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TEORÍA CUÁNTICA DE CAMPOS RELATIVISTAS. FORMALIZACIÓN TEÓRICA

QUANTUM THEORY OF RELATIVISTIC FIELDS. THEORETICAL FORMALIZATION

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Teoría Cuántica de Campos Relativistas. Formalización Teórica

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RESUMEN

El propósito del presente manuscrito, es sentar las bases de la teoría cuántica de campos relativistas, deducida referencialmente en artículos iniciales y condensada en trabajos más recientes. Llámese también teoría cuántica en espacios curvos. Por tanto, para todos los efectos constantes en este artículo y en los trabajos relacionados que le preceden a éste, entiéndase por teoría cuántica en espacios o campos curvos, a la teoría cuántica de campos o espacios relativistas y viceversa, y lo propio a sus equivalentes. Esta teoría, en sentido estricto, surge como iniciativa de conciliación entre la relatividad general y la mecánica cuántica, a escala subatómica. El desarrollo matemático vinculante a la teoría en proposición, ha sido ampliamente desarrollado en trabajos previos, sin embargo, en este manuscrito, me ocuparé única y exclusivamente de los fundamentos teóricos, sin descuidar el lenguaje matemático.

Palabras clave: Campos de gauge, supersimetrías, agujeros negros cuánticos, gravedad cuántica, superpembranas

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Quantum Theory of Relativistic Fields. Theoretical Formalization

ABSTRACT

The purpose of this manuscript is to lay the foundations of the quantum theory of relativistic fields, deduced referentially in initial articles and condensed in more recent works. Also called quantum theory in curved spaces. Therefore, for all the constant effects in this article and in the related works that precede it, quantum theory in curved spaces or fields is understood to mean the quantum theory of relativistic fields or spaces and vice versa, and the same to their equivalents. This theory, in the strict sense, arises as an initiative to reconcile general relativity and quantum mechanics, at the subatomic scale. The mathematical development binding to the theory in proposition has been extensively developed in previous works, however, in this manuscript, I will deal solely and exclusively with the theoretical foundations, without neglecting mathematical language.

Keywords: Gauge fields, supersymmetries, quantum black holes, quantum gravity, supermembranes

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INTRODUCCIÓN

Es bien sabido y hasta la saciedad, que la relatividad general y la mecánica cuántica, son incompatibles, esencialmente, a propósito de lo que describen y miden. La gravedad esencialmente, al igual que los objetos interestelares, se comportan de manera distinta en relación a las interacciones que se suscitan en el espacio cuántico de que se trate, esto es, entre partículas subatómicas. Mientras que, a nivel cosmológico, la métrica es puramente determinista, pues, a escala microscópica o cuántica, poco o nada se tiene de certeza, verbigracia, el principio de indeterminación de Heisenberg, que postula que las variables dinámicas de una partícula, esto es, momento angular, momento lineal, posición, etc, son perturbativas, por lo que, no obedecen a la mecánica clásica.

Sin embargo, luego de extensos trabajos puramente matemáticos aunque marginalmente teóricos, y a través de un paralelismo parcial con la teoría de Witten, he deducido una teoría cuántica de campos, en la que, extrapolamos las ecuaciones einstenianas de campo a espacios cuánticos indiscriminados, diseñando supersimetrías que describen el comportamiento de las partículas subatómicas, especialmente en tratándose de aquellas, cuya masa es superlativa o superior, según sea el caso, o en su defecto, de aquellas que se aproximan, igual o superan la velocidad de la luz, éstas últimas, tengan o no masa. El núcleo medular de esta teoría, radica principalmente en la gravedad, entendida, no como fuerza fundamental, sino como una distorsión del espacio – tiempo cuántico provocada por una partícula cosmológica o en su defecto, por la interacción de las propias partículas subatómicas en un espacio cuántico indiscriminado. En el apartado de resultados y discusión, se explicarán cada uno de los postulados que contiene la teoría cuántica de campos relativistas, entendida como una alternativa a la teoría del todo.

RESULTADOS Y DISCUSIÓN.

En este punto, paso a explicar el modelo:

En términos generales, los campos cuánticos relativistas o curvos, comportan la distorsión tensorial del espacio – tiempo a escala subatómica. En este punto, corresponde explicar, las posibles reglas por las que, se despliega este fenómeno:

1.Por la existencia de una partícula cosmológica. Por partícula cosmológica, entendemos en sentido estricto, al gravitón, esto es, una partícula fundamental, que al contrario de lo teorizado en 1930, tratase de una partícula de naturaleza bosónica o posiblemente tratase de una partícula compañera en relación a otra partícula – gauge de origen fermiónico, con una masa superior a los 1,6 × 10−69 kg, cuyo campo permea un espacio cuántico indiscriminado, al igual que el mecanismo bosónico de Higgs, cuya diferencia, radica en que esta partícula, a diferencia del bosón de Higgs, que transfiere o dota de masa a las demás partículas subatómicas a través del contacto con el campo cuántico de Higgs, pues ésta, es decir, el gravitón, transfiere o dota de gravedad a las partículas que gozan de masa y energía cinética o potencial suficientes para entrar en contacto con el campo cuántico del gravitón (el cual es curvo), lo que provoca que éstas, llamadas también superpartículas, deformen el espacio – tiempo en el que se interrelacionan, afectando la interacción cuántica con las demás partículas fermiónicas o bosónicas, según sea el caso, que ineludiblemente carezcan de masa o tengan menos masa que la superpartícula perturbativa. Las superpartículas, cabe aclarar, pueden ser de origen bosónico o fermiónico, según sea el caso.

Por tanto, la conexión entre la geometría del espacio – tiempo y la materia, está dada por la métrica $\int_{\mathcal{T}_{\mu\nu}} \eta_{\mu\nu} + \hbar_{\mu\nu} \equiv g_{\mu\nu}$, en la que, $\hbar_{\mu\nu}$ es la materia, es decir, la superpartícula, $\eta_{\mu\nu}$, el campo cuántico del gravitón, $g_{\mu\nu}$, la gravedad soportada y transferida por el gravitón a través de su campo cuántico, el cual permea el espacio y finalmente, la constante $\mathcal{T}_{\mu\nu}$ que es el tensor stress–energy. Al encontrarnos así, en un campo cuántico curvo o relativista, es decir, perturbativo – armónico, la regla de oro de Fermi es aplicable dada la transición de estados por la transferencia de permeo, lo que, en hamiltoniano (estado inicial y estados finales), se dirá:

$$\mathcal{H}_0 u_k(x) = E_k u_k(x) = \hbar\omega_k u_k(x)$$

Aquí los valores de energía se miden a través de la frecuencia $\omega_k = E_k/\hbar$. Siendo así, tenemos:

$$\psi(x,t) = \sum_k c_k(0) e^{-i\omega_k t} u_k(x)$$

Hasta lograr el estado de equilibrio:

$$\psi(x,0) = u_i(x)$$
$$\mathcal{H} = \mathcal{H}_0 + \hat{V}(x)$$

$$u_i(x) = \sum_h d_h(0)v_h \;\rightarrow\; \psi'(x,t) = \sum_h d_h(0)e^{-iE_h^v t/\hbar}v_h(x).$$

$$\psi'(x,t) = \sum_k c_k(t)e^{-i\omega_k t}u_k(x)$$

$$i\hbar\frac{\partial\psi'}{\partial t} = \mathcal{H}_0\psi' + \hat{V}\psi'$$

$$i\hbar\sum_k \left[\dot{c}_k(t)e^{-i\omega_k t}u_k(x) - i\omega c_k(t)e^{-i\omega_k t}u_k(x)\right] = \sum_k c_k(t)e^{-i\omega_k t}\left(\mathcal{H}_0 u_k(x) + \hat{V}[u_k(x)]\right)$$

$$\sum_k \left[i\hbar\dot{c}_k(t)e^{-i\omega_k t}u_k(x) + \hbar\omega c_k(t)e^{-i\omega_k t}u_k(x)\right] = \sum_k \left[c_k(t)e^{-i\omega_k t}\hbar\omega_k u_k(x) + c_k(t)e^{-i\omega_k t}\hat{V}[u_k(x)]\right]$$

$$\sum_k i\hbar\dot{c}_k(t)e^{-i\omega_k t}u_k(x) = \sum_k c_k(t)e^{-i\omega_k t}\hat{V}[u_k(x)]$$

$$\sum_k i\hbar\dot{c}_k(t)e^{-i\omega_k t}\int_{-\infty}^{\infty} u_h^*(x)u_k(x)dx = \sum_k c_k(t)e^{-i\omega_k t}\int_{-\infty}^{\infty} u_h^*(x)\hat{V}[u_k(x)]dx$$

$$\sum_k i\hbar\dot{c}_k(t)e^{-i\omega_k t}\int_{-\infty}^{\infty} u_h^*(x)u_k(x)dx = i\hbar\dot{c}_h(t)e^{-i\omega_h t}$$

$$V_{hk} = \int_{-\infty}^{\infty} u_h^*(x)\hat{V}[u_k(x)]dx$$

$$\dot{c}_h(t) = -\frac{i}{\hbar}\sum_k c_k(t)e^{i(\omega_h-\omega_k)t}V_{hk}$$

$$c_h(t) = -\frac{i}{\hbar}\sum_k \int_0^t c_k(t')e^{i(\omega_h-\omega_k)t'}V_{hk}dt' + c_h(0)$$

$$c_h(t) = -\frac{i}{\hbar}\sum_k c_k(0)\int_0^t e^{i(\omega_h-\omega_k)t'}V_{hk}dt' + c_h(0)$$

$$c_h(t) = -\frac{i}{\hbar}\int_0^t e^{i(\omega_h-\omega_i)t'}V_{hi}dt'$$

$$c_h(t) = -\frac{i}{\hbar}V_{hi}\int_0^t e^{i\Delta\omega_h t'}dt' = -\frac{V_{hi}}{\hbar\Delta\omega_h}\left(1 - e^{i\Delta\omega_h t}\right)$$

$$P(i\rightarrow h) = \frac{4|V_{hi}|^2}{\hbar^2\Delta\omega_h^2}\sin\left(\frac{\Delta\omega_h t}{2}\right)^2$$

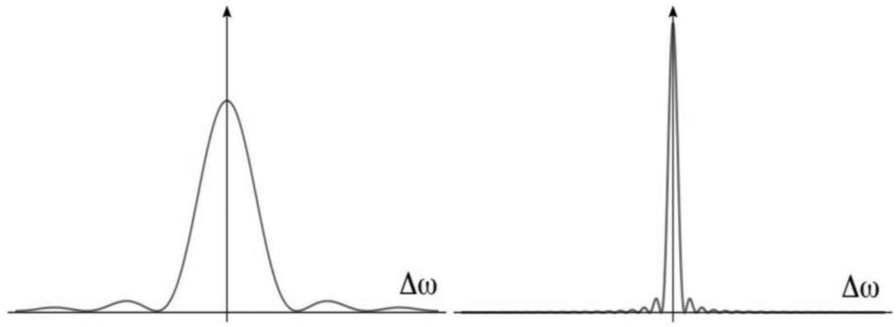
$$P(i\rightarrow h) = \frac{2\pi|V_{hi}|^2 t}{\hbar^2}\delta(\Delta\omega_h)$$

Y la tasa de transición por dispersión energética, se mide:

$$W_{ih} = \frac{dP(i\rightarrow h)}{dt}$$

$$W_{ih} = \frac{2\pi}{\hbar^2}|V_{hi}|^2\delta(\Delta\omega_h)$$

$$W_{ih} = \frac{2\pi}{\hbar}|V_{hi}|^2\rho(E_h)\Big|_{E_h=E_i}$$



En tanto que, en relación al tiempo deformado, la tasa de transición está dada por:

$$V(t) = V\Theta(t - t_0) = \begin{cases} 0 & t < t_0 \\ V & t \geq t_0 \end{cases}$$

$$b_k = -\frac{i}{\hbar}\int_{t_0}^{t} d\tau e^{i\omega_{kl}(\tau - t_0)} V_{k\ell}(\tau)$$

$$\begin{aligned} b_k &= -\frac{i}{\hbar}V_{k\ell}\int_0^t d\tau e^{i\omega_{k\ell}\tau} \\ &= -\frac{V_{k\ell}}{E_k - E_\ell}[\exp\,(i\omega_{kl}t) - 1] \\ &= -\frac{2iV_{k\ell}e^{i\omega_{kt}t/2}}{E_k - E_\ell}\sin\,(\omega_{kt}t/2) \end{aligned}$$

$$e^{i\theta} - 1 = 2ie^{i\theta/2}\sin\,(\theta/2).$$

$$\begin{aligned} P_k &= |b_k|^2 \\ &= \frac{4|V_{k\ell}|^2}{|E_k - E_\ell|^2}\sin^2\left(\frac{\omega_{k\ell}t}{2}\right) \end{aligned}$$

$$\Delta = (E_k - E_t)/2$$

$$P_k = \frac{V^2}{\Delta^2}\sin^2\,(\Delta t/\hbar)$$

$$P_k = \frac{V^2}{V^2 + \Delta^2}\sin^2\left(\sqrt{\Delta^2 + V^2}t/\hbar\right)$$

$$P_k = \frac{V^2t^2}{\hbar^2}\operatorname{sinc}^2(\Delta t/2\hbar)$$

$$\operatorname{sinc}(x) = \frac{\sin\,(x)}{x}.$$

$$\lim_{x\to 0}\operatorname{sinc}(x) = 1,$$

$$\lim_{\Delta\to 0}P_k = \frac{V^2t^2}{\hbar^2}$$

$$\Delta E \cdot \Delta t \geq 2\pi\hbar$$

$$\Delta p \cdot \Delta x \geq 2\pi\hbar.$$

$$\lim_{t\to\infty}\frac{\sin^2\,(ax/2)}{ax^2} = \frac{\pi}{2}\delta(x)$$

$$\lim_{t\to\infty}P_k(t) = \frac{2\pi}{\hbar}|V_{k\ell}|^2\delta(E_k - E_\ell)t$$

$$w_k(t) = \frac{\partial P_k(t)}{\partial t} = \frac{2\pi|V_{k\ell}|^2}{\hbar}\delta(E_k - E_\ell)$$

$$\frac{dP}{dt} = -wP.$$

$$V(t) = V\cos\,\omega t\Theta(t)$$

$$V_{k\ell}(t) = V_{k\ell}\cos\,\omega t$$
$$= \frac{V_{k\ell}}{2}\left[e^{-i\omega t} + e^{i\omega t}\right]$$
$$b_k = \frac{-i}{\hbar}\int_{t_0}^{t} d\tau V_{k\ell}(\tau)e^{i\omega_{kl}\tau}$$
$$= \frac{-iV_{k\ell}}{2\hbar}\int_{0}^{t} d\tau\left[e^{i(\omega_{kl}-\omega)\tau} + e^{i(\omega_{kl}+\omega)\tau}\right]$$
$$= \frac{-iV_{k\ell}}{2\hbar}\left[\frac{e^{i(\omega_{kl}-\omega)t}-1}{\omega_{k\ell}-\omega} + \frac{e^{i(\omega_{kt}+\omega)t}-1}{\omega_{k\ell}+\omega}\right]$$
$$e^{\theta} - 1 = 2ie^{i\theta/2}\sin\,(\theta/2)$$
$$b_k = \frac{V_{k\ell}}{\hbar}[\underbrace{\frac{e^{i(\omega_{k\ell}-\omega)t/2}\sin\,[(\omega_{k\ell}-\omega)t/2]}{\omega_{k\ell}-\omega}}_{\text{estado inicial}} + \underbrace{\frac{e^{i(\omega_{k\ell}+\omega)t/2}\sin\,[(\omega_{k\ell}+\omega)t/2]}{\omega_{k\ell}+\omega}}_{\text{estado final}}]$$
$$P_{k\ell} = |b_k|^2 = \frac{|V_{kl}|^2}{\hbar^2(\omega_{kt}-\omega)^2}\sin^2\left[\frac{1}{2}(\omega_{k\ell}-\omega)t\right]$$
$$P_{k\ell} = |b_k|^2 = \frac{|V_{kl}|^2}{\hbar^2(\omega_{kt}-\omega)^2 + |V_{kl}|^2}\sin^2\left[\frac{1}{2\hbar}\sqrt{|V_{kl}|^2 + (\omega_{k\ell}-\omega)^2}t\right]$$

$$w_{k\ell} = \frac{\pi}{2\hbar^2}|V_{k\ell}|^2[\delta(\omega_{k\ell}-\omega) + \delta(\omega_{k\ell}+\omega)]$$
$$\lim_{\Delta\omega\to 0} P_k(t) = \frac{|V_{k\ell}|^2}{4\hbar^2}t^2$$
$$t \ll \frac{2\hbar}{V_{k\ell}}$$
$$t > \frac{1}{\omega} \approx \frac{1}{\omega_{k\ell}}$$
$$V_{k\ell} \ll \hbar\omega_{k\ell}$$

Calculada la transición de estados, a propósito de la interacción entre el campo gravitónico y las partículas elementales susceptibles o repercutibles, que entran en contacto con dicho campo y por ende, engendran gravedad deformando la geometría espacio – tiempo en la que interactúan (superpartículas), procedemos en este punto, a proponer la ecuación maestra de un campo cuántico gravitónico o del gravitón:

En un campo bosónico, el lagrangiano será:

$$\mathcal{L}_{SM_{gravitón-campo\ bosónico}} = -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc}\partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \frac{1}{2} i g_s^2 \left(\bar{q}_i^g \gamma^\mu q_j^g\right) g_\mu^a +$$

$$G^a \partial^2 G^a + g_s f^{abc}\partial_\mu G^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_\sigma^c} M^2 Z_\mu^0 Z_\mu^0 -$$

$$\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 -$$

$$\frac{1}{2c^2} M \phi^0 \phi^0 - \beta_h \left[\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right] + \frac{2M^4}{g^2}\alpha_h - i g c_w [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- -$$

$$W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - i g s_w [\partial_\nu A_\mu (W_\mu^+ W_\nu^- -$$

$$W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - \frac{1}{2} g^2 W_\mu^+ W_\mu^- W_\nu^+ W_\nu^- +$$

$$\frac{1}{2} g^2 W_\mu^+ W_\nu^- W_\mu^+ W_\nu^- + g^2 c_v^2 (Z_\mu^0 W_\mu^+ Z_\nu^0 W_\nu^- - Z_\mu^0 Z_\mu^0 W_\nu^+ W_\nu^-) + g^2 s_\varphi^2 (A_\mu W_\mu^+ A_\nu W_\nu^- -$$

$$A_\mu A_\mu W_\nu^+ W_\nu^-) + g^2 s_\omega c_v [A_\mu Z_\nu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - 2 A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 +$$

$$H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8} g^2 \alpha_h [H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2 \phi^+\phi^- + 4H^2 \phi^+\phi^- +$$

$$2(\phi^0)^2 H^2] - g M W_\mu^+ W_\mu^- H - \frac{1}{2} g \frac{M}{c_g^2} Z_\mu^0 Z_\mu^0 H - \frac{1}{2} i g [W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W_\mu^- (\phi^0 \partial_\mu \phi^+ -$$

$$\phi^+ \partial_\mu \phi^0)] + \frac{1}{2} g [W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - W_\mu^- (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \frac{1}{2} g \frac{1}{c_v} Z_\mu^0 (H \partial_\mu \phi^0 -$$

$$\phi^0 \partial_\mu H) - i g \frac{g_e^2}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + i g s_w M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - i g \frac{1 - 2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- -$$

$$\phi^- \partial_\mu \phi^+) + i g s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \frac{1}{4} g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+\phi^-] -$$

$$\frac{1}{4} g^2 \frac{1}{c_s^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+\phi^-] - \frac{1}{2} g^2 \frac{s_c^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) -$$

$$\frac{1}{2} i g^2 \frac{s_m^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2} g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) + \frac{1}{2} i g^2 s_w A_\mu H (W_\mu^+ \phi^- -$$

$$W_\mu^- \phi^+) - g^2 \frac{A_\mu}{c_\omega} (2c_\omega^2 - 1) Z_\mu^0 A_\mu \phi^+ \phi^- - g^1 s_\omega^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma\partial + m_e^\lambda) e^\lambda -$$

$$\bar{v}^\lambda \gamma \partial v^\lambda - \bar{u}_j^\lambda (\gamma\partial + m_u^\lambda) u_j^\lambda - \bar{d}_j^\lambda (\gamma\partial + m_d^\lambda) d_j^\lambda + i g s_w A_\mu \left[-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) -\right.$$

$$\left.\frac{1}{3}(d_j^\lambda \gamma^\mu d_j^\lambda)\right] + \frac{ig}{4c_w} Z_\mu^0 \left[(v^\lambda \gamma^\mu (1 + \gamma^5) v^\lambda) + (e^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + \left(\overline{u}_j^\lambda \gamma^\mu \left(\frac{4}{3} s_w^2 -\right.\right.\right.$$

$$\left.\left.1 - \gamma^5\right) u_j^\lambda\right) + \left(d_j^\lambda \gamma^\mu \left(1 - \frac{8}{3} s_w^2 - \gamma^5\right) d_j^\lambda\right)\Big] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(v^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 +$$

$$\gamma^5) C_{\lambda\kappa} d_j^k)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) v^\lambda) + (d_j^k C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m^\lambda}{M} [-\phi^+ (\bar{v}^\lambda (1 -$$

$$\gamma^5) e^\lambda) + \phi^- (\tilde{e}^\lambda (1 + \gamma^5) v^\lambda)] - \frac{g}{2} \frac{m_\lambda^\lambda}{M} [H (\tilde{e}^\lambda e^\lambda) + i \phi^0 (\tilde{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 -$$

$$\gamma^5) d_j^\kappa) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 -$$

$$\gamma^5) u_j^k] - \frac{g}{2} \frac{m_n^\lambda}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_\lambda^\lambda}{M} H (d_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_j^\lambda}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_\lambda^\lambda}{M} \phi^0 (d_j^\lambda \gamma^5 d_j^\lambda) +$$

$$\bar{X}^+ (\partial^2 - M^2) X^+ + \bar{X}^- (\partial^2 - M^2) X^- + \bar{X}^0 \left(\partial^2 - \frac{M^2}{c_0^2}\right) X^0 + \bar{Y} \partial^2 Y + i g c_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- -$$

$$\partial_\mu \bar{X}^+ X^0) + i g s_w W_\mu^+ (\partial_\mu \bar{Y} X^- - \partial_\mu \bar{X}^+ Y) + i g c_w W_\mu^- (\partial_\mu \bar{X} - X^0 - \partial_\mu \bar{X}^0 X^+) +$$

$$i g s_w W_\mu^- (\partial_\mu \bar{X} - Y - \partial_\mu \bar{Y} X^+) + i g c_w Z_\mu^0 (\partial_\mu \bar{X} + X^+ - \partial_\mu \bar{X} - X^-) + i g s_w A_\mu (\partial_\mu \bar{X}^+ X^+ -$$

$$\partial_\mu \bar{X}^- X^-) - \frac{1}{2} g M \left[\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{\epsilon_0^2} \bar{X}^0 X^0 H\right] + \frac{1 - 2c_m^2}{2\omega_w} i g M [\bar{X}^+ X^0 \phi^+ -$$

$$\bar{X}^- X^0 \phi^-] + \frac{1}{2c_z} \mathrm{igM} [\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] + i g M s_w [\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] +$$

$$\frac{1}{2} \mathrm{ig} M [\bar{X}^+ X^+ \phi^0 - \bar{X}^- X^- \phi^0]$$

En un campo fermiónico, el lagrangiano será:

$$
\begin{aligned}
\mathcal{L}_{SM_{gravitón-campo\ fermiónico}} = & -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc}\partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4} g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\
& \frac{1}{2} i g_s^2 (\bar{q}_i^\sigma \gamma^\mu q_j^\sigma) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\
& M^2 W_\mu^+ W_\mu^- - \frac{1}{2} \partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_\nu^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2} \partial_\mu H \partial_\mu H - \\
& \frac{1}{2} m_h^2 H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \frac{1}{2c_0^2} M\phi^0\phi^0 - \beta_h \Big[\frac{2M^2}{g^2} + \\
& \frac{2M}{g}H + \frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)\Big] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z_\mu^0(W_\mu^+W_\nu^- - \\
& W_\nu^+W_\mu^-) - Z_\nu^0(W_\mu^+\partial_\nu W_\mu^- - W_\mu^-\partial_\nu W_\mu^+) + Z_\mu^0(W_\nu^+\partial_\nu W_\mu^- - \\
& W_\nu^-\partial_\nu W_\mu^+)] - igs_w[\partial_\nu A_\mu(W_\mu^+W_\nu^- - W_\nu^+W_\mu^-) - A_\nu(W_\mu^+\partial_\nu W_\mu^- - \\
& W_\mu^-\partial_\nu W_\mu^+) + A_\mu(W_\nu^+\partial_\nu W_\mu^- - W_\nu^-\partial_\nu W_\mu^+)] - \frac{1}{2} g^2 W_\mu^+W_\mu^-W_\nu^+W_\nu^- + \\
& \frac{1}{2} g^2 W_\mu^+W_\nu^-W_\mu^+W_\nu^- + g^2c_w^2(Z_\mu^0W_\mu^+Z_\nu^0W_\nu^- - Z_\mu^0Z_\mu^0W_\nu^+W_\nu^-) + \\
& g^2s_w^2(A_\mu W_\mu^+A_\nu W_\nu^- - A_\mu A_\mu W_\nu^+W_\nu^-) + g^2s_wc_w[A_\mu Z_\nu^0(W_\mu^+W_\nu^- - \\
& W_\nu^+W_\mu^-) - 2A_\mu Z_\mu^0W_\nu^+W_\nu^-] - g\alpha[H^3+H\phi^0\phi^0+2H\phi^+\phi^-] - \\
& \frac{1}{8} g^2\alpha_h[H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2] - \\
& gMW_\mu^+W_\mu^-H - \frac{1}{2} g\frac{M}{c^2}Z_\mu^0Z_\mu^0H - \frac{1}{2} ig[W_\mu^+(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
& W_\mu^-(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2} g[W_\mu^+(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W_\mu^-(H\partial_\mu\phi^+ - \\
& \phi^+\partial_\mu H)] + \frac{1}{2} g\frac{1}{c_\omega}Z_\mu^0(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\frac{s_\omega^2}{c_\omega}MZ_\mu^0(W_\mu^+\phi^- - W_\mu^-\phi^+) + \\
& igs_wMA_\mu(W_\mu^+\phi^- - W_\mu^-\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z_\mu^0(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + \\
& igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4} g^2W_\mu^+W_\mu^-[H^2+(\phi^0)^2+2\phi^+\phi^-] - \\
& \tfrac{1}{4} g^2\frac{1}{c_w^2}Z_\mu^0Z_\mu^0[H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-] - \frac{1}{2} g^2\frac{s_w^2}{c_w}Z_\mu^0\phi^0(W_\mu^+\phi^- + \\
& W_\mu^-\phi^+) - \frac{1}{2} ig^2\frac{s_\mu^2}{c_w}Z_\mu^0H(W_\mu^+\phi^- - W_\mu^-\phi^+) + \frac{1}{2} g^2s_wA_\mu\phi^0(W_\mu^+\phi^- + \\
& W_\mu^-\phi^+) + \frac{1}{2} ig^2s_wA_\mu H(W_\mu^+\phi^- - W_\mu^-\phi^+) - g^2c_v(2c_w^2-1)Z_\mu^0A_\mu\phi^+\phi^- - \\
& g^1s_w^2A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial+m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}_j^\lambda(\gamma\partial+m_u^\lambda)u_j^\lambda - \\
& \bar{d}_j^\lambda(\gamma\partial+m_d^\lambda)d_j^\lambda + igs_wA_\mu\Big[-(\bar{e}^\lambda\gamma^\mu e^\lambda)+\frac{2}{3}(\bar{u}_j^\lambda\gamma^\mu u_j^\lambda)-\frac{1}{3}(\bar{d}_j^\lambda\gamma^\mu d_j^\lambda)\Big] + \\
& \frac{ig}{4c_w}Z_\mu^0\Big[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda)+(\bar{e}^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda)+\Big(\bar{u}_j^\lambda\gamma^\mu\Big(\frac{4}{3} s_w^2 - \\
& 1-\gamma^5)u_j^\lambda\Big)+\Big(\bar{d}_j^\lambda\gamma^\mu\Big(1-\frac{8}{3} s_w^2-\gamma^5\Big)d_j^\lambda\Big)\Big] + \frac{ig}{2\sqrt{2}}W_\mu^+[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\vec{s}^s) + \\
& (\bar{u}_j^\lambda\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d_j^\kappa)] + \frac{ig}{2\sqrt{2}}W_\mu^-[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda)+(\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger\gamma^\mu(1+ \\
& \gamma^5)u_j^\lambda)] + \frac{ig}{2\sqrt{2}}\frac{m_\lambda^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda)+\phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \\
& \frac{2}{2}\frac{m_\lambda^\lambda}{M}[H(\bar{e}^\lambda e^\lambda)+i\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)] + \frac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}_j^\lambda C_{\lambda\kappa}(1-\gamma^5)d_j^\kappa) + \\
& m_u^\lambda(\bar{u}_j^\lambda C_{\lambda\kappa}(1+\gamma^5)d_j^\kappa] + \frac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger(1+\gamma^5)u_j^\kappa) - m_u^\kappa(\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger(1- \\
& \gamma^5)u_j^\kappa] - \frac{2}{2}\frac{m_\lambda^\lambda}{M}H(\bar{u}_j^\lambda u_j^\lambda) - \frac{2}{2}\frac{m_\lambda^\lambda}{M}H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2}\frac{m_\lambda^\lambda}{M}\phi^0(\bar{u}_j^\lambda\gamma^5u_j^\lambda) - \\
& \frac{ig}{2}\frac{m_\lambda^\lambda}{M}\phi^0(\bar{d}_j^\lambda\gamma^5 d_j^\lambda) + X^+(\partial^2-M^2)X^+ + X^-(\partial^2-M^2)X^- + X^0(\partial^2 - \\
& \frac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_wW_\mu^+(\partial_\mu\bar{X}^0X^- - \partial_\mu\bar{X}^+X^0) + igs_wW_\mu^+(\partial_\mu\bar{Y}X^- - \\
& \partial_\mu\bar{X} + Y) + igc_wW_\mu^-(\partial_\mu\bar{X} - X^0 - \partial_\mu\bar{X}^0X^+) + igs_wW_\mu^-(\partial_\mu\bar{X}^-Y - \\
& \partial_\mu\bar{Y}X^+) + igc_wZ_\mu^0(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) + igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \\
& \partial_\mu\bar{X}^-X^-) - \frac{1}{2} gM\Big[\bar{X}^+X^+H + \bar{X}^-X^-H + \frac{1}{c_c^2}\bar{X}^0X^0H\Big] + \\
& \frac{1-2c_w^2}{2c_w}igM[\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-] + \frac{1}{2c_w}igM[\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-] + \\
& igMs_w[\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-] + \frac{1}{2} igM[\bar{X} + X^+\phi^0 - \bar{X}^-X^-\phi^0]
\end{aligned}
$$

Cuya solución a las ecuaciones maestras es:

$$S=\int d^4x\mathcal{L}$$

$$\delta S=\delta\int d^4x\mathcal{L}(\phi_{\mathrm{a}}(\mathrm{x}),\partial_\mu\phi_{\mathrm{a}}(\mathrm{x}))=0$$

$$\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}=0,(a=1,2,3,\dots,n)$$

$$\pi_a(x)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}$$

$$[\phi_a(\mathbf{x},\mathrm{t}),\phi_b(\mathbf{y},\mathrm{t})]=[\pi_a(\mathbf{x},\mathrm{t}),\pi_b(\mathbf{y},\mathrm{t})]=0$$

$$[\phi_a(\mathbf{x},\mathrm{t}),\pi_b(\mathbf{y},\mathrm{t})]=i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y})$$

$$\mathcal{L}\to\mathcal{L}+\alpha\partial_\mu\mathcal{J}^\mu$$

$$\alpha\Delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi_a}(\alpha\Delta\phi_a)+\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\partial_\mu(\alpha\Delta\phi_a).$$

$$\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a\right)-\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\Delta\phi_a$$

$$=\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\Delta\phi_a$$

$$+\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\Delta\phi_a)-\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\Delta\phi_a$$

$$=\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\partial_\mu(\alpha\Delta\phi_a)$$

$$\alpha\Delta\mathcal{L}=\alpha\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\Delta\phi_a+\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a\right)$$

$$\alpha\Delta\mathcal{L}=\alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a\right)$$

$$\partial_\mu j^\mu=0,\ \text{for}\ j^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a-\mathcal{J}^\mu$$

$$\frac{\partial}{\partial t}j^0+\vec{\nabla}\cdot\vec{j}=0$$

$$\int d^3x\left(\frac{\partial}{\partial t}j^0+\vec{\nabla}\cdot\vec{j}\right)=0$$

$$Q(\mathrm{t})=\int d^3xj^0$$

$$\frac{dQ(\mathrm{t})}{dt}=\frac{d}{dt}\int d^3xj^0$$

$$\frac{dQ(\mathrm{t})}{dt}=0$$

$$Q(\mathrm{t})=\int d^3xj^0(\mathbf{x},\mathrm{t})$$

$$\psi'(\mathrm{x})=U_\theta\psi(\mathrm{x})=e^{-i\theta}\psi(\mathrm{x})$$

$$\psi'(\mathrm{x})=e^{-i\theta(\mathrm{x})}\psi(\mathrm{x})$$

$$\begin{aligned}\mathcal{L} \to \mathcal{L}' &= \bar{\psi}'(\mathrm{x})\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi'(\mathrm{x}) \\ &= e^{i\theta(\mathrm{x})}\bar{\psi}(\mathrm{x})\left(i\gamma^{\mu}\partial_{\mu} - m\right)e^{-i\theta(\mathrm{x})}\psi(\mathrm{x}) \\ &= e^{i\theta(\mathrm{x})}\bar{\psi}(\mathrm{x})\gamma^{\mu}e^{-i\theta(\mathrm{x})}\psi(\mathrm{x})\partial_{\mu}\theta(\mathrm{x}) \\ &\quad + e^{i\theta(\mathrm{x})}\bar{\psi}(\mathrm{x})e^{-i\theta(\mathrm{x})}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi'(\mathrm{x}) \\ &= \bar{\psi}(\mathrm{x})\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi'(\mathrm{x}) + \bar{\psi}(\mathrm{x})\gamma^{\mu}\psi(\mathrm{x})\partial_{\mu}\theta(\mathrm{x}) \\ &= \mathcal{L} + j^{\mu}(\mathrm{x})\partial_{\mu}\theta(\mathrm{x})\end{aligned}$$

$$\partial_{\mu} \to D_{\mu} \equiv \partial_{\mu} - iqA_{\mu}(\mathrm{x}),$$

$$\mathcal{L} \to \mathcal{L} + qj^{\mu}A_{\mu}.$$

$$\begin{aligned}\mathcal{L}(\psi, \bar{\psi}, A_{\mu}) &= \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi \\ &= \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + q\bar{\psi}\gamma_{\mu}\psi A_{\mu}\end{aligned}$$

$$A_{\mu} \to A'_{\mu} = A_{\mu} - \frac{1}{q}\partial_{\mu}\theta(\mathrm{x})$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi$$

$$\left(D_{\mu}\psi(\mathrm{x})\right)' = e^{-i\theta(\mathrm{x})}D_{\mu}\psi(\mathrm{x}).$$

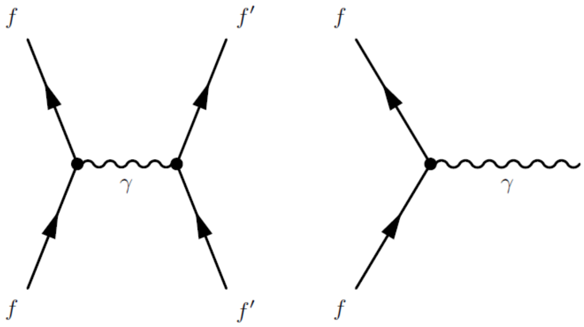
$$\mathcal{L}_{\gamma} = \frac{m^2}{2}A_{\mu}A^{\mu}$$

$$A^{\mu}A_{\mu} \to (A^{\mu} - \partial^{\mu}\theta)\left(A_{\mu} - \partial_{\mu}\theta\right) \neq A^{\mu}A_{\mu}$$

$$\mathcal{L} = -\frac{1}{4}\left(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}\right)(\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu})$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}_{QED} = \bar{\psi}(\mathrm{x})\left(i\gamma^{\mu}D_{\mu} - m\right)\psi(\mathrm{x}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$



$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2 \\ &= \bar{\psi}_1(\mathrm{x})\left(i\gamma^{\mu}\partial_{\mu} - m_1\right)\psi_1(\mathrm{x}) \\ &\quad + \bar{\psi}_2(\mathrm{x})\left(i\gamma^{\mu}\partial_{\mu} - m_2\right)\psi_2(\mathrm{x}).\end{aligned}$$

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$

$$\bar{\psi} = \begin{pmatrix}\overline{\psi_1} & \overline{\psi_2}\end{pmatrix}$$

$$M = \begin{pmatrix}m_1 & 0 \\ 0 & m_2\end{pmatrix}$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - M\right)\psi$$

$$\psi \to \psi' = U\psi$$

$$\bar{\psi} \to \psi' = \bar{\psi}U^{\dagger}$$

$$H = a_0I + a_1\tau_1 + a_2\tau_2 + a_3\tau_3 = a_0I + \boldsymbol{a}\cdot\boldsymbol{\tau}$$

$$U = e^{ia_0}e^{i\boldsymbol{a}\cdot\boldsymbol{\tau}}$$

$$\psi' = e^{i\boldsymbol{a}\cdot\boldsymbol{\tau}}\psi$$

$$\psi' = e^{-ig\frac{\tau}{2}\theta(\mathrm{x})}\psi$$

$$\bar{\psi}' = e^{ig\frac{\tau}{2}\theta(\mathrm{x})}\bar{\psi}$$

$$\partial_\mu \psi' = (\partial_\mu U)\psi + U(\partial_\mu \psi)$$

$$\begin{aligned}
\mathcal{L}' &= \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi \\
&= e^{ig\frac{\tau}{2}\theta(\mathrm{x})}\bar{\psi}(i\gamma^\mu \partial_\mu - M)e^{-ig\frac{\tau}{2}\theta(\mathrm{x})}\psi \\
&= \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi + (\partial_\mu \theta(\mathrm{x}))\bar{\psi}\left(\gamma^\mu g\frac{\tau}{2}\right)\psi \\
&= \mathcal{L} + (\partial_\mu \theta(\mathrm{x}))\bar{\psi}\left(\gamma^\mu g\frac{\tau}{2}\right)\psi.
\end{aligned}$$

$$\partial_\mu \to D_\mu = \partial_\mu - ig\vec{A}_\mu$$

$$(D_\mu \psi)' = D'_\mu \psi' = U(D_\mu \psi)$$

$$\begin{aligned}
D'_\mu \psi' &= (\partial_\mu - ig\overrightarrow{A'}_\mu)\psi' \\
&= (\partial_\mu U)\psi + U(\partial_\mu \psi) - igA'_\mu(U\psi),
\end{aligned}$$

$$\begin{aligned}
U(D_\mu \psi) &= U(\partial_\mu - ig\vec{A}_\mu)\psi \\
&= U(\partial_\mu \psi) - igU(A_\mu \psi)
\end{aligned}$$

$$igA'_\mu(U\psi) = (\partial_\mu U)\psi + igU(A_\mu \psi)$$

$$igA'_\mu(UU^{-1}U\psi) = (\partial_\mu U)U^{-1}U\psi + igU(A_\mu U^{-1}U\psi)$$

$$igA'_\mu \psi' = (\partial_\mu U)U^{-1}\psi' + igU(A_\mu U^{-1}\psi')$$

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$$

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&\to \partial_\mu\left[UA_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1}\right] - \partial_\nu\left[UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\right] \\
&= \partial_\mu(UA_\nu U^{-1}) - \frac{i}{g}(\partial_\mu \partial_\nu U)U^{-1} - \frac{i}{g}(\partial_\nu U)(\partial_\mu U^{-1}) - \partial_\nu(UA_\mu U^{-1}) + \frac{i}{g}(\partial_\nu \partial_\mu U)U^{-1} + \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) \\
&= \frac{i}{g}(\partial_\mu U\partial_\nu U^{-1} - \partial_\nu U\partial_\mu U^{-1})\partial_\mu UA_\nu U^{-1} + U\partial_\mu A_\nu U^{-1} + UA_\nu \partial_\mu U^{-1} - \partial_\nu UA_\mu U^{-1} - U\partial_\nu A_\mu U^{-1} - UA_\mu \partial_\nu U^{-1} \\
&= \frac{i}{g}(\partial_\mu U\partial_\nu U^{-1} - \partial_\nu U\partial_\mu U^{-1}) + \partial_\mu UA_\nu U^{-1} + UA_\nu \partial_\mu U^{-1} - \partial_\nu UA_\mu U^{-1} - UA_\mu \partial_\nu U^{-1} + U(\partial_\mu A_\nu - \partial_n uA_\mu)U^{-1}
\end{aligned}$$

$$\begin{aligned}
&ig(A_\mu A_\nu - A_\nu A_\mu) \\
&\to ig\left[\left(UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\right)\left(UA_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1}\right) - \left(UA_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1}\right)\left(UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\right)\right] \\
&= ig\left[UA_\mu A_\nu U^{-1} - \frac{i}{g}(\partial_\mu UA_\nu U^{-1} - UA_\mu \partial_\nu U^{-1}) + \frac{1}{g^2}\partial_\mu U\partial_\nu U^{-1} - UA_\nu A_\mu U^{-1} + \frac{i}{g}(\partial_\nu UA_\mu U^{-1} - UA_\nu \partial_\mu U^{-1}) \right. \\
&\left. - \frac{1}{g^2}\partial_\nu U\partial_\mu U^{-1}\right]
\end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= -\frac{1}{2}\mathrm{Tr}(\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}) \\
&= -\frac{1}{2}\sum_{i,j=1}^{3} \mathrm{Tr}\left(\frac{\tau^i}{2}F^i_{\mu\nu}\frac{\tau^j}{2}F^{j\mu\nu}\right) = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{YM} &= \mathcal{L}_F + \mathcal{L}_{\text{gauge}} \\
&= \bar{\psi}(i\gamma^\mu D_\mu - M)\psi - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}
\end{aligned}$$

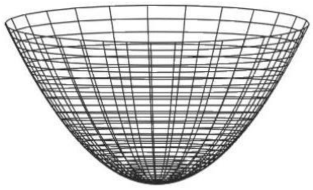
$$\mathcal{L} = \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{breaking}}.$$

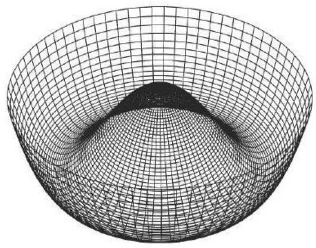
$$\mathcal{L}_o = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2\phi^\dagger \phi.$$

$$\phi(x) \leftrightarrow \phi^\dagger(x),$$

$$\mathcal{L}_I = -\frac{\lambda}{4}(\phi^\dagger \phi)^2.$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - V(\varphi_1^2 + \varphi_2^2),$$





$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \rightarrow \begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{pmatrix}\cos\,\theta & -\sin\,\theta\\ \sin\,\theta & \cos\,\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.$$

$$V(\varphi_1, \varphi_2) = V(\varphi_{01}, \varphi_{02}) + \sum_{a=1,2}\left(\frac{\partial V}{\partial\varphi_a}\right)_0(\varphi_0 - \varphi_{0a})$$

$$+\frac{1}{2}\sum_{a,b=1,2}\left(\frac{\partial^2 V}{\partial\varphi_a\partial\varphi_b}\right)_0(\varphi_a - \varphi_{0a})(\varphi_b - \varphi_{0b}) + \cdots$$

$$m_{ab}^2 = \begin{pmatrix}m^2 & 0\\ 0 & m^2\end{pmatrix},$$

$$V(\varphi_1^2 + \varphi_2^2) = -\frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2.$$

$$\left(\frac{\partial V}{\partial\varphi_1}\right)_0 = -\mu^2\varphi_{01} + \lambda\varphi_{01}(\varphi_1^2 + \varphi_2^2) = 0$$

$$\left(\frac{\partial V}{\partial\varphi_2}\right)_0 = -\mu^2\varphi_{02} + \lambda\varphi_{02}(\varphi_1^2 + \varphi_2^2) = 0$$

$$\varphi_{01}^2 + \varphi_{02}^2 = \frac{\mu^2}{\lambda} = v^2$$

$$\left(\phi^\dagger\phi\right)_0 = |\phi_0|^2 = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}$$

$$\frac{\partial^2 V}{\partial\varphi_1^2} = -\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2) + 2\lambda\varphi_1^2$$

$$\frac{\partial^2 V}{\partial\varphi_2^2} = -\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2) + 2\lambda\varphi_2^2$$

$$\frac{\partial^2 V}{\partial\varphi_1\partial\varphi_2} = 2\lambda\varphi_1\varphi_2$$

$$m_{ab}^2 = \begin{pmatrix}2\lambda v^2 & 0\\ 0 & 0\end{pmatrix}$$

$$\eta(x) = \varphi_1(x) - \sqrt{\frac{\mu^2}{\lambda}}, \xi = \varphi_2$$

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)\right]$$

$$-\lambda\sqrt{\frac{\mu^2}{\lambda}}\eta(\eta^2 + \xi^2) - \frac{\lambda}{4}(\eta^2 + \xi^2)^2 + \frac{\mu^4}{4\lambda}$$

$$\mathcal{L} = |D^\mu\phi|^2 - \mu^2|\phi|^2 - \lambda\left(\phi^\dagger\phi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\phi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

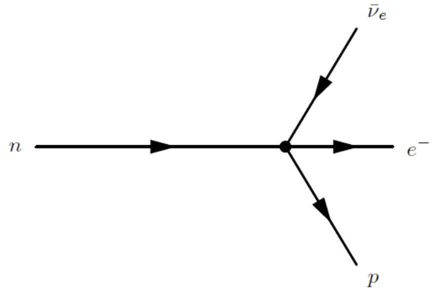
$$D_\mu = \partial_\mu - iqA_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\mu A_\nu$$

$$\phi(x) \rightarrow \phi'(x) = e^{-i\alpha}\phi(\mathrm{x})$$

$$\phi(x) \to \phi'(x) = e^{-i\alpha(x)}\phi(\mathrm{x})$$
$$A_\mu \to A'_\mu = A_\mu - \frac{1}{q}\partial_\mu \alpha(x)$$
$$|\phi_0|^2 = \frac{\mu^2}{2|\lambda|} = \frac{v^2}{2}$$
$$\langle\phi\rangle_0 = \frac{v}{\sqrt{2}}$$
$$\phi' = \phi - \langle\phi\rangle_0$$
$$\phi(x) = \frac{1}{\sqrt{2}}(v+\eta)e^{i\xi/v}$$
$$\approx \frac{v+\eta+i\xi}{\sqrt{2}}$$
$$\mathcal{L}_{so} = \frac{1}{2}\left[(\partial_\mu\eta)(\partial^\mu\eta) + 2\mu^2\eta^2\right] + \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)$$
$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + qvA_\mu(\partial^\mu\xi) + \frac{q^2v^2}{2}A_\mu A^\mu + \cdots$$
$$\phi(x) \to \phi'(x) = e^{i\xi(x)/v}\phi(x) = \frac{1}{\sqrt{2}}(v+\eta)$$
$$A_\mu(x) \to B_\mu(x) = A_\mu(x) - \frac{1}{qv}\partial_\mu\xi(x)$$
$$D_\mu\phi(x) \to D'_\mu\phi'(x) = (\partial_\mu - iqB_\mu)\frac{1}{\sqrt{2}}(v+\eta)$$
$$F_{\mu\nu}(A) \to F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$\mathcal{L}_{so} = \frac{1}{2}\left|\partial_\mu\eta - iqB_\mu(v+\eta)\right|^2 - \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4$$
$$-\frac{1}{4}F_{\mu\nu}(b)F^{\mu\nu}(B)$$
$$= \frac{1}{2}\left[(\partial_\mu\eta)(\partial^\mu\eta) - 2\mu^2\eta^2\right] - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B)$$
$$+\frac{q^2v^2}{2}B_\mu B^\mu + \frac{1}{2}q^2B_\mu B^\mu\eta(\eta+2v) - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4$$
$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$
$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - V(\phi^\dagger\phi)$$
$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk}A^j_\mu A^k_\nu,$$
$$D_\mu\phi = \left(\partial_\mu - ig\frac{\tau^i}{2}A^i_\mu\right)\phi, (i = 1,2,3)$$
$$V(\phi^\dagger\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 . (\mu^2 > 0)$$
$$\phi(x) = \frac{1}{\sqrt{2}}e^{i\tau^i\xi^i(x)/2v}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$
$$\phi(x) \to \phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$
$$\vec{A}_\mu \to \vec{B}_\mu = U\vec{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$
$$U(x) = \frac{1}{\sqrt{2}}e^{i\tau^i\xi^i(x)/2v}$$
$$D_\mu\phi \to (D_\mu\phi)' = \left(\partial_\mu - ig\frac{\tau^i}{2}B^i_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$$
$$F^i_{\mu\nu}(A)F^{i\mu\nu}(A) \to F^i_{\mu\nu}(B)F^{i\mu\nu}(B) = F^i_{\mu\nu}(A)F^{i\mu\nu}(A)$$

$$F_{\mu\nu}^i(B) = \partial_\mu B_\nu^i - \partial_\nu B_\mu^i + g\varepsilon_{ijk} B_\mu^j B_\nu^k$$
$$\mathcal{L} = \left(D_\mu \phi\right)'^\dagger (D^\mu \phi)' - \frac{1}{4} F_{\mu\nu}^i F^{i\mu\nu} + \mu^2 \phi'^\dagger \phi' - \lambda\left(\phi'^\dagger \phi'\right)^2$$
$$\left[\left(D_\mu \phi\right)'\right]^{\dagger a} (D^\mu \phi)'_a = \frac{1}{2}\partial_\mu H \partial^\mu H + g^2 B_\mu^i B^{j\mu} \left(\frac{\tau^i}{2}\right)_b^a \left(\frac{\tau^j}{2}\right)_a^c b'^b b'_c = \frac{1}{2}\partial_\mu H \partial^\mu H + g^2 B_\mu^i B^{j\mu} (v+H)^2$$
$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu H \partial^\mu H - 2\mu^2 H^2\right) - \frac{1}{4} F_{\mu\nu}^i(B) F^{i\mu\nu}(B) + \frac{g^2 v^2}{8} B_\mu^i B^{i\mu} + \frac{g^2}{8} B_\mu^i B^{i\mu} H(2v+H) - \lambda v H^3 - \frac{\lambda}{4} H^4 - \frac{v^4}{4}$$
$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu$$



$$SU(2)_L \times U(1)_Y \overset{\text{SSB}}{\rightarrow} U(1)_{EM}$$
$$L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, R = e_R$$
$$SU(2)_L: L \rightarrow L' = e^{-i\alpha^i(x)\frac{\tau^i}{2}} L, R \rightarrow R' = R$$
$$U(1)_Y: L \rightarrow L' = e^{\frac{i}{2}\beta(x)} L, R \rightarrow R' = e^{i\beta(x)} R$$
$$\mathcal{L}_F = \bar{L} i\gamma^\mu \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot \vec{A}_\mu + \frac{i}{2} g' B_\mu\right) L$$
$$+ \bar{R} i\gamma^\mu (\partial_\mu + ig' B_\mu) R$$
$$D_\mu = \partial_\mu - ig\frac{\vec{\tau}}{2}\cdot \vec{A}_\mu - ig'\frac{Y}{2} B_\mu$$
$$\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu}^i F^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$
$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\varepsilon_{ijk} A_\mu^j A_\nu^k$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$$
$$\mathcal{L}_s = \left(D_\mu \phi\right)^\dagger (D^\mu \phi) - V\left(\phi^\dagger \phi\right)$$
$$D_\mu \phi = \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot \vec{A}_\mu - \frac{i}{2} g' B_\mu\right)\phi$$
$$V\left(\phi^\dagger \phi\right) = m^2 \phi^\dagger \phi + \lambda\left(\phi^\dagger \phi\right)^2$$
$$\mathcal{L}_Y = -G_e\left(\bar{L}\phi R + \bar{R}\phi^\dagger L\right) + h.c.$$
$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_G + \mathcal{L}_s + \mathcal{L}_Y$$
$$\phi_0 = \langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$
$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = e^{i\vec{\tau}\cdot\vec{\xi}/2v} \begin{pmatrix} 0 \\ (v+H)/\sqrt{2} \end{pmatrix}$$
$$\langle 0|\xi_i|0\rangle = \langle 0|H|0\rangle = 0$$
$$U(\xi) = e^{-i\vec{\tau}\cdot\vec{\xi}/2v}$$

$$\phi' = U(\xi)\phi = \begin{pmatrix} 0 \\ (v+H)/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(v+H)\chi$$

$$L' = U(\xi)L$$

$$\vec{A}'_\mu = U(\xi)\vec{A}_\mu U(\xi)^{-1} - \frac{i}{g}\left(\partial_\mu U(\xi)\right)U^\dagger(\xi)$$

$$R' = R$$

$$B'_\mu = B_\mu$$

$$\mathcal{L}_F = \bar{L}' i\gamma^\mu \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu + \frac{i}{2}g' B'_\mu\right) L'$$

$$+\bar{R}' i\gamma^\mu \left(\partial_\mu + ig' B'_\mu\right) R'$$

$$\mathcal{L}_G = -\frac{1}{4}F'^i_{\mu\nu}F'^i i\mu\nu - \frac{1}{4}B'_{\mu\nu}B'\mu\nu$$

$$\mathcal{L}_s = \left(D_\mu\phi\right)'(D^\mu\phi)' - V\left(\phi'^\dagger\phi'\right)$$

$$\mathcal{L}_Y = -G_e\left(\bar{L}'\phi' R' + \bar{R}'\phi'^\dagger L'\right) + h.c.$$

$$\mathcal{L}_s = \left(D_\mu\phi\right)'(D^\mu\phi)' - V\left(\phi'^\dagger\phi'\right)$$

$$\left(D_\mu\phi\right)' = \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu - \frac{i}{2}g' B'_\mu\right)\phi'$$

$$= \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu - \frac{i}{2}g' B'_\mu\right)\frac{1}{\sqrt{2}}(v+H)\chi$$

$$\mathcal{L}_{\text{mass}} = \frac{v^2}{2}\chi^\dagger\left(g\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu + \frac{g'}{2}B'_\mu\right)\left(g\frac{\vec{\tau}}{2}\cdot\vec{A}'^\mu + \frac{g'}{2}B'^\mu\right)\chi$$

$$= \frac{v^2}{8}\left(g^2\vec{A}'_\mu\cdot\vec{A}'^\mu + g'^2 B'_\mu B'^\mu - 2gg' B'_\mu A'^{3\mu}\right)$$

$$= \frac{v^2}{8}\left(g^2 A'^1_\mu A'^{1\mu} + g^2 A'^2_\mu A'^{2\mu} + \left(gA'^3_\mu - g' B'_\mu\right)^2\right),$$

$$W^\pm_\mu = \frac{A'^1_\mu \mp iA'^2_\mu}{\sqrt{2}}$$

$$M_W = \frac{1}{2}gv$$

$$\frac{v^2}{8}\begin{pmatrix} A'^3_\mu & B'_\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} A'^{3\mu} \\ B'^\mu \end{pmatrix}$$

$$\frac{v^2}{8}\begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} g^2+g'^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A_\mu \end{pmatrix} = \frac{v^2}{8}(g^2+g'^2)Z_\mu Z^\mu$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\ \theta_W & -\sin\ \theta_W \\ \sin\ \theta_W & \cos\ \theta_W \end{pmatrix}\begin{pmatrix} A'^3_\mu \\ B'_\mu \end{pmatrix}$$

$$\tan\ \theta_W = \frac{g'}{g}$$

$$\sin\ \theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \sin\ \theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$

$$M_Z = \frac{1}{2}v\sqrt{g^2+g'^2}$$

$$M_z = \frac{M_W}{\cos\ \theta_w}$$

$$M_W \approx \frac{37.22}{\sin\ \theta_W}GeV > 37GeV$$

$$M_Z = \frac{M_W}{\cos\ \theta_W} \approx \frac{74.44}{\sin\ 2\theta_W}GeV > 74GeV.$$

$$M_W = 79.8 \pm 0.8\text{GeV}, M_Z = 90.8 \pm 0.6\text{GeV}$$

$$M_W = 80.4335 \pm 0.094 GeV, M_Z = 91.1876 \pm 0.021 GeV$$

$$V(\phi'^{\dagger}\phi') = -\frac{\mu^2}{2}(v+H)^2\chi^{\dagger}\chi + \frac{\lambda}{4}(v+H)^4(\chi^{\dagger}\chi)^2$$

$$= \frac{-\mu^2 v^2}{4} + \frac{1}{2}(2\mu^2)H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4.$$

$$M_H = \sqrt{2\mu^2}$$

$$\mathcal{L}_s = (D_\mu\phi)'(D^\mu\phi)' - V(\phi'^{\dagger}\phi')$$

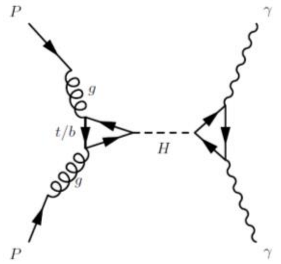
$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}M_H^2 H^2 - \lambda v H^3 - \frac{\lambda}{4}H^4 + \frac{g^2}{8}(H^2 + 2Hv)\left[\frac{1}{\cos^2\theta_W}Z_\mu Z^\mu + 2W_\mu^+ W^{-\mu}\right] + M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu$$

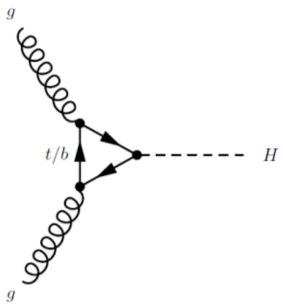
$$\mathcal{L}_Y = -G_e(\bar{L}'\phi' R' + \bar{R}'\phi' \dagger L') + h.c. = -G_e\left(\bar{e}'_L \frac{1}{\sqrt{2}}(v+H)e'_R + \bar{e}'_R \frac{1}{\sqrt{2}}(v+H)e'_L\right) + h.c.$$

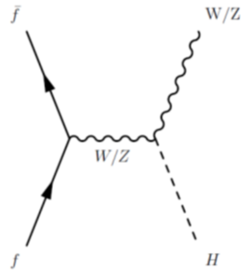
$$= -\frac{G_e v}{\sqrt{2}}\bar{e}'e' - \frac{G_e}{\sqrt{2}}H\bar{e}'e'.$$

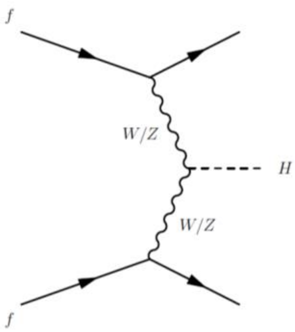
$$m_e = \frac{G_e}{\sqrt{2}}v$$

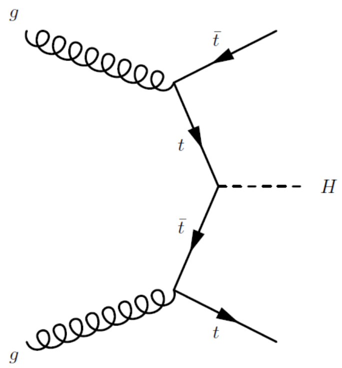
$$\frac{G_e}{\sqrt{2}} = \frac{m_e}{v},$$

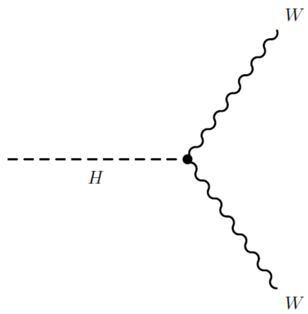


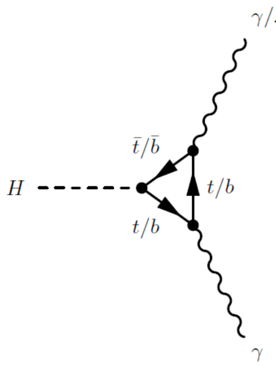


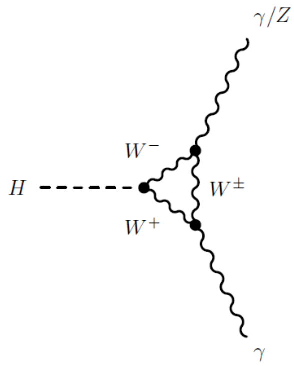


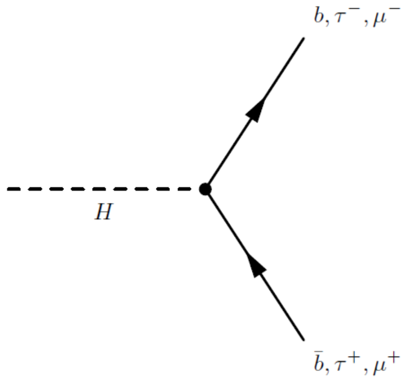












$$i\frac{\partial\psi}{\partial t} + \frac{1}{2m}\nabla^2\psi - V(\mathbf{r},t)\psi = 0$$

$$E = \sqrt{P^2 + m^2},$$

$$\mathbf{p}^2 + m^2 = \mathbf{E}^2$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi = 0$$

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} -\eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$(\Box + m^2)\psi = 0,$$

$$(\square + m^2)\psi^* = 0.$$
$$\psi^* \square \psi - \psi \square \psi^* = 0$$
$$\psi^* \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi - \psi\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi^* = 0$$
$$\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi^* \nabla^2 \psi - \psi \frac{\partial^2 \psi^*}{\partial t^2} + \psi \nabla^2 \psi^* = 0$$
$$\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = 0$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
$$\mathbf{J} = \frac{1}{2im}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$
$$\rho = \frac{i}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$$
$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi$$
$$i\frac{\partial \psi}{\partial t} = \left[-i\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m\right]\psi$$
$$= \left[-i \sum_{k=1}^{N} \hat{\alpha}_k \frac{\partial}{\partial x^k} + \hat{\beta} m\right]\psi = \hat{H}_D \psi$$
$$\psi = \begin{pmatrix} \psi_1(\boldsymbol{x}, t) \\ \psi_2(\boldsymbol{x}, t) \\ \vdots \\ \psi_N(\boldsymbol{x}, t) \end{pmatrix}$$
$$\rho = \psi^\dagger \psi = \sum_{i=1}^{N} \psi_i^\dagger \psi_i$$
$$-\frac{\partial^2 \psi}{\partial t^2} = (-\nabla^2 + m^2)\psi = 0$$
$$-\frac{\partial^2 \psi}{\partial t^2} = -\sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial \psi}{\partial x^i \partial x^j}$$
$$-im \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \psi}{\partial x^i \partial x^j} + \hat{\beta}^2 m^2 \psi$$
$$\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = \{\hat{\alpha}_i, \hat{\alpha}_j\} = 2\delta_{ij} I_N$$
$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0$$
$$\hat{\alpha}^2 = \hat{\beta}^2 = I_N$$
$$\hat{\alpha}_i = -\hat{\beta} \hat{\alpha}_i \hat{\beta}$$
$$\mathrm{Tr}\hat{\alpha}_i = -\mathrm{Tr}\hat{\beta}\hat{\alpha}_i\hat{\beta} = -\mathrm{Tr}\hat{\beta}^2\hat{\alpha}_i = -\mathrm{Tr}\hat{\alpha}_i$$
$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}, \hat{\beta} = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$
$$i\psi^\dagger \frac{\partial \psi}{\partial t} = -i \sum_{k=1}^{N} \psi^\dagger \hat{\alpha}_k \frac{\partial \psi}{\partial x^k} + \psi^\dagger \hat{\beta} m \psi.$$
$$-i\psi \frac{\partial \psi^\dagger}{\partial t} = i \sum_{k=1}^{N} \psi \hat{\alpha}_k \frac{\partial \psi^\dagger}{\partial x^k} + \psi \hat{\beta} m \psi^\dagger$$

$$i\psi^{\dagger}\frac{\partial\psi}{\partial t}+i\psi\frac{\partial\psi^{\dagger}}{\partial t}=-i\sum_{k=1}^{N}\psi^{\dagger}\hat{\alpha}_k\frac{\partial\psi}{\partial x^k}+\psi^{\dagger}\hat{\beta}m\psi$$
$$-i\sum_{k=1}^{N}\psi\hat{\alpha}_k\frac{\partial\psi^{\dagger}}{\partial x^k}-\psi\hat{\beta}m\psi^{\dagger}$$
$$\frac{\partial}{\partial t}(\psi^{\dagger}\psi)+\sum_{k=1}^{N}\frac{\partial}{\partial x^k}(\psi^{\dagger}\hat{\alpha}_k\psi)=0$$
$$\frac{\partial\rho}{\partial t}+\nabla\cdot J=0$$
$$\rho=\psi^{\dagger}\psi=\sum_{i=1}^{4}\psi_i^{*}\psi$$
$$\frac{\partial}{\partial t}\int d^3x\,\psi^{\dagger}\psi=0$$
$$\left(i\frac{\partial}{\partial t}+i\sum_{k=1}^{3}\hat{\alpha}_k\frac{\partial}{\partial x^k}-\hat{\beta}m\right)\psi=0$$
$$\hat{\beta}\left(i\frac{\partial}{\partial t}+i\sum_{k=1}^{3}\hat{\beta}\hat{\alpha}_k\frac{\partial}{\partial x^k}-m\right)\psi=0$$
$$i\left(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}\right)\psi-m\psi=0$$
$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}I_4$$
$$\gamma^{\mu}=\begin{pmatrix}0&\hat{\sigma}^{\mu}\\-\hat{\sigma}^{\mu}&0\end{pmatrix},\gamma^0=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix}$$
$$(i\gamma^{\mu}\partial_{\mu}-m)\psi=0$$
$$\mathcal{L}(\mathrm{x})=\bar{\psi}(\mathrm{x})(i\gamma^{\mu}\partial_{\mu}-m)\psi(\mathrm{x})$$
$$\frac{\partial\mathcal{L}}{\partial\psi(\mathrm{x})}=-\bar{\psi}(\mathrm{x})m$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi(\mathrm{x}))}=i\gamma^{\mu}\bar{\psi}(\mathrm{x})$$
$$\frac{\partial\mathcal{L}}{\partial\psi}-\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}=-\bar{\psi}(\mathrm{x})m-i\gamma^{\mu}\partial_{\mu}\bar{\psi}(\mathrm{x})=0$$
$$\bar{\psi}(\mathrm{x})(i\gamma^{\mu}\partial_{\mu}+m)=0$$
$$\psi^{\dagger}(\mathrm{x})\gamma^0)(i\gamma^{\mu}\partial_{\mu}+m)=0$$
$$\gamma^0(-i(\gamma^0\gamma^{\mu}\gamma^0)\partial_m u+m)\psi=0$$
$$(-i(\gamma^{\mu}\gamma^0)\partial_{\mu}+m\gamma^0)\psi=0$$
$$(i\gamma^{\mu}\partial_{\mu}-m)\psi=0$$

Los operadores perturbativos del gravitón y por ende, del campo gravitónico, están dados por:

$$\hat{H}_{\mathrm{int},l}=\frac{L}{\pi^2}\sqrt{\frac{M\hbar}{\omega_l}}\frac{(-1)^{\frac{l-1}{2}}}{l^2}(\hat{b}_l+\hat{b}_l^{\dagger})\ddot{\hat{h}}$$

Satisfaciendo la relación de dispersión, así:

$\hat{h}=\sum_{\mathbf{k}}h_{q,\mathbf{k}}(\hat{a}_{\mathbf{k}}+\hat{a}_{\mathbf{k}}^{\dagger})$, donde $h_{q,\mathbf{k}}=\frac{1}{c}\sqrt{\frac{8\pi G\hbar}{V\nu_{\mathbf{k}}}}$ es igual a $\hat{a}_{\mathbf{k}}(t)(\hat{a}_{\mathbf{k}}^{\dagger}(t))$

En tanto que, el efecto cuántico gravitacional estimulado por el campo gravitónico, viene dado por la escala de Planck, la misma que se expresa así:

$$\Gamma_{\text{efecto cuántico gravitacional estimulado}} = \frac{ML^2\omega_l^2 h^2}{4l^4\pi^5\hbar} = \frac{v_s^2}{4l^2\pi^3\hbar}Mh^2.$$

$$\Gamma_{\text{efecto cuántico gravitacional estimulado}} = \frac{2\pi}{\hbar^2}\left|\langle\alpha|\langle 1|\hat{H}_{\text{int}}\left|\alpha\right\rangle|0\rangle\right|^2 D(\omega)$$

$$\Gamma_{\text{efecto cuántico gravitacional estimulado}} = \frac{|\alpha|^2}{l^4}\frac{8GML^2\omega_l^4}{\pi^4 c^5}$$

$$\Gamma_{\text{efecto cuántico gravitacional estimulado}} = \frac{1}{l^4}\frac{ML^2\omega_l^2 h_0^2}{4\pi^5\hbar}$$

$$P(\omega,t) \approx D(\omega)\frac{(h_0^2\omega^3ML^2)t}{2\hbar\pi^3} = \Gamma_{\text{efecto cuántico gravitacional estimulado}}\, t$$

$$\Gamma_{\text{efecto cuántico gravitacional estimulado}} = D(\omega)\frac{(h_0^2\omega^3ML^2)}{2\hbar\pi^3} = \frac{Vh_0^2\omega^5ML^2}{4\hbar\pi^5c^3} = \frac{h_0^2Mv_s^2}{4\hbar\pi^3}.$$

Más, el efecto cuántico gravitacional espontáneo, es decir, cuando no interviene el campo gravitónico, viene dado por la misma escala, pero así:

$$\Gamma_{\text{efecto cuántico gravitacional espontáneo}} = \frac{8GML^2\omega_l^4}{l^4\pi^4c^5} = \frac{8\pi G\rho v_s^4R^2}{Lc^5}$$

Calculándose la densidad y distribución de la masa de la superpartícula, sea por permeabilidad del campo gravitónico o en su defecto, por gravedad cuántica intrínseca, en cualquiera de los casos anteriores, se usará la siguiente métrica:

$$M = \frac{\pi^2\hbar\omega^3}{v_S^2\chi(h,\omega,t)^2}$$

$$m\ddot{\xi}_n + m\omega_D^2(2\xi_n - \xi_{n-2} - \xi_{n+2}) = 0.$$

$$\xi_n(t) = e^{-i\omega t}\left(Ae^{ikna/2} + Be^{-ikna/2}\right) + \text{ H.c.}$$

$$\mu_l = \int_{-L/2}^{L/2}\rho(x)dx\chi_l(x)^2 = \frac{M}{L}\int_{-L/2}^{L/2}dx\chi_l(x)^2 = M/2$$

Para una onda monocromática $h(t) = h_0\sin(\nu t)$ tenemos que $\chi \sim \frac{1}{2}h_0\nu^2 t\text{sinc}\left(\frac{t\Delta}{2}\right)$ en la que $\Delta = |\nu - \omega|$, cuya aproximación es $\Delta \ll \omega, \nu$ y cuyo límite es $t\Delta \to \infty$.

En cuanto a la expansión de Taylor es fase $\Phi(s)$ en relación a $h(s)e^{i\omega s}$, desde un punto estacionario $s = s^*$ donde $\Phi'(s)|_{s=s^*} = 0$, evaluamos la integral de Gauss para una solución aproximada de $\chi(h,\omega,t)$.

Calculando lo anterior, tenemos:

$$\nu(t) = \left(\frac{1}{\nu_0^{8/3}} - \frac{8}{3}kt\right)^{-3/8}$$

$$k = \frac{k_f}{(2\pi)^{8/3}} = \frac{96}{5}\pi\left(\frac{\pi GM_c}{c^3}\right)^{5/3}\frac{1}{(2\pi)^{8/3}} = \frac{48}{5}\left(\frac{GM_c}{2c^3}\right)^{5/3}$$

En la que la masa $M_c = (m_1m_2)^{3/5}/(m_1+m_2)^{1/5}$ es efectiva para un sistema binario con masas m_1 and m_2, lo que comporta un campo gravitónico.

Estimamos el tiempo τ $[\omega - \Delta\omega, \omega + \Delta\omega]$, lo que da como resultado:

$$\tau = t(\nu = \omega + \Delta\omega) - t(\nu = \omega - \Delta\omega)$$

$$\tau = \frac{2\Delta\omega}{k\omega^{11/3}}$$

$$\chi = \left|\int_0^t ds e^{i\omega s} h_0 \nu^2 \sin(\nu s)\right| \approx h_0 \nu^2 \frac{t}{2} \text{sinc}\left(\frac{\delta t}{2}\right)$$

Aquí asumimos que $\omega + \nu \gg \omega - \nu = \delta$, por lo que:

$$2\Delta\omega = \frac{8}{T}$$

Según la escala del tiempo, tenemos:

$$\tau = 2\sqrt{\frac{2}{k}}\omega^{-11/6}$$

$$\chi \approx h_0 \omega^2 \left|\int_0^\tau ds e^{i\omega s} \sin(\omega s)\right|$$

$$\chi(\tau) \approx h_0 \frac{\omega^2 \tau}{2}$$

$$\chi \approx h_0 \sqrt{\frac{2}{k}}\omega^{1/6} = h_0 \sqrt{\frac{5}{24}\left(\frac{2c^3}{GM_c}\right)^{5/6}} \omega^{1/6}$$

$$M = \frac{\pi^2 \hbar \omega^3}{v_s^2 \chi^2} \approx \frac{\pi^2 \hbar k}{2 v_s^2 h_0^2} \omega^{8/3} = \frac{24\pi^2}{5} \frac{\hbar}{h_0^2 v_s^2} \left(\frac{GM_c}{2c^3}\right)^{5/3} \omega^{8/3}$$

En tanto que la densidad de la energía de la superpartícula, se calculará así:

$$\left.\frac{d\xi_n}{dn}\right|_{n=\pm N} = 0$$

$$\xi_n(t) = \sum_{l=0,2..}^{N-1} \chi_l(t) \cos[l\pi n/(2N)] + \sum_{l=1,3..}^{N} \chi_l(t) \sin[l\pi n/(2N)].$$

$$\sum_{n=-N}^{N} \cos\left[\frac{l\pi n}{2(N+1)}\right] \cos\left[\frac{l'\pi n}{2(N+1)}\right] = \frac{N+1}{2}\delta_{ll'},$$

$$\sum_{n=-N}^{N} \sin\left[\frac{l\pi n}{2(N+1)}\right] \sin\left[\frac{l'\pi n}{2(N+1)}\right] = \frac{N+1}{2}\delta_{ll'},$$

$$\sum_{n=-N}^{N} \cos\left[\frac{l\pi n}{2(N+1)}\right] \sin\left[\frac{l'\pi n}{2(N+1)}\right] = 0.$$

$$\xi_n(t) = \sum_{l=0,2..}^{N-1} \chi_l(t) \cos\left[\frac{l\pi n}{2(N+1)}\right] + \sum_{l=1,3..}^{N} \chi_l(t) \sin\left[\frac{l\pi n}{2(N+1)}\right].$$

$$\ddot{\chi}_l + \omega_l^2 \chi_l = 0$$

$$E = \frac{m}{2}\sum_{n=-N}^{N} \dot{\xi}_n^2 + \frac{m\omega_D^2}{2}\sum_{n=-N}^{N-2} (\xi_{n+2} - \xi_n)^2$$

$$= \frac{M}{4}\sum_{l=0}^{N} \dot{\chi}_l^2 + \frac{M}{4}\sum_{l=0}^{N} \omega_l^2 \chi_l^2.$$

$$\int_{-L/2}^{L/2} dx \chi_l(x)^2 = L/2$$

$$\hat{H}_{\omega_l} = \hbar\omega_l \left(\hat{b}_l^\dagger \hat{b}_l + \frac{1}{2}\right)$$

$$|\psi_M\rangle = (2\pi t_m)^{-\frac{1}{4}} \int dx e^{-\frac{x^2}{4t_m}} |x\rangle$$

$$\hat{H}_{\text{int}}^M dt = \sqrt{dt}\hat{p}\hat{N}$$

$$\hat{M}_{\hat{N}}(y) = \langle y| e^{-i\hat{H}_{\text{int}}^M dt} |\psi_M\rangle$$

$$= (2\pi t_m)^{-\frac{1}{4}} \exp\left\{\left[-\frac{(y - \hat{N}\sqrt{dt})^2}{4t_m}\right]\right\}.$$

$$\hat{M}_{\hat{N}}(r) = (2\pi t_m/dt)^{-\frac{1}{4}} \exp\left\{\left[-\frac{dt(r - \hat{N})^2}{4t_m}\right]\right\}$$

$$r(t) \approx \langle\hat{N}(t)\rangle + \sqrt{t_m}\zeta(t), \zeta(t), \langle\zeta(t)\zeta(t')\rangle = \delta(t - t')\otimes\rho(t + dt)$$

En este punto, cabe precisar, que las ondas gravitacionales causadas por efecto de la distorsión de la geometría del espacio – tiempo cuántico, a propósito de las interacciones de las superpartículas engendradas por la permeabilidad del campo gravitónico con la materia y en consecuencia, el potencial gravitacional del gravitón, en relación a la densidad de los estados inicial y final, se calculan así:

$$f(x) = -m\nabla\phi = -m\nabla\left(\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}x^2 + \cdots\right)$$

$$= m\frac{\ddot{h}_{xx}}{4}\nabla(x^2) = m\frac{\ddot{h}_{xx}}{2}x$$

$$f_n = m\frac{\ddot{h}_{xx}}{2}(x_n + \xi_n)$$

$$H_I = -m\frac{\ddot{h}_{xx}}{2}\sum_{n=-N}^{N} (x_n\xi_n + \xi_n^2/2)$$

$$-m\frac{\ddot{h}_{xx}}{2}\sum_{n=-N}^{N} x_n\xi_n \approx -\frac{ML\ddot{h}_{xx}}{\pi^2}\sum_{l=1,3\ldots}^{N} (-1)^{\frac{l-1}{2}}\frac{1}{l^2}\chi_l(t)$$

$$-m\frac{\ddot{h}_{xx}}{4}\sum_{n=-N}^{N} \xi_n^2 = -\frac{M\ddot{h}_{xx}}{8}\sum_{l=0}^{N} \chi_l^2$$

$$\hat{H}_I = \sum_{l=0}^{N} \hat{H}_I^l = -\frac{ML\ddot{h}_{xx}}{\pi^2}\sum_{l=1,3\ldots}^{N} (-1)^{\frac{l-1}{2}}\frac{1}{l^2}\hat{\chi}_l$$

$$-\frac{M\ddot{h}_{xx}}{8}\sum_{l=0}^{N} \hat{\chi}_l^2.$$

$$\hat{H}_I^{l,} = -\frac{ML\ddot{h}_{xx}}{\pi^2}(-1)^{\frac{l-1}{2}}\frac{1}{l^2}\sqrt{\frac{\hbar}{M\omega_l}}(\hat{b}_l + \hat{b}_l^\dagger)$$

$$-\frac{\ddot{h}_{xx}}{8}\frac{\hbar}{\omega_l}(\hat{b}_l + \hat{b}_l^\dagger)^2$$

$$\hat{H}_l^{l,} = -\frac{\ddot{h}_{xx}}{8}\frac{\hbar}{\omega_l}\left(\hat{b}_l + \hat{b}_l^\dagger\right)^2$$

$$\frac{\hat{H}_{\text{int}}}{\hbar} = \sqrt{(-1)^{l-1}\frac{8\pi G M v^3 L^2}{\omega_l c^2 V \pi^4 l^4}}\left(\hat{b}_l + \hat{b}_l^\dagger\right)\left(\hat{a}e^{-ivt} + \hat{a}^\dagger e^{ivt}\right),$$

$$N = \frac{h_0^2 c^5}{32\pi G \hbar v^2}$$

$$P_{0\rightleftharpoons 1} \approx \frac{L^2}{4\pi^4}\frac{M}{\omega\hbar}h_0^2 v^4 t^2 \text{sinc}^2\left(\frac{\omega - v}{2}t\right)$$

$$P(t) = \frac{L^2}{\pi^4}\frac{M}{\omega\hbar}|\chi(t)|^2 = \frac{h_0^2 \omega t^2 M v_s^2}{4\pi^2\hbar}$$

$$\Gamma_{\text{mc.}} = \frac{dP(t)}{dt} = \frac{h_0^2 \omega t M v_s^2}{2\pi^2\hbar} = \frac{h_0^2 N_c M v_s^2}{\pi\hbar}$$

$$h_0 = \sqrt{\frac{\pi k_B T}{M v_s^2 Q N_c}},$$

$$P(v,\omega,t) \approx |\beta(v,\omega,t)|^2 \approx \frac{(h_0^2\omega^3 M L^2)\sin^2\left[\frac{1}{2}t(v-\omega)\right]}{\hbar(v-\omega)^2\pi^4}$$

$$P(\omega,t) \approx \sum_v |\beta(v,\omega,t)|^2$$

$$= \int_{\omega-\delta/2}^{\omega+\delta/2} dv D(v)|\beta(v,\omega,t)|^2$$

$$= D(\omega)\frac{(h_0^2\omega^3 M L^2)}{\hbar\pi^4}\int_{\omega-\delta/2}^{\omega+\delta/2} dv \frac{\sin^2\left(\frac{1}{2}t(v-\omega)\right)}{(v-\omega)^2}$$

$$= D(\omega)\frac{(h_0^2\omega^3 M L^2)}{\hbar\pi^4}\Xi(t)$$

$$\sum_v = \int dv D(v) = \int dv \frac{V v^2}{2\pi^2 c^3}$$

$$\lim_{t\delta \gg 1} \Xi(t) \to \frac{1}{2}\pi t$$

$$h_c \equiv 2\pi\sqrt{\frac{\pi k_B T}{M v_s^2 Q}}$$

En el que, el operador – oscilador y los conmutadores del gravitón en relación a la partícula permeada por el campo gravitónico y viceversa, se calculan de la siguiente manera:

$$\hat{H} = \hbar\omega\hat{b}^\dagger\hat{b} + \frac{1}{\pi^2}L\sqrt{\frac{M\hbar}{\omega}}\ddot{h}(t)\left(\hat{b} + \hat{b}^\dagger\right)$$

$$\hat{U}_{int} = \hat{T}e^{-i\int_0^t ds\left(g(s)\hat{b}(s) + g^*(s)\hat{b}^\dagger(s)\right)}$$

$$g(t) = \frac{1}{\pi^2}L\sqrt{\frac{M}{\hbar\omega}}\ddot{h}(t)$$

$$\hat{U} = e^{\Omega(t)}$$

$$\Omega(t) = \int_0^t dt_1 \hat{A}(t_1) + \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2\left[\hat{A}(t_1), \hat{A}(t_2)\right] + \cdots$$

$$\hat{A}(t) = -i\big(g(t)\hat{b}(t) + g^*(t)\hat{b}^\dagger(t)\big), \big[\hat{A}(t_1), \hat{A}(t_2)\big] = -\big(g(t_1)g^*(t_2)e^{-i\omega(t_1-t_2)} - g(t_2)g^*(t_1)e^{+i\omega(t_1-t_2)}\big)$$

$$\hat{U}_{int} = e^{-i\int_0^t ds\left(g(s)\hat{b}(s)+g^*(s)\hat{b}^\dagger(s)\right)}e^{-i\varphi}$$

$$\varphi = \int_0^t dt_1 \int_0^{t_1} \mathrm{Im}\big[g(t_1)g^*(t_2)e^{-i\omega(t_1-t_2)}\big]$$

$$\hat{U} = e^{-i\varphi}e^{-i\omega t\hat{b}^\dagger\hat{b}}\hat{D}(\beta)$$

$$\hat{D}(\beta) = e^{\beta\hat{b}^\dagger - \beta^*\hat{b}}$$

$$\beta = -i\int_0^t ds g^*(s)e^{i\omega s}$$

$$e^{-i\varphi}\big|\beta e^{-i\omega t}\big\rangle$$

$$|\beta| = \frac{L}{\pi^2}\sqrt{\frac{M}{\omega\hbar}}\chi(h,\omega,t)$$

$$\chi(h,\omega,t) = \left|\int_0^t ds\ddot{h}(s)e^{i\omega s}\right|$$

2. Efecto cuántico gravitacional implícito, intrínseco o endógeno. Este fenómeno cuántico, se materializa cuando una partícula, a propósito de su masa hamiltoniana y densidad lagrangiana, se tiene como supermasiva, por lo que, aunque no se vea permeada por el campo de gauge gravitónico y en consecuencia, no interactúe con el gravitón, es capaz, por sí misma, de distorsionar el espacio – tiempo e incluso, colapsar, provocando un agujero negro de mecánica cuántica. Por ejemplo, la partícula subatómica más pesada que se conoce es el quark top, cuya masa es de aproximadamente 173 GeV (gigaelectronvoltios). El bosón de Higgs, que también es bastante pesado, tiene una masa de 125 GeV.

Paso a calcular, extrapolando las ecuaciones de campo de Einstein:

$$L(\phi_a, \partial\phi_a) = \int dV_x \mathcal{L}(\phi_a, \partial\phi_a)$$

$$S = \int dv_x \mathcal{L}(\phi_a, \partial\phi_a)$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0$$

$$\Pi^a = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}$$

$$[\phi_a(\vec{x},t), \phi_b(\vec{x}',t)] = [\Pi^a(\vec{x},t), \Pi^b(\vec{x}',t)] = 0$$

$$[\phi_a(\vec{x},t), \Pi^b(\vec{x}',t)] = i\delta_a^b\delta(\vec{x}-\vec{x}')$$

$$\phi_a(\vec{x})|\phi_a'\rangle = \phi_a'|\phi_a'\rangle$$

$$i\frac{d}{dt}|\psi(t)\rangle = H(\phi,\Pi)|\psi(t)\rangle$$

$$H = \int dV_x\mathcal{H} = \int dV_x(\Pi^a\partial_0\phi_a - \mathcal{L})$$

$$\Theta^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L}$$

$$\partial_\mu\Theta_\nu^\mu = 0$$

$$\int dv_x\partial_\mu\Theta_\nu^\mu = 0$$

$$\frac{d}{dt}\int dV_x\Theta_\nu^0 = 0$$

$$P_\nu = \int dV_x\Theta_\nu^0$$

$$x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu$$

$$\phi_a \longrightarrow \phi_a' = \phi_a + \delta_0\phi_a$$

$$\delta S = G(t_2) - G(t_1),$$
$$G(t) = \int dV_x(\Pi^a \delta\phi_a - \Theta_\nu^0 \delta x^\nu)$$
$$\mathcal{L}(\phi'_a, \partial'\phi'_a) = \mathcal{L}(\phi_a, \partial\phi_a)$$
$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a - \Theta_\nu^\mu \delta x^\nu$$
$$\partial_\mu J^\mu = 0$$
$$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$$
$$(\partial^\mu\partial_\mu\phi + m^2)\phi = 0$$
$$g_{\vec{k}}(\vec{x}, t) = \frac{1}{\sqrt{(2\pi)^{n-1}2\omega_k}} e^{i(\vec{k}\vec{x} - \omega_k t)}$$
$$\phi = \int d^{n-1}k \left(a_{\vec{k}}^\dagger g_{\vec{k}}^* + a_{\vec{k}} g_{\vec{k}}\right)$$
$$[a_{\vec{k}}, a_{\vec{k}'}] = \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger\right] = 0$$
$$\left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] = \delta(\vec{k} - \vec{k}')$$
$$H = \frac{1}{(2\pi)^{n-1}}\int d^{n-1}k\omega_k a_{\vec{k}}^\dagger a_{\vec{k}}$$
$$\left[H, a_{\vec{k}}^\dagger\right] = \omega_k a_{\vec{k}}^\dagger,$$
$$[H, a_{\vec{k}}] = -\omega_k a_{\vec{k}}$$
$$|\vec{k}\rangle = a_{\vec{k}}^\dagger |0\rangle.$$
$$a_{\vec{k_1}}^\dagger a_{\vec{k_2}}^\dagger a_{\vec{k_3}}^\dagger |0\rangle = |\vec{k_1}, \vec{k_2}, \vec{k_3}\rangle$$
$$N_{\vec{k}} = a_{\vec{k}}^\dagger a_{\vec{k}},$$
$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
$$ds^2 = g^{\mu\nu} dx_\mu dx_\nu$$
$$S\left(\phi'_a, \nabla'\phi'_a, g'_{\mu\nu}(x')\right) = S(\phi_a, \nabla\phi_a, g_{\mu\nu}(x))$$
$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$$
$$\partial_\mu \longrightarrow \nabla_\mu$$
$$d^n x \longrightarrow |g|^{1/2} d^n x = dv_x$$
$$S = \int d^n x \mathcal{L}(\phi, \nabla\phi, g_{\mu\nu})$$
$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0$$
$$\delta S = \int dv_x \left(\frac{\partial L}{\partial\phi_a}\delta\phi_a + \frac{\partial L}{\partial(\nabla_\mu\phi_a)}\nabla_\mu\delta\phi_a\right).$$
$$\frac{\partial L}{\partial(\nabla_\mu\phi_a)}\nabla_\mu\phi_a = \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi_a)}\delta\phi_a\right) - \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi_a)}\right)\delta\phi_a$$
$$\delta S = \int dv_x \left(\frac{\partial L}{\partial\phi_a} - \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi_a)}\right)\right)\delta\phi_a$$
$$\nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi_a)}\right) - \frac{\partial L}{\partial\phi_a} = 0$$
$$x^\mu \longrightarrow x'^\mu = x^\mu - \varepsilon^\mu(x)$$
$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\sigma\lambda}(x)$$
$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x - \varepsilon) = g'_{\mu\nu}(x) - \varepsilon^\rho \partial_\rho g'_{\mu\nu}(x)$$

$$g'_{\mu\nu}(x') = (\delta^{\sigma}_{\mu} - \varepsilon^{\sigma}_{,\mu})(\delta^{\lambda}_{\nu} - \varepsilon^{\lambda}_{,\nu})g_{\mu\nu}(x)$$
$$g'_{\mu\nu}(x) - \varepsilon^{\rho} g'_{\mu\nu,\rho}(x) = g_{\mu\nu}(x) + g_{\mu\lambda}(x)\varepsilon^{\lambda}_{,\nu} + g_{\nu\sigma}(x)\varepsilon^{\sigma}_{,\mu}$$
$$\delta_0 g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x)$$
$$\delta_0 g_{\mu\nu}(x) = \varepsilon_{\mu;\nu} + \varepsilon_{\nu;\mu} = \mathcal{L}_{\varepsilon} g_{\mu\nu}$$
$$S(g_{\mu\nu} + \delta_0 g_{\mu\nu}) = S(g_{\mu\nu}) + \int d^n x \frac{\delta S}{\delta g_{\mu\nu}} \delta_0 g_{\mu\nu}$$
$$\delta S = S(g_{\mu\nu} + \delta_0 g_{\mu\nu}) - S(g_{\mu\nu}) = \int d^n x \frac{\delta S}{\delta g_{\mu\nu}} \delta_0 g_{\mu\nu} = 0$$
$$T^{\mu\nu} \equiv -2|g|^{-1/2} \frac{\delta S}{\delta g_{\mu\nu}},$$
$$-\int dv_x T^{\mu\nu} \varepsilon_{\nu;\mu} = 0$$
$$\nabla_{\mu}(T^{\mu\nu}\varepsilon_{\nu}) = T^{\mu\nu}_{;\mu}\varepsilon_{\nu} + T^{\mu\nu}\varepsilon_{\nu;\mu}$$
$$\delta S = -\int dv_x \nabla_{\mu}(T^{\mu\nu}\varepsilon_{\nu}) + \int dv_x (\nabla_{\mu} T^{\mu\nu})\varepsilon_{\nu}$$
$$\int dv_x (\nabla_{\mu} T^{\mu\nu})\varepsilon_{\nu} = 0,$$
$$\nabla_{\mu} T^{\mu\nu} = 0.$$
$$\delta(g^{\mu\sigma} g_{\sigma\nu}) = \delta(\delta^{\mu}_{\nu}) = 0,$$
$$-\delta(g_{\lambda\nu}) = \delta(g^{\mu\sigma}) g_{\sigma\nu} g_{\lambda\mu},$$
$$\frac{\delta}{\delta g_{\mu\nu}} = -g^{\mu\lambda} g^{\nu\sigma} \frac{\delta}{\delta g^{\lambda\sigma}}.$$
$$T_{\mu\nu} = T^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu} = -\frac{2}{|g|^{1/2}} \frac{\delta S}{\delta g_{\alpha\beta}} g_{\alpha\nu} g_{\beta\nu} = 2|g|^{-1/2} \frac{\delta S}{\delta g^{\mu\nu}}.$$
$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + \delta x^{\mu}$$
$$\phi_a(x) \longrightarrow \phi'_a(x) = \phi(x) + \delta_0 \phi_a(x)$$
$$\delta S = S' - S = \int dv'_x L(\phi'(x')_a, \nabla' \phi'_a(x), g_{\mu\nu}) - \int dv_x L(\phi(x)_a, \nabla \phi_a(x), g_{\mu\nu})$$
$$S' = \int_{V'} dv_x L(\phi_a(x) + \delta_0 \phi(x), \nabla \phi_a(x) + \nabla \delta_0 \phi_a(x), g_{\mu\nu}) =$$
$$\int_{V'} dv_x L(\phi_a(x), \nabla \phi_a(x), g_{\mu\nu}) + \int_V dv_x \left(\frac{\partial L}{\partial \phi_a} \delta_0 \phi_a + \frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \nabla_{\mu} \delta_0 \phi_a \right)$$
$$\int_{\partial V} d\sigma_{\mu} \delta x^{\mu} L(\phi_a, \nabla \phi_a) = \int_{V'} dv_x L(\phi_a, \nabla \phi_a) - \int_V dv_x L(\phi_a, \nabla \phi_a)$$
$$\delta S = S' - S = \int_{\partial V} d\sigma_{\mu} \delta x^{\mu} L(\phi_a, \partial \phi_a) + \int_V dv_x \left(\frac{\partial L}{\partial \phi_a} \delta_0 \phi_a + \frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \nabla_{\mu} \delta_0 \phi_a \right)$$
$$\frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \nabla_{\mu}(\delta_0 \phi_a) = \nabla_{\mu} \left(\frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \delta_0 \phi_a \right) - \nabla_{\mu} \left(\frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \right) \delta_0 \phi_a$$
$$\delta S = \int_{\partial V} d\sigma_{\mu} \delta x^{\mu} L(\phi_a, \partial \phi_a) + \int_V dv_x \nabla_{\mu} \left(\frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \delta_0 \phi_a \right)$$
$$\delta S = \int_V dv_x \nabla_{\mu} \left(\delta x^{\mu} L + \frac{\partial L}{\partial(\nabla_{\mu} \phi_a)} \delta_0 \phi_a \right)$$
$$\phi'_a(x) = \phi'_a(x - \delta x) = \phi'(x') - \nabla_{\mu} \phi_a(x) \delta x^{\mu}$$
$$\delta S = \int d^{n-1} x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \delta_0 \phi_a - \Theta^0_{\nu} \delta x^{\nu} \right) \Bigg|_{t_1}^{t_2}$$
$$G(t) = \int d^{n-1} x (\Pi^a \delta \phi_a - \Theta^0_{\nu} \delta x^{\nu})$$
$$\delta S = G(t_2) - G(t_1)$$
$$i\delta_0 F = [F, G]$$

$$[\phi_a(\vec{x},t),\phi_b(\vec{x}',t)] = [\Pi_a(\vec{x},t),\Pi_b(\vec{x}',t)] = 0$$
$$[\phi_a(\vec{x},t),\Pi_b(\vec{x}',t)] = i\delta_{a,b}\delta^{(n-1)}(\vec{x}'-\vec{x})$$
$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) = 0$$
$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a$$
$$\frac{d}{dt}G(t) = \int d^{n-1}x\frac{\partial}{\partial t}(\Pi_a\delta\phi_a) = \int d^{n-1}x\partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi_a)}\delta\phi_a\right) = 0$$
$$p_\lambda = \int d^{n-1}x\Theta^0_\nu$$
$$x^\mu \longrightarrow x'^\mu = x^\mu - \varepsilon\xi^\mu(x),$$
$$\nabla_\mu(T^{\mu\nu}\xi_\nu) = \frac{1}{|g|^{1/2}}\partial_\mu\left(|g|^{1/2}T^{\mu\nu}\xi_\nu\right) = 0$$
$$P_\xi = \int dV_x T^{0\nu}\xi_\nu$$
$$P_\lambda = \int dV_x T^0_\nu$$
$$S(\phi_a,g_{\mu\nu}) = S(\tilde{\phi}_a,\tilde{g}_{\mu\nu}) + \int_{\partial V} dV_x \ldots,$$
$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x),$$
$$\tilde{\phi}_a(x) = \Omega^{2p}(x)\phi_a(x),$$
$$\Omega^2(x) = 1+\lambda(x),$$
$$\delta_0 g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = \lambda g_{\mu\nu}$$
$$\delta_0\phi_a = \tilde{\phi}_a - \phi_a = p\lambda(x)\phi_a(x)$$
$$\delta S = \int d^n x\left(\frac{\delta S}{\delta\phi_a}\delta_0\phi_a + \frac{\delta S}{\delta g_{\mu\nu}}\delta_0 g_{\mu\nu}\right) = \int d^n x\left(\frac{\delta S}{\delta\phi_a}p\lambda\phi_a + \frac{\delta S}{\delta g_{\mu\nu}}\lambda g_{\mu\nu}\right)$$
$$\delta S = \int d^n x\left(\frac{\delta S}{\delta g_{\mu\nu}}\lambda g_{\mu\nu}\right)$$
$$0 = \frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu} = -\frac{1}{2}|g|^{1/2}T^{\mu\nu}g_{\mu\nu}$$
$$S = \int d^n x\frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$
$$S = \int d^n x\frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$$
$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = (1+\lambda(x))g_{\mu\nu}(x)$$
$$\phi(x) \longrightarrow \tilde{\phi}(x) = \left(1-\frac{1}{2}\lambda(x)\right)\phi(x)$$
$$|g|^{1/2} \longrightarrow |\tilde{g}|^{1/2} = (1+2\lambda(x))|g|^{1/2}$$
$$g^{\mu\nu} \longrightarrow \tilde{g}^{\mu\nu} = (1-\lambda(x))g^{\mu\nu}$$
$$\Gamma^\alpha_{\beta\gamma} \longrightarrow \tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \frac{1}{2}\left(\delta^\alpha_\gamma\lambda_{,\beta} + \delta^\alpha_\beta\lambda_{,\gamma} - g_{\beta\gamma}\lambda^{,\alpha}\right),$$
$$R \longrightarrow \tilde{R} = (1-\lambda)R + 3\,\square\,\lambda(x)$$
$$\begin{aligned}\tilde{\mathcal{L}} =& \frac{1}{2}|\tilde{g}|^{1/2}\left(\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi}^2\right)\\ =& \frac{1}{2}(1+2\lambda)|g|^{1/2}\Big[(1-\lambda)g^{\mu\nu}\partial_\mu((1-1/2\lambda)\phi)\partial_\nu((1-1/2\lambda)\phi)\\ &-\frac{1}{6}(1+2\lambda)(1-\lambda)^2R\phi^2 - \frac{1}{2}(1+2\lambda)\,\square\,\lambda\phi^2\Big]\end{aligned}$$
$$\tilde{\mathcal{L}} = \frac{1}{2}|g|^{1/2}\left[g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi - \phi\partial_\mu\phi\partial_\nu\lambda) - \frac{1}{6}R\phi^2 - \frac{1}{2}\,\square\,\lambda\phi^2\right]$$

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2}|g|^{1/2}\left[g^{\mu\nu}\phi\partial_\mu\phi\partial_\nu\lambda + \frac{1}{2}\square\lambda\phi^2\right]$$
$$\tilde{\mathcal{L}} = \mathcal{L} - \partial_\mu\left(|g|^{1/2}g^{\mu\nu}\phi^2\partial_\nu\lambda\right)$$
$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\phi \longrightarrow \tilde{\phi} = \Omega^{-1}\phi$$
$$\tilde{\mathcal{L}} = \mathcal{L} - \partial_\mu\left(\frac{1}{2}|g|^{1/2}g^{\mu\nu}\phi^2\partial_\nu \log\Omega\right)$$
$$(\square + m^2 + \xi R)\phi = 0.$$
$$T^{\mu\nu} = \nabla^\mu\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi + \frac{1}{2}g^{\mu\nu}m^2\phi^2 - \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi^2 + \xi[g^{\mu\nu}\square(\phi^2) - \nabla^\mu\nabla^\nu(\phi^2)]$$
$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$$
$$\delta|g|^{1/2} = \frac{1}{2}|g|^{1/2}g^{\mu\nu}\delta g_{\mu\nu}$$
$$\delta R = \delta\left(R_{\mu\nu}g^{\mu\nu}\right) = \delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$$
$$\delta R_{\mu\nu} = \delta\Gamma^\lambda_{\mu\lambda;\nu} - \delta\Gamma^\lambda_{\mu\nu;\lambda}$$
$$\delta\Gamma^\lambda_{\mu\nu;\lambda} = g^{\rho\sigma}\delta g_{\rho\mu;\sigma\nu}$$
$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}\left(\delta g_{\rho\sigma;\mu\nu} - \delta g_{\rho\mu;\sigma\nu}\right)$$
$$\delta S = \frac{1}{2}\int d^n x|g|^{1/2}\left[\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - m^2\phi^2 - \xi R\phi^2\right) - \delta g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - \xi\left(-R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}\left(\delta g_{\rho\sigma;\mu\nu} - \delta g_{\rho\mu;\sigma\nu}\right)\right)\phi^2\right].$$
$$\int d^n x|g|^{1/2}g^{\rho\sigma}g^{\mu\nu}\delta g_{\rho\sigma;\mu\nu}\phi^2 = \int d^n x|g|^{1/2}g^{\rho\sigma}\delta g_{\rho\sigma}\square(\phi^2)$$
$$\int d^n x|g|^{1/2}g^{\rho\sigma}g^{\mu\nu}\delta g_{\rho\mu;\sigma\nu}\phi^2 = \int d^n x|g|^{1/2}g^{\sigma\mu}g^{\lambda\nu}\delta g_{\mu\nu}\nabla_\sigma\nabla_\lambda(\phi^2)$$
$$\delta S = -\frac{1}{2}\int d^n x|g|^{1/2}T^{\mu\nu}\delta g_{\mu\nu}$$
$$\partial^\mu\partial_\mu\phi + m^2\phi = 0$$
$$(f_1, f_2) = i\int dV_x\left(f_1^*(\vec{x},t)\partial_0 f_2(\vec{x},t) - \partial_0 f_1^*(\vec{x},t)f_2(\vec{x},t)\right) = i\int dV_x\left(f_1^*\overleftrightarrow{\partial}_0 f_2\right)$$
$$(f_1, f_2)^* = (f_2, f_1) = -(f_1^*, f_2^*)$$
$$(f_1, f_2) = i\int d^{n-1}x|g|^{1/2}g^{0\nu}f_1^*\overleftrightarrow{\partial}_\nu f_2$$
$$\frac{d}{dt}(f_1, f_2) = i\int d^{n-1}\partial_0\left(|g|^{1/2}g^{0\nu}f_1^*\overleftrightarrow{\partial}_\nu f_2\right)$$
$$= i\int d^{n-1}|g|^{1/2}\nabla_\mu\left(g^{\mu\nu}f_1^*\overleftrightarrow{\partial}_\nu f_2\right) - i\int d^{n-1}\partial_i\left(|g|^{1/2}g^{i\nu}f_1^*\overleftrightarrow{\partial}_\nu f_2\right)$$
$$\nabla_\mu\left(g^{\mu\nu}f_1^*\overleftrightarrow{\partial}_\nu f_2\right) = g^{\mu\nu}\nabla_\mu(f_1^*\partial_\nu f_2 - \partial_\nu f_1^* f_2) =$$
$$g^{\mu\nu}\left(\partial_\mu f_1^*\partial_\nu f_2 + f_1^*\nabla_\mu\partial_\nu f_2 - \nabla_\mu\partial_\nu f_1^* f_2 - \partial_\nu f_1^*\nabla_\mu f_2\right) =$$
$$f_1^*\square f_2 - \square f_1^* f_2 = f_1^*(-m^2 - \xi R)f_2 - f_2(-m^2 - \xi R)f_1^* = 0$$
$$\frac{d}{dt}(f_1, f_2) = 0$$
$$(f_1, f_2) = i\int d\sigma|g|^{1/2}n^\mu f_1^*\overleftrightarrow{\partial}_\mu f_2$$
$$(f_1, f_2)_{\sigma'} - (f_1, f_2)_\sigma = i\int d\sigma'|g|^{1/2}n'^\mu f_1^*\overleftrightarrow{\partial}_\mu f_2 - i\int d\sigma|g|^{1/2}n^\mu f_1^*\overleftrightarrow{\partial}_\mu f_2 = i\int dv_x\nabla^\mu\left(f_1^*\overleftrightarrow{\partial}_\mu f_2\right) = 0$$
$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$$
$$a(t) = \begin{cases} a_1, & t \to -\infty, \\ a_2, & t \to +\infty. \end{cases}$$

$$\Box\, \phi = \frac{1}{|g|^{1/2}} \partial_\mu \left(|g|^{1/2} \partial^\mu \phi \right) = 0$$

$$\frac{1}{a(t)^3} \partial_t \left(a(t)^3 \partial_t \phi \right) - \frac{1}{a(t)^2} \sum_{i=1}^{3} \partial_i \phi = 0$$

$$\phi = \sum_{\vec{k}} \left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x) \right),$$

$$f_{\vec{k}} = V^{-1/2} e^{i\vec{k}\vec{x}} \psi_k(\tau),$$

$$\tau = \int_{t_0}^{t} a(t')^{-3} dt'$$

$$\frac{d^2 \psi_k(\tau)}{d\tau^2} + k^2 a^4 \psi_k = 0$$

$$\psi_k \sim e^{-i\frac{k}{a_1}t}$$

$$f_{\vec{k}} \sim \frac{1}{\sqrt{2V a_1^3 \omega_{1k}}} e^{i\left(\vec{k}\vec{x} - \omega_{1k} t\right)}$$

$$x^i \longrightarrow x^i = a_1 x^i$$

$$k'^i = k^i / a_1, \left|\overrightarrow{k'}\right| = k/a_1 = \omega_{1k}$$

$$\begin{aligned}
\left(f_{\vec{k}}, f_{\overrightarrow{k'}} \right) &= i \int d^3x |g|^{1/2} g^{0\nu} f_{\vec{k}} \overleftrightarrow{\partial_\nu} f_{\overrightarrow{k'}} \\
&= i \int d^3x \frac{1}{2V(\omega_{1k}\omega_{1k'})^{1/2}} (-i)(\omega_{1k'} + \omega_{1k'}) e^{i(\omega_{1k} - \omega_{1k'})t} e^{i\left(\overrightarrow{k'} - \vec{k}\right)\vec{x}} \\
&= \delta_{\vec{k},\overrightarrow{k'}}
\end{aligned}$$

$$\begin{aligned}
\left(f_{\vec{k}}, f_{\vec{k}'}^* \right) &= i \int d^3x |g|^{1/2} g^{0\nu} f_{\vec{k}} \overleftrightarrow{\partial_\nu} f_{\vec{k}'}^* \\
&= i \int d^3x \frac{1}{2V(\omega_{1k}\omega_{1k'})^{1/2}} i(\omega_{1k'} - \omega_{1k}) e^{i(\omega_{1k} + \omega_{1k'})t} e^{-i\left(\overrightarrow{k'} + \vec{k}\right)\vec{x}} \\
&= 0
\end{aligned}$$

$$\left[A_{\vec{k}}, A_{\vec{k}'} \right] = \left[A_{\vec{k}}^\dagger, A_{\vec{k}'}^\dagger \right] = 0$$

$$\left[A_{\vec{k}}, A_{\vec{k}'}^\dagger \right] = \delta_{\vec{k},\vec{k}'}$$

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\Pi(\vec{x},t), \Pi(\vec{x}',t)] = 0$$

$$i\delta^{(3)}(\vec{x} - \vec{x}') = [\phi(\vec{x},t), \Pi(\vec{x}',t)].$$

$$\begin{aligned}
[\phi(\vec{x},t), \phi(\vec{x}',t)] = \sum_{\vec{k},\vec{k}'} \Bigg(&\left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x) \right) \left(A_{\overrightarrow{k'}} f_{\overrightarrow{k'}}(x') + A_{\overrightarrow{k'}}^\dagger f_{\overrightarrow{k'}}^*(x') \right) - \\
&\left(A_{\overrightarrow{k'}} f_{\overrightarrow{k'}}(x') + A_{\overrightarrow{k'}}^\dagger f_{\overrightarrow{k'}}^*(x') \right) \left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x) \right) \Bigg)
\end{aligned}$$

$$\begin{aligned}
= \sum_{\vec{k},\overrightarrow{k'}} \Big(&\left[A_{\vec{k}}, A_{\overrightarrow{k'}} \right] f_{\vec{k}}(x) f_{\overrightarrow{k'}}(x') + \left[A_{\vec{k}}^\dagger, A_{\overrightarrow{k'}} \right] f_{\overrightarrow{k'}}(x') f_{\vec{k}}^*(x) \Big) + \left[A_{\vec{k}}, A_{\overrightarrow{k'}}^\dagger \right] f_{\vec{k}}(x) f_{\overrightarrow{k'}}^*(x') \\
&+ \left[A_{\vec{k}}^\dagger, A_{\overrightarrow{k'}}^\dagger \right] f_{\overrightarrow{k'}}^*(x') f_{\vec{k}}^*(x) \Big) = 0
\end{aligned}$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = a^3 \partial_t \phi = \partial_\tau \phi$$

$$\begin{aligned}
[\phi(\vec{x},t), \Pi(\vec{x}',t)] = a_1^3 \sum_{\vec{k},\overrightarrow{k'}} \Bigg(&\left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x) \right) \left(A_{\overrightarrow{k'}} \partial_t f_{\overrightarrow{k'}}(x') + A_{\overrightarrow{k'}}^\dagger \partial_t f_{\overrightarrow{k'}}^*(x') \right) - \\
&\left(A_{\overrightarrow{k'}} \partial_t f_{\overrightarrow{k'}}(x') + A_{\overrightarrow{k'}}^\dagger \partial_t f_{\overrightarrow{k'}}^*(x') \right) \left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x) \right) \Bigg)
\end{aligned}$$

$$= a_1^3 \sum_{\vec{k},\vec{k}'} \left(\delta_{\vec{k},\vec{k}'} f_{\vec{k}}(x) \partial_t f_{\vec{k}'}^*(x') - \delta_{\vec{k},\vec{k}'} f_{\vec{k}}^*(x) \partial_t f_{\vec{k'}}(x')\right)$$

$$= a_1^3 \sum_{\vec{k}} \left(f_{\vec{k}}(x) \partial_t f_{\vec{k}}^*(x) - f_{\vec{k}}^*(x) \partial_t f_{\vec{k}}(x)\right)$$

$$= i \frac{1}{2V} \sum_{\vec{k}} \cos\left(\vec{k}(\vec{x} - \vec{x}')\right) = i\delta^{(3)}(\vec{x} - \vec{x}')\Bigg)$$

$$A_{\vec{k}}|0\rangle = 0, \forall \vec{k}$$

$$\psi_k^{(\pm)}(\tau) \sim \frac{1}{(2a_2^3\omega_{2k})^{-1/2}} e^{\mp i\omega_{2k}a_2^3\tau}$$

$$\psi_k(\tau) = \alpha_k \psi_k^{(+)}(\tau) + \beta_k \psi_k^{(-)}(\tau)$$

$$\psi_k(\tau) \sim \frac{1}{(2a_2^3\omega_{2k})^{-1/2}} \left(\alpha_k e^{-i\omega_{2k}a_2^3\tau} + \beta_k e^{i\omega_{2k}a_2^3\tau}\right)$$

$$\psi_k \partial_\tau \psi_k^* - \psi_k^* \partial_\tau \psi_k = i$$

$$1 = |\alpha_k|^2 - |\beta_k|^2$$

$$f_{\vec{k}} \sim \frac{1}{(2Va_2^3\omega_{2k})^{1/2}} e^{i\vec{k}\vec{x}} \left(\alpha_k e^{-i\omega_{2k}t} + \beta_k e^{i\omega_{2k}t}\right)$$

$$\phi = \sum_{\vec{k}} \left(a_{\vec{k}} g_{\vec{k}}(x) + a_{\vec{k}}^\dagger g_{\vec{k}}^*(x)\right),$$

$$g_{\vec{k}}(x) \sim \frac{1}{\sqrt{2Va_1^3\omega_{2k}}} e^{i(\vec{k}\vec{x} - \omega_{2k}t)}$$

$$\phi = \sum_{\vec{k}} \left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x)\right)$$

$$= \sum_{\vec{k}} \frac{1}{(2Va_2^3\omega_{2k})^{1/2}} \Big(A_{\vec{k}} \alpha_k e^{i(\vec{k}\vec{x} - \omega_{2k}t)} + A_{\vec{k}} \beta_k e^{i(\vec{k}\vec{x} + \omega_{2k}t)} + A_{\vec{k}}^\dagger \alpha_k^* e^{-i(\vec{k}\vec{x} - \omega_{2k}t)} +$$

$$= A_{\vec{k}}^\dagger \beta_k^* e^{-i(\vec{k}\vec{x} + \omega_{2k}t)}\Big)$$

$$\sum_{\vec{k}} \left(\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger\right) g_{\vec{k}}(x) + \left(\alpha_k^* A_{\vec{k}}^\dagger + \beta_k A_{-\vec{k}}\right) g_{\vec{k}}^*(x)\right)$$

$$a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger$$

$$a_{\vec{k}}^\dagger = \alpha_k^* A_{\vec{k}}^\dagger + \beta_k A_{-\vec{k}}$$

$$\left[a_{\vec{k}}, a_{\vec{k}}^\dagger\right] = \left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger\right)\left(\alpha_k^* A_{\vec{k}}^\dagger + \beta_k A_{-\vec{k}}\right) - \left(\alpha_k^* A_{\vec{k}}^\dagger + \beta_k A_{-\vec{k}}\right)\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger\right)$$

$$= \delta_{\vec{k},\vec{k}'}(|\alpha_k|^2 - |\beta_k|^2) = \delta_{\vec{k},\vec{k}'}$$

$$\langle N_{\vec{k}} \rangle_{t\to-\infty} = \langle 0|A_{\vec{k}}^\dagger A_{\vec{k}}|0\rangle = 0$$

$$\langle N_{\vec{k}} \rangle_{t\to\infty} = \langle 0|a_{\vec{k}}^\dagger a_{\vec{k}}|0\rangle$$

$$= \langle 0| \left(\alpha_k^* A_{\vec{k}}^\dagger + \beta_k A_{-\vec{k}}\right)\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger\right)|0\rangle = |\beta_k|^2$$

$$\Box\, \phi + (m^2 + \xi R)\phi = 0$$

$$\phi = \sum_{\vec{k}} \left(A_{\vec{k}} f_{\vec{k}}(x) + A_{\vec{k}}^\dagger f_{\vec{k}}^*(x)\right)$$

$$f_{\vec{k}} \sim \frac{1}{\sqrt{2Va_1^3\omega_{1k}}} e^{i(\vec{k}\vec{x} - \omega_{1k}t)}$$

$$\phi = \sum_{\vec{k}} \left(a_{\vec{k}} g_{\vec{k}}(x) + a_{\vec{k}}^{\dagger} g_{\vec{k}}^{*}(x)\right)$$

$$g_{\vec{k}} \sim \frac{1}{\sqrt{2Va_2^3\omega_k}} e^{i(\vec{k}\vec{x}-\omega_{2k}t)}$$

$$a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{\dagger}$$

$$a_{\vec{k}}^{\dagger} = \alpha_k^* A_{\vec{k}}^{\dagger} + \beta_k A_{-\vec{k}}$$

$$1 = |\alpha_k|^2 - |\beta_k|^2$$

$$\left\{A_{\vec{k}}^{\dagger}, A_{\vec{k}'}^{\dagger}\right\}_{\pm} = \left\{A_{\vec{k}}, A_{\vec{k}'}\right\}_{\pm} = 0$$

$$\left\{A_{\vec{k}}, A_{\vec{k}'}^{\dagger}\right\}_{\pm} = \delta_{\vec{k},\vec{k}'}$$

$$\{a_{\vec{k}}, a_{\vec{k'}}\}_{\pm} = \left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{\dagger}\right)\left(\alpha_{k'} A_{\vec{k'}} + \beta_{k'}^* A_{-\vec{k'}}^{\dagger}\right) \pm$$

$$\left(\alpha_{k'} A_{\vec{k'}} + \beta_{k'}^* A_{-\vec{k'}}^{\dagger}\right)\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{\dagger}\right)$$

$$=\left(\alpha_k \beta_{k'}^* \pm \beta_k^* \alpha_{k'}\right)\delta_{\vec{k},-\vec{k'}}$$

$$\left\{a_{\vec{k}}, a_{\vec{k'}}^{\dagger}\right\}_{\pm} =\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{\dagger}\right)\left(\alpha_{k'}^* A_{\vec{k'}}^{\dagger} + \beta_{\prime k} A_{-\vec{k'}}\right) \pm$$

$$\left(\alpha_{k'}^* A_{\vec{k'}}^{\dagger} + \beta_{\prime k} A_{-\vec{k'}}\right)\left(\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{\dagger}\right)$$

$$=(|\alpha_k|^2 \pm |\beta_k|^2)\delta_{\vec{k},\vec{k'}}.$$

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2\right)$$

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$$

$$\phi(x) \longrightarrow \tilde{\phi}(x) = \Omega^{-1}(x)\phi(x)$$

$$\frac{\delta S}{\delta\phi} = \frac{\delta\tilde{S}}{\delta\phi} = \frac{\delta\tilde{S}}{\delta\tilde{\phi}}\Omega^{-1}$$

$$\left(\Box + \frac{1}{6}R\right)\phi = \Omega^3\left(\tilde{\Box} + \frac{1}{6}\tilde{R}\right)\tilde{\phi}$$

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$$

$$\eta = \int_{-\infty}^{t} a^{-1}(t')dt'$$

$$ds^2 = a(t)^2(d\eta^2 - dx^2 - dy^2 - dz^2)$$

$$\tilde{f}_{\vec{k}}(x) = \frac{1}{(2Vk)^{1/2}} e^{i(\vec{k}\vec{x}-k\eta)}$$

$$f_{\vec{k}}(x) = a^{-1}(t)\tilde{f}_{\vec{k}}(x) = \frac{1}{(2Va^3(t)\omega_k(t))^{1/2}} e^{i\left(\vec{k}\vec{x}-\int_{-\infty}^{t}\omega_k(t')dt'\right)},$$

$$\phi = \sum_{\vec{k}} \left(A_{\vec{k}} f_{\vec{k}} + A_{\vec{k}}^{\dagger} f_{\vec{k}}^{*}\right),$$

$$A_{\vec{k}}|0\rangle = 0$$

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\,\theta d\varphi^2$$

$$\frac{D}{D\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0$$

$$\int_a^b \mathcal{L}d\lambda, \mathcal{L} = \frac{1}{2}g^{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$p_\mu = g_{\mu\nu}\frac{dx^\nu}{d\lambda} = \frac{\partial\mathcal{L}}{\partial(dx^\mu/d\lambda)}$$

$$E = p_t = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$$
$$L = r^2 \frac{d\varphi}{d\lambda}$$
$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$
$$E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = 0$$
$$\frac{dr}{d\lambda} = \pm E$$
$$\frac{dt}{d\lambda} \mp \left(1 - \frac{2M}{r}\right)^{-1}\frac{dr}{d\lambda} = 0$$
$$\frac{dr^*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1}$$
$$r^* = r + 2M\ln(r - 2M)$$
$$\frac{d}{d\lambda}(t \mp r^*) = 0$$
$$\frac{du}{d\lambda} = \frac{dt}{d\lambda} - \frac{dr^*}{d\lambda}$$
$$\frac{dr^*}{d\lambda} = \frac{dr^*}{dr}\frac{dr}{d\lambda} = -\left(1 - \frac{2M}{r}\right)^{-1} E$$
$$\frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} E$$
$$\frac{du}{d\lambda} = \frac{2}{\left(1 - \frac{2M}{r}\right)} E$$
$$r - 2M = E$$
$$\frac{du}{d\lambda} = 2E - \frac{4M}{\lambda}$$
$$u(\lambda) = 2E\lambda - 4M\ln(\lambda/K_1)$$

$$u(v) = -4M\ln(\lambda/K_1)$$
$$v_0 - v = K_2\lambda$$
$$u(v) = -4M\ln\left(\frac{v_0 - v}{K_1K_2}\right)$$
$$\square\,\phi = \frac{1}{|g|^{1/2}}\partial_\mu\left(|g|^{1/2} g^{\mu\nu}\partial_\nu\phi\right) = 0$$
$$\phi = \int_0^\infty d\omega\left(a_\omega f_\omega + a_\omega^\dagger f_\omega^*\right)$$
$$\left(f_{\omega_1}, f_{\omega_2}\right) = \delta(\omega_1 - \omega_2)$$
$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$
$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^{(3)}(\vec{x} - \vec{x}')$$
$$\left[a_{\omega_1}, a_{\omega_2}\right] = \left[a_{\omega_1}^\dagger, a_{\omega_2}^\dagger\right] = 0$$
$$\left[a_{\omega_1}, a_{\omega_2}^\dagger\right] = \delta(\omega_1 - \omega_2)$$
$$f_\omega \sim \frac{1}{\sqrt{\omega}r} e^{-i\omega v} S(\theta, \varphi)$$
$$\square\, f_\omega = \frac{1}{1 - \frac{2m}{r}}\partial_t^2 f_\omega - \left(1 - \frac{2M}{r}\right)\frac{2}{r}\partial_r f_\omega - \left(1 - \frac{2M}{r}\right)\partial_r^2 f_\omega$$
$$= -\frac{2i\omega M}{r^2 - 2Mr} = O(r^{-2})$$
$$\left(p_{\omega_1}, p_{\omega_2}\right) = \delta(\omega_1 - \omega_2)$$

$$p_\omega \sim \frac{1}{\sqrt{\omega} r} e^{-i\omega u} S(\theta,\varphi)$$

$$\square\, p_\omega = O(r^{-2})$$

$$\left(q_{\omega_1}, q_{\omega_2}\right) = \delta(\omega_1, \omega_2)$$

$$\left(q_{\omega_1}, p_{\omega_2}\right) = \left(q_{\omega_1}, p^*_{\omega_2}\right) = 0$$

$$\phi = \int_0^\infty d\omega\left(b_\omega p_\omega + c_\omega q_\omega + b^\dagger_\omega p^*_\omega + c^\dagger_\omega q^*_\omega\right)$$

$$\left[b_{\omega_1}, b^\dagger_{\omega_2}\right] = \left[c_{\omega_1}, c^\dagger_{\omega_2}\right] = \delta(\omega_1 - \omega_2),$$

$$a_\omega|0\rangle = 0, \forall\omega$$

$$p_\omega = \int_0^\infty d\omega'\left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right)$$

$$(p_\omega, \phi) = \left(p_\omega, \int_0^\infty d\omega'\left(b_{\omega'} p_{\omega'} + c_{\omega'} q_{\omega'} + b^\dagger_{\omega'} p^*_{\omega'} + c^\dagger_{\omega'} q^*_{\omega'}\right)\right)$$

$$= \int_0^\infty d\omega' b_{\omega'} \delta(\omega - \omega') = b_\omega$$

$$(p_\omega, \phi) = \left(\left(\int_0^\infty d\omega'\left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right)\right), \int_0^\infty d\omega''\left(a_{\omega''} f_{\omega''} + a^\dagger_{\omega''} f^*_{\omega''}\right)\right)$$

$$= \int_0^\infty d\omega' \int_0^\infty d\omega'' \left(\alpha_{\omega\omega'} a_{\omega''} \delta(\omega' - \omega'') - \beta_{\omega\omega'} a^\dagger_{\omega''} \delta(\omega' - \omega'')\right)$$

$$= \int_0^\infty d\omega'\left(\alpha_{\omega\omega'} a_{\omega'} - \beta_{\omega\omega'} a^\dagger_{\omega'}\right)$$

$$\left(p_{\omega_1}, p_{\omega_2}\right) = \left(\int_0^\infty d\omega'\left(\alpha_{\omega_1\omega'} f_{\omega'} + \beta_{\omega_1\omega'} f^*_{\omega'}\right), \int_0^\infty d\omega''\left(\alpha_{\omega_2\omega''} f_{\omega''} + \beta_{\omega_2\omega'2} f^*_{\omega'2}\right)\right)$$

$$= \int_0^\infty \left(\alpha^*_{\omega_1\omega'} \alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'} \beta_{\omega_2\omega'}\right)$$

$$(f_{\omega'}, p_\omega) = \int_0^\infty d\omega'' \left(\alpha_{\omega\omega''}(f_{\omega'}, f_{\omega''}) + \beta_{\omega\omega''}(f_{\omega'}, f^*_{\omega''})\right)$$

$$= \int_0^\infty d\omega''\left(\alpha_{\omega\omega''} \delta(\omega' - \omega'')\right) = \alpha_{\omega\omega'}$$

$$-(f^*_{\omega'}, p_\omega) = -\int_0^\infty d\omega'' \left(\alpha_{\omega\omega''}(f^*_{\omega'}, f_{\omega''}) + \beta_{\omega\omega''}(f^*_{\omega'}, f^*_{\omega''})\right)$$

$$= -\int_0^\infty d\omega'' \left(\beta_{\omega\omega''}\left(-\delta(\omega' - \omega'')\right)\right) = \beta_{\omega\omega'}$$

$$u(v) = -4M\ln\left(\frac{v - v_0}{K}\right), K = K_1 K_2 > 0$$

$$p_\omega \sim \frac{1}{\sqrt{\omega}} \frac{e^{-i\omega u(v)}}{r} S(\theta,\varphi)$$

$$f_{\omega'} \sim \frac{1}{\sqrt{\omega'}} \frac{e^{-i\omega v}}{r} S(\theta,\varphi)$$

$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega) = i\int dV_x \frac{1}{1-\frac{2M}{r}}\left(f^*_{\omega'}\partial_t p_\omega - \partial_t f^*_{\omega'} p_\omega\right)$$

$$= C\int_{2M}^{\infty} \frac{r^2}{1-\frac{2M}{r}} \frac{e^{i\omega' v} e^{-i\omega u(v)}}{r^2}\left(\sqrt{\frac{\omega'}{\omega}} + \sqrt{\frac{\omega}{\omega'}}\right) dr$$

$$= -C\int_{-\infty}^{v_0} dv \sqrt{\frac{\omega'}{\omega}} e^{i\omega' v} e^{-i\omega u(v)},$$

$$\beta_{\omega\omega'} = -(f^*_{\omega'}, p_\omega) = C\int_{-\infty}^{v_0} dv \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v} e^{-i\omega u(v)}$$

$$\alpha_{\omega\omega'} = -C\int_{\infty}^{0} ds \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' s} e^{i\omega' v_0} e^{i\omega 4M \ln(s/K)}$$

$$\beta_{\omega\omega'} = C\int_{-\infty}^{0} ds \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' s} e^{-i\omega' v_0} e^{i\omega 4M \ln(-s/K)}$$

$$\alpha_{\omega\omega'} = iC\int_{-\infty}^{0} ds' \sqrt{\frac{\omega'}{\omega}} e^{\omega' s'} e^{i\omega' v_0} e^{i\omega 4M \log(is'/K)}$$

$$\beta_{\omega\omega'} = -iC\int_{-\infty}^{0} ds' \sqrt{\frac{\omega'}{\omega}} e^{\omega' s'} e^{-i\omega' v_0} e^{i\omega 4M \log(-is'/K)}$$

$$\log\left(\frac{is'}{K}\right) = \ln\left(\frac{|s'|}{K}\right) - i\frac{\pi}{2}$$

$$\log\left(\frac{-is'}{K}\right) = \ln\left(\frac{|s'|}{K}\right) + i\frac{\pi}{2}$$

$$\alpha_{\omega\omega'} = iCe^{i\omega' v_0} e^{2\omega M\pi}\int_{-\infty}^{0} ds' \sqrt{\frac{\omega'}{\omega}} e^{\omega' s'} e^{i\omega 4M \ln(|s'|/K)}$$

$$\beta_{\omega\omega'} = -iCe^{-i\omega' v_0} e^{-2\omega M\pi}\int_{-\infty}^{0} ds' \sqrt{\frac{\omega'}{\omega}} e^{\omega' s'} e^{i\omega 4M \ln(|s'|/K)}$$

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi\omega M}|\beta_{\omega\omega'}|^2$$

$$\left(p_{\omega_1}, p_{\omega_2}\right) = \Gamma(\omega_1)\delta(\omega_1-\omega_2)$$

$$\left(p_{\omega_1}, p_{\omega_2}\right) = \left(p^{(1)}_{\omega_1}, p^{(1)}_{\omega_2}\right) + \left(p^{(2)}_{\omega_1}, p^{(2)}_{\omega_2}\right)$$

$$\left(p^{(1)}_{\omega_1}, p^{(1)}_{\omega_2}\right) = \left(1-\Gamma(\omega_1)\right)\delta(\omega_1-\omega_2)$$

$$\Gamma(\omega_1)\delta(\omega_1-\omega_2) = \int_0^{\infty} d\omega'\left(\alpha^*_{\omega_1\omega'}\alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'}\beta_{\omega_2\omega'}\right)$$

$$b_\omega = \left(p^{(2)}_\omega, \phi\right) = \int_0^{\infty} d\omega'\left(\alpha_{\omega\omega'}a_{\omega'} + \beta_{\omega\omega'}a^\dagger_{\omega'}\right)$$

$$\langle N\rangle = \langle 0|b^\dagger_\omega b_\omega|0\rangle = \int_0^{\infty} d\omega'\beta_{\omega\omega'}\langle\omega'|\int_0^{\infty} d\omega''\beta^*_{\omega\omega''}|\omega''\rangle = \int_0^{\infty} d\omega'|\beta_{\omega\omega'}|^2$$

$$\Gamma(\omega)\delta(0) = \int_0^{\infty} d\omega'\left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2\right) = \left(e^{8\pi M\omega}-1\right)\int_0^{\infty} d\omega'|\beta_{\omega\omega'}|^2$$

$$\delta(\omega_1-\omega_2) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T/2}^{T/2} dt\, e^{i(\omega_1-\omega_2)t}$$

$$\Gamma(\omega)\lim_{T\to\infty}\frac{T}{2\pi}=(e^{8\pi M\omega}-1)\int_0^\infty d\omega'|\beta_{\omega\omega'}|^2$$

$$\langle N\rangle=\lim_{T\to\infty}\frac{T}{2\pi}\Gamma(\omega)\frac{1}{e^{8\pi M\omega}-1}$$

$$\frac{\Gamma(\omega)}{2\pi}\frac{1}{e^{8\pi M\omega}-1}$$

$$T=\frac{1}{8\pi k_B M}=\frac{\kappa}{2\pi}$$

$$p=\int d\omega A(\omega)e^{i\gamma(\omega)}p_\omega$$

$$T=\frac{\hbar c^3}{8\pi GMk_B}\approx 10^{-7}\left(\frac{M_\odot}{M}\right)K$$

$$\frac{dE}{dt}=4\pi r_s^2\sigma T^4$$

$$\frac{dM}{dt}=-\beta\frac{m_p^3}{t_p}\frac{1}{M^2},$$

$$M(t)\ =\left(M_0^3-3\beta\frac{m_p^3}{t_p}t\right)^{1/3}$$

$$\Delta t\ =\frac{t_p}{3}\beta\left(\frac{M_0}{m_p}\right)^3$$

3. Supersimetrías, agujeros negros de mecánica cuántica y supermembranas – Modelo de Supergravedad Cuántica. Debido a la permeabilidad del campo gravitónico en espacios infinitos e indiscriminados, las superpartículas comportan la relación entre bosones y fermiones, a propósito de la interacción con el gravitón, el cual, como se ha dicho también, puede ser una partícula supersimétrica o compañera de otra partícula – gauge de naturaleza fermiónica, todo esto, sin perjuicio del efecto cuántico gravitacional implícito respecto de aquellas partículas subatómicas supermasivas, en las que, por la distorsión del espacio – tiempo cuántico, crean supersimetrías. En este punto, es importante precisar que los campos cuánticos, no son irreductibles, es decir, que coexisten y se afectan entre sí, pues, bosones y fermiones son en sí y a propósito de la supersimetría, partículas compañeras que se afectan unas a otras, según el espectro gravitacional involucrado.

Ahora bien, cuando una superpartícula, deforma el espacio – tiempo cuántico, sea por efecto cuántico gravitacional extrínseco (campo gravitónico – gravitón) o intrínseco (en relación a las partículas supermasivas), según sea el caso, produce diversos fenómenos físicos, desde microagueros o agujeros negros cuánticos, que por su naturaleza son pequeños y que resultan de la colisión de partículas, producto de la distorsión geométrica del espacio – tiempo cuántico, así como supermembranas, es decir, distintas capas dimensionales que se desdoblan desde la singularidad de la curvatura y que dan lugar a diferentes estados transicionales de la materia, lo que equivale a la supergravedad a escala cuántica.

El modelo matemático, comporta:

La métrica de un gravitón está dada por G_{MN} con $M,N=0,1,\cdots,D-1$. En un espacio de D dimensiones, resulta en $\frac{1}{2}(D-1)(D-2)-1$. Para lograr la supersimetría, el gravitón tiene un compañero, esto es, el gravitino $\Psi_{M,\alpha}$, la cual es una partícula – gauge, las cuales interactúan en superespacios. Los cálculos de los espinores, calibre, escalares, tensores y vectores en un superespacio – tiempo infinito, de gauge y curvado bajo parámetros de transformación supersimétricos, por acoplamiento del gravitón y el gravitino, en un campo de gauge gravitónico, comprenden:

$$(\Gamma^M\Psi_M)_\alpha=0$$

Dimensión espacio-tiempo D	Espinor dimensión $\tilde{D}$	Gravitón $\frac{1}{2}(D-1)(D-2)-1$	Gravitino $\frac{1}{2}(D-3)\tilde{D}$	Diferencia

4	4	2	2	0
5	8	5	8	3
6	8	9	12	3
7	16	14	32	18
8	16	20	40	20
9	16	27	48	21
10	16	35	56	21
11	32	44	128	84
12	64	54	288	234

Ahora bien, esta relación híbrida, que da como resultado, la dinámica del campo gravitónico, matemáticamente comporta lo que sigue, siguiendo el sistema de gauge de Yang – Mills – Einstein – Newton – Dirac – Transformaciones de Supergravedad y reparación de gauge.

$$G_{MN}, C_{MNP}, \Psi_{M,\alpha}$$

$$S_{11d} = \frac{1}{16\pi\mathcal{G}_{(11)}}\left[\int d^{11}x\sqrt{\|G\|}\left(R - \frac{1}{2}|G|^2 + \frac{1}{6}\int C \wedge G \wedge G - \frac{i}{2}\bar{\psi}_M\Gamma^{MNP}D_N\left(\frac{\omega + \hat{\omega}}{2}\right)\psi_P \right.\right.$$
$$\left.\left. - \frac{i}{384}(\bar{\psi}_M\Gamma^{MNABCD}\psi_N + 12\bar{\psi}^A\Gamma^{BC}\psi^D)\left(G_{ABCD} + \hat{G}_{ABCD}\right)\right)\right]$$

$$16\pi\mathcal{G}_{(11)} = \frac{\left(2\pi\ell_p\right)^9}{2\pi}$$

$$D_M(\omega)\psi_N \equiv \partial_M\psi_N - \frac{1}{4}\omega_{MAB}\Gamma^{AB}\psi_N$$

$$\omega_{MAB} = \omega_{MAB}^{(0)}(E) + \frac{i}{16}[\bar{\psi}_N\Gamma_{MAB}{}^{NP}\psi_P - 2(\bar{\psi}_M\Gamma_B\psi_A - \bar{\psi}_M\Gamma_A\psi_B + \bar{\psi}_B\Gamma_M\psi_A)]$$

$$\hat{\omega}_{MAB} = \omega_{MAB} - \frac{i}{16}\bar{\psi}_N\Gamma_{MAB}{}^{NP}\psi_P.$$

$$G_{LMNP} \equiv 4\partial_{[L}C_{MNP]}$$

$$\hat{G}_{LMNP} \equiv G_{LMNP} + \frac{3}{2}i\bar{\psi}_L\Gamma_{MN}\psi_P.$$

$$|G|^2 = |dC|^2 \equiv \frac{1}{4!}G_{LMNP}G^{LMNP}$$

$$\delta E_M^A = \frac{i}{2}\bar{\epsilon}\Gamma^A\Psi_M$$

$$\delta C_{MNP} = -\frac{3}{2}i\bar{\epsilon}\Gamma_{[MN}\Psi_{P]}$$

$$\delta C = d\Lambda$$

$$G_{\mu\nu}^{(11)} = G_{\mu\nu}^{(10)} + e^{2\gamma}A_\mu A_\nu, G_{\mu 10}^{(11)} = e^{2\gamma}A_\mu, G_{1010}^{(11)} = e^{2\gamma}$$

$$C_{\mu\nu\rho}^{(11)} = C_{\mu\nu\rho}^{(10)}, C_{\mu\nu 10}^{(11)} = B_{\mu\nu}.$$

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \|A - B^T D^{-1} C\|\|D\|$$

$$\sqrt{\|G^{(11)}\|} = \sqrt{\|G^{(10)}\|}e^\gamma$$

$$\frac{2\pi}{\left(2\pi\ell_p\right)^9}\int d^{11}x\sqrt{\|G^{(11)}\|}R \rightarrow \frac{(2\pi)^2 R_{10}}{\left(2\pi\ell_p\right)^9}\int d^{10}x\sqrt{\|G^{(10)}\|}\left(e^\gamma\left(R - \frac{1}{2}|d\gamma|^2\right) - \frac{1}{2}e^{3\gamma}|dA|^2\right)$$

$$-\frac{2\pi}{(2\pi\ell_p)^9}\frac{1}{2}\int d^{11}x\sqrt{\|G^{(11)\|}\|G|^2} \to -\frac{(2\pi)^2R_{10}}{(2\pi\ell_p)^9}\frac{1}{2}\int d^{10}x\sqrt{\|G^{(10)\|}}\left(e^{\gamma}\left|dC^{(10)}\right|^2+e^{-\gamma}|dB|^2\right)+\cdots$$

$$-\frac{2\pi}{(2\pi\ell_p)^9}\frac{1}{6}\int C\wedge G\wedge G \to -\frac{(2\pi)^2R_{10}}{(2\pi\ell_p)^9}\frac{1}{2}\int B\wedge dC^{(10)}\wedge dC^{(10)}$$

$$S_{IIA} = \frac{2\pi}{(2\pi\ell_s)^8}\int d^{10}x\sqrt{\|G^{(10)}\|}\left[e^{-2\Phi}\left(R+|d\Phi|^2-\frac{1}{2}|dB|^2\right)-\left(\frac{1}{2}|dA|^2+\frac{1}{2}|dC|^2\right)\right]$$

$$-\frac{2\pi}{(2\pi\ell_s)^8}\frac{1}{2}\int B\wedge dC\wedge dC+\cdots,$$

$$G_{\mu\nu}\to e^{-\gamma}G_{\mu\nu}, \Phi=\frac{3}{2}\gamma,$$

$$G^{(11)}_{\mu\nu}=e^{-\gamma}G^{(10)}_{\mu\nu}=e^{-\frac{2}{3}\Phi}G^{(10)}_{\mu\nu}.$$

$$\frac{L}{\ell_p}=e^{-\frac{1}{2}\gamma}\frac{L}{\ell_s}=e^{-\frac{1}{3}\Phi}\frac{L}{\ell_s}.$$

$$e^{\langle\Phi\rangle}=g_s$$

$$\ell_p=g_s^{\frac{1}{3}}\ell_s$$

$$\frac{(2\pi)^2R_{10}}{(2\pi\ell_p)^9}=\frac{2\pi}{(2\pi\ell_s)^8}\frac{1}{g_s^2},$$

$$R_{10}=g_s\ell_s$$

$$\int A_M dX^M$$

$$\int A_M(X(\tau))\frac{dX^M}{d\tau}d\tau=\int A_\tau d\tau$$

$$A_\tau\equiv A_M\frac{dX^M}{d\tau}$$

$$\int B_{MN}dX^M\wedge dX^N=\int B_{\mu\nu}d\xi^\mu\wedge d\xi^\mu$$

$$B_{\mu\nu}\equiv B_{MN}\frac{dX^M}{d\xi^\mu}\frac{dX^N}{d\xi^\nu}$$

$$\int C_{MNP}dX^M\wedge dX^N\wedge dX^P=\int C_{\mu\nu\lambda}d\xi^\mu\wedge d\xi^\nu\wedge d\xi^\lambda=\frac{1}{6}\int d^3\xi\epsilon^{\mu\nu\lambda}C_{\mu\nu\lambda}$$

$$C_{\mu\nu\lambda}\equiv C_{MNP}\frac{dX^M}{d\xi^\mu}\frac{dX^N}{d\xi^\nu}\frac{dX^P}{d\xi^\lambda}$$

$$S_{M2}^{\text{bosonic}}=\int d^3\xi\left(\frac{1}{2}\sqrt{|g|}g^{\mu\nu}G_{MN}\partial_\mu X^M\partial_\nu X^N-\frac{1}{2}\sqrt{|g|}+\frac{1}{6}\epsilon^{\mu\nu\lambda}C_{MNP}\partial_\mu X^M\partial_\nu X^N\partial_\lambda X^P\right)$$

$$g_{\mu\nu}=G_{MN}\partial_\mu X^M\partial_\nu X^N$$

$$\Pi_\mu^M\equiv\partial_\mu X^M-i\bar\theta\Gamma^M\partial_\mu\theta$$

$$S_{M2}^{\text{susy}}=\int d^3\xi\left(\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\Pi_\mu^M\Pi_\nu^N-\frac{1}{2}\sqrt{|g|}\right.$$

$$\left.+\frac{i}{2}\epsilon^{\mu\nu\lambda}\bar\theta\Gamma_{MN}\partial_\mu\theta\left[\Pi_\nu^M\Pi_\lambda^N+i\Pi_\nu^M\bar\theta\Gamma^N\partial_\lambda\theta-\frac{1}{3}\bar\theta\Gamma^M\partial_\nu\theta\bar\theta\Gamma^N\partial_\lambda\theta\right]\right).$$

$$\delta X^M=-i\bar\theta\Gamma^M\varepsilon$$

$$\delta\theta=\varepsilon$$

$$\delta g_{\mu\nu}=0$$

$$\Pi_\mu\equiv\Pi_\mu^M\Gamma_M$$

$$\tau^\mu\equiv\frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\lambda}\Pi_\nu\Pi_\lambda$$

$$\Gamma\equiv\frac{1}{6\sqrt{|g|}}\epsilon^{\mu\nu\lambda}\Pi_\mu\Pi_\nu\Pi_\lambda$$

$$\delta X^M = i\bar{\theta}\Gamma^M(1+\Gamma)\kappa$$
$$\delta\theta = (1+\Gamma)\kappa$$
$$\delta\left(\sqrt{|g|}g^{\mu\nu}\right) = ig^{\sigma(\mu}\epsilon^{\nu)\lambda\rho}\bar{\kappa}(1+\Gamma)\partial_\sigma\theta\Pi_\lambda\Pi_\rho + \frac{2i}{3\sqrt{|g|}}\epsilon^{\sigma\tau(\mu}\epsilon^{\nu)\lambda\rho}\bar{\kappa}\Pi^\alpha\partial_\alpha\theta\left(\Pi_\sigma^M\Pi_{\lambda M}\Pi_\tau^N\Pi_{\rho N} + \Pi_\sigma^M\Pi_{\lambda M}g_{\tau\rho} + g_{\sigma\lambda}g_{\tau\rho}\right)$$
$$g_{\mu\nu} = \Pi_{\mu M}\Pi_\nu^M$$
$$\Gamma^2 = 1$$
$$\tau^\mu = g^{\mu\nu}\Pi_\nu\Gamma = g^{\mu\nu}\Gamma\Pi_\nu$$
$$\{\tau^\mu,\tau^\nu\} = 2g^{\mu\nu}.$$
$$A^M \equiv \partial_\mu\left\{\sqrt{|g|}g^{\mu\nu}\Pi_\nu^M - i\epsilon^{\mu\nu\lambda}(\bar{\theta}\Gamma^{MN}\partial_\nu\theta)\left(\Pi_{\lambda N} + \frac{i}{2}\bar{\theta}\Gamma_N\partial_\lambda\theta\right)\right\} = 0$$
$$(1-\Gamma)g^{\mu\nu}\Pi_\mu^M\Gamma_M\partial_\nu\theta = 0$$
$$B = (1-\Gamma)\tau^\mu\partial_\mu\theta = 0$$
$$\Pi_\mu^M A_M = -2i\sqrt{|g|}\partial_\mu\bar{\theta}B$$
$$X^M \to (X^\mu, X^I), \mu = 0,1,2;\ I = 3,4,\cdots,10,$$
$$X^\mu = \xi^\mu, \mu = 0,1,2.$$
$$\delta X^M = \eta^\nu(\xi)\partial_\nu X^M + \Lambda_N^M X^N$$
$$\delta X^\mu = \eta^\mu + \Lambda_\nu^\mu\xi^\nu + \Lambda_I^\mu X^I$$
$$\eta^\mu = -\Lambda_\nu^\mu\xi^\nu - \Lambda_I^\mu X^I$$
$$\delta X^I = -\left(\Lambda_\nu^\mu\xi^\nu + \Lambda_I^\mu X^I\right)\partial_\mu X^I + \Lambda_J^I X^J$$
$$\xi^\mu \to \xi^\mu + \ell_\nu^\mu\xi^\nu$$
$$\delta\phi = \ell_\nu^\mu\xi^\nu\partial_\mu\phi$$
$$\delta\theta = -\Lambda^\mu{}_\nu\xi^\nu\partial_\mu\theta + \frac{1}{4}\Lambda_{\mu\nu}\Gamma^{\mu\nu}\theta + \frac{1}{4}\Lambda_{IJ}\Gamma^{IJ}\theta,$$
$$(1+\Gamma^*)\theta = 0$$
$$\Gamma^* \equiv \Gamma^1\Gamma^2\cdots\Gamma^8$$
$$S_{M2,\text{ bosonic}}^{\text{static gauge}} = -T_{M2}\int d^3\xi\sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{T_{M2}}\partial_\mu X^I\partial_\nu X^I\right)} \sim -\frac{1}{2}\int \partial_\mu X^I\partial^\mu X^I + \frac{1}{T_{M2}}O(\partial X)^4 + \cdots,$$
$$\delta\Psi_M \equiv D_M(\omega)\epsilon - \frac{i}{288}\left(\Gamma_M^{PQRS} + 8\Gamma^{QRS}\delta_M^P\right)G_{PQRS}\epsilon = 0$$
$$G_7 = \star G_4 - \frac{1}{2}C_3\wedge G_4$$
$$ds^2 = f_{(1)}(r)dy^\mu dy_\mu + f_{(2)}(r)dx^I dx^I$$
$$G_{012r} = f_{(3)}(r),$$
$$r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^8)^2}$$
$$H_{M2}(r) = 1 + \frac{(r_{M2})^6}{r^6}$$
$$f_{(1)}(r) = H_{M2}(r)^{-\frac{2}{3}}$$
$$f_{(2)}(r) = H_{M2}(r)^{\frac{1}{3}}$$
$$f_{(3)}(r) = -\frac{\partial}{\partial r}\left(H_{M2}(r)^{-1}\right)$$
$$G_{J_1J_2\cdots J_7} = 6(r_{M2})^6\epsilon_{IJ_1J_2\cdots J_7}\frac{x^I}{r^8}$$
$$G_{\theta_1\theta_2\cdots\theta_7} = 6(r_{M2})^6\epsilon_{\theta_1\theta_2\cdots\theta_7}$$
$$Q^{(e)} = 6(r_{M2})^6\Omega_7 = 2\pi^4(r_{M2})^6,$$
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi\mathcal{G}_{(D)}T_{\mu\nu}$$
$$R_{00} = \frac{D-3}{D-2}8\pi\mathcal{G}_{(D)}\rho$$

$$\nabla^2 g_{00} = -2\frac{D-3}{D-2}8\pi\mathcal{G}_{(D)}\rho$$
$$\nabla^2\phi = 4\pi\mathcal{G}_{(D)}\rho$$
$$g_{00} = -\left(1+4\frac{D-3}{D-2}\phi\right)$$
$$\phi(x) = -\frac{4\pi\mathcal{G}_{(D)}M}{(D-3)\Omega_{D-2}}\frac{1}{r^{D-3}}$$
$$g_{00} = -\left(1-\frac{16\pi\mathcal{G}_{(D)}M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}\right)$$
$$R_{00} = \frac{D-p-3}{D-2}8\pi\mathcal{G}_{(D)}\rho$$
$$g_{00} = -\left(1+4\frac{D-p-3}{D-2}\phi\right)$$
$$\phi(x) = -\frac{4\pi\mathcal{G}_{(D)}T_p}{(D-p-3)\Omega_{D-p-2}}\frac{1}{r^{D-p-3}}$$
$$g_{00} = -\left(1-\frac{16\pi\mathcal{G}_{(D)}T_p}{(D-2)\Omega_{D-p-2}}\frac{1}{r^{D-p-3}}\right)$$
$$(r_{M2})^6 = \frac{8\pi}{3}\frac{\mathcal{G}_{(11)}}{\Omega_7}n_2T_{M2}$$
$$(r_{M2})^6 = 128\pi^4 n_2\ell_p^9 T_{M2}$$
$$ds^2 = g_{(1)}(r)dy^\mu dy_\mu + g_{(2)}(r)dx^I dx^I$$
$$r = \sqrt{(x^1)^2+(x^2)^2+\cdots+(x^5)^2}$$
$$H_{M5}(r) = 1+\frac{(r_{M5})^3}{r^3}$$
$$g_{(1)}(r) = H_{M5}(r)^{-\frac{1}{3}}$$
$$g_{(2)}(r) = H_{M5}(r)^{\frac{2}{3}}$$
$$G_{\theta_1\theta_2\theta_3\theta_4} = 3(r_{M5})^3\epsilon_{\theta_1\theta_2\theta_3\theta_4}$$
$$Q^{(m)} = \int_{S^4} G = 3(r_{M5})^3\Omega_4 = 8\pi^2(r_{M5})^3$$
$$(r_{M5})^3 = 32\pi^6 n_5\ell_p^9 T_{M5}$$
$$T_{M2}\sim\frac{1}{\ell_p^3}, T_{M5}\sim\frac{1}{\ell_p^6}$$
$$\frac{1}{16\pi G_{(11)}}Q^{(e)}Q^{(m)} = 2\pi n,$$
$$\frac{2\pi}{(2\pi\ell_p)^9}Q^{(e)}Q^{(m)} = (2\pi)^8\ell_p^9 T_{M2}T_{M5}$$
$$T_{M2}T_{M5} = \frac{(2\pi)^2}{(2\pi\ell_p)^9}$$
$$T_{M2} = \frac{2\pi}{(2\pi\ell_p)^3}, T_{M5} = \frac{2\pi}{(2\pi\ell_p)^6}$$
$$T_p = \frac{1}{g_s}\frac{2\pi}{(2\pi\ell_s)^{p+1}}.$$
$$T_{F1} = \frac{2\pi}{(2\pi\ell_s)^2}, T_{NS5} = \frac{1}{g_s^2}\frac{2\pi}{(2\pi\ell_s)^6}.$$
$$T_{M2} = \frac{1}{4\pi^2\ell_p^3}$$

$$
\begin{aligned}
T_{M2}^{\text{wrapped}} &= T_{M2} \times 2\pi R_{10} \\
&= \frac{1}{4\pi^2 g_s \ell_s^3} \times 2\pi g_s \ell_s \\
&= \frac{1}{2\pi \ell_s^2},
\end{aligned}
$$

$$
\begin{aligned}
T_{M2} &= \frac{1}{4\pi^2 \ell_p^3} \\
&= \frac{1}{4\pi^2 \left(g_s^{\frac{1}{3}} \ell_s\right)^3} \\
&= \frac{1}{g_s} \frac{1}{4\pi^2 \ell_s^3} \\
&= T_{D2}.
\end{aligned}
$$

$$
T_{M5} = \frac{1}{32\pi^5 \ell_p^6}
$$

$$
\begin{aligned}
T_{M5}^{\text{wrapped}} &= T_{M5} \times 2\pi R_{10} \\
&= \frac{g_s \ell_s}{16\pi^4 g_s^2 \ell_s^6} \\
&= \frac{1}{g_s} \frac{2\pi}{(2\pi \ell_s)^5} \\
&= T_{D4},
\end{aligned}
$$

$$
\begin{aligned}
T_{M5} &= \frac{1}{32\pi^5 \ell_p^6} \\
&= \frac{1}{g_s^2} \frac{1}{32\pi^5 \ell_s^6} \\
&= \frac{1}{g_s^2} \frac{2\pi}{(2\pi \ell_s)^6} \\
&= T_{NS5},
\end{aligned}
$$

$$
T_0 = \frac{1}{g_s \ell_s} = \frac{1}{R_{10}}.
$$

$$
p = \frac{2\pi n}{L}.
$$

$$
E^2 = p_1^2 + \cdots p_9^2 + p_{10}^2
$$

$$
ds_{\text{Taub-NUT}}^2 = U(\vec{x}) d\vec{x} \cdot d\vec{x} + \frac{1}{U(\vec{x})} (dy + \vec{A} \cdot d\vec{x})^2
$$

$$
\vec{B} = \vec{\nabla} \times \vec{A}
$$

$$
\vec{\nabla} U = -\vec{B}.
$$

$$
U(\vec{x}) = 1 + \frac{R}{2r}
$$

$$
\vec{B} = \frac{R}{2} \frac{\vec{x}}{r^3}
$$

$$
\frac{R}{2r} dr^2 + \frac{2r}{R} dy^2
$$

$$
\tilde{r} = \sqrt{2rR}
$$

$$
d\tilde{r}^2 + \frac{\tilde{r}^2}{R^2} dy^2
$$

$$T_{KK6} = \frac{2\pi}{\left(2\pi\ell_p\right)^9} \times 2\pi R_{10} \int \ d^3x \vec{\nabla}^2 U$$
$$= \frac{2\pi}{\left(2\pi\ell_p\right)^9} \times (2\pi R_{10})^2$$
$$= \frac{1}{g_s}\frac{2\pi}{(2\pi\ell_s)^7} = T_{D6}$$
$$g_s \to \frac{1}{g_s}, \ell_s \to \sqrt{g_s}\ell_s$$
$$T_{p,q} = \frac{1}{2\pi\ell_s^2}\sqrt{p^2 + \frac{q^2}{g_s^2}}$$
$$g_s(\mathrm{IIB}) = \frac{R_{10}}{R_9}, \ell_s(\mathrm{IIB}) = \sqrt{\frac{\ell_p^3}{R_{10}}}.$$
$$T_{M2}^{\mathrm{wrapped}} = T_{M2}\sqrt{p(2\pi R_{10})^2 + q(2\pi R_9)^2}$$
$$= \frac{1}{2\pi\ell_s^2}\sqrt{p^2 + \frac{q^2}{g_s^2}} = T_{p,q}.$$
$$S_{M2}^{\mathrm{bosonic}} = -\frac{1}{(2\pi)^2\ell_p^3}\int \ d^3\xi\sqrt{-\det\left(\eta_{\mu\nu} + (2\pi)^2\ell_p^3\partial_\mu X^I\partial_\nu X^I\right)}$$
$$S_{D2}^{\mathrm{bosonic}} = -\frac{1}{(2\pi\alpha')^2 g_{YM}^2}\int \ d^3\xi\sqrt{-\det\left(\eta_{\mu\nu} + (2\pi\alpha')^2\partial_\mu X^i\partial_\nu X^i + 2\pi\alpha' F_{\mu\nu}\right)}$$
$$g_{YM}^2 = \frac{g_s}{\sqrt{\alpha'}}$$
$$\frac{1}{g_{YM}^2}\left(-\frac{1}{2}\partial_\mu X^i\partial^\mu X^i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$
$$\mathcal{L} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{(2\pi\alpha')^2 g_{YM}^2}\sqrt{-\det\left(\eta_{\mu\nu} + (2\pi\alpha')^2\partial_\mu X^i\partial_\nu X^i + (2\pi\alpha')^2 g_{YM}^4 B_\mu B_\nu\right)}$$
$$B_\mu \to -\frac{1}{g_{YM}}\partial_\mu X^8$$
$$\mathcal{L} = -\frac{1}{(2\pi)^2\ell_p^3}\sqrt{-\det\left(\eta_{\mu\nu} + (2\pi)^2\ell_p^3\partial_\mu X^I\partial_\nu X^I\right)}$$
$$X^8 \sim X^8 + 2\pi g_{YM}$$
$$S_{M2}^{\mathrm{susy}} \overset{\ell_p\to 0}{=} \int \ d^3\xi\left(-\frac{1}{2}\partial_\mu X^I\partial^\mu X^I + \frac{i}{2}\bar{\psi}^A\gamma^\mu\partial_\mu\psi^A\right)$$
$$S_{D2}^{\mathrm{susy}} \overset{\ell_s\to 0}{=} \frac{1}{g_{YM}^2}\int \ d^3\xi\left(-\frac{1}{2}\partial_\mu X^i\partial^\mu X^i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}^A\gamma^\mu\partial_\mu\psi^A\right)$$
$$S_{nD2} \overset{\ell_s\to 0}{=} \frac{1}{g_{YM}^2}\int \ d^3\xi\mathrm{Tr}\left(-\frac{1}{2}D_\mu\boldsymbol{X}^i D^\mu\boldsymbol{X}^i + \frac{1}{4}\left[\boldsymbol{X}^i,\boldsymbol{X}^j\right]^2 - \frac{1}{4}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} + \frac{i}{2}\boldsymbol{\psi}^A\gamma^\mu D_\mu\boldsymbol{\psi}^A - \boldsymbol{\psi}^A\Gamma_{AB}^i\left[\boldsymbol{X}^i,\boldsymbol{\psi}^B\right]\right)$$
$$D_\mu\boldsymbol{X}^i = \partial_\mu\boldsymbol{X}^i - i\left[\boldsymbol{A}_\mu,\boldsymbol{X}^i\right]$$
$$\boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - i\left[\boldsymbol{A}_\mu,\boldsymbol{A}_\nu\right]$$
$$X^9 = \frac{N}{2r}, F_{\theta\phi} = -r^2\partial_r X^9,$$
$$\frac{1}{2\pi}\int \ F = N$$
$$\frac{\partial\boldsymbol{X}^i}{\partial x^9} = \pm\frac{i}{2}\epsilon^{ijk}\left[\boldsymbol{X}^j,\boldsymbol{X}^k\right], i,j,k \in 1,2,3$$
$$\left[\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^j\right] = 2i\epsilon^{ijk}\boldsymbol{\alpha}^k$$

$$X^i = \pm\frac{1}{2x^9}\alpha^i, i = 1,2,3$$
$$R^2 = \frac{(2\pi\alpha')^2}{N}\mathrm{Tr}\sum_{i=1,2,3}(X^i)^2$$
$$\sum_{i=1,2,3}(\alpha^i)^2 = N^2 - 1,$$
$$R = \frac{\pi\alpha'\sqrt{N^2-1}}{x^9}$$
$$X^{10} \sim \frac{N}{r^2}$$
$$\mathcal{R}^+ = \left(\frac{n+1}{4}, \frac{n-1}{4}\right), \mathcal{R}^- = \left(\frac{n-1}{4}, \frac{n+1}{4}\right).$$
$$G_5 \equiv \mathcal{P}_{\mathcal{R}^+} - \mathcal{P}_{\mathcal{R}^-}$$
$$[A_1, A_2, A_3, A_4] \equiv \sum_{\text{permutations } \sigma} \mathrm{sign}(\sigma) A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)}A_{\sigma(4)}$$
$$\frac{\partial \boldsymbol{X}^i}{\partial x^{10}} = \frac{1}{4!}\frac{b}{8\pi\ell_p^3}\epsilon^{ijkl}[\boldsymbol{G}_5, \boldsymbol{X}^j, \boldsymbol{X}^k, \boldsymbol{X}^l]$$
$$\rho_s(\Gamma^i) \equiv \mathbb{1}\otimes\cdots\otimes\Gamma^i\otimes\cdots\otimes\mathbb{1}, s = 1,2,\cdots,n$$
$$\sum_{s=1}^{n}\rho_s(\Gamma^i) = (\Gamma^i\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}) + (\mathbb{1}\otimes\Gamma^i\otimes\cdots\otimes\mathbb{1}) + \cdots + (\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\Gamma^i)$$
$$\boldsymbol{G}^i = \mathcal{P}_{\mathcal{R}^+}\sum_{s=1}^{n}\rho_s(\Gamma^i P_-)\mathcal{P}_{\mathcal{R}^-} + \mathcal{P}_{\mathcal{R}^-}\sum_{s=1}^{n}\rho_s(\Gamma^i P_+)\mathcal{P}_{\mathcal{R}^+}$$
$$X^i(x^{10}) = i\hat{R}(x^{10})\boldsymbol{G}^i.$$
$$\epsilon^{ijkl}\boldsymbol{G}_5\boldsymbol{G}^i\boldsymbol{G}^j\boldsymbol{G}^k = -2(n+2)\boldsymbol{G}^i,$$
$$\hat{R}(x^{10}) = \sqrt{\frac{2\pi\ell_p^3}{(n+2)bx^{10}}}.$$
$$R^2 = \frac{1}{N}\left|\mathrm{Tr}\sum_{i=1}^{4}(X^i)^2\right|.$$
$$R = \sqrt{N}|\hat{R}|.$$
$$x^{10} = \frac{2\pi\ell_p^3 N}{(n+2)bR^2}.$$
$$E = T_{M2}\int d^2\sigma\mathrm{Tr}\left[\left(\frac{d\boldsymbol{X}^i}{dx^{10}} + \frac{b}{8\pi\ell_p^3}\epsilon^{ijkl}\boldsymbol{G}_5\boldsymbol{X}^j\boldsymbol{X}^k\boldsymbol{X}^l\right)^2 + \left(1 - \frac{b}{16\pi\ell_p^3}\epsilon^{ijkl}\left\{\frac{d\boldsymbol{X}^i}{dx^{10}}, \boldsymbol{G}_5\boldsymbol{X}^j\boldsymbol{X}^k\boldsymbol{X}^k\right\}\right)^2\right]^{\frac{1}{2}}$$
$$E|_{BH} = T_{M2}\int d^2\sigma\left(1 - \frac{b}{16\pi\ell_p^3}\epsilon^{ijkl}\left\{\frac{d\boldsymbol{X}^i}{dx^{10}}, \boldsymbol{G}_5\boldsymbol{X}^j\boldsymbol{X}^k\boldsymbol{X}^k\right\}\right)$$
$$E = NT_{M2}L\int dx^{10} + T_{M5}L\int 2\pi^2R^3dR$$
$$\boldsymbol{H}^{KLM} \equiv [\boldsymbol{X}^K, \boldsymbol{X}^L, \boldsymbol{X}^M] \equiv \{[\boldsymbol{X}^K, \boldsymbol{X}^L], \boldsymbol{X}^M\} + \{[\boldsymbol{X}^L, \boldsymbol{X}^M], \boldsymbol{X}^K\} + \{[\boldsymbol{X}^M, \boldsymbol{X}^K], \boldsymbol{X}^L\}$$
$$S = -T_{M2}\int d^3\sigma\mathrm{Tr}\left[1 + (\partial_a\boldsymbol{X}^M)^2 - \frac{b^2}{12}(\boldsymbol{H}^{KLM})^2 + \frac{b^2}{48}[\partial_a\boldsymbol{X}^{[K}, \boldsymbol{H}^{LMN]}]^2\right]^{\frac{1}{2}}$$
$$\mathcal{L}_{CS} = \frac{k}{4\pi}\mathrm{Tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} - \frac{2i}{3}\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A}\right)$$
$$\mathcal{L}_{\mathcal{N}=1} = \frac{k}{4\pi}\mathrm{Tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} - \frac{2i}{3}\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A} - i\bar{\chi}\chi\right)$$

$$\delta \boldsymbol{A}_\mu = i\bar{\epsilon}\gamma_\mu \chi$$
$$\delta\chi = -\frac{1}{2}\gamma^{\mu\nu}\boldsymbol{F}_{\mu\nu}\epsilon.$$
$$\mathcal{L}^{\text{matter}}_{\mathcal{N}=1} = -\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{i}{2}\bar{\psi}^a\gamma^\mu\partial_\mu\psi^a + \frac{1}{2}C^aC^a + t_{abcd}\phi^a\phi^b\left(\frac{1}{3}\phi^cC^d - \frac{1}{2}\bar{\psi}^c\psi^d\right)$$
$$\delta\phi^a = i\bar{\epsilon}\psi^a$$
$$\delta\psi^a = -(\gamma^\mu\partial_\mu\phi^a - C^a)\epsilon$$
$$\delta C^a = -i\bar{\epsilon}\gamma^\mu\partial_\mu\psi^a$$
$$\partial_\mu\phi^a \to \partial_\mu\phi^a - iA^I_\mu(T^I)_{ab}\phi^b$$
$$\phi^a\bar{\chi}^IT^I_{ab}\psi^b$$
$$\mathcal{L}_{\mathcal{N}=2} = \frac{k}{4\pi}\mathrm{Tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} - \frac{2i}{3}\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A} - i\bar{\chi}\chi + 2\boldsymbol{D}\boldsymbol{\sigma}\right)$$
$$\delta\boldsymbol{A}_\mu = \frac{i}{2}(\bar{\epsilon}\gamma_\mu\chi - \bar{\chi}\gamma_\mu\epsilon)$$
$$\delta\boldsymbol{\sigma} = -\frac{1}{2}(\bar{\epsilon}\chi - \bar{\chi}\epsilon)$$
$$\delta\boldsymbol{D} = \frac{1}{2}\left(\bar{\epsilon}\gamma^\mu D_\mu\chi + D_\mu\bar{\chi}\gamma^\mu\epsilon\right) - \frac{1}{2}\left(\bar{\epsilon}\chi,\sigma] + [\bar{\chi},\sigma]\epsilon\right)$$
$$\delta\chi = \left(-\frac{1}{2}\gamma^{\mu\nu}\boldsymbol{F}_{\mu\nu} + i\gamma^\mu D_\mu\sigma - i\boldsymbol{D}\right)\epsilon.$$
$$-\frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi^A + i\bar{\psi}_A\gamma^\mu\partial_\mu\psi^A + F_AF^A + \left(F^AW_{,A} + c.c.\right)$$
$$\delta\phi^A = i\bar{\epsilon}\psi^A$$
$$\delta\psi^A = -\gamma^\mu\partial_\mu\phi^A + F^A\epsilon^*$$
$$\delta F^A = -i\overline{\epsilon^*}\gamma^\mu\partial_\mu\psi^A$$
$$-\sigma^I\sigma^J(\phi_AT^IT^J\phi^A) + D^I(\phi_AT^I\phi^A) - i\sigma^I(\bar{\psi}_AT^I\psi^A) - \phi_A\bar{\chi}^IT^I\psi^A + \phi^AT^I\bar{\psi}_A\chi^I$$
$$\mathcal{L}_{N=2,\text{ gauged}} = \mathcal{L}_{CS} + \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{scalar-fermion}} - V(\phi),$$
$$\mathcal{L}_{\text{scalar-fermion}} = -\frac{4\pi i}{k}(\phi_AT^I\phi^A)(\bar{\psi}_BT^I\psi^B) - \frac{8\pi i}{k}(\bar{\psi}_AT^I\phi^A)(\phi_BT^I\psi^B)$$
$$V(\phi) = \frac{16\pi^2}{k^2}(\phi_AT^I\phi^A)(\phi_BT^J\phi^B)(\phi_CT^IT^J\phi^C)$$
$$W(Q,\tilde{Q}) = \alpha(\tilde{Q}T^IQ)(\tilde{Q}T^IQ)$$
$$\Gamma_{012}\epsilon = \epsilon$$
$$X^I = X^I_aT^a$$
$$\Psi = \Psi_aT^a.$$
$$\delta X^I_d = i\bar{\epsilon}\Gamma^I\Psi_d$$
$$\delta\Psi_d = \partial_\mu X^I_d\Gamma^\mu\Gamma^I\epsilon.$$
$$\delta X^I_d = i\bar{\epsilon}\Gamma^I\Psi_d$$
$$\delta\Psi_d = \partial_\mu X^I_d\Gamma^\mu\Gamma^I\epsilon - \frac{1}{3!}X^I_aX^J_bX^K_cf^{abc}_d\Gamma^{IJK}\epsilon$$
$$[T^a,T^b,T^c] = f^{abc}{}_dT^d$$
$$[\delta_1,\delta_2]X^I_d = -2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1\partial_\mu X^I_d - \left(2i\bar{\epsilon}_2\Gamma^{JK}\epsilon_1X^J_aX^K_bf^{abc}_d\right)X^I_c$$
$$\delta X^I_d = \tilde{\Lambda}^c_dX^I_c, \tilde{\Lambda}^c_d = -2i\bar{\epsilon}_2\Gamma^{JK}\epsilon_1X^J_aX^K_bf^{abc}_d$$
$$\delta X^I = \alpha_{JK}[X^I,X^J,X^K]$$
$$\delta X = [X,A,B]$$
$$\delta[X,Y,Z] = [\delta X,Y,Z] + [X,\delta Y,Z] + [X,Y,\delta Z]$$
$$[A,B,[X,Y,Z]] = [[A,B,X],Y,Z] + [X,[A,B,Y],Z] + [X,Y,[A,B,Z]]$$
$$f^{abc}{}_gf^{efg}{}_d = f^{efa}{}_gf^{gbc}{}_d + f^{agc}{}_df^{efb}{}_g + f^{abg}{}_df^{efc}{}_g$$
$$D_\mu X^I_d = \partial_\mu X^I_d - \tilde{A}^c_{\mu\ d}X^I_c$$
$$\delta\tilde{A}_\mu{}^c{}_d = \partial_\mu\tilde{\Lambda}^c{}_d + \tilde{A}_\mu{}^c{}_e\tilde{\Lambda}^e{}_d - \tilde{\Lambda}^c{}_e\tilde{A}_\mu{}^e{}_d$$

$$\tilde{F}_{\mu\nu}{}^{a}{}_{b} = \partial_\nu \tilde{A}_\mu{}^{a}{}_{b} - \partial_\mu \tilde{A}_\nu{}^{a}{}_{b} - \tilde{A}_\mu{}^{a}{}_{c}\tilde{A}_\nu{}^{c}{}_{b} + \tilde{A}_\nu{}^{a}{}_{c}\tilde{A}_\mu{}^{c}{}_{b}$$

$$\delta X_a^I = i\bar{\epsilon}\Gamma^I\Psi_a$$

$$\delta\Psi_a = D_\mu X_a^I \Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X_b^I X_c^J X_d^K f^{bcd}{}_a\Gamma^{IJK}\epsilon$$

$$\delta\tilde{A}_\mu{}^{b}{}_{a} = i\bar{\epsilon}\Gamma_\mu\Gamma_I X_c^I\Psi_d f^{cdb}{}_a.$$

$$[\delta_1,\delta_2]X_a^I = v^\lambda\partial_\lambda X_a^I + \left(\tilde{\Lambda}^b{}_a - v^\lambda\tilde{A}_\lambda{}^{b}{}_{a}\right)X_b^I$$

$$[\delta_1,\delta_2]\Psi_a = v^\lambda\partial_\lambda\Psi_a + \left(\tilde{\Lambda}^b{}_a - v^\lambda\tilde{A}_\lambda{}^{b}{}_{a}\right)\Psi_b$$

$$[\delta_1,\delta_2]\tilde{A}_\mu{}^{b}{}_{a} = v^\lambda\partial_\lambda\tilde{A}_\mu{}^{b}{}_{a} + \tilde{D}_\mu\left(\tilde{\Lambda}^b{}_a - v^\lambda\tilde{A}_\lambda{}^{b}{}_{a}\right),$$

$$\Gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma_{IJ}X_c^I X_d^J\Psi_b f^{cdb}{}_a = 0$$

$$\tilde{F}_{\mu\nu}{}^{b}{}_{a} + \varepsilon_{\mu\nu\lambda}\left(X_c^J D^\lambda X_d^J + \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d\right)f^{cdb}{}_a = 0.$$

$$D^2X_a^I - \frac{i}{2}\bar{\Psi}_c\Gamma_J^I X_d^J\Psi_b f_a^{cdb} + \frac{1}{2}X_b^J X_c^K X_e^I X_f^J X_g^K f_a^{bcd}{}_a f_d^{efg} = 0$$

$$\langle X, Y\rangle = h^{ab}X_aY_b$$

$$f^{abcd} = f^{[abcd]}$$

$$\mathcal{L} = -\frac{1}{2}D_\mu X^{aI}D^\mu X_a^I + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu\Psi_a + \frac{i}{4}\bar{\Psi}_b\Gamma_{IJ}X_c^I X_d^J\Psi_a f^{abcd} - V + \mathcal{L}_{CS}$$

$$V = \frac{1}{12}X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K f^{abcd}f^{efg}{}_d$$

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right),$$

$$\tilde{A}_\mu{}^{b}{}_{a} = A_{\mu cd}f^{cdb}{}_a$$

$$f^{abcd} = \frac{2\pi}{k}\varepsilon^{abcd}, h^{ab} = \delta^{ab}$$

$$\tilde{A}_\mu{}^{a}{}_{b} = \tilde{A}_\mu^{+a}{}_b + \tilde{A}_\mu^{-a}{}_b$$

$$\mathcal{L}_{CS} = \frac{k}{8\pi}\epsilon^{\mu\nu\lambda}\left(\tilde{A}_\mu^{+a}{}_b\partial_\nu\tilde{A}_\lambda^{+b}{}_a + \frac{2}{3}\tilde{A}_\mu^{+a}{}_b\tilde{A}_\nu^{+b}{}_c\tilde{A}_\lambda^{+c}{}_a\right) - \frac{k}{8\pi}\epsilon^{\mu\nu\lambda}\left(\tilde{A}_\mu^{-a}{}_b\partial_\nu\tilde{A}_\lambda^{-b}{}_a + \frac{2}{3}\tilde{A}_\mu^{-a}{}_b\tilde{A}_\nu^{-b}{}_c\tilde{A}_\lambda^{-c}{}_a\right)$$

$$A_\mu^L = \frac{1}{2}\epsilon_i^{jk}\sigma_k\tilde{A}_{\mu j}^{+i}, A_\mu^R = \frac{1}{2}\epsilon_i^{jk}\sigma_k\tilde{A}_{\mu j}^{-i}$$

$$\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left[\left(A_\mu^L\partial_\nu A_\lambda^L - \frac{2i}{3}A_\mu^L A_\nu^L A_\lambda^L\right) - \left(A_\mu^R\partial_\nu A_\lambda^R - \frac{2i}{3}A_\mu^R A_\nu^R A_\lambda^R\right)\right].$$

$$X_{\alpha\dot{\beta}}^I = \left(\frac{1}{2}X_4^I\mathbb{1} + \frac{i}{2}X_i^I\sigma^i\right)_{\alpha\dot{\beta}}, i = 1,2,3$$

$$X^{\dagger I\alpha\dot{\beta}} = \epsilon^{\alpha\dot{\alpha}}\epsilon^{\beta\dot{\beta}}X_{\beta\dot{\alpha}}^I$$

$$D_\mu X^I = \partial_\mu X^I - iA_\mu^L X^I + iX^I A_\mu^R$$

$$V(X) = \frac{8}{3}\mathrm{Tr}\left(X^{[I}X^{J\dagger}X^{K]}X^{K\dagger}X^J X^{I\dagger}\right)$$

$$\mathcal{L}_{su(2)} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right)$$

$$\int d^3x\mathcal{L}_{su(n)} \to \int d^3x\mathcal{L}_{su(n)} + 2\pi kw$$

$$\delta Z_d^A = i\bar{\epsilon}^{AB}\psi_{Bd}$$

$$\delta\psi_{Bd} = \gamma^\mu D_\mu Z_d^A\epsilon_{AB} + f^{ab}{}_{cd}Z_a^C Z_b^A\bar{Z}_C^c\epsilon_{AB} + f^{ab}{}_{cd}Z_a^C Z_b^D\bar{Z}_B^c\epsilon_{CD}$$

$$\delta\tilde{A}_\mu{}^{c}{}_{d} = -i\bar{\epsilon}_{AB}\gamma_\mu Z_a^A\psi^{Bb}f^{ca}{}_{bd} + i\bar{\epsilon}^{AB}\gamma_\mu\bar{Z}_A^b\psi_{Ba}f^{ca}{}_{bd},$$

$$[\delta_1,\delta_2]Z_d^A = v^\lambda D_\lambda Z_d^A + \tilde{\Lambda}_d^a Z_a^A$$

$$v^\lambda = \frac{i}{2}\bar{\epsilon}_2^{CD}\gamma^\lambda\epsilon_{1CD},$$

$$\tilde{\Lambda}^a{}_d = \Lambda^c{}_b f^{ab}{}_{cd}, \Lambda^c{}_b = i(\bar{\epsilon}_2^{DE}\epsilon_{1CE} - \bar{\epsilon}_1^{DE}\epsilon_{2CE})\bar{Z}_D^c Z_b^C,$$

$$\gamma^\mu D_\mu\psi_{Cd} = f^{ab}{}_{cd}\psi_{Ca}Z_b^D\bar{Z}_D^c + 2f^{ab}{}_{cd}\psi_{Da}Z_b^D\bar{Z}_C^c - \varepsilon_{CDEF}f^{ab}{}_{cd}\psi^{Dc}Z_a^E Z_b^F$$

$$\tilde{F}_{\mu\nu}{}^{c}{}_{d} = -\varepsilon_{\mu\nu\lambda}\left(D^\lambda Z_a^A\bar{Z}_A^b - Z_a^A D^\lambda\bar{Z}_A^b - i\bar{\psi}^{Ab}\gamma^\lambda\psi_{Aa}\right)f^{ca}{}_{bd}$$

$$[T^a, T^b; \bar{T}_c] = f^{ab}{}_{cd} T^d$$

$$f^{ef}{}_{gb} f^{cb}{}_{ad} + f^{fe}{}_{ab} f^{cb}{}_{gd} + f^{*}_{ga}{}^{fb} f^{ce}{}_{bd} + f^{*}_{ag}{}^{eb} f^{cf}{}_{bd} = 0.$$

$$f^{*cd}_{ab} = \left(f^{ab}{}_{cd}\right)^*$$

$$\Lambda^{*b}_{a} = (\Lambda^a{}_b)^*$$

$$\langle \bar{X}, Y \rangle = \bar{X}^a Y_a$$

$$\delta_\Lambda \bar{Z}^d_A = \tilde{\Lambda}^{*d}_a \bar{Z}^a_A$$

$$\tilde{\Lambda}^{*b}_a = \left(\tilde{\Lambda}^a{}_b\right)^* = \Lambda^*{}_d{}^c f^*_{ac}{}^{db}.$$

$$\tilde{\Lambda}^{*a}_b = -\tilde{\Lambda}^a{}_b.$$

$$f^{ab}{}_{cd} = f^{*ab}_{cd}$$

$$f^{ge}{}_{fd} f^{ab}{}_{cg} = f^{ae}{}_{fg} f^{gb}{}_{cd} + f^{be}{}_{fg} f^{ag}{}_{cd} - f^{*eg}_{cf} f^{ab}{}_{gd}$$

$$\begin{aligned}
\mathcal{L} = & -D^\mu \bar{Z}^a_A D_\mu Z^A_a - i\bar{\psi}^{Aa}\gamma^\mu D_\mu \psi_{Aa} - V + \mathcal{L}_{CS} \\
& -i f^{ab}{}_{cd} \bar{\psi}^{Ad} \psi_{Aa} Z^B_b \bar{Z}^c_B + 2i f^{ab}{}_{cd} \bar{\psi}^{Ad} \psi_{Ba} Z^B_b \bar{Z}^c_A \\
& + \frac{i}{2} \varepsilon_{ABCD} f^{ab}{}_{cd} \bar{\psi}^{Ad} \psi^{Bc} Z^C_a Z^D_b - \frac{i}{2} \varepsilon^{ABCD} f^{cd}{}_{ab} \bar{\psi}_{Ac} \psi_{Bd} \bar{Z}^a_C \bar{Z}^b_D.
\end{aligned}$$

$$V = \frac{2}{3} \Upsilon^{CD}_{Bd} \bar{\Upsilon}^{Bd}_{CD}$$

$$\Upsilon^{CD}_{Bd} = f^{ab}{}_{cd} Z^C_a Z^D_b \bar{Z}^c_B - \frac{1}{2} \delta^C_B f^{ab}{}_{cd} Z^E_a Z^D_b \bar{Z}^c_E + \frac{1}{2} \delta^D_B f^{ab}{}_{cd} Z^E_a Z^C_b \bar{Z}^c_E$$

$$\mathcal{L}_{CS} = \frac{1}{2} \varepsilon^{\mu\nu\lambda} \left(f^{ab}{}_{cd} A_\mu{}^c{}_b \partial_\nu A_\lambda{}^d{}_a + \frac{2}{3} f^{ac}{}_{dg} f^{ge}{}_{fb} A_\mu{}^b{}_a A_\nu{}^d{}_c A_\lambda{}^f{}_e \right)$$

$$\frac{\delta \mathcal{L}_{CS}}{\delta A_{\lambda\, b\, b}} f^{ac}_{db} = \frac{1}{2} \varepsilon^{\lambda\mu\nu} \tilde{F}^c_{\mu\nu\, d}$$

$$[X, Y; Z] = -\frac{2\pi}{k} \left(X Z^\dagger Y - Y Z^\dagger X \right)$$

$$\begin{aligned}
\delta X_{dl} &= [X, Y; Z]_{dl} = f^{aibj}{}_{ckdl} \Lambda^{ck}{}_{bj} X_{ai} \\
&= -\frac{2\pi}{k} \left(X_{dk} Z^{\dagger kb} Y_{bl} - Y_{dk} Z^{\dagger kb} X_{bl} \right).
\end{aligned}$$

$$f^{aibj}{}_{ckdl} = -\frac{2\pi}{k} \left(\delta^a{}_d \delta^b{}_c \delta^i{}_k \delta^j{}_l - \delta^a{}_c \delta^b{}_d \delta^i{}_l \delta^j{}_k \right)$$

$$\delta X_{dl} = \tilde{\Lambda}^{ai}_{dl} X_{ai} = -\frac{2\pi}{k} \left(\delta^i_l \Lambda^{aj}_{dj} - \delta^a_d \Lambda^{bi}_{bl} \right) X_{ai}$$

$$f^{ab}{}_{cd} = -\frac{2\pi}{k} \left(J^{ab} J_{cd} + \left(\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c \right) \right)$$

$$\delta X_d = \tilde{\Lambda}^a_d X_a = -\frac{2\pi}{k} \left[(\Lambda^a_d + \Lambda^a_d) - \delta^a_d \Lambda^b_b \right] X_a.$$

$$Z^A_d = Z^A_{\alpha\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_d$$

$$\left(\bar{\sigma}^a \sigma^b \bar{\sigma}^c - \bar{\sigma}^c \sigma^b \bar{\sigma}^a \right)^{\dot{\alpha}\alpha} = -2\epsilon^{abcd} \bar{\sigma}^{\dot{\alpha}\alpha}_d$$

$$\begin{aligned}
\delta Z^A_d &= i\bar{\epsilon}^{AD} \Psi_{Dd} + i\bar{\eta} \Psi^A_d \\
\delta \Psi^d_D &= \gamma^\mu \epsilon_{AD} D_\mu Z^{Ad} + \gamma^\mu \eta D_\mu \bar{Z}^d_D \\
& \quad + \epsilon^{abcd} Z^A_a Z^B_b \bar{Z}_{Dc} \epsilon_{AB} - \epsilon^{abcd} Z^A_a Z^B_b \bar{Z}_{Bc} \epsilon_{AD} \\
& \quad - \epsilon^{abcd} Z^A_a \bar{Z}_{Ab} \bar{Z}_{Dc} \eta - \frac{1}{3} \epsilon_{ABCD} \epsilon^{abcd} \eta^* Z^A_a Z^B_b Z^C_c
\end{aligned}$$

$$\begin{aligned}
[\delta_1, \delta_2] \Psi_{Dd} &= v^\mu D_\mu \Psi_{Dd} + \tilde{\Lambda}^a_d \Psi_{Da} \\
& \quad + \frac{i}{2} \epsilon^{CB}_{[2} \epsilon_{1]CD} E_{Bd} - \frac{i}{4} \epsilon^{BE}_2 \gamma^\mu \epsilon_{1BE} \gamma_\mu E_{Dd} \\
& \quad + i\bar{\eta}_{[2} \epsilon_{1]CD} E^C_d - \frac{i}{2} \left(\bar{\eta}_{[2} \eta^*_{1]} + \bar{\eta}^*_{[2} \gamma^\mu \eta_{1]} \gamma_\mu \right) E_{Dd}
\end{aligned}$$

$$\begin{aligned}
\delta \tilde{A}^{ad}_\mu &= -i\epsilon^{abcd} \bar{\epsilon}_{BC} \gamma_\mu \Psi^B_b Z^C_c - i\epsilon^{abcd} \bar{\epsilon}^{BC} \gamma_\mu \Psi_{Bb} \bar{Z}_{Cc} \\
& \quad + i\epsilon^{abcd} \bar{\eta}^* \gamma_\mu \Psi_{Bb} Z^B_c + i\epsilon^{abcd} \bar{\eta} \gamma_\mu \Psi^B_b \bar{Z}_{Bc}
\end{aligned}$$

$$\delta Z_a^A = g^{AB}\bar{Z}_{Ba}$$
$$\delta\Psi_{Ba} = -\frac{1}{2}\epsilon_{BCDE}g^{DE}\Psi_a^C$$
$$\delta\epsilon^{AB} = g^{AB}\eta^* + \frac{1}{2}\epsilon^{ABCD}g_{CD}^*\eta$$
$$\delta\eta = -\frac{1}{2}g^{AB}\epsilon_{AB},$$
$$D_\mu Z_a^A = \partial_\mu Z_a^A - \tilde{A}_\mu{}^b{}_a Z_b^A - iB_\mu\delta_a^b Z_b^A$$
$$B_\mu \to B_\mu + \partial_\mu\theta$$
$$\delta B_\mu = 0$$
$$\delta\mathcal{L}_{\mathrm{SU}(n)\times\mathrm{SU}(n)}^{\mathrm{gauged}} = -\frac{1}{2}G_{\mu\nu}\bar{\epsilon}_{AB}\gamma^{\mu\nu}\psi^{Aa}Z_a^B + \frac{1}{2}G_{\mu\nu}\bar{\epsilon}^{AB}\gamma^{\mu\nu}\psi_{Aa}\bar{Z}_B^a$$
$$= -\frac{1}{2}\varepsilon^{\mu\nu\lambda}G_{\mu\nu}\bar{\epsilon}_{AB}\gamma_\lambda\psi^{Aa}Z_a^B + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\bar{\epsilon}^{AB}G_{\mu\nu}\bar{\epsilon}\gamma_\lambda\psi_{Aa}\bar{Z}_B^a,$$
$$\mathcal{L}_{\mathrm{U}(n)\times\mathrm{U}(n)} = \mathcal{L}_{\mathrm{SU}(n)\times\mathrm{SU}(n)}^{\mathrm{gauged}} + \frac{k'}{8\pi}\epsilon^{\mu\nu\lambda}G_{\mu\nu}Q_\lambda$$
$$\delta Q_\lambda = \frac{4\pi}{k'}\bar{\epsilon}_{AB}\gamma_\lambda\psi^{Aa}Z_a^B - \frac{4\pi}{k'}\bar{\epsilon}^{AB}\gamma_\lambda\psi_{Aa}\bar{Z}_B^a$$
$$[\delta_1,\delta_2]B_\mu = v^\nu G_{\nu\mu}\ v^\nu = \frac{i}{2}\left(\bar{\epsilon}_2^{CD}\gamma^\nu\epsilon_{CD}^1\right)$$
$$H_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$$
$$[\delta_1,\delta_2]Q_\mu = \frac{k'}{4\pi}v^\nu\varepsilon_{\mu\nu\lambda}\left(iZ_a^A D^\lambda\bar{Z}_A^a - iD^\lambda Z_a^A\bar{Z}_A^a - \bar{\psi}_a^A\gamma^\lambda\psi_A^a\right) + D_\mu\Lambda$$
$$H_{\mu\nu} = -\frac{k'}{4\pi}\varepsilon_{\mu\nu\lambda}\left(iZ_a^A D^\lambda\bar{Z}_A^a - iD^\lambda Z_a^A\bar{Z}_A^a - \bar{\psi}_a^A\gamma^\lambda\psi_A^a\right)$$
$$[\delta_1,\delta_2]Q_\mu = v^\nu H_{\nu\mu} + D_\mu\Lambda$$
$$\mathcal{L}_{\mathrm{U}(1)\times\mathrm{U}(1)CS} = \frac{k'}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu^L\partial_\nu A_\lambda^L - \frac{k'}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu^R\partial_\nu A_\lambda^R$$
$$k' = nk,$$
$$\delta A_\lambda^R = \delta A_\lambda^L = \frac{2\pi}{nk}\bar{\epsilon}_{AB}\gamma_\lambda\psi^{Aa}Z_a^B - \frac{2\pi}{nk}\bar{\epsilon}^{AB}\gamma_\lambda\psi_{Aa}\bar{Z}_B^a$$
$$[\cdot,\cdot,\cdot]: \mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}$$
$$[\cdot,\cdot;\cdot]: \mathcal{V}\otimes\mathcal{V}\otimes\overline{\mathcal{V}}\to\mathcal{V}$$
$$[\cdot,\cdot;\cdot]: \overline{\mathcal{V}}\otimes\overline{\mathcal{V}}\otimes\mathcal{V}\to\overline{\mathcal{V}}$$
$$\varphi_{U,\bar{V}}(X) = [X,U;\bar{V}]\ \varphi_{U,\bar{V}}(\bar{X}) = -[\bar{X},\bar{V};U]$$
$$\varphi_{U,\bar{V}}([X,Y;\bar{Z}]) = \left[\varphi_{U,\bar{V}}(X),Y;\bar{Z}\right] + \left[X,\varphi_{U,\bar{V}}(Y);\bar{Z}\right] + \left[X,Y;\varphi_{U,\bar{V}}(\bar{Z})\right]$$
$$[[X,Y;\bar{Z}],U;\bar{V}] = [[X,U;\bar{V}],Y;\bar{Z}] + [X,[Y,U;\bar{V}];\bar{Z}] - [X,Y;[\bar{Z},\bar{V},U]]$$
$$\langle[\bar{X},\bar{V};U],Y\rangle = \langle\bar{X},[Y,U;\bar{V}]\rangle$$
$$\left[\varphi_{U,\bar{V}},\varphi_{Y,\bar{Z}}\right](X) = \varphi_{\varphi_{U,\bar{V}}(Y),Z}(X) - \varphi_{Y,\varphi_{U,\bar{V}}(\bar{Z})}(X)$$
$$\left(\varphi_{Y,\bar{Z}},\varphi_{U,\bar{V}}\right) = \langle\bar{Z},[Y,U;\bar{V}]\rangle$$
$$\varphi: \mathcal{V}\times\overline{\mathcal{V}}\to\mathcal{G},$$
$$\varphi^*(g)_{U,\bar{V}} = \langle\bar{V},g(U)\rangle,$$
$$\varphi^*(g)_{U,\bar{V}} = \left(\varphi_{U,\bar{V}},g\right)$$
$$[W,U;\bar{V}] = \varphi_{U,\bar{V}}(W).$$
$$\left[g,\varphi_{U,\bar{V}}\right] = \varphi_{g(U),\bar{V}} + \varphi_{U,g(\bar{V})}$$
$$\varphi_{U,\bar{V}}(g)_r = h_{rs}(T^s)^a{}_b U_a\bar{V}_e g^{be}$$
$$f^{abc}{}_d = (T^r)^b{}_e(T^s)^a{}_d h_{rs}g^{ce}.$$
$$f^{abc}{}_d = \sum_R c_R(T^r)^b{}_e(T^s)^a{}_d\kappa_{rs}^R g^{ce}$$
$$[X,Y,Z] = [[X,Y],Z]$$
$$\int d^4\theta Q^\dagger e^V Q, \int d^4\theta\tilde{Q}^\dagger e^{-V}\tilde{Q}$$

$$ds^2 = U dx^a dx_a + U^{-1}(d\phi + \omega^a dx_a)^2$$
$$U = U_\infty + \frac{q}{2|\vec{x}|}$$
$$U^{ij}\partial^a_{(i)}\partial_{(j)a}U_{kl} = 0$$
$$\partial^a_{(j)}\omega^b_{ki} - \partial^b_{(k)}\omega^a_{ji} = \epsilon^{abc}\partial_{(j)c}U_{ki}$$
$$\left\{x_a^{(1)}\right\} = (x^3, x^4, x^5), \left\{x_a^{(2)}\right\} = (x^7, x^8, x^9)$$
$$\tau = -\frac{U_{12}}{U_{11}} + i\frac{\sqrt{\det U}}{U_{11}}$$
$$ds^2 = U_{ij}dx_a^{(i)}dx_a^{(j)} + U^{ij}\left(d\phi_i + \omega^a_{ik}dx_a^{(k)}\right)\left(d\phi_j + \omega^b_{jl}dx_b^{(l)}\right)$$
$$U_1 = U_\infty + \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix}, h_1 = \frac{1}{2|\vec{x}^{(1)}|}, U_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$U_2' = U_\infty' + \begin{pmatrix} h_2 & kh_2 \\ kh_2 & k^2h_2 \end{pmatrix}, h_2 = \frac{1}{2|\vec{x}^{(1)} + k\vec{x}^{(2)}|}, U_\infty' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$G^i_j = \begin{pmatrix} 1 & 0 \\ -k^{-1} & k^{-1} \end{pmatrix}$$
$$U_2 = U_\infty + \begin{pmatrix} 0 & 0 \\ 0 & h_2 \end{pmatrix}, h_2 = \frac{1}{2|\vec{x}^{(2)}|}, U_\infty = k^{-2}\begin{pmatrix} k^2+1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$ds^2 = \sum_{i=1,2}\left(\frac{1}{2|\vec{x}^{(i)}|}d\vec{x}^{(i)} \cdot d\vec{x}^{(i)} + 2|\vec{x}^{(i)}|\left(d\phi_i' + \frac{1}{2}\cos\vartheta_i d\varphi_i\right)^2\right)$$
$$ds^2 = \sum_{i=1,2}\left(dr_i^2 + r_i^2\left(d\phi_i' + \frac{1}{2}\cos\vartheta_i d\varphi_i\right)^2 + \frac{r_i^2}{4}\left(d\vartheta_i^2 + \sin^2\vartheta_i d\varphi_i^2\right)\right)$$
$$z^1 = r_1\cos\frac{\vartheta_1}{2}e^{-i\phi_1' - \frac{i}{2}\varphi_1} \quad z^2 = r_2\cos\frac{\vartheta_2}{2}e^{i\phi_2' + \frac{i}{2}\varphi_2}$$
$$z^3 = r_1\sin\frac{\vartheta_1}{2}e^{-i\phi_1' + \frac{i}{2}\varphi_1} \quad z^4 = r_2\sin\frac{\vartheta_2}{2}e^{i\phi_2' - \frac{i}{2}\varphi_2}$$
$$ds^2 = \sum_{A=1}^{4}|dz^A|^2$$
$$e^{\pi(\Gamma_{34}+\Gamma_{56}+\Gamma_{78}+\Gamma_{910})/k}$$
$$(\Gamma_{34} + \Gamma_{56} + \Gamma_{78} + \Gamma_{910})\epsilon = 0$$
$$\begin{aligned}\mathcal{L} = &-\mathrm{Tr}\left(D^\mu \bar{Z}_A, D_\mu Z^A\right) - i\mathrm{Tr}\left(\bar{\psi}^A, \gamma^\mu D_\mu \psi_A\right) - V + \mathcal{L}_{CS} \\ &+ \frac{2\pi i}{k}\mathrm{Tr}(\bar{\psi}^A\psi_A\bar{Z}_B Z^B - \bar{\psi}^A Z^B \bar{Z}_B \psi_A) - \frac{4\pi i}{k}\mathrm{Tr}(\bar{\psi}^A\psi_B\bar{Z}_A Z^B - \bar{\psi}^A Z^B\bar{Z}_A\psi_B) \\ &- \frac{2\pi i}{k}\varepsilon_{ABCD}\mathrm{Tr}(\bar{\psi}^A Z^C \psi^B Z^D) + \frac{2\pi i}{k}\varepsilon^{ABCD}\mathrm{Tr}(\bar{\psi}_A\bar{Z}_C\psi_B\bar{Z}_D),\end{aligned}$$
$$V = \frac{1}{3}\mathrm{Tr}(4Z^A\bar{Z}_A Z^B\bar{Z}_C Z^C\bar{Z}_B - 4Z^A\bar{Z}_B Z^C\bar{Z}_A Z^B Z_C - Z^A\bar{Z}_A Z^B\bar{Z}_B Z^C\bar{Z}_C - \bar{Z}_A Z^A\bar{Z}_B Z^B\bar{Z}_C Z^C)$$
$$D_\mu Z^A = \partial_\mu Z^A - iA^L_\mu Z^A + iZ^A A^R_\mu$$
$$\mathcal{L}_{CS} = \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\left(\mathrm{Tr}\left(A^L_\mu\partial_\nu A^L_\lambda - \frac{2}{3}iA^L_\mu A^L_\nu A^L_\lambda\right) - \mathrm{Tr}\left(A^R_\mu\partial_\nu A^R_\lambda - \frac{2}{3}iA^R_\mu A^R_\nu A^R_\lambda\right)\right),$$
$$\delta Z^A = i\bar{\epsilon}^{AB}\Psi_B,$$
$$\delta\psi_B = \gamma^\mu\epsilon_{AB}D_\mu Z^A + \frac{2\pi}{k}(Z^C\bar{Z}_B Z^D - Z^D\bar{Z}_B Z^C)\epsilon_{CD} - \frac{2\pi}{k}(Z^A\bar{Z}_C Z^C - Z^C\bar{Z}_C Z^A)\epsilon_{AB}$$
$$\delta A^L_\mu = -\frac{2\pi}{k}(\bar{\epsilon}_{AB}\gamma_\mu Z^B\bar{\psi}^A - \bar{\epsilon}^{AB}\gamma_\mu\psi_A\bar{Z}_B)$$
$$\delta A^R_\mu = -\frac{2\pi}{k}(\bar{\epsilon}_{AB}\gamma_\mu\bar{\psi}^A Z^B - \bar{\epsilon}^{AB}\gamma_\mu\bar{Z}_B\psi_A).$$
$$ds^2 = \frac{R^2}{4}ds^2_{\mathrm{AdS}_4} + R^2 ds^2_{S^7/\mathbb{Z}_k}$$

$$ds^2_{S^7/\mathbb{Z}_k} = \frac{1}{k^2}(d\phi + k\omega)^2 + ds^2_{\mathbb{CP}^3}$$
$$ds^2_{\mathrm{dP}^3} = \frac{1}{|z|^4}\left(|z|^2 dz_i d\bar{z}_i - \bar{z}_i z_j dz_i d\bar{z}_j\right)$$
$$J \sim id\left(\frac{z_i}{|z|}\right) \wedge d\left(\frac{\bar{z}_i}{|z|}\right)$$
$$ds^2_{IIA} = \frac{R^3}{k}\left(\frac{1}{4}ds^2_{AdS_4} + ds^2_{\mathbb{CP}P^3}\right)$$
$$e^{2\Phi} = \frac{R^3}{k^3} \sim \frac{1}{n^2}\left(\frac{n}{k}\right)^{\frac{5}{2}}$$
$$\Phi^i = \mathrm{diag}\left(x_1^i, x_2^i, \cdots x_n^i\right)$$
$$(\mathbb{R}^6)^n/\mathbb{S}_n \equiv \mathrm{Sym}_n(\mathbb{R}^6).$$
$$Z^A = \mathrm{diag}\left(z_1^A, z_2^A\right)$$
$$g_{12}\colon \mathrm{diag}\left(z_1^A, z_2^A\right) \cong \mathrm{diag}\left(z_2^A, z_1^A\right)$$
$$\mathcal{L} = -D_\mu z_1^A D^\mu z_{1A} - D_\mu z_2^A D^\mu z_{2A} + \frac{k}{2\pi}\varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu Q_\lambda$$
$$\mathcal{L} = -D_\mu z_1^A D^\mu z_{1A} - D_\mu z_2^A D^\mu z_{2A} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda} + \frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}\sigma\partial_\mu H_{\nu\lambda}$$
$$w_1^A = e^{i\sigma/k} z_1^A, w_2^A = e^{-i\sigma/k} z_2^A$$
$$\mathcal{L} = -\partial_\mu w_1^A \partial^\mu w_{1A} - \partial_\mu w_2^A \partial^\mu w_{2A}$$
$$\Psi_\gamma = e^{i\oint_\gamma^{\square} A \square}\Psi_0$$
$$\Psi_\gamma = e^{i\int_D F}\Psi_0$$
$$e^{i\int_{D-D'} F} = 1$$
$$\frac{1}{4\pi}\int_\Sigma H \in \mathbb{Z},$$
$$\frac{1}{4\pi}\int \varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda} \in 2\mathbb{Z}$$
$$\frac{1}{4\pi}\int \varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda} \in \mathbb{Z}$$
$$g_{\mathrm{SU}(2)} = \begin{cases} z_1^A \cong e^{\pi i/k} z_1^A & z_2^A \cong e^{-\pi i/k} z_2^A \qquad \mathrm{SU}(2)\times\mathrm{SU}(2) \\ z_1^A \cong e^{2\pi i/k} z_1^A & z_2^A \cong e^{-2\pi i/k} z_2^A \quad (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2 \end{cases}$$
$$(\mathbb{C}^4 \times \mathbb{C}^4)/\mathbb{D}_{4k} \quad \text{for} \quad \mathrm{SU}(2)\times\mathrm{SU}(2)$$
$$(\mathbb{C}^4 \times \mathbb{C}^4)/\mathbb{D}_{2k} \quad \text{for} \quad (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$$
$$Z^C Z_B Z^D - Z^D Z_B Z^C = 0$$
$$Z^A = \mathrm{diag}(z_1^A, \dots, z_n^A)$$
$$A_\mu^L = \mathrm{diag}\left(A_\mu^{L1}, \dots, A_\mu^{Ln}\right), A_\mu^R = \mathrm{diag}\left(A_\mu^{R1}, \dots, A_\mu^{Rn}\right)$$
$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^n D_\mu z_i^A D^\mu z_A^i + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_{i=1}^n B_\mu^i \partial_\nu Q_\lambda^i$$
$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu z_A^i + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{i\mu} H_{i\nu\lambda} + \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i \sigma_i \partial_\mu H_{i\nu\lambda}$$
$$\mathcal{L} = -\frac{1}{2}\sum_i \partial_\mu w_i^A \partial^\mu w_A^i$$
$$\frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\int \partial_\mu H_{i\nu\lambda} \in \mathbb{Z}$$
$$w_i^A \cong e^{2\pi i/k} w_i^A$$
$$\mathcal{M}_k = (\mathbb{C}^4/\mathbb{Z}_k)^n/\mathbb{S}_n \equiv \mathrm{Sym}_n(\mathbb{C}^4/\mathbb{Z}_k)$$
$$L_{CS} = \frac{k}{4\pi}\mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A - \tilde{A} \wedge d\tilde{A} - \frac{2}{3}\tilde{A} \wedge \tilde{A} \wedge \tilde{A}\right)$$
$$\delta\Phi = -\Lambda\Phi + \Phi\tilde{\Lambda}$$

$$D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi - \Phi \tilde{A}_\mu$$
$$\frac{k}{4\pi}\mathrm{Tr}\left(D_\mu \Phi^\dagger D^\mu \Phi\right)$$
$$\frac{k}{4\pi}\mathrm{Tr}\left|A_\mu \Phi - \Phi \tilde{A}_\mu\right|^2$$
$$\frac{k}{4\pi}v^2\mathrm{Tr}\left(A_\mu - \tilde{A}_\mu\right)^2,$$
$$B = \frac{1}{2}(A - \tilde{A}), C = \frac{1}{2}(A + \tilde{A}).$$
$$\mathcal{L} = \frac{k}{\pi}\mathrm{Tr}\left(B \wedge F^{(C)} + \frac{1}{3}B \wedge B \wedge B - v^2 B \wedge {}^*B\right)$$
$$F^{(C)} + B \wedge B - 2v^{2*}B = 0$$
$$B = -\frac{1}{2v^2}F^* F^{(C)} - \frac{1}{2v^2}\,{}^*(B \wedge B)$$
$$= -\frac{1}{2v^2}\,{}^*F^{(C)} - \frac{1}{8v^6}\,{}^{*\,*}F^{(C)} \wedge {}^*F^{(C)}\Big) + \cdots$$
$$\mathcal{L} = \frac{k}{\pi}\left(-\frac{1}{4v^2}F^{(C)} \wedge {}^*F^{(C)} - \frac{1}{24v^6}F^*(C) \wedge {}^*F^{(C)} \wedge {}^*F^{(C)} + \cdots\right)$$
$$\mathcal{L} = -\frac{1}{2}D_\mu X^{aI} D^\mu X_a^I + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4}f_{abcd}\bar{\Psi}^b \Gamma^{IJ} X^{cI} X^{dJ} \Psi^a$$
$$-\frac{1}{12}\left(f_{abcd}X^{aI}X^{bJ}X^{cK}\right)\left(f_{efg}{}^{d}X^{eI}X^{fJ}X^{gK}\right)$$
$$+\frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_{abcd}A_\mu^{ab}\partial_\nu A_\lambda^{cd} + \frac{2}{3}f_{aef}{}^{g}f_{bcdg}A_\mu^{ab}A_\nu^{cd}A_\lambda^{ef}\right)$$
$$D_\mu X^{aI} = \partial_\mu X^{aI} + f_{bcd}^{a}A_\mu^{cd}X^{bI}$$
$$V(X) = \frac{1}{12}\sum_{I,J,K=1}^{8}\left(\varepsilon_{abcd}\varepsilon_{efg}{}^{d}X^{aI}X^{bJ}X^{cK}X^{eI}X^{fJ}X^{gK}\right)$$
$$= \frac{1}{2}\sum_{i<j}^{7}\left(\varepsilon_{abcd}\varepsilon_{efg}{}^{d}X^{ai}X^{bj}X^{c(8)}X^{ei}X^{fj}X^{g(8)}\right)$$
$$+\frac{1}{2}\sum_{i<j<k}^{7}\left(\varepsilon_{abcd}\varepsilon_{efg}{}^{d}X^{ai}X^{bj}X^{ck}X^{ei}X^{fj}X^{gk}\right)$$
$$= \frac{1}{2}v^2\sum_{i<j}^{7}\left(\varepsilon_{AB4D}\varepsilon_{EF4}{}^{D}X^{Ai}X^{Bj}X^{Ei}X^{Fj}\right) + vO(X^5) + O(X^6).$$
$$\frac{1}{2}v^2\sum_{i<j=1}^{7}\left(\varepsilon_{ABC}\varepsilon_{EF}^{C}X^{Ai}X^{Bj}X^{Ei}X^{Fj}\right)$$
$$\frac{i}{4}\varepsilon_{abcd}\bar{\Psi}^b\Gamma^{IJ}X^{cI}X^{dJ}\Psi^a = \frac{i}{2}v\varepsilon_{ABC}\bar{\Psi}^B\Gamma_i X^{Ci}\Psi^A + O(X^2\Psi^2)$$
$$A_\mu^{A4} \equiv A_\mu^A \text{ and } \frac{1}{2}\varepsilon_{BC}^{A}A_\mu^{BC} \equiv B_\mu^A$$
$$\frac{1}{2}\epsilon^{\mu\nu\lambda}\varepsilon_{abcd}A_\mu^{ab}\partial_\nu A_\lambda^{cd} = 2\epsilon^{\mu\nu\lambda}\varepsilon_{ABC}A_\mu^{AB}\partial_\nu A_\lambda^{C} = 4\epsilon^{\mu\nu\lambda}B_\mu^A\partial_\nu A_{\lambda A}$$
$$\frac{1}{3}\epsilon^{\mu\nu\lambda}\varepsilon_{aef}{}^{g}\varepsilon_{bcdg}A_\mu^{ab}A_\nu^{cd}A_\lambda^{ef} = -4\epsilon^{\mu\nu\lambda}\varepsilon_{ABC}B_\mu^A A_\nu^B A_\lambda^C - \frac{4}{3}\epsilon^{\mu\nu\lambda}\varepsilon_{ABC}B_\mu^A B_\nu^B B_\lambda^C.$$
$$D_\mu X^{AI} = \partial_\mu X^{AI} + \varepsilon_{bcd}^{A}A_\mu^{cd}X^{bI}$$
$$= \partial_\mu X^{AI} + 2\varepsilon_{BC}^{A}A_\mu^{C}X^{BI} + 2B_\mu^A X^{4(I)}$$
$$D_\mu X^{4I} = \partial_\mu X^{4I} - 2B_{\mu A}X^{AI}$$

$$\mathcal{L}_{\text{kinetic}} = -2v^2 B_\mu^A B_A^\mu - 2B_\mu^A X^{4I} D'^\mu X_A^I - 2v B_\mu^A D'^\mu X_A^{(8)}$$
$$-2B_{\mu A} X^{AI} B_B^\mu X^{BI} - 2B_\mu^A B_A^\mu X^{4I} X_{4I} + 2B_A^\mu X^{AI} \partial_\mu X^{4I} + \cdots,$$
$$D'_\mu X^{AI} = \partial_\mu X^{AI} - 2\varepsilon_{BC}^A A_\mu^B X^{CI}$$
$$\mathcal{L}_{\text{CS}} = 2\epsilon^{\mu\nu\lambda} B_\mu^A F'_{\nu\lambda A} - \frac{4}{3}\epsilon^{\mu\nu\lambda}\varepsilon_{ABC} B_\mu^A B_\nu^B B_\lambda^C + \cdots,$$
$$F'^A_{\nu\lambda} = \partial_\nu A_\lambda^A - \partial_\lambda A_\nu^A - 2\varepsilon_{BC}^A A_\nu^B A_\lambda^C.$$
$$\mathcal{L} = -2v^2 B_\mu^A B_A^\mu - 2v B_\mu^A D'^\mu X_A^{(8)} + 2\epsilon^{\mu\nu\lambda} B_\mu^A F'_{\nu\lambda A}$$
$$B_\mu^A = \frac{1}{2v^2}\epsilon_\mu^{\nu\lambda} F'^A_{\nu\lambda} - \frac{1}{2v} D'_\mu X^{A(8)}.$$
$$-\frac{1}{v^2} F'^A_{\mu\nu} F'^{\mu\nu}_A - \frac{1}{2}\partial_\mu X^{4I}\partial^\mu X_4^I - \frac{1}{2} D_\mu X^{Ai} D^\mu X_A^i + O(BX\partial X) + O(B^2X^2) + O(B^3)$$
$$A \to \frac{1}{2}A$$
$$D'_\mu X^{AI} \to D_\mu X^{AI} \equiv \partial_\mu X^{AI} - \varepsilon_{BC}^A A_\mu^B X^{CI}$$
$$F'^A_{\mu\nu} \to \frac{1}{2}F_{\mu\nu}^A \equiv \frac{1}{2}\left(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A - \varepsilon_{BC}^A A_\mu^B A_\nu^C\right).$$
$$-\frac{1}{4v^2} F_{\mu\nu}^A F_A^{\mu\nu} - \frac{1}{2}\partial_\mu X^{4I}\partial^\mu X_4^I - \frac{1}{2} D_\mu X^{Ai} D^\mu X_A^i + \frac{1}{v} O(X\partial X(F/v + DX))$$
$$+\frac{1}{v} O(X^2(F/v + DX)^2) + \frac{1}{v^3} O((F/v + DX)^3)$$
$$\frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a \to \frac{i}{2}\bar{\Psi}^A \Gamma^\mu D_\mu \Psi_A + \frac{i}{2}\bar{\Psi}^4 \Gamma^\mu \partial_\mu \Psi_4$$
$$\mathcal{L} = \mathcal{L}_{\text{SU}(2)} + \mathcal{L}_{\text{U}(1)}$$
$$\mathcal{L}_{\text{U}(1)} = -\frac{1}{2}\partial_\mu X^{4I}\partial^\mu X_4^I + \frac{i}{2}\bar{\Psi}^4 \Gamma^\mu \partial_\mu \Psi_4$$
$$\mathcal{L}_{\text{SU}(2)} = \frac{1}{v^2}\mathcal{L}_0 + \frac{1}{v^3}\mathcal{L}_1 + O\left(\frac{1}{v^4}\right)$$
$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu A} F^{\mu\nu A} - \frac{1}{2} D_\mu X^{Ai} D^\mu X_A^i + \frac{1}{4}\left(\varepsilon_{ABC} X^{Ai} X^{Bj}\right)\left(\varepsilon_{DE}^C X^{Di} X^{Ej}\right)$$
$$+\frac{i}{2}\bar{\Psi}^A \not{D}\Psi_A + \frac{i}{2}\varepsilon_{ABC}\bar{\Psi}^A \Gamma^i X^{Bi} \Psi^C$$
$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - \varepsilon_{BC}^A A_\mu^B A_\nu^C \text{ and } D_\mu^{AB} = \partial_\mu \delta^{AB} + \varepsilon_C^{AB} A_\mu^C$$
$$\mathcal{L} = -\text{Tr}\left(D^\mu \bar{Z}_A, D_\mu Z^A\right) + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\left(\text{Tr}\left(A_\mu^L \partial_\nu A_\lambda^L - \frac{2}{3} i A_\mu^L A_\nu^L A_\lambda^L\right) - \text{Tr}\left(A_\mu^R \partial_\nu A_\lambda^R - \frac{2}{3} i A_\mu^R A_\nu^R A_\lambda^R\right)\right)$$
$$-\frac{1}{3}\text{Tr}\left(4Z^A \bar{Z}_A Z^B \bar{Z}_C Z^C \bar{Z}_B - 4Z^A \bar{Z}_B Z^C \bar{Z}_A Z^B Z_C - Z^A \bar{Z}_A Z^B \bar{Z}_B Z^C \bar{Z}_C - \bar{Z}_A Z^A \bar{Z}_B Z^B \bar{Z}_C Z^C\right)$$
$$\hat{D}_\mu Z^A = \partial_\mu Z^A - i A_\mu^L Z + i Z^A A_\mu^R$$
$$Z^A = v\delta^{A4} + z^A$$
$$Z^A = v\delta^{A4}\mathbb{1}_{N\times N} + \frac{1}{\sqrt{2}} X^A + i\frac{1}{\sqrt{2}} X^{A+4}$$
$$A_\mu^+ = \frac{1}{2}\left(A_\mu^L + A_\mu^R\right), A_\mu^- = \frac{1}{2}\left(A_\mu^L - A_\mu^R\right)$$
$$\hat{D}_\mu Z^A = D_\mu Z^A - i\left\{A_\mu^-, Z^A\right\}$$
$$D_\mu Z^A = \partial_\mu Z^A - i\left[A_\mu^+, Z^A\right]$$
$$F_{\mu\nu}^+ = \partial_\mu A_\nu^+ - i\left[A_\mu^+, A_\nu^+\right].$$
$$S_{CS} = \int d^3x \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\text{Tr}\left(A_\mu^- F_{\nu\lambda}^+ - \frac{2i}{3} A_\mu^- A_\nu^- A_\lambda^-\right)$$
$$Z^A = Z_0^A T^0 + i Z_a^A T^a,$$
$$A_\mu^L = A_\mu^{L0} T^0 + A_\mu^{La} T^a,$$
$$A_\mu^R = A_\mu^{R0} T^0 + A_\mu^{Ra} T^a,$$

$$Z^A = \left(\frac{X_0^A}{\sqrt{2}} + v\delta^{A,4}\right)T^0 + i\frac{X_0^{A+4}}{\sqrt{2}}T^0 + i\frac{X_a^A}{\sqrt{2}}T^a - \frac{X_a^{A+4}}{\sqrt{2}}T^a.$$

$$\hat{D}_\mu Z^A = \frac{\partial_\mu X_0^A}{\sqrt{2}}T^0 - \frac{D_\mu X_a^{A+4}}{\sqrt{2}}T^a + \frac{i\partial_\mu X_0^{A+4}}{\sqrt{2}}T^0 + \frac{iD_\mu X_a^A}{\sqrt{2}}T^a$$
$$-2ivA_{\mu a}^- T^a\delta^{A4} - i\sqrt{2}A_{\mu a}^- X_0^A T^a + \sqrt{2}A_{\mu a}^- X_0^{A+4}T^a - A_\mu^{-a} d_{abc}T^c X_b^{A+4}$$
$$+iA_\mu^{-a} d_{abc}T^c X_b^A - 2ivA_{\mu 0}^- T^0\delta^{A4} - i\sqrt{2}A_{\mu 0}^- X_0^A T^0 + \sqrt{2}A_{\mu 0}^- X_0^{A+4}T^0$$
$$+\sqrt{2}A_{\mu 0}^- X_a^A T^a + i\sqrt{2}A_{\mu 0}^- X_a^{A+4}T^a,$$

$$\mathrm{Tr}\left|\hat{D}_\mu Z^A\right|^2 = N\frac{\left(\partial_\mu X_0^A\right)^2}{2} + \left(\frac{\left(D_\mu X\right)_c^{A+4}}{\sqrt{2}}\right)^2 + \left(2vA_{\mu c}^-\delta^{A4} - \frac{\left(D_\mu X\right)_c^A}{\sqrt{2}}\right)^2$$
$$+N\left(\frac{1}{\sqrt{2}}\partial_\mu X_0^{A+4} - 2vA_{\mu 0}^-\delta^{A4}\right)^2$$
$$-\frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\frac{1}{v}\frac{1}{2\sqrt{2}}\left(D_\mu X\right)_a^4 F_{\nu\lambda}^{+a} - \frac{nk}{2\pi}\epsilon^{\mu\nu\lambda}\frac{1}{v}\frac{1}{2\sqrt{2}}\left(\partial_\mu X_0^8\right)F_{\nu\lambda}^{+0}$$

$$S = \int d^3x\left[\frac{k}{2\pi}e^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu^- F_{\nu\lambda}^+ - \frac{2i}{3}A_\mu^- A_\nu^- A_\lambda^-\right) - \mathrm{Tr}\left|\hat{D}_\mu Z^A\right|^2\right]$$
$$= \int d^3x\left[\frac{k}{2\pi}e^{\mu\nu\lambda}\left(A_{\mu a}^- - \frac{1}{2v}\frac{1}{\sqrt{2}}\left(D_\mu X\right)_a^4\right)F_{\nu\lambda}^{+a} + \frac{nk}{2\pi}e^{\mu\nu\lambda}\left(A_{\mu 0}^- - \frac{1}{2v}\frac{1}{\sqrt{2}}\left(\partial_\mu X\right)_0^8\right)F_{\nu\lambda}^{+0}\right.$$
$$-\left(2vA_{\mu a}^- - \frac{1}{\sqrt{2}}\left(D_\mu X\right)_a^4\right)^2 - n\left(\frac{1}{\sqrt{2}}\partial_\mu X_0^{A+4} - 2vA_{\mu 0}^-\delta^{A4}\right)^2$$
$$\left.-\frac{1}{2}\left(D_\mu X\right)_a^{I'}\left(D^\mu X\right)_a^{I'} - \frac{1}{2}n\partial_\mu X_0^A\partial^\mu X_0^A\right]$$

$$A_{\mu a}^- \to A_{\mu a}^- + \frac{1}{2v}\frac{1}{\sqrt{2}}\left(D_\mu X\right)_a^4 \text{ and } A_{\mu 0}^- \to A_{\mu 0}^- + \frac{1}{2v}\frac{1}{\sqrt{2}}\left(\partial_\mu X_0^8\right)$$

$$S = \int d^3x\left(\frac{k}{2\pi}e^{\mu\nu\lambda}\left(A_{\mu a}^- F_{\nu\lambda}^{a+} + NA_{\mu 0}^- F_{\nu\lambda 0}^+\right) - 4v^2A_{\mu a}^- A_a^{-\mu} - 4nv^2A_{\mu 0}^- A_0^{-\mu}\right.$$
$$\left.-\frac{1}{2}\left(D_\mu X\right)_a^{I'}\left(D^\mu X\right)_a^{I'} - \frac{1}{2}n\partial_\mu X_0^{I'}\partial^\mu X_0^{I'}\right),$$

$$A_\mu^- = \frac{k}{16\pi v^2}\epsilon_{\mu\nu\lambda}F^{+\nu\lambda}$$

$$S = \int d^3x\left[-\mathrm{Tr}\left(\frac{k^2}{32\pi^2v^2}F^{+\mu\nu}F_{\mu\nu}^+\right) - \frac{1}{2}\left(D_\mu X\right)_a^{I'}\left(D^\mu X\right)_a^{I'} - \frac{1}{2}n\partial_\mu X_0^{\tilde{I}'}\partial^\mu X_0^{\tilde{I}'}\right]$$

$$\frac{k^2}{32\pi^2v^2} = \frac{1}{4g_{YM}^2}$$

$$-V_6 \to -\frac{g_{YM}^2}{4}\mathrm{Tr}\left(\left[X^{I'}, X^{J'}\right]\left[X^{J'}, X^{I'}\right]\right)$$

$$Z^4 \to Z^4e^{2\pi i/k} \simeq Z^4\left(1 + 2\pi i\frac{1}{k} + \cdots\right) \simeq Z^4 + 2\pi i\frac{Z^4}{k}$$

$$Z^4T_{M2}^{-\frac{1}{2}} \to Z^4T_{M2}^{-\frac{1}{2}} + 2\pi iR$$

$$T^a = \left\{-\frac{i}{\sqrt{2}}\sigma_1, -\frac{i}{\sqrt{2}}\sigma_2, -\frac{i}{\sqrt{2}}\sigma_3, \frac{1}{\sqrt{2}}\mathbb{1}_{2\times 2}\right\},$$

$$f^{abcd} = \frac{2\pi}{k}\epsilon^{abcd} \text{ and } \mathrm{Tr}\left(T_bT^a\right) = \delta_b^a$$

$$\mathcal{L}_{\mathfrak{u}(n)\times\mathfrak{u}(n)} = \mathcal{L}_{\mathfrak{su}(n)\times\mathfrak{su}(n)}^{\text{gauged}} + \frac{nk}{4\pi}\varepsilon^{\mu\nu\lambda}B_\mu\partial_\nu Q_\lambda$$

$$\mathcal{L}_{\mathfrak{u}(n)\times\mathfrak{u}(n)} = \mathcal{L}_{\mathfrak{su}(n)\times\mathfrak{su}(n)}^{\text{gauged}} + \frac{nk}{8\pi}\varepsilon^{\mu\nu\lambda}B_\mu H_{\nu\lambda} + \frac{n}{8\pi}\sigma\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda}$$

$$\mathcal{L}_{\mathrm{u}(n)\oplus\mathrm{u}(n)} = \mathcal{L}^{\mathrm{gauged}}_{\mathrm{su}(n)\oplus\mathrm{su}(n)} + \frac{nk}{8\pi}\varepsilon^{\mu\nu\lambda}B_\mu H_{\nu\lambda} - \frac{n}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu\sigma H_{\nu\lambda}$$

$$B_\mu = \frac{1}{k}\partial_\mu\sigma$$

$$\sigma \to \sigma + k\theta$$

$$\mathcal{L}_{\mathrm{u}(n)\oplus\mathrm{u}(n)}\left(Z^A, \psi_A, \tilde{A}^a_{\mu b}, B_\mu, Q_\mu\right) \cong \mathcal{L}_{\mathrm{su}(n)\oplus\mathrm{su}(n)}\left(e^{i\sigma/k}Z^A, e^{i\sigma/k}\psi_A, \tilde{A}^a_{\mu b}\right)$$

$$\int dF^{L/R} \in \frac{2\pi}{n}\mathbb{Z}$$

$$\int dH = \int \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda} \in \frac{4\pi}{n}\mathbb{Z}$$

$$\hat{Z}^A \cong e^{2\pi i/k}\hat{Z}^A \text{ and } \hat{\psi}_A \cong e^{2\pi i/k}\hat{\psi}_A$$

$$Z^A = \mathrm{diag}(z^A_1, \dots, z^A_n)$$

$$\det\left(g^{l_B}_{\mathrm{U}(1)}g_0\right) = \det\left(g^{l_B}_{\mathrm{U}(1)}\right) = e^{2\pi i n l_B/k}$$

$$\det\left(g_1^{l_1} \dots g_n^{l_n}\right) = e^{2\pi i(l_1+\dots+l_n)/k}.$$

$$l = nl_B \bmod k$$

$$l = pk \bmod n.$$

$$r^A_1 = z^A_1 + z^A_2, r^A_2 = i\left(z^A_1 - z^A_2\right),$$

$$g_{12}: r^A_1 \cong r^A_1, \qquad r^A_2 \cong -r^A_2$$

$$g_{12}g^2_{\mathrm{SU}(2)}: r^A_1 \cong -r^A_1, \qquad r^A_2 \cong r^A_2$$

$$g_{\mathrm{SU}(2)}g_{12}: r^A_1 \cong r^A_2, \qquad r^A_2 \cong r^A_1$$

$$\mathcal{L} = -\frac{1}{2}\sum_{A=1}^{4}\partial_\mu R^A\partial^\mu R^A - \frac{1}{2}\sum_{A=1}^{4}(R^A)^2\left(\partial_\mu\theta^A - B_\mu\right)(\partial^\mu\theta^A - B^\mu) + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}B_\mu\partial_\nu Q_\lambda$$

$$\Pi_{R^A} = \partial_0 R^A$$

$$\Pi_{\theta^A} = (R^A)^2(\partial_0\theta^A - B_0)$$

$$\Pi_{Q_j} = \frac{k}{4\pi}\epsilon^{ij}B_i$$

$$\Pi_{Q_0} = \Pi_{B_0} = 0$$

$$H = \int d^2x\left\{\frac{1}{2}\sum_A \Pi^2_{R_A} + \frac{1}{2}\sum_A (R^A)^{-2}\Pi^2_{\theta^A} + \frac{1}{2}\sum_A (\partial_i R^A)^2 + \frac{1}{2}\sum_A (R^A)^2(\partial_i\theta^A - B_i)^2\right.$$
$$\left. - \frac{k}{4\pi}F_{12}Q_0 + \left(\sum_A \Pi_{\theta^A} + \frac{k}{4\pi}H_{12}\right)B_0\right\},$$

$$F_{12} = 0$$

$$H_{12} = -\frac{4\pi}{k}\sum_A \Pi_{\theta^A}$$

$$\int d^2x H_{12} \in 4\pi\mathbb{Z}$$

$$\sum_A P_{\theta^A} \in k\mathbb{Z}$$

$$\mathcal{L} = -\mathrm{Tr}\left(D_\mu Z^A D^\mu \bar{Z}_A\right) - i\mathrm{Tr}\left(\bar{\psi}^A\gamma^\mu D_\mu\psi\right) + \mathcal{L}_{\mathrm{Yukawa}} - V$$
$$+\frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A^L_\mu\partial_\nu A^L_\lambda - \frac{2i}{3}A^L_\mu A^L_\nu A^L_\lambda\right) - \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A^R_\mu\partial_\nu A^R_\lambda - \frac{2i}{3}A^R_\mu A^R_\nu A^R_\lambda\right)$$

$$H = \int d^2x\mathrm{Tr}\left(\Pi_{Z^A}\Pi_{\bar{Z}_A}\right) + \mathrm{Tr}\left(D_i Z^A D^i\bar{Z}_A\right) - \mathcal{L}_{\mathrm{Yukawa}} + V$$
$$+\mathrm{Tr}\left(iZ^A\Pi_{Z^A} - i\Pi_{\bar{Z}_A}\bar{Z}_A + i\psi_A\psi^A - \frac{k}{2\pi}F^L_{12}\right)A^L_0 + \mathrm{Tr}\left(i\bar{Z}_A\Pi_{\bar{Z}_A} - i\Pi_{Z^A}Z^A - i\psi^A\psi_A + \frac{k}{2\pi}F^R_{12}\right)A^R_0$$

$$\frac{k}{2\pi}F_{12}^{L} = iZ^{A}\Pi_{Z^{A}} - i\Pi_{\bar{Z}_{A}}\bar{Z}_{A} + i\psi_{A}\psi^{A}$$
$$\frac{k}{2\pi}F_{12}^{R} = i\Pi_{Z^{A}}Z^{A} - i\bar{Z}_{A}\Pi_{\bar{Z}_{A}} + i\psi^{A}\psi_{A}$$
$$Z^{A} = \begin{pmatrix} z_1^A & & & \\ & z_2^A & & \\ & & \ddots & \\ & & & z_n^A \end{pmatrix}$$
$$F_{12}^{L} = F_{12}^{R} = -\frac{2\pi}{k}\begin{pmatrix} \sum_A \Pi_{\theta_1^A} & & & \\ & \sum_A \Pi_{\theta_2^A} & & \\ & & \ddots & \\ & & & \sum_A \Pi_{\theta_n^A} \end{pmatrix},$$
$$F = \star \frac{Q_M}{2} d\left(\frac{1}{|x - x_0|}\right)$$
$$e^{2\pi i Q_M} = 1$$
$$Q_M = \vec{q}\cdot\vec{H} + q\mathbb{1}$$
$$|\vec{\mu}^1\rangle = |\vec{\lambda}^1\rangle, |\vec{\mu}^2\rangle = |\vec{\lambda}^1 - \vec{\alpha}_1\rangle, |\vec{\mu}^3\rangle = |\vec{\lambda}^1 - \vec{\alpha}_1 - \vec{\alpha}_2\rangle, \dots, |\vec{\mu}^n\rangle = |\vec{\lambda}^1 - \vec{\alpha}_1 - \cdots - \vec{\alpha}_{n-1}\rangle$$
$$q_i = \langle\vec{\mu}^i|Q_M|\vec{\mu}^i\rangle = \vec{q}\cdot\vec{\mu}^i + q, i = 1, \dots, n$$
$$w_i = \vec{q}\cdot\vec{\alpha}_i = \vec{q}\cdot(\vec{\mu}^i - \vec{\mu}^{i+1}) = q_i - q_{i+1}, i = 1, \dots, n-1$$
$$\vec{q} = (q_1 - q_2)^L\vec{\lambda}^1 + (q_2 - q_3)^L\vec{\lambda}^2 \dots + (q_{n-1} - q_n)^L\vec{\lambda}^{n-1}$$
$$q = \frac{1}{n}(q_1 + \cdots + q_n)$$
$$\vec{q} = q_1\vec{\alpha}_1 + (q_1 + q_2)\vec{\alpha}_2 + \cdots + (q_1 + q_2 + \cdots + q_{n-1})\vec{\alpha}_{n-1}$$
$$\Phi = \mathrm{diag}(v, -v) + \frac{Q_M}{4\pi r} + \cdots$$
$$\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&2&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}$$
$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},$$
$$\frac{1}{2\pi}\int_{S^2} F^L = \frac{1}{2\pi}\int_{S^2} F^R = Q_M \in \mathfrak{u}(1)^n$$
$$\mathcal{M}_{Q_M}(x) \to e^{ik/2\pi \mathrm{t}\mathrm{tr}\int (D\omega_L \wedge F^L - D\omega_R \wedge F^R)}\mathcal{M}_{Q_M}(x)$$
$$= e^{ik\mathrm{tr}((\omega_L(x) - \omega_R(x))Q_M)}\mathcal{M}_{Q_M}(x).$$
$$\vec{\Lambda} = k\vec{q} \oplus -k\vec{q}$$
$$\mathcal{M}_{\vec{\Lambda}}(x),$$
$$\delta\psi_A = \gamma^{\mu}D_{\mu}Z^{B}\epsilon_{BA}$$
$$D_{\mu}Z^1 = 0$$
$$\frac{k}{2\pi}\varepsilon_{\mu\nu\lambda}F_{L}^{\nu\lambda} = iZ^1 D_{\mu}\bar{Z}_1 = D_{\mu}(iZ^1\bar{Z}_1)$$
$$\frac{k}{2\pi}\varepsilon_{\mu\nu\lambda}F_{R}^{\nu\lambda} = iD_{\mu}\bar{Z}_1 Z^1 = D_{\mu}(i\bar{Z}_1 Z^1)$$
$$Z^1\bar{Z}_1 = -\frac{2\pi i}{k}\frac{Q_M}{|x - x_0|}$$
$$\mathrm{Tr}(Z^A\bar{Z}_B Z^C \dots.)$$
$$\mathrm{Tr}(\mathcal{M}_{\vec{\Lambda}} Z^{A_1} Z^{A_2} Z^{A_3} \dots . Z^{A_p})$$

$$\vec{\Lambda} = p\vec{\lambda}^{n-1}$$
$$Q_B^i = \frac{k}{4\pi}\int H_{12}^i + \Sigma_A \Pi_{\theta_i^A}$$
$$Q_Q^i = \frac{k}{4\pi}\int F_{12}^i.$$
$$\mathcal{M}_i = e^{i\sigma_i(x)}$$
$$Z = \int [dz_i^A][dA_i^L][dA_i^R]e^{i\int d^3x\mathcal{L}}$$
$$\frac{1}{8\pi}\int d^3y\varepsilon^{\mu\nu\lambda}\partial_\mu H_{i\nu\lambda}(y) \to \frac{1}{8\pi}\int d^3y\left(\varepsilon^{\mu\nu\lambda}\partial_\mu H_{i\nu\lambda}(y) + 8\pi p\delta(x-y)\right)$$
$$e^{i\sigma_i(x)} = e^{i\int_\gamma d\sigma_i} = e^{ik\int_\gamma B_i}$$
$$D_\mu e^{i\sigma_i} = \left(i\partial_\mu\sigma_i - ikB_{\mu i}\right)e^{i\sigma_i} = 0$$
$$J_{\mu B}^A = \mathrm{Tr}\left(Z^A D_\mu \bar{Z}_B - D_\mu Z^A \bar{Z}_B + i\psi^A\gamma_\mu\psi_B\right)$$
$$J_\mu^{AB} = \mathrm{Tr}\left(\left(\mathcal{M}_{2\vec{\lambda}^{n-1}}\right)\left(Z^A D_\mu Z^B - D_\mu Z^A Z^B + i\varepsilon^{ABCD}\psi_C\gamma_\mu\psi_D\right)\right)$$
$$\mathrm{Tr}\left(\mathcal{M}_{2\overrightarrow{\lambda^{n-1}}} D_\mu Z^A \psi_A\right)$$
$$\mathrm{Tr}\left(\mathcal{M}_{\vec{\lambda}^{n-1}} Z^A\right),$$
$$S = -T_{M2}\int d^3\sigma\sqrt{-\det\left(\partial_\mu x^M \partial_\nu x^N g_{MN}\right)}$$
$$+\frac{T_{M2}}{3!}\int d^3\sigma\epsilon^{\mu\nu\lambda}\partial_\mu x^M\partial_\nu x^N\partial_\lambda x^P C_{MNP}$$
$$S_C = \frac{1}{3!}e^{\mu\nu\lambda}\int d^3x\left(aT_{M2}C_{\mu\nu\lambda} + 3bC_{\mu IJ}\mathrm{Tr}(D_\nu X^I, D_\lambda X^J)\right.$$
$$+12cC_{\mu\nu IJKL}\mathrm{Tr}(D_\lambda X^I, [X^J, X^K, X^L])$$
$$\left.+12dC_{[\mu IJ}C_{\nu KL]}\mathrm{Tr}(D_\lambda X^I, [X^J, X^K, X^L]) + \cdots\right),$$
$$S_C = \frac{1}{3!}\epsilon^{\mu\nu\lambda}\int d^3x\left(nT_{M2}C_{\mu\nu\lambda}\right.$$
$$+\frac{3}{2}G_{\mu\nu IJ}\mathrm{Tr}(X^I, D_\lambda X^J) - \frac{3}{2}C_{\mu IJ}\mathrm{Tr}\left(X^I, \tilde{F}_{\nu\lambda}X^J\right)$$
$$\left.-cG_{\mu\nu\lambda IJKL}\mathrm{Tr}(X^I, [X^J, X^K, X^L])\right),$$
$$S_F = \frac{1}{4}\epsilon^{\mu\nu\lambda}\int d^3x\mathrm{Tr}\left(X^I, \tilde{F}_{\mu\nu}X^J\right)C_{\lambda IJ}$$
$$S_{CG} = -\frac{c}{2\cdot 3!}\epsilon^{\mu\nu\lambda}\int d^3x\mathrm{Tr}(X^I, [X^J, X^K, X^L])(C_3 \wedge G_4)_{\mu\nu\lambda IJKL}$$
$$S_{\mathrm{flux}} = S_C + S_F + S_{CG}$$
$$= \frac{1}{3!}e^{\mu\nu\lambda}\int d^3x\left(nT_{M2}C_{\mu\nu\lambda} + \frac{3}{2}G_{\mu\nu IJ}\mathrm{Tr}(X^I, D_\lambda X^J)\right.$$
$$\left.-c\left(G_7 + \frac{1}{2}C_3\wedge G_4\right)_{\mu\nu\lambda IJKL}\mathrm{Tr}(X^I, [X^J, X^K, X^L])\right).$$
$$\mathcal{L}_{\mathrm{flux}} = c\tilde{G}_{IJKL}\mathrm{Tr}(X^I, [X^J, X^K, X^L])$$
$$\tilde{G}_{IJKL} = -\frac{1}{3!}\epsilon^{\mu\nu\lambda}\left(G_7 + \frac{1}{2}C_3\wedge G_4\right)_{\mu\nu\lambda IJKL}$$
$$= \frac{1}{4!}\epsilon_{IJKLMNPQ}G^{MNPQ}$$
$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=8} + \mathcal{L}_{\mathrm{mass}} + \mathcal{L}_{\mathrm{flux}}$$
$$\mathcal{L}_{\mathrm{mass}} = -\frac{1}{2}m^2\delta_{IJ}\mathrm{Tr}(X^I, X^J) - \frac{ic}{16}\mathrm{Tr}(\bar{\Psi}\Gamma^{IJKL}, \Psi)\tilde{G}_{IJKL}$$
$$m^2 = \frac{c^2}{32\cdot 4!}G^2$$

$$\delta' X_a^I = 0$$

$$\delta' \tilde{A}_\mu{}^b{}_a = 0$$

$$\delta' \Psi_a = \frac{c}{8} \Gamma^{IJKL} \Gamma^M \epsilon X_a^M \tilde{G}_{IJKL}.$$

$$G_{MN[IJ} G_{KL]}^{MN} = 0$$

$$G = \mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10})$$

$$S_C = \frac{1}{3!} e^{\mu\nu\lambda} \int d^3x \left(nT_{M2} C_{\mu\nu\lambda} + \frac{3}{2} C_\mu^A{}_B \operatorname{Tr}(D_\nu \bar{Z}_A, D_\lambda Z^B) + \frac{3}{2} C_{\mu A}^B \operatorname{Tr}(D_\nu Z^A, D_\lambda \bar{Z}_B) \right.$$

$$\left. + \frac{3c}{2} C_{\mu\nu AB}^{CD} \operatorname{Tr}\left([D_\lambda \bar{Z}_D, [Z^A, Z^B; \bar{Z}_C]) + \frac{3c}{2} C_{\mu\nu}^{AB}{}_{CD}^{C} \operatorname{Tr}\left([D_\lambda Z^D, [\bar{Z}_A, \bar{Z}_B; Z^C] \right) \right) \right.$$

$$S_F = \frac{1}{8} \epsilon^{\mu\nu\lambda} \int d^3x C_\mu{}^A{}_B \operatorname{Tr}\left(\bar{Z}_A, \tilde{F}_{\nu\lambda} Z^B\right) + C_{\mu A}{}^B \operatorname{Tr}\left(Z^A, \tilde{F}_{\nu\lambda} \bar{Z}_B\right)$$

$$S_{CG} = -\frac{c}{8 \cdot 3!} \epsilon^{\mu\nu\lambda} \int d^3x (C_3 \wedge G_4)_{\mu\nu AB}^{CD} \operatorname{Tr}(\bar{Z}_D, [Z^A, Z^B; \bar{Z}_C])$$

$$\begin{aligned} S_{\text{flux}} &= S_C + S_F + S_{CG} \\ &= \frac{1}{3!} \epsilon^{\mu\nu\lambda} \int d^3x \left(nT_{M2} C_{\mu\nu\lambda} \right. \\ &\quad + \frac{3}{4} G_{\mu\nu}{}^A{}_B \operatorname{Tr}(\bar{Z}_A, D_\lambda Z^B) + \frac{3}{4} G_{\mu\nu A}{}^B \operatorname{Tr}(Z^A, D_\lambda \bar{Z}_B) \\ &\quad \left. - \frac{c}{4} \left(G_7 + \frac{1}{2} C_3 \wedge G_4 \right)_{\mu\nu\lambda AB}{}^{CD} \operatorname{Tr}\left([\bar{Z}_D, [Z^A, Z^B; \bar{Z}_C] \right) \right). \end{aligned}$$

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=6} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{flux}}$$

$$\mathcal{L}_{\text{flux}} = \frac{c}{4} \operatorname{Tr}\left([\bar{Z}_D, [Z^A, Z^B; \bar{Z}_C] \right) \tilde{G}_{AB}^{CD}$$

$$\begin{aligned} \tilde{G}_{AB}^{CD} &= -\frac{1}{3!} \epsilon^{\mu\nu\lambda} \left(G_7 + \frac{1}{2} C_3 \wedge G_4 \right)_{\mu\nu\lambda AB}^{CD} \\ &= \frac{1}{4} \epsilon_{ABEF} \epsilon^{CDGH} G^{EF} \end{aligned}$$

$$\mathcal{L}_{\text{mass}} = -m^2 \operatorname{Tr}(\bar{Z}_A, Z^A) + \frac{ic}{4} \operatorname{Tr}(\bar{\psi}^A, \psi_F) \tilde{G}_{AE}^{EF}$$

$$\delta' \psi_{Ad} = \frac{c}{4} \epsilon_{DF} Z_d^F \tilde{G}_{AE}^{ED}$$

$$\tilde{G}_{AE}^{EB} \tilde{G}_{BF}^{FC} = \frac{16m^2}{c^2} \delta_A^C$$

$$\tilde{G}_{AB}^{CD} = \frac{1}{2} \delta_B^C \tilde{G}_{AE}^{ED} - \frac{1}{2} \delta_A^C \tilde{G}_{BE}^{ED} - \frac{1}{2} \delta_B^D \tilde{G}_{AE}^{EC} + \frac{1}{2} \delta_A^D \tilde{G}_{BE}^{EC}$$

$$m^2 = \frac{1}{32 \cdot 4!} c^2 G^2$$

$$\tilde{G}_{AB}^{BC} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}$$

$$R_{mn} - \frac{1}{2} g_{mn} R = \frac{1}{2 \cdot 3!} G_{mpqr} G_n^{pqr} - \frac{1}{4 \cdot 4!} g_{mn} G^2$$

$$d \star G_4 - \frac{1}{2} G_4 \wedge G_4 = 0.$$

$$g_{mn} = \begin{pmatrix} e^{2\omega} \eta_{\mu\nu} & 0 \\ 0 & g_{IJ} \end{pmatrix}$$

$$\omega = \omega(x^I) = \omega\left(X^I / T_{M2}^{\frac{1}{2}} \right) g_{IJ} = g_{IJ}(x^I) = g_{IJ}\left(X^I / T_{M2}^{\frac{1}{2}} \right)$$

$$S_1 = -T_{M2}\int d^3x\sqrt{-\det\left(e^{2\omega}\eta_{\mu\nu} + \partial_\mu x^I\partial_\nu x^J g_{IJ}\right)}$$
$$= -T_{M2}\int d^3x e^{3\omega}\left(1 + \frac{1}{2}e^{-2\omega}\partial_\mu x^I\partial^\mu x^J g_{IJ} + \cdots\right)$$
$$= -\int d^3x\left(T_{M2}e^{3\omega} + \frac{1}{2}e^{\omega}\partial_\mu X^I\partial^\mu X^J g_{IJ} + \cdots\right).$$
$$e^{2\omega(x)} = e^{2\omega\left(X^I/\sqrt{T_{M2}}\right)} = 1 + \frac{2}{T_{M2}}\omega_{IJ}X^I X^J + \cdots$$
$$g_{IJ}(x) = g_{IJ}\left(X^I/\sqrt{T_{M2}}\right) = \delta_{IJ} + \cdots$$
$$S_1 = -\int d^3x\left(T_{M2} + 3\omega_{IJ}X^I X^J + \frac{1}{2}\partial_\mu X^I\partial^\mu X^J\delta_{IJ} + \cdots\right)$$
$$\partial_I\partial^I e^{2\omega} = \frac{1}{3\cdot 4!}G^2$$
$$e^{2\omega} = 1 + \frac{1}{48\cdot 4!}G^2\delta_{IJ}x^I x^J$$
$$S_1 = -\int d^3x\frac{1}{32\cdot 4!}G^2X^2$$
$$S_2 = \frac{T_{M2}}{3!}\int d^3x\epsilon^{\mu\nu\lambda}C_{\mu\nu\lambda}$$
$$\partial_I\partial^I C_0 = \frac{1}{2\cdot 4!}G^2$$
$$C_0 = \frac{1}{32\cdot 4!}G^2\delta_{IJ}x^I x^J$$
$$S_2 = -\int d^3x\frac{1}{32\cdot 4!}G^2X^2$$
$$m^2 = \frac{1}{8\cdot 4!}G^2$$
$$S_{\text{ABJM}} = \int d^3x\left[\frac{k}{4\pi}e^{\mu\nu\lambda}\text{Tr}\left(A_\mu^L\partial_\nu A_\lambda^L + \frac{2i}{3}A_\mu^L A_\nu^L A_\lambda^L - A_\mu^R\partial_\nu A_\lambda^R - \frac{2i}{3}A_\mu^R A_\nu^R A_\lambda^R\right) - \text{Tr}\left(D_\mu\bar{Z}_A D^\mu Z^A\right)\right.$$
$$\left. + \frac{4\pi^2}{3k^2}\text{Tr}(Z^A\bar{Z}_A Z^B\bar{Z}_B Z^C\bar{Z}_C + \bar{Z}_A Z^A\bar{Z}_B Z^B\bar{Z}_C Z^C + 4Z^A\bar{Z}_B Z^C\bar{Z}_A Z^B\bar{Z}_C - 6Z^A\bar{Z}_B Z^B\bar{Z}_A Z^C\bar{Z}_C)\right]$$
$$V = |M^\alpha|^2 + |N^\alpha|^2$$
$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k}\left(2Q^{[\alpha}\bar{Q}_\beta Q^{\beta]} + R^\beta\bar{R}_\beta Q^\alpha - Q^\alpha\bar{R}_\beta R^\beta + 2Q^\beta\bar{R}_\beta R^\alpha\right)$$
$$N^\alpha = -\mu R^\alpha + \frac{2\pi}{k}\left(2R^{[\alpha}\bar{R}_\beta R^{\beta]} + Q^\beta\bar{Q}_\beta R^\alpha - R^\alpha\bar{Q}_\beta Q^\beta + 2R^\beta\bar{Q}_\beta Q^\alpha\right),$$
$$\frac{1}{2}M_B^C\epsilon_{CD}Z^D + ([Z^C, Z^D; Z_B] + [Z^E, Z^C; \bar{Z}_E]\delta_B^D)\epsilon_{CD} = 0$$
$$M_B^C = \begin{pmatrix} 2\mu & 0 & 0 & 0 \\ 0 & 2\mu & 0 & 0 \\ 0 & 0 & -2\mu & 0 \\ 0 & 0 & 0 & -2\mu \end{pmatrix}$$
$$\frac{1}{4}M_B^C Z^D - \frac{1}{4}M_B^D Z^C + [Z^C, Z^D; Z_B] + \frac{1}{2}[Z^E, Z^C; \bar{Z}_E]\delta_B^D - \frac{1}{2}[Z^E, Z^D; \bar{Z}_E]\delta_B^C = 0$$
$$\frac{1}{2}M_B^C Z^B = [Z^D, Z^E; \bar{Z}_E]$$
$$R^\alpha = \frac{2\pi}{\mu k}\left(R^\alpha\bar{R}_\beta R^\beta - R^\beta\bar{R}_\beta R^\alpha\right),$$
$$R^\alpha = fG^\alpha$$
$$G^\alpha = G^\alpha\bar{G}_\beta G^\beta - G^\beta\bar{G}_\beta G^\alpha$$

$$(G^1)_{m,l} = \sqrt{m-1}\delta_{m,l}$$
$$(G^2)_{m,l} = \sqrt{(n-m)}\delta_{m+1,l}$$
$$(\bar{G}_1)_{m,l} = \sqrt{m-1}\delta_{m,l}$$
$$(\bar{G}_2)_{m,l} = \sqrt{(n-l)}\delta_{l+1,m}$$
$$J^\alpha_\beta = G^\alpha \bar{G}_\beta \text{ and } \bar{J}^\beta_\alpha = \bar{G}_\alpha G^\beta$$
$$J_i = (\tilde{\sigma}_i)^\alpha{}_\beta G^\beta \bar{G}_\alpha = (\tilde{\sigma}_i)^\alpha{}_\beta J^\beta{}_\alpha \equiv (\sigma_i)_\beta{}^\alpha J^\beta{}_\alpha$$
$$\bar{J}_i = (\tilde{\sigma}_i)^\alpha{}_\beta \bar{G}_\alpha G^\beta = (\tilde{\sigma}_i)^\alpha{}_\beta \bar{J}_\alpha{}^\beta \equiv (\sigma_i)_\beta{}^\alpha \bar{J}_\alpha{}^\beta,$$
$$[J_i, J_j] = 2i\epsilon_{ijk}J_k \text{ and } [\bar{J}_i, \bar{J}_j] = 2i\epsilon_{ijk}\bar{J}_k$$
$$J_i G^\alpha - G^\alpha \bar{J}_i = (\tilde{\sigma}_i)^\alpha{}_\beta G^\beta$$
$$x_i \simeq \frac{J_i}{n} \text{ and } \bar{x}_i \simeq \frac{\bar{J}_i}{n}$$
$$g^\alpha \simeq \frac{G^\alpha}{\sqrt{n}} \text{ and } g^*_\alpha \simeq \frac{\bar{G}_\alpha}{\sqrt{n}}$$
$$x_i = (\tilde{\sigma}_i)^\alpha{}_\beta g^\beta g^*_\alpha$$
$$\bar{x}_i = (\tilde{\sigma}_i)^\alpha{}_\beta g^*_\alpha g^\beta,$$
$$S = -\int d^3x \mathrm{Tr}\left(D_\mu Z^A D^\mu \bar{Z}_A\right)$$
$$S_{phys} = -T_{M2}\int d^3x \mathrm{Tr}\left(D_\mu z^A D^{\mu -} \bar{z}_A\right)$$
$$R^2_{ph} = \frac{2}{n}\mathrm{Tr}(z^A \bar{z}_A) = 8\pi^2 n f^2 \ell_p^3$$
$$R_{11} = \frac{R_{ph}}{k}$$
$$R_{ph} = 2n\mu\lambda$$
$$0 = \left(\partial_2 X^{L'}_a - \frac{1}{3!}\varepsilon^{I'J'K'L'} f^{cdb}{}_a X^{I'}_c X^{J'}_d X^{K'}_b\right)\Gamma_2 \Gamma^{L'}\epsilon$$
$$\frac{dX^{I'}}{dx^2} = -\frac{1}{3!}\varepsilon^{I'J'K'L'}\left[X^{J'}, X^{K'}, X^{L'}\right]$$
$$0 = \gamma^2 \partial_2 Z^\alpha \epsilon_{\alpha B} + \left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right]\epsilon_{\alpha B} + \left[Z^\gamma, Z^\delta; \bar{Z}_B\right]\epsilon_{\gamma\delta}$$
$$0 = \gamma^2 \partial_2 Z^\alpha \epsilon_{\alpha\beta'} + \left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right]\epsilon_{\alpha\beta'}$$
$$\frac{dZ^\alpha}{dx^2} = \left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right] = \frac{2\pi}{k}\left(Z^\gamma Z^\dagger_\gamma Z^\alpha - Z^\alpha Z^\dagger_\gamma Z^\gamma\right)$$
$$0 = \gamma^2 \partial_2 Z^\alpha \epsilon_{\alpha\beta} + \left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right]\epsilon_{\alpha\beta} + \left[Z^\gamma, Z^\delta; \bar{Z}_\beta\right]\epsilon_{\gamma\delta}$$
$$\begin{aligned}\left[Z^\gamma, Z^\delta; \bar{Z}_\beta\right]\epsilon_{\gamma\delta} &= 2\left[Z^1, Z^2; \bar{Z}_\beta\right]\epsilon_{12} \\ &= 2\varepsilon_{\beta\alpha}\left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right]\epsilon_{12} \\ &= -2\left[Z^\gamma, Z^\alpha; \bar{Z}_\gamma\right]\epsilon_{\alpha\beta},\end{aligned}$$
$$Z^\alpha = f(x^2)G^\alpha$$
$$\frac{df}{dx^2} = -\frac{2\pi}{k}f^3$$
$$f = \sqrt{\frac{k}{2\pi}\frac{2}{\sqrt{x^2}}}$$
$$f^{abc}_d = f^{[abc]}_d, \left(f^{abc}_d\right)^* = f^d_{abc}$$
$$f^{ab}{}_{cd} = -f^{ba}{}_{dc} = -f^{ab}{}_{cd}, \left(f^{ab}{}_{cd}\right)^* = f^{cd}{}_{ab}$$
$$\epsilon^{AB} = -\epsilon^{BA}, \epsilon^{AB}\omega_{AB} = 0$$
$$\epsilon_{AB} = \omega_{AC}\omega_{BD}\epsilon^{CD}$$
$$(Z^A_a)^* = \bar{Z}^a_A = -J^{ab}\omega_{AB}Z^B_b$$
$$(\Psi_{Aa})^* = \Psi^{Aa} = -J^{ab}\omega^{AB}\Psi_{Bb},$$

$$\delta Z^A_d = i\epsilon^{AD}\Psi_{Dd}$$
$$\delta\Psi_{Dd} = \gamma^\mu\epsilon_{AD}D_\mu Z^A_d + h^{abc}{}_d Z^A_a Z^B_b Z^C_c \epsilon_{AB}\omega_{DC} + j^{abc}{}_d Z^A_a Z^B_b Z^C_c \epsilon_{DC}\omega_{AB},$$
$$D_\mu Z^A_d = \partial_\mu Z^A_d - \tilde{A}_\mu{}^a{}_d Z^A_a$$
$$\left(h^{abcd}\right)^* = h_{abcd} = \omega_{ae}\omega_{bf}\omega_{cg}\omega_{dh}h^{efgh}$$
$$\left(j^{abcd}\right)^* = j_{abcd} = \omega_{ae}\omega_{bf}\omega_{cg}\omega_{dh}j^{efgh},$$
$$[\delta_1,\delta_2]Z^A_d = v^\mu D_\mu Z^A_d + \tilde{\Lambda}^a_d Z^A_a$$
$$\tilde{\Lambda}^a_d = ih^{abc}{}_d Z^B_b Z^C_c \omega_{DC}\bar{\epsilon}^{DF}_{[2}\epsilon_{1]BF}$$
$$j^{abc}{}_d = \frac{1}{2}\left(h^{bca}{}_d - h^{acb}{}_d\right)$$
$$\delta\Psi_{Dd} = \gamma^\mu\epsilon_{AD}D_\mu Z^A_d + h^{abc}{}_d Z^A_a Z^B_b Z^C_c \epsilon_{AB}\omega_{DC} - h^{acb}{}_d Z^A_a Z^B_b Z^C_c \epsilon_{DC}\omega_{AB}$$
$$[\delta_1,\delta_2]\Psi_{Dd} = v^\mu D_\mu\Psi_{Dd} + \tilde{\Lambda}^a_d\Psi_{Da} - \frac{i}{2}\bar{\epsilon}^{AC}_{[1}\epsilon_{2]AD}E_{Cd} + \frac{i}{4}(\bar{\epsilon}^{AB}_1\gamma_\nu\epsilon_{2AB})\gamma^\nu E_{Dd},$$
$$E_{Dd} = \gamma^\mu D_\mu\Psi_{Dd} - h^{abc}{}_d\left(\Psi_{Dc}Z^A_a Z^B_b + \Psi_{Db}Z^A_a Z^B_c\right)\omega_{AB} + 2h^{abc}{}_d\left(\Psi_{Ab}Z^A_a Z^C_c + \Psi_{Ac}Z^A_a Z^C_b\right)\omega_{DC} = 0.$$
$$\delta\tilde{A}_\mu{}^a{}_d = -i\left(h^{acb}{}_d + h^{abc}{}_d\right)\omega^{BE}\bar{\epsilon}_{EC}\gamma_\mu\Psi_{Bb}Z^C_c.$$
$$h^{abc}{}_g\left(h^{edg}{}_f + h^{egd}{}_f\right)Z^A_a Z^B_b Z^C_c Z^D_d\omega_{AD}\omega_{BC} = 0$$
$$h^{abc}{}_g\left(h^{edg}{}_f + h^{egd}{}_f\right)Z^A_a Z^B_b Z^C_c Z^D_d\bar{\xi}_{AB[1}\gamma^\mu\xi_{2]CD} = 0.$$
$$\delta Z^A_d = i\epsilon^{AD}\Psi_{Dd}$$
$$\delta\Psi_{Dd} = \gamma^\mu\epsilon_{AD}D_\mu Z^A_d + f^{ab}{}_{cd}Z^A_a Z^B_b\bar{Z}^c_A\epsilon_{BD} + f^{ab}{}_{cd}Z^A_a Z^B_b\bar{Z}^c_D\epsilon_{AB}$$
$$\delta\tilde{A}_\mu{}^a{}_d = -if^{ab}{}_{cd}\left(\bar{\epsilon}^{BC}\gamma_\mu\Psi_{Bb}\bar{Z}^c_C + \bar{\epsilon}_{BC}\gamma_\mu\Psi^{Cc}Z^B_b\right)$$
$$h^{abe}{}_d J_{ce} = f^{ab}{}_{cd} = -f^{ba}{}_{cd} = -f^{ab}{}_{dc}.$$
$$h^{abc}{}_d = g^{acb}{}_d - g^{bca}{}_d,$$
$$g^{acbd} = g^{cabd} = g^{bdac}.$$
$$g^{(acb)d} = 0$$
$$J_{gj}\left(g^{afbg}g^{jchd} + g^{afgd}g^{hjbc} + g^{afhg}g^{jdbc} + g^{afgc}g^{bjhd}\right) = 0.$$
$$\delta Z^A_d = i\epsilon^{AD}\Psi_{Dd}$$
$$\delta\Psi_{Dd} = \gamma^\mu\epsilon_{AD}D_\mu Z^A_d - g^{abc}{}_d Z^A_a Z^B_b Z^C_c\epsilon_{DB}\omega_{AC} + 2g^{abc}{}_d Z^A_a Z^B_b Z^C_c\epsilon_{AC}\omega_{DB}$$
$$\delta\tilde{A}_\mu{}^a{}_d = 3ig^{bca}{}_d\omega^{BE}\bar{\epsilon}_{EC}\gamma_\mu\Psi_{Bb}Z^C_c.$$
$$\tilde{\Lambda}^a{}_d = -\frac{3i}{2}g^{bca}{}_d Z^B_b Z^C_c\omega_{DC}\bar{\epsilon}^{DF}_{[2}\epsilon_{1]BF}.$$
$$\mathcal{L} = -D^\mu\bar{Z}^a_A D_\mu Z^A_a - i\bar{\Psi}^{Aa}\gamma^\mu D_\mu\Psi_{Aa} - V + \mathcal{L}_{CS}$$
$$-3ig^{acbd}\omega_{AB}\omega_{CD}\left(Z^A_a Z^B_b\bar{\Psi}^C_c\Psi^D_d - 2Z^A_a Z^D_b\bar{\Psi}^C_c\Psi^B_d\right)$$
$$V = \frac{12}{5}\bar{\Upsilon}^d_{ABC}\Upsilon^{ABC}_d$$
$$\Upsilon^{ABC}_d = g^{abc}{}_d\left(Z^A_a Z^B_b Z^C_c + \frac{1}{4}\omega^{BC}Z^A_a Z^D_b Z_{Dc}\right)$$
$$\delta Z^A_d = \tilde{\Lambda}^a{}_d Z^A_a = g^{bca}{}_d\Lambda_{bc}Z^A_a$$
$$g_1^{aibjckdl} = \left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right)J^{ij}J^{kl}$$
$$g_2^{aibjckdl} = \left(J^{ik}J^{jl} + J^{jk}J^{il}\right)\delta^{ab}\delta^{cd}$$
$$g_3^{(\pm)aibjckdl} = \left(\delta^{ac}\delta^{bd} \pm \delta^{ad}\delta^{bc}\right)\left(J^{ik}J^{jl} \pm J^{jk}J^{il}\right)$$
$$g^{aibjckdl} = -\frac{2\pi}{k}\left[g_1^{aibjckdl} - g_2^{aibjckdl}\right]$$
$$g^{aibjckdl} = -\frac{2\pi}{k}\left[g_3^{(+)aibjckdl} + g_3^{(-)aibjckdl}\right]$$
$$g^{aibjckdl} = -\frac{2\pi}{k}\left[\left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right)J^{ij}J^{kl} - \delta^{ab}\delta^{cd}\left(J^{ik}J^{jl} + J^{jk}J^{il}\right)\right]$$

$$\delta Z^{Adl} = -\frac{2\pi}{k}\left[\left(\delta^{ba}\delta^{cd} - \delta^{bd}\delta^{ca}\right)J^{jk}J^{il} - \delta^{bc}\delta^{ad}\left(J^{ji}J^{kl} + J^{ki}J^{jl}\right)\right]\Lambda_{bjck}Z_{ai}^{A}$$

$$= -\frac{2\pi}{k}\left[\left(\Lambda_{ajdk} - \Lambda_{djak}\right)J^{jk}J^{il} - \delta^{ad}\left(J^{ji}J^{kl} + J^{ki}J^{jl}\right)\Lambda_{bjbk}\right]Z_{ai}^{A}$$

$$g^{aibjckdl} = -\frac{2\pi}{k}\left[J^{ik}J^{jl}\delta^{ac}\delta^{bd} + J^{il}J^{jk}\delta^{ad}\delta^{bc}\right]$$

$$g^{aibjckdl} = -\frac{2\pi}{k}\left[g_1^{aibjckdl} - g_2^{aibjckdl} + \alpha\varepsilon^{abcd}J^{ij}J^{kl}\right]$$

$$g^{aibjckdl} = -\frac{2\pi}{k}\left[g_1^{aibjckdl} - g_2^{aibjckdl} + \beta C^{abcd}J^{ij}J^{kl}\right]$$

$$C^{abcd} = \frac{1}{3!}\varepsilon^{abcdefg}C_{efg}$$

$$g^{aibjckdl} = -\frac{2\pi}{k}\left[\left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} + \frac{1}{2}C^{abcd}\right)J^{ij}J^{kl} - \delta^{ab}\delta^{cd}\left(J^{ik}J^{jl} + J^{jk}J^{il}\right)\right]$$

$$\delta Z^{Adl} = g^{bjckaidl}\Lambda_{bjck}Z_{ai}^{A}$$

$$= -\frac{2\pi}{k}\left[J^{il}\left(\delta^{ba}\delta^{cd} - \delta^{bd}\delta^{ca} + \frac{1}{2}C^{bcad}\right)J^{jk}\Lambda_{bjck} - \delta^{ad}\left(J^{ji}J^{kl} + J^{ki}J^{jl}\right)\Lambda_{bjbk}\right].$$

$$\mathcal{P}_{14}^{abcd} = \frac{1}{3}\left(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \frac{1}{2}C^{abcd}\right)$$

$$\mathcal{P}_{14}^{abcd}C_{bce} = 0$$

$$g^{aibjckdl} = -\frac{2\pi}{k}\left[\delta^{ab}\delta^{cd}\left(J^{ik}J^{jl} + J^{jk}J^{il}\right) - \frac{1}{6}\Gamma_{mn}^{ab}\Gamma_{mn}^{cd}J^{ij}J^{kl}\right],$$

$$\delta Z^{Adl} = g^{bjckaidl}\Lambda_{bjck}Z_{ai}^{A}$$

$$= -\frac{2\pi}{k}\left[\delta^{ad}\left(J^{ji}J^{kl} + J^{ki}J^{jl}\right)\Lambda_{bjbk} - \frac{1}{6}J^{jl}\Gamma_{mn}^{ad}\Gamma_{mn}^{bc}J^{jk}\Lambda_{bjck}\right]Z_{ai^*}^{A}$$

$$g^{abcd} = h_{mn}\left(\tau^{ma}{}_{e}J^{be}\right)\left(\tau^{nc}{}_{f}J^{df}\right)$$

$$Z_a^A = Z_{a1}^A + iZ_{a2}^A$$

$$\bar{Z}_A^a = \bar{Z}_A^{a1} - i\bar{Z}_A^{a2}.$$

$$\Xi_{Aa} = \Psi_{Aa1} + i\Psi_{Aa2}$$

$$\Xi^{*Aa} = \Psi^{Aa1} - i\Psi^{Aa2}$$

$$\bar{Z}_A^{ai} = -\omega_{AB}J^{ab}\delta^{ij}Z_{bj}^B$$

$$\Psi^{Aai} = -\omega^{AB}J^{ab}\delta^{ij}\Psi_{Bbj}\ [\text{ for } \mathrm{Sp}(n)\times\mathrm{SO}(2)]$$

$$\bar{Z}_A^{ai} = -\omega_{AB}\delta^{ab}\varepsilon^{ij}Z_{bj}^B$$

$$\Psi^{Aai} = -\omega^{AB}\delta^{ab}\varepsilon^{ij}\Psi_{Bbj}\ [\text{ for } \mathrm{SO}(4)\times\mathrm{SU}(2)]$$

$$\bar{Z}_A^a = -\omega_{AB}J^{ab}\left(Z_{b1}^B - iZ_{b2}^B\right)$$

$$g^{aibjckdl} = \frac{4\pi}{3k}\left[\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right)J^{ab}J^{cd} - \delta^{ij}\delta^{kl}\left(J^{ac}J^{bd} + J^{bc}J^{ad}\right)\right]$$

$$f^{ab}{}_{cd} = -\frac{2\pi}{k}\left[J^{ab}J_{cd} + \left(\delta_c^a\delta_d^b - \delta_d^a\delta_c^b\right)\right]$$

$$\delta Z_{dl}^A = -\omega^{AD}\bar{\eta}\Psi_{Ddl}$$

$$\delta\Psi_{Ddl} = -i\gamma^\mu\omega_{AD}\eta D_\mu Z_{dl}^A + if^{ab}{}_{cd}\left(\omega_{AB}\omega_{CD} - \omega_{AC}\omega_{BD}\right)\times\left(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{jk}\varepsilon_{il} + i\delta_{ij}\varepsilon_{kl}\right)Z_{ai}^A Z_{bj}^B Z_k^{Cc}\eta$$

$$\delta\tilde{A}_\mu^{\mathrm{aidl}} = if^{abcd}\left(\bar{\eta}\gamma_\mu\Psi_{Bbj}Z_{ck}^B - \bar{\eta}\gamma_\mu\Psi_{Bck}Z_{bj}^B\right)\left(\delta^{jk}\varepsilon^{il} + \varepsilon^{jk}\delta^{il}\right)$$

$$g^{\mathrm{aibjckdl}} \to \alpha\varepsilon^{abcd}\varepsilon^{ij}\varepsilon^{kl}$$

$$\delta Z_{dl}^D \to \alpha\varepsilon^{abcd}\varepsilon^{jk}\varepsilon^{il}\Lambda_{bjck}Z_{ai}^A$$

$$\bar{Z}_{Aa} = -i\omega_{AB}\delta_{ab}\left(Z_{b1}^B - iZ_{b2}^B\right)$$

$$g^{aibjckdl} = \frac{4\pi}{3k}\varepsilon^{abcd}\varepsilon^{ij}\varepsilon^{kl}$$

$$f^{abcd} = -\frac{2\pi}{k}\varepsilon^{abcd}$$

$$\delta Z^A_{dl} = -\omega^{AD}\bar{\eta}\Psi_{Ddl}$$
$$\delta\Psi_{Ddl} = -i\gamma^\mu\omega_{AD}\eta D_\mu Z^A_{dl} - \frac{4\pi}{k}\varepsilon^{abcd}\omega_{AB}\omega_{CD}\delta_{ik}\delta_{jl}Z^A_{ai}Z^B_{bj}Z^C_{ck}\eta$$
$$\delta\tilde{A}_\mu{}^{aidl} = i\frac{4\pi}{k}\varepsilon^{abcd}\varepsilon^{il}\bar{\eta}\gamma_\mu\Psi_{Bbj}Z^B_{cj}$$
$$\eta_{A\dot{A}} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \eta^{A\dot{A}} = \varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}\eta_{B\dot{B}} = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}.$$
$$\delta Z^A_d = i\bar{\eta}^{A\dot{A}}\Psi_{\dot{A}d}.$$
$$\begin{pmatrix} Z^A_a \\ 0 \end{pmatrix} \to Z^A_a \qquad \begin{pmatrix} \bar{Z}^a_A \\ 0 \end{pmatrix} \to \bar{Z}^a_A$$
$$\begin{pmatrix} 0 \\ \Psi_{\dot{A}a} \end{pmatrix} \to \Psi_{Aa} \qquad \begin{pmatrix} 0 \\ \Psi^{\dot{A}a} \end{pmatrix} \to \Psi^{Aa},$$
$$\begin{pmatrix} 0 & \eta^{A\dot{B}} \\ -(\eta^T)^{\dot{A}B} & 0 \end{pmatrix} \to \epsilon^{AB}, \begin{pmatrix} 0 & \eta_{A\dot{B}} \\ -(\eta^T)_{\dot{A}B} & 0 \end{pmatrix} \to \epsilon_{AB}$$
$$\delta Z^A_d = i\bar{\eta}^{A\dot{D}}\Psi_{\dot{D}d}$$
$$\delta\Psi_{\dot{D}d} = \gamma^\mu\eta_{A\dot{D}}D_\mu Z^A_d + f^{ab}{}_{cd}Z^A_a Z^B_b \bar{Z}^c_A \eta_{B\dot{D}}$$
$$\delta\tilde{A}_\mu{}^a{}_d = -if^{ab}{}_{cd}\left(\bar{\eta}^{C\dot{B}}\gamma_\mu\Psi_{\dot{B}b}\bar{Z}^c_C + \bar{\eta}_{B\dot{C}}\gamma_\mu\Psi^{\dot{C}c}Z^B_b\right)$$
$$(Z^A_a)^* = \bar{Z}^a_A = -J^{ab}\varepsilon_{AB}Z^B_b$$
$$(\Psi_{\dot{A}a})^* = \Psi^{\dot{A}a} = -J^{ab}\varepsilon^{\dot{A}\dot{B}}\Psi_{\dot{B}b},$$
$$\begin{pmatrix} \varepsilon_{AB} & 0 \\ 0 & \varepsilon_{\dot{A}\dot{B}} \end{pmatrix} \to \omega_{AB}, \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \varepsilon^{\dot{A}\dot{B}} \end{pmatrix} \to \omega^{AB}$$
$$\delta Z^A_d = i\bar{\eta}^{A\dot{D}}\Psi_{\dot{D}d}$$
$$\delta\Psi_{\dot{D}d} = \gamma^\mu\eta_{A\dot{D}}D_\mu Z^A_d + g^{abc}{}_d Z^A_a Z^B_b Z^C_c \eta_{B\dot{D}}\varepsilon_{AC}$$
$$\delta\tilde{A}_\mu{}^a{}_d = -3ig^{bca}{}_d\varepsilon^{\dot{B}\dot{E}}\bar{\eta}_{C\dot{E}}\gamma_\mu\Psi_{\dot{B}b}Z^C_c,$$
$$\begin{pmatrix} Z^A_a \\ Z^{\dot{A}}_{\dot{a}} \end{pmatrix} \to Z^A_a \qquad \begin{pmatrix} \bar{Z}^a_A \\ Z^{\dot{a}}_{\dot{A}} \end{pmatrix} \to \bar{Z}^a_A$$
$$\begin{pmatrix} \Psi_{A\dot{a}} \\ \Psi_{\dot{A}a} \end{pmatrix} \to \Psi_{Aa} \qquad \begin{pmatrix} \Psi^{\dot{A}a} \\ \Psi^{\dot{A}a} \end{pmatrix} \to \Psi^{Aa},$$
$$\delta Z^A_d = i\bar{\eta}^{A\dot{D}}\Psi_{\dot{D}d}$$
$$\delta Z^{\dot{A}}_{\dot{d}} = -i\bar{\eta}^{D\dot{A}}\Psi_{D\dot{d}}$$
$$\delta\Psi_{\dot{D}d} = \gamma^\mu\eta_{A\dot{D}}D_\mu Z^A_d + f^{ab}{}_{cd}Z^A_a Z^B_b \bar{Z}^c_A \eta_{B\dot{D}} + f^{\dot{a}b}{}_{cd}Z^{\dot{A}}_{\dot{a}}Z^B_b\bar{Z}^{\dot{c}}_{\dot{A}}\eta_{B\dot{D}} - 2f^{\dot{a}b}{}_{cd}Z^{\dot{A}}_{\dot{a}}Z^B_b\bar{Z}^{\dot{c}}_{\dot{D}}\eta_{B\dot{A}}$$
$$\delta\Psi_{D\dot{d}} = -\gamma^\mu\eta_{D\dot{A}}D_\mu Z^{\dot{A}}_{\dot{d}} - f^{\dot{a}\dot{b}}{}_{cd}Z^{\dot{A}}_{\dot{a}}Z^{\dot{B}}_{\dot{b}}\bar{Z}^c_{\dot{A}}\eta_{D\dot{B}} - f^{a\dot{a}}{}_{cd}Z^A_a Z^{\dot{B}}_{\dot{b}}\bar{Z}^c_A\eta_{D\dot{B}} + 2f^{a\dot{a}}{}_{cd}Z^A_a Z^{\dot{B}}_{\dot{b}}\bar{Z}^c_D\eta_{A\dot{B}},$$
$$f^{ab}{}_{cd} = h_{mn}\tau^{ma}{}_c\tau^{nb}{}_d, f^{a\dot{a}}{}_{c\dot{d}} = f^{\dot{b}a}{}_{dc} = h_{mn}\tau^{ma}{}_c\tau^{n\dot{b}}{}_{\dot{d}},$$
$$(Z^A_a)^* = \bar{Z}^a_A = -J^{ab}\varepsilon_{AB}Z^B_b$$
$$\left(Z^{\dot{A}}_{\dot{a}}\right)^* = \bar{Z}^{\dot{a}}_{\dot{A}} = -J^{\dot{a}\dot{b}}\varepsilon_{\dot{A}\dot{B}}Z^{\dot{B}}_{\dot{b}},$$
$$\delta Z^A_d = i\bar{\eta}^{A\dot{D}}\Psi_{\dot{D}d}$$
$$\delta Z^{\dot{A}}_{\dot{d}} = -i\bar{\eta}^{D\dot{A}}\Psi_{D\dot{d}}$$
$$\delta\Psi_{\dot{D}d} = \gamma^\mu\eta_{A\dot{D}}D_\mu Z^A_d + g^{abc}{}_d Z^A_a Z^B_b Z^C_c\eta_{B\dot{D}}\varepsilon_{AC} - 3g^{\dot{a}\dot{b}c}{}_d Z^{\dot{A}}_{\dot{a}}Z^{\dot{B}}_{\dot{b}}Z^C_c\eta_{C\dot{A}}\omega_{\dot{D}\dot{B}}$$
$$\delta\Psi_{D\dot{d}} = -\gamma^\mu\eta_{D\dot{A}}D_\mu Z^{\dot{A}}_{\dot{d}} - g^{\dot{a}\dot{b}\dot{c}}{}_{\dot{d}}Z^{\dot{A}}_{\dot{a}}Z^{\dot{B}}_{\dot{b}}Z^{\dot{C}}_{\dot{c}}\eta_{D\dot{B}}\varepsilon_{\dot{A}\dot{C}} + 3g^{ab\dot{c}}{}_{\dot{d}}Z^A_a Z^B_b Z^{\dot{C}}_{\dot{c}}\eta_{A\dot{C}}\varepsilon_{DB},$$
$$g^{abcd} = h_{mn}J^{eb}J^{fd}\tau^{ma}{}_e\tau^{ncb}{}_f, g^{ab\dot{c}\dot{d}} = h_{mn}J^{eb}J^{\dot{f}\dot{d}}\tau^{ma}{}_e\tau^{n\dot{c}}{}_{\dot{f}},$$

$$-\frac{1}{4g^2_{YM}}\boldsymbol{F}^{\mu\nu}\boldsymbol{F}_{\mu\nu} \to \frac{1}{2}\epsilon^{\mu\nu\lambda}\boldsymbol{B}_\mu\boldsymbol{F}_{\nu\lambda} - \frac{1}{2}\left(D_\mu\boldsymbol{\phi} - g_{YM}\boldsymbol{B}_\mu\right)^2$$
$$\delta\boldsymbol{\phi} = g_{YM}\boldsymbol{M}, \delta\boldsymbol{B}_\mu = D_\mu\boldsymbol{M}$$
$$L = \mathrm{Tr}\left(\frac{1}{2}\epsilon^{\mu\nu\lambda}\boldsymbol{B}_\mu\boldsymbol{F}_{\nu\lambda} - \frac{1}{2}\left(D_\mu\boldsymbol{\phi} - g_{YM}\boldsymbol{B}_\mu\right)^2 - \frac{1}{2}D_\mu\boldsymbol{X}^i D^\mu\boldsymbol{X}^i - \frac{g^2_{YM}}{4}\left[\boldsymbol{X}^i, \boldsymbol{X}^j\right]^2 + \text{ fermions}\right)$$

$$-\frac{1}{2}\hat{D}_\mu \boldsymbol{X}^I \hat{D}^\mu \boldsymbol{X}^I = -\frac{1}{2}\left(\partial_\mu \boldsymbol{X}^I - \left[\boldsymbol{A}_\mu, \boldsymbol{X}^I\right] - g_{YM}^I \boldsymbol{B}_\mu\right)^2$$

$$\frac{g_{YM}{}^2}{4}\left[\boldsymbol{X}^i, \boldsymbol{X}^j\right]^2 = \frac{1}{12}(g_{YM}{}^I[\boldsymbol{X}^J, \boldsymbol{X}^K] + g_{YM}{}^J[\boldsymbol{X}^K, \boldsymbol{X}^I] + g_{YM}{}^K[\boldsymbol{X}^I, \boldsymbol{X}^J])^2.$$

$$g_{YM}^I \to X_+^I(x)$$

$$L_C = \left(C_I^\mu - \partial^\mu X_-^I\right)\partial_\mu X_+^I$$

$$\delta X_-^I = \lambda^I, \delta C_\mu^I = \partial_\mu \lambda^I$$

$$L = \mathrm{Tr}\left(\frac{1}{2}e^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}\hat{D}_\mu X^I \hat{D}_\mu X^I - \frac{1}{12}\left(X_+^I[X^J, X^K] + X_+^J[X^K, X^I] + X_+^K[X^I, X^J]\right)^2\right)$$

$$+(C^{\mu I} - \partial^\mu X_-^I)\partial_\mu X_+^I + L_{\text{gauge-fixing}} + \mathcal{L}_{\text{fermions}}.$$

$$X^{IJK} \equiv X_+^I[\boldsymbol{X}^J, \boldsymbol{X}^K] + X_+^J[\boldsymbol{X}^K, \boldsymbol{X}^I] + X_+^K[\boldsymbol{X}^I, \boldsymbol{X}^J]$$

$$f^{+abc} = f^{abc}, f^{-abc} = f^{+-ab} = f^{abcd} = 0$$

$$\left[T^+, T^{+i}, T_{\vec{m}}^a\right] = m^i T_{\vec{m}}^a$$

$$\left[T_{\vec{m}}^a, T_{\vec{n}}^b, T_{\vec{p}}^c\right] = -if^{abc}T^- \delta_{\vec{m}+\vec{n}+\vec{p},\vec{0}}$$

$$\left[T^+, T^{+i}, T_{\vec{m}}^a\right] = m^i T_{\vec{m}}^a$$

$$\left[T^+, T_{\vec{m}}^a, T_{\vec{n}}^b\right] = m^i T^{-i} h^{ab}\delta_{\vec{m},-\vec{n}} + if^{ab}{}_c T_{\vec{m}+\vec{n}}^c$$

$$\left[T_{\vec{m}}^a, T_{\vec{n}}^b, T_{\vec{p}}^c\right] = -if^{abc}T^- \delta_{\vec{m}+\vec{n}+\vec{p},\vec{0}}$$

$$\langle T^+, T^- \rangle = 1$$

$$\left\langle T^{+i}, T^-{}_j \right\rangle = \delta_j^i$$

$$\left\langle T_{\vec{m}}^a, T_{\vec{n}}^b \right\rangle = h^{ab}\delta_{\vec{m},-\vec{n}},$$

$$\left[X^I, X^{J\dagger}, X^K\right] = \frac{1}{3}\left(X^{[I}X^{J]\dagger}X^K - X^{[I}X^{K\dagger}X^{J]} + X^K X^{[I\dagger}X^{J]}\right)$$

$$\mathrm{STr}\left(AB^\dagger CD^\dagger\right) = \frac{1}{12}\mathrm{Tr}\left[A\left(B^\dagger CD^\dagger + B^\dagger DC^\dagger + C^\dagger DB^\dagger + C^\dagger BD^\dagger + D^\dagger BC^\dagger + D^\dagger CB^\dagger\right) + \text{ h.c. }\right]$$

$$\mathrm{STr}\left[X^{IJK}X^{IJL\dagger}X^{MNK}X^{MNL\dagger}\right] = 2\mathrm{STr}\left[X^{IJM}X^{KLM\dagger}X^{IKN}X^{JLN\dagger}\right]$$

$$= \frac{1}{3}\mathrm{STr}\left[X^{IJK}X^{IJK\dagger}X^{LMN}X^{LMN\dagger}\right],$$

$$X^{IJK} = X^{[I}X^{J\dagger}X^{K]}$$

$$(DX)^4{:}k^2\mathrm{STr}\left[\mathbf{a}D^\mu X^I D_\mu X^{J\dagger} D^\nu X^J D_\nu X^{I\dagger} + \mathbf{b}D^\mu X^I D_\mu X^{I\dagger} D^\nu X^J D_\nu X^{J\dagger}\right]$$

$$X^{IJK}(DX)^3{:}k^2\varepsilon^{\mu\nu\lambda}\ \mathrm{STr}\left[\mathbf{c}X^{IJK}D_\mu X^{I\dagger}D_\nu X^J D_\lambda X^{K\dagger}\right]$$

$$(X^{IJK})^2(DX)^2{:}k^2\mathrm{STr}\left[\mathbf{d}X^{IJK}X^{IJK\dagger}D_\mu X^L D^\mu X^{L\dagger} + \mathbf{e}X^{IJK}X^{IJL\dagger}D_\mu X^K D^\mu X^{L\dagger}\right]$$

$$(X^{IJK})^4{:}k^2\mathrm{STr}\left[\mathbf{f}X^{IJK}X^{IJK\dagger}X^{LMN}X^{LMN\dagger}\right],$$

$$D^\mu X^8 \to \frac{1}{v}\mathbf{f}^\mu, \qquad D^\mu X^i \to \frac{1}{v}D^\mu X^i, \qquad X^{ij8} \to -\frac{1}{4v}X^{ij},$$

$$D^\mu X^{8\dagger} \to -\frac{1}{v}\mathbf{f}^\mu, \qquad X^{ijk} \to O\left(\frac{1}{v^3}\right)$$

$$D^\mu X^{i\dagger} \to -\frac{1}{v}D^\mu X^i, \qquad X^{ij8\dagger} \to \frac{1}{4v}X^{ij},$$

$$X^{ijk\dagger} \to O\left(\frac{1}{v^3}\right),$$

$$S_{\mathbf{a}}^{b} = \mathbf{a}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[D^\mu X^i D_\mu X^j D^\nu X^i D_\nu X^j + 2D^\mu X^i D_\nu X^i \mathbf{f}^\mu \mathbf{f}_\nu + \mathbf{f}^\mu \mathbf{f}_\mu \mathbf{f}^\nu \mathbf{f}_\nu\right]$$

$$S_{\mathbf{b}}^{b} = \mathbf{b}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[D^\mu X^i D_\mu X^i D^\nu X^j D_\nu X^j + 2D^\mu X^i D_\mu X^i \mathbf{f}^\nu \mathbf{f}_\nu + \mathbf{f}^\mu \mathbf{f}_\mu \mathbf{f}^\nu \mathbf{f}_\nu\right]$$

$$S_{\mathbf{c}}^{b} = \mathbf{c}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[\frac{3}{4}\varepsilon^{\mu\nu\lambda} D_\mu X^i \mathbf{f}_\nu D_\lambda X^j X^{ij}\right]$$

$$S_{\mathbf{d}}^{b} = \mathbf{d}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[\frac{3}{16} D^\mu X^i D_\mu X^i X^{jk} X^{jk} + \frac{3}{16}\mathbf{f}^\mu \mathbf{f}_\mu X^{ij} X^{ij}\right]$$

$$S_{\mathbf{e}}^{b} = \mathbf{e}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[\frac{1}{8} D^\mu X^i X^{ij} X^{kj} D_\mu X^k + \frac{1}{16}\mathbf{f}^\mu \mathbf{f}_\mu X^{ij} X^{ij}\right]$$

$$S_{\mathbf{f}}^{b} = \mathbf{f}\left(\frac{k}{v^2}\right)^2 \int d^3x \mathrm{STr}\left[\frac{9}{256} X^{ij} X^{ji} X^{kl} X^{lk}\right]$$

$$(2\pi)^2 \ell_p^3 \left(\frac{k}{2\pi v^2}\right)^2 = \frac{(2\pi\alpha')^2}{g_{YM}^2}$$

$$\begin{aligned} S_{\alpha'^2}^{b} = &\frac{(2\pi\alpha')^2}{g_{YM}^2} \int d^3x \mathrm{STr}\Big[\frac{1}{4} F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} - \frac{1}{16} F^{\mu\nu} F_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} - \frac{1}{4} D_\mu X^i D^\mu X^i D_\nu X^j D^\nu X^j \\ &+ \frac{1}{2} D_\mu X^i D^\nu X^i D_\nu X^j D^\mu X^j + \frac{1}{4} X^{ij} X^{jk} X^{kl} X^{li} - \frac{1}{16} X^{ij} X^{ij} X^{kl} X^{kl} \\ &- F_{\mu\nu} F^{\nu\rho} D_\rho X^i D^\mu X^i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} D_\rho X^i D^\rho X^i - \frac{1}{8} F_{\mu\nu} F^{\mu\nu} X^{kl} X^{kl} \\ &- \frac{1}{4} D_\mu X^i D^\mu X^i X^{kl} X^{kl} - X^{ij} X^{jk} D^\mu X^k D_\mu X^i - F_{\mu\nu} D^\nu X^i D^\mu X^j X^{ij}\Big] \end{aligned}$$

$$\mathrm{STr}\left[\boldsymbol{X}^{ij} \boldsymbol{X}^{jk} \boldsymbol{X}^{kl} \boldsymbol{X}^{li}\right] = \frac{1}{2}\mathrm{STr}\left[\boldsymbol{X}^{ij} \boldsymbol{X}^{ij} \boldsymbol{X}^{kl} \boldsymbol{X}^{kl}\right]$$

$$\mathbf{a} = \frac{1}{2}, \qquad \mathbf{b} = -\frac{1}{4}, \quad \mathbf{c} = \frac{4}{3}$$

$$\mathbf{d} = -\frac{4}{3}, \quad \mathbf{e} = 8, \qquad \mathbf{f} = \frac{16}{9}$$

$$\mathrm{STr}\left(T^a T^b T^c T^d\right) = m h^{(ab} h^{cd)}$$

$$\mathrm{STr}\left(T^i T^j T^k T^l\right) = 2\mathrm{STr}\left(\frac{\sigma^i}{2}\frac{\sigma^j}{2}\frac{\sigma^k}{2}\frac{\sigma^l}{2}\right) = \frac{1}{4}\delta^{(ij}\delta^{kl)}.$$

$$\begin{aligned} S_{\ell_p^3}^{b} = (2\pi)^2 \ell_p^3 \int d^3x \mathrm{STr}\Big[&\frac{1}{4} D^\mu X^I D_\mu X^J D^\nu X^J D_\nu X^I - \frac{1}{8} D^\mu X^I D_\mu X^I D^\nu X^J D_\nu X^J \\ &+ \frac{1}{6}\varepsilon^{\mu\nu\lambda} X^{IJK} D_\mu X^I D_\nu X^J D_\lambda X^K \\ &+ \frac{1}{4} X^{IJK} X^{IJL} D^\mu X^K D_\mu X^L - \frac{1}{24} X^{IJK} X^{IJK} D^\mu X^L D_\mu X^L \\ &+ \frac{1}{288} X^{IJK} X^{IJK} X^{LMN} X^{LMN}\Big] \end{aligned}$$

$$X^{IJK} = [X^I, X^J, X^K].$$

$$\hat{D}_\mu \hat{X}^I = \partial_\mu \hat{X}^I - [\hat{A}_\mu, \hat{X}^I] - \hat{B}_\mu \hat{X}_+^I$$

$$\hat{X}^{IJK} = \hat{X}_+^I [\hat{X}^J, \hat{X}^K] + \hat{X}_+^J [\hat{X}^K, \hat{X}^I] + \hat{X}_+^K [\hat{X}^I, \hat{X}^J].$$

$$(\hat{D}\hat{X})^4{:}\frac{1}{4}\mathrm{STr}\left(\hat{D}^\mu\hat{X}^I\hat{D}_\mu\hat{X}^J\hat{D}^\nu\hat{X}^J\hat{D}_\nu\hat{X}^I-\frac{1}{2}\hat{D}^\mu\hat{X}^I\hat{D}_\mu\hat{X}^I\hat{D}^\nu\hat{X}^J\hat{D}_\nu\hat{X}^J\right)$$

$$\hat{X}^{IJK}(\hat{D}\hat{X})^3{:}\frac{1}{6}\varepsilon^{\mu\nu\lambda}\mathrm{STr}(\hat{X}^{IJK}\hat{D}_\mu\hat{X}^I\hat{D}_\nu\hat{X}^J\hat{D}_\lambda\hat{X}^K)$$

$$\left(\hat{X}^{IJK}\right)^2(\hat{D}\hat{X})^2{:}\frac{1}{4}\mathrm{STr}\left(\hat{X}^{IJK}\hat{X}^{IJL}\hat{D}^\mu\hat{X}^K\hat{D}_\mu\hat{X}^L-\frac{1}{6}\hat{X}^{IJK}\hat{X}^{IJK}\hat{D}^\mu\hat{X}^L\hat{D}_\mu\hat{X}^L\right)$$

$$\left(\hat{X}^{IJK}\right)^4{:}\frac{1}{24}\mathrm{STr}\left(\hat{X}^{IJM}\hat{X}^{KLM}\hat{X}^{IKN}\hat{X}^{JLN}-\frac{1}{12}\hat{X}^{IJK}\hat{X}^{IJK}\hat{X}^{LMN}\hat{X}^{LMN}\right).$$

$$\mathrm{STr}(\hat{X}^{IJK}\hat{X}^{IJL}\hat{X}^{MNK}\hat{X}^{MNL})=\mathrm{STr}\left(\frac{4}{3}\hat{X}^{IJM}\hat{X}^{KLM}\hat{X}^{IKN}\hat{X}^{JLN}+\frac{1}{9}\hat{X}^{IJK}\hat{X}^{IJK}\hat{X}^{LMN}\hat{X}^{LMN}\right)$$

$$S^b_{\mathrm{BLG},\ell_p^3}=\ell_p^3\int d^3x\mathrm{STr}\Bigg[\frac{1}{4}\left(D^\mu X^I D_\mu X^J D^\nu X^J D_\nu X^I-\frac{1}{2}D^\mu X^I D_\mu X^I D^\nu X^J D_\nu X^J\right)$$
$$+\frac{1}{6}\varepsilon^{\mu\nu\lambda}(X^{IJK}D_\mu X^I D_\nu X^J D_\lambda X^K)$$
$$+\frac{1}{4}\left(X^{IJK}X^{IJL}D^\mu X^K D_\mu X^L-\frac{1}{6}X^{IJK}X^{IJK}D^\mu X^L D_\mu X^L\right)$$
$$+\frac{1}{24}\left(X^{IJM}X^{KLM}X^{IKN}X^{JLN}-\frac{1}{12}X^{IJK}X^{IJK}X^{LMN}X^{LMN}\right)\Bigg]$$

$$\delta X^I=i\bar{\epsilon}\Gamma^I\Psi$$

$$\delta\Psi=\Gamma^\mu\Gamma_I\partial_\mu X^I\epsilon+\frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon$$

$$\delta H_{\mu\nu\lambda}=3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi$$

$$\Gamma^\mu\partial_\mu\Psi=0,\partial_\mu\partial^\mu X^I=0,\partial_{[\mu}H_{\nu\lambda\rho]}=0$$

$$D_\mu X^I_a=\partial_\mu X^I_a-\tilde{A}^b_{\mu a}X^I_b$$

$$\delta X^I_a=i\bar{\epsilon}\Gamma^I\Psi_A$$

$$\delta\Psi_a=\Gamma^\mu\Gamma^I D_\mu X^I_a\epsilon+\frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_a\epsilon-\frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_b X^I_c X^J_d f^{cdb}{}_a\epsilon$$

$$\delta H_{\mu\nu\lambda a}=3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_a+i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_b X^I_c\Psi_d f^{cdb}{}_a$$

$$\delta\tilde{A}^b_{\mu a}=i\bar{\epsilon}\Gamma_{\mu\lambda}C^\lambda_c\Psi_d f^{cdb}{}_a$$

$$\delta C^\mu_a=0.$$

$$D^2X^I_a=\frac{i}{2}\bar{\Psi}_C C^\nu_B\Gamma_\nu\Gamma^I\Psi_d f^{cdb}_a+C^\nu_b C_{\nu g}X^J_c X^J_e X^I_f f^{efg}{}_d f^{cdb}{}_a$$

$$D_{[\mu}H_{\nu\lambda\rho]a}=-\frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^\sigma_b X^I_c D^\tau X^I_d f^{cdb}{}_a-\frac{i}{8}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^\sigma_b\bar{\Psi}_c\Gamma^\tau\Psi_d f^{cdb}{}_a$$

$$\Gamma^\mu D_\mu\Psi_a=-X^I_c C^\nu_b\Gamma_\nu\Gamma^I\Psi_d f^{cdb}{}_a$$

$$\tilde{F}_{\mu\nu}{}^b{}_a=-C^\lambda_c H_{\mu\nu\lambda d}f^{cdb}{}_a,$$

$$C^\rho_c D_\rho X^I_D f^{cdb}_a=0,\qquad D_\mu C^\nu_a=0$$

$$C^\rho_c D_\rho\Psi_D f^{cdb}_a=0,\qquad C^\mu_c C^\nu_d f^{bcd}_a=0$$

$$C^\rho_c D_\rho H_{\mu\nu\lambda a}f^{cdb}_a=0.$$

$$[\epsilon]=-\frac{1}{2},[\Psi]=[X]+\frac{1}{2},[X]$$

$$T_{\mu\nu}=D_\mu X^I_a D_\nu X^{Ia}-\frac{1}{2}\eta_{\mu\nu}D_\lambda X^I_a D^\lambda X^{Ia}$$
$$+\frac{1}{4}\eta_{\mu\nu}C^\lambda_b X^I_a X^J_c C_{\lambda g}X^I_f X^J_e f^{cdba}f^{efg}{}_d+\frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho a}$$
$$-\frac{i}{2}\bar{\Psi}_a\Gamma_\mu D_\nu\Psi^a+\frac{i}{2}\eta_{\mu\nu}\bar{\Psi}_a\Gamma^\lambda D_\lambda\Psi^a-\frac{i}{2}\eta_{\mu\nu}\bar{\Psi}_a C^\lambda_b X^I_c\Gamma_\lambda\Gamma^I\Psi_d f^{abcd}$$

$$f^{+AB}{}_C=f^{AB}{}_C,f^{ABC}=f^{ABC}$$

$$\langle C^\lambda_A\rangle=g\delta^\lambda_5\delta^+_A$$

$$\tilde{F}_{\alpha\beta}{}^B{}_A=-gH_{\alpha\beta5}f^{DB}{}_A,$$

$$0 = \tilde{D}^{\alpha}\tilde{D}_{\alpha}X_A^I - g\frac{i}{2}\bar{\Psi}_C\Gamma_5\Gamma^I\Psi_d f^{CD}{}_A - g^2 X_C^J X_E^J X_F^I f^{EF}{}_D f^{CD}{}_A$$
$$0 = \tilde{D}_{[\alpha}H_{\beta\gamma]5A}$$
$$0 = \tilde{D}^{\alpha}H_{\alpha\beta 5A} + \frac{1}{2}gf^{CD}{}_A\left(X_C^I\tilde{D}_\beta X_D^I + \frac{i}{2}\bar{\Psi}_C\Gamma_\beta\Psi_D\right)$$
$$0 = \Gamma^{\mu}\tilde{D}_{\mu}\Psi_A + gX_C^I\Gamma_5\Gamma^I\Psi_D f^{CD}{}_A$$
$$0 = \partial_5 X_D^I = \partial_5\Psi_D = \partial_5 H_{\mu\nu\lambda D},$$
$$\delta X_A^I = i\bar{\epsilon}\Gamma^I\Psi_A$$
$$\delta\Psi_A = \Gamma^{\alpha}\Gamma^I\tilde{D}_\alpha X_A^I\epsilon + \frac{1}{2}\Gamma_{\alpha\beta}\Gamma_5 H_A^{\alpha\beta 5}\epsilon - \frac{1}{2}\Gamma_5\Gamma^{IJ}X_C^I X_D^J f^{CD}{}_A\epsilon$$
$$\delta\tilde{A}_{\alpha}^{B}{}_{A}^{B} = i\bar{\epsilon}\Gamma_\alpha\Gamma_5\Psi_d f^{DB}{}_A^C.$$
$$g = g_{YM}^2, H_{\alpha\beta 5}^A = -\frac{1}{g_{YM}^2}F_{\alpha\beta}^A, \tilde{A}_{\alpha A}^B = A_{\alpha C}f^{CD}{}_A,$$
$$0 = \partial^\mu\partial_\mu X_\pm^I$$
$$0 = \partial_{[\mu}H_{\nu\lambda\rho]\pm}$$
$$0 = \Gamma^\mu\partial_\mu\Psi_\pm$$
$$\delta X_\pm^I = i\bar{\epsilon}\Gamma^I\Psi_\pm$$
$$\delta\Psi_\pm = \Gamma^\mu\Gamma^I\partial_\mu X_\pm^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H_\pm^{\mu\nu\lambda}\epsilon$$
$$\delta H_{\mu\nu\lambda\pm} = 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi_\pm.$$

3.1. Agujeros negros cuánticos. En cuanto a los agujeros negros de mecánica cuántica, en esta sección calcularemos su densidad, volumen, diámetro y radio, radiación y termodinámica entrópica – Gibbons - Hawking - Penrose, condiciones inflacionarias, pelo y dualidad AdS/CFT, topología (verbigracia, el grupo de Poincaré, grupo de Lorentz, y parametrización hiperbolide, etc), simetrías y métricas riemanniana, euclídea, einsteniana, teoremas de Gubser-Klevanov-Polyakov y Witten, de Kerr, de Schwarzschild, de Klein – Gordon, Brown-York, de Calabi Yau, etc, así como su masa, tensores de stress – energía, coordenadas de movimiento en un espacio curvo, operadores, generadores y osciladores, el horizonte de eventos y la singularidad, ésta última la cual está dada por la transformación de la información que absorbe el microagujero negro, lo que por ósmosis, supone la formación de superespacios y por ende, de supermembranas infinitas en supersimetrías bosón – fermión, todo esto, en relación a un campo cuántico de gauge curvado o deformado geométricamente en distorsión e indiscriminado y permeado por un campo de gauge gravitónico. Téngase en cuenta, que el agujero negro cuántico, se forma por el colapso de una superpartícula, sea por el salto de masa que provoca un escalar exponencial de energía, esto es, superior a cero respecto del estado de vacío, o por su naturaleza supermasiva, lo que hace que la partícula – estrella, colapse en sí misma, distorsionando así, el espacio – tiempo cuántico.

El modelo matemático, en un espacio Anti-de Sitter, viene dado por lo que sigue:

$$T_H = \frac{\hbar c^3}{8\pi k_B GM}$$
$$S = \frac{k_B}{\ell_P^2}\frac{A}{4}$$
$$\langle\Psi|\phi(x_1)\phi(x_2)|\Psi\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{\ell_P^2}$$
$$S(A) = k_B\left(\frac{\gamma_c}{\gamma}\frac{A}{4\ell_P^2} - \frac{1}{2}\ln\frac{A}{\ell_P^2}\right)$$
$$\Box\,\phi(x) = 0$$
$$(f,g) = -i\int_\Sigma d\Sigma^\mu(f\nabla_\mu g^* - g^*\nabla_\mu f)$$

$$[\phi(t_0,\vec{x}),\pi(t_0,\vec{x'})] = i\hbar\delta(\vec{x}-\vec{x'})$$
$$\mathcal{S} = \mathcal{S}_0 \oplus \overline{\mathcal{S}}_0$$
$$(f,f) > 0\ \forall f \in \mathcal{S}_0$$
$$(f,g^*) = 0\ \forall f,g \in \mathcal{S}_0$$
$$a(f) = (\phi,f)$$
$$a^\dagger(f) = -a(f^*)$$
$$[a(f),a^\dagger(g)] = \hbar(g,f)$$
$$[a(f),a(g)] = -\hbar(g^*,f),[a^\dagger(f),a^\dagger(g)] = -\hbar(g,f^*)$$
$$[a(f),a(g)] = 0 = [a^\dagger(f),a^\dagger(g)].$$
$$a^\dagger(f_1)\dots a^\dagger(f_n)|0\rangle$$
$$a^\dagger(u_{i_1}^{in})\dots a^\dagger(u_{i_n}^{in})|in\rangle$$
$$u_j^{out} = \sum_i (\alpha_{ji}u_i^{in} + \beta_{ji}u_i^{in*})$$
$$(u_i^{in},u_j^{in}) = \delta_{ij}$$
$$\alpha_{ij} = (u_i^{out},u_j^{in}),\beta_{ij} = -(u_i^{out},u_j^{in*})$$
$$\sum_k (\alpha_{ik}\alpha_{jk}^* + \beta_{ik}\beta_{jk}^*) = \delta_{ij}$$
$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0$$
$$u_i^{in} = \sum_j (\alpha_{ji}^*u_i^{out} - \beta_{ji}u_i^{out*})$$
$$a_i^{\text{in}} = \sum_j (\alpha_{ji}a_j^{\text{out}} + \beta_{ji}^*a_j^{\text{out}\,\dagger})$$
$$a_i^{\text{out}} = \sum_j (\alpha_{ij}^*a_j^{in} - \beta_{ij}^*a_j^{in\dagger})$$
$$N_i^{\text{out}} \equiv \hbar^{-1}a_i^{\text{out}\,\dagger}a_i^{\text{out}}$$
$$\langle\text{ in }|N_j^{\text{out}}\mid\text{ in }\rangle = \hbar^{-1}\langle\text{ in }|a_j^{\text{out}\,\dagger}a_j^{\text{out}}\mid\text{ in }\rangle =$$
$$= \hbar^{-1}\langle\text{ in }|\sum_i (-\beta_{ji}a_i^{\text{in}})\sum_k (-\beta_{jk}^*a_k^{\text{in}\,\dagger})|in\rangle =$$
$$= \sum_i |\beta_{ji}|^2.$$
$$\sum_k \alpha_{ik}\alpha_{jk}^* = \delta_{ij}$$
$$\langle\text{ in }|N_{i_1i_2}^{\text{out}}\mid\text{ in }\rangle = \sum_k \beta_{i_1k}\beta_{i_2k}^* = -\sum_k (u_{i_1}^{\text{out}},u_k^{\text{in}\,*})(u_{i_2}^{\text{out}\,*},u_k^{\text{in}}) =$$
$$= \sum_k \left(\int_\Sigma d\Sigma_1^\mu u_{i_1}^{\text{out}}(x_1)\overleftrightarrow{\partial}_\mu u_k^{\text{in}}(x_1)\right)\left(\int_\Sigma d\Sigma_2^\nu u_{i_2}^{\text{out}\,*}(x_2)\overleftrightarrow{\partial}_\nu u_k^{\text{in}\,*}(x_2)\right).$$
$$\langle in|\phi(x_1)\phi(x_2)|in\rangle = \hbar\sum_k u_k^{in}(x_1)u_k^{in^*}(x_2)$$
$$\langle in|N_{i_1i_2}^{out}|in\rangle = \hbar^{-1}\int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu\left[u_{i_1}^{out}(x_1)\overleftrightarrow{\partial}_\mu\right]\left[u_{i_2}^{out*}(x_2)\overleftrightarrow{\partial}_\nu\right]\langle in|\phi(x_1)\phi(x_2)|in\rangle$$
$$\langle in|N_{i_1i_2}^{out}|in\rangle = \hbar^{-1}\int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu\left[u_{i_1}^{out}(x_1)\overleftrightarrow{\partial}_\mu\right]\left[u_{i_2}^{out*}(x_2)\overleftrightarrow{\partial}_\nu\right]\langle in|:\phi(x_1)\phi(x_2):|in\rangle$$
$$(\phi_1,\phi_2) = -i\int_\Sigma d\Sigma^\mu(\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1)$$
$$\phi = \sum_i (a_i^{in}u_i^{in} + a_i^{in\dagger}u_i^{in*}).$$

$$\phi = \sum_i \left(a_i^{\text{out}} u_i^{\text{out}} + a_i^{\text{out}\,\dagger} u_i^{\text{out}\,*}\right) + \left(a_i^{\text{int}} u_i^{\text{int}} + a_i^{\text{int}\,\dagger} u_i^{\text{int}\,*}\right).$$

$$u_{wlm}^{in}\big|_{I^-} = \frac{1}{\sqrt{4\pi w}} \frac{e^{-iwv}}{r} Y_l^m(\theta,\phi)$$

$$u_{wlm}^{out}\big|_{I^+} = \frac{1}{\sqrt{4\pi w}} \frac{e^{-iwu}}{r} Y_l^m(\theta,\phi)$$

$$\beta_{wlm,w'l'm'} = -\left(u_{wlm}^{out}, u_{w'l'm'}^{in*}\right) = i \int_{I^-}^{in*} dvr^2 d\Omega \left(u_{wlm}^{out} \partial_v u_{w'l'm'}^{in} - u_{w'l'm'}^{in} \partial_v u_{wlm}^{out}\right)$$

$$u_{wlm}^{out}\big|_{I^-} = \frac{t_l(w)}{\sqrt{4\pi w}} \frac{e^{-iwu(v)}}{r} Y_l^m(\theta,\phi)\Theta(v_H - v)$$

$$u = v_H - \kappa^{-1}\ln \kappa|v_H - v|$$

$$\beta_{wlm,w'l'm'} = \frac{-(-)^m t_l(w)}{2\pi} \sqrt{\frac{w'}{w}} \int_{-\infty}^{v_H} dve^{-iw(v_H - \kappa^{-1}\ln \kappa|v_H - v|) - iw'v} \delta_{ll'}\delta_{m-m'}$$

$$\beta_{wlm,w'l'm'} = \frac{-(-)^m t_l(w)}{2\pi\kappa} \sqrt{\frac{w'}{w}} \frac{e^{-i(w+w')v_H}}{(-\kappa^{-1}w'i + \epsilon)^{1+\kappa^{-1}wi}} \Gamma(1 + \kappa^{-1}wi)\delta_{ll'}\delta_{m-m'}$$

$$\int_0^{+\infty} dw' \beta_{w_1 w'} \beta_{w_2 w'}^*$$

$$\int_0^{+\infty} \frac{dw'}{w'} e^{-\kappa^{-1}w_1 i\ln(-\kappa^{-1}w' - i\epsilon)} e^{\kappa^{-1}w_2 i\ln(\kappa^{-1}w' - i\epsilon)} = 2\pi\kappa e^{-\pi\kappa^{-1}w_1}\delta(w_1 - w_2)$$

$$\int_0^{+\infty} dw' \beta_{w_1 w'} \beta_{w_2 w'}^* = \frac{|t_l(w_1)|^2}{e^{2\pi\kappa^{-1}w_1} - 1}\delta(w_1 - w_2)$$

$$\frac{dN_{lm}(w)}{dwdt} \equiv \frac{1}{2\pi}\langle in|N_{wlm}^{out}|in\rangle = \frac{1}{2\pi}\frac{\Gamma_l(w)}{e^{2\pi\kappa^{-1}w} - 1}$$

$$\frac{1}{e^{\hbar w/k_B T} - 1}$$

$$T_H = \frac{\hbar\kappa}{2\pi k_B}$$

$$u_{wlm}^{out}(t,r,\theta,\phi)\big|_{I^+} = u_w^{out}(u)\frac{Y_l^m(\theta,\phi)}{r}$$

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = \frac{4}{\hbar}\int_{I^-} r_1^2 dv_1 d\Omega_1 \int_{I^-} r_2^2 dv_2 d\Omega_2 u_{w_1}^{out} u_{w_2}^{out*} \times$$

$$\frac{Y_{l_1}^{m_1}(\theta_1,\phi_1)}{r_1}\frac{Y_{l_2}^{m_2*}(\theta_2,\phi_2)}{r_2}\partial_{v_1}\partial_{v_2}\langle in|\phi(x_1)\phi(x_2)|in\rangle.$$

$$\langle in|\phi(x_1)\phi(x_2)|in\rangle = \hbar\int_0^\infty dw \sum_{l,m} \frac{e^{-iwv_1}}{\sqrt{4\pi w}}\frac{Y_l^m(\theta_1,\phi_1)}{r_1}\frac{e^{iwv_2}}{\sqrt{4\pi w}}\frac{Y_l^{m*}(\theta_2,\phi_2)}{r_2}$$

$$u_w^{out}|_{I^-} = t_l(w)\frac{e^{-iwu(v)}}{\sqrt{4\pi w}}\Theta(v_H - v)$$

$$\partial_{v_1}\partial_{v_2}\int_0^\infty dw \frac{e^{-iw(v_1 - v_2)}}{4\pi w} = \lim_{\epsilon\to 0^+} -\frac{1}{4\pi}\frac{1}{(v_1 - v_2 - i\epsilon)^2}$$

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2\sqrt{w_1 w_2}}\int_{-\infty}^{v_H} dv_1 dv_2 \frac{e^{-iw_1 u(v_1) + iw_2 u(v_2)}}{(v_1 - v_2 - i\epsilon)^2}\delta_{l_1 l_2}\delta_{m_1 m_2}$$

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2\sqrt{w_1 w_2}} \times$$

$$\times\int_{-\infty}^{\infty} du_1 du_2 \frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{[v(u_1) - v(u_2) - i\epsilon]^2} e^{-iw_1 u_1 + iw_2 u_1}\delta_{l_1 l_2}\delta_{m_1 m_2}$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = \frac{-t_{l_1}(w_1)t^*_{l_2}(w_2)}{4\pi^2\sqrt{w_1w_2}}\int_{-\infty}^{+\infty} du_1du_2\frac{\left(\frac{\kappa}{2}\right)^2 e^{-iw_1u_1+iw_2u_2}}{\left[\sinh\frac{\kappa}{2}(u_1-u_2-i\epsilon)\right]^2}\delta_{l_1l_2}\delta_{m_1m_2}$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = -\frac{t_{l_1}(w_1)t^*_{l_2}(w_2)\delta(w_1-w_2)}{2\pi\sqrt{w_1w_2}}\int_{-\infty}^{+\infty} dze^{-i\frac{w_1+w_2}{2}z}\frac{\left(\frac{\kappa}{2}\right)^2\delta_{l_1l_2}\delta_{m_1m_2}}{\left[\sinh\frac{\kappa}{2}(z-i\epsilon)\right]^2}$$

$$\langle in|N^{out}_{wlm}|in\rangle = \frac{|t_l(w)|^2}{e^{2\pi\kappa^{-1}w}-1}$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = -\frac{t_{l_1}(w_1)t^*_{l_2}(w_2)}{4\pi^2\sqrt{w_1w_2}}\int_{-\infty}^{v_H} dv_1dv_2e^{-iw_1u(v_1)+iw_2u(v_2)}\times$$

$$\left[\frac{1}{(v_1-v_2)^2}-\frac{\frac{du}{dv}(v_1)\frac{du}{dv}(v_2)}{[u(v_1)-u(v_2)]^2}\right]\delta_{l_1l_2}\delta_{m_1m_2}$$

$$v=\frac{au+b}{cu+d}$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = -\frac{t_{l_1}(w_1)t^*_{l_2}(w_2)}{4\pi^2\sqrt{w_1w_2}}\int_{I^+} du_1du_2e^{-iw_1u_1+iw_2u_2}\times$$

$$\left[\frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{\left(v(u_1)-v(u_2)\right)^2}-\frac{1}{[u_1-u_2]^2}\right]\delta_{l_1l_2}\delta_{m_1m_2}$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = -\frac{t_{l_1}(w_1)t^*_{l_2}(w_2)\delta(w_1-w_2)}{2\pi\sqrt{w_1w_2}}\int_{-\infty}^{+\infty} dze^{-i\frac{w_1+w_2}{2}z}\times$$

$$\left[\frac{\left(\frac{\kappa}{2}\right)^2}{\left(\sinh\frac{\kappa}{2}z\right)^2}-\frac{1}{z^2}\right]\delta_{l_1l_2}\delta_{m_1m_2}$$

$$\langle in|N^{out}_{wlm}|in\rangle = -\frac{|t_l(w)|^2}{2\pi w}\int_{-\infty}^{+\infty} dze^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)^2}{\left(\sinh\frac{\kappa}{2}z\right)^2}-\frac{1}{z^2}\right]$$

$$=\frac{|t_l(w)|^2}{e^{2\pi w\kappa^{-1}}-1}$$

$$\gamma^\mu\nabla_\mu\psi=0$$

$$(\psi_1,\psi_2)=\int_\Sigma d\Sigma^\mu\bar{\psi}_1\gamma_\mu\psi_2$$

$$\langle in|N^{out}_{i_1i_2}|in\rangle = \hbar^{-1}\int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu\left[\bar{u}^{out}_{i_2}(x_2)\gamma_\nu\right]_b\left[\gamma_\mu u^{out}_{i_1}(x_1)\right]^a\langle in|\bar{\psi}_a(x_1)\psi^b(x_2)|in\rangle$$

$$u^{\text{out}}_{w\kappa_jm_j}(t,r,\theta,\phi)\Big|_{I^+}\sim\frac{e^{-iwu}}{\sqrt{4\pi}r}\begin{pmatrix}\eta(\hat{r})^{m_j}_{\kappa_j}\\(\hat{r}\vec{\sigma})\eta(\hat{r})^{m_j}_{\kappa_j}\end{pmatrix}$$

$$u^{\text{out}}_{w\kappa_jm_j}(t,r,\theta,\phi)\Big|_{I^-}\sim t_{\kappa_j}(w)\frac{e^{-iwu(v)}}{\sqrt{4\pi}r}\begin{pmatrix}\eta(\hat{r})^{m_j}_{\kappa_j}\\-(\hat{r}\vec{\sigma})\eta(\hat{r})^{m_j}_{\kappa_j}\end{pmatrix}\Theta(v_H-v)\sqrt{\frac{du(v)}{dv}},$$

$$\langle in|\bar{\psi}_a(x_1)\psi^b(x_2)|in\rangle=\hbar\sum_k\bar{v}^{in}_{k,a}(x_1)v^{in,b}_k(x_2)$$

$$v^{in}_k\to v^{in}_{w\kappa_jm}(t,r,\theta,\phi)\Big|_{I^-}\sim\frac{e^{iwv}}{\sqrt{4\pi}r}\begin{pmatrix}\eta(\hat{r})^{m_j}_{\kappa_j}\\-(\hat{r}\vec{\sigma})\eta(\hat{r})^{m_j}_{\kappa_j}\end{pmatrix}$$

$$\langle in|N^{out}_{i_1 i_2}|in\rangle = -i\frac{t_{\kappa_{j_1}}(w_1)t^*_{\kappa_{j_2}}(w_2)}{4\pi^2}\delta_{m_{j_1}m_{j_2}}\delta_{\kappa_{j_1}\kappa_{j_2}} \times$$

$$\times \int_{-\infty}^{v_H} dv_1 dv_2 \sqrt{\frac{du(v_1)}{dv}\frac{du(v_2)}{dv}}\frac{e^{-iw_1 u(v_1)+iw_2 u(v_2)}}{(v_1 - v_2 - i\epsilon)}$$

$$\langle in|N^{out}_{i_1 i_2}|in\rangle = -i\frac{t_{\kappa_{j_1}}(w_1)t^*_{\kappa_{j_2}}(w_2)}{4\pi^2}\delta_{m_{j_1}m_{j_2}}\delta_{\kappa_{j_1}\kappa_{j_2}} \times$$

$$\times \int_{-\infty}^{\infty} du_1 du_2 e^{-iw_1 u_1 + iw_2 u_2}\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left[\frac{\kappa}{2}(u_1 - u_2 - i\epsilon)\right]}.$$

$$\langle in|N^{out}_{i_1 i_2}|in\rangle = -\frac{i}{2\pi}t_{\kappa_{j_1}}(w_1)t^*_{\kappa_{j_2}}(w_2)\delta_{m_{j_1}m_{j_2}}\delta_{\kappa_{j_1}\kappa_{j_2}}\delta(w_1 - w_2) \times$$

$$\times \int_{-\infty}^{+\infty} dz e^{-i\frac{w_1+w_2}{2}z}\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left[\frac{\kappa}{2}(z - i\epsilon)\right]}.$$

$$\frac{-i}{2\pi}\int_{-\infty}^{+\infty} dz e^{-iwz}\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left[\frac{\kappa}{2}(z - i\epsilon)\right]} = \frac{1}{e^{2\pi w\kappa^{-1}} + 1}$$

$$\langle in|N^{out}_{wm_j\kappa_j}|in\rangle = \frac{\left|t_{\kappa_j}(w)\right|^2}{e^{2\pi\kappa^{-1}w} + 1}$$

$$\langle in|N^{out}_{i_1 i_2}|in\rangle = -i\frac{t_{\kappa_{j_1}}(w_1)t_{\kappa_{j_2}}(w_2)^*}{4\pi^2}\int_{-\infty}^{v_H} dv_1 dv_2 \sqrt{\frac{du(v_1)}{dv}\frac{du(v_2)}{dv}} \times$$

$$e^{-iw_1 u(v_1)+iw_2 u(v_2)}\left[\frac{1}{[v_1 - v_2]} - \frac{\sqrt{\frac{du(v_1)}{dv}\frac{du(v_2)}{dv}}}{[u(v_1) - u(v_2)]}\right]\delta_{\kappa_{j_1}\kappa_{j_2}}\delta_{m_{j_1}m_{j_2}}$$

$$\langle in|N^{out}_{i_1 i_2}|in\rangle = -i\frac{t_{\kappa_{j_1}}(w_1)t_{\kappa_{j_2}}(w_2)^*}{4\pi^2}\int_{I^+} du_1 du_2 \sqrt{\frac{dv(u_1)}{du}\frac{dv(u_2)}{du}} \times$$

$$e^{-iw_1 u_1 + iw_2 u_2}\left[\frac{\sqrt{\frac{dv(u_1)}{du}\frac{dv(u_2)}{du}}}{[v(u_1) - v(u_2)]} - \frac{1}{[u_1 - u_2]}\right]\delta_{\kappa_{j_1}\kappa_{j_2}}\delta_{m_{j_1}m_{j_2}}$$

$$\langle in|N^{out}_{w\kappa_j m_j}|in\rangle = -i\frac{\left|t_{\kappa_j}(w)\right|^2}{2\pi}\int_{-\infty}^{+\infty} dz e^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}z\right)} - \frac{1}{z}\right]$$

$$\frac{-i}{2\pi}\int_{-\infty}^{+\infty} dz e^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}z\right)} - \frac{1}{z}\right] = \frac{1}{e^{2\pi w\kappa^{-1}} + 1}$$

$$b^\mu = a\left[X\left(\frac{\partial}{\partial T}\right)^\mu + T\left(\frac{\partial}{\partial X}\right)^\mu\right],$$

$$t = a^{-1}\tanh^{-1} T/X, \xi = a^{-1}\ln\, a\sqrt{X^2 - T^2}, Y = Y_0, Z = Z_0,$$

$$T = \frac{e^{a\xi}}{a}\sinh\, at, X = \frac{e^{a\xi}}{a}\cosh\, at, Y = Y_0, Z = Z_0$$

$$\left(e^{-2a\xi}\left(-\partial_t^2 + \partial_\xi^2\right) + \partial_Y^2 + \partial_Z^2\right)\phi(t,\xi,Y,Z) = 0$$

$$\left[\left(-\partial_t^2 + \partial_\xi^2\right) - e^{2a\xi}\left(k_Y^2 + k_Z^2\right)\hbar^{-2}\right]\phi(t,\xi) = 0$$

$$u^R_{w,\vec{k}_\perp} = \frac{e^{-iwt}}{2\pi^2\sqrt{a}}\sinh^{\frac{1}{2}}\left(\frac{\pi w}{a}\right)K_{iw/a}\left(\frac{|\vec{k}_\perp|}{a\hbar}e^{a\xi}\right)e^{i\vec{k}_\perp\cdot\vec{X}_\perp}$$

$$u^M_{k_X,\vec{k}_\perp} = \frac{1}{\sqrt{2(2\pi)^3k_0}}e^{-ik_0T+i(k_XX+\vec{k}_\perp\cdot\vec{X}_\perp)}$$

$$\beta_{w\vec{k}_\perp,k'_X\vec{k}'_\perp} = -\left[2\pi a k'_0\left(e^{2\pi w/a}-1\right)\right]^{-1/2}\left(\frac{k'_0+k'_X}{k'_0-k'_X}\right)^{-iw/2a}\delta\left(\vec{k}_\perp-\vec{k}'_\perp\right)$$

$$\int_{-\infty}^{+\infty} d\vec{k}'\beta_{w_1\vec{k}_\perp,\vec{k}'}\beta^*_{w_2\vec{k}_\perp,\vec{k}'}$$

$$\int_{-\infty}^{+\infty} dk'_X(2\pi a k'_0)^{-1}\left(\frac{k'_0+k'_X}{k'_0-k'_X}\right)^{-i(w_1-w_2)/a} = \delta(w_1-w_2)$$

$$\int_{-\infty}^{+\infty} d\vec{k}'\beta_{w_1\vec{k}_{\perp1},\vec{k}'}\beta^*_{w_2\vec{k}_{\perp2},\vec{k}'} = \frac{1}{e^{2\pi w_1/a}-1}\delta(w_1-w_2)\delta\left(\vec{k}_{\perp1}-\vec{k}_{\perp2}\right)$$

$$g\int d\tau m(\tau)\phi(x(\tau))$$

$$g\int d\tau\langle E\psi|m(\tau)\phi(x(\tau))|E_0 0_M\rangle$$

$$g^2|\langle E|m(0)|E_0\rangle\Box|^2 F(E-E_0),$$

$$F(w) = \int_{-\infty}^{+\infty} d\tau_1\int_{-\infty}^{+\infty} d\tau_2 e^{-iw(\tau_1-\tau_2)}\langle 0_M|\phi\left(x(\tau_1)\right)\phi\left(x(\tau_2)\right)|0_M\rangle$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\langle 0_M|\phi\left(x(\tau_1)\right)\phi\left(x(\tau_2)\right)|0_M\rangle$$

$$\dot{F}(w) = -\int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau/\hbar}\left[\frac{\hbar}{4\pi^2(\Delta\tau-i\epsilon)^2}\right] = -\frac{w}{2\pi}\theta(-w)$$

$$T = \frac{1}{a}\sinh\ a\tau, X = \frac{1}{a}\cosh\ a\tau$$

$$\langle 0_M|\phi\left(x(\tau_1)\right)\phi\left(x(\tau_2)\right)|0_M\rangle = \frac{-\hbar\left(\frac{a}{2}\right)^2}{4\pi^2\sinh^2\frac{a}{2}(\Delta\tau-i\epsilon)}$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\frac{-\hbar\left(\frac{a}{2}\right)^2}{4\pi^2\sinh^2\frac{a}{2}(\Delta\tau-i\epsilon)} = \frac{1}{2\pi}\frac{w}{e^{2\pi w/\hbar a}-1}$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\left[\langle 0_M|\Phi\left(x(\tau_1)\right)\Phi\left(x(\tau_2)\right)|0_M\rangle - \langle 0_R|\Phi\left(x(\tau_1)\right)\Phi\left(x(\tau_2)\right)|0_R\rangle\right]$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\left[\frac{-\hbar\left(\frac{a}{2}\right)^2}{4\pi^2\sinh^2\frac{a}{2}\Delta\tau}+\frac{\hbar}{4\pi^2(\Delta\tau)^2}\right] = \frac{1}{2\pi}\frac{\hbar w}{e^{2\pi wa}-1}$$

$$\langle {}_M 0|N^R_{i_1,i_2}|0_M\rangle = \frac{4}{\hbar}\int_{H_0^+} dv_1 d\vec{x}_{\perp1}dv_2 d\vec{x}_{\perp2}u^R_{i_1}(x_1)u^{R*}_{i_2}(x_2)$$

$$\partial_{v_1}\partial_{v_2}[\langle {}_M 0|\phi(x_1)\phi(x_2)|0_M\rangle - \langle {}_R 0|\phi(x_1)\phi(x_2)|0_R\rangle].$$

$$\langle {}_M 0|\partial_{V_1}\phi(x_1)\partial_{V_2}\phi(x_2)|0_M\rangle = \frac{1}{4\pi^2}\int d\vec{k}_\perp G^M_{\vec{k}_\perp}(x_1,x_2)e^{i\vec{k}_\perp(\vec{x}_{\perp1}-\vec{x}_{\perp2})}$$

$$G^M_{\vec{k}_\perp}(x_1,x_2)\Big|_{H_0^+} = \partial_{V_1}\partial_{V_2}\frac{\hbar}{2\pi}K_0\left(|\vec{k}_\perp|\sqrt{-(V_1-V_2)(U_1-U_2)}\right)\Big|_{H_0^+}$$

$$= -\frac{\hbar}{4\pi}\frac{1}{(V_1-V_2)^2}$$

$$\langle {}_M 0|\partial_{V_1}\phi(x_1)\partial_{V_2}\phi(x_2)|0_M\rangle\Big|_{H_0^+} = -\frac{\hbar}{4\pi}\frac{1}{(V_1-V_2)^2}\delta(\vec{x}_{\perp1}-\vec{x}_{\perp2})$$

$$u^R_{w,\vec{k}_\perp} = \frac{e^{i\gamma(w,|\vec{k}_\perp|)}}{(2\pi)^{3/2}\sqrt{2w}}\left(e^{-iwu} + re^{-iwv}\right)e^{i\vec{k}_\perp\vec{x}_\perp}$$

$$\langle {}_R0|\partial_{v_1}\phi(x_1)\partial_{v_2}\phi(x_2)|0_R\rangle\Big|_{H^+} = \frac{1}{4\pi^2}\int d\vec{k}_\perp G^R_{\vec{k}_\perp}(x_1,x_2)\Big|_{H^+} e^{-i\vec{k}_\perp\vec{x}_\perp}$$

$$= -\frac{\hbar}{4\pi}\frac{1}{(v_1 - v_2)^2}\delta(\vec{x}_{\perp 1} - \vec{x}_{\perp 2})$$

$$\langle {}_M0|N^R_{i_1,i_2}|0_M\rangle = -\frac{|r|^2}{4\pi^2\sqrt{w_1w_2}}\int_{V_1,V_2>0} e^{-iw_1v_1+iw_2v_2}$$

$$\left[\frac{dV_1dV_2}{(V_1 - V_2)^2} - \frac{dv_1dv_2}{(v_1 - v_2)^2}\right]\delta\left(\vec{k}_{\perp 1} - \vec{k}_{\perp 2}\right)$$

$$\langle {}_M0|N^R_{i_1,i_2}|0_M\rangle = -\frac{|r|^2}{4\pi^2\sqrt{w_1w_2}}\int_{-\infty}^{\infty} dv_1dv_2 e^{-iw_1v_1}e^{iw_2v_2}\times$$

$$\times\left[\frac{(a/2)^2}{\sinh^2\frac{a}{2}(v_1 - v_2)} - \frac{1}{(v_1 - v_2)^2}\right]\delta\left(\vec{k}_{\perp 1} - \vec{k}_{\perp 2}\right)$$

$$\langle {}_M0|N^R_{i_1,i_2}|0_M\rangle = -\frac{1}{4\pi^2 w_1}\int_{-\infty}^{\infty} d(\Delta v)e^{-iw_1\Delta v}\left[\frac{(a/2)^2}{\sinh^2\frac{a}{2}\Delta v} - \frac{1}{(\Delta v)^2}\right]\times$$

$$\times\,\delta(w_1 - w_2)\delta\left(\vec{k}_{\perp 1} - \vec{k}_{\perp 2}\right) = \frac{1}{e^{2\pi w_1/a} - 1}\delta\left(\vec{k}_1 - \vec{k}_2\right)$$

$$\langle {}_M0|N^R_{i_1,i_2}|0_M\rangle = -\frac{1}{4\pi^2\sqrt{w_1w_2}}\int_{U_1,U_2<0} e^{-iw_1u_1+iw_2u_2}$$

$$\left[\frac{dU_1dU_2}{(U_1 - U_2)^2} - \frac{du_1du_2}{(u_1 - u_2)^2}\right]\delta\left(\vec{k}_{\perp 1} - \vec{k}_{\perp 2}\right)$$

$$\left(\Box - \frac{m^2}{\hbar^2} - \xi R\right)\phi = 0$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\left\langle\phi\left(x(\tau_1)\right)\phi\left(x(\tau_2)\right)\right\rangle$$

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{H^2\eta_1\eta_2\hbar}{4\pi^2[-(\eta_1 - \eta_2 - i\epsilon)^2 + (\vec{x}_1 - \vec{x}_2)^2]}$$

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2)$$

$$t = \tau\;\vec{x} = \vec{x}_0$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\frac{-\hbar H^2}{16\pi^2\sinh^2\frac{H}{2}(\Delta\tau - i\epsilon)}$$

$$\langle 0_C|\phi\left(x(\tau_1)\phi\left(x(\tau_2)|0_C\rangle = -\frac{\hbar}{4\pi^2(\Delta\tau - i\epsilon)^2}\right.\right.$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i(w)\Delta\tau/\hbar}\left[\frac{-\hbar\left(\frac{H}{2}\right)^2}{4\pi^2\sinh^2\frac{H}{2}\Delta\tau} + \frac{\hbar}{4\pi^2(\Delta\tau)^2}\right]$$

$$\frac{dN_{lm}(w)}{dwdt} = \frac{1}{2\pi}\Gamma_{lm}(w)\frac{1}{e^{2\pi\kappa^{-1}w} \pm 1}$$

$$\int_0^{+\Lambda}\frac{dw'}{w'}e^{-\kappa^{-1}w_1 i\ln(-\kappa^{-1}w' - i\epsilon)}e^{\kappa^{-1}w_2 i\ln(\kappa^{-1}w' - i\epsilon)} = e^{-\pi\kappa^{-1}w_1}2\pi\delta_\sigma[\kappa^{-1}(w_1 - w_2)]$$

$$\delta_\sigma[\kappa^{-1}(w_1 - w_2)] = \frac{\sin\left[\frac{\kappa^{-1}(w_1 - w_2)}{\sigma}\right]}{\pi\kappa^{-1}(w_1 - w_2)}$$

$$\sigma = \frac{1}{\ln[\kappa^{-1}\Lambda]}$$

$$u_{jnlm}^{out} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} dw e^{2\pi iwn/\epsilon} u_{wlm}^{out}$$

$$\beta_{jn,w'} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} dw e^{2\pi iwn/\epsilon} \beta_{ww'}$$

$$\langle in|N_{jn}^{out,\sigma}|in\rangle \approx \frac{|t_l(w_j)|^2}{e^{2\pi\kappa^{-1}w_j} - 1} \frac{\sin\left[\left(\frac{2\pi n}{\epsilon} - v_H\right)\frac{\pi\kappa\sigma}{2}\right]}{\left[\left(\frac{2\pi n}{\epsilon} - v_H\right)\frac{\pi\kappa\sigma}{2}\right]}$$

$$I^B(w\kappa^{-1}, \alpha\kappa) = -\frac{1}{2\pi w}\int_{-\alpha}^{+\alpha} dz e^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)^2}{\left(\sinh\frac{\kappa}{2}z\right)^2} - \frac{1}{z^2}\right]$$

$$I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa) \equiv \frac{-1}{2\pi w}\int_{-\alpha}^{+\alpha} dz e^{-iwz}\frac{\left(\frac{\kappa}{2}\right)^2}{\left[\sinh\frac{\kappa}{2}(z - i\epsilon)\right]^2}$$

$$\frac{-1}{2\pi w}\int_{-\alpha}^{+\alpha} dz e^{-iwz}\frac{1}{(z - i\epsilon)^2}$$

$$I^B(w\kappa^{-1}, \alpha\kappa) = -\frac{Si(\alpha w)}{\pi} - \frac{\kappa}{4\pi w}\{e^{i\alpha w}({}_2F_1[1, -iw\kappa^{-1}; 1 - iw\kappa^{-1}; e^{-\alpha\kappa}]$$
$$- {}_2F_1[1, iw\kappa^{-1}; 1 + iw\kappa^{-1}; e^{\alpha\kappa}]) + e^{-i\alpha w}({}_2F_1[1, iw\kappa^{-1}; 1 + iw\kappa^{-1}; e^{-\alpha\kappa}]$$
$$- {}_2F_1[1, -iw\kappa^{-1}; 1 - iw\kappa^{-1}; e^{\alpha\kappa}])\} + \frac{1}{2\pi\alpha w}\cos(\alpha w)\left[\alpha\kappa\frac{(1 + e^{\alpha\kappa})}{(e^{\alpha\kappa} - 1)} - 2\right]$$

$$I^B(w\kappa^{-1}, \alpha\kappa) = \left(\frac{1}{12\pi}\alpha\kappa - \frac{1}{720\pi}(\alpha\kappa)^3 + O[(\alpha\kappa)^5]\right)\frac{\kappa}{w}$$
$$-\left(\frac{1}{72\pi}(\alpha\kappa)^3 + O[(\alpha\kappa)^5]\right)\frac{w}{\kappa} + (O[(\alpha\kappa)^5])\left(\frac{w}{\kappa}\right)^3 + \cdots$$

$$\lim_{w\kappa^{-1}\to 0}\frac{I^B(w\kappa^{-1}, \alpha\kappa)}{\left(e^{2\pi w\kappa^{-1}} - 1\right)^{-1}} = \frac{\alpha\kappa}{6} \ll 1$$

$$\frac{I^B\left(w_{\text{typical}}\,\kappa^{-1}, \alpha\kappa\right)}{\left(e^{2\pi w_{\text{typical}}\,\kappa^{-1}} - 1\right)^{-1}} \sim 0.3\alpha\kappa \ll 1$$

$$\frac{I^B(w\kappa^{-1}, \alpha\kappa)}{\left(e^{2\pi w\kappa^{-1}} - 1\right)^{-1}} \approx \frac{\alpha\kappa\left(e^{2\pi w\kappa^{-1}} - 1\right)}{12\pi w\kappa^{-1}}$$

$$I_{i\epsilon\to 0}^B(w\kappa^{-1}, \alpha\kappa) = \frac{e^{\alpha\kappa(1 - iw\kappa^{-1})} + e^{i\alpha w}}{2\pi w\kappa^{-1}(e^{\alpha\kappa} - 1)} + \frac{1}{2\pi(i + w\kappa^{-1})}\{e^{\alpha\kappa(1 - iw\kappa)} \times$$
$${}_2F_1[1, 1 - iw\kappa^{-1}; 2 - iw\kappa^{-1}; e^{\alpha\kappa}] - e^{-\alpha\kappa(1 - iw\kappa)} \times$$
$${}_2F_1[1, 1 - iw\kappa^{-1}; 2 - iw\kappa^{-1}; e^{-\alpha\kappa}]\}$$

$$I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa) = \left(\frac{1}{\pi\alpha\kappa} + \frac{\alpha\kappa}{12\pi} - \frac{(\alpha\kappa)^3}{720\pi} + O((\alpha\kappa)^5)\right)\frac{\kappa}{w} - \frac{1}{2}$$
$$+\left(\frac{\alpha\kappa}{2\pi} - \frac{(\alpha\kappa)^3}{72\pi} + O((\alpha\kappa)^5)\right)\frac{w}{\kappa} - \left(\frac{(\alpha\kappa)^3}{72\pi} + O((\alpha\kappa)^5)\right)\left(\frac{w}{\kappa}\right)^3 + \cdots$$

$$\lim_{\kappa^{-1}w\to 0}\frac{I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa)}{\left(e^{2\pi w\kappa^{-1}} - 1\right)^{-1}} = \frac{2}{\alpha\kappa} \gg 1$$

$$I^F(w\kappa^{-1},\alpha\kappa) \equiv \frac{-i}{2\pi}\int_{-\alpha}^{+\alpha} dz e^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)}{\left(\sinh\frac{\kappa}{2}z\right)^2} - \frac{1}{z}\right] =$$

$$\frac{Si(\alpha w)}{\pi} + \frac{1}{2\pi(1+4w^2\kappa^{-2})}\{(-i+2w\kappa^{-1})\big(e^{-\alpha\kappa/2+i\alpha w} \times$$

$${}_2F_1\left[1,\frac{1}{2}-iw\kappa^{-1};\frac{3}{2}-iw\kappa^{-1};e^{-\alpha\kappa}\right] - {}_2F_1\left[1,\frac{1}{2}-iw\kappa^{-1};\frac{3}{2}-iw\kappa^{-1};e^{\alpha\kappa}\right] \times$$

$$e^{\alpha\kappa/2-i\alpha w}\big) + (i+2w\kappa^{-1})\left(e^{-\alpha\kappa/2-i\alpha w}\,{}_2F_1\left[1,\frac{1}{2}+iw\kappa^{-1};\frac{3}{2}+iw\kappa^{-1};e^{-\alpha\kappa}\right]\right.$$

$$\left.-e^{\alpha\kappa/2+i\alpha w}\,{}_2F_1\left[1,\frac{1}{2}+iw\kappa^{-1};\frac{3}{2}+iw\kappa^{-1};e^{\alpha\kappa}\right]\right)\}$$

$$I^F(w\kappa^{-1},\alpha\kappa) = \left(\frac{(\alpha\kappa)^3}{72\pi} + O[(\alpha\kappa)^5]\right)\frac{w}{\kappa} + O[(\alpha\kappa)^5]\left(\frac{w}{\kappa}\right)^3 + \cdots$$

$$\frac{I^F(w\kappa^{-1},\alpha\kappa)}{\left(e^{2\pi w\kappa^{-1}}+1\right)^{-1}} \sim \frac{(\alpha\kappa)^3}{36\pi}\frac{w}{\kappa} \ll 1$$

$$\frac{I^F\left(w_{\text{typical}}\,\kappa^{-1},\alpha\kappa\right)}{\left(e^{2\pi w_{\text{typical}}\,\kappa^{-1}}+1\right)^{-1}} \sim 3\cdot 10^{-3}(\alpha\kappa)^3 \ll 1$$

$$\frac{I^F(w\kappa^{-1},\alpha\kappa)}{\left(e^{2\pi w\kappa^{-1}}+1\right)^{-1}} \approx \frac{\alpha^3 w\kappa^2\left(e^{2\pi w\kappa^{-1}}+1\right)}{72\pi}$$

$$(V_1-V_2)^2 \sim e^{2av_1}(v_1-v_2)^2$$

$$-\frac{1}{4\pi^2 w}\int_{|\Delta v|>\alpha} d(\Delta v)e^{-iw\Delta v}\left[\frac{(a/2)^2}{\sinh^2\frac{a}{2}\Delta v} - \frac{1}{(\Delta v)^2}\right]$$

$$\approx \frac{1}{e^{2\pi w/a}-1} - \frac{\alpha a}{12\pi w/a} + O(\alpha^3 a^3)$$

$$(U_1-U_2)^2 \sim e^{-2au_1}(u_1-u_2)^2.$$

$$E^2 f^2(E,p) - p^2 g^2(E,p) = m^2$$

$$U{:}\,p_\mu \equiv (-E,p_i) \to \Pi_\mu \equiv (-\epsilon,\Pi_i) = (-f(E,p)E, g(E,p)p_i).$$

$$L(p^\mu) = [U^{-1}\quad L\quad U](p^\mu)$$

$$E \;= \frac{\epsilon}{1+\ell_P\hbar^{-1}\epsilon}$$

$$p_i \;= \frac{\Pi_i}{1+\ell_P\hbar^{-1}\epsilon}$$

$$U{:}\,(p_\mu,x^\nu) \to (\Pi_\mu,X^\nu),$$

$$X^\mu = X^\mu(x^\nu,p_\rho)$$

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d}\phi_j(x'),$$

$$\langle\phi_j(x_1)\phi_j(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\Delta_j/d}\left|\frac{\partial x'}{\partial x}\right|_{x_2}^{\Delta_j/d}\langle\phi_j(x_1')\phi_j(x_2')\rangle.$$

$$\langle\phi_j(x_1)\phi_j(x_2)\rangle = \frac{C_j}{(x_1-x_2)^{2\Delta_j}},$$

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{4\pi^2}\frac{\hbar}{(x_1-x_2)^2},$$

$$\langle\partial_\pm\phi(x_1)\partial_\pm\phi(x_2)\rangle = -\frac{1}{4\pi}\frac{\hbar}{\left(x_1^\pm - x_2^\pm\right)^2}$$

$$U{:}\,\langle\Phi_j(x_1)\Phi_j(x_2)\rangle \to \langle\phi_j(x_1)\phi_j(x_2)\rangle$$

$$U^{-1}\colon \langle \Phi_j(x_1)\Phi_j(x_2)\rangle = \frac{\langle \phi_j(x_1)\phi_j(x_2)\rangle}{1-\ell_P^2\hbar^{-1}\langle \phi_j(x_1)\phi_j(x_2)\rangle}$$

$$\langle \Phi_j(x_1)\Phi_j(x_2)\rangle \to \frac{\left|\frac{\partial x'}{\partial x}\right|^{\Delta_j}/d(x_1)\left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d}(x_2)\langle \phi_j(x_1')\phi_j(x_2')\rangle}{1-\ell_P^2\hbar^{-1}\left|\frac{\partial x'}{\partial x}\right|\Delta_j/d(x_1)\left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d}(x_2)\langle \phi_j(x_1')\phi_j(x_2')\rangle}.$$

$$\langle \Phi(x_1)\Phi(x_2)\rangle = \frac{\hbar}{4\pi^2(x_1-x_2)^2-\ell_P^2}$$

$$\frac{1}{4\pi^2(x_1-x_2)^2-\ell_P^2} \to \frac{\left|\frac{\partial x'}{\partial x}\right|^{1/4}(x_1)\left|\frac{\partial x'}{\partial x}\right|^{1/4}(x_2)}{4\pi^2(x_1-x_2)^2-\ell_P^2\left|\frac{\partial x'}{\partial x}\right|^{1/4}(x_1)\left|\frac{\partial x'}{\partial x}\right|^{1/4}(x_2)}$$

$$\langle \Phi(x_1)\Phi(x_2)\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{\ell_P^2} = -\frac{1}{G}.$$

$$\langle \Psi|\phi(x_1)\phi(x_2)|\Psi\rangle|_{x_1\to x_2} \sim \langle \phi(x_1)\phi(x_2)\rangle\big|_{x_1\to x_2},$$

$$\langle \Psi|\phi(x_1)\phi(x_2)|\Psi\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{\ell_P^2} = -\frac{1}{G}$$

$$\langle \partial_\pm\Phi(x_1)\partial_\pm\Phi(x_2)\rangle = \frac{-\hbar}{4\pi\left(x_1^\pm-x_2^\pm\right)^2+\ell_P^2}$$

$$\langle \partial_\pm\Phi(x_1)\partial_\pm\Phi(x_2)\rangle \to \frac{\frac{dx'^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)\langle \phi_\pm(x_1')\phi_\pm(x_2')\rangle}{1-\ell_P^2\hbar^{-1}\frac{dx^{\pm\pm}}{dx^\pm}(x_1)\frac{dx^{\pm\pm}}{dx^\pm}(x_2)\langle \phi_\pm(x_1')\phi_\pm(x_2')\rangle}.$$

$$\frac{-\hbar}{4\pi\left(x_1^\pm-x_2^\pm\right)^2+\ell_P^2} \to \frac{-\hbar\frac{dx^\pm}{dx^\pm}(x_1)\frac{dx^\pm}{dx^\pm}(x_2)}{4\pi\left(x_1^\pm-x_2^{\pm\pm}\right)^2+\ell_P^2\frac{dx^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)},$$

$$\frac{-\hbar}{4\pi\left(x_1^\pm-x_2^\pm\right)^2+\ell_P^2} \to \frac{-\hbar e^{\pm 2\chi}}{4\pi\left(x_1'^\pm-x_2'^\pm\right)^2+\ell_P^2 e^{\pm 2\chi}}$$

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{m\hbar}{4\pi^2\sqrt{(x_1-x_2)^2}}K_1\left(m\sqrt{(x_1-x_2)^2}\right)$$

$$\langle \Phi(x_1)\Phi(x_2)\rangle \equiv \frac{\langle \phi_m(x_1)\phi_m(x_2)\rangle}{1-\ell_P^2\hbar^{-1}\langle \phi_m(x_1)\phi_m(x_2)\rangle}$$

$$= \frac{m\hbar K_1\left(m\sqrt{(x_1-x_2)^2}\right)}{4\pi^2\sqrt{(x_1-x_2)^2}-m\ell_P^2 K_1\left(m\sqrt{(x_1-x_2)^2}\right)}$$

$$\langle \Phi(x_1)\Phi(x_2)\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{\ell_P^2}$$

$$\langle \phi(x_1)\phi(x_2)\rangle = \int_{-\infty}^{+\infty} ds e^{-im^2 s}K(x_1,x_2;s) = \frac{\hbar}{4\pi^2(x_1-x_2)^2}$$

$$\langle \Phi(x_1)\Phi(x_2)\rangle = \int_{-\infty}^{+\infty} ds e^{-im^2 s}e^{i\ell_P^2/(4\pi^2 s)}K(x_1,x_2;s)$$

$$\langle 0_M|\Phi(x_1)\Phi(x_2)|0_M\rangle = \frac{\hbar}{4\pi^2(x_1-x_2)^2-\ell_P^2}$$

$$\dot{F}_{\ell_P}(w) = -\frac{\hbar}{4\pi^2}\int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau} \times$$

$$\times\left[\frac{1}{(2/a)^2\sinh^2\left(\frac{a}{2}\Delta\tau\right)+\ell_P^2/4\pi^2}-\frac{1}{(\Delta\tau)^2+\ell_P^2/4\pi^2}\right]$$

$$\dot{F}_{\ell_P}(w) = \frac{\hbar}{2\pi}\left[\frac{we^{\pi w/a}}{(e^{2\pi w/a}-1)}\frac{\sinh\left[\frac{w}{a}(\theta-\pi)\right]}{\frac{w}{a}\sin\,\theta}+\frac{\pi e^{-w\ell_P}}{\ell_P}\right]$$

$$\dot{F}_{\ell_P}(w) \approx \frac{\hbar}{2\pi}\left[\frac{w}{e^{2\pi w/a}-1}-\frac{\ell_P a^2}{32\pi}+O(\ell_P/a^2)^3\right]$$

$$-\frac{1}{4\pi^2}\int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}\frac{\hbar}{(2/a)^2\sinh^2\left(\frac{a}{2}\Delta\tau\right)+\ell_P^2/4\pi^2}$$

$$\frac{w\hbar}{2\pi}\frac{e^{\pi w/a}}{(e^{2\pi w/a}-1)}\frac{\sinh\left[\frac{w}{a}(\theta-\pi)\right]}{\frac{w}{a}\sin\,\theta}$$

$$U^{-1}\!:\langle\Phi(x_1)\Phi(x_2)\rangle = \frac{\langle\phi(x_1)\phi(x_2)\rangle}{1-\ell_P^2\hbar^{-1}\langle\phi(x_1)\phi(x_2)\rangle}$$

$$= \frac{H^2\eta_1\eta_2\hbar}{4\pi^2[-(\eta_1-\eta_2-i\epsilon)^2+(\vec{x}_1-\vec{x}_2)^2]-\ell_P^2H^2\eta_1\eta_2}$$

$$\dot{F}_{\ell_P}(w) = -\frac{\hbar}{4\pi^2}\int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau} \times$$

$$\times\left[\frac{1}{(2/H)^2\sinh^2\left(\frac{H}{2}\Delta\tau\right)+\ell_P^2/4\pi^2}-\frac{1}{(\Delta\tau)^2+\ell_P^2/4\pi^2}\right]$$

$$\dot{F}_{\ell_P}(w) = \frac{\hbar}{2\pi}\left[\frac{we^{\pi w/H}}{(e^{2\pi w/H}-1)}\frac{\sinh\left[\frac{w}{H}(\theta-\pi)\right]}{\frac{w}{H}\sin\,\theta}+\frac{\pi e^{-w\ell_P}}{\ell_P}\right]$$

$$\dot{F}_{\ell_P}(w) \approx \frac{\hbar}{2\pi}\left[\frac{w}{e^{2\pi w/H}-1}-\frac{\ell_P H^2}{32\pi}+O(\ell_P/H^2)^3\right]$$

$$\phi(x^\mu) = \sum_{l,m}\frac{\phi_l(t,r)}{r}Y_{lm}(\theta,\varphi)$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}-V_l(r)\right)\phi_l(t,r) = 0$$

$$\langle\partial_u\phi_l(x_1)\partial_u\phi_l(x_2)\rangle = -\frac{\hbar}{4\pi}\frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{[v(u_1)-v(u_2)]^2+\ell_P^2\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)},$$

$$\langle in|N_{wlm}^{out}|in\rangle = -\frac{|t_l(w)|^2}{2\pi w}\int_{-\infty}^{+\infty} dze^{-iwz}\left[\frac{\left(\frac{\kappa}{2}\right)^2}{\left(\sinh\frac{\kappa}{2}z\right)^2+\left(\frac{\kappa}{2}\right)^2\frac{\ell_P^2}{4\pi^2}}-\frac{1}{z^2+\frac{\ell_P^2}{4\pi^2}}\right]$$

$$\langle in|N_{wlm}^{out}|in\rangle = |t_l(w)|^2\left[\frac{e^{\pi w/\kappa}}{(e^{2\pi w/\kappa}-1)}\frac{\sinh\left[\frac{w}{\kappa}(\theta-\pi)\right]}{\frac{w}{\kappa}\sin\,\theta}+\frac{\pi e^{-w\ell_P}}{w\ell_P}\right]$$

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S^\phi}{\delta g^{\mu\nu}}$$

$$T_{00} = \rho = 1/2\dot{\phi}^2+V(\phi)$$

$$T_{ii} = pg_{ii} = g_{ii}\big(1/2\dot{\phi}^2-V(\phi)\big)$$

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

$$h_{\vec{k}}(t,\vec{x}) = \sqrt{\frac{16\pi G}{2(2\pi)^3 k^3}} e^{i\vec{k}\vec{x}}(H - ike^{-Ht})e^{i(kH^{-1}e^{-Ht})}$$

$$P_t(k) = 4\Delta_h^2(k) = \frac{8}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2$$

$$\int_0^\infty \frac{dk}{k}\frac{\sin(k|\Delta\vec{x}|)}{|\Delta\vec{x}|}\Delta_h^2(k,t) = \langle 0_{dS}|h(t,\vec{x})h(t,\vec{x}+\Delta\vec{x})|0_{dS}\rangle$$

$$\int_0^\infty \frac{dk}{k}\Delta_h^2(k,t) = \langle 0_{dS}|h(t,\vec{x})h(t,\vec{x})|0_{dS}\rangle$$

$$P_{\mathcal{R}}(k) = \frac{1}{2M_P^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2$$

$$n_s - 1 \equiv \mathrm{dln}\ P_{\mathcal{R}}/\mathrm{dln}\ k, n_t \equiv \mathrm{dln}\ P_t/\mathrm{dln}\ k$$

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

$$\langle 0_{dS}|N_i^C|0_{dS}\rangle = \sum_j |\beta_{ij}|^2$$

$$= \frac{1}{\hbar}\int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu \left(u_i^C(x_1)\overleftrightarrow{\partial}_\mu\right)\left(u_i^{C*}(x_2)\overleftrightarrow{\partial}_\nu\right) \times$$

$$[\langle 0_{dS}|\phi(x_1)\phi(x_2)|0_{dS}\rangle - \langle 0_C|\phi(x_1)\phi(x_2)|0_C\rangle]$$

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau}[\langle 0_{dS}|\phi(x(\tau))\phi(x(\tau+\Delta\tau))|0_{dS}\rangle$$

$$-\langle 0_C|\phi(x(\tau))\phi(x(\tau+\Delta\tau))|0_C\rangle]$$

$$\int_0^\infty \frac{dk}{k}\Delta_\phi^2(k,t;dS) = \langle 0_{dS}|\phi(t,\vec{x})\phi(t,\vec{x})|0_{dS}\rangle$$

$$\int_0^\infty \frac{dk}{k}\Delta_\phi^2(k,t) \equiv \langle 0_{dS}|\phi(t,\vec{x})\phi(t,\vec{x})|0_{dS}\rangle - \langle 0_C|\phi(t,\vec{x})\phi(t,\vec{x})|0_C\rangle$$

$$u_{\vec{k}}^{dS}(\vec{x},t) = \frac{1}{\sqrt{(2\pi)^3 a(t)^3}} v_k(t)e^{i\vec{k}\vec{x}}$$

$$v_k(t) = \frac{1}{2}\sqrt{\frac{\pi}{H}}H_\nu^{(1)}(kH^{-1}\exp(-Ht))$$

$$\langle 0_{dS}|\phi(t,\vec{x})\phi(t,\vec{x})|0_{dS}\rangle = \hbar(4\pi^2 a(t)^3)^{-1}\int_0^\infty |v_k(t)|^2 k^2 dk$$

$$\Delta_\phi^2(k,t;dS) = \hbar(4\pi^2 a(t)^3)^{-1}k^3|v_k(t)|^2$$

$$\Delta_\phi^2(\bar{k};dS) = \frac{\hbar H^2}{8\pi}\left|H_n^{(1)}(\bar{k}H^{-1})\right|^2 \approx \frac{\hbar H^2}{2\pi}$$

$$u_w^C(\tilde{r},\tilde{t}) = \frac{e^{-iw\tilde{t}}}{\sqrt{4\pi}}\frac{N_\nu(w)}{\tilde{r}}\left[P_{\nu-1/2}^{i\frac{w}{H}}(H\tilde{r}) - \alpha_\nu(w)Q_{\nu-1/2}^{i\frac{w}{H}}(H\tilde{r})\right]$$

$$u_w^C(\tilde{r}=0) = e^{-iw\tilde{t}}\frac{\tilde{N}_\nu(w/H)}{\sqrt{4\pi w}}H\beta_\nu(w/H)$$

$$\beta_\nu(z) = -\frac{2^{1/2-\nu}\sin(\pi\mu_\nu^*)\sin\left(\frac{\pi\mu_\nu}{2}\right)}{\pi^2\cosh\ \pi z}\Gamma(\mu_\nu)[z|\Gamma(\mu_{-\nu}/2)|^2 +$$

$$2\pi\left(\frac{m^2}{\hbar^2 H^2} - 2\right)\Im\left(\frac{\Gamma(\mu_{-\nu})_2F_1\left(\frac{3}{2}-\nu,\frac{1}{2}-\nu+iz;2+iz;-1\right)}{\Gamma(2+iz)\cos\frac{\pi\mu_\nu}{2}}\right)\Bigg]$$

$$\langle 0_C|\phi(t,\vec{x})\phi(t,\vec{x})|0_C\rangle = \frac{\hbar H^2}{4\pi}\int_{\frac{m}{\hbar}}^\infty \frac{dw}{w}\left|\tilde{N}_\nu\right|^2|\beta_\nu|^2$$

$$\Delta_\phi^2(\bar{k};C) = \hbar\frac{H^2}{4\pi}\frac{\bar{k}^2}{\bar{k}^2+m^2\hbar^{-2}}\left|\tilde{N}_\nu(\bar{k}H^{-1})\right|^2\left|\beta_\nu(\bar{k}H^{-1})\right|^2\Big]$$

$\frac{m^2}{H^2\hbar^2}$	10^{-1}	10^{-3}	10^{-5}
$\frac{\Delta_\phi^2(k)}{\Delta_\phi^2(\bar{k},dS)}$	0.2212×10^{-1}	0.2525×10^{-3}	0.2529×10^{-5}

$$\left|\tilde{N}_{3/2}(w/H)\right|^2 = \frac{|\Gamma(1-iw/H)|^2}{\pi}\sinh^2\frac{\pi w}{2H}$$

$$\Delta_{\phi_{m=0}}^2(\bar{k};dS) = \frac{\hbar H^2}{4\pi^2}\left(1+\frac{\bar{k}^2}{H^2}\right) = \Delta_{\phi_{m=0}}^2(\bar{k};C)$$

$$\Delta_\phi^2(\bar{k}) \equiv \Delta_\phi^2(\bar{k};dS) - \Delta_\phi^2(\bar{k};C)$$

$$\Delta_\phi^2(\bar{k}) = \hbar H^2\left[\frac{1}{8\pi}\left|H_\nu^{(1)}(\bar{k}H^{-1})\right|^2 - \right.$$

$$\left.\frac{1}{4\pi}\frac{\bar{k}^2}{\bar{k}^2+m^2\hbar^{-2}}\left|\tilde{N}_\nu(\bar{k}H^{-1})\right|^2\left|\beta_\nu(\bar{k}H^{-1})\right|^2\Big]\right]$$

$$\left.\frac{\Delta_\phi^2(\bar{k})}{\Delta_\phi^2(\bar{k},dS)}\right|_{\bar{k}=H} \approx 0.25\frac{m^2}{H^2\hbar^2}$$

$$\ddot{v}_{\vec{k}} + (\vec{k}^2 - \ddot{a}/a)v_{\vec{k}} = 0.$$

$$\ddot{v}_{\vec{k}} + (k^2 - (2+3\epsilon)/\tau^2)v_{\vec{k}} = 0.$$

$$\Delta_h^2(k,t) \equiv 4\pi k^3\left|u_{\vec{k}}(\vec{x},t)\right|^2$$

$$\Delta_h^2(k) \approx \left(\frac{H}{2\pi}\right)^2$$

$$\langle h^2(\vec{x},t)\rangle = \int_0^\infty \frac{dk}{k}\Delta_h^2(k,t)$$

$$\langle h^2(\vec{x},t)\rangle \approx \int_0^\infty \frac{dk}{k}\frac{16\pi Gk^3}{4\pi^2a^3}\left[\frac{a}{k}\left[1+\frac{(2+3\epsilon)}{2k^2\tau^2}\right]+\cdots\right]$$

$$(16\pi Gk^3/4\pi^2a^3)[1/w + \dot{a}^2/2a^2w^3 + \ddot{a}/2aw^3]$$

$$\langle h^2(\vec{x},t)\rangle = \int_0^\infty \frac{dk}{k}\frac{16\pi Gk^3}{4\pi^2a^3}\left[-a\frac{\tau\pi}{2}\left|H_\nu^{(1)}(-k\tau)\right|^2\right.$$

$$\left.-\frac{a}{k}\left[1+\frac{(2+3\epsilon)}{2k^2\tau^2}\right]\right].$$

$$H_{3/2}^{(1)}(x) = i\exp(ix)\sqrt{2/\pi x}(-1+i/x)$$

$$\Delta_h^2(k) = 4GH^2\left[\frac{1+\epsilon}{2}\left|H_\nu^{(1)}(-k\tau=1+\epsilon)\right|^2 - \frac{(4-\epsilon)}{2\pi}\right]$$

$$\Delta_h^2(k) = \alpha 16\pi G(H(t_k)/2\pi)^2\epsilon(t_k) + O(\epsilon^2)$$

$$P_t(k) = \frac{8\alpha}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2\epsilon(t_k)$$

$$(k^3/4\pi^2a^3)[m^2\dot{a}^2/2a^2w^5 + m^2\ddot{a}/4aw^5 - 5m^4\dot{a}^2/8a^2w^7].$$

$$\Delta_\phi^2(k) = 4\hbar GH^2\left[\frac{1+\epsilon}{2}\left|H_\nu^{(1)}(-k\tau=1+\epsilon)\right|^2 - \frac{(4-\epsilon-5/2m^2/H^2)}{2\pi}\right]$$

$$P_{\mathcal{R}} = \frac{1}{2M_p^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2\left(\alpha\epsilon(t_k)+3\beta\eta(t_k)\right)$$

$$n_t = 2(\epsilon - \eta)$$
$$n_s - 1 = -6\epsilon + 2\eta + \frac{4\alpha\epsilon^2 + (6\beta - 2\alpha)\epsilon\eta - 3\beta(n_t'/2 - 4\epsilon(\epsilon - \eta))}{\alpha\epsilon + 3\beta\eta}$$
$$r = 1 - n_s + \frac{96}{25}n_t + \frac{11}{5}\sqrt{(1 - n_s)^2 + \frac{96}{25}n_t^2}$$
$$r = \frac{\alpha p^2}{(3\beta(p-1)/2 + \alpha p/4)N}, 1 - n_s = \frac{p}{2N}$$
$$S = k_B \frac{A}{4\ell_P^2}$$
$$a_1 + a_2 + \cdots + a_N = 0 \bmod \kappa.$$
$$2m_I = -a_I, \bmod \kappa.$$
$$A = 8\pi\gamma\ell_P^2 \sum_I \sqrt{j_I(j_I + 1)}$$
$$S = k_B \ln \text{n}(A_0)$$
$$A(\vec{j}) := 2\sum_I \sqrt{j_I(j_I + 1)}$$
$$A_0 - \delta \leq A(\vec{j}) \leq A_0 + \delta,$$
$$\sum_I m_I = 0.$$
$$\sum_{I=1}^{N} 2\sqrt{|m_I|(|m_I| + 1)} \leq A$$
$$\sum_{I=1}^{N} m_I = 0, A(\vec{m}) := 2\sum_{i=I}^{N} \sqrt{|m_I|(|m_I| + 1)} \leq A_0$$
$$N_{DL}(A) = \sum_{A' < A} d_{DL}(A')$$
$$\sum_{I=1}^{N} m_I = 0, A(\vec{m}) := 2\sum_{i=I}^{N} \sqrt{|m_I|(|m_I| + 1)} = A_0$$
$$A_0 - \delta \leq A(\vec{j}) := \sum_I \sqrt{j_I(j_I + 1)} \leq A_0 + \delta,$$
$$\sum_I m_I = 0.$$
$$N(A) = \theta(A - \sqrt{3})(2N(A - \sqrt{3}) + 2N(A - 2\sqrt{2}) + \cdots +$$
$$+2N\left(A - 2\sqrt{|m_I|(|m_I| + 1)}\right) + \cdots + 2\left\lfloor\sqrt{A^2 + 1} - 1\right\rfloor\Big),$$
$$P(s) := \int_0^\infty N(A)e^{-As}\,\mathrm{d}A$$
$$= 2\int_{\sqrt{3}}^\infty \left\lfloor\sqrt{A^2 + 1} - 1\right\rfloor e^{-As}\,\mathrm{d}A + 2\int_{\sqrt{3}}^\infty \left(\sum_{k=1}^\infty N(A - \sqrt{k(k+2)})\right)e^{-As}\,\mathrm{d}A$$
$$= \frac{2}{s}\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)}} + 2\sum_{k=1}^\infty \int_{\sqrt{3}}^\infty N(A - \sqrt{k(k+2)})e^{-As}\,\mathrm{d}A$$
$$= \frac{2}{s}\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)}} + 2\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)}} \int_{-\sqrt{k(k+2)}}^\infty N(A)e^{-As}\,\mathrm{d}A$$
$$= \frac{2}{s}\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)}} + 2P(s)\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)}}$$

$$P(s)=\frac{2\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}}{s\left(1-2\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\right)}$$

$$N(A)=\sum_{s_i}\ \mathrm{Res}[P(s)e^{sA}]$$

$$\left(1-2\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\right)=0.$$

$$N(A)\approx Ce^{\pi\gamma_M A}$$

$$S=k_B\ln\,\mathfrak{n}(A)=k_B\ln\,(N(A)+1)=k_B\left(\frac{\gamma_M}{\gamma}\frac{A}{4\ell_P^2}+O(A^0)\right)$$

$$N(A,p)\ =(1-\delta(0,p))\theta(A-2\sqrt{|p|(|p|+1)})$$

$$+\sum_{k=1}^{\infty}\ \left(N(A-\sqrt{k(k+2)},p-k/2)+N_{\leq}(A-\sqrt{k(k+2)},p+k/2)\right)$$

$$P(s,\omega):=\int_0^{\infty}\ e^{-sA}\left[\sum_{p\in\mathbb{Z}/2}\ e^{i\omega p}N(A,p)\right]\mathrm{d}A$$

$$=\int_0^{\infty}\ e^{-sA}\left[\sum_{p\in\mathbb{Z}/2}\ e^{i\omega p}(1-\delta(0,p))\theta(A-2\sqrt{|p|(|p|+1)})\right]\mathrm{d}A$$

$$+\int_0^{\infty}\ e^{-sA}\left[\sum_{p\in\mathbb{Z}/2}\ e^{i\omega p}\sum_{\ell\in\mathbb{Z}_*}\ N\left(A-\sqrt{|\ell|(|\ell|+2)},p-\frac{\ell}{2}\right)\right]\mathrm{d}A$$

$$=\frac{2}{s}\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\cos\frac{k\omega}{2}+2P(s,\omega)\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\cos\frac{k\omega}{2}$$

$$P(s,\omega)=\frac{2}{s}\left(\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\cos\frac{k\omega}{2}\right)\left(1-2\sum_{k=1}^{\infty}\ e^{-s\sqrt{k(k+2)}}\cos\frac{k\omega}{2}\right)^{-1}.$$

$$N(A,0)=\frac{C}{\sqrt{2\pi\beta_M A}}e^{\pi\gamma_M A}$$

$$S=k_B\left(\frac{\gamma_M}{\gamma}\frac{A}{4\ell_P^2}-\frac{1}{2}\ln\,(A/\ell_P^2)+O(A^0)\right).$$

$$\sum_{k=1}^{k_{\max}}\ n_k\sqrt{k(k+2)}=A_0$$

$$d_{GM}\left(n_1,\dots,n_{k_{\max}}\right)=\frac{(\sum_k\ n_k)!}{\prod_k\ n_k!}\prod_k\ (k+1)^{n_k},$$

$$\hat{n}_k=\frac{n_k}{\sum_k\ n_k}=(k+1)e^{-\lambda\sqrt{k(k+2)}}$$

$$\sum_{k=1}^{k_{\max}}\ (k+1)e^{-\lambda\sqrt{k(k+2)}}=1$$

$$d_{GM}(A)=\frac{c}{\sqrt{A}}e^{\lambda A}$$

$$S(A)=k_B\left(\frac{\lambda}{\pi\gamma}\frac{A}{4\ell_P^2}-\frac{1}{2}\ln\,\frac{A}{\ell_P^2}+O(A^0)\right).$$

$$\gamma_{GM}=0.27398563$$

$$\Delta A=\gamma\chi\ell_P^2$$

$$\chi \approx 8.80$$

$$(S_t, p_t) = (S_0 - 2t, p_0 + 3t)$$

$$3S + 2p = K$$

$$\hat{k}_{md} = \sum_k k\hat{n}_k.$$

$$\frac{k_{md}}{p_{md}} = \hat{k}_{md}.$$

$$S(p, K) = \frac{K}{3} - \frac{2}{3}p.$$

$$\frac{k_{md}(p, K)}{p_{md}(K)} = \frac{\frac{K}{3} - \frac{2}{3}p_{md}(K)}{p_{md}(K)} = \hat{k}_{md},$$

$$p_{md}(K) = \frac{K}{3\hat{k}_{md} + 2}.$$

$$\hat{A}_{md} = 4\pi\gamma\ell_{\mathrm{P}}^2 \sum_k \hat{n}_k \sqrt{k(k+2)}$$

$$A_{md}(K) = p_{md}(K)\hat{A}_{md} = \frac{K\hat{A}_{md}}{3\hat{k}_{md} + 2}$$

$$\Delta A = A_{md}(K+2) - A_{md}(K) = \hat{A}_{md}(p_{md}(K+2) - p_{md}(K)) = \frac{2\hat{A}_{md}}{3\hat{k}_{md} + 2}$$

$$\Delta A_{\mathrm{GM}} = \chi_{\mathrm{GM}}\gamma_{\mathrm{GM}} = \frac{8\pi\gamma_{\mathrm{GM}}\ell_{\mathrm{P}}^2 \sum_k \sqrt{k(k+2)}(k+1)e^{-\lambda_{\mathrm{GM}}\sqrt{k(k+2)}}}{3\left(\sum_k k(k+1)e^{-\lambda_{\mathrm{GM}}\sqrt{k(k+2)}}\right) + 2}$$

$$\Delta A_{\mathrm{DL}} = \chi_{\mathrm{DL}}\gamma_{\mathrm{M}} = \frac{8\pi\gamma_{\mathrm{M}}\ell_{\mathrm{P}}^2 \sum_k 2\sqrt{k(k+2)}e^{-\lambda_{\mathrm{DL}}\sqrt{k(k+2)}}}{3\left(\sum_k 2ke^{-\lambda_{\mathrm{DL}}\sqrt{k(k+2)}}\right) + 2}$$

$$\chi_{\mathrm{GM}} = 8.789242, \chi_{\mathrm{DL}} = 8.784286$$

$$\frac{|\chi_{\mathrm{GM}} - 8\ln 3|}{8\ln 3} = 0.000039 = 0.004\%$$

$$\frac{|\chi_{\mathrm{DL}} - 8\ln 3|}{8\ln 3} = 0.00052 = 0.05\%$$

$$\frac{|\chi_{\mathrm{GM}} - \chi_{\mathrm{DL}}|}{\chi_{\mathrm{GM}}} = 0.00056 = 0.06\%$$

$$A(\vec{m}) := 2\sum_I^N \sqrt{|m_I|(|m_I| + 1)} = \sum_I^N \sqrt{(k_I + 1)^2 - 1} = \sum_{k_I}^{k_{\max}} n_k\sqrt{(k+1)^2 - 1},$$

$$\sum_{k_I=1}^{k_{\max}} n_{k_I}\sqrt{(k_I + 1)^2 - 1} = \sum_{i=1}^{r} q_i\sqrt{p_i}$$

$$\sqrt{(k+1)^2 - 1} = y\sqrt{p_i}$$

$$\left(k_m^i + 1 + y_m^i\sqrt{p_i}\right) = \left(k_i^1 + 1 + y_1^i\sqrt{p_i}\right)^m,$$

$$\sum_{i=1}^{r}\sum_{m=1}^{\infty} n_{k_m^i} y_m^i \sqrt{p_i} = \sum_{i=1}^{r} q_i\sqrt{p_i}$$

$$\sum_{m=1}^{\infty} y_m^i n_{k_m^i} = q_i, i = 1, \ldots, r$$

$$S_{q_i}^i = \left\{\alpha_i = \left\{\left(n_{k_m^i}\right)\right\}_{m=1}^{M_i} : \sum_{m=1}^{M_i} y_m^i n_{k_m^i} = q_i\right\}$$

$$R(\alpha)=\frac{\left(\sum_{i=1}^{r}\ \sum_{m}\ n_{k_m^i}\right)!}{\prod_{i=1}^{r}\ \prod_{m}\ n_{k_m^i}!},$$

$$P_{DL}(\alpha)=\frac{2^N}{M}\sum_{l=0}^{M-1}\ \prod_{I=1}^{N}\ \cos\left(2\pi lk_I/M\right)$$

$$\prod_{I=1}^{N}\ \left(z^{k_I}+z^{-k_I}\right)$$

$$P_{DL}(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\ d\theta\prod_{k}\ (2\cos\ k\theta)^{n_k}$$

$$d_{DL}(A)=\sum_{\alpha\in\mathcal{S}_A}\ R(\alpha)P_{DL}(\alpha)$$

$$R(\alpha)=\frac{\left(\sum_{i=1}^{r}\ \sum_{m}\ n_{k_m^i}\right)!}{\prod_{i=1}^{r}\ \prod_{m}\ n_{k_m^i}!}$$

$$\left[\frac{k_1}{2}\right]\otimes\left[\frac{k_2}{2}\right]=\bigoplus_{k_3=0}^{\infty}\ N_{k_1k_2}^{k_3}\left[\frac{k_3}{2}\right]$$

$$\chi_{k_1}\chi_{k_2}=\sum_{k_3}\ N_{k_1k_2}^{k_3}\chi_{k_3}$$

$$\left\langle\chi_{k_i}\mid\chi_{k_j}\right\rangle_{SU(2)}:=\int_0^{2\pi}\frac{d\theta}{\pi}\sin^2\ \theta\chi_{k_i}\chi_{k_j}=\delta_{ij}$$

$$P_{GM}(\alpha)\ =\sum_{j=0}^{\infty}\left\langle\prod_{k}\ \chi_k^{n_k}\mid\chi_{k_j}\right\rangle_{SU(2)}$$

$$=\sum_{j=0}^{\infty}\ \int_0^{2\pi}\frac{d\theta}{\pi}\sin^2\ \theta\prod_{k}\ \left(\frac{\sin\ (k+1)\theta}{\sin\ \theta}\right)^{n_k}\frac{\sin\left(k_j+1\right)\theta}{\sin\ \theta}$$

$$C_{k+2}=XC_{k+1}-C_k,k=0,1,\ldots$$

$$C_k=U_k(X/2),k=0,1,\ldots$$

$$\left[\frac{k_1}{2}\right]\otimes\left[\frac{k_2}{2}\right]\otimes\cdots\otimes\left[\frac{k_N}{2}\right]=\bigoplus_{k=0}^{\infty}\ \left(C_{k_2}C_{k_3}\cdots C_{k_N}\right)_{k_1k}\left[\frac{k}{2}\right]$$

$$\sum_{k=0}^{\infty}\ \left(C_{k_2}C_{k_3}\cdots C_{k_N}\right)_{k_1k}$$

$$P_{GM}(\alpha)=\frac{2}{\pi}\int_0^{\pi}\ \mathrm{d}\theta\cos\ \frac{\theta}{2}\left[\cos\ \frac{\theta}{2}-\cos\ \left(K+\frac{3}{2}\right)\theta\right]\prod_{k}\ \left(\frac{\sin\ (k+1)\theta}{\sin\ \theta}\right)^{n_k}$$

$$\left\langle\eta_i\mid\eta_j\right\rangle_{U(1)}=\frac{1}{2\pi}\int_0^{2\pi}\ d\theta e^{-im_i\theta}e^{im_j\theta}=\delta_{ij}$$

$$P(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\ d\theta\prod_{k}\ \left(\frac{\sin\ (k+1)\theta}{\sin\ \theta}\right)^{n_k}$$

$$\prod_{k}\left(\sum_{\alpha=0}^{k}\ z^{k-2\alpha}\right)^{n_k}=\prod_{k}\ \left(\frac{z^{k+1}-z^{-k-1}}{z-z^{-1}}\right)^{n_k}$$

$$P_{GM}(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\ d\theta\prod_{k}\ \left(\frac{\sin\ (k+1)\theta}{\sin\ \theta}\right)^{n_k}$$

$$d_{GM}(A)=\sum_{\alpha\in\mathcal{S}_a}\ R(\alpha)P_{GM}(\alpha)$$

$$\Delta A_{DL} := \gamma_{DL}\chi_{DL}$$
$$\Delta A_{GM} := \gamma_{GM}\chi_{DL}$$
$$\sum_{I=0}^{N} m_I = 0, A(\vec{j}) := 2\sum_{I=1}^{N} \sqrt{j_I(j_I+1)} = A$$
$$R(A) = \sum_{\alpha\in\mathcal{S}_A} \frac{\left(\sum_i^r \sum_m n_{k_m^i}\right)!}{\prod_{i=1}^r \prod_m n_{k_m^i}!},$$
$$\sum_{m=1}^{\infty} n_{k_m^i} y_m^i = q_i, i = 1, \dots r$$
$$\sum_{\alpha\in\mathcal{S}_A} 1$$
$$\sum_{\alpha\in\mathcal{S}_A} \frac{1}{\prod_{i=1}^r \prod_m n_{k_m^i}!}$$
$$n_{k_1^i} y_1^i + n_{k_2^i} y_2^i + \cdots = q_i$$
$$f(x) = \left(x^{0\cdot y_1^i} + x^{1\cdot y_1^i} + x^{2\cdot y_1^i} + \cdots\right)\left(x^{0\cdot y_2^i} + x^{1\cdot y_2^i} + x^{2\cdot y_2^i} + \cdots\right)\dots,$$
$$f(x) = \frac{1}{1-x^{y_1^i}}\frac{1}{1-x^{y_2^i}}\dots = \prod_{m=1}^{\infty} \frac{1}{1-x^{y_m^i}}$$
$$f(x_1, x_2, \dots) = \prod_{i=1}^{\infty}\prod_{m=1}^{\infty} \frac{1}{1-x_i^{y_i^i}}$$
$$\sum_{\alpha\in\mathcal{S}_A} 1 = \left[x_1^{q_1}, x_2^{q_2}, \dots, x_r^{q_r}\right] f(x_1, x_2, \dots)$$
$$f(x) \longrightarrow g(x) = \left(\frac{1}{0!}x^{0\cdot y_1^i} + \frac{1}{1!}x^{1\cdot y_1^i} + \frac{1}{2!}x^{2\cdot y_1^i} + \cdots\right)\left(\frac{1}{0!}x^{0\cdot y_2^i} + \frac{1}{1!}x^{1\cdot y_2^i} + \cdots\right)\dots$$
$$g(x) = \prod_{m=1}^{\infty} e^{x^{y_m^i}}$$
$$\sum_{\alpha\in\mathcal{S}_A} \frac{1}{\prod_{i=1}^r \prod_m n_{k_m^i}!} = \left[x_1^{q_1}, x_2^{q_2}, \dots, x_r^{q_r}\right] g(x_1, x_2, \dots)$$
$$g(x_1, x_2, \dots) = \prod_{i=1}^{\infty}\prod_{m=1}^{\infty} e^{x^{y_m^i}} = \exp\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)$$
$$\exp\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right) = 1 + \left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right) + \frac{1}{2!}\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)^2 + \cdots,$$
$$G(x_1, x_2, \dots) = 0!\,1 + 1!\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right) + 2!\frac{1}{2!}\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)^2 +$$
$$= \frac{1}{1-\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}}$$
$$R(A) = \left[x_1^{q_1}, x_2^{q_2}, \dots, x_r^{q_r}\right] G(x_1, x_2, \dots)$$
$$\left[x_1^{q_1}, x_2^{q_2}, \dots, x_r^{q_r}\right] G_N(x_1, x_2, \dots),$$
$$G_N(x_1, x_2, \dots) = \left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)^N.$$

$$G(x_1,x_2,\dots)=\sum_{J=1}^{\infty} G_J(x_1,x_2,\dots)=1+\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)+\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)^2+\cdots=$$

$$=\frac{1}{\left(1-\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} x_i^{y_m^i}\right)}$$

$$\prod_{I=1}^{N}\left(z^{k_I}+z^{-k_I}\right),$$

$$\prod_{I=1}^{N}\sum_{\alpha=0}^{k_I} z^{k_I-2\alpha}=\prod_{I=1}^{N}\frac{z^{k_I+1}-z^{-k_I-1}}{z-z^{-1}}$$

$$G^{DL}(z,x_1,x_2,\dots)=\left(1-\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\left(z^{k_m^i}+z^{-k_m^i}\right)x_i^{y_m^i}\right)^{-1}$$

$$G^{GM}(z,x_1,x_2,\dots)=\left(1-\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{\alpha=0}^{k_m^i} z^{k_m^i-2\alpha}\right)x_i^{y_m^i}\right)^{-1}$$

$$d_{DL}(A)=\left[z^0,x_1^{q_1},x_2^{q_2},\dots,x_r^{q_r}\right]G^{DL}(x_1,x_2,\dots)$$

$$d_{GM}(A)=\left[z^0,x_1^{q_1},x_2^{q_2},\dots,x_r^{q_r}\right]G^{GM}(x_1,x_2,\dots)$$

$$G^{DL}(z,x_1,x_2)=$$

$$=\frac{1}{1-(z^2+z^{-2})x_1^2-(z^{16}+z^{-16})x_1^{12}-(z+z^{-1})x_2-(z^6+z^{-6})x_2^4-(z^{25}+z^{-25})x_2^{15}}.$$

$$d_{DL}=991809938488860909241077458398212$$

$$d_{DL}(A)=\sum_{n\in\mathbb{N}} d_{DL}(A_n)\delta(A-An)$$

$$\mathfrak{n}(A)=1+\int_0^A dA\, d_{DL}(A)$$

$$\mathcal{L}[\delta(A-A_n);s]=e^{-A_n s}$$

$$\mathcal{L}[d_{DL}(A);s]=\sum_{n\in\mathbb{N}} d_{DL}(A_n)e^{-A_n s}$$

$$d_{DL}(A)=\mathcal{L}^{-1}\left[\sum_{n\in\mathbb{N}} d_{DL}(A_n)e^{-A_n s};A\right]$$

$$\int_0^A F(A)dA=\mathcal{L}^{-1}\left[\frac{1}{s}f(s);A\right]$$

$$\int_0^A d_{DL}(A)dA=\mathcal{L}^{-1}\left[\frac{1}{s}\sum_{n\in\mathbb{N}} d_{DL}(A_N)e^{-A_n s};A\right]$$

$$G^{DL}\left(1,e^{-s\sqrt{p_1}},e^{-s\sqrt{p_2}},\dots\right)=\sum_{n\in\mathbb{N}} d_{DL}(A_N)e^{-A_n s}+1$$

$$x_1^{q_1},x_2^{q_2}\dots x_r^{q_r}\longrightarrow e^{-s\left(q_1\sqrt{p_1}+q_2\sqrt{p_2}+\cdots+q_r\sqrt{p_r}\right)}=e^{-sA_n}$$

$$G^{DL}\left(1,e^{-s\sqrt{p_1}},e^{-s\sqrt{p_2}},\dots\right)=\left(1-2\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} e^{-s y_m^i\sqrt{p_i}}\right)^{-1}$$

$$y_m^i\sqrt{p_i}=\sqrt{k_m^i\left(k_m^i+2\right)}$$

$$GDL\left(1,e^{-s\sqrt{p_1}},e^{-s\sqrt{p_2}},\dots\right)=\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i\left(k_m^i+2\right)}}\right)^{-1}$$

$$\int_0^A d_{DL}(A)dA = \mathcal{L}^{-1}\left[\frac{2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)};A\right]$$

$$\mathfrak{n}(A) = 1+\int_0^A d_{DL}(A)dA = \mathcal{L}^{-1}\left[\frac{1}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)};A\right]$$

$$[z^0]G_{DL}(z,x_1,x_2,\dots) = \frac{1}{4\pi}\int_0^{4\pi} dwG(e^{iw/2},x_1,x_2,\dots)$$

$$G(1,x_1,x_2,\dots) \longrightarrow [z^0]G(z,x_1,x_2,\dots).$$

$$\int_0^A d_{DL}(A)dA = \mathcal{L}^{-1}\left[\frac{1}{4\pi}\int_0^{4\pi} dw\frac{2\sum_{k=1}^{\infty} \cos(wk/2)e^{-s\sqrt{k_m^i(k_m^i+2)}}}{s\left(1-2\sum_{k=1}^{\infty} \cos(wk/2)e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)};A\right]$$

$$\mathfrak{n}(A) = \mathcal{L}^{-1}\left[\frac{1}{4\pi}\int_0^{4\pi} dw\frac{1}{s\left(1-2\sum_{k=1}^{\infty} \cos(wk/2)e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)};A\right]$$

$$\frac{1}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)}.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)^{-1};A\right] \sim \sum_{s_i} \operatorname{Res}\left[\frac{1}{s}\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k_m^i(k_m^i+2)}}\right)^{-1} e^{sa}\right]$$

$$f(s) = \frac{2}{s(e^{s/2}-3)}.$$

$$\mathcal{L}^{-1}[f(s);A] = 3^{\lfloor 2A\rfloor},$$

$$\mathcal{L}^{-1}[f(s);A] \propto 3^{2A}$$

$$\lambda_p : z \mapsto \operatorname{diag}(z^p, z^{-p}),$$

$$N^{\mathcal{P}} = \sum_{r_i} N_{j_1 j_2}^{r_1} N_{r_1 j_3}^{r_2} \dots N_{r_{N-2} j_{N-1}}^{j_N}$$

$$N^{\mathcal{P}} = \left\langle \chi_{j_1}\cdots\chi_{j_N} | \chi_0 \right\rangle_{SU(2)} = \int_0^{2\pi} \frac{d\theta}{\pi} \sin^2\theta \prod_{I=1}^{N} \frac{\sin[(j_I+1)\theta]}{\sin\theta}$$

$$N_{U(1)}^{\mathcal{P}} = \left\langle \tilde{\eta}_{p_1} \dots \tilde{\eta}_{p_N} \mid \eta_0 \right\rangle_{U(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \prod_{I}^{N} 2\cos p_I\theta$$

$$\gamma^\mu \nabla_\mu \psi = 0$$

$$\gamma^a V_a^i \partial_i \Phi + \frac{1}{r}\left[\frac{\gamma^2}{\sin^{1/2}\theta}\partial_\theta \sin^{1/2}\theta + \frac{\gamma^3}{\sin\theta}\partial_\phi\right]\Phi = 0$$

$$\partial_t \Phi = -\gamma^0\gamma^1\left[\partial_{r^*} + \frac{e^\rho}{r}\left(\frac{\gamma^2\gamma^1}{\sin^{1/2}\theta}\partial_\theta \sin^{1/2}\theta + \frac{\gamma^3\gamma^1}{\sin\theta}\partial_\phi\right)\right]\Phi$$

$$\partial_t \Phi = -\gamma^0\gamma^1\left[\partial_{r^*} - \frac{e^\rho}{r}\gamma^0 K\right]\Phi$$

$$K=\gamma^0\left(\frac{\gamma^1\gamma^2}{\sin^{1/2}\ \theta}\partial_\theta \sin^{1/2}\ \theta+\frac{\gamma^1\gamma^3}{\sin\ \theta}\partial_\phi\right)$$

$$\chi^+_{m_j\kappa_j}\ =\begin{bmatrix}\eta(\hat{r})^{m_j}_{\kappa_j}\\ 0\end{bmatrix}$$

$$\chi^-_{m_j\kappa_j}\ =\begin{bmatrix}0\\ \eta(\hat{r})^{m_j}_{-\kappa_j}\end{bmatrix}$$

$$\psi_{w\kappa_j m_j}(x)=\frac{e^{-\rho/2}e^{-iwt}}{r}\begin{bmatrix}G_{w\kappa_j}(r)\eta(\hat{r})^{m_j}_{\kappa_j}\\ -iF_{w\kappa_j}(r)\sigma^1\eta(\hat{r})^{m_j}_{\kappa_j}\end{bmatrix}$$

$$\partial_{r^*}G_{w\kappa_j}\ =-\frac{e^\rho}{r}\kappa_j G_{w\kappa_j}+wF_{w\kappa_j}$$

$$\partial_{r^*}F_{w\kappa_j}\ =\frac{e^\rho}{r}\kappa_j F_{w\kappa_j}-wG_{w\kappa_j}$$

$$\eta(\hat{r})^{m_j}_{\kappa_j<0}=\begin{bmatrix}\sqrt{\frac{j+m_j}{2j}}\\ \sqrt{\frac{j-m_j}{2j}}Y^{m_j-1/2}_{j-1/2}(\theta,\phi)\\ Y^{m_j+1/2}_{j-1/2}(\theta,\phi)\end{bmatrix}$$

$$\eta(\hat{r})^{m_j}_{\kappa_j>0}=\begin{bmatrix}\sqrt{\frac{j+1-m_j}{2j+2}}Y^{m_j-1/2}_{j+1/2}(\theta,\phi)\\ -\sqrt{\frac{j+1+m_j}{2j+2}}Y^{m_j+1/2}_{j+1/2}(\theta,\phi)\end{bmatrix}$$

$$\psi(x)=\sum_{\kappa_j m_j}\int\ dw\left[a_{w\kappa_j m_j}u_{w\kappa_j m_j}(x)+b^\dagger_{w\kappa_j m_j}v_{w\kappa_j m_j}(x)\right]$$

$$\int_{I^+}d\Omega du r^2\bar{u}^{out}\gamma_+u^{out}=\int_{I^-}d\Omega r^2 dv\bar{u}^{out}\gamma_-u^{out}$$

$$u^{out}(v)|_{I^+}=\sqrt{du(v)/dv}\Theta(v_H-v)u^{out}(u)\big|_{I^-}$$

$$\langle in|N_{i_1i_2}|in\rangle=\sum_k\int_{I^-}dv_2r_2^2d\Omega_2\left(\bar{u}^{out,L}_{i_2}(x_2)\frac{[\gamma^0-\gamma^1]}{2}v^{in,L}_k(x_2)\right)\times$$

$$\times\int_{I^-}dv_1r_1^2d\Omega_1\left(\bar{v}^{in,L}_k(x_1)\frac{[\gamma^0-\gamma^1]}{2}u^{out,L}_{i_1}(x_1)\right)$$

$$\int\ d\Omega_2\bar{u}^{\text{out},L}_{i_2}(x_2)\frac{[\gamma^0-\gamma^1]}{2}v^{\text{in,L}}_k(x_2)=\frac{t^*_{j2}(w_2)}{2\pi r_2^2}\sqrt{\frac{du(v)}{dv}}\Theta(v_H-v)\times$$

$$\times e^{iw_2u(v_2)+iwv_2}\delta_{m_{j_2}m_k}\delta_{j_2j_k}$$

$$\langle in|N_{i_1i_2}|in\rangle=\delta_{m_{j_1}m_{j_2}}\delta_{j_1j_2}\frac{t_{j_1}(w_1)t^*_{j_2}(w_2)}{4\pi^2}\int_{-\infty}^{v_H}dv_1dv_2$$

$$\sqrt{\frac{du(v_1)}{dv}\frac{du(v_2)}{dv}}e^{-iw_1u(v_1)+iw_2u(v_2)}\int_0^\infty\ dwe^{-iw(v_1-v_2)}$$

$$\int_0^\infty\ dwe^{-iw(v_1-v_2)}=\lim_{\epsilon\to0}\frac{-i}{(v_1-v_2-i\epsilon)}$$

$$ds^2=-dT^2+dW^2+dX^2+dY^2+dZ^2$$

$$H^{-1}\sinh H\tau = T$$
$$H^{-1}\cosh H\tau\cos \xi = W$$
$$H^{-1}\cosh Ht\sin \xi\cos \theta = X$$
$$H^{-1}\cosh Ht\sin \xi\cos \theta\cos \phi = Y$$
$$H^{-1}\cosh Ht\sin \xi\cos \theta\sin \phi = Z$$
$$ds^2 = -d\tau^2 + H^{-2}\cosh^2 H\tau[d\xi^2 + \sin^2 \xi(d\theta^2 + \sin^2 \theta)]$$
$$ds^2 = \frac{1}{\cos^2 \tilde{T}}\left(-d\tilde{T}^2 + d\Omega_3^2\right)$$
$$T = H^{-1}\sinh Ht + \frac{1}{2}He^{Ht}|\vec{x}|^2, X = e^{Ht}x, Y = e^{Ht}y$$
$$Z = e^{Ht}z, W = H^{-1}\cosh Ht - \frac{1}{2}He^{Ht}|\vec{x}|^2$$
$$ds^2 = -dt^2 + e^{2Ht}d\vec{x}^2, -\infty < t, x, y, z < \infty$$
$$T = (H^{-2} - \tilde{r})^{1/2}\sinh H\tilde{t}$$
$$X = (H^{-2} - \tilde{r})^{1/2}\cosh (H\tilde{t})$$
$$Y = \tilde{r}\sin \tilde{\theta}\cos \tilde{\phi}$$
$$Z = \tilde{r}\sin \tilde{\theta}\sin \tilde{\phi}$$
$$W = \tilde{r}\cos \tilde{\theta}$$
$$ds^2 = -(1 - \tilde{r}^2H^2)d\tilde{t}^2 + \frac{1}{(1 - \tilde{r}^2H^2)}d\tilde{r}^2 + \tilde{r}^2\left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta}d\tilde{\phi}\right)$$
$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$
$$\Box \phi - m^2\phi - \xi R\phi = 0$$
$$\phi(t, x, y, z) = \frac{1}{(2\pi)^{3/2}}\int d^3k u_k(t)e^{i\vec{k}\vec{x}}$$
$$\partial_t^2 u_k(t) + 3H\partial_t u_k(t) + \left(\frac{k^2}{a^2} + m^2 + \xi R\right)u_k(t) = 0$$
$$u_k(t) = a^{-3/2}v_k(t)$$
$$\partial_t^2 v_k(t) + \left(-\frac{9}{4}H^2 + \frac{k^2}{a^2(t)} + m^2 + 12\xi H^2\right)v_k = 0$$
$$ds^2 = \frac{1}{H\eta^2}(-d\eta^2 + d\vec{x}^2)$$
$$\eta^2\partial_\eta^2 v_k(\eta) + \eta\partial_\eta v_k(\eta) + (k^2\eta^2 - \nu^2)v_k(\eta) = 0$$
$$v_k(k\eta) = AJ_\nu(k\eta) + BY_\nu(k\eta)$$
$$v_k(\eta k) \longrightarrow \frac{1}{\sqrt{2\pi k}}e^{ik\eta}$$
$$v_k(k\eta) = \frac{1}{2}\sqrt{\frac{\pi}{H}}(J_\nu(k\eta) + iY_\nu(k\eta)) = \frac{1}{2}\sqrt{\frac{\pi}{2H}}H_\nu^{(1)}(k\eta)$$
$$u_{\vec{k}}^{dS} = \frac{1}{\sqrt{(2\pi)^3a(t)^3}}v_k(t)$$
$$G_{dS}(x, x') = \hbar\frac{H^2(1/4 - \nu)}{16\pi^2\cos \pi\nu}\,{}_2F_1\left[3/2 + \nu, 3/2 - \nu; 2; \left(1 + \cos \sqrt{H^2\sigma}\right)/2\right]$$
$$G_{dS}(x, x') = \frac{\hbar^2H^2\eta\eta'}{4\pi^2[-(\eta - \eta' - i\epsilon)^2 + (\vec{x} - \vec{x}')^2]}$$
$$ds^2 = -(1 - \tilde{r}^2H^2)d\tilde{t}^2 + \frac{1}{(1 - \tilde{r}^2H^2)}d\tilde{r}^2 + \tilde{r}^2d\Omega^2$$
$$\phi(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi}) = \frac{1}{\sqrt{4\pi}}\frac{1}{\tilde{r}}\sum_{l,m}\int dw u_{w,l}^C(\tilde{r}, \tilde{t})Y_{l,m}(\tilde{\theta}, \tilde{\phi})$$
$$(1 - \tilde{r}^2H^2)\partial_{\tilde{r}}^2 u_{w,l}^C(\tilde{r}) - 2H^2\tilde{r}\partial_{\tilde{r}} u_{w,l}^C(\tilde{r}) +$$
$$+\left(2H - \frac{l(l+1)}{\tilde{r}^2} - m^2 - 12\xi H^2 + \frac{w^2}{1 - \tilde{r}^2H^2}\right)u_{w,l}^C(\tilde{r}) = 0$$

$$\partial_{r^*}^2 u_{w,l}^C(r^*) + V(r^*)u_{w,l}^C(r^*) = -w^2 u_{w,l}^C(r^*)$$

$$u_{w,l=0}^C(r^*) = N_\nu(w)\left[P_{\nu-1/2}^{i\frac{w}{H}}(\tanh\ Hr^*) - \alpha_\nu(w)Q_{\nu-1/2}^{i\frac{w}{H}}(\tanh\ Hr^*)\right]$$

$$\alpha_\nu(w) = \frac{1}{2}\left[2i + 4\left(i + e^{-i\pi\nu}e^{\pi w/H}\right)^{-1}\right]$$

$$P_\nu^{i\frac{w}{H}}(\tanh\ Hz) \sim \frac{1}{\Gamma\left[1-\frac{iw}{H}\right]}e^{iwz}$$

$$Q_\nu^{i\frac{w}{H}}(\tanh\ Hz) \sim A(w)e^{iwx} + B(w)e^{-iwz}$$

$$A(w) = \frac{-i\pi}{4\Gamma(1-iw/H)}\left(\coth\frac{\pi w}{2H} + \tanh\ \frac{\pi w}{2H}\right)$$

$$B(w) = \frac{-\pi}{4\Gamma(1-iw/H)}\frac{\coth\frac{\pi w}{2H}}{\sinh^2\frac{\pi w}{2H}}$$

$$\left(u_{w_1,l=0}^C(v), u_{w_2,l=0}^C(v)\right)$$

$$= -i\int_{-\infty}^{\infty} dv d\tilde{\Omega}\tilde{r}^2\left(u_{w_1,l=0}^C(v)\partial_v u_{w_2,l=0}^{C*}(v) - u_{w_2,l=0}^{C*}(v)\partial_v u_{w_1,l=0}^C(v)\right)$$

$$= N_\nu^2(w_1 + w^2)\left|\Gamma^{-1}\left(1-\frac{iw}{H}\right) - \alpha_\nu(w)A(w)\right|^2 \int_{\infty}^{\infty} dv e^{-i(w_i - w_2)v}$$

$$= \delta(w_1 - w_2)$$

$$\left|\tilde{N}_n(w/H)\right|^2 = \frac{1}{4\pi}\frac{|\Gamma(1-iw/H)|^2}{\left|1+\frac{i\pi}{4}\alpha_n(w)\left(\coth\frac{\pi w}{2H} + \tanh\ \frac{\pi w}{2H}\right)\right|^2}$$

$$u_{w,l=0}^C(\tilde{t},\tilde{r}) = \frac{1}{\sqrt{4\pi w}}\frac{1}{\tilde{r}}\tilde{N}_\nu(w/H)e^{-iw\tilde{t}}\left[P_{\nu-1/2}^{i\frac{w}{H}}(H\tilde{r}) - \alpha_\nu(w)Q_{\nu-1/2}^{i\frac{w}{H}}(H\tilde{r})\right]$$

$$u_{w,l=0}^C(r^*) = N\sin\ wr^*$$

$$\left(u_{w_1,l=0}^C(v), u_{w_2,l=0}^C(v)\right)$$

$$= -i\int_{-\infty}^{\infty} dv d\tilde{\Omega}\tilde{r}^2\left(u_{w_1,l=0}^C(v)\partial_v u_{w_2,l=0}^{C*}(v) - u_{w_2,l=0}^{C*}(v)\partial_v u_{w_1,l=0}^C(v)\right)$$

$$= N\frac{(w_1 + w^2)}{4}\int_{\infty}^{\infty} dv e^{-i(w_i - w_2)v} = \delta(w_1 - w_2)$$

$$\langle 0_C(x)|\phi(x)\phi(x+\Delta(x))|0_C(x)\rangle$$

$$= \lim_{\epsilon\to 0}\frac{\hbar}{4\pi^2}\int_0^{\infty} dw e^{-iw(\Delta t - i\epsilon)}\frac{1}{\tilde{r}}\sin\ \left[\frac{w}{H}\tanh^{-1}(\tilde{r}H)\right]$$

$$= \lim_{\epsilon\to 0}\frac{\hbar}{4\pi^2}\frac{1}{\tilde{r}}\frac{\chi(\tilde{r})}{\chi(\tilde{r})^2 - (\Delta\tilde{t} - i\epsilon)^2}$$

$$\langle 0_C(x)|\phi(x)\phi(x+\Delta(x))|0_C(x)\rangle = \frac{\hbar}{4\pi^2[(\Delta\tilde{t} - i\epsilon)^2 - \tilde{r}^2]}$$

$$\langle in|N_{j_1n_1,j_2n_2}^{out,\sigma}|in\rangle = \int_0^{\Lambda} dw'\beta_{j_1n_1,w'}\beta_{j_2n_2,w'}^* =$$

$$\frac{1}{\epsilon}\int_{j_1\epsilon}^{(j_1+1)\epsilon} dw_1\int_{j_2\epsilon}^{(j_2+1)\epsilon} dw_2 e^{2\pi i w_1 n_1/\epsilon}e^{-2\pi i w_2 n_2/\epsilon}\int_0^{\Lambda} dw'\beta_{w_1w'}\beta_{w_2w'}^*$$

$$\langle in|N_{j_1n_1,j_2n_2}^{out,\sigma}|in\rangle = \frac{1}{\epsilon}\int_{j_1\epsilon}^{(j_1+1)\epsilon} dw_1\int_{j_2\epsilon}^{(j_2+1)\epsilon} dw_2 e^{i\frac{2\pi w_1 n_1}{\epsilon}}e^{-i\frac{2\pi w_2 n_2}{\epsilon}}$$

$$t_l(w_1)t_l^*(w_2)\frac{e^{-i(w_1-w_2)v_H}}{2\pi\sqrt{w_1w_2}}e^{-\pi\kappa^{-1}\omega_1}i^{-i\kappa^{-1}(w_1-w_2)}$$

$$\Gamma(1+i\kappa^{-1}w_1)\Gamma(1-i\kappa^{-1}w_2)\delta_\sigma[\kappa^{-1}(w_1-w_2)].$$

$$\langle in|N^{out,\sigma}_{j_1n_1,j_2n_2}|in\rangle \approx \delta_{j_1j_2}\frac{\left|t_l(w_j)\right|^2\left|\Gamma(1+i\kappa^{-1}w_j)\right|^2}{2\pi w_j}e^{-\pi\kappa^{-1}w_j}e^{\frac{2\pi(n_1-n_2)w_j}{\epsilon}}I_{n_1n_2}(\sigma)$$

$$I_{n_1n_2}(\sigma)=\frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2}dx_1\int_{-\epsilon/2}^{\epsilon/2}dx_2e^{i\left[\frac{2\pi n_1}{\epsilon}-\nu_H\right]x_1-i\left[\frac{2\pi n_2}{\epsilon}-\nu_H\right]x_2-\pi\kappa^{-1}(x_1+x_2)/2}\delta_\sigma[\kappa^{-1}(x_1-x_2)]$$

$$I_{n_1n_2}(\sigma)\approx\frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2}dx_1e^{\frac{2\pi(n_1-n_2)x_1}{\epsilon}}\int_{x_1-\epsilon/2}^{x_1+\epsilon/2}dye^{i\left[\frac{2\pi n_2}{\epsilon}-\nu_H\right]y}\delta_\sigma[\kappa^{-1}y]$$

$$I_{n_1n_2}(\sigma)\approx\kappa\delta_{n_1n_2}\frac{\sin\left[\left(\frac{2\pi n_2}{\epsilon}-\nu_H\right)\frac{\pi\kappa\sigma}{2}\right]}{\left[\left(\frac{2\pi n_2}{\epsilon}-\nu_H\right)\frac{\pi\kappa\sigma}{2}\right]}$$

$$\int_{-\infty}^{\infty}d\log[w/\kappa]e^{-i\kappa^{-1}(w_1-w_2)\log[w/\kappa]}\rightarrow\int_{-\log[\Lambda/\kappa]}^{\log[\Lambda/\kappa]}d\log[w/\kappa]e^{-i\kappa^{-1}(w_1-w_2)\log[w/\kappa]}$$

$$\int_{-\log[\Lambda/\kappa]}^{\log[\Lambda/\kappa]}d\lambda e^{-i\kappa^{-1}(w_1-w_2)\lambda}\rightarrow\int_{-\infty}^{\infty}d\lambda e^{-i\kappa^{-1}(w_1-w_2)\lambda}e^{-(\lambda/\tilde{\Lambda})^2}$$

$$\delta_{\tilde{\sigma}}[\kappa^{-1}(w_1-w_2)]=\frac{\exp\left[\frac{\kappa^{-1}(w_1-w_2)}{2\tilde{\sigma}}\right]^2}{2\tilde{\sigma}\sqrt{\pi}}$$

$$I_{n_1n_2}(\tilde{\sigma})=\kappa\delta_{n_1n_2}e^{-\left[\left(\frac{2\pi n_2}{\epsilon}-\nu_H\right)\kappa\tilde{\sigma}\right]^2}$$

$$\langle in|N^{out,\sigma}_{j_1n_1,j_2n_2}|in\rangle=\delta_{j_1j_2}\delta_{n_1n_2}\frac{\left|t_l(w_j)\right|^2}{e^{2\pi\kappa^{-1}w_j}-1}e^{-\left[\left(\frac{2\pi n_2}{\epsilon}-\nu_H\right)\kappa\tilde{\sigma}\right]^2}$$

$$I=\int_{-\infty}^{\infty}dze^{-iwz}\left[\frac{(k/2)^2}{\sinh^2 kz/2+(k/2)^2\alpha^2}\right]$$

$$I_1=\int_0^{\infty}dx\frac{x^{-iw/k}}{(x-1)^2+k^2\alpha^2x}$$

$$I_1=\frac{2\pi i}{1-e^{2\pi w/k}}[\mathrm{Res}[a+ib]+\mathrm{Res}[a-ib]]$$

$$\mathrm{Res}[a\pm ib]=\pm\frac{(a\pm ib)^{-iw/k}}{2ib}$$

$$(a+ib)=e^{w/k\theta}$$

$$(a-ib)=\left(e^{w/k\theta}-e^{w/k(2\pi-\theta)}\right)$$

$$I_1=\frac{\pi}{1-e^{2\pi w/k}}\frac{\left(e^{w/k\theta}-e^{w/k(2\pi-\theta)}\right)}{\alpha\sqrt{1-\left(\frac{k\alpha}{2}\right)^2}}.$$

$$I_2=-\int_{-\infty}^{\infty}dze^{-iwz}\frac{1}{z^2+\alpha^2}$$

$$I_2=(-2\pi i\mathrm{Res}[i\alpha])^*,$$

$$I_2=-\frac{\pi}{\alpha}e^{-w\alpha}$$

$$I=\frac{1}{(1-e^{2\pi w/k})}\frac{\pi}{\alpha}\left[\frac{\left(e^{w/k\theta}-e^{w/k(2\pi-\theta)}\right)}{\sqrt{1-\left(\frac{k\alpha}{2}\right)^2}}+\left(e^{2\pi w/k}-1\right)e^{-w\alpha}\right]$$

$$I=w\left[\frac{we^{\pi w/k}}{(1-e^{2\pi w/k})}\frac{\sinh\left[\frac{w}{a}(\beta-\pi)\right]}{\frac{w}{a}\sin\beta}-\frac{\pi e^{-w\alpha}}{\alpha}\right]$$

$$\frac{1}{w_k(t)}+(W_k(t)^{-1})^{(2)}$$

$$(W_k(t)^{-1})^{(2)} = \frac{m^2 a'^2}{2a^2 w^5} + \frac{m^2 a''}{4aw^5} - \frac{5m^4 a'^2}{8a^2 w^7} + \frac{a'^2}{2a^2 w^3} + \frac{a''}{2aw^3}$$

$$a' = \frac{\dot{a}}{a}$$

$$a'' = -\frac{\dot{a}^2}{a^3} + \frac{\ddot{a}}{a^2}$$

$$(W_k(t)^{-1})^{(2)} = \frac{m^2 \dot{a}^2 a}{2\tilde{w}^5} - \frac{m^2 \dot{a} a}{4\tilde{w}^5} + \frac{m^2 \ddot{a} a^2}{4\tilde{w}^5} - \frac{5m^4 \dot{a}^2 a^3}{8\tilde{w}^7} + \frac{1}{2}\frac{\ddot{a}}{\tilde{w}^3}$$

$$a = -\frac{1+\epsilon}{H\tau}, \frac{\dot{a}}{a} = -\frac{1+\epsilon}{\tau}, \frac{\ddot{a}}{a} = \frac{2+3\epsilon}{\tau^2},$$

$$(W_k(t)^{-1})^{(2)} = \frac{a^3 m^2}{\tilde{w}^5}\left(\frac{3+5\epsilon}{4\tau^2}\right) - \frac{5}{8}\frac{m^4}{\tilde{w}^7}\frac{1+2\epsilon}{\tau^2} a^5 + \frac{a}{2\tilde{w}^3}\frac{2+3\epsilon}{\tau^2}$$

$$\tilde{w} = a\sqrt{\frac{k}{a^2} + m^2} = \frac{1}{\tau}\sqrt{(1+\epsilon)^2 + \frac{m^2}{H^2}(1+\epsilon)} \approx \frac{1}{\tau}\sqrt{1+2\epsilon + \frac{m^2}{H^2}}$$

$$w_k^{-1} + (W_k(t)^{-1})^{(2)} \approx \frac{a^3 m^2}{\tilde{w}^5}\frac{3}{4\tau^2} + \frac{a}{2\tilde{w}^3}\frac{2+3\epsilon}{\tau^2} + \frac{a}{\tilde{w}} \approx$$

$$\approx -\frac{3}{4}\frac{m^2}{H^3}\left(1 - 5\epsilon - \frac{5}{2}\frac{m^2}{H^2}\right) - \frac{2+5\epsilon}{2H}\left(1 - 3\epsilon - \frac{3}{2}\frac{m^2}{H^2}\right)$$

$$\approx \frac{1}{H}\left(-2 + \frac{\epsilon}{2} + \frac{5}{4}\frac{m^2}{H^2}\right)$$

Para un espacio – tiempo AdS, tenemos:

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} + \eta_{\sigma\nu} M_{\rho\mu}$$

$$i[P_\mu, M_{\sigma\rho}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho$$

$$[P_\mu, P_\nu] = 0$$

$$i[D, P_\mu] = P_\mu$$

$$[M_{\mu\nu}, D] = 0$$

$$K_\mu : x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}$$

$$i[M_{\mu\nu}, K_\rho] = \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu$$

$$[D, K_\mu] = iK_\mu$$

$$[P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu} D)$$

$$[K_\mu, K_\nu] = 0$$

$$J_{\mu\nu} = M_{\mu\nu}, J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu), J_{(d+1)d} = D$$

$$J_{ab} = \begin{pmatrix} J_{\mu\nu} & J_{\mu d} & J_{\mu(d+1)} \\ -J_{\mu d} & 0 & D \\ -J_{\mu(d+1)} & -D & 0 \end{pmatrix}$$

$$P_\mu : x_\mu \to x_\mu + a_\mu \Rightarrow d$$

$$M_{\mu\nu} : x_\mu \to \Lambda_\mu^\nu x_\nu \Rightarrow \frac{d(d-1)}{2}$$

$$D : x^\mu \to \lambda x^\mu \Rightarrow 1$$

$$K_\mu : x_\mu \to \frac{x_\mu + a_\mu x^2}{1 + 2x_\nu a^\nu + a^2 x^2} \Rightarrow d$$

$$x \to \lambda x \Rightarrow \phi(x) \to \phi(x)' = \lambda^\Delta \phi(\lambda x)$$

$$[D, P_\mu] = -iP_\mu \Rightarrow D(P_\mu \phi) = -i(\Delta + 1)(P_\mu \phi)$$

$$\langle\phi(0)\phi(x)\rangle \equiv \frac{1}{(x^2)^{\Delta}}$$
$$ds^2 = dX^2 + dY^2 + dZ^2$$
$$X^2 + Y^2 + Z^2 = L^2$$
$$ds^2 = L^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$
$$ds^2 = dZ^2 + dX^2 + dY^2$$
$$-Z^2 + X^2 + Y^2 = -L^2$$

$$ds^2 = -dZ^2 + dX^2 + dY^2$$
$$-Z^2 + X^2 + Y^2 = -L^2$$
$$X = L\sinh\ \rho\cos\ \varphi, Y = L\sinh\ \rho\sin\ \varphi, Z = L\cosh\ \rho$$
$$ds^2 = L^2(d\rho^2 + \sinh^2\ \rho d\varphi^2)$$
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}$$
$$R_{\mu\nu\theta\sigma} = -\frac{1}{l^2}(g_{\mu\theta}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\theta}), R_{\mu\nu} = -\frac{3}{l^2}g_{\mu\nu}, R = -\frac{12}{l^2}$$
$$ds^2 = -dX_0^2 - dX_4^2 + dX_1^2 + dX_2^2 + dX_3^2$$
$$-X_0^2 - X_4^2 + X_1^2 + X_2^2 + X_3^2 = -l^2$$
$$X_0 = l\cosh\ \rho\cos\ \tau$$
$$X_4 = l\cosh\ \rho\sin\ \tau$$
$$X_1 = l\sinh\ \rho\sin\ \theta\sin\ \varphi$$
$$X_2 = l\sinh\ \rho\sin\ \theta\cos\ \varphi$$
$$X_3 = l\sinh\ \rho\cos\ \theta$$
$$\frac{ds^2}{l^2} = -\cosh^2\ \rho d\tau^2 + d\rho^2 + \sinh^2\ \rho(d\theta^2 + \sin^2\ \theta d\varphi^2)$$
$$\frac{ds^2}{l^2} \approx -d\tau^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$
$$r := \sinh\ \rho, t := l\tau$$
$$ds^2 = \left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$
$$\frac{ds^2}{l^2} = \frac{1}{\cos^2\ \chi}[-d\tau^2 + d\chi^2 + \sin^2\ \chi(d\theta^2 + \sin^2\ \theta d\varphi^2)]$$
$$X_0 = \frac{lr}{2}\left(\vec{x}_i^2 - t^2 + \frac{1}{r^2} + 1\right), \quad X_i = lrx_i\ (i = 1,2),$$
$$X_3 = \frac{lr}{2}\left(\vec{x}_i^2 - t^2 + \frac{1}{r^2} - 1\right), \quad X_4 = lrt,$$
$$\frac{ds^2}{l^2} = r^2(-dt^2 + d\vec{x}^2) + \frac{dr^2}{r^2}$$
$$ds^2 = \frac{l^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$$
$$Z_{\text{gauge}} = Z_{\text{string}}$$
$$\mathcal{L} = -\frac{N}{2g_{YM}^2}F_{\mu\nu}^M F_M^{\mu\nu}$$
$$Z_{CFT} = e^{-W}$$
$$\frac{l^4}{l_s^4} \sim g_{YM}^2 N \sim g_s N \gg 1$$
$$Z_{\text{cuerdas}} \approx e^{-I_{SUGRA}^E}$$
$$Z_{\text{cuerdas}} \approx e^{-I_{SUGRA}^E} = e^{-W} = Z_{CFT}$$
$$\left\langle e^{\int d^3x\phi_0(\vec{x})\mathcal{O}(\vec{x})}\right\rangle_{CFT} = e^{-I_{bulk}^E[\phi|_{\partial AdS}\to\phi_0]}$$
$$\left\langle e^{\int d^3x h_{ab}^0 T^{ab}}\right\rangle_{CFT} = e^{-I_{bulk}^E\left[h_{\mu\nu}|_{\partial AdS}\to h_{ab}^0\right]}$$

$$\left(\nabla^\mu\nabla_\mu - m^2\right)\Phi(z,x^n) = 0$$
$$\Phi(x^n,z) = \int \frac{d^2\vec{k}}{(2\pi)^2} d\omega f_k(z) e^{ik_\mu x^\mu}$$
$$\frac{d^2 f_k}{dz^2} - \frac{2}{z}\frac{df_k}{dz} - \left(k^2 + \frac{m^2 l^2}{z^2}\right) f_k = 0$$
$$f_k(z) = a_1 z^{3/2} K_\nu(kz) + a_2 z^{3/2} I_\nu(kz),$$
$$\frac{d^2 f_k}{dz^2} - k^2 f_k = 0$$
$$f_k(z) = a_1 z^{3/2} K_\nu(kz), K_\nu(kz) = \frac{\pi}{2}\frac{I_{-\nu}(kz) - I_\nu(kz)}{\sin(\nu\pi)}.$$
$$f_k(z) = a_1 z^{3/2} \frac{\pi}{2\sin \pi\nu}\left[\frac{1}{\Gamma(1-\nu)}\left(\frac{kz}{2}\right)^{-\nu} - \frac{1}{\Gamma(1+\nu)}\left(\frac{kz}{2}\right)^{\nu}\right]$$
$$f_k(z) = \phi_0 z^{\Delta_-} + \phi_1 z^{\Delta_+}$$
$$\Delta_\pm(\Delta_\pm - 3) - ml^2 = z^2k^2 \overset{z=0}{\rightarrow} \Delta_\pm(\Delta_\pm - 3) - ml^2 = 0,$$
$$\Delta_\pm = \frac{3}{2} \pm \nu, \nu = \sqrt{\frac{9}{4} + m^2 l^2}$$
$$m_{BF}^2 \leq m^2 < m_{BF}^2 + \frac{1}{l^2} \Rightarrow 0 \leq \nu < 1$$
$$\phi(r,x^n) = \frac{\alpha(x^n)}{r^{3-\Delta}} + \frac{\beta(x^n)}{r^\Delta} + \cdots,$$
$$\Delta = \frac{3}{2} + \nu, \nu = \sqrt{\frac{9}{4} + m^2 l^2}$$
$$\phi = \frac{\alpha}{r} + \frac{\beta}{r^2} + \cdots$$
$$\partial_r\phi = -\frac{\alpha}{r^2} - \frac{2\beta}{r^3} + \cdots = \frac{\alpha'}{r^2} + \cdots$$
$$a\phi + b\partial_r\phi = a\left(\frac{\alpha}{r} + \frac{\beta}{r^2}\right) + b\left(-\frac{\alpha}{r^2} - \frac{2\beta}{r^3}\right) = \frac{\alpha'}{r} + \frac{\beta'}{r^2} + \cdots$$
$$I_{CFT} \rightarrow I_{CFT} + p\int d^3x\mathcal{O}(x)$$
$$I = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + I_B$$
$$\delta I = \frac{1}{2\kappa}\int d^4x\sqrt{-g}G_{\alpha\beta}\delta g^{\alpha\beta} + \int d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} + \delta I_B$$
$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$
$$\delta I_B = -\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} = -\oint_{\partial\mathcal{M}}\epsilon v^\mu n_\nu\sqrt{-h}d^3x$$
$$g^{\alpha\beta}\delta R_{\alpha\beta} = v^\mu_{;\mu}, v^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta}, \epsilon = n^\mu n_\mu = \pm 1$$
$$I_B = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}K$$
$$I = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K$$
$$ds^2 = -N(r)dt^2 + H(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$
$$\text{Krets} = R^{\alpha\beta\gamma\sigma}R_{\alpha\beta\gamma\sigma}$$
$$I[g_{\mu\nu}] = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3xK\sqrt{-h}$$
$$R_{\mu\nu} = 0$$

$$ds^2 = -\left(1-\frac{\mu}{r}\right)dt^2 + \left(1-\frac{\mu}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\ \theta d\phi^2)$$
$$M = \frac{4\pi\mu}{\kappa} = \frac{\mu}{2G}$$
$$I[g_{\mu\nu}, A_\mu] = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3xK\sqrt{-h}$$
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}T^{EM}_{\mu\nu}$$
$$\nabla_\mu F^{\mu\nu} = 0$$
$$T^{EM}_{\mu\nu} = F_{\mu\alpha}F_\nu^{\cdot\alpha} - \frac{1}{4}g_{\mu\nu}F^2$$
$$A \equiv A_\mu dx^\mu = \left(\frac{q}{r} - \frac{q}{r_+}\right)dt, F = -\frac{q}{r^2}dr \wedge dt$$
$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$
$$f(r) = 1 - \frac{\mu}{r} + \frac{q^2}{4r^2} = \frac{(r-r_-)(r-r_+)}{r^2}$$
$$M = \frac{4\pi\mu}{\kappa} = \frac{\mu}{2G}$$
$$Q \equiv \frac{1}{\kappa}\oint\ d^2 \star F = -\frac{q}{4G}$$
$$r_\pm = G\left(M \pm \sqrt{M^2 - 4Q^2}\right)$$
$$\Phi = A_t|_{r=\infty} - A_t|_{r=r_+} = \frac{4GQ}{r^+}$$
$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + l^2 d\Sigma_k^2\right)$$
$$ds^2 = -N(r)dt^2 + H(r)dr^2 + S(r)d\Sigma_k^2$$
$$\begin{cases} d\theta^2 + \sin^2\ \theta d\varphi^2 & \text{for } k = +1 \\ \frac{1}{l^2}\sum_{i=1}^{2} dx_i^2 & \text{for } k = 0 \\ d\theta^2 + \sinh^2\ \theta d\varphi^2 & \text{for } k = -1 \end{cases}$$
$$\mathbb{R}\times H^2, \mathbb{R}\times\mathbb{R}^2, \mathbb{R}\times S^2$$
$$d\Sigma_k^2 = \frac{dy^2}{1-ky^2} + (1-ky^2)dz^2$$
$$I[g_{\mu\nu}] = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3xK\sqrt{-h}$$
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}$$
$$ds^2 = -\left(k - \frac{\mu}{r} + \frac{r^2}{l^2}\right)dt^2 + \left(k - \frac{\mu}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Sigma_k^2$$
$$k - \frac{\mu}{r_h} + \frac{r_h^2}{l^2} = 0$$
$$M = \frac{\sigma_k\mu}{\kappa}$$
$$I[g_{\mu\nu}, \phi] = \int\ d^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$
$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{\partial V}{\partial\phi} = 0$$
$$\left.\frac{dV}{d\phi}\right|_{\phi=0} = 0, V(0) = -\frac{3}{\kappa l^2}, \left.\frac{d^2V}{d\phi^2}\right|_{\phi=0} < 0$$

$$V(\phi)_{AdS}=\left(-\frac{1}{l^2}+\alpha\phi\right)(4+2\cosh\,\phi)-6\alpha\sinh\,\phi$$
$$V(\phi)_{flat}=2\alpha\phi(2+\cosh\,\phi)-6\alpha\sinh\,\phi$$
$$E_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\kappa T_{\mu\nu}^{\phi}$$
$$T_{\mu\nu}^{\phi}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2+V(\phi)\right]$$
$$ds^2=\Omega(x)\left[-f(x)dt^2+\frac{\eta^2dx^2}{f(x)}+\frac{dy^2}{l^2}+\frac{dz^2}{l^2}\right]$$
$$E_t^t-E_x^x=0\rightarrow\phi'^2=\frac{3\Omega'^2-2\Omega''\Omega}{\Omega^2}$$
$$E_t^t-E_y^y=0\rightarrow f''+\frac{\Omega'f'}{\Omega}=0$$
$$E_t^t+E_y^y=0\rightarrow V(\phi)=-\frac{1}{\Omega^2\eta^2}(f\Omega''+f'\Omega')$$
$$\Omega(x)=\frac{\nu^2x^{\nu-1}}{\eta^2(x^\nu-1)^2}$$
$$\phi'^2=\frac{(\nu-1)^2}{x^2}-\frac{4\nu(\nu-1)x^{\nu-2}}{x^\nu-1}+\frac{4\nu^2x^{\nu-1}}{(x^\nu-1)^2}+\frac{2(\nu-1)}{x^2}+\frac{4\nu(1-\nu-x^\nu)x^{\nu-2}x^{\nu-2}}{(x^\nu-1)^2}$$
$$\phi'^2=\frac{\nu^2-1}{2\kappa x^2}\rightarrow\int_\phi^{\phi=0}d\phi=\sqrt{\frac{\nu^2-1}{2\kappa}\int_x^1\frac{dx}{x}}$$
$$\phi(x)=l_\nu^{-1}\ln\,x,l_\nu^{-1}=\sqrt{\frac{\nu^2-1}{2\kappa}}$$
$$(f'\Omega)'\ =0$$
$$f(x)\ =\frac{c_2\eta^2}{\nu^2}\int\frac{(x^\nu-1)^2}{x^{\nu-1}}dx+c_1$$
$$f(x)\ =c_1+\frac{c_2\eta^2}{\nu^2}\left(\frac{x^{2+\nu}}{2+\nu}+\frac{x^{2-\nu}}{2-\nu}-x^2\right)$$
$$f(x)=\frac{1}{l^2}+\alpha\left[\frac{1}{\nu^2-4}-\frac{x^2}{\nu^2}\left(1+\frac{x^{-\nu}}{\nu-2}-\frac{x^\nu}{\nu+2}\right)\right]$$
$$V(\phi)\ =\frac{\Lambda(\nu^2-4)}{6\kappa\nu^2}\left[\frac{\nu-1}{\nu+2}e^{-\phi l_\nu(\nu+1)}+\frac{\nu+1}{\nu-2}e^{\phi l_\nu(\nu-1)}+4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right]$$
$$+\frac{\alpha}{\kappa\nu^2}\left[\frac{\nu-1}{\nu+2}\sinh\,\phi l_\nu(\nu+1)-\frac{\nu+1}{\nu-2}\sinh\,\phi l_\nu(\nu-1)+4\frac{\nu^2-1}{\nu^2-4}\sinh\,\phi l_\nu\right]$$
$$V(\phi)=\frac{\Lambda}{\kappa}-\frac{\phi^2}{l^2}+\frac{\kappa\Lambda}{18}\frac{(\nu^2-3)}{\nu^2-1}\phi^4-\frac{l_\nu^3}{90}(\Lambda\nu^2-4\Lambda-6\alpha)\phi^5+O(\phi^6)$$
$$V(0)=\frac{\Lambda}{\kappa},\left.\frac{dV}{d\phi}\right|_{\phi=0}=0,\left.\frac{d^2V}{d\phi^2}\right|_{\phi=0}=-\frac{2}{l^2}$$
$$ds^2=\Omega(x)\left[-f(x)dt^2+\frac{\eta^2dx^2}{f(x)}+d\theta^2+\sin^2\,\theta d\varphi^2\right]$$
$$f(x)=\frac{1}{l^2}+\alpha\left[\frac{1}{\nu^2-4}-\frac{x^2}{\nu^2}\left(1+\frac{x^{-\nu}}{\nu-2}-\frac{x^\nu}{\nu+2}\right)\right]+\frac{x}{\Omega(x)}$$
$$\Omega(x)|_{\nu=1}=r^2\Rightarrow x=1\pm\frac{1}{\eta r}$$
$$-g_{tt}=\Omega(x)f(x)=k-\frac{\mu}{r}+\frac{r^2}{l^2},\mu=\mp\frac{\alpha+3\eta^2}{3\eta^3}$$

$$I[g_{\mu\nu}, A_\mu, \phi] = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}e^{\gamma\phi}F^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$

$$\nabla_\mu\left(e^{\gamma\phi}F^{\mu\nu}\right) = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{\partial V}{\partial\phi} - \frac{1}{4}\gamma e^{\gamma\phi}F^2 = 0$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left[T_{\mu\nu}^{\phi} + T_{\mu\nu}^{EM}\right]$$

$$T_{\mu\nu}^{\phi} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right], T_{\mu\nu}^{EM} = e^{\gamma\phi}\left(F_{\mu\alpha}F_{\nu}^{\cdot\alpha} - \frac{1}{4}g_{\mu\nu}F^2\right)$$

$$V(\phi) = \left(\frac{\Lambda}{3} + \alpha\phi\right)(4 + 2\cosh(\phi)) - 6\alpha\sinh(\phi).$$

$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2 dx^2}{x^2 f(x)} + d\theta^2 + \sin^2\theta d\varphi^2\right]$$

$$f(x) = \frac{1}{l^2} + \alpha\left[\frac{(x^2-1)}{2x} - \ln x\right] + \eta^2\frac{(x-1)^2}{x} - \frac{q^2\eta^2}{2x^2}(x-1)^3$$

$$\Omega(x) = \frac{x}{\eta^2(x-1)^2}, \phi(x) = \ln x$$

$$A = q\left(\frac{1}{x} - \frac{1}{x_+}\right)dt, F = -\frac{q}{x^2}dx \wedge dt$$

$$I_g = -\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Xi(l, \mathcal{R}, \nabla\mathcal{R})$$

$$ds^2 = -\left(k + \frac{r^2}{l^2}\right)dt^2 + \left(k + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\Sigma_k^2$$

$$h_{ab}dx^a dx^b = -\left(k + \frac{R^2}{l^2}\right)dt^2 + R^2 d\Sigma_k^2$$

$$ds^2 = \frac{r_b^2}{l^2}\left(-dt^2 + l^2 d\Sigma_k^2\right)$$

$$ds^2 = -\left(1 - \frac{\mu}{r} + \frac{r^2}{l^2}\right)dt^2 + \left(1 - \frac{\mu}{r} + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$T = \left.\frac{f'}{4\pi}\right|_{r_+} = \beta^{-1} = \frac{1}{4\pi}\left(\frac{3r_+^2 + l^2}{l^2 r_+}\right)$$

$$I_{bulk}^E = \frac{12\pi\beta}{\kappa l^2}\int_{r_+}^{R} r^2 dr = \frac{4\pi\beta}{\kappa l^2}(R^3 - r_+^3)$$

$$h_{ab}dx^a dx^b = -\left(1 - \frac{\mu}{R} + \frac{R^2}{l^2}\right)dt^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$n_a = \frac{\delta_a^r}{\sqrt{g^{rr}}}, K_{ab} = \frac{\sqrt{g^{rr}}}{2}\partial_r h_{ab}, K = \frac{1}{l^2R^2}\left(1 - \frac{\mu}{R} + \frac{R^2}{l^2}\right)^{-1/2}\left(-\frac{3l^2\mu}{2} + 3R^3 + 2Rl^2\right)$$

$$I_{GH}^E = -\frac{4\pi\beta}{\kappa l^2}\left(-\frac{3l^2\mu}{2} + 3R^3 + 2Rl^2\right)$$

$$I_{bh}^E = I_{bulk}^E + I_{GH}^E = \frac{4\pi\beta}{\kappa l^2}\left(\frac{3l^2\mu}{2} - 2R^3 - 2Rl^2 - r_+^3\right).$$

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$I_{AdS}^E = I_{bulk}^E + I_{GH}^E = \frac{4\pi\beta_0}{\kappa l^2}(-2R^3 - 2l^2R)$$

$$\beta_0\sqrt{1+\frac{R^2}{l^2}}=\beta\sqrt{1+\frac{R^2}{l^2}-\frac{\mu}{R}}$$

$$I^E=I^E_{bh}-I^E_{AdS}=\frac{4\pi\beta}{\kappa l^2}\left[\left(\frac{3l^2\mu}{2}-2R^3-2Rl^2-r_+^3\right)-\frac{\beta_0}{\beta}(-2R^3-2l^2R)\right]$$

$$F=\beta^{-1}I^E=\frac{4\pi}{\kappa l^2}\left(\frac{l^2\mu}{2}-r_+^3\right)$$

$$I_g=-\frac{1}{\kappa}\int_{\partial M}d^3x\sqrt{-h}\left(\frac{2}{l}+\frac{l\mathcal{R}}{2}\right).$$

$$I^E_{bulk}+I^E_{GH}=\frac{4\pi\beta}{\kappa l^2}\left(\frac{3l^2\mu}{2}-2r_b^3-2r_bl^2-r_+^3\right).$$

$$I^E_g=\frac{4\pi\beta}{\kappa l^2}\left(1+\frac{l^2}{R^2}-\frac{\mu l^2}{R^3}\right)^{\frac{1}{2}}(2R^3+kl^2R)\Bigg|_{R=r_b}=\frac{4\pi\beta}{\kappa l^2}(2r_b^3+2l^2r_b-\mu l^2),$$

$$I^E=I^E_{bulk}+I^E_{GH}+I^E_g=\frac{4\pi\beta}{\kappa l^2}\left(\frac{l^2\mu}{2}-r_+^3\right).$$

$$F_{\text{curve}}=\frac{2\pi r_+}{\kappa},F_{SAdS}=\frac{4\pi}{\kappa l^2}\left(\frac{l^2\mu}{2}-r_+^3\right),$$

$$E=-T^2\frac{\partial I^E}{\partial T}=\frac{\mu}{2G}$$

$$T_{curve}=\frac{1}{4\pi r_h},T_{SAdS}=\frac{1}{4\pi r_+}\left(1+\frac{3r_+^2}{l^2}\right).$$

$$\int_{\partial\mathcal{M}}d^3xh^{ab}T_{ab}$$

$$ds^2=h_{ab}dx^adx^b=-N(R)dt^2+S(R)d\Sigma_k^2.$$

$$\tau^{ab}\equiv\frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{ab}}$$

$$ds^2=-\left(1-\frac{\mu}{r}+\frac{r^2}{l^2}\right)dt^2+\left(1-\frac{\mu}{r}+\frac{r^2}{l^2}\right)^{-1}dr^2+r^2d\Omega^2.$$

$$ds^2=-\left(1-\frac{\mu}{R}+\frac{R^2}{l^2}\right)dt^2+R^2d\Omega^2$$

$$I=\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(R-2\Lambda)+\frac{1}{\kappa}\int_{\partial M}d^3x\sqrt{-h}K-\frac{1}{\kappa}\int_{\partial M}d^3x\sqrt{-h}\left(\frac{2}{l}+\frac{l\mathcal{R}}{2}\right),$$

$$\tau_{ab}=-\frac{1}{8\pi G}\left(K_{ab}-h_{ab}K-\frac{2}{l}h_{ab}+lG_{ab}\right).$$

$$ds^2_{borde}=\frac{R^2}{l^2}(-dt^2+l^2d\Omega^2)$$

$$ds^2_{dual}=\gamma_{ab}dx^adx^b=-dt^2+l^2d\Omega^2$$

$$\langle\tau^{dual}_{ab}\rangle=\lim_{R\to\infty}\frac{R}{l}\tau_{ab}=\frac{\mu}{16\pi Gl^2}(3\delta^0_a\delta^0_b+\gamma_{ab})$$

$$I[g_{\mu\nu},\phi]=\int_{\mathcal{M}}d^4x\sqrt{-g}\left[\frac{R}{2\kappa}-\frac{1}{2}(\partial\phi)^2-V(\phi)\right]+\frac{1}{\kappa}\int_{\partial\mathcal{M}}d^3xK\sqrt{-h}$$

$$\mathcal{H}_\perp=\frac{2\kappa}{\sqrt{g}}\left[\pi_{ij}\pi^{ij}-\frac{1}{2}(\pi^i_i)^2\right]-\frac{1}{2\kappa}\sqrt{g}\,{}^{(3)}R$$

$$+\frac{1}{2}\left(\frac{\pi_\phi^2}{\sqrt{g}}+\sqrt{g}g^{ij}\phi_{,i}\phi_{,j}\right)+\sqrt{g}V(\phi)$$

$$\mathcal{H}_i=-2\pi^j_{i|j}+\pi_\phi\phi_{,i}$$

$$ds^2=-(N^\perp)^2dt^2+g_{ij}(dx^i+N^idt)(dx^j+N^jdt)$$

$$H[\xi] = \int_{\partial\mathcal{M}} d^3x\left(\xi^{\perp}\mathcal{H}_{\perp} + \xi^{i}\mathcal{H}_{i}\right) + Q[\xi]$$

$$\delta Q[\xi] = \oint d^2S_l\left[\frac{G^{ijkl}}{2\kappa}\left(\xi^{\perp}\delta g_{ij|k} - \xi^{\perp}{}_{,k}\delta g_{ij}\right) + 2\xi_k\delta\pi^{kl}\right.$$
$$\left. +\left(2\xi^{k}\pi^{jl} - \xi^{l}\pi^{jk}\right)\delta g_{jk} - \left(\sqrt{g}\xi^{\perp}g^{lj}\phi_{,j} + \xi^{l}\pi_{\phi}\right)\delta\phi\right]$$

$$G^{ijkl} \equiv \frac{1}{2}\sqrt{g}\left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right)$$

$$\xi^{\perp} = N^{\perp}\xi^{t}, \xi^{i} = {}^{(3)}\xi^{i} + N^{i}\xi^{t}$$

$$\delta M \equiv \delta Q[\partial_t] = \oint d^2S_l\left[\frac{G^{ijkl}}{2\kappa}\left(\xi^{\perp}\delta g_{ij|k} - \xi^{\perp}{}_{,k}\delta g_{ij}\right) - \sqrt{g}\xi^{\perp}g^{lj}\phi_{,j}\,\delta\phi\right]$$

$$\delta M = \delta M_G + \delta M_{\phi}$$

$$\delta M_G = \oint d^2S_l\frac{G^{ijkl}}{2\kappa}\left(\xi^{\perp}\delta g_{ij|k} - \xi^{\perp}{}_{,k}\delta g_{ij}\right)$$

$$\delta M_{\phi} = -\oint d^2S_l\sqrt{g}\xi^{\perp}g^{lj}\phi,{}_{,j}\delta\phi$$

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = -\left(k + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{k + \frac{r^2}{l^2}} + r^2d\Sigma_k^2$$

$$h_{ab}dx^{a}dx^{b} = \frac{r^2}{l^2}\left(-dt^2 + l^2d\Sigma_k^2\right)$$

$$m_{BF}^2 + \frac{1}{l^2} > m^2 \geq m_{BF}^2, m_{BF}^2 = -\frac{9}{4l^2}$$

$$ds^2 = -N(r)dt^2 + H(r)dr^2 + S(r)d\Sigma_k^2$$

$$V(\phi) = -\frac{3}{\kappa l^2} - \frac{\phi^2}{l^2} + O(\phi^4)$$

$$\phi(r) = \frac{\alpha}{r} + \frac{\beta}{r^2} + O(r^{-3})$$

$$N(r) = -g_{tt} = \frac{r^2}{l^2} + k - \frac{\mu}{r} + O(r^{-2})$$

$$S(r) = r^2 + O(r^{-2})$$

$$NS'^2H - 2NS''HS + (NH)'S'S - 2\kappa NHS^2\phi'^2 = 0$$

$$H(r) = g_{rr} = \frac{l^2}{r^2} + \frac{l^4}{r^4}\left(-k - \frac{\alpha^2\kappa}{2l^2}\right) + \frac{l^5}{r^5}\left(\frac{\mu}{l} - \frac{4\kappa\alpha\beta}{3l^3}\right) + O(r^{-6})$$

$$g_{rr} = \frac{l^2}{r^2} + \frac{al^4}{r^4} + \frac{bl^5}{r^5} + O(r^{-6})$$

$$\beta = C\alpha^2$$

$$V(\phi) = -\frac{3}{\kappa l^2} - \frac{\phi^2}{l^2} + \lambda\phi^3 + O(\phi^4)$$

$$\phi(r) = \frac{\alpha}{r} + \frac{\beta}{r^2} + \frac{\gamma\ln(r)}{r^2} + O(r^{-3})$$

$$H(r) = g_{rr} = \frac{l^2}{r^2} + \frac{l^4}{r^4}\left(-k - \frac{\kappa\alpha^2}{2l^2}\right) + \frac{l^5}{r^5}\left(\frac{\mu}{l} - \frac{4\kappa\alpha\beta}{3l^3} + \frac{2\kappa\alpha\gamma}{9l^3}\right) + \frac{l^5\ln r}{r^5}\left(-\frac{4\kappa\alpha\gamma}{3l^3}\right) + O\left[\frac{\ln(r)^2}{r^6}\right]$$

$$H(r) = \frac{l^2}{r^2} + \frac{l^4a}{r^4} + \frac{l^5b}{r^5} + \frac{l^5c\ln r}{r^5} + O\left[\frac{\ln(r)^2}{r^6}\right]$$

$$a = -k - \frac{\alpha^2\kappa}{2l^2}, b = \frac{\mu}{l} - \frac{4\kappa\alpha\beta}{3l^3} + \frac{2\kappa\alpha\gamma}{9l^3}, c = -\frac{4\kappa\gamma\alpha}{3l^3}$$

$$\partial_r\left(\frac{\phi'S\sqrt{N}}{\sqrt{H}}\right) - S\sqrt{NH}\frac{\partial V}{\partial\phi} = 0$$

$$\frac{3\alpha^2l^2\lambda + \gamma}{l^2} + O(r^{-1}) = 0$$

$$\xi^r = r\eta^r(x^m) + O(r^{-1})$$
$$\xi^m = O(1)$$
$$\phi'(x) = \phi(x) + \xi^\mu\partial_\mu\phi(x) = \frac{\alpha'}{r} + \frac{\beta'}{r^2} + \frac{\gamma'\ln(r)}{r^2} + O(r^{-3}),$$
$$\alpha' = \alpha - \eta^r\alpha + \xi^m\partial_m\alpha$$
$$\beta' = \beta - \eta^r(2\beta - \gamma) + \xi^m\partial_m\beta$$
$$\gamma' = \gamma - 2\gamma\eta^r + \xi^m\partial_m\gamma$$
$$\alpha'\frac{\partial\gamma}{\partial\alpha} - \gamma' = 0 = \alpha\frac{\partial\gamma}{\partial\alpha} - \gamma + \eta^r\left(2\gamma - \alpha\frac{\partial\gamma}{\partial\alpha}\right) + \xi^m\left(\frac{\partial\alpha}{\partial x^m}\frac{\partial\gamma}{\partial\alpha} - \frac{\partial\gamma}{\partial x^m}\right)$$
$$\alpha'\frac{\partial\beta}{\partial\alpha} - \beta' = 0 = \alpha\frac{\partial\beta}{\partial\alpha} - \beta + \eta^r\left(2\beta - \gamma - \alpha\frac{\partial\beta}{\partial\alpha}\right) + \xi^m\left(\frac{\partial\alpha}{\partial x^m}\frac{\partial\beta}{\partial\alpha} - \frac{\partial\beta}{\partial x^m}\right)$$
$$\beta(\alpha) = \left(-C_\gamma\ln(\alpha) + C\right)\alpha^2$$
$$\phi(r) = \frac{\alpha}{r^{\Delta_-}} + \frac{\beta}{r^{\Delta_+}} + \cdots$$
$$I_{CFT} \to I_{CFT} - \int d^3x W[\mathcal{O}(x)]$$
$$I_g^{ct} = -\frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{2}{l} + \frac{\mathcal{R}l}{2}\right)$$
$$I = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{(\partial\phi)^2}{2} - V(\phi)\right) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{2}{l} + \frac{\mathcal{R}l}{2}\right) + I_\phi$$
$$I_\phi^{ct} = \frac{1}{6\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\phi n^\nu\partial_\nu\phi - \frac{\phi^2}{2l}\right)$$
$$I_\phi = -\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[\frac{\phi^2}{2l} + \frac{W(\alpha)}{l\alpha^3}\phi^3\right]$$
$$\delta I = \int d^3x\sqrt{-h}\left[\frac{1}{r}\left(-\sqrt{g^{rr}}\phi' - \frac{\phi}{l} - \frac{3W(\alpha)\phi^2}{l\alpha^3}\right)\left(1 + \frac{1}{r}\frac{d^2W(\alpha)}{d\alpha^2}\right) + \left(\frac{3W(\alpha)}{\alpha} - \beta\right)\frac{\phi^3}{l\alpha^3}\right]\delta\alpha.$$
$$\lim_{r\to\infty}\delta I = 0$$
$$I_\phi = -\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[\frac{\phi^2}{2l} + \frac{\phi^3}{l\alpha^3}\left(W - \frac{\alpha\gamma}{3}\right) - \frac{\phi^3 C_\gamma}{3l}\ln\left(\frac{\phi}{\alpha}\right)\right]$$
$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2dx^2}{f(x)} + d\Sigma_k^2\right]$$
$$I_{bulk}^E = \int_0^{1/T} d\tau\int_{x_+}^{x_b} d^3x\sqrt{g^E}V(\phi) = \frac{\sigma_k}{2\eta\kappa T}\frac{d(\Omega f)}{dx}\bigg|_{x_+}^{x_b}$$
$$\Omega(x) \to S(r), f(x) \to \frac{N(r)}{S(r)}, dx \to \frac{\sqrt{NH}}{\eta S}dr$$
$$I_{bulk}^E = \frac{\sigma_k}{2\kappa T}\frac{S}{\sqrt{NH}}\frac{dN}{dr}\bigg|_{r_+}^{r_b}$$
$$ds^2 = h_{ab}dx^adx^b = \Omega(x_0)[-f(x_0)dt^2 + d\Sigma_k]$$
$$n_a = \frac{\delta_a^x}{\sqrt{g^{xx}}}\bigg|_{x=x_0}, K_{ab} = \frac{\sqrt{g^{xx}}}{2}\partial_x g_{ab}\bigg|_{x=x_0}, K = \frac{1}{2\eta}\left(\frac{f}{\Omega}\right)^{1/2}\left[\frac{(\Omega f)'}{\Omega f} + \frac{2\Omega'}{\Omega}\right]\bigg|_{x_0}$$
$$I_{GH}^E = -\frac{\sigma_k}{\kappa T}\frac{\Omega f}{2\eta}\left[\frac{(\Omega f)'}{\Omega f} + \frac{2\Omega'}{\Omega}\right]\bigg|_{x_b} = -\frac{\sigma_k}{2T\kappa}\left(\frac{S}{\sqrt{NH}}\frac{dN}{dr} + \frac{2N}{\sqrt{NH}}\frac{dS}{dr}\right)\bigg|_{r_b}$$
$$I_g^{ct} = \frac{2\sigma_k}{\kappa Tl}\left(\Omega^{3/2}f^{1/2} + \frac{l^2k}{2}f^{1/2}\Omega^{1/2}\right)\bigg|_{x_b} = \frac{2\sigma_k}{\kappa Tl}S\sqrt{N}\left(1 + \frac{l^2k}{2S}\right)\bigg|_{r_b}$$
$$T = \frac{N'}{4\pi\sqrt{NH}}\bigg|_{r_+}$$

$$I^E_{bulk}+I^E_{GH}+I^{ct}_g=-\frac{1}{T}\left[\frac{\sigma_k S(r_+)T}{4G}\right]-\frac{\sigma_k}{2\kappa T}\left[\frac{2N}{\sqrt{NH}}\frac{dS}{dr}-\frac{4}{l}S\sqrt{N}\left(1+\frac{l^2k}{2S}\right)\right]\Bigg|_{r_b}$$

$$I^E_{bulk}+I^E_{GH}+I^{ct}_g=-\frac{\mathcal{A}}{4G}-\frac{\sigma_k}{T}\left(-\frac{\mu}{\kappa}+\frac{4\alpha\beta}{3l^2}+\frac{r\alpha^2}{2l^2}\right)\Bigg|_{r_b}$$

$$I^{ct}_\phi=\int_{\partial\mathcal{M}}d^3x\sqrt{h^E}\left[\frac{\phi^2}{2l}+\frac{W(\alpha)}{l\alpha^3}\phi^3\right]=\frac{\sigma_k}{T}\left(\frac{W}{l^2}+\frac{\alpha\beta}{l^2}+\frac{r\alpha^2}{2l^2}\right)\Bigg|_{r_\infty}$$

$$I^E=I^E_{bulk}+I^E_{GH}+I^{ct}_g+I^{ct}_\phi=-\frac{\mathcal{A}}{4G}+\frac{\sigma_k}{T}\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{\alpha}{3}\frac{dW}{d\alpha}\right)\right]$$

$$F=I^ET=M-TS$$

$$M=-T^2\frac{\partial I^E}{\partial T}=\sigma_k\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{\alpha}{3}\frac{dW}{d\alpha}\right)\right]$$

$$S=-\frac{\partial(I^ET)}{\partial T}=\frac{\mathcal{A}}{4G}$$

$$I^E_{bulk}+I^E_{GH}+I^{ct}_g+I^{ct}_\phi=-\frac{\mathcal{A}}{4G}+\frac{\sigma_k}{T}\left\{\frac{\mu}{\kappa}+\frac{1}{l^2}\left[W(\alpha)-\frac{\alpha}{3}\frac{dW}{d\alpha}+\frac{2\alpha\gamma}{9}-\frac{\alpha\gamma}{3}\ln r\right]\right\}$$

$$\bar{I}^{ct}_\phi=\int_{\partial\mathcal{M}}d^3x\sqrt{h^E}\left\{\frac{\phi^3\gamma}{3\alpha^2 l}\left[\ln\left(\frac{\alpha}{\phi}\right)-1\right]\right\}=\frac{\sigma_k}{T}\left[-\frac{\alpha\gamma}{3l^2}+\frac{\alpha\gamma\ln r}{3l^2}+O(r^{-1}\ln r)\right]$$

$$I^E=I^E_{bulk}+I^E_{GH}+I^{ct}_g+I^{ct}_\phi+\bar{I}^{ct}_\phi=-\frac{\mathcal{A}}{4G}+\frac{\sigma_k}{T}\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{\alpha}{3}\frac{dW}{d\alpha}-\frac{\alpha\gamma}{9}\right)\right]$$

$$M=-T^2\frac{\partial I^E}{\partial T}=\sigma_k\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{\alpha}{3}\frac{dW}{d\alpha}-\frac{\alpha\gamma}{9}\right)\right]$$

$$S=-\frac{\partial(I^ET)}{\partial T}=\frac{\mathcal{A}}{4G}$$

$$\tau_{ab}=-\frac{1}{\kappa}\left(K_{ab}-h_{ab}K+\frac{2}{l}h_{ab}-lG_{ab}\right)-\frac{h_{ab}}{l}\left[\frac{\phi^2}{2}+\frac{W(\alpha)}{\alpha^3}\phi^3\right].$$

$$\tau_{tt}=\frac{l}{R}\left[\frac{\mu}{8\pi Gl^2}+\frac{1}{l^4}\left(W-\frac{\alpha\beta}{3}\right)\right]+O(R^{-2})$$

$$\tau_{\theta\theta}=\frac{l}{R}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W-\frac{\alpha\beta}{3}\right)\right]+O(R^{-2})$$

$$\tau_{\phi\phi}=\frac{l\sin^2\theta}{R}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W-\frac{\alpha\beta}{3}\right)\right]+O(R^{-2})$$

$$\langle\tau^{dual}_{ab}\rangle=\frac{3\mu}{16\pi Gl^2}\delta^0_a\delta^0_b+\frac{\gamma_{ab}}{l^2}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W(\alpha)-\frac{\alpha\beta}{3}\right)\right]$$

$$\langle\tau^{dual}\rangle=-\frac{3}{l^4}\left[W(\alpha)-\frac{\alpha\beta}{3}\right]$$

$$\tau_{ab}=-\frac{1}{\kappa}\left(K_{ab}-h_{ab}K+\frac{2}{l}h_{ab}-lG_{ab}\right)-\frac{h_{ab}}{l}\left[\frac{\phi^2}{2}+\frac{\phi^3}{\alpha^3}\left(W-\frac{\alpha\gamma}{3}\right)+\frac{\phi^3\gamma}{3\alpha^2}\ln\left(\frac{\alpha}{\phi}\right)\right]$$

$$\tau_{tt}=\frac{l}{R}\left[\frac{\mu}{8\pi Gl^2}+\frac{1}{l^4}\left(W-\frac{\alpha\beta}{3}-\frac{\alpha\gamma}{9}\right)\right]+O\left[\frac{(\ln R)^3}{R^2}\right]$$

$$\tau_{\theta\theta}=\frac{l}{R}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W-\frac{\alpha\beta}{3}-\frac{\alpha\gamma}{9}\right)\right]+O\left[\frac{(\ln R)^3}{R^2}\right]$$

$$\tau_{\phi\phi}=\frac{l\sin^2\theta}{R}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W-\frac{\alpha\beta}{3}-\frac{\alpha\gamma}{9}\right)\right]+O\left[\frac{(\ln R)^3}{R^2}\right]$$

$$\langle\tau^{dual}_{ab}\rangle=\frac{3\mu}{16\pi Gl^2}\delta^0_a\delta^0_b+\frac{\gamma_{ab}}{l^2}\left[\frac{\mu}{16\pi G}-\frac{1}{l^2}\left(W(\alpha)-\frac{\alpha\beta}{3}-\frac{\alpha\gamma}{9}\right)\right]$$

$$\langle\tau^{dual}\rangle=-\frac{3}{l^4}\left(W-\frac{\alpha\beta}{3}-\frac{\alpha\gamma}{9}\right)$$

$$\langle\tau^{\mathrm{dual}}\rangle=0\Rightarrow\gamma=-3l^2\lambda\alpha^2, W(\alpha)=\alpha^3[C+l^2\lambda\ln\alpha]$$

$$-5/4l^2>m^2\geqslant-9/4l^2$$

$$\phi(r)=\frac{\alpha}{r}+\frac{\beta}{r^2}+O(r^{-3})$$
$$\delta M_G=\frac{\sigma_k}{\kappa}[r\delta a+l\delta b+O(1/r)]$$
$$\delta M_\phi=\frac{\sigma_k}{l^2}[r\alpha\delta\alpha+\alpha\delta\beta+2\beta\delta\alpha+O(1/r)]$$
$$\delta M=\frac{\sigma_k}{\kappa l^2}[r(l^2\delta a+\kappa\alpha\delta\alpha)+l^3\delta b+\kappa(\alpha\delta\beta+2\beta\delta\alpha)+O(1/r)]$$
$$\frac{k+a}{\kappa}+\frac{\alpha^2}{2l^2}=0$$
$$\delta M=\frac{\sigma_k}{\kappa l^2}[l^3\delta b+\kappa(\alpha\delta\beta+2\beta\delta\alpha)]$$
$$M=\sigma_k\left[\frac{lb}{\kappa}+\frac{1}{l^2}\left(\alpha\frac{dW(\alpha)}{d\alpha}+W(\alpha)\right)\right]$$
$$\frac{lc}{\kappa}-4\alpha^3\lambda=0$$
$$\delta M_G=\left\{\frac{l\delta b}{\kappa}+\frac{\delta a}{\kappa}r+\frac{l\delta c}{\kappa}\ln(r)+O\left(\frac{\ln(r)^2}{r}\right)\right\}\sigma_k$$
$$\delta M_\phi=\left[\frac{\alpha\delta\beta+2\beta\delta\alpha+3\alpha^2l^2\lambda\delta\alpha}{l^2}+r\frac{\alpha\delta\alpha}{l^2}\right.$$
$$\left.-12\lambda\alpha^2\delta\alpha\ln(r)+O\left(\frac{\ln(r)^2}{r}\right)\right]\sigma_k.$$
$$\delta M=\left[\frac{l\delta b}{\kappa}+\frac{\alpha\delta\beta+2\beta\delta\alpha+3\alpha^2l^2\lambda\delta\alpha}{l^2}\right]\sigma_k$$
$$M=\left[\frac{lb}{\kappa}+\frac{1}{l^2}\left(\alpha\frac{dW}{d\alpha}+W(\alpha)+\alpha^3l^2\lambda\right)\right]\sigma_k$$
$$M=\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W(\alpha)-\frac{1}{3}\alpha\frac{dW}{d\alpha}+\frac{1}{3}\alpha^3l^2\lambda\right)\right]\sigma_k$$
$$W(\alpha)=\alpha^3[C+l^2\lambda\ln(\alpha)]$$
$$d\Sigma_k^2=\frac{dy^2}{1-ky^2}+(1-ky^2)d\phi^2$$
$$E=\int d\sigma^i\tau_{ij}\xi^j=\int dyd\phi Su^i\tau_{ij}\xi^j$$
$$ds^2=\sigma_{ij}dx^idx^j=Sd\Sigma_k^2$$
$$E=\sigma_k\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{\alpha}{3}\frac{dW}{d\alpha}\right)\right]$$
$$E=\sigma_k\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{1}{3}\alpha\frac{dW}{d\alpha}-\frac{\alpha\gamma}{9}\right)\right]=\sigma_k\left[\frac{\mu}{\kappa}+\frac{1}{l^2}\left(W-\frac{1}{3}\alpha\frac{dW}{d\alpha}-\frac{\alpha^3C_\gamma}{9}\right)\right].$$
$$V(\phi)=\frac{\Lambda(\nu^2-4)}{6\kappa\nu^2}\left[\frac{\nu-1}{\nu+2}e^{-\phi l_\nu(\nu+1)}+\frac{\nu+1}{\nu-2}e^{\phi l_\nu(\nu-1)}+4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right]$$
$$+\frac{\Upsilon}{\kappa\nu^2}\left[\frac{\nu-1}{\nu+2}\sinh\phi l_\nu(\nu+1)-\frac{\nu+1}{\nu-2}\sinh\phi l_\nu(\nu-1)+4\frac{\nu^2-1}{\nu^2-4}\sinh\phi l_\nu\right]$$
$$\phi(x)=l_\nu^{-1}\ln x$$
$$f(x)=\frac{1}{l^2}+\Upsilon\left[\frac{1}{\nu^2-4}-\frac{x^2}{\nu^2}\left(1+\frac{x^{-\nu}}{\nu-2}-\frac{x^\nu}{\nu+2}\right)\right]+\frac{x}{\Omega(x)}$$
$$\Omega(x)=r^2+O(r^{-3})$$
$$x=1+\frac{1}{\eta r}+\frac{m}{r^3}+\frac{n}{r^4}+\frac{p}{r^5}+O(r^{-6})$$
$$\Omega(x)=r^2-\frac{24m\eta^3+\nu^2-1}{12\eta^2}-\frac{24n\eta^4-\nu^2+1}{12\eta^3r}+\frac{720m^2\eta^6-480\eta\eta^5+\nu^4-20\nu^2+19}{240\eta^4r^2}$$
$$+O(r^{-3})$$

$$x = 1 + \frac{1}{\eta r} - \frac{(\nu^2 - 1)}{24\eta^3 r^3}\left[1 - \frac{1}{\eta r} - \frac{9(\nu^2 - 9)}{80\eta^2 r^2}\right] + O(r^{-6})$$
$$-g_{tt} = f(x)\Omega(x) = \frac{r^2}{l^2} + 1 + \frac{\Upsilon + 3\eta^2}{3\eta^3 r} + O(r^{-3})$$
$$g_{rr} = \frac{\Omega(x)\eta^2}{f(x)}\left(\frac{dx}{dr}\right) = \frac{l^2}{r^2} - \frac{l^4}{r^4} - \frac{l^2(\nu^2 - 1)}{4\eta^2 r^4} - \frac{l^2(3\eta^2 l^2 + \Upsilon l^2 - \nu^2 + 1)}{3\eta^3 r^5} + O(r^{-6})$$
$$\phi(x) = l_\nu^{-1}\ln\, x = \frac{1}{l_\nu \eta r} - \frac{1}{2 l_\nu \eta^2 r^2} - \frac{\nu^2 - 9}{24\eta^3 r^3} + O(r^{-4})$$
$$M = \sigma\left[\frac{\mu}{\kappa} + \frac{1}{l^2}\left(W - \frac{\alpha}{3}\frac{dW}{d\alpha}\right)\right]$$
$$M = -\frac{\sigma}{\kappa}\left(\frac{3\eta^2 + \Upsilon}{3\eta^3}\right)$$
$$I_{CFT} \to I_{CFT} + \frac{l_\nu}{6}\int d^3x\mathcal{O}^3$$
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$t = \rho\sinh(a\tau), x = \rho\cosh(a\tau)$$
$$ds^2 = -a^2\rho^2 d\tau^2 + d\rho^2 + dy^2 + dz^2$$
$$ds^2 = N(r)dt_E^2 + H(r)dr^2 + S(r)d\Sigma_k^2$$
$$ds^2 = g_{rr}\frac{4N}{[(N)']^2}\left[\rho^2\frac{[(N)']^2}{4Ng_{rr}}d\tau_E^2 + d\rho^2\right]$$
$$T = \frac{1}{\beta} = \left.\frac{(N^2)'}{4\pi\sqrt{N^2 g_{rr}}}\right|_H$$
$$T_{flat} = \frac{1}{4\pi r_h}, T_{RN-flat} = \frac{1}{4\pi r_+}\left(1 - \frac{q^2}{4r_+^2}\right)$$
$$T_{Sch-AdS} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{l^2}\right), T_{RN-AdS} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{l^2} - \frac{q^2}{4r_+^2}\right)$$
$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2 dx^2}{f(x)} + d\theta^2 + d\Sigma_k^2\right]$$
$$f(x) = \frac{1}{l^2} + \alpha\left[\frac{1}{\nu^2 - 4} - \frac{x^2}{\nu^2}\left(1 + \frac{x^{-\nu}}{\nu - 2} - \frac{x^\nu}{\nu + 2}\right)\right] + \frac{kx}{\Omega(x)}$$
$$f'\Omega(x) = \frac{\alpha}{\eta^2} + 2k + k\nu\frac{x^\nu + 1}{x^\nu - 1}$$
$$T = \left.\frac{f'}{4\pi\eta}\right|_{x_h} = \frac{1}{4\pi\eta\Omega(x_h)}\left(\frac{\alpha}{\eta^2} + 2k + k\nu\frac{x_h^\nu + 1}{x_h^\nu - 1}\right)$$
$$ds^2 = \Omega(x)\left(-f(x)dt^2 + \frac{\eta^2 dx^2}{f(x)} + d\theta^2 + \sin^2\theta d\varphi^2\right)$$
$$\Omega(x) = \frac{1}{\eta^2(x - 1)^2}, f(x) = \frac{1}{l^2} + \frac{1}{3}\alpha(x - 1)^3 + \eta^2 x(x - 1)^2$$
$$x = 1 + \frac{1}{\eta r}, x = 1 - \frac{1}{\eta r}.$$
$$\Omega(x)f(x) = F(r) = 1 - \frac{\mu}{r} + \frac{r^2}{l^2}, \mu = \frac{\alpha + 3\eta^2}{3\eta^3}$$
$$I[g_{\mu\nu}] = I_{bulk} + I_{GH} - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{2}{l} + \frac{\mathcal{R}l}{2}\right)$$
$$E_t^t - E_\phi^\phi = 0 \Rightarrow 0 = f'' + \frac{\Omega' f'}{\Omega} + 2\eta^2$$
$$E_t^t + E_\phi^\phi = 0 \Rightarrow 2\kappa V(\phi) = -\frac{(f\Omega'' + f'\Omega')}{\Omega^2\eta^2} + \frac{2}{\Omega}$$

$$I^E_{bulk} = \frac{4\pi\beta}{\eta^3\kappa l^2}\left[-\frac{1}{(x_b-1)^3}+\frac{1}{(x_h-1)^3}\right] = \frac{4\pi\beta}{\kappa l^2}\left(r_b^3 - r_h^3\right)$$
$$ds^2 = h_{ab}dx^a dx^b = \Omega(x)[-f(x)dt^2 + d\theta^2 + \sin^2\ \theta d\phi^2]$$
$$n_a = \frac{\delta^x_a}{\sqrt{g^{xx}}}, K_{ab} = \frac{\sqrt{g^{xx}}}{2}\partial_x h_{ab}$$
$$I^E_{GH} = -\frac{2\pi\beta}{\kappa}\left[-\frac{6}{l^2\eta^3(x-1)^3} - \frac{4}{\eta(x-1)} - \left(\frac{\alpha+3\eta^2}{\eta^3}\right)\right]\Bigg|_{x_b} = -\frac{2\pi\beta}{\kappa}\left(\frac{6r_b^3}{l^2} + 4r_b - 3\mu\right)$$
$$I^E_g = \frac{2\pi\beta}{\kappa}\left[\frac{4}{l^2\eta^3(x_b-1)^3} + \frac{4}{\eta(x_b-1)} - 2\mu\right] = \frac{2\pi\beta}{\kappa}\left(\frac{4r_b^3}{l^2} + 4r_b - 2\mu\right)$$
$$I^E = I^E_{bulk} + I^E_{GH} + I^E_g = \frac{4\pi\beta}{\kappa l^2}\left[\frac{1}{\eta^3(x_h-1)^3} + \frac{\mu l^2}{2}\right] = \frac{4\pi\beta}{\kappa l^2}\left(-r_h^3 + \frac{\mu l^2}{2}\right)$$
$$I^E_\phi = \int_{\partial\mathcal{M}} d^3x^E\sqrt{h^E}\left(\frac{\phi^2}{2l} - \frac{l_\nu}{6l}\phi^3\right) = \frac{4\pi\beta}{\kappa}\left[-\frac{\nu^2-1}{4l^2\eta^3(x_b-1)} + \frac{\nu^2-1}{3l^2\eta^3}\right]$$
$$I^E_{bulk} + I^E_{surf} + I^E_g = -\frac{1}{T}\left(\frac{AT}{4G}\right) + \frac{4\pi\beta}{\kappa}\left[\frac{\nu^2-1}{4l^2\eta^3(x_b-1)} + \frac{12\eta^2l^2 + 4\alpha l^2 - 4\nu^2 + 4}{12l^2\eta^3}\right]$$
$$I^E = \beta\left(-\frac{AT}{4G} + \frac{4\pi}{\kappa}\frac{3\eta^2+\alpha}{3\eta^3}\right)$$
$$M = \frac{1}{2G}\left(\frac{\alpha+3\eta^2}{3\eta^3}\right)$$
$$T = \frac{f'(x)}{4\pi\eta}\Bigg|_{x=x_h} = \frac{1}{4\pi\eta\Omega(x_h)}\left[\frac{\alpha}{\eta^2} + 2 + \nu\frac{x_h^\nu+1}{x_h^\nu-1}\right], S = \frac{A}{4G} = \frac{4\pi\Omega(x_h)}{4G},$$
$$\frac{\partial M}{\partial\eta}\frac{d\eta}{dx_h} = T\left(\frac{\partial S}{\partial x_h} + \frac{\partial S}{\partial\eta}\frac{d\eta}{dx_h}\right).$$
$$M = -\frac{1}{2G}\left(\frac{\alpha+3\eta^2}{3\eta^3}\right)$$
$$T = -\frac{1}{4\pi\eta\Omega(x_h)}\left[\frac{\alpha}{\eta^2} + 2 + \nu\frac{x_h^\nu+1}{x_h^\nu-1}\right]$$
$$C = \frac{\partial M}{\partial T}$$
$$I[g_{\mu\nu}] = \int_{\mathcal{M}} d^4x(R-2\Lambda)\sqrt{-g} + 2\int_{\partial\mathcal{M}} d^3xK\sqrt{-h} - \int_{\partial\mathcal{M}} d^3x\frac{4}{l}\sqrt{-h}$$
$$ds^2 = -\left(-\frac{\mu_b}{r} + \frac{r^2}{l^2}\right)dt^2 + \left(-\frac{\mu_b}{r} + \frac{r^2}{l^2}\right)^{-1}dr^2 + \frac{r^2}{l^2}(dx_1^2 + dx_2^2)$$
$$I^E_b = \frac{2LL_b\beta_b}{l^4}\left(-r_h^3 + \frac{\mu_b l^2}{2}\right) = -\frac{LL_b\beta_b r_h^3}{l^4}$$
$$T = \beta_b^{-1} = \frac{(-g_{tt})'}{4\pi}\Bigg|_{r=r_h} = \frac{3r_h}{4\pi l^2}$$
$$E = -T^2\frac{\partial I^E_b}{\partial T} = \frac{2LL_b\mu_b}{l^2}$$
$$S = -\frac{\partial\left(I^E_b T\right)}{\partial T} = \frac{LL_b r_h^2}{4l^2G} = \frac{\mathcal{A}}{4G}$$
$$ds^2_{dual} = \frac{l^2}{R^2}ds^2 = \gamma_{ab}dx^a dx^b = -dt^2 + dx_1^2 + dx_2^2$$
$$\langle\tau^{dual}_{ab}\rangle = \lim_{R\to\infty}\frac{R}{l}\tau_{ab} = \frac{\mu_b}{16\pi G_N l^2}[3\delta^0_a\delta^0_b + \gamma_{ab}]$$
$$E = Q_{\xi_t} = \int\ d\Sigma^i\tau_{ij}\xi^j = \frac{LL_b}{l^2\kappa}\left[\mu_b + \frac{l^2}{4R} + O(R^{-2})\right]$$

$$ds^2 = -\frac{r^2}{l^2}d\tau^2 + \left(-\frac{\mu_s}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 + \left(-\frac{\mu_s}{r} + \frac{r^2}{l^2}\right)d\theta^2 + \frac{r^2}{l^2}dx_2^2$$

$$-\frac{\mu_s}{r_s} + \frac{r_s^2}{l^2} = 0$$

$$L_s = \frac{4\pi\sqrt{g_{\theta\theta}g_{rr}}}{(g_{\theta\theta})'}\bigg|_{r=r_s} = \frac{4\pi l^2}{3r_s}$$

$$I_s^E = -\frac{LL_s\beta_s\mu_s}{l^2}$$

$$M = -\frac{LL_s\mu_s}{l^2}$$

$$\Delta I = I_b^E - I_s^E = \frac{L}{2\kappa l^4}\left(\frac{4\pi l^2}{3}\right)^3 L_b\beta_b\left(L_s^{-3} - \beta_b^{-3}\right) = \frac{L}{2\kappa l^4}\left(\frac{4\pi l^2}{3}\right)^3 L_b\beta_b\left(\frac{1}{L_s^3} - T^3\right)$$

$$\frac{\mathcal{A}}{Tl^3} = \frac{L}{l}\left(\frac{4\pi}{3}\right)^2 L_sT$$

$$I[g_{\mu\nu},\phi] = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[R - \frac{(\partial\phi)^2}{2} - V(\phi)\right] + 2\int_{\partial\mathcal{M}} d^3xK\sqrt{-h}$$

$$V(\phi) = \frac{\Lambda(\nu^2-4)}{3\nu^2}\left[\frac{\nu-1}{\nu+2}e^{-\phi l_\nu(\nu+1)} + \frac{\nu+1}{\nu-2}e^{\phi l_\nu(\nu-1)} + 4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right]$$

$$+\frac{2\alpha}{\nu^2}\left[\frac{\nu-1}{\nu+2}\sinh\,\phi l_\nu(\nu+1) - \frac{\nu+1}{\nu-2}\sinh\,\phi l_\nu(\nu-1) + 4\frac{\nu^2-1}{\nu^2-4}\sinh\,\phi l_\nu\right]$$

$$V(\phi) = \frac{2\Lambda}{27}\left(5e^{-\phi\sqrt{2}} + 10e^{\phi\sqrt{2}/2} + 16e^{-\phi\sqrt{2}/4}\right)$$

$$+\frac{4\alpha}{45}[\sinh(\phi\sqrt{2}) - 10\sinh(\phi\sqrt{2}/2) + 16\sinh(\phi\sqrt{2}/4)]$$

$$\Omega(x) = \frac{9x^2}{\eta^2(x^3-1)^2}$$

$$\phi(x) = 2\sqrt{2}\ln\,x$$

$$f(x) = \frac{1}{l^2} + \alpha\left[\frac{1}{5} - \frac{x^2}{9}\left(1 + x^{-3} - \frac{x^3}{5}\right)\right]$$

$$ds^2 = \frac{R^2}{l^2}[-dt^2 + dx_1^2 + dx_2^2]$$

$$R^2 \equiv \frac{1}{\eta^2(x-1)^2}$$

$$ds_{dual}^2 = \frac{l^2}{R^2}ds^2 = \gamma_{ab}dx^a dx^b = -dt^2 + dx_1^2 + dx_2^2$$

$$I_{BH}^E = \beta_b\left(-\frac{\mathcal{A}T}{4G_N} + \frac{2LL_b}{l^2}\frac{\alpha}{3\eta^3}\right) = -\frac{LL_b\alpha\beta_b}{3l^2\eta^3}$$

$$\mathcal{A} = \frac{LL_b\Omega(x_h)}{l^2}, T = \frac{\alpha}{4\pi\eta^3\Omega}$$

$$M_b = \frac{2LL_b\mu_b}{l^2}, \mu_b = \frac{\alpha}{3\eta^3}$$

$$ds^2 = \Omega_s(x)\left[-\frac{d\tau^2}{l^2} + \frac{\lambda^2dx^2}{f(x)} + f(x)d\theta^2 + \frac{dx_2^2}{l^2}\right]$$

$$\Omega_s(x) = \frac{9x^2}{\lambda^2(x^3-1)^2}$$

$$L_s = \frac{4\pi\lambda}{f'}\bigg|_{x=x_s} = \frac{4\pi\lambda^3\Omega_s}{\alpha}$$

$$I_{\text{soliton}}^{E} = -\frac{L\beta_s\Omega_s(x_s)}{4l^2G_N} + \frac{2LL_s\beta_s}{l^2}\frac{\alpha}{3\lambda^3} = -\frac{LL_s\beta_s}{l^2}\left(\frac{\alpha}{3\lambda^3}\right)$$

$$M_{\text{soliton}} = -\frac{LL_s\mu_s}{l^2}, \mu_s = \frac{\alpha}{3\lambda^3}$$

$$I^{\phi} = -\int d^3x\sqrt{-h}\left(\frac{\phi^2}{2l} - \frac{l_v}{6l}\phi^3\right), \tau_{ab}^{\phi} = -\frac{2}{\sqrt{-h}}\frac{\delta I^{\phi}}{\delta h^{ab}}$$

$$\tau_{ab} = -\frac{1}{\kappa}\left(K_{ab} - h_{ab}K + \frac{2}{l}h_{ab} - lE_{ab}\right) - \frac{h_{ab}}{l}\left(\frac{\phi^2}{2} - \frac{l_v}{6}\phi^3\right)$$

$$\tau_{\tau\tau} = \frac{\alpha(x-1)}{3\lambda^2 l} + O[(x-1)^2]$$

$$\tau_{\theta\theta} = \frac{2\alpha(x-1)}{3\lambda^2 l} + O[(x-1)^2]$$

$$\tau_{x_2x_2} = -\frac{\alpha(x-1)}{3\lambda^2 l} + O[(x-1)^2]$$

$$ds_{dual}^2 = \frac{l^2}{R^2}ds^2 = -d\tau^2 + d\theta^2 + dx_2^2$$

$$\langle\tau_{ab}^{dual}\rangle = \lim_{R\to\infty}\frac{R}{l}\tau_{ab} = \lim_{x\to x_b}\left[-\frac{1}{\lambda l(x-1)}\right]\tau_{ab} = \frac{1}{l^2}\left(\frac{\alpha}{3\lambda^3}\right)[-3\delta_a^{\theta}\delta_b^{\theta} + \gamma_{ab}]$$

$$M = \oint_{\Sigma} d^2y\sqrt{\sigma}m^a\tau_{ab}\xi^b = \frac{LL_sf^{1/2}\Omega}{\sqrt{-g_{\tau\tau}}}(\partial_\tau)^i\tau_{ij}(\partial_\tau)^j = -\frac{LL_s}{l^2}\left[\frac{\alpha}{3\lambda^3} + O(x-1)\right] \square,$$

$$ds^2 = \sigma_{ij}dx^idx^j = \Omega(x)\left[f(x)d\theta^2 + \frac{dx_2^2}{l^2}\right]$$

$$r_b^2 = \frac{\Omega(x_h,\eta)}{l^2}, r_s^2 = \frac{\Omega(x_s,\lambda)}{l^2}$$

$$E = M_{bh} - M_{soliton} = \frac{LL_b}{l^2}(2\mu_b + \mu_s)$$

$$\Delta F = \beta_b^{-1}(I_{BH}^E - I_{\text{soliton}}^E) = \frac{TL\alpha}{3l^2}\left(\frac{L_s\beta_s}{\lambda^3} - \frac{L_b\beta_b}{\eta^3}\right)$$

$$\Delta F = \frac{4\pi LL_s}{3l^2}\left[\frac{\Omega(\lambda,x_s)}{L_s} - T\Omega(\eta,x_h)\right] = \frac{4\pi L}{3l^2}\Omega(\lambda,x_s)\left(1 - \frac{r_b^3}{r_s^3}\right)$$

$$\frac{\mathcal{A}}{Tl^3} = \frac{\alpha L}{4\pi l^5}\frac{\beta_b^2L_s}{\eta^3} = \frac{L\mathcal{L}}{l}\left(\frac{\lambda}{\eta}\right)$$

$$\mathcal{L} = \frac{16\pi^2}{\alpha^2l^4}\left[\frac{9x_h^2}{(x_h^3-1)^2}\right]^3$$

$$\frac{\mathcal{A}}{Tl^3} = \frac{L\mathcal{L}(\alpha,l)}{l}\frac{r_b}{r_s}$$

$$I = \int d^{n+1}x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{(\partial\phi)^2}{2} - V(\phi)\right] + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^nx\sqrt{-h}K + I_g + I_\phi$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{(\partial\phi)^2}{2} + V\right]$$

$$G = -\frac{R(n-1)}{2}, T = -(n-1)\left[\frac{(\partial\phi)^2}{2} + V\frac{(n+1)}{(n-1)}\right].$$

$$G_{\mu\nu} = \kappa T_{\mu\nu} \to \frac{R}{2\kappa} = \frac{(\partial\phi)^2}{2} + V\frac{(n+1)}{(n-1)}$$

$$I_{bulk}^E = -\frac{2}{n-1}\int d^{n+1}x\sqrt{g^E}V(\phi)$$

$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2 dx^2}{f(x)} + d\Sigma_k^2\right]$$
$$E_t^t - E_x{}^x = 0 \Longrightarrow 2\kappa\phi'^2 = \frac{D-2}{2\Omega^2}[3(\Omega')^2 - 2\Omega\Omega'']$$
$$E_t^t - \frac{1}{D-2}g^{ab}E_{ab} = 0 \Longrightarrow f'' + \frac{D-2}{2\Omega}\Omega' f' + 2k\eta^2 = 0$$
$$E_t^t + \frac{1}{D-2}g^{ab}E_{ab} = 0 \Longrightarrow 2\kappa V = -\frac{D-2}{2\eta^2\Omega^2}\left[f\Omega'' + \frac{D-4}{2\Omega}f(\Omega')^2 + \Omega' f'\right] + \frac{k(D-2)}{\Omega}$$
$$\frac{d}{dx}\left[\Omega^{(D-2)/2}f'\right] + 2\eta^2 k\Omega^{(D-2)/2} = 0$$
$$-\frac{2\eta^2\Omega^{D/2}(2\kappa V)}{D-2} = f\Omega''\Omega^{(D-4)/2} + \Omega'\left(f\Omega^{(D-4)/2}\right)' - 2\eta^2 k\Omega^{(D-2)/2}$$
$$2\kappa V = -\frac{D-2}{2\eta^2\Omega^{D/2}}\left[\Omega^{\frac{D-4}{2}}(f\Omega)'\right]'$$
$$d\Sigma_k^2 = \nu_{ij}dx^i dx^j$$
$$I_{bulk}^E = \frac{\beta\sigma_{k,n-1}}{2\kappa\eta}\left[\Omega^{\frac{D-4}{2}}(f\Omega)'\right]_{x_h}^{x_b}$$
$$h_{ab}dx^a dx^b = \Omega(x)[-f(x)dt^2 + d\Sigma_k^2]$$
$$n_a = \frac{\delta_a^x}{\sqrt{g^{xx}}}, K_{ab} = \frac{\sqrt{g^{xx}}}{2}\partial_x h_{ab}, K = \frac{1}{2\eta}\left(\frac{f}{\Omega}\right)^{1/2}\left[\frac{(\Omega f)'}{\Omega f} + (D-2)\frac{\Omega'}{\Omega}\right].$$
$$I_{GH}^E = -\frac{\beta\sigma_{k,n-1}}{2\kappa\eta}\Omega^{(D-2)/2}f\left[\frac{(f\Omega)'}{f\Omega} + (D-2)\frac{\Omega'}{\Omega}\right]$$
$$I_g = -\frac{1}{\kappa}\int d^n x\sqrt{-h}\left[\frac{n-1}{l} + \frac{l\mathcal{R}}{2(n-2)} + \frac{l^3}{2(n-4)(n-2)^2}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{n\mathcal{R}^2}{4(n-1)}\right)\right]$$
$$\mathcal{R}_{ij} = \frac{(n-2)k}{\Omega}\sigma_{ij}, \mathcal{R} = \frac{k(n-2)(n-1)}{\Omega}, \mathcal{R}_{ab}\mathcal{R}^{ab} = \frac{(n-2)^2(n-1)k^2}{\Omega^2}$$
$$\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{n\mathcal{R}^2}{4(n-1)} = -\frac{k^2}{4\Omega^2}(n-2)^2(n-1)(n-4)$$
$$I_g^E = \frac{\beta\sigma_{k,n-1}}{\kappa}\frac{(D-2)}{l}\sqrt{\Omega^{D-1}f}\left(1 + \frac{l^2 k}{2\Omega} - \frac{l^4 k^2}{8\Omega^2}\right)$$
$$\beta^{-1} = T = \left.\frac{f'}{4\pi\eta}\right|_{x_h}, S = \frac{\mathcal{A}}{4G}$$
$$I_{bulk}^E + I_{GH}^E + I_g^E = -\frac{1}{T}\left(\frac{\mathcal{A}T}{4G}\right)$$
$$-\frac{\sigma_{D-2,k}}{2\kappa T}\Omega^{(D-2)/2}(D-2)\left.\left[\frac{f\Omega'}{\eta\Omega} - \frac{\sqrt{\Omega f}}{2l}\left(1 + \frac{l^2 k}{2\Omega} - \frac{l^4 k^2}{8\Omega^2}\right)\right]\right|_{x_b}$$
$$ds^2 = -N(r)dt^2 + H(r)dr^2 + S(r)d\Sigma_k^2$$
$$\Omega(x) \to S(r), f(x) \to \frac{N(r)}{S(r)}, \frac{\sqrt{NH}}{\eta S}dr \to dx$$
$$2\kappa V = -\frac{D-2}{2\eta^2\Omega^{D/2}}\left[\Omega^{\frac{D-4}{2}}(f\Omega)'\right]' \to 2\kappa V = -\frac{D-2}{2S^{\frac{D-2}{2}}\sqrt{NH}}\frac{d}{dr}\left(\frac{S^{\frac{D-2}{2}}}{\sqrt{NH}}\frac{dN}{dr}\right)$$
$$I_{bulk}^E = \frac{\beta\sigma_{k,n-1}}{2\kappa\eta}\left[\Omega^{\frac{D-4}{2}}(f\Omega)'\right]_{x_h}^{x_b} \to I_{bulk}^E = \left.\frac{\beta\sigma_{k,n-1}}{2\kappa}\frac{dN}{dr}\frac{S^{(n-1)/2}}{\sqrt{NH}}\right|_{r_h}^{r_b}$$
$$h_{ab}dx^a dx^b = -N(R)dt^2 + S(R)d\Sigma_k^2$$
$$n_\mu = \frac{\delta_\mu^r}{\sqrt{g^{rr}}}, K_{\mu\nu} = \frac{\sqrt{g^{rr}}}{2}\partial_r h_{\mu\nu}, K = \frac{1}{2\sqrt{H}}\left[\frac{N'}{N} + (n-1)\frac{S'}{S}\right]$$

$$I_{GH}^{E}=-\frac{\beta\sigma_{k,n-1}}{2\kappa\eta}\Omega^{\frac{D-2}{2}}f\left[\frac{(f\Omega)'}{f\Omega}+(D-2)\frac{\Omega'}{\Omega}\right]\Bigg|_{x_b}\rightarrow I_{GH}^{E}=-\frac{\sigma_{k,n-1}}{2\kappa T}\frac{S^{\frac{D-2}{2}}}{\sqrt{NH}}\left[\frac{dN}{dr}+(D-2)\frac{N}{S}\frac{dS}{dr}\right]\Bigg|_{r_b}$$

$$I_{g}^{E}=\frac{\beta\sigma_{k,n-1}(n-1)}{l\kappa}\Omega^{\frac{D-1}{2}}f^{\frac{1}{2}}\left(1+\frac{l^2k}{2\Omega}-\frac{l^4k^2}{8\Omega^2}\right)_{x_b}\rightarrow I_{g}^{E}$$

$$=\frac{\beta\sigma_{k,n-1}(n-1)}{l\kappa}S^{\frac{D-2}{2}}N^{\frac{1}{2}}\left(1+\frac{l^2k}{2S}-\frac{l^4k^2}{8S^2}\right)_{r_b}$$

$$\beta^{-1}=T=\frac{N'}{4\pi\sqrt{NH}}\Bigg|_{r_h},S=\frac{\mathcal{A}}{4G}$$

$$I_{bulk}^{E}+I_{GH}^{E}+I_{g}^{E}=-\frac{1}{T}\left(\frac{\mathcal{A}T}{4G}\right)$$

$$-\frac{\sigma_{D-2,k}}{2\kappa T}S^{(D-2)/2}(D-2)\left[\frac{NS'}{S\sqrt{NH}}-\frac{2\sqrt{N}}{l}\left(1+\frac{l^2k}{2S}-\frac{l^4k^2}{8S^2}\right)\right]\Bigg|_{r_b}$$

$$2\kappa\phi'^2=\frac{D-2}{2\Omega^2}[3(\Omega')^2-2\Omega\Omega''];\quad\frac{2\kappa\phi'^2}{D-2}=\frac{1}{2S^2}[S'^2-2SS'']+\frac{S'}{2S}\frac{(NH)'}{NH}$$

$$\frac{d}{dx}\left[\Omega^{\frac{D-2}{2}}\frac{df}{dx}\right]=-2\eta^2k\Omega^{\frac{D-2}{2}};\qquad\frac{d}{dr}\left[\frac{S^{D/2}}{\sqrt{NH}}\frac{d}{dr}\left(\frac{N}{S}\right)\right]=-2k\sqrt{NH}S^{\frac{D-4}{2}}$$

$$2\kappa V=-\frac{D-2}{2\eta^2\Omega^{D/2}}\left[\Omega^{\frac{D-4}{2}}(f\Omega)'\right]';\quad 2\kappa V=-\frac{D-2}{2S^{\frac{D-2}{2}}\sqrt{NH}}\frac{d}{dr}\left(\frac{S^{\frac{D-2}{2}}}{\sqrt{NH}}\frac{dN}{dr}\right)$$

$$\partial_x\left[\Omega^{\frac{D-4}{2}}f\phi'\right]=\eta^2\Omega^{D/2}\frac{\partial V}{\partial\phi};\ \partial_r\left(S^{\frac{D-2}{2}}\phi'\sqrt{\frac{N}{H}}\right)=\sqrt{NH}S^{\frac{D-2}{2}}\frac{\partial V}{\partial\phi}$$

$$\tau^{ab}\equiv\frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{ab}}$$

$$I_{GH}+I_g+I_\phi=\frac{1}{\kappa}\int d^nx\sqrt{-h}K-\frac{1}{\kappa}\int d^nx\sqrt{-h}\left[\frac{n-1}{l}+\frac{l\mathcal{R}}{2(n-2)}\right]-\int d^nx\sqrt{-h}\Psi$$

$$\tau_{ab}=-\frac{1}{\kappa}\left(K_{ab}-h_{ab}K+\frac{n-1}{l}h_{ab}-\frac{l}{n-2}G_{ab}\right)-h_{ab}[\Psi].$$

$$h_{ab}dx^adx^b=-N(R)dt^2+S(R)d\Sigma_k^2$$

$$G_{ab}=\mathcal{R}_{ab}-\frac{1}{2}\mathcal{R}h_{ab},G_{tt}=\frac{(n-2)(n-1)}{2}\frac{kN}{S},G_{ij}=-\frac{(n-2)(n-3)}{2}k\nu_{ij}$$

$$\tau_{tt}=-\frac{(n-1)}{\kappa}\left[\frac{NS'}{2S\sqrt{H}}-\frac{N}{l}\left(1+\frac{l^2k}{2S}\right)\right]+N[\Psi]$$

$$\tau_{ij}=\frac{\nu_{ij}}{\kappa}\left[\frac{S}{2\sqrt{H}}\left(\frac{N'}{N}+\frac{S'}{S}(n-2)\right)-\frac{(n-1)S}{l}-\frac{lk(n-3)}{2}\right]-\nu_{ij}S[\Psi]$$

$$h_{ab}dx^adx^b=-L^2dt^2+\sigma_{ij}\left(dy^i+L^idt\right)\left(dy^j+L^jdt\right)$$

$$E=Q_{\frac{\partial}{\partial t}}=\oint_{\Sigma}d^{D-2}y\sqrt{\sigma}u^a\tau_{ab}\xi^b=\left(\oint_{\Sigma}d^2y\sqrt{\nu}\right)\frac{S^{\frac{D-2}{2}}\tau_{tt}}{\sqrt{N}}=\frac{\sigma_{k,n-1}S^{\frac{D-2}{2}}}{\sqrt{N}}\tau_{tt}$$

$$\sigma_{ij}dx^idx^j=Sd\Sigma_k^2$$

$$\tau_{tt}=-\frac{(n-1)}{\kappa}\left[\frac{f^{3/2}\Omega'}{2\eta\sqrt{\Omega}}-\frac{\Omega f}{l}\left(1+\frac{l^2k}{2\Omega}\right)\right]+\frac{\Omega f}{\kappa}[\Psi],$$

$$\tau_{ij}=\frac{\nu_{ij}}{\kappa}\left[\frac{(\Omega f)'}{2\eta\sqrt{\Omega f}}+\frac{(n-2)}{2\eta}\frac{\Omega'\sqrt{f}}{\sqrt{\Omega}}-\frac{(n-1)\Omega}{l}-\frac{lk(n-3)}{2}\right]-\frac{\nu_{ij}\Omega}{\kappa}[\Psi].$$

$$E = Q_{\frac{\partial}{\partial t}} = \oint_{\Sigma} d^{D-2}y\sqrt{\sigma}u^a\tau_{ab}\xi^b = \frac{\sigma_{k,n-1}\Omega^{\frac{D-2}{2}}}{\sqrt{\Omega f}}\tau_{tt}$$

$$\eta^2 V(\phi) = -\frac{f\Omega''}{\eta^2\Omega^2} - \frac{f'\Omega'}{\eta^2\Omega^2}$$

$$\eta^2 V(\phi) = -\frac{1}{x^{2\nu-2}\nu^4}\left[\frac{1}{l^2} + \frac{\alpha}{2}\left(\frac{x^{2+\nu}-1}{2+\nu} + \frac{x^{2-\nu}-1}{2+\nu} - x^2 + 1\right)\right]\left[\frac{x^{\nu-1}(\nu-1)^2\nu^2 - x^{\nu-1}(\nu-1)\nu^2}{x^2(x^\nu-1)^2\eta^2}\right.$$
$$\left. - \frac{4x^{2\nu-1}(\nu-1)\nu^3 + 2x^{2\nu-1}\nu^4}{(x^\nu-1)^4\eta^2x^2} + \frac{2x^{2\nu-1}\nu^3 - 4x^{2\nu-1}(\nu-1)\nu^3]}{x^2\eta^2(x^\nu-1)^3}\right](x^\nu-1)^4\eta^4$$
$$- \frac{1}{2x^{2\nu-2}\nu^4}\left(\frac{x^{\nu-1}(\nu-1)\nu^2}{x\eta^2(x^\nu-1)^2} - \frac{2x^{2\nu-1}\nu^3}{x\eta^2(x^\nu-1)^3}\right)(x^{1+\nu} + x^{1-\nu} - 2x)$$

$$V(\phi) = -\frac{f(x)\nu^2}{x^{2\nu-2}\nu^4}\left(\frac{x^{3\nu-3}\nu^2 + 4x^{2\nu-3}\nu^2 + x^{\nu-3}\nu^2 + 3x^{3\nu-3}\nu - 3x^{\nu-3}\nu}{(x^\nu-1)^4\eta^2}\right)(x^\nu-1)^4\eta^2$$
$$- \frac{f(x)\nu^2}{x^{2\nu-2}\nu^4}\left(\frac{2x^{3\nu-3} - 4x^{2\nu-3} + 2x^{\nu-3}}{(x^\nu-1)^4\eta^2}\right)(x^\nu-1)^4\eta^2$$
$$- \frac{\alpha\eta^4(x^\nu-1)^4}{2x^{2\nu-2}\nu^4}\left(-\frac{\nu^2}{\eta^2(x^\nu-1)^3}\right)(x^{2\nu-2}(\nu+1) + x^{\nu-2}(\nu-1))(x^{1+\nu} + x^{1-\nu} - 2x)$$

$$V(\phi) = -\frac{f(x)x^{-\nu}}{x\nu^2}(x^{2\nu}(\nu+1)(\nu+2) + 4x^\nu(\nu^2-1) + (\nu-1)(\nu-2))$$
$$+ \frac{\alpha(x^\nu-1)}{2\nu^2x^{\nu-1}}\frac{x^{2\nu-2}}{x^{\nu-1}}(x^{1+\nu} + x^{1-\nu} - 2x)(x^{-\nu}(\nu-1) + (\nu+1))$$

$$V(\phi) = -\frac{f(x)}{2x\nu^2}(2x^\nu(\nu+1)(\nu+2) + 8(\nu^2-1) + 2x^{-\nu}(\nu-1)(\nu-2))$$
$$+ \frac{\alpha}{2\nu^2x^{\nu-1}}(x^\nu-1)(1 + \nu + x^{-\nu}(\nu-1))(1 + x^{2\nu} - 2x^\nu)$$

$$V(\phi) = -\frac{f(x)}{2x\nu^2}(2x^\nu(\nu+1)(\nu+2) + 8(\nu^2-1) + 2x^{-\nu}(\nu-1)(\nu-2))$$
$$- \frac{\alpha}{2\nu^2x^{\nu-1}}(x^\nu-1)^2(2 - x^\nu(\nu+1) + x^{-\nu}(\nu-1))$$

$$V(\phi) = -\frac{e^{-l_\nu\phi}}{2\nu^2}\left[\frac{1}{l^2}+\frac{\alpha}{2}\left(\frac{e^{(2+\nu)l_\nu\phi}-1}{2+\nu}+\frac{e^{(2-\nu)l_\nu\phi}-1}{2-\nu}-x^2+1\right)\right][8(\nu^2-1)$$

$$+2(\nu+1)(\nu+2)e^{\nu l_\nu\phi}+$$

$$2(\nu-1)(\nu-2)e^{-\nu l_\nu\phi}]-\frac{\alpha e^{l_\nu\phi}}{2\nu^2}\left(\exp\frac{\nu l_\nu\phi}{2}-\exp-\frac{\nu l_\nu\phi}{2}\right)^2[2-e^{\nu l_\nu\phi}(\nu+1)+e^{-\nu l_\nu\phi}(\nu-1)]$$

$$V(\phi) = -\frac{(\nu^2-4)}{l^2\nu^2}\left[\frac{\nu-1}{\nu+2}e^{-\phi l_\nu(\nu+1)}+\frac{\nu+1}{\nu-2}e^{\phi l_\nu(\nu-1)}+4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right]$$

$$-\frac{\alpha}{2\nu^2}\left[\frac{e^{(\nu+2)l_\nu\phi}}{2+\nu}-\frac{e^{(2-\nu)l_\nu\phi}}{\nu-2}-e^{2l_\nu\phi}+\frac{\nu^2}{\nu^2-4}\right][4(\nu^2-1)e^{-l_\nu\phi}+(\nu+1)(\nu+2)e^{(\nu-1)l_\nu\phi}+$$

$$(\nu-1)(\nu-2)e^{-(\nu+1)l_\nu\phi}]$$

$$-\frac{\alpha}{2\nu^2}(e^{\nu l_\nu\phi}-2+e^{-\nu l_\nu\phi})[2e^{l_\nu\phi}-e^{(\nu+1)l_\nu\phi}(\nu+1)+e^{(1-\nu)l_\nu\phi}(\nu-1)]$$

$$V(\phi) = V_\Lambda(\phi)-\frac{\alpha}{2(\nu^2-4)}[\nu^2(e^{l_\nu\phi(\nu-1)}-e^{l_\nu\phi(\nu+1)}-e^{l_\nu\phi(\nu-1)}+e^{-l_\nu\phi(\nu+1)}+4e^{-l_\nu\phi}-4e^{l_\nu\phi})+$$

$$3\nu(e^{\phi l_\nu(\nu-1)}+e^{\phi l_\nu(\nu+1)}-e^{-\phi l_\nu(\nu-1)}-e^{-\phi l_\nu(\nu+1)})$$

$$+2(e^{\phi l_\nu(\nu-1)}-e^{\phi l_\nu(\nu+1)}-e^{-\phi l_\nu(\nu-1)}+e^{-\phi l_\nu(\nu+1)})-4e^{-l_\nu\phi}+4e^{l_\nu\phi}]$$

$$V(\phi) = V_\Lambda(\phi)-\frac{\alpha}{2(\nu^2-4)}[(2\nu^2+6\nu+4)\sinh\,\phi l_\nu(\nu-1)-(2\nu^2-6\nu+4)\sinh\,\phi l_\nu(\nu+1)$$

$$+8(1-\nu^2)\sinh\,\phi l_\nu]$$

$$V(\phi) = \frac{\Lambda(\nu^2-4)}{3\nu^2}\left[\frac{\nu-1}{\nu+2}e^{-\phi l_\nu(\nu+1)}+\frac{\nu+1}{\nu-2}e^{\phi l_\nu(\nu-1)}+4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right]$$

$$+\alpha\left[\frac{\nu-1}{\nu+2}\sinh\,\phi l_\nu(\nu+1)-\frac{\nu+1}{\nu-2}\sinh\,\phi l_\nu(\nu-1)+4\frac{\nu^2-1}{\nu^2-4}\sinh\,l_{\nu\phi}\right]$$

$$dx^\mu\wedge dx^\nu = dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu$$

$$dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p} = \sum_\sigma (-1)^{|\sigma|}dx^{\sigma(\mu_1)}\otimes\ldots\otimes dx^{\sigma(\mu_p)}$$

$$H = \frac{1}{p!}H_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$$

$$A = A_\mu dx^\mu$$

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$$

$$\star(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}) = \frac{\sqrt{-g}}{(D-p)!}\epsilon^{\mu_1\ldots\mu_p}{}_{\nu_{p+1}\ldots\nu_D}dx^{\nu_{p+1}}\wedge\ldots\wedge dx^{\nu_D}$$

$$dV = d^Dx\sqrt{-g} = \frac{\sqrt{-g}}{D!}\epsilon_{\mu_1\ldots\mu_D}dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_D}$$

$$S = -m_1c\int ds_1 - m_2c\int ds_2 - \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R[g]$$

$$\Delta p_i^\mu = p_i^\mu(\tau\to+\infty)-p_i^\mu(\tau\to-\infty)$$

$$= G\Delta p_i^{(1)\mu}+G^2\Delta p_i^{(2)\mu}+\cdots+G^5\Delta p_i^{(5)\mu}+\cdots$$

$$\frac{d}{dx}\vec{I}(x,D) = \hat{M}(x,D)\vec{I}(x,D)$$

$$\mathcal{I}(\phi_1,\phi_2,\ldots,\phi_n;x) := \int_1^x dx'\phi_1(x')\mathcal{I}(\phi_2,\ldots,\phi_n;x')$$

$$\left[\left(x\frac{d}{dx}-1\right)^4-x^4\left(x\frac{d}{dx}+1\right)^4\right]\varpi(x) = 0.$$

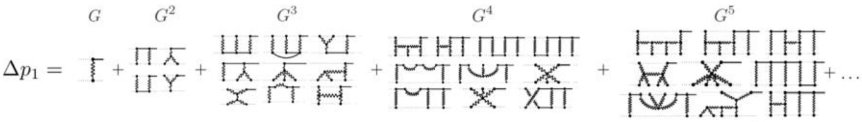
$$\theta^{(5)} = \frac{M^5\Gamma}{b^5c^{10}}(\theta^{(5,0)}+\nu\theta^{(5,1)}+\nu^2\theta^{(5,2)}+\nu^3\Gamma^{-2}\theta^{(5,3)})$$

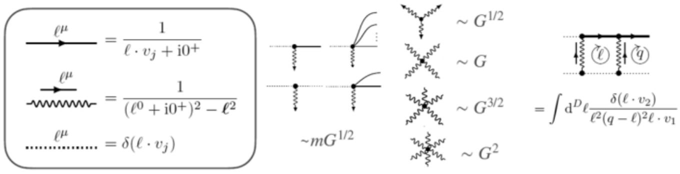
$$P_{\mathrm{rad}}^\mu = -\Delta p_1^\mu-\Delta p_2^\mu$$

$$S_i = -\frac{m_i}{2}\int \quad \mathrm{d}\tau g_{\mu\nu}[x_i(\tau)]\dot{x}_i^{\mu}(\tau)\dot{x}_i^{\nu}(\tau)$$

$$\Delta p_1^{(5)\mu} \quad = m_1 m_2 \left(m_2^4 \Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1 m_2^3 \Delta p_{1\mathrm{SF}}^{(5)\mu}\right.$$

$$\left. + m_1^2 m_2^2 \Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1^3 m_2 \Delta p_{\frac{1\mathrm{SF}}{(5)\mu}} + m_1^4 \Delta p_{\frac{0\mathrm{SF}}{(5)\mu}}\right)$$





$$\mathcal{J}_{\{n\}}^{\{\sigma\}} = \int_{\ell_1 \cdots \ell_4} \frac{\delta^{(\bar{n}_1 - 1)}(\ell_1 \cdot v_1) \prod_{i=2}^{L} \delta^{(\bar{n}_i - 1)}(\ell_i \cdot v_2)}{\prod_{i=1}^{4} D_i^{n_i}(\sigma_i) \prod_{I<J} D_{IJ}^{n_{IJ}}}$$

$$D_1 = \ell_1 \cdot v_2 + \sigma_1 \mathrm{i}0^+, D_i = \ell_i \cdot v_1 + \sigma_i \mathrm{i}0^+$$

$$D_{1j} = \left(\ell_1 - \ell_j\right)^2 + \sigma_{4+j}\operatorname{sign}\left(\ell_1^0 - \ell_j^0\right)\mathrm{i}0^+, D_{q1} = (\ell_1 + q)^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{J}(x,\epsilon) = \epsilon \hat{A}(x)\vec{J}(x,\epsilon),$$

$$\vec{J} = \mathcal{P}e^{\epsilon \int_1^x dx \hat{A}(x)}\vec{J}$$

$$\left[\hat{\theta}^3 - 2x^2\left(2 + 4\hat{\theta} + 3\hat{\theta}^2 + \hat{\theta}^3\right) + x^4(2+\hat{\theta})^3\right] I_1\big|_{\epsilon=0} = 0$$

$$(\mathrm{d} - A^{\mathrm{u}}(x))W^{\mathrm{u}}(x) = 0$$

$$\alpha_1 = \frac{\varpi_0^2}{x(\varpi_0 \varpi_1' - \varpi_0' \varpi_1)},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{\tilde{I}}_{\mathrm{CY3}} \quad = \sum_{i=-2}^{1} \epsilon^i \tilde{M}_{\mathrm{CY3}}^i(x)\vec{\tilde{I}}_{\mathrm{CY3}},$$

$$\vec{\tilde{I}}_{\mathrm{CY3}} \quad = T_{\epsilon\text{-scalings}}\,(W^{\mathrm{ss}})^{-1}\vec{I}_{\mathrm{CY3}}$$

$$G_1'(x) = -\frac{96x(x^4+1)\varpi_0(x)^2}{(x-1)^2(x+1)^2(x^2+1)^2\alpha_1(x)}.$$

$$I_i^{\mathrm{canonicalization}} \quad = f(\epsilon)g(x)\sum_k \ c_k \mathfrak{I}_k$$

$$G_8'(x) = \frac{\varpi_{\mathrm{K3}}(x)G_3(x)\varpi_0(x)\alpha_1'(x)}{\alpha_1(x)^2}.$$

$$\ell_i^{\mathrm{P}} = \left(\ell_i^0, \ell_i\right) \sim (v,1), \ell_i^{\mathrm{R}} = \left(\ell_i^0, \ell_i\right) \sim (v,v).$$

$$\Delta p_{1\mathrm{SF}}^{(5)\mu} \quad = \frac{1}{b^5}\left(\hat{b}^{\mu} c_b(\gamma) + \check{v}_2^{\mu} c_v(\gamma) + \check{v}_1^{\mu} c_v'(\gamma)\right)$$

$$\Delta p_{\frac{1SF}{1SF}}^{(5)\mu} \quad = \frac{1}{b^5}\left(\hat{b}^{\mu} \bar{c}_b(\gamma) + \check{v}_2^{\mu} \bar{c}_v(\gamma) + \check{v}_1^{\mu} \bar{c}_v'(\gamma)\right)$$

$$c_w(\gamma) = c_{w,\,\mathrm{even}}\,(\gamma) + c_{w,\,\mathrm{odd}}\,(\gamma),$$

$$\bar{c}_w(\gamma) = \bar{c}_{w,\,\mathrm{even}}\,(\gamma) + \bar{c}_{w,\,\mathrm{odd}}\,(\gamma),$$

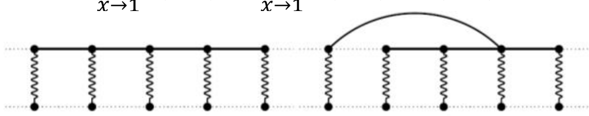
$$c_{w,z}(\gamma) \quad = \sum_{\alpha} \ d_{w,z}^{(\alpha)}(\gamma)F_{w,z}^{(\alpha)}(\gamma)$$

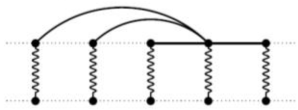
$$+\sum_{\alpha} \ d_{w,z}^{(\alpha,\,\mathrm{tail}\,)}(\gamma)F_{w,z}^{(\alpha,\,\mathrm{tail}\,)}(\gamma)\log\,(\gamma - 1)$$

$$P_{\text{rad}}^{(5)\mu} = \frac{M^6\nu^2}{b^5}\Big(\big[r_1(\gamma)\hat{b}^\mu + r_2(\gamma)\big(v_1^\mu - v_2^\mu\big)\big]\frac{m_1 - m_2}{M}$$
$$+\big[v_1^\mu + v_2^\mu\big]\big(r_3(\gamma) + \nu r_4(\gamma)\big)\Big).$$
$$\mathbf{p}_{\text{out}} = \mathbf{p}_{\text{in}} + \Delta\mathbf{p}_1 + \frac{E_1}{E}\mathbf{P}_{\text{recoil}} + \mathcal{O}(G^6)$$
$$\theta = \Gamma\sum_{n=1}^{5}\left(\frac{GM}{b}\right)^n\sum_{m=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\nu^m\theta^{(n,m)}(\gamma)$$
$$+\frac{1}{\Gamma}\sum_{n=4}^{5}\left(\frac{GM}{b}\right)^n\nu^{n-2}\theta^{(n,n-2)}(\gamma) + \mathcal{O}(G^6).$$
$$E_{\text{rad}}^{(5)} = \frac{M^6\nu^2}{\Gamma b^5}\left[(1-\gamma)r_2(\gamma) + (1+\gamma)r_3(\gamma) + \mathcal{O}(\nu)\right]$$
$$= \frac{M^6\nu^2\pi}{5\Gamma b^5 v^3}\Bigg[122 + \frac{3583}{56}v^2 + \frac{297\pi^2}{4}v^3 - \frac{71471}{504}v^4$$
$$+\left(\frac{9216}{7} - \frac{24993\pi^2}{224}\right)v^5$$
$$+\left(\frac{2904562807}{6899200} + \frac{99\pi^2}{2} - \frac{10593}{70}\log\frac{v}{2}\right)v^6$$
$$+\left(\frac{7296}{7} - \frac{2927\pi^2}{28}\right)v^7$$
$$+\left(\frac{4924457539}{29429400} + \frac{8301\pi^2}{112} - \frac{491013}{3920}\log\frac{v}{2}\right)v^8$$
$$+\left(\frac{99524416}{40425} - \frac{46290891\pi^2}{157696}\right)v^9 + \mathcal{O}(v^{10}, \nu)\Bigg].$$
$$G(a_1, \dots, a_n; y) = \int_0^y \frac{dy'}{y' - a_1}G(a_2, \dots, a_n; y')$$
$$\mathcal{I}\left(\frac{1+x^2}{x}\varpi_{0,\,\text{K3}}(x); x\right)$$
$$\mathcal{I}\left(\frac{1}{(1-x^2)x\varpi_{0,\,\text{K3}}(x)}, \frac{1+x^2}{x}\varpi_{0,\,\text{K3}}(x); x\right)$$
$$\mathcal{I}\left(\frac{1}{(1-x^2)x\varpi_{0,\,\text{K3}}(x)}, \frac{1}{(1-x^2)x\varpi_{0,\,\text{K3}}(x)}, \frac{1+x^2}{x}\varpi_{0,\,\text{K3}}(x); x\right)$$
$$\theta^{(5,1)} = \frac{4}{5v^8} - \frac{137}{5v^6} + \frac{3008}{45v^5} + \frac{\frac{41\pi^2}{4} - \frac{3427}{6}}{v^4} + \frac{\frac{84\pi^2}{5} - \frac{4096}{1575}}{v^3} + \frac{-\frac{12544\log(2v)}{45} + \frac{3593\pi^2}{72} - \frac{445867}{432}}{v^2} + \frac{\frac{2144536}{11025} + \frac{453\pi^2}{35}}{v}$$
$$+\left(-\frac{7552\log(2v)}{1575} + \frac{246527\pi^2}{1440} - \frac{1111790903}{756000}\right) + \left(\frac{19424344}{363825} + \frac{1787\pi^2}{672}\right)v$$
$$+\left(-\frac{1762784\log(2v)}{11025} - \frac{184881\pi^2}{2240} + \frac{56424801733}{49392000}\right)v^2 + \left(\frac{16004496043}{104053950} - \frac{835619\pi^2}{59136}\right)v^3 + \mathcal{O}(v^4),$$
$$\theta^{(5,3)} = \frac{128}{3v} + \left(\frac{14528}{175} + \frac{37\pi^2}{10}\right)v - \frac{22016v^2}{225} + \left(\frac{3262832}{33075} + \frac{893\pi^2}{112}\right)v^3 + \frac{2877184v^4}{7875} + \left(\frac{15803\pi^2}{960} - \frac{4464536}{14553}\right)v^5$$
$$+\frac{970766528v^6}{1819125} + \left(\frac{11234077\pi^2}{394240} - \frac{258810752887}{780404625}\right)v^7 + \frac{83694772064v^8}{70945875}$$
$$+\left(\frac{119425757\pi^2}{2795520} - \frac{5667010769993}{5533778250}\right)v^9 + \frac{11015320038116v^{10}}{5462832375} + \mathcal{O}(v^{11})$$

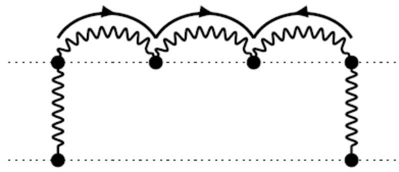
$$r_1(\gamma) = c_b(\gamma) - \bar{c}_b(\gamma)$$
$$r_2(\gamma) = \frac{c_v(\gamma) - \bar{c}_v(\gamma) - c_v'(\gamma) + \bar{c}_v'(\gamma)}{2(\gamma - 1)}$$
$$r_3(\gamma) = -\frac{c_v(\gamma) + \bar{c}_v(\gamma) + c_v'(\gamma) + \bar{c}_v'(\gamma)}{2(\gamma + 1)}$$
$$\left(\Gamma = \sqrt{1 + 2\nu(\gamma - 1)}, \nu = \mu/M = m_1 m_2/M^2, M = m_1 + m_2\right)$$
$$E_{\text{rad}}^{(5)} = \frac{M^6 \nu^2}{\Gamma b^5}[(1-\gamma) r_2(\gamma) + (1+\gamma) r_3(\gamma) + \mathcal{O}(\nu)]$$
$$\mathbf{P}_{\text{recoil}}^{(5)} = \frac{M^5 \nu^2 (m_1 - m_2)}{b^5}\left(r_1(\gamma)\hat{\mathbf{b}} + (r_2(\gamma) - r_3(\gamma))\frac{\mathbf{p}_{\text{in}}}{\mu} + \mathcal{O}(\nu)\right)$$
$$r_1(\gamma) = -\frac{64}{3v^2} - \frac{16192}{525} - \frac{37\pi^2}{20} - \frac{30208v}{225} + \left(-\frac{856768}{33075} - \frac{3429\pi^2}{1120}\right)v^2 - \frac{22016v^3}{2625} + \left(-\frac{1117888}{4851} - \frac{80723\pi^2}{13440}\right)v^4$$
$$-\frac{123897344v^5}{606375} + \left(-\frac{36746586176}{780404625} - \frac{22515319\pi^2}{2365440}\right)v^6 - \frac{16343148032v^7}{70945875} + \left(-\frac{169791059264}{869593725} - \frac{33021283\pi^2}{2562560}\right)v^8$$
$$-\frac{584895938048v^9}{1820944125} + \left(-\frac{53344474395584}{517408266375} - \frac{7894087273\pi^2}{492011520}\right)v^{10} + \mathcal{O}(v^{11}),$$
$$\frac{r_2(\gamma) - r_3(\gamma)}{\pi} = -\frac{53}{3v^3} + \frac{72997}{5040v} + \frac{1491}{400} - \frac{1509\pi^2}{140} + \frac{20211v}{640} + \left(\frac{75661\pi^2}{4480} - \frac{2678867}{16800}\right)v^2$$
$$+\left(\frac{41053\log\left(\frac{v}{2}\right)}{2450} - \frac{503\pi^2}{70} - \frac{123069432361}{4346496000}\right)v^3 + \left(\frac{6139957\pi^2}{394240} - \frac{15259259693}{124185600}\right)v^4$$
$$+\left(\frac{223443793\log\left(\frac{v}{2}\right)}{15523200} - \frac{69203\pi^2}{6720} + \frac{64307545227137}{22375761408000}\right)v^5 + \left(\frac{252585041\pi^2}{5857280} - \frac{16101198460801}{56504448000}\right)v^6$$
$$+\left(\frac{313945836331\log\left(\frac{v}{2}\right)}{8879270400} - \frac{346561\pi^2}{16896} - \frac{204989896406483131}{1828419360768000}\right)v^7 + \left(\frac{918930349\pi^2}{11714560} - \frac{298588187736453 7}{8524099584000}\right)v^8$$
$$+\left(\frac{16292858440697\log\left(\frac{v}{2}\right)}{307814707200} - \frac{288050119\pi^2}{8785920} - \frac{431015900890630345739}{1676186519175168000}\right)v^9$$
$$+\left(\frac{326238718783\pi^2}{2788065280} - \frac{984733411903585726 67}{217581903931392000}\right)v^{10} + \mathcal{O}(v^{11}).$$
$$\mathcal{S}_{\text{bulk}} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\left[\sqrt{-g}g^{\alpha\beta}g^{\gamma\delta}g^{\mu\nu}\left(\Gamma_{\alpha\gamma\delta}\Gamma_{\mu\beta\nu} - \Gamma_{\alpha\gamma\mu}\Gamma_{\delta\beta\nu}\right) + \frac{1}{2}G^\mu G^\nu \eta_{\mu\nu}\right]$$
$$G^\mu = \partial_{\nu_1}h^{\mu\nu_1} - \frac{1}{2}\partial^\mu h + \frac{1}{2}\partial_{\nu_1}hh^{\mu\nu_1} + \frac{1}{2}\partial^\mu h_{\nu_1\nu_2}h^{\nu_1\nu_2} - \partial^{\nu_1}h^{\mu\nu_2}h_{\nu_1\nu_2} - \frac{1}{4}\partial^\mu hh$$
$$+\partial_{\nu_1}h^{\nu_3\mu}h^{\nu_1\nu_2}h_{\nu_2\nu_3} + \frac{1}{2}\partial_{\nu_1}h^{\nu_1\nu_2}h_{\nu_2\nu_3}h^{\nu_3\mu} - \frac{1}{4}\partial_{\nu_2}hh_{\nu_2\nu_3}h^{\nu_3\mu} + \partial_{\nu_1}h_{\nu_2\nu_3}h^{\nu_1\nu_2}h^{\nu_3\mu}$$
$$-\frac{1}{2}\partial^{\nu_2}h_{\nu_1\nu_2}h^{\mu\nu_1}h + \frac{1}{4}\partial^\mu h_{\nu_1\nu_2}h^{\nu_1\nu_2}h - \frac{1}{2}\partial^{\nu_1}h^{\mu\nu_2}h_{\nu_1\nu_2}h + \frac{1}{8}\partial_{\nu_1}h^{\mu\nu_1}h^2.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{J}(x;\epsilon) = \epsilon\hat{A}(x)\vec{J}(x;\epsilon)$$
$$\varpi_0^{[0]} = x + \frac{x^5}{16} + \frac{81x^9}{4096} + \cdots,$$
$$\varpi_1^{[0]} = \log(x)\left(x + \frac{x^5}{16} + \frac{81x^9}{4096} + \cdots\right) + \frac{x^5}{16} + \frac{189x^9}{8192} + \cdots,$$
$$\alpha_1^{[0]} = 1 - \frac{x^4}{4} - \frac{93x^8}{1024} + \cdots.$$

$$\varpi_0^{[1]} = 1 - (1-x) + \frac{(1-x)^3}{3} + \frac{(1-x)^4}{3} + \cdots,$$
$$\varpi_1^{[1]} = 1 - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{6} + \frac{(1-x)^4}{3} + \cdots,$$
$$\alpha_1^{[1]} = 1 - 2\cdot(1-x) + 2\cdot(1-x)^2 - \frac{(1-x)^3}{3} + \frac{(1-x)^4}{2} + \cdots.$$
$$\varpi_{0,\,\mathrm{K3}}^{[0]} = \frac{4}{\pi^2}K^2(x^2) \qquad = 1 + \frac{x^2}{2} + \frac{11x^4}{32} + \cdots,$$
$$\varpi_{0,\,\mathrm{K3}}^{[1]} = \frac{4}{\pi^2}K^2(1-x^2) \quad = 1 - (1-x) + \frac{7(1-x)^2}{8} - \frac{3(1-x)^3}{4} + \cdots$$
$$\hat{A} = \sum_{j=0}^{2} \hat{A}_{-1,j}\frac{\log^j(1-x)}{1-x} + \sum_{i,j\geq 0} \hat{A}_{i,j}(1-x)^i \log^j(1-x)$$
$$\vec{J}(x;\epsilon) = \mathcal{P}e^{\epsilon\int_1^x \mathrm{d}x\hat{A}(x)}\vec{j}$$
$$\vec{J}(x;\epsilon) = \sum_{k,m,n} \vec{f}_{k,m,n}(\epsilon)(1-x)^{k+m\epsilon}\log^n(1-x)$$
$$\frac{\mathrm{d}\hat{S}_0(x;\epsilon)}{\mathrm{d}x} = \epsilon\sum_{j=0}^{2} \hat{A}_{-1,j}\frac{\log^j(1-x)}{1-x}\hat{S}_0(x;\epsilon)$$
$$\vec{J} = \sum_{i,j}^{N} \left(\hat{F}_{i,j}(\epsilon)(1-x)^i\log^j(1-x)\right)\cdot \hat{S}_0(x;\epsilon)\cdot\vec{j} + \mathcal{O}((1-x)^{N+1})$$
$$\lim_{x\to 1}\vec{I}(x;\epsilon) = \lim_{x\to 1}\hat{T}(x;\epsilon)^{-1}\vec{J}(x;\epsilon)$$

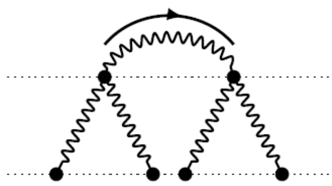




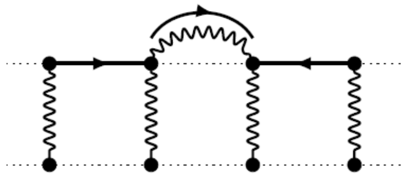
$$-i\delta(x) = \frac{1}{x+\mathrm{i}0^+} - \frac{1}{x-\mathrm{i}0^+}$$
$$M[\ldots,\omega_\times,\ldots] = M[\ldots,\omega_+,\ldots] - M[\ldots,\omega_-,\ldots]$$



$$M[g_+g_+g_-] =$$
$$= -\frac{1}{12288\pi^4} + \frac{12\gamma_E - 41 - 6\log(2) - 12\log(\pi)}{36864\pi^4}\epsilon + \mathcal{O}(\epsilon^2),$$



$$M[0,g_+,0] =$$
$$= -\frac{1}{65536\pi^2} - \frac{4\gamma_E + 3 + 26\log(2) + 4\log(\pi)}{65536\pi^2}\epsilon + \mathcal{O}(\epsilon^2),$$



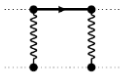
$$M[\omega_+, g_+, \omega_-] = - = \frac{1}{8192\pi^4\epsilon^2} - \frac{2\gamma_E - 3 - 3\log(2) - 2\log(\pi)}{4096\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$

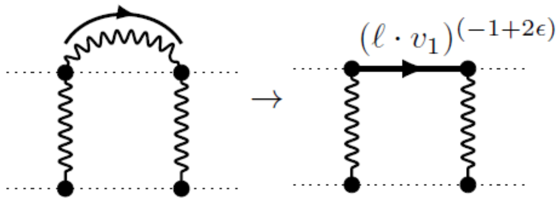
$$\int_\ell = \int \frac{d^D\ell}{(2\pi)^D}$$

$$\int_\ell \frac{\delta(\ell\cdot v_1)}{(\ell^2+i0)^{\nu_1}((\ell-q)^2+i0)^{\nu_2}(\ell\cdot v_2+i0)^{\nu_3}} = (4\pi)^{\frac{1-D}{2}}(-1)^{\nu_1+\nu_2+\nu_3} i^{\nu_3}|q|^{D-1-2\nu_1-2\nu_2-\nu_3}$$

$$\times \frac{\Gamma\left(\nu_1+\nu_2+\frac{\nu_3}{2}-\frac{D-1}{2}\right)\Gamma\left(\frac{\nu_3}{2}\right)\Gamma\left(\frac{D-1}{2}-\nu_1-\frac{\nu_3}{2}\right)\Gamma\left(\frac{D-1}{2}-\nu_2-\frac{\nu_3}{2}\right)}{2\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(D-1-\nu_1-\nu_2-\nu_3)}$$

$$\int_\ell \frac{\delta(\ell\cdot v_1)}{(\ell^2+i0)^{\nu_1}((\ell-q)^2+i0)^{\nu_2}} = (4\pi)^{\frac{1-D}{2}}(-1)^{\nu_1+\nu_2}|q|^{D-1-2\nu_1-2\nu_2}\frac{\Gamma\left(\nu_1+\nu_2-\frac{D-1}{2}\right)\Gamma\left(\frac{D-1}{2}-\nu_1\right)\Gamma\left(\frac{D-1}{2}-\nu_2\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D-1-\nu_1-\nu_2)}$$

$M[\omega_+] =$  $= -i(4\pi)^{\epsilon-1}|q|^{-2(\epsilon+1)}\frac{\Gamma(-\epsilon)^2\Gamma(\epsilon+1)}{2\Gamma(-2\epsilon)}.$



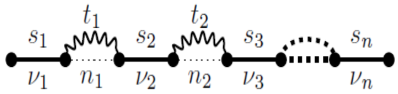
$$\int_\ell \frac{\delta(\ell\cdot v)}{(\ell-k)^2+i0^+(k-\ell)\cdot v} = -(+i)^{1-2\epsilon}(k\cdot v+i0)^{1-2\epsilon}(4\pi)^{\frac{2\epsilon-3}{2}}\Gamma\left(\frac{2\epsilon-1}{2}\right)$$

$$\frac{1}{(\omega+i0^+)^\alpha} = e^{-i\frac{\pi}{2}\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty du\, u^{\alpha-1}e^{-u(0-i\omega)}$$

$$\frac{1}{(\omega-i0^+)^\alpha} = e^{i\frac{\pi}{2}\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty du\, u^{\alpha-1}e^{-u(0+i\omega)}$$

$$\frac{1}{(\ell^2)^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty du\, u^{\alpha-1}e^{-u\ell^2}$$

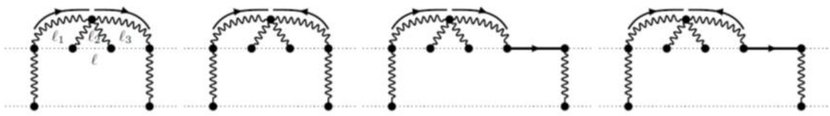
$$\frac{1}{(\omega+i0^+)^\alpha(\omega-i0^+)^\beta} = e^{-i\pi\alpha}\frac{\sin[\pi\beta]}{\sin[\pi(\alpha+\beta)]}\frac{1}{(\omega-i0^+)^{\alpha+\beta}} + e^{i\pi\beta}\frac{\sin[\pi\alpha]}{\sin[\pi(\alpha+\beta)]}\frac{1}{(\omega+i0^+)^{\alpha+\beta}}$$

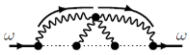


$= \frac{\sin(\pi\alpha_+)}{\sin(\pi[\alpha_+ + \alpha_-])}$ 

$+\frac{\sin(\pi\alpha_-)}{\sin(\pi[\alpha_+ + \alpha_-])}$

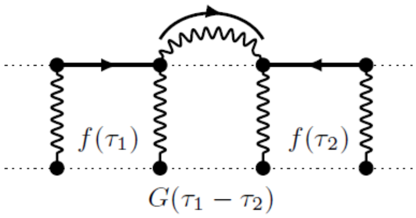
$$\alpha_\pm = \sum_{i|s_i=\pm i0^+}\nu_i + \sum_{a|t_a=\pm i0^+}(2n_a - 3 + 2\epsilon)$$



 $= f(\epsilon)\big(-\mathrm{i}(\omega + \mathrm{i}0^+)\big)^{1-6\epsilon}$,

$$= f(\epsilon)(\omega + \mathrm{i}0^+)^{\frac{1-6\epsilon}{2}}(\omega - \mathrm{i}0^+)^{\frac{1-6\epsilon}{2}}.$$

$$f(\epsilon) = \int_{\ell_1,\ell_2,\ell_3} \frac{1}{(\ell_1-\ell_2)^2(\ell_2-\ell_3)^2(\ell_1^2+1)(\ell_3^2+1)},$$



$$A = M[\omega_+, g_+, \omega_-](|b|,\epsilon) = \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 f(\tau_1)G(\tau_1-\tau_2)f(\tau_2)$$

$$M[\omega_+,\omega_+,g_+] = \frac{1}{2}M_1[\omega_+,\omega_+,g_+]$$

$$M[\omega_-,\omega_-,g_+] = \frac{1}{2}M_1[\omega_-,\omega_-,g_+]$$

$$M[\omega_+,\omega_+,g_+](|b|,\epsilon) = \frac{1}{2}\int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 f(\tau_1)f(\tau_1)G(\tau_1-\tau_2)$$

$$M[\omega_-,\omega_-,g_+](|b|,\epsilon) = \frac{1}{2}\int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 f(-\tau_1)f(-\tau_1)G(\tau_1-\tau_2)$$

$$= \frac{1}{2}\int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 f(\tau_2)f(\tau_2)G(\tau_1-\tau_2)$$

$$B = M[\omega_+,\omega_+,g_+](|b|,\epsilon) + M[\omega_-,\omega_-,g_+](|b|,\epsilon)$$

$$= \frac{1}{2}\int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\big(f^2(\tau_1)+f^2(\tau_2)\big)G(\tau_1-\tau_2).$$

$$f(\tau) = \frac{c_0}{\epsilon} + c_1(\tau) + \mathcal{O}(\epsilon)$$

$$A = \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\left(\frac{c_0^2}{\epsilon^2} + \frac{c_0}{\epsilon}\big(c_1(\tau_1)+c_1(\tau_2)\big) + \mathcal{O}(\epsilon)\right)G(\tau_1-\tau_2)$$

$$= \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\left(\frac{c_0^2}{\epsilon^2} + 2\frac{c_0}{\epsilon}c_1(\tau_1)\right)G(\tau_1-\tau_2) + \mathcal{O}(\epsilon)$$

$$B = \frac{1}{2}\int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\left(\frac{c_0^2}{\epsilon^2} + 2\frac{c_0}{\epsilon}c_1(\tau_1) + \frac{c_0^2}{\epsilon^2} + 2\frac{c_0}{\epsilon}c_1(\tau_2) + \mathcal{O}(\epsilon)\right)G(\tau_1-\tau_2)$$

$$= \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\left(\frac{c_0^2}{\epsilon^2} + 2\frac{c_0}{\epsilon}c_1(\tau_1)\right)G(\tau_1-\tau_2) + \mathcal{O}(\epsilon)$$

$$M[\omega_+,g_+,\omega_-] = -M[\omega_+,\omega_+,g_+] - M[\omega_-,\omega_-,g_+] + \mathcal{O}(\epsilon^0)$$

$$M[\omega_+,g_+\omega_+,\omega_-] = -M[\omega_+,\omega_+,g_+\omega_+] - M[\omega_-,\omega_-,g_+\omega_+] + \mathcal{O}(\epsilon^0)$$

$$M[0,g_+,0] = -\frac{1}{65536\pi^2} - \frac{(3+26\log(2))\epsilon}{65536\pi^2} + \mathcal{O}(\epsilon^2)$$

$$M[g_+,0,0] = -\frac{1}{8192\pi^4} - \frac{(17+6\log(2))\epsilon}{8192\pi^4} + \mathcal{O}(\epsilon^2)$$

$$M[0,g_+\omega_+,\omega_-] = -\frac{1}{4096\pi^4\epsilon^2} - \frac{6+3\log(2)}{2048\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$

$$M[0, g_+\omega_+, \omega_+] = \frac{1}{4096\pi^4\epsilon^2} + \frac{6 + 3\log(2)}{2048\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+, \omega_-, \omega_-] = -\frac{3}{16384\pi^4\epsilon^2} - \frac{13 + 9\log(2)}{8192\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+, \omega_-, \omega_+] = \frac{5}{16384\pi^4\epsilon^2} + \frac{19 + 15\log(2)}{8192\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+, \omega_+, \omega_-] = \frac{1}{16384\pi^4\epsilon^2} + \frac{-1 + 3\log(2)}{8192\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+, \omega_+, \omega_+] = \frac{1}{16384\pi^4\epsilon^2} + \frac{7 + 3\log(2)}{8192\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[\omega_+, g_+, \omega_-] = \frac{1}{8192\pi^4\epsilon^2} + \frac{3 + 3\log(2)}{4096\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[\omega_+, g_+, \omega_+] = \frac{1}{8192\pi^4\epsilon^2} + \frac{7 + 3\log(2)}{4096\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, 0, \omega_-] = -\frac{1}{4096\pi^4\epsilon^2} - \frac{6 + 3\log(2)}{2048\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, 0, \omega_+] = \frac{1}{4096\pi^4\epsilon^2} + \frac{6 + 3\log(2)}{2048\pi^4\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_-, 0] = \frac{1}{8192\pi^2\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_+, 0] = -\frac{1}{8192\pi^2\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[0, g_+, \omega_-] = \frac{i}{768\pi^3} + \frac{55i\epsilon}{2304\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[0, g_+, \omega_+] = \frac{i(-1 + 4\log(2))\epsilon}{768\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[0, g_+\omega_+, 0] = \frac{i}{1024\pi^3} + \frac{i(4 + 2\log(2))\epsilon}{256\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[g_+, 0, \omega_-] = \frac{i}{1024\pi^3} + \frac{i(25 + 2\log(2))\epsilon}{1536\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[g_+, 0, \omega_+] = -\frac{i}{3072\pi^3} + \frac{i(-41 + 30\log(2))\epsilon}{4608\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[g_+, \omega_-, 0] = \frac{i}{2048\pi^3} + \frac{i(2 + 3\log(2))\epsilon}{512\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[g_+, \omega_+, 0] = \frac{i}{2048\pi^3} + \frac{i(4 + 3\log(2))\epsilon}{512\pi^3} + \mathcal{O}(\epsilon^2)$$
$$M[g_+\omega_+, 0,0] = \frac{i}{1024\pi^3} + \frac{i(7 + 6\log(2))\epsilon}{512\pi^3} + \frac{i(272 + 9\pi^2 + 48\log(2)(7 + 3\log(2)))\epsilon^2}{2048\pi^3} + \mathcal{O}(\epsilon^3)$$
$$M[\omega_+, g_+\omega_+, \omega_-] = -\frac{i}{2048\pi^3\epsilon^2} - \frac{i(1 + 2\log(2))}{1024\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[\omega_+, g_+\omega_+, \omega_+] = \frac{i}{2048\pi^3\epsilon^2} + \frac{i(1 + 2\log(2))}{1024\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_-, \omega_-] = \frac{i}{4096\pi^3\epsilon^2} + \frac{i(1 + 2\log(2))}{2048\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_-, \omega_+] = -\frac{3i}{4096\pi^3\epsilon^2} - \frac{3i(1 + 2\log(2))}{2048\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_+, \omega_-] = -\frac{3i}{4096\pi^3\epsilon^2} - \frac{3i(1 + 2\log(2))}{2048\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_+\omega_+, \omega_+, \omega_+] = \frac{i}{4096\pi^3\epsilon^2} + \frac{i(1 + 2\log(2))}{2048\pi^3\epsilon} + \mathcal{O}(\epsilon^0)$$
$$M[g_-g_+, 0] = \frac{1}{30720\pi^4\epsilon} + \frac{58 + 15\log(2)}{115200\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_+g_+,0] = -\frac{1}{30720\pi^4\epsilon} - \frac{58 + 15\log(2)}{115200\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_-g_+\omega_+,\omega_+] = -\frac{1}{4096\pi^4\epsilon} - \frac{7 + 2\log(2)}{2048\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_+g_+\omega_+,\omega_-] = \frac{3}{4096\pi^4\epsilon} + \frac{17 + 6\log(2)}{2048\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_+g_+\omega_+,\omega_+] = -\frac{1}{4096\pi^4\epsilon} - \frac{3 + 2\log(2)}{2048\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_-g_+,\omega_+] = \frac{i}{8192\pi^3\epsilon} + \frac{i(7 + 10\log(2))}{8192\pi^3} + \mathcal{O}(\epsilon^1)$$

$$M[g_+g_+,\omega_-] = \frac{i}{8192\pi^3\epsilon} + \frac{i(7 + 10\log(2))}{8192\pi^3} + \frac{i(342 + 5\pi^2 + 60\log(2)(7 + 5\log(2)))\epsilon}{49152\pi^3} + \mathcal{O}(\epsilon^2)$$

$$M[g_+g_+,\omega_+] = -\frac{i}{8192\pi^3\epsilon} - \frac{i(7 + 10\log(2))}{8192\pi^3} - \frac{i(342 + 29\pi^2 + 60\log(2)(7 + 5\log(2)))\epsilon}{49152\pi^3} + \mathcal{O}(\epsilon^2)$$

$$M[g_-g_+\omega_+,0] = 0$$

$$M[g_+g_+\omega_+,0] = -\frac{i\epsilon}{384\pi^3} + \frac{i(-34 + 9\log(2))\epsilon^2}{576\pi^3} + \mathcal{O}(\epsilon^3)$$

$$M[g_-g_+g_+] = -\frac{1}{12288\pi^4} - \frac{(41 + 6\log(2))\epsilon}{36864\pi^4} + \mathcal{O}(\epsilon^2)$$

$$M[g_+g_+g_+] = \frac{1}{4096\pi^4} + \frac{(41 + 6\log(2))\epsilon}{12288\pi^4} + \frac{(587 - 6\pi^2 + 3\log(2)(41 + 3\log(2)))\epsilon^2}{18432\pi^4} + \mathcal{O}(\epsilon^3)$$

$$M[\mathrm{tail}_+\mathrm{tail}_+] = -\frac{3}{4096\pi^4\epsilon} - \frac{3(8 + \log(2))}{2048\pi^4} - \frac{3(90 + \log(2)(16 + \log(2)))\epsilon}{2048\pi^4} + \mathcal{O}(\epsilon^2)$$

$$M[\ \mathrm{tail}\ _+\mathrm{tail}\ _-] = -\frac{1}{4096\pi^4\epsilon} - \frac{8+\log(2)}{2048\pi^4} + \mathcal{O}(\epsilon^1)$$

$$M[g_-g_+g_+\omega_+] = \frac{i}{4096\pi^3} + \frac{i(1 + 4\log(2))\epsilon}{1024\pi^3} + \mathcal{O}(\epsilon^2)$$

$$M[g_+g_+g_+\omega_+] = -\frac{i}{4096\pi^3} - \frac{i(1 + 4\log(2))\epsilon}{1024\pi^3} + \mathcal{O}(\epsilon^2)$$

$$M[\mathrm{tail}_+\mathrm{tail}_+\omega_+] = \frac{i}{1024\pi^3\epsilon} + \frac{i(1 + 2\log(2))}{128\pi^3} + \mathcal{O}(\epsilon^1)$$

$$M[\ \mathrm{tail}\ _+\mathrm{tail}\_\ _-\omega_+] = 0$$

$$G_1'(x) = -\frac{96x(x^4 + 1)\varpi_0(x)^2}{(x-1)^2(x+1)^2(x^2+1)^2\alpha_1(x)}$$

$$G_2'(x) = -\frac{16(7x^{12} + 314x^{10} + 329x^8 + 1340x^6 + 329x^4 + 314x^2 + 7)\varpi_0(x)^2}{3(x-1)^3x(x+1)^3(x^2+1)^3}$$

$$+\frac{16(7x^8+136x^6+42x^4+136x^2+7)\varpi_0(x)^2\alpha_1'(x)}{3(x-1)^2(x+1)^2(x^2+1)^2\alpha_1(x)}-\frac{2(5x^8+28x^6+262x^4+28x^2+5)\varpi_0(x)^2\alpha_1'(x)^2}{3(x-1)x(x+1)(x^2+1)\alpha_1(x)^2}$$
$$-\frac{xG_1(x)^2\alpha_1(x)^2}{(x-1)(x+1)(x^2+1)\varpi_0(x)^2},$$
$$G_3'(x)=-\frac{xG_1(x)\alpha_1(x)^2}{(x-1)(x+1)(x^2+1)\varpi_0(x)^2},$$
$$G_4'(x)=-\frac{16(7x^8+136x^6+42x^4+136x^2+7)\varpi_0(x)^2}{3(x-1)^2x(x+1)^2(x^2+1)^2\alpha_1(x)}+\frac{4(5x^8+28x^6+262x^4+28x^2+5)\varpi_0(x)^2\alpha_1'(x)}{3(x-1)x^2(x+1)(x^2+1)\alpha_1(x)^2}$$
$$+\frac{G_2(x)}{x\alpha_1(x)},$$
$$G_5'(x)=\varpi_{0,\,\mathrm{K3}}(x)\varpi_0''(x),$$
$$G_6'(x)=\varpi_{0,\,\mathrm{K3}}(x)\varpi_0'(x),$$
$$G_7'(x)=\frac{\varpi_{0,\,\mathrm{K3}}(x)G_3(x)\varpi_0'(x)}{\alpha_1(x)},$$
$$G_8'(x)=\frac{\varpi_{0,\,\mathrm{K3}}(x)G_3(x)\varpi_0(x)\alpha_1'(x)}{\alpha_1(x)^2},$$
$$G_9'(x)=\frac{\varpi_{0,\,\mathrm{K3}}(x)\varpi_0'(x)}{\alpha_1(x)}-\frac{\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)\alpha_1'(x)}{2\alpha_1(x)^2},$$
$$G_{10}'(x)=-\frac{(3x^4-2x^2+3)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{32x^2}+\frac{4x^3(3x^2-5)G_5(x)}{32x^2}+\frac{(27x^4-10x^2+3)G_6(x)}{32x^2},$$
$$G_{11}'(x)=-\frac{3(x-1)(x+1)(x^2+1)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{8x^2}+\frac{12x^5G_5(x)}{8x^2}+\frac{(27x^4-4x^2-3)G_6(x)}{8x^2},$$
$$G_{12}'(x)=-\frac{(x^{12}+2x^{10}-73x^8+236x^6-73x^4+2x^2+1)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{16(x-1)^2x^2(x+1)^2(x^2+1)^2}-\frac{(x^2-3)(3x^2-1)\varpi_{0,\,\mathrm{K3}}(x)G_1(x)\alpha_1(x)}{32(x-1)(x+1)(x^2+1)\varpi_0(x)}$$
$$+\left[\frac{3(x-1)(x+1)(x^2+1)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{16x^2\alpha_1(x)}-\frac{(x^2-3)(3x^2-1)G_5(x)}{32x\alpha_1(x)}-\frac{(9x^4-10x^2-3)G_6(x)}{32x^2\alpha_1(x)}\right]G_3(x)$$
$$+\frac{(x^2-3)(3x^2-1)(x^4-16x^2+1)G_5(x)}{48(x-1)x(x+1)(x^2+1)}+\frac{(9x^{12}-143x^{10}-85x^8+624x^6-551x^4+47x^2+3)G_6(x)}{24(x-1)^2x^2(x+1)^2(x^2+1)^2}$$
$$-\frac{(9x^4-10x^2-3)G_7(x)}{16x^2}+\frac{(9x^4-10x^2-3)G_8(x)}{32x^2}-\frac{2(x^4-16x^2+1)G_{10}(x)}{3(x-1)x(x+1)(x^2+1)}$$
$$G_{13}'(x)=-\frac{15(x^2+1)\varpi_0(x)}{(x-1)^2(x+1)^2}-\frac{1}{\varpi_{0,\,\mathrm{K3}}(x)}\left[\frac{13(x^2-3)(3x^2-1)G_5(x)}{6(x-1)x(x+1)}+\frac{13(9x^4-10x^2-3)G_6(x)}{6(x-1)x^2(x+1)}\right.$$
$$\left.-\frac{208G_{10}(x)}{3(x-1)x(x+1)}\right],$$
$$G_{14}'(x)=-\frac{1}{\varpi_{0,\,\mathrm{K3}}(x)}\left[-\frac{3(x^2+1)G_5(x)}{4x}-\frac{(9x^4-4x^2+3)G_6(x)}{4(x-1)x^2(x+1)}+\frac{2G_{11}(x)}{(x-1)x(x+1)}\right],$$
$$G_{15}'(x)=-\frac{(x^8-12x^6-2x^4-12x^2+1)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{4(x-1)x^2(x+1)(x^2+1)}-\frac{3\varpi_{0,\,\mathrm{K3}}(x)G_1(x)\alpha_1(x)}{8\varpi_0(x)}$$
$$+\left[\frac{(3x^4-2x^2+3)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{4x^2\alpha_1(x)}-\frac{3(x-1)(x+1)(x^2+1)G_5(x)}{8x\alpha_1(x)}-\frac{(9x^4-4x^2+3)G_6(x)}{8x^2\alpha_1(x)}\right.$$
$$\left.+\frac{G_{11}(x)}{x\alpha_1(x)}\right]G_3(x)+\frac{(x^4-16x^2+1)G_5(x)}{4x}+\frac{(9x^8-89x^6+32x^4-11x^2+3)G_6(x)}{6(x-1)x^2(x+1)(x^2+1)}-\frac{3(3x^4+1)G_7(x)}{4x^2}$$
$$+\frac{3(3x^4+1)G_8(x)}{8x^2}-\frac{2(x^4-16x^2+1)G_{11}(x)}{3(x-1)x(x+1)(x^2+1)},$$

$$G'_{16}(x) = -\frac{3(x-1)(x+1)(x^2+1)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{16x^2\alpha_1(x)} + \frac{(x^2-3)(3x^2-1)G_5(x)}{32x\alpha_1(x)} + \frac{(9x^4-10x^2-3)G_6(x)}{32x^2\alpha_1(x)}$$
$$+ \frac{(9x^4-10x^2-3)G_9(x)}{16x^2} - \frac{G_{10}(x)}{x\alpha_1(x)},$$
$$G'_{17}(x) = -\frac{(3x^4-2x^2+3)\varpi_{0,\,\mathrm{K3}}(x)\varpi_0(x)}{4x^2\alpha_1(x)} - \frac{3(x-1)(x+1)(x^2+1)G_5(x)}{8x\alpha_1(x)} + \frac{(9x^4-4x^2+3)G_6(x)}{8x^2\alpha_1(x)}$$
$$+ \frac{3(3x^4+1)G_9(x)}{4x^2} - \frac{G_{11}(x)}{x\alpha_1(x)},$$
$$G'_{18}(x) = \frac{\varpi_0(x)}{(x-1)^2},$$
$$G'_{19}(x) = \frac{\varpi_0(x)}{x},$$
$$G'_{20}(x) = \frac{\varpi_0(x)}{(x+1)^2},$$
$$\varpi_{0,K3} = \left(\frac{2}{\pi}\right)^2 K^2(1-x^2) \text{ and } f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

4. Espacio Cuántico Relativista o Curvo. La curvatura en un campo de gauge indiscriminado, se forma por la gravedad que le resulta permeada, transferida por la interacción del gravitón y su compañera, con las partículas subatómicas de origen bosónico o fermiónico, las mismas que se vuelven superpartículas, lo que distorsiona el espacio – tiempo cuántico. Como ha quedado explícito en este trabajo, el efecto cuántico gravitacional, puede ser intrínseco o endógeno, en tratándose de las partículas supermasivas y exógeno o extrínseco, esto es, por interacción de la partícula de que se trate con el campo gravitónico de gauge. Así es como concluimos que la supergravedad cuántica deriva de una gravedad diseñada en espacios de calibre arbitrarios.

El Modelo Matemático bajo la métrica Friedmann-Lemaître-Robertson-Walker, queda expresado así:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}\times\mathcal{M}} d^d x\sqrt{g}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2\},$$
$$Z[J] \equiv e^{-\frac{1}{\hbar}W[J]} := \int \mathcal{D}\phi e^{-\frac{1}{\hbar}S[\phi]+\frac{1}{\hbar}(J,\phi)}$$
$$\langle\phi(x_1)\phi(x_2)\dots\phi(x_N)\rangle = \frac{\hbar^N}{Z[0]}\frac{\delta}{\delta J(x_N)}\cdots\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}Z[J]\bigg|_{J=0}$$
$$\Gamma[\phi_J] := W[J] + (J,\phi_J),$$
$$\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)} = J(x)$$
$$Z[J] = e^{-\frac{1}{\hbar}S[\phi_J]+\frac{1}{\hbar}(J,\phi_J)}\int \mathcal{D}\phi e^{-\frac{1}{\hbar}(\delta S - J,\phi) - \frac{1}{2\hbar}(\phi,(\delta^2 S,\phi))}$$
$$\delta S(x) := \frac{\delta S[\phi]}{\delta\phi(x)}\bigg|_{\phi=\phi_J} \quad \delta^2 S(x,y) := \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_J}.$$
$$\Gamma[\phi_J] = S[\phi_J] - \hbar\log\int \mathcal{D}\phi e^{-\frac{1}{2}(\phi,(\delta^2 S,\phi))} + \mathcal{O}(\hbar^2),$$
$$\frac{\delta S[\phi]}{\delta\phi(x)}\bigg|_{\phi=\phi_{\text{clás}}} = J(x),$$
$$\phi(x) = \sum_n c_n\varphi_n(x), \text{ con } c_n = (\varphi_n,\phi),$$
$$(\phi, A\phi) = \sum_{n,m} c_n c_m(\varphi_n, A\varphi_m) = \sum_n \lambda_n c_n^2,$$

$$\mathcal{D}\phi = \prod_n dc_n$$

$$\int \mathcal{D}\phi e^{-\frac{1}{2}(\phi, A\phi)} \sim \int \left(\prod_n dc_n\right) e^{-\frac{1}{2}\sum_n \lambda_n c_n^2} = \prod_n \int dc_n e^{-\frac{1}{2}\lambda_n c_n^2}$$

$$\sim \prod_n \lambda_n^{-1/2}$$

$$\Gamma^{(1-\text{ loop })}[\phi] = \frac{1}{2}\log \text{ Det}A.$$

$$\Gamma^{(1-\text{ loop })}[\psi] = -\log \text{ Det}A.$$

$$A = -\partial_\beta^2 - \Delta + \xi R + m^2,$$

$$\Delta = \frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_j\right)$$

$$\log \text{ Det}A := -\frac{d}{ds}\zeta_A(s)\Big|_{s=0},$$

$$\zeta_A(s) := \text{Tr}A^{-s}$$

$$\zeta_A(s) = \sum_{n\in\mathbb{N}} \lambda_n^{-s}$$

$$\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-t}$$

$$\sum_n f(n) = \sum_m \tilde{f}(m),$$

$$\sum_{k\in\mathbb{Z}} e^{-(k+c)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{k\in\mathbb{Z}} e^{-\pi^2k^2/t}e^{2\pi ikc}$$

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\text{Tr}e^{-tA}$$

$$K_A(t) := \text{Tr}e^{-tA} = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}$$

$$\log \text{ Det}A := -\int_0^\infty \frac{dt}{t}K_A(t)$$

$$\zeta_A(s) := \sum_n (\lambda_n/\mu^2)^{-s}$$

$$\zeta(s) = \sum_{\lambda_n>0} \lambda_n^{-s} + (-1)^{-s}\sum_{\lambda_n<0} (-\lambda_n)^{-s}$$

$$\eta(s) = \sum_{\lambda_n>0} \lambda_n^{-s} - \sum_{\lambda_n<0} (-\lambda_n)^{-s},$$

$$\zeta_D'(0) = -i\frac{\pi}{2}[\zeta_{D^2}(0) + \eta_D(0)] + \frac{1}{2}\zeta_{D^2}'(0).$$

$$a(A,B) := \log \frac{\text{Det}AB}{\text{Det}A\text{Det}B},$$

$$\zeta_R(s) := \sum_{n=1}^{\infty} n^{-s}$$

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-nt}$$

$$\zeta_R(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt \frac{t^{s-1}}{e^t - 1}$$

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n$$

$$\zeta_R(s)=\frac{1}{\Gamma(s)}\int_1^{\infty}dt\frac{t^{s-1}}{e^t-1}+\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{B_n}{n!}\frac{1}{s+n-1}$$

$$\zeta_R(s)=\frac{1}{s-1}+\mathcal{O}(1)$$

$$\zeta_H(s,q):=\sum_{n=0}^{\infty}(n+q)^{-s},q\neq 0,-1,-2,\ldots$$

$$ds^2=-dt^2+a^2(\eta)\left[\frac{dr^2}{1-\kappa r^2}+r^2(d\theta^2+\text{sen}^2\theta d\phi^2)\right]$$

$$x^0=a\cos\,\chi\cos\,\theta$$

$$x^2=a\text{sen}\chi\text{sen}\phi$$

$$x^3=a\cos\,\chi\text{sen}\theta$$

$$\frac{ds^2}{a^2}=d\chi^2+\cos^2\,\chi d\theta^2+\text{sen}^2\chi d\phi^2$$

$$g^1=\frac{1}{a^2};g^2=\frac{1}{a^2\cos^2\,\chi};g^3=\frac{1}{a^2\text{sen}^2\chi}$$

$$\triangle_{S^3}=\frac{1}{a^2}\left[\frac{1}{\cos\,\chi\text{sen}\chi}\partial_\chi(\cos\,\chi\text{sen}\chi\partial_\chi)+\frac{1}{\cos^2\,\chi}\partial_\theta^2+\frac{1}{\text{sen}^2\chi}\partial_\phi^2\right]$$

$$\chi_k(h)=\sum_{\alpha=1}^{s}a_\alpha\chi^{(\alpha)}(h).$$

$$a_\alpha=\frac{1}{|H|}\sum_{h\in H}\chi_k(h)\chi^{(\alpha)^*}(h),$$

$$a_1=\frac{1}{|H|}\sum_{h\in H}\chi_k(h)$$

$$\chi_k(\theta_L,\theta_R)=\frac{\text{sen}(k\theta_L/2)}{\text{sen}(\theta_L/2)}\frac{\text{sen}(k\theta_R/2)}{\text{sen}(\theta_R/2)}$$

$$\chi_k(h)=k\frac{\text{sen}(k\theta_h/2)}{\text{sen}(\theta_h/2)}$$

$$d_k^{(H)}=\frac{k}{|H|}\sum_{h\in H}\frac{\text{sen}(k\theta_h/2)}{\text{sen}(\theta_h/2)}$$

$$f(\chi,\theta,\phi)=X(\chi)\Theta(\theta)\Phi(\phi),$$

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m_1^2$$

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=m_2^2$$

$$-\frac{1}{\cos\,\chi\text{sen}\chi}\frac{d}{d\chi}\left(\cos\,\chi\text{sen}\chi\frac{dX}{d\chi}\right)+\left(\frac{m_1^2}{\text{sen}^2\chi}+\frac{m_2^2}{\cos^2\,\chi}\right)X=a^2\lambda X,$$

$$\Phi(\phi)=e^{im_1\phi}$$

$$\Theta(\theta)=e^{im_2\theta}$$

$$X(\chi)=\cos^\alpha\,\chi\text{sen}^\beta\chi\tilde{X}(\chi)$$

$$\tilde{X}''-\left[\frac{m_1^2-\frac{1}{4}}{\text{sen}^2\chi}+\frac{m_2^2-\frac{1}{4}}{\cos^2\,\chi}-(a^2\lambda+1)\right]\tilde{X}=0.$$

$$\tilde{X}_{n,m_1,m_2}(\chi)=(\text{sen}\chi)^{|m_1|+\frac{1}{2}}(\cos\,\chi)^{|m_2|+\frac{1}{2}}P_n^{(|m_1|,|m_2|)}(\cos\,2\chi),$$

$$a^2\lambda=(2n+|m_1|+|m_2|+1)^2-1$$

$$f_{k,m_1,m_2}(\chi,\theta,\phi) = e^{im_1\phi}e^{im_2\theta}\operatorname{sen}^{|m_1|}\chi\cos^{|m_2|}\ \chi P^{(|m_1|,|m_2|)}_{(k-|m_1|-|m_2|-1)/2}(\cos\ 2\chi)$$

$$d_k = 1 + \sum_{n=0}^{\frac{k-1}{2}-1} 4(k-1-2n) = k^2$$

$$d_k = \sum_{n=0}^{\frac{k-2}{2}} 4(k-1-2n) = k^2$$

$$\theta \to \theta + \frac{2\pi m}{p}, \phi \to \phi + \frac{2\pi m}{p}$$

$$k = 2n + |m_1| + |m_2| + 1$$

$$lp = m_1 + m_2$$

$$2b = 2n + |m_1| + |m_2|$$

$$(2q+1)l = m_1 + m_2$$

$$d^{(\mathrm{I})}_{2b+1} = \sum_{\substack{l=0 \\ l\ \text{par}}}^{l_{\text{máx}}} [(2q+1)l + 1]$$

$$d^{(\mathrm{II})}_{2b+1} = d^{(\mathrm{I})}_{2b+1}$$

$$d^{(\mathrm{III})}_{2b+1} = b + 2\sum_{j=1}^{b} \left\lfloor \frac{j}{2q+1} \right\rfloor$$

$$d^{(\mathrm{III},-)}_{2b+1} = \begin{cases} \left\lfloor \frac{2b}{2q+1} \right\rfloor + 1, & \text{si } r \text{ es par} \\ \left\lfloor \frac{2b}{2q+1} \right\rfloor, & \text{si } r \text{ es impar} \end{cases}$$

$$d^{(\mathrm{IV})}_{2b+1} = \begin{cases} d^{(\mathrm{III})}_{2b+1} - \left\lfloor \frac{2b}{2q+1} \right\rfloor, & \text{si } r \text{ es par} \\ d^{(\mathrm{III})}_{2b+1} - \left\lfloor \frac{2b}{2q+1} \right\rfloor + 1, & \text{si } r \text{ es impar} \end{cases}$$

$$d_k^{k\ \text{impar}} = \begin{cases} k\left(\left\lfloor \frac{k-1}{2q+1} \right\rfloor + 1\right), & \text{si } r \text{ es par} \\ k\left\lfloor \frac{k-1}{2q+1} \right\rfloor, & \text{si } r \text{ es impar} \end{cases}$$

$$d_k^{(p=2q+1)} = \begin{cases} k\left(\frac{k-r}{2q+1} + 1\right), & \text{si } r \text{ es impar} \\ k\left(\frac{k-r}{2q+1}\right), & \text{si } r \text{ es par} \end{cases}$$

$$k = 2n + |m_1| + |m_2| + 1$$

$$2lq = m_1 + m_2$$

$$2b = 2n + |m_1| + |m_2|,$$

$$2lq = m_1 + m_2.$$

$$d^{(\mathrm{II})}_{2b+1} = \sum_{l=1}^{\lfloor b/q \rfloor} (2lq + 1).$$

$$d^{(\mathrm{IV})}_{2b+1} = b + 2\sum_{n=0}^{b-1} \left\lfloor \frac{b-n}{q} \right\rfloor$$

$$d_k^{(p=2q)} = \begin{cases} k\left(1 + 2\left\lfloor \frac{k}{2q} \right\rfloor\right), & \text{si } k \text{ es impar} \\ 0, & \text{si } k \text{ es par} \end{cases}$$

$$\frac{\operatorname{sen}(k(\theta+2\pi)/2)}{\operatorname{sen}((\theta+2\pi)/2)}=\frac{\operatorname{sen}(k\pi+k\theta/2)}{\operatorname{sen}(\pi+\theta/2)}=(-1)^{k+1}\frac{\operatorname{sen}(k\theta/2)}{\operatorname{sen}(\theta/2)}$$

$$d_k^{(H^*)}=\frac{k}{|H^*|}[1+(-1)^{k+1}]\sum_{h\in H}\frac{\operatorname{sen}(k\theta_h/2)}{\operatorname{sen}(\theta_h/2)}$$

$$d_k^{(p)}=\frac{k}{2p}\left[\lim_{\theta\to 0}\frac{\operatorname{sen}(k\theta/2)}{\operatorname{sen}(\theta/2)}+\sum_{m=1}^{p-1}\frac{\operatorname{sen}(km\pi/p)}{\operatorname{sen}(m\pi/p)}+p\frac{\operatorname{sen}(k\pi/2)}{\operatorname{sen}(\pi/2)}\right].$$

$$\sum_{m=1}^{p-1}\frac{\operatorname{sen}(km\pi/p)}{\operatorname{sen}(m\pi/p)}=p-k+2p\left\lfloor\frac{k}{2p}\right\rfloor$$

$$d_k^{(p)}=\begin{cases}k\left(\left\lfloor\frac{k}{2p}\right\rfloor+\frac{1+(-1)^{\lfloor(k-1)/2\rfloor}}{2}\right), & \text{si } k \text{ es impar}\\ 0, & \text{si } k \text{ es par}\end{cases}$$

$$d_{2n+1}^{(p)}=\begin{cases}(2n+1)\left\lfloor\frac{n}{p}\right\rfloor, & \text{si } n \text{ es impar}\\ (2n+1)\left(\left\lfloor\frac{n}{p}\right\rfloor+1\right), & \text{si } n \text{ es par}\end{cases}$$

$$d_{2n+1}^{(T^*)}=(2n+1)\left\{2\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-n+1\right\}$$

$$d_{2n+1}^{(O^*)}=(2n+1)\left\{\left\lfloor\frac{n}{4}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-n+1\right\}$$

$$d_{2n+1}^{(I^*)}=(2n+1)\left\{\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-n+1\right\}$$

$$S[\phi]=\frac{1}{2}\int_{\mathbb{R}\times\mathcal{M}}d^4x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{a^2}\phi^2\right),$$

$$A=-\frac{1}{\sqrt{g}}\partial_\mu\left(g^{\mu\nu}\sqrt{g}\partial_\nu\right)+\frac{1}{a^2}$$

$$\zeta_{S^1\times\mathcal{M}}(s)=\mu^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}d_k^{(H)}\left[\left(\frac{2\pi l}{\beta}\right)^2+\left(\frac{k}{a}\right)^2\right]^{-s}$$

$$\zeta_{S^3}(s)=\mu^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}k^2\left[\left(\frac{2\pi l}{\beta}\right)^2+\left(\frac{k}{a}\right)^2\right]^{-s}.$$

$$\zeta_{S^3}(s)=(\mu a)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}k^2\left[\left(\frac{2\pi al}{\beta}\right)^2+k^2\right]^{-s}.$$

$$\zeta_{S^3}^{l=0}(s)=(\mu a)^{2s}\sum_{k=1}^{\infty}k^{-(2s-2)}=(\mu a)^{2s}\zeta_R(2s-2).$$

$$\Gamma_{S^3}^{l=0}=\frac{\zeta_R(3)}{4\pi^2}$$

$$\zeta_{S^3}^{l\neq 0}(s)=(\mu a)^{2s}\{\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}\left[k^2+\left(\frac{2l\pi a}{\beta}\right)^2\right]^{-(s-1)}$$

$$-\sum_{l=1}^{\infty}\left(\frac{2l\pi a}{\beta}\right)^2\sum_{k\in\mathbb{Z}}\left[k^2+\left(\frac{2l\pi a}{\beta}\right)^2\right]^{-s}\Bigg\}$$

$$\zeta_1(s)=\frac{(\mu a)^{2s}}{\Gamma(s-1)}\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}\int_0^\infty dt\,t^{s-1-1}e^{-[k^2+(2\pi al/\beta)^2]t}$$

$$\zeta_1(s)=\frac{(\mu a)^{2s}\sqrt{\pi}}{\Gamma(s-1)}\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}\int_0^\infty dt\,t^{s-\frac{3}{2}-1}e^{-\pi^2k^2/t}e^{-(2l\pi a)^2t/\beta^2}$$

$$\zeta_1(s)=\frac{(\mu a)^{2s}(s-1)\sqrt{\pi}}{\Gamma(s)}\{\Gamma\left(s-\frac{3}{2}\right)\left(\frac{2\pi a}{\beta}\right)^{3-2s}\zeta_R(2s-3)$$

$$+2\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2al}{\beta k}\right)^{\frac{3}{2}-s}K_{s-\frac{3}{2}}(4\pi^2lka/\beta)\Bigg\}$$

$$\zeta_2(s)=-\frac{(\mu a)^{2s}\sqrt{\pi}}{\Gamma(s)}\{\Gamma\left(s-\frac{1}{2}\right)\left(\frac{2\pi a}{\beta}\right)^{3-2s}\zeta_R(2s-3)$$

$$+2\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{2\pi al}{\beta}\right)^{2}\left(\frac{2al}{\beta k}\right)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(4\pi^2lka/\beta)\Bigg\}$$

$$\Gamma_{S^3}=\frac{\zeta_R(3)}{4\pi^2}-\frac{\pi^4}{45}\left(\frac{a}{\beta}\right)^3$$

$$+\frac{1}{4\pi^2}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2+2\frac{4\pi^2kla}{\beta}+\left(\frac{4\pi^2kla}{\beta}\right)^2\right]e^{-4\pi^2kla/\beta}.$$

$$\zeta_{S^3}(s)=\left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}k^2\left[\left(\frac{\beta k}{2\pi a}\right)^2+l^2\right]^{-s}$$

$$\zeta_{S^3}(s)=\frac{1}{\Gamma(s)}\left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty}k^2\int_0^{\infty}dt\,t^{s-1}e^{-[(\beta k/2\pi a)^2+l^2]t}$$

$$\zeta_{S^3}(s)=\frac{\sqrt{\pi}}{\Gamma(s)}\left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{k=1}^{\infty}k^2\sum_{l\in\mathbb{Z}}\int_0^{\infty}dt\,t^{s-\frac{3}{2}}e^{-(\beta k/2\pi a)^2t}e^{-(\pi l)^2/t}$$

$$\zeta_{S^3}^{l=0}(s)=\frac{(\mu a)^{2s}\beta}{2\sqrt{\pi}a}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta_R(2s-3)$$

$$\zeta_{S^3}^{l\neq 0}(s)=\frac{4\sqrt{\pi}}{\Gamma(s)}\left(\frac{2\pi}{\mu\beta}\right)^{-2s}\sum_{k=1}^{\infty}k^2\sum_{l=1}^{\infty}\left(\frac{2\pi^2al}{\beta k}\right)^{s-\frac{1}{2}}K_{\frac{1}{2}-s}(kl\beta/a)$$

$$\Gamma_{S^3}=\frac{\beta}{2a}\zeta_R(-3)-\sqrt{\frac{2\beta}{\pi a}}\sum_{k=1}^{\infty}k^2\sum_{l=1}^{\infty}\sqrt{\frac{k}{l}}K_{\frac{1}{2}}(kl\beta/a),$$

$$\Gamma_{S^3}=\frac{\beta}{240a}-\sum_{k=1}^{\infty}k^2\sum_{l=1}^{\infty}\frac{e^{-kl\beta/a}}{l}=\frac{\beta}{240a}+\sum_{k=1}^{\infty}k^2\log\left(1-e^{-k\beta/a}\right).$$

$$\sum_{n=0}^{\infty}f(n)=\oint_{C}dz\frac{f(z)}{e^{2\pi iz}-1}$$

$$\oint_{C'}dz\frac{f(z)}{e^{2\pi iz}-1}=i\int_{\epsilon}^{\infty}dy\frac{f(iy)}{1-e^{-2\pi y}}+\frac{1}{2}\mathrm{Res}\left(\frac{f(z)}{e^{2\pi iz}-1},0\right)$$

$$+i\int_{\epsilon}^{\infty}dy\frac{f(-iy)}{1-e^{2\pi y}}$$

$$\int_{\epsilon}^{\infty}dy\frac{f(iy)}{1-e^{-2\pi y}}=\int_{\epsilon}^{\infty}dyf(iy)\frac{1-e^{-2\pi y}+e^{-2\pi y}}{1-e^{-2\pi y}}$$

$$=-i\int_0^{\infty}dxf(x+i\epsilon)+\int_{\epsilon}^{\infty}dy\frac{f(iy)}{e^{2\pi y}-1}$$

$$\sum_{n=0}^{\infty}f(n)=\frac{1}{2}f(0)+\int_0^{\infty}dxf(x)+i\int_0^{\infty}dy\frac{f(iy)-f(-iy)}{e^{2\pi y}-1}$$

$$\int_{C_\epsilon^{(n)}} dz \frac{z^2 \log\left(1-e^{-z\beta/a}\right)}{e^{2\pi i z}-1},$$

$$\int_0^\infty dx x^2 \log\left(1-e^{-x\beta/a}\right) = -\frac{\pi^4 a^3}{45\beta^3}$$

$$i\int_0^\infty dy \frac{g(iy)-g(-iy)}{e^{2\pi y}-1} = \int_0^\infty dy \frac{y^2(\pi-\beta x/a)}{e^{2\pi y}-1}$$

$$+2\pi\sum_{n=0}^\infty n \int_{2\pi n a/\beta}^{2\pi(n+1)a/\beta} dy \frac{y^2}{e^{2\pi y}-1}$$

$$\int_0^\infty dy \frac{y^2(\pi-\beta x/a)}{e^{2\pi y}-1} = \frac{\zeta_R(3)}{4\pi^2} - \frac{\beta}{240a}$$

$$\frac{1}{e^{2\pi x-1}} = e^{-2\pi x}\frac{1}{1-e^{-2\pi x}} = e^{-2\pi x}\sum_{k=0}^\infty e^{-2\pi k x},$$

$$\Gamma_{S^3} = -\frac{\pi^4 a^3}{45\beta^3} + \frac{\zeta_R(3)}{4\pi^2} + \frac{1}{4\pi^2}\sum_{l,k=1}^\infty \frac{1}{k^3}\left[2+2\frac{4\pi^2 lka}{\beta} + \left(\frac{4\pi^2 lka}{\beta}\right)^2\right] e^{-4\pi^2 lka/\beta}$$

$$\zeta_{\mathcal{M}}(s) = (\mu a)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^\infty d_k^{(H)}\left[\left(\frac{2\pi a l}{\beta}\right)^2 + k^2\right]^{-s},$$

$$d_k^{(Z_{2q+1})} = \begin{cases} \dfrac{k^2}{2q+1} - \dfrac{rk}{2q+1} + k, & \text{si } r \text{ es impar} \\ \dfrac{k^2}{2q+1} - \dfrac{rk}{2q+1}, & \text{si } r \text{ es par} \end{cases}$$

$$\zeta_{S^3/\mathbb{Z}_{2q+1}}(s) = \frac{\zeta_{S^3}(s)}{2q+1} + \delta\zeta_{2q+1}(s)$$

$$\delta\zeta_{2q+1}(s) = -\frac{(\mu a)^{2s}}{p}\sum_{l\in\mathbb{Z}}\sum_{n=0}^\infty \left\{\sum_{r=1}^q 2r(np+2r)\left[(np+2r)^2 + \left(\frac{2\pi a l}{\beta}\right)^2\right]^{-s}\right.$$

$$\left. + \sum_{r=0}^{q-1} 2(r-q)(np+2r+1)\left[(np+2r+1)^2 + \left(\frac{2\pi a l}{\beta}\right)^2\right]^{-s}\right\}$$

$$\delta\zeta_{2q+1}(s) = -2\left(\frac{\mu a}{p}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{n=0}^\infty \left\{\sum_{r=1}^q r\left(n+\frac{2r}{p}\right)\left[\left(n+\frac{2r}{p}\right)^2 + \left(\frac{2\pi a l}{p\beta}\right)^2\right]^{-s}\right.$$

$$\left. - \sum_{r=1}^q r\left(n+1-\frac{2r}{p}\right)\left[\left(n+1-\frac{2r}{p}\right)^2 + \left(\frac{2\pi a l}{p\beta}\right)^2\right]^{-s}\right\}$$

$$\delta\zeta_{2q+1}^{l=0}(s) = -2\left(\frac{\mu a}{p}\right)^{2s}\sum_{r=1}^q r\left[\zeta_H\left(2s-1,\frac{2r}{p}\right) - \zeta_H\left(2s-1,1-\frac{2r}{p}\right)\right]$$

$$\delta\zeta_{2q+1}^{l=0}(s) = -4\left(\frac{\mu a}{p}\right)^{2s}\frac{\Gamma(2-2s)(2\pi)^{2s-1}}{\Gamma(s)\Gamma(1-s)}\sum_{r=1}^q r\sum_{n=1}^\infty n^{2s-2}\operatorname{sen}\left(\frac{4\pi n r}{p}\right)$$

$$\delta\zeta_{2q+1}^{l\neq 0}(s) = -4\left(\frac{\mu a}{p}\right)^{2s}\sum_{r=1}^q r\sum_{n\in\mathbb{Z}}\sum_{l=1}^\infty \left(n+\frac{r}{p}\right)\left[\left(n+\frac{r}{p}\right)^2 + \left(\frac{2\pi a l}{p\beta}\right)^2\right]^{-s}$$

$$\delta\zeta_{2q+1}^{l\neq 0}(s) = -\frac{(\mu a)^{2s} p^{1-2s}}{(1-s)} \sum_{r=1}^{q} r$$

$$\times \frac{d}{d\alpha} \sum_{n\in\mathbb{Z}} \sum_{l=1}^{\infty} \left[\left(n + \frac{2r + 2\alpha - 2}{p} \right)^2 + \left(\frac{2\pi a l}{p\beta} \right)^2 \right]^{-s+1}$$

$$\delta\zeta_{2q+1}^{l\neq 0}(s) = -\frac{16(\mu a)^{2s}\pi^{\frac{3}{2}}}{\Gamma(s)p^{2s}} \sum_{r=1}^{q} r \sum_{l,n=1}^{\infty} n\mathrm{sen}\left(\frac{4\pi n r}{p}\right)$$

$$\times \left(\frac{p\beta n}{al}\right)^{s-\frac{3}{2}} K_{s-\frac{3}{2}}(4\pi^2 a l n / p\beta)$$

$$\Gamma_{S^3/Z_{p=2q+1}} = \frac{\zeta_R(3)}{4\pi^2 p} + \frac{1}{\pi} \sum_{r=1}^{q} r \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{sen}\left(\frac{4\pi n r}{p}\right) - \frac{\pi^4}{45p}\left(\frac{a}{\beta}\right)^3$$

$$+ \frac{1}{4\pi^2 p} \sum_{k,l=1}^{\infty} \frac{1}{k^3} \left[2 + 2\frac{4\pi^2 akl}{\beta} + \left(\frac{4\pi^2 akl}{\beta}\right)^2 \right] e^{-4\pi^2 akl/\beta}$$

$$+ \frac{2}{\pi p} \sum_{r=1}^{q} r \sum_{n,l=1}^{\infty} \frac{1}{n^2} \mathrm{sen}\left(\frac{4\pi n r}{p}\right) \left(1 + \frac{4\pi^2 aln}{p\beta}\right) e^{-4\pi^2 anl/p\beta}$$

$$\zeta_{S^3/Z_2}(s) = (\mu a)^{2s} \sum_{l\in\mathbb{Z}} \sum_{n=0}^{\infty} (2n+1)^2 \left[\tilde{\omega}_l^2 + (2n+1)^2\right]^{-s}$$

$$\zeta_{S^3/Z_2}(s) = (\mu a)^{2s} \sum_{l\in\mathbb{Z}} \sum_{k=1}^{\infty} \left\{ k^2\left(\tilde{\omega}_l^2 + k^2\right)^{-s} - (2k)^2\left[\tilde{\omega}_l^2 + (2k)^2\right]^{-s} \right\}.$$

$$\Gamma_{S^3/Z_2}(\beta) = \Gamma_{S^3}(\beta) - 4\Gamma_{S^3}(2\beta).$$

$$d_{2k+1}^{(Z_{2q})} = \left(2\left[\frac{2k+1}{2q}\right] + 1\right)(2k+1)$$

$$d_{2k+1}^{(Z_{2q})} = \left(2\frac{k-r}{q} + 1\right)(2k+1) = \frac{(2k+1)^2}{q} + \frac{q-2r-1}{q}(2k+1)$$

$$\zeta_{S^3/Z_{2q}}(s) = \frac{\zeta_{S^3/Z_2}(s)}{q} + \delta\zeta_{2q}(s)$$

$$\delta\zeta_{2q}(s) = 2\left(\frac{\mu a}{2q}\right)^{2s} \sum_{r=0}^{q-1} [(q-r-1) - r]$$

$$\times \sum_{l\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(n + \frac{2r+1}{2q}\right) \left[\left(n + \frac{2r+1}{2q}\right)^2 + \left(\frac{\pi a l}{q\beta}\right)^2\right]^{-s}$$

$$\delta\zeta_{2q}(s) = 2\left(\frac{\mu a}{2q}\right)^{2s} \sum_{r=1}^{q-1} r \sum_{l\in\mathbb{Z}} \sum_{n=0}^{\infty} \left\{ \left(n + 1 - \frac{2r+1}{2q}\right) \left[\left(n + 1 - \frac{2r+1}{2q}\right)^2 + \left(\frac{\pi a l}{q\beta}\right)^2\right]^{-s} \right.$$

$$\left. - \left(n + \frac{2r+1}{2q}\right) \left[\left(n + \frac{2r+1}{2q}\right)^2 + \left(\frac{\pi a l}{q\beta}\right)^2\right]^{-s} \right\}$$

$$\delta\zeta_{2q}^{l=0}(s) = -2\left(\frac{\mu a}{2q}\right)^{2s} \sum_{r=1}^{q-1} r\left\{\zeta_H\left(2s-1, \frac{2r+1}{2q}\right)\right.$$

$$\left. -\zeta_H\left(2s-1, 1 - \frac{2r+1}{2q}\right)\right\}$$

$$\delta\zeta_{2q}^{l=0}(s) = -2\left(\frac{\mu a}{2q}\right)^{2s}\left[\sum_{r=0}^{q-1}(2r+1)\zeta_H\left(2s-1,\frac{2r+1}{2q}\right) - q^{2s}\zeta_H\left(2s-1,\frac{1}{2}\right)\right]$$

$$\delta\zeta_{2q}^{l\neq 0}(s) = -\frac{16\pi^{3/2}}{q\Gamma(s)}\left(\frac{\mu a}{2q}\right)^{2s}\sum_{r=1}^{q-1} r \sum_{l,n=1}^{\infty} n\mathrm{sen}(2\pi n\frac{2r+1}{2q})$$

$$\times\left(\frac{q\beta n}{al}\right)^{s-\frac{3}{2}} K_{s-\frac{3}{2}}\left(\frac{2\pi^2 aln}{q\beta}\right)$$

$$\Gamma_{S^3/Z_{p=2q}} = -\frac{6\zeta_R(3)}{4\pi^2 p} + \frac{1}{\pi}\sum_{r=1}^{q-1} r\sum_{n=1}^{\infty}\frac{1}{n^2}\mathrm{sen}\left(2\pi n\frac{2r+1}{p}\right) - \frac{\pi^4}{45p}\left(\frac{a}{\beta}\right)^3$$

$$+\frac{1}{2\pi^2 p}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2+2\frac{4\pi^2 akl}{\beta}+\left(\frac{4\pi^2 akl}{\beta}\right)^2\right]e^{-4\pi^2 akl/\beta}$$

$$-\frac{2}{\pi^2 p}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2+2\frac{2\pi^2 akl}{\beta}+\left(\frac{2\pi^2 akl}{\beta}\right)^2\right]e^{-2\pi^2 akl/\beta}$$

$$+\frac{4}{\pi p}\sum_{r=1}^{q-1} r\sum_{n,l=1}^{\infty}\frac{1}{n^2}\mathrm{sen}\left(2\pi n\frac{2r+1}{p}\right)\left(1+\frac{4\pi^2 aln}{\beta p}\right)e^{-4\pi^2 aln/\beta p}.$$

$$d_{2k+1}^{(D_p^*)} = \begin{cases}(2k+1)\left\lfloor\frac{k}{p}\right\rfloor, & \text{si } k \text{ es impar},\\ (2k+1)\left(\left\lfloor\frac{k}{p}\right\rfloor+1\right), & \text{si } k \text{ es par},\end{cases}$$

$$d_{2k+1}^{(D_p^*)} = \frac{1}{2}d_{2k+1}^{(Z_{2p})} + \frac{(-1)^k}{2}(2k+1)$$

$$\zeta_{S^3/D_p^*}(s) = \frac{1}{2}\zeta_{S^3/Z_{2p}}(s) + \delta\zeta(s),$$

$$\delta\zeta(s) = \frac{(\mu a)^{2s}}{2}\sum_{l\in\mathbb{Z}}\sum_{k=0}^{\infty}(-1)^k(2k+1)\left[(2k+1)^2+\left(\frac{2\pi al}{\beta}\right)^2\right]^{-s}.$$

$$\delta\zeta(s) = \frac{(\mu a)^{2s}}{2}\sum_{l\in\mathbb{Z}}\left\{\sum_{k=0}^{\infty}(4k+1)\left[(4k+1)^2+\left(\frac{2\pi al}{\beta}\right)^2\right]^{-s}\right.$$

$$\left.-\sum_{k=0}^{\infty}(4k+3)\left[(4k+3)^2+\left(\frac{2\pi al}{\beta}\right)^2\right]^{-s}\right\}$$

$$\delta\zeta^{l=0}(s) = \frac{(\mu a)^{2s}4^{1-2s}}{2}\left[\zeta_H\left(2s-1,\frac{1}{4}\right)-\zeta_H\left(2s-1,\frac{3}{4}\right)\right].$$

$$\delta\zeta^{l\neq 0}(s) = (\mu a)^{2s}\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}(4k+1)\left[(4k+1)^2+\left(\frac{2\pi al}{\beta}\right)^2\right]^{-s}$$

$$\delta\zeta^{l\neq 0}(s) = \frac{(\mu a)^{2s}}{2(1-s)}\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}\frac{d}{d\alpha}\left[(4k+\alpha)^2+\left(\frac{2\pi al}{\beta}\right)^2\right]^{-s+1}\Bigg|_{\alpha=1}$$

$$\delta\zeta^{l\neq 0}(s) = -\frac{\sqrt{\pi}(\mu a)^{2s}}{8\Gamma(s)}\frac{d}{d\alpha}\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}}e^{i\frac{\pi}{2}\alpha k}\int_0^{\infty}dt\,t^{s-\frac{3}{2}-1}e^{-t(2\pi al/\beta)^2-(\pi k)^2/16t}\Bigg|_{\alpha=1}$$

$$\delta\zeta^{l\neq 0}(s) = \frac{\pi^{\frac{3}{2}}(\mu a)^{2s}}{4\Gamma(s)}\sum_{\substack{k,l=1\\ k\text{ impar}}}^{\infty}(-1)^{\left\lfloor\frac{k}{2}\right\rfloor}k\left(\frac{8al}{k\beta}\right)^{\frac{3}{2}-s}K_{\frac{3}{2}-s}(\pi^2 kla/\beta)$$

$$\Gamma_{S^3/D_p^*} = -\frac{3\zeta_R(3)}{16\pi^2 p} + \frac{1}{2\pi}\sum_{r=1}^{p-1} r \sum_{n=1}^{\infty} \frac{1}{n^2}\mathrm{sen}\left(2\pi n\frac{2r+1}{2p}\right) - \frac{G}{\pi}$$
$$-\frac{\pi^4}{180p}\left(\frac{a}{\beta}\right)^3 - \frac{2}{\pi}\sum_{\substack{k,l=1 \\ k \text{ impar}}}^{\infty} (-1)^{\left\lfloor\frac{k}{2}\right\rfloor}\frac{1}{k^2}\left(1+\frac{\pi^2 akl}{\beta}\right)e^{-\pi^2 kla/\beta}$$
$$+\frac{1}{2\pi^2 p}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2+2\frac{4\pi^2 akl}{\beta}+\left(\frac{4\pi^2 kla}{\beta}\right)^2\right]e^{-4\pi^2 akl/\beta}$$
$$-\frac{1}{2\pi^2 p}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2+2\frac{2\pi^2 akl}{\beta}+\left(\frac{2\pi^2 kla}{\beta}\right)^2\right]e^{-2\pi^2 akl/\beta}$$
$$+\frac{1}{\pi p}\sum_{r=1}^{p-1} r \sum_{n,l=1}^{\infty}\frac{1}{n^2}\mathrm{sen}\left(2\pi n\frac{2r+1}{2p}\right)\left(1+\frac{2\pi^2 aln}{\beta p}\right)e^{-2\pi^2 lna/\beta p}$$
$$d_{2n+1}^{(T^*)} = d_{2n+1}^{(Z_6)} + \frac{1}{2}d_{2n+1}^{(Z_4)} - \frac{1}{2}d_{2n+1}^{(Z_2)}$$
$$d_{2n+1}^{(O^*)} = \frac{1}{2}d_{2n+1}^{(Z_8)} + \frac{1}{2}d_{2n+1}^{(Z_6)} + \frac{1}{2}d_{2n+1}^{(Z_4)} - \frac{1}{2}d_{2n+1}^{(Z_2)}$$
$$d_{2n+1}^{(I^*)} = \frac{1}{2}d_{2n+1}^{(Z_{10})} + \frac{1}{2}d_{2n+1}^{(Z_6)} + \frac{1}{2}d_{2n+1}^{(Z_4)} - \frac{1}{2}d_{2n+1}^{(Z_2)}$$
$$\Gamma = \beta E_0 + \sum_{k=1}^{\infty} d_k \log\left(1-e^{-\beta\lambda_k}\right)$$
$$\zeta_{S^3/H}(s) = \left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty} d_k\left[\left(\frac{\lambda_k\beta}{2\pi}\right)^2 + l^2\right]^{-s}$$
$$\zeta_{S^3/H}(s) = \frac{1}{\Gamma(s)}\left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{k=1}^{\infty} d_k\int_0^{\infty} dt t^{s-1}e^{-t[l^2+(\lambda_k\beta/2\pi)^2]}$$
$$\zeta_{S^3/H}^{l=0}(s) = \frac{\beta\mu^{2s}}{2\sqrt{\pi}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{k=1}^{\infty} d_k\lambda_k^{1-2s},$$
$$\zeta_{S^3/H}^{l\neq 0}(s) = \frac{2\sqrt{\pi}}{\Gamma(s)}\left(\frac{2\pi}{\mu\beta}\right)^{-2s}\sum_{k=0}^{\infty} d_k\sum_{l=1}^{\infty} 2\left(\frac{2\pi^2 l}{\lambda_k\beta}\right)^{s-\frac{1}{2}} K_{\frac{1}{2}-s}(\lambda_k\beta l).$$
$$E_0 = \frac{1}{2}\lim_{s\to 0}\sum_{k=1}^{\infty} d_k\left(\lambda_k^2\right)^{\frac{1}{2}-s},$$
$$\Gamma_{S^3/H} = \beta E_0 - 2\sqrt{\pi}\sum_{k=1}^{\infty} d_k\sum_{l=1}^{\infty}\left(\frac{2\pi^2 l}{\lambda_k\beta}\right)^{-1/2} K_{\frac{1}{2}}(\lambda_k\beta l).$$
$$E_{0,S^3/Z_{2q+1}} = \frac{E_{0,S^3}}{2q+1} + \delta E_{0,2q+1},$$
$$\delta E_{0,2q+1} = \frac{1}{2}\lim_{s\to 0} a^{2s-1}\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{q-1}\left(1-\frac{2r+1}{2q+1}\right)[n(2q+1)+2r+1]^{2-2s} - \sum_{r=0}^{q}\frac{2r}{2q+1}[n(2q+1)+2r]^{2-2s}\right\}$$
$$\delta E_{0,2q+1} = \frac{2q+1}{a\pi^3}\sum_{r=1}^{q} r\sum_{n=1}^{\infty}\frac{1}{n^3}\,\mathrm{sen}\left(\frac{4\pi nr}{2q+1}\right).$$

$$E_{0,S^3/Z_{2q+1}} = -\frac{(2q+1)^4 + 10(2q+1)^2 - 14}{720a(2q+1)}.$$

$$E_{0,S^3/Z_2} = \frac{1}{2}\lim_{s\to 0} a^{2s-1}\sum_{k=0}^{\infty}(2k+1)^{3-2s} = -\frac{7}{240a}$$

$$E_{0,S^3/Z_{2q}} = \frac{E_{0,S^3/Z_2}}{q} + \delta E_{0,2q}$$

$$\delta E_{0,2q} = \frac{1}{2}\lim_{s\to 0} a^{2s-1}\sum_{r=0}^{q-1}[(q-r-1)-r]\sum_{n=0}^{\infty}(2nq+2r+1)^{2-2s}$$

$$E_{0,S^3/Z_{2q}} = -\frac{(2q)^4 + 10(2q)^2 - 14}{720a(2q)}$$

$$E_{0,S^3/Z_p} = -\frac{p^4 + 10p^2 - 14}{720ap}$$

$$\Gamma_{S^3/Z_p} = \frac{14 - 10p^2 - p^4}{720p}\frac{\beta}{a} + \mathcal{O}\left(e^{-\beta/a}\right)$$

$$E_{0,S^3/D_p^*} = \frac{1}{2}E_{0,S^3/Z_{2p}} + \delta E_0$$

$$\delta E_0 = \frac{1}{4}\lim_{s\to 0} a^{2s-1}\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2-2s}$$

$$E_{0,S^3/D_p^*} = -\frac{8p^4 + 20p^2 + 180p - 7}{1440ap}$$

$$E_{0,S^3/T^*} = -\frac{3761}{360\times 24a}, E_{0,S^3/O^*} = -\frac{11321}{360\times 48a}$$

$$E_{0,S^3/I^*} = -\frac{43553}{360\times 120a}$$

$$E_{S^3/H} \sim \frac{\pi^4 a^3}{15|H|\beta^4}$$

$$S_{S^3/H} \sim \frac{4\pi^4}{45|H|}\left(\frac{a}{\beta}\right)^3 + S_{0,S^3/H},$$

$$S_{\mathrm{top},S^3/H} = \lim_{\beta\to 0}\left[S_{S^3/H} - \frac{1}{|H|}S_{S^3}\right].$$

$$S = \frac{1}{2}\int_{\mathbb{R}\times S^3/H} d^4x\sqrt{g}\left\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{R}{6}\phi^2 + \left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\phi^2\right\},$$

$$\zeta_{S^3/H}(s) = \mu^{2s}\sum_{k=1}^{\infty}\sum_{l\in\mathbb{Z}} d_k^{(H)}\left[\left(\frac{k}{a}\right)^2 + m^2 + \left(\frac{2\pi l}{\beta}\right)^2\right]^{-s},$$

$$\zeta_{S^3}(s) = \left(\frac{\mu\beta}{2\pi}\right)^{2s}\sum_{k=1}^{\infty}\sum_{l\in\mathbb{Z}} k^2\left[\left(\frac{k\beta}{2\pi a}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2 + l^2\right]^{-s},$$

$$\zeta_{S^3}(s) = \left(\frac{\mu\beta}{2\pi}\right)^{2s}\frac{\pi^{\frac{1}{2}}}{\Gamma(s)}\sum_{k=1}^{\infty}\sum_{l\in\mathbb{Z}} k^2\int_0^{\infty} dt\, t^{s-\frac{1}{2}-1} e^{-\left[\left(\frac{k\beta}{2\pi a}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right]t - (\pi l)^2/t}$$

$$\zeta_{S^3}(s) = \left(\frac{\mu\beta}{2\pi}\right)^{2s}\frac{\pi^{\frac{1}{2}}}{\Gamma(s)}\left\{\Gamma\left(s - \frac{1}{2}\right)\sum_{k=1}^{\infty} k^2\left[\left(\frac{k\beta}{2\pi a}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right]^{\frac{1}{2}-s}\right.$$

$$\left. + 4\sum_{k,l=1}^{\infty} k^2(\pi l)^{s-\frac{1}{2}}\left[\left(\frac{k\beta}{2\pi a}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right]^{\frac{1}{4}-\frac{s}{2}} K_{\frac{1}{2}-s}\left(\sqrt{k^2 + (am)^2}\,\beta l/a\right)\right\}$$

$$\Gamma_{S^3}^{l\neq 0}(\beta) = \sum_{k=1}^{\infty} k^2 \log\left(1 - e^{-\frac{\beta}{a}\sqrt{k^2+(am)^2}}\right)$$

$$\zeta_{S^3}^{l=0}(s) = \frac{(\mu a)^{2s}\beta}{4\pi^{\frac{1}{2}}a}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{k\in\mathbb{Z}}\left\{[k^2+(am)^2]^{\frac{3}{2}-s} - (am)^2[k^2+(am)^2]^{\frac{1}{2}-s}\right\}$$

$$\zeta_{S^3}^{l=0}(s) = \frac{(\mu a)^{2s}}{\Gamma(s)}\frac{\beta}{4a}\left\{\frac{1}{2}(am)^{4-2s}\Gamma(s-2)\right.$$
$$+4\left(s-\frac{3}{2}\right)\sum_{k=1}^{\infty}\left(\frac{\pi k}{am}\right)^{s-2} K_{s-2}(2\pi amk)$$
$$\left.-4(am)^2\sum_{k=1}^{\infty}\left(\frac{\pi k}{am}\right)^{s-1} K_{s-1}(2\pi amk)\right\}$$

$$\Gamma_{S^3}^{l=0}(\beta) = -\frac{(am)^4}{32}\frac{\beta}{a}\left(\frac{3}{2}+2\log\frac{\mu}{m}\right)$$
$$+\frac{(am)^3}{2\pi}\frac{\beta}{a}\sum_{k=1}^{\infty}\frac{1}{k}K_1(2\pi amk) + \frac{3(am)^2}{4\pi^2}\frac{\beta}{a}\sum_{k=1}^{\infty}\frac{1}{k^2}K_2(2\pi amk).$$

$$\Gamma_{S^3}(\beta) = \frac{3(am)^2}{4\pi^2}\frac{\beta}{a}\sum_{k=1}^{\infty}\frac{1}{k^2}K_2(2\pi amk) + \frac{(am)^3}{2\pi}\frac{\beta}{a}\sum_{k=1}^{\infty}\frac{1}{k}K_1(2\pi amk)$$
$$+\sum_{k=1}^{\infty} k^2 \log\left(1 - e^{-\frac{\beta}{a}\sqrt{k^2+(am)^2}}\right)$$

$$E_0 = \frac{3(am)^2}{4\pi^2 a}\sum_{k=1}^{\infty}\frac{1}{k^2}K_2(2\pi amk) + \frac{(am)^3}{2\pi a}\sum_{k=1}^{\infty}\frac{1}{k}K_1(2\pi amk)$$

$$\zeta_{S^3}^{l=0}(s) = (\mu a)^{2s}\frac{\beta}{2\pi^{\frac{1}{2}}a}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{k=1}^{\infty} k^{3-2s}\left[1+\left(\frac{am}{k}\right)^2\right]^{\frac{1}{2}-s}$$

$$\zeta_{S^3}^{l=0}(s) = (\mu a)^{2s}\frac{\beta}{2\pi^{\frac{1}{2}}a}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{3}{2}-s\right)(am)^{2n}}{n!\,\Gamma\left(\frac{3}{2}-n-s\right)}\zeta_R(2s+2n-3)$$

$$\Gamma_{S^3}^{l=0} = -\frac{(am)^4}{16}\frac{\beta}{a}[\log(\mu a/2)+1+\gamma] + \frac{\beta}{2a}\sum_{\substack{n=0\\ n\neq 2}}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)(am)^{2n}}{n!\,\Gamma\left(\frac{3}{2}-n\right)}\zeta_R(2n-3)$$

$$\Gamma_{S^3}(\beta) = \frac{\beta}{240a} - \frac{(am)^2}{48}\frac{\beta}{a} - \frac{(am)^4}{16}\frac{\beta}{a}\left[\log\left(\frac{ma}{2}\right)+\frac{1}{4}+\gamma\right]$$
$$+\frac{\beta}{2a}\sum_{n=3}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)(am)^{2n}}{n!\,\Gamma\left(\frac{3}{2}-n\right)}\zeta_R(2n-3) + \sum_{k=1}^{\infty} k^2\log\left(1-e^{-\frac{\beta}{a}\sqrt{k^2+(am)^2}}\right)$$

$$\zeta_{S^3}(s) = \frac{\mu^{2s}a^2}{2}\sum_{k,l\in\mathbb{Z}}\left[\left(\frac{k}{a}\right)^2 + m^2 + \left(\frac{2\pi l}{\beta}\right)^2\right]^{-s+1}$$
$$-\frac{\mu^{2s}a^2}{2}\sum_{k,l\in\mathbb{Z}}\left[m^2+\left(\frac{2\pi l}{\beta}\right)^2\right]\left[\left(\frac{k}{a}\right)^2 + m^2 + \left(\frac{2\pi l}{\beta}\right)^2\right]^{-s},$$

$$\zeta_{S^3}(s) = \frac{\pi^{\frac{1}{2}}(\mu a)^{2s}}{2}\sum_{l\in\mathbb{Z}}\left[\left(a^2\lambda_{l,0}\right)^{-s+\frac{3}{2}}\frac{\Gamma\left(s-\frac{3}{2}\right)}{2\Gamma(s)}\right.$$

$$+\frac{4}{\Gamma(s-1)}\sum_{k=1}^{\infty}(k\pi)^{s-\frac{3}{2}}\left(a^2\lambda_{l,0}\right)^{-\frac{s}{2}+\frac{3}{4}}K_{s-\frac{3}{2}}\left(2k\pi\sqrt{a^2\lambda_{l,0}}\right)$$

$$\left.-\frac{4}{\Gamma(s)}\sum_{k=1}^{\infty}(k\pi)^{s-\frac{1}{2}}\left(a^2\lambda_{l,0}\right)^{-\frac{s}{2}+\frac{5}{4}}K_{s-\frac{1}{2}}\left(2k\pi\sqrt{a^2\lambda_{l,0}}\right)\right]$$

$$\zeta_{S^3}^{k=0}(s) = (\mu a)^{2s}\frac{\pi^{\frac{1}{2}}}{4}\frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)}\left[(am)^{3-2s}\right.$$

$$\left.+2\left(\frac{2\pi a}{\beta}\right)^{3-2s}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{5}{2}-s\right)}{n!\,\Gamma\left(\frac{5}{2}-n-s\right)}\left(\frac{m\beta}{2\pi}\right)^{2n}\zeta_R(2s+2n-3)\right]$$

$$\Gamma_{S^3}^{k=0}(\beta) = -\frac{\pi}{6}(am)^3 - \frac{\pi}{3}\left(\frac{2\pi a}{\beta}\right)^3\sum_{\substack{n=0\\n\neq 2}}^{\infty}\frac{\Gamma\left(\frac{5}{2}\right)}{n!\,\Gamma\left(\frac{5}{2}-n\right)}\left(\frac{m\beta}{2\pi}\right)^{2n}\zeta_R(2n-3)$$

$$-\frac{(am)^4}{32}\frac{\beta}{a}\left[2\log\left(\mu\beta/2\pi\right)+2\gamma-2\log\ 2\right]$$

$$\Gamma_{S^3} = -\frac{\pi}{6}(am)^3 - \frac{\pi^4}{45}\left(\frac{a}{\beta}\right)^3 + \frac{\pi^2}{12}(am)^2\frac{a}{\beta}$$

$$+\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^3}\left[1+2\pi mak+\frac{1}{2}(2\pi mak)^2\right]e^{-2\pi mak}+\mathcal{O}(m\beta)$$

$$\zeta_{S^3}^{k=0}(s) = \frac{\pi(\mu a)^{2s}}{4\Gamma(s)}\left(\frac{2\pi a}{\beta}\right)^{3-2s}\left[\left(\frac{m\beta}{2\pi}\right)^{4-2s}\Gamma(s-2)+4\sum_{l=1}^{\infty}\left(\frac{2\pi^2 l}{m\beta}\right)^{s-2}K_{2-s}(m\beta l)\right]$$

$$\Gamma_{S^3}^{k=0}(\beta) = -\frac{(am)^4}{32}\frac{\beta}{a}\left[\frac{3}{2}+2\log\left(\frac{\mu}{m}\right)\right]-(am)^2\frac{a}{\beta}\sum_{l=1}^{\infty}\frac{1}{l^2}K_2(m\beta l).$$

$$\Gamma_{S^3}(\beta) = -(am)^2\frac{a}{\beta}\sum_{l=1}^{\infty}\frac{1}{l^2}K_2(m\beta l)+\mathcal{O}\left(e^{-a/\beta}\right)$$

$$\zeta_{S^3/Z_{2q+1}}(s) = \mu^{2s}\sum_{l\in\mathbb{Z}}\sum_{n=0}^{\infty}\left[\sum_{\substack{r=0\\r\text{ par}}}^{p-1}(np+r)n+\sum_{\substack{r=1\\r\text{ impar}}}^{p-2}(np+r)(n+1)\right]\lambda_{l,np+r}^{-s}$$

$$=\frac{1}{p}\zeta_{S^3}(s)+\delta\zeta_{2q+1}(s)$$

$$\delta\zeta_{2q+1}(s) = -\frac{\mu^{2s}}{p}\sum_{l\in\mathbb{Z}}\sum_{n=0}^{\infty}\left\{\sum_{r=1}^{q}2r(np+2r)\lambda_{l,np+2r}^{-s}\right.$$

$$\left.+\sum_{r=0}^{q-1}2(r-q)(np+2r+1)\lambda_{l,np+2r+1}^{-s}\right\}$$

$$\delta\zeta_{2q+1}(s) = -2\left(\frac{\mu a}{p}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{n=0}^{\infty}\sum_{r=1}^{q} r$$

$$\times\left\{\left(n+\frac{2r}{p}\right)\left[\left(n+\frac{2r}{p}\right)^2+\left(\frac{am}{p}\right)^2+\left(\frac{2\pi al}{p\beta}\right)^2\right]^{-s}\right.$$

$$\left.-\left(n+1-\frac{2r}{p}\right)\left[\left(n+1-\frac{2r}{p}\right)^2+\left(\frac{am}{p}\right)^2+\left(\frac{2\pi al}{p\beta}\right)^2\right]^{-s}\right\}$$

$$\delta\zeta_{2q+1}^{l=0}(s) = -2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{\mu a}{p}\right)^{2s}\sum_{n=0}^{\infty}\sum_{r=1}^{q} r\left\{\left(n+\frac{2r}{p}\right)\left[\left(n+\frac{2r}{p}\right)^2+\left(\frac{am}{p}\right)^2\right]^{\frac{1}{2}-s}\right.$$

$$\left.-\left(n+1-\frac{2r}{p}\right)\left[\left(n+1-\frac{2r}{p}\right)^2+\left(\frac{am}{p}\right)^2\right]^{\frac{1}{2}-s}\right\}$$

$$\delta\zeta_{2q+1}^{l=0}(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)}\left(\frac{\mu a}{p}\right)^{2s}\textstyle\sum_{n\in\mathbb{Z}}\sum_{r=1}^{q} r\frac{d}{d\alpha}\left[\left(n+\frac{2r}{p}+\alpha\right)^2+\left(\frac{am}{p}\right)^2\right]^{\frac{3}{2}-s}\Bigg|_{\alpha=0}$$

$$\delta\zeta_{2q+1}^{l=0}(s) = \frac{8(am)^2}{p^2\Gamma(s)}\left(\frac{\pi\mu^2}{pm^2}\right)^{s}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\sum_{r=1}^{q} r\mathrm{sen}(4\pi nr/p)K_{2-s}(2\pi amn/p)$$

$$\delta\zeta_{2q+1}^{l=0}(s) = \frac{1}{s-1}\left(\frac{\mu a}{p}\right)^{2s}\sum_{n\in\mathbb{Z}}\sum_{r=1}^{q} r\frac{d}{d\alpha}\left[\left(n+\frac{2r}{p}+\alpha\right)^2+\left(\frac{am}{p}\right)^2\right]^{-s+1}\Bigg|_{\alpha=0}$$

$$\delta\zeta_{2q+1}^{l=0}(s) = \frac{-8}{\Gamma(s)}\left(\frac{am}{p}\right)^{\frac{3}{2}}\left(\frac{\pi\mu^2 a}{pm}\right)^{s}\sum_{r=1}^{q} r\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\mathrm{sen}(4\pi rn/p)K_{\frac{3}{2}-s}(2\pi amn/p)$$

$$\delta\Gamma_{S^3/Z_{p=2q+1}}^{l=0} = \frac{1}{\pi}\sum_{r=1}^{q} r\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\pi amn}{p}+1\right)\mathrm{sen}(4\pi rn/p)e^{-2\pi amn/p}$$

$$\zeta_{S^3/Z_2}(s) = \mu^{2s}\sum_{l\in\mathbb{Z}}\left\{\sum_{k=1}^{\infty} k^2\left[\left(\frac{k}{a}\right)^2+m^2+\left(\frac{2\pi l}{\beta}\right)^2\right]^{-s}\right.$$

$$\left.-\sum_{k=1}^{\infty}(2k)^2\left[\left(\frac{2k}{a}\right)^2+m^2+\left(\frac{2\pi l}{\beta}\right)^2\right]^{-s}\right\}$$

$$= \zeta_{S^3}(s;\beta,a,m) - 2^{2-2s}\zeta_{S^3}(s;2\beta,a,m/2),$$

$$\Gamma_{S^3/Z_2}(\beta,a,m) = \Gamma_{S^3}(\beta,a,m) - 4\Gamma_{S^3}(2\beta,a,m/2)$$

$$\zeta_{S^3/Z_{2q}}(s) = \frac{1}{q}\zeta_{S^3/Z_2}(s) + \delta\zeta_{2q}(s),$$

$$\delta\zeta_{2q}(s) = \mu^{2s}\sum_{n=0}^{\infty}\sum_{r=0}^{q-1}\left(\frac{q-r-1}{q}-\frac{r}{q}\right)(2nq+2r+1)$$

$$\times\sum_{l\in\mathbb{Z}}\left[\left(\frac{2nq+2r+1}{a}\right)^2+m^2+\left(\frac{2\pi l}{\beta}\right)^2\right]^{-s}$$

$$\delta\zeta_{2q}(s) = 2q\left(\frac{\mu a}{2q}\right)^{2s}\sum_{l\in\mathbb{Z}}\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{q-1}\frac{r}{q}\left(n+1-\frac{2r+1}{2q}\right)\right.$$

$$\times\left[\left(n+1-\frac{2r+1}{2q}\right)^2+\left(\frac{am}{2q}\right)^2+\left(\frac{2\pi al}{2q\beta}\right)^2\right]^{-s}$$

$$\left.-\sum_{r=0}^{q-1}\frac{r}{q}\left(n+\frac{2r+1}{2q}\right)\left[\left(n+\frac{2r+1}{2q}\right)^2+\left(\frac{am}{2q}\right)^2+\left(\frac{2\pi al}{2q\beta}\right)^2\right]^{-s}\right\}$$

$$\delta\zeta_{2q}^{l=0}(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{\mu a}{2q}\right)^{2s}\sum_{n\in\mathbb{Z}}\sum_{r=1}^{q-1}r\frac{d}{d\alpha}\left[\left(n+\frac{2r+1}{2q}+\alpha\right)^2+\left(\frac{am}{2q}\right)^2\right]^{\frac{1}{2}-s}\Bigg|_{\alpha=0}$$

$$\delta\zeta_{2q}^{l=0}(s) = \frac{2\pi am}{\Gamma(s)}\left(\frac{\pi a\mu^2}{2qm}\right)^{s}\sum_{r=1}^{q-1}r\sum_{n=1}^{\infty}n^s\mathrm{sen}\left(2\pi n\frac{2r+1}{2q}\right)K_{1-s}(\pi amn/q)$$

$$\delta\zeta_{2q}^{l=0}(s) = \frac{2}{s-1}\left(\frac{\mu a}{2q}\right)^{2s}\sum_{n\in\mathbb{Z}}\sum_{r=1}^{q-1}r\frac{d}{d\alpha}\left[\left(n+\frac{2r+1}{2q}+\alpha\right)^2+\left(\frac{am}{2q}\right)^2\right]^{-s+1}\Bigg|_{\alpha=0}$$

$$\delta\zeta_{2q}^{l=0}(s) = \frac{-8}{\Gamma(s)}\left(\frac{am}{2q}\right)^{\frac{3}{2}}\left(\frac{\pi\mu^2 a}{2qm}\right)^{s}\sum_{r=1}^{q-1}r\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\mathrm{sen}\left(2\pi n\frac{2r+1}{2q}\right)K_{\frac{3}{2}-s}(\pi amn/q)$$

$$\delta\Gamma_{S^3/Z_{2q}}^{l=0} = \frac{1}{\pi}\sum_{r=1}^{q-1}r\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\pi amn}{2q}+1\right)\mathrm{sen}\left(2\pi n\frac{2r+1}{2q}\right)e^{-\pi amn/q}$$

$$\zeta_{S^3/D_p^*}(s) = \frac{1}{2}\zeta_{S^3/Z_{2p}}(s)+\delta\zeta(s)$$

$$\delta\zeta(s) = \frac{\mu^{2s}}{2}\sum_{l\in\mathbb{Z}}\sum_{k=0}^{\infty}\left[(4k+1)\lambda_{l,4k+1}^{-s}-(4k+3)\lambda_{l,4k+3}^{-s}\right]$$

$$\delta\zeta(s) = \frac{(\mu a)^{2s}}{4(-s+1)}\sum_{k,l\in\mathbb{Z}}\frac{d}{d\alpha}\left[\left(\frac{2\pi al}{\beta}\right)^2+(am)^2+(4k+1)^2\right]^{-s+1}\Bigg|_{\alpha=1}$$

$$\delta\zeta^{l=0}(s) = -\frac{2(am)^2}{\Gamma(s)}\left(\frac{\pi\mu^2 a}{4m}\right)^{s}\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\mathrm{sen}(\pi k/2)K_{2-s}(\pi amk/2)$$

$$\delta\zeta^{l=0}(s) = \frac{\pi^{\frac{3}{2}}(\mu a)^{2s}}{16\Gamma(s)}\sum_{k=1}^{\infty}k\mathrm{sen}(\pi k/2)\left(\frac{4am}{\pi k}\right)^{\frac{3}{2}-s}K_{\frac{3}{2}-s}(\pi amk/2).$$

$$\delta\Gamma_{S^3/D_p^*}^{l=0} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\mathrm{sen}(\pi k/2)\left(1+\frac{\pi amk}{2}\right)e^{-\pi amk/2},$$

$$\Gamma = \beta E_0^{(H)}+\sum_{k=1}^{\infty}d_k^{(H)}\log\left(1-e^{-\beta\sqrt{\lambda_k^2+m^2}}\right),$$

$$S_{S^3/H} = \sum_{k=1}^{\infty}d_k^{(H)}\left[\frac{\frac{\beta}{a}\sqrt{k^2+(am)^2}}{e^{\frac{\beta}{a}\sqrt{k^2+(am)^2}}-1}-\log\left(1-e^{-\frac{\beta}{a}\sqrt{k^2+(am)^2}}\right)\right],$$

$$S_{S^3/H}\sim\frac{4\pi^4}{45|H|}\left(\frac{a}{\beta}\right)^3-\frac{\pi^2}{6|H|}(am)^2\frac{a}{\beta}+S_{0,S^3/H}(am),$$

$$\langle T_\mu^\mu\rangle = -\frac{cR}{12}$$

$$\langle T_\mu^\mu\rangle = cW_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}-aE_4$$

$$W^2 = \frac{1}{16\pi^2}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2\right)$$

$$E_4 = \frac{1}{16\pi^2}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right)$$

$$\langle T_\mu^\mu \rangle = \sum_i c_i I_i - (-1)^{d/2} a E_d$$

$$a = \frac{(-1)^{d/2}}{d}\frac{dF}{d\log r}$$

$$\frac{dg^i}{dt} = -\mu\frac{dg^i(\mu)}{d\mu} =: -\beta^i(g(\mu)),$$

$$\frac{\partial}{\partial t} = -\beta^i(g)\frac{\partial}{\partial g_i}$$

$$\dot{C}(g(t)) = -\beta^i(g)\frac{\partial}{\partial g_i}C(g(t)) \leq 0,$$

$$\dot{C}(g(t)) = -G_{ij}(g)\beta^i(g)\beta^j(g),$$

$$S_{\text{ent}} = \lim_{q\to 1} S_q := \lim_{q\to 1}\frac{q\Gamma_{S^3} - \Gamma_{qS^3}}{1-q}$$

$$S_{\text{ren}} = am\frac{dS_{\text{ent}}}{d(am)} - S_{\text{ent}}$$

$$\tilde{F} = \Gamma - \frac{n}{3}g\frac{d\Gamma}{dg}$$

$$S_{\text{hol},S^3/H} := \lim_{\beta\to 0}\left[S_{S^3/H}(am,\beta/a) - \frac{1}{|H|}S_{S^3}(am,\beta/a)\right]$$

$$S_{\text{hol},S^3/H} = \frac{1}{|H|}\Gamma_{S^3} - \Gamma_{S^3/H}$$

$$S_{\text{hol},S^3/Z_{2q+1}} = -\frac{1}{\pi}\sum_{r=1}^{q} r\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\pi amn}{2q+1}+1\right)\text{sen}\left(\frac{4\pi rn}{2q+1}\right)e^{-\frac{2\pi amn}{2q+1}},$$

$$S_{\text{hol},S^3/Z_{2q}} = -\frac{1}{8\pi^2 q}\sum_{k=1}^{\infty}\frac{1}{k^3}\left[1 + 2\pi amk + \frac{1}{2}(2\pi amk)^2\right]e^{-2\pi amk}$$

$$+\frac{1}{\pi^2 q}\sum_{k=1}^{\infty}\frac{1}{k^3}\left[1 + \pi mak + \frac{1}{2}(\pi amk)^2\right]e^{-\pi amk}$$

$$-\frac{1}{\pi}\sum_{r=1}^{q-1} r\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\pi amn}{2q}+1\right)\text{sen}\left(2\pi\frac{2r+1}{2q}n\right)e^{-\frac{2\pi amn}{2q}}$$

$$S_{\text{hol},S^3/D_p^*} = \frac{1}{2}S_{\text{hol},S^3/Z_{2q}} + \frac{1}{\pi}\sum_{\substack{k=1 \\ k\ \text{impar}}}^{\infty}\frac{(-1)^{\lfloor k/2\rfloor}}{k^2}\left(1+\frac{\pi amk}{2}\right)e^{-\frac{\pi amk}{2}}.$$

$$\zeta_{S^3}(s) = (\mu a)^{2s}\sum_{k=1}^{\infty}k^2\left[k^2 - \frac{1}{4} + (am)^2\right]^{-s}.$$

$$\zeta_{S^3}(s) = (\mu a)^{2s}\sum_{n=0}^{\infty}\frac{\Gamma(-s+1)}{n!\,\Gamma(-s+1-n)}\left[(am)^2 - \frac{1}{4}\right]^n \zeta_R(2s+2n-2),$$

$$\zeta_{S^3}(s) = \frac{(\mu a)^{2s}}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(s+n)}{n!}\left[(am)^2 - \frac{1}{4}\right]^n \zeta_R(2s+2n-2),$$

$$\Gamma_{S^3}(am) = \frac{\zeta_R(3)}{4\pi^2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left[(am)^2 - \frac{1}{4}\right]^{n+1}\zeta_R(2n).$$

$$\Gamma_{S^3}(0)=\frac{\zeta_R(3)}{4\pi^2}-\frac{1}{8}\sum_{n=0}^{\infty}\frac{\zeta_R(2n)}{(n+1)2^{2n}}$$

$$\sum_{n=1}^{\infty}\frac{\zeta_R(2n)}{(n+1)2^{2n}}=\frac{1}{2}+\log\left(\frac{B^{14}}{2}\right)$$

$$\Gamma_{S^3}(0)=-\frac{3\zeta_R(3)}{16\pi^2}-\frac{\log\ 2}{8}\approx 0{,}0638$$

$$\frac{d\Gamma_{S^3}}{d(am)^2}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[(am)^2-\frac{1}{4}\right]^n\zeta_R(2n)$$

$$\sum_{n=1}^{\infty}\frac{\zeta_R(2n)}{2^{2n-1}}=1$$

$$\zeta_{S^3/Z_2}(s)=(\mu a)^{2s}\sum_{\substack{k=1\\ k\ \text{impar}}}^{\infty}k^2\left[k^2-\frac{1}{4}+(am)^2\right]^{-s}$$

$$\zeta_{S^3/Z_2}(s)=\zeta_{S^3}(s)-4^{1-s}(\mu a)^{2s}\sum_{k=1}^{\infty}k^2\left[k^2-\frac{1}{16}+\frac{(am)^2}{4}\right]^{-s}$$
$$=\zeta_{S^3}(s)|_{am}-2^{2-2s}\zeta_{S^3}(s)|_{\tilde{m}},$$

$$\Gamma_{S^3/Z_2}(am)=\Gamma_{S^3}(am)-4\Gamma_{S^3}(\tilde{m})$$

$$S_{\text{hol},S^3/Z_2}(am)=4\Gamma_{S^3}(\tilde{m})-\frac{1}{2}\Gamma_{S^3}(am)$$

$$\frac{d}{d(am)^2}S_{\text{hol},S^3/Z_2}(am)=\frac{d}{d\tilde{m}^2}\Gamma_{S^3}(\tilde{m})$$

$$\frac{d}{d(am)^2}S_{\text{hol},S^3/Z_2}(am)\Big|_{am=0}=\frac{d}{d\tilde{m}^2}\Gamma_{S^3}(\tilde{m})\Big|_{\tilde{m}^2=3/16}$$

$$\frac{d}{d\tilde{m}^2}\Gamma_{S^3}(\tilde{m})\Big|_{\tilde{m}^2=3/16}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\zeta_R(2n)}{4^{2n}}=-\frac{\pi}{16}.$$

$$\frac{d}{d(am)^2}S_{\text{hol},S^3/Z_2}(am)\Big|_{am=0}=-\frac{\pi}{4}$$

$$\zeta_{\pm}(s)=(\mu a)^{2s}\sum_{k=1}^{\infty}k^2(k\pm M)^{-s}$$

$$\zeta_{\pm}(s)=(\mu a)^{2s}[\zeta_H(s-2{,}1\pm M)\mp 2M\zeta_H(s-1{,}1\ \pm M)$$
$$+M^2\zeta_H(s,1\pm M)]$$

$$\Gamma_{S^3}(m)=-\frac{1}{2}\{\zeta_H'(-2{,}1+M)+\zeta_H'(-2{,}1-M)-2M[\zeta_H'(-1{,}1+M)$$
$$-\zeta_H'(-1{,}1-M)]+M^2[\zeta_H'(0{,}1+M)+\zeta_H'(0{,}1-M)]\}$$

$$\partial_p S^{(4)}_{\text{top},S^3/Z_p}\Big|_{p=1}=2F^{(3)}_{S^3}$$

$$S^{(4)}_{\text{top},S^3/Z_p}=\frac{1}{4}\int_0^{\infty}dx\Re\frac{p\text{senh}z+\cosh\ z\text{senh}pz}{z\text{senh}^2 z\text{senh}\frac{pz}{2}}$$
$$-\frac{1}{4p}\int_0^{\infty}dx\Re\frac{\text{senh}z(1+\cosh\ z)}{z\text{senh}^2 z\text{senh}^2\frac{z}{2}}$$

$$\partial_p S^{(4)}_{\mathrm{top},S^3/Z_p}\Big|_{p=1} = \frac{1}{4}\int_0^\infty dx\Re\frac{\mathrm{senhz}}{z\mathrm{senh}^2 z\mathrm{senh}^2\frac{z}{2}} + \frac{1}{4}\int_0^\infty dx\Re\frac{\mathrm{senhz}(1+\cosh\ z)}{z\mathrm{senh}^2 z\mathrm{senh}^2\frac{z}{2}}$$

$$+\frac{1}{4}\int_0^\infty dx\Re\frac{\cosh^2\ z - \coth\frac{z}{2}\mathrm{senhz}(1+\cosh\ z)}{\mathrm{senh}^2 z\mathrm{senh}^2\frac{z}{2}}$$

$$= I_1 + I_2 + I_3$$

$$\cosh^2\ z - \coth\frac{z}{2}\mathrm{senhz}(1+\cosh\ z) = -1 - 2\cosh\ z$$

$$I_3 = -\frac{1}{4}\int_0^\infty dx\Re\frac{1+2\cosh\ z}{\mathrm{senh}^2 z\mathrm{senh}^2\frac{z}{2}}$$

$$I_3 = -\frac{1}{4}\int_0^\infty dx\Re\frac{1+2\cosh\left(x+i\frac{\pi}{2}\right)}{\mathrm{senh}^2\left(x+i\frac{\pi}{2}\right)\mathrm{senh}^2\left(\frac{x}{2}+i\frac{\pi}{4}\right)}$$

$$\Re\frac{1+2\cosh\left(x+i\frac{\pi}{2}\right)}{\mathrm{senh}^2\left(x+i\frac{\pi}{2}\right)\mathrm{senh}^2\left(\frac{x}{2}+i\frac{\pi}{4}\right)} = 2\left(\frac{3}{\cosh^4\ x} - \frac{2}{\cosh^2\ x}\right),$$

$$\int_0^\infty \frac{dx}{\cosh^2\ x} = 1, \int_0^\infty \frac{dx}{\cosh^4\ x} = \frac{2}{3}$$

$$\partial_p S^{(4)}_{\mathrm{top},S^3/Z_p}\Big|_{p=1} = I_1 + I_2 = \frac{1}{4}\int_0^\infty dx\Re\frac{2+\cosh\ z}{z\mathrm{senhz}\mathrm{senh}^2\frac{z}{2}}$$

$$\partial_p S^{(4)}_{\mathrm{top},S^3/Z_p}\Big|_{p=1} = \frac{1}{4}\int_0^\infty dx\frac{2\pi - 6x\mathrm{senh}x - \pi\mathrm{senh}^2 x}{\left(x^2+\frac{\pi^2}{4}\right)\cosh^3\ x}$$

$$F^{(3)} = -\frac{1}{4}\int_0^\infty dx\Re\frac{\cosh\frac{z}{2}(1+\cosh\ z)}{z\mathrm{senh}^2 z\mathrm{senh}^2\frac{z}{2}}$$

$$F^{(3)} = -\frac{\pi}{16}\int_0^\infty dx\frac{1-\cosh\ 2x}{\left(x^2+\frac{\pi^2}{4}\right)\cosh^3\ x}$$

$$\frac{1}{2}\int_0^\infty dx\frac{4-2\cosh^2\frac{\pi x}{2}}{(x^2+1)\cosh^3\frac{\pi x}{2}} - \frac{1}{2}\int_0^\infty dx\frac{3x\mathrm{senh}\frac{\pi x}{2}}{(x^2+1)\cosh^3\frac{\pi x}{2}} = \tilde{I}_1 + \tilde{I}_2$$

$$\frac{1}{1+x^2} = \int_0^\infty dt e^{-t}\cos\ tx, \frac{x}{1+x^2} = \int_0^\infty dt e^{-t}\mathrm{sent}x$$

$$\tilde{I}_1 = \int_0^\infty dt e^{-t}\int_0^\infty dx\frac{2-\cosh^2\frac{\pi x}{2}}{\cosh^3\frac{\pi x}{2}}\cos\ tx = \frac{3\zeta_R(3)}{2\pi^2}$$

$$\tilde{I}_2 = -\frac{3}{2}\int_0^\infty dt e^{-t}\int_0^\infty dx\frac{\mathrm{senh}\frac{\pi x}{2}}{\cosh^3\frac{\pi x}{2}}\mathrm{sent}x = -\frac{3\zeta_R(3)}{2\pi^2}$$

$$\triangle^{(d)} + \frac{d(d-2)}{4a^2} + m^2$$

$$a^2\lambda_n = \left(n+\frac{d-1}{2}\right)^2 - \frac{1}{4} + (am)^2\ n\in\mathbb{N}$$

$$d_n = \frac{(2n+d-1)(n+d-2)!}{n!\,(d-1)!},$$

$$d_n = \frac{2n(n+k-1)!}{(2k)!\,(n-k)!}$$

$$\frac{(n+k-1)!}{(n-k)!} = (n^2-(k-1)^2)(n^2-(k-2)^2)\cdots(n^2-1)n$$

$$\frac{(n+k-1)!}{(n-k)!}=\sum_{l=1}^{k} c_l^{(2k+1)} n^{2l-1},$$

$$\zeta_{S^{2k+1}}(s)=\frac{2(\mu a)^{2s}}{(2k)!}\sum_{n=1}^{\infty}\sum_{l=1}^{k} c_l^{(2k+1)} n^{2l}[n^2+(am)^2-1/4]^{-s}$$

$$\Gamma_{S^{2k+1}}=-\frac{1}{2}\frac{d}{ds}[\zeta_+(s)+\zeta_-(s)]\Big|_{s=0}$$

$$\zeta_\pm(s)=\frac{2(\mu a)^{2s}}{(2k)!}\sum_{n=1}^{\infty}\sum_{l=1}^{k} c_l^{(2k+1)} n^{2l}(n\pm M)^{-s}$$

$$n^{2l}=\sum_{p=0}^{2l}\frac{(2l)!}{p!\,(2l-p)!}(n\pm M)^{2l-p}(\mp M)^p$$

$$\zeta_\pm(s)=\frac{2(\mu a)^{2s}}{(2k)!}\sum_{l=1}^{k} c_l^{(2k+1)}\sum_{p=0}^{2l}\frac{(2l)!}{p!\,(2l-p)!}(\mp M)^p\zeta_H(s-2l+p,1\pm M)$$

$$\Gamma_{S^{2k+1}}(m)=-\frac{1}{(2k)!}\sum_{l=1}^{k} c_l^{(2k+1)}\sum_{p=0}^{2l}\frac{(2l)!}{p!\,(2l-p)!}M^p$$
$$\times[\zeta_H'(p-2l,1-M)+(-1)^p\zeta_H'(p-2l,1+M)]$$

$$\zeta_{S^{2k+1}/Z_2}(s)=\frac{1+(-1)^{k+1}}{2}\zeta_{S^{2k+1}}(s)+(-1)^k\Delta\zeta(s)$$

$$\zeta_\pm(s)=\frac{2^{1-2s}(\mu a)^{2s}}{(2k)!}\sum_{l=1}^{k} c_l^{(2k+1)}2^{2l}\sum_{p=0}^{2l}\frac{(2l)!}{p!\,(2l-p)!}\frac{(\mp M)^p}{2^p}\zeta_H\left(s-2l+p,1\pm\frac{M}{2}\right)$$

$$S_{\mathrm{hol},S^{2k+1}/Z_2}=\frac{(-1)^{k+1}}{(2k)!}\sum_{l=1}^{k} c_l^{(2k+1)}\sum_{p=0}^{2l}\frac{(2l)!}{p!\,(2l-p)!}M^p$$
$$\times\left\{\frac{1}{2}[\zeta_H'(p-2l,1-M)+(-1)^p\zeta_H'(p-2l,1+M)]\right.$$
$$\left.-2^{2l-p}\left[\zeta_H'\left(p-2l,1-\frac{M}{2}\right)+(-1)^p\zeta_H'\left(p-2l,1+\frac{M}{2}\right)\right]\right\}$$

$$d_n=\frac{2}{(2k-1)!}\left(n+k-\frac{1}{2}\right)\frac{(n+2k-2)!}{n!}$$

$$\frac{(n+2k-2)!}{n!}=\frac{\left(\tilde{n}+k-\frac{3}{2}\right)!}{\left(\tilde{n}-k+\frac{1}{2}\right)!}=\prod_{l=0}^{k-2}\left[\tilde{n}^2-\left(l+\frac{1}{2}\right)^2\right]$$

$$\frac{(n+2k-2)!}{n!}=\sum_{l=0}^{k-1} c_l^{(2k)}\tilde{n}^{2l}$$

$$\zeta_{S^{2k}}(s)=\frac{2(\mu a)^{2s}}{(2k-1)!}\sum_{l=0}^{k-1} c_l^{(2k)}\sum_{n=k}^{\infty}\left(n-\frac{1}{2}\right)^{1+2l-2s}\left[1+\frac{(am)^2-1/4}{\left(n-\frac{1}{2}\right)^2}\right]^{-s},$$

$$\zeta_{S^{2k}}(s)=\frac{2(\mu a)^{2s}}{(2k-1)!}\sum_{l=0}^{k-1} c_l^{(2k)}\sum_{p=0}^{\infty}\frac{\Gamma(-s+1)}{p!\,\Gamma(-s+1-p)}\left[(am)^2-\frac{1}{4}\right]^p$$
$$\times\zeta_H\left(2s+2p-2l-1,\frac{3}{2}\right)$$

$$\zeta_{S^{2k}}(0)=\frac{2}{(2k-1)!}\sum_{l=0}^{k-1} c_l^{(2k)}\left\{\zeta_H\left(-2l-1,\frac{3}{2}\right)+\frac{1}{2(l+1)}\left[\frac{1}{4}-(am)^2\right]^{l+1}\right\}$$

$$\zeta_{S^{2k}/Z_2}(s)=\frac{4(\mu a)^{2s}}{(2k-1)!}\sum_{l=0}^{k-1}c_l^{(2k)}2^{-2s+2l}\sum_{n=0}^{\infty}\left(n-\frac{1}{4}\right)^{1+2l-2s}\left[1+\frac{(am)^2-1/4}{4\left(n-\frac{1}{4}\right)^2}\right]^{-s}$$

$$\zeta_{S^{2k}/Z_2}(s)=\frac{(\mu a)^{2s}}{(2k-1)!}\sum_{l=0}^{k-1}c_l^{(2k)}\sum_{p=0}^{\infty}\frac{\Gamma(s+p)}{p!\,\Gamma(s)}2^{2+2l-2s-2p}\left[\frac{1}{4}-(am)^2\right]^p$$
$$\times\zeta_H\left(2s+2p-2l-1,\frac{3}{4}\right)$$

$$\zeta_{S^{2k}/Z_2}(0)=\frac{1}{(2k-1)!}\sum_{l=0}^{k-1}c_l^{(2k)}\left\{2^{2+2l}\zeta_H\left(-2l-1,\frac{3}{4}\right)+\frac{1}{2(l+1)}\left[\frac{1}{4}-(am)^2\right]^{l+1}\right\}$$

$$\zeta^{\text{hol}}(s):=\zeta_{S^{2k}/Z_2}(s)-\frac{1}{2}\zeta_{S^{2k}}(s)$$

$$S_{\text{hol},S^{2k}/Z_2}=\frac{(-1)^{k+1}}{(2k-1)!}\sum_{l=0}^{k-1}c_l^{(2k)}\left\{\frac{\left[\frac{1}{4}-(am)^2\right]^{l+1}}{2(l+1)}[\log\,2+\psi(3/4)-\psi(3/2)]\right.$$
$$-2^{2+2l}\left[-\log\,2\zeta_H\left(-2l-1,\frac{3}{4}\right)+\zeta_H'\left(-2l-1,\frac{3}{4}\right)\right]$$
$$-\sum_{\substack{p=1\\p\neq l+1}}^{\infty}\frac{\left[\frac{1}{4}-(am)^2\right]^p}{2p}\left[2^{2+2l-2p}\zeta_H\left(2p-2l-1,\frac{3}{4}\right)\right.$$
$$\left.\left.-\zeta_H\left(2p-2l-1,\frac{3}{2}\right)\right]+\zeta_H'\left(-2l-1,\frac{3}{2}\right)\right\}$$

$$S[\psi]=\frac{1}{2}\int d^dx\sqrt{g}\bar{\psi}(\partial\!\!\!/+m)\psi,$$

$$\lambda_n^{\pm}=\pm\frac{i}{a}\left(n+\frac{d}{2}\right)+m,d_n\equiv d_n^{\pm}=2^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{d+n-1}{n},$$

$$\lambda_n^{+}=\frac{i}{a}\left(n+\frac{d}{2}\right)+m,n\text{ par },\lambda_n^{-}=-\frac{i}{a}\left(n+\frac{d}{2}\right)+m,n\text{ impar },$$

$$\zeta_{\text{hol}}^{(+)}(s):=\zeta_{S^3/Z_2}(s)-\frac{1}{2}\zeta_{S^3}(s).$$

$$\zeta_{\text{hol}}^{(+)}(s)=\frac{\mu^s}{2}\left\{\sum_{\substack{n=0\\n\text{ par}}}^{\infty}d_n[(\lambda_n^+)^{-s}-(\lambda_n^-)^{-s}]-\sum_{\substack{n=0\\n\text{ impar}}}^{\infty}d_n[(\lambda_n^+)^{-s}-(\lambda_n^-)^{-s}]\right\},$$

$$\zeta_{\text{hol}}^{(+)}(s)=(\mu a)^s2^{1-s}\left\{e^{-i\frac{\pi}{2}s}\delta\zeta_H(s,am)-e^{i\frac{\pi}{2}s}\delta\zeta_H(s,-am)\right\}$$

$$\delta\zeta_H(s,am):=\tilde{\delta}\zeta_H(s-2,am)+\text{iam}\tilde{\delta}\zeta_H(s\;-1,am)$$
$$-\frac{(am)^2+1/4}{4}\tilde{\delta}\zeta_H(s,am)$$

$$\text{con}\tilde{\delta}\zeta_H(s,am):=\zeta_H\left(s,\frac{3}{4}-\frac{iam}{2}\right)-\zeta_H\left(s,\frac{5}{4}-\frac{iam}{2}\right)$$

$$\Gamma_{\text{hol}}^{(+)}(am)=-i\frac{\pi}{8}+2\delta\zeta_H'(0,am)-2\delta\zeta_H'(0,-am)$$

$$\Gamma_{\text{hol}}^{(+)}(0)=-i\frac{\pi}{8}$$

$$\Gamma_{\text{hol}}^{(-)}(am)=-i\frac{\pi}{8}-2\delta\zeta_H'(0,am)+2\delta\zeta_H'(0,-am)$$

$$Z^{S^3/Z_2}:=\frac{1}{2}\left(Z^{(+)}+Z^{(-)}\right)$$

$$\Gamma^{S^3/Z_2} \equiv -\log Z^{S^3/Z_2} = -\log\left(\frac{e^{-\Gamma^{(+)}} + e^{-\Gamma^{(-)}}}{2}\right)$$

$$S_{\mathrm{hol},S^3/Z_2} = \log\cos\Im\left(\Gamma^{S^3/Z_2,(+)}\right)$$

$$\Gamma^{S^d/Z_2} := -\log\left(\frac{e^{-\Gamma^{(+)}} + e^{-\Gamma^{(-)}}}{2}\right)$$

$$\lambda_n^{\pm} = \pm\frac{i}{a}n + m, n = k+1, k+2, \dots,$$

$$d_n = \frac{2^{k+1}}{(2k+1)!}\frac{(n+k)!}{(n-k-1)!}$$

$$\frac{(n+k)!}{(n-k-1)!} = (n+k)(n+k-1)(n+k-2)\dots(n-k+1)(n-k)$$
$$= n(n^2-1)(n^2-2^2)\dots(n^2-k^2)$$

$$d_n = \frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)} n^{1+2l}$$

$$\zeta_{S^{2k+2}}(s) = (\mu a)^s \frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\left\{\sum_{n=1}^{\infty} n^{1+2l}(in+am)^{-s} + \right.$$
$$\left. +(-1)^{-s}\sum_{n=0}^{\infty} n^{1+2l}(in-am)^{-s}\right\}$$

$$\sum_{n=1}^{\infty} n^{1+2l}(in+am)^{-s} = i^{-s}\sum_{n=1}^{\infty}(n-iam+iam)^{1+2l}(n-iam)^{-s}$$
$$= i^{-s}\sum_{p=0}^{2l+1}\binom{2l+1}{p}(iam)^{2l+1-p}\zeta_H(s-p,1-iam)$$

$$\zeta_{S^{2k+2}}(s) = (\mu a)^s \frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\sum_{p=0}^{2l+1}\binom{2l+1}{p}(iam)^{2l+1-p}$$
$$\times\{i^{-s}\zeta_H(s-p,1-iam) + (-i)^{-s}(-1)^{1-s-p}\zeta_H(s-p,iam)\}$$

$$\Gamma^{S^{2k+2}} = \frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\sum_{p=0}^{2l+1}\binom{2l+1}{p}(\mathrm{iam})^{2l+1-p}$$
$$\times\{\log(\mu a)[\zeta_H(-p,1-\mathrm{iam}) + (-1)^{1+p}\zeta_H(-p,\mathrm{iam})]$$
$$+\zeta_H'(-p,1-\mathrm{iam}) + (-1)^{1+p}\zeta_H'(-p,\mathrm{iam})$$
$$-i\pi[\zeta_H(-p,1-\mathrm{iam}) - (-1)^{1+p}\zeta_H(-p,\mathrm{iam})]\}$$

$$\Gamma^{S^{2k+2}} = \frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\sum_{p=0}^{2l+1}\binom{2l+1}{p}(\mathrm{iam})^{2l+1-p}$$
$$\times\{2(-1)^{1+p}\zeta_H(-p,\mathrm{iam})\log(\mu a) + \zeta_H'(-p,1-\mathrm{iam})$$
$$+(-1)^{1+p}\zeta_H'(-p,\mathrm{iam})\}$$

$$\Gamma^{S^{2k+2}/Z_2,(\pm)} = \frac{2^{k+2}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\sum_{p=0}^{2l+1}\binom{2l+1}{p}2^p(iam)^{2l+1-p}$$
$$\times\left\{\log\frac{\mu a}{2}\left[\zeta_H\left(-p,\frac{1}{4}\mp\frac{1}{4}-\frac{iam}{2}\right) - (-1)^p\zeta_H\left(-p,\frac{1}{4}\pm\frac{1}{4}+\frac{iam}{2}\right)\right]\right.$$
$$+\zeta_H'\left(-p,\frac{1}{4}\mp\frac{1}{4}-\frac{iam}{2}\right) - (-1)^p\zeta_H'\left(-p,\frac{1}{4}\pm\frac{1}{4}+\frac{iam}{2}\right)$$
$$\left.-\frac{i\pi}{2}\left[\zeta_H\left(-p,\frac{1}{4}\mp\frac{1}{4}-\frac{iam}{2}\right) + (-1)^p\zeta_H\left(-p,\frac{1}{4}\pm\frac{1}{4}+\frac{iam}{2}\right)\right]\right\}$$

$$S_{\mathrm{hol},S^{2k+2}/Z_2} = \log\,\cos\,\Im\left(\Gamma^{S^{2k+2}/Z_2,(+)}\right).$$

$$\mathcal{A}^{(2k+2)}(m) = \frac{2^{k+2}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)} \sum_{p=0}^{2l+1} \binom{2l+1}{p} (iam)^{2l+1-p}$$

$$\times (-1)^{1+p}\zeta_H(-p, iam)$$

$$(iam)^{2l+1-p}\zeta_H(-p, iam) = -\frac{1}{p+1}\sum_{q=0}^{p+1}\binom{p+1}{q}(iam)^{2l+2-q}B_q$$

$$-\frac{1}{p+1}\sum_{q=0}^{p+1}\binom{p+1}{q}(\mathrm{iam})^{2l+2-q}B_q = -\frac{1}{p+1}\binom{p+1}{1}(\mathrm{iam})^{2l+1}B_1$$

$$\frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)} \sum_{p=0}^{2l+1} \binom{2l+1}{p}(-1)^{1+p}(\,\mathrm{iam}\,)^{2l+1}$$

$$-\frac{2^{k+1}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}(\mathrm{iam})^{2l+1}(1-1)^{2l+1}$$

$$\mathcal{A}^{(2k+2)} = \frac{2^{k+2}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\zeta_R(-2l-1),$$

$$\mathcal{A}^{(2k+2)} = -\frac{2^{k+2}}{(2k+1)!}\sum_{l=0}^{k} c_l^{(2k+2)}\frac{B_{2l+2}}{2l+2}$$

$$\lambda_n^{\pm} = \pm\frac{i}{a}\left(n+\frac{1}{2}\right) + m, n = k, k+1, k+2, \dots,$$

$$d_n = \frac{2^k}{(2k)!}\frac{(n+k)!}{(n-k)!}$$

$$\frac{(n+k)!}{(n-k)!} = (n+k)(n+k-1)(n+k-2)\dots(n-k+1)$$

$$= \left(n+\frac{1}{2}+k-\frac{1}{2}\right)\left(n+\frac{1}{2}+k-\frac{3}{2}\right)\dots\left(n+\frac{1}{2}-k+\frac{1}{2}\right)$$

$$= \left[\left(n+\frac{1}{2}\right)^2 - \left(k-\frac{1}{2}\right)^2\right]\left[\left(n+\frac{1}{2}\right)^2 - \left(k-\frac{3}{2}\right)^2\right]\dots\left[\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right]$$

$$d_n = \frac{2^k}{(2k)!}\sum_{l=0}^{k} c_l^{(2k+1)}\left(n+\frac{1}{2}\right)^{2l}$$

$$\zeta_{S^{2k+1}}(s) = (\mu a)^s\frac{2^k}{(2k)!}\sum_{l=0}^{k} c_l^{(2k+1)}\Big\{i^{-s}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{2l}\left(n+\frac{1}{2}-iam\right)^{-s}$$

$$+(-i)^{-s}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{2l}\left(n+\frac{1}{2}+iam\right)^{-s}\Big\}$$

$$\zeta_{S^{2k+1}}(s) = (\mu a)^s\frac{2^k}{(2k)!}\sum_{l=0}^{k} c_l^{(2k+1)}\sum_{p=0}^{2l}\binom{2l}{p}(iam)^{2l-p}$$

$$\times\left\{i^{-s}\zeta_H\left(s-p,\frac{1}{2}-iam\right) + (-i)^{-s}(-1)^p\zeta_H\left(s-p,\frac{1}{2}+iam\right)\right\}$$

$$\Gamma^{S^{2k+1}} = \frac{2^k}{(2k)!}\sum_{l=0}^{k} c_l^{(2k+1)} \sum_{p=0}^{2l} \binom{2l}{p} (iam)^{2l-p}$$
$$\times\left\{\zeta_H'\left(-p,\frac{1}{2}-iam\right)+(-1)^p\zeta_H'\left(-p,\frac{1}{2}+iam\right)\right.$$
$$\left.-i\pi\zeta_H\left(-p,\frac{1}{2}-iam\right)\right\}$$
$$\Gamma^{S^{2k+1}/Z_2,(\pm)} = \frac{2^k}{(2k)!}\sum_{l=0}^{k} c_l^{(2k+1)} \sum_{p=0}^{2l} \binom{2l}{p} 2^p (iam)^{2l-p}$$
$$\times\left\{\zeta_H'\left(-p,\frac{1}{2}-\frac{iam}{2}\mp\frac{1}{4}\right)+(-1)^p\zeta_H'\left(-p,\frac{1}{2}+\frac{iam}{2}\pm\frac{1}{4}\right)\right.$$
$$\left.-i\pi\zeta_H\left(-p,\frac{1}{2}-\frac{iam}{2}\mp\frac{1}{4}\right)\right\}$$

Bajo la métrica de Yukawa para curvatura, obtenemos:

$$m_* \lesssim g M_{\mathrm{P}}^{D/2-1}$$
$$Y_{ijk} = e^{\phi_4/2}\mathrm{Vol}_X^{1/4}\Theta_{ijk}^{1/4}W_{ijk}$$
$$|Y_{ijk}| \simeq e^{-\phi_4}\left(\frac{m_{\mathrm{bos/fer}}^i}{M_{\mathrm{P}}}\frac{m_{\mathrm{bos/fer}}^j}{M_{\mathrm{P}}}\frac{m_{\mathrm{bos/fer}}^k}{M_{\mathrm{P}}}\right)^{1/2}$$
$$Y_* \sim \frac{1}{u^r}$$
$$m_{\mathrm{bos/fer}} \sim g_* M_s \sim g_*^2 M_{\mathrm{P}}, Y_* \sim g_*, g_* \sim e^{\phi_4}$$
$$\sum_\alpha N_\alpha([\Pi_\alpha]+[\Pi_{\alpha*}]) = 4[\Pi_{O6}].$$
$$J_c \equiv B + iJ = (b^a + it^a)\omega_a$$
$$\Omega_c \equiv C_3 + ie^{-\phi}\mathrm{Re}\Omega = (\zeta^K + iu^K)\alpha_K$$
$$K_K \equiv -\log\left(\mathrm{Vol}_{X_6}\right) = -\log\left(\frac{i}{48}\mathcal{K}_{abc}(T^a-\bar{T}^a)(T^b-\bar{T}^b)(T^c-\bar{T}^c)\right)$$
$$K_Q \equiv -2\log\mathcal{H} = -2\log\left(\frac{i}{8\ell_s^6}\int_{X_6} e^{-2\phi}\Omega\wedge\bar{\Omega}\right) = 4\phi_4$$
$$\ell_K \equiv -\frac{1}{2}\frac{\partial K}{\partial u^K} = -\frac{1}{2\mathcal{H}}\int_{\Sigma_K^-} e^{-\phi}\mathrm{Im}\Omega$$
$$[\Pi_\alpha] = P_{\alpha J}[\Sigma_+^J] + Q_\alpha^K[\Sigma_K^-], [\Pi_{\alpha*}] = P_{\alpha J}[\Sigma_-^J] - Q_\alpha^K[\Sigma_K^-]$$
$$2\pi i f_{\alpha\alpha} = P_{\alpha K}U^K$$
$$\frac{1}{2}M_{\mathrm{P}}^2\int_{X_4} G_{KL}(d\zeta^K - 2Q_\alpha^K A^\alpha)\wedge *\left(d\zeta^L - 2Q_\beta^L A^\beta\right)$$
$$M_{\alpha\beta}^2 = 4G_{KL}Q_\alpha^K Q_\beta^L g_\alpha g_\beta M_{\mathrm{P}}^2$$
$$V_D = \frac{1}{2}\sum_\alpha g_\alpha^2\left(\xi_\alpha + \sum_i q_\alpha^i K_{i\bar{\imath}}\Phi_i\bar{\Phi}_{\bar{\imath}}\right)^2, \text{ with } \pi\xi_\alpha = Q_\alpha^K \ell_K M_{\mathrm{P}}^2$$
$$q_\alpha^i g_\alpha^2\xi_\alpha + q_\beta^i g_\beta^2\xi_\beta.$$
$$\theta_{\alpha\beta}^1 + \theta_{\alpha\beta}^2 + \theta_{\alpha\beta}^3 \in 2\mathbb{Z},$$

$$m_{\alpha\beta}^2(\text{ scalar })_{\vec{k},\pm r} = 2\pi\left[\lambda_{\alpha\beta}^{\vec{k}} \pm \left|\theta_{\alpha\beta}^r\right|\right]e^{2\phi_4}M_{\mathrm{P}}^2,$$
$$\lambda_{\alpha\beta}^{\vec{k}} = \sum_r k_r\left|\theta_{\alpha\beta}^r\right| + \frac{1}{2}\sum_p\left|\theta_{\alpha\beta}^p\right|, k_r \in \mathbb{Z}_{>0}$$

$$m_{\alpha\beta}^2(\text{ vector })_{\vec{k}} = 2\pi\lambda_{\alpha\beta}^{\vec{k}} e^{2\phi_4} M_{\mathrm{P}}^2$$
$$Y_{ijk} = h_{qu} \sum_{\vec{n}} d_{\vec{n}} e^{2\pi i\left[\int_{D_{\vec{n}}} J_c - \int_{\partial D_{\vec{n}}} \nu\right]}$$
$$Y_{ijk} = e^{\phi_4/2} \prod_{r=1}^{3} (\mathrm{Im}T^r)^{1/4} \left[\Theta^{(r)}\right]^{1/4} e^{\frac{1}{2}H^{(r)}} W_{ijk}^{(r)}$$
$$\Theta^{(r)} = 2\pi \frac{\Gamma(1-|\chi_{ab}^r|)}{\Gamma(|\chi_{ab}^r|)} \frac{\Gamma(1-|\chi_{bc}^r|)}{\Gamma(|\chi_{bc}^r|)} \frac{\Gamma(1-|\chi_{ca}^r|)}{\Gamma(|\chi_{ca}^r|)}$$
$$W_{ijk}^{(r)} = \vartheta \begin{bmatrix} \delta_{ijk}^r \\ 0 \end{bmatrix} (\nu^r, |I_{ab}^r I_{bc}^r I_{ca}^r| T^r)$$
$$H^{(r)} = 2\pi i \frac{\nu^r \mathrm{Im}\nu^r}{\mathrm{Im}T^r} |I_{ab}^r I_{bc}^r I_{ca}^r|^{-1}$$
$$Y_{ijk} = e^{\phi_4/2} \mathrm{Vol}_X^{1/4} \Theta_{ijk}^{1/4} W_{ijk}$$
$$Y_{ijk} = e^{K/2} \left(K_{i\bar{\imath}} K_{j\bar{\jmath}} K_{k\bar{k}}\right)^{-1/2} W_{ijk}$$
$$K_{i\bar{\imath}} K_{j\bar{\jmath}} K_{k\bar{k}} = e^{3(K/2-\phi_4)} \Theta_{ijk}^{-1/2}$$
$$K_{i\bar{\imath}} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}} \prod_{r=1}^{3} \left(\frac{\Gamma(|\chi_i^r|)}{\Gamma(1-|\chi_i^r|)}\right)^{1/2}$$
$$K_{i\bar{\imath}} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}} \left[\frac{\Gamma(|\theta_{\alpha\beta}^1|)}{\Gamma(1-|\theta_{\alpha\beta}^1|)} \frac{\Gamma(|\theta_{\alpha\beta}^2|)}{\Gamma(1-|\theta_{\alpha\beta}^2|)} \frac{\Gamma(1-|\theta_{\alpha\beta}^3|)}{\Gamma(|\theta_{\alpha\beta}^3|)}\right]^{1/2} \overset{|\theta_{\alpha\beta}^r| \ll 1}{\simeq} \left[\frac{e^{K-2\phi_4}|\theta_{\alpha\beta}^3|}{2\pi|\theta_{\alpha\beta}^1||\theta_{\alpha\beta}^2|}\right]^{1/2}$$
$$K = -2\log\left(\mathrm{Vol}_X^{1/2}\mathcal{H}\right) = -2\log\left(\hat{\mathrm{V}}_X^{1/2}\hat{\mathcal{H}} - f(\Phi^i, \bar{\Phi}^{\bar{\imath}})\right) = -2\log\left(\hat{\mathrm{V}}_X^{1/2}\hat{\mathcal{H}} - \frac{1}{2}k_{i\bar{\imath}}\Phi^i\bar{\Phi}^{\bar{\imath}} + \cdots\right),$$
$$K = -2\log\left(\hat{V}_X^{1/2}\hat{\mathcal{H}}\right) + \frac{k_{i\bar{\imath}}}{\hat{V}_X^{1/2}\hat{\mathcal{H}}} \Phi^i\bar{\Phi}^{\bar{\imath}} + \cdots$$
$$k_{i\bar{\imath}} = \frac{e^{-\phi_4}}{\sqrt{2\pi}} \prod_{r=1}^{3} \left(\frac{\Gamma(|\chi_i^r|)}{\Gamma(1-|\chi_i^r|)}\right)^{1/2}$$
$$k_{i\bar{\imath}} \simeq \frac{M_{\mathrm{P}}}{m_{\mathrm{bos/fer}}^i}$$
$$K_{i\bar{\imath}} = e^{K/2} h_{i\bar{\imath}} \prod_{r=1}^{2} \left(\frac{\Gamma(|\chi_i^r|)}{\Gamma(1-|\chi_i^r|)}\right)^{1/2} \simeq e^{K/2} h_{i\bar{\imath}}$$
$$h_{i\bar{\imath}} \lesssim g_\alpha^{-1} g_\beta^{-1}$$
$$|Y_{ijk}|^2 = A\mathrm{Vol}_X^{1/2} |W_{ijk}|^2$$
$$|Y_{ijk}|^2 = Be^{\phi_4} \Theta_{ijk}^{1/2}$$
$$m_{\mathrm{bos/fer}} \sim |Y|^2 M_{\mathrm{P}}$$
$$|Y_{ijk}|^2 \simeq Be^{-2\phi_4} \frac{m_{\mathrm{bos/fer}}^i}{M_{\mathrm{P}}} \frac{m_{\mathrm{bos/fer}}^j}{M_{\mathrm{P}}} \frac{m_{\mathrm{bos/fer}}^k}{M_{\mathrm{P}}}$$
$$\left(\frac{m_{\mathrm{bos/fer}}^i}{M_{\mathrm{P}}}\right)^{1/w_i} \sim \left(\frac{m_{\mathrm{bos/fer}}^j}{M_{\mathrm{P}}}\right)^{1/w_j} \sim \left(\frac{m_{\mathrm{bos/fer}}^k}{M_{\mathrm{P}}}\right)^{1/w_k} \sim e^{\phi_4}, w_i, w_j, w_k \geq 1$$
$$m_{\mathrm{bos/fer}}^{ijk} \sim |Y_{ijk}|^{\frac{2w_{ijk}}{w_i+w_j+w_k-2}} M_{\mathrm{P}}$$
$$|Y_{ijk}|^2 \simeq Be^{-2\phi_4} h_{i\bar{\imath}}^{-1} \frac{m_{\mathrm{bos/fer}}^j}{M_{\mathrm{P}}} \frac{m_{\mathrm{bos/fer}}^k}{M_{\mathrm{P}}} = Bh_{i\bar{\imath}}^{-1} \frac{m_{\mathrm{bos/fer}}^j}{M_s} \frac{m_{\mathrm{bos/fer}}^k}{M_s}.$$
$$m_{\mathrm{mono}} \simeq g^{-2} m_{\mathrm{ele}}$$
$$T_{\mathrm{string}} = 2\pi|\xi_a| \simeq g_a^{-2} |m_{\mathrm{ele}}^2|$$
$$m_{\alpha\beta,+}^2 = g_\alpha^2 T_\alpha^+ - g_\beta^2 T_\beta^-, m_{\alpha\beta,-}^2 = -g_\alpha^2 T_\alpha^- + g_\beta^2 T_\beta^+$$

$$m_{\alpha\beta,\min}^2 = \min_K \left(|g_\alpha^2 Q_\alpha^K - g_\beta^2 Q_\beta^K| T_K \right)$$

$$\nabla g_\alpha^{-2} \cdot \nabla T_K \neq 0,\ \text{y/o}\ \nabla g_\beta^{-2} \cdot \nabla T_K \neq 0$$

$$m_{\alpha\beta,\min} \lesssim m_{\text{bos/fer}}^{\min} \lesssim m_{\alpha\beta,\pm}$$

$$u^I \sim \lambda \to \infty$$

$$u^K \ell_K = 2$$

$$K_Q = -n\log\,\lambda + \cdots$$

$$M_s \sim M_{\text{P}} \lambda^{-n/4}$$

$$e^{-\phi}\text{Re}\Omega = s\alpha_0 + u^1\alpha_1 + u^2\alpha_2$$

$$K_Q = -\log\,s - \log\,\kappa(u^1, u^2)$$

$$\kappa(u^1, u^2) \equiv \frac{1}{6}[\kappa_{111}(u^1)^3 + 3\kappa_{112}(u^1)^2u^2 + 3\kappa_{122}u^1(u^2)^2 + \kappa_{222}(u^2)^3]$$

$$\ell_0 = \frac{1}{2s}, \ell_1 = \frac{\partial_{u^1}\kappa(u^1,u^2)}{2\kappa(u^1,u^2)}, \ell_2 = \frac{\partial_{u^2}\kappa(u^1,u^2)}{2\kappa(u^1,u^2)}$$

$$[\Pi_\alpha] = \frac{1}{2}[A] + \frac{1}{2}f_\alpha^i[D_i] - \left(\frac{1}{2}\kappa_{ijk}f_\alpha^j f_\alpha^k + \frac{1}{24}c_{2i}\right)[C^i] + \left(\frac{1}{6}\kappa_{ijk}f_\alpha^i f_\alpha^j f_\alpha^k + \frac{1}{24}c_{2i}f_\alpha^i\right)[B]$$

$$\frac{2\pi}{g_\alpha^2} = s - \left(\frac{1}{2}\kappa_{ijk}f_\alpha^j f_\alpha^k + \frac{1}{24}c_{2i}\right)u^i,$$

$$\ell_i f_\alpha^i = \frac{1}{2s}\left(\frac{1}{6}\kappa_{ijk}f_\alpha^i f_\alpha^j f_\alpha^k + \frac{1}{24}c_{2i}f_\alpha^i\right)$$

$$M_{\alpha\alpha}^2 \sim g_\alpha^2 Q_\alpha^K Q_\alpha^L \frac{\partial_K\partial_L(s\kappa)}{s\kappa} M_{\text{P}}^2 \sim \frac{M_{\text{P}}^2}{su^2}.$$

$$|Y|^2 \sim 1/s$$

$$\text{I)}\ s \sim u \sim \lambda,\ \text{II)}\ s \sim u^3 \sim \lambda.$$

$$\text{I)}\ \frac{m_{\text{Stu}}}{M_{\text{P}}} \sim \left(\frac{m_{\text{KK}}}{M_{\text{P}}}\right)^{3/2} \sim \left(\frac{m_{\text{bos/fer}}}{M_{\text{P}}}\right)^{3/2} \sim \left(\frac{M_s}{M_{\text{P}}}\right)^{3/2} \sim \left(\frac{T_{\text{D4}}^{1/2}}{M_{\text{P}}}\right)^3 \quad m_{\text{bos/fer}} \sim |Y|^2 M_{\text{P}}$$

$$\text{II)}\ \frac{m_{\text{Stu}}}{M_{\text{P}}} \sim \left(\frac{m_{\text{KK}}}{M_{\text{P}}}\right)^{5/4} \sim \left(\frac{m_{\text{bos/fer}}}{M_{\text{P}}}\right)^{5/4} \sim \left(\frac{T_{\text{D4}}^{1/2}}{M_{\text{P}}}\right)^{5/3} \sim \left(\frac{M_s}{M_{\text{P}}}\right)^{5/3}, m_{\text{bos/fer}} \sim |Y|^{4/3} M_{\text{P}}.$$

$$\frac{m_*^2}{M_{\text{P}}^2} \sim \left(\frac{T_{\text{D4}}}{M_{\text{P}}^2}\right)^w$$

$$\frac{m_{\text{KK/w}}}{M_{\text{P}}} \sim \frac{T_{\text{D4}}^{1/2}}{M_{\text{P}}} \sim 1/\sqrt{\lambda}$$

$$\frac{m_{\text{bos/fer}}^{\min}}{M_{\text{P}}} \sim 1/\sqrt{\lambda}$$

$$\frac{m_{\text{KK/w}}}{M_{\text{P}}} \sim \frac{T_{\text{D4}}}{M_{\text{P}}^2} \sim 1/\lambda$$

$$m_{\text{bos/fer}}^{\min} \sim M_{\text{P}}/\lambda,\ \text{or}\ m_{\text{bos/fer}}^{\min} \sim M_{\text{P}}/1\sqrt{\lambda}$$

$$M_{\alpha\beta}^2 = g_\alpha g_\beta Q_\alpha^K Q_\beta^L \frac{\partial_K\partial_L\mathcal{H}}{\mathcal{H}} M_{\text{P}}^2$$

$$\frac{m_{\text{bos/fer}}^{\min}}{M_{\text{P}}} \sim \left(\frac{m_{\text{Stu}}}{M_{\text{P}}}\right)^{2/3} \sim 1/\lambda$$

$$\frac{m_{\text{bos/fer}}^{\min}}{M_{\text{P}}} \sim \left(\frac{m_{\text{Stu}}}{M_{\text{P}}}\right)^{1/2} \sim 1/\sqrt{\lambda}$$

$$\frac{m_{\text{bos/fer}}^{\min}}{M_{\text{P}}} \sim \frac{m_{\text{Stu}}}{M_{\text{P}}} \sim 1/\sqrt{\lambda}$$

$$|Y| \sim \frac{1}{u^r},\ \text{with}\ r = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

$$\Omega = \ell_s^3 R_1 R_3 R_5 dz_1 \wedge dz_2 \wedge dz_3$$
$$= \ell_s^3 (R_1 dy_1 + \mathrm{i} R_2 dy_2) \wedge (R_3 dy_3 + \mathrm{i} R_4 dy_4) \wedge (R_5 dy_5 + \mathrm{i} R_6 dy_6)$$
$$\alpha_0 = 4 dy_1 \wedge dy_3 \wedge dy_5 \ell_s^3, \alpha_1 = -4 dy_1 \wedge dy_4 \wedge dy_6 \ell_s^3$$
$$\alpha_2 = -4 dy_2 \wedge dy_3 \wedge dy_6 \ell_s^3, \alpha_3 = -4 dy_2 \wedge dy_4 \wedge dy_5 \ell_s^3$$
$$\beta^0 = -2 dy_2 \wedge dy_4 \wedge dy_6 \ell_s^3, \beta^1 = 2 dy_2 \wedge dy_3 \wedge dy_5 \ell_s^3$$
$$\beta^2 = 2 dy_1 \wedge dy_4 \wedge dy_5 \ell_s^3, \beta^3 = 2 dy_1 \wedge dy_3 \wedge dy_6 \ell_s^3$$
$$s \equiv u^{(0)} = \frac{1}{4} e^{-\phi} R_1 R_3 R_5, u^{(1)} = \frac{1}{4} e^{-\phi} R_1 R_4 R_6$$
$$u^{(2)} = \frac{1}{4} e^{-\phi} R_2 R_3 R_6, u^{(3)} = \frac{1}{4} e^{-\phi} R_2 R_4 R_5$$
$$\mathcal{H} = \frac{\mathrm{i}}{32 \ell_s^6} \int_{\mathbf{T}^6} e^{-2\phi} \Omega \wedge \bar{\Omega} = 4 \sqrt{u^{(0)} u^{(1)} u^{(2)} u^{(3)}}$$
$$\ell_K = \frac{1}{2 u^K}$$
$$m_{\mathrm{KK},i} = \frac{2 M_s}{R_{2i-1}} = \frac{M_{\mathrm{P}}}{A_i^{1/2} 2 \sqrt{s u^{(i)}}}, m_{\mathrm{w},i} = \frac{1}{2} R_{2i} M_s = \frac{A_i^{1/2} M_{\mathrm{P}}}{2 \sqrt{s u^{(i)}}}$$
$$M_s^2 = \frac{M_{\mathrm{P}}^2}{4 \sqrt{s u^{(1)} u^{(2)} u^{(3)}}}$$
$$[\Pi_\alpha] = 2[(n_\alpha^1, m_\alpha^1)(n_\alpha^2, m_\alpha^2)(n_\alpha^3, m_\alpha^3)], n_\alpha^i, m_\alpha^i \in \mathbb{Z}$$
$$= n_\alpha^1 n_\alpha^2 n_\alpha^3 \Sigma_+^0 - n_\alpha^1 m_\alpha^2 m_\alpha^3 \Sigma_+^1 - m_\alpha^1 n_\alpha^2 m_\alpha^3 \Sigma_+^2 - m_\alpha^1 m_\alpha^2 n_\alpha^3 \Sigma_+^3$$
$$+ \frac{1}{2} (m_\alpha^1 m_\alpha^2 m_\alpha^3 \Sigma_0^- - m_\alpha^1 n_\alpha^2 n_\alpha^3 \Sigma_1^- - n_\alpha^1 m_\alpha^2 n_\alpha^3 \Sigma_2^- - n_\alpha^1 n_\alpha^2 m_\alpha^3 \Sigma_3^-)$$
$$[\Pi_{\alpha^*}] = 2[(n_\alpha^1, -m_\alpha^1)(n_\alpha^2, -m_\alpha^2)(n_\alpha^3, -m_\alpha^3)].$$
$$\Pi_a = 2(k,1)(k,1)(k,-1),$$
$$\Pi_b = 2(0,1)(1,0)(0,-1),$$
$$\Pi_c = 2(0,1)(0,-1)(1,0)$$
$$\frac{\pi \xi_a}{M_{\mathrm{P}}^2} = -\frac{1}{4s} - \frac{k^2}{4u^{(1)}} - \frac{k^2}{4u^{(2)}} + \frac{k^2}{4u^{(3)}}$$
$$2s = e^{-\phi_4} \frac{1}{\sqrt{\tau_1 \tau_2 \tau_3}}, 2u^{(i)} = e^{-\phi_4} \sqrt{\frac{\tau_j \tau_k}{\tau_i}}$$
$$\theta_1 + \theta_2 = \theta_3, \text{ donde } \pi \theta_i = \tan^{-1} \frac{\tau_i}{k}$$
$$\frac{2\pi}{g_a^2} = 2\left(k^3 s + k u^{(1)} + k u^{(2)} - k u^{(3)}\right), \frac{2\pi}{g_b^2} = 2u^{(2)}, \frac{2\pi}{g_c^2} = 2u^{(3)}$$
$$q_a g_a^2 \xi_a = q_a \tan[\pi(\theta_3 - \theta_2 - \theta_1)] M_s^2$$
$$M_{aa}^2 = \frac{1}{2} g_a^2 \left(\frac{1}{s^2} + \frac{k^4}{(u^{(1)})^2} + \frac{k^4}{(u^{(2)})^2} + \frac{k^4}{(u^{(3)})^2} \right) M_{\mathrm{P}}^2$$
$$\frac{m_{\mathrm{bos/fer},i}}{M_{\mathrm{P}}} \simeq \frac{1}{2\sqrt{k s u^{(i)}}}$$

Sector	$m_{\mathrm{bos/fer},1}/M_{\mathrm{P}}$	$m_{\mathrm{bos/fer},2}/M_{\mathrm{P}}$	$m_{\mathrm{bos/fer},3}/M_{\mathrm{P}}$
aa^*	$1/\sqrt{k s u^{(1)}}$	$1/\sqrt{k s u^{(2)}}$	$1/\sqrt{k s u^{(3)}}$
ab	e^{ϕ_4}	$1/\sqrt{k s u^{(2)}}$	e^{ϕ_4}
bc	e^{ϕ_4}	e^{ϕ_4}	e^{ϕ_4}
ca	e^{ϕ_4}	e^{ϕ_4}	$1/\sqrt{k s u^{(3)}}$

$$m_{\mathrm{KKD}_a,i} = \frac{2M_s}{R_{2i-1}\sqrt{k^2+\tau_i^2}} \simeq \frac{M_\mathrm{P}}{A_i^{1/2}2k\sqrt{su^{(i)}}}, m_{\mathrm{wD}_a,i} = \frac{R_{2i}}{2\sqrt{k^2+\tau_i^2}}M_s \simeq \frac{A_i^{1/2}M_\mathrm{P}}{2k\sqrt{su^{(i)}}}.$$

$$s, u^{(2)} \sim \lambda, u^{(1)}, u^{(3)}$$

$$T_{\mathrm{D}4,0}^{1/2}, T_{\mathrm{D}4,2}^{1/2} \sim \lambda^{-1/2}M_\mathrm{P}, M_s \sim \lambda^{-1/2}M_\mathrm{P}$$

$$K_{ca} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left[\frac{\Gamma\left(\frac{1}{2}-\theta_1\right)}{\Gamma\left(\frac{1}{2}+\theta_1\right)}\frac{\Gamma\left(\frac{1}{2}-\theta_2\right)}{\Gamma\left(\frac{1}{2}+\theta_2\right)}\frac{\Gamma(\theta_3)}{\Gamma(1-\theta_3)}\right]^{1/2}$$

$$K_{ab} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left[\frac{\Gamma\left(\frac{1}{2}+\theta_1\right)}{\Gamma\left(\frac{1}{2}-\theta_1\right)}\frac{\Gamma(\theta_2)}{\Gamma(1-\theta_2)}\frac{\Gamma\left(\frac{1}{2}-\theta_3\right)}{\Gamma\left(\frac{1}{2}+\theta_3\right)}\right]^{1/2}$$

$$K_{bc} = e^{K/2-\phi_4}\sqrt{\tau_1/2\pi} = 2e^{K/2}\sqrt{u^{(2)}u^{(3)}/2\pi}$$

$$Y_{abc} = B\sqrt{2\pi g_b g_c}\left[\frac{\Gamma(1-\theta_2)}{\Gamma(\theta_2)}\frac{\Gamma(1-\theta_3)}{\Gamma(\theta_3)}\frac{\Gamma\left(\frac{1}{2}+\theta_2\right)}{\Gamma\left(\frac{1}{2}-\theta_2\right)}\frac{\Gamma\left(\frac{1}{2}+\theta_3\right)}{\Gamma\left(\frac{1}{2}-\theta_3\right)}\right]^{1/4}$$

$$Y_{abc} \sim \left(\frac{\theta_2\theta_3}{u^{(2)}u^{(3)}}\right)^{1/4} \sim \frac{e^{-\phi_4}}{\sqrt{su^{(2)}u^{(3)}}} \sim \lambda^{-1/3}$$

$$Y_{abc} \sim \left(\frac{\theta_2}{u^{(2)}}\right)^{1/4} \sim \frac{e^{-\phi_4/2}}{\sqrt{s^{1/2}u^{(2)}}} \sim \lambda^{-1/2}$$

Limit	Scenario	g_a	g_b	g_c	Y_{abc}
$s \sim \left(u^{(i)}\right)^3 \sim \lambda$	STU-like II)	$\lambda^{-1/2}$	$\lambda^{-1/6}$	$\lambda^{-1/6}$	$\lambda^{-1/3}$
4.20	quasi-EFT IIa)	$\lambda^{-1/2}$	$\lambda^{-1/2}$	const.	$\lambda^{-1/2}$
(4.20)′	quasi-EFT IIa)	$\lambda^{-1/2}$	const.	const.	const.

$$\Pi_a = 8(1,0)(k,1)(k,-1),$$

$$\Pi_b = 2(0,1)(1,0)(0,-1),$$

$$\Pi_c = 2(0,1)(0,-1)(1,0),$$

$$\frac{2\pi}{g_a^2} = 2\left(k^2 s + u^{(1)}\right)$$

$$\frac{\pi\xi_a}{M_\mathrm{P}^2} = \frac{k}{4}\left(-\frac{1}{u^{(2)}} + \frac{1}{u^{(3)}}\right)$$

$$q_a g_a^2 \xi_a = q_a \tan\left[\pi(\theta_3-\theta_2)\right]M_s^2$$

$$M_{aa}^2 = \frac{1}{2}g_a^2\left(\frac{k^2}{(u^{(2)})^2} + \frac{k^2}{(u^{(3)})^2}\right)M_\mathrm{P}^2$$

Sector	$m_{\mathrm{bos/fer},1}/M_\mathrm{P}$	$m_{\mathrm{bos/fer},2}/M_\mathrm{P}$	$m_{\mathrm{bos/fer},3}/M_\mathrm{P}$
aa^*	e^{ϕ_4}	e^{ϕ_4} or $1/\sqrt{ksu^{(2)}}$	e^{ϕ_4} or $1/\sqrt{ksu^{(3)}}$
ab	e^{ϕ_4}	$1/\sqrt{ksu^{(2)}}$	e^{ϕ_4}
bc	e^{ϕ_4}	e^{ϕ_4}	e^{ϕ_4}

ca	e^{ϕ_4}	e^{ϕ_4}	$1/\sqrt{ksu^{(3)}}$

$$
\begin{aligned}
s \sim \lambda, u^{(i)} &= \text{const.} \longrightarrow e^{\phi_4} \sim \lambda^{-1/4},\\
s, u^{(2)} = u^{(3)} \sim \lambda, u^{(1)} &= \text{const.} \Longrightarrow e^{\phi_4} \sim \lambda^{-3/4},\\
u^{(2)} = u^{(3)} \sim \lambda, s, u^{(1)} &= \text{const.} \longrightarrow e^{\phi_4} \sim \lambda^{-1/2},\\
s, u^{(1)} \sim \lambda, u^{(2)} = u^{(3)} &= \text{const.} \longrightarrow e^{\phi_4} \sim \lambda^{-1/2},\\
s \sim \lambda, u^{(2)} = u^{(3)} \sim \lambda^{1/2}, u^{(1)} &= \text{const.} \Longrightarrow e^{\phi_4} \sim \lambda^{-1/2}.
\end{aligned}
$$

Limit	Scenario	$\#T_{\mathrm{D4}}^{1/2}$	$\#m_{\mathrm{KK}}^{\mathrm{bulk}}$	$\#m_{\mathrm{KK}}^{\mathrm{brane}}$	$\#m_{\mathrm{bos/fer}}$	g_a	g_b	g_c	Y_{abc}
(4.33)	EFT I)	1	6	6	2 or 4	$\lambda^{-\frac{1}{2}}$	const.	const.	$\lambda^{-1/4}$
(4.34)	EFT IIa)	-	6	4	2 or 4	$\lambda^{-1/2}$	$\lambda^{-1/2}$	$\lambda^{-1/2}$	$\lambda^{-3/4}$
(4.35)	EFT IIb)	-	2	2	-	const.	$\lambda^{-1/2}$	$\lambda^{-1/2}$	$\lambda^{-1/2}$
(4.36)	EFT IIc)	-	2	2	-	$\lambda^{-1/2}$	const.	const.	const.
(4.37)	non-EFT	-	4	4	2 or 4	$\lambda^{-1/2}$	$\lambda^{-1/4}$	$\lambda^{-1/4}$	$\lambda^{-1/2}$

$$
K_{ca} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left[\frac{\Gamma\left(\frac{1}{2}-\theta_2\right)}{\Gamma\left(\frac{1}{2}+\theta_2\right)}\frac{\Gamma(\theta_3)}{\Gamma(1-\theta_3)}\right]^{1/2}
$$

$$
K_{ab} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left[\frac{\Gamma(\theta_2)}{\Gamma(1-\theta_2)}\frac{\Gamma\left(\frac{1}{2}-\theta_3\right)}{\Gamma\left(\frac{1}{2}+\theta_3\right)}\right]^{1/2}
$$

$$
K_{bc} = e^{K/2-\phi_4}\sqrt{\tau_1/2\pi} = 2e^{K/2}\sqrt{u^{(2)}u^{(3)}/2\pi} = e^{K/2}g_b^{-1}g_c^{-1}
$$

$$
Y_{abc} = B\sqrt{2\pi g_b g_c}\left[\frac{\Gamma(1-\theta_2)}{\Gamma(\theta_2)}\frac{\Gamma(1-\theta_3)}{\Gamma(\theta_3)}\frac{\Gamma\left(\frac{1}{2}+\theta_2\right)}{\Gamma\left(\frac{1}{2}-\theta_2\right)}\frac{\Gamma\left(\frac{1}{2}+\theta_3\right)}{\Gamma\left(\frac{1}{2}-\theta_3\right)}\right]^{1/4}
$$

$p_{\mathrm{KK}}/p_{\mathrm{bos/fer}}$	$p_{\mathrm{bos/fer}} = 0$	$p_{\mathrm{bos/fer}} = 2$	$p_{\mathrm{bos/fer}} = 4$	$p_{\mathrm{bos/fer}} = 6$
$p = 2$	EFT IIb), IIc) (PS)	quasi-EFT IIa) (TM)	X	X
$p = 4$	$\mathcal{N} = 2$ sectors	non-EFT (PS)	non-EFT (PS)*	X
$p = 6$	$\mathcal{N} = 2$ sectors	EFT IIa) (PS)	EFT IIa) (PS)*	STU-like II) (TM)

$$
\begin{aligned}
m_n^{(t)} &= n_t^{1/p_t} m_0^{(t)},\\
\Lambda_{QG} &= N_t^{1/p_t} m_0^{(t)},
\end{aligned}
$$

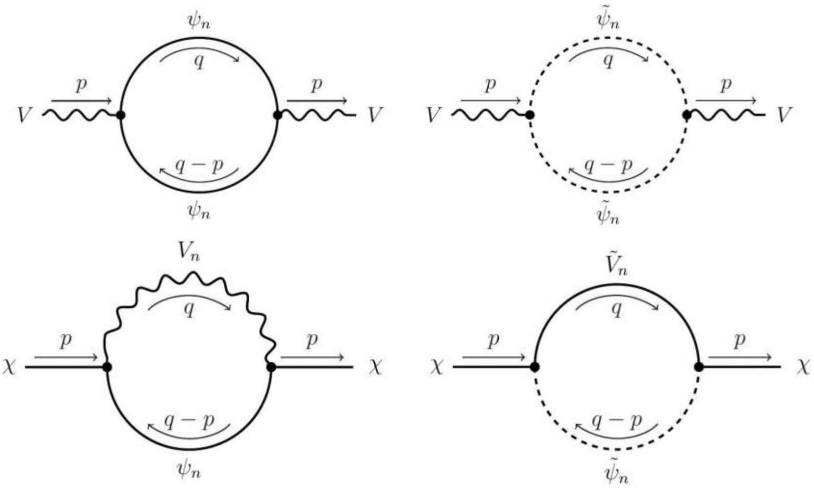
$$\Lambda_{QG} \lesssim \frac{M_{\mathrm{P}}^{(D)}}{N^{1/(D-2)}}$$

$$\Lambda_{QG} \simeq \left(m_0^{(t)}\right)^{p_t/(D-2+p_t)}$$

$$N_g \simeq \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(\frac{M_s}{m_{\mathrm{bos/fer}}^i}\right)^2$$

$$M_s^{(p_{\mathrm{bos/fer}}+2)} \simeq \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(m_{\mathrm{bos/fer}}^i\right)^2$$

$$N_{aa^*} \simeq \prod_i \left(\frac{M_s}{m_{\mathrm{bos/fer}}^i}\right)^2 \simeq k^3 s^2 e^{2\phi_4} \simeq k^3 s$$



$$\delta\left(g_{\alpha\beta}^{-2}\right) \simeq q_{\alpha\beta}^2 \sum_{k=1}^{N_g} \log\left(\Lambda_{QG}^2/m_k^2\right) \simeq q_{\alpha\beta}^2 N_g = q_{\alpha\beta}^2 \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(\frac{M_s}{m_{\mathrm{bos/fer}}^i}\right)^2$$

$$\delta\left(g_{\alpha\beta}^{-2}\right) \simeq q_{\alpha\beta}^2 \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \frac{1}{\theta_i}$$

$$g_{\alpha\beta}^{-2} \simeq q_{\alpha\beta}^2 N \simeq q_{\alpha\beta}^2 \left(\frac{M_{\mathrm{P}}}{M_s}\right)^2 = q_{\alpha\beta}^2 e^{-2\phi_4},$$

$$g_{\alpha\beta}^{-2} \simeq \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(\frac{M_{\mathrm{P}}}{m_{\mathrm{bos/fer}}^i}\right)^{4/(2+p_{\mathrm{bos/fer}})}$$

$$\delta(g_a^{-2}) \simeq (2)^2 N_g \simeq 4k^3 s$$

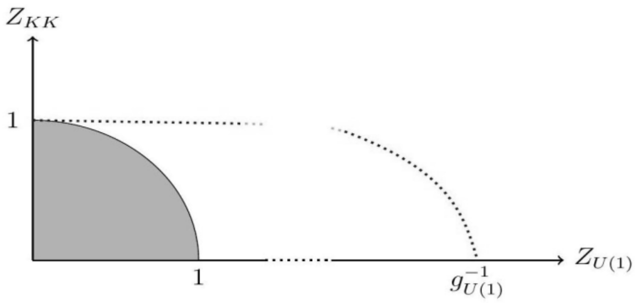
$$\delta(g_a^{-2}) \simeq \left(\frac{M_s}{m_{\mathrm{bos/fer},2}}\right)^2 \simeq ku, \delta(g_a^{-2}) \simeq \left(\frac{M_s}{m_{\mathrm{bos/fer},3}}\right)^2 \simeq ku.$$

$$K_{\alpha\beta} \simeq \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(\frac{M_s}{m_{\mathrm{bos/fer}}^i}\right)^2$$

$$K_{\alpha\beta} \simeq \prod_{i=1}^{p_{\mathrm{bos/fer}}/2} \left(\frac{M_{\mathrm{P}}}{m_{\mathrm{bos/fer}}^i}\right)^{4/(2+p_{\mathrm{bos/fer}})}$$

$$K_{\alpha\beta} \simeq g_{\alpha\beta}^{-2} \simeq N_g \simeq N \simeq \left(\frac{M_{\mathrm{P}}}{M_s}\right)^2 = e^{-2\phi_4}$$

$$m_{\mathrm{KK}}(\lambda) \simeq \frac{1}{\lambda^{w/2}}$$
$$K_\lambda \simeq \frac{1}{m_{\mathrm{KK}}^2}(\partial_\lambda m_{\mathrm{KK}})^2 \simeq \frac{1}{\lambda^2}$$
$$K_Q(\lambda) \sim -\log \lambda$$
$$Y_{ijk} \simeq \left(\frac{m_i}{M_s}\right)^{\gamma_i}\left(\frac{m_j}{M_s}\right)^{\gamma_j}\left(\frac{m_k}{M_s}\right)^{\gamma_k}$$
$$Y_{ijk} \simeq e^{\phi_4/2}\left(\frac{m^i_{\mathrm{bos/fer}}}{M_s}\right)^{1/2}\left(\frac{m^j_{\mathrm{bos/fer}}}{M_s}\right)^{1/2}\left(\frac{m^k_{\mathrm{bos/fer}}}{M_s}\right)^{1/2}$$
$$Y_{ijk} \simeq e^{\phi_4/2}\left(\frac{m^i_{\mathrm{bos/fer}}}{M_s}\right)^{1/2}$$
$$Y_{ijk} \simeq e^{\phi_4/2}N^{-1/2} \simeq e^{\phi_4} \simeq g_{\alpha\beta}$$



$$\alpha' m^2_{\alpha\beta} = \frac{L^2}{4\pi\alpha'} + N_{bos}(\theta^r_{\alpha\beta}) + \frac{(r+v_\theta)^2}{2} - \frac{1}{2} + E_{\alpha\beta}$$
$$E_{\alpha\beta} = \sum_{r=1}^{3}\frac{1}{2}|\theta^r_{\alpha\beta}|(1-|\theta^r_{\alpha\beta}|).$$
$$\chi = r + v_\vartheta = \left(\theta^1_{\alpha\beta} - \frac{1}{2}, \theta^2_{\alpha\beta} - \frac{1}{2}, \theta^3_{\alpha\beta} + \frac{1}{2}, -\frac{1}{2}\right),$$

State	Mass ^{2}
$\chi = (\theta^1_{\alpha\beta} - 1, \theta^2_{\alpha\beta}, \theta^3_{\alpha\beta}, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(-\lvert\theta^1_{\alpha\beta}\rvert + \lvert\theta^2_{\alpha\beta}\rvert + \lvert\theta^3_{\alpha\beta}\rvert)$
$\chi = (\theta^1_{\alpha\beta}, \theta^2_{\alpha\beta} - 1, \theta^3_{\alpha\beta}, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(\lvert\theta^1_{\alpha\beta}\rvert - \lvert\theta^2_{\alpha\beta}\rvert + \lvert\theta^3_{\alpha\beta}\rvert)$
$\chi = (\theta^1_{\alpha\beta}, \theta^2_{\alpha\beta}, \theta^3_{\alpha\beta} + 1, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(\lvert\theta^1_{\alpha\beta}\rvert + \lvert\theta^2_{\alpha\beta}\rvert - \lvert\theta^3_{\alpha\beta}\rvert)$
$\chi = (\theta^1_{\alpha\beta} + 1, \theta^2_{\alpha\beta}, \theta^3_{\alpha\beta}, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(3\lvert\theta^1_{\alpha\beta}\rvert + \lvert\theta^2_{\alpha\beta}\rvert + \lvert\theta^3_{\alpha\beta}\rvert)$
$\chi = (\theta^1_{\alpha\beta}, \theta^2_{\alpha\beta} + 1, \theta^3_{\alpha\beta}, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(\lvert\theta^1_{\alpha\beta}\rvert + 3\lvert\theta^2_{\alpha\beta}\rvert + \lvert\theta^3_{\alpha\beta}\rvert)$
$\chi = (\theta^1_{\alpha\beta}, \theta^2_{\alpha\beta}, \theta^3_{\alpha\beta} - 1, 0)$	$\alpha' m^2_{\alpha\beta} = \frac{1}{2}(\lvert\theta^1_{\alpha\beta}\rvert + \lvert\theta^2_{\alpha\beta}\rvert + 3\lvert\theta^3_{\alpha\beta}\rvert)$

$$\left(\alpha_{-|\theta^1_{\alpha\beta}|}\right)^{k_1}\left(\alpha_{-|\theta^2_{\alpha\beta}|}\right)^{k_2}\left(\alpha_{-|\theta^2_{\alpha\beta}|}\right)^{k_3}$$
$$\chi = (\theta^1_{\alpha\beta}, \theta^2_{\alpha\beta}, \theta^3_{\alpha\beta}, \pm 1)\ \alpha' m^2_{\alpha\beta} = \frac{1}{2}(|\theta^1_{\alpha\beta}| + |\theta^2_{\alpha\beta}| + |\theta^3_{\alpha\beta}|)$$
$$K_Q = -\log s - \log \frac{1}{6}\kappa_{ijk}u^i u^j u^k$$
$$s = e^{-\phi}V_Y, u^i = e^{-\phi}t^i$$
$$[\Pi_\alpha] = \frac{1}{2}[A] + \frac{1}{2}f^i_\alpha[D_i] - \left(\frac{1}{2}\kappa_{ijk}f^j_\alpha f^k_\alpha + \frac{1}{24}c_{2i}\right)[C^i] + \left(\frac{1}{6}\kappa_{ijk}f^i_\alpha f^j_\alpha f^k_\alpha + \frac{1}{24}c_{2i}f^i_\alpha\right)[B]$$
$$[\Pi_\alpha]\cdot[\Pi_\beta] = \frac{1}{6}\kappa_{ijk}(f^i_\beta - f^i_\alpha)(f^j_\beta - f^j_\alpha)(f^k_\beta - f^k_\alpha) + \frac{1}{12}c_{2i}(f^i_\beta - f^i_\alpha),$$

$$r_i^f + r_j^f + r_k^b = (0,0,0,\pm 1)$$
$$\sum_i r_i^b = (1,1,1,0)$$
$$\theta_{ab} = \left(\theta_{ab}^1, \theta_{ab}^2, \theta_{ab}^3, 0\right)$$
$$\theta_{bc} = \left(\theta_{bc}^1, \theta_{bc}^2, \theta_{bc}^3, 0\right)$$
$$\theta_{ca} = \left(\theta_{ca}^1, \theta_{ca}^2, \theta_{ca}^3, 0\right)$$
$$\theta_{ab} = \left(\left|\theta_{ab}^1\right|, \left|\theta_{ab}^2\right|, -\left|\theta_{ab}^3\right|, 0\right)$$
$$\theta_{bc} = \left(\left|\theta_{bc}^1\right|, -\left|\theta_{bc}^2\right|, \left|\theta_{bc}^3\right|, 0\right)$$
$$\theta_{ca} = \left(-|\theta_{ca}^1|, |\theta_{ca}^2|, |\theta_{ca}^3|, 0\right)$$
$$r_{ab}^b = (0,0,1,0), r_{bc}^b = (0,1,0,0), r_{ca}^b = (1,0,0,0),$$
$$r_{ab}^f = \frac{1}{2}(-,-,+,-), r_{bc}^f = \frac{1}{2}(-,+,-,-), r_{ca}^f = \frac{1}{2}(+,-,-,-).$$
$$K_{ab} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left(\frac{\Gamma\left(\left|\theta_{ab}^1\right|\right)\Gamma\left(\left|\theta_{ab}^2\right|\right)\Gamma\left(1-\left|\theta_{ab}^3\right|\right)}{\Gamma\left(1-\left|\theta_{ab}^1\right|\right)\Gamma\left(1-\left|\theta_{ab}^2\right|\right)\Gamma\left(\left|\theta_{ab}^3\right|\right)}\right)^{1/2}$$
$$K_{bc} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left(\frac{\Gamma\left(\left|\theta_{bc}^1\right|\right)\Gamma\left(1-\left|\theta_{bc}^2\right|\right)\Gamma\left(\left|\theta_{bc}^3\right|\right)}{\Gamma\left(1-\left|\theta_{bc}^1\right|\right)\Gamma\left(\left|\theta_{bc}^2\right|\right)\Gamma(1-\theta_{bc}^3\ |)}\right)^{1/2}$$
$$K_{ca} = \frac{e^{K/2-\phi_4}}{\sqrt{2\pi}}\left(\frac{\Gamma(1-|\theta_{ca}^1|)\Gamma(|\theta_{ca}^2|)\Gamma(|\theta_{ca}^3|)}{\Gamma(|\theta_{ca}^1|)\Gamma(1-|\theta_{ca}^2|)\Gamma(1-|\theta_{ca}^3|)}\right)^{1/2}$$
$$K_{\alpha\beta} \simeq \left[\frac{e^{K-2\phi_4}\left|\theta_{\alpha\beta}^i\right|}{2\pi\left|\theta_{\alpha\beta}^j\right|\left|\theta_{\alpha\beta}^k\right|}\right]^{1/2}$$
$$K_{\alpha\beta} \simeq e^{K/2}\frac{M_{\mathrm{P}}}{m_{\mathrm{bos/fer}}}$$
$$\Omega^{(3,0)} = \frac{1}{3!}\Omega_{\mu\nu\rho}\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu \wedge \mathrm{d}z^\rho$$
$$\frac{1}{2!}\omega_{\mu\nu\bar{\sigma}}\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu \wedge \mathrm{d}z^{\bar{\sigma}} \leftrightarrow \mathrm{d}z^{\bar{\sigma}}\left(\omega_{\bar{\sigma}}^{\mu} = \bar{\Omega}^{\mu\nu\rho}\omega_{\nu\rho\bar{\sigma}}\right)\partial_\mu$$
$$\kappa = \int_X \Omega \wedge \omega^\mu \wedge \omega^\nu \wedge \omega^\rho \Omega_{\mu\nu\rho}$$
$$\mathbf{248} = \mathbf{Adj}_{E_8} \to (\mathbf{Adj}_H, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{Adj}_G) \bigoplus_i \left(\mathbf{R}_H^{(i)}, \mathbf{R}_G^{(i)}\right)$$
$$\kappa\left(a^{(i)}, b^{(j)}, c^{(k)}\right) = \int_X \Omega \wedge \widetilde{\Omega}\left(a^{(i)}, b^{(j)}, c^{(k)}\right)$$
$$S_{\mathrm{eff}} = \int \mathrm{d}^4x\left[\int \mathrm{d}^4\theta K\left(\Phi^a, \bar{\Phi}^{\bar{b}}\right) + \frac{1}{4g^2}\left(\int \mathrm{d}^2\theta \mathrm{tr}\mathcal{W}_\alpha\mathcal{W}^\alpha + \int \mathrm{d}^2\theta W(\Phi^a)\right) + \text{ h.c.}\right]$$
$$K\left(\Phi^a, \bar{\Phi}^{\bar{b}}\right) \supset N_{a\bar{b}}\Phi^a\bar{\Phi}^{\bar{b}} + \cdots$$
$$N_{a\bar{b}} \sim (a,b) = \int_X a \wedge \bar{\star}_V b$$
$$Y\left(a^{(i)\prime}, b^{(j)\prime}, c^{(k)\prime}\right) = \frac{\int_X \Omega \wedge \widetilde{\Omega}\left(a^{(i)\prime}, b^{(j)\prime}, c^{(k)\prime}\right)}{\int_X \Omega \wedge \bar{\Omega}}$$
$$248 \to (78,1) \oplus (1,8) \oplus (\overline{27}, \overline{3}) \oplus (27,3)$$
$$[\alpha] \longmapsto [\Omega(\alpha)]$$
$$(a,b) = \int_X a \wedge \bar{\star}_g b \overset{\mathrm{def}}{=} \langle a,b\rangle_{\mathrm{WP}}$$
$$\widetilde{\Omega}(a,b,c) \overset{\mathrm{def}}{=} a^\mu \wedge b^\nu \wedge c^\rho \Omega_{\mu\nu\rho}$$

$$Y_{ijk}=\frac{\int_X \Omega\wedge\tilde{\Omega}(a_i,a_j,a_k)}{\sqrt{N_iN_jN_k}\int_X \Omega\wedge\bar{\Omega}}$$

$$\langle a,b\rangle_{\mathrm{WP}}=\int_{X_t}\rho(a)\wedge\overline{{}^{g_t}\mathcal{H}\rho(b)},$$

$$\rho\left(\frac{\partial}{\partial t}\right)=\left[\left\{\frac{\partial f_{jk}^{\mu}(z_k,t)}{\partial t}\frac{\partial}{\partial z_j^{\mu}}\right\}\right], \text{donde } z_k=f_{kj}(z_j,t)$$

$$\langle a,b\rangle_{\mathrm{WP}}=-\frac{\int_{X_t}\Omega(\mathcal{H}\rho(a))\wedge\overline{\Omega(\mathcal{H}\rho(b))}}{\int_{X_t}\Omega\wedge\bar{\Omega}}\int_{X_t}\mathrm{vol}_{g_t}$$

$$\zeta_t: X\overset{\simeq}{\to}X_t$$

$$\left[\{\bar{\partial}\xi_j\}_j\right]\in\check{H}^1(X,T_X)\simeq H^1(X,T_X)$$

$$\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}=\Omega'+\Omega(\phi)\in\Gamma(X,\Omega^{n,0})\oplus\Gamma(X,\Omega^{n-1,1})$$

$$(\alpha,\beta)=\int_X\alpha\wedge\bar{\beta}$$

$$\frac{\langle a,a\rangle_{\mathrm{WP}}}{\int_X \mathrm{vol}_g}=\frac{\left(\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0},\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)}{(\Omega,\Omega)}+\frac{\left|\left(\Omega,\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)\right|^2}{(\Omega,\Omega)^2}$$

$$\partial\Omega'+\bar{\partial}\Omega'+\partial(\Omega(\phi))=0$$

$$\Omega(\phi)+\bar{\partial}\psi=\mathcal{H}\Omega(\phi)$$

$$\bar{\partial}\Omega'+\partial(\Omega(\phi))+\partial\bar{\partial}\psi-\bar{\partial}\partial\psi=\bar{\partial}(\Omega'+\partial\psi)+\partial\mathcal{H}(\Omega(\phi))=0$$

$$\left(\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0},\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)=(\Omega',\Omega')+(\Omega(\phi),\Omega(\phi))$$

$$(\Omega',\Omega')=(c\Omega-\partial\psi,c\Omega-\partial\psi)=|c|^2(\Omega,\Omega)+(\partial\psi,\partial\psi)$$

$$(\Omega(\phi),\Omega(\phi))=(\mathcal{H}\Omega(\phi)-\bar{\partial}\psi,\mathcal{H}\Omega(\phi)-\bar{\partial}\psi)=(\mathcal{H}\Omega(\phi),\mathcal{H}\Omega(\phi))+(\bar{\partial}\psi,\bar{\partial}\psi)$$

$$\left(\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0},\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)=|c|^2(\Omega,\Omega)+(\mathcal{H}\Omega(\phi),\mathcal{H}\Omega(\phi))$$

$$\left(\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0},\Omega\right)=(\Omega',\Omega)=(c\Omega-\partial\psi,\Omega)=c(\Omega,\Omega)$$

$$-\frac{\left(\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0},\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)}{(\Omega,\Omega)}+\frac{\left|\left(\Omega,\left.\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\right|_{t=0}\right)\right|^2}{(\Omega,\Omega)^2}=-\frac{(\mathcal{H}\Omega(\phi),\mathcal{H}\Omega(\phi))}{(\Omega,\Omega)}$$

$$X=\begin{bmatrix} n_1 & q_1^1 & \cdots & q_K^1\\ \vdots & \vdots & \ddots & \vdots\\ n_m & q_1^N & \cdots & q_K^N\end{bmatrix}$$

$$H^1(T_X)\simeq\bigoplus_{l=1}^{K}H^0\big(X,\mathcal{O}_X(q_l)\big)/\sim\ \ni F$$

$$\frac{\partial p_j}{\partial Z^k}\xi^k+F_j=0,\ \text{for all } j\in\{1,\cdots,K\}$$

$$\mathrm{Jac}(p)^+=\overline{\mathrm{Jac}(p)}\cdot(\mathrm{Jac}(p)\cdot\overline{\mathrm{Jac}(p)})^{-1}$$

$$\xi^k=(\mathrm{Jac(p)}^+)^{kj}F_j+\sum_i c_i\chi_i^k,\ \text{for } c_i\in\mathbb{C}$$

$$\rho\left(\frac{\partial}{\partial t}\right)=[\bar{\partial}\xi]$$

$$\kappa_{ijk}=\int_X\Omega_{\alpha\beta\gamma}\frac{\partial\xi_i^{\alpha}}{\partial\bar{z}^{\mu}}\frac{\partial\xi_j^{\beta}}{\partial\bar{z}^{\nu}}\frac{\partial\xi_k^{\gamma}}{\partial\bar{z}^{\delta}}\Omega\wedge d\bar{z}^{\mu}\wedge d\bar{z}^{\nu}\wedge d\bar{z}^{\delta}$$

$$\begin{array}{ccc} A & \xrightarrow{\zeta_t} & A_t \\ \uparrow & & \uparrow \\ X & \xrightarrow{\zeta_t} & X_t \end{array}$$

$$\Omega_t \wedge \mathrm{d}p_t^1 \wedge \cdots \wedge \mathrm{d}p_t^K = \mathrm{d}Z_t^1 \wedge \cdots \wedge \mathrm{d}Z_t^M$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}Z_t^1 \wedge \cdots \wedge \mathrm{d}Z_t^M\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\det\left(\delta_k^j + t\frac{\partial \xi^j}{\partial Z^k}\right)_{j,k}\Bigg|_{t=0} \mathrm{d}Z^1 \wedge \cdots \wedge \mathrm{d}Z^M +$$

$$+ \sum_{j,k=1}^{M} \frac{\partial \xi^j}{\partial \bar{Z}^k} \mathrm{d}Z^1 \wedge \cdots \wedge \widehat{\mathrm{d}Z^J} \wedge \mathrm{d}\bar{Z}^k \wedge \cdots \wedge \mathrm{d}Z^M$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(dp_t^j\right)\Big|_{t=0} = \mathrm{d}\left(\frac{\partial p^j}{\partial Z^k}\xi^k + F^j\right) = 0$$

$$\frac{\mathrm{d}\Omega_t}{\mathrm{d}t}\Big|_{t=0} \wedge \mathrm{d}p^1 \wedge \cdots \wedge \mathrm{d}p^K\Big|_X = \mathrm{TrJac}(\xi)\mathrm{d}Z^1 \wedge \cdots \wedge \mathrm{d}Z^K|_X +$$

$$+ \sum_{j,k=1}^{M} \frac{\partial \xi^j}{\partial \bar{Z}^k} \mathrm{d}Z^1 \wedge \cdots \wedge \widehat{\mathrm{d}Z^J} \wedge \mathrm{d}\bar{Z}^k \wedge \cdots \wedge \mathrm{d}Z^M\Bigg|_X$$

$$\alpha = f(Z)\mathrm{d}Z^1 \wedge \cdots \wedge \mathrm{d}Z^n, \beta = \sum_{j,k=1}^{n} g_k^j(Z)\mathrm{d}Z^1 \wedge \cdots \wedge \widehat{\mathrm{d}Z^J} \wedge \cdots \wedge \mathrm{d}Z^n \wedge \mathrm{d}\bar{Z}^k$$

$$f(Z) = \mathrm{TrJac}(\xi)/\det\left(\frac{\partial p^j}{\partial Z^{n+k}}\right)_{j,k=1}^{K}$$

$$g_k^j(Z) = \frac{(-1)^{n-j}}{\det\left(\frac{\partial p^{j'}}{\partial Z^{n+k'}}\right)_{j',k'=1}^{K}}\left[\frac{\partial \xi^j}{\partial \bar{Z}^k} + \sum_{l=1}^{K} \frac{\partial \xi^j}{\partial \bar{Z}^{n+l}}\frac{\partial \bar{Z}^{n+l}}{\partial \bar{Z}^k}\right]$$

$$i^* \mathrm{d}\bar{Z}^{n+l} = \sum_{k=1}^{n} \frac{\partial \bar{Z}^{n+l}}{\partial \bar{Z}^k} \mathrm{d}\bar{Z}^k$$

$$\int_{X_t} \mathrm{vol}_{g_t} f \simeq \frac{1}{N}\sum_{k=1}^{N} f\left(p_k^{(t)}\right)$$

$$x_0^3 + x_1^3 + x_2^3 - 3\psi x_3 x_4 x_5 = 0, x_3^3 + x_4^3 + x_5^3 - 3\psi x_0 x_1 x_2 = 0$$

$$\kappa = \left[\frac{\Gamma(3/5)^3\Gamma(1/5)}{\Gamma(2/5)^3\Gamma(4/5)}\right]^{1/2} \approx 1.09236$$

$$\mathbb{Z}_5: \quad z_j \to \alpha^j z_j$$

$$\mathbb{Z}_5': \quad z_j \to z_{j+1}$$

$$Y_{5,5,5} = 1.550 \pm 0.002$$

$$X = \begin{bmatrix} 3 & 3 & 1 & 0 \\ 3 & 0 & 1 & 3 \end{bmatrix}$$

$$p^1 = \frac{1}{3}(x_0^3 + x_1^3 + x_2^3 + x_3^3) = 0$$

$$p^2 = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

$$p^3 = \frac{1}{3}(y_0^3 + y_1^3 + y_2^3 + y_3^3) = 0$$

$$\begin{cases}(x_0,x_1,x_2,x_3) \to (x_0,\alpha^2 x_1,\alpha x_2,\alpha x_3)\\ (y_0,y_1,y_2,y_3) \to (y_0,\alpha y_1,\alpha^2 y_2,\alpha^2 y_3)\end{cases}$$

$$\kappa_{ijk} = \int_{X/\mathbb{Z}_3} \Omega \wedge \Omega(\lambda_i,\lambda_j,\lambda_k)$$

$$p_2 = x_0 y_0 + x_1 y_1 + (1+\epsilon)(x_2 y_2 + x_3 y_3) = 0, \epsilon \in \mathbb{R}$$

$$\eta = \phi + \bar{\partial}_E \mathfrak{s} \in \mathcal{H}^{0,1}_{\bar{\partial}}(X,T_X), \mathfrak{s} \in C^\infty(T_X)$$

$$\ell(\theta) := \left(\bar{\partial}_E^\dagger \eta_\theta, \bar{\partial}_E^\dagger \eta_\theta\right) = \int_{X_t} \bar{\partial}_E^\dagger \eta_\theta \wedge \bar{\star}_E\, \bar{\partial}_E^\dagger \eta_\theta$$

$$= \int_{X_t} \mathrm{vol}_{g_t} g_{\mu\bar{\nu}} \left(\bar{\partial}_E^\dagger \eta_\theta\right)^\mu \left(\overline{\bar{\partial}_E^\dagger \eta_\theta}\right)^{\bar{\nu}}$$

$$25\frac{\Gamma(4/5)^5\Gamma(2/5)^5}{\Gamma(1/5)^5\Gamma(3/5)^5} \approx 0.1922$$

$$(')[\alpha] \longmapsto [\Omega(\alpha)]$$

$$H^1(T_X) \simeq H^{0,1}(T_X) \simeq H^{n,n-1}(T_X^*)^* \simeq H^{n-1}(\Omega^n \otimes T_X^*)^*$$

$$H^{n-1}(\Omega^n \otimes T_X^*)^* \simeq H^{1,n-1}(\mathcal{O}_X)^* \simeq H^{n-1,1}(\mathcal{O}_X^*) \simeq H^{n-1,1}(\mathcal{O}_X)$$

$$\Omega(\bar{\partial}\phi) = \sum_{\mu=1}^{n} (-1)^{\mu-1} f \alpha_{\bar{\nu}}^{\mu} \mathrm{d}\bar{z}^\nu \wedge \hat{\mu} = \sum_{\mu=1}^{n} (-1)^{\mu-1} f \frac{\partial \phi}{\partial \bar{z}^\nu} \mathrm{d}\bar{z}^\nu \wedge \hat{\mu}$$

$$\bar{\partial}\Omega(\alpha) = \sum_{\mu=1}^{n} (-1)^{\mu-1} f \frac{\partial \alpha_{\bar{\nu}}^{\mu}}{\partial \bar{z}^\sigma} \mathrm{d}\bar{z}^\sigma \wedge \mathrm{d}\bar{z}^\nu \wedge \hat{\mu} = 0$$

$$(')\langle a,b\rangle_{\mathrm{WP}} = -\frac{\int_{X_t} \Omega(\mathcal{H}\rho(a)) \wedge \overline{\Omega(\mathcal{H}\rho(b))}}{\int_{X_t} \Omega \wedge \bar{\Omega}} \int_{X_t} \mathrm{vol}_{g_t}$$

$$\Omega(\alpha) = \sum_{\mu,\nu=1}^{n} (-1)^{\mu-1} \alpha \frac{\mu}{\nu} f \, \mathrm{d}\bar{z}^\nu \wedge \hat{\mu}$$

$$\Omega(\alpha) \wedge \overline{\Omega(\beta)} = \sum_{\mu,\mu',\nu,\nu'=1}^{n} (-1)^{\mu+\mu'} |f|^2 \alpha \frac{\mu}{\bar{\nu}} \overline{\beta_{\nu'}^{\mu'}} \mathrm{d}\bar{z}^\nu \wedge \hat{\mu} \wedge \mathrm{d}z^{\nu'} \wedge \widehat{\mu'}$$

$$\Omega(\alpha) \wedge \overline{\Omega(\beta)} = \left(\sum_{\mu,\nu=1}^{n} (-1)^{\mu+\nu} \alpha_{\bar{\nu}}^{\mu} \overline{\beta_{\bar{\mu}}^{\nu}} (-1)^{\mu+\nu-1}\right) \Omega \wedge \bar{\Omega} = -\alpha_{\bar{\nu}}^{\mu} \overline{\beta_{\bar{\mu}}^{\nu}} \Omega \wedge \bar{\Omega}$$

$$\int_{X_t} \Omega(\alpha) \wedge \overline{\Omega(\beta)} = -\frac{1}{\kappa} \int_{X_t} \alpha \wedge \overline{\star_{g_t}} \beta$$

$$\partial\Omega(\alpha) = \partial\left(\sum_{\mu=1}^{n} (-1)^{\mu-1} f(z) \alpha_{\bar{\nu}}^{\mu} \mathrm{d}\bar{z}^\nu \wedge \hat{\mu}\right)$$

$$= \sum_{\mu=1}^{n} \frac{\partial}{\partial z^\mu} \left(f(z) \alpha_{\bar{\nu}}^{\mu}\right) \mathrm{d}\bar{z}^\nu \wedge \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n = 0$$

$$g_{\mu\bar{\rho}} g^{\sigma\bar{\nu}} \alpha_{\bar{\nu}}^{\mu} = \alpha \frac{\sigma}{\bar{\rho}}$$

$$\frac{\partial}{\partial z^\mu}\left(g_{\sigma\bar{\nu}} g^{\mu\bar{\rho}} \alpha \frac{\sigma}{\bar{\rho}} \det g\right) \mathrm{d}\bar{z}^\nu = \kappa \frac{\partial}{\partial z^\mu}\left(|f(z)|^2 g_{\sigma\bar{\nu}} g^{\mu\bar{\rho}} \alpha \frac{\sigma}{\bar{\rho}}\right) \mathrm{d}\bar{z}^\nu$$

$$= \kappa \overline{f(z)} \frac{\partial}{\partial z^\mu}\left(f(z) \alpha \frac{\mu}{\nu}\right) \mathrm{d}\bar{z}^\nu = 0$$

5. La brecha de masa y la curvatura en campos y supercampos cuánticos de Yang – Mills. La brecha de masa, que se constituye esencialmente en un cambio de estado de energía de una partícula, respecto del estado de vacío, de tal suerte que, es ésta excitación, es decir, esta modificación de energía superior a cero, la que puede curvar la geometría del espacio – tiempo cuántico en el que interactúe la partícula excitada.

El modelo matemático, queda expuesto de la siguiente manera:

$$u \wedge * v = \langle u, v \rangle_k d\omega$$
$$\langle u, v \rangle = \int_M u \wedge * v = \int_M \langle u, v \rangle_k d\omega.$$
$$\langle A, B \rangle = -\mathrm{Tr}_{\mathrm{Mat}(\bar{N},\mathbb{C})}[AB]$$
$$A \cdot \Omega := A^{\Omega} = \Omega^{-1} d\Omega + \Omega^{-1} A \Omega$$
$$S_{\mathrm{YM}}(A) = \int_M |dA + A \wedge A|^2 d\omega$$
$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^1 \wedge dx^2, dx^3 \wedge dx^1, dx^2 \wedge dx^3\}$$
$$*(dx^0 \wedge dx^1) = dx^2 \wedge dx^3, \quad *(dx^0 \wedge dx^2) = dx^3 \wedge dx^1, \quad *(dx^0 \wedge dx^3) = dx^1 \wedge dx^2$$
$$*(dx^2 \wedge dx^3) = dx^0 \wedge dx^1, \quad *(dx^3 \wedge dx^1) = dx^0 \wedge dx^2, \quad *(dx^1 \wedge dx^2) = dx^0 \wedge dx^3$$
$$\vec{x} \cdot \vec{y} = -x^0 y^0 + \sum_{i=1}^{3} x^i y^i$$
$$-(x^0 - y^0)^2 + \sum_{i=1}^{3} \left(x^i - y^i\right)^2 \geq 0$$
$$\mathbb{H}_{\mathrm{YM}}(\mathfrak{g}) = \{1\} \oplus \bigoplus_{n \geq 1} \mathcal{H}(\rho_n)$$
$$\mathbb{H}_{\mathrm{YM}}(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathcal{H}(\rho_n)$$
$$\left\langle \sum_{n=0}^{\infty} v_n, \sum_{n=0}^{\infty} u_n \right\rangle := \sum_{n=0}^{\infty} \langle v_n, u_n \rangle$$
$$\{(a^0 + sb^0, a^1, a^2 + sb^2, a^3 + tb^3)^T \in \mathbb{R}^4 : s, t \in I\}$$
$$\inf_{\vec{0} \neq \vec{v} \in TS} \frac{|v|^2}{v^{0,2}} < 1$$
$$\inf_{\vec{0} \neq \vec{v} \in TS} \frac{|v|^2}{v^{0,2}} > 1$$
$$\lambda\left(S, f_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right) + \mu\left(\tilde{S}, g_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right)$$
$$\left\langle \left(S, f_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right), \left(\tilde{S}, g_\beta \otimes \rho\left(E^\beta\right), \{\hat{g}_a\}_{a=0}^{3}\right) \right\rangle$$
$$:= \int_{S \cap \tilde{S}} [f_\alpha \overline{g_\beta}] \cdot d|\hat{\rho}| \cdot \mathrm{Tr}\left[-\rho(E^\alpha)\rho\left(E^\beta\right)\right]$$
$$\equiv \sum_{\alpha=1}^{N} C(\rho) \int_{I^2} [f_\alpha \cdot \overline{g_\alpha}](\sigma(\hat{s})) \left| \sum_{0 \leq a < b \leq 3} \hat{\rho}_\sigma^{ab}(\hat{s}) \left[\det \tilde{J}_{ab}^{\sigma}(\hat{s})\right] \right| d\hat{s},$$
$$\{(S_0 + \vec{a}, \rho(E^\alpha), \{e_a\}_{a=0}^{3}) : \vec{a} \in \mathbb{R}\}$$
$$\left\langle (S_0 + \vec{a}, \rho(E^\alpha), \{e_a\}_{a=0}^{3}), \left(S_0 + \vec{b}, \rho(E^\alpha), \{e_a\}_{a=0}^{3}\right) \right\rangle = 0$$
$$\{\vec{a}_1, A_1\}\{\vec{a}_2, A_2\} = \{\vec{a}_1 + A_1 \vec{a}_2, A_1 A_2\}$$
$$\{\vec{a}_1, \Lambda_1\}\{\vec{a}_2, \Lambda_2\} = \{\vec{a}_1 + Y(\Lambda_1)\vec{a}_2, \Lambda_1 \Lambda_2\}$$
$$\left(S, f_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right) \longmapsto U(\vec{a}, \Lambda)\left(S, f_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right),$$
$$U(\vec{a}, \Lambda)\left(S, f_\alpha \otimes \rho(E^\alpha), \{\hat{f}_a\}_{a=0}^{3}\right)$$
$$:= \left(\Lambda S + \vec{a}, e^{-i\left[\vec{a} \cdot \left(\hat{H}(\rho_n)\Lambda \hat{f}_0 + \hat{P}(\rho_n)\Lambda \hat{f}_1\right)\right]} f_\alpha(\Lambda^{-1}(\cdot - \vec{a})) \otimes \rho(E^\alpha), \{\Lambda \hat{f}_a\}_{a=0}^{3}\right)$$

$$e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)Y(\Lambda)\hat{f}_0+\hat{P}(\rho_n)Y(\Lambda)\hat{f}_1)]}f_\alpha[Y(\Lambda^{-1})(\vec{x}-\vec{a})]\otimes\rho(E^\alpha)$$
$$\equiv e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)Y(\Lambda)\hat{f}_0+\hat{P}(\rho_n)Y(\Lambda)\hat{f}_1)]}f_\alpha[\sigma(\hat{s})]\otimes\rho(E^\alpha)$$
$$U(\vec{a},1)\left(S,f_\alpha\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$
$$\equiv e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}\left(S+\vec{a},f_\alpha(\cdot-\vec{a})\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$
$$\left\langle U(\vec{a},\Lambda)\left(S,f_\alpha\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right),U(\vec{a},\Lambda)\left(\tilde{S},g_\beta\otimes\rho(E^\beta),\{\hat{f}_a\}_{a=0}^3\right)\right\rangle$$
$$:=\int_{[\Lambda S+\vec{a}]\cap[\Lambda\tilde{S}+\vec{a}]}d|\hat{\rho}|e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)\Lambda\hat{f}_0+\hat{P}(\rho_n)\Lambda\hat{f}_1)]}e^{i[\vec{a}\cdot(\hat{H}(\rho_n)\Lambda\hat{f}_0+\hat{P}(\rho_n)\Lambda\hat{f}_1)]}$$
$$\times[f_\alpha\overline{g_\beta}](\Lambda^{-1}(\cdot-\vec{a}))\cdot\mathrm{Tr}[-\rho(E^\alpha)\rho(E^\beta)]$$
$$=\int_{\Lambda(S\cap\tilde{S})+\vec{a}}[f_\alpha\overline{g_\beta}](\Lambda^{-1}(\cdot-\vec{a}))\cdot d|\hat{\rho}|\cdot\mathrm{Tr}[-\rho(E^\alpha)\rho(E^\beta)]$$
$$=\int_{S\cap\tilde{S}}[f_\alpha\overline{g_\beta}](\cdot)\cdot d|\hat{\rho}|\cdot\mathrm{Tr}[-\rho(E^\alpha)\rho(E^\beta)]$$
$$\sum_{u=1}^{\infty}\left(S_u,f_\alpha^u\otimes\rho(E^\alpha),\{\hat{f}_a^u\}_{a=0}^3\right)$$
$\sum_{u=1}^{\infty}\left(S,f_\alpha^u\otimes\rho_n(E^\alpha),\{\hat{f}_a^u\}_{a=0}^3\right)\{\phi^{\alpha,n}(g):1\le\alpha\le\underline{N}\}\{\hat{H}(\rho_n),\hat{P}(\rho_n)\}_{n\ge1},\vec{a}=a^0\hat{f}_0+a^1\hat{f}_1.$
$\{\hat{H}(\rho),\hat{P}(\rho)\}\ \hat{H}(\rho)^2-\hat{P}(\rho)^2=m(\rho)^2, m(\rho)\ge 0.1, m_0:=\inf_{n\in\mathbb{N}}m(\rho_n)>0$
$$\frac{\hat{P}(\rho_n)^2}{\hat{H}(\rho_n)^2}-1=-\frac{m(\rho_n)^2}{\hat{H}(\rho_n)^2}$$
$$0>\frac{\hat{P}(\rho_n)^2}{\hat{H}(\rho_n)^2}-1\longrightarrow 0$$
$\bigoplus_{n=0}^{\infty}\mathcal{H}(\rho_n)$, $\mathrm{SL}(2,\mathbb{C})$ $D^{(j,k)}$, j,k $D^{(j,k)}(-1)=(-1)^{2(j+k)}$ $\mathrm{SU}(2),\Lambda\in\mathrm{SU}(2)\mapsto D^{(j,0)}(\Lambda)$
$\hat{f}_a=\Lambda e_a\equiv Y(\Lambda)e_a$ $\{\hat{g}_a\}_{a=0}^3$ $\hat{f}_a=\hat{g}_a$ $Y(\Lambda)e_a=Y(\tilde{\Lambda})e_a$ $\tilde{\Lambda}\in\mathrm{SL}(2,\mathbb{C})$. $Y(\Lambda)=Y(\tilde{\Lambda}),\Lambda=\pm\tilde{\Lambda}$. $j+k$, $D^{(j,k)}(\pm1)=1, D^{(j,k)}$
$$\vec{k}=(k^0,k^1,k^2,k^3),k^a\in\{0\}\cup\mathbb{N}$$
$$D^{\vec{k}}=\left(\frac{\partial}{\partial x^0}\right)^{k_0^0}\left(\frac{\partial}{\partial x^1}\right)^{k_1^1}\left(\frac{\partial}{\partial x^2}\right)^{k^2}\left(\frac{\partial}{\partial x^3}\right)^{k_3^3},\vec{x}^{\vec{k}}=(x^0)^{k^0}(x^1)^{k^1}(x^2)^{k^2}(x^3)^{k_3^3}$$
$$|\vec{k}|=\sum_{a=0}^{3}|k^a|$$
$$\|f\|_{r,s}:=\sum_{|\vec{k}|\le r}\sum_{|\vec{l}|\le s}\sup_{\vec{x}\in\mathbb{R}^4}\left|\vec{x}^{\vec{k}}D^{\vec{l}}f(\vec{x})\right|.$$
$$\hat{s}=(s,\bar{s})\mapsto\sigma(s,\bar{s})=s\hat{f}_0+\bar{s}\hat{f}_1,s,\bar{s}\in\mathbb{R}$$
$$d\hat{s}=dsd\bar{s},\tilde{\eta}:\vec{v}\in\mathbb{R}^4\mapsto\vec{v},\left(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1\right)\in\mathbb{R},\tilde{f}\{\hat{f}_0,\hat{f}_1\}:\mathbb{R}^4\to\mathbb{C}$$
$$\vec{x}\in\mathbb{R}^4\longmapsto\tilde{f}^{\{\hat{f}_0,\hat{f}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\vec{x}):=\int_{S^b}\frac{e^{-i\tilde{\eta}(\cdot)}}{2\pi}\tilde{f}(\vec{x}+\cdot)d|\hat{\rho}|$$
$$=\int_{\hat{s}\in\mathbb{R}^2}\frac{e^{-i[\sigma(\hat{s})\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{2\pi}\tilde{f}(\vec{x}+\sigma(\hat{s}))\cdot|\hat{\rho}_\sigma|(\hat{s})d\hat{s}$$
$$\tilde{f}^{\{\hat{f}_0,\hat{f}_1\}}\notin\mathcal{P},\tilde{f}\equiv 0.$$
$$\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1S^b.$$
$$\vec{x}=\textstyle\sum_{a=0}^3 x^a\hat{f}_a$$
$$\tilde{f}^{\{\hat{f}_0,\hat{f}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\vec{x})$$
$$=e^{-i[x^0\hat{H}(\rho_n)-x^1\hat{P}(\rho_n)]}\tilde{f}^{\{\hat{f}_0,\hat{f}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(x^2\hat{f}_2+x^3\hat{f}_3)$$

$$\phi^{\alpha,n}(\tilde{f})1 := \left(S_0, \tilde{f}^{\{e_0,e_1\}} \otimes \rho_n(F^\alpha), \{e_a\}_{a=0}^3\right)$$

$$\equiv \left(S_0, \tilde{f}^{\{e_0,e_1\}}\left(\hat{H}(\rho_n), \hat{P}(\rho_n)\right) \otimes \rho_n(F^\alpha), \{e_a\}_{a=0}^3\right) \in \mathcal{H}(\rho_n),$$

$$\left\{a_0 1 + \sum_{n,u=1}^{\infty} \left(S_{n,u}, f_{n,\alpha}^u \otimes \rho_n(E^\alpha), \{\hat{f}_a^{n,u}\}_{a=0}^3\right) : a_0 \in \mathbb{C}, f_{n,\alpha}^u \in \mathcal{P}_{S_{n,u}}, S_{n,u} \in \mathcal{S}\right\}$$

$$\int_{I^2} |f \circ \sigma|^2(\hat{s}) \left| \sum_{0 \le a < b \le 3} \hat{\rho}_\sigma^{ab}(\hat{s})\left[\det \hat{J}_{ab}^\sigma(\hat{s})\right]\right| d\hat{s} < \infty$$

$$\phi^{\alpha,n}(f) \sum_{u=1}^{\infty} \left(S_u, g_\beta^u \otimes \rho_m(E^\beta), \{\hat{f}_a^u\}_{a=0}^3\right)$$

$$:= \begin{cases} \displaystyle\sum_{u=1}^{\infty} \left(S_u, f_n^{\{\hat{f}_0^u, \hat{f}_1^u\}} A(\Lambda^u)_\gamma^\alpha \cdot g_\beta^u \otimes \rho_n([F^\gamma, E^\beta]), \{\hat{f}_a^u\}_{a=0}^3\right), & m = n \\ 0, & m \neq n \end{cases}$$

$$\{\phi^{\alpha,n}(f) : 1 \le \alpha \le \underline{N}\}$$

$$\phi^{\alpha,n}(f) \sum_{m=0}^{\infty} v_m := \sum_{m=0}^{\infty} \phi^{\alpha,n}(f) v_m$$

$$= a_0 \left(S_0, f_n^{\{e_0,e_1\}} \otimes \rho_n(E^\alpha), \{e_a\}_{a=0}^3\right) + \phi^{\alpha,n}(f) v_n$$

$$\phi^{\alpha,n}(f)\left(S, g_\beta \otimes \rho_n(E^\beta), \{\hat{f}_a\}_{a=0}^3\right)$$

$$:= \left(S, f_n^{\{\hat{f}_0, \hat{f}_1\}} \cdot A(\Lambda)_\gamma^\alpha g_\beta \otimes \rho_n([F^\gamma, E^\beta]), \{\hat{f}_a\}_{a=0}^3\right)$$

$$\phi^{\alpha,n}(g)^*\left(S, f_\beta \otimes \rho_n(E^\beta), \{\hat{f}_a\}_{a=0}^3\right)$$

$$= -\left(S, \overline{g^{\{\hat{f}_0, \hat{f}_1\}}}\, \overline{A(\Lambda)_\gamma^\alpha} \cdot f_\beta \otimes \rho_n([F^\gamma, E^\beta]), \{\hat{f}_a\}_{a=0}^3\right)$$

$$+ \left\langle \left(S, f_\beta \otimes \rho_n(E^\beta), \{\hat{f}_a\}_{a=0}^3\right), \phi^{\alpha,n}(g)1 \right\rangle 1,$$

$$\langle \mathrm{ad}(\rho(E^\alpha))\rho(E^\beta), \rho(E^\gamma)\rangle = -\langle \rho(E^\beta), \mathrm{ad}(\rho(E^\alpha))\rho(E^\gamma)\rangle.$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

$$[\lambda_7, \lambda_4] = \lambda_2, [\lambda_7, \lambda_5] = \lambda_1, \mathrm{ad}(\lambda_4)\mathrm{ad}(\lambda_5)\lambda_7 = -\lambda_6$$

$$\mathrm{ad}(\lambda_7)\mathrm{ad}(\lambda_4)\mathrm{ad}(\lambda_5)\lambda_7 = \lambda_3 - \sqrt{3}\lambda_8, [\lambda_5, \lambda_4] = \lambda_3 + \sqrt{3}\lambda_8$$

$$\breve{\varphi}_\delta(t) := \begin{cases} \frac{1}{\delta} t, & 0 \le t \le \delta \\ 1, & \delta < t \le 1 - \delta \\ \frac{1}{\delta}(1 - t), & 1 - \delta < t \le 1 \end{cases}$$

$$\|f - \tilde{g}_\epsilon\|_{L^2} = \left[\int_{I^2} \left|1 - g_\delta^{n-1}\right|^2(\hat{t}) |f|^2(\hat{t}) d\hat{t}\right]^{1/2}$$

$$\le M \left\|1 - g_\delta^{n-1}\right\|_{L^2} < \epsilon$$

$$\psi^{\alpha_1,m_1}(g_1) \cdots \psi^{\alpha_k,m_k}(g_k)\mathcal{D} \subset \mathcal{D}$$

$$\mathrm{span}\{\mathrm{ad}(F^{\alpha_1}) \cdots \mathrm{ad}(F^{\alpha_k}) F^\beta : 1 \le \beta \le \underline{N}, 1 \le \alpha_i \le \underline{N}, i = 1,2,\cdots,k\} = \mathfrak{g}.$$

$$E^{\gamma}=\sum_{\beta=1}^{N(\gamma)} d_{m,\beta}^{\gamma}\operatorname{ad}\left(F^{\alpha_1^{\gamma,\beta}}\right)\cdots\operatorname{ad}\left(F_{m-1}^{\alpha_m^{\gamma,\beta}}\right)F_m^{\alpha_m^{\gamma,\beta}}$$

$$F_i\in\mathcal{P}:\vec{x}\in\mathbb{R}^4\mapsto\frac{1}{c_n}p_1(x^0)\frac{1}{d_n}p_1(x^1)f_i(x^2,x^3), i=1,\cdots,m.$$

$$F_i^{\{e_0,e_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(0,0,x^2,x^3)=\frac{1}{c_n}\hat{p}_1\left(\hat{H}(\rho_n)\right)\frac{1}{d_n}\hat{p}_1\left(\hat{P}(\rho_n)\right)f_i(x^2,x^3)=f_i(x^2,x^3)$$

$$\left[F_1^{\{e_0,e_1\}}\cdots F_m^{\{e_0,e_1\}}\right]\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(0,0,x^2,x^3)=\prod_{i=1}^{m} f_i(x^2,x^3)$$

$$\sum_{\beta=1}^{N(\gamma)} d_{m,\beta}^{\gamma}\phi^{\alpha_1^{\gamma,\beta},n}(F_1)\cdots\phi^{\alpha_m^{\gamma,\beta},n}(F_m)1=\left(S,\prod_{i=1}^{m} f_i\otimes\rho_n(E^{\gamma}),\{e_a\}_{a=0}^{3}\right)$$

$$C_m=\operatorname{span}\{\phi^{\alpha_1,n}(F_1)\cdots\phi^{\alpha_m,n}(F_m)1: F_i\in\mathcal{P},1\le\alpha_i\le\underline{N}\}$$

$$\operatorname{span}\{\operatorname{ad}(F^{\alpha_1})\cdots\operatorname{ad}(F^{\alpha_{\tilde{n}-1}})F^{\alpha_{\tilde{n}}}:1\le\alpha_i\le\underline{N},1\le\tilde{n}\le\underline{n}\}=\mathfrak{g}$$

$$\tilde{E}^{\gamma}=\sum_{\xi=1}^{N(\gamma)} d_{\tilde{n}}(\gamma,\xi)\operatorname{ad}\left(F^{\alpha_1(\gamma,\xi)}\right)\cdots\operatorname{ad}\left(F^{\alpha_{\tilde{n}-1}(\gamma,\xi)}\right)F^{\alpha_{\tilde{n}}(\gamma,\xi)}$$

$$\|f-g_1\cdots g_{\tilde{n}}\|_{L^2}<\frac{\epsilon}{C(\rho_n)}$$

$$\sum_{\xi=1}^{N(\gamma)} d_{\tilde{n}}(\gamma,\xi)\phi^{\alpha_1(\gamma,\xi),n}(G_1)\cdots\phi^{\alpha_{\tilde{n}-1}(\gamma,\xi),n}(G_{\tilde{n}-1})\phi^{\alpha_{\tilde{n}}(\gamma,\xi),n}(G_{\tilde{n}})1$$

$$=\left(S,\prod_{i=1}^{\tilde{n}} g_i\otimes\rho_n(\tilde{E}^{\gamma}),\{e_a\}_{a=0}^{3}\right)$$

$$\left|\left(S,f\otimes\rho(\tilde{E}^{\gamma}),\{e_a\}_{a=0}^{3}\right)-\left(S,\prod_{i=1}^{\tilde{n}} g_i\otimes\rho_n(\tilde{E}^{\gamma}),\{e_a\}_{a=0}^{3}\right)\right|<\epsilon$$

$$c_{\alpha}^{\gamma,\beta}=A(\Lambda)_{\delta}^{\alpha}\operatorname{Tr}\left[-\left[\operatorname{ad}\left(\rho_n(F^{\delta})\right)\rho_n(E^{\gamma})\right]\rho_n(E^{\beta})\right]$$

$$T(f):=\left\langle\phi^{\alpha,n}(f)\left(\tilde{S},\tilde{g}_{\gamma}\otimes\rho_n(E^{\gamma}),\{\hat{f}_a\}_{a=0}^{3}\right),\left(S,g_{\beta}\otimes\rho_n(E^{\beta}),\{\hat{f}_a\}_{a=0}^{3}\right)\right\rangle$$

$$=c_{\alpha}^{\gamma,\beta}\int_{I^2}d\hat{t}\left[f^{\{\hat{f}_0,\hat{f}_1\}}\cdot\tilde{g}_{\gamma}\cdot\overline{g_{\beta}}\right](\sigma(\hat{t}))\cdot|\hat{\rho}_{\sigma}|(\hat{t})$$

$$h=\left[\tilde{g}_{\gamma}\cdot\overline{g_{\beta}}\right]\circ\sigma\cdot|\hat{\rho}_{\sigma}|c_{\alpha}^{\gamma,\beta}.$$

$$T(f)=\int_{I^2} f^{\{\hat{f}_0,\hat{f}_1\}}(\sigma(\hat{t}))h(\hat{t})d\hat{t}$$

$$\vec{x}\longmapsto f^{\{\hat{f}_0,\hat{f}_1\}}(\vec{x})\equiv f^{\{\hat{f}_0,\hat{f}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\vec{x})$$

$$=\int_{\hat{s}\in\mathbb{R}^2}\frac{e^{-i[\tilde{\sigma}(\hat{s})\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{2\pi}f(\vec{x}+\tilde{\sigma}(\hat{s}))|\hat{\rho}_{\tilde{\sigma}}|(\hat{s})d\hat{s}$$

$$T(f)=\int_{I^2}\{\hat{f}_0,\hat{f}_1\}(\sigma(\hat{t}))h(\hat{t})d\hat{t}$$

$$=\int_{\hat{s}\in\mathbb{R}^2,\hat{t}\in I^2} d\hat{t}d\hat{s}f(\sigma(\hat{t})+\tilde{\sigma}(\hat{s}))|\hat{\rho}_{\tilde{\sigma}}|(\hat{s})|\hat{\rho}_{\sigma}|(\hat{t})\cdot\frac{e^{-i[\tilde{\sigma}(\hat{s})\cdot\vec{\alpha}]}}{2\pi}\left[\tilde{g}_{\gamma}\cdot\overline{g_{\beta}}\right]\circ\sigma(\hat{t})\cdot c_{\alpha}^{\gamma,\beta}$$

$$(\hat{s},\hat{t})\in\mathbb{R}^2\times I^2\longmapsto|\hat{\rho}_{\tilde{\sigma}}|(\hat{s})|\hat{\rho}_{\sigma}|(\hat{t})\cdot\frac{e^{-i[\tilde{\sigma}(\hat{s})\cdot\vec{\alpha}]}}{2\pi}\left[\tilde{g}_{\gamma}\cdot\overline{g_{\beta}}\right]\circ\sigma(\hat{t})\cdot c_{\alpha}^{\gamma,\beta},$$

$$f \in \mathcal{P}$$

$$\mapsto \int_{\hat{s}\in\mathbb{R}^2,\hat{t}\in I^2} d\hat{t}d\hat{s}f(\sigma(\hat{t})+\tilde{\sigma}(\hat{s}))|\hat{\rho}_{\tilde{\sigma}}|(\hat{s})|\hat{\rho}_{\sigma}|(\hat{t})\cdot\frac{e^{-i[\tilde{\sigma}(\hat{s})\cdot\vec{a}]}}{2\pi}[\tilde{g}_\gamma\cdot\overline{g_\beta}]\circ\sigma(\hat{t})$$

$$\times\operatorname{Tr}[-[\operatorname{ad}(\cdot)\rho_n(E^\gamma)]\rho_n(E^\beta)]$$

$$T(f)$$

$$=\int_{\mathbb{R}^4} f(\vec{x})\frac{e^{i[x^0\hat{H}(\rho_n)-x^1\hat{P}(\rho_n)]}}{2\pi}[\tilde{g}_\gamma\cdot\overline{g_\beta}](x^2,x^3)d\vec{x}\cdot\operatorname{Tr}[-[\operatorname{ad}(\rho_n(F^\alpha))\rho_n(E^\gamma)]\rho_n(E^\beta)]$$

$$\vec{x}=(x^0,x^1,x^2,x^3)\mapsto\frac{e^{i[x^0\hat{H}(\rho_n)-x^1\hat{P}(\rho_n)]}}{2\pi}[\tilde{g}_\gamma\cdot\overline{g_\beta}](x^2,x^3),$$

$$U(\vec{a},\Lambda)\phi^{\alpha,n}(f)U(\vec{a},\Lambda)^{-1}\left(S,g_\beta\otimes\rho_n(E^\beta),\{\hat{f}_a\}_{a=0}^3\right)$$

$$=A(\Lambda^{-1})^\alpha_\gamma\phi^{\gamma,n}(f(\Lambda^{-1}(\cdot-\vec{a})))\left(S,g_\beta\otimes\rho_n(E^\beta),\{\hat{f}_a\}_{a=0}^3\right)$$

$$U(\vec{a},\Lambda)\phi^{\alpha,n}(f)U(\vec{a},\Lambda)^{-1}1$$

$$=\left(\Lambda S_0+\vec{a},e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)\hat{g}_0+\hat{P}(\rho_n)\hat{g}_1)]}f^{\{\hat{g}_0,\hat{g}_1\}}(\Lambda^{-1}(\cdot-\vec{a}))\otimes\rho_n(F^\alpha),\{\hat{g}_a\}_{a=0}^3\right)$$

$$T(\rho_n,\vec{a})=e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]},T(\rho_n,\vec{a})^{-1}=e^{i[\vec{a}\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}$$

$$U(\vec{a},\Lambda)\phi^{\alpha,n}(f)U(\vec{a},\Lambda)^{-1}\left(S,g_\beta\otimes\rho_n(E^\beta),\{\hat{f}_a\}_{a=0}^3\right)$$

$$=U(\vec{a},\Lambda)\phi^{\alpha,n}(f)\left(\Lambda^{-1}(S-\vec{a}),T(\rho_n,\vec{a})^{-1}g_\beta(\Lambda\cdot+\vec{a})\otimes\rho_n(E^\beta),\mathcal{D}\right)$$

$$=U(\vec{a},\Lambda)\left(\Lambda^{-1}(S-\vec{a}),[T(\rho_n,\vec{a})^{-1}f^e](\cdot)d^\alpha_\gamma\cdot g_\beta(\Lambda\cdot+\vec{a})\otimes\operatorname{ad}(\rho_n(F^\gamma))\rho_n(E^\beta),\mathcal{D}\right)$$

$$=\left(S,T(\rho_n,\vec{a})T(\rho_n,\vec{a})^{-1}f^e(\Lambda^{-1}(\cdot-\vec{a}))d^\alpha_\gamma g_\beta(\cdot)\otimes\operatorname{ad}(\rho_n(F^\gamma))\rho_n(E^\beta),\{\hat{f}_a\}_{a=0}^3\right).$$

$$f^C(\Lambda^{-1}(\vec{x}-\vec{a}))\equiv f^{\{\Lambda^{-1}\hat{f}_0,\Lambda^{-1}\hat{f}_1\}}(\Lambda^{-1}(\vec{x}-\vec{a})),$$

$$f^{\{\hat{g}_0,\hat{g}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\Lambda^{-1}(\vec{x}-\vec{a}))$$

$$:=\int_{\hat{s}\in\mathbb{R}^2}\frac{e^{-i[\hat{\sigma}(\hat{s})\cdot(\hat{H}(\rho_n)\hat{g}_0+\hat{P}(\rho_n)\hat{g}_1)]}}{2\pi}f(\vec{y}+\hat{\sigma}(\hat{s}))|\hat{\rho}_{\hat{\sigma}}|(\hat{s})d\hat{s}$$

$$=\int_{\hat{s}\in\mathbb{R}^2}\frac{e^{-i[\sigma(\hat{s})\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{2\pi}f(\vec{y}+\Lambda^{-1}\sigma(\hat{s}))|\hat{\rho}_\sigma|(\hat{s})d\hat{s}$$

$$=\int_{\hat{s}\in\mathbb{R}^2}\frac{e^{-i[\sigma(\hat{s})\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{2\pi}f(\Lambda^{-1}(\vec{x}+\sigma(\hat{s})-\vec{a}))|\hat{\rho}_\sigma|(\hat{s})d\hat{s}$$

$$=f(\Lambda^{-1}(\cdot-\vec{a}))^{\{\hat{f}_0,\hat{f}_1\}}\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\vec{x})$$

$$\left(S,f(\Lambda^{-1}(\cdot-\vec{a}))^{\{\hat{f}_0,\hat{f}_1\}}A(\Lambda^{-1})^\alpha_\gamma A(\tilde{\Lambda})^\gamma_\delta\cdot g_\beta\otimes\operatorname{ad}\left(\rho_n(F^\delta)\right)\rho_n(E^\beta),\{\tilde{\Lambda}e_a\}_{a=0}^3\right)$$

$$=A(\Lambda^{-1})^\alpha_\gamma\phi^{\gamma,n}[f(\Lambda^{-1}(\cdot-\vec{a}))]\left(S,g_\beta\otimes\rho_n(E^\beta),\{\tilde{\Lambda}e_a\}_{a=0}^3\right)$$

$$U(\vec{a},\Lambda)\phi^{\alpha,n}(f)U(\vec{a},\Lambda)^{-1}1=U(\vec{a},\Lambda)\phi^{\alpha,n}(f)1$$

$$=U(\vec{a},\Lambda)\left(S_0,f^{\{e_0,e_1\}}\otimes\rho_n(F^\alpha),\{e_a\}_{a=0}^3\right.$$

$$=\left(\Lambda S_0+\vec{a},e^{-i[\vec{a}\cdot(\hat{H}(\rho_n)\hat{g}_0+\hat{P}(\rho_n)\hat{g}_1)]}f^{\{e_0,e_1\}}(\Lambda^{-1}(\cdot-\vec{a}))\otimes\rho_n(F^\alpha),\{\hat{g}_a\}_{a=0}^3\right)$$

$$f(\cdot)^{\{\hat{g}_0,\hat{g}_1\}}\mapsto f(\cdot-\Lambda\vec{a})^{\{\hat{g}_0,\hat{g}_1\}},$$

$$(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})=-(x^0-y^0)^2+\sum_{i=1}^3(x^i-y^i)^2\leq(\geq)0.$$

$$(S,f_\alpha\otimes\rho(E^\alpha))\equiv\left(S,f_\alpha\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$

$$[\phi^{\alpha,n}(f),\phi^{\beta,n}(g)]:=\phi^{\alpha,n}(f)\phi^{\beta,n}(g)-\phi^{\alpha,n}(g)\phi^{\beta,n}(f),$$

$$[\phi^{\alpha,n}(f)^*,\phi^{\beta,n}(g)^*]:=\phi^{\alpha,n}(f)^*\phi^{\beta,n}(g)^*-\phi^{\alpha,n}(g)^*\phi^{\beta,n}(f)^*.$$

$$[\phi^{\alpha,n}(f),\phi^{\beta,n}(g)]\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right)=0,$$

$$[\phi^{\alpha,n}(f)^*,\phi^{\beta,n}(g)^*]1=0,[\phi^{\alpha,n}(f)^*,\phi^{\beta,n}(g)^*]\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right)=0,$$

$$\lfloor \phi^{\alpha,n}(f), \phi^{\beta,n}(g)^* \rfloor_{\pm} := \phi^{\alpha,n}(f)\phi^{\beta,n}(g)^* \pm \phi^{\alpha,n}(g)\phi^{\beta,n}(f)^*$$

$$\lfloor \phi^{\alpha,n}(f)^*, \phi^{\beta,n}(g) \rfloor_{\pm} := \phi^{\alpha,n}(f)^*\phi^{\beta,n}(g) \pm \phi^{\alpha,n}(g)^*\phi^{\beta,n}(f)$$

$$\lfloor \phi^{\alpha,n}(f), \phi^{\beta,n}(g)^* \rfloor_{\pm}\left(S, h_\gamma \otimes \rho_n(E^\gamma)\right)$$

$$= -A(\Lambda)^\alpha_\delta \overline{A(\Lambda)^\beta_\mu}\left(S,\ \mathrm{B}^{\pm}[f^e \cdot \overline{g^e} \pm g^e \cdot \overline{f^e}] \cdot h_\gamma \otimes \mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(g)1\right\rangle \phi^{\alpha,n}(f)1$$

$$\pm\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(f)1\right\rangle \phi^{\alpha,n}(g)1$$

$$\lfloor \phi^{\alpha,n}(f)^*, \phi^{\beta,n}(g) \rfloor_{\pm}\left(S, h_\gamma \otimes \rho_n(E^\gamma)\right)$$

$$= -\overline{A(\Lambda)^\alpha_\delta} A(\Lambda)^\beta_\mu\left(S,\ \mathrm{B}^{\pm}[\overline{f^e} \cdot g^e \pm \overline{g^e} \cdot f^e] \cdot h_\gamma \otimes \mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, g^e A(\Lambda)^\beta_\mu \cdot h_\gamma \otimes \mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right), \phi^{\alpha,n}(f)1\right\rangle 1$$

$$\pm\left\langle\left(S, f^e A(\Lambda)^\beta_\mu \cdot h_\gamma \otimes \mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right), \phi^{\alpha,n}(g)1\right\rangle 1$$

$$\phi^{\alpha,n}(f)\phi^{\beta,n}(g)^*\left(S, h_\gamma \otimes \rho_n(E^\gamma)\right)$$

$$=\phi^{\alpha,n}(f)\overline{A(\Lambda)^\beta_\mu}\Big[\left(S, -\overline{g^e}\cdot h_\gamma \otimes \mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(g)1\right\rangle 1\Big]$$

$$= -A(\Lambda)^\alpha_\delta \overline{A(\Lambda)^\beta_\mu}\left(S, f^e \cdot \overline{g^e} \cdot h_\gamma \otimes \mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(g)1\right\rangle \phi^{\alpha,n}(f)1$$

$$\phi^{\alpha,n}(g)\phi^{\beta,n}(f)^*\left(S, h_\gamma \otimes \rho_n(E^\gamma)\right)$$

$$= -A(\Lambda)^\alpha_\delta A(\Lambda)^\beta_\mu\left(S, g^e \cdot \overline{f^e} \cdot h_\gamma \otimes \mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(f)1\right\rangle \phi^{\alpha,n}(g)1$$

$$-A(\Lambda)^\alpha_\delta \overline{A(\Lambda)^\beta_\mu}\left(S, [f^e \cdot \overline{g^e} \pm g^e \cdot \overline{f^e}] \cdot h_\gamma \otimes \mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}(\rho_n(F^\mu))\rho_n(E^\gamma)\right)$$

$$+\left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(g)1\right\rangle \phi^{\alpha,n}(f)1 \pm \left\langle\left(S, h_\gamma\otimes\rho_n(E^\gamma)\right), \phi^{\beta,n}(f)1\right\rangle \phi^{\alpha,n}(g)1$$

$$g^{\{\hat f_0,\hat f_1\}}(\hat H,\hat P)(\vec x) = \int_{\hat s\in\mathbb{R}^2} \frac{e^{-i[\vec y(\hat s)\cdot(\hat H\hat f_0+\hat P\hat f_1)]}}{2\pi} g(\vec x+\vec y(\hat s))\left|\hat\rho_{\vec y}\right|(\hat s)d\hat s$$

$$\overline{f^{\{\hat f_0,\hat f_1\}}}(\hat H,\hat P)(\vec x) = \int_{\hat t\in\mathbb{R}^2} \frac{e^{i[\vec y(\hat t)\cdot(\hat H\hat f_0+\hat P\hat f_1)]}}{2\pi} \bar f(\vec x+\vec y(\hat t))\left|\hat\rho_{\vec y}\right|(\hat t)d\hat t$$

$$\left[g^{\{\hat f_0,\hat f_1\}}\cdot\overline{f\{\hat f_0,\hat f_1\}}\right](\hat H,\hat P)(\vec x)$$

$$= \int_{\hat s,\hat t\in\mathbb{R}^2} \frac{e^{-i[\vec y(\hat s)\cdot(\hat H\hat f_0+\hat P\hat f_1)]}}{(2\pi)^2} g_{\vec x}(\vec y(\hat s)+\vec y(\hat t))\bar f_{\vec x}(\vec y(\hat t))\left|\hat\rho_{\vec y}\right|(\hat s)\left|\hat\rho_{\vec y}\right|(\hat t)d\hat s d\hat t$$

$$= \int_{\hat t\in\mathbb{R}^2,\hat s\in D} \frac{e^{-i[\vec y(\hat s)\cdot(\hat H\hat f_0+\hat P\hat f_1)]}}{(2\pi)^2} g_{\vec x}(\vec y(\hat s)+\vec y(\hat t))\bar f_{\vec x}(\vec y(\hat t))\left|\hat\rho_{\vec y}\right|(\hat s)\left|\hat\rho_{\vec y}\right|(\hat t)d\hat s d\hat t$$

$$\left[f^{\{\hat f_0,\hat f_1\}}\cdot\overline{g^{\{\hat f_0,\hat f_1\}}}\right](\hat H,\hat P)(\vec x)$$

$$= \int_{\hat t\in\mathbb{R}^2,\hat s\in -D} \frac{e^{-i[\vec y(\hat s)\cdot(\hat H\hat f_0+\hat P\hat f_1)]}}{(2\pi)^2} f_{\vec x}(\vec y(\hat s)+\vec y(\hat t))\bar g_{\vec x}(\vec y(\hat t))\left|\hat\rho_{\vec y}\right|(\hat s)\left|\hat\rho_{\vec y}\right|(\hat t)d\hat s d\hat t$$

$$\frac{e^{-i[(u-v)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2} g_{\vec{x}}(u)\bar{f}_{\vec{x}}(v) \mp \frac{e^{-i[(v-u)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2} f_{\vec{x}}(v)\bar{g}_{\vec{x}}(u) = 0$$

$$(u-v)\cdot\left(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1\right) = c\left(\hat{P}(\rho_n)\hat{f}_0+\hat{H}(\rho_n)\hat{f}_1\right)\cdot\left(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1\right) = 0$$

$$e^{-i[(u-v)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]} = \cos\left[(u-v)\cdot\left(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1\right)\right] = 1$$

$$\left[\frac{e^{-i[(u-v)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2} - \frac{e^{-i[(v-u)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2}\right] f_{\vec{x}}(v)\bar{g}_{\vec{x}}(u)$$

$$\left[-\frac{e^{-i[(u-v)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2} + \frac{e^{-i[(v-u)\cdot(\hat{H}(\rho_n)\hat{f}_0+\hat{P}(\rho_n)\hat{f}_1)]}}{(2\pi)^2}\right] f_{\vec{x}}(v)\bar{g}_{\vec{x}}(u),$$

$$\begin{aligned}(f,g)\in\mathcal{P}\times\mathcal{P}\\ &\longmapsto\left\langle \phi^{\alpha,n}(f)\phi^{\beta,n}(g)^*\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right),\left(\tilde{S},\tilde{h}_\gamma\otimes\rho_n(E^\gamma)\right)\right\rangle\\ &\quad-\left\langle\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right),\phi^{\beta,n}(g)1\right\rangle\left\langle\phi^{\alpha,n}(f)1,\left(\tilde{S},\tilde{h}_\gamma\otimes\rho_n(E^\gamma)\right)\right\rangle\end{aligned}$$

$$\begin{aligned}\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} W(\vec{x},\vec{y})f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})d\vec{x}d\vec{y}\\ &=\left\langle \phi^{\alpha,n}(f)\phi^{\beta,n}(g)^*\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right),\left(\tilde{S},\tilde{h}_\gamma\otimes\rho_n(E^\gamma)\right)\right\rangle\\ &\quad-\left\langle\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right),\phi^{\beta,n}(g)1\right\rangle\left\langle\phi^{\alpha,n}(f)1,\left(\tilde{S},\tilde{h}_\gamma\otimes\rho_n(E^\gamma)\right)\right\rangle.\end{aligned}$$

$$\left(x^0\hat{f}_0+x^1\hat{f}_1,y^0\hat{f}_0+y^1\hat{f}_1\right)\longmapsto$$

$$\int_{(x^2,x^3)\in\mathbb{R}^2}\int_{(y^2,y^3)\in\mathbb{R}^2} W(\vec{x},\vec{y})f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})dx^2dx^3dy^2dy^3,$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Re}W(\vec{x},\vec{y})[f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})]d\vec{x}d\vec{y} := \int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} W(\vec{x},\vec{y})[f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})]d\vec{x}d\vec{y}$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Im}W(\vec{x},\vec{y})[f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})]d\vec{x}d\vec{y} := \int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} W(\vec{x},\vec{y})[f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})]d\vec{x}d\vec{y}$$

$$\begin{aligned}\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} W(\vec{x},\vec{y})[f(\vec{x})\otimes_{\mathbb{R}} g(\vec{y})]d\vec{x}d\vec{y}\\ &=\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4}\mathrm{Re}W(\vec{x},\vec{y})[\underline{f}(\vec{x})\underline{g}(\vec{y})+\bar{f}(\vec{x})\bar{g}(\vec{y})]d\vec{x}d\vec{y}\\ &\quad+i\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4}\mathrm{Im}W(\vec{x},\vec{y})[\bar{f}(\vec{x})\underline{g}(\vec{y})-\underline{f}(\vec{x})\bar{g}(\vec{y})]d\vec{x}d\vec{y}\end{aligned}$$

$$\mathrm{Re}W(\vec{x},\vec{y})-\mathrm{Re}W(\vec{y},\vec{x})=0,$$

$$\mathrm{Im}W(\vec{x},\vec{y})+\mathrm{Im}W(\vec{y},\vec{x})=0,$$

$$\begin{aligned}\phi^{\alpha,n}(f)\phi^{\beta,n}(g)^*\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right)-\left\langle\left(S,h_\gamma\otimes\rho_n(E^\gamma)\right),\phi^{\beta,n}(g)1\right\rangle\phi^{\alpha,n}(f)1\\ =-A(\Lambda)^\alpha_\delta A(\Lambda)^\beta_\mu\left(S,\left[f^{\{\hat{f}_0,\hat{f}_1\}}\cdot\overline{g^{\{\hat{f}_0,\hat{f}_1\}}}\right\}\cdot h_\gamma\otimes\mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}\left(\rho_n(F^\mu)\right)\rho_n(E^\gamma)\right).\\ -A(\Lambda)^\alpha_\delta\overline{A(\Lambda)^\beta_\mu}\left(S,A^\pm\cdot h_\gamma\otimes\mathrm{ad}\left(\rho_n(F^\delta)\right)\mathrm{ad}\left(\rho_n(F^\mu)\right)\rho_n(E^\gamma)\right),\end{aligned}$$

$$\begin{aligned}A^\pm(\vec{z})=\int_{\hat{s},\hat{t}\in\mathbb{R}^2}\frac{e^{-i[(\vec{y}(\hat{s})-\vec{y}(\hat{t}))\cdot(\hat{H}\hat{f}_0+\hat{P}\hat{f}_1)]}}{(2\pi)^2}f_{\vec{z}}(\vec{y}(\hat{s}))\bar{g}_{\vec{z}}(\vec{y}(\hat{t}))\left|\hat{\rho}_{\vec{y}}\right|(\hat{s})\left|\hat{\rho}_{\vec{y}}\right|(\hat{t})d\hat{s}d\hat{t}\\ \pm\int_{\hat{s},\hat{t}\in\mathbb{R}^2}\frac{e^{-i[(\vec{y}(\hat{t})-\vec{y}(\hat{s}))\cdot(\hat{H}\hat{f}_0+\hat{P}\hat{f}_1)]}}{(2\pi)^2}g_{\vec{z}}(\vec{y}(\hat{t}))\bar{f}_{\vec{z}}(\vec{y}(\hat{s}))\left|\hat{\rho}_{\vec{y}}\right|(\hat{s})\left|\hat{\rho}_{\vec{y}}\right|(\hat{t})d\hat{s}d\hat{t}\end{aligned}$$

$$\mathrm{Re}W(\vec{x},\vec{y}) - \mathrm{Re}W(\vec{y},\vec{x}) = 0,$$
$$\mathrm{Im}W(\vec{x},\vec{y}) + \mathrm{Im}W(\vec{y},\vec{x}) = 0$$

$$(f,g) \in \mathcal{P}\times\mathcal{P} \mapsto \left\langle \phi^{\alpha,n}(f)^*\phi^{\beta,n}(g)\left(S,h_\gamma \otimes \rho_n(E^\gamma)\right),\left(\tilde{S},\tilde{h}_\gamma \otimes \rho_n(E^\gamma)\right)\right\rangle$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \tilde{W}(\vec{x},\vec{y}) f(\vec{x}) \otimes_{\mathbb{R}} g(\vec{y}) d\vec{x}d\vec{y}$$
$$= \left\langle \phi^{\alpha,n}(f)^*\phi^{\beta,n}(g)\left(S,h_\gamma \otimes \rho_n(E^\gamma)\right),\left(\tilde{S},\tilde{h}_\gamma \otimes \rho_n(E^\gamma)\right)\right\rangle$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \tilde{W}(\vec{x},\vec{y})[f(\vec{x}) \otimes_{\mathbb{R}} g(\vec{y})] d\vec{x}d\vec{y}$$
$$= \int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Re}\tilde{W}(\vec{x},\vec{y})[\underline{f}(\vec{x})\underline{g}(\vec{y}) + \bar{f}(\vec{x})\bar{g}(\vec{y})] d\vec{x}d\vec{y}$$
$$+i\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Im}\tilde{W}(\vec{x},\vec{y})[\underline{f}(\vec{x})\bar{g}(\vec{y}) - \bar{f}(\vec{x})\underline{g}(\vec{y})] d\vec{x}d\vec{y}$$

$$\mathrm{Re}\tilde{W}(\vec{x},\vec{y}) - \mathrm{Re}\tilde{W}(\vec{y},\vec{x}) = 0$$
$$\mathrm{Im}\tilde{W}(\vec{x},\vec{y}) + \mathrm{Im}\tilde{W}(\vec{y},\vec{x}) = 0$$

$$(f,g) \in \mathcal{P}\times\mathcal{P} \mapsto \left\langle \phi^{\alpha,n}(f)^*\phi^{\beta,n}(g)1,\left(\tilde{S},\tilde{h}_\gamma \otimes \rho_n(E^\gamma)\right)\right\rangle.$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \check{W}(\vec{x},\vec{y}) f(\vec{x}) \otimes_{\mathbb{R}} g(\vec{y}) d\vec{x}d\vec{y}$$
$$= \left\langle \phi^{\alpha,n}(f)^*\phi^{\beta,n}(g)1,\left(\tilde{S},\tilde{h}_\gamma \otimes \rho_n(E^\gamma)\right)\right\rangle$$

$$\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \check{W}(\vec{x},\vec{y})[f(\vec{x}) \otimes_{\mathbb{R}} g(\vec{y})] d\vec{x}d\vec{y}$$
$$= \int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Re}\check{W}(\vec{x},\vec{y})[\underline{f}(\vec{x})\underline{g}(\vec{y}) + \bar{f}(\vec{x})\bar{g}(\vec{y})] d\vec{x}d\vec{y}$$
$$+i\int_{\vec{x}\in\mathbb{R}^4}\int_{\vec{y}\in\mathbb{R}^4} \mathrm{Im}\check{W}(\vec{x},\vec{y})[\underline{f}(\vec{x})\bar{g}(\vec{y}) - \bar{f}(\vec{x})\underline{g}(\vec{y})] d\vec{x}d\vec{y}$$

$$\mathrm{Re}\check{W}(\vec{x},\vec{y}) - \mathrm{Re}\check{W}(\vec{y},\vec{x}) = 0,$$
$$\mathrm{Im}\check{W}(\vec{x},\vec{y}) + \mathrm{Im}\check{W}(\vec{y},\vec{x}) = 0,$$
$$\phi^{\alpha,n}(f)\phi^{\beta,n}(g)^*1 \pm \phi^{\alpha,n}(g)\phi^{\beta,n}(f)^*1 = 0.$$
$$W(\vec{y},\vec{x}) = \bar{W}(\vec{x},\vec{y}), \tilde{W}(\vec{y},\vec{x}) = \bar{\tilde{W}}(\vec{x},\vec{y}), \check{W}(\vec{y},\vec{x}) = \bar{\check{W}}(\vec{x},\vec{y}),$$
$$\phi^{\alpha,n}(f)\phi^{\beta,n}(g) \pm \phi^{\beta,n}(g)\phi^{\alpha,n}(f) = 0$$
$$\phi^{\alpha,n}(f)^*\phi^{\beta,n}(g) \pm \phi^{\beta,n}(g)\phi^{\alpha,n}(f)^* = 0$$
$$\left\langle f\sqrt{\phi_\kappa}, g\sqrt{\phi_\kappa}\right\rangle = \int_{\mathbb{R}^4} fg\cdot\phi_\kappa d\lambda$$
$$\left\{\frac{h_i(\kappa x^0)h_j(\kappa x^1)h_k(\kappa x^2)h_l(\kappa x^3)}{\sqrt{i!\,j!\,k!\,l!}}\sqrt{\phi_\kappa(\vec{x})}\,\middle|\, \vec{x} = (x^0,x^1,x^2,x^3) \in \mathbb{R}^4, i,j,k,l \geq 0\right\}$$
$$\left\langle \sum_{0\leq a<b\leq 3} f_{ab} \otimes dx^a \wedge dx^b, \sum_{0\leq a<b\leq 3} \hat{f}_{ab} \otimes dx^a \wedge dx^b\right\rangle = \sum_{0\leq a<b\leq 3} \left\langle f_{ab}, \hat{f}_{ab}\right\rangle.$$
$$df = \sum_{i=1}^{3} \partial_0 f_i \otimes dx^0 \wedge dx^i + \sum_{1\leq i<j\leq 3} (\partial_i f_j - \partial_j f_i) dx^i \wedge dx^j$$
$$c_\gamma^{\alpha\beta} = -\mathrm{Tr}\left[E^\gamma\left[E^\alpha,E^\beta\right]\right], E^\alpha, E^\beta, E^\gamma \in \mathfrak{g}.$$

$$dA + A \wedge A = \sum_{\gamma=1}^{N} [\sum_{j=1}^{3} a_{0:j,\gamma} \otimes dx^0 \wedge dx^j + \sum_{1 \le i < j \le 3} a_{i;j,\gamma} \otimes dx^i \wedge dx^j$$

$$+ \sum_{1 \le i < j \le 3} \sum_{1 \le \alpha,\beta \le N} a_{i,\alpha} a_{j,\beta} c_\gamma^{\alpha\beta} \otimes dx^i \wedge dx^j \Bigg] \otimes E^\gamma$$

$$\frac{1}{Z} e^{-\frac{1}{2}\int_{\mathbb{R}^4} |dA+A\wedge A|^2 d\omega} D[dA]$$

$$Z = \int_{\{dA: A \in \mathcal{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) \otimes \mathfrak{g}\}} e^{-\frac{1}{2}\int_{\mathbb{R}^4} |dA+A\wedge A|^2 d\omega} D[dA]$$

$$\int_{\mathbb{R}^4} |dA + A \wedge A|^2 d\omega = \sum_{1 \le i < j \le 3} \int_{\mathbb{R}^4} \left[\sum_{\alpha=1}^{N} a_{i;j,\alpha}^2 + \sum_{\gamma=1}^{N} \sum_{\substack{\alpha,\beta \\ \alpha,\beta}} a_{i,\alpha} a_{j,\beta} a_{i,\hat{\alpha}} a_{j,\hat{\beta}} c_\gamma^{\alpha\beta} c_\gamma^{\hat{\alpha}\hat{\beta}} \right.$$

$$\left. +2 \sum_{\gamma=1}^{N} \sum_{\alpha,\beta} a_{i;j,\gamma} a_{i,\alpha} a_{j,\beta} c_\gamma^{\alpha\beta} \right] d\omega + \sum_{j=1}^{3} \int_{\mathbb{R}^4} \sum_{\alpha=1}^{N} a_{0:j,\alpha}^2 d\omega.$$

$$\exp\left[-\frac{1}{2} \sum_{\alpha=1}^{N} \int_{\mathbb{R}^4} d\omega \sum_{1 \le i < j \le 3} a_{i;j,\alpha}^2 + \sum_{j=1}^{3} a_{0:j,\alpha}^2 \right] D[dA]$$

$$\langle z^r, z^{r'} \rangle = \frac{1}{\pi} \int_{\mathbb{C}} z^r \cdot \overline{z^{r'}} e^{-|z^2|} dx dp, z = x + \sqrt{-1} p$$

$$\left\{ \frac{z^n}{\sqrt{n!}} : n \ge 0 \right\}$$

$$\Psi_\kappa : \frac{h_i(\kappa \cdot)}{\sqrt{i!}} \frac{h_j(\kappa \cdot)}{\sqrt{j!}} \frac{h_k(\kappa \cdot)}{\sqrt{k!}} \frac{h_l(\kappa \cdot)}{\sqrt{l!}} \sqrt{\phi_\kappa} \mapsto \frac{z_0^i}{\sqrt{i!}} \frac{z_1^j}{\sqrt{j!}} \frac{z_2^k}{\sqrt{k!}} \frac{z_3^l}{\sqrt{l!}}.$$

$$\Psi_\kappa : f_{i,\alpha} \otimes dx^i \otimes E^\alpha \mapsto \Psi_\kappa(f_{i,\alpha}) \otimes dx^i \otimes E^\alpha.$$

$$\left\langle \sum_{0 \le a < b \le 3} f_{ab} \otimes dx^a \wedge dx^b, \sum_{0 \le a < b \le 3} \hat{f}_{ab} \otimes dx^a \wedge dx^b \right\rangle = \sum_{0 \le a < b \le 3} \langle f_{ab}, \hat{f}_{ab} \rangle.$$

$$\mathfrak{d}_a \left[z_a^p \prod_{\substack{b \ne a \\ b=0,\cdots,3}} z_b^{q_b} \right] = \left[\frac{p}{2} z_a^{p-1} - \frac{1}{2} z_a^{p+1} \right] \cdot \prod_{\substack{b \ne a \\ b=0,\cdots,3}} z_b^{q_b}$$

$$\mathfrak{d} : H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3) \to H^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4)$$

$$\mathfrak{d} \sum_{i=1}^{3} f_i \otimes dx^i = \sum_{i=1}^{3} [\mathfrak{d}_0 f_i] \otimes dx^0 \wedge dx^i + \sum_{1 \le i < j \le 3} [\mathfrak{d}_i f_j - \mathfrak{d}_j f_i] \otimes dx^i \wedge dx^j$$

$$\Psi_\kappa : \mathcal{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) \otimes \mathfrak{g} \to H^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4) \otimes \mathfrak{g}$$

$$\Psi_\kappa : \sum_{1 \le i < j \le 3} f_{i,j,\alpha} \otimes dx^i \wedge dx^j \otimes E^\alpha \mapsto \sum_{1 \le i < j \le 3} \Psi_\kappa[f_{i,j,\alpha}] \otimes dx^i \wedge dx^j \otimes E^\alpha.$$

$$\langle dA, dA \rangle = \kappa^2 \langle \mathfrak{d}\Psi_\kappa[A], \mathfrak{d}\Psi_\kappa[A] \rangle$$

$$\frac{1}{Z} e^{-\frac{1}{2}\int_{\mathcal{C}^4} |\kappa\mathfrak{d}A + A\wedge A|^2 d\lambda_4} D[\mathfrak{d}A]$$

$$Z = \int_{\{\mathfrak{d}A: A \in H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3) \otimes \mathfrak{g}\}} e^{-\frac{1}{2}\int_{\mathbb{C}^4} |\kappa\mathfrak{d}A + A\wedge A|^2 d\lambda_4} D[\mathfrak{d}A]$$

$$\mathbb{H} := \{[\mathfrak{d}_0 H^2(\mathbb{C}^4)] \otimes [* \Lambda^2(\mathbb{R}^3)]\} \oplus \{H^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^3)\}$$
$$\subset H^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4)$$

$$\frac{1}{Z} e^{-\frac{1}{2}\int_{\mathcal{C}^4} |\kappa\mathfrak{d}A + A\wedge A|^2 d\lambda_4} D[\mathfrak{d}A] := \frac{y^\kappa d\tilde{\mu}_{\kappa^2}^{\times N}}{\int_{\mathbb{B}\otimes\mathfrak{g}} y^\kappa d\tilde{\mu}_{\kappa^2}^{NN}} = \frac{y^\kappa d\tilde{\mu}_{\kappa^2}^{\times N}}{\mathbb{E}[y^\kappa]},$$

$$\mathbb{E}_{\text{YM}}^{\kappa}[F]:=\frac{1}{\int_{\mathbb{B}\otimes\mathfrak{g}}y^{\kappa}d\tilde{\mu}_{\kappa^2}^{\times N}}\int_{\mathbb{B}\otimes\mathfrak{g}}Fy^{\kappa}d\tilde{\mu}_{\kappa^2}^{\times N}$$

$$\beta(c)=\frac{\partial c}{\partial[\ln\,\tilde{N}_n]},$$

$$\operatorname{Tr}[\rho(E^{\alpha})\rho(E^{\beta})]=C(\rho)\operatorname{Tr}[E^{\alpha}E^{\beta}]$$

$$\mathcal{E}(\rho):=-\sum_{\alpha=1}^{N}\rho(E^{\alpha})\rho(E^{\alpha})$$

$$\sum_{\alpha=1}^{N}\sum_{j=1}^{3}\mathfrak{d}_0 A_{j,\alpha}\otimes dx^0\wedge dx^j\otimes E^{\alpha}\in\mathbb{B}\otimes\mathfrak{g}$$

$$\frac{1}{\kappa}\sum_{\alpha=1}^{N}\frac{\kappa^2}{4}\int_{\hat{s}\in[-\delta,1+\delta]^2}d\hat{s}\sum_{j=1}^{3}|J_{0j}^{\sigma}|(\hat{s})\kappa[\psi\cdot\mathfrak{d}_0 A_{j,\alpha}](\kappa\sigma(\hat{s})/2)\otimes\rho(E^{\alpha})$$

$$c\sum_{\alpha=1}^{N}\int_{\hat{s}\in I_{\delta}^2}d\hat{s}\sum_{j=1}^{3}|J_{0j}^{\sigma}|(\hat{s})[\mathfrak{d}_0 A_{j,\alpha}](\sigma(\hat{s}))\otimes\rho(E^{\alpha})$$

$$\left\langle v_{R[a,T]}^{\kappa,\rho}\right\rangle^2=-\mathbb{E}\left[\left(\cdot,v_{R[a,T]}^{\kappa,\rho}\right)_{\#}^2 y^{\kappa}\right]$$

$$-\mathbb{E}\left[\left(\cdot,v_{R[a,T]}^{\kappa,\rho}\right)_{\#}\left(\cdot,v_{R_{\delta}[a,T]}^{\kappa,\rho}\right)_{\#}y^{\kappa}\right],$$

$$A^{\rho}:=\sum_{\alpha=1}^{N}\sum_{i=1}^{3}a_{i,\alpha}\otimes dx^i\otimes\rho(E^{\alpha})\in\mathcal{S}_{\kappa}(\mathbb{R}^4)\otimes\Lambda^1(\mathbb{R}^3)\otimes\rho(\mathfrak{g})$$

$$\frac{1}{Z}\int_{\{dA\in\mathcal{S}_{\kappa}(\mathbb{R}^4)\otimes\Lambda^2(\mathbb{R}^4)\otimes\mathfrak{g}\}}\exp\left[c\int_{R[a]}d[A^{\rho}]\right]e^{-\frac{1}{2}S_{\text{YM}}(A)}D[dA]$$

$$:=\mathbb{E}_{\text{YM}}^{\kappa}\left[\exp\left[\left(\cdot,v_{R[a]}^{\kappa,\rho}\right)_{\sharp}\right]\right]$$

$$Z=\int_{\{dA\in\mathcal{S}_{\kappa}(\mathbb{R}^4)\otimes\Lambda^2(\mathbb{R}^4)\otimes\mathfrak{g}\}}e^{-\frac{1}{2}S_{\text{YM}}(A)}DA$$

$$-\mathbb{E}\left[\left(\cdot,v_{R[a]}^{\kappa,\rho_n}\right)_{\sharp}\left(\cdot,v_{R_{\delta}[a]}^{\kappa,\rho_n}\right)_{\sharp}y^{\kappa}\right]=\frac{|a|}{4}\otimes\mathcal{E}(\rho_n)-\epsilon(n,\kappa),$$

$$\frac{c}{\kappa^4}C(\rho_n)\leq\operatorname{Tr}\epsilon(n,\kappa)\leq\frac{\bar{c}}{\kappa^4}C(\rho_n)$$

$$-\frac{1}{C(\rho_n)}\mathbb{E}\left[\left(\cdot,v_{R[a]}^{\kappa,\rho_n}\right)_{\sharp}\left(\cdot,v_{R_{\delta}[a]}^{\kappa,\rho_n}\right)_{\sharp}y^{\kappa}\right]=\frac{|a|}{4}\otimes\frac{C_2(\rho_n)}{C(\rho_n)}\mathbb{I}_n-\frac{1}{C(\rho_n)}\epsilon(n,\kappa),$$

$$\tilde{N}_n\frac{C_2(\rho_n)}{C(\rho_n)}-\frac{1}{C(\rho_n)}\operatorname{Tr}\epsilon(n,\kappa)\equiv N-\frac{1}{C(\rho_1)}\operatorname{Tr}\epsilon(1,\kappa)$$

$$G_n^{(2)}(c,e):=\frac{\tilde{N}_n}{e}-\operatorname{Tr}\epsilon(1/c)$$

$$=\frac{\tilde{N}_n}{e}-c^4\bar{\lambda}+f(c^5)$$

$$\left[e\frac{\partial}{\partial e}+\beta(c)\frac{\partial}{\partial c}+2\gamma(c)\right]G_n^{(2)}(c,e)=0.$$

$$G_n^{(2)}(c,e)=\frac{\tilde{N}_n}{e}-c^4\bar{\lambda}+f(c^5)$$

$$\frac{\partial}{\partial e}G_n^{(2)}(c,e)=-\frac{\tilde{N}_n}{e^2},\frac{\partial}{\partial c}G_n^{(2)}(c,e)=-4c^3\bar{\lambda}+\tilde{f}(c^4),$$

$$-\frac{\tilde{N}_n}{e}-4\beta(c)c^3\bar{\lambda}+2\gamma(c)G_n^{(2)}(c,e)+\beta(c)\tilde{f}(c^4)=0$$

$$-\frac{c}{4}\tilde{f}(c^4) + f(c^5) - 4c^3\lambda(c)\bar{\lambda} + \lambda(c)\tilde{f}(c^4) = 0$$

$$\frac{1}{c^4}\left|\tilde{f}(c^4)\right| + \frac{1}{c^5}\left|f(c^5)\right| \leq \tilde{C}_3$$

$$\lambda(c) = \frac{1}{-4c^3\bar{\lambda} + \tilde{f}(c^4)}\left[\frac{c}{4}\tilde{f}(c^4) - f(c^5)\right]$$

$$\frac{\hat{P}^2}{\hat{H}^2} - 1 = -\frac{m^2}{\hat{H}^2}$$

$$E^\alpha = \sum_{\beta=1}^{l} a_{\alpha,\beta}H_\beta, 1 \leq \alpha \leq l$$

$$\lambda_\rho \equiv \left(\lambda_\rho(H_1), \cdots, \lambda_\rho(H_l)\right),$$

$$\mathcal{E}(\rho) = -\sum_{\alpha=1}^{N} \rho(E^\alpha)\rho(E^\alpha) = C_2(\rho)I, C_2(\rho) \geq 0$$

$$\langle -\rho(E^\alpha)\rho(E^\alpha)v, v\rangle = \langle \rho(E^\alpha)v, \rho(E^\alpha)v\rangle \geq 0$$

$$\langle C_2(\rho)v, v\rangle = \sum_{\alpha=1}^{N} \langle \rho(E^\alpha)v, \rho(E^\alpha)v\rangle \geq \sum_{\alpha=1}^{l}\left|\sum_{\beta=1}^{l} a_{\alpha,\beta}\lambda_\rho(H_\beta)v\right|_2^2$$

$$= \sum_{\beta=1}^{l}\sum_{\gamma=1}^{l}\sum_{\alpha=1}^{l} \lambda_\rho(H_\beta)a_{\alpha,\beta}a_{\alpha,\gamma}\lambda_\rho(H_\gamma) \geq c\left|\lambda_\rho\right|_2^2$$

$$\left\{\rho_n : \mathfrak{g} \to \mathrm{End}\left(\mathbb{C}^{\tilde{N}_n}\right)\right\}_{n=1}^{\infty}, \text{ such that } 0 < C_2(\rho_n) \leq C_2(\rho_{n+1})$$

$$\hat{H}(\rho_n)^2 := \frac{\tilde{N}_n}{4}C_2(\rho_n) = \frac{N}{4}C(\rho_n) > 0$$

$$\frac{\partial c}{\partial[\ln\ \tilde{N}]} = -\frac{c}{4} + \lambda(c), |\lambda(c)| \leq \tilde{C}_4c^2$$

$$\frac{dc}{d[\ln\ \tilde{N}]} = -\frac{c}{4} + \lambda(c) \Longrightarrow \frac{dc}{c - 4\lambda(c)} = -\frac{d[\ln\ \tilde{N}]}{4}$$

$$\frac{1}{c}\frac{dc}{1 + \mu(c)} = -\frac{d[\ln\ \tilde{N}]}{4} \Longrightarrow \left[\frac{1}{c}\sum_{k=0}^{\infty} (-1)^k\mu(c)^k\right]dc = -\frac{d[\ln\ \tilde{N}]}{4}$$

$$\ln\ c + \tilde{\mu}(c) = -\frac{1}{4}\ln\ \tilde{N} + C$$

$$ce^{\tilde{\mu}(c)} = \frac{1}{\hat{C}}\tilde{N}^{-1/4} \Longrightarrow c(1 + \bar{\mu}(c)) = \frac{1}{\hat{C}}\tilde{N}^{-1/4}$$

$$\left|\frac{1}{\hat{C}}\frac{1}{1 + \bar{\mu}(c)}\right| \leq \frac{1}{\hat{C}}\left[1 + \tilde{C}_8|\bar{\mu}(c)|\right] \leq \frac{1}{\hat{C}}\left[1 + \tilde{C}_8\tilde{C}_7c\right]$$

$$\leq \frac{1}{\hat{C}}\left[1 + \tilde{C}_7\tilde{C}_8\frac{1}{\hat{C}}\tilde{N}^{-1/4}\left(1 + \tilde{C}_8\tilde{C}_7\right)\right] = \frac{1}{\hat{C}} + \frac{\tilde{N}^{-1/4}}{\hat{C}^2}\tilde{C}_7\tilde{C}_8\left(1 + \tilde{C}_7\tilde{C}_8\right)$$

$$\tilde{N}^{1/4}|\bar{\mu}(c)| \leq c\tilde{N}^{1/4}\tilde{C}_7 \leq \frac{\tilde{C}_7}{\hat{C}} + \frac{\tilde{C}_7}{\hat{C}^2}\tilde{C}_7\tilde{C}_8\left(1 + \tilde{C}_7\tilde{C}_8\right)$$

$$\kappa = \frac{1}{c} = \hat{C}\tilde{N}^{1/4}(1 + \bar{\mu}(c))$$

$$= \hat{C}\tilde{N}^{1/4} + \tilde{R}(c)$$

$$|\tilde{R}(c)| \leq \tilde{C}_7 + \frac{\tilde{C}_7}{\hat{C}}\tilde{C}_7\tilde{C}_8\left(1 + \tilde{C}_7\tilde{C}_8\right) = \tilde{C}_7 + \frac{1}{\hat{C}}\tilde{C}_9$$

$$0 < m_1^2 := \mathrm{Tr}\epsilon(1, \kappa_1) = \frac{\tilde{N}_1}{4}C_2(\rho_1) + \mathrm{Tr}\mathbb{E}\left[\left(\cdot, v_{R[a]}^{\kappa_1,\rho_1}\right)_\sharp\left(\cdot, v_{R_\delta[a]}^{\kappa_1,\rho_1}\right)_\sharp y^\kappa\right]$$

$$-\mathrm{Tr}\mathbb{E}\left[\left(\cdot, v_{R[a]}^{\kappa,\rho_n}\right)_\sharp\left(\cdot, v_{R_\delta[a]}^{\kappa,\rho_n}\right)_\sharp y^\kappa\right] = \frac{\tilde{N}_n}{4}C_2(\rho_n) - \mathrm{Tr}\epsilon(n, \kappa)$$

$$\frac{4}{NC(\rho_n)}\mathrm{Tr}\mathbb{E}\left[-\left(\cdot,v^{\kappa,\rho_n}_{R[a]}\right)_\#\left(\cdot,v^{\kappa,\rho_n}_{R_\delta[a]}\right)_\# y^\kappa\right]-1=-\frac{4\mathrm{Tr}\epsilon(n,\kappa)}{NC(\rho_n)}.$$

$$m_n^2:=\mathrm{Tr}[\epsilon(n,\kappa_n)]$$

$$0<\frac{c}{\kappa_n^4}<\frac{1}{C(\rho_n)}\mathrm{Tr}[\epsilon(n,\kappa_n)]\leq\frac{\bar{c}}{\kappa_n^4}$$

$$\hat{P}(\rho_n)^2:=-\mathrm{Tr}\mathbb{E}\left[\left(\cdot,v^{\kappa_n,\rho_n}_{R[a]}\right)_\sharp\left(\cdot,v^{\kappa_n,\rho_n}_{R_\delta[a]}\right)_\sharp y^\kappa\right]$$

$$=\frac{\tilde{N}_n}{4}C_2(\rho_n)-\mathrm{Tr}\epsilon(n,\kappa_n)=\frac{\tilde{N}_n}{4}C_2(\rho_n)-m_n^2>0,$$

$$\frac{\hat{P}(\rho_n)^2}{\hat{H}(\rho_n)^2}=1-\frac{m_n^2}{\hat{H}(\rho_n)^2}\longrightarrow 1$$

$$\hat{H}(\rho_n)^2-\hat{P}(\rho_n)^2=m_n^2>0,$$

$$U(\vec{a},1)\left(S,f_\alpha\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$

$$:=e^{-i[\hat{f}_0\cdot\hat{H}(\vec{a},\rho)+\hat{f}_1\cdot\hat{P}(\vec{a},\rho)]}\left(S+\vec{a},f_\alpha(\cdot-\vec{a})\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$

$$=e^{i[a^0\hat{H}(\rho)-a^1\hat{P}(\rho)]}\left(S+\vec{a},f_\alpha(\cdot-\vec{a})\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$

$$\hat{P}\sum_{n=0}^\infty v_n:=\sum_{n=1}^\infty\hat{P}(\rho_n)v_n,\hat{H}\sum_{n=0}^\infty v_n:=\sum_{n=1}^\infty\hat{H}(\rho_n)v_n,$$

$$\hat{f}(p)=\mathcal{F}[f](p):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ipx}f(x)dx$$

$$\hat{f}(\vec{p})=\mathcal{F}[f](\vec{p}):=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}e^{-i\vec{p}\cdot\vec{x}}f(\vec{x})d\vec{x}$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^2\int_{S_0}e^{-i[q^2x^2+q^3x^3]}F_\alpha\left(\hat{H}(\rho),\hat{P}(\rho),x^2,x^3\right)dx^2dx^3\otimes\rho(E^\alpha)$$

$$=\hat{F}_\alpha\left(\hat{H}(\rho),\hat{P}(\rho),q^2,q^3\right)\otimes\rho(E^\alpha)$$

$$\hat{F}\left(\hat{H}(\rho)\hat{f}_0+\hat{P}(\rho)\hat{f}_1+q^2\hat{f}_2+q^3\hat{f}_3\right)$$

$$:=\frac{e^{-i[a^0}\hat{H}(\rho)-a^1\hat{P}(\rho)]}{2\pi}\int_{\hat{s}\in\mathbb{R}^2}e^{-i(sq^2+\bar{s}q^3)}f^{\{\hat{f}_0,\hat{f}_1\}}(\hat{H}(\rho),\hat{P}(\rho))\left(s\hat{f}_2+\bar{s}\hat{f}_3\right)d\hat{s}$$

$$\equiv e^{-i[a^0\hat{H}(\rho)-a^1\hat{P}(\rho)]}\hat{f}\left(\hat{H}(\rho)\hat{f}_0+\hat{P}(\rho)\hat{f}_1+q^2\hat{f}_2+q^3\hat{f}_3\right),$$

$$\hat{F}_\alpha\left(\hat{H}(\rho)\hat{f}_0+\hat{P}(\rho)\hat{f}_1+q^2\hat{f}_2+q^3\hat{f}_3\right)\otimes\rho(E^\alpha)\left(S,f_\alpha\otimes\rho(E^\alpha),\{\hat{f}_a\}_{a=0}^3\right)$$

$$e^{-i\vec{a}\cdot\vec{\alpha}}e^{-i[q^2a^2+q^3a^3]}\hat{F}_\alpha\left(\hat{H}(\rho),\hat{P}(\rho),q^2,q^3\right)\otimes\rho(E^\alpha)$$

$$\vec{\alpha}=(\hat{H}(\rho),\hat{P}(\rho),0,0)$$

$$\langle\phi^{\alpha,n}(f)1,\phi^{\beta,n}(g)1\rangle:=C(\rho_n)\mathrm{Tr}\left[-F^\alpha F^\beta\right]\int_{S_0}\left[f^{\{e_0,e_1\}}\overline{g^{\{e_0,e_1\}}}\right]\left(\hat{H}(\rho_n),\hat{P}(\rho_n)\right)(\hat{s})d\hat{s}$$

$$\phi^{\alpha,n}(f)1=\int_{\vec{x}\in\mathbb{R}^4}d\vec{x}f(\vec{x})\phi^{\alpha,n}(\vec{x})1$$

$$U(\vec{a},1)\phi^{\alpha,n}(\vec{x})U(\vec{a},1)^{-1}=\phi^{\alpha,n}(\vec{x}+\vec{a})$$

$$\frac{1}{2\pi}e^{i[x^0\hat{H}(\rho_n)-x^1\hat{P}(\rho_n)]}\delta\left(\cdot-(x^2,x^3)\right)\otimes\rho_n(F^\alpha)$$

$$\int_{\vec{z}\in\mathbb{R}^4}p_\kappa^{\vec{x}}(\vec{z})p_\kappa^{\vec{y}}(\vec{z})d\vec{z}=\frac{\kappa^2}{4(2\pi)}\exp\left[-\kappa^2|\vec{x}-\vec{y}|^2/8\right]$$

$$\left\langle\frac{1}{2\pi}e^{i[x^0\hat{H}-x^1\hat{P}]}p_\kappa^{x^+}\otimes\rho_n(F^\alpha),\frac{1}{2\pi}e^{i[y^0\hat{H}-y^1\hat{P}]}p_\kappa^{y^+}\otimes\rho_n(F^\beta)\right\rangle$$

$$\frac{1}{(2\pi)^2}e^{i[\hat{H}(x^0-y^0)-\hat{P}(x^1-y^1)]}\int_{\vec{z}\in\mathbb{R}^4}p_\kappa^{x^+}(\vec{z})p_\kappa^{y^+}(\vec{z})d\vec{z}\cdot\langle\rho_n(F^\alpha),\rho_n(F^\beta)\rangle$$

$$=\frac{\kappa^2}{4(2\pi)^3}e^{i[\hat{H}(x^0-y^0)-\hat{P}(x^1-y^1)]}\exp[-\kappa^2|x^+-y^+|^2/8]\cdot\langle\rho_n(F^\alpha),\rho_n(F^\beta)\rangle$$

$$\frac{1}{(2\pi)^2}e^{i[\hat{H}(x^0-y^0)-\hat{P}(x^1-y^1)]}\cdot\delta(x^+-y^+)\cdot C(\rho_n)\mathrm{Tr}[-F^\alpha F^\beta],x^+,y^+\in S_0$$

$$\frac{1}{(2\pi)^2}\int_{\vec{x},\vec{y}\in\mathbb{R}^4}f(\vec{x})\overline{g(\vec{y})}\left\langle e^{i(x^0\hat{H}-x^1\hat{P})}p_\kappa^{x^+}\otimes\rho_n(F^\alpha),e^{i(y^0\hat{H}-y^1\hat{P})}p_\kappa^{y^+}\otimes\rho_n(F^\beta)\right\rangle d\vec{x}d\vec{y}$$

$$=\frac{\kappa^2}{4}\int_{\hat{s},\hat{t}\in S_0}f^{\{e_0,e_1\}}(\hat{s})\overline{g^{\{e_0,e_1\}}}(\hat{t})\frac{1}{2\pi}\exp[-\kappa^2|\hat{s}-\hat{t}|^2/8]d\hat{s}d\hat{t}\cdot\langle\rho_n(F^\alpha),\rho_n(F^\beta)\rangle$$

$$\int_{\hat{s}\in S_0}\left[f^{\{e_0,e_1\}}\overline{g^{\{e_0,e_1\}}}\right](\hat{s})d\hat{s}\cdot\langle\rho_n(F^\alpha),\rho_n(F^\beta)\rangle$$

$$\lim_{\kappa\to\infty}\int_{\vec{z}\in\mathbb{R}^4}p_\kappa^{\vec{x}}(\vec{z})f(\vec{z})d\vec{z}=0$$

$$A_r^n=\psi^{\alpha_1,n}(f_1)\cdots\psi^{\alpha_r,n}(f_r),B_s^n=\psi^{\beta_1,n}(g_1)\cdots\psi^{\beta_s,n}(g_s)$$

$$\langle A_r^nP_0B_s^n1,1\rangle\equiv\langle A_r^nB_s^n1,1\rangle-\langle A_r^n1,1\rangle\langle B_s^n1,1\rangle$$

$$h_\theta=\begin{cases}f_\theta, & \text{if }\psi^{\alpha_\theta,n}(f_\theta)=\phi^{\alpha_\theta,n}(f_\theta)\\ -\bar{f}_\theta, & \text{if }\psi^{\alpha_\theta,n}(f_\theta)=\phi^{\alpha_\theta,n}(f_\theta)^*\end{cases}$$

$$h_\theta=\begin{cases}g_{\theta-r}, & \text{if }\psi^{\beta_{\theta-r},n}(g_{\theta-r})=\phi^{\beta_{\theta-r},n}(g_{\theta-r})\\ -\bar{g}_{\theta-r}, & \text{if }\psi^{\beta_{\theta-r},n}(g_{\theta-r})=\phi^{\beta_{\theta-r},n}(g_{\theta-r})^*\end{cases}$$

$$\tilde{h}_\theta=\begin{cases}g_\theta, & \text{if }\psi^{\alpha_\theta,n}(g_\theta)=\phi^{\alpha_\theta,n}(g_\theta)\\ -\bar{g}_\theta, & \text{if }\psi^{\alpha_\theta,n}(g_\theta)=\phi^{\alpha_\theta,n}(g_\theta)^*\end{cases}$$

$$\tilde{h}_\theta=\begin{cases}f_{\theta-s}, & \text{if }\psi^{\alpha_\theta,n}(f_{\theta-s})=\phi^{\alpha_\theta,n}(f_{\theta-s})\\ -\bar{f}_{\theta-s}, & \text{if }\psi^{\alpha_\theta,n}(f_{\theta-s})=\phi^{\alpha_\theta,n}(f_{\theta-s})^*\end{cases}$$

$$\tilde{h}_\theta=\begin{cases}f_{\theta-s}, & \text{if }\psi^{\beta_{\theta-r},n}(f_{\theta-s})=\phi^{\beta_{\theta-r},n}(f_{\theta-s})\\ -\bar{f}_{\theta-s}, & \text{if }\psi^{\beta_{\theta-r},n}(f_{\theta-s})=\phi^{\beta_{\theta-r},n}(f_{\theta-s})^*\end{cases}$$

$$\psi^{\alpha_\theta,n}(h)=\phi^{\alpha_\theta,n}(h)^*,1\leq\theta\leq r$$

$$\psi^{\beta_{\theta-r},n}(h)=\phi^{\beta_{\theta-r},n}(h)^*,r+1\leq\theta\leq r+s$$

$$\chi(\theta)=\begin{cases}-1, & \theta\\ 1, & \text{otherwise}\end{cases}$$

$$\int_Q\{h_\theta\}_{\theta=1}^{r+s}:=\prod_{l=1}^{n(Q)}\left\{\int_{S_0}\left[\prod_{\theta\in A_l}\int_{y_\theta^-\in\mathbb{R}^2}\frac{e^{i\chi(\theta)[y_\theta^0\hat{H}-y_\theta^1\hat{P}]}}{2\pi}h_\theta(y_\theta^-,y^+)dy_\theta^-\right]dy^+\right\}$$

$$=\prod_{l=1}^{n(Q)}\left\{\int_{S_0}\left[\prod_{\theta\in A_l}\int_{y_\theta^0,y_\theta^1\in\mathbb{R}}\frac{e^{i\chi(\theta)[y_\theta^0\hat{H}-y_\theta^1\hat{P}]}}{2\pi}h_\theta(y_\theta^0,y_\theta^1,y^2,y^3)dy_\theta^0dy_\theta^1\right]dy^2dy^3\right\}$$

$$\langle A_r^nP_0B_s^n1,1\rangle=\sum_{Q\in\Gamma}c_Q\int_Q\{h_\theta\}_{\theta=1}^{r+s},$$

$$C_r^n=\psi^{\alpha_1,n}(g_1)\cdots\psi^{\alpha_s,n}(g_s)\psi^{\alpha_{s+1},n}(f_1)\cdots\psi^{\alpha_r,n}(f_{r-s})$$

$$D_s^n=\psi^{\beta_1,n}(f_{r-s+1})\cdots\psi^{\beta_s,n}(f_r)$$

$$\mathcal{W}^n:f_1\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}f_r\otimes_{\mathbb{R}}g_1\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}g_s\mapsto\langle A_r^nP_0B_s^n1,1\rangle.$$

$$\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r,\{\vec{x}_\theta\}_{\theta=r+1}^{r+s})\equiv\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^s,\{\vec{x}_\theta\}_{\theta=s+1}^{s+r})$$
$$:=\mathcal{W}^n(\vec{x}_1,\cdots,\vec{x}_r,\vec{x}_{r+1},\cdots,\vec{x}_{r+s})$$

$$\langle A_r^n B_s^n 1,1\rangle - \langle A_r^n 1,1\rangle\langle B_s^n 1,1\rangle \equiv \langle A_r^n P_0 B_s^n 1,1\rangle$$

$$= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \bigotimes_{\tau=1}^{r+s} p_\tau(\vec{x}_\tau) \cdot \prod_{\tau=1}^{r+s} d\vec{x}_\tau$$

$$= c_R \int_{S_0} \left[\prod_{\tau=1}^{r} \int_{x_\tau^-\in\mathbb{R}^2} E(x_\tau^-) h_\tau(x_\tau^-, x^+) dx_\tau^- \cdot \prod_{\theta=r+1}^{r+s} \int_{x_\theta^-\in\mathbb{R}^2} E(x_\theta^-) h_\theta(x_\theta^-, x^+) dx_\theta^-\right] dx^+$$

$$+ \sum_{\substack{Q\neq R\\ Q\in\Gamma}} c_Q \int_Q \{h_\theta\}_{\theta=1}^{r+s}$$

$$E(x_\tau^-) = E(x_\tau^0, x_\tau^1) = \frac{e^{i\chi(\tau)[x_\tau^0\hat{H} - x_\tau^1\hat{P}]}}{2\pi}$$

$$\mathcal{W}^n: g_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} g_s \otimes_{\mathbb{R}} f_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} f_r \longmapsto \langle C_r^n P_0 D_s^n 1,1\rangle,$$

$$\langle C_r^n D_s^n 1,1\rangle - \langle C_r^n 1,1\rangle\langle D_s^n 1,1\rangle \equiv \langle C_r^n P_0 D_s^n 1,1\rangle$$

$$= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^s, \{\vec{x}_\theta\}_{\theta=s+1}^{s+r}) \bigotimes_{\tau=1}^{s+r} \tilde{p}_\tau(\vec{x}_\tau) \cdot \prod_{\tau=1}^{s+r} d\vec{x}_\tau$$

$$= c_R \int_{S_0} \left[\prod_{\tau=1}^{s} \int_{x_\tau^-\in\mathbb{R}^2} E(x_\tau^-) \tilde{h}_\tau(x_\tau^-, x^+) dx_\tau^- \cdot \prod_{\theta=s+1}^{s+r} \int_{x_\theta^-\in\mathbb{R}^2} E(x_\theta^-) \tilde{h}_\theta(x_\theta^-, x^+) dx_\theta^-\right] dx^+$$

$$+ \sum_{\substack{Q\neq R\\ Q\in\Gamma}} c_Q \int_Q \{\tilde{h}_\theta\}_{\theta=1}^{s+r}.$$

$$\psi^{\beta_1,n}(g_1(\cdot - \vec{a}))\cdots\psi^{\beta_{s-1},n}(g_{s-1}(\cdot - \vec{a}))\left(S_0 + \vec{a}, g_s^{\{e_0,e_1\}}(\cdot - \vec{a}) \otimes \rho_n(F^{\beta_s}), \{e_a\}_{a=0}^3\right),$$

$$P_0 U(\vec{a},1) B_s^n 1 = P_0 \psi^{\beta_1,n}(g_1)_{U(\vec{a})} \psi^{\beta_2,n}(g_2)_{U(\vec{a})} \cdots \psi^{\beta_{s-1},n}(g_{s-1})_{U(\vec{a})} U(\vec{a},1) \psi^{\beta_s,n}(g_s) 1$$

$$= e^{i[a^0\hat{H} - a^1\hat{P}]} P_0 B_s^{n,\vec{a}} 1 = e^{i[a^0\hat{H} - a^1\hat{P}]}\left[B_s^{n,\vec{a}} 1 - \langle B_s^n 1,1\rangle 1\right]$$

$$\vec{a} = \sum_{b=0}^{3} a^b e_b$$

$$\langle A_r^n P_0 U(\vec{a},1) B_s^n 1,1\rangle = e^{i[a^0\hat{H} - a^1\hat{P}]}\left[\left\langle A_r^n B_s^{n,\vec{a}} 1,1\right\rangle - \langle A_r^n 1,1\rangle\langle B_s^n 1,1\rangle\right]$$

$$e^{i[a^0\hat{H} - a^1\hat{P}]} \int_{\mathbb{R}^2} \frac{e^{i[s\hat{H} - t\hat{P}]}}{2\pi} f(s,t,x^2,x^3) ds dt$$

$$= \int_{\mathbb{R}^2} \frac{e^{i[(s+a^0)\hat{H} - (t+a^1)\hat{P}]}}{2\pi} f(s,t,x^2,x^3) ds dt$$

$$= \int_{\mathbb{R}^2} \frac{e^{i[s\hat{H} - t\hat{P}]}}{2\pi} f(s - a^0, t - a^1, x^2, x^3) ds dt$$

$$= f\left(\cdot - (a^0, a^1, 0, 0)\right)^{\{e_0,e_1\}}(\hat{H},\hat{P})(0,0,x^2,x^3)$$

$$\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r+1}^{r+s}) := \mathcal{W}^n(\vec{x}_1, \cdots, \vec{x}_r, \vec{x}_{r+1} + \vec{a}, \cdots, \vec{x}_{r+s} + \vec{a})$$

$$H^n(\vec{a})$$

$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \bigotimes_{\tau=1}^{r} p_\tau(\vec{x}_\tau) \cdot \bigotimes_{\theta=r+1}^{r+s} p_\theta(\vec{x}_{r+\theta} - \vec{a}) \cdot \prod_{\tau=1}^{r+s} d\vec{x}_\tau$$

$$\equiv \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r+1}^{r+s}) \bigotimes_{\tau=1}^{r+s} p_\tau(\vec{x}_\tau) \cdot \prod_{\tau=1}^{r+s} d\vec{x}_\tau$$

$$= c_R \int_{S_0} \left[\prod_{\tau=1}^{r} \int_{x_\tau^- \in \mathbb{R}^2} E(x_\tau^-) h_\tau(x_\tau^-, x^+) dx_\tau^- \cdot \prod_{\theta=r^+}^{r+s} \int_{x_\theta^- \in \mathbb{R}^2} E(x_\theta^-) h_\theta^{\vec{a}}(x_\theta^-, x^+) dx_\theta^- \right] dx^+$$

$$+ \sum_{\substack{Q\neq R\\ Q\in\Gamma}} c_Q \int_Q \{h_\theta^{\vec{a}}\}_{\theta=1}^{r+s}$$

$$E(x_\tau^-) = E(x_\tau^0, x_\tau^1) = \frac{e^{i\chi(\tau)[x_\tau^0\hat{H} - x_\tau^1\hat{P}]}}{2\pi}$$

$$\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r+1}^{r+s}) = W^n(\vec{\xi}_1, \cdots, \vec{\xi}_{r-1}, \vec{\xi}_r - \vec{a}, \vec{\xi}_{r+1}, \cdots, \vec{\xi}_{r+s-1})$$

$$\widehat{W}^n(\vec{p}_1, \cdots, \vec{p}_{r+s})$$

$$= (2\pi)^4 \delta\left(\sum_{\tau=1}^{r+s} \vec{p}_\tau\right) \widehat{W}^n(\vec{p}_1, \vec{p}_1 + \vec{p}_2, \cdots, \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_{r+s-1}).$$

$$\vec{x}_{\delta_i} - \vec{x}_{\epsilon_i} = \vec{\xi}_{\delta_i} + \vec{\xi}_{\delta_i+1} + \cdots + \vec{\xi}_{\epsilon_i-1}$$

$$\vec{x}_{\delta_i} - \vec{x}_{\epsilon_i} = -(\vec{\xi}_{\epsilon_i} + \vec{\xi}_{\epsilon_i+1} + \cdots + \vec{\xi}_{\delta_i-1})$$

$$\sum_{\tau=1}^{r} \chi(\tau)[x_\tau^0\hat{H} - x_\tau^1\hat{P}] + \sum_{\theta=r+1}^{r+s} \chi(\theta)[x_\theta^0\hat{H} - x_\theta^1\hat{P}] = -\sum_{i=1}^{r+s-1} c_i[\xi_i^0\hat{H} - \xi_i^1\hat{P}]$$

$$\prod_{\tau=1}^{r} E(x_\tau^-) \prod_{\theta=r+1}^{r+s} E(x_\theta^-) = \frac{1}{(2\pi)^{r+s}} \exp\left[-i \sum_{j=1}^{r+s-1} c_j[\xi_j^0\hat{H} - \xi_j^1\hat{P}]\right]$$

$$W^n(\vec{\xi}_1, \cdots, \vec{\xi}_{r+s-1}) := \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta\}_{\theta=r+1}^{r+s})$$

$$\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r+1}^{r+s}) = W^n(\vec{\xi}_1, \cdots, \vec{\xi}_{r-1}, \vec{\xi}_r - \vec{a}, \vec{\xi}_{r+1}, \cdots, \vec{\xi}_{r+s-1}),$$

$$\int_{\mathbb{R}^2} e^{-ia^-\cdot q^-} W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t - a^-, \vec{\xi}_{t+1}, \cdots, \vec{\xi}_{r+s-1}) da^-$$

$$W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t - a^-, \vec{\xi}_{t+1}, \cdots, \vec{\xi}_{r+s-1}) = e^{-ic_t a^-\cdot\vec{\alpha}^n} W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t, \vec{\xi}_{t+1}, \cdots, \vec{\xi}_{r+s-1})$$

$$\int_{\mathbb{R}^2} e^{-iq^-\cdot a^-} W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t - a^-, \vec{\xi}_{t^+}, \cdots, \vec{\xi}_{r+s-1}) da^-$$

$$= \int_{\mathbb{R}^2} e^{-ia^-\cdot q^-} e^{-ic_t a^-\cdot\vec{\alpha}^n} W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t, \vec{\xi}_{t^+}, \cdots, \vec{\xi}_{r+s-1}) da^-$$

$$= 2\pi W^n(\vec{\xi}_1, \cdots, \vec{\xi}_t, \cdots, \vec{\xi}_{r+s-1}) \cdot \delta\left(q^- + c_t(\hat{H}(\rho_n), \hat{P}(\rho_n), 0{,}0)\right)$$

$$a^- \mapsto F^-(a^-) \equiv W^n(\vec{\xi}_1, \cdots, \vec{\xi}_r - a^-, \vec{\xi}_{r+1}, \cdots, \vec{\xi}_{r+s-1}),$$

$$a^- \mapsto F^+(a^-) = \mathcal{W}^n(\{\vec{x}_\theta + a^-\}_{\theta=1}^s, \{\vec{x}_\tau\}_{\tau=s+1}^{s+r})$$

$$\equiv W^n(\vec{\xi}_1, \cdots, \vec{\xi}_s + a^-, \vec{\xi}_{s+1}, \cdots, \vec{\xi}_{r+s-1})$$

$$\bar{k} + \underline{k} = \bar{l} + \underline{l}.$$

$$G(a) = \begin{cases} a + \bar{k}, & 1 \le a \le \bar{l} \\ a + \bar{k} + \underline{k}, & \bar{l} + 1 \le a \le \bar{k} \\ a + \underline{k} + \bar{k} - \bar{l} \equiv a + \underline{l}, & r + 1 \le a \le r + \underline{k} \end{cases}$$

$$\tilde{G}(a) = \begin{cases} a + \bar{k}, & 1 \le a \le \bar{k} \\ a + \bar{k} - \bar{l}, & r + 1 \le a \le r + \bar{l} - \bar{k} \\ a + \bar{k} - \bar{l} + \underline{k} \equiv a + \underline{l}, & r + \bar{l} - \bar{k} + 1 \le a \le r + \underline{k} \end{cases}$$

$$\mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^{r},\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) = \pm\mathcal{W}^n(\{\vec{x}_\theta\}_{\theta=r+1}^{r+s},\{\vec{x}_\tau\}_{\tau=1}^{r})$$
$$\mathcal{W}_{\kappa_r,\pi_s}^n : f_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} f_r \otimes_{\mathbb{R}} g_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} g_s \mapsto \langle A_r^n P_0 B_s^n 1,1\rangle,$$
$$\mathcal{W}_{\pi_s,\kappa_r}^n : g_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} g_s \otimes_{\mathbb{R}} f_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} f_r \mapsto \langle C_r^n P_0 D_s^n 1,1\rangle,$$

$$\int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}^n(\{\vec{x}_\tau\}_{\tau=1}^{r},\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \bigotimes_{\tau=1}^{r+s} p_\tau(\vec{x}_\tau)\cdot\prod_{\tau=1}^{r+s} d\vec{x}_\tau$$
$$= \sum_{\substack{(\kappa_r,\bar{\kappa}_r)\in\Omega_r\\(\pi_s,\pi_s)\in\Omega_s}} \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \mathcal{W}_{\kappa_r,\pi_s}^n(\{\vec{x}_\tau\}_{\tau=1}^{r},\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \prod_{\tau=1}^{r+s} q_\tau^{\{\kappa_r,\pi_s\}}(\vec{x}_\tau)\cdot\prod_{\tau=1}^{r+s} d\vec{x}_\tau$$
$$\mathcal{W}_{\kappa_r,\pi_s}^n(\{\vec{x}_\tau\}_{\tau=1}^{r},\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) = \pm\mathcal{W}_{\pi_s,\kappa_r}^n(\{\vec{x}_\theta\}_{\theta=r+1}^{r+s},\{\vec{x}_\tau\}_{\tau=1}^{r})$$
$$T_1^n := \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\tau=1}^{r} d\vec{x}_\tau \cdot \varphi_1(\{\vec{x}_\tau\}_{\tau=1}^{r}) \cdot \prod_{\tau=1}^{r} \psi^{\alpha_\tau,n}(\vec{x}_\tau)$$
$$T_2^n := \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\theta=r+1}^{r+s} d\vec{x}_\theta \cdot \varphi_2(\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \cdot \prod_{\theta=r+1}^{r+s} \psi^{\beta_{\theta-r},n}(\vec{x}_\theta)$$
$$\mathcal{T}_1^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r})$$
$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\theta=r+1}^{r+s} d\vec{x}_\theta \cdot \varphi_1(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\{\vec{x}_\theta\}_{\theta=r+1}^{r+s}) \cdot \prod_{\theta=r+1}^{r+s} \psi^{\beta_{\theta-r},n}(\vec{x}_\theta)$$
$$\mathcal{T}_2^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r})$$
$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\tau=1}^{s} d\vec{x}_\tau \cdot \varphi_2(\{\vec{x}_\tau\}_{\tau=1}^{s}) \cdot \prod_{\tau=1}^{r} \psi^{\alpha_\tau,n}(\vec{x}_\tau)$$
$$T_1^n(\vec{a}) := U(\vec{a},1)T_1^n U(\vec{a},1)^{-1}$$
$$T_2^n(\vec{a}) := U(\vec{a},1)T_2^n U(\vec{a},1)^{-1}$$
$$\mathcal{T}_1^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{a}), \mathcal{T}_2^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{a})$$
$$T_1 := \sum_{n=1}^{\infty} c_{1,n}T_1^n, T_2 := \sum_{n=1}^{\infty} c_{2,n}T_2^n$$
$$\mathcal{T}_1(\{\vec{x}_\tau\}_{\tau=s+1}^{r}) := \sum_{n=1}^{\infty} c_{1,n}\mathcal{T}_1^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r})$$
$$\mathcal{T}_2(\{\vec{x}_\tau\}_{\tau=s+1}^{r}) := \sum_{n=1}^{\infty} c_{2,n}\mathcal{T}_2^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r})$$
$$\left|\hat{H}^k T_j(\cdot,\vec{a})1\right|^2 \le \sum_{n=1}^{\infty} C(\rho_n)^{2k}\left|c_{j,n}\right|^2\left|T_j^n(\cdot)1\right|^2 < \infty$$
$$\langle T_1(\vec{y})T_2(\vec{x})1,1\rangle$$
$$= \sum_{n=1}^{\infty} c_{1,n}c_{2,n}\langle U(\vec{y},1)T_1^n U(\vec{y},1)^{-1}U(\vec{x},1)T_2^n U(\vec{x},1)^{-1}1,1\rangle$$
$$= \sum_{n=1}^{\infty} c_{1,n}c_{2,n}\langle T_1^n U(\vec{x}-\vec{y},1)T_2^n 1,1\rangle$$
$$\langle \mathcal{T}_2(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{x})\mathcal{T}_1(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{y})1,1\rangle$$
$$= \sum_{n=1}^{\infty} c_{1,n}c_{2,n}\langle \mathcal{T}_2^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{x})\mathcal{T}_1^n(\{\vec{x}_\tau\}_{\tau=s+1}^{r},\vec{y})1,1\rangle$$

$$h_{12}^{n}(\vec{a}) := \langle T_1^n(\vec{y}) P_0 T_2^n(\vec{x})1,1\rangle$$
$$= \langle T_1^n U(\vec{y},1)^{-1} U(\vec{x},1) T_2^n 1,1\rangle - \langle T_1^n 1,1\rangle\langle T_2^n 1,1\rangle$$
$$= \langle T_1^n U(\vec{a},1) T_2^n 1,1\rangle - \langle T_1^n 1,1\rangle\langle T_2^n 1,1\rangle.$$
$$h_{21}^{n}\ (\vec{a})$$
$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\tau=s^+}^{r} d\vec{x}_\tau \left\langle \mathcal{T}_2^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}, \vec{x}\right) P_0 \mathcal{T}_1^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}, \vec{y}\right)1,1\right\rangle$$
$$= \left\langle \int_{\mathbb{R}^{4(r-s)}} \prod_{\tau=s^+}^{r} d\vec{x}_\tau \mathcal{T}_2^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right) U(-\vec{a},1) \mathcal{T}_1^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle$$
$$- \int_{\mathbb{R}^{4(r-s)}} \prod_{\tau=s^+}^{r} d\vec{x}_\tau \left\langle \mathcal{T}_1^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle \left\langle \mathcal{T}_2^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle$$
$$h_{12}^{n}(\vec{a})$$
$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\tau=1}^{r+s} d\vec{x}_\tau \mathcal{W}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+\vec{a}\}_{\theta=r^+}^{r+s}\right) \varphi_1\left(\{\vec{x}_\tau\}_{\tau=1}^{r}\right) \varphi_2\left(\{\vec{x}_\theta\}_{\theta=r^+}^{r+s}\right)$$
$$h_{21}^{n}(\vec{a})$$
$$:= \int_{\mathbb{R}^4\times\cdots\times\mathbb{R}^4} \prod_{\tau=1}^{r+s} d\vec{x}_\tau \mathcal{W}^n\left(\{\vec{x}_\theta+\vec{a}\}_{\theta=r^+}^{r+s}, \{\vec{x}_\tau\}_{\tau=1}^{r}\right) \varphi_1\left(\{\vec{x}_\tau\}_{\tau=1}^{r}\right) \varphi_2\left(\{\vec{x}_\theta\}_{\theta=r^+}^{r+s}\right)$$
$\langle T_1^n U(\vec{a},1) T_2^n 1,1\rangle - \langle T_1^n 1,1\rangle\langle T_2^n 1,1\rangle$ and
$$\left\langle \int_{\mathbb{R}^{4(r-s)}} \prod_{\tau=s^+}^{r} d\vec{x}_\tau \mathcal{T}_2^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right) U(-\vec{a},1) \mathcal{T}_1^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle$$
$$- \int_{\mathbb{R}^{4(r-s)}} \prod_{\tau=s^+}^{r} d\vec{x}_\tau \left\langle \mathcal{T}_1^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle \left\langle \mathcal{T}_2^n\left(\{\vec{x}_\tau\}_{\tau=s^+}^{r}\right)1,1\right\rangle$$
$$h_{12}(\vec{a}) := \sum_{n=1}^{\infty} c_{1,n} c_{2,n} h_{12}^{n}(\vec{a}),\ h_{21}(\vec{a}) := \sum_{n=1}^{\infty} c_{1,n} c_{2,n} h_{21}^{n}(\vec{a}).$$
$$\frac{\hat{H}(\rho_n)}{m_n} e_0 + \frac{\hat{P}(\rho_n)}{m_n} e_1, \frac{\hat{P}(\rho_n)}{m_n} e_0 + \frac{\hat{H}(\rho_n)}{m_n} e_1$$
$$\mathcal{W}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}\right) = 0$$
$$\mathcal{W}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}\right)$$
$$= e^{-i\vec{a}\cdot c_r m_n \tilde{f}_0^n} \prod_{\theta=1}^{r+s} E(x_\theta^-) \cdot \mathcal{W}_0^n\left(\{x_\tau^+\}_{\tau=1}^{r}, \{x_\theta^+ + a^+\}_{\theta=r+1}^{r+s}\right),$$
$$E(x_\theta^-) = E\left(x_\theta^0, x_\theta^1\right) = \frac{e^{i\chi(\theta)\left[x_\theta^0 \hat{H} - x_\theta^1 \hat{P}\right]}}{2\pi}\ (0^- \equiv (0,0))$$
$$\mathcal{W}_0^n\left(\{x_\tau^+\}_{\tau=1}^{r}, \{x_\theta^+ + a^+\}_{\theta=r+1}^{r+s}\right) = \mathcal{W}^n\left(\{0^-, x_\tau^+\}_{\tau=1}^{r}, \{0^-, x_\theta^+ + a^+\}_{\theta=r+1}^{r+s}\right)$$
$$\mathcal{W}_{\kappa_r,\pi_s}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}\right) = \pm \mathcal{W}_{\pi_s,\kappa_r}^n\left(\{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}, \{\vec{x}_\tau\}_{\tau=1}^{r}\right)$$
$$\mathcal{W}_{\kappa_r,\pi_s}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}\right) = e^{ia^0 c_r m_n} \mathcal{W}_{\kappa_r,\pi_s}^n\left(\{\vec{x}_\tau\}_{\tau=1}^{r}, \{\vec{x}_\theta+a^+\}_{\theta=r+1}^{r+s}\right),$$
$$\mathcal{W}_{\pi_s,\kappa_r}^n\left(\{\vec{x}_\theta+\vec{a}\}_{\theta=r+1}^{r+s}, \{\vec{x}_\tau\}_{\tau=1}^{r}\right) = e^{-ia^0 c_s m_n} \mathcal{W}_{\pi_s,\kappa_r}^n\left(\{\vec{x}_\theta+a^+\}_{\theta=r+1}^{r+s}, \{\vec{x}_\tau\}_{\tau=1}^{r}\right)$$
$$|h_{12}(\vec{a})| = \left|\sum_{n=1}^{\infty} c_{1,n} c_{2,n} h_{12}^{n}(\vec{a})\right|$$
$$\leq \frac{C}{m_0^l + |a^+|^k}\left(\|\varphi_1\|_{p,q}\|\varphi_2\|_{p,q}\right),$$
$f_1 \otimes_{\mathbb{R}} f_2 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} f_r$ and $g_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} g_s$
$$\varphi\left(\{\vec{x}_\theta\}_{\theta=1}^{r+s}\right) = \varphi_1\left(\{\vec{x}_\tau\}_{\tau=1}^{r}\right) \otimes_{\mathbb{R}} \varphi_2\left(\{\vec{x}_\theta\}_{\theta=r^+}^{r+s}\right)$$

$$\tilde{\varphi}_1(\{q_\tau^-, x_\tau^+\}_{\tau=1}^r) := \int_{\mathbb{R}^{2r}} \prod_{\tau=1}^{r} \frac{e^{-i\chi(\tau)x_\tau^-\cdot q_\tau^-}}{2\pi} \cdot \varphi_1(\{x_\tau^-, x_\tau^+\}_{\tau=1}^r) \prod_{\tau=1}^{r+s} dx_\tau^-$$

$$\tilde{\varphi}_2(\{q_\theta^-, x_\theta^+\}_{\theta=r+1}^{r+s}) := \int_{\mathbb{R}^{2s}} \prod_{\theta=r+1}^{r+s} \frac{e^{-i\chi(\theta)x_\theta^-\cdot q_\theta^-}}{2\pi} \cdot \varphi_2(\{x_\theta^-, x_\theta^+\}_{\theta=r+1}^{r+s}) \prod_{\theta=r+1}^{r+s} dx_\theta^-$$

$$\tilde{\varphi}(\{q_\tau^-, x_\tau^+\}_{\tau=1}^{r+s}) := \tilde{\varphi}_1(\{q_\tau^-, x_\tau^+\}_{\tau=1}^r)\tilde{\varphi}_2(\{q_\theta^-, x_\theta^+\}_{\theta=r+1}^{r+s})$$

$$h_{12}^n(\vec{a}) = e^{-ic_r m_n \vec{a}\cdot\tilde{f}_0^n} \int_{\mathbb{R}^{2(r+s)}} \prod_{\tau=1}^{r+s} dx_\tau^+ \mathcal{W}_0^n(\{x_\tau^+\}_{\tau=1}^r, \{x_\theta^+ + a^+\}_{\theta=r^+}^{r+s}) \tilde{\varphi}(\{H_n^-, x_\tau^+\}_{\tau=1}^{r+s})$$

$$|\tilde{\varphi}(\{q_\theta^-, x_\theta^+\}_{\theta=1}^{r+s})| \le C(\rho_n)^{\bar{n}} \frac{\|\varphi\|_{\hat{k},\hat{l}}}{\sum_{\theta=1}^{r+s} \left(|q_\theta^0|^2 + |q_\theta^1|^2\right)^{l/2} + \left(|x_\theta^2|^2 + |x_\theta^3|^2\right)^{k/2}}$$

$$|\tilde{\varphi}(\{H_n^-, x_\theta^+\}_{\theta=1}^{r+s})| \le C(\rho_n)^{\bar{n}} \frac{\|\varphi\|_{\hat{k},\hat{l}}}{\sum_{\theta=1}^{r+s} m_n^l + (x_\theta^{2,2} + x_\theta^{3,2})^{k/2}}$$

$$e^{-ic_r m_n \vec{a}\cdot\tilde{f}_0^n} \sum_{|\vec{m}|\le\mathcal{N}} \int_{\mathbb{R}^{2(r+s)}} \prod_{\tau=1}^{r+s} dx_\tau^+ D^{\vec{m}} G_{\vec{m}}^n(\{x_\tau^+\}_{\tau=1}^{r+s}; a^+)\tilde{\varphi}(\{H_n^-, x_\theta^+\}_{\theta=1}^{r+s})$$

$$= h_{12}^n(\vec{a})$$

$$\sum_{|\vec{m}|\le\mathcal{N}} D^{\vec{m}} G_{\vec{m}}^n(\{\{x_\tau^+\}_{\tau=1}^{r+s}; a^+) \equiv \mathcal{W}_0^n(\{x_\tau^+\}_{\tau=1}^r, \{x_\theta^+ + a^+\}_{\theta=r+1}^{r+s})$$

$$\sum_{|\vec{m}|\le\mathcal{N}} |G_{\vec{m}}^n|(\{x_\tau^+\}_{\tau=1}^{r+s}; a^+) \le C(\rho_n)^{\tilde{k}} \left[|a^+|^\alpha + \left(\sum_{\tau=1}^{r+s} |x_\tau^+|^2\right)^{\gamma/2}\right]$$

$$\{R > R_0 - \epsilon\} := \left\{\{x_\tau^+\}_{\tau=1}^{r+s} : \sum_{\tau=1}^{r+s} |x_\tau^+|^2 > (R_0 - \epsilon)^2\right\} \subset \mathbb{R}^{2(r+s)}$$

$$\sum_{|\vec{m}|\le\mathcal{N}} \int_{\mathbb{R}^{2(r+s)}} \prod_{\tau=1}^{r+s} dx_\tau^+ D^{\vec{m}} G_{\vec{m}}^n(\{x_\tau^+\}_{\tau=1}^{r+s}; a^+)\tilde{\varphi}(\{q_\theta^-, x_\theta^+\}_{\theta=1}^{r+s})$$

$$= \sum_{|\vec{m}|\le\mathcal{N}} \int_{\{R>R_0-\epsilon\}} \prod_{\tau=1}^{r+s} dx_\tau^+ D^{\vec{m}} G_{\vec{m}}^n(\{x_\tau^+\}_{\tau=1}^{r+s}; a^+)\tilde{\varphi}(\{q_\theta^-, x_\theta^+\}_{\theta=1}^{r+s})$$

$$|h_{12}^n(\vec{a})| \le \sum_{|\vec{m}|\le\mathcal{N}} \int_{\{R>R_0-\epsilon\}} \prod_{\tau=1}^{r+s} dx_\tau^+ \left|G_{\vec{m}}^n(\{x_\tau^+\}_{\tau=1}^{r+s}; a^+) D^{\vec{m}} \tilde{\varphi}(\{H_n^-, x_\theta^+\}_{\theta=1}^{r+s})\right|$$

$$\le C(\rho_n)^{\tilde{K}} \int_{\{R>R_0-\epsilon\}} [|a^+|^\alpha + R^\gamma] \frac{\|\varphi\|_{p,q}}{(r+s)|m_n|^l + R^{k+\bar{l}}} \frac{R^{2(r+s)}}{R} dR d\Omega$$

$$C(\rho_n)^{\tilde{K}} \frac{\|\varphi_1\|_{p,q}\|\varphi_2\|_{p,q}}{m_n^l + |a^+|^k}$$

$$\mathcal{W}_{\kappa_r,\pi_s}^n(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r^+}^{r+s}) \longrightarrow 0$$

$$|h_{12}(\vec{a})| \le \sum_{n=1}^{\infty} |c_{1,n}c_{2,n}||h_{12}^n(\vec{a})| \le \frac{C}{m_0^l + |a^+|^k}(\|\varphi_1\|_{p,q}\|\varphi_2\|_{p,q})$$

$$\{y_j^+\}^{|A_i|} = (\underbrace{H_n^-, y_i^+, H_n^-, y_i^+, \cdots, H_n^-, y_i^+}_{|A_i| \text{ copies of } (H_n^-, y_i^+)}).$$

$$\tilde{\varphi}_1(\{H_n^-, x_\tau^+\}_{\tau=1}^r) \text{ and } \tilde{\varphi}_2(\{H_n^-, x_\theta^+\}_{\theta=r+1}^{r+s}).$$

$$\tilde{\varphi}_{Q,1}^n\left(\{x_\theta^+\}_{\theta=n_{z-1}+1}^r\right)$$
$$:= \int_{\mathbb{R}^{2(z-1)}} \prod_{i=1}^{z-1} dy_i^+ \tilde{\varphi}_1\left(\{y_j^+\}^{|A_1|}, \{y_j^+\}^{|A_2|}, \cdots, \{y_j^+\}^{|A_{z-1}|}, \{H_n^-, x_\theta^+\}_{\theta=n_{z-1}+1}^r\right)$$

$$\tilde{\varphi}_{Q,2}^n\left(\{x_\theta^+\}_{\theta=r^+}^{n_z}\right)$$
$$:= \int_{\mathbb{R}^{2(n(Q)-z)}} \prod_{i=z^+}^{n(Q)} dy_i^+ \tilde{\varphi}_2\left(\{H_n^-, x_\theta^+\}_{\theta=r^+}^{n_z}, \{y_j^+\}^{|A_{z+}|}, \{y_j^+\}^{|A_{z+2}|}, \cdots, \{y_j^+\}^{|A_{n(Q)}|}\right)$$

$$\tilde{\varphi}_{Q,1}^n(x_r^+) := \tilde{\varphi}_{Q,1}^n\left(\{x_r^+\}_{\theta=n_{z-1}+1}^r\right), \tilde{\varphi}_{Q,2}^n(x_{r+1}^+) := \tilde{\varphi}_{Q,2}^n\left(\{x_{r+1}^+\}_{\theta=r+1}^{n_z}\right).$$

$$h_{12}(\vec{a}) = e^{-ic_r m_n \vec{a}\cdot\tilde{f}_0^n} \sum_{Q\in\Omega_r} c_Q \int_{\mathbb{R}^2} dx^+ \tilde{\varphi}_{Q,1}^n(x^+)\tilde{\varphi}_{Q,2}^n(x^+ - a^+)$$
$$= e^{-ic_r m_n \vec{a}\cdot\tilde{f}_0^n} \sum_{Q\in\Omega_r} c_Q \int_{\mathbb{R}^2} dx_r^+ \int_{\mathbb{R}^2} dx_{r^+}^+ \tilde{\varphi}_{Q,1}^n(x_r^+)\delta\left(a^+ + x_{r^+}^+ - x_r^+\right)\tilde{\varphi}_{Q,2}^n\left(x_{r^+}^+\right),$$

$$|h_{12}(\vec{a})| = \left|\sum_{n=1}^{\infty} c_{1,n} c_{2,n} h_{12}^n(\vec{a})\right|$$
$$\leq C \frac{e^{-m_0|a^+|}}{|a^+| - \epsilon} \|\varphi_1\|_{p_1,q_1} \|\varphi_2\|_{p_2,q_2}$$

$$D(\varphi_1, \varphi_2) = \{\vec{x}_\tau - \vec{y}_\theta \in \mathbb{R}^4 \mid \{\vec{x}_\tau\}_{\tau=1}^r \in \mathrm{supp}\varphi_1, \{\vec{y}_\theta\}_{\theta=r+1}^{r+s} \in \mathrm{supp}\varphi_2\}$$

$$\left(\frac{\partial^2}{\partial a^{0,2}} - \hat{\mathcal{P}}\right)\Psi_n(a^0, a) := \left(\sum_{i=0}^{3} \frac{\partial^2}{\partial a^{i,2}}\right)\Psi_n(a^0, a) = h_{12}^n(a^0, a^+) g(a^1).$$

$$\Psi_n(a^0, a) = e^{ic_r m_n a^0} \tilde{G}(n; a)$$

$$\tilde{G}(n; a)$$
$$= \sum_{Q\in\Omega_r} c_Q \int_{\mathbb{R}} d\xi^1 \int_{\mathbb{R}^{2\times 2}} dx_r^+ dx_{r^+}^+ \tilde{\varphi}_{Q,1}^n(x_r^+) G_n\left(a^1 - \xi^1; x_r^+, x_{r^+}^+ + a^+\right)\tilde{\varphi}_{Q,2}^n\left(x_{r^+}^+\right) g(\xi^1),$$

$$\left(-(c_r m_n)^2 + \sum_{i=1}^{3} \frac{\partial^2}{\partial a^{i,2}}\right) G_n(a^1 - \xi^1; x_r^+, x_{r+1}^+ + a^+) = \delta(a^1 - \xi^1, x_r^+ - x_{r+1}^+ - a^+)$$
$$\equiv \delta(a - \xi_r)$$

$$G_n(a^1 - \xi^1; x_r^+, x_{r+1}^+ + a^+) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} dq \frac{e^{iq\cdot(a-\xi_r)}}{\omega^2 + |q|^2}$$

$$G_n(a^1 - \xi^1; x_r^+, x_{r+1}^+ + a^+) = -\frac{1}{(2\pi)^{1/2}} \frac{1}{iR} \int_0^{\infty} \lambda \frac{e^{iR\lambda} - e^{-iR\lambda}}{\omega^2 + \lambda^2} d\lambda$$
$$= -\frac{1}{(2\pi)^{1/2}} \frac{1}{iR} \int_{-\infty}^{\infty} \frac{\lambda e^{iR\lambda}}{(\lambda - i\omega)(\lambda + i\omega)} d\lambda$$

$$G_n(a^1 - \xi^1; x_r^+, x_{r+1}^+ + a^+) = -\frac{2\pi i}{(2\pi)^{1/2}} \frac{1}{iR} \frac{i\omega e^{-R\omega}}{2i\omega} = -\sqrt{2\pi} \frac{e^{-R|c_r m_n|}}{2R}$$

$$R^2 = |a^1 - \xi^1|^2 + |a^2 + \xi_r^2|^2 + |a^3 + \xi_r^3|^2$$
$$\left(G_n(a - \xi) := G_n(a^1 - \xi^1; x^-, x^+ + a^+)\right)$$

$$|h_{12}^n(\vec{a})g(a^1)| = \left|\left[\frac{\partial^2}{\partial a^{0,2}} - \hat{\mathcal{P}}\right]\Psi_n(a^0,a)\right|$$

$$= \left|\sum_{Q\in\Omega_r} c_Q \int_{\mathbb{R}} d\xi^1 \int_{\mathbb{R}^4} d\vec{x}\tilde{\varphi}_{Q,1}^n(x^-)G_n(a-\xi)\left[(c_r m_n)^2 + \hat{\mathcal{P}}\right]\tilde{\varphi}_{Q,2}^n(x^+)g(\xi^1)\right|$$

$$\leq \sum_{Q\in\Omega_r} |c_Q| \left\{\int_{\mathbb{R}} d\xi^1 \left[|g(\xi^1)| + \frac{1}{m_0^2}|g''(\xi^1)|\right] \cdot \sqrt{\frac{\pi}{2}} \frac{e^{-m_0|a^+ + x^+ - y^+|}}{|a^+ + x^+ - y^+|}\right.$$

$$\left. \times \int_{\mathbb{R}^2} dy^+ \left|\tilde{\varphi}_{Q,1}^n(y^+)\right| \cdot \int_{\mathbb{R}^2} dx^+ \left|\left[(c_r m_n)^2 + \hat{\mathcal{P}}\right]\tilde{\varphi}_{Q,2}^n(x^+)\right|\right\}$$

$$\leq C(\rho_n)^{\tilde{k}} e^{m_0\epsilon} \|g\|_{p_3,q_3} \|\varphi_1\|_{p_1,q_1} \|\varphi_2\|_{p_2,q_2} \cdot \frac{e^{-m_0|a^+|}}{|a^+| - \epsilon}$$

$$|h_{12}^n(\vec{a})| \leq C(\rho_n)^{\tilde{k}} e^{m_0\epsilon} \|\varphi_1\|_{p_1,q_1} \|\varphi_2\|_{p_2,q_2} \cdot \frac{e^{-m_0|a^+|}}{|a^+| - \epsilon}$$

$$\left|\mathcal{W}^n\left(\{\vec{x}_\tau\}_{\tau=1}^r, \{\vec{x}_\theta + \vec{a}\}_{\theta=r^+}^{r+s}\right)\right| \leq C(n)e^{-m_0|a^+|}$$

$$J_{ab}^\sigma(s,t) = \begin{pmatrix} \sigma_a'(s,t) & \dot{\sigma}_a(s,t) \\ \sigma_b'(s,t) & \dot{\sigma}_b(s,t) \end{pmatrix}, a \neq b$$

$$\rho_\sigma^{ab} = \frac{1}{\sqrt{\det\left[1 + W_{ab}^{cd,T} W_{ab}^{cd}\right]}} \equiv \frac{|J_{ab}^\sigma|}{\sqrt{\det\left[J_{ab}^{\sigma,T} J_{ab}^\sigma + J_{cd}^{\sigma,T} J_{cd}^\sigma\right]}}$$

$$\int_S d\rho := \sum_{0\leq a<b\leq 3} \int_{I^2} \rho_\sigma^{ab}(s,t)|J_{ab}^\sigma|(s,t)dsdt$$

$$\hat{J}_{ab}^\sigma(s,t) = \begin{cases} J_{ab}^\sigma(s,t), & a \neq 0 \\ \begin{pmatrix} i\sigma_a'(s,t) & i\dot{\sigma}_a(s,t) \\ \sigma_b'(s,t) & \dot{\sigma}_b(s,t) \end{pmatrix}, & a = 0 \end{cases}$$

$$\hat{\rho}_\sigma^{ab} := \frac{\det\hat{J}_{ab}^\sigma}{\sqrt{\det\left[\hat{J}_{ab}^{\sigma,T}\hat{J}_{ab}^\sigma + \hat{J}_{cd}^{\sigma,T}\hat{J}_{cd}^\sigma\right]}},$$

$$\int_S d\hat{\rho} := \sum_{0\leq a<b\leq 3} \int_{I^2} \rho_\sigma^{ab}(s,t)[\det J_{ab}^\sigma(\hat{s})]d\hat{s}$$

$$\int_S d|\hat{\rho}| := \int_{I^2} \left|\sum_{0\leq a<b\leq 3} \rho_\sigma^{ab}(\hat{s})[\det J_{ab}^\sigma](\hat{s})\right| d\hat{s}$$

$$\rho_\sigma(\hat{s}) = \sum_{0\leq a<b\leq 3} \rho_\sigma^{ab}(\hat{s}) \left[\det \underline{J}_{ab}^\sigma\right](\hat{s})$$

$$|\rho_\sigma|(\hat{s}) = \left|\sum_{0\leq a<b\leq 3} \rho_\sigma^{ab}(\hat{s}) \left[\det \underline{J}_{ab}^\sigma\right](\hat{s})\right|$$

$$\sum_{0\leq a<b\leq 3} \rho_{\hat{\sigma}}^{ab}\left[\det \mathcal{J}_{ab}^{\hat{\sigma}}\right] = \sqrt{[\sigma'\cdot\sigma'][\dot{\sigma}\cdot\dot{\sigma}] - [\sigma'\cdot\dot{\sigma}]^2}$$

$$\sum_{0\leq a<b\leq 3} \hat{\rho}_{\hat{\sigma}}^{ab}\left[\det \hat{J}_{ab}^{\hat{\sigma}}\right] = \sum_{0\leq a<b\leq 3} \hat{\rho}_\sigma^{ab}\left[\det \hat{J}_{ab}^\sigma\right]$$

$$\Lambda: se_0 + \bar{s}e_1 \mapsto \left[\operatorname{sgn}(\bar{s})\sqrt{\bar{s}^2 - s^2}\right] e_1,$$

$$\vec{x}+\vec{v}=s\hat{f}_0+\bar{s}\hat{f}_1+x^2\hat{f}_2+x^3\hat{f}_3$$
$$\longrightarrow_{\bar{\Lambda}} (s,\bar{s},x^2,x^3)\equiv se_0+\bar{s}e_1+x^2e_2+x^3e_3 \longrightarrow_{\Lambda} \left(0,\operatorname{sgn}(\bar{s})\sqrt{\bar{s}^2-s^2},x^2,x^3\right)$$
$$\longrightarrow_{R_3} \left(0,\operatorname{sgn}(\bar{s})\sqrt{x^{2,2}+\bar{s}^2-s^2},0,x^3\right) \longrightarrow_{R_2} - \left(0,\operatorname{sgn}(\bar{s})\sqrt{x^{2,2}+\bar{s}^2-s^2},0,x^3\right)$$
$$\longrightarrow_{R_3^{-1}} - \left(0,\operatorname{sgn}(\bar{s})\sqrt{\bar{s}^2-s^2},x^2,x^3\right) \longrightarrow_{\Lambda^{-1}} - (s,\bar{s},x^2,x^3)$$
$$\longrightarrow_{R_1} (-s,-\bar{s},x^2,x^3) \longrightarrow_{\bar{\Lambda}^{-1}} - s\hat{f}_0-\bar{s}\hat{f}_1+x^2\hat{f}_2+x^3\hat{f}_3.$$
$$\tilde{\Lambda}(\vec{x},\vec{v}):=\bar{\Lambda}^{-1}R_1\Lambda^{-1}R_3^{-1}R_2R_3\Lambda\bar{\Lambda},$$
$$v\cdot u=\sinh(\phi-\theta)=\begin{pmatrix}\cosh(\phi-\theta)\\ \sinh(\phi-\theta)\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$
$$v\cdot u=-\cosh(\phi-\theta)=\begin{pmatrix}\cosh(\phi-\theta)\\ \sinh(\phi-\theta)\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}$$
$$\Lambda(\theta)=\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}$$
$$\begin{aligned}\Lambda(\theta)\begin{pmatrix}0\\1\end{pmatrix}\cdot\Lambda(\phi)\begin{pmatrix}1\\0\end{pmatrix}&=\begin{pmatrix}\sinh\theta\\ \cosh\theta\end{pmatrix}\cdot\begin{pmatrix}\cosh\phi\\ \sinh\phi\end{pmatrix}\\ &=-\sinh\theta\cosh\phi+\cosh\theta\sinh\phi=\sinh(\phi-\theta)\\ &=\begin{pmatrix}\cosh(\phi-\theta)\\ \sinh(\phi-\theta)\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$
$$\Lambda(\theta)\begin{pmatrix}1\\0\end{pmatrix}\cdot\Lambda(\phi)\begin{pmatrix}1\\0\end{pmatrix}=-\cosh(\theta-\phi)=\begin{pmatrix}\cosh(\phi-\theta)\\ \sinh(\phi-\theta)\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}.$$

6. Las hiperpartículas. En relación a este tipo de partículas, se refiere a aquellas partículas subatómicas las mismas que, alcanzar, igual o superan la velocidad de la luz (energía cinética), con o sin masa, supermasiva o no, de cuyo resultado, se distorsiona el espacio – tiempo. El efecto cuántico gravitacional inherente a esta especie de partículas, es híbrido, es decir, deriva de la permeabilidad del campo de gauge gravitónico, esto es, por interacción con el gravitón, o en su defecto, deriva de su propio movimiento.

Las ecuaciones de movimiento son las siguientes:

$$S\left(\mathcal{P}^{(n)}_{|0\rangle,t},\mathcal{P}^{(n)}_{|1\rangle,t}\right)=\frac{2\left|\langle X\rangle_{\mathcal{P}^{(n)}_{|0\rangle,t}}-\langle X\rangle_{\mathcal{P}^{(n)}_{|1\rangle,t}}\right|}{\Delta X_{\mathcal{P}^{(n)}_{|0\rangle,t}}+\Delta X_{\mathcal{P}^{(n)}_{|1\rangle,t}}}$$
$$S\left(\mathcal{P}^{(1)}_{|0\rangle,t},\mathcal{P}^{(1)}_{|1\rangle,t}\right)=\frac{2\sqrt{t}|\mu_0-\mu_1|}{\sqrt{\mu_0}+\sqrt{\mu_1}}$$
$$\mathcal{P}^{(N)}_{|j\rangle,t}(k)=\left(L^{*N}_{\mu_j t}\right)(k)=L_{N\mu_j t}(k)=\mathcal{P}^{(1)}_{|j\rangle,Nt}(k)$$
$$t_S^{(1)}/t_S^{(N)}=N$$
$$S\left(\mathcal{P}^{(N)}_{|0\rangle,t},\mathcal{P}^{(N)}_{|1\rangle,t}\right)=\sqrt{N}S\left(\mathcal{P}^{(1)}_{|0\rangle,t},\mathcal{P}^{(1)}_{|1\rangle,t}\right)\sim t\sqrt{N}$$
$$\widetilde{C_p}(|00\rangle\langle 00|)=|00\rangle\langle 00|$$
$$\widetilde{C_p}(|10\rangle\langle 10|)=p|00\rangle\langle 00|+(1-p)|11\rangle\langle 11|$$
$$\mathcal{P}^{(N)}_{|0\rangle,t}(k)=L_{N\mu_0 t}(k)$$
$$\begin{aligned}\mathcal{P}^{(N)}_{|1\rangle,t}(k)&=\sum_{q=0}^{N}\mathcal{T}^{(N)}_p(q)\left(L^{*q}_{\mu_1 t}*L^{*N-q}_{\mu_0 t}\right)(k)\\ &=\sum_{q=0}^{N}\mathcal{T}^{(N)}_p(q)L_{(q\mu_1+(N-q)\mu_0)t}(k)\end{aligned}$$

$$\mathcal{W}_{\mu_1,\mu_0,\lambda,t}(k) = e^{-\lambda t} e^{-\mu_1 t} \frac{(\mu_1 t)^k}{k!}$$

$$+ \int_0^t dt' \lambda e^{-\lambda t'} e^{-(\mu_1 t' + \mu_0 (t-t'))} \frac{\left(\mu_1 t' + \mu_0 (t-t')\right)^k}{k!}$$

$$\mathcal{P}^{(N)}_{|0\rangle,t}(k) = L_{N\mu_0 t}(k)$$

$$\mathcal{P}^{(N)}_{|1\rangle,t}(k) = \sum_{q=0}^{N} \mathcal{T}^{(N)}_p(q) \left(\mathcal{W}^{*q}_{\mu_1,\mu_0,\lambda,t} * L_{(N-q)\mu_0 t}\right)(k)$$

$$\mathcal{S}^{(N)}\left(P_{|0\rangle,t}, P_{|1\rangle,t}\right) = \frac{2\left|\langle K\rangle_{P_{|1\rangle,t}} - \langle K\rangle_{P_{|0\rangle,t}}\right|}{\sqrt{\Delta K^2_{\mathcal{P}^{(N)}_{|0\rangle,t}} + } + \sqrt{\Delta K^2_{\mathcal{P}^{(N)}_{|1\rangle,t}}}}$$

$$\times \left|\langle Q\rangle_{\mathcal{T}^{(N)}_{|0\rangle}} + \langle Q\rangle_{\mathcal{T}^{(N)}_{|1\rangle}} - N\right|$$

$$\Delta K^2_{\mathcal{P}^{(N)}_{|j\rangle,t}} = \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}} \Delta K^2_{P_{|j\rangle,t}}$$

$$+ \left(N - \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\right) \Delta K^2_{P_{|\bar{j}\rangle,t}}$$

$$+ \left(\langle K\rangle_{P_{|j\rangle,t}} - \langle K\rangle_{P_{|\bar{j}\rangle,t}}\right)^2 \Delta Q^2_{\mathcal{T}^{(N)}_{|j\rangle}}$$

$$\mathcal{T}^{(N)}_{curve,p}(q) = (1-p)^q p\theta(0 \le q \le N-2) + (1-p)^{N-1}\theta(q = N)$$

$$\mathcal{T}^{(N=2\tilde{N})}_{\text{curve},p}(q) = (1-p)\left(\mathcal{T}^{(\tilde{N})}_{\text{flat},p} * \mathcal{T}^{(\tilde{N})}_{\text{flat},p}\right)(q) + p\theta(q=0)$$

$$= (1-p)^{q+1}p^2((q+1)\theta(0 \le q \le \tilde{N}-2) + (2\tilde{N}-3-q)\theta(\tilde{N}-1 \le q \le 2(\tilde{N}-2)))$$

$$+2(1-p)^q p\theta(\tilde{N} \le q \le 2\tilde{N}-2) + (1-p)^{2\tilde{N}-1}\theta(q = 2\tilde{N}) + p\theta(q=0)$$

$$\mathcal{T}^{(N=2\tilde{N}+1)}_{\text{curve},p}(q) = (1-p)\left(\mathcal{T}^{(\tilde{N}+1)}_{\text{flat},p} * \mathcal{T}^{(\tilde{N})}_{\text{flat},p}\right)(q) + p\theta(q=0)$$

$$= (1-p)^{q+1}p^2((q+1)\theta(0 \le q \le \tilde{N}-2) + (2\tilde{N}-3-q)\theta(\tilde{N}-1 \le q \le 2(\tilde{N}-2))$$

$$+\theta(\tilde{N}-1 \le q \le 2\tilde{N}-3)) + (1-p)^q p(2\theta(\tilde{N}+1 \le q \le 2\tilde{N}-1) + \theta(q=\tilde{N}))$$

$$+(1-p)^{2\tilde{N}}\theta(q = 2\tilde{N}+1) + p\theta(q=0)$$

$$\mathcal{P}^{(N)}_{|j\rangle,t}(k) = \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q) \left(P^{*q}_{|j\rangle,t} * P^{*(N-q)}_{|\bar{j}\rangle,t}\right)(k)$$

$$\langle K\rangle_{\mathcal{P}^{(N)}_{|j\rangle,t}} = \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q) \langle K\rangle_{\mathcal{L}^{(N)}_{|j\rangle,q,t}}$$

$$= \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q) \left(q\langle K\rangle_{P_{|j\rangle,t}} + (N-q)\langle K\rangle_{P_{|\bar{j}\rangle,t}}\right)$$

$$= \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}} \langle K\rangle_{P_{|j\rangle,t}} + \left(N - \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\right)\langle K\rangle_{P_{|\bar{j}\rangle,t}}$$

$$\Delta K^2_{\mathcal{P}^{(N)}_{(j),t}} = \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\langle K^2\rangle_{\mathcal{L}^{(N)}_{|j\rangle,q,t}} - \left(\sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\langle K\rangle_{\mathcal{L}^{(N)}_{|j\rangle,q,t}}\right)^2$$

$$= \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\left(\langle K^2\rangle_{\mathcal{L}^{(N)}_{|j\rangle,q,t}} - \langle K\rangle^2_{\mathcal{L}^{(N)}_{|j\rangle,q,t}}\right) + \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\langle K\rangle^2_{\mathcal{L}^{(N)}_{|j\rangle,q,t}} - \left(\sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\langle K\rangle_{\mathcal{L}^{(N)}_{|j\rangle,q,t}}\right)^2$$

$$= \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\Delta K^2_{\mathcal{L}^{(N)}_{|j\rangle,q,t}} + \sum_{q=0}^{N} \mathcal{T}^{(N)}_{|j\rangle}(q)\left(q\langle K\rangle_{P_{|j\rangle,t}} + (N-q)\langle K\rangle_{P_{|j\rangle,t}}\right)^2$$

$$-\left[\langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\langle K\rangle_{P_{|j\rangle,t}} + \left(N - \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\right)\langle K\rangle_{P_{|j\rangle,t}}\right]^2$$

$$\Delta K^2_{\mathcal{P}^{(N)}_{|j\rangle,t}} = \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\Delta K^2_{P_{|j\rangle,t}} + \left(N - \langle Q\rangle_{\mathcal{T}^{(N)}_{|j\rangle}}\right)\Delta K^2_{P_{|\bar{j}\rangle,t}} + \left(\langle K\rangle_{P_{|j\rangle,t}} - \langle K\rangle_{P_{|\bar{j}\rangle,t}}\right)^2\Delta Q^2_{\mathcal{T}^{(N)}_{|j\rangle}}$$

$$\mathcal{S}^{(N)}\left(P_{|0\rangle,t}, P_{|1\rangle,t}\right) = \frac{2\left|\langle K\rangle_{P_{|1\rangle,t}} - \langle K\rangle_{P_{|0\rangle,t}}\right|}{\sqrt{\Delta K^2_{\mathcal{P}^{(N)}_{|0\rangle,t}}} + \sqrt{\Delta K^2_{\mathcal{P}^{(N)}_{|1\rangle,t}}}}\left|\langle Q\rangle_{\mathcal{T}^{(N)}_{|0\rangle}} + \langle Q\rangle_{\mathcal{T}^{(N)}_{|1\rangle}} - N\right|$$

$$\mathcal{F}_\eta\left(P_{|0\rangle}, P_{|1\rangle}\right) = \frac{\epsilon_{|0\rangle} + \epsilon_{|1\rangle}}{2} = \frac{P_{|0\rangle}(x \geq \eta) + P_{|1\rangle}(x < \eta)}{2}$$

$$e^{-\mu_0 Nt}\frac{(\mu_0 Nt)^{k_\eta(t)}}{k_\eta(t)!} \leq \epsilon_{|0\rangle} \leq e^{-\mu_0 Nt}\frac{(\mu_0 Nt)^{k_\eta(t)}}{k_\eta(t)!}\frac{k_\eta(t)}{k_\eta(t) - \mu_0 Nt}$$

$$e^{-\mu_1 Nt}\frac{(\mu_1 Nt)^{k_\eta(t)-1}}{\left(k_\eta(t)-1\right)!} \leq \epsilon_{|1\rangle} \leq e^{-\mu_1 Nt}\frac{(\mu_1 Nt)^{k_\eta(t)-1}}{\left(k_\eta(t)-1\right)!}\frac{\mu_1 Nt}{\mu_1 Nt - \left(k_\eta(t)-1\right)}$$

$$L_{\mu_0 Nt}(k \geq \eta(t)) \approx e^{-\mu_0 Nt}\frac{(\mu_0 Nt)^{k_\eta(t)}}{k_\eta(t)!} \approx e^{-\mu_0 Nt\gamma_0},$$

$$L_{\mu_1 Nt}(k < \eta(t)) \approx e^{-\mu_1 Nt}\frac{(\mu_1 Nt)^{k_\eta(t)}}{\left(k_\eta(t)\right)!} \approx e^{-\mu_1 Nt\gamma_1},$$

$$\mathcal{P}^{(1)}_{|j\rangle,t}(\theta) = G_{z_j(t),\sqrt{|z_j(t)|}}(\theta)$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \kappa\bar{\psi}\gamma^0\gamma^5\psi + \frac{(\bar{\psi}\psi)^2}{M^2}.$$

$$\mathcal{L} = -\frac{1}{4}M^2\Delta^2 + \int\frac{d^4k}{(2\pi)^4}\ln\,\det\mathcal{G}^{-1}$$

$$\mathcal{L} = -\frac{1}{4}M^2(\Delta + m)^2 + \int\frac{d^4k}{(2\pi)^4}\ln\,U_+U_-$$

$$V_{\text{eff}}(\Delta) = \frac{1}{4}M^2(\Delta + m)^2 + \frac{\Delta^4}{32\pi^2}(1 + 4\ln\,\Delta/\Lambda_{UV})$$

$$-\frac{\kappa^2\Delta^2}{4\pi^2}(1 - 2\ln\,\Delta/\Lambda_{UV}) + V_0$$

$$\Delta_0 = \Lambda_{UV}\exp\left(-\pi^2M^2/2\kappa^2\right)$$

$$\mathcal{L}_1(\Delta, T) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\ln\,\beta^4U_{n+}U_{n-}$$

$$V_{\text{eff}}(\Delta, T) = V_{\text{eff}}(T=0) - \frac{2}{\pi^2\beta^4}\left[I_2(\beta\Delta) + \kappa^2\beta^2I_0(\beta\Delta)\right]$$

$$I_n(x) = \int_0^\infty ds s^n\ln\left[1 + e^{-\sqrt{s^2+x^2}}\right]$$

$$(1+z)^3 = \frac{w_0(1+w_\Delta)}{w_\Delta(1+w_0)}e^{\frac{1}{w_0}-\frac{1}{w_\Delta}}$$

$$V(\Delta) \simeq V(\Delta_0) + \frac{1}{2}V''(\Delta_0)(\Delta - \Delta_0)^2 = \frac{1}{2}(\kappa\delta\Delta/\pi)^2$$
$$\mathcal{M} = -\frac{1}{\beta}\sum_{i\omega_n}\int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}[i\mathcal{G}(p)i\mathcal{G}(p+k)]$$
$$\xi^{00} = \left.\frac{d\mathcal{M}}{dk_0^2}\right|_{k=0} \quad \xi^{33} = \left.\frac{d\mathcal{M}}{d|k|^2}\right|_{k=0}$$
$$n_A = -\frac{\partial V(\Delta_0, T)}{\partial \kappa} = \frac{\kappa\Delta_0^2}{2\pi^2}\left(1 + \frac{\pi^2 M^2}{\kappa^2}\right) + \frac{\kappa T^2}{3}$$
$$n_\Delta = \left.\frac{\partial V(\Delta, T)}{\partial \Delta}\right|_{\Delta_0} = \frac{1}{2}M^2\Delta_0.$$
$$S_E = \frac{27\pi^2\left[\int_0^{\Delta_m} d\Delta\sqrt{2V(\Delta)}\right]^4}{2\big(V(0) - V(\Delta_m)\big)^3} \simeq 2.8 \times 10^7$$

CONCLUSIONES.

En mérito a lo anterior, se concluye que la teoría hipotética contenida en este manuscrito, reúne los requisitos de rigor científico.

Los postulados que instituyen la presente teoría son:

La existencia hipotética del gravitón y su partícula compañera.

La existencia hipotética de un campo de gauge gravitónico, el cual, permea el espacio – tiempo cuántico, dotándolo de gravedad, esto, a través de las interacciones del gravitón, ésta última, entendiéndose como partícula fundamental.

La existencia hipotética de agujeros negros de mecánica cuántica, formados por el colapso de aquellas partículas, entendidas como supermasivas.

La distorsión del espacio – tiempo cuántico, deriva del efecto cuántico gravitacional endógeno o exógeno, en relación a la partícula entendida como supermasiva.

La brecha de masa de una partícula, respecto del estado de vacío, superior a cero, comporta energía de potencial y en consecuencia, supone la distorsión del campo de calibre o de gauge en el que interactúa.

Las hiperpartículas, son partículas subatómicas hipotéticas, las cuales y sin perjuicio de que estén dotadas o no de masa, la energía cinética de éstas, es decir, el movimiento acelerado que le permita alcanzar, igual o superar la velocidad de la luz, verbigracia, el taquión, causa la deformación del espacio – tiempo cuántico.

La teoría cuántica de campos relativistas, permite el acoplamiento de las ecuaciones de campo de Einstein a campos de gauge indiscriminados.

Las partículas distorsionadas, es decir, aquellas que se afectan por el efecto cuántico gravitacional de la superpartícula, no se desplazan en trayectorias orbitales de carácter cosmológicas, al contrario, éstas, se difuminan en torno a la partícula deformante, siguiendo la tipología de la distorsión, esto es, los pliegues y el diámetro de la curvatura.

El desarrollo matemático contenido en los manuscritos que preceden, aplican por acoplamiento a esta teoría.

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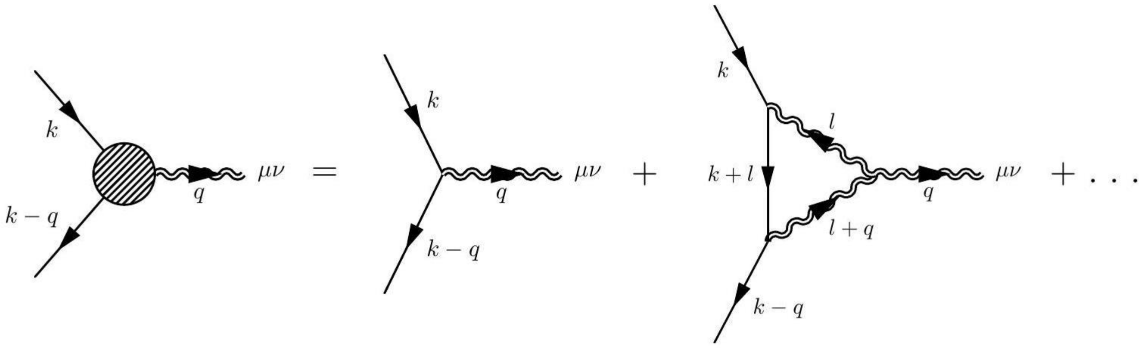
APÉNDICE A.

Espacios de gauge, reglas de Feynman y métrica de Schwarzschild-Tangherlini para campos cuánticos relativistas o curvos. Fenómenos gravitacionales y sistemas de referencia bajo Relatividad General, a propósito de la interacción y permeo del campo gravitónico y el gravitón en espacios cuánticos deformados, tanto bosónicos como fermiónicos y en relación a las partículas y antipartículas supermasivas propiamente dichas y las hiperpartículas.



$$[G_N] = [\text{mass}]^{-(D-2)}.$$

$$\kappa^2 = 32\pi G_N$$

$$R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\mu\nu\rho,\sigma} - \Gamma_{\beta\mu\rho}\Gamma^{\beta}_{\nu\sigma} + \Gamma_{\beta\mu\sigma}\Gamma^{\beta}_{\nu\rho}$$

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right)$$

$$R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\nu\sigma}$$
$$R = g^{\mu\nu}R_{\mu\nu}$$

$$S_{EH} = \int d^D x\sqrt{-g}R$$

$$S_{\phi} = \int d^D x\sqrt{-g}\mathcal{L}_{\phi}$$

$$\mathcal{L}_{\phi} = \frac{1}{2}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)$$

$$S_{(c)} = \frac{2}{\kappa^2}S_{EH} + S_{\phi}$$

$$G^{\mu\nu} = -\frac{\kappa^2}{4}T^{\mu\nu}$$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}$$

$$\delta S_{EH} = -\int d^D x\sqrt{-g}G^{\mu\nu}\delta g_{\mu\nu}$$

$$D_\mu G^{\mu\nu} = 0$$

$$\delta S_\phi = -\frac{1}{2}\int d^D x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}$$

$$d\tau^2 = \left(1 - \frac{\mu}{r^n}\right)dt^2 - \frac{1}{1 - \frac{\mu}{r^n}}dr^2 - r^2 d\Omega_{D-2}^2$$

$$\mu = \frac{16\pi G_N M}{(D-2)\Omega_{D-2}}$$

$$\Omega_d = \frac{2\sqrt{\pi}^{d+1}}{\Gamma((d+1)/2)}$$

$$S_c = \int d^D x\sqrt{-g}\left(\frac{2R}{\kappa^2} + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)\right)$$

$$Z_\omega = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\det\left(\frac{\delta G}{\delta\epsilon}\right)\delta(G_\sigma - \omega_\sigma)e^{iS_c}$$

$$\begin{aligned} Z &= \int \mathcal{D}\omega_\sigma Z_\omega \exp\left(i\int d^D x\frac{1}{\kappa^2\xi}\eta^{\sigma\rho}\omega_\sigma\omega_\rho\right) \\ &= \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}e^{iS_{(\mathrm{c})} + i\frac{2}{\kappa^2}S_{(\mathrm{gf})} + iS_{(\mathrm{gh})}} \end{aligned}$$

$$S_{gf} = \frac{1}{2\xi}\int d^D x\eta^{\sigma\rho}G_\sigma G_\rho$$

$$\begin{aligned} S &= S_{(\mathrm{c})} + \frac{2}{\kappa^2}S_{(\mathrm{gf})} \\ &= \frac{2}{\kappa^2}\left(S_{\mathrm{EH}} + S_{(\mathrm{gf})}\right) + S_\phi \end{aligned}$$

$$\begin{aligned} \tilde{\phi}(q) &= \int d^D x e^{iqx}\phi, \\ \tilde{h}(q) &= \int d^D x e^{iqx}\hbar. \end{aligned}$$

$$\begin{aligned} \kappa^2\int d^D x\eta^{\mu\alpha}\eta^{\nu\beta}h_{\mu\nu}h_{\alpha\beta}\eta^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} &= -\kappa^2\int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D p}{(2\pi)^D}\frac{d^D q}{(2\pi)^D} \\ \times (2\pi)^D\delta^D\left(l_{(1)} + l_{(2)} + p + q\right)&\eta^{\mu\alpha}\eta^{\nu\beta}\tilde{h}_{\mu\nu}(l_1)\tilde{h}_{\alpha\beta}(l_2)\tilde{\phi}(p)\tilde{\phi}(q)\eta^{\rho\sigma}p_\sigma q_\rho \end{aligned}$$

$$V^{\mu\nu}\alpha\beta = -i4\kappa^2\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha})p_\sigma q^\sigma$$

$$p^\mu = \hbar l^\mu$$

$$h_{\mu\nu}(x) = \int \frac{d^D q}{(2\pi)^D} e^{-iqx}\tilde{h}_{\mu\nu}(q)$$
$$\tilde{h}_{\mu\nu}(q) = \int d^D x e^{iqx} h_{\mu\nu}(x)$$

$$\eta^{\parallel}_{\mu\nu} = \frac{k_\mu k_\nu}{m^2}$$
$$\eta^{\perp}_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}.$$

$$q^\mu_\parallel = \eta_\parallel{}^\mu{}_\nu q^\nu$$
$$q^\mu_\perp = \eta_\perp{}^\mu{}_\nu q^\nu$$

$$q_\parallel = \sqrt{q^\mu_\parallel q^\parallel_\mu}$$

$$\hat{g}_{\mu\nu}(\hat{x}) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu}$$

$$x^\mu = \hat{x}^\mu + \hat{\epsilon}^\mu(\hat{x})$$

$$\hat{x}^\mu = x^\mu + \epsilon^\mu(x)$$

$$\epsilon(x) + \hat{\epsilon}(\hat{x}) = 0$$

$$\frac{\partial x^\alpha}{\partial \hat{x}^\mu} = \delta^\alpha_\mu + \frac{\partial \hat{\epsilon}^\alpha(\hat{x})}{\partial \hat{x}^\mu}$$

$$\begin{aligned}\hat{g}_{\mu\nu}(\hat{x}) &= g_{\mu\nu}(x) + g_{\alpha\nu}(x)\frac{\hat{\partial}\hat{\epsilon}^\alpha(\hat{x})}{\hat{\partial}\hat{x}^\mu} + g_{\mu\beta}(x)\frac{\hat{\partial}\hat{\epsilon}^\beta(\hat{x})}{\hat{\partial}\hat{x}^\nu} + g_{\alpha\beta}(x)\frac{\hat{\partial}\hat{\epsilon}^\alpha(\hat{x})}{\hat{\partial}\hat{x}^\mu}\frac{\hat{\partial}\hat{\epsilon}^\beta(\hat{x})}{\hat{\partial}\hat{x}^\nu}\\ &= g_{\alpha\beta}(x)\left(\delta^\alpha_\mu\delta^\beta_\nu + \frac{\hat{\partial}\hat{\epsilon}^\alpha(\hat{x})}{\hat{\partial}\hat{x}^\mu}\delta^\beta_\nu + \delta^\alpha_\mu\frac{\hat{\partial}\hat{\epsilon}^\beta(\hat{x})}{\hat{\partial}\hat{x}^\nu} + \frac{\hat{\partial}\hat{\epsilon}^\alpha(\hat{x})}{\hat{\partial}\hat{x}^\mu}\frac{\hat{\partial}\hat{\epsilon}^\beta(\hat{x})}{\hat{\partial}\hat{x}^\nu}\right)\end{aligned}$$

$$\begin{aligned}g_{\mu\nu}(x) &= g_{\mu\nu}(\hat{x} + \hat{\epsilon}(\hat{x}))\\ &= \sum_{n=0..\infty}\frac{1}{n!}\hat{\epsilon}^\sigma(\hat{x}) \ldots \hat{\partial}_\sigma \ldots g_{\mu\nu}(\hat{x})\\ &= g_{\mu\nu}(\hat{x}) + \hat{\epsilon}^\sigma(\hat{x})\hat{\partial}_\sigma g_{\mu\nu}(\hat{x}) + \frac{1}{2}\hat{\epsilon}^\sigma(\hat{x})\hat{\epsilon}^\rho(\hat{x})\hat{\partial}_\sigma\hat{\partial}_\rho g_{\mu\nu}(\hat{x}) + \cdots\\ &= \sum_{n=0..\infty}\frac{1}{n!}\left(\hat{\epsilon}^\sigma(\hat{x})\hat{\partial}^{(g)}_\sigma\right)^n g_{\mu\nu}(\hat{x})\end{aligned}$$

$$\hat{g}_{\mu\nu} = \left(\delta^\alpha_\mu\delta^\beta_\nu + \partial_\mu\hat{\epsilon}^\alpha\delta^\beta_\nu + \delta^\alpha_\mu\partial_\nu\hat{\epsilon}^\beta + \partial_\mu\hat{\epsilon}^\alpha\partial_\nu\hat{\epsilon}^\beta\right)\sum_{n=0..\infty}\frac{1}{n!}\left(\hat{\epsilon}^\sigma\partial^{(g)}_\sigma\right)^n g_{\alpha\beta}$$

$$\hat{h}_{\mu\nu} = \partial_\mu \hat{\epsilon}_\nu + \partial_\nu \hat{\epsilon}_\mu + \partial_\mu \hat{\epsilon}_\alpha \partial_\nu \hat{\epsilon}^\alpha + \left(\delta^\alpha_\mu \delta^\beta_\nu + \partial_\mu \hat{\epsilon}^\alpha \delta^\beta_\nu + \delta^\alpha_\mu \partial_\nu \hat{\epsilon}^\beta + \partial_\mu \hat{\epsilon}^\alpha \partial_\nu \hat{\epsilon}^\beta\right) \sum_{n=0..\infty} \frac{1}{n!} \left(\hat{\epsilon}^\sigma \partial^{(g)}_\sigma\right)^n h_{\alpha\beta}$$

$$\hat{h}_{\mu\nu} \approx h_{\mu\nu} + \partial_\nu \hat{\epsilon}_\mu + \partial_\mu \hat{\epsilon}_\nu$$

$$S = \int d^D x \sqrt{-g} \left(\frac{2R}{\kappa^2} + \mathcal{L}_\phi\right) + \int d^D x \frac{1}{\kappa^2 \xi} \eta^{\sigma\rho} G_\sigma G_\rho$$

$$g^{\mu\nu} \Gamma^\sigma_{\mu\nu} = 0$$

$$\partial_\mu \left(h^\mu_\sigma - \frac{1}{2} \eta^\mu_\sigma h^\nu_\nu\right) = 0$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$g_{\mu\nu} = e^{\pi_{\mu\nu}}$$

$$g^{\mu\nu} = e^{-\pi^{\mu\nu}}$$

$$\sqrt{-g} g_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$$

$$G_\sigma = (1-\alpha)\partial_\mu \left(h^\mu_\sigma - \frac{1}{2} \eta^\mu_\sigma h^\nu_\nu\right) + \alpha g^{\mu\nu} \Gamma_{\sigma\mu\nu}$$

$$\eta^{\mu\nu} \partial_\mu g_{\sigma\nu} - \frac{1}{2} \partial_\sigma \eta^{\mu\nu} g_{\mu\nu} = 0$$

$$I^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha\right)$$

$$\mathcal{P}^{\mu\nu}_{\alpha\beta} = I^{\mu\nu}_{\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta_{\alpha\beta}$$

$$R = \sum_{n=1..\infty} R_{h^n}$$

$$R = \sum_{n=1..\infty} R_{G^n}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu_\rho h^{\rho\nu} - h^\mu_\rho h^\rho_\sigma h^{\sigma\nu} + \cdots$$

$$\frac{1}{1+x} = \sum_{n=0..\infty} (-x)^n = 1 - x + x^2 - x^3 + \cdots$$

$$\hat{h}^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}.$$

$$\hat{h}_{\mu\nu} = -h_{\mu\nu} - \hat{h}^\sigma_\mu h_{\sigma\nu}$$

$$(g^{\mu\nu})_{h^3} = -h^\mu_\rho h^\rho_\sigma h^{\sigma\nu}$$

$$\sqrt{-g} = \exp\left(\frac{1}{2}\mathrm{trln}\left(\eta^{\mu}_{\nu}+h^{\mu}_{\nu}\right)\right)$$
$$= \exp\left(\frac{1}{2}\left(h^{\mu}_{\mu}-\frac{1}{2}h^{\mu}_{\nu}h^{\nu}_{\mu}+\frac{1}{3}h^{\mu}_{\nu}h^{\nu}_{\rho}h^{\rho}_{\mu}-\frac{1}{4}h^{\mu}_{\nu}h^{\nu}_{\rho}h^{\rho}_{\sigma}h^{\sigma}_{\mu}+\cdots\right)\right)$$
$$= 1+\frac{1}{2}h-\frac{1}{4}\mathcal{P}^{\rho\sigma}_{\mu\nu}h^{\mu\nu}h_{\rho\sigma}+\cdots$$

$$\sqrt{-g}_{h^2} = -\frac{1}{4}\mathcal{P}^{\rho\sigma}_{\mu\nu}h^{\mu\nu}h_{\rho\sigma}$$

$$\sqrt{-g}_{h^2}\left(h^{(a)}_{\mu\nu},h^{(b)}_{\mu\nu}\right) = -\frac{1}{4}\mathcal{P}^{\rho\sigma}_{\mu\nu}h^{(a)\mu\nu}h^{(b)}_{\rho\sigma}.$$

$$G^{\mu\nu} = \sum_{n=1..\infty} G^{\mu\nu}_{h^n}$$

$$G^{\mu\nu}_{h}\left(h_{\mu\nu}\right) = G^{\mu\nu}_{h}\left(\sum_{n=0..\infty} h^{G^n}_{\mu\nu}\right)$$
$$= \sum_{n=1..\infty} G^{\mu\nu}_{h}\left(h^{G^n}_{\mu\nu}\right)$$

$$G^{\mu\nu}_{h^2}\left(h_{\mu\nu},h_{\mu\nu}\right) = G^{\mu\nu}_{h^2}\left(\sum_{n=0..\infty} h^{G^n}_{\mu\nu},\sum_{m=0..\infty} h^{G^m}_{\mu\nu}\right)$$
$$= \sum_{n=1..\infty}\sum_{m=1..\infty} G^{\mu\nu}_{h^2}\left(h^{G^n}_{\mu\nu},h^{G^m}_{\mu\nu}\right)$$
$$= G^{\mu\nu}_{h^2}\left(h^{G}_{\mu\nu},h^{G}_{\mu\nu}\right)+2G^{\mu\nu}_{h^2}\left(h^{G}_{\mu\nu},h^{G^2}_{\mu\nu}\right)+\cdots$$

$$(x_1+x_2+\cdots+x_n)^n$$

$$G^{\mu\nu} \approx G^{\mu\nu}_{h}\left(h^{G}_{\mu\nu}\right)$$
$$+G^{\mu\nu}_{h}\left(h^{G^2}_{\mu\nu}\right)+G^{\mu\nu}_{h^2}\left(h^{G}_{\mu\nu},h^{G}_{\mu\nu}\right)$$
$$+G^{\mu\nu}_{h}\left(h^{G^3}_{\mu\nu}\right)+2G^{\mu\nu}_{h^2}\left(h^{G}_{\mu\nu},h^{G^2}_{\mu\nu}\right)+G^{\mu\nu}_{h^3}\left(h^{G}_{\mu\nu},h^{G}_{\mu\nu},h^{G}_{\mu\nu}\right)$$

$$S_{\mathrm{EH}}+S_{(\mathrm{gf})} = \int d^D x\left(\sqrt{-g}R+\frac{1}{2\xi}\eta^{\rho\sigma}G_{\rho}G_{\sigma}\right)$$

$$S_{EH} = \int d^D x\sqrt{-g}R$$
$$= \int d^D x\sqrt{-g}g^{\mu\nu}\left(\Gamma^{\rho}_{\rho\mu,\nu}-\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}+\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\right)$$
$$= \int d^D x\sqrt{-g}g^{\mu\nu}\left(\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\right)$$

$$S_{EH} = \int d^D x\sqrt{-g}\frac{1}{4}\left(2g^{\sigma\gamma}g^{\rho\delta}g^{\alpha\beta}-g^{\gamma\delta}g^{\alpha\beta}g^{\rho\sigma}-2g^{\sigma\alpha}g^{\gamma\rho}g^{\delta\beta}+g^{\rho\sigma}g^{\alpha\gamma}g^{\beta\delta}\right)g_{\alpha\beta,\rho}g_{\gamma\delta,\sigma}$$

$$G_{\sigma} \approx \mathcal{P}^{\mu\nu}_{\rho\sigma}h^{\rho}_{\mu\nu}-\alpha\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h^{\mu\nu}h_{\alpha\beta,\rho}$$

$$\Gamma_{\rho\mu\nu} = \Gamma_{\rho\mu\nu}^{\sigma\alpha\beta} g_{\alpha\beta,\sigma}$$

$$\Gamma_{\sigma\mu\nu}^{\rho\alpha\beta} = I_{\sigma\kappa}^{\alpha\beta} I_{\mu\nu}^{\rho\kappa} - \frac{1}{2} I_{\mu\nu}^{\alpha\beta} \delta_{\sigma}^{\rho}$$

$$G_{\sigma} G^{\sigma} \approx h_{\mu\nu}^{,\rho} \mathcal{P}_{\rho\kappa}^{\mu\nu} \mathcal{P}_{\alpha\beta}^{\kappa\sigma} h_{,\sigma}^{\alpha\beta} - 2\alpha \mathcal{P}_{\gamma\delta}^{\rho\kappa} h_{,\rho}^{\gamma\delta} \Gamma_{\kappa\mu\nu}^{\sigma\alpha\beta} h^{\mu\nu} h_{\alpha\beta,\sigma}$$

$$\begin{aligned}(S_{\mathrm{EH}})_{h^2} &= \int d^D x \frac{1}{4} (2\eta^{\sigma\gamma}\eta^{\rho\delta}\eta^{\alpha\beta} - \eta^{\gamma\delta}\eta^{\alpha\beta}\eta^{\rho\sigma} - 2\eta^{\sigma\alpha}\eta^{\gamma\rho}\eta^{\delta\beta} + \eta^{\rho\sigma}\eta^{\alpha\gamma}\eta^{\beta\delta}) h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} \\ &= \int d^D x \frac{1}{4} (2\eta^{\sigma\gamma}\eta^{\rho\delta}\eta^{\alpha\beta} - \eta^{\gamma\delta}\eta^{\alpha\beta}\eta^{\rho\sigma} - 2\eta^{\sigma\alpha}\eta^{\gamma\rho}\eta^{\delta\beta} + \eta^{\rho\sigma}\eta^{\alpha\gamma}\eta^{\beta\delta}) h_{\alpha\beta,\sigma} h_{\gamma\delta,\rho}\end{aligned}$$

$$2\mathcal{P}_{\kappa\rho}^{\alpha\beta} \mathcal{P}_{\gamma\delta}^{\kappa\sigma} h_{\alpha\beta}^{\rho} h_{,\sigma}^{\gamma\delta} = \left(2\eta^{\sigma\alpha}\eta^{\gamma\rho}\eta^{\delta\beta} - 2\eta^{\sigma\gamma}\eta^{\rho\delta}\eta^{\alpha\beta} + \frac{1}{2}\eta^{\gamma\delta}\eta^{\alpha\beta}\eta^{\rho\sigma}\right) h_{\alpha\beta,\sigma} h_{\gamma\delta,\rho}$$

$$(S_{\mathrm{EH}})_{h^2} = \frac{1}{4} \int d^D x h_{\gamma\delta}^{,\rho} \left(\delta_{\sigma}^{\rho} \mathcal{P}_{\alpha\beta}^{\gamma\delta} - 2\mathcal{P}_{\rho\kappa}^{\gamma\delta} \mathcal{P}_{\alpha\beta}^{\sigma\kappa}\right) h_{,\sigma}^{\alpha\beta}$$

$$(S_{\mathrm{EH}})_{h^3} = \frac{1}{2} \int d^D x U_{(\mathrm{c})}^{\mu\nu\ \alpha\beta\rho\ \gamma\delta\sigma} h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma}$$

$$\begin{gathered} U_{(\mathrm{c})}^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} = \\ 2 I_{\phi\epsilon}^{\mu\nu} \mathcal{P}_{\rho\sigma}^{\alpha\beta} \mathcal{P}_{\gamma\delta}^{\sigma\phi} h_{\alpha\beta}^{\epsilon} h^{\gamma\delta,\rho} - \mathcal{P}_{\alpha\beta}^{\mu\rho} \mathcal{P}_{\gamma\delta}^{\nu\sigma} \eta_{\rho\sigma} h_{,\kappa}^{\alpha\beta} h^{\gamma\delta,\kappa} \\ + \mathcal{P}_{\rho\sigma}^{\mu\nu} \left(h_{,\beta}^{\rho\alpha} h_{,\alpha}^{\sigma\beta} - \frac{1}{2} h_{\beta}^{\alpha,\rho} h_{\alpha}^{\beta,\sigma} - h_{,\alpha}^{\rho\sigma} h_{,\beta}^{\alpha\beta}\right)\end{gathered}$$

$$\begin{aligned} U_{(\mathrm{c})}^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} = \frac{1}{2} h_{\nu}^{\mu} h_{,\mu} h^{,\nu} - \frac{1}{4} h h_{,\rho} h^{,\rho} + h_{\nu}^{\mu} h_{\mu,\rho}^{\nu} h^{,\rho} - h_{\nu}^{\mu} h_{\mu}^{\nu,\sigma} h_{\sigma,\rho}^{\rho} + \frac{1}{4} h h_{\nu,\rho}^{\mu} h_{\mu}^{\nu,\rho} \\ - h_{\mu}^{\nu} h_{\sigma,\nu}^{\mu} h^{,\sigma} - h_{\nu}^{\mu} h^{,\nu} h_{\mu,\rho}^{\rho} + \frac{1}{2} h h_{\sigma,\rho}^{\rho} h^{,\sigma} - h_{\nu}^{\mu} h_{\mu,\sigma}^{\rho} h_{\rho}^{\nu,\sigma} - \frac{1}{2} h h_{\nu,\mu}^{\rho} h_{\rho}^{\mu,\nu} \\ + h^{\mu\nu} h_{\mu,\rho}^{\sigma} h_{\nu,\sigma}^{\rho} - \frac{1}{2} h_{\nu}^{\mu} h_{\sigma,\mu}^{\rho} h_{\rho}^{\sigma,\nu} + 2 h_{\nu}^{\mu} h_{\rho}^{\sigma,\nu} h_{\mu,\sigma}^{\rho}\end{aligned}$$

$$\left(S_{(\mathrm{gf})}\right)_{h^3} = \frac{1}{2\xi} \int d^D x U_{(\mathrm{gf})}^{\mu\nu\ \alpha\beta\rho\ \gamma\delta\sigma} h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma}$$

$$\begin{aligned} U_{gf}^{\mu\nu\alpha\beta\rho\gamma\delta\sigma} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma} &= -2\alpha \mathcal{P}_{\rho\sigma}^{\alpha\beta} h_{\alpha\beta}^{,\sigma} \beta_{\kappa\gamma\delta}^{\rho\mu\nu} h^{\gamma\delta,\kappa} \\ &= \alpha \mathcal{P}_{\alpha\beta}^{\rho\sigma} h_{,\sigma}^{\alpha\beta} (-h_{\rho}^{\mu,\nu} - h_{\rho}^{\nu,\mu} + h_{,\rho}^{\mu\nu})\end{aligned}$$

$$U^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma = U_{(\mathrm{c})}^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma + \frac{1}{\xi} U_{(\mathrm{gf})}^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma,$$

$$S_{\mathrm{EH}} + S_{(\mathrm{gf})} \approx \frac{1}{4} \int d^D x h_{\mu\nu}^{\rho,} \left(\delta_{\sigma}^{\rho} \mathcal{P}_{\alpha\beta}^{\mu\nu} - 2\left(1 - \frac{1}{\xi}\right) \mathcal{P}_{\rho\kappa}^{\mu\nu} \mathcal{P}_{\alpha\beta}^{\sigma\kappa}\right) h_{,\sigma}^{\alpha\beta} + \frac{1}{2} \int d^D x U^{\mu\nu} \alpha\beta\rho\gamma\delta\sigma h_{\mu\nu} h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma}$$

$$\begin{aligned} \alpha \mathcal{P}_{\alpha\beta}^{\rho\sigma} h_{,\sigma}^{\alpha\beta} h_{,\rho}^{\mu\nu} &= \alpha \mathcal{P}^{\alpha\beta\rho\sigma} I^{\gamma\delta\mu\nu} h_{\alpha\beta,\rho} h_{\mu\nu,\sigma} \\ &= \frac{1}{2} \alpha (\mathcal{P}^{\alpha\beta\rho\sigma} I^{\gamma\delta\mu\nu} + \mathcal{P}^{\gamma\delta\rho\sigma} I^{\alpha\beta\mu\nu}) h_{\alpha\beta,\rho} h_{\mu\nu,\sigma}\end{aligned}$$

$$\frac{1}{2}\alpha\left(\mathcal{P}^{\alpha\beta\rho\sigma}I^{\gamma\delta\mu\nu}+\mathcal{P}^{\gamma\delta\rho\sigma}I^{\alpha\beta\mu\nu}\right)$$

$$U_{(\mathrm{gf})}^{\mu\nu\ \alpha\beta\rho}\gamma\delta\sigma=-\alpha\left(I^{\mu\nu\rho\kappa}I_{\kappa\lambda}^{\alpha\beta}\mathcal{P}^{\lambda\sigma\gamma\delta}+I^{\mu\nu\sigma\kappa}I_{\kappa\lambda}^{\gamma\delta}\mathcal{P}^{\lambda\rho\alpha\beta}\right)+\frac{1}{2}\alpha\left(\mathcal{P}^{\alpha\beta\rho\sigma}I^{\gamma\delta\mu\nu}+\mathcal{P}^{\gamma\delta\rho\sigma}I^{\alpha\beta\mu\nu}\right)$$

$$S_{\mathrm{EH}}=-\int\ d^{D}xh_{\mu\nu}\sum_{n=1..\infty}\frac{1}{(n+1)}\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)$$

$$S_{(\mathrm{gf})}=-\frac{1}{\xi}\int\ d^{D}xh_{\mu\nu}\sum_{n=1..\infty}\frac{1}{(n+1)}\mathcal{H}_{h^{n}}^{\mu\nu}(h,h,\dots,h)$$

$$\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)$$

$$\int\ d^{D}xh_{\mu\nu}\frac{1}{(n+1)}\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)$$

$$\delta\int\ d^{D}xh_{\mu\nu}\frac{1}{(n+1)}\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)=\int\ d^{D}x\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)\delta h_{\mu\nu}$$

$$\begin{aligned}(S_{\mathrm{EH}})_{h^{3}}&=-\frac{1}{3}\int\ d^{D}xh_{\mu\nu}\mathcal{G}_{h^{2}}^{\mu\nu}(h,h)\\&=\frac{1}{2}\int\ d^{D}xh_{\mu\nu}U_{(\mathrm{c})}^{\mu\nu\ \alpha\beta\rho}\gamma\delta\sigma h_{\alpha\beta,\rho}h_{\gamma\delta,\sigma}\end{aligned}$$

$$=\frac{1}{6}\int\ d^{D}x\left(h_{\mu\nu}U_{(\mathrm{c})}^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma h_{\alpha\beta,\rho}h_{\gamma\delta,\sigma}+h_{\mu\nu,\sigma}U_{(\mathrm{c})}^{\alpha\beta\gamma\delta\rho\mu\nu\sigma}h_{\alpha\beta}h_{\gamma\delta,\rho}+h_{\mu\nu,\rho}U_{(\mathrm{c})}^{\gamma\delta\mu\nu\rho\ \alpha\beta\sigma}h_{\alpha\beta,\sigma}h_{\gamma\delta}\right)$$

$$=\frac{1}{6}\int\ d^{D}xh_{\mu\nu}\left(U_{(\mathrm{c})}^{\mu\nu\ \alpha\beta\rho}\gamma\delta\sigma h_{\alpha\beta,\rho}h_{\gamma\delta,\sigma}-U_{(\mathrm{c})}^{\alpha\beta\ \gamma\delta\rho}\mu\nu\sigma\partial_{\sigma}\left(h_{\alpha\beta}h_{\gamma\delta,\rho}\right)-U_{(\mathrm{c})}^{\gamma\delta\mu\nu\alpha\beta\sigma}\partial_{\rho}\left(h_{\alpha\beta,\sigma}h_{\gamma\delta}\right)\right)$$

$$\mathcal{G}_{h^{2}}^{\mu\nu}(h,h)=-\frac{1}{2}\left(U_{(\mathrm{c})}^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma h_{\alpha\beta,\rho}h_{\gamma\delta,\sigma}-2U_{(\mathrm{c})}^{\alpha\beta}\gamma\delta\rho\mu\nu\sigma h_{\alpha\beta,\sigma}h_{\gamma\delta,\rho}-2U_{(\mathrm{c})}^{\alpha\beta}\gamma\delta\rho\mu\nu\sigma h_{\alpha\beta}h_{\gamma\delta,\rho\sigma}\right)$$

$$\delta S_{EH}=-\int\ d^{D}x\delta h_{\mu\nu}\sum_{n=1..\infty}\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)$$

$$\delta S_{EH}=-\int\ d^{D}x\sqrt{-g}G^{\mu\nu}\delta h_{\mu\nu}$$

$$\sum_{n=1..\infty}\mathcal{G}_{h^{n}}^{\mu\nu}(h,h,\dots,h)=\sqrt{-g}G^{\mu\nu}$$

$$\begin{aligned}\mathcal{G}_{h^{n}}^{\mu\nu}&=\left(\sqrt{-g}G^{\mu\nu}\right)_{h^{n}}\\&=\sum_{j=0..n}G_{h^{n-j}}^{\mu\nu}(\sqrt{-g})_{h^{j}}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_{h^{1}}^{\mu\nu}&=G_{h^{1}}^{\mu\nu}\\\mathcal{G}_{h^{2}}^{\mu\nu}&=G_{h^{2}}^{\mu\nu}+\frac{1}{2}hG_{h^{1}}^{\mu\nu}\\\mathcal{G}_{h^{3}}^{\mu\nu}&=G_{h^{3}}^{\mu\nu}+\frac{1}{2}hG_{h^{2}}^{\mu\nu}-\frac{1}{4}\mathcal{P}_{\alpha\beta}^{\rho\sigma}h^{\alpha\beta}h_{\rho\sigma}G_{h^{1}}^{\mu\nu}\end{aligned}$$

$$\mathcal{G}^{\mu\nu} = \sqrt{-g}G^{\mu\nu}$$

$$\delta S_{(\mathrm{gf})} = \frac{1}{\xi}\int d^D x \eta^{\rho\sigma} G_\rho \delta G_\sigma = -\frac{1}{\xi}\int d^D x \sqrt{-g} H^{\mu\nu}\delta h_{\mu\nu}$$

$$\mathcal{H}^{\mu\nu}_{h^n} = \left(\sqrt{-g}H^{\mu\nu}\right)_{h^n}$$

$$\mathcal{H}^{\mu\nu} = \sqrt{-g}H^{\mu\nu}$$

$$G^{\mu\nu} = \frac{1}{2}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)R_{\alpha\beta}$$
$$= \frac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)g^{\gamma\delta}F^{\rho\sigma}_{\alpha\beta\ \gamma\delta}\left(g_{\phi\epsilon,\rho\sigma} + g^{\kappa\lambda}\Gamma_{\kappa\rho\sigma}\Gamma_{\lambda\phi\epsilon}\right)$$

$$F^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = I^{\alpha\beta}_{\mu\nu}I^{\gamma\delta}_{\rho\sigma} + I^{\alpha\beta}_{\rho\sigma}I^{\gamma\delta}_{\mu\nu} - 2I^{\alpha\beta}_{\epsilon\zeta}I^{\zeta\eta}_{\mu\nu}I^{\gamma\delta}_{\eta\theta}I^{\theta\epsilon}_{\rho\sigma}.$$

$$G^{\mu\nu}_a = \frac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)g^{\gamma\delta}F^{\rho\sigma\ \phi\epsilon}_{\alpha\beta\ \gamma\delta}\partial_\rho\partial_\sigma g_{\phi\epsilon},$$
$$G^{\mu\nu}_b = \frac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)g^{\gamma\delta}F^{\rho\sigma\ \phi\epsilon}_{\alpha\beta\ \gamma\delta}g^{\kappa\lambda}\Gamma_{\kappa\rho\sigma}\Gamma_{\lambda\phi\epsilon}.$$

$$G^{\mu\nu}_h = \left(G^{\mu\nu}_a\right)_h = \frac{1}{2}\mathcal{P}^{\mu\nu\alpha\beta}\eta^{\gamma\delta}F^{\rho\sigma\phi\epsilon}_{\alpha\beta\gamma\delta}h_{\rho\sigma,\phi\epsilon} = \frac{1}{2}Q^{\mu\nu\alpha\beta\gamma\delta}h_{\alpha\beta,\gamma\delta}$$

$$Q^{\mu\nu\alpha\beta\gamma\delta} = \mathcal{P}^{\mu\nu\rho\sigma}\eta^{\phi\epsilon}F^{\alpha\beta\gamma\delta}_{\rho\sigma}$$
$$= \eta^{\mu\nu}\mathcal{P}^{\alpha\beta\gamma\delta} - 2I^{\mu\nu}_{\sigma\rho}\mathcal{P}^{\rho\phi\alpha\beta}\eta_{\phi\epsilon}\mathcal{P}^{\epsilon\sigma\gamma\delta}$$

$$\left(G^{\mu\nu}_a\right)_{h^2} = -\left(\mathcal{P}^{\mu\nu}_{\rho\kappa}h^\rho_\sigma\mathcal{P}^{-1\sigma\kappa}_{\alpha\beta} + \frac{1}{2(D-2)}h^{\mu\nu}\eta_{\alpha\beta}\right)Q^{\alpha\beta}_{\gamma\delta}h_\epsilon h^{\gamma\delta,\phi\epsilon} - \frac{1}{2}F^{\mu\nu}_{\gamma\delta\phi}\rho_{\rho\sigma}h_{\rho\sigma}h^{\gamma\delta,\phi\epsilon}$$
$$\left(G^{\mu\nu}_b\right)_{h^2} = \frac{1}{2}Q^{\mu\nu\ \alpha\beta}\gamma\delta$$
$$\eta^{\rho\sigma}\Gamma_{\rho\alpha\beta}\Gamma_{\sigma\gamma\delta}$$

$$\left(S_{\mathrm{EH}} + S_{(\mathrm{gft})}\right)_{h^2} = \frac{1}{4}\int d^D x h^\rho_{\mu\nu}\left(\delta^\rho_\sigma\mathcal{P}^{\mu\nu}_{\alpha\beta} - 2\left(1 - \frac{1}{\xi}\right)\mathcal{P}^{\mu\nu}_{\rho\kappa}\mathcal{P}^{\sigma\kappa}_{\alpha\beta}\right)h^{\alpha\beta}_{,\sigma}$$

$$\frac{2}{\kappa^2}\left(S_{\mathrm{EH}} + S_{(\mathrm{gf})}\right)_{h^2} = \frac{1}{2}\int d^D x h^{\rho\rho}_{\mu\nu}\left(\delta^\rho_\sigma\mathcal{P}^{\mu\nu}_{\alpha\beta} - 2\left(1 - \frac{1}{\xi}\right)\mathcal{P}^{\mu\nu}_{\rho\kappa}\mathcal{P}^{\sigma\kappa}_{\alpha\beta}\right)h^{\alpha\beta}_{,\sigma}$$

$$\frac{2}{\kappa^2}\left(S_{\mathrm{EH}} + S_{(\mathrm{gf})}\right)_{h^2} = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\tilde{h}^\dagger_{\mu\nu}p^2\left(\mathcal{P}^{\mu\nu}_{\alpha\beta} - 2\left(1 - \frac{1}{\xi}\right)\mathcal{P}^{\mu\nu}_{\rho\kappa}\frac{p^\rho p_\sigma}{p^2}\mathcal{P}^{\sigma\kappa}_{\alpha\beta}\right)\tilde{h}^{\alpha\beta}$$

$$\Delta^{\mu\nu}_{\alpha\beta} = \mathcal{P}^{\mu\nu}_{\alpha\beta} - 2\left(1 - \frac{1}{\xi}\right)\mathcal{P}^{\mu\nu}_{\rho\kappa}\frac{p^\rho p_\sigma}{p^2}\mathcal{P}^{\sigma\kappa}_{\alpha\beta}$$

$$I_{\alpha\beta}^{\mu\nu} = \frac{1}{2}\left(\delta_{\nu}^{\mu}\delta_{\beta}^{\alpha} + \delta_{\beta}^{\mu}\delta_{\alpha}^{\nu}\right)$$

$$T_{\alpha\beta}^{\mu\nu} = \frac{1}{4}\eta^{\mu\nu}\eta_{\alpha\beta}$$

$$C_{\alpha\beta}^{\mu\nu} = \frac{1}{2}\left(\eta^{\mu\nu}\frac{p_\alpha p_\beta}{p^2} + \frac{p^\mu p^\nu}{p^2}\eta_{\alpha\beta}\right)$$

$$\mathcal{J}_{\alpha\beta}^{\mu\nu} = I_{\rho\kappa}^{\mu\nu}\frac{p_\sigma p^\rho}{p^2}I_{\alpha\beta}^{\sigma\kappa}$$

$$K_{\alpha\beta}^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}\frac{p_\alpha p_\beta}{p^2}$$

$$\mathcal{P} = I - 2T$$

$$\Delta = \mathcal{P} + 2\left(1 - \frac{1}{\xi}\right)\mathcal{P}\mathcal{J}\mathcal{P}$$

$$G = \alpha_1 I + \alpha_2 T + \alpha_3 C + \alpha_4 \mathcal{J} + \alpha_5 K$$

$$\Delta G = I$$

$$A_{\alpha\beta}^{\mu\nu} = \frac{1}{2}\left(\eta^{\mu\nu}\frac{p_\alpha p_\beta}{p^2} - \frac{p^\mu p^\nu}{p^2}\eta_{\alpha\beta}\right)$$

$$TT = \frac{D}{4}T$$

$$\mathcal{J}\mathcal{J} = \frac{1}{2}\mathcal{J} + \frac{1}{2}K$$

$$\mathcal{J}\mathcal{T}\mathcal{J} = \frac{1}{4}K$$

$$T_{\alpha\beta}^{\mu\nu}C_{\gamma\delta}^{\alpha\beta} = \frac{1}{2}T_{\gamma\delta}^{\mu\nu} + \frac{D}{8}C_{\gamma\delta}^{\mu\nu} + \frac{D}{8}A_{\gamma\delta}^{\mu\nu}$$

$$\begin{aligned}
\Delta_{\alpha\beta}^{\mu\nu}G_{\gamma\delta}^{\alpha\beta} &= \alpha_1 I_{\gamma\delta}^{\mu\nu} \\
&+ \left(\alpha_1\left(-4 + \frac{2}{\xi}\right) + \alpha_2\left(-1 + \frac{1}{2\xi}\right)(D-2) - \alpha_3\frac{1}{\xi}\right)T_{\gamma\delta}^{\mu\nu} \\
&+ \left(\alpha_1\left(2 - 2\frac{1}{\xi}\right) + \alpha_2\frac{D-2}{4}\left(1 - \frac{1}{\xi}\right) + \alpha_3\left(-\frac{D-2}{2} + \frac{D}{4\xi}\right) - (\alpha_4 + \alpha_5)\frac{1}{2\xi}\right)C_{\gamma\delta}^{\mu\nu} \\
&+ \left(\alpha_1 2\left(-1 + \frac{1}{\xi}\right) + \alpha_4\frac{1}{\xi}\right)\mathcal{J}_{\gamma\delta}^{\mu\nu} \\
&+ \left(\alpha_3(D-2)\left(\frac{1}{2} - \frac{1}{\xi}\right) + \alpha_5\frac{1}{\xi}\right)K_{\gamma\delta}^{\mu\nu} \\
&+ \left(\alpha_2\frac{D-2}{4}\left(-1 + \frac{1}{\xi}\right) + \alpha_3\left(-\frac{D-2}{2} + \frac{D-4}{4\xi}\right) - (\alpha_4 + \alpha_5)\frac{1}{2\xi}\right)A_{\gamma\delta}^{\mu\nu}
\end{aligned}$$

$$2\left(-1 + \frac{1}{\xi}\right) + \alpha_4\frac{1}{\xi} = 0$$

$$
\begin{aligned}
\alpha_1 &= 1\\
\alpha_2 &= -\frac{4}{D-2}\\
\alpha_3 &= 0\\
\alpha_4 &= -2(1-\xi)\\
\alpha_5 &= 0
\end{aligned}
$$

$$G = \mathcal{P}^{-1} - 2(1-\xi)\mathcal{J}$$

$$\mathcal{P}^{-1} = I - \frac{4}{D-2}T$$

$$\Delta = \mathcal{P} - 2\left(1-\frac{1}{\xi}\right)\mathcal{P}\mathcal{J}\mathcal{P}$$

$$G = \mathcal{P}^{-1} - 2(1-\xi)\mathcal{J}$$

$$
\begin{aligned}
\Delta &= \Delta_{(\mathrm{c})} + \frac{1}{\xi}\Delta_{(\mathrm{gf})}\\
\Delta_{(\mathrm{c})} &= \mathcal{P} - 2\mathcal{P}\mathcal{J}\mathcal{P}\\
\Delta_{(\mathrm{gf})} &= 2\mathcal{P}\mathcal{J}\mathcal{P}
\end{aligned}
$$

$$
\begin{aligned}
G &= G_{(\mathrm{c})} + \xi G_{(\mathrm{gf})}\\
G_{(\mathrm{c})} &= \mathcal{P}^{-1} - 2\mathcal{J}\\
G_{(\mathrm{gf})} &= 2\mathcal{J}
\end{aligned}
$$

$$\Delta_{(\mathrm{c})}{}^{\mu\nu}_{\alpha\beta} = Q^{\mu\nu}_{\alpha\beta}\frac{\sigma\rho}{p_\sigma p_\rho}$$

$$p_\mu \Delta^{\mu\nu}_{(\mathrm{c})\,\alpha\beta} = p_\mu Q^{\mu\nu}_{\alpha\beta}\frac{\sigma\rho}{p^2} = 0$$

$$p_\mu \Delta_{(\mathrm{c})}{}^{\mu\nu}_{\alpha\beta} = p_\mu \mathcal{P}^{\mu\nu}_{\alpha\beta} - 2p_\mu \mathcal{P}^{\mu\nu}_{\rho\kappa}\frac{p^\rho p_\sigma}{p^2}\mathcal{P}^{\sigma\kappa}_{\alpha\beta}$$

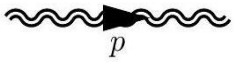
$$2p_\mu \mathcal{P}^{\mu\nu}_{\rho\kappa}\frac{p^\rho p_\sigma}{p^2}\mathcal{P}^{\sigma\kappa}_{\alpha\beta} = p_\mu \mathcal{P}^{\mu\nu}_{\alpha\beta}$$

$$\Delta G = \Delta_{(\mathrm{c})}G_{(\mathrm{c})} + \Delta_{(\mathrm{gf})}G_{(\mathrm{gf})} + \xi\Delta_{(\mathrm{c})}G_{(\mathrm{gf})} + \frac{1}{\xi}\Delta_{(\mathrm{gf})}G_{(\mathrm{c})}$$

$$\mathcal{J}\mathcal{P}\mathcal{J} = \mathcal{J}^2 - 2\mathcal{J}T\mathcal{J} = \frac{1}{2}\mathcal{J}$$

$$
\begin{aligned}
\Delta_{(\mathrm{gf})}\,G_{(\mathrm{c})} &= 2\mathcal{P}\mathcal{J}\mathcal{P}(\mathcal{P}^{-1} - 2\mathcal{J})\\
&= 2\mathcal{P}(\mathcal{J} - 2\mathcal{J}\mathcal{P}\mathcal{J}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{(\mathrm{c})}G_{(\mathrm{c})} + \Delta_{(\mathrm{gf})}G_{(\mathrm{gf})} &= (\mathcal{P} - 2\mathcal{P}\mathcal{J}\mathcal{P})(\mathcal{P}^{-1} - 2\mathcal{J}) + 4\mathcal{P}\mathcal{J}\mathcal{P}\mathcal{J}\\
&= I - 4\mathcal{P}\mathcal{J} + 8\mathcal{P}\mathcal{J}\mathcal{P}\mathcal{J} = I
\end{aligned}
$$

$\mu\nu$  $\alpha\beta = \frac{i}{p^2 + i\epsilon} G^{\mu\nu}_{\alpha\beta}$

$$G^{\mu\nu}_{\alpha\beta} = I^{\mu\nu}_{\alpha\beta} - \frac{1}{D-2}\eta^{\mu\nu}\eta_{\alpha\beta} - 2(1-\xi)I^{\mu\nu}_{\rho\kappa}\frac{p^\rho p_\sigma}{p^2}I^{\kappa\sigma}_{\alpha\beta}$$

$$S_\phi = \frac{1}{2}\int d^D x\sqrt{-g}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)$$

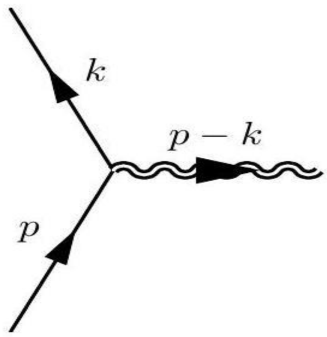
$$\left(S_\phi\right)_h = -\frac{1}{2}\int d^D x h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{1}{4}\int d^D x h^\mu_\mu\left(\phi^{,\nu}\phi_{,\nu} - m^2\phi^2\right)$$

$$S_\phi \approx \frac{1}{2}\int d^D x\left(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)$$
$$-\frac{1}{2}\int d^D x\left(h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}h^\mu_\mu\left(\phi^{,\nu}\phi_{,\nu} - m^2\phi^2\right)\right)$$
$$+\frac{1}{2}\int d^D x\left(h_{\mu\nu}h^{\alpha\gamma}\mathcal{P}^{\mu\nu}_{\alpha\beta}\phi_{,\gamma}\phi^{,\beta} - \frac{1}{4}h_{\mu\nu}h^{\alpha\beta}\mathcal{P}^{\mu\nu}_{\alpha\beta}\left(\phi^{,\nu}\phi_{,\nu} - m^2\phi^2\right)\right)$$

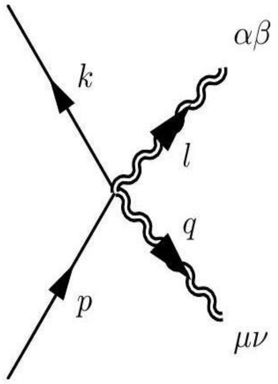
$$V^{\mu\nu}_{\phi^2 h}(p,k,m) = I^{\mu\nu}_{\alpha\beta}p^\alpha k^\beta - \frac{pk - m^2}{2}\eta^{\mu\nu}$$

$$V^{\mu\beta}_{\phi^2 h^2}(p,k,m) = \left(I^{\mu\nu\ \eta}_{\ \ \ \lambda} I^{\alpha\beta}{}_{\kappa}{}_{\kappa}{}^{\sigma\rho\kappa}{}_{\eta} - \frac{1}{4}\left(\eta^{\mu\nu}I^{\alpha\beta\sigma\rho} + \eta^{\alpha\beta}I^{\mu\nu\sigma\rho}\right)\right)p_\sigma k_\rho - \frac{pk - m^2}{4}\mathcal{P}^{\mu\nu\ \alpha\beta}$$

$$\frac{i}{p^2 - m^2 + i\epsilon}$$



$\mu\nu = -i\kappa V^{\mu\nu}_{\phi^2 h}(p,k,m)$



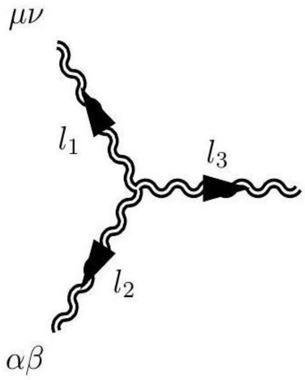
$$= 2i\kappa^2 V^{\mu\nu}_{\phi^2 h^2}(p,k,m)$$

$$S_{h^3} = \frac{1}{\kappa^2}\int d^D x h_{\mu\nu} U^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma h_{\alpha\beta,\rho} h_{\gamma\delta,\sigma}$$

$$\kappa_{\mu\nu} = \int \frac{d^D l}{(2\pi)^D} e^{-ilx}\tilde{h}_{\mu\nu}(l)$$

$$S_{h^3} = -\kappa\int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D}(2\pi)^D\delta^D\left(l_{(1)}+l_{(2)}+l_{(3)}\right)\tilde{f}^{(1)}_{\mu\nu}U^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma\tilde{h}^{(2)}_{\alpha\beta}\tilde{h}^{(3)}_{\gamma\delta}l_{(2)\rho}l_{(3)\sigma}$$

$$S_{h^3} = -\frac{\kappa}{3}\int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D}(2\pi)^D\delta^D\left(l_{(1)}+l_{(2)}+l_{(3)}\right)\left(U^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma l_{(2)\rho}l_{(3)\sigma}+U^{\alpha\beta}\gamma\delta\rho\mu\nu\sigma l_{(3)\rho}l_{(1)\sigma}+U^{\gamma\delta}\mu\nu\rho\alpha\beta\sigma l_{(1)\rho}l_{(2)\sigma}\right)\tilde{h}^{(1)}_{\mu\nu}\tilde{h}^{(2)}_{\alpha\beta}\tilde{h}^{(3)}_{\gamma\delta}$$



$$\gamma\delta = 2i\kappa V^{\mu\nu\ \alpha\beta\ \gamma\delta}_{h^3}(l_1,l_2,l_3)$$

$$V^{\alpha\beta\gamma\delta\sigma}_{h^3}\left(l_{(1)},l_{(2)},l_{(3)}\right) = -\left(U^{\mu\nu\alpha\beta\rho\gamma\delta\sigma}l_{(2)\rho}l_{(3)\sigma}+U^{\alpha\beta\gamma\delta\rho\mu\nu\sigma}l_{(3)\rho}l_{(1)\sigma}+U^{\gamma\delta\mu\nu\rho\alpha\beta\sigma}l_{(1)\rho}l_{(2)\sigma}\right)$$

$$S_{\mathrm{EH}} + S_{(\mathrm{gf})} = -\int d^D x h_{\mu\nu}\sum_{n=1..\infty}\frac{1}{(n+1)}\left(\mathcal{G}^{\mu\nu}_{h^n}(h,h,\ldots,h)+\frac{1}{\xi}\mathcal{H}^{\mu\nu}_{h^n}(h,h,\ldots,h)\right)$$

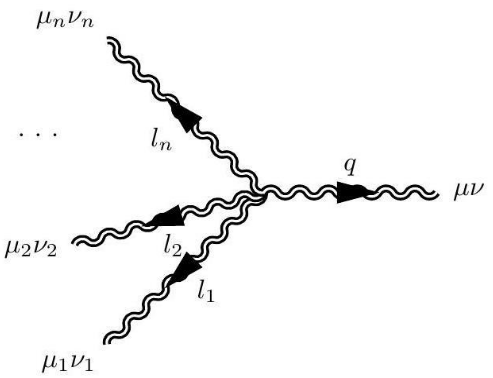
$$\mathcal{G}^{\mu\nu}_{h^3}\left(h_{\mu\nu},h_{\mu\nu},h_{\mu\nu}\right) = \mathcal{G}^{\mu\nu}_{h^3}\left(\int \frac{d^D l_{(1)}}{(2\pi)^D}e^{-ixl_{(1)}}\tilde{h}^{(1)}_{\mu\nu},\int \frac{d^D l_{(2)}}{(2\pi)^D}e^{-ixl_{(2)}}\tilde{h}^{(2)}_{\mu\nu},\int \frac{d^D l_{(3)}}{(2\pi)^D}e^{-ixl_{(3)}}\tilde{h}^{(3)}_{\mu\nu}\right)$$

$$\mathcal{G}^{\mu\nu}_{h^3}\left(h_{\mu\nu},h_{\mu\nu},h_{\mu\nu}\right) = \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D}\mathcal{G}^{\mu\nu}_{h^3}\left(e^{-ixl_{(1)}}\tilde{h}^{(1)}_{\mu\nu},e^{-ixl_{(2)}}\tilde{h}^{(2)}_{\mu\nu},e^{-ixl_{(3)}}\tilde{h}^{(3)}_{\mu\nu}\right)$$

$$\mathcal{G}^{\mu\nu}_{h^3}(h_{\mu\nu},h_{\mu\nu},h_{\mu\nu}) = \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D} e^{-ix(l_{(1)}+l_{(2)}+l_{(3)})}\tilde{\mathcal{G}}^{\mu\nu}_{h^3}\left(\tilde{h}^{(1)}_{\mu\nu},\tilde{h}^{(2)}_{\mu\nu},\tilde{h}^{(3)}_{\mu\nu}\right)$$

$$\int d^D x e^{iqx}\mathcal{G}^{\mu\nu}_{h^3}(h_{\mu\nu},h_{\mu\nu},h_{\mu\nu}) = \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D}(2\pi)^D\delta^D\left(l_{(1)}+l_{(2)}+l_{(3)}\right)\tilde{\mathcal{G}}^{\mu\nu}_{h^3}\left(\tilde{h}^{(1)}_{\mu\nu},\tilde{h}^{(2)}_{\mu\nu},\tilde{h}^{(3)}_{\mu\nu}\right)$$

$$\sum_{n=1..\infty} -\frac{2\kappa^{n-1}}{n+1}\tilde{G}^{(0)}_{\mu\nu}\left(\tilde{G}^{\mu\nu}_{h^n}\left(\tilde{G}^{(1)}_{\mu\nu},\ldots,\tilde{G}^{(n)}_{\mu\nu}\right)+\frac{1}{\xi}\tilde{\mathcal{H}}^{\mu\nu}_{h^n}\left(\tilde{G}^{(1)}_{\mu\nu},\ldots,\tilde{G}^{(n)}_{\mu\nu}\right)\right)$$



$$= in!\,\kappa^{n-1}V^{\mu\nu\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}_{h^{n+1}}(q,l_1,l_2,\ldots,l_n)$$

$$V^{\mu\nu}_{h^{n+1}}{}_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}(q,l_{(1)},l_{(2)},\ldots,l_{(n)})\tilde{h}^{(1)}_{\mu_1\nu_1}\tilde{h}^{(2)}_{\mu_2\nu_2}\ldots\tilde{h}^{(n)}_{\mu_n\nu_n} = -2\left(\tilde{\mathcal{G}}^{\mu\nu}_{h^n}\left(\tilde{h}^{(1)}_{\mu_1\nu_1},\tilde{h}^{(2)}_{\mu_2\nu_2},\ldots,\tilde{h}^{(n)}_{\mu_n\nu_n}\right)+\frac{1}{\xi}\tilde{\mathcal{H}}^{\mu\nu}_{h^n}\left(\tilde{h}^{(1)}_{\mu_1\nu_1},\tilde{h}^{(2)}_{\mu_2\nu_2},\ldots,\tilde{h}^{(n)}_{\mu_n\nu_n}\right)\right)$$

$$\begin{aligned} S &= \frac{2}{\kappa^2}\left(S_{EH}+S_{gf}\right)+\int d^D x\sqrt{-g}\mathcal{L}_\phi \\ &= \int d^D x\sqrt{-g}\left(\frac{2R}{\kappa^2}+\mathcal{L}_\phi\right)+\int d^D x\frac{1}{\kappa^2\xi}\eta^{\sigma\rho}G_\sigma G_\rho \end{aligned}$$

$$\delta S_{(\mathrm{gf})} = \frac{1}{\xi}\int d^D x\eta^{\rho\sigma}G_\rho\delta G_\sigma = -\frac{1}{\xi}\int d^D x\sqrt{-g}H^{\mu\nu}\delta h_{\mu\nu}$$

$$G_\sigma = (1-\alpha)\partial_\mu\left(h^\mu_\sigma-\frac{1}{2}\eta^\mu_\sigma h^\nu_\nu\right)+\alpha g^{\mu\nu}\Gamma_{\sigma\mu\nu}$$

$$\begin{aligned} G_\sigma &= (1-\alpha)\eta^{\mu\nu}\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h_{\alpha\beta,\rho}+\alpha g^{\mu\nu}\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h_{\alpha\beta,\rho} \\ &= \left(\eta^{\mu\nu}+\alpha(g^{\mu\nu}-\eta^{\mu\nu})\right)\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h_{\alpha\beta,\rho} \\ &= \left(\eta^{\mu\nu}+\alpha\hat{h}^{\mu\nu}\right)\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h_{\alpha\beta,\rho} \end{aligned}$$

$$\delta\hat{h}^{\mu\nu} = \delta(g^{\mu\nu}) = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$$

$$\delta G_\sigma = \left(\eta^{\mu\nu}+\alpha\hat{h}^{\mu\nu}\right)\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}\delta h_{\alpha\beta,\rho}-\alpha\Gamma^{\rho\alpha\beta}_{\sigma\mu\nu}h_{\alpha\beta,\rho}g^{\mu\gamma}g^{\nu\delta}\delta h_{\gamma\delta}$$

$$\delta S_{gf} = \frac{1}{\xi}\int d^D x \eta^{\rho\sigma} G_\rho \delta G_\sigma$$

$$= \frac{1}{\xi}\int d^D x G^\sigma \left((\eta^{\mu\nu} + \alpha \hat{h}^{\mu\nu}) \Gamma^{\rho\gamma\delta}_{\sigma\mu\nu} \delta h_{\gamma\delta,\rho} - \alpha \Gamma^{\rho\alpha\beta}_{\sigma\mu\nu} h_{\alpha\beta,\rho} g^{\mu\gamma} g^{\nu\delta} \delta h_{\gamma\delta} \right)$$

$$= -\frac{1}{\xi}\int d^D x \left(\Gamma^{\rho\gamma\delta}_{\sigma\mu\nu} \partial_\rho \left(G^\sigma (\eta^{\mu\nu} + \alpha \hat{h}^{\mu\nu}) \right) + \alpha G^\sigma \Gamma^{\rho\alpha\beta}_{\sigma\mu\nu} h_{\alpha\beta,\rho} g^{\mu\gamma} g^{\nu\delta} \right) \delta h_{\gamma\delta}$$

$$\sqrt{-g} H^{\mu\nu} = \mathcal{H}^{\mu\nu} = \alpha G^\rho \Gamma_{\rho\alpha\beta} g^{\alpha\mu} g^{\beta\nu} + \Gamma^{\sigma\mu\nu}_{\rho\alpha\beta} \partial_\sigma \left(G^\rho (\eta^{\alpha\beta} + \alpha \hat{h}^{\alpha\beta}) \right)$$

$$G^{\mu\nu} + \frac{1}{\xi} H^{\mu\nu} = -\frac{\kappa^2}{4} T^{\mu\nu}$$

$$G^{\mu\nu} = -\frac{\kappa^2}{4} T^{\mu\nu}$$

$$D_\mu H^{\mu\nu} = 0$$

$$\sqrt{-g} D_\mu H^{\mu\nu} = \partial_\mu (\mathcal{H}^{\mu\nu}) + \Gamma^\nu_{\alpha\beta} \mathcal{H}^{\alpha\beta}$$

$$\sqrt{-g} D_\mu H^{\mu\nu} \approx \partial_\mu \mathcal{H}^{\mu\nu}$$

$$H^{\mu\nu} \approx \Gamma^{\sigma\mu\nu}_{\rho\alpha\beta} \eta^{\alpha\beta} \partial_\sigma G^\rho$$

$$D_\mu H^{\mu\nu} \approx \Gamma^{\sigma\mu\nu}_{\rho\alpha\beta} \eta^{\alpha\beta} \partial_\sigma \partial_\mu G^\rho$$

$$D_\mu H^{\mu\nu} \approx \frac{1}{2} \partial_\sigma \partial^\sigma G^\nu$$

$$G^{\mu\nu} = -\frac{\kappa^2}{4} T^{\mu\nu}$$

$$G_\sigma = 0.$$

$$G^{\mu\nu} + \frac{1}{\xi} H^{\mu\nu} = -\frac{\kappa^2}{4} T^{\mu\nu}$$

$$G^{\mu\nu} = \sum_{n=1..\infty} (G^{\mu\nu})_{h^n}$$

$$= G^{\mu\nu}_h + G^{\mu\nu}_{\text{non-linear}}$$

$$G^{\mu\nu}_h + \frac{1}{\xi} H^{\mu\nu}_h = -\frac{\kappa^2}{4} T^{\mu\nu} - G^{\mu\nu}_{\text{non-linear}} - \frac{1}{\xi} H^{\mu\nu}_{\text{non-linear}}$$

$$\tau^{\mu\nu} = T^{\mu\nu} + \frac{4}{\kappa^2} G^{\mu\nu}_{\text{non-linear}}$$

$$G_h^{\mu\nu} = -\frac{\kappa^2}{4}\tau^{\mu\nu}$$

$$Q^{\mu\nu\alpha\beta\gamma\delta}h_{\alpha\beta,\gamma\delta} = -\frac{\kappa^2}{2}\tau^{\mu\nu}$$

$$0 = \partial_\mu Q^{\mu\nu\alpha\beta\gamma\delta}h_{\alpha\beta,\gamma\delta} = -\frac{\kappa^2}{2}\partial_\mu\tau^{\mu\nu}$$

$$H_h^{\mu\nu} = -H_{\text{non-linear}}^{\mu\nu}$$

$$H_h^{\mu\nu} = \mathcal{P}_{\sigma\kappa}^{\mu\nu}\mathcal{P}_{\gamma\delta}^{\rho\kappa}\partial^\sigma\partial_\rho h^{\gamma\delta}$$

$$G_{(\mathrm{c})}{}^{\mu\nu}_{\alpha\beta}H_h^{\alpha\beta} = 0$$

$$G_{(c)}^{\mu\nu}{}^{\mu\nu}_{\alpha\beta}H_{\text{non-linear}}^{\alpha\beta} = 0$$

$$G_{(\mathrm{gf})}^{\alpha\beta}{}^{\mu\nu}_{\alpha\beta}\tau^{\alpha\beta} = 0$$

$$G_{(\mathrm{c})}^{\mu\nu}{}^{\mu\nu}_{\alpha\beta}H_{\text{non-linear}}^{\alpha\beta} = 0$$

$$\mathcal{G}^{\mu\nu} + \frac{1}{\xi}\mathcal{H}^{\mu\nu} = -\frac{\kappa^2}{4}\sqrt{-g}T^{\mu\nu}$$

$$\mathcal{G}^{\mu\nu} = -\frac{\kappa^2}{4}\sqrt{-g}T^{\mu\nu}$$

$$\mathcal{H}^{\mu\nu} = 0$$

$$\mathcal{G}_h^{\mu\nu} = G_h^{\mu\nu}$$

$$\mathcal{H}_h^{\mu\nu} = H_h^{\mu\nu}$$

$$\mathcal{H}_h^{\mu\nu} = -\mathcal{H}_{\text{non-linear}}^{\mu\nu},$$

$$\mathcal{H}_{\text{non-linear}}^{\mu\nu} = H_{\text{non-linear}}^{\mu\nu}$$

$$\tau^{\mu\nu} = \sqrt{-g}T^{\mu\nu} + \frac{4}{\kappa^2}\mathcal{G}_{\text{non-linear}}^{\mu\nu}.$$

$$\left(\eta^{\rho\sigma}\mathcal{P}^{\mu\nu\alpha\beta} - 2\left(1-\frac{1}{\xi}\right)\mathcal{P}^{\mu\nu\rho\phi}\eta_{\phi\epsilon}\mathcal{P}^{\alpha\beta\sigma\epsilon}\right)h_{\alpha\beta,\rho\sigma} = -\frac{\kappa^2}{2}\tau^{\mu\nu} - 2\frac{1}{\xi}H_{\text{non-linear}}^{\mu\nu}$$

$$q^2\Delta^{\mu\nu\alpha\beta}\tilde{h}_{\alpha\beta} = \frac{\kappa^2}{2}\tilde{\tau}^{\mu\nu} + 2\frac{1}{\xi}\tilde{H}_{\text{non-linear}}^{\mu\nu}$$

$$\tilde{h}_{\alpha\beta} = \frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\frac{\kappa^2}{2}\tilde{\tau}^{\mu\nu} + \frac{2}{\xi}\tilde{H}_{\text{non-linear}}^{\mu\nu}\right)$$

$$\tilde{h}_{\alpha\beta} = \frac{1}{q^2}\left(\frac{\kappa^2}{2}G_{(c_{\alpha\beta\mu\nu}}\tilde{\tau}^{\mu\nu} + 2G_{(\mathrm{gf})\ \alpha\beta\mu\nu}\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)$$
$$= \frac{\mathcal{P}^{-1}\ {}_{\alpha\beta\mu\nu}}{q^2}\left(\frac{\kappa^2}{2}\tilde{\tau}^{\mu\nu} + 2\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)$$

$$\sum_{n=1..\infty}\tilde{h}^{G^n}_{\alpha\beta} = \frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\frac{\kappa^2}{2}\sum_{n=0..\infty}\tilde{\tau}^{\mu\nu}_{G^n} + 2\frac{1}{\xi}\sum_{n=2..\infty}\left(\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)_{G^n}\right)$$

$$\tilde{h}^{G^1}_{\alpha\beta}(q) = \frac{G_{\alpha\beta\mu\nu}}{q^2}\frac{\kappa^2}{2}\tilde{T}^{\mu\nu}(q)$$

$$\tilde{h}^{G^n}_{\alpha\beta} = \frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\frac{\kappa^2}{2}\tilde{\tau}^{\mu\nu}_{G^{n-1}} + \frac{2}{\xi}\left(\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)_{G^n}\right)$$

$$\tilde{h}^{G^n}_{\alpha\beta} = 2\frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\tilde{G}^{\mu\nu}_{\text{non-linear}} + \frac{1}{\xi}\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)_{G^n}$$

$$\left(G^{\mu\nu} + \frac{1}{\xi}H^{\mu\nu}\right)_{G^n} = 0$$

$$h^G_{\mu\nu} = \frac{\kappa^2}{2}\int\frac{d^Dq}{(2\pi)^D}e^{-iqx}\frac{G_{\mu\nu\alpha\beta}}{q^2}\tilde{T}^{\alpha\beta}$$

$$\tilde{h}^{G^2}_{\alpha\beta} = 2\frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\tilde{G}^{\mu\nu}_{\text{non-linear}} + \frac{1}{\xi}\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)_{G^2}$$

$$\left(G^{\mu\nu}_{\text{non-linear}}\right)_{G^2} = G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^G_{\mu\nu}\right)$$

$$G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^G_{\mu\nu}\right) = G^{\mu\nu}_{h^2}\left(\int\frac{d^Dl_1}{(2\pi)^D}e^{-il_1x}\tilde{h}^G_{\mu\nu}(l_1), \int\frac{d^Dl_2}{(2\pi)^D}e^{-il_2x}\tilde{h}^G_{\mu\nu}(l_2)\right)$$

$$G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^G_{\mu\nu}\right) = \int\frac{d^Dl_{(1)}}{(2\pi)^D}\frac{d^Dl_{(2)}}{(2\pi)^D}G^{\mu\nu}_{h^2}\left(e^{-il_1x}\tilde{h}^G_{\mu\nu}(l_{(1)}), e^{-il_2x}\tilde{h}^G_{\mu\nu}(l_{(2)})\right)$$

$$G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^G_{\mu\nu}\right) = \int\frac{d^Dl_{(1)}}{(2\pi)^D}\frac{d^Dl_{(2)}}{(2\pi)^D}e^{-i\left(l_{(1)}+l_{(2)}\right)x}\tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l_{(1)}), \tilde{h}^G_{\mu\nu}(l_{(2)})\right)$$
$$= \int\frac{d^Dq}{(2\pi)^D}e^{-iqx}\int\frac{d^Dl}{(2\pi)^D}\tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^G_{\mu\nu}(q-l)\right)$$

$$\left(\tilde{G}^{\mu\nu}_{\text{non-linear}}\right)_{G^2} = \int\frac{d^Dl}{(2\pi)^D}\tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^G_{\mu\nu}(q-l)\right)$$

$$\tilde{h}^{G^2}_{\mu\nu} = 2\frac{G_{\mu\nu\alpha\beta}}{q^2}\int\frac{d^Dl}{(2\pi)^D}\left(\tilde{G}^{\alpha\beta}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^G_{\mu\nu}(q-l)\right) + \frac{1}{\xi}\tilde{H}^{\alpha\beta}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^G_{\mu\nu}(q-l)\right)\right)$$

$$\int \frac{d^D l}{(2\pi)^D} \tilde{G}^{\alpha\beta}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^G_{\mu\nu}(q-l)\right)$$

$$= \int \frac{d^D l}{(2\pi)^D} \tilde{G}^{\alpha\beta}_{h^2}\left(\frac{\kappa^2}{2}\frac{G_{\mu\nu\alpha\beta}\tilde{T}^{\alpha\beta}(l)}{l^2}, \frac{\kappa^2}{2}\frac{G_{\mu\nu\alpha\beta}\tilde{T}^{\alpha\beta}(l-q)}{(l-q)^2}\right)$$

$$= \frac{\kappa^4}{4}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2(l-q)^2} \tilde{G}^{\alpha\beta}_{h^2}\left(G_{\mu\nu\alpha\beta}\tilde{T}^{\alpha\beta}(l), G_{\mu\nu\alpha\beta}\tilde{T}^{\alpha\beta}(l-q)\right)$$

$$\left(G^{\mu\nu}_{\text{non-linear}}\right)_{G^3} = 2G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^{G^2}_{\mu\nu}\right) + G^{\mu\nu}_{h^3}\left(h^G_{\mu\nu}, h^G_{\mu\nu}, h^G_{\mu\nu}\right)$$

$$G^{\mu\nu}_{h^2}\left(h^G_{\mu\nu}, h^{G^2}_{\mu\nu}\right) = \int \frac{d^D q}{(2\pi)^D} e^{-iqx} \int \frac{d^D l}{(2\pi)^D} \tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^{G^2}_{\mu\nu}(q-l)\right)$$

$$G^{\mu\nu}_{h^3}\left(h^G_{\mu\nu}, h^G_{\mu\nu}, h^G_{\mu\nu}\right)$$

$$= \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\frac{d^D l_{(3)}}{(2\pi)^D} e^{-i(l_{(1)}+l_{(2)}+l_{(3)})x} \tilde{G}^{\mu\nu}_{h^3}\left(\tilde{h}^G_{\mu\nu}(l_{(1)}), \tilde{h}^G_{\mu\nu}(l_{(2)}), \tilde{h}^G_{\mu\nu}(l_{(3)})\right)$$

$$= \int \frac{d^D q}{(2\pi)^D} e^{-iqx} \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D} \tilde{G}^{\mu\nu}_{h^3}\left(\tilde{h}^G_{\mu\nu}(l_{(1)}), \tilde{h}^G_{\mu\nu}(l_{(2)}-l_{(1)}), \tilde{h}^G_{\mu\nu}(q-l_{(2)})\right)$$

$$\tilde{h}^{G^3}_{\alpha\beta} = 2\frac{G_{\alpha\beta\mu\nu}}{q^2}\left(\left(\tilde{G}^{\mu\nu}_{\text{non-linear}}\right)_{G^3} + \frac{1}{\xi}\left(\tilde{H}^{\mu\nu}_{\text{non-linear}}\right)_{G^3}\right).$$

$$\left(\tilde{G}^{\mu\nu}_{\text{non-linear}}\right)_{G^3} = 2\int \frac{d^D l}{(2\pi)^D} \tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^G_{\mu\nu}(l), \tilde{h}^{G^2}_{\mu\nu}(q-l)\right)$$
$$+ \int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D} \tilde{G}^{\mu\nu}_{h^3}\left(\tilde{h}^G_{\mu\nu}(l_{(1)}), \tilde{h}^G_{\mu\nu}(l_{(2)}-l_{(1)}), \tilde{h}^G_{\mu\nu}(q-l_{(2)})\right)$$

$$\begin{aligned} T^{\mu\nu} &= \frac{k^\mu k^\nu}{m}\delta^{D-1}(x_\perp) \\ &= m\eta^{\mu\nu}_{\parallel}\delta^{D-1}(x_\perp) \end{aligned}$$

$$\tilde{T}^{\mu\nu}(q) = \int d^D x e^{iqx} T^{\mu\nu} = 2\pi\delta(q_\parallel) m\eta^{\mu\nu}_{\parallel}$$

$$\begin{aligned} \tilde{h}^{G^1}_{\alpha\beta}(q) &= \frac{G_{\alpha\beta\mu\nu}}{q^2}\frac{\kappa^2}{2} 2\pi\delta(q_\parallel) m\eta^{\mu\nu}_{\parallel} \\ &= \frac{\kappa^2 m}{2}\frac{2\pi\delta(q_\parallel)}{q_\perp^2} G_{\alpha\beta\mu\nu}\eta^{\mu\nu}_{\parallel} \end{aligned}$$

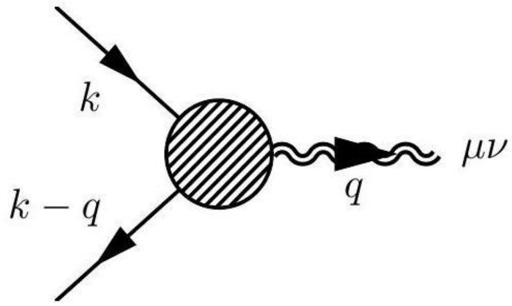
$$G_{\alpha\beta\mu\nu}\tilde{T}^{\mu\nu} = \mathcal{P}^{-1}{}_{\alpha\beta\mu\nu}\tilde{T}^{\mu\nu}$$

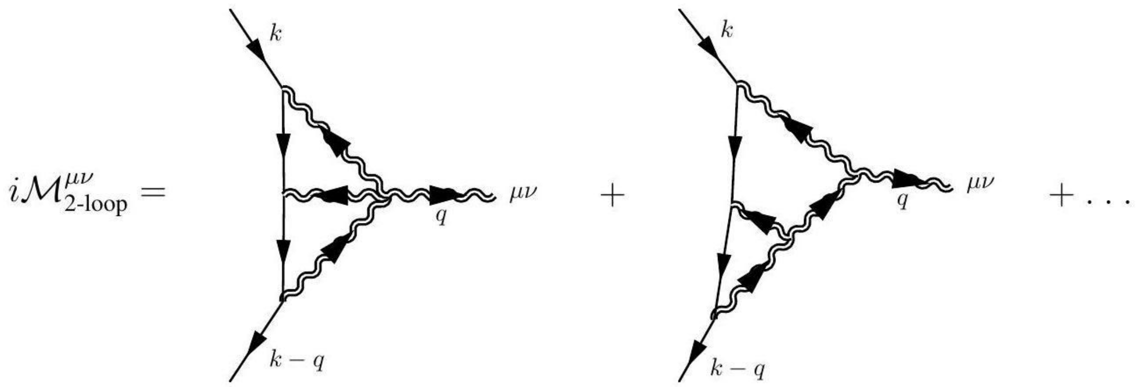
$$\begin{aligned} \mathcal{P}^{-1}{}_{\alpha\beta\mu\nu}\tilde{\eta}^{\mu\nu}_{\parallel} &= \eta^{\mu\nu}_{\parallel} - \frac{1}{D-2}\eta^{\mu\nu} \\ &= \frac{D-3}{D-2}\left(\eta^{\parallel}_{\mu\nu} - \frac{1}{D-3}\eta^{\perp}_{\mu\nu}\right) \end{aligned}$$

$$\tilde{h}^G_{\mu\nu} = \frac{\kappa^2 m}{2}\frac{2\pi\delta\left(q_{\parallel}\right)}{q_{\perp}^2}\frac{D-3}{D-2}\left(\eta^{\parallel}_{\mu\nu} - \frac{1}{D-3}\eta^{\perp}_{\mu\nu}\right)$$

$$h^G_{\mu\nu} = -\frac{\mu}{\sqrt{-x_{\perp}^2}^{D-3}}\left(\eta^{\parallel}_{\mu\nu} - \frac{1}{D-3}\eta^{\perp}_{\mu\nu}\right)$$

$$(G_\sigma)^G = G^h_\sigma\left(h^G_{\mu\nu}\right) = \partial^\rho \mathcal{P}^{\mu\nu}_{\rho\sigma} h^G_{\mu\nu}$$



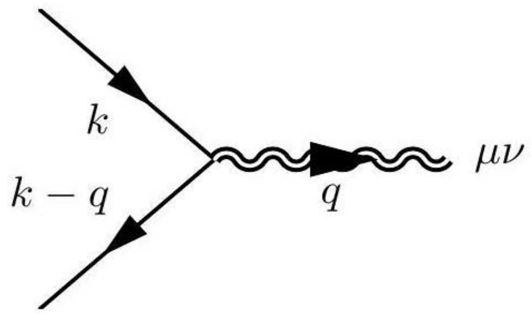


$$2\pi\delta(kq)\mathcal{M}^{\mu\nu}_{\text{vertex}} = -\kappa\tilde{\tau}^{\mu\nu} - \frac{4}{\kappa\xi}\tilde{H}^{\mu\nu}_{\text{non-linear}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\kappa}{2}\int \frac{d^D q\,\delta(kq)e^{-iqx}}{(2\pi)^{D-1}}\frac{G_{\mu\nu\alpha\beta}}{q^2}\mathcal{M}^{\alpha\beta}_{\text{vertex}}$$

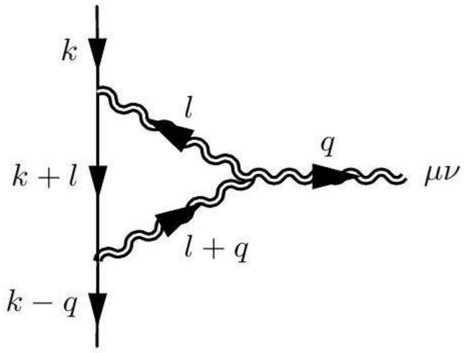
$$\tilde{h}_{\mu\nu} = \frac{\mathcal{P}^{-1}_{\mu\nu\alpha\beta}}{q^2}\left(\frac{\kappa^2}{2}\tilde{\tau}^{\alpha\beta} + 2\tilde{H}^{\alpha\beta}_{\text{non-linear}}\right)$$

$$\begin{aligned} V^{\mu\nu}_{\phi^2 h}(k, k-q, m) &= I^{\mu\nu}_{\alpha\beta}k^\alpha(k-q)^\beta - \frac{k(k-q)-m^2}{2}\eta^{\mu\nu} \\ &= I^{\mu\nu}_{\alpha\beta}k^\alpha(k-q)^\beta + \frac{kq}{2}\eta^{\mu\nu} \end{aligned}$$



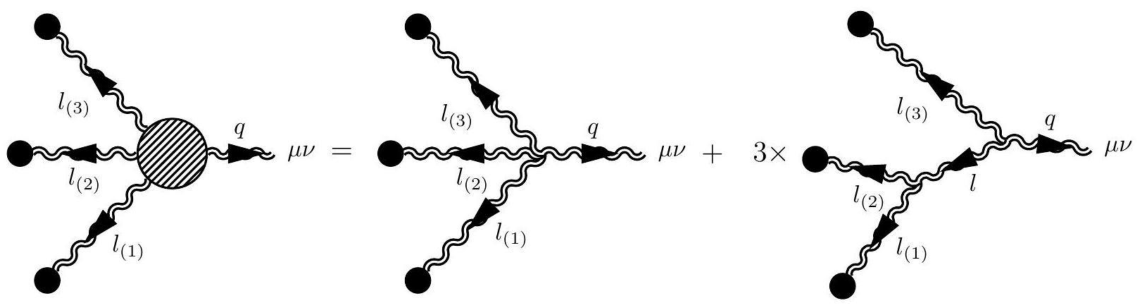
$$\mathcal{M}^{\mu\nu}_{\text{tree}} = -\kappa k^{\mu} k^{\nu} = -\kappa m^{2} \eta^{\mu\nu}_{\parallel}$$

$$\begin{aligned}\tilde{\tau}^{\mu\nu} &= -\frac{2\pi\delta(kq)}{\kappa}\mathcal{M}^{\mu\nu}_{\text{tree}} \\ &= 2\pi\delta(q_{\parallel}) m \eta^{\mu\nu}_{\parallel}\end{aligned}$$



$$i2\kappa V^{\mu\nu\ \alpha\beta\ \gamma\delta}_{h^3}(q,l,l+q)\tilde{f}_{\alpha\beta}(l)\tilde{f}_{\gamma\delta}(l+q) = -i4\kappa\left(\tilde{\mathcal{G}}^{\mu\nu}_{h^2}\left(\tilde{f}^{(l)}_{\alpha\beta},\tilde{f}^{(l+q)}_{\gamma\delta}\right) + \frac{1}{\xi}\tilde{\mathcal{H}}^{\mu\nu}_{h^2}\left(\tilde{f}^{(l)}_{\alpha\beta},\tilde{f}^{(l+q)}_{\gamma\delta}\right)\right)$$

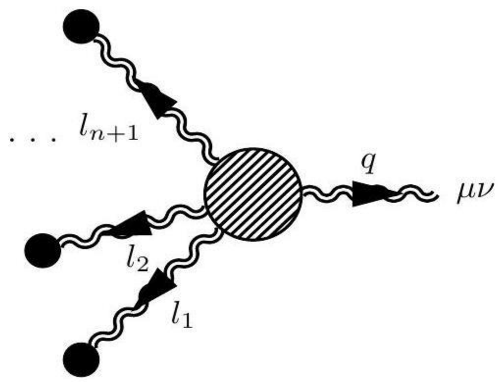
$$\mathcal{M}^{\mu\nu}_{1-\text{ loop}} \sim \int \frac{d^D l}{(2\pi)^D} V^{\mu\nu\alpha\beta\gamma\delta}_{h^3}(q,l,l+q)\tilde{h}^{G}_{\alpha\beta}(l)\tilde{h}^{G}_{\gamma\delta}(l+q)$$

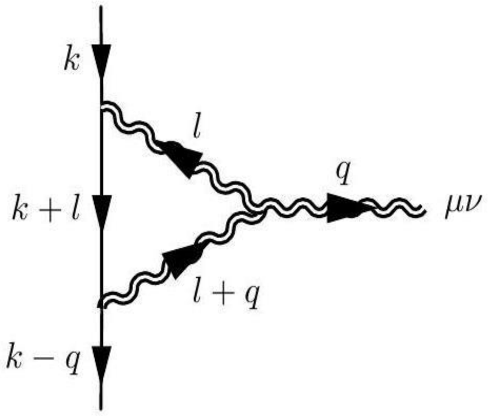


$$\int \frac{d^D l}{(2\pi)^D}\left(\tilde{G}^{\mu\nu}_{h^2}\left(\tilde{h}^{G}_{\mu\nu}(l),\tilde{h}^{G^2}_{\mu\nu}(q-l)\right) + \frac{1}{\xi}\tilde{H}^{\mu\nu}_{h^2}\left(\tilde{h}^{G}_{\mu\nu}(l),\tilde{h}^{G^2}_{\mu\nu}(q-l)\right)\right)$$

$$\int \frac{d^D l_{(1)}}{(2\pi)^D}\frac{d^D l_{(2)}}{(2\pi)^D}\tilde{G}^{\mu\nu}_{h^3}\left(\tilde{h}^{G}_{\mu\nu}(l_{(1)}),\tilde{h}^{G}_{\mu\nu}(l_{(2)}-l_{(1)}),\tilde{h}^{G}_{\mu\nu}(q-l_{(2)})\right)$$

$$i\mathcal{M}^{\mu\nu}_{\text{n-loop}} \sim$$





$$I = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 + i\epsilon} \frac{1}{(l+q_\perp)^2 + i\epsilon} \frac{1}{(l+k)^2 - m^2 + i\epsilon}$$

$$\frac{1}{(l+k)^2 - m^2 + i\epsilon} \approx \frac{1}{2kl + i\epsilon}$$
$$= \frac{1}{2m} \frac{1}{l_\parallel + i\epsilon}$$

$$\frac{1}{l_\parallel + i\epsilon} = \frac{1}{l_\parallel} - i\pi\delta(l_\parallel)$$

$$I = -\frac{i}{4m} \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel) \frac{1}{l_\perp^2} \frac{1}{(l_\perp + q_\perp)^2}$$

$$N_{D-1} = \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel) \frac{1}{l_\perp^2} \frac{1}{(l_\perp + q_\perp)^2}$$
$$= \int \frac{d^{D-1} l_\perp}{(2\pi)^{D-1}} \frac{1}{l_\perp^2} \frac{1}{(l_\perp + q_\perp)^2}$$

$$N_d = -\frac{\Omega_{d-2} \sqrt{-q_\perp^2}^{\,d-4}}{4(4\pi)^{d-2} \sin\left(\frac{\pi}{2} d\right)}$$

$$I^{\mu} = \int \frac{d^D l}{(2\pi)^D} \frac{l^{\mu}}{(l^2+i\epsilon)((l+q_\perp)^2+i\epsilon)((l+k)^2-m^2+i\epsilon)}$$
$$I^{\mu\nu} = \int \frac{d^D l}{(2\pi)^D} \frac{l^{\mu}l^{\nu}}{(l^2+i\epsilon)((l+q_\perp)^2+i\epsilon)((l+k)^2-m^2+i\epsilon)}$$

$$I^{\mu} = Aq_\perp^{\mu} + Bk^{\mu}$$

$$\begin{aligned} I_\perp^{\mu} &= Aq_\perp^{\mu} \\ &= -\frac{i}{4m}\int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{l_\perp^{\mu}}{l_\perp^2(l_\perp+q_\perp)^2} \end{aligned}$$

$$N_{D-1}^{\mu} = \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{l_\perp^{\mu}}{l_\perp^2(l_\perp+q_\perp)^2}$$

$$2q_\mu^\perp N_{D-1}^{\mu} = \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{1}{l_\perp^2} - \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{1}{(l_\perp+q_\perp)^2} - q_\perp^2\int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{1}{l_\perp^2(l_\perp+q_\perp)^2} = -q_\perp^2 N_{D-1}$$

$$N_{D-1}^{\mu} = -\frac{1}{2}q_\perp^{\mu}N_{D-1}$$

$$\begin{aligned} I_\parallel &= mB \\ &= \frac{1}{2m}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2(l+q_\perp)^2}\frac{l_\parallel}{l_\parallel+i\epsilon} \end{aligned}$$

$$I_\parallel = i\frac{N_D}{2m}$$

$$\begin{aligned} I^{\mu} &= -\frac{i}{4m}N_{D-1}^{\mu} + \frac{i}{2m^2}N_D k^{\mu} \\ &= \frac{i}{8m}N_{D-1}q_\perp^{\mu} + \frac{i}{2m^2}N_D k^{\mu} \end{aligned}$$

$$I^{\mu\nu} = -\frac{i}{4m}N_{D-1}^{\mu\nu} - \frac{i}{4m^2}N_D\left(q_\perp^{\mu}k^{\nu} + k^{\mu}q_\perp^{\nu}\right)$$

$$\begin{aligned} N_{D-1}^{\mu\nu} &= \int \frac{d^D l}{(2\pi)^D} 2\pi\delta(l_\parallel)\frac{l_\perp^{\mu}l_\perp^{\nu}}{l_\perp^2(l_\perp+q_\perp)^2} \\ &= \frac{q_\perp^2}{4(D-2)}N_{D-1}\left((D-1)\frac{q_\perp^{\mu}q_\perp^{\nu}}{q_\perp^2} - \eta_\perp^{\mu\nu}\right) \end{aligned}$$

$$\begin{aligned} 2\pi\delta(kq)i\mathcal{M}_{1-\text{loop}}^{\mu\nu} &= -i\frac{4}{\kappa}\left(\tilde{G}_{1-\text{loop}} + \frac{1}{\xi}\tilde{H}_{1-\text{loop}}\right) \\ = 2\pi\delta(kq)m^4(2i\kappa)(-i\kappa)^2 i^3 &\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2(l+q_\perp)^2((l+k)^2-m^2+i\epsilon)} \\ &\times f_{\alpha\beta}f_{\gamma\delta}V_{h^3}^{\alpha\beta}\gamma\delta\mu\nu(l,-l-q,q) \end{aligned}$$

$$f_{\alpha\beta} = \frac{k^{\mu}k^{\nu}}{m^2}\mathcal{P}^{-1}{}_{\mu\nu\alpha\beta} = \eta_\parallel^{\mu\nu}\mathcal{P}^{-1}{}_{\mu\nu\alpha\beta}$$

$$-2(1-\xi)I^{\mu\nu}_{\rho\kappa}\frac{l^\rho l_\sigma}{l^2}I^{\kappa\sigma}_{\alpha\beta}$$

$$\int\frac{d^Dl}{(2\pi)^D}\frac{lk}{l^2l^2(l+q)^2((l+k)^2-m^2+i\epsilon)}=\frac{1}{2}\int\frac{d^Dl}{(2\pi)^D}\frac{((l+k)^2-m^2)-l^2}{l^2l^2(l+q)^2((l+k)^2-m^2+i\epsilon)}$$
$$=\frac{1}{2}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2l^2(l+q)^2}-\frac{1}{2}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(l+q)^2((l+k)^2-m^2+i\epsilon)}$$

$$f_{\mu\nu}=\frac{D-3}{D-2}\eta^{\parallel}_{\mu\nu}-\frac{1}{D-2}\eta^{\perp}_{\mu\nu}$$

$$f_{\alpha\beta}f_{\gamma\delta}\tau^{\alpha\beta\gamma\delta\nu}_{h^3}(l,-l-q,q)=f_{\alpha\beta}f_{\gamma\delta}\left(U^{\mu\nu\alpha\beta\rho\gamma\delta\sigma}l_\rho(l+q)_\sigma+U^{\alpha\beta\gamma\delta\mu\nu\sigma}(l+q)_\rho q_\sigma-U^{\gamma\delta\mu\nu\rho\alpha\beta\sigma}q_\rho l_\sigma\right)$$

$$f_{\alpha\beta}f_{\gamma\delta}\tau^{\alpha\beta\gamma\delta\mu\nu}_{h^3}(l,-l-q,q)=f_{\alpha\beta}f_{\gamma\delta}\left(U^{\mu\nu\alpha\beta\rho\gamma\delta\sigma}l_\rho(l+q)_\sigma+U^{\alpha\beta\gamma\delta\rho\mu\nu\sigma}q_\rho q_\sigma\right)$$

$$2\pi\delta(kq)i\mathcal{M}^{\mu\nu}_{1-\text{loop}}=-2\pi\delta(kq)2m^4\kappa^3f_{\alpha\beta}f_{\gamma\delta}\left(U^{\mu\nu\alpha\beta\rho\gamma\delta\sigma}\left(I_{\rho\sigma}+I_\rho q_\sigma\right)+U^{\alpha\beta\gamma\delta\rho\mu\nu\sigma}q_\rho q_\sigma I\right)$$

$$I_{\rho\sigma}+I_\rho q_\sigma=i\frac{q^2N_{D-1}}{16(D-2)m}\left((D-3)\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}+\eta^\perp_{\rho\sigma}\right)+i\frac{N_D}{4m^2}\left(k_\rho q^\perp_\sigma-q^\perp_\rho k_\sigma\right)$$

$$2\pi\delta(kq)\mathcal{M}^{\mu\nu}_{1-\text{ loop}}=2\pi\delta(q_\parallel)\frac{m^2\kappa^3q^2_\perp N_{D-1}}{2}f_{\alpha\beta}f_{\gamma\delta}\left(-\frac{1}{4(D-2)}U^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma M^\perp_{\rho\sigma}+U^{\alpha\beta\gamma\delta\rho\mu\nu\sigma}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}\right)$$

$$M^\perp_{\rho\sigma}=(D-3)\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}+\eta^\perp_{\rho\sigma}$$

$$U^{\mu\nu\ \alpha\beta\rho}_{gf}\gamma\delta\sigma f_{\alpha\beta}f_{\gamma\delta}M^\perp_{\rho\sigma}$$

$$-\mathcal{P}^{\rho\sigma}_{\alpha\beta}h^{\alpha\beta}_{,\sigma}h^{\mu,\nu}_{\rho}\to-\mathcal{P}^{\rho\sigma}_{\alpha\beta}f^{\alpha\beta}f^\mu_\rho M^{\perp}{}^{\nu}_{\sigma}$$

$$U^{\mu\nu\ \alpha\beta\rho}_{gf}\gamma\delta\sigma f_{\alpha\beta}f_{\gamma\delta}M^\perp_{\rho\sigma}=\alpha\mathcal{P}^{\rho\sigma}_{\alpha\beta}f^{\alpha\beta}\left(-f^\mu_\rho M^{\perp\nu}_\sigma-f^\nu_\rho M^\perp_\sigma+f^{\mu\nu}M^\perp_{\rho\sigma}\right)$$
$$=\alpha\eta^{\rho\sigma}_\parallel\left(-f^\mu_\rho M^\perp_\sigma{}^{\nu}_{\sigma}-f^\nu_\rho M^\perp_\sigma{}^{\mu}_{\sigma}+f^{\mu\nu}M^\perp_{\rho\sigma}\right)=0$$

$$U^{\alpha\beta\ \gamma\delta\rho\mu\nu\sigma}_{gf}f_{\alpha\beta}f_{\gamma\delta}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}$$

$$-\alpha\left(I^{\alpha\beta\rho\kappa}I^{\gamma\delta}_{\kappa\lambda}\mathcal{P}^{\lambda\sigma\mu\nu}+I^{\alpha\beta\sigma\kappa}I^{\mu\nu}_{\kappa\lambda}\mathcal{P}^{\lambda\rho\gamma\delta}\right)f_{\alpha\beta}f_{\gamma\delta}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}=-\alpha\left(f^{\rho\kappa}f_{\kappa\lambda}\mathcal{P}^{\lambda\sigma\mu\nu}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}+f^{\sigma\kappa}I^{\mu\nu}_{\kappa\lambda}\eta^{\lambda\rho}_\parallel\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}\right)$$
$$=-\alpha\frac{1}{(D-2)^2}\mathcal{P}^{\rho\sigma\mu\nu}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}$$

$$\frac{1}{2}\alpha\left(\mathcal{P}^{\gamma\delta\rho\sigma}I^{\mu\nu\alpha\beta}+\mathcal{P}^{\mu\nu\rho\sigma}I^{\gamma\delta\alpha\beta}\right)f_{\alpha\beta}f_{\gamma\delta}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}=\frac{1}{2}\alpha\left(\eta^{\rho\sigma}_\parallel f^{\mu\nu}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}+\mathcal{P}^{\mu\nu\rho\sigma}f_{\alpha\beta}f^{\alpha\beta}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}\right)$$
$$=\frac{1}{2}\alpha f_{\alpha\beta}f^{\alpha\beta}\mathcal{P}^{\mu\nu\rho\sigma}\frac{q^\perp_\rho q^\perp_\sigma}{q^2_\perp}$$

$$U_{gf}^{\alpha\beta}\gamma\delta\rho\mu\nu\sigma f_{\alpha\beta}f_{\gamma\delta}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2} = \alpha\left(-\frac{1}{(D-2)^2}+\frac{1}{2}f_{\alpha\beta}f^{\alpha\beta}\right)\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$
$$= \alpha\frac{D-3}{2(D-2)}\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$
$$= \alpha\frac{D-3}{4(D-2)}\left(2\frac{q_\perp^\mu q_\perp^\nu}{q_\perp^2}-\eta^{\mu\nu}\right)$$

$$\tilde{H}_{1-\text{ loop}}^{\mu\nu} = -\alpha 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 q^2 N_{D-1}}{16}\frac{D-3}{D-2}\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}.$$

$$U_{(c)}^{\mu\nu}\alpha\beta\rho\gamma\delta\sigma f_{\alpha\beta}f_{\gamma\delta}M_{\rho\sigma}^\perp$$

$$U_{(c)}^{\alpha\beta}\gamma\delta\rho\mu\nu\sigma f_{\alpha\beta}f_{\gamma\delta}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$

$$U_{(c)}^{\mu\nu\ \alpha\beta\rho}\gamma\delta\sigma h_{\alpha\beta,\rho}h_{\gamma\delta,\sigma} = 2I_{\phi\epsilon}^{\mu\nu}\mathcal{P}_{\rho\sigma}^{\alpha\beta}\mathcal{P}_{\gamma\delta}^{\sigma\phi}h_{\alpha\beta}^{,\epsilon}h^{\gamma\delta,\rho} - \mathcal{P}_{\alpha\beta}^{\mu\rho}\mathcal{P}_{\gamma\delta}^{\nu\sigma}\eta_{\rho\sigma}h_{,\kappa}^{\alpha\beta}h^{\gamma\delta,\kappa}$$
$$+\mathcal{P}_{\rho\sigma}^{\mu\nu}\left(h_{,\beta}^{\rho\alpha}h_{,\alpha}^{\sigma\beta}-\frac{1}{2}h_\beta^{\alpha,\rho}h_\alpha^{\beta,\sigma}-h_{,\alpha}^{\rho\sigma}h_{,\beta}^{\alpha\beta}\right)$$

$$2I_{\phi\epsilon}^{\mu\nu}\mathcal{P}_{\rho\sigma}^{\alpha\beta}\mathcal{P}_{\gamma\delta}^{\sigma\phi}f_{\alpha\beta}f^{\gamma\delta}M_\perp^{\epsilon\rho} = 2I_{\phi\epsilon}^{\mu\nu}\eta_{\rho\sigma}^\parallel\eta_\parallel^{\sigma\phi}M_\perp^{\epsilon\rho} = 0$$

$$-\mathcal{P}_{\alpha\beta}^{\mu\rho}\mathcal{P}_{\gamma\delta}^{\nu\sigma}\eta_{\rho\sigma}f^{\alpha\beta}f^{\gamma\delta}M_\kappa^{\perp\kappa} = -\eta_\parallel^{\mu\rho}\eta_\parallel^{\nu\sigma}\eta_{\rho\sigma}M_\kappa^{\perp\kappa} = -\eta_\parallel^{\mu\nu}M_\kappa^{\perp\kappa}$$

$$M^{\perp}{}_{\kappa}^{\kappa} = D-3+D-1 = 2(D-2)$$

$$-\mathcal{P}_{\alpha\beta}^{\mu\rho}\mathcal{P}_{\gamma\delta}^{\nu\sigma}\eta_{\rho\sigma}f^{\alpha\beta}f^{\gamma\delta}M^{\perp}{}_{\kappa}^{\kappa} = -2(D-2)\eta_\parallel^{\mu\nu}$$

$$f^{\rho\alpha}f^{\sigma\beta}M_{\alpha\beta}^\perp = \frac{1}{(D-2)^2}M_\perp^{\rho\sigma}$$

$$-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta}M_\perp^{\rho\sigma} = -\frac{1}{2}\frac{(D-3)^2+D-1}{(D-2)^2}M_\perp^{\rho\sigma}$$

$$-f^{\rho\sigma}f^{\alpha\beta}M_{\alpha\beta}^\perp = \frac{1}{D-2}f^{\rho\sigma}M^{\perp}{}_{\kappa}^{\kappa} = 2f^{\rho\sigma}$$

$$f^{\rho\alpha}f^{\sigma\beta}M_{\alpha\beta}-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta}M^{\rho\sigma}-f^{\rho\sigma}f^{\alpha\beta}M_{\alpha\beta} = -\frac{1}{2}\frac{D-3}{D-2}M^{\rho\sigma}+2f^{\rho\sigma}$$

$$\mathcal{P}_{\rho\sigma}^{\mu\nu}M^{\rho\sigma} = -(D-2)\eta_\parallel^{\mu\nu}+(D-3)\left(\frac{q_\perp^\mu q_\perp^\nu}{q_\perp^2}-\eta_\perp^{\mu\nu}\right)$$

$$\mathcal{P}_{\rho\sigma}^{\mu\nu}\left(f^{\rho\alpha}f^{\sigma\beta}M_{\alpha\beta}-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta}M^{\rho\sigma}-f^{\rho\sigma}f^{\alpha\beta}M_{\alpha\beta}\right)$$
$$= \frac{D+1}{2}\eta_\parallel^{\mu\nu}-\frac{1}{2}\frac{(D-3)^2}{D-2}\left(\frac{q_\perp^\mu q_\perp^\nu}{q_\perp^2}-\eta_\perp^{\mu\nu}\right)$$

$$U^{\mu\nu\ \alpha\beta\rho}{}_{\gamma\delta\sigma} f_{\alpha\beta} f_{\gamma\delta} M_{\rho\sigma} = -\frac{3(D-3)}{2}\eta_{\parallel}^{\mu\nu} - \frac{1}{2}\frac{(D-3)^2}{D-2}\left(\frac{q_\perp^\mu q_\perp^\nu}{q_\perp^2} - \eta_\perp^{\mu\nu}\right)$$

$$U^{\alpha\beta}{}_{\gamma\delta\rho\mu\nu\sigma} f_{\alpha\beta} f_{\gamma\delta} \frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2} = \frac{1}{2}\frac{D-4}{D-2}\eta_{\parallel}^{\mu\nu} - \frac{1}{2}\frac{D-1}{D-2}\left(\eta_\perp^{\mu\nu} - \frac{q_\perp^\mu q_\perp^\nu}{q_\perp^2}\right)$$

$$\tilde{G}^{\mu\nu}_{1-\text{ loop}} = 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 q^2 N_{D-1}}{64}\left(\frac{D-7}{D-2}\eta_\parallel^{\mu\nu} - \frac{(D-3)(3D-5)}{(D-2)^2}\left(\eta_\perp^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\right).$$

$$2\pi\delta(kq)\mathcal{M}^{\mu\nu}_{1-\text{ loop}} = -\frac{4}{\kappa}\left(\tilde{G}^{\mu\nu}_{1-\text{ loop}} + \frac{1}{\xi}\tilde{H}^{\mu\nu}_{1-\text{ loop}}\right)$$

$$\tilde{G}^{\mu\nu}_{1-\text{ loop}} = 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 q^2 N_{D-1}}{64}\left(\frac{D-7}{D-2}\eta_\parallel^{\mu\nu} - \frac{(D-3)(3D-5)}{(D-2)^2}\left(\eta_\perp^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\right)$$

$$\tilde{H}^{\mu\nu}_{1-\text{ loop}} = -\alpha 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 q^2 N_{D-1}}{16}\frac{D-3}{D-2}\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$

$$G^{\mu\nu\ \mu\nu}_{(\text{c})\ \alpha\beta}\tilde{H}^{\alpha\beta}_{\text{non-linear}} = 0$$

$$G^{\alpha\beta}_{c^{\alpha\beta}_{\mu\nu}}\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2} = \frac{q_\mu^\perp q_\nu^\perp}{q_\perp^2} - 2\mathcal{J}^{\alpha\beta}_{\mu\nu}\mathcal{P}^{\rho\sigma}_{\alpha\beta}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$

$$\begin{aligned}\mathcal{J}^{\alpha\beta}_{\mu\nu}\mathcal{P}^{\rho\sigma}_{\alpha\beta}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2} &= \mathcal{J}^{\alpha\beta}_{\mu\nu}\mathcal{P}^{\rho\sigma}_{\alpha\beta}\mathcal{J}^{\gamma\delta}_{\rho\sigma}\eta_{\gamma\delta} \\ &= \frac{1}{2}\mathcal{J}^{\gamma\delta}_{\mu\nu}\eta_{\gamma\delta} \\ &= \frac{1}{2}\frac{q_\mu^\perp q_\nu^\perp}{q_\perp^2}\end{aligned}$$

$$\tilde{h}^{G^2}_{\mu\nu} = 2\frac{G^{(\text{c})}_{\mu\nu\alpha\beta}}{q^2}\tilde{G}^{\alpha\beta}_{1-\text{ loop}} + 2\frac{G^{(\text{gf})}_{\mu\nu\alpha\beta}}{q^2}\tilde{H}^{\alpha\beta}_{1-\text{ loop}}$$

$$\frac{G^{(\text{c})}_{\mu\nu\alpha\beta}}{q^2}2\tilde{G}^{\alpha\beta}_{1-\text{loop}} = 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 N_{D-1}}{32}\left(4\frac{(D-3)^2}{(D-2)^2}\eta^\parallel_{\mu\nu} + \frac{(D-3)(3D-5)}{(D-2)^2}\frac{q_\mu q_\nu}{q^2} - \frac{D-7}{(D-2)^2}\eta^\perp_{\mu\nu}\right)$$

$$\frac{G^{(\text{gf})}_{\mu\nu\alpha\beta}}{q^2}2\tilde{H}^{\alpha\beta}_{1-\text{loop}} = -2\pi\delta(q_\parallel)\alpha\frac{\kappa^4 m^2 N_{D-1}}{8}\frac{D-3}{D-2}\frac{q_\mu q_\nu}{q^2}$$

$$\mathcal{P}^{\mu\nu\rho\sigma}\frac{q_\rho^\perp q_\sigma^\perp}{q_\perp^2}$$

$$\tilde{h}^{G^2}_{\mu\nu} = 2\pi\delta(q_\parallel)\frac{\kappa^4 m^2 N_{D-1}}{32}\frac{(D-3)^2}{(D-2)^2}\left(4\eta^\parallel_{\mu\nu} + \left(\frac{3D-5}{D-3} - 4\alpha\frac{D-2}{D-3}\right)\frac{q_\mu q_\nu}{q^2} - \frac{D-7}{(D-3)^2}\eta^\perp_{\mu\nu}\right)$$

$$N_{D-1} = \frac{\Omega_{D-3}\sqrt{-q_\perp^2}^{\,D-5}}{4(4\pi)^{D-3}\cos\left(\frac{\pi}{2}D\right)}$$

$$\int \frac{d^d q_\perp}{(2\pi)^d} e^{-ix_\perp q_\perp} (-q_\perp^2)^{\frac{n}{2}} = \frac{2^n}{\sqrt{\pi}^d} \frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \frac{1}{(-x_\perp^2)^{\frac{d+n}{2}}}$$

$$\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} N_{D-1} = \left(\frac{1}{\Omega_{D-2}(D-3)\sqrt{-x_\perp^2}}\right)^2$$

$$\frac{\partial}{\partial x_\perp^\mu} \frac{\partial}{\partial x_\perp^\nu} \int \frac{d^d q_\perp}{(2\pi)^d} e^{-ix_\perp q_\perp} (-q_\perp^2)^{\frac{n}{2}} = -\int \frac{d^d q_\perp}{(2\pi)^d} e^{-ix_\perp q_\perp} q_\mu^\perp q_\nu^\perp (-q_\perp^2)^{\frac{n}{2}}$$

$$\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} N_{D-1} \frac{q_\mu^\perp q_\nu^\perp}{q_\perp^2} = \left(\frac{1}{\Omega_{D-2}(D-3)\sqrt{-x_\perp^2}^{D-3}}\right)^2 \frac{1}{D-5}\left(2(D-3)\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2} - \eta_{\mu\nu}^\perp\right)$$

$$h_{\mu\nu}^{G^2} = \frac{\mu^2}{r^{2(D-3)}}\left(\frac{1}{2}\eta_{\mu\nu}^{\parallel} - \frac{(4\alpha-3)D-8\alpha+5}{4(D-5)}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2} - \frac{2(1-\alpha)D^2-(13-10\alpha)D+25-12\alpha}{4(D-3)^2(D-5)}\eta_{\mu\nu}^\perp\right)$$

$$G_\sigma^{G^2} = \mathcal{P}_{\rho\sigma}^{\mu\nu} h_{G^2}{}_{\mu\nu}^{\rho} - \alpha\Gamma_{\sigma\alpha\beta}^{\rho\mu\nu} h_G^{\alpha\beta} h_{\mu\nu,\rho}^G$$

$$N_{D-1} = \frac{(-1)^n}{8(16\pi)^n n!} \frac{(-q_\perp^2)^n}{\sqrt{-q_\perp^2}}$$

$$N_{D-1} = \frac{1}{\epsilon}\frac{(-1)^{n+1}\Omega_{2+2n}(-q_\perp^2)^n}{(4\pi)^{3+2n}} + \frac{(-1)^{n+1}n!}{16\pi^2(4\pi)^n(1+2n)!}(-q_\perp^2)^n \ln(-q_\perp^2 r_0^2) + \cdots$$

$$N_{D-1} = \frac{(-1)^{n+1}n!}{16\pi^2(4\pi)^n(1+2n)!}(-q_\perp^2)^n \ln(-q_\perp^2 r_0^2)$$

$$\ln(-q^2) = \frac{1}{\epsilon}((-q^2)^\epsilon - 1)$$

$$\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} \frac{N_{D-1}}{q_\perp^2} = \frac{(-1)^n n!}{16\pi^2(4\pi)^n(1+2n)!}\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} (-q_\perp^2)^{n-1} \ln(-q_\perp^2)$$
$$= \frac{(-1)^n n!}{16\pi^2(4\pi)^n(1+2n)!}\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} \frac{1}{\epsilon}((-q_\perp^2)^{n-1+\epsilon} - (-q_\perp^2)^{n-1})$$

$$\int \frac{d^{D-1} q_\perp e^{-iq_\perp x_\perp}}{(2\pi)^{D-1}} \frac{N_{D-1}}{q_\perp^2} = \frac{(-1)^n n!}{16\pi^2(4\pi)^n(1+2n)!} \frac{4^{n-1}}{\pi^{2+n}(-x_\perp^2)^{2n+1}} \frac{1}{\epsilon}\left(\frac{4^\epsilon\Gamma(2n+1+\epsilon)}{\Gamma(1-n-\epsilon)(-x_\perp^2)^\epsilon} - \frac{\Gamma(2n+1)}{\Gamma(1-n)}\right)$$

$$\frac{1}{\epsilon}\left(\frac{4^\epsilon\Gamma(2n+1+\epsilon)}{\Gamma(1-n-\epsilon)(-x_\perp^2)^\epsilon} - \frac{\Gamma(2n+1)}{\Gamma(1-n)}\right) = \Gamma(2n+1)\left(\frac{\psi(1-n)}{\Gamma(1-n)} + \frac{\psi(2n+1) - \ln\left(-\frac{x_\perp^2}{4}\right)}{\Gamma(1-n)}\right)$$

$$\frac{1}{\epsilon}\left(\frac{4^\epsilon\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)(-x_\perp^2)^\epsilon} - 1\right) = 2\psi(1) - \ln\left(-\frac{x_\perp^2}{4}\right)$$

$$\frac{1}{\epsilon}\left(\frac{4^{\epsilon}\Gamma(1+\epsilon)r_0^{2\epsilon}}{\Gamma(1-\epsilon)(-x_\perp^2)^{\epsilon}}-1\right)=-\ln\left(-\frac{x_\perp^2 e^{2\gamma}}{4r_0^2}\right)$$

$$\int\frac{d^5q\delta(q_\parallel)e^{-iqx}}{(2\pi)^4}N_4\frac{q_\mu q_\nu}{q^2}=-\frac{1}{32\pi^4\sqrt{-x_\perp^2}}\left(\eta_{\mu\nu}^\perp-6\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}-\left(\eta_{\mu\nu}^\perp-4\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)\ln\left(-\frac{x_\perp^2 e^{2\gamma}}{4r_0^2}\right)\right)$$

$$h_{\mu\nu}^{(2)}=\frac{\mu^2}{r^4}\left(\frac{1}{2}\eta_{\mu\nu}^\parallel-\frac{2(6\alpha-5)\ln\frac{r}{r_0}-1}{16}\eta_{\mu\nu}^\perp+\frac{(6\alpha-5)\left(4\ln\frac{r}{r_0}-1\right)}{8}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)$$

$$x^\mu\to x^\mu+\beta\frac{\mu^2}{r^4}x_\perp^\mu+\cdots$$

$$h_{\mu\nu}^{G^2}\to h_{\mu\nu}^{G^2}-\epsilon_{\mu,\nu}-\epsilon_{\nu,\mu}$$

$$G_\sigma^{G^2}\to G_\sigma^{G^2}+\partial^2\epsilon_\sigma$$

$$\partial^2\frac{1}{r^{D-3}}\propto\delta^{D-1}(x_\perp)$$

$$\partial^2\frac{x_\mu^\perp}{r^{D-1}}\propto\partial_\mu\partial^2\frac{1}{r^{D-3}}\propto\partial_\mu\delta^{D-1}(x_\perp)$$

$$h_{\mu\nu}^{G^2}\to h_{\mu\nu}^{G^2}-2\beta\frac{\mu^2}{r^4}\left(-4\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}+\eta_{\mu\nu}^\perp\right)$$

$$h_{\mu\nu}^{G^2}\to h_{\mu\nu}^{G^2}+\ln(\gamma)\frac{6\alpha-5}{8}\frac{\mu^2}{r^4}\left(-4\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}+\eta_{\mu\nu}^\perp\right)$$

$$\epsilon^\sigma=\beta\frac{x_\perp^\sigma}{r^{D-1}}$$

$$-D+1=m(-D+3),$$

$$d\tau^2=\left(1-\frac{\mu}{R^n}\right)dt^2-\frac{1}{1-\frac{\mu}{R^n}}dR^2-R^2d\Omega_{D-2}^2$$

$$\begin{gathered}x^0=t\\ x^1=r\cos(\theta_1)\\ x^2=r\sin(\theta_1)\cos(\theta_2)\\ x^3=r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\ \dots\\ x^{D-1}=r\sin(\theta_1)\sin(\theta_2)\dots\sin(\theta_{D-2})\end{gathered}$$

$$dR^2=\frac{dR^2}{dr^2}\left(\frac{x_\perp dx_\perp}{r}\right)^2$$

$$d\Omega_{D-2}^2=-\frac{1}{r^2}\left(dx_\perp^2+\left(\frac{x_\perp dx_\perp}{r}\right)^2\right)$$

$$d\tau^2 = \left(1-\frac{\mu}{R^n}\right)dt^2 - \frac{1}{1-\frac{\mu}{R^n}}\frac{dR^2}{dr^2}\left(\frac{x_\perp dx_\perp}{r}\right)^2 + \frac{R^2}{r^2}\left(dx_\perp^2 + \left(\frac{x_\perp dx_\perp}{r}\right)^2\right)$$

$$g_{\mu\nu} = \left(1-\frac{\mu}{R^n}\right)\eta_{\mu\nu}^{\parallel} + \frac{1}{1-\frac{\mu}{R^n}}\frac{dR^2}{dr^2}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2} + \frac{R^2}{r^2}\left(\eta_{\mu\nu}^{\perp} - \frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)$$

$$R = r\left(1 + a\frac{\mu}{r^n} + b\left(a\frac{\mu}{r^n}\right)^2 + \cdots\right)$$

$$g_{\mu\nu} \approx \eta_{\mu\nu} - \frac{\mu}{r^n}\eta_{\mu\nu}^{\parallel} + 2a\frac{\mu}{r^n}\eta_{\mu\nu}^{\perp} - (2na-1)\frac{\mu}{r^n}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}$$

$$h_{\mu\nu}^{G} = -\frac{\mu}{r^n}\left(\eta_{\mu\nu}^{\parallel} - 2a\eta_{\mu\nu}^{\perp} + (2na-1)\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)$$

$$\mathcal{P}_{\rho\sigma}^{\mu\nu} h^{G,\rho}{}_{\mu\nu} = 0$$

$$\partial_\sigma h_{\mu\nu}^{G} = -n\frac{\mu}{r^{n+1}}\frac{x_\sigma^\perp}{r}\left(\eta_{\mu\nu}^{\parallel} - 2a\eta_{\mu\nu}^{\perp} + (2na-1)\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right) - \frac{1}{r^{n+1}}(2na-1)\lambda_{\sigma\mu\nu}$$

$$\begin{aligned}\lambda_{\sigma\mu\nu} &= r\partial_\sigma \frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2} \\ &= \left(-\eta_{\sigma\mu}^{\perp} + \frac{x_\sigma^\perp x_\mu^\perp}{x_\perp^2}\right)\frac{x_\nu^\perp}{r} + \left(-\eta_{\sigma\nu}^{\perp} + \frac{x_\sigma^\perp x_\nu^\perp}{x_\perp^2}\right)\frac{x_\mu^\perp}{r}\end{aligned}$$

$$G_\sigma^G = \frac{\mu}{r^{n+1}}(2na-1)\frac{x_\sigma^\perp}{r}$$

$$\begin{aligned}1-\frac{\mu}{R^n} &= 1 - \frac{\mu}{r^n} + \frac{1}{2}\left(\frac{\mu}{r^n}\right)^2 + \cdots \\ \frac{R^2}{r^2} &= 1 + \frac{1}{n}\frac{\mu}{r^n} + (2b+1)\frac{1}{4n^2}\left(\frac{\mu}{r^n}\right)^2 + \cdots \\ \frac{dR^2}{dr^2} &= 1 - \frac{n-1}{n}\frac{\mu}{r^n} - (2(2n-1)b-(n-1)^2)\frac{1}{4n^2}\left(\frac{\mu}{r^n}\right)^2 + \cdots \\ \frac{1}{1-\frac{\mu}{R^n}} &= 1 + \frac{\mu}{r^n} + \frac{1}{2}\left(\frac{\mu}{r^n}\right)^2 + \cdots \\ \frac{1}{1-\frac{\mu}{R^n}}\frac{dR^2}{dr^2} &= 1 + \frac{1}{n}\frac{\mu}{r^n} + (1-n(n-2)-2(2n-1)b)\frac{1}{4n^2}\left(\frac{\mu}{r^n}\right)^2 + \cdots\end{aligned}$$

$$g_{\mu\nu} \approx \eta_{\mu\nu} - \frac{\mu}{r^n}\left(\eta_{\mu\nu}^{\parallel} - \frac{1}{n}\eta_{\mu\nu}^{\perp}\right) + \left(\frac{\mu}{r^n}\right)^2\left(\frac{1}{2}\eta_{\mu\nu}^{\parallel} + \frac{2b+1}{4n^2}\eta_{\mu\nu}^{\perp} - \frac{4b+n-2}{4n}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)$$

$$h_{\mu\nu}^{G^2} = \frac{\mu^2}{r^{2n}}\left(\frac{1}{2}\eta_{\mu\nu}^{\parallel} + \frac{2b+1}{4n^2}\eta_{\mu\nu}^{\perp} - \frac{4b+n-2}{4n}\frac{x_\mu^\perp x_\nu^\perp}{x_\perp^2}\right)$$

$$G_\sigma^{G^2} = \mathcal{P}_{\rho\sigma}^{\mu\nu} h_{\mu\nu}^{G^2,\rho} - \alpha\Gamma_{\sigma\alpha\beta}^{\rho\mu\nu} h_G^{\alpha\beta} h_{\mu\nu,\rho}^{G}$$

$$\partial_\rho h^{G^2}_{\mu\nu} = \frac{2n\mu^2}{r^{2n+1}}\frac{x^\perp_\rho}{r}\left(\frac{1}{2}\eta^{\parallel}_{\mu\nu} + \frac{2b+1}{4n^2}\eta^\perp_{\mu\nu} - \frac{4b+n-2}{4n}\frac{x^\perp_\mu x^\perp_\nu}{x^2_\perp}\right) - \frac{\mu^2}{r^{2n+1}}\frac{4b+n-2}{4n}\lambda_{\rho\mu\nu}$$

$$\alpha\Gamma^{\rho\mu\nu}_{\sigma\alpha\beta}h^{\alpha\beta}_G h^G_{\mu\nu,\rho} = -\alpha\frac{\mu^2}{r^{2n+1}}\frac{n+1}{2}\frac{x^\perp_\sigma}{r}$$

$$\mathcal{P}^{\mu\nu}_{\rho\sigma}h^{G^2,\rho} = -\frac{\mu^2}{r^{2n+1}}\frac{n^2+1+(n-2)b}{2n}\frac{x^\perp_\sigma}{r}$$

$$G^{G^2}_\sigma = -\frac{\mu^2}{2nr^{2n+1}}\frac{x^\perp_\sigma}{r}(n^2+1+(n-2)b-\alpha n(n+1))$$

$$b = \frac{-(1-\alpha)n^2+\alpha n-1}{n-2}$$

$$R = r\left(1+\frac{\mu}{2nr^n}+\left(b_0+b_1\ln\left(\frac{r}{r_0}\right)\right)\left(\frac{\mu}{2nr^n}\right)^2+\cdots\right)$$

$$\frac{1}{1-\frac{\mu}{R^n}}\frac{dR^2}{dr^2} = (\ldots) + 2b_1\left(\frac{\mu}{2nr^n}\right)^2$$

$$h^{G^2}_{\mu\nu} = (\ldots) + 2b_1\frac{x^\perp_\mu x^\perp_\nu}{x^2_\perp}\left(\frac{\mu}{2nr^n}\right)^2$$

$$\partial_\rho h^{G^2}_{\mu\nu} = (\ldots) + b_1\frac{\mu^2}{4n^2r^{2n+1}}\left(8n\frac{x^\perp_\sigma}{r}\frac{x^\perp_\mu x^\perp_\nu}{x^2_\perp} - 2\frac{x^\perp_\sigma}{r}\eta^\perp_{\mu\nu} + 2\lambda_{\sigma\mu\nu}\right)$$

$$h^{G^2}_{\mu\nu} = \frac{\mu^2}{r^{2n}}\left(\frac{1}{2}\eta^{\parallel}_{\mu\nu} + \frac{2b+1}{4n^2}\eta^\perp_{\mu\nu} + \left(-\frac{4b+n-2}{4n}+\frac{b_1}{2n^2}\right)\frac{x^\perp_\mu x^\perp_\nu}{x^2_\perp}\right)$$

$$\mathcal{P}^{\mu\nu}_{\rho\sigma}h^{G^2,\rho} = \frac{\mu^2}{r^{2n+1}}\left(-\frac{n^2+1+(n-2)b}{2n}+\frac{3n-2}{4n^2}b_1\right)\frac{x^\perp_\sigma}{r}$$

$$G^{G^2}_\sigma = -\frac{\mu^2}{2nr^{2n+1}}\left((n-2)b - \frac{3n-2}{2n}b_1 - \alpha n(n+1) + n^2 + 1\right)$$

$$b_1 = 5 - 6\alpha$$

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