}(0)\mid +\rangle^{\otimes 2}\right|^2 =: 2\nu$$

$$\nu = \frac{G^2m^4}{8\hbar^2}\left[2\left(\frac{1}{\sqrt{\pi}\sigma}-\tilde f(L)\right)^2\right.$$
$$\left.+(\tilde f(d)+\tilde f(2L+d)-2\tilde f(L+d))^2\right]$$

$$\frac{1}{2}(|(0,0,0)\rangle + |(0,L,0)\rangle)\otimes(|(d,0,0)\rangle + |(d,L,0)\rangle)$$

$$\bar\nu = \frac{G^2m^4}{4\hbar^2}\left[\left(\frac{1}{\sqrt{\pi}\sigma}-\tilde f(L)\right)^2 + 2\left(f(d)-\tilde f\left(\sqrt{a^2+L^2}\right)\right)^2\right]$$
$$\bar E_\pm := \frac{Gm^2}{2\hbar}\left(\frac{1}{\sqrt{\pi}\sigma}-\tilde f(L)\pm\sqrt{2}\left(\tilde f(d)-\tilde f\left(\sqrt{L^2+d^2}\right)\right)\right)$$

$$\frac{Gm^2}{2\hbar}\left(\frac{1}{\sqrt{\pi}\sigma}-\sqrt{2}\tilde f(d)\right)$$

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Apéndice H.

1. Agujeros negros cuánticos, supermembranas, superespacios, dimensión temporal y supergravedad cuántica para campos cuánticos relativistas o curvos.

$$S_{EH}[g] = \frac{c^3}{16\pi G_N^{(d)}} \int_{\mathcal{M}} d^d x \sqrt{|g|} R(g) + (-1)^d \frac{c^3}{8\pi G_N^{(d)}} \int_{\partial\mathcal{M}} d^{d-1}\Sigma\mathcal{K} + S_{\text{materia}}$$

$$d^{d-1}\Sigma \equiv n^2 d^{d-1}\Sigma_\rho n^\rho$$

$$d^{d-1}\Sigma_\rho = \frac{1}{(d-1)!\sqrt{|g|}} \epsilon_{\rho\mu_1\cdots\mu_{d-1}} dx^{\mu_1} \wedge \dots dx^{\mu_{d-1}}$$

$$\vec{F} = -\frac{8(d-3)\pi G_N^{(d)} mM}{(d-2)\omega_{d-2}} \frac{\vec{x}_{d-1}}{|\vec{x}_{d-1}|^{d-1}}$$

$$\mathcal{Z} = \int Dg e^{+iS_{EH}/\hbar}$$

$$\frac{S_{EH}}{\hbar} = \frac{2\pi}{\ell_{\text{Planck}}^{d-2}} \int d^d x \dots$$

$$\frac{\ell_{\text{Planck}}^{d-2}}{2\pi} = \frac{16\pi G_N^{(d)}\hbar}{c^3}$$

$$\ell_{\text{Planck}} = \frac{\ell_{\text{Planck}}}{2\pi}.$$

$$\lambda_{\text{Compton}} = \frac{\hbar}{Mc}$$

$$R_s = \left(\frac{16\pi M G_N^{(d)} c^{-2}}{(d-2)\omega_{(d-2)}}\right)^{\frac{1}{d-3}}$$

$$M_{\text{Planck}} = \left(\frac{\hbar^{d-3}}{G_N^{(d)} c^{d-5}}\right)^{\frac{1}{d-2}}$$

$$\frac{c^3}{G_N^{(d)}\hbar} = \left(\frac{M_{\text{Planck}}\, c}{\hbar}\right)^{d-2}$$

$$M \sim M_{\text{Planck}} \Rightarrow \lambda_{\text{Compton}} \sim R_s \sim \ell_{\text{Planck}}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \Rightarrow R_{\mu\nu}$$

$$ds^2 = W(r)(dct)^2 - W^{-1}(r)dr^2 - R^2(r)d\Omega^2_{(2)}$$
$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\ \theta d\varphi^2$$

$$ds^2 = W(dct)^2 - W^{-1}dr^2 - r^2 d\Omega^2_{(2)}, W = 1 + \frac{\omega}{r}$$

$$M = -\frac{\omega c^2}{2G_N^{(4)}}, \Rightarrow \omega = -R_S$$

$$M = \frac{1}{8\pi G_N^{(4)}} \int_{S^2_\infty} d^2 S_i \left(\partial_j g_{ij} - \partial_i g_{jj}\right)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_{(N)}^{(4)}}{c^4} T_{\text{materia } \mu\nu}$$

$$ds^2 = \left(1 - \frac{R_s}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2_{(2)}$$

$$v = ct + r + R_S \log \left|1 - \frac{R_s}{r}\right|$$

$$ds^2 = \frac{4R_S^3 e^{-r/R_S}}{r}\left[(dcT)^2 - dX^2\right] - r^2 d\Omega^2_{(2)}$$

$$\left(\frac{r}{R_S} - 1\right) e^{r/R_S} = X^2 - c^2 T^2$$
$$\frac{ct}{R_S} = \ln\left(\frac{X + cT}{X - cT}\right) = 2\text{arcth}(cT/X)$$

$$ds^2 = W^{\frac{2M}{\omega} - 1} W dt^2 - W^{1 - \frac{2M}{\omega}}\left[W^{-1} dr^2 + r^2 d\Omega^2_{(2)}\right]$$
$$\varphi = \varphi_0 + \frac{\Sigma}{\omega} \ln\ W$$
$$W = 1 + \frac{\omega}{r}, \omega = \pm 2\sqrt{M^2 + \Sigma^2}$$

$$A = \int_{r=R_S} d\theta d\varphi r^2 = 4\pi R_S^2$$

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu k^\nu)\left(\nabla_\mu k_\nu\right)\Big|_{\text{horizonte}}$$

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 - r^2 d\Omega^2_{(2)}$$

$$\kappa = \frac{1}{2}\frac{\partial_r g_{tt}}{\sqrt{-g_{tt} g_{rr}}},$$

$$\kappa = \frac{c^4}{4G_N^{(4)} M}$$

$$dE = TdS$$

$$dM \sim \frac{1}{G_N^{(4)}} \kappa dA$$

$$dM = \frac{1}{8\pi G_N^{(4)}} \kappa dA$$

$$M = \frac{1}{4\pi G_N^{(4)}} \kappa A$$

$$T = \frac{\hbar \kappa}{2\pi c}$$

$$S = \frac{Ac^3}{4\hbar G_N^{(4)}}$$

$$S = \frac{1}{32\pi^2} \frac{A}{\ell_{\text{Planck}}^2}$$

$$T = \frac{\hbar c^3}{8\pi G_N^{(4)} M}, S = \frac{4\pi G_N^{(4)} M^2}{\hbar c}$$

$$dMc^2 = TdS, Mc^2 = 2TS$$

$$C^{-1} = \frac{\partial T}{\partial M} = \frac{-\hbar c^3}{8\pi G_N^{(4)} M^2} < 0$$

$$S(E) = \log \rho(E)$$

$$\rho(M) \sim e^{M^2}$$

$$\mathcal{Z} = \text{Tr} e^{-(H - \mu_i C_i)/T}$$

$$W = E - TS - \mu_i C_i$$

$$e^{-\beta W} = \mathcal{Z}$$

$$S = \frac{1}{T}(E - \mu_i C_i) + \log \mathcal{Z}$$

$$\mathcal{Z} = \int Dg e^{-\tilde{S}_{EH}/\hbar}$$

$$S_{EH}[g] = \frac{c^3}{16\pi G_N^{(4)}} \int_{\mathcal{M}} d^4x \sqrt{|g|} R + \frac{c^3}{8\pi G_N^{(4)}} \int_{\partial \mathcal{M}} (\mathcal{K} - \mathcal{K}_0)$$

$$\mathcal{Z} = e^{-\tilde{S}_{EH}(\text{ on-shell })}$$

$$\left(\frac{r}{R_S}-1\right)e^{r/R_S}=X^2-T^2$$

$$\left(\frac{r}{R_S}-1\right)e^{r/R_S}=X^2+\mathcal{T}^2>0$$

$$\frac{X+T}{X-T}=e^{t/R_S}$$

$$\frac{X-i\mathcal{T}}{X+\mathcal{T}}=e^{-2i\mathrm{Arg}(X+i\mathcal{T})}=e^{-i\tau/R_S}$$

$$n_\mu=-\frac{\delta_{\mu r}}{\sqrt{-n^2}}=-\sqrt{-g_{rr}}\delta_{\mu r}$$

$$ds_{(3)}^2=h_{\mu\nu}dx^\mu dx^\nu=g_{tt}dt^2-r^2d\Omega_{(2)}^2\big|_{r=r_0}$$

$$\nabla_\mu n_\nu=-\sqrt{-g_{rr}}\{\delta_{\mu r}\delta_{\nu r}\partial_r\log\sqrt{-g_{rr}}-\Gamma_{\mu\nu}^r\}$$

$$\mathcal{K}=h^{\mu\nu}\nabla_\mu n_\nu=\frac{1}{\sqrt{-g_{rr}}}\left\{\frac{1}{2}\partial_r\log g_{tt}+\frac{2}{r}\right\}\bigg|_{r=r_0}$$

$$\mathcal{K}_0=\frac{2}{r}\bigg|_{r=r_0}$$

$$g_{tt}\sim 1-\frac{2M}{r},g_{rr}\sim-\left(1+\frac{2M}{r}\right)$$

$$(\mathcal{K}-\mathcal{K}_0)|_{r=r_0}\sim-\frac{M}{r_0^2}$$

$$\begin{aligned}\frac{i}{8\pi}\int_{r_0\to\infty}d^3x\sqrt{|h|}(\mathcal{K}-\mathcal{K}_0)&=\lim_{r_0\to\infty}\frac{i}{8\pi}\int_0^{-i\beta}dt\int_{S^2}d\Omega^2r_0^2\sqrt{g_{tt}(r_0)}(\mathcal{K}-\mathcal{K}_0)\\&=\lim_{r_0\to\infty}\frac{\beta}{2}r_0^2(\mathcal{K}-\mathcal{K}_0)=-\frac{\beta M}{2}\end{aligned}$$

$$S=\beta M+\log\mathcal{Z}=\frac{\beta M}{2}=4\pi M^2$$

$$S_{EM}[g,A]=S_{EH}[g]+\frac{1}{c}\int d^dx\sqrt{|g|}\left[-\frac{1}{4}F^2\right]$$

$$F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$$

$$A'_\mu=A_\mu+\partial_\mu\Lambda$$

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\frac{8\pi G_N^{(4)}}{c^3}T_{\mu\nu}&=0\\\nabla_\mu F^{\mu\nu}&=0\end{aligned}$$

$$T_{\mu\nu} = \frac{-2c}{\sqrt{|g|}} \frac{\delta S_M[A]}{\delta g^{\mu\nu}} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F^2$$

$$\nabla_{\mu}\, {}^{\star}F^{\mu\nu}$$

$$A \equiv A_{\mu} dx^{\mu}, F = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \equiv dA$$

$$d^{\star}F = 0, dF = 0. \left(\partial_{[\alpha} F_{\beta\gamma]} = 0\right)$$

$$\frac{1}{c^2} \int d^d x \sqrt{|g|} [-A_{\mu} j^{\mu}]$$

$$\nabla_{\mu} j^{\mu} = 0, (d^{\star} j = 0)$$

$$\partial_{\mu} \mathrm{j}^{\mu} = 0$$

$$\nabla_{\mu} F^{\mu\nu} = \frac{1}{c} j^{\nu}, \left(d^{\star}F = \frac{1}{c} \star j\right)$$

$$j^{\mu}(y) = qc \int_{\gamma} dX^{\mu} \frac{1}{\sqrt{|g|}} \delta^{(4)}(y - X(\xi))$$

$$j^{\mu}(y^0, \vec{y}) = qc \int dX^0 \frac{dX^{\mu}}{dX^0} \frac{1}{\sqrt{|g|}} \delta^{(3)}(\vec{y} - \vec{X}) \delta(y^0 - X^0) = qV^{\mu} \frac{\delta^{(3)}\left(\vec{y} - \vec{X}(y^0)\right)}{\sqrt{|g|}}$$

$$j^{\mu}(y^0, \vec{y}) = qc\delta^{\mu 0} \frac{\delta^{(3)}(\vec{y})}{\sqrt{|g|}}$$

$$-\frac{q}{c} \int_{\gamma(\xi)} A_{\mu} \dot{x}^{\mu} d\xi = -\frac{q}{c} \int_{\gamma} A$$

$$S_{M,q}[X^{\mu}(\xi)] = -Mc \int d\xi \sqrt{g_{\mu\nu}(X) \dot{X}^{\mu} \dot{X}^{\nu}} - \frac{q}{c} \int A_{\mu} \dot{X}^{\mu}$$

$$0 = \int_V d^{\star} j$$

$$\int_V d^{\star} j = \int_{x^0 = x_2^0} {}^{\star} j - \int_{x^0 = x_1^0}^{\star} j = 0$$

$$q = \frac{1}{c} \int_{x^0 = \text{ constant}} {}^{\star} j,$$

$$q = \int_{S_{\infty}^2} {}^{\star} F$$

$$S_{EM}[g, A] = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{|g|} \left[R - \frac{1}{4} F^2\right]$$

$$q = \frac{1}{16\pi G_N^{(d)}} \int_{S_\infty^{d-2}} {}^\star F$$

$$E_r = F_{0r} \sim \frac{4G_N^{(4)} q}{r^2}$$

$$F_{tr} \sim \pm \frac{1}{R^2(r)}$$

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2 d\Omega_{(2)}^2$$

$$F_{tr} = -\frac{4G_N^{(4)} q}{r^2}$$

$$f(r) = r^{-2}(r - r_+)(r - r_-)$$

$$r_\pm = G_N^{(4)} M \pm r_0, r_0 = G_N^{(4)} (M^2 - 4q^2)^{1/2}$$

$$A_\mu = \delta_{\mu t} \frac{-4G_N^{(4)} q}{r}$$

$$A = 4\pi r_+^2$$

$$A_{\text{extreme}} = 4\pi r_+^2 = 4\pi \left(G_N^{(4)} M\right)^2$$

$$F_{12} = -G_N^{(4)} \frac{M_1 M_2}{r_{12}^2} + 4G_N^{(4)} \frac{q_1 q_2}{r_{12}^2}$$

$$ds^2 = H^{-2} dt^2 - H^2 d\vec{x}_3^2$$

$$A_\mu = -2\text{sign}(q)(H^{-1} - 1)\delta_{\mu t},$$

$$H = 1 + \frac{G_N^{(4)} M}{|\vec{x}_3|}$$

$$\partial_{\underline{i}} \partial_{\underline{i}} H = 0$$

$$ds^2 = H^{-2} dt^2 - H^2 d\vec{x}_3^2,$$

$$A_\mu = \delta_{\mu t} \alpha (H^{-1} - 1), \alpha = \pm 2$$

$$\partial_{\underline{i}} \partial_{\underline{i}} H = 0.$$

$$H(\vec{x}_3) = 1 + \sum_{i=1}^{N} \frac{2G_N^{(4)} |q_i|}{|\vec{x}_3 - \vec{x}_{3,i}|}$$

$$ds^2 = \frac{\rho^2}{R_{AdS}^2} dt^2 - R_{AdS}^2 \frac{d\rho^2}{\rho^2} - R_{AdS}^2 d\Omega_{(2)}^2$$

$$A_t = -\frac{2\rho}{R_{AdS}}, F_{\rho t} = -\frac{2}{R_{AdS}}$$

$$ds^2 = H^{-2}Wdt^2 - H^2\left[W^{-1}d\rho^2 + \rho^2 d\Omega_{(2)}^2\right]$$
$$A_\mu = \delta_{\mu t}\alpha(H^{-1} - 1)$$
$$H = 1 + \frac{h}{\rho}, W = 1 + \frac{\omega}{\rho}, \omega = h\left[1 - \left(\frac{\alpha}{2}\right)^2\right]$$

$$\alpha = -\frac{4G_N^{(4)}q}{r_\pm}, h = r_\pm, \omega = \pm 2r_0$$

$$S[g, A^I] = \frac{1}{16\pi G_N^{(4)}}\int d^4x\sqrt{|g|}\left[R - \frac{1}{4}\sum_{I=1}^{I=N}(F^I)^2\right]$$

$$ds^2 = H^{-2}Wdt^2 - H^2\left[W^{-1}d\rho^2 + \rho^2 d\Omega_{(2)}^2\right]$$
$$A_\mu^i = \delta_{\mu t}\alpha^i(H^{-1} - 1)$$
$$H = 1 + \frac{h}{\rho}, W = 1 + \frac{\omega}{\rho}, \omega = h\left[1 - \sum_{i=1}^{i=N}\left(\frac{\alpha^i}{2}\right)^2\right]$$

$$\alpha^i = -\frac{4G_N^{(4)}q^i}{r_\pm}, h = r_\pm, \omega = \pm 2r_0$$

$$r_\pm = G_N^{(4)}M \pm r_0, r_0 = G_N^{(4)}\sqrt{M^2 - 4\sum_{i=1}^{i=N} q_i^2}$$

$$dM = \frac{1}{8\pi G_N^{(4)}}\kappa dA + \Phi dq$$

$$\kappa = \frac{1}{G_N^{(4)}}\frac{\sqrt{M^2 - 4q^2}}{\left(M + \sqrt{M^2 - 4q^2}\right)^2}$$

$$T = \frac{1}{2\pi G_N^{(4)}}\frac{\sqrt{M^2 - 4q^2}}{\left(M + \sqrt{M^2 - 4q^2}\right)^2}, S = \pi G_N^{(4)}\left(M + \sqrt{M^2 - 4q^2}\right)^2$$

$$\tilde{\vec{E}} = \vec{B}, \tilde{\vec{B}} = -\vec{E}$$

$$\tilde{F} = aF + b^\star F, \Rightarrow^\star \tilde{F} = -bF + a^\star F, a^2 + b^2 \neq 0$$

$$\vec{F} \equiv \begin{pmatrix} F \\ ^\star F \end{pmatrix}, \star\vec{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{F}$$

$$\nabla_\mu \vec{F}^{\mu\nu} = 0$$

$$\tilde{\vec{F}} = M\vec{F}, M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$\int_{S^2_\infty} {}^\star\vec{F} = \begin{pmatrix} 16\pi G_N^{(4)} q \\ p \end{pmatrix} \equiv 16\pi G_N^{(4)}\vec{q}, \vec{q} = \begin{pmatrix} q \\ p/16\pi G_N^{(4)} \end{pmatrix}$$

$$\tilde{A}_\mu(x) = -\int_0^1 d\lambda\lambda x^\nu \frac{\epsilon_{\mu\nu}^{\rho\sigma}}{\sqrt{|g|}}\partial_\rho A_\sigma(\lambda x)$$

$$G_{\mu\nu} - \vec{F}_\mu^{T\rho}\vec{F}_{\nu\rho}$$

$$\vec{F} \equiv \begin{pmatrix} e^{-2}F \\ {}^\star F \end{pmatrix}, G_{\mu\nu} + \left(\vec{F}_\mu^\rho\right)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{F}_{\nu\rho}$$

$$M = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad e' = a^{-1}e$$
$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e' = \frac{1}{e}$$

$$\begin{cases} \tilde{F} &= \cos\ \xi F + \sin\ \xi^\star F \\ \star\,\tilde{F} &= -\sin\ \xi F + \cos\ \xi^\star F \end{cases}$$

$$\begin{cases} \tilde{F}_{tr} = \dfrac{-4G_N^{(4)}\cos\ \xi q}{r^2} \\ \tilde{F}_{\theta\varphi} = 4G_N^{(4)}\sin\ \xi q \sin\ \theta \end{cases}$$

$$\tilde{q} = \cos\ \xi q, \tilde{p} = -16G_N^{(4)}\sin\ \xi q, \Rightarrow \tilde{q}^2 + \left(\frac{\tilde{p}}{16\pi G_N^{(4)}}\right) = q^2$$

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2 d\Omega_{(2)}^2$$
$$F_{tr} = -\frac{4G_N^{(4)}q}{r^2}, F_{\theta\varphi} = -\frac{p}{4\pi}\sin\ \theta,$$
$$f(r) = r^{-2}(r - r_+)(r - r_-),$$
$$r_\pm = G_N^{(4)}M \pm r_0, r_0 = G_N^{(4)}\left\{M^2 - 4\left[q^2 + \left(\frac{p}{16\pi G_N^{(4)}}\right)^2\right]\right\}^{1/2}$$

$$[Q^\alpha, M_{ab}] = \Gamma_s(M_{ab})^\alpha{}_\beta Q^\beta$$

$$\{Q^\alpha, Q^\beta\} = i(\gamma^a C^{-1})^{\alpha\beta}P_a$$

$$[M_{ab}, M_{cd}] = -M_{eb}\Gamma_v(M_{cd})^e{}_a - M_{ae}\Gamma_v(M_{cd})^e{}_b$$
$$[P_a, M_{bc}] = -P_e\Gamma_v(M_{bc})^e{}_a,$$
$$[Q^\alpha, M_{ab}] = \Gamma_s(M_{ab})^\alpha{}_\beta Q^\beta,$$
$$\{Q^\alpha, Q^\beta\} = i(\gamma^a C^{-1})^{\alpha\beta}P_a.$$

$$A_\mu = e^a{}_\mu P_a + \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + \bar{\psi}_{\mu\alpha}Q^\alpha$$

$$\Lambda = \sigma^a P_a + \frac{1}{2}\sigma^{ab} M_{ab} + \bar{\epsilon}_\alpha Q^\alpha$$

$$\delta A_\mu = \partial_\mu \Lambda + \Lambda, A_\mu \equiv \mathbb{D}_\mu \Lambda$$

$$S[e^a{}_\mu, \omega_\mu{}^{ab}, \psi_\mu] = \int d^4x e\left[R(e,\omega) + 2e^{-1}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\nu \nabla_\rho \psi_\sigma\right]$$

$$R(e,\omega) = e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(\omega)$$

$$\begin{aligned} \omega_{abc} &= -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}, \\ \Omega_{\mu\nu}{}^a &= \Omega_{\mu\nu}{}^a(e) + \frac{1}{2}T_{\mu\nu}{}^a, \\ \Omega_{\mu\nu}{}^a(e) &= \partial_{[\mu} e^a{}_{\nu]}, T_{\mu\nu}{}^a = i\bar{\psi}_\mu \gamma^a \psi_\nu. \end{aligned}$$

$$\begin{cases} \delta_\xi x^\mu &= \xi^\mu \\ \delta_\xi e^a{}_\mu &= -\xi^\nu \partial_\nu e^a{}_\mu - \partial_\mu \xi^\nu e^a{}_\nu \\ \delta_\xi \psi_\mu &= -\xi^\nu \partial_\nu \psi_\mu - \partial_\mu \xi^\nu \psi_\nu \end{cases}$$

$$\begin{cases} \delta_\sigma e^a{}_\mu &= \sigma^a{}_b e^b{}_\mu \\ \delta_\sigma \psi_\mu &= \frac{1}{2}\sigma^{ab}\gamma_{ab}\psi_\mu \end{cases}$$

$$\begin{cases} \delta_\epsilon e^a{}_\mu &= -i\bar{\epsilon}\gamma\psi_\mu \\ \delta_\epsilon \psi_\mu &= \nabla_\mu \epsilon \end{cases}$$

$$\begin{aligned} [M_{ab}, M_{cd}] &= -M_{eb}\Gamma_v(M_{cd})^e{}_a - M_{ae}\Gamma_v(M_{cd})^e_b, \\ [P_a, M_{bc}] &= -P_e \Gamma_v(M_{bc})^e{}_a, \\ [Q^{\alpha i}, M_{ab}] &= \Gamma_s(M_{ab})^\alpha{}_\beta Q^{\beta i}, \\ \{Q^{\alpha i}, Q^{\beta j}\} &= i\delta^{ij}(\gamma^a C^{-1})^{\alpha\beta} P_a - i(C^{-1})^{\alpha\beta} Q^{ij} - \gamma_5 (C^{-1})^{\alpha\beta} P^{ij} \end{aligned}$$

$$A_\mu = e^a{}_\mu P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + \frac{1}{2}A^{ij}{}_\mu Q^{ij} + \bar{\psi}^i{}_{\mu\alpha} Q^{i\alpha}$$

$$\frac{1}{n!}(\gamma^{a_1\cdots a_n} C^{-1})^{\alpha\beta} Z^{ij}_{a_1\cdots a_n}$$

$$\left[Z^{kl}_{c_1\cdots c_n}, M_{ab}\right] = -n\Gamma_v(M_{ab})^e{}_{[c_1} Z^{kl}_{|e|c_2\cdots c_n]}$$

$$C^{-1}, \gamma_5 C^{-1}, \gamma_5 \gamma_a C^{-1}, \gamma_{abc} C^{-1}, \gamma_{abcd} C^{-1}$$

$$\gamma_a C^{-1}, \gamma_{ab} C^{-1}, \gamma_5 \gamma_{ab} C^{-1}, \gamma_5 \gamma_{abc} C^{-1}$$

$$\begin{aligned} \{Q^{\alpha i}, Q^{\beta j}\} = & i\delta^{ij}(\gamma^a C^{-1})^{\alpha\beta} P_a + i(C^{-1})^{\alpha\beta} Z^{[ij]} + \gamma_5 (C^{-1})^{\alpha\beta} \tilde{Z}^{[ij]} \\ & + (\gamma^a C^{-1})^{\alpha\beta} Z_a^{(ij)} + i(\gamma_5 \gamma^a C^{-1})^{\alpha\beta} Z_a^{[ij]} \\ & + i\left(\gamma^{ab} C^{-1}\right)^{\alpha\beta} Z_{ab}^{(ij)} + \left(\gamma_5 \gamma^{ab} C^{-1}\right)^{\alpha\beta} \tilde{Z}_{ab}^{(ij)} \end{aligned}$$

$$\delta g_{\mu\nu} = -2\nabla_{(\mu} k_{\nu)}$$

$$\delta_\epsilon B \sim \epsilon F = 0,$$
$$\delta_\epsilon F \sim \begin{Bmatrix} \partial\epsilon + B\epsilon \\ \partial B\epsilon + B\epsilon \end{Bmatrix} = 0$$

$$\delta_\epsilon \psi_\mu = \nabla_\mu \epsilon = 0$$

$$(1 - \gamma^0\gamma^1)\epsilon = 0$$

$$k^\mu \sim \bar{\epsilon}\gamma^\mu \epsilon$$

$$\delta_\epsilon |s > \sim \bar{\epsilon}^i_\alpha Q^{i\alpha}|s >= 0$$

$$\bar{\epsilon}\mathfrak{M}_\epsilon = 0,$$
$$\mathfrak{M} \equiv i\delta^{ij}\gamma^a P_a + iZ^{[ij]} + \gamma_5 \tilde{Z}^{[ij]} + \gamma^a Z_a^{(ij)} + i\gamma_5\gamma^a Z_a^{[ij]} + i\gamma^{ab} Z_{ab}^{(ij)} + \gamma_5\gamma^{ab}\tilde{Z}_{ab}^{(ij)}$$

$$\mathfrak{M} = ip\gamma^0(1 \pm \gamma^0\gamma^1)$$

$$\mathfrak{M} = i\gamma^0 M\left(\delta^{ij} + \frac{Q}{M}\gamma^0\epsilon^{ij}\right)$$

$$(\delta^{ij} \pm \gamma^0\epsilon^{ij})\epsilon^j = 0$$

$$M = |Z|$$

$$M = |Z_1| = |Z_2|$$

$$M = |Z_1| \neq |Z_2|.$$

$$M = |Z_1| = |Z_2| = |Z_3| = |Z_4|,$$

$$M = |Z_1| = |Z_2| \neq |Z_{3,4}|,$$

$$M = |Z_1| \neq |Z_{2,3,4}|.$$

$$(\delta^{ij} + \gamma^0\alpha^{ij})$$

$$\mathfrak{M} = i\gamma^0 M\left(\delta^{ij} + \frac{Z^{(p)}}{M}\gamma^0\gamma^1 \cdots \gamma^p \alpha^{ij}\right)$$

$$M \geq |Z_i|, i = 1, \dots, [N/2]$$

$$\left\{ e^a{}_\mu, \psi_\mu = \begin{pmatrix} \psi_\mu^1 \\ \psi_\mu^2 \end{pmatrix}, A_\mu \right\}$$

$$S = \int d^4x e\{R(e,\omega) + 2e^{-1}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\nabla_\rho\psi_\sigma - \mathcal{F}^2 + \mathcal{J}_{(m)}{}^{\mu\nu}(\mathcal{J}_{(e)\mu\nu} + \mathcal{J}_{(m)\mu\nu})\}$$

$$\begin{cases} \mathcal{F}_{\mu\nu} = \tilde{F}_{\mu\nu} + \mathcal{J}_{(m)\mu\nu} \\ \tilde{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{J}_{(e)\mu\nu} \\ F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \end{cases}$$

$$\begin{cases} \mathcal{J}_{(e)\mu\nu} = i\bar{\psi}_\mu \sigma^2 \psi_\nu \\ \mathcal{J}_{(m)\mu\nu} = -\frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\rho \gamma_5 \sigma^2 \psi_\sigma \end{cases}$$

$$T_{\mu\nu}{}^{a} = i\bar{\psi}_\mu \gamma^a \psi_\nu \left(\equiv i\bar{\psi}_{j\mu} \gamma^a \psi_\nu^j\right)$$

$$\begin{cases} \delta_\epsilon e^a{}_\mu = -i\bar{\epsilon}\gamma^a \psi_\mu \\ \delta_\epsilon A_\mu = -i\bar{\epsilon}\sigma^2 \psi_\mu \\ \delta_\epsilon \psi_\mu = \tilde{\nabla}_\mu \epsilon \end{cases}$$

$$\tilde{\nabla}_\mu = \nabla_\mu + \frac{1}{4} \tilde{F} \gamma_\mu \sigma^2$$

$$\begin{cases} \tilde{F}'_{\mu\nu} = \cos\,\theta \tilde{F}_{\mu\nu} + \sin\,\theta^\star \tilde{F}_{\mu\nu} \\ \psi'_\mu = e^{\frac{i}{2}\theta\gamma_5} \psi_\mu \end{cases}$$

$$\left\{e^a{}_\mu, A^{(n)}{}_\mu, \phi, a, \psi^i_\mu, \lambda^i\right\}$$

$$S = \int d^4x \sqrt{|g|} \left\{R + 2(\partial\phi)^2 + \frac{1}{2} e^{4\phi} (\partial a)^2 - e^{-2\phi} \sum_{n=1}^{6} F^{(n)} F^{(n)} + a \sum_{n=1}^{6} F^{(n)\star} F^{(n)}\right\}$$

$$\tau = a + ie^{-2\phi}$$

$$\tilde{F}^{(n)}_{\mu\nu} = \partial_\mu \tilde{A}^{(n)}_\nu - \partial_\nu \tilde{A}^{(n)}_\mu$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1$$

$$\begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ F^{(n)}{}_{\mu\nu} \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ F^{(n)}{}_{\mu\nu} \end{pmatrix}$$

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

$$M^2 + \frac{|Z_1 Z_2|^2}{M^2} - |Z_1|^2 - |Z_2|^2 \geq 0$$

$$\partial_{\underline{i}} \partial_{\underline{i}} \mathcal{H}_1 = \partial_{\underline{i}} \partial_{\underline{i}} \mathcal{H}_2 = 0$$

$$\sum_{n=1}^{N} \left(k^{(n)}\right)^2 = 0, \sum_{n=1}^{N} \left|k^{(n)}\right|^2 = \frac{1}{2}$$

$$\begin{aligned} e^{-2U} &= 2\Im\,\mathrm{m}\left(\mathcal{H}_1 \overline{\mathcal{H}}_2\right) \\ \partial_{[\underline{i}]} \omega_{\underline{j}]} &= \epsilon_{ijk} \Re\mathrm{e}\left(\mathcal{H}_1 \partial_{\underline{k}} \overline{\mathcal{H}}_2 - \overline{\mathcal{H}}_2 \partial_{\underline{k}} \mathcal{H}_1\right) \end{aligned}$$

$$
\begin{aligned}
ds^2 &= e^{2U}\left(dt^2 + \omega_{\underline{i}} dx^{\underline{i}}\right)^2 - e^{-2U} d\vec{x}^2 \\
\lambda &= \frac{\mathcal{H}_1}{\mathcal{H}_2} \\
A_t^{(n)} &= 2e^{2U}\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_2\right) \\
\tilde{A}_t^{(n)} &= -2e^{2U}\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_1\right)
\end{aligned}
$$

$$
\mathcal{H}_1 = i\mathcal{H}_2 = \frac{1}{\sqrt{2}} V^{-1}
$$

$$
\Upsilon = -2\frac{\sum_n \overline{\Gamma^{(n)}}^2}{M}
$$

$$
M^2 + |\Upsilon|^2 - 4\sum_n \left|\Gamma^{(n)}\right|^2 = 0
$$

$$
\frac{1}{2}\left|Z_{1,2}\right|^2 = \sum_n \left|\Gamma^{(n)}\right|^2 \pm \left[\left(\sum_n \left|\Gamma^{(n)}\right|^2\right)^2 - \left|\sum_n \Gamma^{(n)2}\right|^2\right]^{\frac{1}{2}}
$$

$$
A = 4\pi(|M|^2 - |\Upsilon|^2) = 4\pi\left||Z_1|^2 - |Z_2|^2\right|
$$

$$
A = 8\pi\sqrt{\det\left[\begin{pmatrix}\vec{p}^t \\ \vec{q}^t\end{pmatrix}(\vec{p}\vec{q})\right]}
$$

$$
\begin{pmatrix}\vec{p} \\ \vec{q}\end{pmatrix}' = R \otimes S \begin{pmatrix}\vec{p} \\ \vec{q}\end{pmatrix}
$$

$$
\left\{\hat{\hat{Q}}^{\hat{\hat{\alpha}}}, \hat{\hat{Q}}^{\hat{\hat{\beta}}}\right\} = i\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}}\hat{\hat{\mathcal{C}}}^{-1}\right)^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{P}}_{\hat{\hat{a}}} + \frac{1}{2}\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\hat{\hat{a}}_2}\hat{\hat{\mathcal{C}}}^{-1}\right)^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{\mathcal{Z}}}^{(2)}_{\hat{\hat{a}}_1\hat{\hat{a}}_2} + \frac{i}{5!}\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_5}\hat{\hat{\mathcal{C}}}^{-1}\right)^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{\mathcal{Z}}}^{(5)}_{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_5}
$$

$$
+\frac{1}{6!}\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_6}\hat{\hat{\mathcal{C}}}^{-1}\right)\Big)^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{\mathcal{Z}}}^{(6)}_{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_6} + \frac{i}{9!}\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_9}\hat{\hat{\mathcal{C}}}^{-1}\right)\Bigg)^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\mathcal{Z}^{(9)}_{a_1\cdots a_9}
$$

$$
+\frac{1}{10!}\left(\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_{10}}\hat{\hat{\mathcal{C}}}^{-1}\right)\hat{\hat{\alpha}}^{\hat{\alpha}\hat{\beta}}\hat{\hat{\mathcal{Z}}}^{(10)}_{\hat{\hat{a}}_1\cdots\cdots\hat{\hat{a}}_{10}}.
$$

$$
\left\{\hat{\hat{e}}_{\hat{\hat{\mu}}}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}\hat{\hat{L}}}, \hat{\hat{\psi}}_{\hat{\hat{\mu}}}\right\}
$$

$$
\hat{\hat{S}} = \frac{1}{16\pi G_N^{(11)}}\int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\left[\hat{\hat{R}} - \frac{1}{2\cdot 4!}\hat{\hat{G}}^2 - \frac{1}{(144)^2}\frac{1}{\sqrt{|\hat{\hat{g}}}}\hat{\hat{\epsilon}}\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{C}}\hat{\hat{C}}\right]
$$

$$
\hat{\hat{G}} = 4\partial\hat{\hat{C}}
$$

$$\delta_{\hat{\hat{\chi}}}\hat{\hat{C}} = 3\partial\hat{\hat{\chi}}$$

$$\begin{cases} \delta_{\hat{\hat{\epsilon}}}\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}} & = & -\frac{i}{2}\bar{\hat{\hat{\epsilon}}}\,\hat{\hat{\Gamma}}^{\hat{\hat{a}}}\,\hat{\hat{\psi}}_{\hat{\hat{\mu}}}\,, \\ \delta_{\hat{\hat{\epsilon}}}\hat{\hat{\psi}}_{\hat{\hat{\mu}}} & = & 2\nabla_{\hat{\hat{\mu}}}\hat{\hat{\epsilon}} + \frac{i}{144}\left(\hat{\hat{\Gamma}}^{\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}{}_{\hat{\hat{\mu}}} - 8\hat{\hat{\Gamma}}^{\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}\hat{\hat{\eta}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{\alpha}}}\right)\hat{\hat{\epsilon}}\,\hat{\hat{G}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}\,, \\ \delta_{\hat{\hat{\epsilon}}}\hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}} & = & \frac{3}{2}\bar{\hat{\hat{\epsilon}}}\,\hat{\hat{\Gamma}}_{[\hat{\hat{\mu}}\hat{\hat{\nu}}}\,\hat{\hat{\psi}}_{\hat{\hat{\rho}}]}\,. \end{cases}$$

$$\partial\left({}^{\star}\hat{\hat{G}} + \frac{35}{2}\hat{\hat{C}}\hat{\hat{G}}\right)$$

$$*\,\hat{\hat{G}} = 7(\partial\hat{\hat{\hat{C}}} - 10\hat{\hat{C}}\partial\hat{\hat{C}}) \equiv \hat{\hat{\hat{G}}}$$

$$\delta_{\hat{\hat{\chi}}}\hat{\hat{\hat{C}}} = 6\partial\hat{\hat{\chi}}$$

$$\delta_{\hat{\hat{\chi}}}\hat{\hat{\hat{C}}} = -30\partial\hat{\hat{\chi}}\hat{\hat{C}}$$

$$\hat{\hat{P}}_{\hat{\hat{a}}} = (\hat{P}_{\hat{a}}, \hat{Z}^{(0)}), \hat{\hat{Z}}^{(2)}_{\hat{\hat{a}}\hat{\hat{b}}} = \left(\hat{Z}^{(2)}_{\hat{a}\hat{b}}, \hat{Z}^{(1)}_{\hat{a}}\right), \hat{\hat{Z}}^{(5)}_{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_5} = \left(\hat{Z}^{(5)}_{\hat{a}_1\cdots\hat{a}_5}, \hat{Z}^{(4)}_{\hat{a}_1\cdots\hat{a}_4}\right)$$

$$\hat{\hat{Z}}^{(6)}_{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_6} = \left(\hat{Z}^{(6)}_{\hat{a}_1\cdots\hat{a}_6}, \cdot\right), \hat{\hat{Z}}^{(9)}_{\hat{\hat{a}}_1\cdots\hat{\hat{a}}_9} = \left(\cdot, \hat{Z}^{(8)}_{\hat{a}_1\cdots\hat{a}_8}\right)$$

$$\left\{\hat{Q}^{\hat{\alpha}}, \hat{Q}^{\hat{\beta}}\right\} = i\left(\hat{\Gamma}^{\hat{a}}\hat{C}^{-1}\right)^{\hat{\alpha}\hat{\beta}}\hat{P}_{\hat{a}} + \sum_{n=0,1,4,8}\frac{c_n}{n!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\hat{\Gamma}_{11}\hat{C}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}^{(n)}_{\hat{a}_1\cdots\hat{a}_n}$$
$$+ \sum_{n=2,5,6}\frac{c_n}{n!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n}\hat{\Gamma}_{11}\hat{C}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}^{(n)}_{\hat{a}_1\cdots\hat{a}_n}$$

$$\{\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}, \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{C}^{(1)}{}_{\hat{\mu}}, \}.$$

$$\frac{c_5}{5!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_5}\hat{\Gamma}_{11}\hat{C}^{-1}\right)^{\hat{\alpha}\hat{\beta}}\hat{Z}^{(5)}_{\hat{a}_1\cdots\hat{a}_5} + \frac{c_9}{9!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_9}\hat{C}^{-1}\right)\hat{\alpha}^{\hat{\beta}\hat{\beta}}\hat{Z}^{(9)}_{\hat{a}_1\cdots\hat{a}_9}$$

$$\begin{aligned} \hat{\hat{g}}_{\hat{\mu}\hat{\nu}} &= e^{-\frac{2}{3}\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}} - e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}\hat{\mu}^{(1)}\hat{\nu}, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{C}^{(3)}\hat{\mu}\hat{\nu}\hat{\rho} \\ \hat{\hat{g}}_{\hat{\mu}\underline{z}} &= -e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}\hat{\mu}, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\underline{z}} &= \hat{B}_{\hat{\mu}\hat{\nu}}, \\ \hat{\hat{g}}_{\underline{zz}} &= -e^{\frac{4}{3}\hat{\phi}}. \end{aligned}$$

$$\left(\hat{e}_{\hat{\hat{\mu}}}\hat{\hat{a}}^{\hat{u}} = \begin{pmatrix} e^{-\frac{1}{3}\hat{\phi}_{\hat{e}}^{\hat{\mu}}\;\hat{a}} & e^{\frac{2}{3}\hat{\phi}}\hat{C}^{(1)}\hat{\mu} \\ 0 & e^{\frac{2}{3}\hat{\phi}} \end{pmatrix}\right),$$
$$(\hat{\hat{e}}_{\hat{\hat{a}}}\hat{\hat{\mu}}) = \begin{pmatrix} e^{\frac{1}{3}\hat{\phi}}{}^{\hat{e}}\hat{e}_{\hat{a}}^{\hat{\mu}} & -e^{\frac{1}{3}\hat{\phi}}\hat{C}^{(1)}\hat{a} \\ 0 & e^{-\frac{2}{3}\hat{\phi}} \end{pmatrix}.$$

$$\hat{S} = \frac{2\pi t^{(11)}_{\text{lanck}}}{16\pi G_N^{(1)}} \int \ d^{10}\hat{x}\sqrt{|\hat{g}|}\Big\{e^{-2\hat{\phi}}\Big[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2\Big]$$
$$-\Big[\frac{1}{4}\big(\hat{G}^{(2)}\big)^2 + \frac{1}{2\cdot 4!}\big(\hat{G}^{(4)}\big)^2\Big] - \frac{1}{144}\frac{1}{\sqrt{|\hat{g}|}}\hat{\epsilon}\partial\hat{C}^{(3)}\partial\hat{C}^{(3)}\hat{B}\Big\}$$

$$\hat{g}_{\hat{\mu}\hat{\nu}} \to e^{\frac{2}{3}\hat{\phi}_0}\hat{\eta}_{\hat{\mu}\hat{\nu}}$$

$$\hat{g}_{\hat{\mu}\hat{\nu}} \to e^{\frac{2}{3}\hat{\phi}_0}\hat{g}_{\hat{\mu}\hat{\nu}}, \quad \hat{C}^{(1)}\hat{\mu} \to e^{\frac{1}{3}\hat{\phi}_0}\hat{C}^{(1)}\hat{\mu},$$
$$\hat{B}_{\hat{\mu}\hat{\nu}} \to e^{\frac{2}{3}\hat{\phi}_0}\hat{B}_{\hat{\mu}\hat{\nu}}, \quad \hat{C}^{(3)}\hat{\mu}\hat{\nu}\hat{\rho} \to e^{\hat{\phi}_0}\hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}$$

$$\hat{S} = \frac{g_A^2}{16\pi G_{NA}^{100}} \int \ d^{10}\hat{x}\sqrt{|\hat{g}|}\Big\{e^{-2\hat{\phi}}\Big[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2\Big]$$
$$-\Big[\frac{1}{4}\big(\hat{G}^{(2)}\big)^2 + \frac{1}{2\cdot 4!}\big(\hat{G}^{(4)}\big)^2\Big] - \frac{1}{144}\frac{1}{\sqrt{|\hat{g}|}}\hat{\epsilon}\partial\hat{C}^{(3)}\partial\hat{C}^{(3)}\hat{B}\Big\}$$

$$g_A = e^{\hat{\phi}_0}$$

$$\frac{2\pi\ell^{(11)}_{\text{Planck}}\ e^{\frac{8}{3}\hat{\phi}_0}}{16\pi G_N^{(11)}} = \frac{g_A^2}{16\pi G_{NA}^{(10)}}$$

$$G_N^{(10)} = \frac{G_N^{(11)}}{2\pi\ell^{(11)}_{\text{Planck}}\ g_A^{2/3}}$$

$$R_{11} = \frac{1}{2\pi}\lim_{r\to\infty}\int \ \sqrt{|\hat{g}_{\underline{zz}}|}dz = \ell^{(11)}_{\text{Planck}}\ e^{\frac{2}{3}\hat{\phi}_0} = \ell^{(11)}_{\text{Planck}}\ g_A^{2/3}$$

$$G_{NA}^{(10)} = \frac{G_N^{(11)}}{2\pi R_{11}} = \frac{G_N^{(11)}}{V_{11}}$$

$$G_{NA}^{(10)} = \frac{\left(\ell^{(11)}_{\text{Planck}}\right)^8}{32\pi^2 g_A^{2/3}}$$

$$G_{NA}^{(10)} = 8\pi^6 g_A^2 \ell_s^8$$

$$\ell^{(11)}_{\text{Planck}} = 2\pi\ell_s g_A^{1/3}$$
$$R_{11} = \ell_s g_A$$

$$\hat{G}^{(10-k)} = (-1)^{[k/2]\star}\hat{G}^{(k)}$$

$$\hat{G} = d\hat{C} - \hat{H}\wedge\hat{C}$$

$$d\hat{G} - \hat{H}\wedge\hat{G} = 0, d^\star\hat{G} + \hat{H}\wedge^\star\hat{G}$$

$$\hat{H}^{(7)} = e^{-2\hat{\phi}\star}\hat{H}$$

$$dH = 0, d\left(e^{2\phi\star}\hat{H}^{(7)}\right)$$

$$d\left(e^{-2\phi\star}\hat{H}\right) + \frac{1}{2}\star\hat{G}\wedge\hat{G} = 0, d\hat{H}^{(7)} + \frac{1}{2}\star\hat{G}\wedge\hat{G}$$

$$\hat{H}^{(7)} = d\hat{B}^{(6)} - \frac{1}{2}\sum_{n=1}^{n=4}\star\hat{G}^{(2n+2)}\wedge\hat{C}^{(2n-1)}$$

$$\begin{cases}\hat{\hat{\epsilon}} = e^{-\frac{1}{6}(\hat{\phi}^{-}-\hat{\phi}_0)}\hat{\epsilon}\\ \hat{\hat{\psi}}_{\hat{a}} = e^{\frac{1}{6}(\hat{\phi}-\hat{\phi}_0)}\left(2\hat{\psi}_{\hat{a}} - \frac{1}{3}\hat{\Gamma}_{\hat{a}}\hat{\lambda}\right)\\ \hat{\hat{\psi}}_z = \frac{2i}{3}e^{\frac{1}{6}(\hat{\phi}-\hat{\phi}_0)}\hat{\Gamma}_{11}\hat{\lambda}\end{cases}$$

$$\delta_{\hat{\epsilon}}\hat{e}_{\hat{\mu}}^{\hat{a}} = -i\bar{\hat{\epsilon}}\hat{\Gamma}^{\hat{a}}\hat{\psi}_{\hat{\mu}},$$

$$\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} = \left\{\partial_{\hat{\mu}} - \frac{1}{4}\left(\psi_{\hat{\mu}} + \frac{1}{2}\Gamma_{11}\hat{H}_{\mu}\right)\right\}\hat{\epsilon} + \frac{i}{8}e^{\hat{e}}\sum_{n=1,2}\frac{1}{(2n)!}\hat{\epsilon}^{(2n)}\hat{\Gamma}_{\hat{\mu}}\left(-\hat{\Gamma}_{11}\right)^n\hat{\epsilon},$$

$$\delta_{\hat{\epsilon}}\hat{\beta}_{\hat{\mu}\hat{v}} = -2i\bar{\hat{\epsilon}}\hat{\Gamma}_{[\hat{\mu}}\hat{\Gamma}_{11}\hat{\psi}_{\hat{v}]},$$

$$\delta_{\hat{\epsilon}}\hat{C}^{(1)}{}_{\hat{\mu}} = -e^{\hat{\phi}\hat{\epsilon}}\hat{\Gamma}_{11}\left(\hat{\psi}_{\hat{\mu}} - \frac{1}{2}\hat{\Gamma}_{\hat{\mu}}\hat{\lambda}\right),$$

$$\delta_{\hat{\epsilon}}\hat{C}^{(3)}{}_{\hat{\mu}\hat{v}\hat{\rho}} = 3e^{\hat{\phi}}\bar{\epsilon}\hat{\Gamma}_{\hat{\mu}\hat{v}}\left(\hat{\psi}_{\hat{\rho}]} - \frac{1}{3!}\hat{\Gamma}_{\hat{\rho}]}\hat{\lambda}\right) + 3\hat{C}^{(1)}{}_{[\hat{\mu}}\delta_{\hat{\epsilon}}\hat{B}_{\hat{\mu}\hat{v}]},$$

$$\delta_{\hat{\epsilon}}\hat{\lambda} = \left(\partial\partial\phi\hat{\phi} + \frac{1}{12}\hat{\Gamma}_{11}\hat{H}\right)\hat{\epsilon} + \frac{i}{4}e^{\hat{\phi}}\sum_{n=1,2}\frac{5-2n}{(2n)!}\hat{\zeta}^{(2n)}\left(-\hat{\Gamma}_{11}\right)^n\hat{\epsilon},$$

$$\delta_{\hat{\epsilon}}\hat{\phi} = -\frac{i}{2}\hat{\epsilon}\hat{\epsilon}$$

$$\left\{\hat{Q}^{i\hat{\alpha}}, \hat{Q}^{j\hat{\beta}}\right\} = i\delta^{ij}\left(\hat{\Gamma}^{\hat{a}}\hat{\mathcal{C}}^{-1}\right)^{\hat{\alpha}\hat{\beta}}\hat{P}_{\hat{a}} + \left(\hat{\Gamma}^{\hat{a}}\hat{\mathcal{C}}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}_{\hat{a}}^{(1)(ij)} + \frac{i}{3!}\left(\hat{\Gamma}^{\hat{a}_1\hat{a}_2\hat{a}_3}\hat{\mathcal{C}}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}_{\hat{a}_1\hat{a}_2\hat{a}_3}^{(3)[ij]}$$
$$+\frac{i}{5!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_5}\hat{\mathcal{C}}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}_{\hat{a}_1\cdots\hat{a}_5}^{(5)(ij)} + \frac{i}{7!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_7}\hat{\mathcal{C}}^{-1}\right)\hat{\alpha}\hat{\alpha}\hat{Z}_{\hat{a}_1\cdots\hat{a}_7}^{(7)[\iota\jmath]}$$
$$+\frac{i}{9!}\left(\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_9}\hat{\mathcal{C}}^{-1}\right)\hat{\alpha}\hat{\beta}\hat{Z}_{\hat{a}_1\cdots\hat{a}_9}^{(9)(ij)}.$$

$$\hat{Z}_{\hat{a}}^{(1)(ij)} = \hat{Z}_{\hat{a}}^{(1)0}\delta^{ij}\hat{Z}_{\hat{a}}^{(1)1}\sigma^1 + \hat{Z}_{\hat{a}}^{(1)3}\sigma^3$$

$$\hat{Z}_{\hat{a}_1\hat{a}_2\hat{a}_3}^{(3)} = \hat{Z}_{\hat{a}_1\hat{a}_2\hat{a}_3}^{(i)}i\sigma^2$$

$$\{\hat{\jmath}_{\hat{\mu}\hat{v}}, \hat{\mathcal{B}}_{\hat{\mu}\hat{v}}, \hat{\varphi}\}$$

$$\left\{C^{\hat{(0)}}, C^{\hat{(2)}}\hat{\mu}\hat{v}, C^{\hat{(4)}}\hat{\mu}\hat{v}\hat{\rho}\hat{\sigma}\right\}$$

$$\begin{cases}\hat{\mathcal{H}} = 3\partial\mathcal{B}\\ \hat{G}^{(1)} = \partial\hat{C}^{(0)}\\ \hat{G}^{(3)} = 3\left(\partial\hat{C}^{(2)} - \partial\hat{\mathcal{B}}\hat{C}^{(0)}\right)\\ \hat{G}^{(5)} = 5\left(\partial\hat{C}^{(4)} - 6\partial\hat{\mathcal{B}}\hat{C}^{(2)}\right)\end{cases}$$

$$\hat{G}^{(5)} = +^{\star}\hat{G}^{(5)}$$

$$\left(\hat{G}^{(5)}\right)^2 = \left({}^{\star}\hat{G}^{(5)}\right)^2 = -\left(\hat{G}^{(5)}\right)^2 \Rightarrow = 0$$

$$S_{\text{NSD}} = \frac{g_B^2}{16\pi G_{NB}^{(10)}} \int \; d^{10}\hat{x}\sqrt{|\hat{\jmath}|}\left\{e^{-2\hat{\varphi}}\left[\hat{R}(\hat{\jmath}) - 4(\partial\hat{\varphi})^2 + \frac{1}{2\cdot 5!}\hat{\mathcal{H}}^2\right]\right.$$
$$+\frac{1}{2}\left(\hat{G}^{(0)}\right)^2 + \frac{1}{2\cdot 3!}\left(\hat{G}^{(3)}\right)^2 + \frac{1}{4\cdot 3!}\left(\hat{G}^{(5)}\right)^2$$
$$\left.-\frac{1}{192}\frac{1}{\sqrt{|\hat{\jmath}|}} \in \partial\hat{C}^{(4)}\partial\hat{C}^{(2)}\hat{\mathcal{B}}\right\}$$

$$g_B = e^{\hat{\varphi}_0}$$

$$\delta_{\hat{\varepsilon}}\hat{e}_{\hat{\mu}}^{\hat{a}} = -i\bar{\hat{\varepsilon}}\hat{\Gamma}^{\hat{a}}\hat{\zeta}_{\hat{\mu}},$$
$$\delta_{\hat{\varepsilon}}\hat{\zeta}_{\hat{\mu}} = \nabla_{\hat{\mu}}\hat{\varepsilon} - \frac{1}{8}\hat{\mathcal{H}}_{\hat{\mu}}\sigma_3\hat{\varepsilon} + \frac{1}{8}e^{\hat{\varphi}}\sum_{n=1,2,3}\frac{1}{(2n-1)!}\hat{l}^{(2n-1)}\hat{\Gamma}_{\hat{\mu}}\mathcal{P}_n\hat{\varepsilon},$$
$$\delta_{\hat{\varepsilon}}\hat{\mathcal{B}}_{\hat{\mu}\hat{\nu}} = -2i\bar{\hat{\varepsilon}}\sigma^3\hat{\Gamma}_{[\hat{\mu}}\hat{\zeta}_{\hat{\nu}]},$$
$$\delta_{\hat{\varepsilon}}\hat{C}^{(2n-2)}{}_{\hat{\mu}_1\cdots\hat{\mu}_{2n-2}} = i(2n-2)e^{-\hat{\varphi}}\bar{\hat{\varepsilon}}\mathcal{P}_n\hat{\Gamma}_{[\hat{\mu}_1\cdots\hat{\mu}_{2n-3}}\left(\hat{\zeta}_{\hat{\mu}_{2n-2}]} - \frac{1}{2(2n-2)}\hat{\Gamma}_{\hat{\mu}_{2n-2}]\hat{\chi})}\right)$$
$$+\frac{1}{2}(2n-2)(2n-3)\hat{C}^{(2n-4)}{}_{[\hat{\mu}_1\cdots\hat{\mu}_{2n-4}}\delta_{\hat{\varepsilon}}\hat{\mathcal{B}}_{\hat{\mu}_{2n-3}\hat{\mu}_{2n-4}},$$
$$\delta_{\hat{\varepsilon}}\hat{\chi} = \left(\partial\hat{\varphi} - \frac{1}{12}\hat{\mathcal{H}}\sigma^3\right)\hat{\varepsilon} + \frac{1}{2}e^{\hat{\varphi}}\sum_{n=1,2,3}\frac{(n-3)}{(2n-1)!}\hat{\zeta}^{(2n-1)}\mathcal{P}_n\varepsilon,$$
$$\delta_{\hat{\varepsilon}}\hat{\varphi} = -\frac{i}{2}\bar{\hat{\varepsilon}}\hat{\chi}$$

$$\mathcal{P}_n = \begin{cases}\sigma^1, & n\\ i\sigma^2, & n\end{cases}$$

$$\hat{\jmath}_{E\mu\nu} = e^{-\varphi/2}\jmath_{\mu\nu}$$

$$\begin{cases}\vec{\hat{\mathcal{B}}} = \begin{pmatrix}\hat{C}^{(2)}\\ \hat{\mathcal{B}}\end{pmatrix}\\ \hat{D} = \hat{C}^{(4)} - 3\hat{\mathcal{B}}\hat{C}^{(2)}\end{cases}$$

$$\begin{cases}\vec{\hat{\mathcal{H}}} = 3\partial\vec{\hat{\mathcal{B}}}\\ \hat{F} = \hat{G}^{(5)} = +{}^{\star}\hat{F}\\ = 5\left(\partial\hat{D} - \vec{\hat{\mathcal{B}}}^T\eta\vec{\hat{\mathcal{H}}}\right)\end{cases}$$

$$\eta = i\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = -\eta^{-1} = -\eta^T$$

$$\Lambda\eta\Lambda^T = \eta, \Rightarrow \eta\Lambda\eta^T = (\Lambda^{-1})^T, \Lambda \in SL(2,\mathbb{R})$$

$$\hat{\mathcal{M}} = e^{\hat{\varphi}}\begin{pmatrix}|\hat{\tau}|^2 & \hat{C}^{(0)}\\ \hat{C}^{(0)} & 1\end{pmatrix}, \hat{\mathcal{M}}^{-1} = e^{\hat{\varphi}}\begin{pmatrix}1 & -\hat{C}^{(0)}\\ -\hat{C}^{(0)} & |\hat{\tau}|^2\end{pmatrix}$$

$$\hat{\tau} = \hat{C}^{(0)} + ie^{-\hat{\varphi}}$$

$$\hat{\mathcal{M}}' = \Lambda\hat{\mathcal{M}}\Lambda^T,$$
$$\hat{\mathcal{B}}' = \Lambda\hat{\vec{\mathcal{B}}}$$

$$\hat{\tau}' = \frac{a\hat{\tau} + b}{c\hat{\tau} + d}$$

$$\hat{S}_{\text{NSD}} = \frac{g_B^2}{16\pi G_N^{100}} \int \ d^{10}\hat{x}\sqrt{|\hat{\jmath}_E|}\left\{\hat{R}(\hat{\jmath}_E) + \frac{1}{4}\text{Tr}\left(\partial\hat{\mathcal{M}}\hat{\mathcal{M}}^{-1}\right)^2\right.$$
$$\left.+\frac{1}{2\cdot 3!}\hat{\vec{\mathcal{H}}}^T\hat{\mathcal{M}}^{-1}\hat{\vec{\mathcal{H}}} + \frac{1}{4\cdot 5!}\hat{F}^2 - \frac{1}{2^7\cdot 3^3}\frac{1}{\sqrt{|\hat{\jmath}_E|}} \in \hat{D}\hat{\vec{\mathcal{H}}}^T\eta\hat{\vec{\mathcal{H}}}\right\}$$

$$g_B' = 1/g_B$$

$$\hat{\jmath}' = |c\hat{\lambda} + d|\hat{\jmath}$$

$$\hat{\jmath}' = e^{-\hat{\varphi}}\hat{\jmath}$$

$$R' = R/g_B$$

$$\begin{aligned}\{Q^{i\alpha}, Q^{j\beta}\} =& i(\Gamma^a\mathcal{C}^{-1})^{\alpha\beta}\left(\delta^{ij}P_a + \sigma^{1ij}Z_a^{(1)1} + \sigma^{3ij}Z_a^{(1)3}\right)\\
&+(\mathcal{C}^{-1})^{\alpha\beta}\left(\delta^{ij}Z^{(0)0} + \sigma^{1ij}Z^{(0)1} + \sigma^{3ij}Z^{(0)3}\right)\\
&+\frac{i}{2!}(\Gamma^{a_1a_2}\mathcal{C}^{-1})^{\alpha\beta}\sigma^{2ij}Z^{(2)}_{a_1a_2} + \frac{1}{3!}\left(\Gamma^{a_1a_2a_3}\mathcal{C}^{-1}\right)^{\alpha\beta}\sigma^{2ij}Z^{(3)}_{a_1a_2a_3}\\
&+\frac{1}{4!}(\Gamma^{a_1\cdot a_4}\mathcal{C}^{-1})^{\alpha\beta}\left(\sigma^{1ij}Z^{(4)1}_{a_1\cdot a_4} + \sigma^{3ij}Z^{(4)3}_{a_1\cdot a_4}\right)\\
&+\frac{i}{5!}(\Gamma^{a_1\cdot a_5}\mathcal{C}^{-1})^{\alpha\beta}\left(\delta^{ij}Z^{(5)0}_{a_1\cdot a_5} + \sigma^{1ij}Z^{(5)1}_{a_1\cdot a_5} + \sigma^{3ij}Z^{(5)3}_{a_1\cdot a_5}\right)\\
&+\frac{i}{6!}(\Gamma^{a_1\cdots a_6}\mathcal{C}^{-1})^{\alpha\beta}\sigma^{2ij}\left(Z^{(6)}_{a_1\cdots a_6} + Z^{(6)!}_{a_1\cdots a_6}\right)\\
&+\frac{1}{7!}(\Gamma^{a_1\cdots a_7}\mathcal{C}^{-1})^{\alpha\beta}\sigma^{2ij}\left(Z^{(7)}_{a_1\cdots a_7} + Z^{(7)!}_{a_1\cdots a_7}\right)\\
&+\frac{1}{8!}(\Gamma^{a_1\cdot a_8}\mathcal{C}^{-1})^{\alpha\beta}\left(\sigma^{1ij}Z^{(8)1}_{a_1\cdot a_8} + \sigma^{3ij}Z^{(8)3}_{a_1\cdot a_8}\right)\end{aligned}$$

$$\begin{aligned}&\hat{g}_{\mu\nu} = g_{\mu\nu} - k^2A^{(1)}{}_\mu A^{(1)}{}_\nu, g_{\mu\nu} = \hat{g}_{\mu\nu} - \hat{g}_{\mu\underline{x}}\hat{g}_{\nu\underline{x}}/\hat{g}_{\underline{xx}},\\
&\hat{B}_{\mu\nu} = B_{\mu\nu} + A^{(1)}{}_{[\mu}A^{(2)}{}_{\nu]}, B_{\mu\nu} = \hat{B}_{\mu\nu} + \hat{g}_{[\mu|\underline{x}|}\hat{B}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}},\\
&\hat{\phi} = \phi + \frac{1}{2}\log\ k, \phi\ = \hat{\phi} - \frac{1}{4}\log|\hat{g}_{\underline{xx}}|,\\
&\hat{g}_{\mu\underline{x}} = -k^2A^{(1)}{}_\mu, A^{(1)}{}_\mu = \hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}},\\
&\hat{B}_{\mu\underline{x}} = -A^{(2)}{}_\mu, A^{(2)}{}_\mu = -\hat{B}_{\mu\underline{x}},\\
&\hat{g}_{\underline{xx}} = -k^2, k = |\hat{g}_{\underline{xx}}|^{1/2}\end{aligned}$$

$$\begin{aligned}&\hat{C}^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}} = C^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}} + (2n-1)A^{(1)}{}_{[\mu_1}C^{(2n-2)}{}_{\mu_2\cdots\mu_{2n-1}]},\\
&\hat{C}^{(2n+1)}{}_{\mu_1\cdots\mu_{2n}\underline{x}} = C^{(2n)}{}_{\mu_1\cdots\mu_{2n}},\\
&C^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}} = \hat{C}^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}} - (2n-1)\hat{g}_{[\mu_1|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_2\cdots\mu_{2n-1}]\underline{x}}/\hat{g}_{\underline{xx}},\\
&C^{(2n)}{}_{\mu_1\cdots\mu_{2n}} = \hat{C}^{(2n+1)}{}_{\mu_1\cdots\mu_{2n}\underline{x}}\end{aligned}$$

$$\begin{aligned}
&\hat{\jmath}_{\mu\nu} = g_{\mu\nu} - k^{-2} A^{(2)}{}_{\mu} A^{(2)}{}_{\nu}, g_{\mu\nu} = \hat{\jmath}_{\mu\nu} - \hat{\jmath}_{\mu\underline{y}} \hat{\jmath}_{\nu\underline{y}} / \hat{\jmath}_{\underline{yy}},\\
&\hat{\mathcal{B}}_{\mu\nu} = B_{\mu\nu} + A^{(1)}{}_{[\mu} A^{(2)}{}_{\nu]}, B_{\mu\nu} = \hat{\mathcal{B}}_{\mu\nu} + \hat{\jmath}_{[\mu|\underline{y}|} \hat{\mathcal{B}}_{\nu]\underline{y}} / \hat{\jmath}_{\underline{yy}},\\
&\hat{\varphi} = \phi - \frac{1}{2}\log\, k, \phi\ = \hat{\varphi} - \frac{1}{4}\log\, \left|\hat{\jmath}_{\underline{yy}}\right|,\\
&\hat{\jmath}_{\mu\underline{y}} = -k^{-2} A^{(2)}{}_{\mu}, A^{(1)}{}_{\mu} = \hat{\mathcal{B}}_{\mu\underline{y}},\\
&\hat{\mathcal{B}}_{\mu\underline{y}} = A^{(1)}{}_{\mu}, A^{(2)}{}_{\mu} = \hat{\jmath}_{\mu\underline{y}} / J_{\underline{yy}},\\
&\hat{\jmath}_{\underline{yy}} = -k^{-2}, k = \left|\hat{\jmath}_{\underline{yy}}\right|^{-1/2}
\end{aligned}$$

$$\begin{aligned}
\hat{C}^{(2n)}{}_{\mu_1\cdots\mu_{2n}} &= C^{(2n)}{}_{\mu_1\cdots\mu_{2n}} - (2n) A^{(2)}{}_{[\mu_1} C^{(2n-1)}{}_{\mu_2\cdots\mu_{2n}]},\\
\hat{C}^{(2n)}{}_{\mu_1\cdots\mu_{2n-1}\underline{y}} &= -C^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}},\\
C^{(2n)}{}_{\mu_1\cdots\mu_{2n}} &= \hat{C}^{(2n)}{}_{\mu_1\cdots\mu_{2n}} + (2n)\hat{\jmath}_{[\mu_1|\underline{y}} \hat{C}^{(2n)}{}_{\mu_2\cdots\mu_{2n}]\underline{y}} / \hat{\jmath}_{\underline{y}},\\
C^{(2n-1)}{}_{\mu_1\cdots\mu_{2n-1}} &= -\hat{C}^{(2n)}{}_{\mu_1\cdots\mu_{2n-1}\underline{y}}
\end{aligned}$$

$$\begin{aligned}
&\hat{\jmath}_{\mu\nu} = \hat{g}_{\mu\nu} - \left(\hat{g}_{\mu\underline{x}} \hat{g}_{\nu\underline{x}} - \hat{B}_{\mu\underline{x}} \hat{B}_{\nu\underline{x}}\right) / \hat{g}_{\underline{xx}}, \hat{\jmath}_{\mu\underline{y}} = \hat{B}_{\mu\underline{x}} / \hat{g}_{\underline{xx}},\\
&\hat{\mathcal{B}}_{\mu\nu} = \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{x}} \hat{B}_{\nu]\underline{x}} / \hat{g}_{\underline{xx}}, \hat{\mathcal{B}}_{\mu\underline{y}} = \hat{g}_{\mu\underline{x}} / \hat{g}_{\underline{xx}},\\
&\hat{\varphi} = \hat{\phi} - \frac{1}{2}\log\left|\hat{g}_{\underline{xx}}\right|, \hat{\jmath}_{\underline{yy}} = 1/\hat{g}_{\underline{xx}},\\
&\hat{C}^{(2n)}{}_{\mu_1\ldots\mu_{2n}} = \hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} + 2n\hat{B}_{[\mu_1|\underline{x}|} \hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n}]}\\
&-2n(2n-1)\hat{B}_{[\mu_1|\underline{x}|} \hat{g}_{\mu_2|\underline{x}|} \hat{C}^{(2n-1)}{}_{\mu_3\ldots\mu_{2n}]\underline{x}} / \hat{g}_{\underline{xx}},\\
&\hat{C}^{(2n)}{}_{\mu_1\ldots\mu_{2n-1}\underline{y}} = -\hat{C}^{(2n-1)}{}_{\mu_1\ldots\mu_{2n-1}}\\
&+(2n-1)\hat{g}_{[\mu_1|\underline{x}|} \hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n-1}]\underline{x}} / \hat{g}_{\underline{xx}}
\end{aligned}$$

$$\begin{aligned}
&\hat{g}_{\mu\nu} = \hat{\jmath}_{\mu\nu} - \left(\hat{\jmath}_{\mu\underline{y}} \hat{\jmath}_{\underline{y}} - \hat{\mathcal{B}}_{\mu\underline{y}} \hat{\mathcal{B}}_{\nu\underline{y}}\right) / \hat{\jmath}_{\underline{yy}}, \hat{g}_{\mu\underline{x}} = \hat{\mathcal{B}}_{\mu\underline{\underline{y}}} / \hat{\jmath}_{\underline{yy}},\\
&\hat{B}_{\mu\nu} = \hat{\mathcal{B}}_{\mu\nu} + 2\hat{\jmath}_{[\mu|\underline{1}} \hat{\mathcal{B}}_{\nu]\underline{y}} / J_{\underline{y}}, \hat{B}_{\mu\underline{x}} = \hat{\jmath}_{\mu\underline{y}} / \hat{\jmath}_{\underline{yy}},\\
&\hat{\phi} = \hat{\varphi} - \frac{1}{2}\log\,\left|\hat{y}_{\underline{yy}}\right|, \hat{g}_{\underline{xx}} = 1/\hat{\jmath}_{\underline{yy}},\\
&\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n+1}} = -\hat{C}^{(2n+2)}{}_{\mu_1\ldots\mu_{2n+1}\underline{y}} + (2n+1)\hat{\mathcal{B}}_{[\mu_1|\underline{y}|} \hat{C}^{(2n)}{}_{\mu_2\ldots\mu_{2n+1}]}\\
&-2n(2n+1)\hat{\mathcal{B}}_{[\mu_1|\underline{y}|} \hat{\jmath}_{\mu_2|\underline{y}} \hat{C}^{(2n)}{}_{\mu_3\ldots\mu_{2n+1}]\underline{1}} / \hat{\jmath}_{\underline{yy}},\\
&\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} = \hat{C}^{(2n)}{}_{\mu_1\ldots\mu_{2n}}\\
&+2n\hat{\jmath}_{[\mu_1|\underline{y}|} \hat{C}^{(2n)}{}_{\mu_2\ldots\mu_{2n}]\underline{y}} / \hat{\jmath}_{\underline{yy}}
\end{aligned}$$

$$\hat{\jmath}_{\underline{yy}} = 1/\hat{g}_{\underline{xx}}, \hat{g}_{\underline{xx}} = 1/\hat{\jmath}_{\underline{yy}}$$

$$\hat{g}_{\underline{xx}} \to (R_A/\ell_s)^2, \hat{\jmath}_{\underline{yy}} \to (R_B/\ell_s)^2$$

$$R_{A,B} = \ell_s^2 / R_{B,A}$$

$$g_{A,B} = g_{B,A} / R_{B,A}$$

$$g \ll 1, \ell_s \ll 1$$

$$\ell_s/R_c \ll 1$$

$$S_{NG}^{(p)}[X^\mu(\xi)] = -T_{(p)} \int d^{p+1}\xi \sqrt{|g_{ij}|}$$

$$g_{ij} = g_{\mu\nu}(X)\partial_i X^\mu \partial_j X^\nu$$

$$S_P^{(p)}[X^\mu, \gamma_{ij}] = -\frac{T_{(p)}}{2} \int d^{p+1}\xi \sqrt{|\gamma|}[\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + (1-p)]$$

$$\gamma_{ij} = g_{ij}$$

$$\gamma_{ij} = \Omega(\xi) g_{ij}$$

$$S_{NG}^{(p)}[X^\mu(\xi)] = -\frac{T_{(p)}}{K_0^\alpha} \int d^{p+1}\xi K(X)^\alpha \sqrt{|g_{ij}|} - \frac{\mu}{K_0^\alpha (p+1)!} \int d^{p+1}\xi \epsilon^{i_1 \cdots i_{p+1}} A_{(p+1)i_1 \cdots i_{p+1}}$$

$$A_{(p+1)i_1 \cdots i_{p+1}} = A_{(p+1)\mu_1 \cdots \mu_{p+1}}(X)\partial_{i_1} X^{\mu_1} \ldots \partial_{i_{p+1}} X^{\mu_{p+1}}$$

$$\delta A_{(p+1)} = (p+1)\partial \Lambda_{(p)}$$

$$S_{NG}^{(p)}[X^\mu(\xi)] = -\frac{T_{(p)}}{K_0^\alpha} \int d^{p+1}\xi K^\alpha(X) \sqrt{|g_{ij} + F_{ij}|} + \cdots, F_{ij} = 2\partial_{[i} V_{j]}$$

$$S_{NG}^{(p)}[X^\mu(\xi)] = -\frac{T_{(p)}}{K_0^\alpha} \int d^{p+1}\xi K^\alpha(X) \sqrt{|g_{ij}|} \left\{1 - \frac{1}{2}\mathcal{F}^2 + \cdots\right\}$$

$$S = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{|g|} \left[R + 2(\partial \log K)^2 + \frac{(-1)^{p+1}}{2 \cdot (p+2)!} K^\beta F_{(p+2)}^2\right]$$

$$F_{(p+2)} = (p+2)\partial A_{(p+1)}$$

$$S = -T \int d^2\xi \sqrt{|\hat{g}_{ij}|} - \frac{T}{2} \int d^2\xi \epsilon^{ij} \hat{B}_{ij}$$

$$S^{(p)} = -T_{(p)} e^{2\phi_0} \int d^{p+1}\xi e^{-2\phi} \sqrt{|g_{ij}|} + \cdots$$

$$S = -T_{S5} e^{2\hat{\phi}_0} \int d^6\xi e^{-2\hat{\phi}} \sqrt{|\hat{g}_{ij}|} - \frac{T_{S5} e^{2\hat{\phi}_0}}{6!} \int d^6\xi \epsilon^{i_1 \cdots i_6} \hat{B}_{i_1 \cdots i_6}^{(6)}$$

$$S = -T_{Dp} e^{\hat{\phi}_0} \int d^{p+1}\xi e^{-\hat{\phi}} \sqrt{|\hat{g}_{ij} + 2\pi\alpha' \mathcal{F}_{ij}|} - \frac{T_{SDp} e^{\hat{\phi}_0}}{6!} \int d^6\xi \epsilon^{i_1 \cdots i_{p+1}} \hat{C}_{i_1 \cdots i_{p+1}}^{(p+1)}$$

$$\mathcal{F}_{ij} = F_{ij} + \frac{1}{2\pi\alpha'} \hat{B}_{ij}, F_{ij} = 2\partial_{[i} V_{j]}$$

$$R_{A,B} = \ell_s^2 / R_{B,A}$$
$$g_{A,B} = g_B \ell_s / R_{B,A}$$

$$g' = 1/g$$
$$R_i' = R_i/\sqrt{g}$$

$$M' = g^{1/2}M$$

$$M_{F1} = \frac{R_9}{\ell_s^2}$$

$$M_{D1} = M_{F1}' = g^{1/2}M_{F1} = g^{1/2}\frac{R_9}{\ell_s^2} = \frac{R_9'}{g'\ell_s^2}$$

$$M_{D0} = M_{D1}' = \frac{R_9}{g\ell_s^2} = \frac{\ell_s^2/R_9'}{g'\ell_s/R_9'\ell_s^2} = \frac{1}{g'\ell_s}$$

$$M_{D0} = M_{D1}' = \frac{R_9}{g\ell_s^2} = \frac{R_9}{g'\ell_s/R_8'\ell_s^2} = \frac{R_8R_9}{g'\ell_s^3}$$

$$M_{Dp} = \frac{R_{10-p}\ldots R_9}{g\ell_s^{p+1}}$$

$$M_{S5} = g^{1/2}M_{D5}' = g^{1/2}\frac{R_5\ldots R_9}{g\ell_s^6} = g'^{-1/2}\frac{R_5'/g'^{1/2}\ldots R_9'/g^{1/2}}{g'^{-1}\ell_s^6} = \frac{R_5\ldots R_9}{g^2\ell_s^6}$$

Objeto Supermasivo	Masa	Masa	Objeto Masivo
F1m	R_9^{-1}		
D0	$g_A^{-1}\ell_s^{-1}$	R_{10}^{-1}	WM$(+,-^{10})$
F1w	$R_9\ell_s^{-2}$	$R_{10}R_9\left(\ell_{\text{Planck}}^{(11)}\right)^{-3}$	M2$(+,-^{8},+^{2})$
D2	$R_9R_8g_A^{-1}\ell_s^{-3}$	$R_9R_8\left(\ell_{\text{Planck}}^{(11)}\right)^{-3}$	M2$(+,-^{7},+^{2},-)$
D4	$R_9\ldots R_6g_A^{-1}\ell_s^{-5}$	$R_{10}R_9\ldots R_5\left(\ell_{\text{Planck}}^{(11)}\right)^{-6}$	M5$(+,-^{5},+^{5})$
S5A	$R_9\ldots R_5g_A^{-2}\ell_s^{-6}$	$R_9\ldots R_5\left(\ell_{\text{Planck}}^{(11)}\right)^{-6}$	M5$(+,-^{4},+^{5},-)$

D6	$R_9 \dots R_4 g_A^{-1} \ell_s^{-7}$	$R_{10}^2 R_9 \dots R_4 \left(\ell_{\text{Planck}}^{(11)}\right)^{-9}$	KK7M$(+,-^3,+^6,-^\star)$
KK6A	$R_9^2 R_8 \dots R_4 g_A^{-2} \ell_s^{-8}$	$R_{10} R_9^2 \dots R_4 \left(\ell_{\text{Planck}}^{(11)}\right)^{-9}$	KK7M$(+,\ ^3,+^5,+^\star,+)$
D8	$R_9 \dots R_2 g_A^{-1} \ell_s^{-9}$	$R_{10}^3 R_9 \dots R_4 \left(\ell_{\text{Planck}}^{(11)}\right)^{-12}$	KK9M$(+,-,+^8,+^\star)$
KK8A	$R_9^3 R_8 \dots R_2 g_A^{-3} \ell_s^{-11}$	$R_{10} R_9^3 R_8 \dots R_2 \left(\ell_{\text{Planck}}^{(11)}\right)^{-12}$	KK9M$(+,-,+^7,+^\star,+)$
KK9A	$R_9^3 R_8 \dots R_1 g_A^{-4} \ell_s^{-12}$	$R_{10} R_9^3 R_8 \dots R_1 \left(\ell_{\text{Planck}}^{(11)}\right)^{-12}$	KK9M$(+,+^8,+^\star,-)$

Objeto Supermasivo	Masa	Objeto Masivo	Masa
F1m	R_9^{-1}	KK6A	$R_9^2 R_8 \dots R_4 g_B^{-2} \ell_s^{-8}$
F1w	$R_9 \ell_s^{-2}$	D7	$R_9 \dots R_3 g_B^{-1} \ell_s^{-8}$
D1	$R_9 g_B^{-1} \ell_s^{-2}$	Q7	$R_9 \dots R_3 g_B^{-3} \ell_s^{-8}$
D3	$R_9 \dots R_7 g_B^{-1} \ell_s^{-4}$	D9	$R_9 \dots R_1 g_B^{-1} \ell_s^{-10}$
D5	$R_9 \dots R_5 g_B^{-1} \ell_s^{-6}$	Q9	$R_9 \dots R_1 g_B^{-4} \ell_s^{-10}$
S5B	$R_9 \dots R_5 g_B^{-2} \ell_s^{-6}$		

$$S = -T_{M2} \int d^3\xi \sqrt{|\hat{\hat{g}}_{ij}|} - \frac{T_{M2}}{3!} \int d^3\xi \epsilon^{i_1 \cdots i_3} \hat{\hat{C}}_{i_1 \cdots i_3}$$

$$\begin{aligned} \ell_s &= \ell_{\text{Planck}}^{(11)} / R_{10}^{1/2} \\ g_A &= R_{10}^{3/2} / \ell_{\text{Planck}}^{(11)} \end{aligned}$$

$$M_{M2} = M_{F1A} = \frac{R_9}{\ell_s^2} = \frac{R_9 R_{10}}{\left(\ell_{\text{Planck}}^{(11)}\right)^3}$$

$$M_{M2}=\frac{R_8R_9}{\left(\ell_{\text{Planck}}^{(11)}\right)^3}=\frac{R_8R_9}{g_A\ell_s^3}$$

$$M_{M5}=M_{D4}=\frac{R_6\dots R_9}{g_A\ell_s^5}=\frac{R_6\dots R_{10}}{\left(\ell_{\text{Planck}}^{(11)}\right)^6}$$

$$M_{M5}=\frac{R_5\dots R_9}{\left(\ell_{\text{Planck}}^{(11)}\right)^6}=\frac{R_5\dots R_9}{g_A^2\ell_s^6}$$

$$M_{D0}=\frac{1}{g_A\ell_s}=\frac{1}{R_{10}}$$

Objeto Súper	Masa
WM	0
M2	$R_{10}R_9\left(\ell_{\text{Planck}}^{(11)}\right)^{-3}$
M5	$R_{10}\dots R_6\left(\ell_{\text{Planck}}^{(11)}\right)^{-6}$
KK7M	$R_{10}^2R_9\dots R_4\left(\ell_{\text{Planck}}^{(11)}\right)^{-9}$
KK9M	$R_{10}^3R_9\dots R_4\left(\ell_{\text{Planck}}^{(11)}\right)^{-12}$

$$S=\frac{1}{16\pi G_N^{(d)}}\int d^dx\sqrt{|g|}\left[R+2(\partial\varphi)^2+\frac{(-1)^{p+1}}{2\cdot(p+2)!}e^{-2a\varphi}F_{(p+2)}^2\right]$$

$$F_{(p+2)}=dA_{(p+1)},F_{(p+2)\mu_1\dots\mu_{p+2}}=(p+2)\partial_{[\mu_1}A_{(p+1)\mu_2\dots\mu_{p+2}]}$$

$$ds^2=f\left[Wdt^2-d\vec{y}_p^2\right]-g^{-1}\left[W^{-1}d\rho^2-\rho^2d\Omega_{(\tilde{p}+2)}^2\right]$$

$$A_{t\underline{ty^1}\dots\underline{y^p}}=\alpha(H^{-1}-1)$$

$$\tilde{p}\equiv d-p-4$$

$$H=1+\frac{h}{\rho^{\tilde{p}+1}},W=1+\frac{\omega}{\rho^{\tilde{p}+1}}$$

$$ds^2 = H^{\frac{2x-2}{p+1}}\left[Wdt^2 - d\vec{y}_p^2\right] - H^{\frac{-(2x-2)}{\tilde{p}+1}}\left[W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(\tilde{p}+2)}\right]$$
$$e^{-2a\varphi} = e^{-2a\varphi_0}H^{2x}, A_{\underline{t}\underline{y}^1\ldots\underline{y}^p} = e^{a\varphi_0}\alpha(H^{-1}-1),$$
$$H = 1 + \frac{h}{\rho^{\tilde{p}+1}}, W = 1 + \frac{\omega}{\rho^{\tilde{p}+1}},$$
$$\omega = h\left[1 - \frac{a^2}{4x}\alpha^2\right],$$
$$x = \frac{\frac{a^2}{2}c}{1+\frac{a^2}{2}c}, c = \frac{(p+1)+(\tilde{p}+1)}{(p+1)(\tilde{p}+1)}$$

$$ds^2 = H^{\frac{2x-2}{p+1}}\left(dt^2 - d\vec{y}_p^2\right) - H^{\frac{-(2x-2)}{\tilde{p}+1}}d\vec{x}^2_{(\tilde{p}+3)},$$
$$e^{-2a\varphi} = e^{-2a\varphi_0}H^{2x}, A_{\underline{t}\underline{t}\underline{g}^1\ldots\underline{y}^p} = e^{a\varphi_0}\alpha(H^{-1}-1)$$
$$\partial_{\underline{m}}\partial_{\underline{m}}H = 0,$$
$$x = \frac{\frac{a^2}{2}}{1+\frac{a^2}{2}}, c = \frac{(p+1)+(\tilde{p}+1)}{(p+1)(\tilde{p}+1)}, \alpha^2 = \frac{4x}{a^2}$$

$$H = 1 + \frac{h}{\left|\vec{x}_{(\tilde{p}+3)}\right|^{\tilde{p}+1}}$$

$$H = 1 + \sum_{I=1}^{N}\frac{h_I}{\left|\vec{x}_{(\tilde{p}+3)} - \vec{x}_{(\tilde{p}+3)I}\right|^{\tilde{p}+1}}$$

$$ds^2 = Wdt^2 - d\vec{y}_p^2 - W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(\tilde{p}+2)}$$
$$W = 1 + \frac{\omega}{\rho^{\tilde{p}+1}}$$

$$S = S_a + S_p$$

$$S_p[X^\mu, \gamma_{ij}] = -\frac{T}{2}\int d^{p+1}\xi\sqrt{|\gamma|}\left[e^{-2b\varphi}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} - (p-1)\right]$$
$$-\frac{\mu}{(p+1)!}\int d^{p+1}\xi A_{(p+1)\mu_1\cdots\mu_{p+1}}\partial_{i_1}X^{\mu_1}\ldots\partial_{i_{p+1}}X^{\mu_{p+1}}$$

$$Y^i(\xi) = \xi^i$$

$$X^m(\xi) = 0$$

$$a = -(p+1)b$$

$$\mu = T/\alpha$$

$$H = \epsilon + \frac{h}{\left|\vec{x}_{(\tilde{p}+3)}\right|^{\tilde{p}+1}}$$

$$h=\frac{16\pi G_N^{(d)}T}{(\tilde{p}+1)\alpha^2\omega_{(\tilde{p}+2)}}$$

$$\begin{aligned}
d\hat{\hat{s}}_E^2 &= H_{M2}^{-2/3}[Wdt^2-d\vec{y}_2^2]-H_{M2}^{1/3}[W^{-1}d\rho^2+\rho^2 d\Omega_{(7)}^2]\\
\hat{\hat{C}}_{t\underline{y}^1\underline{y}^2} &= \alpha(H_{M2}^{-1}-1),\\
H_{M2} &= 1+\frac{h_{M2}}{\rho^6}, W=1+\frac{\omega}{\rho^6},\\
\omega &= h_{M2}[1-\alpha^2]
\end{aligned}$$

$$\begin{aligned}
d\hat{\hat{s}}^2 &= H_{M2}^{-2/3}[dt^2-d\vec{y}_2^2]-H_{M2}^{1/3}d\vec{x}_8^2,\\
\hat{\hat{C}}_{ty_1y_2} &= \pm(H_{M2}^{-1}-1),\\
H_{M2} &= 1+\frac{h_{M2}}{|\vec{x}_8|^6}
\end{aligned}$$

$$h_{M2}=\frac{16\pi G_N^{(11)}T_{M2}}{6\omega_{(7)}}$$

$$T_{M2}=\frac{M_{M2}}{(2\pi)^2R_9R_{10}}=\frac{1}{(2\pi)^2\left(\ell_{\text{Planck}}^{(11)}\right)^3}=\frac{2\pi}{\left(\ell_{\text{Planck}}^{(11)}\right)^3}$$

$$h_{M2}=\frac{\left(\ell_{\text{Planck}}^{(11)}\right)^6}{6\omega_{(7)}}$$

$$\begin{aligned}
d\hat{\hat{s}}^2 &= H_{M5}^{-1/3}[Wdt^2-d\vec{y}_5^2]-H_{M5}^{2/3}[W^{-1}d\rho^2+\rho^2 d\Omega_{(4)}^2]\\
\tilde{\hat{\hat{C}}}_{tt\underline{y}^1\cdots\underline{y}^5} &= \alpha(H_{M5s}^{-1}-1),\\
H_{M5} &= 1+\frac{h_{M5}}{\rho^3}, W=1+\frac{\omega}{\rho^3},\\
\omega &= h_{M5}[1-\alpha^2]
\end{aligned}$$

$$\begin{aligned}
d\hat{\hat{s}}^2 &= H_{M5}^{-1/3}[dt^2-d\vec{y}_5^2]-H_{M5}^{2/3}d\vec{x}_5^2,\\
\tilde{\hat{\hat{C}}}_{t\underline{t}^1\cdots\underline{y}^5} &= \pm(H_{M5}^{-1}-1),\\
H_{M5} &= 1+\frac{h_{M5}}{|\vec{x}_5|^3}
\end{aligned}$$

$$h_{M5}=\frac{\left(\ell_{\text{Planck}}^{(11)}\right)^3}{3\omega_{(4)}}$$

$$S=\frac{1}{16\pi G_N^{(d)}}\int d^dx\sqrt{|g_s|}\left\{e^{-2\phi}\left[R_s-4(\partial\phi)^2+\frac{(-1)^{p_1+1}}{2\cdot(p_1+2)!}F_{(p_1+2)}^2\right]+\frac{(-1)^{p_2+1}}{2\cdot(p_2+2)!}F_{(p_2+2)}^2\right\}$$

$$g_{s\mu\nu}=e^{\frac{4}{(d-2)}\phi}g_{\mu\nu}$$

$$S = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{|g|} \Big[R + \frac{4}{(d-2)} (\partial \phi)^2$$

$$+ \frac{(-1)^{p_1+1}}{2\cdot(p_1+2)!} e^{-4\frac{(p_1+1)}{(d-2)}\phi} F^2_{(p_1+2)} + \frac{(-1)^{p_2+1}}{2\cdot(p_2+2)!} e^{2\frac{(\tilde{p}_2-p_1)}{(d-2)}\phi} F^2_{(p_2+2)} \Big\}$$

$$\phi = \sqrt{\frac{(d-2)}{2}}\varphi$$

$$a_1 = \frac{2(p_1+1)}{\sqrt{2(d-2)}}, \quad (\mathrm{NS-NS})$$
$$a_2 = \frac{-(\tilde{p}_2 - p_2)}{\sqrt{2(d-2)}}. \quad (\mathrm{RR})$$

$$a_3 = -\frac{2(\tilde{p_1}+1)}{\sqrt{2(d-2)}}$$

$$\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{F1}^{-3/4}[Wdt^2 - dy^2] - H_{F1}^{1/4}[W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(7)}],\\
d\hat{s}_S^2 &= H_{F1}^{-1}[Wdt^2 - dy^2] - [W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(7)}],\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{F1},\\
\hat{B}_{\underline{ty}} &= \alpha(H_{F1}^{-1} - 1),\\
H_{F1} &= 1 + \frac{h_{F1}}{\rho^6}, W = 1 + \frac{\omega}{\rho^6},\\
\omega &= h_{F1}[1-\alpha^2]
\end{aligned}$$

$$\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{F1}^{-3/4}[dt^2 - dy^2] - H_{F1}^{1/4} d\vec{x}_8^2\\
d\hat{s}_S^2 &= H_{F1}^{-1}[dt^2 - dy^2] - d\vec{x}_8^2,\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{F1},\\
\hat{B}_{\underline{ty}} &= \pm(H_{F1}^{-1} - 1),\\
H_{F1} &= 1 + \frac{h_{F1}}{|\vec{x}_8|^6}
\end{aligned}$$

$$h_{F1} = \frac{2^5 \pi^6 \ell_s^6 g^2}{3\omega_{(7)}}$$

$$\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{S5}^{-1/4}[Wdt^2 - d\vec{y}_5^2] - H_{S5}^{3/4}[W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(3)}]\\
d\hat{s}_S^2 &= [Wdt^2 - d\vec{y}_5^2] - H_{S5}[W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(3)}]\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{S5}^{-1},\\
\hat{B}^{(6)}{}_{\underline{tty^1 \ldots y^5}} &= \alpha e^{-2\hat{\phi}_0}(H_{S5}^{-1} - 1)\\
H_{S5} &= 1 + \frac{h_{ps}}{\rho^2}, W = 1 + \frac{\omega}{\rho^2},\\
\omega &= h_{S5}[1-\alpha^2]
\end{aligned}$$

$$
\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{S5}^{-1/4}[dt^2 - d\vec{y}_5^2] - H_{S5}^{3/4}d\vec{x}_4^2,\\
d\hat{s}_S^2 &= [dt^2 - d\vec{y}_5^2] - H_{S5}d\vec{x}_4^2,\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{S5}^{-1},\\
\tilde{\hat{B}}^{(6)}{}_{\underline{t}ty^1\cdots \underline{y}^5} &= \pm e^{-2\hat{\phi}_0}(H_{S5}^{-1} - 1),\\
H_{S5} &= 1 + \frac{h_{S5}}{|\vec{x}_4|^2}
\end{aligned}
$$

$$
h_{S5} = \ell_s^2
$$

$$
\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{D_p}^{-\frac{(7-p)}{8}}\left[Wdt^2 - d\vec{y}_p^2\right] - H_{D_p}^{\frac{(p+1)}{8}}\left[W^{-1}d\rho^2 + \rho^2 d\Omega_{(8-p)}^2\right]\\
d\hat{s}_S^2 &= H_{D_p}^{-1/2}\left[Wdt^2 - d\vec{y}_p^2\right] - H_{Dp}^{1/2}\left[W^{-1}d\rho^2 + \rho^2 d\Omega_{(8-p)}^2\right],\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{D_p}^{\frac{(p-3)}{2}},\\
\hat{C}^{(p+1)}{}_{\underline{ty}^1\cdots \underline{y}^p} &= \alpha e^{-\hat{\phi}_0}\left(H_{Dp}^{-1} - 1\right),\\
H_{Dp} &= 1 + \frac{h_{Dp}}{\rho^{7-p}}, W = 1 + \frac{\omega}{\rho^{7-p}},\\
\omega &= h_{D_p}[1 - \alpha^2]
\end{aligned}
$$

$$
H \sim h\log\,\rho, W \sim \omega\log\,\rho
$$

$$
H \sim h\rho, W \sim \omega\rho
$$

$$
\begin{aligned}
d\tilde{\hat{s}}_E^2 &= H_{Dp}^{\frac{p-7}{8}}\left[dt^2 - d\vec{y}_p^2\right] - H_{Dp}^{\frac{p+1}{8}}d\vec{x}_{9-p}^2,\\
d\hat{s}_S^2 &= H_{Dp}^{-1/2}\left[dt^2 - d\vec{y}_p^2\right] - H_{Dp}^{1/2}d\vec{x}_{9-p}^2,\\
e^{-2(\hat{\phi}-\hat{\phi}_0)} &= H_{Dp}^{\frac{(p-3)}{2}},\\
\hat{C}^{(p+1)}\underline{ty}^1\cdots \underline{y}^p &= \pm e^{-\hat{\phi}_0}\left(H_{Dp}^{-1} - 1\right),\\
H_{Dp} &= 1 + \frac{h_{Dp}}{\left|\vec{x}_{9-p}\right|^{7-p}}
\end{aligned}
$$

$$
H_{D7} = 1 + h_{D7}\log|\vec{x}_2|
$$

$$
H_{D8} = 1 + h_{D8}|x|
$$

$$
h_{Dp} = \frac{(2\pi\ell_s)^{(7-p)}g}{(7-p)\omega_{(8-p)}}
$$

$$
h_{D7} =, h_{D8} = \frac{g}{4\pi\ell_s}
$$

$$\begin{cases} d\hat{\hat{s}}^2 = H_{M2}^{-2/3}\left[dt^2 - dy^2 - e^{\frac{4}{3}\hat{\phi}_0}dz^2\right] - H_{M2}^{1/3}d\vec{x}_8^2 \\ \hat{\hat{C}}_{ty\underline{z}} = \pm e^{\frac{2}{3}\hat{\phi}_0}(H_{M2}^{-1} - 1) \\ H_{M2} = 1 + \dfrac{h_{M2}}{|\vec{x}_8|^6} \end{cases}$$

$$h_{M2} = \frac{\left(\ell_{\text{Planck}}^{(11)}\right)^6}{6\omega_{(7)}} = \frac{\left(2\pi\ell_s g^{1/3}\right)^6}{6\omega_{(7)}} = \frac{(2\pi\ell_s)^6 g^2}{6\omega_{(7)}} = h_{D2}$$

$$H_{M2} = 1 + h_{M2}\sum_{n=-\infty}^{n=+\infty}\frac{1}{(|\vec{x}_7|^2 + (z + 2\pi n R_{11})^2\ |)^3}$$

$$u_n = \frac{(z - 2\pi n R_{11})}{|\vec{x}_7|}, u_n \in \left[\frac{2\pi n R_{11}}{|\vec{x}_7|}, \frac{2\pi(n+1)R_{11}}{|\vec{x}_7|}\right]$$

$$\begin{aligned} H_{M2} &= 1 + \frac{h_{M2}}{|\vec{x}_7|^6}\sum_{n=-\infty}^{n=+\infty}\frac{1}{(1+u_n^2)^3} \sim 1 + \frac{h_{M2}}{|\vec{x}_7|^6}\frac{1}{2\pi R_{11}/|\vec{x}_7|}\int_{-\infty}^{+\infty}\frac{du}{(1+u^2)^3} \\ &= 1 + \frac{h_{M2}\omega_{(5)}}{2\pi R_{11}\omega_{(4)}}\frac{1}{|\vec{x}_7|^5} \end{aligned}$$

$$\frac{h_{M2}\omega_{(5)}}{2\pi R_{11}\omega_{(4)}} = h_{D2}$$

$$H_p = 1 + h_p\sum_{n=-\infty}^{n=+\infty}\frac{1}{(|\vec{x}_{n+1}|^2 + (z + 2\pi n R)^2\ |)^{n/2}} \sim 1 + \frac{h_p\omega_{(n-1)}}{2\pi R\omega_{(n-2)}}\frac{1}{|\vec{x}_{n+1}|^{n-1}}$$

$$h_p' = \frac{h_p\omega_{(n-1)}}{2\pi R\omega_{(n-2)}}$$

$$h_p' = \frac{h_p\omega_{(n-1)}}{V^m\omega_{(n-m-1)}}, V^m = (2\pi)^m R_1 \dots R_m$$

$$\frac{h_{Dp}\omega_{(6-p)}}{2\pi R\omega_{(5-p)}} = \frac{(2\pi\ell_s)^{7-p}g}{2\pi R}\frac{\omega_{(6-p)}}{(7-p)\omega_{(8-p)}\omega_{(6-p)}}$$

$$\frac{\omega_{(n-1)}}{n\omega_{(n+1)}\omega_{(n-2)}} = \frac{1}{(n-1)\omega_{(n)}}$$

$$\begin{cases} \hat{\hat{e}}_{\underline{i}^2}{}^{j} = H_{M2}^{-1/3}\delta_i^j \\ \hat{\hat{e}}_{\underline{m}}^{n} = H_{M2}^{1/6}\delta_m^n \end{cases}$$

$$\begin{cases} \hat{\hat{\omega}}_{\underline{\underline{m}}}{}^{nl} &= -\frac{1}{3}H_{M2}^{-1}\partial_{\underline{q}}H_{M2}\eta_m^{[n}\eta^{p]q} \\ \hat{\hat{\omega}}_{\underline{\underline{2}}}{}^{mj} &= \frac{2}{3}H_{M2}^{-3/2}\partial_{\underline{q}}H_{M2}\eta_i^{[m}\eta^{j]q} \\ \hat{\hat{G}}_{\underline{m}ty^1y^2} &= \mp H_{M2}^{-2}\partial_{\underline{m}}H_{M2} \end{cases}$$

$$\begin{cases} \delta_{\hat{\hat{\epsilon}}}\hat{\hat{\psi}}_{\underline{i}} = \frac{1}{3}H_{M2}^{-3/2}\partial_{\underline{n}}H_{M2}\hat{\hat{\Gamma}}_{(i)}^{n}\left(1 \mp \frac{i}{2}\epsilon(i)jk\hat{\hat{\Gamma}}^{(i)jk}\right)\hat{\hat{\epsilon}} = 0, \\ \delta_{\hat{\hat{\epsilon}}}\hat{\hat{\psi}}_{\underline{\underline{m}}} \end{cases}$$

$$= 2\partial_{\underline{m}}\hat{\hat{\epsilon}} - \frac{1}{6}H_{M2}^{-1}\partial_{\underline{n}}H_{M2}\left[\hat{\hat{\Gamma}}^{mn} \mp i\left(\hat{\hat{\Gamma}}^{mn} + 2\delta^{mn}\right)\hat{\hat{\Gamma}}012\right]\hat{\hat{\epsilon}}$$

$$\left(1 \mp i\hat{\hat{\Gamma}}^{012}\right)\hat{\hat{\epsilon}} = 0$$

$$\delta_{\hat{\hat{\epsilon}}}\hat{\hat{\psi}}_{\underline{m}} = 2\left(\partial_{\underline{m}} + \frac{1}{6}H_{M2}^{-1}\partial_{\underline{m}}H_{M2}\right)\hat{\hat{\epsilon}}$$

$$\hat{\hat{\epsilon}} = H_{M2}^{-1/6}\hat{\hat{\epsilon}}_0, \left(1 \mp i\hat{\hat{\Gamma}}^{012}\right)\hat{\hat{\epsilon}}_0$$

$$\begin{cases} \hat{\hat{e}}_{\underline{i}}^{\,j} = H_{M5}^{-1/6}\delta_i{}^{j}, \\ \hat{\hat{e}}_{\underline{m}}{}^{n} = H_{M5}^{1/3}\delta_m{}^{n} \end{cases}$$

$$\begin{cases} \hat{\hat{\omega}}_{\underline{m}}{}^{nl} &= -\frac{2}{3}H_{M5}^{-1}\partial_{\underline{q}}H_{M2}\eta_m^{[n}\eta^{p]q} \\ \hat{\hat{\omega}}_{\underline{m}}{}^{mj} &= \frac{1}{3}H_{M5}^{-3/2}\partial_{\underline{q}}H_{M2}\eta_i^{[m}\eta^{j]q} \\ \hat{\hat{\hat{G}}}_{\underline{m_1}\cdots\underline{m_4}} &= \pm\epsilon_{m_1\cdots m_5}\partial_{\underline{m_5}}H_{M5} \end{cases}$$

$$\hat{\hat{\epsilon}} = H_{M5}^{-1/12}\hat{\hat{\epsilon}}_0, \left(1 \mp \hat{\hat{\Gamma}}^{012345}\right)\hat{\hat{\epsilon}}_0$$

$$\begin{cases} \delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} &= \left\{\partial_{\hat{\mu}} - \frac{1}{4}\ddot{\psi}_{\hat{\mu}} + \frac{i}{8}e^{\hat{\phi}}\frac{1}{(p+2)!}\hat{\epsilon}^{(p+2)}\hat{\Gamma}_{\hat{\mu}}\left(-\hat{\Gamma}_{11}\right)^{\frac{p+2}{2}}\right\}\hat{\epsilon} \\ \delta_{\hat{\epsilon}}\hat{\lambda} &= \left\{\partial\hat{\phi} - \frac{i}{4}e^{\hat{\phi}}\frac{(p-3)}{(p+2)!}\hat{\zeta}^{(p+2)}\left(-\hat{\Gamma}_{11}\right)^{\frac{p+2}{2}}\right\}\hat{\epsilon} \end{cases}$$

$$\begin{cases} \delta_{\hat{\varepsilon}}\hat{\zeta}_{\hat{\mu}} &= \left\{\partial_{\hat{\mu}} - \frac{1}{4}\psi_{\hat{\mu}} + \frac{1}{8}e^{\hat{\varphi}}\frac{1}{(p+2)!}\hat{\epsilon}^{(p+2)}\hat{\Gamma}_{\hat{\mu}}\mathcal{P}_{\frac{p+3}{2}}\right\}\hat{\varepsilon} \\ \delta_{\hat{\varepsilon}}\hat{\chi} &= \left\{\partial\hat{\varphi} + \frac{1}{4}e^{\hat{\varphi}}\left(\frac{(p-3)}{(p+2)!}\hat{\epsilon}^{(p+2)}\mathcal{P}_{\frac{p+3}{2}}\right\}\hat{\varepsilon} \end{cases}$$

$$\mathcal{P}_n = \begin{cases} \sigma^1, n \text{ par} \\ i\sigma^2, n \text{ impar} \end{cases}$$

$$\begin{cases} \ddot{\psi}_{\underline{i}} & = -\frac{1}{2} H_{Dp}^{-3/2} \partial_{\underline{n}} H_{Dp} \eta_{ij} \Gamma^{nj} \\ \psi_{\underline{m}} & = \frac{1}{2} H_{Dp}^{-1} \partial_{\underline{n}} H_{Dp} \eta_{mq} \hat{\Gamma}^{nq} \\ \hat{G}^{(p+2)} & = \mp e^{-\hat{\phi}_0} H_{Dp}^{\frac{p}{4}-2} \partial_{\underline{m}} H_{Dp} \hat{\Gamma}^{m} \hat{\Gamma}^{01\cdots p} \end{cases}$$

$$\text{IIA: } \hat{\epsilon} = H_{Dp}^{-1/8} \hat{\epsilon}_0, \quad \left[1 \mp i \hat{\Gamma}^{01\cdots p} \left(-\hat{\Gamma}_{11}\right)^{\frac{p+2}{2}}\right] \hat{\epsilon}_0 = 0$$

$$\text{IIB: } \hat{\varepsilon} = H_{Dp}^{-1/8} \hat{\varepsilon}_0, \quad \left(1 \pm i \hat{\Gamma}^{01\cdots p} \mathcal{P}_{\frac{p+3}{2}}\right) \hat{\varepsilon}_0 = 0$$

$$\begin{cases} \delta_{\hat{\epsilon}} \hat{\psi}_{\hat{\mu}} & = \left\{\partial_{\hat{\mu}} - \frac{1}{4}\left(\psi_{\hat{\mu}} + \frac{1}{2} \hat{H}_{\hat{\mu}} \mathcal{O}\right)\right\} \hat{\epsilon} \\ \delta_{\hat{\epsilon}} \hat{\lambda} & = \left\{\partial \partial \hat{\phi} - \frac{1}{12} \hat{H} \mathcal{O}\right\} \epsilon \end{cases}$$

$$\psi_{\underline{i}} = -H_{F1}^{-3/2} \partial_{\underline{m}} H_{F1} \Gamma_i^m$$
$$H_{\underline{i}} = \mp 2 H_{F1}^{-1} \partial_{\underline{m}} H_{F1} \Gamma^{01}$$
$$H_{\underline{m}} = \pm \epsilon_{ij} H_{F1}^{-3/2} \partial_{\underline{m}} H_{F1} \Gamma^{mj}$$

$$\hat{\epsilon} = H_{F1}^{1/4} \hat{\epsilon}_0, \left(1 \pm \hat{\Gamma}^{01} \mathcal{O}\right) \hat{\epsilon}_0$$

$$\begin{cases} \delta_{\hat{\epsilon}} \hat{\psi}_{\hat{\mu}} = \left\{\partial_{\hat{\mu}} - \frac{1}{4}\left(\widehat{\omega}_{\hat{\mu}} + \frac{1}{7!} e^{2\hat{\phi}} \hat{H}^{(7)} \hat{a}_1 \cdots \hat{a}_7 \right.\right. \\ \qquad \left.\left. \hat{\Gamma}_{\hat{\mu} \hat{a}_1 \cdots \hat{a}_7} \mathcal{O}\right)\right\} \hat{\epsilon} \\ \delta_{\hat{\epsilon}} \hat{\lambda} = \left\{\partial \partial \hat{\phi} + \frac{1}{2} \hat{H}^{(7)} \mathcal{O}\right\} \hat{\epsilon} \end{cases}$$

$$\hat{\epsilon} = \hat{\epsilon}_0, \left(1 \pm \hat{\Gamma}^{0\cdots 5} \mathcal{O}\right) \hat{\epsilon}_0$$

$$\begin{cases} d\tilde{s}_E^2 & = H_{F1}^{-6/7} dt^2 - H_{F1}^{1/7} d\vec{x}_8^2 \\ ds_S^2 & = H_{F1}^{-1} dt^2 - d\vec{x}_8^2 \\ A_t & = \pm\left(H_{F1}^{-1} - 1\right) \\ e^{-2(\phi - \phi_0)} & = H_{F1}^{1/2} \\ K/K_0 & = H_{F1}^{-1/2} \end{cases}$$

$$\begin{cases} d\tilde{s}_E^2 & = H_{S5}^{-2/3} dt^2 - H_{S5}^{1/3} d\vec{x}_4^2 \\ ds_S^2 & = dt^2 - H_{S5} d\vec{x}_4^2, \\ A_t & = \pm\left(H_{S5}^{-1} - 1\right), \\ e^{-2(\phi - \phi_0)} & = H_{S5}^{-1}, \\ K/K_0 & = 1 \end{cases}$$

$$
\begin{cases}
d\tilde{s}_E^2 &= H_{Dp}^{-\frac{7-p}{8-p}} dt^2 - H_{Dp}^{\frac{1}{8-p}} d\vec{x}_{9-p}^2 \\
ds_S^2 &= H_{Dp}^{-1/2} dt^2 - H_{Dp}^{1/2} d\vec{x}_4^2 \\
A_t &= \pm\left(H_{Dp}^{-1} - 1\right) \\
e^{-2(\phi-\phi_0)} &= H_{Dp}^{\frac{p-6}{4}} \\
K/K_0 &= H_{Dp}^{-p/4}
\end{cases}
$$

$5-$ pluridimensión $\|+| + + + + + - - -$

$$P_p\epsilon = \left(1 \pm \Gamma^{01\cdots p}\mathcal{O}_p\right)\epsilon$$

$$\left[P_p, P_{p'}\right] = 0$$

$$\mathrm{D}_p \perp \mathrm{D}_{(p+4)}(p)$$

$$q_{F1} \sim \int_{S^7} e^{-2\varphi\star}\hat{\mathcal{H}}$$

$$de^{-2\varphi\star}\hat{\mathcal{H}} = 0$$

$$q_{F1} \sim \int_{S^7} \left(e^{-2\varphi\star}\hat{\mathcal{H}} - {}^{\star}\hat{G}^{(3)}\hat{C}^{(0)} - \hat{G}^{(5)}\hat{C}^{(2)}\right)$$

$$\int_{S^5} \hat{G}^{(5)} \int_{S^2} \hat{C}^{(2)}$$

$$q_{F1} \sim \int_{S^2} dV^{(1)}$$

$$\mathrm{F1} \perp \mathrm{D}_p(0)$$

$$\mathrm{D}_p \perp \mathrm{D}_{p+2}(p-1), p \geq 1$$

$$\mathrm{D}_p \perp \mathrm{D}_{p+4}(p)$$

$$\mathrm{F1} \perp \ \mathrm{S5\ B}(0)$$

$$\mathrm{D}_p \perp \mathrm{D}_p(p-2), p \geq 2$$

$$\mathrm{D}_p \perp \ \mathrm{S5}(p-1), p \geq 1$$

$\mathrm{F1}\|\mathrm{S5},\ \mathrm{F1} \perp \mathrm{D}_p(0),$
$\mathrm{S5} \perp \ \mathrm{S5}(1), \mathrm{S5} \perp \ \mathrm{S5}(1), \mathrm{S5} \perp \mathrm{D}_p(p-1)(p > 1),$
$\mathrm{D}_p \perp \mathrm{D}_{p'}(m) p + p' = 4 + 2m,$
$\mathrm{W}\|\ \mathrm{F1},\ \mathrm{W}\|\mathrm{S5},\ \mathrm{W}\|\mathrm{D}_p,$
$\mathrm{KK} \perp D_p(p-2).$

M2 ⊥ M2(0), M2 ⊥ M5(1), M5 ⊥ M5(1), M5 ⊥ M5(3),
W||M2, W||M5,
KK||M2, KK ⊥ M2(0), KK||M5, KK ⊥ M5(1), KK ⊥ M5(3),
W||KK, W ⊥ KK(2), W ⊥ KK(4).

$$d\hat{s}_s^2 = H_{Dp}^{-1/2} H_{F1}^{-1} dt^2 - H_{Dp}^{+1/2} H_{F1}^{-1} dy^2 - H_{Dp}^{-1/2} d\vec{z}_p^2 - H_{Dp}^{+1/2} d\vec{x}_{8-p}^2,$$

$$e^{-2(\hat{\phi}^{\hat{\phi}}\hat{\phi}_0)} = H_{D_p^2}^{\frac{(p-3)}{2}} H_{F1},$$

$$\hat{C}^{(p+1)} t_{\underline{t}} \cdots\cdots \underline{\underline{z}}^p = \pm e^{-\hat{\phi}_0}\left(H_{Dp}^{-1} - 1\right),$$

$$\hat{B}_{t\underline{t}} = \pm(H_{F1}^{-1} - 1),$$

$$H_{Dp,F1} = 1 + \frac{h_{Dp,F1}}{|\vec{x}_{8-p}|^{6-p}}$$

$$\begin{cases} ds^2 = H^{\alpha}\left[W dt^2 - d\vec{y}_{p-1}^2 - dz^2\right] - H^{\beta}[W^{-1}d\rho^2 + \rho^2 d\Omega^2] \\ e^{-2(\phi-\phi_0)} = H^{\gamma} \\ A_{(p+1)\underline{ty^1\cdots y^{p-1}z}} = \alpha(H^{-1} - 1) \\ W = 1 + \frac{\omega}{\rho^n}, H = 1 + \frac{h}{\rho^n} \end{cases}$$

$$\begin{pmatrix} t \\ z \end{pmatrix} \to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

$$W dt^2 - dz^2 \to dt^2 - dz^2 + \cosh^2\gamma (W-1)(dt + \tanh^2\gamma dz)^2$$

$$H_W^{-1} dt^2 - H_W[dz - (H_W^{-1} - 1)dt]^2, H_W = 1 + \frac{h_W}{\rho^n}$$

$$ds^2 = H^{\alpha}\left\{H_W^{-1}dt^2 - H_W[dz - (H_W^{-1} - 1)dt]^2 - d\vec{y}_{p-1}^2\right\} - H^{\beta} d\vec{x}^2$$

$$ds^2 = H_W^{-1} dt^2 - H_W[dz + \alpha(H_W^{-1} - 1)dt]^2 - d\vec{x}_{d-2}^2,$$

$$H_W = 1 + \frac{h_W}{|\vec{x}_{d-2}|^{d-4}}, \alpha = \pm 1$$

$$H_W = 1 + \frac{h_W}{|\vec{x}_{d-2}|^{d-4}} \delta(u_\alpha). u_\alpha = \frac{1}{\sqrt{2}}(t - \alpha z)$$

$$h_W = -\alpha \frac{\sqrt{2}|p^z| 8\pi G_N^{(d)}}{(d-4)\omega_{(d-3)}}$$

$$\delta(u_\alpha) \sim -\alpha \frac{\sqrt{2}}{2\pi R_z}$$

$$h_W = \frac{|N| 8 G_N^{(d)}}{R_z^2 (d-4)\omega_{(d-3)}}$$

$$\left\{\begin{array}{c} d\hat{s}_S^2 = H_{D1}^{-1/2} H_{D5}^{-1/2} \{H_W^{-1} dt^2 - H_W [dy^1 + \alpha_W (H_W^{-1} - 1) dt]^2\} \\ -H_{D1}^{1/2} H_{D5}^{-1/2} d\vec{y}_4^2 - H_{D1}^{1/2} H_{D5}^{1/2} d\vec{x}_4^2, \\ e^{-2(\hat{\phi}-\hat{\phi}_0)} \\ \hat{C}_{D5}/H_{D1} \\ t\underline{\hat{t}y}^{(2)} = \alpha_{D1}(H_{D1}^{-1} - 1) \\ \hat{C}^{(6)}{}_{t\underline{t}^1 \cdots \underline{y}^5} = \alpha_{D5}(H_{D5}^{-1} - 1) \end{array}\right.$$

$$H_i = 1 + \frac{r_i^2}{|\vec{x}_4|^2}, i = D1, D5, W$$

$$r_{D5}^2 = N_{D5} h_{D5} = N_{D5} \ell_s^2 g$$

$$r_{D1}^2 = N_{D1} h_{D1} \frac{\omega_{(5)}}{V^4 \omega_{(1)}} = \frac{N_{D1} \ell_s^6 g}{V}, V \equiv R_2 \ldots R_5$$

$$r_W^2 = h_W \frac{\omega_{(5)}}{V^4 \omega_{(1)}} = \frac{N_W \ell_s^8 g^2}{R^2 V}$$

$$\begin{aligned} d\tilde{s}_E^2 &= (H_{D1} H_{D5} H_W)^{-2/3} dt^2 - (H_{D1} H_{D5} H_W)^{1/3} d\vec{x}_4^2 \\ ds_S^2 &= (H_{D1} H_{D5})^{-1/2} H_W^{-1} dt^2 - (H_{D1} H_{D5})^{1/2} d\vec{x}_4^2 \\ A^{(D1,D5,W)_t} &= \alpha_{D1,D5,W}\left(H_{D1,D5,W}^{-1} - 1\right) \\ K_V/K_{V0} &= H_{D1}/H_{D5}, e^{-2(\phi-\phi_0)} = K_R/K_{R0} = (H_{D1} H_{D5})^{-1/4} H_W^{1/2} \end{aligned}$$

$$\begin{aligned} A &= \omega_{(3)} \left(\lim_{|\vec{x}_4| \to 0} |\vec{x}_4|^6 H_{D1} H_{D5} H_W \right)^{1/2} = 2\pi^2 (r_{D1} r_{D5} r_W)^{1/2} \\ &= 2\pi^2 \sqrt{N_{D1} N_{D5} N_W} \frac{\ell_s^8 g^2}{RV} \end{aligned}$$

$$S = \frac{A}{4 G_N^{(5)}}, G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 RV} = \frac{\pi}{4} \frac{\ell_s^8 g^2}{RV}$$

$$S = 2\pi \sqrt{N_{D1} N_{D5} N_W}$$

$$M = \frac{N_{D1} R}{g \ell_s} + \frac{N_{D5} RV}{g \ell_s^6} + \frac{N_W}{R}$$

$$ds^2 = H^{-2} dt^2 - H d\vec{x}_4^2, H = H_{D1} = H_{D5} = H_W$$

$$\rho(E) \sim e^{\sqrt{\pi(c-24E_0)EL/3}}$$

$$\rho(E) = e^{2\pi\sqrt{N_{D1} N_{D5} N_W}}$$

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}, e_\mu{}^a e_\nu^b \eta_{ab} = g_{\mu\nu}$$

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma_{\mu\rho}{}^{\nu} \xi^\rho,$$
$$\mathcal{D}_\mu \xi^a = \partial_\mu \xi^a + \omega_{\mu b}{}^{a} \xi^b,$$
$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_\mu{}^{ab} \Gamma_{ab} \psi$$

$$[\nabla_\mu, \nabla_\nu] \xi^\rho = R_{\mu\nu\sigma}{}^{\rho}(\Gamma) \xi^\sigma + T_{\mu\nu}{}^{\sigma} \nabla_\sigma \xi^\rho$$
$$[\mathcal{D}_\mu, \mathcal{D}_\nu] \xi^a = R_{\mu\nu b}{}^{a}(\omega) \xi^b$$

$$R^\sigma_{\mu\nu\rho}(\Gamma) = 2\partial_{[\mu} \Gamma^\sigma_{\nu]\rho} + 2\Gamma_{[\mu|\lambda}{}^{\sigma} \Gamma_{\nu]\rho}$$
$$R_{\mu\nu a}{}^{b}(\omega) = 2\partial_{[\mu} \omega^b_{\nu]a} - 2\omega_{[\mu|a}{}^{c} \omega^b_{|\nu]c}$$

$$\nabla_\mu e^\mu_a = 0$$

$$\omega_{\mu a}{}^{b} = \Gamma_{\mu a}{}^{b} + e_a{}^{\nu} \partial_\mu e^b_\nu$$

$$R_{\mu\nu\rho}{}^{\sigma}(\Gamma) = e_\rho{}^{a} e^\sigma{}_b R_{\mu\nu a}{}^{b}(\omega)$$

$$\nabla_\mu g_{\rho\sigma} = 0$$

$$\Gamma^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\} + K^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g) + K^\rho_{\mu\nu}$$

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\sigma} \{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\}$$

$$K^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \{T_{\mu\sigma\nu} + T_{\nu\sigma\mu} - T_{\mu\nu\sigma}\}$$

$$\omega_{abc} = \omega_{abc}(e) + K_{abc}, \omega_{abc}(e) = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}, \Omega^c_{ab} = e_a{}^{\mu} e_b{}^{\nu} \partial_{[\mu} e^c{}_{\nu]}$$

$$\left\{ \hat{\hat{\Gamma}} \hat{a}, \hat{\hat{\Gamma}}^{\hat{b}} \right\} = +2\hat{\eta}^{\hat{a}\hat{b}}$$

$$\hat{\hat{\Gamma}}_{\hat{1}\hat{0}} = i\hat{\hat{\Gamma}}^{\hat{0}} \ldots \hat{\hat{\Gamma}}^{\hat{p}} \equiv -i\hat{\Gamma}_{1\hat{1}}$$

$$\hat{\hat{\Gamma}}^{\hat{a}\star} = -\hat{\hat{\Gamma}}^{\hat{a}}$$

$$\hat{\hat{\Gamma}}^{\hat{0}\dagger} = +\hat{\hat{\Gamma}}^{\hat{0}}.$$
$$\hat{\hat{\Gamma}}^{\hat{\imath}\dagger} = -\hat{\hat{\Gamma}}^{\hat{\imath}}, \hat{\imath} = 1, \ldots, 10.$$

$$\hat{\hat{\Gamma}}^{\hat{0}T} = -\hat{\hat{\Gamma}}^{\hat{0}}.$$
$$\hat{\hat{\Gamma}}\hat{\imath}T = +\hat{\hat{\Gamma}}^{\hat{\imath}}, \hat{\imath} = 1, \ldots, 10.$$

$$\hat{\hat{\Gamma}}^{\hat{}}_{\hat{0}} \hat{\Gamma}^{\hat{a}\hat{}} \hat{\hat{\hat{0}}} = \hat{\hat{\Gamma}}^{\hat{a}\dagger}$$

$$\hat{\hat{\mathcal{D}}} = i\hat{\hat{\Gamma}}^{0}$$

$$\hat{\hat{\mathcal{D}}} \hat{\hat{\Gamma}}^{\hat{a}_1 \ldots \hat{a}_n} \hat{\hat{\mathcal{D}}}^{-1} = (-1)^{[n/2]} \left(\hat{\hat{\Gamma}}^{\hat{a}_1 \ldots \hat{a}_n} \right)^\dagger$$

$$\hat{\hat{C}} = \hat{\hat{D}} = i\hat{\hat{\Gamma}}^0$$

$$\hat{\hat{C}}^T = \hat{\hat{C}}^\dagger = \hat{\hat{C}}^{-1} = -\hat{\hat{C}}$$

$$\hat{\hat{C}}\hat{\hat{\Gamma}}\hat{\hat{a}}\hat{\hat{C}}^{-1} = -\hat{\hat{\Gamma}}\hat{\hat{a}}^{\hat{2}}$$

$$\hat{\hat{C}}\hat{\hat{\Gamma}}^{\hat{a}_1\ldots\hat{\hat{a}}_n}\hat{\hat{C}}^{-1} = (-1)^{n+[n/2]}\left(\hat{\hat{\Gamma}}^{\hat{a}_1\ldots\hat{\hat{a}}_n}\right)^T$$

$$\bar{\hat{\hat{\lambda}}} = \hat{\hat{\lambda}}^\dagger\hat{\hat{D}}$$

$$\hat{\hat{\lambda}}^c = \hat{\hat{\lambda}}^T\hat{\hat{C}}$$

$$\bar{\hat{\hat{\lambda}}} = \hat{\hat{\lambda}}^c$$

$$\bar{\hat{\hat{\epsilon}}}\hat{\hat{a}}^{\hat{a}_1\cdots\hat{\hat{a}}_n}\hat{\hat{\psi}} = (-1)^{n+[n/2]}\bar{\hat{\hat{N}}}^{\hat{a}_1\cdots\hat{\hat{a}}_n}\hat{\hat{\epsilon}}$$

$$\left(\bar{\hat{\hat{\epsilon}}}\hat{\hat{\Gamma}}^{\hat{a}_1\cdots\hat{\hat{a}}_n}\hat{\hat{\psi}}\right)^\dagger = (-1)^{[n/2]}\bar{\hat{\hat{\Gamma}}}^{\hat{\hat{a}}_1\ldots\hat{\hat{a}}_n}\hat{\hat{\epsilon}}$$

$$\hat{\hat{\Gamma}}^{\hat{\hat{a}}_1\ldots\hat{\hat{a}}_n} = i\frac{(-1)^{[n/2]+1}}{(11-n)!}\hat{\hat{\epsilon}}^{\hat{\hat{a}}_1\ldots\hat{\hat{a}}_n\hat{\hat{b}}_1\ldots\hat{\hat{b}}_{11-n}}\hat{\hat{\Gamma}}_{\hat{\hat{b}}_1\ldots\hat{\hat{b}}_{11-n}}$$

$$\begin{cases}\hat{\hat{\Gamma}}^{\hat{a}} = \hat{\Gamma}^{\hat{a}}, \hat{a} = 0, \ldots, 9\\ \hat{\hat{\Gamma}}^{10} = +i\hat{\Gamma}^0 \ldots \hat{\Gamma}^9\end{cases}$$

$$\hat{\Gamma}_{11} = -\hat{\Gamma}^0 \ldots \hat{\Gamma}^9 = i\hat{\hat{\Gamma}}^{10}$$

$$\hat{\Gamma}_{11}\hat{\psi}^{(\pm)} = \pm\hat{\psi}^{(\pm)}$$

$$\hat{\Gamma}_{11} = \mathbb{I}_{16\times16} \otimes \sigma^3 = \begin{pmatrix}\mathbb{I}_{16\times16} & 0\\ 0 & -\mathbb{I}_{16\times16}\end{pmatrix}$$

$$\hat{\psi} = \begin{pmatrix}\hat{\psi}^{(+)}\\ \hat{\psi}^{(-)}\end{pmatrix}$$

$$\Gamma_{11}\hat{\Gamma}^{\hat{a}_1\cdots\hat{a}_n} = \frac{(-1)^{[(10-n)/2]+1}}{(10-n)!}\hat{\epsilon}^{\hat{a}_1\cdots\hat{a}_n\hat{b}_1\cdots\hat{b}_{10-n}}\hat{\Gamma}_{\hat{b}_1\cdots\hat{b}_{10-n}}$$

$$\begin{cases}\hat{\Gamma}^a = \Gamma^a \otimes \sigma^2, a = 0, \ldots, 8\\ \hat{\Gamma}^9 = \mathbb{I}_{16\times16} \otimes i\sigma^1\end{cases}$$

$$\Gamma^8 = \Gamma^0 \cdots \Gamma^7$$

$$\Gamma_{(8)9} = i\Gamma^8 = i\Gamma^0 \cdots \Gamma^7$$

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{abcd}\gamma^{abcd}$$

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1\cdots a_n b_1\cdots b_{4-n}}\gamma_{b_1\cdots b_{4-n}}\gamma_5$$

$$n^\mu n_\mu = \varepsilon, \begin{cases} \varepsilon = +1, & \Sigma \text{ espacio} \\ \varepsilon = -1, & \Sigma \text{ tiempo} \end{cases}$$

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu$$

$$\mathcal{K}_{\mu\nu} \equiv h_\mu{}^\alpha h_\nu{}^\beta \nabla_{(\alpha} n_{\beta)}$$

$$\mathcal{K}_{\mu\nu} = \frac{1}{2}£_n h_{\mu\nu}$$

$$\mathcal{K} = h^{\mu\nu}\mathcal{K}_{\mu\nu} = h^{\mu\nu}\nabla_\mu n_\nu$$

$$\begin{cases} x^1 = \rho_{n-1}\sin\ \varphi \\ x^2 = \rho_{n-1}\cos\ \varphi \\ x^3 = \rho_{n-2}\cos\ \theta_1 \\ \vdots\ \vdots \\ x^k = \rho_{n-k+1}\cos\ \theta_{k-2}, 3 \le k \le n+1 \end{cases}$$

$$\begin{cases} \rho_l = \left[(x^1)^2 + \cdots + \left(x^{n+1-l}\right)^2\right]^{1/2} = r\prod_{m=1}^{l}\ \sin\ \theta_{n-m}, \\ \rho_0 = r = [(x^1)^2 + \cdots + (x^{n+1})^2]^{1/2} \end{cases}$$

$$d\Omega^n \equiv d\varphi\prod_{i=1}^{n-1}\ \sin^i\ \theta_i d\theta_i$$

$$d\Omega^n = \frac{1}{n!\,r^{n+1}}\epsilon_{\mu_1\dots\mu_{n+1}}x^{\mu_{n+1}}dx^{\mu_1}\dots dx^{\mu_n}$$

$$\begin{cases} d^{n+1}x = r^n dr d\Omega^n, \\ r^n d\Omega^n = d^n y\sqrt{|g|} \end{cases}$$

$$\omega_{(n)} = \int_{S^n} d\Omega^n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$\Gamma(x+1) = x\Gamma(x), \Gamma(0) = 1, \Gamma(1/2) = \pi^{1/2}$$

$$\begin{aligned} d\vec{x}^2 &= d\rho_0^2 + \rho_0^2 d\theta_{n-1}^2 + \cdots + \rho_{n-2}^2 d\theta_1^2 + \rho_{n-1}^2 d\varphi^2 \\ &= dr^2 + r^2\{d\theta_{n-1}^2 + \sin^2\ \theta_{n-1}[d\theta_{n-2}^2 + \sin^2\ \theta_{n-2}(d\theta_{n-3}^2 + \sin^2\ \theta_{n-3}(\cdots \\ &\quad \cdots\sin^2\ \theta_2(d\theta_1^2 + \sin^2\ \theta_1 d\varphi^2)\cdots]\} \\ &= dr^2 + r^2 d\Omega_{(n)}^2 \end{aligned}$$

$$\boldsymbol{P} = P^A\boldsymbol{G}_A$$

$$[\boldsymbol{P}, \boldsymbol{Q}] = \boldsymbol{PQ} - (-1)^{pq}\boldsymbol{Q}$$

$$[\boldsymbol{P},\boldsymbol{Q}]=P^AQ^B[\boldsymbol{G}_A,\boldsymbol{G}_B],\ \text{si } P^A \text{ o } Q^B$$
$$[\boldsymbol{P},\boldsymbol{Q}]=P^AQ^B\{\boldsymbol{G}_A,\boldsymbol{G}_B\},\ \text{si } P^A \text{ y } Q^B$$

$$\begin{aligned}[\boldsymbol{G}_A,\boldsymbol{G}_B] &= \boldsymbol{G}_A\boldsymbol{G}_B-\boldsymbol{G}_B\boldsymbol{G}_A\\ \{\boldsymbol{G}_A,\boldsymbol{G}_B\} &= \boldsymbol{G}_A\boldsymbol{G}_B+\boldsymbol{G}_B\boldsymbol{G}_A\end{aligned}$$

$$\mathrm{D}\boldsymbol{Z}=\mathrm{d}\boldsymbol{Z}+[\boldsymbol{A},\boldsymbol{Z}]$$

$$\boldsymbol{A}\to\boldsymbol{A}'=g(\boldsymbol{A}-g^{-1}\,\mathrm{d}g)g^{-1}$$

$$\delta\boldsymbol{A}=-\mathrm{D}\boldsymbol{\lambda}$$

$$\langle\cdots\rangle_r:\underbrace{\mathfrak{g}\times\cdots\times\mathfrak{g}}_{r}\to\mathbb{C}$$

$$\langle\cdots\boldsymbol{P}\boldsymbol{Q}\cdots\rangle_r=(-1)^{pq}\langle\cdots\boldsymbol{Q}\boldsymbol{P}\cdots\rangle_r$$

$$\langle(g\boldsymbol{Z}_1g^{-1})\cdots(g\boldsymbol{Z}_rg^{-1})\rangle_r=\langle\boldsymbol{Z}_1\cdots\boldsymbol{Z}_r\rangle_r,$$

$$\langle[\boldsymbol{\lambda},\boldsymbol{Z}_1]\boldsymbol{Z}_2\cdots\boldsymbol{Z}_r\rangle_r+\cdots+\langle\boldsymbol{Z}_1\cdots\boldsymbol{Z}_{r-1}[\boldsymbol{\lambda},\boldsymbol{Z}_r]\rangle_r$$

$$\langle[\boldsymbol{A},\boldsymbol{Z}_1]\boldsymbol{Z}_2\cdots\boldsymbol{Z}_r\rangle_r+(-1)^{p_1}\langle\boldsymbol{Z}_1[\boldsymbol{A},\boldsymbol{Z}_2]\boldsymbol{Z}_3\cdots\boldsymbol{Z}_r\rangle_r+ \\ +\cdots+(-1)^{p_1+\cdots+p_{r-1}}\langle\boldsymbol{Z}_1\cdots\boldsymbol{Z}_{r-1}[\boldsymbol{A},\boldsymbol{Z}_r]\rangle_r$$

$$\langle\mathrm{D}(\boldsymbol{Z}_1\cdots\boldsymbol{Z}_r)\rangle_r=\mathrm{d}\langle\boldsymbol{Z}_1\cdots\boldsymbol{Z}_r\rangle_r$$

$$\langle\mathrm{D}(\boldsymbol{Z}_1\cdots\boldsymbol{Z}_r)\rangle_r=\langle(\mathrm{D}\boldsymbol{Z}_1)\boldsymbol{Z}_2\cdots\boldsymbol{Z}_r\rangle_r+(-1)^{p_1}\langle\boldsymbol{Z}_1(\mathrm{D}\boldsymbol{Z}_2)\boldsymbol{Z}_3\cdots\boldsymbol{Z}_r\rangle_r+ \\ +\cdots+(-1)^{p_1+\cdots+p_{r-1}}\langle\boldsymbol{Z}_1\cdots\boldsymbol{Z}_{r-1}(\mathrm{D}\boldsymbol{Z}_r)\rangle_r$$

$$\begin{aligned}[\boldsymbol{P}_a,\boldsymbol{P}_b] &= 0\\ [\boldsymbol{J}_{ab},\boldsymbol{P}_c] &= \eta_{cb}\boldsymbol{P}_a-\eta_{ca}\boldsymbol{P}_b\\ [\boldsymbol{J}_{ab},\boldsymbol{J}_{cd}] &= \eta_{cb}\boldsymbol{J}_{ad}-\eta_{ca}\boldsymbol{J}_{bd}+\eta_{db}\boldsymbol{J}_{ca}-\eta_{da}\boldsymbol{J}_{cb}\end{aligned}$$

$$\langle\boldsymbol{J}_{ab}\boldsymbol{P}_c\rangle=\varepsilon_{abc}$$

$$\langle[\boldsymbol{A},\boldsymbol{J}_{ab}]\boldsymbol{P}_c\rangle+\langle\boldsymbol{J}_{ab}[\boldsymbol{A},\boldsymbol{P}_c]\rangle$$

$$\boldsymbol{A}=e^a\boldsymbol{P}_a+\frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}$$

$$\omega_a^e\varepsilon_{ebc}+\omega_b^e\varepsilon_{aec}+\omega_c^e\varepsilon_{abe}=0$$

$$\mathrm{D}_\omega\varepsilon_{abc}=0$$

$$\delta_{abcd}^{efgh}=0$$

$$\begin{aligned}0 &= \omega^d{}_e\delta_{abcd}^{efgh}\\ &= \omega^d{}_e\left(\delta_a^e\delta_{bcd}^{fgh}-\delta_b^e\delta_{acd}^{fgh}+\delta_c^e\delta_{abd}^{fgh}-\delta_d^e\delta_{abc}^{fgh}\right)\\ &= \omega^d{}_a\delta_{bbd}^{fgh}-\omega^d{}_b\delta_{acd}^{fgh}+\omega^d{}_c\delta_{abd}^{fgh}\\ &= \omega^e{}_a\delta_{ebc}^{fgh}+\omega^e{}_b\delta_{aec}^{fgh}+\omega^e{}_c\delta_{abe}^{fgh}.\end{aligned}$$

$$\omega_a^e\varepsilon_{ebc}+\omega_b^e\varepsilon_{aec}+\omega^e{}_c\varepsilon_{abe}=0$$

$$\mathcal{Q}_{\mathrm{CS}}^{(2n+1)} \equiv (n+1)\int_0^1 dt\langle \boldsymbol{A}(t\,\mathrm{d}\boldsymbol{A} + t^2\boldsymbol{A}^2)^n\rangle$$

$$\mathcal{Q}_{\mathrm{CS}}^{(3)} = \left\langle \boldsymbol{A}\mathrm{d}\boldsymbol{A} + \frac{2}{3}\boldsymbol{A}^3\right\rangle$$

$$\mathcal{Q}_{\mathrm{CS}}^{(5)} = \left\langle \boldsymbol{A}(\mathrm{d}\boldsymbol{A})^2 + \frac{3}{2}\boldsymbol{A}^3\,\mathrm{d}\boldsymbol{A} + \frac{3}{5}\boldsymbol{A}^5\right\rangle$$

$$\mathrm{d}\mathcal{Q}_{\mathrm{CS}}^{(2n+1)} = \langle \boldsymbol{F}^{n+1}\rangle$$

$$\mathrm{d}\delta_{\mathrm{gauge}}\,\mathcal{Q}_{\mathrm{CS}}^{(2n+1)} = 0$$

$$\mathcal{Q}_{\mathrm{CS}}^{(2n+1)}(\boldsymbol{A}') = \mathcal{Q}_{\mathrm{CS}}^{(2n+1)}(\boldsymbol{A}) + (-1)^{n+1}\frac{n!\,(n+1)!}{(2n+1)!}\langle (g^{-1}\,\mathrm{d}g)^{2n+1}\rangle + \mathrm{d}\Omega_{\mathrm{fin}}^{(2n)}$$

$$\delta_{\mathrm{gauge}}\,\mathcal{Q}_{\mathrm{CS}}^{(2n+1)} = \mathrm{d}\Omega^{(2n)}$$

$$L_{\mathrm{YM}} = -\frac{1}{4}\langle \boldsymbol{F}\wedge\star\,\boldsymbol{F}\rangle$$

$$L_{\mathrm{CS}}^{(2n+1)} = (n+1)k\int_0^1 dt\langle \boldsymbol{A}(t\,\mathrm{d}\boldsymbol{A} + t^2\boldsymbol{A}^2)^n\rangle$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = \boldsymbol{J}_{ab}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = \eta_{cb}\boldsymbol{P}_a - \eta_{ca}\boldsymbol{P}_b$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = \eta_{cb}\boldsymbol{J}_{ad} - \eta_{ca}\boldsymbol{J}_{bd} + \eta_{db}\boldsymbol{J}_{ca} - \eta_{da}\boldsymbol{J}_{cb}$$

$$\boldsymbol{A} = \frac{1}{\ell}e^a\boldsymbol{P}_a + \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}$$

$$\boldsymbol{\lambda} = \frac{1}{\ell}\lambda^a\boldsymbol{P}_a + \frac{1}{2}\lambda^{ab}\boldsymbol{J}_{ab}$$

$$\delta e^a = \lambda^a{}_b e^b - \mathrm{D}_\omega\lambda^a$$

$$\delta\omega^{ab} = -\mathrm{D}_\omega\lambda^{ab} + \frac{1}{\ell^2}(\lambda^a e^b - \lambda^b e^a)$$

$$\boldsymbol{F} = \frac{1}{\ell}T^a\boldsymbol{P}_a + \frac{1}{2}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right)\boldsymbol{J}_{ab}$$

$$T^a = \mathrm{D}_\omega e^a$$

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb}$$

$$\left\langle \boldsymbol{J}_{a_1a_2}\cdots\boldsymbol{J}_{a_{2n-1}a_{2n}}\boldsymbol{P}_{a_{2n+1}}\right\rangle = \frac{2^n}{n+1}\varepsilon_{a_1\cdots a_{2n+1}}$$

$$\boldsymbol{e} = \frac{1}{\ell}e^a\boldsymbol{P}_a$$

$$\boldsymbol{\omega} = \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}$$

$$\boldsymbol{T} = \mathrm{d}\boldsymbol{e} + [\boldsymbol{\omega}, \boldsymbol{e}]$$

$$\boldsymbol{R} = \mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega}^2$$

$$\begin{aligned} \boldsymbol{T} &= \frac{1}{\ell} T^a \boldsymbol{P}_a \\ \boldsymbol{R} &= \frac{1}{2} R^{ab} \boldsymbol{J}_{ab} \end{aligned}$$

$$L_{\mathrm{CS}}^{(2n+1)} = (n+1)k \int_0^1 dt \langle (\boldsymbol{R} + t^2 \boldsymbol{e}^2)^n \boldsymbol{e} \rangle + \mathrm{d}B_{\mathrm{CS}}^{(2n)}$$

$$\begin{aligned} L_{\mathrm{CS}}^{(2n+1)} = & \frac{k}{\ell} \varepsilon_{a_1 \cdots a_{2n+1}} \int_0^1 dt \left(R^{a_1 a_2} + \frac{t^2}{\ell^2} e^{a_1} e^{a_2} \right) \times \cdots \times \\ & \times \left(R^{a_{2n-1} a_{2n}} + \frac{t^2}{\ell^2} e^{a_{2n-1}} e^{a_{2n}} \right) e^{a_{2n+1}} + \mathrm{d}B_{\mathrm{CS}}^{(2n)} \end{aligned}$$

$$Q_{\boldsymbol{A} \leftarrow \overline{\boldsymbol{A}}}^{(2n+1)} \equiv (n+1) \int_0^1 dt \langle \boldsymbol{\Theta} \boldsymbol{F}_t^n \rangle$$

$$\begin{aligned} \boldsymbol{\Theta} &\equiv \boldsymbol{A} - \overline{\boldsymbol{A}} \\ \boldsymbol{A}_t &\equiv \overline{\boldsymbol{A}} + t \boldsymbol{\Theta} \\ \boldsymbol{F}_t &\equiv \mathrm{d}\boldsymbol{A}_t + \boldsymbol{A}_t^2 \end{aligned}$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = \int_0^1 dt \frac{d}{dt} \langle \boldsymbol{F}_t^{n+1} \rangle$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = (n+1) \int_0^1 dt \left\langle \boldsymbol{F}_t^n \frac{d}{dt} \boldsymbol{F}_t \right\rangle$$

$$\frac{d}{dt} \boldsymbol{F}_t = \mathrm{D}_t \boldsymbol{\Theta}$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = (n+1) \int_0^1 dt \langle \boldsymbol{F}_t^n \mathrm{D}_t \boldsymbol{\Theta} \rangle$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = (n+1) \int_0^1 dt \langle \mathrm{D}_t (\boldsymbol{F}_t^n \boldsymbol{\Theta}) \rangle$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = (n+1) \mathrm{d} \int_0^1 dt \langle \boldsymbol{F}_t^n \boldsymbol{\Theta} \rangle$$

$$\langle \boldsymbol{F}^{n+1} \rangle - \left\langle \overline{\boldsymbol{F}}^{n+1} \right\rangle = \mathrm{d} Q_{\boldsymbol{A} \leftarrow \overline{\boldsymbol{A}}}^{(2n+1)}$$

$$\langle \boldsymbol{F}^{n+1} \rangle = \mathrm{d} Q_{\boldsymbol{A} \leftarrow 0}^{(2n+1)}$$

$$S_{\mathrm{T}}[\boldsymbol{A}, \overline{\boldsymbol{A}}] = k \int_M Q_{\boldsymbol{A} \leftarrow \overline{\boldsymbol{A}}}^{(2n+1)}$$

$$\begin{aligned} \delta_{\mathrm{dif}} \boldsymbol{A} &= -\pounds_\xi \boldsymbol{A} \\ \delta_{\mathrm{dif}} \overline{\boldsymbol{A}} &= -\pounds_\xi \overline{\boldsymbol{A}} \end{aligned}$$

$$\delta_{\text{gauge}}\boldsymbol{A} = -\mathrm{D}\boldsymbol{\lambda}$$
$$\delta_{\text{gauge}}\overline{\boldsymbol{A}} = -\overline{\mathrm{D}}\boldsymbol{\lambda}$$

$$S_{\mathrm{T}}^{(2n+1)}[\overline{\boldsymbol{A}},\boldsymbol{A}] = -S_{\mathrm{T}}^{(2n+1)}[\boldsymbol{A},\overline{\boldsymbol{A}}]$$

$$\boldsymbol{\Theta} \to -\boldsymbol{\Theta}$$
$$\boldsymbol{A}_t \to \boldsymbol{A}_{1-t}$$
$$\boldsymbol{F}_t \to \boldsymbol{F}_{1-t}$$

$$\int_0^1 f(t)dt = \int_0^1 f(1-t)dt$$

$$\delta S_{\mathrm{T}}^{(2n+1)} = (n+1)k\int_M \left(\langle\delta\boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle\delta\overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right) + \int_{\partial M}\Xi$$

$$\Xi \equiv n(n+1)k\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} = (n+1)\int_0^1 dt\langle\boldsymbol{\Theta}\boldsymbol{F}_t^n\rangle$$

$$\delta\boldsymbol{\Theta} = \delta\boldsymbol{A} - \delta\overline{\boldsymbol{A}}$$
$$\delta\boldsymbol{A}_t = \delta\overline{\boldsymbol{A}} + t\delta\boldsymbol{\Theta}$$
$$\delta\boldsymbol{F}_t = \mathrm{D}_t\delta\boldsymbol{A}_t$$

$$\delta Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} = (n+1)\int_0^1 dt\langle\delta\boldsymbol{\Theta}\boldsymbol{F}_t^n\rangle + n(n+1)\int_0^1 dt\langle\boldsymbol{\Theta}\mathrm{D}_t\delta\boldsymbol{A}_t\boldsymbol{F}_t^{n-1}\rangle$$

$$\langle\boldsymbol{\Theta}\mathrm{D}_t\delta\boldsymbol{A}_t\boldsymbol{F}_t^{n-1}\rangle = \langle\mathrm{D}_t\boldsymbol{\Theta}\delta\boldsymbol{A}_t\boldsymbol{F}_t^{n-1}\rangle + \mathrm{d}\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\frac{d}{dt}\boldsymbol{F}_t = \mathrm{D}_t\boldsymbol{\Theta}$$
$$\frac{d}{dt}\delta\boldsymbol{A}_t = \delta\boldsymbol{\Theta}$$

$$n\langle\mathrm{D}_t\boldsymbol{\Theta}\delta\boldsymbol{A}_t\boldsymbol{F}_t^{n-1}\rangle = \frac{d}{dt}\langle\delta\boldsymbol{A}_t\boldsymbol{F}_t^n\rangle - \langle\delta\boldsymbol{\Theta}\boldsymbol{F}_t^n\rangle$$

$$n\langle\boldsymbol{\Theta}\mathrm{D}_t\delta\boldsymbol{A}_t\boldsymbol{F}_t^{n-1}\rangle = \frac{d}{dt}\langle\delta\boldsymbol{A}_t\boldsymbol{F}_t^n\rangle - \langle\delta\boldsymbol{\Theta}\boldsymbol{F}_t^n\rangle + n\,\mathrm{d}\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\delta Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} = (n+1)\int_0^1 dt\frac{d}{dt}\langle\delta\boldsymbol{A}_t\boldsymbol{F}_t^n\rangle + n(n+1)\mathrm{d}\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\delta Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} = (n+1)\left(\langle\delta\boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle\delta\overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right) + n(n+1)\mathrm{d}\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\langle\boldsymbol{F}^n\boldsymbol{G}_A\rangle = 0$$
$$\left\langle\overline{\boldsymbol{F}}^n\boldsymbol{G}_A\right\rangle = 0$$

$$\left.\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle\right|_{\partial M}$$

$$\mathrm{d}\star J=0$$

$$\star J_{\text{gauge}}=n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$
$$\star J_{\text{dif}}=n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\mathrm{I}_\xi\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$Q_{\text{gauge}}(\boldsymbol{\lambda})=n(n+1)k\int_{\partial\Sigma}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$
$$Q_{\text{dif}}(\xi)=n(n+1)k\int_{\partial\Sigma}\int_0^1 dt\langle\mathrm{I}_\xi\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\delta_{\boldsymbol{\lambda}}Q_{\text{gauge}}(\boldsymbol{\eta})=-Q_{\text{gauge}}([\boldsymbol{\lambda},\boldsymbol{\eta}])$$
$$\delta_{\boldsymbol{\lambda}}Q_{\text{dif}}(\xi)=-Q_{\text{gauge}}(\pounds_\xi\boldsymbol{\lambda})$$

$$\{Q_{\boldsymbol{\eta}},Q_{\boldsymbol{\lambda}}\}=Q_{[\boldsymbol{\eta},\lambda]}$$

$$\mathrm{C}\left(S_{\mathrm{T}}^{(2n+1)}\right)=-S_{\mathrm{T}}^{(2n+1)}$$
$$\mathrm{PT}\left(S_{\mathrm{T}}^{(2n+1)}\right)=-S_{\mathrm{T}}^{(2n+1)}$$

$$\mathrm{CPT}\left(S_{\mathrm{T}}^{(2n+1)}\right)=S_{\mathrm{T}}^{(2n+1)}$$

$$\delta L_{\mathrm{T}}^{(2n+1)}=(n+1)k\left(\langle\delta\boldsymbol{A}\boldsymbol{F}^n\rangle-\left\langle\delta\overline{\boldsymbol{A}}\overline{\boldsymbol{F}}^n\right\rangle\right)+\mathrm{d}\Xi$$
$$\Xi=n(n+1)k\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\delta_{\text{gauge}}\boldsymbol{A}=-\mathrm{D}\boldsymbol{\lambda}$$
$$\delta_{\text{gauge}}\overline{\boldsymbol{A}}=-\overline{\mathrm{D}}\boldsymbol{\lambda}$$

$$\delta_{\text{gauge}}L_{\mathrm{T}}^{(2n+1)}=-(n+1)k\,\mathrm{d}\left(\langle\boldsymbol{\lambda}\boldsymbol{F}^n\rangle-\left\langle\boldsymbol{\lambda}\overline{\boldsymbol{F}}^n\right\rangle\right)+\mathrm{d}\Xi_{\text{gauge}}$$

$$\star J'_{\text{gauge}}\equiv(n+1)k\left(\langle\boldsymbol{\lambda}\boldsymbol{F}^n\rangle-\left\langle\boldsymbol{\lambda}\overline{\boldsymbol{F}}^n\right\rangle\right)-\Xi_{\text{gauge}}$$

$$\Xi_{\text{gauge}}=-n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle+(n+1)k\left(\langle\boldsymbol{\lambda}\boldsymbol{F}^n\rangle-\left\langle\boldsymbol{\lambda}\overline{\boldsymbol{F}}^n\right\rangle\right)$$

$$\star J'_{\text{gauge}}=n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\delta_{\text{dif}}\boldsymbol{A}=-\pounds_\xi\boldsymbol{A}$$
$$\delta_{\text{dif}}\overline{\boldsymbol{A}}=-\pounds_\xi\overline{\boldsymbol{A}}$$

$$\delta_{\text{dif}} L_{\mathrm{T}}^{(2n+1)} = -(n+1)k\left(\langle \pounds_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle \pounds_\xi \overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right) + \mathrm{d}\Xi_{\text{dif}}$$

$$\pounds_\xi \boldsymbol{A} = \mathrm{I}_\xi \boldsymbol{F} + \mathrm{D}\mathrm{I}_\xi \boldsymbol{A}$$

$$\langle \pounds_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle = \langle \mathrm{I}_\xi \boldsymbol{F}\boldsymbol{F}^n\rangle + \langle \mathrm{D}\mathrm{I}_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle$$

$$\langle \pounds_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle = \mathrm{d}\langle \mathrm{I}_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle$$

$$\delta_{\text{dif}} L_{\mathrm{T}}^{(2n+1)} = -(n+1)k\,\mathrm{d}\left(\langle \mathrm{I}_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle \mathrm{I}_\xi \overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right) + \mathrm{d}\Xi_{\text{dif}}$$

$$\begin{aligned}\delta_{\text{dif}}\, L_{\mathrm{T}}^{(2n+1)} &= -\pounds_\xi L_{\mathrm{T}}^{(2n+1)} \\ &= -\mathrm{d}\mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)},\end{aligned}$$

$$\star J'_{\text{dif}} \equiv (n+1)k\left(\langle \mathrm{I}_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle \mathrm{I}_\xi \overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right) - \Xi_{\text{dif}} - \mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)}$$

$$\begin{aligned}\Xi_{\text{dif}} + \mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)} &= -n(n+1)k\,\mathrm{d}\int_0^1 dt\langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle + \\ &+(n+1)k\left(\langle \mathrm{I}_\xi \boldsymbol{A}\boldsymbol{F}^n\rangle - \left\langle \mathrm{I}_\xi \overline{\boldsymbol{A}\boldsymbol{F}}^n\right\rangle\right)\end{aligned}$$

$$\star J'_{\text{dif}} = n(n+1)k\,\mathrm{d}\int_0^1 dt\langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$S_{\mathrm{T}}^{(2n+1)}[\boldsymbol{A},\overline{\boldsymbol{A}}] = S_{\mathrm{CS}}^{(2n+1)}[\boldsymbol{A}] - S_{\mathrm{CS}}^{(2n+1)}[\overline{\boldsymbol{A}}] + \int_{\partial M} \mathcal{B}^{(2n)}$$

$$-S_{\mathrm{CS}}^{(2n+1)}[\overline{\boldsymbol{A}}] = -\int_M L_{\mathrm{CS}}^{(2n+1)}(\overline{\boldsymbol{A}}) = \int_{-M} L_{\mathrm{CS}}^{(2n+1)}(\overline{\boldsymbol{A}})$$

$$L_{\mathrm{T}}^{(2n+1)}(\boldsymbol{A},\overline{\boldsymbol{A}}) = (n+1)k\int_0^1 dt\langle \boldsymbol{\Theta}\boldsymbol{F}_t^n\rangle$$

$$\mathrm{d}Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\overline{\boldsymbol{A}}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\tilde{\boldsymbol{A}}\leftarrow\boldsymbol{A}}^{(2n+1)} = 0$$

$$\begin{aligned}\mathrm{d}Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\overline{\boldsymbol{A}}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\tilde{\boldsymbol{A}}\leftarrow\boldsymbol{A}}^{(2n+1)} =& \langle \boldsymbol{F}^{n+1}\rangle - \left\langle \overline{\boldsymbol{F}}^{n+1}\right\rangle + \left\langle \overline{\boldsymbol{F}}^{n+1}\right\rangle + \\ & -\langle \tilde{\boldsymbol{F}}^{n+1}\rangle + \langle \tilde{\boldsymbol{F}}^{n+1}\rangle - \langle \boldsymbol{F}^{n+1}\rangle \\ =& 0.\end{aligned}$$

$$Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} + Q_{\overline{\boldsymbol{A}}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + Q_{\tilde{\boldsymbol{A}}\leftarrow\boldsymbol{A}}^{(2n+1)} = \mathrm{d}Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}}^{(2n)}$$

$$Q_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} = Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}}^{(2n+1)} + Q_{\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}}^{(2n+1)} + \mathrm{d}Q_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}}^{(2n)}$$

$$t^i \geq 0, i = 0, \ldots, r+1$$
$$\sum_{i=0}^{r+1} t^i = 1$$

$$\boldsymbol{A}_t = \sum_{i=0}^{r+1} t^i \boldsymbol{A}_i$$

$$\boldsymbol{F}_t = \mathrm{d}\boldsymbol{A}_t + \boldsymbol{A}_t^2$$

$$T_{r+1} = (\boldsymbol{A}_0 \boldsymbol{A}_1 \cdots \boldsymbol{A}_{r+1})$$

$$\begin{aligned} \mathrm{d}&: \Omega^a(M) \times \Omega^b(T_{r+1}) \rightarrow \Omega^{a+1}(M) \times \Omega^b(T_{r+1}) \\ \mathrm{d}_t&: \Omega^a(M) \times \Omega^b(T_{r+1}) \rightarrow \Omega^a(M) \times \Omega^{b+1}(T_{r+1}) \end{aligned}$$

$$l_t: \Omega^a(M) \times \Omega^b(T_{r+1}) \rightarrow \Omega^{a-1}(M) \times \Omega^{b+1}(T_{r+1})$$

$$\begin{aligned} l_t \boldsymbol{A}_t &= 0 \\ l_t \boldsymbol{F}_t &= \mathrm{d}_t \boldsymbol{A}_t \end{aligned}$$

$$\begin{aligned} \mathrm{d}^2 &= 0 \\ \mathrm{d}_t^2 &= 0 \\ [l_t, \mathrm{d}] &= \mathrm{d}_t \\ [l_t, \mathrm{d}_t] &= 0 \\ \{\mathrm{d}, \mathrm{d}_t\} &= 0 \end{aligned}$$

$$\int_{\partial T_{r+1}} \frac{l_t^p}{p!} \pi = \int_{T_{r+1}} \frac{l_t^{p+1}}{(p+1)!} \mathrm{d}\pi + (-1)^{p+q}\, \mathrm{d} \int_{T_{r+1}} \frac{l_t^{p+1}}{(p+1)!} \pi$$

$$\pi = \sum_p \alpha_p \left\langle \boldsymbol{A}_t^{a_p} \boldsymbol{F}_t^{b_p} (\mathrm{d}_t \boldsymbol{A}_t)^{c_p} (\mathrm{d}_t \boldsymbol{F}_t)^{d_p} \right\rangle$$

$$\begin{aligned} a_p + 2b_p + c_p + 2d_p &= m \\ c_p + d_p &= q \end{aligned}$$

$$(p+1)\mathrm{d}_t t_t^p \pi = l_t^{p+1}\, \mathrm{d}\pi - \mathrm{d} l_t^{p+1} \pi$$

$$\left[l_t^{p+1}, \mathrm{d}\right] = (p+1)\mathrm{d}_t l_t^p$$

$$\begin{aligned} [l_t^2, \mathrm{d}] &= l_t[l_t, \mathrm{d}] + [l_t, \mathrm{d}] l_t \\ &= l_t\, \mathrm{d}_t + \mathrm{d}_t l_t \\ &= 2\, \mathrm{d}_t l_t. \end{aligned}$$

$$\begin{aligned} \left[l_t^{k+2}, \mathrm{d}\right] &= l_t\left[l_t^{k+1}, \mathrm{d}\right] + [l_t, \mathrm{d}] l_t^{k+1} \\ &= (k+1) l_t\, \mathrm{d}_t l_t^k + \mathrm{d}_t l_t^{k+1} \\ &= (k+2)\mathrm{d}_t l_t^{k+1} \end{aligned}$$

$$(p+1) \int_{\partial T_{r+1}} l_t^p \pi = \int_{T_{r+1}} l_t^{p+1}\, \mathrm{d}\pi - \int_{T_{r+1}} \mathrm{d} l_t^{p+1}$$

$$\mathrm{d} \int_{T_s} \alpha = (-1)^s \int_{T_s} \mathrm{d}\alpha$$

$$(p+1)\int_{\partial T_{r+1}} l_t^p \pi = \int_{T_{r+1}} l_t^{p+1}\, \mathrm{d}\pi + (-1)^{p+q}\, \mathrm{d}\int_{T_{r+1}} l_t^{p+1}\pi$$

$$\pi = \langle \boldsymbol{F}_t^{n+1}\rangle$$

$$\int_{\partial T_{p+1}} \frac{l_t^p}{p!}\langle \boldsymbol{F}_t^{n+1}\rangle = (-1)^p\, \mathrm{d}\int_{T_{p+1}} \frac{l_t^{p+1}}{(p+1)!}\langle \boldsymbol{F}_t^{n+1}\rangle$$

$$\int_{\partial T_1} \langle \boldsymbol{F}_t^{n+1}\rangle = \mathrm{d}\int_{T_1} l_t\langle \boldsymbol{F}_t^{n+1}\rangle$$

$$\boldsymbol{A}_t = t^0\boldsymbol{A}_0 + t^1\boldsymbol{A}_1$$

$$\partial T_1 = (\boldsymbol{A}_1) - (\boldsymbol{A}_0)$$

$$\int_{\partial T_1} \langle \boldsymbol{F}_t^{n+1}\rangle = \langle \boldsymbol{F}_1^{n+1}\rangle - \langle \boldsymbol{F}_0^{n+1}\rangle$$

$$l_t\langle \boldsymbol{F}_t^{n+1}\rangle = (n+1)\langle (l_t\boldsymbol{F}_t)\boldsymbol{F}_t^n\rangle$$

$$\begin{aligned} l_t\boldsymbol{F}_t &= \mathrm{d}_t\boldsymbol{A}_t \\ &= \mathrm{d}t^0\boldsymbol{A}_0 + \mathrm{d}t^1\boldsymbol{A}_1 \\ &= \mathrm{d}t^1(\boldsymbol{A}_1 - \boldsymbol{A}_0). \end{aligned}$$

$$\langle \boldsymbol{F}_1^{n+1}\rangle - \langle \boldsymbol{F}_0^{n+1}\rangle = (n+1)\mathrm{d}\int_{T_1} \mathrm{d}t^1\langle (\boldsymbol{A}_1 - \boldsymbol{A}_0)\boldsymbol{F}_t^n\rangle$$

$$\langle \boldsymbol{F}_1^{n+1}\rangle - \langle \boldsymbol{F}_0^{n+1}\rangle = \mathrm{d}Q^{(2n+1)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}$$

$$\begin{aligned} Q^{(2n+1)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} &= \int_{(\boldsymbol{A}_0\boldsymbol{A}_1)} l_t\langle \boldsymbol{F}_t^{n+1}\rangle \\ &= (n+1)\int_0^1 \mathrm{d}t^1\langle (\boldsymbol{A}_1 - \boldsymbol{A}_0)\boldsymbol{F}_t^n\rangle. \end{aligned}$$

$$\int_{\partial T_2} l_t\langle \boldsymbol{F}_t^{n+1}\rangle = -\mathrm{d}\int_{T_2} \frac{l_t^2}{2}\langle \boldsymbol{F}_t^{n+1}\rangle$$

$$\boldsymbol{A}_t = t^0\boldsymbol{A}_0 + t^1\boldsymbol{A}_1 + t^2\boldsymbol{A}_2$$

$$\partial T_2 = (\boldsymbol{A}_1\boldsymbol{A}_2) - (\boldsymbol{A}_0\boldsymbol{A}_2) + (\boldsymbol{A}_0\boldsymbol{A}_1)$$

$$\int_{\partial T_2} l_t\langle \boldsymbol{F}_t^{n+1}\rangle = \int_{(\boldsymbol{A}_1\boldsymbol{A}_2)} l_t\langle \boldsymbol{F}_t^{n+1}\rangle - \int_{(\boldsymbol{A}_0\boldsymbol{A}_2)} l_t\langle \boldsymbol{F}_t^{n+1}\rangle + \int_{(\boldsymbol{A}_0\boldsymbol{A}_1)} l_t\langle \boldsymbol{F}_t^{n+1}\rangle$$

$$\int_{\partial T_2} l_t\langle \boldsymbol{F}_t^{n+1}\rangle = Q^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1} - Q^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} + Q^{(2n+1)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}$$

$$l_t^2\langle \boldsymbol{F}_t^{n+1}\rangle = n(n+1)\langle (\mathrm{d}_t\boldsymbol{A}_t)^2\boldsymbol{F}_t^{n-1}\rangle$$

$$\int_{T_2} \frac{l_t^2}{2}\langle \boldsymbol{F}_t^{n+1}\rangle = \mathcal{Q}^{(2n)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}$$

$$\mathcal{Q}^{(2n)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} \equiv n(n+1)\int_0^1 dt\int_0^t ds\langle(\boldsymbol{A}_2-\boldsymbol{A}_1)(\boldsymbol{A}_1-\boldsymbol{A}_0)\boldsymbol{F}_t^{n-1}\rangle$$

$$\boldsymbol{A}_t = \boldsymbol{A}_0 + s(\boldsymbol{A}_2-\boldsymbol{A}_1) + t(\boldsymbol{A}_1-\boldsymbol{A}_0)$$

$$\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1} - \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} + \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} = -\mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}$$

$$\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} = \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1} + \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} + \mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}$$

$$L_{\mathrm{T}}^{(2n+1)}(\boldsymbol{A},\overline{\boldsymbol{A}}) = k\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}$$

$$\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}} = \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}} + \mathcal{Q}^{(2n+1)}_{\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}} + \mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}}$$

$$\mathcal{Q}^{(2n)}_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}} \equiv n(n+1)\int_0^1 dt\int_0^t ds\big\langle(\boldsymbol{A}-\tilde{\boldsymbol{A}})(\tilde{\boldsymbol{A}}-\overline{\boldsymbol{A}})\boldsymbol{F}_{st}^{n-1}\big\rangle$$

$$\boldsymbol{A}_{st} = \overline{\boldsymbol{A}} + s(\boldsymbol{A}-\tilde{\boldsymbol{A}}) + t(\tilde{\boldsymbol{A}}-\overline{\boldsymbol{A}})$$

$$\boldsymbol{A} = \boldsymbol{a}_0 + \boldsymbol{a}_1,$$
$$\overline{\boldsymbol{A}} = \overline{\boldsymbol{a}}_0 + \overline{\boldsymbol{a}}_1$$

$$\mathcal{Q}^{(2+1}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} = \mathcal{Q}^{(2+1)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0} + \mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} + \mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1}$$

$$\mathcal{Q}^{(2n+1)}_{a_0\leftarrow\overline{\boldsymbol{a}}_0+\bar{a}_1} = \mathcal{Q}^{(2n+1)}_{a_0\leftarrow\overline{\boldsymbol{a}}_0} + \mathcal{Q}^{(2n+1)}_{\overline{\boldsymbol{a}}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} + \mathrm{d}\mathcal{Q}^{(2n)}_{a_0\leftarrow\overline{\boldsymbol{a}}_0\leftarrow\bar{a}_0+\overline{\boldsymbol{a}}_1}$$

$$\begin{aligned}\mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} =& \mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0} + \mathcal{Q}^{(2n+1)}_{\overline{\boldsymbol{a}}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} + \mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0} + \\ &+\mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} + \mathrm{d}\mathcal{Q}^{(2n)}_{a_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1}\end{aligned}$$

$$\begin{aligned}\mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} =& \mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0} - \mathcal{Q}^{(2n+1)}_{\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1\leftarrow\overline{\boldsymbol{a}}_0} + \mathcal{Q}^{(2n+1)}_{\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0} + \\ &+\mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1} + \mathrm{d}\mathcal{Q}^{(2n}_{\boldsymbol{a}_0+\boldsymbol{a}_1\leftarrow\boldsymbol{a}_0\leftarrow\overline{\boldsymbol{a}}_0+\overline{\boldsymbol{a}}_1}.\end{aligned}$$

$$\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}} = \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}} + \mathcal{Q}^{(2n+1)}_{\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}} + \mathrm{d}\mathcal{Q}^{(2n)}_{\boldsymbol{A}\leftarrow\tilde{\boldsymbol{A}}\leftarrow\overline{\boldsymbol{A}}}$$

$$\mathcal{Q}^{(2n+1)}_{\mathrm{CS}}(\boldsymbol{A}) = \mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow 0}$$

$$\mathcal{Q}^{(2n+1)}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}} = \mathcal{Q}^{(2n+1)}_{\mathrm{CS}}(\boldsymbol{A}) - \mathcal{Q}^{(2n+1)}_{\mathrm{CS}}(\overline{\boldsymbol{A}}) + \mathrm{d}\mathcal{B}^{(2n)}$$

$$\mathcal{B}^{(2n)} = -n(n+1)\int_0^1 dt\int_0^t ds\big\langle\boldsymbol{A}\overline{\boldsymbol{A}}\boldsymbol{F}_{st}^{n-1}\big\rangle$$

$$\boldsymbol{A}_{st} = s\boldsymbol{A} + (1-t)\overline{\boldsymbol{A}}$$
$$\boldsymbol{F}_{st} = \overline{\boldsymbol{F}} + \overline{\mathrm{D}}(s\boldsymbol{A}-t\overline{\boldsymbol{A}}) + (s\boldsymbol{A}-t\overline{\boldsymbol{A}})^2$$

$$
\begin{aligned}
[\boldsymbol{P}_a, \boldsymbol{P}_b] &= \boldsymbol{J}_{ab} \\
[\boldsymbol{J}_{ab}, \boldsymbol{P}_c] &= \eta_{cb}\boldsymbol{P}_a - \eta_{ca}\boldsymbol{P}_b \\
[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] &= \eta_{cb}\boldsymbol{J}_{ad} - \eta_{ca}\boldsymbol{J}_{bd} + \eta_{db}\boldsymbol{J}_{ca} - \eta_{da}\boldsymbol{J}_{cb}
\end{aligned}
$$

$$\langle \boldsymbol{J}_{a_1a_2} \cdots \boldsymbol{J}_{a_{2n-1}a_{2n}} \boldsymbol{P}_{a_{2n+1}} \rangle = \frac{2^n}{n+1} \varepsilon_{a_1 \cdots a_{2n+1}}$$

$$L_{\mathrm{G}}^{(2n+1)} = k \mathcal{Q}_{\boldsymbol{A} \leftarrow \bar{\boldsymbol{A}}}^{(2n+1)}$$

$$
\begin{aligned}
\bar{A} &= \bar{\omega} \\
A &= e + \omega
\end{aligned}
$$

$$
\begin{aligned}
\overline{\boldsymbol{F}} &= \overline{\boldsymbol{R}} \\
\boldsymbol{F} &= \boldsymbol{R} + e^2 + \boldsymbol{T}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{R} &= \mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega}^2 \\
\boldsymbol{T} &= \mathrm{d}\boldsymbol{e} + [\boldsymbol{\omega}, \boldsymbol{e}]
\end{aligned}
$$

$$\tilde{\boldsymbol{A}} = \omega$$

$$\mathcal{Q}_{e+\omega \leftarrow \bar{\omega}}^{(2n+1)} = \mathcal{Q}_{e+\omega \leftarrow \omega}^{(2n+1)} + \mathcal{Q}_{\omega \leftarrow \bar{\omega}}^{(2n+1)} + \mathrm{d}\mathcal{Q}_{e+\omega \leftarrow \omega \leftarrow \bar{\omega}}^{(2n)}$$

$$\mathcal{Q}_{e+\omega \leftarrow \boldsymbol{\omega}}^{(2n+1)} = (n+1) \int_0^1 dt \langle \boldsymbol{e} \boldsymbol{F}_t^n \rangle$$

$$
\begin{aligned}
\boldsymbol{A}_t &= \boldsymbol{\omega} + t\boldsymbol{e} \\
\boldsymbol{F}_t &= \boldsymbol{R} + t^2\boldsymbol{e}^2 + t\boldsymbol{T}
\end{aligned}
$$

$$\mathcal{Q}_{\boldsymbol{e}+\boldsymbol{\omega} \leftarrow \boldsymbol{\omega}}^{(2n+1)} = (n+1) \int_0^1 dt \langle \boldsymbol{e}(\boldsymbol{R} + t^2\boldsymbol{e}^2)^n \rangle$$

$$\mathcal{Q}_{\omega \leftarrow \bar{\omega}}^{(2n+1)} = 0$$

$$\mathcal{Q}_{e+\omega \leftarrow \omega \leftarrow \bar{\omega}}^{(2n)} = n(n+1) \int_0^1 dt \int_0^t ds \langle \boldsymbol{e}\boldsymbol{\theta}\boldsymbol{F}_{st}^{n-1} \rangle$$

$$\theta \equiv \omega - \bar{\omega}$$

$$
\begin{aligned}
\boldsymbol{A}_{st} &= \overline{\boldsymbol{\omega}} + s\boldsymbol{e} + t\boldsymbol{\theta} \\
\boldsymbol{F}_{st} &= \overline{\boldsymbol{R}} + \mathrm{D}_{\overline{\boldsymbol{\omega}}}(s\boldsymbol{e} + t\boldsymbol{\theta}) + s^2\boldsymbol{e}^2 + st[\boldsymbol{e}, \boldsymbol{\theta}] + t^2\boldsymbol{\theta}^2
\end{aligned}
$$

$$\mathcal{Q}_{\boldsymbol{e}+\boldsymbol{\omega} \leftarrow \boldsymbol{\omega} \leftarrow \overline{\boldsymbol{\omega}}}^{(2n)} = n(n+1) \int_0^1 dt \int_0^t ds \left\langle \boldsymbol{e}\boldsymbol{\theta}\left(\overline{\boldsymbol{R}} + t\mathrm{D}_{\bar{\omega}}\boldsymbol{\theta} + s^2\boldsymbol{e}^2 + t^2\boldsymbol{\theta}^2\right)^{n-1} \right\rangle$$

$$
\begin{aligned}
L_{\mathrm{G}}^{(2n+1)} = (n+1)k \int_0^1 dt \langle \boldsymbol{e}(\boldsymbol{R} + t^2\boldsymbol{e}^2)^n \rangle + \\
+ n(n+1)k \, \mathrm{d} \int_0^1 dt \int_0^t ds \left\langle \boldsymbol{e}\boldsymbol{\theta}\left(\overline{\boldsymbol{R}} + t\mathrm{D}_{\overline{\boldsymbol{\omega}}}\boldsymbol{\theta} + s^2\boldsymbol{e}^2 + t^2\boldsymbol{\theta}^2\right)^{n-1} \right\rangle
\end{aligned}
$$

$$\langle \boldsymbol{J}_{ab}(\boldsymbol{R}+\boldsymbol{e}^2)^{n-1}\boldsymbol{T}\rangle = 0$$
$$\langle \boldsymbol{P}_a(\boldsymbol{R}+\boldsymbol{e}^2)^{n}\rangle = 0$$

$$\mathcal{R}_{abc} \equiv \langle \boldsymbol{F}^{n-1}\boldsymbol{J}_{ab}\boldsymbol{P}_c\rangle$$

$$\mathcal{R}_{abc}T^c = 0$$
$$\mathcal{R}_{abc}\left(R^{ab}+\frac{1}{\ell^2}e^a e^b\right) = 0$$

$$\mathcal{R}_{abc} = \frac{2}{n+1}\varepsilon_{abca_1\cdots a_{2n-2}}\left(R^{a_1a_2}+\frac{1}{\ell^2}e^{a_1a_2}\right)\cdots$$
$$\cdots\left(R^{a_{2n-3}a_{2n-2}}+\frac{1}{\ell^2}e^{a_{2n-3}a_{2n-2}}\right)$$

$$\left.\int_0^1 dt\langle(\delta\overline{\boldsymbol{\omega}}+t\delta\boldsymbol{\theta}+t\delta\boldsymbol{e})(\boldsymbol{\theta}+\boldsymbol{e})\boldsymbol{F}_t^{n-1}\rangle\right|_{\partial M}$$

$$\boldsymbol{A}_t = \overline{\boldsymbol{\omega}}+t(\boldsymbol{e}+\boldsymbol{\theta})$$
$$\boldsymbol{F}_t = \overline{\boldsymbol{R}}+t\mathrm{D}_{\overline{\boldsymbol{\omega}}}(\boldsymbol{e}+\boldsymbol{\theta})+t^2(\boldsymbol{e}^2+[\boldsymbol{e},\boldsymbol{\theta}]+\boldsymbol{\theta}^2)$$

$$\delta\overline{\boldsymbol{\omega}}|_{\partial M} = 0$$

$$\left.\int_0^1 dt\left\langle t(\delta\boldsymbol{\theta}\boldsymbol{e}-\boldsymbol{\theta}\delta\boldsymbol{e})\left(\overline{\boldsymbol{R}}+t^2\boldsymbol{e}^2+t^2\boldsymbol{\theta}^2\right)^{n-1}\right\rangle\right|_{\partial M}$$

$$\delta\theta^{[ab}e^{c]} = \theta^{[ab}\delta e^{c]}$$

$$\overline{\boldsymbol{\omega}} \rightarrow \overline{\boldsymbol{\omega}}+\overline{\boldsymbol{e}}_{\mathrm{g}}$$

$$\mathcal{Q}^{(2n+1)}_{\omega\leftarrow\bar{\omega}} = 0 \rightarrow \mathcal{Q}^{(2n+1)}_{\omega+e_g\leftarrow\bar{\omega}+\bar{e}_g} \neq 0$$

$$\lambda_{\alpha_1}\cdots\lambda_{\alpha_n} = \lambda_{\gamma(\alpha_1,\ldots,\alpha_n)}$$

$$K^{\rho}_{\alpha_1\cdots\alpha_n} = \begin{cases}1, & \text{cuando } \rho=\gamma(\alpha_1,\ldots,\alpha_n)\\ 0, & \infty\end{cases}$$

$$K_{\alpha_1\cdots\alpha_n}{}^{\rho} = K_{\alpha_1\cdots\alpha_{n-1}}{}^{\sigma}K_{\sigma\alpha_n}{}^{\rho} = K_{\alpha_1\sigma}{}^{\rho}K_{\alpha_2\cdots\alpha_n}{}^{\sigma}$$

$$(\lambda_\alpha)_\mu{}^\nu = K_{\mu\alpha}{}^\nu$$

$$(\lambda_\alpha)_\mu{}^\sigma(\lambda_\beta)_\sigma{}^\nu = K_{\alpha\beta}{}^\sigma(\lambda_\sigma)_\mu{}^\nu = \left(\lambda_{\gamma(\alpha,\beta)}\right)_\mu{}^\nu$$

$$S_p\times S_q = \left\{\lambda_\gamma \mid \lambda_\gamma=\lambda_{\alpha_p}\lambda_{\alpha_q}, \text{con } \lambda_{\alpha_p}\in S_p, \lambda_{\alpha_q}\in S_q\right\}$$

$$0_S\lambda_\alpha = \lambda_\alpha 0_S = 0_S$$

$$\left[\boldsymbol{T}_{a_0},\boldsymbol{T}_{b_0}\right] = C_{a_0b_0}{}^{c_0}\boldsymbol{T}_{c_0}+C_{a_0b_0}{}^{c_1}\boldsymbol{T}_{c_1}$$
$$\left[\boldsymbol{T}_{a_0},\boldsymbol{T}_{b_1}\right] = C_{a_0b_1}{}^{c}\boldsymbol{T}_{c_1}$$
$$\left[\boldsymbol{T}_{a_1},\boldsymbol{T}_{b_1}\right] = C_{a_1b_1}{}^{c}\boldsymbol{T}_{c_0}+C_{a_1b_1}{}^{c_1}\boldsymbol{T}_{c_1}$$

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}$$

$$\begin{aligned}\left[\boldsymbol{T}_{(A,\alpha)}, \boldsymbol{T}_{(B,\beta)}\right] &\equiv \lambda_\alpha \lambda_\beta [\boldsymbol{T}_A, \boldsymbol{T}_B] \\ &= \lambda_{\gamma(\alpha,\beta)} C_{AB}{}^{C} \boldsymbol{T}_C \\ &= C_{AB}^{C} \boldsymbol{T}_{(C,\gamma(\alpha,\beta))}\end{aligned}$$

$$K_{\alpha\beta}^{\rho} = \begin{cases} 1, & \text{cuando } \rho = \gamma(\alpha,\beta) \\ 0, & \infty \end{cases}$$

$$\left[\boldsymbol{T}_{(A,\alpha)}, \boldsymbol{T}_{(B,\beta)}\right] = K_{\alpha\beta}{}^{\rho} C_{AB}{}^{C} \boldsymbol{T}_{(C,\rho)}$$

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}$$

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = -(-1)^{\mathfrak{q}(A)\mathfrak{q}(B)} C_{(B,\beta)(A,\alpha)}{}^{(C,\gamma)}$$

$$\begin{aligned} K_{i,N+1}^{j} &= K_{N+1,i}^{j} = 0 \\ K_{i,N+1}{}^{N+1} &= K_{N+1,i}^{N+1} = 1 \\ K_{N+1,N+1}^{j} &= 0 \\ K_{N+1,N+1}{}^{N+1} &= 1 \end{aligned}$$

$$\begin{aligned} \left[\boldsymbol{T}_{(A,i)}, \boldsymbol{T}_{(B,j)}\right] &= K_{ij}{}^{k} C_{AB}{}^{C} \boldsymbol{T}_{(C,k)} + K_{ij}{}^{N+1} C_{AB}{}^{C} \boldsymbol{T}_{(C,N+1)} \\ \left[\boldsymbol{T}_{(A,N+1)}, \boldsymbol{T}_{(B,j)}\right] &= C_{AB}{}^{C} \boldsymbol{T}_{(C,N+1)} \\ \left[\boldsymbol{T}_{(A,N+1)}, \boldsymbol{T}_{(B,N+1)}\right] &= C_{AB}{}^{C} \boldsymbol{T}_{(C,N+1)} \end{aligned}$$

$$\left[\boldsymbol{T}_{(A,i)}, \boldsymbol{T}_{(B,j)}\right] = C_{(A,i)(B,j)}{}^{(C,k)} \boldsymbol{T}_{(C,k)}$$

$$C_{(A,i)(B,j)}{}^{(C,k)} = K_{ij}{}^{k} C_{AB}{}^{C}$$

$$\boldsymbol{T}_{(A,N+1)} = 0_S \boldsymbol{T}_A = \boldsymbol{0}$$

$$C_{(A,i)(B,j)}{}^{(C,k)} = \begin{cases} 0, & \text{cuando } i+j \neq k \\ C_{AB}^{C}, & \text{cuando } i+j = k \end{cases}$$

$$S_{\mathrm{E}}^{(N)} = \{\lambda_\alpha, \alpha = 0, \dots, N, N+1\}$$

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{cuando } \alpha+\beta \leq N \\ \lambda_{N+1}, & \text{cuando } \alpha+\beta \geq N+1 \end{cases}$$

$$K_{\alpha\beta}^{\gamma} = \begin{cases} \delta_{\alpha+\beta}^{\gamma}, & \text{cuando } \alpha+\beta \leq N \\ \delta_{N+1}^{\gamma}, & \text{cuando } \alpha+\beta \geq N+1 \end{cases}$$

$$C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = \begin{cases} \delta_{\alpha+\beta}^{\gamma} C_{AB}{}^{C}, & \text{cuando } \alpha+\beta \leq N \\ \delta_{N+1}^{\gamma} C_{AB}^{C}, & \text{cuando } \alpha+\beta \geq N+1 \end{cases}$$

$$C_{(A,i)(B,j)}{}^{(C,k)} = \delta_{i+j}^{k} C_{AB}{}^{C}$$

$$\lambda^\alpha \lambda^\beta = \lambda^{\alpha+\beta}$$

$$\lambda^{\alpha} = 0 \text{ cuando } \alpha > N$$

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r$$

$$S_p \times S_q \subset \bigcap_{r \in i_{(p,q)}} S_r$$

$$W_p \equiv S_p \otimes V_p$$

$$\mathfrak{G}_{\mathrm{R}} \equiv \bigoplus_{p \in I} W_p$$

$$\begin{aligned}[W_p, W_q] &\subset (S_p \times S_q) \otimes [V_p, V_q] \\ &\subset \bigcap_{s \in i_{(p,q)}} S_s \otimes \bigoplus_{r \in i_{(p,q)}} V_r \\ &\subset \bigoplus_{r \in i_{(p,q)}} \left(\bigcap_{s \in i_{(p,q)}} S_s \right) \otimes V_r.\end{aligned}$$

$$\bigcap_{s \in i_{(p,q)}} S_s \subset S_r$$

$$[W_p, W_q] \subset \bigoplus_{r \in i_{(p,q)}} S_r \otimes V_r$$

$$[W_p, W_q] \subset \bigoplus_{r \in i_{(p,q)}} W_r$$

$$C_{(a_p,\alpha_p)(b_q,\beta_q)}{}^{(c_r,\gamma_r)} = K_{\alpha_p \beta_q}{}^{\gamma_r} C_{a_p b_q}{}^{c_r}$$

$$\begin{aligned}[V_0, V_0] &\subset V_0, \\ [V_0, V_1] &\subset V_1, \\ [V_1, V_1] &\subset V_0.\end{aligned}$$

$$\begin{aligned}S_0 &= \left\{ \lambda_{2m}, \text{con } m = 0, \dots, \left[\left[\frac{N}{2} \right]\right] \right\} \cup \{\lambda_{N+1}\}, \\ S_1 &= \left\{ \lambda_{2m+1}, \text{con } m = 0, \dots, \left[\left[\frac{N-1}{2} \right]\right] \right\} \cup \{\lambda_{N+1}\}\end{aligned}$$

$$\begin{aligned}S_0 \times S_0 &\subset S_0, \\ S_0 \times S_1 &\subset S_1, \\ S_1 \times S_1 &\subset S_0\end{aligned}$$

$$\mathfrak{G}_{\mathrm{R}} = W_0 \oplus W_1$$

$$\begin{aligned}W_0 &= S_0 \otimes V_0 \\ W_1 &= S_1 \otimes V_1\end{aligned}$$

$$C_{(a_p,\alpha_p)(b_q,\beta_q)}{}^{(c_r,\gamma_r)} = \begin{cases} \delta^{\gamma_r}_{\alpha_p+\beta_q} C_{a_p b_q}{}^{c_r}, & \text{cuando } \alpha_p + \beta_q \leq N \\ \delta^{\gamma_r}_{N+1} C_{a_p b_q} c_r, & \text{cuando } \alpha_p + \beta_q \geq N+1 \end{cases}$$

$$C_{(a_p,\alpha_p)(b_q,\beta_q)}{}^{(c_r,\gamma_r)} = \begin{cases} \delta^{\gamma_r}_{\alpha_p+\beta_q} C^{c_r}_{a_p b_q b_q}, & \text{cuando } \alpha_p + \beta_q \leq N \\ \delta^{\gamma_r}_{N+1} C_{a_p b_q}, & \text{cuando } \alpha_p + \beta_q \geq N+1 \end{cases}$$

$$N_0 = 2\left[\frac{N}{2}\right]$$

$$N_1 = 2\left[\frac{N-1}{2}\right] + 1$$

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4\},$$
$$S_1 = \{\lambda_1, \lambda_3, \lambda_4\}$$

$$\begin{aligned} [V_0, V_0] &\subset V_0, \\ [V_0, V_1] &\subset V_1, \\ [V_0, V_2] &\subset V_2, \\ [V_1, V_1] &\subset V_0 \oplus V_2, \\ [V_1, V_2] &\subset V_1, \\ [V_2, V_2] &\subset V_0 \oplus V_2 \end{aligned}$$

$$S_p = \left\{\lambda_{2m+p}, \text{con } m = 0, \dots, \left[\frac{N-p}{2}\right]\right\} \cup \{\lambda_{N+1}\}, p = 0,1,2$$

$$\begin{aligned} S_0 \times S_0 &\subset S_0, \\ S_0 \times S_1 &\subset S_1, \\ S_0 \times S_2 &\subset S_2, \\ S_1 \times S_1 &\subset S_0 \cap S_2, \\ S_1 \times S_2 &\subset S_1, \\ S_2 \times S_2 &\subset S_0 \cap S_2 \end{aligned}$$

$$C_{(a_p,\alpha_p)(b_q,\beta_q)}{}^{(c_r,\gamma_r)} = \begin{cases} \delta^{\gamma_q}_{\alpha_p+\beta_q} C_{a_p b_c}{}^{c_r}, & \text{cuando } \alpha_p + \beta_q \leq N \\ \delta^{\gamma_r}_{N+1} C_{a_p b_q}, & \text{cuando } \alpha_p + \beta_q \geq N+1 \end{cases}$$

$$C_{(a_p,\alpha_p)(b_q,\beta_q)}{}^{(c_r,\gamma_r)} = \begin{cases} \delta^{\gamma_r}_{\alpha_p+\beta_q} C^{c_r}_{a_p b_q b_r}, & \text{cuando } \alpha_p + \beta_q \leq N \\ \delta^{\gamma_r}_{N+1} C_{a_p b_q}, & \text{cuando } \alpha_p + \beta_q \geq N+1 \end{cases}$$

$$N_p = 2\left[\frac{N-p}{2}\right] + p, p = 0,1,2$$

$$[\boldsymbol{S}_{\kappa\lambda}, \boldsymbol{S}_{\mu\nu}] = -i(\delta_{\mu\lambda}\boldsymbol{S}_{\kappa\nu} - \delta_{\mu\kappa}\boldsymbol{S}_{\lambda\nu} + \delta_{\nu\lambda}\boldsymbol{S}_{\mu\kappa} - \delta_{\nu\kappa}\boldsymbol{S}_{\mu\lambda})$$

	a	b
a	a	b
b	b	a

$$[\boldsymbol{G}_i, \boldsymbol{G}_j] = i\varepsilon_{ijk}\boldsymbol{G}_k,$$

$$\boldsymbol{J}_i = a\boldsymbol{G}_i,$$
$$\boldsymbol{K}_i = b\boldsymbol{G}_i,$$

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = i\varepsilon_{ijk}\boldsymbol{J}_k,$$
$$[\boldsymbol{J}_i, \boldsymbol{K}_j] = i\varepsilon_{ijk}\boldsymbol{K}_k,$$
$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = i\varepsilon_{ijk}\boldsymbol{J}_k$$

$$\boldsymbol{M}_{ij} = \varepsilon_{ijk}\boldsymbol{J}_k,$$
$$\boldsymbol{M}_{i4} = \boldsymbol{K}_i,$$

$$[\boldsymbol{M}_{\kappa\lambda}, \boldsymbol{M}_{\mu\nu}] = -i(\delta_{\mu\lambda}\boldsymbol{M}_{\kappa\nu} - \delta_{\mu\kappa}\boldsymbol{M}_{\lambda\nu} + \delta_{\nu\lambda}\boldsymbol{M}_{\mu\kappa} - \delta_{\nu\kappa}\boldsymbol{M}_{\mu\lambda})$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = \boldsymbol{J}_{ab},$$
$$[\boldsymbol{J}^{ab}, \boldsymbol{P}_c] = \delta^{ab}_{ec}\boldsymbol{P}^e,$$
$$[\boldsymbol{P}_a, \boldsymbol{Z}_{b_1\cdots b_5}] = -\frac{1}{5!}\varepsilon_{ab_1\cdots b_5 c_1\cdots c_5}\boldsymbol{Z}^{c_1\cdots c_5},$$
$$[\boldsymbol{J}^{ab}, \boldsymbol{J}_{cd}] = \delta^{abf}_{\mathrm{ecd}}\boldsymbol{J}^e{}_f,$$
$$[\boldsymbol{J}^{ab}, \boldsymbol{Z}_{c_1\cdots c_5}] = \frac{1}{4!}\delta^{abe_1\cdots e_4}_{dc_1\cdots c_5}\boldsymbol{Z}^d_{e_1\cdots e_4},$$
$$[\boldsymbol{Z}^{a_1\cdots a_5}, \boldsymbol{Z}_{b_1\cdots b_5}] = \eta^{[a_1\cdots a_5][c_1\cdots c_5]}\varepsilon_{c_1\cdots c_5 b_1\cdots b_5 e}\boldsymbol{P}^e + \delta^{a_1\cdots a_5 e}_{db_1\cdots b_5}\boldsymbol{J}^d{}_e$$
$$-\frac{1}{3!\,3!\,5!}\varepsilon_{c_1\cdots c_{11}}\delta^{a_1\cdots a_5 c_4 c_5 c_6}_{d_1 d_2 d_3 b_1\cdots b_5}{}^{[c_1c_2c_3][d_1d_2d_3]}\boldsymbol{Z}^{c_7\cdots c_{11}},$$
$$[\boldsymbol{P}_a, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_a\boldsymbol{Q},$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{ab}\boldsymbol{Q},$$
$$[\boldsymbol{Z}_{abcde}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{abcde}\boldsymbol{Q},$$
$$\{\boldsymbol{Q}, \overline{\boldsymbol{Q}}\} = \frac{1}{8}\left(\Gamma^a\boldsymbol{P}_a - \frac{1}{2}\Gamma^{ab}\boldsymbol{J}_{ab} + \frac{1}{5!}\Gamma^{abcde}\boldsymbol{Z}_{abcde}\right)$$

Superespacios de $\mathfrak{G}_{\mathrm{R}}$	Generadores
$S_0 \otimes V_0$	$\boldsymbol{J}_{ab} = \lambda_0\boldsymbol{J}_{ab}^{((0.5)}$ $\boldsymbol{Z}_{ab} = \lambda_2\boldsymbol{J}_{ab}^{(0.\mathrm{sp})}$ $\mathbf{0}$ $= \lambda_3\boldsymbol{J}_{ab}^{(\mathrm{osp})}$
$S_1 \otimes V_1$	$\boldsymbol{Q} = \lambda_1\boldsymbol{Q}^{(\mathrm{osp})}$

$$\mathbf{0} = \lambda_3 \boldsymbol{Q}^{(0\text{sp})}$$

$$\boldsymbol{P}_a = \lambda_2 \boldsymbol{P}_a^{(\text{osp})}$$

$$\boldsymbol{Z}_{\text{abcde}} = \lambda_2 \boldsymbol{Z}_{\text{abcde}}^{(0\text{sp})}$$

$S_2 \otimes V_2$

$$\mathbf{0} = \lambda_3 \boldsymbol{P}_a^{(\text{osp})}$$

$$0 = \lambda_3 \boldsymbol{Z}_{\text{abcde}}^{(0\text{sp})}$$

$$[\boldsymbol{J}^{ab}, \boldsymbol{J}_{cd}] = \delta_{ecd}^{abf} \boldsymbol{J}_f^e,$$
$$[\boldsymbol{J}^{ab}, \boldsymbol{P}_c] = \delta_{ec}^{ab} \boldsymbol{P}^e,$$
$$[\boldsymbol{J}^{ab}, \boldsymbol{Z}_{cd}] = \delta_{ecd}^{aff} \boldsymbol{Z}_f^e,$$
$$\left[\boldsymbol{J}^{ab}, \boldsymbol{Z}_{c_1\cdots c_5}\right] = \frac{1}{4!} a_{dc_1\cdots c_5}^{abe_4} \boldsymbol{Z}_{e_1\cdots e_4}^d,$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{ab}\boldsymbol{Q},$$
$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = \mathbf{0},$$
$$[\boldsymbol{P}_a, \boldsymbol{Z}_{bc}] = \mathbf{0},$$
$$\left[\boldsymbol{P}_a, \boldsymbol{Z}_{b_1\cdots b_5}\right] = \mathbf{0},$$
$$[\boldsymbol{Z}_{ab}, \boldsymbol{Z}_{cd}] = \mathbf{0},$$
$$\left[\boldsymbol{Z}_{ab}, \boldsymbol{Z}_{c_1\cdots c_5}\right] = \mathbf{0},$$
$$\left[\boldsymbol{Z}_{a_1\cdots a_5}, \boldsymbol{Z}_{b_1\cdots b_5}\right] = \mathbf{0},$$
$$[\boldsymbol{P}_a, \boldsymbol{Q}] = \mathbf{0},$$
$$[\boldsymbol{Z}_{ab}, \boldsymbol{Q}] = \mathbf{0},$$
$$[\boldsymbol{Z}_{abcde}, \boldsymbol{Q}] = \mathbf{0},$$
$$\{\boldsymbol{Q}, \overline{\boldsymbol{Q}}\} = \frac{1}{8}\left(\Gamma^a \boldsymbol{P}_a - \frac{1}{2}\Gamma^{ab}\boldsymbol{Z}_{ab} + \frac{1}{5!}\Gamma^{abcde}\boldsymbol{Z}_{abcde}\right)$$

$$[\boldsymbol{P}_a, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_a \boldsymbol{Q}',$$
$$[\boldsymbol{Z}_{ab}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{ab} \boldsymbol{Q}',$$
$$[\boldsymbol{Z}_{abcde}, \boldsymbol{Q}] = -\frac{1}{2}\Gamma_{abcde} \boldsymbol{Q}'$$

Superespacios de $\mathfrak{G}_{\text{R}}$ Generadores

$S_0 \otimes V_0$

$$\boldsymbol{J}_{ab} = \lambda_0 \boldsymbol{J}_{ab}^{(\text{osp})}$$

$$\boldsymbol{Z}_{ab}$$

$$= \lambda_2 \boldsymbol{J}_{ab}^{(\text{osp})}$$

$$\mathbf{0}$$

$$= \lambda_4 \boldsymbol{J}_{ab}^{(\text{osp})}$$

$$\boldsymbol{Q} = \lambda_1 \boldsymbol{Q}^{(\text{osp })}$$

$S_1 \otimes V_1$ $\quad$ $\boldsymbol{Q}' = \lambda_3 \boldsymbol{Q}^{((\text{osp})}$

$$\mathbf{0} = \lambda_4 \boldsymbol{Q}^{((\text{s.p})}$$

$$\boldsymbol{P}_a = \lambda_2 \boldsymbol{P}_a^{(\text{osp})}$$

$$\boldsymbol{Z}_{\text{abcde}} = \lambda_2 \boldsymbol{Z}_{\text{abcde}}^{(\text{osp})}$$

$S_2 \otimes V_2$

$$\mathbf{0} = \lambda_4 \boldsymbol{P}_a^{(\text{osp})}$$

$$\mathbf{0} = \lambda_4 \boldsymbol{Z}_{\text{abcde}}^{(\text{osp})}$$

$$\lambda_\alpha \lambda_\beta = \lambda_{(\alpha+\beta)\text{mod}4}$$

$$S_0 = \{\lambda_0, \lambda_2\},$$
$$S_1 = \{\lambda_1, \lambda_3\},$$
$$S_2 = \{\lambda_0, \lambda_2\}.$$

$S_{\text{E}}^{(2)}$				$\mathbb{Z}_4$			
0	1	2	3	0	1	2	3
1	2	3	3	1	2	3	0
2	3	3	3	2	3	0	1
3	3	3	3	3	0	1	2

Subespacios de $\mathfrak{G}_{\mathrm{R}}$	Generadores
$S_0 \otimes V_0$	$\boldsymbol{J}_{ab} = \lambda_0 \boldsymbol{J}_{ab}^{((\mathrm{osp})\,)}$ $\boldsymbol{Z}_{ab} = \lambda_2 \boldsymbol{J}_{ab}^{(\mathrm{osp})\,)}$
$S_1 \otimes V_1$	$\boldsymbol{Q} = \lambda_1 \boldsymbol{Q}^{((0.5\mathrm{p})\,)}$ $\boldsymbol{Q}' = \lambda_3 \boldsymbol{Q}^{((\mathrm{asp}\,)}$
$S_2 \otimes V_2$	$\boldsymbol{P}'_a = \lambda_0 \boldsymbol{P}_a^{(\mathrm{osp}\,)}$ $abccde = \lambda_0 \boldsymbol{Z}_{abcde}^{(\mathrm{osp}\,)}$ $\boldsymbol{P}_a = \lambda_2 \boldsymbol{P}_a^{(\mathrm{osp})\,)}$ $a_{bccee} = \lambda_2 \boldsymbol{Z}_{abcde}^{(\mathrm{osp}\,)}$

$$\{\boldsymbol{Q},\boldsymbol{Q}\} \sim \boldsymbol{P} + \boldsymbol{Z}_2 + \boldsymbol{Z}_5,$$
$$\{\boldsymbol{Q}',\boldsymbol{Q}'\} \sim \boldsymbol{P} + \boldsymbol{Z}_2 + \boldsymbol{Z}_5,$$
$$\{\boldsymbol{Q},\boldsymbol{Q}'\} \sim \boldsymbol{P}' + \boldsymbol{J} + \boldsymbol{Z}'_5$$

$$[\boldsymbol{P},\boldsymbol{P}] \sim \boldsymbol{J}$$
$$[\boldsymbol{P}',\boldsymbol{P}'] \sim \boldsymbol{J}$$
$$[\boldsymbol{P},\boldsymbol{P}'] \sim \boldsymbol{Z}_2$$

$$[\boldsymbol{Z}_2,\boldsymbol{Z}_2] \sim \boldsymbol{J},$$
$$[\boldsymbol{Z}_2,\boldsymbol{Z}_5] \sim \boldsymbol{Z}'_5,$$
$$[\boldsymbol{Z}_2,\boldsymbol{Z}'_5] \sim \boldsymbol{Z}_5,$$
$$[\boldsymbol{Z}_5,\boldsymbol{Z}_5] \sim \boldsymbol{P}' + \boldsymbol{J} + \boldsymbol{Z}'_5,$$
$$[\boldsymbol{Z}_5,\boldsymbol{Z}'_5] \sim \boldsymbol{P} + \boldsymbol{Z}_2 + \boldsymbol{Z}_5,$$
$$[\boldsymbol{Z}'_5,\boldsymbol{Z}'_5] \sim \boldsymbol{P}' + \boldsymbol{J} + \boldsymbol{Z}'_5$$

$$[\boldsymbol{P},\boldsymbol{Q}] \sim \boldsymbol{Q}',$$
$$[\boldsymbol{Z}_2,\boldsymbol{Q}] \sim \boldsymbol{Q}',$$
$$[\boldsymbol{Z}_5,\boldsymbol{Q}] \sim \boldsymbol{Q}'$$
$$[\boldsymbol{P},\boldsymbol{Q}'] \sim \boldsymbol{Q},$$
$$[\boldsymbol{Z}_2,\boldsymbol{Q}'] \sim \boldsymbol{Q},$$
$$[\boldsymbol{Z}_5,\boldsymbol{Q}'] \sim \boldsymbol{Q},$$
$$[\boldsymbol{P}',\boldsymbol{Q}] \sim \boldsymbol{Q},$$
$$[\boldsymbol{J},\boldsymbol{Q}] \sim \boldsymbol{Q},$$
$$[\boldsymbol{Z}'_5,\boldsymbol{Q}] \sim \boldsymbol{Q},$$
$$[\boldsymbol{P}',\boldsymbol{Q}'] \sim \boldsymbol{Q}'$$
$$[\boldsymbol{J},\boldsymbol{Q}'] \sim \boldsymbol{Q}'$$
$$[\boldsymbol{Z}'_5,\boldsymbol{Q}'] \sim \boldsymbol{Q}'$$

$$\left|\boldsymbol{T}_{(A_1,\alpha_1)} \cdots \boldsymbol{T}_{(A_n,\alpha_n)}\right| \equiv \alpha_\gamma K_{\alpha_1 \cdots \alpha_n}{}^{\gamma} \left|\boldsymbol{T}_{A_1} \cdots \boldsymbol{T}_{A_n}\right|$$

$$\sum_{p=1}^{n} X^{(p)}_{A_0\cdots A_n} = 0$$

$$X^{(p)}_{A_0\cdots A_n} = (-1)^{q(A_0)\left(q(A_1)+\cdots+q(A_{p-1})\right)} C^{B}_{A_0 A_p} \times$$
$$\times \left|\boldsymbol{T}_{A_1}\cdots \boldsymbol{T}_{A_{p-1}}\boldsymbol{T}_B \boldsymbol{T}_{A_{p+1}}\cdots \boldsymbol{T}_{A_n}\right|,$$

$$X^{(p)}_{(A_0,\alpha_0)\cdots(A_n,\alpha_n)} = (-1)^{q(A_0,\alpha_0)\left(q(A_1,\alpha_1)+\cdots+q(A_{p-1},\alpha_{p-1})\right)} \times$$
$$\times C_{(A_0,\alpha_0)(A_p,\alpha_p)}{}^{(B,\beta)} \mid \boldsymbol{T}_{(A_1,\alpha_1)}\cdots \boldsymbol{T}_{(A_{p-1},\alpha_{p-1})}$$
$$\times \boldsymbol{T}_{(B,\beta)}\boldsymbol{T}_{(A_{p+1},\alpha_{p+1})}\cdots \boldsymbol{T}_{(A_n,\alpha_n)} \mid,$$

$$X^{(p)}_{(A_0,\alpha_0)\cdots(A_n,\alpha_n)} = \alpha_\gamma K_{\alpha_0\cdots\alpha_n}{}^{\gamma} X^{(p)}_{A_0\cdots A_n}$$

$$\sum_{p=1}^{n} X^{(p)}_{(A_0,\alpha_0)\cdots(A_n,\alpha_n)} = 0$$

$$\left|\boldsymbol{T}_{(A_1,\alpha_1)}\cdots \boldsymbol{T}_{(A_n,\alpha_n)}\right| = \sum_{m=0}^{M} \alpha_\gamma^{\beta_1\cdots\beta_m} K_{\beta_1\cdots\beta_m\alpha_1\cdots\alpha_n}{}^{\gamma}\left|\boldsymbol{T}_{A_1}\cdots \boldsymbol{T}_{A_n}\right|$$

$$\mathrm{STr}\left(\boldsymbol{T}_{(A_1,\alpha_1)}\cdots \boldsymbol{T}_{(A_n,\alpha_n)}\right) = K_{\gamma\alpha_1\cdots\alpha_n}{}^{\gamma}\mathrm{Str}\left(\boldsymbol{T}_{A_1}\cdots \boldsymbol{T}_{A_n}\right)$$

$$\left|\boldsymbol{T}_{(a_{p_1},\alpha_{p_1})}\cdots \boldsymbol{T}_{(a_{p_n},\alpha_{p_n})}\right| = \alpha_\gamma K^{\gamma}_{\alpha_{p_1}\cdots\alpha_{p_n}}\left|\boldsymbol{T}_{a_{p_1}}\cdots \boldsymbol{T}_{a_{p_n}}\right|$$

$$\left|\boldsymbol{T}_{(A_1,i_1)}\cdots \boldsymbol{T}_{(A_n,i_n)}\right| \equiv \alpha_j K_{i_1\cdots i_n}{}^{j}\left|\boldsymbol{T}_{A_1}\cdots \boldsymbol{T}_{A_n}\right|$$

$$Y^{(p)}_{(A_0,i_0)\cdots(A_n,i_n)} = (-1)^{q(A_0,i_0)\left(q(A_1,i_1)+\cdots+q(A_{p-1},i_{p-1})\right)} \times$$
$$\times C_{(A_0,i_0)(A_p,i_p)}{}^{(B,j)} \mid \boldsymbol{T}_{(A_1,i_1)}\cdots \boldsymbol{T}_{(A_{p-1},i_{p-1})}$$
$$\times \boldsymbol{T}_{(B,j)}\boldsymbol{T}_{(A_{p+1},i_{p+1})}\cdots \boldsymbol{T}_{(A_n,i_n)} \mid.$$

$$Y^{(p)}_{(A_0,i_0)\cdots(A_n,i_n)} = \alpha_k K_{i_0 i_p}{}^{j} K_{i_1\cdots i_{p-1} j i_{p+1}\cdots i_n}{}^{k} X^{(p)}_{A_0\cdots A_n}$$

$$K_{i_0 i_p}{}^{j} K_{i_1\cdots i_{p-1} j i_{p+1}\cdots i_n}{}^{k} = K_{i_0 i_p}{}^{\gamma} K_{i_1\cdots i_{p-1}\gamma i_{p+1}\cdots i_n}{}^{k} +$$
$$- K_{i_0 i_p}{}^{N+1} K_{i_1\cdots i_{p-1}(N+1) i_{p+1}\cdots i_n}{}^{k}$$
$$= K_{i_0\cdots i_n}{}^{k} - K_{i_0 i_p}{}^{N+1} K_{i_1\cdots i_{p-1}(N+1) i_{p+1}\cdots i_n}{}^{k}.$$

$$K_{i_0 i_p}{}^{j} K_{i_1\cdots i_{p-1} j i_{p+1}\cdots i_n}{}^{k} = K_{i_0\cdots i_n}{}^{k}$$

$$Y^{(p}_{(A_0,i_0)\cdots(A_n,i_n)} = \alpha_k K_{i_0\cdots i_n}{}^{k} X^{(p)}_{A_0\cdots A_n}$$

$$\sum_{p=1}^{n} Y^{(p)}_{(A_0,i_0)\cdots(A_n,i_n)} = 0$$

$$\left|\boldsymbol{T}_{(a_{p_1},i_{p_1})}\cdots \boldsymbol{T}_{(a_{p_n},i_{p_n})}\right| = \alpha_j K_{i_{p_1}\cdots i_{p_n}}{}^{j}\left|\boldsymbol{T}_{a_{p_1}}\cdots \boldsymbol{T}_{a_{p_n}}\right|$$

$$\mathrm{STr}\left(\boldsymbol{T}_{(A_1,i_1)}\cdots\boldsymbol{T}_{(A_n,i_n)}\right) = K_{j_1 i_1}{}^{2} K_{j_2 i_2}{}^{j_3}\cdots K_{j_{n-1} i_{n-1}}{}^{j_n}\times$$
$$\times K_{j_n i_n}{}^{1}\mathrm{Str}\left(\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_n}\right).$$

$$K_{j_1 i_1\cdots i_n}{}^{j_1} = K_{j_1 i_1}{}^{2} K_{\gamma_2 i_2}{}^{3}\cdots K_{\gamma_{n-1} i_{n-1}}{}^{\gamma_n} K_{\gamma_n i_n}{}^{j_1}$$
$$= K_{j_1 i_1}{}^{2} K_{j_2 i_2}{}^{2}\cdots K_{j_{n-1} i_{n-1}}{}_{n} K_{j_n i_n}{}_{1},$$

$$\mathrm{STr}\left(\boldsymbol{T}_{(A_1,i_1)}\cdots\boldsymbol{T}_{(A_n,i_n)}\right) = K_{j_1 i_1\cdots i_n}{}^{j_1}\mathrm{Str}\left(\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_n}\right)$$

$$\mathrm{STr}\left(\boldsymbol{T}_{(A_1,i_1)}\cdots\boldsymbol{T}_{(A_n,i_n)}\right) = K_{i_1\cdots i_n}{}^{0}\mathrm{Str}\left(\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_n}\right)$$

$$\mathrm{STr}\left(\boldsymbol{T}_{(A_1,0)}\cdots\boldsymbol{T}_{(A_n,0)}\right) = \mathrm{Str}\left(\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_n}\right)$$

$$\{\boldsymbol{Q},\overline{\boldsymbol{Q}}\} = 2\gamma^a\boldsymbol{P}_a$$

$$\{\boldsymbol{Q},\overline{\boldsymbol{Q}}\} = \frac{1}{8}\left(\Gamma^a\boldsymbol{P}_a - \frac{1}{2}\Gamma^{ab}\boldsymbol{Z}_{ab} + \frac{1}{5!}\Gamma^{abcde}\boldsymbol{Z}_{abcde}\right)$$

$$[\boldsymbol{J}^{ab},\boldsymbol{Z}_{cd}] = \delta^{abf}_{ecd}\boldsymbol{Z}^{e}_{f},$$
$$[\boldsymbol{J}^{ab},\boldsymbol{Z}_{c_1\cdots c_5}] = \frac{1}{4!}\delta^{abe_1\cdots e_4}_{dc_1\cdots c_5}\boldsymbol{Z}^{d}_{e_1\cdots e_4}$$

$$\lambda_\alpha\lambda_\beta = \begin{cases}\lambda_{\alpha+\beta}, & \text{cuando } \alpha+\beta\leq 2\\ \lambda_3, & \text{cuando } \alpha+\beta\geq 3\end{cases}$$

$$\left\langle\boldsymbol{T}_{(A_1,i_1)}\cdots\boldsymbol{T}_{(A_n,i_n)}\right\rangle = \alpha_j K_{i_1\cdots i_n}{}^{j}\left\langle\boldsymbol{T}_{A_1}\cdots\boldsymbol{T}_{A_n}\right\rangle$$

$$K_{i_1\cdots i_n}{}^{j} = \delta^{j}_{i_1+\cdots+i_n}$$

$$\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_6b_6}\right\rangle_{\mathrm{M}} = \alpha_0\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_6b_6}\right\rangle_{osp}$$
$$\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\right\rangle_{\mathrm{M}} = \alpha_2\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\right\rangle_{\mathrm{osp}},$$
$$\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{cd}\right\rangle_{\mathrm{M}} = \alpha_2\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{J}_{cd}\right\rangle_{\mathrm{osp}},$$
$$\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{c_1\cdots c_5}\right\rangle_{\mathrm{M}} = \alpha_2\left\langle\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{c_1\cdots c_5}\right\rangle_{osp},$$
$$\left\langle\boldsymbol{Q}\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_4b_4}\overline{\boldsymbol{Q}}\right\rangle_{\mathrm{M}} = \alpha_2\left\langle\boldsymbol{Q}\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_4b_4}\overline{\boldsymbol{Q}}\right\rangle_{\mathrm{osp}}$$

$$G = \begin{bmatrix} C_{\alpha\beta} & 0\\ 0 & 1\end{bmatrix}$$

$$\boldsymbol{P}_a = \begin{bmatrix}\frac{1}{2}(\Gamma_a)^\alpha & 0\\ 0 & 0\end{bmatrix},$$
$$\boldsymbol{J}_{ab} = \begin{bmatrix}\frac{1}{2}(\Gamma_{ab})^\alpha & 0\\ 0 & 0\end{bmatrix},$$
$$\boldsymbol{Z}_{abcde} = \begin{bmatrix}\frac{1}{2}(\Gamma_{abcde})^\alpha & 0\\ 0 & 0\end{bmatrix}$$
$$\boldsymbol{Q}^\gamma = \begin{bmatrix}0 & C^{\gamma\alpha}\\ \delta^\gamma_\beta & 0\end{bmatrix}$$

$$A = \begin{bmatrix} A^{\alpha}{}_{\beta} & A^{\alpha} \\ A_{\beta} & 0 \end{bmatrix}$$

$$\{A_1\} = A_1$$

$$\{A_1 \cdots A_n\} = \frac{1}{n}\sum_{p=1}^{n} A_p\{A_1 \cdots \hat{A}_p \cdots A_n\}$$

$$\langle A_1 \cdots A_6\rangle = \mathrm{STr}\{A_1 \cdots A_6\}$$

$$\langle A_1 \cdots A_6\rangle = \mathrm{Tr}\{A_1 \cdots A_6\}$$

$$\langle A_1 \cdots A_6\rangle = \mathrm{Tr}(\{A_1 \cdots A_5\}A_6)$$

$$\langle \chi\zeta A_1 \cdots A_4\rangle = -\frac{2}{5}\bar{\chi}\{A_1 \cdots A_4\}\zeta$$

$$\{A_1A_2\} = 2\mathrm{Tr}(A_1A_2)\mathbb{1} + A_1A_2\Gamma_{[4]},$$

$$\{A_1A_2A_3\} = \sum_{\langle ijk\rangle}\left(\mathrm{Tr}(A_iA_j)A_k - \frac{4}{3}[A_iA_jA_k]\right)\Gamma_{[2]} + A_1A_2A_3\Gamma_{[6]},$$

$$\{A_1 \cdots A_4\} = \sum_{\langle ijkl\rangle}\left(\frac{1}{2}\mathrm{Tr}(A_iA_j)\mathrm{Tr}(A_kA_l) - \frac{2}{3}\mathrm{Tr}(A_iA_jA_kA_l)\right)\mathbb{1} +$$

$$+\sum_{\langle ijkl\rangle}\left(\frac{1}{2}\mathrm{Tr}(A_iA_j)A_kA_l - \frac{4}{3}A_i[A_jA_kA_l]\right)\Gamma_{[4]} +$$

$$+(A_1 \cdots A_4)\Gamma_{[8]},$$

$$\{A_1 \cdots A_5\} = \sum_{\langle ijklm\rangle}\left(\frac{1}{2}\mathrm{Tr}(A_iA_j)\mathrm{Tr}(A_kA_l)A_m - \frac{2}{3}\mathrm{Tr}(A_iA_jA_kA_l)A_m +\right.$$

$$\left.-\frac{4}{3}\mathrm{Tr}(A_iA_j)[A_kA_lA_m] + \frac{32}{15}[A_iA_jA_kA_lA_m]\right)\Gamma_{[2]} +$$

$$+\sum_{\langle ijklm\rangle}\left(\frac{1}{6}\mathrm{Tr}(A_iA_j)A_kA_lA_m - \frac{2}{3}A_iA_j[A_kA_lA_m]\right)\Gamma_{[6]} +$$

$$+(A_1 \cdots A_5)\Gamma_{[10]}$$

$$\mathrm{Tr}(A_{i_1} \cdots A_{i_n}) = (A_{i_1})^{c_1}_{c_1}(A_{i_1})^{c_2}{}_{c_3} \cdots (A_{i_n})^{c_n}$$

$$[A_{i_1} \cdots A_{i_n}]^{ab} = (A_{i_1})^{a}{}_{c_1}(A_{i_2})^{c_1}_{1}{}_{c_2} \cdots (A_{i_n})^{c_{n-1}b}$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\text{osp}} = \frac{1}{2}\varepsilon_{a_1\cdots a_{11}} L_1^{a_1a_2}\cdots L_5^{a_9a_{10}} B_1^{a_{11}},$$

$$\langle \boldsymbol{J}^6\rangle_{\text{osp}} = \frac{1}{3}\sum_{\langle i_1\cdots i_6\rangle}\left[\frac{1}{4}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) + \right.$$

$$\left. -\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) + \frac{16}{15}\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}L_{i_6})\right],$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}\rangle_{\text{osp}} = \frac{1}{3}\varepsilon_{a_1\cdots a_{11}}\sum_{\langle i_1\cdots i_5\rangle}\left[-\frac{5}{4}L_{i_1}^{a_1a_2}\cdots L_{i_4}^{a_7a_8}(L_{i_5})_{bc}B_5^{bca_9a_{10}a_{11}} + \right.$$

$$+10L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}L_{i_3}^{a_5a_6}(L_{i_4})^{a_7}(L_{i_5})^{a_8}{}_c B_5^{ba_9a_{10}a_{11}} +$$

$$+\frac{1}{4}L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}L_{i_3}^{a_5a_6}B_5^{a_7\cdots a_{11}}\text{Tr}(L_{i_4}L_{i_5}) +$$

$$\left. -L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}[L_{i_3}L_{i_4}L_{i_5}]_5^{a_5a_6}B_5^{a_7\cdots a_{11}}\right],$$

$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\text{osp}} = -\frac{1}{240}\varepsilon_{a_1\cdots a_8abc}L_1^{a_1a_2}\cdots L_4^{a_7a_8}(\bar{\chi}\Gamma^{abc}\zeta) +$$

$$+\frac{1}{60}\sum_{\langle i_1\cdots i_4\rangle}\left[\frac{3}{4}\text{Tr}(L_{i_1}L_{i_2})L_{i_3}^{a_1a_2}L_{i_4}^{a_3a_4}(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta) + \right.$$

$$-2L_{i_1a_2}^{a_1}[L_{i_2}L_{i_3}L_{i_4}]^{a_3a_4}(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta) +$$

$$\left. +\frac{3}{4}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\bar{\chi}\zeta - \text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\bar{\chi}\zeta\right]$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_1^c\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\rangle_{\text{osp}}$$

$$\langle \boldsymbol{J}^6\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_6^{a_6b_6}\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_6b_6}\rangle_{\text{osp}}$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_5^{c_1\cdots c_5}\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{c_1\cdots c_5}\rangle_{\text{osp}}$$

$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_4^{a_4b_4}\bar{\chi}_\alpha\zeta^\beta\left\langle \boldsymbol{Q}^\alpha\boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_4b_4}\overline{\boldsymbol{Q}}_\beta\right\rangle_{\text{osp}}$$

$$\text{Tr}(A_{i_1}\cdots A_{i_n}) = (A_{i_1})^{c_1}{}_{c_2}(A_{i_1})^{c_2}\cdots(A_{i_n})^{c_n}{}_{c_1},$$

$$[A_{i_1}\cdots A_{i_n}]^{ab} = (A_{i_1})^a{}_{c_1}(A_{i_2})^{c_1}{}_{c_2}\cdots(A_{i_n})^{c_{n-1}b}.$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\text{osp}} = \frac{1}{2}\varepsilon_{a_1\cdots a_{11}} L_1^{a_1a_2}\cdots L_5^{a_9a_{10}} B_1^{a_{11}},$$

$$\langle \boldsymbol{J}^6\rangle_{\text{osp}} = \frac{1}{3}\sum_{\langle i_1\cdots i_6\rangle}\left[\frac{1}{4}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) + \right.$$

$$\left. -\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) + \frac{16}{15}\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}L_{i_6})\right]$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}\rangle_{\text{osp}} = \frac{1}{3}\varepsilon_{a_1\cdots a_{11}} \sum_{\langle i_1\cdots i_5\rangle} \Big[-\frac{5}{4}L_{i_1}^{a_1a_2}\cdots L_{i_4}^{a_7a_8}(L_{i_5})_{bc}B_5^{bca_9a_{10}a_{11}} +$$
$$+10L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}L_{i_3}^{a_5a_6}(L_{i_4})^{a_7}{}_b(L_{i_5})^{a_8}{}_cB_5^{bca_9a_{10}a_{11}} +$$
$$+\frac{1}{4}L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}L_{i_3}^{a_5a_6}B_5^{a_7\cdots a_{11}}\text{Tr}(L_{i_4}L_{i_5}) +$$
$$-L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}[L_{i_3}L_{i_4}L_{i_5}]^{a_5a_6}B_5^{a_7\cdots a_{11}}\Big],$$
$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\text{osp}} = -\frac{1}{240}\varepsilon_{a_1\cdots a_8abc}L_1^{a_1a_2}\cdots L_4^{a_7a_8}(\bar{\chi}\Gamma^{abc}\zeta) +$$
$$+\frac{1}{60}\sum_{\langle i_1\cdots i_4\rangle}\Big[\frac{3}{4}\text{Tr}(L_{i_1}L_{i_2})L_{i_3}^{a_1a_2}L_{i_4}^{a_3a_4}(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta) +$$
$$-2L_{i_1}^{a_1a_2}[L_{i_2}L_{i_3}L_{i_4}]^{a_3a_4}(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta) +$$
$$+\frac{3}{4}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\bar{\chi}\zeta - \text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\bar{\chi}\zeta\Big],$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_1^c\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\rangle_{\text{osp}},$$
$$\langle \boldsymbol{J}^6\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_6^{a_6b_6}\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_6b_6}\rangle_{\text{osp}},$$
$$\langle \boldsymbol{J}^5\boldsymbol{Z}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_5^{c_1\cdots c_5}\langle \boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_5b_5}\boldsymbol{Z}_{c_1\cdots c_5}\rangle_{\text{osp}},$$
$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\text{osp}} = L_1^{a_1b_1}\cdots L_4^{a_4b_4}\bar{\chi}_\alpha\zeta^\beta\left\langle \boldsymbol{Q}^\alpha\boldsymbol{J}_{a_1b_1}\cdots\boldsymbol{J}_{a_4b_4}\overline{\boldsymbol{Q}}_\beta\right\rangle_{\text{osp}}.$$

$$\langle \boldsymbol{J}^6\rangle_{\text{M}} = \frac{1}{3}\alpha_0\sum_{\langle i_1\cdots i_6\rangle}\Big[\frac{1}{4}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) +$$
$$-\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\text{Tr}(L_{i_5}L_{i_6}) + \frac{16}{15}\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}L_{i_6})\Big]$$
$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\text{M}} = \frac{1}{2}\alpha_2\varepsilon_{a_1\cdots a_{11}}L_1^{a_1a_2}\cdots L_5^{a_9a_{10}}B_1^{a_{11}}$$
$$\langle \boldsymbol{J}^5\boldsymbol{Z}_2\rangle_{\text{M}} =$$

$$\alpha_2\sum_{\langle i_1\cdots i_5\rangle}\Big[\frac{1}{2}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4})\text{Tr}(L_{i_5}B_2) +$$
$$-\frac{4}{3}\text{Tr}(L_{i_1}L_{i_2})\text{Tr}(L_{i_3}L_{i_4}L_{i_5}B_2) +$$

$$-\frac{2}{3}\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4})\text{Tr}(L_{i_5}B_2) +$$

$$+\frac{32}{15}\text{Tr}(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}B_2)\Big]$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}_5\rangle_{\mathrm{M}} = \frac{1}{3}\alpha_2\varepsilon_{a_1\cdots a_{11}}\sum_{\langle i_1\cdots i_5\rangle}\Big[-\frac{5}{4}L_{i_1}^{a_1a_2}\cdots L_{i_4}^{a_7a_8}\big(L_{i_5}\big)_{bc}B_5^{bca_9a_{10}a_{11}} + \\ +10L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}L_{i_3}^{a_5a_6}\big(L_{i_4}\big)^{a_7}{}_b\big(L_{i_5}\big)^{a_8}{}_cB_5^{bca_9a_{10}a_{11}} + \\ +\frac{1}{4}\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)L_{i_3}^{a_1a_2}L_{i_4}^{a_3a_4}L_{i_5}^{a_5a_6}B_5^{a_7\cdots a_{11}} + \\ -L_{i_1}^{a_1a_2}L_{i_2}^{a_3a_4}\big[L_{i_3}L_{i_4}L_{i_5}\big]^{a_5a_6}B_5^{a_7\cdots a_{11}}\Big],$$

$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\mathrm{M}} = -\frac{\alpha_2}{240}\varepsilon_{a_1\cdots a_8abc}L_1^{a_1a_2}\cdots L_4^{a_7a_8}\big(\bar{\chi}\Gamma^{abc}\zeta\big) + \\ +\frac{\alpha_2}{60}\sum_{\langle i_1\cdots i_4\rangle}\Big[\frac{3}{4}\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)L_{i_3}^{a_1a_2}L_{i_4}^{a_3a_4}\big(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta\big) + \\ -2L_{i_1}^{a_1a_2}\big[L_{i_2}L_{i_3}L_{i_4}\big]^{a_3}_{3a_4}\big(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta\big) + \\ +\frac{3}{4}\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)\mathrm{Tr}\big(L_{i_3}L_{i_4}\big)\bar{\chi}\zeta - \mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}\big)\bar{\chi}\zeta\Big],$$

$$\langle \boldsymbol{J}^6\rangle_{\mathrm{M}} = L_1^{a_1b_1}\cdots L_6^{a_6b_6}\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_6b_6}\rangle_{\mathrm{M}},$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle_{\mathrm{M}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_1^c\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_5b_5}\boldsymbol{P}_c\rangle_{\mathrm{M}},$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}_2\rangle_{\mathrm{M}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_2^{cd}\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_{55}b_5}\boldsymbol{Z}_{cd}\rangle_{\mathrm{M}}$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}_5\rangle_{\mathrm{M}} = L_1^{a_1b_1}\cdots L_5^{a_5b_5}B_5^{c_1\cdots c_5}\big\langle \boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_5}\,\boldsymbol{c}_{c_1c_5}\big\rangle_{\mathrm{M}},$$

$$\langle \boldsymbol{Q}\boldsymbol{J}^4\overline{\boldsymbol{Q}}\rangle_{\mathrm{M}} = L_1^{a_1b_1}\cdots L_4^{a_4b_4}\bar{\chi}_\alpha\zeta^\beta\langle \boldsymbol{Q}^\alpha\boldsymbol{J}_{a_1b_1}\cdots \boldsymbol{J}_{a_4b_4}\boldsymbol{Q}_\beta\rangle_{\mathrm{M}}.$$

$$\langle\cdots\rangle'_{\mathrm{M}} = \langle\cdots\rangle_{6=6} + \beta_{4+2}\langle\cdots\rangle_{6=4+2} + \beta_{2+2+2}\langle\cdots\rangle_{6=2+2+2}$$

$$\langle \boldsymbol{J}^6\rangle'_{\mathrm{M}} = \frac{1}{3}\alpha_0\sum_{\langle i_1\cdots i_6\rangle}\Big[\frac{1}{4}\gamma_5\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)\mathrm{Tr}\big(L_{i_3}L_{i_4}\big)\mathrm{Tr}\big(L_{i_5}L_{i_6}\big) + \\ -\kappa_{15}\mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}\big)\mathrm{Tr}\big(L_{i_5}L_{i_6}\big) + \frac{16}{15}\mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}L_{i_6}\big)\Big]$$

$$\langle \boldsymbol{J}^5\boldsymbol{P}\rangle'_{\mathrm{M}} = \frac{1}{2}\alpha_2\varepsilon_{a_1\cdots a_{11}}L_1^{a_1a_2}\cdots L_5^{a_9a_{10}}B_1^{a_{11}}$$

$$\langle \boldsymbol{J}^5\boldsymbol{Z}_2\rangle'_{\mathrm{M}} = \alpha_2\sum_{\langle i_1\cdots i_5\rangle}\Big[\frac{1}{2}\gamma_5\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)\mathrm{Tr}\big(L_{i_3}L_{i_4}\big)\mathrm{Tr}\big(L_{i_5}B_2\big) + \\ -\frac{4}{3}\kappa_{15}\mathrm{Tr}\big(L_{i_1}L_{i_2}\big)\mathrm{Tr}\big(L_{i_3}L_{i_4}L_{i_5}B_2\big) + \\ -\frac{2}{3}\kappa_{15}\mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}\big)\mathrm{Tr}\big(L_{i_5}B_2\big) + \\ +\frac{32}{15}\mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}L_{i_5}B_2\big)\Big],$$

$$
\begin{aligned}
\langle J^5 \boldsymbol{Z}_5\rangle'_{\mathrm{M}} =& \frac{1}{3}\alpha_2 \varepsilon_{a_1\cdots a_{11}} \sum_{\langle i_1\cdots i_5\rangle} \Big[-\frac{5}{4} L_{i_1}^{a_1a_2} \cdots L_{i_4}^{a_4a_8} \big(L_{i_5}\big)_{bc} b_5^{bca_9a_{10}a_{11}} + \\
&+10 L_{i_1}^{a_1a_2} L_{i_2a_4}^{a_3} L_{i_3}^{a_3a_6} \big(L_{i_4}\big)^{a_7}{}_b \big(L_{i_5}\big)^{a_8}{}_c B_5^{bca_9a_{10}a_{11}} + \\
&+\frac{1}{4}\kappa_{15} \mathrm{Tr}\big(L_{i_1}L_{i_2}\big) L_{i_3}^{a_1a_2} L_{i_4a_4a_4}^{a_{55}L_5a_6} B_5^{a_7\cdots a_{11}} + \\
&-L_{i_1a_2a_2}^{L_{i_2}^{a_3a_4}[L_{i_3}L_{i_4}L_{i_5}a_5a_6} B_5^{a_7\cdots a_{11}}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{Q} \boldsymbol{J}^4 \overline{\boldsymbol{Q}}\rangle'_{\mathrm{M}} =& -\frac{\alpha_2}{240} \varepsilon_{a_1\cdots a_8abc} L_1^{a_1a_2} \cdots L_4^{a_7a_8} \big(\bar{\chi}\Gamma^{abc}\zeta\big) + \\
&+\frac{\alpha_2}{60} \sum_{\langle i_1\cdots i_4\rangle} \Big\{\frac{3}{4}\kappa_9 \mathrm{Tr}\big(L_{i_1}L_{i_2}\big) L_{i_3}^{a_1a_2} L_{i_4}^{a_3a_4} \big(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta\big) + \\
&-2L_{i_1}^{a_2a_2} \big[L_{i_2}L_{i_3}L_{i_4}\big]_3^{a_3a_4} \big(\bar{\chi}\Gamma_{a_1\cdots a_4}\zeta\big) + \\
&+\frac{3}{4}(5\gamma_9 - 4) \mathrm{Tr}\big(L_{i_1}L_{i_2}\big) \mathrm{Tr}\big(L_{i_3}L_{i_4}\big) \bar{\chi}\zeta + \\
&-\kappa_3 \mathrm{Tr}\big(L_{i_1}L_{i_2}L_{i_3}L_{i_4}\big) \bar{\chi}\zeta\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\kappa_n &= 1 + \frac{1}{n}\beta_{4+2} \mathrm{Tr}(\mathbb{1}), \\
\gamma_n &= \kappa_n + \frac{1}{15}\beta_{2+2+2} [\mathrm{Tr}(\mathbb{1})]^2
\end{aligned}
$$

$$
\begin{aligned}
\beta_{4+2} &= \frac{1}{\mathrm{Tr}(\mathbb{1})} n(\kappa_n - 1), \\
\beta_{2+2+2} &= \frac{15}{[\mathrm{Tr}(\mathbb{1})]^2} (\gamma_n - \kappa_n)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_m &= 1 + \frac{n}{m}(\kappa_n - 1) \\
\gamma_m &= \gamma_n + \Big(\frac{n}{m} - 1\Big)(\kappa_n - 1)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{4+2} = 0 \quad &\Leftrightarrow \quad \kappa_n = 1 \\
\beta_{2+2+2} = 0 \quad &\Leftrightarrow \quad \gamma_n = \kappa_n.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5 + \overline{\boldsymbol{\psi}} \\
\overline{\boldsymbol{A}} &= \overline{\boldsymbol{\omega}} + \overline{\boldsymbol{e}} + \overline{\boldsymbol{b}}_2 + \overline{\boldsymbol{b}}_5 + \bar{\chi}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{2}\omega^{ab} \boldsymbol{J}_{ab}, \\
\boldsymbol{e} &= \frac{1}{\ell} e^a \boldsymbol{P}_a, \\
\boldsymbol{b}_2 &= \frac{1}{2} b_2^{ab} \boldsymbol{Z}_{ab}, \\
\boldsymbol{b}_5 &= \frac{1}{5!} b_5^{abcde} \boldsymbol{Z}_{abcde}, \\
\overline{\boldsymbol{\psi}} &= \bar{\psi}_\alpha \boldsymbol{Q}^\alpha
\end{aligned}
$$

$$
\boldsymbol{F} = \boldsymbol{R} + \boldsymbol{F}_P + \boldsymbol{F}_2 + \boldsymbol{F}_5 + \mathrm{D}_\omega \overline{\boldsymbol{\psi}}
$$

$$
\begin{aligned}
\boldsymbol{R} &= \frac{1}{2} R^{ab} \boldsymbol{J}_{ab}, \\
\boldsymbol{F}_P &= \left(\frac{1}{\ell} T^a + \frac{1}{16} \bar{\psi} \Gamma^a \psi\right) \boldsymbol{P}_a, \\
\boldsymbol{F}_2 &= \frac{1}{2}\left(\mathrm{D}_\omega b^{ab} - \frac{1}{16} \bar{\psi} \Gamma^{ab} \psi\right) \boldsymbol{Z}_{ab}, \\
\boldsymbol{F}_5 &= \frac{1}{5!}\left(\mathrm{D}_\omega b^{a_1 \cdots a_5} + \frac{1}{16} \bar{\psi} \Gamma^{a_1 \cdots a_5} \psi\right) \boldsymbol{Z}_{a_1 \cdots a_5}, \\
\mathrm{D}_\omega \overline{\boldsymbol{\psi}} &= \mathrm{D}_\omega \bar{\psi} \boldsymbol{Q}
\end{aligned}
$$

$$
\mathrm{D}_\omega \psi = \mathrm{d}\psi + \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi,
$$

$$
\mathrm{D}_\omega \bar{\psi} = \mathrm{d}\bar{\psi} - \frac{1}{4} \omega^{ab} \bar{\psi} \Gamma_{ab}
$$

$$
\begin{aligned}
\delta_{\text{gauge}} \boldsymbol{A} &= -\mathrm{D}\boldsymbol{\lambda}, \\
\delta_{\text{gauge}} \overline{\boldsymbol{A}} &= -\overline{\mathrm{D}}\boldsymbol{\lambda}
\end{aligned}
$$

$$
\boldsymbol{\lambda} = \frac{1}{2} \lambda^{ab} \boldsymbol{J}_{ab} + \frac{1}{\ell} \kappa^a \boldsymbol{P}_a + \frac{1}{2} \kappa^{ab} \boldsymbol{Z}_{ab} + \frac{1}{5!} \kappa^{abcde} \boldsymbol{Z}_{abcde} + \bar{\varepsilon} \boldsymbol{Q}
$$

- $\boldsymbol{\lambda} = (1/\ell) \kappa^a \boldsymbol{P}_a$,

$$
\begin{aligned}
\delta e^a &= -\mathrm{D}_\omega \kappa^a, \\
\delta b_2^{ab} &= 0, \\
\delta b_5^{abcde} &= 0, \\
\delta \omega^{ab} &= 0, \\
\delta \psi &= 0
\end{aligned}
$$

- $\boldsymbol{Z}_2 \colon \boldsymbol{\lambda} = (1/2) \kappa^{ab} \boldsymbol{Z}_{ab}$,

$$
\begin{aligned}
\delta e^a &= 0, \\
\delta b_2^{ab} &= -\mathrm{D}_\omega \kappa^{ab}, \\
\delta b_5^{abcde} &= 0, \\
\delta \omega^{ab} &= 0, \\
\delta \psi &= 0.
\end{aligned}
$$

- $\boldsymbol{Z}_5 \colon \boldsymbol{\lambda} = (1/5!) \kappa^{abcde} \boldsymbol{Z}_{abcde}$,

$$
\begin{aligned}
\delta e^a &= 0, \\
\delta b_2^{ab} &= 0, \\
\delta b_5^{abcde} &= -\mathrm{D}_\omega \kappa^{abcde}, \\
\delta \omega^{ab} &= 0, \\
\delta \psi &= 0.
\end{aligned}
$$

- $\boldsymbol{\lambda} = (1/2) \lambda^{ab} \boldsymbol{J}_{ab}$,

$$
\begin{aligned}
\delta e^a &= \lambda^a{}_b e^b, \\
\delta b_2^{ab} &= -2\lambda^{[a}{}_c b_2^{b]c}, \\
\delta b_5^{abcde} &= 5\lambda^{[a}{}_f b_5^{bced]}, \\
\delta \omega^{ab} &= -\mathrm{D}_\omega \lambda^{ab}, \\
\delta \psi &= \frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi.
\end{aligned}
$$

- $\boldsymbol{\lambda} = \bar{\varepsilon}\boldsymbol{Q}$,

$$
\begin{aligned}
\delta e^a &= \frac{\ell}{8}\bar{\varepsilon}\Gamma^a\psi, \\
\delta b_2^{ab} &= -\frac{1}{8}\bar{\varepsilon}\Gamma^{ab}\psi, \\
\delta b_5^{abcde} &= \frac{1}{8}\bar{\varepsilon}\Gamma^{abcde}\psi, \\
\delta \omega^{ab} &= 0, \\
\delta \psi &= -\mathrm{D}_\omega \varepsilon
\end{aligned}
$$

$$
L_{\mathrm{M}}^{(11)}(\boldsymbol{A}, \overline{\boldsymbol{A}}) = \mathcal{Q}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{A}}}^{(11)}
$$

$$
L_{\mathrm{M}}^{(11)}(\boldsymbol{A}, \overline{\boldsymbol{A}}) = \mathcal{Q}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{\omega}}}^{(11)} + \mathcal{Q}_{\overline{\boldsymbol{\omega}}\leftarrow\overline{\boldsymbol{A}}}^{(11)} + \mathrm{d}\mathcal{Q}_{\boldsymbol{A}\leftarrow\overline{\boldsymbol{\omega}}\leftarrow\overline{\boldsymbol{A}}}^{(10)}
$$

$$
\mathcal{Q}_{A\leftarrow\bar{\omega}}^{(11)} = \mathcal{Q}_{\boldsymbol{A}\leftarrow\omega}^{(11)} + \mathcal{Q}_{\omega\leftarrow\bar{\omega}}^{(11)} + \mathrm{d}\mathcal{Q}_{\boldsymbol{A}\leftarrow\omega\leftarrow\bar{\omega}}^{(10)}
$$

$$
L_{\mathrm{M}}^{(11)}(\boldsymbol{A}, \overline{\boldsymbol{A}}) = \mathcal{Q}_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}}^{(11)} - \mathcal{Q}_{\overline{\boldsymbol{A}}\leftarrow\bar{\omega}}^{(11)} + \mathcal{Q}_{\boldsymbol{\omega}\leftarrow\overline{\boldsymbol{\omega}}}^{(11)} + \mathrm{d}\mathcal{B}_{\mathrm{M}}^{(10)}
$$

$$
\mathcal{B}_{\mathrm{M}}^{(10)} = \mathcal{Q}_{\boldsymbol{A}\leftarrow\boldsymbol{\omega}\leftarrow\bar{\omega}}^{(10)} + \mathcal{Q}_{\boldsymbol{A}\leftarrow\bar{\omega}\leftarrow\overline{\boldsymbol{A}}}^{(10)}
$$

$$
\begin{aligned}
\boldsymbol{A}_0 &= \boldsymbol{\omega} \\
\boldsymbol{A}_1 &= \boldsymbol{\omega} + \boldsymbol{e} \\
\boldsymbol{A}_2 &= \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 \\
\boldsymbol{A}_3 &= \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5 \\
\boldsymbol{A}_4 &= \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{b}_2 + \boldsymbol{b}_5 + \overline{\boldsymbol{\psi}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_0}^{(11} &= \mathcal{Q}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_3}^{(11)} + \mathcal{Q}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0}^{(11)} + \mathrm{d}\mathcal{Q}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0}^{(10)} \\
\mathcal{Q}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_0}^{(11)} &= \mathcal{Q}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_2}^{(11)} + \mathcal{Q}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} + \mathrm{d}\mathcal{Q}_{\boldsymbol{A}_3\leftarrow\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} \\
\mathcal{Q}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_0} &= \mathcal{Q}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1} + \mathcal{Q}_{\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0} + \mathrm{d}\mathcal{Q}_{\boldsymbol{A}_2\leftarrow\boldsymbol{A}_1\leftarrow\boldsymbol{A}_0}
\end{aligned}
$$

$$
\mathcal{Q}_{\boldsymbol{A}_4\leftarrow\boldsymbol{A}_0}^{(11)} = 6\left[H_a e^a + \frac{1}{2}H_{ab}b_2^{ab} + \frac{1}{5!}H_{abcde}b_5^{abcde} - \frac{5}{2}\bar{\psi}\mathcal{R}\mathrm{D}_\omega\psi\right]
$$

$$
\begin{aligned}
H_a &\equiv \langle \boldsymbol{R}^5\boldsymbol{P}_a \rangle_{\mathrm{M}}, \\
H_{ab} &\equiv \langle \boldsymbol{R}^5\boldsymbol{Z}_{ab} \rangle_{\mathrm{M}}, \\
H_{abcce} &\equiv \langle \boldsymbol{R}^5\boldsymbol{Z}_{abcde} \rangle_{\mathrm{M}}, \\
\mathcal{R}^\alpha{}_\beta &\equiv \left\langle \boldsymbol{Q}^\alpha\boldsymbol{R}^4\overline{\boldsymbol{Q}}_\beta \right\rangle_{\mathrm{M}},
\end{aligned}
$$

$$H_a = \frac{\alpha_2}{64} R_a^{(5)},$$
$$H_{ab} = \alpha_2 \left[\frac{5}{2}\left(R^4 - \frac{3}{4}R^2R^2\right)R_{ab} + 5R^2R_{ab}^3 - 8R_{ab}^5\right]$$
$$H_{abcde} = -\frac{5}{16}\alpha_2 \left[5R_{[ab}R_{cde]}^{(4)} - 40R^f_{[a}R^g{}_b R_{cde]fg}^{(3)} + \right.$$
$$\left. -R^2R_{abcde}^{(3)} + 4R_{abcdefg}^{(2)}(R^3)^{fg}\right],$$
$$\mathcal{R} = -\frac{\alpha_2}{40}\left\{\left(R^4 - \frac{3}{4}R^2R^2\right)\mathbb{1} + \frac{1}{96}R_{abc}^{(4)}\Gamma^{abc} + \right.$$
$$\left. -\frac{3}{4}\left[R^2R^{ab} - \frac{8}{3}(R^3)^{ab}\right]R^{cd}\Gamma_{abcd}\right\}$$

$$R^n = R^{a_1}_{a_{a_2}\cdots} \cdots R^{a_n}_{a_1}$$
$$R^n_{ab} = R_{ac_1}{}^{c_1 c}{}_{c_2} \cdots R^{c_{n-1}}{}_b$$
$$R^{(n)}_{a_1 \cdots a_{d-2n}} = \varepsilon_{a_1 \cdots a_{d-2n} b_1 \cdots b_{2n}} R^{b_1 b_2} \ldots R^{b_{2n-1} b_{2n}}$$

$$\mathcal{Q}^{(11)}_{\omega \leftarrow \bar{\omega}} = 3 \int_0^1 dt \theta^{ab} L_{ab}(t)$$

$$L_{ab}(t) = \langle \boldsymbol{R}_t^5 \boldsymbol{J}_{ab} \rangle_{\mathrm{M}}$$
$$\boldsymbol{R}_t = \frac{1}{2} R_t^{ab} \boldsymbol{J}_{ab},$$
$$R_t^{ab} = \bar{R}^{ab} + t\mathrm{D}_{\bar{\omega}}\theta^{ab} + t^2\theta^a{}_c\theta^{cb}$$

$$L_{ab}(t) = \alpha_0 \left[\frac{5}{2}\left(R_t^4 - \frac{3}{4}R_t^2R_t^2\right)(R_t)_{ab} + 5R_t^2(R_t^3)_{ab} - 8\left(R_t^5\right)_{ab}\right]$$

$$\mathcal{Q}^{(11)}_{\omega \leftarrow \bar{\omega}} = \mathcal{Q}^{(11)}_{\omega \leftarrow 0} - \mathcal{Q}^{(11)}_{\bar{\omega} \leftarrow 0} + \mathrm{d}\mathcal{Q}^{(10)}_{\omega \leftarrow \boldsymbol{A}_3 \leftarrow \bar{\omega}}$$

$$\langle \boldsymbol{F}^5 \boldsymbol{G}_A \rangle_{\mathrm{M}} = 0$$

$$H_a = 0,$$
$$H_{ab} = 0,$$
$$H_{abcde} = 0,$$
$$\mathcal{R}\mathrm{D}_\omega \psi = 0$$

$$L_{ab} - 10(\mathrm{D}_\omega \bar{\psi})\mathcal{Z}_{ab}(\mathrm{D}_\omega \psi) + 5H_{abc}\left(T^c + \frac{1}{16}\bar{\psi}\Gamma^c\psi\right) +$$
$$+\frac{5}{2}H_{abcd}\left(\mathrm{D}_\omega b^{cd} - \frac{1}{16}\bar{\psi}\Gamma^{cd}\psi\right) + \frac{1}{24}H_{abc_1\cdots c_5}\left(\mathrm{D}_\omega b^{c_1\cdots c_5} + \frac{1}{16}\bar{\psi}\Gamma^{c_1\cdots c_5}\psi\right) = 0$$

$$L_{ab} \equiv \langle \boldsymbol{R}^5 \boldsymbol{J}_{ab} \rangle_{\mathrm{M}},$$
$$(\mathcal{Z}_{ab})^\alpha \equiv \left\langle \boldsymbol{Q}^\alpha \boldsymbol{R}^3 \boldsymbol{J}_{ab} \overline{\boldsymbol{Q}}_\beta \right\rangle_{\mathrm{M}},$$
$$H_{abc} \equiv \langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{P}_c \rangle_{\mathrm{M}},$$
$$H_{abcd} \equiv \langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{Z}_{cd} \rangle_{\mathrm{M}},$$
$$H_{abcdefg} \equiv \langle \boldsymbol{R}^4 \boldsymbol{J}_{ab} \boldsymbol{Z}_{cdefg} \rangle_{\mathrm{M}}.$$

$$
\begin{aligned}
H_c &= \frac{1}{2} R^{ab} H_{abc}, \\
H_{cd} &= \frac{1}{2} R^{ab} H_{abcd}, \\
H_{cdefg} &= \frac{1}{2} R^{ab} H_{abcdefg}, \\
\mathcal{R}^{\alpha}_{\beta} &= \frac{1}{2} R^{ab} (\mathcal{Z}_{ab})^{\alpha}{}_{\beta}
\end{aligned}
$$

$$
\begin{aligned}
L_{ab} &= \alpha_0 \left[\frac{5}{2} \left(R^4 - \frac{3}{4} R^2 R^2 \right) R_{ab} + 5 R^2 R^3_{ab} - 8 R^5_{ab} \right] \\
\mathcal{Z}_{ab} &= \frac{\alpha_2}{40} \Big\{ 2 \left(R^3_{ab} - \frac{3}{4} R^2 R_{ab} \right) \mathbb{1} - \frac{1}{48} R^{(3)}_{abcde} \Gamma^{cde} + \\
&\quad - \frac{3}{4} \left(R_{ab} R^{cd} - \frac{1}{2} R^2 \delta^{cd}_{ab} \right) R^{ef} \Gamma_{cdef} + \\
&\quad - \left[\delta^{cg}_{ab} R_{gh} R^{hd} R^{ef} - R^c_a R^d_b R^{ef} + \frac{1}{2} \delta^{ef}_{ab} (R^3)^{cd} \right] \Gamma_{cdef} \Big) \\
H_{abc} &= \frac{\alpha_2}{32} R^{(4)}_{abc}
\end{aligned}
$$

$$
\begin{aligned}
H_{abcd} =& \alpha_2 \delta^{ef}_{ab} g^{gh}_{cd} \left[\frac{3}{4} R^2 R_{ef} R_{gh} - R^3_{ef} R_{gh} - R_{ef} R^3_{gh} + \right. \\
& - \frac{4}{5} \left(R_{eh} R^3_{fg} + R^3_{eh} R_{fg} - R^2_{eh} R^2_{fg} \right) + \frac{1}{2} R^2 R_{eh} R_{fg} + \\
& \left. + \frac{1}{8} \eta_{[ef][gh]} \left(R^4 - \frac{3}{4} R^2 R^2 \right) - \eta_{fg} \left(R^2 R^2_{eh} - \frac{8}{5} R^4_{eh} \right) \right], \\
H_{abc_1 \cdots c_5} =& \frac{\alpha_2}{80} \delta^{\mathrm{clefg}}_{c_1 \ldots c_5} \left(- \frac{5}{3} R^{(3)}_{abccde} R_{fg} + 10 R^{(2)}_{\mathrm{abcdepq}} R^p{}_f R^q{}_g + \right. \\
& - \frac{1}{6} R_{ab} R^{(3)}_{\mathrm{cdefg}} + \frac{1}{4} R^2 R^{(2)}_{\mathrm{abcdefg}} - \frac{2}{3} R^{(1)}_{\mathrm{abcdefgpq}} (R^3)^{pq} + \\
& + \frac{1}{3} R^p{}_a R^q{}_b R^{(2)}_{\mathrm{clefgpq}} - \frac{1}{3} R^q{}_a R^{(2)}_{bcdefgp} R^p{}_q + \frac{1}{3} R^q{}_b R^{(2)}_{\mathrm{acclefgp}} R^p{}_q + \\
& \left. - \frac{10}{3} \eta_{ga} R^{(3)}_{bcdep} R^p{}_f + \frac{10}{3} \eta_{gb} R^{(3)}_{\mathrm{acdep}} R^p{}_f - \frac{5}{24} \eta_{[ab][cd]} R^{(4)}_{\mathrm{efg}} \right).
\end{aligned}
$$

$$
\langle \boldsymbol{F}^n \boldsymbol{G}_A \rangle = 0
$$

$$
\mathrm{d}s^2 = e^{-2\xi|z|} \left(\mathrm{d}z^2 + \tilde{g}^{(d)}_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \right) + \gamma^{(10-d)}_{\kappa\lambda} \mathrm{d}y^\kappa \mathrm{d}y^\lambda
$$

$$
\begin{aligned}
R^{ab} &= \tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b + 2\xi\theta(z) \left(\tilde{e}^a \kappa^b - \tilde{e}^b \kappa^a \right) - \kappa^a \kappa^b, \\
R^{aZ} &= -2 e^{\xi|z|} \xi \delta(z) E^Z \tilde{e}^a - 2\xi\theta(z) \tilde{T}^a + \mathrm{D}_{\tilde{\omega}} \kappa^a, \\
T^a &= \kappa^a E^Z + e^{-\xi|z|} \tilde{T}^a, \\
T^Z &= -e^{-\xi|z|} \kappa^a \tilde{e}_a
\end{aligned}
$$

$$
\begin{aligned}
\xi e^{\xi|z|} \delta(z) E^Z \varepsilon_{abcd} \left(\tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b \right) \tilde{e}^c &= \mathcal{T}_d \\
\varepsilon_{abcd} \left(\tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b \right) \left(\tilde{R}^{cd} - \xi^2 \tilde{e}^c \tilde{e}^d \right) &= \mathcal{T}
\end{aligned}
$$

$$
\begin{aligned}
\xi e^{\xi|z|} \delta(z) E^Z \varepsilon_{abcd} \left(\tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b \right) \tilde{e}^c &= \mathcal{T}_d, \\
\varepsilon_{abcd} \left(\tilde{R}^{ab} - \xi^2 \tilde{e}^a \tilde{e}^b \right) \left(\tilde{R}^{cd} - \xi^2 \tilde{e}^c \tilde{e}^d \right) &= \mathcal{T},
\end{aligned}
$$

$$\mathcal{T}_d = 2E^Z e^{\xi|z|}\xi\delta(z)\varepsilon_{abcd}\left(\frac{1}{2}\kappa^a\kappa^b - \xi\theta(z)\left(\tilde{e}^a\kappa^b - \tilde{e}^b\kappa^a\right)\right)\tilde{e}^c +$$
$$+\varepsilon_{abcd}\left[\tilde{R}^{ab} - \xi^2\tilde{e}^a\tilde{e}^b + 2\xi\theta(z)\left(\tilde{e}^a\kappa^b - \tilde{e}^b\kappa^a\right) - \kappa^a\kappa^b\right]\times$$
$$\times\left(\frac{1}{2}\mathrm{D}_{\tilde{\omega}}\kappa^c - \xi\theta(z)\tilde{T}^c\right),$$

$$\mathcal{T} = -4\varepsilon_{abcd}\left(\tilde{R}^{ab} - \xi^2\tilde{e}^a\tilde{e}^b + \xi\theta(z)\left(\tilde{e}^a\kappa^b - \tilde{e}^b\kappa^a\right) - \frac{1}{2}\kappa^a\kappa^b\right)\times$$
$$\times\left(\xi\theta(z)\left(\tilde{e}^c\kappa^d - \tilde{e}^d\kappa^c\right) - \frac{1}{2}\kappa^c\kappa^d\right)$$

$$\delta L = E(\phi)\delta\phi + \mathrm{d}\Xi(\phi,\delta\phi)$$

$$\delta_{\text{gauge}} L = \mathrm{d}\Omega$$

$$\star J_{\text{gauge}} = \Omega - \Xi_{\text{gauge}}$$

$$\begin{aligned}\delta_{\text{dif}} L &= -\pounds_\xi L\\ &= -\left(\mathrm{dI}_\xi + \mathrm{I}_\xi\mathrm{d}\right)L\\ &= -\mathrm{dI}_\xi L,\end{aligned}$$

$$\star J_{\text{dif}} = -\Xi_{\text{dif}} - \mathrm{I}_\xi L$$

$$\Xi = n(n+1)k\int_0^1 dt\langle\delta\boldsymbol{A}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\star J_{\text{gauge}} = \Omega - \Xi_{\text{gauge}}$$

$$\Omega = 0$$

$$\begin{aligned}\delta_{\text{gauge}}\boldsymbol{A} &= -\mathrm{D}\boldsymbol{\lambda},\\ \delta_{\text{gauge}}\overline{\boldsymbol{A}} &= -\overline{\mathrm{D}}\boldsymbol{\lambda}\end{aligned}$$

$$\delta_{\text{gauge}}\boldsymbol{A}_t = -\mathrm{D}_t\boldsymbol{\lambda}$$

$$\Xi_{\text{gauge}} = -n(n+1)k\int_0^1 dt\langle\mathrm{D}_t\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\Xi_{\text{gauge}} = -n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle + n(n+1)k\int_0^1 dt\langle\boldsymbol{\lambda}\mathrm{D}_t\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\Xi_{\text{gauge}} = -n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle + (n+1)k\int_0^1 dt\frac{d}{dt}\langle\boldsymbol{\lambda}\boldsymbol{F}_t^{n}\rangle$$

$$\Xi_{\text{gauge}} = -n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle + (n+1)k\left(\langle\boldsymbol{\lambda}\boldsymbol{F}^n\rangle - \left\langle\boldsymbol{\lambda}\overline{\boldsymbol{F}}^n\right\rangle\right)$$

$$\star J_{\text{gauge}} = n(n+1)k\,\mathrm{d}\int_0^1 dt\langle\boldsymbol{\lambda}\boldsymbol{\Theta}\boldsymbol{F}_t^{n-1}\rangle$$

$$\star J_{\mathrm{dif}} = -\Xi_{\mathrm{dif}} - \mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)}$$

$$\delta_{\mathrm{dif}} \boldsymbol{A} = -£_\xi \boldsymbol{A},$$
$$\delta_{\mathrm{dif}} \overline{\boldsymbol{A}} = -£_\xi \overline{\boldsymbol{A}}$$

$$\delta_{\mathrm{dif}} \boldsymbol{A}_t = -£_\xi \boldsymbol{A}_t$$

$$\Xi_{\mathrm{dif}} = -n(n+1)k \int_0^1 dt \langle £_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle$$

$$£_\xi \boldsymbol{A}_t = \mathrm{I}_\xi \boldsymbol{F}_t + \mathrm{D}_t \mathrm{I}_\xi \boldsymbol{A}_t$$

$$\Xi_{\mathrm{dif}} = -n(n+1)k \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{F}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle - n(n+1)k \int_0^1 dt \langle \mathrm{D}_t \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle$$

$$\begin{aligned} \Xi_{\mathrm{dif}} = -\mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)} &+ (n+1)k \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{\Theta} \boldsymbol{F}_t^{n} \rangle + \\ &-n(n+1)k \, \mathrm{d} \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle + \\ &+n(n+1)k \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{A}_t \mathrm{D}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle. \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \boldsymbol{F}_t &= \mathrm{D}_t \boldsymbol{\Theta} \\ \frac{d}{dt} \mathrm{I}_\xi \boldsymbol{A}_t &= \mathrm{I}_\xi \boldsymbol{\Theta} \end{aligned}$$

$$\begin{aligned} \Xi_{\mathrm{dif}} = -\mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)} &+ (n+1)k \int_0^1 dt \frac{d}{dt} \langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{F}_t^{n} \rangle + \\ &-n(n+1)k \, \mathrm{d} \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle, \end{aligned}$$

$$\begin{aligned} \Xi_{\mathrm{dif}} + \mathrm{I}_\xi L_{\mathrm{T}}^{(2n+1)} &= -n(n+1)k \, \mathrm{d} \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle + \\ &+(n+1)k \left(\langle \mathrm{I}_\xi \boldsymbol{A} \boldsymbol{F}^n \rangle - \left\langle \mathrm{I}_\xi \overline{\boldsymbol{A}} \overline{\boldsymbol{F}}^n \right\rangle \right). \end{aligned}$$

$$\star J_{\mathrm{dif}} = n(n+1)k \, \mathrm{d} \int_0^1 dt \langle \mathrm{I}_\xi \boldsymbol{A}_t \boldsymbol{\Theta} \boldsymbol{F}_t^{n-1} \rangle$$

$$\delta_{b_1 \cdots b_n}^{a_1 \cdots a_n} \equiv \det \begin{bmatrix} \delta_{b_1}^{a_1} & \delta_{b_2}^{a_1} & \cdots & \delta_{b_n}^{a_1} \\ \delta_{b_1}^{a_2} & \delta_{b_2}^{a_2} & \cdots & \delta_{b_n}^{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{b_1}^{a_n} & \delta_{b_2}^{a_n} & \cdots & \delta_{b_n}^{a_n} \end{bmatrix}$$

$$\delta_{b_1 \cdots b_r a_{r+1} \cdots a_n}^{a_1 \cdots a_r a_{r+1} \cdots a_n} = \frac{(d-r)!}{(d-n)!} \delta_{b_1 \cdots b_r}^{a_1 \cdots a_r}$$

$$\delta^{a_1\cdots a_n}_{a_1\cdots a_n} = \frac{d!}{(d-n)!}$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} A^{b_1\cdots b_n} = n!\, A^{a_1\cdots a_n},$$
$$\delta^{a_n}_{b_1\cdots b_n} A_{a_1\cdots a_n} = n!\, A_{b_1\cdots b_n}$$

$$\varepsilon_{a_1\cdots a_d} = \delta^{1\cdots\cdots}_{a_1\cdots a_d},$$
$$\varepsilon^{a_1\cdots a_d} = \delta^{a_1\cdots a_d}_{1\cdots d}$$

$$\varepsilon^{a_1\cdots a_d}\varepsilon_{b_1\cdots b_d} = \delta^{a_1\cdots a_d}_{b_1\cdots b_d}$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = \sum_{p=1}^{n} (-1)^{p+1}\delta^{a_1}_{b_p} \mathcal{V}^{a_2\cdots a_n}_{b_1\cdots b_p\cdots b_n}$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = \sum_{p=1}^{n-1}\sum_{q=p+1}^{n} (-1)^{p+q+1}\delta^{a_1 a_2}_{b_p b_q}\delta^{a_2\cdots\cdots}_{b_1\cdots \hat{b}_p\cdots \hat{b}_q\cdots b_n}$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = (-1)^{r(r+1)/2}\sum_{p_1=1}^{n-r+1}\sum_{p_2=p_1+1}^{n-r+2}\cdots\sum_{p_{r-1}=p_{r-2}+1}^{n-1}$$
$$\sum_{p_r=p_{r-1}+1}^{n} (-1)^{p_1+\cdots+p_r}\delta_{b_{p_1\cdots b_{p_r}}^{a1_1\cdots a_r}}^{\delta^{a_{r+1}\cdots a_n}_{b_1\cdots b_{p_1}\cdots b_{p_r}\cdots b_n}.}$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = (-1)^{r(r+1)/2}\sum_{p_1=1}^{n-r+1}\sum_{p_2=p_1+1}^{n-r+2}\cdots\sum_{p_{r-1}=p_{r-2}+1}^{n-1}$$
$$\sum_{p_r=p_{r-1}+1}^{n} (-1)^{p_1+\cdots+p_r}\delta^{a_{p_1}\cdots a_{p_r}}_{b_1\cdots b_r}\delta^{a_1\cdots \hat{a}_{p_1}\cdots \hat{a}_{p_r}\cdots a_n}_{b_{r+1}\cdots b_n}.$$

$$\delta^{a_1\cdots a_n}_{b_1\cdots b_n} = \binom{n}{p}\delta^{[a_1\cdots a_p}_{b_1\cdots b_p}\delta^{a_{p+1}\cdots a_n]}_{b_{p+1}\cdots b_n} = \binom{n}{p}\delta^{a_1\cdots a_p}_{[b_1\cdots b_p}\delta^{a_{p+1}\cdots a_n}_{b_{p+1}\cdots b_n]}$$

$$\eta_{[a_1\cdots a_p][b_1\cdots b_p]} \equiv \delta^{c_1\cdots c_p}_{a_1\cdots a_p}\left(\eta_{b_1 c_1}\cdots \eta_{b_p c_p}\right),$$
$$\eta^{[a_1\cdots a_p][b_1\cdots b_p]} \equiv \delta^{a_1\cdots a_p}_{c_1\cdots c_p}\left(\eta^{b_1 c_1}\cdots \eta_p b_p c_p\right)$$

$$\eta_{[b_1\cdots b_p][a_1\cdots a_p]} = \eta_{[a_1\cdots a_p][b_1\cdots b_p]},$$
$$\eta^{b_1\cdots b_p][a_1\cdots a_p]} = \eta^{a_1\cdots a_p][b_1\cdots b_p]}.$$

$$\eta_{[a_1\cdots a_p][b_1\cdots b_p]}A^{b_1\cdots b_p} = p!\, A_{a_1\cdots a_p},$$
$$\eta^{[a_1\cdots a_p][b_1\cdots b_p]}A_{b_1\cdots b_p} = p!\, A^{a_1\cdots a_p}.$$

$$\eta^{[a_1\cdots a_p][c_1\cdots c_p]}\eta_{[c_1\cdots c_p][b_1\cdots b_p]} = p!\,\delta^{a_1}_{b_1\cdots b_p\cdots a_p}$$

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$$

$$\Gamma_{a_1\cdots a_p} \equiv \Gamma_{[a_1} \cdots \Gamma_{a_p]}$$
$$= \frac{1}{p!}\delta^{b_1\cdots b_p}_{a_1\cdots a_p}\Gamma_{b_1} \cdots \Gamma_{b_p}$$

$$\Gamma_* \equiv \Gamma_0 \cdots \Gamma_{d-1}$$
$$= \frac{1}{d!}\varepsilon^{a_1\cdots a_d}\Gamma_{a_1} \cdots \Gamma_{a_d}$$
$$= \frac{1}{d!}\varepsilon^{a_1\cdots a_d}\Gamma_{a_1\cdots a_d}$$

$$\Gamma_*^2 = (-1)^{(d-2)(d+1)/2}$$
$$\Gamma_*\Gamma_a = (-1)^{d+1}\Gamma_a\Gamma_*,$$
$$\Gamma_*\Gamma_{a_1\cdots a_p} = (-1)^{p(d+1)}\Gamma_{a_1\cdots a_p}\Gamma_*.$$

$$\Gamma_{a_1\cdots a_{d-k}} = \frac{1}{k!}(-1)^{k(k-1)/2}\varepsilon_{a_1\cdots a_d}\Gamma^{a_{d-k+1}\cdots a_d}\Gamma_*$$

$$\Gamma_a^T = \xi C\Gamma_a C^{-1}$$

$$C^T = \lambda C$$

$d$mod8	λ	ξ	S	A
0	+1	+1	1,4	2,3
0	+1	-1	3,4	1,2
1	+1	+1	1,4	2,3
2	+1	+1	1,4	2,3
2	-1	-1	1,2	3,4
3	-1	-1	1,2	3,4
4	-1	+1	2,3	1,4
4	-1	-1	1,2	3,4
5	-1	+1	2,3	1,4
6	-1	+1	2,3	1,4
6	+1	-1	3,4	1,2

7 +1 -1 3,4 1,2

$$\left(C\Gamma_{a_1\cdots a_p}\right)^T = (-1)^{p(p-1)/2}\lambda\xi^p\left(C\Gamma_{a_1\cdots a_p}\right)$$

$$\mathrm{Tr}\left(\Gamma^{a_1\cdots a_p}\Gamma_{b_1\cdots b_q}\right) = (-1)^{p(p-1)/2} m\delta^{a_1\cdots a_p}_{b_1\cdots b_q}$$

$$M = \sum_{p\geq 0}\frac{1}{p!}k_{a_1\cdots a_p}\Gamma^{a_1\cdots a_p}$$

$$k_{a_1\cdots a_p} = \frac{1}{m}(-1)^{p(p-1)/2}\mathrm{Tr}\left(M\Gamma_{a_1\cdots a_p}\right)$$

$$\Gamma_{a_1\cdots a_i}\Gamma^{b_1\cdots b_j} = \sum_{k=|i-j|}^{i+j}\frac{i!\,j!}{s!\,t!\,u!}\delta^{[b_1}_{[a_i}\cdots\delta^{b_s}_{a_{t+1}}\Gamma_{a_1\cdots a_t]}{}^{b_{s+1}\cdots b_j]}$$

$$\begin{aligned} s &= \frac{1}{2}(i+j-k)\\ t &= \frac{1}{2}(i-j+k)\\ u &= \frac{1}{2}(-i+j+k) \end{aligned}$$

$$\Gamma^{a_1\cdots a_i}\Gamma_{b_1\cdots b_j} = \sum_{s=0}^{\min(i,j)}\frac{1}{t!\,u!}(-1)^{s(s-1)/2}\delta^{a_1\cdots a_i e_1\cdots e_u}_{d_1\cdots d_t b_1\cdots b_j}\Gamma^{d_1\cdots d_t}{}_{e_1\cdots e_u}$$

$$\begin{aligned} t &= i-s,\\ u &= j-s \end{aligned}$$

$$\begin{aligned}\left[\Gamma^{a_1\cdots a_i},\Gamma_{b_1\cdots b_j}\right] = &\sum_{s=0}^{\min(i,j)}\frac{1}{t!\,u!}(-1)^{s(s-1)/2}\times\\ &\times\left[1-(-1)^{ij-s^2}\right]\delta^{a_1\cdots a_i e_1\cdots e_u}_{d_1\cdots d_t b_1\cdots b_j}\Gamma^{d_1\cdots d_t}_{e_1\cdots e_u},\end{aligned}$$

$$\begin{aligned}\left\{\Gamma^{a_1\cdots a_i},\Gamma_{b_1\cdots b_j}\right\} = &\sum_{s=0}^{\min(i,j)}\frac{1}{t!\,u!}(-1)^{s(s-1)/2}\times\\ &\times\left[1+(-1)^{ij-s^2}\right]\delta^{a_1\cdots a_i e_1\cdots e_u}_{d_1\cdots d_t b_1\cdots b_j}\Gamma^{d_1\cdots d_t}{}_{e_1\cdots e_u}.\end{aligned}$$

$$\Gamma_{a_1\cdots a_i}\Gamma_{b_1\cdots b_j} = \sum_{s=0}^{\min(i,j)} D_{a_1\cdots a_i b_1\cdots b_j}(s)$$

$$D_{a_1\cdots a_i b_1\cdots b_j}(s) = (-1)^{s(i-s)+s(s-1)/2} \sum_{p_1=1}^{1+j-s} \cdots \sum_{p_s=p_{s-1}+1}^{j} \sum_{q_1=1}^{1+i-s} \cdots \sum_{q_s=q_{s-1}+1}^{i} (-1)^{p_1+\cdots+p_s+q_1+\cdots+q_s} \eta_{[a_{q_1}\cdots a_{q_s}][b_{p_1}\cdots b_{p_s}]}\Gamma_{a_1\cdots \hat{a}_{q_1}\cdots \hat{a}_{q_s}\cdots a_i b_1\cdots \hat{b}_{p_1}\cdots \hat{b}_{p_s}\cdots b_j}.$$

$$A_i B_j = \sum_{s=0}^{\min(i,j)} \binom{i}{s}\binom{j}{s}(-1)^{s(s-1)/2}\eta_{[b_1\cdots b_s][c_1\cdots c_s]} A^{a_1\cdots a_{i-s} b_1\cdots b_s} B^{c_1\cdots c_s a_{i-s+1}\cdots a_{i+j-2s}}\Gamma_{a_1\cdots a_{i+j-2s}},$$

$$A_i B_j = \sum_{s=0}^{\min(i,j)} \binom{i}{s}\binom{j}{s}(-1)^{s(s-1)/2}\eta_{[b_1\cdots b_s][c_1\cdots c_s]} A^{a_1\cdots a_{i-s} b_1\cdots b_s} B^{c_1\cdots c_s a_{i-s+1}\cdots a_{i+j-2s}}\Gamma_{a_1\cdots a_{i+j-2s}},$$

$$A_i = A^{a_1\cdots a_i}\Gamma_{a_1\cdots a_i},$$
$$B_j = B^{b_1\cdots b_j}\Gamma_{b_1\cdots b_j}$$

$$\psi' = \exp\left(\frac{1}{4}\lambda^{ab}\Gamma_{ab}\right)\psi$$

$$\delta\psi = \frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi$$

$$\bar{\psi}_\alpha = \psi^\beta C_{\beta\alpha}$$

$$\psi^\alpha = \bar{\psi}_\beta C^{\beta\alpha}$$

$$C^{\alpha\gamma}C_{\gamma\beta} = C_{\beta\gamma}C^{\gamma\alpha} = \delta^\alpha_\beta$$

$$\bar{\chi}\zeta = \bar{\zeta}\chi$$
$$\bar{\chi}S\zeta = -\bar{\zeta}S\chi$$
$$\bar{\chi}A\zeta = \bar{\zeta}A\chi$$

$$AdS_2 \times S^2 \times S^4 \times \Sigma$$

$$(a;q)_0 := 1, (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), (q)_n := \prod_{k=1}^{n}(1-q^k),$$
$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), (q)_\infty := \prod_{k=1}^{\infty}(1-q^k)$$

$$\langle W_{\mathcal{R}_1} \cdots W_{R_k} \rangle^G (t;q)$$
$$= \frac{1}{|\mathrm{Weyl}(G)|} \frac{(q)_\infty^{2\mathrm{rank}(G)}}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^{\mathrm{rank}(G)}} \oiiint \prod_{\alpha \in \mathrm{root}(G)} ds \frac{(s^\alpha; q)_\infty (q s^\alpha; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s^\alpha; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s^\alpha; q\right)_\infty} \prod_{i=1}^{k} \chi_{\mathcal{R}_i}$$

$$\mathcal{I}^G (t;q) := \mathrm{Tr}(-1)^F q^{J + \frac{H+C}{4}} t^{H-C}$$

$$\langle \mathcal{W}_{\mathcal{R}_1} \cdots \mathcal{W}_{\mathcal{R}_k} \rangle^G (t;q) = \frac{\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^G (t;q)}{\mathcal{I}^G (t;q)}$$

$$\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^G_{\frac{1}{2}\mathrm{BPS}} (\mathfrak{q})$$
$$= \frac{1}{|\mathrm{Weyl}(G)|} \frac{1}{(1 - \mathfrak{q}^2)^{\mathrm{rank}(G)}} \oiiint \prod_{\alpha \in \mathrm{root}(G)} ds \frac{(1 - s^\alpha)}{(1 - \mathfrak{q}^2 s^\alpha)} \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{g}}$$

$$\frac{1}{N!} \oiiint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \frac{\prod_{i \neq j} 1 - \frac{s_i}{s_j}}{\prod_{i,j} 1 - \mathrm{t} \frac{s_i}{s_j}} P_\mu (s;\mathrm{t}) P_\lambda (s^{-1};\mathrm{t}) = \frac{\delta_{\mu\lambda}}{(\mathrm{t};\mathrm{t})_{N - l(\mu)} \prod_{j \geq 1} (\mathrm{t};\mathrm{t})_{m_j(\mu)}}$$

$$\frac{1}{|\mathrm{Weyl}(G)|} \oiiint \prod_{\alpha \in \mathrm{root}(G)} ds (1 - s^\alpha) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathfrak{g}}$$

$$\langle T_B T_B \rangle^G (t;q) = \sum_{v \in \mathrm{Rep}(B)} \frac{1}{|\mathrm{Weyl}(B)|} \frac{(q)_\infty^{2\mathrm{rank}(G)}}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^{\mathrm{rank}(G)}} \oint \prod_{\alpha \in \mathrm{root}(G)} ds$$
$$\times \frac{\left(q^{\frac{|\alpha(B)|}{2}} s^\alpha; q\right)_\infty \left(q^{1 + \frac{|\alpha(B)|}{2}} s^\alpha; q\right)_\infty}{\left(q^{\frac{1 + |\alpha(B)|}{2}} t^2 s^\alpha; q\right)_\infty \left(q^{\frac{1 + |\alpha(B)|}{2}} t^{-2} s^\alpha; q\right)_\infty} Z_{\mathrm{bubb}}^{(B,v)} (t, s; q).$$

$$(z_e, z_m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$(z_e, z_m) = (0{,}0), (z_e, z_m) = (1{,}0)$$

$$(z_e, z_m) = (0{,}0), (z_e, z_m) = (0{,}1)$$

$$(z_e, z_m) = (0{,}0), (z_e, z_m) = (1{,}1)$$

$$\chi_{\mathrm{sp}}^{\mathrm{sp}(2N+1)} = \prod_{i=1}^{N} \left(s_i^{\frac{1}{2}} + s_i^{-\frac{1}{2}} \right)$$

$$\chi_{\square}^{\mathrm{so}\,(2N+1)} = 1 + \sum_{i=1}^{N} (s_i + s_i^{-1}).$$

$$\chi_\lambda^{\mathrm{so}\,(2N+1)} = \frac{\det\left(s_j^{\lambda_i+N-i+1/2} - s_j^{-(\lambda_i+N-i+1/2)}\right)}{\det\left(s_j^{N-i+1/2} - s_j^{-(N-i+1/2)}\right)}$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^{SO(2N+1)} = \int d\mu^{SO(2N+1)}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\frac{\bar{P}_n(s)^2-\bar{P}_{2n}(s)}{2}\right]\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{so}(2\mathfrak{N}+1)}(s)$$

$$d\mu^{SO(2N+1)} = \frac{1}{2^N N!}\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}(1-s_i)(1-s_i^{-1}) \times \prod_{1\le i<j\le N}(1-s_is_j)(1-s_i^{-1}s_j^{-1})(1-s_is_j^{-1})(1-s_i^{-1}s_j)$$

$$f_n(q,t) = \frac{q^{n/2}(t^{2n}+t^{-2n})-2q^n}{1-q^n}$$

$$P_m(s) := \sum_{i=1}^{N}(s_i^m + s_i^{-m})$$

$$\bar{P}_m(s) := 1 + P_m(s) = 1 + \sum_{i=1}^{N}(s_i^m + s_i^{-m})$$

$$\bar{M}_n(s) = \frac{\bar{P}_n(s)^2-\bar{P}_{2n}(s)}{2} = P_n(s) + \frac{P_n(s)^2 - P_{2n}(s)}{2}$$

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\bar{M}_n(s)\right) = \sum_\lambda \frac{1}{z_\lambda}f_\lambda(q,t)\bar{M}_\lambda(s)$$

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i}m_i!,\ f_\lambda(q,t) = \prod_{i=1}^{\ell(\lambda)} f_{\lambda_i}(q,t),\ \bar{M}_\lambda(s) = \prod_{i=1}^{\ell(\lambda)}\bar{M}_{\lambda_i}(s)$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^{SO(2N+1)} = \sum_\lambda\frac{1}{z_\lambda}f_\lambda(q,t)\int d\mu^{SO(2N+1)}\bar{M}_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{so}(2N+1)}(s)$$

$$\int d\mu^{SO(2N+1)}\bar{M}_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{so}(2N+1)}(s)$$

$$\int d\mu^{SO(2N+1)}\bar{P}_\mu(s) = \sum_{\nu\in R_{2N+1}(|\mu|)}\chi_\nu^S(\mu) + \sum_{\nu\in W_{2N+1}(|\mu|)}\chi_\nu^S(\mu)$$

$$R_n(p) = \{\lambda\vdash p \mid \ell(\lambda)\le n \text{ and } \forall\lambda_i \text{ is even}\}$$
$$W_n(p) = \{\lambda\vdash p \mid \ell(\lambda) = n \text{ and } \forall\lambda_i \text{ is odd}\}$$

$$s_\lambda = \sum_{\mu\vdash\lambda}\frac{\chi_\lambda^S(\mu)}{z_\mu}p_\mu$$

$$\bar{M}_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{so}\,(2N+1)}(s)=\sum_{\mu}a_{\lambda,\mathcal{R}}^{\mu}\bar{P}_\mu(s)$$

$$\int d\mu^{SO(2N+1)}\bar{M}_\lambda(s)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{\mathrm{soo}\,(2N+1)}(s)=\sum_{\mu}a_{\lambda,\mathcal{R}}^{\mu}\int d\mu^{SO(2N+1)}\bar{P}_\mu(s)$$

$$=\sum_{\mu}a_{\lambda,\mathcal{R}}^{\mu}\left(\sum_{\nu\in R_{2N+1}(|\mu|)}\chi_\nu^S(\mu)+\sum_{\nu\in W_{2N+1}(|\mu|)}\chi_\nu^S(\mu)\right)$$

$$\left(\chi_{\mathrm{sp}}^{\mathrm{so}(2N+1)}\right)^2=\prod_{i=1}^{N}(1+s_i)(1+s_i^{-1})$$

$$d\mu^{SO(2N+1)}\left(\chi_{\mathrm{sp}}^{\mathrm{so}(2N+1)}\right)^2=d\mu^{USp(2N)}$$

$$d\mu^{USp(2N)}=\frac{1}{2^N N!}\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}(1-s_i^2)(1-s_i^{-2})$$
$$\times\prod_{1\leq i<j\leq N}(1-s_is_j)(1-s_i^{-1}s_j^{-1})(1-s_is_j^{-1})(1-s_i^{-1}s_j)$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2N+1)}=\int d\mu^{USp(2N)}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\bar{M}_n(s)\right]$$
$$=\int d\mu^{USp(2N)}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)\left(P_n(s)+\frac{P_n(s)^2-P_{2n}(s)}{2}\right)\right]$$

$$\mathcal{I}^{SO(3)}(t;q)=\mathcal{I}^{USp(2)}(t;q)$$
$$=-\frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(q;q)_\infty^2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1<p_2}}\frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}}t^2\right)\left(1-q^{p_2-\frac{1}{2}}t^2\right)}.$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(3)}(t;q)$$
$$=\frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{(s^{\pm};q)_\infty(qs^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}\left(s^{\frac{1}{2}}+s^{-\frac{1}{2}}\right)^2$$

$$\langle T_{\left(\frac{1}{2}\right)}T_{\left(\frac{1}{2}\right)}\rangle^{USp(2)/\mathbb{Z}_2}(t;q)$$
$$=\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{\left(q^{\frac{1}{2}}s^{\pm 2};q\right)_\infty\left(q^{\frac{3}{2}}s^{\pm 2};q\right)_\infty}{(qt^2s^{\pm 2};q)_\infty(qt^{-2}s^{\pm 2};q)_\infty}$$

$$\langle W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Spin}(3)}(t;q) = \left\langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \right\rangle^{USp(2)/\mathbb{Z}_2} (t;q)$$

$$\langle W_{\text{sp}} W_{\text{sp}} \rangle^{Spin(3)}(t;q) = \langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \rangle^{USp(2)/\mathbb{Z}_2} (t;q)$$

$$= \langle W_{\square} W_{\square} \rangle^{SU(2)}(t;q) = \frac{\left(q^{\frac{1}{2}} t^{22}; q\right)_{\infty}}{(qt^{\pm 4}; q)_{\infty}} \sum_{m\in\mathbb{Z}\setminus\{0,n\}} \frac{t^{2m} - t^{-2m}}{t^2 - t^{-2}} \frac{q^{\frac{m-1}{2}}}{1-q^m}.$$

$$\langle W_{\text{sp}} W_{\text{sp}} \rangle^{Spin(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2}\right)} T_{\left(\frac{1}{2}\right)} \right\rangle^{USS(2)/\mathbb{Z}_2}_{\frac{1}{2}\text{BPS}} (\mathfrak{q})$$

$$= \langle W_{\square} W_{\square} \rangle^{SU(2)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2}{1-\mathfrak{q}^4} = \frac{1}{1-\mathfrak{q}^2}.$$

$$\langle \underbrace{W_{\text{sp}} W_{\text{sp}} \cdots W_{\text{sp}}}_{2k} \rangle^{Spin(3)}(t;q) = \langle \underbrace{W_{\square} W_{\square} \cdots W_{\square}}_{2k} \rangle^{SU(2)}(t;q)$$

$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Sin}(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{2+3\mathfrak{q}^2+\mathfrak{q}^4}{1-\mathfrak{q}^4}$$

$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Sin}(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{5+9\mathfrak{q}^2+5\mathfrak{q}^4+\mathfrak{q}^6}{1-\mathfrak{q}^4}$$

$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{Spin(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{14+28\mathfrak{q}^2+20\mathfrak{q}^4+7\mathfrak{q}^6+\mathfrak{q}^8}{1-\mathfrak{q}^4}$$

$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Sin}(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{42+90\mathfrak{q}^2+75\mathfrak{q}^4+35\mathfrak{q}^6+9\mathfrak{q}^8+\mathfrak{q}^{10}}{1-\mathfrak{q}^4}$$

$$\langle \underbrace{W_{\text{sp}} \cdots W_{\text{sp}}}_{2k} \rangle^{\text{Spin}(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \mathcal{I}^{SO(3)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) \sum_{i=0}^{k} a^{\mathfrak{so}(3)}_{ksp}(i)\mathfrak{q}^{2i}$$

$$= \frac{1}{1-\mathfrak{q}^4} \sum_{i=0}^{k} a^{\mathfrak{so}(3)}_{ksp}(i)\mathfrak{q}^{2i}$$

$$a^{\text{so}(3)}_{ksp}(i) = (2i+1)\frac{(2k)!}{(k-i)!\,(k+i+1)!}$$

$$= C_{k+i+1,2i+1}$$

$$C_{n,m} = \frac{m}{n}\binom{2n-m-1}{n-1}$$

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

$$= \prod_{1\le i\le j\le k-1} \frac{i+j+2}{i+j}$$

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k=0}^{\infty} a^{\text{so}(3)}_{ksp}(0)x^k$$

$$\frac{1}{x^{i+1}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{2i+1} = \sum_{k=0}^{\infty} a_{k\mathrm{sp}}^{\mathrm{so}(3)}(i)x^k$$

$$\sum_{k=0}^{\infty} x^k \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{Sin}(3)}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^4} \cdot \frac{2(1-\sqrt{1-4x})}{4x-(1-\sqrt{1-4x})^2\mathfrak{q}^2}$$

$$\begin{aligned}&\langle W_{\square} W_{\square}\rangle^{SO(3)}(t;q)\\ &= \frac{1}{2}\frac{(q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint \frac{ds}{2\pi i s}\frac{(s^{\pm};q)_\infty (qs^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}(1+s+s^{-1})^2\end{aligned}$$

$$\begin{aligned}&\langle W_{\square} W_{\square}\rangle^{SO(3)}(t;q) = \langle W_{\square\square} W_{\square\square}\rangle^{SU(2)}(t;q)\\ &= \frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(qt^{\pm 4};q)_\infty}\left[\frac{3}{2}\sum_{m\in\mathbb{Z}\backslash\{0\}}\left(\frac{t^{2m}-t^{-2m}}{t-t^{-2}}\frac{q^{\frac{m-1}{2}}}{1-q^m}\right) - \frac{2}{1-q} - \frac{q^{\frac{1}{2}}(t^2+t^{-2})}{1-q^2}\right]\end{aligned}$$

$$\begin{aligned}\langle W_{\square} W_{\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) &= \langle W_{\square} W_{\square\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SU(2)}(\mathfrak{q})\\ &= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{1-\mathfrak{q}^4}\\ &= \frac{1-\mathfrak{q}^6}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}\end{aligned}$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle^{SO(3)}(t;q) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2(k-i)} \rangle^{SO(3)}(t;q)$$

$$\begin{aligned}\langle W_{\square}\rangle^{SO(3)}(t;q) &= \langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle^{SO(3)}(t;q) - \mathcal{I}^{SO(3)}(t;q)\\ &= -\frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(q;q)_\infty^2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+1<p_2}}\frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}}t^2\right)\left(1-q^{p_2-\frac{1}{2}}t^2\right)}\end{aligned}$$

$$\begin{aligned}\langle W_{\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) &= \langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) - \mathcal{I}_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q})\\ &= \frac{\mathfrak{q}^2}{1-\mathfrak{q}^4}.\end{aligned}$$

$$\begin{aligned}\langle W_{\square} W_{\square} W_{\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) &= \langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) - 3\langle W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q})\\ &+3\langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q}) - \mathcal{I}_{\frac{1}{2}\mathrm{BPS}}^{SO(3)}(\mathfrak{q})\\ &= \frac{1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6}{1-\mathfrak{q}^4}.\end{aligned}$$

$$\langle W_\square W_\square W_\square W_\square \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{3 + 6\mathfrak{q}^2 + 6\mathfrak{q}^4 + 3\mathfrak{q}^6 + \mathfrak{q}^8}{1 - \mathfrak{q}^4}$$

$$\langle W_\square W_\square W_\square W_\square W_\square \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{6 + 15\mathfrak{q}^2 + 15\mathfrak{q}^4 + 10\mathfrak{q}^6 + 4\mathfrak{q}^8 + \mathfrak{q}^{10}}{1 - \mathfrak{q}^4}$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{15 + 36\mathfrak{q}^2 + 40\mathfrak{q}^4 + 29\mathfrak{q}^6 + 15\mathfrak{q}^8 + 5\mathfrak{q}^{10} + \mathfrak{q}^{12}}{1 - \mathfrak{q}^4}$$

$$\langle \underbrace{W_\square \cdots W_\square}_{k} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{k} a_k^{\mathfrak{so}(3)}(i)\mathfrak{q}^{2i}}{1 - \mathfrak{q}^4}$$

$$a_k^{\mathrm{so}\,(3)}(i) = c_k^{(i)} - c_k^{(i+1)}$$

$$(1 + x + x^2)^n = \sum_{i=-k}^{k} c_k^{(i)} x^{k+i}$$

$$R_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} C_i$$

$$\sum_{n=1}^{\infty} R_n x^n = \frac{1}{2x}\left(1 - \frac{\sqrt{1 - 3x}}{\sqrt{1 + x}}\right)$$

$$\langle W_{(k)} W_{(k)} \rangle^{SO(3)}(t; q) = \frac{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty}{(q t^{\pm 4}; q)_\infty} \left[\frac{2k + 1}{2} \sum_{m \in \mathbb{Z}\setminus\{0\}} \left(\frac{t^{2m} - t^{-2m}}{t - t^{-2}} \frac{q^{\frac{m-1}{2}}}{1 - q^m}\right) - \sum_{m=1}^{2k} (2k - m + 1)\left(\frac{t^{2m} - t^{-2m}}{t - t^{-2}} \frac{q^{\frac{m-1}{2}}}{1 - q^m}\right)\right]$$

$$\langle W_{(k)} W_{(k)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + \cdots + \mathfrak{q}^{4k}}{1 - \mathfrak{q}^4} = \frac{1 - \mathfrak{q}^{4k+2}}{(1 - \mathfrak{q}^2)(1 - \mathfrak{q}^4)}$$

$$\langle W_{(\infty)} W_{(\infty)} \rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1 - \mathfrak{q}^2)(1 - \mathfrak{q}^4)}$$

$$\langle \underbrace{W_{\square\square} \cdots W_\square}_{k} \rangle^{SO(3)}(t; q) = \sum_{k_1 + k_2 + k_3 = k} \binom{k}{k_1, k_2, k_3} (-3)^{k_2} \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4k_1 + 2k_2} \rangle^{SO(3)}(t; q)$$

$$\langle W_{\square\square}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{\mathrm{sp}}\, W_{\mathrm{sp}}\, W_{\mathrm{sp}}\, W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 3\langle W_{\mathrm{sp}}\, W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{\mathfrak{q}^4}{1-\mathfrak{q}^4},$$

$$\langle W_{\square} W_{\square} W_{\square\square}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{12}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 9\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{10}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$+30\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{8}\rangle \frac{1\ \ \mathrm{BPS}}{2\, SO(3)}(\mathfrak{q}) - 45\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$+30\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4}\rangle^{SO(\mathfrak{q})}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 9\langle W_{\mathrm{sp}}\, W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{1+3\mathfrak{q}^2+5\mathfrak{q}^4+4\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+\mathfrak{q}^{12}}{1-\mathfrak{q}^4}.$$

$$\langle \underbrace{W_{\square\square\square} \cdots W_{\square\square}}_{k}\rangle^{SO(3)}(t;q)$$

$$= \sum_{k_1+k_2+k_3+k_4=k} \binom{k}{k_1,k_2,k_3,k_4} (-1)^{k_2+k_4} 5^{k_2} 6^{k_3} \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6k_2+4k_1+2k_2}\rangle^{SO(3)}(t;q).$$

$$\langle W^{\square}_{\square\square}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 5\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 6\langle W_{\mathrm{sp}}\, W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{\mathfrak{q}^6}{1-\mathfrak{q}^4},$$

$$\langle W_{\square}^{W_{\square\square}W_{\square\square}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{18}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 15\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{16}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 93\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{14}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$-308\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{12}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 588\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{10}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 651\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{8}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$+399\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{6}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - 123\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{4}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) + 18\langle W_{\mathrm{sp}}\, W_{\mathrm{sp}}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) - \mathcal{I}^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{1+3\mathfrak{q}^2+5\mathfrak{q}^4+7\mathfrak{q}^6+6\mathfrak{q}^8+5\mathfrak{q}^{10}+4\mathfrak{q}^{12}+3\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18}}{1-\mathfrak{q}^4}.$$

$$\chi^{\mathrm{so}(3)}_{(k)} = \sum_{n=0}^{k} (-1)^n \binom{2k-n}{n} \chi_{\mathrm{sp}}^{\mathrm{so}(3)^{2k-2n}}$$

$$\langle W_{(k)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2k}}{1-\mathfrak{q}^4}$$

$$\langle W_{(k)} W_{(l)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2(l-k)}(1-\mathfrak{q}^{4k+2})}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\langle W_{(k)}W_{(k)}W_{(k)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2-3\mathfrak{q}^{2k+2}+\mathfrak{q}^{6k+4}}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)}$$

$$\langle W_{(\infty)}W_{(\infty)}W_{(\infty)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^3}$$

$$\langle W_{(k)}W_{(k)}W_{(k)}W_{(k)}\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{2k+1-3\mathfrak{q}^2-(2k+1)\mathfrak{q}^4+4\mathfrak{q}^{4k+4}-\mathfrak{q}^{8k+6}}{(1-\mathfrak{q}^2)^3(1-\mathfrak{q}^4)}$$

$$\begin{aligned}&\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(5)}(t;q)\\&=\frac{1}{8}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}\\&\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}\prod_{i=1}^{2}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)^2\end{aligned}$$

$$\begin{aligned}&\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(4)/\mathbb{Z}_2}(t;q)\\&=\frac{1}{2}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{\left(q^{\frac{1}{2}}s_i^{\pm 2};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm 2};q\right)_\infty}{(qt^2s_i^{\pm 2};q)_\infty(qt^{-2}s_i^{\pm 2};q)_\infty}\\&\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty(qt^2s_1^{\pm}s_2^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty(qt^{-2}s_1^{\pm}s_2^{\pm};q)_\infty}\end{aligned}$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{Spin(5)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(4)/\mathbb{Z}_2}(t;q)$$

$$\begin{aligned}\langle W_{\mathrm{sp}}\,W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(4)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\&=\frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}\\&=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}\end{aligned}$$

$$\langle \underbrace{V_{\text{sp}} \cdots W_{\text{sp}}}_{4} \rangle^{SSin(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{3+6\mathrm{q}^2+8\mathrm{q}^4+9\mathrm{q}^6+6\mathrm{q}^8+3\mathrm{q}^{10}+\mathrm{q}^{12}}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}$$

$$\langle \underbrace{S_{\text{sp}} \cdots W_{\text{sp}}}_{6} \rangle^{SSin(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{1}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}(14+40\mathrm{q}^2+66\mathrm{q}^4+85\mathrm{q}^6$$
$$+81\mathrm{q}^8+59\mathrm{q}^{10}+34\mathrm{q}^{12}+15\mathrm{q}^{14}+5\mathrm{q}^{16}+\mathrm{q}^{18})$$

$$\langle \underbrace{S_{\text{sp}} \cdots W_{\text{sp}}}_{8} \rangle^{\text{Sin}(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{1}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}(84+300\mathrm{q}^2+581\mathrm{q}^4+840\mathrm{q}^6+945\mathrm{q}^8+842\mathrm{q}^{10}$$
$$+616\mathrm{q}^{12}+378\mathrm{q}^{14}+195\mathrm{q}^{16}+83\mathrm{q}^{18}+28\mathrm{q}^{20}+7\mathrm{q}^{22}+\mathrm{q}^{24})$$

$$\langle \underbrace{W_{\text{sp}} \cdots W_{\text{sp}}}_{2k} \rangle^{Sp(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{\sum_{i=0}^{3k} a_{k\,sp}^{\text{so}\,(5)}(i)\mathrm{q}^{2i}}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}$$

$$a_{k\,\text{sp}}^{a_{\text{sp}}\,(0)} = C_k C_{k+2} - C_{k+1}^2$$
$$= \frac{24(2k+1)!\,(2k-1)!}{(k-1)!\,k!\,(k+2)!\,(k+3)!}$$
$$= \prod_{1\le i\le j\le k-1} \frac{i+j+4}{i+j},$$

$${}_3F_2\left(1,\frac{1}{2},\frac{3}{2};3,4;16x\right) = \sum_{k=0}^{\infty} a_{ksp}^{sso(5)}(0)x^k$$

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!}$$

$$a_{ksp}^{\text{so}\,(5)}(1) = \frac{60(2k)!\,(2k+2)!}{(k-1)!\,k!\,(k+3)!\,(k+4)!}$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{\sum_{i=0}^{2k} a_k^{\text{son}(5)}(i)\mathrm{q}^{2i}}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}$$

$$\langle W_{\square} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{\mathrm{q}^4}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)},$$

$$\langle W_{\square}W_{\square} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{1+\mathrm{q}^2+\mathrm{q}^4+\mathrm{q}^6+\mathrm{q}^8}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}$$
$$= \frac{1-\mathrm{q}^{10}}{(1-\mathrm{q}^2)(1-\mathrm{q}^4)(1-\mathrm{q}^8)},$$

$$\langle W_{\square}W_{\square}W_{\square} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{\mathrm{q}^2+3\mathrm{q}^4+3\mathrm{q}^6+3\mathrm{q}^8+2\mathrm{q}^{10}+\mathrm{q}^{12}}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{1}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}(3+3\mathrm{q}^2+9\mathrm{q}^4+15\mathrm{q}^6+12\mathrm{q}^8$$
$$+12\mathrm{q}^{10}+6\mathrm{q}^{12}+\mathrm{q}^{16})$$

$$\langle W_{\square}W_{\square}W_{\square}W_{\square}W_{\square} \rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathrm{q}) = \frac{1}{(1-\mathrm{q}^4)(1-\mathrm{q}^8)}(1+10\mathrm{q}^2+24\mathrm{q}^4+36\mathrm{q}^6+44\mathrm{q}^8$$
$$+41\mathrm{q}^{10}+31\mathrm{q}^{12}+19\mathrm{q}^{14}+10\mathrm{q}^{16}+4\mathrm{q}^{18}+\mathrm{q}^{20})$$

$$a_{k\square}^{\mathfrak{so}(5)}(0)=\sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor}C_iC_{i+1}\binom{k}{2i}-\sum_{i=0}^{\left\lfloor\frac{k+1}{2}\right\rfloor}C_i^2\binom{k}{2i-1}$$
$$=-k\,{}_3F_2\left(\frac{3}{2},\frac{1}{2}-\frac{k}{2};3,3;16\right)+\,{}_3F_2\left(\frac{3}{2},\frac{1}{2}-\frac{k}{2};2,3;16\right)$$

$$\langle\underbrace{\langle W_\square\cdots W_\square}_{k}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\sum_{i=0}^{3k}a_k^{\mathfrak{k}\mathfrak{N}\mathfrak{s}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_\square\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\mathfrak{q}^2+\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2},$$
$$\langle W_\square W_\square\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+3\mathfrak{q}^4+2\mathfrak{q}^6+3\mathfrak{q}^8+\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^3},$$
$$\langle W_\square W_\square W_\square\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+6\mathfrak{q}^2+9\mathfrak{q}^4+16\mathfrak{q}^6+15\mathfrak{q}^8$$
$$+15\mathfrak{q}^{10}+9\mathfrak{q}^{12}+6\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18})$$
$$\langle W_\square W_\square W_\square W_\square\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(6+22\mathfrak{q}^2+54\mathfrak{q}^4+82\mathfrak{q}^6+15\mathfrak{q}^8$$
$$+15\mathfrak{q}^{10}+9\mathfrak{q}^{12}+6\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18})$$

$$\langle\underbrace{W_{(2)}\cdots W_{(2)}}_{k}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\sum_{i=0}^{4k}a_k^{\mathfrak{son}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2},$$
$$\langle W_{\square\square}W_{\square\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+3\mathfrak{q}^8+2\mathfrak{q}^{10}+3\mathfrak{q}^{12}+\mathfrak{q}^{14}+\mathfrak{q}^{16}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$\langle W_{\square\square}W_{\square\square}W_{\square\square}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$\times(1+3\mathfrak{q}^2+9\mathfrak{q}^4+13\mathfrak{q}^6+20\mathfrak{q}^8+21\mathfrak{q}^{10}$$
$$+22\mathfrak{q}^{12}+18\mathfrak{q}^{14}+15\mathfrak{q}^{16}+9\mathfrak{q}^{18}+6\mathfrak{q}^{20}+2\mathfrak{q}^{22}+\mathfrak{q}^{24})$$

$$\langle W_{(k)}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\mathfrak{q}^{4k-4}+\mathfrak{q}^{4k}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$
$$=\frac{\mathfrak{q}^{4k-4}}{(1-\mathfrak{q}^4)^2}.$$

$$\langle W_{(k)}W_{(k)}\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(5)}(\mathfrak{q})=\frac{\sum_{i=0}^{8k}a_2^{\mathfrak{so}\mathfrak{d}(5)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{\square\square\square} W_{\square\square}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+\mathfrak{q}^2+2\mathfrak{q}^4+3\mathfrak{q}^6+4\mathfrak{q}^8+4\mathfrak{q}^{10}$$
$$+6\mathfrak{q}^{12}+5\mathfrak{q}^{14}+5\mathfrak{q}^{16}+4\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24})$$

$$\langle W_{\square\square\square} W_{\square\square\square}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(1+\mathfrak{q}^2+2\mathfrak{q}^4+3\mathfrak{q}^6+5\mathfrak{q}^8+5\mathfrak{q}^{10}$$
$$+8\mathfrak{q}^{12}+8\mathfrak{q}^{14}+10\mathfrak{q}^{16}+9\mathfrak{q}^{18}+10\mathfrak{q}^{20}+7\mathfrak{q}^{22}+7\mathfrak{q}^{24}$$
$$+4\mathfrak{q}^{26}+3\mathfrak{q}^{28}+\mathfrak{q}^{30}+\mathfrak{q}^{32})$$

$$\langle W_{(\infty)} W_{(\infty)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= 1+\mathfrak{q}^2+3\mathfrak{q}^4+4\mathfrak{q}^6+9\mathfrak{q}^8+11\mathfrak{q}^{10}+21\mathfrak{q}^{12}+26\mathfrak{q}^{14}+44\mathfrak{q}^{16}+54\mathfrak{q}^{18}+84\mathfrak{q}^{20}+\cdots$$

$$\langle W_{(\infty)} W_{(\infty)}\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{24}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)^2(1-\mathfrak{q}^{12})}$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(7)}(t;q)$$
$$= \frac{1}{48}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}tt^{\pm};q\right)_\infty^3}\int\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty}\prod_{i=1}^{3}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)^2.$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(6)/\mathbb{Z}_2}(t;q)$$
$$= \frac{1}{6}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}\pm^{\pm 2};q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}\frac{\left(q^{\frac{1}{2}}s_i^{\pm 2};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm 2};q\right)_\infty}{(qt^2 s_i^{\pm 2};q)_\infty(qt^{-2}s_i^{\pm 2};q)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty(qt^2 s_i^{\pm}s_j^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty(qt^{-2}s_i^{\pm}s_j^{\pm};q)_\infty}.$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(7)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\rangle^{USp(6)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+2\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)}.$$

$$\langle\underbrace{W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}}_{2k}\rangle^{\mathrm{SPS}}_{\frac{1}{2}\mathrm{BPin}(7)}(\mathfrak{q}) = \frac{\sum_{i=0}^{6k} a_{k\,\mathrm{sp}}^{\mathfrak{so}(7)}(i)\,\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{1}{(1-q^4)(1-q^8)(1-q^{12})}(4+9q^2+15q^4 +25q^6+29q^8+32q^{10}+33q^{12}+26q^{14}+20q^{16} +13q^{18}+6q^{20}+3q^{22}+q^{24}),$$

$$\langle W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}W_{\rm sp}\rangle^{\rm Spin(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{1}{(1-q^4)(1-q^8)(1-q^{12})}(30+105q^2+235q^4 +435q^6+650q^8+855q^{10}+1010q^{12}+1055q^{14} +1006q^{16}+865q^{18}+665q^{20}+470q^{22}+299q^{24} +170q^{26}+89q^{28}+40q^{30}+15q^{32}+5q^{34}+q^{36}).$$

$$a_{k\rm sp}^{\rm so\,(7)}(0)=\prod_{1\le i\le j\le k-1}\frac{i+j+6}{i+j}$$

$${}_4F_3\left(1,\frac{1}{2},\frac{5}{2},\frac{3}{2};4,5,6;64x\right)=\sum_{k=0}^{\infty}a_{k\rm sp}^{\rm so\,(7)}(0)x^k$$

$$\langle\underbrace{W_{\square}\cdots W_{\square}}_{k}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{\sum_{i=0}^{3k}\ a_k^{\rm son(7)}(i)q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\langle W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{q^6}{(1-q^4)(1-q^8)(1-q^{12})},$$

$$\langle W_{\square}W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}}{(1-q^4)(1-q^8)(1-q^{12})}=\frac{1-q^{14}}{(1-q^2)(1-q^4)(1-q^8)(1-q^{12})},$$

$$\langle W_{\square}W_{\square}W_{\square}\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{q^4+3q^6+3q^8+3q^{10}+3q^{12}+3q^{14}+2q^{16}+q^{18}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\left\langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\right\rangle^{SO(7)}_{\frac{1}{2}\rm SPS}(q)=\frac{\sum_{i=0}^{5k}\ a^{\rm so(7)}_{k\substack{\square\\\square}}(i)q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\underbrace{\left\langle W_{\substack{\square\\\square\\\square}}\cdots W_{\substack{\square\\\square\\\square}}\right\rangle}_{k}{}^{SO(7)}_{\frac{1}{2}\rm SPS}(q)=\frac{\sum_{i=0}^{6k}\ a^{\rm so(7)}_{k\substack{\square\\\square\\\square}}(i)q^{2i}}{(1-q^4)(1-q^8)(1-q^{12})}$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{q^2+q^6+q^{10}}{(1-q^4)(1-q^8)(1-q^{12})},$$

$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(7)}_{\frac{1}{2}\rm BPS}(q)=\frac{1}{(1-q^4)(1-q^8)(1-q^{12})}\times(1+q^2+3q^4+2q^6+5q^8+3q^{10} +5q^{12}+2q^{14}+3q^{16}+q^{18}+q^{20}).$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\right\rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\left\langle W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\right\rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times(1+\mathfrak{q}^2+3\mathfrak{q}^4+4\mathfrak{q}^6+7\mathfrak{q}^8+6\mathfrak{q}^{10}$$
$$+9\mathfrak{q}^{12}+6\mathfrak{q}^{14}+7\mathfrak{q}^{16}+4\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24}).$$

$$\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{3lk}\ a^{50}_{k(l)}(7)}{}(i)\mathfrak{q}^{2i}$$

$$\langle W_{\square\square}\rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times(1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+4\mathfrak{q}^8+3\mathfrak{q}^{10}+5\mathfrak{q}^{12}$$
$$+3\mathfrak{q}^{14}+5\mathfrak{q}^{16}+2\mathfrak{q}^{18}+3\mathfrak{q}^{20}+\mathfrak{q}^{22}+\mathfrak{q}^{24})$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle^{\text{Spin}(2N+1)}(t;q)$$
$$=\frac{1}{2^N N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm};q\right)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\prod_{i=1}^N\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)^2.$$

$$\langle T_{\left(\frac{1}{2}\right)^N}T_{\left(\frac{1}{2}\right)^N}\rangle^{USp(2N)/\mathbb{Z}_2}(t;q)$$
$$=\frac{1}{N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}\frac{\left(q^{\frac{1}{2}}s_i^{\pm 2};q\right)_\infty\left(\left(q^{\frac{3}{2}}s_i^{\pm 2};q\right)_\infty^{\pm 2};q\right)_\infty}{}(qt^{-2}s_i^{\pm 2};q)_\infty$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{(t^2s_i^{\pm}s_j^{\mp};q)_\infty\left(qt^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qt^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}.$$

$$\langle W_{\text{sp}}W_{\text{sp}}\rangle^{Spin(2N+1)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\langle T_{\left(\frac{1}{2}^N\right)}T_{\left(\frac{1}{2}^N\right)}\rangle^{USp(2N)/\mathbb{Z}_2}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})$$
$$=\prod_{n=1}^N\frac{1}{(1-\mathfrak{q}^{2n})}.$$

$$\mathcal{I}_{\frac{1}{2}\mathrm{BPS}}^{Spin(2N+1)}(\mathfrak{q}) = \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{4n}}$$

$$\langle \mathcal{W}_{\mathrm{sp}} \mathcal{W}_{\mathrm{sp}} \rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{Spin}(2N+1)}(\mathfrak{q}) = \prod_{n=1}^{N} (1+\mathfrak{q}^{2n})$$

$$\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{\mathrm{Spin}(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{\frac{N(N+1)k}{2}} a_{k\,\mathrm{sp}}^{\mathrm{so}\,(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$a_{k\mathrm{sp}}^{\mathrm{so}\,(2N+1)}(0) = \det\left(C_{2N-i-j+k}\right) = \prod_{1\le i\le j\le k-1} \frac{i+j+2N}{i+j},$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{S(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{Nk} a_{k\square}^{so(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$\langle W_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\mathfrak{q}^{2N}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})},$$

$$\langle W_{\square} W_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\cdots+\mathfrak{q}^{4N}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})} = \frac{1-\mathfrak{q}^{4N+2}}{(1-\mathfrak{q}^2)\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$\langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \mathfrak{q}^{2N},$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4N+2}}{1-\mathfrak{q}^2}$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS},c}^{SO(2N+1)}(\mathfrak{q}) = \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) - \langle \mathcal{W}_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q})^2 = \frac{1-\mathfrak{q}^{4N}}{1-\mathfrak{q}^2}.$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{k} \rangle^{\frac{1}{2}\mathrm{BPS}\ SO(2N+1)}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)k} a_{k\square\square}^{\mathrm{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$\langle W_{\square} \rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\mathfrak{q}^2+\mathfrak{q}^6+\cdots+\mathfrak{q}^{4N-2}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})} = \frac{\mathfrak{q}^2(1-\mathfrak{q}^{NN})}{(1-\mathfrak{q}^4)\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}.$$

$$\left\langle \mathcal{W}_{\square}{}_{\square} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{SO(2N+1)}(\mathfrak{q}) = \frac{\mathfrak{q}^2(1-\mathfrak{q}^{4N})}{1-\mathfrak{q}^4}$$

$$\langle \underbrace{W_{(l)} \cdots W_{(l)}}_{k} \rangle^{SO(2N+1)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{Nlk} a_{k(l)}^{\mathrm{so}(2N+1)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N} (1-\mathfrak{q}^{4n})}$$

$$(z_e, z_m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (1,0)$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (0,1)$$

$$(z_e, z_m) = (0,0), (z_e, z_m) = (1,1)$$

$$\chi_{\square}^{\mathrm{usp}(2N)} = \sum_{i=1}^{N} (s_i + s_i^{-1})$$

$$\chi_{\square}^{\mu\mathrm{sp}(2N)} = \frac{\det\left(s_j^{\lambda_i+N-i+1} - s_j^{-\lambda_i-N+i-1}\right)}{\det\left(s_j^{N-i+1} - s_j^{-N+i-1}\right)}$$

$$\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^{USp(2N)} = \int d\mu^{USp(2N)} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n(q,t) L_n(s)\right) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\boldsymbol{usp}(2N)}(s),$$

$$L_n(s) = \frac{P_n(s)^2 + P_{2n}(s)}{2}$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n(q,t) L_n(s)\right) = \sum_{\lambda} \frac{1}{z_\lambda} f_\lambda(q,t) L_\lambda(s)$$

$$\langle W_{\mathcal{R}_1} \cdots W_{\mathcal{R}_k} \rangle^{USp(2N)} = \sum_{\lambda} \frac{1}{z_\lambda} f_\lambda(q,t) \int d\mu^{USp(2N)} L_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathrm{usp}(2N)}(s)$$

$$L_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mathbf{usp}(2N)}(s) = \sum_{\mu} b_{\lambda,\mathcal{R}}^{\mu} P_\mu(s)$$

$$\int d\mu^{USp(2N)} P_\mu(s) = \sum_{\nu \in R_{2N}^{c}(|\mu|)} \chi_\nu^{S}(\mu),$$

$$R_n^c(p) = \{\lambda \vdash p \mid \ell(\lambda) \leq n \text{ and } \forall \lambda_i' \text{ is even}\}$$

$$\int d\mu^{USp(2N)} L_\lambda(s) \prod_{i=1}^{k} \chi_{\mathcal{R}_i}^{\mu\mathrm{sp}(2N)}(s) = \sum_{\mu} b_{\lambda,\mathcal{R}}^{\mu} \int d\mu^{USp(2N)} P_\mu(s) = \sum_{\mu} b_{\lambda,\mathcal{R}}^{\mu} \sum_{\nu \in R_{2N}^{c}(|\mu|)} \chi_\nu^{S}(\mu).$$

$$\left\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\right\rangle^{USp(2N)} = \sum_{\lambda}\frac{1}{z_\lambda}f_\lambda(q,t)\sum_{\mu} b^{\mu}_{\lambda,\mathcal{R}}\sum_{\nu\in R^c_{2N}(|\mu|)}\chi^S_\nu(\mu)$$

$$\langle W_{\square}W_{\square}\rangle^{USp(2)}(t;q)$$
$$= \frac{1}{2}\frac{(q)^2_\infty}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{(s^{\pm 2};q)_\infty(qs^{\pm 2};q)_\infty}{\left(q^{\frac{1}{2}}t^2s^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^{\pm 2};q\right)_\infty}(s+s^{-1})^2$$

$$\left\langle T_{(1)}T_{(1)}\right\rangle^{SO(3)}(t;q)$$
$$= \frac{(q)^2_\infty}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{\left(q^{\frac{1}{2}}s^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s^{\pm};q\right)_\infty}{(qt^2s^{\pm};q)_\infty(qt^{-2}s^{\pm};q)_\infty}$$

$$\langle W_{\square}W_{\square}\rangle^{USp(2)}(t;q)$$
$$= \left\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\right\rangle^{\mathrm{Spin}(3)}(t;q)$$
$$= \left\langle T_{(1)}T_{(1)}\right\rangle^{SO(3)}(t;q) = \left\langle T_{\left(\frac{1}{2}\right)}T_{\left(\frac{1}{2}\right)}\right\rangle^{USp(2)/\mathbb{Z}_2}(t;q)$$

$$\langle W_{\square}W_{\square}\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{(1)}T_{(1)}\right\rangle^{SO(3)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2}{1-\mathfrak{q}^4}$$
$$= \frac{1}{1-\mathfrak{q}^2}$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{USp(2)}(t;q)$$
$$= \frac{1}{2}\frac{(q)^2_\infty}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}\frac{(s^{\pm 2};q)_\infty(qs^{\pm 2};q)_\infty}{\left(q^{\frac{1}{2}}t^2s^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^{\pm 2};q\right)_\infty}(1+s^2+s^{-2})^2$$

$$\left\langle W_{(2k)}\right\rangle^{USp(2)}(t;q) = \left\langle W_{(k)}\right\rangle^{SO(3)}(t;q).$$

$$\left\langle W_{(2k)}\right\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2k}}{(1-\mathfrak{q}^4)}$$

$$\left\langle W_{(k)}W_{(k)}\right\rangle^{USp(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{2k+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$\left\langle W_{(\infty)}W_{(\infty)}\right\rangle^{USS(2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}$$

$$
\begin{aligned}
&\langle W_{\square} W_{\square}\rangle^{USp(4)}(t;q) \\
&= \frac{1}{8}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{\left(s_i^{\pm 2};q\right)_\infty\left(qs_i^{\pm 2};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm 2};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm 2};q\right)_\infty} \\
&\times\frac{\left(s_1^{\pm}s_2^{\mp};q\right)_\infty\left(s_1^{\pm}s_2^{\pm};q\right)_\infty\left(qs_1^{\pm}s_2^{\mp};q\right)_\infty\left(qs_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2 s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2 s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}\left[\sum_{i=1}^{2}\left(s_i+s_i^{-1}\right)\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
&\left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(5)}(t;q) \\
&= \frac{1}{2}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{}{\left(q^{\frac{1}{2}}s_1^{\pm};q\right)_\infty\left(s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm};q\right)_\infty;q\Big)_\infty\left(q^{\frac{1}{2}}t^2 s_2^{\pm};qs_\infty^{\pm};q\right)_\infty\left(qt^{-2}s_1^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_2^{\pm};q\right)_\infty} \\
&\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(qt^2 s_1^{\pm}s_2^{\mp};q\right)_\infty\left(qt^2 s_1^{\pm}s_2^{\pm};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}.
\end{aligned}
$$

$$
\begin{aligned}
\langle W_{\square}W_{\square}\rangle^{USp(4)}(t;q) &= \left\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\right\rangle^{\mathrm{Spin}(5)}(t;q) \\
= \left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(5)}(t;q) &= \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{USp(4)/\mathbb{Z}_2}(t;q).
\end{aligned}
$$

$$
\begin{aligned}
\langle W_{\square}W_{\square}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) &= \left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)},
\end{aligned}
$$

$$
\left\langle W_{\substack{\square\\\square}}\right\rangle^{USp(4)}(t;q) = \left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(5)}(t;q)
$$

$$
\langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{USp(4)}(t;q) = \langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(5)}(t;q)
$$

$$
\langle\underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{USp(4)}(t;q) = \langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{k}\rangle^{SO(5)}(t;q)
$$

$$
\langle\underbrace{W_{(2l)}\cdots W_{(2l)}}_{k}\rangle^{USp(4)}(t;q) = \langle\underbrace{W_{(l^2)}\cdots W_{(l^2)}}_{k}\rangle^{SO(5)}(t;q)
$$

$$
\langle\underbrace{W_{(2l)}\cdots W_{(2l)}}_{k}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3lk} a_k^{\mathrm{usp}\,(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}
$$

$$
\begin{aligned}
&\langle W_{(\infty)}W_{(\infty)}\rangle^{USp(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle W_{(\infty^2)}W_{(\infty^2)}\right\rangle^{SO(5)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) \\
&= 1+\mathfrak{q}^2+4\mathfrak{q}^4+5\mathfrak{q}^6+13\mathfrak{q}^8+16\mathfrak{q}^{10}+33\mathfrak{q}^{12}+41\mathfrak{q}^{14}+73\mathfrak{q}^{16}+90\mathfrak{q}^{18}+145\mathfrak{q}^{20}+\cdots.
\end{aligned}
$$

$$\langle W_{(\infty)} W_{(\infty)} \rangle^{USp(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \left\langle W_{(\infty^2)} W_{(\infty^2)} \right\rangle^{SO(5)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})$$
$$= \frac{1-\mathfrak{q}^{16}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^3(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)^2}.$$

$$\langle \underbrace{W_{(l^2)} \cdots W_{(l^2)}}_{k} \rangle^{USp(4)}(t;q) = \langle \underbrace{W_{(l)} \cdots W_{(l)}}_{k} \rangle^{SO(5)}(t;q)$$

$$\langle W_{\square} W_{\square} \rangle^{USp(6)}(t;q)$$
$$= \frac{1}{48} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^3} \oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm 2};q)_\infty (q s_i^{\pm 2};q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm 2};q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm 2};q\right)_\infty}$$
$$\times \prod_{i<j} \frac{(s_i^{\pm} s_j^{\mp};q)_\infty (s_i^{\pm} s_j^{\pm};q)_\infty (q s_i^{\pm} s_j^{\mp};q)_\infty (q s_i^{\pm} s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\pm};q\right)_\infty} \left[\sum_{i=1}^{3} (s_i + s_i^{-1})\right]^2.$$

$$\left\langle T_{(1,0,0)} T_{(1,0,0)} \right\rangle^{SO(7)}(t;q)$$
$$= \frac{1}{8} \frac{(q)_\infty^6}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^3} \oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i} \frac{\left(q^{\frac{1}{2}\delta_{i,1}} s_i^{\pm};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i,1}} s_i^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i,1}}{2}} t^2 s_i^{\pm};q\right)_\infty \left(q^{\frac{1+\delta_{i,1}}{2}} t^{-2} s_i^{\pm};q\right)_\infty}$$
$$\times \prod_{i<j} \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}} t^2 s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}} t^2 s_i^{\pm} s_j^{\pm};q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}} t^{-2} s_i^{\pm} s_j^{\mp};q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}} t^{-2} s_i^{\pm} s_j^{\pm};q\right)_\infty}.$$

$$\langle W_{\square} W_{\square} \rangle^{USp(6)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \langle T_{(1,0,0)} T_{(1,0,0)} \rangle^{SO(7)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)\left(1-\mathfrak{q}^8(1-\mathfrak{q}^{12})\right)}$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{2k} \rangle^{USp(6)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{5k} a_{k\square}^{\text{usp}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle W_{\square} W_{\square} W_{\square} W_{\square} \rangle^{USp(6)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}(3+6\mathfrak{q}^2+9\mathfrak{q}^4+12\mathfrak{q}^6+14\mathfrak{q}^8$$
$$+15\mathfrak{q}^{10}+12\mathfrak{q}^{12}+9\mathfrak{q}^{14}+6\mathfrak{q}^{16}+3\mathfrak{q}^{18}+\mathfrak{q}^{20})$$

$$\det\begin{pmatrix} F_0 & F_1 & F_2 \\ F_1 & F_2 & F_3 \\ F_2 & F_3 & F_4 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{a_{k\square}^{\text{usp}\,(6)}(0)}{(2k)!} x^{2k}$$

$$F_m(x) := \sum_{j=0}^{m} \binom{m}{j} \left(I_{2j-m}(2x) - I_{2j-m+2}(2x)\right)$$

$$I_k(2x) := \sum_{n=0}^{\infty} \frac{x^{2n+k}}{n!\,(n+k)!}$$

$$\left\langle W_{\substack{\Box\\\Box}} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{\mathfrak{q}^4 + \mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\left\langle W_{\substack{\Box\\\Box}} W_{\substack{\Box\\\Box}} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + 2\mathfrak{q}^4 + 2\mathfrak{q}^6 + 3\mathfrak{q}^8 + 2\mathfrak{q}^{10} + 3\mathfrak{q}^{12} + \mathfrak{q}^{14} + \mathfrak{q}^{16}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.$$

$$\left\langle W_{\substack{\Box\\\Box\\\Box}} W_{\substack{\Box\\\Box\\\Box}} \right\rangle_{\frac{1}{2}\mathrm{BPS}}^{USS(6)}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + \mathfrak{q}^4 + 2\mathfrak{q}^6 + 2\mathfrak{q}^8 + 2\mathfrak{q}^{10} + 2\mathfrak{q}^{12} + \mathfrak{q}^{14} + \mathfrak{q}^{16} + \mathfrak{q}^{18}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle \underbrace{W_{\Box\Box} \cdots W_{\Box\Box}}_{k} \rangle^{USp(6)}(t;q) = \langle \underbrace{W_{\substack{\Box\\\Box}} \cdots W_{\substack{\Box\\\Box}}}_{k} \rangle^{SO(7)}(t;q)$$

$$\langle W_{\Box\Box} \rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{\mathfrak{q}^2 + \mathfrak{q}^6 + \mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})},$$

$$\langle W_{\Box\Box} W_{\Box\Box} \rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})} \times (1 + \mathfrak{q}^2 + 3\mathfrak{q}^4 + 2\mathfrak{q}^6 + 5\mathfrak{q}^8 + 3\mathfrak{q}^{10} + 5\mathfrak{q}^{12} + 2\mathfrak{q}^{14} + 3\mathfrak{q}^{16} + \mathfrak{q}^{18} + \mathfrak{q}^{20})$$

$$\langle \underbrace{W_{(2l)} \cdots W_{(2l)}}_{k} \rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{\sum_{i=0}^{5lk} a_{k(2l)}^{\mathfrak{usp}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle W_{\Box\Box\Box} \rangle_{\frac{1}{2}\mathrm{BPS}}^{USp(6)}(\mathfrak{q}) = \frac{\mathfrak{q}^4 + \mathfrak{q}^8 + 2\mathfrak{q}^{12} + \mathfrak{q}^{16} + \mathfrak{q}^{20}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$

$$\langle W_\Box W_\Box \rangle^{USp(2N)}(t;q) = \frac{1}{2^N N!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \frac{(s_i^{\pm 2}; q)_\infty (q s_i^{\pm 2}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm 2}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm 2}; q\right)_\infty}$$

$$\times \prod_{i<j} \frac{(s_i^{\pm} s_j^{\mp}; q)_\infty (s_i^{\pm} s_j^{\pm}; q)_\infty (q s_i^{\pm} s_j^{\mp}; q)_\infty (q s_i^{\pm} s_j^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_i^{\pm} s_j^{\pm}; q\right)_\infty} \left[\sum_{i=1}^{N} (s_i + s_i^{-1})\right]^2.$$

$$\langle T_{(1,0^{N-1})} T_{(1,0^{N-1})} \rangle^{SO(2N+1)}(t;q)$$

$$= \frac{1}{2^{N-1}(N-1)!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \frac{\left(q^{\frac{1}{2}\delta_{i,1}} s_i^{\pm}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i,1}} s_i^{\pm}; q\right)_\infty}{\left(\frac{1+\delta_{i,1}}{2} t^2 s_i^{\pm}; q\right)_\infty \left(q^{\frac{1+\delta_{i,1}}{2}} t^{-2} s_i^{\pm}; q\right)_\infty}$$

$$\times \prod_{i<j} \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{1+\frac{1}{2}\delta^{\pm}_{i+j,1,1} s_i^{\pm} s_j^{\pm}; q}\right)_\infty}{\left(q^{\frac{1+\delta_{i+j,1}}{2}} t^2 s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}} t^2 s_i^{\pm} s_j^{\pm}; q\right)_\infty \left(q^{\frac{1+\delta^2_{i+j,1}}{2}} t^{-2} s_i^{\pm} s_j^{\mp}; q\right)_\infty \left(q^{\frac{1+\delta_{i+j,1}}{2}} t^{-2} s_i^{\pm} s_j^{\pm}; q\right)_\infty}.$$

$$\langle W_{\square} W_{\square} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{(1,0^{N-1})} T_{(1,0^{N-1})} \right\rangle^{SO(2N+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$= \frac{1}{(1-\mathfrak{q}^2)\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}.$$

$$\mathcal{I}^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{4n}}$$

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4N}}{1-\mathfrak{q}^2}$$

$$\langle \underbrace{W_{\square} \cdots W_{\square}}_{2k} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)k} a_{k\square}^{\mathrm{usp}\,(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}$$

$$\det\left(F_{i+j-2}(x)\right) = \sum_{k=0}^{\infty} \frac{a_{k\square}^{\mathrm{usp}\,(2N)}(0)}{(2k)!} x^{2k}$$

$$\langle \underbrace{W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cdots W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{k} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{2(N-1)k} a_{k\begin{smallmatrix}\square\\\square\end{smallmatrix}}^{\mathrm{usp}(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}$$

$$\langle \underbrace{W_{\square\square} \cdots W_{\square\square}}_{k} \rangle^{USp(2N)}(t;q) = \langle \underbrace{W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cdots W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{k} \rangle^{SO(2N+1)}(t;q)$$

$$\langle W_{\square\square} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2 + \mathfrak{q}^6 + \cdots + \mathfrak{q}^{4N-2}}{\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}$$

$$= \frac{\mathfrak{q}^2(1-\mathfrak{q}^{4N})}{(1-\mathfrak{q}^4)\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}.$$

$$\langle \underbrace{W_{(2l)} \cdots W_{(2l)}}_{k} \rangle^{USp(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-1)lk} a_{k(2l)}^{\mathrm{usp}(2N)}(i)\mathfrak{q}^{2i}}{\prod_{n=1}^{N}(1-\mathfrak{q}^{4n})}$$

$$\left(z_{e,S}, z_{e,C}; z_{m,S}, z_{m,C}\right) \in (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

$$
\begin{aligned}
&\mathrm{Spin}(2N),\\
&SO(2N)=\mathrm{Spin}(2N)/\mathbb{Z}_2^V,\\
&Ss(2N)=\mathrm{Spin}(2N)/\mathbb{Z}_2^S,\\
&Sc(2N)=\mathrm{Spin}(2N)/\mathbb{Z}_2^C,\\
&SO(2N)/\mathbb{Z}_2=\mathrm{Spin}(2N)/\left(\mathbb{Z}_2^S\times\mathbb{Z}_2^C\right).
\end{aligned}
$$

$$
\begin{aligned}
Ss(2N)_+:&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(1{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}1),\\
Ss(2N)_-:&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(1{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}1;0{,}1).
\end{aligned}
$$

$$
\begin{aligned}
Ss(2N)_+:&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(1{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;1{,}0),\\
Ss(2N)_-:&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(1{,}0;0{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}1;1{,}0).
\end{aligned}
$$

$$
\begin{aligned}
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(0{,}0;0{,}0)\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(n_{SS},n_{SC};1{,}0),\\
&\left(z_{e,S},z_{e,C};z_{m,S},z_{m,C}\right)=(n_{CS},n_{CC};0{,}1),
\end{aligned}
$$

$$
(z_e,z_m)\in\mathbb{Z}_4\times\mathbb{Z}_4
$$

$$
\begin{aligned}
&\mathrm{Spin}(2N),\\
&SO(2N)=\mathrm{Spin}(2N)/\mathbb{Z}_2,\\
&SO(2N)/\mathbb{Z}_2=\mathrm{Spin}(2N)/\mathbb{Z}_4.
\end{aligned}
$$

$$
(z_e,z_m)=(0{,}0),(z_e,z_m)=(n,1)
$$

$$
\chi_{\mathrm{sp}}^{\mathfrak{so}(2N)}=\frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]
$$

$$
\chi_{\overline{\mathrm{sp}}}^{\mathfrak{so}(2N)}=\frac{1}{2}\left[\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)-\prod_{i=1}^{N}\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]
$$

$$
\chi_{\square}^{\mathfrak{so}\,(2N)}=\sum_{i=1}^{N}\left(s_i+s_i^{-1}\right)
$$

$$
\chi_{\lambda}^{\mathfrak{so}(2N)}=\frac{\det\left(s_j^{\lambda_i+N-i}+s_j^{-\lambda_i-N+i}\right)+\det\left(s_j^{\lambda_i+N-i}-s_j^{-\lambda_i-N+i}\right)}{\det\left(s_j^{N-i}+s_j^{-N+i}\right)}
$$

$$\langle W_{\mathcal{R}_1}\cdots W_{\mathcal{R}_k}\rangle^{SO(2N)} = \int d\mu^{SO(2N)}\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n(q,t)M_n(s)\right)\prod_{i=1}^{k}\chi_{\mathcal{R}_i}^{so(2N)}(s),$$

$$d\mu^{SO(2N)} = \frac{1}{2^{N-1}N!}\prod_{i=1}^{N}\frac{ds_i}{2\pi i s_i}\prod_{1\le i<j\le N}(1-s_is_j)(1-s_i^{-1}s_j^{-1})(1-s_is_j^{-1})(1-s_i^{-1}s_j)$$

$$M_n(s) = \frac{P_n(s)^2+P_{2n}(s)}{2}$$

$$\mathcal{I}^{SO(4)}(t;q) = \mathcal{I}^{SU(2)}(t;q)\times\mathcal{I}^{SU(2)}(t;q) = \frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}{(q;q)_\infty^4}\left(\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1<p_2}}\frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2-2}}{\left(1-q^{p_1-\frac{1}{2}}t^2\right)\left(1-q^{p_2-\frac{1}{2}}t^2\right)}\right)^2.$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(4)}(t;q) = \frac{1}{4}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}$$
$$\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}$$
$$\times\left(s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}+s_1^{-\frac{1}{2}}s_2^{-\frac{1}{2}}\right)^2.$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(4)/\mathbb{Z}_2}(t;q) = \frac{1}{2}\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}$$
$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{(qt^2s_1^{\pm}s_2^{\mp};q)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty(qt^{-2}s_1^{\pm}s_2^{\mp};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}.$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{Spin(4)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(4)/\mathbb{Z}_2}(t;q) = \mathcal{I}^{SU(2)}(t;q)\langle W_\Box W_\Box\rangle^{SU(2)}(t;q),$$

$$\langle W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2}\right)} \right\rangle^{SO(4)/\mathbb{Z}_2}_{\frac{1}{2}\text{BPS}} (\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2}$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)}.$$

$$\langle \underbrace{W_{\text{sp}} \cdots W_{\text{sp}}}_{2k} \rangle^{Spin(4)}(t;q) = \mathcal{I}^{SU(2)}(t;q) \langle \underbrace{W_{\square} \cdots W_{\square}}_{2k} \rangle^{SU(2)}(t;q)$$

$$\langle \underbrace{W_{\text{sp}} \cdots W_{\text{sp}}}_{2k} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \mathcal{I}^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) \sum_{i=0}^{k} a^{\text{so}(4)}_{ksp}(i)\mathfrak{q}^{2i}$$
$$= \frac{1}{(1-\mathfrak{q}^4)^2} \sum_{i=0}^{k} a^{\text{so}(4)}_{ksp}(i)\mathfrak{q}^{2i}$$

$$a^{\text{so}(4)}_{ksp}(i) = \frac{(2i+1)(2k)!}{(k-i)!\,(k+i+1)!}$$

$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{2+3\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2},$$
$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{Spin(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{5+9\mathfrak{q}^2+5\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)^2},$$
$$\langle W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} W_{\text{sp}} \rangle^{Spin(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{14+28\mathfrak{q}^2+20\mathfrak{q}^4+7\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}.$$

$$\langle W^{2k}_{\text{sp}} W^{2m}_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \left(\mathcal{I}^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})\right)^{-1} \langle W^{2k}_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) \langle W^{2m}_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})$$
$$= (1-\mathfrak{q}^4)^2 \langle W^{2k}_{\text{sp}} \rangle^{\text{Sin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) \langle W^{2m}_{\text{sp}} \rangle^{\text{Sin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}).$$

$$\langle W^{2k}_{\text{sp}} W^{2m}_{\text{sp}} \rangle^{\text{Spin}(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)^2} \left(\sum_{i=0}^{k} a^{\text{so}(4)}_{ksp}(i)\mathfrak{q}^{2i} \right) \left(\sum_{j=0}^{m} a^{\text{so}(4)}_{msp}(j)\mathfrak{q}^{2j} \right)$$

$$\langle W_{\square} W_{\square} \rangle^{SO(4)}(t;q) = \frac{1}{4} \frac{(q)^4_\infty}{\left(q^{\frac{1}{2}} t^{\pm 2}; q\right)^2_\infty} \oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i}$$
$$\times \frac{(s_1^{\pm} s_2^{\mp}; q)_\infty (s_1^{\pm} s_2^{\pm}; q)_\infty (q s_1^{\pm} s_2^{\mp}; q)_\infty (q s_1^{\pm} s_2^{\pm}; q)_\infty}{\left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 s_1^{\pm} s_2^{\pm}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\mp}; q\right)_\infty \left(q^{\frac{1}{2}} t^{-2} s_1^{\pm} s_2^{\pm}; q\right)_\infty}$$
$$\times (s_1 + s_2 + s_1^{-1} + s_2^{-1})^2.$$

$$\left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(4)}(t;q)=\frac{(q)_\infty^4}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}$$

$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{\left(qt^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(qt^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(qt^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}.$$

$$\langle W_\square W_\square\rangle^{SO(4)}(t;q)=\langle W_\square W_\square\rangle^{SU(2)}(t;q)^2$$

$$\left\langle W_\square W_\square\right\rangle^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^2)^2}$$

$$\langle\underbrace{W_\square\cdots W_\square}_{2k}\rangle^{SO(4)}(t;q)=\langle\underbrace{W_\square\cdots W_\square}_{k}\rangle^{SU(2)}(t;q)^2$$

$$\langle\underbrace{W_\square\cdots W_\square}_{2k}\rangle^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{2k}a_k^{\mathfrak{so}(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_\square W_\square W_\square W_\square\rangle^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{4+12\mathfrak{q}^2+13\mathfrak{q}^4+6\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_\square W_\square W_\square W_\square W_\square W_\square\right\rangle^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{25+90\mathfrak{q}^2+131\mathfrak{q}^4+100\mathfrak{q}^6+43\mathfrak{q}^8+10\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square W_\square W_\square\rangle^{SO(4)}_{\frac{1}{2}\text{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)^2}(196+784\mathfrak{q}^2+1344\mathfrak{q}^4+1316\mathfrak{q}^6$$
$$+820\mathfrak{q}^8+336\mathfrak{q}^{10}+89\mathfrak{q}^{12}+14\mathfrak{q}^{14}+\mathfrak{q}^{16}).$$

$$a_{k\square}^{\mathrm{so}(4)}(0)=C_k^2$$

$$\langle W_\square W_\square\rangle^{SO(4)^-}(t;q)=\frac{1}{2}\frac{(q)_\infty^2(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}$$

$$\times\frac{(s^{\pm};q)_\infty(-s;q)_\infty(qs^{\pm};q)_\infty(-qs^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^2s^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s^{\pm};q\right)_\infty}(s+s^{-1})^2.$$

$$\left\langle T_{(1)}T_{(1)}\right\rangle^{SO(4)^-}(t;q)=\frac{(q)_\infty^2(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\frac{ds}{2\pi i s}$$

$$\times\frac{\left(q^{\frac{1}{2}}s^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}s;q\right)_\infty\left(q^{\frac{3}{2}}s^{\pm};q\right)_\infty\left(-q^{\frac{3}{2}}s^{\pm};q\right)_\infty}{(qt^2s^{\pm};q)_\infty(-qt^2s^{\pm};q)_\infty(qt^{-2}s^{\pm};q)_\infty(-qt^{-2}s^{\pm};q)_\infty}.$$

$$\langle W_\square W_\square\rangle^{SO(4)^-}(t;q) = \langle T_{(1)}T_{(1)}\rangle^{SO(4)^-}(t;q) = \langle W_\square W_\square\rangle^{SU(2)}(t^2;q^2).$$

$$\langle W_\square W_\square\rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{(1)}T_{(1)}\rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^4}.$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k}\rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{k} a_{k\square}^{\mathfrak{so}(4)^-}(i)\mathfrak{q}^{4i}}{1-\mathfrak{q}^8}$$

$$a_{k\square}^{\mathfrak{so}(4)^-}(i) = (2i+1)\frac{(2k)!}{(k-i)!\,(k+i+1)!},$$

$$\langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{O(4)^+}(t;q) = \frac{1}{2}\Big[\langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{SO(4)}(t;q) + \langle W_{\lambda_1}\cdots W_{\lambda_k}\rangle^{SO(4)^-}(t;q)\Big].$$

$$\langle W_\square W_\square\rangle^{O(4)^+}(t;q) = \langle T_{(1)}T_{(1)}\rangle^{O(4)^+}(t;q)$$

$$\langle \underbrace{W_\square \cdots W_\square}_{2k}\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{2k+1} a_{k\square}^{O(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_\square W_\square\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)} = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)},$$

$$\langle W_\square W_\square W_\square W_\square\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{3+6\mathfrak{q}^2+9\mathfrak{q}^4+9\mathfrak{q}^6+6\mathfrak{q}^8+3\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_\square W_\square W_\square W_\square W_\square W_\square\rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(15+45\mathfrak{q}^2+80\mathfrak{q}^4+95\mathfrak{q}^6+85\mathfrak{q}^8 +55\mathfrak{q}^{10}+20\mathfrak{q}^{12}+5\mathfrak{q}^{14}).$$

$$a_{k\square}^{\mathfrak{o}(4)}(0) = \frac{1}{2}(C_k^2+C_k)$$

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} = (1,1),\ \begin{smallmatrix}\bar{\square}\\\square\end{smallmatrix} = (1,-1)$$

$$\langle \underbrace{W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\cdots W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{k}\rangle^{SO(4)}(t;q) = \langle \underbrace{W_{\begin{smallmatrix}\bar\square\\\square\end{smallmatrix}}\cdots W_{\begin{smallmatrix}\bar\square\\\square\end{smallmatrix}}}_{k}\rangle^{SO(4)}(t;q) = \mathcal{I}^{SU(2)}(t;q)\langle \underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{SU(2)}(t;q).$$

$$\langle \underbrace{W_\square \cdots W_\square}_{k}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{k} a_{k\begin{smallmatrix}\square\\\square\end{smallmatrix}}^{\mathfrak{so}(4)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2},$$
$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2},$$
$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6}{(1-\mathfrak{q}^4)^2}$$
$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{3+6\mathfrak{q}^2+6\mathfrak{q}^4+3\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}$$
$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{6+15\mathfrak{q}^2+15\mathfrak{q}^4+10\mathfrak{q}^6+4\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)^2},$$
$$\left\langle W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}W_{\substack{\square\\\square}}\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{15+36\mathfrak{q}^2+40\mathfrak{q}^4+29\mathfrak{q}^6+15\mathfrak{q}^8+5\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)^2}$$

$$\left\langle \left(W_{\substack{\square\\\square}}\right)^k\left(W_{\substack{\bar{\square}\\\square}}\right)^m\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=(1-\mathfrak{q}^4)^2\left\langle\left(W_{\substack{\square\\\square}}\right)^k\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\left\langle\left(W_{\substack{\bar{\square}\\\square}}\right)^m\right\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$

$$\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SO(4)}(t;q)=\langle\underbrace{W_{(l)}\cdots W_{(l)}}_{k}\rangle^{SU(2)}(t;q)^2$$

$$\langle\underbrace{W_{\square\square}\cdots W_{\square\square}}_{k}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{2k}a_{k\square\square}^{\mathfrak{so}(4)^-}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)^2}$$
$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{(1+\mathfrak{q}^2+\mathfrak{q}^4)^2}{(1-\mathfrak{q}^4)^2}$$
$$\langle W_{\square\square}W_{\square\square}W_{\square\square}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{(1+3\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6)^2}{(1-\mathfrak{q}^4)^2}$$

$$a_{k\square\square}^{\mathfrak{so}(4)^-}(0)=R_k^2$$
$$a_{k\square\square}^{\mathfrak{so}(4)^-}(1)=2R_kR_{k+1}$$

$$\langle(W_{\square\square})^k\rangle^{SO(4)}=\left\langle\left(W_{\substack{\square\\\square}}\right)^k\left(W_{\substack{\bar{\square}\\\square}}\right)^k\right\rangle^{SO(4)}$$

$$\langle W_{(2k)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^{4k}}{(1-\mathfrak{q}^4)^2}$$

$$\langle W_{(k)}W_{(k)}\rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\left(1-\mathfrak{q}^{2k+2}\right)^2}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)^2}$$

$$\langle W_{(\infty)} W_{(\infty)} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)^2}$$

$$\langle \underbrace{W_{(l)} \cdots W_{(l)}}_{k} \rangle^{SO(4)^-}(t;q) = \langle \underbrace{W_{(l)} \cdots W_{(l)}}_{k} \rangle^{SU(2)}(t^2;q^2)$$

$$\langle W_{(2l)} \rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{4l}}{1-\mathfrak{q}^8},$$

$$\langle W_{(l)} W_{(l)} \rangle^{SO(4)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4l+4}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\langle W_{(2l)} \rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{4l}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{(l)} W_{(l)} \rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^2+\mathfrak{q}^4-\mathfrak{q}^{2l+2}-\mathfrak{q}^{2l+6}+\mathfrak{q}^{4l+6}}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\langle W_{(\infty)} W_{(\infty)} \rangle^{O(4)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^2+\mathfrak{q}^4}{(1-\mathfrak{q}^2)^2(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\begin{aligned}\langle \underbrace{W_{(l,l)} \cdots W_{(l,l)}}_{k} \rangle^{SO(4)}(t;q) &= \langle \underbrace{W_{(l,-l)} \cdots W_{(l,-l)}}_{k} \rangle^{SO(4)}(t;q) \\ &= \mathcal{I}^{SU(2)}(t;q) \langle \underbrace{W_{(2l)} \cdots W_{(2l)}}_{k} \rangle^{SU(2)}(t;q).\end{aligned}$$

$$\langle W_{(l,l)} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(l,-l)} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^{2l}}{(1-\mathfrak{q}^4)^2},$$

$$\langle W_{(l,l)} W_{(l,l)} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle W_{(l,-l)} W_{(l,-l)} \rangle^{SO(4)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1-\mathfrak{q}^{4l+2}}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2}.$$

$$\langle W^k_{(l,l)} W^k_{(l,-l)} \rangle^{SO(4)} = \langle W^k_{(2l)} \rangle^{SO(4)}$$

$$\begin{aligned}\mathcal{I}^{SO(6)}(t;q) &= \mathcal{I}^{SU(4)}(t;q) \\ &= -\frac{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}{(q;q)^2_\infty} \sum_{\substack{p_1,p_2,p_3,p_4 \in \mathbb{Z} \\ p_1<p_2<p_3<p_4}} \frac{\left(q^{\frac{1}{2}}t^{-2}\right)^{p_1+p_2+p_3+p_4-8}}{(1-q^{p_1-1}t^4)(1-q^{p_2-1}t^4)(1-q^{p_3-1}t^4)(1-q^{p_4-1}t^4)}\end{aligned}$$

$$\begin{aligned}\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(6)}(t;q) &= \frac{1}{24} \frac{(q)^6_\infty}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)^3_\infty} \oint \prod_{i=1}^{3} \frac{ds_i}{2\pi i s_i} \\ &\times \prod_{i\neq j} \frac{(s_i^{\pm}s_j^{\mp};q)_\infty (s_i^{\pm}s_j^{\pm};q)_\infty (q s_i^{\pm}s_j^{\mp};q)_\infty (q s_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty} \\ &\times (1+s_1s_2+s_1s_3+s_2s_3)(1+s_1^{-1}s_2^{-1}+s_1^{-1}s_3^{-1}+s_2^{-1}s_3^{-1})\end{aligned}$$

$$\left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(6)/\mathbb{Z}_2}(t;q) = \frac{1}{6}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i<j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qt^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(qt^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}.$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{Spin(6)}(t;q) = \langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\rangle^{SO(6)/\mathbb{Z}_2}(t;q)$$
$$= \langle W_{\square}W_{\bar{\square}}\rangle^{SU(4)}(t;q).$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{Spin(6)}(t;q) = \left\langle T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}T_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}\right\rangle^{SO(6)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)}$$

$$\langle\underbrace{W_{\mathrm{sp}}W_{\mathrm{sp}}\cdots W_{\mathrm{sp}}W_{\mathrm{sp}}}_{2k}\rangle^{Spin(6)}(t;q) = \langle\underbrace{W_{\square}W_{\bar{\square}}\cdots W_{\square}W_{\bar{\square}}}_{2k}\rangle^{SU(4)}(t;q)$$

$$\langle\underbrace{W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\cdots W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}}_{2k}\rangle^{Spin(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{3k}a_{k\ \mathrm{sp}}^{\mathrm{so}\,(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}$$

$$\left\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\right\rangle^{Spin(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{2+4\mathfrak{q}^2+6\mathfrak{q}^4+7\mathfrak{q}^6+5\mathfrak{q}^8+3\mathfrak{q}^{10}+\mathfrak{q}^{12}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{Spin(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(6+18\mathfrak{q}^2+35\mathfrak{q}^4+50\mathfrak{q}^6$$
$$+53\mathfrak{q}^8+45\mathfrak{q}^{10}+29\mathfrak{q}^{12}+14\mathfrak{q}^{14}+5\mathfrak{q}^{16}+\mathfrak{q}^{18}).$$

$$\det\begin{pmatrix} I_0(2x) & I_1(2x) & I_2(2x) & I_3(2x)\\ I_1(2x) & I_0(2x) & I_1(2x) & I_2(2x)\\ I_2(2x) & I_1(2x) & I_0(2x) & I_1(2x)\\ I_3(2x) & I_2(2x) & I_1(2x) & I_0(2x)\end{pmatrix} = \sum_{k=0}^{\infty}\frac{a_{k\ \mathrm{sp}}^{\mathrm{so}(6)}(0)}{(k!)^2}x^{2k}$$

$$\langle W_{\square}W_{\square}\rangle^{SO(6)}(t;q) = \frac{1}{24}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i\neq j}\frac{\left(s_i^{\pm}s_j^{\mp};q\right)_\infty\left(s_i^{\pm}s_j^{\pm};q\right)_\infty\left(qs_i^{\pm}s_j^{\mp};q\right)_\infty\left(qs_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times(s_1+s_2+s_3+s_1^{-1}+s_2^{-1}+s_3^{-1})^2.$$

$$\left\langle T_{(1,0,0)}T_{(1,0,0)}\right\rangle^{SO(6)}(t;q)=\frac{1}{4}\frac{(q)_\infty^6}{\left(q^{\frac{1}{2}}t2;q\right)_\infty^3}\oint\prod_{i=1}^{3}\frac{ds_i}{2\pi i s_i}$$

$$\times\prod_{i<j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty}$$

$$\times\frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1'}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1,1}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}.$$

$$\langle W_\square W_\square\rangle^{SO(6)}(t;q)=\left\langle T_{(1,0,0)}T_{(1,0,0)}\right\rangle^{SO(6)}(t;q)$$
$$=\left\langle W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}\right\rangle^{SU(4)}(t;q).$$

$$\langle W_\square W_\square\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\left\langle T_{(1,0,0)}T_{(1,0,0)}\right\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$=\frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}$$
$$=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)^2}.$$

$$\langle\underbrace{W_\square\cdots W_\square}_{2k}\rangle^{SO(6)}(t;q)=\langle\underbrace{W_{\substack{\square\\\square}}\cdots W_{\substack{\square\\\square}}}_{2k}\rangle^{SU(4)}(t;q)$$

$$\langle\langle\underbrace{W_\square\cdots W_\square}_{2k}\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{4k}a_{k\square}^{\mathfrak{son}(6)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}$$

$$\langle W_\square W_\square W_\square W_\square\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(3+7\mathfrak{q}^2+15\mathfrak{q}^4+18\mathfrak{q}^6+20\mathfrak{q}^8$$
$$+14\mathfrak{q}^{10}+9\mathfrak{q}^{12}+3\mathfrak{q}^{14}+\mathfrak{q}^{16}),$$
$$\langle W_\square W_\square W_\square W_\square W_\square W_\square\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)}(16+60\mathfrak{q}^2+149\mathfrak{q}^4+249\mathfrak{q}^6$$
$$+334\mathfrak{q}^8+347\mathfrak{q}^{10}+301\mathfrak{q}^{12}+206\mathfrak{q}^{14}+119\mathfrak{q}^{16}$$
$$+53\mathfrak{q}^{18}+20\mathfrak{q}^{20}+5\mathfrak{q}^{22}+\mathfrak{q}^{24}).$$

$$\langle W_\square W_\square\rangle^{SO(6)^-}(t;q)=\frac{1}{8}\frac{(q)_\infty^4(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}$$

$$\times\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}\frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty(-qs_i^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$

$$\times\frac{(s_1^{\pm}s_2^{\mp};q)_\infty(s_1^{\pm}s_2^{\pm};q)_\infty(qs_1^{\pm}s_2^{\mp};q)_\infty(qs_1^{\pm}s_2^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_1^{\pm}s_2^{\pm};q\right)_\infty}$$

$$\times(s_1+s_2+s_1^{-1}+s_2^{-1})^2.$$

$$\left\langle T_{(1,0)}T_{(1,0)}\right\rangle^{SO(6)^-}(t;q)=\frac{1}{2}\frac{(q)_\infty^4(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^2\left(-q^{\frac{1}{2}}t^{\pm};q\right)_\infty}\oint\prod_{i=1}^{2}\frac{ds_i}{2\pi i s_i}$$

$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm};q\right)_\infty(s_2^{\pm};q)_\infty\left(-q^{\frac{1}{2}}s_1^{\pm};q\right)_\infty(-s_2^{\pm};q)_\infty}{(qt^2s_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^2s_2^{\pm};q\right)_\infty(-qt^2s_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^2s_2^{\pm};q\right)_\infty}$$

$$\times\frac{\left(q^{\frac{3}{2}}s_1^{\pm};q\right)_\infty(qs_2^{\pm};q)_\infty\left(-q^{\frac{3}{2}}s_1^{\pm};q\right)_\infty(-qs_2^{\pm};q)_\infty}{(qt^{-2}s_1^{\pm};q)_\infty\left(q^{\frac{1}{2}}t^{-2}s_2^{\pm};q\right)_\infty(-qt^{-2}s_1^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_2^{\pm};q\right)_\infty}$$

$$\times\frac{\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\mp};q\right)_\infty\left(q^{\frac{3}{2}}s_1^{\pm}s_2^{\pm};q\right)_\infty}{(qt^2s_1^{\pm}s_2^{\mp};q)_\infty(qt^2s_1^{\pm}s_2^{\pm};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\mp};q)_\infty(qt^{-2}s_1^{\pm}s_2^{\pm};q)_\infty},$$

$$\begin{aligned}\langle W_\square W_\square\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})&=\langle T_{(1,0)}T_{(1,0)}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})\\&=\frac{1+\mathfrak{q}^2+\mathfrak{q}^6+\mathfrak{q}^8}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}\\&=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^8)}.\end{aligned}$$

$$\langle\underbrace{W_\square\cdots W_\square}_{2k}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\sum_{i=0}^{4k}a_k^{so(6)^-}(i)\mathfrak{q}^{2i}}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)^8},$$

$$\langle W_\square W_\square W_\square W_\square\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{3+5\mathfrak{q}^2+3\mathfrak{q}^4+6\mathfrak{q}^6+8\mathfrak{q}^8+4\mathfrak{q}^{10}+3\mathfrak{q}^{12}+3\mathfrak{q}^{14}+\mathfrak{q}^{16}}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}$$

$$\begin{aligned}\langle W_\square W_\square W_\square W_\square W_\square W_\square\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})&=\frac{1}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(14+30\mathfrak{q}^2+31\mathfrak{q}^4+49\mathfrak{q}^6\\&+66\mathfrak{q}^8+55\mathfrak{q}^{10}+49\mathfrak{q}^{12}+46\mathfrak{q}^{14}+29\mathfrak{q}^{16}+15\mathfrak{q}^{18}\\&+10\mathfrak{q}^{20}+5\mathfrak{q}^{22}+\mathfrak{q}^{24})\end{aligned}$$

$$a_k^{so(6)^-}(0)=C_kC_{k+2}-C_{k+1}^2$$

$$\left\langle W_{\lambda_1}\cdots W_{\lambda_k}\right\rangle^{O(6)^+}(t;q)=\frac{1}{2}\left[\left\langle W_{\lambda_1}\cdots W_{\lambda_k}\right\rangle^{SO(6)^+}(t;q)+\left\langle W_{\lambda_1}\cdots W_{\lambda_k}\right\rangle^{SO(6)^-}(t;q)\right]$$

$$\begin{aligned}\langle W_\square W_\square\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})&=\frac{1+\mathfrak{q}^2+\mathfrak{q}^4+\mathfrak{q}^6+\mathfrak{q}^8+\mathfrak{q}^{10}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}\\&=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},\\ \langle W_\square W_\square W_\square W_\square\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})&=\frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}(3+6\mathfrak{q}^2+9\mathfrak{q}^4+12\mathfrak{q}^6\\&+15\mathfrak{q}^8+15\mathfrak{q}^{10}+12\mathfrak{q}^{12}+9\mathfrak{q}^{14}+6\mathfrak{q}^{16}+3\mathfrak{q}^{18}).\end{aligned}$$

$$\langle \underbrace{W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \cdots W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}_{k} \rangle^{SO(6)}(t;q) = \langle \underbrace{W_{\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}} W_{\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}}}_{k} \rangle^{SU(4)}(t;q),$$

$$\Big\langle \underbrace{W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \cdots W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}}_{2k} \Big\rangle^{SO(6)}(t;q) = \langle \underbrace{W_{\square\square} \cdots W_{\square\square}}_{2k} \rangle^{SU(4)}(t;q)$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times (1 + 2\mathfrak{q}^2 + 4\mathfrak{q}^4 + 6\mathfrak{q}^6 + 7\mathfrak{q}^8 + 7\mathfrak{q}^{10}$$
$$+ 6\mathfrak{q}^{12} + 5\mathfrak{q}^{14} + 3\mathfrak{q}^{16} + \mathfrak{q}^{18}),$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} W_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \Big\rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times (1 + \mathfrak{q}^2 + 2\mathfrak{q}^4 + 3\mathfrak{q}^6 + 3\mathfrak{q}^8 + 3\mathfrak{q}^{10}$$
$$+ 3\mathfrak{q}^{12} + 2\mathfrak{q}^{14} + \mathfrak{q}^{16} + \mathfrak{q}^{18}).$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1+\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + 2\mathfrak{q}^4 + \mathfrak{q}^8}{(1+\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{(1-\mathfrak{q}^4)^2(1-\mathfrak{q}^8)},$$

$$\Big\langle W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \Big\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + 2\mathfrak{q}^4 + \mathfrak{q}^6 + 2\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2(1-\mathfrak{q}^8)}.$$

$$\langle \underbrace{W_{(l)} \cdots W_{(l)}}_{2k} \rangle^{SO(6)}(t;q) = \langle \underbrace{W_{(l^2)} \cdots W_{\overline{(l^2)}}}_{k} \rangle^{SU(4)}(t;q)$$

$$\langle W_{\square\square} \rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^4 + \mathfrak{q}^8}{(1+\mathfrak{q}^4)(1-\mathfrak{q}^6)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square\square} W_{\square\square} \rangle^{SO(6)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}$$
$$\times (1 + \mathfrak{q}^2 + 3\mathfrak{q}^4 + 4\mathfrak{q}^6 + 6\mathfrak{q}^8 + 6\mathfrak{q}^{10} + 7\mathfrak{q}^{12}$$
$$+ 6\mathfrak{q}^{14} + 4\mathfrak{q}^{16} + 4\mathfrak{q}^{18} + \mathfrak{q}^{20} + \mathfrak{q}^{22})$$

$$\langle W_{\square\square}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{SO(6)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1+2\mathfrak{q}^8-2\mathfrak{q}^{10}+\mathfrak{q}^{12}-\mathfrak{q}^{14}-\mathfrak{q}^{18}}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)}.$$

$$\langle W_{\square\square}\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{\mathfrak{q}^4+\mathfrak{q}^8}{(1+\mathfrak{q}^6)(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)},$$

$$\langle W_{\square\square}W_{\square\square}\rangle^{O(6)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})=\frac{1+\mathfrak{q}^2+2\mathfrak{q}^4+2\mathfrak{q}^6+4\mathfrak{q}^8+3\mathfrak{q}^{10}+4\mathfrak{q}^{12}+2\mathfrak{q}^{14}+2\mathfrak{q}^{16}+\mathfrak{q}^{18}}{(1-\mathfrak{q}^4)(1-\mathfrak{q}^8)(1-\mathfrak{q}^{12})}.$$

$$\langle W_{\mathrm{sp}}W_{\mathrm{sp}}\rangle^{\mathrm{Spin}(2N)}(t;q)=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}$$
$$\times\prod_{i\neq j}\frac{(s_i^\pm s_j^\mp;q)_\infty(s_i^\pm s_j^\pm;q)_\infty(qs_i^\pm s_j^\mp;q)_\infty(qs_i^\pm s_j^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^\pm s_j^\mp;q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^\pm s_j^\pm;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q\right)_\infty}$$
$$\times\frac{1}{4}\left[\prod_{i=1}^N\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^N\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]^2.$$

$$\langle W_{\mathrm{sp}}W_{\overline{\mathrm{sp}}}\rangle^{\mathrm{Spin}(2N)}(t;q)=\frac{1}{2^{N-1}N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}tt^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}$$
$$\times\prod_{i\neq j}\frac{(s_i^\pm s_j^\mp;q)_\infty(s_i^\pm s_j^\pm;q)_\infty(qs_i^\pm s_j^\mp;q)_\infty(qs_i^\pm s_j^\pm;q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^\pm s_j^\mp;q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^\pm s_j^\pm;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\pm;q\right)_\infty}$$
$$\times\frac{1}{4}\left[\prod_{i=1}^N\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)+\prod_{i=1}^N\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]\left[\prod_{i=1}^N\left(s_i^{\frac{1}{2}}+s_i^{-\frac{1}{2}}\right)-\prod_{i=1}^N\left(s_i^{\frac{1}{2}}-s_i^{-\frac{1}{2}}\right)\right]$$

$$\langle T_{\left(\frac{1}{2}N\right)}T_{\left(\frac{1}{2}N\right)}\rangle^{SO(2N)/\mathbb{Z}_2}(t;q)=\frac{1}{N!}\frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}\oint\prod_{i=1}^N\frac{ds_i}{2\pi i s_i}$$
$$\times\prod_{i<j}\frac{(s_i^\pm s_j^\mp;q)_\infty\left(q^{\frac{1}{2}}s_i^\pm s_j^\pm;q\right)_\infty(qs_i^\pm s_j^\mp;q)_\infty\left(q^{\frac{3}{2}}s_i^\pm s_j^\pm;q\right)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^\pm s_j^\mp;q\right)_\infty\left(qt^2s_i^\pm s_j^\pm;q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^\pm s_j^\mp;q\right)_\infty\left(qt^{-2}s_i^\pm s_j^\pm;q\right)_\infty}.$$

$$\langle W_{\mathrm{sp}} W_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle T_{\left(\frac{1}{2}^N\right)} T_{\left(\frac{1}{2}^N\right)} \rangle^{SO(2N)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{2n}}$$

$$\langle W_{\mathrm{sp}} W_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{\left(\frac{1}{2}^N\right)} T_{\left(\frac{1}{2}^{N}\right)} \right\rangle^{SO(2N)/\mathbb{Z}_2}_{\frac{1}{2}\mathrm{BPS}} (\mathfrak{q})$$
$$= \prod_{n=1}^{N} \frac{1}{1-\mathfrak{q}^{2n}}$$

$$\mathcal{I}^{Spin(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{1-\mathfrak{q}^{2N}} \prod_{n=1}^{N-1} \frac{1}{1-\mathfrak{q}^{4n}}$$

$$\langle \mathcal{W}_{\mathrm{sp}} \mathcal{W}_{\mathrm{sp}} \rangle^{\mathrm{Spin}(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\mathrm{sp}} \mathcal{W}_{\overline{\mathrm{sp}}} \rangle^{\mathrm{Spin}(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \prod_{n=1}^{N-1} (1+\mathfrak{q}^{2n}).$$

$$\langle \underbrace{W_{\mathrm{sp}} \cdots W_{\mathrm{sp}}}_{2k} \rangle^{\mathrm{Spin}(2N=4n)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \underbrace{W_{\mathrm{sp}} \cdots W_{\overline{\mathrm{sp}}}}_{2k} \rangle^{\mathrm{Spin}(2N=4n+2)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{\sum_{i=0}^{\frac{N(N+1)k}{2}} a^{\mathfrak{sp}(2N)}_{k\mathfrak{sp}}(i)\mathfrak{q}^{2i}}{1-\mathfrak{q}^{2N}\prod_{n=1}^{N-1} 1-\mathfrak{q}^{4n}},$$

$$\langle W_{\square} W_{\square} \rangle^{SO(2N)}(t;q) = \frac{1}{2^{N-1}N!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i}$$
$$\times \prod_{i\neq j} \frac{(s_i^{\pm}s_j^{\mp};q)_\infty (s_i^{\pm}s_j^{\pm};q)_\infty (q s_i^{\pm}s_j^{\mp};q)_\infty (q s_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}}t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty} \left[\sum_{i=1}^{N} (s_i + s_i^{-1})\right]^2.$$

$$\left\langle T_{(1,0^{N-1})} T_{(1,0^{N-1})} \right\rangle^{SO(2N)} (t;q) = \frac{1}{2^{N-2}(N-1)!} \frac{(q)_\infty^{2N}}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^N}$$
$$\times \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i<j} \frac{\left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^2 s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times \frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{1+\frac{1}{2}\delta_{i+j,1}} s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm}s_j^{\mp};q\right)_\infty \left(q^{\frac{1}{2}(1+\delta_{i+j,1})} t^{-2} s_i^{\pm}s_j^{\pm};q\right)_\infty}.$$

$$\langle W_{\square} W_{\square}\rangle^{SO(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1+\mathfrak{q}^2+\cdots\mathfrak{q}^{2N-4}+2\mathfrak{q}^{2N-2}+\mathfrak{q}^{2N}+\cdots\mathfrak{q}^{4N-4}}{(1-\mathfrak{q}^{2N})\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^{2(N-1)})\prod_{n=1}^{N-2}(1-\mathfrak{q}^{4n})}.$$

$$\langle \underbrace{W_{\square}\cdots W_{\square}}_{2k}\rangle^{SO(2N)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{\sum_{i=0}^{(2N-2)k} a_k^{\mathfrak{so}(2N)}(i)\mathfrak{q}^{2i}}{(1-\mathfrak{q}^{2N})\prod_{n=1}^{N-1}(1-\mathfrak{q}^{4n})}.$$

$$\langle W_{\square} W_{\square}\rangle^{SO(2N)^-}(t;q) = \frac{1}{2^{N-1}(N-1)!}\frac{(q)_\infty^{2N-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}$$
$$\times\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi i s_i}\ \frac{(s_i^{\pm};q)_\infty(-s_i^{\pm};q)_\infty(qs_i^{\pm};q)_\infty(-qs_i^{\pm};q)_\infty}{t^2s_i^{\pm};q)_\infty\left(-q^{\frac{1}{2}}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{(s_i^{\pm}s_j^{\mp};q)_\infty(s_i^{\pm}s_j^{\pm};q)_\infty(qs_i^{\pm}s_j^{\mp};q)_\infty(qs_i^{\pm}s_j^{\pm};q)_\infty}{\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}\left[\sum_{i=1}^{N-1}(s_i+s_i^{-1})\right]^2.$$

$$\langle T_{(1,0^{N-2})}T_{(1,0^{N-2})}\rangle^{SO(2N)^-}(t;q)$$
$$=\frac{1}{2^{N-2}(N-2)!}\frac{(q)_\infty^{2N-2}(-q;q)_\infty^2}{\left(q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty^{N-1}\left(-q^{\frac{1}{2}}t^{\pm 2};q\right)_\infty}\oint\prod_{i=1}^{N-1}\frac{ds_i}{2\pi i s_i}$$
$$\times\frac{\left(q^{\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty\left(-q^{1+\frac{1}{2}\delta_{i,1}}s_i^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i,1})}t^2s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}t^2s_i^{\pm};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i,1})}t^{-2}s_i^{\pm};q\right)_\infty\left(-q^{\frac{1}{2}(1+\delta_{i,1})}t^{-2}s_i^{\pm};q\right)_\infty}$$
$$\times\prod_{i<j}\frac{\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^2s_i^{\pm}s_j^{\pm};q\right)_\infty}$$
$$\times\frac{\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{1+\frac{1}{2}\delta_{i+j,1}}s_i^{\pm}s_j^{\pm};q\right)_\infty}{\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^{-2}s_i^{\pm}s_j^{\mp};q\right)_\infty\left(q^{\frac{1}{2}(1+\delta_{i+j,1})}t^{-2}s_i^{\pm}s_j^{\pm};q\right)_\infty}.$$

$$\langle W_\square W_\square \rangle^{SO(2N)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle T_{(1,0^{N-2})} T_{(1,0^{N-2})} \right\rangle^{SO(2N)^-}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1 + \mathfrak{q}^2 + \cdots + \mathfrak{q}^{2N-4}}{\prod_{n=1}^{N-1}(1 - \mathfrak{q}^{4n})}$$
$$= \frac{1 - \mathfrak{q}^{2(N-1)}}{1 - \mathfrak{q}^2} \prod_{n=1}^{N-1} \frac{1}{1 - \mathfrak{q}^{4n}}$$

$$\langle W_\square W_\square \rangle^{O(2N)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 + \mathfrak{q}^2 + \cdots + \mathfrak{q}^{4N-2}}{\prod_{n=1}^{N}(1 - \mathfrak{q}^{4n})}$$
$$= \frac{1}{1 - \mathfrak{q}^2} \prod_{n=1}^{N-1} \frac{1}{1 - \mathfrak{q}^{4n}}.$$

$$\mathcal{I}^{O(2N)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \prod_{n=1}^{N} \frac{1}{1 - \mathfrak{q}^{4n}}$$

$$\langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{O(2N)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1 - \mathfrak{q}^{4N}}{1 - \mathfrak{q}^2}$$

$$\mathcal{I}^{SO(2\infty+1)}(t;q) = \mathcal{I}^{USp(2\infty)}(t;q) = \mathcal{I}^{SO(2\infty)}(t;q) = \mathcal{I}^{O(2\infty)^+}(t;q)$$
$$= \prod_{n,m,l=0}^{\infty} \frac{\left(1 - q^{n+m+l+\frac{3}{2}} t^{-4m+4l\pm 2}\right)^2}{(1 - q^{n+m+l+1} t^{-4m+4l\pm 4})(1 - q^{n+m++1} t^{-4m+4l})(1 - q^{n+m+l+3} t^{-4m+4l})}.$$

$$i^{AdS_5 \times \mathbb{RP}^5}(t;q) = \frac{q^{\frac{1}{2}}(t^2 + t^{-2}) - q - q^2}{(1 - qt^4)(1 - qt^{-4})} - \frac{q^{\frac{1}{2}}(t^2 + t^{-2})}{\left(1 + q^{\frac{1}{2}} t^2\right)\left(1 + q^{\frac{1}{2}} t^{-2}\right)(1 - q)}$$

$$i^X(t;q) := \mathrm{Tr}(-1)^F q^{\frac{h+j}{2}} t^{2(q_2 - q_3)}$$

$$i^X_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) := \mathrm{Tr}(-1)^F \mathfrak{q}^{2(q_2 - q_3)}$$

$$\langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \frac{1}{1 - \mathfrak{q}^2}.$$

$$\langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{USp(2\infty)}(t;q)$$
$$= \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{SO(2\infty)}(t;q) = \langle \mathcal{W}_\square \mathcal{W}_\square \rangle^{O(2\infty)^+}(t;q) = \frac{1 - q}{\left(1 - q^{\frac{1}{2}t^2}\right)\left(1 - q^{\frac{1}{2}} t^{-2}\right)}.$$

$$i^{\mathrm{string}}(t;q) = -q + q^{\frac{1}{2}} t^2 + q^{\frac{1}{2}} t^{-2}$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\text{BPS}}(q) = \langle \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\text{BPS}}(q) = \langle \mathcal{W}_{\square\square} \rangle^{O(2\infty)^+}_{\frac{1}{2}\text{BPS}}(q)$$
$$= \frac{q^2}{(1-q^4)}$$
$$= q^2 + q^6 + q^{10} + q^{14} + q^{18} + \cdots .$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\text{BPS}}(q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\text{BPS}}(q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{O(2\infty)^+}_{\frac{1}{2}\text{BPS}}(q)$$
$$= \frac{1+q^2+q^4}{(1-q^4)^2}$$
$$= 1 + q^2 + 3q^4 + 2q^6 + 5q^8 + 3q^{10} + 7q^{12} + 4q^{14} + 9q^{16} + \cdots .$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}_{\frac{1}{2}\text{BPS},c}(q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{\frac{1}{2}\text{BPS},c}(q) \langle \mathcal{W}_{\square} \mathcal{W}_{\square} \rangle^{O(2\infty)^+}_{\frac{1}{2}\text{BPS},c}$$
$$= \frac{1}{(1-q^2)(1-q^4)} = 1 + q^2 2q^4 + 2q^6 + 3q^8 + 3q^{10} + 4q^{12} + 4q^{14} + 5q^{16} + 5q^{18} + \cdots$$

$$\langle \mathcal{W}_\lambda \mathcal{W}_\lambda \rangle^{G}_{\frac{1}{2}\text{BPS},c}(q) := \langle \mathcal{W}_\lambda \mathcal{W}_\lambda \rangle^{G}_{\frac{1}{2}\text{BPS}}(q) - \langle \mathcal{W}_\lambda \rangle^{G}_{\frac{1}{2}\text{BPS}}(q)^2 .$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{q^{\frac{1}{2}}(t^2+t^{-2}) - q - q^2}{(1-qt^4)(1-qt^4)}$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}(t;q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{1}{(1-qt^4)(1-qt^{-4})}\Big(1 + (t^2+t^{-2})q^{\frac{1}{2}} + (3+t^4+t^{-4})q - 3(t^2+t^{-2})q^{\frac{3}{2}}$$
$$-(t^2+t^{-2})q^2 - 3(t^2+t^{-2})q^{\frac{5}{2}} + (3+t^4+t^{-4})q^3 + (t^2+t^{-2})q^{\frac{7}{2}} + q^4\Big).$$

$$\left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{SO(2\infty+1)}_{c}(t;q) = \langle \mathcal{W}_{\square\square} \mathcal{W}_{\square\square} \rangle^{USp(2\infty)}_{c}(t;q) = \left\langle \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mathcal{W}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \right\rangle^{O(2\infty)^+}_{c}(t;q)$$
$$= \frac{(1-q)\left(1+q-q^{\frac{3}{2}}(t^2+t^{-2})\right)}{\left(1-q^{\frac{1}{2}}t^2\right)\left(1-q^{\frac{1}{2}}t^{-2}\right)(1-qt^4)(1-qt^{-4})}.$$

$$\langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(2\infty+1)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Sin(4\infty)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \left\langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\right\rangle^{S^{\rm Sin(4\infty+2)}}_{\frac{1}{2}\rm BPS}(\mathfrak{q})$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-\mathfrak{q}^{4n-2}}.$$

$$\langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(\infty)}_{\frac{1}{2}\rm BPS}(\mathfrak{q}) = \sum_{n\geq 0} d^{(H)}_{\{\rm sp,sp\}}(n)\mathfrak{q}^{2n}$$

$$i^{\text{fat string}}_{\frac{1}{2}\,\text{BPS}}(\mathfrak{q}) = \frac{\mathfrak{q}^2}{1-\mathfrak{q}^4} = \mathfrak{q}^2 + \mathfrak{q}^6 + \mathfrak{q}^{10} + \cdots$$

$$S_{\rm D5} = T_5 \int d^6\sigma \sqrt{\det(g+2\pi\alpha' F)} - iT_5 \int 2\pi\alpha' F \wedge C_{(4)}$$

$$S_{{\rm D5}AdS_2\times\mathbb{RP}^4} = T_5 \int d^6\sigma \sqrt{\det g} = T_5 {\rm vol}(AdS_2){\rm vol}(\mathbb{RP}^4)$$

$$ds^2_{AdS_2} = \frac{1}{r^2}(-dt^2 + dr^2), ds^2_4 = g_{ij} d\sigma^i d\sigma^j, i,j = 1,2,3,4$$

$$S = T_5 \int d^6\sigma \sqrt{g^{(4)}} \frac{1}{2}\frac{1}{r^2}\left[r^2(\partial_t\phi)^2 - r^2(\partial_r\phi)^2 + \left(\nabla_i\phi\nabla^i\phi - 4\phi^2\right)\right].$$

$$\phi(t,r,\Theta) = \sum_w \phi_w(t,r) Y^w(\Theta).$$

$$S = T_5 \sum_w \frac{2}{3}\pi^2 \int d^2\sigma \frac{1}{r^2}(r^2(\partial_t\phi_w)^2 - r^2(\partial_r\phi_w)^2 - w(w+1)\phi_w^2)$$

$$h = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}$$

$$\langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(2\infty+1)}(t;q) = \langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(4\infty)}(t;q) = \langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(4\infty+2)}(t;q)$$

$$= \prod_{n=0}^{\infty}\prod_{m=0}^{\infty} \frac{(1-q^{1+n+m}t^{4n-4m})(1-q^{2+n+m}t^{4n-4m})}{\left(1-q^{\frac{1}{2}+n+m}t^{2+4n-4m}\right)\left(1-q^{\frac{1}{2}+n+m}t^{-2+4n+4m}\right)}$$

$$\langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(2\infty+1)}(q) = \langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(4\infty)}(q) = \langle \mathcal{W}_{\rm sp}\mathcal{W}_{\rm sp}\rangle^{\rm Spin(4\infty+2)}(q)$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)^{2n-1}}{\left(1-q^{n-\frac{1}{2}}\right)^{2n}}$$

$$= 1 + 2q^{1/2} + 2q^2 + 6q^{3/2} + 7q^2 + 10q^{5/2} + 21q^3 + 22q^{7/2} + \cdots.$$

$$i^{\text{fat string}}(t;q) = \frac{q^{\frac{1}{2}}(t^2+t^{-2}) - q - q^2}{(1-qt^4)(1-qt^{-4})}.$$

$$\langle \mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{\mathfrak{q}^4}{(1-\mathfrak{q}^4)}$$

$$\langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1+\mathfrak{q}^2+\mathfrak{q}^8}{(1-\mathfrak{q}^4)^2}.$$

$$\langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_{\frac{1}{2}\mathrm{BPS},c}(\mathfrak{q}) == \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q}) = \langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_{\frac{1}{2}\mathrm{BPS}}(\mathfrak{q})$$
$$= \frac{1}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^4)},$$

$$\langle \mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}(t;q) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}(t;q) = \langle \mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{q(1+t^4+t^{-4}) - q^{\frac{3}{2}}(t^2+t^{-2}) - q^2}{(1-qt^4)(1-qt^{-4})}$$

$$\langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}(t;q) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}(t;q) = \langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}(t;q)$$
$$= \frac{1}{(1-qt^4)(1-qt^{-4})}\Big(1+(t^2+t^{-2})q^{\frac{1}{2}} + q - (t^2+t^{-2})q^{\frac{3}{2}} + (t^8+t^4+t^{-4}+t^{-8})q^2$$
$$-(2t^6+5t^2+5t^{-2}+2t^{-6})q^{\frac{5}{2}} + q^3 + 3(t^2+t^{-2})q^{\frac{7}{2}} + q^4\Big).$$

$$\langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{SO(2\infty+1)}_c(t;q) = \left\langle \mathcal{W}_{\substack{\Box\\\Box}}\mathcal{W}_{\substack{\Box\\\Box}}\right\rangle^{USp(2\infty)}_c(t;q) = \langle \mathcal{W}_{\Box\Box}\mathcal{W}_{\Box\Box}\rangle^{O(2\infty)^+}_c(t;q)$$
$$= \frac{(1-q)\left(1+q-q^{\frac{3}{2}}(t^2+t^{-2})\right)}{\left(1-q^{\frac{1}{2}}t^2\right)\left(1-q^{\frac{1}{2}}t^{-2}\right)(1-qt^4)(1-qt^{-4})}.$$

$$\chi_\lambda^{\mathrm{usp}(2N)} = \det\left(E_{\lambda'_i-i+j} - E_{\lambda'_i-i-j}\right)_{1\le i,j\le l(\lambda')}$$
$$\chi_\lambda^{\mathrm{so}(2N+1)} = \det\left(\bar{H}_{\lambda_i-i+j} - \bar{H}_{\lambda_i-i-j}\right)_{1\le i,j\le l(\lambda)}$$

$$E_k = e_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1})$$
$$\bar{H}_k = h_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1},1)$$

$$P_k = p_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1})$$
$$\bar{P}_k = p_k(s_1,\dots,s_N,s_1^{-1},\dots,s_N^{-1},1)$$

$$p_k(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i^k$$

$$\chi_{\Box}^{\mathrm{usp}\,(2N)} = P_1, \qquad \chi_{\Box}^{\mathrm{so}\,(2N+1)} = \bar{P}_1,$$

$$\chi_{\Box\Box}^{\mathrm{usp}\,(2N)} = \frac{P_2}{2} + \frac{P_1^2}{2}, \qquad \chi_{\Box\Box}^{\mathrm{so}\,(2N+1)} = \frac{\bar{P}_2}{2} + \frac{\bar{P}_1^2}{2} - 1,$$

$$\chi_{\substack{\Box\\\Box}}^{\mathrm{usp}\,(2N)} = -\frac{P_2}{2} + \frac{P_1^2}{2} - 1, \qquad \chi_{\substack{\Box\\\Box}}^{\mathrm{so}\,(2N+1)} = -\frac{\bar{P}_2}{2} + \frac{\bar{P}_1^2}{2},$$

$$\chi_{\Box\Box\Box}^{\mathrm{usp}\,(2N)} = \frac{P_3}{3} + \frac{P_2 P_1}{2} + \frac{P_1^3}{6}, \qquad \chi_{\Box\Box\Box}^{\mathrm{so}\,(2N+1)} = \frac{\bar{P}_3}{3} + \frac{\bar{P}_2 \bar{P}_1}{2} + \frac{\bar{P}_1^3}{6} - \bar{P}_1,$$

$$\chi_{\substack{\Box\Box\\\Box}}^{\mathrm{usp}\,(2N)} = -\frac{P_3}{3} + \frac{P_1^3}{3} - P_1, \qquad \chi_{\substack{\Box\Box\\\Box}}^{\mathrm{so}\,(2N+1)} = -\frac{\bar{P}_3}{3} + \frac{\bar{P}_1^3}{3} - \bar{P}_1,$$

$$\chi_{\substack{\Box\\\Box\\\Box}}^{\mathrm{usp}\,(2N)} = \frac{P_3}{3} - \frac{P_2 P_1}{2} + \frac{P_1^3}{6} - P_1, \qquad \chi_{\substack{\Box\\\Box\\\Box}}^{\mathrm{so}\,(2N+1)} = \frac{\bar{P}_3}{3} - \frac{\bar{P}_2 \bar{P}_1}{2} + \frac{\bar{P}_1^3}{6}.$$

$$T_p\mathcal{M}^{\mathbb{C}} = T_p\mathcal{M}^{+} \oplus T_p\mathcal{M}^{-},$$

$$T_p\mathcal{M}^{\pm} = \left\{Z \in T_p\mathcal{M}^{\mathbb{C}} / \mathcal{J}_p Z = \pm iZ\right\}$$

$$\mathcal{J}_p(\mathcal{P}^{\pm}Z) = \mathcal{J}_p Z^{\pm} = \pm i(\mathcal{P}^{\pm}Z) = \pm i Z^{\pm}$$

$$\mathcal{N}(u,v) \equiv [\mathcal{J}u, \mathcal{J}v] - \mathcal{J}[u, \mathcal{J}v] - \mathcal{J}[\mathcal{J}u, v] - [u,v]$$

$$\phi(p) \equiv z^{\mu} \equiv x^{\mu} + iy^{\mu}, \psi(p) \equiv w^{\mu} \equiv u^{\mu} + iv^{\mu}, 1 \leq \mu, \nu \leq n$$

$$\frac{\partial u^{\nu}}{\partial x^{\mu}} = \frac{\partial v^{\nu}}{\partial y^{\mu}}, \frac{\partial u^{\nu}}{\partial y^{\mu}} = -\frac{\partial v^{\nu}}{\partial x^{\mu}}.$$

$$\mathcal{J}_p \frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial y^{\mu}}, \mathcal{J}_p \frac{\partial}{\partial y^{\mu}} = -\frac{\partial}{\partial x^{\mu}}, \text{ entonces}$$

$$\mathcal{J}_p \frac{\partial}{\partial u^{\mu}} = \frac{\partial}{\partial v^{\mu}}, \mathcal{J}_p \frac{\partial}{\partial v^{\mu}} = -\frac{\partial}{\partial u^{\mu}}$$

$$(\mathcal{J}_p) = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \forall p \in \mathcal{M}$$

$$(\mathcal{J}_p) = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}, \forall p \in \mathcal{M}$$

$$\alpha = \sum_{p+q=k} \alpha^{(p,q)}, \alpha^{(p,q)} = \frac{1}{p!\,q!} \alpha_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}_1} \dots \wedge d\bar{z}^{\bar{\nu}_q}$$

$$d = \partial + \bar{\partial},$$

$$\partial \alpha^{(p,q)} = \alpha^{(p+1,q)}, \bar{\partial}\alpha^{(p,q)} = \alpha^{(p,q+1)},$$

$$\alpha^{(p+1,q)} = \frac{1}{p!\,q!} \partial_{\mu} \alpha^{(p,q)}_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q} dz^{\mu} \wedge dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}_1} \dots \wedge d\bar{z}^{\bar{\nu}_q}.$$

$$\alpha^{(p,q+1)} = \frac{1}{p!\,q!} \partial_{\bar{\nu}} \alpha^{(p,q z^{\mu_1}}_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}} \wedge d\bar{z}^{\bar{\nu}_1} \dots \wedge d\bar{z}^{\bar{\nu}_q}.$$

$$\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2$$

$$\mathcal{G}_p(\mathcal{J}_p u, \mathcal{J}_p v) = \mathcal{G}_p(u, v)\ \forall p \in \mathcal{M}, u, v \in T_p\mathcal{M}$$

$$\mathcal{G}_p(u, v) \equiv g_p(u, v) + g_p(\mathcal{J}_p u, \mathcal{J}_p v)\ \forall p \in \mathcal{M}, u, v \in T_p\mathcal{M}$$

$$\mathcal{G}_p(\mathcal{J}_p u, u) = \mathcal{G}_p(\mathcal{J}_p^2 u, \mathcal{J}_p u) = -\mathcal{G}_p(u, \mathcal{J}_p u) = -\mathcal{G}_p(\mathcal{J}_p u, u)$$

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{G}\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = \mathcal{G}\left(\mathcal{J}_p\frac{\partial}{\partial z^\mu}, \mathcal{J}_p\frac{\partial}{\partial z^\nu}\right) = -\mathcal{G}\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right)$$
$$\mathcal{G}_{\overline{\mu}\overline{\nu}} \equiv \mathcal{G}\left(\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}, \frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) = \mathcal{G}\left(\mathcal{J}_p\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}, \mathcal{J}_p\frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) = -\mathcal{G}\left(\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}, \frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right)$$
$$\mathcal{G}_{\mu\bar{\nu}} \equiv \mathcal{G}\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) = \mathcal{G}\left(\mathcal{J}_p\frac{\partial}{\partial z^\mu}, \mathcal{J}_p\frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) = \mathcal{G}\left(\frac{\partial}{\partial \bar{z}^{\bar{\nu}}}, \frac{\partial}{\partial \bar{z}^{\bar{\mu}}}\right) = \mathcal{G}_{\bar{\nu}\mu}$$

$$\mathcal{G} = \mathcal{G}_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^{\bar{\nu}} + \mathcal{G}_{\bar{\nu}\mu} d\bar{z}^{\bar{\nu}} \otimes dz^\mu$$

$$\omega_p(u, v) \equiv \mathcal{G}_p(\mathcal{J}_p u, v)\ \forall u, v \in T_p\mathcal{M}.$$

$$\omega_p(u, v) = \mathcal{G}_p(\mathcal{J}_p^2 u, \mathcal{J}_p v) = \mathcal{G}_p(-u, \mathcal{J}_p v) = -\mathcal{G}_p(\mathcal{J}_p v, u) = -\omega(u, v).$$

$$\omega_p(\mathcal{J}_p u, \mathcal{J}_p v) = \mathcal{G}_p(\mathcal{J}_p^2 u, \mathcal{J}_p v) = \mathcal{G}_p(-u, \mathcal{J}_p v) = \omega(u, v)$$

$$\omega = i\mathcal{G}_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}}$$

$$[\omega] \in H_{\bar{\partial}}^{(1,1)}(\mathcal{M}; \mathbb{C})$$

$$\omega = i\partial_\mu \partial_{\bar{\nu}} \mathcal{K}(z, \bar{z}) dz^\mu \wedge d\bar{z}^{\bar{\nu}}$$

$$\mathcal{K}' = \mathcal{K} + f + f',$$

$$\omega(\mathcal{K}') = i\partial\bar{\partial}(\mathcal{K} + f + f') = i\partial\bar{\partial}\mathcal{K} = \omega(\mathcal{K}).$$

$$\Gamma^\rho_{\mu\nu} = \mathcal{G}^{\rho\bar{\rho}} \partial_\mu \mathcal{G}_{\bar{\rho}\nu}, \Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} = \mathcal{G}^{\bar{\rho}\rho} \partial_{\bar{\mu}} \mathcal{G}_{\rho\bar{\nu}}$$
$$R_{\mu\bar{\nu}} = \frac{1}{2} \partial_\mu \partial_{\bar{\nu}} (\log\ \det\mathcal{G}).$$

$$\mathfrak{R} \equiv iR_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}}$$

$$\Psi_{(i)} = e^{-(qf_{(i,j)} + \bar{q}\bar{f}_{(i,j)})} \Psi_{(j)},$$

$$\mathcal{K}_{(i)} = \mathcal{K}_{(j)} + f_{(i,j)} + \bar{f}_{(i,j)}.$$

$$\mathcal{Q} \equiv (2i)^{-1}\left(dz^\mu \partial_\mu \mathcal{K} - d\bar{z}^{\bar{\nu}} \partial_{\bar{\nu}} \mathcal{K}\right)$$

$$\mathcal{Q}_{(i)} = \mathcal{Q}_{(j)} - \frac{i}{2} \partial f_{(i,j)},$$

$$\mathfrak{D}_\mu \equiv \nabla_\mu + iq\mathcal{Q}_\mu, \mathfrak{D}_{\bar{\nu}} \equiv \nabla_{\bar{\nu}} - i\bar{q}\mathcal{Q}_{\bar{\nu}}$$

$$D_\mu \equiv \partial_\mu + iq\mathcal{Q}_\mu, D_{\bar{\nu}} \equiv \partial_{\bar{\nu}} - i\bar{q}\mathcal{Q}_{\bar{\nu}}$$

$$\Omega \equiv \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \to \begin{cases} \langle \Omega \mid \bar{\Omega} \rangle & \equiv \overline{\mathcal{X}}^\Lambda \mathcal{F}_\Lambda - \mathcal{X}^\Lambda \overline{\mathcal{F}}_\Lambda = -ie^{-\mathcal{K}} \\ \partial_{\bar{\nu}} \Omega & = 0 \\ \langle \partial_\mu \Omega \mid \Omega \rangle & = 0 \end{cases}$$

$$\mathcal{V} \equiv \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \to \begin{cases} \langle \mathcal{V} \mid \overline{\mathcal{V}} \rangle & \equiv \overline{\mathcal{L}}^\Lambda \mathcal{M}_\Lambda - \mathcal{L}^\Lambda \overline{\mathcal{M}}_\Lambda = -i \\ \mathfrak{D}_{\bar{\nu}} \mathcal{V} & = \left(\partial_{\bar{\nu}} + \frac{1}{2} \partial_{\overline{\mathcal{L}}} \mathcal{K} \right) \mathcal{V} = 0, \\ \langle \mathfrak{D}_\mu \mathcal{V} \mid \mathcal{V} \rangle & = 0 \end{cases}$$

$$\mathcal{U}_\mu \equiv \mathfrak{D}_\mu \mathcal{V} = \begin{pmatrix} f^\Lambda{}_\mu \\ h_{\Sigma\mu} \end{pmatrix}, \overline{\mathcal{U}}_{\bar{\nu}} = \overline{\mathcal{U}_\nu},$$

$$\mathfrak{D}_{\bar{\nu}} \mathcal{U}_\mu = \mathcal{G}_{\mu\bar{\nu}} \mathcal{V} \langle \mathcal{U}_\mu \mid \overline{\mathcal{U}}_{\bar{\nu}} \rangle = i\mathcal{G}_{\mu\bar{\nu}}$$
$$\langle \mathcal{U}_\mu \mid \overline{\mathcal{V}} \rangle = 0, \langle \mathcal{U}_\mu \mid \mathcal{V} \rangle = 0$$

$$\langle \mathfrak{D}_\mu \mathcal{U}_\nu \mid \mathcal{V} \rangle = \langle \mathcal{U}_\nu \mid \mathcal{U}_\mu \rangle = 0.$$

$$\mathcal{A} = i\langle \mathcal{A} \mid \overline{\mathcal{V}} \rangle \mathcal{V} - i\langle \mathcal{A} \mid \mathcal{V} \rangle \overline{\mathcal{V}} + i\langle \mathcal{A} \mid \mathcal{U}_\mu \rangle \mathcal{G}^{\mu\bar{\nu}} \overline{\mathcal{U}}_{\bar{\nu}} - i\langle \mathcal{A} \mid \overline{\mathcal{U}}_{\bar{\nu}} \rangle \mathcal{G}^{\mu\bar{\nu}} \mathcal{U}_\mu$$

$$\mathcal{C}_{\mu\nu\rho} \equiv \langle \mathfrak{D}_\mu \mathcal{U}_\nu \mid \mathcal{U}_\rho \rangle \ \to \ \mathfrak{D}_\mu \mathcal{U}_\nu = i\mathcal{C}_{\mu\nu\rho} \mathcal{G}^{\rho\bar{\epsilon}} \overline{\mathcal{U}}_{\bar{\epsilon}}$$

$$\mathfrak{D}_{\bar{\mu}} \mathcal{C}_{\nu\rho\epsilon} = 0, \mathfrak{D}_{[\mu} \mathcal{C}_{\nu]\rho\epsilon} = 0$$

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma, h_{\Lambda\mu} = \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_\mu.$$

$$\begin{aligned} \mathcal{L}^\Lambda \Im m \mathcal{N}_{\Lambda\Sigma} \overline{\mathcal{L}}^\Sigma &= -\frac{1}{2}, \\ \mathcal{L}^\Lambda \Im m \mathcal{N}_{\Lambda\Sigma} f^\Sigma{}_\mu &= \mathcal{L}^\Lambda \Im m \mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma{}_{\bar{\nu}} = 0 \\ f^\Lambda{}_\mu \Im m \mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma{}_{\bar{\nu}} &= -\frac{1}{2} \mathcal{G}_{\mu\bar{\nu}}. \end{aligned}$$

$$\begin{aligned} (\partial_\mu \mathcal{N}_{\Lambda\Sigma}) \mathcal{L}^\Sigma &= -2i\Im m(\mathcal{N})_{\Lambda\Sigma} f^\Sigma{}_\mu \\ \partial_\mu \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_\nu &= -2\mathcal{C}_{\mu\nu\rho} \mathcal{G}^{\rho\bar{\rho}} \Im m \mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma{}_{\bar{\rho}} \\ \mathcal{C}_{\mu\nu\rho} &= f^\Lambda{}_\mu f^\Sigma{}_\nu \partial_\rho \overline{\mathcal{N}}_{\Lambda\Sigma} \\ \mathcal{L}^\Sigma \partial_{\bar{\nu}} \mathcal{N}_{\Lambda\Sigma} &= 0, \\ \partial_{\bar{\nu}} \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_\mu &= 2i\mathcal{G}_{\mu\bar{\nu}} \Im m \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma. \\ U^{\Lambda\Sigma} \equiv f^\Lambda{}_\mu \mathcal{G}^{\mu\bar{\nu}} \bar{f}^\Sigma{}_{\bar{\nu}} &= -\frac{1}{2} \Im m(\mathcal{N})^{-1|\Lambda\Sigma} - \overline{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma \end{aligned}$$

$$\begin{aligned} \mathcal{T}_\Lambda &\equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im_{\mathrm{m}} \mathcal{N}_{\Sigma\Lambda}, \\ \mathcal{T}^\mu{}_\Lambda &\equiv -\bar{f}_\Lambda{}^\mu = -\mathcal{G}^{\mu\bar{\nu}} \bar{f}^\Sigma{}_{\bar{\nu}} \Im \mathrm{m} \mathcal{N}_{\Sigma\Lambda}. \end{aligned}$$

$$\begin{aligned} \partial_\mu \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{T}_{\mu(\Lambda} \mathcal{T}_{\Sigma)}, \\ \partial_{\bar{\nu}} \mathcal{N}_{\Lambda\Sigma} &= 4\overline{\mathcal{C}}_{\bar{\nu}\bar{\rho}\bar{\epsilon}} \mathcal{T}^{\bar{\nu}}{}_{(\Lambda} \mathcal{T}^{\bar{\rho}}{}_{\Sigma)}. \end{aligned}$$

$$e^{-\mathcal{K}} = -2\Im m \mathcal{N}_{\Lambda\Sigma} \mathcal{X}^\Lambda \overline{\mathcal{X}}^\Sigma$$

$$\partial_\mu \mathcal{X}^\Lambda [2\mathcal{F}_\Lambda - \partial_\Lambda(\mathcal{X}^\Sigma \mathcal{F}_\Sigma)] = 0.$$

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})$$

$$\mathcal{N}_{\Lambda\Sigma} = \overline{\mathcal{F}}_{\Lambda\Sigma} + 2i \frac{\Im\,\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\Im\mathrm{sm}\mathcal{F}_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^\Omega \Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\mathcal{X}^{\Omega'}}.$$

$$\mathcal{C}_{\mu\nu\rho} = e^{\mathcal{K}} \partial_\mu \mathcal{X}^\Lambda \partial_\nu \mathcal{X}^\Sigma \partial_\rho \mathcal{X}^\Omega \mathcal{F}_{\Lambda\Sigma\Omega},$$

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, J_{\nu\rho}] &= (\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu), \\ [J_{\mu\nu}, J_{\rho\gamma}] &= -(\eta_{\mu\rho} J_{\nu\gamma} + \eta_{\nu\gamma} J_{\mu\rho} - \eta_{\mu\gamma}, J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\gamma}), \end{aligned}$$

$$[T_r, T_s] = f_{rs}^t T_t,$$

$$[P_\mu, T_s] = [J_{\mu\nu}, T_s] = 0$$

$$\begin{aligned} [Q_\alpha^L, J_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^{\beta} Q_\beta^L, \\ [Q_\alpha^L, P_\mu] &= [\bar{Q}_{\dot{\alpha}}^L, P_\mu] = 0 \\ \{Q_\alpha^L, \bar{Q}_{\dot{\beta}M}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_M^L, \\ \{Q_\alpha^L, Q_\beta^M\} &= \epsilon_{\alpha\beta} Z^{LM}, \end{aligned}$$

$$[Q_\alpha^L, T_r] = S_r{}^L{}_M Q_\alpha^M \neq 0,$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu + \cdots$$

$$\begin{aligned} \delta_\epsilon B &\sim \bar{\epsilon} F, \\ \delta_\epsilon F &\sim B\epsilon. \end{aligned}$$

$$\begin{aligned} \delta_\epsilon B &\sim \bar{\epsilon} F, \\ \delta_\epsilon F &\sim \partial\epsilon + B\epsilon. \end{aligned}$$

$$S_{EH}[\mathbf{e}, \omega] = \int \star \mathbf{R}(\omega) \wedge \mathbf{e} \wedge \mathbf{e}.$$

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \{ R + \mathcal{G}_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + 2\Im m \mathcal{N}_{\Lambda\Sigma}(\phi) F^\Lambda{}_{\mu\nu} F^{\Sigma\mu\nu} \\ -2\Re e \mathcal{N}_{\Lambda\Sigma}(\phi) F^\Lambda{}_{\mu\nu} \star F^{\Sigma\mu\nu} \}, \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= G_{\mu\nu} + \mathcal{G}_{ij} \left[\partial_\mu \phi^i \partial_\nu \phi^j - \frac{1}{2} \partial_\rho \phi^i \partial^\rho \phi^j \right] + 8\Im m \mathcal{N}_{\Lambda\Sigma} F_\mu^{\Lambda+\rho} F_{\nu\rho}^{\Sigma-} = 0, \\ \mathcal{E}_i &= \nabla_\mu (\mathcal{G}_{ij} \partial^\mu \phi^j) - \frac{1}{2} \partial_i \mathcal{G}_{jk} \partial_\rho \phi^j \partial^\rho \phi^k + \partial_i [\tilde{F}_\Lambda^{\nu\mu*} F_{\mu\nu}^\Lambda] = 0, \\ \mathcal{E}_\Lambda^\mu &= \nabla_\nu \star \tilde{F}_\Lambda^{\nu\mu} = 0, \end{aligned}$$

$$\tilde{F}_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}} \frac{\delta S}{\delta \star F^{\Lambda\mu\nu}} = \Re e \mathcal{N}_{\Lambda\Sigma} F_{\mu\nu}^\Sigma + \Im m \mathcal{N}_{\Lambda\Sigma}^* F_{\mu\nu}^\Sigma.$$

$$\mathcal{B}^{\Lambda\mu} \equiv \nabla_\nu \star F^{\Lambda\nu\mu} = 0.$$

$$\mathcal{E}_{\mu}^{M} \equiv \begin{pmatrix} \mathcal{B}_{\mu}^{\Lambda} \\ \mathcal{E}_{\Lambda\mu} \end{pmatrix}$$

$$\mathcal{E}_{\mu}^{M} = 0 \rightarrow m^{M}{}_{N}\mathcal{E}_{\mu}^{N} = 0, m^{M}{}_{N} \in \mathrm{GL}(2n_{v} + 2, \mathbb{R}).$$

$$F_{\mu}^{M} \equiv \begin{pmatrix} F^{\Lambda} \\ \tilde{F}_{\Lambda} \end{pmatrix}, F'^{M} = m^{M}{}_{N}F^{N}.$$

$$\tilde{F}'_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S'}{\delta \star F'^{\Lambda}{}_{\mu\nu}}.$$

$$\mathrm{i}: \mathrm{Diff}(\mathcal{M}_{\text{escalar}}) \rightarrow \mathrm{GL}(2n_{v} + 2, \mathbb{R})$$

$$\{\phi, F^{M}, \mathcal{N}_{\Sigma\Lambda}(\phi)\} \xrightarrow{\xi} \{\xi(\phi), (\mathrm{i}(\xi))^{M}{}_{N}F^{N}, \mathcal{N}'_{\Sigma\Lambda}(\xi(\phi))\}.$$

$$\mathrm{i}: \mathrm{Diff}(\mathcal{M}_{\text{escalar}}) \rightarrow \mathrm{Sp}(2n_{v} + 2, \mathbb{R})$$

$$\mathcal{N}'(\xi(\phi)) = (A\mathcal{N}(\phi) + B)(C\mathcal{N}(\phi) + D)^{-1}$$

$$m \equiv \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \mathrm{Sp}(2n_{v} + 2, \mathbb{R})$$

$$\mathrm{i}: \mathrm{Isometrías}\left(\mathcal{M}_{\text{escalar}}, \mathcal{G}_{ij}\right) \rightarrow \mathrm{Sp}(2n_{v} + 2, \mathbb{R}).$$

$$S = \int d^{4}x\sqrt{|g|}\{R + h_{uv}(q)\partial_{\mu}q^{u}\partial^{\mu}q^{v} + \mathcal{G}_{i\bar{j}}(z,\bar{z})\partial_{\mu}z^{i}\partial^{\mu}\bar{z}^{\bar{j}}$$
$$+2\Im_{\Lambda\Sigma}(z,\bar{z})F^{\Lambda}{}_{\mu\nu}F^{\Sigma\mu\nu} - 2\Re_{\Lambda\Sigma}(z,\bar{z})F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\mu\nu}\}$$

$$\mathcal{F} = -\frac{1}{3!}\kappa_{ijk}^{0}z^{i}z^{j}z^{k} + \frac{ic}{2} + \frac{i}{(2\pi)^{3}}\sum_{\{d_{i}\}} n_{\{d_{i}\}}Li_{3}\left(e^{2\pi i d_{i}z^{i}}\right)$$

$$\mathcal{F}_{\mathrm{P}} = -\frac{1}{3!}\kappa_{ijk}^{0}z^{i}z^{j}z^{k} + \frac{ic}{2},$$

$$\mathcal{F}_{\mathrm{NP}} = \frac{i}{(2\pi)^{3}}\sum_{\{d_{i}\}} n_{\{d_{i}\}}Li_{3}\left(e^{2\pi i d_{i}z^{i}}\right).$$

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa_{ijk}^{0}\frac{\mathcal{X}^{i}\mathcal{X}^{j}\mathcal{X}^{k}}{\mathcal{X}^{0}} + \frac{ic(\mathcal{X}^{0})^{2}}{2} + \frac{i(\mathcal{X}^{0})^{2}}{(2\pi)^{3}}\sum_{\{d_{i}\}} n_{\{d_{i}\}}Li_{3}\left(e^{2\pi i d_{i}\frac{\mathcal{X}^{i}}{\mathcal{X}^{0}}}\right)$$

$$z^{i} = \frac{\mathcal{X}^{i}}{\mathcal{X}^{0}}$$

$$\mathcal{B} \equiv \mathcal{M} - I^{-}(\mathcal{J}^{+}),$$

$$\nabla^{\mu}\xi^{2} = -2\kappa\xi^{\mu}.$$

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J + \Phi\delta Q$$

$$\delta A \geq 0.$$

$$T = \frac{\kappa}{2\pi}.$$

$$S_{\text{bh}} = \frac{A}{4}.$$

$$\delta_\epsilon B \sim \bar{\epsilon} F = 0,$$
$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0,$$

$$\mathbf{g} = \left(1 - \frac{2M}{r}\right) dt \otimes dt - \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\ \theta d\phi \otimes d\phi)$$

$$\delta_\epsilon \Psi_\mu\big|_{\text{Schw.}} = 0$$

$$\mathbf{g} = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt \otimes dt - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\ \theta d\phi \otimes d\phi)$$

$$T = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - q^2}}{2\pi^2 r_+^2} = 0.$$

$$\delta_\epsilon \Psi_\mu\big|_{\text{RN}}\ \text{extr.} = 0$$

$$\mathbf{g} = e^{2U(\tau)} dt \otimes dt - e^{-2U(\tau)} \gamma_{\underline{mn}} dx^{\underline{m}} \otimes dx^{\underline{n}},$$
$$\gamma_{\underline{mn}} dx^{\underline{m}} \otimes dx^{\underline{n}} = \frac{r_0^2}{\sinh^2\ r_0\tau}\left[\frac{r_0^2}{\sinh^2\ r_0\tau} d\tau \otimes d\tau + h_{S^2}\right]$$
$$h_{S^2} = d\theta \otimes d\theta + \sin^2\ \theta d\phi \otimes d\phi,$$

$$\mathbf{g} = e^{2U(\tau)} dt \otimes dt - e^{-2U(\tau)}[\delta_{ab} dx^a \otimes dx^b],$$

$$\lim_{\tau\to-\infty} e^{-2U} = \frac{A}{4\pi} \lim_{\tau\to-\infty} \tau^2, \lim_{\tau\to-\infty} \tau \frac{d\phi^i}{d\tau} = 0, i = 1, \ldots, n_v$$

$$\lim_{\tau\to-\infty} \phi^i = \phi_h^i, \mathcal{G}^{ij}(\phi_h)\partial_j V_{\text{bh}}(\phi_h) = 0$$

$$\partial_j V_{\text{bh}}(\phi_h) = 0,$$

$$S = \pi V_{\text{bh}}(\phi_h(\mathcal{Q})) = 0,$$

$$I_{\text{FGK}}[U, z^i] = \int d\tau\{(\dot{U})^2 + \mathcal{G}_{ij}\dot{z}^i\dot{\bar{z}}^{\bar{j}} - e^{2U} V_{\text{bh}}(z, \bar{z}, \mathcal{Q})\},$$

$$(\dot{U})^2 + \mathcal{G}_{ij}\dot{z}^i\dot{\bar{z}}^j + e^{2U} V_{\text{bh}}(z, \bar{z}, \mathcal{Q}) = r_0^2.$$

$$V_{\text{bh}}(z, \bar{z}, \mathcal{Q}) \equiv \frac{1}{2}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N,$$

$$(\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix},$$

$$(\mathcal{M}_{MN}(\mathcal{N})) \equiv \begin{pmatrix} I+RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}$$

$$X \equiv \frac{1}{\sqrt{2}} e^{U+i\alpha}$$

$$\mathcal{F}_\Lambda \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{X}^\Lambda} \text{ y } \mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}, \text{ se tiene } \mathcal{F}_\Lambda = \mathcal{F}_{\Lambda\Sigma}\mathcal{X}^\Sigma$$

$$(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} = e^{\mathcal{K}/2} \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix}$$

$$\frac{\mathcal{M}^\Lambda}{X} = \mathcal{F}_{\Lambda\Sigma} \frac{\mathcal{L}^\Lambda}{X}.$$

$$\mathcal{R}^M \equiv \Re\mathrm{e}(\mathcal{V}^M/X), \mathcal{I}^M \equiv \Im\mathrm{m}(\mathcal{V}^M/X),$$

$$\mathcal{R}^M = -\mathcal{M}_{MN}(\mathcal{F})\mathcal{I}^M$$

$$d\mathcal{R}^M = -\mathcal{M}_{MN}(\mathcal{F})d\mathcal{I}^M$$

$$\frac{\partial \mathcal{I}^M}{\partial \mathcal{R}_N} = \frac{\partial \mathcal{I}^N}{\partial \mathcal{R}_M} = -\frac{\partial \mathcal{R}^M}{\partial \mathcal{I}_N} = -\frac{\partial \mathcal{R}^N}{\partial \mathcal{I}_M} = -\mathcal{M}^{MN}(\mathcal{F}).$$

$$H^M \equiv \mathcal{I}^M(X, z, \bar{X}, \bar{z})$$

$$z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0}, e^{-2U} = \frac{1}{2|X|^2} = \tilde{H}_M(H)H^M.$$

$$\dot{\alpha} = 2|X|^2 \dot{H}^M H_M - \left[\frac{1}{2i} \dot{z}^i \partial_i \mathcal{K} + c.c.\right].$$

$$\mathbf{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U},$$

$$\tilde{H}_M = \frac{1}{2}\frac{\partial \mathrm{W}}{\partial H^M} \equiv \frac{1}{2}\partial_M \mathrm{W}, H^M = \frac{1}{2}\frac{\partial \mathrm{W}}{\partial \tilde{H}^M}.$$

$$-I_{\text{H-FGK}}[H] = \int d\tau \left\{\frac{1}{2} g_{MN} \dot{H}^M \dot{H}^N - V\right\}$$

$$r_0^2 = \frac{1}{2} g_{MN} \dot{H}^M \dot{H}^N + V$$

$$g_{MN} \equiv \partial_M \partial_N \log \mathrm{W} - 2\frac{H_M H_N}{\mathrm{W}}$$

$$V(H) \equiv \left\{-\frac{1}{4} g_{MN} + \frac{H_M H_N}{2\mathbf{W}^2}\right\} Q^M Q^N$$

$$V_{\text{bh}} = -\mathrm{W}V.$$

$$g_{MN}\ddot{H}^N + [PQ, M]\dot{H}^P \dot{H}^Q + \partial_M V = 0$$

$$[PQ,M] \equiv \partial_{(P}g_{Q)M} - \frac{1}{2}\partial_M g_{PQ}$$

$$\tilde{H}_M(\ddot{H}^M - r_0^2 H^M) + \frac{(\dot{H}^M H_M)^2}{\mathrm{W}} = 0,$$

$$\dot{H}^M H_M = 0,$$

$$\tilde{H}_M(\ddot{H}^M - r_0^2 H^M) = 0,$$

$$H^M = A^M - \frac{B^M}{\sqrt{2}}\tau,$$

$$H^M = A^M \cosh(r_0\tau) + B^M \sinh(r_0\tau)$$

$$H^M = A^M - \frac{Q^M}{\sqrt{2}}\tau,$$

$$H^0 = H_0 = H_i = 0, p^0 = p_0 = q_i = 0,$$

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, r_0 = 0,$$

$$H^M = H^M(a,b),$$

$$\{H^P = 0, Q^P = 0\} \Rightarrow \mathcal{E}_P = 0,$$

$$\begin{pmatrix} iH^i \\ \tilde{H}_i \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^i \\ \dfrac{\partial F(\mathcal{X})}{\partial \mathcal{X}^i} \end{pmatrix}, \begin{pmatrix} \tilde{H}^0 \\ 0 \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^0 \\ \dfrac{\partial F(\mathcal{X})}{\partial \mathcal{X}^0} \end{pmatrix},$$

$$e^{-2U} = \tilde{H}_i H^i, z^i = i\frac{H^i}{\tilde{H}^0},$$

$$\frac{\partial F(H)}{\partial \tilde{H}^0} = 0$$

$$F(H) = \frac{i}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{\tilde{H}^0} + \frac{ic\left(\tilde{H}^0\right)^2}{2} + \frac{i\left(\tilde{H}^0\right)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}}\right)$$

$$\tilde{H}_i = -i\frac{\partial F(H)}{\partial H^i},$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^i H^j H^k}{\left(\tilde{H}^0\right)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[Li_3\left(e^{-2\pi d_i \frac{H^i}{H^0}}\right) + Li_2\left(e^{-2\pi d_i \frac{H^i}{H^0}}\right)\left[\frac{\pi d_i H^i}{\tilde{H}^0}\right]\right]$$

$$\lim_{|w|\to 0} Li_s(w) = w, \forall s \in \mathbb{N}$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{\left(\tilde{H}^0\right)^3}+c+\frac{1}{4\pi^3}\sum_{\{d_i\}}n_{\{d_i\}}\left[e^{-2\pi d_i\frac{H^i}{H^0}}+e^{-2\pi d_i\frac{H^i}{H^0}}\left[\frac{\pi d_iH^i}{\tilde{H}^0}\right]\right]=0,\Im\,\mathrm{m}z^i\gg 1$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{\left(\tilde{H}^0\right)^3}+c+\frac{1}{4\pi^3}\sum_{\{d_i\}}n_{\{d_i\}}e^{-2\pi d_i\frac{H^i}{\tilde{H}^0}}\left[\frac{\pi d_iH^i}{\tilde{H}^0}\right]$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{\left(\tilde{H}^0\right)^3}+c+\frac{\hat{n}}{4\pi^3}e^{-2\pi\hat{d}_i\frac{H^i}{H^0}}\left[\frac{\pi\hat{d}_iH^i}{\tilde{H}^0}\right]$$

$$e^{-2U}=\mathbf{W}(H)=\alpha\left|\kappa^0_{ijk}H^iH^jH^k\right|^{2/3}$$

$$V_{\mathrm{bh}}=\frac{\mathrm{W}(H)}{4}\partial_{ij}\log\ \mathrm{W}(H)Q^iQ^j,$$

$$z^i=i(3!\,c)^{1/3}\frac{H^i}{\left(\kappa^0_{ijk}H^iH^jH^k\right)^{1/3}},$$

$$\kappa^0_{ijk}\Im\,\mathrm{m}z^i\Im\,\mathrm{m}z^j\Im\,\mathrm{m}z^k>\frac{3c}{2}.$$

$$c>\frac{c}{4}$$

$$e^{-\mathcal{K}}=6c$$

$$c>0\Rightarrow h^{11}>h^{21}$$

$$X^{3,1}\Rightarrow\kappa^0_{111}=48,\kappa^0_{222}=\kappa^0_{333}=8$$
$$Y^{3,1}\Rightarrow\kappa^0_{122}=6,\kappa^0_{222}=18,\kappa^0_{333}=8$$

$$\mathbf{W}(H)=\alpha|48(H^1)^3+8[(H^2)^3+(H^3)^3]|^{2/3},$$
$$\mathbf{W}(H)=\alpha|18(H^2)^2[H^1+H^2]+8(H^3)^3|^{2/3}.$$

$$H=a\cosh\,(r_0\tau)+\frac{b}{r_0}\sinh\,(r_0\tau),b=s_b\sqrt{r_0^2a^2+\frac{p^2}{2}},$$

$$z^1=i(3!\,c)^{1/3}\lambda^{-1/3}=s_{2,3}z^{2,3}$$

$$\lambda=[48+8(s_2+s_3)]\text{ para }X^{3,1}$$
$$\lambda=[18+18s_2+8s_3]\text{ para }Y^{3,1}.$$

$$ds^2=\left[\frac{1}{2}(3!\,c)^{1/3}\left[a\cosh\,(r_0\tau)+\frac{b}{r_0}\sinh\,(r_0\tau)\right]^2\right]^{-1}dt^2$$
$$-\frac{1}{2}(3!\,c)^{1/3}\left[a\cosh\,(r_0\tau)+\frac{b}{r_0}\sinh\,(r_0\tau)\right]^2\left[\frac{r_0^4}{\sinh^4\ r_0\tau}d\tau^2+\frac{r_0^2}{\sinh^2\ r_0\tau}d\Omega^2_{(2)}\right]$$

$$2\pi i d_i z^i \sim -\frac{1}{3}\sum_{i=1}^{3} d_i, d_i \geq 1$$

$$s_2 = s_3 = -1, \text{ para } X^{3,1},$$
$$s_2 = -s_3 = 1, \text{ para } Y^{3,1}.$$

$$\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$$

$$a = -s_b \frac{\Im m z^1}{\sqrt{3c}}.$$

$$M = r_0 \sqrt{1 + \frac{3cp^2}{2r_0^2 (\Im\, \mathrm{m} z^1)^2}},$$

$$S_\pm = r_0^2 \pi \left(\sqrt{1 + \frac{3cp^2}{2r_0^2 (\Im\, \mathrm{m} z^1)^2}} \pm 1 \right)^2.$$

$$S_+ S_- = \frac{\pi^2 \alpha^2}{4} p^4 \lambda^{4/3}.$$

$$e^{-2U} = \mathbf{W}(H) = \alpha |H^1 H^2 H^3|^{2/3},$$

$$z^i = i c^{1/3} \frac{H^i}{(H^1 H^2 H^3)^{1/3}}.$$

$$H^i = a^i \cosh (r_0 \tau) + \frac{b^i}{r_0} \sinh (r_0 \tau), b^i = s_b^i \sqrt{r_0^2 (a^i)^2 + \frac{(p^i)^2}{2}}.$$

$$a^i = -s_b^i \frac{\Im m z_\infty^i}{\sqrt{3c}}.$$

$$M = \frac{r_0}{3} \sum_i \sqrt{1 + \frac{3c(p^i)^2}{2r_0^2 \left(\Im m z_\infty^i\right)^2}}$$

$$S_\pm = r_0^2 \pi \prod_i \left(\sqrt{1 + \frac{3c(p^i)^2}{2r_0^2 \left(\Im\, \mathrm{m} z_\infty^i\right)^2}} \pm 1 \right)^{2/3}$$

$$S_+ S_- = \frac{\pi^2 \alpha^2}{4} \prod_i \left(p^i\right)^{4/3}$$

$$-\frac{1}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{\left(\tilde{H}^0\right)^3} + \frac{\hat{n}}{4\pi^3} e^{-2\pi \hat{d}_l \frac{H^l}{H^0}} \left[\frac{\pi \hat{d}_n H^n}{\tilde{H}^0} \right]$$

$$\tilde{H}^0 = \frac{\pi \hat{d}_l H^l}{W_a\left(s_a \sqrt{\frac{3\hat{n}\left(\hat{d}_n H^n\right)^3}{2\kappa^0_{ijk} H^i H^j H^k}}\right)},$$

$$\tilde{H}_i = \frac{1}{2}\kappa^0_{ijk}\frac{H^j H^k}{\pi \hat{d}_l H^l} W_a\left(s_a \sqrt{\frac{3\hat{n}\left(\hat{d}_m H^m\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right).$$

$$e^{-2U} = \mathrm{W}(H) = \frac{\kappa^0_{ijk} H^i H^j H^k}{2\pi \hat{d}_m H^m} W_a\left(s_a \sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right)$$

$$z^i = i\frac{H^i}{\pi \hat{d}_m H^m} W_a\left(s_a \sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right).$$

$$s_0 \equiv \mathrm{sign}\left[\kappa^0_{ijk}\frac{H^i H^j H^k}{\hat{d}_m H^m}\right],$$
$$s_{-1} \equiv -1$$

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, r_0 = 0.$$

$$S = \frac{1}{2}\kappa^0_{ijk}\frac{p^i p^j p^k}{\hat{d}_m p^m} W_a(s_a \beta),$$
$$\beta = \sqrt{\frac{3\hat{n}\left(\hat{d}_l p^l\right)^3}{2\kappa^0_{pqr} p^p p^q p^r}},$$

$$M = \dot{U}(0) = \frac{1}{2\sqrt{2}}\left[\frac{3\kappa^0_{ij} p^i a^j a^k}{\kappa^0_{pqr} a^p a^q a^r}\left[1 - \frac{1}{1 + W_a(s_a \alpha)}\right] - \frac{d_l p^l}{d_n a^n}\left[1 - \frac{3}{2\left(1 + W_a(s_a \alpha)\right)}\right]\right],$$
$$\alpha = \sqrt{\frac{3\hat{n}(d_l a^l)^3}{2\kappa^0_{pqr} a^p a^q a^r}}.$$

$$W_a(x) e^{-2W_a(x)} \gg e^{-2W_a(x)}.$$

$$e^{-2U} = \frac{\kappa^0_{ijk} H^i H^j H^k}{2\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right),$$
$$z^i = i\frac{H^i}{\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right).$$

$$\frac{\kappa^0_{ijk}H^iH^jH^k}{2\pi\hat{d}_nH^n} > 0\ \forall\tau \in (-\infty, 0],$$
$$\frac{\kappa^0_{ijk}a^ia^ja^k}{2\pi\hat{d}_ma^m}W_0(\alpha) = 1,$$

$$e^{-2U} \overset{\tau\to-\infty}{\to} \frac{\kappa^0_{ijk}p^ip^jp^k}{8\pi\hat{d}_mp^m}W_0(\beta)\tau^2.$$

$$L = \bigoplus_{k=0}^{N} L_k,$$

$$u_j \circ u_k \in L_{(j+k)\mathrm{mod}(N+1)}.$$

$u_0 \circ v_0 \in L_0, \forall u_0, v_0 \in L_0,$

$u_0 \circ u_1 \in L_1, \forall u_0 \in L_0, v_1 \in L_1,$

$u_1 \circ v_1 \in L_0, \forall u_1, v_1 \in L_1,$

- Supersimetrización: $\forall x_i \in L_i, \forall x_j \in L_j i, j = 0,1, x_i \circ x_j = -(-1)^{ij}x_j \circ x_i.$
- Identidades de Jacobi: $\forall x_k \in L_k, \forall x_l \in L_l, \forall x_m \in L_m, k, l, m = 0,1,\ x_k \circ (x_l \circ x_m)(-1)^{km} + x_l \circ (x_m \circ x_k)(-1)^{lk} + x_m \circ (x_k \circ x_l)(-1)^{ml} = 0.$

$$x_\mu \circ x_\nu \equiv x_\mu x_\nu - (-1)^{g_\mu g_\nu}x_\nu x_\mu = c^\omega_{\mu\nu}x_\omega$$

$$Li_w(z) = \sum_{j=1}^{\infty}\frac{z^j}{j^w}, z, w \in \mathbb{C}.$$

$$Li_{w-1}(z) = z\frac{\partial Li_w(z)}{\partial z}.$$

$$Li_1(z) = -\log(1-z)$$

$$Li_0(z) = \frac{z}{1-z}, Li_{-n}(z) = \left(z\frac{\partial}{\partial z}\right)^n \frac{z}{1-z}.$$

$$Li_w(z) = \int_0^z \frac{Li_{w-1}(s)}{s}ds$$

$$z = W(z)e^{W(z)}, \forall z \in \mathbb{C}.$$

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}, \forall z \notin \{0, -1/e\}, \left.\frac{dW(z)}{dz}\right|_{z=0}$$

$$\lim_{x\to-1/e}\frac{dW_0(x)}{dx} = \infty, \lim_{x\to-1/e}\frac{dW_{-1}(x)}{dx} = -\infty.$$

$$X \times S$$

$$\mu^{1,1} \in \Omega^{0,1}(X,\ \mathrm{T}_X) \otimes C^\infty(S)$$

$$\mu^{1,1} = \mu_j^i(z,\bar{z},t)\mathrm{d}\bar{z}_i \frac{\partial}{\partial z_j}$$

$$\gamma^{1,0} \in \Omega^{1,0}(X) \otimes C^\infty(S), \gamma^{1,2} \in \Omega^{1,2}(X) \otimes C^\infty(S)$$

$$\bar{\partial}\mu^{1,1} + \frac{1}{2}[\mu^{1,1}, \mu^{1,1}] + \Omega^{-1} \vee (\partial\gamma^{1,0} \wedge \partial\gamma^{1,2}) = 0$$

$$\Omega^{0,\cdot}(X,V) = \left(\Omega^{0,j}(X,V)[-j], \bar{\partial}\right)$$

$$\partial_\Omega(\mu) \wedge \Omega = L_\mu(\Omega)$$

$$\partial_\Omega : \Omega^{0,\cdot}(X,\ \mathrm{T}_X) \to \Omega^{0,\cdot}(X)$$

$$\Omega^{0,\cdot}(X,\ \mathrm{T}_X) \overset{\partial_\Omega}{\to} \Omega^{0,\cdot}(X)$$

$$[\mu, \mu'] = L_\mu \mu'$$
$$[\mu, \nu] = L_\mu \nu$$

$$(\mathrm{Sym}(\mathcal{L}^\vee[-1]), \delta_{\mathcal{L}})$$

$$\Phi^* : \mathrm{C}^\cdot(\mathcal{L}') \to \mathrm{C}^\cdot(\mathcal{L})$$

$$\Psi_\infty : \nu \mapsto 1 - e^{-\nu}, \mu \mapsto e^{-\nu}\mu$$

$$[\mu]_1 = \bar{\partial}\mu + \partial_\Omega \mu$$
$$[\mu_1, \mu_2]_2 = \partial_\Omega(\mu_1 \wedge \mu_2)$$
$$[\nu, \mu_1, \mu_2]_3 = \partial_\Omega(\nu\mu_1 \wedge \mu_2)$$

$$[\nu_1, \dots, \nu_{k-2}, \mu_1, \mu_2]_k = \partial_\Omega(\nu_1 \cdots \nu_k \mu_1 \wedge \mu_2)$$
$$[\nu_1, \dots, \nu_{k-3}, \mu_1, \mu_2, \gamma]_k = \nu_1 \cdots \nu_{k-3}(\mu \wedge \mu') \vee \partial\gamma.$$
$$[\nu_1, \dots, \nu_{k-2}, \mu, \gamma]_k = \nu_1 \cdots \nu_{k-2}\mu \vee \partial\gamma.$$

$$[x \otimes a]_1 = [x]_1^{\mathcal{L}} \otimes a + (-1)^{|x|} x \otimes \ \mathrm{d}_{\mathcal{A}} a$$
$$[x_1 \otimes a_1, \dots, x_k \otimes a_k]_k = [x_1, \dots, x_k]_k^{\mathcal{L}} \otimes (a_1 \cdots a_k), k \geqslant 2.$$

$$\mathrm{F}_A = [A]_1 + \frac{1}{2}[A,A]_2 + \frac{1}{3!}[A,A,A]_3 + \cdots$$

$$(A,B) \in \mathcal{L}[1] \oplus \mathcal{L}^![-2]$$

$$\begin{matrix} -n & & -n+1 & -1 & & 0 & \\ \Omega^0(X;S) & \overset{\partial}{\to} & \Omega^1(X;S) & & \mathrm{PV}^1(X;S) & \overset{\partial_\Omega}{\to} & \mathrm{PV}^0(X;S). \end{matrix}$$

$$\Omega^i(X;S) = \Omega^{i,\cdot,\cdot}(X;S)$$
$$= \bigoplus_{j,k} \mathrm{PV}^{i,j}(X) \otimes \Omega^k(S)[-j-k]$$

$$n = \dim_{\mathbb{C}}(X) + \dim_{\mathbb{R}}(S) - 1$$

$$\int_{X\times S}^{\Omega} \mu \vee \gamma + \int_{X\times S}^{\Omega} \nu\beta$$

$$S_{BF,0} = \int^{\Omega} \left[\beta \wedge (\bar{\partial} + \mathrm{d}_S)\nu + \gamma \wedge (\bar{\partial} + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}[\mu,\mu] \vee \gamma + [\mu,\nu]\beta\right]\left[\beta \wedge (\bar{\partial} + \mathrm{d}_S)\nu + \gamma \wedge (\bar{\partial} + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}[\mu,\mu] \vee \gamma + [\mu,\nu]\beta\right].$$

$$S_{BF,\infty} = \int^{\Omega} \left[\beta \wedge (\bar{\partial} + \mathrm{d}_S)\nu + \gamma \wedge (\bar{\partial} + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}\frac{1}{1-\nu}\mu^2 \vee \partial\gamma\right]\left[\beta \wedge (\bar{\partial} + \mathrm{d}_S)\nu + \gamma \wedge (\bar{\partial} + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}\frac{1}{1-\nu}\mu^2 \vee \partial\gamma\right].$$

$$\mu \mapsto e^{-\nu}\mu, \nu \mapsto 1 - e^{-\nu}, \beta \mapsto (\beta - \mu \vee \gamma)e^{\nu}, \gamma \mapsto e^{\nu}\gamma$$

$$\Omega^0(X;S)_\beta \xrightarrow{\partial} \Omega^1(X;S)_\gamma \qquad\qquad \mathrm{PV}^1(X;S)_\mu \xrightarrow{\partial_\Omega} \mathrm{PV}^0(X;S)_\nu.$$

$$S_{BF} + gJ$$

$$\{S_{BF} + gJ, S_{BF} + gJ\} = 0$$

$$\{S_{BF}, J\} = \{J, J\} = 0$$

$$J = \frac{1}{6}\gamma \wedge \partial\gamma \wedge \partial\gamma$$

$$\gamma^{1,i;j} \in \Omega^{1,i}(X) \otimes \Omega^{j}(S)$$

$$\deg(J) = 6$$

$$\{\beta \wedge \partial_\Omega \mu, J\} = \frac{1}{2}\partial\beta \wedge \partial\gamma \wedge \partial\gamma = 0$$

$$\left\{\frac{1}{2}\frac{1}{1-\nu}\partial\gamma \vee \mu^2, \frac{1}{6}\gamma \wedge \partial\gamma \wedge \partial\gamma\right\} = \frac{1}{2}(\mu \vee \partial\gamma) \wedge \partial\gamma \wedge \partial\gamma$$

$$\mathrm{Sym}^3(\boxminus) \cong \boxplus \oplus \nexists$$

$$\wedge^3(\exists) \cong \boxminus \oplus \boxminus$$

$$\gamma \mapsto \sqrt{g}\gamma, \beta \mapsto \sqrt{g}\beta$$

$$\frac{1}{\sqrt{g}}(S_{BF,\infty} + J)$$

$$\tilde{J} = \frac{1}{6}e^{\nu}\gamma \wedge \partial(e^{\nu}\gamma) \wedge \partial(e^{\nu}\gamma)$$

$$\bar{\partial}\nu + \mathrm{d}_S\nu + \partial_\Omega\mu = 0$$

$$\bar{\partial}\mu + \mathrm{d}_S\mu + \frac{1}{2}\frac{1}{1-\nu}\partial_\Omega(\mu^2) + \frac{1}{2}(\partial\gamma \wedge \partial\gamma) \vee (g\Omega^{-1}) = 0$$

$$(\bar{\partial} + \mathrm{d}_S)\gamma + \partial\beta + \frac{1}{1-\nu}(\mu \vee \partial\gamma) = 0$$

$$(\bar{\partial} + \mathrm{d}_S)\beta + \frac{1}{2}\frac{1}{(1-\nu)^2}\mu^2 \vee \partial\gamma = 0$$

- $\mu = \sum_{i,j} \mu^{i;j}\ \mu^{i;j} \in \mathrm{PV}^{1,i}(X) \otimes \Omega^j(\mathbb{R}), i = 0, \dots, 5, j = 0,1$
- $\nu = \sum_{i,j} \nu^{i;j}\ \nu^{i;j} \in \mathrm{PV}^{0,i}(X,\ \mathrm{T}_X) \otimes \Omega^j(\mathbb{R}), i = 0, \dots, 5, j = 0,1$
- $\gamma = \sum_{i,j} \gamma^{i;j}\ \gamma^{i;j} \in \Omega^{1,i}(X) \otimes \Omega^j(\mathbb{R}), i = 0, \dots, 5, j = 0,1$
- $\beta = \sum_{i,j} \beta^{i;j}\ \beta^{i;j} \in \Omega^{0,i}(X) \otimes \Omega^j(\mathbb{R}), i = 0, \dots, 5, j = 0,1$

$$\bar{\partial}\mu^{1;0} + \frac{1}{2}[\mu^{1;0}, \mu^{1;0}] + \left(\frac{1}{2}\partial\gamma^{1;0} \wedge \partial\gamma^{1;0} + \partial\gamma^{2;0} \wedge \partial\gamma^{0;0}\right) \vee (g\Omega^{-1}) = 0$$

$$\alpha \overset{\text{def}}{=} \partial\gamma^{0;0}$$

$$\Omega_X^{2,\,\text{hol}} \overset{\wedge\alpha}{\to} \Omega_X^{4,\,\text{hol}} \cong {}_{\Omega}\mathcal{T}_X^{\text{hol}}$$

$$\bar{\partial}\xi + \frac{1}{2}[\xi, \xi] = \alpha \vee \rho$$

$$\mu^{1;0} = \mu_i^j(z, \bar{z}, t)\mathrm{d}\bar{z}_j\partial_{z_i}$$
$$\mu^{0;1} = \mu_i^t(z, \bar{z}, t)\mathrm{d}t\partial_{z_i}$$

$$\mu^{0;0} = \mu_i(z, \bar{z}, t)\partial_{z_i}$$

$$\beta^{3;0} = \beta^{ijk}(z, \bar{z}, t)\mathrm{d}\bar{z}_i\ \mathrm{d}\bar{z}_j\ \mathrm{d}\bar{z}_k, \beta^{2;1} = \beta_t^{ij}(z, \bar{z}, t)\mathrm{d}\bar{z}_i\ \mathrm{d}\bar{z}_j\ \mathrm{d}t$$
$$\gamma^{2;0} = \gamma^{ijk}(z, \bar{z}, t)\mathrm{d}z_i\ \mathrm{d}\bar{z}_j\ \mathrm{d}\bar{z}_k, \gamma^{1;1} = \gamma_t^{ij}(z, \bar{z}, t)\mathrm{d}z_i\ \mathrm{d}\bar{z}_j\ \mathrm{d}t.$$

$$\mu^i\partial_{z_i} \in \mathrm{Vect}(\mathbb{C}^5) \cong \mathcal{O}(\mathbb{C}^5)\partial_{z_i}, \nu \in \mathcal{O}(\mathbb{C}^5)$$
$$\beta \in \mathcal{O}(\mathbb{C}^5), \gamma^i\ \mathrm{d}z_i \in \Omega^1(\mathbb{C}^5) \cong \mathcal{O}(\mathbb{C}^5)\mathrm{d}z_i$$

$$\boldsymbol{\mu}^i_{(m_j)}: \mu^i \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\mu^i$$
$$\boldsymbol{\nu}_{(m_j)}: \nu \mapsto \partial_{z_1}^{m_1}\partial_{z_2 m_2}^{\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\nu}$$
$$\boldsymbol{\gamma}^i_{(m_j)}: \gamma^i \mapsto \partial_{z_1}^{m_1}\partial_{z_2 m_2}^{\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\gamma^i}$$
$$\boldsymbol{\beta}_{(m_j)}: \beta \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\beta$$

$$i(q_1, \dots, q_5) = \frac{\sum_{i=1}^5 q_i}{\prod_{i=1}^5 (1 - q_i)} + \frac{\sum_{i=1}^5 q_i^{-1}}{\prod_{i=1}^5 \left(1 - q_i^{-1}\right)}$$

$$q_1^{m_1+1} \cdots q_i^{m_i} \cdots q_5^{m_5+1}.$$

$$q_1^{m_1} \cdots q_i^{m_i+1} \cdots q_5^{m_5}$$

$$\sum_{i=1}^{5}\left(\sum_{(m_i)\in\mathbb{Z}_{\geqslant 0}^5} q_1^{m_1}\cdots q_i^{m_i+1}\cdots q_5^{m_5} - \sum_{(m_i)\in\mathbb{Z}_{\geqslant 0}^5} q_1^{m_1+1}\cdots q_i^{m_i}\cdots q_5^{m_5+1}\right)$$

$$-\frac{\sum_{i=1}^{5} q_1\cdots \hat{q}_i\cdots q_5}{\prod_{i=1}^{5}(1-q_i)} + \frac{\sum_{i=1}^{5} q_i}{\prod_{i=1}^{5}(1-q_i)}$$

$$\prod_{i=1}^{5}\prod_{(m_i)\in\mathbb{Z}_{\geqslant 0}^5}\frac{1-q_1^{m_1+1}\cdots q_i^{m_i}\cdots q_5^{m_5+1}}{1-q_1^{m_1}\cdots q_i^{m_i+1}\cdots q_5^{m_5}}$$

$$E(5,10)_+ = \mathrm{Vect}_0(\mathbb{C}^5)$$

$$E(5,10)_- = \Omega_{cl}^2(\mathbb{C}^5)$$

$$[\alpha,\alpha'] = \Omega^{-1}\vee(\alpha\wedge\alpha')$$

$$\delta = \{S_{BF,\infty} + J, -\}$$

$$\delta^{(1)} = \bar{\partial} + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu\to\nu} + \partial|_{\beta\to\gamma}$$

$$\mathrm{Vect}(\mathbb{C}^5)\xrightarrow{\partial_\Omega}\mathcal{O}(\mathbb{C}^5)$$
$$\mathcal{O}(\mathbb{C}^5)\xrightarrow{\partial}\Omega^1(\mathbb{C}^5)$$

$$\mu = \mu\otimes 1 \in \Pi\mathrm{Vect}_0(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R})$$

$$[\gamma] = [\gamma]\otimes 1 \in \left(\Omega^1(\mathbb{C}^5)/\mathrm{d}\left(\mathcal{O}(\mathbb{C}^5)\right)\right)\otimes\Omega^0(\mathbb{R})$$

$$(\mu,[\gamma],b)\in\mathrm{Vect}_0(\mathbb{C}^5)\oplus\Pi\Omega^1(\mathbb{C}^5)/\partial\mathcal{O}(\mathbb{C}^5)\oplus\mathbb{C}$$

$$\big[[\gamma],[\gamma']\big] = \Omega^{-1}\vee(\partial\gamma\wedge\partial\gamma')\in\mathrm{Vect}_0(\mathbb{C}^5)$$

$$\partial:\Omega^1(\mathbb{C}^5)/\mathrm{d}\Big((\mathbb{C}^5)\overset{\cong}{\Longrightarrow}\Omega_{cl}^2(\mathbb{C}^5)\Big)$$

$$_K\leftrightarrow\left(\Pi\mathcal{E},\delta^{(1)}\right)\overset{q}{\rightleftarrows}(E(5,10)\oplus\mathbb{C}_b,0),$$

$$\widetilde{K}\partial\gamma + \partial K\gamma = \gamma$$

$$\gamma - \widetilde{K}\partial\gamma = \partial K\gamma$$

$$[\mu,\mu',[\gamma]]_3 = \varphi(\mu,\mu',[\gamma])$$

$$\varphi: E(5,10)\times E(5,10)\times E(5,10) \to \mathbb{C}_b$$
$$\varphi(\mu,\mu',\alpha) = \langle\mu\wedge\mu',\alpha\rangle|_{z=0}$$

$$\mathrm{C}^{\mathrm{even}}(\mathcal{L}) = \mathrm{C}^{2\cdot,+}(\mathcal{L})\oplus\mathrm{C}^{2\cdot+1,-}(\mathcal{L})$$
$$\mathrm{C}^{\mathrm{odd}}(\mathcal{L}) = \mathrm{C}^{2\cdot,-}(\mathcal{L})\oplus\mathrm{C}^{2\cdot+1,+}(\mathcal{L})$$

$$\mathfrak{t}_{11d} = V \oplus \Pi S$$

$$\mathrm{Sym}^2(S) \cong V \oplus \wedge^2 V \oplus \wedge^5 V$$

$$[\psi, \psi'] = \Gamma_{\wedge^1}(\psi, \psi')$$

$$\mathfrak{siso}_{11d} = \mathfrak{so}(11, \mathbb{C}) \ltimes \mathfrak{t}_{11d}$$

$$c_{M2} \in \mathrm{C}^{2,+}(\mathfrak{siso}_{11d}; \Omega^{\cdot}(\mathbb{R}^{11})[2])$$

$$c_{M2}(\psi, \psi') = \Gamma_{\wedge^2}(\psi, \psi') \in \Omega^2(\mathbb{R}^{11})$$

$$V = L \oplus L^{\vee} \oplus \mathbb{C}_t, S = \wedge^{\cdot} L$$

$$\mathfrak{sl}(L) \oplus \wedge^2 L \oplus \wedge^2 L^{\vee} \oplus L \oplus L^{\vee} \oplus \mathbb{C}$$

$$S = \wedge^{\cdot}(L) = \mathbb{C} \oplus L \oplus \wedge^2 L \oplus \wedge^3 L \oplus \wedge^4 L \oplus \wedge^5 L$$

$$\mathrm{Stab}(Q) = \mathfrak{sl}(L) \oplus \wedge^2 L^{\vee} \oplus L^{\vee} \subset \mathfrak{so}(11, \mathbb{C})$$

$$L \oplus \mathrm{Stab}(Q) \oplus \Pi(\wedge^2 L^{\vee}) \oplus \mathbb{C}$$

$$[Q, -]: \mathfrak{so}(11, \mathbb{C}) \to S$$

$$[\psi, \psi']_2 = \psi \wedge \psi' \in \wedge^4 L^{\vee} \cong L_v$$

$$[v, v', \psi]_3 = 4\langle v \wedge v', \psi \rangle \in \mathbb{C}_b$$

$$\left[z_i \wedge z_j, z_k \wedge z_l\right]_2 = \epsilon_{ijklm} \partial_{z_m}$$

$$\left[\partial_{z_i}, \partial_{z_j}, z_k \wedge z_\ell\right]_3 = 4\left(\delta_k^i \delta_\ell^j - \delta_\ell^i \delta_k^j\right)$$

$$H^{\cdot}(\mathfrak{m2brane}^{\,Q}) \oplus \left(L^{\vee} \xrightarrow{\mathbb{B}} \Pi L^{\vee}\right)$$

$$[v, \lambda] = \langle v, \lambda \rangle \in \mathbb{C}_b$$
$$[v, \psi] = \langle v, \psi \rangle \in \Pi L_{\tilde{\lambda}}$$

$$\mathfrak{g} \rightsquigarrow \mathfrak{m2brane}^{\,Q}$$

$$H: \Omega^2(\mathbb{R}^{11}) \to \Omega^1(\mathbb{R}^{11})$$

$$\tilde{\psi} = \psi - H\Gamma_{\wedge^2}(Q, \psi) \in \Pi S \oplus \Pi \Omega^1$$

$$[v, \tilde{\psi}] = -L_v(H\Gamma_{\wedge^2}(Q, \psi)) = -\langle v, \Gamma_{\wedge^2}(Q, \psi) \rangle - \mathrm{d}\langle v, H\Gamma_{\wedge^2}(Q, \psi) \rangle$$

$$v \otimes \psi \mapsto \langle v, H\Gamma_{\wedge^2}(Q, \psi) \rangle \in L_\lambda$$

$$K \circlearrowright (\mathfrak{g}, \delta) \underset{i}{\overset{q}{\rightleftarrows}} (H^{\cdot}(\mathfrak{m2brane}^{\,Q}), 0)$$

$$H^{\cdot}(\mathfrak{m2brane}^{\,Q}) \rightsquigarrow \mathcal{L}(\mathbb{C}^5 \times \mathbb{R})$$

$$
\begin{aligned}
L^\vee &\mapsto 0 \\
\wedge^2 L_1^\vee &\mapsto 0 \\
z_i \wedge z_j \in \wedge^2 L_2^\vee &\mapsto \frac{1}{2}\left(z_i \, \mathrm{d}z_j - z_j \, \mathrm{d}z_i\right) \in \Omega^{1,0}(\mathbb{C}^5) \mathbin{\widehat{\otimes}} \Omega^0(\mathbb{R}) \\
A_{ij} \in \mathfrak{sl}(5) &\mapsto \sum_{ij} A_{ij} z_i \partial_{z_j} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \mathbin{\widehat{\otimes}} \Omega^0(\mathbb{R}) \\
\partial_{z_j} \in L &\mapsto \partial_{z_i} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \mathbin{\widehat{\otimes}} \Omega^0(\mathbb{R}^5) \\
1 \in \mathbb{C}_b &\mapsto 1 \in \Omega^{0,0}(\mathbb{C}^5) \mathbin{\hat{\otimes}} \Omega^0(\mathbb{R}).
\end{aligned}
$$

$$\left[z_i \wedge z_j, z_k \wedge z_l\right] = \epsilon_{ijklm} \partial_{z_m}$$

$$\left[\partial_{z_i}, z_j \, \mathrm{d}z_k - z_k \, \mathrm{d}z_j\right] = \delta_j^i \, \mathrm{d}z_k - \delta_k^i \, \mathrm{d}z_j$$

$$\Phi^{(2)}\left(\partial_{z_i}, z_j \wedge z_k\right) = \frac{1}{2}\left(\delta_j^i z_k - \delta_k^i z_j\right).$$

$$\left[\Phi^{(1)}\left(\partial_{z_i}\right), \Phi^{(1)}\left(z_j \wedge z_k\right)\right] = \partial \Phi^{(2)}\left(\partial_{z_i}, z_j \wedge z_k\right)$$

$$
\begin{aligned}
\Phi^{(1)}\left[\partial_{z_i}, \partial_{z_j}, z_k \wedge z_l\right]_3 &= \left[\Phi^{(1)}\left(\partial_{z_i}\right), \Phi^{(1)}\left(\partial_{z_i}\right), \Phi^{(1)}\left(z_k \wedge z_l\right)\right]_3 \\
&+ \left[\partial_{z_i}, \Phi^{(2)}\left(\partial_{z_j}, z_k \wedge z_l\right)\right] + \left[\partial_{z_j}, \Phi^{(2)}\left(\partial_{z_i}, z_k \wedge z_l\right)\right]
\end{aligned}
$$

$$H^{\cdot}(\mathfrak{m2brane}^{Q}) \to E\widehat{(5{,}10)}$$

$$L \oplus \mathrm{Stab}(Q) \oplus \Pi(\wedge^3 L) \oplus \mathbb{C}_b \to E(5{,}10) \oplus \mathbb{C}_{b'}$$

$$
\begin{aligned}
L_1^\vee &\mapsto 0 \\
\wedge^2 L_1^\vee &\mapsto 0 \\
z_i \wedge z_j \in \wedge^2 L_2 &\mapsto \mathrm{d}z_i \wedge \mathrm{d}z_j \in \Omega_{cl}^2(\mathbb{C}^5) \\
A_{ij} \in \mathfrak{sl}(5) &\mapsto \sum_{ij} A_{ij} z_i \partial_{z_j} \in \mathrm{Vect}_0(\mathbb{C}^5) \\
\partial_{z_i} \in L &\mapsto \partial_{z_i} \in \mathrm{Vect}_0(\mathbb{C}^5) \\
b \in \mathbb{C}_b &\mapsto b \in \mathbb{C}_{b'}.
\end{aligned}
$$

$$\mu_2(\psi, \psi', v, v') = \langle v \wedge v', \Gamma(\psi, \psi')\rangle$$

$$A \in \Pi\Omega^{0,\cdot}(\mathbb{C}^2) \mathbin{\widehat{\otimes}} \Omega^{\cdot}(\mathbb{R}^7),$$

$$\bar{\partial} A + \mathrm{d}_{\mathbb{R}^7} A + \partial_{z_1} A \wedge \partial_{z_2} A = 0$$

$$\mathbb{C}^2 \times \mathbb{R}^7$$

$$Q + Q_{nm}$$

$$Q_{nm} \in \wedge^2 (L^\vee)$$

$$\mathbb{C}^5 \times \mathbb{R} = \mathbb{C}_{z_i}^2 \times \mathbb{C}_{w_a}^3 \times \mathbb{R}$$

$$Q_{nm} = \mathrm{d}z_1 \wedge \mathrm{d}z_2$$

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2_{\geqslant 0}} \frac{1}{1-q^{-n_1+n_2}}$$

$$\boldsymbol{A}_{(n_1,n_2)}: A \mapsto \partial^{n_1}_{w_1}\partial^{n_2}_{w_2}A(0)$$

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2_{\geqslant 0}} q^{-n_1+n_2}$$

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2_{\geqslant 0}} \frac{1}{1-q^{-n_1+n_2}}$$

$$q_1q_2 = 1, q_3q_4q_5 = 1$$

$$i(q) = \frac{1}{(1-q)(1-q^{-1})}$$

$$\gamma_{nm} = \frac{1}{2}(z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R})$$

$$\mathrm{d}z_1 \wedge \mathrm{d}z_2 \in \Omega^2_{cl}(\mathbb{C}^5)$$

$$[f_l\partial_{z_l}, \mathrm{d}z_1 \wedge \mathrm{d}z_2] = \partial f_i \wedge \mathrm{d}z_j - \partial f_j \wedge \mathrm{d}z_i$$
$$[g_a\partial_{w_a}, \mathrm{d}z_1 \wedge \mathrm{d}z_2] = 0$$
$$[h^{ab}\,\mathrm{d}w_a \wedge \mathrm{d}w_b, \mathrm{d}z_1 \wedge \mathrm{d}z_2] = \epsilon_{abc}h^{ab}\partial_{w_c}.$$

$$\partial_{z_1}f + \partial_{z_2}g = 0$$

$$H^{\cdot}(E(5,10), [\mathrm{d}z_1 \wedge \mathrm{d}z_2, -]) \simeq \mathrm{Vect}_0(\mathbb{C}^2)$$

$$0 \to \mathbb{C} \to \mathcal{O}(\mathbb{C}^2) \to \mathrm{Vect}_0(\mathbb{C}^2) \to 0$$

$$H^{\cdot}(\widehat{E(5,10)}, [\mathrm{d}z_1 \wedge \mathrm{d}z_2, -]) \simeq \mathcal{O}(\mathbb{C}^2)$$

$$\varphi(\mu, \mu', \alpha) = \langle \mu \wedge \mu', \alpha\rangle|_{z=0}$$

$$(f_i\partial_{z_i}, g_j\partial_{z_j}) \mapsto (f_1g_2 - f_2g_1)(z_1 = z_2 = 0)$$

$$Z \times M$$

$$\alpha \in \Pi\Omega^{0,\cdot}(Z)\,\widehat{\otimes}\,\Omega^{\cdot}(M)$$

$$\{\alpha^I(z,\bar{z})\mathrm{d}\bar{z}_I, \alpha^J(z,\bar{z})\mathrm{d}\bar{z}_J\}_{pb} = (\partial_{z_1}\alpha^I\partial_{z_2}\alpha^J \pm \partial_{z_2}\alpha^I\partial_{z_1}\alpha^J)\mathrm{d}\bar{z}_I \wedge \mathrm{d}\bar{z}_J$$

$$\frac{1}{2}\int_{Z\times M}(\alpha \wedge \mathrm{d}\alpha) \wedge \omega_Z^{2,0} + \frac{1}{6}\int_{Z\times M}\alpha \wedge \{\alpha,\alpha\}_{pb} \wedge \omega_Z^{2,0}$$

$$\mu_z \in \mathrm{PV}^{1,\cdot}(\mathbb{C}^2_z) \otimes \mathrm{PV}^{0,\cdot}(\mathbb{C}^3_w) \otimes \Omega^{\cdot}(\mathbb{R})$$
$$\mu_w \in \mathrm{PV}^{0,\cdot}(\mathbb{C}^2_z) \otimes \mathrm{PV}^{1,\cdot}(\mathbb{C}^3_w) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\frac{1}{2}\int_{\mathbb{C}^2\times\mathbb{C}^3\times\mathbb{R}} \frac{1}{1-\nu}(\partial\gamma \vee \mu^2) \wedge (\mathrm{d}^2 z \wedge\ \mathrm{d}^3 w)$$

$$\frac{1}{6}\int_{\mathbb{C}^2\times\mathbb{C}^3\times\mathbb{R}} \gamma\partial\gamma\partial\gamma$$

$$\int\ \frac{1}{1-\nu}\left(\frac{1}{2}\partial^w\gamma_w \vee \mu_w^2 + \partial^z\gamma_w \vee \mu_w\mu_z + \partial^w\gamma_z \vee \mu_w\mu_z + \frac{1}{2}\partial^z\gamma_z \vee \mu_z^2\right) \wedge (\mathrm{d}^2 z \wedge\ \mathrm{d}^3 w)$$
$$+\frac{1}{2}\int\ \frac{1}{1-\nu}(\mathrm{d}^2 z \vee \mu_z^2) \wedge (\mathrm{d}^2 z \wedge\ \mathrm{d}^3 w)$$

$$\frac{1}{6}\int\ (\gamma_w\partial^z\gamma_w\partial^z\gamma_w + \gamma_w\partial^w\gamma_w\partial^z\gamma_z + \gamma_w\partial^w\gamma_z\partial^w\gamma_z) + \frac{1}{2}\int\ (\gamma_w\partial^w\gamma_w) \wedge \mathrm{d}^2 z$$

$$\frac{1}{2}\int\ (\ \mathrm{d}^2 z \vee \mu_z^2) \wedge (\mathrm{d}^2 z \wedge\ \mathrm{d}^3 w) + \frac{1}{2}\int\ (\gamma_w \wedge \partial^w\gamma_w) \wedge \mathrm{d}^2 z$$

$$\Omega_Z^{1,\cdot}\ \widehat{\otimes}\ \Omega_W^{0,\cdot} \overset{\omega_Z^{2,0}\otimes\mathbb{1}}{\rightarrow} \mathrm{PV}_Z^{1,\cdot}\ \widehat{\otimes}\ \mathrm{PV}_W^{0,\cdot}$$

$$\Omega_Z^{0,\cdot}\ \widehat{\otimes}\ \Omega_W^{1,\cdot} \overset{\mathbb{1}\otimes\partial^w}{\rightarrow} \Omega_Z^{0,\cdot}\ \widehat{\otimes}\ \Omega_W^{2,\cdot} \overset{\mathbb{1}\otimes\Omega_W}{\rightarrow} \mathrm{PV}_Z^{0,\cdot}\ \widehat{\otimes}\ \mathrm{PV}_W^{1,\cdot}$$

$$\Omega_Z^{0,\cdot}\ \widehat{\otimes}\ \Omega_W^{\cdot,\cdot}\ \hat{\otimes}\ \Omega_L^{\cdot} = \bigoplus_{k=0}^{3} \Omega_Z^{0,\cdot}\ \hat{\otimes}\ \Omega_W^{k,\cdot}\ \hat{\otimes}\ \Omega_L^{\cdot}$$

$$\mu_z = (1-\tilde{\alpha}^3)\left(\partial_{z_1} \wedge \partial_{z_2}\right) \vee \partial^z\alpha^0, \mu_w = \left(\partial_{w_1} \wedge \partial_{w_2} \wedge \partial_{w_3}\right) \vee \alpha^2, \nu = \tilde{\alpha}^3$$
$$\beta = \alpha^0, \gamma_w = \alpha^1, \gamma_z = 0$$

$$\int\ \sum_{k=0}^{3} \alpha^k\left(\bar{\partial} + \mathrm{d}_{\mathbb{R}}\right)\alpha^{3-k}$$

$$\int\ \alpha^0\partial^w\alpha^2 - \int\ \alpha^0\partial^z\alpha^0\partial^z\alpha^3$$

$$\int\ \frac{1}{2}\alpha^1\partial^w\alpha^1$$

$$\frac{1}{2}\int\ \frac{1}{1-\tilde{\alpha}^3}\partial^w\alpha^1(\tilde{\alpha}^2)^2\ \mathrm{d}^2 z + \int\ \alpha^2\partial^z\alpha^0\partial^z\alpha^1 + \frac{1}{2}\int\ (1-\alpha^3)\partial^z\alpha^0\partial^z\alpha^0$$

$$\frac{1}{6}\int\ \alpha^1\partial^z\alpha^1\partial^z\alpha^1$$

$$S_{pCS}(\alpha) + \int\ \frac{1}{2}\frac{1}{1-\tilde{\alpha}^3}\partial^w\alpha^1(\tilde{\alpha}^2)^2\ \mathrm{d}^2 z$$

$$\frac{1}{6}\int\ \frac{1}{1-\tilde{\alpha}^3}\alpha^2(\tilde{\alpha}^2)^2$$

$$\frac{1}{2}\int\ \frac{1}{1-\tilde{\alpha}^3}\partial^w\alpha^1(\tilde{\alpha}^2)^2 + \frac{1}{6}\int\ \frac{1}{1-\tilde{\alpha}^3}\partial^w(\alpha^2)\alpha^2(\tilde{\alpha}^2)^2$$

$$\mathrm{PV}^{i,j}(X)=\Omega^{0,j}\left(X,\wedge^{i}\ \mathrm{T}_X\right)$$

$$\partial_\Omega:\mathrm{PV}^{i,\cdot}(X)\to \mathrm{PV}^{i-1,\cdot}(X)$$

$$\left(\mathrm{PV}^{\cdot\cdot}(X)[[u]][2],\bar{\partial}+u\partial_\Omega\right)$$

$$(\partial_\Omega\otimes 1)\delta_{\Delta\subset X\times X}\in[\mathrm{PV}^{\cdot\cdot}(X)]^{\hat{\otimes}2}$$

$$I_{BCOV}(\Sigma)=\mathrm{Tr}_X\langle\exp\ \Sigma\rangle_0=\sum_{n\geqslant 0}\mathrm{Tr}_X\left\langle\Sigma^{\otimes n}\right\rangle_0$$

$$\left\langle u^{k_1}\mu_1\otimes\cdots\otimes u^{k_m}\mu_m\right\rangle_0:=\left(\int_{\overline{\mathcal{M}}_{0,m}}\psi_1^{k_1}\cdots\psi_m^{k_m}\right)\mu_1\cdots\mu_m=\binom{m-3}{k_1,\cdots,k_m}\mu_1\cdots\mu_m$$

$$\Sigma\mapsto[u(\exp\ (\Sigma/u)-1)]_+$$

$$\left(\bigoplus_{i+j\leqslant d-1}u^i\mathrm{PV}^{j,\cdot}(X)[2],\bar{\partial}+u\partial_\Omega\right)$$

$$\begin{array}{r}\mathrm{PV}^{1,\cdot}\overset{u\partial_\Omega}{\to}u\mathrm{PV}^{0,\cdot}\\ \mathrm{PV}^{2,\cdot}\overset{u\partial_\Omega}{\to}u\mathrm{PV}^{1,\cdot}\overset{u\partial_\Omega}{\to}u^2\mathrm{PV}^{0,\cdot}\\ \mathrm{PV}^{3,\cdot}\overset{u\partial_\Omega}{\to}u\mathrm{PV}^{2,\cdot}\overset{u\partial_\Omega}{\to}u^2\mathrm{PV}^{1,\cdot}\overset{u\partial_\Omega}{\to}u^3\mathrm{PV}^{0,\cdot}\end{array}$$

$$\alpha=\sum_n\alpha_n u^n\in\mathcal{E}_{mKS}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)$$

$$I_{IIA}=\int_{\mathbb{C}^4\times\mathbb{R}^2}\alpha_0^3+\cdots$$

$$\eta\in\mathrm{PV}^{0,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2),\mu+u\nu\in\mathrm{PV}^{1,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)\oplus u\mathrm{PV}^{0,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)$$
$$\Pi\in\mathrm{PV}^{3,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2),\sigma\in\mathrm{PV}^{3,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)$$

$$I_{IIA}=\frac{1}{2}\mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2}\frac{1}{1-\nu}\mu^2\wedge\Pi+\mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2}\frac{1}{1-\nu}\eta\wedge\mu\wedge\sigma+\frac{1}{2}\mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2}\frac{1}{1-\nu}\eta\wedge\Pi^2+\cdots$$

$$\begin{array}{l}\mathrm{PV}^{0,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)_\eta\\ u^{-1}\Omega^{0,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)_\beta\overset{u\partial}{\to}\Omega^{1,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)_\gamma\\ \Omega^{0,\cdot}(\mathbb{C}^4)\otimes\Omega^{\cdot}(\mathbb{R}^2)_\theta\end{array}$$

$$\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega}\eta\theta+\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega}\mu\vee\gamma+\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega}\nu\beta$$

$$\tilde{I}_{IIA}=\frac{1}{2}\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega}\frac{1}{1-\nu}\mu^2\vee\partial\gamma+\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega}\frac{1}{1-\nu}(\eta\wedge\mu)\vee\partial\theta+\frac{1}{2}\int_{\mathbb{C}^4\times\mathbb{R}^2}\frac{1}{1-\nu}\eta\wedge\partial\gamma\wedge\partial\gamma$$

$$M\times V\to M\times V_{\mathbb{R}}$$

$$\mathbb{C}^4\times\mathbb{C}\times\mathbb{R}_t\to\mathbb{C}^4\times\mathbb{R}_x\times\mathbb{R}_t\cong\mathbb{C}^4\times\mathbb{R}^2$$

$$\nu_{11d} \in \mathrm{PV}^{0,\cdot}(\mathbb{C}^5) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\nu(z_i, x, t) = \nu_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}$$

$$\beta(z_i, x, t) = \beta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

$$\mu_{11d} = \mu_{11d}^0 + \theta_{11d}\partial_{z_5}$$

$$\mu_{11d}^0 \in \mathrm{PV}^{1,\cdot}(\mathbb{C}^4) \otimes \Omega^{0,\cdot}(\mathbb{C}_{z_5}) \otimes \Omega^{\cdot}(\mathbb{R}_t)$$
$$\theta_{11d} \in \Omega^{0,\cdot}(\mathbb{C}^4) \otimes \Omega^{0,\cdot}(\mathbb{C}_{z_5}) \otimes \Omega^{\cdot}(\mathbb{R}_t).$$

$$\mu(z_i, x, t) = \mu_{11d}^0(z_i, x, y = 0, t)\big|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

$$\theta(z_i, x, t) = \theta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}$$

$$\gamma_{11d} = \gamma_{11d}^0 + \eta_{11d}\, \mathrm{d}z_5$$

$$\gamma_{11d}^0 \in \Omega^{1,\cdot}(\mathbb{C}^4) \otimes \Omega^{0,\cdot}(\mathbb{C}_{z_5}) \otimes \Omega^{\cdot}(\mathbb{R}_t)$$
$$\eta_{11d} \in \mathrm{PV}^{0,\cdot}(\mathbb{C}^4) \otimes \Omega^{0,\cdot}(\mathbb{C}_{z_5}) \otimes \Omega^{\cdot}(\mathbb{R}_t)$$

$$\gamma(z_i, x, t) = \gamma_{11d}^0(z_i, x, y = 0, t)\big|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}$$

$$\eta(z_i, x, t) = \eta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}$$

$$\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}\mu^2 \vee \partial\gamma + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}(\theta \wedge \mu) \vee \partial\eta$$

$$\int_{\mathbb{C}^4\times\mathbb{R}^2} \eta \wedge \partial\gamma \wedge \partial\gamma$$

$$\tilde{\theta} = \frac{1}{1-\nu}\theta, \tilde{\eta} = (1-\nu)\eta, \tilde{\beta} = \beta + \frac{1}{1-\nu}\eta \wedge \theta$$

$$\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}\mu^2 \vee \partial\gamma + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}}{}^4} \frac{1}{1-\nu}\tilde{\eta} \wedge \partial\gamma \wedge \partial\gamma + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}}^4} (\tilde{\theta} \wedge \mu) \vee \partial\left(\frac{1}{1-\nu}\tilde{\eta}\right)$$
$$+ \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}(\tilde{\eta} \wedge \tilde{\theta})\partial_\Omega\mu$$

$$- \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \left(\frac{1}{1-\nu}\tilde{\eta}\right)\partial_\Omega(\tilde{\theta}\mu) + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \left(\frac{1}{1-\nu}\tilde{\eta}\right)\tilde{\theta}\partial_\Omega\mu$$

−	+	−	+
$\mathrm{PV}^{1,\bullet}(\mathbb{C}^5)$ $\xrightarrow{u\partial_\Omega}$	$u\mathrm{PV}^{0,\bullet}(\mathbb{C}^5)$		

$$\mathrm{PV}^{3,\bullet}(\mathbb{C}^5) \xrightarrow{u\partial_\Omega} u\mathrm{PV}^{2,\bullet}(\mathbb{C}^5) \xrightarrow{u\partial_\Omega} u^2\mathrm{PV}^{1,\bullet}(\mathbb{C}^5) \xrightarrow{u\partial_\Omega} u^3\mathrm{PV}^{0,\bullet}(\mathbb{C}^5)$$

$$I_{\mathrm{typeI}} = \mathrm{Tr}_{\mathbb{C}^5}\frac{1}{1-\nu}\mu^2 \vee \sigma + \cdots$$

$$\begin{array}{ccccc} - & & + & & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & & \\ & & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde{\beta}} & \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde{\gamma}}. \end{array}$$

$$\tilde{I}_{\mathrm{typeI}} = \frac{1}{2}\int_{\mathbb{C}^5}^{\Omega} \frac{1}{1-\nu}\mu^2 \vee \partial\tilde{\gamma}$$

$$\begin{array}{ccc} - & & + \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu \\ \Omega^{0,\bullet}(\mathbb{C}^5)_\beta & \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma. \end{array}$$

$$\mathcal{M}_{t=0}, \mathcal{M}_{t=1} \subset \mathcal{E}_\partial$$

$$\mathcal{M}_{t=0} \overset{\unrhd}{\times} \mathcal{E}_\partial \mathcal{M}_{t=1}.$$

$$\mathcal{M}_{t=0}\colon \gamma|_{t=0} = \beta|_{t=0} = 0$$

$$\mathcal{M}_{t=1}\colon \gamma|_{t=1} = \beta|_{t=1} = 0$$

$$\mathbb{C}^5 \times [0,1] \to \mathbb{C}^5$$

$$\begin{array}{ccc} - & & + \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu. \end{array}$$

$$\begin{array}{ccccc} - & & + & & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & & \\ \Omega^{0,\bullet}(\mathbb{C}^5)_\beta & \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma & & \\ & \overset{1}{\searrow} & & \overset{1}{\searrow} & \\ & & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde{\beta}} & \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde{\gamma}}. \end{array}$$

$$\mathcal{M}_{t=0} \hookrightarrow \tilde{\mathcal{M}}_{t=0} \to \mathcal{E}_\partial$$

$$\begin{array}{ccccc} - & & + & & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & & \\ & & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde{\beta}} & \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde{\gamma}} \end{array}$$

$$\mathbb{C}^5 \times S^1 \to \mathbb{C}^5$$

$$\mu + \epsilon\mu' \in \Pi\mathrm{PV}^{1,\cdot}(\mathbb{C}^5)[\epsilon]$$

$$\begin{array}{ccccccc} \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & & & & \\ & & \epsilon\Omega^{0,\bullet}(\mathbb{C}^5)_{\beta'} & \xrightarrow{\partial_\Omega} & \epsilon\Omega^{1,\bullet}(\mathbb{C}^5)_{\gamma'}. & & \\ & & \epsilon\mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_{\mu'} & \xrightarrow{\partial_\Omega} & \epsilon\mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_{\nu'} & & \\ & & & & \Omega^{0,\bullet}(\mathbb{C}^5)_\beta & \xrightarrow{\partial_\Omega} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma. \end{array}$$

$$\int_{\mathbb{C}^5}^{\Omega} \left(\beta' \wedge \bar{\partial}\nu + \beta \wedge \bar{\partial}\nu' + \gamma' \wedge \bar{\partial}\mu + \gamma \wedge \bar{\partial}\mu' + \beta' \wedge \partial_\Omega \mu + \beta \wedge \partial_\Omega \mu'\right)$$
$$+ \int_{\mathbb{C}^5}^{\Omega} \left(\frac{1}{2}\frac{1}{1-\nu}\mu^2 \vee \partial\gamma' + \frac{1}{1-\nu}(\mu \wedge \mu') \vee \partial\gamma' + \frac{1}{2}\frac{\nu'}{(1-\nu)^2}\mu^2 \vee \partial\gamma\right)$$
$$+\frac{1}{2}\int_{\mathbb{C}^5} \gamma' \wedge \partial\gamma \wedge \partial\gamma$$

$$X \times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2 \times \mathbb{R}$$

$$\alpha, \eta \in \Pi\Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$A_{\mathrm{grav}}, B_{\mathrm{grav}} \in \Pi\Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\int_{\mathbb{C}^2\times\mathbb{R}}^{\Omega} \left(\eta\bar{\partial}\alpha + B_{\mathrm{grav}}\bar{\partial}A_{\mathrm{grav}} + B\bar{\partial}A + \psi\bar{\partial}\chi\right)$$
$$+ \int_{\mathbb{C}^2\times\mathbb{R}}^{\Omega} \left(\frac{1}{2}\eta\{\alpha,\alpha\} + B_{\mathrm{grav}}\{\alpha, A_{\mathrm{grav}}\} + B\{\alpha, A\} + \psi\{\alpha,\chi\}\right)$$
$$+\frac{1}{6}\int_{\mathbb{C}^2\times\mathbb{R}} B_{\mathrm{grav}}\,\partial B_{\mathrm{grav}}\,\partial B_{\mathrm{grav}}$$

$$\begin{aligned} \mathrm{PV}^{0,\cdot}(X \times \mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) &\simeq H^{\cdot}(X, \mathcal{O}) \otimes \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \\ &= \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \oplus \Pi\bar{\Omega}_X \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \end{aligned}$$

$$\nu_{11d} = \nu + \bar{\Omega}_X \tilde{\nu}$$

$$\begin{aligned} \Pi\mathrm{PV}^{1,\cdot}(X \times \mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) &\simeq \Pi H^{\cdot}(X, \mathcal{O}) \otimes \mathrm{PV}^{1,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \\ &\oplus \Pi H^{\cdot}(X,\ \mathrm{T}_X) \otimes \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \\ &= \Pi\mathrm{PV}^{1,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \oplus \bar{\Omega}_X \mathrm{PV}^{1,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \\ &\oplus H^1(X,\ \mathrm{T}_X) \otimes \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \oplus \Pi H^2(X,\ \mathrm{T}_X) \otimes \mathrm{PV}^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \end{aligned}$$

$$\begin{aligned} \mu_{11d} &= \mu + \bar{\Omega}_X \tilde{\mu} \\ &\quad + e^i \chi_i + f^a A_a + (\Omega_X^{-1} \vee \omega^2) A_{\mathrm{grav}} \end{aligned}$$

$$H^2(X, \Omega_X^2)_\perp \subset H^2(X, \Omega_X^2) \cong H^2(X,\ \mathrm{T}_X)$$

$$\begin{aligned} \beta_{11d} &= \beta + \bar{\Omega}_X \tilde{\beta} \\ \gamma_{11d} &= \gamma + \bar{\Omega}_X \tilde{\gamma} + e_i \psi^i + f_a B^a + \omega \wedge B_{\mathrm{grav}} \end{aligned}$$

$$\alpha, \chi \in \Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\eta = (\mathrm{d}^2 z)^{-1} \vee \partial\tilde{\gamma}, \psi = (\mathrm{d}^2 z)^{-1} \vee \partial\gamma \in \Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\begin{array}{cccccc} \alpha, A_{\mathrm{grav}} & \in \Pi\Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}), \eta, B_{\mathrm{grav}} & & & & \in \Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}) \\ \chi, \chi_i & \in & \Omega^0 \cdot (\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}), & \psi, \psi^i & \in\ \Omega^{0,}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}), & i = 1, \ldots, h^{2,1} \\ A_a & \in & \Pi\Omega^{0,\cdot}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}), & B^a & \in & \Pi\Omega^{0,}(\mathbb{C}^2) \otimes \Omega^{\cdot}(\mathbb{R}), a = 1, \ldots, h^{1,1} - 1. \end{array}$$

$$\int_{\mathbb{C}^2\times\mathbb{R}}^{\Omega} \left(\frac{1}{2}\partial\alpha \wedge \partial\alpha \wedge \eta + \partial A_{\mathrm{grav}} \wedge \partial A_{\mathrm{grav}} \wedge B_{\mathrm{grav}}\right)$$
$$+ \int_{\mathbb{C}^2\times\mathbb{R}}^{\Omega} \left(\partial\alpha \wedge \partial\chi \wedge \psi + \partial\alpha \wedge \partial\chi_i \wedge \psi^i + \partial\alpha \wedge \partial A_a \wedge B^a\right)$$

$$\frac{1}{6}\int_{\mathbb{C}^2\times\mathbb{R}} B_{\text{grav}}\,\partial B_{\text{grav}}\,\partial B_{\text{grav}}$$

$$H^{\cdot}(X,\Omega^{\cdot}) = \bigoplus_{i,j} H^i\left(X,\Omega_X^j\right)$$

$$H^{\cdot}(X,\Omega^{\cdot})\otimes\mathcal{O}(\mathbb{C}^2)$$

$$[[\omega]\otimes f,[\omega]\otimes g] = [\omega^2]\otimes\{f,g\}\in H^{2,2}(X,\Omega^2)\otimes\mathcal{O}(\mathbb{C}^2)$$

$$\mathbb{C}_w^4\times\mathbb{C}_z\times\mathbb{R}$$

$$I_{M2}(\gamma) = N\int_{\mathbb{C}_z}\gamma+\cdots$$

$$I_{D2}(\gamma) = N\int_{\mathbb{R}\times\mathbb{C}_z}\gamma+\cdots$$

$$\bar{\partial}\mu+\frac{1}{2}[\mu,\mu]+\partial\gamma\partial\gamma = N\Omega^{-1}\delta_{w=0}$$

$$\partial_\Omega\mu = 0$$

$$F_{M2} = \frac{6}{(2\pi i)^4}\frac{\sum_{a=1}^4 \bar{w}_a\,\mathrm{d}\bar{w}_1\cdots\widehat{\mathrm{d}}_a\cdots\mathrm{d}\bar{w}_4}{\|w\|^8}\partial_z$$

$$\bar{\partial}(NF_{M2})+\frac{1}{2}[NF_{M2},NF_{M2}] = N\Omega^{-1}\delta_{w=0}$$
$$\partial_\Omega(NF_{M2}) = 0$$

$$[F_{M2},F_{M2}] = 0$$

$$\frac{\partial}{\partial z},z\frac{\partial}{\partial z}-\frac{1}{4}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a},z\left(z\frac{\partial}{\partial z}-\frac{1}{2}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a}\right)\in\mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R})$$

$$\sum_{a,b=1}^4 B_{ab}w_a\frac{\partial}{\partial w_b}\in\mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}),(B_{ab})\in\mathfrak{sl}(4)$$

$$(\wedge^2\,\mathbb{C}^4)_{+1}\oplus(\wedge^2\,\mathbb{C}^4)_{-1}$$

$$\frac{1}{2}(w_a\,\mathrm{d}w_b-w_b\,\mathrm{d}w_a)\in\Omega^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}),a,b = 1,2,3,4$$

$$\frac{1}{2}z(w_a\,\mathrm{d}w_b-w_b\,\mathrm{d}w_a)\in\Omega^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}),a,b = 1,2,3,4$$

$$i_{M2}\colon\mathfrak{osp}(6\mid 1)\hookrightarrow E(5,10)$$

$$\mathrm{d}w_a\wedge\,\mathrm{d}w_b,a,b = 1,2,3,4$$

$$z\,\mathrm{d}w_a\wedge\,\mathrm{d}w_b+\frac{1}{2}\,\mathrm{d}z\wedge(w_a\,\mathrm{d}w_b-w_b\,\mathrm{d}w_a),a,b = 1,2,3,4$$

$$(\mathbb{C}^5 \times \mathbb{R}) \setminus \{w = 0\} \cong (\mathbb{C}^4_w \setminus 0) \times \mathbb{C}_z \times \mathbb{R}$$

$$[F, -]\colon \mathrm{PV}^{i,\cdot}(\mathbb{C}^4_w \setminus 0) \otimes \mathrm{PV}^{j,\cdot}(\mathbb{C}_z) \otimes \Omega^{\cdot}(\mathbb{R}) \to \mathrm{PV}^{i,\cdot+3}(\mathbb{C}^4_w \setminus 0) \otimes \mathrm{PV}^{j,\cdot}(\mathbb{C}_z) \otimes \Omega^{\cdot}(\mathbb{R})$$
$$[F, -]\colon \Omega^{i,\cdot}(\mathbb{C}^4_w \setminus 0) \otimes \Omega^{j,\cdot}(\mathbb{C}_z) \otimes \Omega^{\cdot}(\mathbb{R}) \to \Omega^{i,\cdot+3}(\mathbb{C}^4_w \setminus 0) \otimes \Omega^{j,\cdot}(\mathbb{C}_z) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\delta^{(1)} = \bar{\partial} + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu \to \nu} + \partial|_{\beta \to \gamma}$$

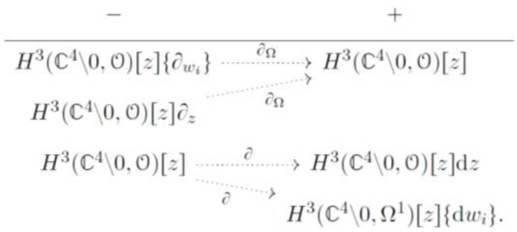
+	−
$H^\bullet(\mathbb{C}^4\backslash 0, \mathrm{T}) \otimes H^\bullet(\mathbb{C}, \mathcal{O})$	$H^\bullet(\mathbb{C}^4\backslash 0, \mathcal{O}) \otimes H^\bullet(\mathbb{C}, \mathcal{O})$
$H^\bullet(\mathbb{C}^4\backslash 0, \mathcal{O}) \otimes H^\bullet(\mathbb{C}, \mathrm{T})$	
$H^\bullet(\mathbb{C}^4\backslash 0, \mathcal{O}) \otimes H^\bullet(\mathbb{C}, \mathcal{O})$	$H^\bullet(\mathbb{C}^4\backslash 0, \mathcal{O}) \otimes H^\bullet(\mathbb{C}, \Omega^1)$
	$H^\bullet(\mathbb{C}^4\backslash 0, \Omega^1) \otimes H^\bullet(\mathbb{C}, \mathcal{O})$

$$\mathbb{C}[z] \hookrightarrow H^{\cdot}(\mathbb{C}, \mathcal{F})$$

$$\mathbb{C}[w_1, \dots, w_4] \hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathcal{O})$$
$$\mathbb{C}[w_1, \dots, w_4]\{\partial_{w_i}\} \hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathrm{T})$$
$$\mathbb{C}[w_1, \dots, w_4]\{\mathrm{d}w_i\} \hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \Omega^1)$$

$$(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}] \hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathcal{O})$$
$$(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}]\{\partial_{w_i}\} \hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathrm{T})$$
$$(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}]\{\mathrm{d}w_i\} \hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \Omega^1)$$

$$H^{\cdot}\big(\mathcal{L}(\mathbb{C}^5 \times \mathbb{R} \setminus \{w = 0\}), \bar{\partial}\big)$$



$$[F] = (w_1 \cdots w_4)^{-1}\partial_z \in H^3(\mathbb{C}^4 \setminus 0, \mathcal{O})[z]\partial_z$$

$$[[F], z(w_a\, \mathrm{d}w_b - w_b\, \mathrm{d}w_a)] = (w_1 \cdots w_4)^{-1}(w_a\, \mathrm{d}w_b - w_b\, \mathrm{d}w_a) = 0$$

$$\{w_1 = w_2 = t = 0\} \subset \mathbb{C}^3_z \times \mathbb{C}^2_w \times \mathbb{R}$$

$$I_{M5} = N \int_{\mathbb{C}^3_z} \partial_\Omega^{-1}\mu \vee \Omega + \cdots$$

$$N \int_{\mathbb{C}^2 \times \mathbb{R}} \partial_\Omega^{-1}\mu \vee \Omega_{\mathbb{C}^4} + \cdots$$

$$\bar{\partial}\partial\gamma + \partial_\Omega\left(\frac{1}{1-\nu}\mu\right) \wedge \partial\gamma = N\delta_{w_1 = w_2 = t = 0}$$

$$(\bar{\partial} + \mathrm{d}_{\mathbb{R}})\mu + \partial\gamma\partial\gamma = 0$$

$$F_{M5} = \frac{1}{(2\pi i)^3}\frac{\bar{w}_1\,\mathrm{d}\bar{w}_2 \wedge\,\mathrm{d}t - \bar{w}_2\,\mathrm{d}\bar{w}_1 \wedge\,\mathrm{d}t + t\,\mathrm{d}\bar{w}_1 \wedge\,\mathrm{d}\bar{w}_2}{(\|w\|^2 + t^2)^{5/2}} \wedge\,\mathrm{d}w_1 \wedge\,\mathrm{d}w_2$$

$$\bar{\partial}(NF_{M5}) + \mathrm{d}_{\mathbb{R}}(NF_{M5}) = N\delta_{w_1=w_2=t=0}$$
$$(NF_{M5}) \wedge (NF_{M5}) = 0.$$

$$\bar{\partial}F + \mathrm{d}_{\mathbb{R}}F = N\delta_{w_1=w_2=t=0}$$

$$(\mathbb{C}^2 \times \mathbb{R}) \setminus 0 \simeq S^4 \times \mathbb{R}$$

$$\oint_{S^4} NF = N$$

$$\mathbb{C}^5 \times \mathbb{R} = \mathbb{C}_z^3 \times \mathbb{C}_w^2 \times \mathbb{R}_t$$

$$\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$$

$$\frac{\partial}{\partial z_i} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), i = 1,2,3$$

$$A_{ij}z_i\frac{\partial}{\partial z_j} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \left(A_{ij}\right) \in \mathfrak{sl}(3)$$

$$\sum_{i=1}^{3} z_i\frac{\partial}{\partial z_i} - \frac{3}{2}\sum_{a=1}^{2} w_a\frac{\partial}{\partial w_a} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R})$$

$$z_j\left(\sum_{i=1}^{3} z_i\frac{\partial}{\partial z_i} - 2\sum_{a=1}^{2} w_a\frac{\partial}{\partial w_a}\right) \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R})$$

$$w_1\frac{\partial}{\partial w_2}, w_2\frac{\partial}{\partial w_1}, \frac{1}{2}\left(w_1\frac{\partial}{\partial w_1} - w_2\frac{\partial}{\partial w_2}\right) \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R})$$

$$L \otimes R \oplus \wedge^2 L \otimes R \cong \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \wedge^2 \mathbb{C}^3 \otimes \mathbb{C}$$

$$z_i\,\mathrm{d}w_a \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), a = 1,2, i = 1,2,3$$

$$\frac{1}{2}w_a\left(z_i\,\mathrm{d}z_j - z_j\,\mathrm{d}z_i\right) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), a = 1,2, k = 1,2,3$$

$$i_{M5}\colon \mathfrak{osp}(6 \mid 1) \hookrightarrow E.$$

$$\mathrm{d}z_i \wedge\,\mathrm{d}w_a, i = 1,2,3, a = 1,2.$$

$$w_a\,\mathrm{d}z_i \wedge\,\mathrm{d}z_j + \frac{1}{2}\,\mathrm{d}w_a \wedge \left(z_i\,\mathrm{d}z_j - z_j\,\mathrm{d}z_i\right), i,j = 1,2,3, a = 1,2$$

$$\Omega^{0,\cdot}(\mathbb{C}^5) \otimes \Omega^{\cdot}(\mathbb{R})$$

$$\Omega^{0,\cdot}(\mathbb{C}^5) \otimes \Omega^{\cdot}(\mathbb{R}) \cong \Omega^{\cdot}(\mathbb{C}^5 \times \mathbb{R})/(\mathrm{d}z_1, \dots, \mathrm{d}z_5)$$

$$\mathbb{C}^5 \times \mathbb{R} \setminus \mathbb{C}^3 \cong \mathbb{C}_z^3 \times (\mathbb{C}_w^2 \times \mathbb{R} \setminus 0)$$

$$\Omega^{\cdot}(\mathbb{C}^5 \times \mathbb{R} \setminus \mathbb{C}^3)/(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3, \mathrm{d}w_1, \mathrm{d}w_2)$$

$$\Omega^{0,\cdot}(\mathbb{C}_z^3) \otimes \left(\Omega^{\cdot}(\mathbb{C}_w^2 \times \mathbb{R} \setminus 0)/(\mathrm{d}w_1, \mathrm{d}w_2)\right)$$

$$\mathbb{C}[w_1, w_2] \hookrightarrow H^0\left(\Omega^{\cdot}(\mathbb{C}_w^2 \times \mathbb{R} \setminus 0)/(\mathrm{d}w_1, \mathrm{d}w_2)\right)$$

$$w_1^{-1} w_2^{-1} \mathbb{C}[w_1, w_2] \hookrightarrow H^2\left(\Omega^{\cdot}(\mathbb{C}_w^2 \times \mathbb{R} \setminus 0)/(\mathrm{d}w_1, \mathrm{d}w_2)\right)$$

$$\frac{\bar{w}_1\, \mathrm{d}\bar{w}_2 \wedge\, \mathrm{d}t - \bar{w}_2\, \mathrm{d}\bar{w}_1 \wedge\, \mathrm{d}t + t\, \mathrm{d}\bar{w}_1 \wedge\, \mathrm{d}\bar{w}_2}{(\|w\|^2 + t^2)^{5/2}}$$

$$\delta^{(1)} = \bar{\partial} + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu \to \nu} + \partial|_{\beta \to \gamma}.$$

$$\mathbb{C}^5 \times \mathbb{R} \setminus \{w_1 = w_2 = t = 0\} \cong \mathbb{C}_z^3 \times (\mathbb{C}_w^2 \times \mathbb{R} \setminus 0)$$

$+$		$-$
$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{w_i}\}$	$\xrightarrow{\partial_\Omega}$	$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]$
$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{z_i}\}$	$\xrightarrow[\partial_\Omega]{}$	
$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]$	$\xrightarrow{\partial}$	$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\mathrm{d}z_i\}$
	$\xrightarrow[\partial]{}$	$w_1^{-1} w_2^{-1} \mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\mathrm{d}w_i\}.$

$$\left[[F], f^i(z, w)\mathrm{d}z_i\right] = \epsilon_{ijk} w_1^{-1} w_2^{-1} \partial_{z_j} f^i(z, w) \partial_{z_k}$$

$$\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{z_i}\} \subset H^0(\mathbb{C}^3, \mathrm{T}) \otimes H^2\left(\Omega^{\cdot}(\mathbb{C}^2 \times \mathbb{R} \setminus 0)/(\mathrm{d}w_1, \mathrm{d}w_2)\right)$$

$$\left[[F], w_a\left(z_i\, \mathrm{d}z_j - z_j\, \mathrm{d}z_i\right)\right] = 2\epsilon_{ijk}(w_1^{-1} w_2^{-1}) \cdot w_a \partial_{z_k} = 0$$

$$\partial_t A_i = \Delta A_i + \left[A_j, 2\partial_j A_i - \partial_i A_j + \left[A_j, A_i\right]\right] + (C_{\mathrm{A}}^{\varepsilon} A)_i + \xi_i^{\varepsilon}$$

$$\lim_{\varepsilon \downarrow 0} |C_{\mathrm{A}}^{\varepsilon} - \bar{C}_{\mathrm{A}}^{\varepsilon}| = 0$$

$$\left|\mathbf{E}W_\ell[\mathcal{F}_s(A^a)] - \mathbf{E}W_\ell\left[\mathcal{F}_s\left(A^b\right)\right]\right| \gtrsim t^{\frac{10}{9}}$$

$$\chi^{\varepsilon}(t, x) = \varepsilon^{-5} \chi(\varepsilon^{-2} t, \varepsilon^{-1} x)$$

$$E = \mathfrak{g}^3 \oplus \mathbf{V}$$

$$\mathfrak{g} \ni v \mapsto \mathrm{Ad}_g v \in \mathfrak{g} \ \text{ and } \ \mathfrak{g}^3 \ni (v_1, v_2, v_3) \mapsto \left(\mathrm{Ad}_g v_1, \mathrm{Ad}_g v_2, \mathrm{Ad}_g v_3\right) \in \mathfrak{g}^3$$

$$C_{\mathrm{YM}}^{\varepsilon} \in L_G(\mathfrak{g}), C_{\mathrm{Higgs}}^{\varepsilon} \in L_G(\mathbf{V})$$

$$C_{\mathrm{A}}^{\varepsilon} = C_{\mathrm{YM}}^{\varepsilon} + \dot{C}_{\mathrm{A}} \in L(\mathfrak{g}^3) \ \text{ and } \ C_{\Phi}^{\varepsilon} = C_{\mathrm{Higgs}}^{\varepsilon} + \dot{C}_{\Phi} \in L(\mathbf{V}).$$

$$\begin{aligned} \partial_t A_i &= \Delta A_i + \left[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\right] \\ &\quad - \mathbf{B}\big((\partial_i \Phi + A_i \Phi) \otimes \Phi\big) + (C_{\mathrm{A}}^{\varepsilon} A)_i + \xi_i^{\varepsilon} \\ \partial_t \Phi &= \Delta \Phi + 2A_j \partial_j \Phi + A_j^2 \Phi - |\Phi|^2 \Phi + C_{\Phi}^{\varepsilon} \Phi + \xi_{\mathrm{H}}^{\varepsilon} \\ (A(0), \Phi(0)) &= (a, \varphi) \in \mathcal{C}^{\infty}, \end{aligned}$$

$$\langle \mathbf{B}(u \otimes v), h \rangle_{\mathfrak{g}} = \langle u, hv \rangle_{\mathbf{V}}.$$

$$\partial_t X = \Delta X + X \partial X + X^3 + C^{\varepsilon} X + \chi^{\varepsilon} * \xi$$

$$X = (A, \Phi) : [0, T] \times \mathbf{T}^3 \to E$$

$$C = \{C^{\varepsilon}\}_{\varepsilon \in (0,1)} = \{C_{\mathrm{A}}^{\varepsilon}, C_{\Phi}^{\varepsilon}\}_{\varepsilon \in (0,1)}$$

$$\mathfrak{G}^{\varrho} \stackrel{\text{def}}{=} \mathcal{C}^{\varrho}(\mathbf{T}^3, G)$$

$$g \cdot A \stackrel{\text{def}}{=} \mathrm{Ad}_g(A) - (\mathrm{d}g) g^{-1}, \text{ and } g \cdot \Phi \stackrel{\text{def}}{=} g\Phi$$

$$g \cdot \mathrm{SYMH}(C, (a, \varphi)) \stackrel{\text{law}}{=} \mathrm{SYMH}(C, g(0) \cdot (a, \varphi))$$

$$g^{-1}(\partial_t g) = \partial_j (g^{-1} \partial_j g) + [A_j, g^{-1} \partial_j g]$$

$$\check{C} \in L_G(\mathfrak{g})$$

$$[\mathrm{SYMH}(C, (a, \varphi))] \stackrel{\text{law}}{=} [\mathrm{SYMH}(C, g(0) \cdot (a, \varphi))]$$

$$\begin{aligned} \partial_s A_i &= \Delta A_i + \left[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\right] \\ A(0) &= a \end{aligned}$$

$$\mathrm{d}y_t = y_t \, \mathrm{d}\ell_A, y_0 = 1$$

$$\ell_A(t) = \int_0^t \langle A(\ell_s), \dot{\ell}_s \rangle \mathrm{d}s$$

$$W_{\ell}(A) = \mathrm{Trhol}(A, \ell)$$

$$\mathring{C}_{\mathrm{A}} = \check{C} + c \in L_G(\mathfrak{g}^3), C^{\varepsilon} = \left(C_{\mathrm{YM}}^{\varepsilon} + \dot{C}_{\mathrm{A}}, C_{\Phi}^{\varepsilon}\right) \in L_G(\mathfrak{g}^3) \oplus L_G(\mathbf{V})$$

$$\mathrm{SYMH}(C, x) = \left(A^{(1)}, \Phi^{(1)}\right), \mathrm{SYMH}(C, g \cdot x) = \left(A^{(2)}, \Phi^{(2)}\right)$$

$$\left| \mathbf{E} W_{\ell}\left[\mathcal{F}_s\left(A_t^{(1)}\right)\right] - \mathbf{E} W_{\ell}\left[\mathcal{F}_s\left(A_t^{(2)}\right)\right] \right| \geq \sigma t^{1+r}$$

$$\left| \mathbf{E} W_{\ell}[\mathcal{F}_s(A_t)] - \mathbf{E} W_{\ell}[\mathcal{F}_s(\tilde{A}_t)] \right| \geq \sigma t^{1+r}$$

$$\partial_t \tilde{X} = \Delta \tilde{X} + \tilde{X} \partial \tilde{X} + \tilde{X}^3 + \chi^{\varepsilon} * \xi + C^{\varepsilon} \tilde{X} + (c \, \mathrm{d}\tilde{g} \tilde{g}^{-1}, 0), \tilde{X}(0) = \tilde{x}$$

$$\partial_t \tilde{g} = \Delta \tilde{g} - (\partial_j \tilde{g}) \tilde{g}^{-1} (\partial_j \tilde{g}) + [\tilde{A}_j, (\partial_j \tilde{g}) \tilde{g}^{-1}] \tilde{g}$$

$$(\partial_t g) g^{-1} = \partial_j \left((\partial_j g) g^{-1}\right) + [B_j, (\partial_j g) g^{-1}]$$

$$\partial_t Y = \Delta Y + Y\partial Y + Y^3 + C^\varepsilon Y + (C^\varepsilon \mathrm{d}g g^{-1}, 0) + \mathrm{Ad}_g(\chi^\varepsilon * \xi)$$

$$\mathbf{E}W_\ell\left[\mathcal{F}_s\left(A_t^{(1)}\right)\right] = \mathbf{E}W_\ell[\mathcal{F}_s(\tilde{A}_t)] + O(t^M)$$

$$\left|\mathbf{E}W_\ell[\mathcal{F}_s(\tilde{A}_t)] - \mathbf{E}W_\ell\left[\mathcal{F}_s\left(A_t^{(2)}\right)\right]\right| \geq \sigma t^{1+r}$$

$$\mathcal{P}_t \star f = \int_0^t P_{t-s} f_s \,\mathrm{d}s$$

$$|f|_{\mathcal{C}^\beta} = \sup_{s\in(0,1)} s^{-\beta/2} |P_s f|_\infty$$

$$|f|_{\mathcal{C}^\beta(\mathbf{R}^3)} \stackrel{\text{def}}{=} \max_{|k|<\lfloor\beta\rfloor} |\partial^k f|_\infty + \max_{|k|=\lfloor\beta\rfloor} |\partial^k f|_{\mathcal{C}^{\beta-\lfloor\beta\rfloor}} < \infty,$$

$$|f|_{\mathcal{C}^\eta} = \sup_{x\neq y} |x-y|^{-\eta} |f(x) - f(y)|$$

$$O = [-1,2] \times \mathbf{T}^3$$

$$|\xi|_{\mathcal{C}^\beta(O)} = \sup_{z\in O} \sup_{\varphi\in\mathcal{B}^r} \sup_{\lambda\in(0,1]} \lambda^{-\beta} |\langle \xi, \varphi_z^\lambda\rangle|$$

$$\varphi_{(s,y)}^\lambda(t,x) = \lambda^{-5}\varphi((t-s)\lambda^{-2}, (x-y)\lambda^{-1}).$$

$$\begin{aligned} \partial_t X &= \Delta X + X\partial X + X^3 + \chi^\varepsilon * \xi + C^\varepsilon X + (ch, 0), \\ \partial_t h_i &= \Delta h_i - [h_j, \partial_j h_i] + \left[[A_j, h_j], h_i\right] + \partial_i [A_j, h_j], \end{aligned}$$

$$\omega \in (-1/2, 0), \kappa \stackrel{\text{def}}{=} \frac{1}{100}(\omega + 1/2) \wedge \frac{1}{100}(-\omega) \in \left(0, \frac{1}{200}\right)$$

$$|f_1 f_2|^{\gamma_n}_{\mathcal{D}_\alpha^{\gamma,\eta}} \lesssim |f_1|_{\mathcal{D}_1^{\gamma_1,\eta_1}} |f_2|_{\mathcal{D}_2^{\gamma_2,\eta_2}}$$

$$|\partial f|_{\mathcal{D}_{\alpha-1}^{\gamma-1,\eta-1}} \lesssim |f|_{\mathcal{D}_\alpha^{\gamma,\eta}}$$

$$\begin{aligned} \mathcal{X} &= PX(0) + \mathbf{\Psi} + \mathcal{P}\mathbf{1}_+\{\mathcal{X}\partial\mathcal{X} + \mathcal{X}^3 + \dot{C}\mathcal{X} + c\mathcal{H}\} \\ &\stackrel{\text{def}}{=} PX(0) + \mathbf{\Psi} + \mathcal{P}\mathbf{1}_+\{Q^{\mathrm{YMH}}(\mathcal{X}) + c\mathcal{H}\} \\ \mathcal{H} &= Ph(0) + \mathcal{P}\mathbf{1}_+(\mathcal{H}\partial\mathcal{H} + \mathcal{X}\mathcal{H}^2) + \mathcal{P}'\mathbf{1}_+(\mathcal{X}\mathcal{H}) \end{aligned}$$

$$\mathbf{\Psi} = \mathcal{P}^{\mathbf{1}_+\xi}\mathbf{1}_+\Xi \in \mathcal{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa, -\frac{1}{2}-\kappa}$$

$$\mathcal{X} = \mathcal{Y} + \mathbf{\Psi},$$

$$\begin{aligned} \mathcal{Y} &= PX(0) + \mathcal{P}\mathbf{1}_+\{\mathcal{X}^3 + c\mathcal{H} + \check{C}\mathcal{X}\} + \mathcal{P}^{\Psi\partial\Psi}(\Psi\partial\Psi) \\ &+ \mathcal{P}\mathbf{1}_+(\mathcal{Y}\partial\Psi + \mathbf{\Psi}\partial\mathcal{Y} + \mathcal{Y}\partial\mathcal{Y}) \\ &= PX(0) + \mathcal{P}\mathbf{1}_+\{\tilde{Q}^{\mathrm{YMH}}(\mathcal{Y}) + c\mathcal{H}\} + \mathcal{P}^{\Psi\partial\Psi}(\Psi\partial\Psi), \end{aligned}$$

$$\tilde{Q}^{\mathrm{YMH}}(\mathcal{Y}) = \mathcal{X}^3 + \dot{C}\mathcal{X} + \mathcal{Y}\partial\Psi + \mathbf{\Psi}\partial\mathcal{Y} + \mathcal{Y}\partial\mathcal{Y}$$

$$\tilde{Q}^{\mathrm{YMH}}:\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}\to\mathcal{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}$$

$$\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}\times\mathcal{D}_{-\frac{3}{2}-\kappa}^{\frac{1}{2}+2\kappa,-\frac{3}{2}-\kappa}\to\mathcal{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}.$$

$$(\mathcal{Y},\mathcal{H})\in\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}\times\mathcal{D}_{0}^{1+2\kappa,0}$$

$$|\mathcal{P}\mathbf{1}_{+}f|_{\mathcal{D}^{\gamma+2,\bar{\eta}}}\lesssim t^{\theta/2}|f|_{\mathcal{D}^{\gamma,\eta}},|\mathcal{P}'\mathbf{1}_{+}f|_{\mathcal{D}^{\gamma+1,\bar{\eta}-1}}\lesssim t^{\theta/2}|f|_{\mathcal{D}^{\gamma,\eta}}$$

$$|\mathcal{P}^{w}\mathbf{1}_{+}f|_{\mathcal{D}^{\gamma+2,\bar{\eta}}}\lesssim t^{\theta/2}\big(|f|_{\mathcal{D}^{\gamma,\eta}}+|w|_{\mathcal{C}^{\eta\wedge\alpha}(O)}\big)$$

$$\tau^{-1/q}\lesssim 2\ +\|Z\|_{\frac{3}{2}+2\kappa;O}+|X(0)|_{\mathcal{C}^{3}}+|h(0)|_{\mathcal{C}^{3}}$$
$$+|\mathcal{P}\star\mathbf{1}_{+}\xi|_{\mathcal{C}([-1,3],\mathcal{C}^{-1/2-\kappa})}+|\mathcal{P}\star(\Psi\partial\Psi)|_{\mathcal{C}([-1,3],\mathcal{C}^{-2\kappa})}$$

$$|\mathcal{P}'\mathbf{1}_{+}(\mathcal{X}\mathcal{H})|_{\mathcal{D}_{0}^{1+2\kappa,-\kappa}}\lesssim t^{1/4}|\mathcal{X}\mathcal{H}|_{\mathcal{D}_{-\frac{1}{2}-\kappa}^{\frac{1}{2}+\kappa,-\frac{1}{2}-\kappa}}$$
$$\lesssim t^{1/4}|\mathcal{X}|_{\mathcal{D}^{2}_{-\frac{1}{2}-\kappa}}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}|\mathcal{H}|_{\mathcal{D}_{0}^{1+2\kappa,0}}\lesssim t^{1/4}$$

$$|\mathcal{P}\mathbf{1}_{+}(\mathcal{H}\partial\mathcal{H})|_{\mathcal{D}_{0}^{1+2\kappa,0}}\lesssim t,|\mathcal{P}\mathbf{1}_{+}(\mathcal{X}\mathcal{H}^{2})|_{\mathcal{D}_{0}^{\frac{3}{2}+\kappa,-\kappa}}\lesssim t^{3/4}$$

$$\left|\mathcal{P}\mathbf{1}_{+}\big(\tilde{Q}^{\mathrm{YMH}}(\mathcal{Y})\big)\right|_{\mathcal{D}_{0}^{\frac{3}{2}+2\kappa,\omega}}\lesssim t^{1/4-\kappa/2}\left|\tilde{Q}^{\mathrm{YMH}}(\mathcal{Y})\right|_{\mathcal{D}_{-\frac{3}{2}-\kappa}^{\kappa,\omega-\frac{3}{2}-2\kappa}}\lesssim t^{1/4-\kappa/2}$$

$$\left|\mathcal{P}^{\Psi\partial\Psi}(\mathbf{\Psi}\partial\mathbf{\Psi})\right|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\lesssim t^{-(\kappa+\omega)/2}$$

$$|\mathcal{P}\mathcal{H}|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}}+2\kappa,\omega}^{\frac{3}{2}}\lesssim|\mathcal{P}Ph(0)|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}+\left|\mathcal{P}O_{\mathcal{D}_{0}^{1+2\kappa,-\kappa}}\big(t^{1/4}\big)\right|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}$$
$$\lesssim t^{1-\frac{\omega}{2}}|\mathrm{Ph}(0)|_{\mathcal{D}_{0}^{0+,0}}+t^{1/4+1-\frac{\kappa}{2}-\frac{\omega}{2}}\lesssim t^{1-\omega/2}$$

$$B_{0}=PX(0)+\mathbf{\Psi},h_{0}=Ph(0)$$

$$B_{n}=\mathcal{P}\big(\dot{C}B_{n-4}\mathbf{1}_{n\geq4}\big)+\sum_{k_{1}+k_{2}=n-1}\mathcal{P}\big(B_{k_{1}}\partial B_{k_{2}}\big)+\sum_{k_{1}+k_{2}+k_{3}=n-2}\mathcal{P}\big(B_{k_{1}}B_{k_{2}}B_{k_{3}}\big)$$

$$\mathcal{X}=\sum_{i=0}^{n}B_{i}+c\mathbf{1}_{n=5}\mathcal{P}h_{0}+r_{n},\mathcal{H}=h_{0}+q_{0}$$

$$r_{0}=O_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\big(t^{-\omega/2-\kappa/2}\big),q_{0}=O_{\mathcal{D}_{0}^{1+2\kappa,-\kappa}}\big(t^{1/4}\big)$$

$$\eta(0)=-1/2-\kappa,\qquad\eta(n)=-1/2+2\kappa\ (1\leq n\leq5)$$
$$b(0)=0,\qquad b(n)=(1/4-\kappa/2)n-3\kappa/2\ (1\leq n\leq5)$$

$$|B_n|_{\mathcal{D}^{\frac{3}{2}+2\kappa,\eta(n)}_{\alpha(n)}} \lesssim t^{b(n)} \ \forall 0 \le n \le 5$$

$$r_0 = O_{\mathcal{D}^{\frac{3}{2}+2\kappa,\omega}_{-\kappa}}\left(t^{-\omega/2-\kappa/2}\right), r_n = O_{\mathcal{D}^{\frac{3}{2}+2\kappa,\omega}_{0}}\left(t^{(n+1)/4-\kappa_n}\right) \ \forall 1 \le n \le 5,$$

$$\frac{1}{2}\eta(n) + b(n) = -\frac{1}{4} - \frac{\kappa}{2} + \left(\frac{1}{4} - \frac{\kappa}{2}\right) n \ \forall n \ge 0$$

$$\begin{aligned}
&\left|\mathcal{P}\left(B_{k_1}\partial B_{k_2}\right)\right|^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}_{\mathcal{D}^{2}_{\alpha(n)}} \\
&\lesssim t^{\frac{1}{2}\left(\eta(k_1)+\eta(k_2)+\frac{3}{2}-2\kappa\right)} \left|B_{k_1}\partial B_{k_2}\right|_{\mathcal{D}^{\kappa,\eta(k_1)+\eta(k_2)-1}_{\alpha(k_1)+\alpha(k_2)-1}} \\
&\lesssim t^{\frac{1}{2}\left(\eta(k_1)+\eta(k_2)+\frac{3}{2}-2\kappa\right)} \left|B_{k_1}\right|_{\mathcal{D}^{\frac{3}{2}+2\kappa,\eta(k_1)}_{\alpha(k_1)}}\Big| \left|\partial B_{k_2}\right|_{\mathcal{D}^{\frac{1}{2}+2\kappa(k_2)-1}_{\alpha(k_2)-1}} \\
&\lesssim t^{\frac{1}{2}\left(\eta(k_1)+\eta(k_2)+\frac{3}{2}-2\kappa\right)} \cdot t^{b(k_1)} \cdot t^{b(k_2)} = t^{b(n)}.
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{P}\left(B_{k_1}B_{k_2}B_{k_3}\right)\right|_{\mathcal{D}^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}_{\alpha(n)}} &\lesssim t^{\frac{1}{2}\left(\sum_i \eta(k_i)+\frac{5}{2}-2\kappa\right)} \prod_{i=1}^{3} \left|B_{k_i}\right|_{\mathcal{D}^{\frac{3}{2}+2\kappa,\eta(k_i)}_{\alpha(k_i)}} \\
&\lesssim t^{\frac{1}{2}\left(\sum_i \eta(k_i)+\frac{5}{2}-2\kappa\right)} \cdot t^{\sum_i b(k_i)} = t^{b(n)}.
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{P}(B_{n-4})\right|_{\mathcal{D}^{\frac{3}{2}+2\kappa,-\frac{1}{2}+2\kappa}_{\alpha(n)}} &\lesssim t^{\frac{1}{2}\left(\eta(n-4)+\frac{5}{2}-2\kappa\right)} \left|B_{n-4}\right|_{\mathcal{D}^{\frac{3}{2}+2\kappa,\eta(n-4)}_{\alpha(n-4)}} \\
&\lesssim t^{\left(\frac{1}{4}-\frac{\kappa}{2}\right)n+\frac{\kappa}{2}} \le t^{b(n)}
\end{aligned}$$

$$\sum_{i=0}^{n} B_i + c\mathbf{1}_{n=5}\mathcal{P}h_0 + r_n = B_0 + \mathcal{P}\mathbf{1}_+\{Q^{\mathrm{YMH}}(\mathcal{X}) + c\mathcal{H}\}$$

$$r_n = \mathcal{P}\mathbf{1}_+\left\{Q^{\mathrm{YMH}}\left(\sum_{i=0}^{n-1} B_i + r_{n-1}\right) + c(h_0 + q_0 - \mathbf{1}_{n=5}h_0)\right\} - \sum_{i=1}^{n} B_i$$

$$\begin{aligned}
r_n = &\mathcal{P}\mathbf{1}_+\Bigg(\sum_{k_1+k_2\ge n} B_{k_1}\partial B_{k_2} + \sum_{k_1+k_2+k_3\ge n-1} B_{k_1}B_{k_2}B_{k_3} \\
&+ r_{n-1}\partial r_{n-1} + \left(\sum_{i=0}^{n-1} B_i\right)\partial r_{n-1} + r_{n-1}\partial\left(\sum_{i=0}^{n-1} B_i\right) \\
&+ r_{n-1}^3 + 3r_{n-1}^2\left(\sum_{i=0}^{n-1} B_i\right) + 3r_{n-1}\left(\sum_{i=0}^{n-1} B_i\right)^2 + c(h_0 + q_0 - \mathbf{1}_{n=5}h_0) \\
&+ \mathring{C}\left(\sum_{i=(n-3)\vee 0}^{n-1} B_i + r_{n-1}\right)\Bigg),
\end{aligned}$$

$$\left|\mathcal{P}\left(B_{k_1}\partial B_{k_2}\right)\right|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(\eta(k_1)+\eta(k_2)+1-\omega)} \cdot t^{b(k_1)} \cdot t^{b(k_2)}$$

$$\leq t^{-\omega/2-\kappa+(1/4-\kappa/2)n} = t^{(n+1)/4-\kappa_n}$$

$$\left|\mathcal{P}\left(B_{k_1}B_{k_2}B_{k_3}\right)\right|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(\sum_i \eta(k_i)+2-\omega)} \cdot t^{\sum_i b(k_i)}$$

$$\leq t^{\frac{1}{4}-\frac{\omega}{2}-\frac{3}{2}\kappa+(1/4-\kappa/2)(n-1)} = t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}B_i|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(\eta(i)+2-\omega)}|B_i|_{\mathcal{D}_{\alpha(i)}^{\frac{3}{2}+2\kappa,\eta(i)}}$$

$$\lesssim t^{\frac{1}{2}(\eta(i)+2-\omega)} \cdot t^{b(i)} = t^{\frac{3}{4}-\frac{\omega}{2}-\frac{\kappa}{2}+\left(\frac{1}{4}-\frac{\kappa}{2}\right)i} \leq t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}(B_0\partial r_{n-1})|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{1/4-\kappa/2}|B_0\partial r_{n-1}|_{\mathcal{D}_{-\frac{3}{2}-2\kappa}^{\kappa,\omega-\frac{3}{2}-\kappa}}$$

$$\lesssim t^{1/4-\kappa/2}|B_0|_{\mathcal{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}|\partial r_{n-1}|_{\mathcal{D}_{-1-\kappa}^{\frac{1}{2}+2\kappa,\omega-1}}$$

$$\lesssim t^{1/4-\kappa/2}t^{n/4-\kappa_{n-1}} = t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}(r_{n-1}\partial B_0)|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{(n+1)/4-\kappa_n}$$

$$\left|\mathcal{P}(B_0^2 r_{n-1})\right|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{1/2-\kappa}\left|B_0^2 r_{n-1}\right|_{\mathcal{D}_{-1-3\kappa}^{\frac{1}{2},\omega-1-2\kappa}}$$

$$\lesssim t^{1/2-\kappa}|B_0|^2_{\mathcal{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}|r_{n-1}|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}$$

$$\lesssim t^{1/2-\kappa}t^{n/4-\kappa_{n-1}} = t^{\frac{1}{4}-\frac{\kappa}{2}}t^{(n+1)/4-\kappa_n} \leq t^{(n+1)/4-\kappa_n}$$

$$\left|\mathcal{P}(B_0 r_{n-1}^2)\right|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}\left|B_0 r_{n-1}^2\right|_{\mathcal{D}}^{1,2\omega-\frac{1}{2}-3\kappa}$$

$$\lesssim t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}|B_0|_{\mathcal{D}_{-\frac{1}{2}-\kappa}^{\frac{3}{2}+2\kappa,-\frac{1}{2}-\kappa}}|r_{n-1}|^2_{\mathcal{D}_{-\kappa}^{\frac{3}{2}}+2\kappa,\omega}$$

$$\lesssim t^{\frac{\omega}{2}+\frac{3}{4}-\frac{\kappa}{2}}\left(t^{n/4-\kappa_{n-1}}\right)^2 \leq t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}(r_{n-1}\partial r_{n-1})|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{(1+\omega)/2}|r_{n-1}\partial r_{n-1}|_{\mathcal{D}_{-1-2\kappa}^{\frac{1}{2}+\kappa,2\omega-1}}$$

$$\lesssim t^{(1+\omega)/2}|r_{n-1}|_{\mathcal{D}_{-\kappa}^{\frac{3}{2}+2\kappa,\omega}}\Big| \, |\partial r_{n-1}|^{\frac{1}{2}+2}_{\mathcal{D}_{-1-\kappa}^{1+2\kappa-1}}$$

$$\lesssim t^{(1+\omega)/2}\left(t^{n/4-\kappa_{n-1}}\right)^2 \leq t^{(n+1)/4-\kappa_n}$$

$$\left|\mathcal{P}(r_{n-1}^3)\right|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{1+\omega}\left|r_{n-1}^3\right|_{\mathcal{D}_{-3\kappa}^{\frac{3}{2},3\omega}} \lesssim t^{1+\omega}|r_{n-1}|^3_{\mathcal{D}_{-\kappa}^{3}} + 2\kappa,\omega$$

$$\lesssim t^{1+\omega}\left(t^{n/4-\kappa_{n-1}}\right)^3 \leq t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}q_0|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(2-\kappa-\omega)}|q_0|_{\mathcal{D}_0^{\frac{1}{2}+2\kappa,-\kappa}} \lesssim t^{\frac{1}{2}(2-\kappa-\omega)} \cdot t^{\frac{1}{4}} \leq t^{(n+1)/4-\kappa_n}$$

$$|\mathcal{P}h_0|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}(2-\omega)}|h_0|_{\mathcal{D}_0^{\frac{1}{2}+2\kappa,0}} \lesssim t^{(n+1)/4-\kappa_n}$$

$$\Psi = \mathcal{R}\Psi = \mathcal{P} \star \mathbf{1}_+\xi$$

$$(\mathcal{R}\mathcal{X})(t) = X(0) + \Psi_t + \mathcal{P}_t \star (\Psi\partial\Psi) + O_{L^\infty}\left(t^{\frac{1}{4}-3\kappa/2}\right).$$

$$\mathcal{R}B_1 = \mathcal{P} \star (PX(0)P'X(0) + PX(0)\partial\Psi + P'X(0)\Psi + \Psi\partial\Psi)$$

$$|\Psi|_{L^\infty} \lesssim t^{-\frac{1}{4}-\kappa/2}, |\partial\Psi|_{L^\infty} \lesssim t^{-\frac{3}{4}-\kappa/2}$$

$$|r_1|_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}} \lesssim t^{\frac{1}{2}-\kappa_1}, |(\mathcal{R}r_1)(t)|_{L^\infty} \lesssim t^{\frac{1}{2}-\kappa_1+\omega/2} = t^{\frac{1}{4}-3\kappa/2}$$

$$\mathcal{N}(A)\colon (0,\infty) \to \mathcal{C}^\infty(\mathbf{T}^3, \mathfrak{g}^3 \otimes (\mathfrak{g}^3)^3), \mathcal{N}_s(A) \stackrel{\text{def}}{=} P_sA \otimes \nabla P_sA$$

$$\mathcal{L} = \mathbf{T}^3 \times \{v \in \mathbf{R}^3 : |v| \leq 1/4\}$$

$$|f|_{\gamma-\text{gr}} = \sup_{\ell\in\mathcal{L}} \frac{|\int_\ell f|}{|\ell|^\gamma},$$

$$\int_\ell f = \int_0^1 |v| f(y+tv)\text{d}t \in F$$

$$[\,[A;B]\,]_{\gamma,\delta} \stackrel{\text{def}}{=} \sup_{s\in(0,1)} s^\delta|\mathcal{N}_sA - \mathcal{N}_sB|_{\gamma-\text{gr}}, [\,[A]\,]_{\gamma,\delta} = [\,[A;0]\,]_{\gamma,\delta}$$

$$[\,[A;B]\,]_{\beta,\delta} \stackrel{\text{def}}{=} \sup_{s\in(0,1)} s^\delta|\mathcal{N}_sA - \mathcal{N}_sB|_{\mathcal{C}^\beta}$$

$$\|A\|_{\alpha,\theta} \stackrel{\text{def}}{=} \sup_{s\in(0,1)} |P_sA|_{\alpha-\text{gr};<s^\theta}$$

$$|A|_{\alpha-\text{gr};<r} \stackrel{\text{def}}{=} \sup_{\ell\in\mathcal{L},|\ell|<r} \frac{|A(\ell)|}{|\ell|^\alpha}$$

$$|A|_{\mathcal{C}^\eta} \lesssim \|A\|_{\alpha,\theta}$$

$$\Sigma(A) \stackrel{\text{def}}{=} \|A\|_{\alpha,\theta} + [\,[A]\,]_{\gamma,\delta} < \infty$$

$$\Sigma(A,B) \stackrel{\text{def}}{=} \|A-B\|_{\alpha,\theta} + [\,[A;B]\,]_{\gamma,\delta} < \infty$$

$$\alpha \in (0,1/2), \theta > 0, \gamma \in (1/2,1], \delta \in (0,1)$$
$$\mu \stackrel{\text{def}}{=} \gamma - 1 + 2(1-\delta) \in (-1/2,0), \text{ and } \eta + \mu > -1$$

$$|\mathcal{F}_sA|_\infty \lesssim s^{\frac{\eta}{2}}\Sigma(A), |\partial\mathcal{F}_sA|_\infty \lesssim s^{\frac{\eta}{2}-\frac{1}{2}}\Sigma(A)$$

$$|P_sA|_\infty \lesssim s^{\frac{\eta}{2}}|A|_{\mathcal{C}^\eta}, |\partial P_sA|_\infty \lesssim s^{\frac{\eta}{2}-\frac{1}{2}}|A|_{\mathcal{C}^\eta}$$

$$\lambda \stackrel{\text{def}}{=} (1-\zeta)\eta/2 - \theta(1-\alpha)\zeta < 0$$

$$|P_s A|_{\gamma-\text{gr}} \lesssim s^{\lambda}\|A\|_{\alpha,\theta}$$

$$0 < \nu \leq \min\left\{\frac{\eta}{2}+\frac{\mu}{2}+\frac{1}{2}, 1+3\eta/2, \mu+\frac{1}{2}\right\}$$

$$|\mathcal{P}_s \star \mathcal{N}A|_{\gamma-\text{gr}} \lesssim \int_0^s |\mathcal{N}_r A|_{\gamma-\text{gr}}\,\mathrm{d}r \lesssim \int_0^s r^{-\delta}[\,[A]\,]_{\gamma,\delta}\mathrm{d}r \lesssim s^{1-\delta}[\,[A]\,]_{\gamma,\delta}$$

$$\mathcal{F}_s A = P_s A + \mathcal{P}_s \star \mathcal{N}A + R_s A$$

$$|R_s A|_{\infty} \lesssim s^{\nu}(\Sigma(A)+\Sigma(A)^3)$$

$$R_s = \int_0^s P_{s-r}\{(P_r A + \mathcal{P}_r \star \mathcal{N}A + R_r)\partial(P_r A + \mathcal{P}_r \star \mathcal{N}A + R_r)$$
$$+(\mathcal{F}_r A)^3\}\mathrm{d}r - \mathcal{P}_s \star \mathcal{N}A$$

$$|\mathcal{P}_s \star \mathcal{N}A|_{\infty} \lesssim \int_0^s |P_{s-r}(\mathcal{N}_r A)|_{\infty}\mathrm{d}r \lesssim \int_0^s (s-r)^{\frac{\gamma-1}{2}}|\mathcal{N}_r A|_{\mathcal{C}^{\gamma-1}}\,\mathrm{d}s$$
$$\lesssim \int_0^s (s-r)^{\frac{\gamma-1}{2}}|\mathcal{N}_r A|_{\gamma-\text{gr}}\,\mathrm{d}s$$
$$\lesssim \int_0^s (s-r)^{\frac{\gamma-1}{2}} r^{-\delta}[\,[A]\,]_{\gamma,\delta}\mathrm{d}s \asymp s^{\mu/2}[\,[A]\,]_{\gamma,\delta},$$

$$|\partial\mathcal{P}_s \star \mathcal{N}A|_{\infty} \lesssim s^{\mu/2-\frac{1}{2}}[\,[A]\,]_{\gamma,\delta}$$

$$\int_0^s \left\{r^{\frac{\eta}{2}+\frac{\mu}{2}-\frac{1}{2}} + r^{\mu-\frac{1}{2}} + r^{3\eta/2}\right\}\mathrm{d}r \asymp s^{\frac{\eta}{2}+\frac{\mu}{2}+\frac{1}{2}} + s^{\mu+\frac{1}{2}} + s^{1+3\eta/2}$$

$$|R|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{s\in(0,T)}\left\{s^{-\nu}|R_s|_{\infty} + s^{-\nu+\frac{1}{2}}|\partial R_s|_{\infty}\right\}$$

$$s^{-\nu}|R|_{\infty} \lesssim \Sigma(A)+\Sigma(A)^3 + s^{\kappa'}|R|_{\mathcal{B}}(\Sigma(A)+\Sigma(A)^2)$$
$$+s^{\kappa''}|R|_{\mathcal{B}}^2(1+\Sigma(A)) + s^{1+2\nu}|R|_{\mathcal{B}}^3$$

$$\kappa' = \min\left\{\frac{\eta}{2}+\frac{1}{2}, 1+\eta\right\} = \frac{\eta}{2}+\frac{1}{2}$$
$$\kappa'' = \min\left\{\frac{1}{2}+\nu, 1+\frac{\eta}{2}+\nu\right\} = \frac{1}{2}+\nu$$

$$|R|_{\mathcal{B}} \lesssim \Sigma(A)+\Sigma(A)^3 + T^{\kappa}\left(|R|_{\mathcal{B}} + |R|_{\mathcal{B}}^3\right)(1+\Sigma(A)^2)$$

$$\mathcal{F}_s A = P_s A + O_{\Omega_{\gamma-\text{gr}[\mathfrak{g},\mathfrak{g}]}\left(s^{\nu}(\Sigma(A)+\Sigma(A)^3)\right), P_s A = O_{\Omega_{\gamma-\text{gr}}}(s^{\lambda}) \ldots\ldots}$$

$$\mathcal{F}_s\tilde{A} = \mathcal{F}_s A + P_s r + O_{L^{\infty}[\mathfrak{g},\mathfrak{g}]}\left(s^{\eta/2+\frac{1}{2}}|r|_{\infty}\right)$$

$$\mathcal{F}_s A + Q_s = P_s(A+r) + \int_0^s P_{s-u}\{(\mathcal{F}_u A + Q_u)\partial(\mathcal{F}_u A + Q_u) + (\mathcal{F}_u A + Q_u)^3\}\mathrm{d}u$$

$$Q_s = P_s r + \int_0^s P_{s-u}(Q_u \partial \mathcal{F}_u A + (\mathcal{F}_u A)\partial Q_u + Q_u \partial Q_u$$
$$+(\mathcal{F}_u A)^2 Q_u + (\mathcal{F}_u A) Q_u^2 + Q_u^3)\mathrm{d}u$$

$$\int_0^s \Big\{ |Q_u|_\infty u^{\eta/2-\frac{1}{2}} + u^{\eta/2}|Q_u|_{\mathcal{C}^1} + |Q_u|_\infty |Q_u|_{\mathcal{C}^1}$$
$$+u^\eta |Q_u|_\infty + u^{\eta/2}|Q_u|_\infty^2 + |Q_u|_\infty^3 \}\mathrm{d}u$$
$$\lesssim |Q|_\infty s^{\eta/2+\frac{1}{2}} + s^{\eta/2+\frac{1}{2}}|Q|_{L^\infty_{1/2}\mathcal{C}^1} + s^{1/2}|Q|_\infty |Q|_{L^\infty_{1/2}\mathcal{C}^1}$$
$$+s^{\eta+1}|Q|_\infty + s^{\eta/2+1}|Q|_\infty^2 + s|Q|_\infty^3,$$

$$|Q|_\infty + |Q|_{L^\infty_{1/2}\mathcal{C}^1} \lesssim |r|_\infty$$

$$Q_s = P_s r + O_{L^\infty}\left(s^{\eta/2+\frac{1}{2}}|r|_\infty\right)$$

$$\tilde{\mathcal{X}} = B + c\bar{h} + O_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}}\left(t^{3/2-\kappa_5}\right)$$

$$\tilde{\mathcal{X}} = B + c\mathcal{P}\mathrm{Ph}(0) + O_{\mathcal{D}_0^{\frac{3}{2}+2\kappa,\omega}}\left(t^{3/2-\kappa_5}\right),$$

$$\tilde{A} = A + ctP_t h(0) + O_{L^\infty}\left(t^{5/4-7\kappa/2}\right).$$

$$\mathcal{F}_s \tilde{A} = \mathcal{F}_s A + P_s\left(ctP_t h(0) + O_{L^\infty}\left(t^{5/4-7\kappa/2}\right)\right) + O_{L^\infty[\mathfrak{g},\mathfrak{g}]}\left(s^{\eta/2+\frac{1}{2}}t\right)$$

$$|f|_{p\text{-var}} \overset{\text{def}}{=} \sup_{P\subset[0,1]}\left(\sum_{[s,t]\in P}|f(t)-f(s)|^p\right)^{1/p}$$

$$\mathrm{d}J^\gamma(x) = J^\gamma(x)\mathrm{d}\gamma(x), J^\gamma(0) = \mathrm{id}$$

$$J^{\gamma+\zeta}(1) = J^\gamma(1) + \int_0^1 \mathrm{d}\boldsymbol{\zeta}(x) + \int_0^1\int_0^x \{\mathrm{d}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\gamma}(y) + \mathrm{d}\boldsymbol{\gamma}(x)\mathrm{d}\boldsymbol{\zeta}(y)\}$$
$$+O\{v(w^2+w^{L-1}) + w^L + w^{L+1} + v^{L+1} + v^2(1+w+v+w^{L-3})\}$$

$$J^\gamma(1) = \mathrm{id} + I^\gamma + \int_0^1 \cdots \int_0^{x_{L-1}} J^\gamma(x_L)\mathrm{d}\gamma(x_L)\mathrm{d}\gamma(x_{L-1})\ldots\mathrm{d}\gamma(x_1)$$

$$I^{\gamma+\zeta} = I^\gamma + \int_0^1 \mathrm{d}\boldsymbol{\zeta}(x) + \int_0^1\int_0^x \{\mathrm{d}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\gamma}(y) + \mathrm{d}\boldsymbol{\gamma}(x)\mathrm{d}\boldsymbol{\zeta}(y)\}$$
$$+O\{v^2 + v(w^2+w^{L-1})\} + O\{v^2(w+v+w^{L-3}+v^{L-3})\}$$

$$|\ell_a|_{\frac{1}{\gamma}\text{-var}} \le |\ell_a|_{\gamma-\text{Höl}} \le |a|_{\gamma-\text{gr}}$$

$$W_\ell(\mathcal{F}_s\tilde{A}) = W_\ell(\mathcal{F}_sA) + t\mathrm{Tr}\int_\ell \mathrm{ch}(0) + t\mathrm{Tr}\int_{[0,1]^2} \mathrm{d}\ell_{A(0)}(x_1)\mathrm{d}\ell_{\mathrm{ch}(0)}(x_2)$$
$$+O\left(t^{5/4-7\kappa/2} + ts + ts^\lambda\|\Psi_t^{\mathrm{YM}}\|_{\alpha,\theta} + ts^\lambda\|\mathcal{P}_t \star (\Psi\partial\Psi)^{\mathrm{YM}}\|_{\alpha,\theta} + |A(0)|_{L^\infty}s^{\eta/2+\frac{1}{2}}t\right.$$
$$\left.+s^\nu t(\Sigma(A) + \Sigma(A)^3) + t(u^2 + u^{L-1}) + s^{\nu L} + u^L + u^{L+1} + t^2u\right)$$

$$\gamma(x) = \int_0^x (\mathcal{F}_sA)_1(y,0,0)\mathrm{d}y = \ell_{\mathcal{F}_sA}(x)$$

$$W_\ell(\mathcal{F}_sA) = \mathrm{Tr}J^\gamma(1), W_\ell(\mathcal{F}_s\tilde{A}) = \mathrm{Tr}J^{\gamma+\zeta}(1)$$

$$D_h \stackrel{\mathrm{def}}{=} ctP_{t+s}h(0) = cth(0) + O_{L^\infty}(t(t+s)),$$
$$D_{\mathrm{err}} \stackrel{\mathrm{def}}{=} O_{L^\infty[\mathfrak{g},\mathfrak{g}]}\left(s^{\eta/2+\frac{1}{2}}t\right) + O_{L^\infty}\left(t^{5/4-7\kappa/2}\right).$$

$$W_\ell(\mathcal{F}_s\tilde{A}) = W_\ell(\mathcal{F}_sA) + \mathrm{Tr}\left(\int_0^1 \mathrm{d}\zeta(x)\right)$$
$$+\mathrm{Tr}\int_0^1\int_0^x \{\mathrm{d}\zeta(x)\mathrm{d}\gamma(y) + \mathrm{d}\gamma(x)\mathrm{d}\zeta(y)\}$$
$$+O\{v(w^2 + w^{L-1}) + w^L + w^{L+1} + v^{L+1} + v^2(1 + w + v + w^{L-3})\}$$

$$\mathrm{Tr}\left(\int_0^1 \mathrm{d}\zeta(x)\right) = t\mathrm{Tr}\int_\ell \mathrm{ch}(0) + O\left(t^{5/4-7\kappa/2} + ts\right)$$

$$\mathrm{Tr}\int_0^1\int_0^x \{\mathrm{d}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\gamma}(y) + \mathrm{d}\boldsymbol{\gamma}(x)\mathrm{d}\boldsymbol{\zeta}(y)\} = \mathrm{Tr}\int_{[0,1]^2}\{\mathrm{d}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\gamma}(y)\}$$

$$\mathcal{F}_sA = P_sA + O_{\gamma-\mathrm{gr}}\left(s^\nu(\Sigma(A) + \Sigma(A)^3)\right)$$

$$A = A(0) + \Psi_t^{\mathrm{YM}} + \mathcal{P}_t \star (\Psi\partial\Psi)^{\mathrm{YM}} + O_{L^\infty}\left(t^{1/4-3\kappa/2}\right)$$

$$\int_0^1 \mathrm{d}\gamma(x) = \int_0^1 \mathrm{d}\ell_{A(0)}(x) + O\left(s^\lambda\|\Psi_t^{\mathrm{YM}}\|_{\alpha,\theta} + s^\lambda\|\mathcal{P}_t \star (\Psi\partial\Psi)^{\mathrm{YM}}\|_{\alpha,\theta}\right)$$

$$+O\left(t^{1/4-3\kappa/2} + s^\nu(\Sigma(A) + \Sigma(A)^3)\right)$$

$$\int_0^1 \mathrm{d}\boldsymbol{\zeta}(y) = t\int_0^1 \mathrm{d}\ell_{\mathrm{ch}(0)}(y) + O\left(t^{5/4-7\kappa/2} + s^{\eta/2+\frac{1}{2}}t\right)$$

$$\mathrm{Tr}\int_{[0,1]^2}\{\mathrm{d}\boldsymbol{\zeta}(x)\mathrm{d}\gamma(y)\} = t\mathrm{Tr}\left(\int_{[0,1]^2} \mathrm{d}\ell_{A(0)}(x_1)\mathrm{d}\ell_{ch(0)}(x_2)\right)$$
$$+O\left(ts^\lambda\|\Psi_t^{\mathrm{YM}}\|_{\alpha,\theta} + ts^\lambda\|\mathcal{P}_t \star (\Psi\partial\Psi)^{\mathrm{YM}}\|_{\alpha,\theta} + t^{5/4-7\kappa/2}\right)$$
$$+O\left(ts^\nu(\Sigma(A) + \Sigma(A)^3) + s^{\eta/2+\frac{1}{2}}t|A(0)|_{L^\infty}\right)$$

$$w = |\gamma|_{\frac{1}{\gamma}-\mathrm{var}} \leq |\mathcal{F}_s A|_{\gamma-\mathrm{gr}} = |\mathcal{P}_s A|_{\gamma-\mathrm{gr}} + O(s^\nu) = O(s^\lambda)\|A\|_{\alpha,\theta} + O(s^\nu)$$

$$v = |\zeta|_{\frac{1}{\gamma}-\mathrm{var}} \leq |D_h + D_{\mathrm{err}}|_{\gamma-\mathrm{gr}} \lesssim s^{\eta/2+\frac{1}{2}}t + t = O(t)$$

$$[\,[A+B]\,]_{\gamma,\delta} \lesssim [\,[A]\,]_{\gamma,\delta} + |B|_{\mathcal{C}_{\bar{\eta}}}\left(|A|_{\mathcal{C}^\eta} + |B|_{\mathcal{C}^{\bar{\eta}}}\right)$$

$$\mathcal{N}_s(A+B) = \mathcal{N}_s A + \mathcal{N}_s B + P_s A \otimes \nabla P_s B + P_s B \otimes \nabla P_s A$$

$$Z_{s,t} = s^\delta \mathcal{N}_s \Psi_t$$

$$\mathbf{E}\left[\left|\sup_{(s,t)\neq(\bar{s},\bar{t})} \frac{\left|Z_{s,t} - Z_{\bar{s},\bar{t}}\right|_{\gamma-\mathrm{gr}}}{(|t-\bar{t}| + |s-\bar{s}|)^{\bar{\kappa}}}\right|^p\right]^{1/p} < \infty$$

$$\left(\mathbf{E} \mid Z_{s,t} - Z_{\bar{s},\bar{t}}\tau^p_{\gamma-\mathrm{gr}}\right)^{1/p} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa/2}$$

$$\mathbf{E}\left|\sup_{t\in[0,1]} t^{-\bar{\kappa}}[\,[\Psi_t]\,]_{\gamma,\delta}\right|^p < \infty$$

$$\left|C_{s,\bar{s};,t,\bar{t}}(x)\right| \lesssim (|s-\bar{s}| + |t-\bar{t}|)^\kappa |x|^{4\delta-4-2\kappa}$$

$$\left|\nabla C_{s,\bar{s};,t\bar{t}}\right| \lesssim (|s-\bar{s}| + |t-\bar{t}|)^\kappa |x-y|^{4\delta-5-2\kappa}$$

$$\left|\nabla C_{r,s}(x)\right| \lesssim |x|^{-2}, \left|\nabla\left(C_{r,r} - C_{r,s}\right)(x)\right| \lesssim |r-s|^\kappa |x|^{-2-2\kappa}$$

$$d((x,v),(\bar{x},\bar{v})) \overset{\mathrm{def}}{=} |x-\bar{x}| \vee |x+v-(\bar{x}+\bar{v})|$$

$$\left(\mathbf{E}\left|\int_\ell \left(Z_{s,t} - Z_{\bar{s},\bar{t}}\right|^p\right)^{1/p} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^\kappa \left|\ell\right|^{2\delta-1-\kappa}$$

$$\left(\mathbf{E}\left|\left(\int_\ell - \int_{\bar{\ell}}\right)\left(Z_{s,t} - Z_{\bar{s},\bar{t}}\right)\right|^p\right)^{1/p} \lesssim (|t-\bar{t}| + |s-\bar{s}|)^{\kappa/2} d(\ell,\bar{\ell})^{2\delta-3/2-\kappa}$$

$$\begin{aligned}
\mathbf{E}\left|\int_\ell \left(Z_{s,t} - Z_{\bar{s},\bar{t}}\right)\right|^2 &= |\ell|^2 \int_{[0,1]^2} C((r-\bar{r})v)\mathrm{d}r\,\mathrm{d}\bar{r} \\
&\lesssim (|t-\bar{t}| + |s-\bar{s}|)^\kappa |\ell|^{4\delta-2-2\kappa} \int_{[0,1]^2} |r-\bar{r}|^{4\delta-4-2\kappa}\,\mathrm{d}r\,\mathrm{d}\bar{r} \\
&\lesssim (|t-\bar{t}| + |s-\bar{s}|)^\kappa |\ell|^{4\delta-2-2\kappa}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} \mid \left(\int_\ell - \int_{\bar{\ell}}\right)\left(Z_{s,t} - Z_{\bar{s},\bar{t}}\right|^2 &= |\ell|^2 \int_{[0,1]^2} \{C((r-\bar{r})v) - C(rv - \bar{r}\bar{v}) \\
&\quad - C(r\bar{v} - \bar{r}v) + C((r-\bar{r})\bar{v})\}\mathrm{d}r\,\mathrm{d}\bar{r} \\
&= |\ell|^2 \int_{[0,1]^2} \{2C((r-\bar{r})v) - 2C(rv - \bar{r}\bar{v})\}\mathrm{d}r\,\mathrm{d}\bar{r},
\end{aligned}$$

$$(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}\int_0^h r^{4\delta-4-2\kappa}\,\mathrm{d}r$$
$$\asymp (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-2-2\kappa}h^{4\delta-3-2\kappa}$$
$$= (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$(|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-3-2\kappa}|v-\bar{v}|\int_h^1 r^{4\delta-5-2\kappa}\,\mathrm{d}r$$
$$\asymp (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell|^{4\delta-3-2\kappa}|v-\bar{v}|h^{4\delta-4-2\kappa}$$
$$= (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$\mathbf{E}\,|\left(\int_\ell-\int_{\bar{\ell}}\right)\left(Z_{s,t}-Z_{\bar{s},\bar{t}}\right|^2 \lesssim (|t-\bar{t}|+|s-\bar{s}|)^{\kappa}|\ell||v-\bar{v}|^{4\delta-3-2\kappa}$$

$$\left(\mathbf{E}\left|\int_\ell A\right|^p\right)^{1/p} \le M_p|\ell|^{\alpha}$$

$$\left(\mathbf{E}\left|\left(\int_\ell-\int_{\bar{\ell}}\right)A\right|^p\right)^{1/p} \le M_p d(\ell,\bar{\ell})^{\beta}$$

$$\left(\mathbf{E}|A|^p_{\gamma-\mathrm{gr}}\right)^{1/p} \le \lambda M_p$$

$$\sup_{\ell\in D}\frac{\left|\int_\ell A\right|}{|\ell|^{\gamma}} \lesssim \sup_{N\ge 1}\sup_{\substack{a\in D_N\\ |a|\le K2^{-N/\omega}}}\frac{\left|\int_a A\right|}{2^{-\gamma N/\omega}}+\sup_{N\ge 1}\sup_{\substack{a,b\in D_N\\ d(a,b)\le K2^{-N}}}\frac{\left|\left(\int_a-\int_b\right)A\right|}{2^{-\gamma N/\omega}}.$$

$$\mathbf{E}\left|\sup_{\ell\in D}\frac{\left|\int_\ell A\right|}{|\ell|^{\gamma}}\right|^p \lesssim M_p^p\sum_{N\ge 1}\left\{2^{N(6-p(\alpha-\gamma)/\omega)}+2^{N(12-p(\beta-\gamma/\omega))}\right\}$$

$$\alpha=\frac{1}{2}-\varepsilon,\theta=\varepsilon,\gamma=\frac{1}{2}+\varepsilon,\delta=1-\varepsilon,\nu=-\bar{\eta}=\varepsilon/2$$

$$\|f\|_{\alpha,\theta} \lesssim |f|_{\mathcal{C}^{\bar{\eta}}}$$

$$\mathcal{P}\star(\Psi\partial\Psi)=\lim_{\varepsilon\downarrow 0}\mathcal{P}\star(\Psi_\varepsilon\partial\Psi_\varepsilon),$$

$$\mathbf{E}\|\Psi_t\|^p_{\alpha,\theta}+\mathbf{E}|\mathcal{P}_t\star(\Psi\partial\Psi)|^p_{\mathcal{C}^{\bar{\eta}}}+\mathbf{E}[\,[\Psi]\,]^p_{\gamma,\delta}=O(t^{p\varepsilon}).$$

$$Q_t=\Big\{\|Z\|_{\frac{3}{2}+2\kappa;O}+|\mathcal{P}\star\mathbf{1}_+\xi|_{\mathcal{C}([-1,3],\mathcal{C}^{-1/2-\kappa})}+|\mathcal{P}\star(\Psi\partial\Psi)|_{\mathcal{C}([-1,3],\mathcal{C}^{-2\kappa})}$$
$$+t^{-\varepsilon}\|\Psi_t\|_{\alpha,\theta}+t^{-\varepsilon}|\mathcal{P}_t\star(\Psi\partial\Psi)|_{\mathcal{C}^{\bar{\eta}}}+t^{-\varepsilon}[\,[\Psi_t]\,]_{\gamma,\delta}<M\Big\}$$

$$\{\|Z\|_{\frac{3}{2}+2\kappa;O}+|\mathcal{P}\star\mathbf{1}_+\xi|_{\mathcal{C}([-1,3],\mathcal{C}^{-1/2-\kappa})}+|\mathcal{P}\star(\Psi\partial\Psi)|_{\mathcal{C}([-1,3],\mathcal{C}^{-2\kappa})}<M\}\supset Q_t\,,$$

$$\mathbf{E}W_\ell(\mathcal{F}_s\tilde{A})\mathbf{1}_{Q_t} - \mathbf{E}W_\ell(\mathcal{F}_sA)\mathbf{1}_{Q_t} = \mathbf{P}[Q_t]\Bigg\{t\mathrm{Tr}\left(\int_\ell \mathrm{ch}(0)\right)$$

$$+t\mathrm{Tr}\left(\int_{[0,1]^2} \mathrm{d}\ell_{A(0)}(x_1)\mathrm{d}\ell_{ch(0)}(x_2)\right) + O\left(t^{1+r+\beta/6} + t^{1+3r/2} + t^{1+r+\nu\beta}\right)\Bigg\}$$

$$\|A\|_{\alpha,\theta} \leq |A(0)|_\infty + \|\Psi_t\|_{\alpha,\theta} + \|\mathcal{P}_t \star (\Psi\partial\Psi)\|_{\alpha,\theta} + O\left(t^{1/4-3\kappa/2}\right) \lesssim t^r$$

$$\beta = -\frac{r}{4\lambda} > 0$$

$$u = s^\lambda\|A\|_{\alpha,\theta} \lesssim s^\lambda t^r = t^{3r/4}$$

$$[\,[A]\,]_{\gamma,\delta} \lesssim [\,[\Psi_t]\,]_{\gamma,\delta} + |\Psi|_{\mathcal{C}^\eta}\left\{|A(0)|_{L^\infty} + |\mathcal{P}_t(\Psi\partial\Psi)|_{\mathcal{C}^{\bar{\eta}}} + O\left(t^{1/4-3\kappa/2}\right)\right\} \lesssim t^r$$

$$\begin{aligned} t^{5/4-7\kappa/2} + ts &\lesssim t^{1+3r/2}, \\ ts^\lambda\left(\|\Psi_t\|_{\alpha,\theta} + \|\mathcal{P}_t \star (\Psi\partial\Psi)\|_{\alpha,\theta}\right) &\lesssim s^\lambda t^{1+\varepsilon} \lesssim t^{1+3r/2}, \\ |A(0)|_{L^\infty} s^{\eta/2+\frac{1}{2}}t &\lesssim t^{r+1}t^{\left(\frac{\eta}{2}+\frac{1}{2}\right)\beta} \lesssim t^{1+r+\beta/6}, \\ s^\nu t(\Sigma(A) + \Sigma(A)^3) &\lesssim t^{1+r+\nu\beta}, \\ t(u^2 + u^{L-1}) &\lesssim t^{1+3r/2}, \end{aligned}$$

$$\left|\mathbf{E}\{W_\ell(\mathcal{F}_s\tilde{A}) - W_\ell(\mathcal{F}_sA)\}\mathbf{1}_{Q_t}\right| \gtrsim t^{1+r}$$

$$A(0) = t^r\mathrm{ch}(0).$$

$$\int_\ell \mathrm{ch}(0) = \int_\ell c_1^{(1)}h_1(0) = c_1^{(1)}\zeta(1).$$

$$\left|\mathbf{E}W_\ell(\mathcal{F}_s\tilde{A})\mathbf{1}_{Q_t} - \mathbf{E}W_\ell(\mathcal{F}_sA)\mathbf{1}_{Q_t}\right| \gtrsim t$$

$$\left|\mathbf{E}W_\ell(\mathcal{F}_s\tilde{A})\mathbf{1}_{Q_t} - \mathbf{E}W_\ell(\mathcal{F}_sA)\mathbf{1}_{Q_t}\right| \gtrsim t^{1+r}\left|\mathrm{Tr}\left(\left\{c_1^{(1)}\zeta(1)\right\}^2\right)\right| - o(t^{1+r}) \gtrsim t^{1+r}$$

$$u(x,y,z) = \begin{cases} e^{\psi(y)X} & \text{if } y \in \left[0,\frac{1}{4}\right] \\ 1 & \text{if } y \in \left[\frac{1}{4},\frac{3}{4}\right] \\ e^{\psi(y-1)X} & \text{if } y \in \left[\frac{3}{4},1\right] \end{cases}$$

$$h_2(x,0,0) \overset{\mathrm{def}}{=} (\partial_2 u)u^{-1}(x,0,0) = X, x \in [0,1]$$

$$\int_\ell ch = \int_0^1 c_1^{(2)}h_2(x,0,0)\mathrm{d}x = c_1^{(2)}X$$

$$\begin{aligned} \delta\psi_{\bar{a}}^+ &= D_{\bar{a}}\epsilon^+ + \frac{1}{16}F_\#\gamma_{\bar{a}}\epsilon^-, \quad \delta\rho^+ = \gamma^a D_a\epsilon^+, \\ \delta\psi_a^- &= D_a\epsilon^- + \frac{1}{16}F_\#^T\gamma_a\epsilon^+, \quad \delta\rho^- = \gamma^{\bar{a}}D_{\bar{a}}\epsilon^-. \end{aligned}$$

$$R_{a\bar{b}} + \frac{1}{16}\Phi^{-1}\langle F, \Gamma_{a\bar{b}} F\rangle = 0, S = 0, \Gamma^A D_A F = 0,$$

$$\gamma^b D_b \psi_{\bar{a}}^+ - D_{\bar{a}}\rho^+ = \frac{1}{16}\gamma^b F_\# \gamma_{\bar{a}} \psi_b^-, \qquad \gamma^a D_a \rho^+ - D^{\bar{a}} \psi_{\bar{a}}^+ = -\frac{1}{16} F_\# \rho^-,$$

$$\gamma^{\bar{b}} D_{\bar{b}} \psi_a^- - D_a \rho^- = \frac{1}{16}\gamma^{\bar{b}} F_\#^T \gamma_a \psi_{\bar{b}}^+, \qquad \gamma^{\bar{a}} D_{\bar{a}} \rho^- - D^a \psi_a^- = -\frac{1}{16} F_\#^T \rho^+.$$

$$\left\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1\ldots\mu_n}, \psi_\mu^\pm, \lambda^\pm\right\}$$

$$\psi_\mu = \psi_\mu^+ + \psi_\mu^- \ \gamma^{(10)}\psi_\mu^\pm = \mp\psi_\mu^\pm$$
$$\lambda = \lambda^+ + \lambda^- \ \gamma^{(10)}\lambda^\pm = \pm\lambda^\pm$$

$$\psi_\mu = \begin{pmatrix}\psi_\mu^+ \\ \psi_\mu^-\end{pmatrix} \quad \gamma^{(10)}\psi_\mu^\pm = \psi_\mu^\pm$$
$$\lambda = \begin{pmatrix}\lambda^+ \\ \lambda^-\end{pmatrix} \quad \gamma^{(10)}\lambda^\pm = -\lambda^\pm.$$

$$\rho^\pm := \gamma^\mu \psi_\mu^\pm - \lambda^\pm,$$

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2\right) - \frac{1}{4}\sum_n \frac{1}{n!}\left(F_{(n)}^{(B)}\right)^2\right],$$

$$F^{(B)} = \sum_n F_{(n)}^{(B)} = \sum_n \mathrm{e}^B \wedge \mathrm{d}A_{(n-1)},$$

$$F_{(n)}^{(B)} = (-)^{[n/2]} * F_{(10-n)}^{(B)},$$

$$\begin{aligned}S_{\mathrm{F}} = -\frac{1}{2\kappa^2}\int \sqrt{-g}\big[&\mathrm{e}^{-2\phi}\big(2\bar{\psi}^{+\mu}\gamma^\nu\nabla_\nu\psi_\mu^+ - 4\bar{\psi}^{+\mu}\nabla_\mu\rho^+ - 2\bar{\rho}^+\nabla\rho^+ \\ &-\frac{1}{2}\bar{\psi}^{+\mu} H\!\!\!/ \psi_\mu^+ - \bar{\psi}_\mu^+ H^{\mu\nu\lambda}\gamma_\nu\psi_\lambda^+ - \frac{1}{2}\rho^+ H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi_\lambda^+ + \frac{1}{2}\rho^+ H/\rho^+\big) \\ &+\mathrm{e}^{-2\phi}\big(2\bar{\psi}^{-\mu}\gamma^\nu\nabla_\nu\psi_\mu^- - 4\bar{\psi}^{-\mu}\nabla_\mu\rho^- - 2\bar{\rho}^-\nabla\rho^- \\ &+\frac{1}{2}\bar{\psi}^{-\mu} HH\psi_\mu^- + \bar{\psi}_\mu^- H^{\mu\nu\lambda}\gamma_\nu\psi_\lambda^- + \frac{1}{2}\rho^- H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi_\lambda^- - \frac{1}{2}\rho^- H/\rho^-\big) \\ &-\frac{1}{4}\mathrm{e}^{-\phi}\big(\bar{\psi}_\mu^+\gamma^\nu F\!\!\!/^{(B)}\gamma^\mu\psi_\nu^- + \rho^+ F^{(B)}\rho^-\big)\big].\end{aligned}$$

$$\mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}\mathrm{e}^{2\phi}\sum_n \frac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}}F_\nu^{(B)\lambda_1\ldots\lambda_{n-1}} = 0$$

$$\nabla^\mu\left(\mathrm{e}^{-2\phi}H_{\mu\nu\lambda}\right) - \frac{1}{2}\sum_n \frac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} = 0$$

$$\nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}\mathcal{R} - \frac{1}{48}H^2 = 0$$

$$\mathrm{d}F^{(B)} - H \wedge F^{(B)} = 0$$

$$\gamma^{\nu}\left[\left(\nabla_{\nu}\mp\frac{1}{24}H_{\nu\lambda\rho}\gamma^{\lambda\rho}-\partial_{\nu}\phi\right)\psi_{\mu}^{\pm}\pm\frac{1}{2}H_{\nu\mu}{}^{\lambda}\psi_{\lambda}^{\pm}\right]-\left(\nabla_{\mu}\mp\frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\rho^{\pm}$$
$$=\frac{1}{16}\mathrm{e}^{\phi}\sum_{n}(\pm)^{[(n+1)/2]}\gamma^{\nu}\psi_{(n)}^{(B)}\gamma_{\mu}\psi_{\nu}^{\mp},$$
$$\left(\nabla_{\mu}\mp\frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}-2\partial_{\mu}\phi\right)\psi^{\mu\pm}-\gamma^{\mu}\left(\nabla_{\mu}\mp\frac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda}-\partial_{\mu}\phi\right)\rho^{\pm}$$
$$=\frac{1}{16}\mathrm{e}^{\phi}\sum_{n}(\pm)^{[(n+1)/2]}F_{(n)}^{(B)}\rho^{\mp},$$

$$\epsilon=\epsilon^{+}+\epsilon^{-}\quad\gamma^{(10)}\epsilon^{\pm}=\mp\epsilon^{\pm}$$

$$\epsilon=\begin{pmatrix}\epsilon^{+}\\ \epsilon^{-}\end{pmatrix}\quad\gamma^{(10)}\epsilon^{\pm}=\epsilon^{\pm}$$

$$\delta e_{\mu}^{a}=\bar{\epsilon}^{+}\gamma^{a}\psi_{\mu}^{+}+\bar{\epsilon}^{-}\gamma^{a}\psi_{\mu}^{-}$$
$$\delta B_{\mu\nu}=2\bar{\epsilon}^{+}\gamma_{[\mu}\psi_{\nu]}^{+}-2\bar{\epsilon}^{-}\gamma_{[\mu}\psi_{\nu]}^{-},$$
$$\delta\phi-\frac{1}{4}\delta\log(-g)=-\frac{1}{2}\bar{\epsilon}^{+}\rho^{+}-\frac{1}{2}\bar{\epsilon}^{-}\rho^{-},$$
$$(\mathrm{e}^{B}\wedge\delta A)_{\mu_{1}\dots\mu_{n}}^{(n)}=\frac{1}{2}\left(\mathrm{e}^{-\phi}\bar{\psi}_{\nu}^{+}\gamma_{\mu_{1}\dots\mu_{n}}\gamma^{\nu}\epsilon^{-}-\mathrm{e}^{-\phi}\bar{\epsilon}^{+}\gamma_{\mu_{1}\dots\mu_{n}}\rho^{-}\right)$$
$$\mp\frac{1}{2}\left(\mathrm{e}^{-\phi}\bar{\epsilon}^{+}\gamma^{\nu}\gamma_{\mu_{1}\dots\mu_{n}}\psi_{\nu}^{-}+\mathrm{e}^{-\phi}\bar{\rho}^{+}\gamma_{\mu_{1}\dots\mu_{n}}\epsilon^{-}\right),$$

$$\delta\psi_{\mu}^{\pm}=\left(\nabla_{\mu}\mp\frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\epsilon^{\pm}+\frac{1}{16}\mathrm{e}^{\phi}\sum_{n}(\pm)^{[(n+1)/2]}F_{(n)}^{(B)}\gamma_{\mu}\epsilon^{\mp},$$
$$\delta\rho^{\pm}=\gamma^{\mu}\left(\nabla_{\mu}\mp\frac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda}-\partial_{\mu}\phi\right)\epsilon^{\pm}.$$

$$B_{(i)}=B_{(j)}-\mathrm{d}\Lambda_{(ij)}$$

$$\Lambda_{(ij)}+\Lambda_{(jk)}+\Lambda_{(ki)}=\mathrm{d}\Lambda_{(ijk)},$$

$$A_{(i)}=\mathrm{e}^{\mathrm{d}\Lambda_{(ij)}}\wedge A_{(j)}+\mathrm{d}\hat{\Lambda}_{(ij)},$$

$$B_{(i)}'=B_{(i)}-\mathrm{d}\lambda_{(i)},A_{(i)}'=\mathrm{e}^{\mathrm{d}\lambda_{(i)}}A_{(i)},$$

$$\Omega_{\mathrm{cl}}^{2}(M)\longrightarrow G_{\mathrm{NS}}\longrightarrow\mathrm{Diff}(M),$$

$$\delta_{v+\lambda}g=\mathcal{L}_{v}g,\delta_{v+\lambda}\phi=\mathcal{L}_{v}\phi,\delta_{v+\lambda}B_{(i)}=\mathcal{L}_{v}B_{(i)}-\mathrm{d}\lambda_{(i)},$$

$$\mathrm{d}\lambda_{(i)}=\mathrm{d}\lambda_{(j)}-\mathcal{L}_{v}\,\mathrm{d}\Lambda_{(ij)}$$

$$\lambda_{(i)}=\lambda_{(j)}-i_{v}\,\mathrm{d}\Lambda_{(ij)},$$

$$0\longrightarrow T^{*}M\longrightarrow E\longrightarrow TM\longrightarrow 0$$

$$v_{(i)}+\lambda_{(i)}=v_{(j)}+\left(\lambda_{(j)}-i_{v_{(j)}}\mathrm{d}\Lambda_{(ij)}\right),$$

$$\langle V,V\rangle=i_{v}\lambda,$$

$$\tilde{E} = \det T^*M \otimes E$$

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB},\ \eta = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\tilde{F} = \{(x, \{\hat{E}_A\}): x \in M, \text{and } \{\hat{E}_A\}\ \tilde{E}_x\}.$$

$$V^A \mapsto V'^A = M^A_B V^B, \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A.$$

$$V^M = \begin{cases} v^\mu & \text{for } M = \mu \\ \lambda_\mu & \text{for } M = \mu + d^{\cdot} \end{cases}$$

$$E^{\otimes n}_{(p)} = (\det T^*M)^p \otimes E \otimes \cdots \otimes E.$$

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$$

$$V^A \Gamma_A \Psi_{(i)} = i_v \Psi_{(i)} + \lambda_{(i)} \wedge \Psi_{(i)}$$

$$\Psi_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)}} \wedge \Psi_{(j)}$$

$$S^{\pm}_{(p)} = (\det T^*M)^p \otimes S^{\pm}(E)$$

$$\langle \Psi, \Psi' \rangle = \sum_n (-)^{[(n+1)/2]} \Psi^{(d-n)} \wedge \Psi'^{(n)} \in \Gamma((\det T^*M)^{2p})$$

$$\hat{E}_A = \begin{cases} \hat{E}_a = (\det e)(\hat{e}_a + i_{\hat{e}_a} B) & \text{para } A = a \\ E^a = (\det e) e^a & \text{para } A = a + d \end{cases}$$

$$\langle \hat{E}_A, \hat{E}_B \rangle = (\det e)^2 \eta_{AB}$$

$$\begin{aligned} V^{(B)} &= v^a (\det e) \hat{e}_a + \lambda_a (\det e) e^a \\ &= v_{(i)} + \lambda_{(i)} - i_{v_{(i)}} B_{(i)}, \end{aligned}$$

$$M = (\det A)^{-1} \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix},$$

$$\hat{E}_A = \begin{cases} \hat{E}_a = \mathrm{e}^{-2\phi} (\det e)(\hat{e}_a + i_{\hat{e}_a} B) & \text{para } A = a \\ E^a = \mathrm{e}^{-2\phi} (\det e) e^a & \text{para } A = a + d \end{cases}$$

$$V^{(B,\phi)} = \mathrm{e}^{2\phi} \left(v_{(i)} + \lambda_{(i)} - i_{v_{(i)}} B_{(i)} \right).$$

$$\Psi^{(B)} = \sum_n \frac{1}{n!} \Psi^{(B)}_{a_1 \ldots a_n} e^{a_1} \wedge \cdots \wedge e^{a_n} = \mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)},$$

$$\Psi^{(B,\phi)} = \mathrm{e}^{p\phi} \mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)}.$$

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \zeta - i_w\, \mathrm{d}\lambda$$

$$\begin{aligned} \mathcal{L}_v w^\mu &= v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu, \\ \mathcal{L}_v \zeta_\mu &= v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu) \zeta_\nu + p(\partial_\nu v^\nu) \zeta^\mu. \end{aligned}$$

$$\partial_M = \begin{cases} \partial_\mu & \text{para } M = \mu \\ 0 & \text{para } M = \mu + d \end{cases}$$

$$L_V W^M = V^N \partial_N W^M + (\partial^M V^N - \partial^N V^M) W_N + p(\partial_N V^N) W^M$$

$$m \cdot W = \begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} \begin{pmatrix} w \\ \zeta \end{pmatrix} - p\mathrm{tra} \begin{pmatrix} w \\ \zeta \end{pmatrix}$$

$$\begin{aligned} L_V \alpha^{M_1 \dots M_n} = & V^N \partial_N \alpha^{M_1 \dots M_n} + (\partial^{M_1} V^N - \partial^N V^{M_1}) \alpha_N{}^{M_2 \dots M_n} \\ & + \dots + (\partial^{M_n} V^N - \partial^N V^{M_n}) \alpha^{M_1 \dots M_{n-1}}{}_N + p(\partial_N V^N) W^M \end{aligned}$$

$$L_V \Psi = V^N \partial_N \Psi + \frac{1}{4} (\partial_M V_N - \partial_N V_M) \Gamma^{MN} \Psi + p(\partial_M V^M) \Psi$$

$$\begin{aligned} [\,[V, W]\,] & = \frac{1}{2} (L_V W - L_W V) \\ & = [v, w] + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda - \frac{1}{2} \, \mathrm{d}(i_v \zeta - i_w \lambda), \end{aligned}$$

$$[\,[U, V]\,]^M = U^N \partial_N V^M - V^N \partial_N U^M - \frac{1}{2} (U_N \partial^M V^N - V_N \partial^M U^N).$$

$$(\mathrm{d}\Psi)_{(i)} = \frac{1}{2} \Gamma^M \partial_M \Psi_{(i)} = \mathrm{d}\Psi_{(i)},$$

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B$$

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M \delta^A{}_B,$$

$$\Omega_M^{AB} = -\Omega_M{}^{BA}.$$

$$\begin{aligned} D_M \alpha^{A_1 \dots A_n} = & \partial_M \alpha^{A_1 \dots A_n} + \Omega_M{}^{A_1}{}_B \alpha^{B A_2 \dots A_n} \\ & + \dots + \Omega_M{}^{A_n}{}_B \alpha^{A_1 \dots A_{n-1} B} - p \Lambda_M \alpha^{A_1 \dots A_n}. \end{aligned}$$

$$D_M \Psi = \left(\partial_M + \frac{1}{4} \Omega_M^{AB} \Gamma_{AB} - p \Lambda_M \right) \Psi$$

$$W = W^A \hat{E}_A = w^a \hat{E}_a + \zeta_a E^a$$

$$(D_M^\nabla W^A) \hat{E}_A = \begin{cases} (\nabla_\mu w^a) \hat{E}_a + (\nabla_\mu \zeta_a) E^a & \text{para } M = \mu \\ 0 & \text{para } M = \mu + d \end{cases}$$

$$T(V) \cdot \alpha = L_V^D \alpha - L_V \alpha$$

$$T_{ABC} = -3 \tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - \Phi^{-2} \langle \hat{E}_A, L_{\Phi^{-1} \hat{E}_B} \hat{E}_C \rangle$$

$$T \in \Gamma(\Lambda^3 E \oplus E)$$

$$T^M{}_{PN} = (T_1)^M{}_{PN} - (T_2)_P \delta^M{}_N,$$

$$\begin{aligned} (T_1)_{MNP} & = -3 \tilde{\Omega}_{[MNP]} = -3 \Omega_{[MNP]} \\ (T_2)_M & = -\tilde{\Omega}_Q{}_M = \Lambda_M - \Omega_Q{}_M \end{aligned}$$

$$\Gamma^M D_M\Psi = \Gamma^M\left(\partial_M\Psi + \frac{1}{4}\Omega_{MNP}\Gamma^{NP}\Psi - \frac{1}{2}\Lambda_M\Psi\right)$$
$$= \Gamma^M\partial_M\Psi + \frac{1}{4}\Omega_{[MNP]}\Gamma^{MNP}\Psi - \frac{1}{2}(\Lambda_M - \Omega_{N\ M})\Gamma^M\Psi$$
$$= 2\,\mathrm{d}\Psi - \frac{1}{12}(T_1)_{[MNP]}\Gamma^{MNP}\Psi - \frac{1}{2}(T_2)_M\Gamma^M\Psi.$$

$$\Gamma^M D_M\Psi = 2\,\mathrm{d}\Psi$$

$$L_{\Phi-1}^{\hat{E}_A}\hat{E}_B = \left(L_{\Phi^{-1}\hat{E}_A}\Phi\right)\Phi^{-1}\hat{E}_B + \Phi\left(L_{\Phi^{-1}\hat{E}_A}\left(\Phi^{-1}\hat{E}_B\right)\right)$$

$$L_{\Phi^{-1}\hat{E}_A}\Phi = \begin{cases} -\mathrm{e}^{-2\phi}(\det e)\left(i_{\hat{e}_a}i_{\hat{e}_b}\,\mathrm{d}e^b + 2i_{\hat{e}_a}\,\mathrm{d}\phi\right) & \text{para } A = a \\ 0 & \text{para } A = a + d \end{cases}$$

$$L_{\Phi^{-1}\hat{E}_A}\Phi^{-1}\hat{E}_B = \begin{pmatrix} [\hat{e}_a, \hat{e}_b] + i_{\hat{e}_a,\hat{e}_b}B - i_{\hat{e}_a}i_{\hat{e}_b}H & \mathcal{L}_{\hat{e}_a}e^b \\ -\hat{e}_{\hat{e}_b}e^a & 0 \end{pmatrix}_{AB}$$

$$T_1 = -4H, T_2 = -4\,\mathrm{d}\phi,$$

$$R(U,V,W) = [D_U, D_V]W - D_{[\,[U,V]\,]}W$$

$$\left[D_{fU}, D_{gV}\right]hW - D_{[\,[fU,gV]\,]}hW$$
$$= \mathrm{fgh}\left([D_U, D_V]W - D_{[\,[U,V]\,]}W\right) - \frac{1}{2}h\langle U,V\rangle D_{(f\,\mathrm{d}g - g\,\mathrm{d}f)}W,$$

$$E = C_+ \oplus C_-,$$

$$\left\langle \hat{E}_a^+, \hat{E}_b^+\right\rangle = \Phi^2\eta_{ab},$$
$$\left\langle \hat{E}_{\bar{a}}^-, \hat{E}_{\bar{b}}^-\right\rangle = -\Phi^2\eta_{\bar{a}\bar{b}},$$
$$\left\langle \hat{E}_a^+, \hat{E}_{\bar{a}}^-\right\rangle = 0.$$

$$\hat{E}_A = \begin{cases} \hat{E}_a^+ & \text{para } A = a \\ \hat{E}_{\bar{a}}^- & \text{para } A = \bar{a} + d \end{cases}$$

$$\left\langle \hat{E}_A, \hat{E}_B\right\rangle = \Phi^2\eta_{AB}, \text{ where } \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}$$

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{para } A = a \\ -\hat{E}^{-\bar{a}} & \text{para } A = \bar{a} + d' \end{cases}$$

$$\hat{E}_a^+ = \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}_a^+ + e_a^+ + i_{\hat{e}_a^+}B\right),$$
$$\hat{E}_{\bar{a}}^- = \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}_{\bar{a}}^- - e_{\bar{a}}^- + i_{\hat{e}_{\bar{a}}}B\right),$$

$$\Phi = \mathrm{e}^{-2\phi}\sqrt{-g}$$

$$g = \eta_{ab}e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}}e^{-\bar{a}} \otimes e^{-\bar{b}},$$
$$g(\hat{e}_a^+, \hat{e}_b^+) = \eta_{ab}, g\left(\hat{e}_{\bar{a}}^-, \hat{e}_{\bar{b}}^-\right) = \eta_{\bar{b}\bar{b}}.$$

$$G = \Phi^{-2}\left(\eta^{ab}\hat{E}_a^+ \otimes \hat{E}_b^+ + \eta^{\bar{a}\bar{b}}\hat{E}_{\bar{a}}^- \otimes \hat{E}_{\bar{b}}^-\right)$$

$$G_{MN}=\frac{1}{2}\begin{pmatrix} g-Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}_{MN}$$

$$\Gamma^{(+)}=\frac{1}{d!}\epsilon^{a_1\dots a_d}\Gamma_{a_1}\dots\Gamma_{a_d}, \qquad \Gamma^{(-)}=\frac{1}{d!}\epsilon^{\bar{a}_1\dots\bar{a}_d}\Gamma_{\bar{a}_1}\dots\Gamma_{\bar{a}_d}.$$

$$\Gamma_a\cdot\Psi^{(B)}=i_{\hat{e}_a^+}\Psi^{(B)}+e_a^+\wedge\Psi^{(B)}, \Gamma_{\bar{a}}\cdot\Psi^{(B)}=i_{\hat{e}_a^-}\Psi^{(B)}-e_a^-\wedge\Psi^{(B)}$$

$$\Gamma^{(+)}\Psi_{(n)}^{(B)}=(-)^{[n/2]}*\Psi_{(n)}^{(B)}, \qquad \Gamma^{(-)}\Psi_{(n)}^{(B)}=(-)^{d}(-)^{[n+1/2]}*\Psi_{(n)}^{(B)}$$

$$\Gamma^{(-)}\Gamma^{A}\Gamma^{(-)-1}=G_B^A\Gamma^B$$

$$DG=0, D\Phi=0$$

$$W=w_+^a\hat{E}_a^+ + w_-^{\bar{a}}\hat{E}_{\bar{a}}^-,$$

$$D_MW^A=\begin{cases}\partial_M w_+^a+\Omega_M{}^a{}_b w_+^b & \text{para } A=a \\ \partial_M w_-^{\bar{a}}+\Omega_M{}^{\bar{a}}{}_{\bar{b}} w_-^{\bar{b}} & \text{para } A=\bar{a}\end{cases},$$

$$\Omega_{Mab}=-\Omega_{Mba}, \Omega_{M\bar{a}\bar{b}}=-\Omega_{M\bar{b}\bar{a}}.$$

$$\nabla_\mu v^\nu=\left(\partial_\mu v^a+\omega_\mu^{+a}{}_b v^b\right)(\hat{e}_a^+)^\nu=\left(\partial_\mu v^{\bar{a}}+\omega_\mu^{-\bar{a}}\bar{b}v^{\bar{b}}\right)(\hat{e}_{\bar{a}}^-)^\nu.$$

$$D_M^\nabla W^a=\begin{cases}\nabla_\mu w_+^a & \text{para } M=\mu \\ 0 & \text{para } M=\mu+d\end{cases}, D_M^\nabla W^{\bar{a}}=\begin{cases}\nabla_\mu w_-^{\bar{a}} & \text{para } M=\mu \\ 0 & \text{para } M=\mu+d\end{cases}.$$

$$W=w_+^a\hat{E}_a^+ + w_-^{\bar{a}}\hat{E}_{\bar{a}}^- = (w_+^a+w_-^a)\hat{E}_a+(w_{+a}-w_{-a})E^a,$$

$$T_1=-4H, T_2=-4\,\mathrm{d}\phi$$

$$D_MW^A=D_M^\nabla W^A+\Sigma_M{}^A{}_B W^B$$

$$\Sigma_{Mab}=-\Sigma_{Mba}, \Sigma_{M\bar{a}\bar{b}}=-\Sigma_{M\bar{b}\bar{a}}.$$

$$(T_1)_{ABC}=-4H_{ABC}-3\Sigma_{[ABC]}, (T_2)_A=-4\,\mathrm{d}\phi_A-\Sigma_C{}^C{}_A.$$

$$\mathrm{d}x^\mu=\frac{1}{2}\Phi^{-1}\left(\hat{e}_a^{+\mu}\hat{E}^{+a}-\hat{e}_{\bar{a}}^{-\mu}\hat{E}^{-\bar{a}}\right),$$

$$\mathrm{d}\phi=\frac{1}{2}\partial_a\phi\left(\Phi^{-1}\hat{E}^{+a}\right)-\frac{1}{2}\partial_{\bar{a}}\phi\left(\Phi^{-1}\hat{E}^{-\bar{a}}\right).$$

$$\Lambda^3T^*M\hookrightarrow\Lambda^3E\simeq\Lambda^3C_+\oplus(\Lambda^2C_+\otimes C_-)\oplus(C_+\otimes\Lambda^2C_-)\oplus\Lambda^3C_-,$$

$$d\phi_A = \begin{cases} \frac{1}{2}\partial_a\phi & A = a \\ \frac{1}{2}\partial_{\bar{a}}\phi & A = \bar{a} + d \end{cases}, H_{ABC} = \begin{cases} \frac{1}{8}H_{abc} & (A,B,C) = (a,b,c) \\ \frac{1}{8}H_{ab\bar{c}} & (A,B,C) = (a,b,\bar{c}+d) \\ \frac{1}{8}H_{a\bar{b}\bar{c}} & (A,B,C) = (a,\bar{b}+d,\bar{c}+d) \\ \frac{1}{8}H_{\bar{a}\bar{b}\bar{c}} & (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d) \end{cases}$$

$$\Sigma_{[abc]} = -\frac{1}{6}H_{abc}, \quad \Sigma_{\bar{a}bc} = -\frac{1}{2}H_{\bar{a}bc}, \quad \Sigma_a{}^a{}_b = -2\partial_b\phi,$$
$$\Sigma_{[\bar{a}\bar{b}\bar{c}]} = +\frac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \quad \Sigma_{a\bar{b}\bar{c}} = +\frac{1}{2}H_{a\bar{b}\bar{c}}, \quad \Sigma_{\bar{a}}{}_{\bar{b}} = -2\partial_{\bar{b}}\phi.$$

$$D_a w_+^b = \nabla_a w_+^b - \frac{1}{6}H_a{}^b{}_c w_+^c - \frac{2}{9}\left(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi\right)w_+^c + A_a^{+b}{}_c w_+^c,$$
$$D_{\bar{a}} w_+^b = \nabla_{\bar{a}} w_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_c w_+^c,$$
$$D_a w_-^{\bar{b}} = \nabla_a w_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}},$$
$$D_{\bar{a}} w_-^{\bar{b}} = \nabla_{\bar{a}} w_-^{\bar{b}} + \frac{1}{6}H_{\bar{a}}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}} - \frac{2}{9}\left(\delta_{\bar{a}}{}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi\right)w_-^{\bar{c}} + A_{\bar{a}}^{-\bar{b}}{}_{\bar{c}} w_-^{\bar{c}},$$

$$A^+_{abc} = -A^+_{acb}, \quad A^+_{[abc]} = 0, \quad A_a^{+a}{}_b = 0,$$
$$A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}, \quad A^-_{[\bar{a}\bar{b}\bar{c}]} = 0, \quad A_{\bar{a}}^{-\bar{a}}{}_{\bar{b}} = 0$$

$$D_{\bar{a}} w_+^b = \nabla_{\bar{a}} w_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_c w_+^c,$$
$$D_a w_-^{\bar{b}} = \nabla_a w_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}},$$

$$D_a w_+^a = \nabla_a w_+^a - 2(\partial_a\phi)w_+^a,$$
$$D_{\bar{a}} w_-^{\bar{a}} = \nabla_{\bar{a}} w_-^{\bar{a}} - 2(\partial_{\bar{a}}\phi)w_-^{\bar{a}}.$$

$$D_M\epsilon^+ = \partial_M\epsilon^+ + \frac{1}{4}\Omega_M{}^{ab}\gamma_{ab}\epsilon^+,$$
$$D_M\epsilon^- = \partial_M\epsilon^- + \frac{1}{4}\Omega_M{}^{\bar{a}\bar{b}}\gamma_{\bar{a}\bar{b}}\epsilon^-.$$

$$D_{\bar{a}}\epsilon^+ = \left(\nabla_{\bar{a}} - \frac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\epsilon^+,$$
$$D_a\epsilon^- = \left(\nabla_a + \frac{1}{8}H_{a\bar{b}}\gamma^{\bar{b}\bar{c}}\right)\epsilon^-,$$
$$\gamma^a D_a\epsilon^+ = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon^+,$$
$$\gamma^{\bar{a}} D_{\bar{a}}\epsilon^- = \left(\gamma^{\bar{a}}\nabla_{\bar{a}} + \frac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi\right)\epsilon^-.$$

$$R_{a\bar{b}}w_+^a = [D_a, D_{\bar{b}}]w_+^a$$

$$R_{\bar{a}b}w_-^{\bar{a}} = [D_{\bar{a}}, D_b]w_-^{\bar{a}}$$

$$\frac{1}{2}R_{a\bar{b}}\gamma^{a}\epsilon^{+} = [\gamma^{a}D_{a}, D_{\bar{b}}]\epsilon^{+},$$
$$\frac{1}{2}R_{\bar{a}\gamma}\gamma^{\bar{a}}\epsilon^{-} = [\gamma^{\bar{a}}D_{\bar{a}}, D_{b}]\epsilon^{-}.$$

$$-\frac{1}{4}S\epsilon^{+} = \left(\gamma^{a}D_{a}\gamma^{b}D_{b} - D^{\bar{a}}D_{\bar{a}}\right)\epsilon^{+}$$

$$-\frac{1}{4}S\epsilon^{-} = \left(\gamma^{\bar{a}}D_{\bar{a}}\gamma^{\bar{b}}D_{\bar{b}} - D^{a}D_{a}\right)\epsilon^{-}$$

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4}H_{acd}H_{b}^{cd} + 2\nabla_{a}\nabla_{b}\phi + \frac{1}{2}\mathrm{e}^{2\phi}\nabla^{c}\left(\mathrm{e}^{-2\phi}H_{cab}\right),$$

$$S = \mathcal{R} + 4\nabla^{2}\phi - 4(\partial\phi)^{2} - \frac{1}{12}H^{2}$$

$$\mathrm{d}s_{10}^{2} = \mathrm{d}s^{2}\left(\mathbb{R}^{9-d,1}\right) + \mathrm{d}s_{d}^{2}$$

$$\{g, B, \phi\} \in \frac{O(10,10)}{O(9,1)\times O(1,9)} \times \mathbb{R}^{+}$$

$$\delta_{V}G = L_{V}G, \delta_{V}\Phi = L_{V}\Phi$$

$$\psi_{\bar{a}}^{+} \in \Gamma\left(C_{-}\otimes S^{\mp}(C_{+})\right), \psi_{a}^{-} \in \Gamma\left(C_{+}\otimes S^{+}(C_{-})\right),$$

$$\rho^{+} \in \Gamma\left(S^{\pm}(C_{+})\right), \rho^{-} \in \Gamma\left(S^{+}(C_{-})\right)$$

$$\epsilon^{+} \in \Gamma\left(S^{\mp}(C_{+})\right), \epsilon^{-} \in \Gamma\left(S^{+}(C_{-})\right)$$

$$F \in \Gamma\left(S_{(1/2)}^{\pm}\right),$$

$$F^{(B)} = \mathrm{e}^{B_{(i)}} \wedge F_{(i)} = \mathrm{e}^{B_{(i)}} \wedge \sum_{n} \mathrm{d}A_{(i)}^{(n-1)}.$$

$$\Gamma^{A} = \begin{cases}\gamma^{a}\otimes 1 & \text{para } A = a\\ \gamma^{(10)}\otimes\gamma^{\bar{a}}\gamma^{(10)} & \text{para } A = \bar{a}+d\end{cases}$$

$$S_{(1/2)} \simeq S(C_{+})\otimes S(C_{-})$$

$$F_{\#}: S(C_{-}) \to S(C_{+}).$$

$$F_{\#}^{T} = (CF_{\#}C^{-1})^{T},$$

$$\not{F}^{(B,\phi)} = \sum_{n}\frac{1}{n!}F_{a_{1}\dots a_{n}}^{(B,\phi)}\gamma^{a_{1}\dots a_{n}}.$$

$$F_{\#} = \Lambda^{+}\not{F}^{(B,\phi)}(\Lambda^{-})^{-1}$$

$$F_{(i)} = \mathrm{e}^{-B_{(i)}} \wedge F^{(B)} = \mathrm{e}^{-\phi}\mathrm{e}^{-B_{(i)}} \wedge F^{(B,\phi)}$$
$$= \mathrm{e}^{-\phi}\mathrm{e}^{-B_{(i)}} \wedge \sum_n \left[\frac{1}{32(n!)}(-)^{[n/2]}\mathrm{tr}\big(\gamma_{(n)}(\Lambda^+)^{-1}F_\#\Lambda^-\big)\right].$$

$$\Gamma^{(-)}F = -F,$$

$$\delta\psi_{\bar a}^+ = D_{\bar a}\epsilon^+ + \frac{1}{16}F_\#\gamma_{\bar a}\epsilon^-,$$
$$\delta\psi_a^- = D_a\epsilon^- + \frac{1}{16}F_\#^T\gamma_a\epsilon^+$$
$$\delta\rho^+ = \gamma^a D_a\epsilon^+,$$
$$\delta\rho^- = \gamma^{\bar a} D_{\bar a}\epsilon^-,$$

$$\tilde\delta\hat E_a^+ = (\delta\log\ \Phi)\hat E_a^+ - \big(\delta\Lambda_{a\bar b}^+\big)\hat E^{-\bar b},$$
$$\tilde\delta\hat E_{\bar a}^- = (\delta\log\ \Phi)\hat E_{\bar a}^- - (\delta\Lambda_{\bar a b}^-)\hat E^{+b},$$

$$\delta\Lambda_{a\bar a}^+ = \bar\epsilon^+\gamma_a\psi_{\bar a}^+ + \bar\epsilon^-\gamma_{\bar a}\psi_a^-,$$
$$\delta\Lambda_{a\bar a}^- = \bar\epsilon^+\gamma_a\psi_{\bar a}^+ + \bar\epsilon^-\gamma_{\bar a}\psi_a^-,$$

$$\delta\log\ \Phi = -2\delta\phi + \frac{1}{2}\delta\log(-g) = \bar\epsilon^+\rho^+ + \bar\epsilon^-\rho^-$$

$$\tilde\delta e_\mu^{+a} = \bar\epsilon^+\gamma_\mu\psi^{+a} + \bar\epsilon^-\gamma^a\psi_\mu^-,$$
$$\tilde\delta e_\mu^{-\bar a} = \bar\epsilon^+\gamma^{\bar a}\psi_\mu^+ + \bar\epsilon\gamma_\mu\psi^{-\bar a},$$

$$\tilde\delta g_{\mu\nu} = 2\bar\epsilon^+\gamma_{(\mu}\psi_{\nu)}^+ + 2\bar\epsilon\gamma_{(\mu}\psi_{\nu)}^-$$

$$\tilde\delta e_\mu^{+a} = \delta e_\mu^{+a} - \big(\bar\epsilon^+\gamma^a\psi^{+b} - \bar\epsilon^+\gamma^b\psi^{+a}\big)e_{\mu b}^+,$$
$$\tilde\delta e_\mu^{-\bar a} = \delta e_\mu^{+\bar a} - \big(\bar\epsilon^-\gamma^{\bar a}\psi^{-\bar b} - \bar\epsilon\gamma^{\bar b}\psi^{-\bar a}\big)e_{\mu\bar b}^-.$$

$$\delta G_{a\bar a} = \delta G_{\bar a a} = 2(\bar\epsilon^+\gamma_a\psi_{\bar a}^+ + \bar\epsilon\gamma_{\bar a}\psi_a^-)$$

$$\frac{1}{16}(\delta A_\#) = (\gamma^a\epsilon^+\bar\psi_a^- - \rho^+\bar\epsilon^-) \mp (\psi_{\bar a}^+\bar\epsilon^-\gamma^{\bar a} + \epsilon^+\bar\rho^-)$$

$$R_{a\bar b} + \frac{1}{16}\Phi^{-1}\langle F, \Gamma_{a\bar b}F\rangle = 0$$

$$S = 0$$

$$\frac{1}{2}\Gamma^A D_A F = \mathrm{d}F = 0$$

$$S_B = \frac{1}{2\kappa^2}\int \left(\Phi S + \frac{1}{4}\big\langle F, \Gamma^{(-)}F\big\rangle\right)$$

$$S_F = -\frac{1}{2\kappa^2}\int \; 2\Phi[\bar{\psi}^{+\bar{a}}\gamma^b D_b\psi^+_{\bar{a}} + \bar{\psi}^{-a}\gamma^{\bar{b}} D_{\bar{b}}\psi^-_a$$
$$+2\bar{\rho}^+ D_{\bar{a}}\psi^{+\bar{a}} + 2\bar{\rho}^- D_a\psi^{-a}$$
$$-\bar{\rho}^+\gamma^a D_a\rho^+ - \bar{\rho}^-\gamma^{\bar{a}} D_{\bar{a}}\rho^-$$
$$-\frac{1}{8}(\bar{\rho}^+ F_\# \rho^- + \bar{\psi}^+_{\bar{a}}\gamma^a F_\# \gamma^{\bar{a}}\psi^-_a)\Big].$$

$$\gamma^b D_b\psi^+_{\bar{a}} - D_{\bar{a}}\rho^+ = +\frac{1}{16}\gamma^b F_\# \gamma_{\bar{a}}\psi^-_b,$$
$$\gamma^{\bar{b}} D_{\bar{b}}\psi^-_a - D_a\rho^- = +\frac{1}{16}\gamma^{\bar{b}} F^T_\# \gamma_a\psi^+_{\bar{b}},$$
$$\gamma^a D_a\rho^+ - D^{\bar{a}}\psi^+_{\bar{a}} = -\frac{1}{16}F_\# \rho^-,$$
$$\gamma^{\bar{a}} D_{\bar{a}}\rho^- - D^a\psi^-_a = -\frac{1}{16}F^T_\# \rho^+,$$

$$\omega_{(k)} = \frac{1}{k!}\omega_{\mu_1\dots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\,\mathrm{d}x^{\mu_k}$$
$$\omega_{(k)}\wedge\eta_{(l)} = \frac{1}{(k+l)!}\left(\frac{(k+l)!}{k!\,l!}\omega_{[\mu_1\dots\mu_k}\eta_{\mu_{k+1}\dots\mu_{k+l}]}\right)\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\,\mathrm{d}x^{\mu_{k+l}}$$
$$*\,\omega_{(k)} = \frac{1}{(10-k)!}\left(\frac{1}{k!}\sqrt{-g}\epsilon_{\mu_1\dots\mu_{10-k}\nu_1\dots\nu_k}\omega^{\nu_1\dots\nu_k}\right)\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\,\mathrm{d}x^{\mu_{10-k}}$$

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \gamma^{\mu_1\dots\mu_k} = \gamma^{[\mu_1}\dots\gamma^{\mu_k]}$$

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T, C^T = -C$$

$$C\gamma^{\mu_1\dots\mu_k}C^{-1} = (-)^{[(k+1)/2]}(\gamma^{\mu_1\dots\mu_k})^T,$$
$$\bar{\epsilon}\gamma^{\mu_1\dots\mu_k}\chi = (-)^{[(k+1)/2]}\bar{\chi}\gamma^{\mu_1\dots\mu_k}\epsilon,$$

$$\gamma^{(10)} = \gamma^0\gamma^1\dots\gamma^9 = \frac{1}{10!}\epsilon_{\mu_1\dots\mu_{10}}\gamma^{\mu_1\dots\mu_{10}}$$

$$\gamma_{\mu_1\dots\mu_k}\gamma^{(10)} = (-)^{[k/2]}\frac{1}{(10-k)!}\sqrt{-g}\epsilon_{\mu_1\dots\mu_k\nu_1\dots\nu_{10-k}}\gamma^{\nu_1\dots\nu_{10-k}},$$

$$\gamma^{(k)}\gamma^{(10)} = (-)^{[k/2]} * \gamma^{(10-k)}$$

$$\Psi = \sum_k \frac{1}{k!}\Psi_{\mu_1\dots\mu_k}\gamma^{\mu_1\dots\mu_k}$$

$$F = \{(x,\{\hat{e}_a\}): x\in M \text{ and } \{\hat{e}_a\}\; T_xM\}.$$

$$v^a \mapsto v'^a = A^a{}_b v^b, \hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b(A^{-1})^b{}_a.$$

$$\mathcal{L}_v w = -\mathcal{L}_w v = [v,w],$$

$$\mathcal{L}_v\alpha^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} = v^\mu\partial_\mu\alpha^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}$$
$$+(\partial_\mu v^{\mu_1})\alpha^{\mu\mu_2\dots\mu_p}_{\nu_1\dots\nu_q} + \cdots + (\partial_\mu v^{\mu_p})\alpha^{\mu_1\dots\mu_{p-1}\mu}_{\nu_1\dots\nu_q}$$
$$-(\partial_{\nu_1}v^\mu)\alpha^{\mu_1\dots\mu_p}_{\mu\nu_2\dots\nu_q} - \cdots - \left(\partial_{\nu_q}v^\mu\right)\alpha^{\mu_1}_{\nu_1\dots\nu_{q-1}\mu}.$$

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w].$$

$$T^{\mu}{}_{\nu\lambda} = \omega_{\nu}{}^{\mu}{}_{\lambda} - \omega_{\lambda}{}^{\mu}{}_{\nu},$$

$$T^{a}{}_{bc} = \omega_{b}{}^{a}{}_{c} - \omega_{c}{}^{a}{}_{b} + [\hat{e}_b, \hat{e}_c]^{a}.$$

$$(i_v T)\alpha = \mathcal{L}_v^{\nabla}\alpha - \mathcal{L}_v \alpha,$$

$$\mathcal{R}(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w,$$
$$\mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho} v^{\rho} = [\nabla_{\mu}, \nabla_{\nu}]v^{\lambda} - T^{\rho}{}_{\mu\nu}\nabla_{\rho}v^{\lambda}.$$

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\lambda\mu\nu}{}^{\lambda}.$$

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$$

$$P = \{(x, \{\hat{e}_a\}) \in F : g(\hat{e}_a, \hat{e}_b) = \delta_{ab}\},$$

$$g|_x \in GL(d, \mathbb{R})/O(d)$$

$$\nabla_{\partial/\partial x^{\mu}}\hat{e}_a = \omega_{\mu}{}^{b}{}_{a}\hat{e}_b.$$

$$\nabla_{\mu}A^{\alpha}{}_{\beta} := \partial_{\mu}A^{\alpha}{}_{\beta} - \Gamma^{\rho}_{\beta\mu}A^{\alpha}_{\rho} + \Gamma^{\alpha}_{\rho\mu}A^{\rho}{}_{\beta}.$$

$$\Gamma^{\rho}_{\mu\nu} := \left\{ {\rho \atop \mu\nu} \right\} + K^{\rho}_{\mu\nu} + L^{\rho}{}_{\mu\nu},$$

$$\left\{ {\rho \atop \mu\nu} \right\} := \frac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}),$$

$$K^{\rho}{}_{\mu\nu} := \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}),$$

$$L^{\rho}{}_{\mu\nu} := \frac{1}{2}(Q^{\rho}{}_{\mu\nu} - Q^{\rho}_{\mu}{}_{\nu} - Q_{\nu}{}^{\rho}{}_{\mu}).$$

$$R^{\mu}_{\nu\rho\sigma} := \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\rho},$$

$$T^{\mu}_{\nu\rho} := 2\Gamma^{\mu}_{[\rho\nu]} \equiv \Gamma^{\mu}_{\rho\nu} - \Gamma^{\mu}_{\nu\rho},$$

$$Q_{\mu\nu\rho} := \nabla_{\mu}g_{\nu\rho} \equiv \partial_{\mu}g_{\nu\rho} - 2\Gamma^{\lambda}_{(\nu|\mu}g_{\rho)\lambda} \neq 0.$$

$$R^{\mu}_{\nu\rho\sigma} = -R^{\mu}_{\nu\sigma\rho},$$

$$T^{\mu}_{\nu\rho} = -T^{\mu}_{\rho\nu},$$

$$Q_{\mu\nu\rho} = Q_{\mu\rho\nu}.$$

$$R^{\mu}_{[\nu\rho\sigma]} = \nabla_{[\nu}T^{\mu}_{\rho\sigma]} + T^{\mu}_{\alpha[\nu}T^{\alpha}_{\rho\sigma]},$$
$$\nabla_{[\alpha}R^{\mu}_{|\nu|\rho\sigma]} = -R^{\mu}_{\nu\tau[\alpha}T^{\tau}_{\rho\sigma]}$$

$$\partial_\mu := \left(\frac{\partial}{\partial x^\mu}\right)_p,$$

$$dx^\mu \partial_\nu = \delta^\mu_\nu.$$

$$e_A := e^\mu_A \partial_\mu, e^A := e^A_\mu \, dx^\mu,$$

$$g_{\mu\nu} = \eta_{AB} e^A e^B{}_\nu, \eta_{AB} = g_{\mu\nu} e^\mu_A e^\nu_B.$$

$$\begin{aligned}[e_A, e_B] &:= e_A e_B - e_B e_A \\ &= (e^\mu_A \partial_\mu)(e^\nu_B \partial_\nu) - (e^\nu_B \partial_\nu)(e^\mu_A \partial_\mu) \\ &= [e^\mu_A e^C_\nu (\partial_\mu e^\nu_B) - e^\nu_B e^C_\mu (\partial_\nu e^\mu_B)] e_C \\ &= e^\mu_A e^\nu_B [\partial_\nu e^C_\mu - \partial_\mu e^C_\nu] e_C \\ &= f^C_{AB} e_C,\end{aligned}$$

$$f^C_{AB} := e^\mu_A e^\nu_B [\partial_\nu e^C_\mu - \partial_\mu e^C_\nu]$$

$$d\omega = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu$$

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

$$d\omega(u, v) = u\omega(v) - v\omega(u) - \omega([u, v]_{\mathcal{L}})$$

$$\begin{aligned} d\omega(u, v) &:= \partial_\mu \omega_\nu (u^\mu v^\nu - u^\nu v^\mu), \\ u\omega(v) &:= u^\mu v^\nu \partial_\mu \omega_\nu + u^\mu \omega_\nu \partial_\mu v^\nu, \\ \omega([u, v]_{\mathcal{L}}) &:= \omega_\nu (u^\mu \partial_\mu v^\nu - v^\mu \partial_\mu u^\nu),\end{aligned}$$

$$\begin{aligned}\{de^C(e_A, e_B)\} e_C &= \{e_A[e^C(e_B)] - e_B[e^C(e_A)] \\ &\quad - e^C([e_A, e_B]_{\mathcal{L}})\} e_C \\ &= -e^C\left([e_A, e_B]^L_{\mathcal{L}} e_L\right) e_C \\ &= -[e_A, e_B]_{\mathcal{L}}.\end{aligned}$$

$$\nabla_{e_A} e_B = \gamma^C_{AB} e_C,$$

$$\gamma_{\lambda\nu\mu} := e^A_\mu e^B_\lambda \nabla_A (e_\nu)_B$$

$$\begin{aligned}&= -e^A_\mu (e_\nu)_B \nabla_A e^B_\lambda \\ &= -e^A_\mu e^B_\nu \nabla_A (e_\lambda)_B = -\gamma_{\nu\lambda\mu},\end{aligned}$$

$$\gamma^C_{AB} = \omega^C_B(e_A) \;\Leftrightarrow\; \omega^C_B = \gamma^C_{AB} e^A$$

$$\begin{aligned}[\nabla_\mu, \partial_\nu] &= \nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu \\ &= (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) \partial_\lambda \\ &= T^\lambda_{\mu\nu} \partial_\lambda,\end{aligned}$$

$$T(v, u) := \nabla_v u - \nabla_u v - [v, u]_{\mathcal{L}}.$$

$$T(e_A,e_B)=\nabla_{e_A}e_B-\nabla_{e_B}e_A-[e_A,e_B]_{\mathcal{L}}$$
$$=\left[\omega^C_B(e_A)-\omega^C_A(e_B)+\mathrm{d}e^C(e_A,e_B)\right]e_C$$
$$=\left[\left(\omega^C_D\wedge e^D+\mathrm{d}e^C\right)(e_A,e_B)\right]e_C.$$

$$T=\Omega^C\otimes e_C$$

$$de^C:=-\omega^C_A\wedge e^A$$
$$=-\frac{1}{2}(\gamma^C_{AB}-\gamma^C_{BA})e^A\wedge e^B$$
$$=-\frac{1}{2}e^\mu_A e^\nu_B(\partial_\nu e^C_\mu-\partial_\mu e^C_\nu)e^A\wedge e^B$$
$$=-\frac{1}{2}f^C_{AB}e^A\wedge e^B,$$

$$\eta_{AB}=\eta_{\mu\nu}e^\mu_A e^\nu_B.$$

$$\Lambda^\mu_\nu:x^\mu\longrightarrow x'^\mu=\Lambda^\mu_\nu(x)x^\nu,$$

$$\eta_{\mu\nu}x^\mu x^\nu=-t^2+x^2+y^2+z^2.$$

$$\Lambda^\alpha{}_\beta=\mathcal{G}\cdot\begin{bmatrix}\gamma & -\gamma\mathcal{R}^i{}_j\frac{v^j}{c}\\ -\gamma\mathcal{R}^i{}_j{}^j\frac{v^j}{c} & \mathcal{R}^i{}_j\left(\delta^i_j+(\gamma-1)\frac{v^iv^j}{v^2}\right)\end{bmatrix},$$

$$\mathbb{1}:=\mathrm{diag}(1,1,1,1)$$
$$\mathbb{P}:=\mathrm{diag}(1,-1,-1,-1)$$
$$\mathbb{T}:=\mathrm{diag}(-1,1,1,1)$$

$$\bar{e}^A{}_\mu=\Lambda^A_B e^B{}_\mu,$$

$$g_{\mu\nu}=\eta_{AB}\bar{e}^A_\mu\bar{e}^B{}_\nu\ \eta_{AB}=g_{\mu\nu}\bar{e}^\mu_A\bar{e}^\nu_B.$$

$$\Lambda^\alpha{}_\beta=\delta^\alpha_\beta+\omega^\alpha{}_\beta+\mathcal{O}\left[\left(\omega^\alpha_\beta\right)^2\right].$$

$$(J_{AB})^C_D:=2i\eta_{[B|D}\delta^C_{A]}=i\left(\eta_{BD}\delta^C_A-\eta_{AD}\delta^C_B\right).$$

$$\Lambda=e^{\frac{i}{2}\omega_{AB}J^{AB}}.$$

$$\omega_\mu:J_{AB}\in\mathfrak{L}\longrightarrow\omega_\mu:=\frac{1}{2}\omega^{AB}{}_\mu J_{AB},$$

$$\mathcal{D}_\mu:=\partial_\mu-\omega_\mu=\partial_\mu-\frac{i}{2}\omega^{AB}{}_\mu J_{AB},$$

$$\mathcal{D}_\mu e^C=\partial_\mu e^C-\frac{i}{2}\omega^{AB}{}_\mu\left[i\left(\eta_{BD}\delta^C_A-\eta_{AD}\delta^C_B\right)\right]e^D$$
$$=\partial_\mu e^C+\frac{1}{2}\left[\omega^A{}_{D\mu}\delta^C_A+\omega^B{}_{D\mu}\delta^C_B\right]e^D$$
$$=\partial_\mu e^C+\omega^C{}_{D\mu}e^D.$$

$$
\begin{aligned}
\mathcal{D}_\mu\left(e_\lambda^C dx^\lambda\right) &= \mathcal{D}_\mu\left(e_\lambda^C\right)\mathrm{d}x^\lambda + e_\lambda^C \mathcal{D}_\mu\left(\mathrm{d}x^\lambda\right) \\
&= \mathcal{D}_\mu\left(e_\lambda^C\right)\mathrm{d}x^\lambda + e_\lambda^C\left(\delta_\mu^\lambda + e_E^\lambda e_\mu^D \omega^E{}_{D\rho}\,\mathrm{d}x^\rho\right) \\
&= \mathcal{D}_\mu\left(e_\lambda^C\right)\mathrm{d}x^\lambda + e_\mu^C, \\
\mathcal{D}_\mu\left(e_\lambda^C dx^\lambda\right) &= \partial_\mu\left(e_\lambda^C dx^\lambda\right) + \omega^C{}_{D\mu} e_\lambda^D\,\mathrm{d}x^\lambda \\
&= \partial_\mu\left(e_\lambda^C\right) dx^\lambda + e_\mu^C + \omega^C{}_{D\mu} e_\lambda^D\,\mathrm{d}x^\lambda.
\end{aligned}
$$

$\mathcal{D}_\mu\left(e_\lambda^C\right) = \partial_\mu\left(e_\lambda^C\right) + \omega^C{}_{D\mu} e_\lambda^D.$

$\tilde{\nabla}_\mu X_B^A := \partial_\mu + \omega^A{}_{C\mu} X^C{}_B - \omega^C{}_{B\mu} X_C^A.$

$$
\begin{aligned}
\nabla V &= \left(\nabla_\mu V^\nu\right)\mathrm{d}x^\mu \otimes \partial_\nu \\
&= \left(\partial_\mu V + \Gamma_{\mu\lambda}^\nu V^\lambda\right)\mathrm{d}x^\mu \otimes \partial_\nu.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\nabla} V &= \left(\tilde{\nabla}_\mu V^A\right)\mathrm{d}x^\mu \otimes e_A \\
&= \left(\partial_\mu V^A + \omega^A{}_{B\mu} V^B\right)\mathrm{d}x^\mu \otimes e_A \\
&= \left[\partial_\mu\left(e_\lambda^A V^\lambda\right) + \omega^A{}_{B\mu} e_\lambda^B V^\lambda\right]\mathrm{d}x^\mu \otimes \left(e_A^\nu \partial_\nu\right) \\
&= \left[\partial_\mu V^\nu + \left(e_A^\nu \partial_\mu e_\lambda^A + \omega^A{}_{B\mu} e_A^\nu e_\lambda^B\right)V^\lambda\right]\mathrm{d}x^\mu \otimes \partial_\nu \\
&= \left[\partial_\mu V^\nu + \left(e_A^\nu \mathcal{D}_\mu e_\lambda^A\right)V^\lambda\right]\mathrm{d}x^\mu \otimes \partial_\nu.
\end{aligned}
$$

$$
\Gamma_{\mu\nu}^\lambda \equiv e_A^\lambda \mathcal{D}_\mu e^A{}_\nu
$$

$$
\omega_{B\mu}^A = e_\lambda^A e_B^\nu \Gamma_{\mu\nu}^\lambda + e_\sigma^A \partial_\mu e_B^\sigma \equiv e_\nu^A \nabla_\mu e_B^\nu
$$

$\omega^{AB} = \omega^{AB}{}_\mu \mathrm{d}x^\mu,$

$$
\nabla_\mu e_\nu^A = \partial_\mu e_\nu^A - \Gamma_{\mu\nu}^\lambda e_\lambda^A + \omega_{B\mu}^A e_\nu^B = 0;
$$

$$
\begin{aligned}
0 = \nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\mu}^\sigma g_{\sigma\nu} - \Gamma_{\lambda\nu}^\sigma g_{\mu\sigma} \\
&= \partial_\lambda\left(e_\mu^A e_\nu^B \eta_{AB}\right) - e_A^\sigma g_{\sigma\nu} \mathcal{D}_\lambda e_\mu^A - e_A^\sigma g_{\mu\sigma} \mathcal{D}_\lambda e_\nu^A \\
&= -e_\nu^A e_\mu^D\left(\omega_{AD\lambda} - \omega_{DA\lambda}\right),
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu x'^A &= \partial_\mu\left(\Lambda_B^A(x) x^B\right) \\
&= \left(\partial_\mu x^B\right)\Lambda_B^A(x) + x^B\left(\partial_\mu \Lambda_B^A(x)\right), \\
\partial_\mu x'^A &= e'{}_\mu \partial'_C x'^A = e'^A{}_\mu = e_\mu^C \Lambda_C^A(x).
\end{aligned}
$$

$$
e_\mu^A = \partial_\mu x^A + \dot{\omega}^A{}_{B\mu} x^B \equiv \mathcal{D}_\mu x^A,
$$

$$
\dot{\omega}^A{}_{B\mu} := \Lambda_C^A(x)\partial_\mu \Lambda_B^C(x)
$$

$\omega^A{}_{B\mu} = \underbrace{\Lambda_C^A(x)\omega'^C{}_{D\mu}\Lambda_C^D}_{\text{non inertial}} + \underbrace{\Lambda_C^A \partial_\mu \Lambda_B^C(x)}_{\text{inertial}}.$

$f_{AB}^{C} = \dot{\omega}_{BA}^{C} - \dot{\omega}_{AB}^{C}.$

as

$\dot{\omega}_{BC}^{A} = \frac{1}{2}\left(f_{B}{}^{A}{}_{C} + f_{C}{}^{A}{}_{B} - f_{BC}^{A}\right).$

$$R_{B\mu\nu}^{A} = \partial_{\nu}\dot{\omega}_{B\mu}^{A} - \partial_{\mu}\dot{\omega}_{B\nu}^{A} + \dot{\omega}_{E\nu}^{A}\dot{\omega}_{B\mu}^{E} -$$

$\dot{\omega}_{E\mu}^{A}\dot{\omega}_{B\nu}^{E} \equiv 0,$

$T_{\nu\mu}^{A} = \partial_{\nu}e_{\mu}^{A} - \partial_{\mu}e_{\nu}^{A} + \dot{\omega}_{E\nu}^{A}e_{\mu}^{E} - \dot{\omega}_{E\mu}^{A}e_{\nu}^{E}.$

$g_{\mu\nu}(\varphi(p)) = \eta_{\mu\nu}, \partial_{\lambda}g_{\mu\nu}(\varphi(p)) = 0.$

$\frac{\mathrm{d}^2\xi^{\alpha}}{\mathrm{d}s^2} = 0,$

$\frac{\mathrm{d}^2x^{\lambda}}{\mathrm{d}s^2} + \mathring{\Gamma}_{\mu\nu}^{\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} = 0,$

$\mathring{\Gamma}_{\mu\nu}^{\lambda} := \frac{\partial x^{\lambda}}{\partial \xi^{\sigma}}\frac{\partial^2\xi^{\sigma}}{\partial x^{\mu}\partial x^{\nu}},$

$\left[\mathring{\nabla}_{\mu}, \mathring{\nabla}_{\nu}\right]v^{\alpha} = \mathring{R}_{\beta\mu\nu}^{\alpha}v^{\beta}.$

$\mathring{R}_{\mu\nu\alpha\beta} = -\mathring{R}_{\nu\mu\alpha\beta},$

$\mathring{R}_{\mu\nu\alpha\beta} = \mathring{R}_{\alpha\beta\mu\nu}.$

$T^{\mu} = \frac{\partial x^{\mu}}{\partial t}, S^{\mu} = \frac{\partial x^{\mu}}{\partial s}.$

$V^{\mu} = T^{\nu}\mathring{\nabla}_{\nu}T^{\mu},$

$A^{\mu} = T^{\nu}\mathring{\nabla}_{\nu}V^{\mu}.$

$A^{\mu} = \mathring{R}^{\mu}{}_{\lambda\alpha\beta}T^{\lambda}T^{\alpha}S^{\beta},$

$S_{\mathrm{GR}} := \frac{c^4}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(\mathcal{L}_{\mathrm{GR}} + \mathcal{L}_{\mathrm{m}}),$

$\mathring{G}_{\mu\nu} := \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \frac{8\pi G}{c^4}T_{\mu\nu},$

$$T^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}}$$

$$\mathrm{d}e^C + \overset{\circ}{\omega}{}^A_B \wedge e^B = 0,$$

$$\overset{\circ}{\omega}_{AB} + \overset{\circ}{\omega}_{BA} = \mathrm{d}g_{AB},$$

$$\mathrm{d}\overset{\circ}{\omega}{}^A_B + \overset{\circ}{\omega}{}^A_C \wedge \overset{\circ}{\omega}{}^C{}_B = \frac{1}{2}\overset{\circ}{r}_{BCD} e^C \wedge e^D,$$

$$\overset{\circ}{\omega}{}^A_{B\mu} := e^A_\nu \overset{\circ}{\nabla}_\mu e^\nu_B,$$

$$\overset{\circ}{f}{}^A_{BC} := \dot{\gamma}^A_{BC} - \dot{\gamma}^A_{CB},$$

$$\mathrm{d}g_{AB} = \partial_C g_{AB} e^C,$$

$$\dot{\gamma}^A_{BC} = \frac{1}{2}\left(\overset{\circ}{f}{}^A_{BC} - g_{CL} g^{AM} \overset{\circ}{f}{}^L_{BM} - g_{BL} g^{AM} \overset{\circ}{f}{}^L_{CM}\right)$$

$$+\overset{\circ}{\Gamma}{}^A_{BC}.$$

$$\begin{aligned}\overset{\circ}{r}{}^A_{BCD} =& \partial_D \overset{\circ}{\gamma}{}^A_{BC} - \partial_C \overset{\circ}{\gamma}{}^A_{BD} + \overset{\circ}{\gamma}{}^A_{CM} \overset{\circ}{\gamma}{}^M_{DB} \\ & -\dot{\gamma}^A_{DM} \overset{\circ}{\gamma}{}^M_{CB} - \dot{\gamma}^A_{MB} \dot{\gamma}^M_{CD}.\end{aligned}$$

$$x^A \longrightarrow \bar{x}^A = x^A + \varepsilon^A(x^\mu),$$

$$P_A := \partial_A.$$

$$[P_A, P_B] \equiv [\partial_A, \partial_B] = 0.$$

$$\delta \bar{x}^A = \varepsilon(x^\mu)^B \partial_B x^A = \varepsilon(x^\mu)^A.$$

$$\delta_\varepsilon \Psi = \epsilon^A(x^\mu) \partial_A \Psi.$$

$$\partial_\varepsilon(\partial_\mu \Psi) = \varepsilon^A \partial_A(\partial_\mu \Psi).$$

$$\partial_\varepsilon(\partial_\mu \Psi) = \underbrace{\varepsilon^A(x^\mu)\partial_A(\partial_\mu \Psi)}_{\text{correct}} + \underbrace{\left(\partial_\mu \varepsilon^A(x^\mu)\right)\partial_A \Psi}_{\text{spurious}},$$

$$e'_\mu \Psi \equiv \partial_\mu \Psi = \partial_\mu + B^A_\mu \partial_A \Psi,$$

$$\delta_\varepsilon B^A_\mu = -\partial_\mu \varepsilon^A(x^\mu).$$

$$\partial_\varepsilon(e'_\mu \Psi) = \underbrace{\varepsilon^A(x^\mu)\partial_A(\partial_\mu \Psi)}_{\text{correct}},$$

$$e_\mu = \Psi = e^A_\mu \partial_A \Psi, e^A_\mu = \partial_\mu x^A + B^A_\mu,$$

$$B^A_\mu \longrightarrow \Lambda^A_B(x) B^B_\mu.$$

$$e_\mu \Psi = \partial_\mu + \dot{\omega}^A_{B\mu} x^B \partial_A \Psi + B^A_\mu \partial_A \Psi,$$

$$e_\mu^A = \partial_\mu x^A + \dot{\omega}^A_{B\mu} x^B + B_\mu^A = \dot{\mathcal{D}}_\mu x^A + B_\mu^A.$$

$$\delta_\varepsilon B_\mu^A = -\dot{\mathcal{D}}_\mu \varepsilon^A(x^\mu).$$

$$e'^A_\mu \longrightarrow e_\mu^A,$$

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}.$$

$$\partial_\mu \Psi \to \mathcal{D}'_\mu \Psi = \partial_\mu \Psi$$

$$+\frac{1}{2} e'^A{}_\mu \left(f'_B{}^C{}_A + f'_A{}^C{}_B - f'^C_{BA}\right) S_C^B \Psi,$$

$$\partial_\mu \Psi \to \mathcal{D}_\mu \Psi = \partial_\mu \Psi$$
$$+\frac{1}{2} e^A{}_\mu \left(f_B^C{}^C{}_A + f_A C_B^C - f^C_{BA}\right) S_C^B \Psi,$$

$$\underbrace{\begin{Bmatrix} e'^A \longrightarrow e^A \\ \partial_\mu \longrightarrow \mathcal{D}_\mu \end{Bmatrix}}_{\text{grav. coupling prescription in TG}} \quad \Leftrightarrow \quad \underbrace{\eta_{\mu\nu} \longrightarrow g_{\mu\nu}}_{\text{grav. coupling prescription in GR}}$$

$$\frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = 0,$$

$$\frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = \underbrace{\Lambda_B^A(x) \frac{\mathrm{d}u^B}{\mathrm{d}\sigma}}_{\text{correct}} + \underbrace{\frac{\mathrm{d}\Lambda_B^A(x)}{\mathrm{d}\sigma} u^B}_{\text{spurious}}.$$

$$\frac{\mathrm{d}u'^A}{\mathrm{d}\sigma} = 0 \longrightarrow \frac{\mathrm{d}u^B}{\mathrm{d}\sigma} + \dot{\omega}^A_{B\mu} u^B u^\mu = 0.$$

$$\dot{\gamma}(\tau) := \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau} \partial_\mu.$$

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} := \nabla_\gamma Y^\mu \equiv \frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} Y^\alpha \frac{\mathrm{d}\gamma^\beta}{\mathrm{d}\tau} = 0,$$

$$\nabla_\gamma \dot{\gamma} \equiv \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0,$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -K^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau},$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \dot{\Gamma}^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -L^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}.$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 \frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2} \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}.$$

$$[e_\mu, e_\nu] = \hat{T}^A_{\nu\mu} \partial_A,$$

$$\hat{T}^A_{\mu\nu} = \partial_\nu B_\mu^A - \partial_\mu B_\nu^A + \dot{\omega}^A_{B\nu} B_\mu^B - \dot{\omega}^A_{B\mu} B_\nu^B$$

$$\dot{\mathcal{D}}_\mu(\dot{\mathcal{D}}_\nu x^A) - \dot{\mathcal{D}}_\nu(\dot{\mathcal{D}}_\mu x^A) \equiv 0$$

$$\hat{T}^A_{\mu\nu} = \partial_\nu e^A_\mu - \partial_\mu e^A{}_\nu + \dot{\omega}^A_{B\nu} e^B{}_\mu - \dot{\omega}^A_{B\mu} e^B{}_\nu.$$

$$\hat{T}^\lambda_{\mu\nu} = e^\lambda_A \hat{T}^\lambda_{\mu\nu} := \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}.$$

$$e^A_{(\mathrm{r})\mu} := \lim_{G\to 0} e^A{}_\mu.$$

$$\hat{T}^A_{BC}\left(e^A_\mu, \dot{\omega}^A_{B\mu}\right) = \dot{\omega}^A_{BC} - \dot{\omega}^A_{BC} - f^A_{BC}\left(e_{(\mathrm{r})}\right) = 0,$$

$$\dot{\omega}^A_{BC} = \frac{1}{2} e^C_{(\mathrm{r})\mu}\left[f_B{}^A{}_C\left(e_{(\mathrm{r})}\right) + f_C{}^A_B\left(e_{(\mathrm{r})}\right) - f^A_{BC}\left(e_{(\mathrm{r})}\right)\right].$$

$$\dot{\omega}^C_{AB} - \dot{\omega}^C_{BA} = f^C_{AB} + T^C_{AB}.$$

$$\frac{1}{2}\left(f^C_B{}_A + f_A C^C_B - f^C_{BA}\right) = \dot{\omega}^C_{BA} - \hat{K}^C_{BA},$$

$$\hat{K}^C_{BA} = \frac{1}{2}\left(\hat{T}^C_B{}_A + \hat{T}^C_A{}_B - \hat{T}^C_{BA}\right),$$

$$\dot{\omega}^C_{B\mu} - \hat{K}^C_{B\mu} = \overset{\circ}{\omega}{}^C_{B\mu},$$

$$\hat{S}^{\mu\nu}_A := \hat{K}^{\mu\nu}{}_A - e_A{}^\nu \hat{T}^\mu + e^\mu_A \hat{T}^\nu,$$

$$\begin{aligned}\hat{T} &:= \frac{1}{2}\hat{S}^{\mu\nu}_A \hat{T}^A_{\mu\nu} \\ &= \frac{1}{4}\hat{T}^\rho_{\mu\nu}\hat{T}^{\mu\nu}_\rho + \frac{1}{2}\hat{T}^\rho_{\mu\nu}\hat{T}^{\nu\mu}_\rho - \hat{T}_\mu \hat{T}^\mu,\end{aligned}$$

$$\hat{R} = \overset{\circ}{R} + \hat{T} + \frac{2}{e}\partial_\mu\left(e\hat{T}^\mu\right) = 0,$$

$$\overset{\circ}{R} = -\hat{T} - \underbrace{\frac{2}{e^2\partial_\mu(eT^\mu)}}_{\text{boundary term}}..$$

$$S_{\mathrm{TEGR}} = -\frac{c^4}{16\pi G}\int \mathrm{d}^4x e \underbrace{\mathcal{L}_{\mathrm{TEGR}}}_{-\hat{T}} + \int \mathrm{d}^4x e \mathcal{L}_m,$$

$$\hat{G}_{\mu\nu} := \frac{1}{e}\partial_\lambda\left(e\hat{S}_{\mu\nu}{}^\lambda\right) - \frac{4\pi G}{c^4}\mathrm{t}_{\mu\nu} = \frac{4\pi G}{c^4}T_{\mu\nu},$$

$$\mathrm{t}_{\mu\nu} = \frac{c^4}{4\pi G}\hat{S}_{\lambda\nu}{}^\rho \Gamma^\lambda_{\rho\mu} - g_{\mu\nu}\frac{c^4}{16\pi G}\hat{T}$$

$$\hat{S}_A{}^{\mu\nu} = -\frac{8\pi G}{c^4 e}\frac{\partial \mathcal{L}_{\mathrm{TEGR}}}{\partial\left(\partial_\nu e^A{}_\mu\right)}.$$

$$\hat{G}_{\mu\nu} := \frac{1}{e}e^A{}_\mu g_{\nu\rho}\partial_\sigma\left(e\hat{S}^{\rho\sigma}_A\right) - \hat{S}^\sigma_B{}_\nu \hat{T}^B_{\sigma\mu}$$

$$\hat{T}_{\text{gen}} := -\tfrac{c_1}{4}\hat{T}_{\alpha\mu\nu}\hat{T}^{\alpha\mu\nu} - \tfrac{c_2}{2}\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu} + c_3\hat{T}_{\alpha}\hat{T}^{\alpha},$$

$$\hat{T}_{[\mu\nu]} = e^{A}_{[\mu}g_{\nu]\rho}\hat{T}^{\rho}_{A} = 0.$$

$$\mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},0\right), \mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},\dot{\omega}^{A}_{B\mu}\right),$$

$$\mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},\dot{\omega}^{A}_{B\mu}\right) + \partial_{\mu}\left[\frac{ec^4}{8\pi G}\hat{T}^{\mu}\left(e^{A}_{\mu},\dot{\omega}^{A}_{B\mu}\right)\right] = \mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},0\right) + \partial_{\mu}\left[\frac{ec^4}{8\pi G}\hat{T}^{\mu}\left(e^{A}_{\mu},0\right)\right],$$

$$\hat{T}^{\mu}\left(e^{A}_{\mu},\dot{\omega}^{A}_{B\mu}\right) = \hat{T}^{\mu}\left(e^{A}_{\mu},0\right) - \dot{\omega}^{\mu}.$$

$$\mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},\dot{\omega}^{A}_{B\mu}\right) = \mathcal{L}_{\text{TEGR}}\left(e^{A}_{\mu},0\right) + \partial_{\mu}\left[\tfrac{ec^4}{8\pi G}\dot{\omega}^{\mu}\right].$$

$$g_{\nu\lambda}\overset{\overset{\triangledown}{\nabla}}{\mu}{}^{\lambda} = \overset{\vec{\nabla}}{\mu}v_{\nu} - v^{\lambda}\mu\nu\overset{Q}{\lambda}$$

$$T^{\lambda}\vec{\nabla}_{\lambda}v \cdot w = T^{\lambda}v^{\mu}w^{\nu}\mathring{Q}_{\lambda\mu\nu}$$

$$T^{\lambda}\vec{\nabla}_{\lambda}\left(\frac{v \cdot w}{|v||w|}\right) \neq 0$$

$$a^{\mu} := u^{\lambda}\overset{\diamond}{\nabla}_{\lambda}u^{\mu}$$

$$\tilde{a}_{\mu} := u^{\lambda}\overset{\diamond}{\nabla}_{\lambda}u_{\mu} = a_{\mu} + \lambda\overset{\diamond}{v}\mu u^{\lambda}u^{\nu}$$

$$\begin{aligned} u_{\mu}a^{\mu} &= u_{\mu}u^{\lambda}\overset{\diamond}{\nabla}_{\lambda}u^{\mu} \\ &= u^{\lambda}\overset{\overset{\triangledown}{\nabla}}{\lambda}\left(u_{\mu}u^{\mu}\right) - u^{\mu}u^{\lambda}\overset{\diamond}{\nabla}_{\lambda}u_{\mu} \\ &= \overset{\diamond}{Q}_{\lambda\mu\nu}u^{\lambda}u^{\mu}u^{\nu} + 2u_{\mu}a^{\mu} - \tilde{a}_{\mu}u^{\mu}, \end{aligned}$$

$$a^{\mu}u_{\mu} = \tilde{a}_{\mu}u^{\mu} - \overset{\diamond}{Q}_{\lambda\mu\nu}u^{\lambda}u^{\mu}u^{\nu}.$$

$$\left(\tilde{a}_{\mu} - a_{\mu}\right)u^{\mu} = \overset{\diamond}{Q}_{\lambda\mu\nu}u^{\lambda}u^{\mu}u^{\nu}$$

$$a^{\mu} = 0, \tilde{a}_{\mu} = \overset{\diamond}{Q}_{\lambda\nu\mu}u^{\lambda}u^{\nu};$$

$$\overset{\diamond}{Q}_{(\lambda\mu\nu)} = 0, \overset{\diamond}{Q}_{(\lambda\mu)\nu} = 0$$

$$S_{\text{STEGR}} := \int d^4x\sqrt{-g}\left[\frac{c^4}{16\pi G}\underbrace{\mathcal{L}_{\text{STEGR}}}_{\overset{\diamond}{Q}} + \mathcal{L}_{m}\right],$$

$$\mathring{Q} := g^{\mu\nu}\left(\mathring{L}^{\alpha}{}_{\beta\mu}\mathring{L}^{\beta}{}_{\nu\alpha} - \mathring{L}^{\alpha}{}_{\beta\alpha}\mathring{L}^{\beta}{}_{\mu\nu}\right)$$
$$= \frac{1}{4}\left(\mathring{Q}_{\alpha}\mathring{Q}^{\alpha} - \mathring{Q}_{\alpha\beta\gamma}\mathring{Q}^{\alpha\beta\gamma}\right)$$
$$+\frac{1}{2}\left(\mathring{Q}_{\alpha\beta\gamma}\mathring{Q}^{\beta\alpha\gamma} - \mathring{Q}_{\alpha}\mathring{Q}^{\alpha}\right),$$

$$\mathring{Q} = \mathring{R} + \mathring{\nabla}_{\mu}\left(\mathring{Q}^{\mu} - \bar{\mathring{Q}}^{\mu}\right),$$

$$\mathring{\nabla}_{\mu}\left(\mathring{Q}^{\mu} - \bar{\mathring{Q}}^{\mu}\right) \equiv \frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}\left(\mathring{Q}^{\mu} - \bar{\mathring{Q}}^{\mu}\right)\right]$$

$$\mathring{Q}_{\text{gen}} := c_1\mathring{Q}_{\alpha\beta\gamma}\grave{Q}^{\alpha\beta\gamma} + c_2\mathring{Q}_{\alpha\beta\gamma}\mathring{Q}^{\beta\alpha\gamma} + c_3\mathring{Q}_{\alpha}\grave{Q}^{\alpha}$$
$$+c_4\overset{\varrho}{Q}_{\alpha}\mathring{Q}^{\alpha} + c_5\mathring{Q}_{\alpha}\overset{\varrho}{Q}^{\alpha},$$

$$\mathring{P}^{\alpha}_{\mu\nu} := \frac{1}{2\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathring{Q})}{\partial\grave{Q}^{\mu\nu}_{\alpha}}$$
$$= \frac{1}{4}\mathring{Q}^{\alpha}{}_{\mu\nu} - \frac{1}{4}\mathring{Q}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{1}{4}g_{\mu\nu}\mathring{Q}^{\alpha\beta}{}_{\beta}$$
$$+\frac{1}{4}\left[\mathring{Q}^{\beta\alpha}_{\beta}g_{\mu\nu} + \frac{1}{2}\delta^{\alpha}_{(\mu}\mathring{Q}_{\nu)}{}^{\beta}{}_{\beta}\right].$$

$$\mathring{Q} := \mathring{Q}_{\alpha\mu\nu}\mathring{P}^{\alpha\mu\nu}.$$

$$\frac{1}{\sqrt{-g}}\mathring{q}_{\mu\nu} := \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathring{Q})}{\partial g^{\mu\nu}} - \frac{1}{2}\mathring{Q}g_{\mu\nu}$$
$$= \frac{1}{4}\left(2\mathring{Q}_{\alpha\beta\mu}\mathring{Q}^{\alpha\beta}{}_{\nu} - \mathring{Q}_{\mu\alpha\beta}\mathring{Q}_{\nu}{}^{\alpha\beta}\right)$$
$$-\frac{1}{4}\left(2\mathring{Q}_{\alpha}{}^{\beta}{}_{\beta}\mathring{Q}^{\alpha}{}_{\mu\nu} - \mathring{Q}_{\mu}{}^{\beta}{}_{\beta}\mathring{Q}_{\nu}{}^{\beta}{}_{\beta}\right)$$
$$-\frac{1}{2}\left(\mathring{Q}_{\alpha\beta\mu} - \mathring{Q}_{\nu}{}^{\beta\alpha}{}_{\alpha}\mathring{Q}^{\alpha}{}_{\mu\nu}\right).$$

$$\mathring{G}_{\mu\nu} := -2\nabla_{\alpha}\left(\sqrt{-g}\mathring{P}^{\alpha}_{\mu\nu}\right)$$
$$+\mathring{q}_{\mu\nu} - \frac{\sqrt{-g}}{2}\mathring{Q}_{g_{\mu\nu}} = \frac{8\pi G}{c^4}T_{\mu\nu},$$

$$\nabla_{\mu}\nabla_{\nu}\left(\sqrt{-g}\mathring{P}^{\mu\nu}_{\alpha}\right) = 0,$$

$$\Gamma^{\alpha}_{\mu\nu} := (e^{-1})^{\alpha}{}_{\beta}\partial_{\mu}e^{\beta}{}_{\nu}.$$

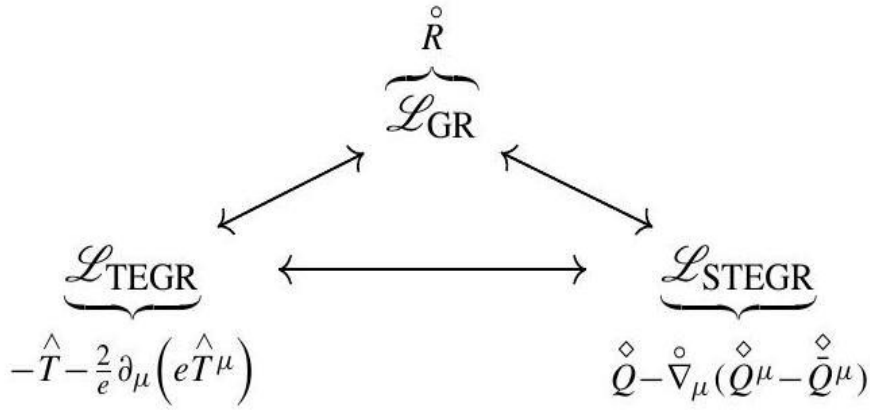
$$T^{\alpha}_{\mu\nu} := (e^{-1})^{\alpha}_{\beta}\partial_{[\nu}e^{\beta}_{\mu]} = 0,$$

$$\partial_\mu e^\beta{}_\nu = \partial_\nu e^\beta{}_\mu \;\Leftrightarrow\; e^\alpha_\beta \equiv e'^\alpha_\beta := \partial_\beta \xi^\alpha,$$

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\lambda}\partial_\mu\partial_\nu\xi^\lambda.$$

$$\xi^\alpha := M^\alpha_\beta x^\beta + \xi^\alpha_0,$$

$$\mathring{\nabla}_\mu = \partial_\mu, \mathring{L}^\lambda_{\mu\nu} = -\mathring{\Gamma}^\lambda_{\mu\nu}.$$



$$\nabla_\lambda R^\alpha_{\beta\mu\nu} + \nabla_\mu R^\alpha_{\beta\nu\lambda} + \nabla_\nu R^\alpha_{\beta\lambda\mu} = T^\rho_{\mu\lambda}R^\alpha_{\beta\nu\rho} + T^\rho_{\nu\lambda}R^\alpha_{\beta\mu\rho} + T^\rho_{\nu\mu}R^\alpha_{\beta\lambda\rho},$$

$$\mathring{\nabla}_\lambda \mathring{R}^\alpha_{\beta\mu\nu} + \mathring{\nabla}_\mu \mathring{R}^\alpha_{\beta\nu\lambda} + \mathring{\nabla}_\nu \mathring{R}^\alpha_{\beta\lambda\mu} = 0.$$

$$\partial_\lambda \mathring{R}^\lambda_{\beta\mu\nu} + \partial_\mu \mathring{R}^\lambda_{\beta\nu\lambda} + \partial_\nu \mathring{R}^\lambda_{\beta\lambda\mu} = 0.$$

$$\partial_\lambda \mathring{R}^\lambda_{\beta\mu\nu} - \partial_\mu \mathring{R}^\lambda_{\beta\lambda\nu} + \partial_\nu \mathring{R}^\lambda_{\beta\lambda\mu} = 0.$$

$$-\partial_\lambda \mathring{R}^\lambda_\nu - \partial_\beta \mathring{R}^{\;\beta}_{\;\nu} + \partial_\nu \mathring{R} = 0,$$

$$\partial_\mu \mathring{R}^\mu_\nu - \frac{1}{2}\partial_\nu \mathring{R} = 0.$$

$$\partial_\mu\left(\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}\right) = 0 \Rightarrow \mathring{\nabla}_\mu\left(\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}\right) = 0,$$

$$\mathring{\nabla}_\mu \mathring{G}^{\mu\nu} = 0, \Leftrightarrow \mathring{\nabla}_\mu T^{\mu\nu} = 0.$$

$$\hat{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} + \hat{\nabla}_\nu R^\alpha_{\beta\lambda\mu} + \hat{\nabla}_\mu R^\alpha_{\beta\nu\lambda} = 0,$$

$$\hat{\mathcal{K}}^\alpha_{\beta\mu\nu} := \mathring{\nabla}_\mu \hat{K}^\alpha_{\beta\nu} - \mathring{\nabla}_\nu \mathring{K}^\alpha_{\beta\mu} + \hat{K}^\alpha_{\sigma\mu}\hat{K}^\sigma_{\beta\nu} - \hat{K}^\alpha_{\sigma\nu}\hat{K}^\sigma_{\beta\mu},$$

$$\hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} = -\hat{\mathcal{K}}_\beta{}^\alpha{}_{\mu\nu}, \hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} = -\hat{\mathcal{K}}^\alpha{}_{\beta\nu\mu}.$$

$$\hat{\nabla}_\lambda \mathring{R}^\lambda_{\beta\mu\nu} + \hat{\nabla}_\mu \mathring{R}^\lambda_{\beta\nu\lambda} + \hat{\nabla}_\nu \mathring{R}^\lambda_{\beta\lambda\mu}$$
$$+\hat{\nabla}_\lambda \hat{\mathcal{K}}^\lambda_{\beta\mu\nu} + \hat{\nabla}_\mu \hat{\mathcal{K}}^\lambda_{\beta\nu\lambda} + \hat{\nabla}_\nu \hat{\mathcal{K}}^\lambda_{\beta\lambda\mu} = 0.$$

$$\hat{\nabla}_\mu \left(\mathring{R}^\mu_\nu + \hat{\mathcal{K}}^\mu_\nu\right) - \frac{1}{2}\hat{\nabla}_\nu(\mathring{R} + \hat{\mathcal{K}}) = 0,$$

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} = -\hat{\mathcal{K}}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \hat{\mathcal{K}},$$

$$\hat{\mathcal{K}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{K}} = 0,$$

$$\hat{\mathcal{K}}_{\mu\nu} = \mathring{\nabla}_\alpha \hat{K}^\alpha{}_{\mu\nu} - \mathring{\nabla}_\nu \hat{K}^\alpha{}_{\mu\alpha} + \hat{K}^\sigma{}_{\mu\nu} \hat{K}^\alpha{}_{\sigma\alpha} - \hat{K}^\sigma{}_{\mu\alpha} \hat{K}^\alpha{}_{\sigma\nu}$$
$$= \mathring{\nabla}_\alpha \hat{K}^\alpha_{\mu\nu} + \mathring{\nabla}_\nu \hat{T}_\mu - \hat{K}_{\sigma\mu\nu} \hat{T}^\sigma - \hat{K}^\sigma_{\mu\alpha} \hat{K}^\alpha_{\sigma\nu}$$
$$= \mathring{\nabla}_\alpha \hat{S}_\nu{}^\alpha_\mu + \mathring{\nabla}_\alpha \hat{T}^\alpha g_{\mu\nu} - \hat{K}^\alpha_{\sigma\nu} \hat{S}_\alpha{}^\sigma_\mu,$$

$$\hat{K}^\alpha{}_{\mu\alpha} = -\hat{T}_\mu,$$

$$\hat{K}^\alpha{}_{\alpha\mu} = 0,$$

$$\hat{K}^\mu_{\nu\lambda} = \hat{S}^{\mu\nu}_\lambda + \delta^\nu_\lambda \hat{T}^\mu - \delta^\mu_\lambda \hat{T}^\nu.$$

$$\hat{\mathcal{K}} = 2\mathring{\nabla}_\lambda \hat{T}^\lambda + \hat{T} = \frac{2}{e}\partial_\lambda\left(e\hat{T}^\lambda\right) + \hat{T}.$$

$$\mathring{\nabla}_\alpha \hat{S}_{\nu\mu}{}^\alpha + \hat{K}^\alpha_{\sigma\nu} \hat{S}_\alpha{}^\sigma{}_\mu + \frac{1}{2} g_{\mu\nu} \hat{T} = 0,$$

$$\partial_\lambda R^\alpha_{\beta\mu\nu} + \partial_\nu R^\alpha_{\beta\lambda\mu} + \partial_\mu R^\alpha_{\beta\nu\lambda} = 0,$$

$$\mathring{\mathcal{L}}^\alpha{}_{\beta\mu\nu} = \mathring{\nabla}_\mu \mathring{L}^\alpha{}_{\beta\nu} - \mathring{\nabla}_\nu \mathring{L}^\alpha{}_{\beta\mu} + \mathring{L}^\alpha_\alpha \mathring{L}^\sigma{}_{\beta\nu} - \mathring{L}^\alpha{}_{\sigma\nu} \mathring{L}^\sigma{}_{\beta\mu},$$

$$\overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}{}^\beta{}_{\alpha\mu\nu}, \qquad \overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}{}^\alpha{}_{\beta\nu\mu},$$

$$\partial_\lambda \mathring{R}_{\beta\mu\nu} + \partial_\mu \mathring{R}^\lambda_{\beta\nu\lambda} + \partial_\nu \mathring{R}_{\beta\lambda\mu}$$
$$+\partial_\lambda \mathring{\mathcal{L}}^\lambda_{\beta\mu\nu} + \partial_\mu \mathring{\mathcal{L}}^\lambda_{\beta\nu\lambda} + \partial_\nu \mathring{\mathcal{L}}^\lambda_{\beta\lambda\mu} = 0.$$

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} = -\mathring{\mathcal{L}}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \mathring{\mathcal{L}},$$

$$\mathring{\mathcal{L}}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathring{\mathcal{L}} = 0,$$

$$\begin{aligned}\mathring{\mathcal{L}}_{\mu\nu} &= \mathring{\nabla}_\alpha \mathring{\mathcal{L}}^\alpha{}_{\mu\nu} - \mathring{\nabla}_\nu \mathring{\mathcal{L}}^\alpha{}_{\mu\alpha} + \mathring{\mathcal{L}}^\sigma{}_{\mu\nu} \mathring{\mathcal{L}}^\alpha{}_{\sigma\alpha} - \mathring{L}^\sigma{}_{\mu\alpha} \mathring{\mathcal{L}}^\alpha{}_{\sigma\nu} \\ &= \mathring{\nabla}_\alpha \mathring{\mathcal{L}}^\alpha{}_{\mu\nu} + \frac{1}{2} \circ_\nu{}_\nu \mathring{Q}_\mu - \frac{1}{2} \mathring{Q}_\alpha \mathring{\mathcal{L}}^\alpha{}_{\mu\nu} \\ &- \frac{1}{4}\left[\mathring{Q}_\mu{}^\sigma{}_\alpha \mathring{Q}_\nu{}^\alpha{}_\sigma + 2\mathring{Q}^\alpha{}_{\sigma\nu}\left(\mathring{Q}^\sigma{}_{\alpha\mu} - \mathring{Q}_\alpha{}^\sigma{}_\mu\right)\right],\end{aligned}$$

$$\begin{aligned}\mathring{L}^\alpha_{\mu\alpha} &= -\frac{1}{2}\mathring{Q}_\mu \\ \mathring{L}^\alpha{}_{\mu\nu} &= 2P^\alpha_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\left(\mathring{Q}^\alpha - \mathring{\tilde{Q}}^\alpha\right) \\ &- \frac{1}{4}\left(\delta^\alpha_\mu \mathring{Q}_\nu + \delta^\alpha_\nu \mathring{Q}_\mu\right)\end{aligned}$$

$$\begin{aligned}\mathring{\mathcal{L}} &= \mathring{\nabla}_\alpha \left(\mathring{Q}^\alpha - \mathring{Q}^\alpha\right) + \frac{1}{4}\mathring{Q}_{\alpha\beta\gamma}\mathring{Q}^{\alpha\beta\gamma} - \frac{1}{2}\mathring{Q}_{\alpha\beta\gamma}\mathring{Q}^{\gamma\beta\alpha} \\ &- \frac{1}{4}\mathring{Q}_\alpha \mathring{Q}^\alpha + \frac{1}{2}\mathring{Q}_\alpha \mathring{Q}^\alpha \\ &= \mathring{\nabla}_\alpha \left(\mathring{Q}^\alpha - \mathring{Q}^\alpha\right) - \mathring{Q}.\end{aligned}$$

$$\partial_\alpha \mathring{Q}^\alpha = \mathring{\nabla}_\alpha \mathring{Q}^\alpha + \mathring{L}^\alpha{}_{\sigma\alpha}\mathring{Q}^\sigma = \mathring{\nabla}_\alpha \mathring{Q}^\alpha - \frac{1}{2}\mathring{Q}_\alpha \mathring{Q}^\alpha,$$

$$\begin{aligned}&2\partial_\alpha P^\alpha{}_{\mu\nu} + \frac{1}{2}\mathring{Q}_{\alpha\mu\nu}\left(\mathring{Q}^\alpha - \mathring{\tilde{Q}}^\alpha\right) + \frac{1}{2} g_{\mu\nu}\partial_\alpha\left(\mathring{Q}^\alpha - \mathring{\tilde{Q}}^\alpha\right) \\ &+ \frac{1}{2}\mathring{L}^\sigma{}_{\mu\nu}\mathring{Q}_\sigma + \frac{1}{4}\mathring{Q}_\mu{}^\alpha{}_\sigma \mathring{Q}_\nu{}^\sigma{}_\alpha + \frac{1}{2}\mathring{Q}^\alpha{}_{\sigma\mu}\left(\mathring{Q}^\sigma{}_{\nu\alpha} - \mathring{Q}_\alpha{}^\sigma{}_\nu\right) \\ &- \frac{1}{2} g_{\mu\nu}\mathring{\nabla}_\alpha\left(\mathring{Q}^\alpha - \mathring{Q}_\alpha\right) + \frac{1}{2} g_{\mu\nu}\mathring{Q},\end{aligned}$$

$$\frac{2}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\mathring{P}^\alpha_{\mu\nu}\right) - \frac{1}{\sqrt{-g}}\mathring{q}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\mathring{Q} = 0. \ldots$$

$$\mathrm{d}s^2 = -e^{\nu(t,r)}\mathrm{d}t^2 + e^{\lambda(t,r)}\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2,$$

$$-e^{\nu(t,r)} \approx -1 + \tfrac{2M}{r},$$

$$\mathrm{d}s^2 = -\left(1 - \tfrac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \tfrac{2M}{r}\right)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2,$$

$$\dot{f}(t,r) := \tfrac{\mathrm{d}f(t,r)}{\mathrm{d}t}, f'(t,r) := \tfrac{\mathrm{d}f(t,r)}{\mathrm{d}r}.$$

$$\overset{\circ}{G}_{\mu\nu} \equiv \overset{\circ}{R}_{\mu\nu} = 0,$$

$$\overset{\circ}{G}_{tr} \equiv \frac{\dot{\lambda}(t,r)}{r} = 0, \Rightarrow \ \lambda = \lambda(r).$$

$$\overset{\circ}{G}_{rr} \equiv -e^{\lambda(r)} + r\nu'(t,r) + 1 = 0,$$

$$\overset{\circ}{G}_{tt} \equiv e^{-\lambda(r)}(r\lambda'(r) - 1) + 1 = 0.$$

$$\left[e^{-\lambda(r)}r\right]' = 1 \Rightarrow e^{-\lambda(r)} = 1 - \frac{C_1}{r},$$

$$\lambda'(r) + \nu'(r) = 0, \Rightarrow \ \lambda(r) + \nu(r) = C_2,$$

$$-e^{\nu(r)} = 1 - \frac{2M}{r}, e^{\lambda(r)} = \frac{1}{1-\frac{2M}{r}}.$$

$$e_\mu^A = \begin{pmatrix} \sqrt{-e^{\nu(r)}} & 0 & 0 & 0 \\ 0 & \sqrt{e^{\lambda(r)}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\,\theta \end{pmatrix}.$$

$$\hat{T}_{tr}^{t} = -\frac{1}{2}\nu'(r) = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1},$$

$$\hat{T}_{r\varphi}^{\varphi} = \frac{1}{r}.$$

$$\hat{K}_{ttr} = \frac{1}{2}e^{\nu(r)}\nu'(r) = \frac{M}{r^2}$$
$$\hat{K}_{\varphi r\varphi} = r$$

$$\hat{S}_{\hat{t}}^{tr} = \frac{2e^{-\lambda(r)}\sqrt{e^{-\nu(r)}}}{r} = \frac{2}{r}\sqrt{1 - \frac{2M}{r}},$$

$$\hat{S}_{\hat{\varphi}}^{r\varphi} = -\frac{e^{-\lambda(r)}(r\nu'(r)+2)}{2r^2} = \frac{M-r}{r^3}.$$

$$\hat{T} = -\frac{2e^{-\lambda(r)}(r\nu'(r)+1)}{r^2} = -\frac{2}{r^2},$$

$$\overset{\circ}{Q}_{r\mu\nu} = \begin{pmatrix} -e^{\nu(r)}\nu'(r) & 0 & 0 \\ 0 & e^{\lambda(r)}\lambda'(r) & 0 \\ 0 & 0 & 2r \end{pmatrix}$$
$$= \begin{pmatrix} -\dfrac{2M}{r^2} & 0 & 0 \\ 0 & -\dfrac{2M}{r^2\left(1 - \dfrac{2M}{r}\right)^2} & 0 \\ 0 & 0 & 2r \end{pmatrix},$$

$$\overset{\circ}{P}{}^{t}_{tr} = \frac{r\lambda'(r) - rv'(r) + 4}{8r} = \frac{1}{8}(\lambda'(r) + v'(r)),$$
$$\overset{\circ}{P}{}^{r}_{rr} = \frac{e^{v(r)-\lambda(r)}}{r} = \frac{1}{r}\left(1 - \frac{2M}{r}\right)^2,$$
$$\overset{\circ}{P}{}^{r}_{\varphi\varphi} = -\frac{1}{4}re^{-\lambda(r)}(rv'(r) + 2) = \frac{M - r}{2},$$
$$\overset{\circ}{P}{}^{\varphi}_{r\varphi} = \frac{1}{8}(\lambda'(r) + v'(r)) = 0,$$

$$\frac{\overset{\circ}{q}_{\mu\nu}}{\sqrt{-g}} = \begin{pmatrix} \frac{2e^{v(r)-\lambda(r)}v'(r)}{r} & 0 & 0 \\ 0 & \frac{2rv'(r)+2}{r^2} & 0 \\ 0 & 0 & -\frac{rv'(r)+2}{e^{\lambda(r)}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4M}{r^3}\left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & \frac{2}{r^2\left(1-\frac{2M}{r}\right)} & 0 \\ 0 & 0 & \frac{2M}{r} - 2 \end{pmatrix}.$$

$$e_\mu^A g_{\nu\rho}\partial_\sigma \hat{S}_A{}^{\rho\sigma} + e^{-1}e_\mu^A g_{\nu\rho}\partial_\sigma e$$
$$-\hat{S}_B{}^{\sigma}{}_{\nu}T^B{}_{\sigma\mu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0.$$

$$\partial_\sigma \hat{S}_{\mu\nu}{}^{\sigma} - \hat{S}_{\alpha\nu}{}^{\sigma}\Gamma^\alpha{}_{\nu\sigma} - \hat{S}_{\alpha\mu}{}^{\sigma}\Gamma^\alpha{}_{\nu\sigma} - \hat{S}_\mu{}^{\rho\sigma}\Gamma^\alpha{}_{\rho\sigma}g_{\nu\alpha}$$
$$-\hat{S}_\alpha{}^{\sigma}{}_{\nu}\hat{T}^\alpha_{\sigma\mu} + \Gamma^\alpha_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\sigma} + \frac{1}{2}\hat{T}g_{\mu\nu} = 0.$$

$$\overset{\circ}{\nabla}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} - \hat{K}^\alpha{}_{\mu\sigma}\hat{S}_{\alpha\nu}{}^{\sigma} - \hat{K}^\alpha{}_{\nu\sigma}\hat{S}_{\mu\alpha}{}^{\sigma}$$
$$+\hat{K}^\sigma{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\alpha} - \hat{K}_{\nu\rho\sigma}\hat{S}_\mu{}^{\rho\sigma} + \hat{T}_\sigma \hat{S}_{\mu\nu}{}^{\sigma}$$
$$-\hat{S}_\alpha{}^{\sigma}{}_{\nu}\hat{T}^\alpha{}_{\sigma\mu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0.$$

$$-\hat{K}^\alpha{}_{\nu\sigma}\hat{S}_{\mu\alpha}{}^{\sigma} - \hat{K}_{\nu\rho\sigma}\hat{S}_\mu{}^{\rho\sigma} = 0,$$

$$\hat{K}^\sigma{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^{\alpha} + \hat{T}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} = 0.$$

$$\overset{\circ}{\nabla}\hat{S}_{\mu\nu}{}^{\sigma} + \hat{K}^\alpha_{\mu\sigma}\hat{S}_{\alpha\ \nu} - \hat{T}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\ \nu} + \frac{1}{2}\hat{T}g_{\mu\nu} = \overset{\circ}{\nabla}_\sigma \hat{S}_{\mu\nu}{}^{\sigma} + \hat{K}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\ \nu\ \nu} + \frac{1}{2}g_{\mu\nu}\hat{T} = 0$$

$$\frac{\partial_\alpha\sqrt{-g}}{\sqrt{-g}} = \overset{\circ}{\Gamma}{}^{\sigma}_{\alpha\sigma} = -\overset{\circ}{L}{}^{\sigma}_{\alpha\sigma} = \frac{1}{2}\overset{\circ}{Q}_\alpha.$$

$$
\begin{aligned}
&2\partial_\alpha P^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\mu\nu}\left(\overset{\diamond}{Q}{}^\alpha - \overset{\diamond}{\tilde{Q}}{}^\alpha\right) + \frac{1}{2}g_{\mu\nu}\partial_\alpha\left(\overset{\diamond}{Q}{}^\alpha - \overset{\diamond}{\tilde{Q}}{}^\alpha\right) \\
&+\frac{1}{2}\overset{\diamond}{L}{}^\sigma{}_{\mu\nu}\overset{\diamond}{Q}_\sigma + \frac{1}{4}\overset{\diamond}{Q}_\mu{}^\alpha{}_\sigma\overset{\diamond}{Q}_\nu{}^\sigma{}_\alpha + \frac{1}{2}\overset{\diamond}{Q}{}^\alpha{}_{\sigma\mu}\left(\overset{\diamond}{Q}{}^\sigma{}_{\nu\alpha} - \overset{\diamond}{Q}_\alpha{}^\sigma{}_\nu\right) \\
&-\frac{1}{2}g_{\mu\nu}\overset{\circ}{\nabla}_\alpha\left(\overset{\diamond}{Q}{}^\alpha - \overset{\diamond}{\tilde{Q}}{}^\alpha\right) + \frac{1}{2}g_{\mu\nu}{}^Q = 0.
\end{aligned}
$$

$$\mathbb{P}(x_n, t_n \mid x_1, t_1; \dots; x_{n-1}, t_{n-1}) = \mathbb{P}(x_n, t_n \mid x_{n-1}, t_{n-1})$$

$$\hat{\mathcal{E}}(t)(\hat{\rho}(0)) = \hat{\rho}(t)$$

$$\hat{\mathcal{E}}(t_1 + t_2) = \hat{\mathcal{E}}(t_1)\hat{\mathcal{E}}(t_2),$$

$$\frac{d\hat{\rho}}{dt}(t) = \hat{\mathcal{L}}\hat{\rho}(t)$$

$$\hat{H} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{SB}}$$

$$\hat{H}_{\mathrm{S}} = \frac{\hat{P}^2}{2M} + V(\hat{Q}),$$

$$\hat{H}_{\mathrm{B}}^{(k)} = \sum_{k=1}^{N}\left(\frac{\hat{p}_k^2}{2m_k} + \frac{m_k\omega_k^2\hat{q}_k^2}{2}\right),$$

$$\hat{H}_{\mathrm{SB}}^{(k)} = \sum_{k=1}^{N}\left(g_k\hat{q}_k\hat{Q} + \frac{g_k^2}{2m_k\omega_k^2}\hat{Q}^2\right).$$

$$\frac{d^2\hat{Q}}{dt^2}(t) + V'(\hat{Q}(t)) + \frac{1}{M}\int_0^t k(t-t')\hat{P}(t')dt' + k(t)\hat{Q}(0) = \hat{f}(t)$$

$$k(t) = \sum_{i=1}^{N}\frac{g_k^2}{m_k\omega_k^2}\cos(\omega_k t)$$

$$\hat{f}(t) = \sum_{i=1}^{N}\left(g_k\hat{q}_k(0)\cos(\omega_k t) + \frac{g_k\hat{p}_k(0)}{m_k\omega_k}\sin(\omega_k t)\right)$$

$$\langle\hat{f}(t)\rangle = \mathrm{Tr}\big(\hat{f}(t)\rho_{\mathrm{th}}\big) = 0$$

$$\left\langle\{\hat{f}(t), \hat{f}(t')\}\right\rangle = \sum_{k=1}^{N}\frac{\hbar g_k^2}{m_k\omega_k}\coth\left(\frac{\hbar\omega_k}{2k_B T}\right)\cos\big(\omega_k(t-t')\big),$$

$$\hat{\Theta}\hat{\boldsymbol{q}}\hat{\Theta}^{-1} = \hat{\boldsymbol{q}}, \hat{\Theta}\hat{\boldsymbol{p}}\hat{\Theta}^{-1} = -\hat{\boldsymbol{p}},$$

$$\hat{A}_R(t) = \hat{\Theta}\hat{A}(-t)\hat{\Theta}^{-1}$$

$$\hat{\boldsymbol{q}}_R(t) = \hat{\Theta}\hat{\boldsymbol{q}}(-t)\hat{\Theta}^{-1} = \hat{\boldsymbol{q}}(t), \hat{\boldsymbol{p}}_R(t) = \hat{\Theta}\hat{\boldsymbol{p}}(-t)\hat{\Theta}^{-1} = -\hat{\boldsymbol{p}}(t).$$

$$M\frac{d^2\hat{Q}}{dt^2}(-t)+V'(\hat{Q}(-t))+\frac{1}{M}\int_0^{-t}k(-t-t')\hat{P}(t')dt'+k(-t)\hat{Q}(0)=\hat{f}(-t)$$

$$\frac{d\hat{\rho}_R}{dt}=-\hat{\Theta}\frac{d\hat{\rho}}{dt}\hat{\Theta}^{-1}=-\hat{\Theta}\hat{\mathcal{L}}\hat{\Theta}^{-1}\hat{\Theta}\hat{\rho}\hat{\Theta}^{-1}=-\hat{\mathcal{L}}_R\hat{\rho}_R.$$

$$\hat{\rho}(t)=\hat{\mathcal{E}}(t)\hat{\rho}(0),\hat{\mathcal{E}}(t)=\exp\left(-\hat{\mathcal{L}}t\right).$$

$$\hat{\Theta}\hat{\mathcal{E}}(t)\hat{\Theta}^{-1}=\exp\left(-\hat{\Theta}\hat{\mathcal{L}}\hat{\Theta}^{-1}t\right)=\exp\left(-\hat{\mathcal{L}}_Rt\right).$$

$$\hat{\Theta}\hat{\mathcal{E}}(t)\hat{\Theta}^{-1}=\exp\left(\hat{\mathcal{L}}t\right)=\hat{\mathcal{E}}(t)^{-1}.$$

$$\hat{\Theta}\hat{U}(t)\hat{\Theta}^{-1}=\hat{U}(-t)$$

$$\frac{d\hat{\rho}}{dt}=\hat{\mathcal{L}}(t)\hat{\rho}(t)$$

$$\frac{d\hat{\rho}_R}{dt}(t)=-\hat{\Theta}\frac{d\hat{\rho}}{dt}(-t)\hat{\Theta}^{-1}=-\hat{\mathcal{L}}_R(-t)\hat{\rho}_R(t)$$

$$\hat{\rho}(t)=\hat{\mathcal{E}}(t,0)\hat{\rho}(0),\hat{\mathcal{E}}(t_2,t_1)=\hat{\mathcal{T}}\exp\left(-\int_{t_1}^{t_2}\hat{\mathcal{L}}(t')dt'\right),$$

$$\hat{\Theta}\hat{\mathcal{E}}(t_1,t_2)\hat{\Theta}^{-1}=\hat{\mathcal{T}}\exp\left(-\int_{t_1}^{t_2}\hat{\Theta}\hat{\mathcal{L}}(t')\hat{\Theta}^{-1}dt'\right)=\hat{\mathcal{T}}\exp\left(-\int_{t_1}^{t_2}\hat{\mathcal{L}}_R(t')dt'\right).$$

$$\hat{\Theta}\hat{\mathcal{E}}(t_1,t_2)\hat{\Theta}^{-1}=\mathcal{T}\exp\left(-\int_{t_2}^{t_1}\hat{\mathcal{L}}(t')dt'\right)=\mathcal{E}(t_2,t_1)^{-1}.$$

$$\int_0^{\infty}k(t')dt'<\infty$$

$$\int_0^{\tau_B}k(t')dt'\approx\int_0^{\infty}k(t')dt'$$

$$\int_0^{t}k(t-t')\hat{P}(t')dt'=\int_0^{t}k(t')\hat{P}(t-t')dt'\approx\hat{P}(t)\int_0^{\tau_B}k(t')dt'=\hat{P}(t)\int_0^{\infty}k(t')dt'$$

$$\int_0^{t}k(t')dt'=\mathrm{sgn}(t)\int_0^{|t|}k(t')dt',$$

$$\int_0^{t}k(t-t')\hat{P}(t')dt'\approx\mathrm{sgn}(t)\hat{P}(t)\int_0^{\infty}k(t')dt'$$

$$M\frac{d^2\hat{Q}}{dt^2}+V'(\hat{Q}(t))+\mathrm{sgn}(t)\gamma\hat{P}(t)=\hat{f}(t).$$

$$\int_0^{\infty}k(t')dt'=M\gamma$$

$$\langle\{\hat{f}(t),\hat{f}(t')\}\rangle = \frac{\gamma M\hbar}{\pi}\int_0^{\Lambda}\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left(\omega(t-t')\right)d\omega$$

$$\langle\{\hat{f}(t),\hat{f}(t'\}\rangle = 2\gamma Mk_BT\delta(t-t').$$

$$\mathrm{Tr_S}(\hat{Y}\hat{\mu}(t)) = \mathrm{Tr_S}\left(\hat{\rho}_S\hat{Y}(t)\right)$$

$$\dot{\hat{\mu}}(t) = -\frac{i}{\hbar}\left[\hat{H}_S,\hat{\mu}(t)\right] + \frac{i}{2\hbar}[\{\gamma\mathrm{sgn}(t)\hat{P},\hat{\mu}(t)\},\hat{Q}] + \frac{i}{2\hbar}[\{\hat{f}(t),\hat{\mu}(t)\},\hat{Q}]$$

$$\hat{\rho}(t) = \langle\hat{\mu}(t)\rangle := \mathrm{Tr_B}(\hat{\mu}(t)\hat{\rho}_{\mathrm{th}}),$$

$$\dot{\hat{\rho}}(t) = -\frac{i}{\hbar}\left[\hat{H}_S,\hat{\rho}(t)\right] + \frac{i\mathrm{sgn}(t)}{2\hbar}[\{\gamma\hat{P},\hat{\rho}(t)\},\hat{Q}] - \frac{\Gamma(t)}{\hbar^2}[[\hat{\rho}(t),\hat{Q}],\hat{Q}].$$

$$\Gamma(t) = \int_0^t\langle\{\hat{f}(t),\hat{f}(t')\}\rangle dt'$$

$$\int_0^t\langle\{\hat{f}(t),\hat{f}(t')\}\rangle dt' = \mathrm{sgn}(t)\int_0^{|t|}\langle\{\hat{f}(|t|),\hat{f}(t')\}\rangle dt'$$

$$\lim_{t\to\infty}\int_0^{|t|}\langle\{\hat{f}(|t|),\hat{f}(t')\}\rangle dt' = 2\gamma Mk_BT.$$

$$\dot{\hat{\rho}}(t) = -\frac{i}{\hbar}\left[\hat{H}_S,\hat{\rho}(t)\right] + \frac{i\mathrm{sgn}(t)}{2\hbar}[\{\gamma\hat{P},\hat{\rho}(t)\},\hat{Q}] - \mathrm{sgn}(t)\frac{2\gamma Mk_BT}{\hbar^2}[[\hat{\rho}(t),\hat{Q}],\hat{Q}].$$

$$\frac{d\hat{\rho}}{dt}(t) = \left(i\hat{\mathcal{L}}_H + \mathrm{sgn}(t)\hat{\mathcal{L}}_D\right)\hat{\rho}(t)$$

$$\hat{\mathcal{L}}_H\hat{\rho}(t) = -\frac{1}{\hbar}\left[\hat{H}_S,\hat{\rho}(t)\right]$$
$$\hat{\mathcal{L}}_D\hat{\rho}(t) = \frac{i}{2\hbar}[\{\gamma\hat{P},\hat{\rho}(t)\},\hat{Q}] - \frac{2\gamma Mk_BT}{\hbar^2}[[\hat{\rho}(t),\hat{Q}],\hat{Q}].$$

$$\frac{d\hat{\rho}_R}{dt}(t) = -\hat{\Theta}\frac{d\hat{\rho}}{dt}(t)\hat{\Theta}^{-1} = \left(i\hat{\mathcal{L}}_H + \mathrm{sgn}(t)\hat{\mathcal{L}}_D\right)\hat{\rho}_R(t)$$

$$\hat{\rho}(t) = \hat{\mathcal{E}}(t)\hat{\rho}(0), \hat{\mathcal{E}}(t) = \exp\left(i\hat{\mathcal{L}}_Ht + \hat{\mathcal{L}}_D|t|\right)$$

$$\rho(p,q,t) = \frac{1}{\sqrt{\pi N(t)}}\exp\left(\frac{-(q+\mathrm{sgn}(t)A(t)p)^2}{N(t)} - B(t)p^2\right)$$

$$N(t) = \frac{mk_BT}{\hbar^2}\left(1-e^{-2\gamma|t|}\right) + \frac{e^{-2\gamma|t|}}{\sigma^2}$$
$$A(t) = \frac{i\hbar}{2\sigma^2m\gamma}e^{-\gamma|t|}\left(1-e^{-\gamma|t|}\right) - \frac{ik_BT}{2\hbar\gamma}\left(1-e^{-\gamma|t|}\right)^2$$
$$B(t) = \frac{\hbar^2}{4\sigma^2m^2\gamma^2}\left(1-e^{-\gamma|t|)}\right)^2 + \frac{\sigma^2}{4} + \frac{k_BT}{m\gamma^2}\left(2\gamma|t| - 3 + 4e^{-\gamma|t|} - e^{-2\gamma|t|}\right).$$

$$\psi(x,0)=\frac{1}{(\sigma^2\pi)^{1/4}}\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

$$S_{\mathrm{vN}}(\xi)=\frac{1-\xi}{2\xi}\log\left(\frac{1+\xi}{1-\xi}\right)-\log\left(\frac{2\xi}{1+\xi}\right)$$

$$\hat{H}=\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{B}}+\hat{H}_{\mathrm{SB}}$$

$$\hat{H}_{\mathrm{I}}(t)=e^{\frac{i}{\hbar}(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{B}})t}\hat{H}_{\mathrm{SB}}e^{\frac{-i}{\hbar}(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{B}})t}$$

$$\frac{d}{dt}\hat{\rho}_{\mathrm{I}}(t)=-\frac{i}{\hbar}\left[\hat{H}_{\mathrm{I}}(t),\hat{\rho}_{\mathrm{I}}(t)\right]$$

$$\frac{d}{dt}\hat{\rho}_{\mathrm{S}}(t)=-\frac{1}{\hbar^2}\int_0^t \mathrm{Tr}_{\mathrm{B}}\left(\left[\hat{H}_{\mathrm{I}}(t),\left[\hat{H}_{\mathrm{I}}(s),\hat{\rho}_{\mathrm{I}}(s)\right]\right]\right)ds$$

$$\frac{d}{dt}\hat{\rho}_{\mathrm{S}}(t)=-\frac{1}{\hbar^2}\int_0^t \mathrm{Tr}_{\mathrm{B}}\left(\left[\hat{H}_{\mathrm{I}}(t),\left[\hat{H}_{\mathrm{I}}(s),\hat{\rho}_{\mathrm{S}}(t)\otimes\hat{\rho}_{\mathrm{B}}\right]\right]\right)ds$$

$$\hat{H}_{\mathrm{I}}(t)=\sum_{\alpha,\omega} e^{-i\omega t}\hat{A}_\alpha(\omega)\otimes\hat{B}_\alpha(t)$$

$$\frac{d}{dt}\hat{\rho}_{\mathrm{S}}(t)=$$
$$-\frac{1}{\hbar^2}\sum_{\alpha,\beta,\omega}\left[\Gamma_{\alpha\beta}(\omega,t)\hat{A}_\alpha^\dagger(\omega)\hat{A}_\beta(\omega)\hat{\rho}_{\mathrm{S}}(t)+\Gamma_{\beta\alpha}^*(\omega,t)\hat{\rho}_{\mathrm{S}}(t)\hat{A}_\alpha^\dagger(\omega)\hat{A}_\beta(\omega)\left(\Gamma_{\alpha\beta}(\omega,t)+\Gamma_{\beta\alpha}^*(\omega,t)\right)\hat{A}_\beta(\omega)\hat{\rho}_{\mathrm{S}}(t)\hat{A}_\alpha^\dagger(\omega)\right].$$

$$\Gamma_{\alpha\beta}(\omega,t)=\int_0^t e^{i\omega s}\mathrm{Tr}\left(\hat{B}_\alpha^\dagger(t)\hat{B}_\beta(t-s)\hat{\rho}_{\mathrm{B}}\right)ds$$

$$\Gamma_{\alpha\beta}(\omega,t)+\Gamma_{\beta\alpha}^*(\omega,t)=\mathrm{sgn}(t)\int_{-|t|}^{|t|} e^{i\omega s}\mathrm{Tr}\left(\hat{B}_\alpha^\dagger(s)\hat{B}_\beta(0)\hat{\rho}_B\right)ds.$$

$$\Gamma_{\alpha\beta}(\omega,t)+\Gamma_{\beta\alpha}^*(\omega,t)\approx\mathrm{sgn}(t)\gamma_{\alpha\beta}(\omega)$$

$$\gamma_{\alpha\beta}(\omega)=\int_{-\infty}^{\infty} e^{i\omega s}\mathrm{Tr}\left(\hat{B}_\alpha^\dagger(s)\hat{B}_\beta(0)\hat{\rho}_B\right)ds$$

$$\eta_{\alpha\beta}(\omega)=\frac{1}{2i}\left(\Gamma_{\alpha\beta}(\omega,t)-\Gamma_{\beta\alpha}^*(\omega,t)\right).$$

$$\frac{d}{dt}\hat{\rho}_{\mathrm{S}}(t)=-\frac{i}{\hbar}\left[\hat{H}_{\mathrm{S}},\hat{\rho}_{\mathrm{S}}\right]+\frac{\mathrm{sgn}(t)}{\hbar^2}\sum_{\alpha,\beta,\omega}\hat{\mathcal{D}}_{\alpha\beta}(\omega)\hat{\rho}_{\mathrm{S}}(t)$$

$$\hat{H}_{\mathrm{S}}=\frac{1}{\hbar}\sum_{\alpha,\beta,\omega}\eta_{\alpha\beta}(\omega)\hat{A}_\alpha^\dagger(\omega)\hat{A}_\beta(\omega)$$

$$\hat{\mathcal{D}}_{\alpha\beta}(\omega)\hat{\rho}_{S}(t) = \gamma_{\alpha\beta}(\omega)\left(\hat{A}_{\beta}(\omega)\hat{\rho}_{S}(t)\hat{A}_{\alpha}^{\dagger}(\omega) - \frac{1}{2}\{\hat{A}_{\alpha}^{\dagger}(\omega)\hat{A}_{\beta}(\omega), \hat{\rho}_{S}(t)\}\right).$$

$$\hat{\Theta}\gamma_{\alpha\beta}(\omega)\hat{\Theta}^{-1} = \int_{-\infty}^{\infty} e^{i\omega s}\mathrm{Tr}\big(\hat{\Theta}\hat{B}_{\alpha}^{\dagger}(s)\hat{B}_{\beta}(0)\hat{\Theta}^{-1}\hat{\rho}_{B}\big)ds = \gamma_{\alpha\beta}(\omega).$$

$$\frac{d}{dt}\rho_{n}(t) = \sum_{n'\neq n}\big(W_{n,n'}\rho_{n'}(t) - W_{n',n}\rho_{n}(t)\big),$$

$$\langle n|\hat{A}_{\alpha}^{\dagger}(\omega)\hat{A}_{\beta}(\omega)|n'\rangle = \delta_{n,n'}\langle n|\hat{A}_{\alpha}|m\rangle\langle m|\hat{A}_{\beta}|n\rangle,$$

$$\frac{d}{dt}\rho_{n}(t) = \frac{\mathrm{sgn}(t)}{\hbar^{2}}\sum_{n\neq n'}\big(W_{n,n'}\rho_{n'}(t) - W_{n',n}\rho_{n}(t)\big),$$

$$W_{n',n} = \sum_{\alpha,\beta}\gamma_{\alpha\beta}(\varepsilon_{n} - \varepsilon_{n'})\langle n|\hat{A}_{\alpha}|n'\rangle\langle n'|\hat{A}_{\beta}|n\rangle$$

$$\hat{H} = \hat{H}_{0} + \lambda\hat{H}_{SB}$$

$$\frac{d}{dt}\rho_{\varepsilon,k}(t) = \frac{\lambda^{2}}{\hbar^{2}}\sum_{\varepsilon',k'}\int_{0}^{t}\Lambda_{\varepsilon,k,\varepsilon',k'}(t-s)\big(\rho_{\varepsilon',k'}(s) - \rho_{\varepsilon,k}(s)\big)ds,$$

$$\Lambda_{\varepsilon,k,\varepsilon',k'}(t) = 2\left|\langle\varepsilon,k|\hat{H}_{SB}|\varepsilon',k'\rangle\right|^{2}\cos\left(\frac{\varepsilon-\varepsilon'}{\hbar}t\right)$$

$$\int_{0}^{t}\Lambda_{\varepsilon,k,\varepsilon',k'}(t-s)ds = 2\hbar\frac{\left|\langle\varepsilon,k|\hat{H}_{SB}|\varepsilon',k'\rangle\right|^{2}}{\varepsilon-\varepsilon'}\sin\left(\frac{\varepsilon-\varepsilon'}{\hbar}t\right).$$

$$\frac{\hbar}{\varepsilon-\varepsilon'}\sin\left(\frac{\varepsilon-\varepsilon'}{\hbar}t\right) \approx \mathrm{sgn}(t)\pi\hbar\delta(\varepsilon-\varepsilon')$$

$$\frac{d}{dt}\rho_{\varepsilon,k}(t) = \mathrm{sgn}(t)\sum_{k'}\big(W_{k,k'}^{\varepsilon}\rho_{\epsilon,k'}(t) - W_{k',k}^{\epsilon}\rho_{\epsilon,k}(t)\big)$$

$$W_{k,k'}^{\varepsilon} = \frac{2\lambda^{2}}{\hbar}\left|\langle\varepsilon,k|\hat{H}_{SB}|\varepsilon,k'\rangle\right|^{2}\eta(\varepsilon)$$

$$\psi_{R}(x,a+t) = \psi^{*}(x,a-t)$$

$$\frac{dW}{dt} = -\frac{p}{M}\frac{dW}{dx} + \sum_{n=0}^{\infty}\frac{(-\hbar^{2})^{n}V^{(2n+1)}(x)}{(2n+1)!\,2^{2n}}\frac{d^{2n+1}W}{dp^{2n+1}} + \mathrm{sgn}(t)\gamma\frac{d}{dp}(pW) + \mathrm{sgn}(t)2\gamma Mk_{B}T\frac{d^{2}W}{dp^{2}}$$

$$\frac{dW}{dt} = -\frac{p}{M}\frac{dW}{dx} + \frac{dV}{dx}\frac{dW}{dp} + \mathrm{sgn}(t)\gamma\frac{d}{dp}(pW) + \mathrm{sgn}(t)2\gamma Mk_{B}T\frac{d^{2}W}{dp^{2}}$$

$$\dot{\hat{\mu}}(t) = -\frac{i}{\hbar}\big[\hat{H}_{S},\hat{\mu}(t)\big] + \frac{i}{2\hbar}[\{\gamma\mathrm{sgn}(t)\hat{P},\hat{\mu}(t)\},\hat{Q}] + \frac{i}{2\hbar}[\{\hat{f}(t),\hat{\mu}(t)\},\hat{Q}],$$

$$\hat{\rho}(t) = \langle \hat{\mu}(t) \rangle := \mathrm{Tr_B}(\hat{\mu}(t)\hat{\rho}_{\mathrm{th}}),$$

$$\hat{A}\hat{\mu}(t) = -\frac{i}{\hbar}\left[\hat{H}_s, \hat{\mu}(t)\right] + \frac{i}{2M\hbar}[\{\gamma \mathrm{sgn}(t)\hat{P}, \hat{\mu}(t)\}, \hat{Q}],$$
$$\hat{B}\hat{\mu}(t) = \frac{i}{\hbar}[\hat{\mu}(t), \hat{Q}],$$
$$\hat{\alpha}(t)\hat{\mu}(t) = \frac{1}{2}\{\hat{f}(t), \hat{\mu}(t)\},$$

$$\hat{\eta}(t) = \exp(-\hat{A}t)\hat{\mu}(t).$$

$$\dot{\hat{\eta}}(t) = \hat{B}(t)\hat{\alpha}(t)\hat{\eta}(t),$$

$$\frac{d}{dt}\langle \hat{\eta}(t) \rangle = \hat{B}(t)\langle \hat{\alpha}(t)\hat{\mu}(0) \rangle + \int_0^t \hat{B}(t)\hat{B}(t')\langle \hat{\alpha}(t)\hat{\alpha}(t')\hat{\mu}(t') \rangle dt'$$

$$\frac{d\hat{\rho}}{dt}(t) = \hat{A}\hat{\rho}(t) + \int_0^t \langle \hat{\alpha}(t)\hat{\alpha}(t')\hat{I} \rangle \hat{B}\hat{B}(t'-t)\hat{\rho}(t')dt'$$

$$\frac{d\hat{\rho}}{dt}(t) = \hat{A}\hat{\rho}(t) - \int_0^t \langle \hat{\alpha}(t)\hat{\alpha}(t')\hat{I} \rangle dt' \frac{1}{\hbar}[[\hat{\rho}(t), \hat{Q}], \hat{Q}].$$

$$\dot{\hat{\rho}}(t) = -\frac{i}{\hbar}\left[\hat{H}_S, \hat{\rho}(t)\right] + \frac{i}{2\hbar}[\{\gamma \mathrm{sgn}(t)\hat{P}, \hat{\rho}(t)\}, \hat{Q}] - \frac{\Gamma(t)}{\hbar^2}[[\hat{\rho}(t), \hat{Q}], \hat{Q}],$$

$$\sum_{k=1}^{m_1} \Pr(1 \mid [1,k]) = 1$$
$$\sum_{k=1}^{m_2} \Pr(1 \mid [2,k]) = 1$$
$$\cdots$$
$$\sum_{k=1}^{m_{n-1}} \Pr(1 \mid [n-1,k]) = 1$$
$$\sum_{k=1}^{m_n} \Pr(1 \mid [n,k]) = \begin{cases} 0, & \mathrm{NCHV}, \\ 1, & \mathrm{Q} \end{cases}$$

$$\sum_{i \in V} \Pr(1 \mid i) - \sum_{(i,j) \in E} \Pr(1{,}1 \mid i,j) \overset{\mathrm{NCHV}}{\leqslant} \alpha(G) \overset{\mathrm{Q}}{\leqslant} \vartheta(G)$$

$$\alpha(G) = n-1, \vartheta(G) = n, \text{ and } \chi(\bar{G}) = n$$

$$p_1 := \sum_{k=1}^{19} \Pr(1 \mid k) = 1$$
$$p_2 := \sum_{k=20}^{38} \Pr(1 \mid k) = 1$$

$$p_3 := \sum_{k=39}^{57} \Pr(1 \mid k) = \begin{cases} 0, & \mathrm{NCHV} \\ 1, & \mathrm{Q} \end{cases}$$

$$|\boldsymbol{a}\rangle = \begin{pmatrix} a_1 & a_2 & \cdots \\ & & a_d \end{pmatrix}^\dagger$$
$$\leftrightarrow \{|\alpha_1, \Delta t\rangle, |\alpha_2, 2\Delta t\rangle, \cdots, |\alpha_d, d\Delta t\rangle\}$$

$$\langle J \mid \boldsymbol{a} \rangle = \sum_{k=1}^{b} \langle J_k \mid \boldsymbol{a} \rangle$$

$$p_1 = 0.9939(15), p_2 = 0.9980(2), p_3 = 0.9983(2)$$

$$\sum_{i \in V} \Pr(1 \mid i) = 2.9902(4)$$

$$\alpha(G) + \sum_{(i,j) \in E} \Pr(1,1 \mid i,j) = 2.651(4)$$

$$\begin{aligned} & \max_{\mathbf{B}} \vartheta = \operatorname{tr}(\mathrm{BJ}) \\ & \quad \mathbf{B} \geqslant 0, \operatorname{tr}(\mathbf{B}) = 1 \\ & \quad B_{ij} = 0, \forall (i,j) \in E(G) \end{aligned}$$

$$h_0 = \tilde{\alpha} \operatorname{Re}\left(e^{-i\phi} \sum_{k=1} c_k e^{ik\varphi} \right)$$

$$h_1 = \tilde{\alpha} \operatorname{Re}\left(e^{-i\phi + \pi/2} \sum_{k=1} c_k e^{ik\varphi} \right)$$

$$h_2 = \tilde{\alpha} \operatorname{Re}\left(e^{-i\phi + \pi/2} \sum_{k=4} c_k e^{ik\varphi} \right)$$

$$P(1,1 \mid i,j) = P(1 \mid i) P(1 \mid j, i = 1)$$

$$\hat{\mathcal{H}}^{(N)}/\hbar = \omega_0 \hat{a}^\dagger \hat{a} + \omega_{\mathrm{a}} \left(\hat{S}_z + \frac{N}{2} \right) + \frac{2g}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \hat{S}_x$$

$$g > \frac{\sqrt{\omega_{\mathrm{a}} \omega_0}}{2}$$

$$\hat{\mathcal{H}}_{\text{spin}} = \hat{\mathcal{H}}_{\text{Fe}} + \hat{\mathcal{H}}_{\text{Er}} + \hat{\mathcal{H}}_{\text{Fe-Er}}$$

$$\hat{\mathcal{H}}_{\mathrm{Fe}} = \sum_{s=\mathrm{A,\,B}} \sum_{i=1}^{N_0} \mu_0 \mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Fe}}^{x} S_{i,x}^{s} H_x^{\mathrm{DC}} + J_{\mathrm{Fe}} \sum_{i,i'} \mathrm{S}_i^{\mathrm{A}} \cdot \mathbf{S}_{i'}^{\mathrm{B}}$$
$$-D_{\mathrm{Fe}}^{y} \sum_{i,j'} \left(S_{i,z}^{\mathrm{A}} S_{i',x}^{\mathrm{B}} - S_{i',z}^{\mathrm{B}} S_{i,x}^{\mathrm{A}} \right)$$
$$- \sum_{s=A,B} \sum_{i=1}^{N_0} \left[A_{\mathrm{Fe}}^{x} \left(S_{i,x}^{s} \right)^2 + A_{\mathrm{Fe}}^{z} \left(S_{i,z}^{s} \right)^2 \right],$$
$$\hat{\mathcal{H}}_{\mathrm{Er}} = \sum_{s=\mathrm{A,\,B}} \sum_{i=1}^{N_0} \mu_0 \mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Er}}^{x} \mathfrak{a}_{i,x}^{s} H_x^{\mathrm{DC}} + J_{\mathrm{Er}} \sum_{i,i'} \mathfrak{Z}_i^{\mathrm{A}} \cdot \mathfrak{Z}_{i'}^{\mathrm{B}}$$
$$- \sum_{s=\mathrm{A,\,B}} \sum_{i=1}^{N_0} \left[A_{\mathrm{Er}}^{x} \left(\mathfrak{Z}_{i,x}^{s} \right)^2 + A_{\mathrm{Er}}^{z} \left(\Xi_{i,z}^{s} \right)^2 \right],$$
$$\hat{\mathcal{H}}_{\mathrm{Fe-Er}} = \sum_{i=1}^{N_0} \sum_{s,s'=\mathrm{A,\,B}} \left[\mathrm{J}_i^{s} \cdot \mathrm{S}_i^{s'} + \mathrm{D}^{s,s'} \cdot \left(\mathfrak{Z}_i^{s} \times \mathrm{S}_i^{s'} \right) \right]$$

$$\langle \mathfrak{B}_{\parallel}^{s} \rangle = -\frac{1}{2} \tanh \left(\frac{\mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \left| \overline{\mathbf{B}}_{\mathrm{Er}}^{s} \right|}{2 k_{\mathrm{B}} T} \right)$$
$$\langle S_{\parallel}^{s} \rangle = -B_S \left(\frac{S \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \left| \overline{\mathbf{B}}_{\mathrm{Fe}}^{s} \right|}{k_{\mathrm{B}} T} \right)$$

$$\hbar \frac{d}{dt} \delta \mathfrak{B}^{s} = -\delta \mathfrak{s}^{s} \times \mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \overline{\mathbf{B}}_{\mathrm{Er}}^{s} \left(\mathfrak{g}^{\mathrm{A/B}}, \mathbf{S}^{\mathrm{A/B}} \right) - \bar{\mathfrak{s}}^{s} \times \mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \mathbf{B}_{\mathrm{Er}}^{s} \left(\delta \mathfrak{s}^{\mathrm{A/B}}, \delta \mathbf{S}^{\mathrm{A/B}} \right)$$
$$\hbar \frac{d}{dt} \delta \mathbf{S}^{s} = -\delta \mathbf{S}^{s} \times \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \overline{\mathbf{B}}_{\mathrm{Er}}^{s} \left(\mathfrak{g}^{\mathrm{A/B}}, \mathbf{S}^{\mathrm{A/B}} \right) - \overline{\mathbf{S}}^{s} \times \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \mathbf{B}_{\mathrm{Er}}^{s} \left(\delta \mathfrak{s}^{\mathrm{A/B}}, \delta \mathbf{S}^{\mathrm{A/B}} \right)$$

$$\frac{\hat{\mathcal{H}}_{\mathrm{Dicke}}}{\hbar} \sim \omega_0 \hat{a}^{\dagger} \hat{a} + \omega_{\mathrm{a}} \hat{\Sigma}_x^{+} + \frac{i g_z}{\sqrt{N_0}} \left(\hat{a}^{\dagger} - \hat{a} \right) \hat{\Sigma}_z^{-} + \cdots$$

$$\hbar \frac{d\mathfrak{s}^{s}}{dt} = -\mathfrak{B}^{s} \times \mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \mathbf{B}_{\mathrm{Er}}^{s} \left(\mathfrak{g}^{\mathrm{A/B}}, \mathbf{S}^{\mathrm{A/B}} \right)$$
$$\hbar \frac{d\mathbf{S}^{s}}{dt} = -\mathbf{S}^{s} \times \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \mathbf{B}_{\mathrm{Fe}}^{s} \left(\mathfrak{h}^{\mathrm{A/B}}, \mathbf{S}^{\mathrm{A/B}} \right)$$

$$\mathbf{B}_{\mathrm{Er}}^{s} = \mathbf{B}^{\mathrm{DC}} + \frac{2 z_{\mathrm{Er}} J_{\mathrm{Er}}}{\mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Er}}} \mathfrak{s}^{\bar{s}} + \frac{2}{\mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Er}}} \sum_{s'=\mathrm{A,\,B}} \left[J \mathbf{S}^{s} - \left(\mathbf{D}^{s,s'} \times \mathbf{S}^{s} \right) - \mathbf{A}_{\mathrm{Er}} \cdot \mathfrak{s}^{s} \right]$$
$$\mathbf{B}_{\mathrm{Fe}}^{s} = \mathbf{B}^{\mathrm{DC}} + \frac{2 z_{\mathrm{Fe}} \mathrm{J}_{\mathrm{Fe}}}{\mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Fe}}} \boldsymbol{s}^{\bar{s}} - \frac{z_{\mathrm{Fe}}}{\mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Fe}}} \mathbf{D}_{\mathrm{Fe}} \times \boldsymbol{s}^{\bar{s}} + \frac{2}{\mu_{\mathrm{B}} \mathfrak{g}_{\mathrm{Fe}}} \sum_{s'=\mathrm{A,\,B}} \left[J \mathbf{S}^{s} - \left(\mathbf{D}^{s,s'} \times \mathbf{S}^{s} \right) - \mathbf{A}_{\mathrm{Fe}} \cdot \mathfrak{s}^{s} \right]$$

$$\mathcal{H}_{\mathrm{Er}}^{s} = \mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \mathfrak{Z}^{s} \cdot \overline{\mathbf{B}}_{\mathrm{Er}}^{s} = \mathfrak{g}_{\mathrm{Er}} \mu_{\mathrm{B}} \mathfrak{Z}_{\parallel}^{s} \left| \overline{\mathbf{B}}_{\mathrm{Er}}^{s} \right|$$
$$\mathcal{H}_{\mathrm{Fe}}^{s} = \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \mathbf{S}^{s} \cdot \overline{\mathbf{B}}_{\mathrm{Fe}}^{s} = \mathfrak{g}_{\mathrm{Fe}} \mu_{\mathrm{B}} \mathrm{S}_{\parallel}^{s} \left| \overline{\mathbf{B}}_{\mathrm{Fe}}^{s} \right|$$

$$\langle \mathfrak{h}_{\parallel}^{s} \rangle = -\frac{1}{2}\tanh\left(\frac{\mathrm{g}_{\mathrm{Er}}\mu_{\mathrm{B}}\left|\overline{\mathbf{B}}_{\mathrm{Er}}^{s}\right|}{2k_{\mathrm{B}}T}\right)$$

$$\langle S_{\parallel}^{s} \rangle = -B_{S}\left(\frac{S\mathrm{g}_{\mathrm{Fe}}\mu_{\mathrm{B}}\left|\overline{\mathbf{B}}_{\mathrm{Fe}}^{s}\right|}{k_{\mathrm{B}}T}\right)$$

$$\hbar\frac{d}{dt}\delta\mathfrak{s}^{s} = -\delta\mathfrak{s}^{s}\times\mathrm{g}_{\mathrm{Er}}\mu_{\mathrm{B}}\overline{\mathbf{B}}_{\mathrm{Er}}^{s}\left(\mathrm{g}^{\mathrm{A/B}},\mathbf{S}^{\mathrm{A/B}}\right) - \bar{\mathfrak{s}}^{s}\times\mathrm{g}_{\mathrm{Er}}\mu_{\mathrm{B}}\mathbf{B}_{\mathrm{Er}}^{s}\left(\delta 3^{\mathrm{A/B}},\delta\mathbf{S}^{\mathrm{A/B}}\right)$$

$$\hbar\frac{d}{dt}\delta\mathbf{S}^{s} = -\delta\mathbf{S}^{s}\times\mathrm{g}_{\mathrm{Fe}}\mu_{\mathrm{B}}\overline{\mathbf{B}}_{\mathrm{Er}}^{s}\left(\mathfrak{s}^{\mathrm{A/B}},\mathbf{S}^{\mathrm{A/B}}\right) - \overline{\mathbf{S}}^{s}\times\mathrm{g}_{\mathrm{Fe}}\mu_{\mathrm{B}}\mathbf{B}_{\mathrm{Er}}^{s}\left(\delta\mathbf{3}^{\mathrm{A/B}},\delta\mathbf{S}^{\mathrm{A/B}}\right)$$

$$C = \sum_{i}\frac{w_i}{N_i}\sum_{j\in i\mathrm{th\ mode}}\frac{\left(f_{ij}-\tilde{f}_{ij}\right)^2}{\tilde{f}_{ij}^2}$$

$$\overline{\mathbf{S}}_{0}^{\mathrm{A}} = \begin{pmatrix} S\sin\ \beta_0 \\ 0 \\ -S\cos\ \beta_0 \end{pmatrix}, \overline{\mathbf{S}}_{0}^{\mathrm{B}} = \begin{pmatrix} S\sin\ \beta_0 \\ 0 \\ S\cos\ \beta_0 \end{pmatrix}$$

$$\beta_0 = -\frac{1}{2}\arctan\left[\frac{z_{\mathrm{Fe}}D_{\mathrm{Fe}}^{y}}{z_{\mathrm{Fe}}J_{\mathrm{Fe}} - A_{\mathrm{Fe}}^{x} + A_{\mathrm{Fe}}^{z}}\right]$$

$$\widehat{\mathcal{H}}_{\mathrm{Fe}} \approx \sum_{k=0,\pi}\hbar\omega_k a_k^{\dagger}a_k + \mathrm{const}$$

$$\omega_k = \mathrm{g}_{\mathrm{Fe}}\mu_{\mathrm{B}}/\hbar\sqrt{(b\cos\ k - a)(d\cos\ k + c)}$$

$$a = [S/(\mathrm{g}_{\mathrm{Fe}}^{x}\mu_{\mathrm{B}})]\left[-A_{\mathrm{Fe}}^{z} - A_{\mathrm{Fe}}^{x} - (z_{\mathrm{Fe}}J_{\mathrm{Fe}} + A_{\mathrm{Fe}}^{z} - A_{\mathrm{Fe}}^{x})\cos\ (2\beta_0) + z_{\mathrm{Fe}}D_{\mathrm{Fe}}^{y}\sin\ (2\beta_0)\right]$$

$$b = [S/(\mathrm{g}_{\mathrm{Fe}}^{x}\mu_{\mathrm{B}})]z_{\mathrm{Fe}}J_{\mathrm{Fe}}$$

$$c = [S/(\mathrm{g}_{\mathrm{Fe}}^{x}\mu_{\mathrm{B}})]\left[(z_{\mathrm{Fe}}J_{\mathrm{Fe}} + 2A_{\mathrm{Fe}}^{z} - 2A_{\mathrm{Fe}}^{x})\cos\ (2\beta_0) + z_{\mathrm{Fe}}D_{\mathrm{Fe}}^{y}\sin\ (2\beta_0)\right]$$

$$d = [S/(\mathrm{g}_{\mathrm{Fe}}^{x}\mu_{\mathrm{B}})]\left[-z_{\mathrm{Fe}}J_{\mathrm{Fe}}\cos\ (2\beta_0) - z_{\mathrm{Fe}}D_{\mathrm{Fe}}^{y}\sin\ (2\beta_0)\right]$$

$$\delta\mathbf{S}_{i}^{\mathrm{A}} \approx \sqrt{\frac{S}{2N_0}}\begin{bmatrix} -(T_0 - T_\pi)\cos\ \beta_0 \\ (Y_0 - Y_\pi) \\ -(T_0 - T_\pi)\sin\ \beta_0 \end{bmatrix}$$

$$\delta\mathbf{S}_{i}^{\mathrm{B}} \approx \sqrt{\frac{S}{2N_0}}\begin{bmatrix} (T_0 + T_\pi)\cos\ \beta_0 \\ (Y_0 + Y_\pi) \\ -(T_0 + T_\pi)\sin\ \beta_0 \end{bmatrix}$$

$$T_k = \left(\frac{b\cos\ k - a}{d\cos\ k - a}\right)^{1/4}\frac{a_{-k}^{\dagger} + a_k}{\sqrt{2}}$$

$$Y_k = \left(\frac{d\cos\ k + c}{b\cos\ k - a}\right)^{1/4 i\left(a_{-k}^{\dagger} - a_k\right)}\frac{\sqrt{2}}{}$$

$$\mathfrak{I}_i^s = \frac{1}{N_0}\sum_{j=1}^{N_0} \mathfrak{I}_j^s$$
$$\equiv \frac{1}{N_0}\Sigma_j^s$$

$$\widehat{\mathcal{H}}_{\mathrm{Er}} \approx \mu_0\mu_{\mathrm{B}}\mathfrak{g}_{\mathrm{Er}}^x H_x^{\mathrm{DC}}\Sigma_x^+ + z_{\mathrm{Er}}J_{\mathrm{Er}}\sum_{i=1}^{N_0} \mathfrak{s}_i^{\mathrm{A}} \cdot \sum_{i'=1}^{N_0} \frac{\mathfrak{s}_{i'}^{\mathrm{B}}}{N_0}$$
$$-\sum_i \sum_{\xi=x,z} A_{\mathrm{Er}}^{\xi} \sum_{s=\mathrm{A,\,B}} \left(\frac{1}{N_0}\sum_j 3_{j,\xi}^s\right)^2$$
$$= \mu_0\mu_{\mathrm{B}}\mathfrak{g}_{\mathrm{Er}}^x H_x^{\mathrm{DC}}\Sigma_x^+ + \frac{z_{\mathrm{Er}}J_{\mathrm{Er}}}{N_0}\Sigma^{\mathrm{A}} \cdot \Sigma^{\mathrm{B}} - \sum_{s=\mathrm{A,\,B}\xi=x,z} \sum_{\mathrm{Er}} \frac{A_{\mathrm{Er}}^{\xi}}{N_0}\left(\Sigma_\xi^s\right)^2$$

$$\widehat{\mathcal{H}}_{\mathrm{Fe-Er}} = 4S\left(J\sin\,\beta_0 + D_y\cos\,\beta_0\right)\Sigma_x^+ + (-4SD_x\cos\,\beta_0)\Sigma_y^-$$
$$+\sqrt{\frac{S}{N_0}}\left[\left(J\cos\,\beta_0 - D_y\sin\,\beta_0\right)T_\pi\Sigma_x^+ + JY_0\Sigma_y^+\right.$$
$$\left.+(D_x\sin\,\beta_0)T_\pi\Sigma_y^- + D_xY_\pi\Sigma_z^- - \left(J\sin\,\beta_0 + D_y\cos\,\beta_0\right)T_0\Sigma_z^+\right]$$

$$\widehat{\mathcal{H}}_{\mathrm{Dicke}} \approx \sum_{m=\mathrm{qFM,qAFM}} \hbar\omega_m a_m^\dagger a_m + E_x\Sigma_x^+ + E_y\Sigma_y^- + \mu_0\mu_{\mathrm{B}}\mathfrak{g}_{\mathrm{Er}}^x H_x^{\mathrm{DC}}\Sigma_x^+$$
$$+\frac{z_{\mathrm{Er}}J_{\mathrm{Er}}}{N_0}\Sigma^{\mathrm{A}} \cdot \Sigma^{\mathrm{B}} - \sum_{\xi=x,zs=A,\,\mathrm{B}} \sum_0 \frac{A_{\mathrm{Er}}^{\xi}}{N_0}\left(\Sigma_\xi^s\right)^2 + \frac{\hbar g_x}{\sqrt{N_0}}\left(a_{\mathrm{qAFM}}^\dagger + a_{\mathrm{qAFM}}\right)\Sigma_x^+$$
$$+\frac{i\hbar g_y}{\sqrt{N_0}}\left(a_{\mathrm{qFM}}^\dagger - a_{\mathrm{qFM}}\right)\Sigma_y^+ + \frac{\hbar g_{y'}}{\sqrt{N_0}}\left(a_{\mathrm{qAFM}}^\dagger + a_{\mathrm{qAFM}}\right)\Sigma_y^-$$
$$+\frac{i\hbar g_z}{\sqrt{N_0}}\left(a_{\mathrm{qAFM}}^\dagger - a_{\mathrm{qAFM}}\right)\Sigma_z^- + \frac{\hbar g_{z'}}{\sqrt{N_0}}\left(a_{\mathrm{qFM}}^\dagger + a_{\mathrm{qFM}}\right)\Sigma_z^+$$

$$\hbar g_x = \sqrt{xS}\left(J\cos\,\beta_0 - D_y\sin\,\beta_0\right)\left(\frac{b+a}{d-c}\right)^{1/4} = h\sqrt{x}$$
$$\hbar g_y = \sqrt{xS}J\left(\frac{d+c}{b-a}\right)^{1/4} = h\sqrt{x}$$
$$\hbar g_{y'} = \sqrt{xS}D_x\sin\,\beta_0\left(\frac{b+a}{d-c}\right)^{1/4} = h\sqrt{x}$$
$$\hbar g_z = \sqrt{xS}D_x\left(\frac{d-c}{b+a}\right)^{1/4} = h\sqrt{x}$$
$$\hbar g_{z'} = \sqrt{xS}\left(-J\sin\,\beta_0 - D_y\cos\,\beta_0\right)\left(\frac{b-a}{d+c}\right)^{1/4} = h\sqrt{x}$$

$$x = \tanh\left[\frac{\left|E_x + \mu_0\mu_{\mathrm{B}}\mathfrak{g}_{\mathrm{Er}}^x \mathrm{He}_x^{\mathrm{DC}}\right|}{2k_{\mathrm{B}}T}\right]$$

$$\varepsilon \propto \frac{2\pi}{\hbar}\sum_{\alpha}\left|\sum_{(v_i,c_i,i=1,2,3)}\langle e_3h_3|\mathbf{e}\cdot\hat{p}\left|e_1h_1e_2h_2\right\rangle_{\alpha}\right|^2\Gamma$$

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