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TEORÍA CUÁNTICA DE CAMPOS RELATIVISTAS. CUANTIZACIÓN. VOLUMEN I

QUANTUM THEORY OF RELATIVISTIC FIELDS. QUANTIZATION. VOLUME I

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Teoría Cuántica de Campos Relativistas. Cuantización. Volumen I

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RESUMEN

En este trabajo, me propondré formalizar matemáticamente la Teoría Cuántica de Campos Relativistas (TCCR), en escenarios de gravedad y supergravedad cuánticas. Se profundizará principalmente en la necesidad ineludible de encontrar y determinar simetrías y supersimetrías isométricas y homeomorfas, esto a propósito de campos de gauge asimétricos y entrópicos. La finalidad de este artículo, es dotar a la TCCR, de un modelo formal que reconcilie la relatividad especial y general con la mecánica cuántica, desde dos conceptos esenciales, siendo éstos, gravedad y supergravedad cuánticas, y la relación con sus consecuencias lógicas. Intentaremos aquí, compactar un modelo que permita predecir y detectar el comportamiento de una partícula blanca y oscura respectivamente, así como sus interacciones con el supergravitón y el gravitón, según corresponda, esto, cuando existe interferencia gravitónica, pero quiero en lo principal, describir un modelo de simulación por deformación del espacio – tiempo cuántico, cuando interactúa, exclusivamente una partícula supermasiva o una partícula estrella, debido a la inmensa densidad de su masa y/o energía y entender por tanto, a la gravedad, como una cualidad propia de las partículas subatómicas cuyo centro de masa – energía es extremo, lo que permite la distorsión del plano cuántico, a tal punto de hacerlo, en dimensiones altas o en dimensiones sin desprendimiento y la formación de materia y energía oscuras, integradas éstas últimas, con la existencia de partículas oscuras, en extremo densas, en la que la gravedad, es el único elemento transdimensional y multisectorial.

Palabras Clave: teoría cuántica de campos relativistas, supergravedad cuántica, gravedad cuántica, dimensiones D, supersimetría

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Quantum Theory of Relativistic Fields. Quantization. Volume I

ABSTRACT

In this work, I will propose to mathematically formalize the Quantum Theory of Relativistic Fields (TCCR), in quantum gravity and supergravity scenarios. The unavoidable need to find and determine isometric and homeomorphic symmetries and supersymmetries will be deepened, this with regard to asymmetric and entropic gauge fields. The purpose of this article is to provide the TCCR with a formal model that reconciles special and general relativity with quantum mechanics, from two essential concepts, these being, quantum gravity and supergravity, and the relationship with their logical consequences. We will try here to compact a model that allows predicting and detecting the behavior of a white and dark particle respectively, as well as their interactions with the supergraviton and the graviton, as appropriate, this, when there is gravitonic interference, but I want to describe a simulation model by deformation of quantum space-time, when it interacts, exclusively, a supermassive particle or a star particle, due to the immense density of its mass and/or energy and therefore understand gravity as a quality of subatomic particles whose center of mass-energy is extreme, which allows the distortion of the quantum plane, to the point of doing so, in high dimensions or in dimensions without detachment and the formation of dark matter and energy, The latter are integrated with the existence of dark, extremely dense particles, in which gravity is the only transdimensional and multisectoral element.

Keywords: quantum theory of relativistic fields, quantum supergravity, quantum gravity, D dimensions, supersymmetry

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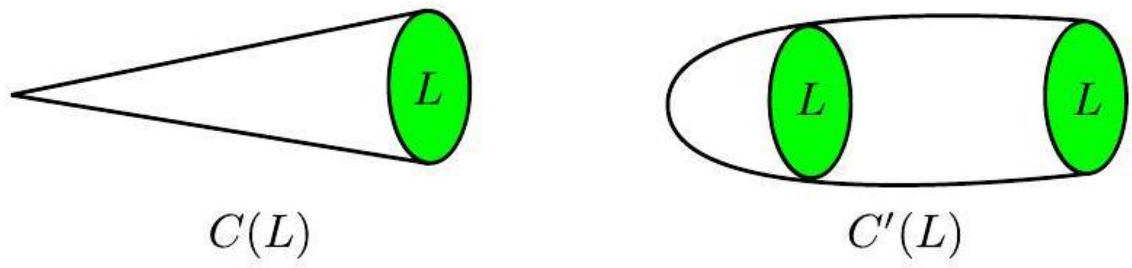
INTRODUCCIÓN

La Teoría Cuántica de Campos Relativistas, plantea algunas hipótesis teóricas, las mismas que serán abordadas en este trabajo, entre ellas, la existencia de partículas supermasivas y partículas estrella respectivamente, la existencia de agujeros negros a escala cuántica, la existencia de ondas cuánticas gravitacionales por colapso de una partícula estrella u oscura o por colisión de dos partículas con centros de masa y/o energía extremadamente densos, pero sobre todo, la supergravedad y gravedad cuánticas, entendidas como un fenómeno de distorsión y deformación del espacio – tiempo cuántico, por interacción de las antes referidas partículas, con o sin intervención gravitónica y en un marco de dualidad holográfica y simetrías de gauge compactas e isométricas. El objetivo de este trabajo es cuantizar los fenómenos antes referidos, robusteciendo la física de partículas que le es inherente a este sistema y por ende, la aplicación de la mecánica relativista a la física cuántica y viceversa.

RESULTADOS Y DISCUSIÓN

El desarrollo matemático expuesto a continuación, complementa el sistema de ecuaciones contenido en el volumen II de este manuscrito.

Modelo cohomológico de Supergravedad Cuántica Relativista con o sin intervención supergravitónica (también se puede referir como interferencia o intervención gravitónica, en un sentido más amplio): Ecuaciones de Campo.



$$S_{\mathrm{Sym}} = N \int B_2 \wedge dC_2$$

$$\Gamma^{(1)} = \frac{H_2(X, \partial X; \mathbb{Z})}{H_2(X; \mathbb{Z})}$$

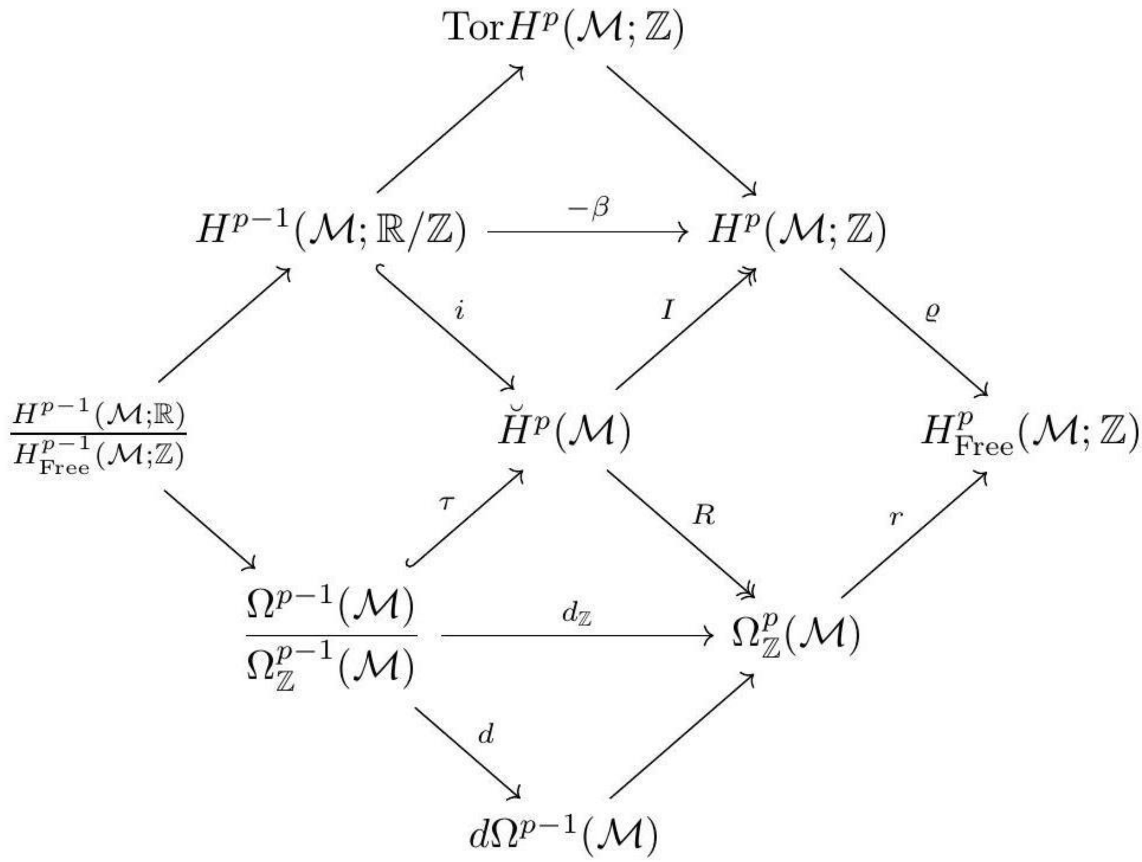
$$\breve{G}_4 = \breve{F} \star \breve{v}_2 + \breve{B}_2 \star \breve{t}_2.$$

$$\frac{S_{\mathrm{top}}}{2\pi} = -\frac{1}{6} \int_{M_{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4$$

$$k\alpha_p = d\beta_{p-1}$$

$$C_3 \supset \tilde{B}_2 \wedge \beta_1 + \tilde{A}_1 \wedge \alpha_2, G_4 \supset \left(d\tilde{A}_1 + k\tilde{B}_2\right) \wedge \alpha_2 + d\tilde{B}_2 \wedge \beta_1$$

$$S \supset \int \; -\frac{1}{2} g_{\tilde{A}_1\tilde{A}_1}\left(d\tilde{A}_1 + k\tilde{B}_2\right) \wedge * \left(d\tilde{A}_1 + k\tilde{B}_2\right) - \frac{1}{2} g_{\tilde{B}_2\tilde{B}_2} d\tilde{B}_2 \wedge * \, d\tilde{B}_2$$



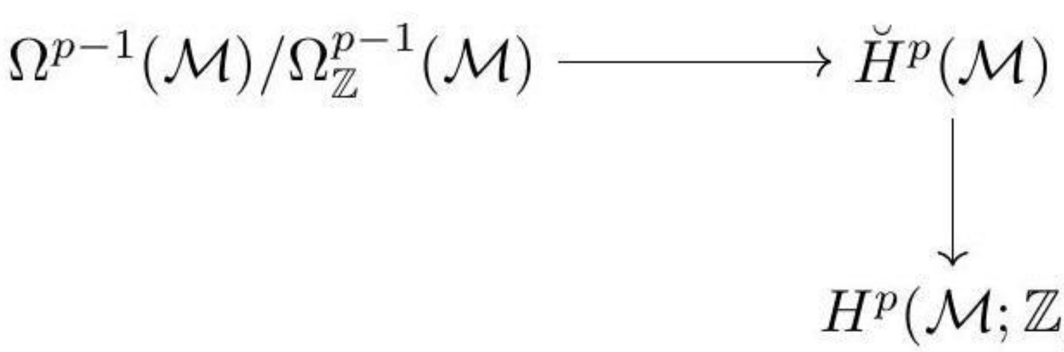
$$r(R(\breve{a})) = \varrho(I(\breve{a}))$$

$$I(\breve{a}) = I(i(u)) = -\beta(u)$$

$$\ldots \to H^{p-1}(\mathcal{M};\mathbb{Z}) \xrightarrow{\varrho} H^{p-1}(\mathcal{M};\mathbb{R}) \to H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} H^p(\mathcal{M};\mathbb{Z}) \xrightarrow{\varrho} H^p(\mathcal{M};\mathbb{R}) \to \cdots$$

$$R(\breve{a}) = R(\tau([\omega])) = d_{\mathbb{Z}}[\omega]$$

$$\frac{H^{p-1}(\mathcal{M};\mathbb{R})}{H^{p-1}_{\mathrm{Free}}(\mathcal{M};\mathbb{Z})} \cong \frac{\Omega^{p-1}_{\mathrm{closed}}(\mathcal{M})}{\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})}$$



$$\breve{a} = \breve{\Phi} + \tau([\omega]),$$

$$\star\colon \check{H}^p(\mathcal{M}) \times \check{H}^q(\mathcal{M}) \to \check{H}^{p+q}(\mathcal{M}).$$

$$\check{a} \star \check{b} = (-)^{pq}\check{b} \star \check{a}, I(\check{a} \star \check{b}) = I(\check{a}) \smile I(\check{b}), R(\check{a} \star \check{b}) = R(\check{a}) \wedge R(\check{b})$$

$$\tau([\omega]) \star \check{b} = \tau([\omega \wedge R(\check{b})]), i(u) \star \check{b} = i(u \smile I(\check{b})),$$

$$\int_{\mathcal{F}} : \check{H}^p(\mathcal{M}) \to \check{H}^{p-\dim(\mathcal{F})}(\mathcal{B}),$$

$$\int_{\mathcal{F}} R(\check{a}) = R\left(\int_{\mathcal{F}} \check{a}\right), \int_{\mathcal{F}} I(\check{a}) = I\left(\int_{\mathcal{F}} \check{a}\right).$$

$$\int_{\mathcal{F}} i(u) = i\left(\int_{\mathcal{F}} u\right), \int_{\mathcal{F}} \tau([\omega]) = \tau\left(\left[\int_{\mathcal{F}} \omega\right]\right).$$

$$\int_{\mathcal{M}} \check{a} = \int_{\mathcal{M}} I(\check{a}) = \int_{\mathcal{M}} R(\check{a}) \in \mathbb{Z}, \check{a} \in \check{H}^{\dim(\mathcal{M})}(\mathcal{M})$$

$$\int_{\mathcal{M}} \check{a} = \int_{\mathcal{M}} u \in \mathbb{R}/\mathbb{Z}, \check{a} \in \check{H}^{\dim(\mathcal{M})+1}(\mathcal{M}), u \in H^{\dim(\mathcal{M})}(\mathcal{M}; \mathbb{R}/\mathbb{Z}), \check{a} = i(u).$$

$$e^{iS_{\text{top}}} = \exp\ 2\pi i \int_{\mathcal{M}_{11}} \left[-\frac{1}{6} C_3 \wedge G_4 \wedge G_4 - C_3 \wedge X_8\right],$$

$$X_8 = \frac{1}{192}[p_1(T\mathcal{M}_{11})^2 - 4p_2(T\mathcal{M}_{11})].$$

$$\frac{S_{\text{top}}}{2\pi} = \int_{\mathcal{M}_{11}} \check{I}_{12} \text{ mod} 1$$

$$\check{I}_{12} = -\frac{1}{6}\check{G}_4 \star \check{G}_4 \star \check{G}_4 - \frac{1}{192}\check{G}_4 \star \check{p}_1(T\mathcal{M}_{11}) \star \check{p}_1(T\mathcal{M}_{11}) + \frac{1}{48}\check{G}_4 \star \check{p}_2(T\mathcal{M}_{11}).$$

$$\mathcal{M}_{11} = \mathcal{W}_{11-n} \times L_n.$$

$$p_{\mathcal{W}}\colon \mathcal{M}_{11} \to \mathcal{W}_{11-n}, p_L\colon \mathcal{M}_{11} \to L_n$$

$$\text{if}\, \lambda \in \Omega^r(\mathcal{W}_{11-n})\ \omega \in \Omega^s(L_n), \lambda \wedge \omega\ p_{\mathcal{W}}^*(\lambda) \wedge p_L^*(\omega),$$

$$\begin{aligned} &\text{if}\, a \in H^r(\mathcal{W}_{11-n}; \mathbb{Z})\ b \in H^s(L_n; \mathbb{Z}), \quad && a \smile b\ p_{\mathcal{W}}^*(a) \smile p_L^*(b), \\ &\text{if}\, \check{a} \in \check{H}^r(\mathcal{W}_{11-n})\ \check{b} \in \check{H}^s(L_n), && \check{a} \star \check{b}\ p_{\mathcal{W}}^*(\check{a}) \star p_L^*(\check{b}). \end{aligned}$$

$$R(\check{a} \star \check{b}) = R(\check{a}) \wedge R(\check{b}), I(\check{a} \star \check{b}) = I(\check{a}) \smile I(\check{b}), \check{a} \in \check{H}^r(\mathcal{W}_{11-n}), \check{b} \in \check{H}^s(L_n)$$

$$\begin{aligned} &H^p(L_n; \mathbb{Z})\colon v_{p(\alpha)}, \alpha \in \{1, \dots b^p\} \\ &H^p(L_n; \mathbb{Z})\colon t_{p(i)}, i \in \mathcal{I}_p. \end{aligned}$$

$$\text{Tor}\, H^*(\mathcal{W}_{11-n}; \mathbb{Z}) = 0$$

$$a_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \sigma_{4-p}^{(\alpha_p)} \smile v_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \rho_{4-p}^{(i_p)} \smile t_{p(i_p)}.$$

$$\sigma_{4-p}^{(\alpha_p)}, \rho_{4-p}^{(i_p)} \in H^{4-p}(\mathcal{W}_{11-d}; \mathbb{Z})$$

$$\check{H}^*(\mathcal{W}_{11-n}), \check{H}^*(L_n)^{10}, \check{F}_{4-p}^{(\alpha_p)}, \check{B}_{4-p}^{(i_p)} \in \check{H}_{4-p}(\mathcal{W}_{11-n}), \check{v}_{p(\alpha_p)}, \check{t}_{p(i_p)} \in \check{H}_p(L_n)$$

$$\sigma_{4-p}^{(\alpha_p)} = I\left(\check{F}_{4-p}^{(\alpha_p)}\right), \rho_{4-p}^{(i_p)} = I\left(\check{B}_{4-p}^{(\alpha_p)}\right), v_{p(\alpha_p)} = I\left(\check{v}_{p(\alpha_p)}\right), t_{p(i_p)} = I\left(\check{t}_{p(i_p)}\right)$$

$$\check{F}_{4-p}^{(\alpha_p)}, \check{B}_{4-p}^{(i_p)} \in \check{H}_{4-p}(\mathcal{W}_{11-n}), \check{v}_{p(\alpha_p)}, \check{t}_{p(i_p)} \in \check{H}_p(L_n)$$

$$\breve{a}_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \breve{F}_{4-p}^{(\alpha_p)} \star \breve{v}_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \breve{B}_{4-p}^{(i_p)} \star \breve{t}_{p(i_p)}.$$

$$I(\breve{a}_4) = a_4$$

$$\breve{G}_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \breve{F}_{4-p}^{(\alpha_p)} \star \breve{v}_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \breve{B}_{4-p}^{(i_p)} \star \breve{t}_{p(i_p)} + \tau([\omega_3]), \omega_3 \in \Omega^3(\mathcal{M}_{11}).$$

$$\breve{F}_{4-p}^{(\alpha_p)}, \breve{B}_{4-p}^{(i_p)} \in \breve{H}_{4-p}(\mathcal{W}_{11-n})$$

$$\breve{B}, \breve{B}' \in \breve{H}^{4-p}(\mathcal{W}_{11-d})$$

$$I(\breve{B}) = I(\breve{B}')$$

$$I(\breve{B} - \breve{B}') = 0$$

$$\tau(\mathrm{b}) \star \breve{t}_{p(i_p)} = \tau\left(\mathrm{b} \wedge R\left(\breve{t}_{p(i_p)}\right)\right) = 0$$

$$\breve{B} \star \breve{t}_{p(i_p)} = \breve{B}' \star \breve{t}_{p(i_p)}, \text{ so } \breve{B}_{4-p}^{(i_p)} \star \breve{t}_{p(i_p)}$$

$$I\left(\breve{B}_{4-p}^{(i_p)}\right) \smile I\left(\breve{t}_{p(i_p)}\right)$$

$$H^{4-p}(\mathcal{W}_{11-d}; \mathbb{Z}) \otimes \mathrm{Tor} H^p(L_p; \mathbb{Z})$$

$$H^{4-p}(\mathcal{W}_{11-d}; \mathbb{Z}) = 0 \text{) to } H^{4-p}(\mathcal{W}_{11-d}; \mathrm{Tor}\, H^p(L_p; \mathbb{Z}))$$

$$I\left(\breve{B}_{4-p}^{(i_p)}\right) \smile I\left(t_{p(i_p)}\right)$$

$$H^{4-p}\left(\mathcal{W}_{11-d}; \mathbb{Z}_{n_{(i_p)}}\right)$$

$$K\left(\mathbb{Z}_{n_{(i_p)}}, 4-p\right) = B^{4-p}\mathbb{Z}_{n_{(i_p)}}$$

$$H^1\left(\mathcal{W}_{11-d}; \mathbb{Z}_{n_{(i_p)}}\right)$$

$$\int_{X \times Y} \breve{a} \star \breve{b} = (-1)^{(q - \dim(Y))\dim(X)} \left(\int_X \breve{a}\right) \star \left(\int_Y \breve{b}\right).$$

$$\int_{X \times Y} \breve{a} \star \breve{b} = \begin{cases} \left(\left(\int_X u\right)\left(\int_Y R(\breve{b})\right)\right) & \text{if } p = \dim(X) + 1 \\ (-1)^p \left(\int_X R(\breve{a})\right)\left(\int_Y v\right) & \text{if } p = \dim(X) \\ 0 & \text{otherwise} \end{cases}$$

$$\breve{G}_4^3 = \sum \text{ monomials involving } \breve{t} \text{ and } \breve{v} + \sum \text{ monomials involving } \breve{v} \text{ and } \tau([\omega_3]),$$

$$\Phi(\mathcal{T}_3)\Phi(\mathcal{T}_6) = e^{2\pi i\, \mathrm{L}(\mathcal{T}_3, \mathcal{T}_6)} \Phi(\mathcal{T}_6)\Phi(\mathcal{T}_3)$$

Γ	G_Γ	Γ^{ab}	$L_{S^3/\Gamma}$
A_{n-1}	$SU(n)$	$\mathbb{Z}_n$	$\frac{1}{n}$
D_{2n}	$\mathrm{Spin}(4n)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}\begin{pmatrix} n & n-1 \\ n-1 & n \end{pmatrix}$
D_{2n+1}	$\mathrm{Spin}(4n+2)$	$\mathbb{Z}_4$	$\frac{2n-1}{4}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{1}{2}$
$2I$	E_8	0	0

$$\mathrm{L}\colon \mathrm{Tor}\, H_3(X_{10};\mathbb{Z}) \times \mathrm{Tor}\, H_6(X_{10};\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

$$\mathrm{L}(\mathcal{T}_3,\mathcal{T}_6) := \frac{1}{n}\mathcal{T}_3 \cdot C_7 \bmod 1.$$

$$\mathrm{Tor}H^k(\mathcal{M}_d;\mathbb{Z}) \times \mathrm{Tor}H^{d+1-k}(\mathcal{M}_d;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$$\mathrm{Tor}\, H_p(X_{10};\mathbb{Z}) = H_{p-1}(\mathcal{M}_7;\mathbb{Z}) \otimes H_1(S^3/\Gamma;\mathbb{Z}),$$

$$H_q(S^3/\Gamma;\mathbb{Z}) = H^{3-q}(S^3/\Gamma;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0,3 \\ \Gamma^{ab} := \dfrac{\Gamma}{[\Gamma,\Gamma]} & \text{for } q = 1 \\ 0 & \text{for } q = 2 \end{cases}$$

$$\mathrm{L}(\Sigma_2 \times \mathcal{T}_1, \Sigma_2 \times \mathcal{T}_1) = (\Sigma_1 \cdot \Sigma_2)\mathrm{L}_{S^3/\Gamma}(\mathcal{T}_1,\mathcal{T}_1)$$

$$\Psi(\Sigma_2)\Psi(\Sigma_5) = e^{2\pi i \ell^{-1}\Sigma_2\cdot\Sigma_5}\Psi(\Sigma_5)\Psi(\Sigma_2),$$

$$\ell^{-1} := \mathrm{L}_{S^3/\Gamma}(\mathcal{T}_1,\mathcal{T}_1)$$

$$S_{\mathrm{Sym}} = \ell \int B_2 \wedge dC_5$$

$$\mathrm{Tor}\big(H^3(\mathcal{W}_8;\mathbb{Z})\big) = 0$$

$$H^2\big(\mathcal{W}_8;\Gamma^{ab}\big) = H^2(\mathcal{W}_8;\mathbb{Z}) \otimes H^2(S^3/\Gamma;\mathbb{Z})$$

$$a_4 \in H^4(M^{11};\mathbb{Z})$$

$$a_4 = \sigma_4 \smile 1 + \sum_i \rho_2^{(i)} \smile t_{2(i)} + \sigma_1 \smile \mathrm{vol}(S^3/\Gamma)$$

$$H^3(S^3/\Gamma;\mathbb{Z}) = \mathbb{Z}, t_{2(i)}$$

$$H^2(S^3/\Gamma;\mathbb{Z}) = \Gamma^{ab}$$

$$H^0(S^3/\Gamma;\mathbb{Z}) = \mathbb{Z}$$

$$\check{G}_4 = \check{\gamma}_4 \star \check{1} + \sum_i \check{B}_2^{(i)} \star \check{t}_{2(i)} + \check{\xi}_1 \star \check{v}_3 + \tau([\omega_3]),$$

$$\omega_3 \in \Omega^3(\mathcal{M}^{11}), \check{\xi}_1 \in \check{H}^1(\mathcal{W}_8;\mathbb{Z}), \check{v}_3 \in \check{H}^3(S^3/\Gamma;\mathbb{Z}), I(\check{\xi}_1) = \sigma_1$$

$$I(\check{v}) = \mathrm{vol}(S^3/\Gamma)$$

$$I(\check{\gamma}_4) = \sigma_4$$

$$R(\check{t}_{2(i)}) = 0$$

$$\begin{aligned}\frac{S_{\mathrm{top}}}{2\pi} &= -\frac{1}{6}\int_{M_{11}} \check{G}_4 \star \check{G}_4 \star \check{G}_4 \\ &= \frac{1}{2}\int_{\mathcal{W}_8} \check{\gamma}_4^2 \star \check{\xi}_1 \int_{S^3/\Gamma} \check{v} - \frac{1}{2}\int_{\mathcal{W}_8} \check{\gamma}_4 \star \check{B}_2^2 \int_{S^3/\Gamma} \check{t}_2^2 - \frac{1}{6}\int \check{\tau}(w_3)\end{aligned}$$

$$\mathrm{CS}[S^3/\Gamma, \check{t}_2] = \frac{1}{2}\int_{S^3/\Gamma} \check{t}_2 \star \check{t}_2$$

$$S_{\mathrm{Sym}} = \frac{1}{2}\int_{\mathcal{W}_8} \check{\gamma}_4 \star \check{\gamma}_4 \star \check{\xi}_1 - \mathrm{CS}[S^3/\Gamma, \check{t}_2]\int_{\mathcal{W}_8} \check{\gamma}_4 \star \check{B}_2 \star \check{B}_2$$

$$S_{\mathrm{Sym}} = \frac{1}{2}\int_{\mathcal{W}_8} \check{\gamma}_4 \star \check{\gamma}_4 \star \check{\xi}_1 - \sum_{i,j} \mathrm{CS}[S^3/\Gamma]_{ij}\int_{\mathcal{W}_8} \check{\gamma}_4 \star \check{B}_2^{(i)} \star \check{B}_2^{(j)}$$

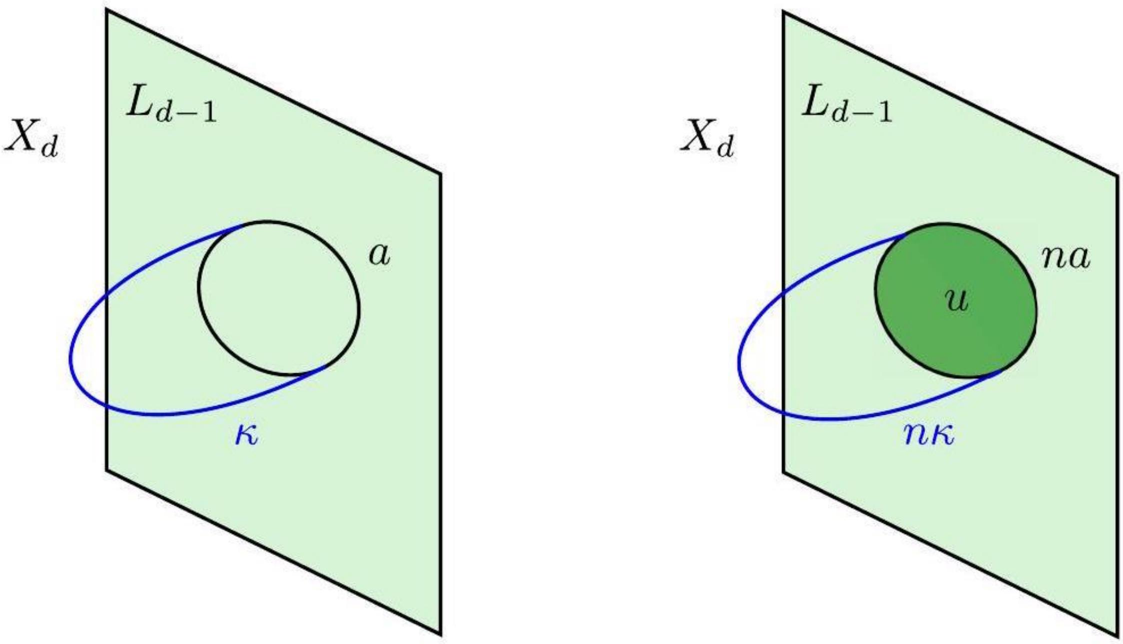
$$\mathrm{CS}[S^3/\Gamma]_{ij} = \frac{1}{2}\int_{S^3/\Gamma} \check{t}_{2(i)} \star \check{t}_{2(j)}$$

$$\mathrm{CS}[S^3/\Gamma]_{ij} + \mathrm{CS}[S^3/\Gamma]_{ji} = \int_{S^3/\Gamma} \check{t}_{2(i)} \star \check{t}_{2(j)}$$

$$\cdots \to H_{d-2}(X_d;\mathbb{Z}) \to H_{d-2}(X_d, L_{d-1};\mathbb{Z}) \to H_{d-3}(L_{d-1};\mathbb{Z}) \to H_{d-3}(X_d;\mathbb{Z}) \to \cdots$$

$$H_{d-3}(X_d;\mathbb{Z}) = 0$$

$$H_{d-2}(X_d;\mathbb{Z}) \xrightarrow{A} H_{d-2}(X_d, L_{d-1};\mathbb{Z}) \xrightarrow{f} H^2(L_{d-1};\mathbb{Z}) \to 0.$$



$$na_2 = 0, a_2 = f(\kappa), A(Z) = n\kappa$$

$$H_{d-2}(X_d, L_{d-1}; \mathbb{Z}) \cong H^2(X_d; \mathbb{Z})$$

$$\mathrm{Tor}\, H_1(X_d; \mathbb{Z}) = 0$$

$$H^2(X_d; \mathbb{Z}) \cong \mathrm{Hom}(H_2(X_d; \mathbb{Z}), \mathbb{Z})$$

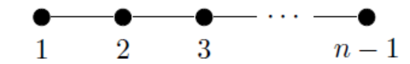
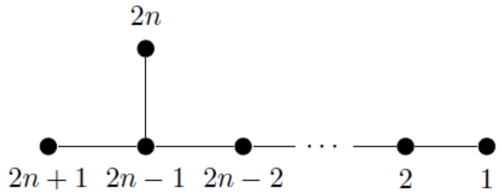
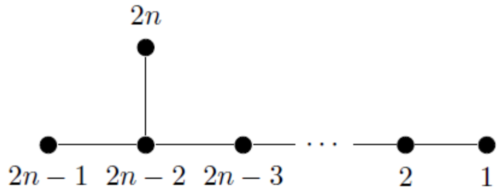
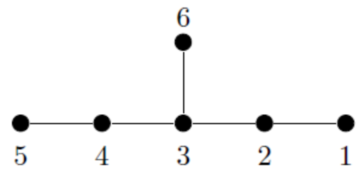
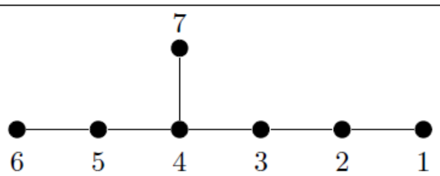
$$H_{d-2}(X_d; \mathbb{Z}) \xrightarrow{A} \mathrm{Hom}(H_2(X_d; \mathbb{Z}), \mathbb{Z}) \xrightarrow{f} H^2(L_{d-1}; \mathbb{Z}) \to 0.$$

$$A: H_{d-2}(X_d; \mathbb{Z}) \otimes H_2(X_d; \mathbb{Z}) \to \mathbb{Z}.$$

$$\int_{L_3} \check{t}_2 \star \check{t}_2 = \mathrm{L}_{L_3}(\mathrm{PD}[t_2], \mathrm{PD}[t_2]) = \left[\frac{Z \cdot Z}{n^2}\right]_{\mathrm{mod}1}$$

$$\mathrm{CS}[L_3, \check{t}_2] = \frac{1}{2}\int_{L_3} \check{t}_2 \star \check{t}_2 = \left[\frac{Z \cdot Z}{2n^2}\right]_{\mathrm{mod}1}.$$

$$\mathrm{L}_{S^3/\Gamma}(\mathrm{PD}[s_2], \mathrm{PD}[t_2]) = \int_{S^3/\Gamma} \check{s}_2 \star \check{t}_2 = \mathrm{CS}[S^3/\Gamma, \check{s}_2 + \check{t}_2] - \mathrm{CS}[S^3/\Gamma, \check{s}_2] - \mathrm{CS}[S^3/\Gamma, \check{t}_2] \ \mathrm{mod}1,$$

Γ	Dynkin diagram	Z
A_{n-1}		$\sum_{i=1}^{n-1} iS_i$
D_{2n}		$\sum_{i=1}^{2n-1}(1-(-1)^i)S_i$
D_{2n+1}		$\frac{1}{2}\sum_{i=1}^{2n-1}(1-(-1)^i)S_i$, $\frac{1}{2}\sum_{i=1}^{2n-2}\left(1-(-1)^i\right)S_i + S_{2n}$
E_6		$\sum_{i=1}^{5} iS_i + S_{2n} + 3S_{2n+1}$
E_7		$S_1 + S_3 + S_7$

$$\check{s}_2 \star \check{t}_2 = -\check{s}_2 \star i\left(\beta^{-1}(I(\check{t}_2))\right) = -i\left(I(\check{s}_2) \smile \beta^{-1}(I(\check{t}_2))\right),$$

$$\int_{S^3/\Gamma} i\left(I(\check{s}_2) \smile \beta^{-1}(I(\check{t}_2))\right) = i\left(\int_{S^3/\Gamma} I(\check{s}_2) \smile \beta^{-1}(I(\check{t}_2))\right) = \int_{S^3/\Gamma} I(\check{s}_2) \smile \beta^{-1}(I(\check{t}_2)),$$

$$\int_{S^3/\Gamma} \check{s}_2 \star \check{t}_2 = -\int_{S^3/\Gamma} I(\check{s}_2) \smile \beta^{-1}\big(I(\check{t}_2)\big) = \mathrm{L}_{S^3/\Gamma}(\mathrm{PD}[s_2], \mathrm{PD}[t_2]) \bmod 1.$$

Γ	G_Γ	Γ^{ab}	$-\mathrm{CS}[S^3/\Gamma, \check{t}_2]$
A_{n-1}	$SU(n)$	$\mathbb{Z}_n$	$\frac{n-1}{2n}$
D_{2n}	$\mathrm{Spin}(4n)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{4}\begin{pmatrix} n & n-1 \\ n-1 & n \end{pmatrix}$
D_{2n+1}	$\mathrm{Spin}(4n+2)$	$\mathbb{Z}_4$	$\frac{2n+1}{8}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{5}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{3}{4}$
$2I$	E_8	0	0

$$n_{\mathrm{inst}} = -\mathrm{CS}[S^3/\Gamma, \check{t}_2] \bmod 1.$$

$$\Gamma^{(2)} = \frac{H_2(X, L_5; \mathbb{Z})}{H_2(X; \mathbb{Z})} \cong H_1(L_5; \mathbb{Z})$$

$$\Gamma^{(1)} \cong \mathbb{Z}^{b_4}/\mathcal{M}_{4,2}\mathbb{Z}^{b_2}$$

$$\Psi(\Sigma_2)\Psi(\Sigma_3) = e^{2\pi i \ell^{-1}\Sigma_2 \cdot \Sigma_3}\Psi(\Sigma_3)\Psi(\Sigma_2)$$

$$S_{\mathrm{BF}} = \ell \int B_2 \wedge dC_3$$

$$\mathcal{A}_{B^3} = \frac{qp(p-1)(p-2)}{6\mathrm{gcd}(p,q)^3} B_2^3$$

$$\mathcal{A}_{FB^2} = \frac{p(p-1)}{2\mathrm{gcd}(p,q)^2} F_I B_2^2$$

$$H^*(L_5; \mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}^{b^2} \oplus \mathrm{Tor}H^2(L_5; \mathbb{Z}), \mathbb{Z}^{b^2} \oplus \mathrm{Tor}H^3(L_5; \mathbb{Z}), \mathrm{Tor}H^2(L_5; \mathbb{Z}), \mathbb{Z}\}$$

$$\begin{aligned} \breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \sum_{\alpha=1}^{b^2} \breve{F}_2^{(\alpha)} \star \breve{v}_{2(\alpha)} + \sum_{\alpha=1}^{b^2} \breve{\xi}_{1(\alpha)} \star \breve{v}_3^{(\alpha)} \\ + \sum_i \breve{B}_2^{(i)} \star \breve{t}_{2(i)} + \sum_m \breve{b}_1^{(m)} \star \breve{t}_{3(m)} + \sum_i \breve{\psi}_{0(i)} \star \breve{t}_4^{(i)} + \tau([\omega_3]) \end{aligned}$$

$$-\frac{1}{6}\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4$$
$$= -\sum_{\alpha} \int_{W_6} \breve{\gamma}_4 \star \breve{F}_2^{(\alpha)} \star \xi_{1(\alpha)} - \sum_{i,j,k} \left[\frac{1}{6}\int_{L_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{t}_{2(k)}\right] \int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{B}_2^{(k)}$$
$$-\sum_{i,j,\alpha} \left[\frac{1}{2}\int_{L_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{v}_{2(\alpha)}\right] \int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{F}_2^{(\alpha)}$$
$$-\sum_{i,\alpha,\beta} \left[\frac{1}{2}\int_{L_5} \breve{t}_{2(i)} \star \breve{v}_{2(\alpha)} \star \breve{v}_{2(\beta)}\right] \int_{W_6} \breve{B}_2^{(i)} \star \breve{F}_2^{(\alpha)} \star \breve{F}_2^{(\beta)}$$
$$+\sum_{m,n} \left[\frac{1}{2}\int_{L_5} \breve{t}_{3(m)} \star \breve{t}_{3(n)}\right] \int_{W_6} \breve{\gamma}_4 \star \breve{b}_1^{(m)} \star \breve{b}_1^{(n)} + \sum_{m,\alpha} \left[\int_{L_5} \breve{t}_{3(m)} \star \breve{v}_3^{(\alpha)}\right] \int_{W_6} \breve{\gamma}_4 \star \breve{b}_1^{(m)} \star \breve{\xi}_{1(\alpha)}$$
$$-\sum_{i,j} \left[\int_{L_5} \breve{t}_{2(i)} \star \breve{t}_4^{(j)}\right] \int_{W_6} \breve{\gamma}_4 \star \breve{B}_2^{(i)} \star \breve{\psi}_{0(j)} - \sum_{\alpha,j} \left[\int_{L_5} \breve{v}_{2(\alpha)} \star \breve{t}_4^{(j)}\right] \int_{W_6} \breve{\gamma}_4 \star \breve{F}_2^{(\alpha)} \star \breve{\psi}_{0(j)}$$

$$\int_{L_5} v_{2(\alpha)} \star v_3^{(\beta)} = \delta_\alpha^\beta$$

$$p_1(T\mathcal{M}_{11}) = p_1(TW_6) + p_1(TL_5)$$
$$p_2(T\mathcal{M}_{11}) = p_2(TW_6) + p_2(TL_5) + p_1(TW_6) \smile p_1(TL_5)$$

$$X_8 = -\frac{1}{96} p_1(TW_6) \smile p_1(TL_5).$$

$$-\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{X}_8 = \frac{1}{96}\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{p}_1(TW_6) \star \breve{p}_1(TL_5) = \frac{1}{96}\sum_i \int_{L_5} \breve{t}_{2(i)} \star \breve{p}_1(TL_5) \int_{W_6} \breve{B}_2^{(i)} \star \breve{p}_1(TW_6).$$

$$p_1(TW_6) \smile a_2 = 4a_2 \smile a_2 \smile a_2 \bmod 24 \text{ for any } a_2 \in H^2(W_6;\mathbb{Z}).$$

$$\mathrm{Ind}(\not{D}_A) = \int_{W_6} \left[\frac{1}{6} F \wedge F \wedge F - \frac{1}{24} F \wedge p_1(TW_6)\right] = \int_{W_6} \left[\frac{1}{6} a_2 \smile a_2 \smile a_2 - \frac{1}{24} a_2 \smile p_1(TW_6)\right],$$

$$\int_{W_6} [4a_2 \smile a_2 \smile a_2 - a_2 \smile p_1(TW_6)] \in 24\mathbb{Z}$$

$$\int_{W_6} \breve{B}_2^{(i)} \star \breve{p}_1(TW_6) = \int_{W_6} B_2^{(i)} \smile p_1(TW_6) = 24M^{(i)} + 4\int_{W_6} B_2^{(i)} \smile B_2^{(i)} \smile B_2^{(i)}$$
$$= 24M^{(i)} + 4\int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} \star \breve{B}_2^{(i)},$$

$$\int_{W_6} \breve{a}_6 = \int_{W_6} I(\breve{a}_6)$$

$$I\left(\breve{B}_2^{(i)}\right) = B_2^{(i)}$$

$$I(\breve{p}_1(TW_6)) = \breve{p}_1(TW_6)$$

$$-\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{X}_8 = \frac{1}{24}\sum_i \int_{L_5} \breve{t}_{2(i)} \star \breve{p}_1(TL_5) \int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} + \sum_i M^{(i)} \int_{L_5} \breve{t}_{2(i)} \star \frac{\breve{p}_1(TL_5)}{4}.$$

$$\int_{L_5} \breve{t}_{2(i)} \star \breve{v}_{2(\alpha)} \star \breve{v}_{2(\beta)} = 0$$

$$S_{\mathrm{Sym}} = \sum_{i,j,k} \Omega_{ijk} \int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{B}_2^{(k)} + \sum_{i,j,\alpha} \Omega_{ij\alpha} \int_{W_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{F}_2^{(\alpha)}$$

$$\Omega_{ijk} = -\frac{1}{6}\int_{L_5} \check{t}_{2(i)} \star \check{t}_{2(j)} \star \check{t}_{2(k)} + \frac{1}{24}\delta_{i,j}\delta_{i,k}\int_{L_5} \check{t}_{2(i)} \star \check{p}_1(TL_5)$$
$$\Omega_{ij\alpha} = -\frac{1}{2}\int_{L_5} \check{t}_{2(i)} \star \check{t}_{2(j)} \star \check{v}_{2(\alpha)}$$

$$\Omega_{ijk} = \left[-\frac{1}{6}\frac{Z_{(i)} \cdot Z_{(j)} \cdot Z_{(k)}}{n_{(i)}n_{(j)}n_{(k)}} + \frac{1}{24}\delta_{i,j}\delta_{i,k}\frac{Z_{(i)} \cdot p_1(TX_6)}{n_{(i)}}\right]_{\mathrm{mod}1},$$
$$\Omega_{ij\alpha} = \left[-\frac{1}{2}\frac{Z_{(i)} \cdot Z_{(j)} \cdot D_{(\alpha)}}{n_{(i)}n_{(j)}}\right]_{\mathrm{mod}1},$$

$$w_0 = (0,0), w_p = (0,p), w_x = (-1,k_x), w_y = (1,k_y)$$

$$q = p - (k_x + k_y)$$

$$D_I = D_{w_x}$$

$$S_a = (0,a), a = 1,\cdots,p-1$$

$$Z = \sum_{a=1}^{p-1} aS_a$$

$$p_1(TX_6) = -c_2(TX_6 \otimes \mathbb{C})$$

$$TX_6 \otimes \mathbb{C}$$

$$TX_6 \otimes \mathbb{C} = TX_6 \oplus \overline{TX_6}$$

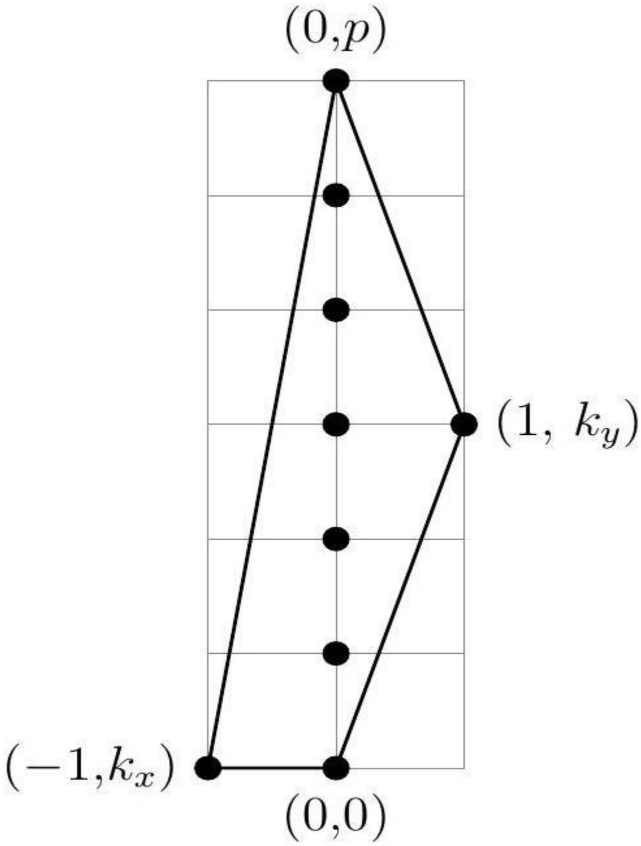
$$c(TX_6 \otimes \mathbb{C}) = c(TX_6)c(\overline{TX_6})$$

$$c(TX_6) = \prod_{i=1}^{n} (1 + D_i)$$

$$c(TX_6 \otimes \mathbb{C}) = c(TX_6)c(\overline{TX_6}) = \left(\prod_{i=1}^{n} (1 + D_i)\right)\left(\prod_{i=1}^{n} (1 - D_i)\right) = \prod_{i=1}^{n} (1 - D_i^2).$$

$$p_1(TX_6) = \sum_i D_i^2$$

$$Z \cdot Z \cdot Z = p(p-1)(p^2 + pq - 2q), Z \cdot p_1 = 4p(p-1)$$
$$Z \cdot Z \cdot D_I = -p(p-1)$$



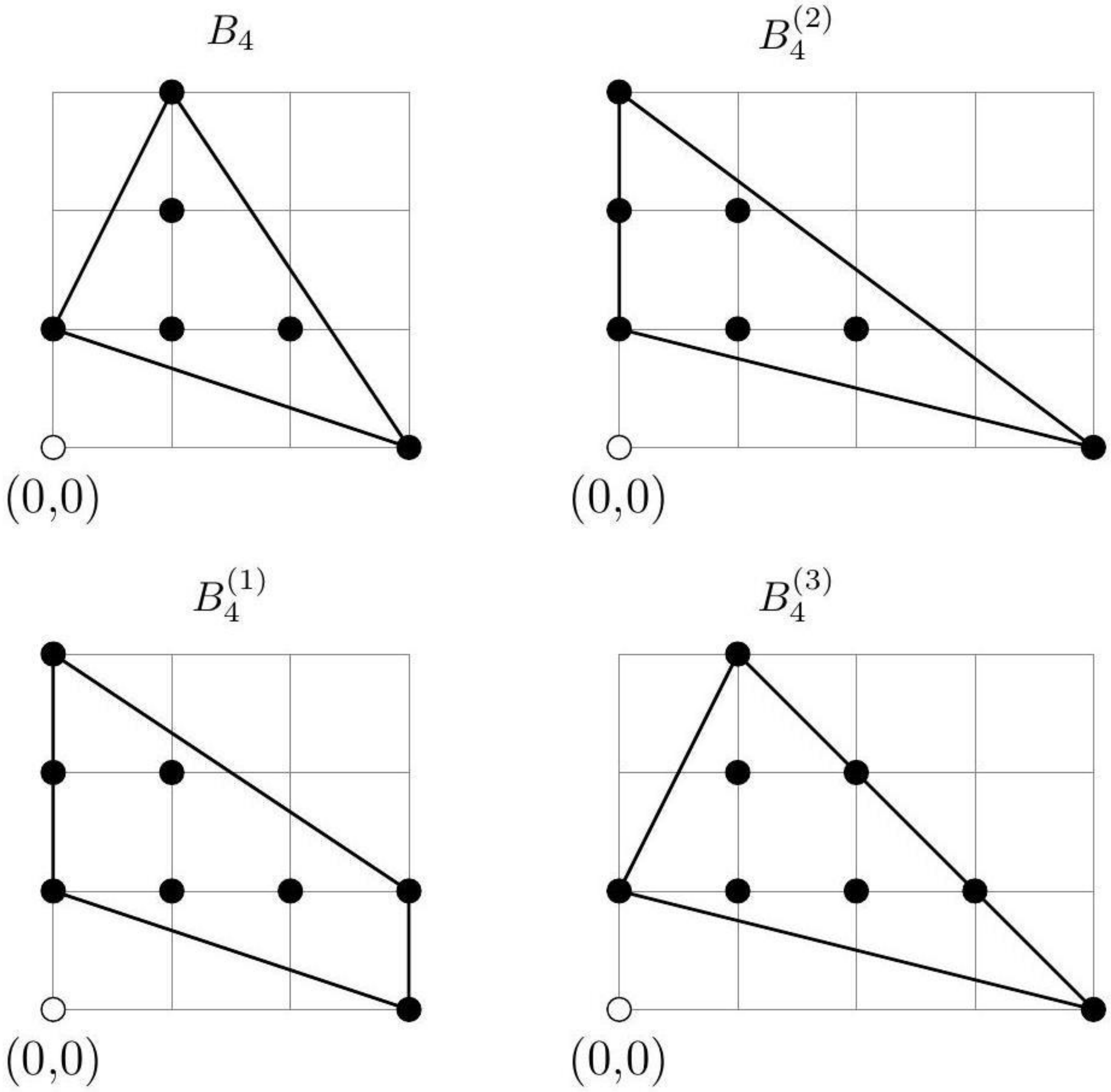
$$p = 6, q = 6 - (k_x + k_y) = 3$$

$$Z \cdot D_I \cdot D_I = 0$$

$$\int_{L_5} \check{t}_2 \star \frac{\check{p}_1(TL_5)}{4} = \left[\frac{Z \cdot p_1}{4\mathrm{gcd}(p,q)}\right]_{\mathrm{mod}1} = 0.$$

$$-\frac{1}{6} Z \cdot Z \cdot Z + \frac{1}{24}\mathrm{gcd}(p,q) Z \cdot p_1 = \frac{qp(p-1)(p-2)}{6} - \mathrm{gcd}(p,q)^3 (p-1) \frac{P(P+1)(P-1)}{6},$$

$$S_{\mathrm{Sym}} = \int_{W_6} \left[\frac{qp(p-1)(p-2)}{6\mathrm{gcd}(p,q)^3} B_2^3 + \frac{p(p-1)}{2\mathrm{gcd}(p,q)^2} B_2^2 F_I\right].$$



$$\frac{qp(p-1)(p-2)}{6\mathrm{gcd}(p,q)^3}3\mathrm{gcd}(p,q)\int_{W_6}B_2^2b_2\in\mathbb{Z},$$
$$\frac{qp(p-1)(p-2)}{6\mathrm{gcd}(p,q)^3}3\mathrm{gcd}(p,q)^2\int_{W_6}B_2b_2^2\in\mathbb{Z},$$
$$\frac{qp(p-1)(p-2)}{6\mathrm{gcd}(p,q)^3}\mathrm{gcd}(p,q)^3\int_{W_6}b_2^3\in\mathbb{Z}.$$

Theory	$\Gamma^{(1)}$	Toric Fan
B_N	$\mathbb{Z}_{N(N-3)+3}$	$(N-1,0,1),(1,N-1,1),(0,1,1)$
$B_N^{(1)}$	$\mathbb{Z}_{N-1}$	$((N-1,0,1),(N-1,1,1)(0,N-1-k,1)),\ k=0,\cdots,N-2$
$B_N^{(2)}$	$\mathbb{Z}_N$	$((N,0,1),(0,N-1-k,1)),\ k=0,\cdots,N-2$
$B_N^{(3)}$	$\mathbb{Z}_{N-1}$	$((0,1,1),(N-k,k,1)),\ k=0,\cdots,N-1$

$$S_{\text{Sym}}^{(B_N)} = \int_{\mathcal{W}_6} \frac{(N-1)(N-2)}{6(N(N-3)+3)} B_2^3$$

$$S_{\text{Sym}}^{\left(B_N^{(1)}\right)} = \int_{\mathcal{W}_6} \frac{(N-3)(N-2)}{6(N-1)} B_2^3.$$

$$S_{\text{Sym}}^{\left(B_N^{(2)}\right)} = \int_{\mathcal{W}_6} \frac{(N-2)(N-1)}{6N} B_2^3.$$

$$Z_{B_N^{(2)}} = (-N+1)\sum_{i=1}^{N-2} v_{1,i} + (-N+2)\sum_{i=1}^{N-3} v_{2,i} + \cdots.$$

$$S_{\text{Sym}}^{\left(B_N^{(1)}\right),\,\text{mixed}} = \int_{\mathcal{W}_6} \frac{N-2}{2(N-1)} B_2^2 F,$$

$$\sum_{\ell} c_\ell = 0, \sum_{\ell} p_\ell c_\ell = 0, \sum_{\ell} q_\ell c_\ell = 0$$

$$\int_{M_4} \omega_\ell \wedge \beta_j = \Omega_{\ell j} \text{Vol}_4$$

$$\begin{aligned} F_3 &= q^\ell \beta_\ell + f_2^\ell \wedge \omega_\ell + g_3 + \cdots \\ H_3 &= p^\ell \beta_\ell + h_2^\ell \wedge \omega_\ell + h_3 + \cdots \\ F_5 &= f_5 + f_4^\ell \wedge \omega_\ell + g_2^\ell \wedge \beta_\ell + f_1 \wedge \text{vol}_4 \end{aligned}$$

$$da_1^\ell = \sum_j \Omega_{\ell j}\left(q^\ell h_2^j - p^\ell f_2^j\right)$$

$$S_{11}^{\text{top}} = \int F_5 \wedge H_3 \wedge F_3$$

$$\begin{aligned} df_5 &= h_3 \wedge f_3 \rightarrow f_5 = dc_4 + b_2 \wedge f_3 \\ df_4^\ell &= \left(h_3 \wedge f_2^\ell - g_3 \wedge h_2^\ell\right) \rightarrow f_4^\ell = dc_3^\ell + \left(b_2 f_2^\ell - c_2 h_2^\ell\right) \end{aligned}$$

$$\begin{aligned} df_1 &= g_2^\ell q_\ell - g_2^\ell p_\ell \rightarrow f_1 = dc_0 + b_1^\ell q_\ell - c_1^\ell p_\ell \\ dg_2^\ell &= \left(q^\ell h_3 - p^\ell g_3\right) \rightarrow g_2^\ell = dc_1^\ell + \left(q^\ell b_2 - p^\ell c_2\right). \end{aligned}$$

$$S_{7d}^{\text{singl}} = \int dc_3^j \Omega_{j\ell} Q^{\ell I} \wedge \sigma_{IJ} \mathcal{F}_3^J$$

$$Q^{\ell I} = \begin{pmatrix} p_1 & q_1 \\ \vdots & \vdots \\ p_L & q_L \end{pmatrix}, \mathcal{F}_3^J = \begin{pmatrix} h_3 \\ g_3 \end{pmatrix}, \sigma_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S_{7d}^{\text{singl}} = \int dC_3 \Omega Q \sigma \mathcal{F} = \int dC_3 \Omega B^{-1} B Q A A^{-1} \sigma \mathcal{F} = \int d\tilde{C}_3 Q_{\text{SNF}} \tilde{\mathcal{F}}_3$$

$$S_{7d}^{\text{vec}} = \int dc_4 \wedge \left(q^\ell \Omega_{\ell j} h_2^j - p^\ell \Omega_{\ell j} f_2^j\right)$$

$$S_{7d}^{\text{vec}} = \int dc_4 \wedge \Omega_{j\ell} Q^{\ell I} \sigma_{IJ} \mathcal{F}_2^{Ij} = \int dc_4 \wedge \text{Tr}(Q^T \Omega \mathcal{F}_2^T \sigma)$$

$$\mathcal{F}_2 = \begin{pmatrix} h_2^1 & \dots & h_2^L \\ f_2^1 & \dots & f_2^L \end{pmatrix}$$

$$\begin{aligned} S_{7d}^{\text{vec}} &= \int dc_4 \wedge \text{Tr}((B^{-1} B Q A A^{-1})^T \Omega \mathcal{F}_2^T \sigma) = \int dc_4 \wedge \text{Tr}\left((A^{-1})^T Q_{\text{SNF}}^T \tilde{\mathcal{F}}_2^T \sigma\right) = \\ &= \int dc_4 \wedge \text{Tr}\left(\sigma \tilde{\mathcal{F}}_2 Q_{\text{SNF}} A^{-1}\right) \end{aligned}$$

$$\tilde{\mathcal{F}}_2 = (B^{-1})^T \Omega \mathcal{F}_2^T$$

$$\mathrm{Tr}\left(\sigma \tilde{\mathcal{F}}_2 Q_{\mathrm{SNF}} A^{-1}\right) = 0$$

$$S_{7d}^{\mathrm{anom}} = \int \ (q^\ell b_2 - p^\ell c_2) \wedge \Omega_{\ell j}\left(f_2^j h_3 - h_2^j f_3\right) = \int \ \mathcal{C}_2 \sigma Q^T \Omega \mathcal{F}_2^T \sigma \tilde{\mathcal{F}}_3 = \int \ \tilde{\mathcal{C}}_2 Q_{\mathrm{SNF}}^T \tilde{\mathcal{F}}_2^T A \tilde{\mathcal{F}}_3$$

$$g_2^\ell = d\tilde{c}_1^\ell + Q\sigma \mathcal{C}_2 = d\tilde{c}_1^\ell + B^{-1} Q_{\mathrm{SNF}} \widetilde{\mathcal{C}_2}.$$

$$f_1 = dc_0 + \mathrm{Tr}(\sigma \mathcal{F}_2 \Omega Q) = dc_0 + \mathrm{Tr}\left(\sigma \tilde{\mathcal{F}}_2 Q_{\mathrm{SNF}} A^{-1}\right)$$

$$S_{6d}^{CS} = \sum_\ell \int \ \tilde{Q}^{\ell I} \sigma_{IJ} \hat{C}^J e^{Q^{\ell I}\sigma_{IJ}\hat{F}^{J\ell}} \sqrt{\hat{\mathcal{A}}(R_{\mathcal{T}})/\hat{\mathcal{A}}(R_{\mathcal{N}})}$$

$$\sqrt{\hat{\mathcal{A}}(R_{\mathcal{T}})/\hat{\mathcal{A}}(R_{\mathcal{N}})} = 1 - \frac{1}{48}\left(p_1(R_{\mathcal{T}}) - p_1(R_{\mathcal{N}})\right) + \cdots.$$

$$\tilde{Q} = \begin{pmatrix} \tilde{p}_1 & \tilde{q}_1 \\ \vdots & \vdots \\ \tilde{p}_L & \tilde{q}_L \end{pmatrix}, \tilde{Q}^{\ell[I} Q^{J],\ell} = \frac{1}{2}\epsilon^{IJ} \ \forall \ell.$$

$$\begin{aligned}
&\hat{F}^\ell = Q^{\ell I}\sigma_{IJ}\left(d\hat{a}^{\ell J} + \mathcal{C}_2^J\right) \\
&\hat{C}_0 = c_0 \\
&\hat{C}_2 = \mathcal{C}_2 \\
&\hat{C}_4 = c_4 - \frac{1}{2} Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J \wedge Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J \\
&\hat{C}_6 = \mathcal{C}_6 + c_4 \wedge Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J + \frac{1}{6}\tilde{Q}^{\ell I}\sigma_{IJ}\mathcal{C}_2^J \wedge Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J \wedge Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J \\
&\hat{C}_6 = \mathcal{C}_8 + \frac{1}{24}\left(\tilde{Q}^{\ell I}\sigma_{IJ}\mathcal{C}_2^J\right) \wedge \left(Q^{\ell I}\sigma_{IJ}\mathcal{C}_2^J\right)^3,
\end{aligned}$$

$$S_{6d}^{CS} = \sum_l \int \ \frac{1}{6}\tilde{Q}\sigma\mathcal{C}_2 \wedge Q\sigma\mathcal{C}_2 \wedge Q\sigma\mathcal{C}_2 + \frac{1}{48}\tilde{Q}\sigma\mathcal{C}_2 \wedge p_1(R_{\mathcal{T}}) + \cdots,$$

$$S_{\mathrm{Sym}}^{CS} = \sum_l \int \ \frac{1}{6}\tilde{Q}A\widetilde{\mathcal{C}_2} \wedge QA\widetilde{\mathcal{C}_2} \wedge QA\widetilde{\mathcal{C}_2} + \frac{1}{48}\tilde{Q}A\widetilde{\mathcal{C}_2} \wedge p_1(R_{\mathcal{T}})$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$Q_{\mathrm{SNF}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_{7d}^{\mathrm{bulk}} = 2\int \ d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{f}_3 + \cdots$$

$$S_{\mathrm{Sym}}^{\mathrm{mixed}} = \frac{1}{4}\int \ d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{c}_2$$

$$\tilde{c}_2 p_1(R_{\mathcal{T}}) = 4\tilde{c}_2\tilde{c}_2\tilde{c}_2 \bmod 24$$

$$\tilde{Q} = \begin{pmatrix} 1 + x_1 & x_1 \\ -1 - x_2 & x_2 \\ -1 + x_3 & x_3 \\ 1 - x_4 & x_4 \end{pmatrix},$$

$$S_{\mathrm{Sym}}^{\mathrm{CS}} = \frac{(x_1 + x_4)}{4}\int \ \tilde{c}_2^3$$

$$S_{7d}^{\text{add}} = \int \left(d\tilde{g}_2^2 - 2\tilde{f}_3\right)\left(m\tilde{g}_2^2 \wedge \tilde{g}_2^2 + m'\frac{1}{48}p_1(R_{\mathcal{T}})\right)$$

$$S_{\text{Sym}}^{\text{cubic}} = S_{\text{Sym}}^{\text{CS}} + S_{6d}^{\text{add}} = 0.$$

$$Q = \begin{pmatrix} -p & 1 \\ q & 1 \\ 0 & -1 \\ (p-q) & -1 \end{pmatrix}, Q_{\text{SNF}} = \begin{pmatrix} 1 & 0 \\ 0 & \text{gcd}(p,q) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_{7d}^{\text{bulk}} = -p\int d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{f}_3 + \cdots$$

$$S_{6d}^{\text{bulk}} = -\frac{p}{2\text{gcd}(p,q)^2}\int d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{c}_2$$

$$S_{7d}^{\text{add}} = n\left(d\tilde{g}_2^2 - \text{gcd}(p,q)\tilde{f}_3\right) \wedge d\tilde{a}_1 \wedge \tilde{g}_2^2$$

$$S_{\text{Sym}}^{\text{mixed}} = \frac{p(p-1)}{2\text{gcd}(p,q)^2}\int d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{c}_2$$

$$\tilde{Q} = \begin{pmatrix} 1+px_1 & x_1 \\ 1-px_2 & x_2 \\ -1 & x_3 \\ -1 & x_4 \end{pmatrix},$$

$$S_{\text{Sym}}^{\text{CS}} = \left(\frac{qp}{3\text{gcd}(p,q)^3} + \frac{(x_1+x_2)}{4}\right)\int \tilde{c}_2\tilde{c}_2\tilde{c}_2$$

$$S_{7d}^{\text{add}} = \int \left(d\tilde{g}_2^2 - \text{gcd}(p,q)\tilde{f}_3\right) \wedge \left(m\tilde{g}_2^2 \wedge \tilde{g}_2^2 + m'\frac{1}{48}p_1(R_{\mathcal{T}})\right)$$

$$g_2^2 = d\tilde{c}_1^2 + \text{gcd}(p,q)\tilde{c}_2$$

$$d\tilde{g}_2^2 - \text{gcd}(p,q)\tilde{f}_3$$

$$\tilde{g}_2^{\ell} = (B^{-1})_j^{\ell} g_2^j$$

$$\tilde{c}_2 \to \tilde{c}_2 + \text{gcd}(p,q)d\lambda_1$$

$$4m + m' = 2(p-3)\frac{p^2q}{\text{gcd}(p,q)^3}$$

$$S_{\text{Sym}} = S_{\text{Sym}}^{\text{CS}} + S_{6d}^{\text{add}} = \frac{qp(p-2)(p-1)}{6\text{gcd}(p,q)^3}\int \tilde{c}_2\tilde{c}_2\tilde{c}_2$$

$$Q = \begin{pmatrix} N-1 & N-2 \\ -1 & -(N-1) \\ -(N-2) & 1 \end{pmatrix}, Q_{\text{SNF}} = \begin{pmatrix} 1 & 0 \\ 0 & N(N-3)+3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\tilde{Q} = \begin{pmatrix} 1 & 1 \\ x_1 & (N-1)x_1 - 1 \\ -1-x_2(N-1) & x_2 \end{pmatrix},$$

$$S_{\text{Sym}}^{\text{CS}} = \left(\frac{x_2(N(N-3)+3)+2}{6(N(N-3)+3)} + \frac{x_2}{12}\right)\int \tilde{c}_2\tilde{c}_2\tilde{c}_2$$

$$S^{\text{add}} = \int \left(d\tilde{g}_2^2 - (N(N-3)+3)\tilde{f}_3\right) \wedge \left(m\tilde{g}_2^2 \wedge \tilde{g}_2^2 + m'\frac{1}{48}p_1(R_{\mathcal{T}})\right)$$

$$\tilde{c}_2 \to \tilde{c}_2 + (N(N-3)+3)d\lambda_1$$

$$4m + m' = 2(1-x_2)$$

$$S_{\text{Sym}}^{\text{cubic}} = S_{\text{Sym}}^{\text{CS}} + S_{6d}^{\text{add}} = \frac{(N-2)(N-1)}{6(N(N-3)+3)} \int \tilde{c}_2 \tilde{c}_2 \tilde{c}_2$$

$$Q = \begin{pmatrix} -1 & -(N-1) \\ N-2 & N-1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 0 \\ \vdots & \vdots \end{pmatrix}, Q_{\text{SNF}} = \begin{pmatrix} 1 & 0 \\ 0 & N-1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}$$

$$\tilde{Q} = \begin{pmatrix} x_1 & x_1(N-1)+1 \\ 1 & 1 \\ x_2 & 1 \\ x_3 & -1 \\ \vdots & \vdots \end{pmatrix}$$

$$S_{\text{Sym}}^{\text{CS}} = \left(\frac{x_1(N-1)+2}{6(N-1)} + \frac{x_1+1}{12} \right) \int \tilde{c}_2 \tilde{c}_2 \tilde{c}_2$$

$$S^{\text{add}} = \int \left(d\tilde{g}_2^2 - (N-1)\tilde{f}_3 \right) \wedge \left(m\tilde{g}_2^2 \wedge \tilde{g}_2^2 + m' \frac{1}{48} p_1(R_{\mathcal{T}}) \right)$$

$$4m + m' = 2p - 7 - x_1$$

$$S_{\text{Sym}}^{\text{cubic}} = S_{\text{Sym}}^{\text{CS}} + S_{6d}^{\text{add}} = \frac{(N-3)(N-2)}{6(N-1)} \int \tilde{c}_2 \tilde{c}_2 \tilde{c}_2.$$

$$S_{7d}^{\text{bulk}} = -(N-1) \int d\tilde{a} \wedge \tilde{c}_2 \wedge \tilde{f}_3 + \cdots$$

$$d\tilde{a}_1 = \left(\tilde{h}_2^2 \right)$$

$$\tilde{c}_2 \to \tilde{c}_2/(N-1)$$

$$S_{6d}^{\text{bulk}} = -\frac{1}{2(N-1)} \int d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{c}_2$$

$$S_{7d}^{\text{add}} = n \int \left(d\tilde{g}_2^2 - (N-1)\tilde{f}_3 \right) \wedge d\tilde{a}_1 \wedge \tilde{g}_2^2$$

$$g_2^2 = d\tilde{c}_1^2 + (N-1)\tilde{c}_2$$

$$\tilde{c}_2 \to \tilde{c}_2/(N-1)$$

$$S_{\text{Sym}}^{\text{mixed}} = \frac{(N-2)}{2(N-1)} \int d\tilde{a}_1 \wedge \tilde{c}_2 \wedge \tilde{c}_2$$

$$Q = \begin{pmatrix} -1 & -N \\ N-1 & N \\ -1 & 0 \\ -1 & 0 \\ \vdots & \vdots \end{pmatrix}, Q_{\text{SNF}} = \begin{pmatrix} 1 & 0 \\ 0 & N \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}$$

$$\tilde{Q} = \begin{pmatrix} x_1 & x_1 N + 1 \\ 1 & 1 \\ x_2 & -1 \\ \vdots & \vdots \end{pmatrix}$$

$$S_{\text{Sym}}^{\text{CS}} = \left(\frac{x_1 N + 2}{6N} + \frac{x_1}{12} \right) \int \tilde{c}_2 \tilde{c}_2 \tilde{c}_2$$

$$S^{\text{add}} = \int \left(d\tilde{g}_2^2 - N\tilde{f}_3 \right) \wedge \left(m\tilde{g}_2^2 \wedge \tilde{g}_2^2 + m' \frac{1}{48} p_1(R_{\mathcal{T}}) \right)$$

$$4m+m'=2p-7-x_1.$$

$$S_{\text{Sym}}^{\text{cubic}}=S_{\text{Sym}}^{\text{CS}}+S_{6d}^{\text{add}}=\frac{(N-1)(N-2)}{6N}\int\ \tilde{c}_2\tilde{c}_2\tilde{c}_2.$$

$$ds^2=dx_6^2+N(d\rho^2+d\Omega_3),e^{\phi}=e^{\phi_0-\rho},H=N\text{vol}_3,$$

$$S_{\text{top}}=\int\ F_5\wedge H_3\wedge F_3,$$

$$F_5=f_5+f_2\wedge\text{vol}_3$$
$$F_3=f_3$$

$$H_3=h_3+N\text{vol}_3$$

$$df_3=0\ \rightarrow h_3=db_2$$
$$dh_3=0\ \rightarrow f_3=dc_2$$
$$df_2=Nf_3\ \rightarrow f_2=dc_1+Nc_2$$
$$df_5=Nh_3\wedge f_3\ \rightarrow f_5=dc_4-Nc_2\wedge db_2$$

$$S_{\text{top}_{8d}}=N\int\ dc_4\wedge dc_2-N\int\ db_2\wedge c_2\wedge dc_2$$

$$S_{\text{top}_{8d}}=-\int\ db_2\wedge f_2\wedge dc_2$$

$$b_2\rightarrow b_2+\Lambda_2,\oint\Lambda_2\in\mathbb{Z}$$

$$S_{\text{top}_{7d}}=-\frac{N}{2}\int\ db_2\wedge c_2\wedge c_2$$

$$S_{\text{top}_{8d}}=-N\int\ db_2\wedge f_2\wedge dc_2-m\int\ db_2\wedge f_2\wedge(Nf_3-df_2)$$

$$S_{\text{Sym}}=-\frac{N}{2}\int\ db_2\wedge c_2\wedge c_2+m\frac{N^2}{2}\int\ db_2\wedge c_2\wedge c_2$$

$$S_{\text{Sym}}=\frac{N-1}{2N}\int\ db_2\wedge\tilde{c}_2\wedge\tilde{c}_2$$

$$\frac{\tilde{c}_2}{N}=c_2$$

$$\int_{\mathcal{C}_4}G_4=\frac{1}{2}\int_{\mathcal{C}_4}\ w_4(T\mathcal{M}_{11})\ \text{mod}1$$

$$\mathcal{M}_{11}=\mathcal{W}_{11-n}\times L_n$$

$$w(T\mathcal{M}_{11})=w(T\mathcal{W}_{11-n})\smile w(TL_n)$$

$$w_{4-i}(T\mathcal{W}_{11-n})\smile w_i(TL_n),i=0,1,2,3,4$$

$$w_1(TL_n)=0=w_2(TL_n).$$

$$w_1(T\mathcal{W}_{11-d})=0=w_2(T\mathcal{W}_{11-n}).$$

$$w_{4-i}(T\mathcal{W}_{11-n})\smile w_i(TL_n)\ \text{with}\ i=1,2,3$$

$$\text{Sq}^1(w_2)=w_3+w_1\smile w_2,$$

$$v_4=w_1^4+w_2^2+w_1\smile w_3+w_4$$

$$\mathrm{Sq}^2(w_3) = w_2 \smile w_3 + w_1 \smile w_4 + w_5,$$

$$\mathfrak{P}(w_2) = \rho_4(p_1) + \theta_2(w_1 \mathrm{Sq}^1(w_2) + w_4),$$

$$ds^2(S^3/\mathbb{Z}_N) = \frac{1}{4}(d\theta^2 + \sin^2\,\theta d\phi^2) + \frac{1}{N^2} D\psi^2, D\psi = d\psi - \frac{N}{2}\cos\,\theta d\phi$$

$$\omega_2 = \frac{1}{4\pi}\sin\,\theta d\theta \wedge d\phi, \int_{S^2} \omega_2 = 1$$

$$S_{10\text{ d,top}} = 2\pi \int_{\mathcal{M}_{10}} I_{10}^{(0)}, dI_{10}^{(0)} = I_{11} = -\frac{1}{2} H_3 \wedge F_4 \wedge F_4 - H_3 \wedge X_8$$

$$dH_3 = 0, dF_4 = -H_3 \wedge F_2$$

$$F_2 = N\omega_2$$

$$H_3 = \widetilde{H}_3 + f_1 \wedge \omega_2, F_4 = -\gamma_4 - \tilde{F}_2 \wedge \omega_2$$

$$d\widetilde{H}_3 = 0, df_1 = 0, d\gamma_4 = 0, d\tilde{F}_2 = N\widetilde{H}_3$$

$$I_9 = -\tilde{F}_2 \wedge \widetilde{H}_3 \wedge \gamma_4$$

$$I_9^{\text{extra}} = \left(N\widetilde{H}_3 - d\tilde{F}_2\right)\tilde{F}_2\gamma_4 = N\widetilde{H}_3\tilde{F}_2\gamma_4 - d\left(\tilde{F}_2\tilde{F}_2\gamma_4\right)$$

$$I_9^{\text{improved}} = (N-1)\widetilde{H}_3\tilde{F}_2\gamma_4 - d\left(\tilde{F}_2\tilde{F}_2\gamma_4\right)$$

$$I_8^{(0)} = \frac{1}{2} N(N-1)\tilde{F}_2\tilde{F}_2\gamma_4 = \frac{(N-1)}{2N} B_2 B_2 \gamma_4,$$

$$ds_{11}^2 = e^{-\frac{2}{3}\Phi} ds_{10}^2 + e^{\frac{4}{3}\Phi}(dy + C_1)^2, y \sim y + 2\pi R$$
$$C_3^{11\text{ d}} = C_3 + B_2 \wedge dy$$

$$d\tilde{F}_4 = -H_3 F_2, \mathcal{I}_{11} = -\frac{1}{2} H_3 \tilde{F}_4 \tilde{F}_4$$

$$X_4 = S^2 \times S^2$$

$$I_{ab} = \int_{X_4} \omega_{2a}\omega_{2b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$F_2 = n^a \omega_{2a}, H_3 = \widetilde{H}_3 + f_1^a \omega_{2a}, \tilde{F}_4 = \tilde{u}_0 V_4 + F_2^a \omega_{2a} + \gamma_4$$

$$\int_{X_4} V_4 = 1$$

$$n^a = \begin{pmatrix} p \\ p \end{pmatrix}$$

$$dF_2^a = -n^a \widetilde{H}^3, d\tilde{u}_0 = -I_{ab} n^a f_1^b$$

$$I_7 = -\int_{X_4} \mathcal{I}_{11} = \tilde{u}_0 \widetilde{H}_3 \gamma_4 + I_{ab} F_2^a f_1^b \gamma_4 + \frac{1}{2} I_{ab} F_2^a F_2^b \widetilde{H}_3$$

$$dF_2^a = -p\widetilde{H}^3$$

$$F_I = F_2^1 - F_2^2$$

$$M_b^a = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, n'^a = \begin{pmatrix} 0 \\ p \end{pmatrix}, I'_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$I_7 = -\int_{X_4} \mathcal{I}_{11} = \tilde{u}_0 \widetilde{H}_3 \gamma_4 + I'_{ab}(F_I, F_2^1)^a f_1'^b \gamma_4 - F_I F_2^1 \widetilde{H}_3 + F_2^1 F_2^1 \widetilde{H}_3$$

$$dF_2^1 = -p\widetilde{H}_3, d\tilde{u}_0 = pf_1'^1 - 2pf_1'^2$$

$$\widetilde{H}_3 = d\tilde{B}_2, \gamma_4 = dc_3$$

$$I_7^{\text{improved}} = -F_I F_2^1 \widetilde{H}_3 + F_2^1 F_2^1 \widetilde{H}_3 + m\left(dF_2^1 + p\widetilde{H}_3\right) F_2^1 F_2^1 + n\left(dF_2^1 + p\widetilde{H}_3\right) F_2^1 F_I$$

$$I_6^{(0)} = -\frac{(1+mp)}{3p^2} B_2^3 - \frac{np-1}{2p} B_2^2 F_I$$

$$I_6^{0-1} = -\frac{p-1}{2p} B_2^2 F_I$$

$$\check{b} \star \tau([\omega]) = (-1)^q \tau([R(\check{b}) \wedge \omega]), \check{b} \star i(u) = (-1)^q i(I(\check{b}) \smile u)$$

$$\check{b} \in \check{H}^q(\mathcal{M}), \omega \in \Omega^{p-1}(\mathcal{M}), u \in H^{p-1}(\mathcal{M}; \mathbb{R}/\mathbb{Z})$$

$$CS[S^3/\Gamma, k\check{t}_2] = k^2 CS[S^3/\Gamma, \check{t}_2] \text{mod} 1,$$

$$p_q(A \oplus B) = \sum\ r_{2q-j}(A) \smile r_j(B), r_{2s} = p_s, r_{2s+1} = \text{Bock}(w_{2s}) \smile \text{Bock}(w_{2s}) + p_s \smile \text{Bock}(w_1),$$

$$e^{2\pi i \int_{w_2} B_2}, e^{2\pi i \int_{w_1} C_1}, e^{2\pi i \int_{w_3} C_3}$$

$$H_3 = dB_2, F_2 = dC_1, F_4 = dC_3 - C_1 \wedge dB_2$$

Modelo de Gravedad Cuántica Relativista con o sin intervención gravitónica.

$$\varphi = -(e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}),$$

$$C_3 \rightarrow C_{mnp}, C_{mnp}, C_{mn\underline{p}}, C_{mnp}$$

$$\begin{aligned}
\delta\Phi_{\underline{mnp}} &= 3\partial_{[\underline{m}}\Lambda_{\underline{np}]} \\
\delta V_{\underline{mn}} &= \frac{1}{2i}\left(\Lambda_{\underline{mn}} - \bar{\Lambda}_{\underline{mn}}\right) - 2\partial_{[\underline{m}} U_{\underline{n}]} \\
\delta\Sigma_{\alpha\underline{m}} &= -\frac{1}{4}\bar{D}^2 D_\alpha U_{\underline{m}} + \partial_{\underline{m}} \Upsilon_\alpha \\
\delta X &= \frac{1}{2i}\left(D^\alpha \Upsilon_\alpha - \bar{D}_{\dot{\alpha}} \overline{\Upsilon\Upsilon}^{\dot{\alpha}}\right)
\end{aligned}$$

$$\delta V_{\underline{\underline{m}}} = \lambda_{\underline{\underline{m}}} + \bar{\lambda}_{\underline{\underline{m}}}, \delta V_{\underline{mn}} = -i\varphi_{\underline{mnp}}\left(\lambda^{\underline{p}} - \bar{\lambda}_{\underline{p}}\right),$$

$$\delta H_{\alpha\dot{\alpha}} = D_\alpha \bar{L}_{\dot{\alpha}} - \bar{D}_{\dot{\alpha}} L_\alpha$$

$$\delta\Psi_{\underline{m}\alpha} = \Xi_{\underline{m}\alpha} + D_\alpha \Omega_{\underline{m}}$$

$$\begin{aligned}
\delta\Phi_{\underline{mnp}} &= -\frac{i}{2}\psi_{\underline{mnpq}}\bar{D}^2\bar{\Omega}^{\underline{q}} \\
\delta V_{\underline{mn}} &= \frac{1}{2i}\varphi_{\underline{mnp}}\left(\Omega^{\underline{p}} - \bar{\Omega}^{\underline{p}}\right) \\
\delta\Sigma_{\alpha\underline{m}} &= -\Xi_{\underline{m}\alpha} \\
\delta X &= D^\alpha L_\alpha + \bar{D}_{\dot{\alpha}}\bar{L}^{\dot{\alpha}} \\
\delta V_{\underline{m}} &= -\frac{1}{2}\left(\Omega_{\underline{m}} + \bar{\Omega}_{\underline{m}}\right) \\
\delta\Psi_{\underline{m}\alpha} &= 2i\partial_{\underline{m}} L_\alpha
\end{aligned}$$

$$E_{\underline{mnpq}} := 4\partial_{[\underline{m}}\Phi_{\underline{npq}]}$$

$$F_{\underline{mnp}} := \frac{1}{2i}\left(\Phi_{\underline{mnp}} - \bar{\Phi}_{\underline{mnp}}\right) - 3\partial_{[\underline{m}}V_{\underline{np}]}$$

$$W_{\underline{mn}\alpha} := -\frac{1}{4}\bar{D}^2 D_\alpha V_{\underline{mn}} + 2\partial_{[\underline{m}}\Sigma_{|\alpha|\underline{n}]}$$

$$H_{\underline{m}} := \frac{1}{2i}\left(D^\alpha \Sigma_{\alpha\underline{m}} - \bar{D}_{\dot{\alpha}}\bar{\Sigma}^{\dot{\alpha}}_{\underline{m}}\right) - \partial_{\underline{m}}X$$

$$G := -\frac{1}{4}\bar{D}^2 X$$

$$W_{\gamma\beta\alpha} := \frac{i}{16}\bar{D}^2\partial_{(\gamma}{}^{\dot{\gamma}}D_\beta H_{\alpha)\dot{\gamma}}$$

$$R := -\frac{1}{24}\bar{D}^2\bar{G} + \frac{i}{24}\bar{D}^2\partial_{\alpha\dot{\alpha}}H^{\dot{\alpha}\alpha}$$

$$G_{\alpha\dot{\alpha}} := -\frac{i}{6}\partial_{\alpha\dot{\alpha}}(G - \bar{G}) + \left[\frac{1}{2}\Box - \frac{1}{32}\{D^2, \bar{D}^2\}\right]H_{\alpha\dot{\alpha}} + \left[\frac{1}{4}\partial_{\alpha\dot{\alpha}}\partial^{\beta\dot{\beta}} + \frac{1}{12}\Delta_{\alpha\dot{\alpha}}\Delta^{\beta\dot{\beta}}\right]H_{\dot{\beta}\beta}$$

$$X_{\underline{m}\alpha\dot{\alpha}} := \frac{1}{2i}\left(\bar{D}_{\dot{\alpha}}\Psi_{\underline{m}\alpha} + D_\alpha\bar{\Psi}_{\underline{m}\dot{\alpha}}\right) + \partial_{\underline{m}}H_{\alpha\dot{\alpha}}$$

$$\Psi_{\underline{mn}\alpha} := 2\partial_{[\underline{m}}\Psi_{\underline{n}]\alpha}$$

$$\hat{H}_{\underline{m}} := H_{\underline{m}} + \frac{1}{2i}\left(D^\alpha\Psi_{\underline{m}\alpha} - \bar{D}_{\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}_{\underline{m}}\right)$$

$$\Delta_{\alpha\dot{\alpha}} := -\frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]$$

$$\hat{W}_{\underline{mn}\alpha} := W_{\underline{mn}\alpha} + \Psi_{\underline{mn}\alpha}$$

$$\mathcal{W}_{\alpha\underline{m}} := -\frac{1}{4}\bar{D}^2 D_\alpha \mathcal{V}_{\underline{m}}$$

$$F_{\alpha\underline{mnp}} := D_\alpha F_{\underline{mnp}} - 3i\varphi_{\underline{q}[\underline{mn}}\partial_{\underline{p}]}D_\alpha\mathcal{V}\underline{q}$$

$$\delta_\Omega X_{\underline{m\alpha\dot{\alpha}}} = \frac{1}{2i}\left(\bar{D}_{\dot{\alpha}}D_\alpha\Omega_{\underline{m}} + D_\alpha\bar{D}_{\dot{\alpha}}\bar{\Omega}_{\underline{m}}\right)$$

$$\delta_\Omega\hat{H}_{\underline{m}} = \frac{1}{2i}\left(D^2\Omega_{\underline{m}} - \bar{D}^2\bar{\Omega}_{\underline{m}}\right)$$

$$\delta_\Omega\mathcal{W}_{\alpha\underline{m}} = \frac{1}{8}\bar{D}^2 D_\alpha\left(\Omega_{\underline{m}} + \bar{\Omega}_{\underline{m}}\right)$$

$$\delta_\Omega\hat{W}_{\underline{mn}\alpha} = \frac{i}{8}\varphi_{\underline{mnp}}\bar{D}^2 D_\alpha(\Omega^{\underline{p}} - \bar{\Omega}^{\underline{p}}) + 2\partial_{[\underline{m}}D_{|\alpha|}\Omega_{\underline{n}]}$$

$$\delta_\Omega F_{\alpha\underline{mnp}} = -\frac{1}{4}\psi_{\underline{mnpq}}D_\alpha\bar{D}^2\bar{\Omega}^{\underline{q}} + 3i\varphi_{\underline{qmn}}\partial_{\underline{p}]}D_\alpha\Omega^{\underline{q}}$$

$$\delta_\Omega E_{\underline{mnpq}} = 2i\psi_{\underline{[mnp|r|}}\partial_{\underline{q}]}\bar{D}^2\bar{\Omega}^{\underline{r}}$$

$$\lambda_{\underline{m}\alpha} = \varphi_{\underline{m}}\underline{np}\hat{W}_{\alpha\underline{np}} - \frac{i}{6}\psi_{\underline{m}}\underline{npq}F_{\alpha\underline{npq}} + D_\alpha\hat{H}_{\underline{m}} + 2i\mathcal{W}_{\alpha\underline{m}} - 2\bar{D}^{\dot{\alpha}}X_{\underline{m}\alpha\dot{\alpha}}$$

$$\hat{T}^{\hat{A}} = \mathcal{D}\hat{E}^{\hat{A}} = \mathrm{d}\hat{E}^{\hat{A}} + \hat{E}^{\hat{B}} \wedge \hat{\Omega}^{\hat{A}}_{\hat{B}} = \frac{1}{2}\hat{E}^{\hat{C}} \wedge \hat{E}^{\hat{B}}\hat{T}_{\hat{B}\hat{C}}{}^{\hat{A}}$$

$$\hat{R}^{\hat{B}}_{\hat{A}} = \mathrm{d}\hat{\Omega}^{\hat{B}}_{\hat{A}} + \hat{\Omega}^{\hat{C}}_{\hat{A}} \wedge \hat{\Omega}^{\hat{B}}_{\hat{C}}$$

$$\mathcal{D}\hat{T}^{\hat{A}} = \hat{E}^{\hat{B}} \wedge \hat{R}^{\hat{A}}_{\hat{B}}, \mathcal{D}\hat{R}^{\hat{A}}_{\hat{B}} = 0$$

$$\hat{T}_{\hat{N}\hat{M}}{}^{\hat{A}} = 2\hat{\mathcal{D}}_{[\hat{N}}\hat{E}_{\hat{M}]}{}^{\hat{A}} = 2\partial_{[\hat{N}}\hat{E}_{\hat{M}]}{}^{\hat{A}} - 2\hat{E}_{[\hat{N}}{}^{\hat{B}}\hat{\Omega}_{\hat{M}]\hat{B}}{}^{\hat{A}}(-)^{mb}$$

$$\hat{R}_{\hat{N}\hat{M}\hat{A}}{}^{\hat{B}} = 2\partial_{[\hat{N}}\hat{\Omega}_{\hat{M}]\hat{A}}{}^{\hat{B}} - 2\hat{\Omega}_{[\hat{N}|\hat{A}|}{}^{\hat{C}}\hat{\Omega}_{\hat{M}]\hat{C}}{}^{\hat{B}}$$

$$\mathcal{D}_{[\hat{D}}\hat{T}_{\hat{C}\hat{B}]}\hat{A} + \hat{T}_{[\hat{D}\hat{C}}\hat{\hat{F}}\hat{T}_{|\hat{F}|\hat{B}]}\hat{A} = \hat{R}_{[\hat{D}\hat{C}\hat{B}]}\hat{A}$$

$$\mathcal{D}_{[\hat{E}}\hat{R}_{\hat{D}\hat{C}\hat{B}]}\hat{A} + \hat{T}_{[\hat{E}\hat{D}}{}^{\hat{F}}\hat{R}_{|\hat{F}|\hat{C}\hat{B}]}\hat{A} = 0$$

$$\hat{\Omega}_{\hat{a}\hat{b}} = -\hat{\Omega}_{\hat{b}\hat{a}}, \hat{\Omega}_{\hat{\alpha}}^{\hat{\beta}} = \frac{1}{4}\hat{\Omega}_{\hat{a}\hat{b}}\left(\hat{\Gamma}^{\hat{a}\hat{b}}\right)_{\hat{\alpha}}^{\hat{\beta}}$$

$$\hat{C}_3 = \frac{1}{3!}\mathrm{d}z^{\hat{M}} \wedge \mathrm{d}z^{\hat{N}} \wedge \mathrm{d}z^{\hat{P}}\hat{C}_{\hat{P}\hat{N}\hat{M}}$$

$$\mathrm{d}\hat{G}_4 = 0 = \frac{1}{4!}\mathcal{D}_{[\hat{E}}\hat{G}_{\hat{D}\hat{C}\hat{B}\hat{A}]} + \frac{1}{3!\,2!}\hat{T}_{[\hat{E}\hat{D}}{}^{\hat{F}}\hat{G}_{|\hat{F}|\hat{C}\hat{B}\hat{A}]}$$

$$\hat{e}^{-1}\mathcal{L} = -\frac{1}{2}\hat{\mathcal{R}} + \frac{1}{2}\hat{\psi}_{\hat{m}}{}^{\hat{\alpha}}\left(\Gamma^{\hat{m}\hat{n}\hat{p}}\right)_{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{n}}\hat{\psi}_{\hat{p}}{}^{\hat{\beta}} - \frac{1}{4\cdot 4!}\hat{G}_{\hat{m}\hat{n}\hat{p}\hat{q}}\hat{G}^{\hat{m}\hat{n}\hat{p}\hat{q}}$$
$$-\frac{1}{12}\varepsilon^{\hat{m}_1\cdots\hat{m}_{11}}\hat{C}_{\hat{m}_1\hat{m}_2\hat{m}_3}\hat{G}_{\hat{m}_4\cdots\hat{m}_7}\hat{G}_{\hat{m}_8\cdots\hat{m}_{11}} + \cdots$$

$$\hat{G}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = 0 = \hat{G}_{\hat{a}\hat{\beta}\hat{\gamma}\hat{\delta}} = \hat{G}_{\hat{a}\hat{b}\hat{c}\hat{\delta}}$$
$$\hat{G}_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}} = 2\left(\hat{\Gamma}_{\hat{a}\hat{b}}\right)_{\hat{\gamma}\hat{\delta}}$$

$$\hat{T}_{\hat{\gamma}\hat{\beta}}^{\hat{a}} = 2\left(\hat{\Gamma}^{\hat{a}}\right)_{\hat{\gamma}\hat{\beta}}, \hat{T}_{\hat{\gamma}\hat{\beta}}^{\hat{a}} = 0 = \hat{T}_{\hat{\gamma}\hat{\beta}}^{\hat{\alpha}},$$
$$\hat{T}_{\hat{c}\hat{b}}{}^{\hat{a}} = 0.$$

$$\hat{T}_{\hat{a}\hat{\beta}}{}^{\hat{\gamma}} = -\frac{1}{36}\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}}\left(\hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}\right)_{\hat{\beta}}{}^{\hat{\gamma}} - \frac{1}{288}\left(\hat{\Gamma}_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\right)_{\hat{\beta}}{}^{\hat{\gamma}}\hat{G}^{\hat{b}\hat{c}\hat{e}\hat{e}}$$
$$\hat{T}_{\hat{a}\hat{b}}{}^{\hat{\alpha}} = -\frac{1}{84}\left(\hat{\Gamma}^{\hat{c}\hat{d}}\right)^{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{\beta}}\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}}$$

$$\left(\hat{\Gamma}^{\hat{a}\hat{b}\hat{c}}\right)_{\hat{\alpha}}{}^{\hat{\beta}}\hat{T}_{\hat{b}\hat{c},\hat{\beta}} = 0$$

$$\delta\hat{e}_{\hat{m}}{}^{\hat{a}} = -\varepsilon^{\hat{\alpha}}\left(\hat{\Gamma}^{\hat{a}}\right)_{\hat{\alpha}\hat{\beta}}\Psi_{\hat{M}}{}^{\hat{\beta}}$$
$$\delta\hat{\psi}_{\hat{m}}{}^{\hat{\alpha}} = 2\hat{\mathcal{D}}_{\hat{m}}\varepsilon^{\hat{\alpha}} + 2\hat{e}_{\hat{m}}{}^{\hat{a}}\varepsilon^{\hat{\beta}}\left(\frac{1}{36}\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}}\left(\hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}\right)_{\hat{\beta}}{}^{\hat{\alpha}} + \frac{1}{288}\left(\hat{\Gamma}_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\right)_{\hat{\beta}}{}^{\hat{\alpha}}\hat{G}^{\hat{b}\hat{c}\hat{d}\hat{e}}\right),$$
$$\delta\hat{C}_{\hat{m}\hat{n}\hat{p}} = -3\varepsilon^{\hat{\alpha}}\left(\hat{\Gamma}_{[\hat{m}\hat{n}}\right)_{|\hat{\alpha}\hat{\beta}|}\hat{\Psi}_{\hat{p}]}{}^{\hat{\beta}}$$

$$\mathring{E}^{\hat{a}} = \mathrm{d}x^{\hat{a}} - \theta^{\hat{\alpha}}\left(\Gamma^{\hat{a}}\right)_{\hat{\alpha}\hat{\beta}}\mathrm{d}\theta^{\hat{\beta}}, \mathring{E}^{\hat{\alpha}} = \mathrm{d}\theta^{\hat{\alpha}}$$

$$\hat{E}_{\hat{M}}^{\hat{A}} = \mathring{E}_{\hat{M}}^{\hat{A}} + \mathring{E}_{\hat{M}}^{\hat{B}}\boldsymbol{H}_{\hat{B}}^{\hat{A}}$$

$$\delta\boldsymbol{H}_{\hat{B}}{}^{\hat{A}} = -\boldsymbol{L}_{\hat{B}}{}^{\hat{A}} + D_{\hat{B}}\boldsymbol{\xi}^{\hat{A}} + \boldsymbol{\xi}^{\hat{C}}\mathring{T}_{\hat{C}\hat{B}}{}^{\hat{A}}, \delta\boldsymbol{\Omega}_{\hat{M}\hat{B}}{}^{\hat{A}} = \partial_{\hat{M}}\boldsymbol{L}_{\hat{B}}{}^{\hat{A}}$$

$$\hat{T}_{\hat{C}\hat{B}}{}^{\hat{A}} = \mathring{T}_{\hat{C}\hat{B}}{}^{\hat{A}} + \boldsymbol{T}_{\hat{C}\hat{B}}{}^{\hat{A}}$$

$$\boldsymbol{T}_{\hat{C}\hat{B}}{}^{\hat{A}} = 2\hat{D}_{[\hat{C}}\boldsymbol{H}_{\hat{B}]}{}^{\hat{A}} + 2\boldsymbol{\Omega}_{[\hat{C}\hat{B}]}{}^{\hat{A}} + \mathring{T}_{\hat{C}\hat{B}}{}^{\hat{D}}\boldsymbol{H}_{\hat{D}}{}^{\hat{A}} - 2\boldsymbol{H}_{[\hat{C}}{}^{\hat{D}}\mathring{T}_{|\hat{D}|\hat{B}]}{}^{\hat{A}},$$

$$\boldsymbol{R}_{\hat{B}}^{\hat{A}} = \mathrm{d}\boldsymbol{\Omega}_{\hat{B}}^{\hat{A}}$$

$$\boldsymbol{R}_{\hat{D}\hat{C}\hat{B}}^{\hat{A}} = 2D_{[\hat{D}}\boldsymbol{\Omega}_{\hat{C}]\hat{B}}{}^{\hat{A}} + \mathring{T}_{\hat{D}\hat{C}}{}^{\hat{F}}\boldsymbol{\Omega}_{\hat{F}\hat{B}}{}^{\hat{A}}.$$

$$\hat{D}_{[\hat{D}}\boldsymbol{T}_{\hat{C}\hat{B}]}{}^{\hat{A}} + \mathring{T}_{[\hat{D}\hat{C}}{}^{\hat{F}}\boldsymbol{T}_{|\hat{F}|\hat{B}]}{}^{\hat{A}} + \boldsymbol{T}_{[\hat{D}\hat{C}}{}^{\hat{F}}\mathring{T}_{|\hat{F}|\hat{B}]}{}^{\hat{A}} = \boldsymbol{R}_{[\hat{D}\hat{C}\hat{B}]}{}^{\hat{A}}, \hat{D}_{[\hat{E}}\boldsymbol{R}_{\hat{D}\hat{C}]\hat{B}}{}^{\hat{A}} = 0$$

$$\mathring{G}_{\hat{\alpha}\hat{\beta}\hat{c}\hat{d}} = 2\left(\Gamma_{\hat{c}\hat{d}}\right)_{\hat{\alpha}\hat{\beta}}$$

$$\boldsymbol{C} = \frac{1}{3!}\mathring{E}^{\hat{A}} \wedge \mathring{E}^{\hat{B}} \wedge \mathring{E}^{\hat{C}}\boldsymbol{C}_{\hat{C}\hat{B}\hat{A}}$$

$$\boldsymbol{G}_{\hat{D}\hat{C}\hat{B}\hat{A}} = 4D_{[\hat{D}}\boldsymbol{C}_{\hat{C}\hat{B}\hat{A}]} + 6\mathring{T}_{[\hat{D}\hat{C}}{}^{\hat{F}}\boldsymbol{C}_{|\hat{F}|\hat{B}\hat{A}]} - 4\boldsymbol{H}_{[\hat{D}|}{}^{\hat{F}}\mathring{G}_{\hat{F}|\hat{C}\hat{B}\hat{A}]}.$$

$$\delta\boldsymbol{C}_{\hat{M}\hat{N}\hat{P}} = 3\partial_{[\hat{M}}\boldsymbol{\Lambda}_{\hat{N}\hat{P}]} + \boldsymbol{\xi}^{R}\mathring{G}_{\hat{R}\hat{M}\hat{N}\hat{P}}$$

$$\frac{1}{4!}D_{[\hat{E}}\boldsymbol{G}_{\hat{D}\hat{C}\hat{B}\hat{A}]}+\frac{1}{2!}\frac{1}{3!}\mathring{T}_{[\hat{E}\hat{D}}^{\hat{F}}\boldsymbol{G}_{|\hat{F}|\hat{C}\hat{B}\hat{A}]}+\frac{1}{2!}\frac{1}{3!}\boldsymbol{T}_{[\hat{E}\hat{D}}{}^{\hat{F}}{}^{\hat{F}}\mathring{G}_{|\hat{F}|\hat{C}\hat{B}\hat{A}]}=0.$$

$$(\hat{\Gamma}_{\hat{a}})_{(\hat{\alpha}\hat{\beta}}(\hat{\Gamma}^{\hat{a}\hat{b}})_{\hat{\gamma}\hat{\delta})}=0$$

$$\psi^{\hat{\alpha}}=(\psi^{\alpha I},\bar{\psi}_{\dot{\alpha}I})$$

$$\begin{aligned}\psi^{\alpha I}&=\eta^{I}\psi^{\alpha}+i(\Gamma_{\underline{m}}\eta)^{I}\psi^{\underline{m}\alpha}\\ \psi_{\dot{\alpha}I}&=\eta_{I}\psi_{\dot{\alpha}}+i(\Gamma^{\underline{m}}\eta)_{I}\psi_{\underline{m}\dot{\alpha}}\end{aligned}$$

$$\varphi_{\underline{mnp}}=i\eta^{\mathrm{T}}\Gamma_{\underline{mnp}}\eta,\psi_{\underline{mnpq}}=\eta^{\mathrm{T}}\Gamma_{\underline{mnpq}}\eta$$

$$\theta^{\hat{\mu}}=(\theta^{\mu},\theta^{\underline{m}\mu},\bar{\theta}_{\dot{\mu}},\bar{\theta}_{\underline{m}\dot{\mu}})$$

$$\theta\underline{\underline{m\mu}}=0=\bar{\theta}_{\underline{m}\dot{\mu}}$$

$$\mathrm{d}\theta\underline{\underline{m}}^{\underline{m}}=0=\mathrm{d}\bar{\theta}_{\underline{m}\dot{\mu}}$$

$$\begin{aligned}\mathring{E}^{a}&=\mathrm{d}x^{a}+\theta^{\alpha}(\gamma^{a})_{\alpha\dot{\beta}}\mathrm{d}\bar{\theta}^{\dot{\beta}}+\bar{\theta}_{\dot{\alpha}}(\gamma^{a})^{\dot{\alpha}\beta}\mathrm{d}\theta_{\beta}\\ \mathring{E}^{\underline{m}}&=\mathrm{d}y\underline{\underline{m}}\\ \mathring{E}^{\alpha}&=\mathrm{d}\theta^{\alpha}\\ \mathring{E}^{\underline{m}\alpha}&=0\end{aligned}$$

$$\hat{E}^{\hat{A}}=(\hat{E}^{a},\hat{E}^{a},\hat{E}^{\alpha},\hat{E}^{a\alpha},\hat{E}_{\dot{\alpha}},\hat{E}_{\underline{a}\dot{\alpha}})$$

$$\boldsymbol{H}_{\alpha}{}^{\dot{\beta}\beta}:=iD_{\alpha}H^{\dot{\beta}\beta},\boldsymbol{H}_{\alpha}{}^{\beta}:=\delta_{\alpha}{}^{\beta}H,\boldsymbol{H}_{\alpha\dot{\beta}}:=0$$

$$H=\frac{1}{12}D_{\alpha}\bar{D}_{\dot{\alpha}}H^{\dot{\alpha}\alpha}-\frac{i}{6}\partial_{\alpha\dot{\alpha}}H^{\dot{\alpha}\alpha}-\frac{1}{6}G+\frac{1}{3}\bar{G}.$$

$$\begin{aligned}\delta\boldsymbol{H}_{\alpha}{}^{\dot{\beta}\beta}&=D_{\alpha}\boldsymbol{\xi}^{\dot{\beta}\beta}+4i\delta_{\alpha}{}^{\beta}\bar{\boldsymbol{\varepsilon}}^{\dot{\beta}}\\ \delta\boldsymbol{H}_{\alpha}{}^{\beta}&=D_{\alpha}\boldsymbol{\varepsilon}^{\beta}-\boldsymbol{L}_{\alpha}{}^{\beta}=\frac{1}{2}\delta_{\alpha}{}^{\beta}D_{\gamma}\boldsymbol{\varepsilon}^{\gamma}\\ \delta\boldsymbol{H}_{\alpha\dot{\beta}}&=D_{\alpha}\bar{\boldsymbol{\varepsilon}}_{\dot{\beta}}=0\end{aligned}$$

$$\begin{aligned}\boldsymbol{\xi}_{\alpha\dot{\alpha}}&:=-i(D_{\alpha}\bar{L}_{\dot{\alpha}}+\bar{D}_{\dot{\alpha}}L_{\alpha})\\ \boldsymbol{\varepsilon}_{\alpha}&:=-\frac{1}{4}\bar{D}^{2}L_{\alpha}\\ \boldsymbol{L}_{\alpha\beta}&:=D_{(\alpha}\boldsymbol{\varepsilon}_{\beta)}\end{aligned}$$

$$\boldsymbol{T}_{CB}^{A}=2D_{[C}\boldsymbol{H}_{B]}^{A}+2\boldsymbol{\Omega}_{[CB]}^{A}+\mathring{T}_{CB}^{D}\boldsymbol{H}_{D}^{A}-2\boldsymbol{H}_{[C}^{D}\mathring{T}_{|D|B]}^{A},$$

$$\begin{aligned}&\boldsymbol{T}_{\underline{\alpha\beta}}{}^{c}=0,\\ &\boldsymbol{T}_{\underline{\alpha\beta\gamma}}=0,\boldsymbol{T}_{\underline{\alpha}b}{}^{c}=\boldsymbol{T}_{a\underline{\beta}}{}^{c}=0\\ &\boldsymbol{T}_{ab}{}^{c}=0,\end{aligned}$$

$$\boldsymbol{H}_{\beta\dot{\beta}}^{\dot{\alpha}\alpha}=\frac{1}{2}[D_{\beta},\bar{D}_{\dot{\beta}}]H^{\dot{\alpha}\alpha}-\delta_{\beta}{}^{\alpha}\delta_{\dot{\beta}}^{\dot{\alpha}}\left[\frac{1}{3}(G+\bar{G})+\frac{1}{6}[D_{\gamma},\bar{D}_{\dot{\gamma}}]H^{\dot{\gamma}\gamma}\right]$$

$$\delta\boldsymbol{H}_{b}{}^{a}=\partial_{b}\boldsymbol{\xi}^{a}-\boldsymbol{L}_{b}{}^{a},$$

$$\begin{aligned}\boldsymbol{\Omega}_{\dot{\gamma}b}^{a}&=-\bar{D}_{\dot{\gamma}}\boldsymbol{H}_{b}^{a}+\partial_{b}\boldsymbol{H}_{\dot{\gamma}}^{a}-2i\boldsymbol{H}_{b}^{\beta}(\sigma^{a})_{\beta\dot{\gamma}}.\\ \boldsymbol{\Omega}_{\gamma b}^{a}&=-D_{\gamma}\boldsymbol{H}_{b}^{a}+\partial_{b}\boldsymbol{H}_{\gamma}^{a}+2i\boldsymbol{H}_{b}{}^{\dot{\beta}}(\sigma^{a})_{\gamma\dot{\beta}}.\end{aligned}$$

$$\boldsymbol{\Omega}_{\dot{\gamma}\beta}{}^{\alpha}=-\bar{D}_{\dot{\gamma}}\boldsymbol{H}_{\beta}{}^{\alpha}-2i\boldsymbol{H}_{\beta\dot{\gamma}}{}^{\alpha}.$$

$$\boldsymbol{H}^{\alpha}_{\beta\dot{\beta}} = \frac{i}{8}\bar{D}^2 D_\beta H^{\alpha}_{\dot{\beta}} - i\delta^{\alpha}_{\beta}\bar{D}_{\dot{\beta}}\bar{H}$$
$$\boldsymbol{\Omega}_{\dot{\gamma},\beta\alpha} = \frac{1}{4}\bar{D}^2 D_{(\beta}H_{\alpha)\dot{\gamma}}$$
$$\boldsymbol{\Omega}_{\dot{\gamma},\dot{\beta}\dot{\alpha}} = -2\epsilon_{\dot{\gamma}(\dot{\beta}}\bar{D}_{\dot{\alpha})}\bar{H}$$

$$\delta\boldsymbol{H}^{\alpha}_{\beta\dot{\beta}} = \partial_{\beta\dot{\beta}}\varepsilon^{\alpha}, \delta\boldsymbol{\Omega}_{\gamma,\beta\alpha} = D_\gamma \boldsymbol{L}_{\beta\alpha}, \delta\boldsymbol{\Omega}_{\dot{\gamma},\beta\alpha} = \bar{D}_{\dot{\gamma}}\boldsymbol{L}_{\beta\alpha}$$

$$\left(\boldsymbol{H}^{\hat{A}}_{\hat{B}}\right)\Big|_{4|4+7} = \begin{pmatrix} \boldsymbol{H}_B{}^A & \boldsymbol{H}_B{}^{\underline{m}} \\ \boldsymbol{H}_{\underline{n}}{}^A & \boldsymbol{H}_{\underline{n}}{}^{\underline{\underline{m}}} \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{H}_B{}^A & A_B{}^{\underline{m}} \\ \boldsymbol{\chi}_{\underline{n}}{}^A & \boldsymbol{H}_{\underline{n}}{}^{\underline{m}} \end{pmatrix}.$$

$$\boldsymbol{H}_{\underline{m}}{}^{\dot{\alpha}\alpha} = \boldsymbol{\chi}^{\dot{\alpha}\alpha}_{\underline{m}} = -\frac{1}{2}\left(\bar{D}^{\dot{\alpha}}\Psi^{\alpha}_{\underline{m}} - D^{\alpha}\bar{\Psi}^{\dot{\alpha}}_{\underline{m}}\right)$$

$$\delta\boldsymbol{\chi}^{\dot{\alpha}\alpha}_{\underline{m}} = \partial_{\underline{m}}\boldsymbol{\xi}^{\dot{\alpha}\alpha} - \boldsymbol{L}^{\dot{\alpha}\alpha}_{\underline{m}}$$

$$\boldsymbol{L}^{\dot{\alpha}\alpha}_{\underline{m}} = \frac{1}{2}\bar{D}^{\dot{\alpha}}D^{\alpha}\Omega_{\underline{m}} - \frac{1}{2}D^{\alpha}\bar{D}^{\dot{\alpha}}\bar{\Omega}_{\underline{m}}$$

$$2\chi_{\underline{m},\alpha} = \psi_{\underline{m},\alpha} = \frac{i}{4}\left[\bar{D}^2\Psi_{\underline{m}\alpha} + \frac{2i}{3}\left(D_\alpha\hat{H}_{\underline{m}} + 2D^{\dot{\alpha}}X_{\underline{m}\alpha\dot{\alpha}}\right) - \frac{8}{3}\mathcal{W}_{\alpha\underline{m}}\right] + d_1\lambda_{\underline{m}\alpha}$$

$$\delta\boldsymbol{\chi}^{\alpha}_{\underline{m}} = \partial_{\underline{m}}\varepsilon^{\alpha}$$

$$\boldsymbol{H}_{\alpha}\underline{\underline{m}} = iD_\alpha\mathcal{V}\underline{\underline{m}}.$$

$$\delta\boldsymbol{H}_{\alpha}\underline{\underline{m}} = iD_\alpha\lambda\underline{\underline{m}} - \frac{i}{2}D_\alpha\left(\Omega^{\underline{\underline{m}}} + \bar{\Omega}\underline{\underline{m}}\right) = D_\alpha\boldsymbol{\xi}\underline{\underline{m}} + 2i\varepsilon_{\underline{m}\alpha}$$

$$\xi^{\underline{\underline{m}}} := \frac{i}{2}\left(\Omega^{\underline{\underline{m}}} - \bar{\Omega}\underline{\underline{m}}\right) + i(\lambda\underline{\underline{m}} - \bar{\lambda}\underline{\underline{m}}), \varepsilon_{\underline{\underline{m}}\alpha} := -\frac{1}{2}D_\alpha\Omega_{\underline{m}}$$

$$\boldsymbol{H}_{\alpha\dot{\alpha}}\underline{\underline{m}} = \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]\mathcal{V}\underline{\underline{m}} = -\Delta_{\alpha\dot{\alpha}}\mathcal{V}\underline{\underline{m}}$$

$$\delta\boldsymbol{H}_{\alpha\dot{\alpha}}\underline{\underline{m}} = i\partial_{\alpha\dot{\alpha}}\left(\lambda^{\underline{\underline{m}}} - \bar{\lambda}\underline{\underline{m}}\right) + \frac{1}{2}\Delta_{\alpha\dot{\alpha}}(\Omega^{\underline{\underline{m}}} + \bar{\Omega}^{\underline{\underline{m}}}) = \partial_{\alpha\dot{\alpha}}\boldsymbol{\xi}^{\underline{\underline{m}}} - \boldsymbol{L}_{\alpha\dot{\alpha}}\underline{\underline{m}}$$

$$\boldsymbol{H}_{\underline{n}}\underline{\underline{m}} = \frac{1}{2}g_{\underline{n}}\underline{\underline{m}} = \frac{1}{4}\varphi_{\underline{rs}(\underline{n}}F^{\underline{m})\underline{rs}} - \frac{1}{36}\delta^{\underline{\underline{m}}}_{\underline{\underline{n}}}\varphi^{\underline{pqr}}F_{\underline{pqr}}$$

$$\delta\boldsymbol{H}_{\underline{n}}\underline{\underline{m}} = \partial_{\underline{n}}\boldsymbol{\xi}^{\underline{\underline{m}}} - \boldsymbol{L}_{\underline{n}}\underline{\underline{m}}, \boldsymbol{L}_{\underline{nm}} := \partial_{[\underline{n}}\boldsymbol{\xi}_{\underline{m}]}.$$

$$g_{\alpha\underline{mn}} := D_\alpha\left[g_{\underline{mn}} - 2i\partial_{(\underline{m}}\mathcal{V}_{\underline{n})}\right]$$

$$F_{\underline{m}} := -\frac{1}{12}\psi_{\underline{mnpq}}F\underline{npq}, F_{\alpha\underline{m}} := -\frac{1}{12}\psi_{\underline{mnpq}}F_{\alpha}\underline{npq}$$

$$\boldsymbol{H}_b{}^{\underline{n}\alpha} := \frac{1}{2}\psi_b{}^{\underline{n}\alpha}$$
$$\psi_{\beta\dot{\beta},\underline{m}\alpha} := D_{(\alpha}X_{|\underline{m}|\beta)\dot{\beta}} + \epsilon_{\beta\alpha}\left(\frac{1}{3}\bar{D}_{\dot{\beta}}\hat{H}_{\underline{m}} + \frac{2i}{3}\overline{\mathcal{W}}_{\dot{\beta}\underline{m}} + \frac{1}{6}D^{\gamma}X_{\underline{m}\gamma\dot{\beta}}\right) + d_2\epsilon_{\beta\alpha}\bar{\lambda}_{\underline{m}\dot{\beta}}$$

$$\boldsymbol{H}_{\underline{\underline{m}}}{}^{\underline{\underline{n}}\alpha} := \frac{1}{2}\psi_{\underline{\underline{m}}}{}^{\underline{\underline{n}}\alpha}$$
$$\psi_{\underline{\underline{m}},\underline{\underline{n}}\alpha} := \frac{i}{2}g_{\alpha\underline{mn}} - \frac{1}{2}\left(\hat{W}_{\alpha\underline{mn}} - \frac{1}{6}\varphi_{\underline{mnp}}\varphi\underline{\underline{prs}}\hat{W}_{\alpha\underline{rs}}\right) - \frac{i}{72}\varphi_{\underline{mnp}}\psi\underline{\underline{pqrs}}F_{\alpha\underline{qrs}}$$
$$+\frac{1}{6}\varphi_{\underline{mnp}}D_\alpha\hat{H}^{\underline{p}} + d_3\varphi_{\underline{mnp}}\lambda^{\underline{p}}{}_{\alpha}.$$

$$\delta\psi_{\beta\dot{\beta},\underline{m}\alpha} = 2\partial_{\beta\dot{\beta}}\varepsilon_{\underline{m}\alpha}, \delta\psi_{\underline{m},\underline{n}\alpha} = 2\partial_{\underline{m}}\varepsilon_{\underline{n}\alpha}$$

$$\boldsymbol{H}_{\alpha}{}^{\underline{n}\beta} = \frac{1}{2}\psi_{\alpha}{}^{\underline{n}\beta} := -\frac{1}{4}\delta_{\alpha}{}^{\beta}(F^{\underline{n}} - i\hat{H}^{\underline{n}})\boldsymbol{H}^{\dot{\alpha},\underline{n}\beta} := \frac{1}{2}\psi^{\dot{\alpha},\underline{n}\beta} := -\frac{i}{2}X^{\underline{n}\dot{\alpha}\beta}$$

$$\delta\boldsymbol{H}_{\alpha}{}^{\underline{n}\beta} = D_{\alpha}\varepsilon^{\underline{n}\beta} - \frac{1}{4}\delta_{\alpha}{}^{\beta}\varphi\underline{npq}\boldsymbol{L}_{\underline{pq}}, \delta\boldsymbol{H}^{\dot{\alpha},\underline{n}\beta}$$

$$= \bar{D}^{\dot{\alpha}}\varepsilon^{\underline{n}\beta} + \frac{1}{2}\boldsymbol{L}_{\underline{n}}^{\beta\dot{\alpha}}.$$

$$\boldsymbol{T}_{\hat{C}\hat{B}}^{\alpha} \sim \frac{1}{4}\delta_{\hat{B}}^{\beta}\boldsymbol{\Omega}_{\hat{C}de}(\gamma^{de})_{\beta}^{\alpha} - \hat{C} \leftrightarrow \hat{B}$$

$$\boldsymbol{T}_{\hat{C}\hat{B}}{}^{\underline{m}\alpha} \sim +\frac{1}{2}\delta_{\hat{B}\dot{\beta}}\boldsymbol{\Omega}_{\hat{C}d}{}^{\underline{m}}(\sigma_d)^{\dot{\beta}\alpha} + \frac{1}{4}\delta_{\hat{B}}^{\beta}\varphi^{\underline{mnp}}\boldsymbol{\Omega}_{\hat{C}\underline{np}} - \hat{C} \leftrightarrow \hat{B}$$

$$\delta\boldsymbol{H}_{\hat{B}}{}^{\underline{m}\alpha}|_{\text{Lorentz}} = -\frac{1}{2}\delta_{\hat{B}\dot{\beta}}\boldsymbol{L}_d{}^{\underline{m}}(\sigma_d)^{\dot{\beta}\alpha} - \frac{1}{4}\delta_{\hat{B}}{}^{\beta}\varphi\underline{mnp}\boldsymbol{L}_{\underline{np}}$$

$$\boldsymbol{\Omega}_{\hat{c}\hat{b}\hat{a}} = -\frac{1}{2}\left(K_{\hat{c}\hat{b}\hat{a}} + K_{\hat{a}\hat{b}\hat{c}} + K_{\hat{a}\hat{c}\hat{b}}\right)$$

$$\boldsymbol{T}_{\beta\underline{m}}^{a} = D_{\beta}\boldsymbol{H}_{\underline{m}}^{a} - \partial_{\underline{m}}\boldsymbol{H}_{\beta}^{a} + \boldsymbol{\Omega}_{\beta\underline{m}}^{a} + 2i\boldsymbol{H}_{\underline{m},\dot{\gamma}}(\sigma^{a})_{\beta}^{\dot{\gamma}}.$$

$$\boldsymbol{\Omega}_{\beta\underline{\underline{m}}}^{a} = -\boldsymbol{T}_{\underline{m}\beta}^{a} + \partial_{\underline{m}}\boldsymbol{H}_{\beta}^{a} - D_{\beta}\boldsymbol{H}_{\underline{m}}^{a} - 2i\boldsymbol{H}_{\underline{m},\dot{\gamma}}(\sigma^{a})_{\beta}^{\dot{\gamma}}.$$

$$\boldsymbol{\Omega}_{\beta a}\underline{\underline{m}} = -\boldsymbol{T}_{a\beta}\underline{\underline{m}} + \partial_{a}\boldsymbol{H}_{\beta}\underline{\underline{m}} - D_{\beta}\boldsymbol{H}_{a}\underline{\underline{m}} - 2i\delta\underline{\underline{mn}}\boldsymbol{H}_{a,\underline{n}\beta}.$$

$$\boldsymbol{T}_{\underline{m}\beta,a} + \boldsymbol{T}_{a\beta,\underline{m}} = \partial_{a}\boldsymbol{H}_{\beta,\underline{m}} - D_{\beta}\boldsymbol{H}_{a,\underline{m}} + \partial_{\underline{m}}\boldsymbol{H}_{\beta,a} - D_{\beta}\boldsymbol{H}_{\underline{m},a} - 2i\boldsymbol{H}_{\underline{m}\dot{\gamma}}(\sigma_a)_{\beta}^{\dot{\gamma}} - 2i\boldsymbol{H}_{a,\underline{m}\beta},$$

$$(\sigma^{a})_{\alpha\dot{\alpha}}\left(\boldsymbol{T}_{\underline{m}\beta,a} + \boldsymbol{T}_{a\beta,\underline{m}}\right) = i\epsilon_{\beta\alpha}(2d_1 + d_2)\bar{\lambda}_{\underline{m}\dot{\alpha}}.$$

$$\boldsymbol{T}_{a\beta}\underline{m} = id_4(\sigma_a)_{\beta\dot{\beta}}\bar{\lambda}^{\dot{\beta}\underline{m}}$$

$$\boldsymbol{T}_{\underline{m}\beta}{}^{a} = i\left(d_1 + \frac{1}{2}d_2 - d_4\right)(\sigma^{a})_{\beta\dot{\beta}}\bar{\lambda}^{\dot{\beta}}\underline{m}$$

$$\boldsymbol{\Omega}_{\beta a}\underline{m} = (\sigma_a)_{\beta\dot{\beta}}\left(\overline{\mathcal{W}}^{\dot{\beta}\underline{m}} + id_4\bar{\lambda}^{\dot{\beta}\underline{m}}\right) - 2i\boldsymbol{H}_{a,\beta}\underline{m}.$$

$$\boldsymbol{T}_{\underline{n}\beta}\underline{m} = \partial_{\underline{n}}\boldsymbol{H}_{\beta}\underline{m} - D_{\beta}\boldsymbol{H}_{\underline{n}}\underline{m} - \boldsymbol{\Omega}_{\beta\underline{n}}\underline{m} - 2i\delta\underline{mp}\boldsymbol{H}_{\underline{n},\underline{p}}\beta.$$

$$\boldsymbol{T}_{\beta\underline{\underline{n}}}\underline{\underline{m}} = -\boldsymbol{T}_{\underline{n}\beta}\underline{\underline{m}} = id_5\varphi_{\underline{n}}\underline{\underline{mp}}\lambda_{\underline{p}\beta}$$

$$\boldsymbol{\Omega}_{\beta\underline{\underline{mn}}} = i(d_5 - d_3)\varphi_{\underline{mn}}\lambda_{\underline{p}\beta} + iD_{\beta}\partial_{[\underline{m}}\mathcal{V}_{\underline{n}]} + \frac{i}{2}\left(\hat{W}_{\beta\underline{\underline{mn}}} - \frac{1}{6}\varphi_{\underline{mnp}}\varphi\underline{\underline{pqr}}\hat{W}_{\beta\underline{q}}\right)$$

$$-\frac{1}{72}\varphi_{\underline{\underline{mnp}}}\psi\underline{\underline{pqrs}}F_{\beta\underline{\underline{qrs}}} - \frac{i}{6}\varphi_{\underline{\underline{mnp}}}D_{\beta}\hat{H}_{\underline{\underline{p}}}$$

$$0 = \boldsymbol{T}_{\alpha\beta}\underline{\underline{m}} = 2D_{(\alpha}\boldsymbol{H}_{\beta)}\underline{\underline{m}}$$

$$0 = \boldsymbol{T}_{\alpha\dot{\beta}}\underline{\underline{m}} = D_{\alpha}\boldsymbol{H}_{\dot{\beta}}\underline{\underline{m}} + \bar{D}_{\dot{\beta}}\boldsymbol{H}_{\alpha}\underline{\underline{m}} + 2i\delta\underline{\underline{mn}}\boldsymbol{H}_{\alpha,\underline{n}\dot{\beta}}$$

$$\boldsymbol{T}_{\underline{m}\beta}^{\alpha} = \partial_{\underline{m}}\boldsymbol{H}_{\beta}^{\alpha} - D_{\beta}\boldsymbol{H}_{\underline{m}}^{\alpha} + \boldsymbol{\Omega}_{\underline{m}\beta}^{\alpha},$$

$$\boldsymbol{T}_{\underline{m}\beta}^{\alpha} =: i\delta_{\beta}^{\alpha}S_{\underline{m}} + S_{\underline{m}\beta}^{\alpha}$$

$$\begin{aligned} iS_{\underline{m}} &= \partial_{\underline{m}}H + \frac{1}{2}D^{\gamma}H_{m\gamma} \\ &= -\frac{i}{6}\partial_{\alpha\dot{\alpha}}X_{\underline{m}}^{\alpha\dot{\alpha}} - \frac{1}{24}\bar{D}^{2}H_{\underline{m}} + \frac{1}{24}D^{2}H_{\underline{m}} - \frac{i}{6}D^{\alpha}\mathcal{W}_{\alpha\underline{m}} + \frac{1}{4}d_1D^{\alpha}\lambda_{\underline{m}\alpha} \\ &= \frac{i}{6}J(\mathcal{V})_{\underline{m}} - \frac{i}{36}\boldsymbol{G}_{\underline{mnpq}}\varphi\underline{npq} + \frac{1}{4}d_1D^{\alpha}\lambda_{\underline{m}\alpha} \end{aligned}$$

$$\begin{aligned} S_{\underline{m}\beta\alpha} &= -D_{(\beta}\boldsymbol{H}_{|\underline{m}|\alpha)} + \boldsymbol{\Omega}_{\underline{m}\beta\alpha} \\ &= -\frac{1}{12}D_{(\beta}\bar{D}^{\dot{\gamma}}X_{|\underline{m}|\alpha)\dot{\gamma}} - \frac{i}{4}\partial_{(\beta}{}^{\dot{\gamma}}X_{\underline{m}\alpha)\dot{\gamma}} + \frac{i}{12}D_{(\beta}\mathcal{W}_{\alpha)\underline{m}} - \frac{1}{2}d_1D_{(\beta}\lambda_{\underline{m}\alpha)} \\ &= -\frac{i}{12}\varphi_{\underline{m}}\underline{np}\boldsymbol{G}_{\alpha\beta\underline{np}} + \left(\frac{1}{24} - \frac{d_1}{2}\right)D_{(\beta}\lambda_{|\underline{m}|\alpha)} \end{aligned}$$

$$\boldsymbol{T}_{\underline{m}}^{\dot{\beta}\alpha} \equiv -iS_{\underline{m}c}(\bar{\sigma}^{c})^{\dot{\beta}\alpha} = -\bar{D}^{\dot{\beta}}\boldsymbol{H}_{\underline{m}}^{\alpha}$$
$$= -\frac{1}{2}d_{1}\bar{D}^{\dot{\beta}}\lambda_{\underline{m}}^{\alpha} + \frac{1}{12}\bar{D}^{\dot{\beta}}D^{\alpha}\hat{H}_{\underline{m}} + \frac{1}{12}\bar{D}^{2}X_{\underline{m}}^{\dot{\beta}\alpha}$$
$$= -\frac{1}{2}d_{1}\bar{D}^{\dot{\beta}}\lambda_{\underline{m}}^{\alpha} + \frac{1}{12}\left[-\frac{1}{2}\bar{D}^{\dot{\beta}}\lambda_{\underline{m}}^{\alpha} - \frac{1}{2}D^{\alpha}\bar{\lambda}_{\underline{m}}^{\dot{\beta}} + 2\tilde{G}^{\dot{\beta}\alpha}{}_{\underline{m}} + \frac{i}{6}\psi_{\underline{m}}{}^{\overset{npq}{\rightarrow}}\boldsymbol{G}^{\dot{\beta}\alpha}{}_{\underline{npq}}\right]$$

$$\boldsymbol{T}_{\gamma\beta}{}^{\underline{m\alpha}} = 2D_{(\gamma}\boldsymbol{H}_{\beta)}{}^{\underline{m\alpha}} + \frac{1}{2}\varphi\underline{mnp}\delta_{(\beta}{}^{\alpha}\boldsymbol{\Omega}_{\gamma)}\underline{np}$$
$$\boldsymbol{T}_{\gamma\dot{\beta}}{}^{\underline{m\alpha}} = D_{\gamma}\boldsymbol{H}_{\dot{\beta}}{}^{\underline{m\alpha}} + \bar{D}_{\dot{\beta}}\boldsymbol{H}_{\gamma}\underline{\underline{m\alpha}} + 2i\boldsymbol{H}_{\gamma\dot{\beta}}{}^{\underline{m\alpha}} - \frac{1}{2}\boldsymbol{\Omega}_{\gamma}{}^{d\underline{m}}(\sigma_{d})^{\alpha}{}_{\dot{\beta}} + \frac{1}{4}\varphi\underline{mnp}\delta_{\gamma}{}^{\alpha}\boldsymbol{\Omega}_{\dot{\beta}}\underline{np}$$
$$\boldsymbol{T}_{\dot{\gamma}\dot{\beta}}\underline{\underline{m}}^{\alpha} = 2\bar{D}_{(\dot{\gamma}}\boldsymbol{H}_{\dot{\beta})}{}^{\underline{m\alpha}} - \boldsymbol{\Omega}_{(\dot{\gamma}}{}^{d\underline{m}}\left(\sigma_{|d|}\right)^{\alpha}{}_{\dot{\beta})}$$

$$\boldsymbol{T}_{\gamma\beta}{}^{\underline{m\alpha}} = 3i(d_{5} - d_{3})\delta_{(\gamma}^{\alpha}\lambda_{\beta)}^{\underline{m}}$$
$$\boldsymbol{T}_{\gamma\dot{\beta}}{}^{\underline{m\alpha}} = i\delta_{\gamma}^{\alpha}\bar{\lambda}_{\dot{\beta}}^{\underline{m}}\left(-\frac{3}{2}(d_{2} - d_{3} + d_{5}) - d_{4}\right)$$
$$\boldsymbol{T}_{\dot{\gamma}\dot{\beta}\underline{m}}{}^{\alpha} = 0$$

$$\boldsymbol{T}_{b\gamma}{}^{\underline{m\alpha}} = \partial_{b}\boldsymbol{H}_{\gamma}\underline{\underline{m}}^{\alpha} - D_{\gamma}\boldsymbol{H}_{b}\underline{\underline{m}}^{\alpha} + \boldsymbol{\Omega}_{b,\gamma}\underline{\underline{m}}^{\alpha},$$
$$\boldsymbol{T}_{b\dot{\gamma}}\underline{\underline{m}}^{\alpha} = \partial_{b}\boldsymbol{H}_{\dot{\gamma}}\underline{\underline{m}}^{\alpha} - \bar{D}_{\dot{\gamma}}\boldsymbol{H}_{b}\underline{\underline{m}}^{\alpha} + \boldsymbol{\Omega}_{b,\dot{\gamma}}\underline{\underline{m}}^{\alpha},$$

$$G_{\underline{mnpq}} = \mathrm{Re}E_{\underline{mnpq}} + 2\psi_{\underline{r}[\underline{mnp}}\partial_{\underline{q}]}\hat{H}^{\underline{r}}$$

$$(\sigma^{a})^{\alpha\dot{\alpha}}\boldsymbol{G}_{a\underline{mnp}} = \frac{1}{2}\left[D_{\alpha}F_{\dot{\alpha}\underline{mnp}} - \bar{D}_{\dot{\alpha}}F_{\alpha\underline{mnp}} - 6\varphi_{\underline{q}[\underline{mn}}\partial_{\underline{p}]}X^{\underline{q}}{}_{\alpha\dot{\alpha}} + \psi_{\underline{mnpq}}\partial_{\alpha\dot{\alpha}}\hat{H}^{\underline{q}}\right]$$
$$\boldsymbol{G}_{ab\underline{mn}} = -(\sigma_{ab})^{\alpha\beta}G_{\alpha\beta\underline{mn}} + \text{h.c.}$$
$$= \frac{i}{2}(\sigma_{ab})^{\alpha\beta}\left[D_{(\alpha}\hat{W}_{\beta)\underline{mn}} + i\varphi_{\underline{mn}\underline{\underline{p}}}\partial_{(\alpha}\dot{\gamma}_{|\underline{p}|\beta)\dot{\gamma}}\right] + \text{h.c.}$$
$$\boldsymbol{G}_{abc\underline{m}} = \epsilon_{abcd}\tilde{G}_{\underline{m}}^{d}$$
$$= \frac{1}{8}\epsilon_{abcd}(\bar{\sigma}^{d})^{\dot{\alpha}\alpha}\left([D_{\alpha},\bar{D}_{\dot{\alpha}}]\hat{H}_{\underline{m}} - D^{2}X_{\underline{m}\alpha\dot{\alpha}} - \bar{D}^{2}X_{\underline{n}\alpha\dot{\alpha}}\right)$$
$$\boldsymbol{G}_{abcd} = 3i\epsilon_{abcd}(R - \bar{R})$$

$$(\sigma^{a})_{\alpha\dot{\alpha}}G_{a\underline{mn}} = \frac{1}{2}D_{\alpha}g_{\dot{\alpha}\underline{mn}} - \frac{1}{2}\bar{D}_{\dot{\alpha}}g_{\alpha\underline{mn}} - 2\partial_{(\underline{m}}X_{\underline{n})\alpha\dot{\alpha}}$$
$$(\sigma^{a})_{\alpha\dot{\alpha}}G_{a\underline{m}} = \frac{1}{2}D_{\alpha}F_{\dot{\alpha}\underline{m}} - \frac{1}{2}\bar{D}_{\dot{\alpha}}F_{\alpha\underline{m}} - \varphi_{\underline{m}}\underline{\underline{np}}\partial_{\underline{n}}X_{\underline{p}\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}}\hat{H}_{\underline{m}}$$

$$\boldsymbol{G}_{ab\underline{mn}} = -\frac{1}{6}(\sigma_{b})_{\alpha\dot{\alpha}}\varphi_{\underline{mnp}}\bar{\lambda}^{\underline{p}\dot{\alpha}}$$

$$C_{\underline{mnp}} = \frac{1}{2}\left(\Phi_{\underline{mnp}} + \bar{\Phi}_{\underline{mnp}}\right) + \frac{1}{2}\psi_{\underline{mnpq}}\hat{H}^{\underline{q}}$$
$$(\sigma^{a})_{\alpha\dot{\alpha}}C_{a\underline{mn}} = \frac{1}{2}[D_{\alpha},\bar{D}_{\dot{\alpha}}]V_{\underline{mn}} - \varphi_{\underline{mnp}}\partial_{\alpha\dot{\alpha}}\mathcal{V}^{\underline{p}} + \varphi_{\underline{mnp}}X^{\underline{p}}{}_{\alpha\dot{\alpha}}$$
$$C_{ab\underline{m}} = -(\sigma_{ab})^{\alpha\beta}C_{\alpha\beta\underline{m}} + \text{h.c.},$$
$$= -(\sigma_{ab})^{\alpha\beta}\left[-\frac{i}{2}\left(D_{(\alpha}\Sigma_{\beta)\underline{m}} + D_{(\alpha}\Psi_{|\underline{m}|\beta)}\right)\right] + \text{h.c.}$$
$$C_{abc} = \epsilon_{abcd}\tilde{C}^{d}$$
$$= -\frac{1}{2}\epsilon_{abcd}(\bar{\sigma}^{d})^{\dot{\alpha}\alpha}\left[-\frac{1}{4}([D_{\alpha},\bar{D}_{\dot{\alpha}}]X + D^{2}H_{\alpha\dot{\alpha}} + \bar{D}^{2}H_{\alpha\dot{\alpha}})\right].$$

$$\hat{E}^{\alpha I} = \eta^{I}\hat{E}^{\alpha} + i\left(\Gamma_{\underline{m}}\eta\right)^{I}\hat{E}^{\underline{m}\alpha}$$
$$\hat{E}_{\dot{\alpha}I} = \eta_{I}\hat{E}_{\dot{\alpha}} + i(\Gamma\underline{m}\eta)_{I}\hat{E}_{\underline{m}\dot{\alpha}}$$

$$D_{\alpha I} = \eta_{I}D_{\alpha} + i(\Gamma\underline{m}\eta)_{I}D_{\underline{m}\alpha}$$
$$\bar{D}^{\dot{\alpha}I} = \eta^{I}\bar{D}^{\dot{\alpha}} + i\left(\Gamma_{\underline{m}}\eta\right)^{I}\bar{D}^{\underline{m}\dot{\alpha}}$$

$$D_{[\hat{D}}\boldsymbol{T}_{\hat{C}\hat{B}]}^{A} + \mathring{T}_{[\hat{D}\hat{C}}{}^{\hat{f}}\boldsymbol{T}_{|\hat{f}|\hat{B}]}^{A} + \boldsymbol{T}_{[\hat{D}\hat{C}}{}^{\hat{\gamma}}\mathring{T}_{|\hat{\gamma}|\hat{B}]}{}^{A} = \boldsymbol{R}_{[\hat{D}\hat{C}\hat{B}]}^{A}.$$

$$D_{[D}\boldsymbol{T}^{A}_{CB]} + \mathring{T}^{f}_{[DC}\boldsymbol{T}^{A}_{|f|B]} + \boldsymbol{T}^{F}_{[DC}\mathring{T}^{A}_{|F|B]} = \boldsymbol{R}^{A}_{[DCB]}$$
$$\partial_{\underline{m}}\boldsymbol{T}^{A}_{CB} + 2D_{[C}\boldsymbol{T}^{A}_{B]\underline{m}} + \mathring{T}^{f}_{CB}\boldsymbol{T}^{A}_{f\underline{m}} + 2\boldsymbol{T}^{F}_{\underline{m}[C}\mathring{T}^{A}_{|F|B]} = 2\boldsymbol{R}^{A}_{\underline{m}[CB]} + \boldsymbol{R}^{A}_{CB\underline{m}}$$
$$2\partial_{[\underline{n}}\boldsymbol{T}^{A}_{\underline{m}]B} + D_{B}\boldsymbol{T}^{A}_{\underline{nm}} + \underset{\boldsymbol{T}}{\underline{nm}}^{F}\,{}^{F}\mathring{T}^{A}_{FB} = \boldsymbol{R}_{\underline{nm}B}{}^{A} + 2\boldsymbol{R}^{A}_{B[\underline{nm}]}$$
$$\partial_{[\underline{n}}\boldsymbol{T}^{A}_{\underline{mp}]} = \boldsymbol{R}^{A}_{\underline{\underline{nmp}]}}$$

$$D_{[\hat{D}}\boldsymbol{T}_{\hat{C}\hat{B}]}\underline{a} + \mathring{T}^{\hat{f}}_{[\hat{D}\hat{C}}\boldsymbol{T}_{|\hat{f}|\hat{B}]}\underline{a} + \boldsymbol{T}_{[\hat{D}\hat{C}}{}^{\hat{\gamma}}\mathring{T}_{|\hat{\gamma}|\hat{B}]}\underline{a} = \boldsymbol{R}_{[\hat{D}\hat{C}\hat{B}]}\underline{a}.$$

$$D_{[D}\boldsymbol{T}_{CB]}{}^{\underline{a}} + \mathring{T}^{f}_{[DC}\boldsymbol{T}_{|f|B]}{}^{\underline{a}} + \boldsymbol{T}_{[DC}T_{|\hat{\gamma}|B]}^{\hat{\gamma}\quad\underline{a}} = \boldsymbol{R}_{[DCB]}{}^{\underline{a}}$$
$$\partial_{\underline{m}}\boldsymbol{T}_{CB}{}^{\underline{a}} + 2D_{[C}\boldsymbol{T}_{B]\underline{m}}{}^{\underline{a}} + \mathring{T}^{f}_{CB}\boldsymbol{T}_{f\underline{m}}{}^{\underline{a}} + 2\boldsymbol{T}_{\underline{m}[C}T_{|\hat{\gamma}|B]}^{\hat{\gamma}\quad\underline{a}} = 2\boldsymbol{R}_{\underline{m}[CB]}A + \boldsymbol{R}_{CB\underline{m}}{}^{\underline{a}}$$
$$2\partial_{[\underline{n}}\boldsymbol{T}_{\underline{m}]B}{}^{\underline{a}} + D_{B}\boldsymbol{T}_{\underline{nm}}{}^{\underline{a}} + \boldsymbol{T}_{\underline{nm}}{}^{\hat{\gamma}}\mathring{T}_{\hat{\gamma}B^{\underline{a}}} = \boldsymbol{R}_{\underline{nm}B^{\underline{a}}} + 2\boldsymbol{R}_{B[\underline{nm}]}{}^{\underline{a}}$$
$$\partial_{[\underline{n}}\boldsymbol{T}_{\underline{mp}]} = \boldsymbol{R}_{\underline{[nmp]}}{}^{\underline{a}}.$$

$$D_{[\hat{D}}\boldsymbol{T}_{\hat{C}\hat{B}]}\underline{\underline{m\alpha}} + \mathring{T}^{\hat{f}}_{[\hat{D}\hat{C}}\boldsymbol{T}_{|\hat{f}|\hat{B}]}\underline{\underline{m\alpha}} = \boldsymbol{R}_{[\hat{D}\hat{C}\hat{B}\}}\overset{m\alpha}{\rightarrow}$$

$$D_{[D}\boldsymbol{T}_{CB]}\underline{\underline{m\alpha}} + \mathring{T}^{f}_{[DC}\boldsymbol{T}_{|f|B]}\underline{\underline{m\alpha}} = \boldsymbol{R}_{[DCB]}\underline{\underline{m\alpha}}$$
$$\partial_{\underline{n}}\boldsymbol{T}_{CB}\underline{\underline{m\alpha}} + 2D_{[C}\boldsymbol{T}_{B]\underline{n}}\underline{\underline{m\alpha}} + \mathring{T}^{f}_{CB}\boldsymbol{T}_{f\underline{n}}\underline{\underline{m\alpha}} = 2\boldsymbol{R}_{\underline{n}[CB]}\underline{\underline{m\alpha}} + \boldsymbol{R}_{CB\underline{n}}\underline{\underline{m\alpha}}$$
$$2\partial_{[\underline{n}}\boldsymbol{T}_{\underline{p}]}\underline{\underline{m\alpha}} + D_{B}\boldsymbol{T}_{\underline{np}}\underline{\underline{m\alpha}} = \boldsymbol{R}_{\underline{np}}B^{\underline{\underline{m\alpha}}} + 2\boldsymbol{R}_{B\underline{np}}\underline{\underline{m\alpha}}$$
$$\partial_{[\underline{n}}\boldsymbol{T}_{\underline{pq}]}\underline{\underline{m\alpha}} = \boldsymbol{R}_{\underline{npq}]}\underline{\underline{m\alpha}}$$

$$D_{[\hat{E}}\boldsymbol{G}_{\hat{D}\hat{C}\hat{B}\hat{A}]} + 2\mathring{T}_{[\hat{E}\hat{D}}{}^{\hat{F}}\boldsymbol{G}_{|\hat{F}|\hat{C}\hat{B}\hat{A}]} + 2\boldsymbol{T}_{[\hat{E}\hat{D}}{}^{\hat{F}}\mathring{G}_{|\hat{F}|\hat{C}\hat{B}\hat{A}]} = 0$$

$$D_{[E}\boldsymbol{G}_{DCBA]} + 2\mathring{T}^{f}_{[ED}\boldsymbol{G}_{|f|CBA]} + 2\boldsymbol{T}_{[ED}{}^{\hat{F}}\mathring{G}_{|\hat{F}|CBA]} = 0$$
$$D_{[\underline{m}}\boldsymbol{G}_{DCBA]} + 2\mathring{T}^{f}_{[\underline{m}D}\boldsymbol{G}_{|f|CBA]} + 2\boldsymbol{T}_{[\underline{m}D}{}^{\hat{F}}\mathring{G}_{|\hat{F}|CBA]} = 0$$
$$D_{[\underline{m}}\boldsymbol{G}_{\underline{n}CBA]} + 2\mathring{T}^{f}_{[\underline{mn}}\boldsymbol{G}_{|f|CBA]} + 2\boldsymbol{T}_{[\underline{mn}}{}^{\hat{F}}\mathring{G}_{|\hat{F}|CBA]} = 0$$
$$D_{[\underline{m}}\boldsymbol{G}_{\underline{np}BA]} + 2\mathring{T}_{\underline{\underline{mn}}}{}^{f}\boldsymbol{G}_{|f|\underline{p}BA]} + 2\boldsymbol{T}_{[\underline{mn}}{}^{\hat{F}}\mathring{G}_{|\hat{F}|\underline{p}BA]} = 0$$
$$D_{[\underline{m}}\boldsymbol{G}_{\underline{npq}A]} + 2\boldsymbol{T}_{[\underline{mn}}{}^{\hat{F}}\mathring{G}_{|\hat{F}|\underline{pq}A]} = 0$$
$$\partial_{[\underline{m}}\boldsymbol{G}_{\underline{npqr}]} = 0$$

$$\boldsymbol{R}^{\alpha}_{\beta} = \frac{1}{4}\boldsymbol{R}_{ab}(\gamma^{ab})^{\alpha}_{\beta}$$
$$\boldsymbol{R}^{\alpha}_{\underline{m}\beta} = -\frac{1}{4}\delta^{\alpha}_{\beta}\varphi_{\underline{mnp}}\boldsymbol{R}\underline{np}$$
$$\boldsymbol{R}^{\underline{m}\alpha}_{\beta} = \frac{1}{4}\delta^{\alpha}_{\beta}\varphi^{\underline{mnp}}\boldsymbol{R}_{\underline{np}}$$
$$\boldsymbol{R}_{\underline{n}}\beta^{\underline{m}\alpha} = \frac{1}{4}\delta^{\underline{m}}_{\underline{n}}\boldsymbol{R}_{cd}(\gamma^{cd})^{\alpha}_{\beta} + \frac{1}{4}\delta^{\alpha}_{\beta}\left[\psi_{\underline{n}}\underline{mpq}\boldsymbol{R}_{\underline{pq}} + \boldsymbol{R}^{\underline{m}}_{\underline{n}} - \boldsymbol{R}^{\underline{m}}_{\underline{n}}\right]$$
$$\boldsymbol{R}^{\underline{n}\dot{\beta},\alpha} = -\frac{1}{2}(\bar{\sigma}^{a})^{\dot{\beta}\alpha}\boldsymbol{R}^{\underline{n}}_{a}$$
$$\boldsymbol{R}^{\dot{\beta},\underline{m}\alpha} = \frac{1}{2}(\bar{\sigma}^{b})^{\dot{\beta}\alpha}\boldsymbol{R}_{b}\underline{m}$$
$$\boldsymbol{R}^{\underline{n}\dot{\beta},\underline{m}\alpha} = -\frac{1}{2}(\bar{\sigma}^{d})^{\dot{\beta}\alpha}\varphi\underline{mnp}\boldsymbol{R}_{d\underline{p}}$$

$$\boldsymbol{T}^{c}_{\underline{\alpha\beta}} = \boldsymbol{T}_{\underline{\alpha\beta}}\underline{m} = 0$$

$$\boldsymbol{T}^{c}_{\underline{a\beta}} = \boldsymbol{T}^{c}_{\underline{m\beta}} = \boldsymbol{T}_{\underline{a\beta}}{}^{\underline{m}} = \boldsymbol{T}_{\underline{n\beta}}\underline{\underline{m}}^{=0}, \boldsymbol{T}_{\alpha\underline{\beta\gamma}}{}^{=\boldsymbol{T}_{\alpha\underline{\beta}}}\underline{\underline{m}}^{\gamma} = \boldsymbol{T}_{\alpha\underline{\beta},\underline{m}\dot{\gamma}} = 0.$$

$$\boldsymbol{T}^{c}_{ab} = \boldsymbol{T}^{c}_{a\underline{\underline{m}}} = \boldsymbol{T}^{c}_{\underline{\underline{mn}}} = \boldsymbol{T}_{ab}{}^{\underline{m}} = \boldsymbol{T}_{a\underline{\underline{n}}}{}^{\underline{m}} = \boldsymbol{T}_{\underline{np}}{}^{\underline{m}} = 0.$$

$$\boldsymbol{G}_{\underline{\alpha\beta\gamma\delta}} = 0, \boldsymbol{G}_{\alpha\underline{\beta\gamma\delta}} = \boldsymbol{G}_{\underline{m}\underline{\beta\gamma\delta}} = 0, \boldsymbol{G}_{ab\underline{\gamma\delta}} = \boldsymbol{G}_{a\underline{m}\underline{\gamma\delta}} = \boldsymbol{G}_{\underline{mn}\underline{\gamma\delta}} = 0.$$

$$G_{\underline{\alpha mnp}} = 0$$

$$G_{\alpha b\underline{mn}} = -\frac{1}{6}(\sigma_b)_{\alpha\dot{\alpha}}\varphi_{\underline{mnp}}\bar{\lambda}^{\underline{p}\dot{\alpha}}$$

$$\begin{aligned}(\sigma^c)_{(\delta|\dot{\beta}|}\boldsymbol{G}_{\gamma)cba} &= 0 = (\sigma^c)_{\delta(\dot{\beta}}\boldsymbol{G}_{\dot{\gamma})cba}\\ (\sigma^b)_{(\delta|\dot{\beta}|}\boldsymbol{G}_{\gamma)ab\underline{m}} &= 0 = (\sigma^b)_{\delta(\dot{\beta}}\boldsymbol{G}_{\dot{\gamma})ab\underline{m}}\\ (\sigma^b)_{(\delta|\dot{\beta}|}\boldsymbol{G}_{\gamma)b\underline{mn}} &= 0 = (\sigma^b)_{\delta(\dot{\beta}}\boldsymbol{G}_{\dot{\gamma})b\underline{mn}}\end{aligned}$$

$$\boldsymbol{G}_{\underline{\delta}abc} = \boldsymbol{G}_{\underline{\delta}ab\underline{m}} = 0, \boldsymbol{G}_{\alpha\underline{bmn}} = -\frac{1}{6}(\sigma_b)_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_{\underline{mn}},$$

$$G_{\alpha b\underline{mn}} = -\frac{1}{6}\varphi_{\underline{mnp}}(\sigma_b)_{\alpha\dot{\alpha}}\bar{\lambda}^{\underline{p}\dot{\alpha}}.$$

$$\begin{aligned}\boldsymbol{T}_{b\gamma,\alpha} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\beta\dot{\beta},\gamma,\alpha}\\ \boldsymbol{T}_{\beta\dot{\beta},\gamma,\alpha} &= -\frac{i}{4}\left(\epsilon_{\beta\alpha}G_{\gamma\dot{\beta}} - 3\epsilon_{\gamma\beta}G_{\alpha\dot{\beta}} - 3\epsilon_{\gamma\alpha}G_{\beta\dot{\beta}}\right)\\ \boldsymbol{T}_{b\dot{\gamma},\dot{\alpha}} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\beta\dot{\beta},\dot{\gamma},\dot{\alpha}}\\ \boldsymbol{T}_{\beta\dot{\beta},\dot{\gamma},\dot{\alpha}} &= -\frac{i}{4}\left(\epsilon_{\dot{\beta}\dot{\alpha}}G_{\beta\dot{\gamma}} - 3\epsilon_{\dot{\gamma}\dot{\beta}}G_{\beta\dot{\alpha}} - 3\epsilon_{\dot{\gamma}\dot{\alpha}}G_{\beta\dot{\beta}}\right)\\ \boldsymbol{T}_{b\gamma,\dot{\alpha}} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\beta\dot{\beta},\gamma,\dot{\alpha}}\\ \boldsymbol{T}_{\beta\dot{\beta},\gamma,\dot{\alpha}} &= 2i\epsilon_{\gamma\beta}\epsilon_{\dot{\beta}\dot{\alpha}}R^{\dagger}, R^{\dagger} = \bar{R}\\ \boldsymbol{T}_{b\dot{\gamma},\alpha} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\beta\dot{\beta},\dot{\gamma},\alpha}\\ \boldsymbol{T}_{\beta\dot{\beta},\dot{\gamma},\alpha} &= 2i\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon_{\beta\alpha}R.\end{aligned}$$

$$\begin{aligned}\boldsymbol{T}_{\underline{m}\beta,\alpha} &= -i\epsilon_{\beta\alpha}S_{\underline{m}} + S_{\underline{m}\beta\alpha}\\ \boldsymbol{T}_{\underline{m}\dot{\beta},\alpha} &= -i(\sigma^c)_{\alpha\dot{\beta}}S_{\underline{m}c} = -iS_{\underline{m}\alpha\dot{\beta}}\\ \boldsymbol{T}_{\underline{m}\dot{\beta},\dot{\alpha}} &= -i\epsilon_{\dot{\beta}\dot{\alpha}}S_{\underline{m}} + \bar{S}_{\underline{m}\dot{\beta}\dot{\alpha}}\end{aligned}$$

$$\left(S_{\underline{m}\alpha\beta}\right)^{*} = -\bar{S}_{\underline{m}\dot{\alpha}\dot{\beta}}, \bar{S}_{\underline{m}c} := \left(S_{\underline{m}c}\right)^{*}$$

$$\bar{S}_{\underline{m}} := \left(S_{\underline{m}}\right)^{*} = S_{\underline{m}}$$

$$S_{\underline{m}}, S_{\underline{m}\alpha\beta} = -\left(\bar{S}_{\underline{m}\dot{\alpha}\dot{\beta}}\right)^{*}, \text{and } S_{\underline{m}c} = \left(\bar{S}_{\underline{m}c}\right)^{*}$$

$$\begin{aligned}\boldsymbol{T}_{\underline{m}}\gamma,\underline{n}\beta &= -\frac{i}{4}\boldsymbol{R}_{\gamma\beta\underline{mn}} + \frac{1}{2}\epsilon_{\gamma\beta}Z_{\underline{m},\underline{n}}\\ \boldsymbol{T}_{\underline{m}}\dot{\gamma},\underline{n}\dot{\beta} &= \frac{i}{4}\boldsymbol{R}_{\dot{\gamma}\dot{\beta}\underline{mn}} - \frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\bar{Z}_{\underline{m},\underline{n}}\\ \boldsymbol{T}_{\underline{m}}\gamma,\underline{n}\dot{\beta} &= \frac{i}{4}\boldsymbol{R}_{\gamma\dot{\beta}\underline{mn}} - iX_{\gamma\dot{\beta}\underline{m},\underline{n}}\\ \boldsymbol{T}_{\underline{m}}\dot{\gamma},\underline{n}\beta &= -\frac{i}{4}\boldsymbol{R}_{\beta\dot{\gamma}\underline{mn}} - iX_{\beta\dot{\gamma}\underline{m},\underline{n}}\end{aligned}$$

$$\begin{aligned}\bar{Z}_{\underline{m},\underline{n}} := \left(Z_{\underline{m},\underline{n}}\right)^{*}, &\qquad X_{\beta\dot{\gamma}\underline{m},\underline{n}} = \left(X_{\gamma\dot{\beta}\underline{m},\underline{n}}\right)^{*}\\ \boldsymbol{R}_{\dot{\gamma}\dot{\beta},\underline{mn}} = -\left(\boldsymbol{R}_{\gamma\beta\underline{mn}}\right)^{*}, &\quad \boldsymbol{R}_{\dot{\gamma}\beta,\underline{mn}} = -\left(\boldsymbol{R}_{\gamma\dot{\beta}\underline{mn}}\right)^{*}\end{aligned}$$

$$X_{a\underline{m},\underline{n}} =: \tilde{X}_{a\underline{mn}} + \frac{1}{7}\delta_{\underline{mn}}X_a + \frac{1}{3}\varphi_{\underline{mn}\underline{\underline{p}}}X_{a\underline{p}}$$

$$\delta\underline{\underline{mn}}X_{a\underline{m},\underline{n}} = X_a, \tilde{X}_{a\underline{mn}} = \tilde{X}_{a\underline{nm}}, X_{a[\underline{m},\underline{n}]} = \frac{1}{3}\varphi_{\underline{mnp}}X_a\underline{\underline{p}}.$$

$$\boldsymbol{G}_{a\underline{mnp}} = -6X_{a[\underline{m},}\,\underline{\varphi}_{\underline{np}]\underline{q}}$$

$$X_a = -\frac{1}{36}\varphi\underline{\underline{mnp}}\boldsymbol{G}_{a\underline{mnp}}$$

$$\tilde{X}_{a\underline{mn}} = -\frac{1}{8}\varphi\left(\underline{\underline{m}}\underline{\underline{pq}}\boldsymbol{G}_{|a|\underline{n})\underline{pq}} + \frac{1}{56}\delta_{\underline{mn}}\phi\underline{\underline{pqr}}\boldsymbol{G}_{a\underline{pqr}}\right.$$

$$X_{a\underline{m}} = \frac{1}{48}\psi_{\underline{m}}\underline{\underline{npq}}\boldsymbol{G}_{a\underline{npq}}.$$

$$iX_{\alpha\dot{\beta}\underline{m}} = \frac{1}{16}\left(D_\alpha\bar{\lambda}_{\underline{m}\dot{\beta}} + \bar{D}_{\dot{\beta}}\lambda_{\underline{m}\alpha}\right) - \frac{3i}{4}\left(S_{\underline{m}\alpha\dot{\beta}} + \bar{S}_{\underline{m}\alpha\dot{\beta}}\right)$$

$$\boldsymbol{R}_{\alpha\dot{\beta}\underline{mn}} = \varphi_{\underline{mn}\underline{p}}\left[\frac{i}{12}\left(D_\alpha\bar{\lambda}_{\underline{p}\dot{\beta}} - \bar{D}_{\dot{\beta}}\lambda_{\underline{p}\alpha}\right) - \left(S_{\underline{p}\alpha\dot{\beta}} - \bar{S}_{\underline{p}\alpha\dot{\beta}}\right)\right]$$

$$\varphi\underline{\underline{pmn}}\boldsymbol{R}_{\alpha\beta\underline{mn}} = 0 = \varphi\underline{\underline{pmn}}\boldsymbol{R}_{\dot{\alpha}\dot{\beta}\underline{mn}},$$

$$Z_{\underline{m},\underline{n}} =: \frac{1}{2}\tilde{\tilde{\bar{R}}}_{\underline{mn}} + \frac{1}{14}\delta_{\underline{mn}}\tilde{\tilde{\bar{R}}} + \frac{1}{6}\varphi_{\underline{mnp}}\bar{R}\underline{p} + L_{[\underline{mn}]_{14}}$$

$$\delta^{\underline{mn}}Z_{\underline{m},\underline{n}} = \frac{1}{2}\tilde{\tilde{\bar{R}}}, \tilde{\tilde{\bar{R}}}_{\underline{mn}} = \tilde{\tilde{\bar{R}}}_{\underline{nm}}, Z_{[\underline{m},\underline{n}]} = \frac{1}{6}\varphi_{\underline{mnp}}\bar{R}\underline{p} + L_{[\underline{mn}]_{14}}.$$

$$L_{[\underline{mn}]_{14}} = \bar{L}_{[\underline{mn}]_{14}}$$

$$R_{\underline{m}} - \bar{R}_{\underline{m}} = 6iS_{\underline{m}} + \frac{1}{4}D^\alpha\lambda_{\alpha\underline{m}} - \frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_{\underline{m}}$$

$$\boldsymbol{R}_{\alpha\beta\underline{mn}} = -\left(\boldsymbol{R}_{\dot{\alpha}\dot{\beta}\underline{mn}}\right)^*, \tilde{X}_{a\underline{mn}}, X_a, L_{[\underline{mn}]_{14}}, \mathrm{Re}\left(R_{\underline{m}}\right), \tilde{\tilde{\bar{R}}}_{\underline{mn}}, \text{and } \tilde{\tilde{\bar{R}}}.$$

$$\begin{aligned}
\boldsymbol{T}_{\gamma b,\underline{m}\alpha} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\gamma,\beta\dot{\beta},\underline{m}\alpha} \\
&= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\left[2i\epsilon_{\gamma\beta}\bar{S}_{\underline{m}\alpha\dot{\beta}} + \epsilon_{\gamma\alpha}\left[\frac{i}{4}\bar{S}_{\underline{m}\beta\dot{\beta}} + \frac{3i}{4}S_{\underline{m}\beta\dot{\beta}} + \frac{1}{16}\left(D_\beta\bar{\lambda}_{\dot{\beta}\underline{m}} - \bar{D}_{\dot{\beta}}\lambda_{\beta\underline{m}}\right)\right]\right] \\
\boldsymbol{T}_{\dot{\gamma} b,\underline{m}\dot{\alpha}} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\dot{\gamma},\beta\dot{\beta},\underline{m}\dot{\alpha}} \\
&= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\left[2i\epsilon_{\dot{\gamma}\dot{\beta}}S_{\underline{m}\beta\dot{\alpha}} + \epsilon_{\dot{\gamma}\dot{\alpha}}\left[\frac{i}{4}S_{\underline{m}\beta\dot{\beta}} + \frac{3i}{4}\bar{S}_{\underline{m}\beta\dot{\beta}} + \frac{1}{16}\left(\bar{D}_{\dot{\beta}}\lambda_{\beta\underline{m}} - D_\beta\bar{\lambda}_{\dot{\beta}\underline{m}}\right)\right]\right] \\
\boldsymbol{T}_{\gamma b,\underline{m}\dot{\alpha}} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\gamma,\beta\dot{\beta},\underline{m}\dot{\alpha}} \\
&= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\left[i\epsilon_{\gamma\beta}\epsilon_{\dot{\beta}\dot{\alpha}}S_{\underline{m}} + 3\epsilon_{\gamma\beta}\bar{S}_{\underline{m}\beta\dot{\alpha}} - \epsilon_{\dot{\beta}\dot{\alpha}}S_{\underline{m}\gamma\beta}\right] \\
\boldsymbol{T}_{\dot{\gamma} b,\underline{m}\alpha} &= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\boldsymbol{T}_{\dot{\gamma},\beta\dot{\beta},\underline{m}\alpha} \\
&= -\frac{1}{2}(\bar{\sigma}_b)^{\dot{\beta}\beta}\left[i\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon_{\beta\alpha}S_{\underline{m}} + 3\epsilon_{\dot{\gamma}\dot{\beta}}S_{\underline{m}\beta\alpha} - \epsilon_{\beta\alpha}\bar{S}_{\underline{m}\dot{\gamma}\dot{\beta}}\right]
\end{aligned}$$

$$\boldsymbol{R}_{\hat{\delta}\hat{\gamma}\hat{b}\hat{a}} \overset{4|4+7}{\rightarrow} \boldsymbol{R}_{\underline{\delta\gamma\hat{b}}\hat{a}}$$

$$\boldsymbol{R}_{\dot{\delta}\dot{\gamma}ba} = -(\sigma_{ba})^{\beta\alpha}\boldsymbol{R}_{\dot{\delta}\dot{\gamma}\beta\alpha} + (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}}\boldsymbol{R}_{\dot{\delta}\dot{\gamma}}\dot{\beta}\dot{\alpha},$$

$$\boldsymbol{R}_{\dot{\delta}\dot{\gamma}\beta\alpha} = 0 = \boldsymbol{R}_{\delta\gamma\dot{\beta}\dot{\alpha}},$$

$$\boldsymbol{R}_{\dot{\delta}\dot{\gamma}\dot{\beta}\dot{\alpha}} = 4\left(\epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\dot{\gamma}\dot{\beta}} + \epsilon_{\dot{\gamma}\dot{\alpha}}\epsilon_{\dot{\delta}\dot{\beta}}\right)R, \quad \boldsymbol{R}_{\delta\gamma\beta\alpha} = 4\left(\epsilon_{\delta\alpha}\epsilon_{\gamma\beta} + \epsilon_{\gamma\alpha}\epsilon_{\delta\beta}\right)R^\dagger$$
$$\boldsymbol{R}_{\delta\dot{\gamma}\dot{\beta}\dot{\alpha}} = -\left(\epsilon_{\dot{\gamma}\dot{\beta}}G_{\delta\dot{\alpha}} + \epsilon_{\dot{\gamma}\dot{\alpha}}G_{\delta\dot{\beta}}\right), \quad \boldsymbol{R}_{\delta\dot{\gamma}\beta\alpha} = -\left(\epsilon_{\delta\beta}G_{\alpha\dot{\gamma}} + \epsilon_{\delta\alpha}G_{\beta\dot{\gamma}}\right).$$

$$\boldsymbol{R}_{\gamma\beta,\underline{m}a} = -8(\sigma_{ac})_{\beta\gamma}\bar{S}^c_{\underline{m}}, \boldsymbol{R}_{\dot{\gamma}\dot{\beta},\underline{m}a} = -8(\bar{\sigma}_{ac})_{\dot{\beta}\dot{\gamma}}S^c_{\underline{m}}$$

$$\boldsymbol{R}_{\gamma\dot{\beta},\underline{m}a} = 2i\left[(\sigma_a)_{\beta\dot{\beta}}S^{\beta}_{\underline{m}\gamma} - (\sigma_a)_{\gamma\dot{\gamma}}\bar{S}^{\dot{\gamma}}_{\underline{m}\dot{\beta}}\right]$$

$$\boldsymbol{R}_{\alpha\dot{\beta}\underline{m}n} = \varphi_{\underline{mn}}\left[\frac{i}{12}\left(D_\alpha\bar{\lambda}_{\dot{\beta}\underline{p}} - \bar{D}_{\dot{\beta}}\lambda_{\alpha\underline{p}}\right) - \left(S_{\underline{p}\alpha\dot{\beta}} - \bar{S}_{\underline{p}\alpha\dot{\beta}}\right)\right].$$

$$\text{- } \boldsymbol{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} \overset{4|4+7}{\rightarrow} \boldsymbol{G}_{\hat{a}\hat{b}\hat{c}\hat{d}}$$

$$\boldsymbol{G}_{abcd} = 3i\epsilon_{abcd}(R-\bar{R})$$
$$\boldsymbol{G}_{abc\underline{m}} = 3i\epsilon_{abcd}\left(\bar{S}_{\underline{m}}{}^{d}-S_{\underline{m}}{}^{d}\right)$$
$$\boldsymbol{G}_{ab\underline{mn}} = -(\sigma_{ab})^{\alpha\beta}\left[\varphi_{\underline{mnp}}\left(-\frac{i}{12}D_{\alpha}\lambda_{\beta\underline{p}}+2iS_{\underline{p}\alpha\beta}\right)-\frac{1}{2}\boldsymbol{R}_{\alpha\beta\underline{mn}}\right]$$
$$+(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\left[\varphi_{\underline{mnp}}\left(-\frac{i}{12}\bar{D}_{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}\underline{p}}+2i\bar{S}_{\underline{p}\dot{\alpha}\dot{\beta}}\right)+\frac{1}{2}\boldsymbol{R}_{\dot{\alpha}\dot{\beta}\underline{mn}}\right]$$
$$\boldsymbol{G}_{a\underline{mnp}} = -6X_{a[\underline{m},}\varphi_{\underline{np}]\underline{q}}$$

$$\boldsymbol{G}_{\underline{mnpq}} = \frac{1}{24\times 7}\psi_{\underline{mnpq}}\mathcal{G}+\frac{1}{42}\varphi_{[\underline{mnp}}\mathcal{G}_{\underline{q}]}+\psi_{[\underline{mnp}}{}^{\underline{r}}\mathcal{G}_{\underline{q}]\underline{r}}$$
$$\mathcal{G}_{\underline{mn}} = \mathcal{G}_{\underline{nm}};\ \delta^{\underline{\underline{mn}}}\mathcal{G}_{\underline{mn}} = 0$$

$$\left(G_{\alpha\dot{\beta}}\right)^{*} = G_{\beta\dot{\alpha}}$$
$$\bar{S}_{\underline{m}} = \left(S_{\underline{m}}\right)^{*} = S_{\underline{m}}$$
$$\varphi_{\underline{mnp}}\boldsymbol{R}_{\alpha\beta\underline{np}} = 0 = \varphi_{\underline{mnp}}\boldsymbol{R}_{\dot{\alpha}\dot{\beta}\underline{n}}$$
$$\boldsymbol{R}_{\alpha\dot{\beta}\underline{mn}} = \varphi_{\underline{mnp}}\left[\frac{i}{12}\left(D_{\alpha}\bar{\lambda}_{\dot{\beta}\underline{p}}-\bar{D}_{\dot{\beta}}\lambda_{\alpha\underline{p}}\right)-\left(S_{\underline{p}\alpha\dot{\beta}}-\bar{S}_{\underline{p}\alpha\dot{\beta}}\right)\right]$$
$$iX_{\alpha\dot{\beta}\underline{m}} = \frac{1}{16}\left(D_{\alpha}\bar{\lambda}_{\dot{\beta}\underline{m}}+\bar{D}_{\dot{\beta}}\lambda_{\alpha\underline{m}}\right)-\frac{3i}{4}\left(S_{\underline{m}\alpha\dot{\beta}}+\bar{S}_{\underline{m}\alpha\dot{\beta}}\right)$$
$$R_{\underline{m}}-\bar{R}_{\underline{m}} = 6iS_{\underline{m}}+\frac{1}{4}D^{\alpha}\lambda_{\alpha\underline{m}}-\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}{}_{\underline{m}}$$

$$\boldsymbol{T}_{\hat{b}\hat{c},\hat{\alpha}} \overset{4|4+7}{\rightarrow} \boldsymbol{T}_{\hat{b}\hat{c},\underline{\alpha}},\boldsymbol{T}_{\hat{b}\hat{c},\underline{m\alpha}}$$

$$\boldsymbol{T}_{\gamma\dot{\gamma},\beta\dot{\beta},\alpha} = -2\epsilon_{\dot{\gamma}\dot{\beta}}W_{\gamma\beta\alpha}-\frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\left(\epsilon_{\gamma\alpha}\bar{D}_{\dot{\delta}}G^{\dot{\delta}}_{\beta}+\epsilon_{\beta\alpha}\bar{D}_{\dot{\delta}}G^{\dot{\delta}}_{\gamma}\right)$$
$$+\frac{1}{2}\epsilon_{\gamma\beta}\left(\bar{D}_{\dot{\gamma}}G_{\alpha\dot{\beta}}+\bar{D}_{\dot{\beta}}G_{\alpha\dot{\gamma}}\right)$$
$$\boldsymbol{T}_{\gamma\dot{\gamma},\beta\dot{\beta},\dot{\alpha}} = -2\epsilon_{\gamma\beta}\bar{W}_{\dot{\gamma}\dot{\beta}\dot{\alpha}}-\frac{1}{2}\epsilon_{\gamma\beta}\left(\epsilon_{\dot{\gamma}\dot{\alpha}}D_{\delta}G^{\delta}_{\dot{\beta}}+\epsilon_{\dot{\beta}\dot{\alpha}}D_{\delta}G^{\delta}_{\dot{\gamma}}\right)$$
$$+\frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\left(D_{\gamma}G_{\beta\dot{\alpha}}+D_{\beta}G_{\gamma\dot{\alpha}}\right)$$

$$\boldsymbol{T}_{a\underline{m},\beta} = -i(\bar{\sigma}_{a})^{\dot{\alpha}\alpha}X_{\underline{m}\alpha\beta\dot{\alpha}}+i(\sigma_{a})_{\beta\dot{\alpha}}X^{\dot{\alpha}}_{\underline{m}}$$

$$\left(X_{\underline{m}\alpha\beta\dot{\gamma}}\right)^{*} = -\bar{X}_{\underline{m}\dot{\alpha}\dot{\beta}\gamma},\left(X_{\underline{m}\dot{\alpha}}\right)^{*} = \bar{X}_{\underline{m}\alpha}$$

$$\boldsymbol{T}_{a\underline{m},\dot{\beta}} = \left(\boldsymbol{T}_{a\underline{m},\beta}\right)^{*} = -i(\bar{\sigma}_{a})^{\dot{\alpha}\alpha}\bar{X}_{\underline{m}\dot{\alpha}\dot{\beta}\alpha}-i(\sigma_{a})_{\alpha\dot{\beta}}\bar{X}^{\alpha}_{\underline{m}}$$

$$X_{\underline{m}\alpha\beta\dot{\gamma}} = \frac{i}{4}D_{(\alpha}S_{|\underline{m}|\beta)\dot{\gamma}}-\bar{D}_{\dot{\gamma}}S_{\underline{m}\alpha\beta},\bar{X}_{\underline{m}\dot{\alpha}\dot{\beta}\gamma} = \frac{i}{4}\bar{D}_{(\dot{\alpha}}\bar{S}_{|\underline{m}\gamma|\dot{\beta})}-D_{\gamma}\bar{S}_{\underline{m}\dot{\alpha}\dot{\beta}}$$
$$X_{\underline{m}\dot{\alpha}} = -\frac{i}{4}\left[\bar{D}_{\dot{\alpha}}S_{\underline{m}}+\frac{1}{2}D^{\beta}S_{\underline{m}\beta\dot{\alpha}}\right],\bar{X}_{\underline{m}\alpha} = \frac{i}{4}\left[D_{\alpha}S_{\underline{m}}-\frac{1}{2}\bar{D}_{\dot{\beta}}\bar{S}_{\underline{m}\alpha}{\dot{\beta}}\right]$$

$$D_{\alpha}\bar{S}_{\underline{m}b} = \frac{i}{4}(\sigma_{b})_{\alpha\dot{\beta}}\bar{\rho}^{\dot{\beta}}_{\underline{m}},\bar{\rho}_{\underline{m}\dot{\alpha}} = \left(\rho_{\underline{m}\alpha}\right)^{*}$$

$$\boldsymbol{T}_{\underline{mn}}{}^{\alpha} = \frac{1}{6}\varphi_{\underline{mnp}}U^{\underline{p}\alpha}+T_{[\underline{mn}]_{14}}{}^{\alpha}$$
$$\boldsymbol{T}_{\underline{mn}}{}^{\dot{\alpha}} = \left(\boldsymbol{T}_{\underline{mn}}{}^{\alpha}\right)^{*} = \frac{1}{6}\varphi_{\underline{mnp}}\bar{U}^{\underline{p}\dot{\alpha}}+\bar{T}_{[\underline{mn}]_{14}}{}^{\dot{\alpha}}.$$

$$\left(U_{\underline{m}\alpha}\right)^{*} = \bar{U}_{\underline{m}\dot{\alpha}},\left(T^{\alpha}_{[\underline{mn}]_{14}}\right)^{*} = \bar{T}^{\dot{\alpha}}_{[\underline{mn}]_{14}}$$

$$U_{\underline{m}\alpha} = -\frac{3i}{2}\rho_{\underline{m}\alpha}-D_{\alpha}S_{\underline{m}}+\bar{D}^{\dot{\beta}}\bar{S}_{\underline{m}\alpha\dot{\beta}}-\frac{i}{48}[D^{2}-2\bar{D}^{2}]\lambda_{\underline{m}\alpha}-\frac{i}{24}[D_{\alpha}\bar{D}_{\dot{\beta}}+2\bar{D}_{\dot{\beta}}D_{\alpha}]\bar{\lambda}_{\underline{m}}{}^{\dot{\beta}},$$

$$\boldsymbol{T}_{ab,\underline{m}\gamma} = -(\sigma_{ab})^{\alpha\beta}\left[T_{\underline{m}\alpha\beta\gamma}+\epsilon_{\gamma\alpha}T_{\underline{m}\beta}\right]-(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}T_{\dot{\alpha}\dot{\beta}\gamma\underline{m}}$$
$$\boldsymbol{T}_{ab,\underline{m}\dot{\gamma}} = \left(\boldsymbol{T}_{ab,\underline{m}\gamma}\right)^{*} = (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\left[-\bar{T}_{\underline{m}\dot{\alpha}\dot{\beta}\dot{\gamma}}+\epsilon_{\dot{\gamma}\dot{\alpha}}\bar{T}_{\underline{m}\dot{\beta}}\right]-(\sigma_{ab})^{\alpha\beta}\bar{T}_{\alpha\beta\dot{\gamma}\underline{m}}$$

$$\left(T_{\underline{m}\alpha\beta\gamma}\right)^{*} = -\bar{T}_{\underline{m}\dot{\alpha}\dot{\beta}\dot{\gamma}}, \left(T_{\underline{m}\alpha}\right)^{*} = \bar{T}_{\underline{m}\dot{\alpha}}, \left(T_{\dot{\alpha}\dot{\beta}\gamma\underline{m}}\right)^{*} = -\bar{T}_{\alpha\beta\dot{\gamma}\underline{m}}$$

$$T_{\underline{m}\alpha\beta\gamma} = iD_{(\alpha}S_{|\underline{m}|\beta\gamma)}, \bar{T}_{\underline{m}\dot{\alpha}\dot{\beta}\dot{\gamma}} = -i\bar{D}_{(\dot{\alpha}}\bar{S}_{|\underline{m}|\dot{\beta}\dot{\gamma})}$$

$$T_{\underline{m}\alpha} = \frac{1}{2}\left[D_{\alpha}S_{\underline{m}} - \frac{i}{2}\rho_{\underline{m}\alpha}\right], \bar{T}_{\underline{m}\dot{\alpha}} = \frac{1}{2}\left[\bar{D}_{\dot{\alpha}}S_{\underline{m}} + \frac{i}{2}\bar{\rho}_{\underline{m}\dot{\alpha}}\right].$$

$$T_{\dot{\alpha}\dot{\beta}\gamma\underline{m}} = \frac{1}{2}\bar{D}_{(\dot{\alpha}}\bar{S}_{|\underline{m}\gamma|\dot{\beta})} - iD_{\gamma}\bar{S}_{\underline{m}\dot{\alpha}\dot{\beta}}, \bar{T}_{\alpha\beta\dot{\gamma}\underline{m}} = -\frac{1}{2}D_{(\alpha}\bar{S}_{|\underline{m}|\beta)\dot{\gamma}} + i\bar{D}_{\dot{\gamma}}S_{\underline{m}\alpha\beta}.$$

$$\boldsymbol{T}_{a\underline{m},\underline{n}\beta} = -i(\bar{\sigma}_{a})^{\dot{\gamma}\alpha}Y_{\underline{m},\underline{n}\alpha\beta\dot{\gamma}} + i(\sigma_{a})_{\beta\dot{\alpha}}Y_{\underline{m},\underline{n}}{}^{\dot{\alpha}}$$

$$= -i(\bar{\sigma}_{a})^{\dot{\gamma}\alpha}\left[\frac{1}{2}\tilde{Q}_{\underline{mn}\alpha\beta\dot{\gamma}} + \frac{1}{14}\delta_{\underline{mn}}\tilde{Q}_{\alpha\beta\dot{\gamma}} + \frac{1}{6}\varphi_{\underline{mnp}}Q^{\underline{p}}{}_{\alpha\beta\dot{\gamma}} + K_{[\underline{mn}]_{14}\alpha\beta\dot{\gamma}}\right]$$

$$+i(\sigma_{a})_{\beta\dot{\alpha}}\left[\frac{1}{2}\tilde{P}_{\underline{mn}}{}^{\dot{\alpha}} + \frac{1}{14}\delta_{\underline{mn}}\tilde{P}^{\dot{\alpha}} + \frac{1}{6}\varphi_{\underline{mnp}}P^{\underline{p}\dot{\alpha}} + M^{\dot{\alpha}}_{[\underline{mn}]_{14}}\right]$$

$$\tilde{Q}_{\underline{mn}\alpha\beta\dot{\gamma}} = iD_{(\alpha}\tilde{X}_{\beta)\dot{\gamma}\underline{mn}},$$

$$Q_{\underline{m}\alpha\beta\dot{\gamma}} = \frac{1}{16}D_{(\alpha}\bar{D}_{|\dot{\gamma}}\lambda_{\underline{m}|\beta)} - \frac{3i}{4}D_{(\alpha}S_{|\underline{m}|\beta)\dot{\gamma}},$$

$$\tilde{P}_{\underline{mn}\dot{\alpha}} = \frac{1}{12}\bar{D}_{\dot{\alpha}}\left[\bar{\tilde{R}}_{\underline{mn}} - \tilde{R}_{\underline{mn}}\right] - \frac{i}{6}D^{\beta}\tilde{X}_{\beta\dot{\alpha}\underline{mn}}, \tilde{P}_{\beta)\dot{\gamma}} = \frac{1}{12}\bar{D}_{\dot{\alpha}}\left[\bar{\tilde{R}} - \tilde{R}\right] - iD^{\beta}X_{\beta\dot{\alpha}}$$

$$P_{\underline{m}\dot{\alpha}} = \frac{3}{8}\bar{\rho}_{\underline{m}\dot{\alpha}} - \frac{i}{2}\bar{D}_{\dot{\alpha}}S_{\underline{m}} - \frac{i}{8}D^{\beta}S_{\underline{m}\beta\dot{\alpha}} - \frac{1}{96}\left[D^{2} + \bar{D}^{2}\right]\bar{\lambda}_{\underline{m}\dot{\alpha}} - \frac{1}{96}\left[D^{\beta}\bar{D}_{\dot{\alpha}} + 2\bar{D}_{\dot{\alpha}}D^{\beta}\right]\lambda_{\underline{m}\beta}$$

$$M_{[\underline{mn}]_{14}\dot{\alpha}} = -\frac{i}{48}\bar{D}^{\dot{\beta}}\boldsymbol{R}_{\beta\dot{m}n} - \frac{i}{4}T_{[\underline{mn}]_{14}\dot{\alpha}}$$

$$\boldsymbol{T}_{\underline{mn},\underline{p}}\alpha \overset{SL_{7}}{=} V_{[\underline{mnp}],\alpha} + W_{\underline{mnp}}\alpha$$

$$V_{[\underline{mnp}],\alpha} = \frac{1}{42}\varphi_{\underline{mnp}}V_{\alpha} + \frac{1}{24}\psi_{\underline{mnpq}}V^{\underline{q}}{}_{\alpha} + \frac{3}{4}\varphi\underline{\underline{q}}_{[\underline{mn}}V_{\underline{p}]\underline{q},\alpha},$$

$$V_{\underline{mn}\alpha} = V_{\underline{nm}\alpha}, \delta\underline{\underline{mn}}V_{\underline{mn}\alpha} = 0.$$

$$W_{\underline{mn}|\underline{p}\alpha} = J_{\underline{mn}|\underline{p},\alpha} + \delta_{\underline{pm}}\Upsilon_{\underline{n}\alpha} - \delta_{\underline{pn}}\Upsilon_{\underline{m}\alpha}$$

$$J_{\underline{mn}|\underline{p}\alpha} = -J_{\underline{nm}|\underline{p}\alpha}, J_{[\underline{mn}|\underline{p}]\alpha} = 0, \delta\underline{np}J_{\underline{mn}|\underline{p}\alpha} = 0.$$

$$J_{\underline{mnp},\alpha} = J^{14}_{\underline{mnp},\alpha} + J^{27}_{\underline{mnp},\alpha} + J^{64}_{\underline{mnp},\alpha}$$

$$J^{14}_{\underline{mn}|\underline{p},\alpha} = \varphi_{\underline{mn}}\underline{q}J_{\underline{pq},\alpha} - \frac{1}{2}\varphi_{\underline{np}}\underline{\underline{q}}J_{\underline{qm},\alpha} + \frac{1}{2}\varphi_{\underline{mp}}\underline{\underline{q}}J_{\underline{qn},\alpha}$$

$$J^{27}_{\underline{mn}|\underline{p},\alpha} = \varphi_{\underline{mn}}\underline{q}I_{\underline{qp},\alpha} - \frac{1}{2}\varphi_{\underline{np}}\underline{q}I_{\underline{qm},\alpha} + \frac{1}{2}\varphi_{\underline{mp}}\underline{\underline{q}}I_{\underline{qn},\alpha}$$

$$\varphi_{\underline{m}}\underline{pq}J^{14}_{\underline{pq}|\underline{n},\alpha} = 9J_{\underline{mn},\alpha}, \quad \varphi_{\underline{m}}\underline{pq}J^{14}_{\underline{np}|\underline{q},\alpha} = -\frac{9}{2}J_{\underline{mn},\alpha}$$

$$\varphi_{\underline{m}}\underline{\underline{pq}}J^{27}_{\underline{pq}|\underline{n},\alpha} = 7I_{\underline{mn},\alpha}, \quad \varphi_{\underline{m}}\underline{\underline{pq}}J^{27}_{\underline{np}|\underline{q},\alpha} = -\frac{7}{2}I_{\underline{mn},\alpha}.$$

$$Z_{\underline{mn}|\underline{p},\alpha} = -Z_{\underline{nm}|\underline{p},\alpha}, \delta\underline{np}Z_{\underline{mn}|\underline{p},\alpha} = 0, Z_{[\underline{mn}|\underline{p}],\alpha} = 0,$$

$$\varphi_{\underline{q}}\underline{\underline{mn}}Z_{\underline{mn}|\underline{p},\alpha} = 0, \varphi_{\underline{q}}\underline{\underline{np}}Z_{\underline{mn}|\underline{p},\alpha} = 0.$$

$$\varphi_{\underline{m}}\underline{pq}\boldsymbol{T}_{\underline{np},\underline{q},\alpha} = \frac{1}{7}\delta_{\underline{mn}}V_{\alpha} + \left(V_{\underline{mn},\alpha} - \frac{7}{2}I_{\underline{mn},\alpha}\right) - \frac{9}{2}J_{\underline{mn},\alpha} + \varphi_{\underline{mn}}\underline{p}\left(\frac{1}{6}V_{\underline{p}\alpha} - \Upsilon_{\underline{p}\alpha}\right)$$

$$\varphi_{\underline{m}}\underline{\underline{pq}}\boldsymbol{T}_{\underline{pq},\underline{n},\alpha} = \frac{1}{7}\delta_{\underline{mn}}V_{\alpha} + \left(V_{\underline{mn},\alpha} + 7I_{\underline{mn},\alpha}\right) + 9J_{\underline{mn},\alpha} + \varphi_{\underline{mn}}\underline{\underline{p}}\left(-\frac{1}{6}V_{\underline{p}\alpha} + 2\Upsilon_{\underline{p}\alpha}\right)$$

$$V_\alpha = iD_\alpha\tilde{\tilde{R}}$$
$$V_{\underline{m}\alpha} = -\frac{3i}{2}\rho_{\underline{m}\alpha} - \bar{D}^{\dot\beta}\left[\frac{3}{2}S_{\underline{m}\alpha\dot\beta} + \bar{S}_{\underline{m}\alpha\dot\beta}\right] + 2D_\alpha S_{\underline{m}}$$
$$+\frac{i}{24}[D^2 + 4\bar{D}^2]\lambda_{\underline{m}\alpha} + \frac{1}{12}[D_\alpha\bar{D}_{\dot\beta} + 2\bar{D}_{\dot\beta}D_\alpha]\bar{\lambda}_{\underline{m}}{}^{\dot\beta} + \frac{1}{3}\varphi_{\underline{m}\underline{np}}\partial_{\underline{n}}\lambda_{\underline{p}\alpha}$$
$$V_{\underline{mn}\alpha} = \frac{i}{18}D_\alpha\left(\tilde{\tilde{R}}_{\underline{mn}} - \tilde{R}_{\underline{mn}}\right) + \frac{4}{9}\bar{D}^{\dot\beta}\tilde{X}_{\alpha\dot\beta\underline{mn}} + \frac{1}{9}\partial_{(\underline{m}}\lambda_{\underline{n})}\Big|_{\text{traceless}}{}_{,\alpha}$$
$$\Upsilon_{\underline{m}\alpha} = \frac{i}{12}D_\alpha\bar{R}_{\underline{m}} - \frac{3i}{16}\rho_{\underline{m}\alpha} - \frac{1}{8}\bar{D}^{\dot\beta}\left[\frac{3}{2}S_{\underline{m}\alpha\dot\beta} + \bar{S}_{\underline{m}\alpha\dot\beta}\right] + \frac{1}{4}D_\alpha S_{\underline{m}}$$
$$+\frac{i}{192}[D^2 + 4\bar{D}^2]\lambda_{\underline{m}\alpha} + \frac{i}{96}[D_\alpha\bar{D}_{\dot\beta} + 2\bar{D}_{\dot\beta}D_\alpha]\bar{\lambda}_{\underline{m}}{}^{\dot\beta} + \frac{1}{24}\varphi_{\underline{m}\underline{np}}\partial_{\underline{n}}\lambda_{\underline{p}\alpha}$$
$$J_{\underline{mn}\alpha} = \frac{i}{9}D_\alpha L_{[\underline{mn}]_{14}} + \frac{1}{54}D^\beta R_{\beta\alpha\underline{mn}}$$
$$I_{\underline{mn}\alpha} = -\frac{i}{14}D_\alpha\left[\frac{17}{18}\tilde{\tilde{R}}_{\underline{mn}} + \frac{1}{18}\tilde{R}_{\underline{mn}}\right] + \frac{2}{63}\bar{D}^{\dot\beta}\tilde{X}_{\alpha\dot\beta mn} + \frac{1}{126}\partial_{\underline{m}}\lambda_{\underline{n}}\Big)_{\text{traceless}}{}_{,\alpha}$$

$$\boldsymbol{R}_{\hat{d}\hat{\gamma}\hat{b}\hat{a}} \overset{4|4+7}{\rightarrow} \boldsymbol{R}_{\hat{d}\hat{\gamma}\hat{b}\hat{a}}$$

$$\boldsymbol{R}_{\alpha bcd} = i\left[(\sigma_b)_{\alpha\dot\beta}\boldsymbol{T}^{\dot\beta}_{cd} - (\sigma_c)_{\alpha\dot\beta}\boldsymbol{T}^{\dot\beta}_{db} - (\sigma_d)_{\alpha\dot\beta}\boldsymbol{T}^{\dot\beta}_{bc}\right]$$
$$\boldsymbol{R}_{\dot\alpha bcd} = -i\left[(\sigma_b)_{\beta\dot\alpha}\boldsymbol{T}^{\beta}_{cd} - (\sigma_c)_{\beta\dot\alpha}\boldsymbol{T}^{\beta}_{db} - (\sigma_d)_{\beta\dot\alpha}\boldsymbol{T}^{\beta}_{bc}\right].$$

$$\boldsymbol{R}_{\alpha\underline{m},ab} = -i\boldsymbol{T}_{ab,\underline{m}\alpha} - 2i(\sigma_{[a})_{|\alpha\dot\beta|}\boldsymbol{T}_{b]\underline{m}}{}^{\dot\beta}, \quad \boldsymbol{R}_{\dot\alpha\underline{m},ab} = i\boldsymbol{T}_{ab,\underline{m}\dot\alpha} + 2i(\sigma_{[a})_{|\beta\dot\alpha|}\boldsymbol{T}_{b]\underline{m}}{}^{\beta}$$
$$\boldsymbol{R}_{\alpha a,b\underline{m}} = i\boldsymbol{T}_{ab,\underline{m}\alpha} + 2i(\sigma_{(a})_{|\alpha\dot\beta|}\boldsymbol{T}_{b)\underline{m}}{}^{\dot\beta}, \quad \boldsymbol{R}_{\dot\alpha a,b\underline{m}} = -i\boldsymbol{T}_{ab,\underline{m}\dot\alpha} - 2i\left(\sigma_{(a)}|_{|\beta\dot\alpha|}\boldsymbol{T}_{b)\underline{m}}{}^{\beta}\right.$$
$$\boldsymbol{R}_{\alpha a,\underline{mn}} = i(\sigma_a)_{\alpha\dot\beta}\boldsymbol{T}_{\underline{mn}}{}^{\dot\beta} + 2i\boldsymbol{T}_{a[\underline{m},\underline{n}]\alpha}, \quad \boldsymbol{R}_{\dot\alpha a,\underline{mn}} = -i(\sigma_a)_{\beta\dot\alpha}\boldsymbol{T}_{\underline{mn}}{}^{\beta} - 2i\boldsymbol{T}_{a[\underline{m},\underline{n}]\dot\alpha}$$
$$\boldsymbol{R}_{\alpha\underline{m},a\underline{n}} = i(\sigma_a)_{\alpha\dot\beta}\boldsymbol{T}_{\underline{mn}}{}^{\dot\beta} - 2i\boldsymbol{T}_{a(\underline{m},\underline{n})\alpha}, \quad \boldsymbol{R}_{\dot\alpha\underline{m},a\underline{n}} = -i(\sigma_a)_{\beta\dot\alpha}\boldsymbol{T}_{\underline{mn}}{}^{\beta} + 2i\boldsymbol{T}_{a(\underline{m},\underline{n})\dot\alpha}$$
$$\boldsymbol{R}_{\alpha\underline{m},\underline{np}} = i\left[\boldsymbol{T}_{\underline{mn},\underline{p}\alpha} - \boldsymbol{T}_{\underline{np},\underline{m}\alpha} + \boldsymbol{T}_{\underline{pm},\underline{n}\alpha}\right], \quad \boldsymbol{R}_{\dot\alpha\underline{m},\underline{np}} = -i\left[\boldsymbol{T}_{\underline{mn},\underline{p}\dot\alpha} - \boldsymbol{T}_{\underline{np},\underline{m}\dot\alpha} + \boldsymbol{T}_{\underline{pm},\underline{n}\dot\alpha}\right].$$

$$\bar{D}_{\dot\alpha}R = 0 = D_\alpha\bar{R}$$
$$D^\beta G_{\beta\dot\alpha} = \bar{D}_{\dot\alpha}\bar{R}$$
$$D_\alpha\bar{S}_{\underline{m}b} = \frac{i}{4}(\sigma_b)_{\alpha\dot\beta}\bar{\rho}_{\underline{m}}{}^{\dot\beta} \Leftrightarrow D_\alpha\bar{S}_{\underline{m}\beta\dot\gamma} = \frac{i}{2}\epsilon_{\alpha\beta}\bar{\rho}_{\underline{m}\dot\gamma}$$
$$D_\alpha\left(R_{\underline{m}} - \bar{R}_{\underline{m}}\right) = 6iD_\alpha S_{\underline{m}} + 4i\bar{D}^{\dot\beta}\bar{S}_{\underline{m}\alpha\dot\beta} - \frac{1}{8}D^2\lambda_{\underline{m}\alpha} - \frac{1}{4}D_\alpha\bar{D}_{\dot\beta}\bar{\lambda}_{\underline{m}}{}^{\dot\beta}$$
$$\frac{1}{42}D_\alpha\mathcal{G}_{\underline{m}} = V_{\underline{m}\alpha} = -\frac{3i}{2}\rho_{\underline{m}\alpha} - \bar{D}^{\dot\beta}\left[\frac{3}{2}S_{\underline{m}\alpha\dot\beta} + \bar{S}_{\underline{m}\alpha\dot\beta}\right] + 2D_\alpha S_{\underline{m}} + \frac{i}{24}[D^2 + 4\bar{D}^2]\lambda_{\underline{m}\alpha}$$
$$+\frac{1}{12}[D_\alpha\bar{D}_{\dot\beta} + 2\bar{D}_{\dot\beta}D_\alpha]\bar{\lambda}_{\underline{m}}{}^{\dot\beta} + \frac{1}{3}\varphi_{\underline{m}\underline{np}}\partial_{\underline{n}}\lambda_{\underline{p}\alpha}$$
$$\bar{D}_{\dot\alpha}\left(\frac{1}{2}\tilde{R} + \frac{1}{6}\tilde{\tilde{R}}\right) = \frac{i}{3}\partial^{\underline{m}}\bar{\lambda}_{\underline{m}\dot\alpha} + 3iD^\beta X_{\beta\dot\alpha}$$
$$D_\alpha\left(\tilde{\tilde{R}} - \frac{i}{48}\mathcal{G}\right) = 0$$
$$D_\alpha\mathcal{G}_{\underline{mn}} = -\frac{2i}{21}D_\alpha\left(4\tilde{\tilde{R}}_{\underline{mn}} - \tilde{R}_{\underline{mn}}\right) - \frac{16}{21}\bar{D}^{\dot\beta}\tilde{X}_{\alpha\dot\beta\underline{mn}} - \frac{4}{21}\partial_{(\underline{m}}\lambda_{\underline{n})}\Big)_{\text{traceless}}{}_{,\alpha}$$

$$\boldsymbol{R}_{[\hat{d}\hat{c},\hat{b}]\hat{a}} = 0$$

$$\boldsymbol{R}_{[dc,b]a} = 0$$
$$\boldsymbol{R}_{dc,\beta\alpha} = D_\beta\boldsymbol{T}_{dc,\alpha} + \partial_d\boldsymbol{T}_{c\beta,\alpha} - \partial_c\boldsymbol{T}_{d\beta,\alpha}, \boldsymbol{R}_{dc,\dot\beta\dot\alpha} = \bar{D}_{\dot\beta}\boldsymbol{T}_{dc,\dot\alpha} + \partial_d\boldsymbol{T}_{c\dot\beta,\dot\alpha} - \partial_c\boldsymbol{T}_{d\dot\beta,\dot\alpha}$$
$$0 = D_\beta\boldsymbol{T}_{dc,\dot\alpha} + \partial_d\boldsymbol{T}_{c\beta,\dot\alpha} - \partial_c\boldsymbol{T}_{d\beta,\dot\alpha}, 0 = \bar{D}_{\dot\beta}\boldsymbol{T}_{dc,\alpha} + \partial_d\boldsymbol{T}_{c\dot\beta,\alpha} - \partial_c\boldsymbol{T}_{d\dot\beta,\alpha}$$

$$R_{\delta\dot\delta,\gamma\dot\gamma,\beta\dot\beta,\alpha\dot\alpha} = (\sigma^d)_{\delta\dot\delta}(\sigma^c)_{\gamma\dot\gamma}(\sigma^b)_{\beta\dot\beta}(\sigma^a)_{\alpha\dot\alpha}\boldsymbol{R}_{dc,ba}$$
$$= 4\left[\epsilon_{\dot\delta\dot\gamma}\epsilon_{\dot\beta\dot\alpha}X_{(\delta\gamma)(\beta\alpha)} + \epsilon_{\delta\gamma}\epsilon_{\beta\alpha}\bar{X}_{(\dot\delta\dot\gamma)(\dot\beta\dot\alpha)} - \epsilon_{\dot\delta\dot\gamma}\epsilon_{\beta\alpha}\Psi_{(\delta\gamma)(\dot\beta\dot\alpha)} - \epsilon_{\delta\gamma}\epsilon_{\dot\beta\dot\alpha}\bar{\Psi}_{(\dot\delta\dot\gamma)(\beta\alpha)}\right]$$

$$\Psi_{(\delta\gamma)(\dot\beta\dot\alpha)} = \bar{\Psi}_{(\dot\beta\dot\alpha)(\dot\delta\dot\gamma)}, \epsilon^{\beta\delta}X_{(\alpha\beta)(\gamma\delta)} = \epsilon_{\alpha\gamma}\Lambda, \bar{\Lambda} = \Lambda$$

$$X_{(\delta\gamma)(\beta\alpha)} = -D_{(\alpha}W_{\beta)\delta\gamma} + \frac{1}{2}\epsilon_{(\delta|(\beta}D_{\alpha)|}\bar{D}^{\dot{\gamma}}G_{\gamma)\dot{\gamma}} + \frac{i}{2}\epsilon_{(\beta|(\delta}\partial_{\gamma)|}{}^{\dot{\gamma}}G_{\alpha)\dot{\gamma}}$$

$$\bar{\Psi}_{(\dot{\delta}\dot{\gamma})}{}^{(\beta\alpha)} = -\frac{1}{4}\left[D^{(\beta}\bar{D}_{(\dot{\delta}}G^{\alpha)}{}_{\dot{\gamma})} - \bar{D}_{(\dot{\delta}}D^{(\beta}G^{\alpha)}{}_{\dot{\gamma})}\right]$$

$$\boldsymbol{R}_{\underline{m}c,ba} = -(\sigma_{ba})^{\beta\alpha}\boldsymbol{R}_{\underline{m}c,\beta\alpha} - (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}}\boldsymbol{R}_{\underline{m}c,\dot{\beta}\dot{\alpha}}$$

$$R_{\underline{m},\gamma\dot{\gamma},\beta\alpha} = (\sigma^{c})_{\gamma\dot{\gamma}}\boldsymbol{R}_{\underline{m}c,\beta\alpha} = (\sigma^{c})_{\gamma\dot{\gamma}}\left(\partial_{\underline{m}}\boldsymbol{T}_{c\beta,\alpha} + \partial_{c}\boldsymbol{T}_{\beta\underline{m},\alpha} - D_{\beta}\boldsymbol{T}_{c\underline{m},\alpha}\right)$$
$$= -\frac{i}{4}\left[\epsilon_{\gamma\alpha}\partial_{\underline{m}}G_{\beta\dot{\gamma}} - 3\epsilon_{\beta\gamma}\partial_{\underline{m}}G_{\alpha\dot{\gamma}} - 3\epsilon_{\beta\alpha}\partial_{\underline{m}}G_{\gamma\dot{\gamma}}\right] + \partial_{\gamma\dot{\gamma}}\left(i\epsilon_{\beta\alpha}S_{\underline{m}} - S_{\underline{m}\beta\alpha}\right)$$
$$+\frac{1}{2}\epsilon_{\beta(\gamma}D^{2}S_{|\underline{m}|\alpha)\dot{\gamma}} + 2iD_{\beta}\bar{D}_{\dot{\gamma}}S_{\underline{m}\gamma\alpha} + \frac{1}{2}\epsilon_{\gamma\alpha}\left[D_{\beta}\bar{D}_{\dot{\gamma}}S_{\underline{m}} - \frac{1}{4}D^{2}S_{\underline{m}\beta\dot{\gamma}}\right]$$

$$R_{\underline{m},\gamma\dot{\gamma},\dot{\beta}\dot{\alpha}} = (\sigma^{c})_{\gamma\dot{\gamma}}\boldsymbol{R}_{\underline{m}c,\dot{\beta}\dot{\alpha}} = (\sigma^{c})_{\gamma\dot{\gamma}}\left(\partial_{\underline{m}}\boldsymbol{T}_{c\dot{\beta},\dot{\alpha}} + \partial_{c}\boldsymbol{T}_{\dot{\beta}\underline{m},\dot{\alpha}} - \bar{D}_{\dot{\beta}}\boldsymbol{T}_{c\underline{m},\dot{\alpha}}\right)$$
$$= \frac{i}{4}\left[\epsilon_{\dot{\gamma}\dot{\alpha}}\partial_{\underline{m}}G_{\gamma\dot{\beta}} - 3\epsilon_{\dot{\beta}\dot{\gamma}}\partial_{\underline{m}}G_{\gamma\dot{\alpha}} - 3\epsilon_{\dot{\beta}\dot{\alpha}}\partial_{\underline{m}}G_{\gamma\dot{\gamma}}\right] + \partial_{\gamma\dot{\gamma}}\left(-i\epsilon_{\dot{\beta}\dot{\alpha}}S_{\underline{m}} + \bar{S}_{\underline{m}\dot{\beta}\dot{\alpha}}\right)$$
$$+\frac{1}{2}\epsilon_{\dot{\beta}(\dot{\gamma}}\bar{D}^{2}\bar{S}_{|\underline{m}\gamma|\dot{\alpha})} - 2i\bar{D}_{\dot{\beta}}D_{\gamma}\bar{S}_{\underline{m}\dot{\gamma}\dot{\alpha}} + \frac{1}{2}\epsilon_{\dot{\gamma}\dot{\alpha}}\left[-\bar{D}_{\dot{\beta}}D_{\gamma}S_{\underline{m}} - \frac{1}{4}\bar{D}^{2}\bar{S}_{\underline{m}\gamma\dot{\beta}}\right]$$

$$\boldsymbol{R}_{\underline{mn},ba} = -(\sigma_{ba})^{\beta\alpha}\boldsymbol{R}_{\underline{mn},\beta\alpha} - (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}}\boldsymbol{R}_{\underline{mn},\dot{\beta}\dot{\alpha}}$$
$$\boldsymbol{R}_{\underline{mn},\beta\alpha} = 2\partial_{[\underline{m}}\boldsymbol{T}_{\underline{n}]\beta,\alpha} + D_{\beta}\boldsymbol{T}_{\underline{mn},\alpha}$$
$$\boldsymbol{R}_{\underline{mn},\dot{\beta}\dot{\alpha}} = 2\partial_{[\underline{m}}\boldsymbol{T}_{\underline{n}]\dot{\beta},\dot{\alpha}} + \bar{D}_{\dot{\beta}}\boldsymbol{T}_{\underline{mn},\dot{\alpha}}$$

$$\boldsymbol{R}_{dc,\underline{np}} = \frac{1}{6}\varphi_{\underline{np}}\mathcal{T}^{\underline{q}}R_{dc\underline{q}} + R_{dc,[\underline{np}]_{14}}$$

$$\mathcal{R}_{dc\underline{m}} = 6i\partial_{[d}\left(S_{|\underline{m}|c]} + \bar{S}_{|\underline{m}|c]}\right) - \frac{1}{4}(\bar{\sigma}_{[c})^{\dot{\gamma}\gamma}\partial_{d]}\left(D_{\gamma}\bar{\lambda}_{\underline{m}\dot{\gamma}} - \bar{D}_{\dot{\gamma}}\lambda_{\underline{m}\gamma}\right)$$
$$R_{dc,[\underline{mn}]_{14}} = R_{[\underline{mn}]_{14},dc}$$
$$= -(\sigma_{dc})^{\beta\alpha}\left[2\partial_{[\underline{m}}S_{\underline{n}]_{14}\beta,\alpha} + D_{\beta}T_{[\underline{mn}]_{14},\alpha}\right] - (\bar{\sigma}_{dc})^{\dot{\beta}\dot{\alpha}}\left[2\partial_{[\underline{m}}\bar{S}_{\underline{n}]_{14}\dot{\beta},\dot{\alpha}} + \bar{D}_{\dot{\beta}}\bar{T}_{[\underline{mn}]_{14},\dot{\alpha}}\right].$$

$$\boldsymbol{R}_{\underline{n}a,b\underline{m}} = \frac{1}{7}\delta_{\underline{nm}}\mathcal{S}_{ab} + \tilde{\mathcal{S}}_{ab\underline{nm}} + \frac{1}{6}\varphi_{\underline{nmp}}\mathcal{S}_{ab\underline{p}} + \mathcal{S}_{ab[\underline{nm}]_{14}}$$
$$\tilde{\mathcal{S}}_{ab\underline{nm}} = \tilde{\mathcal{S}}_{ab\underline{mn}},\ \delta\underline{nm}\boldsymbol{R}_{\underline{n}a,b\underline{m}} = \mathcal{S}_{ab}$$

$$\mathcal{S}_{ab} = i\eta_{ab}\partial^{\underline{n}}S_{\underline{n}} - 3(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\gamma}}\partial^{\underline{n}}\bar{S}_{\underline{n}\dot{\alpha}\dot{\gamma}} + (\sigma_{ab})^{\beta\gamma}\partial^{\underline{n}}S_{\underline{n}\beta\gamma}$$
$$-2i\partial_{a}X_{b} - \frac{i}{24}\eta_{ab}D^{2}(\tilde{R} - \bar{\tilde{R}}) - \frac{1}{2}\eta_{ab}D^{\rho}\bar{D}^{\dot{\rho}}X_{\rho\dot{\rho}} - (\sigma_{ab})^{\beta\gamma}D_{\beta}\bar{D}^{\dot{\rho}}X_{\gamma\dot{\rho}}$$
$$-\frac{1}{2}(\bar{\sigma}_{a})^{\dot{\gamma}\gamma}(\bar{\sigma}_{b})^{\dot{\alpha}\beta}D_{\beta}\bar{D}_{(\dot{\alpha}}X_{|\gamma|\dot{\gamma})}$$

$$\tilde{\mathcal{S}}_{ab\underline{nm}} = \partial_{(\underline{n}}\left[i\eta_{|ab|}S_{\underline{m})} - 3(\bar{\sigma}_{|ab|})^{\dot{\alpha}\dot{\beta}}\bar{S}_{\underline{m})\dot{\alpha}\dot{\beta}} + (\sigma_{|ab|})^{\alpha\beta}S_{\underline{m})\alpha\beta}\right]_{(\underline{nm})\text{traceless}}$$
$$-2i\partial_{a}\tilde{X}_{b\underline{nm}} - \frac{1}{2}(\bar{\sigma}_{a})^{\dot{\gamma}\gamma}(\bar{\sigma}_{b})^{\dot{\alpha}\beta}D_{\beta}\bar{D}_{(\dot{\alpha}}\tilde{X}_{|\gamma|\dot{\gamma})\underline{nm}} - \frac{1}{12}\eta_{ab}D^{\rho}\bar{D}^{\dot{\rho}}\tilde{X}_{\rho\dot{\rho}\underline{m}}$$
$$+\frac{1}{6}(\sigma_{ab})^{\alpha\beta}D_{\alpha}\bar{D}^{\dot{\rho}}\tilde{X}_{\beta\dot{\rho}\underline{m}} + \frac{i}{24}\eta_{ab}D^{2}\left(\tilde{R}_{\underline{nm}} - \bar{\tilde{R}}_{\underline{nm}}\right)$$

$$\mathcal{S}_{ab\underline{m}} = \varphi_{\underline{mnp}}\partial_{\underline{n}}\left[i\eta_{ab}S_{\underline{p}} - 3(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\bar{S}_{\underline{p}\dot{\alpha}\dot{\beta}} + (\sigma_{ab})^{\alpha\beta}S_{\underline{p}\alpha\beta}\right] - \eta_{ab}D^{2}\left[\frac{1}{2}S_{\underline{m}} - \frac{i}{48}\bar{D}^{\dot{\beta}}\bar{\lambda}_{\underline{m}\dot{\beta}}\right]$$
$$+(\bar{\sigma}_{b})^{\dot{\beta}\beta}\partial_{a}\left[\frac{1}{8}\left(D_{\beta}\bar{\lambda}_{\underline{m}\dot{\beta}} + \bar{D}_{\dot{\beta}}\lambda_{\underline{m}\beta}\right) - \frac{3i}{2}\left(S_{\underline{m}\beta\dot{\beta}} + \bar{S}_{\underline{m}\beta\dot{\beta}}\right)\right]$$
$$+(\bar{\sigma}_{b})^{\dot{\beta}\beta}(\bar{\sigma}_{a})^{\dot{\alpha}\alpha}D_{\beta}\left[\frac{i}{16}\bar{D}_{(\dot{\beta}}D_{|\alpha}\bar{\lambda}_{\underline{m}|\dot{\alpha})} + \frac{3}{4}\bar{D}_{(\dot{\beta}}\bar{S}_{|\underline{m}\alpha|\dot{\alpha})}\right]$$
$$-\frac{1}{8}(\sigma_{a}\bar{\sigma}_{b})^{\alpha\beta}D_{\beta}\left[3i\rho_{\underline{m}\alpha} - \bar{D}^{\dot{\beta}}\bar{S}_{\underline{m}\alpha\dot{\beta}} - \frac{i}{12}\bar{D}^{2}\lambda_{\underline{m}\alpha} + \frac{i}{12}\bar{D}^{\dot{\beta}}D_{\alpha}\bar{\lambda}_{\underline{m}\dot{\beta}}\right]$$

$$\mathcal{S}_{ab[\underline{nm}]_{14}} = \partial_{[\underline{n}}\left[i\eta_{|ab|}S_{\underline{m}]_{14}} + \frac{3}{2}(\bar{\sigma}_{|b}\sigma_{a|})^{\dot{\alpha}\dot{\gamma}}\bar{S}_{\underline{m}]_{14}\dot{\alpha}\dot{\gamma}} - 3(\bar{\sigma}_{|ab|})^{\dot{\alpha}\dot{\gamma}}\bar{S}_{\underline{m}]_{14}\dot{\alpha}\dot{\gamma}} + (\sigma_{|ab|})^{\beta\gamma}S_{\underline{m}]_{14}\beta\gamma}\right]$$
$$+\frac{1}{16}D^{2}\left((\bar{\sigma}_{ab})^{\dot{\beta}\dot{\gamma}}\boldsymbol{R}_{\dot{\beta}\dot{n}\dot{m}} - \frac{1}{3}(\sigma_{ab})^{\beta\gamma}\boldsymbol{R}_{\beta\gamma\underline{nm}}\right) + \frac{1}{4}(\sigma_{a}\bar{\sigma}_{b})^{\gamma\beta}D_{\beta}T_{[\underline{nm}]_{14},\gamma}$$

$$\boldsymbol{R}_{\underline{n}c,\underline{pq}} = \frac{1}{6}\varphi_{\underline{pqr}}\mathcal{R}_{\underline{n}c,\underline{r}} + R_{\underline{n}c,[\underline{pq}]_{14}}$$

$$\mathcal{R}_{\underline{nc},\underline{m}} =: \frac{1}{7}\delta_{\underline{nm}}\mathcal{R}_c + \tilde{\mathcal{R}}_{c\underline{nm}} + \frac{1}{6}\varphi_{\underline{nm}\underline{\underline{p}}}\mathcal{R}_{c\underline{p}} + \mathcal{R}_{c[\underline{nm}]_{14}}$$

$$\tilde{\mathcal{R}}_{\underline{nm}} = \tilde{\mathcal{R}}_{\underline{mn}}, \delta\underline{\underline{nm}}\mathcal{R}_{\underline{nc},\underline{m}} = \mathcal{R}_c$$

$$\begin{aligned}
\mathcal{R}_c =& -\partial_c\tilde{\bar{R}} + \frac{i}{12}(\sigma_c)_{\alpha\dot{\beta}}D^\alpha\bar{D}^{\dot{\beta}}(\tilde{\bar{R}} - \tilde{R}) - 2D^2X_c + i\partial^{\underline{n}}(5\bar{S}_{\underline{n}c} + 3S_{\underline{n}c}) \\
& -\frac{1}{8}(\bar{\sigma}_c)^{\dot{\gamma}\gamma}\partial^{\underline{n}}(D_\gamma\bar{\lambda}_{\underline{n}\dot{\gamma}} - \bar{D}_{\dot{\gamma}}\lambda_{\underline{n}\gamma}) \\
\tilde{\mathcal{R}}_{c\underline{nm}} =& -\partial_c\tilde{\bar{R}}_{\underline{nm}} + \frac{i}{12}(\sigma_c)_{\alpha\dot{\beta}}D^\alpha\bar{D}^{\dot{\beta}}(\tilde{\bar{R}}_{\underline{nm}} - \tilde{R}_{\underline{nm}}) - \frac{4}{3}D^2\tilde{X}_{c\underline{nm}} \\
& + \left[i\partial_{\underline{n}}(5\bar{S}_{\underline{m}c} + 3S_{\underline{m}c}) - \frac{1}{8}(\bar{\sigma}_c)^{\dot{\gamma}\gamma}\partial_{\underline{n}}(D_\gamma\bar{\lambda}_{\underline{m}\dot{\gamma}} - \bar{D}_{\dot{\gamma}}\lambda_{\underline{m}\gamma})\right]_{(\underline{nm})\text{traceless}} \\
\mathcal{R}_{\alpha\dot{\beta}\underline{m}} =& (\sigma^a)_{\alpha\dot{\beta}}\mathcal{R}_{a\underline{m}} \\
=& 2\partial_{\alpha\dot{\beta}}\bar{R}_{\underline{m}} - \frac{3i}{2}D_\alpha\bar{\rho}_{\underline{m}\dot{\beta}} - 2D_\alpha\bar{D}_{\dot{\beta}}S_{\underline{m}} + \frac{5}{2}D^2S_{\underline{m}\alpha\dot{\beta}} \\
& + \frac{i}{24}D_\alpha\bar{D}^2\bar{\lambda}_{\underline{m}\dot{\beta}} + \frac{i}{6}D^2\bar{D}_{\dot{\beta}}\lambda_{\underline{m}\alpha} + \frac{i}{12}D_\alpha\bar{D}_{\dot{\beta}}D^\beta\lambda_{\underline{m}\beta} \\
& + \varphi_{\underline{m}}\underline{np}\partial_{\underline{n}}\left[5i\bar{S}_{\underline{p}\alpha\dot{\beta}} + 3iS_{\underline{p}\alpha\dot{\beta}} + \frac{1}{4}\left(D_\alpha\bar{\lambda}_{\underline{p}\dot{\beta}} - \bar{D}_{\dot{\beta}}\lambda_{\underline{p}\alpha}\right)\right] \\
\mathcal{R}_{c[\underline{nm}]_{14}} =& 2\partial_cL_{[\underline{nm}]_{14}} + \partial_{[\underline{n}}(5i\bar{S} + 3iS)_{\underline{m}]_{14}} + (\bar{\sigma}_c)^{\dot{\gamma}\gamma}D_\gamma\left[\frac{1}{2}T_{\underline{nm}]_{14},\dot{\gamma}} + \frac{1}{24}\bar{D}^{\dot{\rho}}\boldsymbol{R}_{\dot{\rho}\dot{\gamma}\underline{nm}}\right] \\
& + (\bar{\sigma}_c)^{\dot{\gamma}\gamma}\left[\frac{1}{8}D^\rho\bar{D}_{\dot{\gamma}}\boldsymbol{R}_{\rho\gamma\underline{nm}} - \frac{1}{8}\partial_{[\underline{n}}(D_{|\gamma|}\bar{\lambda}_{\underline{m}]_{14}\dot{\gamma}} - \bar{D}_{|\dot{\gamma}|}\lambda_{\underline{m}]_{14}\gamma})\right]
\end{aligned}$$

$$\begin{aligned}
R_{\underline{nc},[\underline{pq}]_{14}} =& R_{[\underline{pq}]_{14},\underline{nc}} \\
=& -(\bar{\sigma}_c)^{\dot{\alpha}\beta}\left[D_\beta\boldsymbol{T}_{\underline{pq},\underline{n}\dot{\alpha}} - 2i\partial_{[\underline{p}}\left(\tilde{X}_{|\beta\dot{\alpha}\underline{n}|\underline{q}]} + \frac{1}{7}\delta_{\underline{q}]\underline{n}}\tilde{X}_{\beta\dot{\alpha}}\right)\right. \\
& \left. + i\partial_{[\underline{p}}\varphi_{\underline{q}]\underline{\underline{r}}}\left(\frac{i}{12}D_\beta\bar{\lambda}_{\underline{r}\dot{\alpha}} + \bar{S}_{\underline{r}\beta\dot{\alpha}}\right)\right]_{[\underline{pq}]_{14}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{R}_{[\underline{mn}]_7,[\underline{pq}]_7} &= \boldsymbol{R}_{[\underline{pq}]_7,[\underline{mn}]_7} \in (\mathbf{7}\times\mathbf{7})_{\text{symmetric}} = \mathbf{1} + \mathbf{27} \\
\boldsymbol{R}_{[\underline{mn}]_7,[\underline{pq}]_{14}} &= \boldsymbol{R}_{[\underline{pq}]_{14},[\underline{mn}]_7} \in \mathbf{7}\times\mathbf{14} = \mathbf{7} + \mathbf{27} + \mathbf{64} \\
\boldsymbol{R}_{[\underline{mn}]_{14},[\underline{pq}]_{14}} &= \boldsymbol{R}_{[\underline{pq}]_{14},[\underline{mn}]_{14}} \in (\mathbf{14}\times\mathbf{14})_{\text{symmetric}} = \mathbf{1} + \mathbf{27} + \mathbf{77'}
\end{aligned}$$

$$\boldsymbol{R}_{\underline{pq},\underline{mn}} = \frac{1}{6}\varphi_{\underline{mn}}{}^{\underline{r}}\mathcal{R}_{\underline{pq},\underline{r}} + R_{\underline{pq},[\underline{mn}]_{14}}$$

$$\begin{aligned}
\mathcal{R}_{\underline{np},\underline{m}} =& 4\partial_{[\underline{n}}\left[\frac{1}{2}\tilde{\bar{R}}_{\underline{p}]\underline{m}} + \frac{1}{7}\delta_{\underline{p}]\underline{m}}\tilde{\bar{R}} + \frac{1}{6}\varphi_{\underline{p}]\underline{mq}}\bar{R}\underline{q} + L_{\underline{p}]\underline{m}\ 14}\right] \\
& -2D^\alpha\left[\frac{1}{42}\varphi_{\underline{npm}} + \frac{1}{24}\psi_{\underline{npmq}}V^{\underline{q}}{}_\alpha + \frac{3}{4}\varphi^{\underline{q}}\left[\underline{np}V_{\underline{m}]q\alpha}\right.\right. \\
& + \varphi_{\underline{np}}J_{\underline{qm}\alpha} - \frac{1}{2}\varphi_{\underline{pm}}{}^{\underline{q}}J_{\underline{qn}\alpha} + \frac{1}{2}\varphi_{\underline{nm}}{}^{\underline{q}}J_{\underline{qp}\alpha} \\
& + \varphi_{\underline{np}}I_{\underline{qm}\alpha} - \frac{1}{2}\varphi_{\underline{pm}}I_{\underline{qn}\alpha} + \frac{1}{2}\varphi_{\underline{nm}}I_{\underline{qp}\alpha} \\
& \left. + Z_{\underline{np}\underline{m}\alpha}\right]
\end{aligned}$$

$$R_{\underline{pq},[\underline{mn}]_{14}} = R_{[\underline{pq}]_7,[\underline{mn}]_{14}} + R_{[\underline{pq}]_{14},[\underline{mn}]_{14}}$$

$$R_{[\underline{pq}]_7,[\underline{mn}]_{14}} = R_{[\underline{mn}]_{14},[\underline{pq}]_7} = \frac{1}{6}\varphi_{\underline{pq}}{}^{\underline{r}}\mathcal{R}_{[\underline{mn}]_{14},\underline{r}}.$$

$$\psi\underline{\underline{mnpq}}\boldsymbol{R}_{\underline{mn},\underline{pq}} = 0 \;\Rightarrow\; R_{[\underline{mn}]_{14},}\,\underline{[mn]}_{14} = \frac{1}{3}\varphi\underline{\underline{mnp}}\mathcal{R}_{\underline{mn},\underline{p}}$$

$$\psi_{(\underline{r}}\underline{\underline{mnp}}\boldsymbol{R}_{|\underline{mn},\underline{p}|\underline{q})} = 0$$

$$\Rightarrow \left(\boldsymbol{R}_{(14\times 14)|_{27}}\right)_{\underline{qr}} = -\frac{1}{21}\delta_{\underline{rq}}\varphi\underline{mnp}\mathcal{R}_{\underline{mn},\underline{p}} + \frac{1}{6}\psi_{(\underline{r}}\underline{mnp}\varphi_{\underline{q})\underline{p}}\mathcal{R}_{\underline{mn},\underline{s}}$$

$$-\frac{2}{3}\varphi_{(\underline{r}}\underline{\underline{ps}}\Pi^{\mathbf{14}}\underline{\underline{q}}\Big)\underline{\underline{p}} - \mathcal{R}_{\underline{\underline{ij}},\underline{\underline{ij}}}$$

$$D_{\delta}\bar{W}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = 0 = \bar{D}_{\dot{\delta}}W_{\gamma\beta\alpha}$$

$$D^{\alpha}W_{\alpha\beta\gamma} = \frac{1}{2}\bar{D}_{\dot{\gamma}}D_{(\beta}G_{\gamma)}^{\dot{\gamma}}$$

$$\bar{D}_{\dot{\gamma}}D_{(\gamma}\bar{S}^{\underline{m}}{}_{\beta)\dot{\alpha}} = 0$$

$$2iD^{2}\bar{S}_{\underline{m}\dot{\gamma}\dot{\alpha}} = -D^{\beta}\bar{D}_{(\dot{\gamma}}\bar{S}_{|\underline{m}\beta|\dot{\alpha})} - \frac{1}{2}\bar{D}_{(\dot{\gamma}}D^{\beta}\bar{S}_{|\underline{m}\beta|\dot{\alpha})}$$

$$8i\partial_{\underline{m}}R^{\dagger} = -D^{2}S_{\underline{m}} - \left(D^{\beta}\bar{D}^{\dot{\alpha}} + \frac{1}{2}\bar{D}^{\dot{\alpha}}D^{\beta}\right)\bar{S}_{\underline{m}\beta\dot{\alpha}}$$

$$4i\varphi_{\underline{m}}\overset{\underline{np}}{\rightarrow}\partial_{\underline{n}}S_{\underline{p}} = -\frac{3i}{2}D^{\alpha}\rho_{\underline{m}\alpha} - D^{2}S_{\underline{m}} + D^{\alpha}\bar{D}^{\dot{\beta}}\bar{S}_{\underline{m}\alpha\dot{\beta}} + \frac{i}{24}\left(D^{\alpha}\bar{D}^{2}\lambda_{\underline{m}\alpha} + \bar{D}_{\dot{\beta}}D^{2}\bar{\lambda}_{\underline{m}}{}^{\dot{\beta}}\right)$$

$$D^{\alpha}T_{[\underline{mn}]_{14},\alpha} = 4i\partial_{\underline{m}}S_{\underline{n}]_{14}}$$

$$2i\varphi_{\underline{m}}\overset{\underline{np}}{\rightarrow}\partial_{\underline{n}}S_{\underline{p}\alpha\dot{\beta}} = -\frac{3i}{2}\bar{D}_{\dot{\beta}}\rho_{\underline{m}\alpha} - \bar{D}_{\dot{\beta}}D_{\alpha}S_{\underline{m}} + \frac{1}{2}\bar{D}^{2}\bar{S}_{\underline{m}\alpha\dot{\beta}} - \frac{i}{48}\bar{D}_{\dot{\beta}}D^{2}\lambda_{\underline{m}\alpha} + \frac{i}{48}\bar{D}^{2}D_{\alpha}\bar{\lambda}_{\underline{m}\dot{\beta}}$$

$$+\frac{1}{12}\bar{D}_{\dot{\beta}}\partial_{\alpha}{}^{\dot{\gamma}}\bar{\lambda}_{\underline{m}\dot{\gamma}}$$

$$\bar{D}_{\dot{\beta}}T_{[\underline{mn}]_{14},\alpha} = 2i\partial_{[\underline{m}}S_{\underline{n}]_{14}\alpha\dot{\beta}}$$

$$3iD_{(\alpha}\rho_{|\underline{m}|\beta)} + 3i\partial_{(\alpha}\dot{\gamma}_{|\underline{m}|\beta)\dot{\gamma}} - \frac{1}{2}\left(7D_{(\alpha}\bar{D}^{\dot{\gamma}} + 3\bar{D}^{\dot{\gamma}}D_{(\alpha)}\bar{S}_{|\underline{m}|\beta)\dot{\gamma}} - 4\varphi_{\underline{m}}\overset{\underline{np}}{\rightarrow}\partial_{\underline{n}}S_{\underline{m}\alpha\beta}\right.$$

$$= \frac{i}{24}D_{(\alpha}\bar{D}^{\dot{\gamma}}D_{\beta)}\bar{\lambda}_{\underline{m}\dot{\gamma}} - \frac{i}{12}\left(2D_{(\alpha}\bar{D}^{2} - 3\bar{D}^{2}D_{(\alpha)}\lambda_{|\underline{m}|\beta)}\right.$$

$$(\bar{\sigma}_{c})^{\dot{\alpha}\beta}\left(2\partial_{[b}\boldsymbol{T}_{a]\beta,\underline{m}\dot{\alpha}} + D_{\beta}\boldsymbol{T}_{ba,\underline{m}\dot{\alpha}}\right) = (\sigma_{ba})^{\beta\alpha}\boldsymbol{R}_{\underline{m}c,\beta\alpha} + (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}}\boldsymbol{R}_{\underline{m}c,\dot{\beta}\dot{\alpha}}$$

$$\Rightarrow \partial_{(\alpha}{}^{\dot{\beta}}\left[\epsilon_{\beta)\gamma}\left(i\epsilon_{\dot{\beta}\dot{\gamma}}S_{\underline{m}} + 3\bar{S}_{\underline{m}\dot{\beta}\dot{\gamma}}\right) + S_{|\underline{m}|\beta)\gamma}\epsilon_{\dot{\beta}\dot{\gamma}} - \frac{1}{2}\epsilon_{\gamma(\alpha}D^{2}\bar{S}_{|\underline{m}|\beta)\dot{\gamma}} + 2iD_{\gamma}\bar{D}_{\dot{\gamma}}S_{\underline{m}\alpha\beta} = R_{\underline{m},\gamma\dot{\gamma},\alpha\beta},\right.$$

$$0 = i\partial^{\underline{n}}\bar{S}_{\underline{n}\alpha\dot{\beta}} - \frac{3}{8}D^{2}X_{\alpha\dot{\beta}} + \frac{i}{24}D_{\alpha}\bar{D}_{\dot{\beta}}(\tilde{\bar{R}} - \tilde{R})$$

$$0 = i\partial_{(\underline{n}}\bar{S}_{\underline{m})_{\text{traceless}}\alpha\dot{\beta}} - \frac{1}{6}D^{2}\tilde{X}_{\alpha\dot{\beta}\underline{\underline{nm}}} + \frac{i}{24}D_{\alpha}\bar{D}_{\dot{\beta}}\left(\tilde{\bar{R}}_{\underline{nm}} - \tilde{R}_{\underline{nm}}\right)$$

$$0 = \varphi_{\underline{m}}\underline{\underline{np}}\partial_{\underline{n}}\bar{S}_{\underline{p}\alpha\dot{\beta}} + \frac{3}{8}D_{\alpha}\bar{\rho}_{\underline{m}\dot{\beta}} - \frac{i}{2}D_{\alpha}\bar{D}_{\dot{\beta}}S_{\underline{m}} - \frac{i}{8}D^{2}S_{\underline{m}\alpha\dot{\beta}}$$

$$+\frac{1}{48}\left[D^{2}\bar{D}_{\dot{\beta}}\lambda_{\underline{m}\alpha} - D_{\alpha}\bar{D}_{\dot{\beta}}D^{\beta}\lambda_{\underline{m}\beta} - \frac{1}{2}D_{\alpha}\bar{D}^{2}\bar{\lambda}_{\underline{m}\dot{\beta}}\right]$$

$$0 = 3i\partial_{[\underline{n}}\bar{S}_{\underline{m}]_{14}\alpha\dot{\beta}} + \frac{3}{4}D_{\alpha}T_{[\underline{nm}]_{14},\dot{\beta}}$$

$$-\frac{1}{16}\left[3D^{\gamma}\bar{D}_{\dot{\beta}} + 2\bar{D}_{\dot{\beta}}D^{\gamma}\right]\boldsymbol{R}_{\alpha\gamma\underline{nm}} + \frac{1}{16}D_{\alpha}\bar{D}^{\dot{\gamma}}\boldsymbol{R}_{\dot{\gamma}\dot{\beta}\underline{nm}}$$

$$\mathcal{R}_{\underline{m},c\underline{n}} = -(\bar{\sigma}_{c})^{\dot{\alpha}\beta}\varphi_{\underline{n}}\underline{\underline{pq}}\left[D_{\beta}\boldsymbol{T}_{\underline{pq},\underline{m}\dot{\alpha}} - 2i\partial_{\underline{p}}\left(\tilde{X}_{\beta\dot{\alpha}\underline{mq}} + \frac{1}{7}\delta_{\underline{mq}}\tilde{X}_{\beta\dot{\alpha}}\right)\right.$$

$$\left.+i\varphi_{\underline{qm}}\partial_{\underline{p}}\left(\frac{i}{12}D_{\beta}\bar{\lambda}_{\underline{r}\dot{\alpha}} + \bar{S}_{\underline{r}\beta\dot{\alpha}}\right)\right]$$

$$2\boldsymbol{R}_{\underline{m}[a,b]\underline{n}} = (\sigma_{ab})^{\beta\alpha}\boldsymbol{R}_{\underline{mn},\beta\alpha} + (\bar{\sigma}_{ab})^{\dot{\beta}\dot{\alpha}}\boldsymbol{R}_{\underline{mn},\dot{\beta}\dot{\alpha}}$$

$$\left[2\partial_{[\underline{\underline{n}}}\boldsymbol{T}_{\underline{\underline{p}}]\beta,\underline{m}\alpha} + D_{\beta}\boldsymbol{T}_{\underline{np},\underline{m}\alpha}\right]_{(\alpha\beta)} = 0$$

$$\partial_{[\hat{e}}\boldsymbol{G}_{\hat{d}\hat{c}\hat{b}\hat{a}]} = 0.$$

$$\partial_{[\underline{m}}\boldsymbol{T}_{cb]}^{\underline{\alpha}} = \partial_{[\underline{n}}\boldsymbol{T}_{\underline{m}b]}^{\underline{\alpha}} = \partial_{[\underline{n}}\boldsymbol{T}_{\underline{mp}]}^{\underline{\alpha}} = 0$$

$$\partial_{[d}\boldsymbol{T}_{cb]}\underline{\underline{m\alpha}} = \partial_{[\underline{n}}\boldsymbol{T}_{cb]}\underline{\underline{m\alpha}} = \partial_{[\underline{n}}\boldsymbol{T}_{\underline{p}b]}\underline{\underline{m\alpha}} = \partial_{[\underline{n}}\boldsymbol{T}_{\underline{pq}]}{}^{\underline{m\alpha}} = 0$$

$$\Gamma_{\mu}^{*} = -\eta(-1)^{t}B\Gamma_{\mu}B^{-1}, \Gamma_{\mu}^{T} = -\eta C\Gamma_{\mu}C^{-1}$$

$$B^{T} = \epsilon\eta^{t}(-1)^{\frac{t(t-1)}{2}}B, C^{T} = -\epsilon C, \epsilon = -\sqrt{2}\cos\left(\frac{\pi}{4}(1+\eta D)\right)$$

$$\hat{\Gamma}_{\hat{a}}^{*} = \hat{B}\hat{\Gamma}_{\hat{a}}\hat{B}^{-1}, \hat{\Gamma}_{\hat{a}}^{T} = -\hat{C}\hat{\Gamma}_{\hat{a}}\hat{C}^{-1}$$

$$\hat{B}^{\dagger}\hat{B} = \mathbf{1} = \hat{B}^{*}\hat{B} \Rightarrow \hat{B}^{T} = \hat{B}, \hat{C}^{T} = -\hat{C}, \hat{B} = -\hat{C}\hat{\Gamma}_{0}$$

$$A_{\hat{\alpha}} = -\hat{C}_{\hat{\alpha}\hat{\beta}}A^{\hat{\beta}}, A^{\hat{\alpha}} = -\hat{C}^{\hat{\alpha}\hat{\beta}}A_{\hat{\beta}}$$

$$\sigma^{0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma^{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(\bar{\sigma}^{a})^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^{a})_{\beta\dot{\beta}}$$

$$\gamma^{a} = \begin{bmatrix} \mathbf{0}_2 & i\sigma^{a} \\ i\bar{\sigma}^{a} & \mathbf{0}_2 \end{bmatrix}$$

$$C_{4\mathrm{D}} = -i\sigma^{3} \otimes \sigma^{2} = \begin{bmatrix} -\epsilon^{\alpha\beta} & \mathbf{0}_2 \\ \mathbf{0}_2 & -\epsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix}$$

$$B_{4\mathrm{D}} = \sigma^{2} \otimes \sigma^{2} = \begin{bmatrix} \mathbf{0}_2 & -\epsilon^{\alpha\beta} \\ -\epsilon_{\dot{\alpha}\dot{\beta}} & \mathbf{0}_2 \end{bmatrix}$$

$$\gamma_{5} := i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = -\sigma^{3} \otimes \sigma^{0} = \begin{bmatrix} \delta_{\alpha}{}^{\beta} & \mathbf{0}_2 \\ \mathbf{0}_2 & -\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{bmatrix}$$

$$\Gamma^{[\underline{a}_1} \dots \Gamma^{\underline{a}_7]} =: \Gamma^{a_1 \dots a_7} = -i\epsilon\underline{a_1 \dots a_7}, \epsilon\underline{12 \dots 7} = 1$$

$$(\Gamma^{\underline{a}})^{T} = -\Gamma^{\underline{a}}, (\Gamma^{\underline{a}})^{*} = -\Gamma^{\underline{a}}$$

$$\Rightarrow (\Gamma^{\underline{a}})_{I}^{J} = -(\Gamma^{\underline{a}})_{J}^{I} = (\Gamma^{\underline{a}})_{IJ} = -(\Gamma^{\underline{a}})_{JI} = -(\Gamma^{\underline{a}})_{IJ}^{*}$$

$$\hat{\Gamma}^{a} := \gamma^{a} \otimes \mathbf{1}_8 \Longrightarrow (\hat{\Gamma}^{a})_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{bmatrix} \mathbf{0}_{16} & i(\sigma^{a})_{\alpha\dot{\beta}}\delta_{IJ} \\ i(\bar{\sigma}^{a})^{\dot{\alpha}\beta}\delta^{IJ} & \mathbf{0}_{16} \end{bmatrix}$$

$$\hat{\Gamma}^{a} := -\gamma_{5} \otimes \Gamma^{a} \Longrightarrow (\hat{\Gamma}^{a})_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{bmatrix} -\delta_{\alpha}{}^{\beta}(\Gamma^{a})_{I}{}^{J} & \mathbf{0}_{16} \\ \mathbf{0}_{16} & \delta^{\dot{\alpha}}{}_{\dot{\beta}}(\Gamma^{a})^{I}{}_{J} \end{bmatrix}$$

$$\hat{C} := C_{4\mathrm{D}} \otimes C_{7\mathrm{D}} \Rightarrow \hat{C}^{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -\epsilon^{\alpha\beta}\delta^{IJ} & \mathbf{0}_{16} \\ \mathbf{0}_{16} & -\epsilon_{\dot{\alpha}\dot{\beta}}\delta_{IJ} \end{bmatrix}$$

$$\hat{C}^{-1} = -\hat{C} \Rightarrow \hat{C}_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -\epsilon_{\alpha\beta}\delta_{IJ} & \mathbf{0}_{16} \\ \mathbf{0}_{16} & -\epsilon^{\dot{\alpha}\dot{\beta}}\delta^{IJ} \end{bmatrix}$$

$$\hat{\Gamma}^{ab} = \gamma^{ab} \otimes \mathbf{1}_8 = -2\begin{bmatrix} \sigma^{ab} & \mathbf{0}_2 \\ \mathbf{0}_2 & \bar{\sigma}^{ab} \end{bmatrix} \otimes \mathbf{1}_8$$

$$\hat{\Gamma}^{a\underline{b}} = \frac{1}{2}[\gamma_5, \gamma^{a}] \otimes \Gamma^{\underline{b}} = \begin{bmatrix} \mathbf{0}_2 & i\sigma^{a} \\ -i\bar{\sigma}^{a} & \mathbf{0}_2 \end{bmatrix} \otimes \Gamma^{\underline{b}}$$

$$\hat{\Gamma}^{\underline{ab}} = \mathbf{1}_4 \otimes \Gamma^{\underline{ab}} = \begin{bmatrix} \delta_{\alpha}{}^{\beta} & \mathbf{0}_2 \\ \mathbf{0}_2 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{bmatrix} \otimes \Gamma^{\underline{ab}}$$

$$\sigma^{ab} = \tfrac{1}{4}(\sigma^{a}\bar{\sigma}^{b} - \sigma^{b}\bar{\sigma}^{a}) \text{ and } \bar{\sigma}^{ab} = \tfrac{1}{4}(\bar{\sigma}^{a}\sigma^{b} - \bar{\sigma}^{b}\sigma^{a})$$

$$[x^{\hat{m}}] = [d\hat{x}^{\hat{m}}] = -1, [\theta^{\hat{\mu}}] = [d\theta^{\hat{\mu}}] = -\frac{1}{2}$$

$$[\hat{E}^{\hat{a}}] = -1 \quad \Longrightarrow \quad [\hat{E}_{\hat{m}}{}^{\hat{a}}] = 0, \quad [\hat{E}_{\hat{\mu}}{}^{\hat{a}}] = -\frac{1}{2}$$

$$[\hat{E}^{\hat{\alpha}}] = -\frac{1}{2} \quad \Longrightarrow \quad [\hat{E}_{\hat{m}}{}^{\hat{\alpha}}] = \frac{1}{2}, \quad [\hat{E}_{\hat{\mu}}{}^{\hat{\alpha}}] = 0$$

$$\hat{\Sigma} = \hat{e}_{\hat{M}_1} \otimes \dots \otimes \hat{e}_{\hat{M}_p} \otimes \hat{e}^{\hat{N}_1} \otimes \dots \otimes \hat{e}^{\hat{N}_q}\hat{\Sigma}^{\hat{M}_1 \dots \hat{M}_p}{}_{\hat{N}_1 \dots \hat{N}_q}$$

$$[\hat{\Sigma}] = [\hat{\Sigma}^{\hat{M}_1 \dots \hat{M}_p}\hat{N}_1 \dots \hat{N}_q] - n_v - \frac{1}{2}n_s + m_v + \frac{1}{2}m_s$$

$n_v =$ vector $\hat{N}$'s, $n_s =$ spinor $\hat{N}$'s

$m_v =$ vector $\hat{M}$'s, $m_s =$ spinor $\hat{M}$'s

$$[\Omega^{\hat{a}}_{\hat{b}}] = 0 \quad \Longrightarrow \quad [\hat{\Omega}^{\hat{a}}_{\hat{c}\hat{b}}] = 1, \quad [\hat{\Omega}^{\hat{a}}_{\hat{\gamma}\hat{b}}] = \frac{1}{2},$$

$$[\hat{T}^{\hat{a}}] = -1 \quad \Longrightarrow \quad [\hat{T}^{\hat{a}}_{\hat{b}\hat{c}}] = 1, \quad [\hat{T}^{\hat{a}}_{\hat{b}\hat{\gamma}}] = \frac{1}{2}, \quad [\hat{T}^{\hat{a}}_{\hat{\beta}\hat{\gamma}}] = 0,$$

$$[\hat{T}^{\hat{\alpha}}] = -\frac{1}{2} \quad \Longrightarrow \quad [\hat{T}^{\hat{\alpha}}_{\hat{b}\hat{c}}] = \frac{3}{2}, \quad [\hat{T}^{\hat{\alpha}}_{\hat{b}\hat{\gamma}}] = 1, \quad [\hat{T}^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}] = \frac{1}{2},$$

$$[\hat{R}^{\hat{a}}_{\hat{b}}] = 0 \quad \Longrightarrow \quad [\hat{R}^{\hat{a}}_{\hat{d}\hat{c}\hat{b}}] = 2, \quad [\hat{R}^{\hat{a}}_{\hat{d}\hat{\gamma}\hat{b}}] = \frac{3}{2}, \quad [\hat{R}^{\hat{a}}_{\hat{\delta}\hat{\gamma}\hat{b}}] = 1.$$

$$[\hat{C}] = -3 \quad \Longrightarrow \quad [\hat{C}_{\hat{a}\hat{b}\hat{c}}] = 0, \quad [\hat{C}_{\hat{a}\hat{b}\hat{\gamma}}] = -\frac{1}{2}, [\hat{C}_{\hat{a}\hat{\beta}\hat{\gamma}}] = -1, [\hat{C}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}] = -\frac{3}{2}$$

$$[\hat{G}] = -3 \ \Longrightarrow \ [\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}}] = 1, [\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{\delta}}] = \frac{1}{2}, [\hat{G}_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}] = 0, [\hat{G}_{\hat{a}\hat{\beta}\hat{\gamma}\hat{\delta}}] = -\frac{1}{2}, [\hat{G}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}] = -1.$$

$$[X] = -1, [\Sigma_{\alpha\underline{m}}] = -\frac{1}{2}, [V_{\underline{mn}}] = -1, [\Phi_{\underline{mnp}}] = 0, [\mathcal{V}\underline{m}] = -1,$$
$$[H_{\alpha\dot{\alpha}}] = -1, [\Psi_{\underline{m}\alpha}] = -\frac{1}{2}$$

$$\psi \otimes \eta, \psi_{\underline{m}} \otimes (i\Gamma\underline{m}\eta)$$

$$\varphi_{\underline{mnp}} = i\eta^T \Gamma_{\underline{mnp}}\eta$$

$$\Psi_{\hat{\alpha}} = \begin{bmatrix} \Psi_{\alpha I} \\ \Psi^{\dot{\alpha} I} \end{bmatrix}, \Psi_{\alpha I} = \eta_I \Psi_\alpha + i(\Gamma^{\underline{m}}\eta)_I \Psi_{\underline{m}\alpha}, \Psi^{\dot{\alpha} I} = \eta^I \Psi^{\dot{\alpha}} + i(\Gamma_{\underline{m}}\eta)^I \Psi^{\underline{m}\dot{\alpha}}$$

$$\eta^T \Gamma^{\underline{m}} \eta \ = 0$$
$$\eta^T \Gamma^{\underline{m}} \Gamma^{\underline{n}} \eta \ = \delta^{\underline{mn}}$$
$$\eta^T \Gamma^{\underline{m}} \Gamma^{\underline{n}} \Gamma^{\underline{p}} \eta \ = \eta^T \Gamma^{\underline{mnp}} \eta = -i\varphi\underline{mnp}$$
$$\eta^T \Gamma^{\underline{mnpq}} \eta \ = \psi\underline{mnpq} = \frac{1}{3!}\epsilon\underline{\underline{mnpqrst}}\varphi_{\underline{rst}} = (\star\,\varphi)\underline{mnpq}$$
$$\eta^T \Gamma^{\underline{m}} \Gamma^{\underline{n}} \Gamma^{\underline{p}} \Gamma^{\underline{q}} \eta \ = \psi\underline{mnpq} + \delta^{\underline{mn}}\delta\underline{\underline{pq}} - \delta\underline{\underline{mp}}\delta\underline{nq} + \delta\underline{mq}\delta\underline{np}$$

$$\Psi_\alpha = \eta^I \Psi_{\alpha I}, \quad \Psi_{\underline{m}\alpha} = i(\Gamma_{\underline{m}}\eta)^I \Psi_{\alpha I}$$
$$\Psi^{\dot{\alpha}} = \eta_I \Psi^{\dot{\alpha} I}, \quad \Psi^{\underline{m}\dot{\alpha}} = i(\Gamma^{\underline{m}}\eta)_I \Psi^{\dot{\alpha} I}$$

$$A^{\hat{\alpha}} B_{\hat{\alpha}} := -A^{\hat{\alpha}} \hat{C}_{\hat{\alpha}\hat{\beta}} B^{\hat{\beta}} = A^\alpha B_\alpha + A_{\dot{\alpha}} B^{\dot{\alpha}} + A^{\underline{m}\alpha} B_{\underline{m}\alpha} + A_{\underline{m}\dot{\alpha}} B^{\underline{m}\dot{\alpha}}$$

$$\overset{\circ}{\dot{T}}{}^{a}_{\alpha\dot{\beta}} \ = 2i(\sigma^a)_{\alpha\dot{\beta}}, \overset{\circ}{\dot{T}}{}^{a}_{\underline{m}\alpha,\underline{n}\dot{\beta}} = 2i\delta_{\underline{mn}}(\sigma^a)_{\alpha\dot{\beta}}$$
$$\overset{\circ}{\dot{T}}_{\alpha,\underline{n}\beta}\beta^{\underline{m}} \ = 2i\delta^{\underline{m}}_{\underline{n}}\epsilon_{\alpha\beta}, \overset{\circ}{\dot{T}}{}^{\dot{\alpha},\underline{n}\dot{\beta},\underline{m}} = -2i\delta^{\underline{mn}}\epsilon^{\dot{\alpha}\dot{\beta}}$$
$$\overset{\circ}{\dot{T}}_{\underline{n}}\beta, \underline{p}\gamma^{\underline{m}} \ = 2i\varphi^{\underline{m}}\underline{np}\epsilon_{\beta\gamma}, \overset{\circ}{\dot{T}}{}^{\underline{n}}\dot{\beta}, \underline{p}\dot{\gamma}, \underline{m} = -2i\varphi^{\underline{mnp}}\epsilon^{\dot{\beta}\dot{\gamma}}$$

$$\overset{\circ}{\dot{G}}_{ab\gamma}{}^{\delta} = -4(\sigma_{ab})_\gamma{}^\delta, \overset{\circ}{\dot{G}}_{ab}{}^{\dot{\gamma}}{}_{\dot{\delta}} = -4(\bar{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\delta}}$$
$$\overset{\circ}{\dot{G}}_{ab,\underline{n}\gamma}{}^{\underline{m}\delta} = -4\delta^{\underline{m}}_{\underline{n}}(\sigma_{ab})_\gamma{}^\delta, \overset{\circ}{\dot{G}}_{ab}{}^{\underline{n}\dot{\gamma}}{}_{\underline{m}\dot{\delta}} = -4\delta^{\underline{n}}_{\underline{m}}(\bar{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\delta}},$$
$$\dot{G}_{a\underline{m},\gamma,\underline{n}\dot{\delta}} = -2(\sigma_a)_{\gamma\dot{\delta}}\delta_{\underline{mn}}, \dot{G}_{a\underline{m},\underline{n}\gamma,\dot{\delta}} = 2(\sigma_a)_{\gamma\dot{\delta}}\delta_{\underline{mn}}$$
$$\dot{G}_{a\underline{\underline{m}}}{}^{\dot{\gamma},\underline{n}\delta} = 2(\bar{\sigma}_a)^{\dot{\gamma}\delta}\delta\underline{\underline{n}}, \dot{G}_{a\underline{\underline{n}}}{}^{\underline{n}}\dot{\gamma}, \delta = -2(\bar{\sigma}_a)^{\dot{\gamma}\delta}\delta\underline{\underline{n}}$$

$$\overset{\circ}{\dot{G}}_{a\underline{m},\underline{n}\gamma,\underline{p}\dot{\delta}} = -2(\sigma_a)_{\gamma\dot{\delta}}\varphi_{\underline{mnp}}, \overset{\circ}{\dot{G}}_{a\underline{m}}{}^{\underline{n}\dot{\gamma},\underline{p}\delta} = 2(\bar{\sigma}_a)^{\dot{\gamma}\delta}\varphi_{\underline{m}}\underline{np},$$
$$\bar{G}_{\underline{mn},\gamma}{}^{\underline{p}\delta} = 2\delta_\gamma{}^\delta\varphi_{\underline{mn}}\underline{\underline{p}}, \qquad \overset{\circ}{\dot{G}}_{\underline{mn},\underline{p}\gamma}{}^{\delta} = -2\delta_\gamma{}^\delta\varphi_{\underline{mnp}}$$

$$\overset{\circ}{\dot{G}}_{\underline{mn}}\overset{\dot{\gamma}}{p}_{\underline{p}\dot{\delta}} = 2\delta^{\dot{\gamma}}{}_{\dot{\delta}}\varphi_{\underline{mnp}}, \overset{\circ}{\dot{G}}_{\underline{mn}}{}^{\underline{p}}{}_{\dot{\delta}} = -2\delta^{\dot{\gamma}}{}_{\dot{\delta}}\varphi_{\underline{mn}}{}^{\underline{p}}$$

$$\overset{\circ}{\dot{G}}_{\underline{mn},\underline{p}\gamma^q}{}^{\delta} = 2\delta_\gamma{}^\delta\left[\psi_{\underline{mnp}}\underline{\underline{q}} + 2\delta_{\underline{p}[\underline{\underline{m}}}\delta^{\underline{q}}_{\underline{n}]}\right], \overset{\circ}{\dot{G}}_{\underline{mn}}{}^{\underline{p}}{}_{\underline{q}}\dot{\delta} = 2\delta^{\dot{\gamma}}{}_{\dot{\delta}}\left[\psi_{\underline{mn}}\underline{\underline{p}}^{\underline{q}} + 2\delta^{\underline{p}}_{[\underline{m}}\delta_{\underline{n}]\underline{q}}\right]$$

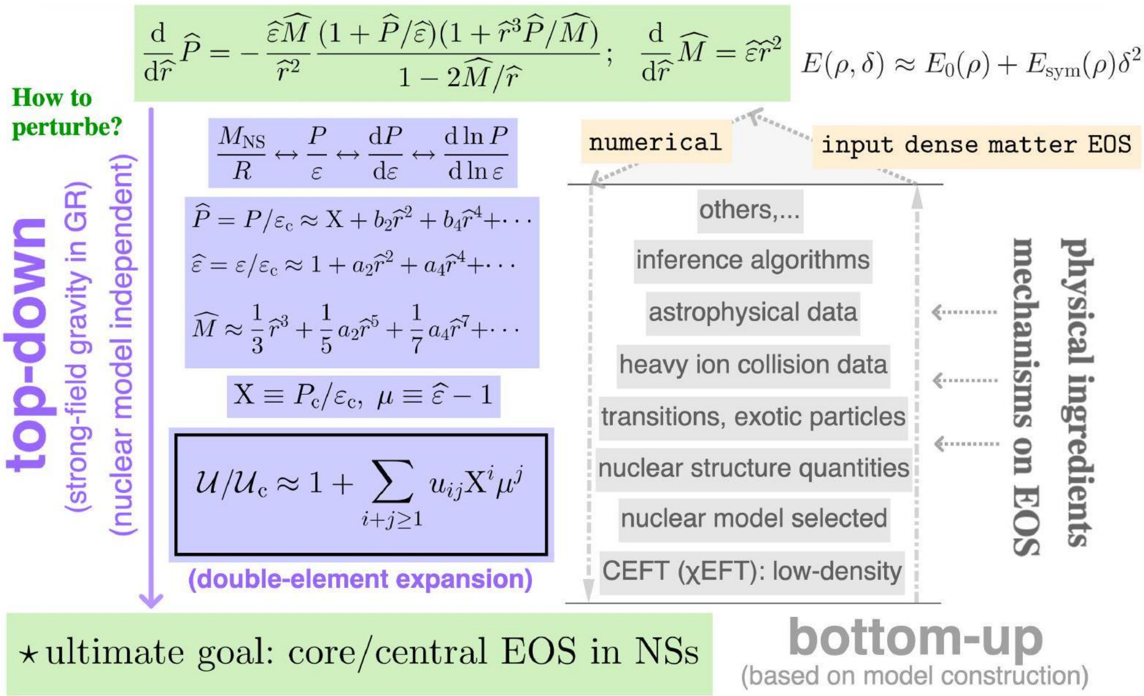
Morfología y fenomenología de la partícula estrella o blanca en supergravedad cuántica relativista.

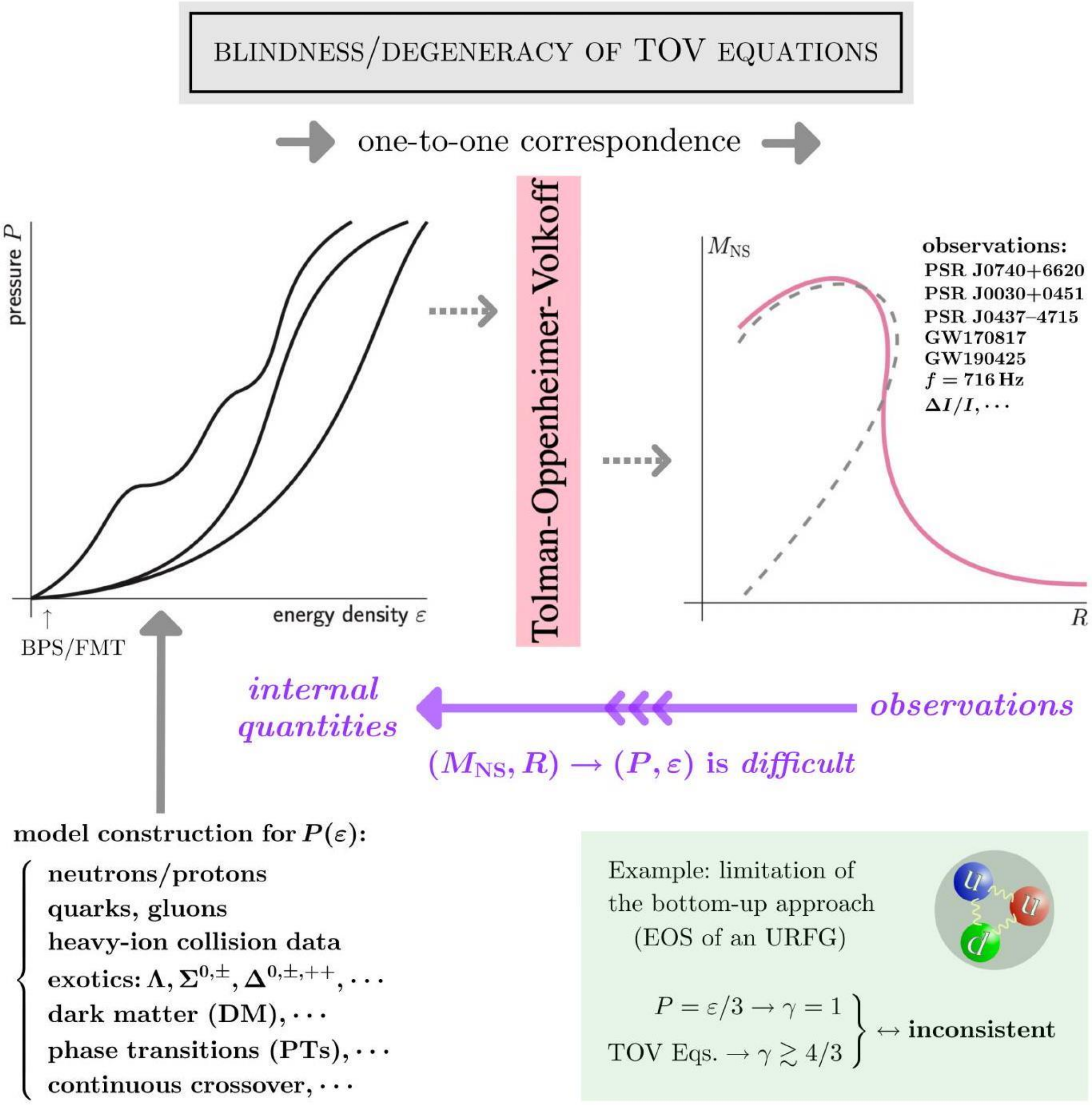
$$s^2 \equiv c^2 \frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}P}{\mathrm{d}\varepsilon}.$$

$$\xi \equiv \frac{GM_{\mathrm{NS}}}{Rc^2} = \frac{M_{\mathrm{NS}}}{R},$$

$$\phi \equiv P/\varepsilon.$$

$$\gamma \equiv \frac{\mathrm{dln}\ P}{\mathrm{dln}\ \varepsilon} = \frac{s^2}{P/\varepsilon} = \frac{s^2}{\phi},$$





$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\varepsilon}{r^2}\left(1+\frac{P}{\varepsilon}\right)\left(1+\frac{4\pi r^3 P}{M}\right)\left(1-\frac{2GM}{r}\right)^{-1}, \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\varepsilon$$

$$W = \frac{1}{G}\frac{1}{\sqrt{4\pi G\varepsilon_c}} = \frac{1}{\sqrt{4\pi\varepsilon_c}}, Q = \frac{1}{\sqrt{4\pi G\varepsilon_c}} = \frac{1}{\sqrt{4\pi\varepsilon_c}}$$

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} = -\frac{\hat{\varepsilon}\hat{M}}{\hat{r}^2}\frac{(1+\hat{P}/\hat{\varepsilon})(1+\hat{r}^3\hat{P}/\hat{M})}{1-2\hat{M}/\hat{r}}, \frac{\mathrm{d}\hat{M}}{\mathrm{d}\hat{r}} = \hat{r}^2\hat{\varepsilon}$$

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} = -\frac{\hat{M}\hat{\varepsilon}}{\hat{r}^2}, \frac{\mathrm{d}\hat{M}}{\mathrm{d}\hat{r}} = \hat{r}^2\hat{\varepsilon}$$

$$P(R) = 0 \leftrightarrow \hat{P}(\hat{R}) = 0.$$

$$M_{\mathrm{NS}} = \hat{M}_{\mathrm{NS}}W, \text{with } \hat{M}_{\mathrm{NS}} \equiv \hat{M}(\hat{R}) = \int_0^{\hat{R}} \mathrm{d}\hat{r}\hat{r}^2\hat{\varepsilon}(\hat{r})$$

$$\hat{P}(\hat{r}) = \frac{1}{\hat{r}^2}\frac{2\zeta^2}{1+6\zeta+\zeta^2}, \hat{\varepsilon}(\hat{r}) = \frac{1}{\hat{r}^2}\frac{2\zeta}{1+6\zeta+\zeta^2}, \hat{M}(\hat{r}) = \frac{2\zeta\hat{r}}{1+6\zeta+\zeta^2}.$$

$$\hat{M} \sim \int \hat{\varepsilon}\,\mathrm{d}\mathbf{x} \sim \int \hat{\varepsilon}\hat{r}^2\,\mathrm{d}\hat{r} \sim \hat{r}$$

$$P = \varepsilon/3.$$

$$\hat{M}_{\mathrm{NS}} = 3\hat{R}/14 \leftrightarrow M_{\mathrm{NS}} = 3R/14G$$

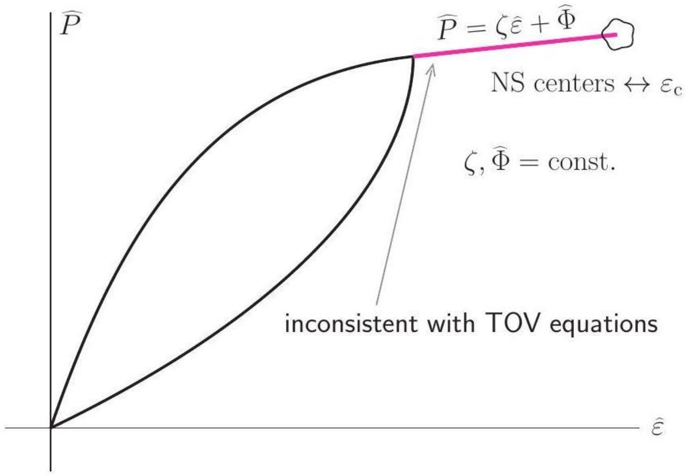
$$R/\mathrm{nm} \approx 6.9 M_{\mathrm{NS}}/M_{\odot},$$

$$\hat{P} = \zeta\hat{\varepsilon} + \hat{\Phi}, \text{ where } \hat{\Phi} = \hat{P}_c - \zeta\hat{\varepsilon}_c = \hat{P}_c - \zeta$$

$$\hat{p}(\hat{r}) \approx \frac{1}{\hat{r}^2}\frac{2\zeta^2}{1+6\zeta+\zeta^2} + \frac{2(1+2\zeta)\hat{\Phi}}{(1+3\zeta)(2+\zeta)'}$$
$$\hat{\varepsilon}(\hat{r}) \approx \frac{1}{\hat{r}^2}\frac{2\zeta}{1+6\zeta+\zeta^2} - \frac{3(1+\zeta)\hat{\Phi}}{(1+3\zeta)(2+\zeta)}$$
$$\hat{M}(\hat{r}) \approx \frac{2\zeta\hat{r}}{1+6\zeta+\zeta^2} - \frac{(1+\zeta)\hat{\Phi}\hat{r}^3}{(1+3\zeta)(2+\zeta)}$$

$$\hat{p} = \hat{\varepsilon} + \hat{\Phi}: \hat{p}(\hat{r}) = \frac{1}{4\hat{r}^2} + \frac{\hat{\Phi}}{2}, \hat{\varepsilon}(\hat{r}) = \frac{1}{4\hat{r}^2} - \frac{\hat{\Phi}}{2}, \hat{M}(\hat{r}) = \frac{\hat{r}}{4} - \frac{\hat{\Phi}\hat{r}^3}{6}.$$

$$\xi = \hat{M}/\hat{R} \approx 4^{-1}\left(1 - 32\hat{M}^2\hat{\Phi}/3\right) \approx 4^{-1}\left(1 - 2\hat{R}^2\hat{\Phi}/3\right)$$



$$\hat{M}(\hat{r}) = \int_0^{\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(x)$$

$$\hat{M}(\hat{r}) \to -\int_0^{-\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(-x)$$

$$\hat{M}(-\hat{r}) = \int_0^{-\hat{r}} \mathrm{d}x x^2 \hat{\varepsilon}(x)$$

$$\hat{P}(\hat{r}) = -\int_0^{-\hat{r}} \mathrm{d}x \frac{\hat{\varepsilon}(x)\hat{M}(x)}{x^2} \frac{[1+\hat{P}(-x)/\hat{\varepsilon}(x)]\left[1+x^3\hat{P}(-x)/\hat{M}(x)\right]}{1-2\hat{M}(x)/x}$$
$$\hat{P}(-\hat{r}) = -\int_0^{-\hat{r}} \mathrm{d}x \frac{\hat{\varepsilon}(x)\hat{M}(x)}{x^2} \frac{[1+\hat{P}(x)/\hat{\varepsilon}(x)]\left[1+x^3\hat{P}(x)/\hat{M}(x)\right]}{1-2\hat{M}(x)/x}$$

$$\hat{\varepsilon}(\hat{r}) \approx 1 + a_2\hat{r}^2 + a_4\hat{r}^4 + a_6\hat{r}^6 + \cdots$$
$$\hat{P}(\hat{r}) \approx \mathrm{X} + b_2\hat{r}^2 + b_4\hat{r}^4 + b_6\hat{r}^6 + \cdots$$
$$\hat{M}(\hat{r}) \approx \frac{1}{3}\hat{r}^3 + \frac{1}{5}a_2\hat{r}^5 + \frac{1}{7}a_4\hat{r}^7 + \frac{1}{9}a_6\hat{r}^9 + \cdots$$

$$\mathrm{X} \equiv \hat{P}_c \equiv P_c/\varepsilon_c$$

$$\hat{M}_{\mathrm{NS}} = M_{\mathrm{NS}}/W \approx 0.18, \hat{R} = R/Q \approx 1.1,$$

$$\mu \equiv \hat{\varepsilon} - \hat{\varepsilon}_c = \hat{\varepsilon} - 1 < 0$$

$$\mathcal{U} \mid \mathcal{U}_c \approx 1 + \sum_{i+j\geq 1} u_{ij} \mathrm{X}^i \mu^j$$

$$\xi = \tau(\mathrm{X}) \approx \tau_1 \mathrm{X} + \tau_2 \mathrm{X}^2 + \cdots$$

$$b_2 = -\frac{1}{6}(1 + 4X + 3X^2)$$

$$b_4 = -\frac{2a_2}{15} + \left(\frac{1}{12} - \frac{3}{10}a_2\right)X + \frac{1}{3}X^2 + \frac{1}{4}X^3$$

$$b_6 = -\frac{1}{216} - \frac{a_2^2}{30} - \frac{a_2}{54} - \frac{5a_4}{63} + \left(\frac{a_2}{45} - \frac{4a_4}{21} - \frac{1}{54}\right)X + \left(\frac{2a_2}{15} - \frac{1}{18}\right)X^2 - \frac{1}{6}X^3 - \frac{1}{8}X^4$$

$$s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{r}} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}} = \frac{b_2 + 2b_4\hat{r}^2 + \cdots}{a_2 + 2a_4\hat{r}^2 + \cdots}$$

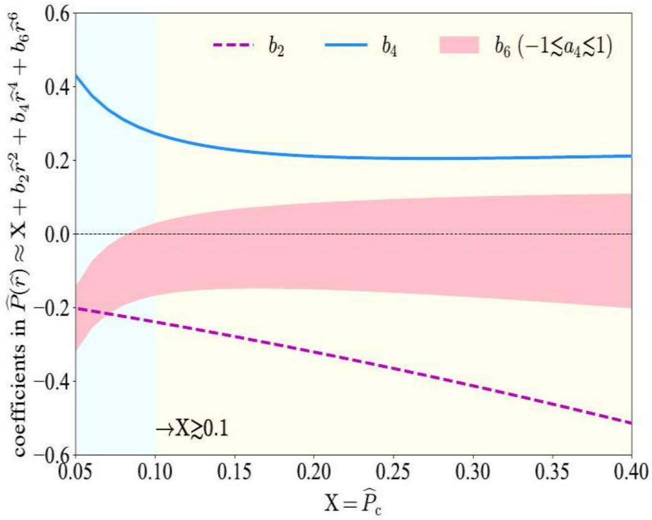
$$a_2 = b_2/s_c^2.$$

$$b_4 = -\frac{1}{2}b_2\left(\mathrm{X} + \frac{4 + 9\mathrm{X}}{15s_c^2}\right)$$

$$\hat{P}/\hat{\varepsilon} \approx \mathrm{X} - \frac{1}{6}\frac{1+\Psi}{4+\Psi}\left[1 + \frac{7+\Psi}{4+\Psi}\cdot 4\mathrm{X} + \frac{\Psi^2 + 14\Psi + 88}{(4+\Psi)^2}\cdot 3\mathrm{X}^2\right]\hat{r}^2$$

$$\hat{r}^3\hat{P}/\hat{M} \approx 3\mathrm{X} - \frac{1}{10}\frac{11+5\Psi}{4+\Psi}\left[1 + \frac{5\Psi^2 + 40\Psi + 53}{5\Psi^2 + 31\Psi + 44}\cdot 4\mathrm{X} + \frac{5\Psi^3 + 69\Psi^2 + 402\Psi + 392}{(11+5\Psi)(4+\Psi)^2}\cdot 3\mathrm{X}^2\right]\hat{r}^2$$

$$2\hat{M}/\hat{r} \approx \frac{2}{3}\hat{r}^2\left[1 - \frac{3}{10\mathrm{X}}\frac{1}{4+\Psi}\left[1 + \frac{3}{4+\Psi}\cdot 4\mathrm{X} - \frac{\Psi^2 + 18\Psi + 8}{(4+\Psi)^2}\cdot 3\mathrm{X}^2\right]\hat{r}^2\right]$$



$$\hat{P}/\hat{\varepsilon} \approx \mathrm{X} - \frac{1}{24}\left(1 + 7\mathrm{X} + \frac{33}{2}\mathrm{X}^2\right)\hat{r}^2$$

$$\hat{r}^3\hat{P}/\hat{M} \approx 3\mathrm{X} - \frac{11}{40}\left(1 + \frac{53}{11}\mathrm{X} + \frac{147}{22}\mathrm{X}^2\right)\hat{r}^2$$

$$2\hat{M}/\hat{r} \approx \frac{2}{3}\hat{r}^2\left[1 - \frac{3}{40\mathrm{X}}\left(1 + 3\mathrm{X} - \frac{3}{2}\mathrm{X}^2\right)\hat{r}^2\right]$$

$$\frac{1}{\omega^2}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}\theta}{\mathrm{d}\omega}\right) + \theta^n = 0$$

$$\omega = \frac{\hat{r}}{\sqrt{(n+1)X}};$$

$$P = K\varepsilon^{1+1/n} = K\varepsilon_{\mathrm{c}}^{1+1/n}\theta^{n+1} = P_{\mathrm{c}}\theta^{n+1},$$

$$\theta(\omega) = 1 - \frac{1}{6}\omega^2 + \frac{n}{120}\omega^4 + \cdots$$

$$P/P_{\mathrm{c}} = \theta^{n+1}(\omega) \approx 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{n}{n+1}\frac{1}{45\mathrm{X}^2}\hat{r}^4 + \cdots$$

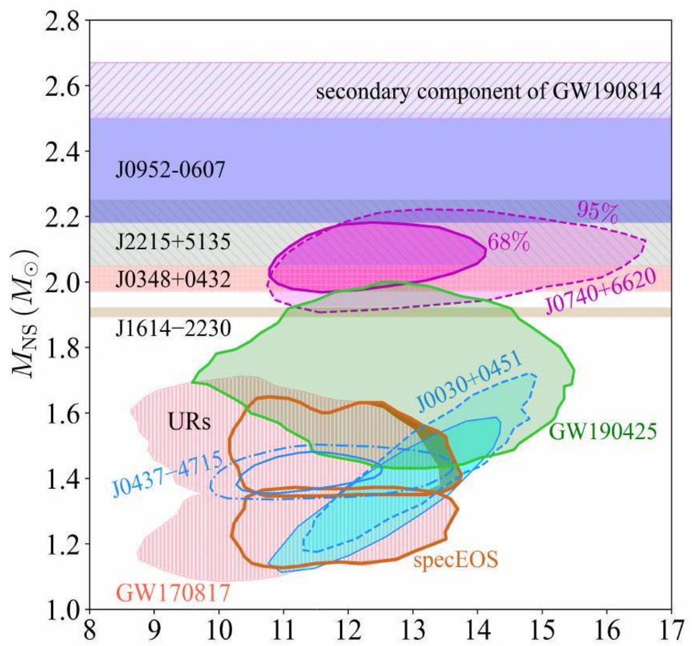
$$P/P_{\mathrm{c}} \approx 1 + \frac{b_2}{\mathrm{X}}\hat{r}^2 + \frac{b_4}{\mathrm{X}}\hat{r}^4 + \cdots \approx 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{1}{45\mathrm{X}s_{\mathrm{c}}^2}\hat{r}^4$$

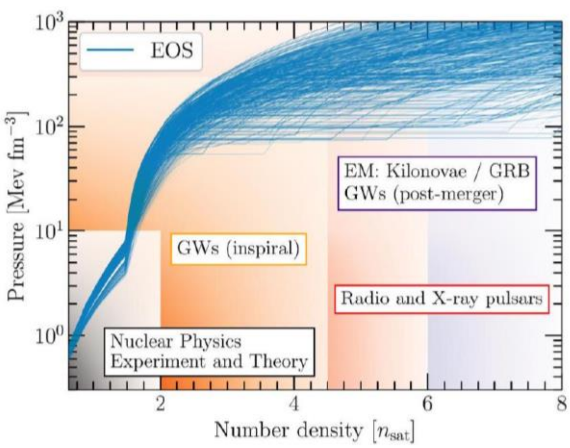
$$P/P_{\mathrm{c}} \approx 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{3}{4+\Psi}\frac{1}{45\mathrm{X}^2}\hat{r}^4 \overset{\Psi=0}{\rightarrow} 1 - \frac{1}{6\mathrm{X}}\hat{r}^2 + \frac{1}{60\mathrm{X}^2}\hat{r}^4$$

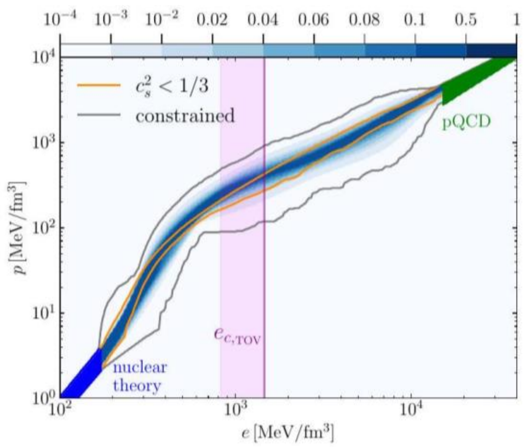
$$n = \frac{3}{1+\Psi};$$

$$\hat{p} = \sum_{k=1} d_k\hat{\varepsilon}^k \approx d_1\hat{\varepsilon} + d_2\hat{\varepsilon}^2 + d_3\hat{\varepsilon}^3 + \cdots$$

$$\mathrm{X} = \sum_{k=1} d_k, s_{\mathrm{c}}^2 = \left.\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}}\right|_{\hat{r}=0\leftrightarrow\hat{\varepsilon}_{\mathrm{c}}=1} = \sum_{k=1} kd_k.$$



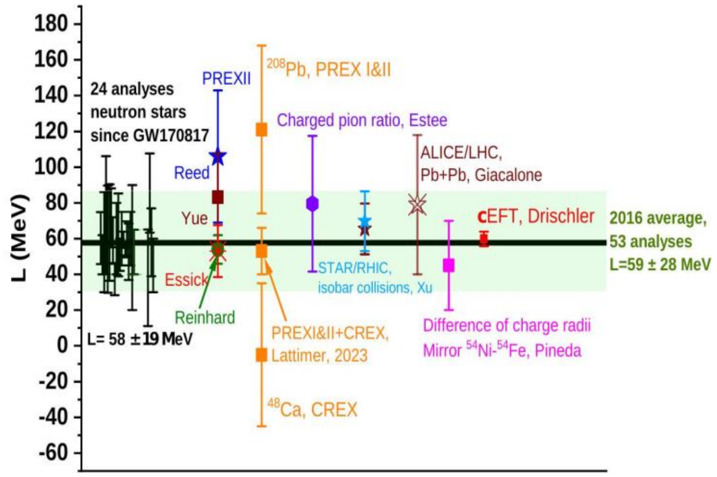


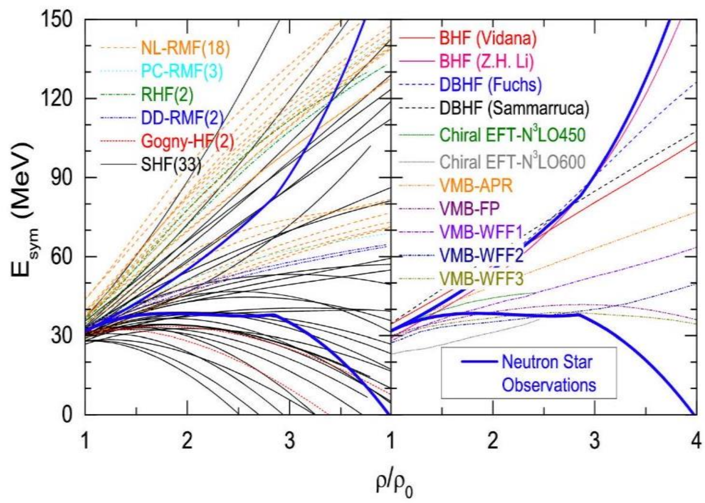


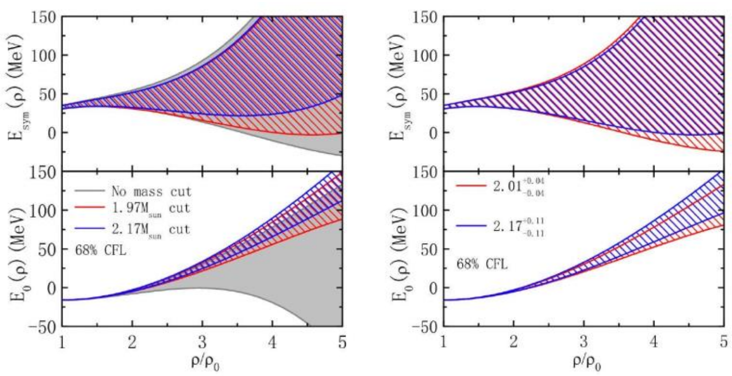
$$E_0(\rho) \approx E_0(\rho_0) + \frac{1}{2}K_0\chi^2 + \frac{1}{6}J_0\chi^3 + \mathcal{O}(\chi^4), \chi \equiv \frac{\rho - \rho_0}{3\rho_0}$$

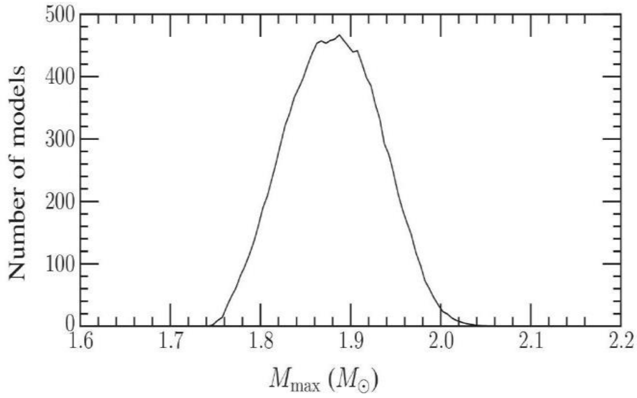
$$E_{\text{sym}}(\rho) \equiv \frac{1}{2}\frac{\partial^2 E(\rho,\delta)}{\partial \delta^2}\bigg|_{\delta=0} \approx S + L\chi + \frac{1}{2}K_{\text{sym}}\chi^2 + \frac{1}{6}J_{\text{sym}}\chi^3 + \mathcal{O}(\chi^4)$$

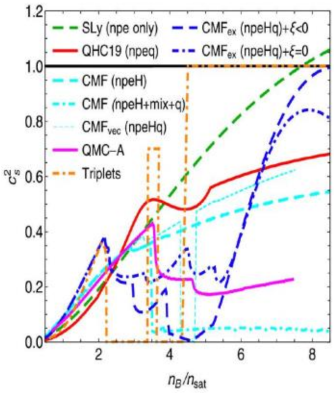
$$P(\rho,\delta) = \rho^2 \frac{\partial E(\rho,\delta)}{\partial \rho}$$

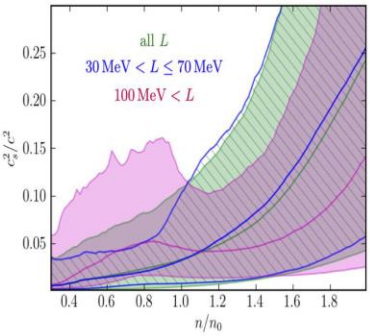


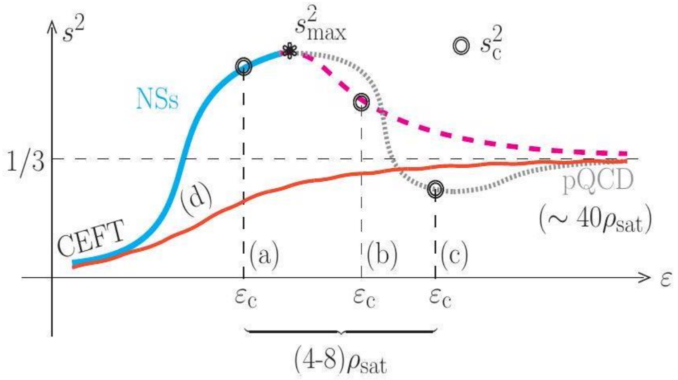












$$r \sim (P/\varepsilon)^{1/2}/\sqrt{G\varepsilon} \sim 1/\sqrt{G\varepsilon}$$

$$R \sim P^{\sigma},$$

$$R \sim \left(\mathrm{X}^{1/2}/\sqrt{\varepsilon_{\mathrm{c}}}\right) \cdot \vartheta(\mathrm{X}),$$

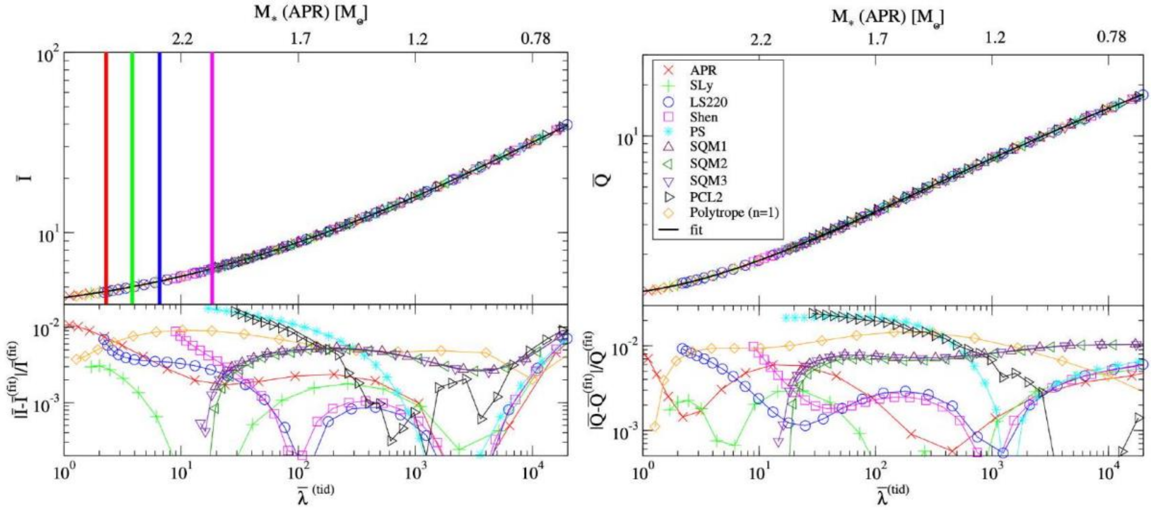
$$\hat{R} = \left(\frac{6X}{1+3X^2+4X}\right)^{1/2} \sim \left(\frac{X}{1+3X^2+4X}\right)^{1/2}, \text{from } \hat{P}(\hat{R}) = 0$$

$$\vartheta(X) = \left(\frac{1}{1+3X^2+4X}\right)^{1/2}$$

$$R = \hat{R}Q = \left(\frac{3}{2\pi G}\right)^{1/2} \nu_{\mathrm{c}} \sim \nu_{\mathrm{c}}, \text{ with } \nu_{\mathrm{c}} \equiv \frac{\mathrm{X}^{1/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}\left(\frac{1}{1+3\mathrm{X}^2+4\mathrm{X}}\right)^{1/2},$$

$$M_{\mathrm{NS}} \approx \frac{1}{3}\hat{R}^3 W = \left(\frac{6}{\pi G^3}\right)^{1/2} \Gamma_{\mathrm{c}} \sim \Gamma_{\mathrm{c}}, \text{ with } \Gamma_{\mathrm{c}} \equiv \frac{\mathrm{X}^{3/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}\left(\frac{1}{1+3\mathrm{X}^2+4\mathrm{X}}\right)^{3/2},$$

$$\xi = \frac{M_{\mathrm{NS}}}{R} \approx \frac{2\Pi_{\mathrm{c}}}{G} \sim \Pi_{\mathrm{c}}, \text{ with } \Pi_{\mathrm{c}} = \frac{\mathrm{X}}{1 + 3\mathrm{X}^2 + 4\mathrm{X}}.$$



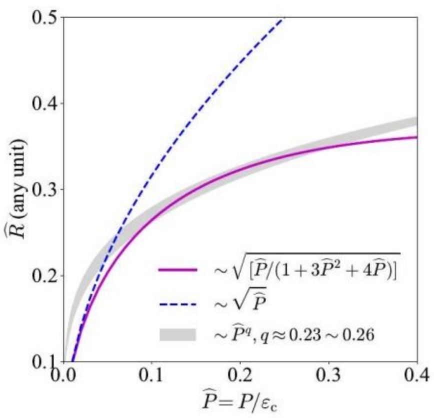
$$\vartheta \approx 1, M_{\mathrm{NS}} \sim \frac{\mathrm{X}^{3/2}}{\sqrt{\varepsilon_{\mathrm{c}}}} \sim P_{\mathrm{c}}^{3/2}\varepsilon_{\mathrm{c}}^{-2}, R \sim \frac{\mathrm{X}^{1/2}}{\sqrt{\varepsilon_{\mathrm{c}}}} \sim P_{\mathrm{c}}^{1/2}\varepsilon_{\mathrm{c}}^{-1}, \xi \approx 2\mathrm{X}.$$

$$P_{\mathrm{c}} \sim M_{\mathrm{NS}}^2/R^4.$$

$$I = -\frac{2}{3G}\int_0^R \mathrm{d}r r^3 \omega(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}j(r)\right) = \frac{8\pi}{3}\int_0^R \mathrm{d}r r^4 [\varepsilon(r) + P(r)] \exp\,[\lambda(r)] j(r)\omega(r)$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}}{\mathrm{d}r}\omega(r)\right) + 4r^3\omega(r)\frac{\mathrm{d}}{\mathrm{d}r}j(r) = 0$$

$$M_{\mathrm{NS}}^{\max} \sim D_{\mathrm{M}}\varepsilon_{\mathrm{c}}^{-1/2},$$



$$\left.\frac{\mathrm{d}M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_{\mathrm{c}}}\right|_{M_{\mathrm{NS}}=M_{\mathrm{NS}}^{\max}=M_{\mathrm{TOV}}} = 0, \left.\frac{\mathrm{d}^2 M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_{\mathrm{c}}^2}\right|_{M_{\mathrm{NS}}=M_{\mathrm{NS}}^{\max}=M_{\mathrm{TOV}}} < 0.$$

$$s_c^2 = X\left(1 + \frac{1+\Psi}{3}\frac{1+3X^2+4X}{1-3X^2}\right),$$

$$\Psi = \frac{2\varepsilon_{\mathrm{c}}}{M_{\mathrm{NS}}}\frac{\mathrm{d}M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_{\mathrm{c}}} = 2\frac{\mathrm{dln}\ M_{\mathrm{NS}}}{\mathrm{dln}\ \varepsilon_{\mathrm{c}}} \geq 0$$

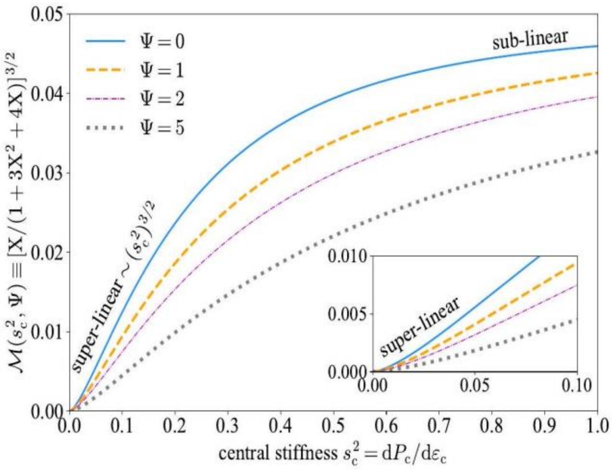
$$s_c^2 = \mathrm{X}\left(1 + \frac{1}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right).$$

$$s_{\mathrm{c}}^2 \le 1 \leftrightarrow \mathrm{X} \le 0.374 \equiv \mathrm{X}_+.$$

$$\frac{\mathrm{d}R}{\mathrm{d}\varepsilon_{\mathrm{c}}} = \frac{\mathrm{d}R}{\mathrm{d}M_{\mathrm{NS}}}\frac{\mathrm{d}M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_{\mathrm{c}}} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathrm{c}}}\left[\left(\frac{3}{2\pi G}\right)^{1/2} v_{\mathrm{c}}\right] = \left(\frac{R}{\varepsilon_{\mathrm{c}}}\right)\cdot\left(\frac{\Psi}{6}-\frac{1}{3}\right).$$

$$R \sim \varepsilon_{\mathrm{c}}^{\Psi/6-1/3}.$$

$$\begin{aligned}M_{\mathrm{NS}} &\sim \frac{1}{\sqrt{\varepsilon_{\mathrm{c}}}}\left(\frac{X(s_c^2,\Psi)}{1+3X^2(s_c^2,\Psi)+4X(s_c^2,\Psi)}\right)^{3/2} \equiv \frac{\mathcal{M}(s_c^2,\Psi)}{\sqrt{\varepsilon_{\mathrm{c}}}}\\ &= \frac{1}{\sqrt{\varepsilon_{\mathrm{c}}}}\cdot\frac{3\sqrt{3}s_c^3}{(4+\Psi)^{3/2}}\left[1-18\frac{5+2\Psi}{(4+\Psi)^2}s_c^2+\frac{81}{2}\frac{148+126\Psi+29\Psi^2}{(4+\Psi)^4}s_c^4+\cdots\right]\\ &\to \frac{1}{\sqrt{\varepsilon_{\mathrm{c}}}}\frac{3\sqrt{3}s_c^3}{8}\left(1-\frac{45s_c^2}{8}+\frac{2997s_c^4}{128}+\cdots\right)\end{aligned}$$



$$\mathrm{X} \approx \mathrm{X}_+(\Psi) + L_1\varphi + L_2\varphi^2 + \mathcal{O}(\varphi^3),$$

$$L_1 = \frac{[1-3\mathrm{X}_+^2(\Psi)]\mathrm{X}_+(\Psi)[1+\mathrm{X}_+(\Psi)][1+3\mathrm{X}_+(\Psi)]}{1+8\mathrm{X}_+(\Psi)+8\mathrm{X}_+^2(\Psi)-12\mathrm{X}_+^3(\Psi)-21\mathrm{X}_+^4(\Psi)},$$

$$L_2 = -\frac{2[1-3\mathrm{X}_+^2(\Psi)]\mathrm{X}_+^2(\Psi)[1-\mathrm{X}_+(\Psi)][1+\mathrm{X}_+(\Psi)]^2[1+3\mathrm{X}_+(\Psi)]^2[2+9\mathrm{X}_+(\Psi)+18\mathrm{X}_+^2(\Psi)+9\mathrm{X}_+^3(\Psi)]}{[1+8\mathrm{X}_+(\Psi)+8\mathrm{X}_+^2(\Psi)-12\mathrm{X}_+^3(\Psi)-21\mathrm{X}_+^4(\Psi)]^3},$$

$$\mathcal{M}(s_c^2,\Psi)/\mathcal{M}(1,\Psi) \approx 1 + T_1\varphi + T_2\varphi^2 + \mathcal{O}(\varphi^3), \mathcal{M}(1,\Psi) = \left(\frac{X_+(\Psi)}{1+3X_+^2(\Psi)+4X_+(\Psi)}\right)^{3/2}$$

$$\begin{aligned}T_1 &= \frac{3}{2}\frac{[1-3\mathrm{X}_+^2(\Psi)]^2}{1+8\mathrm{X}_+(\Psi)+8\mathrm{X}_+^2(\Psi)-12\mathrm{X}_+^3(\Psi)-21\mathrm{X}_+^4(\Psi)}\\ T_2 &= \frac{3}{8}\frac{[1-3\mathrm{X}_+^2(\Psi)]^2}{[1+8\mathrm{X}_+(\Psi)+8\mathrm{X}_+^2(\Psi)-12\mathrm{X}_+^3(\Psi)-21\mathrm{X}_+^4(\Psi)]^3}\times[1-24\mathrm{X}_+(\Psi)-282\mathrm{X}_+^2(\Psi)-820\mathrm{X}_+^3(\Psi)\\ &\quad -504\mathrm{X}_+^4(\Psi)+1344\mathrm{X}_+^5(\Psi)+1746\mathrm{X}_+^6(\Psi)-324\mathrm{X}_+^7(\Psi)-945\mathrm{X}_+^8(\Psi)]\end{aligned}$$

$$\frac{\mathrm{d}^2M_{\mathrm{NS}}}{\mathrm{d}\varepsilon_{\mathrm{c}}^2} \sim \left(1-\frac{s_c^2}{\mathrm{X}}\right)\left[\left(\frac{s_c^2}{\mathrm{X}}-\frac{\mathrm{d}s_c^2}{\mathrm{dX}}\right)+\mathrm{X}\left(1-\frac{s_c^2}{\mathrm{X}}\right)\frac{12\mathrm{X}^2+12\mathrm{X}+4}{9\mathrm{X}^4+12\mathrm{X}^3-4\mathrm{X}-1}\right]$$

$$\left.\frac{\mathrm{d}s_c^2}{\mathrm{dX}}\right|_{M_{\mathrm{NS}}^{\max}} < \sigma_c^2 \equiv \frac{\mathrm{d}}{\mathrm{dX}}\left[\mathrm{X}\left(1+\frac{1}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right)\right] = \frac{2}{3}\frac{9\mathrm{X}^4-3\mathrm{X}^2+4\mathrm{X}+2}{(3\mathrm{X}^2-1)^2},$$

$$R_{\max}/\mathrm{nm} \approx A_{\mathrm{R}}^{\max}v_{\mathrm{c}} + B_{\mathrm{R}}^{\max} \approx 1.05_{-0.03}^{+0.03}\times 10^3\left(\frac{v_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) + 0.64_{-0.25}^{+0.25},$$

$$M_{\mathrm{NS}}^{\max}/M_{\odot} \approx A_{\mathrm{M}}^{\max}\Gamma_{\mathrm{c}} + B_{\mathrm{M}}^{\max} \approx 1.73_{-0.03}^{+0.03}\times 10^3\left(\frac{\Gamma_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) - 0.106_{-0.035}^{+0.035}$$

$$\frac{M_{\mathrm{NS}}^{\max}}{M_{\odot}} \approx \frac{1.65\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}\left(\frac{R_{\max}}{\mathrm{nm}}-0.64\right)-0.106.$$

radius (nm)	$10^3\mathcal{V}_{\mathrm{c}}$	ε_{c}	P_{c}	s_{c}^2
$12.39^{+1.30}_{-0.98}$	$11.2^{+1.2}_{-0.9}$	901^{+214}_{-287}	218^{+93}_{-125}	$0.45^{+0.14}_{-0.18}$
$13.7^{+2.6}_{-1.5}$	$12.4^{+2.5}_{-1.4}$	656^{+187}_{-339}	124^{+53}_{-99}	$0.32^{+0.08}_{-0.14}$
$12.90^{+1.25}_{-0.97}$	$11.7^{+1.2}_{-0.9}$	794^{+181}_{-235}	173^{+69}_{-89}	$0.39^{+0.09}_{-0.13}$
$12.49^{+1.28}_{-0.88}$	$11.3^{+1.3}_{-0.9}$	879^{+208}_{-312}	208^{+94}_{-140}	$0.44^{+0.14}_{-0.21}$
$12.76^{+1.49}_{-1.02}$	$11.5^{+1.8}_{-1.2}$	822^{+255}_{-383}	184^{+105}_{-157}	$0.40^{+0.15}_{-0.22}$

$$P_{\mathrm{c}}(\varepsilon_{\mathrm{c}}) \approx f_{\mathrm{M}}^{2/3}\varepsilon_{\mathrm{c}}^{4/3}\cdot\left(1+4f_{\mathrm{M}}^{2/3}\varepsilon_{\mathrm{c}}^{1/3}+19f_{\mathrm{M}}^{4/3}\varepsilon_{\mathrm{c}}^{2/3}+100f_{\mathrm{M}}^{2}\varepsilon_{\mathrm{c}}+\cdots\right),$$
$$P_{\mathrm{c}}(\varepsilon_{\mathrm{c}}) \approx f_{\mathrm{R}}^{2}\varepsilon_{\mathrm{c}}^{2}\cdot\left(1+4f_{\mathrm{R}}^{2}\varepsilon_{\mathrm{c}}+19f_{\mathrm{R}}^{4}\varepsilon_{\mathrm{c}}^{2}+100f_{\mathrm{R}}^{6}\varepsilon_{\mathrm{c}}^{3}+\cdots\right),$$

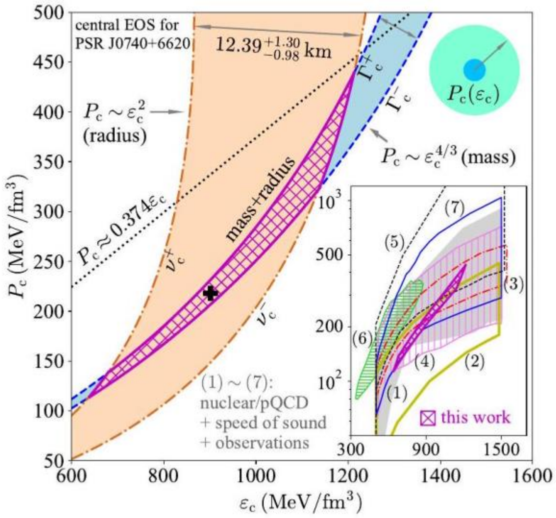
$$f_{\mathrm{M}}/\left[\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}\right]=\left(M_{\mathrm{NS}}^{\max}/M_{\odot}+0.106\right)/1730$$

$$f_{\mathrm{R}}/\left[\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}\right]=\left(R_{\max}/\mathrm{nm}-0.64\right)/1050$$

$$\mathrm{X}\approx 0.24^{+0.05}_{-0.07},\ \text{under } R_{\max}\approx 12.39^{+1.30}_{-0.98}\ \mathrm{nm}.$$

$$\gamma_{\mathrm{c}}^{(\mathrm{M})}=\frac{s_{\mathrm{c}}^2}{\mathrm{X}}\approx\frac{4}{3}\left(1+f_{\mathrm{M}}^{2/3}\varepsilon_{\mathrm{c}}^{1/3}+\frac{11}{2}f_{\mathrm{M}}^{4/3}\varepsilon_{\mathrm{c}}^{3/3}+34f_{\mathrm{M}}^{2}\varepsilon_{\mathrm{c}}+\cdots\right), f_{\mathrm{M}}\approx\left(\frac{M_{\mathrm{NS}}^{\max}+0.106}{1730}\right)\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2},$$

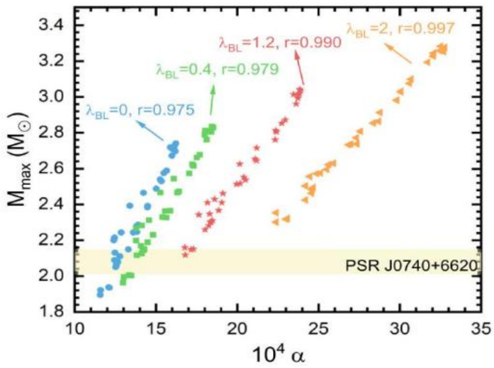
$$\gamma_{\mathrm{c}}^{(\mathrm{R})}\approx 2\left(1+2f_{\mathrm{R}}^{2}\varepsilon_{\mathrm{c}}+11f_{\mathrm{R}}^{4}\varepsilon_{\mathrm{c}}^{2}+68f_{\mathrm{R}}^{6}\varepsilon_{\mathrm{c}}^{3}+\cdots\right), f_{\mathrm{R}}\approx\left(\frac{M_{\mathrm{NS}}^{\max}-0.64}{1050}\right)\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}.$$

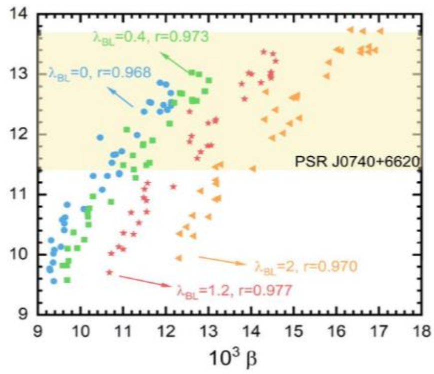


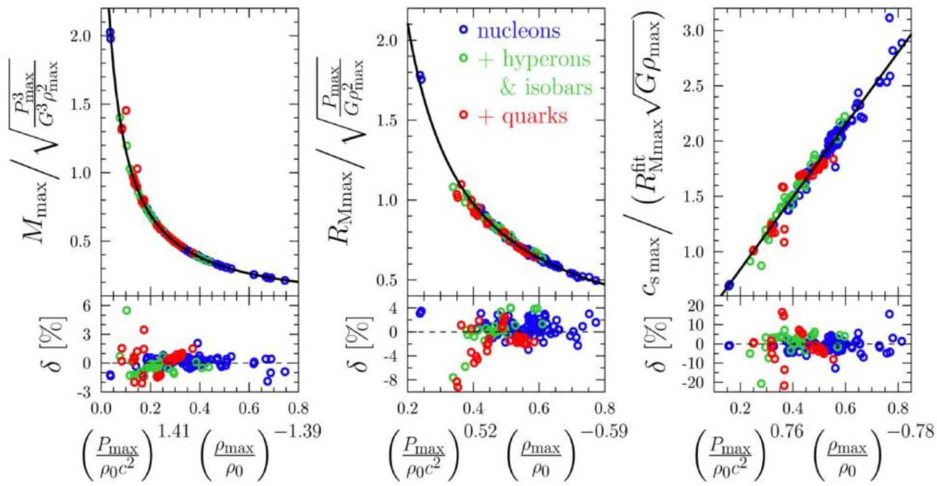
$$P_{\mathrm{c}}(\varepsilon_{\mathrm{c}})\approx 0.012\varepsilon_{\mathrm{c}}^{4/3}\cdot\left(1+0.047\varepsilon_{\mathrm{c}}^{1/3}+0.0026\varepsilon_{\mathrm{c}}^{2/3}+0.00016\varepsilon_{\mathrm{c}}+\cdots\right)$$

$$\delta P=P_{\mathrm{t}}-P=\frac{\lambda_{\mathrm{BL}}}{3}\frac{(\varepsilon+3P)(\varepsilon+P)r^3}{r-2M},$$

$$M_{\mathrm{NS}}^{\max}\sim\alpha\equiv\Gamma_{\mathrm{c}}\left(1-\frac{\lambda_{\mathrm{BL}}}{2\pi}\right)^{-3/2}, R\sim\beta\equiv\nu_{\mathrm{c}}\left(1-\frac{\lambda_{\mathrm{BL}}}{2\pi}\right)^{-1/2},$$







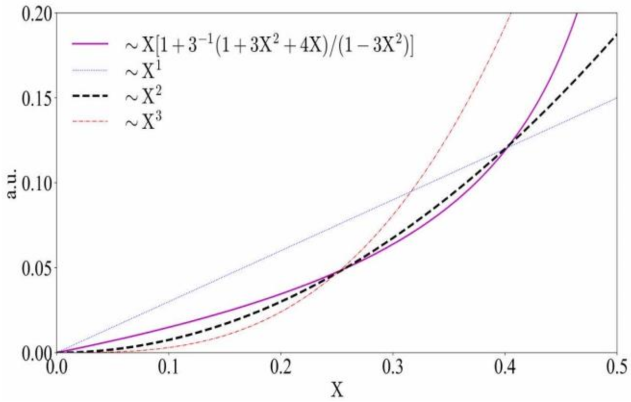
$$M_{\mathrm{NS}}^{\max} \approx \frac{a_{\mathrm{M}} \varepsilon_{\mathrm{c}}^{-1/2} \mathrm{X}^{3/2}}{b_{\mathrm{M}} + c_{\mathrm{M}} \mathrm{X}^{p_{\mathrm{M}}} \varepsilon_{\mathrm{c}}^{q_{\mathrm{M}}}}, R_{\max} \approx \frac{a_{\mathrm{R}} \varepsilon_{\mathrm{c}}^{-1/2} \mathrm{X}^{1/2}}{b_{\mathrm{R}} + c_{\mathrm{R}} \mathrm{X}^{p_{\mathrm{R}}} \varepsilon_{\mathrm{c}}^{q_{\mathrm{R}}}}, \mathrm{X} = P_{\mathrm{c}}/\varepsilon_{\mathrm{c}},$$

$$s^2 = \mathrm{d}\hat{P}/\mathrm{d}\hat{\varepsilon} = b_2/a_2 = \mathrm{X}.$$

$$s_{\mathrm{c}}^2 = \frac{b_2}{a_2} = \frac{\mathrm{X} + b_4 \hat{R}^4}{1 + a_4 \hat{R}^4}, s_{\mathrm{c}}^2 - \mathrm{X} = \frac{(b_4 - a_4 \mathrm{X})\hat{R}^4}{1 + a_4 \hat{R}^4}$$

$$s_{\mathrm{surf}}^2 = \frac{b_2 + 2b_4 \hat{R}^2}{a_2 + 2a_4 \hat{R}^2} = \frac{X - b_4 \hat{R}^4}{1 - a_4 \hat{R}^4}$$

$$s_{\mathrm{c}}^2 - s_{\mathrm{surf}}^2 = \frac{\mathrm{X} + b_4 \hat{R}^4}{1 + a_4 \hat{R}^4} - \frac{\mathrm{X} - b_4 \hat{R}^4}{1 - a_4 \hat{R}^4} = \frac{2(b_4 - a_4 \mathrm{X})\hat{R}^4}{(1 - a_4 \hat{R}^4)(1 + a_4 \hat{R}^4)} = \frac{2(s_{\mathrm{c}}^2 - \mathrm{X})}{1 - a_4 \hat{R}^4}$$



$$\mathcal{O} = a[\mathcal{O}]\left(\frac{M_{NS}^{max}}{M_{\odot}}\right)^{b[\mathcal{O}]}\left(\frac{R_{max}}{nm}\right)^{c[\mathcal{O}]}$$

$$s_c^2 = X\left(1+\frac{1}{3}\frac{1+3X^2+4X}{1-3X^2}\right) \sim X^k, \text{with } k \approx 2,$$

$$P_c \sim (M_{NS}^{max})^{1+q^{-1}} R_{max}^{-3-q^{-1}} \sim M_{NS}^{max,3}/R_{max}^5, \text{with } q \approx 1/2.$$

$$s_c \sim X^{k/2} \sim (M_{NS}^{max}/R_{max})^{k/2q} \approx (M_{NS}^{max}/R_{max})^2, \text{with } q \approx 1/2, k \approx 2.$$

$$\frac{\rho_c}{\rho_{sat}} \approx \frac{7.35\times 10^3 X}{1+3X^2+4X}\left(\frac{R_{max}}{nm} - 0.64\right)^{-2}$$

$$\frac{\rho_c}{\rho_{sat}} \approx \bar{d}_0\left[1-\left(\frac{R_{max}}{nm}\right)\right] + \bar{d}_1\left(\frac{R_{max}}{nm}\right)^2$$

$$\rho_c/\rho_{sat} \sim R_{max}^{-2}\cdot[1+\text{corrections of } R_{max}^{-1}]$$

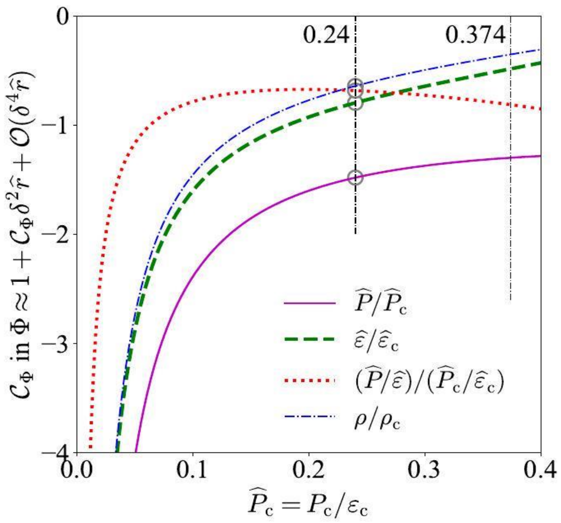
$$\rho_c/\rho_{sat} \approx 2\times 10^4\left(\frac{X}{1+3X^2+4X}\right)^3\left(\frac{M_{NS}^{max}}{M_{\odot}}+0.106\right)^{-2} \sim M_{NS}^{max,-2}\cdot\left[1+\text{ corrections of } M_{NS}^{max,-1}\right]$$

$$\hat{\rho} \equiv \rho/\rho_c \approx 1+\left(\frac{b_2/s_c^2}{1+X}\right)\hat{r}^2 + \frac{1}{1+X}\left(a_4 - \frac{b_2^2/2s_c^2}{1+X}\right)\hat{r}^4$$

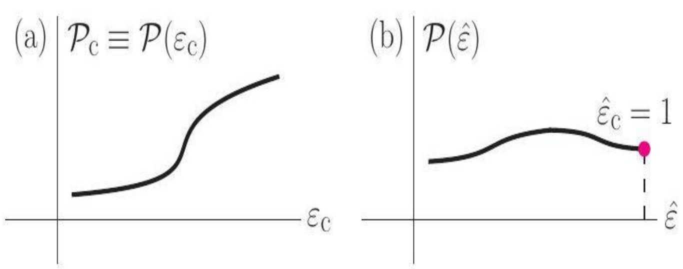
$$\rho/\rho_c \approx \hat{\varepsilon} - \mu\left(1+\frac{4}{3}\mu\right)X(1-X),$$

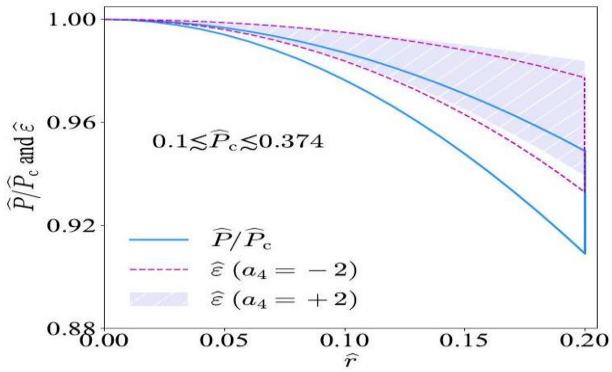
$$\hat{p}/X \approx 1+\frac{b_2}{X}\hat{r}^2, \hat{\varepsilon}/\hat{\varepsilon}_c \approx 1+\frac{b_2}{s_c^2}\hat{r}^2,$$

$$\phi/X \approx 1+b_2\left(\frac{1}{X}-\frac{1}{s_c^2}\right)\hat{r}^2 = 1+\frac{b_2}{X}\left[1-\left(1+\frac{1}{3}\frac{1+3X^2+4X}{1-3X^2}\right)^{-1}\right]\hat{r}^2,$$



$$a_4 \le -\frac{b_2}{2s_c^2}\frac{1}{\hat{R}^2}, \text{ and } a_4 \le \frac{b_2}{2s_c^2}\left(\frac{b_2}{1+\mathrm{X}} - \frac{1}{\hat{R}^2}\right),$$





$$\hat{p}/\mathrm{X} = P/P_c \approx 1 + \frac{4}{3}\mu + \frac{16}{15}\mu^2 + \frac{4}{15}\mu^3 + \left(\frac{4}{3} - \frac{4}{5}\mu - \frac{268}{135}\mu^2\right)\mu\mathrm{X}$$
$$+\left[2 + \left(\frac{28}{3} - \frac{256a_4}{3}\right)\mu + \left(\frac{370}{27} - \frac{30208}{315}a_4\right)\mu^2\right]\mu\mathrm{X}^2$$
$$+\left[4 + \left(\frac{1280a_4}{3} - \frac{262}{15}\right)\mu + \left(\frac{68608}{105}a_4 + \frac{2048}{3}a_6 - \frac{1496}{27}\right)\mu^2\right]\mu\mathrm{X}^3 + \mathcal{O}(\mathrm{X}^4, \mu^4)$$

$$\gamma_c \equiv \frac{\mathrm{dln}\ P}{\mathrm{dln}\ \varepsilon}\bigg|_{\text{center}} = s_c^2/\mathrm{X} = 1,$$

$$\phi \approx \frac{\mathrm{X}}{1+\mu}\left[1 + \frac{4\mu}{3} + \frac{16\mu^2}{15} + \left(\frac{4}{3} - \frac{4\mu}{5}\right)\mu\mathrm{X}\right] \approx \mathrm{X}\left[1 + \frac{\mu}{3}(4\mathrm{X}+1) + \frac{\mu^2}{15}(11 - 32\mathrm{X})\right],$$

$$\mu \approx \frac{3}{4X+1}\left(\frac{\phi}{X} - 1\right) + \frac{9}{5}\frac{32X - 11}{4X+1}\left(\frac{\phi}{X} - 1\right)^2.$$

$$s^2/\phi_c \approx \frac{4}{3} + \frac{32}{5}\left(1 - \frac{19}{4}\mathrm{X}\right)\left(\frac{\phi}{\phi_c} - 1\right) - \frac{876}{25}\left(1 - \frac{3439}{219}\mathrm{X}\right)\left(\frac{\phi}{\phi_c} - 1\right)^2, \phi_c \equiv \mathrm{X}.$$

$$\varepsilon(\rho,\delta) = [E(\rho,\delta) + M_{\mathrm{N}}]\rho + \varepsilon_\ell(\rho,\delta),$$

$$\mu_{\mathrm{n}} - \mu_{\mathrm{p}} = \mu_{\mathrm{e}} \approx \mu_\mu \approx 4\delta E_{\mathrm{sym}}(\rho),$$

$$\begin{aligned}
\xi \ &\approx A_\xi \Pi_{\mathrm{c}} + B_\xi \approx 2.31^{+0.03}_{-0.03}\Pi_{\mathrm{c}} - 0.032^{+0.003}_{-0.003}\\
M_{\mathrm{NS}}/M_\odot \ &\approx A_{\mathrm{M}} + B_{\mathrm{M}} \approx 1242^{+15}_{-15}\left(\frac{\Gamma_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) - 0.08^{+0.02}_{-0.02}\\
R/\mathrm{nm} \ &\approx A_{\mathrm{R}}\nu_{\mathrm{c}} + B_{\mathrm{R}} \approx 572^{+25}_{-25}\left(\frac{\nu_{\mathrm{c}}}{\mathrm{fm}^{3/2}/\mathrm{MeV}^{1/2}}\right) + 4.22^{+0.35}_{-0.35}
\end{aligned}$$

$$\xi \lesssim 0.264^{+0.005}_{-0.005} \equiv \xi_{\mathrm{GR}},$$

$$M_{\mathrm{NS}} \sim \frac{1}{\sqrt{\varepsilon_{\mathrm{c}}}}\left(\frac{\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}}\right)^{3/2} \cdot (1 + \kappa_1\mathrm{X} + \kappa_2\mathrm{X}^2 + \cdots)$$

$$\xi \sim \frac{X}{1+3X^2+4X}\cdot\left(1+\frac{18}{25}X\right)$$

$$\varepsilon_{\mathrm{c}} = \left(\frac{\xi - B_\xi}{A_\xi}\right)^3\left(\frac{A_{\mathrm{M}}}{M_{\mathrm{NS}}/M_\odot - B_{\mathrm{M}}}\right)^2 = \underbrace{\left(\frac{X}{1+3X^2+4X}\right)^3}_{\text{upper bounded}}\left(\frac{A_{\mathrm{M}}}{M_{\mathrm{NS}}/M_\odot - B_{\mathrm{M}}}\right)^2$$

$$\Psi \approx a_\Psi\left(\frac{M_{\mathrm{NS}}}{M_\odot}\right) + b_\Psi \approx -1.62^{+0.13}_{-0.13}\left(\frac{M_{\mathrm{NS}}}{M_\odot}\right) + 5.12^{+0.22}_{-0.22}$$

$$M_{\mathrm{NS}}/M_\odot \sim \varepsilon_{\mathrm{c}}^{\Psi/2}.$$

$$M_{\mathrm{NS}} \approx 1.4M_\odot\colon M_{\mathrm{NS}}/M_\odot \sim \varepsilon_{\mathrm{c}}^{1.43\pm0.14}$$

$$\text{at } M_{\mathrm{NS}} \approx 1.4M_\odot\colon R \sim \varepsilon_{\mathrm{c}}^{0.14\pm0.05}.$$

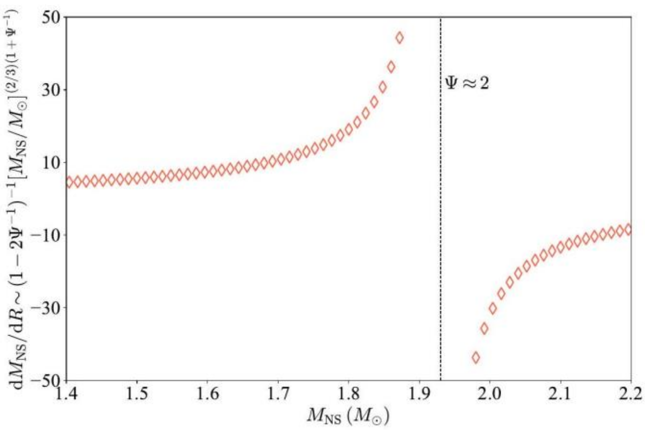
$$\mathrm{d}R/\mathrm{d}\varepsilon_{\mathrm{c}} = \mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \cdot \mathrm{d}M_{\mathrm{NS}}/\mathrm{d}\varepsilon_{\mathrm{c}}$$

$$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} = \mathrm{d}R/\mathrm{d}\varepsilon_{\mathrm{c}} \cdot (\mathrm{d}M_{\mathrm{NS}}/\mathrm{d}\varepsilon_{\mathrm{c}})^{-1}$$

$$\frac{\mathrm{d}R}{\mathrm{d}M_{\mathrm{NS}}} = \text{ב} \times \left(1-\frac{2}{\Psi}\right)\cdot\varepsilon_{\mathrm{c}}^{-3^{-1}(1+\Psi)} = \text{ג} \times \left(1-\frac{2}{\Psi}\right)\cdot\left(\frac{M_{\mathrm{NS}}}{M_\odot}\right)^{-(2/3)(1+\Psi^{-1})}$$

$$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \sim (1-2\Psi^{-1})\cdot R^{-2(1+\Psi^{-1})/(1-2\Psi^{-1})},$$

$$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \approx 3^{-1}\xi^{-1}(1-2\Psi^{-1})$$



$$\mathrm{Y} \equiv \frac{\varepsilon_{\mathrm{c}}}{\varepsilon_0} \lesssim \frac{21.71}{(M_{\mathrm{NS}}/M_\odot + 0.08)^2} \equiv \mathrm{Y}_+ \sim M_{\mathrm{NS}}^{-2},\ \text{under X} \lesssim 0.374 \leftrightarrow \Delta_{\mathrm{c}} \gtrsim \Delta_{\mathrm{GR}} \approx -0.041$$

$$\mathrm{Y} \lesssim 51\left(M_{\mathrm{NS}}/M_{\odot}\right)^{-2},$$

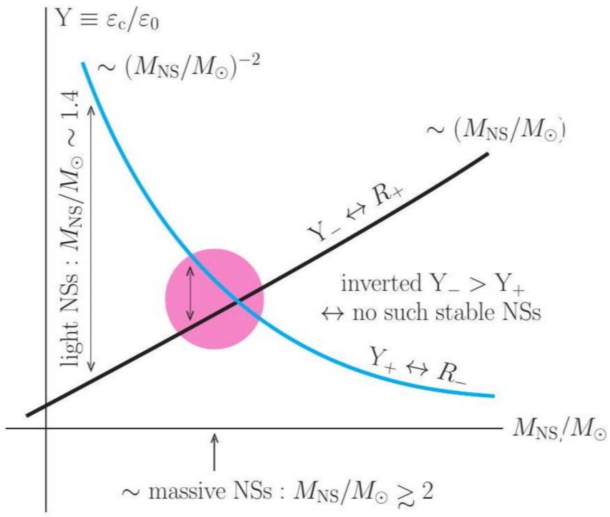
$$R/\mathrm{nm} = \frac{1.477 M_{\mathrm{NS}}/M_{\odot}}{A_{\xi}\Pi_{\mathrm{c}} + B_{\xi}} \gtrsim 5.58 M_{\mathrm{NS}}/M_{\odot}$$

$$Y_{+} \leftrightarrow R_{-},$$

$$\mathrm{d}M_{\mathrm{NS}}/\mathrm{d}\varepsilon_{\mathrm{c}} > 0 \leftrightarrow \mathrm{dY}/\mathrm{d}\left(M_{\mathrm{NS}}/M_{\odot}\right) > 0,$$

$$R/\mathrm{nm} = \frac{\Sigma M_{\mathrm{NS}}/M_{\odot}}{A_{\xi}\left(\frac{M_{\mathrm{NS}}/M_{\odot} - B_{\mathrm{M}}}{A_{\mathrm{M}}}\sqrt{Y\varepsilon_{0}}\right)^{2/3} + B_{\xi}} \approx \frac{1.477 M_{\mathrm{NS}}/M_{\odot}}{0.106\left(M_{\mathrm{NS}}/M_{\odot} + 0.08\right)^{2/3} Y^{1/3} - 0.032},$$

$$Y_{-} \leftrightarrow R_{+},$$



$$\begin{aligned} \mathrm{X}(s_{\mathrm{c}}^{2}, \Psi) &\approx \frac{3s_{\mathrm{c}}^{2}}{4+\Psi}\left[1 - \frac{12(1+\Psi)}{(4+\Psi)^{2}}s_{\mathrm{c}}^{2} + \frac{18(1+\Psi)(4+13\Psi)}{(4+\Psi)^{4}}s_{\mathrm{c}}^{4} + \cdots\right] \\ &\rightarrow \frac{3s_{\mathrm{c}}^{2}}{4}\left(1 - \frac{3}{4}s_{\mathrm{c}}^{2} + \frac{9}{32}s_{\mathrm{c}}^{4} + \cdots\right) \end{aligned}$$

$$\begin{aligned} \Pi_{\mathrm{c}}(s_{\mathrm{c}}^{2}, \Psi) &\approx \frac{3s_{\mathrm{c}}^{2}}{4+\Psi}\left[1 - \frac{12(5+2\Psi)}{(4+\Psi)^{2}}s_{\mathrm{c}}^{2} + \frac{9(344+298\Psi+71\Psi^{2})}{(4+\Psi)^{4}}s_{\mathrm{c}}^{4} + \cdots\right] \\ &\rightarrow \frac{3s_{\mathrm{c}}^{2}}{4}\left(1 - \frac{15}{4}s_{\mathrm{c}}^{2} + \frac{387}{32}s_{\mathrm{c}}^{4} + \cdots\right) \end{aligned}$$

$$R/\mathrm{nm} \gtrsim 3.59 M_{\mathrm{NS}}/M_{\odot} + 4.51,$$

$$R/\mathrm{nm} = \frac{(A_{\mathrm{M}}B_{\mathrm{R}}\Pi_{\mathrm{c}} - A_{\mathrm{R}}B_{\mathrm{M}})\Sigma}{(A_{\mathrm{M}}\Sigma - A_{\xi}A_{\mathrm{R}})\Pi_{\mathrm{c}} - A_{\mathrm{R}}B_{\xi}},$$

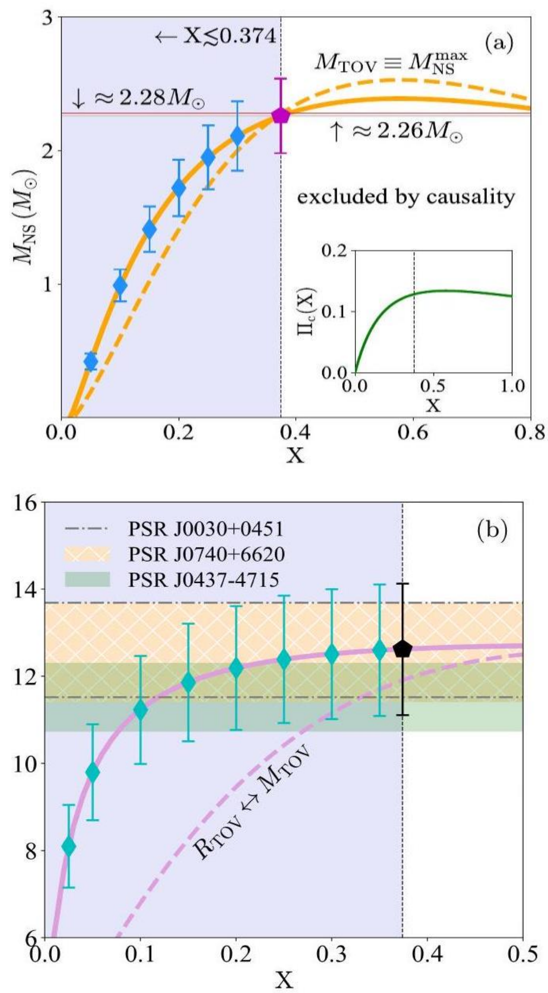
$$\frac{R\Sigma^{-1}}{\mathrm{nm}} \approx \underbrace{\frac{A_{\mathrm{M}}B_{\mathrm{R}}}{A_{\mathrm{M}}\Sigma - A_{\xi}A_{\mathrm{R}}}}_{\text{no }\Pi_{\mathrm{c}}\text{ factor}} \times [1 + \underbrace{\overbrace{\left(\frac{A_{\mathrm{R}}B_{\xi}}{A_{\mathrm{M}}\Sigma - A_{\xi}A_{\mathrm{R}}} - \frac{A_{\mathrm{R}}B_{\mathrm{M}}}{A_{\mathrm{M}}B_{\mathrm{R}}}\right)}^{\text{negative: }-0.027\pm0.007}}_{\text{upper bounded}}\frac{1}{\Pi_{\mathrm{c}}} + \mathcal{O}\left(\frac{1}{\Pi_{\mathrm{c}}^{2}}\right)].$$

$$\left(\frac{A_{\mathrm{R}}B_{\xi}}{A_{\mathrm{M}}\Sigma - A_{\xi}A_{\mathrm{R}}} - \frac{A_{\mathrm{R}}B_{\mathrm{M}}}{A_{\mathrm{M}}B_{\mathrm{R}}}\right) < 0,$$

$$\frac{R\Sigma^{-1}}{\text{nm}} \approx \frac{A_\text{M} B_\text{R}}{A_\text{M}\Sigma - A_\xi A_\text{R}}$$

$$\left(\frac{M_\text{NS}}{M_\odot}\right) = \frac{(A_\xi \Pi_\text{c} + B_\xi)(A_\text{M} B_\text{R} \Pi_\text{c} - A_\text{R} B_\text{M})}{(A_\text{M}\Sigma - A_\xi A_\text{R})\Pi_\text{c} - A_\text{R} B_\xi} \approx \frac{A_\xi A_\text{M} B_\text{R}}{A_\text{M}\Sigma - A_\xi A_\text{R}} \Pi_\text{c} \approx \left(\frac{R\Sigma^{-1}}{\text{nm}}\right) A_\xi \Pi_\text{c}$$

$$M_\text{NS}/M_\odot \lesssim 2.26 \pm 0.28, R/\text{nm} \lesssim 12.62 \pm 1.51.$$



$$\xi_\text{max} \equiv \xi_\text{TOV} \approx A_\xi^\text{max} \Pi_\text{c} + B_\xi^\text{max} \approx 2.59\Pi_\text{c} - 0.05.$$

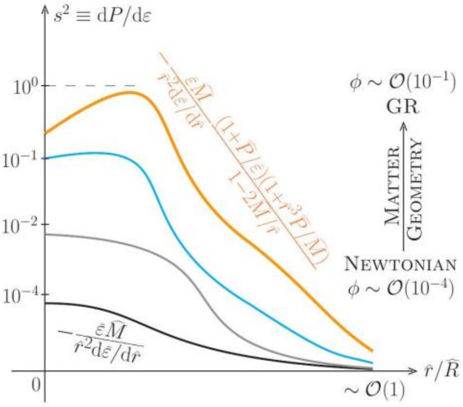
$$M_\text{TOV}/M_\odot \approx 2.28 \pm 0.28, R_\text{TOV}/\text{nm} \approx 11.91 \pm 1.51.$$

$$s^2 \equiv \text{d}P/\text{d}\varepsilon = \text{d}\hat{P}/\text{d}\hat{\varepsilon} = \phi f(\phi), \phi = P/\varepsilon,$$

$$f \approx f_0 + f_1\phi + f_2\phi^2 + \cdots,$$

$$P(\rho_0) \approx P_0(\rho_0) + P_\text{sym}(\rho_0)\delta^2 \approx 3^{-1} L \rho_0 \delta^2 \lesssim 3\text{MeV/fm}^3,$$

$$f_0 \approx s^2/\phi \gtrsim 1 \sim 2.$$



$$\psi(\hat{\varepsilon}) \approx C_1 + C_0\frac{f_1}{f_0-1}\hat{\varepsilon}^{f_0-1}$$

$$\hat{\varepsilon}_* \approx \left(-\frac{2C_0f_1}{f_0-1}\right)^{-\frac{1}{f_0-1}}, s^2(\hat{\varepsilon}_*) \approx C_0f_0\hat{\varepsilon}_*^{f_0-1}\left(1+\frac{C_0f_1}{f_0-1}\hat{\varepsilon}_*^{f_0-1}\right) = \frac{f_0(1-f_0)}{4f_1}, f_0 > 1,$$

$$\left[\frac{\mathrm{d}^2s^2}{\mathrm{d}\hat{\varepsilon}^2}\right]_{\hat{\varepsilon}_*} \approx C_0f_0\hat{\varepsilon}_*^{f_0-3}\left[f_0^2-3f_0+2+4C_0\left(f_0-\frac{3}{2}\right)f_1\hat{\varepsilon}_*^{f_0-1}\right] = -C_0f_0(f_0-1)^2\left(-\frac{2C_0f_1}{f_0-1}\right)^{-\frac{f_0-3}{f_0-1}} < 0$$

$$-\left(\frac{1}{2}\right)^{\frac{f_0-3}{1-f_0}}(C_0f_0)^{\frac{2}{f_0-1}}(1-f_0)^2 \lesssim \left[\frac{\mathrm{d}^2s^2}{\mathrm{d}\hat{\varepsilon}^2}\right]_{\hat{\varepsilon}_*} \lesssim -C_0f_0(1-f_0)^2\left(\frac{1}{2}\frac{1}{f_0-1}\right)^{\frac{f_0-3}{1-f_0}}$$

$$s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2\,\mathrm{d}\hat{\varepsilon}/\mathrm{d}\hat{r}}\frac{(1+\hat{P}/\hat{\varepsilon})(1+\hat{r}^3\hat{P}/\widehat{M})}{1-2\widehat{M}/\hat{r}}$$

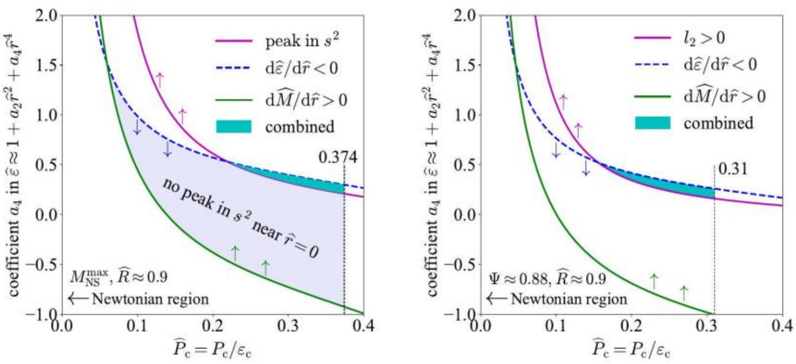
$$s^2 = \frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{\varepsilon}} = -\frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2\,\mathrm{d}\hat{\varepsilon}/\mathrm{d}\hat{r}}.$$

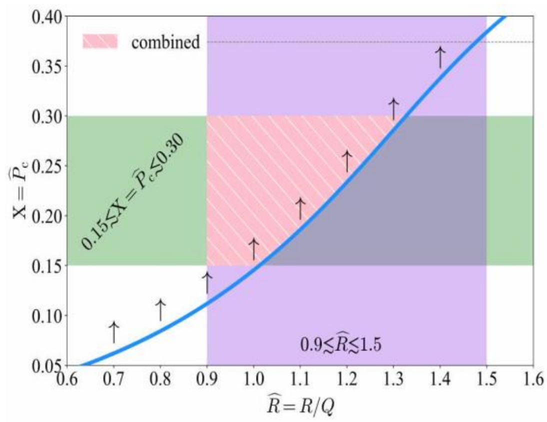
$$s^2 \approx s_c^2 + l_2\hat{r}^2, l_2 = \frac{2s_c^2}{b_2}(b_4 - s_c^2a_4)$$

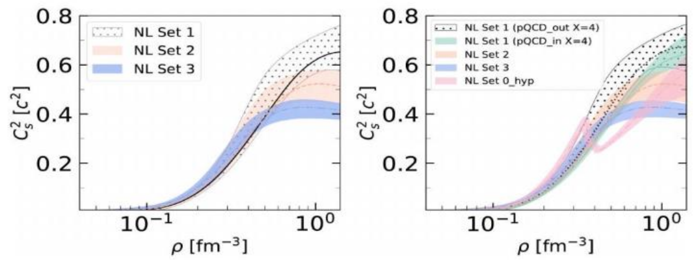
$$a_4 > \frac{1}{12}\frac{1+3X^2+4X}{s_c^2}\left(X+\frac{4+9X}{15s_c^2}\right) \approx \frac{1}{80X^2}\left(1+\frac{17}{4}X+\frac{9}{2}X^2-\frac{13}{4}X^3-\frac{49}{2}X^4+\cdots\right).$$

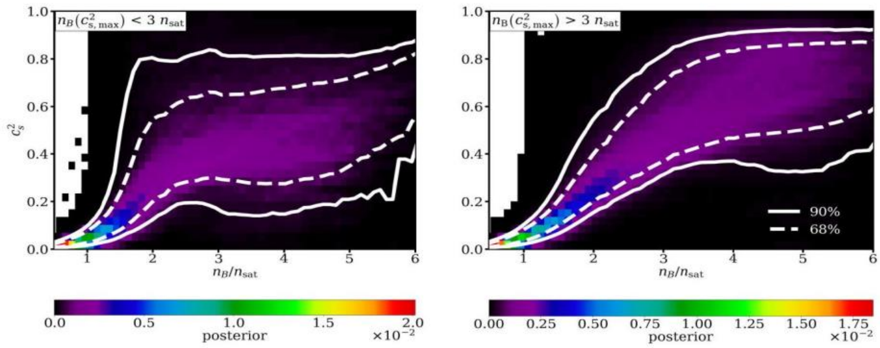
$$s^2 = \frac{1}{12}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{s_c^2}\left(\mathrm{X}+\frac{4+9\mathrm{X}}{15s_c^2}\right) \lesssim a_4 \lesssim \frac{1}{\hat{R}^4} \text{ with } \hat{R} \sim \mathcal{O}(1).$$

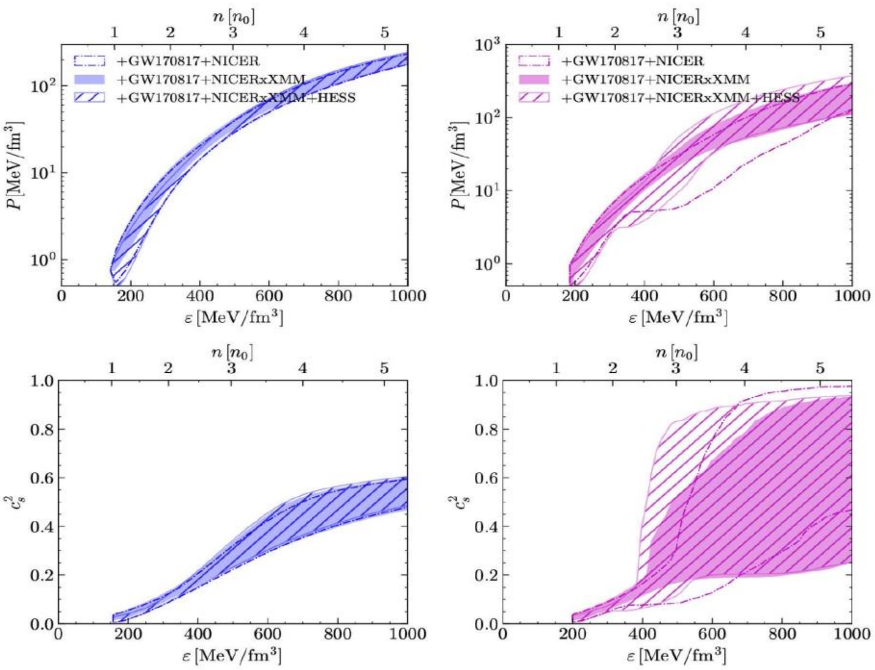
$$s^2 \approx \Delta\hat{P}/\Delta\hat{\varepsilon}$$



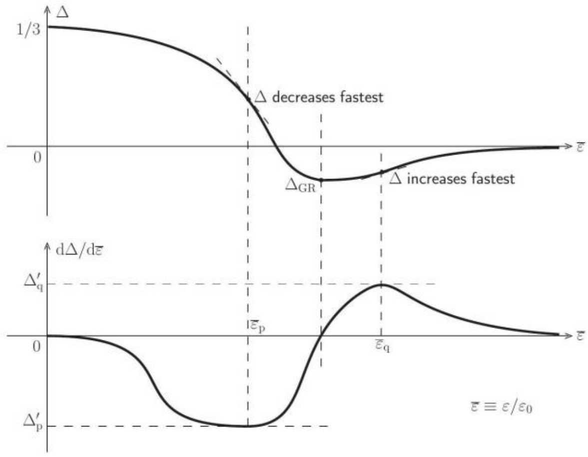

combined
$0.15 \lesssim X = \widehat{P}_c \lesssim 0.30$
$0.9 \lesssim \widehat{R} \lesssim 1.5$
$X = \widehat{P}_c$
$\widehat{R} = R/Q$


NL Set 1
NL Set 2
NL Set 3
NL Set 1 (pQCD_out X=4)
NL Set 1 (pQCD_in X=4)
NL Set 0_hyp
$C_s^2\ [c^2]$
$\rho\ [\mathrm{fm}^{-3}]$


$n_B(c^2_{s,\max}) < 3\ n_{sat}$
$n_B(c^2_{s,\max}) > 3\ n_{sat}$
c_s^2
n_B/n_{sat}
90%
68%
posterior
$\times 10^{-2}$


+GW170817+NICER
+GW170817+NICERxXMM
+GW170817+NICERxXMM+HESS
$n\,[n_0]$
$P\,[\mathrm{MeV/fm^3}]$
$\varepsilon\,[\mathrm{MeV/fm^3}]$
c_s^2

$$s^2 = s^2_{\mathrm{deriv}} + s^2_{\mathrm{non-deriv}} = -\bar{\varepsilon}\frac{\mathrm{d}\Delta}{\mathrm{d}\bar{\varepsilon}} + \frac{1}{3} - \Delta, \bar{\varepsilon} \equiv \varepsilon/\varepsilon_0$$



$$\Delta' \approx \Delta'_{\mathrm{p}} + 2^{-1}\Delta'''_{\mathrm{p}}\left(\bar{\varepsilon} - \bar{\varepsilon}_{\mathrm{p}}\right)^2, \Delta'_{\mathrm{p}} < 0, \Delta'''_{\mathrm{p}} > 0$$

$$\mathrm{d}s^2/\mathrm{d}\bar{\varepsilon} = -2\Delta'_{\mathrm{p}} + \left(3\bar{\varepsilon}_{\mathrm{p}}\bar{\varepsilon} - 2\bar{\varepsilon}^2 - \bar{\varepsilon}_{\mathrm{p}}^2\right)\Delta'''_{\mathrm{p}}$$

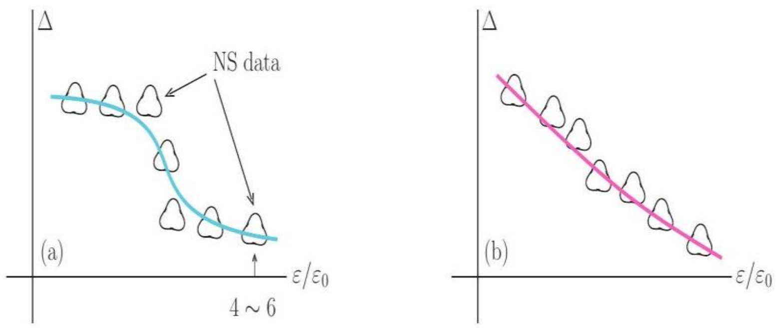
$$\bar{\varepsilon}_{\mathrm{p}}^* = \frac{3\bar{\varepsilon}_{\mathrm{p}}\Delta'''_{\mathrm{p}} + \sqrt{\bar{\varepsilon}_{\mathrm{p}}^2\Delta'''_{\mathrm{p}}2 - 16\Delta'_{\mathrm{p}}\Delta'''_{\mathrm{p}}}}{4\Delta'''_{\mathrm{p}}} \approx \bar{\varepsilon}_{\mathrm{p}}\left(1 - \frac{2}{\bar{\varepsilon}_{\mathrm{p}}^2}\frac{\Delta'_{\mathrm{p}}}{\Delta'''_{\mathrm{p}}}\right) > \bar{\varepsilon}_{\mathrm{p}} :: \Delta'_{\mathrm{p}}$$

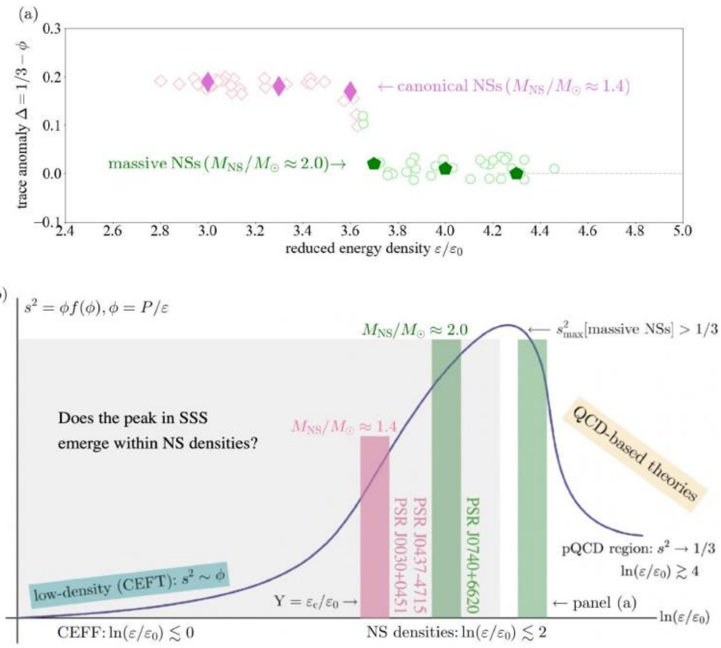
$$s^2\left(\bar{\varepsilon}_{\mathrm{p}}^*\right) \approx \frac{1}{3} - \Delta_{\mathrm{p}} - \bar{\varepsilon}_{\mathrm{p}}\Delta'_{\mathrm{p}}, \left.\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}_{\mathrm{p}}^*} \approx \Delta'''_{\mathrm{p}}\bar{\varepsilon}_{\mathrm{p}}\left(\frac{8}{\bar{\varepsilon}_{\mathrm{p}}^2}\frac{\Delta'_{\mathrm{p}}}{\Delta'''_{\mathrm{p}}} - 1\right) < 0.$$

$$s^2_{\mathrm{deriv}}\left(\bar{\varepsilon}_{\mathrm{p}}^*\right) \approx -\bar{\varepsilon}_{\mathrm{p}}\Delta'_{\mathrm{p}}, s^2_{\text{non-deriv}}\left(\bar{\varepsilon}_{\mathrm{p}}^*\right) \approx \frac{1}{3} - \Delta_{\mathrm{p}}, :: \frac{s^2\left(\bar{\varepsilon}_{\mathrm{p}}^*\right)}{s^2_{\mathrm{deriv}}\left(\bar{\varepsilon}_{\mathrm{p}}^*\right)} = 1 - \frac{3^{-1} - \Delta_{\mathrm{p}}}{\bar{\varepsilon}_{\mathrm{p}}\Delta'_{\mathrm{p}}} > 1.$$

$$\left.\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}_{\mathrm{q}}^*} \approx \Delta'''_{\mathrm{q}}\bar{\varepsilon}_{\mathrm{q}}\left(\frac{8}{\bar{\varepsilon}_{\mathrm{q}}^2}\frac{\Delta'_{\mathrm{q}}}{\Delta'''_{\mathrm{q}}} - 1\right) > 0, \text{at } \bar{\varepsilon}_{\mathrm{q}}^* \approx \bar{\varepsilon}_{\mathrm{q}}\left(1 - \frac{2}{\bar{\varepsilon}_{\mathrm{q}}^2}\frac{\Delta'_{\mathrm{q}}}{\Delta'''_{\mathrm{q}}}\right) > \bar{\varepsilon}_{\mathrm{q}},$$

$$s^2(\phi) \approx 3\phi - \frac{a}{2b}\left(a + \sqrt{a^2 - 4b\phi}\right) \approx 2\phi\left(1 - \frac{b}{2a^2}\phi\right) + \mathcal{O}(\phi^3), \text{and } s^2/\phi \approx 2\left(1 - \frac{b}{2a^2}\phi\right).$$



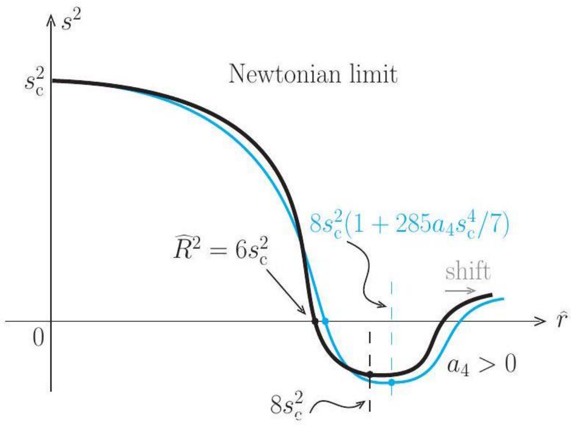


$$s^2 \approx s_c^2 + l_2^N \hat{r}^2 + l_4^N \hat{r}^4 \approx s_c^2 + \left(12a_4 s_c^4 - \frac{4}{15}\right)\hat{r}^2 + \left(144 a_4^2 s_c^6 + 18 a_6 s_c^4 - \frac{62}{35} a_4 s_c^2 + \frac{1}{60 s_c^2}\right)\hat{r}^4$$

$$a_2 = b_2/s_c^2 \approx -1/6s_c^2 \lesssim \mathcal{O}(10^{k-1}), 12a_4 s_c^4 \lesssim \mathcal{O}(10^{2-k})$$

$$144 a_4^2 s_c^6 \sim 18 a_6 s_c^4 \lesssim \mathcal{O}(10^{5-k}), (62/35)a_4 s_c^2 \lesssim \mathcal{O}(10^1), 1/60 s_c^2 \lesssim \mathcal{O}(10^{k-2})$$

$$s^2 \approx s_c^2 - (4/15)\hat{r}^2 + (60 s_c^2)^{-1}\hat{r}^4.$$

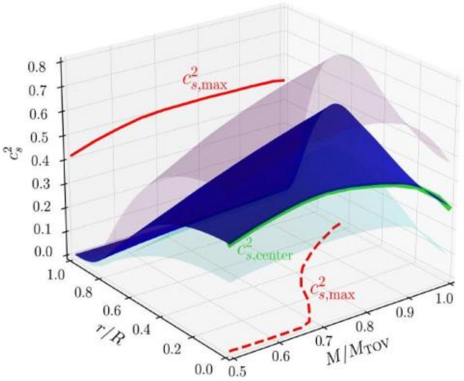


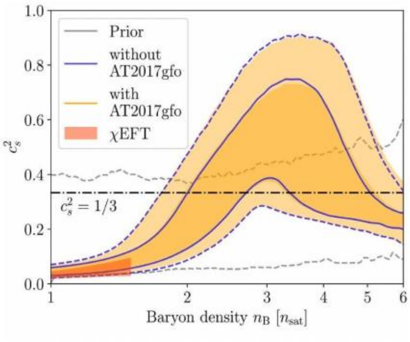
$$\begin{aligned} s^2 \approx\, & s_c^2 + \left(12a_4 s_c^4 - \frac{4}{15}\right)\hat{r}^2 + \left(144 a_4^2 s_c^6 + 18 a_6 s_c^4 - \frac{62}{35} a_4 s_c^2 + \frac{1}{60 s_c^2}\right)\hat{r}^4 \\ & + \left[1728 a_4^3 s_c^8 + 432 a_4 a_6 s_c^6 + \left(24 a_8 - \frac{744}{35} a_4^2\right) s_c^4 - \frac{52}{15} a_6 s_c^2 + \frac{1}{35} a_4\right]\hat{r}^6 \\ \approx\, & s_c^2 - (4/15)\hat{r}^2 + (60 s_c^2)^{-1}\hat{r}^4 + (a_4/35)\hat{r}^6 \end{aligned}$$

$$\begin{aligned} \hat{r}^2_{\min} &\approx 8s_c^2(1 + 285 a_4 s_c^4/7 + 1416 a_6 s_c^6) \\ s^2_{\min} &\approx -\frac{s_c^2}{15}(1 + 288 a_4 s_c^4/7 + 9344 a_6 s_c^6) \\ \hat{R}^2 &\approx 6 s_c^2 (1 + 144 a_4 s_c^4/7 + 1620 a_6 s_c^6) \end{aligned}$$

$$\left(\frac{\mathrm{d}s^2}{\mathrm{d}\hat{\varepsilon}}\right)_{\mathrm{N}} = -\frac{3\hat{\varepsilon}}{\hat{r}^3}\left(\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}\hat{r}}\right)^2\left(\frac{\hat{r}^3\hat{\varepsilon}}{3} - \widehat{M}\right) + \frac{\hat{\varepsilon}\widehat{M}}{\hat{r}^2}\left(\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}\hat{r}}\right)^{-3}\left[\frac{\mathrm{d}^2\hat{\varepsilon}}{\mathrm{d}\hat{r}^2} - \frac{1}{\hat{r}}\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}\hat{r}}\left(1 + \frac{\hat{r}}{\hat{\varepsilon}}\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}\hat{r}}\right)\right],$$

$$s_c^2 \approx 0.45^{+0.14}_{-0.18}$$





$$s_c^2(\mathrm{X} \le 1/3) \le 7/9 \approx 0.778$$

$$s_c^2 \approx \frac{4}{3}\mathrm{X}\left(1 + \mathrm{X} + \frac{3}{2}\mathrm{X}^2 + 3\mathrm{X}^3\right) + \mathcal{O}(\mathrm{X}^5) \approx \frac{4}{3}H\left(1 + 5H + \frac{57}{2}H^2 + 175H^3\right) + \mathcal{O}(H^5)$$

$$X = -\frac{4\xi - \tau + \sqrt{4\xi^2 - 8\tau\xi + \tau^2}}{6\xi} \approx \frac{\xi}{\tau} + 4\left(\frac{\xi}{\tau}\right)^2 + 19\left(\frac{\xi}{\tau}\right)^3 + 100\left(\frac{\xi}{\tau}\right)^4 + \cdots.$$

$$X \approx \frac{\xi}{2} + \xi^2;$$

$$X \approx \frac{\xi}{2} + \frac{133\xi^2}{120};$$

$$X \approx \frac{\xi}{2} + \frac{5\xi^2}{4}.$$

$$s_c^2 \approx \frac{4 + \Psi}{3\tau}\xi + \frac{4}{3}\frac{5 + 2\Psi}{\tau^2}\xi^2 + \frac{38 + 19\Psi}{\tau^3}\xi^3 + \frac{100}{3}\frac{7 + 4\Psi}{\tau^4}\xi^4 + \cdots$$

$$\xi\text{-scaling}\ \frac{M_{\mathrm{NS}}/R}{\xi \approx A_\xi \Pi_c + B_\xi}\mathrm{X}\ \underset{s_c^2 = \mathrm{X}\left(1 + \frac{1+\Psi}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}^2}{1-3\mathrm{X}^2}\right)}{\overset{\mathcal{M}_{NS}/\mathcal{M}_\odot \leftrightsquigarrow \psi}{\rightarrow}}\ s_c^2$$

$$s_c^2 \approx 0.47^{+0.09}_{-0.09},$$

$$a_4 = s_c^{-2}\left(b_4 - a_2^2 \sum_{k=1}^{K} 2^{-1}k(k-1)d_k\right),$$

$$s^2(\hat{r}) \approx s_c^2[1 + (2/b_2)(b_4 - s_c^2 a_4)\hat{r}^2] + \mathcal{O}(\hat{r}^4) \approx s_c^2 + 2a_2 D\hat{r}^2 + \mathcal{O}(\hat{r}^4), D = \sum_{k=1}^{K} 2^{-1}k(k-1)d_k$$

$$s^2(\hat{\varepsilon}) \approx s_c^2 + 2D(\hat{\varepsilon} - 1) = s_c^2 + 2D\mu$$

$$D < 0 \leftrightarrow D\mu > 0 \leftrightarrow s_c^2 < s^2(\hat{\varepsilon}) \leftrightarrow \mathfrak{I}$$

$$0 \le d_1 + 2d_2\hat{\varepsilon} + 3d_3\hat{\varepsilon}^2 + \cdots \le 1$$

$$\left.\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}\hat{\varepsilon}^2}\right|_{\hat{\varepsilon}=\hat{\varepsilon}_c=1} = \sum_{k=1}^{K} k(k-1)d_k = 2D < \sigma_c^2 s_c^2$$

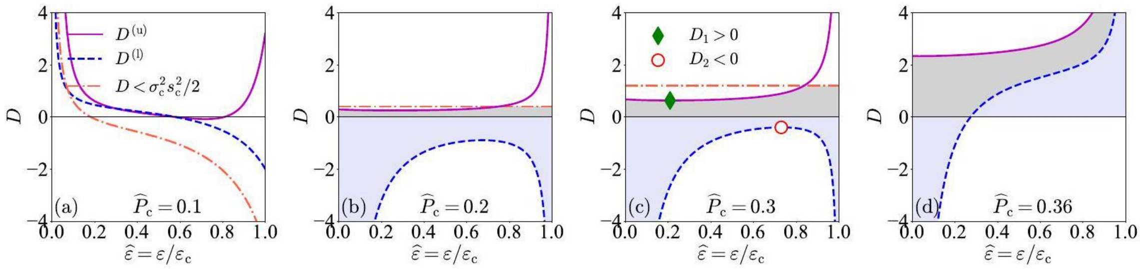
$$d_2 = -\mathrm{X} + s_c^2 - \sum_{k=3}^{K} (k-1)d_k$$

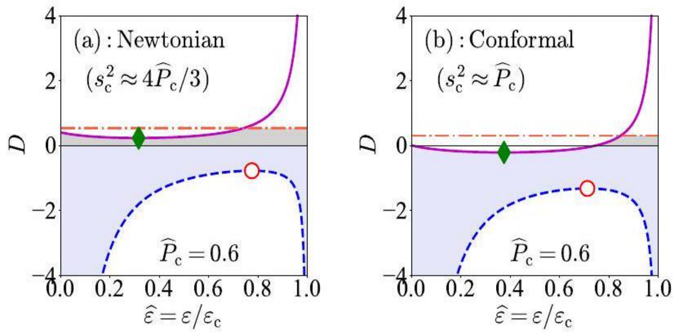
$$d_3 = -2\mathrm{X} + s_c^2 - \sum_{k=4}^{K} (k-2) d_k$$

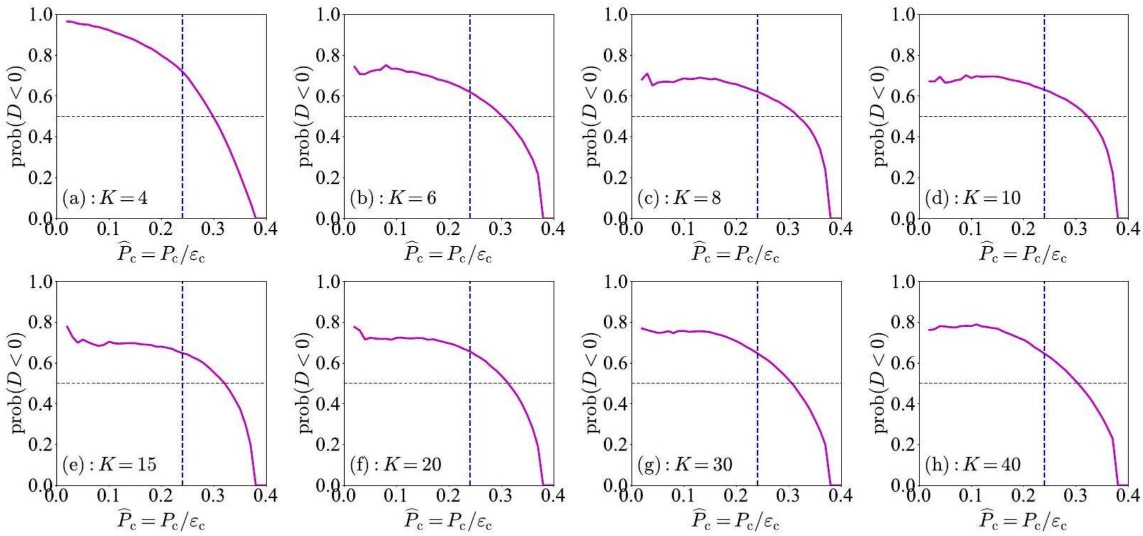
$$D = \sum_{k=1}^{K} \frac{k(k-1)}{2} d_k = d_2 + 3d_3 + \sum_{k=4}^{K} \frac{k(k-1)}{2} d_k = 2s_c^2 - 3\mathrm{X} + \sum_{k=4}^{K} \frac{(k-2)(k-3)}{2} d_k$$

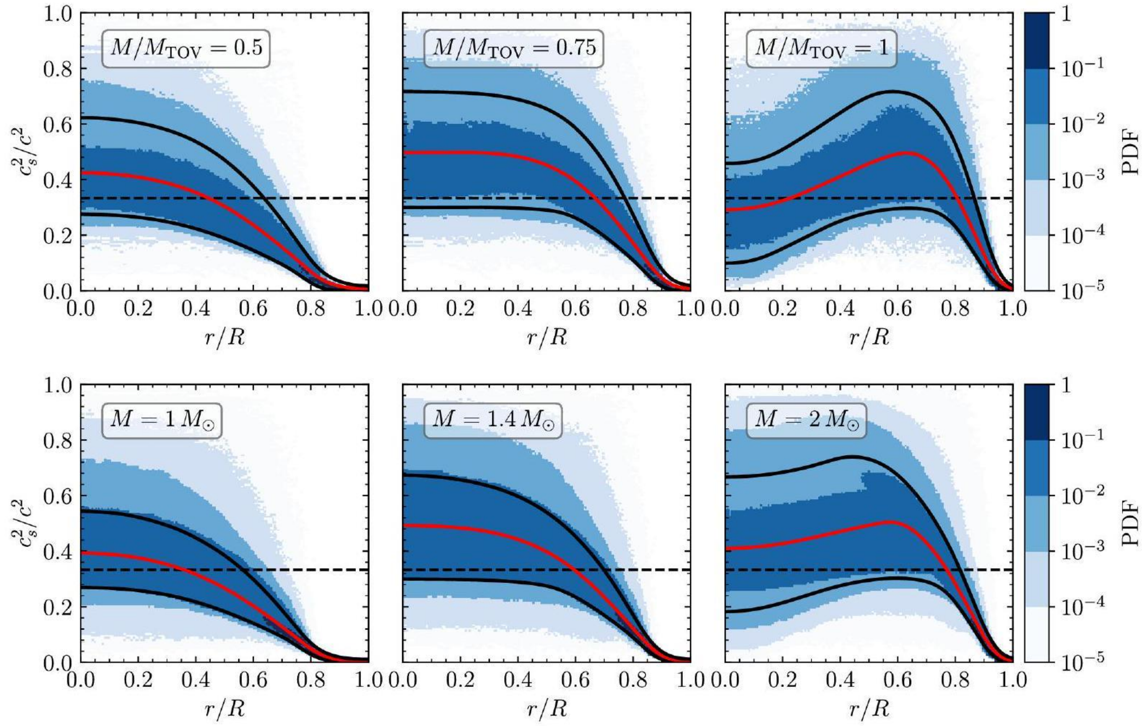
$$\mathrm{prob}(D<0) \approx \frac{\#[0 \le s^2 \le 1 \text{ and } 2D < \sigma_c^2 s_c^2 \text{ and } D < 0]}{\#[0 \le s^2 \le 1 \text{ and } 2D < \sigma_c^2 s_c^2]}$$

$$d_4^{(\mathrm{l})}(\hat{\varepsilon}) = \frac{\hat{\varepsilon}(\mathrm{X} - s_c^2) - \mathrm{X} + \hat{\varepsilon}^{-1}}{4\hat{\varepsilon}^2 - 3\hat{\varepsilon} - 1}, d_4^{(\mathrm{u})}(\hat{\varepsilon}) = \frac{\hat{\varepsilon}(\mathrm{X} - s_c^2) - \mathrm{X}}{4\hat{\varepsilon}^2 - 3\hat{\varepsilon} - 1}$$

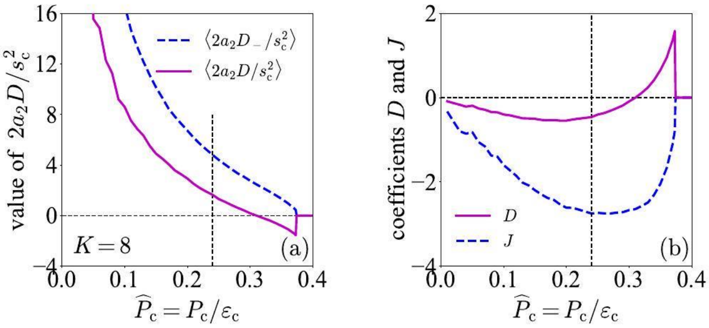






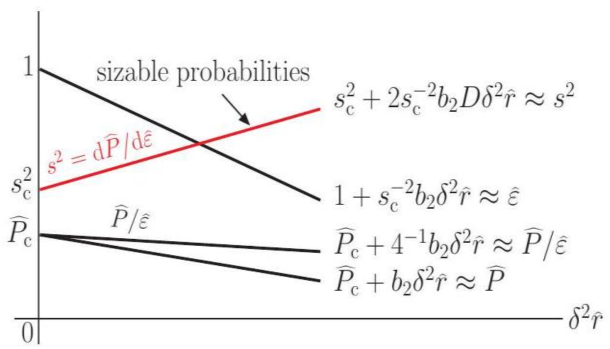


$$u_{\rm c} = 2s_{\rm c}^2 - 3{\rm X} \approx -\frac{1-2\Psi}{3}{\rm X}, \Psi > 0$$



$$\langle 2a_2D/s_{\rm c}^2\rangle = (2a_2/s_{\rm c}^2)\sum_{k=\pm}{\rm prob}(D_k)D_k$$

$$s^2/s_{\rm c}^2 \approx 1 + \frac{9.4{\rm X}\langle a_2D/s_{\rm c}^2\rangle}{1+3{\rm X}^2+4{\rm X}}\left(\frac{r}{R_{\rm max}}\right)^2$$



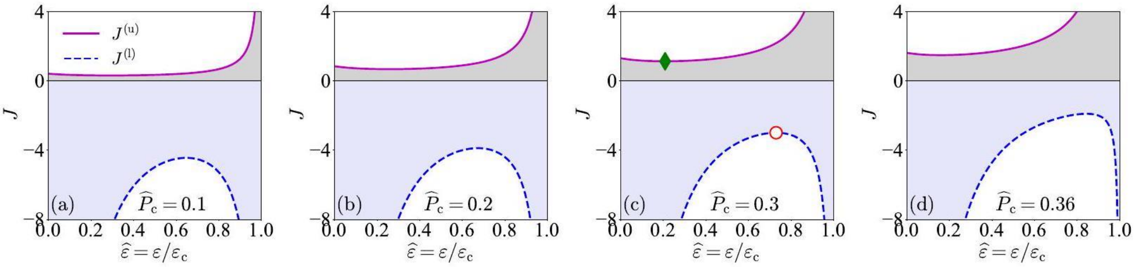
$$s^2/s_{\rm c}^2 \approx 1 + \frac{2}{b_2}(b_4 - s_{\rm c}^2a_4)\hat{r}^2 + \underbrace{\frac{3}{b_2}\left[(b_6 - s_{\rm c}^2a_6) - \frac{4}{3}\frac{a_4}{a_2}(b_4 - s_{\rm c}^2a_4)\right]\hat{r}^4}_{\text{term relevant for estimating the peak}}.$$

$$J = \sum_{k=1}^{K} \frac{k(k-1)(k-2)}{6} d_k = d_3 + 4d_4 + 10d_5 + \cdots$$

$$s^2(\hat{r}) \approx s_c^2 + 2a_2 D\hat{r}^2 + (3a_2^2 J + 2a_4 D)\hat{r}^4$$

$$\hat{r}_{\mathrm{pk}} = \left(-\frac{a_2 D}{3a_2^2 J + 2a_4 D}\right)^{1/2}$$

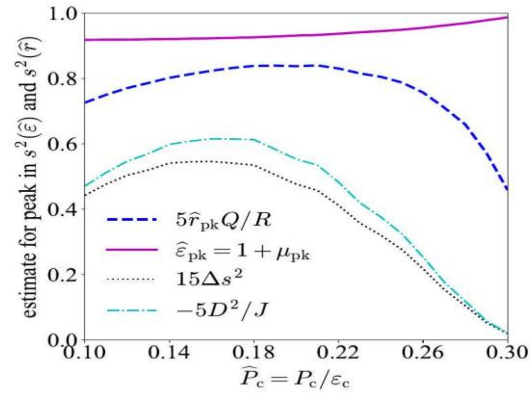
$$s^2(\mu) \approx s_c^2 + 2D\mu + 3J\mu^2 - \frac{2}{a_2^2}\left(3a_4 J + \frac{a_6}{a_2} D\right)\mu^3$$

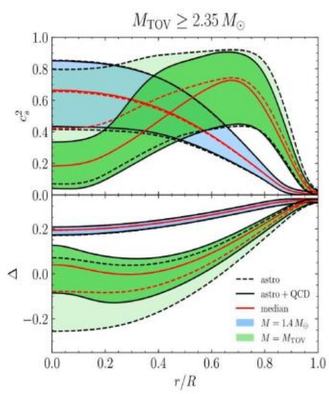


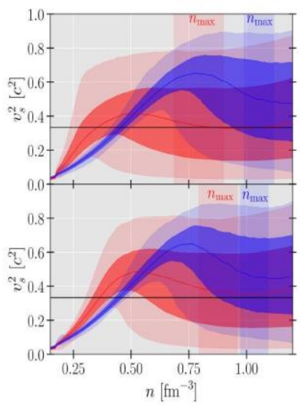
$$\hat{P}(\mu) \approx \mathrm{X} + s_c^2\mu + \frac{1}{2}\frac{\mathrm{d}s^2}{\mathrm{d}\mu}\bigg|_{\mu=0}\mu^2 + \frac{1}{6}\frac{\mathrm{d}^2 s^2}{\mathrm{d}\mu^2}\bigg|_{\mu=0}\mu^3 \approx \mathrm{X} + s_c^2\mu + D\mu^2 + J\mu^3.$$

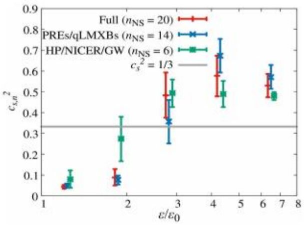
$$\mu_{\mathrm{pk}} \approx -D/3J.$$

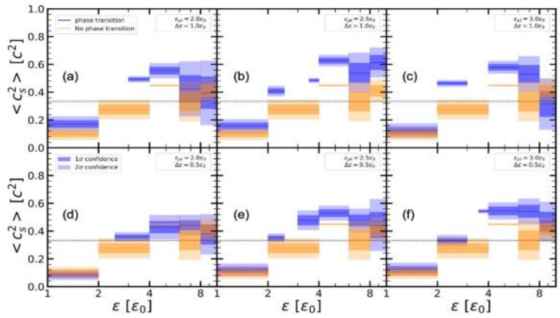
$$\hat{r}_{\mathrm{pk}} \approx \sqrt{-\frac{1}{3a_2}\frac{D}{J}} \cdot \left(1 - \frac{a_4}{3a_2^{3/2}}\frac{D}{J}\right),$$











$$-t(\hat{\varepsilon}) = -t(\mu) \approx -t_c + (2D + t_c)\mu + (3J - D - t_c)\mu^2$$

$$\mu_{\rm pk}^{(-t)} = \hat{\varepsilon}_{\rm pk}^{(-t)} - 1 = \frac{1}{2}\frac{2D + t_c}{D + t_c - 3J}$$

$$-t_{\rm pk} \equiv -t\left(\mu_{\rm pk}^{(-t)}\right) = \frac{4D^2 + 12Jt_c - 3t_c^2}{4D - 12J + 4t_c}$$

$$\mu_{\rm pk} - \mu_{\rm pk}^{(-t)} = \frac{2}{9}\left(\frac{D}{J}\right)^2 \cdot \frac{1 + t_c/D + 3Jt_c/2D^2}{1 - D/3J - t_c/3J} > 0$$

$$\left| \begin{aligned} \frac{\mathrm{d}s^2}{\mathrm{d}\hat{\varepsilon}} = \frac{Y}{1 - 2\hat{M}/\hat{r}} \Bigg\{ \left(\frac{\mathrm{d}s^2}{\mathrm{d}\hat{\varepsilon}}\right)_{\rm N} - \frac{\hat{\varepsilon}\hat{M}}{1 - 2\hat{M}/\hat{r}} \frac{2}{\hat{r}^4} \left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)^2 (\hat{r}^3\hat{\varepsilon} - \hat{M}) \\ - \frac{\hat{\varepsilon}}{\hat{M}}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1} \left[\hat{r}\hat{M}s^2 + \hat{p}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}(3\hat{M} - \hat{r}^3\hat{\varepsilon})\right] - \frac{\hat{M}}{\hat{r}^2}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{p}}{\hat{\varepsilon}}\right)^{-1}\left(s^2 - \frac{\hat{p}}{\hat{\varepsilon}}\right)\Bigg\}, \end{aligned} \right|$$

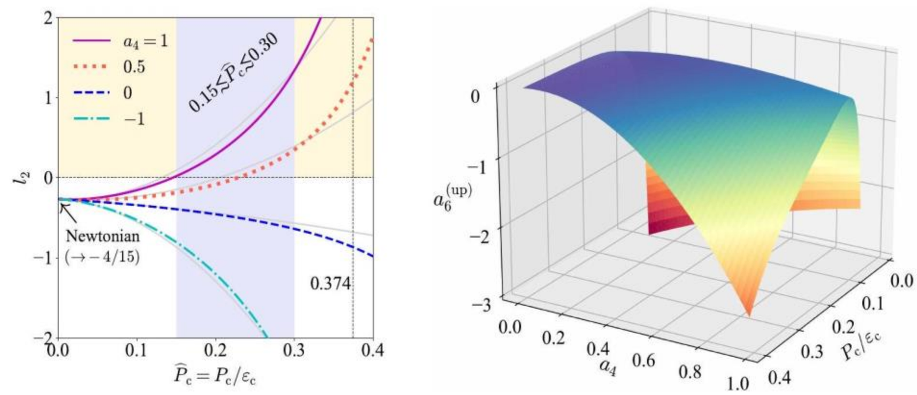
$$\hat{r}^3\hat{\varepsilon} - \hat{M} \approx \frac{2\hat{r}^3}{3}\left(1 + \frac{6}{5}a_2\hat{r}^2\right) > 0$$

$$-\frac{Y}{1 - 2\hat{M}/\hat{r}}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)^2 \left[\frac{\hat{\varepsilon}\hat{M}}{1 - 2\hat{M}/\hat{r}}\frac{2}{\hat{r}^4}(\hat{r}^3\hat{\varepsilon} - \hat{M}) + \frac{\hat{p}\hat{\varepsilon}}{\hat{M}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1}(3\hat{M} - \hat{r}^3\hat{\varepsilon})\right] < 0$$

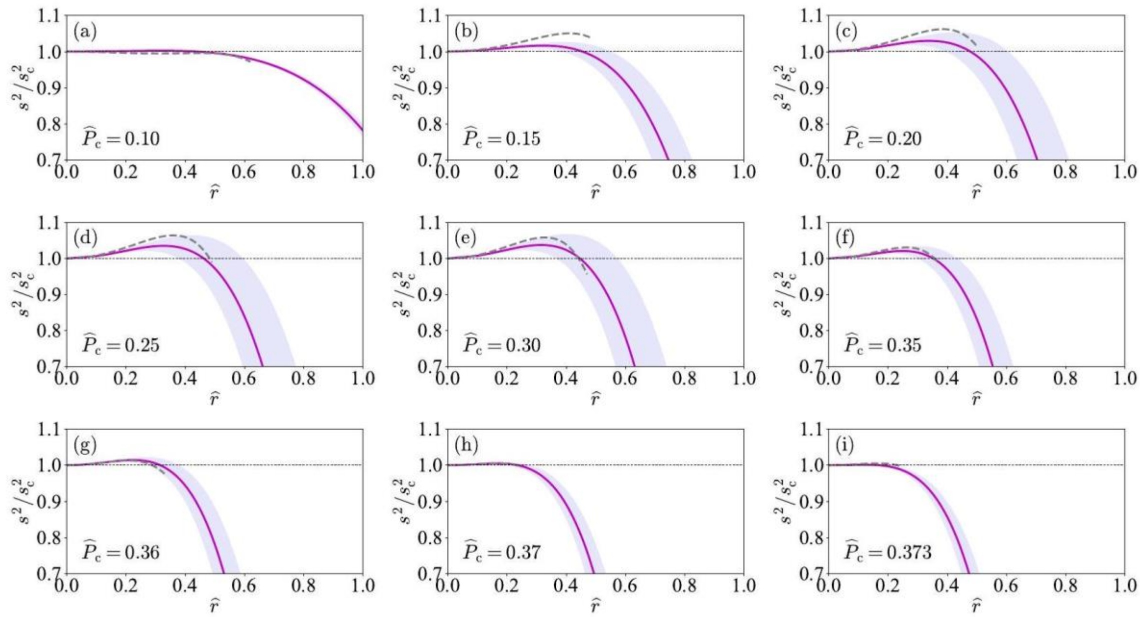
$$\begin{aligned} -\frac{Y}{1 - 2\hat{M}/\hat{r}}\left[\frac{\hat{\varepsilon}}{\hat{M}}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{r}^3\hat{p}}{\hat{M}}\right)^{-1}\hat{r}\hat{M}s^2 + \frac{\hat{M}}{\hat{r}^2}\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\left(1 + \frac{\hat{p}}{\hat{\varepsilon}}\right)^{-1}s^2\right] \\ = -\frac{\hat{\varepsilon}\hat{r}s^2}{1 - 2\hat{M}/\hat{r}}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\varepsilon}}\right)\left(1 + \frac{2\hat{P}}{\hat{\varepsilon}} + \frac{\hat{M}}{\hat{\varepsilon}\hat{r}^3}\right) > 0. \end{aligned}$$

$$\begin{aligned} s^2 \approx & s_c^2 + \left[\left(12a_4s_c^4 - \frac{4}{15}\right) - \left(48a_4s_c^4 + s_c^2 + \frac{3}{5}\right)X\right]\hat{r}^2 \\ & + \left[\left(144a_4^2s_c^6 + 18a_6s_c^4 - \frac{62}{35}a_4s_c^2 + \frac{1}{60s_c^2} + \frac{1}{12}s_c^2 - \frac{1}{18}\right)\right. \\ & \left. + \left(\frac{1}{15s_c^2} + \frac{1}{15} - 12a_4s_c^4 - 72a_6s_c^4 - 1152a_4^2s_c^6 + \frac{116}{35}a_4s_c^2\right)X\right]\hat{r}^4. \end{aligned}$$

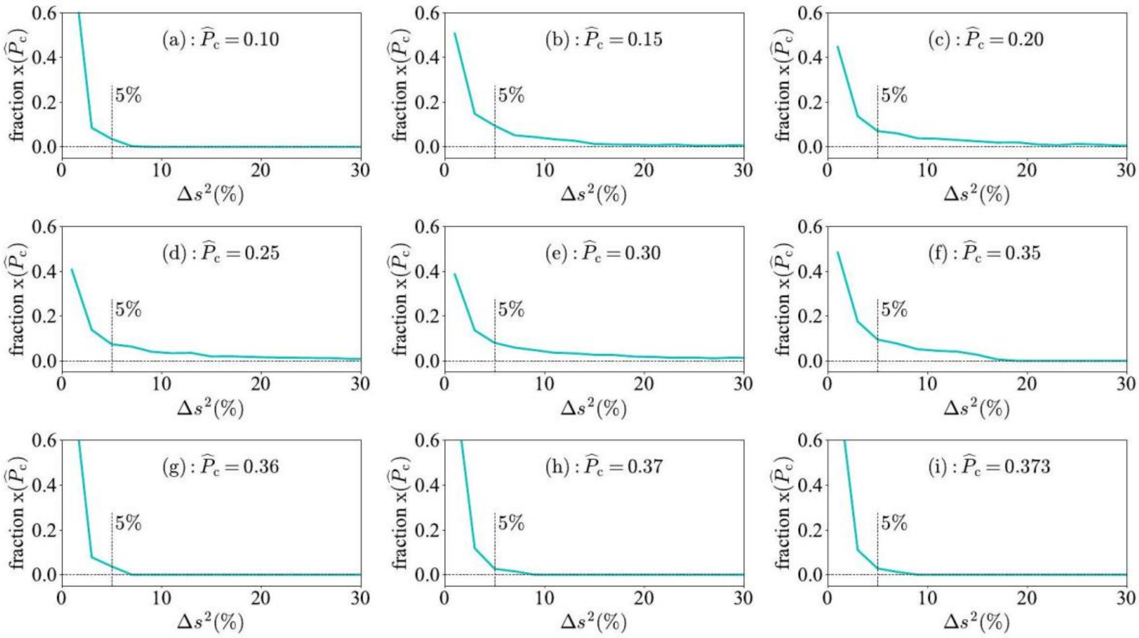
$$\begin{aligned} s^2 \approx & \frac{4}{3}X + \frac{4}{3}X^2 + \left[-\frac{4}{15} - \frac{3}{5}X + \left(\frac{64a_4}{3} - \frac{4}{3}\right)X^2\right]\hat{r}^2 \\ & + \left[\frac{1}{80X} - \frac{13}{720} + \left(\frac{229}{1440} - \frac{248a_4}{105}\right)X + \left(\frac{157}{360} + \frac{72a_4}{35} + 32a_6\right)X^2\right]\hat{r}^4 \end{aligned}$$



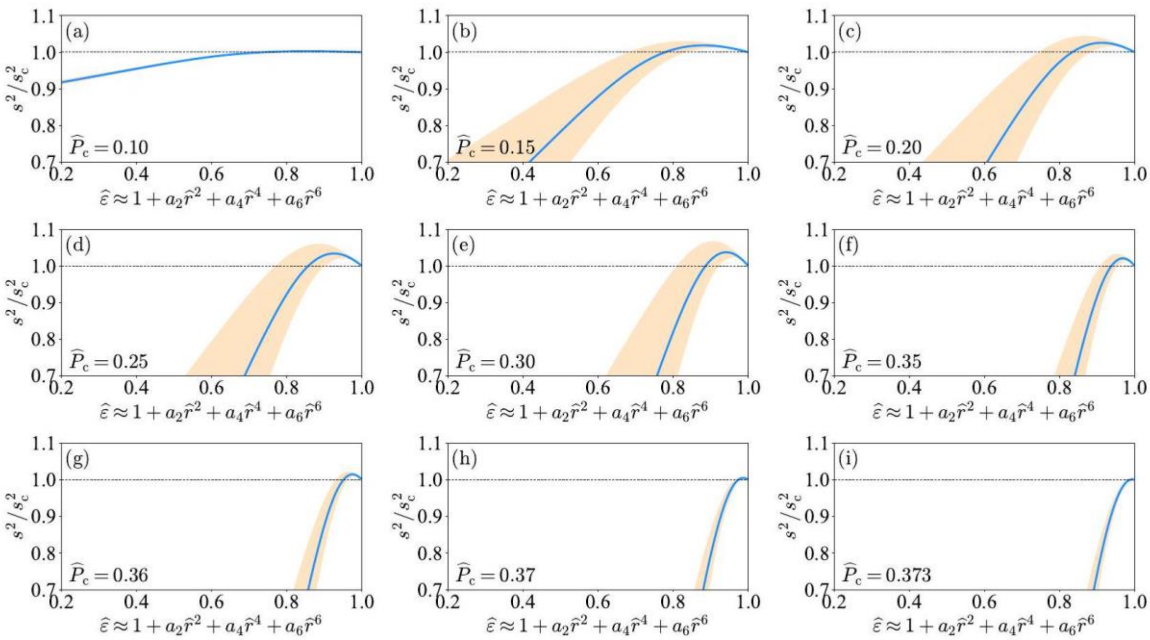
$$a_6 < \frac{b_6}{s_c^2} + \frac{4}{3}\frac{a_4}{b_2}(s_c^2 a_4 - b_4) \equiv a_6^{(up)},$$

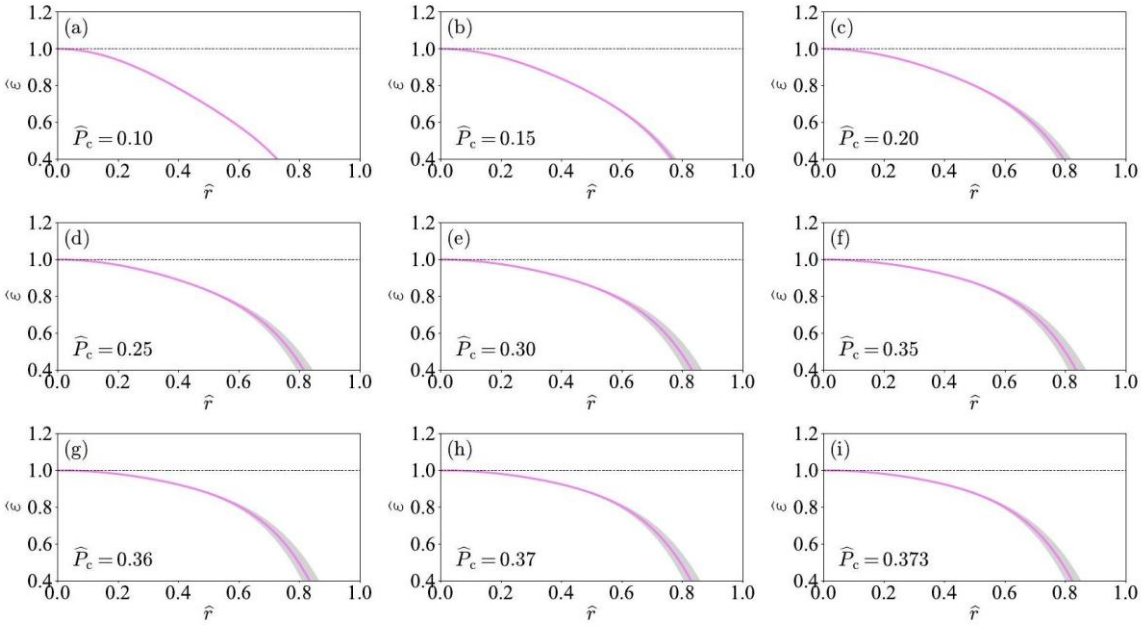


$$\hat{r}_{pk} = \sqrt{-l_2/2l_4}.$$



$$s_{\max}^2 \equiv s^2(\hat{r}_{\mathrm{pk}}) = s_c^2 - l_2^2/4l_4.$$



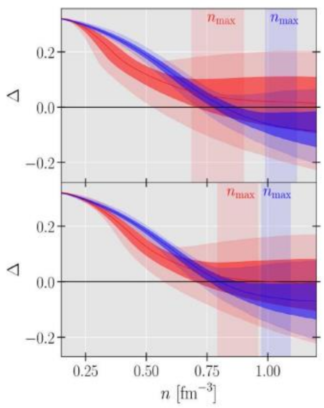


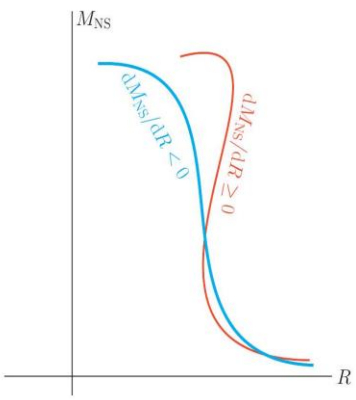
$$\Delta s^2 \equiv s_{\max}^2/s_c^2 - 1 = -l_2^2/4l_4 s_c^2$$

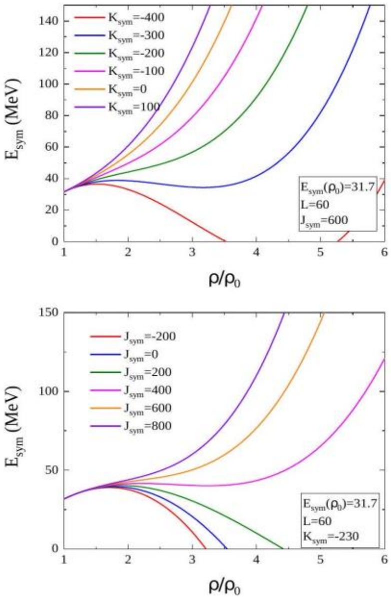
$$l_2 + 2l_4\hat{r}^2 + 3l_6\hat{r}^4 = 0$$

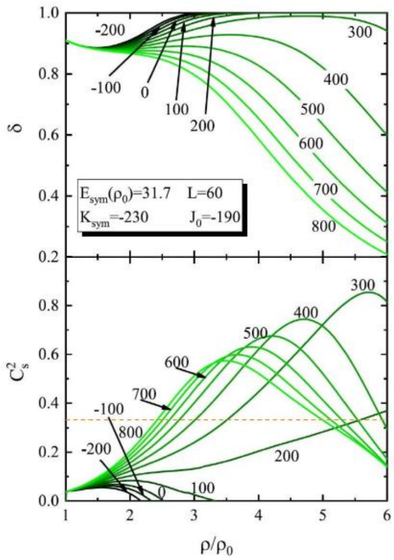
$$l_6 = \frac{4}{b_2}\left[(b_8 - s_c^2 a_8) - \frac{3}{2}\frac{a_4}{a_2}(b_6 - s_c^2 a_6) - \left(\frac{3}{2}\frac{a_6}{a_2} - 2\left(\frac{a_4}{a_2}\right)^2\right)(b_4 - s_c^2 a_4)\right]$$

$$b_8 = -\frac{1}{648}(1 + 3X^2 + 4X)\left(1 - 3X - \frac{27}{2}X^3\right) - \left(\frac{19}{1620} + \frac{X}{54} + \frac{X^2}{90} + \frac{7X^3}{120}\right)a_2 - \left(\frac{4}{225} + \frac{X}{150}\right)a_2^2 - \left(\frac{11}{756} - \frac{X}{252} - \frac{X^2}{12}\right)a_4 - \frac{3a_2a_4}{70} - \left(\frac{1}{18} + \frac{5X}{36}\right)a_6$$

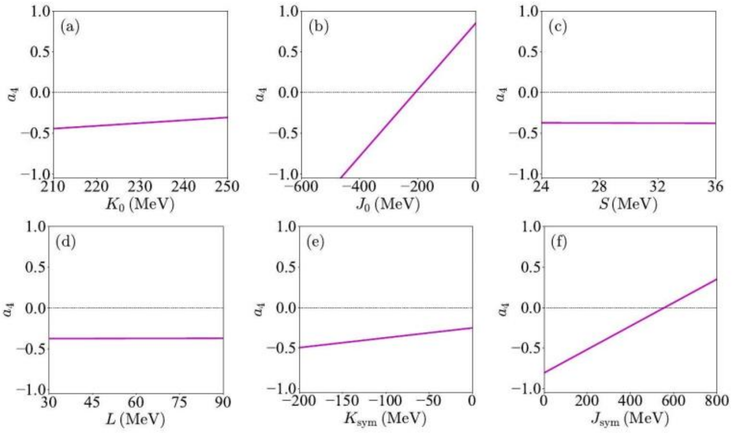




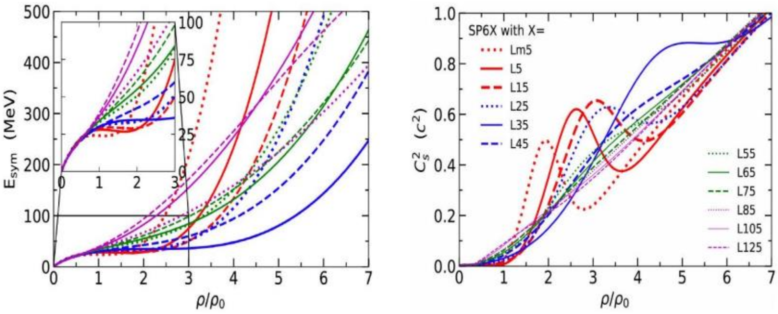




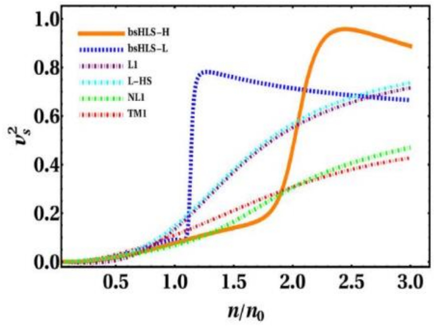
$$\hat{\varepsilon} \approx (\rho_c/\varepsilon_c)\left[M_N + E_0(\rho_0) + 2^{-1}K_0\chi^2 + 6^{-1}J_0\chi^3 + \left(S + L\chi + 2^{-1}K_{sym}\chi^2 + 6^{-1}J_{sym}\chi^3\right)\delta^2\right].$$



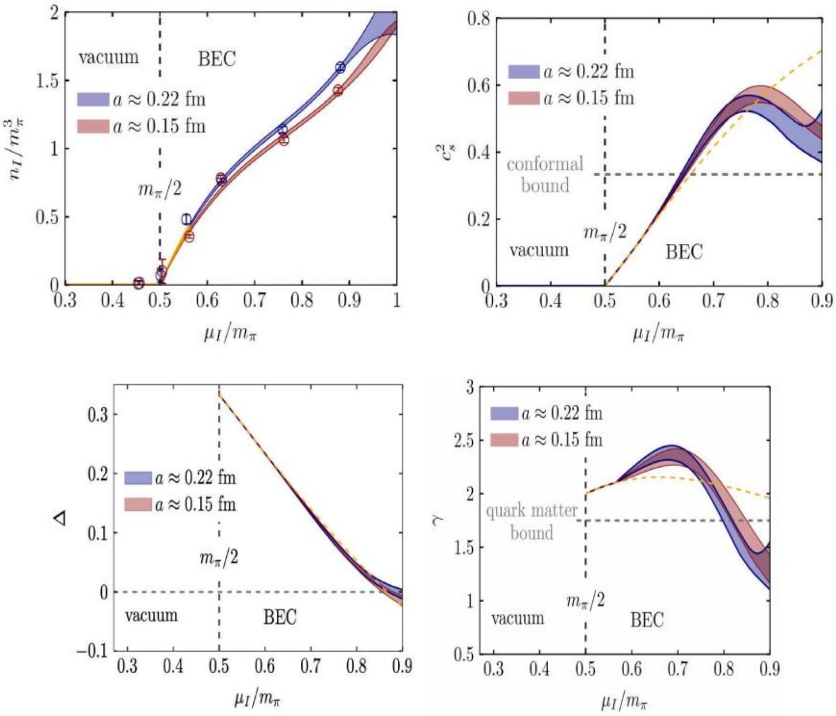
$$\begin{aligned} a_4 \approx & -\frac{\beta_3}{54(\beta_3\beta_2-1)^2\hat{\rho}_0^3}\Bigg\{+2\left((\beta_3\beta_2-1)\beta_1^2+\frac{2\beta_2\beta_3}{3}\right)\left[\left(\frac{J_0}{\bar{M}_N}\right)+\left(\frac{J_{sym}}{\bar{M}_N}\right)\delta^2\right] \\ & -3\left((\beta_3\beta_2-1)\beta_1^2+\beta_2\beta_3\right)\left[\left(\frac{J_0}{\bar{M}_N}\right)-3\left(\frac{K_0}{\bar{M}_N}\right)+\left(\left(\frac{J_{sym}}{\bar{M}_N}\right)-3\left(\frac{K_{sym}}{\bar{M}_N}\right)\right)\delta^2\right]\hat{\rho}_0 \\ & +\left((\beta_3\beta_2-1)\beta_1^2+2\beta_2\beta_3\right)\left[\left(\frac{J_0}{\bar{M}_N}\right)-6\left(\frac{K_0}{\bar{M}_N}\right)+\left(\left(\frac{J_{sym}}{\bar{M}_N}\right)-6\left(\frac{K_{sym}}{\bar{M}_N}\right)+18\left(\frac{L}{\bar{M}_N}\right)\right)\delta^2\right]\hat{\rho}_0^2 \\ & -54\beta_2\beta_3\left[(\beta_3\beta_2-1)+\frac{1}{162}\left(\frac{J_0}{\bar{M}_N}\right)-\frac{1}{18}\left(\frac{K_0}{\bar{M}_N}\right)\right. \\ & \left.+\left(\frac{1}{162}\left(\frac{J_{sym}}{\bar{M}_N}\right)-\frac{1}{18}\left(\frac{K_{sym}}{\bar{M}_N}\right)+\frac{1}{3}\left(\frac{L}{\bar{M}_N}\right)-\left(\frac{S}{\bar{M}_N}\right)\right)\delta^2\right]\hat{\rho}_0^3\Bigg\} \end{aligned}$$



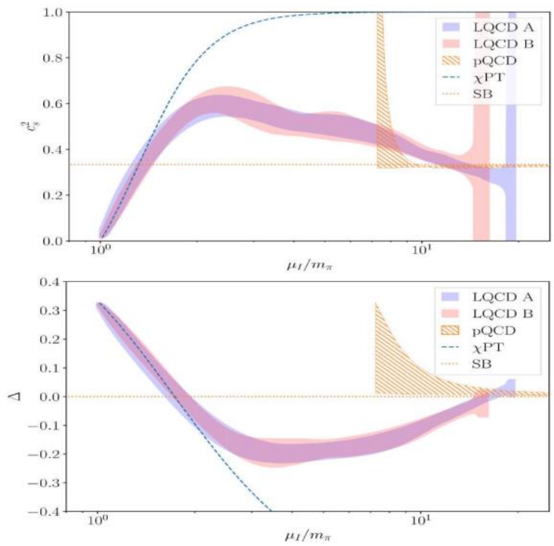
E_sym (MeV)
ρ/ρ0
SP6X with X=
Lm5
L5
L15
L25
L35
L45
L55
L65
L75
L85
L105
L125
C_s^2 (c^2)

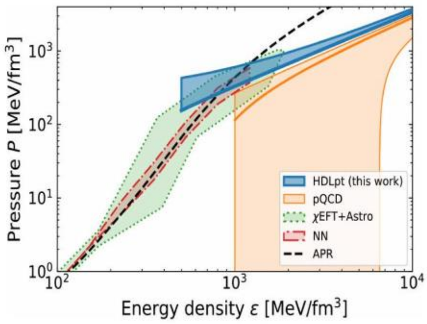


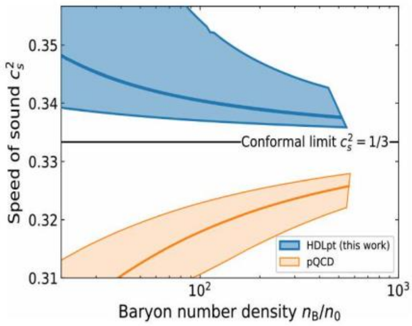
bsHLS–H
bsHLS–L
L1
L–HS
NL1
TM1
v_s^2
n/n_0

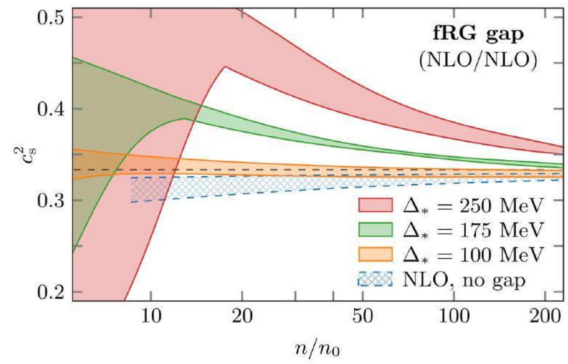


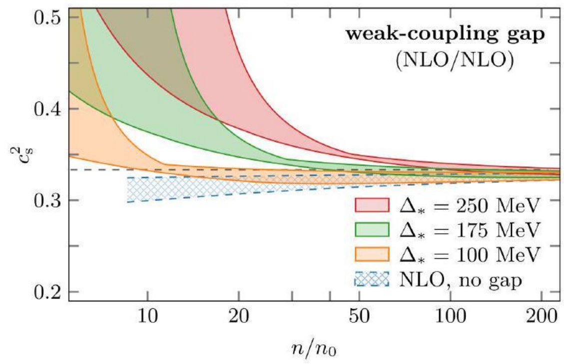
vacuum
BEC
$a \approx 0.22$ fm
$a \approx 0.15$ fm
$m_\pi/2$
n_I/m_π^3
μ_I/m_π
c_s^2
conformal
bound
Δ
γ
quark matter
bound

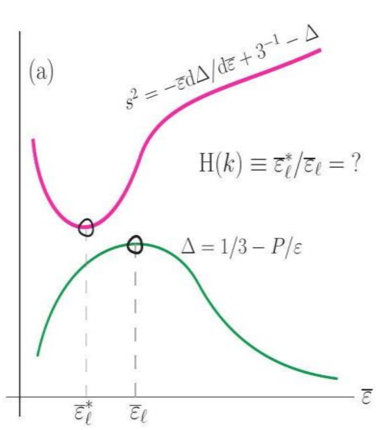


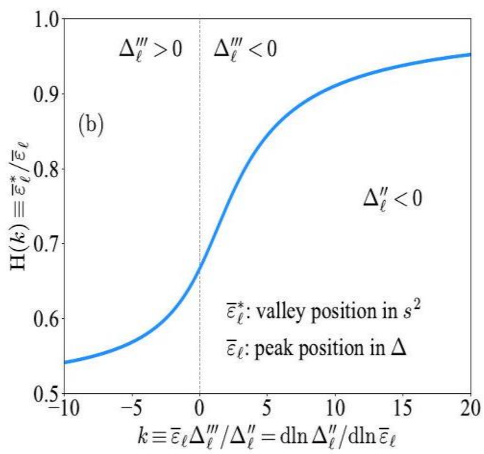












$$\Delta(\bar{\varepsilon}) \approx \Delta_\ell + \frac{1}{2}\Delta_\ell''(\bar{\varepsilon} - \bar{\varepsilon}_\ell)^2, \Delta_\ell \equiv \Delta(\bar{\varepsilon}_\ell) > 0, \Delta_\ell'' \equiv \Delta''(\bar{\varepsilon}_\ell) < 0,$$

$$\bar{\varepsilon}_\ell^* \approx 2\bar{\varepsilon}_\ell/3 \leftrightarrow \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 2/3.$$

$$\mathrm{H}(k) \equiv \left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right) = \frac{3}{4}\left(1-\frac{1}{k}\right) + \frac{\sqrt{k^2-2k+9}}{4k},$$

$$k \equiv \frac{\bar{\varepsilon}_\ell \Delta_\ell'''}{\Delta_\ell''} = \frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell} = \left[2-3\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right)\right]\left[1-3\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right)+2\left(\frac{\bar{\varepsilon}_\ell^*}{\bar{\varepsilon}_\ell}\right)^2\right]^{-1},$$

$$\left.\frac{\mathrm{d}^2 s^2}{\mathrm{d}\bar{\varepsilon}^2}\right|_{\bar{\varepsilon}=\bar{\varepsilon}_\ell^*} = -\Delta_\ell''\sqrt{k^2-2k+9} > 0, \text{ in relation to } \Delta_\ell'' < 0$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 2^{-1}(1-k^{-1}-2k^{-2}-2k^{-3}+2k^{-4}+10k^{-5}); \text{in relation to } k.$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx 1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}-\frac{5}{k^5}, \text{in relation to } k;$$
$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \frac{2}{3}\left(1+\frac{1}{18}k+\frac{1}{162}k^2-\frac{1}{1458}k^3-\frac{5}{13122}k^4-\frac{1}{39366}k^5\right), \text{in relation to } k \approx 0.$$

$$\frac{\bar{\varepsilon}_\ell^* \leftrightarrow s^2}{\bar{\varepsilon}_\ell \leftrightarrow \Delta} : \begin{cases} 1/2 \le \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \le 2/3 & \Delta_\ell''' \ge 0; \\ 2/3 \le \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \le 1 & \Delta_\ell''' \le 0 \end{cases}$$

$$s^2(\bar{\varepsilon}_\ell^*) = \frac{1}{3} - \Delta_\ell - \frac{1}{96}\frac{\Delta_\ell''\bar{\varepsilon}_\ell^2}{k^2}\left(3+k-\sqrt{k^2-2k+9}\right)^2\left(6-k+\sqrt{k^2-2k+9}\right), k = \frac{\bar{\varepsilon}_\ell \Delta_\ell'''}{\Delta_\ell''}.$$

$$s^2(\bar{\varepsilon}_\ell^*) \approx \frac{1}{3} - \Delta_\ell + \bar{\varepsilon}_\ell^2\Delta_\ell''\left(\frac{k}{12}-\frac{1}{8}\right) = \frac{1}{3} - \Delta_\ell + \frac{\bar{\varepsilon}_\ell^3\Delta_\ell'''}{12}\left(1-\frac{3}{2k}\right) \approx \frac{1}{3} - \Delta_\ell + \frac{\bar{\varepsilon}_\ell^3\Delta_\ell'''}{12}, \text{in relation to } k,$$

$$s^2(\bar{\varepsilon}_\ell^*) \approx \frac{1}{3} - \Delta_\ell - \frac{1}{96}\frac{\Delta_\ell''\bar{\varepsilon}_\ell^2}{k^2}\left(80-\frac{96}{k}\right) \approx \frac{1}{3} - \Delta_\ell - \frac{5\Delta_\ell''}{6}\left(\frac{\Delta_\ell''}{\Delta_\ell'''}\right)^2, \text{in relation to } k$$

$$s^2(\bar{\varepsilon}_\ell^*) \approx \frac{1}{3} - \Delta_\ell - \left(\frac{1}{6}-\frac{5k}{81}\right)\bar{\varepsilon}_\ell^2\Delta_\ell'' \approx \frac{1}{3} - \Delta_\ell - \frac{\bar{\varepsilon}_\ell^2\Delta_\ell''}{6}, k \approx 0$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \mathrm{H}(k)\left(1+\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{6\Delta_\ell''}\frac{(2-5\mathrm{H}(k))(1-\mathrm{H}(k))^2}{\mathrm{H}(k)\sqrt{k^2-2k+9}}\right)$$

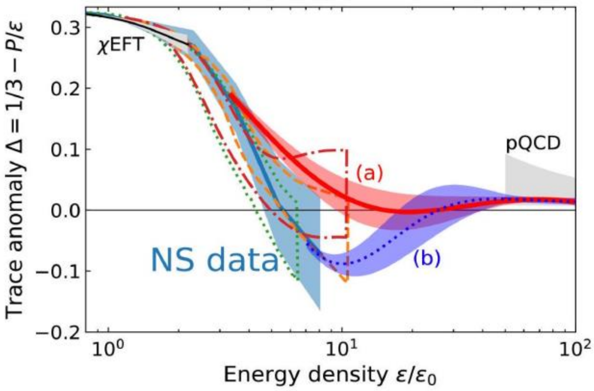
$$\begin{aligned}
\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell &\approx \mathrm{H}(k)\left[1+\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\frac{1}{k}\left(1-\frac{1}{k}-\frac{20}{k^2}+\cdots\right)\right] \\
&\approx \frac{1}{2}\left[1-\frac{1}{k}\left(1-\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\right)-\frac{2}{k^2}\left(1+\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\right)-\frac{2}{k^3}\left(1+\frac{7\bar{\varepsilon}_\ell^2\Delta_\ell''''}{16\Delta_\ell''}\right)+\cdots\right] \\
&\approx \frac{1}{2}\left[1-\frac{1}{k}-\frac{2}{k^2}-\frac{2}{k^3}+\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{24\Delta_\ell''}\frac{1}{k}\left(1-\frac{2}{k}-\frac{21}{k^2}\right)+\cdots\right] \\
&= \frac{1}{2}\left[1-\frac{1}{k}-\frac{2}{k^2}-\frac{2}{k^3}+\frac{1}{24}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(1-\frac{2}{k}-\frac{21}{k^2}\right)+\cdots\right].
\end{aligned}$$

$$\begin{aligned}
\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell &\approx 1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}-\frac{\bar{\varepsilon}_\ell^2\Delta_\ell''''}{2\Delta_\ell''}\frac{1}{k^3}\left(1-\frac{8}{3k}-\frac{8}{3k^2}\right)+\cdots \\
&\approx 1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{2}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(\frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)^{-2}\left(1-\frac{8}{3k}-\frac{8}{3k^2}\right)+\cdots
\end{aligned}$$

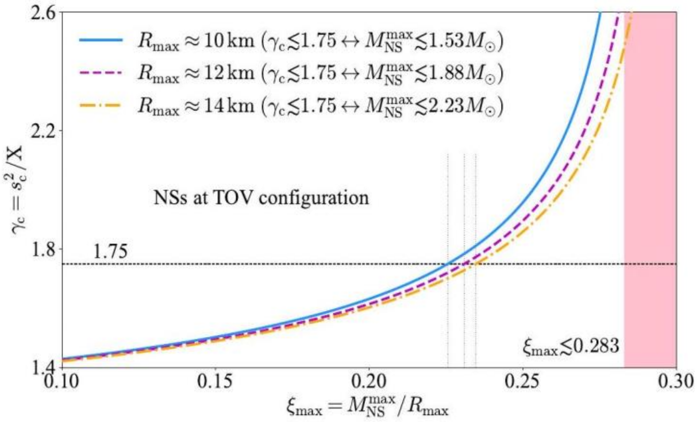
$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \approx \frac{2}{3}\left[1+\frac{1}{18}k+\frac{1}{162}k^2-\frac{1}{1458}k^3-\frac{1}{81}\left(\frac{\mathrm{dln}\ \Delta_\ell'''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(\frac{\mathrm{dln}\ \Delta_\ell''}{\mathrm{dln}\ \bar{\varepsilon}_\ell}\right)\left(1+\frac{1}{36}k-\frac{2}{27}k^2-\frac{4}{243}k^3\right)\right].$$

$$\phi = P/\varepsilon = \hat{P}/\hat{\varepsilon} \le \mathrm{X} \le 0.374.$$

$$p^{(\omega)} \approx \varepsilon^{(\omega)} \approx \frac{1}{2}g_\omega^2\omega^2 \approx \frac{1}{2}\left(\frac{g_\omega}{m_\omega}\right)^2\rho^2$$



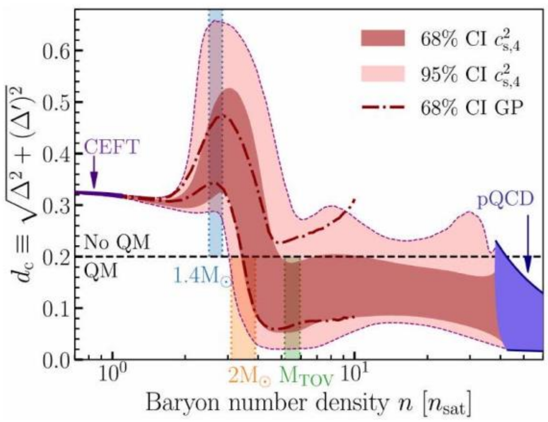
$$\Delta \geq \Delta_{\mathrm{GR}} \approx -0.04.$$

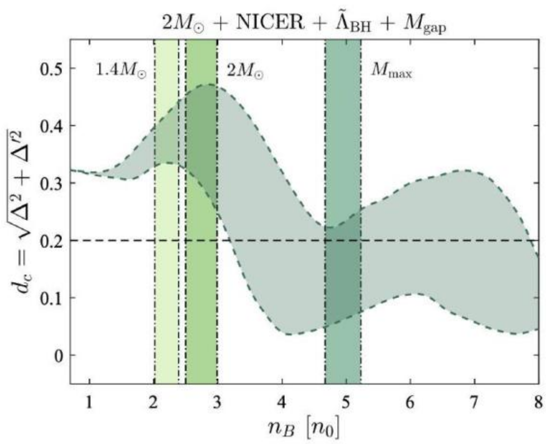


$$\gamma_c = \frac{2}{3}\frac{(6\Lambda-1)\sqrt{1-8\Lambda+4\Lambda^2}+1-10\Lambda+6\Lambda^2}{(4\Lambda-1)\sqrt{1-8\Lambda+4\Lambda^2}+1-8\Lambda+4\Lambda^2} \approx \frac{4}{3}\left(1+\Lambda+\frac{11}{2}\Lambda^2+34\Lambda^3\right)+\cdots$$

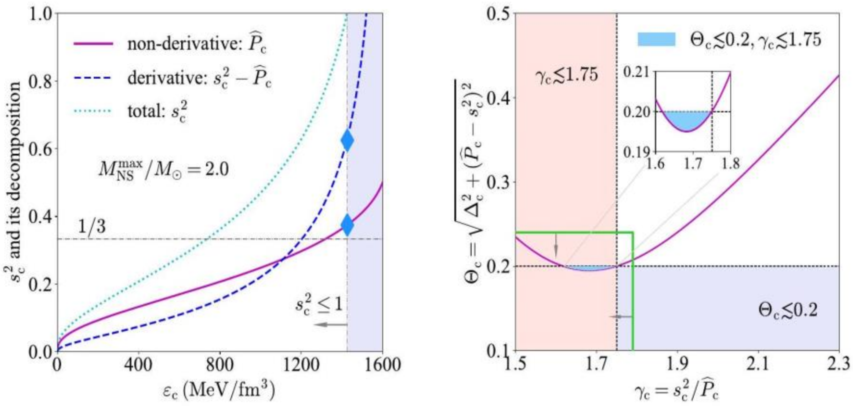
$$\Lambda \approx \frac{105}{173}\frac{\xi_{\max}/\Sigma + 0.106/(R_{\max}/\mathrm{nm})}{1-0.64/(R_{\max}/\mathrm{nm})} \lesssim \frac{105}{173}\frac{\xi_{\max}}{\Sigma},$$

$$\gamma_c = 1 + \frac{1+\Psi}{3}\frac{1+3X^2+4X}{1-3X^2} \geq \frac{4+\Psi}{3} \geq 4/3.$$





$$\Theta \equiv \sqrt{\Delta^2 + (P/\varepsilon - s^2)^2} = \sqrt{(1/3-\phi)^2 + (\phi - s^2)^2}$$



$$\Theta_c \approx 1/3 - \eta + 2\eta^2/3 + 13\eta^3/6 + 41\eta^4/8 + \cdots,$$

$$\Theta_c \approx 0.38 \pm 0.05, R \approx 12^{+1}_{-1}\ \mathrm{nm},$$

$$\gamma/\gamma_c \approx 1 + \frac{b_2}{s_c^2}\left(1 + \frac{2D}{s_c^2} - \frac{s_c^2}{X}\right)\hat{r}^2 \approx 1 - \frac{3D}{16X^2}\hat{r}^2$$

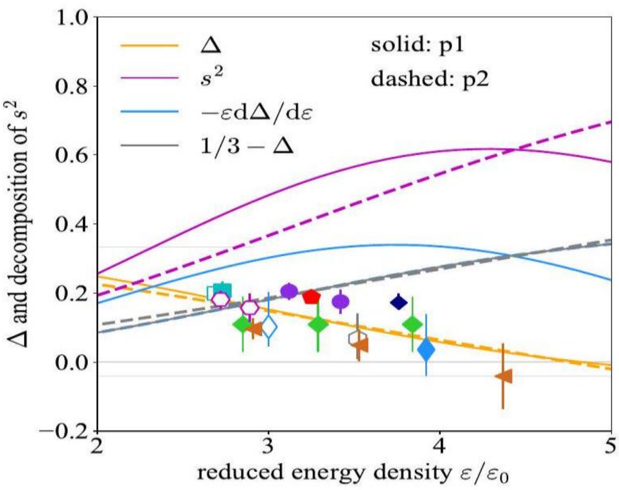
$$\Theta/\Theta_c \approx 1 + \frac{b_2}{s_c^2}\frac{3t_c(1 + 3s_c^2 - 6X - 6D)}{1 + 9s_c^2 - 6X(1 + 3s_c^2) + 18X^2}\hat{r}^2 \approx 1 + \frac{1 - 6D}{8}\hat{r}^2$$

$$\gamma/\gamma_c \approx 1 + \left(\frac{t_c}{X} + \frac{2D}{s_c^2}\right)\mu \approx 1 + \frac{3D}{2X}\mu$$

$$\Theta/\Theta_c \approx 1 + \frac{3t_c + 9s_c^2(t_c + 2D) - 18X(t_c + D)}{1 + 9s_c^2 - 6X(1 + 3s_c^2) + 18X^2}\mu \approx 1 + (6D - 1)X\mu$$

$$\frac{\Delta\Theta}{\Delta\gamma} \approx \left(4 - \frac{2D}{3}\right)X^2 \approx \left(4 - \frac{2D}{3}\right)X^2$$

$$\Delta \approx \frac{1}{3}(1 - ft\bar{\varepsilon}^a)\exp(-t\bar{\varepsilon}^a),$$



$$\Delta \approx \frac{1}{3}\exp(-k_1\bar{\varepsilon}^2) - k_2\bar{\varepsilon},$$

$$X = \frac{P_c}{\varepsilon_c} = \frac{1 - \sqrt{1 - 2\xi}}{3\sqrt{1 - 2\xi} - 1} \approx \frac{\xi}{2}\left(1 + 2\xi + \frac{17}{4}\xi^2 + \frac{37}{4}\xi^3 + \cdots\right), \xi = M_{NS}/R.$$

$$R_{Buch}/\mathrm{nm} \gtrsim 3.32 M_{NS}/M_\odot.$$

$$R_{Buch}^{EDC}/\mathrm{nm} \gtrsim 3.94 M_{NS}/M_\odot.$$

$$R_{\text{Buch}}^{(+)} \gtrsim \frac{(1+3\mathrm{X}_+)^2}{2\mathrm{X}_+(1+2\mathrm{X}_+)} M_{\text{NS}} \leftrightarrow \frac{R_{\text{Buch}}^{(+)}}{\text{nm}} \gtrsim 1.477 \frac{(1+3\mathrm{X}_+)^2}{2\mathrm{X}_+(1+2\mathrm{X}_+)} \left(\frac{M_{\text{NS}}}{M_\odot}\right)$$

$$R_{\text{Buch}}^{\mathrm{X}_+\approx 0.374}/\text{nm} \gtrsim 5.01 M_{\text{NS}}/M_\odot.$$

$$\mathrm{X} = \frac{\xi}{2-5\xi} \approx \frac{\xi}{2}\left(1+\frac{5}{2}\xi+\frac{25}{4}\xi^2+\frac{125}{8}\xi^3+\cdots\right), \text{or } \frac{R_{\text{BF}}^{(+)}}{\text{nm}} \gtrsim 1.477\left(\frac{5}{2}+\frac{1}{2\mathrm{X}_+}\right)\left(\frac{M_{\text{NS}}}{M_\odot}\right)$$

$$R_{\text{BF}}/\text{nm} \gtrsim 3.69 M_{\text{NS}}/M_\odot;\ R_{\text{BF}}^{\text{EDC}}/\text{nm} \gtrsim 4.43 M_{\text{NS}}/M_\odot;\ R_{\text{BF}}^{\mathrm{X}_+\approx 0.374}/\text{nm} \gtrsim 5.67 M_{\text{NS}}/M_\odot;$$

$$R/\text{nm} \gtrsim 4.18 M_{\text{NS}}/M_\odot.$$

$$R/\text{nm} \gtrsim 4.51 M_{\text{NS}}/M_\odot.$$

$$R/\text{nm} \gtrsim 4.34 M_{\text{NS}}/M_\odot,$$

$$R/\text{nm} \gtrsim 3.6 + 3.9 M_{\text{NS}}/M_\odot.$$

$$\xi_{\max} \equiv \frac{M_{\text{NS}}^{\max}}{R_{\max}} = \frac{M_{\text{NS}}^{\max}/M_\odot}{R_{\max}/\text{nm}}\left(\frac{M_\odot}{\text{nm}}\right) < \frac{1.73\times 10^3 \Gamma_{\text{c}}}{1.05\times 10^3 v_{\text{c}}}\left(\frac{M_\odot}{\text{nm}}\right) \approx \frac{2.44\mathrm{X}}{1+3\mathrm{X}^2+4\mathrm{X}} \equiv \xi_{\max}^{(\text{up})}.$$

$$\xi_{\max} \lesssim \Pi_{\text{c}}(0.374)\left(\frac{M_\odot}{\text{nm}}\right)\frac{A_{\text{M}}^{\max}}{A_{\text{R}}^{\max}}\left[1+\frac{\text{nm}}{R_{\max}}\left(\frac{B_{\text{M}}^{\max}A_{\text{R}}^{\max}}{A_{\text{M}}\Pi_{\text{c}}(0.374)}-B_{\text{R}}^{\max}\right)\right] \approx 0.313_{-0.01}^{+0.01}\cdot\left(1-\frac{1.14_{-0.3}^{+0.3}\ \text{nm}}{R_{\max}}\right)$$

$$\xi_{\max} \equiv \xi_{\text{TOV}} \lesssim 0.283_{-0.014}^{+0.014},$$

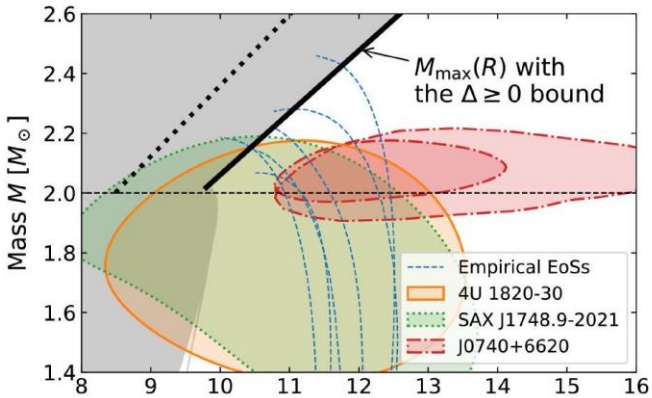
$$1+z = \left(1-\frac{2M_{\text{NS}}}{R_{\text{GR}}}\right)^{-1} = \left(1-\frac{2\xi}{\sqrt{1+z}}\right)^{-1}, R_{\text{GR}} = \sqrt{1+z}R, \xi = M_{\text{NS}}/R$$

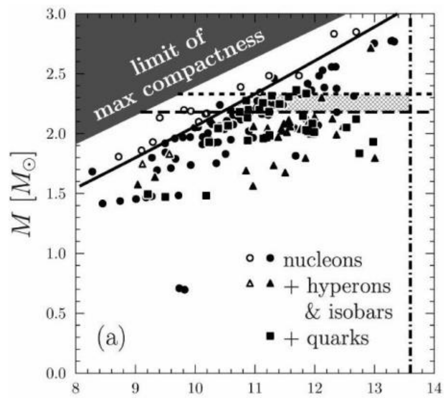
$$\frac{\Delta I}{I} \approx \frac{28\pi P_{\text{t}} R^3}{3M_{\text{NS}}}\frac{1.67\xi-0.6\xi^2}{\xi}\left[1+\frac{2P_{\text{t}}(1+5\xi-14\xi^2)}{\rho_{\text{t}} M_{\text{N}}\xi^2}\right]^{-1},$$

$$R_{\max}/\text{nm} \gtrsim 4.73 M_{\text{NS}}^{\max}/M_\odot + 1.14.$$

$$R_{\max}/\text{nm} \gtrsim 4.83 M_{\text{NS}}^{\max}/M_\odot + 0.04.$$

$$R_{\max}/\text{nm} \gtrsim 3.75 M_{\text{NS}}^{\max}/M_\odot + 2.27,$$





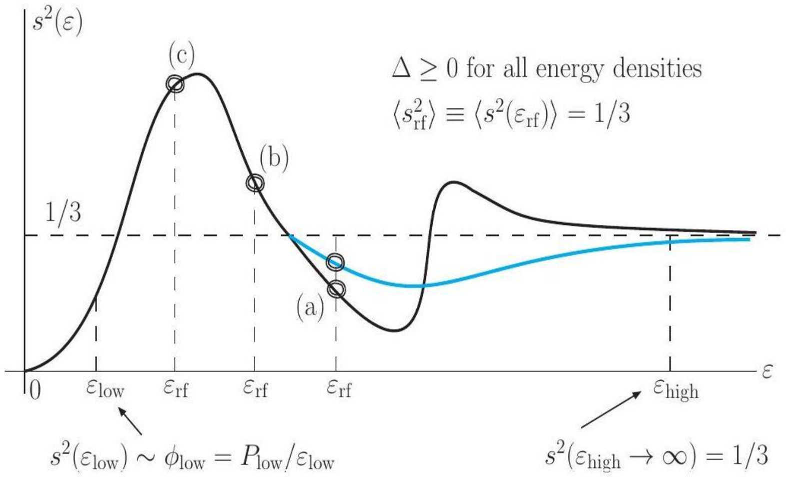
$$X = \frac{2}{15}\left\{\sqrt{\frac{3}{\xi}}\tan\left[\arctan\sqrt{\frac{1}{3}\frac{\xi}{1-2\xi}}+\frac{1}{2}\ln\left(\frac{1}{6}+\sqrt{\frac{1-2\xi}{3\xi}}\right)-\frac{1}{2}\ln\left(\frac{1}{\sqrt{3\xi}}-\frac{5}{6}\right)\right]-\frac{5}{2}\right\}$$
$$\approx \frac{\xi}{2}\left(1+\frac{133}{60}\xi+\frac{599}{112}\xi^2+\frac{17915}{1344}\xi^3+\cdots\right)$$

$$\varepsilon_c \leq \varepsilon_{\text{ult}} \equiv 6.32\left(\frac{M_{\text{NS}}^{\max}}{M_{\odot}} + 0.106\right)^{-2} \text{ GeV/fm}^3$$

$$P_c \leq P_{\text{ult}} \equiv 2.36\left(\frac{M_{\text{NS}}^{\max}}{M_{\odot}} + 0.106\right)^{-2} \text{ GeV/fm}^3$$

$$\varepsilon_{\text{ult}} \approx 7.62\left(\frac{M_{\odot}}{M_{\text{NS}}^{\max}}\right)^2 \text{ GeV/fm}^3, p_{\text{ult}} \approx 5.12\left(\frac{M_{\odot}}{M_{\text{NS}}^{\max}}\right)^2 \text{ GeV/fm}^3,$$

$$\langle s^2(\varepsilon)\rangle \equiv \frac{1}{\varepsilon}\int_0^{\varepsilon} \mathrm{d}\varepsilon' s^2(\varepsilon') = \frac{p}{\varepsilon} = \phi$$



$$\langle s_{\text{rf}}^2\rangle \equiv \frac{1}{\varepsilon_{\text{rf}}}\int_0^{\varepsilon_{\text{rf}}} \mathrm{d}\varepsilon s^2(\varepsilon) = \frac{1}{3} \leftrightarrow \int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\text{low}}}^{\varepsilon_{\text{rf}}} \mathrm{d}\varepsilon s^2(\varepsilon) = \frac{1}{3}\varepsilon_{\text{low}} + \frac{1}{3}(\varepsilon_{\text{rf}} - \varepsilon_{\text{low}}).$$

$$\langle s_{\text{low}}^2\rangle \equiv \frac{1}{\varepsilon_{\text{low}}}\int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3} \leftrightarrow \int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3}\varepsilon_{\text{low}}.$$

$$\alpha \equiv \langle s^2(\varepsilon_{\text{low}} \to \varepsilon_{\text{rf}})\rangle \equiv \frac{1}{\varepsilon_{\text{rf}} - \varepsilon_{\text{low}}}\int_{\varepsilon_{\text{low}}}^{\varepsilon_{\text{rf}}} \mathrm{d}\varepsilon s^2(\varepsilon) \geq \frac{1}{3}$$

$$\alpha = \frac{1}{3} + \frac{3^{-1} - \langle s_{\text{low}}^2\rangle}{1 - \varepsilon_{\text{low}}/\varepsilon_{\text{rf}}}\frac{\varepsilon_{\text{low}}}{\varepsilon_{\text{rf}}} > \frac{1}{3}$$

$$\langle s_{\text{high}}^2\rangle \equiv \frac{1}{\varepsilon_{\text{high}}}\int_0^{\varepsilon_{\text{high}}} \mathrm{d}\varepsilon s^2(\varepsilon) \leq \frac{1}{3}$$

$$\leftrightarrow \frac{1}{\varepsilon_{\text{high}}}\left[\int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\text{low}}}^{\varepsilon_{\text{rf}}} \mathrm{d}\varepsilon s^2(\varepsilon) + \int_{\varepsilon_{\text{rf}}}^{\varepsilon_{\text{high}}} \mathrm{d}\varepsilon s^2(\varepsilon)\right] \leq \frac{1}{3}$$

$$\leftrightarrow \int_{\varepsilon_{\text{rf}}}^{\varepsilon_{\text{high}}} \mathrm{d}\varepsilon s^2(\varepsilon) < \frac{1}{3}\varepsilon_{\text{low}} - \int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon) + \frac{1}{3}(\varepsilon_{\text{high}} - \varepsilon_{\text{rf}})\mathfrak{E} \int_{\varepsilon_{\text{low}}}^{\varepsilon_{\text{rf}}} \mathrm{d}\varepsilon s^2(\varepsilon) > \frac{1}{3}(\varepsilon_{\text{rf}} - \varepsilon_{\text{low}})$$

$$\leftrightarrow \langle s^2(\varepsilon_{\text{rf}} \to \varepsilon_{\text{high}})\rangle \equiv \frac{1}{\varepsilon_{\text{high}} - \varepsilon_{\text{rf}}}\int_{\varepsilon_{\text{rf}}}^{\varepsilon_{\text{high}}} \mathrm{d}\varepsilon s^2(\varepsilon) < \frac{1}{3} + \frac{3^{-1}\varepsilon_{\text{low}} - \int_0^{\varepsilon_{\text{low}}} \mathrm{d}\varepsilon s^2(\varepsilon)}{\varepsilon_{\text{high}} - \varepsilon_{\text{rf}}}$$

$$\langle s^2(\varepsilon_{\text{rf}} \to \varepsilon_{\text{high}})\rangle < \frac{1}{3},$$

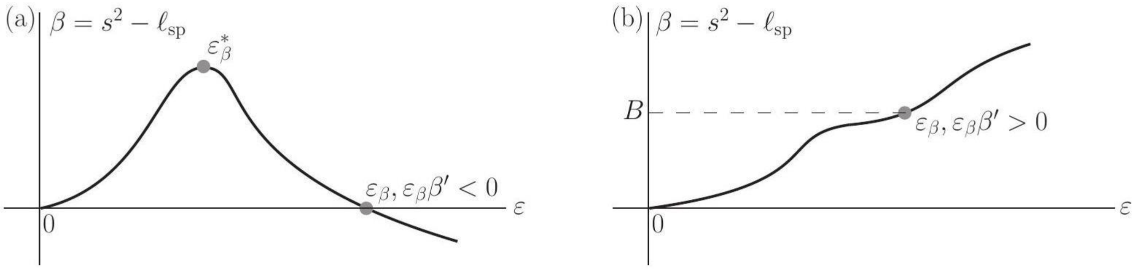
$$\langle s^2(\varepsilon_{\text{low}} \to \varepsilon_{\text{rf}})\rangle \gtrsim \mathrm{X}_{\text{rf}} + \frac{\mathrm{X}_{\text{rf}} - \mathrm{X}_{\text{low}}}{\varepsilon_{\text{rf}}/\varepsilon_{\text{low}} - 1} \approx \frac{(\varepsilon_{\text{rf}}/\varepsilon_{\text{low}})\mathrm{X}_{\text{rf}} - \mathrm{X}_{\text{low}}}{\varepsilon_{\text{rf}}/\varepsilon_{\text{low}} - 1}$$

$$s^2 = \frac{\mathrm{d}p}{\mathrm{d}\varepsilon} = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho}\right) = \frac{2\rho}{\mu}\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho} + \frac{\rho^2}{\mu}\frac{\mathrm{d}^2(\varepsilon/\rho)}{\mathrm{d}\rho^2},$$

$$\ell_{\mathrm{sp}} = \frac{2\rho}{\mu}\frac{\mathrm{d}(\varepsilon/\rho)}{\mathrm{d}\rho} = 2\left(1-\frac{\varepsilon}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\varepsilon}\right) = 2\left(1-\frac{\varepsilon}{\rho}\frac{\rho}{P+\varepsilon}\right) = \frac{2P}{P+\varepsilon} = \frac{2}{1+\phi^{-1}} = 2\frac{\Delta-1/3}{\Delta-4/3}$$

$$\beta = s^2 - \ell_{\mathrm{sp}} = -\varepsilon\mathrm{d}\Delta/\mathrm{d}\varepsilon + (3^{-1}-\Delta)[1+2/(\Delta-4/3)].$$

$$K_{\mathrm{NM}}(\rho) \equiv 9\rho^2\frac{\mathrm{d}^2(\varepsilon/\rho)}{\mathrm{d}\rho^2} = 9\mu\left(s^2-\frac{2P}{P+\varepsilon}\right) = 9\mu\left(s^2-\frac{2}{1+\phi^{-1}}\right), \phi = P/\varepsilon$$



$$\beta(\varepsilon) \approx \beta'(\varepsilon-\varepsilon_\beta) + \frac{1}{2}\beta''(\varepsilon-\varepsilon_\beta)^2,\ \beta' \equiv \left.\frac{\mathrm{d}\beta}{\mathrm{d}\varepsilon}\right|_{\varepsilon=\varepsilon_\beta} < 0,\ \beta'' \equiv \left.\frac{\mathrm{d}^2\beta}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=\varepsilon_\beta}$$

$$\Delta(\varepsilon) \approx \Delta_\beta + \Delta_\beta'(\varepsilon-\varepsilon_\beta) + 2^{-1}\Delta_\beta''(\varepsilon-\varepsilon_\beta)^2.$$

$$\begin{aligned}\beta(\varepsilon) \approx & -\varepsilon_\beta\Delta_\beta' - \frac{1}{3}\frac{2-3\Delta_\beta-9\Delta_\beta^2}{4-3\Delta_\beta} \\ & -\frac{1}{(4-3\Delta_\beta)^2}\left[2(7-24\Delta_\beta+9\Delta_\beta^2)\Delta_\beta' + (16-24\Delta_\beta+9\Delta_\beta^2)\varepsilon_\beta\Delta_\beta''\right](\varepsilon-\varepsilon_\beta) \\ & +\frac{3}{2}\frac{1}{(4-3\Delta_\beta)^3}\left[36\Delta_\beta'^2 - (40-126\Delta_\beta+108\Delta_\beta^2-27\Delta_\beta^3)\Delta_\beta''\right](\varepsilon-\varepsilon_\beta)^2\end{aligned}$$

$$\Delta_\beta' = -\frac{1}{3\varepsilon_\beta}\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)}{4-3\Delta_\beta}$$

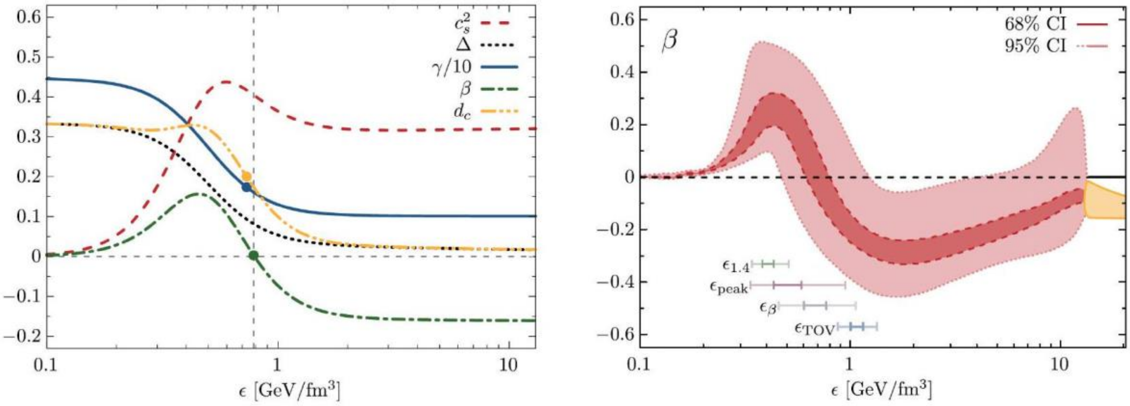
$$\Delta_\beta'' = -\frac{\beta'}{\varepsilon_\beta} + \frac{2}{3\varepsilon_\beta^2}\frac{(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2}{(4-3\Delta_\beta)^3}$$

$$\Delta(\varepsilon) \approx \Delta_\beta - \frac{\delta\varepsilon}{3\varepsilon_\beta}\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)}{4-3\Delta_\beta} - \frac{\delta\varepsilon^2}{2\varepsilon_\beta^2}\left[\varepsilon_\beta\beta' - \frac{2}{3}\frac{(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2}{(4-3\Delta_\beta)^3}\right]$$

$$\varepsilon_\Delta/\varepsilon_\beta = 1 - \frac{\Delta_\beta'}{\Delta_\beta''\varepsilon_\beta} = \frac{3(2+3\Delta_\beta)(1-3\Delta_\beta)(10-24\Delta_\beta+9\Delta_\beta^2) - 3\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}{2(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2 - 3\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}$$

$$\Delta_{\min} = \Delta_\beta - \frac{1}{2}\frac{\Delta_\beta'^2}{\Delta_\beta''} = -\frac{(2+3\Delta_\beta)(1-3\Delta_\beta)^2(8-78\Delta_\beta+27\Delta_\beta^2) + 18\varepsilon_\beta\beta'\Delta_\beta(4-3\Delta_\beta)^3}{12(2+3\Delta_\beta)(7-3\Delta_\beta)(1-3\Delta_\beta)^2 - 18\varepsilon_\beta\beta'(4-3\Delta_\beta)^3}.$$

$$\Delta(\varepsilon) \approx -\frac{1}{6}\frac{\delta\varepsilon}{\varepsilon_\beta} + \frac{1}{2}\left(\frac{7}{48} - \varepsilon_\beta\beta'\right)\frac{\delta\varepsilon^2}{\varepsilon_\beta^2}, \varepsilon_\Delta/\varepsilon_\beta \approx \frac{15-48\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}, \Delta_{\min} \approx \frac{2}{3}\frac{1}{48\varepsilon_\beta\beta'-7}.$$



$$\Delta_\beta \approx \frac{2}{3}\frac{29-240\varepsilon_\beta\beta'+128\varepsilon_\beta^2\beta''}{159-744\varepsilon_\beta\beta'+320\varepsilon_\beta^2\beta''},$$

$$\frac{\varepsilon_{\rm pk}}{\varepsilon_\beta}\approx\frac{10-32\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}\left[1+12\Delta_\beta\cdot\frac{3+8\varepsilon_\beta\beta'}{(5-16\varepsilon_\beta\beta')(7-48\varepsilon_\beta\beta')}\right]$$

$$s_{\rm pk}^2\equiv s^2(\varepsilon_{\rm pk})\approx\frac{1}{32}\frac{121-672\varepsilon_\beta\beta'+256\varepsilon_\beta^2\beta'^2}{7-48\varepsilon_\beta\beta'}\left[1-\frac{3}{4}\frac{2001-34464\varepsilon_\beta\beta'+108800\varepsilon_\beta^2\beta'^2}{(7-48\varepsilon_\beta\beta')(121-672\varepsilon_\beta\beta'+256\varepsilon_\beta^2\beta'^2)}\cdot\Delta_\beta\right],$$

$$\left.\frac{\mathrm{d}^2s^2}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=\varepsilon_{\rm pk}}\approx-\frac{28-192\varepsilon_\beta\beta'-75\Delta_\beta}{64\varepsilon_\beta^2}<0.$$

$$\frac{\varepsilon_{\rm deriv,pk}}{\varepsilon_\beta}\approx\frac{3}{4}\frac{10-32\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}\left[1+12\Delta_\beta\cdot\frac{3+8\varepsilon_\beta\beta'}{(5-16\varepsilon_\beta\beta')(7-48\varepsilon_\beta\beta')}\right]$$

$$\frac{\varepsilon_{\rm pk}-\varepsilon_\beta^*}{\varepsilon_\beta}\approx\frac{3+16\varepsilon_\beta\beta'}{7-48\varepsilon_\beta\beta'}+\frac{\beta'}{\varepsilon_\beta\beta''}$$

$$\Delta_\beta=\frac{4}{3}\frac{s_\beta^2-1/2}{s_\beta^2-2}=\frac{4}{3}\frac{\gamma_\beta-3/2}{\gamma_\beta},\ \text{or}\ s_\beta^2=\frac{2-6\Delta_\beta}{4-3\Delta_\beta},\gamma_\beta=\frac{6}{4-3\Delta_\beta}.$$

$$\gamma(\varepsilon)\approx\frac{6}{4-3\Delta_\beta}\left[1-\frac{\delta\varepsilon}{\varepsilon_\beta}\frac{(2-9\Delta_\beta)-8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(1-3\Delta_\beta)(4-3\Delta_\beta)^2}\right],\beta'<0$$

$$s^2/s_\beta^2\approx1+\frac{2}{s_\beta^2}\frac{\delta\varepsilon}{\varepsilon_\beta}\frac{3(2-3\Delta_\beta)+8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(4-3\Delta_\beta)^3},\beta'<0$$

$$\begin{aligned}\Delta(\varepsilon)\approx{}&\Delta_\beta-\frac{\delta\varepsilon}{3\varepsilon_\beta}\frac{2+12B-(3+9B)\Delta_\beta-9\Delta_\beta^2}{4-3\Delta_\beta}\\&-\frac{\delta\varepsilon^2}{2\varepsilon_\beta^2}\left[\varepsilon_\beta\beta'-\frac{2}{3}\frac{(7-3\Delta_\beta)(1-3\Delta_\beta)[2+12B-(3+9B)\Delta_\beta-9\Delta_\beta^2]}{(4-3\Delta_\beta)^3}\right]\end{aligned}$$

$$\gamma(\varepsilon)\approx\frac{6}{4-3\Delta_\beta}\left[1-\frac{2(1+6B)-9(1+5B)\Delta_\beta-8\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(1-3\Delta_\beta)(4-3\Delta_\beta)^2}\frac{\delta\varepsilon}{\varepsilon_\beta}\right]$$

$$s^2/s_\beta^2\approx1-\frac{1}{s_\beta^2}\frac{18(1+3B)\Delta_\beta-12(1+6B)-16\varepsilon_\beta\beta'(4-9\Delta_\beta)}{(4-3\Delta_\beta)^3}\frac{\delta\varepsilon}{\varepsilon_\beta},s_\beta^2=B+\frac{2-6\Delta_\beta}{4-3\Delta_\beta}$$

$$\frac{6\Delta_\beta-2}{4-3\Delta_\beta}\le B\le\frac{2+3\Delta_\beta}{4-3\Delta_\beta}$$

$$2(1+6B)-9(1+5B)\Delta_\beta-8\varepsilon_\beta\beta'(4-9\Delta_\beta)<0$$

$$0 < B < \min\left[\frac{8\varepsilon_\beta\beta'(4-9\Delta_\beta)-(2-9\Delta_\beta)}{12-45\Delta_\beta}, \frac{2+3\Delta_\beta}{4-3\Delta_\beta}\right], \text{where } \varepsilon_\beta\beta' > 0$$

$$s^2/s_\beta^2 \gtrsim 1 + \frac{\delta\varepsilon}{\varepsilon_\beta}\frac{1}{s_\beta^2}\frac{16(1+6B)-36(1+4B)\Delta_\beta}{(4-3\Delta_\beta)^3}$$

mass and energy compactification $\xi = M_{\mathrm{NS}}/R \leftrightarrow$ central radius $\mathrm{X} = \phi_{\mathrm{c}} = \hat{P}_{\mathrm{c}} = P_{\mathrm{c}}/\varepsilon_{\mathrm{c}}$

$$\leftrightarrow \text{ average SSS from surface to center } \mathrm{X} = \langle s_c^2\rangle = \frac{1}{\varepsilon_{\mathrm{c}}}\int_0^{\varepsilon_{\mathrm{c}}} \mathrm{d}\varepsilon' s^2(\varepsilon')$$

$$\leftrightarrow \text{ central stiffness } s_c^2 = \mathrm{X}\left(1 + \frac{1+\Psi}{3}\frac{1+3\mathrm{X}^2+4\mathrm{X}}{1-3\mathrm{X}^2}\right)$$

$$s_c^2 = \mathrm{X}[1 + 3^{-1}(1+3\mathrm{X}^2+4\mathrm{X})/(1-3\mathrm{X}^2)] \text{ and } s_c^2 \le 1$$

$$\Delta = 1/3 - \phi = 1/3 - P/\varepsilon$$

$$\Delta \gtrsim \Delta_{\mathrm{GR}} \approx -0.041.$$

$$\Pi_{\mathrm{c}} = \mathrm{X}/(1+3\mathrm{X}^2+4\mathrm{X}) \lesssim 0.128;$$

$$R_{\max}/\mathrm{nm} \gtrsim 4.73 M_{\mathrm{NS}}^{\max}/M_\odot + 1.14$$

$$\xi_{\mathrm{TOV}} \equiv \xi_{\max} \equiv M_{\mathrm{NS}}^{\max}/R_{\max} \lesssim 0.313(1 - 1.14\ \mathrm{nm}/R_{\max})$$

$$\varepsilon_{\mathrm{c}} \lesssim \varepsilon_{\mathrm{ult}} \approx 6.32\left(M_{\mathrm{NS}}^{\max}/M_\odot + 0.106\right)^{-2}\mathrm{GeV/fm^3}$$

$$P_{\mathrm{c}} \lesssim P_{\mathrm{utl}} \approx 2.36\left(M_{\mathrm{NS}}^{\max}/M_\odot + 0.106\right)^{-2}\mathrm{GeV/fm^3},$$

$$M_{\mathrm{NS}}^{\max}/M_\odot \approx 2.3,\ P_{\mathrm{ult}} \approx 408\mathrm{MeV/fm^3}.$$

$$\rho_{\mathrm{c}}/\rho_{\mathrm{sat}} \approx 7350\mathrm{X}/(1+3\mathrm{X}^2+4\mathrm{X}) \cdot (R_{\max}/\mathrm{nm} - 0.64)^{-2},$$

$$\beta = -\varepsilon\mathrm{d}\Delta/\mathrm{d}\varepsilon + (3^{-1} - \Delta)[1 + 2/(\Delta - 4/3)]$$

$$\varepsilon_\beta \approx 1\mathrm{GeV} \triangle \int \beta(\varepsilon_\beta)$$

$$-\varepsilon_\beta\beta' \approx \mathcal{O}(1), \Delta_\beta \lesssim 0.1$$

$$(s^2(r) > s_c^2 \equiv s^2(r=0))$$

$$\phi \sim \mathrm{X} = P_{\mathrm{c}}/\varepsilon_{\mathrm{c}}$$

$$\Delta s^2 = s_{\max}^2/s_c^2 - 1$$

$$s_{\max}^2 \equiv s^2(r_{\mathrm{pk}})$$

$$s^2 \approx s_c^2 - 4\hat{r}^2/15 + \hat{r}^4/60s_c^2$$

$$\hat{r}^2 = 6s_c^2$$

$$\mathrm{d}s^2/\mathrm{d}\hat{r} \approx -(8\hat{r}/15)(1 - \hat{r}^2/8s_c^2)$$

$$\Theta = [(3^{-1} - \phi)^2 + (s^2 - \phi)^2]^{1/2}$$

$$s^2/\phi = (4/3 + 4X/3 + 32\mu/15)X/\phi$$

$$s^2(\phi)/\mathrm{X} \approx 4/3 + (32/5)(1 - 19\mathrm{X}/4)(\phi/\mathrm{X} - 1) - (876/25)(1 - 3439\mathrm{X}/219)(\phi/\mathrm{X} - 1)^2$$

$$\Theta_{\mathrm{N}} \approx 3^{-1} - \phi + 3\phi^2/2 \gtrsim 0.2 \blacksquare \phi \approx 0.$$

$$M_{\mathrm{NS}}^{\max} \equiv M_{\mathrm{TOV}}$$

$\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}}$ depends of Ψ and $\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \sim (1-2\Psi^{-1})M_{\mathrm{NS}}^{-(2/3)(1+\Psi^{-1})}$ or $\mathrm{d}R/\mathrm{d}M_{\mathrm{NS}} \sim (1-2\Psi^{-1})R^{-2(1+\Psi^{-1})/(1-2\Psi^{-1})} \sim \Psi - 2$

$$M_{\mathrm{NS}}/M_{\odot} \approx 1.4 - 2.2, \text{ i.e., } M_{\mathrm{NS}} \sim \varepsilon_{\mathrm{c}}^{\Psi/2} \approx \varepsilon_{\mathrm{c}} \text{ for } \Psi \approx 2.$$

$$\Delta(\bar{\varepsilon}) \approx \Delta_\ell + 2^{-1}\Delta_\ell''(\bar{\varepsilon} - \bar{\varepsilon}_\ell)^2 + 6^{-1}\Delta_\ell'''(\bar{\varepsilon} - \bar{\varepsilon}_\ell)^3$$

$$\Delta_\ell \equiv \Delta(\bar{\varepsilon}_\ell), \Delta_\ell'' = \Delta''(\bar{\varepsilon}_\ell)$$

$$\Delta_\ell''' = \Delta'''(\bar{\varepsilon}_\ell), \bar{\varepsilon}_\ell$$

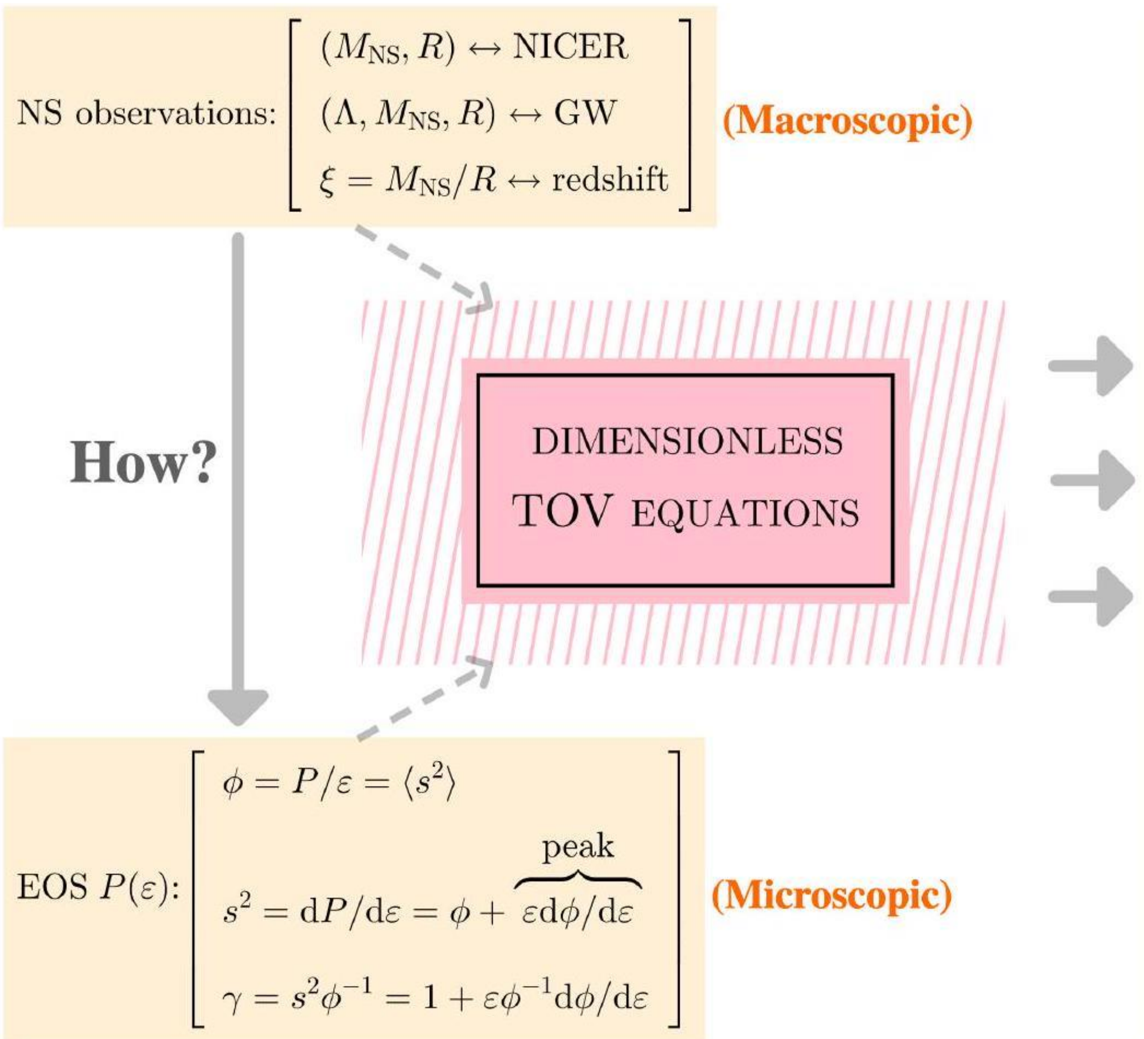
$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \equiv \mathrm{H}(k) \equiv (3/4)(1-k^{-1}) + (4k)^{-1}\sqrt{k^2 - 2k + 9}$$

$$k \equiv \bar{\varepsilon}_\ell \Delta_\ell'''/\Delta_\ell'' = \mathrm{dln}\ \Delta_\ell''/\mathrm{dln}\ \bar{\varepsilon}_\ell$$

$$1/2 \lesssim \bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell \lesssim 1$$

$$\bar{\varepsilon}_\ell^*/\bar{\varepsilon}_\ell$$

$$\Delta_\ell'''' \equiv \Delta_\ell^{(4)}$$

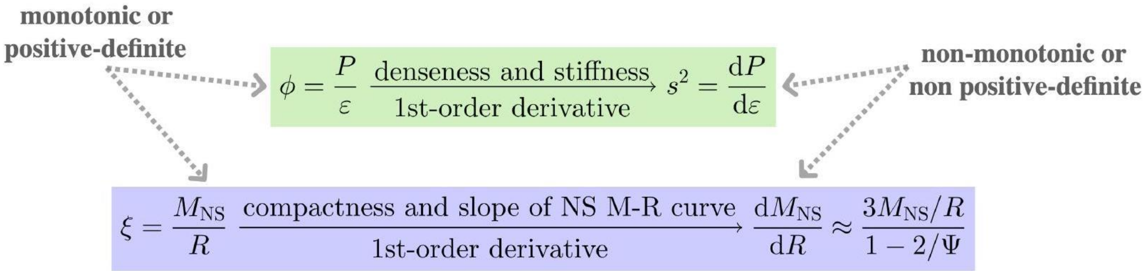


$$M_{\mathrm{NS}} \sim \frac{\Pi_{\mathrm{c}}^{3/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}, R \sim \frac{\Pi_{\mathrm{c}}^{1/2}}{\sqrt{\varepsilon_{\mathrm{c}}}}, \xi \sim \Pi_{\mathrm{c}}$$
$$\Pi_{\mathrm{c}} = \mathrm{X}/[1 + 3\mathrm{X}^2 + 4\mathrm{X}]$$

$$\xi = M_{\mathrm{NS}}/R \leftrightarrow \mathrm{X} = \langle s_c^2 \rangle \leftrightarrow s_c^2$$

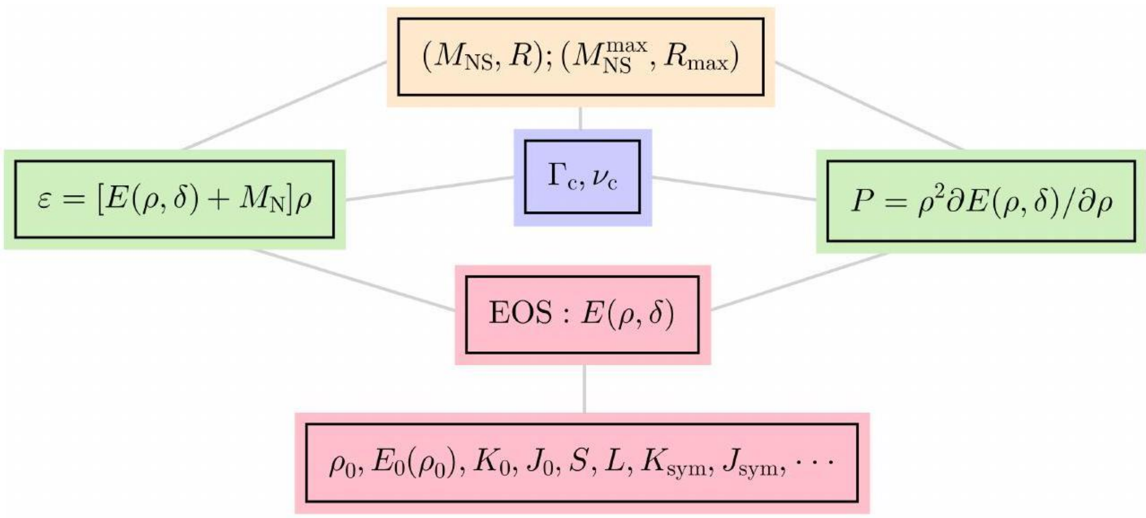
$$s_{\mathrm{c}}^2 = \mathrm{X}\left[1 + \frac{1+\Psi}{3}\frac{1 + 3\mathrm{X}^2 + 4\mathrm{X}}{1 - 3\mathrm{X}^2}\right]$$
$$\Psi = 2\,\mathrm{dln}\ M_{\mathrm{NS}}/\mathrm{dln}\ \varepsilon_{\mathrm{c}}$$

$$\Delta \equiv 1/3 - P/\varepsilon \gtrsim -0.04$$



$$k_2 = \frac{8}{5}\xi^5(1-2\xi)^5[2 - y_R + 2\xi(y_R - 1)] \times \{6\xi[2 - y_R + \xi(5y_R - 8)]$$
$$+4\xi^3[13 - 11y_R + \xi(3y_R - 2) + 2\xi^2(1 + y_R)]$$
$$+3(1 - 2\xi)^2 \ln(1 - 2\xi)[2 - y_R + 2\xi(y_R - 1)]\}$$

$$vy' + y^2 + ye^{\lambda}[1 + 4\pi r^2(P - \varepsilon)] + r^2 Q = 0, Q = 4\pi e^{\lambda}\left(5\varepsilon + 9P + \frac{\varepsilon + P}{\mathrm{d}P/\mathrm{d}\varepsilon}\right) - \frac{6e^{\lambda}}{r^2} - v'^2$$



$$L_{\mathrm{ChS}}^{(5)} = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde}\left(\frac{2}{3}R^{ab}e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e\right)$$

$$\varepsilon_{abcde} R^{cd} T^e = 0,$$
$$\alpha_3 l^2 \varepsilon_{abcde} R^{bc} R^{de} = -\frac{\delta L_{\mathrm{M}}}{\delta h^a},$$
$$\varepsilon_{abcde}(2\alpha_3 R^{bc} e^d e^e + \alpha_1 l^2 R^{bc} R^{de} + 2\alpha_3 l^2 D_\omega k^{bc} R^{de}) = -\frac{\delta L_{\mathrm{M}}}{\delta e^a},$$
$$2\varepsilon_{abcde}\left(\alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{cd} T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 R^{cd} D_\omega h^e + \alpha_3 l^2 R^{cd} k_f^e e^f\right) = -\frac{\delta L_{\mathrm{M}}}{\delta \omega^{ab}}.$$

$$de^a + \omega^a_b e^b = 0,$$
$$\varepsilon_{abcde} R^{cd} D_\omega h^e = 0,$$
$$\alpha_3 l^2 \star (\varepsilon_{abcde} R^{bc} R^{de}) = -\star\left(\frac{\delta L_\mathrm{M}}{\delta h^a}\right),$$
$$\star(\varepsilon_{abcde} R^{bc} e^d e^e) + \frac{1}{2\alpha} l^2 \star (\varepsilon_{abcde} R^{bc} R^{de}) = \kappa_\mathrm{E} T_{ab} e^b,$$

$$\alpha = \alpha_3/\alpha_a, \kappa_\mathrm{E} = \kappa/2\alpha_3, T_{ab} = \star(\delta L_\mathrm{M}/\delta e^a)\star$$

$$ds^2 = -e^{2f(r)}dt^2 + e^{2g(r)}dr^2 + r^2 d\Omega_3^2 = \eta_{ab} e^a e^b$$

$$e^T = e^{f(r)}dt, e^R = e^{g(r)}dr, e^1 = r d\theta_1, e^2 = r\sin\,\theta_1 d\theta_2, e^3 = r\sin\,\theta_1 \sin\,\theta_2 d\theta_3$$

$$\frac{e^{-2g}}{r^2}(g'r + e^{2g} - 1) + \mathrm{sgn}(\alpha) l^2 \frac{e^{-2g}}{r^3} g'(1 - e^{-2g}) = \frac{\kappa_\mathrm{E}}{12}\rho,$$
$$\frac{e^{-2g}}{r^2}(f'r - e^{2g} + 1) + \mathrm{sgn}(\alpha) l^2 \frac{e^{-2g}}{r^3} f'(1 - e^{-2g}) = \frac{\kappa_\mathrm{E}}{12} p,$$
$$\frac{e^{-2g}}{r^2}\{(-f'g'r^2 + f''r^2 + (f')^2 r^2 + 2f'r - 2g'r - e^{2g} + 1)$$
$$+\mathrm{sgn}(\alpha) l^2 (f'' + (f')^2 - f'g' - e^{-2g} f'' - e^{-2g}(f')^2 + 3e^{-2g} f'g')\} = \frac{\kappa_\mathrm{E}}{4} p.$$

$$D_\omega(\star T_a) = 0$$

$$f'(r) = -\frac{p'(r)}{\rho(r) + p(r)}$$

$$e^{-2g(r)} = 1 + \mathrm{sgn}(\alpha)\frac{r^2}{l^2} - \mathrm{sgn}(\alpha)\sqrt{\frac{r^4}{l^4} + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{6\pi^2 l^2}\mathcal{M}(r)},$$

$$\mathcal{M}(r) = 2\pi^2 \int_0^r \rho(\bar{r})\bar{r}^3 d\bar{r}$$

$$\frac{df(r)}{dr} = f'(r) = \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E} p(r) r^3 + 12r\left(1 - e^{-2g(r)}\right)}{12 l^2 e^{-2g(r)}\left(1 - e^{-2g(r)} + \mathrm{sgn}(\alpha)\frac{r^2}{l^2}\right)}$$

$$\frac{dp(r)}{dr} = p'(r) = -\mathrm{sgn}(\alpha)\frac{(\rho(r) + p(r))\left(\kappa_\mathrm{E} p(r) r^3 + 12r\left(1 - e^{-2g(r)}\right)\right)}{12 l^2 e^{-2g(r)}\left(1 - e^{-2g(r)} + \mathrm{sgn}(\alpha)\frac{r^2}{l^2}\right)}$$

$$\frac{dp(r)}{dr} = -\frac{\kappa_\mathrm{E}\mathcal{M}(r)\rho(r)}{12\pi^2 r^3}\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{6\pi^2 r^4} l^2 \mathcal{M}(r)\right)^{-1/2}$$
$$\times\left[\frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} - \frac{12\mathrm{sgn}(\alpha)\pi^2 r^4}{\kappa_\mathrm{E} l^2 \mathcal{M}(r)}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{6\pi^2 r^4} l^2 \mathcal{M}(r)}\right)\right]$$
$$\times\left[1 + \mathrm{sgn}(\alpha)\frac{r^2}{l^2}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{6\pi^2 r^4} l^2 \mathcal{M}(r)}\right)\right]^{-1}$$

$$\sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{6\pi^2 r^4} l^2 \mathcal{M}(r)} = 1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{E}}{12\pi^2 r^4} l^2 \mathcal{M}(r) + O(l^4)$$

$$\frac{dp(r)}{dr} \approx -\frac{\frac{\kappa_\mathrm{E}\mathcal{M}(r)\rho(r)}{12\pi^2 r^3}\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right)}{\left(1 + \mathrm{sgn}(\alpha)\frac{\kappa_\mathrm{F}}{6\pi^2 r^4} l^2 \mathcal{M}(r)\right)\left(1 - \frac{\kappa_\mathrm{F}}{12\pi^2 r^2}\mathcal{M}(r)\right)}$$

$$\frac{dp(r)}{dr} = p'(r) \approx -\frac{\kappa_E \mathcal{M}(r)}{12\pi^2 r^3}\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right)\left(1 - \frac{\kappa}{12\pi^2 r^2}\mathcal{M}(r)\right)^{-1}$$

$$\mathcal{M}'(r) = 2\pi^2 r^3 \rho(r)$$

$$f(r) = -\int_r^\infty \frac{\kappa_E \mathcal{M}(\bar{r})}{12\pi^2\bar{r}^3}\left(1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{6\pi^2\bar{r}^4}l^2\mathcal{M}(\bar{r})\right)^{-1/2}$$
$$\times\left[\frac{\pi^2\bar{r}^4 p(\bar{r})}{\mathcal{M}(\bar{r})} - \frac{12\mathrm{sgn}(\alpha)\pi^2\bar{r}^4}{\kappa_E l^2\mathcal{M}(\bar{r})}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{6\pi^2\bar{r}^4}l^2\mathcal{M}(\bar{r})}\right)\right]$$
$$\times\left[1 + \mathrm{sgn}(\alpha)\frac{\bar{r}^2}{l^2}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{6\pi^2\bar{r}^4}l^2\mathcal{M}(\bar{r})}\right)\right]^{-1} d\bar{r}$$

$$\mathcal{M}(r) = M\,, p(r) = \rho(r) = 0$$

$$f(r) = \frac{1}{2}\ln\left[1 + \mathrm{sgn}(\alpha)\frac{r^2}{l^2}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{6\pi^2 r^4}l^2 M}\right)\right],$$

$$e^{2f(r)} = e^{-2g(r)} = 1 + \mathrm{sgn}(\alpha)\frac{r^2}{l^2} - \mathrm{sgn}(\alpha)\sqrt{\frac{r^4}{l^4} + \mathrm{sgn}(\alpha)\frac{\kappa_E}{6\pi^2 l^2}M},$$

$$\rho_0 + p(r) = Ce^{-f(r)}$$

$$\mathcal{M}(r) = \frac{\pi^2}{2}\rho_0 r^4$$

$$e^{-2g(r)} = 1 + \mathrm{sgn}(\alpha)\frac{r^2}{l^2} - \mathrm{sgn}(\alpha)\sqrt{\frac{r^4}{l^4} + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12 l^2}\rho_0 r^4}$$

$$\frac{e^{-2g}}{r^3}(f' + g')[r^2 + \mathrm{sgn}(\alpha)l^2(1 - e^{-2g})] = \frac{\kappa_E}{12}(\rho_0 + p).$$

$$e^f = \frac{\kappa_E}{12}Ce^{-g}\int \frac{r^3 dr}{e^{-3g}[r^2 + \mathrm{sgn}(\alpha)l^2(1 - e^{-2g})]} + C_0 e^{-g}$$

$$\int \frac{r^3 dr}{e^{-3g}[r^2 + \mathrm{sgn}(\alpha)l^2(1 - e^{-2g})]} = \frac{-\mathrm{sgn}(\alpha)l^2 e^{g(r)}}{\sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)}$$

$$e^f = C_1 + C_0 e^{-g}$$

$$C_1 := -\frac{\mathrm{sgn}(\alpha)\kappa_E l^2 C}{12\sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)}$$

$$C = \rho_0\sqrt{1 + \mathrm{sgn}(\alpha)\frac{R^2}{l^2}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)},$$

$$C_1 = -\frac{\mathrm{sgn}(\alpha)\kappa_E l^2\rho_0\sqrt{1 + \mathrm{sgn}(\alpha)\frac{R^2}{l^2}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)}}{12\sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\left(1 - \sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)},$$

$$C_0 = -\frac{1}{\sqrt{1 + \mathrm{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu T^{\mu r} = \frac{f'(r)(\rho(r) + p(r)) + p'(r)}{e^{2g(r)}}$$

$$f' = -\frac{p'}{\rho + p}$$

$$f'(r) = \frac{\kappa_{\mathrm{E}}\mathcal{M}(r)}{12\pi^2 r^3}\left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right)\left(1 - \frac{\kappa_{\mathrm{E}}}{12\pi^2 r^2}\mathcal{M}(r)\right)^{-1}$$

$$p'(r) = -\frac{\kappa_{\mathrm{E}}\mathcal{M}(r)}{12\pi^2 r^3}\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right)\left(1 - \frac{\kappa_{\mathrm{E}}}{12\pi^2 r^2}\mathcal{M}(r)\right)^{-1}$$

$$\hat{T}_a := T_{\mu\nu} e_a^{\mu} dx^{\nu}$$

$$\nabla_\mu T_\nu^\mu = -e_\nu^a \star D_\omega(\star \hat{T}_a)$$

$$\star \hat{T}_a = \frac{\sqrt{-g}}{4!}\epsilon_{\mu\nu\rho\sigma\tau} T_a^{\mu} dx^{\nu} dx^{\rho} dx^{\sigma} dx^{\tau}.$$

$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}) T_\nu{}^{\lambda} + \partial_\lambda T_\nu{}^{\lambda} - T_a{}^{\lambda}(\partial_\lambda e_\nu^a + \omega_{\lambda b}^{a} e_\nu^b),$$

$$\partial_\lambda e_\nu^a + \omega^a{}_{\lambda b} e_\nu^b - \Gamma_{\lambda\nu}{}^{\rho} e_\rho^a = 0,$$

$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}) T_\nu{}^{\rho} + \partial_\lambda T_\nu{}^{\lambda} - \Gamma_{\lambda\nu}{}^{\rho} T_\rho{}^{\lambda}$$
$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \partial_\lambda T_\nu{}^{\lambda} + \Gamma_{\lambda\rho}{}^{\lambda} T_\nu{}^{\rho} - \Gamma_{\lambda\nu}{}^{\rho} T_\rho{}^{\lambda} = \nabla_\lambda T_\nu{}^{\lambda}$$

$$P = \frac{1}{p!} P_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \cdots dx^{\alpha_p}$$

$$\star P = \frac{\sqrt{|g|}}{(d-p)!\,p!}\varepsilon_{\alpha_1 \cdots \alpha_d} g^{\alpha_1 \beta_1} \cdots g^{\alpha_p \beta_p} P_{\beta_1 \cdots \beta_p} dx^{\alpha_{p+1}} \cdots dx^{\alpha_d}$$

$$\hat{T}_a = T_{ab} e^b$$

$$T_{TT} = \rho(r)\,, T_{RR} = T_{ii} = p(r)$$

$$D_\omega(\star \hat{T}_a) = 0$$

$$D_\omega(\star \hat{T}_a) = D_\omega(T_{ab} \star e^b) = \frac{1}{4!}\epsilon_{fbcde}(D_\omega T_a^f) e^b e^c e^d e^e$$

$$D_\omega(\star \hat{T}_a) = \frac{1}{4!}\epsilon_{fbcde}(dT_a^f + \omega_a^g T_g^f + \omega_g^f T_a^g) e^b e^c e^d e^e$$

$$D_\omega(\star \hat{T}_R) = e^{-g}(p' + f'(\rho + p)) e^T e^R e^1 e^2 e^3 = 0$$

$$p' + f'(\rho + p) = 0$$

$$h_a = h_{\mu\nu} e_a^{\mu} dx^{\nu}$$

$\xi_0 = \partial_t$
$\xi_1 = \partial_{\theta_3}$
$\xi_2 = \sin\,\theta_3 \partial_{\theta_2} + \cot\,\theta_2 \cos\,\theta_3 \partial_{\theta_3}$
$\xi_3 = \sin\,\theta_2 \sin\,\theta_3 \partial_{\theta_1} + \cot\,\theta_1 \cos\,\theta_2 \sin\,\theta_3 \partial_{\theta_2} + \cot\,\theta_1 \csc\,\theta_2 \cos\,\theta_3 \partial_{\theta_3}$
$\xi_4 = \cos\,\theta_3 \partial_{\theta_2} - \cot\,\theta_2 \sin\,\theta_3 \partial_{\theta_3}$
$\xi_5 = \sin\,\theta_2 \cos\,\theta_3 \partial_{\theta_1} + \cot\,\theta_1 \cos\,\theta_2 \cos\,\theta_3 \partial_{\theta_2} - \cot\,\theta_1 \csc\,\theta_2 \sin\,\theta_3 \partial_{\theta_3}$
$\xi_6 = \cos\,\theta_2 \partial_{\theta_1} - \cot\,\theta_1 \sin\,\theta_2 \partial_{\theta_2}$

$$h^T = h_{tt}(r)e^T + h_{tr}(r)e^R$$
$$h^R = h_{rt}(r)e^T + h_{rr}(r)e^R$$
$$h^i = h(r)e^i$$

$$\epsilon_{abcde}R^{cd}Dh^e = 0$$

$$Dh^a = dh^a + \omega^a_b h^b$$

$$Dh^T = e^{-g}(-h'_{tt} - f'h_{tt} + f'h_{rr})e^T e^R$$
$$Dh^R = e^{-g}(-h'_{rt} - f'h_{rt} + f'h_{tr})e^T e^R$$
$$Dh^i = \frac{e^{-g}}{r}(rh' + h - h_{rr})e^R e^i - \frac{e^{-g}}{r}h_{rt}e^T e^i$$

$$h_{tr} = h_{rt} = 0$$
$$h_r = (rh)'$$
$$h'_t = f'(h_r - h_t)$$

$$h_t(r) = h_t(f(r))$$

$$\frac{dh_t(f)}{df}f'(r) = f'(h_r - h_t)$$

$$\dot{h}_t + h_t = h_r$$

$$\dot{h}_t := \frac{dh_t(f)}{df}$$

$$h_t^{\mathrm{h}}(f) = Ae^{-f(r)}$$

$$h_r(r) = h_r(f(r)) = \sum_{n=0}^{\infty} B_n e^{nf(r)} + \sum_{m=2}^{\infty} C_m e^{-mf(r)},$$

$$h_t^{\mathrm{p}}(f) = \sum_{n=0}^{\infty} \frac{B_n}{n+1}e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1}e^{-mf(r)}.$$

$$h_t(f(r)) = Ae^{-f(r)} + \sum_{n=0}^{\infty} \frac{B_n}{n+1}e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1}e^{-mf(r)}$$

$$h(r) = \frac{1}{r}\left(\int h_r(r)dr + D\right)$$

$$h(r) = \frac{1}{r}\sum_{n=0}^{\infty}\left(B_n \int e^{nf(r)}dr\right) + \frac{1}{r}\sum_{m=2}^{\infty}\left(C_m \int e^{-mf(r)}dr\right) + \frac{D}{r}$$

$$h_r(r) = h = \aleph$$

$$h(r) = h + \frac{D}{r}$$

$$h_t(r) = Ae^{-f(r)} + h.$$

$$e^{2f(r\to\infty)} = e^{-2g(r\to\infty)} = 1$$

$$h_r(r\to\infty) = h, h(r\to\infty) = h, h_t(r\to\infty) = A + h.$$

$$h_r(r) = h, h(r) = h$$

$$h_t(r) = \begin{cases} \frac{A}{C_0 + C_1 e^{-g(r)}} + h & \text{if } r < R \\ \frac{A}{e^{-g(r)}} + h & \text{if } r \geq R \end{cases}$$

$$e^{-g(r)} = \begin{cases} \sqrt{1 + \text{sgn}(\alpha)\frac{r^2}{l^2} - \text{sgn}(\alpha)\sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha)\frac{\kappa_E}{6\pi^2 l^2}\mathcal{M}(r)}} & \text{if } r < R \\ \sqrt{1 + \text{sgn}(\alpha)\frac{r^2}{l^2} - \text{sgn}(\alpha)\sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha)\frac{\kappa_E}{6\pi^2 l^2}M}} & \text{if } r \geq R \end{cases}$$

Aclaraciones hasta aquí, en relación a este apartado:

1. Todas las figuras arriba colocadas, describen el comportamiento y características de funcionamiento de una partícula blanca o estrella y sus interacciones en relación al campo cuántico en el que incida a propósito de su manifestación física.

2. Todas las magnitudes, están calculadas a escala cuántica, más de existir contradicción (muy probablemente se haga referencia a magnitudes a escala macroscópica), prevalecerá el método de cálculo que corresponda a nivel subatómico, todo esto, dentro de las hipótesis teóricas contenidas en la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) formulada por este autor, tanto en este artículo como en otros trabajos.

Modelo Matemático de Cuantización de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR). Modelo Unificado.

$$\ddot{q} = -U'(q),$$

$$\mathcal{L}(q) := \frac{\dot{q}^2}{2} - U(q)$$

$$S(q) := \int_a^b \mathcal{L}(q)dt$$

$$\left.\frac{d}{ds}\right|_{s=0} \int_a^b \mathcal{L}(q + s\varepsilon)dt = \int_a^b \left(\frac{\partial \mathcal{L}}{\partial q}\varepsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon}\right)dt =$$
$$\int_a^b (-U'(q)\varepsilon + \dot{q}\dot{\varepsilon})dt = -\int_a^b (U'(q) + \ddot{q})\varepsilon dt$$

$$U''(q) < \frac{\pi^2}{(b-a)^2}$$

$$\int_a^b \varepsilon'(t)^2 dt \geq \frac{\pi^2}{(b-a)^2}\int_a^b \varepsilon(t)^2 dt\Big)$$

$$\left.\frac{d}{ds}\right|_{s=0} \int_{\mathbb{R}} \mathcal{L}(q + s\varepsilon)dt := \int_{\mathbb{R}} \left(\frac{\partial \mathcal{L}}{\partial q}\varepsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon}\right)dt$$

$$S(\phi) = \int_D \mathcal{L}(\phi)dxdt$$

$$\mathcal{L}(\phi) = \frac{1}{2}(\phi_t^2 - v^2(\nabla\phi)^2)$$

$$\frac{d}{ds}\Big|_{s=0} \int_{\mathbb{R}^{d+1}} \mathcal{L}(u + s\varepsilon)dxdt = 0$$

$$U: \mathbb{R}^k \to \mathbb{R}$$

$$S(y) := \int_a^b \left(\frac{1}{2}y'^2 + U(y)\right) dt$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq$$

$$q: [a, b] \to \mathbb{R}^n, q(a) = \mathbf{a}, q(b) = \mathbf{b}$$

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}} Dq = 1$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \frac{1}{Z} \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq$$

$$Z := \int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa} Dq}$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n)$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{\frac{iS(q)}{\hbar}} Dq$$

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{\frac{iS(q)}{\hbar}} Dq = 1$$

$$S(q) = \int_a^b \left(\frac{\dot{q}^2}{2} - U(q)\right) dt$$

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n).$$

$$\langle \phi_{j_1}(x_1, t_1) \dots \phi_{j_n}(x_n, t_n) \rangle = \int \phi_{j_1}(x_1, t_1) \dots \phi_{j_n}(x_n, t_n) e^{\frac{iS(\phi)}{\hbar}} D\phi$$

$$\int e^{\frac{iS(\phi)}{\hbar}} D\phi = e^{\frac{iS(q)}{\hbar} Dq} e^{-\frac{1}{2}B(x,x)}$$

$$\int g(x) e^{-\frac{f(x)}{\kappa}} dx, \int g(x) e^{\frac{if(x)}{\hbar}} dx$$

$$B(x, y) = \sum_{j=1}^{d} a_j x_j y_j$$

$$(\det B)^{-\frac{1}{2}} = e^{\frac{\pi i \sigma(Q)}{4}} |\det Q|^{-\frac{1}{2}}$$

$$\mathcal{F}: \mathcal{S}(V) \to \mathcal{S}(V^*)$$

$$\mathcal{F}(g)(p) := (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i(p,x)} dx$$

$$(\mathcal{F}^2 g)(x) = g(-x)$$

$$\mathcal{F}: \mathcal{S}'(V) \to \mathcal{S}'(V^*)$$

$$\mathcal{F}\left(e^{-\frac{1}{2}B(x,x)}\right) = (\det B)^{-\frac{1}{2}} e^{-\frac{1}{2}B^{-1}(p,p)}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx-\frac{1}{2}ax^2} dx = \frac{1}{\sqrt{a}} e^{-\frac{1}{2a}p^2}$$

$$\frac{e^{-\frac{1}{2}a^{-1}p^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+ia^{-1}p} e^{-\frac{1}{2}ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$(2\pi)^{-\frac{d}{2}} \int_V e^{-\frac{1}{2}B(x,x)} dx = (\det B)^{-\frac{1}{2}}$$

$$I_g(\hbar) := \int_V g\left(\hbar^{\frac{1}{2}} x\right) e^{-\frac{1}{2}B(x,x)} dx, \hbar \geq 0$$

$$I_g(0) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0)$$

$$\Delta_B: \mathcal{S}(V) \to \mathcal{S}(V)$$

$$B: \Delta_B = \sum_{j=1}^{d} \partial_{B^{-1}e_j^*} \partial_{e_j}$$

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar), \hbar \geq 0$$

$$I_g(\hbar) = \left(g\left(\hbar^{\frac{1}{2}} x\right), e^{-\frac{1}{2}B(x,x)}\right) =$$
$$\hbar^{-\frac{d}{2}} (\det B)^{-\frac{1}{2}} \left(\hat{g}\left(\hbar^{-\frac{1}{2}} p\right), e^{-\frac{1}{2}B^{-1}(p,p)}\right) = (\det B)^{-\frac{1}{2}} \left(\hat{g}(p), e^{-\frac{\hbar}{2}B^{-1}(p,p)}\right),$$

$$e^{-\frac{\hbar}{2}B^{-1}(p,p)} \to 1 \text{ in } \mathcal{S}'(V^*) \text{ as } \hbar \to 0$$

$$\lim_{\hbar \to 0} I_g(\hbar) = (\det B)^{-\frac{1}{2}} (\hat{g}(p), 1) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0) = I_g(0),$$

$$I_{\ell f}(\hbar) = \hbar I_{\partial_{B^{-1}\ell} f}(\hbar)$$

$$I_{\ell f}(\hbar) = \hbar^{\frac{1}{2}} \left(\ell(x) f\left(\hbar^{\frac{1}{2}} x\right), e^{-\frac{1}{2}B(x,x)}\right) = \hbar^{\frac{1}{2}} \left(f\left(\hbar^{\frac{1}{2}} x\right), \ell(x) e^{-\frac{1}{2}B(x,x)}\right) =$$
$$-\hbar^{\frac{1}{2}} \left(f\left(\hbar^{\frac{1}{2}} x\right), \partial_{B^{-1}\ell} e^{-\frac{1}{2}B(x,x)}\right) = \hbar^{\frac{1}{2}} \left(\partial_{B^{-1}\ell} f\left(\hbar^{\frac{1}{2}} x\right), e^{-\frac{1}{2}B(x,x)}\right) =$$
$$\hbar \left(\left(\partial_{B^{-1}\ell} f\right)\left(\hbar^{\frac{1}{2}} x\right), e^{-\frac{1}{2}B(x,x)}\right) = \hbar I_{\partial_{B^{-1}\ell} f}(\hbar)$$

$$I_g'(\hbar) = \frac{1}{2} \hbar^{-1} I_{Eg}(\hbar),$$

$$E := \sum_{j=1}^{d} e_j^* \partial_{e_j}$$

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar), \hbar > 0.$$

$$I_{e^{-\frac{1}{2}C(x,x)}}(\hbar) = \int_V e^{-\frac{1}{2}(B+\hbar C)(x,x)}dx = (2\pi)^{\frac{d}{2}}\det(B+\hbar C)^{-\frac{1}{2}}$$

$$I'_g(0) = I_{\partial_{B^{-1}f}}(0) = I_{\frac{1}{2}\Delta_B g}(0)$$

$$\frac{1}{2}\Delta_B g(0) = \frac{1}{2}\Delta_B(\ell f)(0) = \sum_j \ell(e_j)\partial_{B^{-1}e_j^*}f(0) = \partial_{B^{-1}\ell}f(0).$$

$$\int_a^b g(x)e^{-\frac{f(x)}{\hbar}}dx = \hbar^{\frac{1}{2}}e^{-\frac{f(c)}{\hbar}}I(\hbar),$$

$$I(0) = \sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}}$$

$$I(\hbar) = \int_{-\infty}^{\infty} g\left(\hbar^{\frac{1}{2}}y\right)e^{-\frac{M}{2}y^2}dy$$

$$p'(0) = \sqrt{f''(0)} > 0.$$

$$I(\hbar) = \hbar^{-\frac{1}{2}}\int g(x)e^{-\frac{p(x)^2}{2\hbar}}dx \sim \int_{-\infty}^{\infty}\tilde{g}\left(\hbar^{\frac{1}{2}}y\right)e^{-\frac{y^2}{2}}dy$$

$$\tilde{g}(z) := g(p^{-1}(z))(p^{-1})'(z) = \frac{g(p^{-1}(z))}{p'(p^{-1}(z))}$$

$$I_N(\hbar) := \int_{-\infty}^{\infty}\tilde{g}_N\left(\hbar^{\frac{1}{2}}y\right)e^{-\frac{y^2}{2}}dy$$

$$\tilde{g}(z) \sim \sum_{n=0}^{\infty} b_n z^n$$

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n}\hbar^n \int_{-\infty}^{\infty} y^{2n}e^{-\frac{y^2}{2}}dy$$

$$\int_{-\infty}^{\infty} y^{2n}e^{-\frac{y^2}{2}}dy = 2^{n+\frac{1}{2}}\int_0^{\infty} u^{n-\frac{1}{2}}e^{-u}du = 2^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right) = (2\pi)^{\frac{1}{2}}(2n-1)!!$$

where $(2n-1)!! := \prod_{1\le j\le n}(2j-1)$.

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n}2^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)\hbar^n$$

$$\int_a^b g(x)e^{\frac{if(x)}{\hbar}}dx = \hbar^{\frac{1}{2}}e^{\frac{if(c)}{\hbar}}I(\hbar)$$

$$I(0) = \sqrt{2\pi}e^{\pm\frac{\pi i}{4}}\frac{g(c)}{\sqrt{|f''(c)|}}$$

$$g(a) = \cdots = g^{(n-1)}(a) = g(b) = \cdots = g^{(n-1)}(b) = 0.$$

$$I(\hbar) := \int_a^b g(x)e^{\frac{if(x)}{\hbar}}dx$$

$$\int_a^b g(x)e^{\frac{ix}{\hbar}}dx = i\hbar\int_a^b g'(x)e^{\frac{ix}{\hbar}}dx$$

$$I_2(\hbar) = \int_a^b g_2(x) e^{\frac{if(x)}{\hbar}} dx$$

$$I(\hbar) = \int_{-\infty}^{\infty} g\left(\hbar^{\frac{1}{2}} y\right) e^{\frac{iM}{2} y^2} dy$$

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right) (i\hbar)^n$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \hbar^{\frac{1}{2}} I(\hbar)$$

$$I(\hbar) = \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy$$

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n$$

$$a_n = (-1)^n \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{y^{4n}}{2^n n!} dy =$$

$$(-1)^n \frac{2^{n+\frac{1}{2}} \Gamma\left(2n + \frac{1}{2}\right)}{n!} = (-1)^n \sqrt{2\pi} \frac{(4n-1)!!}{2^n n!}$$

$$\tilde{I}(\hbar) = \sum_{n \geq 0} a_n \hbar^n$$

$$g(\hbar) = \sum_{n \geq 0} a_n \frac{\hbar^n}{n!}$$

$$I(\hbar) = \int_0^{\infty} g(\hbar u) e^{-u} du = \hbar^{-1} \int_0^{\infty} g(u) e^{-\frac{u}{\hbar}} du$$

$$I(\hbar) = \hbar^{-1} (\mathcal{L}g)(\hbar^{-1})$$

$$I(\hbar) = \int_{-\infty}^{\infty} |v| g(\hbar v^2) e^{-v^2} dv = \hbar^{-\frac{1}{2}} \int_{-\infty}^{\infty} g_*\left(\hbar^{\frac{1}{2}} v\right) e^{-v^2} dv$$

$$g_*(v) = |v| g(v^2) \int_0^{\infty} x^n e^{-x} dx = n!$$

$$\tilde{I} := \sum_{n \geq 0} (-1)^n n! \, \hbar^n$$

$$g(\hbar) = \sum_{n \geq 0} (-1)^n \hbar^n = \frac{1}{1+\hbar}$$

$$I(\hbar) = \int_0^{\infty} \frac{e^{-u}}{1+\hbar u} du = \hbar^{-1} e^{\hbar^{-1}} E_1(\hbar^{-1})$$

$$E_1(x) := \int_x^{\infty} \frac{e^{-u}}{u} du$$

$$\Gamma(s+1) = \int_0^{\infty} t^s e^{-t} dt, s > 0$$

$$\frac{\Gamma(s+1)}{s^{s+1}} = \int_0^{\infty} x^s e^{-sx} dx = \int_0^{\infty} e^{-s(x-\log x)} dx$$

$$\hbar = \frac{1}{s}, f(x) = x - \log\, x, g(x) = 1$$

$$\Gamma(s+1) \sim s^s e^{-s}\sqrt{2\pi s}\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots\right)$$

$$p(x) = \sqrt{2(x - \log\,(1+x))} = x\sqrt{1 - \frac{2x}{3} + \frac{x^2}{2} - \cdots} = x - \frac{x^2}{3} + \frac{7x^3}{36} + \cdots$$

$$p^{-1}(z) = z + \frac{z^2}{3} + \frac{z^3}{36} + \cdots$$

$$(p^{-1})'(z) = 1 + \frac{2z}{3} + \frac{z^2}{12} + \cdots$$

$$(\log\,\Gamma)''(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \int_0^{\infty} te^{-(z+n)t} dt = \int_0^{\infty} \frac{te^{-zt}}{1-e^{-t}} dt$$

$$\sum_{n\geq 0} \frac{B_n t^n}{n!} = \frac{t}{1-e^{-t}},$$

$$(\log\,\Gamma)''(z) \sim \sum_{n\geq 0} B_n z^{-n-1}$$

$$(\log\,\Gamma)'(z) \sim \log\, z + C_1 - \sum_{n\geq 1} \frac{B_n}{n} z^{-n}$$

$$\log\,\Gamma(z+1) \sim z\log\, z - z + C_1 z + \frac{1}{2}\log\, z + C_2 + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}$$

$$(\log\,\Gamma)'(z) \sim \log\, z - \sum_{n\geq 1} \frac{B_n}{n} z^{-n}$$

$$\log\,\Gamma(z+1) \sim z\log\, z + \frac{1}{2}\log\, z + \frac{1}{2}\log\,(2\pi) + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}$$

$$1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots = \exp\left(\sum_{n\geq 2} \frac{B_n}{n(n-1)} s^{-n+1}\right).$$

$$I_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{a\cos\,\theta} d\theta$$

$$I_0(a) = \sum_{n=0}^{\infty} \frac{a^{2n}}{2^{2n} n!^2}$$

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$$\int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I(\hbar)$$

$$I(0) = (2\pi)^{\frac{d}{2}} \frac{g(c)}{\sqrt{\det f''(c)}}$$

$$\int_D g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar)$$

$$I(0) = (2\pi)^{\frac{d}{2}} e^{\frac{\pi i\sigma}{4}} \frac{g(c)}{\sqrt{|\det f''(c)|}}$$

$$f(x_1,\dots,x_n) = f(x_1,\dots,x_{d-1}) \pm x_d^2$$

$$\partial_u f(y,u) = 0$$

$$f_*(y,v) := f(y,u) = f(y, g(y,u))$$

$$\partial_v f_*(y,v) = \partial_u f_*(y,v)\frac{\partial u}{\partial v} = \partial_u f(y,u)\frac{\partial u}{\partial v} = \pm 2v\partial_v g(y,v)$$

$$f(y,u) - f(y,0) = h(y,u)u^2$$

$$\tilde{u} := \sqrt{|h(y,u)|}u$$

$$f(u,y) = f(0,y) \pm u^2$$

$$f = x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_d^2$$

$$f(x,y) = a(y) + b(y)x + c(y)x^2 + d(y)x^3$$

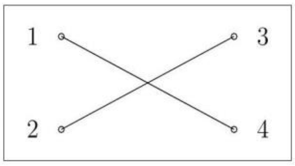
$$I(\hbar) := \int_D g(x)e^{\frac{if(x)}{\hbar}}dx$$

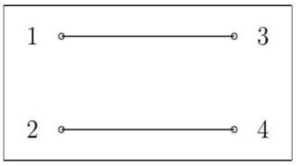
$$I(\hbar) = \hbar^{-\frac{d}{2}}e^{\frac{S(c)}{\hbar}}\int_D g(x)e^{-\frac{S(x)}{\hbar}}dx$$

$$I(\hbar) = a_0 + a_1\hbar + \dots + a_m\hbar^m + \dots$$

$$\int_V P(x)e^{-\frac{B(x,x)}{2}}dx$$

$$|\Pi_k| = \frac{(2k)!}{2^k \cdot k!} = (2k-1)\,!!.$$





$$\int_V \ell_1(x)\dots\ell_N(x)e^{-\frac{B(x,x)}{2}}dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}\sum_{\sigma\in\Pi_{N/2}}\prod_{i\in\{1,\dots,N\}/\sigma}B^{-1}(\ell_i,\ell_{\sigma(i)})$$

$$\int_{-\infty}^{\infty} x^{2k}e^{-\frac{x^2}{2}}dx = (2\pi)^{\frac{1}{2}}(2k-1)!!$$

$$\int_V \ell_1(x)\ell_2(x)e^{-\frac{B(x,x)}{2}}dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}B^{-1}(\ell_1,\ell_2)$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x)e^{-\frac{B(x,x)}{2}}dx =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}\left(B^{-1}(\ell_1,\ell_2)B^{-1}(\ell_3,\ell_4) + B^{-1}(\ell_1,\ell_3)B^{-1}(\ell_2,\ell_4) + B^{-1}(\ell_1,\ell_4)B^{-1}(\ell_2,\ell_3)\right)$$

$$\langle \ell_1 \dots \ell_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D \ell_1(x) \dots \ell_N(x) e^{-\frac{S(x)}{\hbar}} dx$$

$$S(x) = \frac{B(x,x)}{2} - \sum_{i \geq 3} \frac{B_i(x,\dots,x)}{i!},$$

$$\langle \ell_1 \dots \ell_N \rangle = \hbar^{\frac{N}{2}} \int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2} + \sum_{i\geq 3} \hbar^{\frac{i}{2}-1} \frac{B_i(x,\dots,x)}{i!}} dx$$

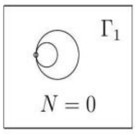
$$\langle \ell_1 \dots \ell_N \rangle = O\left(\hbar^{\left\lceil \frac{N}{2} \right\rceil}\right) \text{ as } \hbar \to 0$$

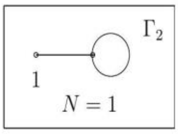
$$B_3 := \sum_i b_i^{13} \otimes b_i^{23} \otimes b_i^{33}, B_4 := \sum_j b_j^{14} \otimes b_j^{24} \otimes b_j^{34} \otimes b_j^{44},$$

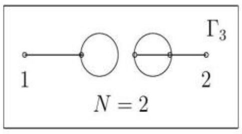
$$F_{\Gamma_3}(\ell_1, \ell_2) =$$
$$\sum_i B^{-1}(\ell_1, b_i^{13}) B^{-1}(b_i^{23}, b_i^{33}) \cdot \sum_{i,j} B^{-1}(b_i^{13}, b_j^{14}) B^{-1}(b_i^{23}, b_j^{24}) B^{-1}(b_i^{33}, b_j^{34}) B^{-1}(b_j^{44}, \ell_2).$$

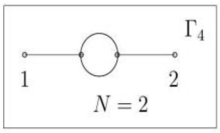
$$\langle \ell_1 \dots \ell_N \rangle = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\Gamma \in G{\geq}3(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

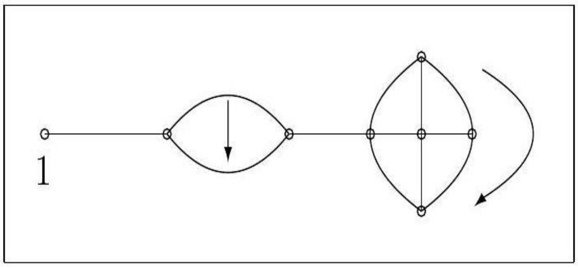
$$\Gamma_0 = \emptyset$$
$$N = 0$$











$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} := \frac{\langle \ell_1 \dots \ell_N \rangle}{\langle \emptyset \rangle}$$

$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

$$S(x) = \frac{B(x,x)}{2} + \tilde{S}(x)$$

$$\tilde{S}(x) := -\sum_{i \geq 0} g_i \frac{B_i(x,\dots,x)}{i!}$$

$$Z = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx$$

$$F_\Gamma = \prod_i g_i^{n_i} \cdot \mathbb{F}_\Gamma$$

$$Z = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

$$b(\Gamma) = \sum_i n_i \left(\frac{i}{2} - 1\right)$$

$$\mathbb{C}\left[g_0 \hbar^{-\frac{3}{2}}, g_1 \hbar^{-1}, g_2 \hbar^{-\frac{1}{2}}; g_j, j \geq 3\right]\left[\left[\hbar^{\frac{1}{2}}\right]\right],$$

$$g_0 \hbar^{-\frac{3}{2}}, g_1 \hbar^{-1}, g_2 \hbar^{-\frac{1}{2}}, g_3, g_4, \ldots$$

$$\mathbb{C}\left[\left[\hbar^{\frac{1}{2}}\right]\right]$$

$$\mathbb{C}\left[\left[g_j, j \geq 0\right]\right]$$

$$\langle e^{\ell} \rangle = \hbar^{-\frac{d}{2}} \int_V e^{\ell(x) - \frac{S(x)}{\hbar}} dx := \sum_{N=0}^{\infty} \frac{\langle \ell^N \rangle}{N!}$$

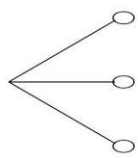
$$e^{-\frac{S(x)}{\hbar}} dx$$

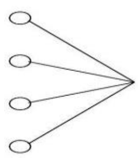
$$g_0 = g_2 = 0, g_1 = \hbar, B_1 = \ell, B_0 = 0, B_2 = 0$$

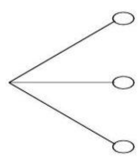
$$y = \hbar^{-\frac{1}{2}} x$$

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

$$Z_{\mathbf{n}} = \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} \left(\hbar^{\frac{i}{2}-1} B_i(y, \ldots, y)\right)^{n_i} dy$$

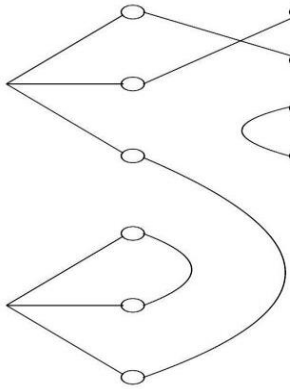






$$Z_{\mathbf{n}} = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} \hbar^{n_i\left(\frac{i}{2}-1\right)} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma)$$

$\sigma:$



$$\mathbb{G}_{\mathbf{n}} = \prod_i \left(S_{n_i} \ltimes S_i^{n_i}\right)$$

$$|\mathbb{G}_{\mathbf{n}}| = \prod_i i!^{n_i}\, n_i!$$

$$N_\Gamma = \frac{\prod_i i!^{n_i}\, n_i!}{|\mathrm{Aut}(\Gamma)|}$$

$$\sum_{\sigma\in\Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma) = \sum_\Gamma \frac{\prod_i i!^{n_i}\, n_i!}{|\mathrm{Aut}(\Gamma)|}\mathbb{F}_\Gamma$$

$$\sum_i n_i\left(\frac{i}{2}-1\right)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+ge^{zx}} = \sum_{n\geq 0} g^n \sum_{\Gamma\in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|}$$

$$\sum_k \sum_{\Gamma\in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{\frac{z^2n^2}{2}}}{n!}$$

$$\sum_{\Gamma\in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k!\, n!}$$

$$Z_0 = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}$$

$$\log\frac{Z}{Z_0} = \sum_{\mathbf{n}}\prod_i \left(g_i\hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\Gamma\in G_c(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

$$\mathbb{F}_{\Gamma_1\Gamma_2} = \mathbb{F}_{\Gamma_1}\mathbb{F}_{\Gamma_2}, b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$$

$$\left|\mathrm{Aut}\left(\Gamma_1^{k_1}\dots\Gamma_l^{k_l}\right)\right| = \prod_j \left|\mathrm{Aut}(\Gamma_j)\right|^{k_j} k_j!.$$

$$\left(\log\frac{Z}{Z_0}\right)_j := \sum_{\mathbf{n}}\prod_i g_i^{n_i} \sum_{\Gamma\in G^{(j)}(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

$$\log\frac{Z}{Z_0} = \sum_{j=0}^{\infty}\left(\log\frac{Z}{Z_0}\right)_j \hbar^{j-1}$$

$$\left(\log \frac{Z}{Z_0}\right)_0 = -S(x_0)$$

$$\left(\log \frac{Z}{Z_0}\right)_1 = \frac{1}{2}\log \frac{\det B}{\det S''(x_0)}$$

$$\frac{B(x,x)}{2} - \sum_{i=0}^{N} g_i \frac{B_i(x,\ldots,x)}{i!} \text{ and } \hbar > 0.$$

$$Z := \hbar^{-\frac{d}{2}} \int_{\mathbf{B}} e^{-\frac{S(x)}{\hbar}} dx$$

$$\frac{Z}{Z_0} = e^{-\frac{S(x_0)}{\hbar}} I(\hbar),$$

$$I(\hbar) \sim \sqrt{\frac{\det B}{\det S''(x_0)}} (1 + a_1\hbar + a_2\hbar^2 + \cdots)$$

$$\log \frac{Z}{Z_0} = -S(x_0)\hbar^{-1} + \frac{1}{2}\log \frac{\det B}{\det S''(x_0)} + O(\hbar)$$

$$\left(\log \frac{Z}{Z_0}\right)_0 \hbar\log \frac{Z}{Z_0}\left(\log \frac{Z}{Z_0}\right)_0 + \hbar\left(\log \frac{Z}{Z_0}\right)_1$$

$$\beta(x) := \sum_{i\geq 1} g_i \frac{B^{-1}B_i(x,\ldots,x,-)}{(i-1)!}$$

$$x_0 = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma\in G^{(0)}(\mathbf{n},1)} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

$$S(x) = \frac{B(x,x)}{2} - \sum_i g_i \frac{B_i(x,\ldots,x)}{i!} - S(x_0) \prod_i g_i^{n_i} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|} \frac{B(x_0,x_0)}{2} g_i \frac{B_i(x_0,\ldots,x_0)}{i!} \sum_i g_i \frac{B_i(x_0,\ldots,x_0)}{i!}$$

$$S(x) := \frac{\frac{x^2}{2}}{40} - gh(x)$$

$$h(x) = \sum_{n\geq 0} c_n x^n c_1 \neq 0$$

$$x = \bar{g}h'(x), \text{ i.e., } x_0 = f(g) \text{ where } x = f(y),\ y = \frac{x}{h'(x)}.$$

$$F(g) = -\frac{f(g)^2}{2} + gh(f(g)).$$

$$F'(g) = -f(g)f'(g) + h(f(g)) + gh'(f(g))f'(g).$$

$$h'(f(g)) = \frac{f(g)}{g}$$

$$F'(g) = h(f(g))$$

$$-S(x_0) = \int_0^g h(f(a))da$$

$$S(x) = \frac{x^2}{2} - ge^x$$

$$\sum_{n\geq 0} g^n \sum_{\Gamma\in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = -S(x_0)$$

$$-S(x_0) = \int_0^g e^{f(a)}da = \int_0^g \frac{f(a)}{a}da$$

$$f(g) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!}g^n$$

$$a_n = \frac{1}{2\pi i}\oint \frac{f(g)}{g^{n+1}}dg = \frac{1}{2\pi i}\oint \frac{x}{(xe^{-x})^{n+1}}d(xe^{-x}) =$$
$$\frac{1}{2\pi i}\oint e^{nx}\frac{1-x}{x^n}dx = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}$$

$$-S(x_0) = \int_0^g \frac{f(a)}{a}da = \sum_{n\geq 1}\frac{n^{n-2}}{n!}g^n$$

$$\sum_{\Gamma\in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}$$

$$S(x) = \frac{x^2}{2} - g\left(x + \frac{x^3}{6}\right)$$

$$x = g\left(1 + \frac{x^2}{2}\right)$$

$$x_0 = \frac{1-\sqrt{1-2g^2}}{g}$$

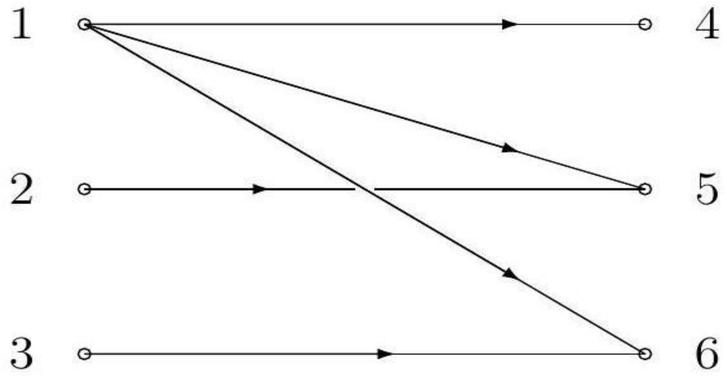
$$-S(x_0) = \int_0^g \left(\frac{1-\sqrt{1-2a^2}}{a} + \frac{\left(1-\sqrt{1-2a^2}\right)^3}{6a^3}\right)da =$$
$$\frac{2}{3}\int_0^g \frac{1-(1+a^2)\sqrt{1-2a^2}}{a^3}da = \frac{(1-2g^2)^{\frac{3}{2}} - (1-3g^2)}{3g^2}$$

$$-S(x_0) = \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{(n+1)!}g^{2n}$$

$$xe^{-y} = a, ye^{-x} = b.$$

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq}a^p b^q, y = b + \sum_{p\geq 1, q\geq 1} d_{pq}a^p b^q.$$

$$c_{pq} = \frac{1}{(2\pi i)^2}\oint\oint \frac{x}{a^{p+1}b^{q+1}}da\wedge db =$$
$$\frac{1}{(2\pi i)^2}\oint\oint \frac{e^{qx+py}}{x^p y^{q+1}}(1-xy)dx\wedge dy = \frac{q^{p-1}p^{q-1}}{(p-1)!\,q!}.$$



$$d_{pq} = \frac{q^{p-1}p^{q-1}}{p!\,(q-1)!} - a\partial_a S(x,y) = x, -b\partial_b S(x,y) = y$$

$$-S(x,y) = b + \int_0^a \frac{x}{u} du = a + b + \sum_{p,q\geq 1} \frac{p^{q-1}q^{p-1}}{p!\,q!} a^p b^q$$

$$p^{q-1}q^{p-1}\frac{(p+q)!}{p!\,q!}$$

$$U(\mathbf{y}) := \sum_{1\leq i<k\leq m} u_{ik}(y_i - y_k)^2.$$

$$K_m(\mathbf{u}) := \det U = \det\left(\delta_{i\ell}\sum_{k\neq\ell} u_{k\ell} - u_{i\ell}\right)_{(j)}$$

$$Q_{\mathbf{p}}(\mathbf{z}) := \sum_{\mathbf{r}:|\mathbf{r}|=|\mathbf{p}|-1} N(\mathbf{p},\mathbf{r}) \prod_{i\leq j} z_{ij}^{r_{ij}}$$

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1 \dots p_m)^{-1} K(p_k z_{k\ell} p_\ell, k \neq \ell) \prod_\ell \left(\sum_k p_k z_{k\ell}\right)^{p_\ell - 1}.$$

$$S(x,y) = \frac{1}{2} x^T B x - \sum_{j=1}^m a_j e^{x_j}$$

$$\sum_i x_i b_{ij} e^{-x_j} = a_j.$$

$$-S(x) = \int X_j \frac{da_j}{a_j}$$

$$Q_{\mathbf{p}}(\mathbf{z}) = \frac{p_j^{-1}}{(2\pi i)^m} \oint X_j \left(\prod_k a_k^{-p_k - 1}\right) d\mathbf{a} =$$

$$\frac{p_j^{-1}}{(2\pi i)^m} \oint X_j \left(\prod_k (X_k e^{-x_k})^{-p_k-1}\right) d(X_1 e^{-x_1}) \wedge \dots \wedge d(X_m e^{-x_m}) =$$

$$\frac{p_j^{-1}}{(2\pi i)^m} \oint_{T\subset\{1,\dots,m\}} (-1)^{|T|} D_T(\mathbf{z}) X_j \left(\prod_{\ell\notin T} X_\ell^{-1}\right)\left(\prod_\ell X_\ell^{-p_\ell}\right) e^{\sum_{k,\ell} p_k z_{k\ell} X_\ell} dX_1 \wedge \dots \wedge dX_m$$

$$= p_j^{-1} \sum_{T\subset\{1,\dots,m\}} (-1)^{|T|} \frac{p_j - 1 + \delta_{jT}^c}{\sum_k p_k z_{kj}} D_T(\mathbf{z}) \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - \delta_{\ell T}}}{(p_\ell - \delta_{\ell T})!} =$$

$$p_j^{-1}\left(\frac{p_j - 1}{\sum_k p_k z_{kj}} \det\left(\delta_{i\ell}\sum_k p_k z_{k\ell} - z_{i\ell}p_\ell\right) + \det\left(\delta_{i\ell}\sum_k p_k z_{k\ell} - z_{i\ell}p_\ell\right)_{(j)}\right)\prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell!}$$

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1 \dots p_m)^{-1} \det\left(\delta_{i\ell}\sum_k p_k z_{k\ell} p_\ell - p_i z_{i\ell} p_\ell\right)_{(j)} \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell!}.$$

$$N_\Gamma = \det U,$$

$$U(\mathbf{y}) = \sum_{i<j} (A_\Gamma)_{ij}(y_i - y_j)^2 = (\Delta_\Gamma \mathbf{y}, \mathbf{y}),$$

$$N_\Gamma = \frac{1}{m}\lambda_1 \dots \lambda_{m-1},$$

$$\hbar^{-1}\left(\log\frac{Z_{S_{\text{eff}}}}{Z_0}\right)_0 = \log\frac{Z_S}{Z_0}$$

$$S_{\text{eff}}(x) = \frac{B(x,x)}{2} - \sum_{i\geq 0}\frac{\mathcal{B}_i(x,\dots,x)}{i!},$$

$$\mathcal{B}_N(x,\dots,x) = \hbar\sum_{\mathbf{n}}\prod_i\left(g_i\hbar^{\frac{i}{2}-1}\right)^{n_i}\sum_{\Gamma\in G_{1\text{PI}}(\mathbf{n},N)}\frac{\mathbb{F}_\Gamma(Bx,\dots,Bx)}{|\text{Aut}(\Gamma)|}.$$

$$L(f)(p) = (p,x_0) - f(x_0).$$

$$f(x) = \frac{ax^2}{2}, a \neq 0$$

$$px - f = px - \frac{x^2}{2}p = \frac{x}{a}\frac{p^2}{2a}$$

$$L\left(\frac{ax^2}{2}\right) = \frac{p^2}{2a}f(x) = \frac{B(x,x)}{2}$$

$$L(f)(p) = \frac{B^{-1}(p,p)}{2}$$

$$\frac{mv^2}{2} - U(x)\,\frac{p^2}{2m} + U(x)$$

$$\hbar^{-\frac{d}{2}}\int_V e^{\frac{i(-(p,x)+S(x))}{\hbar}}dx e^{\frac{iS(x)}{\hbar}}dx - \frac{iL(S)(p)}{\hbar}$$

$$S(x) = \frac{B(x,x)}{2} + O(x^3)$$

$$Z(p) = \hbar^{-\frac{d}{2}}\int_V e^{\frac{(p,x)-S(x)}{\hbar}}dx$$

$$\log\frac{Z(p)}{Z_0} = -\hbar^{-1}S_{\text{eff}}(x_0,p),$$

$$S_{\text{eff}}(x) = L\left(\hbar\log\frac{Z(p)}{Z_0}\right), \hbar\log\frac{Z(p)}{Z_0} = L(S_{\text{eff}}(x))$$

$$S(x) = \frac{x^2}{2} - g\left(x + \frac{x^3}{6}\right)$$

$$(x) = \frac{x^2}{2} - \frac{gx^3}{6}S_{\text{eff}} = S + \hbar S_1 + O(\hbar^2)$$

$$u_t = \frac{1}{2}\Delta_B u \Delta_B u(x,t) = e^{\frac{t\Delta_B}{2}}u(x,0)$$

$$S(x) = \frac{B(x,x)}{2} - \tilde{S}(x)$$

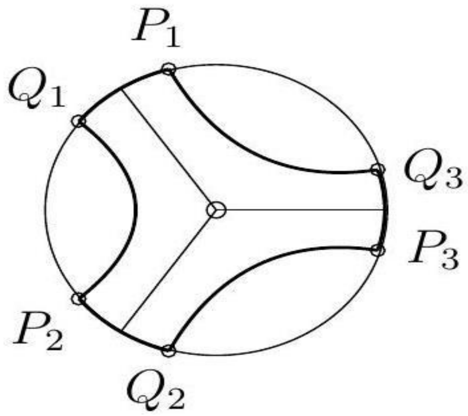
$$S^\circ(x) := \frac{B(x,x)}{2} - \tilde{S}^\circ(x)$$

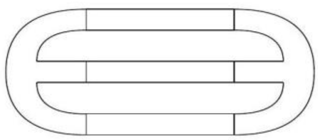
$$\tilde{S}^\circ(x) := e^{\frac{\hbar\Delta_B}{2}}\tilde{S}(x)$$

$$Z_N := \hbar^{-\frac{N^2}{2}}\int_{\mathfrak{h}_N} e^{-\frac{S(A)}{\hbar}}dA$$

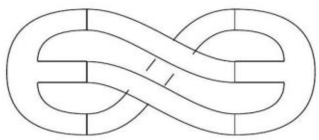
$$\int_{\mathfrak{h}_N} e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = 1$$

$$S(A) := \frac{\mathrm{Tr}(A^2)}{2} - \sum_{m \geq 1} g_m \frac{\mathrm{Tr}(A^m)}{m}$$

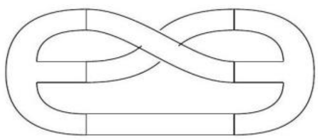




Γ_1 $\quad$ g = 0



Γ_2 $\quad$ g = 0



Γ_3 $\quad$ g = 1

$$B_m(A, \ldots, A) = (m-1)!\,\mathrm{Tr}(A^m).$$

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{\left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i}}{i!^{n_i}\, n_i!} \sum_{\sigma \in \Pi_c(T_{\mathbf{n}})} \mathbb{F}(\sigma)$$

$$\mathrm{Tr}(A^m) = \sum_{i_1,\ldots,i_m=1}^{N} \left(e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}\right).$$

$$B_m = \sum_{s \in S_{m-1}} \sum_{i_1,\ldots,i_m=1}^{N} s\left(e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*\right)$$

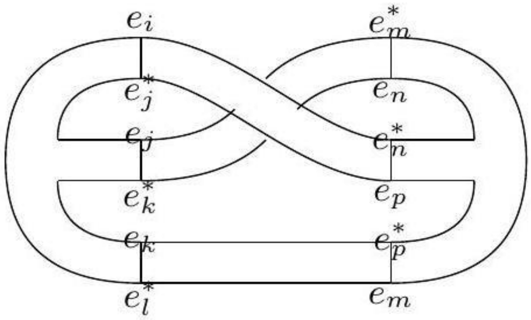
$$\mathbb{F}(\sigma) = \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma)$$

$$\sum_{i_1,\ldots,i_m=1}^{N} e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*$$

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i}\hbar^{n_i\left(\frac{i}{2}-1\right)}}{i!^{n_i}\, n_i!} \sum_{\sigma\in\Pi(T_{\mathbf{n}})} \sum_{s\in\Pi_i S_{i-1}^{n_i}} \tilde{\mathbb{F}}(s\sigma) =$$

$$\sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i}\hbar^{n_i\left(\frac{i}{2}-1\right)}}{i^{n_i} n_i!} \sum_{\sigma} \tilde{\mathbb{F}}(\sigma)$$

$$\prod_i i!^{n_i} \prod_i i^{n_i} \sum_{s,\sigma} \tilde{\mathbb{F}}(s\sigma)\tilde{\mathbb{F}}(\sigma) \left|\prod_i S_{i-1}^{n_i}\right| = \prod_i (i-1)!^{n_i}$$



Contraction nonzero iff
$i = r,\ j = p,\ j = m,\ k = r,$
$k = p,\ i = m,$
that is
$i = r = k = p = j = m.$

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma}\in\tilde{G}_c(\mathbf{n})} \frac{N^{v(\tilde{\Gamma})}}{|\mathrm{Aut}(\tilde{\Gamma})|} =$$

$$\sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\tilde{\Gamma}\in\tilde{G}_c(\mathbf{n})} \frac{N^{v(\tilde{\Gamma})}\hbar^{b(\tilde{\Gamma})}}{|\mathrm{Aut}(\tilde{\Gamma})|}.$$

$$\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}} := \prod_i \left(S_{n_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{n_i}\right) \tilde{\Gamma}_\sigma = \tilde{\Gamma}_{g\sigma g} \in \mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}$$

$$\sigma\mathrm{Aut}(\tilde{\Gamma}_\sigma)|\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}| = \prod_i i^{n_i} n_i\,!$$

$$b(\tilde{\Gamma}) = 2g(\tilde{\Gamma}) - 2 + v(\tilde{\Gamma})$$

$$\hat{Z}_N(\hbar) := Z_N\left(\frac{\hbar}{N}\right),$$

$$\log \hat{Z}_N = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma}\in\tilde{G}_c(\mathbf{n})} \frac{N^{2-2\,\mathrm{g}(\tilde{\Gamma})}}{|\mathrm{Aut}(\tilde{\Gamma})|}$$

$$W_\infty := \lim_{N\to\infty} \frac{\log \hat{Z}_N}{N^2}$$

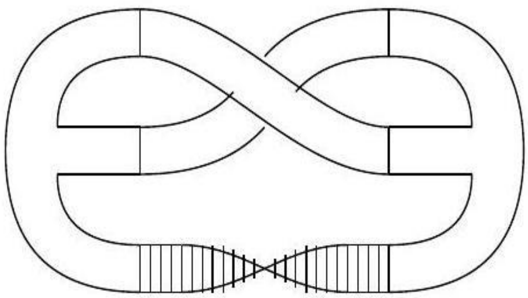
$$W_\infty = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma}\in\tilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|},$$

$$\frac{\log \hat{Z}_N}{N^2} = \sum_{\mathrm{g}\in\mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\,\mathrm{g}}$$

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma}\in\tilde{G}_c(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|},$$

$$Z_N = \hbar^{-\frac{N(N+1)}{4}} \int_{\mathfrak{S}_N} e^{-\frac{S(A)}{\hbar}} dA$$

$$(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il}\delta_{jk} e_i^* = e_i e_i \otimes e_j (e_i \otimes e_j, e_k \otimes e_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$



$$\hat{Z}_N := Z_N \left(\frac{2\hbar}{N}\right) \tilde{G}_c^{\mathrm{tw}}(\mathbf{n})$$

$$\log \hat{Z}_N = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\mathrm{tw}}(\mathbf{n})} \frac{N^{2-2\,\mathrm{g}(\tilde{\Gamma})}}{|\mathrm{Aut}(\tilde{\Gamma})|}$$

$$\mathrm{g} := 1 - \frac{\chi}{2} \mathbb{RP}^2 \frac{1}{2}$$

$$W_\infty := \lim_{N\to\infty} \frac{\log \hat{Z}_N}{N^2}$$

$$W_\infty = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|} \tilde{G}_c^{\mathrm{tw}}(\mathbf{n})$$

$$\frac{\log \hat{Z}_N}{N^2} = \sum_{\mathrm{g} \in \frac{1}{2}\mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\,\mathrm{g}}$$

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|}$$

$$S(A) = \frac{\mathrm{Tr}(A^2)}{2} - s\frac{\mathrm{Tr}(A^{2m})}{2m}$$

$$\int_{\mathfrak{H}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = P_m(N)$$

$$P_m(N) = \sum_{\mathrm{g}\geq 0} \varepsilon_{\mathrm{g}}(m) N^{m+1-2\,\mathrm{g}}$$

$$P_m(x) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} \binom{m}{p} 2^p \frac{x(x-1)\dots(x-p)}{(p+1)!}.$$

$$C_m = \frac{(2m)!}{m!\,(m+1)!} = \frac{1}{m+1}\binom{2m}{m}$$

$$D_m = \sum_{k+l=m-1} D_k D_l, D_0 = 1$$

$$h(x) := \sum_m D_m x^m = 1 + x + \cdots$$

$$h(x) - 1 = xh(x)^2$$

$$h(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

$$\lim_{N \to \infty} \frac{1}{N} \int_{\mathfrak{h}_N} \mathrm{Tr} f\left(\frac{A}{\sqrt{N}}\right) e^{-\frac{\mathrm{Tr}(A^2)}{2}} = \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4 - x^2}dx$$

$$e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA[-2\sqrt{N}, 2\sqrt{N}]$$

$$\frac{1}{2\pi} \int_{-2}^{2} x^{2m}\sqrt{4 - x^2}dx = C_m$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$f(x,t) = \sum_{n \geq 0} H_n(x) \frac{t^n}{n!} = e^{2xt - t^2}$$

$$f'' - 2xf' + 2nf = H_n(x)e^{-x^2/2}$$

$$L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n!\, \delta_{mn}$$

$$H_n(x) e^{-\frac{x^2}{2}} L^2(\mathbb{R})$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x) dx = \frac{(2m)!}{(m-k)!} 2^{2(k-m)}$$

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^{r} \frac{r!}{2^k k!^2 (r-k)!} H_{2k}(x).$$

$$\sum_{n \geq 0} (-1)^n \frac{t^n}{n!} \frac{d^n}{dx^n}$$

$$f_{xx} - 2xf_x + 2tf_t = 0.$$

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x,t) f(x,u) e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2ut - (x-u-t)^2} dx = e^{2ut}$$

$$H_n(x) e^{-\frac{x^2}{2}} L^2(\mathbb{R}) E \subset L^2(\mathbb{R}) \mathbb{C}[x] e^{-\frac{x^2}{2}} e^{ipx - \frac{x^2}{2}} \in E p \in \mathbb{R} \phi \in C_0^{\infty}(\mathbb{R})$$

$$\phi(x) e^{-\frac{x^2}{2}} = \int_{\mathbb{R}} \hat{\phi}(p) e^{ipx - \frac{x^2}{2}} dp \in E$$

$$C_0^{\infty}(\mathbb{R}) E C_0^{\infty}(\mathbb{R}) L^2(\mathbb{R}) E = L^2(\mathbb{R}) \int_{\mathbb{R}} x^{2m} e^{2xt - t^2} e^{-x^2} dx$$

$$\int_{\mathbb{R}} x^{2m} e^{-(x-t)^2} dx = \int_{\mathbb{R}} (y+t)^{2m} e^{-y^2} dy == \sqrt{\pi} \sum_{p} \binom{2m}{2p} \frac{(2m-2p)!}{2^{m-p}(m-p)!} t^{2p}$$

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_r^2(x) H_{2k}(x) e^{-x^2} dx = \frac{2^{r+k} r!^2 (2k)!}{k!^2 (r-k)!}$$

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)f(x,v)e^{-x^2}dx =$$
$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2(ut+uv+tv)-(x-u-t-v)^2}dx = e^{2(ut+tv+uv)}$$

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m})e^{-\frac{\mathrm{Tr}(A^2)}{2}}dA.$$

$$\sigma: \mathfrak{h}_N \to \mathbb{R}^N/S_N$$

$$\sigma_* dA = Ce^{-\sum_i \frac{\lambda_i^2}{2}}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda$$

$$P_m(N) = \frac{NJ_m}{J_0}, J_m := \int_{\mathbb{R}^N}\left(\frac{1}{N}\sum_i \lambda_i^{2m}\right)e^{-\sum_i \frac{\lambda_i^2}{2}}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda.$$

$$\prod_{i<j}(\lambda_i-\lambda_j) = 2^{-\frac{N(N-1)}{2}}\det\big(H_k(\lambda_\ell)\big),$$

$$J_m = 2^{m+\frac{N^2}{2}}\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum_i \lambda_i^2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}}\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum_i \lambda_i^2}\det\left(H_k(\lambda_j)\right)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}}\sum_{\sigma,\tau\in S_N}(-1)^\sigma(-1)^\tau\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum_i \lambda_i^2}\prod_i H_{\sigma(i)}(\lambda_i)H_{\tau(i)}(\lambda_i)d\lambda$$

$$J_m = 2^{m-\frac{N(N-2)}{2}}\sum_{\sigma\in S_N}\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum_i \lambda_i^2}\prod_i H_{\sigma i}(\lambda_i)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}}(N-1)!\,\gamma_0\ldots\gamma_{N-1}\sum_{j=0}^{N-1}\frac{1}{\gamma_j}\int_{-\infty}^{\infty}x^{2m}H_j(x)^2e^{-x^2}dx$$

$$\gamma_i := \int_{-\infty}^{\infty}H_i(x)^2e^{-x^2}dx$$

$$P_m(N) = 2^m\sum_{j=0}^{N-1}\frac{1}{\gamma_j}\int_{-\infty}^{\infty}x^{2m}H_j(x)^2e^{-x^2}dx$$

$$P_m(N) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\sum_{j=0}^{N-1}\sum_{k=0}^{j}\frac{2^m x^{2m}H_{2k}(x)}{2^k k!^2\,(j-k)!}e^{-x^2}dx$$

$$P_m(N) = \frac{(2m)!}{2^m}\sum_{j=0}^{N-1}\sum_{k=0}^{j}\frac{2^k j!}{(m-k)!\,k!^2\,(j-k)!} =$$
$$\frac{(2m)!}{2^m m!}\sum_{j=0}^{N-1}\sum_{k=0}^{j}2^k\binom{m}{k}\binom{j}{k}$$

$$\sum_{k=0}^{j}2^k\binom{m}{k}\binom{j}{k} = C\cdot T\cdot\left((1+z)^m(1+2z^{-1})^j\right).$$

$$P_m(N) = \frac{(2m)!}{2^m m!}C\cdot T\cdot\left((1+z)^m\frac{(1+2z^{-1})^N-1}{2z^{-1}}\right) =$$
$$\frac{(2m)!}{2^m m!}\sum_{p=0}^{m}2^p\binom{m}{p}\binom{N}{p+1}$$

$$\chi(\Gamma) = \sum_{\sigma \in \mathrm{cells}(Y)/\Gamma} \frac{(-1)^{\dim \sigma}}{|\mathrm{Stab}\sigma|}.$$

$$\chi_{\mathrm{orb}}(M/\Gamma) = \sum_{m} \frac{\chi(X_m)}{m}.$$

$$\chi_{\mathrm{orb}}\,(M/\Gamma) = \chi(\Gamma)$$

$$M/\Gamma' \to M/\Gamma \chi_{\mathrm{orb}}\,(M/\Gamma) = \tfrac{1}{[\Gamma:\Gamma']} \chi(M/\Gamma') M/\Gamma'\ \Gamma'\ \chi(M/\Gamma') = \chi(\Gamma')$$

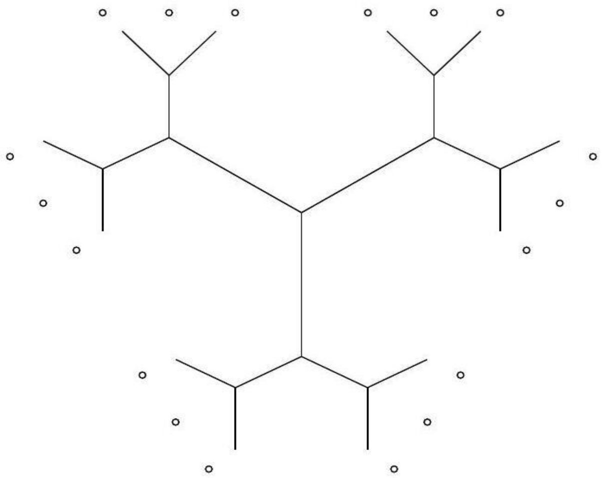
$$\Gamma = SL_2(\mathbb{Z})\mathbb{Z}/4\ \rho = \tfrac{-1+i\sqrt{3}}{2}\mathbb{Z}/6H/\Gamma(-1)\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6} = -\tfrac{1}{12}\chi_{\mathrm{orb}}\,(H/\Gamma) = \chi(\Gamma) - \tfrac{1}{12}$$

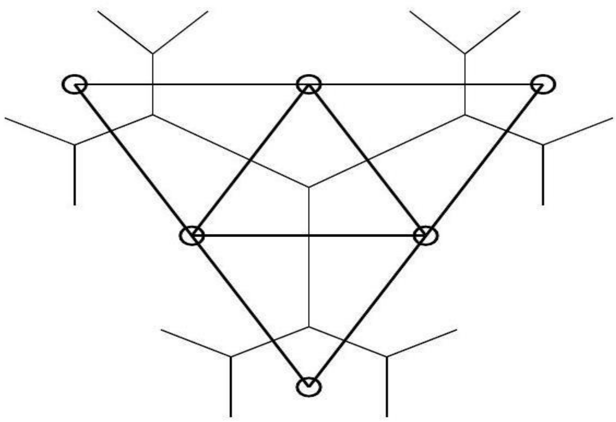
$$\Gamma_{\mathrm{g}}^{1} \to \mathrm{Sp}(2\,\mathrm{g}, \mathbb{Z}/n\mathbb{Z})\Gamma_{\mathrm{g}}^{1} H_1(\Sigma, \mathbb{Z}/n\mathbb{Z})$$

$$\chi(\Gamma_{\mathrm{g}}^{1}) = -\frac{B_{2\,\mathrm{g}}}{2\,\mathrm{g}},$$

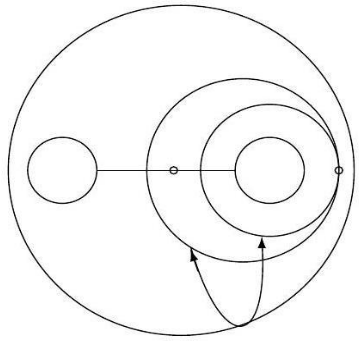
$$M_{\mathrm{g}} = \mathcal{T}_g/\Gamma_g 1 \to \pi_1(\Sigma) \to \Gamma_{\mathrm{g}}^{1} \to \Gamma_{\mathrm{g}} \to 1 \chi(\Gamma_{\mathrm{g}}) = \chi(\Gamma_{\mathrm{g}}^{1})/\chi(\Sigma)\ \chi(\Gamma_{\mathrm{g}}) = \chi_{\mathrm{orb}}\,(M_{\mathrm{g}}) = \tfrac{B_{2\,\mathrm{g}}}{4\,\mathrm{g}(\,\mathrm{g}-1)}(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n)$$

$$B_0 = \{(e^{i\theta}, 1)\}; B_1 = \{(e^{i\theta}, 1), (1, e^{i\theta})\}; B_2 = \{(e^{i\theta}, 1), (1, e^{i\theta}), (e^{i\theta}, e^{i\theta})\}$$





$$\chi(\Gamma_{\mathrm{g}}^{1}) = \sum_{\sigma \in \mathrm{cells}(Y_{\mathrm{g}})/\Gamma_{\mathrm{g}}^{1}} (-1)^{\dim \sigma} \frac{1}{|\mathrm{Stab}\sigma|}$$



$$\chi(\Gamma_g^1) = \sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}.$$

$$\sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}$$

$$\varepsilon_g(n) = \sum_i \binom{2n}{i} \mu_g(n-i).$$

$$\mu_g(n) = \sum_i \binom{n}{i} \lambda_g(n-i).$$

$$\sum_{m_1,\dots,m_{n-i}:\sum_{j=1}^{n-i} m_j = i} (m_1+1).$$

$$\varepsilon(n) = \sum_i \binom{2n}{i} \mu(n-i);$$
$$\mu(n) = \sum_i \binom{n}{i} \lambda(n-i).$$

$$\varepsilon(n) = \binom{2n}{n} f(n)$$

$$\sum_{n\geq 1} (-1)^{n-1} \frac{\lambda(n)}{2n} = f'(0)$$

$$E(z) = \sum_{n\geq 0} \varepsilon(n) z^n$$

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} L\left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right)$$

$$E(z) = \sum_{i,n} \binom{2n}{i}\binom{n-i}{k} z^n = \sum_{p,q\geq 0} \binom{2p+2q}{p}\binom{q}{k} z^{p+q}$$

$$F_r(z) := \sum_{p\geq 0} \binom{2p+r}{p} z^p$$

$$F_r(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z}\right)^r$$

$$F_r = z^{-1}(F_{r-1} - F_{r-2})$$

$$E(z)=\frac{1+\sqrt{1-4z}}{2(1-4z)}\left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right)^k$$

$$E(z)=P(z\partial)|_{z=0}\frac{1}{\sqrt{1-4z}}P(z)=(1+a)^z-1P'(0)=\log(1+a))\binom{z}{j},j\geq 1$$

$$E(z)=\frac{1}{\sqrt{1-4(1+a)z}}-\frac{1}{\sqrt{1-4z}}$$

$$L(u)=\frac{1}{1+u}\left(\frac{1}{\sqrt{1-4au(1+u)}}-1\right)$$

$$\sum_k(-1)^{k-1}\frac{\lambda_k}{2k}=\frac{1}{2}\int_{-1}^0L(u)\frac{du}{u}=\frac{1}{2}\sum_{p\geq 1}\binom{2p}{p}(-1)^{p-1}a^p\int_0^1x^{p-1}(1-x)^{p-1}dx$$

$$\int_0^1x^{p-1}(1-x)^{p-1}dx\frac{(p-1)!^2}{(2p-1)!}$$

$$\sum_k(-1)^{k-1}\frac{\lambda_k}{2k}=\sum_{p\geq 1}(-1)^{p-1}\frac{a^p}{p}=\log(1+a)$$

$$P_n(x):=\sum_{\mathrm{g}}\varepsilon_{\mathrm{g}}(n)x^{n+1-2\,\mathrm{g}}=\frac{(2n)!}{2^nn!}\sum_{p\geq 0}\binom{n}{p}2^p\binom{x}{p+1}$$

$$P_n(x)=\binom{2n}{n}\sum_{q\geq 0}2^{-q}\binom{n}{q}\frac{n!}{(n-q+1)!}x(x-1)\dots(x-n+q)$$

$$x^{-2g}(g\geq 1)\frac{P_n(x)}{x^{n+1}}\binom{2n}{n}f_{\mathrm{g}}(n)Q(x,n)=\left(1-\frac{1}{x}\right)\dots\left(1-\frac{n}{x}\right)Q(x,a)=\frac{\Gamma(x)}{\Gamma(x-a)x^a}\frac{1}{x}\text{ as }x\to+\infty\Gamma(x)Q(x,-1)=1$$

$$\sum_{\mathrm{g}\geq 1}f_{\mathrm{g}}'(0)x^{-2\,\mathrm{g}}$$

$$\frac{1}{2x}\left(1-\frac{1}{x}\right)\dots\left(1-\frac{n}{x}\right)$$

$$\frac{1}{n+1}\left(1-\frac{1}{x}\right)\dots\left(1-\frac{n}{x}\right)$$

$$\frac{d}{da}\bigg|_{a=0}\frac{Q(x,a)}{a+1}=-1+\frac{d}{da}\bigg|_{a=0}Q(x,a).$$

$$\sum_{\mathrm{g}\geq 1}f_{\mathrm{g}}'(0)x^{-2\,\mathrm{g}}=\frac{1}{2x}+\frac{d}{da}\bigg|_{a=0}Q(x,a)=\frac{1}{2x}+\frac{\Gamma'(x)}{\Gamma(x)}-\log x$$

$$f_{\mathrm{g}}'(0)=-\frac{B_{2\,\mathrm{g}}}{2\,\mathrm{g}}$$

$$a^2=1,b^3=1,c^7=1,abc=1$$

$$U(x)=\frac{x^2}{2}-\sum_{j\geq 1}g_j\frac{x^j}{j}$$

$$Z_N(\hbar)=\hbar^{-\frac{N^2}{2}}\int_{\mathfrak{h}_N}e^{-\mathrm{Tr}U(A)}dA$$

$$\lim_{N\to\infty}\frac{\log\hat{Z}_N}{N^2}=W_\infty$$

$$W_\infty = \sum_{\mathbf{n}} \prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \sum_{\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|}.$$

$$\prod_i \left(g_i \hbar^{\frac{i}{2}-1}\right)^{n_i} \prod_i i^{n_i} n_i!$$

$$U(x) = \frac{x^2}{2} + gx^4$$

$$W_\infty = \sum_{n\geq 1} c_n \frac{(-1)^n g^n}{n!},$$

$$c_n = (12)^n \frac{(2n-1)!}{(n+2)!}.$$

$$\hat{Z}_N = \int_{\mathfrak{h}_N} e^{-N\mathrm{Tr}\left(\frac{1}{2}A^2 + gA^4\right)} dA$$

$$\hat{Z}_N = \frac{J_N(g)}{J_N(0)},$$

$$J_N(g) = \int_{\mathbb{R}^N} e^{-N\left(\frac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4\right)} \prod_{i<j} \left(\lambda_i - \lambda_j\right)^2 d\lambda$$

$$W_\infty(g) = E(g) - E(0)E(g) = \lim_{N\to\infty} N^{-2} \log J_N(g)$$

$$K(\lambda_1, \dots, \lambda_N) := -N\left(\frac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4\right) + 2\sum_{i<j} \log|\lambda_i - \lambda_j|$$

$$\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = N\left(\frac{1}{2}\lambda_i + 2g\lambda_i^3\right).$$

$$\tfrac{1}{N}\textstyle\sum_i \delta(x - \lambda_i)\, \mu(x) = f(x,g)dx$$

$$f(x,0) = \tfrac{1}{2\pi}\sqrt{4-x^2}\; f(x,g) = \tfrac{1}{2\pi}\sqrt{4-x^2} + O(g)$$

$$\int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx = \frac{1}{2}y + 2gy^3, |y| \leq 2a$$

$$F(y) = \int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx$$

$$\frac{1}{2}(F_+(y) + F_-(y)) = \frac{1}{2}y + 2gy^3$$

$$\mathrm{Re}F_+(y) = \mathrm{Re}F_-(y) = \frac{1}{2}y + 2gy^3$$

$$\mathrm{Re}G(z) = \frac{1}{2}a(z + z^{-1}) + 2ga^3(z + z^{-1})^3$$

$$G(z) = 4ga^3 z^{-3} + (a + 12ga^3)z^{-1}$$

$$F(y) = \frac{1}{2}y + 2gy^3 - \left(\frac{1}{2} + 4ga^2 + 2gy^2\right)\sqrt{y^2 - 4a^2}.$$

$$f(y,g) = \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2 + 2gy^2\right)\sqrt{4a^2 - y^2}$$

$$yF(y) \to 1, y \to \infty \int f(x,g)dx = 1)zG(z) \to 1/a, z \to \infty$$

$$\frac{1}{a} = a + 12ga^3,$$

$$12ga^4 + a^2 - 1 = 0.$$

$$a = \left(\frac{(1+48g)^{1/2} - 1}{24g}\right)^{1/2}$$

$$E(g) = \int_{-2a}^{2a}\int_{-2a}^{2a} \log|x-y| f(x,g)f(y,g)dxdy - \int_{-2a}^{2a}\left(\frac{1}{2}x^2 + gx^4\right)f(x,g)dx$$

$$2\int_{-2a}^{2a}(\log|x-u| - \log|x|)f(x,g)dx = \frac{1}{2}u^2 + gu^4$$

$$E(g) = \int_{-2a}^{2a}\left(\log|u| - \frac{1}{4}u^2 - \frac{1}{2}gu^4\right)f(u,g)du$$

$$E(g) - E(0) = \log a - \frac{1}{24}(a^2-1)(9-a^2).$$

$$E(g) - E(0) = \sum_{k=1}^{\infty}(-12g)^k \frac{(2k-1)!}{k!\,(k-2)!}.$$

$$\langle q(t_1) \dots q(t_n)\rangle := \frac{\int q(t_1)\dots q(t_n)e^{\frac{iS(q)}{\hbar}Dq}}{\int e^{\frac{iS(q)}{\hbar}Dq}}$$

$$S(q) = \int \mathcal{L}(q)dt, \text{ and } \mathcal{L}(q) = \frac{\dot{q}^2}{2} - U(q)$$

$$\langle q(t_1)\dots q(t_n)\rangle\ \mathcal{G}_n^M(t_1,\dots,t_n)\tau = it\mathcal{G}_n^M(t_1,\dots,t_n)q(t) := q_*(\tau)d\tau = idt, \frac{dq}{dt} = i\frac{dq_*}{d\tau}$$

$$\mathcal{G}_n^M(t_1,\dots,t_n) = \frac{\int q_*(\tau_1)\dots q_*(\tau_n)e^{-\frac{1}{\hbar}\int\left(\frac{1}{2}\left(\frac{dq_*}{d\tau}\right)^2 + U(q_*)\right)d\tau}Dq_*}{\int e^{-\frac{1}{\hbar}\int\left(\frac{1}{2}\left(\frac{dq_*}{d\tau}\right)^2 + U(q_*)\right)d\tau}Dq_*}$$

$$\mathcal{G}_n^M(t_1,\dots,t_n) = \mathcal{G}_n^E(it_1,\dots,it_n)$$

$$\mathcal{G}_n^E(t_1,\dots,t_n) := \frac{\int q(t_1)\dots q(t_n)e^{-\frac{S_E(q)}{\hbar}}Dq}{\int e^{-\frac{S_E(q)}{\hbar}}Dq}$$

$$S_E(q) = \int \mathcal{L}_E(q)d\tau$$

$$\mathcal{L}_E(q) = \frac{\dot{q}^2}{2} + U(q)$$

$$\mathcal{G}_n(t_1,\dots,t_n) = \sum_{\Gamma\in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F_\Gamma(\ell_1,\dots,\ell_n),$$

$$(f,g) = \int_{\mathbb{R}} f(x)g(x)dx$$

$$B(q,q) = \int (\dot{q}^2 + m^2q^2)dt = (Aq,q)$$

$$A = -\frac{d^2}{dt^2} + m^2$$

$$K(x,y) = G(x-y)$$

$$(AG)(x) = \delta(x)$$

$$G(x) = \frac{e^{-m|x|}}{2m}$$

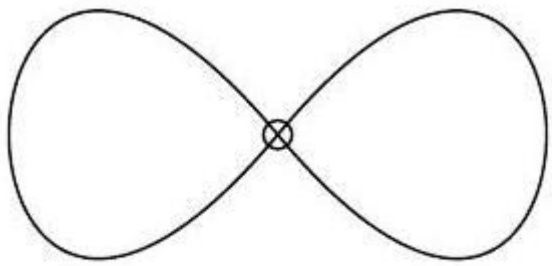
$$L^2(\mathbb{R})\ [\ m^2, +\infty\ )A' = -\frac{d^2}{dt^2} - m^2[-m^2, +\infty)$$

$$U(x) = \frac{m^2x^2}{2} - \sum_{n\geq 3} \frac{g_n x^n}{n!}.$$

$$B_r(q,q,,\ldots,q) = \int q^r(t)dt$$

$$F_\Gamma(\ell_1,\ldots,\ell_n) := \prod_j g_{v(j)} \int G_\Gamma(\mathbf{t},\mathbf{s})d\mathbf{s}$$

$$\int_{\mathbb{R}} G(0)^2 ds$$

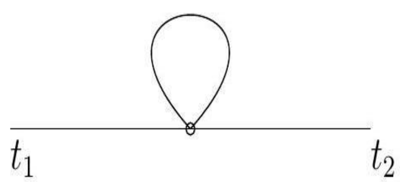


$$U(q) = \frac{m^2q^2}{2}$$

$$\mathcal{G}_{2k}(t_1,\ldots,t_{2k}) = \hbar^k \sum_{\sigma\in\Pi_k} \prod_{i\in\{1,\ldots,2k\}/\sigma} G\big(t_i - t_{\sigma(i)}\big).$$

$$\mathcal{G}_2(t_1,t_2) = \hbar G(t_1 - t_2)$$

$$U(q) = \frac{m^2q^2}{2} - \frac{gq^4}{24}$$



$$F_\Gamma = g\int_{\mathbb{R}} G(s,t_1)G(s,t_2)G(s,s)ds = \frac{g}{8m^3}\int_{\mathbb{R}} e^{-m(|s-t_1|+|s-t_2|)}ds$$

$$F_\Gamma = \frac{g}{8m^3}\left(2\int_0^\infty e^{-m(2s+|t_1-t_2|)}ds + |t_1-t_2|e^{-m|t_1-t_2|}\right) = \frac{g}{8m^4}e^{-m|t_1-t_2|}(1+m|t_1-t_2|)$$

$$\mathcal{G}_2(t_1, t_2) = \tilde{G}(t_1 - t_2),$$

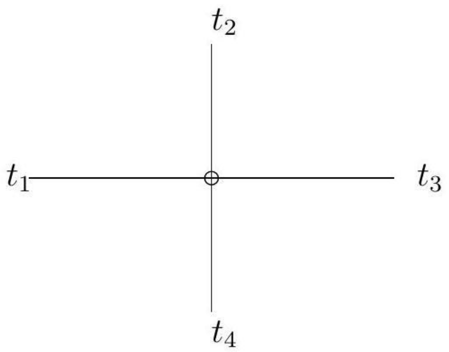
$$\tilde{G}(t) := \frac{1}{2m} e^{-m|t|} + \frac{g}{16m^4} e^{-m|t|}(1 + m|t|) + O(g^2).$$

$$\frac{1}{2}(\dot{q}^2 - M(q)) \frac{1}{2} \Sigma_i \left(\dot{q}_i{}^2 - m_i^2 q_i^2\right) \frac{e^{-m_i|t-s|}}{2m_i} \mathcal{G}_n^c(t_1, \dots, t_n)\, \mathcal{G}_n(t_1, \dots, t_n)$$

$$\mathcal{G}_n(t_1, \dots, t_n) = \sum_{\{1,\dots,n\} = S_1 \sqcup \dots \sqcup S_k} \prod \mathcal{G}_{|S_i|}^c(t_j; j \in S_i).$$

$$\mathcal{G}_2(t_1, t_2) = \mathcal{G}_2^c(t_1, t_2) + \mathcal{G}_1^c(t_1)\mathcal{G}_1^c(t_2)$$

$$U = \frac{m^2 q^2}{2}$$



$$U = \frac{m^2 q^2}{2} - \frac{g q^4}{4} g^2$$

$$\mathcal{G}_4^c(t_1, t_2, t_3, t_4) = g \int_{\mathbb{R}} G(t_1 - s) G(t_2 - s) G(t_3 - s) G(t_4 - s) ds + O(g^2).$$

$$\lim_{z \to \infty} \mathcal{G}_n(t_1, \dots, t_r, t_{r+1} + z, \dots, t_n + z) = \mathcal{G}_r(t_1, \dots, t_r) \mathcal{G}_{n-r}(t_{r+1} \dots t_n)$$

$$\lim_{z \to \infty} \mathcal{G}_n^c(t_1, \dots, t_r, t_{r+1} + z, \dots, t_n + z) = 0$$

$$Z(J) = \int e^{\frac{-S_E(q) + (J,q)}{\hbar}} Dq$$

$$\frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int_{\mathbb{R}^n} \mathcal{G}_n(t_1, \dots, t_n) J(t_1) \dots J(t_n) dt_1 \dots dt_n$$

$$W(J) := \log \frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^c(t_1, \dots, t_n) J(t_1) \dots J(t_n) dt_1 \dots dt_n$$

$$W(J) = \hbar^{-1} W_0(J) + W_1(J) + \hbar W_2(J) + \cdots$$

$$W_0(J) = -S_E(q_J) + (q_J, J),$$

$$S_E^J(q) := S_E(q) - (q, J)$$

$$W_1(J) = -\frac{1}{2} \log \det L_J$$

$$d^2 S_E^J(q_J)(f_1, f_2) = d^2 S_E^0(0)(L_J f_1, f_2)$$

$$\frac{m^2 q^2}{2} + \frac{g q^4}{2}$$

$$S_E^J(q) = \int \left(\frac{\dot{q}^2}{2} + U(q)\right) dt - aq(0)$$

$$\ddot{q} = m^2 q + 2gq^3 - a\delta(t).$$

$$\ddot{q} = m^2 q + 2gq^3$$

$$E = \frac{\dot{q_\pm}^2}{2} - U(q_\pm) = 0$$

$$t - t_\pm = \int \frac{dq}{\sqrt{2U(q)}} = \int \frac{dq}{mq\sqrt{1 + \frac{gq^2}{m^2}}} = \frac{1}{2m} \log \frac{\sqrt{1 + \frac{gq^2}{m^2}} - 1}{\sqrt{1 + \frac{gq^2}{m^2}} + 1}$$

$$q_J(t) = \frac{2mg^{-\frac{1}{2}}}{C^{-1}e^{m|t|} - Ce^{-m|t|}}$$

$$\frac{C(1 + C^2)}{(1 - C^2)^2} = \frac{ag^{\frac{1}{2}}}{4m^2}$$

$$W_0 = -S_E^J(q_J)$$

$$L_J = 1 + \frac{gA^{-1} \circ q_J(t)^2}{2}$$

$$A = -\frac{d^2}{dt^2} + m^2 A^{-1} \circ q_J(t)^2$$

$$K_J(x, y) := \frac{e^{-m|x-y|} q_J(y)^2}{2m},$$

$$1 + \frac{gA^{-1} \circ q_J(t)^2}{2}$$

$$\mathcal{G}_n^{1\,\mathrm{PIa}} := A^{\otimes n} \mathcal{G}_n^{1\,\mathrm{PI}}$$

$$G\left(t_i - s_j\right) \delta\left(t_i - s_j\right))$$

$$S_{\mathrm{eff}}(q) = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^{1PIa}(t_1, \dots, t_n) q(t_1) \dots q(t_n) dt_1 \dots dt_n$$

$$-S_{\mathrm{eff}}(\tilde{q}_J) + (J, \tilde{q}_J) - S_{\mathrm{eff}}(q) + (J, q)$$

$$(q_1, q_2) = \int_{\mathbb{R}} q_1(t) q_2(t) dt = \int_{\mathbb{R}} \hat{q}_1(E) \overline{\hat{q}_2(E)} dE = \int_{\mathbb{R}} \hat{q}_1(E) \hat{q}_2(-E) dE$$

$$B^{-1}(f, f) = \int_{\mathbb{R}} \frac{1}{E^2 + m^2} \hat{f}(E) \hat{f}(-E) dE$$

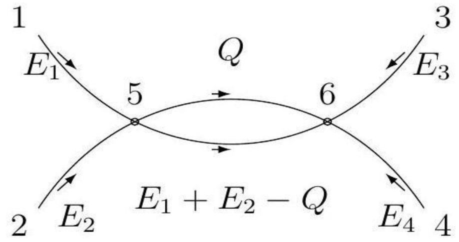
$$\delta_{s_1 = \dots = s_k} \delta_{Q_1 + \dots + Q_k = 0} \frac{1}{E_i^2 + m^2} \frac{1}{Q_k^2 + m^2} \phi_\Gamma(\mathbf{E}, \mathbf{Q}) \hat{F}_\Gamma(\mathbf{E})$$

$$\hat{F}_\Gamma(E_1, \dots, E_n) = \prod_j g_{v(j)} \int_{Y(\mathbf{E})} \phi_\Gamma(\mathbf{E}, \mathbf{Q}) d\mathbf{Q} \cdot \delta(E_1 + \dots + E_n) d\mathbf{E}$$

$$d\mathbf{Q} = \prod_{e \notin T} dQ_e F_\Gamma\left(\delta_{t_1}, \dots, \delta_{t_n}\right) \hat{F}_\Gamma(E_1, \dots, E_n)$$

$$\hat{\mathcal{G}}_n^c(E_1, \dots, E_n) = \sum_{\Gamma \in G_{\geq 3}^*(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \hat{F}_\Gamma(E_1, \dots, E_n)$$

$$\hat{\mathcal{G}}_4^c(E_1, E_2, E_3, E_4) = g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta\left(\sum_i E_i\right) d\mathbf{E} + O(g^2)$$



$$\frac{g^2}{2}\left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 + m^2)((E_1 + E_2 - Q)^2 + m^2)}\right) \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta\left(\sum_i E_i\right) d\mathbf{E}$$

$$\hat{\mathcal{G}}_4^c(E_1, E_2, E_3, E_4) =$$
$$g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \left(1 + \frac{\pi g}{m} \sum_{i=2}^{4} \frac{1}{(E_1 + E_i)^2 + 4m^2}\right) \delta\left(\sum_i E_i\right) d\mathbf{E} + O(g^3)$$

$$\sum_i \left(E_i + E_j\right)^2 = (E_k + E_\ell)^2)$$

$$\frac{1}{E^2 + m^2} = \int_{\mathbb{R}} G(t) e^{iEt} dt$$

$$\int_{\mathbb{R}} G(e^{i\theta} t) e^{iEt} dt = e^{-i\theta} \int_{\mathbb{R}} G(t) e^{ie^{-i\theta}Et} dt = \frac{e^{-i\theta}}{e^{-2i\theta} E^2 + m^2}$$

$$\lim_{\varepsilon \to 0+} \frac{i}{E^2 - m^2 + i\varepsilon} \frac{i}{E^2 - m^2 + i\varepsilon}$$

$$\frac{i}{E^2 - m^2 + i\varepsilon}$$

$$-\frac{g^2}{2}\left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 - m^2 + i\varepsilon)((E_1 + E_2 - Q)^2 - m^2 + i\varepsilon)}\right) \prod_{j=1}^{4} \frac{1}{E_j^2 - m^2 + i\varepsilon} \delta\left(\sum_i E_j\right) d\mathbf{E}$$

$$\left(-\frac{d^2}{dt^2} + m^2\right) f = \delta(t)$$

$$G_L(t) = \sum_{k \in \mathbb{Z}} G(t - kL) = \frac{e^{-m\left(t - \frac{L}{2}\right)} + e^{-m\left(\frac{L}{2} - t\right)}}{2m\left(e^{\frac{mL}{2}} - e^{-\frac{mL}{2}}\right)}, 0 \leq t \leq L$$

$$U(q) = \frac{m^2 q^2}{2} + \sum_{n \geq 3} \frac{g_n q^n}{n!},$$

$$m^2 = m_0^2 + g_2 Z_{m_0, \mathbf{g}, L}(0) / Z_{m_0, 0, L}(0)$$

$$\frac{Z_{m_0, \mathbf{g}, L}}{Z_{m_0, 0, L}} = \sum_{\Gamma \in G_{\geq 2}(0)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma$$

$$g_2 = a, g_3 = g_4 = \cdots = 0$$

$$\log \frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = W_1 = -\frac{1}{2}\log \det M$$

$$M = 1 + a\left(-\frac{d^2}{dt^2} + m_0^2\right)^{-1}$$

$$-\frac{d^2}{dt^2} + m_0^2 C^\infty(\mathbb{R}/L\mathbb{Z}) e^{\frac{2\pi i n t}{L}} \frac{4\pi^2 n^2}{L^2} + m_0^2$$

$$\det M = \prod_{n\in\mathbb{Z}} \left(1 + \frac{a}{\frac{4\pi^2 n^2}{L^2} + m_0^2}\right).$$

$$\sinh(z) = z \prod_{n\geq 1}\left(1 + \frac{z^2}{\pi^2 n^2}\right)$$

$$\frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = \frac{\sinh\left(\frac{m_0 L}{2}\right)}{\sinh\left(\frac{mL}{2}\right)}$$

$$U = \frac{m^2 q^2}{2} \frac{C}{\sinh\left(\frac{mL}{2}\right)} - \frac{d^2}{dt^2}$$

$$-G''(t) = \delta(t).$$

$$G(t) = -\frac{1}{2}|t| + C$$

$$\left\langle e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_n)}{r}}\right\rangle$$

$$\int e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_n)}{r}} e^{-\frac{S(q)}{\hbar}} Dq$$

$$S(q) := \frac{1}{2}\int \dot{q}^2 dt \int e^{-\frac{S(q)}{\hbar}} Dq$$

$$\int e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_n)}{r}} e^{-\frac{S(q)}{\hbar}} Dq = e^{-\frac{\hbar}{2r^2}B^{-1}\left(\sum_j p_j q(t_j), \sum_j p_j q(t_j)\right)} =$$
$$e^{-\frac{\hbar}{2r^2}\sum_{j,\ell} p_\ell p_j G(t_\ell - t_j)} = e^{\frac{\hbar}{2r^2}\sum_{\ell<j} p_\ell p_j |t_\ell - t_j|}$$

$$B(q,q) := \int \dot{q}^2 dt$$

$$\left\langle e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_k)}{r}}\right\rangle = e^{\frac{\hbar}{2r^2}\sum_{\ell<j} p_\ell p_j |t_l - t_j|}$$

$$\sum_{\ell<j} p_\ell p_j (t_\ell - t_j) = \sum_{j=1}^{n-1} (t_j - t_{j+1})(p_{j+1} + \dots + p_n)(p_1 + \dots + p_j) =$$
$$-\sum_j (t_j - t_{j+1})(p_1 + \dots + p_j)^2$$

$$(p_1 + \dots + p_j)^2 \geq \frac{\dot{q}^2}{2} - \frac{d^2}{dt^2}\mathbb{R}/L\mathbb{Z} - G''(t) = \delta(t) - \frac{d^2}{dt^2}\int G'' dt - G''(t)\delta(t) - \frac{d^2}{dt^2}\{q \in C^\infty(\mathbb{R}/L\mathbb{Z}) : \int q dt = 0\} -$$
$$G''(t) = \delta(t) - \frac{1}{L}\int G dt = 0.$$

$$G(t) = \frac{\left(t - \frac{L}{2}\right)^2}{2L} - \frac{L}{24}$$

$$\langle q(0)^2\rangle = \frac{L}{12}$$

$$\left\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}}\right\rangle_0 := e^{-\frac{\hbar}{2r^2}\sum_{\ell,j} p_\ell p_j G(t_\ell - t_j)}$$

$$\left\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}}\right\rangle_0 = e^{\frac{\hbar}{2r^2}\left(\sum_{\ell<j} p_\ell p_j |t_\ell - t_j| + \frac{\left(\sum_j p_j t_j\right)^2}{L}\right)}$$

$$q(t) + \frac{2\pi r N t}{L} q \mapsto q + \frac{2\pi r N t}{L} e^{\frac{2\pi i N}{L}\sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}$$

$$\left\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}}\right\rangle = e^{\frac{\hbar}{2r^2}\left(\sum_{l<j} p_\ell p_j |t_\ell - t_j| + \frac{\left(\sum p_j t_j\right)^2}{L}\right)} \frac{\sum_{N\in\mathbb{Z}} e^{\frac{2\pi i N}{L}\sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}}{\sum_{N\in\mathbb{Z}} e^{-\frac{2\pi^2 r^2 N^2}{\hbar L}}}$$

$$\theta(u,T) := \sum_{N\in\mathbb{Z}} e^{2\pi i u N - \pi T N^2}$$

$$\left\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}}\right\rangle = e^{\frac{\hbar}{2r^2}\left(\sum_j (t_j - t_{j+1})(p_1+\dots+p_j)^2 + \frac{\left(\sum_j p_j t_j\right)^2}{L}\right)} \frac{\theta\left(\frac{\sum_j p_j t_j}{L}, \frac{2\pi r^2}{\hbar L}\right)}{\theta\left(0, \frac{2\pi r^2}{\hbar L}\right)}$$

$$U(q) := \frac{m^2 q^2}{2} - \frac{g q^4}{4!}\hbar = 1$$

$$U(q) := \frac{m^2 q^2}{2} - \frac{g q^3}{3}$$

$$U(x) := \frac{m^2 \sinh^2(gx)}{2g^2} W_0(J) \log\left(Z(J)/Z(0)\right) J(t) = a\delta(t)$$

$$\mathcal{L}: TX \to \mathbb{R} S(q) = \int \mathcal{L}(q,\dot{q})dt$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i}.$$

$$\mathcal{L}(q,v) = \frac{v^2}{2} - U(q) U: X \to \mathbb{R}$$

$$\ddot{q} = -U'(q),$$

$$\ddot{q} = \nabla_{\dot{q}}\dot{q}$$

$$T_q X \to T_q^* X$$

$$H: T^*X \to \mathbb{R} H(q,p) = pv_0 - \mathcal{L}(q,v_0) pv - \mathcal{L}(q,v)$$

$$\mathcal{L} = \frac{v^2}{2} - U(q) \; H(q,p) = \frac{p^2}{2} + U(q)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$\alpha(z) = (p, d\pi(q,p)z)$$

$$\alpha = \sum_i p_i dq_i, \omega = \sum_i dp_i \wedge dq_i.$$

$$\{f,g\} = (df \otimes dg, \omega^{-1})$$

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

$$\frac{d}{dt}f(q(t),p(t)) = \{f,H\}(q(t),p(t))$$

$$\frac{df}{dt} = \{f,H\}$$

$$w_n := \frac{v_n}{\|Av_n\|} \|Aw_n\| = Aw_n \to u \in \mathcal{H}' \|u\| = (0,u) \in \bar{\Gamma}_A$$

$$\langle v, Aw\rangle = \langle Av, w\rangle (v_n, Av_n) \to (0,u)\ u \in \mathcal{H} u_k \in V\ u_k \to u$$

$$\langle Au_k, v_n\rangle = \langle u_k, Av_n\rangle \to \langle u_k, u\rangle, n \to \infty.$$

$$\|u\|^2 = v_n \to v, w_n \to wAv_n \to \bar{A}v, Aw_n \to \bar{A}w$$

$$\langle v, \bar{A}w\rangle = \lim_{n\to\infty}\langle v_n, Aw_n\rangle = \lim_{n\to\infty}\langle Av_n, w_n\rangle = \langle \bar{A}v, w\rangle,$$

$$\langle A^\dagger u_n, v\rangle = \langle u_n, Av\rangle \to \langle u, Av\rangle, n \to \infty$$

$$\langle A^\dagger u_n, v\rangle \to \langle w, v\rangle \langle u, Av\rangle = \langle w, v\rangle\ u \in V^\vee\ w = A^\dagger u$$

$$B(v,w) := (A^\dagger v, w) - (v, A^\dagger w)(\ A^\dagger, L\ )\ (\ A^\dagger, L^\perp\ ),$$

$$\frac{d}{dt}(U(t)v) = iAU(t)v$$

$$B(f,g) = i(\overline{f(2\pi)}g(2\pi) - \overline{f(0)}g(0)).$$

$$B((a,b),(a,b)) = i(|b|^2 - |a|^2),$$

$$P := -i\frac{d}{dx} V = C_0^\infty(\mathbb{R})\ \mathcal{H} = L^2(\mathbb{R})\bar{V} = V^\vee\ H^1(\mathbb{R})\ f \in L^2(\mathbb{R})\ f' \in L^2(\mathbb{R})$$

$$PL^2(\mathbb{R})P^2 = -\frac{d^2}{dx^2}\mathbb{R}_{\geq 0}$$

$$P := -i\frac{d}{dx}V \text{ of } \mathcal{H} = L^2(\mathbb{R}_{\geq 0})\bar{V}\ f \in H^1(\mathbb{R}_{\geq 0})\ f(0) = 0V^\vee\ H^1(\mathbb{R}_{\geq 0})V^\vee/\bar{V}B(f,g) = -i\overline{f(0)}g(0)n_+ = 1, n_-$$

$$A = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\ V = C_0^\infty(\mathbb{R})n + \frac{1}{2}, n \in \mathbb{N}H_n(x)e^{\frac{x^2}{2}}$$

$$A := -\frac{1}{2}\frac{d^2}{dx^2} + U(x)V = C_0^\infty(\mathbb{R})$$

$$\int_M (u\Delta v - v\Delta u)dx = \int_{\partial M}(u\partial_{\mathbf{n}}v - v\partial_{\mathbf{n}}u)d\sigma$$

$$B(f,g) = i\int_{\partial M}(\bar{f}\partial_{\mathbf{n}}g - g\partial_{\mathbf{n}}\bar{f})d\sigma$$

$$B(f,g) = i(\bar{f}g' - \overline{f'}g)\Big|_0^\pi$$

$$f(0) = 0, f'(\pi) - af(\pi) = 0$$

$$\lambda\cos\pi\lambda = a\sin\pi\lambda.$$

$$\lambda\mathrm{cotan}\pi\lambda = a.$$

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+a\chi_{[-1,1]}(x)V=C_0^\infty(\mathbb{R})\subset\mathcal{H}=L^2(\mathbb{R})$$

$$(H-E)u=f$$

$$f(x)=\int_{\mathbb{R}}G(x,y)dy$$

$$(H-E)f=\delta(x-y)$$

$$[\hat{f},\hat{g}]=i\hbar\widehat{\{f,g\}}+O(\hbar^2),\hbar\to 0$$

$$\widehat{p^m}\to\left(-i\hbar\frac{d}{dq}\right)^m$$

$$\hat{H}=-\frac{\hbar^2}{2}\frac{d^2}{dq^2}+U(q)$$

$$\frac{d}{dt}\langle\psi(t),A\psi(t)\rangle=\left\langle\psi(t),\frac{[A,\hat{H}]}{i\hbar}\psi(t)\right\rangle=-\frac{i}{\hbar}\langle\psi(t),[A,\hat{H}]\psi(t)\rangle$$

$$\dot{\psi}=-\frac{i}{\hbar}\hat{H}\psi$$

$$\psi(t)=e^{-\frac{it\hat{H}}{\hbar}}\psi(0),$$

$$A(t):=e^{\frac{it\hat{H}}{\hbar}}A(0)e^{-\frac{it\hat{H}}{\hbar}}$$

$$A'(t)=-\frac{i}{\hbar}[A(t),\hat{H}]$$

$$\langle\psi(t),A\psi(t)\rangle=\langle\psi(0),A(t)\psi(0)\rangle.$$

$$U(q)\to\infty \text{ as } |q|\to\infty\hat{H}=-\frac{\hbar^2}{2}\frac{d^2}{dq^2}+U(q)$$

$$E(\phi):=\langle\phi,\hat{H}\phi\rangle=\int_{\mathbb{R}}\left(\frac{\hbar^2}{2}\phi'(q)^2+U(q)\phi(q)^2\right)dq$$

$$\phi:\mathbb{R}\to\mathbb{R}\int_{\mathbb{R}}\phi(t)^2dt=\langle v|A|w\rangle$$

$$\mathcal{G}_n^{\text{Ham}}(t_1,\dots,t_n):=\langle\Omega,q(t_1)\dots q(t_n)\Omega\rangle$$

$$Z_L^{\text{Ham}}=\text{Tr}\left(e^{-\frac{L\hat{H}}{\hbar}}\right)$$

$$\mathcal{G}_{n,L}^{\text{Ham}}(-it_1,\dots,-it_n)=\frac{\text{Tr}\left(q(-it_n)\dots q(-it_1)e^{-\frac{L\hat{H}}{\hbar}}\right)}{\text{Tr}\left(e^{-\frac{L\hat{H}}{\hbar}}\right)}$$

$$\mathcal{G}_n^{\text{Ham}}(-it_1,\dots,-it_n)=\mathcal{G}_n^E(t_1,\dots,t_n)$$

$$Z_L^{\text{Ham}}=Z_L,\mathcal{G}_{n,L}^{\text{Ham}}=\mathcal{G}_{n,L}$$

$$\hbar=m=U=\frac{q^2}{2}Z_L=\frac{1}{2\sinh\left(\frac{L}{2}\right)}$$

$$\hat{H}=-\frac{1}{2}\frac{d^2}{dq^2}+\frac{q^2}{2}$$

$$H_n(x)e^{-\frac{x^2}{2}}$$

$$Z_L^{\text{Ham}} = e^{-\frac{L}{2}} + e^{-\frac{3L}{2}} + \cdots = \frac{1}{e^{\frac{L}{2}} - e^{-\frac{L}{2}}} = Z_L,$$

$$\widehat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}$$

$$\widehat{H} = a^\dagger a + \frac{1}{2},$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dq} + q\right), a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dq} + q\right)$$

$$[a, a^\dagger] = 1$$

$$\widehat{H}\ (a^\dagger)^n \Omega = e^{-\frac{q^2}{2}} n + \frac{1}{2}, n \in \mathbb{Z}_{\geq 0}$$

$$q(0) = q = \frac{1}{\sqrt{2}}(a + a^\dagger)$$

$$[a^\dagger a, a] = -a, [a^\dagger a, a^\dagger] = a^\dagger$$

$$q(t) = \frac{1}{\sqrt{2}} e^{ita^\dagger a}(a + a^\dagger)e^{-ita^\dagger a} = \frac{1}{\sqrt{2}}(e^{-it}a + e^{it}a^\dagger)$$

$$\mathcal{G}_{n,L}^{\text{Ham}}(-it_1, \ldots, -it_n) = 2^{-\frac{n}{2}} \frac{\text{Tr}\left(\prod_{j=1}^n \ (e^{t_j}a^\dagger + e^{-t_j}a)e^{-L\left(a^\dagger a + \frac{1}{2}\right)}\right)}{\text{Tr}\left(e^{-L\left(a^\dagger a + \frac{1}{2}\right)}\right)}.$$

$$\mathcal{G}_{n,L}^{\text{Ham}}(-it_1, \ldots, -it_n) =$$
$$\sum_{j=2}^{n} \frac{1}{2} \mathcal{G}_{n-2,L}^{\text{Ham}}(-it_2, \ldots, -it_{j-1}, -it_{j+1}, \ldots, -it_n)\left(\frac{e^{t_1 - t_j}}{e^L - 1} - \frac{e^{t_j - t_1}}{e^{-L} - 1}\right) =$$
$$\sum_{j=2}^{n} \mathcal{G}_{n-2,L}^{\text{Ham}}(-it_2, \ldots, -it_{j-1}, -it_{j+1}, \ldots, -it_n) G_L(t_1 - t_j)$$

$$\frac{e^{t-s}}{e^L - 1} - \frac{e^{s-t}}{e^{-L} - 1} = G_L(t - s)\ \langle \Omega, q(t_1) \ldots q(t_n)\Omega \rangle\ q(t_j) U(q) := \frac{m^2 q^2}{2} - V(q)$$

$$V(q) = \sum_{k \geq 3} \frac{g_k q^k}{k!}$$

$$Z_L^{\text{Ham}} = \text{Tr}(e^{-L\widehat{H}}) = \text{Tr}(e^{-L(\widehat{H}_0 - V)}),$$

$$\widehat{H}_0 = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}m^2 q^2$$

$$e^{-L(\widehat{H}_0 - V)} = e^{-L\widehat{H}_0} + \sum_{N \geq 1} \int_{L \geq s_1 \geq \cdots \geq s_N \geq 0} e^{-(L - s_1)\widehat{H}_0} V e^{-(s_1 - s_2)\widehat{H}_0} V \ldots e^{-(s_{n-1} - s_n)\widehat{H}_0} V e^{-s_n \widehat{H}_0} d\mathbf{s}$$

$$e^{A+B} = e^A + \sum_{N \geq 1} \int_{1 \geq s_1 \geq \cdots \geq s_N \geq 0} e^{(1 - s_1)A} B e^{(s_1 - s_2)A} B \ldots e^{(s_{N-1} - s_N)A} B e^{s_N A} d\mathbf{s}$$

$$Z_L^{\text{Ham}} = \sum_{N \geq 0} \sum_{j_1, \ldots, j_N = 3}^{\infty} \frac{g_{j_1} \ldots g_{j_N}}{j_1! \ldots j_N!} \int_{1 \geq s_1 \geq \cdots \geq s_N \geq 0} \text{Tr}\left(q_0(-is_1)^{j_1} \ldots q_0(-is_N)^{j_N} e^{-L\widehat{H}_0}\right) d\mathbf{s}$$

$$q_0(t)\,\frac{m^2q^2}{2}$$

$$\int_{0\le s_1,\dots,s_N\le L}\prod_{v-w}G_L(s_v-s_w)d\mathbf{s}$$

$$\frac{\prod_k g_k^{i_k}}{|\,\mathrm{Aut}\Gamma}Z_L^{\mathrm{Ham}}=Z_L$$

$$\mathrm{Tr}\left(e^{-(L-t_1)\hat{H}}qe^{-(t_1-t_2)\hat{H}}q\dots qe^{-t_n\hat{H}}\right)$$

$$\mathcal{G}_{n,L}^{\mathrm{Ham}}=\mathcal{G}_{n,L}\;q:\mathbb{R}\to\mathbb{R}\mathcal{H}=L^2(\mathbb{R})\;\hat{H}=-\frac{\hbar^2}{2}\frac{d^2}{dq^2}$$

$$q:\mathbb{R}\to S^1=\mathbb{R}/2\pi r\mathbb{Z}\mathcal{H}:=L^2(S^1)\;e^{\frac{iNq}{r}}\;\hbar^2\frac{N^2}{2r^2}$$

$$\left\langle\Omega,e^{\frac{t_1\hat{H}}{\hbar}}e^{\frac{ip_1q}{r}}e^{\frac{(t_2-t_1)\hat{H}}{\hbar}}\dots e^{\frac{ip_nq}{r}}e^{-\frac{t_n\hat{H}}{\hbar}}\Omega\right\rangle=$$

$$e^{\frac{\hbar}{2r^2}\sum_j(t_j-t_{j+1})(p_1+\dots+p_j)^2}$$

$$\mathrm{Tr}\left(e^{-\frac{L\hat{H}}{\hbar}}\right)=\sum_{N\in\mathbb{Z}}e^{-\frac{N^2L\hbar}{2r^2}}$$

$$\mathrm{Tr}\left(e^{\frac{t_1\hat{H}}{\hbar}}e^{\frac{ip_1q}{r}}e^{\frac{(t_2-t_1)\hat{H}}{\hbar}}\dots e^{\frac{ip_nq}{r}}e^{\frac{(L-t_n)\hat{H}}{\hbar}}\right)=\sum_{N\in\mathbb{Z}}e^{\frac{\hbar}{2r^2}\sum_{j=0}^{n}(t_j-t_{j+1})(N-p_1-\dots-p_j)^2},$$

$$e^{\frac{\hbar}{2r^2}\sum_j(t_j-t_{j+1})(p_1+\dots+p_j)^2}\sum_{N\in\mathbb{Z}}e^{-\frac{\hbar}{2r^2}(LN^2+2N\sum_j p_jt_j)}=$$

$$e^{\frac{\hbar}{2r^2}\sum_j(t_j-t_{j+1})(p_1+\dots+p_j)^2}\theta\left(\frac{\hbar}{2\pi ir^2}\sum_j p_jt_j,\frac{L\hbar}{2\pi r^2}\right)$$

$$\theta\left(\frac{u}{iT},\frac{1}{T}\right)=\sqrt{T}e^{\frac{\pi u^2}{T}}\theta(u,T)$$

$$T=\frac{2\pi r^2}{\hbar L}$$

$$q_N(t)=\frac{2\pi tNr}{L}e^{-\frac{2\pi^2N^2r^2}{L\hbar}}$$

$$Z(r,L)=\theta\left(0,\frac{\hbar L}{2\pi r^2}\right)=r\sqrt{\frac{2\pi}{\hbar L}}\theta\left(0,\frac{2\pi r^2}{\hbar L}\right)$$

$$Z(L)\sim\frac{1}{\sqrt{\hbar L}}$$

$$Z(L)=\mathrm{Tr}\left(e^{-\frac{L\hat{H}}{\hbar}}\right)$$

$$\mathbb{R}/2\pi\mathbb{Z}H:=-\frac{\hbar^2}{2}\partial^2+U(x)$$

$$\int_0^{2\pi}U(x)dx$$

$$E_{2m-1}=E_{2m}=\frac{\hbar^2m^2}{2}\Psi_0=\Psi_{2m-1}=\sin\,mx,\Psi_{2m}=\cos\,mx$$

$$U(x) = \begin{cases} Mb, 0 \leq x < a \\ -Ma, a \leq x < 2\pi \end{cases}$$

$$f_p(p) = g_p'(p) = 1, g_p(p) = f_p'(p)$$

$$f_0(x) = \cos\sqrt{\frac{2}{\hbar^2}(E-Mb)}x, g_0(x) = \frac{\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}x}{\sqrt{\frac{2}{\hbar^2}(E-Mb)}}$$

$$f_a(x) = \cos\sqrt{\frac{2}{\hbar^2}(E+Ma)}(x-a), g_a(x) = \frac{\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}(x-a)}{\sqrt{\frac{2}{\hbar^2}(E+Ma)}(x-a)}$$

$$A := \begin{pmatrix} \cos\sqrt{\frac{2}{\hbar^2}(E-Mb)}a & \frac{\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}a}{\sqrt{\frac{2}{\hbar^2}(E-Mb)}} \\ -\sqrt{\frac{2}{\hbar^2}(E-Mb)}\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}a & \cos\sqrt{\frac{2}{\hbar^2}(E-Mb)}a \end{pmatrix}$$

$$B := \begin{pmatrix} \cos\sqrt{\frac{2}{\hbar^2}(E+Ma)}b & \frac{\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b}{\sqrt{\frac{2}{\hbar^2}(E+Ma)}} \\ -\sqrt{\frac{2}{\hbar^2}(E+Ma)}\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b & \cos\sqrt{\frac{2}{\hbar^2}(E+Ma)}b \end{pmatrix}.$$

$$\cos\sqrt{\frac{2}{\hbar^2}(E-Mb)}a\cos\sqrt{\frac{2}{\hbar^2}(E+Ma)}b - \frac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}}\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}a\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b = 1$$

$$a < E < b\sqrt{\frac{2}{\hbar^2}(E-Mb)}$$

$$\cosh\sqrt{\frac{2}{\hbar^2}(Mb-E)}a\cos\sqrt{\frac{2}{\hbar^2}(E+Ma)}b - \frac{E+M\frac{a-b}{2}}{\sqrt{(Mb-E)(E+Ma)}}\sinh\sqrt{\frac{2}{\hbar^2}(Mb-E)}a\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b = 1$$

$\frac{1}{2}\hbar^2 n^2\ [C^{-1}, C]$ (so $\hbar \to 0$)

$$1 - \cos\left(\sqrt{\frac{2}{\hbar^2}(E-Mb)}a + \sqrt{\frac{2}{\hbar^2}(E+Ma)}b\right) =$$

$$\left(1 - \frac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}}\right)\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}a\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b$$

$$E = \frac{1}{2}(\hbar^2 n^2 + \varepsilon),$$

$$|\varepsilon| \ll \frac{1}{n}$$

$$\sqrt{\frac{2}{\hbar^2}(E-Mb)} = \sqrt{n^2+\frac{\varepsilon-2Mb}{\hbar^2}} = n\left(1+\frac{\varepsilon-2Mb}{2\hbar^2n^2}-\frac{(\varepsilon-2Mb)^2}{8\hbar^4n^4}\dots\right)$$
$$\sqrt{\frac{2}{\hbar^2}(E+Ma)} = \sqrt{n^2+\frac{\varepsilon+2Ma}{\hbar^2}} = n\left(1+\frac{\varepsilon+2Ma}{2\hbar^2n^2}-\frac{(\varepsilon+2Ma)^2}{8\hbar^4n^4}\dots\right)$$

$$\sqrt{\frac{2}{\hbar^2}(E-Mb)}a+\sqrt{\frac{2}{\hbar^2}(E+Ma)}b = 2\pi n\left(1+\frac{\varepsilon}{2\hbar^2n^2}-\frac{M^2ab}{2\hbar^4n^4}+\cdots\right).$$

$$LHS = \frac{\pi^2}{2}\left(\frac{\varepsilon}{\hbar^2 n}-\frac{M^2ab}{2\hbar^4n^3}\right)^2+\cdots$$

$$1-\frac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}} = -\frac{\pi^2M^2}{2\hbar^4n^4}+\cdots$$

$$RHS = \frac{\pi^2M^2}{2\hbar^4n^4}\sin^2 na+\cdots$$

$$\frac{\varepsilon}{\hbar^2 n}-\frac{M^2ab}{2\hbar^4n^3} = \pm\frac{M|\sin na|}{\hbar^2n^2}$$

$$\varepsilon = \frac{M^2ab}{2\hbar^2n^2}\pm\frac{M|\sin na|}{n}$$

$$\Lambda_n = \frac{\hbar^2n^2}{2}, n>\Lambda_n^{\pm}(M) = \Lambda_n+\frac{M^2a(2\pi-a)}{8\Lambda_n}\pm\frac{M|\sin na|}{2n}+o\left(\left(M+\frac{1}{n}\right)^2\right), M\to 0, n\to\infty$$

$$\hbar\frac{dF}{dx} = AF$$

$$F(x) = e^{\frac{\phi(x)}{\hbar}}(\psi_0(x)+\hbar\psi_1(x)+\hbar^2\psi_2(x)\dots)$$

$$(\hbar\partial_x+\phi'-A)(\psi_0+\hbar\psi_1+\hbar^2\psi_2+\cdots) = 0$$

$$A\psi_0 = \phi'\psi_0$$

$$\phi(x) = \int \lambda_j(x)dx, \psi_0(x) = f(x)v_j(x)$$

$$\psi_0' = (A-\lambda)\psi_1$$

$$f'v_j+fv_j' = (A-\lambda)\psi_1.$$

$$\psi_1\left(v_j^*, f'v_j+fv_j'\right)$$

$$f' = -(v_j^*, v_j')f$$

$$f(x) = \exp\left(-\int (v_j^*, v_j')dx\right)$$

$$F_j(x) = \exp\left(\frac{\int \lambda_j(x)dx}{\hbar}\right)\left(\exp\left(-\int (v_j^*(x), v_j'(x))dx\right)v_j(x)+O(\hbar)\right)$$

$$\left(-\frac{\hbar^2}{2}\partial_x^2 + U(x)\right)\Psi = E\Psi$$

$$p(x) := \sqrt{2(E - U(x))}$$

$$\hbar^2\partial_x^2\Psi = -p^2\Psi$$

$$\hbar\partial_x\begin{pmatrix}\Psi \\ \hbar\Psi'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -p^2 & 0\end{pmatrix}\begin{pmatrix}\Psi \\ \hbar\Psi'\end{pmatrix}.$$

$$A = \begin{pmatrix}0 & 1 \\ -p^2 & 0\end{pmatrix}$$

$$\lambda_1 = ip, \lambda_2 = -ip$$

$$v_1 = \begin{pmatrix}1 \\ ip\end{pmatrix}, v_2 = \begin{pmatrix}1 \\ -ip\end{pmatrix},$$

$$v_1^* = \frac{1}{2}(1, -ip^{-1}), v_2^* = \frac{1}{2}(1, ip^{-1})$$

$$\Psi_\pm = \exp\left(\pm\frac{i\int\ pdx}{\hbar}\right)\left(\exp\left(-\frac{1}{2}\int\ p^{-1}p'dx\right) + O(\hbar)\right) =$$

$$p^{-\frac{1}{2}}\exp\left(\pm\frac{i\int\ pdx}{\hbar}\right)(1 + O(\hbar))$$

$$\Psi_\pm(x) = \left(2(E - U(x))^{-\frac{1}{4}}\exp\left(\pm\frac{i\int\ \sqrt{2(E - U(x))}dx}{\hbar}\right)(1 + O(\hbar))\right.$$

$$H = -\frac{1}{2}\hbar^2\partial^2 + U(x)n \sim \frac{A}{\hbar}$$

$$E > \sup U(x).$$

$$\int_0^{2\pi}\ \sqrt{2(E - U(x))}dx = 2\pi n\hbar, n \in \mathbb{Z}_{\geq 0}$$

$$\nu(E) \sim \frac{A(E)}{\hbar}, \hbar \to 0, \text{where } A(E) := \frac{1}{\pi}\int_0^{2\pi}\ \sqrt{2(E - U(x))}dx$$

$$E_{\left[\frac{A}{\hbar}\right]}(\hbar) \sim E(A)$$

$$A = \frac{1}{\pi}\int_0^{2\pi}\ \sqrt{2(E - U(x))}dx$$

$$T^*S^1 = S^1 \times \mathbb{R}$$

$$H_{\text{cl}} \leq E,$$

$$H_{\text{cl}} := \frac{1}{2}p^2 + U(x)$$

$$\nu(E) \sim \frac{A(E)}{\hbar}$$

$$-\tfrac{1}{2}\hbar^2\Delta + U(x)U(x) \to +\infty \text{ as } x \to \infty)$$

$$\sqrt{2E}\int_0^{2\pi}\left(1 - \frac{MU_0(x)}{2E} - \frac{M^2U_0(x)^2}{8E^2} + o(M^2)\right)dx = 2\pi n\hbar$$

$$\int_0^{2\pi} U_0(x)dx = I := \frac{1}{2\pi}\int_0^{2\pi} U_0(x)^2 dx$$

$$\sqrt{2E} = n\hbar + \frac{M^2 I}{2(2E)^{\frac{3}{2}}} + o(M^2) = n\hbar\left(1 + \frac{M^2 I}{2n^4\hbar^4} + \cdots\right)$$

$$E = \frac{1}{2}n^2\hbar^2\left(1 + \frac{M^2 I}{n^4\hbar^4} + \cdots\right) = \Lambda_n + \frac{M^2 I}{8\Lambda_n} + \cdots$$

$$\Lambda_n := \frac{1}{2}n^2\hbar^2$$

$$I = a(2\pi - a)\frac{M^2 a(2\pi - a)}{8\Lambda_n}\frac{1}{n} \ll M$$

$$SV = SV_0 \otimes \Lambda V_1, \Lambda V = \Lambda V_0 \otimes SV_1.$$

$$S^i V = \Pi^i(\Lambda^i \Pi V), \Lambda^i V = \Pi^i(S^i \Pi V)$$

$$V = V_0 \oplus V_1$$

$$\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$$

$$\mathcal{O}(V) = k[x_1, \dots, x_n, \xi_1, \dots, \xi_m]$$

$$x_i x_j = x_j x_i, x_i \xi_r = \xi_r x_i, \xi_r \xi_s = -\xi_s \xi_r$$

$$\xi_r^2 = \mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$$

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd)$$

$$C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$$

$$C^\infty_{M_0} \otimes \Lambda(\xi_1, \dots, \xi_m)$$

$$C^\infty_{U_0} \otimes \Lambda V_1^*$$

$$\left(U_0, C^\infty_M|_{U_0}\right)$$

$$y_i = f_{0,i}(x_1, \dots, x_n) + f_{2,i}^{j_1 j_2}(x_1, \dots, x_n)\xi_{j_1}\xi_{j_2} + \cdots,$$
$$\eta_i = a_{1,i}^{j}(x_1, \dots, x_n)\xi_j + a_{3,i}^{j_1 j_2 j_3}(x_1, \dots, x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + \cdots,$$

$$M = N = \mathbb{R}^{1|2}, F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)\ g = g(x)$$

$$g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2$$

$$R := R_0 \oplus R_1 \mathrm{Mat}_{n|m}(R)$$

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

$$\mathrm{sTr}(A) = \sum_{i,j=1}^{n+m} \lambda_{ij} a_{ij}, \lambda_{ij} \in \mathbb{Z}$$

$$\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = n, \mathrm{sTr}(AB) = \mathrm{sTr}(BA) A, B \in \mathrm{Mat}_{n|m}(R)$$

$$\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) + \varepsilon \mathrm{Tr}(A_{11}) \varepsilon = -1$$

$$\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) - \mathrm{Tr}(A_{11}).$$

$$\mathrm{sdet}(e^{C}) = e^{\mathrm{sTr}C} = e^{\mathrm{Tr}(C_{00}) - \mathrm{Tr}(C_{11})}$$

$$C = C_{00} \oplus C_{11}$$

$$\mathrm{sdet}(e^{C}) = \frac{\det(e^{C_{00}})}{\det(e^{C_{11}})}$$

$$A = A_{00} \oplus A_{11}$$

$$\mathrm{sdet}A = \frac{\det A_{00}}{\det A_{11}}$$

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} a_+ + ba\_c & ba_- \\ a\_c & a_- \end{pmatrix}.$$

$$\mathrm{sdet}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \mathrm{sdet}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = 1$$

$$\mathrm{sdet}(A) = \frac{\det a_+}{\det a_-}$$

$$\mathrm{sdet}(A) = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det(A_{11})}\mathrm{Mat}_{n|m}(R)$$

$$\det(A) = \det(A_{00} - A_{01}A_{11}^{-1}A_{10})\det(A_{11}).$$

$$\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B).$$

$$A(t) \in \mathrm{Mat}_{n|m}(R) \text{ is a } C^1$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ber}(A(t)) = \mathrm{sTr}(A'(0)A(0)^{-1})\mathrm{Ber}(A(0)).$$

$$C \in \mathrm{Mat}_{n|m}(R)$$

$$\mathrm{Ber}(e^{C}) = e^{\mathrm{sTr}C}$$

$$A = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & Y \\ X & 1 + XY \end{pmatrix}$$

$$\det(1 - Y(1 + XY)^{-1}X) = \det(1 + XY)$$

$$X: V_0 \to V_1 \otimes R \text{ and } Y: V_1 \to V_0 \otimes R$$

$$\det(1 - Y(1 + XY)^{-1}X) = \sum_{k \geq 0} (-1)^k \mathrm{Tr}\big(Y(1 + XY)^{-1}X|_{\Lambda^k V_0}\big) =$$
$$= \sum_{k \geq 0} (-1)^k \mathrm{sTr}\left(Y(1 + XY)^{-1}|_{\Lambda^k V_1} \circ X|_{\Lambda^k V_0}\right) = \sum_{k \geq 0} (-1)^k \mathrm{sTr}\big(XY(1 + XY)^{-1}|_{\Lambda^k V_1}\big)$$
$$\sum_{k \geq 0} \mathrm{Tr}\big(XY(1 + XY)^{-1}|_{S^k \Pi V_1}\big) = \det(1 - XY(1 + XY)^{-1})^{-1} = \det(1 + XY)$$

$$f'(t) = \mathrm{sTr}(C)f(t)$$

$$f(t) = e^{\mathrm{sTr}(C)t}$$

$$f \in C^\infty(U_0) \otimes \Lambda V_1^*$$

$$U := U_0 \times V_1 \text{ (i.e. } f = \textstyle\sum f_i \otimes \omega_i, f_i \in C^\infty(U_0), \omega_i \in \Lambda V_1^*$$

$$dv = dv_0/dv_1$$

$\int_U f(v)dv$ is $\int_{U_0} (f(v),(dv_1)^{-1})dv_0$

$$\int_U f(v)dv$$

$$dv = \frac{dx_1 \dots dx_n}{d\xi_1 \dots d\xi_m}$$

$\int_U f(v)dv \int_{U_0} f_{\text{top}}(x_1,\dots,x_n)dx_1 \dots dx_n f_{\text{top}}\, \xi_1 \dots \xi_m\, f \in \Lambda V^*, v \in V \frac{\partial f}{\partial v}$

$$A_{00} = \left(\frac{\partial y_i}{\partial x_k}\right), A_{01} = \left(\frac{\partial y_i}{\partial \xi_\ell}\right), A_{10} = \left(\frac{\partial \eta_j}{\partial x_k}\right), A_{11} = \left(\frac{\partial \eta_j}{\partial \xi_\ell}\right).$$

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\text{Ber}(DF(v))|dv$$

$|f(\xi)| := f(\xi)$ is $a > 0$ and $|f(\xi)| := -f(\xi)$ if $a < 0$.

$$g(x_1+z,x_2,\dots,x_n,\xi)\left(1+\frac{\partial z}{\partial x_1}\right) - g(x_1,x_2,\dots,x_n,\xi)$$

$$\xi_j' = \xi_j, j\ \{U_\alpha\} f_\alpha : U_\alpha \to U_\alpha' \mathbb{R}^{n|m}$$

$$g_{\alpha\beta}: f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta) f_\alpha f_\beta^{-1} s_\alpha \in C_M^\infty(U_\alpha) U_\alpha \cap U_\beta s_\beta(z) = s_\alpha(z)\left|\text{Ber}(g_{\alpha\beta})(f_\beta(z))\right|$$

$\int_M \omega\ \phi_\alpha \text{Supp}\phi_\alpha \subset (U_\alpha)_0 \int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$

$$\frac{\Lambda^m B}{m!} = \text{Pf}(B)dv.$$

$$\text{Pf}(A) = \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1,\dots,2m\}, i < \sigma(i)} a_{i\sigma(i)},$$

$$\text{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

$d\xi = d\xi_1 \wedge \dots \wedge d\xi_{2m}\ \xi = (\xi_1,\dots,\xi_{2m}) B(\xi,\xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \text{Pf}(B)$$

$$\frac{1}{m!}\frac{\Lambda^m B}{d\xi}$$

$$V = Y \oplus Y^*$$

$$dv = dydy^*$$

$$S(y,y^*) = (Ay,y^*)$$

$$S(\xi_1,\dots,\xi_m,\eta_1,\dots,\eta_m) = \sum_{i,j} a_{ij}\xi_j\eta_i,$$

$$\int_V e^S (dv)^{-1} = (-1)^{\frac{n(n-1)}{2}} \det A$$

$$S(y,y_*) = \frac{1}{2}B\big((y,y_*),(y,y_*)\big)$$

$$B\big((y,y^*),(w,w^*)\big)=(Ay,w^*)-(Aw,y^*).$$

$$\mathrm{Pf}(B)=(-1)^{\frac{n(n-1)}{2}}\det(A)$$

$$\int_V \lambda_1(\xi)\ldots\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1}=\mathrm{Pf}(-B)\mathrm{Pf}\Big(B^{-1}(\lambda_i,\lambda_j)\Big)$$

$$\int_V \lambda_1(\xi)\ldots\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1}=$$
$$\mathrm{Pf}(-B)\sum_{\sigma\in\Pi_m}\varepsilon_\sigma\prod_{i\in\{1,\ldots,2m\},i<\sigma(i)}B^{-1}(\lambda_i,\lambda_{\sigma(i)})$$

$$B(e_j,e_l)=1\text{ if }j=2i-1,l=2i,\text{ and }B(e_j,e_l)=0$$

$$Y=\mathbb{R}^{\frac{n(n+1)}{2}+\frac{m(m-1)}{2}\Big|mn}$$

$$A=\begin{pmatrix}A_{00} & A_{01}\\ A_{10} & A_{11}\end{pmatrix}$$

$$\int_{Y_+\times\mathbb{R}^{n|m}} f(A)e^{-x^TA_{00}x-2x^TA_{01}\xi-\xi^TA_{11}\xi}dAdx(d\xi)^{-1}=$$
$$=C\int_{Y_+}f(A)\mathrm{Ber}(A)^{-1/2}dA$$

$$\sum_{\sigma\in S_{2n}}(-1)^\sigma X_{\sigma(1)}\ldots X_{\sigma(2n)}=0$$

$$X=\sum_{i=1}^{2n}X_i\xi_i$$

$$V=V_0\oplus V_1$$

$$dv=dv_0(dv_1)^{-1}$$

$$B=B_0\oplus B_1(B_0>0)$$

$$S(v)=\frac{1}{2}B(v,v)-\sum_{r\geq 3}\frac{B_r(v,v,\ldots,v)}{r!}$$

$$I(\hbar)=\int_V \ell_1(v_0)\ldots\ell_n(v_0)\lambda_1(v_1)\ldots\lambda_p(v_1)e^{-\frac{S(v)}{\hbar}}dv$$

$$B_r(v,v,\ldots,v)=\sum_{s=0}^{r}\binom{r}{s}B_{s,r-s}(v_1,\ldots,v_1,v_0,\ldots,v_0)$$

$$I(\hbar)=(2\pi)^{\frac{\dim V_0}{2}}\hbar^{\frac{\dim V_0-\dim V_1}{2}}\frac{\mathrm{Pf}(-B_1)}{\sqrt{\det B_0}}\sum_\Gamma\frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\mathbb{F}_\Gamma\big(\ell_1,\ldots,\ell_n,\lambda_1,\ldots,\lambda_p\big),$$

$$\mathbb{F}_\Gamma=-\mathbb{F}(\sigma)$$

$$S(v)=S_b(v_0)-\frac{1}{2}S_f(v_0)(v_1,v_1)$$

$$I(\hbar)=\hbar^{-\frac{\dim V_1}{2}}\int_{V_0}\ell_1(v_0)\ldots\ell_n(v_0)e^{-\frac{S_b(v_0)}{\hbar}}\mathrm{Pf}\Big(S_f(v_0)\Big)dv_0$$

$$S=\int_{\mathbb{R}}\mathcal{L}dt$$

$$\Lambda(C_0^\infty(\mathbb{R},V)^*)$$

$$\mathcal{L}=\frac{1}{2}\psi\dot{\psi},$$

$$S=\frac{1}{2}\int\ \psi\dot{\psi}dt$$

$$\psi\dot{\psi}\neq\frac{d}{dt}\left(\frac{\psi^2}{2}\right)=\psi\dot{\psi}=-\dot{\psi}\psi\frac{1}{2}(\dot{q}^2-m^2q^2)$$

$$\dot{\psi}=0$$

$$\int\ \psi(t_1)\dots\psi(t_n)e^{\frac{iS(\psi)}{\hbar}D\psi}$$

$$\mathcal{L}=\frac{1}{2}\psi\dot{\psi}$$

$$\langle\psi(t_1)\psi(t_2)\rangle G(t_1-t_2)$$

$$\frac{dG}{dt}=i\delta(t)$$

$$G(t)=\frac{i}{2}\text{sign}(t)+C$$

$$G(t)=\frac{i}{2}\text{sign}(t)$$

$$\langle\psi(t_1)\dots\psi(t_{2n})\rangle=(-1)^\sigma(2n-1)!!\left(\frac{i}{2}\right)^n$$

$$\mathcal{L}=\frac{1}{2}(\psi_1\dot{\psi}_1+\psi_2\dot{\psi}_2-m\psi_1\psi_2)$$

$$\frac{dG}{dt}-MG=i\delta(t)$$

$$M=\begin{pmatrix}0 & m\\ -m & 0\end{pmatrix}$$

$$G(t)=\begin{cases}e^{Mt}Q_-, t<0\\ e^{Mt}Q_+, t>0\end{cases}$$

$$\lim_{t\to+\infty}e^{-iMt}Q_+=0,\ \lim_{t\to-\infty}e^{-iMt}Q_-=0$$

$\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$ is a $\mathbb{Z}/2$-graded complex vector space

$\langle x,y\rangle=\overline{\langle y,x\rangle}$ $\langle x,y\rangle=-\overline{\langle y,x\rangle}$ for odd x,y.

$$T:\mathcal{H}_0\oplus\Pi\mathcal{H}_1\to\mathcal{H}_0\oplus\Pi\mathcal{H}_1$$

$$\langle x,T^\dagger y\rangle=(-1)^{p(x)p(T)}\langle Tx,y\rangle$$

$$\mathcal{L}=\frac{1}{2}((\psi,\dot{\psi})-(\psi,M\psi))$$

$\psi:\mathbb{R}\to\Pi V$, and $M:V\to V$

$$\mathcal{L}_b=\frac{1}{2}(\dot{q}^2-m^2q^2)$$

$$Y := T^*V = V \oplus V^*$$

$$\ddot{q} = -m^2 q,$$

$$\dot{q} = p, \dot{p} = -m^2 q.$$

$$\dot{f} = \{f, H\}; \text{ in this case it is } H = \frac{1}{2}(p^2 + m^2 q^2)$$

$$\dot{\psi} = M\psi.$$

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)} b\{a, c\}$$

$$\dot{f} = \{f, H\}$$

$$\dot{\psi} = M\psi$$

$$H = \frac{1}{2}(\psi, M\psi).$$

$$ab - ba = i(a, b)$$

$$ab + ba = i(a, b)$$

$$ab + ba = Q(a, b), a, b \in V$$

$$\dim V = 2d, \mathrm{Cl}(V_{\mathbb{C}}^*)$$

$$V_{\mathbb{C}} = L \oplus L^*$$

$$\dim V = 2d + 1$$

$$V_{\mathbb{C}} = L \oplus L^* \oplus K,$$

$$\mathcal{H} = \Lambda(L \oplus K)$$

$$\mathcal{H}_+ \oplus \mathcal{H}_-$$

$$H = \frac{1}{2}(\psi, M\psi) V^* a_{ij}$$

$$\hat{H} = \frac{1}{2}\sum_{i,j} a_{ij}\varepsilon_i\varepsilon_j$$

$$\hat{H} = \sum_j m_j(\xi_j\partial_j - \partial_j\xi_j) = \sum_j m_j(2\xi_j\partial_j - 1).$$

$$Z = \mathrm{sTr}(e^{-L\hat{H}}) = \prod_j (e^{m_j L} - e^{-m_j L})$$

$$\psi(0) \in V \otimes \mathrm{End}(\mathcal{H})$$

$$V^* \to \mathrm{End}(\mathcal{H})$$

$$\psi(t) = e^{it\hat{H}}\psi(0)e^{-it\hat{H}}$$

$$\langle\psi(t_1) \ldots \psi(t_n)\rangle, t_1 \geq \cdots \geq t_n V^{\otimes n}\, \psi(t_j)$$

$$\langle\psi(t_1) \ldots \psi(t_n)\rangle = \langle\Omega, \psi(t_1) \ldots \psi(t_n)\Omega\rangle.$$

$$\langle\psi(t_1)\ldots\psi(t_n)\rangle = \frac{\mathrm{sTr}\left(\psi(t_1)\ldots\psi(t_n)e^{-L\hat{H}}\right)}{\mathrm{sTr}\left(e^{-L\hat{H}}\right)}$$

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 + m^2\phi^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 - \mu\psi_1\psi_2\right) + g\phi\psi_1\psi_2$$

$$|\mathbf{v}|^2 = c^2t^2 - \sum_{j=1}^{d-1} x_j^2$$

$$(V_M)_{\mathbb{C}}\ dt^2 - \sum_j dx_j^2 - dt^2 - \sum_j dx_j^2 dt^2 + \sum_j dx_j^2 \text{ on } V_E.$$

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$$

$$(\Box + m^2)\phi = 0,$$

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial x_j^2}.$$

$$\mathcal{L}_E = \frac{1}{2}((d\phi)^2 + m^2\phi^2)$$

$$(-\Delta + m^2)\phi = 0.$$

$$\int \phi(x_1)\ldots\phi(x_n)e^{-S(\phi)}D\phi$$

$$S = \int \mathcal{L}$$

$$A = -\Delta + m^2A^{-1}$$

$$-\Delta G + m^2G = \delta$$

$$G(x) = g(|x|)$$

$$-g'' - \frac{d-1}{r}g' + m^2g = 0$$

$$r^{\frac{2-d}{2}}J_{\pm\frac{2-d}{2}}(imr)$$

$$g = Cr^{\frac{2-d}{2}}\left(J_{\frac{2-d}{2}}(imr) + i^dJ_{-\frac{2-d}{2}}(imr)\right), d \neq 2.$$

$$\frac{e^{-mr}}{2m}$$

$$\int \phi(x_1)\ldots\phi(x_n)e^{-S(\phi)}D\phi$$

$$G_M(x) = g\left(\sqrt{-|x|^2 - i\varepsilon}\right)$$

$$(\Box + m^2)G_M = i\delta$$

$$p^2\hat{G} + m^2\hat{G} = 1$$

$$\hat{G}(p) = \frac{1}{p^2 + m^2}$$

$$\hat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$\frac{1}{2}\psi\frac{d\psi}{dt}\frac{d}{dt}$$

$$\mathbf{D} = \sum_i \Gamma_i \frac{\partial}{\partial x_i}$$

$$\Gamma(v): S_\pm \to S_\mp$$

$$Y = Y_+ \otimes S_+ \oplus Y_- \otimes S_- Y_\pm = \mathrm{Hom}(S_\pm, Y)$$

$$Y' := Y_+ \otimes S_- \oplus Y_- \otimes S_+$$

$$Y = Y_0 \otimes S$$

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D} - M)\psi),$$

$$\mathbf{D}\psi = M\psi.$$

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D}_E + M)\psi)$$

$$\mathbf{p} = \sum_j p_j \Gamma_j G(x) \in \mathrm{Hom}(Y^*, Y) \frac{1}{i\mathbf{p} + M}$$

$$M^\dagger: Y^* \to Y$$

$$(-i\mathbf{p} + M)(i\mathbf{p} + M) = p^2 + M^2$$

$$\hat{G}(p) = (p^2 + M^2)^{-1}(-i\mathbf{p} + M).$$

$$\hat{G}_M(p) = (p^2 - M^2 + i\varepsilon)^{-1}(\mathbf{p} + iM).$$

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$$

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0$$

$$\phi(0,x) = q(x), \phi_t(0,x) = p(x)$$

$$Y := T^* C_0^\infty(\mathbb{R}^{d-1}) := C_0^\infty(\mathbb{R}^{d-1}) \oplus C_0^\infty(\mathbb{R}^{d-1})$$

$$\omega\big((q_1, p_1), (q_2, p_2)\big) = \int_{\mathbb{R}^{d-1}} (p_1(x)q_2(x) - p_2(x)q_1(x))dx$$

$$(q, p) \mapsto q(x), (q, p) \mapsto p(x)$$

$$\rho \in C_0^\infty(\mathbb{R}^{d-1})$$

$$\phi(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} q(x)\rho(x)dx, \phi_t(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} p(x)\rho(x)dx$$

$$\{\phi(\rho_1), \phi(\rho_2)\} = 0, \{\phi_t(\rho_1), \phi_t(\rho_2)\} = 0$$
$$\{\phi(\rho_1), \phi_t(\rho_2)\} = \int_{\mathbb{R}^{d-1}} \rho_1(x)\rho_2(x)dx$$

$$\{\phi(x), \phi(y)\} = 0, \{\phi_t(x), \phi_t(y)\} = 0, \{\phi(x), \phi_t(y)\} = \delta(x - y);$$

$$\{P(\phi)(x), Q(\phi)(y)\} = \sum_{\alpha} \{P, Q\}_{\alpha}(\phi)(x)\partial_x^{\alpha}\delta(x-y)$$

$$\left\{\phi_{tx}(u)\phi_t(u), \frac{1}{3}\phi^3(v)\right\} =$$
$$-(\phi_{tx}(u)\phi^2(u) + 2\phi_{tx}(u)\phi(u)\phi_x(u))\delta(u-v) - \phi_t(u)\phi^2(u)\delta'(u-v).$$

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} (\phi_t^2 + (d_s\phi)^2 + m^2\phi^2)dx$$

$$F_t = \{F, H\}$$

$$\phi_t(t,x) = \frac{d}{dt}\phi(t,x)$$

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0$$

$$\{\phi(t_1,x_1), \phi(t_2,x_2)\} = \mathbf{G}(t_2 - t_1, x_2 - x_1)$$

$$\mathbf{G}(0,x) = 0, \mathbf{G}_t(0,x) = \delta(x)$$

$$E^2 = p^2 + m^2$$

$$\widehat{\mathbf{G}}(E,p) = f_+(p)\delta_{X_m^+} + f_-(p)\delta_{X_m^-}$$

$$\int_{\mathbb{R}} \widehat{\mathbf{G}}(E,p)dE = 0, \int_{\mathbb{R}} \widehat{\mathbf{G}}(E,p)EdE = 1$$

$$f_+(p) + f_-(p) = 0, \sqrt{p^2+m^2}(f_+(p) - f_-(p)) = 1.$$

$$f_+ = -f_- = \frac{1}{2\sqrt{p^2+m^2}}$$

$$\widehat{\mathbf{G}}(E,p) = \frac{1}{2\sqrt{p^2+m^2}}\left(\delta_{X_m^+} - \delta_{X_m^-}\right)$$

$$\{\phi(t_1,x_1), \phi(t_2,x_2)\} = 0$$

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2) - \frac{g}{4}\phi^4$$

$$\phi_{tt} - \Delta_s\phi + m^2\phi + g\phi^3 = 0$$

$$g(t,x) = (at + bx + c, \alpha t + \beta x + \gamma)$$

$$(\phi g)(x)(q,p) = \phi(bx + c, \beta x + \gamma)$$

$$(\phi_t g)(x)(q,p) = (\partial_\alpha\phi)(bx + c, \beta x + \gamma) + a\phi_t(bx + c, \beta x + \gamma).$$

$$SO(\mathbb{R}^{d-1}) \ltimes \mathbb{R}^{d-1}$$

$$\widehat{H}_\pi := i\frac{d}{dt}\bigg|_{t=0} \pi(t,0)$$

$$\widehat{H}_\pi\, V^* \cong V\sigma(\pi)\pi\langle w_1, \pi(v)w_2\rangle, v \in V, \text{for any } w_1, w_2 \in \mathcal{H}$$

$$R = R_0 \oplus R_1 \text{ of } \mathrm{Spin}_+(V)$$

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

$$\mathcal{S} \otimes R^* \to \mathrm{End}\mathcal{D}$$

$$\mathcal{S} \otimes R^* \to \mathbb{C} \text{ defined by } f \mapsto \langle w_1, \phi(f) w_2 \rangle$$

$$\phi(f_1) \dots \phi(f_n)\Omega$$

$$|v_1 - v_2|^2 < 0$$

$$[\phi(f_1), \phi(f_2)] = 0$$

$$\mathcal{H}^{\tilde{\mathbf{P}}} = \mathbb{C}\Omega\phi|_{\mathcal{S}\otimes E} \neq 0$$

$$\mathrm{Spin}_+(V) \to \mathrm{SO}_+(V), \text{ so } \zeta^2 = 1$$

$$\zeta|_E = (-1)^j$$

$$W_n(f_1 \boxtimes \dots \boxtimes f_n) = \langle \Omega, \phi(f_1) \dots \phi(f_n)\Omega \rangle R^{*\otimes n} V^{\otimes n} \; R^{*\otimes n}$$

$$W_n(f_1 \boxtimes \dots \boxtimes f_n) = \int_{V^n} W_n(x_1, \dots, x_n) f_1(x_1) \dots f_n(x_n) dx_1 \dots dx_n$$

$$W_n^{u_1,\dots,u_n}(x_1, \dots, x_n) := (W_n(x_1, \dots, x_n), u_1 \otimes \dots \otimes u_n).$$

$$\phi(f) = \int_V \phi(x) f(x) dx$$

$$W_n(x_1, \dots, x_n) = \langle \Omega, \phi(x_1) \dots \phi(x_n)\Omega \rangle.$$

$$\tilde{\mathcal{D}} := T(\mathcal{S} \otimes R)$$

$$f_1 \otimes f_2 \otimes \dots \otimes f_n, f_i \in \mathcal{S} \otimes R$$

$$\tilde{\mathcal{D}}\langle f_1 \otimes \dots \otimes f_n, g_1 \otimes \dots \otimes g_m \rangle := (-1)^{\sum_{i<j} p(f_i)p(f_j)} W_{n+m}(\bar{f}_n \boxtimes \dots \boxtimes \bar{f}_1 \boxtimes g_1 \boxtimes \dots \boxtimes g_m).$$

$$\langle f_1 \otimes \dots \otimes f_n, g_1 \otimes \dots \otimes g_m \rangle = \langle \phi(f_1) \dots \phi(f_n)\Omega, \phi(g_1) \dots \phi(g_m)\Omega \rangle$$

$$W : T(\mathcal{S} \otimes R) \to \mathbb{C}$$

$$* : T(\mathcal{S} \otimes R) \to T(\mathcal{S} \otimes R)$$

$$(f_1 \otimes \dots \otimes f_n)^* = (-1)^{\sum_{i<j} p(f_i)p(f_j)} \bar{f}_n \otimes \dots \otimes \bar{f}_1$$

$$W_n(f^*) = \overline{W_n(f)}$$

$$W_n^{u_1,\dots,u_n}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = (-1)^{p(u_i)p(u_{i+1})} W_n(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$$
$$\text{if } |x_i - x_{i+1}|^2 < 0.$$

$$W(f^* \otimes f) \geq 0 \text{ for any } f \in T(\mathcal{S} \otimes R)$$

$$W_2(v_1, v_2) = \mathbb{W}(v)$$

$$\mathbb{W}(v) = \langle \Omega, \phi(0)\phi(v)\Omega \rangle =$$
$$= \langle \Omega, \phi(0)\pi(v)\phi(0)\pi(-v)\Omega \rangle = \langle \phi(0)\Omega, \pi(v)\phi(0)\Omega \rangle$$

$$W_1(x) = \langle \Omega, \phi(x)\Omega \rangle$$

$$(R^*)^{\mathrm{Spin}_+(V)} \phi(x) \text{ by } \phi(x) - c)$$

$$\int_{V^2} \mathbb{W}(x_2 - x_1) \overline{f(x_1)} f(x_2) dx_1 dx_2 \geq 0$$

$$\mathbb{W}(x) = W_2(0,x)$$

$$\int_V \widehat{\mathbb{W}}(p)\overline{\hat{f}(p)}\hat{f}(p)dp \geq 0$$

$$E = \sqrt{p^2 + m^2}$$

$$\mathrm{Spin}(d-2) \ltimes \mathbb{R}^{d-2}$$

$$\mathbb{R}^{d-2}\phi(t,x) = \phi(0,x-t), \phi(t,x) = \phi(0,x+t)$$

$$\mathcal{H}_{X_m^+,\rho} \text{ (or } \mathcal{H}_{X_0^{+\pm},\rho} \text{ for } d = 2\text{)}$$

$$(\square + m^2)G_M = i\delta$$

$$\mathbb{W}(x) = G_M(x)$$

$$\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$$

$$\mathbb{W}(x) = G_M(x)$$

$$\mathbb{W}(x) = \overline{G_M(x)}$$

$$G_M(x_2 - x_1) = W_2^T(x_1, x_2)$$

$$(\square + m^2)\mathbb{W} = 0$$

$$\mathrm{Re}\mathbb{W}(x) = \mathrm{Re}G(x)$$

$$(\square + m^2)\mathrm{Im}\mathbb{W}(x)\delta$$

$$G_M(x)\mathbb{W}(-x) = \overline{\mathbb{W}(x)}\widehat{\mathbb{W}}(p)X_m SO_+(V)$$

$$\widehat{\mathbb{W}}(p) = c_+\delta_{X_m^+} + c_-\delta_{X_m^-}$$

$$\widehat{\mathbb{W}}(p) = c\delta_{X_m^+}.$$

$$\mathcal{H}^{(1)} = L^2(X_m^+)$$

$$\langle\phi(x)\phi(y)\rangle = \langle\Omega, \phi(x)\phi(y)\Omega\rangle = G(x-y)$$

$$|x-y|^2 = 0$$

$$\langle\phi(x)\phi(y)\phi(z_1)\dots\phi(z_k)\rangle = G(x-y)\langle\phi(z_1)\dots\phi(z_k)\rangle + \sum_{i\neq j} G(x-z_i)G(y-z_j)\langle\phi(z_1)\dots\hat{\phi}(z_i)\dots\hat{\phi}(z_j)\dots\phi(z_k)\rangle$$

$$\langle:\phi(x)\phi(y):\phi(z_1)\dots\phi(z_k)\rangle = \sum_{i\neq j} G(x-z_i)G(y-z_j)\langle\phi(z_1)\dots\hat{\phi}(z_i)\dots\hat{\phi}(z_j)\dots\phi(z_k)\rangle.$$

$$:\phi(x)\phi(y): = \phi(x)\phi(y) - G(x-y).$$

$$\langle:\phi^2(x):\phi(z_1)\dots\phi(z_k)\rangle = \sum_{i\neq j} G(x-z_i)G(x-z_j)\langle\phi(z_1)\dots\hat{\phi}(z_i)\dots\hat{\phi}(z_j)\dots\phi(z_k)\rangle.$$

$$\phi(x)\phi(y)\phi(z): = \phi(x)\phi(y)\phi(z) - G(x-y)\phi(z) - G(y-z)\phi(x) - G(z-x)\phi(y)$$

$$\phi(x)\phi(y) = G(x-y) + :\phi(x)\phi(y):$$

$$\phi(x)\phi(y) = G(x-y) + \sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y):$$

$\mathbf{n} := (n_1, \dots, n_{d+1}), (x-y)^{\mathbf{n}} := \prod_i (x_i - y_i)^{n_i}, \partial^{\mathbf{n}} := \prod_i \partial_{x_i}^{n_i}$, and $\mathbf{n}! := \prod_i n_i!$

$$\langle :\phi^2(x): \phi(y)\phi(z_1)\dots\phi(z_k)\rangle = 2G(x-y)\left\langle \phi(x)\phi(z_1)\dots\widehat{\phi(z_j)}\dots\phi(z_k)\right\rangle + \sum_{j,m,n \text{ distinct}} G(x-z_j)G(x-z_m)G(y-z_n)\left\langle \phi(z_1)\dots\widehat{\phi(z_j)}\dots\widehat{\phi(z_m)}\dots\widehat{\phi(z_n)}\dots\phi(z_k)\right\rangle.$$

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + :\phi^2(x)\phi(y):.$$

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + \sum_{\mathbf{n},\mathbf{m}} \frac{(x-y)^{\mathbf{n}+\mathbf{m}}}{\mathbf{n}!\,\mathbf{m}!} :\partial^{\mathbf{n}}\phi(y)\cdot\partial^{\mathbf{m}}\phi(y)\cdot\phi(y):.$$

$$:\phi^2(x)::\phi^2(y): = 2G^2(x-y) + 4G(x-y):\phi(x)\phi(y): + :\phi^2(x)\phi^2(y):,$$

$$:\phi^2(x)::\phi^2(y): = 2G^2(x-y) + 4G(x-y)\sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y): + :\phi^2(x)\phi^2(y):,$$

$$|\mathbf{n}| := \sum_i n_i$$

$$:\phi^2(x)::\phi^2(y): = \frac{2}{|x-y|^2} + \frac{4}{|x-y|}:\phi^2(y): + \sum_{j=1}^{3} \frac{4}{|x-y|}(x_j - y_j):\partial_{x_i}\phi(y)\cdot\phi(y): + \text{regular}.$$

$$A(x)B(y) \sim \sum_j F_j(x-y)C_j(y), x \to y$$

$$|F_j(z)| = O(|z|^N), z \to 0$$

$$v := \left.\frac{d}{dt}\right|_{t=0} g^t$$

$$\{Q_y, F\} = y \cdot F := \left.\frac{d}{dt}\right|_{t=0} e^{ty}\cdot F$$

$$\{Q_y, Q_z\} = Q_{[y,z]} + C(y,z)$$

$$C([x,y],z) + C([y,z],x) + C([z,x],y) = 0.$$

$$\mu: \hat{\mathfrak{g}} \to C^\infty(M),$$

$\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ is a 1 -dimensional central extension of $\mathfrak{g}$ with commutator

$$[(y,a),(z,b)] = ([y,z], C(y,z)).$$

$$\mu(y,a) = Q_y + a$$

$C^\infty(M) \otimes \hat{\mathfrak{g}}^*$, i.e., geometrically as a C^∞-map

$$\mu: M \to \hat{\mathfrak{g}}^*.$$

$\mathfrak{g} = \mathbb{R}^{2n}$ and $C(y,z) = \omega(y,z)$

$\hat{\mathfrak{g}}\ \mathbb{R}^{2n} \oplus \mathbb{R}$

$$[(y,a),(z,b)] = ([y,z], C(y,z))$$

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$$

$$\mu: M \to \mathfrak{g}^*.$$

$$H = \frac{1}{2}\int_X (\phi_t^2 + |d_x\phi|^2 + m^2\phi^2)dx$$

$$H = \int_{\mathbb{R}^d} J dx$$

$$J = \frac{1}{2}(\phi_t^2 + |d_x\phi|^2 + m^2\phi^2) = \frac{1}{2}\left(:\phi_t^2: + \sum_{j=1}^{d} :\phi_{x_j}^2: + m^2 :\phi^2:\right)$$

$$J_k = \phi_t \phi_{x_k}$$

$$\{J_k(x), \phi(y)\} = -\phi_{x_k}(x)\delta(x-y), \{J_k(x), \phi_t(y)\} = \phi_t(x)\delta_{x_k}(x-y)$$

$$P_k = \int_{\mathbb{R}^d} J_k(x)dx$$

$$\{P_k, \phi(y)\} = -\phi_{x_k}(y), \{P_k, \phi_t(y)\} = -\phi_{tx_k}(y)$$

$$[\pi(g), \hat{H}] = \hat{H}: \mathcal{H} \to \mathcal{H}$$

$$\pi_*: \mathfrak{g} \to \mathrm{End}(\mathcal{S})$$

$$\hat{p}_j := -i\hbar\partial_{x_j}$$

$$\hat{M}_{kj} := -i\hbar\left(x_k\partial_j - x_j\partial_k\right).$$

$$Q = \int_{\mathbb{R}^d} J(x)dx$$

$$J = \frac{1}{2}\left(:\phi_t^2: + \sum_{j=1}^{d} :\phi_{x_j}^2: + m^2 :\phi^2:\right)$$

$$J_k =: \phi_t \phi_{x_k}:$$

$$\hat{H} = \int_{\mathbb{R}^d} J(x)dx$$

$$\hat{P}_k := \int_{\mathbb{R}^d} J_k(x)dx$$

$$|dx|^2 := (dt)^2 - g_{ij}dx^i dx^j$$

$$\phi: M \to E \cong E^* |d\phi(x)|^2$$

$$d\phi \in T_{\phi(x)}M \otimes E (d\phi)_{ij}$$

$$|d\phi|^2 = \sum_{i,j} (d\phi)_{ij}^2 |\nabla_A\phi|^2 |d\phi + A\phi|^2 m^2 |\phi|^2$$

$$S = S_+ \oplus S_-$$

$$S_M = S_{M+} \oplus S_{M-}$$

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{LC}$$

$$S_M \otimes E$$

$$\nabla^{\text{total}} = \nabla^{LC} \otimes \nabla_A$$

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{\text{total}}$$

$$\nabla_A = d + A$$

$$F_A = dA + \frac{1}{2}[A, A]$$

$$|F_A|^2 \ (\wedge^2 T^*M \otimes \text{ad}E)_x T_x M (\text{ad}E)_x \cong \mathfrak{g}$$

$$\nabla_{A_1} - \nabla_{A_2} \in \Omega^1(M) \otimes \text{ad}E$$

$$\Omega^1(M) \otimes \text{ad}E$$

$$\mathcal{G}_E = C^\infty(M, E)$$

$$A^g = g^{-1}dg + g^{-1}Ag$$

$$\mathcal{M} := \sqcup_{\text{topological types } E} \text{Conn}(E)/\mathcal{G}_E,$$

$$S(\phi) := \int \mathcal{L}(\phi) dx$$

$$V \cong \mathbb{R}^d \ \mathcal{L}(\phi)$$

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \int \phi(x_1) \dots \phi(x_n) e^{-\frac{S_E(\phi)}{\hbar}} D\phi.$$

$$\mathcal{L}_E(\phi) := \frac{1}{2}((d\phi)^2 + m^2\phi^2) + \frac{g}{6}\phi^3$$

$$\hat{G}_0(p) = \frac{1}{p^2 + m^2}$$

$$A(p) = \frac{g^2}{2(p^2 + m^2)^2} \int_V \frac{dq}{(q^2 + m^2)((p - q)^2 + m^2)}.$$

$$y_1 + \cdots + y_n = 1,$$

$$\int_{\Delta_n} \frac{dy}{(a_1 y_1 + \cdots + a_n y_n)^n} = \frac{1}{a_1 \dots a_n}.$$

$$\frac{1}{(a_1 y_1 + \cdots + a_n y_n)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(a_1 y_1 + \cdots + a_n y_n)t} dt$$

$$\int_{\Delta_n} \frac{dy}{(a_1 y_1 + \cdots + a_n y_n)^n} = \frac{1}{(n-1)!} \int_{\Delta_n} \int_0^\infty t^{n-1} e^{-(a_1 y_1 + \cdots + a_n y_n)t} dt dy$$
$$= \frac{1}{(n-1)!} \int_{t\Delta_n} \int_0^\infty e^{-a_1 z_1 + \cdots + a_n z_n} dt dz$$
$$= \int_{z_1, \dots, z_n \geq 0} e^{-a_1 z_1 + \cdots + a_n z_n} dz = \prod_{j=1}^n \int_0^\infty e^{-a_j z_j} dz_j = \frac{1}{a_1 \dots a_n}$$

$$\int_V \frac{dq}{(q^2 + m^2)((p - q)^2 + m^2)} = \int_0^1 \int_V \frac{dq}{((1 - y)q^2 + y(p - q)^2 + m^2)^2} dy =$$
$$\int_0^1 \int_V \frac{dq}{(q^2 + M^2(y, p))^2} dy$$

$$M^2(y,p) := y(1-y)p^2 + m^2$$

$$\int_V \frac{dq}{(q^2+M^2)^2} = C_d \int_0^\infty \frac{r^{d-1}dr}{(r^2+M^2)^2}$$

$$\int_V \frac{dq}{(q^2+M^2)^2} = 2\pi \int_0^\infty \frac{rdr}{(r^2+M^2)^2} = \pi \int_0^\infty \frac{ds}{(s+M^2)^2} = \frac{\pi}{M^2}$$

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \pi \int_0^1 \frac{dy}{y(1-y)p^2+m^2}$$

$$= \frac{2\pi}{p^2\sqrt{\frac{4m^2}{p^2}+1}} \operatorname{arccotanh} \sqrt{\frac{4m^2}{p^2}+1}$$

$$A_\Lambda(p) := \frac{g^2}{2(p^2+m^2)^2} \int_{|q|\le\Lambda} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)}$$

$$A_\Lambda(p) \sim \pi^2 \frac{g^2}{(p^2+m^2)^2} \log\left(\frac{\Lambda}{m}\right), \Lambda \to \infty$$

$$A_\Lambda(p) \sim C_d \frac{g^2}{2(d-4)(p^2+m^2)^2} \Lambda^{d-4}, \Lambda \to \infty$$

$$\hat{G}_{\Lambda,m^2}(p) = \frac{1}{p^2+m^2} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log\left(\frac{\Lambda}{m}\right) + \cdots$$

$$m^2 + Kg^2 \log\left(\frac{\Lambda}{m}\right)$$

$$\hat{G}_{\Lambda,m^2+Kg^2\log\left(\frac{\Lambda}{m}\right)}(p) = \frac{1}{p^2+m^2+Kg^2\log\left(\frac{\Lambda}{m}\right)} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log\left(\frac{\Lambda}{m}\right) + \cdots$$

$$= \frac{1}{p^2+m^2} + (\pi^2 - K) \frac{g^2}{(p^2+m^2)^2} \log\left(\frac{\Lambda}{m}\right) + \cdots$$

$$\mathcal{L}_{E,\Lambda} := \frac{1}{2}\left((d\phi)^2 + \left(m^2 + \pi^2 g^2 \log\left(\frac{\Lambda}{m}\right)\right)\phi^2\right) + \frac{g}{6}\phi^3$$

$$\hat{G}(p) = \frac{1}{p^2+m^2} + \frac{g^2}{2(p^2+m^2)^2} I(p)$$

$$I(p) = \lim_{\Lambda\to\infty}\left(\int_{\mathbb{R}^4} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} - 2\pi^2 \log\left(\frac{\Lambda}{m}\right)\right)$$

$$I(p) = \int_0^1 I(p,y)dy, I(p,y) := \lim_{\Lambda\to\infty}\left(\int_0^\Lambda \frac{r^3dr}{(r^2+M^2(y,p))^2} - 2\pi^2 \log\left(\frac{\Lambda}{m}\right)\right)$$

$$I(p,y) = 2\pi^2\left(\log m - \frac{1}{2}\left(1 + \log\left(y(1-y)p^2+m^2\right)\right)\right)$$

$$I(p) = 2\pi^2\left(\frac{1}{2} + \sqrt{\frac{4m^2}{p^2}+1} \cdot \operatorname{arccotanh} \sqrt{\frac{4m^2}{p^2}+1}\right)$$

$$A_\Lambda(p) \sim C_5 \frac{g^2}{2(p^2+m^2)^2}\Lambda + O(1), \lambda \to \infty$$

$$A_\Lambda(p) \sim \frac{g^2}{4(p^2+m^2)^2}\left(C_6\Lambda^2 + Cp^2 \log\left(\frac{\Lambda}{m}\right) + O(1)\right), \Lambda \to \infty$$

$$m^2 \mapsto m^2 + K\Lambda^2$$

$$Cp^2 \log\left(\frac{\Lambda}{m}\right)$$

$$\frac{1}{2}(d\phi)^2 1 + C' g^2 \log\left(\frac{\Lambda}{m}\right) C'$$

$$\frac{g^3}{\prod_{j=1}^3 (p_j^2+m^2)} J(p_1,p_2,p_3)\delta(p_1+p_2+p_3)$$

$$J(p_1,p_2,p_3) = \int_V \frac{dq}{(q^2+m^2)((q-p_1)^2+m^2)((q-p_1-p_2)^2+m^2)}$$

$$g + C'' g^3 \log\left(\frac{\Lambda}{m}\right)$$

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d + N,$$

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d$$

$$D(\Phi) = \frac{d-2}{2}e(\Phi) - d + N_\Phi$$

$$[\Phi] := D(\Phi) + d,$$

$$[\Phi_1\Phi_2] = [\Phi_1][\Phi_2].$$

$$D(\Gamma) = d - \frac{k(d-2)}{2} + \sum_\Phi D(\Phi),$$

$$D(\phi^n) = \frac{n}{2}(d-2) - d = \left(\frac{n}{2} - 1\right)d - n$$

$$D(\phi^{n-2}(d\phi)^2) = \left(\frac{n}{2} - 1\right)(d-2)$$

$$D(\Gamma) = d - \frac{k}{2}(d-2) + kD(\phi^3) = d - \frac{k}{2}(d-2) + \frac{k}{2}(d-6) = d - 2k$$

$$[(d\phi)^2] = D((d\phi)^2) + d = d, \text{ so } 2[\phi] + 2 = d, \text{ i.e. } [\phi] = \frac{d-2}{2}$$

$$D(\phi^n) = \left(\frac{n}{2} - 1\right)d - n$$

$$d \leq \frac{2n}{n-2}$$

$$D(\phi^{n-2}(d\phi)^2) = \left(\frac{n}{2} - 1\right)(d-2)$$

$$\mathcal{L} = \tfrac{1}{2}(d\phi)^2 + P_3(\phi);$$

$$\mathcal{L} = \tfrac{1}{2}(d\phi)^2 + P_4(\phi);$$

$$\mathcal{L} = \tfrac{1}{2}(d\phi)^2 + P_6(\phi);$$

$$\mathcal{L} = \tfrac{1}{2} g(\phi)(d\phi)^2 + U(\phi),$$

$$\frac{1}{2}(d\phi)^2 + P_m(\phi)$$

$$2[\psi] + 1 = d$$

$$[\psi] = \frac{d-1}{2}$$

$$D(\psi^{2k}) = 2k[\psi] - d = k(d-1) - d = (k-1)(d-1) - 1.$$

$$[\phi^n \psi^2] = n\frac{d-2}{2} + d - 1$$

$$D(\phi^n \psi^2) = n\frac{d-2}{2} - 1$$

$$\nabla_A = d + A$$

$$F_A = dA + \frac{1}{2}[A, A],$$

$$\mathcal{L} := \int_V |F_A|^2 dx$$

$\nabla_A \mapsto g^{-1}\nabla_A g$, i.e., $A \mapsto g^{-1}dg + g^{-1}Ag$, where $g: V \to G$

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{\dim M} g_{ij}(\phi) d\phi^i d\phi^j$$

$$\mathcal{L} = R(g)$$

$$\mathcal{L} = (dh)^2 + \cdots$$

$$\phi_{tt} - \phi_{xx} = 0$$

$$\phi = \frac{1}{\sqrt{2}}\phi_L + \frac{1}{\sqrt{2}}\phi_R$$

$$\phi_L(t,x) = \psi_L(x+t), \phi_R(t,x) = \psi_R(x-t)$$

$$(\partial_t - \partial_x)\phi_L = 0, (\partial_t + \partial_x)\phi_R = 0$$

$$\phi_x + \phi_t = \sqrt{2}\psi_L'(x+t), \phi_x - \phi_t = \sqrt{2}\psi_R'(x-t)$$

$$\{\psi_L'(x), \psi_L'(y)\} = \delta'(x-y), \{\psi_R'(x), \psi_R'(y)\} = -\delta'(x-y)$$

$$\{\psi_L'(x), \psi_R'(y)\} = 0.$$

$$\bar{\partial}_u \phi_L = 0, \partial_u \phi_R = 0,$$

$$\phi_L = \psi_L(u)$$

$$\phi_R = \psi_R(\bar{u}) \int_0^{2\pi} \phi(t,x) dx$$

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n}, \varphi^*(\bar{z}) = \sum_{n \in \mathbb{Z}} \varphi_n^* \bar{z}^{-n}$$

$$a(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}:=i\partial_z\varphi(z), a^*(\bar{z})=\sum_{n\in\mathbb{Z}}a_n^*\bar{z}^{-n-1}:=-i\bar{\partial}_z\varphi^*(\bar{z})$$

$$za=\partial_u\phi_L=\psi_L'(u), \bar{z}a^*=\bar{\partial}_u\phi_R=\psi_R'(\bar{u})$$

$$\{za(z), wa(w)\}=\delta'(u-v)$$

$$\delta'(u-v)=i\sum_{n\in\mathbb{Z}}nz^nw^{-n}$$

$$\delta(w-z):=\sum_{n\in\mathbb{Z}}z^nw^{-n-1}$$

$$\{a(z), a(w)\}=-i\delta'(w-z)$$

$$\left\{\sum_{m\in\mathbb{Z}}a_mz^{-m}, \sum_{n\in\mathbb{Z}}a_{-n}w^n\right\}=-i\sum_{n\in\mathbb{Z}}nz^{-n}w^n.$$

$$\{a_n, a_m\}=-in\delta_{n,-m}$$

$$\{a_n^*, a_m^*\}=in\delta_{n,-m},$$

$$\{a_n, a_m^*\}=0$$

$$\{a^*(z), a^*(w)\}=i\delta'(w-z), \{a(z), a^*(w)\}=0.$$

$$H=\frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}}(\phi_t^2+\phi_x^2)dx$$

$$H=\frac{1}{4}\int_{\mathbb{R}/2\pi\mathbb{Z}}((\bar{z}a^*-za)^2+(\bar{z}a^*+za)^2)dx=\frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}}(\bar{z}^2a^{*2}+z^2a^2)dx$$

$$H=\sum_{n>0}(a_{-n}a_n+a_{-n}^*a_n^*)$$

$$\{a_m, H\}=-ima_m, \{a_m^*, H\}=ima_m^*$$

$$[a_n, a_m]=n\delta_{n,-m}, [a_n^*, a_m^*]=-n\delta_{n,-m}, [a_n, a_m^*]=0.$$

$$a(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}, a^*(\bar{z})=\sum_{n\in\mathbb{Z}}a_n^*\bar{z}^{-n-1}$$

$$[a(z), a(w)]=\delta'(w-z), [a^*(z), a^*(w)]=-\delta'(w-z), [a(z), a^*(w)]=0$$

$$[a_n, a_m]=n\delta_{n,-m}K$$

$$a_n\Omega=0, n>0, K\Omega=\Omega$$

$$\mathcal{F}=\mathbb{C}[X_1, X_2, \dots]$$

$$n\frac{\partial}{\partial X_n}$$

$$[\hat{H}, a_n]=-na_n, [\hat{H}, a_n^*]=na_n^*$$

$$a_n\Omega=0, a_{-n}^*\Omega=0$$

$$\mathcal{F}\otimes\mathcal{F}^*\ \mathcal{A}\oplus\mathcal{A}^*, \mathcal{F}^*:=\mathbb{C}[X_1^*, X_2^*, \dots]X_n^*\ a_{-n}^*\mapsto n\frac{\partial}{\partial X_n^*}n>0$$

$$\mathbb{C}[X_j]\ \mathbb{C}[X_j^*]\|X_j^n\|^2 = j^n n!\,\mathbb{C}[X_j^*]a_i^\dagger = a_{-i} \text{ and } a_i^{*\dagger} = a_{-i}^*\mathcal{D}$$

$$\hat{H} = \sum_{n>0} (a_{-n}a_n + a_n^* a_{-n}^*) + C$$

$$\hat{H} = \hat{H}_L + \hat{H}_R$$

$$\hat{H}_L := \sum_{n>0} a_{-n}a_n + \frac{C}{2}, \hat{H}_L := \sum_{n>0} a_n^* a_{-n}^* + \frac{C}{2}.$$

$$z\partial_z + \tfrac{1}{2}\mathbb{C}[z]$$

$$\hat{H} = \sum_{n>0} (a_{-n}a_n + a_n^* a_{-n}^* + n) = \frac{1}{2}\sum_{n\neq 0} (a_{-n}a_n + a_n^* a_{-n}^*),$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)s \mapsto 1-s$$

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s)$$

$$C = 1+2+3+\cdots := \zeta(-1)$$

$$\zeta(-1) = \frac{\pi^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)}\zeta(2) = -\frac{\pi^{-\frac{3}{2}}}{2\pi^{\frac{1}{2}}}\frac{\pi^2}{6} = -\frac{1}{12}.$$

$$C := -\frac{1}{12}$$

$$\zeta(2\,\mathrm{g}) = (-1)^{\mathrm{g}+1}2^{2\,\mathrm{g}-1}\frac{B_{2\,\mathrm{g}}}{(2\,\mathrm{g})!}\pi^{2\,\mathrm{g}}.$$

$$\zeta(1-2\,\mathrm{g}) = \pi^{\frac{1}{2}-2\,\mathrm{g}}\frac{\Gamma(\,\mathrm{g})}{\Gamma\left(\frac{1}{2}-\mathrm{g}\right)}\cdot(-1)^{\mathrm{g}+1}2^{2\,\mathrm{g}-1}\frac{B_{2\,\mathrm{g}}}{(2\,\mathrm{g})!}\pi^{2\,\mathrm{g}} = -\frac{B_{2\,\mathrm{g}}}{2\,\mathrm{g}}.$$

$$E = E_\tau = \mathbb{C}^\times/q^{\mathbb{Z}} \cong \mathbb{R}/2\pi T\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$$

$$T>0, \tau = iT, q = e^{2\pi i\tau} = e^{-2\pi T}\in(0,1).$$

$$Z(\tau) := \mathrm{Tr}\left(e^{2\pi i\tau\hat{H}}\right).$$

$$\hat{H}(P\otimes Q) = (\deg P + \deg Q + C)P\otimes Q,$$

$$\deg(X_n) = \deg(X_n^*) = n.$$

$$Z(\tau) = \frac{e^{2\pi i\tau\left(C+\frac{1}{12}\right)}}{\eta(\tau)^2},$$

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\cdot\eta(\tau).$$

$$\mathcal{L}(\phi) = \frac{1}{4\pi}\int_E (d\phi)^2 = \frac{1}{4\pi}\int_E d\phi \wedge * \, d\phi$$

$$Z\left(-\frac{1}{\tau}\right) = -i\tau Z(\tau)$$

$$\mathcal{H}_{\text{full}} := \mathcal{H} \otimes L^2(\mathbb{R})L^2(\mathbb{R})\widehat{H}$$

$$\widehat{H}_{\text{full}} := \widehat{H} + \hat{\mu}^2$$

$$\mathcal{H}_{\text{full}} = \int_{\mathbb{R}} \mathcal{H}_\mu d\mu$$

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$$\mathcal{H}_\mu = \mathcal{F}_\mu \otimes \mathcal{F}_\mu^*$$

$$\widehat{H} = \widehat{H}_L + \widehat{H}_R$$

$$\widehat{H}_L = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \frac{1}{24}, \widehat{H}_R = \frac{1}{2}a_0^{*2} + \sum_{n>0} a_n^* a_{-n}^* - \frac{1}{24}$$

$$\mathcal{Z}(\tau) = (-i\tau)^{-\frac{1}{2}}Z(\tau)$$

$$\mathcal{Z}\left(-\frac{1}{\tau}\right) = \mathcal{Z}(\tau)$$

$$[D, a_n] = na_n, [D, a_n^*] = na_n^*$$

$$D(P \otimes Q) = (\text{deg}P - \text{deg}Q)P \otimes Q$$

$$D = \widehat{H}_L - \widehat{H}_R$$

$$Z(\tau) = \text{Tr}\left(e^{-2\pi T\widehat{H}}e^{2\pi isD}\right) = |q|^{-\frac{1}{12}}\text{Tr}\left(q^{\widehat{H}_L}\bar{q}^{\widehat{H}_R}\right),$$

$$q = e^{-2\pi(T+is)} = e^{2\pi i\tau}$$

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}$$

$$\mathcal{Z}(\tau) = \frac{1}{\sqrt{\text{Im}\tau}|\eta(\tau)|^2}$$

$$\tau \mapsto \tfrac{a\tau+b}{c\tau+d} \text{ for } a, b, c, d \in \mathbb{Z}, ad - bc = 1$$

$$\langle\Omega, a(z)a(w)\Omega\rangle = \sum_{n=1}^{\infty} nz^{-n-1}w^{n-1} = \frac{1}{(z-w)^2}$$

$$\langle\Omega, a(z_1) \ldots . a(z_{2k})\Omega\rangle = \sum_{\sigma\in\Pi_{2k}} \frac{1}{\prod_{j\in[1,2k]/\sigma}\left(z_j - z_{\sigma(j)}\right)^2},$$

$$\frac{\langle\tilde{a}(z)\tilde{a}(w)\rangle_E}{\langle\emptyset\rangle_E} = \text{Tr}_{\mathcal{F}}\left(\tilde{a}(z)\tilde{a}(w)e^{-2\pi T\widehat{H}_L}\right)$$

$$[L_n, L_m] = (n-m)L_{m+n}, m, n \in \mathbb{Z}.$$

$$W_{\mathbb{C}} = W \oplus W^*$$

$$\mathcal{A} \oplus \mathcal{A}^*$$

$$[L_n, a(z)] = z^{n+1}a'(z) + (n+1)z^n a(z),$$
$$[L_n^*, a^*(\bar{z})] = \bar{z}^{n+1}a^{*\prime}(\bar{z}) + (n+1)\bar{z}^n a_*(\bar{z}),$$
$$[L_n^*, a(z)] = [L_n, a^*(\bar{z})] = 0,$$

$$[L_n, a_m] = -ma_{m+n}, [L_m^*, a_n^*] = -ma_{m+n}^*, [L_n, a_m^*] = [L_n^*, a_m] = 0.$$

$$L_0 = \widehat{H}_L + C_L, L_0^* = -\widehat{H}_R + C_R$$

$$L_0 = \sum_{k\geq 1} a_{-k}a_k + \text{ const}$$

$$L_n := \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{-k}a_{k+n}$$

$$\mathcal{D} = \mathcal{F} \otimes \mathcal{F}^*$$

$$[L_n, a(z)] = -z^{n+1}a'(z) + (n+1)z^n a(z), [L_n, a^*(z)] = 0$$

$$[L_n, L_m] - (n-m)L_{m+n}$$

$$[L_n, L_m] - (n-m)L_{m+n} = 0$$

$$[L_n, L_{-n}] - 2nL_0 = C(n)$$

$$L_{-n}\Omega = \frac{1}{2}\sum_{0<j<n} X_j X_{n-j}$$

$$L_n L_{-n}\Omega = \frac{1}{4}\sum_{0<j<n} j(n-j)\frac{\partial^2}{\partial X_j \partial X_{n-j}}\sum_{0<j<n} X_j X_{n-j} = \frac{1}{2}\sum_{0<j<n} j(n-j) = \frac{n^3-n}{12}$$

$$C(n) = \frac{n^3-n}{12}$$

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n,-m}C$$

$$L_0 = \sum_{k\geq 1} a_{-k}a_k, L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{-k}a_{k+n}, n \neq 0$$

$$L_0 = \frac{1}{2}\mu^2 + \sum_{k\geq 1} a_{-k}a_k$$

$$L_0^* = -\frac{1}{2}\mu^2 - \sum_{k\geq 1} a_k^* a_{-k}^*, L_n^* = -\frac{1}{2}\sum_{k\in\mathbb{Z}} a_k^* a_{-k+n}^*, n \neq 0$$

$$L_n^\dagger = L_{-n}$$

$$\mathcal{D} = \mathcal{F}^{\otimes \ell} \otimes \mathcal{F}^{*\otimes \ell}$$

$$: a(z)a(w) := a(z)a(w) - \frac{1}{(z-w)^2}$$

$$\langle \Omega, a(z_1) \dots a(z_{i-1}) \colon a(z_i)a(z_{i+1}) \colon a(z_{i+2}) \dots a(z_n)\Omega \rangle =$$
$$\sum_{\sigma\in\Pi_{2k}:\sigma(i)\neq i+1} \frac{1}{\prod_{j\in\Pi_{2k}/\sigma} \left(z_j - z_{\sigma(j)}\right)^2}.$$

$$:a(z)a(w):=\sum_{m,n\in\mathbb{Z}}:a_n a_m:z^{-n-1}w^{-m-1},$$

$$\frac{1}{2}:a(z)^2:=T(z):=\sum_{n\in\mathbb{Z}}L_n z^{-n-2}$$

$$L_n=\frac{1}{2\pi i}\oint\ z^{n+1}T(z)dz$$

$$z^{n+1}T(z)+\overline{z^{n+1}T(z)}$$

$$:a(z_0)a(z_1)\dots a(z_n):=a(z_0):a(z_1)\dots a(z_n):-\sum_{k\in[1,n]}\frac{:\prod_{j\neq k}\ a(z_j):}{(z_0-z_k)^2}$$

$$:a^{(r_1)}(z_1)\dots a^{(r_n)}(z_n):=\partial_{z_1}^{r_1}\dots\partial_{z_n}^{r_n}:a(z_1)\dots.a(z_n):$$

$$P\mapsto P(a)(z)\Omega|_{z=0}$$

$$:a(z_1)\dots a(z_n):\cdots a(w_1)\dots a(w_m):=\sum_{I\subset[1,n],J\subset[1,m],s:I\cong J}\frac{:\prod_{i\notin I}\ a(z_i)\prod_{j\notin J}\ a(w_j):}{\prod_{i\in I}\ \left(z_i-w_{s(i)}\right)^2}.$$

$$:a(z)^n::a(w)^m:=\sum_{k=0}^{\min(m,n)}\ k!\binom{n}{k}\binom{m}{k}\frac{:a(z)^{n-k}a(w)^{m-k}:}{(z-w)^{2k}}.$$

$$a(z)a(w)=\frac{1}{(z-w)^2}+:a(z)a(w):=\frac{1}{(z-w)^2}+\ \mathfrak{J}_{regular\ terms}$$

$$a(z):a(w)^m:=\frac{m:a^{m-1}(w):}{(z-w)^2}+:a(z)a(w)^m:$$
$$=\frac{m:a^{m-1}(w):}{(z-w)^2}+\ \mathfrak{J}_{regular\ terms}$$

$$a(z)T(w)=\frac{a(w)}{(z-w)^2}+\ \mathfrak{J}_{regular\ terms}$$

$$:a(z)^2::a(w)^2:=\frac{2}{(z-w)^4}+\frac{4:a(z)a(w):}{(z-w)^2}+:a(z)^2a(w)^2:=$$
$$\frac{2}{(z-w)^4}+\frac{4:a(w)^2:}{(z-w)^2}+\frac{4:a(w)a'(w)}{z-w}+\ \mathfrak{J}_{regular\ terms}$$

$$T(z)T(w)=\frac{1}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{T'(w)}{z-w}+\ \mathfrak{J}_{regular\ terms}$$

$$T(z)T(w)=\frac{c}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{T'(w)}{z-w}+\ \mathfrak{J}_{regular\ terms}$$

$$P(a)(z)Q(a)(w)=\sum_{j=1}^{N}\ R_j(a)(w)(z-w)^{-j}+\ \mathfrak{J}_{regular\ terms}$$
where $(z-w)^{-j}:=\sum_{k\geq 0}\ \binom{k+j-1}{j-1}z^{-j-k}w^k$

$$Q(a)(w)P(a)(z)(z-w)^{-j}(z-w)^{-j}:=-\sum_{k<0}\ \binom{k+j-1}{j-1}z^{-j-k}w^k$$

$$[P(a)(z),Q(a)(w)]=\sum_{j=1}^{N}\ \frac{1}{(j-1)!}R_j(a)(w)\delta^{(j-1)}(w-z)$$

$$[a(z), a(w)] = \delta'(w-z)$$

$$[a(z), T(w)] = a(w)\delta'(w-z)$$

$$[T(z), T(w)] = \frac{c}{12}\delta'''(w-z) + 2T(w)\delta'(w-z) + T'(w)\delta(w-z)$$

$$a(z)a(w) \sim \frac{1}{(z-w)^2} + \sum_{k=0}^{\infty} :a^{(k)}(w)a(w): \frac{(z-w)^k}{k!}$$

$$P(a)(z)Q(a)(w) \sim \sum_{j=-\infty}^{N} R_j(a)(w)(z-w)^{-j}$$

$$\varphi(z) = -i\int a(z)dz = -i\left(a_0 \log z + \sum_{n\neq 0} \frac{a_{-n}}{n} z^n + a_0^{\vee}\right)$$

$$e^{i\lambda\varphi(z)} = e^{\lambda\int a(z)dz} = e^{\lambda\left(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n\right)} e^{\lambda a_0^{\vee}}$$

$$X(\lambda, z) :=: e^{\lambda\left(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n\right)} : e^{\lambda a_0^{\vee}} =$$
$$= e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}} z^{\lambda\mu} e^{\lambda\partial_\mu},$$

$$X_0(\lambda, z) := e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}}$$

$$e^A e^B = e^B e^A e^{[A,B]}$$

$$\left[\sum_{n>0}\frac{a_n}{n}z^{-n}, \sum_{n>0}\frac{a_{-n}}{n}w^n\right] = \sum_{n>0}\frac{z^{-n}w^n}{n} = -\log\left(1 - \frac{w}{z}\right)$$

$$X_0(\lambda, z)X_0(\nu, w) = \left(1 - \frac{w}{z}\right)^{\lambda\nu} :X_0(\lambda, z)X_0(\nu, w):$$

$$X(\lambda, z)X(\nu, w) = (z-w)^{\lambda\nu} :X(\lambda, z)X(\nu, w):$$

$$X(\lambda_1, z_1)\dots X(\lambda_n, z_n) = \prod_{1\leq j<k\leq n}(z_j - z_k)^{\lambda_j\lambda_k} :X(\lambda_1, z_1)\dots X(\lambda_n, z_n):$$

$$\langle\Omega_{\mu+\lambda}, X(\lambda_1, z_1)\dots X(\lambda_n, z_n)\Omega_\mu\rangle = \prod_{j=1}^{n} z_j^{\lambda_j\mu} \prod_{1\leq j<k\leq n}(z_j - z_k)^{\lambda_j\lambda_k}$$

$$X(\lambda, z)X(\nu, w) = e^{\pi i\lambda\nu}X(\nu, w)X(\lambda, z)$$

$$X(\lambda, z)X(\lambda, w) = e^{\pi i\lambda^2}X(\lambda, w)X(\lambda, z)$$

$$X'(\lambda, z) = \lambda :a(z)X(\lambda, z):$$

$$[a_n, X(\lambda, z)] = \lambda z^n X(\lambda, z)$$

$$[L_n, X(\lambda, z)] = z^{n+1}X'(\lambda, z) + \frac{\lambda^2}{2}(n+1)z^n X(\lambda, z)$$

$$X(z)Y(w) = e^{2\pi i\sqrt{s_X s_Y}}Y(w)X(z)$$

$$\phi(t, x) = \alpha + \mu t + Nrx$$

$$\mathcal{H}_r^{\circ} = \bigoplus_{N,\ell\in\mathbb{Z}} \mathcal{H}_r^{\circ}(N, \ell),$$

$$\mathcal{F}_{\frac{1}{\sqrt{2}}(\ell r^{-1}+Nr)} \otimes \mathcal{F}^*_{\frac{1}{\sqrt{2}}(\ell r^{-1}-Nr)}$$

$$\mathcal{Z}^\circ_r(\tau) = |\eta(\tau)|^{-2}\vartheta_r(\tau,\bar{\tau})$$

$$\vartheta_r(\tau,\bar{\tau}) := \sum_{\ell,N\in\mathbb{Z}} e^{\frac{1}{2}\pi i\tau(\ell r^{-1}+Nr)^2-\frac{1}{2}\pi i\bar{\tau}(\ell r^{-1}-Nr)^2} = \sum_{\ell,N\in\mathbb{Z}} e^{-\pi(\ell^2 r^{-2}+N^2r^2)\mathrm{Im}\tau+2\pi i\ell N\mathrm{Re}\tau}$$

$$\mathcal{Z}^\circ_r(\tau) = \mathcal{Z}^\circ_{r^{-1}}(\tau)$$

$$\vartheta_r\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right) = |\tau|\vartheta_r(\tau,\bar{\tau}),$$

$$Q(\tau) = \begin{pmatrix} r^2\mathrm{Im}\tau & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & r^{-2}\mathrm{Im}\tau \end{pmatrix}$$

$$Q(\tau)^{-1} = |\tau|^{-2}\begin{pmatrix} r^{-2}\mathrm{Im}\tau & i\mathrm{Re}\tau \\ i\mathrm{Re}\tau & r^2\mathrm{Im}\tau \end{pmatrix} = \begin{pmatrix} r^{-2}\mathrm{Im}\tau' & -i\mathrm{Re}\tau' \\ -i\mathrm{Re}\tau' & r^2\mathrm{Im}\tau' \end{pmatrix} = SQ(\tau')S$$

$$\tau' := -\frac{1}{\tau} \text{ and } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\bigoplus_{i=1}^n \mathcal{V}_i \otimes \mathcal{V}_i^*$$

$$\mathcal{V}(pq) \otimes \mathcal{V}(pq)^*$$

$$\mathcal{V}(s) := \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{m\sqrt{2s}}$$

$$\mathcal{V}(2) \otimes \mathcal{V}(2)^*$$

$$\mathcal{W} \otimes \mathcal{W}^*$$

$$\mathcal{W} = \bigoplus_{n\in 2\mathbb{Z}+1} \mathcal{F}_{\frac{n}{\sqrt{2}}}$$

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t,t^{-1}] \oplus \mathbb{C}K$$

$$b(z) := \sum_n (b\otimes t^n)z^{-n-1}$$

$$\mathcal{V}_j \otimes \mathcal{V}^*_{-j}, j = 0,1,2,3, \text{ where } \mathcal{V}_j = \bigoplus_{n\in 4\mathbb{Z}+j} \mathcal{F}_{\frac{n}{2}}$$

$$\xi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \xi_n z^{-n-\frac{1}{2}}$$

$$[\xi(z),\xi(w)]_+ = \delta(z-w)$$

$$\xi_n\xi_m + \xi_m\xi_n = \delta_{m,-n}$$

$$\mathcal{D} := \Lambda \otimes \Lambda^*$$

$$[H,\xi_n] = -\xi_n, [H,\xi_n^*] = \xi_n^*$$

$$H = H_L + H_R$$

$$H_L = \sum_{n>0} n\xi_{-n}\xi_n$$

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} n : \xi_n \xi_{-n+m} :$$

$$I \in C^\infty[0, \varepsilon) I(\hbar) \sim \textstyle\sum_{n=0}^{\infty} a_n \hbar^n \; N \geq 0 I(\hbar) = \sum_{n=0}^{N-1} a_n \hbar^n + O(\hbar^N) \text{ as } \hbar \to \frac{f(x)}{\hbar}$$

Modelo Super de Yang – Mills para campos cuánticos – relativistas o curvos, tanto en supergravedad como en gravedad cuánticas respectivamente, con o sin intervención supergravitónica o gravitónica.

$$S = \int d\tau \left(\dot{x}^m p_m - \frac{1}{2} p^2 + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha \right)$$

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$$

$$\left(\gamma^{(m}\right)^{\alpha\delta} \gamma^{n)}_{\delta\beta} = 2\eta^{mn} \delta^\alpha_\beta$$

$$\delta\omega_\alpha = \Lambda_m (\gamma^m \lambda)_\alpha$$

$$J = -\omega_\alpha \lambda^\alpha, N_{mn} = \frac{1}{2} (\omega \gamma_{mn} \lambda)$$

$$[p_m, x^n] = -\delta^n_m, \{p_\alpha, \theta^\beta\} = \delta^\beta_\alpha, [\omega_\alpha, \lambda^\beta] = -\delta^\beta_\alpha$$

$$p_m(\tau_1) x^n(\tau_2) \sim -\delta^n_m \sigma_{12}$$
$$p_\alpha(\tau_1) \theta^\beta(\tau_2) \sim \delta^\beta_\alpha \sigma_{12}$$
$$\omega_\alpha(\tau_1) \lambda^\beta(\tau_2) \sim -\delta^\beta_\alpha \sigma_{12}$$

$$\frac{1}{2} \operatorname{sign}(\tau_i - \tau_j) = \sigma_{ij}$$

$$[A(\tau), B(\tau)\} \sim A(\tau + \epsilon) B(\tau) \mp B(\tau) A(\tau - \epsilon)$$

$$Q = \lambda^\alpha d_\alpha$$

$$d_\alpha = p_\alpha - \frac{1}{2} (\gamma^m \theta)_\alpha p_m$$

$$\Psi(x, \theta, \lambda) = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$$

$$Q\Psi^{(1)} = 0, \delta\Psi^{(1)} = Q\Lambda$$

$$(\gamma^{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0, \delta A_\alpha = D_\alpha \Lambda,$$

$$D_\alpha = \partial_\alpha + \frac{1}{2} (\gamma^m \theta)_\alpha \partial_m$$

$$\bar{\lambda} \gamma^m \bar{\lambda} = 0, \bar{\lambda} \gamma^m r = 0.$$

$$S = \int d\tau \left(\dot{x}^m p_m - \frac{1}{2} p^2 + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha + \bar{\omega}^\alpha \dot{\bar{\lambda}}_\alpha + s^\alpha \dot{r}_\alpha \right)$$

$$Q = \lambda^\alpha d_\alpha + r_\alpha \bar{\omega}^\alpha$$

$$\{Q, b\} = \frac{p^2}{2}$$

$$b = -\mathbf{\Delta}_m \mathbf{A}^m,$$

$$\mathbf{A}_m = \frac{(\bar{\lambda}\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(\bar{\lambda}\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np}$$
$$\mathbf{\Delta_m} = -p_m - \frac{(\lambda\gamma^{mn} r)}{4(\lambda\bar{\lambda})} \mathbf{A_n}$$

$$p_m \to -\partial_m, d_\alpha \to D_\alpha, \omega_\alpha \to -\partial_{\lambda^\alpha},$$

$$\{Q, b_0\} = \square,$$

$$\Psi = \left\{Q, \frac{b_0(\Psi)}{h}\right\},$$

$$S^{\text{coupled}} = \int d\tau \Big\{\dot{x}^m \tilde{p}_m - \frac{1}{2}\tilde{p}^2 + \dot{\theta}^\alpha \tilde{p}_\alpha + \dot{\lambda}^\alpha \omega_\alpha$$
$$- \frac{1}{2}\eta_I \dot{\eta}^I - g\eta_I\eta_J \left(\dot{\theta}^\alpha \mathbb{A}_\alpha^{IJ} + \Pi^m \mathbb{A}_m^{IJ} + \tilde{d}_\alpha \mathbb{W}^{IJ\alpha} + \frac{1}{2} N^{mn} \mathbb{F}_{mn}^{IJ}\right)\Big\}$$

$$\Pi^m = \dot{x}^m - \frac{1}{2}(\dot{\theta}\gamma^m\theta)$$

$$\eta_I\eta_J \mathbb{A}_\alpha^{IJ} = \eta_I\eta_J T_a^{IJ} \mathbb{A}_\alpha^a$$

$$T_a = \eta_I\eta_J T_a^{IJ}$$

$$[T_a, T_b] = i f_{ab}^c T_c$$

$$\mathbb{A}_\alpha \equiv T_a \mathbb{A}_\alpha^a$$

$$S^{\text{coupled}} = \int d\tau \Big\{\dot{x}^m \tilde{p}_m - \frac{1}{2}\tilde{p}^2 + \dot{\theta}^\alpha \tilde{p}_\alpha + \dot{\lambda}^\alpha \omega_\alpha$$
$$- \frac{i}{2}\eta_I \dot{\eta}^I - g\left(\dot{\theta}^\alpha \mathbb{A}_\alpha + \Pi^m \mathbb{A}_m + \tilde{d}_\alpha \mathbb{W}^\alpha + \frac{1}{2} N^{mn} \mathbb{F}_{mn}\right)\Big\}$$

$$Q^{\text{coupled}} = \lambda^\alpha \tilde{d}_\alpha$$

$$p_m = \frac{\partial L}{\partial \dot{x}^m} = \tilde{p}_m - g\mathbb{A}_m$$
$$p_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = \tilde{p}_\alpha - g\left(\mathbb{A}_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \mathbb{A}_m\right)$$

$$H^{\text{coupled}}(x^m, p_m, \theta^\alpha, p_\alpha, \lambda^\alpha, \omega_\alpha) = \left(\dot{x}^m p_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha \omega_\alpha - L\right)\Big|_{\dot{q}=\dot{q}(p)}$$
$$= \frac{1}{2}p^2 + g\left(p^m \mathbb{A}_m + d_\alpha \mathbb{W}^\alpha + \frac{1}{2} N^{mn} \mathbb{F}_{mn}\right) + g^2\left(\frac{1}{2}\mathbb{A}^2 + \mathbb{A}_\alpha \mathbb{W}^\alpha\right)$$

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha p_m$$

$$H^{\text{coupled}} = H_0 + \mathbb{U}$$

$$\mathbb{U} = g\left(p^m \mathbb{A}_m + d_\alpha \mathbb{W}^\alpha + \frac{1}{2} N^{mn} \mathbb{F}_{mn}\right) + g^2\left(\frac{1}{2}\mathbb{A}^2 + \mathbb{A}_\alpha \mathbb{W}^\alpha\right)$$

$$S^{\text{coupled}} = S - \int d\tau \mathbb{U}$$

$$S = \int d\tau \left(\dot{x}^m p_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha \omega_\alpha - \frac{1}{2}p^2\right)$$

$$Q^{\text{coupled}} = \lambda^\alpha \tilde{d}_\alpha = \lambda^\alpha d_\alpha + g\lambda^\alpha \mathbb{A}_\alpha = Q + \mathbb{V}$$

$$\mathbb{V} = g\lambda^\alpha \mathbb{A}_\alpha$$

$$\{Q^{\text{coupled}}, Q^{\text{coupled}}\} = 0$$

$$\{Q, \mathbb{V}\} = -\frac{1}{2}\{\mathbb{V}, \mathbb{V}\}.$$

$$[H^{\text{coupled}}, Q^{\text{coupled}}] = 0$$

$$[Q, \mathbb{U}] = [H, \mathbb{V}] + [\mathbb{U}, \mathbb{V}].$$

$$(\gamma^{mnpqr})^{\alpha\beta}(D_\alpha \mathbb{A}_\beta + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\}) = 0,$$

$$\begin{aligned} [\nabla_\alpha, \mathbb{A}_m] &= [\partial_m, \mathbb{A}_\alpha] + (\gamma_m \mathbb{W})_\alpha \\ \{\nabla_\alpha, \mathbb{W}^\beta\} &= \frac{1}{4}(\gamma^{mn})_\alpha^\beta \mathbb{F}_{mn} \\ [\nabla_\alpha, \mathbb{F}_{mn}] &= \nabla_{[m}(\gamma_{n]}\mathbb{W})_\alpha \end{aligned}$$

$$\nabla_\alpha = D_\alpha + \mathbb{A}_\alpha, \text{ with } D_\alpha = \partial_\alpha + \tfrac{1}{2}(\theta\gamma^m)_\alpha \partial_m$$

$$\nabla_m = \partial_m + \mathbb{A}_m$$

$$\mathbb{A}_\alpha = \sum_P A_{\alpha P} T^P e^{k_P x} = A_{\alpha i} T^i e^{k_i x} + A_{\alpha ij} T^{ij} e^{k_{ij} x_{ij}} + \cdots,$$

$$A_\alpha^P = \frac{1}{s_P} \sum_{R+Q=P} A_\alpha^{[RQ]},$$

$$A_\alpha^{[PQ]} = -\frac{1}{2}[A_\alpha^P (k^P \cdot A^Q) + A_m^P (\gamma^m W^P)_\alpha - (P \leftrightarrow Q)].$$

$$\mathbb{V} = \sum_P V_P T^P = V_i T^i + V_{ij} T^{ij} + \cdots$$

$$\mathbb{U} = \sum_P U_P T^P e^{k_P x} + \sum_{PQ} D_{PQ} (T^P \odot T^Q) e^{k_{PQ} x},$$

$$\begin{aligned} \{Q, V_i\} &= 0 \\ \{Q, V_{ij}\} &= -V_i V_j \end{aligned}$$

$$\begin{aligned} [Q, U_i] &= [H, V_i], \\ [Q, U_{ij}] &= [H, V_{ij}] + U_i V_j - U_j V_i, \\ [Q, D_{ij}] &= \frac{1}{2}([U_i, V_j] + [U_j, V_i]). \end{aligned}$$

$$\begin{aligned} \{Q, V_P\} &= -\sum_{R+Q=P} V_R V_Q, \\ \{Q, U_P\} &= [H, V_P] + \sum_{R+Q=P} U_R V_Q - U_Q V_R, \end{aligned}$$

$$\{Q, D_P\} = \frac{1}{2} \sum_{R+Q=P} [U_R, V_Q] + [U_Q, V_R].$$

$$U_1(z_1) U_2(z_2) \sim \frac{1}{z_{12}^2} D_{12} + \frac{1}{z_{12}} U_{12}.$$

$$
\begin{aligned}
A_{\alpha}^{12} &= -\frac{1}{2s_{12}}\left[A_{\alpha}^{1}(k^{1}\cdot A^{2}) + A_{m}^{1}(\gamma^{m}W^{2})_{\alpha} - (12)\right],\\
A_{m}^{12} &= -\frac{1}{2s_{12}}\left[A_{m}^{1}(k^{1}\cdot A^{2}) + A_{n}^{1}F_{mn}^{2} - (W^{1}\gamma_{m}W^{2}) - (12)\right],\\
W_{12}^{\alpha} &= -\frac{1}{2s_{12}}\left[W_{1}^{\alpha}(k_{1}\cdot A_{2}) + W_{1}^{m\alpha}A_{m}^{2} + \frac{1}{2}(\gamma^{rs}W^{1})^{\alpha}F_{rs}^{2} - (12)\right],\\
F_{12}^{mn} &= -\frac{1}{2s_{12}}\left[F_{1}^{mn}(k_{1}\cdot A_{2}) + F_{1}^{p|mn}A_{p}^{2} + 2F_{1}^{mp}F_{2p}^{n} + 4\gamma_{\alpha\beta}^{[m}W_{1}^{n]\alpha}W_{2}^{\beta} - (12)\right].
\end{aligned}
$$

$$
\begin{aligned}
V_{1} &= \lambda^{\alpha}A_{1\alpha}\\
V_{12} &= \lambda^{\alpha}A_{12\alpha}
\end{aligned}
$$

$$
\begin{aligned}
U_{1} &= p^{m}A_{1m} + d_{\alpha}W_{1}^{\alpha} + \frac{1}{2}N^{mn}F_{1mn}\\
U_{12} &= p^{m}A_{12m} + d_{\alpha}W_{12}^{\alpha} + \frac{1}{2}N^{mn}F_{12mn}\\
D_{12} &= \frac{1}{2}(A_{1}\cdot A_{2} + W_{1}A_{2} + W_{2}A_{1})
\end{aligned}
$$

$$
V_{P} = \frac{1}{s_{P}}\sum_{RQ=P}\left[U_{R},V_{Q}\right]
$$

$$
\begin{aligned}
\{Q,V_{P}\} =& \frac{1}{s_{P}}\sum_{RQ=P}\left\{\left[[H,V_{R}],V_{Q}\right] + \sum_{R_{1}R_{2}=R}\left[U_{R_{1}}V_{R_{2}},V_{Q}\right] - \left[V_{R_{1}}U_{R_{2}},V_{Q}\right]\right.\\
&\left. - \sum_{Q_{1}Q_{2}=Q}\left[U_{R},V_{Q_{1}}V_{Q_{2}}\right]\right\}\\
=& \frac{1}{s_{P}}\sum_{RQ=P}\left\{-k_{R}\cdot k_{Q}V_{R}V_{Q} - \sum_{R_{1}R_{2}=R}V_{R_{1}}\left[U_{R_{2}},V_{Q}\right] - \sum_{Q_{1}Q_{2}=Q}\left[U_{R},V_{Q_{1}}\right]V_{Q_{2}}\right\}\\
=& -\frac{1}{s_{P}}\left\{\sum_{RQ=P}k_{R}\cdot k_{Q}V_{R}V_{Q} + \sum_{RQ_{1}Q_{2}=P}V_{R}\left[U_{Q_{1}},V_{Q_{2}}\right]\right.\\
&\left. + \sum_{R_{1}R_{2}Q=P}\left[U_{R_{1}},V_{R_{2}}\right]V_{Q}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\{Q,V_{P}\} &= -\frac{1}{s_{P}}\sum_{RQ=P}\{k_{R}\cdot k_{Q} + s_{R} + s_{Q}\}V_{R}V_{Q}\\
&= -\sum_{RQ=P}V_{R}V_{Q}(2.59)
\end{aligned}
$$

$$
k_{P}^{2} = \left(k_{R}+k_{Q}\right)^{2} \Rightarrow s_{P} = k_{R}\cdot k_{Q} + s_{R} + s_{Q}
$$

$$
\tilde{U}_{i} = [b,V_{i}].
$$

$$
\tilde{U}_{i} = -\hat{\mathbf{A}}_{m}(V_{i})\Delta^{m} + \hat{\Delta}^{m}(V_{i})\mathbf{A}_{m}.
$$

$$
\begin{aligned}
\tilde{V}_{P} &= \frac{1}{s_{P}}\sum_{R+Q=P}\left[\tilde{U}_{R},\tilde{V}_{Q}\right],\\
\tilde{U}_{P} &= \left[b,\tilde{V}_{P}\right].
\end{aligned}
$$

$$
\left[U_{i}',V_{j}'\right] = b_{0}\left(V_{i}'V_{j}'\right)
$$

$$
\begin{aligned}
V_{P}' &= \frac{1}{s_{P}}\sum_{RQ=P}b_{0}\left(V_{R}'V_{Q}'\right),\\
U_{P}' &= [b,V_{P}'].
\end{aligned}
$$

$$b_0(V'_R V'_Q) = [U'_R, V'_Q]$$

$$b_0(V'_P) = 0$$

$$A_N(1,2,\dots,N) = g^{N-2} \sum_{\sigma(1,\dots,N)/\mathbb{Z}_N} \mathrm{Tr}(T^{\sigma_1} \dots T^{\sigma_N}) \mathcal{A}_N(\sigma_1, \sigma_2, \dots, \sigma_N),$$

$$\mathcal{A} = \left\langle V_1(\infty) \exp\left\{\int d\tau \mathbb{U}\right\} V_{N-1}(-\infty)\right\rangle$$
$$= \sum \frac{1}{(N-2)!} \left\langle V_1(\infty) \left(\int d\tau \mathbb{U}\right)^{N-3} V_N(0) V_{N-1}(-\infty)\right\rangle,$$

$$\mathcal{A}_N = \sum_{|R_1|+\cdots+|R_k|=N-3} \int d\tau_1 \dots d\tau_k$$
$$\times \langle V_1(\infty)(U_{R_1} + D_{R_1})(\tau_1) \dots (U_{R_k} + D_{R_k})(\tau_k) V_N(0) V_{N-1}(-\infty)\rangle,$$

$$\sigma_{ij} = \frac{1}{2} \mathrm{sign}(\tau_i - \tau_j)$$

$$p_m \to -\partial_m, d_\alpha \to D_\alpha, \omega_\alpha \to -\partial_{\lambda^\alpha}.$$

$$\mathcal{A}_N \propto \int \mathcal{D}x \exp\left\{-\int d\tau p_m \dot{x}^m\right\} \prod_P \exp\{k_P \cdot x(\tau_P)\}.$$

$$k_j \cdot x(\tau_j) = \int d\tau \delta(\tau - \tau_j) k_j \cdot x(\tau)$$

$$\int \mathcal{D}x \exp\left\{\int d\tau \left(\dot{p}_m + \sum_j \delta(\tau - \tau_j) k_{jm}\right) x^m\right\} = \delta\left(\dot{p}_m + \sum_j \delta(\tau - \tau_j) k_{jm}\right).$$

$$\dot{p}_m(\tau) = -\sum_j \delta(\tau - \tau_j) k_{jm}$$

$$p^m(\tau) = -\sum_P \sigma_{\tau P} k_P^m$$

$$p^m(\tau) = -\frac{1}{2}\left(\sum_{\tau_P<\tau} k_P^m - \sum_{\tau_P>\tau} k_P^m\right),$$

$$\exp\left\{\frac{1}{2}\int d\tau p^2\right\} = \prod_i \exp\left\{\frac{1}{2}(\tau_{i+1} - \tau_i)\left(\sum_{j\le i} k_{P_j}\right)^2\right\}$$

$$\exp\left\{\frac{1}{2}\int d\tau p^2\right\} = \prod_i \exp\left\{-\frac{1}{2}\tau_i k_{P_i} \cdot \left(\sum_{j<i} k_{P_j} - \sum_{j>i} k_{P_j}\right)\right\}.$$

$$\langle (\theta\gamma^m\lambda)(\theta\gamma^n\lambda)(\theta\gamma^p\lambda)(\theta\gamma_{mnp}\theta)\rangle = 1.$$

$$\mathcal{A}_N = \sum_{|R_1|+\cdots+|R_k|=N-3} \int d\tau_1 \dots d\tau_k \langle V_1 \mathbb{U}_{R_1} \mathbb{U}_{R_2} \dots \mathbb{U}_{R_k} V_N V_{N-1}\rangle.$$

$$Q\mathbb{U}_P = \partial V_P + \sum_{R+Q=P} (U_{[R} V_{Q]} + [U_{\{R}, V_{Q\}}]).$$

$$Q\mathcal{A}_N=-\sum_{k=1}^{N-3}\sum_{|R_1|+\cdots+|R_k|=N-3}\int d\tau_1\ldots d\tau_k$$

$$\times\sum_{i=1}^{k}\left\langle V_1\mathbb{U}_{P_1}\ldots\mathbb{U}_{P_{i-1}}\left(\partial V_{P_i}+\sum_{R_i+Q_i=P_i}\left(U_{[R_i}V_{Q_i]}+[U_{\{R_i},V_{Q_i}]\right)\right)\mathbb{U}_{P_{i+1}}\ldots\mathbb{U}_{P_k}V_NV_{N-1}\right\rangle$$

$$-\sum_k\sum_i\int d\tau_1\cdots d\tau_k\langle V_1\mathbb{U}_{P_1}\cdots\mathbb{U}_{P_{i-1}}\partial V_{P_i}\mathbb{U}_{P_{i+1}}\cdots\mathbb{U}_{P_k}V_NV_{N-1}\rangle$$

$$=-\sum_k\sum_i\int d\tau_1\cdots d\hat{\tau}_i\cdots d\tau_k\langle V_1\mathbb{U}_{P_1}\cdots\mathbb{U}_{P_{i-1}}(\tau_{i-1})V_{P_i}(\tau_{i-1}-\epsilon)\mathbb{U}_{P_{i+1}}\cdots\mathbb{U}_{P_k}V_NV_{N-1}\rangle$$

$$+\int d\tau_1\cdots d\hat{\tau}_i\cdots d\tau_k\langle V_1\mathbb{U}_{P_1}\cdots\mathbb{U}_{P_{i-1}}V_{P_i}(\tau_{i+1}+\epsilon)\mathbb{U}_{P_{i+1}}(\tau_{i+1})\cdots\mathbb{U}_{P_k}V_NV_{N-1}\rangle$$

$$-\sum_k\sum_i\int d\tau_1\cdots d\tau_{i-1}d\tau d\tau_{i+2}\cdots d\tau_k$$

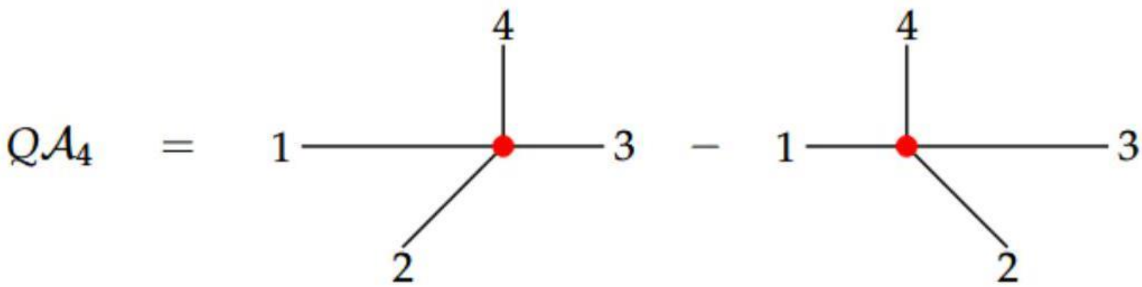
$$\times\langle V_1\mathbb{U}_{P_1}\cdots\left(V_{P_i}(\tau+\epsilon)U_{P_{i+1}}(\tau)-U_{P_i}(\tau)V_{P_{i+1}}(\tau-\epsilon)\right)\cdots\mathbb{U}_{P_k}V_NV_{N-1}\rangle$$

$$V_{P_i}(\tau+\epsilon)U_{P_{i+1}}(\tau)-U_{P_i}(\tau)V_{P_{i+1}}(\tau-\epsilon)$$

$$\begin{aligned}&V_{P_i}(\tau+\epsilon)U_{P_{i+1}}(\tau)-U_{P_i}(\tau)V_{P_{i+1}}(\tau-\epsilon)\\&=V_{P_i}U_{P_{i+1}}(\tau)-U_{P_i}V_{P_{i+1}}(\tau)+\frac{1}{2}\left([U_{P_i},V_{P_{i+1}}]+[U_{P_{i+1}},V_{P_i}]\right)(\tau)\\&=V_{[P_i}U_{P_{i+1}]}+[U_{\{P_i},V_{P_{i+1}\}}].\end{aligned}$$

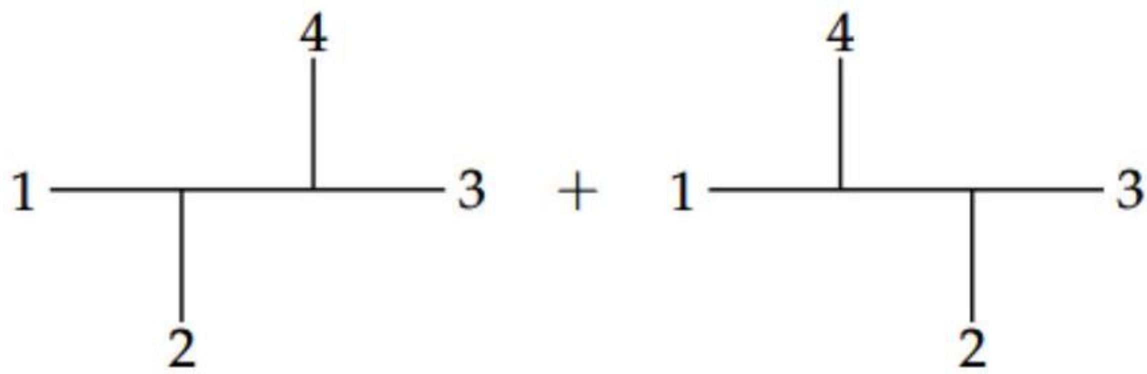
$$\sum_{k=1}^{N-3}\sum_{|R_1|+\cdots+|R_k|=N-3}\int d\tau_1\ldots d\tau_k$$

$$\times\sum_i\sum_{R_i+Q_i=P_i}\langle V_1\mathbb{U}_{P_1}\cdots\mathbb{U}_{P_{i-1}}\left(U_{[R_i}V_{Q_i]}+[U_{\{R_i},V_{Q_i\}}]\right)(\tau_i)\mathbb{U}_{P_{i+1}}\cdots\mathbb{U}_{P_k}V_NV_{N-1}\rangle,$$



$$\mathcal{A}_4=\int_{-\infty}^{\infty}d\tau_2\langle V_1(\infty)U_2(\tau_2)V_4(0)V_3(-\infty)\rangle$$

$$\int\mathcal{D}x\mathcal{D}p\exp\left\{-\int d\tau\left(p_m\dot{x}^m-\frac{1}{2}p^2\right)\right\}\prod_{j=1}^{4}\exp\{k_j\cdot x(\tau_j)\}=\exp\{(\sigma_{2i}s_{2i})\tau_2\}$$



$$\int_0^\infty d\tau_2 \exp\{(\sigma_{2i}s_{2i})\tau_2\} = \int_0^\infty d\tau_2 \exp\{-s_{21}\tau_2\} = \frac{1}{s_{21}}$$
$$\int_0^\infty d\tau_2 \exp\{(\sigma_{2i}s_{2i})\tau_2\} = \int_{-\infty}^0 d\tau_2 \exp\{+s_{23}\tau_2\} = \frac{1}{s_{23}}$$

$$\mathcal{A}_4 = \int d\tau_2 \exp\{(\sigma_{2i}s_{2i})\tau_2\}\langle\sigma_{21}\hat{U}_2(V_1)V_4V_3 + \sigma_{23}V_1\hat{U}_2(V_4)V_3 + \sigma_{24}V_1V_4\hat{U}_2(V_3)\rangle,$$

$$\mathcal{A}_4 = \frac{1}{2s_{21}}\langle-\hat{U}_2(V_1)V_4V_3 + V_1\hat{U}_2(V_4)V_3 + V_1V_4\hat{U}_2(V_3)\rangle + \frac{1}{2s_{23}}\langle-\hat{U}_2(V_1)V_4V_3 - V_1\hat{U}_2(V_4)V_3 + V_1V_4\hat{U}_2(V_3)\rangle.$$

$$\mathcal{A}_4 = -\frac{1}{s_{21}}\langle\hat{U}_2(V_1)V_4V_3\rangle + \frac{1}{s_{23}}\langle V_1V_4\hat{U}_2(V_3)\rangle.$$

$$\mathcal{A}_4 = \frac{1}{s}n_s + \frac{1}{u}n_u,$$

$$\hat{U}_2(V_1)V_4V_3 = -n_s, V_1\hat{U}_2(V_4)V_3 = n_t, V_1V_4\hat{U}_2(V_3) = n_u.$$

$$Qn_s = -s\mathcal{O}, Qn_u = u\mathcal{O}.$$

$$\hat{U}_i(V_j) = -\left((A_i \cdot k_j)V_j + (W_i\gamma^m\lambda)A_{jm} + Q(W_iA_j)\right).$$

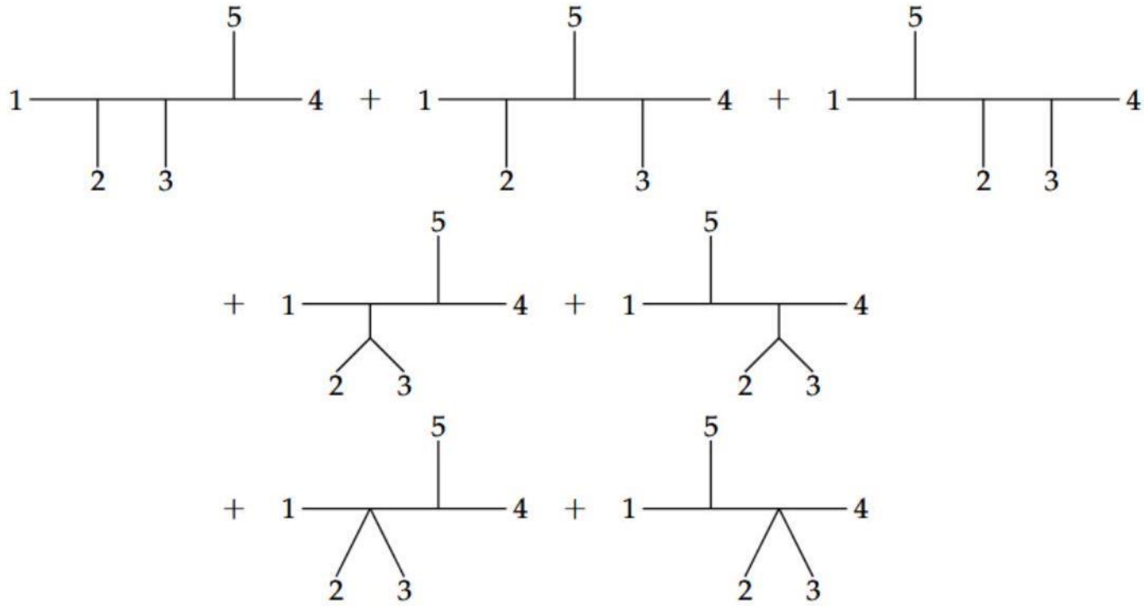
$$Q(A_1 \cdot A_2) = k_1 \cdot A_2V_1 + (\lambda\gamma^m W_1)A_2^m + (1 \leftrightarrow 2).$$

$$\frac{1}{s_{12}}\hat{U}_2(V_1) = \lambda^\alpha A_{\alpha 12} + \frac{1}{s_{12}}QD_{12} = V_{21} + \frac{1}{s_{12}}QD_{12}.$$

$$\mathcal{A}_4 = -\langle V_{12}V_3V_4\rangle - \langle V_1V_{23}V_4\rangle,$$

$$\mathcal{A}_5 = \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \langle V_1(\infty)U_2(\tau_2)U_3(\tau_3)V_5(0)V_4(-\infty)\rangle + \int_{-\infty}^{\infty} d\tau \langle V_1(\infty)(U_{23}(\tau) + D_{23}(\tau))V_5(0)V_4(-\infty)\rangle$$

$$U_P \mapsto \tilde{U}_P,$$



$$\mathcal{A}_5 = \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \langle V_1(\infty)\tilde{U}_2(\tau_2)\tilde{U}_3(\tau_3)V_5(0)V_4(-\infty)\rangle$$
$$+ \int_{-\infty}^{\infty} d\tau \langle V_1(\infty)\tilde{U}_{23}(\tau)V_5(0)V_4(-\infty)\rangle$$

$$\langle p_m(\tau_1)\mathcal{O}(x(\tau_2))\rangle = \sigma_{12}\langle \partial_m \mathcal{O}(x(\tau_2))\rangle.$$

$$\int_0^{\infty} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \langle V_1(\infty)\tilde{U}_2(\tau_2)\tilde{U}_3(\tau_3)V_5(0)V_4(-\infty)\rangle$$
$$= -\frac{1}{2}\int_0^{\infty} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \left\langle \left(\hat{U}_2(V_1)\tilde{U}_3 V_5 V_4 - V_1 \hat{U}_2(\tilde{U}_3 V_5 V_4)\right)\right\rangle$$
$$= -\int_0^{\infty} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \left\langle \hat{U}_2(V_1)\tilde{U}_3 V_5 V_4 \right\rangle$$

$$\frac{1}{s_{12}s_{45}} \left\langle \hat{U}_3\left(\hat{U}_2(V_1)\right) V_5 V_4 \right\rangle.$$

$$\mathcal{A}_5 = \frac{1}{s_{12}s_{45}} \left\langle \hat{U}_3\left(\hat{U}_2(V_1)\right) V_5 V_4 \right\rangle + \frac{1}{s_{12}s_{34}} \left\langle \hat{U}_2(V_1) V_5 \hat{U}_3(V_4) \right\rangle + \frac{1}{s_{15}s_{34}} \left\langle V_1 V_5 \hat{U}_3\left(\hat{U}_2(V_4)\right) \right\rangle$$
$$+ \frac{1}{s_{45}} \left\langle \hat{U}_{23}(V_1) V_5 V_4 \right\rangle + \frac{1}{s_{15}} \left\langle V_1 V_5 \hat{U}_{23}\left(\hat{U}_4\right) \right\rangle.$$

$$\{Q, T_{12}\} = s_{12} V_1 V_2$$
$$\{Q, T_{123}\} = s_{123} T_{12} V_3 + s_{12}(T_{13} V_2 - T_{12} V_3 + V_1 T_{23})$$

$$T_{(12)} = T_{(12)3} = T_{123} + \text{cyclic} = 0$$

$$\hat{U}_i(V_j),\ \hat{U}_i\left(\hat{U}_j(V_k)\right) \text{ and } \hat{U}_{ij}(V_k)$$

$$\left\{Q, \hat{U}_2(V_1)\right\} = s_{12} V_2 V_1$$
$$\left\{Q, \hat{U}_3\left(\hat{U}_2(V_1)\right)\right\} = s_{123}\hat{U}_2(V_1)V_3 + s_{12}\left(\hat{U}_3(V_2)V_1 + V_2\hat{U}_3(V_1) - V_3\hat{U}_2(V_1)\right).$$

$$\left\{Q, \hat{U}_2(V_1) - T_{21}\right\} = 0$$

$$\hat{U}_2(V_1) = T_{21} + [Q, E_{21}].$$

$$\left\{Q, \hat{U}_3\left(\hat{U}_2(V_1)\right)\right\} = \{Q, T_{123} - s_{123} V_3 E_{21} + s_{21}(E_{32} V_1 E_{31} V_2 + E_{21} V_3)\}.$$

$$\hat{\hat{U}}_3\left(\hat{\hat{U}}_2(V_1)\right) = T_{123} - s_{123}V_3E_{21} + s_{21}(E_{32}V_1E_{31}V_2 + E_{21}V_3) + [Q, \cdots].$$

$$s_{23}\hat{\hat{U}}_{23}(V_1) = T_{321} - s_{123}V_1E_{23} + s_{23}(E_{12}V_3 - E_{13}V_2 + E_{23}V_1) + [Q, \cdots].$$

$$\mathcal{A}_5 = \frac{\langle T_{123}V_5V_4\rangle}{s_{12}s_{45}} + \frac{\langle T_{12}T_{34}V_5\rangle}{s_{12}s_{34}} + \frac{\langle V_1T_{432}V_5\rangle}{s_{15}s_{34}} + \frac{\langle T_{321}V_4V_5\rangle}{s_{23}s_{45}} + \frac{\langle V_1T_{234}V_5\rangle}{s_{23}s_{15}}.$$

$$\mathcal{A}'_N = \sum_{j=1}^{N-2} \langle M_{1\ldots j}M_{j+1\ldots N-1}V_N\rangle,$$

$$QM_{1\ldots p} = \sum_{j=1}^{p-1} M_{1\ldots j}M_{j+1\ldots p},$$

$$\mathcal{A}_N = \sum_{|R_1|+\cdots+|R_k|=N-3} \int_{\tau_j>\tau_{j+1}} d\tau_1 \ldots d\tau_k$$
$$\times \langle V_1(\infty)V_N(0)V_{N-1}(-\infty)\left(U_{R_1} + D_{R_1}\right)(\tau_1) \ldots \left(U_{R_k} + D_{R_k}\right)(\tau_k)\rangle.$$

$$\mathcal{A}_N = \sum_{k=1}^{N-3} \sum_{\sum_{i=1}^{k}|P_i|=N-3} \int_{\tau_j>\tau_{j+1}} d\tau_1 \ldots d\tau_k \langle V_1\tilde{U}_{P_1} \ldots \tilde{U}_{P_k}V_NV_{N-1}\rangle$$

$$\mathcal{A}_N = \sum_{k=1}^{N-3} \sum_{\sum_{i=1}^{k}|P_i|=N-3} \sum_{j} (-1)^j \int_{\tau_j>0>\tau_{j+1}} d\tau_1 \ldots d\tau_k$$
$$\times \left\langle \left(\hat{\hat{U}}_{P_1} \ldots \hat{\hat{U}}_{P_j}V_1\right) V_N \left(\hat{\hat{U}}_{P_{j+1}} \ldots \hat{\hat{U}}_{P_k}V_{N-1}\right)\right\rangle.$$

$$\sum_{\sum P_i=j+1,\ldots,N-3} \int_{0>\tau_{j+1}} d\tau_{j+1} \ldots d\tau_k \left(\hat{\hat{U}}_{P_{j+1}} \ldots \hat{\hat{U}}_{P_k}V_{N-1}\right) = \tilde{V}_{j+1\ldots N-1}$$

$$\mathcal{A}_N = -\sum_{j=1}^{N-2} \langle \tilde{V}_{1\ldots j}\tilde{V}_{j+1\ldots N-1}V_N\rangle$$

$$\mathcal{A}_N = -\sum_{j=1}^{N-2} \langle V_{1\ldots j}V_{j+1\ldots N-1}V_N\rangle,$$

$$QV_P = -\sum_{RQ=P} V_RV_Q$$

$$\mathcal{A}'_N = -\sum_{j=1}^{N-2} \langle V'_{1\ldots j}V'_{j+1\ldots N-1}V_N\rangle$$

$$Q(V'_{12} - V_{12}) = 0 \Rightarrow V'_{12} = V_{12} + QE_{12}$$

$$QV'_{123} = QV_{123} + Q(-E_{12}V_3 + V_1E_{23})$$
$$\Rightarrow V'_{123} = V_{123} - E_{12}V_3 + V_1E_{23} + Q(E_{123}),$$

$$V'_P = V_P + Q(E_P) + \gimel.$$

$$\begin{aligned} QV'_{Pm} &= -V'_PV'_m + \cdots \\ &= -V_PV'_m - QE_PV'_m + \cdots \\ &= -V_PV'_m - Q(E_PV'_m) + \cdots, \\ \Rightarrow V'_{Pm} &= V_{Pm} - E_PV'_m + \cdots, \end{aligned}$$

$$\begin{aligned} QV'_{Pmn} &= -V_PV'_{mn} - Q(E_PV'_{mn}) + \cdots \\ \Rightarrow V'_{Pmn} &= V_{Pmn} - E_PV'_{mn} + \cdots \end{aligned}$$

$$QV'_{PQ} = -V_P V'_Q - Q\left(E_P V'_Q\right) + \cdots$$
$$\Rightarrow V'_{PQ} = V_{PQ} - E_P V'_Q + \cdots$$

$$\begin{aligned}
\mathcal{A}'_N &= -\sum_{j=1}^{N-2} \left\langle V'_{1\ldots j} V'_{j+1\ldots N-1} V_N \right\rangle \\
&= -\left\langle V'_P V'_Q V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle V'_{PQ_1} V'_{Q_2} V_N \right\rangle + \cdots \\
&= -\left\langle V_P V'_Q V_N \right\rangle - \left\langle Q E_P V'_Q V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle V'_{PQ_1} V'_{Q_2} V_N \right\rangle + \cdots \\
&= -\left\langle V_P V'_Q V_N \right\rangle + \left\langle E_P Q V'_Q V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle V'_{PQ_1} V'_{Q_2} V_N \right\rangle + \cdots \\
&= -\left\langle V_P V'_Q V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle E_P V'_{Q_1} V'_{Q_2} V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle V'_{PQ_1} V'_{Q_2} V_N \right\rangle + \cdots \\
&= -\left\langle V_P V'_Q V_N \right\rangle - \sum_{Q_1 Q_2} \left\langle V_{PQ_1} V'_{Q_2} V_N \right\rangle + \cdots
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}'_N &= -\sum_{j=1}^{N-2} \left\langle V'_{1\ldots j} V'_{j+1\ldots N-1} V_N \right\rangle \\
&= -\sum_{j=1}^{N-2} \left\langle V_{1\ldots j} V_{j+1\ldots N-1} V_N \right\rangle = \mathcal{A}_N
\end{aligned}$$

$$\mathcal{A}_N = \int \mathcal{D}\Phi \mathcal{N} e^{-S} \mathcal{O}$$

$$\mathcal{A}_N = \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \mathcal{N} f(\lambda, \bar{\lambda}, r, \theta)$$

$$\begin{aligned}
[d\lambda]\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3} &= (\epsilon T^{-1})^{\alpha_1\alpha_2\alpha_2}_{\beta_1\ldots\beta_{11}} d\lambda^{\beta_1} \ldots d\lambda^{\beta_{11}} \\
[d\bar{\lambda}]\bar{\lambda}_{\alpha_1}\bar{\lambda}_{\alpha_2}\bar{\lambda}_{\alpha_3} &= (\epsilon T)^{\beta_1\ldots\beta_{11}}_{\alpha_1\alpha_2\alpha_2} d\bar{\lambda}_{\beta_1} \ldots d\bar{\lambda}_{\beta_{11}} \\
[dr] &= (\epsilon T^{-1})^{\alpha_1\alpha_2\alpha_2}_{\beta_1\ldots\beta_{11}} \bar{\lambda}_{\alpha_1}\bar{\lambda}_{\alpha_2}\bar{\lambda}_{\alpha_3} \left(\frac{\partial}{\partial r_{\beta_1}}\right) \ldots \left(\frac{\partial}{\partial r_{\beta_{11}}}\right).
\end{aligned}$$

$$\mathcal{N} = \exp\left\{-\left\{Q, \bar{\lambda}_\alpha \theta^\alpha\right\}\right\} = \exp\left\{-\bar{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha\right\}$$

$$\left[b, \left\{Q, \bar{\lambda}_\alpha \theta^\alpha\right\}\right] = -\left[Q, \left\{b, \bar{\lambda}_\alpha \theta^\alpha\right\}\right].$$

$$\left[\mathbf{\Delta}_m, \bar{\lambda}_\alpha \theta^\alpha\right] = \frac{(\bar{\lambda}\gamma_m r)}{2(\bar{\lambda}\lambda)} = 0, \left\{\mathbf{A}^m, \bar{\lambda}_\alpha \theta^\alpha\right\} = \frac{(\bar{\lambda}\gamma^m \bar{\lambda})}{2(\bar{\lambda}\lambda)} = 0$$

$$\langle \mathcal{O}_1 [U'_P, \mathcal{O}_2] \rangle = \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \mathcal{N} (\mathcal{O}_1 [U'_P, \mathcal{O}_2])$$

$$U'_P = -\hat{\mathbf{A}}_m (V'_P) \Delta^m + \hat{\Delta}^m (V'_P) \mathbf{A}_m$$

$$\begin{aligned}
\langle \mathcal{O}_1 [U'_P, \mathcal{O}_2] \rangle &= \left\langle \mathcal{O}_1 \left(-\hat{\mathbf{A}}_m (V'_P) \hat{\Delta}^m (\mathcal{O}_2) + \hat{\Delta}^m (V'_P) \mathbf{A}_m (\mathcal{O}_2)\right)\right\rangle \\
&= \left\langle \left(\hat{\mathbf{A}}_m (V'_P) \hat{\Delta}^m (\mathcal{O}_1) - \hat{\Delta}^m (V'_P) \mathbf{A}_m (\mathcal{O}_1)\right) \mathcal{O}_2 \right\rangle \\
&= -\langle [U'_P, \mathcal{O}_1] \mathcal{O}_2 \rangle
\end{aligned}$$

$$\int [d\lambda] \lambda^\alpha (\gamma_{mn})_\alpha^{\ \beta} \partial_{\lambda^\beta} [f(\lambda)] = 0$$

$$\langle b_0 (V'_A V'_B) V'_C V'_D \rangle.$$

$$b_0 (V'_A V'_B V'_C) = b_0 (V'_A V'_B) V'_C + b_0 (V'_B V'_C) V'_A + b_0 (V'_C V'_A) V'_B.$$

$$\langle b_0(V_A'V_B')V_C'V_D'\rangle + \langle b_0(V_B'V_C')V_A'V_D'\rangle + \langle b_0(V_C'V_A')V_B'V_D'\rangle = 0.$$

$$\hat{U}_i(V_j) = -\left(\left(A_i \cdot k_j\right)V_j + (W_i\gamma^m\lambda)A_{jm} + Q\left(W_iA_j\right)\right).$$

$$\hat{U}_3\hat{U}_2V_1,$$
$$\hat{U}_{[23]}V_1,$$
$$\hat{U}_4\hat{U}_3\hat{U}_2V_1,$$

$$L_{21}(z_1) = \lim_{z_2 \to z_1} (z_2 - z_1)U_2(z_2)V_1(z_1).$$

$$L_{2131\cdots n1} = \lim_{z_n \to z_1} (z_n - z_1)U_n(z_n)L_{2131\cdots}(z_1).$$

$$QL_{21} = s_{12}V_1V_2$$
$$QL_{2131} = s_{123}L_{21}V_3 + s_{12}[L_{31}V_2 + V_1L_{32} - L_{21}V_3]$$

$$\mathcal{A}_4 = \frac{n_s}{s} + \frac{n_t}{t},$$

$$L_{12} + L_{21} = -QD_{12}$$

$$D_{12} = (A_1 \cdot A_2 + W_1A_2 + W_2A_1)$$

$$T_{12} = L_{[12]} = L_{21} - L_{\{21\}} = L_{21} + \frac{1}{2}QD_{12}.$$

$$T_{\{12\}} = 0.$$

$$QL_{2131} = s_{123}T_{12}V_3 + s_{12}[T_{13}V_2 + V_1T_{23} - T_{12}V_3]$$
$$-\frac{1}{2}Q(s_{123}D_{12}V_3 + s_{12}[D_{13}V_2 - V_1D_{23} - D_{12}V_3]).$$

$$Q\left(L_{2131} + \frac{1}{2}s_{123}D_{12}V_3 + \frac{1}{2}s_{12}[D_{13}V_2 - V_1D_{23} - D_{12}V_3]\right)$$
$$= s_{123}T_{12}V_3 + s_{12}[T_{13}V_2 + V_1T_{23} - T_{12}V_3]$$

$$T_{123} = L_{2131} + \frac{1}{2}s_{123}D_{12}V_3 + \frac{1}{2}s_{12}[D_{13}V_2 - V_1D_{23} - D_{12}V_3] + Q(\cdots)$$

$$QT_{12} = s_{12}V_1V_2$$
$$QT_{123} = s_{123}T_{12}V_3 + s_{12}[T_{13}V_2 + V_1T_{23} - T_{12}V_3]$$

$$T_{\{12\}} = 0$$
$$T_{\{12\}3} = T_{\{123\}} = 0.$$

$$\int d^8zu(\Phi)\left(-\frac{D^2}{4\,\Box}\right)v(\Phi) = \int d^6zu(\Phi)v(\Phi)$$

$$d^8z = d^4xd^2\theta d^2\bar{\theta} \text{ and } d^6z = d^4xd^2\theta - \tfrac{D^2}{4\Box}$$

$$S_c = \int d^4x\mathcal{L} = \operatorname{tr}\int d^8ze^{-2gV}\bar{\Phi}e^{2gV}\Phi + \frac{1}{2}\operatorname{tr}\int d^6z\mathcal{W}^2 +$$
$$+\int d^8z(\bar{\Psi}_1e^{2gV}\Psi_1 + \bar{\Psi}_2e^{-2gV}\Psi_2) + \lambda\int d^6z\Psi_1\Phi\Psi_2 + \text{ h.c.}$$

$$W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2gV}D_\alpha e^{2gV})$$

$$W_{\text{tree}} = \lambda\int d^6z\Psi_1\Phi\Psi_2$$

$$e^{2gV'} \to e^{ig\bar{\Lambda}}e^{2gV}e^{-ig\Lambda}$$

$$\Phi' = e^{ig\Lambda}\Phi e^{-ig\Lambda}, \bar{\Phi}' = e^{ig\bar{\Lambda}}\bar{\Phi} e^{-ig\bar{\Lambda}}$$
$$\Psi_1' = e^{ig\Lambda}\Psi_1, \Psi_2' = e^{-ig\Lambda}\Psi_2$$

$$\Lambda = \Lambda_a T^a$$

$$(T^a)_{bc} = f^{abc}$$

$[t_a, t_b] = i(T^c)_{ab} t_c, (t^a)^i_j (t^a)^j_k = C_F \delta^i_k, T_{abc} T^{c'}_{ab} = C_A \delta^{c'}_c, \mathrm{tr}[t_a t_b] = T_A \delta_{ab}$ where $C_F = \frac{N^2-1}{2N}, C_A = N$ and $T_F = 1/2$ is the Dynkin index - gauge group $SU(N)$.

$$\mathcal{S}_{GF} = -\frac{1}{16\xi}\mathrm{tr}\int d^8 z D^2 V \bar{D}^2 V$$

$$\mathcal{S}_{FP} = \mathrm{tr}\int d^8 z \left[\bar{c}' c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c + \bar{c}] + \cdots\right]$$

$$\mathcal{S}_0 = \mathcal{S}_c + \mathcal{S}_{GF} + \mathcal{S}_{FP}$$

$$\Phi \to \Phi + \sqrt{\hbar}\phi, \Psi_I \to \Psi_I + \sqrt{\hbar}\psi_I$$

$$\begin{aligned}\mathcal{S}^{(2)} = \mathrm{tr}&\int d^8 z(\bar{\phi}\phi - 2g^2\bar{\Phi}[v,\phi] + 2g^2\bar{\phi}[v,\Phi]) \\ &\int d^8 z(\bar{\psi}_1\psi_1 - g\bar{\Psi}_1 v\psi_1 - g\bar{\psi}_1 v\Psi_1) \\ &\int d^8 z(\bar{\psi}_2\psi_2 + g\bar{\Psi}_2 v\psi_2 + g\bar{\psi}_2 v\Psi_2) \\ &-\lambda\int d^6 z(\psi_1\Phi\psi_2 + \Psi_1\phi\psi_2 + \psi_1\phi\Psi_2 + \text{ h.c.}) \\ &+\mathrm{tr}\int d^8 z\left(-\frac{1}{2}v \,\hat{\square}\, v + cc' + c'c\right)\end{aligned}$$

$$\partial_a\Phi = \partial_a\Psi_I = \partial_a\bar{\Phi} = \partial_a\bar{\Psi}_I = 0$$

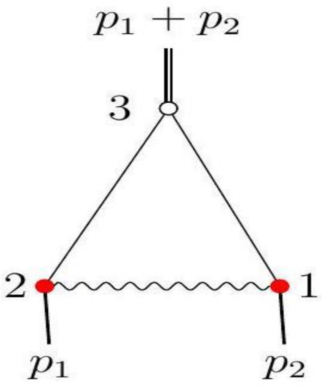
$$\Gamma[\Phi, \Psi_i \mid \bar{\Phi}, \bar{\Psi}_i] = \int d^8 z(\mathbf{K} + \mathbf{A}) + \left(\int d^6 z\mathbf{W} + \text{ h.c.}\right)$$

$$\Gamma[\Phi, \Psi_I \mid \bar{\Phi}, \bar{\Psi}_I] = \sum_{L=1}^{\infty} \hbar^L \Gamma^{(L)}[\Phi, \Psi_I \mid \bar{\Phi}, \bar{\Psi}_I]$$

$$\mathbf{W}[\Phi, \Psi_I] = \sum_{L=1}^{\infty} \hbar^L \mathbf{W}^{(L)}[\Phi, \Psi_I]$$

$$n_{D^2} + 1 = n_{\bar{D}^2},$$

$$\begin{aligned}\mathbf{W}^{(1)} = \lim_{p_1,p_2\to 0} \lambda g^2(2C_F - C_A)\int \prod_{l=1}^{3} d^8 z_l \Phi(z_1)\Psi_1(z_2)\Psi_2(z_3) \\ \left\{\frac{1}{\square_2}\delta_{1,2}\frac{D_1^2\bar{D}_3}{16\,\square_1}\delta_{1,3}\frac{D_2^2}{4}\delta_{2,3}\frac{1}{\square_2}\right\}.\end{aligned}$$



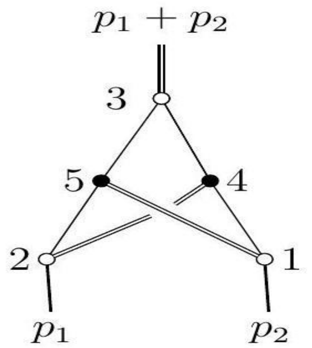
$$\Psi_1(y_1,\theta)\Psi_1(y_2,\theta)\Phi(x,\theta)\simeq[\Psi_1\Phi\Psi_2](x,\theta)$$

$$\mathbf{W}^{(1)}=\frac{\hbar}{(4\pi)^2}g^2(2C_F-C_A)\Upsilon^{(1)}W_{\text{tree}}$$

$$\Upsilon^{(1)}=\lim_{p_1,p_2\to 0}\int d^4q\,\frac{(p_1+p_2)^2}{q^2(q-p_1)^2(q_1+p_2)^2}=\int_0^1 d\tau\,\frac{2\log(\tau)}{\tau^2-\tau+1}$$

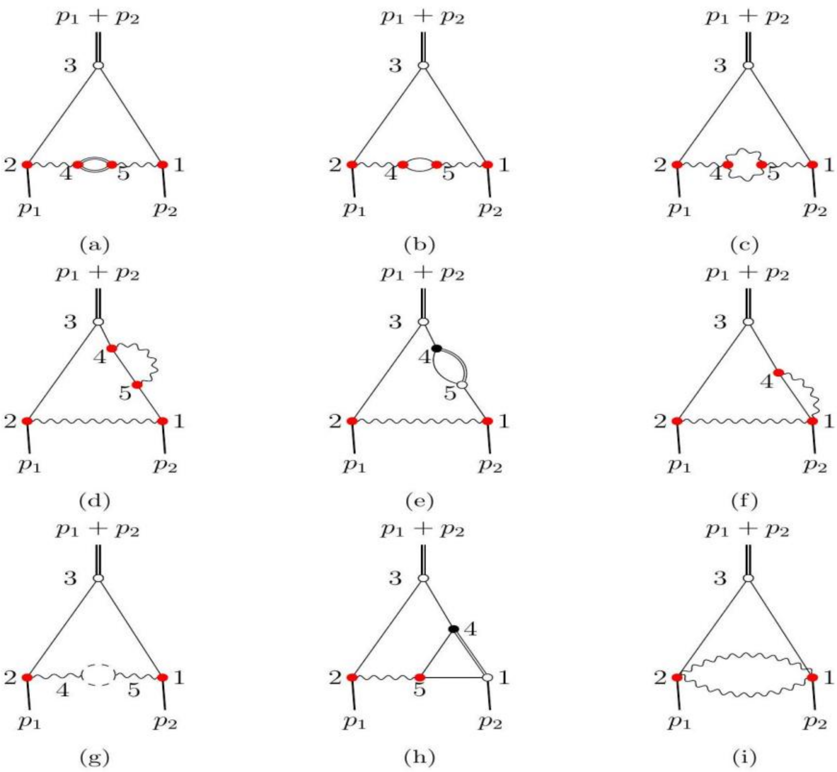
$$\mathbf{W}^{(2)}=\lim_{p_1,p_2\to 0}2|\lambda|^4(C_A-C_F)(C_A-2C_F)\int\prod_{l=1}^{5}d^8z_l\lambda\Psi_1(z_3)\Phi(z_4)\Psi_2(z_5)$$
$$\left\{\frac{1}{\square_1}\delta_{1,3}\frac{D_2^2\bar{D}_3^2}{16\,\square_2}\delta_{3,2}\frac{1}{16\,\square_2}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\square_1}\delta_{1,4}\frac{D_1^2\bar{D}_5^2}{16\,\square_1}\delta_{1,5}\frac{D_2^2}{4\,\square_2}\delta_{2,5}\right\}$$

$$I^{(2)}=\lim_{p_{1,2}\to 0}\int\frac{d^4q_1}{(4\pi)^4}\frac{d^4q_2}{(4\pi)^4}\frac{q_1^2p_1^2+q_2^2p_2^2-2p_1p_2(q_1q_2)}{q_1^2q_2^2(q_1+q_2)^2(q_1-p_1)^2(q_2-p_2)^2(q_1+q_2-p_1-p_2)^2}$$



$$\mathbf{W}^{(2)}=\frac{\hbar^2}{(4\pi)^4}12(C_A-C_F)(C_A-2C_F)|\lambda|^4\zeta(3)\times W_{\text{tree}}$$

$$\mathbf{W}_{div}^{(2),A}+\mathbf{W}_{div}^{(2),B}+\mathbf{W}_{div}^{(2),G}=\lim_{p_1,p_2\to 0}\left(2N_fT_F-C_A\right)(2C_F-C_A)4g^4\lambda$$
$$\times\int\prod_{l=1}^{5}d^8z_l\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{D_1^2\bar{D}_3^2}{16\,\square_1}\delta_{1,3}\frac{D_2^2}{4\,\square_2}\delta_{3,2}\frac{1}{\square_4}\delta_{2,4}\frac{\bar{D}_4^2D_5^2}{16\,\square_4}\delta_{4,5}\frac{\bar{D}_5^2D_4^2}{16\,\square_5}\delta_{5,4}\frac{1}{\square_1}\delta_{5,1}\right\}.$$



$$\mathbf{W}_{div}^{(2),C} \sim \lim_{p_1,p_2\to 0} 4g^4\lambda \int \prod_{l=1}^{5} d^8 z_l \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{D_1^2\bar{D}_3^2}{16\,\square_1}\delta_{1,3}\right.$$

$$\left.\frac{D_2^2}{4\,\square_2}\delta_{3,2}\frac{1}{\square_4}\delta_{2,4}\frac{1}{\square_4}\delta_{4,5}\frac{1}{\square_5}\delta_{5,4}\frac{1}{\square_1}\delta_{5,1}\right\} = 0$$

$$\mathbf{W}_{div}^{(2),I} = 0.$$

$$\mathbf{W}_{div,1}^{(2)} = \mathrm{A} + \mathrm{B} + \mathrm{G} = 4g^4\big(2N_f T_F - C_A\big)(2C_F - C_A) \times J_{1,1}^{(1)}\Upsilon^{(1)} \times W_{\text{tree}}\,,$$

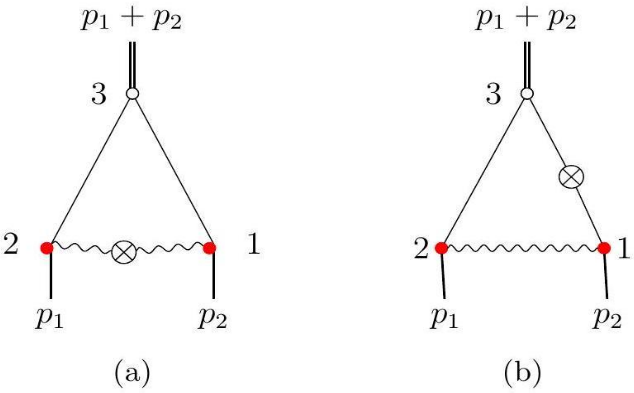
$$J_{1,1}^{(1)}(k) = \left(\frac{1}{\epsilon} + 2 + O(\epsilon^1)\right)(k^2/\mu^2)^{-\epsilon}$$

$$\mathbf{W}_{div,2}^{(2)} = 2\mathrm{D} + 2\mathrm{E} = 2g^2(2C_F - C_A)(|\lambda|^2 - g^2) \times J_{1,1}^{(1)}\Upsilon^{(1)} \times W_{tree}.$$

$$\mathbf{W}_{div}^{(2)} = \Big(g^4\{2N_f T_F - C_A\} + g^2(|\lambda^2| - g^2)\Big)\,4(2C_F - C_A)J_{1,1}^{(1)}\Upsilon^{(1)} \times W_{\text{tree}}$$

$$\beta(g) = \frac{2g^3}{(4\pi)^2}\big(2N_f T_F - C_A\big).$$

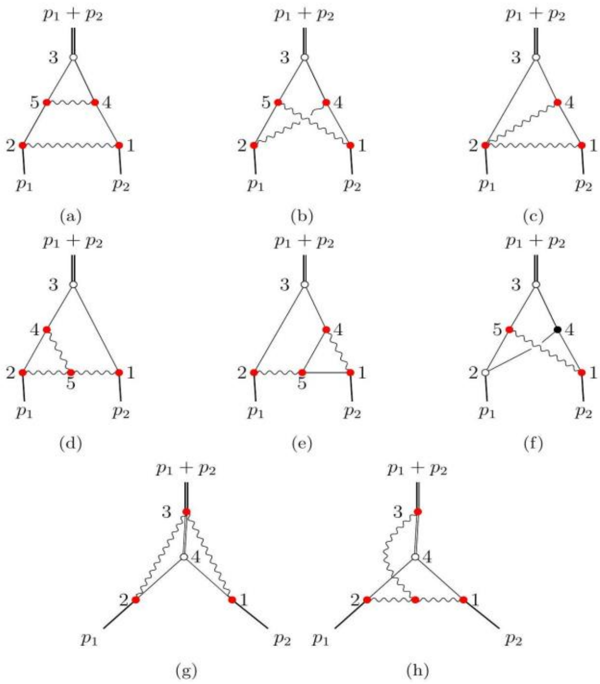
$$\sum_i m_i T_F(R_i) = C_A$$



$$\mathbf{W}^{(2),A}_{fin} = \lim_{p_1,p_2\to 0} g^4(C_A-2C_F)^2 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\Big\{\frac{1}{\Box_1}\delta_{2,1} \frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\frac{D_4^2\bar{D}_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_5^2}{4\,\Box_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{16\,\Box_5}\delta_{5,2}\frac{1}{\Box_4}\delta_{4,5}\Big\}$$

$$J_a^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{q_1^2(p_1+p_2)^2}{q_1^2(q_1-p_1)^2(q_2-p_2)^2q_2^2(q_2-p_2)^2(q_1-p_2)^2} = 6\zeta(3);$$

$$\mathbf{W}^{(2),A}_{\text{fin}} = 6g^4(C_A-2C_F)^2\zeta(3)\times W_{\text{tree}}.$$



$$\mathbf{W}^{(2),B}_{fin} = \lim_{p_1,p_2\to 0} 4g^4(C_A-C_F)(C_A-2C_F) \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\Big\{\frac{1}{\Box_4}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4} \frac{\bar{D}_4^2 D_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_5^2}{4\,\Box_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{15}\delta_{5,2}\frac{1}{\Box_5}\delta_{1,5}\Big\}$$

$$J_b^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{-q_1^2p_1^2-q_2^2p_2^2+2(q_1q_2)(p_1p_2)}{q_1^2(q_1-p_1)^2(q_1+q_2-p_1)^2q_2^2(q_2+p_2)^2(q_1+q_2+p_1)^2}$$

$$\mathbf{W}^{(2),B}_{fin} = -24g^4(C_A-C_F)(C_A-2C_F)\zeta(3)\times W_{tree}$$

$$\mathbf{W}_{fin}^{(2),C} = \lim_{p_1,p_2\to 0} 2g^4(C_A - C_F)(C_A - 2C_F) \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)$$
$$\left\{\frac{1}{\Box_1}\delta_{2,1}\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\frac{\bar{D}_4^2 D_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_2^2}{4\,\Box_3}\delta_{3,2}\frac{1}{\Box_4}\delta_{2,4}\right\}$$

$$J_c^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{q_1^2(p_1+p_2)^2}{q_1^2(q_1+p_1)^2(q_2+p_1)^2(q_2-q_1)^2(q_2-p_2)^2}$$

$$\mathbf{W}_{fin}^{(2),C} = 12g^4(C_A - C_F)(C_A - 2C_F)\zeta(3)\times W_{tree}$$

$$\mathbf{W}_{fin}^{(2),D} = \lim_{p_1,p_2\to 0} 2g^4 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_5}\delta_{5,1}\frac{D_1^2\bar{D}_3^2}{16\,\Box_1}\delta_{1,3}\right.$$
$$\left.\frac{D_4^2}{4\,\Box_3}\delta_{3,4}\frac{\bar{D}_4^2 D_2^2}{16\,\Box_4}\delta_{4,2}\frac{1}{\Box_4}\delta_{2,5}\frac{1}{\Box_4}\delta_{5,4}\right\}$$

$$J_d^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{(p_1+p_2)^2}{q_1^2(q_1+p_1)^2(q_1-p_2)^2(q_2-p_2)^2 q_2^2(q_1-q_2)^2}$$

$$\mathbf{W}_{\text{fin}}^{(2),D} = 2g^4\Upsilon^{(2)}\times W_{\text{tree}}$$

$$\Upsilon^{(2)} = \int_0^1 d\tau\,\frac{2\log^3(\tau)}{\tau^2-\tau+1}$$

$$\mathbf{W}^{(2),E} = \lim_{p_1,p_2\to 0} 2g^4(C_A - 2C_F)^2 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_4}\delta_{1,4}\frac{D_1^2\bar{D}_5^2}{16\,\Box_5}\delta_{1,5}\right.$$
$$\left.\frac{D_5^2\bar{D}_4^2}{16\,\Box_5}\delta_{5,4}\frac{D_4^2\bar{D}_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_2^2}{4\,\Box_3}\delta_{3,2}\frac{1}{\Box_5}\delta_{2,5}\right\}.$$

$$J_e^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{-q_1^2p_1^2+(q_1-q_2)^2p_2^2-2(q_1q_2)(p_1p_2)}{q_1^2(q_1-p_1)^2(q_2-q_1+p_1)^2(q_2-p_2)^2q_2^2(q_2+p_1)^2}$$

$$\mathbf{W}_{\text{fin}}^{(2),E}(\Phi) = -12g^4(C_A - 2C_F)^2\zeta(3)\times W_{\text{tree}}$$

$$\mathbf{W}_{fin}^{(2),F}(\Phi) \sim \lim_{p_1,p_2\to 0} 4g^4 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_4}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\right.$$
$$\left.\frac{\bar{D}_4^2 D_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_5^2}{4\,\Box_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{16\,\Box_5}\delta_{5,2}\frac{1}{\Box_5}\delta_{1,5}\right\}$$

$$J_f^{(2)} = 0$$

$$\mathbf{W}_{fin}^{(2),G} \sim \lim_{p_1,p_2\to 0} 4g^4 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\frac{\bar{D}_4^2 D_2^2}{16\,\Box_4}\delta_{4,2}\right.$$
$$\left.\frac{1}{\Box_3}\delta_{2,3}\frac{1}{\Box_1}\delta_{3,1}\frac{D_3^2}{4\,\Box_4}\delta_{3,4}\right\} = 0$$

$$\mathbf{W}^{(2),H} \sim \lim_{p_1,p_2\to 0} 2g^4 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_5}\delta_{3,5}\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\frac{\bar{D}_4^2 D_2^2}{16\,\Box_4}\delta_{4,2}\right.$$
$$\left.\frac{1}{\Box_3}\delta_{2,5}\frac{1}{\Box_1}\delta_{5,3}\frac{D_3^2}{4\,\Box_4}\delta_{3,4}\right\} = 0.$$

$$\mathbf{W}_{\text{fin}}^{(2)} = \left[2g^4\Upsilon^{(2)} + 6(C_A - 2C_F)(2(C_A - C_F)|\lambda|^4 - (3C_A - 4C_F)g^4)\zeta(3)\right]\times W_{\text{tree}}$$

$$\mathbf{W}_{fin,\mathcal{N}=2}^{(2)} = 2g^4\left(\Upsilon^{(2)} - 3\frac{1}{N^2}\zeta(3)\right)\times W_{tree}$$

$$\mathbf{W}^{(2)} = \mathbf{W}_{div}^{(2)} + \mathbf{W}_{fin}^{(2)} = \Big[\Big(g^4\{2N_f T_F - N\} + g^2(|\lambda^2| - g^2)\Big) J_{1,1}^{(1)} \Upsilon^{(1)} + \frac{1}{2} g^4 \Upsilon^{(2)} + (-3(C_A - C_F))|\lambda|^4 + \frac{3}{2}(3C_A - 4C_F) g^4\Big)\zeta(3)\Big] \times 4(2C_F - C_A) W_{tree}$$

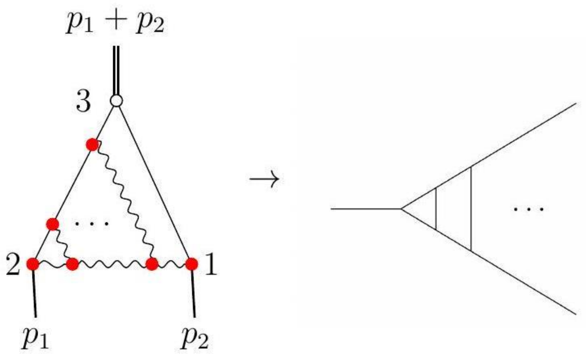
$$\Phi = \left(z_\phi^{1/2}\right)\Phi_R, \bar{\Phi} = \left(z_\phi^{1/2}\right)\bar{\Phi}_R$$
$$\Psi_I = \left(z_\psi^{1/2}\right)\Psi_{I,R}, \bar{\Phi} = \left(z_\psi^{1/2}\right)\bar{\Psi}_{I,R}$$
$$V = z_V^{1/2} V_R, g = z_g g_R, \lambda = z_\lambda^{3/2}\lambda$$

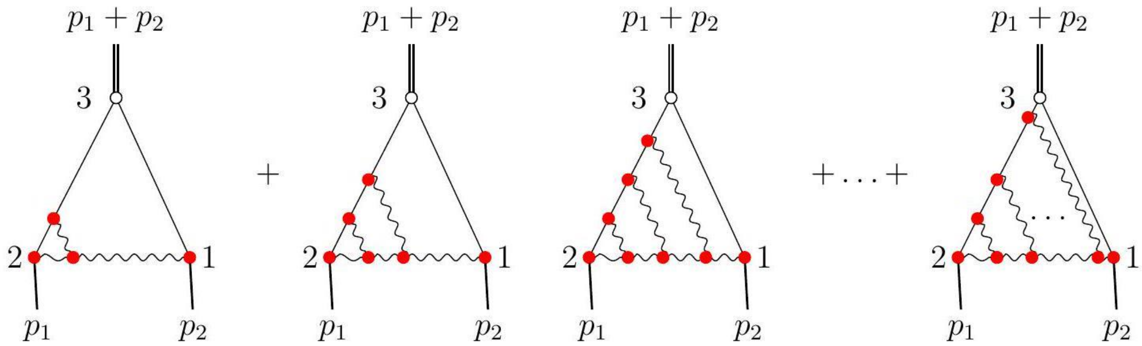
$$z_\phi^{1/2} = z_\psi^{-1}$$

$$\mathbf{W}_R^{(2)} = \Big(\frac{1}{2} g_R^4 \Upsilon^{(2)} + \Big(-3(C_A - C_F)|\lambda|_R^4 + \frac{3}{2}(3C_A - 4C_F) g_R^4\Big)\zeta(3) + 2u\Big) \times 4(2C_F - C_A) W_{\text{tree}\circ R}$$

$$W_{\text{tree}\circ R} = \operatorname{tr}\int d^6 z \lambda_R \Psi_{1,R}\Phi_R\Psi_{2,R}$$

$$u = \Big(g_R^4\{2N_f T_F - C_A\} + g_R^2(|\lambda_R^2| - g_R^2)\Big)\Upsilon^{(1)}$$





$$\mathbf{W}'^{(m)} = \lim_{p_1,p_2\to 0}(-1)^{m+1}\frac{g^{2m}}{4}N^{m-2}\int \prod_{l=1}^{2m+1} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3) \times \Big\{\frac{1}{\square_5}\delta_{5,1}\frac{D_1^2\bar{D}_3^2}{16\,\square_1}\delta_{1,3}\frac{D_4^2}{4\,\square_3}\delta_{3,4}\frac{\bar{D}_4^2 D_6^2}{16\,\square_4}\delta_{4,6}\cdots \cdots\frac{\bar{D}_{2m-2}^2 D_{2m}^2}{\square_{2m-2}}\delta_{2m-2,2m}\frac{\bar{D}_{2m}^2 D_2^2}{\square_{2m}}\delta_{2m,2}\frac{1}{\square_{2m+1}}\delta_{2m+1,2}\cdots \frac{1}{\square_{2m}}\delta_{2m+1,2m}\cdots\frac{1}{\square_4}\delta_{5,4}\Big\}$$

$$\mathbf{W}'^{(m)} = (-1)^{m-1}\frac{g^{2m}}{4}N^{m-2}\Upsilon^{(m)} \times W_{\text{tree}}$$

$$\mathbf{W}'^{lead} = \frac{y}{2N^2}\int_0^1 d\tau \frac{\log(\tau)(1-\tau)}{(1 + y\log^2(\tau))(1+\tau^3)} \times W_{\text{tree}} = \Upsilon^{\text{tot}} \times W_{\text{tree}}$$

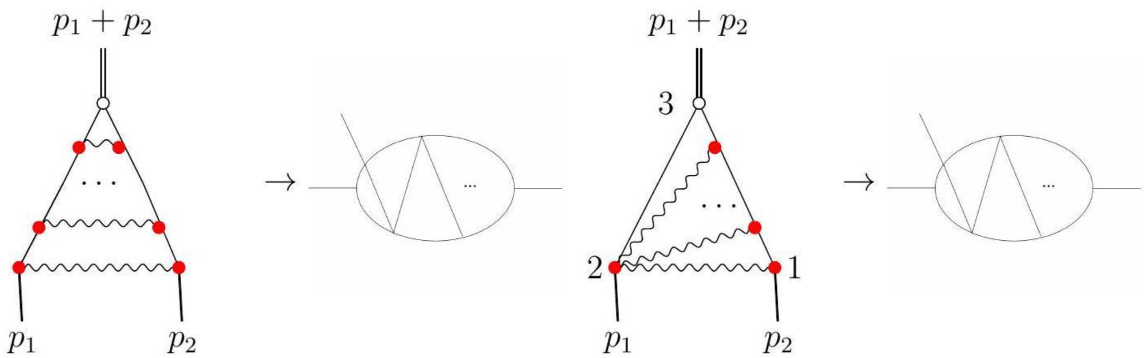
$$\Upsilon^{tot} = \frac{1}{4N^2}\sum_{m=1}^{\infty}((\pi - 2\mathrm{Si}(x))\sin(x) - 2\mathrm{Ci}(x)\cos(x))U_m(1/2)$$

$$\Upsilon^{tot}|_{\hbar\to\infty} \simeq \frac{1}{2N^2}\log\left(\frac{12e^{\gamma_E}}{\sqrt{\hbar}gN}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}\right) + O(\hbar^{-2})$$

$$Z(L+1) = 4C_L\sum_{p=1}^{\infty}\frac{(-1)^{(p-1)(L+1)}}{p^{2(L+1)-3}} = \begin{cases}4C_L\zeta(2L-1) \text{ for } L = 2N+1\\ 4C_L\left(1-2^{2(1-L)}\right)\zeta(2L-1) \text{ for } L = 2N\end{cases}$$

$$C_L = \frac{1}{(L+1)}\binom{2L}{L}$$

$$\mathbf{W}^{\mathrm{sub}} \sim g^{2L}c_L/N^L \times Z(L+1) \times W_{\mathrm{tree}},$$



$$\sigma^\mu = (\sigma_0, -\vec{\boldsymbol{\sigma}}) \text{ and } \bar{\sigma}^\mu = (\sigma_0, \vec{\boldsymbol{\sigma}})$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}, \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}\ \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma.$$

$$\left(\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu\right)^\alpha_\beta = -2\eta_{\mu\nu}\delta^\alpha_\beta \text{ and } \left(\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu\right)^{\dot{\alpha}}_{\dot{\beta}} = -2\eta_{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}$$

$$\mathrm{tr}(\text{ odd number of } \sigma's) = 0$$
$$\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = \mathrm{tr}(\bar{\sigma}^\mu\sigma^\nu) = -2\eta^{\mu\nu}$$

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \bar{D}_{\dot{\alpha}} = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu$$

$$\{D_\alpha, D_\beta\} = 0, \{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma)^\mu_{\alpha\dot{\beta}}\partial_\mu$$
$$D^2\bar{D}_{\dot{\alpha}}D^2 = 0, \bar{D}^2D_\alpha\bar{D}^2 = 0$$
$$D^\alpha\bar{D}^2D_\alpha = \bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}}$$
$$D^2\bar{D}^2 + \bar{D}^2D^2 - 2D^\alpha\bar{D}^2D_\alpha = 16\,\square$$
$$D^2\bar{D}^2D^2 = 16\,\square\,D^2, \bar{D}^2D^2\bar{D}^2 = 16\,\square\,\bar{D}^2$$
$$[D^2, \bar{D}_{\dot{\alpha}}] = -4i\partial_{\alpha\dot{\alpha}}D^\alpha, [\bar{D}^2, D_\alpha] = 4i\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}$$

$$\int d^2\theta = -\frac{1}{4}D^2, \int d^2\bar{\theta} = \frac{1}{4}\bar{D}^2$$
$$d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}d\theta_\alpha d\theta_\beta, \int d\theta_\alpha\theta^\beta = \delta^\beta_\alpha$$

$$J^{(1)}_{\alpha,\beta}(k) = \int\frac{d^dq}{(q-k)^{2\alpha}q^{2\beta}} = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-d/2)}(k^2/\mu^2)^{d/2-\alpha-\beta},$$

$$a(\alpha) = \Gamma(d/2-\alpha)/\Gamma(\alpha)$$

$$J^{(1)}_{1,1}(k) = (k^2/\mu^2)^{-\epsilon}\left(\frac{1}{\epsilon} + 2 + O(\epsilon^1)\right).$$

$$\overset{\Rightarrow}{=} = \langle \phi_a \bar{\phi}_b \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\,\Box} \delta_{ab} \delta^8 (z_1 - z_2)$$

$$-\langle \psi_i \bar{\psi}_j \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\,\Box} \delta_{ij} \delta^8 (z_1 - z_2)$$

$$\sim m = \langle v_a v_b \rangle = \left(-\frac{D^\alpha \bar{D}^2 D_\alpha}{8\,\Box^2} + \xi \frac{\{D^2, \bar{D}^2\}}{16\,\Box^2} \right) \delta_{ab} \delta^8 (z_1 - z_2) \overset{\xi=1}{=} \frac{1}{16\,\Box} \delta_{ab} \delta^8 (z_1 - z_2)$$

$$= \langle c_a' c_b \rangle = \langle c_b' c_a \rangle = \frac{1}{16\,\Box} \delta_{ab} \delta^8 (z_1 - z_2)$$

$$J^{(2)} = \lim_{p_1, p_2 \to 0} \int \frac{d^4 q_1}{(4\pi)^2} \frac{d^4 q_2}{(4\pi)^2} \frac{q_1^2 (p_1 + p_2)^2}{q_1^2 (q_1 - p_1)^2 (q_2 - p_2)^2 q_2^2 (q_2 - p_2)^2 (q_1 - p_2)^2} = 6\zeta(3).$$

$$J^{(3)} = 20\zeta(5), J^{(4)} = \frac{441}{8} \zeta(7),$$

$$\Upsilon_L(u, v) = \frac{\Gamma(1-\epsilon)}{u^{2(1-\epsilon)}} \sum_{l=0}^{\infty} \frac{\epsilon^l}{l!} \Upsilon_L^{(l)}(z_1, z_2)$$

$$\Upsilon_L^{(l)}(z_1, z_2) = \sum_{f=0}^{L} \frac{\left(-\ln(z_1 z_2)\right)^f (2L - f)}{f!\,(L-f)!} \sum_{m=0}^{l} \frac{(-1)^m l!}{m!\,(l-m)!} \mathbf{Z}_m(z_1, z_2; 2L + l - f)$$

$$\mathbf{Z}_m(z_1, z_2; k) = \frac{\Gamma(k-m)}{(z_1 - z_2)} \sum_{\{n_i\}=1}^{\infty} \frac{\left(z_1^{n_0} - z_2^{n_0}\right)}{\left(\sum_0^m n_i\right)^{k-m}} \left(\prod_{i=1}^{m} \frac{z_1^{n_i} + z_2^{n_i}}{n_i} \right)$$

$$\Upsilon^{(l)} = \int_0^1 d\tau \frac{2\log^{2l-1}(\tau)}{\tau^2 - \tau + 1} =$$

$$= \frac{1}{2^{2l-1} 3^{2l}} \left(\psi^{(2l+1)}\left(\frac{2}{3}\right) - \psi^{(2l+1)}\left(\frac{1}{3}\right) - \psi^{(2l+1)}\left(\frac{1}{6}\right) + \psi^{(2l+1)}\left(\frac{5}{6}\right) \right)$$

$$O_2^{I_1 I_2} = \mathrm{tr}\left(\phi^{\{I_1} \phi^{I_2\}}\right)$$

$$O_2(x, t) = O_2^{I_1 I_2} t_{I_1} t_{I_2}, t \cdot t = 0$$

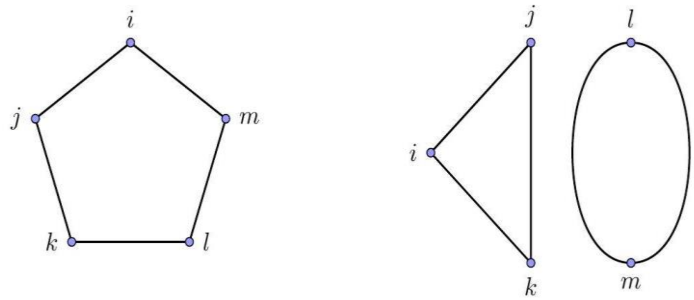
$$G(x_i, t_i) = \langle O_2(x_1, t_1) O_2(x_2, t_2) O_2(x_3, t_3) O_2(x_4, t_4) O_2(x_5, t_5) \rangle.$$

$$G(x_i, t_i) = \frac{x_{13}^2}{x_{12}^4 x_{35}^4 x_{14}^2 x_{34}^2} G(u_i, t_i)$$

$$u_1 = \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}, u_{i+1} = u_i|_{x_i \to x_{i+1}}$$

$$\prod_{i<j} t_{ij}^{a_{ij}}, t_{ij} = t_i \cdot t_j$$

$$a_{ij} = a_{ji}, \sum_{j \neq i} a_{ij} = 2$$



$\mathcal{T}_{(ijklm)}$ $\mathcal{T}_{(ijk)(lm)}$

$$G(x_i, \lambda_i t_i) = \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 \lambda_5^2 G(x_i, t_i),$$

$$\mathcal{T}_{(ijklm)} = t_{ij} t_{jk} t_{kl} t_{lm} t_{mi},$$
$$\mathcal{T}_{(ijk)(lm)} = t_{ij} t_{jk} t_{ki} t_{lm}^2$$

$$t_{ij} = (z_i - z_j)(v_i - v_j),$$

$$G(z_i, \bar{z}_i, t_i)|_{t_{ij}=(z_i-z_j)(v_i-v_j)} = g(\bar{z}_i, v_i).$$

$$g(\bar{z}_i, v_i) = g_{\text{free}}(\bar{z}_i, v_i).$$

$$t_{ij} = x_{ij}^2,$$

$$G(x_i, t_i)|_{t_{ij}=x_{ij}^2} = \lambda$$

$$\langle \mathcal{O}_1(x_1, t_1) \dots \mathcal{O}_n(x_n, t_n) \rangle_{\text{conn}} = \int [d\delta] M(\delta_{ij}, t_{ij}) \prod_{1 \le i < j \le n} (x_{ij}^2)^{-\delta_{ij}} \Gamma(\delta_{ij}),$$

$$\delta_{ij} = \delta_{ji}, \delta_{ii} = -\Delta_i, \sum_j \delta_{ij} = 0.$$

$$\mathcal{O}(x, z) \equiv \mathcal{O}^{\mu_1 \dots \mu_J}(x) z_{\mu_1} \dots z_{\mu_J}, z \cdot z = 0$$

$$\langle \mathcal{O}(x_0, z_0) \cdots \mathcal{O}_n \rangle = \sum_{a_1, \dots, a_J = 1}^{n} \prod_{i=1}^{J} (z_0 \cdot x_{a_i 0}) \int [d\delta] M^{\{a\}}(\delta_{ij}, t_{ij}) \prod_{i=1}^{n} \frac{\Gamma(\delta_i + \{a\}_i)}{(x_{i0}^2)^{\delta_i + \{a\}_i}} \prod_{1 \le i < j \le n} \frac{\Gamma(\delta_{ij})}{(x_{ij}^2)^{\delta_{ij}}},$$

$$\{a\}_i = \boldsymbol{\delta}_i^{a_1} + \cdots + \boldsymbol{\delta}_i^{a_J}, \delta_i = -\sum_{j=1}^{n} \delta_{ij}, \sum_{i,j=1}^{n} \delta_{ij} = J - \Delta_0.$$

$$\sum_{a_1=1}^{n} (\delta_{a_1} + \boldsymbol{\delta}_{a_1}^{a_2} + \boldsymbol{\delta}_{a_1}^{a_3} + \cdots + \boldsymbol{\delta}_{a_1}^{a_J}) M^{a_1 a_2 \dots a_J} = 0$$

$$(2J + d - 4) \sum_{\substack{a,b=1 \\ a \ne b}}^{n} \delta_{ab} [M^{a c_2 \dots c_J}]^{ab} = (J - 1) \sum_{\substack{a,b=1 \\ a \ne b}}^{n} \delta_{ab} [M^{abc_3 \dots c_J}]^{ab}$$

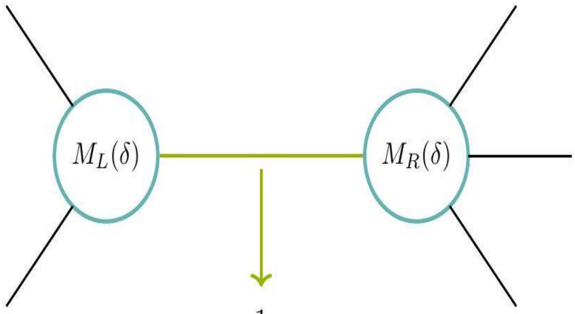
$$[M(\delta_{ij})]^{ab} \equiv M(\delta_{ij} + \boldsymbol{\delta}_i^a \boldsymbol{\delta}_j^b + \boldsymbol{\delta}_j^a \boldsymbol{\delta}_i^b)$$

$$M(\delta_{ij}, t_{ij}) = \frac{1}{N^3} \left(M^{\text{SUGRA}}(\delta_{ij}, t_{ij}) + \frac{1}{\lambda^{3/2}} M^{R^4}(\delta_{ij}, t_{ij}) + \mathcal{O}\left(\frac{1}{\lambda^{5/2}}\right) \right) + \mathcal{O}\left(\frac{1}{N^4}\right)$$

$$\mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) = \sum_p C_{\mu_1 \dots \mu_J}^{(1 \dots k, p)}(x_1, \dots, x_k, y, \partial_y) \mathcal{O}_p^{\mu_1 \dots \mu_J}(y)$$

$$M(\delta_{ij}) \approx \frac{Q_m(\delta_{ij})}{\delta_{LR} - (\tau + 2m)}, \delta_{LR} = \sum_{a=1}^{k} \sum_{b=k+1}^{n} \delta_{ab}$$

$$\langle O_2 O_2 X \rangle \langle X O_2 O_2 O_2 \rangle$$



$$\frac{1}{\delta_{LR} - (\tau + 2m)}$$

$$Q_0(\delta_{ij}) \propto M_L M_R, \text{ (scalar)}$$

$$Q_0 \propto \sum_{a=1}^{k} \sum_{i=k+1}^{n} \delta_{ai} M_L^a M_R^i$$

$$Q_0 \propto \sum_{a,b=1}^{k} \sum_{i,j=k+1}^{n} \delta_{ai}(\delta_{bj} + \boldsymbol{\delta}_b^a \boldsymbol{\delta}_j^i) M_L^{ab} M_R^{ij}$$

$$\delta_{LR} - (\tau + 2m) \to 2(1 - m - \delta_{12})$$

$$\langle O_2 O_2 O_2 O_2 \rangle$$
$$\langle J_\mu O_2 O_2 O_2 \rangle$$
$$\langle T_{\mu\nu} O_2 O_2 O_2 \rangle$$

$$\langle O_2 O_2 O_2 O_2 \rangle = \frac{t_{12}^2 t_{34}^2}{x_{12}^4 x_{34}^4} \mathcal{G}_{2222}(u, v; \sigma, \tau)$$
$$\mathcal{G}_{2222}(u, v; \sigma, \tau) = \mathcal{G}_{2222}^{\text{free}}(u, v; \sigma, \tau) + R(u, v; \sigma, \tau) H(u, v)$$

$$R(u, v; \sigma, \tau) = \tau + \sigma\tau u^2 + \tau u(\tau - \sigma - 1) + \sigma u v(\sigma - \tau - 1) + \sigma v^2 + v(1 - \sigma - \tau)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \sigma = \frac{t_{13} t_{24}}{t_{12} t_{34}}, \tau = \frac{t_{14} t_{23}}{t_{12} t_{34}}$$

$$\mathcal{G}_{2222}^{\text{free}} = 1 + \sigma^2 u^2 + \frac{\tau^2 u^2}{v^2} + \frac{1}{c}\left(\frac{\sigma\tau u^2}{v} + \sigma u + \frac{\tau u}{v}\right)$$

$$\mathcal{M}(s, t) \Gamma\left(\frac{4 - s}{2}\right)^2 \Gamma\left(\frac{4 - t}{2}\right)^2 \Gamma\left(\frac{4 - \tilde{u}}{2}\right)^2 = \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v u^{-\frac{s}{2} - 1} v^{1 - \frac{t}{2}} H(u, v)$$
$$\delta_{ij} = \frac{\Delta_i + \Delta_j - s_{ij}}{2}, s_{12} = s_{34} = s, s_{14} = s_{23} = t$$
$$s_{13} = s_{24} = 8 - s - t = 4 - \tilde{u}$$
$$\mathcal{M}(s, t) = \frac{1}{N^2} \frac{32}{(s - 2)(t - 2)(2 - s - t)} + \frac{1}{N^2 \lambda^{3/2}} 480 \zeta_3 + \cdots$$

$$t_{0i} t_{0j} t_{0k} t_{0\ell} \longrightarrow t_{i\ell} t_{jk} + t_{ik} t_{j\ell} - \frac{1}{3} t_{ij} t_{k\ell}$$
$$Y_{0,ij} Y_{0,k\ell} \longrightarrow t_{ik} t_{j\ell} - t_{i\ell} t_{jk}$$

$$M^{R^4}(\delta_{ij},t_{ij}) = \sum_{\substack{k=1\\ \ell=\bar{k}+1}}^{5} \frac{A^{k\ell}(\delta_{ij},t_{ij})}{\delta_{k\ell}-1} + R(\delta_{ij},t_{ij})$$

$$A^{12}(\delta_{ij},t_{ij}) = \sum_{\text{R-sym}} \sum_{n_i=0}^{n_1+\cdots+n_4\le 4} p^{12}_{\{n_i\}}(t_{ij})\delta_{15}^{n_1}\delta_{23}^{n_2}\delta_{34}^{n_3}\delta_{45}^{n_4},$$
$$R(\delta_{ij},t_{ij}) = \sum_{I} \sum_{n_i=0}^{n_1+\cdots+n_5\le 4} r_{\{n_i\},I}\mathcal{T}^I\delta_{12}^{n_1}\delta_{15}^{n_2}\delta_{23}^{n_3}\delta_{34}^{n_4}\delta_{45}^{n_5},$$

$$\mathcal{C}_{ij} = \frac{1}{2}\left(L_{IJ}^{(i)} + L_{IJ}^{(j)}\right)\left(L^{(i),IJ} + L^{(j),IJ}\right),$$

$$L_{IJ}^{(i)} = t_{i,I}\frac{\partial}{\partial t_i^J} - t_{i,J}\frac{\partial}{\partial t_i^I}.$$

$$R_{ij}^{\lambda_c} = \sum_I a_{ij,I}^{\lambda_c}\mathcal{T}^I$$

$$\mathcal{C}_{12}R_{12}^{\lambda_c} = \lambda_c R_{12}^{\lambda_c}$$

$$\begin{aligned}
R_{12}^{0} &= a_{12,1}^{0}t_{12}^2t_{34}t_{35}t_{45}\\
R_{12}^{-16} &= a_{12,2}^{-16}(t_{12}t_{15}t_{24}t_{34}t_{35} - t_{12}t_{14}t_{25}t_{34}t_{35}) + a_{12,3}^{-16}(t_{12}t_{15}t_{23}t_{34}t_{45} - t_{12}t_{13}t_{25}t_{34}t_{45})\\
&\quad + a_{12,4}^{-16}(t_{12}t_{14}t_{23}t_{35}t_{45} - t_{12}t_{13}t_{24}t_{35}t_{45})\\
R_{12}^{-24} &= a_{12,5}^{-24}\left(t_{15}t_{24}t_{34}t_{35}t_{12} + t_{14}t_{25}t_{34}t_{35}t_{12} - \frac{1}{3}t_{34}t_{35}t_{45}t_{12}^2\right) + a_{12,6}^{-24}t_{12}t_{15}t_{25}t_{34}^2\\
&\quad + a_{12,7}^{-24}\left(t_{15}t_{23}t_{34}t_{45}t_{12} + t_{13}t_{25}t_{34}t_{45}t_{12} - \frac{1}{3}t_{34}t_{35}t_{45}t_{12}^2\right) + a_{12,8}^{-24}t_{12}t_{14}t_{24}t_{35}^2\\
&\quad + a_{12,9}^{-24}\left(t_{14}t_{23}t_{35}t_{45}t_{12} + t_{13}t_{24}t_{35}t_{45}t_{12} - \frac{1}{3}t_{34}t_{35}t_{45}t_{12}^2\right) + a_{12,10}^{-24}t_{12}t_{13}t_{23}t_{45}^2
\end{aligned}$$

$$\begin{aligned}
\sum_{\text{R-sym}} p^{12}_{\{n_i\}}(t_{ij}) &= p^{12,0}_{\{n_i\}}R_{12}^0 + p^{12,-16}_{\{n_i\}}R_{12}^{-16} + p^{12,-24}_{\{n_i\}}R_{12}^{-24}\\
&= p^{12,0}_{\{n_i\},1}t_{12}^2t_{34}t_{35}t_{45} + p^{12,-16}_{\{n_i\},2}(t_{12}t_{15}t_{24}t_{34}t_{35} - t_{12}t_{14}t_{25}t_{34}t_{35})\\
&\quad + p^{12,-16}_{\{n_i\},3}(t_{12}t_{15}t_{23}t_{34}t_{45} - t_{12}t_{13}t_{25}t_{34}t_{45}) + p^{12,-16}_{\{n_i\},4}(t_{12}t_{14}t_{23}t_{35}t_{45} - t_{12}t_{13}t_{24}t_{35}t_{45}) + \cdots
\end{aligned}$$

$$M_{222} = f_{OOO}t_{45}t_{04}t_{05}$$
$$M_{2222}^{\text{str}} = 480\zeta_3(\delta_{23}^4t_{03}^2t_{12}^2 - 2\delta_{12}\delta_{23}^3t_{02}t_{03}t_{12}t_{13} + 2\delta_{12}\delta_{23}^3t_{01}t_{03}t_{12}t_{23} + \cdots)$$

$$M_{22222}^{\text{str}} = \frac{C_{OOO}M_{222}M_{2222}^{\text{str}}}{(\delta_{45}-1)} + \cdots$$

$$\begin{aligned}
M_{22222}^{\text{str}} = 480\zeta_3\frac{C_{OOO}}{(\delta_{45}-1)}\Big(&2\delta_{23}^4t_{12}^2t_{34}t_{35}t_{45} + \frac{2}{3}\delta_{12}\delta_{23}^3t_{12}t_{23}t_{45}(3t_{15}t_{34} + 3t_{14}t_{35} - t_{13}t_{45})\\
&- \frac{2}{3}\delta_{12}\delta_{23}^3t_{12}t_{13}t_{45}(3t_{25}t_{34} + 3t_{24}t_{35} - t_{23}t_{45}) + \cdots\Big) + \cdots
\end{aligned}$$

$$C_{OOO} = 2\sqrt{2}, C_{OOT} = \frac{4\sqrt{2}}{7}, C_{OOJ} = 4\sqrt{2}$$

$$t_{ij} = x_{ij}^2$$

$$G(x_i,t_i)_{\text{conn}} = \int [d\delta]M(\delta_{ij},t_{ij})\prod_{1\le i<j\le n}(x_{ij}^2)^{-\delta_{ij}}\Gamma(\delta_{ij}),$$

$$G(x_i,t_i)_{\text{conn}} = \frac{x_{13}^2t_{12}^2t_{35}^2t_{14}t_{34}}{x_{12}^4x_{35}^4x_{14}^2x_{34}^2t_{13}}\int [d\delta]M(\delta_{ij},\sigma_i)\Gamma_{22222}u_1^{2-\delta_{12}}u_2^{-\delta_{23}}u_3^{2-\delta_{34}}u_4^{-\delta_{45}}u_5^{1-\delta_{15}},$$

$$\Gamma_{22222} = \Gamma(\delta_{12})\Gamma(\delta_{15})\Gamma(\delta_{23})\Gamma(\delta_{34})\Gamma(\delta_{45})\Gamma(1+\delta_{15}-\delta_{23}-\delta_{34})\Gamma(1-\delta_{12}-\delta_{15}+\delta_{34})$$
$$\Gamma(1-\delta_{15}+\delta_{23}-\delta_{45})\Gamma(1+\delta_{12}-\delta_{34}-\delta_{45})\Gamma(1-\delta_{12}-\delta_{23}+\delta_{45})$$

$$M(\delta_{ij},\sigma_i) = \sum_{\{n_i\}} \sigma_1^{n_1}\sigma_2^{n_2}\sigma_3^{n_3}\sigma_4^{n_4}\sigma_5^{n_5} M_{\{n_i\}}(\delta_{ij}) \to \sum_{\{n_i\}} u_1^{n_1}u_2^{n_2}u_3^{n_3}u_4^{n_4}u_5^{n_5} M_{\{n_i\}}(\delta_{ij})$$

$$u_1^{n_1}u_2^{n_2}u_3^{n_3}u_4^{n_4}u_5^{n_5} M_{\{n_i\}}(\delta_{ij}) \to \mathbb{D}_{n_1,\dots,n_5} \circ M_{\{n_i\}}(\delta_{ij}),$$

$$\mathbb{D}_{n_1,\dots,n_5} \circ M_{\{n_i\}}(\delta_{ij}) = M_{\{n_i\}}(\delta_{12}+n_1,\delta_{23}+n_2,\dots) \times (\delta_{12})_{n_1}(\delta_{15})_{n_5}(\delta_{23})_{n_2}(\delta_{34})_{n_3}(\delta_{45})_{n_4}$$
$$(\delta_{15}-\delta_{23}-\delta_{34}+1)_{n_5-n_2-n_3}(\delta_{23}-\delta_{15}-\delta_{45}+1)_{n_2-n_4-n_5}(1-\delta_{12}-\delta_{15}+\delta_{34})_{n_3-n_1-n_5}$$
$$(1+\delta_{12}-\delta_{34}-\delta_{45})_{n_1-n_3-n_4}(1-\delta_{12}-\delta_{23}+\delta_{45})_{n_4-n_1-n_2}$$

$$\prod_{1\le i<j\le 5} (x_{ij}^2)^{-\alpha_{ij}} D_{\tilde{\Delta}_1\dots\tilde{\Delta}_5} \leftrightarrow M^{\alpha_{ij}}(\delta) = \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\sum_i \tilde{\Delta}_i - d}{2}\right)}{\prod_i \Gamma(\tilde{\Delta}_i)} \prod_{i<j} \frac{\Gamma(\delta_{ij}-\alpha_{ij})}{\Gamma(\delta_{ij})}$$

$$\tilde{\Delta}_i + \sum_j \alpha_{ij} = \Delta_i$$

$$D_{11112} = \frac{4\pi^2}{x_{14}^2 x_{35}^2 x_{25}^2} \sum_{i=1}^{5} \frac{\eta_{i5}\hat{I}_4^{(i)}}{N_5}$$

$$u_1 = \frac{(w-1)(\bar{w}-1)z\bar{z}}{(w-z)(\bar{w}-\bar{z})+\lambda}, \quad u_2 = (z-1)(\bar{z}-1)$$
$$u_3 = \frac{(w-z)(\bar{w}-\bar{z})+\lambda}{(w-1)(\bar{w}-1)}, \quad u_4 = \frac{1}{(w-1)(\bar{w}-1)}, u_5 = \frac{w\bar{w}}{(w-z)(\bar{w}-\bar{z})+\lambda}$$

$$\sum_{i=1}^{5} \eta_{i5}\hat{I}_4^{(i)}\bigg|_{\lambda\to 0} = 0$$

$$\mathrm{Li}_2\left(\frac{zw}{(1-z)(1-w)}\right) = \mathrm{Li}_2\left(\frac{z}{1-w}\right) + \mathrm{Li}_2\left(\frac{w}{1-z}\right) - \mathrm{Li}_2(z) - \mathrm{Li}_2(w) - \log(1-z)\log(1-w)$$

$$C_{OOJ} = 2C_{OOO}, C_{OOT} = \frac{2}{7}C_{OOO}$$

$$s_{ij} = \Delta_i + \Delta_j - 2\delta_{ij}$$

$$A(S_{ij},\epsilon_{\mu\nu}) = \Gamma\left(\frac{\Delta_\Sigma}{2}-2\right) \lim_{L\to\infty} L^{15/2} V_5 \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \alpha^{2-\frac{\Delta_\Sigma}{2}} e^{\alpha} M\left(s_{ij} = \frac{L^2}{2\alpha}S_{ij}, t_i\right)$$

$$\frac{L^4}{\ell_s^4} = \lambda = g_{\mathrm{YM}}^2 N, g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}$$

$$A(S_{ij},\epsilon_{\mu\nu}) \sim k_{10}^3 \tilde{A}_{54}^T \cdot S_0 \cdot \left[1 + 2\zeta_3\left(\frac{\alpha'}{2}\right)^3 M_3 + 2\zeta_5\left(\frac{\alpha'}{2}\right)^5 M_5 + O(\alpha'^6)\right] \cdot A_{45}$$

$$\tilde{A}_{54} \equiv \begin{pmatrix} \tilde{A}^{\mathrm{YM}}(1,2,3,5,4) \\ \tilde{A}^{\mathrm{YM}}(1,3,2,5,4) \end{pmatrix}, A_{45} \equiv \begin{pmatrix} A^{\mathrm{YM}}(1,2,3,4,5) \\ A^{\mathrm{YM}}(1,3,2,4,5) \end{pmatrix}$$

$$\epsilon_{\mu\nu} = t_\mu \tilde{t}_\nu$$

$$S_0 \equiv \frac{1}{4}\begin{pmatrix} S_{12}(S_{13}+S_{23}) & S_{12}S_{13} \\ S_{12}S_{13} & S_{13}(S_{12}+S_{23}) \end{pmatrix}$$

$$M_3 \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$m_{12}=\frac{S_{13}S_{24}}{8}(S_{12}+S_{23}+S_{34}+S_{45}+S_{15})$$
$$m_{11}=\frac{S_{34}}{8}[S_{12}(S_{12}+2S_{23}+S_{34})-S_{34}S_{45}-S_{45}^2]-\frac{S_{12}S_{15}}{8}(S_{12}+S_{15})$$

$$m_{21}=m_{12}|_{2\leftrightarrow 3} \text{ and } m_{22}=m_{11}|_{2\leftrightarrow 3}$$

$$\frac{M^{R^4}}{960\sqrt{2}\zeta_3}=\sum_{a=1}^{12} M_a^P P^a+\sum_{a=1}^{10} M_a^T T^a$$

$$M_1^P=M_1^{P,\text{sing}}+M_1^{P,reg}$$

$$\begin{aligned}
M_1^{P,sing} = & -\frac{2(\delta_{12}+\delta_{23}-2)}{\delta_{45}-1}(\delta_{12}(\delta_{23}(4\delta_{23}-5\delta_{34}-7)-5\delta_{15}(\delta_{23}-3)+4\delta_{34}-6)\\
&+\delta_{12}^2(4\delta_{23}-5\delta_{15}+1)+\delta_{15}(4\delta_{23}-9)+(5(3-\delta_{23})\delta_{23}-9)\delta_{34}+(\delta_{23}-3)^2)\\
&-\frac{2(\delta_{23}+\delta_{34}-2)}{\delta_{15}-1}(\delta_{12}(4\delta_{34}-5\delta_{23}(\delta_{23}+\delta_{34}-3)-9)+\delta_{23}(\delta_{34}(4\delta_{34}-5\delta_{45}-7)\\
&+4\delta_{45}-6)+\delta_{23}^2(4\delta_{34}+1)+(5(3-\delta_{34})\delta_{34}-9)\delta_{45}+(\delta_{34}-3)^2)\\
&-\frac{2(\delta_{12}+\delta_{15}-2)}{\delta_{34}-1}(\delta_{12}(\delta_{15}(4\delta_{15}-5\delta_{23}-5\delta_{45}-7)+15\delta_{23}+4\delta_{45}-6)\\
&+\delta_{12}^2(4\delta_{15}-5\delta_{23}+1)+\delta_{15}(5(3-\delta_{15})\delta_{45}+\delta_{15}+4\delta_{23}-6)-9(\delta_{23}+\delta_{45}-1)\big)\\
&-\frac{2(\delta_{15}+\delta_{45}-2)}{\delta_{23}-1}(\delta_{12}(4\delta_{45}-5\delta_{15}(\delta_{15}+\delta_{45}-3)-9)+\delta_{45}^2(4\delta_{15}-5\delta_{34}+1)\\
&+\delta_{45}(\delta_{15}(4\delta_{15}-5\delta_{34}-7)+15\delta_{34}-6)+4\delta_{15}\delta_{34}+\delta_{15}^2-6\delta_{15}-9\delta_{34}+9)\\
&-\frac{2(\delta_{34}+\delta_{45}-2)}{\delta_{12}-1}(\delta_{15}(\delta_{34}(4-5\delta_{45})-5(\delta_{45}-3)\delta_{45}-9)+\delta_{23}(5\delta_{34}(3-\delta_{34}-\delta_{45})\\
&+4\delta_{45}-9)+(4\delta_{34}+1)\delta_{45}^2+(\delta_{34}(4\delta_{34}-7)-6)\delta_{45}+(\delta_{34}-3)^2)
\end{aligned}$$

$$\begin{aligned}
M_1^{P,\text{reg}} = & \, 23(\delta_{12}+\delta_{15}+\delta_{23}+\delta_{34}+\delta_{45})-54(\delta_{12}(\delta_{15}+\delta_{23})+\delta_{45}(\delta_{15}+\delta_{34})+\delta_{23}\delta_{34})\\
&-103(\delta_{12}^2+\delta_{15}^2+\delta_{23}^2+\delta_{34}^2+\delta_{45}^2)+31(\delta_{12}^2(\delta_{15}+\delta_{23})+\delta_{12}(\delta_{15}^2+\delta_{23}^2)+\delta_{45}^2(\delta_{15}+\delta_{34})\\
&+\delta_{45}(\delta_{15}^2+\delta_{34}^2)+\delta_{23}\delta_{34}(\delta_{23}+\delta_{34})\big)+124(\delta_{12}(\delta_{34}+\delta_{45})+\delta_{15}(\delta_{23}+\delta_{34})+\delta_{23}\delta_{45})\\
&-30\big(\delta_{12}\big(\delta_{15}\delta_{34}+\delta_{45}(\delta_{23}+\delta_{34})\big)+\delta_{15}\delta_{23}(\delta_{34}+\delta_{45})+1\big)-8(\delta_{12}^2(\delta_{34}+\delta_{45})+\delta_{12}(\delta_{34}^2+\delta_{45}^2)\\
&+\delta_{15}^2(\delta_{23}+\delta_{34})+\delta_{15}(\delta_{23}^2+\delta_{34}^2)+\delta_{23}\delta_{45}(\delta_{23}+\delta_{45})\big)+18(-\delta_{12}(\delta_{15}(\delta_{23}+\delta_{45})+\delta_{23}\delta_{34})\\
&+\delta_{12}^3-\delta_{34}\delta_{45}(\delta_{15}+\delta_{23})+\delta_{15}^3+\delta_{23}^3+\delta_{34}^3+\delta_{45}^3\big)
\end{aligned}$$

$$M_1^T=M_1^{T,\text{sing}}+M_1^{T,\text{reg}}$$

$$\begin{aligned}
M_1^{T,sing} = & \frac{2(\delta_{23}-\delta_{15}+\delta_{34}-1)^2(-\delta_{12}-\delta_{15}+\delta_{23}+\delta_{34}+1)^2}{\delta_{35}-1}\\
&+\frac{2\delta_{15}^2(\delta_{12}+\delta_{15}-2)^2}{\delta_{34}-1}+\frac{2\delta_{23}^2(\delta_{12}+\delta_{23}-2)^2}{\delta_{45}-1}+\frac{2}{\delta_{12}-1}(\delta_{15}(\delta_{34}(20(1-\delta_{45})\delta_{45}-6)\\
&+2\delta_{23}(2\delta_{34}(5\delta_{45}-3)-6\delta_{45}+3)+3\delta_{45}(2\delta_{45}-1)\big)+2\delta_{15}^2(\delta_{45}(5\delta_{45}-7)+3)\\
&-\delta_{45}(\delta_{23}(20(\delta_{34}-1)\delta_{34}+6)+\delta_{34}(\delta_{34}+15)-11)+(\delta_{23}+1)(2\delta_{23}(\delta_{34}(5\delta_{34}-7)+3)\\
&-4\delta_{34}^2+11\delta_{34}-6)+(\delta_{34}(10\delta_{34}-1)-4)\delta_{45}^2)
\end{aligned}$$

$$\begin{aligned}
M_1^{T,\text{reg}} = & \, \delta_{12}(\delta_{15}(8\delta_{23}+18\delta_{34}-22\delta_{45}+31)-15\delta_{15}^2+\delta_{23}(14\delta_{45}-26\delta_{34}+39)-15\delta_{23}^2\\
&-2\delta_{34}(\delta_{34}-16\delta_{45}+9)-22\delta_{45})+\delta_{12}^2(-7\delta_{15}-7\delta_{23}+3)+3\delta_{15}^2(4\delta_{23}+4\delta_{34}-9\delta_{45}+17)\\
&+\delta_{15}(\delta_{45}(14\delta_{23}+44\delta_{34}-3)-2(\delta_{23}(5\delta_{34}+17)+6\delta_{23}^2+\delta_{34}(4\delta_{34}+31)+3)+15\delta_{45}^2)\\
&-8\delta_{15}^3+\delta_{45}(\delta_{23}(36\delta_{34}-62)-\delta_{34}(27\delta_{34}+7)+24)+23\delta_{23}\delta_{34}^2-15\delta_{23}^2\delta_{34}-3\delta_{23}\delta_{34}\\
&+51\delta_{23}^2-14\delta_{23}+5(3-5\delta_{34})\delta_{45}^2+2\delta_{34}^3+15\delta_{34}^2+20\delta_{34}-11
\end{aligned}$$

$$t_{ij}=y_{ij}^2$$

$$\mathbb{O}_2=O_2+(\rho\sigma^\mu\bar{\rho})J_\mu+(\rho\sigma^\mu\bar{\rho})(\rho\sigma^\nu\bar{\rho})T_{\mu\nu}+\cdots$$

$$J=\frac{1}{2}\mathcal{D}_J\mathbb{O}_2(x,y,\rho,\bar{\rho})\Big|_{\rho,\bar{\rho}=0}$$
$$T=\frac{1}{4}\mathcal{D}_T\mathbb{O}_2(x,y,\rho,\bar{\rho})\Big|_{\rho,\bar{\rho}=0}$$

$$\mathcal{D}_J=\lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}v^{a}\bar{v}^{\dot{a}}\left(\frac{\partial}{\partial\bar{\rho}^{a\dot{\alpha}}}\frac{\partial}{\partial\rho^{\alpha\dot{a}}}+\frac{1}{2}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\right)$$
$$\mathcal{D}_T=\lambda^{\alpha_1}\lambda^{\alpha_2}\bar{\lambda}^{\dot{\alpha}_1}\bar{\lambda}^{\dot{\alpha}_2}\epsilon^{\dot{a}_1\dot{a}_2}\epsilon^{a_1a_2}\times\left(\frac{\partial}{\partial\bar{\rho}^{a_1\dot{\alpha}_1}}\frac{\partial}{\partial\bar{\rho}^{a_2\dot{\alpha}_2}}\frac{\partial}{\partial\rho^{\alpha_1\dot{a}_1}}\frac{\partial}{\partial\rho^{\alpha_2\dot{a}_2}}\right.$$
$$\left.-\frac{\partial}{\partial\bar{\rho}^{a_1\dot{\alpha}_1}}\frac{\partial}{\partial\rho^{\alpha_1\dot{a}_1}}\frac{\partial}{\partial y^{a_2\dot{a}_2}}\frac{\partial}{\partial x^{\alpha_2\dot{\alpha}_2}}-\frac{1}{6}\frac{\partial}{\partial y^{a_1\dot{a}_1}}\frac{\partial}{\partial y^{a_2\dot{a}_2}}\frac{\partial}{\partial x^{\alpha_1\dot{\alpha}_1}}\frac{\partial}{\partial x^{\alpha_2\dot{\alpha}_2}}\right)$$

$$\langle J(1)O_2(2)O_2(3)O_2(4)\rangle=\frac{1}{2}\mathcal{D}_J(X_1)\langle\mathbb{O}(X_1)\mathbb{O}_2(X_2)\mathbb{O}_2(X_3)\mathbb{O}_2(X_4)\rangle\Big|_{\rho,\bar{\rho}=0},$$
$$\langle T(1)O_2(2)O_2(3)O_2(4)\rangle=\frac{1}{4}\mathcal{D}_T(X_1)\langle\mathbb{O}_2(X_1)\mathbb{O}_2(X_2)\mathbb{O}_2(X_3)\mathbb{O}_2(X_4)\rangle\Big|_{\rho,\bar{\rho}=0},$$

$$\langle J(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle=\frac{1}{x_{12}^4x_{34}^4}\sum_{k=2}^{4}\alpha^{(k)}(u,v;y_{ij},Y_{1,ij})\frac{z\cdot x_{1k}}{x_{1k}^2},$$
$$\langle T(1)\mathcal{O}_2(2)\mathcal{O}_2(3)\mathcal{O}_2(4)\rangle=\frac{1}{x_{12}^4x_{34}^4}\sum_{k,l=2}^{4}\beta^{(k,l)}(u,v;y_{ij})\frac{z\cdot x_{1k}}{x_{1k}^2}\frac{z\cdot x_{1l}}{x_{1l}^2},$$

$$Y_{1,ij}=y_{1i}^2y_{1j}^2V_{1,ij},V_{i,jk}=\bar{v}_i\left(y_{ij}^{-1}-y_{ik}^{-1}\right)v_i$$

$$\langle\mathcal{J}(1)O_2(2)O_2(3)O_2(4)\rangle=\sum_{k=2}^{4}\frac{z\cdot x_{1k}}{x_{1k}^2}\int[d\delta]M^k\prod_{i=2}^{4}\frac{\Gamma(\delta_i+\boldsymbol{\delta}_i^k)}{x_{1i}^{2\delta_i}}\prod_{i<j}\frac{\Gamma(\delta_{ij})}{x_{ij}^{2\delta_{ij}}}$$
$$\langle T(1)O_2(2)O_2(3)O_2(4)\rangle=\sum_{k,l=2}^{4}\frac{z\cdot x_{1k}}{x_{1k}^2}\frac{z\cdot x_{1l}}{x_{1l}^2}\int[d\delta]M^{kl}\prod_{i=2}^{4}\frac{\Gamma(\delta_i+\boldsymbol{\delta}_i^k+\boldsymbol{\delta}_i^l)}{x_{1i}^{2\delta_i}}\prod_{i<j}\frac{\Gamma(\delta_{ij})}{x_{ij}^{2\delta_{ij}}}$$

$$\delta_{ij}=\delta_{ji},\delta_{ii}=-\Delta_i,\delta_i=-\sum_{j=2}^{4}\delta_{ij},\sum_{i,j=2}^{4}\delta_{ij}=J-\Delta_1$$

$$\alpha^{(k)}(u,v;y_{ij},Y_{1,ij})=\int\frac{dsdt}{4}u^{\frac{s}{2}}v^{\frac{t-4}{2}}M^k(s,t;y_{ij},Y_{1,ij})\prod_{i=2}^{4}\Gamma(\delta_i+\boldsymbol{\delta}_i^k)\prod_{i<j}\Gamma(\delta_{ij})$$
$$\beta^{(k,l)}(u,v;y_{ij})=\int\frac{dsdt}{4}u^{\frac{s}{2}}v^{\frac{t-4}{2}}M^{kl}(s,t;y_{ij})\prod_{i=2}^{4}\Gamma(\delta_i+\boldsymbol{\delta}_i^k+\boldsymbol{\delta}_i^l)\prod_{i<j}\Gamma(\delta_{ij})$$

$$M^k(s,t;y_{ij},Y_{1,ij})\prod_{i=2}^{4}\Gamma(\delta_i+\boldsymbol{\delta}_i^k)\prod_{i<j}\Gamma(\delta_{ij})=\int_0^{\infty}dudvu^{-\frac{s}{2}-1}v^{1-\frac{t}{2}}\alpha^{(k)}(u,v;y_{ij},Y_{1,ij})$$
$$M^{kl}(s,t;y_{ij})\prod_{i=2}^{4}\Gamma(\delta_i+\boldsymbol{\delta}_i^k+\boldsymbol{\delta}_i^l)\prod_{i<j}\Gamma(\delta_{ij})=\int_0^{\infty}dudvu^{-\frac{s}{2}-1}v^{1-\frac{t}{2}}\beta^{(k,l)}(u,v;y_{ij})$$

$$\int_0^{\infty}du\int_0^{\infty}dvu^{-\frac{s}{2}-1}v^{1-\frac{t}{2}}u^mv^n\frac{\partial^a}{\partial u^a}\frac{\partial^b}{\partial v^b}H(u,v)=\mathcal{M}(s-2m+2a,t-2n+2b)$$
$$\times(-1)^{a+b}\left(m-a-\frac{s}{2}\right)_a\left(n-b+\frac{4-t}{2}\right)_b\prod_{i<j}\Gamma(\tilde{\delta}_{ij}(s-2m-2a,t-2n-2b))$$

$$M^2 = \frac{30\zeta_3}{\lambda^{3/2}}\big((t-4)^2(s+3t-13)t_{24}t_{34}Y_{1,23} - (s+t-4)^2(2s+3t-11)t_{23}t_{34}Y_{1,24} + (s-4)(2s-9)(s+2t-8)t_{23}t_{24}Y_{1,34}\big) - \frac{2(t-4)^2t_{24}t_{34}Y_{1,23}}{(s-2)(s+t-6)} + \frac{2(s+t-4)^2t_{23}t_{34}Y_{1,24}}{(s-2)(t-2)} - \frac{2(s-4)(s+2t-8)t_{23}t_{24}Y_{1,34}}{(t-2)(s+t-6)}$$

$$M^3 = \frac{30\zeta_3}{\lambda^{3/2}}\big((s-4)^2(2s-t-5)t_{23}t_{24}Y_{1,34} + (t-4)^2(s-2t+5)t_{24}t_{34}Y_{1,23} - (s-t)(s+t-4)(2s+2t-7)t_{23}t_{34}Y_{1,24}\big) - \frac{2(s-4)^2t_{23}t_{24}Y_{1,34}}{(t-2)(s+t-6)} + \frac{2(t-4)^2t_{24}t_{34}Y_{1,23}}{(s-2)(s+t-6)} + \frac{2(s-t)(s+t-4)t_{23}t_{34}Y_{1,2,4}}{(s-2)(t-2)}$$

$$M^4 = \frac{30\zeta_3}{\lambda^{3/2}}\big((s+t-4)^2(3s+2t-11)t_{23}t_{34}Y_{1,24} - \big((s-4)^2(3s+t-13)t_{23}t_{24}Y_{1,34}\big) - (t-4)(2t-9)(2s+t-8)t_{24}t_{34}Y_{1,23}\big) + \frac{2(s-4)^2t_{23}t_{24}Y_{1,34}}{(t-2)(s+t-6)} + \frac{2(t-4)(2s+t-8)t_{24}t_{34}Y_{1,23}}{(s-2)(s+t-6)} - \frac{2(s+t-4)^2t_{23}t_{34}Y_{1,24}}{(s-2)(t-2)}$$

$$M^{22} = -\frac{80\zeta_3}{\lambda^{3/2}}(s^2(2t-11) + s(t(2t-39)+118) - 23(t-4)^2)t_{23}t_{24}t_{34} + \frac{8(s-4)(t-4)(s+t-4)t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$M^{23} = -\frac{80\zeta_3}{\lambda^{3/2}}(s^2(2t-11) + s(t-8)(2t-11) + t(13t-35) - 32)t_{23}t_{24}t_{34} + \frac{8(t-4)(s(s+t-8)+2(t+2))t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$M^{24} = -\frac{80\zeta_3}{\lambda^{3/2}}(s^2(2t+13) + s(t(2t+21)-173) + t(13t-173) + 520)t_{23}t_{24}t_{34} + \frac{8(s+t-4)(s(t+2)+2(t-10))t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$M^{33} = -\frac{80\zeta_3}{\lambda^{3/2}}(s^2(2t-11) + s(t(2t-15)+58) + (58-11t)t - 128)t_{23}t_{24}t_{34} + \frac{8(s-4)(t-4)(s+t-4)t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$M^{34} = -\frac{80\zeta_3}{\lambda^{3/2}}(s^2(2t+13) + s(t(2t-27)-35) - 11(t-8)t - 32)t_{23}t_{24}t_{34} + \frac{8(s-4)(s(t+2)+(t-8)t+4)t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$M^{44} = -\frac{80\zeta_3}{\lambda^{3/2}}((2s-11)t^2 + (s(2s-39)+118)t - 23(s-4)^2)t_{23}t_{24}t_{34} + \frac{8(s-4)(t-4)(s+t-4)t_{23}t_{24}t_{34}}{3(s-2)(t-2)(s+t-6)}$$

$$D_{\Delta_1,\dots,\Delta_n} = \int \frac{dz_0 d^d z}{z_0^{d+1}} \prod_{i=1}^{n} \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i)^2}\right)^{\Delta_i}$$

$$D_{\Delta_1,\dots,\Delta_i+1,\dots,\Delta_j+1,\dots,\Delta_n} = \frac{d/2-\Sigma}{\Delta_i\Delta_j}\frac{\partial}{\partial x_{ij}^2}D_{\Delta_1,\dots,\Delta_n},$$

$$\Sigma = \frac{1}{2}\sum_{i=1}^{n}\Delta_i$$

$$\frac{\pi^{d/2}\Gamma\left(\Sigma-\frac{d}{2}\right)\Gamma(\Sigma)}{2\prod_i \Gamma(\Delta_i)}\int_0^1 \prod_{j=1}^{n} da_j\,\delta\left(1-\sum_j a_j\right)\frac{\prod_j a_j^{\Delta_j-1}}{\left(\sum_{i<j} a_i a_j x_{ij}^2\right)^{\Sigma}}$$

$$\Gamma\left(\Sigma-\frac{d}{2}\right)$$

$$I_4[1]=\Phi(z,\bar{z})=\frac{1}{z-\bar{z}}\left[2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})+\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right]$$

$$I_n[P(\{a_i\})]=\Gamma(n-2)\int_0^1\prod_{j=1}^{n}da_j\delta\left(1-\sum_j a_j\right)\frac{P(\{a_i\})}{\left(\sum_{i<j}a_ia_jx_{ij}^2\right)^{n-2}}$$

$$D_{11112}=\frac{4\pi^2}{x_{14}^2x_{35}^2x_{25}^2}\sum_{i=1}^{5}\frac{\eta_{i5}\hat{I}_4^{(i)}}{N_5}$$

$$\rho=N_n\eta^{-1},N_n=2^{n-1}\det\rho$$

$$\rho=\begin{pmatrix}0&u_1&1&1&u_5\\u_1&0&u_2&1&1\\1&u_2&0&u_3&1\\1&1&u_3&0&u_4\\u_5&1&1&u_4&0\end{pmatrix},$$

$$\hat{I}_4^{(5)}=\Phi(u_1u_3,u_2)$$

$$u_1u_3=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}=z\bar{z},u_2=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}=(1-z)(1-\bar{z})$$

$$\partial_z\Phi=\frac{\Phi}{\bar{z}-z}+\frac{\log[(1-z)(1-\bar{z})]}{z(\bar{z}-z)}+\frac{\log(z\bar{z})}{(z-1)(z-\bar{z})}$$
$$\partial_{\bar{z}}\Phi=\frac{\Phi}{z-\bar{z}}+\frac{\log[(1-z)(1-\bar{z})]}{\bar{z}(z-\bar{z})}+\frac{\log(z\bar{z})}{(\bar{z}-1)(\bar{z}-z)}$$

$$I_{Sp(2N)}^{SCI}=\chi_N'\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\prod_{k=1}^{3}\Gamma\left(s_kz_i^{\pm1}z_j^{\pm1};p,q\right)}{\Gamma\left(z_i^{\pm1}z_j^{\pm1};p,q\right)}\prod_{j=1}^{N}\frac{\prod_{k=1}^{3}\Gamma\left(s_kz_j^{\pm2};p,q\right)}{\Gamma\left(z_j^{\pm2};p,q\right)}\frac{dz_j}{2\pi iz_j}$$
$$I_{SO(2N+1)}^{SCI}=\chi_N'\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\prod_{k=1}^{3}\Gamma\left(s_kz_i^{\pm1}z_j^{\pm1};p,q\right)}{\Gamma\left(z_i^{\pm1}z_j^{\pm1};p,q\right)}\prod_{j=1}^{N}\frac{\prod_{k=1}^{3}\Gamma\left(s_kz_j^{\pm1};p,q\right)}{\Gamma\left(z_j^{\pm1};p,q\right)}\frac{dz_j}{2\pi iz_j}$$
$$I_{SO(2N)}^{SCI}=\chi_N'\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\prod_{k=1}^{3}\Gamma\left(s_kz_i^{\pm1}z_j^{\pm1};p,q\right)}{\Gamma\left(z_i^{\pm1}z_j^{\pm1};p,q\right)}\prod_{j=1}^{N}\frac{dz_j}{2\pi iz_j}$$

$$\Gamma(a,b;p,q):=\Gamma(a;p,q)\Gamma(b;p,q),\Gamma(az^{\pm1};p,q):=\Gamma(az;p,q)\Gamma(az^{-1};p,q)$$
$$\Gamma(z_i^{\pm}z_j^{\pm};p,q):=\Gamma\left(z_i^{+1}z_j^{+1};p,q\right)\Gamma\left(z_i^{+1}z_j^{-1};p,q\right)\Gamma\left(z_i^{-1}z_j^{+1};p,q\right)\Gamma\left(z_i^{-1}z_j^{-1};p,q\right)$$

$$\chi_N'=\begin{cases}\dfrac{(p;p)_\infty^N(q;q)_\infty^N}{2^NN!}\displaystyle\prod_{k=1}^{3}\Gamma^N(s_k;p,q) & Sp(2N)\text{ or }SO(2N+1)\\ \dfrac{(p;p)_\infty^N(q;q)_\infty^N}{2^{N-1}N!}\displaystyle\prod_{k=1}^{3}\Gamma^N(s_k;p,q) & SO(2N)\end{cases}$$

$$\Gamma(a,b;p,q)=1\ (\text{ if } ab=pq),\Gamma(z;p,q)\underset{p\to0}{=}\frac{1}{(z;q)_\infty}\underset{q\to0}{=}\frac{1}{1-z}$$

$$I_{Sp(2N)}(u,q,t) = \frac{(q;q)_\infty^N}{2^N N!}\frac{(tu;q)_\infty^N}{(t,u;q)_\infty^N}$$

$$\times \oint_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\left(z_i^{\pm 1}z_j^{\pm 1}, tuz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty}{\left(tz_i^{\pm 1}z_j^{\pm 1}, uz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty} \prod_{j=1}^{N} \frac{\left(z_j^{\pm 2}, tuz_j^{\pm 2};q\right)_\infty}{\left(tz_j^{\pm 2}, uz_j^{\pm 2};q\right)_\infty} \frac{dz_j}{2\pi i z_j},$$

$$I_{SO(2N+1)}(u,q,t) = \frac{(q;q)_\infty^N}{2^N N!}\frac{(tu;q)_\infty^N}{(t,u;q)_\infty^N}$$

$$\times \oint_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\left(z_i^{\pm 1}z_j^{\pm 1}, tuz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty}{\left(tz_i^{\pm 1}z_j^{\pm 1}, uz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty} \prod_{j=1}^{N} \frac{\left(z_j^{\pm 1}, tuz_j^{\pm 1};q\right)_\infty}{\left(tz_j^{\pm 1}, uz_j^{\pm 1};q\right)_\infty} \frac{dz_j}{2\pi i z_j},$$

$$I_{SO(2N)}(u,q,t) = \frac{(q;q)_\infty^N}{2^{N-1}N!}\frac{(tu;q)_\infty^N}{(t,u;q)_\infty^N} \times \oint_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\left(z_i^{\pm 1}z_j^{\pm 1}, tuz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty}{\left(tz_i^{\pm 1}z_j^{\pm 1}, uz_i^{\pm 1}z_j^{\pm 1};q\right)_\infty} \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j}$$

$$I_{U(N)}(t,u;q) = \frac{1}{N!}\frac{(q;q)_\infty^N (tu;q)_\infty^N}{(t;q)_\infty^N (u;q)_\infty^N} \oint_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} \prod_{1\le i\ne j\le N} \frac{(x_i/x_j;q)_\infty (tux_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty (ux_i/x_j;q)_\infty}$$

$$w(\mathbf{x}) = \prod_{1\le i\ne j\le N} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}$$

$$\prod_{i,j=1}^{N} \frac{(tux_i/x_j;q)_\infty}{(ux_i/x_j;q)_\infty} = \sum_{\ell(\lambda)\le N} u^{|\lambda|} b_\lambda P_\lambda(\mathbf{x};q,t) P_\lambda(\mathbf{x}^{-1};q,t)$$

$$\frac{1}{N!}\oint_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} w(\mathbf{x}) P_\lambda(\mathbf{x};q,t) P_\mu(\mathbf{x}^{-1};q,t) = \mathcal{N}_{\lambda,N}\delta_{\lambda\mu}$$

$$\mathcal{N}_{\lambda,N} = \prod_{1\le i<j\le N} \frac{\left(t^{j-i}q^{\lambda_i-\lambda_j+1};q\right)_\infty \left(t^{j-i}q^{\lambda_i-\lambda_j};q\right)_\infty}{\left(t^{j-i+1}q^{\lambda_i-\lambda_j};q\right)_\infty \left(t^{j-i-1}q^{\lambda_i-\lambda_j+1};q\right)_\infty}$$

$$I_N(t,u;q) = \frac{(q;q)_\infty^N}{(t;q)_\infty^N} \sum_{\ell(\lambda)\le N} u^{|\lambda|} b_\lambda \mathcal{N}_{\lambda,N}$$

$$\Delta_n(x_1,x_2,\dots,x_n;q,t;t_0,t_1,t_2,t_3) = \prod_{1\le i\le n} \frac{(x_i^2;q)_\infty}{(t_0x_i,t_1x_i,t_2x_i,t_3x_i;q)_\infty} \prod_{1\le i<j\le n} \frac{(x_ix_j^{\pm};q)_\infty}{(tx_ix_j^{\pm};q)_\infty}.$$

$$\lim_{m\to\infty} (x_1\cdots x_n)^m K_{m^n-\lambda}(\mathbf{x};q,t;t_0,t_1,t_2,t_3) = P_\lambda(\mathbf{x};q,t) \prod_{i=1}^{n} \frac{(t_0x_i,t_1x_i,t_2x_i,t_3x_i;q)_\infty}{(x_i^2;q)_\infty} \prod_{1\le i<j\le n} \frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}$$

$$I_n'(u,q,t;t_0,t_1,t_2,t_3) = \mathcal{X}_n \lim_{m\to\infty} \sum_{\lambda} b_\lambda(q,t) \oint_{\mathbb{T}^n} u^{mn} \Delta_n(\mathbf{x};q,t;t_0,t_1,t_2,t_3) \Delta_n(\mathbf{x}^{-1};q,t;t_0,t_1,t_2,t_3)$$

$$\times K_{m^n-\lambda}(\sqrt{u}\mathbf{x};q,t;t_0,t_1,t_2,t_3) K_{m^n-\lambda}(\sqrt{u}\mathbf{x}^{-1};q,t;t_0,t_1,t_2,t_3) \prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j}$$

$$(t_0,t_1,t_2,t_3) = (+\sqrt{t},-\sqrt{t},+\sqrt{qt},-\sqrt{qt}), \quad \mathcal{X}_n = \frac{(q;q)_\infty^n}{2^n n!\,(t;q)_\infty^n} \quad \text{for } C_n$$

$$(t_0,t_1,t_2,t_3) = (-1,+t,-\sqrt{q},+\sqrt{q}), \quad \mathcal{X}_n = \frac{(q;q)_\infty^n}{2^n n!\,(t;q)_\infty^n} \quad \text{for } B_n$$

$$(t_0,t_1,t_2,t_3) = (-1,+1,-\sqrt{q},+\sqrt{q}), \quad \mathcal{X}_n = \frac{(q;q)_\infty^n}{2^{n-1} n!\,(t;q)_\infty^n} \quad \text{for } D_n$$

$$\sum_{\lambda \text{ is even}} b^{\text{oa}}_{\lambda;m}(q,t)P_\lambda(x;q,t) = (x_1\cdots x_n)^m K_{m^n}(x;q,t;+\sqrt{q},-\sqrt{q},+\sqrt{qt},-\sqrt{qt}),$$

$$\sum_{\lambda\subset m^n} b^{\text{el}}_{\lambda;2m}(q,t)P_\lambda(x;q,t) = (x_1\cdots x_n)^m K_{m^n}(x;q,t;-1,+t,-\sqrt{q},+\sqrt{q}),$$

$$\sum_{\substack{\lambda\subset m^n\\ m_i(\lambda) \text{ is even}}} b^{\text{ol}}_{\lambda;m}(q,t)P_\lambda(x;q,t) = (x_1\cdots x_n)^m K_{m^n}(x;q,t;-1,+1,-\sqrt{q},+\sqrt{q}),$$

$$\sum_{\lambda\subset m^n} a^{\text{odd}\,(\lambda')} b^{\text{el}}_{\lambda;2m}(q,t)P_\lambda(x;q,t) = \prod_{i=1}^{n}\frac{(tax_i,q)_\infty}{(ax_i,q)_\infty}(x_1\cdots x_n)^m K_{m^n}(x;q,t;-1,+1,-\sqrt{q},+\sqrt{q}),$$

$$b_\lambda(s;q,t) := \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t_\lambda(s)}, b_\lambda(q,t) := \prod_{s\in\lambda} b_\lambda(s;q,t),$$

$$b^{\text{oa}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ a(s)\text{ odd}}} b_\lambda(s;q,t), b^{\text{el}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ l(s)\text{ even}}} b_\lambda(s;q,t), b^{\text{ol}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ l(s)\text{ odd}}} \frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)-1}},$$

$$b^{\text{oa}}_{\lambda;m}(q,t) := b^{\text{oa}}_\lambda(q,t)\prod_{\substack{s\in\lambda\\ a'(s)\text{ odd}}} \frac{1-q^{2m-a'(s)+1}t^{l'(s)}}{1-q^{2m-a'(s)}t^{l'(s)+1}},$$

$$b^{\text{ol}}_{\lambda;m}(q,t) := b^{\text{ol}}_\lambda(q,t)\prod_{\substack{s\in\lambda\\ l'(s)\text{ odd}}} \frac{1-q^{m-a'(s)}t^{l'(s)-1}}{1-q^{m-a'(s)-1}t^{l'(s)}},$$

$$b^{\text{el}}_{\lambda;m}(q,t) := b^{\text{el}}_\lambda(q,t)\prod_{\substack{s\in\lambda\\ l'(s)\text{ even}}} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1-q^{m-a'(s)-1}t^{l'(s)+1}},$$

$$b^{\text{oa}}_{\lambda;\infty}(q,t) = b^{\text{oa}}_\lambda(q,t), b^{\text{ol}}_{\lambda;\infty}(q,t) = b^{\text{ol}}_\lambda(q,t), b^{\text{el}}_{\lambda;\infty}(q,t) = b^{\text{el}}_\lambda(q,t),$$

$$\sum_{|\lambda| \text{ is even}} b^{\text{e}}_{\lambda;m\to\infty}(q,t)P_\lambda(x;q,t) = (x_1\cdots x_n)^{m\to\infty} K_{(m\to\infty)^n}(x;q,t;+\sqrt{t},-\sqrt{t},+\sqrt{qt},-\sqrt{qt})$$

$$b^{\text{e}}_{[0]}(q,t) = 1, b^{\text{e}}_{[1,1]}(q,t) = \frac{(1-t)(q-t)}{(qt-1)(1-q)}, b^{\text{e}}_{[2]}(q,t) = \frac{(1-t)}{(1-q)}$$

$$b^{\text{e}}_{[1,1,1,1]}(q,t) = \frac{(1-t)(q-t)t(1-t^3)}{(qt-1)(1-q)(1-qt^3)}, b^{\text{e}}_{[2,1,1]}(q,t) = \frac{(1-t)^2(t-q)(1+qt^2)}{(qt-1)^2(1-q)(1-q^2t^2)}$$

$$b^{\text{e}}_{[3,1]}(q,t) = \frac{(1-t)^2(t-q)q}{(1-q)^2(1-q^3t)}, b^{\text{e}}_{[4]}(q,t) = \frac{(1-t)(1-qt)}{(1-q)(1-q^2)}$$

$$\begin{aligned}I_{Sp(2N)}(u,q,t) = \mathcal{X}_N \sum_{\lambda}\sum_{|\mu|,|\nu| \text{ is even}} & u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t)b^{\text{e}}_\mu(q,t)b^{\text{e}}_\nu(q,t)\\ &\times \oint_{\mathbb{T}^N}\Delta_N(\mathbf{x};q,t;\pm\sqrt{t},\pm\sqrt{qt})\Delta_N(\mathbf{x}^{-1};q,t;\pm\sqrt{t},\pm\sqrt{qt})\\ &\times P_\lambda(\mathbf{x};q,t)P_\mu(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)P_\nu(\mathbf{x}^{-1};q,t)\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}\end{aligned}$$

$$\begin{aligned}I'_{SO(2N+1)}(u,q,t) = \mathcal{X}_N \sum_{\lambda}\sum_{\mu,\nu} & u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t)b^{\text{el}}_\mu(q,t)b^{\text{el}}_\nu(q,t)\\ &\times \oint_{\mathbb{T}^N}\Delta_N(\mathbf{x};q,t;-1,+t,-\sqrt{q},+\sqrt{q})\Delta_N(\mathbf{x}^{-1};q,t;-1,+t,-\sqrt{q},+\sqrt{q})\\ &\times P_\lambda(\mathbf{x};q,t)P_\mu(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)P_\nu(\mathbf{x}^{-1};q,t)\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}\end{aligned}$$

$$\begin{aligned}I_{SO(2N)}(u,q,t) = \mathcal{X}_N \sum_{\lambda}\sum_{m_i(\mu),m_i(\nu) \text{ is even}} & u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t)b^{\text{ol}}_\mu(q,t)b^{\text{ol}}_\nu(q,t)\\ &\times \oint_{\mathbb{T}^N}\Delta_N(\mathbf{x};q,t;-1,+1,-\sqrt{q},+\sqrt{q})\Delta_N(\mathbf{x}^{-1};q,t;-1,+1,-\sqrt{q},+\sqrt{q})\\ &\times P_\lambda(\mathbf{x};q,t)P_\mu(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)P_\nu(\mathbf{x}^{-1};q,t)\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}\end{aligned}$$

$$I_{SO(2N+1)}(u,q,t)=\mathcal{X}_N\sum_{\lambda}\sum_{\mu,\nu}u^{|\lambda|+\frac{|\mu|+|\nu|+odd(\mu')+odd(\nu')}{2}}b_\lambda(q,t)b_\mu^{\mathrm{el}}(q,t)b_\nu^{\mathrm{el}}(q,t)$$
$$\times\oint_{\mathbb{T}^N}\Delta_N(\mathbf{x};q,t;-1,+t,-\sqrt{q},+\sqrt{q})\Delta_N(\mathbf{x}^{-1};q,t;-1,+t,-\sqrt{q},+\sqrt{q})$$
$$\times P_\lambda(\mathbf{x};q,t)P_\mu(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)P_\nu(\mathbf{x}^{-1};q,t)\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}$$

$$\prod_{i=1}^{n}\frac{(tux_i;q)_\infty}{(ux_i;q)_\infty}$$

$$G(x;u)=\prod_{i=1}^{n}\frac{(tux_i;q)_\infty}{(x_iu;q)_\infty}=\sum_{l=0}^{\infty}\frac{(t;q)_l}{(q;q)_l}P_{[l]}(\mathbf{x};q,t)u^l$$

$$w_l(q,t):=\frac{(t;q)_l}{(q;q)_l}$$

$$I_{SO(2N+1)}(u,q,t)=\mathcal{X}_N\sum_{l=0}^{\infty}\sum_{\lambda}\sum_{m_i(\mu),m_i(\nu)\text{is even}}u^{|\lambda|+l+\frac{|\mu|+|\nu|}{2}}b_\lambda(q,t)b_\mu^{\mathrm{ol}}(q,t)b_\nu^{\mathrm{ol}}(q,t)w_l(q,t)$$
$$\times\oint_{\mathbb{T}^N}\Delta_N(\mathbf{x};q,t;-1,+t,-\sqrt{q},+\sqrt{q})\Delta_N(\mathbf{x}^{-1};q,t;-1,+t,-\sqrt{q},+\sqrt{q})$$
$$\times P_\lambda(\mathbf{x};q,t)P_\mu(\mathbf{x};q,t)P_{[l]}(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)P_\nu(\mathbf{x}^{-1};q,t)P_{[l]}(\mathbf{x}^{-1};q,t)\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}$$

$$(a,q)_0=1,(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\frac{(ax;q)_\infty}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n$$
$$(a;q^{-1})_n=(a^{-1};q)_n(-a)^nq^{\binom{n}{2}},(aq^k;q)_{n-k}=\frac{(a;q)_n}{(q;q)_k}$$
$$\frac{(a;q)_\infty(tua;q)_\infty}{(ta;q)_\infty(ua;q)_\infty}=\sum_{k=0}^{\infty}\left[\sum_{n=0}^{k}\frac{(t;q^{-1})_n(t;q)_{k-n}}{(q;q)_n(q;q)_{k-n}}(-1)^nu^{k-n}q^{-\binom{n}{2}}\right]a^k$$

$$c_k(u;q,t):=\sum_{n=0}^{k}\frac{(t;q^{-1})_n(t;q)_{k-n}}{(q;q)_n(q;q)_{k-n}}(-1)^nu^{k-n}q^{-\binom{n}{2}}$$
$$c_{n_1,n_2,\dots,n_m}(u;q,t):=c_{n_1}(u;q,t)c_{n_2}(u;q,t)\cdots c_{n_m}(u;q,t)$$
$$c_{n_1,n_2,\dots,n_i,0,\dots,0}(u;q,t)=c_{n_1,n_2,\dots,n_i}(u;q,t)$$

$$w(\mathbf{x})\sum_{\ell(\lambda)\le N}u^{|\lambda|}b_\lambda P_\lambda(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)$$
$$\prod_{1\le i<j\le N}\left[\sum_{k_{ij}^+=0}^{\infty}c_{k_{ij}^+}(u;q,t)(x_ix_j)^{k_{ij}^+}\right]\prod_{1\le i<j\le N}\left[\sum_{k_{ij}^-=0}^{\infty}c_{k_{ij}^-}(u;q,t)(x_ix_j)^{-k_{ij}^-}\right].$$

$$w(\mathbf{x})\sum_{\ell(\lambda)\le 2}u^{|\lambda|}b_\lambda P_\lambda(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)$$
$$\times\left[\sum_{n^+=0}^{\infty}(c_{n^+}(u,q,t))(x_1x_2)^{n^+}\right]\times\left[\sum_{n^-=0}^{\infty}(c_{n^-}(u,q,t))(x_1x_2)^{-n^-}\right]$$
$$P_\mu(x_1,\dots,x_n;q,t)\cdot x_1\cdots x_n=P_{\mu+1^n}(x_1,\dots,x_n;q,t)$$

$$I_{SO(4)}(u,q,t)=\mathcal{X}_2\sum_{\ell(\lambda)\le 2}\sum_{n}u^{|\lambda|}b_\lambda c_n^2(u,q,t)\mathcal{N}_{\lambda+n^2,2}$$

$$I_{SO(4)}(u,q,t)=\frac{1}{2}\sum_{n}c_n^2(u,q,t)I_{SU(2)}(u,q,t)$$

$$w(\mathbf{x}) \sum_{\ell(\lambda)\leq N} u^{|\lambda|} b_\lambda P_\lambda(\mathbf{x}; q,t) P_\lambda(\mathbf{x}^{-1}; q,t)$$
$$\times \left[\sum_{n_{12}^+, n_{13}^+, n_{23}^+ = 0}^{\infty} c_{n_{12}^+, n_{13}^+, n_{23}^+}(u,q,t) \mathbf{x}^{A(n_{12}^+, n_{13}^+, n_{23}^+)} \right] \left[\sum_{n_{12}^-, n_{13}^-, n_{23}^- = 0}^{\infty} c_{n_{12}^-, n_{13}^-, n_{23}^-}(u,q,t) (\mathbf{x}^{-1})^{A(n_{12}^-, n_{13}^-, n_{23}^-)} \right]$$

$$\mathbf{x}^{A(n_{12}^+, n_{13}^+, n_{23}^+)} := (x_1)^{n_{12}^+ + n_{13}^+} (x_2)^{n_{12}^+ + n_{23}^+} (x_3)^{n_{13}^+ + n_{23}^+}$$
$$(\mathbf{x}^{-1})^{A(n_{12}^-, n_{13}^-, n_{23}^-)} := (x_1^{-1})^{n_{12}^- + n_{13}^-} (x_2^{-1})^{n_{12}^- + n_{23}^-} (x_3^{-1})^{n_{13}^- + n_{23}^-}.$$

$$I_{SO(6)}(u,q,t) = \mathcal{X}_3 \sum_{\ell(\lambda)\leq 3} u^{|\lambda|} b_\lambda \sum_{n_{12}^+ \geq n_{13}^+ \geq n_{23}^+}^{\infty} \sum_{n_{12}^- \geq n_{13}^- \geq n_{23}^-}^{\infty} \frac{1}{r_{[n_{12}^+, n_{13}^+, n_{23}^+]}} \frac{1}{r_{[n_{12}^-, n_{13}^-, n_{23}^-]}} c_{n_{12}^+, n_{13}^+, n_{23}^+} c_{n_{12}^-, n_{13}^-, n_{23}^-}$$
$$\langle r_\mu m_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), r_\nu m_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle,$$

$$\mu = [n_{12}^+ + n_{13}^+, n_{12}^+ + n_{23}^+, n_{13}^+ + n_{23}^+], \nu = [n_{12}^- + n_{13}^-, n_{12}^- + n_{23}^-, n_{13}^- + n_{23}^-]$$
$$r_\lambda := \prod_{i=1}^{N} \sqrt[m_{\lambda_i}]{(\lambda)} \sqrt{m_{\lambda_i}(\lambda)!}$$

$$I_{SO(6)}(u,q,t) = \mathcal{X}_3 \sum_{\ell(\lambda)\leq 3} u^{|\lambda|} b_\lambda \sum_{n_{12}^+ \geq n_{13}^+ \geq n_{23}^+}^{\infty} \sum_{n_{12}^- \geq n_{13}^- \geq n_{23}^-}^{\infty} c_{n_{12}^+, n_{13}^+, n_{23}^+} c_{n_{12}^-, n_{13}^-, n_{23}^-} \langle m_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), m_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle.$$

$$I_{SO(6)}(u,q,t)$$
$$= \mathcal{X}_3 \Big[\mathcal{N}_{[0],3} + u b_{[1]} \mathcal{N}_{[1],3} + u^2 b_{[2]} \mathcal{N}_{[2],3} + (u^2 b_{[1,1]} + c_{1,0,0}^2) \mathcal{N}_{[1,1],3} + \left(u^3 b_{[2,1]} + u c_{1,0,0}^2 \frac{1-t}{1-q} \right) \mathcal{N}_{[2,1],3}$$
$$+ u^3 b_{[3]} \mathcal{N}_{[3],3} + \left(u^3 b_{[1,1,1]} + u c_{1,0,0}^2 \frac{1-t}{1-q} \left(\frac{-qt^2 - qt + t^2 - q + t + 1}{-qt^2 + 1} \right)^2 \right) \mathcal{N}_{[1,1,1],3}$$
$$+ \Bigg(u^4 b_{[2,1,1]} + u^2 c_{1,0,0}^2 \Bigg(b_{[1,1]} \Bigg(\left(\frac{qt + q - t - 1}{qt - 1} \right)^2 + b_{[2]} \left(\frac{q^3t^3 - qt^3 - q^2 + 1}{q^3t^3 - q^2t^2 - qt + 1} \right)^2 \Bigg) + c_{1,1,0}^2$$
$$+ c_{2,0,0}^2 \left(\frac{-qt + q - t + 1}{qt - 1} \right)^2 + 2 c_{1,1,0} c_{2,0,0} \frac{-qt + q - t + 1}{qt - 1} \Bigg) \mathcal{N}_{[2,1,1],3} + u^4 b_{[4]} \mathcal{N}_{[4],3}$$
$$+ (u^4 b_{[3,1]} + u^2 c_{1,0,0}^2 b_{[2]}) \mathcal{N}_{[3,1],3} + (u^4 b_{[2,2]} + u^2 c_{1,0,0}^2 b_{[1,1]} + c_{2,0,0}^2) \mathcal{N}_{[2,2],3} + \mathcal{O}(\mathcal{N}_{[\lambda]>4,3}) \Big]$$

$$b_{[1]} = \frac{1-t}{1-q}, b_{[2]} = \frac{(1-qt)(1-t)}{(1-q^2)(1-q)}, b_{[1,1]} = \frac{(1-t)(1-t^2)}{(1-q)(1-qt)}, b_{[1,1,1]} = \frac{(1-t)(1-t^2)(1-t^3)}{(1-q)(1-qt)(1-qt^2)}$$
$$b_{[2,1]} = \frac{(1-qt)(1-t)(1-t^2)}{(1-q^2)(1-q)(1-qt)}, b_{[3]} = \frac{(1-q^2t)(1-qt)(1-t)}{(1-q^3)(1-q^2)(1-q)}, b_{[3,1]} = \frac{(1-q^2t)(1-t)(1-t^2)}{(1-q^3)(1-q^2)(1-q)}$$
$$b_{[1,1,1,1]} = \frac{(1-t)(1-t^2)(1-t^3)(1-t^4)}{(1-q)(1-qt)(1-qt^2)(1-qt^3)}, b_{[2,1,1]} = \frac{(1-t)(1-t^2)(1-t^3)}{(1-q^2)(1-q)(1-qt^2)}$$
$$b_{[2,2]} = \frac{(1-t)(1-t^2)(1-qt^2)}{(1-q^2)(1-q)(1-q^2t)}, b_{[4]} = \frac{(1-q^3t)(1-q^2t)(1-qt)(1-t)}{(1-q^4)(1-q^3)(1-q^2)(1-q)}$$

$$c_{1,0,0} = \frac{(1-t)}{1-q}(u-1), c_{1,1,0} = \frac{(1-t)^2}{(1-q)^2}(u-1)^2$$
$$c_{2,0,0} = \frac{(1-t)(1-qt)}{(1-q)(1-q^2)} u^2 - \frac{(1-t)^2}{(1-q)^2} u + \frac{(1-t)(1-tq^{-1})}{(1-q)(1-q^2)} q$$

$$I_{SO(2N)}(u,q,t) = \mathcal{X}_N \sum_{\ell(\lambda)\leq N} u^{|\lambda|} b_\lambda$$
$$\times \sum_{n_{12}^+ \geq n_{13}^+ \geq \cdots \geq n_{N-1,N}^+}^{\infty} \sum_{n_{12}^- \geq n_{13}^- \geq \cdots \geq n_{N-1,N}^-}^{\infty} c_{n_{12}^+, n_{13}^+, \cdots, n_{N-1,N}^+} c_{n_{12}^-, n_{13}^-, \cdots, n_{N-1,N}^-}$$
$$\times \sum_{\mu}^{\{n^+\}} \sum_{\nu}^{\{n^-\}} \langle n_\mu m_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\nu m_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle$$

$$\sum_{\mu}^{\{n^+\}} \{(n'^+_{12}, n'^+_{13}, \cdots, n'^+_{N-1,N}) \mid (n'^+_{12}, n'^+_{13}, \cdots, n'^+_{N-1,N}) = \sigma(n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}), \forall\sigma \in S_{N(N-1)/2}\}$$

$$I_{SO(8)}(u,q,t) =$$
$$\mathcal{X}_3 \Big[\mathcal{N}_{[0],4} + ub_{[1]}\mathcal{N}_{[1],4} + u^2 b_{[2]}\mathcal{N}_{[2],4} + (u^2 b_{[1,1]} + c_1^2)\mathcal{N}_{[1,1],4} + \left(u^3 b_{[2,1]} + uc_1^2 \frac{1-t}{1-q}\right)\mathcal{N}_{[2,1],4}$$
$$+u^3 b_{[3]}\mathcal{N}_{[3],4} + \left(u^3 b_{[1,1,1]} + uc_1^2 \frac{1-t}{1-q}\left(\frac{-qt^2 - qt + t^2 - q + t + 1}{-qt^2+1}\right)^2\right)\mathcal{N}_{[1,1,1],4}$$
$$+\left(u^4 b_{[2,1,1]} + u^2 c_1^2 \left(b_{[1,1]}\left(\left(\frac{qt+q-t-1}{qt-1}\right)^2 + b_{[2]}\left(\frac{q^3t^3 - qt^3 - q^2 + 1}{q^3t^3 - q^2t^2 - qt + 1}\right)^2\right) + c_{1,1}^2\right.\right.$$
$$\left.+c_2^2\left(\frac{-qt+q-t+1}{qt-1}\right)^2 + 2c_{1,1}c_2\frac{-qt+q-t+1}{qt-1}\right)\mathcal{N}_{[2,1,1],4} + u^4 b_{[4]}\mathcal{N}_{[4],4}$$
$$+(u^4 b_{[3,1]} + u^2 c_1^2 b_{[2]})\mathcal{N}_{[3,1],4} + (u^4 b_{[2,2]} + u^2 c_1^2 b_{[1,1]} + c_2^2)\mathcal{N}_{[2,2],4}$$
$$+\left(u^2 c_1^2 b_{[1,1]} B^2 + u^4 b_{[1,1,1,1]} + \left(c_{1,1}(A+3) + c_2 B\right)^2\right)\mathcal{N}_{[1,1,1,1],4} + \mathcal{O}(\mathcal{N}_{[\lambda]>4,4})\Big]$$

$$c_1 = \frac{(1-t)}{1-q}(u-1), c_{1,1} = \frac{(1-t)^2}{(1-q)^2}(u-1)^2, c_2 = \frac{(1-t)(1-qt)}{(1-q)(1-q^2)}u^2 - \frac{(1-t)^2}{(1-q)^2}u + \frac{(1-t)(q-t)}{(1-q)(1-q^2)}$$
$$A = \frac{-3qt^3 + qt^2 - t^3 + qt - t^2 + q - t + 3}{qt^3 - 1}$$
$$B = \frac{q^2t^5 - q^2t^4 + qt^5 - q^2t^2 - qt^3 + t^4 + q^2t - qt^2 - t^3 + q - t + 1}{q^2t^5 - qt^3 - qt^2 + 1}$$

$$c_{n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}} \to c_{n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}, n^+_1, n^+_2, \cdots, n^+_N}$$
$$A(n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}) \to A'(n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}, n^+_1, n^+_2, \cdots, n^+_N)$$
$$A(n^+_{12}, n^+_{13}, \cdots, n^+_{N-1,N}) = (n^+_{12} + n^+_{13} + \cdots + n^+_{1,N}, \cdots, n^+_{1,N} + n^+_{2,N} + \cdots + n^+_{N-1,N})$$
$$A'(n^+_{12}, \cdots, n^+_{N-1,N}, n^+_1, \cdots, n^+_N) = (n^+_{12} + \cdots + n^+_{1,N} + 2n^+_1, \cdots, n^+_{1,N} + \cdots + n^+_{N-1,N} + 2n^+_N)$$

$$I_{Sp(2N)}(u,q,t) = \mathcal{X}_N \sum_{\ell(\lambda)\le N} u^{|\lambda|} b_\lambda$$
$$\sum_{n^+_{12}\ge\cdots\ge n^+_{N-1,N}\ge n^+_1\ge\cdots\ge n^+_N}^{\infty} \; \sum_{n^-_{12}\ge\cdots\ge n^-_{N-1,N}\ge n^-_1\ge\cdots\ge n^-_N}^{\infty} c_{n^+_{12},\cdots,n^+_{N-1,N},n^+_1,\cdots,n^+_N} c_{n^-_{12},\cdots,n^-_{N-1,N},n^-_1,\cdots,n^-_N}$$
$$\times \sum_{\mu}^{\{n^+\}} \sum_{\nu}^{\{n^-\}} \langle n_\mu m_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\nu m_\nu(\mathbf{x}) P_\lambda(\mathbf{x})\rangle$$

$$I_{Sp(2N)}\,(u,q,t)$$
$$= \mathcal{X}_N \Bigg[\mathcal{N}_{[0],N} + ub_{[1]}\mathcal{N}_{[1],N} + \left(u^2 b_{[1,1]} + c_1^2\left(1 + \frac{-qt+q-t+1}{qt-1}\right)^2\right)\mathcal{N}_{[1,1],N}$$
$$+(u^2 b_{[2]} + c_1^2)\mathcal{N}_{[2],N} + \left(ub_{[1]}c_{1,0,\cdots,0}^2\left(1 + \frac{q^2 - qt + q - t}{q^2t - 1}\right)^2 + u^3 b_{[2,1]}\right)\mathcal{N}_{[2,1],N}$$
$$+(u^3 b_{[3]} + ub_{[1]}c_1^2)\mathcal{N}_{[3],N} + (u^3 b_{[1,1,1]} + uc_{1,0,\cdots,0}^2 b_{[1]}(W+X)^2)\mathcal{N}_{[1,1,1],N}$$
$$+\left(u^4 b_{[1,1,1,1]} + u^2 b_{[1,1]}c_1^2(M+O)^2 + \left(c_{1,1}(3+2L+A+D) + c_2(D+F)\right)^2\right)\mathcal{N}_{[1,1,1,1],N}$$
$$+\left(u^4 b_{[2,1,1]} + u^2 c_1^2\left(b_{[1,1]}(N+Q)^2 + b_{[2]}(S+T)^2\right)\right.$$
$$\left.+\left(c_{1,1}(2+B+E) + c_2(E+G)\right)^2\right)\mathcal{N}_{[2,1,1],N}$$
$$+\left(u^4 b_{[3,1]} + u^2 c_1^2\left(b_{[1,1]}(1+R)^2 + b_{[2]}U^2(S+T)^2\right) + \left(c_{1,1}(C+1) + c_2(I+1)\right)^2\right)\mathcal{N}_{[2,2],N}$$
$$+\left(u^4 b_{[3,1]} + u^2 c_1^2\left(b_{[1,1]} + b_{[2]}(1+K)^2\right) + \left(c_{1,1} + c_2 H\right)^2\right)^{\mathcal{N}_{[3,1],N}}$$
$$+(u^4 b_{[4]} + u^2 c_1^2 b_{[2]} + c_2^2)\mathcal{N}_{[4],N} + \mathcal{O}(\mathcal{N}_{[\lambda]>4,N})\Bigg],$$

$$A=\frac{2q^2t^5-q^2t^4+qt^5-2q^2t^3+qt^4+t^5-2qt^3-2qt^2+q^2+qt-2t^2+q-t+2}{q^2t^5-qt^3-qt^2+1}$$
$$B=\frac{-q^3t^3+q^3t+t^2-1}{q^3t^3-q^2t^2-qt+1},C=\frac{-qt+q-t+1}{qt-1}$$
$$D=\frac{q^2t^5-q^2t^4+qt^5-q^2t^2-qt^3+t^4+q^2t-qt^2-t^3+q-t+1}{q^2t^5-qt^3-qt^2+1},E=\frac{-qt+q-t+1}{qt-1}$$
$$F=\frac{-q^3t^6+q^3t^5-q^2t^6+q^3t^4+q^2t^5-qt^6+q^2t^4+qt^5-t^6-q^3t^2+qt^4+t^5-q^3t-q^2t^2+t^4+q^3-q^2t-qt^2+q^2-qt-t^2+q-t+1}{q^3t^6-q^2t^5-q^2t^4-q^2t^3+qt^3+qt^2+qt-1}$$
$$G=\frac{q^3t^3-q^3t^2+q^2t^3-q^3t-q^2t^2+qt^3+q^3-q^2t-qt^2+t^3+q^2-qt-t^2+q-t+1}{q^3t^3-q^2t^2-qt+1}$$
$$H=\frac{-q^3t+q^2t^2+q^3-q^2t-qt+t^2+q-t}{q^3t^2-q^2t-qt+1},I=\frac{-q^3t+q^2t^2+q^3-q^2t-qt+t^2+q-t}{q^3t^2-q^2t-qt+1}$$

$$K=\frac{q^3t-q^2t^2-q^3+2q^2t-qt^2-q^2+qt}{q^4t^2-q^3t-qt+1},L=\frac{-3qt^3+qt^2-t^3+qt-t^2+q-t+3}{qt^3-1},$$
$$M=\frac{q^2t^5+q^2t^4-qt^5+2q^2t^3-2qt^4+q^2t^2-3qt^3+t^4+q^2t-3qt^2+t^3-2qt+2t^2-q+t+1}{q^2t^5-qt^3-qt^2+1},$$
$$N=\frac{qt+q-t-1}{qt-1},O=\frac{-q^2t^5-q^2t^3+t^5+q^2t^2+t^3+q^2-t^2-1}{q^2t^5-qt^3-qt^2+1}$$
$$Q=\frac{q^3t-q^2t^2-qt+t^2}{q^3t^3-q^2t^2-qt+1},R=\frac{-qt+q-t+1}{qt-1},S=\frac{q^3t^3-qt^3-q^2+1}{q^3t^3-q^2t^2-qt+1},$$
$$T=\frac{-q^4t^4+q^4t^3-q^3t^4+q^3t^3+q^2t^4-q^2t^3+qt^4+q^3t-qt^3-q^3+q^2t-q^2-qt+q-t+1}{q^4t^4-2q^3t^3+2qt-1},$$
$$U=\frac{q^4t^3+q^4t^2-q^3t^3-q^3t^2-q^2t^3-q^3t-q^2t^2+qt^3-q^3+q^2t+qt^2+q^2+qt+q-t-1}{q^4t^3-2q^3t^2-q^2t^2+q^2t+2qt-1},$$
$$W=\frac{-qt^2-qt+t^2-q+t+1}{-qt^2+1},X=\frac{q^2t^3-t^3-q^2+1}{-q^2t^3+qt^2+qt-1},Y=\frac{q^2-qt+q-t}{q^2t-1}.$$

$$A=\frac{2q^2t^5-q^2t^4+qt^5-2q^2t^3+qt^4+t^5-2qt^3-2qt^2+q^2+qt-2t^2+q-t+2}{q^2t^5-qt^3-qt^2+1},$$
$$B=\frac{-q^3t^3+q^3t+t^2-1}{q^3t^3-q^2t^2-qt+1},\quad C=\frac{-qt+q-t+1}{qt-1},$$
$$D=\frac{q^2t^5-q^2t^4+qt^5-q^2t^2-qt^3+t^4+q^2t-qt^2-t^3+q-t+1}{q^2t^5-qt^3-qt^2+1},\quad E=\frac{-qt+q-t+1}{qt-1},$$
$$F=\frac{-q^3t^6+q^3t^5-q^2t^6+q^3t^4+q^2t^5-qt^6+q^2t^4+qt^5-t^6-q^3t^2+qt^4+t^5-q^3t-q^2t^2+t^4+q^3-q^2t-qt^2+q^2-qt-t^2+q-t+1}{q^3t^6-q^2t^5-q^2t^4-q^2t^3+qt^3+qt^2+qt-1},$$
$$G=\frac{q^3t^3-q^3t^2+q^2t^3-q^3t-q^2t^2+qt^3+q^3-q^2t-qt^2+t^3+q^2-qt-t^2+q-t+1}{q^3t^3-q^2t^2-qt+1},$$
$$H=\frac{-q^3t+q^2t^2+q^3-q^2t-qt+t^2+q-t}{q^3t^2-q^2t-qt+1},\quad I=\frac{-q^3t+q^2t^2+q^3-q^2t-qt+t^2+q-t}{q^3t^2-q^2t-qt+1},$$

$$K=\frac{q^3t-q^2t^2-q^3+2q^2t-qt^2-q^2+qt}{q^4t^2-q^3t-qt+1},\quad L=\frac{-3qt^3+qt^2-t^3+qt-t^2+q-t+3}{qt^3-1},$$

$$M=\frac{q^2t^5+q^2t^4-qt^5+2q^2t^3-2qt^4+q^2t^2-3qt^3+t^4+q^2t-3qt^2+t^3-2qt+2t^2-q+t+1}{q^2t^5-qt^3-qt^2+1},$$

$$N=\frac{qt+q-t-1}{qt-1},\quad O=\frac{-q^2t^5-q^2t^3+t^5+q^2t^2+t^3+q^2-t^2-1}{q^2t^5-qt^3-qt^2+1}$$

$$Q=\frac{q^3t-q^2t^2-qt+t^2}{q^3t^3-q^2t^2-qt+1},\quad R=\frac{-qt+q-t+1}{qt-1},\quad S=\frac{q^3t^3-qt^3-q^2+1}{q^3t^3-q^2t^2-qt+1},$$

$$T=\frac{-q^4t^4+q^4t^3-q^3t^4+q^3t^3+q^2t^4-q^2t^3+qt^4+q^3t-qt^3-q^3+q^2t-q^2-qt+q-t+1}{q^4t^4-2q^3t^3+2qt-1},$$

$$U=\frac{q^4t^3+q^4t^2-q^3t^3-q^3t^2-q^2t^3-q^3t-q^2t^2+qt^3-q^3+q^2t+qt^2+q^2+qt+q-t-1}{q^4t^3-2q^3t^2-q^2t^2+q^2t+2qt-1},$$

$$W=\frac{-qt^2-qt+t^2-q+t+1}{-qt^2+1},\quad X=\frac{q^2t^3-t^3-q^2+1}{-q^2t^3+qt^2+qt-1},\quad Y=\frac{q^2-qt+q-t}{q^2t-1}.$$

$$A'(n_{12}^+,n_{13}^+,\cdots,n_{N-1,N}^+,n_1^+,n_2^+,\cdots,n_N^+)\to A''(n_{12}^+,n_{13}^+,\cdots,n_{N-1,N}^+,n_1^+,n_2^+,\cdots,n_N^+)$$
$$A'(n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+)=(n_{12}^++\cdots+n_{1,N}^++2n_1,\cdots,n_{1,N}^++\cdots+n_{N-1,N}^++2n_N)$$
$$A''(n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+)=(n_{12}^++\cdots+n_{1,N}^++n_1,\cdots,n_{1,N}^++\cdots+n_{N-1,N}^++n_N)$$

$$I_{SO(2N+1)}(u,q,t)=\mathcal{X}_N\sum_{\ell(\lambda)\le N}u^{|\lambda|}b_\lambda \sum_{n_{12}^+\ge\cdots\ge n_{N-1,N}^+\ge n_1^+\ge\cdots\ge n_N^+}^{\infty}\ \sum_{n_{12}^-\ge\cdots\ge n_{N-1,N}^-\ge n_1^-\ge\cdots\ge n_N^-}^{\infty} c_{n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+}c_{n_{12}^-,\cdots,n_{N-1,N}^-,n_1^-,\cdots,n_N^-}\times\sum_{\mu}^{\{n^+\}}\sum_{\nu}^{\{n^-\}}\langle n_\mu m_\mu(\mathbf{x})P_\lambda(\mathbf{x}),n_\nu m_\nu(\mathbf{x})P_\lambda(\mathbf{x})\rangle$$

$$\begin{aligned}I_{SO(2N+1)}&(u,q,t)\\ &=\mathcal{X}_N\Big[\mathcal{N}_{[0],2}+(ub_{[1]}+c_1^2)\mathcal{N}_{[1],N}+(u^2b_{[2]}+c_2^2+ub_{[1]}c_1^2)\mathcal{N}_{[2],N}\\ &+\left(u^2b_{[1,1]}+(c_1+c_{1,1}+lc_2)^2+um^2b_{[1]}c_1^2\right)\mathcal{N}_{[1,1],N}\\ &+\left(u^3b_{[2,1]}+(c_{1,1}+c_{2,1}+kc_3)^2+ub_{[1]}(c_1+c_{1,1}+Yc_2)^2+(u^2b_{[1,1]}+u^2b_{[2]}p^2)c_1^2\right)\mathcal{N}_{[2,1],N}\\ &+(u^3b_{[3]}+c_{3,0,\cdots,0}^2+ub_{[1]}c_{2,0,\cdots,0}^2+u^2b_{[2]}c_{1,0,\cdots,0}^2)\mathcal{N}_{[3],N}\\ &+\Big(u^3b_{[1,1,1]}+\left(3c_{1,1}+c_{1,1,1}+i(c_{1,1}+c_{2,1})+jc_3\right)^2\\ &+ub_{[1]}\left((c_1+c_{1,1})W+Xc_2\right)^2+u^2b_{[1,1]}n^2c_1^2\Big)\mathcal{N}_{[1,1,1],N}+\mathcal{O}(\mathcal{N}_{|\lambda|>3,N})\Big]\end{aligned}$$

$$l=\frac{-qt+q-t+1}{qt-1},m=\frac{qt+q-t-1}{qt-1},n=\frac{-qt^2-qt+t^2-q+t+1}{-qt^2+1}$$
$$i=\frac{2qt^2-qt+t^2-q+t-2}{-qt^2+1},j=\frac{-q^2t^3+q^2t^2-qt^3+q^2t+qt^2-t^3-q^2+qt+t^2-q+t-1}{-q^2t^3+qt^2+qt-1}$$
$$k=\frac{q^2t-q^2+qt-q+t-1}{-q^2t+1},p=\frac{-q^3t^2+qt^2+q^2-1}{-q^3t^2+q^2t+qt-1}$$

$$d_n(q,t):=\frac{(t^{-1};q)_n}{(q;q)_n}t^n,d_{n_1,n_2,\cdots,n_N}(q,t):=d_{n_1}(q,t)d_{n_2}(q,t)\cdots d_{n_N}(q,t)$$

$$I_{SO(2N)}(u,q,t) = \mathcal{X}_N \sum_{\ell(\lambda)\leq N} \sum_{\lambda} \sum_{m_i(\mu),m_i(\nu)\text{ is even}} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t) b_\mu^{\mathrm{ol}}(q,t) b_\nu^{\mathrm{ol}}(q,t)$$
$$\times \sum_{n_{12}^+\geq n_{13}^+\geq\cdots\geq n_{N-1,N}^+}^{\infty} \sum_{n_{12}^-\geq n_{13}^-\geq\cdots\geq n_{N-1,N}^-}^{\infty} d_{n_{12}^+,n_{13}^+,\cdots,n_{N-1,N}^+} d_{n_{12}^-,n_{13}^-,\cdots,n_{N-1,N}^-}$$
$$\times \sum_{\rho}^{\{n^+\}} \sum_{\gamma}^{\{n^-\}} \langle n_\rho m_\rho(\mathbf{x}) P_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\gamma m_\gamma(\mathbf{x}) P_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle,$$
$$I_{Sp(2N)}(u,q,t) = \mathcal{X}_N \sum_{\ell(\lambda)\leq N} \sum_{|\mu|,|\nu|\text{ is even}} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t) b_\mu^{\mathrm{e}}(q,t) b_\nu^{\mathrm{e}}(q,t)$$
$$\sum_{n_{12}^+\geq\cdots\geq n_{N-1,N}^+\geq n_1^+\geq\cdots\geq n_N^+}^{\infty} \sum_{n_{12}^-\geq\cdots\geq n_{N-1,N}^-\geq n_1^-\geq\cdots\geq n_N^-}^{\infty} d_{n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+} d_{n_{12}^-,\cdots,n_{N-1,N}^-,n_1^-,\cdots,n_N^-}$$
$$\times \sum_{\rho}^{\{n^+\}} \sum_{\gamma}^{\{n^-\}} \langle n_\rho m_\rho(\mathbf{x}) P_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\gamma m_\gamma(\mathbf{x}) P_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle,$$
$$I'_{SO(2N+1)}(u,q,t) = \mathcal{X}_N \sum_{\ell(\lambda)\leq N} \sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t) b_\mu^{\mathrm{el}}(q,t) b_\nu^{\mathrm{el}}(q,t)$$
$$\sum_{n_{12}^+\geq\cdots\geq n_{N-1,N}^+\geq n_1^+\geq\cdots\geq n_N^+}^{\infty} \sum_{n_{12}^-\geq\cdots\geq n_{N-1,N}^-\geq n_1^-\geq\cdots\geq n_N^-}^{\infty} d_{n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+} d_{n_{12}^-,\cdots,n_{N-1,N}^-,n_1^-,\cdots,n_N^-}$$
$$\times \sum_{\rho}^{\{n^+\}} \sum_{\gamma}^{\{n^-\}} \langle n_\rho m_\rho(\mathbf{x}) P_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\gamma m_\gamma(\mathbf{x}) P_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle,$$
$$I_{SO(2N+1)}(u,q,t) = \mathcal{X}_N \sum_{\lambda} \sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|+\mathrm{odd}\,(\mu')+\mathrm{odd}\,(\nu')}{2}} b_\lambda(q,t) b_\mu^{\mathrm{el}}(q,t) b_\nu^{\mathrm{el}}(q,t)$$
$$\sum_{n_{12}^+\geq\cdots\geq n_{N-1,N}^+\geq n_1^+\geq\cdots\geq n_N^+}^{\infty} \sum_{n_{12}^-\geq\cdots\geq n_{N-1,N}^-\geq n_1^-\geq\cdots\geq n_N^-}^{\infty} d_{n_{12}^+,\cdots,n_{N-1,N}^+,n_1^+,\cdots,n_N^+} d_{n_{12}^-,\cdots,n_{N-1,N}^-,n_1^-,\cdots,n_N^-}$$
$$\times \sum_{\rho}^{\{n^+\}} \sum_{\gamma}^{\{n^-\}} \langle n_\rho m_\rho(\mathbf{x}) P_\mu(\mathbf{x}) P_\lambda(\mathbf{x}), n_\gamma m_\gamma(\mathbf{x}) P_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) \rangle,$$

$$I_{SO(2N+1)}(u,q,t) = \mathcal{X}_N \sum_{l=0}^{\infty} \sum_{\lambda} \sum_{m_i(\mu),m_i(\nu)\text{ is even}} u^{|\lambda|+l+\frac{|\mu|+|\nu|}{2}} b_\lambda(q,t) b_\mu^{\mathrm{ol}}(q,t) b_\nu^{\mathrm{ol}}(q,t) w_l(q,t)$$
$$\times \sum_{n_{12}^+\geq n_{13}^+\geq\cdots\geq n_{N-1,N}^+}^{\infty} \sum_{n_{12}^-\geq n_{13}^-\geq\cdots\geq n_{N-1,N}^-} d_{n_{12}^+,n_{13}^+,\cdots,n_{N-1,N}^+} d_{n_{12}^-,n_{13}^-,\cdots,n_{N-1,N}^-}$$
$$\times \sum_{\rho} \sum_{\gamma}^{\{n^+\}} \sum^{\{n^-\}} \langle n_\rho m_\rho(\mathbf{x}) P_\mu(\mathbf{x}) P_\lambda(\mathbf{x}) P_{[l]}(\mathbf{x}), n_\gamma m_\gamma(\mathbf{x}) P_\nu(\mathbf{x}) P_\lambda(\mathbf{x}) P_{[l]}(\mathbf{x}) \rangle$$

$$I_{Sp(2N)}^{p=v=u=0} = I_{SO(2N+1)}^{p=v=u=0} = \prod_{j=0}^{N-1} \frac{(qt^{2j+1};q)_\infty}{(t^{2j+2};q)_\infty}$$
$$I_{SO(2N)}^{p=v=u=0} = \frac{(qt^{N-1};q)_\infty}{(t^N;q)_\infty} \prod_{j=0}^{N-2} \frac{(qt^{2j+1};q)_\infty}{(t^{2j+2};q)_\infty}$$

$$\mathrm{sp}_\lambda(x_1,\ldots,x_n) = K_\lambda(x_1,\ldots,x_n;q,q;q^{1/2},-q^{1/2},q,-q)$$
$$\mathrm{o}_\lambda(x_1,\ldots,x_n) = K_\lambda(x_1,\ldots,x_n;q,q;1,-1,q^{1/2},-q^{1/2})$$
$$\mathrm{so}_\lambda(x_1,\ldots,x_n) = K_\lambda(x_1,\ldots,x_n;q,q;-1,-q^{1/2},q^{1/2},q)$$

$$I_{Sp(2N)}^{p=v=0,q=t}(u,q)=\mathcal{X}_N\sum_{\lambda}\sum_{\mu,\nu\text{ is even}}u^{|\lambda|+\frac{|\mu|+|\nu|}{2}}b_\lambda(q,q)b_\mu^{\mathrm{oa}}(q,q)b_\nu^{\mathrm{oa}}(q,q)$$

$$\times\oint_{\mathbb{T}^N}\Delta_{C_N}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}$$

$$I'^{s=v=0,q=t}_{SO(2N+1)}(u,q)=\mathcal{X}_N\sum_{\lambda}\sum_{\mu,\nu}u^{|\lambda|+\frac{|\mu|+|\nu|}{2}}b_\lambda(q,q)b_\mu^{\mathrm{el}}(q,q)b_\nu^{\mathrm{el}}(q,q)$$

$$\times\oint_{\mathbb{T}^N}\Delta_{B_N}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}$$

$$I_{SO(2N)}^{s=v=0,q=t}(u,q)=\mathcal{X}_N\sum_{\lambda}\sum_{m_i(\mu),m_i(\nu)}u^{|\lambda|+\frac{|\mu|+|\nu|}{2}}b_\lambda(q,q)b_\mu^{\mathrm{ol}}(q,q)b_\nu^{\mathrm{ol}}(q,q)$$

$$\times\oint_{\mathbb{T}^N}\Delta_{D_N}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j}$$

$$I_{SO(2N+1)}^{s=v=0,q=t}(u,q)=\mathcal{X}_N\sum_{\lambda}\sum_{\mu,\nu}u^{|\lambda|+\frac{|\mu|+|\nu|+odd(\mu')+odd(\nu')}{2}}b_\lambda(q,q)b_\mu^{\mathrm{el}}(q,q)b_\nu^{\mathrm{el}}(q,q)$$

$$\times\oint_{\mathbb{T}^N}\Delta_{B_N}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j},$$

$$I_{SO(2N+1)}^{s=v=0,q=t}(u,q)=\mathcal{X}_N\sum_{\lambda}\sum_{l}\sum_{m_i(\mu),m_i(\nu)}u^{|\lambda|+l+\frac{|\mu|+|\nu|}{2}}b_\lambda(q,q)b_\mu^{\mathrm{el}}(q,q)b_\nu^{\mathrm{el}}(q,q)w_l(q,q)$$

$$\times\oint_{\mathbb{T}^N}\Delta_{B_N}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_{[l]}(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})s_{[l]}(\mathbf{x}^{-1})\prod_{j=1}^{N}\frac{dx_j}{2\pi i x_j},$$

$$b_\lambda(q,q)=b_\lambda^{\mathrm{ol}}(q,q)=b_\lambda^{\mathrm{oa}}(q,q)=b_\lambda^{\mathrm{el}}(q,q)=w_l(q,q)=1$$

$$\prod_{i=1}^{N}(1-x_i^2)\prod_{1\le i<j\le N}(1-x_ix_j)$$

$$\prod_{i=1}^{N}(1-x_i^2)\prod_{1\le i<j\le N}(1-x_ix_j)=\sum_{\lambda\subset S_N^C}f_\lambda^C s_\lambda:=(-1)^{\frac{N(N+1)}{2}}\left(\sum_{\lambda\subset S_{N-1}^C}f_\lambda^C s_{[(N+1)^N-\lambda]}\right)+\sum_{\lambda\subset S_{N-1}^C}f_\lambda^C s_\lambda,$$

$$f_\lambda=\{+1,-1\},S_N^C$$

$$\prod_{i=1}^{N}(1-x_i^2)\prod_{1\le i<j\le N}(1-x_ix_j)$$

$$\prod_{i=1}^{N-1}(1-x_i^2)\prod_{1\le i<j\le N-1}(1-x_ix_j)=(-1)^{\frac{N(N+1)}{2}}\sum_{\lambda\subset S_{N-1}^C}f_\lambda^C s_\lambda$$

$$1=s_{[\emptyset]}$$
$$1-x_1^2=-s_{[2]}(x_1)+s_{[\emptyset]}(x_1)$$
$$(1-x_1^2)(1-x_2^2)(1-x_1x_2)=-s_{[3,3]}(x_1,x_2)+s_{[3,1]}(x_1,x_2)-s_{[2]}(x_1,x_2)+s_{[\emptyset]}(x_1,x_2)$$

$$S_0^C=\{[\emptyset]\},S_1^C=\{[\emptyset],[2]\}|S_N^C|S_N^CS_N^C$$

$$S_N^C=S_{N-1}^C\cup\{(N+1)^N-\lambda\mid\lambda\in S_{N-1}^C\},\text{ so }|S_N^C|=2^N$$

$$\prod_{i=1}^{N}(1-x_i)\prod_{1\le i<j\le N}(1-x_ix_j)=\sum_{\lambda\subset S_N^B}f_\lambda^B s_\lambda:=(-1)^{\frac{N(N+1)}{2}}\left(\sum_{\lambda\subset S_{N-1}^B}f_{[N^N-\lambda]}^B s s_\lambda\right)+\sum_{\lambda\subset S_{N-1}^B}f_\lambda^B s_\lambda,$$

$$\prod_{i=1}^{N-1}(1-x_i^2)\prod_{1\le i<j\le N-1}(1-x_ix_j)=(-1)^{\frac{N(N+1)}{2}}\sum_{\lambda\subset S_{N-1}^B} f_\lambda^B s_\lambda$$

$$1=s_{[\emptyset]}$$
$$1-x_1=-s_{[1]}(x_1)+s_{[\emptyset]}(x_1)$$
$$(1-x_1)(1-x_2)(1-x_1x_2)=-s_{[2,2]}(x_1,x_2)+s_{[2,1]}(x_1,x_2)-s_{[1]}(x_1,x_2)+s_{[\emptyset]}(x_1,x_2)$$

$S_0^B=\{[\emptyset]\}, S_1^B=\{[\emptyset],[1]\}$ and $|S_N^B|=2^N$. In D_N

$$\prod_{1\le i<j\le N}(1-x_ix_j)=\sum_{\lambda\subset S_N^D} f_\lambda^D s_\lambda := (-1)^{\frac{N(N-1)}{2}}\left(\sum_{\lambda\subset S_{N-1}^D} f_\lambda^D s_{[(N-1)^N-\lambda]}\right)+\sum_{\lambda\subset S_{N-1}^D} f_\lambda^D s_\lambda$$

$$\prod_{1\le i<j\le N-1}(1-x_ix_j)=(-1)^{\frac{N(N-1)}{2}}\sum_{\lambda\subset S_{N-1}^D} s_\lambda$$

$1=s_{[\emptyset]}$,
$1-x_1x_2=-s_{[1,1]}(x_1,x_2)+s_{[\emptyset]}(x_1,x_2)$,
$(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)=-s_{[2,2,2]}(x_1,x_2,x_3)+s_{[2,1,1]}(x_1,x_2,x_3)-s_{[1,1]}(x_1,x_2,x_3)+s_{[\emptyset]}(x_1,x_2,x_3)$,
$S_1^D=\{[\emptyset]\}, S_2^D=\{[\emptyset],[1,1]\}$ and $|S_N^D|=2^{N-1}$

$$I_{Sp(2N)}^{p=v=0,q=t}(u)=\mathcal{X}_N\sum_{\rho\subset S_N^C}\sum_{\gamma\subset S_N^C}\sum_{\lambda}\sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^C f_\gamma^C$$
$$\times\oint_{\mathbb{T}^N}\Delta_{A_{N-1}}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\rho(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})s_\gamma(\mathbf{x}^{-1})\prod_{j=1}^N\frac{dx_j}{2\pi i x_j}$$
$$I_{SO(2N+1)}^{\prime s=v=0,q=t}(u)=\mathcal{X}_N\sum_{\rho\subset S_N^B}\sum_{\gamma\subset S_N^B}\sum_{\lambda}\sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^B f_\gamma^B$$
$$\times\oint_{\mathbb{T}^N}\Delta_{A_{N-1}}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\rho(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})s_\gamma(\mathbf{x}^{-1})\prod_{j=1}^N\frac{dx_j}{2\pi i x_j}$$
$$I_{SO(2N)}^{s=v=0,q=t}(u)=\mathcal{X}_N\sum_{\rho\subset S_N^D}\sum_{\gamma\subset S_N^D}\sum_{\lambda}\sum_{m_i(\mu),m_i(\nu)} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^D f_\gamma^D$$
$$\times\oint_{\mathbb{T}^N}\Delta_{A_{N-1}}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\rho(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})s_\gamma(\mathbf{x}^{-1})\prod_{j=1}^N\frac{dx_j}{2\pi i x_j}$$
$$I_{SO(2N+1)}^{s=v=0,q=t}(u)=\mathcal{X}_N\sum_{\rho\subset S_N^B}\sum_{\gamma\subset S_N^B}\sum_{\lambda}\sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|+odd(\mu')+odd(\nu')}{2}} f_\rho^B f_\gamma^B$$
$$\times\oint_{\mathbb{T}^N}\Delta_{A_{N-1}}s_\lambda(\mathbf{x})s_\mu(\mathbf{x})s_\rho(\mathbf{x})s_\lambda(\mathbf{x}^{-1})s_\nu(\mathbf{x}^{-1})s_\gamma(\mathbf{x}^{-1})\prod_{j=1}^N\frac{dx_j}{2\pi i x_j}$$

$$\frac{1}{N!}\oint_{\mathbb{T}^N}\prod_{i=1}^N\frac{dx_i}{2\pi i x_i}\Delta_{A_{N-1}}(\mathbf{x})s_\lambda(\mathbf{x})s_\mu(\mathbf{x}^{-1})=\delta_{\lambda\mu}$$

$$I_{Sp(2N)}^{p=v=0,q=t}(u) = N!\,\mathcal{X}_N \sum_{\rho\subset S_N^C} \sum_{\gamma\subset S_N^C} \sum_{\lambda,\kappa,\theta,\tau} \sum_{\mu,\nu \text{ is even}} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^C f_\gamma^C c_{\lambda\mu}^{\kappa} c_{\kappa\rho}^{\tau} c_{\lambda\nu}^{\theta} c_{\theta\gamma}^{\tau}$$

$$I'^{s=v=0,q=t}_{SO(2N+1)}(u) = N!\,\mathcal{X}_N \sum_{\rho\subset S_N^B} \sum_{\gamma\subset S_N^B} \sum_{\lambda,\kappa,\theta,\tau} \sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^B f_\gamma^B c_{\lambda\mu}^{\kappa} c_{\kappa\rho}^{\tau} c_{\lambda\nu}^{\theta} c_{\theta\gamma}^{\tau}$$

$$I_{SO(2N)}^{s=v=0,q=t}(u) = N!\,\mathcal{X}_N \sum_{\rho\subset S_N^D} \sum_{\gamma\subset S_N^D} \sum_{\lambda,\kappa,\theta,\tau} \sum_{\substack{m_i(\mu),m_i(\nu)\\ \text{is even}}} u^{|\lambda|+\frac{|\mu|+|\nu|}{2}} f_\rho^D f_\gamma^D c_{\lambda\mu}^{\kappa} c_{\kappa\rho}^{\tau} c_{\lambda\nu}^{\theta} c_{\theta\gamma}^{\tau}$$

$$I_{SO(2N+1)}^{s=v=0,q=t}(u) = N!\,\mathcal{X}_N \sum_{\rho\subset S_N^B} \sum_{\gamma\subset S_N^B} \sum_{\lambda,\kappa,\theta,\tau} \sum_{\mu,\nu} u^{|\lambda|+\frac{|\mu|+|\nu|+odd(\mu')+odd(\nu')}{2}} f_\rho^B f_\gamma^B c_{\lambda\mu}^{\kappa} c_{\kappa\rho}^{\tau} c_{\lambda\nu}^{\theta} c_{\theta\gamma}^{\tau}$$

$$\begin{aligned}
I_{Sp(2)}^{p=v=0,q=t} &= \frac{1}{2}\left(2+2u^2+2u^4+2u^6+\mathcal{O}(u^7)\right),\\
I_{Sp(4)}^{p=v=0,q=t} &= \frac{1}{4}\left(4+4u^2+8u^4+\mathcal{O}(u^5)\right),\\
I_{Sp(6)}^{p=v=0,q=t} &= \frac{1}{8}\left(8+8u^2+0u^3+\mathcal{O}(u^4)\right),\\
I_{SO(3)}^{p=v=0,q=t} &= \frac{1}{2}\left(2+2u^2+2u^4+2u^6+\mathcal{O}(u^7)\right),\\
I_{SO(5)}^{p=v=0,q=t} &= \frac{1}{4}\left(4+4u^2+8u^4+\mathcal{O}(u^5)\right),\\
I_{SO(7)}^{p=v=0,q=t} &= \frac{1}{8}\left(8+8u^2+0u^3+\mathcal{O}(u^4)\right).
\end{aligned}$$

$$\sum_{\substack{\lambda\\ \mathrm{odd}(\lambda)\text{ is even}}} t^{\frac{|\lambda|-\mathrm{odd}(\lambda)}{2}} d_\lambda(t^{-1}) P_\lambda(\mathbf{x};t^{-1}) = \prod_{i\geq 0}\frac{1-x_i^2}{1-tx_i^2}\prod_{i<j}\frac{1-x_ix_j}{1-tx_ix_j}$$

$$\sum_{\lambda} t^{\frac{|\lambda|+odd(\lambda')}{2}} d_\lambda(t^{-1}) P_\lambda(\mathbf{x};t^{-1}) = \prod_{i\geq 0}\frac{1-x_i}{1-tx_i}\prod_{i<j}\frac{1-x_ix_j}{1-tx_ix_j}$$

$$\sum_{\substack{\lambda\\ \lambda'\text{ is even}}} t^{\frac{|\lambda|}{2}} c_\lambda(t^{-1}) P_\lambda(\mathbf{x};t^{-1}) = \prod_{i<j}\frac{1-x_ix_j}{1-tx_ix_j}$$

$$\sum_{\ell(\lambda)\leq N} t^{|\lambda|} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{x}^{-1}) = \prod_{i,j=1}^{N}\frac{1}{1-tx_i/x_j},$$

$$\begin{aligned}
c_\lambda(t) &= \prod_{i\geq 1}(1-t)(1-t^3)\cdots\left(1-t^{m_i(\lambda)-1}\right)\\
d_\lambda(t) &= \prod_{i\geq 1}(1-t)(1-t^3)\cdots\left(1-t^{2\lceil m_i(\lambda)/2\rceil-1}\right)
\end{aligned}$$

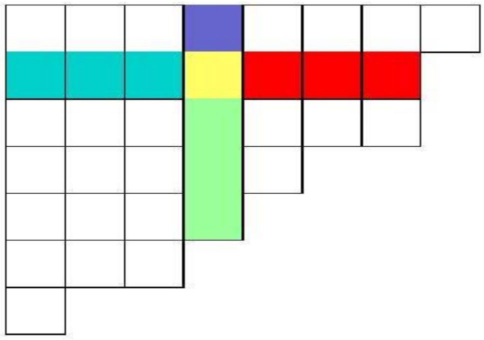
$$s_\mu(\mathbf{x}) P_\nu(\mathbf{x};t) = \sum_\lambda \bar{K}_{\mu\nu}^{\lambda}(t) s_\lambda(\mathbf{x})$$

$$I_{Sp(2N)}^{p=v=0,q=u}(t) = \frac{1}{2^N}\sum_{\lambda,\rho} \sum_{\substack{\mu,\nu\\ \text{odd }(\nu),\text{ odd }(\nu)\\ \text{is even}}} t^{|\lambda|+\frac{|\mu|-odd(\mu)+|\nu|-odd(\nu)}{2}} d_\mu(t^{-1}) d_\nu(t^{-1}) \bar{K}_{\lambda\mu}^{\rho}(t^{-1}) \bar{K}_{\lambda\nu}^{\rho}(t^{-1}),$$

$$I_{SO(2N)}^{s=v=0,q=t}(t) = \frac{1}{2^{N-1}}\sum_{\lambda,\rho} \sum_{\substack{\mu,\nu\\ \mu',\nu'\text{ is even}}} c_\mu(t^{-1}) c_\nu(t^{-1}) t^{|\lambda|+\frac{|\mu|+|\nu|}{2}} \bar{K}_{\lambda\mu}^{\rho}(t^{-1}) \bar{K}_{\lambda\nu}^{\rho}(t^{-1}),$$

$$I_{SO(2N+1)}^{s=v=0,q=t}(t) = \frac{1}{2^N}\sum_{\lambda,\rho}\sum_{\mu,\nu} t^{|\lambda|+\frac{|\mu|+|\nu|+odd(\mu')+odd(\nu')}{2}} d_\mu(t^{-1}) d_\nu(t^{-1}) \bar{K}_{\lambda\mu}^{\rho}(t^{-1}) \bar{K}_{\lambda\nu}^{\rho}(t^{-1}),$$

$$\begin{aligned}
a(s) = a_\lambda(s) &:= \lambda_i - j, \quad & a'(s) = a'_\lambda(s) &:= j-1,\\
l(s) = l_\lambda(s) &:= \lambda'_j - i, \quad & l'(s) = l'_\lambda(s) &:= i-1,
\end{aligned}$$



$$f(\boldsymbol{x}) = \sum_{\mu_1,\ldots,\mu_n \geq 0} a_{\mu_1,\ldots,\mu_n} x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

$$\sigma(f) = f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) = \sum_{\mu_1,\ldots,\mu_n \geq 0} a_{\mu_1,\ldots,\mu_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}.$$

$$\boldsymbol{x}^{\boldsymbol{\mu}} = x_1^{\mu_1} \cdots x_n^{\mu_n}, \deg_{\boldsymbol{x}} \boldsymbol{x}^{\boldsymbol{\mu}} = |\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_n,$$

$$\sigma(\boldsymbol{x}^{\boldsymbol{\mu}}) = \boldsymbol{x}^{\sigma \cdot \boldsymbol{\mu}} = x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n} = x_1^{\mu_{\sigma^{-1}(1)}} \cdots x_n^{\mu_{\sigma^{-1}(n)}}.$$

$$m_\lambda(\boldsymbol{x}) = \sum_{\mu \in \mathbf{S}_n \cdot \lambda} \boldsymbol{x}^{\mu},$$

$$m_\lambda(\boldsymbol{x}) = \frac{1}{|\mathbf{S}_{n,\lambda}|} \sum_{\sigma \in \mathbf{S}_n} \sigma \cdot \boldsymbol{x}^{\lambda} = \frac{1}{|\mathbf{S}_{n,\lambda}|} \sum_{\sigma \in \mathbf{S}_n} \boldsymbol{x}^{\sigma \cdot \lambda},$$

$$m_{[1]} = P_{[1]}, m_{[1]} = \left(\frac{qt + q - t - 1}{qt - 1}\right) P_{[1,1]} + P_{[2]}$$

$$m_{[1]}P_{[1,1]} = \frac{-qt^2 - qt + t^2 - q + t + 1}{-qt^2 + 1} P_{[1,1,1]} + P_{[2,1]}, m_{[1]}P_{[2]} = \left(\frac{-q^3t^2 + qt^2 + q^2 - 1}{-q^3t^2 + q^2t + qt - 1}\right) P_{[2,1]} + P_{[3]}$$

$$m_{[1]}P_{[1,1,1]} = \left(\frac{qt^3 + qt^2 - t^3 + qt - t^2 + q - t - 1}{qt^3 - 1}\right) P_{[1,1,1,1]} + P_{[2,1,1]}$$

$$m_{[1]}P_{[2,1]} = \frac{-q^4t^5 - q^4t^4 + q^3t^5 + q^3t^4 + q^2t^4 + q^3t^2 + q^2t^3 - qt^4 + q^3t - q^2t^2 - qt^3 - q^2t - qt - q + t + 1}{-q^4t^5 + q^3t^4 + q^3t^3 + q^2t^3 - q^2t^2 - qt^2 - qt + 1} P_{[2,1,1]} + \frac{qt + q - t - 1}{qt - 1} P_{[2,2]} + P_{[3,1]},$$

$$m_{[1]}P_{[3]} = \left(\frac{-q^5t^2 + q^2t^2 + q^3 - 1}{-q^5t^2 + q^3t + q^2t - 1}\right) P_{[3,1]} + P_{[4]},$$

$$m_{[1,1]} = P_{[1,1]},$$

$$m_{[1,1]}P_{[1,1]} = \left(\frac{q^2t^5 + q^2t^4 - qt^5 + 2q^2t^3 - 2qt^4 + q^2t^2 - 3qt^3 + t^4 + q^2t - 3qt^2 + t^3 - 2qt + 2t^2 - q + t + 1}{q^2t^5 - qt^3 - qt^2 + 1}\right) P_{[1,1,1,1]} + \left(\frac{qt + q - t - 1}{qt - 1}\right) P_{[2,1,1]} + P_{[2,2]},$$

$$m_{[1,1]} \cdot P_{[2]} = \left(\frac{q^3t^3 - qt^3 - q^2 + 1}{q^3t^3 - q^2t^2 - qt + 1}\right) P_{[2,1,1]} + P_{[3,1]},$$

$$m_{[2]} = \left(\frac{-q \cdot t + q - t + 1}{q \cdot t - 1}\right) P_{[1,1]} + P_{[2]},$$

$$m_{[2]}P_{[1]} = \left(\frac{q^2t^3 - t^3 - q^2 + 1}{-q^2t^3 + qt^2 + qt - 1}\right) P_{[1,1,1]} + \left(\frac{q^2 - qt + q - t}{q^2t - 1}\right) P_{[2,1]} + P_{[3]},$$

$$m_{[2]}P_{[1,1]} = \left(\frac{-q^2t^5 - q^2t^3 + t^5 + q^2t^2 + t^3 + q^2 - t^2 - 1}{q^2t^5 - qt^3 - qt^2 + 1}\right) P_{[1,1,1,1]} + \left(\frac{q^3t - q^2t^2 - qt + t^2}{q^3t^3 - q^2t^2 - qt + 1}\right) P_{[2,1,1]} + \left(\frac{-qt + q - t + 1}{qt - 1}\right) P_{[2,2]} + P_{[3,1]},$$

$$m_{[2]}P_{[2]} = \frac{-q^4t^4 + q^4t^3 - q^3t^4 + q^3t^3 + q^2t^4 - q^2t^3 + qt^4 + q^3t - qt^3 - q^3 + q^2t - q^2 - qt + q - t + 1}{q^4t^4 - 2q^3t^3 + 2qt - 1}P_{[2,1,1]}$$
$$+\frac{q^4t^3 + q^4t^2 - q^3t^3 - q^3t^2 - q^2t^3 - q^3t - q^2t^2 + qt^3 - q^3 + q^2t + qt^2 + q^2 + qt + q - t - 1}{q^4t^3 - 2q^3t^2 - q^2t^2 + q^2t + 2qt - 1}P_{[2,2]}$$
$$+\frac{q^3t - q^2t^2 - q^3 + 2q^2t - qt^2 - q^2 + qt}{q^4t^2 - q^3t - qt + 1}P_{[3,1]} + P_{[4]}$$
$$m_{[1,1,1]} = P_{[1,1,1]}, m_{[1,1,1]}P_{[1]} = \frac{qt^3 + qt^2 - t^3 + qt - t^2 + q - t - 1}{qt^3 - 1}P_{[1,1,1,1]} + P_{[2,1,1]}$$
$$m_{[2,1]} = \frac{2qt^2 - qt + t^2 - q + t - 2}{-qt^2 + 1}P_{[1,1,1]} + P_{[2,1]}$$
$$m_{[2,1]}P_{[1]} = \frac{-2q^2t^5 - q^2t^4 + qt^5 - qt^4 + t^5 + 2t^4 + 2q^2t + q^2 - qt + q - t - 2}{q^2t^5 - qt^3 - qt^2 + 1}P_{[1,1,1,1]}$$
$$+\frac{-q^3t^3 + 2q^3t^2 - 2q^2t^3 + q^3t + t^2 - 2q + 2t - 1}{q^3t^3 - q^2t^2 - qt + 1}P_{[2,1,1]} + \frac{qt + q - t - 1}{qt - 1}P_{[2,2]} + P_{[3,1]}$$
$$m_{[3]} = \frac{-q^2t^3 + q^2t^2 - qt^3 + q^2t + qt^2 - t^3 - q^2 + qt + t^2 - q + t - 1}{-q^2t^3 + qt^2 + qt - 1}P_{[1,1,1]}$$
$$+\frac{q^2t - q^2 + qt - q + t - 1}{-q^2t + 1}P_{[2,1]} + P_{[3]}$$
$$m_{[3]}P_{[1]} = \frac{q^3t^6 - q^3t^4 - t^6 - q^3t^2 + t^4 + q^3 + t^2 - 1}{q^3t^6 - q^2t^5 - q^2t^4 - q^2t^3 + qt^3 + qt^2 + qt - 1}P_{[1,1,1,1]}$$
$$+\frac{-q^3t^2 + q^2t^3 - q^2t^2 + qt^3 + q^3 - q^2t - qt^2 + t^3 + q^2 - qt + q - t}{q^3t^3 - q^2t^2 - qt + 1}P_{[2,1,1]}$$
$$+\frac{-q^3t^2 + q^3 + t^2 - 1}{q^3t^2 - q^2t - qt + 1}P_{[2,2]} + \frac{q^3 - q^2t + q^2 - qt + q - t}{q^3t - 1}P_{[3,1]} + P_{[4]}$$
$$m_{[2,1,1]} = \left(\frac{-3qt^3 + qt^2 - t^3 + qt - t^2 + q - t + 3}{qt^3 - 1}\right)P_{[1,1,1,1]} + P_{[2,1,1]}$$
$$m_{[2,2]} = \left(\frac{q^2t^5 - q^2t^4 + qt^5 - q^2t^2 - qt^3 + t^4 + q^2t - qt^2 - t^3 + q - t + 1}{q^2t^5 - qt^3 - qt^2 + 1}\right)P_{[1,1,1,1]}$$
$$+\left(\frac{-qt + q - t + 1}{qt - 1}\right)P_{[2,1,1]} + P_{[2,2]}$$

$$(t_0, t_1, t_2, t_3) = (-1, +1, -\sqrt{q}, +\sqrt{q})$$

$$(\sqrt{q}a; q)_\infty(-\sqrt{q}a; q)_\infty = (qa^2; q^2)_\infty, (a; q)_\infty(-a; q)_\infty = (a^2; q^2)_\infty, (qa; q^2)_\infty(a; q^2)_\infty = (a; q)_\infty$$

$$\prod_{1\le i\le n} \frac{(x_i^2; q)_\infty}{(+x_i, -x_i, \sqrt{q}x_i, -\sqrt{q}x_i; q)_\infty} = 1$$

$$\Delta_n(x_1, x_2, \dots, x_n; q, t; 1, -1, +\sqrt{q}, -\sqrt{q}) = \prod_{1\le i<j\le n} \frac{(x_ix_j; q)_\infty}{(tx_ix_j^i q)_\infty} \prod_{1\le i<j\le n} \frac{(x_ix_j^{-1}; q)_\infty}{(tx_ix_j^{-1}; q)_\infty},$$
$$\Delta_n(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}; q, t; 1, -1, +\sqrt{q}, -\sqrt{q}) = \prod_{1\le i<j\le n} \frac{(x_i^{-1}x_j^{-1}; q)_\infty}{(tx_i^{-1}x_j^{-1}; q)_\infty} \prod_{1\le i<j\le n} \frac{(x_i^{-1}x_j; q)_\infty}{(tx_i^{-1}x_j; q)_\infty}.$$

$$\Delta_n(\mathbf{x}; q, t; 1, -1, +\sqrt{q}, -\sqrt{q})\Delta_n(\mathbf{x^{-1}}; q, t; 1, -1, +\sqrt{q}, -\sqrt{q}) = \prod_{1\le i<j\le N} \frac{(x_i^{\pm1}x_j^{\pm1}; q)_\infty}{(tx_i^{\pm1}x_j^{\pm1}; q)_\infty}$$

$$\lim_{m\to\infty} u^{\frac{mn}{2}}(x_1 \cdots x_n)^m K_{m^n}(\sqrt{u}\mathbf{x}; q, t; 1, -1, \sqrt{q}, -\sqrt{q}) = \frac{(tux_ix_j; q)_\infty}{(ux_ix_j; q)_\infty}$$
$$\lim_{m\to\infty} u^{\frac{mn}{2}}(x_1 \cdots x_n)^{-m} K_{m^n}(\sqrt{u}\mathbf{x}^{-1}; q, t; 1, -1, \sqrt{q}, -\sqrt{q}) = \frac{(tux_i^{-1}x_j^{-1}; q)_\infty}{(ux_i^{-1}x_j^{-1}; q)_\infty}$$

$$\prod_{i,j=1}^{N}\frac{(tux_i/x_j;q)_\infty}{(ux_i/x_j;q)_\infty}=\sum_{\ell(\lambda)\le N}u^{|\lambda|}b_\lambda P_\lambda(\mathbf{x};q,t)P_\lambda(\mathbf{x}^{-1};q,t)=\sum_{\ell(\lambda)\le N}b_\lambda P_\lambda(\sqrt{u}\mathbf{x};q,t)P_\lambda(\sqrt{u}\mathbf{x}^{-1};q,t)$$

$$\lim_{m\to\infty}\sum_{\ell(\lambda)\le N}u^{mn}(z_1\cdots z_n)^m K_{m^n}(\sqrt{u}\mathbf{z};q,t;1,-1,\sqrt{q},-\sqrt{q})(z_1\cdots z_n)^{-m}K_{m^n}(\sqrt{u}\mathbf{z}^{-1};q,t;1,-1,\sqrt{q},-\sqrt{q})$$

$$\times b_\lambda P_\lambda(\sqrt{u}\mathbf{z};q,t)P_\lambda(\sqrt{u}\mathbf{z}^{-1};q,t)=\prod_{1\le i<j\le N}\frac{(tuz_i^{\pm1}z_j^{\pm1};q)_\infty}{(uz_i^{\pm1}z_j^{\pm1};q)_\infty}\times\frac{(tu;q)_\infty^N}{(u;q)_\infty^N}$$

$$\sum_\pi q^{|\pi|}=\prod_{n=1}^{\infty}(1-q^n)^{-1}$$

$$\sum_\pi q^{|\pi|}=\prod_{n=1}^{\infty}(1-q^n)^{-n}.$$

$$\sum_\pi q^{|\pi|}\overset{?}{=}\prod_{n=1}^{\infty}(1-q^n)^{-\frac{1}{2}n(n+1)}$$

$$(-1)^{\mu_\pi}[-\mathrm{v}_\pi]\in\mathbb{Q}\left(t_1,t_2,t_3,t_4,y^{\frac{1}{2}}\right)\text{ with }t_4=t_1^{-1}t_2^{-1}t_3^{-1}$$

$$\mu_\pi=|\{(i,i,i,j)\in\pi: j>i\}|$$

$$Z_\pi=\sum_{(a_1,a_2,a_3,a_4)\in\pi}t_1^{a_1}t_2^{a_2}t_3^{a_3}t_4^{a_4}$$

$$(-1)^{\mu_{\vec{\pi}}}[-\mathrm{v}_{\vec{\pi}}]\in\mathbb{Q}\left(t_1,t_2,t_3,t_4,w_1,\dots,w_r,y_1^{\frac{1}{2}},\dots,y_r^{\frac{1}{2}}\right)\text{ with }t_4=t_1^{-1}t_2^{-1}t_3^{-1}$$

$$\mathbf{Z}_r^{\mathrm{NP}}=\sum_{\vec{\pi}=(\pi_1,\dots,\pi_r)}(-1)^{\mu_{\vec{\pi}}}[-\mathbf{v}_{\vec{\pi}}]((-1)^r q)^{|\vec{\pi}|}$$

$$\mathrm{Exp}\big(F(z_1,\dots,z_n)\big)=\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}F(z_1^m,\dots,z_n^m)\right)$$

$$\left[z_1^{i_1}\cdots z_n^{i_n}\right]=z_1^{\frac{i_1}{2}}\cdots z_n^{\frac{i_n}{2}}-z_1^{-\frac{i_1}{2}}\cdots z_n^{-\frac{i_n}{2}}$$

$$\mathrm{Z}_r^{\mathrm{NP}}=\mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3][y]}{[t_1][t_2][t_3][t_4]\left[y^{\frac{1}{2}}q\right]\left[y^{\frac{1}{2}}q^{-1}\right]}\right)$$

$$\sum_{n=0}^{\infty}\chi\big(\mathrm{Hilb}^n(\mathbb{C}^3),\hat{\mathcal{O}}^{\mathrm{vir}}\big)(-q)^n=\mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3]}{[t_1][t_2][t_3]\left[\kappa^{\frac{1}{2}}q\right]\left[\kappa^{\frac{1}{2}}q^{-1}\right]}\right),\kappa=t_1t_2t_3$$

$$\sum_{n=0}^{\infty}\chi\big(\mathrm{Quot}_r^n(\mathbb{C}^3),\hat{\mathcal{O}}^{\mathrm{vir}}\big)((-1)^r q)^n=\mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3][\kappa^r]}{[t_1][t_2][t_3][\kappa]\left[\kappa^{\frac{r}{2}}q\right]\left[\kappa^{\frac{r}{2}}q^{-1}\right]}\right)$$

$$0\to\Lambda\to E\to\Lambda^\vee\to 0$$

$$
\begin{array}{ccccccc}
T_A|_M & \xrightarrow{ds} & E|_M \overset{q}{\cong} E^\vee|_M & \xrightarrow{(ds)^\vee} & \Omega_A|_M \\
 & & \downarrow s^\vee & & \| \\
 & & I/I^2|_M & \xrightarrow{d} & \Omega_A|_M
\end{array}
$$

$$\mathcal{O}_A \overset{o\otimes o}{\to} \det(E) \otimes \det(E) \overset{\mathrm{id}\otimes\Lambda^m q}{\to} \det(E) \otimes \det(E^\vee) \to \mathcal{O}_A$$

$$\pi_\Lambda : \det(E) \cong \det(\Lambda) \otimes \det(\Lambda^\vee) \cong \mathcal{O}_A$$

$$o_\Lambda = (-i)^{m/2} \pi_\Lambda^{-1}$$

$$[M]^{\mathrm{vir}} \in A_n\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right)$$

$$\hat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right)$$

$$A_n\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right)$$

$$n = \frac{1}{2}\mathrm{rk}(E^{\bullet})$$

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^b} : a, b \in \mathbb{Z}\right\}$$

$$K_0\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right) = K_0(M) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

$$[M]^{\mathrm{vir}} \in A_n^T\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right) \widehat{\mathcal{O}_M^{\mathrm{vir}}} \in K_0^T(M)_{\mathrm{loc}} K_0^T(M)_{\mathrm{loc}}$$

$$K_0^T(M)_{\mathrm{loc}} := K_0(M) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}\left(t^{\frac{1}{2}}\right)$$

$$\iota_* \hat{\mathcal{O}}_M^{\mathrm{vir}} = \hat{\Lambda}^\bullet \Lambda^\vee \otimes \det(T_A)^{-\frac{1}{2}} \in K_0^T(A)_{\mathrm{loc}},$$

$$\Lambda^\bullet V = \sum_{i=0}^{\mathrm{rk}\, V} (-1)^i \Lambda^i V, \quad \hat{\Lambda}^\bullet V = \frac{\Lambda^\bullet V}{\det(V)^{\frac{1}{2}}}.$$

$$\hat{\mathcal{O}}_M^{\mathrm{vir}} = \sum_{P \in M^T} (-1)^{n_P^\Lambda} \iota_{P*}\left(\frac{1}{\hat{\Lambda}^\bullet(\Omega_A - \Lambda^\vee)|_P}\right)$$

$$n_P^\Lambda \equiv \dim\mathrm{coker}(p_\Lambda \circ ds|_P)^f \ (\mathrm{mod} 2)$$

$$p_\Lambda \circ ds|_P$$

$$T_A|_P \xrightarrow{ds|_P} E|_P \xrightarrow{p_\Lambda} \Lambda|_P$$

$$E|_P = \Lambda|_P \oplus \Lambda^\vee|_P$$

$$N_{r,n}^{\mathrm{glob}} = \chi(M, \hat{\mathcal{O}}_M^{\mathrm{vir}} \otimes \hat{\Lambda}^\bullet \mathcal{H}om(\mathcal{V}, \mathcal{W})),$$

$$\mathsf{G}_r = \sum_{n=0}^{\infty} N_{r,n}^{\mathrm{glob}} q^n$$

$$\nu_n \colon \mathrm{Hilb}^n(\mathbb{C}^4) \to \mathrm{Sym}^n(\mathbb{C}^4)$$

$$R\nu_{n*}(\hat{\mathcal{O}}^{\mathrm{vir}} \otimes \hat{\Lambda}^\bullet \mathcal{H}om(\mathcal{V}, \mathcal{W})).$$

$$\mathrm{Exp}\left(\frac{G_1}{[t_1][t_2][t_3][t_4]}\right)$$

$$(\mathbb{C}^*)^4 = \mathrm{Spec}\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}](p_1, p_2, p_3, p_4) \in \mathbb{C}^4$$

$$(t_1, t_2, t_3, t_4) \cdot (p_1, p_2, p_3, p_4) = (t_1^{-1} p_1, t_2^{-1} p_2, t_3^{-1} p_3, t_4^{-1} p_4),$$

$$H^0(\mathbb{C}^4, \mathcal{O}_{\mathbb{C}^4}) \text{ is } \prod_{i=1}^4 (1 - t_i)^{-1} \prod_{i=1}^4 \left(1 - t_i^{-1}\right)^{-1}$$

$$K_0^T(Y)_{\mathrm{loc}} := K_0^T(Y) \otimes_{K_0^T(\mathrm{pt})} \mathbb{Q}\left(t^{\frac{1}{2}}\right), \text{ where } K_0^T(\mathrm{pt}) = \mathbb{Z}[t^{\pm 1}] \text{ and } t = (t_1, \ldots, t_{\dim T})$$

$$Y^T \hookrightarrow Y \text{ gives } K_0^T(Y^T)_{\mathrm{loc}} \cong K_0^T(Y)_{\mathrm{loc}}$$

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$$

$$[\mathcal{F}] = [\mathcal{F}_0] - [\mathcal{F}_1]$$

$$\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$$

$$\mathcal{H}(\mathcal{F}) := H^{-1}(\mathcal{F})[1] \oplus H^0(\mathcal{F})$$

$$\sigma_{n_1,\ldots,n_k} \colon \prod_{i=1}^k \mathrm{Sym}^{n_i}(X) \to \mathrm{Sym}^N(X)$$

$$U_{n_1,\ldots,n_k} \subset \prod_{i=1}^k \mathrm{Sym}^{n_i}(X)$$

$$\nu_{n_1,\ldots,n_k} \colon \prod_{i=1}^k \mathrm{Hilb}^{n_i}(X) \to \prod_{i=1}^k \mathrm{Sym}^{n_i}(X)$$

$$\begin{array}{ccc} \prod_{i=1}^{k} \mathrm{Hilb}^{n_i}(X)|_{U_{n_1,\dots,n_k}} & \longrightarrow & \mathrm{Hilb}^{N}(X) \\ \downarrow \nu_{n_1,\dots,n_k} & & \downarrow \nu_N \\ U_{n_1,\dots,n_k} & \xrightarrow{\sigma_{n_1,\dots,n_k}} & \mathrm{Sym}^{N}(X) \end{array}$$

$$Y := \prod_{i=1}^{k} \mathrm{Hilb}^{n_i}(\mathbb{C}^d)\Big|_{U_{n_1,\dots,n_k}}$$

$$\mathfrak{m} = (x_1, \dots, x_d) \subseteq \mathbb{C}[x_1, \dots, x_d]$$

$$R = \mathbb{C}[x_1, \dots, x_d]/\mathfrak{m}^n$$

$$\mathrm{Hilb}^n(\mathbb{C}^d, 0) \cong \mathrm{Hilb}^n(X)$$

$$\mathrm{Hilb}\ ^n(X)\pi_0(\mathrm{Hilb}^n(X)) = \pi_0(\mathrm{Gr})\ \pi_1(\mathrm{Hilb}^n(X)) = \pi_1(\mathrm{Gr})\ \mathrm{Hilb}^n(X)$$

$$V \subseteq \prod_{i=1}^{k} (\mathbb{C}^d)^{n_i}$$

$$\prod_{i=1}^{k} S_{n_i}$$

$$\prod_{i=1}^{k} S_{n_i}\pi_1(U_{n_1,\dots,n_k}) = \pi_1(Y)$$

$$L^{\otimes n} \cong \mathcal{O}_Y$$

$$\mathbb{V}(L) \to \mathbb{V}(\mathcal{O}_Y) = Y \times \mathbb{A}^1$$

$$\phi_{m,n} : \mathcal{F}_m \boxtimes \mathcal{F}_n|_{U_{m,n}} \xrightarrow{\cong} \mathcal{F}_{m+n}|_{U_{m,n}},$$

$$\phi_{m,n+p} \circ (\mathrm{id}_{\mathcal{F}_m} \boxtimes \phi_{n,p}) = \phi_{m+n,p} \circ (\phi_{m,n} \boxtimes \mathrm{id}_{\mathcal{F}_p})$$

$$\tau: \mathrm{Sym}^m(X) \times \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(X) \times \mathrm{Sym}^m(X)$$

$$
\begin{array}{ccc}
\mathcal{F}_m \boxtimes \mathcal{F}_n|_{U_{m,n}} & \xrightarrow{\phi_{m,n}} & \mathcal{F}_{m+n}|_{U_{m,n}} \\
\cong \downarrow v & & \cong \downarrow w \\
\tau^*(\mathcal{F}_n \boxtimes \mathcal{F}_m|_{U_{n,m}}) & \xrightarrow{\tau^*\phi_{n,m}} & \tau^*(\mathcal{F}_{n+m}|_{U_{n,m}})
\end{array}
$$

$$\mathcal{F}_m \boxtimes \mathcal{F}_n \cong \tau^*(\mathcal{F}_n \boxtimes \mathcal{F}_m)$$

$$
\begin{array}{ccc}
U_{m,n} & \xrightarrow{\tau} & U_{n,m} \\
& \searrow^{\sigma_{m,n}} & \downarrow \sigma_{n,m} \\
& & \mathrm{Sym}^{m+n}(X)
\end{array}
$$

$$\phi_{n_1,\ldots,n_k} : \boxtimes_{i=1}^{k} \mathcal{F}_{n_i}|_{U_{n_1,\ldots,n_k}} \xrightarrow{\cong} \mathcal{F}_N|_{U_{n_1,\ldots,n_k}}.$$

$$\boxtimes_j \mathrm{Sym}^{m_j}(\mathcal{F}_j) \quad \text{on} \quad \prod_j \mathrm{Sym}^{m_j}\,\mathrm{Sym}^j(X).$$

$$\boxtimes_j \mathrm{Sym}^{m_j}(\mathcal{F}_j)|_U \xrightarrow{\cong} \mathcal{F}_N|_U.$$

$$1 + \sum_{n=1}^{\infty} \chi(\mathrm{Sym}^n(X), \mathcal{F}_n) q^n = \mathrm{Exp}\left(\sum_{n=1}^{\infty} \chi(X, \mathcal{G}_n) q^n \right)$$

$$\phi_{m,n} : \mathcal{F}_m \boxtimes \mathcal{F}_n|_{U_{m,n}} \xrightarrow{\cong} \mathcal{F}_{m+n}|_{U_{m,n}}$$

$$\phi_{m,n+p} \circ \left(\mathrm{id}_{\mathcal{F}_m} \boxtimes \phi_{n,p} \right) = c_{m,n,p} \cdot \phi_{m+n,p} \circ \left(\phi_{m,n} \boxtimes \mathrm{id}_{\mathcal{F}_p} \right)$$

$$\psi_{1^k} = \phi_{1,k-1} \circ \left(\mathrm{id}_{\mathcal{F}_1} \boxtimes \phi_{1,k-2}\right) \circ \cdots \circ \left(\mathrm{id}_{\mathcal{F}_1} \boxtimes \cdots \boxtimes \mathrm{id}_{\mathcal{F}_1} \boxtimes \phi_{1,1}\right)$$

$$\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_1|_{U_{1,\dots,1}} \to \mathcal{F}_k\big|_{U_{1,\dots,1}}$$

$$\lambda_{m,n}\phi_{m,n} \circ (\psi_{1^m} \boxtimes \psi_{1^n}) = \psi_{1^{m+n}}$$

$$\psi_{m,n} \circ \left(\psi_{1^m} \boxtimes \psi_{1^n}\right) = \psi_{1^{m+n}}.$$

$$\psi_{m,n+p} \circ \left(\mathrm{id}_{\mathcal{F}_m} \boxtimes \psi_{n,p}\right) = c'_{m,n,p} \cdot \psi_{m+n,p} \circ \left(\psi_{m,n} \boxtimes \mathrm{id}_{\mathcal{F}_p}\right)$$

$$\psi_{1^m} \boxtimes \psi_{1^n} \boxtimes \psi_{1^p}$$

$$\psi_{1^{m+n+p}} = c'_{m,n,p} \cdot \psi_{1^{m+n+p}}$$

$$A = \bigoplus\nolimits_{n\geqslant 0} A_n$$

$$A_n = \bigoplus_{x\in \mathrm{Sym}^n(X)\setminus\Delta} \mathcal{F}_n\Big|_x$$

$$\mu_{x,y}\colon \mathcal{F}_m|_x \otimes \mathcal{F}_n|_y \xrightarrow{\cong} \mathcal{F}_{m+n}|_{x+y},$$

$$\mu_{m,n}\colon A_m \otimes_{\mathbb{C}} A_n \to A_{m+n}$$

$$\mathcal{F}_1^{\otimes m} \to \mathcal{F}_m \{\mathcal{F}_n[n]\}_{n=1}^{\infty} (-1)^{mn} d_{m,n}$$

$$\begin{array}{ccc} E[m] \otimes F[n] & \longrightarrow & (E \otimes F)[m+n] \\ \downarrow & & \downarrow \\ F[n] \otimes E[m] & \longrightarrow & (F \otimes E)[m+n] \end{array}$$

$$s \otimes s - q(s,s)$$

$$\mathrm{rev}(s_1 \cdots s_m) = s_m \cdots s_1$$

$$C(E) \otimes M \to M$$

$$\mathrm{rev}\colon (C(E)\Lambda)^{\dagger} \cong \Lambda C(E)$$

$$C(E) \to \mathcal{H}om(C(E)\Lambda, C(E)\Lambda)$$

$$\det\Lambda \cong \Lambda C(E) \otimes_{C(E)} C(E)\Lambda$$

$$e_1 \wedge \cdots \wedge e_n \mapsto e_1 \cdots e_n f_1 \cdots f_n \otimes e_1 \cdots e_n.$$

$$e_{[n]} f_I \otimes f_J e_{[n]} = e_{[n]} f_I f_J \otimes e_{[n]} = \begin{cases} 0 \text{ if } I \cap J \neq \varnothing \\ \pm e_{[n]} f_{I\cup J} \otimes e_{[n]} \text{ if } I \cap J = \varnothing. \end{cases}$$

$$e_{[n]} f_I \otimes e_{[n]} = \pm 2^{|I|-n} e_{[n]} f_{[n]} e_{[n]\setminus I} \otimes e_{[n]} = \pm 2^{|I|-n} e_{[n]} f_{[n]} \otimes e_{[n]\setminus I} e_{[n]}$$

$$\Lambda C(E) \otimes_{C(E)} C(E)\Lambda$$

$$e_{[n]} f_{[n]} \otimes e_{[n]}$$

$$e_{[n]} f_{[n]} - e_{[n]} f'_{[n]}$$

$$e_{[n]} f_{[n]} \otimes e_{[n]} = e_{[n]} f'_{[n]} \otimes e_{[n]}$$

$$\phi: S^\dagger \otimes_{C(E)} S \cong \mathcal{O}_X$$

$$C(E) \to \mathcal{H}om(S,S) = S \otimes S^\vee$$

$$S \otimes S^\dagger \cong C(E)$$

$$\begin{aligned}(C(E)\Lambda \otimes L^{-1})^\dagger \otimes_{C(E)} (C(E)\Lambda \otimes L^{-1}) &\cong L^{-2} \otimes \Lambda C(E) \otimes_{C(E)} C(E)\Lambda \\ &\cong \det(\Lambda)^{-1} \otimes \det(\Lambda) \cong \mathcal{O}_X\end{aligned}$$

$$\begin{aligned}\left(S^\dagger \otimes_{C(E)} C(E)\Lambda\right)^\dagger \otimes S^\dagger \otimes_{C(E)} C(E)\Lambda &\cong \Lambda C(E) \otimes_{C(E)} S \otimes S^\dagger \otimes_{C(E)} C(E)\Lambda \\ &\cong \Lambda C(E) \otimes_{C(E)} C(E)\Lambda \cong \det\Lambda\end{aligned}$$

$$(\det\Lambda)_S^{1/2} := \Lambda C(E) \otimes_{C(E)} S$$

$$(\det\Lambda)_S^{1/2} = \Lambda C(E) \otimes_{C(E)} S \to C(E) \otimes_{C(E)} S \to S$$

$$\omega_o = (e_1 f_1 - 1) \cdots (e_n f_n - 1)$$

$$\omega_o s + s\omega_o = 0$$

$$L \otimes \Lambda^\bullet \Lambda^\vee \cong S$$

$$t \otimes f_{i_1} \wedge \cdots \wedge f_{i_k} \mapsto t^\vee \otimes f_{i_1} \cdots f_{i_k} e_1 \cdots e_n$$

$$\cdots \xrightarrow{s_{\Lambda^\vee} \wedge + s_\Lambda \lrcorner} L \otimes \Lambda^{\text{even}} \Lambda^\vee \xrightarrow{s_{\Lambda^\vee} \wedge + s_\Lambda \lrcorner} L \otimes \Lambda^{\text{odd}} \Lambda^\vee \xrightarrow{s_{\Lambda^\vee} \wedge + s_\Lambda \lrcorner} \cdots$$

$$Z(s) = Z(s_\Lambda) \cap Z(s_{\Lambda^\vee})$$

$$\mathcal{H}(S,s) = H^{-1}(S,s)[1] \oplus H^0(S,s)$$

$$(E_1 \oplus E_2, q_1 \oplus q_2)\ o_1 \otimes o_2$$

$$A \mathbin{\widehat{\otimes}} B$$

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$$

$$C(E_1) \mathbin{\widehat{\otimes}} C(E_2) \cong C(E_1 \oplus E_2)$$

$$s_1 \otimes 1 \mapsto (s_1, 0)$$

$$1 \otimes s_2 \mapsto (0, s_2)$$

$$\omega_{o_1} \otimes \omega_{o_2} \mapsto \omega_o$$

$$o_1 \otimes o_2 \; (E_1 \oplus E_2, q_1 \oplus q_2)$$

$C(E_1 \oplus E_2)$ and $C(E_1) \mathbin{\widehat{\otimes}} C(E_2)$

$M_1 \otimes M_2$ is a $C(E_1) \mathbin{\widehat{\otimes}} C(E_2)$

$$(a \otimes b)(m \otimes m') = (-1)^{|b||m|} am \otimes bm'$$

$$(M_1 \otimes M_2)^\dagger \otimes_{C(E_1 \oplus E_2)} (N_1 \otimes N_2) \cong \left(M_1^\dagger \otimes_{C(E_1)} N_1\right) \otimes \left(M_2^\dagger \otimes_{C(E_2)} N_2\right).$$

$$a \otimes b \otimes c \otimes d \mapsto (-1)^{(|a|+|c|)|b|} a \otimes c \otimes b \otimes d$$

$C(E_1) \mathbin{\widehat{\otimes}} C(E_2)$-module $S_1 \otimes S_2 E_1 \oplus E_2$.[5]

$$(S_1 \otimes S_2)^\dagger \otimes_{C(E_1 \oplus E_2)} (S_1 \otimes S_2) \cong \left(S_1^\dagger \otimes_{C(E_1)} S_1\right) \otimes \left(S_2^\dagger \otimes_{C(E_2)} S_2\right) \cong \mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$$

$$S \setminus S_1 = S_1^\vee \otimes_{C(E_1)} S$$

$$s_2(a \otimes b) = (-1)^{|a|} a \otimes s_2 b$$

$$S_1 \otimes (S \backslash S_1) = S_1 \otimes S_1^\vee \otimes_{C(E_1)} S \overset{m \otimes \mathrm{id}_S}{\cong} C(E_1) \otimes_{C(E_1)} S \cong S,$$

$C(E_1) \mathbin{\widehat{\otimes}} C(E_2)$-modules

$$s_2(a \otimes (b \otimes c)) = (-1)^{|a|} a \otimes s_2(b \otimes c) = (-1)^{|a|+|b|} a \otimes b \otimes s_2 c$$

$$\begin{aligned} m \otimes \mathrm{id}_S(s_2(a \otimes b \otimes c)) &= (-1)^{|a|+|b|} m(a \otimes b) \otimes s_2 c \\ &= (-1)^{|m(a \otimes b)|} m(a \otimes b) \otimes s_2 c = s_2(m(a \otimes b) \otimes c) \end{aligned}$$

$$\begin{aligned} \mathcal{O}_X &\cong S^\dagger \otimes_{C(E_1 \oplus E_2)} S \cong \left(S_1 \otimes (S \setminus S_1)\right)^\dagger \otimes_{C(E_1 \oplus E_2)} \left(S_1 \otimes (S \setminus S_1)\right) \\ &\cong \left(S_1^\dagger \otimes_{C(E_1)} S_1\right) \otimes \left((S \setminus S_1)^\dagger \otimes_{C(E_2)} (S \setminus S_1)\right) \cong (S \setminus S_1)^\dagger \otimes_{C(E_2)} (S \setminus S_1) \end{aligned}$$

$$\det(\Lambda_1)^{1/2}_{S_1} \otimes \det(\Lambda_2)^{1/2}_{S_2} \to S_1 \otimes S_2$$
$$\det(\Lambda_1 \oplus \Lambda_2)^{1/2}_{S_1 \otimes S_2} \to S_1 \otimes S_2$$

$$\det(\Lambda_1)^{1/2}_{S_1} \otimes \det(\Lambda_2)^{1/2}_{S_2} \cong \det(\Lambda_1 \oplus \Lambda_2)^{1/2}_{S_1 \otimes S_2}$$

$$\left(E|_{A_0}, q\right) = (E_0 \oplus E', q_0 + q') \; \mathrm{Spec}\hat{\mathcal{O}}_{A, f(p)} \to A$$

$$N = N_{A_0/A} = f^* T_A / T_{A_0} E'|_{Z(s_0)}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{A_0} & \longrightarrow & f^*T_A & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ds_0} & & \downarrow{\scriptstyle ds} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & E_0 & \longrightarrow & f^*E & \longrightarrow & E' & \longrightarrow & 0
\end{array}$$

$$\hat{\mathcal{O}}_{A,f(p)} \cong \mathbb{C}\left[\left[x_1, \dots, x_{d_0}, y_1, \dots y_{d-d_0}\right]\right]$$

$$E \cong \mathcal{O}_A^{2n}$$

$$q\left(a_1, \dots, a_{2n_0}, b_1, \dots, b_{2n-2n_0}\right) = \sum_{i=1}^{n_0} a_i a_{i+n_0} + \sum_{i=1}^{n-n_0} b_i b_{i+n-n_0}$$

$$\mathcal{O}_{A_0}^{2n_0} \to \mathcal{O}_A^{2n}\Big|_{A_0}$$

$$\left(g_1, \dots, g_{2n_0}, h_1, \dots, h_{2d-2d_0}\right)$$

$$\left(x_1, \dots, x_{d_0}\right) + \left(y_1, \dots, y_{d-d_0}\right)^2\left(y_1, \dots, y_{d-d_0}\right)$$

$$E' \cong \mathcal{O}_{A_0}^{2n-2n_0} \subseteq \mathcal{O}_{A_0}^{2n}$$

$$E' \cong \mathcal{O}_{A_0}^{2n-2n_0}$$

$$(g_1, \dots, g_{2n_0}, y_1, \dots, y_{d-d_0}, \overbrace{0, \dots, 0}^{d-d_0})$$

$$\mathbb{C}\left[\left[x_1, \dots, x_{d_0}\right]\right]$$

$$R = \mathbb{C}\left[\left[x_1, \dots, x_{d_0}, y_1, \dots y_{d-d_0}\right]\right]$$

$$\left(0, \dots, 0, h_1', \dots, h_{2d-2d_0}'\right)$$

$$\left(g_1, \dots, g_{2n_0}, y_1 + h_1'', \dots, y_{d-d_0} + h_{d-d_0}'', h_{d-d_0+1}'', \dots, h_{2d-2d_0}''\right)$$

$$\left(g_1', \dots, g_{2n_0}', y_1, \dots, y_{d-d_0}, h_{d-d_0+1}''', \dots, h_{2d-2d_0}'''\right)$$

$$g_i' \in I_{1,0} + I_{0,2} \text{ and } h_i''' \in I_{1,1} + I_{0,2}$$

$$\left(v_1, \dots, v_{n_0}, v_1', \dots, v_{n_0}', w_1, \dots, w_{d-d_0}, w_1', \dots, w_{d-d_0}'\right).$$

$$q(s, v_1) = g_{n_0+1}'$$

$$q(s, v_1) = F(x) + \sum_{i=1}^{d-d_0} y_i F_i(x, y)$$

$$v_1 - \sum_{i=1}^{d-d_0} F_i w_i'$$

$$\left(g_1'', \dots, g_{2n_0}'', y_1, \dots y_{d-d_0}, h_{d-d_0+1}'''', \dots, h_{2d-2d_0}''''\right)$$

$$h_i'''' = y_1 G_i(x, y) + F_i(x, y)$$

$$w_1 + \sum_{i=2}^{d-d_0} G_i w_i'$$

$$q(s, w_i) = F_i(x, y)$$

$$0 = q(s, s) = \sum_{i=1}^{n_0} g_i'' g_{i+n_0}'' + \sum_{i=1}^{d-d_0} y_i q(s, w_i)$$

$$S' = S_0^{\vee} \otimes_{C(E_0)} S\big|_{A_0}$$

$\mathcal{O}_{Z(s_0)}$-modules $(\det N)_{S'}^{1/2}\Big|_{Z(s_0)} \to S'\Big|_{Z(s_0)}$

$$f^*(\mathcal{H}(S, s)) \to \mathcal{H}(f^*(S, s)) \cong \mathcal{H}\big((S_0, s_0) \otimes (S', 0)\big) \cong \mathcal{H}(S_0, s_0) \otimes (S', 0)\big|_{Z(s_0)}$$

$$\mathcal{H}(S_0, s_0) \otimes (\det N)_{S'}^{1/2} \to \mathcal{H}(S_0, s_0) \otimes (S', 0)\big|_{Z(s_0)}$$

$$f^*(\mathcal{H}(S, s)) \xrightarrow{\cong} \mathcal{H}(S_0, s_0) \otimes (\det N)_{S'}^{1/2}.$$

$E = \bar{E}_0 \oplus \bar{E}'$, while $\bar{E}_0|_{A_0} = E_0$ and $\bar{E}'|_{A_0} = E'$.

$$\Lambda = \Lambda_0 \oplus \Lambda' \subseteq Es = (\bar{s}_0, \bar{s}')\Lambda'|_{A_0} = NE'$$

$S \cong C(E)\Lambda$ and $S_0 \cong C(E_0)\Lambda_0|_{A_0}$

$$S \cong C(E)\Lambda \cong C(\bar{E}_0)\Lambda_0 \otimes C(\bar{E}')\Lambda'$$

$$(S, s) \cong (C(\bar{E}_0)\Lambda_0, \bar{s}_0) \otimes (C(\bar{E}')\Lambda', \bar{s}')$$

$$f^*\mathcal{H}(S) \cong f^*\big(\mathcal{H}(\bar{S}_0 \otimes \bar{S}')\big) \overset{\aleph}{\cong} f^*\big(\mathcal{H}(\bar{S}_0) \otimes \mathcal{H}(\bar{S}')\big) \cong f^*\big(\mathcal{H}(\bar{S}_0)\big) \otimes f^*\big(\mathcal{H}(\bar{S}')\big)$$

$$\overset{\gimel}{\cong} \mathcal{H}(f^*\bar{S}_0) \otimes f^*\big(\mathcal{H}(\bar{S}')\big) \cong \mathcal{H}(S_0) \otimes f^*\big(\mathcal{H}(\bar{S}')\big) \to \mathcal{H}(S_0) \otimes \mathcal{H}\big(f^*(\bar{S}')\big) \cong \mathcal{H}(S_0) \otimes S'.$$

$f^*\big(\mathcal{H}(\bar{S}')\big) \to \mathcal{H}\big(f^*(\bar{S}')\big) \cong S'$ and $(\det N)_{S'}^{1/2} \to S'$

$\bar{S}' = C(\bar{E}')\Lambda'$ and $S' = C(E')\Lambda'|_{A_0}$

$\det(\Lambda')|_{A_0} \subseteq C(E')\Lambda'\big|_{A_0}$ As $N = \Lambda'|_{A_0}$

$$f^*(\mathcal{H}(S, s)) \xrightarrow{\cong} \mathcal{H}(S_0, s_0) \otimes (\det N)_{S'}^{1/2}.$$

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 E_i \subseteq f_{i+1}^* E_{i+1}\ E_0 \subseteq (f_2 \circ f_1)^* E_2$$

$$S_j \setminus S_i = S_i^{\vee} \otimes_{C(E_i)} S_j\big|_{A_i}$$

$C\left(E_j\big|_{A_i}/E_i\right) N_{ij}$ of $E_j\big|_{A_i}/E_i$

$$\det(N_{ij})^{1/2}_{S_j\setminus S_i}\Big|_{Z(s_i)} \to S_j \setminus S_i\Big|_{Z(s_i)}$$

$$(S_1 \setminus S_0) \otimes (S_2 \setminus S_1) = S_0^\vee \otimes_{C(E_0)} S_1 \otimes S_1^\vee \otimes_{C(E_1)} S_2 \cong S_0^\vee \otimes_{C(E_0)} C(E_1) \otimes_{C(E_1)} S_2$$
$$\cong S_0^\vee \otimes_{C(E_0)} S_2 = S_2 \setminus S_0,$$

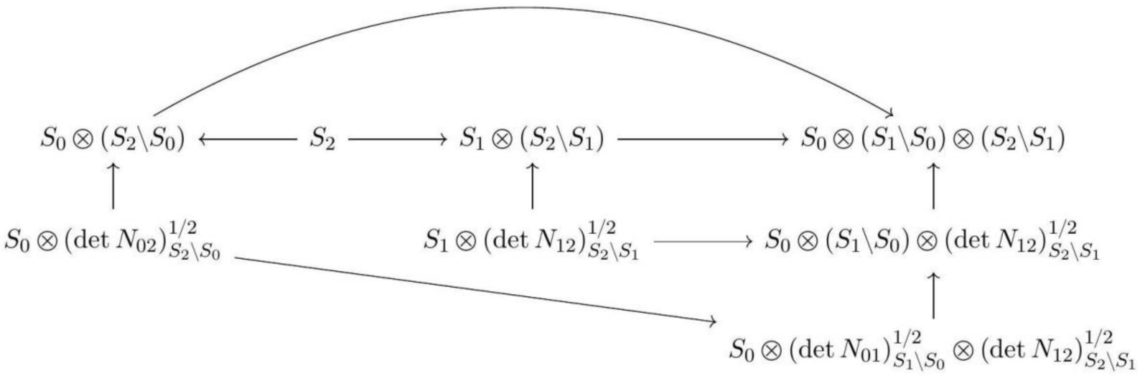
$$(\det N_{01})^{1/2}_{S_1\setminus S_0} \otimes (\det N_{12})^{1/2}_{S_2\setminus S_1}\Big|_{Z(s_0)} \cong (\det N_{02})^{1/2}_{S_2\setminus S_0}\Big|_{Z(s_0)}$$

$$f_1^*(\mathcal{H}(S_1, s_1)) \to \mathcal{H}(S_0, s_0) \otimes (\det N_{01})^{1/2}_{S_1\setminus S_0}$$

$$f_2^*(\mathcal{H}(S_2, s_2)) \to \mathcal{H}(S_1, s_1) \otimes (\det N_{12})^{1/2}_{S_2\setminus S_1}$$

$$\mathcal{H}(S_2, s_2)|_{A_0} \cong \mathcal{H}(S_0, s_0) \otimes (\det N_{01})^{1/2}_{S_1\setminus S_0} \otimes (\det N_{12})^{1/2}_{S_2\setminus S_1}\Big|_{A_0}$$

$$\mathcal{H}(S_2, s_2)|_{A_0} \to \mathcal{H}(S_0, s_0) \otimes (\det N_{02})^{1/2}_{S_2\setminus S_0}$$



$$\mathcal{H}(S, s)|_{A_0} \cong \mathcal{H}((S_1, s_1) \otimes (S_2, s_2)) \otimes (\det N)^{1/2}_{S\setminus S_1\otimes S_2}$$

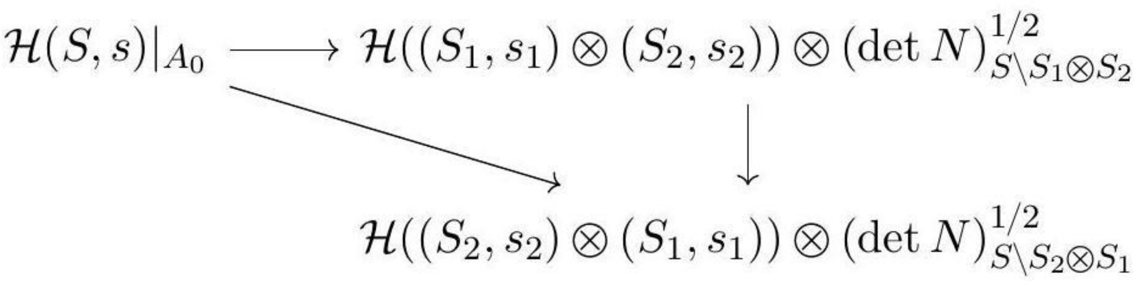
$$\mathcal{H}(S, s)|_{A_0} \cong \mathcal{H}((S_2, s_2) \otimes (S_1, s_1)) \otimes (\det N)^{1/2}_{S\setminus S_2\otimes S_1}.$$

$$S_1 \otimes S_2 \cong S_2 \otimes S_1$$

$$a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \text{ of } C(E_0)\text{-modules}$$

$$S \setminus (S_1 \otimes S_2) \cong S \setminus (S_2 \otimes S_1) \text{ of } C(E|_{A_0}/E_0)\text{-modules}$$

$$(\det N)^{1/2}_{S\setminus S_1\otimes S_2}\Big|Z(s_0) \cong (\det N)^{1/2}_{S\setminus S_2\otimes S_1}\Big|Z(s_0)$$



$$
\begin{array}{ccc}
S|_{A_0} & \searrow & \\
\downarrow & & \\
(S_1 \otimes S_2) \otimes (S\backslash(S_1 \otimes S_2)) & \longrightarrow & (S_2 \otimes S_1) \otimes (S\backslash(S_2 \otimes S_1)) \\
\uparrow & & \uparrow \\
(S_1 \otimes S_2) \otimes C(E') \otimes_{C(E')} S\backslash(S_1 \otimes S_2) & \longrightarrow & (S_2 \otimes S_1) \otimes C(E') \otimes_{C(E')} S\backslash(S_2 \otimes S_1) \\
\uparrow & & \uparrow \\
(S_1 \otimes S_2) \otimes (\det N)^{1/2}_{S\backslash(S_1 \otimes S_2)} & \longrightarrow & (S_2 \otimes S_1) \otimes (\det N)^{1/2}_{S\backslash(S_2 \otimes S_1)}
\end{array}
$$

$$\left[F \xrightarrow{a} E \cong E^\vee \xrightarrow{a^\vee} F^\vee\right]$$

$$E^\bullet|_{M^T} = E^\bullet|^f_{M^T} \oplus E^\bullet|^m_{M^T}.$$

$$N^{\mathrm{vir}} := \left(E^\blacksquare|^m_{M^T}\right)^\vee$$

$$E^\blacksquare|^f_{M^T}\ (\tau^{\geqslant -1}\mathbb{L}_M)|^f_{M^T}$$

$$(\tau^{\geqslant -1}\mathbb{L}_M)|^f_{M^T} \to \tau^{\geqslant -1}\mathbb{L}_{M^T}$$

$$\mathcal{O}_M \cong \det(E^\cdot), o^f: \mathcal{O}_{M^T} \cong \det\left(E^\cdot|^f_{M^T}\right), \text{ and } o^m: \mathcal{O}_{M^T} \cong \det\left(E^\cdot|^m_{M^T}\right)$$

$$[M]^{\mathrm{vir}} \in A^T_*\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right), \hat{\mathcal{O}}^{\mathrm{vir}}_M \in K^T_0(M)_{\mathrm{loc}}$$

$$[M^T]^{\mathrm{vir}} \in A_*\left(M^T, \mathbb{Z}\left[\frac{1}{2}\right]\right), \hat{\mathcal{O}}^{\mathrm{vir}}_{M^T} \in K_0\left(M^T, \mathbb{Z}\left[\frac{1}{2}\right]\right)$$

$$o|_{M^T} = o^f \otimes o^m$$

$$[M]^{\mathrm{vir}} = \iota_*\left(\frac{1}{\sqrt{e}(N^{\mathrm{vir}})} \cap [M^T]^{\mathrm{vir}}\right), \hat{\mathcal{O}}^{\mathrm{vir}}_M = \iota_*\left(\frac{1}{\sqrt{\mathrm{e}}(N^{\mathrm{vir}})} \cdot \hat{\mathcal{O}}^{\mathrm{vir}}_{M^T}\right)$$

$$\Lambda|^m_{M^T} \subset E|^m_{M^T}, \Lambda|^f_{M^T} \subset E|^f_{M^T}$$

$$\left[F \xrightarrow{a} E \cong E^\vee \xrightarrow{a^\vee} F^\vee\right]$$

$$[P]^{\mathrm{vir}} = [P] \text{ and } \hat{\mathcal{O}}^{\mathrm{vir}}_P = \mathcal{O}_P$$

$$\text{If } \mathrm{vd}P < 0, [P]^{\mathrm{vir}} = 0 \text{ and } \hat{\mathcal{O}}^{\mathrm{vir}}_P = 0.$$

$$[M]^{\mathrm{vir}} = \sum_{\substack{P \in M^T \\ \mathrm{vd}P=0}} (-1)^{n^\Lambda_P} \iota_{P*}\left(\frac{1}{e((F-\Lambda)|_P)}\right), \quad \hat{\mathcal{O}}^{\mathrm{vir}}_M = \sum_{\substack{P \in M^T \\ \mathrm{vd}P=0}} (-1)^{n^\Lambda_P} \iota_{P*}\left(\frac{1}{\hat{\Lambda}^\bullet(F^\vee - \Lambda^\vee)|_P}\right),$$

$$(-1)^{n_P^\Lambda} = \begin{cases} 1 & \text{if } a(F|_P^f) \text{ is positively oriented} \\ -1 & \text{otherwise.} \end{cases}$$

$$\dim(\Lambda \cap \Lambda') \equiv \frac{1}{2}\dim(E) \pmod{2}$$

$$E|_P = \Lambda|_P \oplus \Lambda^\vee|_P$$

$$\operatorname{dimcoker}(p_\Lambda \circ a|_P)^f \equiv \operatorname{dimcoker}(p_{\Lambda^\vee} \circ a|_P)^f + \dim(\Lambda^\vee|_P^f) \pmod{2}.$$

$$l = \dim(\Lambda|_P^f) = \dim(F|_P^f)$$

$$\operatorname{dimcoker}(p_\Lambda \vee\circ a|_P)^f = \dim\left(a(F|_P^f) \cap \Lambda|_P^f\right)$$

$$\Lambda|_P^f \text{ and } \Lambda^\vee|_P^f$$

$$\dim\left(\Lambda|_P^f \cap a(F|_P^f)\right) \equiv \dim\left(\Lambda^\vee|_P^f \cap a(F|_P^f)\right) \pmod{2}$$

$$\dim\left(\Lambda|_P^f \cap a(F|_P^f)\right) \not\equiv \dim\left(\Lambda^\vee|_P^f \cap a(F|_P^f)\right) \pmod{2}$$

$$n_P^\Lambda \equiv \operatorname{dimcoker}(p_\Lambda \circ a|_P)^f \pmod{2}$$

$$p_\Lambda: E|_P \to \Lambda|_P,\ E|_P = \Lambda|_P \oplus \Lambda^\vee|_P$$

$$\Lambda|_P^f, a(F|_P^f)$$

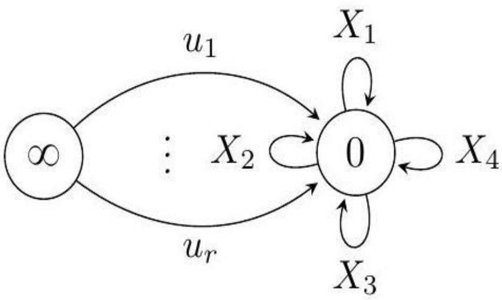
$$\dim\left(a(F|_P^f) \cap \Lambda|_P^f\right) \equiv \frac{1}{2}\dim(E|_P^f) = \dim(\Lambda^\vee|_P^f) \pmod{2}$$

$$\operatorname{dimcoker}(p_{\Lambda^\vee} \circ a|_P)^f = \dim\left(a(F|_P^f) \cap \Lambda|_P^f\right)$$
$$\operatorname{dimcoker}(p_\Lambda \circ a|_P)^f \equiv \operatorname{dimcoker}(p_{\Lambda^\vee} \circ a|_P)^f + \dim(\Lambda^\vee|_P^f) \pmod{2}$$

$$\begin{aligned} n_P^\Lambda &\equiv \dim\left(a(F|_P^f) \cap \Lambda|_P^f\right) - \dim(\Lambda^\vee|_P^f) \\ &= \operatorname{dimcoker}(p_{\Lambda^\vee} \circ a|_P)^f - \dim(\Lambda^\vee|_P^f) \\ &\equiv \operatorname{dimcoker}(p_\Lambda \circ a|_P)^f \pmod{2}. \end{aligned}$$

$$\mathcal{N}_{r,n}^{\text{glob}} \in K_0^{\mathbb{T}'}\left(\operatorname{Quot}_r^n(\mathbb{C}^4)\right)$$

$$N_{r,n}^{\text{glob}} = \chi\left(\operatorname{Quot}_r^n(\mathbb{C}^4), \mathcal{N}_{r,n}^{\text{glob}}\right)$$



$$W = \mathbb{C}^4 \otimes \operatorname{End}(V) \oplus \operatorname{Hom}(\mathbb{C}^r, V)$$

$$g \cdot (X_1, \dots, X_4, u_1, \dots, u_r) = (gX_1g^{-1}, \dots, gX_4g^{-1}, gu_1, \dots, gu_r)$$

$$X_i \in \mathrm{End}(V), u_i \in V.$$

$$\mathbb{C}\langle X_1, \dots, X_4\rangle \cdot \langle u_1, \dots, u_r\rangle = V$$

$$\mathrm{ncQuot}_r^n(\mathbb{C}^4) = [U/\mathrm{GL}(V)] \subset [W/\mathrm{GL}(V)].$$

$$E = \Lambda^2\mathbb{C}^4 \otimes \mathrm{End}(V) \otimes \mathcal{O}_U$$

$$(\Lambda^2\mathbb{C}^4 \otimes \mathrm{End}(V)) \otimes (\Lambda^2\mathbb{C}^4 \otimes \mathrm{End}(V)) \to \Lambda^4\mathbb{C}^4 \overset{\omega^\vee(-)}{\cong} \mathbb{C}$$
$$(\omega_1 \otimes f_1) \otimes (\omega_2 \otimes f_2) \mapsto \omega_1 \wedge \omega_2 \cdot \mathrm{tr}(f_1 \circ f_2).$$

$$s \in H^0(U,E), \left(\sum_{i=1}^{4} e_i \otimes f_i, u_1, \dots, u_r\right) \mapsto \sum_{i,j=1}^{4} (e_i \wedge e_j) \otimes (f_i \circ f_j), f_i \in \mathrm{End}(V)$$

$$\mathbb{C}^3 = \langle e_1, e_2, e_3\rangle$$

$$\Lambda = (\langle e_4\rangle \wedge \mathbb{C}^3) \otimes \mathrm{End}(V) \otimes \mathcal{O}_U$$

$$Z(s) \cong \mathrm{Quot}_r^n(\mathbb{C}^4)$$

$$\left(\textstyle\sum_i e_i \otimes f_i, u_1, \dots, u_r\right)$$

$$[f_i, f_j] = f_i \circ f_j - f_j \circ f_i = 0, \forall i,j = 1, \dots, 4$$

$$Z(s) \cong \mathrm{Quot}_r^n(\mathbb{C}^4)$$

$$q\left(\sum_{i,j=1}^{4} (e_i \wedge e_j) \otimes (f_i \circ f_j), \sum_{i,j=1}^{4} (e_i \wedge e_j) \otimes (f_i \circ f_j)\right)$$
$$= e_1 \wedge e_2 \wedge e_3 \wedge e_4 \otimes \mathrm{tr}\left(\sum_{\sigma \in S_4} (-1)^{|\sigma|} f_{\sigma(1)} \circ f_{\sigma(2)} \circ f_{\sigma(3)} \circ f_{\sigma(4)}\right)$$

$$\lambda = \sum_{i=1}^{3} (e_4 \wedge e_i) \otimes f_i \in (\langle e_4\rangle \wedge \mathbb{C}^3) \otimes \mathrm{End}(V)$$

$$(t_1, \dots, t_4, w_1, \dots, w_r, y_1, \dots, y_r) \cdot (X_1, \dots, X_4, u_1, \dots, u_r)$$
$$= (t_1^{-1}X_1, \dots, t_4^{-1}X_4, w_1^{-1}u_1, \dots, w_r^{-1}u_r)$$

$$\mathcal{V}|_{\mathrm{Quot}_r^n(\mathbb{C}^4)} \cong \pi_*\mathcal{Q}$$

$$\mathrm{Quot}_r^n(\mathbb{C}^4) \times \mathbb{C}^4$$

$$E = \Lambda^2\mathbb{C}^4 \otimes \mathcal{E}nd(\mathcal{V})$$

$$M = \mathrm{Quot}_r^n(\mathbb{C}^4)$$

$$[M]^{\mathrm{vir}} \in A_{rn}^{\mathbb{T}}\left(M, \mathbb{Z}\left[\frac{1}{2}\right]\right), \hat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0^{\mathbb{T}}(M)_{\mathrm{loc}}$$

$$\mathcal{W} = \bigoplus_{i=1}^{r} \mathcal{O}_A \otimes w_i \otimes y_i$$

$$\mathcal{N}_{r,n}^{\mathrm{glob}} = \widehat{\mathcal{O}}^{\mathrm{vir}} \otimes \widehat{\Lambda}^{\bullet}\mathcal{H}om(\mathcal{V},\mathcal{W}) \in K_0^{\mathbb{T}}(\mathrm{Quot}_r^n(\mathbb{C}^4))_{\mathrm{loc}}$$

$$N_{r,n}^{\mathrm{glob}} = \chi\left(\mathrm{Quot}_r^n(\mathbb{C}^4), \mathcal{N}_{r,n}^{\mathrm{glob}}\right) \in \mathbb{Q}\left(t_1^{\frac{1}{2}}, \dots, t_4^{\frac{1}{2}}, w_1^{\frac{1}{2}}, \dots, w_r^{\frac{1}{2}}, y_1^{\frac{1}{2}}, \dots, y_r^{\frac{1}{2}}\right), t_4 = t_1^{-1}t_2^{-1}t_3^{-1}.$$

$$\chi\left(\mathrm{Quot}_r^n(\mathbb{C}^4), \mathcal{N}_{r,n}^{\mathrm{glob}}\right) = \chi\left(\mathrm{Quot}_r^n(\mathbb{C}^4)^{\mathbb{T}}, \tilde{\mathcal{N}}_{r,n}^{\mathrm{glob}}\right)$$

$$\tilde{\mathcal{N}}_{r,n}^{\mathrm{glob}}\ K_0^{\mathbb{T}}(\mathrm{Quot}_r^n(\mathbb{C}^4)^{\mathbb{T}})$$

$$\mathrm{Quot}\ {}_r^n(\mathbb{C}^4)^{\mathbb{T}} \hookrightarrow \mathrm{Quot}\ {}_r^n(\mathbb{C}^4)\ \mathcal{N}_{r,n}^{\mathrm{glob}}$$

$$\mathsf{G}_r = \sum_{n=0}^{\infty} N_{r,n}^{\mathrm{glob}} q^n$$

$$\mathbb{T}' = T_t \times T_w \times \tilde{T}_y$$

$$\mathcal{N}_{r,n}^{\mathrm{glob}} \in K_0^{\mathbb{T}}\left(\mathrm{Quot}_r^n(\mathbb{C}^4)\right)_{\mathrm{loc}}$$

$$K_0^{\mathbb{T}'}\left(\mathrm{Quot}_r^n(\mathbb{C}^4)\right)$$

$$N_{r,n}^{\mathrm{glob}} \in \mathbb{Q}\left(t_1, \dots, t_4, w_1, \dots, w_r, y_1^{\frac{1}{2}}, \dots, y_r^{\frac{1}{2}}\right) \text{ with } t_4 = t_1^{-1}t_2^{-1}t_3^{-1}$$

$$N_{r,n}^{\mathrm{glob}} \in \mathbb{Q}\left(t_1, \dots, t_4, (y_1 \cdots y_r)^{\frac{1}{2}}\right) \text{ with } t_4 = t_1^{-1}t_2^{-1}t_3^{-1}$$

$$\widehat{\Lambda}^{\bullet}\mathcal{H}om(\mathcal{V},\mathcal{W}) = (-1)^{rn}\widehat{\Lambda}^{\bullet}\mathcal{H}om(\mathcal{W},\mathcal{V}).$$

$$\sum_{i\geqslant 0}(-1)^i\Lambda^i E^{\vee} = \sum_{i\geqslant 0}(-1)^i H^i(\Lambda^{\bullet}E^{\vee}, s_{\lrcorner}) \in K_0(A).$$

$$\hat{\mathcal{O}}_M^{\mathrm{vir}} = \mathcal{O}_M^{\mathrm{vir}} \otimes (K_M^{\mathrm{vir}})^{\frac{1}{2}} \in K_0(M).$$

$$S := C(E)\Lambda \otimes L^{-1}$$

$$S \cong L \otimes \Lambda^{\bullet}\Lambda^{\vee},$$

$$\mathcal{H}(S,s) \cdot \det(T_A|_M)^{-\frac{1}{2}} = \hat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0^T(M)_{\mathrm{loc}}$$

$$E = \Lambda \oplus \Lambda^{\vee}\ K_{\mathbb{T}}^0(A)$$

$$\Lambda = \langle e_4 \rangle \wedge \mathbb{C}^3 \cdot \mathcal{E}nd(\mathcal{V})$$

$$\det(\Lambda) = t_4^{-2n^2} \otimes \mathcal{O}_A$$

$$\mathcal{N}_{r,n}^{\text{sheaf}} = \mathcal{H}(S,s) \otimes \Lambda^{\bullet}\mathcal{H}om(\mathcal{V},\mathcal{W}) \otimes (\det(T_A) \otimes \det(\mathcal{H}om(\mathcal{V},\mathcal{W})))^{-\frac{1}{2}},$$

$$0 \to \mathcal{E}nd(\mathcal{V}) \to \mathbb{C}^4 \otimes \mathcal{E}nd(\mathcal{V}) \oplus \bigoplus_{i=1}^{r} \mathcal{V} \otimes w_i^{-1} \to T_A \to 0$$

$$T_A = (\mathbb{C}^4 - 1)\mathcal{E}nd(\mathcal{V}) + \sum_{i=1}^{r} \mathcal{V} \otimes w_i^{-1}$$

$$\det(T_A) = (w_1 \cdots w_r)^{-n} \det(\mathcal{V})^{\otimes r}$$

$$\det(\mathcal{H}om(\mathcal{V},\mathcal{W})) = (y_1 w_1 \cdots y_r w_r)^n \det(\mathcal{V})^{-\otimes r}$$

$$\det(T_A) \otimes \det(\mathcal{H}om(\mathcal{V},\mathcal{W})) \cong (y_1 \cdots y_r)^n \otimes \mathcal{O}_A$$

$$(y_1 \cdots y_r)^{\frac{n}{2}} \otimes \mathcal{O}_A$$

$$\mathcal{N}_{r,n}^{\text{glob}} = [\mathcal{N}_{r,n}^{\text{sheaf}}] \in K_0^{\mathbb{T}}\left(\text{ncQuot}\,_r^n(\mathbb{C}^4)\right)_{\text{loc}}$$

$$\mathcal{N}_{r,n}^{\text{glob}} \in K_0^{\mathbb{T}'}(\text{ncQuot}_r^n(\mathbb{C}^4))$$

$$\text{Quot}\,_1^n(\mathbb{C}^4) = \text{Hilb}^n(\mathbb{C}^4)$$

$$\text{ncQuot}_1^n(\mathbb{C}^4) = \text{ncHilb}^n(\mathbb{C}^4)$$

$$A_m \times A_n \dashrightarrow A_{m+n}.$$

$$O_{m,n} \subseteq A_m \times A_n$$

$$\big((X_1,\dots,X_4,u),(Y_1,\dots,Y_4,v)\big)$$

$$(a_1,\dots,a_4) \in \mathbb{C}^4 \setminus \{0\}$$

$$\textstyle\sum_{i=1}^4 a_i X_i \text{ and } \sum_{i=1}^4 a_i Y_i$$

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

$$f_1(A) = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, f_2(A) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}$$

$$A_{11}B_{12} = B_{12}A_{22} \text{ and } A_{22}B_{21} = B_{21}A_{11}$$

$$\begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix} A = A \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}$$

$$\text{Hilb}^m(\mathbb{C}^4) \times \text{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}} = \big(\text{Hilb}^m(\mathbb{C}^4) \times \text{Hilb}^n(\mathbb{C}^4)\big) \cap O_{m,n}$$

$$\left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix}, \begin{pmatrix} X_3 & 0 \\ 0 & Y_3 \end{pmatrix}, \begin{pmatrix} X_4 & 0 \\ 0 & Y_4 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)$$

$$\mathbb{C}\langle X_1,\dots,X_4\rangle \cdot u = \mathbb{C}^m \text{ and } \mathbb{C}\langle Y_1,\dots,Y_4\rangle \cdot v = \mathbb{C}^n$$

$$\mathbb{C}^{m+n}(a_1,\dots,a_4) \in \mathbb{C}^4 \setminus 0$$

$$A = \begin{pmatrix} X' & 0 \\ 0 & Y' \end{pmatrix}$$

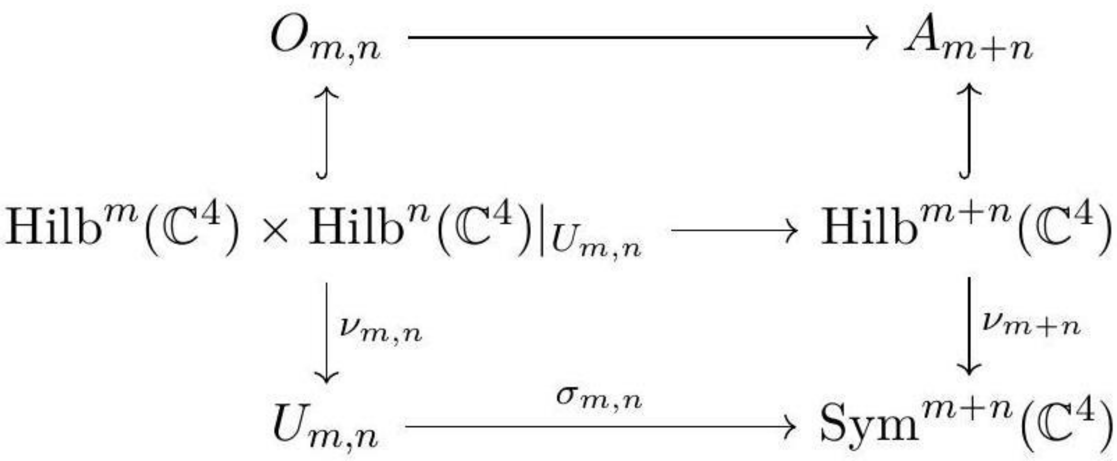
$$S(X) = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4 : \exists v \in \mathbb{C}^m \setminus \{0\} \text{ such that } X_i v = \lambda_i v\}$$

$$S(Y) = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4 : \exists v \in \mathbb{C}^n \setminus \{0\} \text{ such that } Y_i v = \lambda_i v\}$$

$$l(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{i=1}^{4} a_i \lambda_i$$

$$\sum_{i=1}^{4} a_i X_i \sum_{i=1}^{4} a_i Y_i l(S(X)) l(S(Y))$$

$$\big((X_i), u\big), \big((Y_i), v\big) O_{m,n} S(X) \cap S(Y) = \emptyset$$



$$O_{m,n} \to A_{m+n}(P, Q) \in O_{m,n}(X_1^P, \dots, X_4^P, u^P) Q(X_1^Q, \dots, X_4^Q, u^Q)$$

$$\left(\begin{pmatrix} X_1^P & 0 \\ 0 & X_1^Q \end{pmatrix}, \dots, \begin{pmatrix} X_4^P & 0 \\ 0 & X_4^Q \end{pmatrix}, \begin{pmatrix} u^P \\ u^Q \end{pmatrix} \right)$$

$$T_{A_m}\big|_P \oplus T_{A_n}\big|_Q \to T_{A_{m+n}}\Big|_{P+Q}$$

$$T_{A_m}\big|_P \xi_1 = (Y_1, \dots, Y_4, u) T_{A_n}\big|_Q \xi_2 = (Z_1, \dots, Z_4, v)$$

$$A_{m+n}(\xi_1, \xi_2) g_{11} \in \mathrm{End}(\mathbb{C}^m), g_{12} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m), g_{21} \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n), g_{22} \in \mathrm{End}(\mathbb{C}^n)$$

$$[g_{11}, X_i^P] = Y_i, [g_{22}, X_i^Q] = Z_i$$
$$g_{12} X_i^Q = X_i^P g_{12}, g_{21} X_i^P = X_i^Q g_{21}$$
$$g_{11} u^P + g_{12} u^Q = u, g_{21} u^P + g_{22} u^Q = v$$

$$(a_1, \dots, a_4) \in \mathbb{C}^4 \setminus 0$$

$$\sum_i a_i X_i^P \sum a_i X_i^Q$$

$$A_{11} = \sum_i a_i X_i^P, A_{22} = \sum_i a_i X_i^Q, B_{12} = g_{12}$$

$$T_{A_m|P} \oplus T_{A_n}\big|_Q$$

$$E_{m,n} := E_m \boxplus E_n \text{ on } A_m \times A_n$$

$$\Lambda_{m,n} = \Lambda_m \boxplus \Lambda_n$$

$$E_{m,n} \hookrightarrow \sigma_{m,n}^* E_{m+n}$$

$$\mathcal{V}_{m,n} = \mathcal{V}_m \boxplus \mathcal{V}_n \text{ on } A_m \times A_n$$

$$F_{m,n} = \mathbb{C}^4 \otimes \left(\mathcal{Hom}(\mathcal{V}_m, \mathcal{V}_n) \oplus \mathcal{Hom}(\mathcal{V}_n, \mathcal{V}_m)\right)$$

$$\sigma^*(E_{m+n})/E_{m,n}$$

$$\Lambda^2\mathbb{C}^4 \otimes \left(\mathcal{Hom}(\mathcal{V}_m, \mathcal{V}_n) \oplus \mathcal{Hom}(\mathcal{V}_n, \mathcal{V}_m)\right)$$

$$\phi: F_{m,n} \to \sigma^*(E_{m,n})/E_{m,n}$$

$$\sum_i e_i \otimes Y_i \mapsto \sum_{i<j} (e_i \wedge e_j) \otimes \left([X_i, Y_j] + [X_j, Y_i]\right),$$

$$X_i \in \mathcal{E}nd(\mathcal{V}_m \oplus \mathcal{V}_n)$$

$$\mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}$$

$$\phi(F_{m,n})\sigma_{m,n}^*(E_{m+n}/E_{m,n})$$

$$p \in \mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}$$

$$\phi(F_{m,n})\big|_p \subseteq \sigma_{m,n}^*(E_{m+n})/E_{m,n}\big|_p$$

$$\phi(F_{m,n})\big|_p \subseteq \sigma_{m,n}^*(E_{m+n})/E_{m,n}\big|_p$$

$$\sum_{\sigma \in S_4} (-1)^{|\sigma|} \mathrm{tr}\left(\left([X_{\sigma(1)}, Y_{\sigma(2)}] + [X_{\sigma(2)}, Y_{\sigma(1)}]\right)\left([X_{\sigma(3)}, Y_{\sigma(4)}] + [X_{\sigma(4)}, Y_{\sigma(3)}]\right)\right) = 0$$

$$\phi(F_{m,n})\big|_p \mathcal{V}_i|_p W = \mathrm{Hom}(V_m, V_n) \oplus \mathrm{Hom}(V_n, V_m)\phi|_p$$

$$\mathbb{C}^4 \otimes W \to \Lambda^2\mathbb{C}^4 \otimes W$$

$$\mathbb{C}^3 = \langle e_2, e_3, e_4 \rangle$$

$$\mathbb{C}^3 \otimes W \to \mathbb{C}^4 \otimes W \to \Lambda^2\mathbb{C}^4 \otimes W \to e_1 \otimes \mathbb{C}^3 \otimes W$$

$$(Y_2, Y_3, Y_4) \mapsto ([Y_2, X_1], [Y_3, X_1], [Y_4, X_1])$$

$$\mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}, \mathrm{im}(\phi)$$

$$\mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}$$

$$\{x_1 = x_2 = x_3 = 0\} \subset \mathbb{C}^4$$

$$(\phi) = \Lambda_{m+n}/\Lambda_{m,n}$$

$$\mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}} s_{m+n} E_{m+n}$$

$$ds_{m+n}: N_{A_m \times A_n / A_{m+n}} \hookrightarrow \sigma^*_{m,n}(E_{m+n})/E_{m,n}$$

$$ds_{m+n}\left(N_{A_m \times A_n / A_{m+n}}\right)$$

$$\sigma^*_{m,n}(E_{m+n})/E_{m,n}$$

$$\mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}$$

$$ds_{m+n}: N_{A_m \times A_n / A_{m+n}} \to \sigma^*_{m,n}(E_{m+n})/E_{m,n}$$

$$\begin{array}{ccc} T_{A_m \times A_n} & \longrightarrow & T_{A_{m+n}} \\ \Big\downarrow ds_m \oplus ds_n & & \Big\downarrow ds_{m+n} \\ E_m \oplus E_n & \longrightarrow & E_{m+n} \end{array}$$

$$N_{A_m \times A_n / A_{m+n}} \to E_{m+n}/(E_m \oplus E_n)$$

$$0 \to \mathcal{E}nd(\mathcal{V}_{m+n}) \to \mathbb{C}^4 \otimes \mathcal{E}nd(\mathcal{V}_{m+n}) \to T_{A_{m+n}} \to 0$$

$$F_{m,n} \to \mathbb{C}^4 \otimes \mathcal{E}nd(\mathcal{V}_{m+n})$$

$$F_{m,n} \to \mathbb{C}^4 \otimes \mathcal{E}nd(\mathcal{V}_{m+n}) \to T_{A_{m+n}} \to N_{A_m \times A_n / A_{m+n}} \to E_{m+n}/(E_m \oplus E_n)$$

$$\det\left(N_{A_m \times A_n / A_{m+n}}\right) \cong \mathcal{O}$$

$$N := N_{A_m \times A_n / A_{m+n}} \text{In } K^0_{\mathbb{T}'}(O_{m,n})$$

$$N = T_{A_{m+n}}\big|_{O_{m,n}} - \left(T_{A_m} \boxplus T_{A_n}\right)\big|_{O_{m,n}}$$

$$S_m \otimes S_n, S_{m+n}$$

$$C(E_{m,n}) \cong C(E_m) \,\widehat{\otimes}\, C(E_n) \text{ and } C(E_{m+n})$$

$$\begin{aligned} \sigma^*_{m,n}(\mathcal{H}(S_{m+n}, s_{m+n})) &\cong \mathcal{H}\left((S_m, s_m) \otimes (S_n, s_n)\right)\big|_{O_{m,n}} \\ &\cong \mathcal{H}(S_m, s_m) \otimes \mathcal{H}(S_n, s_n)|_{O_{m,n}} \end{aligned}$$

$$\left(\begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{D}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_2 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_3 & 0 \\ 0 & \mathbf{D}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_4 & 0 \\ 0 & \mathbf{D}_4 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)$$

$$\left(\begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{D}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_3 & \mathbf{B}_3 \\ \mathbf{C}_3 & \mathbf{D}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{A}_4 & \mathbf{B}_4 \\ \mathbf{C}_4 & \mathbf{D}_4 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)$$

$$\begin{array}{ccc} \widetilde{Z}_{m,n} & \hookrightarrow & \widetilde{Z}_{m+n} \\ \downarrow & & \downarrow \\ A_m \times A_n & \dashrightarrow & A_{m+n} \end{array}$$

$$0 \to \mathfrak{gl}_{m+n} \to \mathbb{C}^4 \otimes \mathrm{End}(\mathbb{C}^{m+n}) \oplus \mathbb{C}^{m+n} \to T \to 0$$

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, X_1 = \begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{D}_1 \end{pmatrix}$$

$$0 \to E_m \oplus E_n \to E_{m+n} \to E'_{m,n} \to 0$$

$$\sigma^*_{m,n}(\mathcal{H}(S_{m+n}, s_{m+n})) \cong \mathcal{H}((S_m, s_m) \otimes (S_n, s_n))\big|_{O_{m,n}}.$$

$$\phi_{m,n}\colon \mathcal{F}_m \boxtimes \mathcal{F}_n|_{U_{m,n}} \cong \sigma^*_{m,n}\mathcal{F}_{m+n}$$

$$\begin{aligned} \sigma^*_{m,n} R\nu_{m+n*}\mathcal{H}(S_{m+n}, s_{m+n}) &\cong R\nu_{m,n*}(\mathcal{H}(S_m, s_m) \otimes \mathcal{H}(S_n, s_n))\big|_{U_{m,n}} \\ &\cong R\nu_{m*}\mathcal{H}(S_m, s_m) \boxtimes^L R\nu_{n*}\mathcal{H}(S_n, s_n)|_{U_{m,n}}. \end{aligned}$$

$$\left\{\mathcal{H}(R\nu_{n*}(\mathcal{H}(S_n, s_n))\right\}_{n=1}^{\infty}, \quad \left\{\mathcal{H}(R\nu_{n*}(\Lambda^{\bullet}(\mathcal{V}_n^{\vee} \otimes y)))\right\}_{n=1}^{\infty}, \quad \left\{\mathcal{H}(R\nu_{n*}(\mathcal{N}_{1,n}^{\mathrm{sheaf}}))\right\}_{n=1}^{\infty}.$$

$$((X_1, \dots, X_4, u), (Y_1, \dots, Y_4, v), (Z_1, \dots, Z_4, w))$$

$$\begin{array}{ccc} O_{m,n,p} & \longrightarrow & A_{m+n+p} \\ \uparrow & & \uparrow \\ \mathrm{Hilb}^m(\mathbb{C}^4) \times \mathrm{Hilb}^n(\mathbb{C}^4) \times \mathrm{Hilb}^p(\mathbb{C}^4)|_{U_{m,n,p}} & \longrightarrow & \mathrm{Hilb}^{m+n+p}(\mathbb{C}^4) \\ \downarrow{\scriptstyle \nu_{m,n,p}} & & \downarrow{\scriptstyle \nu_{m+n+p}} \\ U_{m,n,p} & \xrightarrow{\sigma_{m,n,p}} & \mathrm{Sym}^{m+n+p}(\mathbb{C}^4) \end{array}$$

$$
\begin{array}{ccccc}
 & & O_{m+n,p} & & \\
 & \overset{\sigma_{m,n}\times \mathrm{id}}{\nearrow} & & \overset{\sigma_{m+n,p}}{\searrow} & \\
O_{m,n,p} & & \xrightarrow{\quad\sigma_{m,n,p}\quad} & & A_{m+n+p} \\
 & \underset{\mathrm{id}\times\sigma_{n,p}}{\searrow} & & \underset{\sigma_{m,n+p}}{\nearrow} & \\
 & & O_{m,n+p} & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \sigma^*_{m,n}\mathcal{H}(S_{m+n}) \otimes \mathcal{H}(S_p) & & \\
 & \nearrow & & \searrow & \\
\sigma^*_{m,n,p}\mathcal{H}(S_{m+n+p}) & & & & \mathcal{H}(S_m) \otimes \mathcal{H}(S_n) \otimes \mathcal{H}(S_p) \\
 & \searrow & & \nearrow & \\
 & & \mathcal{H}(S_m) \otimes \sigma^*_{n,p}\mathcal{H}(S_{n+p}) & &
\end{array}
$$

$$\det\left(N_{A_m\times A_n\times A_p/A_{m+n}\times A_p}\right)\det\left(N_{A_{m+n}\times A_p/A_{m+n+p}}\right)\mathbb{T}'\det\left(N_{A_m\times A_n\times A_p/A_m\times A_{n+p}}\right)\det\left(N_{A_m\times A_{n+p}/A_{m+n+p}}\right)\mathcal{O}$$

$$H^0_{\mathbb{T}'}\left(\mathrm{Hilb}^m(\mathbb{C}^4)\times \mathrm{Hilb}^n(\mathbb{C}^4)\times \mathrm{Hilb}^p(\mathbb{C}^4)|_{U_{m,n,p}}, \mathcal{O}^*\right)$$

$$\begin{aligned}\mathcal{O} = \mathcal{O}\otimes\mathcal{O} &\cong \det\left(N_{A_m\times A_n\times A_p/A_{m+n}\times A_p}\right)\otimes \det\left(N_{A_{m+n}\times A_p/A_{m+n+p}}\right)\\ &\cong \det\left(N_{A_m\times A_n\times A_p/A_{m+n+p}}\right)\cong \mathcal{O}\end{aligned}$$

$$\mathcal{V}_{m+n}|_{O_{m,n}} \cong \mathcal{V}_m \oplus \mathcal{V}_n$$

$$\Lambda^\bullet(\mathcal{V}^\vee_{m+n}\otimes y)|_{O_{m,n}} \cong \Lambda^\bullet(\mathcal{V}^\vee_m\otimes y)\otimes\Lambda^\bullet(\mathcal{V}^\vee_n\otimes y).$$

$$\mathrm{Hilb}^m(\mathbb{C}^4)\times \mathrm{Hilb}^n(\mathbb{C}^4)|_{U_{m,n}}$$

$$\left\{\mathcal{H}\left(R\nu_{n*}(\mathcal{N}^{\mathrm{sheaf}}_{1,n})\right)\right\}$$

$$\left\{\mathcal{H}\left(R\nu_{n*}(\mathcal{N}^{\mathrm{sheaf}}_{1,n})\right)[n]\right\}$$

$$\sum_{n=0}^{\infty}\chi\big(\mathrm{Sym}^n(\mathbb{C}^4), R\nu_{n*}(\mathcal{N}^{\mathrm{sheaf}}_{1,n})[n]\big)q^n = \mathrm{G}_1(-q) = \mathrm{Z}^{\mathrm{NP}}_1(-q)$$

$$\mathrm{Z}^{\mathrm{NP}}_1(-q) = \mathrm{Exp}\left(\frac{G_1}{[t_1][t_2][t_3][t_4]}\right)$$

$$-[t_1t_2][t_1t_3][t_2t_3][y^2] - \frac{[t_1^2t_2^2][t_1^2t_3^2][t_2^2t_3^2][y^2]}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}$$

$$\mathrm{G}_1 = \sum_n N_{1,n}^{\mathrm{glob}} q^n$$

$$\mathrm{G}_1 = \mathrm{Exp}\left(\frac{G_1}{[t_1][t_2][t_3][t_4]}\right)$$

$$N_{1,n}^{\mathrm{glob}} = \chi\big(\mathrm{Hilb}^n(\mathbb{C}^4), \mathcal{N}_{1,n}^{\mathrm{sheaf}}\big) = \chi\left(\mathrm{Sym}^n(\mathbb{C}^4), R\nu_{n*}\big(\mathcal{N}_{1,n}^{\mathrm{sheaf}}\big)\right)$$

$$\left\{\mathcal{H}\left(R\nu_{n*}\big(\mathcal{N}_{1,n}^{\mathrm{sheaf}}\big)\right)\right\}\mathbb{C}^4$$

$$\mathrm{G}_1 = \mathrm{Exp}\left(\sum_{n=1}^{\infty} \chi(\mathbb{C}^4, \mathcal{G}_n) q^n\right)$$

$$\mathcal{G}_n = \frac{\iota_*(\mathcal{G}_n|_0)}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)}$$

$$\mathcal{G}_n|_0 \in K_0^{\mathbb{T}'}(\mathrm{pt})\iota\colon \{0\} \hookrightarrow \mathbb{C}^4$$

$$\chi\big(\mathrm{Hilb}^n(\mathbb{C}^4), \hat{\mathcal{O}}^{\mathrm{vir}}\big) \text{ for } n = 1,2$$

$$\sum_{n=0}^{\infty} \chi\big(\mathrm{Hilb}^n(\mathbb{C}^4), \hat{\mathcal{O}}^{\mathrm{vir}}\big) q^n$$

$$\det\big(T_{A_n}\big)\det\big(T_{A_n}\big) \otimes \det(\mathcal{V}_n^\vee \otimes y) R\nu_{n*}\hat{\mathcal{O}}^{\mathrm{vir}}$$

$$R\nu_{n*}\left(\hat{\mathcal{O}}^{\mathrm{vir}} \otimes \det\left(T_{A_n}\big|_{M_n}\right)^{\frac{1}{2}}\right)$$

$$\coprod_{n=n_1+\cdots+n_r} \prod_{\alpha=1}^{r} \mathrm{Hilb}^{n_\alpha}(\mathbb{C}^4)^{(\mathbb{C}^*)^4}$$

$$P_{\vec{\pi}} = [\mathcal{O}_{\mathbb{C}^4}^{\oplus r} \to Q], Q = \bigoplus_{\alpha=1}^{r} \mathcal{O}_{Z_\alpha}$$

$$\big(T_{A_n} - \Lambda\big)\big|_{P_{\vec{\pi}}} \in K_0^{\mathbb{T}'}(\mathrm{pt}),$$

$$A_n := \mathrm{ncQuot}_r^n(\mathbb{C}^4) K_0^{\mathbb{T}'}(\mathrm{pt})$$

$$\big(T_{A_n} - \Lambda^\vee\big)\big|_{P_{\vec{\pi}}}$$

$$0 \to \mathrm{End}(V) \to \mathbb{C}^4 \otimes \mathrm{End}(V) \oplus \mathrm{Hom}(\mathbb{C}^r, V) \to T_{A_n}\big|_{P_{\vec{\pi}}} \to 0$$

$$T_{A_n}\big|_{P_{\vec{\pi}}} = (\mathbb{C}^4 - 1)\cdot \mathrm{End}(V) + \mathrm{Hom}(\mathbb{C}^r, V),$$
$$\Lambda^\vee|_{P_{\vec{\pi}}} = (\langle e_4 \rangle \wedge \mathbb{C}^3)^\vee \cdot \mathrm{End}(V) \cong \Lambda^2\mathbb{C}^3 \otimes \mathrm{End}(V),$$

$$\begin{aligned} \mathbb{C}^4 &= t_1^{-1} + t_2^{-1} + t_3^{-1} + t_4^{-1} \\ \mathbb{C}^r &= w_1 + \cdots + w_r \\ V &= Z_1 w_1 + \cdots + Z_r w_r \\ \Lambda^2\mathbb{C}^3 &= t_1^{-1}t_2^{-1} + t_1^{-1}t_3^{-1} + t_2^{-1}t_3^{-1} \\ Z_\alpha &= \sum_{(i,j,k,l)\in\pi_\alpha} t_1^i t_2^j t_3^k t_4^l \end{aligned}$$

$$\left(T_{A_n} - \Lambda^\vee\right)\big|_{P_{\vec{\pi}}} =$$

$$\sum_{\alpha,\beta=1}^{r} \frac{w_\beta}{w_\alpha}\left(Z_\beta - (1 - t_1^{-1} - t_2^{-1} - t_3^{-1} - t_4^{-1} + t_1^{-1}t_2^{-1} + t_1^{-1}t_3^{-1} + t_2^{-1}t_3^{-1})Z_\alpha^* Z_\beta\right)$$

$$-\mathcal{H}om(\mathcal{W},\mathcal{V})|_{P_{\vec{\pi}}}^{\vee} = \sum_{\alpha,\beta=1}^{r} \frac{w_\beta}{w_\alpha}\left(-y_\beta Z_\alpha^\vee\right)$$

$$[t^a w^b y^c] = t^{\frac{a}{2}} w^{\frac{b}{2}} y^{\frac{c}{2}} - t^{-\frac{a}{2}} w^{-\frac{b}{2}} y^{-\frac{c}{2}}$$

$$[L_1 + L_2] = [L_1][L_2], [L_1 - L_2] = \frac{[L_1]}{[L_2]}$$

$$\mathrm{v}_{\vec{\pi},\alpha\beta}^{\mathrm{pre}} = \frac{w_\beta}{w_\alpha}\left(Z_\beta - (1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})Z_\alpha^\vee Z_\beta\right), \mathrm{v}_{\vec{\pi}}^{\mathrm{pre}} = \sum_{\alpha,\beta=1}^{r} \mathrm{v}_{\vec{\pi},\alpha\beta}^{\mathrm{pre}}$$

$$\mathrm{v}_{\vec{\pi},\alpha\beta} = \mathrm{v}_{\vec{\pi},\alpha\beta}^{\mathrm{pre}} + \frac{w_\beta}{w_\alpha}\left(-y_\beta Z_\alpha^\vee\right), \mathrm{v}_{\vec{\pi}} = \sum_{\alpha,\beta=1}^{r} \mathrm{v}_{\vec{\pi},\alpha\beta}.$$

$$\mathrm{v}dP_{\vec{\pi}} = 0$$

$$\mathrm{v}_{\vec{\pi}}^{\mathrm{pre}} + \left(\mathrm{v}_{\vec{\pi}}^{\mathrm{pre}}\right)^\vee = \left(\left(T_{A_n} - \Lambda^\vee\right) + \left(T_{A_n} - \Lambda^\vee\right)^\vee\right)\Big|_{P_{\vec{\pi}}}$$

$$\left(T_{A_n} - \Lambda^\vee\right)\big|_{P_{\vec{\pi}}} P_{\vec{\pi}} = \left[\mathcal{O}_{\mathbb{C}^4}^{\oplus r} \to Q\right]\mathcal{F}$$

$$\mathrm{v}_{\vec{\pi}}^{\mathrm{pre}} + \left(\mathrm{v}_{\vec{\pi}}^{\mathrm{pre}}\right)^\vee = R\mathrm{Hom}\left(\mathcal{O}_{\mathbb{C}^4}^{\oplus r}, \mathcal{O}_{\mathbb{C}^4}^{\oplus r}\right) - R\mathrm{Hom}(\mathcal{F},\mathcal{F})$$

$$\left[\left(\mathcal{H}om(\mathcal{W},\mathcal{V})^\vee + \Lambda^\vee - T_{A_n}\right)\big|_{P_{\vec{\pi}}}\right] = (-1)^{n+k_{\vec{\pi}}}[-\mathrm{v}_{\vec{\pi}}]$$

$$k_{\vec{\pi}} = \sum_{\alpha=1}^{r} |\{(i,j,k,l) \in \pi_\alpha : l \neq \min(i,j,k)\}|$$

$$\alpha,\beta \left(T_{A_n} - \Lambda^\vee\right)\big|_{P_{\vec{\pi}}} \mathrm{v}_{\vec{\pi},\alpha\beta}^{\mathrm{pre}}$$

$$\frac{w_\beta}{w_\alpha}(t_4^{-1} - t_4)Z_\alpha^\vee Z_\beta$$

$$R\mathrm{Hom}\left(\mathcal{O}_{\mathbb{C}^4}^{\oplus r}, \mathcal{O}_{\mathbb{C}^4}^{\oplus r}\right) - R\mathrm{Hom}(\mathcal{F},\mathcal{F})\mathrm{Quot}\ {}_{r}^{n}(\mathbb{C}^3)\mathrm{Hilb}^n(\mathbb{C}^4)$$

$$\sum_{\alpha,\beta=1}^{r} \frac{w_\beta}{w_\alpha}\left(Z_\beta + \frac{Z_\alpha^\vee}{t_1t_2t_3t_4} - (1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})(1 - t_4^{-1})Z_\alpha^\vee Z_\beta\right)$$

$$Q = \sum_{\alpha} Z_\alpha w_\alpha$$

$$\dim(t_4^{-1}Q^\vee Q)^m = n^2 - \dim(t_4^{-1}Q^\vee Q)^f = n^2 - \sum_{\alpha=1}^{r} \dim(t_4^{-1}Z_\alpha^\vee Z_\alpha)^f,$$

$$\dim(t_1t_2t_3Z^\vee Z)^f \equiv |\{(i,j,k,l) \in \pi : l \neq \min(i,j,k)\}| \ (\mathrm{mod}2).$$

$$Z = \sum_{(i,j,k,l)\in\pi} t_1^{i-l} t_2^{j-l} t_3^{k-l}$$

$$\left|\left\{\left((i,j,k,l),(i',j',k',l')\right) \in \pi \times \pi : i' - i = j' - j = k' - k = l' - l - 1\right\}\right|.$$

$$|\{(i,j,k,l)\in\xi: i_0-i=j_0-j=k_0-k=l_0-l\pm 1\}| \equiv \begin{cases}1 \pmod 2 & \text{if } l_0\neq \min(i_0,j_0,k_0)\\ 0 \pmod 2 & \text{otherwise}.\end{cases}$$

$$\hat{\mathcal{O}}_M^{\mathrm{vir}} = \sum_{P\in M^T} (-1)^{n_P^\Lambda} \iota_{P*}\left(\frac{1}{\hat{\Lambda}^\bullet(\Omega_A-\Lambda^\vee)|_P}\right),$$

$$n_P^\Lambda \equiv \dim\operatorname{coker}(p_\Lambda\circ ds|_P)^f \pmod 2,$$

$$\hat{\Lambda}^\bullet\Lambda^\vee|_P^m = (-1)^{\operatorname{rk}\Lambda|_P^m}\hat{\Lambda}^\bullet\Lambda|_P^m.$$

$$\hat{\mathcal{O}}_M^{\mathrm{vir}} = (-1)^{\operatorname{rk}\Lambda}\sum_{P\in M^T} (-1)^{n_P^{\Lambda^\vee}} \iota_{P*}\left(\frac{1}{\hat{\Lambda}^\bullet(\Omega_A-\Lambda)|_P}\right),$$

$$n_P^{\Lambda^\vee} \equiv \dim\operatorname{coker}(p_{\Lambda^\vee}\circ ds|_P)^f \pmod 2.$$

$$N_{r,n}^{\mathrm{glob}} = (-1)^{rn}\textstyle\sum_{\vec{\pi}} (-1)^{n_{P_{\vec{\pi}}}^{\Lambda^\vee}+k_{\vec{\pi}}}[-\mathrm{v}_{\vec{\pi}}]$$

$$\begin{aligned} N_{r,n}^{\mathrm{glob}} &= \sum_{\vec{\pi}}(-1)^{n_{P_{\vec{\pi}}}^{\Lambda}}\cdot\hat{\Lambda}^\bullet\mathcal{H}om(\mathcal{V},\mathcal{W})|_{P_{\vec{\pi}}}\cdot\hat{\Lambda}^\bullet(\Lambda^\vee-\Omega_{A_n})|_{P_{\vec{\pi}}}\\ &= (-1)^{(r-1)n}\sum_{\vec{\pi}}(-1)^{n_{P_{\vec{\pi}}}^{\Lambda^\vee}}\cdot\hat{\Lambda}^\bullet\mathcal{H}om(\mathcal{W},\mathcal{V})|_{P_{\vec{\pi}}}\cdot\hat{\Lambda}^\bullet(\Lambda-\Omega_{A_n})|_{P_{\vec{\pi}}},\end{aligned}$$

$$\hat{\Lambda}^\bullet(V_1+V_2) = \hat{\Lambda}^\bullet V_1\cdot\hat{\Lambda}^\bullet V_2,$$

$$(-1)^{(r-1)n}\textstyle\sum_{\vec{\pi}}(-1)^{n_{P_{\vec{\pi}}}^{\Lambda^\vee}}\left[\left(\mathcal{H}om(\mathcal{W},\mathcal{V})^\vee+\Lambda^\vee-T_{A_n}\right)|_{P_{\vec{\pi}}}\right]$$

$$P_{\vec{\pi}} = [\mathcal{O}_{\mathbb{C}^4}^{\oplus r}\twoheadrightarrow Q],\qquad Q=\bigoplus_{\alpha=1}^{r}\mathcal{O}_{Z_\alpha},$$

$$(-1)^{n_{P_{\vec{\pi}}}^{\Lambda^\vee}}, n_P^{\Lambda^\vee}\equiv \operatorname{dimcoker}(p_{\Lambda^\vee}\circ ds|_P)^f \pmod 2.$$

$$n_{P_{\vec{\pi}}}^{\Lambda^\vee}+k_{\vec{\pi}}\equiv\mu_{\vec{\pi}} \pmod 2, \mu_{\vec{\pi}} := \sum_{\alpha=1}^{r}|\{(i,i,i,j)\in\pi_\alpha: j>i\}|$$

$$P=[\mathcal{O}_{\mathbb{C}^4}^{\oplus r}\to Q]\in A_n := \mathrm{ncQuot}_r^n(\mathbb{C}^4)$$

$$0\to\operatorname{End}(V)\to\mathbb{C}^4\otimes\operatorname{End}(V)\oplus\operatorname{Hom}(\mathbb{C}^r,V)\to T_{A_n}\mid P\to 0$$

$$\mathbb{C}^4 \otimes \mathrm{End}(V) \oplus \mathrm{Hom}(\mathbb{C}^r, V) \twoheadrightarrow T_{A_n}|_P \xrightarrow{ds} E|_P = \Lambda^2\mathbb{C}^4 \otimes \mathrm{End}(V).$$

$$\left(\sum_i e_i \otimes f_i^P, u_1^P, \dots, u_r^P\right) P \in A_n = U/\mathrm{GL}(V)$$

$$\left(\sum_{i=1}^{4} e_i \otimes f_i, u_1, \dots, u_r\right) \mapsto \sum_{i,j=1}^{4} e_i \wedge e_j \cdot \left(f_i^P \circ f_j - f_j \circ f_i^P\right)$$

$$\Lambda^\vee|_P = (\langle e_4\rangle \wedge \mathbb{C}^3)^\vee \otimes \mathrm{End}(V) \cong \Lambda^2\mathbb{C}^3 \otimes \mathrm{End}(V)$$

$$\mathbb{C}^3 = \langle e_1, e_2, e_3\rangle \subset \mathbb{C}^4 T_{A_n|P} \to \Lambda^\vee\big|_P$$

$$\mathbb{C}^3 \otimes \mathrm{End}(V) \to \Lambda^2\mathbb{C}^3 \otimes \mathrm{End}(V)$$
$$\sum_{i=1}^{3} e_i \otimes f_i \mapsto \sum_{i,j=1}^{3} e_i \wedge e_j \cdot \left(f_i^P \circ f_j - f_j \circ f_i^P\right)$$

$$\mathrm{End}(V) \to \mathbb{C}^3 \otimes \mathrm{End}(V) \xrightarrow{\theta_1} \Lambda^2\mathbb{C}^3 \otimes \mathrm{End}(V) \xrightarrow{\theta_2} \Lambda^3\mathbb{C}^3 \otimes \mathrm{End}(V),$$

$$\mathrm{Ext}^2(p_*Q, p_*Q)^f = \bigoplus_{\alpha=1}^{r} \mathrm{Ext}^2\left(p_*\mathcal{O}_{Z_\alpha}, p_*\mathcal{O}_{Z_\alpha}\right)^f = 0$$

$$\mathrm{coker}(\theta_1)^f \cong \left(\frac{\Lambda^2\mathbb{C}^3 \otimes \mathrm{End}(V)}{\ker(\theta_2)}\right)^f \cong \mathrm{im}(\theta_2)^f$$

$$\mathrm{dim}\,\mathrm{im}(\theta_2)^f = \dim(\Lambda^3\mathbb{C}^3 \otimes \mathrm{End}(V))^f - \dim_{\mathrm{Ext}}{}^3(p_*Q, p_*Q)^f$$
$$= \sum_{\alpha=1}^{r} \left\{\dim(t_1^{-1}t_2^{-1}t_3^{-1}Z_\alpha^\vee Z_\alpha)^f - \dim\mathrm{Ext}^3\left(p_*\mathcal{O}_{Z_\alpha}, p_*\mathcal{O}_{Z_\alpha}\right)^f\right\}$$

$$\sum_{\alpha=1}^{r} \left(k_{\pi_\alpha} - \mu_{\pi_\alpha}\right) (\mathrm{mod}2)$$

$$\dim\mathrm{Hom}(p_*\mathcal{O}_Z, p_*\mathcal{O}_Z)^f = |\{(i,i,i,j) \in \pi : j \geqslant i\}|$$
$$\dim_{\mathrm{Ext}^1(p_*\mathcal{O}_Z, p_*\mathcal{O}_Z)^f} = \dim\mathrm{Ext}^2(p_*\mathcal{O}_Z, p_*\mathcal{O}_Z)^f = 0$$
$$\dim\mathrm{Ext}^3(p_*\mathcal{O}_Z, p_*\mathcal{O}_Z)^f = |\{(i,i,i,j) \in \pi : j > i\}|$$

$$\begin{aligned}
&\mathrm{Hom}\left(\mathcal{O}_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = \mathbb{C}, && \text{if } (w_1, w_2, w_3) \in \pi \\
&\mathrm{Hom}\left(\mathcal{O}_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = 0, && \text{if } (w_1, w_2, w_3) \notin \pi \\
&\mathrm{Ext}^k\left(\mathcal{O}_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = 0, && \text{if } w_1, w_2, w_3 \geqslant 0, k > 0 \\
&\mathrm{Ext}^3\left(\mathcal{O}_Z, \mathcal{O}_{Z'} \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = \mathbb{C}, && \text{if } (-w_1-1, -w_2-1, -w_3-1) \in \pi \\
&\mathrm{Ext}^3\left(\mathcal{O}_Z, \mathcal{O}_{Z'} \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = 0, && \text{if } (-w_1-1, -w_2-1, -w_3-1) \notin \pi \\
&\mathrm{Ext}^k\left(\mathcal{O}_Z, \mathcal{O}_{Z'} \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)^f = 0, && \text{if } w_1, w_2, w_3 \leqslant -1, k < 3
\end{aligned}$$

$$(\mathbb{C}^*)^3 K_{\mathbb{C}^3} = \mathcal{O}_{\mathbb{C}^3} \otimes t_1t_2t_3$$

$$0 \to I_{Z'} \to \mathcal{O}_{\mathbb{C}^3} \to \mathcal{O}_{Z'} \to 0$$

$$0 \to \mathrm{Hom}\left(\mathcal{O}_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right) \to \mathrm{Hom}\left(\mathcal{O}_{\mathbb{C}^3}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)$$
$$\to \mathrm{Hom}\left(I_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)$$

$$\mathrm{Hom}\left(\mathcal{O}_{Z'}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right) \cong \mathrm{Hom}\left(\mathcal{O}_{\mathbb{C}^3}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\right)$$

$$0 \to R_t \to \cdots \to R_1 \to \mathcal{O}_{\mathbb{C}^3} \to \mathcal{O}_{Z'} \to 0$$

$$(\mathbb{C}^*)^3(\mathbb{C}^*)^3\mathcal{O}_{\mathbb{C}^3} \otimes t_1^{a_1}t_2^{a_2}t_3^{a_3}(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 0} \setminus \pi'$$

$$\mathrm{Hom}\big(\mathcal{O}_{\mathbb{C}^3} \otimes t_1^{a_1}t_2^{a_2}t_3^{a_3}, \mathcal{O}_Z \otimes t_1^{-w_1}t_2^{-w_2}t_3^{-w_3}\big)^f = 0$$

$$(a_1 + w_1, a_2 + w_2, a_3 + w_3) \in \mathbb{Z}^3_{\geqslant 0} \setminus \pi' \subset \mathbb{Z}^3_{\geqslant 0} \setminus \pi$$

$$p_*\mathcal{O}_Z = \bigoplus_{i \geqslant 0} \mathcal{O}_{Z_i} \otimes t_4^i = \bigoplus_{i \geqslant 0} \mathcal{O}_{Z_i} \otimes (t_1t_2t_3)^{-i}$$

$$\mathrm{Hom}\left(\mathcal{O}_{Z_i} \otimes (t_1t_2t_3)^{-i}, \mathcal{O}_{Z_j} \otimes (t_1t_2t_3)^{-j}\right)^f$$

$$\mathrm{Ext}^k\left(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_j} \otimes (t_1t_2t_3)^{i-j}\right)^f = 0$$

$$\mathrm{Ext}^3\left(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_j} \otimes (t_1t_2t_3)^{i-j}\right)^f$$

$$\mathrm{G}_r = \sum_{n=0}^{\infty} N_{r,n}^{\mathrm{glob}} q^n, \mathrm{Z}_r^{\mathrm{NP}} = \sum_{\vec{\pi}=(\pi_1,\ldots,\pi_r)} (-1)^{\mu_{\vec{\pi}}}[-\mathrm{v}_{\vec{\pi}}]((-1)^r q)^{|\vec{\pi}|}$$

$$\mathrm{Z}_1^{\mathrm{NP}} = \mathrm{Exp}\left(\frac{G_1}{[t_1][t_2][t_3][t_4]}\right)$$

$$\mathrm{Z}_1^{\mathrm{NP}} = \mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3]H_1}{[t_1][t_2][t_3][t_4]}\right)$$

$$\mathrm{v}_\pi = \sum_{i,j,k,l \in \mathbb{Z}} c_{ijkl}^\pi t_1^i t_2^j t_3^k y^l$$

$$\sum_{i \in \mathbb{Z}} c_{ii00}^\pi < 0$$

$$Z_\pi = \sum_{(i,j,k,l)\in\pi} t_1^i t_2^j t_3^k t_4^l = \sum_{(i,j,k,l)\in\pi} u^{i-j} v^{k-l} = \sum_{i,j\in\mathbb{Z}} a_{ij}^\pi u^i v^j$$
$$\mathrm{v}_\pi = \sum_{i,j,k\in\mathbb{Z}} b_{ijk}^\pi u^i v^j y^k$$

$$a_{ij}^\pi = |\{(a+i, a, b+j, b) \in \pi : a, b \geqslant 0\}|.$$

$$b_{000}^\pi = a_{00}^\pi + \sum_{i,j\in\mathbb{Z}} \left\{\left(2a_{i,j+1}^\pi a_{ij}^\pi - a_{i,j+1}^\pi a_{i+1,j}^\pi - a_{i+1,j+1}^\pi a_{ij}^\pi\right) - \left(2\left(a_{ij}^\pi\right)^2 - 2a_{ij}^\pi a_{i+1,j}^\pi\right)\right\}$$
$$= a_{00}^\pi - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} \left(a_{ij}^\pi - a_{i+1,j}^\pi - a_{i,j+1}^\pi + a_{i+1,j+1}^\pi\right)^2$$

Now let $s(i,j) = 1$ if $i \geqslant 0$ and $j \geqslant 0$, or if $i < 0$ and $j < 0$. Otherwise let $s(i,j) = -1$. Then

$$\sum_{i,j\in\mathbb{Z}} \left(a_{ij}^\pi - a_{i+1,j}^\pi - a_{i,j+1}^\pi + a_{i+1,j+1}^\pi\right)^2 \geqslant \sum_{i,j\in\mathbb{Z}} \left|a_{ij}^\pi - a_{i+1,j}^\pi - a_{i,j+1}^\pi + a_{i+1,j+1}^\pi\right|$$
$$\geqslant \sum_{i,j\in\mathbb{Z}} s(i,j)\left(a_{ij}^\pi - a_{i+1,j}^\pi - a_{i,j+1}^\pi + a_{i+1,j+1}^\pi\right)$$
$$= 4a_{00}^\pi$$

$$\{(i,j): i,j \geqslant 0\}, \{(i,j): i,j < 0\}, \{(i,j): i \geqslant 0, j < 0\}, \{(i,j): i < 0, j \geqslant 0\}\ \mathbb{Z}^2 a_{00}^\pi$$

$$b_{000}^\pi \leqslant -a_{00}^\pi < 0$$

$$H_1 = \sum_{n=1}^{\infty} H_{1,n}(t_1, t_2, t_3, y) q^n$$

$$N_{1,n}^{\mathrm{glob}} = \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3][t_4]} H_{1,n}(t_1, t_2, t_3, y) + C$$

$$\frac{[t_1^\ell t_2^\ell][t_1^\ell t_3^\ell][t_2^\ell t_3^\ell]}{[t_1^\ell][t_2^\ell][t_3^\ell][t_4^\ell]} H_{1,m}(y^\ell)$$

$$\frac{[t_1^\ell t_2^\ell][t_1^\ell t_3^\ell][t_2^\ell t_3^\ell]}{[t_1^\ell][t_2^\ell][t_3^\ell][t_4^\ell]} = \frac{[t_1^\ell t_2^\ell][t_1^\ell t_3^\ell][t_2^\ell t_3^\ell]}{[t_1^\ell][t_2^\ell][t_3^\ell][(t_1 t_2 t_3)^{-\ell}]}$$

$$\lim_{t_i^{\pm 1} \to \infty} N_{1,n}^{\mathrm{glob}}$$

$$\frac{t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3}{2}} y^{\frac{1}{2}} - t_1^{\frac{i_1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3}{2}} y^{-\frac{1}{2}}}{t_1^{\frac{i_1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3}{2}} - t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3}{2}}}$$

$$\frac{\left(t_1^{\frac{i_1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3}{2}} - t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3}{2}}\right)\left(t_1^{\frac{i_1-1}{2}} t_2^{\frac{i_2-1}{2}} t_3^{\frac{i_3}{2}} - t_1^{-\frac{i_1-1}{2}} t_2^{-\frac{i_2-1}{2}} t_3^{-\frac{i_3}{2}}\right)}{\left(t_1^{\frac{i_1-1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3}{2}} - t_1^{-\frac{i_1-1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3}{2}}\right)\left(t_1^{\frac{i_1}{2}} t_2^{\frac{i_2-1}{2}} t_3^{\frac{i_3}{2}} - t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2-1}{2}} t_3^{-\frac{i_3}{2}}\right)}$$
$$\times \frac{\left(t_1^{\frac{i_1-1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3-1}{2}} - t_1^{-\frac{i_1-1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3-1}{2}}\right)\left(t_1^{\frac{i_1}{2}} t_2^{\frac{i_2-1}{2}} t_3^{\frac{i_3-1}{2}} - t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2-1}{2}} t_3^{-\frac{i_3-1}{2}}\right)}{\left(t_1^{\frac{i_1}{2}} t_2^{\frac{i_2}{2}} t_3^{\frac{i_3-1}{2}} - t_1^{-\frac{i_1}{2}} t_2^{-\frac{i_2}{2}} t_3^{-\frac{i_3-1}{2}}\right)\left(t_1^{\frac{i_1-1}{2}} t_2^{\frac{i_2-1}{2}} t_3^{\frac{i_3-1}{2}} - t_1^{-\frac{i_1-1}{2}} t_2^{-\frac{i_2-1}{2}} t_3^{-\frac{i_3-1}{2}}\right)}$$

$$Z \subset \mathbb{C}^3 = \{x_4 = 0\}, \mathrm{v}_\pi|_{y=t_4}\ \mathrm{V}_\pi^{\mathrm{DT}}$$

$$\mathrm{Z}_1^{\mathrm{NP}}\Big|_{y=t_4} = \sum_{n=0}^{\infty} \chi\left(\mathrm{Hilb}^n(\mathbb{C}^3), \hat{\mathcal{O}}^{\mathrm{vir}}\right)(-q)^n$$

$$\sum_{n=0}^{\infty} \chi\left(\mathrm{Hilb}^n(\mathbb{C}^3), \hat{\mathcal{O}}^{\mathrm{vir}}\right) q^n = \sum_{\pi} [-\mathrm{V}_\pi^{\mathrm{DT}}] q^{|\pi|}$$

$$H_1(y) = \frac{[t_1][t_2][t_3][t_4]}{[t_1 t_2][t_1 t_3][t_2 t_3]} \log\left(\mathrm{Z}_1^{\mathrm{NP}}\right)$$

$$\log\left(\mathrm{Z}_1^{\mathrm{NP}}\right)\Big|_{y=t_4} = \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]\left[\kappa^{\frac{1}{2}} q\right]\left[\kappa^{\frac{1}{2}} q^{-1}\right]}$$

$$H_1(t_4) = \frac{[t_4]}{\left[t_4^{-\frac{1}{2}} q\right]\left[t_4^{-\frac{1}{2}} q^{-1}\right]} = \frac{[t_4]}{\left[t_4^{\frac{1}{2}} q\right]\left[t_4^{\frac{1}{2}} q^{-1}\right]}$$

$$\lim_{L \to \infty} [-\mathrm{v}_{\vec{\pi},\alpha\beta}][-\mathrm{v}_{\vec{\pi},\beta\alpha}]\Big|_{(w_1 = L, w_2 = L^2, \ldots, w_r = L^r)} = \frac{\left(-y_\beta^{\frac{1}{2}}\right)^{|\pi_\alpha|}}{\left(-y_\alpha^{\frac{1}{2}}\right)^{|\pi_\beta|}}$$

$$Z_\alpha = \sum_{(i_1, i_2, i_3)} t_1^{i_1} t_2^{i_2} t_3^{i_3}, Z_\beta = \sum_{(j_1, j_2, j_3)} t_1^{j_1} t_2^{j_2} t_3^{j_3}$$

$$[\pm w_\beta w_\alpha^{-1} t_1^{k_1}t_2^{k_2}t_3^{k_3}] = \left(\left(L^{\beta-\alpha}t_1^{k_1}t_2^{k_2}t_3^{k_3}\right)^{\frac{1}{2}}\left(1-\left(L^{\beta-\alpha}t_1^{k_1}t_2^{k_2}t_3^{k_3}\right)^{-1}\right)\right)^{\pm 1}, \beta-\alpha>0$$

$$\left\{L^{\frac{\beta-\alpha}{2}(-|\pi_\beta|+|\pi_\alpha|)}\frac{\prod_{(i_1,i_2,i_3)} t_1^{-\frac{i_1}{2}}t_2^{-\frac{i_2}{2}}t_3^{-\frac{i_3}{2}}y_\beta^{\frac{1}{2}}}{\prod_{(j_1,j_2,j_3)} t_1^{\frac{j_1}{2}}t_2^{\frac{j_2}{2}}t_3^{\frac{j_3}{2}}}\right\}\left\{(-1)^{|\pi_\alpha|+|\pi_\beta|}L^{\frac{\beta-\alpha}{2}(|\pi_\beta|-|\pi_\alpha|)}\frac{\prod_{(j_1,j_2,j_3)} t_1^{\frac{j_1}{2}}t_2^{\frac{j_2}{2}}t_3^{\frac{j_3}{2}}y_\alpha^{-\frac{1}{2}}}{\prod_{(i_1,i_2,i_3)} t_1^{-\frac{i_1}{2}}t_2^{-\frac{i_2}{2}}t_3^{-\frac{i_3}{2}}}\right\}$$

$$\nu_n\colon \mathrm{Quot}_r^n(\mathbb{C}^4)\to \mathrm{Sym}^n(\mathbb{C}^4)$$

$$N_{r,n}^{\mathrm{glob}} = \chi\left(\mathrm{Sym}^n(\mathbb{C}^4), R\nu_{n*}\left(\mathcal{N}_{r,n}^{\mathrm{glob}}\right)\right)$$

$$R\nu_{n*}\left(\mathcal{N}_{r,n}^{\mathrm{glob}}\right)\in K_0^{\mathbb{T}'}\left(\mathrm{Sym}^n(\mathbb{C}^4)\right)$$

$$\sum_{(i_1,\ldots,i_r)} V_{i_1,\ldots,i_r} w_1^{i_1}\cdots w_r^{i_r}, V_{i_1,\ldots,i_r}\in K_0^{T_x\times\tilde{T}_y}\left(\mathrm{Sym}^n(\mathbb{C}^4)\right)$$

$$(w_1,\ldots,w_r)=(L,\ldots,L^r),\ \text{as}\ L\to\infty$$

$$(w_1,\ldots,w_r)=(L^{n_1},\ldots,L^{n_r}), L^{\pm1}\to\infty.$$

$$\frac{[a]}{\left[a^{\frac{1}{2}}b^{\frac{1}{2}}c\right]\left[a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-1}\right]}+\frac{[b]}{\left[a^{-\frac{1}{2}}b^{\frac{1}{2}}c\right]\left[a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-1}\right]}=\frac{[ab]}{\left[(ab)^{\frac{1}{2}}c\right]\left[(ab)^{\frac{1}{2}}c^{-1}\right]}$$

$$\mathsf{Z}_r^{\mathrm{NP}} = \sum_{\vec{\pi}=(\pi_1,\ldots,\pi_r)} ((-1)^r q)^{|\vec{\pi}|}(-1)^{\mu_{\vec{\pi}}}\prod_{\alpha=1}^{r}[-\mathsf{V}_{\vec{\pi},\alpha\alpha}]\cdot\prod_{1\leqslant\alpha<\beta\leqslant r}[-\mathsf{V}_{\vec{\pi},\alpha\beta}][-\mathsf{V}_{\vec{\pi},\beta\alpha}].$$

$$\mathsf{Z}_1^{\mathrm{NP}} = \mathsf{Z}_1^{\mathrm{NP}}(t_1,t_2,t_3,y,q)$$

$$\mathbf{Z}_r^{\mathrm{NP}} = \sum_{\vec{\pi}=(\pi_1,\ldots,\pi_r)} ((-1)^r q)^{|\vec{\pi}|}\cdot\prod_{\alpha=1}^{r}(-1)^{\mu_{\pi_\alpha}[-\mathsf{v}_{\pi_\alpha}]}\cdot\prod_{1\leqslant\alpha<\beta\leqslant r}\frac{\left(-y_\beta^{\frac{1}{2}}\right)^{|\pi_\alpha|}}{\left(-y_\alpha^{\frac{1}{2}}\right)^{|\pi_\beta|}}$$

$$=\prod_{\alpha=1}^{r}\mathsf{Z}_1^{\mathrm{NP}}\left(t_1,t_2,t_3,y_\alpha,q\prod_{\beta\neq\alpha}y_\beta^{\frac{1}{2}\mathrm{sgn}(\beta-\alpha)}\right)$$

$$\frac{[t_1t_2][t_1t_3][t_2t_3]}{[t_1][t_2][t_3][t_4]}\sum_{\alpha=1}^{r}\frac{[y]}{\left[y^{\frac{1}{2}}q\right]\left[y^{\frac{1}{2}}q^{-1}\right]}\Bigg|_{\left(y_\alpha,q\prod_{\beta\neq\alpha}y_\beta^{\frac{1}{2}\mathrm{sgn}(\beta-\alpha)}\right)}$$

$$\frac{[y_1y_2]}{\left[(y_1y_2)^{\frac{1}{2}}(y_3\cdots y_r)^{\frac{1}{2}}q\right]\left[(y_1y_2)^{\frac{1}{2}}(y_3\cdots y_r)^{-\frac{1}{2}}q^{-1}\right]}$$

$$\frac{[y_1y_2y_3]}{\left[(y_1y_2y_3)^{\frac{1}{2}}(y_4\cdots y_r)^{\frac{1}{2}}q\right]\left[(y_1y_2y_3)^{\frac{1}{2}}(y_4\cdots y_r)^{-\frac{1}{2}}q^{-1}\right]}$$

$$\frac{[y]}{\left[y^{\frac{1}{2}}q\right]\left[y^{\frac{1}{2}}q^{-1}\right]}, y:=y_1\cdots y_r$$

$$[\mathcal{O}_{\mathbb{C}^4}^{\oplus r}\to Q]\vec{\pi}=(\pi_1,\ldots,\pi_r)$$

$$\iota_* \mathcal{O}_{\mathbb{C}^3}^{\oplus r} \iota \colon \mathbb{C}^3 = \{x_4 = 0\} \hookrightarrow \mathbb{C}^4$$

$$[-\mathrm{V}_{\vec{\pi}}]|_{y_1=\cdots=y_r=t_4} = 0$$

$$(-1)^{|\vec{\pi}|+\mu_{\vec{\pi}}} = (-1)^{|\vec{\pi}|}$$

$$\sum_{n=0}^{\infty} \chi\left(\mathrm{Quot}_r^n(\mathbb{C}^3), \hat{\mathcal{O}}^{\mathrm{vir}}\right)((-1)^r q)^n = \mathrm{Z}_r^{\mathrm{NP}}\big|_{(y_1=\cdots=y_r=t_4)} = \mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3][\kappa^r]}{[t_1][t_2][t_3][\kappa]\left[\kappa^{\frac{r}{2}}q\right]\left[\kappa^{\frac{r}{2}}q^{-1}\right]}\right)$$

$$\Delta = \{x_1x_2x_3x_4 = 0\} = \cup_{i=1}^{4} \mathbb{C}_i^3 \subset \mathbb{C}^4, \mathcal{E}_{\vec{r}} := \oplus_{i=1}^{4} \mathcal{O}_{\mathbb{C}_i^3}^{\oplus r_i}$$

$$0 = \mathcal{B}_S \cdot Q_S \equiv \left(u + \frac{i}{2}\right)^2 Q_S(u+i) + \left(u - \frac{i}{2}\right)^2 Q_S(u-i) + (I_0(S) - 2u^2)Q(u).$$

$$Q_S(u) = \prod_{n=1}^{S} (u - v_n)$$

$$\int_{\mathbb{R}} \left[Q_S(u) \overleftarrow{\mathcal{B}}_S Q_J(u) - Q_S(u) \overrightarrow{\mathcal{B}}_J Q_J(u)\right] \mu(u) du = 0,$$

$$\int_{\mathbb{R}} Q_S(u) \left(u + \frac{i}{2}\right)^2 Q_J(u+i)\mu(u)du \overset{u \to u-i}{=} \int_{\mathbb{R}} Q_S(u-i) \left(u - \frac{i}{2}\right)^2 Q_J(u)\mu(u)du$$

$$0 = \int_{\mathbb{R}} Q_S(u)(\overleftarrow{\mathcal{B}}_S - \overleftarrow{\mathcal{B}}_J)Q_J(u)\mu(u)du + \mathtt{res}_\mu = \Delta I_0 \int_{\mathbb{R}} Q_S(u)Q_J(u)\mu(u)du + \mathtt{res}_\mu\,,$$

$$\Delta I_0 \int_{\mathbb{R}} Q_S(u)Q_J(u)\mu(u)du = 0$$

$$\langle Q_S Q_J \rangle_\mu \equiv \int_{\mathbb{R}} Q_S(u)Q_J(u)\mu(u)du \propto \delta_{S,J}$$

$$Q_{12|12}(u) = \mathcal{P}(u)(f^+)^2, \text{ with } f^{\pm} := f\left(u \pm \frac{i}{2}\right),$$

$$\mathcal{P}(u) = \prod_{k=1}^{S} (u - v_k)$$

$$\frac{f(u+i)}{f(u)} = \prod_{k=1}^{S} \left(\frac{\frac{1}{x(u)} - x_k^+}{\frac{1}{x(u)} - x_k^-}\right)\left(\frac{x_k^-}{x_k^+}\right)^{\frac{1}{2}}$$

$$f(u) = \exp\left(g^2 q_1^+ \psi_0(-iu) + \frac{g^4}{2}(iq_2^- \psi_1(-iu) - q_1^+ \psi_2(-iu)) + \mathcal{O}(g^6)\right)$$

$$Q_S(u) = \prod_{n=1}^{S} (u - v_n)e^{\alpha \cdot \sigma(u)}, \sigma(u) = \log\left(f^+ \bar{f}^-\right)$$

$$\sigma(u) = 2g^2 q_1^+ \psi_{0,+}(u) - g^4\left(q_1^+ \psi_{2,+}(u) + q_2^- \psi_{1,-}(u)\right) + \mathcal{O}(g^6)$$

$$\psi_{n,\pm}(u) = \frac{1}{2}\left(\psi_n\left(\frac{1}{2} + iu\right) \pm \psi_n\left(\frac{1}{2} - iu\right)\right).$$

$$0=\mathcal{B}_S\cdot Q_S=B_S^+Q_S(u+i)+B_S^-Q_S(u+i)+T_S(u)Q_S(u)$$

$$B_S^\pm=(x^\pm)^2\pm ig^2q_1^+(1-2\alpha)(x^\pm)^1+g^4(q_2^++(q_1^+)^2(1-2\alpha)^2)(x^\pm)^0$$

$$T_S=I_0(S)-2u^2+g^4\left(\frac{1}{(x^-)^2}+\frac{1}{(x^+)^2}\right)$$

$$\mathcal{P}_S(u)=Q_S(u)|_{\alpha=0},\mathcal{Q}_S(u)=Q_S(u)|_{\alpha=1},\mathbb{Q}_S(u)=Q_S(u)|_{\alpha=\frac{1}{2}}$$

$$\left((x^+)^2+g^4\frac{q_2^+}{2}\right)\mathbb{Q}_S(u+i)+\left((x^-)^2+g^4\frac{q_2^+}{2}\right)\mathbb{Q}_S(u-i)\\-\left(I_0(S)-2u^2+g^4\left(\frac{1}{(x^-)^2}+\frac{1}{(x^+)^2}\right)\right)\mathbb{Q}_S(u)$$

$$\Delta I_0\int_{\mathbb{R}}\mathbb{Q}_S(u)\mathbb{Q}_J(u)\mu(u)du+g^4\frac{\Delta q_2^+}{2}\int_{\mathbb{R}}(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J(u)\mu(u)du+\mathrm{res}_\mu=0$$

$$B_M\cdot F=\left(u+\frac{i}{2}\right)^M F(u+i)+\left(u-\frac{i}{2}\right)^M F(u-i)-2u^MF(u),$$

$$\begin{cases}\Delta I_0\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_1}+g^4\dfrac{\Delta q_2^+}{2}\langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_1}=0\\ \Delta I_0\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_2}+g^4\dfrac{\Delta q_2^+}{2}\langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_2}=0\end{cases}$$

$$\begin{pmatrix}\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_1} & \langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_1}\\ \langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_2} & \langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_2}\end{pmatrix}\cdot\begin{pmatrix}\Delta I_0\\ g^4/4\Delta q_2^+\end{pmatrix}=0.$$

$$\begin{vmatrix}\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_1} & \langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_1}\\ \langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_2} & \langle(B_0\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_2}\end{vmatrix}\propto\delta_{S,J}+\mathcal{O}(g^6).$$

$$\mathcal{B}\cdot\mathbb{Q}=(x^+)^3\left(1+g^2\frac{q_1^-}{x^+}\right)\mathbb{Q}(u+i)+(x^-)^3\left(1+g^2\frac{q_1^-}{x^-}\right)\mathbb{Q}(u-i)-T(u)\mathbb{Q}(u)=0$$

$$T(u)=2u^3-2g^2q_1^-+I_1u+I_0$$

$$\mu_1(u)=\frac{\pi}{2\cosh^2(\pi u)},\mu_2(u)=\frac{\pi^2\tanh(\pi u)}{\cosh^2(\pi u)}$$

$$\begin{pmatrix}\langle\mathcal{P}_S\mathcal{P}_J\rangle_{\mu_1} & \langle\mathcal{P}_Su\mathcal{P}_J\rangle_{\mu_1}\\ \langle\mathcal{P}_S\mathcal{P}_J\rangle_{\mu_2} & \langle\mathcal{P}_Su\mathcal{P}_J\rangle_{\mu_2}\end{pmatrix}\cdot\begin{pmatrix}\Delta I_0\\ \Delta I_1\end{pmatrix}=0$$

$$\mu_\ell(u)=\frac{\ell\pi^\ell/2}{\cosh^2(\pi u)}\tanh^{\ell-1}(\pi u)$$

$$\frac{\mathcal{P}_S(-i/2)}{\mathcal{P}_S(i/2)}-\frac{\mathcal{P}_J(-i/2)}{\mathcal{P}_J(i/2)}$$

$$\mu_\ell(u)=\frac{\ell\pi^\ell/2}{\cosh^2(\pi u)}\tanh^{\ell-1}(\pi u)\left(1+g^2\pi^2\left(-\frac{\ell}{3}+(\ell+2)\tanh^2(\pi u)\right)\right)+\mathcal{O}(g^4)$$

$$\begin{pmatrix}\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_1} & \langle\mathbb{Q}_Su\mathbb{Q}_J\rangle_{\mu_1} & \langle(B_2\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_1}\\ \langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_2} & \langle\mathbb{Q}_Su\mathbb{Q}_J\rangle_{\mu_2} & \langle(B_2\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_2}\\ \langle\mathbb{Q}_S\mathbb{Q}_J\rangle_{\mu_3} & \langle\mathbb{Q}_Su\mathbb{Q}_J\rangle_{\mu_3} & \langle(B_2\cdot\mathbb{Q}_S)\mathbb{Q}_J\rangle_{\mu_3}\end{pmatrix}\begin{pmatrix}\Delta I_0\\ \Delta I_1\\ g^2\Delta q_1^-\end{pmatrix}=0$$

$$B_2\cdot F=\left(u+\frac{i}{2}\right)^2F(u+i)+\left(u-\frac{i}{2}\right)^2F(u-i)-2u^2F(u)$$

$$\begin{pmatrix} \langle Q_S Q_J \rangle_{\mu_1} & g^2 \langle B_0[Q_S, Q_J] \rangle_{\mu_1} \\ \langle Q_S Q_J \rangle_{\mu_2} & g^2 \langle B_0[Q_S, Q_J] \rangle_{\mu_2} \end{pmatrix} \cdot \begin{pmatrix} \Delta I_0 \\ \frac{g^2 q_2^+}{4} \end{pmatrix},$$

$$\begin{pmatrix} \langle Q_S Q_J \rangle_{\mu_1} & g^4 \langle B_0[Q_S, Q_J] \rangle_{\mu_1} \\ \langle Q_S Q_J \rangle_{\mu_2} & g^4 \langle B_0[Q_S, Q_J] \rangle_{\mu_2} \end{pmatrix} \cdot \begin{pmatrix} \Delta I_0 \\ \frac{\Delta q_2^+}{4} \end{pmatrix}$$

$$\mathcal{M}_{2,0}[Q_S, Q_J] = \left(\langle Q_S Q_J \rangle_{\mu_1} \right)$$
$$\mathcal{M}_{2,1}[Q_S, Q_J] = \begin{pmatrix} \langle Q_S Q_J \rangle_{\mu_1} & \langle (B_1 \cdot Q_S) Q_J \rangle_{\mu_1} \\ \langle Q_S Q_J \rangle_{\mu_2} & \langle (B_1 \cdot Q_S) Q_J \rangle_{\mu_2} \end{pmatrix}$$
$$\mathcal{M}_{2,2}[Q_S, Q_J] = \begin{pmatrix} \langle Q_S Q_J \rangle_{\mu_1} & \langle (B_1 \cdot Q_S) Q_J \rangle_{\mu_1} & \langle (B_0 \cdot Q_S) Q_J \rangle_{\mu_1} \\ \langle Q_S Q_J \rangle_{\mu_2} & \langle (B_1 \cdot Q_S) Q_J \rangle_{\mu_2} & \langle (B_0 \cdot Q_S) Q_J \rangle_{\mu_2} \\ \langle Q_S Q_J \rangle_{\mu_3} & \langle (B_1 \cdot Q_S) Q_J \rangle_{\mu_3} & \langle (B_0 \cdot Q_S) Q_J \rangle_{\mu_3} \end{pmatrix}.$$

$$\mathcal{D}_{L,\ell}[Q_S, Q_J] = \sqrt{\det \mathcal{M}_{L,\ell}[Q_S, Q_J] \det \mathcal{M}_{L,\ell}[Q_J, Q_S]},$$

$$\mu_1^{(0)}(u) = \frac{\pi}{2} \frac{1}{\cosh^2(\pi u)}, \mu_2^{(0)}(u) = \pi^2 \frac{\tanh(\pi u)}{\cosh^2(\pi u)}, \text{ and } \mu_3^{(0)}(u) = \frac{3\pi^3}{2} \frac{\tanh^2(\pi u)}{\cosh^2(\pi u)}.$$

$$\mathcal{M}_{2,2}[Q_4, Q_4] = \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{25}{3} & 0 \\ 3 & 0 & \frac{65}{4} \end{pmatrix}$$

$$\mathcal{M}_{2,2}[Q_4, Q_6] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{37}{25} & 0 \\ \frac{1}{5} & 0 & \frac{343}{36} \end{pmatrix}, \mathcal{M}_{2,2}[Q_6, Q_4] = \begin{pmatrix} 0 & 0 & \frac{22}{15} \\ 0 & \frac{7}{15} & 0 \\ \frac{1}{5} & 0 & 17.781 \end{pmatrix},$$

$$\det(\mathcal{M}_{2,2}[Q_4, Q_4]) = \frac{1625}{108}, \det(\mathcal{M}_{2,2}[Q_4, Q_6]) = 0, \det(\mathcal{M}_{2,2}[Q_6, Q_4]) = -\frac{154}{1125}.$$

$$\langle (B_0 \cdot Q_S) Q_J \rangle_{\mu_1} = \begin{cases} 0 & \text{for } S > J \\ 1/(2S+1) & \text{for } S = J \\ \frac{1}{2}(-1)^{(S-J)/2}(q_1^+(J) - q_1^+(S)) & \text{for } S < J \end{cases}$$

$$\mathcal{D}_{2,0}[Q_S, Q_J] = \delta_{S,J} \frac{(2S+1)}{(2S)!} \left(Q_S\left(\frac{i}{2}\right) Q_S\left(-\frac{i}{2}\right) \right)^{-2} \times \mathbb{B}_2(S)$$
$$\mathcal{D}_{2,1}[Q_S, Q_J] = \delta_{S,J} \frac{(2S+1)}{(2S)!} \left(Q_S\left(\frac{i}{2}\right) Q_S\left(-\frac{i}{2}\right) \right)^{-3} \times \frac{\mathbb{B}_2(S)}{q_1^+(S)}$$
$$\mathcal{D}_{2,2}[Q_S, Q_J] = \delta_{S,J} \frac{(2S+1)}{(2S)!} \left(Q_S\left(\frac{i}{2}\right) Q_S\left(-\frac{i}{2}\right) \right)^{-4} \times \frac{\mathbb{B}_2(S)}{q_1^+(S)(q_3^+(S) - 16 H_3(S))}$$

$$\mu_\ell^{(2)}(u) = \mu_\ell^{(0)}(u)\left(1 + g^2\pi^2 a_\ell \tanh^2(\pi u) + g^4\pi^4(b_\ell \tanh^2(\pi u) + c_\ell \tanh^4(\pi u))\right).$$

$$\mathcal{M}_{2,0}[Q_S, Q_J] = \left(\langle \mathcal{P}_S \mathcal{P}_J \rangle_{\mu_1}\right)$$

$$\mathcal{M}_{2,1}[Q_S, Q_J] = \begin{pmatrix} \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{\mu_1} & \langle (B_1 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_1} \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{\mu_2} & \langle (B_1 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_2} \end{pmatrix}$$

$$\mathcal{M}_{2,2}[Q_S, Q_J] = \begin{pmatrix} \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{\mu_1} & \langle (B_1 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_1} & \langle (B_0 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_1} \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{\mu_2} & \langle (B_1 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_2} & \langle (B_0 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_2} \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{\mu_3} & \langle (B_1 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_3} & \langle (B_0 \cdot \mathcal{P}_S)\mathcal{Q}_J \rangle_{\mu_3} \end{pmatrix}$$

$$\mu_1^{(2)}(u) = \frac{\pi}{2}\frac{1}{\cosh^2(\pi u)}\left(1 + 2\pi^2 g^2 \tanh^2(\pi u) - \frac{g^4\pi^4}{3}(11\tanh^2(\pi u) - 15\tanh^4(\pi u))\right),$$

$$\mu_2^{(2)}(u) = \pi^2\frac{\tanh(\pi u)}{\cosh^2(\pi u)}\left(1 + 3\pi^2 g^2 \tanh^2(\pi u) - \frac{g^4\pi^4}{3}(18\tanh^2(\pi u) - 27\tanh^4(\pi u))\right),$$

$$\mu_3^{(2)}(u) = \frac{3\pi^3}{2}\frac{\tanh^2(\pi u)}{\cosh^2(\pi u)}\left(1 + 4\pi^2 g^2 \tanh^2(\pi u) - \frac{g^4\pi^4}{3}(25\tanh^2(\pi u) - 42\tanh^4(\pi u))\right).$$

$$\mathcal{D}_{2,0}[Q_S, Q_J] \propto \delta_{S,J} + O(g^2)$$
$$\mathcal{D}_{2,1}[Q_S, Q_J] \propto \delta_{S,J} + O(g^4)$$
$$\mathcal{D}_{2,2}[Q_S, Q_J] \propto \delta_{S,J} + O(g^6)$$

$$\det\mathcal{M}_{2,2}[Q_4, Q_6] = \begin{vmatrix} 3.2g^4 & -12.1ig^2 + 220.3ig^4 & 152.5g^4 \\ -1.3ig^2 - 4ig^4 & -2.5 + 10.4g^2 + 820.7g^4 & -62.2ig^2 + 873.4g^4 \\ -0.2 - 3.4g^2 + 113.9g^4 & -304.3ig^2 - 830.7ig^4 & -9.5 + 0.2g^2 + 4020.3g^4 \end{vmatrix} = \mathcal{O}(g^6).$$

$$\left(u + \frac{i}{2}\right)^L Q_S(u+i) + \left(u - \frac{i}{2}\right)^L Q_S(u-i) - \left(2u^L + \sum_{k=0}^{L-2} I_k(S)u^k\right)Q_S(u) = 0.$$

$$\mathcal{M}_{L,0}[Q_S, Q_J] = \begin{pmatrix} \langle \mathcal{P}_S \mathcal{P}_J \rangle_1 & \langle u^1 \mathcal{P}_S \mathcal{P}_J \rangle_1 & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{P}_J \rangle_1 \\ \langle \mathcal{P}_S \mathcal{P}_J \rangle_2 & \langle u^1 \mathcal{P}_S \mathcal{P}_J \rangle_2 & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{P}_J \rangle_2 \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathcal{P}_S \mathcal{P}_J \rangle_{L-1} & \langle u^1 \mathcal{P}_S \mathcal{P}_J \rangle_{L-1} & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{P}_J \rangle_{L-1} \end{pmatrix}$$

$$\langle f \rangle_\ell = \int_{-\infty}^{\infty} f(u)\mu_\ell(u)du$$

$$\mu_\ell^{(0)}(u) = \frac{\ell\pi^\ell/2}{\cosh^2(\pi u)}\tanh^{\ell-1}(\pi u)$$

$$\mathcal{M}_{L,\ell}[Q_S, Q_J] =$$
$$\begin{pmatrix} \langle \mathcal{P}_S \mathcal{Q}_J \rangle_1 & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{Q}_J \rangle_1 & \langle (B_{L-1}\mathcal{P}_S)\mathcal{Q}_J \rangle_1 & \dots & \langle (B_{L-\ell}\mathcal{P}_S)\mathcal{Q}_J \rangle_1 \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_2 & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{Q}_J \rangle_2 & \langle (B_{L-1}\mathcal{P}_S)\mathcal{Q}_J \rangle_2 & \dots & \langle (B_{L-\ell}\mathcal{P}_S)\mathcal{Q}_J \rangle_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{L-1} & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{Q}_J \rangle_{L-1} & \langle (B_{L-1}\mathcal{P}_S)\mathcal{Q}_J \rangle_{L-1} & \dots & \langle (B_{L-\ell}\mathcal{P}_S)\mathcal{Q}_J \rangle_{L-1} \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_L & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{Q}_J \rangle_L & \langle (B_{L-1}\mathcal{P}_S)\mathcal{Q}_J \rangle_L & \dots & \langle (B_{L-\ell}\mathcal{P}_S)\mathcal{Q}_J \rangle_L \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathcal{P}_S \mathcal{Q}_J \rangle_{L-1+\ell} & \dots & \langle u^{L-2} \mathcal{P}_S \mathcal{Q}_J \rangle_{L-1+\ell} & \langle (B_{L-1}\mathcal{P}_S)\mathcal{Q}_J \rangle_{L-1+\ell} & \dots & \langle (B_{L-\ell}\mathcal{P}_S)\mathcal{Q}_J \rangle_{L-1+\ell} \end{pmatrix}$$

$$B_M\mathcal{P}_S = \left(u + \frac{i}{2}\right)^M \mathcal{P}_S(u+i) + \left(u - \frac{i}{2}\right)^M \mathcal{P}_S(u-i) - 2u^M\mathcal{P}_S(u)$$

$$\mathcal{D}_{L,\ell}[Q_S, Q_J] \equiv \sqrt{\det\mathcal{M}_{L,\ell}[Q_S, Q_J]\det\mathcal{M}_{L,\ell}[Q_J, Q_S]} \propto \delta_{S,J} + \mathcal{O}\left(g^{2(\ell+1)}\right)$$

$$\mu_\ell(u) = \frac{\ell\pi^\ell/2}{\cosh^2(\pi u)}\tanh^{\ell-1}(\pi u)e^{\nu_\ell(u)}$$

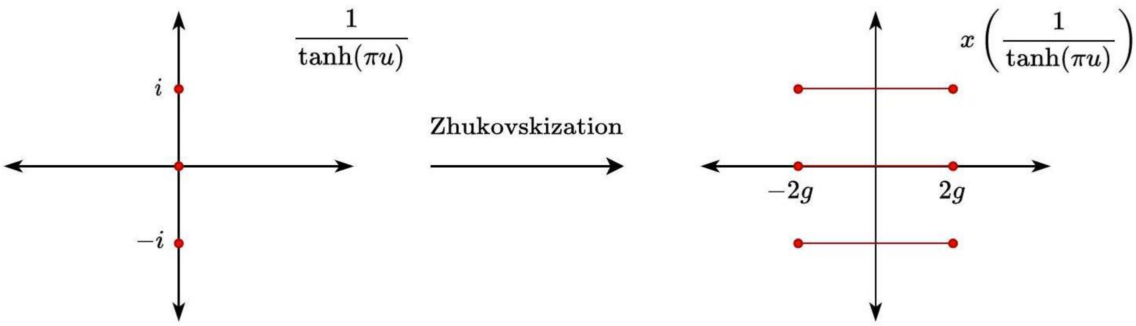
$$\nu_\ell(u) = \sum_{n=1}^{\infty}\left(\binom{2n-1}{n}\frac{\ell+1}{n} - (g\pi)^2\binom{2n}{n}\frac{(\ell+1)(1+4n/3)-1}{n+1}\right)(g\pi)^{2n}\tanh^{2n}(\pi u)$$

$$\det(\mathcal{M}_{L,\ell}) = \int\left(\prod_{n=1}^{L+\ell-1} du_n\,\mu_n(u_n)\right)\times\left(\prod_{n=1}^{L+\ell-1}\mathcal{P}_S(u_n)\right)\times\overleftarrow{\mathbb{V}}\times\left(\prod_{n=1}^{L+\ell-1}\mathcal{Q}_J(u_n)\right),$$

$$\overleftarrow{\mathbb{V}} = \begin{vmatrix} 1 & \dots & u_1^{L-2} & \overleftarrow{B}_{L-1}(u_1) & \dots & \overleftarrow{B}_{L-\ell}(u_1) \\ 1 & \dots & u_2^{L-2} & \overleftarrow{B}_{L-1}(u_2) & \dots & \overleftarrow{B}_{L-\ell}(u_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & u_{L+\ell-1}^{L-2} & \overleftarrow{B}_{L-1}(u_{L+\ell-1}) & \dots & \overleftarrow{B}_{L-\ell}(u_{L+\ell-1}) \end{vmatrix}.$$

$$\mu_\ell(u) = \mu_\ell^{(0)}(u)\sum_{n=0}^{\infty}\sum_{m=0}^{n}(g\pi)^{2n}a_{n,m}\tanh^{2m}(\pi u).$$

$$\det(\mathcal{M}[Q_{L=2,S=4}, Q_{L=3,S=6}]) = \begin{vmatrix} 0.548 & 1.014 & -11.226 \\ 3.436 & 6.360 & -70.433 \\ 16.199 & 30.034 & -332.084 \end{vmatrix} = 0$$



$$\mu_\ell(u) = \frac{\ell(2\pi)^\ell\tanh^{\ell-1}(\pi u)}{\cosh^2(\pi u)}\times\left(\frac{1}{1+\sqrt{1+(2\pi g\tanh(\pi u))^2}}\right)^{\ell+1}\times$$
$$\times\exp\left[g^2\pi^2\left(\ell+\frac{2(\ell+4)}{3}\frac{1}{1+\sqrt{1+(2\pi g\tanh(\pi u))^2}}-\frac{4(\ell+1)}{3}\frac{1}{\sqrt{1+(2\pi g\tanh(\pi u))^2}}\right)\right]$$

$$\mu_\ell(u) = \frac{\ell}{2\pi\sinh^2(\pi u)}\frac{1}{x(\tau)^{\ell+1}}\exp\left[g^2\pi^2\left(\ell+\frac{4+\ell}{3}\frac{\tau}{x(\tau)}+\frac{4(\ell+1)}{3}\frac{1}{1-2x(\tau)/\tau}\right)\right],$$

$$\Delta I_0\langle\mathbb{Q}_S\mathbb{Q}_J\rangle_\mu + g^4\frac{\Delta q_2^+}{4}\langle B_0[\mathbb{Q}_S,\mathbb{Q}_J]\rangle_\mu + \mathrm{res}_\mu = 0$$

$$B_M[F,G] \equiv (B_M\cdot F)G + (B_M\cdot G)F$$

$$\mathrm{res}_\mu = \operatorname*{Res}_{u=i/2}\left[\mathbb{Q}_S(u)\left((x^-)^2-\frac{g^4}{4}(q_2^+(S)+q_2^+(J))\right)\mathbb{Q}_J(u-i)\mu(u)\right]$$
$$+\operatorname*{Res}_{u=-i/2}\left[\mathbb{Q}_S(u)\left((x^+)^2-\frac{g^4}{4}(q_2^+(S)+q_2^+(J))\right)\mathbb{Q}_J(u+i)\mu(u)\right].$$

$$\rho_1(u) = \frac{\pi/2}{\cosh^2(\pi u)}\left(1 - g^2\pi^2(1 - 3\tanh^2(\pi u)) + \frac{2}{3}g^4\pi^4(2 - 15\tanh^2(\pi u) + 15\tanh^4(\pi u))\right).$$

$$\mathbb{M}_{S,J}^{(1)} = \left(\langle \mathbb{Q}_S\mathbb{Q}_J\rangle_{\nu_1}\right), \det\left(\mathbb{M}_{S,J}^{(1)}\right) \propto \delta_{S,J} + \mathcal{O}(g^4).$$

$$\rho_1(u) \equiv \oint \frac{dv}{2\pi i}\frac{\pi/2}{\cosh^2(\pi(u-v))}\frac{1}{x(v)}$$

$$Q_{S=2}^{(1)}(u) = \frac{1}{4} - 3u^2,$$
$$Q_{S=2}^{(2)}(u) = -3iu + \left(\frac{1}{4} - 3u^2\right)\psi_1\left(\frac{1}{2} + iu\right).$$

$$\begin{vmatrix} Q(u+i) & Q(u) & Q(u-i) \\ Q^{(1)}(u+i) & Q^{(1)}(u) & Q^{(1)}(u-i) \\ Q^{(2)}(u+i) & Q^{(2)}(u) & Q^{(2)}(u-i) \end{vmatrix} = 0$$

$$Q_S^{(1)}\left(u+\frac{i}{2}\right)Q_S^{(2)}\left(u-\frac{i}{2}\right) - Q_S^{(1)}\left(u-\frac{i}{2}\right)Q_S^{(2)}\left(u+\frac{i}{2}\right) = \frac{1}{u^2}.$$

$$\hat{Q}_S^{(1)}(u) = \frac{1}{\mathcal{P}_S(i/2)}Q_S^{(1)}(u)$$

$$\hat{Q}_S^{(2)}(u) = \mathcal{P}_S(i/2)Q_S^{(2)}(u).$$

$$\Delta I_0\left\langle \mathbb{Q}_S^{(2)}\mathbb{Q}_J^{(2)}\right\rangle_\mu + g^4\frac{\Delta q_2^+}{4}\left\langle B_0\left[\mathbb{Q}_S^{(2)}, \mathbb{Q}_J^{(2)}\right]\right\rangle_\mu + \mathrm{res}_\mu = 0$$

$$\rho_2(u) = \pi\tanh(\pi u)$$

$$\mathrm{res}_{\rho_2} = \frac{\Delta q_2^+}{4} \times 1 = \frac{\Delta q_2^+}{4} \times \mathcal{P}_S\left(\frac{i}{2}\right)\mathcal{P}_J\left(\frac{i}{2}\right),$$

$$\begin{cases} \Delta I_0 \times \left\langle \hat{\mathbb{Q}}_S^{(1)}\hat{\mathbb{Q}}_J^{(1)}\right\rangle_{\rho_1} + \dfrac{\Delta q_2^+}{4} \times g^4\left\langle B_0\left[\hat{\mathbb{Q}}_S^{(1)}, \hat{\mathbb{Q}}_J^{(1)}\right]\right\rangle_{\rho_1} & = \mathcal{O}(g^6) \\ \Delta I_0 \times \left\langle \hat{\mathbb{Q}}_S^{(2)}\hat{\mathbb{Q}}_J^{(2)}\right\rangle_{\rho_2} + \dfrac{\Delta q_2^+}{4} \times 1 & = \mathcal{O}(g^2) \end{cases}$$

$$\mathbb{M}_{S,J}^{(2)}\begin{pmatrix} \Delta I_0 \\ \frac{\Delta q_2^+}{4} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(g^6) \\ \mathcal{O}(g^2) \end{pmatrix}, \mathbb{M}_{S,J}^{(2)} = \begin{pmatrix} \left\langle \hat{\mathbb{Q}}_S^{(1)}\hat{\mathbb{Q}}_J^{(1)}\right\rangle_{\rho_1} & g^4\left\langle B_0\left[\hat{\mathbb{Q}}_S^{(1)}, \hat{\mathbb{Q}}_J^{(1)}\right]\right\rangle_{\rho_1} \\ \left\langle \hat{\mathbb{Q}}_S^{(2)}\hat{\mathbb{Q}}_J^{(2)}\right\rangle_{\rho_2} & 1 \end{pmatrix}.$$

$$\det\left(\mathbb{M}_{S,J}^{(2)}\right) \propto \delta_{S,J} + \mathcal{O}(g^6)$$

$$\det\mathcal{M}_{L,\ell}[Q_S, Q_J], \det\mathcal{M}_{L,\ell}[Q_J, Q_S], \sqrt{\det\mathcal{M}_{L,\ell}[Q_S, Q_J]\det\mathcal{M}_{L,\ell}[Q_S, Q_J]}$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{S}\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\frac{1 - g^2/x_k^+x_j^-}{1 - g^2/x_k^-x_j^+}e^{2i\theta(v_k,v_j)}$$

$$\prod_{k=1}^{S}\frac{x_k^+}{x_k^-} = 1$$

$$x_k^\pm = x\left(v_k \pm \frac{i}{2}\right) \text{ with } x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2}$$

$$\theta(u,v) = \sum_{r=1}^{\Lambda}\sum_{s=1}^{\Lambda}\sum_{n=0}^{\Lambda-r-s} g^{2(r+s+n)}\beta_{r,s,n}t_r(u)t_s(v)$$
$$\beta_{r,s,n} = 2(-1)^n\frac{\sin\left(\frac{\pi}{2}(r-s)\right)\zeta_{2n+r+s}\Gamma(2n+r+s)\Gamma(2n+r+s+1)}{\Gamma(n+1)\Gamma(n+r+1)\Gamma(n+s+1)\Gamma(n+r+s+1)}$$
$$t_r(u) = \left(\frac{1}{x^+(u)}\right)^r - \left(\frac{1}{x^-(u)}\right)^r$$

$$\mathcal{P}(u) = \prod_{k=1}^{S}(u-v_k)$$

$$q_n^{\pm}[Q_S] = \sum_{k=1}^{S}\left(\left(\frac{i}{x_k^+}\right)^n \pm \left(\frac{-i}{x_k^-}\right)^n\right)$$

$$q_n^+ = \oint\frac{dx}{4\pi i g^{n-1}}(-1)^n\left(1-\frac{g^2}{x^2}\right)\left(\frac{x^n}{g^n}-\frac{g^n}{x^n}\right)\left(i^n\frac{\mathcal{P}'\left(u+\frac{i}{2}\right)}{\mathcal{P}\left(u+\frac{i}{2}\right)}+(-i)^n\frac{\mathcal{P}'\left(u-\frac{i}{2}\right)}{\mathcal{P}\left(u-\frac{i}{2}\right)}\right)$$
$$q_n^- = \oint\frac{dx}{4\pi i g^{n-1}}(-1)^n\left(1-\frac{g^2}{x^2}\right)\left(\frac{x^n}{g^n}-\frac{g^n}{x^n}\right)\left(i^{n-1}\frac{\mathcal{P}'\left(u+\frac{i}{2}\right)}{\mathcal{P}\left(u+\frac{i}{2}\right)}+(-i)^{n-1}\frac{\mathcal{P}'\left(u-\frac{i}{2}\right)}{\mathcal{P}\left(u-\frac{i}{2}\right)}\right)$$

$$\mathbb{B}_{L,S} = \frac{\det(\partial_{v_i}\phi_j)}{\prod_{i\neq j} h(v_i,v_j)}$$

$$e^{i\phi_j} = e^{ip(v_j)L}\prod_{k\neq j} S(v_j,v_k)$$

$$e^{ip(u)} = \frac{u+\frac{i}{2}}{u+\frac{i}{2}}, S(u,v) = \frac{h(u,v)}{h(v,u)} \text{ and } h(u,v) = \frac{u-v}{u-v+i}.$$

$$\mathcal{D}_{L,0}[Q_S,Q_J] = \delta_{S,J}\times\frac{\Gamma(L)}{\Gamma(2S+L)}\left(Q_S\left(\frac{i}{2}\right)Q_S\left(-\frac{i}{2}\right)\right)^{-L}\times\mathbb{B}_L(S),$$
$$\mathcal{D}_{L,1}[Q_S,Q_J] = \delta_{S,J}\times\frac{\Gamma(L)}{\Gamma(2S+L)}\left(Q_S\left(\frac{i}{2}\right)Q_S\left(-\frac{i}{2}\right)\right)^{-(L+1)}\times\frac{\mathbb{B}_L(S)}{q_1^+(S)}.$$

$$\text{res}_\mu = \text{Res}_{u=i/2}\left[\mathbb{Q}_S(u)\left((x^-)^2-\frac{g^4}{4}(q_2^+(S)+q_2^+(J))\right)\mathbb{Q}_J(u-i)\mu(u)\right]+$$
$$\underset{u=-i/2}{\text{Res}}\left[\mathbb{Q}_S(u)\left((x^+)^2-\frac{g^4}{4}(q_2^+(S)+q_2^+(J))\right)\mathbb{Q}_J(u+i)\mu(u)\right]+(S\leftrightarrow J).$$

$$\mu(u) = \frac{\pi/2}{\cosh^2(\pi u)}\left(1+g^2(a_0+a_1\tanh^2(\pi u))+\mathcal{O}(g^4)\right)$$

$$\mu(u) = \sum_{n=-\Lambda_0}^{\Lambda_0}\left(\frac{a_n}{\left(u+\frac{i}{2}\right)^n}+\frac{a_n}{\left(u-\frac{i}{2}\right)^n}\right)+g^2\sum_{n=-\Lambda_1}^{\Lambda_1}\left(\frac{b_n}{\left(u+\frac{i}{2}\right)^n}+\frac{b_n}{\left(u-\frac{i}{2}\right)^n}\right)+\mathcal{O}(g^4)$$

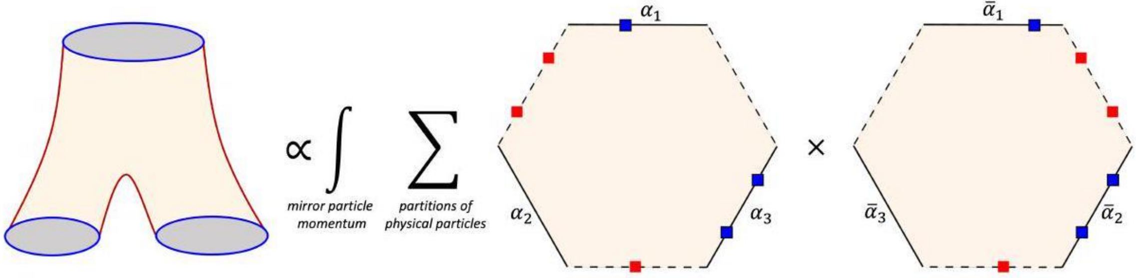
$$\mathbb{Q}_S(u) = \mathcal{P}_S(u)\left(1+g^2q_1^+\psi_0^+-\frac{g^4}{2}(q_1^+\psi_2^++q_2^-\psi_1^-)\right)+\mathcal{O}(g^6)$$

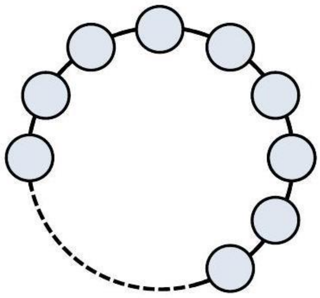
$$\text{res}_\mu = 0+ig^2\mathcal{P}_S\left(\frac{i}{2}\right)\mathcal{P}_J\left(\frac{i}{2}\right)(q_1^+(S)-q_1^+(J))+$$
$$+2g^2(2-a_1)\left(\mathcal{P}_S\left(\frac{i}{2}\right)\mathcal{P}_J'\left(\frac{i}{2}\right)-\mathcal{P}_J\left(\frac{i}{2}\right)\mathcal{P}_S'\left(\frac{i}{2}\right)\right)+\mathcal{O}(g^4)$$

$$\mathcal{P}'_{\mathrm{S}}\left(\frac{i}{2}\right) = \frac{q_1^+}{2i} + g^2\frac{q_3^+}{2i} + g^4\frac{q_5^+}{2i} + \mathcal{O}(g^6)$$

$$\mathcal{P}''_{\mathrm{S}}\left(\frac{i}{2}\right) = \left(\frac{q_2^+}{2} - \frac{(q_1^+)^2}{4}\right) + \frac{g^2}{2}(q_4^+ - q_1^+q_3^+) + \frac{g^4}{2}\left(3q_6^+ - q_1^+q_5^+ - \frac{(q_3^+)^2}{2}\right) + \mathcal{O}(g^6)$$

$$\mathrm{res}_\mu = 0 - ig^2(a_1 - 3)(q_1^+(S) - q_1^+(J)) + \mathcal{O}(g^4).$$





$$\langle \mathcal{O}^k(x_1)\mathcal{O}^l(x_2)\mathcal{O}^m(x_3)\rangle = \frac{C_{\mathcal{O}^k\mathcal{O}^l\mathcal{O}^m}\delta_{(k+l+m)\mathrm{mod}M}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}},$$

$$\frac{\langle \mathcal{O}^k(x_1)\mathcal{O}^l(x_2)\mathcal{O}^m(x_3)\rangle}{\langle \mathcal{V}^0(x_1)\mathcal{V}^0(x_2)\mathcal{V}^0(x_3)\rangle} = \frac{\hat{C}_{\mathcal{O}^k\mathcal{O}^l\mathcal{O}^m}\delta_{(k+l+m)\mathrm{mod}M}}{|x_1-x_2|^{\gamma_1+\gamma_2-\gamma_3}|x_1-x_3|^{\gamma_1+\gamma_3-\gamma_2}|x_2-x_3|^{\gamma_2+\gamma_3-\gamma_1}},$$

$$\gamma = \mathrm{diag}(\mathbb{1}_N, \omega\cdot\mathbb{1}_N, \omega^2\cdot\mathbb{1}_N, \dots, \omega^{M-1}\cdot\mathbb{1}_N), \omega = \exp\frac{2\pi i}{M}$$

$$R_\gamma^{-1}(X,Y,Z) = \gamma^\dagger(X,Y,Z)\gamma = (\omega^{t_X}X, \omega^{t_Y}Y, \omega^{t_Z}Z), t_X, t_Y, t_Z \in \mathbb{Z}\mathrm{mod}M$$

$$R_\gamma^{-1}(X,Y,Z) = \gamma^\dagger(X,Y,Z)\gamma = (\omega X, \omega^{-1}Y, Z),$$

$$X = \begin{pmatrix} 0 & X_{12} & & \\ & 0 & X_{23} & \\ & & \ddots & \ddots \\ X_{M1} & & & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & & & Y_{1M} \\ Y_{21} & 0 & & \\ & Y_{32} & \ddots & \\ & & \ddots & 0 \end{pmatrix}, Z = \begin{pmatrix} Z_{11} & & & \\ & Z_{22} & & \\ & & \ddots & \\ & & & Z_{MM} \end{pmatrix}$$

$$\mathrm{tr}_{MN} ZXXYZ\bar{Z} \dots$$

$$\#(X) - \#(Y) - \#(\bar{X}) + \#(\bar{Y}) = wM, \qquad w \in \mathbb{Z}.$$

$$\mathrm{tr}_{MN}\gamma^k ZXXYZ\bar{Z} \dots,$$

$$R_b: X_{12} \to Z_{11}$$

$$\Delta(R_b) = \mathbb{1} \otimes R_b + R_b \otimes \Omega$$

$$\mathrm{tr}\Phi_V^L$$

$$\left(\begin{array}{cc|cc} L^{\alpha}{}_{\beta}+D & K^{\alpha}{}_{\dot{\beta}} & \bar{Q}^{\alpha}{}_{b} & \bar{Q}^{\alpha}{}_{\dot{b}} \\ P^{\dot{\alpha}}{}_{\beta} & L^{\dot{\alpha}}{}_{\dot{\beta}}-D & Q^{\dot{\alpha}}{}_{b} & Q^{\dot{\alpha}}{}_{\dot{b}} \\ \hline \bar{S}^{a}{}_{\beta} & S^{a}{}_{\dot{\beta}} & R^{a}{}_{b}-J & R^{a}{}_{\dot{b}} \\ \bar{S}^{\dot{a}}{}_{\beta} & S^{\dot{a}}{}_{\dot{\beta}} & R^{\dot{a}}{}_{b} & R^{\dot{a}}{}_{\dot{b}}+J \end{array}\right),$$

$$\mathcal{H}=D-J,$$

$$R_{\gamma}=\begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{\gamma}=\begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{tr}\gamma^{k}\Phi_{V}^{L-K}\Phi_{E}^{K}+\text{ permutations}$$

$$\mathrm{tr}\gamma^{k}\Phi_{V}\Phi_{E}\Phi_{V}\Phi_{V}\Phi_{E}\ldots \leftrightarrow |\downarrow\uparrow\downarrow\downarrow\uparrow\cdots\rangle^{k},$$

$$\mathcal{D}_{2}=\sum_{l=1}^{L}\mathbb{1}-\mathbb{P}_{l,l+1}$$

$$(\Phi_{V},\Phi_{E})\sim(\omega^{p}\Phi_{V},\omega^{q}\Phi_{E})$$

$$e^{ip_{j}L}\prod_{j=1,j\neq i}^{K}S_{j,k}=\omega^{k(p-q)},e^{iP}=\prod_{j=1}^{K}e^{ip_{j}}=\omega^{kp},$$

$$S_{j,k}=\frac{u_{j}-u_{k}-i}{u_{j}-u_{k}+i},e^{ip_{j}}=\frac{u_{j}+i/2}{u_{j}-i/2}$$

$$E=\sum_{j=1}^{K}\frac{1}{u_{i}^{2}+1/4}.$$

$$|\Psi\rangle^{k}=\sum_{1\leq n_{1}<\cdots<n_{K}\leq L}\sum_{\sigma\in S_{K}}e^{i\sum_{l=1}^{K}p_{\sigma(k)}n_{k}}\prod_{\substack{j>l\\ \sigma(j)<\sigma(l)}}S_{\sigma(j),\sigma(l)}|n_{1},\ldots,n_{K}\rangle^{k},$$

$$\mathcal{B}^{k}=\frac{|\Psi\rangle^{k}}{\sqrt{\mathcal{G}\prod_{j}\left(u_{j}^{2}+\frac{1}{4}\right)\prod_{i<j}S_{i,j}}},$$

$$\mathcal{G}=\mathrm{Det}\phi_{jl},\phi_{jl}=-i\frac{\partial\log\left(e^{ip_{j}L}\prod_{j\neq l}S(u_{j},u_{l})\right)}{\partial u_{l}}.$$

$$\mathrm{tr}\gamma^{k}\bar{Y}^{|J_{2}|}X^{J_{1}}+\text{ permutations},(J_{1}>0,J_{2}<0)$$

$$e^{ip_jL}\prod_{j=1,j\neq i}^{|J_2|} S_{j,k}=1, e^{iP}=\prod_{j=1}^{|J_2|} e^{ip_j}=\omega^k$$

$$\mathrm{tr}\gamma^k Y^{J_2}X^{J_1} + \text{ permutations }, (J_1>0, J_2>0)$$

$$e^{ip_jL}\prod_{j=1,j\neq i}^{J_2} S_{j,k}=\omega^{2k}, e^{iP}=\prod_{j=1}^{J_2} e^{ip_j}=\omega^k$$

$$\mathrm{tr}\gamma^k Z^{J_3}X^{J_1} + \text{ permutations }, (J_1>0, J_3>0)$$

$$\mathfrak{B}_{\text{bifundamental vacuum}}:\ e^{ip_jL}\prod_{j=1,j\neq i}^{J_3} S_{j,k}=\omega^{k}, e^{iP}=\prod_{j=1}^{J_3} e^{ip_j}=\omega^k$$

$$\mathfrak{A}_{\text{adjoint vacuum}}:\ e^{ip_jL}\prod_{j=1,j\neq i}^{J_1} S_{j,k}=\omega^{-k}, e^{iP}=\prod_{j=1}^{J_1} e^{ip_j}=1$$

$$\mathcal{O}_j^{k,L}=\mathrm{tr}\left(\gamma^k\bar{Y}X^j\bar{Y}X^{L-j-2}\right)$$

L	Eigenstate	E	u_1	u_2
8	$\sqrt{6}\mathcal{B}_{\pm}^{1,8}=\mathcal{O}_1^{1,8}\pm i\sqrt{3}\mathcal{O}_2^{1,8}-\mathcal{O}_3^{1,8}$	4	$\pm\frac{\sqrt{3}}{2}$	$\pm\frac{1}{2\sqrt{3}}$
10	$2\mathcal{B}_{\pm}^{1,10}=\mathcal{O}_1^{1,10}\pm i\sqrt{2}\mathcal{O}_2^{1,10}-\mathcal{O}_3^{1,10}$	4	$\frac{1}{2}\pm\frac{1}{\sqrt{2}}$	$-\frac{1}{2}\pm\frac{1}{\sqrt{2}}$

L	Eigenstate	E	u_1	u_2
6	$2\mathcal{C}_{\pm}^{1,6}=\sqrt{2}\mathcal{O}_1^{1,6}\pm i\mathcal{O}_2^{1,6}$	4	$\frac{1}{2}\pm\frac{1}{\sqrt{2}}$	$-\frac{1}{2}\pm\frac{1}{\sqrt{2}}$
8	$\sqrt{2}\mathcal{C}_{\pm}^{1,8}=\mathcal{O}_1^{1,10}\pm i\mathcal{O}_2^{1,10}$	4	$1\pm\frac{\sqrt{3}}{2}$	$-1\pm\frac{\sqrt{3}}{2}$

$$u_2=-\frac{1}{2i}\frac{2u_1(\omega^k+1)-i(\omega^k-1)}{2u_1(\omega^k-1)-i(\omega^k+1)}\Rightarrow u_2=\frac{1}{4u_1},$$

$$T=-i\epsilon_{\dot{\alpha}\alpha}P^{\dot{\alpha}\alpha}$$

$$\mathcal{T}=-i\epsilon_{\dot{\alpha}\alpha}P^{\dot{\alpha}\alpha}+\epsilon_{\dot{a}a}R^{\dot{a}a}$$

$$\mathcal{R}_b^a=R_b^a+R_{\dot{b}}^{\dot{a}},\qquad \mathcal{L}_\beta^\alpha=L_\beta^\alpha+L_{\dot{\beta}}^{\dot{\alpha}}$$

$$\mathcal{Q}_\alpha^a=Q_\alpha^a+i\epsilon_{\alpha\dot{\beta}}\epsilon^{a\dot{b}}S_{\dot{b}}^{\dot{\beta}},\quad \mathcal{S}_a^\alpha=S_a^\alpha+i\epsilon_{a\dot{b}}\epsilon^{\alpha\dot{\beta}}Q_{\dot{\beta}}^{\dot{b}}$$

$$\Phi^{a\dot{a}}=\begin{pmatrix}0 & \bar{\Phi}_V & \bar{\Phi}_T & \Phi_L\\ -\bar{\Phi}_V & 0 & \bar{\Phi}_L & \Phi_T\\ -\bar{\Phi}_T & -\bar{\Phi}_L & 0 & \Phi_V\\ -\Phi_L & -\Phi_T & -\Phi_V & 0\end{pmatrix}$$

$$\mathcal{T}:\Phi_L\to\bar{\Phi}_V, \bar{\Phi}_L\to-\bar{\Phi}_V, \Phi_V\to\Phi_L-\bar{\Phi}_L$$

$$\tilde{\mathcal{O}}(t)=e^{t\mathcal{T}}\mathcal{O}(0)e^{-t\mathcal{T}}$$

$$\tilde{\Phi}^{b\dot{b}}(t)=e^{t\epsilon_{a\dot{a}}R^{a\dot{a}}}\Phi^{b\dot{b}}e^{-t\epsilon_{c\dot{c}}R^{c\dot{c}}}=\Phi^{b\dot{b}}+t\epsilon_{a\dot{a}}\left[R^{a\dot{a}},\Phi^{b\dot{b}}\right]+\frac{t^2}{2}\epsilon_{a\dot{a}}\epsilon_{c\dot{c}}\left[R^{c\dot{c}},\left[R^{a\dot{a}},\Phi^{b\dot{b}}\right]\right],$$

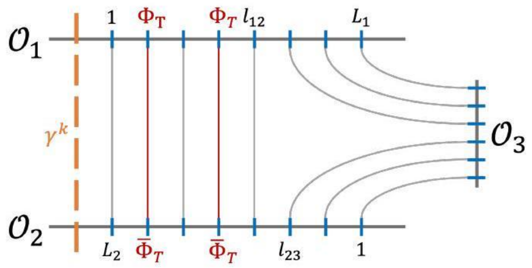
$$\tilde{\Phi}_T(t) = \Phi_T, \qquad \tilde{\bar{\Phi}}_T(t) = \bar{\Phi}_T$$
$$\tilde{\Phi}_L(t) = \Phi_L + t\bar{\Phi}_V, \qquad \tilde{\bar{\Phi}}_L(t) = \bar{\Phi}_L - t\bar{\Phi}_V$$
$$\tilde{\Phi}_V(t) = \Phi_V + t(\Phi_L - \bar{\Phi}_L) + t^2\bar{\Phi}_V, \quad \tilde{\bar{\Phi}}_V(t) = \bar{\Phi}_V$$

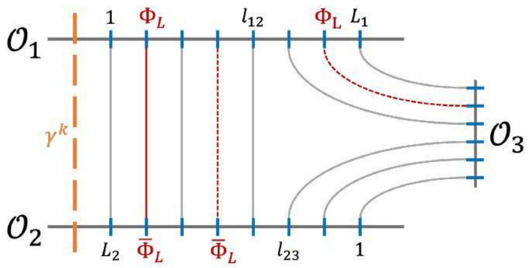
$$\left\langle \tilde{\Phi}_T(t_i)\tilde{\bar{\Phi}}_T(t_j)\right\rangle = \frac{1}{(t_i - t_j)^2}, \left\langle \tilde{\Phi}_L(t_i)\tilde{\bar{\Phi}}_L(t_j)\right\rangle = \frac{1}{(t_i - t_j)^2}$$
$$\langle \tilde{\Phi}_L(t_i)\tilde{\Phi}_V(t_j)\rangle = \frac{1}{t_i - t_j}, \left\langle \tilde{\bar{\Phi}}_L(t_i)\tilde{\Phi}_V(t_j)\right\rangle = -\frac{1}{t_i - t_j}, \langle \tilde{\Phi}_V(t_i)\tilde{\Phi}_V(t_j)\rangle = 1$$

$$\mathrm{tr}(T^a A)\mathrm{tr}(T^b B) = \delta^{ab}\left(\mathrm{tr}(AB) - \frac{1}{N}\mathrm{tr}(A)\mathrm{tr}(B)\right)$$
$$\mathrm{tr}(T^a A T^b B) = \delta^{ab}\left(\mathrm{tr}(A)\mathrm{tr}(B) - \frac{1}{N}\mathrm{tr}(AB)\right)$$

$$\psi^{p,q}_{n,m} \equiv e^{i(np+mq)} + e^{i(nq+mp)}S_{p,q}$$

$$C^{\bullet\bullet\circ}_{\Phi_T,\bar{\Phi}_T} = N_1 N_2 \sum_{1\leq n_1<n_2\leq \ell_{12}} \frac{\psi^{p_1,p_2}_{n_1,n_2} * \psi^{p_3,p_4}_{L_2-n_2+1,L_2-n_1+1}}{t^4_{12}},$$





$$C^{\bullet\bullet\circ}_{\Phi_L,\bar{\Phi}_L} = N_1 N_2 \left(\sum_{1\leq n_1<n_2\leq \ell_{12}} \frac{\psi^{p_1,p_2}_{n_1,n_2}}{t^2_{12}} + \sum_{\substack{1\leq n_1\leq \ell_{12}\\ \ell_{12}<n_2\leq L_1}} \frac{\psi^{p_1,p_2}_{n_1,n_2}}{t_{12}t_{13}} + \sum_{\ell_{12}<n_1<n_2\leq L_1} \frac{\psi^{p_1,p_2}_{n_1,n_2}}{t^2_{13}} \right) \times$$
$$\left(\sum_{1\leq m_1<m_2\leq \ell_{23}} \frac{\psi^{p_3,p_4}_{m_1,m_2}}{t^2_{23}} + \sum_{\substack{1\leq m_1\leq \ell_{23}\\ \ell_{23}<m_2\leq L_2}} \frac{\psi^{p_3,p_4}_{m_1,m_2}}{t_{23}t_{21}} + \sum_{\ell_{23}<m_1<m_2\leq L_2} \frac{\psi^{p_3,p_4}_{m_1,m_2}}{t^2_{21}} \right).$$

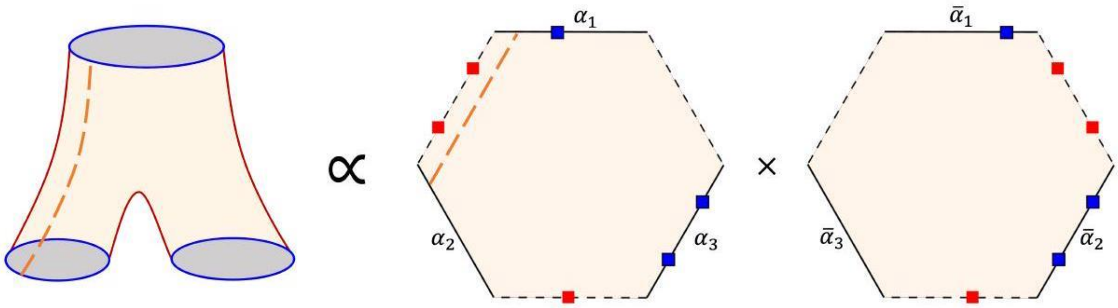
$$\rho_\ell(\alpha,\bar{\alpha}) = \prod_{j\in\bar{\alpha}} e^{ip_j\ell} \prod_{\substack{j,k\\ k\in\alpha}} S_{p_j,p_k}$$

$$\langle \mathcal{B}_1 \mathcal{O}_{L_2}\mathcal{O}_{L_3}\rangle = \sqrt{\frac{L_1L_2L_3}{\mathcal{G}S_{1,2}}} \sum_{\alpha\cup\bar{\alpha}=\{u_1,u_2\}} \frac{(-1)^{|\bar{\alpha}|}t_{23}}{t_{12}t_{13}} \rho_{\ell_{12}}(\alpha,\bar{\alpha})\langle \mathbf{h} \mid \alpha\rangle\langle \mathbf{h} \mid \bar{\alpha}\rangle$$

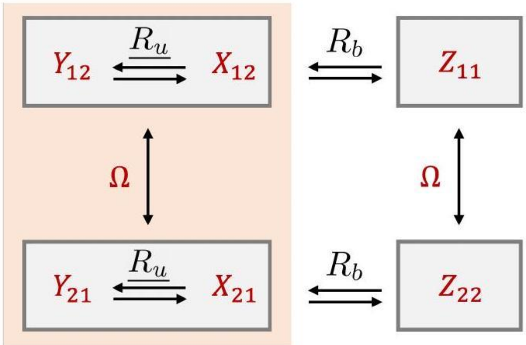
$$\langle \mathbf{h} \mid \{\}\rangle = 1, \langle \mathbf{h} \mid \{\Phi_L(u)\}\rangle = 1, \langle \mathbf{h} \mid \{\Phi_L(u_1),\Phi_L(u_2)\}\rangle = \frac{u_1-u_2}{u_1-u_2-i}.$$

$$\rho^k_\ell(\alpha,\bar{\alpha}) = \prod_{j\in\bar{\alpha}} \omega^{k|\bar{\alpha}|} e^{ip_j\ell} \prod_{\substack{j,k\\ k\in\alpha}} S_{p_j,p_k}$$

$$\langle \mathcal{O}^0\mathcal{O}^0\mathcal{O}^0\rangle \text{ and } \langle \mathcal{O}^1\mathcal{O}^1\mathcal{O}^0\rangle.$$



$$\begin{aligned}&\text{unbroken:} \quad \{R^1{}_1, R^2{}_2, R^{\dot1}{}_{\dot1}, R^{\dot2}{}_{\dot2}, R^1{}_{\dot1}, R^{\dot1}{}_1, R^2{}_{\dot2}, R^{\dot2}{}_2\},\\ &\text{broken:} \quad \{R^1{}_2, R^2{}_1, R^{\dot1}{}_{\dot2}, R^{\dot2}{}_{\dot1}, R^2{}_{\dot1}, R^{\dot1}{}_2, R^1{}_{\dot2}, R^{\dot2}{}_1\},\end{aligned}$$



$$\tilde{Z}(t) = \begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}, \tilde{Y}(t) = \begin{pmatrix} 0 & Y_{12} + t\bar{X}_{12} \\ Y_{21} + t\bar{X}_{21} & 0 \end{pmatrix}$$

$$\tilde{X}(t) = \begin{pmatrix} 0 & X_{12} + t(Y_{12} - \bar{Y}_{12}) + t^2\bar{X}_{12} \\ X_{21} + t(Y_{21} - \bar{Y}_{21}) + t^2\bar{X}_{21} & 0 \end{pmatrix}$$

$$\gamma^\dagger \tilde{X} \gamma = -\tilde{X}, \gamma^\dagger \tilde{Y} \gamma = -\tilde{Y}, \gamma^\dagger \tilde{Z} \gamma = +\tilde{Z}$$

$$\mathcal{O}^1(t) = e^{t\mathcal{T}} \mathcal{O}^1(0) e^{-t\mathcal{T}} = \operatorname{tr}\left(\gamma \tilde{\Phi}^{a_1 \dot{a}_1} \tilde{\Phi}^{a_2 \dot{a}_2} \dots \tilde{\Phi}^{a_L \dot{a}_L}\right)$$

$$\{R^1{}_1 + R^{\dot1}{}_{\dot1}, R^2{}_2 + R^{\dot2}{}_{\dot2}, R^1{}_2 + R^{\dot1}{}_{\dot2}, R^2{}_1 + R^{\dot2}{}_{\dot1}\},$$

$$\langle \mathbf{h} | g | \Psi \rangle = 0, \forall g \in \mathfrak{psu}(2 \mid 2)_D$$

$$\langle \mathcal{B}_1 \mathcal{B}_2 \mathcal{O}_L \rangle \sim \delta_{0,K \bmod M} \sum_{\substack{\alpha \cup \bar{\alpha} = \{u_1, u_2\} \\ \beta \cup \bar{\beta} = \{u_3, u_4\}}} \frac{(-1)^{|\bar{\alpha}| + |\bar{\beta}|}}{t_{12}^2} \rho_{\ell_{12}}(\alpha, \bar{\alpha}) \rho_{\ell_{12}}(\beta, \bar{\beta}) \langle \mathbf{h} \mid \alpha, \beta, \{\} \rangle \langle \mathbf{h} \mid \bar{\alpha}, \{\}, \bar{\beta} \rangle,$$

$$\sqrt{\frac{L_1 L_2 L}{\mathcal{G}_1 S_{1,2} \mathcal{G}_2 S_{3,4}}}$$

Correlators	$L=2$	4	6	8	10
$\langle \mathcal{B}^{1,8}_{\pm}\mathcal{B}^{1,8}_{\pm}\mathcal{O}_L\rangle$	$-4\sqrt{2}$	0	$\sqrt{6}$	$\frac{2\sqrt{2}}{3}$	0
$\langle \mathcal{B}^{1,8}_{\pm}\mathcal{B}^{1,8}_{\mp}\mathcal{O}_L\rangle$	0	0	$\sqrt{6}$	$\frac{2\sqrt{2}}{3}$	0
$\langle \mathcal{B}^{1,10}_{\pm}\mathcal{B}^{1,8}_{\pm}\mathcal{O}_L\rangle$	$-\frac{5+2\sqrt{6}}{\sqrt{2}}$	$2+\sqrt{6}$	$\frac{3}{2}$	$\frac{6\sqrt{2}+\sqrt{3}}{3}$	$\sqrt{\frac{5}{3}}$
$\langle \mathcal{B}^{1,10}_{\pm}\mathcal{B}^{1,8}_{\mp}\mathcal{O}_L\rangle$	$\frac{5-2\sqrt{6}}{\sqrt{2}}$	$2-\sqrt{6}$	$\frac{3}{2}$	$\frac{6\sqrt{2}-\sqrt{3}}{3}$	$\sqrt{\frac{5}{3}}$
$\langle \mathcal{B}^{1,10}_{\pm}\mathcal{B}^{1,10}_{\pm}\mathcal{O}_L\rangle$	$-6\sqrt{2}$	4	$\sqrt{\frac{3}{2}}$	$-3\sqrt{2}$	$3\sqrt{\frac{5}{2}}$
$\langle \mathcal{B}^{1,10}_{\pm}\mathcal{B}^{1,10}_{\mp}\mathcal{O}_L\rangle$	0	4	$3\sqrt{\frac{3}{2}}$	$-\sqrt{2}$	$\sqrt{\frac{5}{2}}$

Correlators	$L=2$	4	6	8	10
$\langle \mathcal{B}^{1,8}_{\pm}\hat{\mathcal{B}}^{1,8}_{\pm}\mathcal{O}_L\rangle$	0	0	$-\frac{1}{\sqrt{6}}$	$\sqrt{2}$	$-\frac{\sqrt{5}}{3\sqrt{2}}$
$\langle \mathcal{B}^{1,8}_{\pm}\hat{\mathcal{B}}^{1,8}_{\mp}\mathcal{O}_L\rangle$	0	0	$\frac{1}{\sqrt{6}}$	$\sqrt{2}$	$\frac{\sqrt{5}}{3\sqrt{2}}$
$\langle \mathcal{B}^{1,10}_{\pm}\hat{\mathcal{B}}^{1,8}_{\pm}\mathcal{O}_L\rangle$	0	0	0	0	$-\sqrt{\frac{5}{2}}$
$\langle \mathcal{B}^{1,10}_{\pm}\hat{\mathcal{B}}^{1,8}_{\mp}\mathcal{O}_L\rangle$	0	0	0	0	$\sqrt{\frac{5}{2}}$
$\langle \mathcal{B}^{1,10}_{\pm}\hat{\mathcal{B}}^{1,10}_{\pm}\mathcal{O}_L\rangle$	0	0	$-\frac{\sqrt{3}}{2\sqrt{2}}$	$\sqrt{2}$	0
$\langle \mathcal{B}^{1,10}_{\pm}\hat{\mathcal{B}}^{1,10}_{\mp}\mathcal{O}_L\rangle$	0	0	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\sqrt{2}$	0

$$\rho^k_\ell(\alpha,\bar{\alpha}) = \prod_{j\in\bar{\alpha}} \omega^{-k|\bar{\alpha}|} e^{ip_j\ell} \prod_{\substack{j,k\\ k\in\alpha}} S_{p_j,p_k},$$

Correlators	$L=2$	4	6	8	10
$\langle \mathcal{C}^{1,6}_{\pm}\hat{\mathcal{C}}^{1,6}_{\pm}\mathcal{O}_L\rangle$	$-4\sqrt{2}$	4	$-\sqrt{\frac{3}{2}}$	0	0
$\langle \mathcal{C}^{1,6}_{\pm}\hat{\mathcal{C}}^{1,6}_{\mp}\mathcal{O}_L\rangle$	0	0	$\sqrt{\frac{3}{2}}$	0	0
$\langle \mathcal{C}^{1,8}_{\pm}\hat{\mathcal{C}}^{1,6}_{\pm}\mathcal{O}_L\rangle$	$3\pm2\sqrt{2}$	$\mp3-2\sqrt{3}$	$\sqrt{3}\pm\sqrt{6}$	$\mp\sqrt{2}$	0
$\langle \mathcal{C}^{1,8}_{\pm}\hat{\mathcal{C}}^{1,6}_{\mp}\mathcal{O}_L\rangle$	$3\mp2\sqrt{2}$	$\pm3-2\sqrt{3}$	$\sqrt{3}\mp\sqrt{6}$	$-\mp\sqrt{2}$	0
$\langle \mathcal{C}^{1,8}_{\pm}\hat{\mathcal{C}}^{1,8}_{\pm}\mathcal{O}_L\rangle$	$-6\sqrt{2}$	8	$-5\sqrt{\frac{3}{2}}$	$3\sqrt{2}$	$-\sqrt{\frac{5}{2}}$
$\langle \mathcal{C}^{1,8}_{\pm}\hat{\mathcal{C}}^{1,8}_{\mp}\mathcal{O}_L\rangle$	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{2}$	$\sqrt{\frac{5}{2}}$

Correlators	$L=2$	4	6	8
$\langle \mathcal{B}^{0,4} \mathcal{O}^{1,L} \mathcal{O}^{1,L} \rangle$	$\sqrt{6}$	$4\sqrt{\frac{2}{3}}$	$5\sqrt{\frac{2}{3}}$	$2\sqrt{6}$

$$\mathcal{B}^{0,4} = \frac{1}{\sqrt{3}}\left(\mathcal{O}_0^{0,4} - \mathcal{O}_1^{0,4}\right).$$

$$Z_{11} \rightarrow \hat{Z}(t) = Z_{11} + t(Y_{12} - \bar{Y}_{12}) + t^2 \bar{Z}_{11}$$
$$Y_{12} \rightarrow \hat{Y}(t) = Y_{12} + t\bar{Z}_{11}$$

$$\mathrm{tr}\gamma \tilde{Z}(t)^L = \cdots + t^2 \mathrm{tr}(\cdots Z_{11} Y_{12} Z_{22} \cdots Z_{22} Y_{21} Z_{11} \cdots) + \cdots$$

$$\mathrm{tr}\gamma \tilde{Z}(t)^3 = \mathrm{tr}\gamma Z^3 + t^2(6\mathrm{tr}\gamma YYZ - 6\mathrm{tr}\gamma Y\bar{Y}Z - 6\mathrm{tr}\gamma \bar{Y}YZ + \mathrm{tr}\gamma \bar{Z}ZZ) + O(t^4)$$

L	Eigenstate	E	u_1
4	$\mathcal{B}^{1,4} = \mathcal{O}_0^{1,4}$	2	$\frac{\sqrt{3}}{2}$
5	$\sqrt{-2(2 \pm \sqrt{2})}\mathcal{B}_{\pm}^{1,5} = (1 \pm \sqrt{2})\mathcal{O}_0^{1,5} + \mathcal{O}_1^{1,5}$	$4 \mp 2\sqrt{2}$	$\frac{1}{2} \pm \frac{1}{\sqrt{2}}$

ℓ_{12}	2	3	4
$\langle \mathcal{B}^{1,4} \mathcal{O}_L \mathcal{O}^{1,2} \rangle$	2	$4\sqrt{2}$	$4\sqrt{3}$
$\langle \mathcal{B}^{1,4} \mathcal{O}_L \mathcal{O}^{1,3} \rangle$	3	$2\sqrt{15}$	$2\sqrt{21}$
$\langle \mathcal{B}^{1,4} \mathcal{O}_L \mathcal{O}^{1,4} \rangle$	4	$4\sqrt{6}$	$8\sqrt{2}$
$\langle \mathcal{B}_{\pm}^{1,5} \mathcal{O}_L \mathcal{O}^{1,2} \rangle$	—	$\sqrt{\frac{3}{2}(10 \pm 7\sqrt{2})}$	$\sqrt{5(10 \pm 7\sqrt{2})}$
$\langle \mathcal{B}_{\pm}^{1,5} \mathcal{O}_L \mathcal{O}^{1,3} \rangle$	$\sqrt{\frac{3}{2}(2 \pm \sqrt{2})}$	$\sqrt{3(10 \pm 7\sqrt{2})}$	$3\sqrt{(10 \pm 7\sqrt{2})}$
$\langle \mathcal{B}_{\pm}^{1,5} \mathcal{O}_L \mathcal{O}^{1,4} \rangle$	$\sqrt{3(2 \pm \sqrt{2})}$	$\sqrt{5(10 \pm 7\sqrt{2})}$	$\sqrt{14(10 \pm 7\sqrt{2})}$

$$C_Y^{\bullet\circ\circ} = N_1 \delta_{0,K \bmod M} \sum_{1 \le n_1 < n_2 \le \ell_{12}} \frac{\psi_{n_1,n_2}^{p_1,p_2}}{t_{12}^2},$$

$$\mathcal{B}^{k,4} = \mathrm{tr}\gamma^k YYYZ$$

$$\langle \mathcal{B}^{k,4} \mathcal{O}_6 \mathcal{O}^{-k,4} \rangle = 2\sqrt{6}$$

$$\langle \mathcal{B}_X^{1,4} \mathcal{B}_{\bar{X}}^{1,4} \mathcal{O}_4 \rangle = 2$$

k	Eigenstate	Energy E
1	$\mathcal{B}_1^{1,6} \sim (1 - i\sqrt{3})\,(\sqrt{17} - 5)\,\mathcal{O}_0^{1,6} - i\,(\sqrt{3} - i)\,(\sqrt{17} - 1)\,\mathcal{O}_1^{1,6} + 4\mathcal{O}_2^{1,6}$	$\frac{1}{2}\,(7 - \sqrt{17})$
	$\mathcal{B}_2^{1,6} \sim i\,(\sqrt{3} + i)\,(5 + \sqrt{17})\,\mathcal{O}_0^{1,6} + (1 + i\sqrt{3})\,(1 + \sqrt{17})\,\mathcal{O}_1^{1,6} + 4\mathcal{O}_2^{1,6}$	$\frac{1}{2}\,(7 + \sqrt{17})$
2	$\mathcal{B}_1^{2,6} \sim (1 + i\sqrt{3})\,(\sqrt{17} - 5)\,\mathcal{O}_0^{2,6} + i\,(\sqrt{3} + i)\,(\sqrt{17} - 1)\,\mathcal{O}_1^{2,6} + 4\mathcal{O}_2^{2,6}$	$\frac{1}{2}\,(7 + \sqrt{17})$
	$\mathcal{B}_2^{2,6} \sim i\,(\sqrt{3} - i)\,(5 + \sqrt{17})\,\mathcal{O}_0^{2,6} - (1 - i\sqrt{3})\,(1 + \sqrt{17})\,\mathcal{O}_1^{2,6} - 4\mathcal{O}_2^{2,6}$	$\frac{1}{2}\,(7 - \sqrt{17})$

L	0	6	12
$\left\langle \mathcal{B}_1^{k,6}\bar{\mathcal{B}}_1^{k,6}\mathcal{O}_L \right\rangle$	1	0.609612	0
$\left\langle \mathcal{B}_2^{k,6}\bar{\mathcal{B}}_3^{k,6}\mathcal{O}_L \right\rangle$	1	1.64039	0
$\left\langle \mathcal{B}_1^{k,6}\bar{\mathcal{B}}_2^{k,6}\mathcal{O}_L \right\rangle$	0	0	0

$\mathbb{Z}_2$ vac.	Sector	Length	Eigenstate	Energy	u_1	u_2
bif.	$\mathrm{SU}(2)_{L,R}$	8	$\sqrt{6}\mathcal{B}_\pm^{1,8} = \mathcal{O}_1^{1,8} \pm i\sqrt{3}\mathcal{O}_2^{1,8} - \mathcal{O}_3^{1,8}$	4	$\pm\frac{\sqrt{3}}{2}$	$\pm\frac{1}{2\sqrt{3}}$
		10	$2\mathcal{B}_\pm^{1,10} = \mathcal{O}_1^{1,10} \pm i\sqrt{2}\mathcal{O}_2^{1,10} - \mathcal{O}_3^{1,10}$	4	$\frac{1}{2} \pm \frac{1}{\sqrt{2}}$	$-\frac{1}{2} \pm \frac{1}{\sqrt{2}}$
	mixed SU(2)	6	$2\mathcal{C}_\pm^{1,6} = \sqrt{2}\mathcal{O}_1^{1,6} \pm i\mathcal{O}_2^{1,6}$	4	$\frac{1}{2} \pm \frac{1}{\sqrt{2}}$	$-\frac{1}{2} \pm \frac{1}{\sqrt{2}}$
		8	$\sqrt{2}\mathcal{C}_\pm^{1,8} = \mathcal{O}_1^{1,10} \pm i\mathcal{O}_2^{1,10}$	4	$1 \pm \frac{\sqrt{3}}{2}$	$-1 \pm \frac{\sqrt{3}}{2}$
		10	$\mathcal{C}_{1\pm}^{1,10} = -0.270598\mathcal{O}_1^{1,8} \mp \frac{1}{2}i\mathcal{O}_2^{1,8} + 0.653281\mathcal{O}_3^{1,8} \pm 0.353553i\mathcal{O}_4^{1,8}$	4	± 0.334089	± 0.748303
			$\mathcal{C}_{2\pm}^{1,10} = -0.653281\mathcal{O}_1^{1,8} \mp \frac{1}{2}i\mathcal{O}_2^{1,8} - 0.270598\mathcal{O}_3^{1,8} \mp 0.353553i\mathcal{O}_4^{1,8}$	4	± 2.51367	± 0.0994562
adj.	mixed SU(2)	3	$\mathcal{B}^{1,3} = \mathcal{O}_0^{1,3}$	4	$\frac{1}{2}$	$-u_1$
		4	$\mathcal{B}^{1,4} = \mathcal{O}_0^{1,4}$	2	$\frac{\sqrt{3}}{2}$	$-u_1$
		5	$\sqrt{-2(2 \pm \sqrt{2})}\mathcal{B}_\pm^{1,5} = (1 \pm \sqrt{2})\mathcal{O}_0^{1,5} + \mathcal{O}_1^{1,5}$	$4 \mp 2\sqrt{2}$	$\frac{1}{2} \pm \frac{1}{\sqrt{2}}$	$-u_1$
		6	$\mathcal{B}_+^{1,6} = 0.850651\mathcal{O}_0^{1,6} + 0.525731\mathcal{O}_1^{1,6}$	$3 - \sqrt{5}$	$\frac{1}{2}\sqrt{5 + 2\sqrt{5}}$	$-u_1$
			$\mathcal{B}_-^{1,6} = 0.525731\mathcal{O}_0^{1,6} - 0.850651\mathcal{O}_1^{1,6}$	$3 + \sqrt{5}$	$\frac{1}{2}\sqrt{5 - 2\sqrt{5}}$	$-u_1$
		7	$\sqrt{3}\mathcal{B}_1^{1,7} = (\mathcal{O}_0^{1,7} - \mathcal{O}_1^{1,7} - \mathcal{O}_2^{1,7})$	4	$\frac{1}{2}$	$-u_1$
			$\mathcal{B}_+^{1,7} = 0.788675\mathcal{O}_0^{1,7} + 0.57735\mathcal{O}_1^{1,7} + 0.211325\mathcal{O}_2^{1,7}$	$4 - 2\sqrt{3}$	$1 + \frac{\sqrt{3}}{2}$	$-u_1$
			$\mathcal{B}_-^{1,7} = 0.211325\mathcal{O}_0^{1,7} - 0.57735\mathcal{O}_1^{1,7} + 0.788675\mathcal{O}_2^{1,7}$	$4 + 2\sqrt{3}$	$1 - \frac{\sqrt{3}}{2}$	$-u_1$
		8	$\mathcal{B}_1^{1,8} = 0.327985\mathcal{O}_0^{1,8} - 0.736976\mathcal{O}_1^{1,8} + 0.591009\mathcal{O}_2^{1,8}$	6.49396	0.240787	$-u_1$
			$\mathcal{B}_2^{1,8} = 0.591009\mathcal{O}_0^{1,8} - 0.327985\mathcal{O}_1^{1,8} - 0.736976\mathcal{O}_2^{1,8}$	3.10992	0.62698	$-u_1$
			$\mathcal{B}_3^{1,8} = 0.736976\mathcal{O}_0^{1,8} + 0.591009\mathcal{O}_1^{1,8} + 0.327985\mathcal{O}_2^{1,8}$	0.396125	2.19064	$-u_1$
		9	$\mathcal{B}_1^{1,9} = 0.392847\mathcal{O}_0^{1,8} - 0.69352\mathcal{O}_1^{1,8} + 0.13795\mathcal{O}_2^{1,8} + 0.587938\mathcal{O}_3^{1,8}$	5.53073	0.334089	$-u_1$
			$\mathcal{B}_2^{1,9} = 0.69352\mathcal{O}_0^{1,8} + 0.587938\mathcal{O}_1^{1,8} + 0.392847\mathcal{O}_2^{1,8} + 0.13795\mathcal{O}_3^{1,8}$	0.304482	2.51367	$-u_1$
			$\mathcal{B}_3^{1,9} = 0.13795\mathcal{O}_0^{1,8} - 0.392847\mathcal{O}_1^{1,8} + 0.587938\mathcal{O}_2^{1,8} - 0.69352\mathcal{O}_3^{1,8}$	7.69552	0.0994562	$-u_1$
			$\mathcal{B}_4^{1,9} = 0.587938\mathcal{O}_0^{1,8} - 0.13795\mathcal{O}_1^{1,8} - 0.69352\mathcal{O}_2^{1,8} - 0.392847\mathcal{O}_3^{1,8}$	2.46927	0.748303	$-u_1$
		10	$\mathcal{B}_1^{1,10} = 0.57735(\mathcal{O}_0^{1,10} - \mathcal{O}_2^{1,10} - \mathcal{O}_3^{1,10})$	2	$\frac{\sqrt{3}}{2}$	$-u_1$
			$\mathcal{B}_2^{1,10} = 0.228013\mathcal{O}_0^{1,10} - 0.57735\mathcal{O}_1^{1,10} + 0.656539\mathcal{O}_2^{1,10} - 0.428525\mathcal{O}_3^{1,10}$	7.06418	0.181985	$-u_1$
			$\mathcal{B}_3^{1,10} = 0.428525\mathcal{O}_0^{1,10} - 0.57735\mathcal{O}_1^{1,10} - 0.228013\mathcal{O}_2^{1,10} + 0.656539\mathcal{O}_3^{1,10}$	4.69459	0.41955	$-u_1$
			$\mathcal{B}_4^{1,10} = 0.656539\mathcal{O}_0^{1,10} + 0.57735\mathcal{O}_1^{1,10} + 0.428525\mathcal{O}_2^{1,10} + 0.228013\mathcal{O}_3^{1,10}$	0.24123	2.83564	$-u_1$

$$
\begin{aligned}
C^{\bullet\bullet\circ}_{\Phi_L,\Phi_L} = C^{\bullet\bullet\circ}_{\Phi_L,\bar{\Phi}_L} - N_1N_2\Bigg(& \sum_{\substack{1\leq n_1<n_2\leq \ell_{12} \\ \ell_{23}<m_1\leq L_2-n_1}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_1+1}\cdot\frac{1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)^2} \\
& + \sum_{\substack{1\leq n_1<\ell_{12} \\ \ell_{12}<n_2\leq L_1 \\ \ell_{23}<m_1\leq L_2-n_1}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_1+1}\cdot\frac{-1}{(t_1-t_2)^2}\cdot\frac{1}{(t_1-t_3)}\cdot\frac{1}{(t_2-t_1)} \\
& + \sum_{\substack{1\leq n_1<n_2\leq \ell_{12} \\ 1\leq m_1\leq \ell_{23}}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_1+1}\cdot\frac{-1}{(t_1-t_2)^2}\cdot\frac{1}{(t_1-t_2)}\cdot\frac{1}{(t_2-t_3)} \\
& + \sum_{\substack{1\leq n_1\leq \ell_{12}<n_2\leq L_1 \\ 1\leq m_1\leq \ell_{23}}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_1+1}\cdot\frac{-1}{(t_1-t_2)}\cdot\frac{1}{(t_1-t_3)}\cdot\frac{1}{(t_2-t_3)} \\
& + \sum_{\substack{1<n_1<n_2\leq \ell_{12} \\ L_2-n_1+1<m_2\leq L_2}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{L_2-n_1+1,m_2}\cdot\frac{1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)^2} \\
& + \sum_{\substack{1<n_1\leq \ell_{12}<n_2\leq L_1 \\ L_2-n_1+1<m_2\leq L_2}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{L_2-n_1+1,m_2}\cdot\frac{-1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)}\cdot\frac{1}{(t_1-t_3)} \\
& + \sum_{\substack{1\leq n_1<n_2\leq \ell_{12}-1 \\ \ell_{23}<m_1\leq L_2-n_2}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_2+1}\cdot\frac{1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)^2} \\
& + \sum_{\substack{1\leq n_1<n_2\leq \ell_{12} \\ 1\leq m_1\leq \ell 23}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{m_1,L_2-n_2+1}\cdot\frac{-1}{(t_1-t_2)^2}\cdot\frac{1}{(t_1-t_2)}\cdot\frac{1}{(t_2-t_3)} \\
& + \sum_{\substack{1\leq n_1<n_2\leq \ell_{12} \\ L_2-n_2+1<m_2\leq L_2}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{L_2-n_2+1,m_2}\cdot\frac{1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)^2} \\
& - \sum_{1\leq n_1<n_2\leq \ell_{12}} \psi^{p_1,p_2}_{n_1,n_2}\,\psi^{p_3,p_4}_{L_2-n_2+1,L_2-n_1+1}\cdot\frac{1}{(t_1-t_2)^2}\cdot\frac{1}{(t_2-t_1)^2}\Bigg).
\end{aligned}
$$

$$
H_\mu = \frac{R}{2}\mathrm{Tr}\left[\left(\frac{\mu}{3R}\right)^2\sum_{i=1}^{3}\left(X^i\right)^2 + \left(\frac{\mu}{6R}\right)^2\sum_{m=4}^{9}\left(X^m\right)^2 + \frac{\mu}{4R}\psi^{\dagger I\alpha}\psi_{I\alpha} + i\,\frac{2\mu}{3R}\epsilon_{ijk}X^iX^jX^k\right]
$$

$$
X^m = 0, X^i = \frac{\mu}{3R}J^i
$$

$$
N = \sum_{k=1}^{K} n_k N_k
$$

Oscillator	Mass	$SO(6)\times SO(3)$ Rep	BPS-fraction
α_{jm}	$\frac{j+1}{3}\mu$	$(1,2j+1)$, $0\leq j\leq N-2$	0
β_{jm}	$\frac{j}{3}\mu$	$(1,2j+1)$, $1\leq j\leq N$	$\frac{1}{4}$
x^a_{jm}	$\frac{j+\frac{1}{2}}{3}\mu$	$(6,2j+1)$, $0\leq j\leq N-1$	$\frac{1}{8}$
$\chi^{\bar{I}}_{jm}$	$\frac{j+\frac{3}{4}}{3}\mu$	$(\bar{4},2j+1)$, $-\frac{1}{2}\leq j\leq N-\frac{3}{2}$	$\frac{1}{16}$
η^{I}_{jm}	$\frac{j+\frac{1}{4}}{3}\mu$	$(4,2j+1)$, $\frac{1}{2}\leq j\leq N-\frac{1}{2}$	$\frac{3}{16}$

$$
\{Q^\dagger_{I\alpha}, Q_{J\beta}\} = 2\delta_{IJ}\delta_{\alpha\beta}H - \frac{\mu}{3}\epsilon^{ijk}\sigma^k_{\alpha\beta}\delta_{IJ}M^{ij} - \frac{i\mu}{6}\delta_{\alpha\beta}(g^{mn})_{IJ}M^{mn},
$$

$$\Theta_{\{\epsilon_j\}} \equiv \frac{1}{2}\{Q, Q^\dagger\}_{\{\epsilon_j\}} = H - \epsilon_1 \frac{\mu}{3} M^{12} + \epsilon_2 \frac{\mu}{6} M^{45} + \epsilon_3 \frac{\mu}{6} M^{67} - \frac{\mu}{6} \epsilon_2 \epsilon_3 M^{89},$$

$$Z \equiv \mathrm{Tr}\left(e^{-\beta\Theta - 2\omega M^{12} - \Delta_{45} M^{45} - \Delta_{67} M^{67} - \Delta_{89} M^{89}}\right)$$

$$\Delta_{45} + \Delta_{67} + \Delta_{89} - 2\omega = 2\pi i$$

$$\mathcal{I} = \mathrm{Tr}_{BPS}\left[e^{2\pi i M^{12} - \beta\Theta - \Delta_{45}(M^{12}+M^{45}) - \Delta_{67}(M^{12}+M^{67}) - \Delta_{89}(M^{12}+M^{89})}\right]$$

$$\mathcal{I} = \sum_{\{n_i, N_i\}, \sum_{k=1}^{K} n_k N_k = N} \mathcal{I}_{n_i, N_i}$$

$$\mathcal{I}_{n_i,N_i} = \int \prod_{k=1}^{K} [dU_k] \exp\left(\sum_{m=1}^{\infty} \sum_{k,l=1}^{K} \frac{1}{m} z(m\Delta_{mn}) \mathrm{Tr}\left(\left(U_k^\dagger\right)^m\right) \mathrm{Tr}(U_l^m)\right)$$

$$z_{kl}(\omega, \Delta_{45}, \Delta_{67}, \Delta_{89}) = \delta_{N_k,N_l} + \frac{\left(e^{-\omega|N_k - N_l|} - e^{-\omega(N_k+N_l)}\right)}{e^{2\omega} - 1} \times$$
$$\left(1 + e^{\omega + \frac{1}{2}(\Delta_{45} - \Delta_{67} - \Delta_{89})}\right)\left(1 + e^{\omega + \frac{1}{2}(\Delta_{67} - \Delta_{45} - \Delta_{89})}\right)\left(1 + e^{\omega + \frac{1}{2}(\Delta_{89} - \Delta_{45} - \Delta_{67})}\right)$$

$$z_{N,1}(\Delta_{45}, \Delta_{67}, \Delta_{89}) = 1 - \left(1 - e^{-\Delta_{45}}\right)\left(1 - e^{-\Delta_{67}}\right)\left(1 - e^{-\Delta_{89}}\right).$$

$$\log\left(\mathcal{I}_{N,1}\right) = -\frac{3N^2}{2\pi^2} \Delta_{45}\Delta_{67}\Delta_{89}$$

$$S_{N,1} = 2\pi \sqrt{\frac{2}{3}} N^2 \sqrt{\frac{M^{12} + M^{45}}{N^2} \times \frac{M^{12} + M^{67}}{N^2} \times \frac{M^{12} + M^{89}}{N^2}}.$$

$$\mathcal{I}_{\mathrm{SYM}} = \int [dU] \exp\left(\sum_{m=1}^{\infty} \frac{1 - \frac{\left(1 - e^{-m\Delta_1}\right)\left(1 - e^{-m\Delta_2}\right)\left(1 - e^{-m\Delta_3}\right)}{\left(1 - e^{-m\omega_1}\right)\left(1 - e^{-m\omega_2}\right)}}{m} \mathrm{Tr}(U^m)\mathrm{Tr}\left(\left(U^\dagger\right)^m\right)\right)$$

$$\log\left(\mathcal{I}_{\mathrm{SYM},0}\right) = \frac{1}{2} N^2 \frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2}$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i$$

$$S_{\mathrm{SYM}} = \mathrm{ext}\left(\log\left(\mathcal{I}_{\mathrm{SYM},0}\right) + \sum_{j=1}^{3} \Delta_j Q_j + \sum_{i=1}^{2} \omega_i J_i\right)$$

$$\Delta_1 = \Delta_2 = \Delta_3 = 2\beta + 2\pi i, \omega_1 = \omega_2 = 3\beta + 2\pi i$$

$$Q^3 + \frac{N^2}{2} J^2 = \left(\frac{N^2}{2} + 3Q\right)\left(3Q^2 - N^2 J\right)$$

$$S(Q, J) = 2\pi\sqrt{3Q^2 - N^2 J}$$

$$Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{1}{2}N^2(J_1 + J_2) = \frac{Q_1Q_2Q_3 + \frac{1}{2}N^2 J_1 J_2}{Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2}$$

$$M^{45}M^{67} + M^{45}M^{89} + M^{67}M^{89} - 3N^2 M^{12} = \frac{M^{45}M^{67}M^{89} + \frac{3}{2}N^2 (M^{12})^2}{M^{45} + M^{67} + M^{89} + \frac{3}{2}N^2}$$

$$\Delta_{mn} = \Delta, M^{45} = M^{67} = M^{89} = Q.$$

$$\log\left(\mathcal{I}_{N,1}\right) = AN^2\frac{\Delta^3}{\omega^2},$$

$$3\Delta - 2\omega = 2\pi i$$

$$S_{N,1} = \mathrm{ext}\left(AN^2\frac{\Delta^3}{\omega^2} + 3\Delta Q + 2\omega M^{12} + \Lambda(3\Delta - 2\omega - 2\pi i)\right)$$

$$\frac{\partial S_{N,1}}{\partial \Delta} = 3AN^2\frac{\Delta^2}{\omega^2} + 3Q + 3\Lambda = 0$$
$$\frac{\partial S_{N,1}}{\partial \omega} = -2AN^2\frac{\Delta^3}{\omega^3} + 2M^{12} - 2\Lambda = 0$$

$$AN^2\left(\frac{\Delta^2}{\omega^2} - \frac{\Delta^3}{\omega^3}\right) + Q + M^{12} = 0$$

$$y \equiv \frac{\Delta}{\omega}, \epsilon \equiv \frac{Q + M^{12}}{AN^2}, \tilde{\epsilon} = \epsilon + \frac{2}{27}.$$

$$y^3 - y^2 - \epsilon = 0$$

$$t \equiv y - \frac{1}{3}$$

$$t^3 - \frac{1}{3}t - \tilde{\epsilon} = 0$$

$$\Delta_{dis} = -(4p^3 + 27q^2).$$

$$u_1 \equiv -\frac{q}{2} + \frac{1}{6\sqrt{3}}\sqrt{4p^3 + 27q^2}$$
$$u_2 \equiv -\frac{q}{2} - \frac{1}{6\sqrt{3}}\sqrt{4p^3 + 27q^2}$$

$$p = -\frac{1}{3} \text{ and } q = -\tilde{\epsilon}$$

$$\Delta_{dis} = \frac{4}{27} - 27\tilde{\epsilon}^2 = -27\epsilon^2 - 4\epsilon.$$

$$u_1 = \frac{\tilde{\epsilon}}{2} + \sqrt{\frac{\tilde{\epsilon}^2}{4} - \frac{1}{27^2}},$$
$$u_2 = \frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} - \frac{1}{27^2}}.$$

$$t_1 = u_1^{\frac{1}{3}} + u_2^{\frac{1}{3}}$$
$$t_2 = \frac{-1 + i\sqrt{3}}{2}u_1^{\frac{1}{3}} - \frac{1 + i\sqrt{3}}{2}u_2^{\frac{1}{3}}$$
$$t_3 = -\frac{1 + i\sqrt{3}}{2}u_1^{\frac{1}{3}} + \frac{-1 + i\sqrt{3}}{2}u_2^{\frac{1}{3}}$$

$$y_i = t_i + \frac{1}{3}.$$

$$\epsilon = \frac{Q + J}{AN^2} \ll 1.$$

$$u_1^{\frac{1}{3}} \approx \frac{1}{3} + \sqrt{\frac{\epsilon}{3}} + \frac{\epsilon}{2} + O\left(\epsilon^{\frac{3}{2}}\right),$$

$$u_2^{\frac{1}{3}} \approx \frac{1}{3} - \sqrt{\frac{\epsilon}{3}} + \frac{\epsilon}{2} + O\left(\epsilon^{\frac{3}{2}}\right).$$

$$y_1 = 1 + \cdots, y_2 = i\sqrt{\epsilon} + \cdots, y_3 = -i\sqrt{\epsilon} + \cdots.$$

$$\Delta_* \approx i\sqrt{\epsilon}\omega_*.$$

$$\omega_* \approx -\frac{\pi i}{1 - \frac{3}{2}i\sqrt{\epsilon}},$$

$$\Delta_* \approx \frac{\pi\sqrt{\epsilon}}{1 - \frac{3}{2}i\sqrt{\epsilon}}.$$

$$S_{N,1} = AN^2\frac{\Delta_*^3}{\omega_*^2} + 3\Delta_*Q + 2\omega_*M^{12} \approx -\pi AN^2\epsilon^{\frac{3}{2}} + 3\pi Q\sqrt{\epsilon} - 2\pi i M^{12} + \cdots$$

$$S_{N,1} = 2\pi\frac{1}{\sqrt{A}N}Q^{\frac{3}{2}}.$$

$$S_{N,1} = 2\pi\sqrt{\frac{2}{3}\frac{1}{N}}(Q + M^{12})^{\frac{3}{2}},$$

$$A = \frac{3}{2}.$$

$$\omega \approx -\pi i + \cdots,$$

$$\log\left(\mathcal{I}_{N,1}\right) \approx -\frac{3}{2\pi^2}N^2\Delta_{45}\Delta_{67}\Delta_{89}$$

$$\log\left(\mathcal{I}_{N,1}\right) = \frac{3}{2}N^2\frac{\Delta^3}{\omega^2},$$

$$S_{N,1} = 2\pi\sqrt{\frac{2}{3}\frac{1}{N}}Q^{\frac{3}{2}}, Q = M^{45} = M^{67} = M^{89}$$

$$M^{12} = 0$$

$$\log\left(\mathcal{I}_{N,1}\right) = \frac{3}{2}N^2\frac{\Delta_{45}\Delta_{67}\Delta_{89}}{\omega^2}.$$

$$S_{N,1} = \max\left(\log\left(\mathcal{I}_{N,1}\right) + M^{45}\Delta_{45} + M^{67}\Delta_{67} + M^{89}\Delta_{89} + 2M^{12}\omega + \Lambda(\Delta_{45} + \Delta_{67} + \Delta_{89} - 2\omega - 2\pi i)\right)$$

$$\Lambda_{mn} \equiv \Lambda + M^{mn}, m, n = \{45,67,89\}$$

$$\frac{\partial S_{N,1}}{\partial \Delta_{mn}} = \frac{3N^2\Delta_{45}\Delta_{67}\Delta_{89}}{2\Delta_{mn}\omega^2} + \Lambda_{mn} = 0$$
$$\frac{\partial S_{N,1}}{\partial \omega} = -3N^2\frac{\Delta_{45}\Delta_{67}\Delta_{89}}{\omega^3} + 2M^{12} - 2\Lambda = 0$$
$$\frac{\partial S_{N,1}}{\partial \Lambda} = \Delta_{45} + \Delta_{67} + \Delta_{89} - 2\omega - 2\pi i = 0$$

$$\Delta_{67} = \Delta_{45}\frac{\Lambda_{45}}{\Lambda_{67}}, \Delta_{89} = \Delta_{45}\frac{\Lambda_{45}}{\Lambda_{89}}.$$

$$\omega = \pm iN\Delta_{45}\sqrt{\frac{3\Lambda_{45}}{2\Lambda_{67}\Lambda_{89}}}$$

$$\Delta_{45} = \frac{2\pi i \Lambda_{67}\Lambda_{89}}{\Lambda_{67}\Lambda_{89} + \Lambda_{45}\Lambda_{89} + \Lambda_{45}\Lambda_{67} - 2iN\sqrt{\frac{3}{2}\Lambda_{45}\Lambda_{67}\Lambda_{89}}}$$

$$\Delta_{67} = \frac{2\pi i \Lambda_{45}\Lambda_{89}}{\Lambda_{67}\Lambda_{89} + \Lambda_{45}\Lambda_{89} + \Lambda_{45}\Lambda_{67} - 2iN\sqrt{\frac{3}{2}\Lambda_{45}\Lambda_{67}\Lambda_{89}}}$$

$$\Delta_{89} = \frac{2\pi i \Lambda_{45}\Lambda_{67}}{\Lambda_{67}\Lambda_{89} + \Lambda_{45}\Lambda_{89} + \Lambda_{45}\Lambda_{67} - 2iN\sqrt{\frac{3}{2}\Lambda_{45}\Lambda_{67}\Lambda_{89}}}$$

$$\omega = -\frac{2\pi N\sqrt{\frac{3}{2}\Lambda_{45}\Lambda_{67}\Lambda_{89}}}{\Lambda_{67}\Lambda_{89} + \Lambda_{45}\Lambda_{89} + \Lambda_{45}\Lambda_{67} - 2iN\sqrt{\frac{3}{2}\Lambda_{45}\Lambda_{67}\Lambda_{89}}}$$

$$M^{12} - \Lambda = i\frac{1}{N}\sqrt{\frac{2}{3}\Lambda_{45}\Lambda_{67}\Lambda_{89}}$$

$$\Lambda = \frac{3}{2}N^2\lambda, \Lambda_{mn} = \frac{3}{2}N^2\lambda_{mn}, M^{12} = \frac{3}{2}N^2 j, M^{mn} = \frac{3}{2}N^2 q_{mn}$$

$$j - \lambda = i\sqrt{\lambda_{45}\lambda_{67}\lambda_{89}}$$

$$(\lambda + q_{45})(\lambda + q_{67})(\lambda + q_{89}) + \lambda^2 - 2j\lambda + j^2 = 0$$

$$\lambda^3 + (q_{45} + q_{67} + q_{89} + 1)\lambda^2 + (q_{45}q_{67} + q_{45}q_{89} + q_{67}q_{89} - 2j)\lambda + q_{45}q_{67}q_{89} + j^2 = 0$$

$$-\frac{3}{2}N^2\frac{\Delta_{45}\Delta_{67}\Delta_{89}}{\omega^2} = M^{45}\Delta_{45} + M^{67}\Delta_{67} + M^{89}\Delta_{89} + 2M^{12}\omega + \Lambda(\Delta_{45} + \Delta_{67} + \Delta_{89} - 2\omega).$$

$$S_{N,1} = -2\pi i\Lambda = -3\pi N^2 i\lambda$$

$$\mathrm{Re}(\lambda) = 0 \Rightarrow \lambda = i\alpha, \alpha \in \mathbb{R}$$

$$-i\alpha^3 - (q_{45} + q_{67} + q_{89} + 1)\alpha^2 + i(q_{45}q_{67} + q_{45}q_{89} + q_{67}q_{89} - 2j)\alpha + q_{45}q_{67}q_{89} + j^2 = 0$$

$$-\alpha\left(\alpha^2 - (q_{45}q_{67} + q_{45}q_{89} + q_{67}q_{89} - 2j)\right) = 0$$

$$(q_{45} + q_{67} + q_{89} + 1)\alpha^2 = q_{45}q_{67}q_{89} + j^2$$

$$q_{45}q_{67} + q_{45}q_{89} + q_{67}q_{89} - 2j = \frac{q_{45}q_{67}q_{89} + j^2}{q_{45} + q_{67} + q_{89} + 1}$$

$$S_{N,1} = 3\pi N^2\sqrt{\frac{q_{45}q_{67}q_{89} + j^2}{q_{45} + q_{67} + q_{89} + 1}}$$

$$-\epsilon_2\Delta_{45} - \epsilon_3\Delta_{67} + \epsilon_2\epsilon_3\Delta_{89} - 2\epsilon_1\omega = 2\pi i$$

$$\log\left(\mathcal{J}_{N,1}\right) = \frac{3N^2\Delta_{45}\Delta_{67}\Delta_{89}}{2\omega^2}$$

$$S_{N,1} = \max\Big(\frac{3N^2\Delta_{45}\Delta_{67}\Delta_{89}}{2\omega^2} + M^{45}\Delta_{45} + M^{67}\Delta_{67} + M^{89}\Delta_{89} + 2M^{12}\omega$$
$$+\Lambda(-\epsilon_2\Delta_{45} - \epsilon_3\Delta_{67} + \epsilon_2\epsilon_3\Delta_{89} - 2\epsilon_1\omega - 2\pi i)\Big)$$

$$-i\alpha^3-(-\epsilon_2 q_{45}-\epsilon_3 q_{67}+\epsilon_2\epsilon_3 q_{89}+1)\alpha^2+i(\epsilon_2\epsilon_3 q_{45}q_{67}-\epsilon_3 q_{45}q_{89}-\epsilon_2 q_{67}q_{89}-2\epsilon_1 j)\,\alpha$$
$$+q_{45}q_{67}q_{89}+j^2=0$$

$$-\alpha\left(\alpha^2-(\epsilon_2\epsilon_3 q_{45}q_{67}-\epsilon_3 q_{45}q_{89}-\epsilon_2 q_{67}q_{89}-2\epsilon_1 j)\right)=0$$

$$(-\epsilon_2 q_{45}-\epsilon_3 q_{67}+\epsilon_2\epsilon_3 q_{89}+1)\alpha^2=q_{45}q_{67}q_{89}+j^2$$

$$\epsilon_2\epsilon_3 q_{45}q_{67}-\epsilon_3 q_{45}q_{89}-\epsilon_2 q_{67}q_{89}-2\epsilon_1 j=\frac{q_{45}q_{67}q_{89}+j^2}{-\epsilon_2 q_{45}-\epsilon_3 q_{67}+\epsilon_2\epsilon_3 q_{89}+1}$$

$$S_{N,1}=3\pi N^2\sqrt{\epsilon_2\epsilon_3 q_{45}q_{67}-\epsilon_3 q_{45}q_{89}-\epsilon_2 q_{67}q_{89}-2\epsilon_1 j}$$

$$S_{N,1}=3\pi N^2\sqrt{\frac{q_{45}q_{67}q_{89}+j^2}{-\epsilon_2 q_{45}-\epsilon_3 q_{67}+\epsilon_2\epsilon_3 q_{89}+1}}$$

$$S_{N,1}\approx 3\pi N^2\sqrt{q_{45}q_{67}q_{89}}\approx 3\pi N^2\sqrt{(q_{45}+j)(q_{67}+j)(q_{89}+j)}$$

$$j=0, q_{45}=q_{67}=q_{89}=q\neq 0:$$

$$(\epsilon_2\epsilon_3-\epsilon_2-\epsilon_3)q^2=\frac{q^3}{1+(\epsilon_2\epsilon_3-\epsilon_2-\epsilon_3)q}$$

$$S_{N,1}=\frac{9\sqrt{3}\pi^2 N^2}{8}$$

$$j=\frac{2}{3}\frac{M^{12}}{N^2}=-2\epsilon_1$$

$$S_{N,1}=6\pi N^2$$

$$12\left(\frac{H}{\mu},\frac{M^{12}}{3},\frac{M^{45}}{6},\frac{M^{67}}{6},\frac{M^{89}}{6}\right)$$

$$\Theta_{\{\epsilon_j\}}=H-\epsilon_1\frac{\mu}{3}M^{12}+\epsilon_2\frac{\mu}{6}M^{45}+\epsilon_3\frac{\mu}{6}M^{67}-\frac{\mu}{6}\epsilon_2\epsilon_3 M^{89}.$$

$$(1,-2\epsilon_1,-\epsilon_2,-\epsilon_3,\epsilon_2\epsilon_3)$$

$$\Theta=H-\frac{\mu}{3}M^{12}+\frac{\mu}{6}M^{45}+\frac{\mu}{6}M^{67}-\frac{\mu}{6}M^{89}.$$

$\beta(4j,4j,0,0,0)$
$x(4j+2,4j,-2,0,0),(4j+2,4j,0,-2,0),(4j+2,4j,0,0,2),$
$\chi(4j+3,4j,-1,-1,1),$
$\eta(4j+1,4j,-1,-1,-1),(4j+1,4j,-1,1,1),(4j+1,4j,1,-1,1).$

$$z_{N,1}^{+++}=1+e^{-2\omega}+e^{\Delta_{45}}+e^{\Delta_{67}}+e^{-\Delta_{89}}+e^{\omega}e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}-\Delta_{89})}$$
$$+e^{-\omega}\left(e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}+\Delta_{89})}+e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}-\Delta_{89})}+e^{\frac{1}{2}(-\Delta_{45}+\Delta_{67}-\Delta_{89})}\right)$$

$$-\Delta_{45}-\Delta_{67}+\Delta_{89}-2\omega=2\pi i$$

$$z_{N,1}^{+++}=1-\left(1-e^{\Delta_{45}}\right)\left(1-e^{\Delta_{67}}\right)\left(1-e^{-\Delta_{89}}\right)$$

$$\mathcal{I}_{N,1}^{+++}=\int\,[dU]\exp\left(\sum_{m=1}^{\infty}\frac{1-\left(1-e^{m\Delta_{45}}\right)\left(1-e^{m\Delta_{67}}\right)\left(1-e^{-m\Delta_{89}}\right)}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}\left(U^{\dagger m}\right)$$

$$\epsilon_1=1,\epsilon_2=1,\epsilon_3=-1$$

$$\Theta = H - \frac{\mu}{3}M^{12} + \frac{\mu}{6}M^{45} - \frac{\mu}{6}M^{67} + \frac{\mu}{6}M^{89}$$

$\beta(4j, 4j, 0,0,0)$
$x(4j + 2,4j, -2,0,0), (4j + 2,4j, 0,2,0), (4j + 2,4j, 0,0, -2),$
$\chi(4j + 3,4j, -1,1, -1),$
$\eta(4j + 1,4j, -1, -1, -1), (4j + 1,4j, -1,1,1), (4j + 1,4j, 1,1, -1).$

$$-\Delta_{45} + \Delta_{67} - \Delta_{89} - 2\omega = 2\pi i$$

$$z_{N,1}^{++-} = 1 + e^{-2\omega} + e^{\Delta_{45}} + e^{-\Delta_{67}} + e^{\Delta_{89}} + e^{\omega}e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}+\Delta_{89})} + e^{-\omega}\left(e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}+\Delta_{89})} + e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}-\Delta_{89})} + e^{\frac{1}{2}(-\Delta_{45}-\Delta_{67}+\Delta_{89})}\right)$$

$$z_{N,1}^{++-} = 1 - (1 - e^{\Delta_{45}})(1 - e^{-\Delta_{67}})(1 - e^{\Delta_{89}})$$

$$\mathcal{I}_{N,1}^{++-} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - (1 - e^{m\Delta_{45}})(1 - e^{-m\Delta_{67}})(1 - e^{m\Delta_{89}})}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}(U^{\dagger m})$$

$$\Theta = H - \frac{\mu}{3}M^{12} - \frac{\mu}{6}M^{45} + \frac{\mu}{6}M^{67} + \frac{\mu}{6}M^{89}$$

$\beta(4j, 4j, 0,0,0)$
$x(4j + 2,4j, 2,0,0), (4j + 2,4j, 0, -2,0), (4j + 2,4j, 0,0, -2)$
$\chi(4j + 3,4j, 1, -1, -1)$
$\eta(4j + 1,4j, -1, -1, -1), (4j + 1,4j, 1, -1,1), (4j + 1,4j, 1,1, -1)$

$$\Delta_{45} - \Delta_{67} - \Delta_{89} - 2\omega = 2\pi i$$

$$z_{N,1}^{+-+} = 1 + e^{-2\omega} + e^{-\Delta_{45}} + e^{\Delta_{67}} + e^{\Delta_{89}} + e^{\omega+\frac{\Delta_{67}+\Delta_{89}-\Delta_{45}}{2}} + e^{-\omega}\left(e^{\frac{\Delta_{45}+\Delta_{67}+\Delta_{89}}{2}} + e^{\frac{\Delta_{67}-\Delta_{45}-\Delta_{89}}{2}} + e^{\frac{\Delta_{89}-\Delta_{45}-\Delta_{67}}{2}}\right)$$

$$z_{N,1}^{+-+} = 1 - (1 - e^{-\Delta_{45}})(1 - e^{\Delta_{67}})(1 - e^{\Delta_{89}})$$

$$\mathcal{I}_{N,1}^{+-+} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - (1 - e^{-m\Delta_{45}})(1 - e^{m\Delta_{67}})(1 - e^{m\Delta_{89}})}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}(U^{\dagger m})$$

$$\Theta = H - \frac{\mu}{3}M^{12} - \frac{\mu}{6}M^{45} - \frac{\mu}{6}M^{67} - \frac{\mu}{6}M^{89}$$

$$\Delta_{45} + \Delta_{67} + \Delta_{89} - 2\omega = 2\pi i$$

$\beta(4j, 4j, 0,0,0)$
$x(4j + 2,4j, 2,0,0), (4j + 2,4j, 0,2,0), (4j + 2,4j, 0,0,2),$
$\chi(4j + 3,4j, 1,1,1)$
$\eta(4j + 1,4j, -1,1,1), (4j + 1,4j, 1, -1,1), (4j + 1,4j, 1,1, -1).$

$$z_{N,1}^{+--} = 1 + e^{-2\omega} + e^{-\Delta_{45}} + e^{-\Delta_{67}} + e^{-\Delta_{89}} + e^{\omega-\frac{\Delta_{45}+\Delta_{67}+\Delta_{89}}{2}} + e^{-\omega}\left(e^{\frac{\Delta_{45}-\Delta_{67}-\Delta_{89}}{2}} + e^{\frac{\Delta_{67}-\Delta_{45}-\Delta_{89}}{2}} + e^{\frac{\Delta_{89}-\Delta_{67}-\Delta_{45}}{2}}\right)$$

$$z_{N,1}^{+--} = 1 - (1 - e^{-\Delta_{45}})(1 - e^{-\Delta_{67}})(1 - e^{-\Delta_{89}})$$

$$\mathcal{I}_{N,1}^{+--} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - (1 - e^{-m\Delta_{45}})(1 - e^{-m\Delta_{67}})(1 - e^{-m\Delta_{89}})}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}(U^{\dagger m})$$

$$\epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = 1:$$

$$\Theta = H + \frac{\mu}{3}M^{12} + \frac{\mu}{6}M^{45} + \frac{\mu}{6}M^{67} - \frac{\mu}{6}M^{89}$$

$\beta(4j, -4j, 0,0,0)$
$x(4j+2, -4j, -2,0,0), (4j+2, -4j, 0, -2,0), (4j+2, -4j, 0,0,2)$
$\chi(4j+3, -4j, -1, -1,1)$
$\eta(4j+1, -4j, -1, -1, -1), (4j+1, -4j, -1,1,1), (4j+1, -4j, 1, -1,1)$

$$-\Delta_{45} - \Delta_{67} + \Delta_{89} + 2\omega = 2\pi i$$

$$z_{N,1}^{+++} = 1 + e^{2\omega} + e^{\Delta_{45}} + e^{\Delta_{67}} + e^{-\Delta_{89}} + e^{-\omega}e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}-\Delta_{89})} + e^{\omega}\left(e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}+\Delta_{89})} + e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}-\Delta_{89})} + e^{\frac{1}{2}(-\Delta_{45}+\Delta_{67}-\Delta_{89})}\right)$$

$$z_{N,1}^{-++} = 1 - \left(1 - e^{\Delta_{45}}\right)\left(1 - e^{\Delta_{67}}\right)\left(1 - e^{-\Delta_{89}}\right)$$

$$\mathcal{I}_{N,1}^{-++} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{m\Delta_{45}}\right)\left(1 - e^{m\Delta_{67}}\right)\left(1 - e^{-m\Delta_{89}}\right)}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}\left(U^{\dagger m}\right)$$

$$\epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = -1:$$

$$\Theta = H + \frac{\mu}{3}M^{12} + \frac{\mu}{6}M^{45} - \frac{\mu}{6}M^{67} + \frac{\mu}{6}M^{89}.$$

$\beta(4j, -4j, 0,0,0)$
$x(4j+2, -4j, -2,0,0), (4j+2, -4j, 0,2,0), (4j+2, -4j, 0,0, -2),$
$\chi(4j+3, -4j, -1,1, -1),$
$\eta(4j+1, -4j, -1, -1, -1), (4j+1, -4j, -1,1,1), (4j+1, -4j, 1,1, -1).$

$$-\Delta_{45} + \Delta_{67} - \Delta_{89} + 2\omega = 2\pi i$$

$$z_{N,1}^{-+-} = 1 + e^{2\omega} + e^{\Delta_{45}} + e^{-\Delta_{67}} + e^{\Delta_{89}} + e^{-\omega}e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}+\Delta_{89})} + e^{\omega}\left(e^{\frac{1}{2}(\Delta_{45}+\Delta_{67}+\Delta_{89})} + e^{\frac{1}{2}(\Delta_{45}-\Delta_{67}-\Delta_{89})} + e^{\frac{1}{2}(-\Delta_{45}-\Delta_{67}+\Delta_{89})}\right) \quad (138)$$

$$z_{N,1}^{-+-} = 1 - \left(1 - e^{\Delta_{45}}\right)\left(1 - e^{-\Delta_{67}}\right)\left(1 - e^{\Delta_{89}}\right)$$

$$\mathcal{I}_{N,1}^{-+-} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{m\Delta_{45}}\right)\left(1 - e^{-m\Delta_{67}}\right)\left(1 - e^{m\Delta_{89}}\right)}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}\left(U^{\dagger m}\right)$$

$$\Theta = H + \frac{\mu}{3}M^{12} - \frac{\mu}{6}M^{45} + \frac{\mu}{6}M^{67} + \frac{\mu}{6}M^{89}$$

$\beta(4j, -4j, 0,0,0)$
$x(4j+2, -4j, 2,0,0), (4j+2, -4j, 0, -2,0), (4j+2, -4j, 0,0, -2),$
$\chi(4j+3, -4j, 1, -1, -1),$
$\eta(4j+1, -4j, -1, -1, -1), (4j+1, -4j, 1, -1,1), (4j+1, -4j, 1,1, -1).$

$$\Delta_{45} - \Delta_{67} - \Delta_{89} + 2\omega = 2\pi i$$

$$z_{N,1}^{--+} = 1 + e^{2\omega} + e^{-\Delta_{45}} + e^{\Delta_{67}} + e^{\Delta_{89}} + e^{-\omega+\frac{\Delta_{67}+\Delta_{89}-\Delta_{45}}{2}} + e^{\omega}\left(e^{\frac{\Delta_{45}+\Delta_{67}+\Delta_{89}}{2}} + e^{\frac{\Delta_{67}-\Delta_{45}-\Delta_{89}}{2}} + e^{\frac{\Delta_{89}-\Delta_{45}-\Delta_{67}}{2}}\right)$$

$$z_{N,1}^{--+} = 1 - \left(1 - e^{-\Delta_{45}}\right)\left(1 - e^{\Delta_{67}}\right)\left(1 - e^{\Delta_{89}}\right)$$

$$\mathcal{I}_{N,1}^{--+} = \int [dU]\exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{-m\Delta_{45}}\right)\left(1 - e^{m\Delta_{67}}\right)\left(1 - e^{m\Delta_{89}}\right)}{m}\right)\mathrm{Tr}(U^m)\mathrm{Tr}\left(U^{\dagger m}\right)$$

$$\epsilon_1 = -1, \epsilon_2 = -1, \epsilon_3 = -1$$

$$\Theta = H + \frac{\mu}{3} M^{12} - \frac{\mu}{6} M^{45} - \frac{\mu}{6} M^{67} - \frac{\mu}{6} M^{89}$$

$$\begin{aligned}
&\beta(4j, -4j, 0,0,0)\\
&x(4j+2, -4j, 2,0,0), (4j+2, -4j, 0,2,0), (4j+2, -4j, 0,0,2),\\
&\chi(4j+3, -4j, 1,1,1),\\
&\eta(4j+1, -4j, -1,1,1), (4j+1, -4j, 1, -1,1), (4j+1, -4j, 1,1, -1).
\end{aligned}$$

$$\Delta_{45} + \Delta_{67} + \Delta_{89} + 2\omega = 2\pi i$$

$$\begin{aligned}
z_{N,1}^{---} =& 1 + e^{2\omega} + e^{-\Delta_{45}} + e^{-\Delta_{67}} + e^{-\Delta_{89}} + e^{-\omega - \frac{\Delta_{45}+\Delta_{67}+\Delta_{89}}{2}}\\
&+ e^{\omega}\left(e^{\frac{\Delta_{45}-\Delta_{67}-\Delta_{89}}{2}} + e^{\frac{\Delta_{67}-\Delta_{45}-\Delta_{89}}{2}} + e^{\frac{\Delta_{89}-\Delta_{67}-\Delta_{45}}{2}}\right)
\end{aligned}$$

$$z_{N,1}^{---} = 1 - \left(1 - e^{-\Delta_{45}}\right)\left(1 - e^{-\Delta_{67}}\right)\left(1 - e^{-\Delta_{89}}\right)$$

$$\mathcal{J}_{N,1}^{---} = \int [dU] e\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{-m\Delta_{45}}\right)\left(1 - e^{-m\Delta_{67}}\right)\left(1 - e^{-m\Delta_{89}}\right)}{m}\right) \mathrm{Tr}\left(U^m\right)\mathrm{Tr}\left(U^{\dagger m}\right)$$

$$\begin{aligned}
\mathcal{J}_{N,1}^{+++} = \mathcal{J}_{N,1}^{-++} = &\int [dU] \exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{m\Delta_{45}}\right)\left(1 - e^{m\Delta_{67}}\right)\left(1 - e^{-m\Delta_{89}}\right)}{m}\right)\\
&\times \mathrm{Tr}\left(U^m\right)\mathrm{Tr}\left(U^{\dagger m}\right)\\
\mathcal{J}_{N,1}^{++-} = \mathcal{J}_{N,1}^{-+-} = &\int [dU] \left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{m\Delta_{45}}\right)\left(1 - e^{-m\Delta_{67}}\right)\left(1 - e^{m\Delta_{89}}\right)}{m}\right)\\
&\times \mathrm{Tr}\left(U^m\right)\mathrm{Tr}\left(U^{\dagger m}\right)\\
\mathcal{J}_{N,1}^{+-+} = \mathcal{J}_{N,1}^{--+} = &\int [dU] \exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{-m\Delta_{45}}\right)\left(1 - e^{m\Delta_{67}}\right)\left(1 - e^{m\Delta_{89}}\right)}{m}\right)\\
&\times \mathrm{Tr}\left(U^m\right)\mathrm{Tr}\left(U^{\dagger m}\right)\\
\mathcal{J}_{N,1}^{+--} = \mathcal{J}_{N,1}^{---} = &\int [dU] \exp\left(\sum_{m=1}^{\infty} \frac{1 - \left(1 - e^{-m\Delta_{45}}\right)\left(1 - e^{-m\Delta_{67}}\right)\left(1 - e^{-m\Delta_{89}}\right)}{m}\right)\\
&\times \mathrm{Tr}\left(U^m\right)\mathrm{Tr}\left(U^{\dagger m}\right)
\end{aligned}$$

$$\log\left(\mathcal{J}_{N,1}\right) = -3N^2 \frac{\Delta_{45}\Delta_{67}\Delta_{89}}{2\pi^2}$$

$$\tilde{\mathcal{O}}_p = \frac{1}{N^{(p-2)/2}} \mathcal{O}_p$$

$$\tilde{\mathcal{O}}_p \mapsto W_p.$$

$$\begin{array}{ccc}
\langle \widetilde{\mathcal{O}}_p \widetilde{\mathcal{O}}_p \rangle \ , \ \langle \widetilde{\mathcal{O}}_{p_1} \widetilde{\mathcal{O}}_{p_2} \widetilde{\mathcal{O}}_{p_3} \rangle & \xrightarrow{N^2 \rightsquigarrow 1 - c/3} & \text{W-algebra OPE coefficients}\\
\text{rational functions of } N^2 & & g_p(c) \ , \ c_{p_1 p_2}{}^{p_3}(c)
\end{array}$$

$$\mathfrak{hs}^{\mathrm{s,\,s}} \cong \mathrm{Wedge}(\mathcal{W}_{\infty}^{\mathrm{s,\,s}})$$

$$J(z,y) = J^{IJ}(z) y_I y_J, G(z,y) = G^I(z) y_I, \tilde{G}(z,y) = \tilde{G}^I(z) y_I, y_I = (1, y)$$

$$\begin{aligned}
J(z_1, y_1) J(z_2, y_2) &= \frac{-k y_{12}^2 \mathbb{1}}{z_{12}^2} + \frac{2 y_{12}\left(1 + \frac{1}{2} y_{12} \partial_{y_2}\right) J(z_2, y_2)}{z_{12}} + \mathfrak{R}\\
T(z_1) T(z_2) &= \frac{\frac{1}{2} c \mathbb{1}}{z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial_{z_2} T(z_2)}{z_{12}} + \mathfrak{R}.
\end{aligned}$$

$$\text{short } \mathfrak{psl}(2|2) \text{ multiplet:} \qquad \begin{array}{ccc} & X & \\ G_X := G^{\downarrow} X & & \widetilde{G}_X := \widetilde{G}^{\downarrow} X \\ & T_X := \widetilde{G}^{\downarrow} G^{\downarrow} X & \end{array} .$$

$$\mathbb{J} = \left\{ \begin{array}{ccc} & J & \\ G = G^{\downarrow} J & & \widetilde{G} = \widetilde{G}^{\downarrow} J \\ & T = \widetilde{G}^{\downarrow} G^{\downarrow} J & \end{array} \right\} .$$

$$\mathbb{W}_p = \left\{ \begin{array}{ccc} & W_p & \\ G_{W_p} := G^{\downarrow} W_p & & \widetilde{G}_{W_p} = \widetilde{G}^{\downarrow} W_p \\ & T_{W_p} := \widetilde{G}^{\downarrow} G^{\downarrow} W_p & \end{array} \right\} .$$

$$\langle W_p(z_1, y_1) W_p(z_2, y_2) \rangle = g_p \frac{y_{12}^p}{z_{12}^p}$$

$$\mathbb{J} \times \mathbb{J}, \mathbb{J} \times \mathbb{W}_p, \mathbb{W}_{p_1} \times \mathbb{W}_{p_2}$$

$$A(z_1)B(z_2) = \sum_{n \in \mathbb{Z}} \frac{1}{z_{12}^n} \{AB\}_n(z_2)$$

$$\{A\{BC\}_p\}_q = (-1)^{|A||B|} \{B\{AC\}_q\}_p + \sum_{\ell=1}^{\infty} \binom{q-1}{\ell-1} \{\{AB\}_\ell C\}_{p+q-\ell}, p, q \in \mathbb{Z}$$

$$\mathrm{Jacobi}_{p,q}(A, B, C) := \{A\{BC\}_p\}_q - (-1)^{|A||B|} \{B\{AC\}_q\}_p - \sum_{\ell=1}^{\infty} \binom{q-1}{\ell-1} \{\{AB\}_\ell C\}_{p+q-\ell} ,$$

$$\mathrm{Jacobi}(A, B, C) := \{\mathrm{Jacobi}_{p,q}(A, B, C) , \ p, q \in \mathbb{N}\} .$$

$$\mathrm{Jacobi}(\mathbb{A}, \mathbb{B}, \mathbb{C}) = \{\mathrm{Jacobi}(A, B, C), A \in \mathbb{A}, B \in \mathbb{B}, C \in \mathbb{C}\}$$

$$\begin{array}{ll} \mathcal{C}_{3,1}^{W_3 W_3} = (W_3 W_3)_0^1 + \cdots, & \mathcal{C}_{4,1}^{W_3 W_5} = (W_3 W_5)_0^1 + \cdots \\ \mathcal{C}_{3,3}^{W_3 W_3} = (W_3 W_3)_0^3 + \cdots, & \mathcal{C}_{4,2}^{W_3 W_3} = (W_3 W_3)_{-1}^2 + \cdots \\ \mathcal{C}_{\frac{7}{2},\frac{1}{2}}^{W_3 W_4} = (W_3 W_4)_0^{\frac{1}{2}} + \cdots & \mathcal{C}_{4,2}^{W_3 W_5} = (W_3 W_5)_0^2 + \cdots \\ \mathcal{C}_{\frac{7}{2},\frac{3}{2}}^{W_3 W_4} = (W_3 W_4)_0^{\frac{3}{2}} + \cdots, & \mathcal{C}_{4,2}^{W_4 W_4} = (W_4 W_4)_0^2 + \cdots \\ \mathcal{C}_{\frac{7}{2},\frac{5}{2}}^{W_3 W_4} = (W_3 W_4)_0^{\frac{5}{2}} + \cdots, & \mathcal{C}_{4,3}^{W_3 W_5} = (W_3 W_5)_0^3 + \cdots \\ \mathcal{C}_{\frac{7}{2},\frac{7}{2}}^{W_3 W_4} = (W_3 W_4)_0^{\frac{7}{2}} + \cdots, & \mathcal{C}_{4,4}^{W_3 W_5} = (W_3 W_5)_0^4 + \cdots \\ \mathcal{C}_{4,0}^{W_3 W_3} = (W_3 W_3)_{-1}^0 + \cdots & \mathcal{C}_{4,4}^{W_4 W_4} = (W_4 W_4)_0^4 + \cdots \\ \mathcal{C}_{4,0}^{W_4 W_4} = (W_4 W_4)_0^0 + \cdots & \end{array}$$

$$\mathcal{C}_{\frac{9}{2},\frac{9}{2}}^{W_3 W_3 W_3} = (W_3 (W_3 W_3)_0^3)_0^3 + \cdots$$

$$W_3 \times W_3 = g_3 \mathbb{1} + {c_{33}}^{4} W_4 \ ,$$

$$W_3 \times W_4 = {c_{34}}^{3} W_3 + {c_{34}}^{5} W_5 \ ,$$

$$W_3 \times W_5 = {c_{35}}^{4} W_4 + {c_{35}}^{6} W_6 + {c_{35}}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + {c_{35}}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \ ,$$

$$W_4 \times W_4 = g_4 \mathbb{1} + {c_{44}}^{4} W_4 + {c_{44}}^{6} W_6 + {c_{44}}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + {c_{44}}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \ ,$$

$$\begin{aligned} W_3 \times W_6 = {} & {c_{36}}^{3} W_3 + {c_{36}}^{5} W_5 + {c_{36}}^{7} W_7 \\ & + {c_{36}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}} + {c_{36}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} + {c_{36}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \ , \end{aligned}$$

$$\begin{aligned} W_4 \times W_5 = {} & {c_{45}}^{3} W_3 + {c_{45}}^{5} W_5 + {c_{45}}^{7} W_7 \\ & + {c_{45}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}} + {c_{45}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}} + {c_{45}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} + {c_{45}}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \ , \end{aligned}$$

$$\begin{aligned} W_3 \times W_7 = {} & {c_{37}}^{4} W_4 + {c_{37}}^{6} W_6 + {c_{37}}^{8} W_8 + {c_{37}}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \\ & + {c_{37}}^{\mathcal{C}^{W_3W_3}_{4,2}} \mathcal{C}^{W_3W_3}_{4,2} + {c_{37}}^{\mathcal{C}^{W_3W_5}_{4,2}} \mathcal{C}^{W_3W_5}_{4,2} + {c_{37}}^{\mathcal{C}^{W_4W_4}_{4,2}} \mathcal{C}^{W_4W_4}_{4,2} \\ & + {c_{37}}^{\mathcal{C}^{W_3W_5}_{4,3}} \mathcal{C}^{W_3W_5}_{4,3} + {c_{37}}^{\mathcal{C}^{W_3W_5}_{4,4}} \mathcal{C}^{W_3W_5}_{4,4} + {c_{37}}^{\mathcal{C}^{W_4W_4}_{4,4}} \mathcal{C}^{W_4W_4}_{4,4} \ , \end{aligned}$$

$$\begin{aligned} W_4 \times W_6 = {} & {c_{46}}^{4} W_4 + {c_{46}}^{6} W_6 + {c_{46}}^{8} W_8 + {c_{46}}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + {c_{46}}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \\ & + {c_{46}}^{\mathcal{C}^{W_3W_5}_{4,1}} \mathcal{C}^{W_3W_5}_{4,1} + {c_{46}}^{\mathcal{C}^{W_3W_3}_{4,2}} \mathcal{C}^{W_3W_3}_{4,2} + {c_{46}}^{\mathcal{C}^{W_3W_5}_{4,2}} \mathcal{C}^{W_3W_5}_{4,2} + {c_{46}}^{\mathcal{C}^{W_4W_4}_{4,2}} \mathcal{C}^{W_4W_4}_{4,2} \\ & + {c_{46}}^{\mathcal{C}^{W_3W_5}_{4,3}} \mathcal{C}^{W_3W_5}_{4,3} + {c_{46}}^{\mathcal{C}^{W_3W_5}_{4,4}} \mathcal{C}^{W_3W_5}_{4,4} + {c_{46}}^{\mathcal{C}^{W_4W_4}_{4,4}} \mathcal{C}^{W_4W_4}_{4,4} \ , \end{aligned}$$

$$\begin{aligned} W_5 \times W_5 = {} & g_5 \mathbb{1} + {c_{55}}^{4} W_4 + {c_{55}}^{6} W_6 + {c_{55}}^{8} W_8 + {c_{55}}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + {c_{55}}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \\ & + {c_{55}}^{\mathcal{C}^{W_3W_3}_{4,0}} \mathcal{C}^{W_3W_3}_{4,0} + {c_{55}}^{\mathcal{C}^{W_4W_4}_{4,0}} \mathcal{C}^{W_4W_4}_{4,0} + {c_{55}}^{\mathcal{C}^{W_3W_3}_{4,2}} \mathcal{C}^{W_3W_3}_{4,2} + {c_{55}}^{\mathcal{C}^{W_3W_5}_{4,2}} \mathcal{C}^{W_3W_5}_{4,2} \\ & + {c_{55}}^{\mathcal{C}^{W_4W_4}_{4,2}} \mathcal{C}^{W_4W_4}_{4,2} + {c_{55}}^{\mathcal{C}^{W_3W_5}_{4,4}} \mathcal{C}^{W_3W_5}_{4,4} + {c_{55}}^{\mathcal{C}^{W_4W_4}_{4,4}} \mathcal{C}^{W_4W_4}_{4,4} \ . \end{aligned}$$

$$c = 6k = 3(1 - \nu)$$

$$W_6' = W_6 + \mu_6 \mathcal{C}^{W_3W_3}_{3,3}, W_7' = W_7 + \mu_7 \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}$$

$$S_p(\nu) = \frac{(\sqrt{\nu}+1)_{p-1} - (\sqrt{\nu}-p+1)_{p-1}}{p(p-1)} \frac{\nu^{\left\lfloor \frac{p-2}{2} \right\rfloor}}{\nu^{\frac{p-2}{2}}}$$

$$\begin{aligned} S_3(\nu) &= 1 \\ S_4(\nu) &= \nu + 1 \\ S_5(\nu) &= \nu + 5 \\ S_6(\nu) &= \nu^2 + 15\nu + 8 \\ S_7(\nu) &= \nu^2 + 35\nu + 84 \\ S_8(\nu) &= \nu^3 + 70\nu^2 + 469\nu + 180 \\ S_9(\nu) &= \nu^3 + 126\nu^2 + 1869\nu + 3044 \end{aligned}$$

$$g_p = p\nu \frac{\prod_{r=1}^{p-1} (\nu - r^2)}{S_p(\nu)} \frac{\nu^{\left\lfloor \frac{p-2}{2} \right\rfloor}}{\nu^{p-1}} .$$

$$\langle W_6 \mathcal{C}^{W_3W_3}_{3,3} \rangle = 0, \left\langle W_7 \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \right\rangle = 0$$

$$c_{q_1 q_2}{}^{p} = q_1 q_2, \text{ for } p \geq q_1, p \geq q_2, q_1 + q_2 = p + 2$$

$$c_{46}{}^{8} = \frac{8}{7} c_{37}{}^{8}, c_{55}{}^{8} = \frac{25}{21} c_{37}{}^{8}.$$

$$\nu \to \infty: g_p = p\nu + \mathcal{O}(\nu^0), c_{q_1 q_2}{}^{p} = q_1 q_2 + \mathcal{O}(\nu^{-1})$$

OPE coefficients in $W_3 \times W_3$ ***and*** $W_3 \times W_4$
$c_{33}{}^{4} = 9$, $c_{34}{}^{3} = \frac{12(\nu - 9)}{\nu + 1}$, $c_{34}{}^{5} = 12$.

OPE coefficients in $W_3 \times W_5$
$c_{35}{}^{4} = \frac{15(\nu - 16)(\nu + 1)}{\nu(\nu + 5)}$, $c_{35}{}^{6} = 15$, $c_{35}{}^{c^{W_3W_3}_{3,1}} = -\frac{10(\nu - 2)}{\nu(\nu + 5)}$, $c_{35}{}^{c^{W_3W_3}_{3,3}} = \frac{15(\nu - 16)(\nu - 9)(\nu + 3)}{\nu(\nu + 5)\left(\nu^2 + 15\nu + 8\right)}$.

OPE coefficients in $W_4 \times W_4$
$c_{44}{}^{4} = \frac{16\left(\nu^2 - 20\nu + 9\right)}{\nu(\nu + 1)}$, $c_{44}{}^{6} = 16$, $c_{44}{}^{c^{W_3W_3}_{3,1}} = -\frac{16}{\nu}$, $c_{44}{}^{c^{W_3W_3}_{3,3}} = \frac{32(\nu - 9)(\nu - 4)}{\nu\left(\nu^2 + 15\nu + 8\right)}$.

OPE coefficients in $W_3 \times W_6$
$c_{36}{}^{3} = 0$, $c_{36}{}^{5} = \frac{18(\nu - 25)(\nu + 5)}{\nu^2 + 15\nu + 8}$, $c_{36}{}^{7} = 18$, $c_{36}{}^{c^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} = -\frac{21(\nu - 1)}{\nu^2 + 15\nu + 8}$, $c_{36}{}^{c^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} = \frac{3(\nu - 25)(\nu - 16)}{\nu\left(\nu^2 + 15\nu + 8\right)}$, $c_{36}{}^{c^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} = \frac{36(\nu - 25)(\nu - 16)(\nu + 7)}{\left(\nu^2 + 15\nu + 8\right)\left(\nu^2 + 35\nu + 84\right)}$.

OPE coefficients in $W_4 \times W_5$

$$c_{45}{}^{3} = \frac{20(\nu-16)(\nu-9)}{\nu(\nu+5)} \ ,$$

$$c_{45}{}^{5} = \frac{20\left(\nu^3-32\nu^2-77\nu+36\right)}{\nu(\nu+1)(\nu+5)} \ ,$$

$$c_{45}{}^{7} = 20 \ ,$$

$$c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}}} = -\frac{10(\nu+12)}{\nu(\nu+5)} \ ,$$

$$c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} = -\frac{20(2\nu+3)}{\nu(\nu+5)} \ ,$$

$$c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} = \frac{10(\nu-16)}{\nu(\nu+5)} \ ,$$

$$c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} = \frac{40(\nu-16)\left(2\nu^2-7\nu-63\right)}{\nu(\nu+5)\left(\nu^2+35\nu+84\right)} \ .$$

OPE coefficients in $W_3 \times W_7$

$$c_{37}{}^{4} = 0 \ ,$$

$$c_{37}{}^{6} = \frac{21(\nu-36)\left(\nu^2+15\nu+8\right)}{\nu\left(\nu^2+35\nu+84\right)} \ ,$$

$$c_{37}{}^{8} = 21 \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = 0 \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_3W_3}_{4,2}} = -\frac{224\left(\nu^5+42\nu^4+264\nu^3-253\nu^2+17226\nu+9720\right)}{3\nu^2(\nu+5)\left(\nu^2+15\nu+8\right)\left(\nu^2+35\nu+84\right)} \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_3W_5}_{4,2}} = -\frac{56\left(2\nu^2+13\nu-90\right)}{5\nu\left(\nu^2+35\nu+84\right)} \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_4W_4}_{4,2}} = -\frac{21\left(\nu^2-\nu+60\right)}{2\nu\left(\nu^2+35\nu+84\right)} \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_3W_5}_{4,3}} = \frac{28(\nu-36)(\nu-25)}{5\nu\left(\nu^2+35\nu+84\right)} \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_3W_5}_{4,4}} = -\frac{42\left(\nu^3-42\nu^2+29\nu+396\right)}{\nu(\nu+5)\left(\nu^2+35\nu+84\right)} + \frac{21}{25}c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,4}} \ ,$$

$$c_{37}{}^{\mathcal{C}^{W_4W_4}_{4,4}} = -\frac{42(\nu-16)(\nu-3)}{\nu\left(\nu^2+35\nu+84\right)} + \frac{21}{25}c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,4}} \ .$$

OPE coefficients in $W_4 \times W_6$

$$c_{46}{}^{4} = \frac{24(\nu-25)(\nu-16)(\nu+1)}{\nu\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{6} = \frac{12(2\nu-1)\left(\nu^3-46\nu^2-387\nu-240\right)}{\nu(\nu+1)\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{8} = 24\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,1}} = -\frac{56(\nu-25)(\nu-2)}{3\nu\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = \frac{36(\nu-25)(\nu-16)(\nu-9)(\nu+3)}{\nu\left(\nu^2+15\nu+8\right)^2}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,1}} = -\frac{84(\nu+23)}{5\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_3}_{4,2}} = \frac{16\left(5\nu^7-499\nu^6-9539\nu^5+119535\nu^4+1843758\nu^3+1311460\nu^2-32542800\nu-18144000\right)}{35(\nu-15)\nu^2(\nu+5)\left(\nu^2+15\nu+8\right)^2}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,2}} = -\frac{12\left(19\nu^2+117\nu+320\right)}{5\nu\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,2}} = -\frac{24(\nu-4)(\nu+5)}{\nu\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,3}} = \frac{96(\nu-25)(\nu-2)}{5\nu\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,4}} = \frac{672(\nu-9)(\nu+2)}{\nu(\nu+5)\left(\nu^2+15\nu+8\right)} + \frac{24}{25}c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,4}}\,,$$

$$c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,4}} = -\frac{24(\nu-16)(\nu+3)}{\nu\left(\nu^2+15\nu+8\right)} + \frac{24}{25}c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,4}}\,.$$

OPE coefficients in $W_5 \times W_5$

$$c_{55}{}^{4} = \frac{25(\nu-16)\left(\nu^3-32\nu^2-77\nu+36\right)}{\nu^2(\nu+5)^2}\,,$$

$$c_{55}{}^{6} = \frac{25\left(\nu^2-59\nu+16\right)}{\nu(\nu+5)}\,,$$

$$c_{55}{}^{8} = 25\,,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,1}} = -\frac{50\left(5\nu^3-103\nu^2-460\nu-864\right)}{9\nu^2(\nu+5)^2}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = \frac{25(\nu-16)(\nu-9)\left(3\nu^3-24\nu^2-115\nu+64\right)}{\nu^2(\nu+5)^2\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,0}} = -\frac{40\left(\nu^3-13\nu^2+210\nu-558\right)}{3(\nu-7)\nu^2(\nu+5)}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,0}} = -\frac{25}{\nu}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,2}} = \frac{40\left(5\nu^7-226\nu^6-2904\nu^5+46954\nu^4+338011\nu^3+1286280\nu^2+7747320\nu+3959280\right)}{21(\nu-15)\nu^2(\nu+5)^3\left(\nu^2+15\nu+8\right)}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,2}} = -\frac{60(\nu-2)}{\nu(\nu+5)}\,,$$

$$c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,2}} = -\frac{25}{\nu}\,.$$

$$\mathrm{grV}_{h,j,r} = \bigoplus_{\Re\in\frac{1}{2}\mathbb{Z}_{\geq 0}} \mathfrak{G}_{h,j,r,\Re},\ \mathfrak{G}_{h,j,r,\Re} = \mathfrak{F}_{h,j,r,\Re}/\mathfrak{F}_{h,j,r,\Re-1}$$

$$\mathcal{O}_1(z_1,y_1)\mathcal{O}_2(z_2,y_2) = \sum_{n\in\mathbb{Z}}\sum_{j}\frac{y_{12}^{j_1+j_2-j}}{z_{12}^{n}}\hat{\mathcal{D}}_{j_1,j_2;j}\left(y_{12},\partial_{y_2}\right)\{\mathcal{O}_1\mathcal{O}_2\}_n^j(z_2,y_2)$$

$$\{\mathcal{O}_1\mathcal{O}_2\}_0^j \in \mathfrak{F}_{h_1+h_2,j,r_1+r_2,\Re_1+\Re_2} \qquad \text{(normal ordered product)}$$

$$\{\mathcal{O}_1\mathcal{O}_2\}_1^j \in \mathfrak{F}_{h_1+h_2-1,j,r_1+r_2,\Re_1+\Re_2-1} \qquad \text{(simple pole in the OPE)}$$

$$\pi: \mathcal{V}_{h,j,r} \to \tilde{\mathcal{V}}_{h,j,r} := \mathcal{V}_{h,j,r}/\mathcal{N}_{h,j,r},\ \pi: v \mapsto v + \mathcal{N}_{h,j,r}$$

$$\widetilde{\mathfrak{F}}_{h,j,r,\mathfrak{R}} = \pi(\mathfrak{F}_{h,j,r,\mathfrak{R}}).$$

$$\partial^k A \in \mathcal{I}, \{AX\}_n \in \mathcal{I}, \{XA\}_n \in \mathcal{I}, \text{ for any } A \in \mathcal{I}, X \in \mathcal{V}, n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}.$$

$$\mathfrak{T}_N \supsetneq \mathfrak{T}_{N+1}.$$

$$(h,j) = (3,1)\,, \quad \mathcal{C}^{W_3W_3}_{3,1}\,, \quad \frac{81(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{5(\nu-11)(\nu-2)\nu}\,,$$

$$(h,j) = (3,3)\,, \quad \mathcal{C}^{W_3W_3}_{3,3}\,, \quad \frac{18(\nu-4)(\nu-1)^2 S_6(\nu)}{\nu(\nu+3)(\nu+7)}\,,$$

$$(h,j) = (\tfrac{7}{2},\tfrac{1}{2})\,, \quad \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}}\,, \quad \frac{24(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{5(\nu-6)\nu^2}\,,$$

$$(h,j) = (\tfrac{7}{2},\tfrac{3}{2})\,, \quad \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}\,, \quad \frac{72(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)}{5(\nu-13)\nu^2}\,,$$

$$(h,j) = (\tfrac{7}{2},\tfrac{5}{2})\,, \quad \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}\,, \quad \frac{144(\nu-9)(\nu-4)(\nu-1)^2}{7\nu(\nu+2)}\,,$$

$$(h,j) = (\tfrac{7}{2},\tfrac{7}{2})\,, \quad \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}\,, \quad \frac{12(\nu-9)(\nu-4)(\nu-1)^2 S_7(\nu)}{\nu^2(\nu+7)(\nu+9)}\,,$$

$$(h,j) = (4,0)\,, \quad \begin{matrix}\mathcal{C}^{W_3W_3}_{4,0}\\ \mathcal{C}^{W_4W_4}_{4,0}\end{matrix}\,, \quad \det = -\frac{162(\nu-25)(\nu-16)(\nu-7)(\nu-4)^3(\nu-1)^4}{(\nu-9)(\nu-6)\nu^3(3\nu-11)}\,,$$

$$(h,j) = (4,1)\,, \quad \mathcal{C}^{W_3W_5}_{4,1}\,, \quad \frac{15(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{2(\nu-7)\nu^3}\,,$$

$$(h,j) = (4,2)\,, \quad \begin{matrix}\mathcal{C}^{W_3W_3}_{4,2}\\ \mathcal{C}^{W_3W_5}_{4,2}\\ \mathcal{C}^{W_4W_4}_{4,2}\end{matrix}\,, \quad \det = -\frac{1749600(\nu-36)(\nu-25)(\nu-16)^2(\nu-9)^3(\nu-4)^4(\nu-1)^6(\nu+5)}{7\nu^7(3\nu-1)\left(\nu^2-12\nu-57\right)^2}\,,$$

$$(h,j) = (4,3)\,, \quad \mathcal{C}^{W_3W_5}_{4,3}\,, \quad \frac{225(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{8\nu^2(\nu+3)}\,,$$

$$(h,j) = (4,4)\,, \quad \begin{matrix}\mathcal{C}^{W_3W_5}_{4,3}\\ \mathcal{C}^{W_4W_4}_{4,3}\end{matrix}\,, \quad \det = \frac{480(\nu-16)(\nu-9)^2(\nu-4)^3(\nu-1)^4 S_8(\nu)}{\nu^5(\nu+5)(\nu+9)(\nu+11)^2}\,,$$

$$W_3 \times W_{p+1} = g_3 \delta_{2,p} \mathbf{1} + c_{3,p+1}{}^{p} W_p + 3(p+1) W_{p+2} + \mu_p^+ \mathcal{C}_{\frac{p}{2}+1,\frac{p}{2}+1} + \mu_p^0 \mathcal{C}_{\frac{p}{2}+1,\frac{p}{2}} + \mu_p^- \mathcal{C}_{\frac{p}{2}+1,\frac{p}{2}-1}.$$

$$c^{p}_{3,p+1}(\nu) = 3(p+1)(\nu - p^2)\frac{S_p(\nu)}{S_{p+1}(\nu)}\frac{1}{\nu^{1-(p \bmod 2)}},$$

$$(\mu_4^+(\nu), \mu_5^+(\nu), \dots) = \left(\frac{15(\nu-16)(\nu-9)(\nu+3)}{\nu S_5(\nu) S_6(\nu)}, \frac{36(\nu-25)(\nu-16)(\nu+7)}{S_6(\nu) S_7(\nu)}, \dots\right),$$

$$(\mu_5^0(\nu), \mu_6^0(\nu), \dots) = \left(\frac{3(\nu-25)(\nu-16)}{\nu S_6(\nu)}, \frac{28(\nu-36)(\nu-24)}{5\nu S_7(\nu)}, \dots\right)$$

$$(\mu_4^-(\nu), \mu_5^-(\nu), \dots) = \left(\frac{-10(\nu-2)}{\nu S_5(\nu)}, \frac{-21(\nu-1)}{S_6(\nu)}, \dots\right)$$

$$\mathcal{N}^{\mathrm{su}(2)}_{(2,0)} = (JJ)^0_0 + \frac{1}{3}T, \mathcal{N}^{\mathrm{su}(3)}_{(\frac{5}{2},\frac{1}{2})} = (JW_3)^{\frac{1}{2}}_0 + \frac{1}{4}T_{W_3}$$

$$\mathcal{N}^{\mathrm{su}(N)}_{(\frac{N}{2}+1,\frac{N}{2}-1)} = \mathcal{C}_{(\frac{N}{2}+1,\frac{N}{2}-1)}\Big|_{\nu=N^2, W_{p>N}=0}$$

$$\mathcal{N}^{\mathrm{su}(N)}_{(\frac{N+3}{2},\frac{N+1}{2})} = \mathcal{C}_{(\frac{N+3}{2},\frac{N+1}{2})}\Big|_{\nu=N^2, W_{p>N}=0}$$

$$
\begin{aligned}
N &= 2: \quad \mathcal{N}_{(2,0)} \\
N &= 3: \quad \mathcal{N}_{\left(\frac{5}{2},\frac{1}{2}\right)};\mathcal{N}_{(3,1)} \\
N &= 4: \quad \mathcal{N}_{(3,1)};\mathcal{N}_{\left(\frac{7}{2},\frac{3}{2}\right)},\mathcal{N}_{\left(\frac{7}{2},\frac{1}{2}\right)};\mathcal{N}_{(4,2)},\mathcal{N}_{(4,0)} \\
N &= 5: \quad \mathcal{N}_{\left(\frac{7}{2},\frac{3}{2}\right)};\mathcal{N}_{(4,2)},\mathcal{N}_{(4,1)},\mathcal{N}_{(4,0)};\mathcal{N}_{\left(\frac{9}{2},\frac{5}{2}\right)},\mathcal{N}_{\left(\frac{9}{2},\frac{3}{2}\right)},\mathcal{N}_{\left(\frac{9}{2},\frac{1}{2}\right)}
\end{aligned}
$$

$$\mathcal{N} = (JJ)_0^0 + \frac{1}{3}T$$

$$\mathcal{N} \times W_p \supset -\frac{p(p+1)}{6}W_p$$

$$\langle \mathcal{C}_{3,1}^{W_3W_3}\mathcal{C}_{3,1}^{W_3W_3}\rangle = \frac{9(\nu-16)(\nu-9)\nu}{5(\nu-11)(\nu-4)(\nu-2)}g_3^2$$

$$\mathcal{C}_{3,1}^{W_3W_3} \times W_4 = -\frac{108(\nu-16)(\nu+1)g_3}{5(\nu-11)(\nu-4)(\nu-2)}W_4 + \cdots$$

$$
\begin{aligned}
\mathcal{C}_{3,1}^{W_3W_3} \times W_5 \ = &-\frac{18\sqrt{\frac{3}{5}}\sqrt{\nu-16}\sqrt{\nu-9}\sqrt{\nu}\sqrt{g_3}\sqrt{g_5}}{(\nu-11)(\nu-4)\sqrt{\nu+5}}W_3 \\
&-\frac{6(5\nu^3-103\nu^2-460\nu-864)g_3}{(\nu-11)(\nu-4)(\nu-2)(\nu+5)}W_5 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}_{3,1}^{W_3W_3} \times W_6 = &-\frac{42\sqrt{6}\sqrt{\nu-25}\sqrt{\nu-16}\sqrt{\nu}\sqrt{\nu+1}g_3\sqrt{g_6}}{5(\nu-11)(\nu-4)\sqrt{\nu^2+15\nu+8}\sqrt{g_4}}W_4 \\
&-\frac{18(11\nu^4-292\nu^3-3223\nu^2-10304\nu-2560)g_3}{5(\nu-11)(\nu-4)(\nu-2)(\nu^2+15\nu+8)}W_6 + \cdots
\end{aligned}
$$

$$c_{34}{}^3 = \frac{12(\nu-9)\nu}{\nu+1}, c_{45}{}^3 = \frac{20(\nu-16)(\nu-9)\nu}{\nu+5},$$

$$\langle \mathcal{C}_{3,3}^{W_3W_3}\mathcal{C}_{3,3}^{W_3W_3}\rangle = \frac{2\nu(\nu^2+15\nu+8)}{(\nu-4)(\nu+3)(\nu+7)}g_3^2$$

$$
\begin{aligned}
c_{35}{}^4 &= \frac{15(\nu-16)(\nu+1)}{\nu+5}, & c_{45}{}^3 &= \frac{20(\nu-16)(\nu-9)\nu}{\nu+5}, \\
c_{46}{}^4 &= \frac{24(\nu-25)(\nu-16)\nu(\nu+1)}{\nu^2+15\nu+8}, & c_{55}{}^4 &= \frac{25(\nu-16)\left(\nu^3-32\nu^2-77\nu+36\right)}{(\nu+5)^2}
\end{aligned}
$$

$$
\begin{aligned}
&W_3 \times W_3 = g_3 \mathbb{1} + c_{33}{}^4 W_4 , \qquad W_3 \times W_4 = c_{34}{}^3 W_3 , \\
&W_4 \times W_4 = g_4 \mathbb{1} + c_{44}{}^4 W_4 + c_{44}{}^{\mathcal{C}_{3,3}^{W_3W_3}} \mathcal{C}_{3,3}^{W_3W_3} ,
\end{aligned}
$$

$$
\begin{aligned}
c_{33}{}^4 &= \frac{4\sqrt{7}}{\sqrt{85}}\frac{g_3}{\sqrt{g_4}} , & c_{34}{}^3 &= \frac{4\sqrt{7}}{\sqrt{85}}\sqrt{g_4} , \\
c_{44}{}^4 &= -\frac{11\sqrt{5}}{3\sqrt{119}}\sqrt{g_4} , & c_{44}{}^{\mathcal{C}_{3,3}^{W_3W_3}} &= \frac{17}{28}\frac{g_4}{g_3} .
\end{aligned}
$$

$$\left\langle \mathcal{C}_{\frac{7}{2},\frac{3}{2}}^{W_3W_4}\mathcal{C}_{\frac{7}{2},\frac{3}{2}}^{W_3W_4}\right\rangle = \frac{6(\nu-25)(\nu-16)(\nu+1)}{5(\nu-13)(\nu-4)(\nu-1)}g_3g_4$$

$$\mathcal{C}^{W_3W_4}_{\frac{7}{2}\frac{3}{2}} \times W_5 = -\frac{24\sqrt{\frac{3}{5}}(\nu-25)\sqrt{\nu-16}(\nu+1)(2\nu+3)\sqrt{g_3}\sqrt{g_5}}{(\nu-13)\sqrt{\nu-9}(\nu-4)(\nu-1)\sqrt{\nu}\sqrt{\nu+5}}W_4$$

$$-\frac{4\sqrt{\frac{3}{5}}\sqrt{\nu-16}(\nu+1)(13\nu^3-454\nu^2-2407\nu-4352)\sqrt{g_3}g_4}{(\nu-4)(\nu-1)(\nu-13)\sqrt{\nu-9}\sqrt{\nu}(\nu+5)^{3/2}\sqrt{g_5}}W_6+\cdots$$

$$c_{46}{}^{4} = \frac{24(\nu-25)(\nu-16)\nu(\nu+1)}{\nu^2+15\nu+8}$$

$$\begin{aligned}
W_3 \times W_3 &= g_3\mathbb{1} + c_{33}{}^4 W_4 \ , \\
W_3 \times W_4 &= c_{34}{}^3 W_3 + c_{34}{}^5 W_5 \ , \\
W_3 \times W_5 &= c_{35}{}^4 W_4 + c_{35}{}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + c_{35}{}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \ , \\
W_4 \times W_4 &= g_4\mathbb{1} + c_{44}{}^4 W_4 + c_{44}{}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + c_{44}{}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} \ , \\
W_4 \times W_5 &= c_{45}{}^3 W_3 + c_{45}{}^5 W_5 + c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}} + c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} + c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \ , \\
W_5 \times W_5 &= g_5\mathbb{1} + c_{55}{}^4 W_4 + c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,1}} \mathcal{C}^{W_3W_3}_{3,1} + c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}} \mathcal{C}^{W_3W_3}_{3,3} + \bar{c}_{55}{}^{\mathcal{C}^{W_4W_4}_{4,0}} \mathcal{C}^{W_4W_4}_{4,0} \\
&\quad + \bar{c}_{55}{}^{\mathcal{C}^{W_3W_5}_{4,2}} \mathcal{C}^{W_3W_5}_{4,2} + \bar{c}_{55}{}^{\mathcal{C}^{W_4W_4}_{4,2}} \mathcal{C}^{W_4W_4}_{4,2} + c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,4}} \mathcal{C}^{W_3W_5}_{4,4} + c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,4}} \mathcal{C}^{W_4W_4}_{4,4} \ ,
\end{aligned}$$

$$c_{33}{}^4 = \frac{10}{\sqrt{91}}\frac{g_3}{\sqrt{g_4}} \ , \qquad c_{34}{}^4 = \frac{10}{\sqrt{91}}\sqrt{g_4} \ , \qquad c_{34}{}^5 = \frac{\sqrt{13}}{\sqrt{14}}\frac{\sqrt{g_3}\sqrt{g_4}}{\sqrt{g_5}} \ ,$$

$$c_{35}{}^4 = \frac{\sqrt{13}}{\sqrt{14}}\frac{\sqrt{g_3}\sqrt{g_5}}{\sqrt{g_4}} \ , \qquad c_{35}{}^{\mathcal{C}^{W_3W_3}_{3,1}} = -\frac{23}{30\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ , \qquad c_{35}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = \frac{1}{5\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ ,$$

$$c_{44}{}^4 = \frac{67}{15\sqrt{91}}\sqrt{g_4} \ , \qquad c_{44}{}^{\mathcal{C}^{W_3W_3}_{3,1}} = -\frac{39}{50}\frac{g_4}{g_3} \ , \qquad c_{44}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = \frac{13}{25}\frac{g_4}{g_3} \ ,$$

$$c_{45}{}^3 = \frac{\sqrt{13}}{\sqrt{14}}\frac{\sqrt{g_4}\sqrt{g_5}}{\sqrt{g_3}} \ , \qquad c_{45}{}^5 = -\frac{87}{10\sqrt{91}}\sqrt{g_4} \ , \qquad c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}}} = -\frac{37}{30\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ ,$$

$$c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} = \frac{3}{10\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ , \qquad c_{45}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} = \frac{23}{30\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ , \qquad c_{55}{}^4 = -\frac{87}{10\sqrt{91}}\frac{g_5}{\sqrt{g_4}} \ ,$$

$$c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,1}} = -\frac{77}{1800}\frac{g_5}{g_3} \ , \qquad c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}} = \frac{173}{300}\frac{g_5}{g_3} \ , \qquad \bar{c}_{55}{}^{\mathcal{C}^{W_4W_4}_{4,0}} = \frac{116}{351}\frac{g_5}{g_4} \ ,$$

$$\bar{c}_{55}{}^{\mathcal{C}^{W_3W_5}_{4,2}} = -\frac{3981}{17875\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ , \qquad \bar{c}_{55}{}^{\mathcal{C}^{W_4W_4}_{4,2}} = -\frac{58012}{83655}\frac{g_5}{g_4} \ , \qquad c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,4}} = -\frac{1}{\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}} \ ,$$

$$c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,4}} = \frac{25}{39}\frac{\sqrt{g_5}}{\sqrt{g_4}} \ .$$

$$\mathcal{C}^{W_3W_4}_{\frac{7}{2}\frac{3}{2}}, \mathcal{C}^{W_3W_3}_{4,0} + \frac{25}{13}\frac{g_3}{g_4}\mathcal{C}^{W_4W_4}_{4,0}, \mathcal{C}^{W_3W_3}_{4,2} + \frac{315\sqrt{2}}{143}\frac{\sqrt{g_3}}{\sqrt{g_5}}\mathcal{C}^{W_3W_5}_{4,2} + \frac{3500}{1859}\frac{g_3}{g_4}\mathcal{C}^{W_4W_4}_{4,2}$$

$$\bar{c}_{55}\mathcal{C}^{W_4W_4}_{4,0} = c_{55}\mathcal{C}^{W_4W_4}_{4,0} - \frac{25}{13}\frac{g_3}{g_4}c_{55}\mathcal{C}^{W_3W_3}_{4,0}$$

$$\bar{c}_{55}\mathcal{C}^{W_3W_5}_{4,2} = c_{55}\mathcal{C}^{W_3W_5}_{4,2} - \frac{315\sqrt{2}}{143}\frac{\sqrt{g_3}}{\sqrt{g_5}}c_{55}\mathcal{C}^{W_3W_3}_{4,2}$$

$$\bar{c}_{55}\mathcal{C}^{W_4W_4}_{4,2} = c_{55}\mathcal{C}^{W_4W_4}_{4,2} - \frac{3500}{1859}\frac{g_3}{g_4}c_{55}\mathcal{C}^{W_3W_3}_{4,2}$$

$$c_{55}\mathcal{C}^{W_3W_5}_{4,4}, c_{55}\mathcal{C}^{W_4W_4}_{4,4}$$

4d $\mathcal{N}=2$ multiplet	Schur operator			
	structure	h	r	R
$\widehat{\mathcal{B}}_{R'}$	$\Psi^{1\dots1}$	R'	0	R'
$\mathcal{D}_{R'(0,j_2)}$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}\Psi^{1\dots1}_{\dot{+}\dots\dot{+}}$	$R'+j_2+1$	$j_2+\frac{1}{2}$	$R'+\frac{1}{2}$
$\overline{\mathcal{D}}_{R'(j_1,0)}$	$\mathcal{Q}^1_{+}\Psi^{1\dots1}_{+\dots+}$	$R'+j_1+1$	$-j_1-\frac{1}{2}$	$R'+\frac{1}{2}$
$\widehat{\mathcal{C}}_{R'(j_1,j_2)}$	$\mathcal{Q}^1_{+}\widetilde{\mathcal{Q}}^1_{\dot{+}}\Psi^{1\dots1}_{+\dots+\dot{+}\dots\dot{+}}$	$R'+j_1+j_2+2$	j_2-j_1	$R'+1$

$$E-(j_1+j_2)-2R=0, r+j_1-j_2=0.$$

$$h=E-R=\frac{1}{2}(E+j_1+j_2).$$

$$\mathcal{V}_{h,r,R}=\left\{(h,r,R)\in\frac{1}{2}\mathbb{Z}_{\geq 0}\times\frac{1}{2}\mathbb{Z}\times\frac{1}{2}\mathbb{Z}_{\geq 0}\right\}.$$

$$\mathcal{F}_{h,r,R}=\bigoplus_{k\geq 0}\mathcal{V}_{h,r,R-k}.$$

$$\mathcal{O}_1(z)\mathcal{O}_2(0)=\sum_{n\in\mathbb{Z}}\frac{1}{z^n}\{\mathcal{O}_1\mathcal{O}_2\}_n(0)$$

$$\{\mathcal{O}_1\mathcal{O}_2\}_0\in\mathcal{F}_{h_1+h_2,r_1+r_2,R_1+R_2}$$
$$\{\mathcal{O}_1\mathcal{O}_2\}_1\in\mathcal{F}_{h_1+h_2-1,r_1+r_2,R_1+R_2-1}$$

$$\mathrm{gr}\mathcal{V}_{h,r}=\bigoplus_{R\in\frac{1}{2}\mathbb{Z}_{\geq 0}}\mathcal{G}_{h,r,R},\mathcal{G}_{h,r,R}=\mathcal{F}_{h,r,R}/\mathcal{F}_{h,r,R-1}$$

$$\hat{\mathcal{C}}_{R'\left(j_1,-\frac{1}{2}\right)}\cong\overline{\mathcal{D}}_{R'+\frac{1}{2}(j_1,0)}$$
$$\hat{\mathcal{C}}_{R'\left(-\frac{1}{2},j_2\right)}\cong\mathcal{D}_{R'+\frac{1}{2}(0,j_2)}$$
$$\hat{\mathcal{C}}_{R'\left(-\frac{1}{2},-\frac{1}{2}\right)}\cong\hat{\mathcal{B}}_{R'+1}$$

$$\mathcal{C}_{[q_1,p,q_2]\left(j_1,-\frac{1}{2}\right)}\cong\overline{\mathcal{D}}_{[q_1,p,q_2+1](j_1,0)}$$
$$\mathcal{C}_{[q_1,p,q_2]\left(-\frac{1}{2},j_2\right)}\cong\mathcal{D}_{[q_1+1,p,q_2](0,j_2)}$$
$$\mathcal{C}_{[q_1,p,q_2]\left(-\frac{1}{2},-\frac{1}{2}\right)}\cong\mathcal{B}_{[q_1+1,p,q_1+1](0,0)}$$

$$\sum_{\mathfrak{R},r}n_{h,j,r,\mathfrak{R}}x^{2r}\xi^{\mathfrak{R}},$$

$$\mathfrak{R}=\xi^3+\xi^2+\xi^{7/2}\chi_1(x)$$

$$\chi_1(x)=x+x^{-1}$$

$$Z_{\mathcal{W}_\infty^{s,s}}(q,a,x,\xi)=\mathrm{Tr}q^h a^j x^{2r}\xi^{\mathfrak{R}}$$

$$x=-\xi^{1/2}$$

h	j	generic ν	$\nu = N^2$ with N equal to 2	3	4	5	6	7	8, 9, …
1	1	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$	0	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$
2	2	$2\xi^2$	ξ^2	ξ^2	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$
2	0	ξ^2	0	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2
$\frac{5}{2}$	$\frac{5}{2}$	$2\xi^{5/2}$	0	$\xi^{5/2}$	$\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$	0	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{5/2}$	0	0	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$
3	3	$4\xi^3$	ξ^3	$2\xi^3$	$3\xi^3$	$3\xi^3$	$4\xi^3$	$4\xi^3$	$4\xi^3$
3	2	ξ^3	0	0	ξ^3	ξ^3	ξ^3	ξ^3	ξ^3
3	1	$\xi^2+3\xi^3$	ξ^2	$\xi^2+\xi^3$	$\xi^2+2\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$
$\frac{7}{2}$	$\frac{7}{2}$	$4\xi^{7/2}$	0	$\xi^{7/2}$	$2\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$4\xi^{7/2}$	$4\xi^{7/2}$
$\frac{7}{2}$	$\frac{5}{2}$	$3\xi^{7/2}$	0	$\xi^{7/2}$	$2\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{3}{2}$	$\xi^{5/2}+4\xi^{7/2}$	0	$\xi^{5/2}+\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+3\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$
$\frac{7}{2}$	$\frac{1}{2}$	$\xi^{5/2}+2\xi^{7/2}$	0	$\xi^{5/2}$	$\xi^{5/2}+\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$
4	4	$7\xi^4$	ξ^4	$2\xi^4$	$4\xi^4$	$5\xi^4$	$6\xi^4$	$6\xi^4$	$7\xi^4$
4	3	$4\xi^4$	0	ξ^4	$2\xi^4$	$3\xi^4$	$4\xi^4$	$4\xi^4$	$4\xi^4$
4	2	$3\xi^3+8\xi^4$	ξ^3	$2\xi^3+\xi^4$	$3\xi^3+4\xi^4$	$3\xi^3+6\xi^4$	$3\xi^3+7\xi^4$	$3\xi^3+8\xi^4$	$3\xi^3+8\xi^4$
4	1	$\xi^3+2\xi^4+\mathfrak{N}$	0	ξ^3	$\xi^3+\mathfrak{N}$	$\xi^3+\xi^4+\mathfrak{N}$	$\xi^3+2\xi^4+\mathfrak{N}$	$\xi^3+2\xi^4+\mathfrak{N}$	$\xi^3+2\xi^4+\mathfrak{N}$
4	0	$\xi^2+2\xi^3+4\xi^4$	ξ^2	$\xi^2+\xi^3+\xi^4$	$\xi^2+2\xi^3+2\xi^4$	$\xi^2+2\xi^3+3\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$

$$\chi_{\mathcal{W}_\infty^{s,s}}(q,a,\xi) = \mathrm{Tr}(-1)^F q^h a^j \xi^{\mathfrak{R}+r}.$$

$$(-1)^{2r} = (-1)^F$$

$$Z_{\mathcal{W}_\infty^{s,s}}(q,a,x,\xi) = \sum_{\substack{h,j,r,\mathfrak{R}\\ h>j}} n_{h,j,r,\mathfrak{R}}\mathfrak{L}_{h,j}(q,a,x)x^{2r}\xi^{\mathfrak{R}} + \sum_{j,r,\mathfrak{R}} n_{j,j,r,\mathfrak{R}}\mathfrak{S}_j(q,a,x)x^{2r}\xi^{\mathfrak{R}}$$

$$\begin{aligned}\mathfrak{L}_{h,j}(q,a,x) = \frac{1}{1-q}\Big[&q^h\chi_{2j}(a) + q^{h+\frac{1}{2}}\chi_{2j-1}(a)\chi_1(x) + q^{h+\frac{1}{2}}\chi_{2j+1}(a)\chi_1(x)\\ &+q^{h+1}\chi_{2j}(a)(1+\chi_2(x)) + q^{h+1}\chi_{2j-2}(a) + q^{h+1}\chi_{2j+2}(a)\\ &+q^{h+\frac{3}{2}}\chi_{2j-1}(a)\chi_1(x) + q^{h+\frac{3}{2}}\chi_{2j+1}(a)\chi_1(x) + q^{h+2}\chi_{2j}(a)\Big]\end{aligned}$$

$$\mathfrak{S}_j(q,a,x) = \frac{1}{1-q}\Big[q^j\chi_{2j}(a) + q^{j+\frac{1}{2}}\chi_{2j-1}(a)\chi_1(x) + q^{j+1}\chi_{2j-2}(a)\Big]$$

$$\chi_{2j}(a) = a^{-2j} + a^{-2j+2} + \dots + a^{2j}, \chi_{2r}(x) = x^{-2r} + x^{-2r+2} + \dots + x^{2r}.$$

$$Z_{\mathcal{W}_\infty^{s,s}}(q,a,x,\xi) = \prod_{p=2}^{\infty}\prod_{n=0}^{\infty} \frac{\left[\prod_{m=-\frac{p}{2}+\frac{1}{2}}^{\frac{p}{2}-\frac{1}{2}}\left(1+x\xi^{\frac{p}{2}}q^{\frac{p}{2}+\frac{1}{2}+n}a^{2m}\right)\right]\left[\prod_{m=-\frac{p}{2}+\frac{1}{2}}^{\frac{p}{2}-\frac{1}{2}}\left(1+x^{-1}\xi^{\frac{p}{2}}q^{\frac{p}{2}+\frac{1}{2}+n}a^{2m}\right)\right]}{\left[\prod_{m=-\frac{p}{2}}^{\frac{p}{2}}\left(1-\xi^{\frac{p}{2}}q^{\frac{p}{2}+n}a^{2m}\right)\right]\left[\prod_{m=-\frac{p}{2}+1}^{\frac{p}{2}-1}\left(1-\xi^{\frac{p}{2}}q^{\frac{p}{2}+1+n}a^{2m}\right)\right]}$$

$$\mathcal{I}(\rho,\sigma,\tau) = \mathrm{Tr}(-1)^F\rho^{\frac{1}{2}(E-2j_1-2R-r)}\sigma^{\frac{1}{2}(E+2j_1-2R-r)}\tau^{\frac{1}{2}(E+2j_2+2R+r)}(e^{-\beta})^{\frac{1}{2}(E-2j_2-2R+r)}$$

$$\sigma = \frac{q}{\tau}, \tau^2 = q\xi$$

$$\mathcal{I}_{\mathrm{Macdonald}}^{4\,\mathrm{d}SU(N)\mathrm{SYM}}(q,a,\xi) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{Schur}}}(-1)^F q^{E-R}\xi^{R+r}a^j$$

$$f^V_{\text{Macdonald}}(q,a,\xi) = -\frac{q(\xi+1)}{1-q} + \chi_1(a)\frac{\sqrt{q}\sqrt{\xi}}{1-q}$$

$$\mathcal{I}^{4\text{ d}SU(N)\text{SYM}}_{\text{Macdonald}}(q,a,\xi) = \int [db]\text{PE}\left[f^V_{\text{Macdonald}}(q,a,\xi)\chi_{\text{adj of }\mathfrak{su}(N)}(b)\right]$$

$$\text{PE}[g(y_1,y_2,\ldots)] = \exp\sum_{k=1}^{\infty}\frac{1}{k}g\left(y_1^k,y_2^k,\ldots\right),$$

$$\text{for } x = -\xi^{1/2}, \mathfrak{N} := \xi^3 + \xi^2 + \xi^{7/2}\chi_1(x) = 0$$

$$\frac{\mathbb{C}^{(2|1)}\times\mathbb{R}^{N-1}}{S_N}$$

$$\text{HS}_{\text{HL}}(\tau,a,x) = \text{Tr}_{\text{HL}}\tau^{2h}a^j x^{2r}$$

$$\text{HS}_{\text{HL}}(\tau,a,x=-1) = \mathcal{I}^{4\text{ d}SU(N)\text{SYM}^{SY}}_{\text{HL}}(\tau,a)$$

$$\mathcal{I}^{4\text{ d}SU(N)\text{SYM}}_{\text{HL}}(\tau,a) = \lim_{q\to 0}\mathcal{I}^{4\text{ d}SU(N)\text{SYM}}_{\text{Macdonald}}(q,a,\xi=\tau^2/q)$$

$$Z_{\text{free 4 d}SU(N)\text{SYM}}(q,a,x,\xi) = \text{Tr}_{\mathcal{H}\text{ Schur}}\, q^{E-R}\xi^R x^{2r}a^j$$

$$f^V(q,a,\xi,x) = \frac{q\sqrt{\xi}}{1-q}\chi_1(x) + \chi_1(a)\frac{\sqrt{q}\sqrt{\xi}}{1-q}$$

$$Z_{\text{free 4 d}SU(N)\text{SYM}}(q,a,x,\xi) = \int [db]\text{PE}\left[f^V(q,a,\xi,x,\alpha)\chi_{\text{adj of }\mathfrak{su}(N)}(b)\right]$$

$$\mathcal{A}^{p+q_1+q_2+j_1+j_2+2}_{[q_1,p,q_2]}\Big|_{q_1-q_2=2(j_2-j_1)} \cong$$
$$\cong \mathcal{C}_{[q_1,p,q_2](j_1,j_2)} \oplus \mathcal{C}_{[q_1+1,p,q_2]\left(j_1-\frac{1}{2},j_2\right)} \oplus \mathcal{C}_{[q_1,p,q_2+1]\left(j_1,j_2-\frac{1}{2}\right)} \oplus \mathcal{C}_{[q_1+1,p,q_2+1]\left(j_1-\frac{1}{2},j_2-\frac{1}{2}\right)}$$

$$\begin{aligned}A_{[q_1,p,q_1]}(q,a,x,\xi) &= (1+\xi)\xi^{\frac{p}{2}+q_1+1}\mathfrak{L}_{\frac{p}{2}+q_1+2,\frac{p}{2}}(q,a,x)\\ &+\xi^{\frac{p}{2}+q_1+\frac{3}{2}}\chi_1(x)\mathfrak{L}_{\frac{p}{2}+q_1+2,\frac{p}{2}}(q,a,x)\end{aligned}$$

$$\begin{aligned}A_{[q_1,p,q_2](j_1,j_2)}(q,a,x,\xi) &= (1+\xi)\xi^{\frac{p+q_1+q_2}{2}+1}x^{2(j_2-j_1)}\mathfrak{L}_{\frac{p+q_1+q_2}{2}+j_1+j_2+2,\frac{p}{2}}(q,a,x)\\ &+\xi^{\frac{p+q_1+q_2}{2}+\frac{3}{2}}x^{2(j_2-j_1)}\chi_1(x)\mathfrak{L}_{\frac{p+q_1+q_2}{2}+j_1+j_2+2,\frac{p}{2};j_2-j_1+\frac{1}{2}}(q,a,x)\end{aligned}$$

$$\begin{aligned}Z_{\mathcal{W}^{s,s}_\infty} = Z_{\text{free 4 d}SU(N\geq 8)\text{SYM}} - \Big[&A_{[0,0,0]} + A_{[0,1,0]} + 3A_{[0,2,0]} + A_{[1,2,1]}\\ &+2A_{[0,1,0]\left(\frac{1}{2}\frac{1}{2}\right)} + A_{[0,1,1]\left(\frac{1}{2},0\right)} + A_{[1,1,0]\left(0,\frac{1}{2}\right)} + 4A_{[0,3,0]} + 3A_{[1,1,1]}\\ &+A_{[0,0,0](1,1)} + A_{[0,0,2](1,0)} + A_{[2,0,0](0,1)} + 3A_{[0,2,0]\left(\frac{1}{2}\frac{1}{2}\right)}\\ &+4A_{[0,2,1]\left(\frac{1}{2},0\right)} + 4A_{[1,2,0]\left(0,\frac{1}{2}\right)} + 6A_{[1,0,1]\left(\frac{1}{2}\frac{1}{2}\right)}\\ &+A_{[1,0,2]\left(\frac{1}{2},0\right)} + A_{[2,0,1]\left(0,\frac{1}{2}\right)} + 8A_{[0,4,0]} + 7A_{[1,2,1]} + 7A_{[2,0,2]}\Big] + \mathcal{O}\left(q^{\frac{9}{2}}\right)\end{aligned}$$

$$\phi(z,y)^I{}_J, b(z)^I{}_J, c(z)^I{}_J.$$

$$\phi(x,y)^I{}_I = b(z)^I{}_I = c(z)^I{}_I = 0.$$

$$\phi(z,y)^I{}_J = \phi_0(z)^I{}_J + y\phi_1(z)^I{}_J.$$

$$\begin{aligned}\phi(y_1,z_1)^I{}_J\phi(y_2,z_2)^K{}_L &= \frac{y_{12}}{z_{12}}\left(\delta^I{}_L\delta^K{}_J - \frac{1}{N}\delta^I{}_J\delta^K{}_L\right) + \mathcal{R}\\ b(z_1)^I{}_J c(z_2)^K{}_L &= \frac{1}{z_{12}}\left(\delta^I{}_L\delta^K{}_J - \frac{1}{N}\delta^I{}_J\delta^K{}_L\right) + \mathcal{R}\end{aligned}$$

$$J_{\mathrm{B}} = (\phi_0)^{I_1}{}_{I_2}(\phi_1)^{I_2}{}_{I_3}c^{I_3}{}_{I_1} - (\phi_0)^{I_1}{}_{I_2}(\phi_1)^{I_3}{}_{I_1}c^{I_2}{}_{I_3} + b^{I_1}{}_{I_2}c^{I_2}{}_{I_3}c^{I_3}{}_{I_1}.$$

$$Q_{\mathrm{B}}X := \{J_{\mathrm{B}}X\}_1,$$

$$J(y) = \frac{1}{2}\phi(y)^I{}_J\phi(y)^J{}_I$$
$$G(y) = \phi(y)^I{}_J b^J{}_I$$
$$\tilde{G}(y) = \phi(y)^I{}_J\partial_z c^J{}_I$$
$$T = -b^I{}_J\partial_z c^I{}_J - \frac{1}{2}(\phi_0)^I{}_J\partial_z(\phi_1)^J{}_I + \frac{1}{2}\partial_z(\phi_0)^I{}_J(\phi_1)^J{}_I$$

$$\phi(y)^{I_1}{}_{I_2}c^{I_2}{}_{I_3}c^{I_3}I_1.$$

$$Q_{\mathrm{B}}\{J(y_1)X(y_2)\}_n = +\{J(y_1)Q_{\mathrm{B}}X(y_2)\}_n$$
$$Q_{\mathrm{B}}\{G(y_1)X(y_2)\}_n = -\{G(y_1)Q_{\mathrm{B}}X(y_2)\}_n$$
$$Q_{\mathrm{B}}\{\tilde{G}(y_1)X(y_2)\}_n = -\{\tilde{G}(y_1)Q_{\mathrm{B}}X(y_2)\}_n$$
$$Q_{\mathrm{B}}\{TX(y_2)\}_n = +\{TQ_{\mathrm{B}}X(y_2)\}_n$$

$$\{b^I{}_J X(y_2)\}_1 = 0$$

$$\{b^I{}_J J_{\mathrm{B}}\}_1 = K^I{}_J,$$

$$K^I{}_J = (\phi_0)^I{}_K(\phi_1)^K{}_J - (\phi_0)^K{}_J(\phi_1)^I{}_K + b^I{}_K c^K{}_J - b^K{}_J c^I{}_K.$$

$$\{b^I{}_J X(y_2)\}_1 = 0, \{K^I{}_J X(y_2)\}_1 = 0$$

$$\{b^I{}_J Y(y_2)\}_1 = 0, \{K^I{}_J Y(y_2)\}_1 = 0$$

$$T_p(x, Y) = \mathrm{Tr}(\phi(x, Y)^p)$$

$$T_{p_1, p_2, \ldots, p_n} = T_{p_1}T_{p_2}\ldots T_{p_n}$$

$$\langle \phi_a^b(x_1, Y_1)\phi_c^d(x_2, Y_2)\rangle = \left(\delta_c^b\delta_a^d - \frac{1}{N}\delta_a^b\delta_c^d\right)\frac{Y_1 \cdot Y_2}{(x_1 - x_2)^2}.$$

$$\langle \mathcal{O}_p T_{q_1, q_2, \ldots, q_n}\rangle = 0$$

$$\mathcal{O}_2 = T_2$$
$$\mathcal{O}_3 = T_3$$
$$\mathcal{O}_4 = T_4 - \frac{2N^2 - 3}{N(N^2 + 1)}T_{2,2}$$
$$\mathcal{O}_5 = T_5 - \frac{5(N^2 - 2)}{N(N^2 + 5)}T_{2,3}$$
$$\mathcal{O}_6 = T_6 - \frac{3N^4 - 11N^2 + 80}{N(N^4 + 15N^2 + 8)}T_{3,3} - \frac{6(N^2 - 4)(N^2 + 5)}{N(N^4 + 15N^2 + 8)}T_{4,2} + \frac{7(N^2 - 7)}{N^4 + 15N^2 + 8}T_{2,2,2}$$

$$\langle \mathcal{O}_p(x_1, Y_1)\mathcal{O}_p(x_2, Y_2)\rangle = \langle \mathcal{O}_p\mathcal{O}_p\rangle (Z_{12})^p$$
$$\langle \mathcal{O}_{p_1}(x_1, Y_1)\mathcal{O}_{p_2}(x_2, Y_2)\mathcal{O}_{p_3}(x_3, Y_3)\rangle = \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\rangle (Z_{12})^{p_{12}}(Z_{23})^{p_{23}}(Z_{13})^{p_{13}}$$

$$Z_{ij} = \frac{Y_i \cdot Y_j}{(x_i - x_j)^2}, p_{12} = \frac{p_1 + p_2 - p_3}{2}, p_{23} = \frac{p_2 + p_3 - p_1}{2}, p_{13} = \frac{p_1 + p_3 - p_2}{2}.$$

$$\langle \mathcal{O}_p\mathcal{O}_p\rangle = p^2(p - 1)\frac{(N - p + 1)_{p-1}(N + 1)_{p-1}}{(N + 1)_{p-1} - (N - p + 1)_{p-1}}$$

$$\langle \mathcal{O}_p\mathcal{O}_{q_1}\mathcal{O}_{q_2}\rangle = 0$$

$$\langle \mathcal{O}_p\mathcal{O}_{q_1}\mathcal{O}_{q_2}\rangle = q_1 q_2\langle \mathcal{O}_p\mathcal{O}_p\rangle$$

$$\langle \mathcal{O}_p \mathcal{O}_{q_1} \mathcal{O}_{q_2} \rangle = \langle \mathcal{O}_p \mathcal{O}_p \rangle \left\{ q_1 q_2 \left[N - \frac{(q_1-1)(q_2-1)}{N} \right] + f(q_1,q_2,N) + f(q_2,q_1,N) \right\}$$

$$f(q_1,q_2,N) = \frac{q_1 q_2 (q_2-1)}{2N(q_1-2)} \left[2N^2 + N(q_1-1)_2 + 2(q_1-2)_2 + \frac{2N(q_1-1)_2 (N)_{q_1}}{(N-q_1+1)_{q_1} - (N)_{q_1}} \right]$$

$$\frac{\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \rangle}{\sqrt{\langle \mathcal{O}_{p_1} \mathcal{O}_{p_1} \rangle \langle \mathcal{O}_{p_2} \mathcal{O}_{p_2} \rangle \langle \mathcal{O}_{p_3} \mathcal{O}_{p_3} \rangle}} = \frac{\sqrt{p_1 p_2 p_3}}{N} + \mathfrak{S}$$

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \rangle = p_1 p_2 p_3 N^{\frac{p_1+p_2+p_3}{2}} - 1 + \mathfrak{S}.$$

$$\tilde{\mathcal{O}}_p = \frac{1}{N^{(p-2)/2}} \mathcal{O}_p.$$

$$\langle \tilde{\mathcal{O}}_p \tilde{\mathcal{O}}_p \rangle = N^{2-p} \langle \mathcal{O}_p \mathcal{O}_p \rangle, \langle \tilde{\mathcal{O}}_{p_1} \tilde{\mathcal{O}}_{p_2} \tilde{\mathcal{O}}_{p_3} \rangle = N^{3-\frac{p_1+p_2+p_3}{2}} \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \rangle.$$

$$\langle \tilde{\mathcal{O}}_p \tilde{\mathcal{O}}_p \rangle (N^2 \rightsquigarrow \nu), \langle \tilde{\mathcal{O}}_{p_1} \tilde{\mathcal{O}}_{p_2} \tilde{\mathcal{O}}_{p_3} \rangle (N^2 \rightsquigarrow \nu).$$

$$W_p \leftrightarrow \tilde{\mathcal{O}}_p = \frac{1}{N^{(p-2)/2}} \mathcal{O}_p$$

$$g_p(\nu) = \langle \tilde{\mathcal{O}}_p \tilde{\mathcal{O}}_p \rangle (N^2 \rightsquigarrow \nu)$$

$$c_{p_1 p_2}{}^{p_3}(\nu) = \frac{\langle \tilde{\mathcal{O}}_{p_1} \tilde{\mathcal{O}}_{p_2} \tilde{\mathcal{O}}_{p_3} \rangle (N^2 \rightsquigarrow \nu)}{\langle \tilde{\mathcal{O}}_{p_3} \tilde{\mathcal{O}}_{p_3} \rangle (N^2 \rightsquigarrow \nu)}.$$

$p = q_1 + q_2$: $(p, q_1, q_2) \in \{(6,3,3), (7,4,3)\}$
$p = q_1 + q_2 - 2$: $(p, q_1, q_2) \in \{(4,3,3), (5,4,3), (6,5,3), (7,6,3), (6,4,4), (7,5,4)\}$
$p = q_1 + q_2 - 4$: $(p, q_1, q_2) \in \{(4,4,4), (5,5,4), (6,6,4), (6,5,5)\}$

$$c_{p_1 p_2}^{p_3}(\nu) = p_1 p_2 + \mathfrak{S}$$

$$X(z) = \sum_m z^{-m-h_X} X_m$$

$$|m| \leq h_X - 1$$

$$T(z) = \sum_{m \in \mathbb{Z}} z^{-m-2} T_m, [T_m, T_n] = (m-n) T_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}$$

$$J(z,y) = \sum_{m \in \mathbb{Z}} z^{-m-1} J_m(y)$$

$$G(z,y) = \sum_{m \in \frac{1}{2}+\mathbb{Z}} z^{-m-\frac{3}{2}} G_m(y), \tilde{G}(z,y) = \sum_{m \in \frac{1}{2}+\mathbb{Z}} z^{-m-\frac{3}{2}} \tilde{G}_m(y)$$

$$\{T_m\}_{m=-1,0,1}, \{J_m(y)\}_{m=0}, \{G_m(y)\}_{m=-\frac{1}{2},\frac{1}{2}}, \{\tilde{G}_m(y)\}_{m=-\frac{1}{2},\frac{1}{2}}$$

$$\mathfrak{hs}^{\mathfrak{s},\mathfrak{s}} \cong (\mathcal{W}_\infty^{\mathfrak{s},\mathfrak{s}})$$

	h	j	allowed values of m
$J_m(y)$	1	1	0
$G_m(y)$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
$\tilde{G}_m(y)$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
T_m	2	0	$-1, 0, 1$

	h	j	allowed values of m
$(W_p)_m(y)$	$\frac{p}{2}$	$\frac{p}{2}$	$-\frac{p}{2}+1, -\frac{p}{2}+2, \ldots, \frac{p}{2}-1$
$(G_{W_p})_m(y)$	$\frac{p}{2}+\frac{1}{2}$	$\frac{p}{2}-\frac{1}{2}$	$-\frac{p}{2}+\frac{1}{2}, -\frac{p}{2}+\frac{3}{2}, \ldots, \frac{p}{2}-\frac{1}{2}$
$(\widetilde{G}_{W_p})_m(y)$	$\frac{p}{2}+\frac{1}{2}$	$\frac{p}{2}-\frac{1}{2}$	$-\frac{p}{2}+\frac{1}{2}, -\frac{p}{2}+\frac{3}{2}, \ldots, \frac{p}{2}-\frac{1}{2}$
$(T_{W_p})_m(y)$	$\frac{p}{2}+1$	$\frac{p}{2}-1$	$-\frac{p}{2}, -\frac{p}{2}+1, \ldots, \frac{p}{2}$

$$[\mathbb{I},\mathbb{J}], [\mathbb{J},\mathbb{W}_p], [\mathbb{W}_{p_1},\mathbb{W}_{p_2}].$$

$$[W_{p_1},W_{p_2}] = \delta_{p_1,p_2}\gamma_{p_1}\left(J-\frac{1}{6}T\right)+\sum_q \kappa_{p_1p_2}{}^q\left(W_q-\frac{(p_1-p_2+q)(-p_1+p_2+q)}{4(q-1)q(q+1)}T_{W_q}\right).$$

$$\max(3,|p_1-p_2|+2)\le q\le p_1+p_2-2, q-p_1-p_2\in 2\mathbb{Z}.$$

$$\gamma_p = 2p^2, \kappa_{p_1p_2}{}^q = p_1p_2$$

$\mathfrak{hs}^{s,s}$ side	$\mathcal{W}_\infty^{s,s}$ side

$$\gamma_p = \lim_{c\to\infty}\left(-\frac{6p}{c}g_p\right)$$

$$\kappa_{p_1p_2}{}^q = \lim_{c\to\infty} c_{p_1p_2}{}^q$$

$$W_p\times W_p\supset g_p\left[\mathbb{1}-\frac{6p}{c}\left(J-\frac{1}{6}T\right)+\frac{18p(p-1)}{c(c-6)}(JJ)_0^2-\frac{9p(p+1)}{c(c+9)}\left((JJ)_0^0+\frac{1}{3}T\right)+\cdots\right]$$

$$c\to\infty, g_p\to\infty, \frac{g_p}{c}$$

$$\lim_{c\to\infty}\left(-\frac{6p}{c}g_p\right)=\lim_{\nu\to\infty}\left(\frac{2p}{\nu-1}g_p\right)=\lim_{\nu\to\infty}\frac{2p^2\nu}{\nu-1}\frac{\prod_{r=1}^{p-1}(\nu-r^2)}{S_p(\nu)}\frac{\nu^{\left\lfloor\frac{p-2}{2}\right\rfloor}}{\nu^{p-1}}=2p^2$$

$$\lim_{\nu\to\infty} c_{p_1p_2}^q = p_1p_2$$

$$W_6' = W_6+\mu_6 C_{3,3}^{W_3W_3}$$

$$\begin{aligned}
(c_{36}{}^3)' &= c_{36}{}^3+\frac{6(\nu-1)(\nu^2+15\nu+8)}{(\nu+3)(\nu+7)}\mu_6,\\
(c_{36}{}^5)' &= c_{36}{}^5+\frac{54(\nu-1)}{\nu+7}\mu_6,\\
(c_{36}{}^7)' &= c_{36}{}^7,\\
(c_{46}{}^4)' &= c_{46}{}^4+\frac{144(\nu-4)(\nu-1)(\nu+1)}{\nu(\nu+3)(\nu+7)}\mu_6,\\
(c_{46}{}^6)' &= c_{46}{}^6+\frac{108(\nu-1)}{\nu+7}\mu_6,\\
(c_{46}{}^8)' &= c_{46}{}^8.
\end{aligned}$$

$$J(z_1,y_1)J(z_2,y_2)=\frac{-ky_{12}^2\mathbb{1}}{z_{12}^2}+\frac{2y_{12}\left(1+\frac{1}{2}y_{12}\partial_{y_2}\right)J(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$T(z_1)J(z_2,y_2)=\frac{J(z_2,y_2)}{z_{12}^2}+\frac{\partial_{z_2}J(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$T(z_1)T(z_2)=\frac{\frac{1}{2}c\mathbb{1}}{z_{12}^4}+\frac{2T(z_2)}{z_{12}^2}+\frac{\partial_{z_2}T(z_2)}{z_{12}}+\mathcal{R}$$

$$T(z_1)G(z_2,y_2)=\frac{\frac{3}{2}G(z_2,y_2)}{z_{12}^2}+\frac{\partial_{z_2}G(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$T(z_1)\tilde{G}(z_2,y_2)=\frac{\frac{3}{2}\tilde{G}(z_2,y_2)}{z_{12}^2}+\frac{\partial_{z_2}\tilde{G}(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$J(z_1,y_1)G(z_2,y_2)=\frac{y_{12}(1+y_{12}\partial_{y_2})G(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$J(z_1,y_1)\tilde{G}(z_2,y_2)=\frac{y_{12}(1+y_{12}\partial_{y_2})\tilde{G}(z_2,y_2)}{z_{12}}+\mathcal{R}$$

$$G(z_1,y_1)G(z_2,y_2)=\mathcal{R}$$

$$\tilde{G}(z_1,y_1)\tilde{G}(z_2,y_2)=\mathcal{R}$$

$$G(z_1,y_1)\tilde{G}(z_2,y_2)=\frac{-2ky_{12}\mathbb{1}}{z_{12}^3}+\frac{2\left(1+\frac{1}{2}y_{12}\partial_{y_2}\right)J(z_2,y_2)}{z_{12}^2}+\frac{\left(1+\frac{1}{2}y_{12}\partial_{y_2}\right)\partial_{z_2}J(z_2,y_2)-y_{12}T(z_2)}{z_{12}}+\mathcal{R}$$

$$A(z_1)B(z_2)=\sum_{n\in\mathbb{Z}}\frac{1}{z_{12}^n}\{AB\}_n(z_2)$$

$$\{AB\}_{-n}(z)=\frac{1}{n!}\{\partial_z^n AB\}_0(z), n>0$$

$$T(z_1)A(z_2,y_2)=\sum_{n\in\mathbb{Z}}\frac{1}{z_{12}^n}\{TA(y_2)\}_n(z_2)$$

$$J(z_1,y_1)A(z_2,y_2)=\sum_{n\in\mathbb{Z}}\frac{1}{z_{12}^n}\{J(y_1)A(y_2)\}_n(z_2)$$

$$\{TA(y_2)\}_3(z_2)=0$$

$$\{TA(y_2)\}_2(z_2)=h_A A(z_2,y_2)$$

$$\{TA(y_2)\}_1(z_2)=\partial_{z_2}A(z_2,y_2)$$

$$\{J(y_1)A(y_2)\}_1(z_2)=y_{12}(2j_A+y_{12}\partial_{y_2})A(z_2,y_2)$$

$$A(z_1,y_1)B(z_2,y_2)=\sum_C c_{AB}{}^C\frac{y_{12}^{j_A+j_B-j_C}}{z_{12}^{h_A+h_B-h_C}}\mathcal{D}_{h_A,h_B;h_C}(z_{12},\partial_{z_2})\widehat{\mathcal{D}}_{j_A,j_B;j_C}(y_{12},\partial_{y_2})C(z_2,y_2)$$

$$j_C\in\{|j_A-j_B|,|j_A-j_B|+1,\dots,j_A+j_B\}.$$

$$\mathcal{D}_{h_A,h_B;h_C}(z_{12},\partial_{z_2})=\sum_{k=0}^{\infty}\frac{(h_C+h_A-h_B)_k}{k!\,(2h_C)_k}z_{12}^k\partial_{z_2}^k$$

$$\widehat{\mathcal{D}}_{j_A,j_B;j_C}(y_{12},\partial_{y_2})=\sum_{k=0}^{\infty}\frac{(-j_C-j_A+j_B)_k}{k!\,(-2j_C)_k}z_{12}^k\partial_{z_2}^k$$

$$(x)_k=\prod_{i=0}^{k-1}(x+i),(x)_0=1$$

$$
\begin{aligned}
&J \times J = -k\mathbb{1} + 2J\,, \quad T \times T = \tfrac{c}{2}\mathbb{1} + 2T\,, \quad T \times J = J\,,\\
&J \times G = G\,, \qquad J \times \widetilde{G} = \widetilde{G}\,,\\
&T \times G = \tfrac{3}{2}G\,, \qquad T \times \widetilde{G} = \tfrac{3}{2}\widetilde{G}\,,\\
&G \times G = 0\,, \qquad \widetilde{G} \times \widetilde{G} = 0\,, \qquad G \times \widetilde{G} = -2k\mathbb{1} + 2J - T\,.
\end{aligned}
$$

$$A(z,y) = \sum_{\alpha=0}^{2j_A} A_\alpha(z)y^\alpha, B(z,y) = \sum_{\alpha=0}^{2j_B} B_\alpha(z)y^\alpha$$

$$j \in \{|j_A - j_B|, |j_A - j_B| + 1, \dots, j_A + j_B\}.$$

$$(AB)_n^j(z,y) = \sum_{\alpha=0}^{2j_A}\sum_{\beta=0}^{2j_B} \mathcal{C}_{j_A,j_B,j,\alpha,\beta}\, y^{j-j_A-j_B+\alpha+\beta} \sum_{p=0}^{\infty} \mathcal{K}_{h_A,h_B,n,p}\,\partial_z^p\{A_\alpha B_\beta\}_{n+p}(z),$$

$$
\begin{aligned}
\mathcal{C}_{j_A,j_B,j,\alpha,\beta} &= \sum_{\ell=0}^{j_A+j_B-j} \frac{(-1)^\ell (j_A - j_B + j - \ell)_\ell^\downarrow}{\ell!\,(j_A + j_B - j - \ell)!\,(2j + \ell + 1)_\ell^\downarrow} \sum_{s=0}^{\ell} \binom{\ell}{s} (\alpha)_{j_A+j_B-j-s}^\downarrow (\beta)_s^\downarrow\\
\mathcal{K}_{h_A,h_B,n,p} &= \frac{(-1)^p (2h_A - n - p)_p}{p!\,(2h_A + 2h_B - 2n - p - 1)_p}
\end{aligned}
$$

$$(x)_n^\downarrow = \prod_{i=0}^{n-1} (x - i), (x)_0^\downarrow = 1.$$

$$A(z_1,y_1)B(z_2,y_2) = \sum_{n,j} \frac{y_{12}^{j_A+j_B-j}}{z_{12}^n} \mathcal{D}_{h_A,h_B;h_A+h_B-n}(z_{12},\partial_{z_2})\widehat{\mathcal{D}}_{j_A,j_B;j}(y_{12},\partial_{y_2})(AB)_n^j(z_2,y_2)$$

$$
\begin{aligned}
(G^\uparrow A)(z,y) &= (GA)_1^{j+\frac{1}{2}}(z,y), \quad (G^\downarrow A)(z,y) = (GA)_1^{j-\frac{1}{2}}(z,y)\\
(\tilde{G}^\uparrow A)(z,y) &= (\tilde{G}A)_1^{j+\frac{1}{2}}(z,y), \quad (\tilde{G}^\downarrow A)(z,y) = (\tilde{G}A)_1^{j-\frac{1}{2}}(z,y)
\end{aligned}
$$

$$
\begin{aligned}
&\{TX(y_2)\}_3(z_2) = 0\\
&\{TX(y_2)\}_2(z_2) = h_X X(z_2,y_2)\\
&\{TX(y_2)\}_1(z_2) = \partial_{z_2} X(z_2,y_2)\\
&\{J(y_1)X(y_2)\}_1(z_2) = y_{12}(2j_X + y_{12}\partial_{y_2})X(z_2,y_2)\\
&\{G(y_1)X(y_2)\}_2(z_2) = 0\\
&\{\tilde{G}(y_1)X(y_2)\}_2(z_2) = 0
\end{aligned}
$$

	X	$G^\downarrow X \equiv G_X$	$\widetilde{G}^\downarrow X \equiv \widetilde{G}_X$	$\widetilde{G}^\downarrow G^\downarrow X \equiv T_X$
h	j_X	$j_X + \frac{1}{2}$	$j_X + \frac{1}{2}$	$j_X + 1$
j	j_X	$j_X - \frac{1}{2}$	$j_X - \frac{1}{2}$	$j_X - 1$
r	r_X	$r_X + \frac{1}{2}$	$r_X - \frac{1}{2}$	r_X

	X	$G^{\downarrow}X \equiv X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}$	$\tilde{G}^{\downarrow}X \equiv X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}$	$G^{\uparrow}X \equiv X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}$	$\tilde{G}^{\uparrow}X \equiv X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}$
h	h_X	$h_X+\frac{1}{2}$	$h_X+\frac{1}{2}$	$h_X+\frac{1}{2}$	$h_X+\frac{1}{2}$
j	j_X	$j_X-\frac{1}{2}$	$j_X-\frac{1}{2}$	$j_X+\frac{1}{2}$	$j_X+\frac{1}{2}$
r	r_X	$r_X+\frac{1}{2}$	$r_X-\frac{1}{2}$	$r_X+\frac{1}{2}$	$r_X-\frac{1}{2}$

	$G^{\downarrow}\tilde{G}^{\downarrow}X \equiv X_{[1,-1,0]}$	$G^{\downarrow}G^{\uparrow}X \equiv X_{[1,0,1]}$	$\tilde{G}^{\downarrow}\tilde{G}^{\uparrow}X \equiv X_{[1,0-1]}$	$G^{\downarrow}\tilde{G}^{\uparrow}X \equiv X_{[1,0,0]}$	$\tilde{G}^{\downarrow}G^{\uparrow}X \equiv X'_{[1,0,0]}$	$G^{\uparrow}\tilde{G}^{\uparrow}X \equiv X_{[1,1,0]}$
h	h_X+1	h_X+1	h_X+1	h_X+1	h_X+1	h_X+1
j	j_X-1	j_X	j_X	j_X	j_X	j_X+1
r	r_X	r_X+1	r_X-1	r_X	r_X	r_X

	$G^{\downarrow}\tilde{G}^{\downarrow}\tilde{G}^{\uparrow}X \equiv X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]}$	$\tilde{G}^{\downarrow}G^{\downarrow}G^{\uparrow}X \equiv X_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]}$	$G^{\uparrow}\tilde{G}^{\downarrow}\tilde{G}^{\uparrow}X \equiv X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}$	$\tilde{G}^{\uparrow}G^{\downarrow}G^{\uparrow}X \equiv X_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}$	$G^{\downarrow}\tilde{G}^{\downarrow}G^{\uparrow}\tilde{G}^{\uparrow}X \equiv X_{[2,0,0]}$
h	$h_X+\frac{3}{2}$	$h_X+\frac{3}{2}$	$h_X+\frac{3}{2}$	$h_X+\frac{3}{2}$	h_X+2
j	$j_X-\frac{1}{2}$	$j_X-\frac{1}{2}$	$j_X+\frac{1}{2}$	$j_X+\frac{1}{2}$	j_X
r	$r_X-\frac{1}{2}$	$r_X+\frac{1}{2}$	$r_X-\frac{1}{2}$	$r_X+\frac{1}{2}$	r_X

$$\{TX(y_2)\}_{n\geq 3}(z_2)=0$$
$$\{TX(y_2)\}_2(z_2)=h_X X(z_2,y_2)$$
$$\{TX(y_2)\}_1(z_2)=\partial_{z_2}X(z_2,y_2)$$
$$\{J(y_1)X(y_2)\}_{n\geq 2}(z_2)=0$$
$$\{J(y_1)X(y_2)\}_1(z_2)=y_{12}(2j_X+y_{12}\partial_{y_2})X(z_2,y_2)$$
$$\{G(y_1)X(y_2)\}_{n\geq 2}(z_2)=0$$
$$\{\tilde{G}(y_1)X(y_2)\}_{n\geq 2}(z_2)=0$$

$$\mathbb{J}=\{J,G,\tilde{G},T\}$$

$$\mathbb{X}=\{X,G_X,\tilde{G}_X,T_X\}$$

$$J\times X=2j_X X, J\times\tilde{G}_X=(2j_X-1)\tilde{G}_X,$$
$$G\times X=G_X, G\times\tilde{G}_X=2j_X X-T_X,$$
$$\tilde{G}\times X=\tilde{G}_X, \tilde{G}\times\tilde{G}_X=0,$$
$$T\times X=j_X X, T\times\tilde{G}_X=\left(j_X+\frac{1}{2}\right)\tilde{G}_X,$$
$$J\times G_X=(2j_X-1)G_X, J\times T_X=(2j_X-1)X+(2j_X-2)T_X,$$
$$G\times G_X=0, G\times T_X=\left(2j_X-\frac{1}{2j_X}\right)G_X,$$
$$\tilde{G}\times G_X=-2j_X X+T_X, \tilde{G}\times T_X=\left(2j_X-\frac{1}{2j_X}\right)\tilde{G}_X,$$
$$T\times G_X=\left(j_X+\frac{1}{2}\right)G_X, T\times T_X=(j_X+1)T_X,$$

$$\mathbb{X}=\left\{X,X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]},X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}\dots,X_{[2,0,0]}\right\}$$

$$J\times X=2j_X X$$
$$G\times X=X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}+X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}$$
$$\tilde{G}\times X=X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}+X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}$$
$$T\times X=h_X X$$

$$J \times X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} = 2\left(j_X - \frac{1}{2}\right) X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]},$$
$$G \times X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} = -\frac{2j_X}{2j_X + 1} X_{[1,0,1]},$$
$$\tilde{G} \times X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} = -X_{[1,-1,0]} - X_{[1,0,0]} + \frac{1}{2j_X + 1} X'_{[1,1,0]} - \frac{2j_X(h_X + j_X + 1)}{2j_X + 1} X,$$
$$T \times X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} = \left(h_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]},$$
$$J \times X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} = 2\left(j_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]},$$
$$G \times X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} = X_{[1,0,1]},$$
$$\tilde{G} \times X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} = -X_{[1,1,0]} + X'_{[1,1,0]} + (h_X - j_X)X,$$
$$T \times X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} = \left(h_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]},$$
$$J \times X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} = 2\left(j_X - \frac{1}{2}\right) X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]},$$
$$G \times X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} = X_{[1,-1,0]} + \frac{1}{2j_X + 1} X_{[1,0,0]} - X'_{[1,1,0]} + \frac{2j_X(h_X + j_X + 1)}{2j_X + 1} X,$$
$$\tilde{G} \times X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} = -\frac{2j_X}{2j_X + 1} X_{[1,0,-1]},$$
$$T \times X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} = \left(h_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]},$$
$$J \times X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} = 2\left(j_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]},$$
$$G \times X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} = X_{[1,0,0]} + X_{[1,1,0]} + (j_X - h_X)X,$$
$$\tilde{G} \times X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} = X_{[1,0,-1]},$$
$$T \times X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} = \left(h_X + \frac{1}{2}\right) X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]}$$
$$J \times X_{[1,-1,0]} = 2(j_X - 1)X_{[1,-1,0]} - \frac{(2j_X - 1)(h_X + j_X + 1)}{2j_X + 1} X,$$
$$G \times X_{[1,-1,0]} = -\frac{(2j_X - 1)(h_X + j_X + 1)}{2j_X} X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} - \frac{2j_X - 1}{2j_X} X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]},$$
$$\tilde{G} \times X_{[1,-1,0]} = \frac{2j_X - 1}{2j_X} X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} - \frac{(2j_X - 1)(h_X + j_X + 1)}{2j_X} X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]},$$
$$T \times X_{[1,-1,0]} = (h_X + 1)X_{[1,-1,0]},$$
$$J \times X_{[1,0,1]} = 2j_X X_{[1,0,1]},$$
$$G \times X_{[1,0,1]} = 0,$$
$$\tilde{G} \times X_{[1,0,1]} = (j_X - h_X)X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} - (h_X + j_X + 1)X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} + X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} + X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]},$$
$$T \times X_{[1,0,1]} = (h_X + 1)X_{[1,0,1]},$$
$$J \times X_{[1,0,0]} = 2j_X X_{[1,0,0]} + \frac{j_X(h_X - j_X)(h_X + j_X + 1)}{h_X(j_X + 1)} X,$$
$$G \times X_{[1,0,0]} = \frac{(2h_X + 1)(h_X - j_X)}{2h_X} X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} + \frac{(h_X + j_X + 1)(h_X - j_X)}{2h_X(j_X + 1)} X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} - \frac{2j_X + 1}{2(j_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]},$$
$$\tilde{G} \times X_{[1,0,0]} = -X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} - \frac{1}{2(j_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} + \frac{h_X - j_X}{2h_X} X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} - \frac{(h_X + j_X + 1)(2h_X j_X + h_X + j_X)}{2h_X(j_X + 1)} X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]},$$
$$T \times X_{[1,0,0]} = (h_X + 1)X_{[1,0,0]},$$

$$J \times X'_{[1,1,0]} = 2j_X X'_{[1,1,0]} - \frac{j_X(h_X - j_X)(h_X + j_X + 1)}{h_X(j_X + 1)} X,$$

$$G \times X'_{[1,1,0]} = -\frac{(h_X - j_X)}{2h_X} X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} + \frac{(h_X + j_X + 1)(2h_X j_X + h_X + j_X)}{2h_X(j_X + 1)} X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} - X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} - \frac{1}{2(j_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]},$$

$$\tilde{G} \times X'_{[1,1,0]} = -\frac{2j_X + 1}{2(j_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} - \frac{(2h_X + 1)(h_X - j_X)}{2h_X} X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} - \frac{(h_X + j_X + 1)(h_X - j_X)}{2h_X(j_X + 1)} X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]},$$

$$T \times X'_{[1,1,0]} = (h_X + 1) X'_{[1,1,0]},$$

$$J \times X_{[1,0,-1]} = 2j_X X_{[1,0,-1]},$$

$$G \times X_{[1,0,-1]} = X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} + X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} + (h_X - j_X) X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} + (h_X + j_X + 1) X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]},$$

$$\tilde{G} \times X_{[1,0,-1]} = 0,$$

$$T \times X_{[1,0,-1]} = (h_X + 1) X_{[1,0,-1]},$$

$$J \times X_{[1,1,0]} = 2(j_X + 1) X_{[1,1,0]} + (j_X - h_X) X,$$

$$G \times X_{[1,1,0]} = (h_X - j_X) X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} + X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]},$$

$$\tilde{G} \times X_{[1,1,0]} = (h_X - j_X) X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} - X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]},$$

$$T \times X_{[1,1,0]} = (h_X + 1) X_{[1,1,0]},$$

$$J \times X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} = 2\left(j_X - \frac{1}{2}\right) X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} - \frac{2j_X(h_X + j_X + 1)}{2j_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]}$$
$$- \frac{2(2j_X - 1)(h_X - j_X)(h_X + j_X + 1)}{(2h_X + 1)(2j_X + 1)} X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]},$$

$$G \times X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} = -\frac{2(h_X + 1)(h_X - j_X)}{2h_X + 1} X_{[1,-1,0]} + \frac{2j_X}{2j_X + 1} X_{[2,0,0]} - \frac{2(h_X + j_X + 1)(2h_X j_X + h_X + j_X)}{(2h_X + 1)(2j_X + 1)} X_{[1,0,0]}$$
$$+ \frac{2(h_X + j_X + 1)(h_X - j_X)}{(2h_X + 1)(2j_X + 1)} X'_{[1,1,0]} - \frac{4j_X(h_X + j_X + 1)(h_X - j_X)}{(2h_X + 1)(2j_X + 1)} X,$$

$$\tilde{G} \times X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} = -\frac{4(h_X + 1)j_X(h_X + j_X + 1)}{(2h_X + 1)(2j_X + 1)} X_{[1,0,-1]},$$

$$T \times X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} = \left(h_X + \frac{3}{2}\right) X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]},$$

$$J \times X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} = 2\left(j_X - \frac{1}{2}\right) X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} + \frac{2j_X(h_X + j_X + 1)}{2j_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]}$$
$$+ \frac{2(2j_X - 1)(h_X - j_X)(h_X + j_X + 1)}{(2h_X + 1)(2j_X + 1)} X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]},$$

$$G \times X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} = \frac{4(h_X + 1)j_X(h_X + j_X + 1)}{(2h_X + 1)(2j_X + 1)} X_{[1,0,1]},$$

$$\tilde{G} \times X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} = -\frac{2(h_X + 1)(h_X - j_X)}{2h_X + 1} X_{[1,-1,0]} + \frac{2j_X}{2j_X + 1} X_{[2,0,0]} - \frac{2(h_X + j_X + 1)(h_X - j_X)}{(2h_X + 1)(2j_X + 1)} X_{[1,0,0]}$$
$$+ \frac{2(h_X + j_X + 1)(2h_X j_X + h_X + j_X)}{(2h_X + 1)(2j_X + 1)} X'_{[1,1,0]} - \frac{4j_X(h_X + j_X + 1)(h_X - j_X)}{(2h_X + 1)(2j_X + 1)} X,$$

$$T \times X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} = \left(h_X + \frac{3}{2}\right) X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]},$$

$$J \times X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} = 2\left(j_X + \frac{1}{2}\right) X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} + (h_X - j_X) X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} + \frac{2(h_X + j_X + 1)(h_X - j_X)}{2h_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]},$$

$$G \times X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} = -X_{[2,0,0]} + \frac{(h_X - j_X)}{2h_X + 1} X_{[1,0,0]} - \frac{2(h_X + 1)(h_X + j_X + 1)}{2h_X + 1} X_{[1,1,0]}$$
$$+ (h_X - j_X) X'_{[1,1,0]} + \frac{2(h_X - j_X)(h_X + j_X + 1)}{2h_X + 1} X,$$

$$\tilde{G} \times X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} = \frac{2(h_X + 1)(h_X - j_X)}{2h_X + 1} X_{[1,0,-1]},$$

$$T \times X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} = \left(h_X + \frac{3}{2}\right) X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]},$$

$$
\begin{aligned}
J \times X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]} &= (j_X - h_X) X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} - \frac{2(h_X - j_X)(h_X + j_X + 1)}{2h_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} + 2\left(j_X + \frac{1}{2}\right) X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]}, \\
G \times X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]} &= -\frac{2(h_X + 1)(h_X - j_X)}{2h_X + 1} X_{[1,0,1]}, \\
\tilde{G} \times X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]} &= -X_{[2,0,0]} + (j_X - h_X) X_{[1,0,0]} - \frac{2(h_X + 1)(h_X + j_X + 1)}{2h_X + 1} X_{[1,1,0]} \\
&\quad - \frac{h_X - j_X}{2h_X + 1} X'_{[1,1,0]} + \frac{2(h_X - j_X)(h_X + j_X + 1)}{2h_X + 1} X, \\
T \times X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]} &= \left(h_X + \frac{3}{2}\right) X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]}, \\
J \times X_{[2,0,0]} &= 2j_X X_{[2,0,0]} + (j_X - h_X) X_{[1,-1,0]} - \frac{(h_X - j_X)(h_X + j_X + 1)}{h_X + 1} X_{[1,0,0]} \\
&\quad - (h_X + j_X + 1) X_{[1,1,0]} + \frac{(h_X - j_X)(h_X + j_X + 1)}{h_X + 1} X'_{[1,1,0]} + \frac{2j_X (h_X - j_X)(h_X + j_X + 1)}{(h_X + 1)(2h_X + 1)} X, \\
G \times X_{[2,0,0]} &= \frac{2(h_X + j_X + 1)(h_X - j_X)}{2h_X + 1} X_{\left[\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]} + \frac{2(h_X + j_X + 1)(h_X - j_X)}{2h_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]} \\
&\quad - \frac{(2h_X + 3)(h_X - j_X)}{2(h_X + 1)} X_{\left[\frac{3}{2},-\frac{1}{2},\frac{1}{2}\right]} - \frac{(2h_X + 3)(h_X + j_X + 1)}{2(h_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]}, \\
\tilde{G} \times X_{[2,0,0]} &= \frac{(2h_X + 3)(h_X - j_X)}{2(h_X + 1)} X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} + \frac{(2h_X + 3)(h_X + j_X + 1)}{2(h_X + 1)} X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} \\
&\quad + \frac{2(h_X + j_X + 1)(h_X - j_X)}{2h_X + 1} X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} + \frac{2(h_X + j_X + 1)(h_X - j_X)}{2h_X + 1} X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]}, \\
T \times X_{[2,0,0]} &= (h_X + 2) X_{[2,0,0]} - \frac{3(h_X - j_X)(h_X + j_X + 1)}{2h_X + 1} X,
\end{aligned}
$$

$$
\langle A(z_1, y_1) B(z_2, y_2) \rangle = \langle AB \rangle \frac{y_{12}^{2j}}{z_{12}^{2h}}
$$

$$
\begin{aligned}
\langle G_X \tilde{G}_Y \rangle &= 2h(-1)^{|X|} \langle XY \rangle \\
\langle \tilde{G}_X G_Y \rangle &= -2h(-1)^{|X|} \langle XY \rangle \\
\langle T_X T_Y \rangle &= -(2h - 1)(2h + 1) \langle XY \rangle
\end{aligned}
$$

$$\left\langle X_{\left[\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right]} Y_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} \right\rangle = \frac{2j(h+j+1)}{2j+1}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]} Y_{\left[\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right]} \right\rangle = -\frac{2j(h+j+1)}{2j+1}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{1}{2},\frac{1}{2},+\frac{1}{2}\right]} Y_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} \right\rangle = (h-j)(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]} Y_{\left[\frac{1}{2},\frac{1}{2},+\frac{1}{2}\right]} \right\rangle = -(h-j)(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{[1,-1,0]} Y_{[1,-1,0]} \right\rangle = -\frac{(2j-1)(h+j+1)^2}{2j+1}\langle XY\rangle$$

$$\left\langle X_{[1,1,0]} Y_{[1,1,0]} \right\rangle = -(h-j)^2\langle XY\rangle$$

$$\left\langle X_{[1,0,+1]} Y_{[1,0,-1]} \right\rangle = -(h-j)(h+j+1)\langle XY\rangle$$

$$\left\langle X_{[1,0,-1]} Y_{[1,0,+1]} \right\rangle = -(h-j)(h+j+1)\langle XY\rangle$$

$$\begin{matrix} \left\langle X_{[1,0,0]} Y_{[1,0,0]} \right\rangle \left\langle X_{[1,0,0]} Y'_{[1,0,0]} \right\rangle \\ \left\langle X'_{[1,0,0]} Y_{[1,0,0]} \right\rangle \left\langle X'_{[1,0,0]} Y'_{[1,0,0]} \right\rangle \end{matrix} = \begin{pmatrix} M_{11} M_{12} \\ M_{21} M_{22} \end{pmatrix} \langle XY\rangle$$

$$\left\langle X_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} Y_{\left[\frac{3}{2},-\frac{1}{2},+\frac{1}{2}\right]} \right\rangle = -\frac{4j(h+1)(h-j)(h+j+1)^2}{(2h+1)(2j+1)}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{3}{2},-\frac{1}{2},+\frac{1}{2}\right]} Y_{\left[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\right]} \right\rangle = \frac{4j(h+1)(h-j)(h+j+1)^2}{(2h+1)(2j+1)}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{3}{2},\frac{1}{2},+\frac{1}{2}\right]} Y_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} \right\rangle = \frac{2(h+1)(h+j+1)(h-j)^2}{2h+1}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{\left[\frac{3}{2},\frac{1}{2},-\frac{1}{2}\right]} Y_{\left[\frac{3}{2},\frac{1}{2},+\frac{1}{2}\right]} \right\rangle = -\frac{2(h+1)(h+j+1)(h-j)^2}{2h+1}(-1)^{|X|}\langle XY\rangle$$

$$\left\langle X_{[2,0,0]} Y_{[2,0,0]} \right\rangle = \frac{(2h+3)(h-j)^2(h+j+1)^2}{2h+1}\langle XY\rangle$$

$$\langle X_{[\frac{1}{2},-\frac{1}{2},+\frac{1}{2}]}\, Y_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}\rangle = \frac{2j(h+j+1)}{2j+1}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}\, Y_{[\frac{1}{2},-\frac{1}{2},+\frac{1}{2}]}\rangle = -\frac{2j(h+j+1)}{2j+1}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{1}{2},\frac{1}{2},+\frac{1}{2}]}\, Y_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}\rangle = (h-j)(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}\, Y_{[\frac{1}{2},\frac{1}{2},+\frac{1}{2}]}\rangle = -(h-j)(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[1,-1,0]}\, Y_{[1,-1,0]}\rangle = -\frac{(2j-1)(h+j+1)^2}{2j+1}\langle XY\rangle\ ,$$

$$\langle X_{[1,1,0]}\, Y_{[1,1,0]}\rangle = -(h-j)^2\langle XY\rangle\ ,$$

$$\langle X_{[1,0,+1]}\, Y_{[1,0,-1]}\rangle = -(h-j)(h+j+1)\langle XY\rangle\ ,$$

$$\langle X_{[1,0,-1]}\, Y_{[1,0,+1]}\rangle = -(h-j)(h+j+1)\langle XY\rangle\ ,$$

$$\begin{pmatrix}\langle X_{[1,0,0]}\, Y_{[1,0,0]}\rangle & \langle X_{[1,0,0]}\, Y'_{[1,0,0]}\rangle \\ \langle X'_{[1,0,0]}\, Y_{[1,0,0]}\rangle & \langle X'_{[1,0,0]}\, Y'_{[1,0,0]}\rangle\end{pmatrix} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]}\, Y_{[\frac{3}{2},-\frac{1}{2},+\frac{1}{2}]}\rangle = -\frac{4j(h+1)(h-j)(h+j+1)^2}{(2h+1)(2j+1)}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{3}{2},-\frac{1}{2},+\frac{1}{2}]}\, Y_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]}\rangle = \frac{4j(h+1)(h-j)(h+j+1)^2}{(2h+1)(2j+1)}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{3}{2},\frac{1}{2},+\frac{1}{2}]}\, Y_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}\rangle = \frac{2(h+1)(h+j+1)(h-j)^2}{2h+1}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}\, Y_{[\frac{3}{2},\frac{1}{2},+\frac{1}{2}]}\rangle = -\frac{2(h+1)(h+j+1)(h-j)^2}{2h+1}(-1)^{|X|}\langle XY\rangle\ ,$$

$$\langle X_{[2,0,0]}\, Y_{[2,0,0]}\rangle = \frac{(2h+3)(h-j)^2(h+j+1)^2}{2h+1}\langle XY\rangle\ ,$$

$$M_{11} = M_{22} = \frac{(h+j+1)(h-j)^2}{2h(j+1)}$$

$$M_{12} = M_{21} = \frac{(h-j)(h+j+1)(h+j+2hj)}{2h(j+1)}$$

$$\frac{1}{2}(p_2-p_1) \le h \le \frac{1}{2}(p_1+p_2) - 1,\, h - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z}.$$

$$h \le \frac{1}{2}(p_1+p_2) - 1,\, h - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z},$$

$$\frac{1}{2}(p_2-p_1) \le j \le \frac{1}{2}(p_1+p_2) - 1,\, j - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z}.$$

$$0 \le h \le \frac{1}{2}(p_1+p_2) - 1,\, h - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z}.$$

$$h \le \frac{1}{2}(p_1+p_2) - 1,\, h - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z}$$

$$\frac{1}{2}(p_2-p_1) \le j \le \frac{1}{2}(p_1+p_2) - 1,\, j - \frac{1}{2}(p_1+p_2) = 0 \bmod \mathbb{Z}$$

$$h - j = 0 \bmod 2$$

$$W_6' = W_6 + \mu_6 C_{3,3}^{W_3 W_3}$$

$$W_7' = W_7 + \mu_7 C_{\frac{7}{2},\frac{7}{2}}^{W_3 W_4}$$

$$\langle W_{p_1}(z_1,y_1)W_{p_2}(z_2,y_2)W_{p_3}(z_3,y_3)\rangle = \lambda_{p_1p_2p_3}\frac{y_{12}^{\frac{1}{2}(p_1+p_2-p_3)}y_{23}^{\frac{1}{2}(p_2+p_3-p_1)}y_{13}^{\frac{1}{2}(p_1+p_3-p_2)}}{z_{12}^{\frac{1}{2}(p_1+p_2-p_3)}z_{23}^{\frac{1}{2}(p_2+p_3-p_1)}z_{13}^{\frac{1}{2}(p_1+p_3-p_2)}},$$

$$\lambda_{p_1p_2p_3} = \sum_{\mathcal{O}} c_{p_1p_2}^{\mathcal{O}} g_{\mathcal{O}p_3}$$

$$\langle \mathcal{O}(z_1,y_1)W_{p_2}(z_2,y_2)\rangle = g_{\mathcal{O}p_3}\frac{y_{12}^{p_3}}{z_{12}^{p_3}}.$$

$$\sum_{\mathcal{O}} c_{33}{}^{\mathcal{O}} g_{\mathcal{O}4} = \sum_{\mathcal{O}'} c_{34}{}^{\mathcal{O}'} g_{\mathcal{O}'3}$$

$$c_{33}{}^{4} g_4 = c_{34}{}^{3} g_3.$$

$$\sum_{\mathcal{O}} c_{36}{}^{\mathcal{O}} g_{\mathcal{O}5} = \sum_{\mathcal{O}'} c_{35}{}^{\mathcal{O}} g_{\mathcal{O}6}, \sum_{\mathcal{O}} c_{37}{}^{\mathcal{O}} g_{\mathcal{O}6} = \sum_{\mathcal{O}'} c_{36}{}^{\mathcal{O}} g_{\mathcal{O}7}$$

$$c_{36}{}^{5} g_5 = c_{35}{}^{6} g_6$$

$$c_{37}{}^{6} g_6 = c_{36}{}^{7} g_7$$

OPE coefficients in $W_3 \times W_3$ ***and*** $W_3 \times W_4$

$$(c_{33}{}^{4})^2 = \frac{36(\nu-9)\nu}{(\nu-4)(\nu-1)(\nu+1)}\frac{(g_3)^2}{g_4} \ ,$$

$$c_{34}{}^{3} = \frac{g_4}{g_3} c_{33}{}^{4} \ ,$$

$$(c_{34}{}^{5})^2 = \frac{60(\nu-16)(\nu+1)}{(\nu-4)(\nu-1)(\nu+5)}\frac{g_3 g_4}{g_5}$$

OPE coefficients in $W_3 \times W_5$
$c_{35}{}^{4} = \frac{g_5}{g_4} c_{34}{}^{5}$, $(c_{35}{}^{6})^2 = \frac{90(\nu-25)\nu(\nu+5)}{(\nu-4)(\nu-1)(\nu^2+15\nu+8)} \frac{g_3 g_5}{g_6}$, $c_{35}{}^{c_{3,1}^{W_3W_3}} = -\frac{(\nu-4)(\nu-2)(\nu-1)}{6(\nu-16)(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2}$, $c_{35}{}^{c_{3,3}^{W_3W_3}} = \frac{(\nu-4)(\nu-1)(\nu+3)}{4\nu(\nu^2+15\nu+8)} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2}$

OPE coefficients in $W_4 \times W_4$
$c_{44}{}^{4} = \frac{4(\nu^2-20\nu+9)}{3(\nu-9)\nu} \frac{c_{33}{}^{4} g_4}{g_3}$, $c_{44}{}^{6} = \frac{(\nu-4)(\nu-1)(\nu+1)}{45(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} c_{35}{}^{6} g_4}{(g_3)^2}$, $c_{4,4}{}^{c_{3,1}^{W_3W_3}} = -\frac{12(\nu+1)}{(\nu-9)\nu} \frac{g_4}{g_3}$, $c_{4,4}{}^{c_{3,3}^{W_3W_3}} = \frac{24(\nu-4)(\nu+1)}{\nu(\nu^2+15\nu+8)} \frac{g_4}{g_3}$

OPE coefficients in $W_4 \times W_5$

$$c_{45}{}^{3} = \frac{g_5}{g_3} c_{34}{}^{5} \ ,$$

$$c_{45}{}^{5} = \frac{5\left(\nu^3 - 32\nu^2 - 77\nu + 36\right)}{3(\nu-9)\nu(\nu+5)} \frac{c_{33}{}^{4} g_4}{g_3} \ ,$$

$$(c_{45}{}^{7})^2 = \frac{140(\nu-36)(\nu-25)(\nu+1)(\nu+5)}{(\nu-9)(\nu-4)(\nu-1)\left(\nu^2+35\nu+84\right)} \frac{g_4 g_5}{g_7} \ ,$$

$$c_{45}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{1}{2}}} = -\frac{(\nu-4)(\nu-1)(\nu+12)}{6(\nu-16)(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2} \ ,$$

$$c_{45}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{3}{2}}} = -\frac{(\nu-4)(\nu-1)(2\nu+3)}{3(\nu-16)(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2} \ ,$$

$$c_{45}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{5}{2}}} = \frac{(\nu-4)(\nu-1)}{6(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2} \ ,$$

$$c_{45}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{7}{2}}} = \frac{2(\nu-4)(\nu-1)\left(2\nu^2 - 7\nu - 63\right)}{3(\nu-9)\nu\left(\nu^2+35\nu+84\right)} \frac{c_{33}{}^{4} c_{34}{}^{5} g_5}{(g_3)^2}$$

OPE coefficients in $W_3 \times W_6$

$$c_{36}{}^{3} = 0 \ ,$$

$$c_{36}{}^{5} = \frac{g_6}{g_5} c_{35}{}^{6} \ ,$$

$$c_{36}{}^{7} = \frac{(\nu-4)(\nu-1)\left(\nu^2+15\nu+8\right)}{60(\nu-25)\nu(\nu+5)} \frac{c_{33}{}^{4} c_{35}{}^{6} c_{45}{}^{7} g_6}{g_3 g_5} \ ,$$

$$c_{36}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{3}{2}}} = -\frac{7(\nu-4)(\nu-1)^2}{30(\nu-25)(\nu-16)(\nu+1)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{g_3 g_4} \ ,$$

$$c_{36}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{5}{2}}} = \frac{(\nu-4)(\nu-1)}{30\nu(\nu+1)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{g_3 g_4} \ ,$$

$$c_{36}{}^{c^{W_3 W_4}_{\frac{7}{2},\frac{7}{2}}} = \frac{2(\nu-4)(\nu-1)(\nu+7)}{5(\nu+1)\left(\nu^2+35\nu+84\right)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{g_3 g_4}$$

OPE coefficients in $W_3 \times W_7$ (partial)

$$c_{37}{}^{4} = 0 \,,$$

$$c_{37}{}^{6} = \frac{(\nu-4)(\nu-1)\left(\nu^2+15\nu+8\right)}{60(\nu-25)\nu(\nu+5)} \frac{c_{33}{}^{4} c_{35}{}^{6} c_{45}{}^{7} g_7}{g_3 g_5} \,,$$

$$c_{37}{}^{C^{W_3W_3}_{3,3}} = 0 \,,$$

$$c_{37}{}^{C^{W_3W_3}_{4,2}} = -\frac{8(\nu-4)(\nu-1)\left(\nu^5+42\nu^4+264\nu^3-253\nu^2+17226\nu+9720\right)}{15(\nu-36)(\nu-25)(\nu-16)\nu(\nu+1)(\nu+5)\left(\nu^2+15\nu+8\right)} \frac{c_{34}{}^{5} c_{45}{}^{7} g_7}{g_3 g_4} \,,$$

$$c_{37}{}^{C^{W_3W_5}_{4,2}} = -\frac{2(\nu-4)(\nu-1)\left(2\nu^2+13\nu-90\right) g_7}{15(\nu-36)(\nu-25)\nu(\nu+5)} \frac{c_{33}{}^{4} c_{45}{}^{7}}{g_3 g_5} \,,$$

$$c_{37}{}^{C^{W_4W_4}_{4,2}} = -\frac{(\nu-9)(\nu-4)(\nu-1)\left(\nu^2-\nu+60\right)}{10(\nu-36)(\nu-25)(\nu-16)(\nu+1)^2} \frac{c_{34}{}^{5} c_{45}{}^{7} g_7}{(g_4)^2} \,,$$

$$c_{37}{}^{C^{W_3W_5}_{4,3}} = \frac{(\nu-4)(\nu-1)}{15\nu(\nu+5)} \frac{c_{33}{}^{4} c_{45}{}^{7} g_7}{g_3 g_5}$$

OPE coefficients in $W_4 \times W_6$ (partial)

$$c_{46}{}^{4} = \frac{(\nu-4)(\nu-1)(\nu+1)}{45(\nu-9)\nu} \frac{c_{33}{}^{4} c_{34}{}^{5} c_{35}{}^{6} g_6}{(g_3)^2} \,,$$

$$c_{46}{}^{6} = \frac{(2\nu-1)\left(\nu^3-46\nu^2-387\nu-240\right)}{(\nu-9)\nu\left(\nu^2+15\nu+8\right)} \frac{c_{33}{}^{4} g_4}{g_3} \,,$$

$$c_{46}{}^{C^{W_3W_3}_{3,1}} = -\frac{7(\nu-4)(\nu-2)(\nu-1)}{45(\nu-16)(\nu-9)\nu} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{(g_3)^2} \,,$$

$$c_{46}{}^{C^{W_3W_3}_{3,3}} = \frac{3(\nu-4)(\nu-1)(\nu+3)}{10\nu\left(\nu^2+15\nu+8\right)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{(g_3)^2} \,,$$

$$c_{46}{}^{C^{W_3W_5}_{4,1}} = -\frac{7(\nu-4)(\nu-1)(\nu+1)(\nu+23)}{30(\nu-25)(\nu-9)\nu(\nu+5)} \frac{c_{33}{}^{4} c_{35}{}^{6} g_4 g_6}{(g_3)^2 g_5} \,,$$

$$c_{46}{}^{C^{W_3W_3}_{4,2}} = \frac{2(\nu-4)(\nu-1)\left(5\nu^7-499\nu^6-9539\nu^5+119535\nu^4+1843758\nu^3+1311460\nu^2-32542800\nu-18144000\right)}{525(\nu-25)(\nu-16)(\nu-15)(\nu-9)\nu^2(\nu+5)\left(\nu^2+15\nu+8\right)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{(g_3)^2} \,,$$

$$c_{46}{}^{C^{W_3W_5}_{4,2}} = -\frac{(\nu-4)(\nu-1)(\nu+1)\left(19\nu^2+117\nu+320\right)}{30(\nu-25)(\nu-9)\nu^2(\nu+5)} \frac{c_{33}{}^{4} c_{35}{}^{6} g_4 g_6}{(g_3)^2 g_5} \,,$$

$$c_{46}{}^{C^{W_4W_4}_{4,2}} = -\frac{4(\nu-4)^2(\nu-1)(\nu+5)}{15(\nu-25)(\nu-16)\nu(\nu+1)} \frac{c_{34}{}^{5} c_{35}{}^{6} g_6}{g_3 g_4} \,,$$

$$c_{46}{}^{C^{W_3W_5}_{4,3}} = \frac{4(\nu-4)(\nu-2)(\nu-1)(\nu+1)}{15(\nu-9)\nu^2(\nu+5)} \frac{c_{33}{}^{4} c_{35}{}^{6} g_4 g_6}{(g_3)^2 g_5}$$

OPE coefficients in $W_5 \times W_5$ (partial)

$$c_{55}{}^{4} = \frac{5\left(\nu^3-32\nu^2-77\nu+36\right)}{3(\nu-9)\nu(\nu+5)}\,\frac{c_{33}{}^{4}g_5}{g_3}\,,$$

$$c_{55}{}^{6} = \frac{(\nu-4)(\nu-1)\left(\nu^2-59\nu+16\right)}{36(\nu-16)(\nu-9)\nu}\,\frac{c_{33}{}^{4}c_{34}{}^{5}c_{35}{}^{6}g_5}{(g_3)^2}\,,$$

$$c_{55}{}^{C^{W_3W_3}_{3,1}} = -\frac{10\left(5\nu^3-103\nu^2-460\nu-864\right)}{3(\nu-16)(\nu-9)\nu(\nu+5)}\,\frac{g_5}{g_3}\,,$$

$$c_{55}{}^{C^{W_3W_3}_{3,3}} = \frac{15\left(3\nu^3-24\nu^2-115\nu+64\right)}{\nu(\nu+5)\left(\nu^2+15\nu+8\right)}\,\frac{g_5}{g_3}\,,$$

$$c_{55}{}^{C^{W_3W_3}_{4,0}} = -\frac{8\left(\nu^3-13\nu^2+210\nu-558\right)}{(\nu-16)(\nu-9)(\nu-7)\nu}\,\frac{g_5}{g_3}\,,$$

$$c_{55}{}^{C^{W_4W_4}_{4,0}} = -\frac{20(\nu+5)}{(\nu-16)(\nu+1)}\,\frac{g_5}{g_4}\,,$$

$$c_{55}{}^{C^{W_3W_3}_{4,2}} = \frac{8\left(5\nu^7-226\nu^6-2904\nu^5+46954\nu^4+338011\nu^3+1286280\nu^2+7747320\nu+3959280\right)}{7(\nu-16)(\nu-15)(\nu-9)\nu(\nu+5)^2\left(\nu^2+15\nu+8\right)}\,\frac{g_5}{g_3}\,,$$

$$c_{55}{}^{C^{W_3W_5}_{4,2}} = -\frac{(\nu-4)(\nu-2)(\nu-1)}{(\nu-16)(\nu-9)\nu}\,\frac{c_{33}{}^{4}c_{34}{}^{5}g_5}{(g_3)^2}\,,$$

$$c_{55}{}^{C^{W_4W_4}_{4,2}} = -\frac{20(\nu+5)}{(\nu-16)(\nu+1)}\,\frac{g_5}{g_4}$$

Constraints among undetermined OPE coefficients
$c_{37}{}^{c^{W_3 W_5}_{4,4}} = \frac{(\nu-9)(\nu-4)(\nu-1)\left(\nu^2+35\nu+84\right)}{100(\nu-36)(\nu-25)(\nu+1)(\nu+5)} \frac{c_{34}{}^5 c_{45}{}^7 g_7}{g_4 g_5} c_{55}{}^{c^{W_3 W_5}_{4,4}} - \frac{(\nu-4)(\nu-1)\left(\nu^3-42\nu^2+29\nu+396\right)}{2(\nu-36)(\nu-25)\nu(\nu+5)^2} \frac{c_{33}{}^4 c_{45}{}^7 g_7}{g_3 g_5}$,
$c_{37}{}^{c^{W_4 W_4}_{4,4}} = \frac{(\nu-9)(\nu-4)(\nu-1)\left(\nu^2+35\nu+84\right)}{100(\nu-36)(\nu-25)(\nu+1)(\nu+5)} \frac{c_{34}{}^5 c_{45}{}^7 g_7}{g_4 g_5} c_{55}{}^{c^{W_3 W_5}_{4,4}} - \frac{2(\nu-9)(\nu-4)(\nu-3)(\nu-1)}{5(\nu-36)(\nu-25)(\nu+1)^2} \frac{c_{34}{}^5 c_{45}{}^7 g_7}{(g_4)^2}$,
$c_{46}{}^8 = \frac{(\nu-4)(\nu-1)\left(\nu^2+15\nu+8\right)}{105(\nu-25)\nu(\nu+5)} \frac{c_{35}{}^6 c_{45}{}^7 g_6}{g_3 g_5} c_{37}{}^8$,
$c_{46}{}^{c^{W_3 W_5}_{4,4}} = \frac{28(\nu-4)(\nu-1)(\nu+1)(\nu+2)}{3(\nu-25)\nu^2(\nu+5)^2} \frac{c_{33}{}^4 c_{35}{}^6 g_4 g_6}{(g_3)^2 g_5} + \frac{(\nu-4)(\nu-1)\left(\nu^2+15\nu+8\right)}{75(\nu-25)\nu(\nu+5)} \frac{c_{34}{}^5 c_{35}{}^6 g_6}{g_3 g_5} c_{55}{}^{c^{W_3 W_5}_{4,4}}$,
$c_{46}{}^{c^{W_4 W_4}_{4,4}} = \frac{(\nu-4)(\nu-1)\left(\nu^2+15\nu+8\right)}{75(\nu-25)\nu(\nu+5)} \frac{c_{34}{}^5 c_{35}{}^6 g_6}{g_3 g_5} c_{55}{}^{c^{W_4 W_4}_{4,4}} - \frac{4(\nu-4)(\nu-1)(\nu+3)}{15(\nu-25)\nu(\nu+1)} \frac{c_{34}{}^5 c_{35}{}^6 g_6}{g_3 g_4}$,
$c_{55}{}^8 = \frac{(\nu-4)(\nu-1)(\nu+5)}{84(\nu-16)(\nu+1)} \frac{c_{34}{}^5 c_{45}{}^7 g_5}{g_3 g_4} c_{37}{}^8$

$$\left(A_r \ldots \left(A_3\left(A_2(A_1 B)^{j_1}_{n_1}\right)^{j_2}_{n_2}\right)^{j_3}_{n_3} \ldots\right)^{j_r}_{n_r}, r \geq 0, A_i \in \mathbb{J}, B \in \mathbb{X}, n_i \leq 0$$

$$A \times B = \sum_{\substack{C \in \mathcal{L}_{X,h_{\max}} \\ h_C < h_A + h_B \\ j_C \in \{|j_A - j_B|, \ldots, j_A + j_B\} \\ r_C = r_A + r_B}} c_{AB}{}^C C, A \in \mathbb{W}_{p_1}, B \in \mathbb{W}_{p_2}$$

$$\text{Jacobi}(\mathbb{J}, \mathbb{W}_{p_1}, \mathbb{W}_{p_2})$$

$$W_{p_1} \times W_{p_2} = c_{p_1 p_2}{}^X X + \cdots$$

$$W_3 \times W_3 = g_3\left[\mathbb{1} - \frac{3J}{k} + \frac{(2k-1)T}{2k(2k+3)} - \frac{6(JJ)_0^0}{k(2k+3)} + \frac{3(JJ)_0^2}{(k-1)k}\right]$$

$$W_3 \times G_{W_3} = g_3\left[-\frac{3G}{2k} - \frac{6(JG)_0^{\frac{1}{2}}}{k(2k+3)} + \frac{3(JG)_0^{\frac{3}{2}}}{(k-1)k}\right]$$

$$W_3 \times \tilde{G}_{W_3} = g_3\left[-\frac{3\tilde{G}}{2k} - \frac{6(J\tilde{G})_0^{\frac{1}{2}}}{k(2k+3)} + \frac{3(J\tilde{G})_0^{\frac{3}{2}}}{(k-1)k}\right]$$

$$W_3 \times T_{W_3} = g_3\left[-\frac{2J}{k} + \frac{2(JJ)_0^2}{(k-1)k} - \frac{5(2k-1)(JJ)_{-1}^1}{2(k-1)k(2k+3)} - \frac{(4k+1)(G\tilde{G})_0^1}{(k-1)k(2k+3)} - \frac{(2k-7)(JT)_0^1}{(k-1)k(2k+3)}\right]$$

$$G_{W_3} \times G_{W_3} = g_3\left[-\frac{6(GG)_0^0}{k(2k+3)}\right]$$

$$G_{W_3} \times \tilde{G}_{W_3} = g_3\left[3\mathbb{1} - \frac{6J}{k} + \frac{4(k+1)T}{k(2k+3)} - \frac{6(JJ)_0^0}{k(2k+3)} + \frac{3(JJ)_0^2}{(k-1)k} - \frac{(2k-7)(JJ)_{-1}^1}{(k-1)k(2k+3)}\right.$$
$$\left. - \frac{6(G\tilde{G})_0^0}{k(2k+3)} + \frac{2(k+4)(G\tilde{G})_0^1}{(k-1)k(2k+3)} - \frac{2(4k+1)(JT)_0^1}{(k-1)k(2k+3)}\right]$$

$$G_{W_3} \times T_{W_3} = g_3\left[-\frac{4G}{k} - \frac{4(JG)_0^{\frac{1}{2}}}{k(2k+3)} + \frac{5(JG)_0^{\frac{3}{2}}}{(k-1)k} - \frac{6(GT)_0^{\frac{1}{2}}}{k(2k+3)} - \frac{9(GJ)_{-1}^{\frac{1}{2}}}{k(2k+3)}\right]$$

$$\tilde{G}_{W_3} \times \tilde{G}_{W_3} = g_3\left[-\frac{6(\tilde{G}\tilde{G})_0^0}{k(2k+3)}\right]$$

$$\tilde{G}_{W_3} \times T_{W_3} = g_3\left[-\frac{4\tilde{G}}{k} - \frac{4(J\tilde{G})_0^{\frac{1}{2}}}{k(2k+3)} + \frac{5(J\tilde{G})_0^{\frac{3}{2}}}{(k-1)k} - \frac{6(\tilde{G}T)_0^{\frac{1}{2}}}{k(2k+3)} - \frac{9(J\tilde{G})_{-1}^{\frac{1}{2}}}{k(2k+3)}\right]$$

$$T_{W_3} \times T_{W_3} = g_3\left[-8\mathbb{1} + \frac{8J}{k} - \frac{4(10k+13)T}{3k(2k+3)} + \frac{8(JJ)_0^0}{k(2k+3)} - \frac{8(k+4)(JJ)_{-1}^1}{3(k-1)k(2k+3)}\right.$$
$$\left. + \frac{12(G\tilde{G})_{-1}^0}{k(2k+3)} - \frac{4(7k+13)(G\tilde{G})_0^1}{3(k-1)k(2k+3)} + \frac{40(k+1)(JT)_0^1}{3(k-1)k(2k+3)} + \frac{27(J(JJ)_{-1}^1)_0^0}{5k(2k+3)} - \frac{6(TT)_0^0}{k(2k+3)}\right]$$

$$W_3 \times W_4 = c_{34}{}^{X}\left[X - \frac{(k-1)T_X}{12(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{2(JX)_0^{\frac{5}{2}}}{k-3}\right]$$

$$\begin{aligned} W_3 \times G_{W_4} = c_{34}{}^{X} \Bigg[& \frac{2G_X}{3} - \frac{2(4k+1)(GX)_0^1}{3(k+4)(2k-1)} - \frac{2k(GX)_0^2}{(k-3)(2k-1)} + \frac{5(JG_X)_0^0}{3(k+4)} \\ & + \frac{(2k-7)(JG_X)_0^1}{3(k+4)(2k-1)} - \frac{2(4k-3)(JG_X)_0^2}{3(k-3)(2k-1)} \Bigg] \end{aligned}$$

$$\begin{aligned} W_3 \times \tilde{G}_{W_4} = c_{34}{}^{X} \Bigg[& \frac{2\tilde{G}_X}{3} - \frac{2(4k+1)(\tilde{G}X)_0^1}{3(k+4)(2k-1)} - \frac{2k(\tilde{G}X)_0^2}{(k-3)(2k-1)} + \frac{5\left(J\tilde{G}_X\right)_0^0}{3(k+4)} \\ & + \frac{(2k-7)\left(J\tilde{G}_X\right)_0^1}{3(k+4)(2k-1)} - \frac{2(4k-3)\left(J\tilde{G}_X\right)_0^2}{3(k-3)(2k-1)} \Bigg] \end{aligned}$$

$$\begin{aligned} W_3 \times T_{W_4} = c_{34}{}^{X} \Bigg[& X + \frac{5(3k+13)T_X}{48(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{12(k+4)} - \frac{(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{3(JX)_0^{\frac{5}{2}}}{2(k-3)} \\ & + \frac{(4k+1)\left(G\tilde{G}_X\right)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{(2k+3)(5k-1)\left(G\tilde{G}_X\right)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} - \frac{(4k+1)\left(\tilde{G}G_X\right)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} \\ & - \frac{(2k+3)(5k-1)\left(\tilde{G}G_X\right)_0^{\frac{3}{2}} + \frac{(38k^2-57k+39)(JX)_{-1}^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{10k(JX)_{-1}^{\frac{1}{2}}}{3(k+4)(2k-1)}}{2(k-3)(k+4)(2k-1)} \\ & - \frac{2(JX)_{-1}^{\frac{5}{2}} + \frac{2(k-1)(JT_X)_0^{\frac{1}{2}}}{3(2k-1)} - \frac{(k-1)(2k+15)(JT_X)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} + \frac{(2k^2-19k-3)(TX)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)}}{} \Bigg] \end{aligned}$$

$$G_{W_3} \times W_4 = c_{34}{}^{X}\left[\frac{G_X}{3} + \frac{2(k-1)(GX)_0^1}{(k+4)(2k-1)} - \frac{2(k-1)(GX)_0^2}{(k-3)(2k-1)} - \frac{(4k+1)(JG_X)_0^1}{3(k+4)(2k-1)} - \frac{4k(JG_X)_0^2}{3(k-3)(2k-1)}\right]$$

$$G_{W_3} \times G_{W_4} = c_{34}{}^{X}\left[\frac{5(GG_X)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(GG_X)_0^{\frac{3}{2}}}{3(2k-1)}\right]$$

$$\begin{aligned} G_{W_3} \times \tilde{G}_{W_4} = c_{34}{}^{X} \Bigg[& 2X - \frac{5(k+3)T_X}{12(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{2(JX)_0^{\frac{5}{2}}}{k-3} \\ & + \frac{(46k^2-419k+213)(JX)_{-1}^{\frac{3}{2}}}{36(k-3)(k+4)(2k-1)} + \frac{10(k-2)(JX)_{-1}^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{2(JX)_{-1}^{\frac{5}{2}}}{3(2k-1)} \\ & + \frac{2(k-1)\left(G\tilde{G}_X\right)_0^{\frac{1}{2}}}{(k+4)(2k-1)} - \frac{(k-1)(2k+15)\left(G\tilde{G}_X\right)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} + \frac{(2k^2+35k+3)\left(\tilde{G}G_X\right)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{(4k+1)\left(\tilde{G}G_X\right)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} \\ & + \frac{(4k+1)(JT_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{(2k+3)(5k-1)(JT_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{3(k-1)(2k+1)(TX)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} \Bigg] \\ & + \frac{5(JG_X)_{-1}^0}{3(k+4)} + \frac{2(k-1)(JG_X)_{-1}^1}{(k+4)(2k-1)} - \frac{2(JG_X)_{-1}^2}{3(2k-1)} - \frac{5(k+1)(GX)_0^1}{3(k+4)(2k-1)} - \frac{(7k-3)(GX)_0^2}{2(k-3)(2k-1)} \\ & + \frac{4(7k-2)(GX)_{-1}^1}{9(k+4)(2k-1)} - \frac{2(GX)_{-1}^2}{3(2k-1)} + \frac{5(GT_X)_0^0}{3(k+4)} + \frac{(2k-7)(GT_X)_0^1}{3(k+4)(2k-1)} + \frac{2(4k+1)(TG_X)_0^1}{3(k+4)(2k-1)} \Bigg] \end{aligned}$$

$$\tilde{G}_{W_3} \times W_4 = c_{34}X\left[\frac{\tilde{G}_X}{3} + \frac{2(k-1)(\tilde{G}X)_0^1}{(k+4)(2k-1)} - \frac{2(k-1)(\tilde{G}X)_0^2}{(k-3)(2k-1)} - \frac{(4k+1)\left(J\tilde{G}_X\right)_0^1}{3(k+4)(2k-1)} - \frac{4k\left(J\tilde{G}_X\right)_0^2}{3(k-3)(2k-1)}\right]$$

$$\tilde{G}_3 \times G_{W_4} = c_{34}X\left[-2X + \frac{5(k+3)T_X}{12(k+4)} - \frac{5(JX)_0^{\frac{1}{2}}}{3(k+4)} + \frac{2(JX)_0^{\frac{3}{2}}}{3(2k-1)} + \frac{2(JX)_0^{\frac{5}{2}}}{k-3}\right.$$

$$- \frac{(46k^2-419k+213)(JX)_{-1}^{\frac{3}{2}}}{36(k-3)(k+4)(2k-1)} - \frac{10(k-2)(JX)_{-1}^{\frac{1}{2}}}{9(k+4)(2k-1)} + \frac{2(JX)_{-1}^{\frac{5}{2}}}{3(2k-1)}$$

$$+ \frac{(2k^2+35k+3)\left(G\tilde{G}_X\right)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{(4k+1)\left(G\tilde{G}_X\right)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{2(k-1)\left(\tilde{G}G_X\right)_0^{\frac{1}{2}}}{(k+4)(2k-1)} - \frac{(k-1)(2k+15)\left(\tilde{G}G_X\right)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)}$$

$$\left. - \frac{(4k+1)(JT_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} - \frac{(2k+3)(5k-1)(JT_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} - \frac{3(k-1)(2k+1)(TX)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)}\right]$$

$$\tilde{G}_3 \times \tilde{G}_{W_4} = c_{34}X\left[\frac{5\left(\tilde{G}\tilde{G}_X\right)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2\left(\tilde{G}\tilde{G}_X\right)_0^{\frac{3}{2}}}{3(2k-1)}\right]$$

$$\tilde{G}_{W_3} \times T_{W_4} = c_{34}X\left[\frac{5\tilde{G}_X}{3} - \frac{5(k+1)(\tilde{G}X)_0^1}{3(k+4)(2k-1)} - \frac{(7k-3)(\tilde{G}X)_0^2}{2(k-3)(2k-1)} + \frac{4(7k-2)(\tilde{G}X)_{-1}^1}{9(k+4)(2k-1)} - \frac{2(\tilde{G}X)_{-1}^2}{3(2k-1)}\right.$$

$$+ \frac{5\left(J\tilde{G}_X\right)_0^0}{3(k+4)} - \frac{5\left(J\tilde{G}_X\right)_0^1}{2(k+4)(2k-1)} - \frac{(11k-6)\left(J\tilde{G}_X\right)_0^2}{3(k-3)(2k-1)} + \frac{2(k-1)\left(J\tilde{G}_X\right)_{-1}^1}{(k+4)(2k-1)} - \frac{2\left(J\tilde{G}_X\right)_{-1}^2}{3(2k-1)}$$

$$\left. + \frac{5\left(\tilde{G}T_X\right)_0^0}{3(k+4)} + \frac{(2k-7)\left(\tilde{G}T_X\right)_0^1}{3(k+4)(2k-1)} + \frac{10f\left(J\left(J\tilde{G}_X\right)_0^1\right)_0^0}{9(k+4)} + \frac{2(4k+1)\left(T\tilde{G}_X\right)_0^1}{3(k+4)(2k-1)}\right]$$

$$
\begin{aligned}
T_{W_3} \times W_4 = c_{34}X \Bigg[& \frac{2X}{3} - \frac{2(JX)_0^{\frac{3}{2}}}{9(2k-1)} - \frac{4(JX)_0^{\frac{5}{2}}}{3(k-3)} + \frac{(98k^2-247k+69)(JX)_{-1}^{\frac{3}{2}}}{36(k-3)(k+4)(2k-1)} - \frac{2(JX)_{-1}^{\frac{5}{2}}}{3(2k-1)} \\
& + \frac{(k-1)(2k+1)\left(G\tilde{G}_X\right)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} - \frac{(k-1)(2k+1)\left(\tilde{G}G_X\right)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} \\
& + \frac{(7k-2)(JG_X)_{-1}^{1}}{3(k+4)(2k-1)} - \frac{2(JG_X)_{-1}^{2}}{3(2k-1)}
\end{aligned}
$$

$$
\begin{aligned}
T_{W_3} \times G_{W_4} = c_{34}X \Bigg[& \frac{10G_X}{9} - \frac{10(k+1)(JG_X)_0^1}{9(k+4)(2k-1)} - \frac{4(7k-3)(JG_X)_0^2}{9(k-3)(2k-1)} + \frac{(7k-2)(JG_X)_{-1}^1}{3(k+4)(2k-1)} - \frac{2(JG_X)_{-1}^2}{3(2k-1)} \\
& + \frac{10(k-2)(GX)_0^1}{9(k+4)(2k-1)} - \frac{2(5k-3)(GX)_0^2}{3(k-3)(2k-1)} + \frac{2(13k-8)(GX)_{-1}^1}{9(k+4)(2k-1)} - \frac{2(GX)_{-1}^2}{3(2k-1)} \\
& + \frac{(4k+1)(GT_X)_0^1}{3(k+4)(2k-1)} + \frac{2(k-1)(TG_X)_0^1}{(k+4)(2k-1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
T_{W_3} \times \tilde{G}_{W_4} = c_{34}X \Bigg[& \frac{10\tilde{G}_X}{9} - \frac{10(k+1)\left(J\tilde{G}_X\right)_0^1}{9(k+4)(2k-1)} - \frac{4(7k-3)\left(J\tilde{G}_X\right)_0^2}{9(k-3)(2k-1)} + \frac{(7k-2)\left(J\tilde{G}_X\right)_{-1}^1}{3(k+4)(2k-1)} - \frac{2\left(J\tilde{G}_X\right)_{-1}^2}{3(2k-1)} \\
& + \frac{10(k-2)(\tilde{G}X)_0^1}{9(k+4)(2k-1)} - \frac{2(5k-3)(\tilde{G}X)_0^2}{3(k-3)(2k-1)} + \frac{2(13k-8)(\tilde{G}X)_{-1}^1}{9(k+4)(2k-1)} - \frac{2(\tilde{G}X)_{-1}^2}{3(2k-1)} \\
& + \frac{(4k+1)\left(\tilde{G}T_X\right)_0^1}{3(k+4)(2k-1)} + \frac{2(k-1)\left(T\tilde{G}_X\right)_0^1}{(k+4)(2k-1)}
\end{aligned}
$$

$$
\begin{aligned}
T_{W_3} \times T_{W_4} = c_{34}X \Bigg[& -\frac{5X}{2} + \frac{5(3k+11)T_X}{9(k+4)} - \frac{20(JX)_0^{\frac{1}{2}}}{9(k+4)} + \frac{5(JX)_0^{\frac{3}{2}}}{9(2k-1)} \\
& + \frac{5(46k^2+431k-237)(JX)_{-1}^{\frac{3}{2}}}{144(k-3)(k+4)(2k-1)} - \frac{10(k-4)(JX)_{-1}^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{40(JX)_{-2}^{\frac{1}{2}}}{9(k+4)} + \frac{10(JX)_{-2}^{\frac{3}{2}}}{9(2k-1)} \\
& + \frac{5(46k^2+193k-111)\left(G\tilde{G}_X\right)_0^{\frac{3}{2}}}{72(k-3)(k+4)(2k-1)} - \frac{2(k-11)\left(G\tilde{G}_X\right)_0^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{5\left(G\tilde{G}_X\right)_{-1}^{\frac{1}{2}}}{3(k+4)} + \frac{2\left(G\tilde{G}_X\right)_{-1}^{\frac{3}{2}}}{3(2k-1)} \\
& - \frac{5(46k^2+193k-111)\left(\tilde{G}G_X\right)_0^{\frac{3}{2}}}{72(k-3)(k+4)(2k-1)} + \frac{2(k-11)\left(\tilde{G}G_X\right)_0^{\frac{1}{2}}}{9(k+4)(2k-1)} + \frac{5\left(\tilde{G}G_X\right)_{-1}^{\frac{1}{2}}}{3(k+4)} - \frac{2\left(\tilde{G}G_X\right)_{-1}^{\frac{3}{2}}}{3(2k-1)} \\
& - \frac{5(18k^2+53k-27)(JT_X)_0^{\frac{3}{2}}}{24(k-3)(k+4)(2k-1)} - \frac{2(3k+2)(JT_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} - \frac{2(JT_X)_{-1}^{\frac{3}{2}}}{3(2k-1)} + \frac{5(JT_X)_{-1}^{\frac{1}{2}}}{6(k+4)} \\
& - \frac{5(22k^2+43k-33)(TX)_0^{\frac{3}{2}}}{24(k-3)(k+4)(2k-1)} - \frac{2(TX)_{-1}^{\frac{3}{2}}}{3(2k-1)} + \frac{5(TT_X)_0^{\frac{1}{2}}}{3(k+4)} \Bigg].
\end{aligned}
$$

$$W_3 \times W_4 = c_{34}{}^X \left[X - \frac{(k-1)T_X}{12(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{2(JX)_0^{\frac{5}{2}}}{k-3}\right],$$

$$W_3 \times G_{W_4} = c_{34}{}^X \left[\frac{2G_X}{3} - \frac{2(4k+1)(GX)_0^1}{3(k+4)(2k-1)} - \frac{2k(GX)_0^2}{(k-3)(2k-1)} + \frac{5(JG_X)_0^0}{3(k+4)} + \frac{(2k-7)(JG_X)_0^1}{3(k+4)(2k-1)} - \frac{2(4k-3)(JG_X)_0^2}{3(k-3)(2k-1)}\right],$$

$$W_3 \times \widetilde{G}_{W_4} = c_{34}{}^X \left[\frac{2\widetilde{G}_X}{3} - \frac{2(4k+1)(\widetilde{G}X)_0^1}{3(k+4)(2k-1)} - \frac{2k(\widetilde{G}X)_0^2}{(k-3)(2k-1)} + \frac{5(J\widetilde{G}_X)_0^0}{3(k+4)} + \frac{(2k-7)(J\widetilde{G}_X)_0^1}{3(k+4)(2k-1)} - \frac{2(4k-3)(J\widetilde{G}_X)_0^2}{3(k-3)(2k-1)}\right],$$

$$W_3 \times T_{W_4} = c_{34}{}^X \left[X + \frac{5(3k+13)T_X}{48(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{12(k+4)} - \frac{(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{3(JX)_0^{\frac{5}{2}}}{2(k-3)} + \frac{(4k+1)(G\widetilde{G}_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{(2k+3)(5k-1)(G\widetilde{G}_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} - \frac{(4k+1)(\widetilde{G}G_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} - \frac{(2k+3)(5k-1)(\widetilde{G}G_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{(38k^2-57k+39)(JX)_{-1}^{\frac{3}{2}}}{12(k-3)(k+4)(2k-1)} + \frac{10k(JX)_{-1}^{\frac{1}{2}}}{3(k+4)(2k-1)} - \frac{2(JX)_{-1}^{\frac{5}{2}}}{3(2k-1)} + \frac{2(k-1)(JT_X)_0^{\frac{1}{2}}}{(k+4)(2k-1)} - \frac{(k-1)(2k+15)(JT_X)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} + \frac{(2k^2-19k-3)(TX)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)}\right],$$

$$G_{W_3} \times W_4 = c_{34}{}^X \left[\frac{G_X}{3} + \frac{2(k-1)(GX)_0^1}{(k+4)(2k-1)} - \frac{2(k-1)(GX)_0^2}{(k-3)(2k-1)} - \frac{(4k+1)(JG_X)_0^1}{3(k+4)(2k-1)} - \frac{4k(JG_X)_0^2}{3(k-3)(2k-1)}\right],$$

$$G_{W_3} \times G_{W_4} = c_{34}{}^X \left[\frac{5(GG_X)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(GG_X)_0^{\frac{3}{2}}}{3(2k-1)}\right],$$

$$G_{W_3} \times \widetilde{G}_{W_4} = c_{34}{}^X \left[2X - \frac{5(k+3)T_X}{12(k+4)} + \frac{5(JX)_0^{\frac{1}{2}}}{3(k+4)} - \frac{2(JX)_0^{\frac{3}{2}}}{3(2k-1)} - \frac{2(JX)_0^{\frac{5}{2}}}{k-3} + \frac{(46k^2-419k+213)(JX)_{-1}^{\frac{3}{2}}}{36(k-3)(k+4)(2k-1)} + \frac{10(k-2)(JX)_{-1}^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{2(JX)_{-1}^{\frac{5}{2}}}{3(2k-1)} + \frac{2(k-1)(G\widetilde{G}_X)_0^{\frac{1}{2}}}{(k+4)(2k-1)} - \frac{(k-1)(2k+15)(G\widetilde{G}_X)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)} + \frac{(2k^2+35k+3)(\widetilde{G}G_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{(4k+1)(\widetilde{G}G_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{(4k+1)(JT_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} + \frac{(2k+3)(5k-1)(JT_X)_0^{\frac{3}{2}}}{6(k-3)(k+4)(2k-1)} + \frac{3(k-1)(2k+1)(TX)_0^{\frac{3}{2}}}{2(k-3)(k+4)(2k-1)}\right],$$

$$G_{W_3} \times T_{W_4} = c_{34}{}^X \left[\frac{5G_X}{3} + \frac{5(JG_X)_0^0}{3(k+4)} - \frac{5(JG_X)_0^1}{2(k+4)(2k-1)} - \frac{(11k-6)(JG_X)_0^2}{3(k-3)(2k-1)} + \frac{5(JG_X)_{-1}^0}{3(k+4)} + \frac{2(k-1)(JG_X)_{-1}^1}{(k+4)(2k-1)} - \frac{2(JG_X)_{-1}^2}{3(2k-1)} - \frac{5(k+1)(GX)_0^1}{3(k+4)(2k-1)} - \frac{(7k-3)(GX)_0^2}{2(k-3)(2k-1)} + \frac{4(7k-2)(GX)_{-1}^1}{9(k+4)(2k-1)} - \frac{2(GX)_{-1}^2}{3(2k-1)} + \frac{5(GT_X)_0^0}{3(k+4)} + \frac{(2k-7)(GT_X)_0^1}{3(k+4)(2k-1)} + \frac{2(4k+1)(TG_X)_0^1}{3(k+4)(2k-1)}\right],$$

$$\widetilde{G}_{W_3} \times W_4 = {c_{34}}^X \left[\frac{\widetilde{G}_X}{3} + \frac{2(k-1)(\widetilde{G}X)^1_0}{(k+4)(2k-1)} - \frac{2(k-1)(\widetilde{G}X)^2_0}{(k-3)(2k-1)} - \frac{(4k+1)(J\widetilde{G}_X)^1_0}{3(k+4)(2k-1)} - \frac{4k(J\widetilde{G}_X)^2_0}{3(k-3)(2k-1)} \right],$$

$$\begin{aligned}
\widetilde{G}_3 \times G_{W_4} = {c_{34}}^X \Bigg[& -2X + \frac{5(k+3)T_X}{12(k+4)} - \frac{5(JX)^{\frac{1}{2}}_0}{3(k+4)} + \frac{2(JX)^{\frac{3}{2}}_0}{3(2k-1)} + \frac{2(JX)^{\frac{5}{2}}_0}{k-3} \\
& - \frac{(46k^2-419k+213)(JX)^{\frac{3}{2}}_{-1}}{36(k-3)(k+4)(2k-1)} - \frac{10(k-2)(JX)^{\frac{1}{2}}_{-1}}{9(k+4)(2k-1)} + \frac{2(JX)^{\frac{5}{2}}_{-1}}{3(2k-1)} \\
& + \frac{(2k^2+35k+3)(G\widetilde{G}_X)^{\frac{3}{2}}_0}{6(k-3)(k+4)(2k-1)} + \frac{(4k+1)(G\widetilde{G}_X)^{\frac{1}{2}}_0}{3(k+4)(2k-1)} + \frac{2(k-1)(\widetilde{G}G_X)^{\frac{1}{2}}_0}{(k+4)(2k-1)} - \frac{(k-1)(2k+15)(\widetilde{G}G_X)^{\frac{3}{2}}_0}{2(k-3)(k+4)(2k-1)} \\
& - \frac{(4k+1)(JT_X)^{\frac{1}{2}}_0}{3(k+4)(2k-1)} - \frac{(2k+3)(5k-1)(JT_X)^{\frac{3}{2}}_0}{6(k-3)(k+4)(2k-1)} - \frac{3(k-1)(2k+1)(TX)^{\frac{3}{2}}_0}{2(k-3)(k+4)(2k-1)} \Bigg],
\end{aligned}$$

$$\widetilde{G}_3 \times \widetilde{G}_{W_4} = {c_{34}}^X \left[\frac{5(\widetilde{G}\widetilde{G}_X)^{\frac{1}{2}}_0}{3(k+4)} - \frac{2(\widetilde{G}\widetilde{G}_X)^{\frac{3}{2}}_0}{3(2k-1)} \right],$$

$$\begin{aligned}
\widetilde{G}_{W_3} \times T_{W_4} = {c_{34}}^X \Bigg[& \frac{5\widetilde{G}_X}{3} - \frac{5(k+1)(\widetilde{G}X)^1_0}{3(k+4)(2k-1)} - \frac{(7k-3)(\widetilde{G}X)^2_0}{2(k-3)(2k-1)} + \frac{4(7k-2)(\widetilde{G}X)^1_{-1}}{9(k+4)(2k-1)} - \frac{2(\widetilde{G}X)^2_{-1}}{3(2k-1)} \\
& + \frac{5(J\widetilde{G}_X)^0_0}{3(k+4)} - \frac{5(J\widetilde{G}_X)^1_0}{2(k+4)(2k-1)} - \frac{(11k-6)(J\widetilde{G}_X)^2_0}{3(k-3)(2k-1)} + \frac{2(k-1)(J\widetilde{G}_X)^1_{-1}}{(k+4)(2k-1)} - \frac{2(J\widetilde{G}_X)^2_{-1}}{3(2k-1)}
\end{aligned}$$

$$+ \frac{5(\widetilde{G}T_X)^0_0}{3(k+4)} + \frac{(2k-7)(\widetilde{G}T_X)^1_0}{3(k+4)(2k-1)} + \frac{10f(J(J\widetilde{G}_X)^1_0)^0_0}{9(k+4)} + \frac{2(4k+1)(T\widetilde{G}_X)^1_0}{3(k+4)(2k-1)} \Bigg],$$

$$\begin{aligned}
T_{W_3} \times W_4 = {c_{34}}^X \Bigg[& \frac{2X}{3} - \frac{2(JX)^{\frac{3}{2}}_0}{9(2k-1)} - \frac{4(JX)^{\frac{5}{2}}_0}{3(k-3)} + \frac{(98k^2-247k+69)(JX)^{\frac{3}{2}}_{-1}}{36(k-3)(k+4)(2k-1)} - \frac{2(JX)^{\frac{5}{2}}_{-1}}{3(2k-1)} \\
& + \frac{(k-1)(2k+1)(G\widetilde{G}_X)^{\frac{3}{2}}_0}{2(k-3)(k+4)(2k-1)} - \frac{(k-1)(2k+1)(\widetilde{G}G_X)^{\frac{3}{2}}_0}{2(k-3)(k+4)(2k-1)} \\
& + \frac{(2k^2-19k-3)(JT_X)^{\frac{3}{2}}_0}{6(k-3)(k+4)(2k-1)} + \frac{(k-1)(2k-13)(TX)^{\frac{3}{2}}_0}{2(k-3)(k+4)(2k-1)} \Bigg],
\end{aligned}$$

$$\begin{aligned}
T_{W_3} \times G_{W_4} = {c_{34}}^X \Bigg[& \frac{10G_X}{9} - \frac{10(k+1)(JG_X)^1_0}{9(k+4)(2k-1)} - \frac{4(7k-3)(JG_X)^2_0}{9(k-3)(2k-1)} + \frac{(7k-2)(JG_X)^1_{-1}}{3(k+4)(2k-1)} - \frac{2(JG_X)^2_{-1}}{3(2k-1)} \\
& + \frac{10(k-2)(GX)^1_0}{9(k+4)(2k-1)} - \frac{2(5k-3)(GX)^2_0}{3(k-3)(2k-1)} + \frac{2(13k-8)(GX)^1_{-1}}{9(k+4)(2k-1)} - \frac{2(GX)^2_{-1}}{3(2k-1)} \\
& + \frac{(4k+1)(GT_X)^1_0}{3(k+4)(2k-1)} + \frac{2(k-1)(TG_X)^1_0}{(k+4)(2k-1)} \Bigg],
\end{aligned}$$

$$\begin{aligned}
T_{W_3} \times \widetilde{G}_{W_4} = {c_{34}}^X \Bigg[& \frac{10\widetilde{G}_X}{9} - \frac{10(k+1)(J\widetilde{G}_X)^1_0}{9(k+4)(2k-1)} - \frac{4(7k-3)(J\widetilde{G}_X)^2_0}{9(k-3)(2k-1)} + \frac{(7k-2)(J\widetilde{G}_X)^1_{-1}}{3(k+4)(2k-1)} - \frac{2(J\widetilde{G}_X)^2_{-1}}{3(2k-1)} \\
& + \frac{10(k-2)(\widetilde{G}X)^1_0}{9(k+4)(2k-1)} - \frac{2(5k-3)(\widetilde{G}X)^2_0}{3(k-3)(2k-1)} + \frac{2(13k-8)(\widetilde{G}X)^1_{-1}}{9(k+4)(2k-1)} - \frac{2(\widetilde{G}X)^2_{-1}}{3(2k-1)} \\
& + \frac{(4k+1)(\widetilde{G}T_X)^1_0}{3(k+4)(2k-1)} + \frac{2(k-1)(T\widetilde{G}_X)^1_0}{(k+4)(2k-1)}
\end{aligned}$$

$$
\begin{aligned}
T_{W_3} \times T_{W_4} = c_{34}{}^{X} \Big[& -\frac{5X}{2} + \frac{5(3k+11)T_X}{9(k+4)} - \frac{20(JX)_0^{\frac{1}{2}}}{9(k+4)} + \frac{5(JX)_0^{\frac{3}{2}}}{9(2k-1)} \\
& + \frac{5(46k^2+431k-237)(JX)_{-1}^{\frac{3}{2}}}{144(k-3)(k+4)(2k-1)} - \frac{10(k-4)(JX)_{-1}^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{40(JX)_{-2}^{\frac{1}{2}}}{9(k+4)} + \frac{10(JX)_{-2}^{\frac{3}{2}}}{9(2k-1)} \\
& + \frac{5(46k^2+193k-111)(G\widetilde{G}_X)_0^{\frac{3}{2}}}{72(k-3)(k+4)(2k-1)} - \frac{2(k-11)(G\widetilde{G}_X)_0^{\frac{1}{2}}}{9(k+4)(2k-1)} - \frac{5(G\widetilde{G}_X)_{-1}^{\frac{1}{2}}}{3(k+4)} + \frac{2(G\widetilde{G}_X)_{-1}^{\frac{3}{2}}}{3(2k-1)} \\
& - \frac{5(46k^2+193k-111)(\widetilde{G}G_X)_0^{\frac{3}{2}}}{72(k-3)(k+4)(2k-1)} + \frac{2(k-11)(\widetilde{G}G_X)_0^{\frac{1}{2}}}{9(k+4)(2k-1)} + \frac{5(\widetilde{G}G_X)_{-1}^{\frac{1}{2}}}{3(k+4)} - \frac{2(\widetilde{G}G_X)_{-1}^{\frac{3}{2}}}{3(2k-1)} \\
& - \frac{5(18k^2+53k-27)(JT_X)_0^{\frac{3}{2}}}{24(k-3)(k+4)(2k-1)} - \frac{2(3k+2)(JT_X)_0^{\frac{1}{2}}}{3(k+4)(2k-1)} - \frac{2(JT_X)_{-1}^{\frac{3}{2}}}{3(2k-1)} + \frac{5(JT_X)_{-1}^{\frac{1}{2}}}{6(k+4)}
\end{aligned}
$$

$$
- \frac{5(22k^2+43k-33)(TX)_0^{\frac{3}{2}}}{24(k-3)(k+4)(2k-1)} - \frac{2(TX)_{-1}^{\frac{3}{2}}}{3(2k-1)} + \frac{5(TT_X)_0^{\frac{1}{2}}}{3(k+4)} \Big]
$$

$$
\begin{aligned}
&\mathcal{A} := (JX)_0^0 - \frac{1}{5}X_{[1,-1,0]}, && (h_{\mathcal{A}}, j_{\mathcal{A}}) = (4{,}0) \\
&\mathcal{B} := (JX)_0^1 + \frac{1}{4}X'_{[1,0,0]} - \frac{1}{4}X_{[1,0,0]}, && (h_{\mathcal{B}}, j_{\mathcal{B}}) = (4{,}1) \\
&\mathcal{C} := (JX)_0^2 - \frac{1}{2}X_{[1,1,0]}, && (h_{\mathcal{C}}, j_{\mathcal{C}}) = (4{,}2)
\end{aligned}
$$

$$W_5 \times W_5 = c_{55}X\left[X + \frac{1}{10}X_{[1,-1,0]} - \frac{1}{2}X_{[1,1,0]} + \frac{3\mathcal{A}}{k+3} - \frac{3\mathcal{C}}{k-3}\right]$$

$$W_5 \times G_{W_5} = c_{55}X\left[X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]} + \frac{1}{2}X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{1}{20}X_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{1}{4}X_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}\right.$$

$$+ \frac{3\mathcal{A}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]} - \frac{3\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{k-3}\Bigg]}{W_5 \times \tilde{G}_{W_5} =} c_{55}X\left[X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]} + \frac{1}{2}X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{1}{20}X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{1}{4}X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}\right.$$

$$\left. + \frac{3\mathcal{A}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{k+3} - \frac{3\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{k-3}\right]$$

$$W_5 \times T_{W_5} = c_{55}X\left[+\frac{2}{5}X + \frac{1}{20}X_{[2,0,0]} + \frac{1}{2}X_{[1,0,0]} - \frac{1}{2}X'_{[1,0,0]} - \frac{6}{5}X_{[1,1,0]}\right.$$

$$\left. - \frac{3\mathcal{A}_{[1,1,0]}}{k+3} - \frac{6\mathcal{C}}{5(k-3)} + \frac{\mathcal{C}_{[1,-1,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,0,0]}}{2(k-3)} - \frac{3\mathcal{C}'_{[1,0,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,1,0]}}{k-3}\right]$$

$$G_{W_5} \times G_{W_5} = c_{55}X\left[X_{[1,0,1]} + \frac{3\mathcal{A}_{[1,0,1]}}{k+3} - \frac{3\mathcal{C}_{[1,0,1]}}{k-3}\right],$$

$$G_{W_5} \times \tilde{G}_{W_5} = c_{55}X\left[2X + \frac{1}{2}X_{[1,0,0]} + \frac{1}{2}X'_{[1,0,0]} + \frac{3}{5}X_{[1,-1,0]} - \frac{1}{5}X_{[2,0,0]} - \frac{3}{2}X_{[1,1,0]}\right.$$

$$+ \frac{3\mathcal{A}}{k+3} + \frac{3\mathcal{A}_{[1,0,0]}}{k+3} - \frac{3\mathcal{A}_{[1,1,0]}}{k+3}$$

$$\left. - \frac{3\mathcal{C}}{k-3} - \frac{\mathcal{C}_{[1,-1,0]}}{k-3} - \frac{3\mathcal{C}_{[1,0,0]}}{2(k-3)} - \frac{3\mathcal{C}'_{[1,0,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,1,0]}}{k-3}\right]$$

$$G_{W_5} \times T_{W_5} = c_{55}{}^{X}\left[\frac{6}{5}X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]} + \frac{6}{5}X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]} - \frac{27}{50}X_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{27}{20}X_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}\right.$$

$$\left. + \frac{12\mathcal{A}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{5(k+3)} - \frac{3\mathcal{A}_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}}{k+3} - \frac{21\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}}{10(k-3)} - \frac{6\mathcal{C}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{5(k-3)} + \frac{3\mathcal{C}_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]}}{2(k-3)} + \frac{3\mathcal{C}_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}}{k-3}\right]$$

$$\tilde{G}_{W_5} \times \tilde{G}_{W_5} = c_{55}{}^{X}\left[X_{[1,0,-1]} + \frac{3\mathcal{A}_{[1,0,-1]}}{k+3} - \frac{3\mathcal{C}_{[1,0,-1]}}{k-3}\right]$$

$$\tilde{G}_{W_5} \times T_{W_5} = c_{55}{}^{X}\left[\frac{6}{5}X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{6}{5}X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]} + \frac{27}{50}X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{27}{20}X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}\right.$$

$$\left. + \frac{12\mathcal{A}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{5(k+3)} + \frac{3\mathcal{A}_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}}{k+3} - \frac{21\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}}{10(k-3)} - \frac{6\mathcal{C}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{5(k-3)} - \frac{3\mathcal{C}_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}}{k-3}\right]$$

$$W_5 \times W_5 = c_{55}{}^{X}\left[X + \frac{1}{10}X_{[1,-1,0]} - \frac{1}{2}X_{[1,1,0]} + \frac{3\mathcal{A}}{k+3} - \frac{3\mathcal{C}}{k-3}\right],$$

$$W_5 \times G_{W_5} = c_{55}{}^{X}\left[X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]} + \frac{1}{2}X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{1}{20}X_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{1}{4}X_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}\right.$$

$$\left. + \frac{3\mathcal{A}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{k+3} - \frac{3\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{k-3}\right],$$

$$W_5 \times \tilde{G}_{W_5} = c_{55}{}^{X}\left[X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]} + \frac{1}{2}X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{1}{20}X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{1}{4}X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}\right.$$

$$\left. + \frac{3\mathcal{A}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{k+3} - \frac{3\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{k-3}\right],$$

$$W_5 \times T_{W_5} = c_{55}{}^{X}\left[+\frac{2}{5}X + \frac{1}{20}X_{[2,0,0]} + \frac{1}{2}X_{[1,0,0]} - \frac{1}{2}X'_{[1,0,0]} - \frac{6}{5}X_{[1,1,0]}\right.$$

$$\left. - \frac{3\mathcal{A}_{[1,1,0]}}{k+3} - \frac{6\mathcal{C}}{5(k-3)} + \frac{\mathcal{C}_{[1,-1,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,0,0]}}{2(k-3)} - \frac{3\mathcal{C}'_{[1,0,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,1,0]}}{k-3}\right],$$

$$
\begin{aligned}
G_{W_5} \times G_{W_5} &= c_{55}{}^X \left[X_{[1,0,1]} + \frac{3\mathcal{A}_{[1,0,1]}}{k+3} - \frac{3\mathcal{C}_{[1,0,1]}}{k-3} \right], \\
G_{W_5} \times \widetilde{G}_{W_5} &= c_{55}{}^X \left[2X + \frac{1}{2} X_{[1,0,0]} + \frac{1}{2} X'_{[1,0,0]} + \frac{3}{5} X_{[1,-1,0]} - \frac{1}{5} X_{[2,0,0]} - \frac{3}{2} X_{[1,1,0]} \right. \\
&\quad + \frac{3\mathcal{A}}{k+3} + \frac{3\mathcal{A}_{[1,0,0]}}{k+3} - \frac{3\mathcal{A}_{[1,1,0]}}{k+3} \\
&\quad \left. - \frac{3\mathcal{C}}{k-3} - \frac{\mathcal{C}_{[1,-1,0]}}{k-3} - \frac{3\mathcal{C}_{[1,0,0]}}{2(k-3)} - \frac{3\mathcal{C}'_{[1,0,0]}}{2(k-3)} + \frac{3\mathcal{C}_{[1,1,0]}}{k-3} \right], \\
G_{W_5} \times T_{W_5} &= c_{55}{}^X \left[\frac{6}{5} X_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]} + \frac{6}{5} X_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]} - \frac{27}{50} X_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]} - \frac{27}{20} X_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]} \right. \\
&\quad \left. + \frac{12\mathcal{A}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{5(k+3)} - \frac{3\mathcal{A}_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}}{k+3} - \frac{21\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},\frac{1}{2}]}}{10(k-3)} - \frac{6\mathcal{C}_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}}{5(k-3)} + \frac{3\mathcal{C}_{[\frac{3}{2},-\frac{1}{2},\frac{1}{2}]}}{2(k-3)} + \frac{3\mathcal{C}_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}}{k-3} \right]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}_{W_5} \times \widetilde{G}_{W_5} &= c_{55}{}^X \left[X_{[1,0,-1]} + \frac{3\mathcal{A}_{[1,0,-1]}}{k+3} - \frac{3\mathcal{C}_{[1,0,-1]}}{k-3} \right], \\
\widetilde{G}_{W_5} \times T_{W_5} &= c_{55}{}^X \left[\frac{6}{5} X_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{6}{5} X_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]} + \frac{27}{50} X_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]} + \frac{27}{20} X_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]} \right. \\
&\quad \left. + \frac{12\mathcal{A}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{5(k+3)} + \frac{3\mathcal{A}_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}}{k+3} - \frac{21\mathcal{C}_{[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]}}{10(k-3)} - \frac{6\mathcal{C}_{[\frac{1}{2},\frac{1}{2},-\frac{1}{2}]}}{5(k-3)} - \frac{3\mathcal{C}_{[\frac{3}{2},-\frac{1}{2},-\frac{1}{2}]}}{2(k-3)} - \frac{3\mathcal{C}_{[\frac{3}{2},\frac{1}{2},-\frac{1}{2}]}}{k-3} \right],
\end{aligned}
$$

$$
T_{W_5} \times T_{W_5} = c_{55}{}^X \left[-\frac{108}{25} X - \frac{54}{25} X_{[1,-1,0]} + \frac{54}{25} X_{[1,1,0]} + \frac{36}{25} X_{[2,0,0]} - \frac{24\mathcal{A}}{5(k+3)} + \frac{18\mathcal{A}_{[1,1,0]}}{5(k+3)} + \frac{3\mathcal{A}_{[2,0,0]}}{k+3} \right.
$$

$$
\left. + \frac{84\mathcal{C}}{25(k-3)} + \frac{12\mathcal{C}_{[1,-1,0]}}{5(k-3)} - \frac{6\mathcal{C}_{[1,1,0]}}{5(k-3)} - \frac{3\mathcal{C}_{[2,0,0]}}{k-3} \right].
$$

$$
\begin{aligned}
T_{W_5} \times T_{W_5} = c_{55}^X & \left[-\frac{108}{25} X - \frac{54}{25} X_{[1,-1,0]} + \frac{54}{25} X_{[1,1,0]} + \frac{36}{25} X_{[2,0,0]} - \frac{24\mathcal{A}}{5(k+3)} + \frac{18\mathcal{A}_{[1,1,0]}}{5(k+3)} + \frac{3\mathcal{A}_{[2,0,0]}}{k+3} \right. \\
& \left. + \frac{84\mathcal{C}}{25(k-3)} + \frac{12\mathcal{C}_{[1,-1,0]}}{5(k-3)} - \frac{6\mathcal{C}_{[1,1,0]}}{5(k-3)} - \frac{3\mathcal{C}_{[2,0,0]}}{k-3} \right]
\end{aligned}
$$

$$
W_6' = W_6 + \mu_6 \mathcal{C}_{3,3}^{W_3 W_3}, W_7' = W_7 + \mu_7 \mathcal{C}_{\frac{7}{2},\frac{7}{2}}^{W_3 W_4}
$$

$$
\left(c_{pq}^{\mathcal{C}_{3,3}^{W_3 W_3}} \right)' = c_{pq}^{\mathcal{C}_{3,3}^{W_3 W_3}} - c_{pq}{}^6 \mu_6
$$

$$
\begin{aligned}
(c_{36}^3)' &= c_{36}^3 + \frac{g_3(2k-1)(k(2k-17)+12)\mu_6}{(k-4)(k-2)(2k+3)} \\
(c_{36}^5)' &= c_{36}^5 + \frac{c_{33}^4 c_{34}^5 k \mu_6}{2(k-4)} \\
(c_{36}^7)' &= c_{36}^7 \\
(c_{46}^4)' &= c_{46}^4 - \frac{24 g_3 (k-1)\mu_6}{k^2 - 6k + 8} \\
(c_{46}^6)' &= c_{46}{}^6 + \frac{3 c_{34}^5 c_{35}^6 k \mu_6}{5(k-4)} \\
(c_{46}^8)' &= c_{46}^8
\end{aligned}
$$

$$\left(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}}\right)' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} + \frac{c_{33}{}^{4}(k-1)\mu_6}{3(k-4)}$$

$$\left(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}}\right)' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} - \frac{c_{33}{}^{4}(k-6)\mu_6}{k-4}$$

$$\left(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}}\right)' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} + \frac{c_{33}{}^{4}(2k-5)\mu_6}{k-4}$$

$$\left(c_{37}{}^{\mathcal{C}^{W_3W_3}_{3,3}}\right)' = c_{37}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - c_{37}{}^{6}\mu_6$$

$$\left(c_{37}{}^{\mathcal{C}^{W_3W_3}_{4,2}}\right)' = c_{37}{}^{\mathcal{C}^{W_3W_3}_{4,2}} + \frac{8c_{35}{}^{6}c_{37}\mathcal{C}^{W_3W_5}_{4,2}k\mu_6}{315(k+7)} + \frac{c_{37}{}^{\mathcal{C}^{W_4W_4}_{4,2}}c_{44}{}^{6}k\mu_6}{35(k+7)}$$

$$\left(c_{55}\mathcal{C}^{W_3W_3}_{3,3}\right)' = c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - c_{55}{}^{6}\mu_6$$

$$\left(c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,2}}\right)' = c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,2}} + \frac{8c_{35}{}^{6}c_{55}\mathcal{C}^{W_3W_5}_{4,2}k\mu_6}{315(k+7)} + \frac{c_{44}{}^{6}c_{55}\mathcal{C}^{W_4W_4}_{4,2}k\mu_6}{35(k+7)}$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,3}}\right)' = -c_{46}{}^{6}\mu_6 + c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - \frac{18g_3(k+4)(4k+1)\mu_6}{c_{33}{}^{4}(k-1)k(2k+3)} + \frac{3c_{33}{}^{4}c_{44}\mathcal{C}^{W_3W_3}_{3,3}k\mu_6}{4(k-4)} - \frac{3c_{34}{}^{5}c_{35}{}^{6}k\mu_6^2}{5(k-4)}$$

$$(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}})' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}} + \frac{c_{33}{}^{4}(k-1)\mu_6}{3(k-4)} ,$$

$$(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}})' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}}} - \frac{c_{33}{}^{4}(k-6)\mu_6}{k-4} ,$$

$$(c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}})' = c_{36}{}^{\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}}} + \frac{c_{33}{}^{4}(2k-5)\mu_6}{k-4} ,$$

$$(c_{37}{}^{\mathcal{C}^{W_3W_3}_{3,3}})' = c_{37}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - c_{37}{}^{6}\mu_6 ,$$

$$(c_{37}{}^{\mathcal{C}^{W_3W_3}_{4,2}})' = c_{37}{}^{\mathcal{C}^{W_3W_3}_{4,2}} + \frac{8c_{35}{}^{6}c_{37}{}^{\mathcal{C}^{W_3W_5}_{4,2}}k\mu_6}{315(k+7)} + \frac{c_{37}{}^{\mathcal{C}^{W_4W_4}_{4,2}}c_{44}{}^{6}k\mu_6}{35(k+7)} ,$$

$$(c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}})' = c_{55}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - c_{55}{}^{6}\mu_6 ,$$

$$(c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,2}})' = c_{55}{}^{\mathcal{C}^{W_3W_3}_{4,2}} + \frac{8c_{35}{}^{6}c_{55}{}^{\mathcal{C}^{W_3W_5}_{4,2}}k\mu_6}{315(k+7)} + \frac{c_{44}{}^{6}c_{55}{}^{\mathcal{C}^{W_4W_4}_{4,2}}k\mu_6}{35(k+7)} ,$$

$$(c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,3}})' = -c_{46}{}^{6}\mu_6 + c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,3}} - \frac{18g_3(k+4)(4k+1)\mu_6}{c_{33}{}^{4}(k-1)k(2k+3)} + \frac{3c_{33}{}^{4}c_{44}{}^{\mathcal{C}^{W_3W_3}_{3,3}}k\mu_6}{4(k-4)} - \frac{3c_{34}{}^{5}c_{35}{}^{6}k\mu_6^2}{5(k-4)} ,$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_3}_{4,2}}\right)' = c_{46}\mathcal{C}^{W_{4,2}W_3} - \frac{36g_3(20k^4-76k^3-1441k^2+2917k+12980)\mu_6}{35c_{33}{}^{4}(k-4)(k-3)k(k+7)(2k+3)} + \frac{2c_{33}{}^{4}c_{44}{}^{6}k\mu_6^2}{35(k-4)(k+7)}$$
$$- \frac{2c_{33}{}^{4}c_{44}\mathcal{C}^{W_3W_3}_{3,3}(k-1)k\mu_6}{105(k-4)(k+7)} + \frac{8c_{34}{}^{5}c_{35}{}^{6}k\mu_6^2}{525(k+7)} + \frac{8c_{35}{}^{6}c_{46}{}^{C_{4,2}W_5}k\mu_6}{315(k+7)} + \frac{c_{44}{}^{6}c_{46}{}^{W_{4,2}W_4}k\mu_6}{35(k+7)},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,1}}\right)' = c_{46}{}^{\mathcal{C}^{W_3W_3}_{3,1}} - \frac{12g_3(k-2)(2k+1)\mu_6}{c_{33}{}^{4}(k-4)k(2k+3)},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,1}}\right)' = c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,1}} - \frac{c_{34}{}^{5}\mu_6}{5},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,4}}\right)' = c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,4}} + \frac{4c_{33}{}^{4}\mu_6}{k-4},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,2}}\right)' = c_{46}{}^{\mathcal{C}^{W_4W_4}_{4,2}} + \frac{2c_{33}{}^{4}\mu_6}{k-4},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,2}}\right)' = c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,2}} + \frac{3c_{34}{}^{5}\mu_6}{5},$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,4}}\right)' = c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,4}} + 2c_{34}{}^{5}\mu_6,$$

$$\left(c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,3}}\right)' = c_{46}{}^{\mathcal{C}^{W_3W_5}_{4,3}} - \frac{6c_{34}{}^{5}\mu_6}{5}.$$

$$(c_{46}c^{W_3W_3}_{4,2})' = c_{46}c^{W_3W_3}_{4,2} - \frac{36g_3\left(20k^4-76k^3-1441k^2+2917k+12980\right)\mu_6}{35c_{33}{}^4(k-4)(k-3)k(k+7)(2k+3)} + \frac{2c_{33}{}^4c_{44}{}^6k\mu_6^2}{35(k-4)(k+7)}$$
$$- \frac{2c_{33}{}^4c_{44}c^{W_3W_3}_{3,3}(k-1)k\mu_6}{105(k-4)(k+7)} + \frac{8c_{34}{}^5c_{35}{}^6k\mu_6^2}{525(k+7)} + \frac{8c_{35}{}^6c_{46}c^{W_3W_5}_{4,2}k\mu_6}{315(k+7)} + \frac{c_{44}{}^6c_{46}c^{W_4W_4}_{4,2}k\mu_6}{35(k+7)},$$
$$(c_{46}c^{W_3W_3}_{3,1})' = c_{46}c^{W_3W_3}_{3,1} - \frac{12g_3(k-2)(2k+1)\mu_6}{c_{33}{}^4(k-4)k(2k+3)},$$
$$(c_{46}c^{W_3W_5}_{4,1})' = c_{46}c^{W_3W_5}_{4,1} - \frac{c_{34}{}^5\mu_6}{5},$$
$$(c_{46}c^{W_4W_4}_{4,4})' = c_{46}c^{W_4W_4}_{4,4} + \frac{4c_{33}{}^4\mu_6}{k-4},$$
$$(c_{46}c^{W_4W_4}_{4,2})' = c_{46}c^{W_4W_4}_{4,2} + \frac{2c_{33}{}^4\mu_6}{k-4},$$
$$(c_{46}c^{W_3W_5}_{4,2})' = c_{46}c^{W_3W_5}_{4,2} + \frac{3c_{34}{}^5\mu_6}{5},$$
$$(c_{46}c^{W_3W_5}_{4,4})' = c_{46}c^{W_3W_5}_{4,4} + 2c_{34}{}^5\mu_6,$$
$$(c_{46}c^{W_3W_5}_{4,3})' = c_{46}c^{W_3W_5}_{4,3} - \frac{6c_{34}{}^5\mu_6}{5}.$$

$$(c_{pq}c^{W_3W_4}_{\frac{7}{2},\frac{7}{2}})' = c_{pq}c^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} - c_{pq}{}^7\mu_7$$

$$(c_{37}{}^4)' = c_{37}{}^4 + \frac{g_3(2k^3-39k^2+97k-60)\mu_7}{2k^3-15k^2+13k+60}$$
$$(c_{37}{}^6)' = c_{37}{}^6 + \frac{2c_{34}{}^5c_{35}{}^6k\mu_7}{5(k-5)}$$
$$(c_{37}{}^8)' = c_{37}{}^8.$$

$$\left(c_{37}\,c^{W_4W_4}_{4,2}\right)' = c_{37}\,c^{W_4W_4}_{4,2} + \frac{c_{33}{}^4\mu_7}{6}, \left(c_{37}\,c^{W_4W_4}_{4,4}\right)' = c_{37}\,c^{W_4W_4}_{4,4} + c_{33}{}^4\mu_7$$
$$\left(c_{37}\,c^{W_3W_5}_{4,2}\right)' = c_{37}\,c^{W_3W_5}_{4,2} + \frac{c_{34}{}^5(k+5)\mu_7}{10(k-5)}$$
$$\left(c_{37}\,c^{W_3W_3}_{4,2}\right)' = c_{37}\,c^{W_3W_3}_{4,2} - \frac{6g_3(4k^5-76k^4-85k^3+2940k^2-10507k+24300)\mu_7}{7c_{33}{}^4k(2k^5-7k^4-89k^3+427k^2-33k-1260)} - \frac{c_{33}{}^4c_{44}\,c^{W_3W_3}_{3,3}(k-1)k\mu_7}{126(k^2+2k-35)}$$
$$\left(c_{37}\,c^{W_3W_5}_{4,3}\right)' = c_{37}\,c^{W_3W_5}_{4,3} - \frac{2c_{34}{}^5(k-10)\mu_7}{5(k-5)}, \left(c_{37}\,c^{W_5}_{4,4}\right)' = c_{37}\,c^{W_3W_5}_{4,4} + \frac{c_{34}{}^5(k-2)\mu_7}{k-5}$$

$$(c_{37}c^{W_4W_4}_{4,2})' = c_{37}c^{W_4W_4}_{4,2} + \frac{c_{33}{}^4\mu_7}{6}, \qquad (c_{37}c^{W_4W_4}_{4,4})' = c_{37}c^{W_4W_4}_{4,4} + c_{33}{}^4\mu_7,$$
$$(c_{37}c^{W_3W_5}_{4,2})' = c_{37}c^{W_3W_5}_{4,2} + \frac{c_{34}{}^5(k+5)\mu_7}{10(k-5)},$$
$$(c_{37}c^{W_3W_3}_{4,2})' = c_{37}c^{W_3W_3}_{4,2} - \frac{6g_3\left(4k^5-76k^4-85k^3+2940k^2-10507k+24300\right)\mu_7}{7c_{33}{}^4k\left(2k^5-7k^4-89k^3+427k^2-33k-1260\right)} - \frac{c_{33}{}^4c_{44}c^{W_3W_3}_{3,3}(k-1)k\mu_7}{126\left(k^2+2k-35\right)},$$
$$(c_{37}c^{W_3W_5}_{4,3})' = c_{37}c^{W_3W_5}_{4,3} - \frac{2c_{34}{}^5(k-10)\mu_7}{5(k-5)}, \qquad (c_{37}c^{W_3W_5}_{4,4})' = c_{37}c^{W_3W_5}_{4,4} + \frac{c_{34}{}^5(k-2)\mu_7}{k-5}.$$

$$c^{W_3W_3}_{3,1} = (W_3W_3)^1_0 + c_{33}{}^4\left[\frac{kT_{W_4}}{15(k+5)} - \frac{5(JW_4)^1_0}{3(k+5)}\right] + g_3\left[\frac{(k-3)(2k-1)(G\tilde{G})^1_0}{2(k-1)k(2k+1)(2k+3)}\right.$$
$$\left.+\frac{(10k^2-77k-48)(JJ)^1_{-1}}{5(k-1)k(2k+1)(2k+3)} + \frac{(2k-1)(3k-2)(JT)^1_0}{(k-1)k(2k+1)(2k+3)} - \frac{3(26k-1)(J(JJ)^0_0)^1_0}{5(k-1)k(2k+1)(2k+3)}\right]$$
$$c^{W_3W_3}_{3,3} = (W_3W_3)^3_0 + c_{33}{}^4\frac{(JW_4)^3_0}{k-4} + g_3\frac{(J(JJ)^2_0)^3_0}{(k-2)(k-1)k}.$$

$$
\begin{aligned}
C_{4,0}^{W_3W_3} ={}& (W_3W_3)^{0}_{-1} - \frac{(k-1)\left(G_{W_3}\tilde{G}_{W_3}\right)^{0}_{0}}{4(k+4)} - \frac{5(J(W_3W_3)^{1}_{0})^{0}_{0}}{2(k+4)} + c_{33}{}^{4}\left[\frac{25(J(JW_4)^{1}_{0})^{0}_{0}}{12(k+4)^2} - \frac{(k-1)\left(JT_{W_4}\right)^{0}_{0}}{6(k+4)^2}\right. \\
&- \frac{(33k^2+73k-20)(J(TJ)^{1}_{0})^{0}_{0}}{k(k+4)(2k+3)(2k+5)(3k+4)} \\
&+ \frac{(54k^4+39k^3+2980k^2+11097k+7880)(J(JJ)^{1}_{-1})^{0}_{0}}{40(k-1)k(k+4)(2k+3)(2k+5)(3k+4)} \\
&\left.+ \frac{3(129k^2+341k+20)(J(J(JJ)^{0}_{0})^{1}_{0})^{0}_{0}}{4(k-1)k(k+4)(2k+3)(2k+5)(3k+4)} + \frac{(2k-1)(k^2-6k-20)(TT)^{0}_{0}}{4k(k+4)(2k+3)(2k+5)(3k+4)}\right]
\end{aligned}
$$

$$
\begin{aligned}
C_{4,2}^{W_3W_3} ={}& (W_3W_3)^{2}_{-1} - \frac{(2k^2+3k-41)\left(G_{W_3}\tilde{G}_{W_3}\right)^{2}_{0}}{4(k^2+5k-17)} + \frac{7(k+7)(J(W_3W_3)^{1}_{0})^{2}_{0}}{4(k^2+5k-17)} - \frac{63(k-3)(J(W_3W_3)^{3}_{0})^{2}_{0}}{20(k^2+5k-17)} \\
&- \frac{3(k^2+5k-38)\left(W_3T_{W_3}\right)^{2}_{0}}{8(k^2+5k-17)} + c_{33}{}^{4}\left[-\frac{(6k^3-21k^2+161k-106)\left(G\tilde{G}_{W_4}\right)^{2}_{0}}{32(2k-1)(3k-1)(k^2+5k-17)}\right. \\
&+ \frac{(6k^3-21k^2+161k-106)\left(\tilde{G}G_{W_4}\right)^{2}_{0}}{32(2k-1)(3k-1)(k^2+5k-17)} + \frac{9(k-1)(2k-31)(3k-2)(JW_4)^{2}_{-1}}{64(2k-1)(3k-1)(k^2+5k-17)} \\
&- \frac{5(93k^2-44k-5)(J(JW_4)^{1}_{0})^{2}_{0}}{24(2k-1)(3k-1)(k^2+5k-17)} + \frac{9(k-3)(11k-13)(J(JW_4)^{2}_{0})^{2}_{0}}{32(2k-1)(3k-1)(k^2+5k-17)} \\
&\left.+ \frac{(30k^3+35k^2-273k+184)\left(JT_{W_4}\right)^{2}_{0}}{48(2k-1)(3k-1)(k^2+5k-17)} - \frac{(2k^3+35k^2-7k+2)(TW_4)^{2}_{0}}{8(2k-1)(3k-1)(k^2+5k-17)}\right] \\
&+ g_3\left[-\frac{(4k^2+27k-271)\left(J(\tilde{G}G)^{1}_{0}\right)^{2}_{0}}{8(k-1)k(2k+3)(k^2+5k-17)} + \frac{(20k^2+93k-263)(J(TJ)^{1}_{0})^{2}_{0}}{4(k-1)k(2k+3)(k^2+5k-17)}\right. \\
&\left.- \frac{(k+67)(JJ)^{2}_{-2}}{4(k-1)k(k^2+5k-17)} + \frac{(k+4)(5k-53)(J(JJ)^{1}_{-1})^{2}_{0}}{2(k-1)k(2k+3)(k^2+5k-17)} - \frac{33(k+4)(J(J(JJ)^{0}_{0})^{1}_{0})^{2}_{0}}{2(k-1)k(2k+3)(k^2+5k-17)}\right]
\end{aligned}
$$

$$
\begin{aligned}
C_{\frac{7}{2},\frac{1}{2}}^{W_3W_4} ={}& (W_3W_4)^{\frac{1}{2}}_{0} + c_{34}{}^{3}\left[-\frac{(k-1)(2k-5)\left(G\tilde{G}_{W_3}\right)^{\frac{1}{2}}_{0}}{15(k+4)(2k-1)(2k+5)} + \frac{(k-1)(2k-5)\left(\tilde{G}G_{W_3}\right)^{\frac{1}{2}}_{0}}{15(k+4)(2k-1)(2k+5)}\right. \\
&\left.- \frac{2(6k^2-5k-64)(JW_3)^{\frac{1}{2}}_{-1}}{9(k+4)(2k-1)(2k+5)} + \frac{(6k+1)f\left(J(JW_3)^{\frac{1}{2}}_{0}\right)^{\frac{1}{2}}_{0}}{(k+4)(2k-1)(2k+5)} - \frac{(k-1)(42k-5)\left(JT_{W_3}\right)^{\frac{1}{2}}_{0}}{60(k+4)(2k-1)(2k+5)}\right]
\end{aligned}
$$

$$
\begin{aligned}
C_{\frac{7}{2},\frac{3}{2}}^{W_3W_4} ={}& (W_3W_4)^{\frac{3}{2}}_{0} + c_{34}^{3}\left[+\frac{4(8k-3)\left(J(JW_3)^{\frac{1}{2}}_{0}\right)^{\frac{3}{2}}_{0}}{9(k-3)k(k+4)} - \frac{5(13k-15)\left(J(JW_3)^{\frac{3}{2}}_{0}\right)^{\frac{3}{2}}_{0}}{54(k-3)k(2k-1)}\right. \\
&- \frac{(k-1)(2k^2-17k+12)\left(G\tilde{G}_{W_3}\right)^{\frac{3}{2}}_{0}}{12(k-3)k(k+4)(2k-1)} + \frac{(k-1)(2k^2-17k+12)\left(\tilde{G}G_{W_3}\right)^{\frac{3}{2}}_{0}}{12(k-3)k(k+4)(2k-1)} \\
&- \frac{(k-1)(6k^2-19k+24)\left(JT_{W_3}\right)^{\frac{3}{2}}_{0}}{12(k-3)k(k+4)(2k-1)} - \frac{(k-1)(14k^2-31k-12)(TW_3)^{\frac{3}{2}}_{0}}{12(k-3)k(k+4)(2k-1)} \\
&\left.- \frac{(126k^3-2069k^2+283k+300)(JW_3)^{\frac{3}{2}}_{-1}}{216(k-3)k(k+4)(2k-1)}\right] + c_{34}{}^{5}\left[\frac{kT_{W_5}}{20(k+6)} - \frac{9(JW_5)^{\frac{3}{2}}_{0}}{5(k+6)}\right],
\end{aligned}
$$

$$\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} = (W_3W_4)^{\frac{5}{2}}_0 + c_{34}{}^3 \left[\frac{2(7k-26)(JW_3)^{\frac{5}{2}}_{-1}}{21(k-3)(2k-1)} - \frac{10\left(J(JW_3)^{\frac{3}{2}}_0\right)^{\frac{5}{2}}_0}{21(k-3)(2k-1)} \right] + c_{34}{}^5 \frac{(JW_5)^{\frac{5}{2}}_0}{5(2k-3)}$$

$$\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} = (W_3W_4)^{\frac{7}{2}}_0 - c_{34}{}^3 \frac{\left(J(JW_3)^{\frac{5}{2}}_0\right)^{\frac{7}{2}}_0}{(k-4)(k-3)} + c_{34} \frac{(JW_5)^{\frac{7}{2}}_0}{k-5}$$

$$\mathcal{C}^{W_3W_5}_{4,1} = (W_3W_5)^1_0 + c_{35}{}^4 \left[-\frac{(k-3)\left(G\tilde{G}_{W_4}\right)^1_0}{24(k+3)(k+5)} + \frac{(k-3)\left(\tilde{G}G_{W_4}\right)^1_0}{24(k+3)(k+5)} \right.$$
$$\left. -\frac{(7k^2-4k-115)(JW_4)^1_{-1}}{16(k-1)(k+3)(k+5)} + \frac{(3k+1)(J(JW_4)^1_0)^1_0}{2(k-1)(k+3)(k+5)} - \frac{(13k-3)\left(JT_{W_4}\right)^1_0}{120(k+3)(k+5)} \right]$$
$$+c_{35}\mathcal{C}^{W_3W_3}_{3,1} \left[\frac{7\left(J\mathcal{C}^{W_3W_3}_{3,1}\right)^1_0}{2(2k+1)} + \frac{(k-3)\left(\mathcal{C}^{W_3W_3}_{3,1}\right)_{[1,0,0]}}{4(2k+1)} - \frac{(k-3)\left(\mathcal{C}^{W_3W_3}_{3,1}\right)'_{[1,0,0]}}{4(2k+1)} \right],$$

$$\hat{\mathcal{C}}^{W_3W_5}_{4,2} = (W_3W_5)^2_0 + c_{35}{}^4 \left[-\frac{(18k^4-217k^3+530k^2-371k-20)\left(G\tilde{G}_{W_4}\right)^2_0}{40(k-4)(k-1)(k+5)(2k-1)(3k-1)} \right.$$
$$+\frac{(18k^4-217k^3+530k^2-371k-20)\left(\tilde{G}G_{W_4}\right)^2_0}{40(k-4)(k-1)(k+5)(2k-1)(3k-1)}$$
$$-\frac{(714k^4-17251k^3+26950k^2-17213k+5540)(JW_4)^2_{-1}}{560(k-4)(k-1)(k+5)(2k-1)(3k-1)}$$
$$+\frac{(283k^2-249k-4)(J(JW_4)^1_0)^2_0}{14(k-4)(k+5)(2k-1)(3k-1)} - \frac{3(57k^2-131k+4)(J(JW_4)^2_0)^2_0}{56(k-4)(k-1)(2k-1)(3k-1)}$$
$$\left. -\frac{(18k^4-105k^3+262k^2-231k-4)\left(JT_{W_4}\right)^2_0}{20(k-4)(k-1)(k+5)(2k-1)(3k-1)} - \frac{(38k^4-197k^3+210k^2+29k-20)(TW_4)^2_0}{10(k-4)(k-1)(k+5)(2k-1)(3k-1)} \right]$$
$$+c_{35}{}^6 \left[\frac{4kT_{W_6}}{105(k+7)} - \frac{28(JW_6)^2_0}{15(k+7)} + c_{35}\mathcal{C}^{W_3W_3}_{3,3} \left[\frac{4kT_{\mathcal{C}^{W_3W_3}_{3,3}}}{105(k+7)} - \frac{28\left(J\mathcal{C}^{W_3W_3}_{3,3}\right)^2_0}{15(k+7)} \right] \right]$$
$$+c_{35}\mathcal{C}^{W_3W_3}_{3,1} \left[\frac{2\left(J\mathcal{C}^{W_3W_3}_{3,1}\right)^2_0}{k-3} + \frac{(k-5)\left(\mathcal{C}^{W_3W_3}_{3,1}\right)_{[1,1,0]}}{2(k-3)} \right],$$

$$\mathcal{C}^{W_3W_5}_{4,3} = (W_3W_5)^3_0 + c_{35}{}^4 \left[\frac{(4k-19)(JW_4)^3_{-1}}{8(k-4)(k-1)} - \frac{3(J(JW_4)^2_0)^3_0}{8(k-4)(k-1)} \right] + c_{35}{}^6 \frac{(JW_6)^3_0}{6(k-2)} + c_{35}\mathcal{C}^{W_3W_3}_{3,3} \frac{\left(J\mathcal{C}^{W_3W_3}_{3,3}\right)^3_0}{6(k-2)}$$

$$\mathcal{C}^{W_3W_5}_{4,4} = (W_3W_5)^4_0 - c_{35}{}^4 \frac{(J(JW_4)^3_0)^4_0}{(k-5)(k-4)} - c_{35}{}^6 \frac{(JW_6)^4_0}{k-6} - c_{35}\mathcal{C}_{3^{W_3W_3}_{3,3}} \frac{\left(J\mathcal{C}^{W_3W_3}_{3,3}\right)^4_0}{k-6}.$$

$$
\begin{aligned}
\hat{\mathcal{C}}_{4,0}^{W_4W_4} &= (W_4W_4)_0^0 + g_4\Bigg[-\frac{2(54k^2-163k-185)\big(J(\tilde{G}G)_0^1\big)_0^0}{5k(2k+1)(2k+3)(2k+5)(3k+4)} \\
&+\frac{4(k-1)(2k^2-19k+20)(G\tilde{G})_{-1}^0}{5k(2k+1)(2k+3)(2k+5)(3k+4)}+\frac{4(98k^2+169k-120)(J(TJ)_0^1)_0^0}{5k(2k+1)(2k+3)(2k+5)(3k+4)} \\
&+\frac{(72k^4+270k^3-1229k^2-3953k-1775)(J(JJ)_{-1}^1)_0^0}{10(k-1)k(2k+1)(2k+3)(2k+5)(3k+4)} \\
&-\frac{3(38k^2+127k-18)(J(J(JJ)_0^0)_0^1)_0^0}{(k-1)k(2k+1)(2k+3)(2k+5)(3k+4)}-\frac{(k-1)(88k^2+114k-145)(TT)_0^0}{15k(2k+1)(2k+3)(2k+5)(3k+4)}\Bigg] \\
&+c_{44}{}^4\left[\frac{(k-1)\big(JT_{W_4}\big)_0^0}{5(k+4)(k+5)}-\frac{5(J(JW_4)_0^1)_0^0}{2(k+4)(k+5)}\right] \\
&+c_{44}\mathcal{C}_{3,1}^{W_3W_3}\left[-\frac{5\big(J\mathcal{C}_{3,1}^{W_3W_3}\big)_0^0}{2(k+3)}-\frac{(k-2)\big(\mathcal{C}_{3,1}^{W_3W_3}\big)_{[1,-1,0]}}{10(k+3)}\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{C}}_{4,2}^{W_4W_4} &= (W_4W_4)_0^2 + g_4\Bigg[\frac{4(k-4)\big(J(\tilde{G}G)_0^1\big)_0^2}{(k-2)k(2k+1)(2k+3)}-\frac{2(7k^2+7k-122)(JJ)_{-2}^2}{7(k-2)(k-1)k(2k+1)} \\
&-\frac{4(14k^2-124k-25)(J(JJ)_{-1}^1)_0^2}{7(k-2)(k-1)k(2k+1)(2k+3)}-\frac{4(4k-7)(J(TJ)_0^1)_0^2}{(k-2)k(2k+1)(2k+3)} \\
&+\frac{6(34k-19)(J(J(JJ)_0^0)_0^1)_0^2}{7(k-2)(k-1)k(2k+1)(2k+3)}\Bigg]+c_{44}{}^4\Bigg[-\frac{3(k-1)(2k^2-19k+20)\big(G\tilde{G}_{W_4}\big)_0^2}{10(k-4)(k+5)(2k-1)(3k-1)} \\
&+\frac{3(k-1)(2k^2-19k+20)\big(\tilde{G}G_{W_4}\big)_0^2}{10(k-4)(k+5)(2k-1)(3k-1)}-\frac{27(28k^3-389k^2+136k+45)(JW_4)_{-1}^2}{280(k-4)(k+5)(2k-1)(3k-1)} \\
&+\frac{(349k^2-442k+153)(J(JW_4)_0^1)_0^2}{14(k-4)(k+5)(2k-1)(3k-1)}-\frac{9(17k-19)(J(JW_4)_0^2)_0^2}{56(k-4)(2k-1)(3k-1)} \\
&-\frac{(k-1)(6k^2-23k+32)\big(JT_{W_4}\big)_0^2}{5(k-4)(k+5)(2k-1)(3k-1)}-\frac{2(k-1)(11k^2-42k+10)(TW_4)_0^2}{5(k-4)(k+5)(2k-1)(3k-1)}\Bigg]+c_{44}{}^6\left[\frac{3kT_{W_6}}{70(k+7)}-\frac{21(JW_6)_0^2}{10(k+7)}\right] \\
&+c_{44}{}^{\mathcal{C}_{3,3}^{W_3W_3}}\left[\frac{3kT_{\mathcal{C}_{3,3}^{W_3W_3}}}{70(k+7)}-\frac{21\big(J\mathcal{C}_{3,3}^{W_3W_3}\big)_0^2}{10(k+7)}\right]+c_{44}\mathcal{C}_{3,1}^{W_3W_3}\left[\frac{2\big(J\mathcal{C}_{3,1}^{W_3W_3}\big)_0^2}{k-3}+\frac{(k-5)\big(\mathcal{C}_{3,1}^{W_3W_3}\big)_{[1,1,0]}}{2(k-3)}\right]
\end{aligned}
$$

$$
\mathcal{C}_{4,4}^{W_4W_4} = (W_4W_4)_0^4 + g_4\frac{(J(J(JJ)_0^2)_0^3)_0^4}{(k-3)(k-2)(k-1)k}-c_{44}{}^4\frac{(J(JW_4)_0^3)_0^4}{(k-5)(k-4)}+c_{44}{}^6\frac{(JW_6)_0^4}{k-6}+c_{44}{}^{\mathcal{C}_{3,3}^{W_3W_3}}\frac{\big(J\mathcal{C}_{3,3}^{W_3W_3}\big)_0^4}{k-6},
$$

$$
\begin{aligned}
\mathcal{C}_{4,2}^{W_3W_5} &= \hat{\mathcal{C}}_{4,2}^{W_3W_5}+\left[\frac{2(k-5)}{5(k-3)}c_{35}\mathcal{C}_{3,1}^{W_3W_3}+\frac{8k}{315(k+7)}c_{35}{}^{\mathcal{C}_{3,3}^{W_3}}\right]\mathcal{C}_{4,2}^{W_3W_3} \\
\mathcal{C}_{4,0}^{W_4W_4} &= \hat{\mathcal{C}}_{4,0}^{W_4W_4}+\frac{k-2}{5(k+3)}c_{44}{}^{\mathcal{C}_{3,1}^{W_3W_3}}\mathcal{C}_{4,0}^{W_3W_3} \\
\mathcal{C}_{4,2}^{W_4W_4} &= \hat{\mathcal{C}}_{4,2}^{W_4W_4}+\left[\frac{2(k-5)}{5(k-3)}c_{44}\mathcal{C}_{3,1}^{W_3W_3}+\frac{k}{35(k+7)}c_{44}{}^{\mathcal{C}_{3,3}^{W_3W_3}}\right]\mathcal{C}_{4,2}^{W_3W_3}
\end{aligned}
$$

$$
\begin{aligned}
\langle\mathcal{C}_{3,1}^{W_3W_3}\mathcal{C}_{3,1}^{W_3,W_3}\rangle &= \frac{9(\nu-16)(\nu-9)\nu}{5(\nu-11)(\nu-4)(\nu-2)}g_3^2 \\
\langle\mathcal{C}_{3,3}^{W_3W_3}\mathcal{C}_{3,3}^{W_3,W_3}\rangle &= \frac{2\nu(\nu^2+15\nu+8)}{(\nu-4)(\nu+3)(\nu+7)}g_3^2
\end{aligned}
$$

$$\left\langle \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{1}{2}} \right\rangle = \frac{2(\nu-16)(\nu+1)}{5(\nu-6)(\nu-4)} g_3 g_4$$

$$\left\langle \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}} \right\rangle = \frac{6(\nu-25)(\nu-16)(\nu+1)}{5(\nu-13)(\nu-4)(\nu-1)} g_3 g_4$$

$$\left\langle \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{5}{2}} \right\rangle = \frac{12\nu(\nu+1)}{7(\nu-4)(\nu+2)} g_3 g_4$$

$$\left\langle \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{7}{2}} \right\rangle = \frac{(\nu+1)(\nu^2+35\nu+84)}{(\nu-4)(\nu+7)(\nu+9)} g_3 g_4$$

$$\langle \mathcal{C}^{W_3W_5}_{4,1} \mathcal{C}^{W_3W_5}_{4,1} \rangle = \frac{(\nu-25)(\nu+5)}{2(\nu-7)(\nu-4)} g_3 g_5$$

$$\langle \mathcal{C}^{W_3W_5}_{4,3} \mathcal{C}^{W_3W_5}_{4,3} \rangle = \frac{15\nu(\nu+5)}{8(\nu-4)(\nu+3)} g_3 g_5$$

$$\langle \mathcal{C}^{W_3W_3}_{4,0} \mathcal{C}^{W_3W_3}_{4,0} \rangle = -\frac{15(\nu-11)(\nu-7)\nu(3\nu^2-17\nu-10)}{16(\nu-9)^2(\nu-6)(\nu-4)(3\nu-11)} g_3^2$$

$$\langle \mathcal{C}^{W_3W_3}_{4,0} \mathcal{C}^{W_4W_4}_{4,0} \rangle = \frac{3(\nu-7)(\nu+1)(29\nu^3-440\nu^2+741\nu+4950)}{4(\nu-9)^3(\nu-6)(\nu-4)(3\nu-11)} g_3 g_4$$

$$\langle \mathcal{C}^{W_4W_4}_{4,0} \mathcal{C}^{W_4W_4}_{4,0} \rangle = \frac{3(\nu+1)^2(2\nu^6-130\nu^5+2915\nu^4-32595\nu^3+217053\nu^2-865215\nu+1559250)}{5(\nu-9)^4(\nu-6)(\nu-4)\nu(3\nu-11)} g_4^2$$

$$\det\begin{pmatrix} \langle \mathcal{C}^{W_3W_3}_{4,0} \mathcal{C}^{W_3W_3}_{4,0} \rangle & \langle \mathcal{C}^{W_3W_3}_{4,0} \mathcal{C}^{W_4W_4}_{4,0} \rangle \\ \langle \mathcal{C}^{W_3W_3}_{4,0} \mathcal{C}^{W_4W_4}_{4,0} \rangle & \langle \mathcal{C}^{W_4W_4}_{4,0} \mathcal{C}^{W_4W_4}_{4,0} \rangle \end{pmatrix} = -\frac{9(\nu-25)(\nu-16)(\nu-7)\nu(\nu+1)^2}{8(\nu-9)^3(\nu-6)(\nu-4)(3\nu-11)} g_3^2 g_4^2$$

$$\langle \mathcal{C}^{W_3W_5}_{4,4} \mathcal{C}^{W_3W_5}_{4,4} \rangle = \frac{(\nu^3+46\nu^2+109\nu+384)}{(\nu-4)(\nu+9)(\nu+11)} g_3 g_5$$

$$\langle \mathcal{C}^{W_3W_5}_{4,4} \mathcal{C}^{W_4W_4}_{4,4} \rangle = \frac{2(\nu-1)(\nu+1)(\nu^2+5\nu+18)}{3(\nu-9)\nu(\nu+9)(\nu+11)g_3} c_{33}{}^{4} c_{34}{}^{5} g_4 g_5$$

$$\langle \mathcal{C}^{W_4W_4}_{4,4} \mathcal{C}^{W_4W_4}_{4,4} \rangle = \frac{2(\nu+1)^2(\nu^4+34\nu^3-35\nu^2-1440\nu-6480)}{(\nu-9)(\nu-4)\nu(\nu+5)(\nu+9)(\nu+11)} g_4^2$$

$$\det\begin{pmatrix} \langle \mathcal{C}^{W_3W_5}_{4,4} \mathcal{C}^{W_3W_5}_{4,4} \rangle & \langle \mathcal{C}^{W_3W_5}_{4,4} \mathcal{C}^{W_4W_4}_{4,4} \rangle \\ \langle \mathcal{C}^{W_3W_5}_{4,4} \mathcal{C}^{W_4W_4}_{4,4} \rangle & \langle \mathcal{C}^{W_4W_4}_{4,4} \mathcal{C}^{W_4W_4}_{4,4} \rangle \end{pmatrix} = \frac{2(\nu+1)^2(\nu^3+70\nu^2+469\nu+180)}{(\nu-9)(\nu-4)(\nu+9)(\nu+11)^2} g_3 g_4^2 g_5.$$

$$\langle \mathcal{C}^{W_3W_3}_{4,2} \mathcal{C}^{W_3W_3}_{4,2} \rangle = -\frac{63(\nu-15)(\nu-9)(\nu+5)(3\nu^3+5\nu^2-138\nu-20)g_3^2}{8(\nu-4)(3\nu-1)(\nu^2-12\nu-57)^2},$$

$$\langle \mathcal{C}^{W_3W_3}_{4,2} \mathcal{C}^{W_3W_5}_{4,2} \rangle = \frac{7c_{33}{}^{4} c_{34}{}^{5}(\nu-15)(\nu-1)(\nu+5)(6\nu^6+167\nu^5+266\nu^4-6207\nu^3+66088\nu^2+65680\nu+14400)g_5}{40(\nu-16)\nu(3\nu-1)(\nu^2-12\nu-57)^2(\nu^2+15\nu+8)},$$

$$\langle \mathcal{C}^{W_3W_3}_{4,2} \mathcal{C}^{W_4W_4}_{4,2} \rangle = \frac{126(\nu-15)(\nu+1)(\nu+5)(\nu^5+77\nu^4+356\nu^3-6722\nu^2-3072\nu+360)g_3g_4}{5(\nu-4)\nu(3\nu-1)(\nu^2-12\nu-57)^2(\nu^2+15\nu+8)},$$

$$\begin{aligned}\langle \mathcal{C}^{W_3W_5}_{4,2} \mathcal{C}^{W_3W_5}_{4,2} \rangle = {} & \frac{2g_3g_5}{35(\nu-16)(\nu-15)(\nu-4)\nu(3\nu-1)(\nu^2-12\nu-57)^2(\nu^2+15\nu+8)^2} \\ & \times (75\nu^{13}-4225\nu^{12}+40586\nu^{11}+1586164\nu^{10}-29938723\nu^9-132686081\nu^8 \\ & +4461908636\nu^7+2651104202\nu^6-136281685134\nu^5-11758354\,64700\nu^4 \\ & -722010916800\nu^3+1328678712000\nu^2+1390037760000\nu+342921600000),\end{aligned}$$

$$\begin{aligned}\langle \mathcal{C}^{W_3W_5}_{4,2} \mathcal{C}^{W_4W_4}_{4,2} \rangle = {} & -\frac{2c_{33}{}^{4} c_{34}{}^{5}(\nu-1)(\nu+1)(\nu+5)g_4g_5}{175(\nu-16)(\nu-15)(\nu-9)\nu^2(3\nu-1)(\nu^2-12\nu-57)^2(\nu^2+15\nu+8)^2 g_3} \\ & \times (275\nu^{11}-8977\nu^{10}-52588\nu^9+4009341\nu^8-13085243\nu^7-495583967\nu^6 \\ & +2173165606\nu^5+10317029523\nu^4+70708858830\nu^3 \\ & +61989055200\nu^2+8993268000\nu-2857680000)\end{aligned}$$

$$\det\begin{pmatrix} \langle \mathcal{C}_{4,2}^{W_3W_3}\mathcal{C}_{4,2}^{W_3W_3}\rangle & \langle \mathcal{C}_{4,2}^{W_3W_3}\mathcal{C}_{4,2}^{W_3W_5}\rangle & \langle \mathcal{C}_{4,2}^{W_3W_3}\mathcal{C}_{4,2}^{W_4W_4}\rangle \\ \langle \mathcal{C}_{4,2}^{W_3W_3}\mathcal{C}_{4,2}^{W_3W_5}\rangle & \langle \mathcal{C}_{4,2}^{W_3W_5}\mathcal{C}_{4,2}^{W_3W_5}\rangle & \langle \mathcal{C}_{4,2}^{W_3W_5}\mathcal{C}_{4,2}^{W_4W_4}\rangle \\ \langle \mathcal{C}_{4,2}^{W_3W_3}\mathcal{C}_{4,2}^{W_4W_4}\rangle & \langle \mathcal{C}_{4,2}^{W_3W_5}\mathcal{C}_{4,2}^{W_4W_4}\rangle & \langle \mathcal{C}_{4,2}^{W_4W_4}\mathcal{C}_{4,2}^{W_4W_4}\rangle \end{pmatrix} = \\ = -\frac{810(\nu-36)(\nu-25)(\nu-16)(\nu+1)^2(\nu+5)^2 g_3^3 g_4^2 g_5}{7(\nu-4)^2(3\nu-1)\left(\nu^2-12\nu-57\right)^2}\,.$$

$$W_5 \times W_6 \supset c_{56}{}^3 W_3, W_6 \times W_6 \supset g_6 \mathbb{1}$$

$$c_{56}{}^3 g_3 = \lambda_{563} = \lambda_{356} = c_{35}{}^6 g_6 \;\Rightarrow\; c_{56}{}^3 = \frac{g_6}{g_3} c_{35}{}^6.$$

$$\beta_p(z_1)\gamma_{p'}(z_2) = -\frac{\delta_{pp'}}{z_{12}} + \Re, b_p(z_1)c_{p'}(z_2) = \frac{\delta_{pp'}}{z_{12}} + \Re, p, p' = 2, \dots, N.$$

$$\begin{aligned} J(z,y) &= J^+(z) + J^0(z)y + J^-(z)y^2 \\ G(z,y) &= G^+(z) + G^-(z)y \\ \tilde{G}(z,y) &= \tilde{G}^+(z) + \tilde{G}^-(z)y \\ W_p(z,y)\big|_{y=0} &= W_p^{\text{h.w.}}(z) \\ G_{W_p}(z,y)\big|_{y=0} &= G_{W_p}^{\text{h.w.}}(z) \end{aligned}$$

$$\begin{aligned} &J^0 = \sum_{p=2}^{N} \left[p\beta_p\gamma_p + (p-1)b_p c_p\right] \\ &G^- = \sum_{p=2}^{N} b_p\gamma_p, \tilde{G}^+ = \sum_{p=2}^{N} \left[p\beta_p\partial_z c_p + (p-1)\partial_z\beta_p c_p\right] \\ &T = \sum_{p=2}^{N} \left[-\frac{1}{2}p\beta_p\partial_z\gamma_p + \left(1-\frac{1}{2}p\right)\partial\beta_p\gamma_p - \frac{1}{2}(p+1)b_p\partial_z c_p + \frac{1}{2}(1-p)\partial_z b_p c_p\right] \\ &J^+ = \beta_2, G^+ = b_2, W_p^{\text{h.w.}} = \beta_p, G_{W_p}^{\text{h.w.}} = b_p \end{aligned}$$

$$c = -3\sum_{p=2}^{N} (2p-1) = -3(N^2-1)$$

	h	m	$h-m$	$h+m$	r
β_p	$\frac{1}{2}p$	$\frac{1}{2}p$	0	p	0
b_p	$\frac{1}{2}(p+1)$	$\frac{1}{2}(p-1)$	1	p	$+\frac{1}{2}$
c_p	$-\frac{1}{2}(p-1)$	$-\frac{1}{2}(p-1)$	0	$1-p$	$-\frac{1}{2}$
γ_p	$1-\frac{1}{2}p$	$-\frac{1}{2}p$	1	$1-p$	0
∂_z	1	0	1	1	0

$$J^- = \beta_2\gamma_2^2 + \gamma_2 b_2 c_2 - \frac{3}{2}\partial_z\gamma_2$$

$$J^- = \beta_2\gamma_2^2 + 3\beta_2\gamma_2\gamma_3 + \gamma_2 b_2 c_2 + 2\gamma_2 b_3 c_3 + \gamma_3 b_3 c_2 + \Lambda\beta_2^2\gamma_3^2 + 2\Lambda\beta_2\gamma_3 b_2 c_3 - 4\partial_z\gamma_2$$

$$g_3 = \frac{40}{3}\Lambda$$

$$\begin{aligned}J^{-} = {} & 5\Lambda_1^2\beta_2^2\gamma_4 b_2 c_4 - \frac{11}{3}\Lambda_1\beta_2\gamma_3 b_3 c_4 - \frac{5}{3}\Lambda_1\beta_2\gamma_4 b_3 c_3 + \frac{7}{3}\Lambda_1\beta_2\gamma_4 b_4 c_4 + 2\Lambda_1\Lambda_2\beta_2\gamma_3 b_2 c_3 \\ & - \frac{5}{2}\Lambda_1\beta_3\gamma_3 b_2 c_4 - \frac{17}{6}\Lambda_1\beta_3\gamma_4 b_2 c_3 + \frac{7}{3}\Lambda_1\beta_4\gamma_4 b_2 c_4 + \frac{17}{2}\Lambda_1\Lambda_2^{-1}\beta_3\gamma_4 b_3 c_4 \\ & -\Lambda_2\gamma_3 b_4 c_3 + \gamma_2 b_2 c_2 + 2\gamma_2 b_3 c_3 + 3\gamma_2 b_4 c_4 + \gamma_3 b_3 c_2 + \gamma_4 b_4 c_2 - \frac{1}{2}\Lambda_1\partial_z b_2 c_4 - \frac{1}{2}\Lambda_1\partial_z\beta_2\gamma_4 \\ & + \frac{5}{3}\Lambda_1^2\beta_2^3\gamma_4^2 + \Lambda_1\Lambda_2\beta_2^2\gamma_3^2 + \frac{7}{3}\Lambda_1\beta_4\beta_2\gamma_4^2 - \frac{16}{3}\Lambda_1\beta_3\beta_2\gamma_3\gamma_4 + \frac{7}{6}\Lambda_1 b_2 b_3 c_3 c_4 - \Lambda_2\beta_4\gamma_3^2 \\ & + \frac{17}{4}\Lambda_1\Lambda_2^{-1}\beta_3^2\gamma_4^2 + \beta_2\gamma_2^2 + 3\beta_3\gamma_2\gamma_3 + 4\beta_4\gamma_2\gamma_4 - \frac{15}{2}\partial_z\gamma_2\end{aligned}$$

$$\begin{aligned} g_3 &= \frac{85}{2}\Lambda_1\Lambda_2, & g_4 &= 595\Lambda_1^2 \\ c_{33}{}^{4} &= -2\Lambda_2, & c_{34}{}^{3} &= -28\Lambda_1 \\ c_{44}{}^{4} &= \frac{55}{3}\Lambda_1, & c_{44}{}^{C^{W_3W_3}_{3,3}} &= \frac{17}{2}\Lambda_1\Lambda_2^{-1}\end{aligned}$$

$$\begin{aligned} c_{33}^{4} &= \frac{4\sqrt{7}}{\sqrt{85}}\frac{g_3}{\sqrt{g_4}}, & c_{34}^{3} &= \frac{4\sqrt{7}}{\sqrt{85}}\sqrt{g_4} \\ c_{44}^{4} &= -\frac{11\sqrt{5}}{3\sqrt{119}}\sqrt{g_4}, c_{44} & C^{W_3W_3}_{3,3} &= -\frac{2\sqrt{7}}{\sqrt{85}}\frac{g_3}{\sqrt{g_4}}\end{aligned}$$

$$\begin{aligned}J^{-} = {} & \frac{467}{14560}\Lambda_1\Lambda_2\Lambda_3\beta_2^4\gamma_5^2 + \frac{261}{6400}\Lambda_1^2\beta_2^3\gamma_4^2 - \frac{3}{560}\Lambda_1\Lambda_2\beta_2^3\gamma_3\gamma_5 + \frac{467}{3640}\Lambda_1\Lambda_2\Lambda_3\beta_2^3\gamma_5 b_2 c_5 \\ & + \frac{783}{6400}\Lambda_1^2\beta_2^2\gamma_4 b_2 c_4 - \frac{3}{280}\Lambda_1\Lambda_2\beta_2^2\gamma_3 b_2 c_5 - \frac{3}{560}\Lambda_1\Lambda_2\beta_2^2\gamma_5 b_2 c_3 + \frac{209}{600}\Lambda_1\Lambda_3\beta_2^2\gamma_4 b_3 c_5 \\ & + \frac{113}{800}\Lambda_1\Lambda_3\beta_2^2\gamma_5 b_3 c_4 + \frac{47}{96}\Lambda_1\Lambda_3\beta_2^2\beta_3\gamma_4\gamma_5 - \frac{134}{273}\Lambda_2\Lambda_3\beta_2^2\beta_4\gamma_5^2 - \frac{134}{273}\Lambda_2\Lambda_3\beta_2^2\gamma_5 b_4 c_5 \\ & + \frac{9}{80}\Lambda_1\Lambda_2\Lambda_3^{-1}\beta_2^2\gamma_3^2 + \beta_2\gamma_2^2 + \frac{23}{18}\Lambda_3^2\beta_2\beta_3^2\gamma_5^2 + \frac{23}{9}\Lambda_3^2\beta_2\beta_3\gamma_5 b_3 c_5 + \frac{7}{80}\Lambda_1\beta_2\beta_4\gamma_4^2 \\ & + \frac{17}{25}\Lambda_1\beta_2\gamma_3 b_3 c_4 + \frac{8}{25}\Lambda_1\beta_2\gamma_4 b_3 c_3 + \frac{7}{80}\Lambda_1\beta_2\gamma_4 b_4 c_4 - \frac{119}{200}\Lambda_1\beta_2\gamma_4 b_5 c_5 + \Lambda_1\beta_2\beta_3\gamma_3\gamma_4 \\ & - \frac{157}{400}\Lambda_1\beta_2\gamma_5 b_5 c_4 - \frac{79}{80}\Lambda_1\beta_2\beta_5\gamma_4\gamma_5 + \Lambda_2\beta_2\gamma_3 b_4 c_5 + \frac{2}{7}\Lambda_2\beta_2\gamma_5 b_4 c_3 + \frac{9}{7}\Lambda_2\beta_2\beta_4\gamma_3\gamma_5 \\ & - \frac{6}{25}\Lambda_1\Lambda_3\beta_2 b_2 b_3 c_4 c_5 + \frac{137}{300}\Lambda_1\Lambda_3\beta_2\beta_3\gamma_4 b_2 c_5 + \frac{209}{400}\Lambda_1\Lambda_3\beta_2\beta_3\gamma_5 b_2 c_4 - \frac{268}{273}\Lambda_2\Lambda_3\beta_2\beta_4\gamma_5 b_2 c_5 \\ & + \frac{9}{40}\Lambda_1\Lambda_2\Lambda_3^{-1}\beta_2\gamma_3 b_2 c_3 + \frac{23}{18}\Lambda_3^2\beta_3^2\gamma_5 b_2 c_5 + \gamma_2 b_2 c_2 + 2\gamma_2 b_3 c_3 + 3\gamma_2 b_4 c_4 + 4\gamma_2 b_5 c_5 + \gamma_3 b_3 c_2 \\ & + 3\beta_3\gamma_2\gamma_3 + \gamma_4 b_4 c_2 + 4\beta_4\gamma_2\gamma_4 + \gamma_5 b_5 c_2 + 5\beta_5\gamma_2\gamma_5 - \frac{1}{5}\Lambda_1 b_2 b_3 c_3 c_4 + \frac{43}{400}\Lambda_1 b_2 b_5 c_4 c_5 \\ & + \frac{12}{25}\Lambda_1\beta_3\gamma_3 b_2 c_4 + \frac{13}{25}\Lambda_1\beta_3\gamma_4 b_2 c_3 + \frac{7}{80}\Lambda_1\beta_4\gamma_4 b_2 c_4 - \frac{39}{80}\Lambda_1\beta_5\gamma_4 b_2 c_5 - \frac{1}{2}\Lambda_1\beta_5\gamma_5 b_2 c_4 \\ & - \frac{3}{7}\Lambda_2 b_2 b_4 c_3 c_5 + \frac{4}{7}\Lambda_2\beta_4\gamma_3 b_2 c_5 + \frac{5}{7}\Lambda_2\beta_4\gamma_5 b_2 c_3 - \frac{5}{2}\Lambda_3\beta_3\beta_5\gamma_5^2 - \frac{3}{2}\Lambda_3 b_3 b_4 c_4 c_5 \\ & + \frac{4}{3}\Lambda_3\beta_3\gamma_3 b_3 c_5 + \frac{17}{6}\Lambda_3\beta_3\gamma_4 b_4 c_5 + \frac{4}{3}\Lambda_3\beta_4\gamma_4 b_3 c_5 + \frac{2}{3}\Lambda_3\beta_3\gamma_5 b_3 c_3 + \Lambda_3\beta_3\gamma_5 b_4 c_4 \\ & - \frac{5}{2}\Lambda_3\beta_3\gamma_5 b_5 c_5 + \frac{5}{2}\Lambda_3\beta_4\gamma_5 b_3 c_4 - \frac{5}{2}\Lambda_3\beta_5\gamma_5 b_3 c_5 + \Lambda_3\beta_3^2\gamma_3\gamma_5 + \frac{23}{6}\Lambda_3\beta_3\beta_4\gamma_4\gamma_5 \\ & + \frac{1000}{273}\Lambda_1^{-1}\Lambda_2\Lambda_3\beta_4^2\gamma_5^2 + \frac{2000}{273}\Lambda_1^{-1}\Lambda_2\Lambda_3\beta_4\gamma_5 b_4 c_5 + \frac{91}{80}\Lambda_2^{-1}\Lambda_1\Lambda_3\beta_3^2\gamma_4^2 + \frac{91}{40}\Lambda_2^{-1}\Lambda_1\Lambda_3\beta_3\gamma_4 b_3 c_4 \\ & + \frac{3}{25}\Lambda_1\partial_z b_2 c_4 + \frac{2}{3}\Lambda_3\partial_z b_3 c_5 + \frac{3}{25}\Lambda_1\partial_z\beta_2\gamma_4 + \frac{2}{3}\Lambda_3\partial_z\beta_3\gamma_5 - 12\partial_z\gamma_2 + \frac{117}{200}\Lambda_1\Lambda_3^{-1}\gamma_3 b_5 c_4 \\ & + \frac{39}{100}\Lambda_1\Lambda_3^{-1}\gamma_4 b_5 c_3 + \frac{39}{40}\Lambda_1\Lambda_3^{-1}\beta_5\gamma_3\gamma_4 + \frac{6}{7}\Lambda_2\Lambda_3^{-1}\beta_4\gamma_3^2 + \frac{6}{7}\Lambda_2\Lambda_3^{-1}\gamma_3 b_4 c_3.\end{aligned}$$

$$\mathcal{C}^{W_3W_4}_{\frac{7}{2},\frac{3}{2}}, \mathcal{C}^{W_3W_3}_{4,0} + \frac{40}{91}\frac{\Lambda_2}{\Lambda_1\Lambda_3}\mathcal{C}^{W_4W_4}_{4,0}, \mathcal{C}^{W_3W_3}_{4,2} + \frac{63}{143}\frac{1}{\Lambda_3}\mathcal{C}^{W_3W_5}_{4,2} + \frac{800}{1859}\frac{\Lambda_2}{\Lambda_1\Lambda_3}\mathcal{C}^{W_4W_4}_{4,2}$$

$$
\begin{array}{lll}
g_3 = \frac{117}{10}\Lambda_1\Lambda_2\Lambda_3^{-1}, & c_{33}{}^{4} = \frac{12}{7}\Lambda_2\Lambda_3^{-1}, & c_{34}{}^{3} = \frac{15}{2}\Lambda_1 \\
c_{34}{}^{5} = \frac{39}{40}\Lambda_1\Lambda_3^{-1}, & g_4 = \frac{819}{16}\Lambda_1^2, & c_{44}{}^{4} = \frac{67}{20}\Lambda_1 \\
c_{44}\mathcal{C}_{3,3}^{W_3W_3} = \frac{91}{40}\Lambda_1\Lambda_3\Lambda_2^{-1}, & c_{44}\mathcal{C}_{3,1}^{W_3W_3} = -\frac{273}{80}\Lambda_1\Lambda_3\Lambda_2^{-1}, & c_{35}{}^{4} = \frac{78}{7}\Lambda_2 \\
c_{35}\mathcal{C}_{3,3}^{W_3W_3} = \Lambda_3, & c_{35}\mathcal{C}_{3,1}^{W_3W_3} = -\frac{23}{6}\Lambda_3, & c_{45}{}^{3} = \frac{195}{4}\Lambda_1\Lambda_3 \\
c_{45}{}^{5} = -\frac{261}{40}\Lambda_1, & c_{45}{}^{\frac{7}{2},\frac{5}{2}} = \frac{3}{2}\Lambda_3, & c_{45}{}^{\mathcal{C}_{\frac{7}{2},\frac{1}{2}}^{W_3W_4}} = -\frac{37}{6}\Lambda_3 \\
c_{45}\mathcal{C}_{\frac{7}{2},\frac{7}{2}}^{W_3W_4} = \frac{23}{6}\Lambda_3, & c_{55}{}^{4} = -\frac{522}{7}\Lambda_2\Lambda_3, & c_{55}\mathcal{C}_{3,1}^{W_3W_3} = -\frac{77}{36}\Lambda_3^2 \\
c_{55}\mathcal{C}_{3,3}^{W_3W_3} = \frac{173}{6}\Lambda_3^2, & c_{55}\mathcal{C}_{4,4}^{W_3W_5} = -5\Lambda_3, & c_{55}\mathcal{C}_{4,4}^{W_4W_4} = \frac{2000}{273}\Lambda_2\Lambda_3\Lambda_1^{-1} \\
\bar{c}_{55}\mathcal{C}_{4,2}^{W_3W_5} = -\frac{3981}{3575}\Lambda_3, & \bar{c}_{55}\mathcal{C}_{4,2}^{W_4W_4} = -\frac{928192}{117117}\frac{\Lambda_2\Lambda_3}{\Lambda_1}, & \bar{c}_{55}\mathcal{C}_{4,0}^{W_4W_4} = \frac{9280}{2457}\frac{\Lambda_2\Lambda_3}{\Lambda_1}
\end{array}
$$

$$\Lambda_1 = \frac{4}{3\sqrt{91}}\sqrt{g_4}, \Lambda_2 = \frac{\sqrt{7}}{6\sqrt{26}}\frac{\sqrt{g_3}\sqrt{g_5}}{\sqrt{g_4}}, \Lambda_3 = \frac{1}{5\sqrt{2}}\frac{\sqrt{g_5}}{\sqrt{g_3}}$$

$$[A_m(y_1), B_n(y_2)] = \sum_C \kappa_{AB}^C P_{h_A,h_B;h_C}(m,n) y_{12}^{j_A+j_B-j_C} \widehat{\mathcal{D}}_{j_A,j_B;j_C}(y_{12}, \partial_{y_2}) C_{m+n}(y_2).$$

$$h_C \leq h_A + h_B - 1, h_C - h_A - h_B \in \mathbb{Z}, j_C \in \{|j_A - j_B|, |j_A - j_B| + 1, \ldots, j_A + j_B\}$$

$$P_{h_A,h_B;h_C}(m,n) = \sum_{k=0}^{h_{ABC}-1} \frac{(m+h_A-1)^{\downarrow}_{h_{ABC}-1-k}}{(h_{ABC}-1-k)!}(-m-n-h_C)^{\downarrow}_k \frac{(h_C+h_A-h_B)_k}{k!\,(2h_C)_k},$$

$$[A, B] = \sum_C \kappa_{AB}^C C$$

$$\kappa_{BA}{}^{C} = (-1)^{|A||B|}\kappa_{AB}{}^{C},$$

$$A(z, y) = \sum_m z^{-m-h_A} A_m(y)$$

$$[A_m(y_1), B_n(y_2)] = \int_0 \frac{dz_2}{2\pi i}\int_{z_2} \frac{dz_1}{2\pi i} z_1^{m+h_A-1} z_2^{n+h_B-1} A(z_1, y_1) B(z_2, y_2)$$

$$
\begin{aligned}
[J_0(y_1), J_0(y_2)] &= 2P_{1,1;1}(0,0) y_{12}\widehat{\mathcal{D}}_{1,1;1} J_0(y_2) \\
[J_0(y_1), G_n(y_2)] &= P_{1,\frac{3}{2};\frac{3}{2}}(0,n) y_{12}\widehat{\mathcal{D}}_{1,\frac{1}{2};\frac{1}{2}} G_n(y_2) \\
[J_0(y_1), \tilde{G}_n(y_2)] &= P_{1,\frac{3}{2};\frac{3}{2}}(0,n) y_{12}\widehat{\mathcal{D}}_{1,\frac{1}{2};\frac{1}{2}} \tilde{G}_n(y_2) \\
[J_0(y_1), T_n] &= 0 \\
[G_m(y_1), G_n(y_2)] &= 0 \\
[G_m(y_1), \tilde{G}_n(y_2)] &= 2P_{\frac{3}{2},\frac{3}{2};1}(m,n)\widehat{\mathcal{D}}_{\frac{1}{2},\frac{1}{2};1} J_{m+n}(y_2) - P_{\frac{3}{2},\frac{3}{2};2}(m,n) y_{12}\widehat{\mathcal{D}}_{\frac{1}{2},\frac{1}{2};0} T_{m+n} \\
[G_m(y_1), T_n] &= \frac{3}{2} P_{\frac{3}{2},2;\frac{3}{2}}(m,n)\widehat{\mathcal{D}}_{\frac{1}{2},0;\frac{1}{2}} G_{m+n}(y_2) \\
[\tilde{G}_m(y_1), \tilde{G}_n(y_2)] &= 0 \\
[\tilde{G}_m(y_1), T_n] &= \frac{3}{2} P_{\frac{3}{2},2;\frac{3}{2}}(m,n)\widehat{\mathcal{D}}_{\frac{1}{2},0;\frac{1}{2}} \tilde{G}_{m+n}(y_2) \\
[T_m, T_n] &= 2P_{2,2;2}(m,n) T_{m+n}
\end{aligned}
$$

$$
\begin{array}{lll}
P_{1,1;1}(m,n) \equiv 1, & P_{1,\frac{3}{2};\frac{3}{2}}(m,n) \equiv 1, & P_{\frac{3}{2},\frac{3}{2};1}(m,n) = \frac{1}{2}(m-n) \\
P_{\frac{3}{2},\frac{3}{2};2}(m,n) \equiv 1, & P_{\frac{3}{2},2;\frac{3}{2}}(m,n) = \frac{1}{3}(2m-n), & P_{2,2;2}(m,n) = \frac{1}{2}(m-n)
\end{array}
$$

$$[J,J]=2J,\quad [J,G]=G,\quad [J,\tilde{G}]=\tilde{G},[J,T]=0,$$
$$[G,G]=0,\qquad [G,\tilde{G}]=2J-T,[G,T]=\frac{3}{2}G,$$
$$[\tilde{G},\tilde{G}]=0,\qquad [\tilde{G},T]=\frac{3}{2}\tilde{G},$$
$$[T,T]=2T.$$

$$[J,J]=-k\mathbb{1}+2J,[J,G]=G,[J,\tilde{G}]=\tilde{G},[J,T]=J,$$
$$[G,G]=0,[G,\tilde{G}]=-2k\mathbb{1}+2J-T,[G,T]=\frac{3}{2}G,$$
$$[\tilde{G},\tilde{G}]=0,[\tilde{G},T]=\frac{3}{2}\tilde{G},$$
$$[T,T]=\frac{c}{2}\mathbb{1}+2T.$$

$$[J_m,J_n]\supset\mathbb{1}_{m+n}:\quad P_{1,1;0}(m,n)=-n$$
$$[G_m,\tilde{G}_n]\supset\mathbb{1}_{m+n}:\quad P_{\frac{3}{2},\frac{3}{2};0}(m,n)=\frac{1}{8}(4n^2-1)$$
$$[T_m,T_n]\supset\mathbb{1}_{m+n}:\quad P_{2,2;0}(m,n)=-\frac{1}{6}(n^2-n)$$

$$\text{for } [J_m,T_n]\supset J_{m+n}: P_{1,2;1}(m,n)=m$$

$$[J,W_p]=pW_p,\qquad [J,\tilde{G}_{W_p}]=(p-1)\tilde{G}_{W_p},$$
$$[G,W_p]=G_{W_p},\qquad [G,\tilde{G}_{W_p}]=pW_p-T_{W_p},$$
$$[\tilde{G},W_p]=\tilde{G}_{W_p},\qquad [\tilde{G},\tilde{G}_{W_p}]=0,$$
$$[T,W_p]=\frac{p}{2}W_p,\qquad [T,\tilde{G}_{W_p}]=\left(\frac{p}{2}+\frac{1}{2}\right)\tilde{G}_{W_p},$$
$$[J,G_{W_p}]=(p-1)G_{W_p},\qquad [J,T_{W_p}]=(p-2)T_{W_p},$$
$$[G,G_{W_p}]=0,\qquad [G,T_{W_p}]=\left(p-\frac{1}{p}\right)G_{W_p},$$
$$[\tilde{G},G_{W_p}]=-pW_p+T_{W_p},\qquad [\tilde{G},T_{W_p}]=\left(p-\frac{1}{p}\right)\tilde{G}_{W_p},$$
$$[T,G_{W_p}]=\left(\frac{p}{2}+\frac{1}{2}\right)G_{W_p},\qquad [T,T_{W_p}]=\left(\frac{p}{2}+1\right)T_{W_p}.$$

$$\text{for } \left[J_m,\left(T_{W_p}\right)_n\right]\supset\left(W_p\right)_{m+n}: P_{1,\frac{p}{2}+1;\frac{p}{2}}(m,n)=m$$

$$[W_{p_1}, W_{p_2}] = \delta_{p_1,p_2}\gamma_{p_1}\left(J - \frac{1}{6}T\right) + \sum_q \kappa_{p_1p_2}{}^q\left(W_q - \frac{(p_1-p_2+q)(-p_1+p_2+q)}{4(q-1)q(q+1)}T_{W_q}\right),$$

$$\left[W_{p_1}, G_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{1}{2}G + \sum_q \kappa_{p_1p_2}{}^q\frac{q-p_1+p_2}{2q}G_{W_q},$$

$$\left[W_{p_1}, \tilde{G}_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{1}{2}\tilde{G} + \sum_q \kappa_{p_1p_2}{}^q\frac{q-p_1+p_2}{2q}\tilde{G}_{W_q},$$

$$\left[W_{p_1}, T_{W_{p_2}}\right] = \sum_q \kappa_{p_1p_2}{}^q\left(-\frac{(-p_1-p_2+q)(-p_1+p_2+q)}{4p_2}W_q\right.$$
$$\left.+\frac{(-p_1+p_2+q-2)(-p_1+p_2+q)(-p_1+p_2+q+2)(p_1+p_2+q)}{16p_2(q-1)q(q+1)}T_{W_q}\right),$$

$$\left[G_{W_{p_1}}, W_{p_2}\right] = \sum_q \kappa_{p_1p_2}{}^q\frac{p_1-p_2+q}{2q}G_{W_q},$$

$$\left[G_{W_{p_1}}, G_{W_{p_2}}\right] = 0,$$

$$\left[G_{W_{p_1}}, \tilde{G}_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\left((p_1-1)J - \frac{1}{6}(p_1+1)T\right)$$
$$+\sum_q \kappa_{p_1p_2}{}^q\left(\frac{1}{2}(p_1+p_2-q)W_q - \frac{(p_1-p_2+q)(-p_1+p_2+q)(p_1+p_2+q)}{8(q-1)q(q+1)}T_{W_q}\right),$$

$$\left[G_{W_{p_1}}, T_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{(p_1-1)(p_1+1)}{2p_1}G + \sum_q \kappa_{p_1p_2}{}^q\frac{-(-p_1-p_2+q)(-p_1+p_2+q)(p_1+p_2+q)}{8p_2q}G_{W_q},$$

$$\left[\tilde{G}_{W_{p_1}}, W_{p_2}\right] = \sum_q \kappa_{p_1p_2}{}^q\frac{p_1-p_2+q}{2q}\tilde{G}_{W_q},$$

$$\left[\tilde{G}_{W_{p_1}}, G_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\left(-(p_1-1)J + \frac{1}{6}(p_1+1)T\right)$$
$$+\sum_q \kappa_{p_1p_2}{}^q\left(\frac{1}{2}(-p_1-p_2+q)W_q + \frac{(p_1-p_2+q)(-p_1+p_2+q)(p_1+p_2+q)}{8(q-1)q(q+1)}T_{W_q}\right),$$

$$\left[\tilde{G}_{W_{p_1}}, \tilde{G}_{W_{p_2}}\right] = 0,$$

$$\left[\tilde{G}_{W_{p_1}}, T_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{(p_1-1)(p_1+1)}{2p_1}\tilde{G} + \sum_q \kappa_{p_1p_2}{}^q\frac{-(-p_1-p_2+q)(-p_1+p_2+q)(p_1+p_2+q)}{8p_2q}\tilde{G}_{W_q},$$

$$\left[T_{W_{p_1}}, W_{p_2}\right] = \sum_q \kappa_{p_1p_2}{}^q\left(\frac{(p_1-p_2+q)(p_1+p_2-q)}{4p_1}W_q + \frac{(p_1-p_2+q-2)(p_1-p_2+q)(p_1-p_2+q+2)(p_1+p_2+q)}{16p_1(q-1)q(q+1)}T_{W_q}\right),$$

$$\left[T_{W_{p_1}}, G_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{(p_1-1)(p_1+1)}{2p_1}G + \sum_q \kappa_{p_1p_2}{}^q\frac{-(-p_1-p_2+q)(p_1-p_2+q)(p_1+p_2+q)}{8p_1q}G_{W_q},$$

$$\left[T_{W_{p_1}}, \tilde{G}_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\frac{(p_1-1)(p_1+1)}{2p_1}G + \sum_q \kappa_{p_1p_2}{}^q\frac{-(-p_1-p_2+q)(p_1-p_2+q)(p_1+p_2+q)}{8p_1q}G_{W_q},$$

$$\left[T_{W_{p_1}}, T_{W_{p_2}}\right] = \delta_{p_1,p_2}\gamma_{p_1}\left(-\frac{(p_1-2)(p_1^2-1)}{p_1}J + \frac{(p_1+2)(p_1^2-1)}{6p_1}T\right)$$
$$+\sum_q \kappa_{p_1p_2}{}^q\left(-\frac{(p_1+p_2-q-2)(p_1+p_2-q)(p_1+p_2-q+2)(p_1+p_2+q)}{16p_1p_2}W_q\right.$$
$$\left.-\frac{(-p_1-p_2+q)(p_1-p_2+q)(-p_1+p_2+q)(p_1+p_2+q-2)(p_1+p_2+q)(p_1+p_2+q+2)}{64p_1p_2(q-1)q(q+1)}T_{W_q}\right),$$

$$\max(3, |p_1-p_2|+2) \le q \le p_1+p_2-2, q-p_1-p_2 \in 2\mathbb{Z}$$

$$\left[\mathbb{J}, \left[\mathbb{W}_{p_1}, \mathbb{W}_{p_2}\right]\right] \pm \text{ cyclic } = 0$$

$$\left[\left(W_{p_1}\right)_m, \left(W_{p_1+2}\right)_n\right] \supset f^J_{p_1,p_1+2}P_{\frac{p_1}{2},\frac{p_1}{2}+1;1}(m,n)J_{m+n}$$

$$P_{h,h+1;1}(m,n) = \frac{\prod_{j=-h+1}^{h+1}(m+j)}{4^{h-1}(1)_{h-1}\left(\frac{3}{2}\right)_{h-1}}$$

$$P_{h+1,h;1}(m,n) = -P_{h,h+1;1}(n,m)$$

$$P_{h_1,h_2;h_1+h_2}(m,n) \equiv 0$$
$$P_{h_1,h_2;|h_1-h_2|}(m,n) = 0 \text{ if } |m| \leq h_1 - 1 \text{ and } |n| \leq h_2 - 1$$

$$\left[\mathbb{W}_{p_1}, \left[\mathbb{W}_{p_2}, \mathbb{W}_{p_3}\right]\right] \pm \text{ cyclic } = 0$$

$$\kappa^q_{p_1 p_2} = q\,\frac{\sqrt{\gamma_{p_1}}\sqrt{\gamma_{p_2}}}{\sqrt{2}\sqrt{\gamma_q}}$$

$$\gamma_p = 2p^2$$

$$\kappa_{p_1 p_2}{}^q = p_1 p_2$$

$$\mathbb{D}_{\text{degree}} = 1$$

$$\left(\frac{1}{2}(\nu - 1)\right)$$

$$\mathbb{D}_{\text{degree}} = \frac{3}{2}$$

$$\left(\frac{3(\nu-4)(\nu-1)}{\nu}\right)$$

$$\mathbb{D}_{\text{degree}} = 2$$

$$V_{2,0} = JJ + T + J'.$$

$$\left(\frac{3}{2}(\nu-4)(\nu-1)\right).$$

$$\mathbb{D}_{\text{degree}} = 2{,}2.$$

$$\begin{pmatrix} \frac{1}{2}(\nu-1)(\nu+1) \\ \frac{4(\nu-9)(\nu-4)(\nu-1)}{\nu(\nu+1)} \end{pmatrix}.$$

$$\mathbb{D}_{\text{degree}} = \frac{5}{2}$$

$$\left(\frac{12(\nu-9)(\nu-4)(\nu-1)}{\nu}\right).$$

$$\mathbb{D}_{\text{degree}} = \frac{5}{2}$$

$$\left(\frac{9}{5}(\nu-4)(\nu-1)\right)$$

$$\mathbb{D}_{\text{degree}} = \frac{5}{2}, \frac{5}{2}.$$

$$\begin{pmatrix} \frac{3(\nu-4)(\nu-1)(\nu+5)}{2\nu} \\ \frac{5(\nu-16)(\nu-9)(\nu-4)(\nu-1)}{\nu^2(\nu+5)} \end{pmatrix}.$$

$$\mathbb{D}_{\text{degree}} = 2{,}3{,}3{,}3$$

$$v_1^{(3,1,0)} = G\tilde{G} + JT + JJ' + J''$$

$$v_2^{(3,1,0)} = V_{3,1} - \frac{3(\nu-4)}{2(3\nu-2)} v_1^{(3,1,0)}$$

$$V_{3,1} = JV_{2,0} + \mathfrak{psl}(2 \mid 2)$$

$$\begin{pmatrix} -\frac{10}{9}(\nu-1)(3\nu-2) \\ \frac{15(\nu-4)(\nu-2)(\nu-1)(\nu+1)}{4(3\nu-2)} \\ \frac{30(\nu-11)(\nu-9)(\nu-4)(\nu-1)}{\nu(\nu+1)} \\ \frac{81(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{5(\nu-11)(\nu-2)\nu} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = 3$$

$$\left(\frac{8(\nu-9)(\nu-4)(\nu-1)}{3\nu}\right).$$

$$\mathbb{D}_{\text{degree}} = 3,3,3,3$$

$$\begin{pmatrix} \frac{3}{4}(\nu-1)(\nu+1)(\nu+3) \\ \frac{2(\nu-9)(\nu-4)(\nu-1)(\nu+7)}{\nu(\nu+1)} \\ \frac{6(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)}{\nu^2(\nu^2+15\nu+8)} \\ \frac{18(\nu-4)(\nu-1)^2(\nu^2+15\nu+8)}{\nu(\nu+3)(\nu+7)} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = \frac{5}{2},\frac{7}{2},\frac{7}{2}$$

$$v_1^{\left(\frac{7}{2},\frac{1}{2},0\right)} = G\tilde{G}_{W_3} + \tilde{G}G_{W_3} + JT_{W_3} + JW_3' + J'W_3 + T_{W_3}'$$

$$\begin{pmatrix} -\frac{105(\nu-4)(\nu-2)(\nu-1)}{2\nu} \\ \frac{15}{16}(\nu-9)(\nu-6)(\nu-4)(\nu-2)(\nu-1) \\ \frac{24(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{5(\nu-6)\nu^2} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = \frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2}$$

$$v_1^{\left(\frac{7}{2},\frac{3}{2},0\right)} = G\tilde{G}_{W_3} + \tilde{G}G_{W_3} + JT_{W_3} + TW_3 + JW_3' + J'W_3 + W_3''$$

$$\begin{pmatrix} -63(\nu-4)(\nu-1) \\ \frac{9(\nu-9)(\nu-4)(\nu-1)^3(\nu+5)}{13\nu-5} \\ \frac{162}{125}(\nu-4)(\nu-1)(13\nu-5) \\ \frac{5(\nu-16)(\nu-13)(\nu-9)(\nu-4)(\nu-1)}{3\nu^2(\nu+5)} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = \frac{7}{2},\frac{7}{2},\frac{7}{2}$$

$$\begin{pmatrix} \frac{63}{50}(\nu-4)(\nu-1)(\nu+5) \\ \frac{25(\nu-16)(\nu-9)(\nu-4)(\nu-1)(\nu+2)}{7\nu^2(\nu+5)} \\ \frac{144(\nu-9)(\nu-4)(\nu-1)^2}{7\nu(\nu+2)} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = \frac{7}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2}$$

$$\begin{pmatrix} \frac{3(\nu-4)(\nu-1)(\nu+5)(\nu+7)}{2\nu} \\ \frac{5(\nu-16)(\nu-9)(\nu-4)(\nu-1)(\nu+9)}{2\nu^2(\nu+5)} \\ \frac{12(\nu-9)(\nu-4)(\nu-1)^2(\nu^2+35\nu+84)}{\nu^2(\nu+7)(\nu+9)} \\ \frac{7(\nu-36)(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)}{\nu^3(\nu^2+35\nu+84)} \end{pmatrix}$$

$$\mathbb{D}_{\text{degree}} = 2,3,3,4,4,4,4$$

$$G\tilde{G}' + \tilde{G}G' + TT + J'J' + JJ'' + T'' + J'''$$

$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + J^{(3)} + J^2T + JJ'' + TJ' + (J')^2 + T'' + T^2$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G_{W_3}\tilde{G}_{W_3} + J^{(3)} + J^2T + JJ'' + TJ' + (J')^2 + W_3W_3' + T'' + T^2$$

$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G_{W_3}\tilde{G}_{W_3} + J^4 + J^{(3)} + J^2T + JJ'' + TJ' + (J')^2 + J^2J' + W_3W_3' + T'' + T^2$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G_{W_3}\tilde{G}_{W_3} + J^4 + J^{(3)} + J^2W_4 + J^2T + JJ'' + W_4J' + TJ'$$
$$+ (J')^2 + J^2J' + JT_{W_4} + W_3W_3' + T'' + T^2$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G_{W_3}\tilde{G}_{W_3} + J^4 + J^{(3)} + J^2W_4 + J^2T + JJ'' + W_4J' + TJ'$$
$$+ (J')^2 + J^2J' + JW_3^2 + JW_4' + JT' + JT_{W_4} + W_3W_3' + T'' + T^2$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G_{W_3}\tilde{G}_{W_3} + J^4 + J^{(3)} + J^2W_4 + J^2T + JJ'' + W_4J' + T\,J'$$
$$+ (J')^2 + J^2J' + JW_3^2 + JW_4' + JT' + JT_{W_4} + W_3W_3' + W_4'' + W_4^2 + W_6' + T'' + T^2 + T_{W_4}'$$

$$\begin{pmatrix} \frac{35}{48}(\nu-1)(3\nu-5) \\ -\frac{5(\nu-4)(\nu-1)(9\nu^2-63\nu+26)}{54(3\nu-5)} \\ -\frac{135(\nu-9)(\nu-4)(\nu-1)(9\nu^3-63\nu^2+182\nu-88)}{16\nu^2(9\nu^2-63\nu+26)} \\ \frac{5(\nu-4)(\nu-1)\nu(9\nu^4-96\nu^3+423\nu^2-684\nu+748)}{24(9\nu^3-63\nu^2+182\nu-88)} \\ \frac{15(\nu-9)(\nu-4)(\nu-1)^2(3\nu^4-92\nu^3+1081\nu^2-5508\nu+10516)}{2\nu(9\nu^4-96\nu^3+423\nu^2-684\nu+748)} \\ \frac{27(\nu-16)(\nu-11)(\nu-9)(\nu-4)(\nu-1)^2(3\nu^2-17\nu-10)}{10\nu(3\nu^4-92\nu^3+1081\nu^2-5508\nu+10516)} \\ \frac{96(\nu-25)(\nu-16)(\nu-9)(\nu-4)^2(\nu-1)^2}{5(\nu-11)\nu^2(3\nu^2-17\nu-10)} \end{pmatrix}.$$

$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + JJ'' + TJ' + (J')^2 + J^2J' + JT'$$
$$G\tilde{G}_{W_4} + G_{W_4}\tilde{G} + W_4J' + JW_4' + JT_{W_4} + T_{W_4}'$$

$$G\tilde{G}_{W_4} + G_{W_4}\tilde{G} + J^2W_4 + W_4J' + JW_4' + JT_{W_4} + T_{W_4}'$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_4}\tilde{G} + J^{(3)} + J^2W_4 + JJ'' + W_4J' + TJ'$$
$$+ (J')^2 + J^2J' + JW_3^2 + JW_4' + JT' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + T_{W_4}'$$
$$\tilde{G}G' + G\tilde{G}' + G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_4}\tilde{G} + J^{(3)} + J^2W_4 + JJ'' + W_4J' + TJ'$$
$$+ (J')^2 + J^2J' + JW_3^2 + JW_4' + JT' + JT_{W_4} + W_3W_3' + W_3W_5 + W_3T_{W_3} + W_4'' + T_{W_4}'$$

$$\begin{pmatrix} -2(\nu-4)(\nu-1)(\nu+1) \\ -\frac{576(\nu-9)(\nu-4)(\nu-3)(\nu-1)}{5\nu(\nu+1)} \\ \frac{72(\nu-11)(\nu-9)(\nu-7)(\nu-4)(\nu-1)}{125(\nu-3)\nu} \\ \frac{81(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{10(\nu-11)\nu} \\ \frac{15(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{2(\nu-7)\nu^3} \end{pmatrix}.$$

$$W_3G' + GJW_3 + GW_3' + GT_{W_3} + G_{W_3}'' + G_{W_3}J^2 + G_{W_3}J' + G_{W_3}T$$
$$W_3\tilde{G}' + \tilde{G}JW_3 + \tilde{G}W_3' + \tilde{G}T_{W_3} + \tilde{G}_{W_3}'' + \tilde{G}_{W_3}J^2 + \tilde{G}_{W_3}J' + \tilde{G}_{W_3}T$$

$$\left(\frac{729}{512}(\nu-9)(\nu-4)(\nu-1)\right)$$

$$v_1^{(4,2,0)} = G\tilde{G}J + J^2T + JJ'' + (J')^2 + J^2J'$$
$$v_2^{(4,2,0)} = G\tilde{G}_{W_4} + G_{W_4}\tilde{G} + W_4J' + JW_4' + JT_{W_4} + W_4'' + W_4T$$
$$v_3^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$

$$v_4^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$
$$v_5^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$
$$v_6^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$
$$v_7^{(4,2,0)} = JW_6 + T_{W_6}$$
$$v_8^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_3^2 + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$
$$v_9^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_3^2 + JW_4' + JT_{W_4} + W_3W_3' + W_3T_{W_3} + W_4'' + W_4T$$
$$v_{10}^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_3^2 + JW_4' + JW_6 + JT_{W_4} + W_3W_3' + W_3W_5 + W_3T_{W_3} + W_4'' + W_4T + T_{W_6}$$
$$v_{11}^{(4,2,0)} = G\tilde{G}J + G\tilde{G}_{W_4} + G_{W_3}\tilde{G}_{W_3} + G_{W_4}\tilde{G} + J^4 + J^2W_4 + J^2T + JJ'' + W_4J' + (J')^2 + J^2J' + JW_3^2 + JW_4' + JW_6 + JT_{W_4} + W_3W_3' + W_3W_5 + W_3T_{W_3} + W_4'' + W_4^2 + W_4T + W_6' + T_{W_6}$$

$$\begin{pmatrix} -\frac{7}{100}(\nu-1)(\nu+1)(3\nu+2) \\ -\frac{1792(\nu-9)(\nu-4)(\nu-1)(15\nu+7)}{225\nu(\nu+1)} \\ -\frac{567(\nu-4)(\nu-1)(45\nu^3-84\nu^2-343\nu-52)}{8\nu(3\nu+2)(15\nu+7)} \\ \frac{63(\nu-4)(\nu-1)(45\nu^5-24\nu^4-720\nu^3-108\nu^2+4147\nu+468)}{400(45\nu^3-84\nu^2-343\nu-52)} \\ \frac{24(\nu-9)(\nu-4)(\nu-1)(22455\nu^6+25023\nu^5-402600\nu^4-151284\nu^3+121781\nu^2+634517\nu+12999708)}{\nu(45\nu^5-24\nu^4-720\nu^3-108\nu^2+4147\nu+468)} \\ \frac{896(\nu-9)(\nu-4)(\nu-1)^2(15\nu^6-80\nu^5-1288\nu^4+4096\nu^3+3841\nu^2-32240\nu-5544)}{9\nu(22455\nu^6+25023\nu^5-402600\nu^4-151284\nu^3+121781\nu^2+634517\nu+12999708)} \\ \frac{15(\nu-25)(\nu-16)(\nu-15)(\nu-9)(\nu-4)(\nu-1)}{7\nu^2(\nu^2+15\nu+8)} \\ \frac{81(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2(15\nu^5+211\nu^4+403\nu^3-3427\nu^2-9286\nu-1260)}{10\nu(15\nu^6-80\nu^5-1288\nu^4+4096\nu^3+3841\nu^2-32240\nu-5544)} \\ \frac{225(\nu-9)(\nu-4)(\nu-1)^2(\nu^2+15\nu+8)(3\nu^3+5\nu^2-138\nu-20)}{7\nu(15\nu^5+211\nu^4+403\nu^3-3427\nu^2-9286\nu-1260)} \\ \frac{150(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2(3\nu^4-148\nu^3+2265\nu^2-4160\nu-32400)}{7(\nu-15)\nu^3(3\nu^3+5\nu^2-138\nu-20)} \\ \frac{1152(\nu-36)(\nu-25)(\nu-16)(\nu-9)(\nu-4)^2(\nu-1)^2}{7\nu^2(3\nu^4-148\nu^3+2265\nu^2-4160\nu-32400)} \end{pmatrix}$$

$v_1^{(4,3,0)} = J^2W_4 + W_4J' + JW_4'$
$v_2^{(4,3,0)} = JW_6 + W_6'$
$v_3^{(4,3,0)} = J^2W_4 + W_4J' + J^2J' + JW_3^2 + JW_4' + W_3W_3'$
$v_4^{(4,3,0)} = J^2W_4 + W_4J' + J^2J' + JW_3^2 + JW_4' + JW_6 + W_3W_3' + W_3W_5 + W_6'$

$$\begin{pmatrix} \frac{16(\nu-9)(\nu-4)(\nu-1)(\nu+7)}{9\nu} \\ \frac{9(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)(\nu+3)}{2\nu^2(\nu^2+15\nu+8)} \\ \frac{27(\nu-4)(\nu-1)^2(\nu^2+15\nu+8)}{2\nu(\nu+7)} \\ \frac{225(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2}{8\nu^2(\nu+3)} \end{pmatrix}$$

$v_1^{(4,4,0)} = J^4$
$v_2^{(4,4,0)} = J^2W_4$
$v_3^{(4,4,0)} = W_8$
$v_4^{(4,4,0)} = JW_6$
$v_5^{(4,4,0)} = JW_3^3 + J^4 + J^2W_4$
$v_6^{(4,4,0)} = JW_6 + J^2W_4 + J^4 + JW_3^3 + W_3W_5$
$v_7^{(4,4,0)} = JW_6 + J^2W_4 + J^4 + JW_3^3 + W_3W_5 + W_4^2$

$$\begin{pmatrix} \frac{3}{2}(\nu-1)(\nu+1)(\nu+3)(\nu+5) \\ \frac{2(\nu-9)(\nu-4)(\nu-1)(\nu+7)(\nu+9)}{\nu(\nu+1)} \\ \frac{8(\nu-49)(\nu-36)(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)}{\nu^3(\nu^3+70\nu^2+469\nu+180)} \\ \frac{3(\nu-25)(\nu-16)(\nu-9)(\nu-4)(\nu-1)(\nu+11)}{\nu^2(\nu^2+15\nu+8)} \\ \frac{9(\nu-4)(\nu-1)^2(\nu+11)(\nu^2+15\nu+8)}{\nu(\nu+3)(\nu+7)} \\ \frac{15(\nu-16)(\nu-9)(\nu-4)(\nu-1)^2(\nu^3+46\nu^2+109\nu+384)}{\nu^3(\nu+5)(\nu+9)(\nu+11)} \\ \frac{32(\nu-9)(\nu-4)^2(\nu-1)^2(\nu^3+70\nu^2+469\nu+180)}{\nu^2(\nu+11)(\nu^3+46\nu^2+109\nu+384)} \end{pmatrix}.$$

h	j	2	3	4	5	6	7	8	9	10
1	1	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ
$\frac{3}{2}$	$\frac{3}{2}$	0	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$	$\xi^{3/2}$
2	0	0	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2
2	2	ξ^2	ξ^2	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	0	0	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{3}{2}$	0	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	0	$\xi^{5/2}$	$\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$	$2\xi^{5/2}$
3	1	ξ^2	$\xi^2+\xi^3$	$\xi^2+2\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$	$\xi^2+3\xi^3$
3	2	0	0	ξ^3	ξ^3	ξ^3	ξ^3	ξ^3	ξ^3	ξ^3
3	3	ξ^3	$2\xi^3$	$3\xi^3$	$3\xi^3$	$4\xi^3$	$4\xi^3$	$4\xi^3$	$4\xi^3$	$4\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	0	$\xi^{5/2}$	$\xi^{5/2}+\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$
$\frac{7}{2}$	$\frac{3}{2}$	0	$\xi^{5/2}+\xi^{7/2}$	$\xi^{5/2}+2\xi^{7/2}$	$\xi^{5/2}+3\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$	$\xi^{5/2}+4\xi^{7/2}$
$\frac{7}{2}$	$\frac{5}{2}$	0	$\xi^{7/2}$	$2\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	0	$\xi^{7/2}$	$2\xi^{7/2}$	$3\xi^{7/2}$	$3\xi^{7/2}$	$4\xi^{7/2}$	$4\xi^{7/2}$	$4\xi^{7/2}$	$4\xi^{7/2}$
4	0	ξ^2	$\xi^2+\xi^3+\xi^4$	$\xi^2+2\xi^3+2\xi^4$	$\xi^2+2\xi^3+3\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$	$\xi^2+2\xi^3+4\xi^4$
4	1	0	ξ^3	ξ^3	$\xi^3+\xi^4$	$\xi^3+2\xi^4$	$\xi^3+2\xi^4$	$\xi^3+2\xi^4$	$\xi^3+2\xi^4$	$\xi^3+2\xi^4$
4	2	ξ^3	$2\xi^3+\xi^4$	$3\xi^3+4\xi^4$	$3\xi^3+6\xi^4$	$3\xi^3+7\xi^4$	$3\xi^3+8\xi^4$	$3\xi^3+8\xi^4$	$3\xi^3+8\xi^4$	$3\xi^3+8\xi^4$
4	3	0	ξ^4	$2\xi^4$	$3\xi^4$	$4\xi^4$	$4\xi^4$	$4\xi^4$	$4\xi^4$	$4\xi^4$
4	4	ξ^4	$2\xi^4$	$4\xi^4$	$5\xi^4$	$6\xi^4$	$6\xi^4$	$7\xi^4$	$7\xi^4$	$7\xi^4$
$\frac{9}{2}$	$\frac{1}{2}$	0	0	$2\xi^{7/2}+\xi^{9/2}$	$2\xi^{7/2}+2\xi^{9/2}$	$2\xi^{7/2}+3\xi^{9/2}$	$2\xi^{7/2}+4\xi^{9/2}$	$2\xi^{7/2}+4\xi^{9/2}$	$2\xi^{7/2}+4\xi^{9/2}$	$2\xi^{7/2}+4\xi^{9/2}$
$\frac{9}{2}$	$\frac{3}{2}$	0	$\xi^{5/2}+2\xi^{7/2}+\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+2\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+4\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+6\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+7\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+7\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+7\xi^{9/2}$	$\xi^{5/2}+3\xi^{7/2}+7\xi^{9/2}$
$\frac{9}{2}$	$\frac{5}{2}$	0	$2\xi^{7/2}+\xi^{9/2}$	$3\xi^{7/2}+4\xi^{9/2}$	$4\xi^{7/2}+7\xi^{9/2}$	$4\xi^{7/2}+9\xi^{9/2}$	$4\xi^{7/2}+10\xi^{9/2}$	$4\xi^{7/2}+11\xi^{9/2}$	$4\xi^{7/2}+11\xi^{9/2}$	$4\xi^{7/2}+11\xi^{9/2}$
$\frac{9}{2}$	$\frac{7}{2}$	0	$\xi^{9/2}$	$3\xi^{9/2}$	$5\xi^{9/2}$	$6\xi^{9/2}$	$7\xi^{9/2}$	$7\xi^{9/2}$	$7\xi^{9/2}$	$7\xi^{9/2}$
$\frac{9}{2}$	$\frac{9}{2}$	0	$2\xi^{9/2}$	$3\xi^{9/2}$	$5\xi^{9/2}$	$6\xi^{9/2}$	$7\xi^{9/2}$	$7\xi^{9/2}$	$8\xi^{9/2}$	$8\xi^{9/2}$
5	0	0	0	$-\xi^3-\xi^4-\xi^5$	$-\xi^3-\xi^4-2\xi^5$	$-\xi^3-2\xi^4-2\xi^5$	$-\xi^3-2\xi^4-2\xi^5$	$-\xi^3-2\xi^4-2\xi^5$	$-\xi^3-2\xi^4-2\xi^5$	$-\xi^3-2\xi^4-2\xi^5$
5	1	$\xi^2+\xi^3$	$\xi^2+2\xi^3+\xi^4+\xi^5$	$\xi^2+2\xi^3+4\xi^4+3\xi^5$	$\xi^2+2\xi^3+6\xi^4+6\xi^5$	$\xi^2+2\xi^3+6\xi^4+9\xi^5$	$\xi^2+2\xi^3+6\xi^4+11\xi^5$	$\xi^2+2\xi^3+6\xi^4+12\xi^5$	$\xi^2+2\xi^3+6\xi^4+12\xi^5$	$\xi^2+2\xi^3+6\xi^4+12\xi^5$
5	2	ξ^3	ξ^4	$\xi^3+4\xi^4+2\xi^5$	$\xi^3+4\xi^4+4\xi^5$	$\xi^3+4\xi^4+6\xi^5$	$\xi^3+4\xi^4+8\xi^5$	$\xi^3+4\xi^4+9\xi^5$	$\xi^3+4\xi^4+9\xi^5$	$\xi^3+4\xi^4+9\xi^5$
5	3	ξ^4	$3\xi^4+2\xi^5$	$5\xi^4+5\xi^5$	$6\xi^4+10\xi^5$	$7\xi^4+14\xi^5$	$7\xi^4+16\xi^5$	$7\xi^4+17\xi^5$	$7\xi^4+18\xi^5$	$7\xi^4+18\xi^5$
5	4	0	ξ^5	$4\xi^5$	$6\xi^5$	$8\xi^5$	$9\xi^5$	$10\xi^5$	$10\xi^5$	$10\xi^5$
5	5	ξ^5	$2\xi^5$	$5\xi^5$	$7\xi^5$	$9\xi^5$	$10\xi^5$	$11\xi^5$	$11\xi^5$	$12\xi^5$

h	j	$N=2$	$N=3$
1	1	ξ	ξ
$\frac{3}{2}$	$\frac{3}{2}$	0	$\xi^{3/2}$
2	0	$\xi+\xi^2+\xi^{3/2}\chi_1$	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	ξ^2	ξ^2
$\frac{5}{2}$	$\frac{1}{2}$	0	$\xi^{3/2}+\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	0	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	0	$\xi^{5/2}$
3	0	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$2\xi^2+\xi^3+\xi^{5/2}\chi_1$	$3\xi^2+3\xi^3+2\xi^{5/2}\chi_1$
3	2	0	0
3	3	ξ^3	$2\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$\xi^{3/2}+\xi^{5/2}+\xi^2\chi_1$	$2\xi^{3/2}+5\xi^{5/2}+\xi^{7/2}+\left(3\xi^2+2\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	0	$3\xi^{5/2}+3\xi^{7/2}+2\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	0	$\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	0	$\xi^{7/2}$
4	0	$\xi+3\xi^2+2\xi^3+\xi^4+\left(\xi^{3/2}+\xi^{5/2}+\xi^{7/2}\right)\chi_1$	$\xi+6\xi^2+8\xi^3+4\xi^4+\left(\xi^{3/2}+5\xi^{5/2}+3\xi^{7/2}\right)\chi_1+\left(\xi^2+\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$\xi^2+\xi^3+\xi^{5/2}\chi_1$	$2\xi^2+8\xi^3+2\xi^4+\left(5\xi^{5/2}+5\xi^{7/2}\right)\chi_1+3\xi^3\chi_2$
4	2	$2\xi^3+\xi^4+\xi^{7/2}\chi_1$	$5\xi^3+4\xi^4+3\xi^{7/2}\chi_1$
4	3	0	ξ^4
4	4	ξ^4	$2\xi^4$

h	j	$N=4$
1	1	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$
2	0	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{3/2}+2\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	$\xi^{5/2}$
3	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$4\xi^2+5\xi^3+3\xi^{5/2}\chi_1$
3	2	ξ^3
3	3	$3\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$2\xi^{3/2}+7\xi^{5/2}+4\xi^{7/2}+\left(3\xi^2+4\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	$4\xi^{5/2}+5\xi^{7/2}+3\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	$2\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	$2\xi^{7/2}$
4	0	$\xi+7\xi^2+14\xi^3+8\xi^4+\left(\xi^{3/2}+7\xi^{5/2}+7\xi^{7/2}\right)\chi_1+\left(\xi^2+2\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$3\xi^2+14\xi^3+6\xi^4+\left(7\xi^{5/2}+10\xi^{7/2}\right)\chi_1+4\xi^3\chi_2$
4	2	$9\xi^3+10\xi^4+6\xi^{7/2}\chi_1$
4	3	$2\xi^4$
4	4	$4\xi^4$

h	j	$N=5$
1	1	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$
2	0	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{3/2}+2\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	$2\xi^{5/2}$
3	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$4\xi^2+6\xi^3+3\xi^{5/2}\chi_1$
3	2	ξ^3
3	3	$3\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$2\xi^{3/2}+7\xi^{5/2}+5\xi^{7/2}+\left(3\xi^2+4\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	$5\xi^{5/2}+7\xi^{7/2}+4\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	$3\xi^{7/2}$
4	0	$\xi+7\xi^2+15\xi^3+10\xi^4+\left(\xi^{3/2}+7\xi^{5/2}+8\xi^{7/2}\right)\chi_1+\left(\xi^2+2\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$3\xi^2+16\xi^3+9\xi^4+\left(7\xi^{5/2}+12\xi^{7/2}\right)\chi_1+4\xi^3\chi_2$
4	2	$10\xi^3+13\xi^4+7\xi^{7/2}\chi_1$
4	3	$3\xi^4$
4	4	$5\xi^4$

h	j	$N=6$
1	1	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$
2	0	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{3/2}+2\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	$2\xi^{5/2}$
3	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$4\xi^2+6\xi^3+3\xi^{5/2}\chi_1$
3	2	ξ^3
3	3	$4\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$2\xi^{3/2}+7\xi^{5/2}+5\xi^{7/2}+\left(3\xi^2+4\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	$5\xi^{5/2}+8\xi^{7/2}+4\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	$3\xi^{7/2}$
4	0	$\xi+7\xi^2+15\xi^3+11\xi^4+\left(\xi^{3/2}+7\xi^{5/2}+8\xi^{7/2}\right)\chi_1+\left(\xi^2+2\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$3\xi^2+16\xi^3+10\xi^4+\left(7\xi^{5/2}+12\xi^{7/2}\right)\chi_1+4\xi^3\chi_2$
4	2	$11\xi^3+15\xi^4+8\xi^{7/2}\chi_1$
4	3	$4\xi^4$
4	4	$6\xi^4$

h	j	$N=7$
1	1	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$
2	0	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{3/2}+2\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	$2\xi^{5/2}$
3	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$4\xi^2+6\xi^3+3\xi^{5/2}\chi_1$
3	2	ξ^3
3	3	$4\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$2\xi^{3/2}+7\xi^{5/2}+5\xi^{7/2}+\left(3\xi^2+4\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	$5\xi^{5/2}+8\xi^{7/2}+4\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	$4\xi^{7/2}$
4	0	$\xi+7\xi^2+15\xi^3+11\xi^4+\left(\xi^{3/2}+7\xi^{5/2}+8\xi^{7/2}\right)\chi_1+\left(\xi^2+2\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$3\xi^2+16\xi^3+10\xi^4+\left(7\xi^{5/2}+12\xi^{7/2}\right)\chi_1+4\xi^3\chi_2$
4	2	$11\xi^3+16\xi^4+8\xi^{7/2}\chi_1$
4	3	$4\xi^4$
4	4	$6\xi^4$

h	j	$N=8$
1	1	ξ
$\frac{3}{2}$	$\frac{3}{2}$	$\xi^{3/2}$
2	0	$\xi+2\xi^2+\xi^{3/2}\chi_1$
2	2	$2\xi^2$
$\frac{5}{2}$	$\frac{1}{2}$	$\xi^{3/2}+2\xi^{5/2}+\xi^2\chi_1$
$\frac{5}{2}$	$\frac{3}{2}$	$\xi^{5/2}$
$\frac{5}{2}$	$\frac{5}{2}$	$2\xi^{5/2}$
3	0	$\xi^2+\xi^3+\xi^{5/2}\chi_1$
3	1	$4\xi^2+6\xi^3+3\xi^{5/2}\chi_1$
3	2	ξ^3
3	3	$4\xi^3$
$\frac{7}{2}$	$\frac{1}{2}$	$2\xi^{3/2}+7\xi^{5/2}+5\xi^{7/2}+\left(3\xi^2+4\xi^3\right)\chi_1+\xi^{5/2}\chi_2$
$\frac{7}{2}$	$\frac{3}{2}$	$5\xi^{5/2}+8\xi^{7/2}+4\xi^3\chi_1$
$\frac{7}{2}$	$\frac{5}{2}$	$3\xi^{7/2}$
$\frac{7}{2}$	$\frac{7}{2}$	$4\xi^{7/2}$
4	0	$\xi+7\xi^2+15\xi^3+11\xi^4+\left(\xi^{3/2}+7\xi^{5/2}+8\xi^{7/2}\right)\chi_1+\left(\xi^2+2\xi^3\right)\chi_2+\xi^{5/2}\chi_3$
4	1	$3\xi^2+16\xi^3+10\xi^4+\left(7\xi^{5/2}+12\xi^{7/2}\right)\chi_1+4\xi^3\chi_2$
4	2	$11\xi^3+16\xi^4+8\xi^{7/2}\chi_1$
4	3	$4\xi^4$
4	4	$7\xi^4$

$$\mathrm{HS}_{j,j}(\tau,a,x)=\tau^{2j}\chi_{2j}(a)+x\tau^{2j+1}\chi_{2j-1}(a)$$
$$\mathrm{HS}_{h,j}(\tau,a,x)=\tau^{2h}\chi_{2j}(a)+x\tau^{2h+1}\left(\chi_{2j-1}(a)+\chi_{2j+1}(a)\right)+x^2\tau^{2h+2}\chi_{2j}(a)$$

$$1+\sum_{n=1}^{\infty}\mathrm{HS}_{n,n}.$$

$$\begin{aligned}&1+\mathrm{HS}_{1,1}+\mathrm{HS}_{\frac{3}{2},\frac{3}{2}}+\mathrm{HS}_{2,0}+\mathrm{HS}_{2,2}+\mathrm{HS}_{\frac{5}{2},\frac{3}{2}}+\mathrm{HS}_{\frac{5}{2},\frac{5}{2}}\\&+\mathrm{HS}_{3,1}+2\mathrm{HS}_{3,3}+\mathrm{HS}_{\frac{7}{2},\frac{3}{2}}+\mathrm{HS}_{\frac{7}{2},\frac{5}{2}}+\mathrm{HS}_{\frac{7}{2},\frac{7}{2}}\\&+\mathrm{HS}_{4,0}+\mathrm{HS}_{4,2}+\mathrm{HS}_{4,3}+2\mathrm{HS}_{4,4}+\mathcal{O}\left(q^{9/2}\right)\end{aligned}$$

$$\begin{aligned}&1+\mathrm{HS}_{1,1}+\mathrm{HS}_{\frac{3}{2},\frac{3}{2}}+\mathrm{HS}_{2,0}+2\mathrm{HS}_{2,2}+\mathrm{HS}_{\frac{5}{2},\frac{1}{2}}+\mathrm{HS}_{\frac{5}{2},\frac{3}{2}}+\mathrm{HS}_{\frac{5}{2},\frac{5}{2}}\\&+2\mathrm{HS}_{3,1}+\mathrm{HS}_{3,2}+3\mathrm{HS}_{3,3}+\mathrm{HS}_{\frac{7}{2},\frac{1}{2}}+2\mathrm{HS}_{\frac{7}{2},\frac{3}{2}}+2\mathrm{HS}_{\frac{7}{2},\frac{5}{2}}+2\mathrm{HS}_{\frac{7}{2},\frac{7}{2}}\\&+2\mathrm{HS}_{4,0}+(1+x)\mathrm{HS}_{4,1}+4\mathrm{HS}_{4,2}+2\mathrm{HS}_{4,3}+4\mathrm{HS}_{4,4}+\mathcal{O}\left(q^{9/2}\right)\end{aligned}$$

$$\begin{aligned}&1+\mathrm{HS}_{1,1}+\mathrm{HS}_{\frac{3}{2},\frac{3}{2}}+\mathrm{HS}_{2,0}+2\mathrm{HS}_{2,2}+\mathrm{HS}_{\frac{5}{2},\frac{1}{2}}+\mathrm{HS}_{\frac{5}{2},\frac{3}{2}}+2\mathrm{HS}_{\frac{5}{2},\frac{5}{2}}\\&+3\mathrm{HS}_{3,1}+\mathrm{HS}_{3,2}+3\mathrm{HS}_{3,3}+2\mathrm{HS}_{\frac{7}{2},\frac{1}{2}}+3\mathrm{HS}_{\frac{7}{2},\frac{3}{2}}+3\mathrm{HS}_{\frac{7}{2},\frac{5}{2}}+3\mathrm{HS}_{\frac{7}{2},\frac{7}{2}}\\&+3\mathrm{HS}_{4,0}+(2+x)\mathrm{HS}_{4,1}+6\mathrm{HS}_{4,2}+3\mathrm{HS}_{4,3}+5\mathrm{HS}_{4,4}+\mathcal{O}\left(q^{9/2}\right)\end{aligned}$$

$$
\begin{aligned}
&1 + \mathrm{HS}_{1,1} + \mathrm{HS}_{\frac{3}{2},\frac{3}{2}} + \mathrm{HS}_{2,0} + 2\mathrm{HS}_{2,2} + \mathrm{HS}_{\frac{5}{2},\frac{1}{2}} + \mathrm{HS}_{\frac{5}{2},\frac{3}{2}} + 2\mathrm{HS}_{\frac{5}{2},\frac{5}{2}} \\
&+3\mathrm{HS}_{3,1} + \mathrm{HS}_{3,2} + 4\mathrm{HS}_{3,3} + 2\mathrm{HS}_{\frac{7}{2},\frac{1}{2}} + 4\mathrm{HS}_{\frac{7}{2},\frac{3}{2}} + 3\mathrm{HS}_{\frac{7}{2},\frac{5}{2}} + 3\mathrm{HS}_{\frac{7}{2},\frac{7}{2}} \\
&+4\mathrm{HS}_{4,0} + (3+x)\mathrm{HS}_{4,1} + 7\mathrm{HS}_{4,2} + 4\mathrm{HS}_{4,3} + 6\mathrm{HS}_{4,4} + \mathcal{O}\left(q^{9/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&1 + \mathrm{HS}_{1,1} + \mathrm{HS}_{\frac{3}{2},\frac{3}{2}} + \mathrm{HS}_{2,0} + 2\mathrm{HS}_{2,2} + \mathrm{HS}_{\frac{5}{2},\frac{1}{2}} + \mathrm{HS}_{\frac{5}{2},\frac{3}{2}} + 2\mathrm{HS}_{\frac{5}{2},\frac{5}{2}} \\
&+3\mathrm{HS}_{3,1} + \mathrm{HS}_{3,2} + 4\mathrm{HS}_{3,3} + 2\mathrm{HS}_{\frac{7}{2},\frac{1}{2}} + 4\mathrm{HS}_{\frac{7}{2},\frac{3}{2}} + 3\mathrm{HS}_{\frac{7}{2},\frac{5}{2}} + 4\mathrm{HS}_{\frac{7}{2},\frac{7}{2}} \\
&+4\mathrm{HS}_{4,0} + (3+x)\mathrm{HS}_{4,1} + 8\mathrm{HS}_{4,2} + 4\mathrm{HS}_{4,3} + 6\mathrm{HS}_{4,4} + \mathcal{O}\left(q^{9/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&1 + \mathrm{HS}_{1,1} + \mathrm{HS}_{\frac{3}{2},\frac{3}{2}} + \mathrm{HS}_{2,0} + 2\mathrm{HS}_{2,2} + \mathrm{HS}_{\frac{5}{2},\frac{1}{2}} + \mathrm{HS}_{\frac{5}{2},\frac{3}{2}} + 2\mathrm{HS}_{\frac{5}{2},\frac{5}{2}} \\
&+3\mathrm{HS}_{3,1} + \mathrm{HS}_{3,2} + 4\mathrm{HS}_{3,3} + 2\mathrm{HS}_{\frac{7}{2},\frac{1}{2}} + 4\mathrm{HS}_{\frac{7}{2},\frac{3}{2}} + 3\mathrm{HS}_{\frac{7}{2},\frac{5}{2}} + 4\mathrm{HS}_{\frac{7}{2},\frac{7}{2}} \\
&+4\mathrm{HS}_{4,0} + (3+x)\mathrm{HS}_{4,1} + 8\mathrm{HS}_{4,2} + 4\mathrm{HS}_{4,3} + 7\mathrm{HS}_{4,4} + \mathcal{O}\left(q^{9/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{O}_1 = \mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1 - \phi_0\phi_0\phi_1\phi_1) \\
&\mathcal{O}_2 = \mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0\phi_1) - \mathrm{Tr}(\phi_0\phi_0)\mathrm{Tr}(\phi_1\phi_1) \\
&\mathcal{O}_3 = \mathrm{Tr}(b\phi_1\phi_0 - b\phi_0\phi_1) \\
&\mathcal{O}_4 = \mathrm{Tr}(\partial c\phi_1\phi_0 - b\phi_0\phi_1) \\
&\mathcal{O}_5 = \mathrm{Tr}(\partial cb - \partial\phi_0\phi_1 + \partial\phi_1\phi_0)
\end{aligned}
$$

$$
\{J_{\mathrm{B}}\mathcal{O}_1\}_1 = \sum_{j=1}^{5} m_{ij}\mathcal{O}_j, m = \begin{pmatrix} 0 & 0 & 0 & 3N & 0 \\ 0 & 0 & 0 & 6 & 0 \\ -2 & 0 & 0 & 0 & 2N \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}
$$

$$
\widetilde{m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, m = U\widetilde{m}U^{-1}, U = \begin{pmatrix} 0 & 3N & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2N & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\mathcal{P}_i := \sum_{j=1}^{5} (U^{-1})_{ij}\mathcal{O}_j, \{J_{\mathrm{B}}\mathcal{P}_i\}_1 = \sum_{j=1}^{5} \widetilde{m}_{ij}\mathcal{P}_i
$$

$$
\{J_{\mathrm{B}}\mathcal{P}_1\}_1 = 0, \{J_{\mathrm{B}}\mathcal{P}_2\}_1 = \mathcal{P}_3, \{J_{\mathrm{B}}\mathcal{P}_3\}_1 = 0, \{J_{\mathrm{B}}\mathcal{P}_4\}_1 = \mathcal{P}_5, \{J_{\mathrm{B}}\mathcal{P}_5\}_1 = 0
$$

$$
\begin{aligned}
&\mathcal{P}_1 = \mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0\phi_1) - \mathrm{Tr}(\phi_0\phi_0)\mathrm{Tr}(\phi_1\phi_1) + \frac{2}{N}\mathrm{Tr}(\phi_0\phi_0\phi_1\phi_1 - \phi_0\phi_1\phi_0\phi_1), \\
&\mathcal{P}_2 = \frac{1}{3N}\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1 - \phi_0\phi_0\phi_1\phi_1), \\
&\mathcal{P}_3 = \mathrm{Tr}(\partial c\phi_1\phi_0 - \partial c\phi_0\phi_1), \\
&\mathcal{P}_4 = \frac{1}{2N}\mathrm{Tr}(b\phi_1\phi_0 - b\phi_0\phi_1), \\
&\mathcal{P}_5 = \mathrm{Tr}(\partial cb + \partial\phi_1\phi_0 - \partial\phi_0\phi_1) + \frac{1}{N}\mathrm{Tr}(\phi_0\phi_0\phi_1\phi_1 - \phi_0\phi_1\phi_0\phi_1).
\end{aligned}
$$

$$
\{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_n, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, \dots, \mathcal{X}_m, \mathcal{Y}_m\}
$$

$$
J_{\mathrm{B}} \cdot \mathcal{Z}_i = 0, i = 1, \dots, n;\ J_{\mathrm{B}} \cdot \mathcal{X}_i = \mathcal{Y}_i, i = 1, \dots, m
$$

$$
\begin{aligned}
&\mathcal{Z}_1 = \mathrm{Tr}(\phi(y)^4) \\
&\mathcal{Z}_2 = 2\mathrm{Tr}(\phi(y)^4) - \mathrm{Tr}(\phi(y)^2)^2
\end{aligned}
$$

$$\begin{aligned}
\mathcal{Z}_1 =& \frac{6}{N}\mathrm{Tr}(\phi_0^2\phi_1\phi_0\phi_1) - \frac{6}{N}\mathrm{Tr}(\phi_0^3\phi_1^2) + \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1^2) - 2\mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0^2\phi_1) + \mathrm{Tr}(\phi_1^2)\mathrm{Tr}(\phi_0^3) \\
&+ y\left[\frac{6}{N}\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1^2) - \frac{6}{N}\mathrm{Tr}(\phi_0^2\phi_1^3) - 2\mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0\phi_1^2) + \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_1^3) + \mathrm{Tr}(\phi_1^2)\mathrm{Tr}(\phi_0^2\phi_1)\right] \\
\mathcal{X}_1 =& \mathrm{Tr}(\phi_0^3\phi_1^2) - \mathrm{Tr}(\phi_0^2\phi_1\phi_0\phi_1) + y[\mathrm{Tr}(\phi_0^2\phi_1^3) - \mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1^2)] \\
\mathcal{Y}_1 =& 2N\mathrm{Tr}(dc\phi_0^2\phi_1) - 2N\mathrm{Tr}(dc\phi_1\phi_0^2) + y[2N\mathrm{Tr}(dc\phi_0\phi_1^2) - 2N\mathrm{Tr}(dc\phi_1^2\phi_0)] \\
\mathcal{X}_2 =& \mathrm{Tr}(b\phi_0^2\phi_1) - \mathrm{Tr}(b\phi_1\phi_0^2) + y[\mathrm{Tr}(b\phi_0\phi_1^2) - \mathrm{Tr}(b\phi_1^2\phi_0)] \\
\mathcal{Y}_2 =& 2\mathrm{Tr}(\phi_0^2\phi_1\phi_0\phi_1) - 2\mathrm{Tr}(\phi_0^3\phi_1^2) \\
&- N\mathrm{Tr}(dcb\phi_0) - N\mathrm{Tr}(dc\phi_0 b) - 2N\mathrm{Tr}(d\phi_1\phi_0^2) + N\mathrm{Tr}(d\phi_0\phi_1\phi_0) + N\mathrm{Tr}(d\phi_0\phi_0\phi_1) \\
&+ y[2\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1^2) - 2\mathrm{Tr}(\phi_0^2\phi_1^3) \\
&- N\mathrm{Tr}(dcb\phi_1) - N\mathrm{Tr}(dc\phi_1 b) + 2N\mathrm{Tr}(d\phi_0\phi_1^2) - N\mathrm{Tr}(d\phi_1\phi_1\phi_0) - N\mathrm{Tr}(d\phi_1\phi_0\phi_1)].
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_1 =& \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0^2\phi_1) - \mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0^3) \\
&+ y[2\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1^2) - \mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0^2\phi_1) - \mathrm{Tr}(\phi_1^2)\mathrm{Tr}(\phi_0^3)] \\
&+ y^2[\mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0\phi_1^2) + \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_1^3) - 2\mathrm{Tr}(\phi_1^2)\mathrm{Tr}(\phi_0^2\phi_1)] \\
&+ y^2[\mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_1^3) - \mathrm{Tr}(\phi_1^2)\mathrm{Tr}(\phi_0\phi_1^2)].
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_1 &= \mathrm{Tr}(\phi(y)^5) \\
\mathcal{Z}_2 &= \mathrm{Tr}(\phi(y)^5) - \frac{5}{6}\mathrm{Tr}(\phi(y)^2)\mathrm{Tr}(\phi(y)^3)
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_1|_{y=0} =& \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(dcb) - 3\mathrm{Tr}(\phi_0 dc)\mathrm{Tr}(\phi_0 b) - 2\mathrm{Tr}(\phi_0 d\phi_1)\mathrm{Tr}(\phi_0^2) - \mathrm{Tr}(\phi_1 d\phi_0)\mathrm{Tr}(\phi_0^2) \\
&+ 3\mathrm{Tr}(\phi_0 d\phi_0)\mathrm{Tr}(\phi_0\phi_1) \\
\mathcal{Z}_2|_{y=0} =& -\frac{5N}{4(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0^2\phi_1^2) + \frac{5N}{4(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1) - \frac{(3N^2+8)}{4(3N^2-2)}\mathrm{Tr}(\phi_0^3\phi_1\phi_0\phi_1) \\
&- \frac{(9N^2-16)}{4(3N^2-2)}\mathrm{Tr}(\phi_0^2\phi_1\phi_0^2\phi_1) - \frac{N}{4}\mathrm{Tr}(\phi_0^3)\mathrm{Tr}(\phi_0\phi_1^2) + \frac{N}{4}\mathrm{Tr}(\phi_0^2\phi_1)^2 + \mathrm{Tr}(\phi_0^4\phi_1^2) \\
\mathcal{Z}_3|_{y=0} =& \frac{1}{100}(3N^2-2)\mathrm{Tr}(\phi_0^2)^2\mathrm{Tr}(\phi_1^2) - \frac{N(11N^2+26)}{20(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0^2\phi_1^2) \\
&+ \frac{1}{100}(2-3N^2)\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1)^2 + \frac{N(7N^2+62)}{40(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1) \\
&- \frac{(13N^2+8)}{5(3N^2-2)}\mathrm{Tr}(\phi_0^3\phi_1\phi_0\phi_1) - \frac{2(N-3)(N+3)}{5(3N^2-2)}\mathrm{Tr}(\phi_0^2\phi_1\phi_0^2\phi_1) \\
&- \frac{N}{8}\mathrm{Tr}(\phi_0^4)\mathrm{Tr}(\phi_1^2) + \frac{N}{4}\mathrm{Tr}(\phi_0^3\phi_1)\mathrm{Tr}(\phi_0\phi_1) + \mathrm{Tr}(\phi_0^4\phi_1^2) \\
=& \frac{1}{8}(3N^2-2)N\mathrm{Tr}(\phi_0^2)^2\mathrm{Tr}(\phi_1^2) - \frac{1}{8}(3N^2-2)N\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1)^2 - \frac{1}{4}N^2\mathrm{Tr}(\phi_0^3)\mathrm{Tr}(\phi_0\phi_1^2) \\
&+ \frac{N^2}{4}\mathrm{Tr}(\phi_0^2\phi_1)^2 + \frac{(16N^3-132N^2+191N-792)N}{4(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0^2\phi_1^2) \\
&+ \frac{(61N^3-132N^2-444N+1848)N}{8(3N^2-2)}\mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0\phi_1\phi_0\phi_1) \\
&+ \frac{(215N^3-1056N^2+260N-1056)}{4(3N^2-2)}\mathrm{Tr}(\phi_0^3\phi_1\phi_0\phi_1) \\
&+ \frac{(145N^3-528N^2-500N+2112)}{4(3N^2-2)}\mathrm{Tr}(\phi_0^2\phi_1\phi_0^2\phi_1) + \frac{1}{8}(31N-132)N\mathrm{Tr}(\phi_0^4)\mathrm{Tr}(\phi_1^2) \\
&- 6(5N-22)\mathrm{Tr}(\phi_0^4\phi_1^2) - \frac{1}{4}(31N-132)N\mathrm{Tr}(\phi_0^3\phi_1)\mathrm{Tr}(\phi_0\phi_1)
\end{aligned}$$

$$\mathcal{Z}_1|_{y=0} = \mathrm{Tr}(\phi_0\phi_1)\mathrm{Tr}(\phi_0^2) - \mathrm{Tr}(\phi_0^2)\mathrm{Tr}(\phi_0^4)$$

$$\begin{aligned}
\mathcal{Z}_1 &= \mathrm{Tr}(\phi(y)^6) \\
\mathcal{Z}_2 &= \mathrm{Tr}(\phi(y)^6) - \frac{1}{2}\mathrm{Tr}(\phi(y)^2)\mathrm{Tr}(\phi(y)^4) \\
\mathcal{Z}_3 &= \mathrm{Tr}(\phi(y)^6) - \frac{1}{3}\mathrm{Tr}(\phi(y)^2)\mathrm{Tr}(\phi(y)^4) - \frac{1}{4}\mathrm{Tr}(\phi(y)^2)^3 \\
\mathcal{Z}_4 &= \mathrm{Tr}(\phi(y)^6) - \frac{3}{4}\mathrm{Tr}(\phi(y)^2)\mathrm{Tr}(\phi(y)^4) - \frac{1}{3}4\mathrm{Tr}(\phi(y)^3)^2 + \frac{1}{8}\mathrm{Tr}(\phi(y)^2)^3
\end{aligned}$$

$$\left(c_{34}^{5}\right)^2 = \frac{60(\nu-16)(\nu+1)g_3 g_4}{(\nu-4)(\nu-1)(\nu+5)g_5}$$

$$X_1X_2 \dots X_n := \{X_1\{X_2\{\dots\{X_{n-1}, X_n\}_0 \dots\}_0\}_0\}_0$$

$$\int \prod_{I=1}^{n-3} \frac{dy_I}{y_I} y^{\alpha' X_I} \prod_{i<j} \left(F_{i,j}(\mathbf{y_I})\right)^{-\alpha' c_{i,j}}$$

$$\int \frac{dy_I}{y_I} y^{\alpha' X_I + q}$$

$$A_n^{\text{dark particle/white particle}} = \int \prod_{I=1}^{n-3} \frac{dy_I}{y_I} y_I^{\alpha' X_I - 1} \prod_{i<j} \left(F_{i,j}(\mathbf{y_I})\right)^{-\alpha' c_{i,j}}$$

$$\mathrm{Res}_{z_1 \to z_2} V(z_1)V(z_2) = V'(z_2)$$

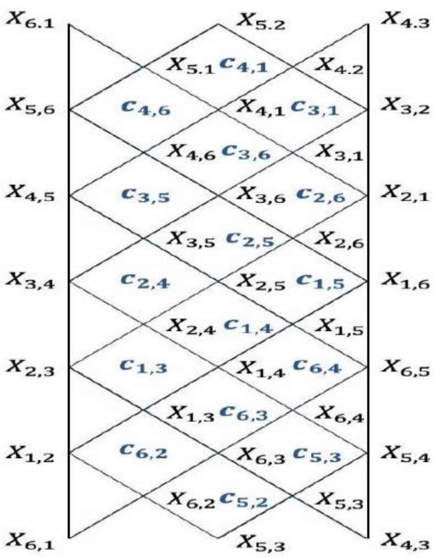
$$A_{2n}^{\text{dark particle/white particle}} = \int_0^\infty \prod_{s=1}^{n} \frac{dy_{2s-1,2s+1}}{y_{2s-1,2s+1}^2} y^{\alpha' X_{2s-1,2s+1}} \prod_{r=1}^{n-3} \frac{dy_{i_r,j_r}}{y_{i_r,j_r}^2} y^{\alpha' X_{i_r,j_r}} \prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}}$$

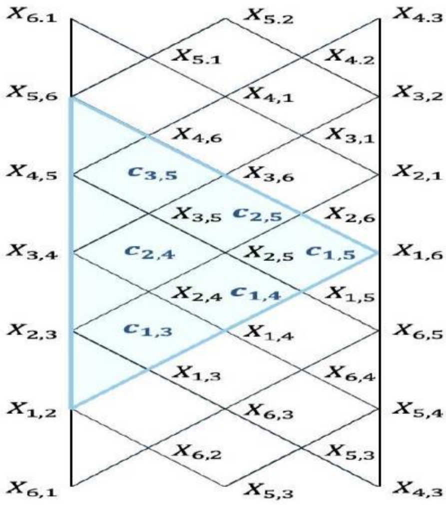
$$\xrightarrow{\text{scaffold}} \int_0^\infty \prod_{r=1}^{n-3} \frac{dy_{i_r,j_r}}{y_{i_r,j_r}^2} y^{\alpha' X_{i_r,j_r}} \left(\prod_{s=1}^{n} \partial_{y_{2s-1,2s+1}}\right) \left(\prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}}\right)\Bigg|_{y_{2s-1,2s+1}=0}$$

$$F_{2i-1,2j-1}\big|_{y_s=0} = F_{2i,2j-1}\big|_{y_s=0} = F_{2i-1,2j}\big|_{y_s=0} = F_{2i,2j}\big|_{y_s=0} = F_{i,j}^{\text{amp}}$$

$$c_{i,j} \equiv -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$$

$$A_n^{\mathrm{Tr}\phi^3} = \int_{z_1 < \cdots < z_n} \frac{dz_1 \cdots dz_n}{\mathrm{SL}(2,\mathbb{R})\mathbb{PT}} \prod_{i<j} z_{i,j}^{-\alpha' c_{i,j}}$$





$$u_{i,j} + \prod_{(k,l)\cap(i,j)} u_{k,l} = 1.$$

$$u_{i,j} = \frac{z_{i-1,j} z_{i,j-1}}{z_{i,j} z_{i-1,j-1}}$$

$$\prod_{i<j} z_{i,j}^{-\alpha' c_{i,j}} = \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

$$\int \frac{dz_1 \cdots dz_n}{\mathrm{SL}(2,\mathbb{R}) \mathbb{PT}}$$

$$M_L(y) = \begin{pmatrix} y & y \\ 0 & 1 \end{pmatrix}, M_R(y) = \begin{pmatrix} y & 0 \\ 1 & 1 \end{pmatrix}$$

$$F_{i-1,j-1} = (1,1) \cdot M_{L/R}(y_1) \cdots M_{L/R}(y_{n-3}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_{i,j} = y_{i,j} \frac{F_{i-1,j} F_{i,j-1}}{F_{i,j} F_{i-1,j-1}}$$

$$u_{i,j} = \frac{F_{i-1,j} F_{i,j-1}}{F_{i-1,j-1} F_{i,j}}$$

$$A^{\mathrm{Tr}\phi^3} = \int_{z_1 < \cdots < z_n} \frac{dz_1 \cdots dz_n}{\mathrm{SL}(2,\mathbb{R}) \mathbb{PT}} \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}} = \int_{\mathbb{R}_{>0}} \prod_{I=1}^{n-3} \frac{dy_I}{y_I} y_I^{\alpha' X_I} \prod_{i<j} \left(F_{i,j}(\mathbf{y_I})\right)^{-\alpha' c_{i,j}}$$

$$\int_{\mathbb{R}_{>0}} \frac{dy_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i} + q}$$

$$A_{2n}^{\text{dark particle/white particle}} = \int \frac{dz_1 dz_2 \cdots dz_{2n}}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n} e^{i p_a \cdot X(z_a)} \right\rangle = \int \frac{dz_1 \cdots dz_{2n}}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{-\alpha' c_{i,j}}$$

$$\prod_C u_C = \prod_i \frac{z_{i,i+1}}{z_{i,i+2}}, \prod_C u_C^{\alpha' X_C} = \prod_i^n z_{i,i+1}^{2\alpha' p_i^2} \prod_i^n z_{i,i+2}^{-\alpha' p_i^2} \prod_{i<j} z_{i,j}^{-\alpha' c_{i,j}},$$

$$A_n^{\text{dark particle/white particle}} = \int \frac{dz_1 \cdots dz_n}{\mathrm{SL}(2,\mathbb{R}) \mathbb{PT}} \mathbb{PT} \prod_{i<j} z_{ij}^{-\alpha' c_{i,j}} = \int_0^\infty \prod_I \frac{dy_I}{y_I} \prod_C u_C^{\alpha' X_C - 1}$$

$$\int \frac{dz_1 \cdots dz_n}{\mathrm{SL}(2,\mathbb{R}) \mathbb{PT}}$$

$$A_n^{\text{dark particle/white particle}} = \int \prod_I \frac{dy_I}{y_I} y_I^{\alpha' X_I - 1} \prod_{i<j} \left(F_{i,j}(\mathbf{y_I})\right)^{-\alpha' c_{i,j}}$$

$$A_{2n,\,\text{deform}}^{\text{Tr}\phi^3} = \int_{z_1<\cdots<z_n} \frac{dz_1 \cdots dz_n}{\text{SL}(2,\mathbb{R})\mathbb{PT}} \prod_C u_C^{\alpha' X_C} \frac{\prod u_{e,e}}{\prod u_{o,o}} = \int \prod_I \frac{dy_I}{y_I^2} y_I^{\alpha' X_I} \prod_{i<j} \left(F_{i,j}(\mathbf{y_I})\right)^{-\alpha' c_{i,j}},$$

$$p_{2i-1} + p_{2i} = k_i, i = 1,2, \dots, n$$

$$X_{2i-1,2i} := (p_{2i-1} + p_{2i})^2 = k_i^2 = 0 \implies c_{2i-1,2i} = \frac{2}{\alpha'},$$

$$\int dz_{2i-1} dz_{2i} z_{2i-1,2i}^{-\alpha' c_{2i-1,2i}} F(z_{2i-1}, z_{2i}, \dots)$$

$$z_{2i-1,2i} = U, z_{2i-1} = V, z_{2i} = U + V, dz_{2i-1} dz_{2i} = dUdV$$

$$\int dUdV U^{-\alpha' c_{2i-1,2i}} F(V, U + V, \dots)$$

$$\int dV \int_0^\delta dU U^{-\alpha' c_{2i-1,2i}} \left(F(V,V) + U \frac{\partial F}{\partial U}\bigg|_{U=0} + \frac{U^2}{2!} \frac{\partial^2 F}{\partial U^2}\bigg|_{U=0} + \cdots \right)$$
$$= \int dV \left[\frac{F|_{U=0}}{-\alpha' c_{2i-1,2i} + 1} + \frac{\frac{\partial F}{\partial U}\Big|_{U=0}}{-\alpha' c_{2i-1,2i} + 2} + \cdots + \frac{1}{(n-1)!} \frac{\frac{\partial^{n-1} F}{\partial U^{n-1}}\Big|_{U=0}}{-\alpha' c_{2i-1,2i} + n} \right]$$

$$\text{Res}_{\alpha' c_{2i-1,2i} \to 2} A_{2n}^{\text{dark particle/white particle}} = \text{Res}_{\alpha' c_{2i-1,2i} \to 2} \int \frac{dz_1 \cdots dz_{2n}}{\text{SL}(2,\mathbb{R})} z_{2i-1,2i}^{-\alpha' c_{2i-1,2i}} \prod_{(a,b) \neq (2i-1,2i)} z_{a,b}^{-\alpha' c_{a,b}}$$

$$\prod_{(a,b) \neq (2i-1,2i)} z_{a,b}^{-\alpha' c_{a,b}}$$

$$\text{Res}_{\alpha' p_{2i-1} p_{2i} \to -1} A_{2n}^{\text{dark particle/white particle}} = \int \frac{dz_1 \cdots dV \cdots dz_{2n}}{\text{SL}(2,\mathbb{R})} \partial_U F \bigg|_{U=0}$$

$$\partial_U F|_{U=0} = \left\langle p_{2i} \cdot i\partial X(z_{2i}) e^{i(p_{2i-1}+p_{2i}) \cdot X(z_{2i})} \prod_{a \neq 2i} e^{i p_a \cdot X(z_a)} \right\rangle$$

$$\int \frac{dz_1 \cdots dz_{2i-2} dz_{2i} \cdots dz_{2n}}{\text{SL}(2,\mathbb{R})} \left\langle \epsilon_i \cdot i\partial X(z_{2i}) e^{i k_i X(z_{2i})} \prod_{a \neq 2i} e^{i p_a \cdot X(z_a)} \right\rangle$$

$$\int \frac{\prod_{a=1}^n dz_{2a}}{\text{SL}(2,\mathbb{R})} \left\langle \prod_{i=1}^n (\epsilon_i \cdot i\partial X(z_i)) e^{i k_i \cdot X(z_i)} \right\rangle$$

$$A_{2n}^{\text{YM}} = \int \frac{\prod_{a=1}^{2n} dz_a}{\text{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n} \epsilon_a \cdot i\partial X(z_a) e^{i p_a \cdot X(z_a)} \right\rangle$$

$$k_i^2 = (p_{2i-1} + p_{2i})^2 \implies -\frac{1}{\alpha'}$$

$$\left(\frac{(\epsilon_1 \cdot p_2)(\epsilon_2 \cdot p_1)}{z_{1,2}^2}, \frac{\epsilon_2 \cdot \epsilon_1}{z_{1,2}^2}, \frac{\epsilon_2 \cdot p_1}{z_{1,2}}, \frac{\epsilon_1 \cdot p_2}{z_{1,2}} \right)$$

$$\left\langle\begin{bmatrix}\left(-2\alpha'^2(\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)+\alpha'(\epsilon_1\cdot\epsilon_2)\right)\\ \times\left(p_2\cdot i\partial^2X+(p_2\cdot i\partial X)(p_2\cdot i\partial X)\right)\\ -2\alpha'(\epsilon_1\cdot p_2)\left(\epsilon_2\cdot i\partial^2X+(\epsilon_2\cdot i\partial X)(p_2\cdot i\partial X)\right)\\ +2\alpha'(\epsilon_2\cdot p_1)(\epsilon_1\cdot i\partial X)(p_2\cdot i\partial X)\\ +(\epsilon_1\cdot i\partial X)(\epsilon_2\cdot i\partial X)\end{bmatrix}e^{ik_1\cdot X(z_2)}\prod_{i=3}^{2n}\epsilon_i\cdot i\partial X(z_i)e^{ip_i\cdot X(z_i)}\right\rangle$$

$$V(z)_{N=2}=\left[B^\mu i\partial^2X_\mu(z)+E^{\mu\nu}i\partial X_\mu(z)i\partial X_\nu(z)\right]e^{ik\cdot X(z)}$$

$$B^\mu=\left(-2\alpha'^2(\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)+\alpha'(\epsilon_1\cdot\epsilon_2)\right)p_2^\mu-2\alpha'(\epsilon_1\cdot p_2)\epsilon_2^\mu$$
$$E^{\mu\nu}=\Big[\left(-2\alpha'^2(\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)+\alpha'(\epsilon_1\cdot\epsilon_2)\right)(p_2^\mu\otimes p_2^\nu)-\alpha'(\epsilon_1\cdot p_2)\left((\epsilon_2^\mu\otimes p_2^\nu)+(p_2^\mu\otimes\epsilon_2^\nu)\right)$$
$$+\alpha'(\epsilon_2\cdot p_1)\left((\epsilon_1^\mu\otimes p_2^\nu)+(p_2^\mu\otimes\epsilon_1^\nu)\right)+\frac{1}{2}\left((\epsilon_1^\mu\otimes\epsilon_2^\nu)+(\epsilon_2^\mu\otimes\epsilon_1^\nu)\right)\Big]$$

$$T_X=-\frac{1}{2}\eta_{\rho\sigma}\partial X^\rho\partial X^\sigma$$

$$\begin{aligned}T_X(z)V_{N=2}(0)=&\frac{1}{2}\eta_{\rho\sigma}i\partial X^\rho i\partial X^\sigma(z)\left(B^\mu i\partial^2X_\mu e^{ik\cdot X}+E^{\mu\nu}i\partial X_\mu i\partial X_\nu e^{ik\cdot X}\right)(0)\\ &=(8\alpha'^2(k\cdot B)+4\alpha'^2\mathrm{Tr}(E))\frac{e^{ik\cdot X}}{z^4}+\left(4\alpha'(B\cdot i\partial X)+8\alpha'^2E_{\mu\nu}k^\mu i\partial X^\nu\right)\frac{e^{ik\cdot X}}{z^3}\\ &+\mathcal{O}\left(\frac{1}{z^2}\right)+\mathcal{O}\left(\frac{1}{z}\right)+\Re.\end{aligned}$$

$$\begin{gathered}8\alpha'^2(k\cdot B)+4\alpha'^2\mathrm{Tr}(E)=0\\ 4\alpha'B^\nu+8\alpha'^2k_\mu E^{\mu\nu}=0\end{gathered}$$

$$B^\mu=\alpha'(\epsilon_1\cdot\epsilon_2)p_2^\mu,E^{\mu\nu}=\alpha'(\epsilon_1\cdot\epsilon_2)(p_2^\mu\otimes p_2^\nu)+\frac{1}{2}\left((\epsilon_1^\mu\otimes\epsilon_2^\nu)+(\epsilon_2^\mu\otimes\epsilon_1^\nu)\right).$$

$$\begin{aligned}\mathrm{Res}_{\alpha'c_{2i-1,2i}\to1}A_{2n}^{\mathrm{YM}}&=\mathrm{Res}_{\alpha'c_{2i-1,2i}\to1}\int\frac{\prod_{a=1}^{2n}dz_a}{\mathrm{SL}(2,\mathbb{R})}\left\langle\prod_{a=1}^{2n}\epsilon_a\cdot i\partial X(z_a)e^{ip_a\cdot X(z_a)}\right\rangle\\ &=\int\frac{\prod_{a=1}^{n}dz_{2a}}{\mathrm{SL}(2,\mathbb{R})}\left\langle\prod_{i=1}^{n}\left[B_i^\mu i\partial^2X_\mu(z_i)+E_i^{\mu\nu}i\partial X_\mu(z_i)i\partial X_\nu(z_i)\right]e^{ik_i\cdot X(z_i)}\right\rangle\end{aligned}$$

$$\begin{aligned}&\epsilon_2\cdot i\partial Xe^{ip_2X}(z_2)\epsilon_1\cdot i\partial Xe^{ip_1X}(z_1)\\ &\sim\left[\frac{2\alpha'(\epsilon_1\cdot\epsilon_2)}{z_{1,2}^2}+\frac{2\alpha'(\epsilon_2\cdot p_1)(\epsilon_1\cdot i\partial X)}{z_{1,2}}+(\epsilon_2\cdot i\partial X)(\epsilon_2\cdot i\partial X)\right.\\ &\left.-\frac{4\alpha'^2(\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)}{z_{1,2}^2}-\frac{2\alpha'(\epsilon_1\cdot p_2)\left(\epsilon_2\cdot(i\partial X+(i\partial X)^2z_{1,2})\right)}{z_{1,2}}\right]\\ &\times\left[1+(p_2\cdot i\partial X)z_{1,2}+\frac{1}{2}((p_2\cdot i\partial X)^2+(p_2\cdot i\partial^2X))z_{1,2}^2\right]\left(\frac{1}{z_{1,2}}\right)e^{i(p_1+p_2)X}(z_1)\end{aligned}$$

$$V(z_1)V(z_2)|_{z_1\to z_2}=\cdots+\frac{V'}{z_{1,2}}+\cdots$$

$$\alpha'c_{i,j}=-\mathbb{N}_0,\forall(i,j)\in\{(i,j)\mid1\le i\le a-2,a\le j\le n-1\}.$$

$$\{(1,3),(3,5),\ldots,(2n-1,1)\}$$

$$A_{2n}^{\text{dark particle/white particle}}=\int_0^\infty\prod_{s=1}^{n}\frac{dy_{2s-1,2s+1}}{y_{2s-1,2s+1}^2}y^{\alpha'X_{2s-1,2s+1}}\prod_{r=1}^{n-3}\frac{dy_{i_r,j_r}}{y_{i_r,j_r}^2}y^{\alpha'X_{i_r,j_r}}\prod_{i<j}F_{i,j}^{-\alpha'c_{i,j}},$$

$$A_n^{\mathrm{YM}}\equiv\int_0^\infty\prod_{t=1}^{n-3}\frac{dy_{i_t,j_t}}{y_{i_t,j_t}^2}y^{\alpha'X_{i_t,j_t}}\left(\prod_{i=1}^{n}\partial_{y_{2i-1,2i+1}}\right)\left(\prod_{i<j}F_{i,j}^{-\alpha'c_{i,j}}\right)\Bigg|_{\substack{y2a-1,2a+1=0\\ a\in\{1,2,\ldots,n\}}}$$

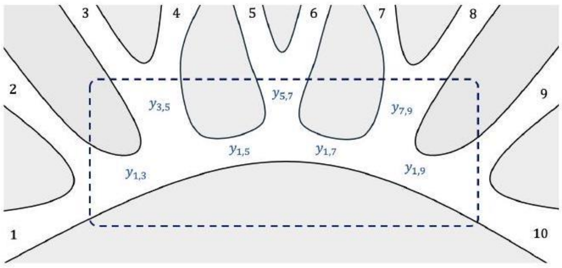
$$\prod_{i,j} F_{i,j}^{-\alpha' c_{i,j} - b_{i,j}} \Bigg|_{y_s=0}$$

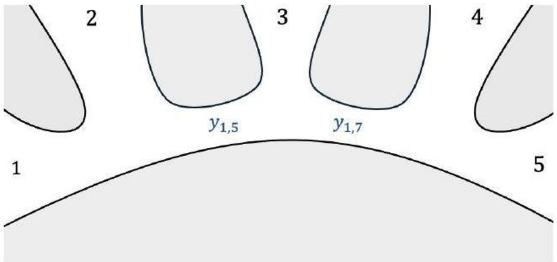
$$F_{2i-1,2j-1}\big|_{y_s=0} = F_{2i,2j-1}\big|_{y_s=0} = F_{2i-1,2j}\big|_{y_s=0} = F_{2i,2j}\big|_{y_s=0} = F_{i,j}^{\text{amp}}$$

$$\mathcal{T} = \{(1,3),(3,5),(5,7),(7,9),(9,1),(1,5),(1,7)\}$$

$$\begin{aligned} F_{1,7} &= (1,1)\cdot M_R(y_{1,3})M_R(y_{1,5})M_L(y_{1,7})M_R(y_{7,9})\cdot \begin{pmatrix}1\\0\end{pmatrix} \\ &= 1 + y_{1,7} + y_{1,5}y_{1,7} + y_{7,9}y_{1,7} + y_{7,9}y_{1,5}y_{1,7} + y_{1,3}y_{1,5}y_{1,7} + y_{1,3}y_{1,5}y_{1,7}y_{7,9} \\ F_{1,8} &= 1 + y_{1,7} + y_{1,5}y_{1,7} + y_{1,3}y_{1,5}y_{1,7} \\ F_{2,7} &= 1 + y_{1,7} + y_{1,5}y_{1,7} + y_{1,7}y_{7,9} + y_{1,5}y_{1,7}y_{7,9} \\ F_{2,8} &= 1 + y_{1,7} + y_{1,5}y_{1,7} \end{aligned}$$

$$F_{1,4}^{\text{amp}} = 1 + y_{1,4} + y_{1,3}y_{1,4} = 1 + y_{1,7} + y_{1,5}y_{1,7},$$





$$\mathcal{T} = \{(1,3),(3,5),\ldots,(2n-1,2n+1),(1,5),(1,7),\ldots,(1,2n-3)\},$$

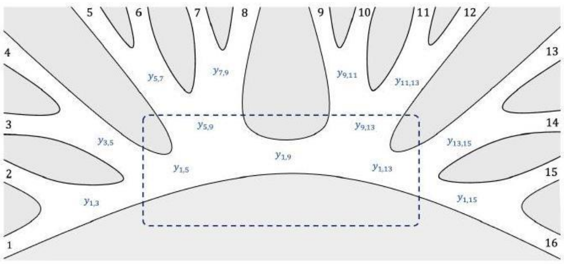
$$\alpha' c_{i,j} = -\mathbb{N}_0, \forall (i,j) \in \mathrm{N}_{\mathrm{YM}} := \{(i,j) \mid i \in \{1,2,\ldots,2a\}, j \in \{2a+3,\ldots,2n-2\}\},$$

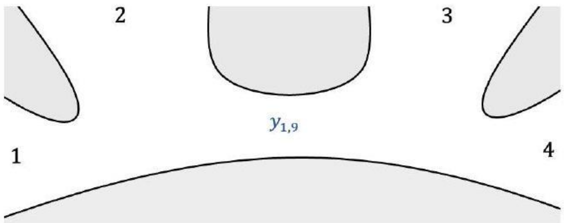
$$F_{i,j}\big|_{y_s=0} = F_{i',j'}^{\text{amp}}$$

$$i \in \{4i'-3, 4i'-2, 4i'-1, 4i'\}, j \in \{4j'-3, 4j'-2, 4j'-1, 4j'\}\}.$$

$$(i,j) \in \{i \in \{1,2,\ldots,4a\}, j \in \{4a+5, 4a+6, \ldots, 4n-4\}\}.$$

$$\alpha' c_{i,j} \in -\mathbb{N}_0, \forall (i,j) \in \{i \in \{1,2,\ldots,4a\}, j \in \{4a+5, 4a+6, \ldots, 4n-4\}\}$$





$$\begin{aligned} &\int \frac{dz_1 dz_2 dz_3 dz_4}{\mathrm{SL}(2,\mathbb{R})} \left\langle \left(B^\mu i\partial^2 X_\mu + E^{\mu\nu} i\partial X_\mu i\partial X_\nu\right) e^{ik\cdot X}(z_1) e^{ip_5\cdot X}(z_2) e^{ip_6\cdot X}(z_3) e^{ip_7\cdot X}(z_4)\right\rangle \\ &= \mathrm{B}(2\alpha' k\cdot p_5 - 1, 2\alpha' p_5\cdot p_6 + 1)(-2\alpha' B\cdot p_5 + 4\alpha'^2 p_5\cdot E\cdot p_5) \\ &+ \mathrm{B}(2\alpha' k\cdot p_5 + 1, 2\alpha' p_5\cdot p_6 - 1)(-2\alpha' B\cdot p_6 + 4\alpha'^2 p_6\cdot E\cdot p_6) \\ &+ \mathrm{B}(2\alpha' k\cdot p_5, 2\alpha' p_5\cdot p_6)(8\alpha' p_5\cdot E\cdot p_6) \end{aligned}$$

$$k = q_1 + q_2, q_1 = p_1 + p_2, q_2 = p_3 + p_4, k^2 = -\frac{1}{\alpha'}, q_i^2 = 0, p_i^2 = \frac{1}{\alpha'}$$

$$\mathcal{T} = \{(1,3),(3,5),(1,5),(1,6)\}$$

$$
\begin{aligned}
A_{\mathbf{1L2,3T}} =& \operatorname{Res}_{\alpha' X_{1,3}} \operatorname{Res}_{\alpha' X_{3,5}} \operatorname{Res}_{\alpha' X_{1,5}=-1} A_7^{\text{dark particle/white particle}} \\
=& \operatorname{Res}_{\alpha' X_{1,3}} \operatorname{Res}_{\alpha' X_{3,5}} \operatorname{Res}_{\alpha' X_{1,5}=-1} \\
& \times \left[\int \left(\frac{dy_{1,3}}{y_{1,3}^2} y_{1,3}^{\alpha' X_{1,3}} \frac{dy_{3,5}}{y_{3,5}^2} y_{3,5}^{\alpha' X_{3,5}} \frac{dy_{1,5}}{y_{1,5}^2} y_{1,5}^{\alpha' X_{1,5}} \right) \frac{dy_{1,6}}{y_{1,6}^2} y_{1,6}^{\alpha' X_{1,6}} \right) \prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}} \right] \\
=& \int \frac{dy_{1,6}}{y_{1,6}^2} y_{1,6}^{\alpha' X_{1,6}} \left(\partial_{y_{1,3}} \partial_{y_{3,5}} \partial_{y_{1,5}} \right) \left(\prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}} \right) \Bigg|_{y_{1,3}=y_{3,5}=y_{1,5}=0}
\end{aligned}
$$

$$(i,j) \in \mathrm{N}, \text{ where } \mathrm{N} = \{(i,j) \mid i = 1,2,3,4, j = 6\}$$

$$B(q_1, q_2, \epsilon_1, \epsilon_2) \cdot p_6 = E^{\mu}(q_1, q_2, \epsilon_1, \epsilon_2) \cdot p_6 = 0$$

$$\frac{1}{\Gamma(2\alpha' k \cdot p_5 + 2\alpha' p_5 \cdot p_6)} = \frac{1}{\Gamma(-1)} = 0.$$

$$V_{\mathrm{ST}}^{-1}(p,z) = e^{-\phi} e^{ip\cdot X}(z), V_{\mathrm{ST}}^{0}(p,z) = p \cdot \psi e^{ip\cdot X}(z)$$

$$A_{2n}^{\mathrm{ST}} = \int \frac{\prod_{a=1}^{2n} dz_{2a}}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n} V_{\mathrm{ST}}^{q_a}(z_a) \right\rangle$$

$$
\begin{aligned}
A_{2n}^{\mathrm{ST}} =& \int \frac{d^{2n} z d^{2n-2}\theta}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n\setminus i,j} e^{ip_a \cdot X(z_a) + \theta_a p_a \psi(z_a)} e^{-\phi(z_i)} e^{ip_i \cdot X(z_i)} e^{-\phi(z_j)} e^{ip_j \cdot X(z_j)} \right\rangle \\
=& \int \frac{d^{2n} z d^{2n-2}\theta}{\mathrm{SL}(2,\mathbb{R})} \frac{(-1)^{i+j+1}}{z_{i,j}} \prod_{a<b} |z_a - z_b + \theta_a \theta_b|^{-\alpha' c_{a,b}} \\
=& \int \frac{d^{2n} z}{\mathrm{SL}(2,\mathbb{R})} \frac{(-1)^{i+j+1}}{z_{i,j}} \prod_{a<b} z_{a,b}^{-\alpha' c_{a,b}} \operatorname{Pf}\left(D_{a,b}^{i,j}\right)
\end{aligned}
$$

$$D_{a,b} = \begin{cases} \dfrac{-\alpha' c_{a,b}}{z_{a,b}}, & a < b \\ -D_{b,a}, & a > b \end{cases}$$

$$
\begin{aligned}
A_{2n}^{\mathrm{ST}} =& \int \frac{d^{2n} z}{\mathrm{SL}(2,\mathbb{R})} \prod_{i=1}^{2n} z_{i,i+1}^{-1} \prod_{j=1}^{2n} z_{j,j+2}^{1/2} \prod_{C} u_C^{\alpha' X_C} \frac{1}{z_{a,b}} \operatorname{Pf}(D^{a,b}) \\
=& \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_{C} u_C^{\alpha' X_C} \prod_{i=1}^{2n} z_{i,i+2}^{\frac{1}{2}} \cdot \frac{1}{z_{a,b}} \operatorname{Pf}(D^{a,b})
\end{aligned}
$$

$$\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\},$$

$$\pi_\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & \hat{a} & \cdots & \hat{b} & \cdots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & \cdots & \cdots & \cdots & i_n & j_n \end{bmatrix},$$

$$D_\alpha^{a,b} = \operatorname{sgn}(\pi_\alpha) D_{i_1,j_1} D_{i_2,j_2} \cdots D_{i_n,j_n}$$

$$\operatorname{Pf}(D^{a,b}) = \sum_{\pi_\alpha \in \Pi} \operatorname{sgn}(\pi_\alpha) \prod_{p \in \alpha} \frac{-\alpha' c_p}{z_p}.$$

$$
\begin{aligned}
A_{2n}^{\mathrm{ST}} =& \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_{C} u_C^{\alpha' X_C} \left[\prod_{i=1}^{2n} z_{i,i+2}^{\frac{1}{2}} \cdot \frac{1}{z_{a,b}} \sum_{\pi_\alpha \in \Pi} \operatorname{sgn}(\pi_\alpha) \prod_{p \in \alpha} \frac{-\alpha' c_p}{z_p} \right] \\
=& \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_{C} u_C^{\alpha' X_C} \left[\sum_{\pi_\alpha \in \Pi} \operatorname{sgn}(\pi_\alpha) \prod_{p \in \alpha} (-\alpha' c_p) \prod_{C} u_C^{n_C^{\pi_\alpha}} \right]
\end{aligned}
$$

$$n_{i_d,j_d}^{\pi_\alpha} \in \left[-1, \frac{j_d - i_d}{2} - 1\right]$$

$$A_{2n}^{\mathrm{ST}} = \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} y_{i_d,j_d}^{\alpha' X_{i_d,j_d}} \prod_{i<j} F_{i,j}^{-\alpha' c_{ij}} \left(\sum_{\pi_\alpha \in \Pi} \mathrm{sgn}(\pi_\alpha) \frac{1}{F_{a,b}} \prod_{p\in\alpha} \frac{-\alpha' c_p}{F_p} \prod_{d=1}^{2n-3} y_{i_d,j_d}^{n_{i_d}^{\pi_\alpha} j_d} \right).$$

Caso 1: Partícula blanca o estrella: $(-1,-1) \to V_{\mathrm{SYM}}^{-2}$

$$V_{\mathrm{ST}}^{-1}(p_2,z_2)V_{\mathrm{ST}}^{-1}(p_1,z_1) = \left(e^{-\ln(z_2-z_1)}\right)\left(e^{-\phi(z_2)}e^{-\phi(z_1)}\right)\left(e^{2\alpha' p_1\cdot p_2 \ln(z_2-z_1)}\right)\left(e^{ip_2\cdot X(z_2)}e^{ip_1\cdot X(z_1)}\right)$$
$$\sim e^{-\ln z_{1,2}}\left(e^{-\phi(z_1)} - z_{1,2}\partial\phi e^{-\phi(z_1)}\right)e^{-\phi(z_1)}e^{-\ln z_{1,2}}\left(e^{ip_2\cdot X(z_1)} + z_{1,2}p_2\cdot i\partial X e^{ip_2\cdot X(z_1)}\right)e^{ip_1\cdot X(z_1)}$$
$$\sim \frac{1}{z_{1,2}^2}\left(1 + p_2\cdot\partial i X z_{1,2}\right)\left(1 - \partial\phi z_{1,2}\right)e^{-2\phi}e^{ik\cdot X}(z_1)$$

$$V_{\mathrm{SYM}}^{-2} = (\epsilon\cdot i\partial X - \partial\phi)e^{-2\phi}e^{ik\cdot X}.$$

Caso 2: Partícula supermasiva u oscura $(0,-1) \to V_{\mathrm{SYM}}^{-1}$

$$V_{\mathrm{ST}}^{0}(p_2,z_2)V_{\mathrm{ST}}^{-1}(p_1,z_1) = p_2\cdot\psi e^{ip_2\cdot X}(z_2)e^{-\phi}e^{ip_1\cdot X}(z_1)$$
$$\sim \left(\frac{1}{z_{1,2}}\right)(p_2\cdot\psi)\left(1 + p_2\cdot i\partial X z_{1,2}\right)e^{-\phi}e^{ik\cdot X}(z_1)$$

$$V_{\mathrm{SYM}}^{-1} = (\epsilon\cdot\psi)e^{-\phi}e^{ik\cdot X}.$$

Caso 3. Hiperpartícula/Suprapartícula $(0,0) \to V_{\mathrm{SYM}}^{0}$

$$V_{\mathrm{ST}}^{0}(p_2,z_2)V_{\mathrm{ST}}^{0}(p_1,z_1) = p_2\cdot\psi e^{ip_2\cdot X}(z_2)p_1\cdot\psi e^{ip_1\cdot X}(z_1)$$
$$\sim \left(\frac{1}{z_{1,2}}\right)\left(\frac{p_1\cdot p_2}{z_{1,2}} + (p_2\cdot\psi)(p_1\cdot\psi)\right)\left(1 + p_2\cdot i\partial X z_{1,2}\right)e^{ik\cdot X}(z_1)$$

$$\left((p_1\cdot p_2)(\epsilon\cdot i\partial X) + (\epsilon\cdot\psi)((k-\epsilon)\cdot\psi)\right)e^{ik\cdot X} = \left(-\frac{1}{2\alpha'}(\epsilon\cdot i\partial X) + (\epsilon\cdot\psi)(k\cdot\psi)\right)e^{ik\cdot X}$$
$$\Longrightarrow V_{\mathrm{SYM}}^{0} = \left((\epsilon\cdot i\partial X) + 2\alpha'(k\cdot\psi)(\epsilon\cdot\psi)\right)e^{ik\cdot X}$$

$$A_n^{\mathrm{SYM}} = \int \frac{\prod_{i=1}^{n} dz_i}{\mathrm{SL}(2,\mathbb{R})}\left\langle \prod_{i=1}^{n} V_{\mathrm{SYM}}^{q_i}(z_i)\right\rangle$$

$$A_n^{\mathrm{SYM}} = \prod_{i=1}^{n} \mathrm{Res}_{\alpha' X_{2i-1,2i+1}} A_{2n}^{\mathrm{ST}}$$
$$= \prod_{i=1}^{n} \mathrm{Res}_{\alpha' X_{2i-1,2i+1}} \left[\int \left(\prod_{s=1}^{n} \frac{dy_{2s-1,2s+1}}{y_{2s-1,2s+1}} y_{2s-1,2s+1}^{\alpha' X_{2s-1,2s+1}}\right)\left(\prod_{r=1}^{n-3} \frac{dy_{i_r,j_r}}{y_{i_r,j_r}} y_{i_r,j_r}^{\alpha' X_{i_r,j_r}}\right) \prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}} \right.$$
$$\left. \times \left(\sum_{\pi_\alpha\in\Pi} \mathrm{sgn}(\pi_\alpha) \frac{1}{F_{a,b}} \prod_{p\in\alpha} \frac{c_p}{F_p} \prod_{s=1}^{n} y_{2s-1,2s+1}^{n_{2s-1,2s+1}^{\pi_\alpha}} \prod_{r=1}^{n-3} y_{i_r,j_r}^{n_{i_r,j_r}^{\pi_\alpha}} \right) \right]$$

$$(i,j) \in \mathrm{N}_{\mathrm{YM}} := \{(i,j) \mid 1 \le i \le 2a, 2a+3 \le j \le 2n-2\}.$$

$$-\alpha' c_{i_{s_h},j_{s_h}} - 1 \in \mathbb{N}_0,$$

$$\prod_{(i,j)\in\mathrm{N}_{\mathrm{YM}}} F_{i,j}^{-\alpha' c_{i,j}}$$

$$\alpha' c_{i,j} \in -\mathbb{N}_0, \forall (i,j) \in \mathrm{N}_{\mathrm{YM}}.$$

$$\epsilon_1\cdot\epsilon_2 = \epsilon_1\cdot\epsilon_3 = \epsilon_2\cdot\epsilon_3 = 0.$$

$$\epsilon_i \cdot \epsilon_j = 0, \forall 1 \le i < j \le n.$$

$$B \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1+u)}$$

$$B = A_{\mathrm{YM}} + (2\alpha')^2 s_{13}\left[\left(\frac{f_{12}f_{34}}{s_{12}^2(1-s_{12})} + \mathrm{cyc}(2,3,4)\right) - \frac{g_1 g_2 g_3 g_4}{s_{12}^2 s_{13}^2 s_{14}^2}\right],$$

$$f_{ij} \equiv (\epsilon_i \cdot \epsilon_j)(k_i \cdot k_j) - (k_i \cdot \epsilon_j)(k_j \cdot \epsilon_i)$$

$$g_i \equiv (k_{i-1} \cdot \epsilon_i) s_{i,i+1} - (k_{i+1} \cdot \epsilon_i) s_{i-1,i}$$

$$\begin{aligned} V^{-1/2}_{\text{dark particle/white particle}}(u,p,z) &= u^\alpha \Theta_\alpha e^{-\phi/2} e^{ip\cdot X}(z) \\ V^{1/2}_{\text{dark particle/white particle}}(u,p,z) &= -2\left[Q_{\mathrm{BRST}}, \xi(z) V^{-1/2}_{\text{dark particle/white particle}}(u,p,z)\right] \\ &= \frac{1}{2\sqrt{\alpha'}} u^\alpha \left(i\partial X_\mu + \frac{\alpha'}{2} k_\nu \psi^\nu \psi_\mu\right)(\gamma^\mu \Theta)_\alpha e^{\phi/2} e^{ip\cdot X}(z), \end{aligned}$$

$$\begin{aligned} &\left\langle V^{-1/2}_{\text{dark particle/white particle}}(z_1) V^{-1/2}_{\text{dark particle/white particle}}(z_2) V^{-1/2}_{\text{dark particle/white particle}}(z_3) V^{-1/2}_{\text{dark particle/white particle}}(z_4)\right\rangle \\ &= u_\alpha u_\beta u_\lambda u_\delta \left(\frac{(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2 z_{1,2} z_{2,3} z_{2,4} z_{3,4}} + \frac{(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2 z_{1,3} z_{3,4} z_{3,2} z_{4,2}} + \frac{(\gamma^\mu C)_{\alpha\delta}(\gamma_\mu C)_{\beta\lambda}}{2 z_{1,4} z_{4,2} z_{4,3} z_{2,3}}\right) \prod_{1\le i<j\le 4} z_{i,j}^{-\alpha' c_{i,j}} \\ &= u_\alpha u_\beta u_\lambda u_\delta \left(\frac{(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2 z_{1,2} z_{2,3} z_{3,4} z_{1,4}} + \frac{(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2 z_{1,3} z_{1,4} z_{2,3} z_{2,4}}\right) \prod_{1\le i<j\le 4} z_{i,j}^{-\alpha' c_{i,j}} \end{aligned}$$

$$(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta} + (\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta} + (\gamma^\mu C)_{\alpha\delta}(\gamma_\mu C)_{\beta\lambda} = 0$$

$$\begin{aligned} A_4^{\text{dark particle/white particle}} &= u_\alpha u_\beta u_\lambda u_\delta \int \frac{d^4 z}{\mathrm{SL}(2,\mathbb{R})\mathbb{PT}} \prod_{i<j} z_{i,j}^{-\alpha' c_{i,j}} \mathbb{PT}\left(\frac{(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2 z_{1,2} z_{2,3} z_{3,4} z_{1,4}} + \frac{(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2 z_{1,3} z_{1,4} z_{2,3} z_{2,4}}\right) \\ &= u_\alpha u_\beta u_\lambda u_\delta \int \frac{dy_{1,3}}{y_{1,3}} \prod_C u_C^{\alpha' X_C} \left(\frac{-(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2} + \frac{-(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2} u_{1,3}\right) \\ &= u_\alpha u_\beta u_\lambda u_\delta \int \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} \prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}} \left(\frac{-(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2} + \frac{-(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2} y_{1,3} \frac{F_{1,2}F_{3,4}}{F_{1,3}F_{2,4}}\right) \end{aligned}$$

$$\begin{aligned} A_6^{\substack{\text{dark particle}\\ \text{/white particle}}} &= \int \frac{d^6 z}{SL(2,\mathbb{R})} \left\langle V_F^{-1/2}(z_1) V^{-1/2}_{\substack{\text{dark particle}\\ \text{/white particle}}}(z_2) V^{-1/2}_{\substack{\text{dark particle}\\ \text{/white particle}}}(z_3) V^{-1/2}_{\substack{\text{dark particle}\\ \text{/white particle}}}(z_4) V^{-1/2}_{\substack{\text{dark particle}\\ \text{/white particle}}}(z_5) V^{1/2}_{\substack{\text{dark particle}\\ \text{/white particle}}}(z_6)\right\rangle \\ &= \int \frac{d^6 z}{SL(2,\mathbb{R})} \prod_{1\le i<j\le 6} z_{i,j}^{-\alpha' c_{i,j}} G(z_t, \dot\alpha_t) u^1_{\dot\alpha_1} u^2_{\dot\alpha_2} u^3_{\dot\alpha_3} u^4_{\dot\alpha_4} u^5_{\dot\alpha_5} u^6_{\dot\alpha_6} \\ &= \int \frac{d^6 z}{SL(2,\mathbb{R}) \prod_{i=1}^6 z_{i,i+1}} \prod_{1\le i<j\le 6} z_{i,j}^{-\alpha' c_{i,j}} \prod_{i=1}^{6} z_{i,i+1} G(z_t, \dot\alpha_t) u^1_{\dot\alpha_1} u^2_{\dot\alpha_2} u^3_{\dot\alpha_3} u^4_{\dot\alpha_4} u^5_{\dot\alpha_5} u^6_{\dot\alpha_6} \end{aligned}$$

$$\begin{aligned} G(z_t, \dot\alpha_t) = \sum_{\pi\in S_5} \mathrm{sign}(\pi) \Bigg\{ \sum_{i=1}^{5} \frac{1}{z_{i,6}} \left(k^{i\nu} - \frac{1}{8} k^6_\mu M^{\mu\nu}_{(i)}\right) \Bigg[-\frac{1}{80} \frac{A_\nu^{\pi,\{\dot\alpha_j\}}}{z_{\pi(1),\pi(2)} z_{\pi(2),\pi(3)} z_{\pi(3),\pi(4)} z_{\pi(4),\pi(5)} z_{\pi(5),\pi(1)}} \\ + \frac{1}{24} \frac{B_\nu^{\pi,\{\dot\alpha_j\}} z_{\pi(4),6}}{z_{\pi(1),\pi(2)} z_{\pi(2),\pi(3)} z_{\pi(3),\pi(4)} z_{\pi(4),\pi(5)} z_{\pi(5),6} z_{\pi(4),\pi(1)}} \Bigg] \Bigg\} \end{aligned}$$

$$M^{\mu\nu}_{(i)}\left(f^{\dot\alpha_6 \dot\alpha_5 \dot\alpha_4 \cdots \dot\alpha_1}\right) = (\gamma^{\mu\nu})^{\dot\alpha_i}{}_{\dot\beta} f^{\dot\alpha_6 \dot\alpha_5 \cdots \dot\alpha_{i+1} \dot\beta \dot\alpha_{i-1} \cdots \dot\alpha_1},$$

$$A_\nu^{\pi,\{\dot\alpha_t\}} := \left(\gamma_\nu\gamma^{\rho\lambda}\right)^{\dot\alpha_6,\dot\alpha_{\pi(5)}}\left(\gamma_\rho\right)^{\dot\alpha_{\pi(4)},\dot\alpha_{\pi(3)}}\left(\gamma_\lambda\right)^{\dot\alpha_{\pi(2)},\dot\alpha_{\pi(1)}},$$
$$B_\nu^{\pi,\{\dot\alpha_t\}} := \left(\gamma_\nu\right)^{\dot\alpha_6,\dot\alpha_{\pi(5)}}\left(\gamma^\rho\right)^{\dot\alpha_{\pi(4)},\dot\alpha_{\pi(3)}}\left(\gamma_\rho\right)^{\dot\alpha_{\pi(2)},\dot\alpha_{\pi(1)}}.$$

$$\left(k^{i\nu} - \frac{1}{8}k_\mu^6 M_{(i)}^{\mu\nu}\right) A_\nu^{\pi,\{\dot\alpha_t\}} u_{\dot\alpha_6}^6 \equiv C_i^{\pi,\{\dot\alpha_t\}}$$
$$\left(k^{i\nu} - \frac{1}{8}k_\mu^6 M_{(i)}^{\mu\nu}\right) B_\nu^{\pi,\{\dot\alpha_t\}} u_{\dot\alpha_6}^6 \equiv D_i^{\pi,\{\dot\alpha_t\}}$$

$$\sum_{i=1}^{5} C_i^{\pi,\{\dot\alpha_t\}} u_{\dot\alpha_5}^5 u_{\dot\alpha_4}^4 u_{\dot\alpha_3}^3 u_{\dot\alpha_2}^2 u_{\dot\alpha_1}^1 = \sum_{i=1}^{5} C_i^{\pi=1,\{\dot\alpha_t\}} u_{\dot\alpha_{\pi-1}(5)}^5 u_{\dot\alpha_{\pi-1}(4)}^4 u_{\dot\alpha_{\pi-1}(3)}^3 u_{\dot\alpha_{\pi-1}(2)}^2 u_{\dot\alpha_{\pi-1}(1)}^1,$$
$$\sum_{i=1}^{5} D_i^{\pi}(\dot\alpha_t) u_{\dot\alpha_5}^5 u_{\dot\alpha_4}^4 u_{\dot\alpha_3}^3 u_{\dot\alpha_2}^2 u_{\dot\alpha_1}^1 = \sum_{i=1}^{5} D_i^{\pi=1,\{\dot\alpha_t\}} u_{\dot\alpha_{\pi-1}(5)}^5 u_{\dot\alpha_{\pi-1}(4)}^4 u_{\dot\alpha_{\pi-1}(3)}^3 u_{\dot\alpha_{\pi-1}(2)}^2 u_{\dot\alpha_{\pi-1}(1)}^1.$$

$$\sum_{i=1}^{5} C_i^{\pi=1,\{\dot\alpha_t\}} = 0, \sum_{i=1}^{5} D_i^{\pi=1,\{\dot\alpha_t\}} = 0$$

$$\frac{1}{z_{\pi(1),\pi(2)} z_{\pi(2),\pi(3)} z_{\pi(3),\pi(4)} z_{\pi(4),\pi(5)} z_{\pi(5),\pi(1)}}$$

$$\sum_{i=1}^{5} \frac{C_i^{\pi=1}}{z_{i,6}} = \sum_{i=1}^{4}\left(\frac{C_i^{\pi=1}}{z_{i,6}} - \frac{C_i^{\pi=1}}{z_{5,6}}\right)$$
$$= C_1^{\pi=1}\frac{z_{1,5}}{z_{1,6}z_{5,6}} + C_2^{\pi=1}\frac{z_{2,5}}{z_{2,6}z_{5,6}} + C_3^{\pi=1}\frac{z_{3,5}}{z_{3,6}z_{5,6}} + C_4^{\pi=1}\frac{z_{4,5}}{z_{4,6}z_{5,6}}$$

$$\frac{z_{k,5}}{z_{k,6}z_{5,6}} \cdot \frac{\prod_{i=1}^{6} z_{i,i+1}}{z_{\pi(1),\pi(2)} z_{\pi(2),\pi(3)} z_{\pi(3),\pi(4)} z_{\pi(4),\pi(5)} z_{\pi(5),\pi(1)}} := \prod_{i<j} z_{i,j}^{g_{i,j}^{\pi,k}} = \prod_{m<n} u_{m,n}^{\beta_{m,n}^{\pi,k}}$$
$$\frac{z_{k,5}}{z_{k,6}z_{5,6}} \cdot \frac{\prod_{i=1}^{6} z_{i,i+1} \cdot z_{\pi(4),6}}{z_{\pi(1),\pi(2)} z_{\pi(2),\pi(3)} z_{\pi(3),\pi(4)} z_{\pi(4),\pi(5)} z_{\pi(5),6} z_{\pi(4),\pi(1)}} := \prod_{i<j} z_{i,j}^{h_{i,j}^{\pi,k}} = \prod_{m<n} u_{m,n}^{\gamma_{m,n}^{\pi,k}}.$$

$$\prod_{i<j} z_{i,j}^{g_{i,j}^{\pi,k}} = \prod_{m<n} u_{m,n}^{\beta_{m,n}^{\pi,k}} = \prod_{t=1}^{3} y_{i_t,j_t}^{\beta_{it',j}^{\pi,k}} \prod_{i<j} F_{i,j}^{g_{ij}^{\pi,k}},$$
$$\prod_{i<j} z_{i,j}^{h_{ij}^{\pi,k}} = \prod_{m<n} u_{m,n}^{\gamma_{m,n}^{\pi,k}} = \prod_{t=1}^{3} y_{i_t,j_t}^{\gamma_{it',j}^{\pi,k}} \prod_{i<j} F_{i,j}^{h_{ij}^{\pi,k}}.$$

$$A_6^{\text{dark particle}/\text{white particle}} = \int_0^\infty \prod_{I=1}^{3} \frac{dy_I}{y_I} y_I^{\alpha' X_I} \prod_{i<j} F_{i,j}^{-\alpha'} c_{i,j}$$
$$\times \left[\sum_{\pi\in S_5}\left(\frac{-1}{80}\sum_{k=1}^{4} C_k^{\pi=1}\prod_{I=1}^{3} y_I^{\beta_I^{\pi,k}} \prod_{i<j} F_{i,j}^{g_{i,j}^{\pi,k}} + \frac{1}{24}\sum_{k=1}^{4} D_k^{\pi=1}\prod_{I=1}^{3} y_I^{\gamma_I^{\pi,k}} \prod_{i<j} F_{i,j}^{h_{i,j}^{\pi,k}}\right)\right.$$
$$\left. \times u_{\dot\alpha_{\pi-1(5)}}^5 u_{\dot\alpha_{\pi-1(4)}}^4 u_{\dot\alpha_{\pi-1(3)}}^3 u_{\dot\alpha_{\pi-1(2)}}^2 u_{\dot\alpha_{\pi-1(1)}}^1\right]$$

$$\text{If } \alpha' c_{1,3}, \alpha' c_{1,4}, \alpha' c_{1,5} \in -\mathbb{N}, \quad \Rightarrow y_{1,3}$$
$$\text{If } \alpha' c_{1,4}, \alpha' c_{2,4}, \alpha' c_{1,5}, \alpha' c_{2,5} \in -\mathbb{N}, \quad \Rightarrow y_{1,4}$$
$$\text{If } \alpha' c_{1,5}, \alpha' c_{2,5}, \alpha' c_{3,5} \in -\mathbb{N}, \quad \Rightarrow y_{1,5}$$

$$\text{Trop}\left(\frac{y_1^a y_2^b}{1 + y_1 + y_2}\right) = at_1 + bt_2 - \max(0, t_1, t_2).$$

$$\mathcal{I} := (\alpha')^{n-3}\int_0^\infty \prod_{I=1}^{n-3} \frac{dy_I}{y_I} y_I^{\alpha' X_I} \prod_{i<j} F_{i,j}^{-\alpha' c_{i,j}}(\mathbf{y}),$$

$$R(\mathbf{y}) := \prod_{I=1}^{n-3} y_I^{X_I} \prod_{i<j} F_{i,j}^{-c_{i,j}}(\mathbf{y}).$$

$$\text{Trop}R(\mathbf{t}) = \sum_{I=1}^{n-3} X_I t_I - \sum_{i<j} c_{i,j}\text{Trop}F_{i,j}(\mathbf{t}),$$

$$\lim_{\alpha'\to 0} \mathcal{I} = \lim_{\alpha'\to 0} \int_{-\infty}^{\infty} (\alpha')^{n-3} \prod_{I=1}^{n-3} dt_I \exp\left(\alpha' \text{Trop}R(\mathbf{t})\right) = \int_{-\infty}^{\infty} \prod_{I=1}^{n-3} dt_I \exp\left(\sum_{I=1}^{n-3} X_I t_I - \sum_{i<j} c_{i,j} f_{i,j}(\mathbf{t})\right),$$

$$\int_0^{\infty} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} F_{1,3}^{-\alpha' c_{1,3}} = \int_{-\infty}^{\infty} dt e^{\alpha' X_{1,3} t}(1+e^t)^{-\alpha' c_{1,3}}$$

$$\lim_{\alpha'\to 0} \alpha' \int_0^{\infty} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} F_{1,3}^{-\alpha' c_{1,3}} = \int_{-\infty}^{\infty} dt e^{X_{1,3} t - c_{1,3} f_{1,3}(t)}$$

$$\begin{aligned}\int_0^{\infty} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+1} F_{1,3}^{-\alpha' c_{1,3}-1} &= \int_{-\infty}^{\infty} dt e^{\alpha' X_{1,3} t + t}(1+e^t)^{-\alpha' c_{1,3}-1} \\ &= \int_0^{\infty} dt e^{\alpha' X_{1,3} t + t} e^{-\alpha' c_{1,3} t - t}(1+e^{-t})^{-\alpha' c_{1,3}-1} \\ &\quad + \int_{-\infty}^{0} dt e^{\alpha' X_{1,3} t + t}(1+e^t)^{-\alpha' c_{1,3}-1}\end{aligned}$$

$$\begin{aligned}A_4^{\text{dark particle}/\text{white particle},F} &= u_\alpha u_\beta u_\lambda u_\delta \left[\frac{-(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2}\left(\frac{1}{(c_{1,3}-X_{1,3})} + \frac{1}{X_{1,3}}\right)\right. \\ &\quad \left. + \frac{-(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2}\left(\frac{1}{(c_{1,3}-X_{1,3})}\right)\right] \\ &= u_\alpha u_\beta u_\lambda u_\delta \left[\frac{-(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu C)_{\lambda\delta}}{2}\left(\frac{-1}{u} + \frac{1}{s}\right) + \frac{-(\gamma^\mu C)_{\alpha\lambda}(\gamma_\mu C)_{\delta\beta}}{2}\left(\frac{-1}{u}\right)\right].\end{aligned}$$

$$p_1 + p_2 = k, \frac{1}{\sqrt{2}} u_\alpha (\gamma^\mu C) u_\beta = \epsilon^\mu, \epsilon \cdot k = 0$$

Caso 1: Partícula blanca $\left(-\frac{1}{2}, -\frac{1}{2}\right) \to V_{\text{SYM}}^{-1}$

$$\begin{aligned}&V_{\text{white particle}}^{-1/2}(u_\alpha, p_2, z_2) V_{\text{white particle}}^{-1/2}(u_\beta, p_1, z_1) = u^\alpha \Theta_\alpha e^{-\phi/2} e^{ip_2\cdot X}(z_2) u^\beta \Theta_\beta e^{-\phi/2} e^{ip_1\cdot X}(z_1) \\ &\sim u_\alpha u_\beta \left(\frac{1}{z_{1,2}^{1/4}}\right)\left(\frac{(\gamma^\mu C)_{\alpha\beta}\psi_\mu}{\sqrt{2} z_{1,2}^{3/4}}\right) e^{-\phi} e^{ik\cdot X}(z_1)\end{aligned}$$

$$V_{\text{SYM}}^{-1} = (\epsilon \cdot \psi) e^{-\phi} e^{ik\cdot X}$$

Caso 2: Partícula oscura $\left(-\frac{1}{2}, \frac{1}{2}\right) \to V_{\text{SYM}}^{0}$

$$\begin{aligned}&V_{\text{dark particle}}^{-1/2}(u_\alpha, p_2, z_2) V_{\text{dark particle}}^{1/2}(u_\beta, p_1, z_1) = u^\alpha \Theta_\alpha e^{-\phi/2} e^{ip_2\cdot X}(z_2)\left(-2\left[Q_{\text{BRST}}, \xi(z) V_{\text{dark particle}}^{-1/2}(u,p,z)\right]\right)(z_1) \\ &= u^\alpha \Theta_\alpha e^{-\phi/2} e^{ip_2\cdot X}(z_2)\left(-2\oint \frac{dw}{2\pi i} e^{\phi}\eta i\partial X \cdot \psi(w)\right)\xi(z_1) u^\beta \Theta_\beta e^{-\phi/2} e^{ip_1\cdot X}(z_1) \\ &\sim u_\alpha u_\beta \left(\frac{(\gamma^\mu C)_{\alpha\beta}}{\sqrt{2} z_{1,2}}\right)\left(i\partial X_\mu + 2\alpha'(k\cdot\psi)\psi_\mu\right) e^{ik\cdot X}(z_1)\end{aligned}$$

$$\Rightarrow V_{\text{SYM}}^{0} = ((\epsilon \cdot i\partial X) + 2\alpha'(k\cdot\psi)(\epsilon\cdot\psi)) e^{ik\cdot X}$$

$$A_{2n}^{\text{dark particle/white particle}} = \int \frac{\prod_{a=1}^{2n} dz_{2a}}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n} V_{\text{dark particle/white particle}}^{q_a}(z_a) \right\rangle$$

$$\rightarrow A_n^{\mathrm{SYM}} = \int \frac{\prod_{i=1}^{n} dz_i}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{i=1}^{n} V_{\mathrm{SYM}}^{q_i}(z_i) \right\rangle$$

$$\int \frac{d^n z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z^{-\alpha' c_{i,j}} \rightarrow \int \frac{d^n z_i d^{2n-2}\theta}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{a,b}} \prod_{i<j} \left|z_{i,j} - \theta_i\theta_j\right|^{-\alpha' c_{i,j}}$$

$$\tilde{u}_{i,j} \equiv \frac{\left|z_{i-1,j} - \theta_{i-1}\theta_j\right|\left|z_{i,j-1} - \theta_i\theta_{j-1}\right|}{\left|z_{i,j} - \theta_i\theta_j\right|\left|z_{i-1,j-1} - \theta_{i-1}\theta_{j-1}\right|}$$

$$\tilde{u}_{i,j} + \prod_{(k,l)\cap(i,j)} \tilde{u}_{k,l} = 1 + \not{f}_{\text{fermion terms}}$$

$$A_{2n}^{\mathrm{ST}} = \int \frac{\prod_{a=1}^{2n} dz_{2a}}{\mathrm{SL}(2,\mathbb{R})} \left\langle \prod_{a=1}^{2n} V_{\mathrm{ST}}^{q_a}(z_a) \right\rangle$$

$V_{\mathrm{ST}}^{-1}(z) = e^{-\phi} e^{ip\cdot X}(z), V_{\mathrm{ST}}^{0}(z) = p \cdot \psi e^{ip\cdot X}(z), p^2 = \frac{1}{2\alpha'}$, and $\sum_a q_a = -2$

$$\left\langle e^{-\phi} e^{ip_1\cdot X}(z_1) e^{-\phi} e^{ip_2\cdot X}(z_2) p_3 \cdot \psi e^{ip_3\cdot X}(z_3) p_4 \cdot \psi e^{ip_4\cdot X}(z_4) \prod_{a=4}^{2n} V_{\mathrm{ST}}^{0}(z_a) \right\rangle$$

$$\left(\frac{1}{z_{1,2}^2} e^{-\phi} e^{ip_1\cdot X}(z_1) e^{-\phi} e^{ip_2\cdot X}(z_2) \right)$$

$$\left(\frac{p_3 \cdot p_4}{z_{3,4}^2} e^{ip_3\cdot X}(z_3) e^{ip_4\cdot X}(z_4), \frac{(p_3\cdot\psi)(p_4\cdot\psi)}{z_{3,4}} e^{ip_3\cdot X}(z_3) e^{ip_4\cdot X}(z_4) \right)$$

$$\begin{aligned} &\langle (\epsilon_1 \cdot i\partial X - \partial\phi) e^{-2\phi} e^{ik_1\cdot X}(z_1) \\ &\left((\epsilon_2 \cdot i\partial X) + 2\alpha'(k_2\cdot\psi)(\epsilon_2\cdot\psi) \right) e^{ik_2\cdot X}(z_2) \prod_{i=3}^{n} V_{\mathrm{SYM}}^{0}(z_i) \rangle \end{aligned}$$

$$\left\langle e^{-\phi} e^{ip_1\cdot X}(z_1) p_2 \cdot \psi e^{ip_2\cdot X}(z_2) e^{-\phi} e^{ip_3\cdot X}(z_3) p_4 \cdot \psi e^{ip_4\cdot X}(z_4) \prod_{a=4}^{2n} V_{\mathrm{ST}}^{0}(z_a) \right\rangle.$$

$$\left(\frac{p_1 \cdot p_2}{z_{12}} e^{-\phi} e^{ip_1\cdot X}(z_1) e^{ip_2\cdot X}(z_2) \right)$$

$$\left\langle (\epsilon\cdot\psi) e^{-\phi} e^{ik_1\cdot X}(z_1) (\epsilon\cdot\psi) e^{-\phi} e^{ik_2\cdot X}(z_2) \prod_{i=3}^{n} V_{\mathrm{SYM}}^{0}(z_i) \right\rangle$$

$$\mathbb{V} = (\mathcal{N} = 4) \oplus \bigoplus_{s} R^{(s)},$$

$$(\text{large } \mathcal{N} = 4) \oplus \bigoplus_{s=1}^{\infty} \hat{R}^{(s)},$$

$$\langle \hat{R}^{(1)} \hat{R}^{(1)} \hat{R}^{(1)} \rangle.$$

$$\mathbb{V} = \bigoplus_{w=1}^{\infty} R^{\left(\frac{w+1}{2}\right)},$$

$$\tilde{\mathbb{V}} = (\mathcal{N} = 4) \oplus H^*(\mathbb{T}^4) \otimes \bigoplus_{s} R^{(s)}$$

$$\bigoplus_{w=1}^{M} \left(R^{\left(\frac{w-1}{2}\right)} \oplus 2 \cdot R^{\left(\frac{w}{2}\right)} \oplus R^{\left(\frac{w+1}{2}\right)}\right)$$

$$c = 6M: \bigoplus_{w=1}^{M} R^{\left(\frac{w+1}{2}\right)}.$$

$$\begin{array}{l} [h = s, j = s] \\ R^{(s)}: 2 \cdot \left[h = s + \frac{1}{2}, j = s - \frac{1}{2}\right] \\ \qquad\qquad [h = s + 1, j = s - 1] \end{array}$$

$$\mathbb{V} = (\mathcal{N} = 4) \oplus \bigoplus_{s=\frac{3}{2},2,\dots} R^{(s)}$$

$$W_m^{(s)}(z) W_n^{(s)}(w) \sim \frac{\langle 0,0 \| s, m; s, n\rangle}{(z-w)^{2s}} \mathbf{1} + \cdots$$

$$|s, m\rangle \otimes |s, n\rangle \to |0,0\rangle.$$

$$\langle 0,0 \| s, m; s, n\rangle = (-1)^{2s} \langle 0,0 \| s, n; s, m\rangle$$

$$W_n^{(s)}(w) W_m^{(s)}(z) \sim \frac{\langle 0,0 \| s, n; s, m\rangle}{(w-z)^{2s}} \mathbf{1} + \cdots \sim W_m^{(s)}(z) W_n^{(s)}(w).$$

$$\begin{array}{lll} W_m^{(s)} & m = -s, \dots, s & (h = s) \\ W_m^{(s)[\alpha]} & m = -\left(s - \frac{1}{2}\right), \dots, \left(s - \frac{1}{2}\right) & \left(h = s + \frac{1}{2}\right) \\ W_m^{(s)\uparrow} & m = -(s-1), \dots, (s-1) & (h = s + 1), \end{array}$$

dark particle: $W_m^{(s)}, (|m| \leq s)\ W_n^{(s)\uparrow}, (|n| \leq s - 1), s = \frac{3}{2}, 2, \frac{5}{2}, \dots,$

white particle: $W_r^{(s)[\alpha]}, \left(|r| \leq s - \frac{1}{2}\right), \alpha \in \{\pm\}, s = \frac{3}{2}, 2, \frac{5}{2}, \dots.$

(bosonic/fermionic∵$\mathcal{N} = 4$) $\times W^{(s_1)} \times W^{(s_2)}$

$$W^{(s_1)} \times W^{(s_2)} \times W^{(s_3)}$$

$W^{(s_1)} \times W^{(s_2)}$ with $s_1 + s_2 \leq P - 1$

$$W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)}$$

$W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)}$ and $W^{\left(\frac{3}{2}\right)} \times W^{(2)}$

$$W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)} \sim n_{\frac{3}{2}}[\mathbf{1} + c_1 K + c_2 L + c_3 \partial K + c_4 \colon KK \colon] \oplus C^2_{\frac{3}{2}\frac{3}{2}} W^{(2)}$$

$$c_1 = \frac{3\sqrt{10}}{c}, c_2 = \frac{3(c-3)}{c(9+c)}, c_3 = \frac{3\sqrt{5}}{\sqrt{2}c}$$

$$c_4^{(0)} = -\frac{36\sqrt{3}}{c(c+9)}, c_4^{(2)} = \frac{18\sqrt{6}}{c(c-6)}$$

$$W^{\left(\frac{3}{2}\right)} \times W^{(2)} \sim C_{\frac{3}{2}2}^{\frac{3}{2}}\left[W^{\left(\frac{3}{2}\right)} + d_1 : KW^{\left(\frac{3}{2}\right)} : + d_2 \partial W^{\left(\frac{3}{2}\right)}\right] \oplus C_{\frac{3}{2},2}^{\frac{5}{2}} W^{\left(\frac{5}{2}\right)} \oplus C_{\frac{3}{2},2}^{\frac{3}{2}\uparrow} W^{\left(\frac{3}{2}\right)\uparrow}$$

$$d_1^{\left(\frac{1}{2}\right)} = -\frac{6\sqrt{6}}{c+30}, d_1^{\left(\frac{3}{2}\right)} = \frac{2\sqrt{\frac{6}{5}}}{c-3}, d_1^{\left(\frac{5}{2}\right)} = \frac{6\sqrt{\frac{14}{5}}}{c-18}, d_2 = \frac{(c+3)}{3(c-3)}$$

$$C_{\frac{3}{2},2}^{\frac{3}{2}}, C_{\frac{3}{2},2}^{\frac{5}{2}}, C_{\frac{3}{2},2}^{\frac{3}{2}\uparrow}$$

$$C_{\frac{3}{2},\frac{3}{2}}^{2} \cdot C_{\frac{3}{2},2}^{\frac{3}{2}} = -18\sqrt{5}\frac{(c+24)(c-3)}{(9+c)(c-6)} n_{\frac{3}{2}}.$$

$$W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)} \times W^{(2)}$$

$$W^{(2)} \times W^{(2)}, W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{5}{2}\right)}, \text{ and } W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)\uparrow},$$

$$\begin{aligned} W^{(2)} \times W^{(2)} \sim & n_2[\mathbf{1} + K + L + \partial K + :KK: + :KKK: + :KL: + :K\partial K: + \partial L] \\ & \oplus C_{2,2}^{2}[W^{(2)} + \partial W^{(2)} + :KW^{(2)}:] \oplus C_{2,2}^{3} W^{(3)} \\ & \oplus C_{2,2}^{2\uparrow} W^{(2)\uparrow} \oplus C_{2,2}^{GG} :GG: \oplus C_{2,2}^{:\frac{3}{2}\frac{3}{2}:} :W^{\left(\frac{3}{2}\right)} W^{\left(\frac{3}{2}\right)}:, \end{aligned}$$

$$\begin{aligned} W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{5}{2}\right)} \sim & \oplus C_{\frac{3}{2},\frac{5}{2}}^{2}[W^{(2)} + \partial W^{(2)} + :KW^{(2)}:] \oplus C_{\frac{3}{2},\frac{5}{2}}^{3} W^{(3)} \\ & \oplus C_{\frac{3}{2},\frac{5}{2}}^{2\uparrow} W^{(2)\uparrow} \oplus C_{\frac{3}{2},\frac{5}{2}}^{GG} :GG: \oplus C_{\frac{3}{2},\frac{5}{2}}^{:\frac{3}{2}:} :W^{\left(\frac{3}{2}\right)} W^{\left(\frac{3}{2}\right)}:. \end{aligned}$$

$$\begin{aligned} W^{\left(\frac{3}{2}\right)} \times W^{\left(\frac{3}{2}\right)\uparrow} \sim & C_{\frac{3}{2},\frac{3}{2}\uparrow}^{0}[K + L + \partial K + :KK: + :KKK: + :KL: + :K\partial K: + \partial L] \\ & \oplus C_{\frac{3}{2},\frac{3}{2}\uparrow}^{2}[W^{(2)} + \partial W^{(2)} + :KW^{(2)}:] \oplus C_{\frac{3}{2},\frac{3}{2}\uparrow}^{3} W^{(3)} \\ & \oplus C_{\frac{3}{2},\frac{3}{2}\uparrow}^{2\uparrow} W^{(2)\uparrow} \oplus C_{\frac{3}{2},\frac{3}{2}\uparrow}^{GG} :GG: \oplus C_{\frac{3}{2},\frac{3}{2}\uparrow}^{\frac{3}{2}:} :W^{\left(\frac{3}{2}\right)} W^{\left(\frac{3}{2}\right)}: \end{aligned}$$

$$\begin{aligned} \left(C_{\frac{3}{2},\frac{3}{2}}^{2}\right)^2 &= -18\sqrt{5}\frac{(c+24)(c-3)}{(c+9)(c-6)}\frac{\left(n_{\frac{3}{2}}\right)^2}{n_2}, \\ \left(C_{2,2}^{2}\right)^2 &= -\frac{112}{\sqrt{5}}\frac{(c^2+54c-90)^2}{c(c+9)(c+24)(c-3)(c-6)} n_2, \\ C_{\frac{3}{2},\frac{5}{2}}^{2} \cdot C_{\frac{3}{2},2}^{\frac{5}{2}} &= -21\sqrt{\frac{15}{2}\frac{(c+45)(c-6)}{c(c+9)(c-18)}} n_{\frac{3}{2}}, \\ C_{\frac{3}{2},\frac{3}{2}\uparrow}^{2} \cdot C_{\frac{3}{2},2}^{\frac{3}{2}\uparrow} &= 3\sqrt{\frac{5}{2}\frac{(c-3)(c-6)}{c(c+9)(c+30)}} n_{\frac{3}{2}}. \end{aligned}$$

$$C_{\frac{3}{2},\frac{5}{2}}^{3} C_{\frac{3}{2},2}^{\frac{5}{2}} = -i\frac{5^{\frac{3}{4}}\sqrt{21}}{2}\sqrt{\frac{(c+24)(c-3)}{c(c+9)(c-6)}}\frac{n_{\frac{3}{2}}}{\sqrt{n_2}} C_{2,2}^{3}.$$

$$\mathbb{V} = (\mathcal{N} = 4) \oplus R^{\left(\frac{3}{2}\right)} \oplus R^{(2)} \oplus \cdots \oplus R^{\left(\frac{N}{2}\right)},$$

$$N = 2 : \mathbb{V} = (\mathcal{N} = 4), \text{ with } c = -3(2^2 - 1) = -9.$$

$$N = 3 : \mathcal{N} = 4 \oplus R^{\left(\frac{3}{2}\right)}, \text{ with } c = -3(3^2 - 1) = -24$$

$$\left\langle W^{\left(\frac{3}{2}\right)}W^{\left(\frac{3}{2}\right)}W^{(2)}\right\rangle.$$

$$C^{2}_{\frac{3}{2},\frac{3}{2}}=0$$

$$N=4{:}\ \mathcal{N}=4\oplus R^{\left(\frac{3}{2}\right)}\oplus R^{(2)},\ \text{with}\ c=-3(4^2-1)=-45$$

$$\left\langle W^{(2)}W^{\left(\frac{3}{2}\right)}W^{\left(\frac{5}{2}\right)}\right\rangle\cdot\left\langle W^{\left(\frac{5}{2}\right)}W^{\left(\frac{3}{2}\right)}W^{(2)}\right\rangle.$$

$$(\mathcal{N}=4)\oplus R^{\left(\frac{3}{2}\right)}$$

$$W^{\left(\frac{3}{2}\right)}\times W^{\left(\frac{3}{2}\right)}\sim n_{\frac{3}{2}}[\mathbf{1}+c_1K+c_2L+c_3\partial K+c_4{:}KK{:}],$$

$$(\mathcal{N}=4)\oplus R^{\left(\frac{3}{2}\right)}\oplus R^{(2)}$$

$$\mathbb{V}_0\equiv(\mathcal{N}=4)\oplus R^{\left(\frac{1}{2}\right)}\oplus\bigoplus_{s\geq 1}N_s\cdot R^{(s)}$$

$$\mathbb{V}_0=\mathbb{V}\left[\frac{1}{2}\right]\oplus\mathbb{V}_1$$

$$W_r^{\left(\frac{1}{2}\right)}(z)W_s^{\left(\frac{1}{2}\right)}(w)\sim n_{\frac{1}{2}}\frac{\epsilon_{rs}}{(z-w)}$$

$$\mathbb{V}_1\equiv(\mathcal{N}=4)\oplus\bigoplus_{s\geq 1}N_sR^{(s)}$$

$$\mathbb{V}_2\equiv(\mathcal{N}=4)_{(c+\hat{c})}\oplus R^{(1)}\oplus\bigoplus_{s\geq 1}N_sR^{(s)}$$

$$K^a+\hat{K}^a,G^{\alpha\beta}+\hat{G}^{\alpha\beta},L+\hat{L}$$

$$(\hat{c}K^a-c\hat{K}^a),(\hat{c}G^{\alpha\beta}-c\hat{G}^{\alpha\beta}),(\hat{c}L-c\hat{L})$$

$$\mathbb{V}_N=(\mathcal{N}=4)\oplus\bigoplus_{s=\frac{3}{2}}^{\frac{N}{2}}R^{(s)}$$

$$\mathbb{V}_N^{(1)}=(\mathcal{N}=4)\oplus R^{\left(\frac{1}{2}\right)}\oplus\bigoplus_{s=\frac{3}{2}}^{\frac{N}{2}}R^{(s)}$$

$$K^3(z)K^3(w)\ \sim\frac{k}{2(z-w)^2},K^+(z)K^-(w)\ \sim\frac{k}{(z-w)^2}+\frac{2K^3(w)}{(z-w)}$$
$$K^3(z)K^{\pm}(w)\ \sim\pm\frac{K^{\pm}(w)}{(z-w)}$$

$$K^a(z)G^{\alpha\beta}(w)\sim\frac{D^{(1/2)}(t^a)^{\alpha}{}_{\gamma}G^{\gamma\beta}(w)}{(z-w)}$$

$$D^{(1/2)}(t^+)^{-}_{-}=D^{(1/2)}(t^-)^{+}_{-}=1,D^{(1/2)}(t^3)^{+}_{+}=\frac{1}{2}=-D^{(1/2)}(t^3)^{-}_{-}.$$

$$G^{\alpha\beta}(z)G^{\gamma\delta}(w) \sim -\frac{\frac{4c}{3}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}}{(z-w)^3} - \frac{8\epsilon^{\beta\delta}D^{\alpha\gamma}(t_a)K^a(w)}{(z-w)^2} - \frac{4\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}L(w) + 4\epsilon^{\beta\delta}D^{\alpha\gamma}(t_a)\partial K^a(w)}{(z-w)}$$

$$D^{+-}(t_3) = 1 = D^{-+}(t_3), D^{++}(t_+) = -1, D^{--}(t_-) = 1$$

$$\begin{array}{lll} W_m^{(s)} & m = -s, \dots, s & (h = s) \\ W_m^{(s)[\alpha]} & m = -\left(s - \frac{1}{2}\right), \dots, \left(s - \frac{1}{2}\right) & \left(h = s + \frac{1}{2}\right) \\ W_m^{(s)\uparrow} & m = -(s-1), \dots, (s-1) & (h = s+1) \end{array}$$

$$\begin{aligned} D^{(j)}(t^3)_{n,m} &= n\delta_{n,m} \\ D^{(j)}(t^+)_{n,m} &= \sqrt{j(j+1) - n(n+1)}\,\delta_{m,n+1} \\ D^{(j)}(t^-)_{n,m} &= \sqrt{j(j+1) - n(n-1)}\,\delta_{m,n-1} \end{aligned}$$

$$K^a(z)W_m^{(s)}(w) \sim \frac{D^{(s)}(t^a)_{mm'}W_{m'}^{(s)}(w)}{(z-w)}$$

$$\begin{aligned} L(z)W_m^{(s)}(w) &\sim \frac{sW_m^{(s)}(w)}{(z-w)^2} + \frac{\partial W_m^{(s)}(w)}{(z-w)} \\ K^a(z)W_r^{(s)[\alpha]}(w) &\sim \frac{D^{(s-1/2)}(t^a)_{rr'}W_{r'}^{(s)[\alpha]}(w)}{(z-w)} \\ L(z)W_r^{(s)[\alpha]}(w) &\sim \frac{\left(s+\frac{1}{2}\right)W_r^{(s)[\alpha]}(w)}{(z-w)^2} + \frac{\partial W_r^{(s)[\alpha]}(w)}{(z-w)} \\ K^a(z)W_n^{(s)\uparrow}(w) &\sim \frac{D^{(s-1)}(t^a)_{nn'}W_{n'}^{(s)\uparrow}(w)}{(z-w)} \\ L(z)W_n^{(s)\uparrow}(w) &\sim \frac{(s+1)W_n^{(s)\uparrow}(w)}{(z-w)^2} + \frac{\partial W_n^{(s)\uparrow}(w)}{(z-w)} \end{aligned}$$

$$\begin{aligned} L(z)W_m^{(s)}(w) &\sim \frac{sW_m^{(s)}(w)}{(z-w)^2} + \frac{\partial W_m^{(s)}(w)}{(z-w)} \\ K^a(z)W_r^{(s)[\alpha]}(w) &\sim \frac{D^{(s-1/2)}(t^a)_{rr'}W_{r'}^{(s)[\alpha]}(w)}{(z-w)} \\ L(z)W_r^{(s)[\alpha]}(w) &\sim \frac{(s+\frac{1}{2})W_r^{(s)[\alpha]}(w)}{(z-w)^2} + \frac{\partial W_r^{(s)[\alpha]}(w)}{(z-w)} \\ K^a(z)W_n^{(s)\uparrow}(w) &\sim \frac{D^{(s-1)}(t^a)_{nn'}W_{n'}^{(s)\uparrow}(w)}{(z-w)} \\ L(z)W_n^{(s)\uparrow}(w) &\sim \frac{(s+1)W_n^{(s)\uparrow}(w)}{(z-w)^2} + \frac{\partial W_n^{(s)\uparrow}(w)}{(z-w)}\ . \end{aligned}$$

$$G^{\alpha\beta}(z)\,W_m^{(s)}(w) \sim \frac{\rho_{\alpha,m}^{(s,-)} W_{m+\frac{\alpha}{2}}^{(s)[\beta]}(w)}{(z-w)}\,,$$

$$G^{\alpha\beta}(z)\,W_r^{(s)[\gamma]}(w) \sim \epsilon^{\beta\gamma}\Bigg[c_1(s)\frac{\rho_{\alpha,r}^{(s-1/2,+)} W_{r+\frac{\alpha}{2}}^{(s)}(w)}{(z-w)^2} + \frac{\rho_{\alpha,r}^{(s-1/2,-)} W_{r+\frac{\alpha}{2}}^{(s)\uparrow}(w) + c_3(s)\,\rho_{\alpha,r}^{(s-1/2,+)}\partial W_{r+\frac{\alpha}{2}}^{(s)}(w)}{(z-w)} + \frac{c_4(s)\,C^{(s-\frac{1}{2},-)}(a,m;\alpha,r)\,:K^a W_m^{(s)}:(w)}{(z-w)}\Bigg]\,,$$

$$G^{\alpha\beta}(z)\,W_n^{(s)\uparrow}(w) \sim c_5(s)\,\frac{\rho_{\alpha,n}^{(s-1,+)} W_{n+\frac{\alpha}{2}}^{(s)[\beta]}(w)}{(z-w)^2} + \frac{c_6(s)\,\big(C^{(s-1,-)}(a,r;\alpha,n) + A_s C^{(s-1,+)}(a,r;\alpha,n)\big) : K^a W_r^{(s)[\beta]} : (w)}{(z-w)} + \frac{c_7(s)\,\rho_{\alpha,n}^{(s-1,+)}\partial W_{n+\frac{\alpha}{2}}^{(s)[\beta](w)} + c_8(s) D^{(s)}(\gamma,m;\alpha,n) : G^{\gamma\beta} W_m^{(s)} :}{(z-w)}\,,$$

$$j \otimes \frac{1}{2} \to \left(j \pm \frac{1}{2}\right):\ \rho_{\alpha,m}^{(j,\pm)} = \left\langle j \pm \frac{1}{2}, m + \frac{\alpha}{2} \middle\| j, m; \frac{1}{2}, \frac{\alpha}{2}\right\rangle.$$

$$|j,r\rangle \otimes \left|\frac{1}{2},\frac{\alpha}{2}\right\rangle \overset{j\pm\frac{1}{2}}{\longleftrightarrow} \left|j+\frac{1}{2},m\right\rangle \otimes |1,a\rangle,$$

$$C^{(j,\pm)}(a,m;\alpha,r) = \delta\left(m + a - \left(r + \frac{\alpha}{2}\right)\right)$$
$$\times\, \epsilon_a \left\langle j \pm \frac{1}{2}, m + a \middle\| \left(j + \frac{1}{2}\right), m; 1, a\right\rangle \cdot \left\langle j \pm \frac{1}{2}, \frac{\alpha}{2} + r \middle\| j, r; \frac{1}{2}, \frac{\alpha}{2}\right\rangle$$

$$\epsilon_+ = -1, \epsilon_0 = \sqrt{2}, \epsilon_- = 1$$

$$|s-1,n\rangle \otimes \left|\frac{1}{2},\frac{\alpha}{2}\right\rangle \overset{s-\frac{1}{2}}{\longleftrightarrow} |s,m\rangle \otimes \left|\frac{1}{2},\frac{\gamma}{2}\right\rangle,$$

$$D^{(s)}(\gamma,m;\alpha,n) = \delta\left(m + \frac{\gamma}{2} - \left(\frac{\alpha}{2} + n\right)\right)$$
$$\times \left\langle s - \frac{1}{2}, n + \frac{\alpha}{2} \middle\| (s-1), n; \frac{1}{2}, \frac{\alpha}{2}\right\rangle \cdot \left\langle s - \frac{1}{2}, \frac{\gamma}{2} + m \middle\| s, m; \frac{1}{2}, \frac{\gamma}{2}\right\rangle$$

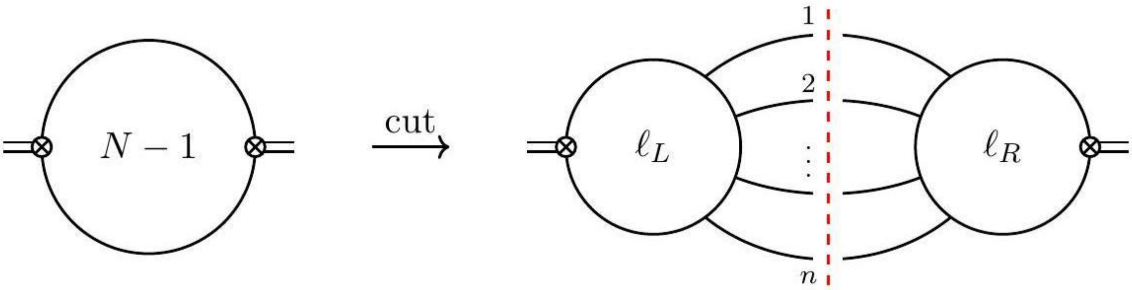
$$\langle J, m_1 + m_2 \| j_1, m_1; j_2, m_2\rangle =$$
$$\sqrt{\frac{(2J+1)(J+j_1-j_2)!\,(J+j_2-j_1)!\,(j_1+j_2-J)!}{(j_1+j_2+J+1)!}}$$
$$\times \sqrt{(J+M)!\,(J-M)!\,(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!}$$
$$\times \sum_{k\geq 0}\left[\frac{(-1)^k}{k!\,(j_1+j_2-J-k)!\,(j_1-m_1-k)!} \times \frac{1}{(j_2+m_2-k)!\,(J-j_2+m_1+k)!\,(J-j_1-m_2+k)!}\right]$$

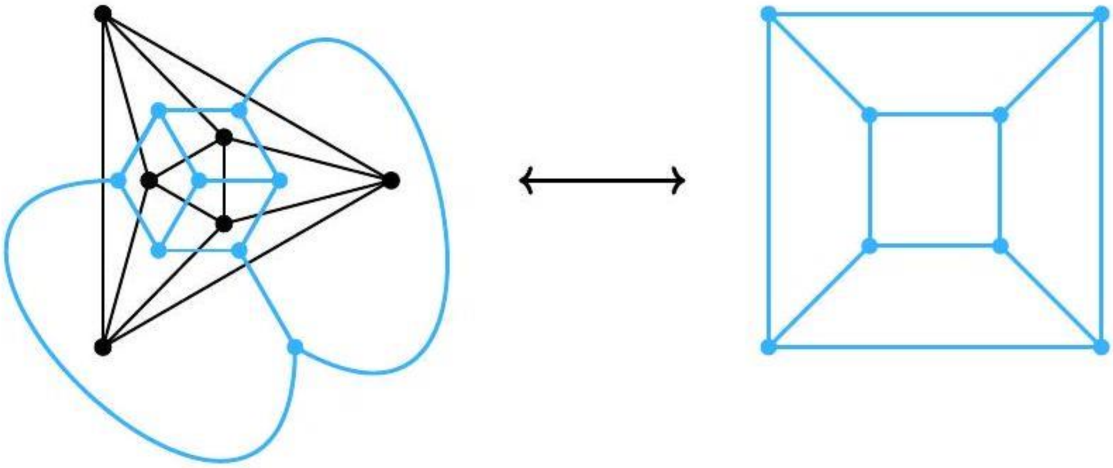
$$A_s = \frac{1}{\sqrt{s(2s+1)}}$$

$$c_1(s) = -4\sqrt{2s(2s+1)}, c_3(s) = -4\sqrt{\frac{2s+1}{2s}}, c_4(s) = -24\frac{2s+1}{c+12(s+1)},$$

$$c_5(s) = \frac{(2s+1)(c+12s+6)}{c+12(s+1)}\sqrt{\frac{8(2s-1)}{s}}, c_6(s) = \frac{24(2s+1)}{c+12(s+1)},$$

$$c_7(s) = \sqrt{\frac{8(2s-1)}{s}}, c_8(s) = -\frac{12(2s+1)}{s(c+12(s+1))}\sqrt{2s(2s-1)}.$$





$$(2\pi)^4\delta\left(q-\sum_{i=1}^{n} p_i\right)F_n(\mathcal{O}) \equiv \int d^4x e^{iq\cdot x}\langle\Phi(1)\cdots\Phi(n)|\mathcal{O}(x)|0\rangle$$

$$\mathbb{F}\overline{\mathbb{F}}_{n,k}^{(\ell)} = \sum_{\ell_L+\ell_R=\ell} \mathcal{C}_n \int \prod_{i=1}^{n} d^4\eta_i d^4\bar{\eta}_i e^{-\sum_i \bar{\eta}_i\eta_i} F_{n,k}^{(\ell_L)}(\eta_i,\lambda_i)\bar{F}_{n,k}^{(\ell_R)}(\bar{\eta}_i,\tilde{\lambda}_i)$$

$$\text{where } \mathcal{C}_n = \mathrm{tr}(T^{a_1}\cdots T^{a_n})\mathrm{tr}(T^{a_n}\cdots T^{a_1})$$

$$\mathbb{F}\overline{\mathbb{F}}_{n}^{(\ell)} = \sum_{k=0}^{n-2} \mathbb{F}\overline{\mathbb{F}}_{n,k}^{(\ell)}, \text{ where } \mathbb{F}\overline{\mathbb{F}}_{n,k}^{(\ell)} = \mathbb{F}\overline{\mathbb{F}}_{n,n-k-2}^{(\ell)}$$

$$\mathrm{F}_n^{(\ell)} = \left(\sum_{\text{planar cut}} \mathrm{Cut}_n\right)\mathcal{I}_{N=n+\ell}, \text{ where } \mathcal{I}_N = \sum_i c_N^{(i)}\mathfrak{M}_N^{(i)}$$

$$\lambda_i \to \sqrt{\omega_i}\lambda_P, \eta_i \to \sqrt{\omega_i}\eta_P, \sum_i \omega_i = 1, i = 1, \dots N$$

$$A_{N+m}(1,2\cdots,N+3)\rightarrow \mathrm{Split}_{1\rightarrow N}\times A_m(P,N+1,\cdots,N+m),$$

$$\lim_{1\|\cdots\|N}\frac{|A_{N+3}|^2}{|A_4|^2}=\lim_{1\|\cdots\|N}\frac{\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1}}{\mathrm{FF}^{(0)}_2}=|\ \mathrm{Split}\ _{1\rightarrow N}|^2$$

$$|i\rangle=\sqrt{\omega_i}(|1\rangle+z_i|N+1\rangle),|\,i]=\sqrt{\omega_i}(|\,1]+\bar{z}_i\mid N+1]),i\in[2,N].$$

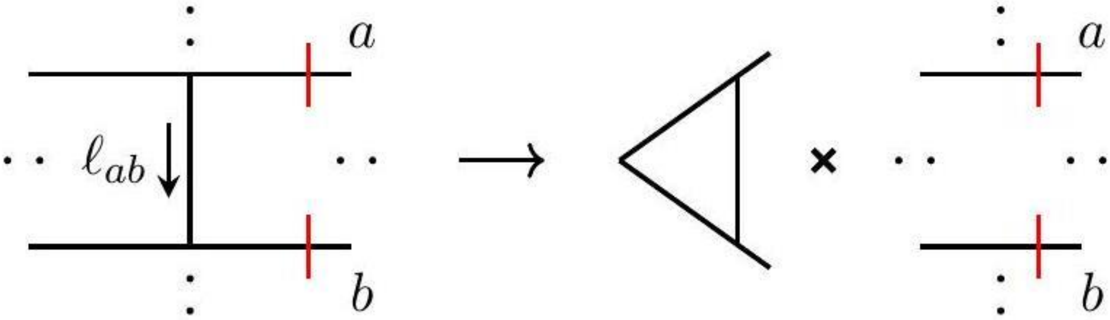
$$s_{i,N+1}/s_{1,N+1}=\omega_i,s_{1i}/s_{1,N+1}=\omega_i|z_i|^2,s_{ij}/s_{1,N+1}=\omega_i\omega_j|z_{ij}|^2,i,j\in[2,N].$$

$$\lim_{\epsilon\rightarrow 0}\frac{\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1}}{2\ \mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1,0}}=\mathcal{G}_N$$

$$\mathrm{F}_{\overline{\mathrm{F}}^{(0)}_{N+1,k}}\rightarrow \mathcal{S}^{(0)}_{N+1}\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N,k}+\mathcal{S}^{(0)}_{N+1}\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N,k-1},\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1,0}\rightarrow \mathcal{S}^{(0)}_{N+1}\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N,0}$$

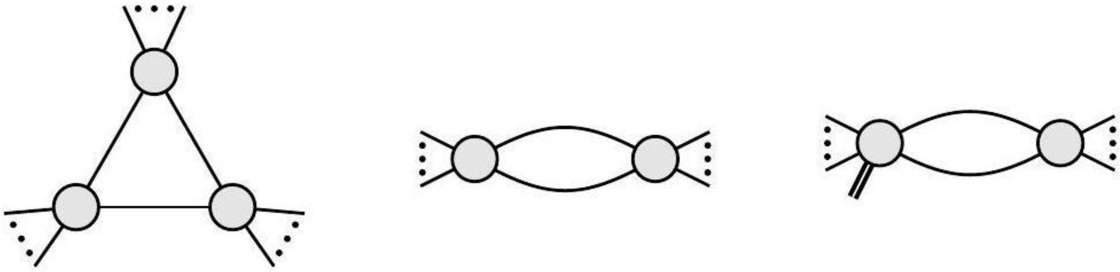
$$\mathcal{S}^{(0)}_{N+1}=\frac{s_{N,1}}{s_{N,N+1}s_{N+1,1}}$$

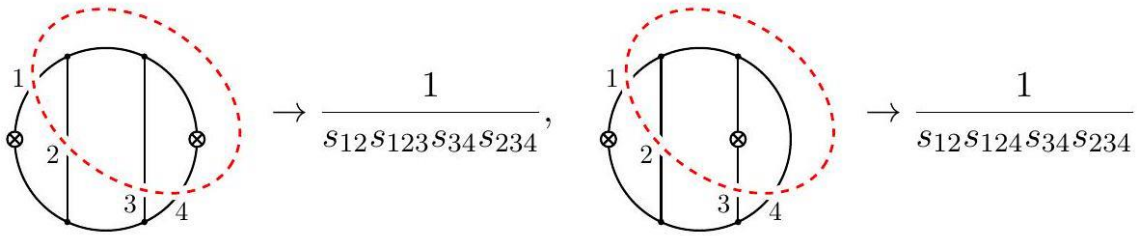
$$\lim_{p_{N+1}\rightarrow 0}\frac{\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1}}{2\ \mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1,0}}=2\times\frac{\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N}}{2\ \mathrm{F}\overline{\mathrm{F}}^{(0)}_{N,0}}.$$

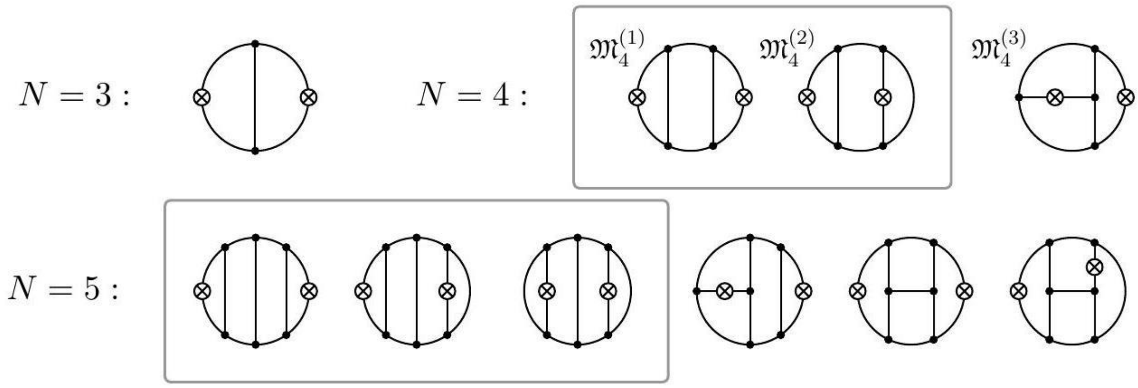


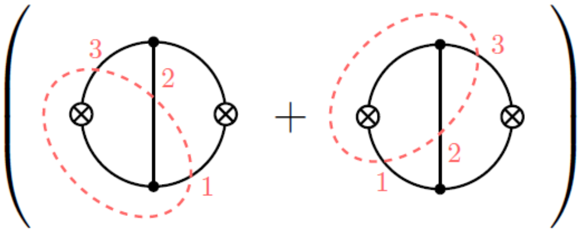
$$\sum_{l\in\mathcal{L}}\lim_{l\rightarrow 0}\mathrm{F}^{(\ell)}_{\mathrm{F}}=\left[\sum_{i=1}^{n}V_{i,i+1}(l_{i,i+1})\right]\times \mathrm{FF}^{(\ell-1)}_n$$

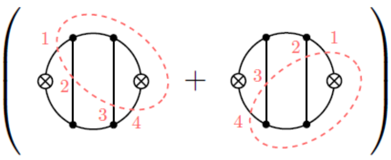
$$V_{a,b}(l):=\frac{p_a\cdot p_b}{(p_a\cdot l)l^2(p_b\cdot l)}.$$

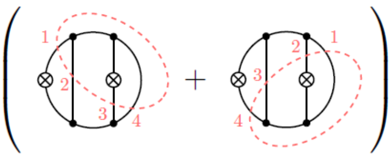


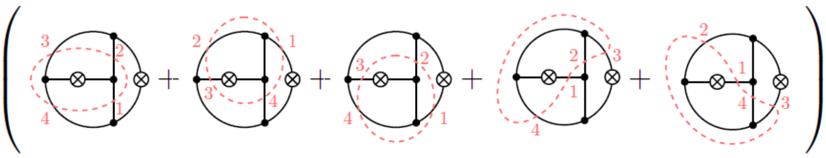




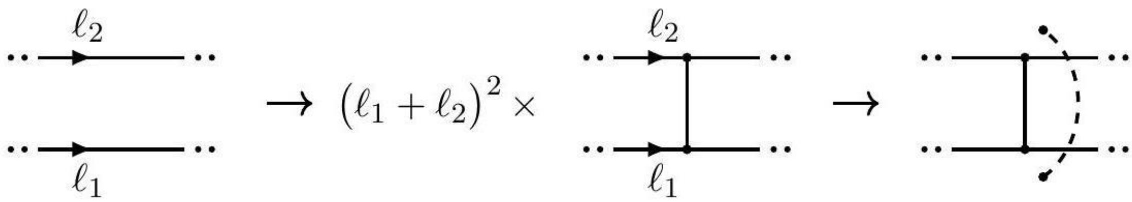
$$\frac{1}{4}\times\sum_{\text{cyc}\{1,2,3\}}$$

$$= \frac{q^2}{s_{12}s_{23}s_{31}},$$

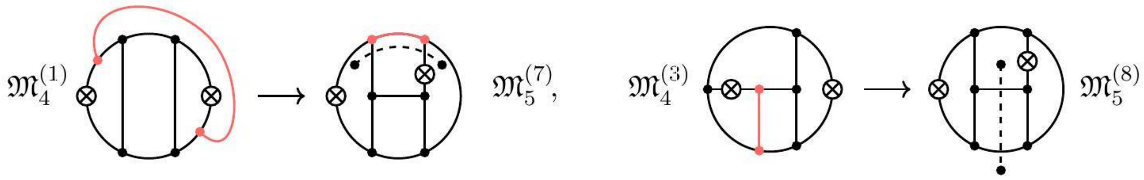
$$\mathfrak{M}_4^{(1)} \to \frac{1}{4}\times\sum_{\text{cyc}\{1,2,3,4\}}$$

$$= \frac{1}{s_{12}s_{34}s_{123}s_{341}} + \text{cyc.}$$

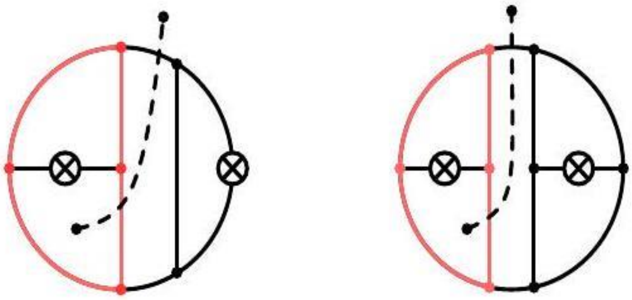
$$\mathfrak{M}_4^{(2)} \to \frac{1}{4}\times\sum_{\text{cyc}\{1,2,3,4\}}$$

$$= \frac{1}{s_{12}s_{34}s_{412}s_{341}} + \text{cyc.}$$

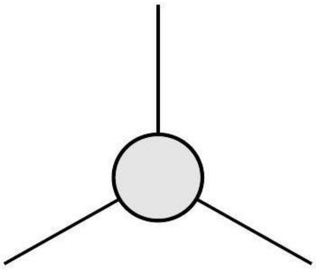
$$\mathfrak{M}_4^{(3)} \to \frac{1}{8}\times\sum_{\text{cyc}\{1,2,3,4\}}$$

$$= \frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{12}s_{23}s_{412}s_{234}} + \text{cyc.}$$

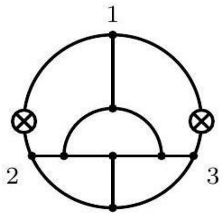
$$\mathrm{F}\overline{\mathrm{F}}_4^{(0)} = c_4^{(1)}\left(\frac{1}{s_{12}s_{34}s_{123}s_{341}} + \text{ cyc.}\right) + c_4^{(2)}\left(\frac{1}{s_{12}s_{34}s_{412}s_{341}} + \text{ cyc.}\right)$$
$$+c_4^{(3)}\left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{12}s_{23}s_{412}s_{234}} + \text{ cyc.}\right)$$

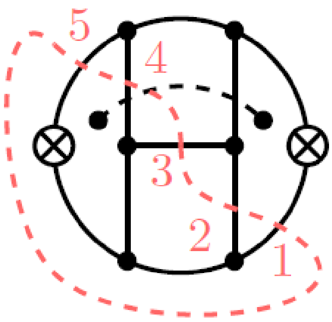




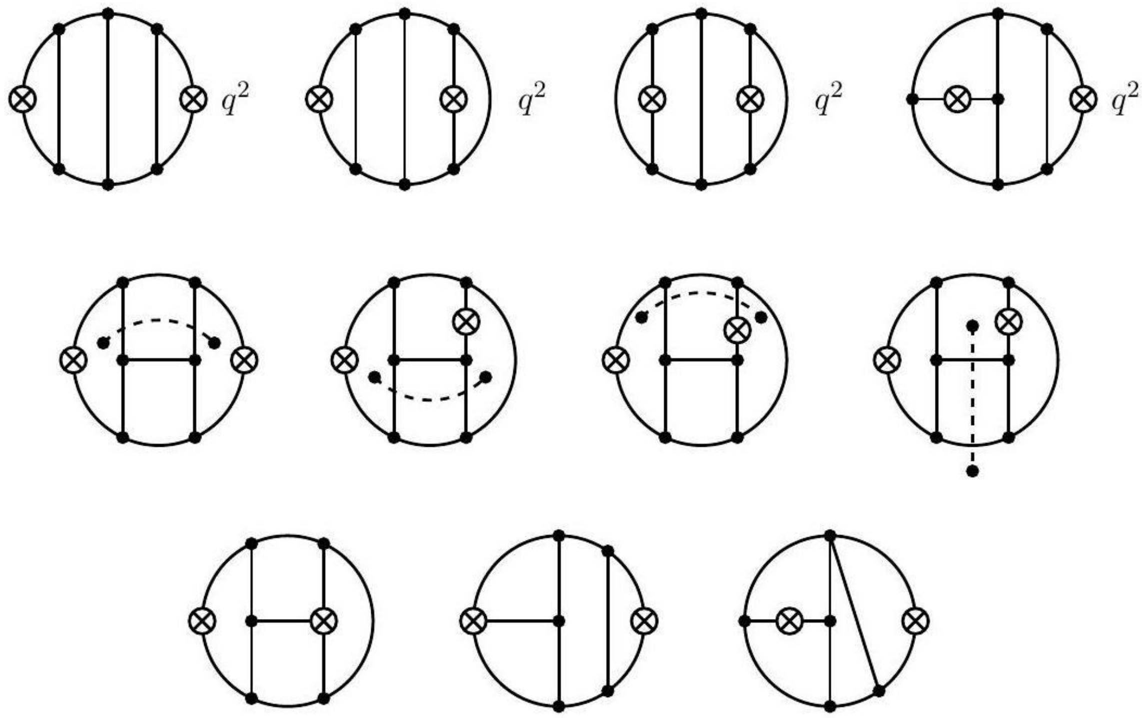


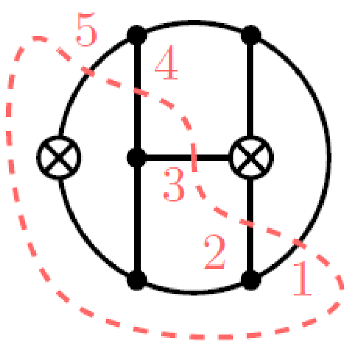






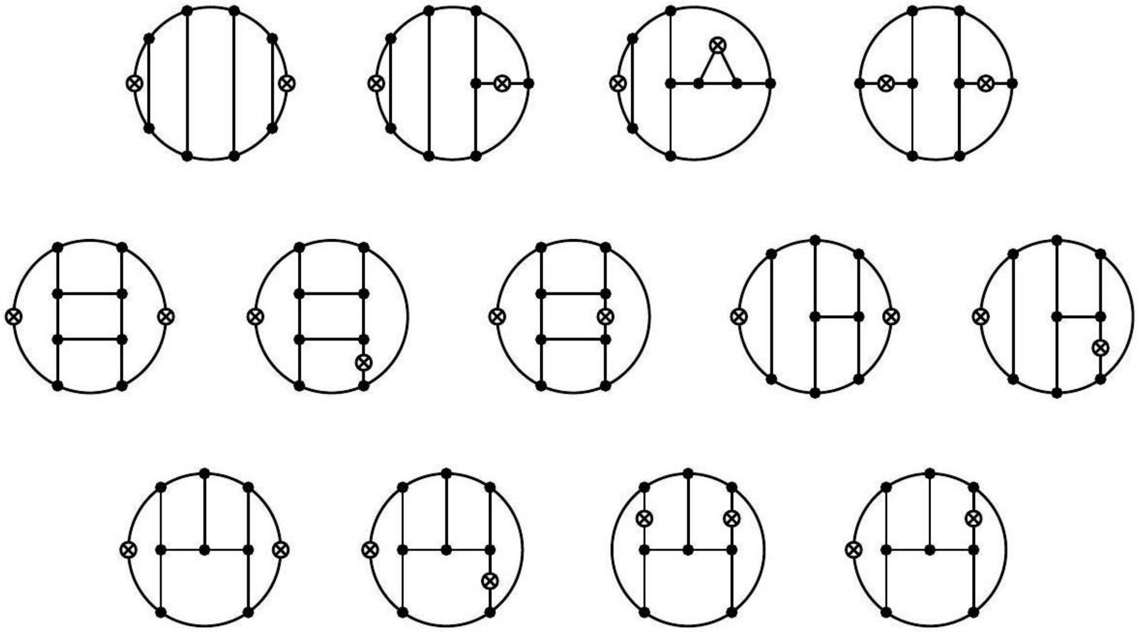
$$= \frac{s_{234}}{s_{12}s_{34}s_{1234}s_{23}s_{45}s_{2345}}.$$

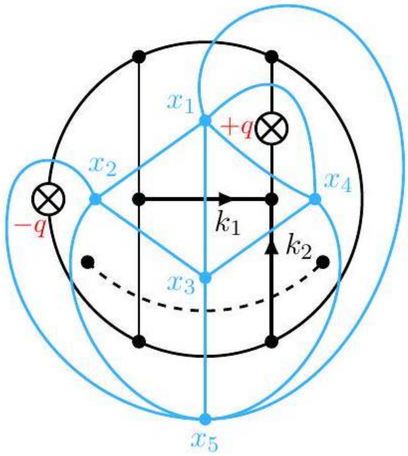


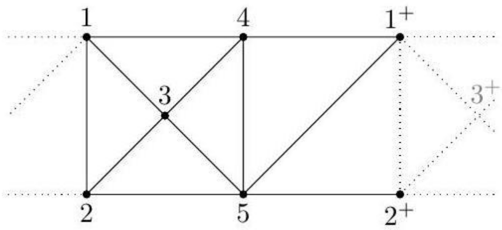


$$= \frac{1}{s_{12}s_{34}s_{1234}s_{45}s_{451}}.$$

$$\left\{c_5^{(1)}, \dots, c_5^{(11)}\right\} = \{2,2,2,2,2,2,0,2,-2,0,-2\}$$

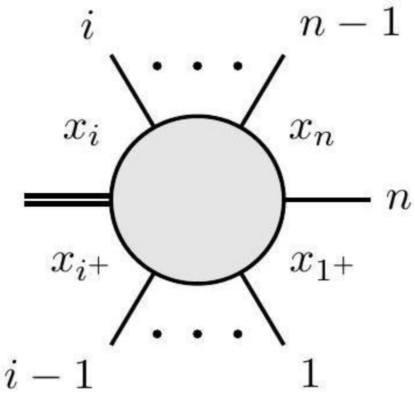


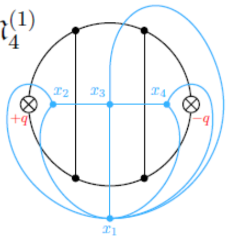
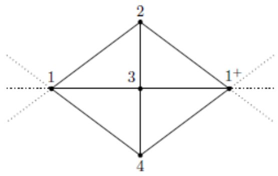
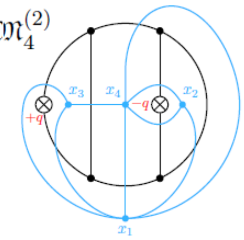
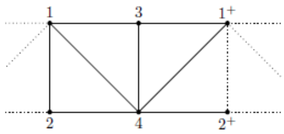
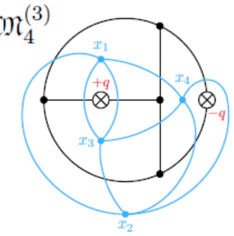
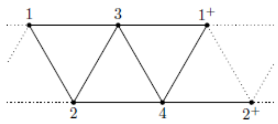




$$\mathfrak{M}_5^{(6)} = \frac{1}{2}\left(\frac{x_{24}^2}{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{35}^2 x_{45}^2 x_{41^+}^2 x_{51^+}^2 x_{52^+}^2} + \text{ all perms}\right).$$

$$\mathrm{F}\bar{\mathrm{F}}_n^{(\ell=N-n)} = \lim_{\substack{x_{i,i+1}^2 \to 0,\\ 1\le i\le n}} \left[\left(\prod_{1\le i\le n} x_{i,i+1}\right) \mathcal{I}_N\right],$$



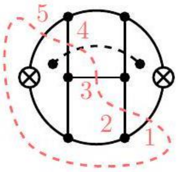
master diagram	periodic dual graph	integrand perm seed
$\mathfrak{M}_4^{(1)}$ 		$\frac{1}{4}\frac{1}{x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{21^+}^2x_{34}^2x_{31^+}^2x_{41^+}^2}$
$\mathfrak{M}_4^{(2)}$ 		$\frac{1}{4}\frac{1}{x_{12}^2x_{13}^2x_{14}^2x_{24}^2x_{34}^2x_{31^+}^2x_{41^+}^2x_{42^+}^2}$
$\mathfrak{M}_4^{(3)}$ 		$\frac{1}{8}\frac{1}{x_{12}^2x_{13}^2x_{23}^2x_{24}^2x_{34}^2x_{31^+}^2x_{41^+}^2x_{42^+}^2}$

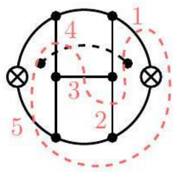
$$\mathcal{I}_3=\frac{2}{x_{12}^2x_{13}^2x_{23}^2x_{21^+}^2x_{31^+}^2x_{11^+}^2}+\frac{2}{x_{23}^2x_{21^+}^2x_{31^+}^2x_{32^+}^2x_{12}^2x_{22^+}^2}+\frac{2}{x_{31^+}^2x_{32^+}^2x_{12}^2x_{13}^2x_{23}^2x_{33^+}^2}.$$

$$\mathrm{F}\bar{\mathrm{F}}_3^{(0)}=\lim_{\substack{x_{i,i+1}^2\to 0,\\ 1\le i\le 3}}\left[\left(x_{12}^2x_{23}^2x_{31^+}^2\right)\mathcal{I}_3\right]=\frac{2(x_{13}^2+x_{21^+}^2+x_{32^+}^2)}{q^2x_{13}^2x_{21^+}^2x_{32^+}^2}=\frac{2}{x_{13}^2x_{21^+}^2x_{32^+}^2},$$

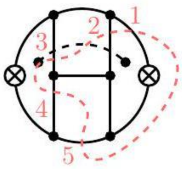
$$\begin{aligned}\mathrm{F}\overline{\mathrm{F}}_4^{(0)}&=2\left(\frac{1}{s_{12}s_{23}s_{34}s_{41}}+\frac{1}{s_{12}s_{23}s_{412}s_{234}}+\frac{1}{s_{12}s_{34}s_{123}s_{341}}+\frac{1}{s_{12}s_{34}s_{412}s_{341}}+\text{cyc.}\right)\\&=\frac{2}{s_{12}s_{23}s_{34}s_{41}}\left(1+\frac{s_{23}s_{41}}{s_{123}s_{341}}+\frac{s_{23}s_{41}}{s_{412}s_{341}}+\frac{s_{34}s_{41}}{s_{412}s_{234}}+\text{cyc.}\right)\end{aligned}$$

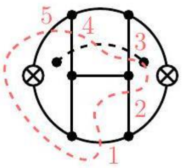
$$\mathrm{FF}_4^{(0)} = \lim_{x_{i,i+1}^2\to 0, 1\le i\le 4}\left[\left(x_{12}^2x_{23}^2x_{34}^2x_{41^+}^2\right)\mathcal{I}_4\right]$$
$$= 2\left(\frac{1}{x_{13}^2x_{24}^2x_{31^+}^2x_{42^+}^2}+\frac{1}{x_{13}^2x_{24}^2x_{43^+}^2x_{21^+}^2}+\frac{1}{x_{13}^2x_{31^+}^2x_{14}^2x_{32^+}^2}+\frac{1}{x_{13}^2x_{31^+}^2x_{43^+}^2x_{32^+}^2}+\widetilde{\text{cyc.}}\right)$$
$$=\frac{2}{x_{13}^2x_{24}^2x_{31^+}^2x_{42^+}^2}\left(1+\frac{x_{31^+}^2x_{42^+}^2}{x_{43^+}^2x_{21^+}^2}+\frac{x_{24}^2x_{42^+}^2}{x_{14}^2x_{32^+}^2}+\frac{x_{24}^2x_{42^+}^2}{x_{43^+}^2x_{32^+}^2}+\widetilde{\text{cyc.}}\right),$$



$$= \frac{s_{234}}{s_{12}s_{34}s_{1234}s_{23}s_{45}s_{2345}},$$


$$= \frac{s_{234}}{s_{34}s_{345}s_{2345}s_{23}s_{123}s_{1234}},$$



$$= \frac{s_{23}}{s_{34}s_{234}s_{2345}s_{12}s_{123}s_{45}},$$


$$= \frac{s_{34}}{s_{23}s_{234}s_{1234}s_{45}s_{345}s_{12}},$$

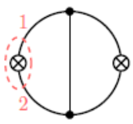
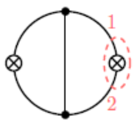
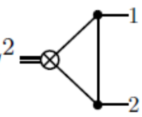
$$\mathrm{F}_n^{(\ell)} = \left(\sum_{\text{planar cut}} \mathrm{Cut}_n\right)\mathcal{I}_{N=n+\ell}, \text{ where } \mathcal{I}_N = \sum_i c_N^{(i)}\mathfrak{M}_N^{(i)}$$

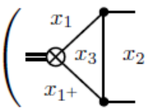
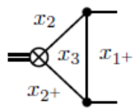
$$\mathrm{FF}_2^{(\ell)} = \mathrm{FF}_{2,0}^{(\ell)} = \sum_{\ell_L+\ell_R=\ell} F_{2,0}^{(\ell_L)}\bar{F}_{2,0}^{(\ell_R)} = \mathrm{Cut}_2(\mathcal{I}_{N=2+\ell}),$$

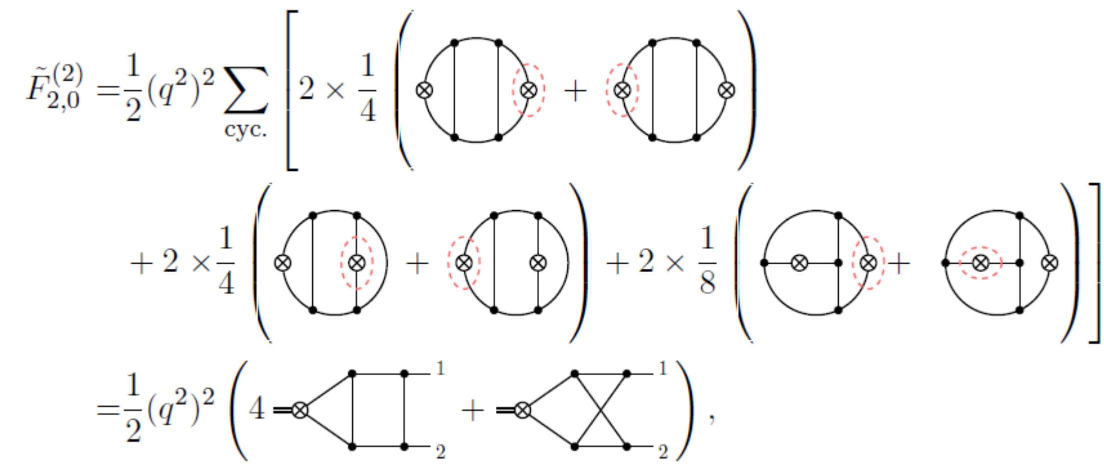
$$\mathrm{Cut}_2(\mathcal{I}_{N=2+\ell})|_{\text{tree}\times\ell\text{-loop}} = F_{2,0}^{(\ell)}\bar{F}_{2,0}^{(0)} + F_{2,0}^{(0)}\bar{F}_{2,0}^{(\ell)} = 2\mathrm{FF}_2^{(0)}\left(\frac{F_{2,0}^{(\ell)}}{F_{2,0}^{(0)}}\right) = \frac{2}{(q^2)^2}\tilde{F}_{2,0}^{(\ell)}$$

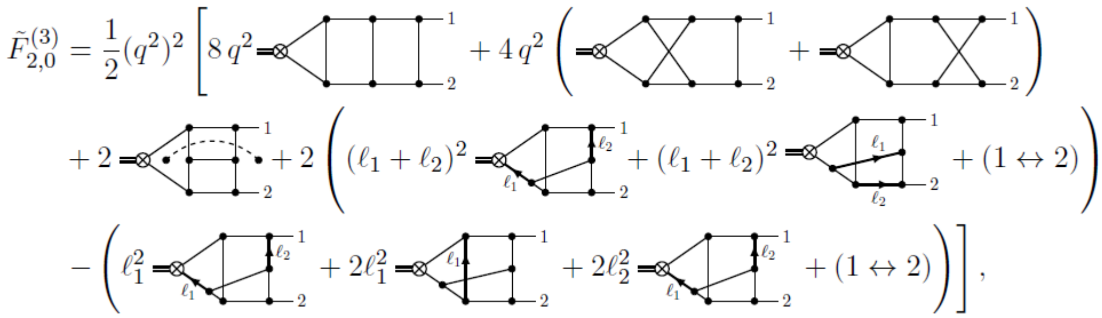
$$F_{n,k}^{(\ell)} = F_{n,0}^{(0)}\tilde{F}_{n,k}^{(\ell)}$$

$$\tilde{F}_{2,0}^{(\ell)} = \frac{1}{2}(q^2)^2[\mathrm{Cut}_2(\mathcal{I}_{N=2+\ell})]\Big|_{\text{tree}\times\ell\text{-loop}}$$
$$=\frac{1}{2}(q^2)^2\lim_{x_{12}^2,x_{21^+}^2\to 0}[(x_{12}^2x_{21^+}^2)\mathcal{I}_{N=2+\ell}]\Big|_{\text{tree}\times\ell\text{-loop}}$$

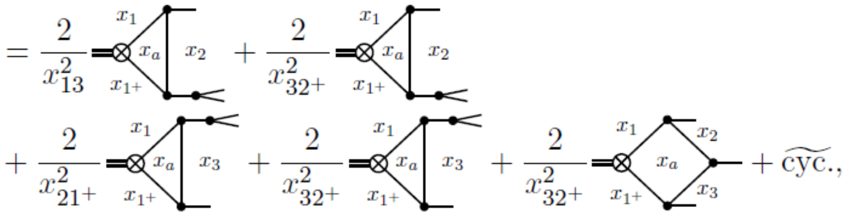
$$\tilde{F}_{2,0}^{(1)} = \frac{1}{2}(q^2)^2\times 2\times\frac{1}{4}\times\sum_{\text{cyc}(1,2)}\frac{1}{q^2}\Bigg($$  $$+$$  $$\Bigg) = q^2$$ 

$$=\frac{1}{2}q^2\left(\frac{1}{x_{13}^2x_{23}^2x_{1^+3}^2}+\frac{1}{x_{23}^2x_{1^+3}^2x_{2^+3}}\right)=\frac{1}{2}q^2\Bigg($$  $$+$$  $$\Bigg),$$





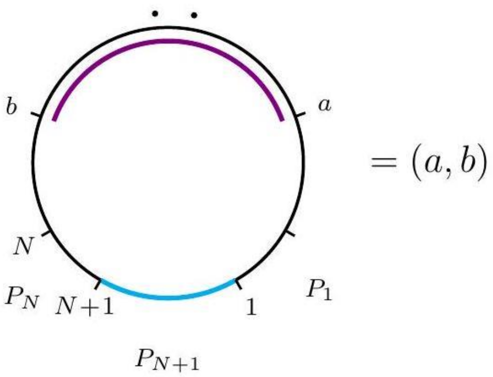
$$\mathrm{F}\bar{\mathrm{F}}_3^{(1)} = \frac{2}{x_{13}^2 x_{21+}^2 x_{1a}^2 x_{2a}^2 x_{1+a}^2} + \frac{2}{x_{21+}^2 x_{32+}^2 x_{1a}^2 x_{2a}^2 x_{1+a}^2}$$
$$+ \frac{2}{x_{13}^2 x_{21+}^2 x_{1a}^2 x_{3a}^2 x_{1+a}^2} + \frac{2}{x_{13}^2 x_{32+}^2 x_{1a}^2 x_{3a}^2 x_{1+a}^2} + \frac{2}{x_{32+}^2 x_{1a}^2 x_{2a}^2 x_{3a}^2 x_{1+a}^2} + \widetilde{\text{cyc.}}$$

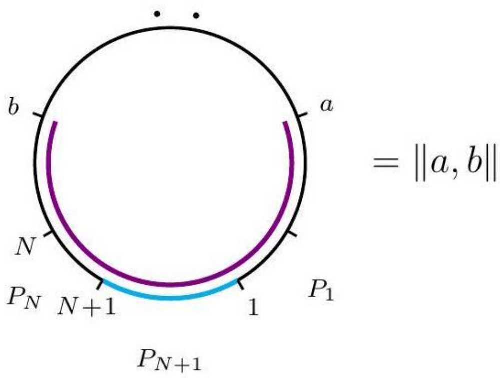


$$\mathrm{E^N C}(z_1 \ldots z_N) = \frac{\langle \mathcal{O}^\dagger \mathcal{E}(z_1) \cdots \mathcal{E}(z_N) \mathcal{O} \rangle}{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}, \mathcal{E}(z) = \int_{-\infty}^{\infty} dv T_{vv}(0, v, \vec{z})$$

$$\mathcal{E}(z)|p_i\rangle = p_i^0 \delta^2(z - \hat{p}_i)|p_i\rangle$$

$$\mathrm{E^N C}(z_1 \ldots z_N) = \sum_{n>N} \sum_{\sigma \in S_n} \int \mathrm{dPS}_n \prod_{i=1}^{N} [p_i^0 \delta^2(z_i - \hat{p}_i)] \times |F_{\mathcal{O}}(\sigma(1), \ldots, \sigma(n))|^2$$





$$p_i = \frac{x_i}{1+|z_i|^2}\begin{pmatrix}1 & z_i \\ \bar{z}_i & |z_i|^2\end{pmatrix}, q = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, i = 1, \cdots, N$$

$$(a,b) := p^2_{a\cdots(b-1)}, \|a,b\| := (b,\bar{a}) = (\underline{b}, a), N+1 \geq b \geq a+1 \geq 2,$$

$$\bar{a} := a+N+1, \underline{a} := a-N-1, \bar{a} - a = a - \underline{a} := q.$$

$$(a,b) = \frac{1}{2}\sum_{i,j\in[a,b-1]} x_i x_j \gamma_{ij}, \|a,b\| = 1 - x_{a\cdots(b-1)} - (a,b).$$

$$\gamma_{ij} := \frac{|z_{ij}|^2}{(1+|z_i|^2)\left(1+|z_j|^2\right)}$$

$$\mathrm{dPS}_{N+1} = \prod_i \frac{d^2 z_i}{(1+|z_i|^2)^2}[x_i dx_i]\delta(\|1,N+1\|)$$

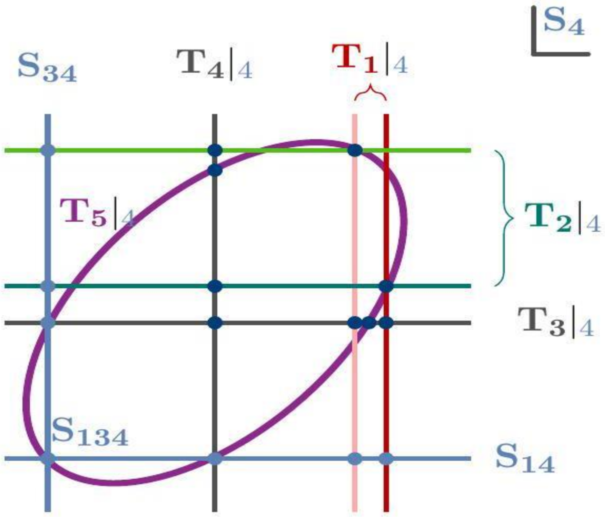
$$\mathrm{E^N C}\big|_{\mathrm{LO}} = \frac{[(1+|z_1|^2)(1+|z_N|^2)]^{-1}}{|z_{12}|^2 \ldots |z_{N-1,N}|^2}\int_0^1 \prod_i [dx_i] I_{\mathrm{E^N C}}(x_i, z_i) + \mathrm{perm}(z_1 \ldots z_N)$$

$$I_{\mathrm{E^N C}}(x_i, z_i) = \frac{x_1 x_N \delta(\|1,N+1\|)}{\|1,N\|\|2,N+1\|}\left[\frac{\mathrm{F}\overline{\mathrm{F}}^{(0)}_{N+1}}{\mathrm{FF}^{(0)}_{N+1,0}}\right]$$

$$\frac{\|1,N\|}{x_N} = 1 - \sum_{i=1}^{N-1} x_i \gamma_{iN}, \frac{\|2,N+1\|}{x_1} = 1 - \sum_{i=2}^{N} x_i \gamma_{1i}$$

$$\begin{aligned} I_{\mathrm{E^3 C}} = \delta(S_4)\Bigg[&\frac{1}{T_1T_2} + \frac{S_1S_3}{T_3T_4} + \frac{S_1S_2^2S_3\gamma_{12}\gamma_{23}}{T_1T_2T_3T_4} \\ &+ \frac{S_1S_2S_3\gamma_{12}}{T_1T_3T_4} + \frac{S_1S_2S_3\gamma_{12}}{T_1T_3T_6} + \frac{S_1S_2S_3\gamma_{12}}{T_1T_3T_5} + \frac{S_1S_2S_3\gamma_{23}}{T_2T_3T_4} + \frac{S_1S_2S_3\gamma_{23}}{T_2T_4T_6} + \frac{S_1S_2S_3\gamma_{23}}{T_2T_4T_5} \\ &+ \frac{S_1S_2S_3\gamma_{12}}{T_1T_5T_6} + \frac{S_1S_2S_3\gamma_{23}}{T_2T_5T_6} + \frac{S_2S_3^2\gamma_{23}}{T_1T_5T_6} + \frac{S_1^2S_2\gamma_{12}}{T_2T_5T_6}\Bigg] \end{aligned}$$

$$\begin{aligned} T_1 = 1 - x_2\gamma_{12} - x_3\gamma_{13}, T_2 = 1 - x_1\gamma_{13} - x_2\gamma_{23} \\ T_3 = 1 - x_1, T_4 = 1 - x_3, T_5 = 1 - x_1 - x_2 - x_3, T_6 = 1 - x_2 \\ S_{1,2,3} = x_{1,2,3}, S_4 = 1 - x_1 - x_2 - x_3 + x_1x_2\gamma_{12} + x_1x_3\gamma_{13} + x_2x_3\gamma_{23} \end{aligned}$$



$$\{(T_1,T_2),(T_1,T_3),(T_3,T_4),(T_1,T_5),(T_3,T_5)\}$$

$$\varphi_1=\delta_4[\operatorname{dlog} T_1|_4\wedge \operatorname{dlog} T_2|_4],\varphi_2=\delta_4[\operatorname{dlog} T_1|_4\wedge \operatorname{dlog} T_3|_4]$$
$$\varphi_3=\delta_4[\operatorname{dlog} T_3|_4\wedge \operatorname{dlog} T_4|_4],\varphi_4=\delta_4[\operatorname{dlog} T_1|_4\wedge \operatorname{dlog} T_5|_4]$$
$$\varphi_5=\delta_4[\operatorname{dlog} T_3|_4\wedge \operatorname{dlog} T_5|_4],\varphi_6=\delta_4\left[\frac{\gamma_{13}\Delta_1\,\mathrm{d}^2x}{T_1T_2}\right]$$
$$\varphi_7=\delta_4\left[\frac{\Delta_1\,\mathrm{d}^2x}{T_1T_3}\right],\varphi_8=\delta_4\left[\frac{\gamma_{23}\,\mathrm{d}^2x}{T_3}\right],\varphi_9=\delta_4\left[\frac{\Delta_1\,\mathrm{d}^2x}{T_1T_5}\right]$$
$$\varphi_{10}=\delta_4\left[\frac{(\gamma_{12}-\gamma_{13})\mathrm{d}^2x}{T_3T_5}\right],\varphi_{11}=\delta_4\left[\frac{\Delta_2\,\mathrm{d}^2x}{T_5}\right]$$

$$\delta_i[\cdot]:=\operatorname{Res}_{S_i=0}[\operatorname{dlog} S_i\wedge\cdot],\delta_{i|j}:=\delta_j\delta_i$$

$$\Delta_1=\sqrt{4\gamma_{12}\gamma_{13}\gamma_{23}+\Delta_2^2},\Delta_2=\sqrt{\gamma_{12}^2+\gamma_{13}^2+\gamma_{23}^2-2\gamma_{12}\gamma_{13}-2\gamma_{12}\gamma_{23}-2\gamma_{13}\gamma_{23}}$$

$$\varphi_{12}=\delta_{4|1}[\operatorname{dlog} T_1|_{41}],\varphi_{13}=\delta_{4|1}[\operatorname{dlog} T_4|_{41}],\varphi_{14}=\delta_{4|1}\left[\frac{\Delta_1\,\mathrm{d}x}{T_1}\right]$$

$$\varphi_{15}=\delta_{4|1|3}$$

$$T_1=\frac{\|1,N\|}{x_1},T_2=\frac{\|2,N+1\|}{x_N},$$

$$T_{N+1}=(1,N+1)=1-x_{1\cdots N},T_A=\|a,a+1\|=1-x_a,a\in[1,N]$$
$$T_{ab}=(a,b),3\le b-a\le N-1,a,b\in[1,N+1]$$
$$T_{AB}=\|a,b\|=1-x_{a\cdots(b-1)}+(a,b),2\le b-a\le N-2,a,b\in[1,N+1]$$
$$S_{1,2,\cdots,N}=x_{1,2,\cdots,N},S_{N+1}=\|1,N+1\|=1-x_{1\cdots N}+(1,N+1)$$

$$\frac{1}{\sqrt{P_6(x_2,\ldots,x_b,\ldots,x_{a-1},\ldots x_{N-1})}}$$

$$\frac{1}{\sqrt{Q_6(x_2,\cdots,x_{N-1})}},\ \text{for } a\le N,\frac{1}{\sqrt{Q_4(x_2,\cdots,x_{N-1})}},\ \text{for } a=N+1$$

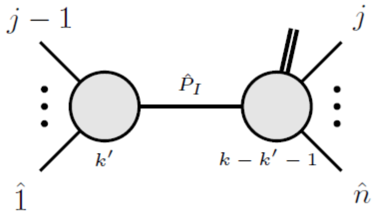
$$\frac{1}{\sqrt{R_6(x_0,x_2\cdots,x_{N-2})}},\ \text{for } b\le a-1,\frac{1}{\sqrt{R_4(x_0,x_2\cdots,x_{N-2})}},\ \text{for } b=a$$

$$b_{i-1}<b_i<a_{i-1},b_{i+1}<a_i<a_{i+1},a_{i-1}\le b_{i+1},b_1=1,a_M=N+1$$

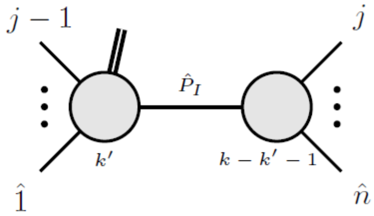
$$\{\|1,3\|,(2,5),(3,6)\}=\{s_{34\cdots N},s_{234},s_{345}\}$$

$$| \hat{n}] = | n] + z \mid 1], |\hat{1}\rangle = |1\rangle - z|n\rangle, \hat{\eta}_n = \eta_n + z\eta_1$$

$$F_{n,k} = \mathrm{Res}_{z=0} \frac{\hat{F}_{n,k}(z)}{z} = - \sum_{z_I \neq 0} \mathrm{Res}_{z=z_I} \frac{\hat{F}_{n,k}(z)}{z},$$

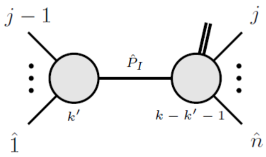
$$\int d^4\eta_I \, \hat{A}_{j,k'} \, \frac{1}{P_I} \, \hat{F}_{n-j+2,k-k'-1} =$$


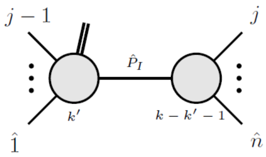
,

$$\int d^4\eta_I \, \hat{F}_{j,k'} \, \frac{1}{P_I} \, \hat{A}_{n-j+2,k-k'-1} =$$


,

$$F_{n,k} = \left(\sum_{I,k'} \int d^4\eta_I \, A_{j+1,k'} \, \frac{1}{P_I} \, F_{n-j+1,k-k'-1} \right) + \left(\sum_{I,k'} \int d^4\eta_I \, F_{j+1,k'} \, \frac{1}{P_I} \, A_{n-j+1,k-k'-1} \right)$$

$= \sum_{I,k'}$


$+ \sum_{I,k'}$


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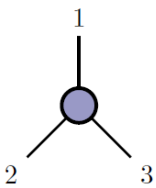
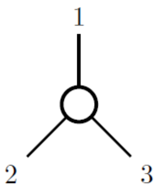
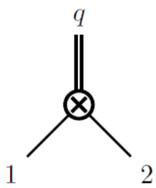


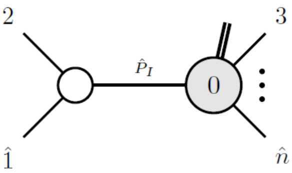
$$\Leftrightarrow \quad A_{3,1}(1,2,3) = \frac{\delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$



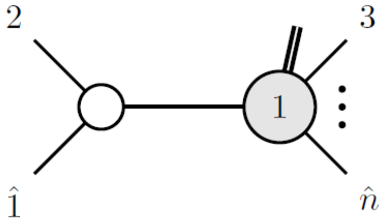
$$\Leftrightarrow \quad A_{3,-1}(1,2,3) = \frac{\delta^{(4)}([12]\eta_3 + [23]\eta_1 + [31]\eta_2)}{[12][23][31]},$$

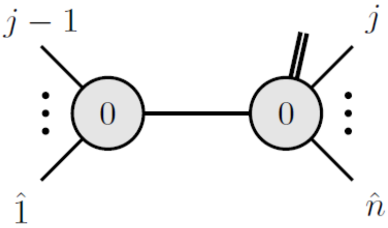


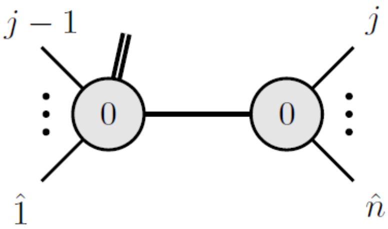
$$\Leftrightarrow \quad F_{2,0}(q,\gamma;1,2) = \frac{\delta^{(8)}(\gamma + \sum_i \lambda_i \eta_i)}{\langle 12\rangle\langle 21\rangle}.$$

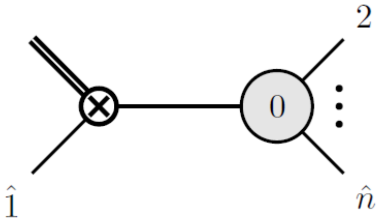
$F_{n,0} =$ 

$$= \int d^4\eta_I \, A_{3,-1}(\hat{1}, 2, \hat{P}_I) \, \frac{1}{P_I^2} \, F_{n-1,0}(-\hat{P}_I, 3, \ldots, \hat{n}) = \frac{\delta^{(8)}(\gamma + \sum_i \lambda_i \eta_i)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}.$$

$F_{n,1} =$ 

$+ \sum_{j=4}^{n}$ 

$+ \sum_{j=3}^{n-1}$ 

$+$ 

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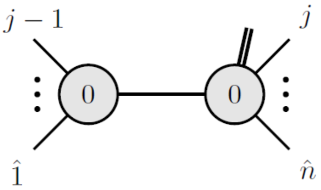
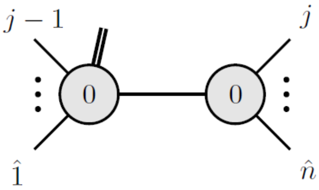
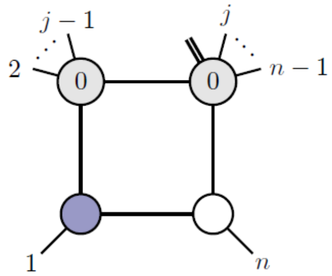
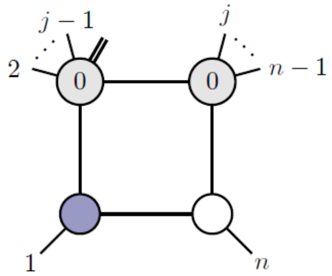


$= F_{n,0} \, R''_{n2j}$, 

$= F_{n,0} \, R'_{n2j}$,

$R''_{n2j} =$ 

, $R'_{n2j} =$ 

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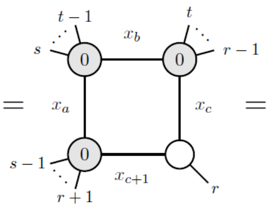
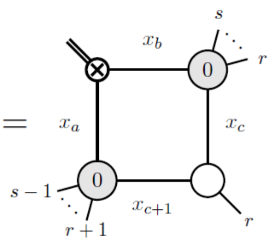
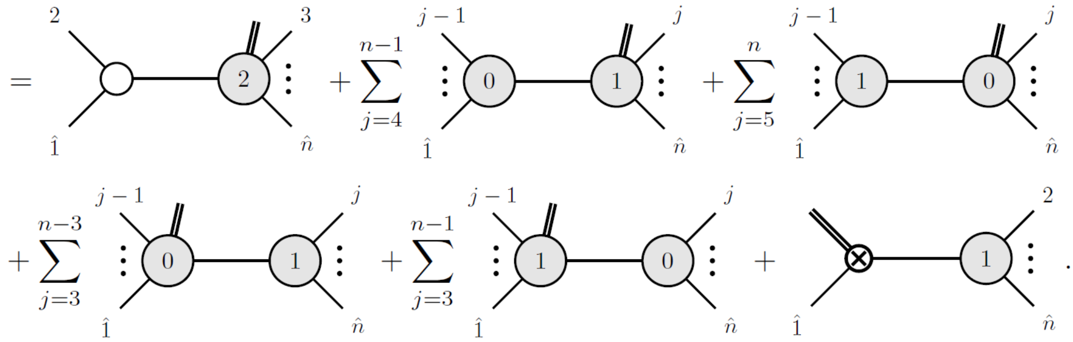
$R^{\square}_{rst} =$ 

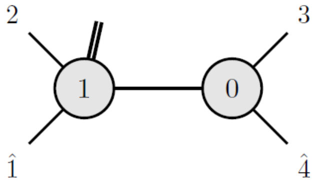
$$= \frac{\langle s-1\, s \rangle \langle t-1\, t \rangle \, \delta^{(4)}\left(\langle r | x_{ca} x_{ab} | \theta_{bc} \rangle + \langle r | x_{cb} x_{ba} | \theta_{ac} \rangle \right)}{x_{ab}^2 \langle r | x_{cb} x_{ba} | s-1 \rangle \langle r | x_{cb} x_{ba} | s \rangle \langle r | x_{ca} x_{ab} | t-1 \rangle \langle r | x_{ca} x_{ab} | t \rangle},$$

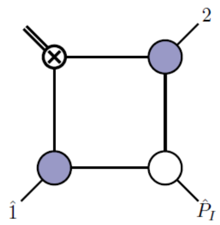
$R'_{rss} =$ 

$$= -\frac{\langle s-1\, s\rangle\, \delta^{(4)}\left(\langle r|x_{ca}x_{ab}|\theta_{bc}\rangle + \langle r|x_{cb}x_{ba}|\theta_{ac}\rangle\right)}{x_{ab}^4 \langle r|x_{cb}x_{ba}|s-1\rangle \langle r|x_{ca}x_{ab}|s\rangle \langle r|x_{ca}x_{bc}|r\rangle}.$$

$$F_{n,1} = F_{n,0}\left(\sum_{j=2}^{n-2}\sum_{t=j+2}^{n} R''_{njt} + \sum_{j=2}^{n-1}\sum_{t=j}^{n-1} R'_{njt}\right).$$

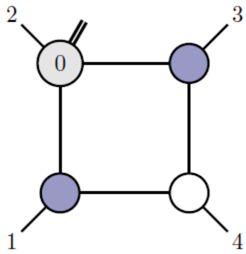
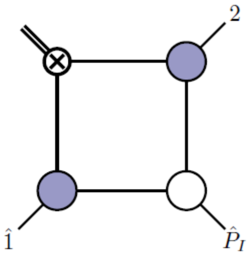
$F_{n,2} =$ 

$F_{4,2} =$  $= \int d^4\eta_I\, F_{3,1}(\hat{1}, 2, \hat{P}_I)\, \frac{1}{P^2}\, A_{3,0}(-\hat{P}_I, 3, \hat{4})$

$$= \underbrace{\int d^4\eta_I\, F_{3,0}(\hat{1}, 2, \hat{P}_I)\, \frac{1}{P^2}\, A_{3,0}(-\hat{P}_I, 3, \hat{4})}_{F_{4,0} \times R'_{423}} \times$$  .

$$0 = \hat{P}_I^2 = (p_3 + p_{\hat{4}})^2 = 2p_3 \cdot p_{\hat{4}} = \langle 4|3\mid \hat{4}] = \langle 4|3\mid 4] + z_I\langle 4|3\mid 1] \Rightarrow z_I = -\frac{\langle 4|3\mid 4]}{\langle 4|3\mid 1]},$$

$$|\hat{1}\rangle = |1\rangle - z_I|4\rangle = \frac{|1\rangle[1|3|4\rangle + |4\rangle[4|3|4\rangle}{[1|3|4\rangle} = \frac{(1+4)3|4\rangle}{[1|3|4\rangle}$$

$F_{4,2} = F_{4,0} \times$  $\times$  $\Big|_{\substack{|\hat{1}\rangle \to (1+4)3|4\rangle \\ |\hat{P}_I\rangle \to (3+4)1|4\rangle}}$,

$$\frac{\mathrm{F}\,\overline{\mathrm{F}}_n}{\mathrm{F}\,\overline{\mathrm{F}}_{n,0}} = \left(\sum_{k=0}^{n-2} M_{n,k} M_{n,n-2-k}\right) / M_{n,n-2}$$

$$u = -\langle \mathrm{Tr}[\phi^2] \rangle.$$

$$\mathcal{J}_{\mathrm{D}}^{(0)} = (-a\tau_3, 0,0,0,0).$$

$$F_{\mathrm{AdS}}(a) = -\log Z_{\mathrm{AdS}}$$

$$\mathcal{R} \to \mathcal{R} + \delta q$$

$$\mathrm{Re}[F_{\mathrm{AdS}}] = -\log |Z_{\mathrm{AdS}}|$$

$$\partial_\delta \mathrm{Re}[F_{\mathrm{AdS}}](m - i\delta^*) = 0$$
$$\partial_\delta^2 \mathrm{Re}[F_{\mathrm{AdS}}](m - i\delta^*) \leqslant 0$$

$$\partial_\mu J_\pm^\mu = \pm \mathcal{O}_3$$

$$Z_{\mathrm{AdS}}(a, \Lambda) = (\Lambda L)^{4a^2} Z_{1-\text{ loop}}(a) Z_{\text{Nekrasov}}(a, \Lambda)$$

$$F_{\mathrm{AdS}} = -4\pi i \mathcal{F}_{\mathrm{AdS}}$$

$$u = \pi i \left(\mathcal{F}(a) - \frac{1}{2} a \partial_a \mathcal{F}(a)\right)$$

$$\tilde{M} = 0, \mathcal{A}_\mu = 0, \mathcal{V}_\mu = 0, T_{\mu\nu} = 0, \bar{T}_{\mu\nu} = 0, g_{\mu\nu} = g_{\mu\nu}^{\mathrm{AdS}_4} \text{ with } \mathcal{R} = -12,$$

$$\mathcal{Q} = \epsilon^A Q_A + \bar{\epsilon}_A \bar{Q}^A,$$

$$D_\mu \epsilon_{A\alpha} = -\frac{i}{2} \tau_{3,A}^B (\sigma_\mu \bar{\epsilon}_B)_\alpha, D_\mu \bar{\epsilon}_A^{\dot{\alpha}} = \frac{i}{2} \tau_{3,A}^B (\bar{\sigma}_\mu \epsilon_B)^{\dot{\alpha}}.$$

$$\begin{aligned} S =& \frac{\mathrm{Im}(\tau)}{4\pi} \int_{\mathrm{AdS}_4} \mathcal{L}_{\mathrm{YM}} + i \frac{\mathrm{Re}(\tau)}{8\pi} \int_{\mathrm{AdS}_4} \mathcal{L}_\theta \\ \mathcal{L}_{\mathrm{YM}} =& \mathrm{Tr}\Big[\frac{1}{2} F^{\mu\nu} F_{\mu\nu} - 4 D_\mu \bar{\phi} D^\mu \phi + 8 \bar{\phi} \phi + i \bar{\lambda}^A \bar{\phi} \lambda_A - i \lambda^A \not{D} \bar{\lambda}_A \\ & - \frac{1}{2} D^{AB} D_{AB} + 4[\phi, \bar{\phi}]^2 - 2\lambda^A [\bar{\phi}, \lambda_A] + 2\bar{\lambda}^A [\phi, \bar{\lambda}_A]\Big] \\ \mathcal{L}_\theta =& \mathrm{Tr}\left[\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right] \end{aligned}$$

$$ds^2 = d\eta^2 + \sinh(\eta)^2 d\Omega_{S^3}^2$$

$$\epsilon_{A\alpha} = \sinh\left(\frac{\eta}{2}\right) \begin{pmatrix} \zeta_\alpha \\ \tilde{\zeta}_\alpha \end{pmatrix}_A$$
$$\bar{\epsilon}_{A\dot{\alpha}} = -i\cosh\left(\frac{\eta}{2}\right) \tau_{3,A}^B \begin{pmatrix} (\zeta \sigma_\perp)_{\dot{\alpha}} \\ (\tilde{\zeta} \sigma_\perp)_{\dot{\alpha}} \end{pmatrix}_B$$

$$S_{\partial} = \frac{\mathrm{Im}(\tau)}{4\pi}\sinh^3(\eta_0)\int_{S^3}\mathcal{L}^{\partial}_{\mathrm{YM}} + i\frac{\mathrm{Re}(\tau)}{8\pi}\sinh^3(\eta_0)\int_{S^3}\mathcal{L}^{\partial}_{\theta}$$

$$\begin{aligned}
\mathcal{L}^{\partial}_{\mathrm{YM}} = \frac{2}{\sinh(\eta_0)}\Big[&\coth(\eta_0)\left(\frac{3+\cosh(2\eta_0)}{\sinh(\eta_0)}(\phi_1^2-\phi_2^2) - 8i\coth(\eta_0)\phi_1\phi_2\right) \\
&-(\cosh(\eta_0)\phi_2 + i\phi_1)\left(4i\frac{D_\perp\phi_1}{\sinh(\eta_0)} + 4\coth(\eta_0)D_\perp\phi_2 + 2iD_{12}\right) \\
&+\cosh^2\left(\frac{\eta_0}{2}\right)\lambda_1\lambda_2 + \sinh^2\left(\frac{\eta_0}{2}\right)\bar{\lambda}_1\bar{\lambda}_2\Big]
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{\partial}_{\theta} = \frac{2}{\sinh(\eta_0)}\Big[&2\coth(\eta_0)\left(2\coth(\eta_0)(\phi_1^2-\phi_2^2) - \frac{3+\cosh(2\eta_0)}{\sinh(\eta_0)}i\phi_1\phi_2\right) \\
&-(\phi_2 + i\cosh(\eta_0)\phi_1)\left(4i\frac{D_\perp\phi_1}{\sinh(\eta_0)} + 4\coth(\eta_0)D_\perp\phi_2 + 2iD_{12}\right) \\
&+\cosh^2\left(\frac{\eta_0}{2}\right)\lambda_1\lambda_2 - \sinh^2\left(\frac{\eta_0}{2}\right)\bar{\lambda}_1\bar{\lambda}_2 - \frac{i}{2}\sinh(\eta_0)(\lambda_1\sigma_\perp\bar{\lambda}_2 - \lambda_2\sigma_\perp\bar{\lambda}_1)\Big]
\end{aligned}$$

$$\phi_1 = \frac{\phi+\bar{\phi}}{2i}, \phi_2 = \frac{\phi-\bar{\phi}}{2}$$

$$\begin{aligned}
\phi_{1,2}(\eta_0) &\underset{\eta_0\to\infty}{\sim} e^{-\eta_0}\left(\phi_{1,2}^{(0)} + \ldots\right) + e^{-2\eta_0}\left(\tilde{\phi}_{1,2}^{(0)} + \ldots\right), \\
\lambda(\eta_0) &\underset{\eta_0\to\infty}{\sim} e^{-\frac{3}{2}\eta_0}\left(\lambda_+^{(0)} + \ldots\right) + e^{-\frac{3}{2}\eta_0}\left(\lambda_-^{(0)} + \ldots\right), \\
\bar{\lambda}(\eta_0) &\underset{\eta_0\to\infty}{\sim} e^{-\frac{3}{2}\eta_0}\left(\bar{\lambda}_+^{(0)} + \ldots\right) + e^{-\frac{3}{2}\eta_0}\left(\bar{\lambda}_-^{(0)} + \ldots\right), \\
A_i(\eta_0) &\underset{\eta_0\to\infty}{\sim} \left(A_i^{(0)} + \ldots\right) + e^{-\eta_0}\left(\tilde{A}_i^{(0)} + \ldots\right),
\end{aligned}$$

$$\mathcal{J}_{\mathrm{D}}(\eta): \begin{cases}
J = 2\phi_1, \\
j = -\frac{i}{\sqrt{2}}\left(\sinh\left(\frac{\eta}{2}\right)\lambda_2 + i\cosh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_2\right), \\
\tilde{j} = \frac{i}{\sqrt{2}}\left(\sinh\left(\frac{\eta}{2}\right)\lambda_1 - i\cosh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_1\right), \\
j_k = -\frac{i}{\sinh^2(\eta)}\left(\cosh(\eta)\frac{1}{2}\varepsilon^{ij}{}_k F_{ij} + \sinh(\eta)F_{k\perp}\right), \\
K = -2\sinh(\eta)D_\perp\phi_2 - 2\cosh(\eta)\left(\phi_2 + \frac{i}{2}D_{12}\right) + 2i\phi_1.
\end{cases}$$

$$\mathcal{J}_{\mathrm{N}}(\eta): \begin{cases}
J = 2\phi_2 \\
j = \frac{1}{\sqrt{2}}\left(\sinh\left(\frac{\eta}{2}\right)\lambda_2 - i\cosh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_2\right), \\
\tilde{j} = -\frac{1}{\sqrt{2}}\left(\sinh\left(\frac{\eta}{2}\right)\lambda_1 + i\cosh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_1\right), \\
j_k = -\frac{1}{\sinh^2(\eta)}\left(\frac{1}{2}\varepsilon^{ij}{}_k F_{ij} + \cosh(\eta)\sinh(\eta)F_{k\perp}\right), \\
K = 2\sinh(\eta)D_\perp\phi_1 + 2\cosh(\eta)\phi_1 - D_{12} + 2i\phi_2.
\end{cases}$$

$$\mathcal{V}(\eta): \begin{cases}
\sigma = 2\cosh(\eta)\phi_1 - 2i\phi_2 \\
\lambda = -\sinh(\eta)\left(\cosh\left(\frac{\eta}{2}\right)\lambda_1 - i\sinh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_1\right) \\
\tilde{\lambda} = -\sinh(\eta)\left(\cosh\left(\frac{\eta}{2}\right)\lambda_2 + i\sinh\left(\frac{\eta}{2}\right)\sigma_\perp\bar{\lambda}_2\right) \\
A_i = A_i \\
D = -2\cosh(\eta)\sinh(\eta)D_\perp\phi_2 - 2i\sinh(\eta)D_\perp\phi_1 - i\sinh^2(\eta)D_{12} \\
+2\sinh^2(\eta)\phi_2 - i\cosh(\eta)\phi_1 - \phi_2
\end{cases}$$

$$\lim_{\eta_0\to\infty} e^{\eta_0}\mathcal{J}_{\mathrm{D}}(\eta_0)=0$$
$$\lim_{\eta_0\to\infty} e^{\eta_0}\mathcal{J}_{\mathrm{N}}(\eta_0)=0$$

$$\mathcal{J}_{\mathrm{D}}^{(0)}=\left(2\phi_1^{(0)},-\sqrt{2}i\lambda_{2+}^{(0)},\sqrt{2}i\lambda_{1-}^{(0)},-i\varepsilon_i^{jk}\partial_j A_k^{(0)},2\left(\tilde{\phi}_2^{(0)}+i\phi_1^{(0)}\right)\right)$$
$$\mathcal{J}_{\mathrm{N}}^{(0)}=\left(2\phi_2^{(0)},\sqrt{2}\lambda_{2-}^{(0)},-\sqrt{2}\lambda_{1+}^{(0)},-\tilde{A}_i^{(0)},-2\left(\tilde{\phi}_1^{(0)}-i\phi_2^{(0)}\right)\right)$$

$$\mathcal{J}_{\mathrm{D}}^{(0)}=(-a,0,0,0,0)$$

$$\mathcal{R}=\mathcal{R}_0+\sum_a \delta^a q_a,$$

$$F_{\mathrm{AdS}}=-\log Z_{\mathrm{AdS}}$$

$$e^{\pi i\tau}=(\Lambda L)^{\frac{b_1}{2}}$$

$$H(x)=G(1+x)G(1-x)$$

$$Z_{\text{pure SYM}}^{\mathrm{D}}(a,\Lambda)=\Lambda^{\frac{b_1}{2}\mathrm{Tr}(a^2)}\prod_{\alpha\in\Delta^+} H(i\alpha\cdot a)\frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)}Z_{\text{Nekrasov}}(a,\Lambda)$$

$$Z_{\text{pureSYM}}^{\mathrm{N}}(\Lambda)=\int da\Lambda^{\frac{b_1}{2}\mathrm{Tr}(a^2)}\prod_{\alpha\in\Delta^+}(\alpha\cdot a)\sinh(\pi\alpha\cdot a)H(i\alpha\cdot a)\cdot Z_{\text{Nekrasov}}(a,\Lambda)$$

$$\epsilon_{A\alpha}=\frac{1}{\sqrt{2}}\sinh\left(\frac{\eta}{2}\right)\delta_{A\alpha},\bar{\epsilon}_A^{\dot{\alpha}}=\frac{i}{\sqrt{2}}\cosh\left(\frac{\eta}{2}\right)\tau_{3,A}^{\alpha}$$

$$V=\mathrm{Tr}[\lambda^A(\mathcal{Q}_{\mathrm{loc}}\lambda^A)^*]$$

$$(\phi)^*=-\bar{\phi},(A_\mu)^*=A_\mu,(D_{AB})^*=-D^{AB}$$

$$\begin{aligned}\mathcal{Q}_{\mathrm{loc}}V|_{\text{bosonic}}=\mathrm{Tr}[&\epsilon^A\epsilon_A\left(F_{\mu\nu}^-+\frac{2}{\epsilon^A\epsilon_A}v_{[\mu}D_{\nu]}-\phi_2+i\frac{\epsilon^A\sigma_{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2\\&+\bar{\epsilon}_A\bar{\epsilon}^A\left(F_{\mu\nu}^+-\frac{2}{\bar{\epsilon}_A\bar{\epsilon}^A}v_{[\mu}D_{\nu]}+\phi_2-i\frac{\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\epsilon}^B\tau_{3,AB}}{\bar{\epsilon}_C\bar{\epsilon}^C}\phi_2\right)^2\\&+\frac{4}{\epsilon^A\epsilon_A+\bar{\epsilon}_A\bar{\epsilon}^A}\left(D_\mu[(\epsilon^A\epsilon_A+\bar{\epsilon}_A\bar{\epsilon}^A)\phi_1]\right)^2+\left(\frac{1}{\epsilon^A\epsilon_a}+\frac{1}{\bar{\epsilon}_A\bar{\epsilon}^A}\right)(v\cdot D\phi_2)^2\\&+\frac{1}{2(\epsilon^A\epsilon_A+\bar{\epsilon}_A\bar{\epsilon}^A)}\left|2\phi_1(\epsilon^C\epsilon_C-\bar{\epsilon}_C\bar{\epsilon}^C)\tau_{3,AB}+(\epsilon^C\epsilon_C+\bar{\epsilon}_C\bar{\epsilon}^C)D_{AB}\right|^2\\&-16(\epsilon^A\epsilon_A+\bar{\epsilon}_A\bar{\epsilon}^A)[\phi_1,\phi_2]^2]\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{\mathrm{loc}}V|_{\text{even}}=\mathrm{Tr}[&\frac{4}{\cosh(\eta)}\left(D_\mu[\cosh(\eta)\phi_1]\right)^2+\frac{1}{2\cosh(\eta)}\left|-2\phi_1\tau_{3,AB}+\cosh(\eta)D_{AB}\right|^2\\&+4\phi_2(-D^2-2)\phi_2-16\cosh(\eta)[\phi_1,\phi_2]^2\\&+\cosh^2\left(\frac{\eta}{2}\right)\left(F_{\mu\nu}^-+i\frac{\epsilon^A\sigma_{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2\\&+\sinh^2\left(\frac{\eta}{2}\right)\left(F_{\mu\nu}^+-i\frac{\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\epsilon}^B\tau_{3,AB}}{\bar{\epsilon}_C\bar{\epsilon}^C}\phi_2\right)^2\\&+D_\nu\left(D^\nu(2\cosh(\eta)\phi_2^2)-4\tilde{F}^{\mu\nu}v_\mu\phi_2\right)],\end{aligned}$$

$$\phi_1=-\frac{a}{2\cosh(\eta)},D_{AB}=-\frac{1}{\cosh^2(\eta)}\tau_{3,AB}a$$

$$\phi_2=\frac{c_1}{\sinh(\eta)^2}+\frac{c_2\cosh(\eta)}{\sinh(\eta)^2},[c_1,a]=[c_2,a]=0$$

$$c_1=c_2=0$$

$$\mathcal{J}_{\mathrm{D}}^{(0)} = \lim_{\eta_0\to\infty} e^{\eta_0}\mathcal{J}_{\mathrm{D}}(\eta_0) = (-a,0,0,0,0)$$
$$\mathcal{J}_{\mathrm{N}}^{(0)} = \lim_{\eta_0\to\infty} e^{\eta_0}\mathcal{J}_{\mathrm{N}}(\eta_0) = 0$$

$$e^{-S}\big|_{\mathrm{loc}} = e^{\pi i\tau \mathrm{Tr}[a^2]}$$

$$\mathrm{Re}[F_{\mathrm{AdS}}(-i\delta,\Lambda)] = -\log | Z_{SU(2)}^{\mathrm{D}}(a,\Lambda)|\big|_{a=-i\delta}$$

$$Z_{SU(2)}^{\mathrm{D}}(a,\Lambda) = \Lambda^{4a^2}\frac{2a}{\sinh(2\pi a)}G(1+2ia)G(1-2ia)Z_{\mathrm{Nekrasov}}(a,\Lambda)$$

$$\phi_1 = -\frac{a\tau_3}{2\cosh(\eta)}$$

$$Z_{\mathrm{Nekrasov}} = 1+\sum_{k=1}^{\infty} I_k(a)\Lambda^{4k}$$

$$I_1(a) = \frac{1}{2(a^2+1)}$$
$$I_2(a) = \frac{8a^2+33}{4(a^2+1)(4a^2+9)^2}$$
$$I_3(a) = \frac{8a^4+99a^2+366}{24(a^2+1)(4a^2+9)^2(a^2+4)^2}$$
$$I_4(a) = \frac{256a^8+7616a^6+91276a^4+521211a^2+1109820}{384(a^2+1)(4a^2+9)^2(a^2+4)^3(4a^2+25)^2}$$

$$Z_{1\text{-loop}} = \frac{2a}{\sinh(2\pi a)}G(1+2ia)G(1-2ia)\Big|_{a=-i\delta}$$

$$F_{\mathrm{AdS}} \simeq -4\pi i L^2\mathcal{F}_{\mathrm{curvature}}(a,\Lambda)+O(\log(L))$$

$$\mathcal{F}_{\mathrm{supercurvature}}(a) = -\frac{i}{2\pi}a^2\left(\log\left(\frac{\Lambda^2}{4a^2}\right)+3+\sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}\right)$$

$$\frac{\partial^2\mathrm{Re}[F_{\mathrm{AdS}}]}{\partial\delta^2}\bigg|_{\delta=\delta^*} \leqslant 0$$

$$\mathrm{Im}[\tau(a)]|_{\delta=\delta^*} \geqslant 0$$

$$A_k = \mathrm{Tr}[\phi^k], \bar{A}_k = \mathrm{Tr}[\bar{\phi}^k].$$

$$u = -\langle A_2\rangle = \pi i\left(\mathcal{F}_{\mathrm{curve}}(a) - \frac{1}{2}a\partial_a\mathcal{F}_{\mathrm{supercurvature}}(a)\right)$$

$$(2-a\partial_a)\mathcal{F}_{\mathrm{curvature}} = \Lambda\partial_\Lambda\mathcal{F}_{\mathrm{supercurvature}}$$

$$u = \partial_{\tau_{\mathrm{UV}}}\mathcal{F}_{\mathrm{supercurvature}}$$

$$S_{\mathcal{N}=2} \to S_{\mathcal{N}=2} + \int_{\mathrm{AdS}_4}\sqrt{g}(\lambda_k\mathcal{C}_k+\bar{\lambda}_k\bar{\mathcal{C}}_k) + \int_{\partial\mathrm{AdS}_4}\sqrt{h}(\lambda_k\mathcal{C}_{\partial,k}+\bar{\lambda}_k\bar{\mathcal{C}}_{\partial,k})$$

$$\mathcal{C} = C - i(w-2)(\tau_3)^{AB}B_{AB} + 2(w-2)(w-3)A$$
$$\bar{\mathcal{C}} = \bar{C} + i(w-2)(\tau_3)^{AB}\bar{B}_{AB} + 2(w-2)(w-3)\bar{A}$$

$$\delta\mathcal{C} = -D_\mu\left(2i\bar{\epsilon}^A\sigma^\mu\Lambda_A + 2s_2(w-2)(\tau_3)^{AB}\bar{\epsilon}_{(A}\bar{\sigma}^\mu\Psi_{B)}\right)$$
$$\delta\bar{\mathcal{C}} = -D_\mu\left(2i\epsilon^A\sigma^\mu\bar{\Lambda}_A - 2s_1(w-2)(\tau_3)^{AB}\epsilon_{(A}\sigma^\mu\bar{\Psi}_{B)}\right)$$

$$\mathcal{C}_\partial = \coth\left(\frac{\eta_0}{2}\right)\left(-i(\tau_3)^{AB}B_{AB} + 2D_\perp A + 4(w-2)A\right.$$
$$\left. - \left(1 + \coth^2\left(\frac{\eta_0}{2}\right) - 6\coth\left(\frac{\eta_0}{2}\right)\coth(\eta_0)\right)A\right)$$
$$\bar{\mathcal{C}}_\partial = \tanh\left(\frac{\eta_0}{2}\right)\left(i(\tau_3)^{AB}\bar{B}_{AB} + 2D_\perp\bar{A} + 4(w-2)\bar{A}\right.$$
$$\left. - \left(1 + \tanh^2\left(\frac{\eta_0}{2}\right) - 6\tanh\left(\frac{\eta_0}{2}\right)\coth(\eta_0)\right)\bar{A}\right)$$

$$\mathcal{C} = D_\mu V^\mu + \delta_Q F, \bar{\mathcal{C}} = D_\mu \bar{V}^\mu + \delta_Q \bar{F}$$

$$V^\mu = 2B_{AB}\frac{\bar{\epsilon}^A\bar{\sigma}^\mu\epsilon^B}{\epsilon^C\epsilon_C} - 2i\tau_{3,AB}\frac{\bar{\epsilon}^A\bar{\sigma}^\mu\epsilon^B}{\epsilon^C\epsilon_C}\left(s_2(2w-3) - s_1\frac{\bar{\epsilon}_C\bar{\epsilon}^C}{\epsilon^D\epsilon_D}\right)A - 2\frac{\bar{\epsilon}_C\bar{\epsilon}^C}{\epsilon^D\epsilon_D}D^\mu A$$
$$\bar{V}^\mu = -2\bar{B}_{AB}\frac{\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\bar{\epsilon}_C\bar{\epsilon}^C} + 2i\tau_{3,AB}\frac{\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\bar{\epsilon}_C\bar{\epsilon}^C}\left(s_1(2w-3) - s_2\frac{\epsilon^C\epsilon_C}{\bar{\epsilon}_D\bar{\epsilon}^D}\right)A - 2\frac{\bar{\epsilon}_C\bar{\epsilon}^C}{\epsilon^D\epsilon_D}D^\mu A$$
$$F = -\frac{\epsilon^A\Lambda_A}{\epsilon^C\epsilon_C} + i\frac{\tau_3^{AB}\epsilon_A\Psi_B}{\epsilon^C\epsilon_C}\left(s_2(w-3) - s_1\frac{\bar{\epsilon}_C\bar{\epsilon}^C}{\epsilon^D\epsilon_D}\right) - D_\mu\left(\frac{\bar{\epsilon}^A\bar{\sigma}^\mu\Psi_A}{\epsilon^C\epsilon_C}\right)$$
$$\bar{F} = -\frac{\bar{\epsilon}^A\bar{\Lambda}_A}{\bar{\epsilon}_C\bar{\epsilon}^C} + i\frac{\tau_3^{AB}\bar{\epsilon}_A\bar{\Psi}_B}{\bar{\epsilon}_C\bar{\epsilon}^C}\left(-s_1(w-3) + s_2\frac{\epsilon^C\epsilon_C}{\bar{\epsilon}_D\bar{\epsilon}^D}\right) + D_\mu\left(\frac{\epsilon^A\sigma^\mu\bar{\Psi}_A}{\bar{\epsilon}_C\bar{\epsilon}^C}\right)$$

$$\int \mathcal{C}_k + \int_\partial \mathcal{C}_{\partial,k} = 32\pi^2 A_k(0) + \delta_Q \int F$$
$$\int \bar{\mathcal{C}}_k + \int_\partial \bar{\mathcal{C}}_{\partial,k} = \delta_Q \int \bar{F}$$

$$\partial_{\lambda_k} F_{\mathrm{AdS}}\big|_{\lambda=0} = 32\pi^2\langle A_k(0)\rangle,$$

$$\lambda_2 = i\frac{\tau_{\mathrm{UV}}}{8\pi}.$$

$$\partial_{\tau_{\mathrm{UV}}} F_{\mathrm{AdS}} = 4\pi i\langle A_2(0)\rangle = -4\pi i u$$

$$F_{\mathrm{AdS}} = -4\pi i \mathcal{F}_{\mathrm{AdS}}$$

$$\log\left(Z^D_{SU(2)}\right)_{\mathrm{pert}} = 2L^2a^2\log(\Lambda^2) + \log\left(G(1+2iaL)G(1-2iaL)\frac{2aL}{\sinh(2\pi aL)}\right)$$

$$\log\left(Z^D_{SU(2)}\right)_{\mathrm{pert}} \underset{L\to\infty}{\sim} 2L^2a^2\left(\log\left(\frac{\Lambda^2}{4a^2}\right) + 3\right) + O(\log(L)),$$

$$\log\left(Z_{\mathrm{Nekrasov}}(a,\Lambda)\right) \underset{L\to\infty}{\sim} 4\pi i L^2\mathcal{F}_{\mathrm{inst,curve}}(a) + \cdots$$

$$F(a,\Lambda) \underset{L\to\infty}{\sim} -4\pi i L^2\mathcal{F}_{\mathrm{supercurve}}(a) + \cdots,$$

$$\tau^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \tau^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ \tau^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$(\sigma^a)_{\alpha\dot{\beta}} = (-i\vec{\tau},1), (\bar{\sigma}^a)^{\dot{\alpha}\beta} = (i\vec{\tau},1), \left(\gamma^{a'}\right)_\alpha^{\ \beta} = \tau^{a'}, \vec{\tau}_A^B = \vec{\tau},$$
$$\sigma^{ab} = \sigma^{[a}\bar{\sigma}^{b]}, \bar{\sigma}^{ab} = \bar{\sigma}^{[a}\sigma^{b]}, \gamma^{a'b'} = \gamma^{[a'}\gamma^{b']}.$$

$$D_\mu\chi_\alpha = \partial_\mu\chi_\alpha + \frac{1}{4}\omega_{\mu,ab}(\sigma^{ab}\chi)_\alpha - iA_\mu\chi_\alpha,$$
$$D_\mu\bar{\chi}^{\dot{\alpha}} = \partial_\mu\bar{\chi}^{\dot{\alpha}} + \frac{1}{4}\omega_{\mu,ab}(\bar{\sigma}^{ab}\bar{\chi})^{\dot{\alpha}} - iA_\mu\bar{\chi}^{\dot{\alpha}},$$

$$\omega_{\mu,ab} = -2\partial_{[\mu}e_{\nu][a}e^\nu_{b]} - e^\nu_{[a}e^\rho_{b]}\partial_{[\nu}e_{\rho]c}$$

$$D_\mu\phi = \partial_\mu\phi - i[A_\mu,\phi]$$

$$ds^2 = d\eta^2 + \sinh(\eta)^2 d\Omega^2_{S^3},$$

$$z_1 = \sin\left(\frac{\theta}{2}\right) e^{i\frac{\psi-\phi}{2}}, z_2 = \cos\left(\frac{\theta}{2}\right) e^{i\frac{\psi+\phi}{2}}, |z_1|^2 + |z_2|^2 = 1,$$

$$d\Omega_{S^3}^2 = \frac{1}{4}(d\theta^2 + \sin^2(\theta) d\phi^2 + (d\psi + \cos(\theta) d\phi)^2).$$

$$\left(e_{\mathrm{AdS}_4}\right)^a{}_\mu = \begin{pmatrix} 0 & \frac{1}{2}\sinh(\eta)\cos(\psi) & \frac{1}{2}\sinh(\eta)\sin(\theta)\sin(\psi) & 0 \\ 0 & -\frac{1}{2}\sinh(\eta)\sin(\psi) & \frac{1}{2}\sinh(\eta)\sin(\theta)\cos(\psi) & 0 \\ \frac{1}{2}\sinh(\eta) & 0 & \frac{1}{2}\sinh(\eta)\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$V^i = -i\varepsilon^{ijk}\partial_j C_k, H = \frac{i}{2}\varepsilon^{ijk}\partial_i B_{jk}$$

$$H = \frac{i}{L}, V_i = A_i^{\mathcal{R}} = 0, g_{ij} = g_{ij}^{S^3},$$

$$D_i\zeta = -\frac{i}{2}\gamma_i\zeta, D_i\tilde{\zeta} = -\frac{i}{2}\gamma_i\tilde{\zeta}$$

$$\begin{aligned}
\delta A &= -i\epsilon^A\Psi_A \\
\delta\Psi_A &= 2\sigma^\mu\bar{\epsilon}_A D_\mu A + B_{AB}\epsilon^B + \frac{1}{2}\sigma^{\mu\nu}\epsilon_A G_{\mu\nu}^- + w\sigma^\mu D_\mu\bar{\epsilon}_A A \\
\delta B_{AB} &= -2i\bar{\epsilon}_{(A}\bar{\sigma}^\mu D_\mu\Psi_{B)} + 2i\epsilon_{(A}\Lambda_{B)} + i(1-w)D_\mu\bar{\epsilon}_{(A}\bar{\sigma}^\mu\Psi_{B)} \\
\delta G_{\mu\nu}^- &= -\frac{i}{2}\bar{\epsilon}^A\bar{\sigma}^\rho\sigma_{\mu\nu}D_\rho\Psi_A - \frac{i}{2}\epsilon^A\sigma_{\mu\nu}\Lambda_A - \frac{i}{4}(1+w)D_\rho\bar{\epsilon}^A\bar{\sigma}^\rho\sigma_{\mu\nu}\Psi_A \\
\delta\Lambda_A &= -\frac{1}{2}D_\rho G_{\mu\nu}^-\sigma^{\mu\nu}\sigma^\rho\bar{\epsilon}_A + D_\mu B_{AB}\sigma^\mu\bar{\epsilon}^B - C\epsilon_A \\
&\quad + \frac{1}{2}(1+w)B_{AB}\sigma^\mu D_\mu\bar{\epsilon}^B + \frac{1}{4}(1-w)G_{\mu\nu}^-\sigma^{\mu\nu}\sigma^\rho D_\rho\bar{\epsilon}_A \\
&\quad + 4D_\rho A\bar{T}^{\mu\nu}\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\epsilon}_A + 4wAD_\rho\bar{T}^{\mu\nu}\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\epsilon}_A \\
\delta C &= -2i\bar{\epsilon}^A\bar{\sigma}^\mu D_\mu\Lambda_A + 4i\bar{T}^{\mu\nu}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\sigma}^\rho D_\rho\Psi_A \\
&\quad - iwD_\mu\bar{\epsilon}^A\bar{\sigma}^\mu\Lambda_A + 4i(w-1)D_\rho\bar{T}^{\mu\nu}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\sigma}^\rho\Psi_A
\end{aligned}$$

$$\begin{aligned}
\delta\bar{A} &= i\bar{\epsilon}^A\bar{\Psi}_A \\
\delta\bar{\Psi}_A &= 2\bar{\sigma}^\mu\epsilon_A D_\mu\bar{A} + \bar{B}_{AB}\bar{\epsilon}^B + \frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\epsilon}_A G_{\mu\nu}^+ + w\bar{\sigma}^\mu D_\mu\epsilon_A\bar{A} \\
\delta\bar{B}_{AB} &= 2i\epsilon_{(A}\sigma^\mu D_\mu\bar{\Psi}_{B)} + 2i\bar{\epsilon}_{(A}\bar{\Lambda}_{B)} - i(1-w)D_\mu\epsilon_{(A}\sigma^\mu\bar{\Psi}_{B)} \\
\delta G_{\mu\nu}^+ &= -\frac{i}{2}\epsilon^A\sigma^\rho\bar{\sigma}_{\mu\nu}D_\rho\bar{\Psi}_A - \frac{i}{2}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\Lambda}_A + \frac{i}{4}(1+w)D_\rho\epsilon^A\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\Psi}_A \\
\delta\bar{\Lambda}_A &= \frac{1}{2}D_\rho G_{\mu\nu}^+\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\epsilon_A - D_\mu\bar{B}_{AB}\bar{\sigma}^\mu\epsilon^B + \bar{C}\bar{\epsilon}_A \\
&\quad - \frac{1}{2}(1+w)\bar{B}_{AB}\bar{\sigma}^\mu D_\mu\epsilon^B - \frac{1}{4}(1-w)G_{\mu\nu}^+\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho D_\rho\epsilon_A \\
&\quad + 4D_\rho\bar{A}T^{\mu\nu}\bar{\sigma}^\rho\sigma_{\mu\nu}\epsilon_A + 4w\bar{A}D_\rho T^{\mu\nu}\bar{\sigma}^\rho\sigma_{\mu\nu}\epsilon_A \\
\delta\bar{C} &= -2i\epsilon^A\sigma^\mu D_\mu\bar{\Lambda}_A + 4iT^{\mu\nu}\epsilon^A\sigma_{\mu\nu}\sigma^\rho D_\rho\bar{\Psi}_A \\
&\quad - iwD_\mu\epsilon^A\sigma^\mu\bar{\Lambda}_A + 4i(w-1)D_\rho T^{\mu\nu}\epsilon^A\sigma_{\mu\nu}\sigma^\rho\bar{\Psi}_A
\end{aligned}$$

$$\begin{aligned}
\delta\phi &= -i\epsilon^A\lambda_A, \\
\delta\bar{\phi} &= i\bar{\epsilon}^A\bar{\lambda}_A, \\
\delta\lambda_A &= \frac{1}{2}\sigma^{\mu\nu}\epsilon_A\left(F_{\mu\nu} + 8\bar{\phi}T_{\mu\nu}\right) + 2\sigma^\mu\bar{\epsilon}_A D_\mu\phi + \sigma^\mu D_\mu\bar{\epsilon}_A\phi + 2i\epsilon_A[\phi,\bar{\phi}] + D_{AB}\epsilon^B, \\
\delta\bar{\lambda}_A &= \frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\epsilon}_A\left(F_{\mu\nu} + 8\phi\bar{T}_{\mu\nu}\right) + 2\bar{\sigma}^\mu\epsilon_A D_\mu\bar{\phi} + \bar{\sigma}^\mu D_\mu\epsilon_A\bar{\phi} - 2i\bar{\epsilon}_A[\phi,\bar{\phi}] + D_{AB}\bar{\epsilon}^B, \\
\delta A_\mu &= i\epsilon^A\sigma_\mu\bar{\lambda}_A - i\bar{\epsilon}^A\bar{\sigma}_\mu\lambda_A, \\
\delta D_{AB} &= -2i\bar{\epsilon}_{(A}\bar{\sigma}^\mu D_\mu\lambda_{B)} + 2i\epsilon_{(A}\sigma^\mu D_\mu\bar{\lambda}_{B)} - 4\left[\phi,\bar{\epsilon}_{(A}\bar{\lambda}_{B)}\right] + 4\left[\bar{\phi},\epsilon_{(A}\lambda_{B)}\right].
\end{aligned}$$

$$A = \mathcal{F}(\phi)$$
$$\Psi_{A\alpha} = \mathcal{F}_I \lambda^I_{A\alpha}$$
$$B_{AB} = \mathcal{F}_I D^I_{AB} + \frac{i}{2}\mathcal{F}_{IJ}\lambda^I_A \lambda^J_B$$
$$G^-_{\mu\nu} = \mathcal{F}_I\left(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}\right) + 2i\mathcal{F}_{IJ}\lambda^{IA}\sigma_{\mu\nu}\lambda^J_A$$
$$\Lambda_{A\alpha} = \mathcal{F}_I\left(D^I_{\bar{\lambda}}\right)_\alpha - i\mathcal{F}_I[\bar{\phi}, \lambda_{A\alpha}]^I + \frac{i}{4}\mathcal{F}_{IJ}\left(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}\right)\left(\sigma^{\mu\nu}\lambda^J_A\right)_\alpha$$
$$+\frac{1}{2}\mathcal{F}_{IJ}D^I_{AB}\lambda^{JB}_\alpha - \frac{i}{12}\mathcal{F}_{IJK}\left(2(\lambda^I_A\lambda^{JB})\lambda^K_{B\alpha} - \lambda^I_{A\alpha}(\lambda^{JB}\lambda^K_B)\right)$$
$$C = -2\mathcal{F}_I\left(D^\mu D_\mu - \frac{\mathcal{R}}{6} + \frac{\tilde{M}}{2}\right)\bar{\phi}^I - 8\mathcal{F}_I\left(F^{+I}_{\mu\nu} + 8\phi^I \bar{T}_{\mu\nu}\right)\bar{T}^{\mu\nu}$$

$$+\mathcal{F}_I\left[\bar{\lambda}^A, \bar{\lambda}_A\right]^I - 2\mathcal{F}_I[[\bar{\phi}, \phi], \bar{\phi}]^I + \frac{1}{4}\mathcal{F}_{IJ}D^{IAB}D^J_{AB}$$
$$-\frac{1}{2}\mathcal{F}_{IJ}\left(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}\right)(F^{-J\mu\nu} + 8\bar{\phi}^J T^{\mu\nu}) + i\mathcal{F}_{IJ}\lambda^{IA}\Big/\bar{\lambda}^J_A$$
$$-\mathcal{F}_{IJ}\left([\bar{\phi}, \lambda^A]^I \lambda^J_A\right) + \frac{i}{4}\mathcal{F}_{IJK}D^{ABI}\lambda^J_A\lambda^K_B$$
$$+\frac{i}{8}\mathcal{F}_{IJK}\left(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}\right)\lambda^{JA}\sigma^{\mu\nu}\lambda^K_A + \frac{1}{24}\mathcal{F}_{IJKL}(\lambda^{IA}\lambda^{JB})(\lambda^K_A\lambda^L_B)$$

$$\bar{A} = \overline{\mathcal{F}}(\bar{\phi})$$
$$\bar{\Psi}^{A\dot{\alpha}} = \overline{\mathcal{F}}_I\bar{\lambda}^{IA\dot{\alpha}}$$
$$\bar{B}^{AB} = \overline{\mathcal{F}}_I D^{IAB} - \frac{i}{2}\overline{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\lambda}^{JB}$$
$$G^+_{\mu\nu} = \overline{\mathcal{F}}_I\left(F^{+I}_{\mu\nu} + 8\phi^I \bar{T}_{\mu\nu}\right) - \frac{i}{8}\overline{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\sigma}_{\mu\nu}\bar{\lambda}^J_A$$
$$\bar{\Lambda}^{A\dot{\alpha}} = -\overline{\mathcal{F}}_I(\bar{\nabla}\lambda^I_A)^{\dot{\alpha}} + i\overline{\mathcal{F}}_I\left[\phi, \bar{\lambda}^{A\dot{\alpha}}\right]^I - \frac{i}{4}\overline{\mathcal{F}}_{IJ}\left(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu}\right)\left(\bar{\sigma}^{\mu\nu}\bar{\lambda}^{JA}\right)^{\dot{\alpha}}$$
$$+\frac{1}{2}\overline{\mathcal{F}}_{IJ}D^{IAB}\bar{\lambda}^{J\dot{\alpha}}_B + \frac{i}{12}\overline{\mathcal{F}}_{IJK}\left(2\left(\bar{\lambda}^{IA}\bar{\lambda}^{JB}\right)\bar{\lambda}^{K\dot{\alpha}}_B - \bar{\lambda}^{IA\dot{\alpha}}\left(\bar{\lambda}^{JB}\bar{\lambda}^K_B\right)\right)$$
$$\bar{C} = -2\overline{\mathcal{F}}_I\left(D^\mu D_\mu - \frac{\mathcal{R}}{6} + \frac{\tilde{M}}{2}\right)\phi^I - 8\overline{\mathcal{F}}_I\left(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}\right)T^{\mu\nu}$$
$$-\overline{\mathcal{F}}_I[\lambda^A, \lambda_A]^I + 2\overline{\mathcal{F}}_I[\phi, [\phi, \bar{\phi}]]^I + \frac{1}{4}\overline{\mathcal{F}}_{IJ}D^{IAB}D^J_{AB}$$
$$-\frac{1}{2}\overline{\mathcal{F}}_{IJ}\left(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu}\right)(F^{+J\mu\nu} + 8\phi^J\bar{T}^{\mu\nu}) - i\overline{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\phi}\lambda^J_A$$
$$+\overline{\mathcal{F}}_{IJ}\left(\left[\phi, \bar{\lambda}^A\right]^I\bar{\lambda}^J_A\right) - \frac{i}{4}\overline{\mathcal{F}}_{IJK}D^{ABI}\bar{\lambda}^J_A\bar{\lambda}^K_B$$
$$+\frac{i}{8}\overline{\mathcal{F}}_{IJK}\left(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu}\right)\bar{\lambda}^{JA}\bar{\sigma}^{\mu\nu}\bar{\lambda}^K_A - \frac{1}{24}\overline{\mathcal{F}}_{IJKL}\left(\bar{\lambda}^{IA}\bar{\lambda}^{JB}\right)\left(\bar{\lambda}^K_A\bar{\lambda}^L_B\right)$$

$$\delta\sigma = -\zeta\tilde{\lambda} + \tilde{\zeta}\lambda$$
$$\delta\lambda = \left(i(D + H\sigma) - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} - i\gamma^i(D_i\sigma + iV_i\sigma)\right)\zeta$$
$$\delta\tilde{\lambda} = \left(-i(D + H\sigma) - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} + i\gamma^i(D_i\sigma - iV_i\sigma)\right)\tilde{\zeta},$$
$$\delta A_i = -i\left(\zeta\gamma_i\tilde{\lambda} + \tilde{\zeta}\gamma_i\lambda\right),$$
$$\delta D = D_i\left(\zeta\gamma^i\tilde{\lambda} - \tilde{\zeta}\gamma^i\lambda\right) - iV_i\left(\zeta\gamma^i\tilde{\lambda} + \tilde{\zeta}\gamma^i\lambda\right) - H(\zeta\tilde{\lambda} - \tilde{\zeta}\lambda) - [\sigma, \zeta\tilde{\lambda} + \tilde{\zeta}\lambda].$$

$$\delta J = i\zeta j + i\tilde{\zeta}\tilde{j}$$
$$\delta j = \tilde{\zeta}K + i\gamma^i\tilde{\zeta}(j_i + iD_iJ) + \tilde{\zeta}[\sigma, J]$$
$$\delta\tilde{j} = \zeta K - i\gamma^i\zeta(j_i - iD_iJ) - \zeta[\sigma, J]$$
$$\delta j_i = i\varepsilon_{ijk}D^j\left(\zeta\gamma^k j - \tilde{\zeta}\gamma^k\tilde{j}\right) + \left[\sigma, \zeta\gamma_i j + \tilde{\zeta}\gamma_i\tilde{j}\right] + i\left[J, \tilde{\zeta}\gamma_i\lambda - \zeta\gamma_i\tilde{\lambda}\right]$$
$$\delta K = -iD_i\left(\zeta\gamma^i j + \tilde{\zeta}\gamma^i\tilde{j}\right) + 2iH(\zeta j + \tilde{\zeta}\tilde{j}) - V_i\left(\zeta\gamma^i j + \tilde{\zeta}\gamma^i\tilde{j}\right) + [\zeta\tilde{\lambda} + \tilde{\zeta}\lambda, J]$$

$$D_i j^i - [\sigma + iD, J] - i[\sigma, K] + [\tilde{\lambda}, \tilde{j}] - [\lambda, j] = 0$$
$$\left[F_{ij}, J\right] = 0$$

$$J=\sigma, j=i\tilde{\lambda}, \tilde{j}=-i\lambda$$
$$j_i=-\frac{i}{2}\varepsilon_{ijk}F^{jk}, K=D+H\sigma$$

$$Q=\epsilon^A Q_A+\bar{\epsilon}_A\bar{Q}^A$$

$$Q^2\lambda_A=iv^\mu\partial_\mu\lambda_A+i[\Phi,\lambda_A]+\frac{i}{4}L_{ab}\sigma^{ab}\lambda_A+\left(\frac{3}{2}w+\Theta\right)\lambda_A+\tilde{\Theta}_{AB}\lambda^B$$

$$v^\mu=2\bar{\epsilon}^A\bar{\sigma}^\mu\epsilon_A$$
$$\Phi^I=2i\bar{\phi}^I\epsilon^A\epsilon_A+2i\phi^I\bar{\epsilon}_A\bar{\epsilon}^A+v^\mu A_\mu^I$$
$$L_{ab}=D_{[a}v_{b]}+v^\mu\omega_{\mu,ab}$$
$$w=-\frac{i}{2}\left(\epsilon^A\sigma^\mu D_\mu\bar{\epsilon}_A+D_\mu\epsilon^A\sigma^\mu\bar{\epsilon}_A\right)$$
$$\Theta=-\frac{i}{4}\left(\epsilon^A\sigma^\mu D_\mu\bar{\epsilon}_A-D_\mu\epsilon^A\sigma^\mu\bar{\epsilon}_A\right)$$
$$\tilde{\Theta}_{AB}=-i\epsilon_{(A}\sigma^\mu D_\mu\bar{\epsilon}_{B)}+iD_\mu\epsilon_{(A}\sigma^\mu\bar{\epsilon}_{B)}$$

$$Q^2=-v^\mu P_\mu+\Phi^I G_I+L_{ab}M^{ab}+wD+\Theta\mathcal{R}_{U(1)}+\tilde{\Theta}_{AB}\mathcal{R}_{SU(2)}^{AB}$$

$$Q^2|_{\mathrm{AdS}}=-v^\mu P_\mu+\Phi^I G_I+L_{ab}M^{ab}+\tilde{\Theta}_{AB}\mathcal{R}_{SU(2)}^{AB}$$
$$\tilde{\Theta}_{AB}=\tau_{3,AB}(\epsilon^C\epsilon_C-\bar{\epsilon}_C\bar{\epsilon}^C)$$

$$D_i\xi_A=-\frac{i}{2}\gamma_i\xi_A, D_i\tilde{\xi}_A=\frac{i}{2}\gamma_i\tilde{\xi}_A$$

$$\epsilon_A=e^{\frac{\eta}{2}}\epsilon_{+A}+e^{-\frac{\eta}{2}}\epsilon_{-A},$$
$$\bar{\epsilon}_A=e^{\frac{\eta}{2}}\bar{\epsilon}_{+A}+e^{-\frac{\eta}{2}}\bar{\epsilon}_{-A}.$$

$$\epsilon_{\pm A}=\mp i\tau_{3A}{}^B\sigma_4\bar{\epsilon}_{\pm B}$$

$$D_i\epsilon_A=\frac{i}{2}\tau_{3A}^B\sigma_i\bar{\epsilon}_B$$

$$D_i^{3d}\epsilon_{\pm A}=\frac{i}{2}\gamma_i^{3d}\epsilon_{\mp A},$$

$$\frac{\left(\gamma_{3d}^i\right)_\alpha^\beta}{\sinh(\eta)}=i\left(\sigma^i\bar{\sigma}^\perp\right)_\alpha^\beta=-i\left(\sigma^\perp\bar{\sigma}^i\right)_\alpha^\beta$$
$$\sinh(\eta)\left(\gamma_i^{3d}\right)_\alpha^\beta=i(\sigma_i\bar{\sigma}^\perp)_\alpha^\beta=-i(\sigma^\perp\bar{\sigma}_i)_\alpha^\beta$$

$$D_i\chi_\alpha=D_i^{3d}\chi_\alpha+\frac{1}{2}\omega_{i,a4}(\sigma^{a4}\chi)_\alpha$$

$$\xi_A=\epsilon_{+A}-\epsilon_{-A}, \tilde{\xi}_A=\epsilon_{+A}+\epsilon_{-A},$$

$$\epsilon_{A\alpha}=\sinh\left(\frac{\eta}{2}\right)\xi_{A\alpha}+\cosh\left(\frac{\eta}{2}\right)\tilde{\xi}_{A\alpha}$$
$$\bar{\epsilon}_{A\dot{\alpha}}=-i\tau_{3,A}^B\left(\cosh\left(\frac{\eta}{2}\right)\xi_A^\beta+\sinh\left(\frac{\eta}{2}\right)\tilde{\xi}_A^\beta\right)\sigma_{\perp,\beta\dot{\alpha}}$$

$$\zeta'=e^{i\theta_{\mathcal{R}}}\zeta, \tilde{\zeta}'=e^{-i\theta_{\mathcal{R}}}\tilde{\zeta}'$$

$$\xi_A'=\left(e^{i\theta_{\mathcal{R}}\tau_3}\right)_A^B\xi_B=\begin{pmatrix}e^{i\theta_{\mathcal{R}}}\xi_1\\ e^{-i\theta_{\mathcal{R}}}\xi_2\end{pmatrix}_A$$

$$\zeta=\xi_1, \tilde{\zeta}=\xi_2$$

$$\epsilon_{A\alpha} = \sinh\left(\frac{\eta}{2}\right)\xi_{A\alpha} = \sinh\left(\frac{\eta}{2}\right)\begin{pmatrix}\zeta_\alpha \\ \tilde{\zeta}_\alpha\end{pmatrix}_A$$

$$\bar{\epsilon}_{A\dot{\alpha}} = -i\tau_{3,A}^{B}\cosh\left(\frac{\eta}{2}\right)(\xi_B\sigma_\perp)_\alpha = -i\tau_{3,A}^{B}\begin{pmatrix}(\zeta\sigma_\perp)_{\dot{\alpha}} \\ (\tilde{\zeta}\sigma_\perp)_{\dot{\alpha}}\end{pmatrix}_B$$

$$Q^2|_{3d} = -v^iP_i + \Phi^I G_I + L_{a'b'}M^{a'b'} + \tilde{\Theta}_{AB}\mathcal{R}_{SU(2)}^{AB},$$

$$v^i = -2\tilde{\zeta}\gamma^i\zeta, \Phi = -4i\zeta\tilde{\zeta}(\phi_2 + i\cosh(\eta)\phi_1) + v^iA_i, \tilde{\Theta}_{AB} = 2\zeta\tilde{\zeta}\tau_{3,AB}$$

$$\int \mathcal{D}\Psi\mathcal{D}\Psi_{\text{ghost}}\, e^{-S[\Psi]-t\mathcal{Q}_{\text{loc}}V-\mathcal{Q}_{\text{BRST}}V_{\text{ghost}}}$$

$$V_{\text{ghost}} = \text{Tr}\left[\bar{c}\left(-G[\Psi-\Psi^{\text{loc}}] + \frac{\kappa}{2}B\right)\right],$$

$$\begin{aligned}
&\mathcal{Q}_{\text{BRST}}c = \frac{i}{2}[c,c]_+\\
&\mathcal{Q}_{\text{BRST}}\bar{c} = B\\
&\mathcal{Q}_{\text{BRST}}B = 0\\
&\mathcal{Q}_{\text{BRST}}\Psi = i[c,\Psi]_\pm\left(\text{except for } \mathcal{Q}_{\text{BRST}}A_\mu = D_\mu c\right),
\end{aligned}$$

$$\begin{aligned}
&\hat{\mathcal{Q}} = \mathcal{Q}_{\text{loc}} + \mathcal{Q}_{\text{BRST}},\\
&\hat{V} = V + V_{\text{ghost}},\\
&V = \sum_{\Psi_{\text{odd}}}\Psi_{\text{odd}}\left(\hat{\mathcal{Q}}\Psi_{\text{odd}}\right)^*,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{Q}_{\text{loc}}c = \Phi^{\text{loc}} - \Phi\\
&\mathcal{Q}_{\text{loc}}\bar{c} = 0\\
&\mathcal{Q}_{\text{loc}}B = iv^\mu\partial_\mu\bar{c} + iv^\mu A_\mu^{\text{loc}}\bar{c}
\end{aligned}$$

$$\int \mathcal{D}\Psi\mathcal{D}\Psi_{\text{ghost}}\, e^{-S[\Psi]-t\hat{\mathcal{Q}}\hat{V}}$$

$$\int \hat{\mathcal{Q}}\hat{V}\Big|_{\text{even}} = \int \mathcal{Q}V\Big|_{\text{even}} + \int B\left(-G[\Psi-\Psi^{\text{loc}}] + \frac{\kappa}{2}B\right)$$

$$\begin{aligned}
&\mathcal{Q}_{\text{BRST}}c = \frac{i}{2}[c,c]_+ - a_0\\
&\mathcal{Q}_{\text{BRST}}\bar{c} = B\\
&\mathcal{Q}_{\text{BRST}}B = -i[a_0,\bar{c}]\\
&\mathcal{Q}_{\text{BRST}}a_0 = 0\\
&\mathcal{Q}_{\text{BRST}}\bar{a}_0 = \bar{c}_0\\
&\mathcal{Q}_{\text{BRST}}c_0 = -i[a_0,B_0]\\
&\mathcal{Q}_{\text{BRST}}\bar{c}_0 = -i[a_0,\bar{a}_0]\\
&\mathcal{Q}_{\text{BRST}}B_0 = c_0\\
&\mathcal{Q}_{\text{BRST}}\Psi = c^I G_I\Psi\ (\text{except for } \mathcal{Q}_{\text{BRST}}A_\mu = D_\mu c)
\end{aligned}$$

$$\mathcal{Q}_{\text{loc}}a_0 = \mathcal{Q}_{\text{loc}}\bar{a}_0 = \mathcal{Q}_{\text{loc}}B_0 = \mathcal{Q}_{\text{loc}}c_0 = \mathcal{Q}_{\text{loc}}\bar{c}_0 = 0$$

$$V_{\text{ghost}} = \text{Tr}\left[\bar{c}\left(-G[\Psi-\Psi^{\text{loc}}] + \frac{\kappa_1}{2}B + iB_0\right) - c\left(\bar{a}_0 - \frac{\kappa_2}{2}a_0\right)\right].$$

$$\hat{V}\Big|_{\text{quad.}} = \begin{pmatrix}\hat{\mathcal{Q}}X_{\text{even}} & X_{\text{odd}}\end{pmatrix}\begin{pmatrix}D_{00} & D_{01}\\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}X_{\text{even}}\\ \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix}, \begin{pmatrix}-\hat{\mathcal{Q}}^2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}D_{00} & D_{01}\\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}X_{\text{even}}\\ \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix}\cdot\begin{pmatrix}X_{\text{even}} & \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix}\begin{pmatrix}D_{00} & D_{01}\\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \hat{\mathcal{Q}}^2\end{pmatrix}\begin{pmatrix}\hat{\mathcal{Q}}X_{\text{even}}\\ X_{\text{odd}}\end{pmatrix}$$

$$Z_{1-\text{loop}} = \sqrt{\frac{\det(K_{\text{odd}})}{\det(K_{\text{even}})}} = \sqrt{\frac{\det_{X_{\text{odd}}}(\hat{\mathcal{Q}}^2)}{\det_{X_{\text{even}}}(\hat{\mathcal{Q}}^2)}} = \sqrt{\frac{\det_{\text{Coker.}\,D_{10}}(\hat{\mathcal{Q}}^2)}{\det_{\text{Ker.}\,D_{10}}(\hat{\mathcal{Q}}^2)}}.$$

$$Z_{1-\text{loop}} = \sqrt{\prod_n \lambda_n^{c_n - k_n}}$$

$$\text{Ind}_{D_{10}}(t) = \text{Tr}_{\text{Ker.}\,D_{10}}\left(e^{-i\hat{Q}^2 t}\right) - \text{Tr}_{\text{Coker.}\,D_{10}}\left(e^{-i\hat{Q}^2 t}\right) = -\sum_n (c_n - k_n) e^{-i\lambda_n t}$$

$$\text{Ind}_{D_{10}}(t) = \sum_{x_0} \frac{\text{Tr}_{X_{\text{even}}}\left(e^{-i\hat{Q}^2 t}\right)\Big|_{x_0} - \text{Tr}_{X_{\text{odd}}}\left(e^{-i\hat{Q}^2 t}\right)\Big|_{x_0}}{\det\left(\delta_\nu^\mu - \partial_\nu x'^\mu\right)}$$

$$X_{\text{even}} = \left(\phi_2, A_\mu\right) X_{\text{odd}} = \left(\Theta_{AB}, \bar{c}, c\right)$$

$$\Theta_{AB} = \epsilon_{(A}\lambda_{B)} + \bar{\epsilon}_{(A}\bar{\lambda}_{B)}$$

$$\Phi = 2i\bar{\epsilon}_A\bar{\epsilon}^A\phi + 2i\epsilon^A\epsilon_A\bar{\phi} - iv^\mu A_\mu$$

$$i\mathcal{L}_v = i\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + i\left(x^3\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^3}\right)$$

$$z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$$

$$z_1' = e^{it}z_1, z_2' = e^{it}z_2$$

$$\det\left(\delta_\nu^\mu - \partial_\nu x'^\mu\right) = \left|1 - e^{it}\right|^4$$

$$\begin{aligned}
&e^{-i\hat{Q}^2 t}A_{z_1} = e^{-i(-1+ia)t}A_{z_1}, && e^{-i\hat{Q}^2 t}\Theta_{11} = e^{-i(2+ia)t}\Theta_{11},\\
&e^{-i\hat{Q}^2 t}A_{z_2} = e^{-i(-1+ia)t}A_{z_2}, && e^{-i\hat{Q}^2 t}\Theta_{12} = e^{-i(ia)t}\Theta_{12},\\
&e^{-i\hat{Q}^2 t}A_{\bar{z}_1} = e^{-i(1+ia)t}A_{\bar{z}_1}, && e^{-i\hat{Q}^2 t}\Theta_{22} = e^{-i(-2+ia)t}\Theta_{22},\\
&e^{-i\hat{Q}^2 t}A_{\bar{z}_2} = e^{-i(1+ia)t}A_{\bar{z}_2}, && e^{-i\hat{Q}^2 t}\bar{c} = e^{-i(ia)t}\bar{c},\\
&e^{-i\hat{Q}^2 t}\phi_2 = e^{-i(ia)t}\phi_2, && e^{-i\hat{Q}^2 t}c = e^{-i(ia)t}c.
\end{aligned}$$

$$\begin{aligned}
\text{Ind}_{D_{10}}(t) &= \text{Tr}_{\text{adj.}}\left(e^{at}\right)\frac{\left(\left(2e^{it} + 2e^{-it} + 1\right) - \left(e^{2it} + e^{-2it} + 3\right)\right)}{|1 - e^{it}|^4}\\
&= -\text{Tr}_{\text{adj.}}\left(e^{at}\right)\frac{1 + e^{2it}}{(1 - e^{it})^2} = -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \frac{1 + e^{2it}}{(1 - e^{it})^2}
\end{aligned}$$

$$\begin{aligned}
\text{Ind}_{D_{10}}^{[a]}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \frac{1 + e^{2it}}{(1 - e^{it})^2}\\
\text{Ind}_{D_{10}}^{[b]}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \frac{1 + e^{-2it}}{(1 - e^{-it})^2}\\
\text{Ind}_{D_{10}}^{[c]}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \left(\frac{1}{(1 - e^{it})^2} + \frac{1}{(1 - e^{-it})^2}\right)\\
\text{Ind}_{D_{10}}^{[d]}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \left(\frac{e^{2it}}{(1 - e^{it})^2} + \frac{e^{-2it}}{(1 - e^{-it})^2}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Ind}_{D_{10}}^{[a]}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at}\left(1 + e^{2it}\right)\left(\sum_{k\geqslant 0} e^{ikt}\right)^2\\
&= -\sum_{\alpha\in\Delta} e^{-i(i\alpha\cdot a)t} - \sum_{\alpha\in\Delta}\sum_{k>0} 2ke^{-i(i\alpha\cdot a - k)t}
\end{aligned}$$

$$Z_{1-\text{loop}}^{[a]} = \prod_{\alpha\in\Delta^+} \alpha\cdot a \prod_{k>0} \left(k^2 + (\alpha\cdot a)^2\right)^k$$

$$Z_{1-\text{loop}}^{[b]} = Z_{1-\text{loop}}^{[a]}$$

$$Z^{[c]}_{1-\text{loop}} = \prod_{\alpha\in\Delta^+} (\alpha\cdot a)^2 \prod_{k>0} (k+i\alpha\cdot a)^{k+1}(k-i\alpha\cdot a)^{k+1},$$
$$Z^{[d]}_{1-\text{loop}} = \prod_{\alpha\in\Delta^+} \prod_{k>0} (k+i\alpha\cdot a)^{k-1}(k-i\alpha\cdot a)^{k-1}.$$

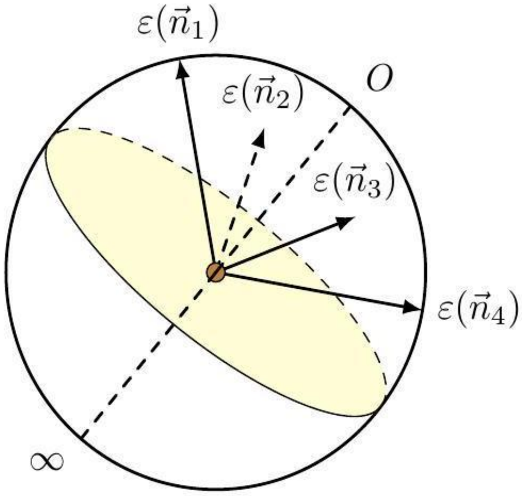
$$\Gamma(z) = \frac{e^{-\gamma_E z}}{z}\prod_{k>0}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}}$$
$$G(z) = (2\pi)^{\frac{z}{2}} e^{-\frac{1+\gamma_E z^2+z}{2}} \prod_{k>0}\left(1+\frac{z}{k}\right)^{k} e^{-z+\frac{z^2}{k}}$$

$$\prod_{k>0}(k+z) = \prod_{k>0} k e^{\frac{z}{k}} \times \frac{e^{-\gamma_E z}}{\Gamma(z+1)}$$
$$\prod_{k>0}(k+z)^k = \prod_{k>0} k^k e^{z-\frac{z^2}{2k}} \times G(1+z)\frac{e^{\frac{1+\gamma_E z^2+z}{2}}}{(2\pi)^{\frac{z}{2}}}$$

$$Z^{[a]}_{1-\text{loop}} = Z^{[b]}_{1-\text{loop}} = \left(e\prod_{k>0} k^{2k}\right)^{\text{rk}(G)} \times \prod_{\alpha\in\Delta^+ k>0} e^{\frac{(\alpha\cdot a)^2}{k}}$$
$$\prod_{\alpha\in\Delta^+} \alpha\cdot a G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-(\gamma_E\alpha\cdot a)^2}$$
$$Z^{[c]}_{1-\text{loop}} = \left(e\prod_{k>0}(-1)^{1+k}k^{2k+2}\right)^{\text{rk}(G)} \times \prod_{\alpha\in\Delta^+ k>0} e^{\frac{(\alpha\cdot a)^2}{k}}$$
$$\prod_{\alpha\in\Delta^+} \alpha\cdot a\sinh(\pi\alpha\cdot a)G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}$$
$$Z^{[d]}_{1-\text{loop}} = \left(e\prod_{k>0}(-1)^{1+k}k^{2k+2}\right)^{\text{rk}(G)} \times \prod_{\alpha\in\Delta^+ k>0} e^{\frac{(\alpha\cdot a)^2}{k}}$$
$$\prod_{\alpha\in\Delta^+} \frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)} G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}$$

$$Z^{[a]}_{1-\text{loop}} = Z^{[b]}_{1-\text{loop}} = \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \alpha\cdot a G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}$$
$$Z^{[c]}_{1-\text{loop}} = \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \alpha\cdot a\sinh(\pi\alpha\cdot a)G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}$$
$$Z^{[d]}_{1-\text{loop}} = \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)} G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}$$

$$\frac{d\sigma}{dx_{12}\cdots dx_{(n-1)n}} \equiv \sum_m \sum_{1\le i_1,\cdots i_n\le m} \int d\sigma_m \times \prod_{1\le k\le n}\frac{E_{i_k}}{q^0}\prod_{1\le j<l\le n}\delta\left(x_{jl}-\frac{1-\cos\theta_{i_j i_l}}{2}\right),$$



$$\mathcal{E} = \int_0^\infty \mathrm{d}t \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\vec{n})$$

$$\mathrm{E^nC}(\theta_{ij}) = \int \prod_{i=1}^{n} \mathrm{d}\Omega_{\vec{n}_i} \prod_{i\neq j} \delta(\vec{n}_i \cdot \vec{n}_j - \cos\,\theta_{ij}) \frac{\int\, d^4x e^{iqx} \langle 0|\mathcal{O}^+(x)\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\mathcal{O}(0)|0\rangle}{(q^0)^n \int\, d^4x e^{iqx}\langle 0|\mathcal{O}^+(x)\mathcal{O}(0)|0\rangle}$$

$$\int \mathrm{d}^4x e^{iqx}\langle X|\mathcal{O}(x)|0\rangle \equiv (2\pi)^4\delta^4(q-q_X)\mathcal{F}_X$$

$$\mathrm{E^nC} = \frac{1}{\sigma_{\mathrm{tot}}} \sum_{(n_1,\cdots,n_n)\in X} \int \mathrm{d}\Pi_X \left(\prod_{i=1}^{n} \delta^2(\vec{n}_i - \hat{p}_{n_i})\frac{E_i}{q^0}\right)|\mathcal{F}_X|^2$$

$$x_i = \frac{2q\cdot p_i}{q^2}\; i = 1,\cdots,n;$$

$$\zeta_{ij} = \frac{q^2(p_i\cdot p_j)}{2(q\cdot p_i)(q\cdot p_j)}\; i,j = 1,\cdots,n;$$

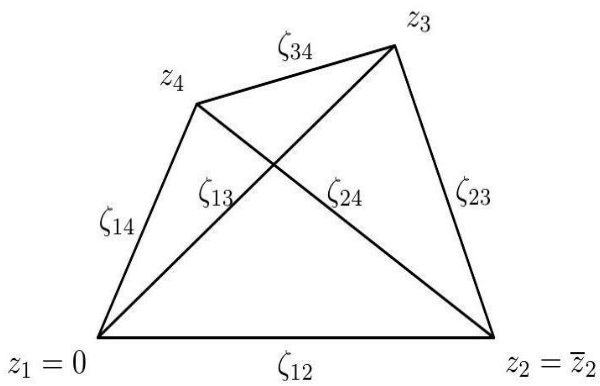
$$\mathrm{E^nC}(\zeta_{ij}) = \frac{1}{\sigma_{\mathrm{tot}}}\int_0^1 \mathrm{d}x_1\cdots\,\mathrm{d}x_n(x_1\cdots x_n)^2\delta(1-Q_n)\left|\mathcal{F}_{n+1}^{(0)}\right|^2$$

$$\delta(1-Q_n) = \delta\left(1 - \sum_i x_i + \sum_{1\le i<j\le n}\zeta_{ij}x_ix_j\right).$$

$$\zeta_{12} = \frac{z_2^2}{1+z_2^2}, \zeta_{23} = \frac{(z_2-z_3)(z_2-\overline{z_3})}{(1+z_2^2)(1+z_3\overline{z_3})}, \zeta_{13} = \frac{z_3\overline{z_3}}{1+z_3\overline{z_3}}$$

$$\zeta_{12} = -\frac{s(1-x_2)^2}{(1+s)^2x_2}, \zeta_{23} = -\frac{s(1-x_1x_2)^2}{(1+s)^2x_1x_2}, \zeta_{13} = -\frac{s(1-x_1)^2}{(1+s)^2x_1}$$

$$\zeta_{ij} = \frac{|z_i-z_j|^2}{(1+|z_i|^2)\left(1+|z_j|^2\right)}, i,j = 1,\cdots,4$$

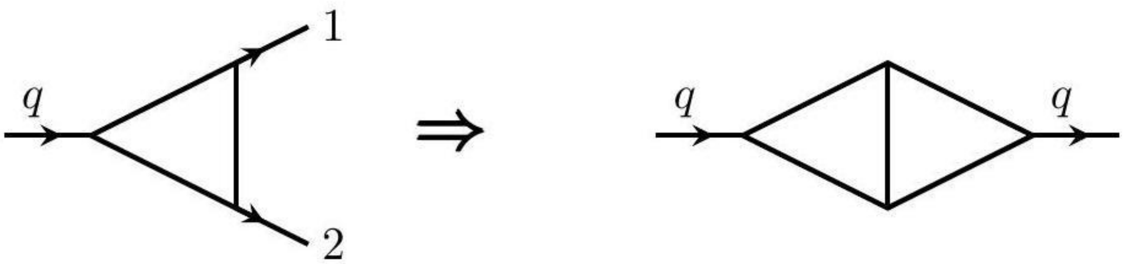


$$\sum_{\text{physical states}} \left|\hat{\mathcal{F}}_{\mathcal{O},n}(1,2,\ldots,n)\right|^2,$$

$$\mathcal{F}^{(0),\text{MHV}}_{\text{tr}(\phi^2_{AB}),n}(1,2,\ldots,n) = \frac{\delta^{(4)}\left(q - \sum_i \lambda_i \tilde{\lambda}_i\right)\delta^{(4)}_{AB}\left(\sum_i \lambda_i \eta_i\right)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle},$$

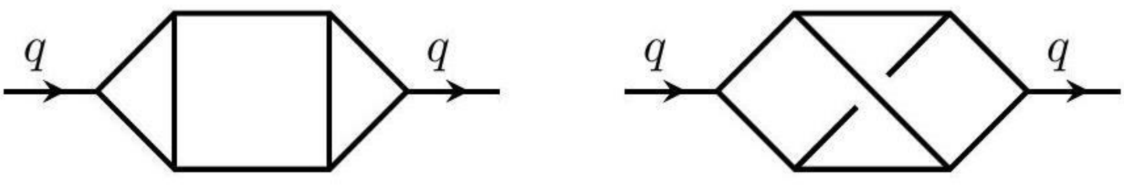
$$\Phi(p,\eta) = g_+(p) + \eta^A \bar{\psi}_A(p) + \frac{1}{2!}\eta^A\eta^B\phi_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\psi^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}\, g_-(p).$$

$$\int d^4x\, e^{iq\cdot x}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle\Big|_{n\text{-particle cut}} = \int d\text{PS}_{n\text{-particle}} \sum_{\text{physical states}} \left|\hat{\mathcal{F}}_{\mathcal{O},n}(1,2,\ldots,n)\right|^2$$



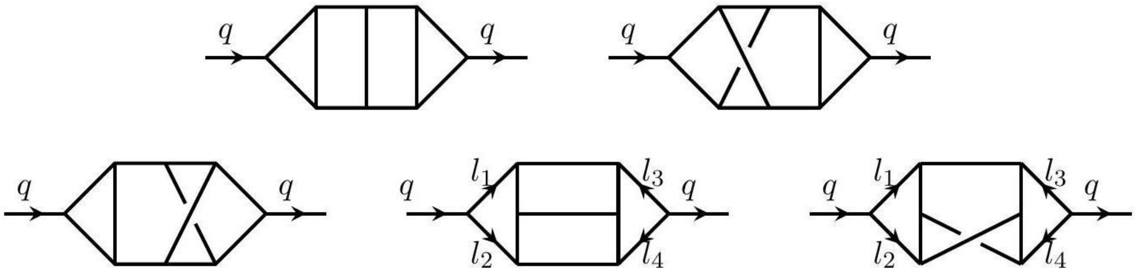
$$\sum_{\text{physical states}} \left|\mathcal{F}_{\mathcal{O},n}(1,2,3)\right|^2 = 2\frac{q^2}{s_{12}s_{23}} + 2\frac{q^2}{s_{12}s_{13}} + 2\frac{q^2}{s_{13}s_{23}},$$

$$\int \prod_{i=1}^{3} d^4\eta_i \left[\mathcal{F}^{\text{MHV}}_{\mathcal{O}_{12},3}(1,2,3)\mathcal{F}^{\text{NMHV}}_{\mathcal{O}_{34},3}(3,2,1) + \mathcal{F}^{\text{NMHV}}_{\mathcal{O}_{12},3}(1,2,3)\mathcal{F}^{\text{MHV}}_{\mathcal{O}_{34},3}(3,2,1)\right] = 2\frac{(q^2)^2}{s_{12}s_{23}s_{31}}$$



$$\frac{2(q^2)^2}{s_{12}s_{23}s_{34}s_{41}} + \left[\frac{(q^2)^2}{s_{12}s_{34}s_{123}s_{341}} + \frac{(q^2)^2}{s_{12}s_{34}s_{234}s_{412}} + \frac{2(q^2)^2}{s_{12}s_{34}s_{123}s_{234}} + \frac{2(q^2)^2}{s_{12}s_{23}s_{234}s_{412}} + \text{cyclic perm.}\right].$$

$$\int \prod_{i=1}^{4} d^4\eta_i \left[\mathcal{F}^{\text{MHV}}_{\mathcal{O}_{12},4}\mathcal{F}^{\text{N}^2\text{MHV}}_{\mathcal{O}_{34},4} + \mathcal{F}^{\text{N}^2\text{MHV}}_{\mathcal{O}_{12},4}\mathcal{F}^{\text{MHV}}_{\mathcal{O}_{34},4} + \mathcal{F}^{\text{NMHV}}_{\mathcal{O}_{12},4}\mathcal{F}^{\text{NMHV}}_{\mathcal{O}_{34},4}\right]$$



$$n_1 = n_2 = n_3 = (q^2)^3$$

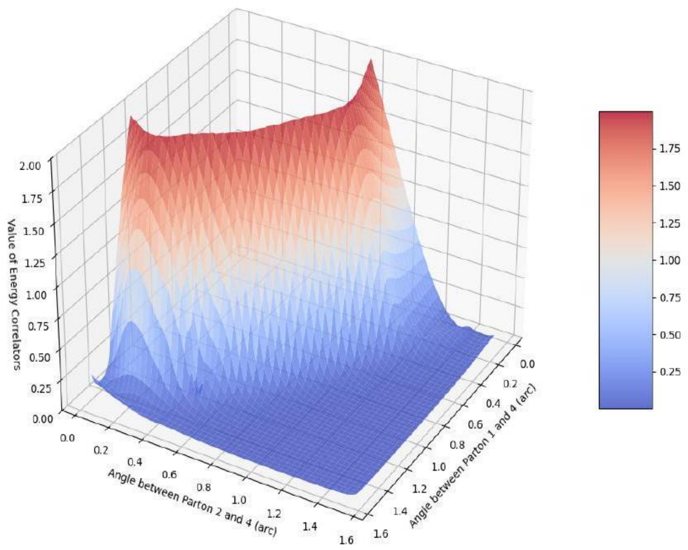
$$n_4 = -n_5 = \frac{(q^2)^2}{2}(\ell_1 \cdot \ell_3 - \ell_1 \cdot \ell_4 + \ell_2 \cdot \ell_4 - \ell_2 \cdot \ell_3 - q^2)$$

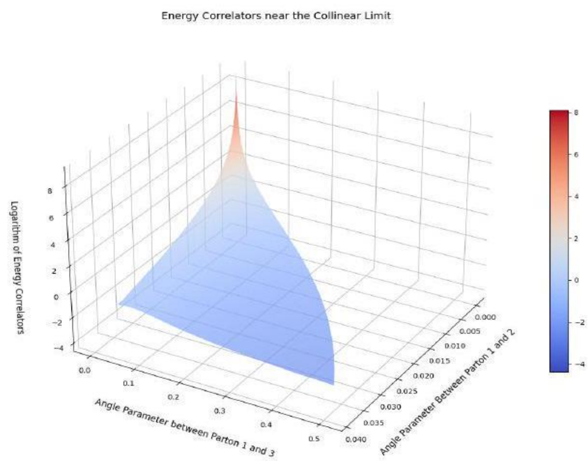
$$\int \prod_{i=1}^{5} d^4\eta_i \left[\mathcal{F}_{\mathcal{O}_{12},5}^{\mathrm{MHV}}\mathcal{F}_{\mathcal{O}_{34},5}^{\mathrm{N^3MHV}} + \mathcal{F}_{\mathcal{O}_{12},5}^{\mathrm{N^3MHV}}\mathcal{F}_{\mathcal{O}_{34},5}^{\mathrm{MHV}} + \mathcal{F}_{\mathcal{O}_{12},5}^{\mathrm{NMHV}}\mathcal{F}_{\mathcal{O}_{34},5}^{\mathrm{N^2MHV}} + \mathcal{F}_{\mathcal{O}_{12},5}^{\mathrm{N^2MHV}}\mathcal{F}_{\mathcal{O}_{34},5}^{\mathrm{NMHV}}\right]$$

$$\mathcal{F}_{\mathcal{O}_{12},n}(1,2,\ldots,n)\mathcal{F}_{\mathcal{O}_{34},n}(n,\ldots,2,1)$$

$$\zeta_{12} = \frac{z_2^2}{1+z_2^2}, \zeta_{13} = \frac{z_2^2|z_3|^2}{1+z_2^2|z_3|^2}, \zeta_{14} = \frac{z_2^2|z_4|^2}{1+z_2^2|z_4|^2}, \zeta_{23} = \frac{z_2^2|1-z_3|^2}{(1+z_2^2)(1+z_2^2|z_3|^2)}$$

$$\zeta_{24} = \frac{z_2^2|1-z_4|^2}{(1+z_2^2)(1+z_2^2|z_4|^2)}, \zeta_{34} = \frac{z_2^2|z_3-z_4|^2}{(1+z_2^2|z_3|^2)(1+z_2^2|z_4|^2)}$$





$$\frac{(q^2)^2}{s_{12}s_{13}s_{24}s_{34}}+\frac{(q^2)^2}{s_{12}s_{13}s_{124}s_{134}}+\frac{4(q^2)^2}{s_{12}s_{34}s_{124}s_{134}}+\frac{4(q^2)^2}{s_{13}s_{24}s_{123}s_{124}}+\frac{2(q^2)^2}{s_{23}s_{24}s_{123}s_{124}}+\frac{(q^2)^2}{s_{24}s_{34}s_{124}s_{134}}.$$

$$\frac{1}{\zeta_{12}\zeta_{13}}\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\frac{\delta(1-x_1-x_2-x_3+\zeta_{12}x_1x_2+\zeta_{23}x_2x_3+\zeta_{13}x_1x_3)}{(-1+x_1\zeta_{13}+x_2\zeta_{23})(-1+x_1\zeta_{12}+x_3\zeta_{23})}$$

$$\int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\frac{\delta(1-x_1-x_2-x_3+\zeta_{12}x_1x_2+\zeta_{23}x_2x_3+\zeta_{13}x_1x_3)}{(-1+x_1\zeta_{13}+x_2\zeta_{23})(-1+x_1\zeta_{12}+x_3\zeta_{23})(-1+x_3)(-1+x_2)}$$

$$\mathcal{D}_1=\frac{s_{134}}{q^2}=-1+x_2,\mathcal{D}_2=\frac{s_{124}}{q^2}=-1+x_3,\mathcal{D}_3=\frac{s_{123}}{q^2}=-1+x_1+x_2+x_3$$
$$\mathcal{D}_4=\frac{s_{34}}{q^2x_3}=-1+x_1\zeta_{13}+x_2\zeta_{23},\mathcal{D}_5=\frac{s_{24}}{q^2x_2}=-1+x_1\zeta_{12}+x_3\zeta_{23}$$

$$P=q_1^{\beta_1}\dots q_k^{\beta_k}N(x_1,\dots,x_n)$$

$$[q_1,\dots q_k]\succ[x_1,\dots,x_n]$$

$$\{x_1\to 1+(\zeta_{12}+\zeta_{13}-2)cx_1,x_2\to cx_2,x_3\to cx_3\}|_{c\to 0}$$
$$\{x_1\to cx_1,x_2\to 1+(\zeta_{12}+\zeta_{23}-2)cx_2,x_3\to cx_3\}|_{c\to 0}$$
$$\{x_1\to cx_1,x_2\to cx_2,x_3\to 1+(\zeta_{13}+\zeta_{23}-2)cx_3\}|_{c\to 0}$$

$$\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\delta(\mathcal{D}_\delta)\underset{\text{counting}}{\overset{\text{power}}{\to}}2$$

$$\{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}_4,\mathcal{D}_5\}\underset{\text{counting}}{\overset{\text{power}}{\to}}\begin{cases}\aleph_1 & \{0,0,1,0,0\}\\ \aleph_2 & \{1,0,1,0,0\}\\ \aleph_3 & \{0,1,1,0,0\}\end{cases}$$

$$I(x_0,y_0)=\int_0^{x_0}\mathrm{d}x\int_0^{y_0}\mathrm{d}y\frac{1}{x+y}=(x_0+y_0)\log(x_0+y_0)-x_0\log(x_0)-y_0\log(y_0)$$

$$I(x_0,y_0)\to\int_0^{x_0}\mathrm{d}(cx)\int_0^{y_0}\mathrm{d}(cy)\frac{1}{cx+cy}\underset{\text{counting}}{\overset{\text{power}}{\to}}1>0$$

$$D_1=-1+x_2,D_2=-1+x_3,D_3=-1+x_1+x_2+x_3,$$
$$D_1=-1+x_2,D_2=-1+x_1+x_2+x_3,D_3=-1+x_1\zeta_{12}+x_3\zeta_{23},$$
$$D_1=-1+x_3,D_2=-1+x_1\zeta_{13}+x_2\zeta_{23},D_3=-1+x_1\zeta_{12}+x_3\zeta_{23}.$$

$$\mathrm{Int}[n_1,n_2,n_3,1]=\int\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\frac{\delta(D_\delta)}{D_1^{n_1}D_2^{n_2}D_3^{n_3}}=\int\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_\delta}\bigg|_{\mathrm{cut}(D_\delta)}$$

$\mathrm{Int}[1,1,0,1],\mathrm{Int}[0,1,1,1,\{x_2\}]=\mathrm{Int}[-1,1,1,1]+\mathrm{Int}[0,1,1,1];$
$\mathrm{Int}[-1,1,1,1],\mathrm{Int}[0,1,1,1],\mathrm{Int}[1,-1,1,1],\mathrm{Int}(1,0,1,1),$
$\mathrm{Int}[0,0,1,1],\mathrm{Int}[-1,1,0,1],\mathrm{Int}[0,1,-1,1],\mathrm{Int}[0,1,0,1],$
$\mathrm{Int}[1,0,-1,1],\mathrm{Int}[1,-1,0,1],\mathrm{Int}[1,0,0,1],\mathrm{Int}[0,0,0,1];$
$\mathrm{Int}[-1,1,1,1],\mathrm{Int}[0,1,1,1],\mathrm{Int}[1,1,-1,1],\mathrm{Int}[1,1,0,1],$
$\mathrm{Int}[-1,0,1,1],\mathrm{Int}[0,0,1,1],\mathrm{Int}[-1,1,0,1],\mathrm{Int}[0,1,-1,1],\mathrm{Int}[0,1,0,1],$
$\mathrm{Int}[1,-1,0,1],\mathrm{Int}[1,0,-1,1],\mathrm{Int}[1,0,0,1],\mathrm{Int}[0,0,0,1]$

$$\mathcal{O}_{\mathrm{IBP}}=\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(a_i\,\cdot),$$

$$\sum_{i=1}^{3}a_i\frac{\partial}{\partial x_i}D_j-b_jD_j=0,j\in\mathbb{D}\mathbb{V}\cup\{\delta\}$$

$$\mathcal{O}_{\mathrm{IBP}}\mathrm{Int}[n_1,n_2,n_3,1]\equiv\int\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\left(\mathcal{O}_{\mathrm{IBP}}\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_\delta}\right)\bigg|_{\mathrm{cut}(D_\delta)}$$

$$\mathcal{O}_{\mathrm{IBP}}\mathrm{Int}[n_1,n_2,n_3,1] = \sum_{i=1}^{3} \left(\mathrm{BT}_{x_i=1} - \mathrm{BT}_{x_i=0}\right)$$

$$D_1 = -1 + x_1\zeta_{12}, D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23}, D_\delta = 1 - x_1 - x_2 + x_1x_2\zeta_{12},$$
$$D_1 = -1 + x_3\zeta_{13}, D_2 = -1 + x_1\zeta_{13}, D_\delta = 1 - x_1 - x_3 + x_1x_3\zeta_{13},$$
$$D_1 = -1 + x_3\zeta_{23}, D_2 = -1 + x_2\zeta_{23}, D_\delta = 1 - x_2 - x_3 + x_2x_3\zeta_{23}.$$

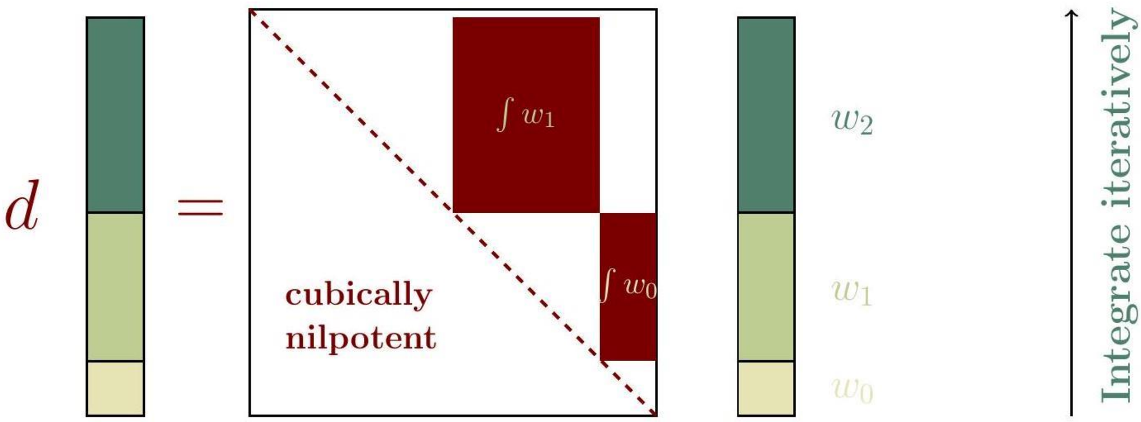
$$\mathrm{Int}_2[sf,\{n_1,n_2,1\}] = \int \mathrm{d}x_1\, \mathrm{d}\hat{x}_i\, \mathrm{d}x_3 \frac{\delta(D_\delta)}{D_1^{n_1}D_2^{n_2}}$$

$$\{\mathrm{Int}[1,1,-1,1], \mathrm{Int}[-1,1,1,1], \mathrm{Int}[1,1,0,1], \mathrm{Int}[0,1,1,1], \mathrm{Int}[1,0,0,1],$$
$$\mathrm{Int}_2[1,\{0,0,1\}], \mathrm{Int}_2[2,\{0,1,1\}], \mathrm{Int}_2[2,\{0,0,1\}], \mathrm{Int}_2[3,\{0,0,1\}],1\}$$

$$\mathcal{O}_{\partial\zeta_{**}} = \frac{\partial}{\partial\zeta_{**}} + \mathcal{O}_{\mathrm{IBP}} \equiv \frac{\partial}{\partial\zeta_{**}} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i$$

$$\frac{\partial}{\partial\zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, j \in \mathbb{D}\mathbb{v} \cup \{\delta\},$$

$$\frac{\partial}{\partial\zeta_{**}} \mathrm{Int}[n_1,n_2,n_3,1] - \mathcal{O}_{\partial\zeta_{**}} \mathrm{Int}[n_1,n_2,n_3,1] = \sum_{i=1}^{3} \left(\mathrm{BT}_{x_i=0}\right).$$



$$s,\ x_1,\ x_2,\ 1-x_1,\ 1-x_2,\ 1+s,\ s+x_1,\ s+x_2,$$
$$1+sx_1,\ s+x_1x_2,\ 1+sx_1x_2,\ 1-x_1x_2,\ 1-x_1^2x_2\,;$$
$$s,\ x_1,\ x_2,\ 1-x_1,\ 1-x_2,\ 1+s,\ s+x_1,\ s+x_2,$$
$$1+sx_1,\ s+x_1x_2,\ 1+sx_1x_2,\ 1-x_1x_2\,;$$
$$z_2,\ z_3,\ \bar{z}_3,\ z_3-z_2,\ \bar{z}_3-z_2,\ \bar{z}_3-z_3,\ z_2^2+1,\ z_2z_3+1,\ z_2\bar{z}_3+1,\ z_3\bar{z}_3+1\,.$$

$$\delta(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_1x_4\zeta_{14} + x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}).$$

$$\mathrm{E}^4\mathrm{C}(p_1,p_2,p_3,p_4) = \sum_{\sigma\in S_4} \sigma\mathcal{E}^4\mathcal{C}(p_1,p_2,p_3,p_4)$$

$$\frac{s_{45}}{x_4} = 1 - x_1\zeta_{14} - x_2\zeta_{24} - x_3\zeta_{34}, \frac{s_{15}}{x_1} = 1 - x_2\zeta_{12} - x_3\zeta_{13} - x_4\zeta_{14}, s_{2345} = 1 - x_1$$
$$s_{1345} = 1 - x_2, s_{1235} = 1 - x_4, s_{1234} = 1 - x_1 - x_2 - x_3 - x_4$$
$$s_{345} = 1 - x_1 - x_2 + x_1x_2\zeta_{12}, s_{145} = 1 - x_2 - x_3 + x_2x_3\zeta_{23}, s_{125} = 1 - x_3 - x_4 + x_3x_4\zeta_{34}$$
$$s_{123} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}, s_{234} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}, s_{1245} = 1 - x_3$$

$$\{x_1 \to cx_1, x_2 \to cx_2, x_3 \to cx_3, x_4 \to 1 + (\zeta_{14} + \zeta_{24} + \zeta_{34} - 3)cx_4\}$$
$$\{x_1 \to cx_1, x_2 \to cx_2, x_3 \to 1 + (\zeta_{13} + \zeta_{23} + \zeta_{34} - 3)cx_3, x_4 \to cx_4\}$$
$$\{x_1 \to cx_1, x_2 \to 1 + (\zeta_{12} + \zeta_{23} + \zeta_{24} - 3)cx_2, x_3 \to cx_3, x_4 \to cx_4\}$$
$$\{x_1 \to 1 + (\zeta_{12} + \zeta_{13} + \zeta_{14} - 3)cx_1, x_2 \to cx_2, x_3 \to cx_3, x_4 \to cx_4\}$$

$$\{0,0,1,0,0,1,1,0,0,1,2,0\}$$
$$\{0,0,0,1,0,1,1,1,0,1,1,1\}$$
$$\{0,0,0,0,0,1,0,1,1,1,1,0\}$$
$$\{0,0,0,0,1,1,0,0,1,2,1,0\}$$

$$dx_1 dx_2 dx_3 dx_4 \delta(D_\delta) \underset{\text{counting}}{\overset{\text{power}}{\to}} \gimel$$

$$\mathrm{Int}[\{n_1, n_2, \cdots, n_{11}, 1\}, N] = \int_0^1 \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_4 N \delta(D_\delta)}{D_1^{n_1} D_2^{n_2} \cdots D_{11}^{n_{11}} D_{12}^{n_{12}}},$$

$$\begin{aligned}\int \frac{x_2 \delta(D_\delta)}{D_1 D_5 D_{10}} &= -\frac{(z_2^2+1)(z_3(-\overline{z_3}) + z_3\overline{z_4} + z_4 z_3 \overline{z_3}\overline{z_4} + z_4\overline{z_3})}{z_2(z_3\overline{z_3}+1)(z_2 z_4 \overline{z_4} + \overline{z_4} - z_2 + z_4)} \int \frac{x_3 \delta(D_\delta)}{D_1 D_5 D_{10}} \\ &+ \frac{(z_2^2+1) z_4 \overline{z_4}}{z_2(z_2 z_4 \overline{z_4} + \overline{z_4} - z_2 + z_4)} \int \frac{\delta(D_\delta)}{D_{10}} + \frac{(z_2^2+1) z_4 \overline{z_4}}{z_2(z_2 z_4 \overline{z_4} + \overline{z_4} - z_2 + z_4)} \int \frac{\delta(D_\delta)}{D_1 D_5} \\ &- \frac{(z_2^2+1)}{z_2(z_2 z_4 \overline{z_4} + \overline{z_4} - z_2 + z_4)} \int \frac{\delta(D_\delta)}{D_5 D_{10}} - \frac{(z_2^2+1)}{z_2(z_2 z_4 \overline{z_4} + \overline{z_4} - z_2 + z_4)} \int \frac{\delta(D_\delta)}{D_1 D_5 D_{10}}\end{aligned}$$

$$D_1 = -1 + x_1\zeta_{14} + x_2\zeta_{24} + x_3\zeta_{34}, D_2 = -1 + x_2\zeta_{12} + x_3\zeta_{13} + x_4\zeta_{14}$$
$$D_3 = -1 + x_1, D_4 = -1 + x_2, D_5 = -1 + x_4, D_6 = -1 + x_1 + x_2 + x_3 + x_4$$
$$D_7 = -1 + x_1 + x_2 - x_1x_2\zeta_{12}, D_8 = -1 + x_2 + x_3 - x_2x_3\zeta_{23}, D_9 = -1 + x_3 + x_4 - x_3x_4\zeta_{34}$$
$$D_{10} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}, D_{11} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}$$

$$\begin{aligned}\mathcal{O}_{\mathrm{IBP}}\,\mathrm{Int}[\{n_1, n_2, \cdots, n_{11}, 1\}, N] &\equiv \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_4 \left(\mathcal{O}_{\mathrm{IBP}} \frac{N}{D_1^{n_1} D_2^{n_2} \cdots D_{11}^{n_{11}} D_\delta}\right)\Bigg|_{\mathrm{cut}\,(D_\delta)} \\ &= \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_4 \frac{P}{D_1^{n_1'} D_2^{n_2'} \cdots D_{11}^{n_{11}} D_\delta}\Bigg|_{\mathrm{cut}(D_\delta)}\end{aligned}$$

$$\mathrm{Int}[\{n_1, n_2, \dots, n_{11}, 1\}, N] = \sum_{m \in \text{ monomials in } D_i} c_i \mathrm{Int}[\{n_1, \cdots, n_i + 1, \cdots, n_{11}, 1\}, mN], i = 1,2$$
$$0 = \sum_{m \in \text{ monomials in } D_\delta} c_i \mathrm{Int}[\{n_1, n_2, \dots, n_{11}, 1\}, mN]$$

$$\mathrm{Int}[\{2,0,0,0,1,0,0,0,0,1,0,1\}, x_3^3], \mathrm{Int}[\{2,0,2,0,0,0,0,0,0,1,0,1\}, x_3x_4^3], \cdots$$

$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,0,1\},1], \mathrm{Int}[\{1,0,0,0,1,0,0,0,0,0,0,1\},1],$$
$$\mathrm{Int}[\{1,0,0,0,1,0,0,0,0,1,0,1\},x_1], \mathrm{Int}[\{1,0,0,0,1,0,0,0,0,1,0,1\},x_2];$$
$$\mathrm{Int}[\{0,0,0,0,0,1,0,0,1,0,0,1\},1], \mathrm{Int}[\{0,0,0,0,0,1,0,0,1,0,0,1\},x_1],$$
$$\mathrm{Int}[\{0,0,0,0,0,1,0,0,1,0,0,1\},x_2], \mathrm{Int}[\{0,0,0,0,0,1,0,0,1,0,0,1\},x_3],$$
$$\mathrm{Int}[\{0,0,0,0,0,1,0,0,1,0,0,1\},x_1^2];$$
$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_2], \mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_3],$$
$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_2^2], \mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_3^2],$$
$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_1x_2], \mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_1x_3],$$
$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_1^2x_2], \mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_1x_2^2],$$
$$\mathrm{Int}[\{0,0,0,0,0,0,0,0,0,1,1,1\},x_2x_4].$$

$$\int \frac{N(x)\mathrm{d}x}{\sqrt{P(x)}}$$

$$\frac{\delta(D_\delta)}{D_7 D_9} = \frac{\delta(1 - x_{1234} - s_{1234})}{s_{125}s_{345}}.$$

$$\{1 - x_{1234} - s_{1234} = 0, s_{125} = 0, s_{345} = 0\}$$

$$x_1 = \frac{\cdots + \sqrt{(\zeta_{14}^2\zeta_{34}^2 - 2\zeta_{14}\zeta_{24}\zeta_{34}^2 + 4\zeta_{12}\zeta_{14}\zeta_{24}\zeta_{34}^2 + \zeta_{24}^2\zeta_{34}^2)x_4^4 + \cdots}}{2(-\zeta_{12}\zeta_{13} + x_4\zeta_{12}\zeta_{13} - x_4\zeta_{12}\zeta_{14} + x_4^2\zeta_{12}\zeta_{14}\zeta_{34})}$$

$$\int \frac{N(x_4)\mathrm{d}x_4}{\sqrt{(\zeta_{14}^2\zeta_{34}^2 - 2\zeta_{14}\zeta_{24}\zeta_{34}^2 + 4\zeta_{12}\zeta_{14}\zeta_{24}\zeta_{34}^2 + \zeta_{24}^2\zeta_{34}^2)x_4^4 + \cdots}},$$

$$\frac{\delta(D_\delta)}{D_1 D_7 D_9} = \frac{\delta(1 - x_{1234} - s_{1234})x_4}{s_{45}s_{125}s_{345}}$$

$$\{1 - x_{1234} - s_{1234} = 0, \quad s_{125} = 0, \quad s_{345} = 0\}$$
$$\{1 - x_{1234} - s_{1234} = 0, \quad s_{345} = 0, \quad s_{45} = 0\}$$
$$\{1 - x_{1234} - s_{1234} = 0, \quad s_{125} = 0, \quad s_{45} = 0\}$$

$$\vec{f}(x) = (f_1(x), f_2(x), \dots, f_n(x))^T,$$

$$\frac{d\vec{f}(x)}{dx} = \mathbf{A}(x)\vec{f}(x)$$

$$f_1'(x) = \sum_{i=1}^{n} \mathbf{A}_{1i}(x) f_i(x)$$
$$= \alpha_{(1,1)}(x)f_1(x) + \alpha_{(1,2)}(x)f_2(x) + \cdots \alpha_{(1,n)}(x)f_n(x)$$

$$f_1''(x) = \frac{d}{dx}f_1'(x) = \sum_{j=1}^{n} \left(\alpha_{(1,j)}'(x)f_j(x) + \alpha_{(1,j)}(x)f_j'(x)\right)$$

$$f_1^{(m)}(x) = \alpha_{(m,1)}(x)f_1(x) + \alpha_{(m,2)}(x)f_2(x) + \cdots \alpha_{(m,n)}(x)f_n(x)$$

$$\vec{g}(x) = \begin{pmatrix} f_1'(x) \\ f_1''(x) \\ \vdots \\ f_1^{(r)}(x) \end{pmatrix} = \mathbf{B}(x)\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} = \mathbf{B}(x)\vec{f}(x)$$

$$f_1(x) = \sum_{i=1}^{n} \mathbf{B}_{1i}^{-1}(x) g_i(x)$$

$$f_1^{(r)}(x) - \sum_{k=0}^{r-1} C_k(x) f_1^{(k)}(x) = 0$$

$$\hat{\mathcal{P}} = 1 - \sum_{i=0}^{n-1} C_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}$$

$$\mathbf{N}_{k\times n}(x)\vec{g}_{n\times 1}(x) = \mathbf{N}(x)\mathbf{B}(x)\vec{f}(x) = \vec{0}_k$$

$$\hat{\mathcal{P}} = \sum_{i=1}^{n} \mathbf{N}_{li}(x)\frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}, \forall l \in \{1,2,\cdots,k\}$$

$$\vec{f}(z_2) = \left(\frac{\delta(D_\delta)}{D_7 D_9}, \frac{x_1\delta(D_\delta)}{D_7 D_9}, \frac{x_1^2\delta(D_\delta)}{D_7 D_9}\right)^T$$

$$\frac{d\vec{f}(z_2)}{dz_2} = \mathbf{A}(z_2)\vec{f}(z_2)$$

$$f_2(z_2) = \frac{x_1\delta(D_\delta)}{D_7D_9}$$

$$\frac{df_2(z_2)}{dz_2} = \mathbf{A}_2(z_2)\vec{f}(z_2)$$

$$\begin{pmatrix} f_2'(z_2) \\ f_2''(z_2) \\ f_2^{(3)}(z_2) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_2 \\ \mathbf{A}_2' + \mathbf{A}_2 \cdot \mathbf{A} \\ \mathbf{A}_2'' + 2(\mathbf{A}_2' \cdot \mathbf{A}) + (\mathbf{A}_2 \cdot \mathbf{A}') + (\mathbf{A}_2 \cdot \mathbf{A} \cdot \mathbf{A}) \end{pmatrix} \begin{pmatrix} f_1(z_2) \\ f_2(z_2) \\ f_3(z_2) \end{pmatrix} \triangleq \mathbf{B}(z_2)\vec{f}(z_2)$$

$$f_2(z_2) - \mathbf{B}_2^{-1}(z_2) \begin{pmatrix} f_2'(z_2) \\ f_2''(z_2) \\ f_2^{(3)}(z_2) \end{pmatrix} = 0$$

$$\hat{\mathcal{P}}_{f_2} = 1 - \mathbf{B}_2^{-1} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}z_2}, \frac{\mathrm{d}^2}{\mathrm{d}z_2^2}, \frac{\mathrm{d}^3}{\mathrm{d}z_2^3} \right)^{\mathbf{T}}.$$

$$\hat{\mathcal{P}}_{f_2} = \left(c_1 \frac{\mathrm{d}^2}{\mathrm{d}z_2^2} + c_2 \frac{\mathrm{d}}{\mathrm{d}z_2} + c_3 \right) \left(\frac{\mathrm{d}}{\mathrm{d}z_2} + c_4 \right).$$

$$\vec{f}(z_2) = \left(\frac{x_2\delta(D_\delta)}{D_6D_{11}}, \frac{x_3\delta(D_\delta)}{D_6D_{11}}, \frac{x_4\delta(D_\delta)}{D_6D_{11}} \right)^T.$$

$$\left(c_1 \frac{\mathrm{d}}{\mathrm{d}z_2} + c_2 \right) \left(\frac{\mathrm{d}}{\mathrm{d}z_2} + c_3 \right) \left(\frac{\mathrm{d}}{\mathrm{d}z_2} + c_4 \right).$$

Propagators	Count	Numerators	Type of Functions
D_{10}	2	$1, x_1$	MPL
D_9	3	$1, x_1, x_2$	MPL
D_3	1	1	MPL
D_8, D_{11}	6	$1, x_1, x_2, x_2^2, x_3^2, x_2x_4$	MPL
D_4, D_{10}	3	$1, x_1, x_3$	MPL
D_4, D_5	1	1	MPL
D_5, D_8	3	$1, x_2, x_3$	MPL
D_6, D_{11}	3	x_2, x_3, x_4	MPL
D_3, D_6	1	1	MPL
D_6, D_9	5	$1, x_1, x_1^2, x_2, x_3$	Elliptic
D_7, D_9	9	$1, x_1, x_1^2, x_2, x_3, x_1x_3, x_2x_3, x_4, x_1x_4$	Elliptic
D_8, D_9	4	$1, x_1, x_2, x_4$	MPL
D_2, D_6	1	1	MPL
D_{10}, D_{11}	9	$x_2, x_1x_2, x_1^2x_2, x_2^2, x_1x_2^2, x_3, x_1x_3, x_3^2, x_2x_4$	Hyperelliptic $g = 2$
D_9, D_{10}	5	$x_1, x_2, x_3, x_1x_4, x_3x_4$	MPL
D_2, D_{10}	3	$1, x_1, x_2$	MPL
D_1, D_5	1	1	MPL
D_1, D_9	3	$1, x_1, x_2$	MPL

Propagators	Count	Numerators	Type of Functions
D_1, D_2	1	1	MPL
D_5, D_6, D_{11}	3	x_2, x_1x_2, x_3	MPL
D_2, D_5, D_6	2	$1, x_1$	MPL
D_5, D_{10}, D_{11}	2	x_1x_2, x_2^2	Hyperelliptic $g = 2$
D_4, D_6, D_{10}	2	x_1, x_2	MPL
D_5, D_8, D_{10}	4	x_1, x_1^2, x_2, x_1x_2	MPL
D_6, D_7, D_{10}	6	$x_1, x_1^2, x_1^3, x_1x_2, x_1x_3, x_2x_4$	Elliptic
D_6, D_9, D_{10}	3	x_1^2, x_2^2, x_1x_2	Elliptic
D_2, D_4, D_{10}	2	$1, x_1$	MPL
D_2, D_5, D_{10}	2	x_1, x_2	MPL
D_2, D_9, D_{10}	4	x_1, x_2, x_3, x_3^2	MPL
D_1, D_8, D_{11}	4	$1, x_1, x_2, x_4$	MPL
D_1, D_4, D_5	2	$1, x_1$	MPL
D_1, D_5, D_8	4	$1, x_1, x_1^2, x_2$	MPL
D_1, D_6, D_{11}	5	$x_2, x_1x_2, x_1^2x_2, x_3, x_4$	MPL
D_1, D_6, D_8	4	$1, x_1, x_1^2, x_2$	Elliptic
D_1, D_4, D_9	2	$1, x_1$	MPL
D_1, D_7, D_9	4	$1, x_1, x_2, x_4$	Elliptic
D_1, D_8, D_9	4	$1, x_1, x_2, x_4$	MPL
D_1, D_2, D_4	2	$1, x_1$	MPL
D_1, D_2, D_5	2	$1, x_1$	MPL
D_1, D_2, D_9	4	$1, x_1, x_1^2, x_2$	MPL
D_1, D_{10}, D_{11}	3	x_2, x_3, x_1x_4	Hyperelliptic g=2
D_1, D_5, D_{10}	2	x_1, x_2	MPL
D_1, D_9, D_{10}	3	x_1, x_2, x_3	MPL
D_1, D_2, D_{10}	2	$1, x_1$	MPL

$$s, x_1, x_2, 1 - x_1, 1 - x_2, 1 + s, 1 - s, s + x_1, s + x_2, 1 + sx_1, 1 + sx_2, x_1 - x_2, s + x_1x_2,$$
$$1 - x_1x_2, 1 + sx_1x_2, 1 - x_1^2x_2, 1 - x_1x_2^2, sx_1 + sx_2 + x_1x_2 - 2sx_1x_2 + s^2x_1x_2 + sx_1^2x_2 + sx_1x_2^2$$

$$-1 + \zeta_{12} + \zeta_{13}, \zeta_{12} - \zeta_{13}, \zeta_{12}^2 - 2\zeta_{12}\zeta_{13} + \zeta_{13}^2 - 2\zeta_{12}\zeta_{23} - 2\zeta_{13}\zeta_{23} + 4\zeta_{12}\zeta_{13}\zeta_{23} + \zeta_{23}^2$$

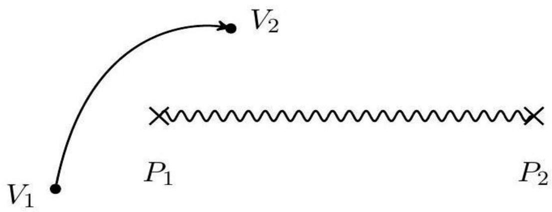
$$\frac{\delta(D_\delta)}{D_8D_9}, \frac{\delta(D_\delta)}{D_4D_9}, \frac{\delta(D_\delta)}{D_5D_8},$$

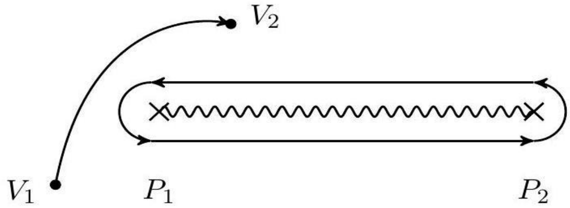
$$I(\zeta_{ij}) = \int_0^1 \mathrm{d}x_k \frac{\sum_S S_1(x_k; \zeta_{ij}) \otimes S_2(x_k; \zeta_{ij})}{P(x_k)}$$

$$\mathcal{S}(\mathrm{Disc}(\mathcal{I})) = S_2 \otimes \cdots \otimes S_n,$$

$$\mathcal{S}(\mathcal{I}) = S_1 \otimes S_2 \otimes \cdots \otimes S_n$$

$$I = \int_0^\infty \frac{(r_2 - r_1)\mathrm{d}x}{(x - r_1)(x - r_2)} = \log\left(\frac{r_1}{r_2}\right)$$





$$\mathcal{S}\left(\mathrm{Disc}\left(I(\zeta_{ij})\right)\right) = \int_C \frac{\otimes S_2(x_k;\zeta_{ij})\mathrm{d}x_k}{P(x_k)}$$

$$I = \int_a^b \frac{A_1(t) \otimes A_2(t) \ldots \otimes A_n(t)}{t-c}\,\mathrm{d}t,$$

$$\begin{aligned}\mathcal{S}(I) =& \frac{c-b}{c-a} \otimes A_1(c) \otimes A_2(c) \ldots \otimes A_n(c)\\ &+\sum_e A(b) \otimes \left(\int_e^b \frac{1}{t-c} A_2(t) \otimes A_3(t) \ldots \otimes A_n(t)\right)\\ &+\sum_e A(a) \otimes \left(\int_a^e \frac{1}{t-c} A_2(t) \otimes A_3(t) \ldots \otimes A_n(t)\right),\end{aligned}$$

$$I_{\text{e.g.}} = \int_a^b \frac{1}{t-c}\log(t-d)\mathrm{d}t$$

$$\begin{aligned}\mathcal{S}(I_{\text{e.g.}}) &= \frac{c-b}{c-a} \otimes (c-d) + (d-b) \otimes \left(\int_d^b \frac{\mathrm{d}t}{t-c}\right) + (d-a) \otimes \left(\int_a^d \frac{\mathrm{d}t}{t-c}\right)\\ &= \frac{c-b}{c-a} \otimes (c-d) + (d-b) \otimes \frac{c-b}{c-d} + (d-a) \otimes \frac{c-d}{c-a}\end{aligned}$$

$$I_{\mathrm{MI}} = \int_a^b \frac{1}{t-c}(t-d) \otimes (t-e)\mathrm{d}t$$

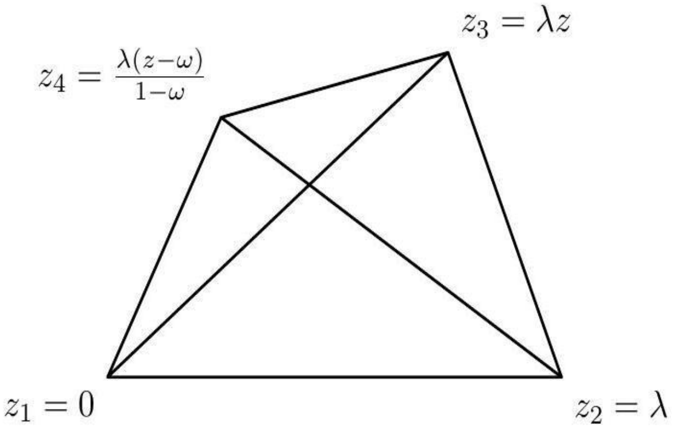
$$\mathcal{S}(I_{\mathrm{MI}}) = \frac{c-b}{c-a} \otimes (a-d) \otimes (a-e)$$

$$+(d-b) \otimes \frac{c-b}{c-d} \otimes (c-e) + (d-b) \otimes (e-b) \otimes \frac{c-b}{c-e} + (d-b) \otimes (e-d) \otimes \frac{c-e}{c-d}$$

$$+(d-a) \otimes \frac{c-d}{c-a} \otimes (c-e) + (d-a) \otimes (e-d) \otimes \frac{c-d}{c-e} + (d-a) \otimes (e-a) \otimes \frac{c-e}{c-a}$$

$$\sum_n I_n^{\infty} = \sum_k I_k^{\text{finite}}$$

$$(z_1, z_2, z_3, z_4) = \left(0, \lambda, \lambda z, \frac{\lambda(z-\omega)}{1-\omega}\right)$$



$\lambda, z, \bar{z}, w - z, \bar{z} - z, -1 + z, \bar{w} - \bar{z}, -1 + \bar{z}, 1 + \lambda^2, w - \bar{w}z, \bar{w} - \bar{z}w, -1 + \bar{z}z, 1 + \lambda^2 z, 1 + \bar{z}\lambda^2, \bar{z}w - \bar{w}z, \bar{w}w - \bar{z}z, 1 +$
$\bar{z}\lambda^2 z, -1 + \bar{z}\lambda^4 z, \bar{w}w - \bar{z}w - \bar{w}z, -1 + w + \lambda^2 w - \lambda^2 z, -1 + \bar{z} + z + \bar{z}\lambda^2 z, \bar{z} + z - \bar{z}z + \bar{z}\lambda^2 z, -1 + \bar{w} + \bar{w}\lambda^2 -$
$\bar{z}\lambda^2, -1 + w + \lambda^2 w - \bar{z}\lambda^2 z, -1 + \bar{w} + \bar{w}\lambda^2 - \bar{z}\lambda^2 z, \bar{w} - w - \bar{z}\lambda^2 w + \bar{w}\lambda^2 z, -1 + w + \bar{z}\lambda^2 w - \bar{z}\lambda^2 z, \bar{w} - z + \bar{w}\lambda^2 z -$
$\bar{z}\lambda^2 z, -\bar{z} + w + \bar{z}\lambda^2 w - \bar{z}\lambda^2 z, -1 + \bar{w} + \bar{w}\lambda^2 z - \bar{z}\lambda^2 z, -\bar{w} + \bar{z} + w - \bar{z}w - z + \bar{w}z, 1 - \bar{w}w - \bar{w}\lambda^2 w + \bar{z}\lambda^2 z, 1 - \bar{w}z -$
$\bar{w}\lambda^2 z + \bar{z}\lambda^2 z, 1 - \bar{w} - w + \bar{z}w + \bar{w}z - \bar{z}z, -1 + \bar{z}w + \bar{z}\lambda^2 w - \bar{z}\lambda^2 z, -1 + w + \lambda^2 w - 2\lambda^2 z + \lambda^2 z^2, -1 + \bar{w} + \bar{w}\lambda^2 -$
$2\bar{z}\lambda^2 + \bar{z}^2\lambda^2, -\lambda^2 w - z + wz + 2\lambda^2 wz - \lambda^2 z^2, \bar{z} - \bar{w}\bar{z} + \bar{w}\lambda^2 - 2\bar{w}\bar{z}\lambda^2 + \bar{z}^2\lambda^2, -w + \bar{w}w - \bar{w}z + wz + \bar{w}\lambda^2 wz -$
$\bar{w}\lambda^2 z^2, -\bar{z}\lambda^2 w - z + wz + 2\bar{z}\lambda^2 wz - \bar{z}\lambda^2 z^2, \bar{z} - \bar{w}\bar{z} + \bar{w}\lambda^2 z - 2\bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z, -\bar{w} + \bar{w}\bar{z} + \bar{w}w - \bar{z}w + \bar{w}\bar{z}\lambda^2 w -$
$\bar{z}^2\lambda^2 w, -\bar{z}w + \bar{w}\bar{z}w + \bar{w}z - \bar{w}\bar{z}z - \bar{w}wz + \bar{z}wz, -\bar{w}w + \bar{z}w + \bar{w}z - \bar{w}\bar{z}z - \bar{z}wz + \bar{w}\bar{z}wz,$
$1 - \bar{w} - w + \bar{w}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w - \bar{w}\lambda^2 z + \bar{z}\lambda^2 z,$
$1 - \bar{w} - w + \bar{w}w - \bar{w}\lambda^4 w + \bar{z}\lambda^4 w + \bar{w}\lambda^4 z - \bar{z}\lambda^4 z,$
$\bar{z} - \bar{z}w - \bar{z}\lambda^2 w - z + \bar{z}\lambda^2 z + wz + \bar{z}\lambda^2 wz - \bar{z}\lambda^2 z^2,$
$-\bar{z}w + \bar{z}w^2 + \bar{z}\lambda^2 w^2 + z - wz - 2\bar{z}\lambda^2 wz + \bar{z}\lambda^2 z^2,$
$1 - w - \lambda^2 w - \bar{z}\lambda^2 w - \bar{z}\lambda^4 w + \lambda^2 z + \bar{z}\lambda^2 z + \bar{z}\lambda^4 z^2,$
$\bar{z} - \bar{w}\bar{z} - \bar{w}z + \bar{w}^2 z + \bar{w}^2\lambda^2 z - 2\bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z,$
$\bar{z} - \bar{w}\bar{z} - z + \bar{w}z + \bar{w}\lambda^2 z - \bar{z}\lambda^2 z - \bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z,$
$-1 + \bar{w} + \bar{w}\lambda^2 - \bar{z}\lambda^2 + \bar{w}\lambda^2 z - \bar{z}\lambda^2 z + \bar{w}\lambda^4 z - \bar{z}^2\lambda^4 z, -\bar{w} + \bar{z} - w + \bar{w}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w + z - \bar{z}z - \bar{w}\lambda^2 z + \bar{z}\lambda^2 z,$
$1 - \bar{z} - \bar{w}w + \bar{z}w - \bar{w}\lambda^2 w + \bar{z}\lambda^2 w - z + \bar{w}z + \bar{w}\lambda^2 z - \bar{z}\lambda^2 z, \bar{z}w - \bar{w}\bar{z}w + \bar{w}z - \bar{w}wz - 2\bar{w}\bar{z}\lambda^2 wz + \bar{z}^2\lambda^2 wz + \bar{w}\bar{z}\lambda^2 z^2,$
$-\bar{z}^2 + 2\bar{z}z + 4\bar{z}\lambda^2 z - 4\bar{z}^2\lambda^2 z - z^2 - 4\bar{z}\lambda^2 z^2 + 4\bar{z}^2\lambda^2 z^2, -1 + \bar{w} + w - \bar{w}w + \bar{w}\bar{z}\lambda^4 wz - \bar{z}^2\lambda^4 wz - \bar{w}\bar{z}\lambda^4 z^2 + \bar{z}^2\lambda^4 z^2,$
$\bar{z}w - \bar{w}\bar{z}w - \bar{w}z + 2\bar{w}\bar{z}z - \bar{z}^2 z + \bar{w}wz - 2\bar{z}wz + \bar{z}^2 wz + \bar{z}z^2 - \bar{w}\bar{z}z^2, -\bar{w} + \bar{z} + w - \bar{z}w + \bar{z}\lambda^2 w - \bar{z}^2\lambda^2 w - z + \bar{w}z -$
$\bar{w}\lambda^2 z + \bar{z}^2\lambda^2 z + \bar{w}\lambda^2 z^2 - \bar{z}\lambda^2 z^2, \bar{z}w - \bar{w}\bar{z}w + \bar{w}z - \bar{z}z - \bar{w}wz + \bar{w}\bar{z}wz - \bar{w}\bar{z}\lambda^2 wz + \bar{z}^2\lambda^2 wz + \bar{w}\bar{z}\lambda^2 z^2 - \bar{z}^2\lambda^2 z^2,$
$-\bar{w}w + \bar{w}\bar{z}w + \bar{z}z - \bar{w}\bar{z}z + \bar{w}wz - \bar{z}wz + \bar{w}\bar{z}\lambda^2 wz - \bar{z}^2\lambda^2 wz - \bar{w}\bar{z}\lambda^2 z^2 + \bar{z}^2\lambda^2 z^2, -w + \bar{w}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w - \bar{w}z -$
$\bar{w}\lambda^2 z + wz + \bar{w}\lambda^2 wz + \bar{z}\lambda^2 wz + \bar{w}\lambda^4 wz - \bar{w}\lambda^2 z^2 - \bar{w}\lambda^4 z^2, -\bar{w} + \bar{w}\bar{z} + \bar{w}w - \bar{z}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w + \bar{w}\bar{z}\lambda^2 w -$
$\bar{z}^2\lambda^2 w + \bar{w}\bar{z}\lambda^4 w - \bar{z}^2\lambda^4 w - \bar{w}\lambda^2 z + \bar{w}\bar{z}\lambda^2 z, -\bar{z}w + \bar{w}\bar{z}w + \bar{w}\bar{z}\lambda^2 w - \bar{z}^2\lambda^2 w + \bar{w}z - \bar{w}\bar{z}z - \bar{w}wz + \bar{z}wz - \bar{w}\lambda^2 wz +$
$\bar{z}^2\lambda^2 wz + \bar{w}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^2 z^2, \bar{w}^2 w - \bar{z}w - \bar{w}w^2 + \bar{z}w^2 - \bar{w}\bar{z}\lambda^2 w^2 + \bar{z}^2\lambda^2 w^2 + \bar{w}z - \bar{w}^2 z + \bar{w}^2\lambda^2 wz - \bar{z}^2\lambda^2 wz -$
$\bar{w}^2\lambda^2 z^2 + \bar{w}\bar{z}\lambda^2 z^2, -\bar{w}w + \bar{z}w + \bar{w}w^2 - \bar{z}w^2 + \bar{w}\lambda^2 w^2 - \bar{z}\lambda^2 w^2 - z + \bar{w}z + wz - \bar{w}wz - 2\bar{w}\lambda^2 wz + 2\bar{z}\lambda^2 wz +$
$\bar{w}\lambda^2 z^2 - \bar{z}\lambda^2 z^2, -\bar{z} + \bar{w}\bar{z} - \bar{w}w + \bar{w}^2 w + \bar{z}w - \bar{w}\bar{z}w + \bar{w}^2\lambda^2 w - 2\bar{w}\bar{z}\lambda^2 w + \bar{z}^2\lambda^2 w + \bar{w}z - \bar{w}^2 z - \bar{w}^2\lambda^2 z + 2\bar{w}\bar{z}\lambda^2 z -$
$\bar{z}^2\lambda^2 z, \bar{w}^2 - 2\bar{w}w - 4\bar{w}\lambda^2 w + 2\bar{w}\bar{z}\lambda^2 w + w^2 + 2\bar{z}\lambda^2 w^2 + \bar{z}^2\lambda^4 w^2 + 2\bar{w}^2\lambda^2 z + 2\bar{w}\lambda^2 wz - 4\bar{w}\bar{z}\lambda^2 wz - 2\bar{w}\bar{z}\lambda^4 wz +$
$\bar{w}^2\lambda^4 z^2, -w + \bar{w}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w + z - \bar{w}z + \lambda^2 z - \bar{w}\lambda^2 z - \lambda^2 wz + \bar{w}\lambda^2 wz + \bar{w}\lambda^4 wz - \bar{z}\lambda^4 wz - \bar{w}\lambda^2 z^2 + \bar{z}\lambda^2 z^2 -$
$\bar{w}\lambda^4 z^2 + \bar{z}\lambda^4 z^2, 1 - \bar{w} - w + \bar{w}w - \bar{w}\lambda^4 w + \bar{z}\lambda^4 w + \lambda^2 z - 2\bar{w}\lambda^2 z + \bar{z}\lambda^2 z - \lambda^2 wz + 2\bar{w}\lambda^2 wz - \bar{z}\lambda^2 wz + 2\bar{w}\lambda^4 wz -$
$2\bar{z}\lambda^4 wz - \bar{w}\lambda^4 z^2 + \bar{z}\lambda^4 z^2, -\bar{w} + \bar{z} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 + \bar{w}w - \bar{z}w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w + \bar{w}\bar{z}\lambda^2 w - \bar{z}^2\lambda^2 w + \bar{w}\bar{z}\lambda^4 w - \bar{z}^2\lambda^4 w -$
$\bar{w}\lambda^2 z + \bar{z}^2\lambda^2 z - \bar{w}\bar{z}\lambda^4 z + \bar{z}^2\lambda^4 z, 1 - \bar{w} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 - w + \bar{w}w - 2\bar{z}\lambda^2 w + 2\bar{w}\bar{z}\lambda^2 w - \bar{w}\lambda^4 w + 2\bar{w}\bar{z}\lambda^4 w - \bar{z}^2\lambda^4 w +$
$\bar{z}\lambda^2 z - \bar{w}\bar{z}\lambda^2 z + \bar{w}\lambda^4 z - 2\bar{w}\bar{z}\lambda^4 z + \bar{z}^2\lambda^4 z, -\bar{w} + \bar{z} - w + \bar{w}w - \bar{z}\lambda^2 w + \bar{w}\bar{z}\lambda^2 w + z - \bar{z}z - \bar{w}\lambda^2 z + 2\bar{z}\lambda^2 z - \bar{w}\bar{z}\lambda^2 z +$
$\bar{w}\lambda^2 wz - \bar{z}\lambda^2 wz + \bar{w}\bar{z}\lambda^4 wz - \bar{z}^2\lambda^4 wz - \bar{w}\bar{z}\lambda^4 z^2 + \bar{z}^2\lambda^4 z^2, -\bar{z} + \bar{w}\bar{z} + \bar{z}w - \bar{w}\bar{z}w + \bar{z}\lambda^2 w - \bar{w}\bar{z}\lambda^2 w + z - \bar{w}z -$
$\bar{w}\lambda^2 z + 2\bar{w}\bar{z}\lambda^2 z - \bar{z}^2\lambda^2 z - wz + \bar{w}wz + \bar{w}\lambda^2 wz - 2\bar{z}\lambda^2 wz + \bar{z}^2\lambda^2 wz + \bar{z}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^2 z^2, -\bar{w}w + \bar{w}\bar{z}w + \bar{z}z - \bar{w}\bar{z}z -$
$\bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z + \bar{w}wz - \bar{z}wz - \bar{z}\lambda^2 wz + 2\bar{w}\bar{z}\lambda^2 wz - \bar{z}^2\lambda^2 wz + \bar{w}\bar{z}\lambda^4 wz - \bar{z}^2\lambda^4 wz + \bar{z}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^4 z^2 +$
$\bar{z}^2\lambda^4 z^2, -1 + \bar{w} + \bar{z} - \bar{w}\bar{z} + w - \bar{w}w - \bar{z}w + \bar{w}\bar{z}w + z - \bar{w}z - \bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z - wz + \bar{w}wz - \bar{z}\lambda^2 wz + 2\bar{w}\bar{z}\lambda^2 wz -$
$\bar{z}^2\lambda^2 wz + \bar{w}\bar{z}\lambda^4 wz - \bar{z}^2\lambda^4 wz + \bar{z}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^2 z^2 - \bar{w}\bar{z}\lambda^4 z^2 + \bar{z}^2\lambda^4 z^2,$
$1 - 2\bar{w} + \bar{w}^2 - 2w + 4\bar{w}w - 2\bar{w}^2 w + 2\bar{w}\lambda^2 w - 2\bar{w}^2\lambda^2 w + w^2 - 2\bar{w}w^2 + \bar{w}^2 w^2 - 2\bar{w}\lambda^2 w^2 + 2\bar{w}^2\lambda^2 w^2 + \bar{w}^2\lambda^4 w^2 +$
$2\bar{z}\lambda^2 z - 2\bar{w}\bar{z}\lambda^2 z - 2\bar{z}\lambda^2 wz + 2\bar{w}\bar{z}\lambda^2 wz - 2\bar{w}\bar{z}\lambda^4 wz + \bar{z}^2\lambda^4 z^2, \bar{w} - \bar{w}^2 - \bar{w}w + \bar{w}^2 w - \bar{w}\lambda^2 w + \bar{w}^2\lambda^2 w - z + \bar{w}z +$
$2\bar{w}\lambda^2 z - 2\bar{w}^2\lambda^2 z - \bar{z}\lambda^2 z + \bar{w}\bar{z}\lambda^2 z + wz - \bar{w}wz - \bar{w}\lambda^2 wz + \bar{w}^2\lambda^2 wz + 2\bar{z}\lambda^2 wz - 2\bar{w}\bar{z}\lambda^2 wz + \bar{w}^2\lambda^4 wz - 2\bar{w}\bar{z}\lambda^4 wz +$
$\bar{z}^2\lambda^4 wz - \bar{z}\lambda^2 z^2 + \bar{w}\bar{z}\lambda^2 z^2 - \bar{w}^2\lambda^4 z^2 + 2\bar{w}\bar{z}\lambda^4 z^2 - \bar{z}^2\lambda^4 z^2, \bar{z} - \bar{w}\bar{z} - w + \bar{w}w - \bar{z}w + \bar{w}\bar{z}w + \bar{w}\lambda^2 w - 2\bar{z}\lambda^2 w +$
$\bar{w}\bar{z}\lambda^2 w + w^2 - \bar{w}w^2 - \bar{w}\lambda^2 w^2 + 2\bar{z}\lambda^2 w^2 - \bar{w}\bar{z}\lambda^2 w^2 - \bar{w}\bar{z}\lambda^4 w^2 + \bar{z}^2\lambda^4 w^2 + \bar{z}\lambda^2 z - 2\bar{w}\bar{z}\lambda^2 z + \bar{z}^2\lambda^2 z - \bar{z}\lambda^2 wz +$
$2\bar{w}\bar{z}\lambda^2 wz - \bar{z}^2\lambda^2 wz + 2\bar{w}\bar{z}\lambda^4 wz - 2\bar{z}^2\lambda^4 wz - \bar{w}\bar{z}\lambda^4 z^2 + \bar{z}^2\lambda^4 z^2,$

$1 - \bar{w} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 - w + \bar{w}w - 2\bar{z}\lambda^2 w + 2\bar{w}\bar{z}\lambda^2 w - \bar{w}\lambda^4 w + 2\bar{w}\bar{z}\lambda^4 w - \bar{z}^2\lambda^4 w + \lambda^2 z - 2\bar{w}\lambda^2 z + \bar{z}\lambda^2 z + \bar{z}\lambda^4 z -$
$2\bar{w}\bar{z}\lambda^4 z + \bar{z}^2\lambda^4 z - \lambda^2 wz + 2\bar{w}\lambda^2 wz - \bar{w}\bar{z}\lambda^2 wz + 2\bar{w}\lambda^4 wz - 2\bar{z}\lambda^4 wz + \bar{w}\bar{z}\lambda^6 wz - \bar{z}^2\lambda^6 wz - \bar{w}\lambda^4 z^2 + \bar{z}\lambda^4 z^2 -$
$\bar{w}\bar{z}\lambda^6 z^2 + \bar{z}^2\lambda^6 z^2,$
$1 - \bar{w} - \bar{w}\lambda^2 + \bar{z}\lambda^2 - w + \bar{w}w - \lambda^2 w + \bar{w}\lambda^2 w - \bar{z}\lambda^2 w + \bar{w}\bar{z}\lambda^2 w - \bar{z}\lambda^4 w + \bar{w}\bar{z}\lambda^4 w + \lambda^2 z - \bar{w}\lambda^2 z + \bar{z}\lambda^2 z - \bar{w}\bar{z}\lambda^2 z -$
$\bar{w}\lambda^4 z + \bar{z}\lambda^4 z - \bar{w}\bar{z}\lambda^4 z + \bar{z}^2\lambda^4 z + \bar{w}\lambda^2 wz - \bar{z}\lambda^2 wz + \bar{w}\lambda^4 wz - \bar{z}\lambda^4 wz + \bar{w}\bar{z}\lambda^4 wz - \bar{z}^2\lambda^4 wz + \bar{w}\bar{z}\lambda^6 wz - \bar{z}^2\lambda^6 wz +$
$\bar{z}\lambda^4 z^2 - \bar{w}\bar{z}\lambda^4 z^2 - \bar{w}\bar{z}\lambda^6 z^2 + \bar{z}^2\lambda^6 z^2,$
$\bar{w}^2 - 2\bar{w}\bar{z} + \bar{z}^2 - 2\bar{w}w + 2\bar{z}w + 2\bar{w}\bar{z}w - 2\bar{z}^2 w - 4\bar{w}\lambda^2 w + 4\bar{z}\lambda^2 w + 4\bar{w}\bar{z}\lambda^2 w - 4\bar{z}^2\lambda^2 w + w^2 - 2\bar{z}w^2 + \bar{z}^2 w^2 +$
$2\bar{w}z - 2\bar{w}^2 z - 2\bar{z}z + 2\bar{w}\bar{z}z + 4\bar{w}\lambda^2 z - 4\bar{z}\lambda^2 z - 4\bar{w}\bar{z}\lambda^2 z + 4\bar{z}^2\lambda^2 z - 2wz + 2\bar{w}wz + 2\bar{z}wz - 2\bar{w}\bar{z}wz + 4\bar{w}\lambda^2 wz -$
$4\bar{z}\lambda^2 wz - 4\bar{w}\bar{z}\lambda^2 wz + 4\bar{z}^2\lambda^2 wz + z^2 - 2\bar{w}z^2 + \bar{w}^2 z^2 - 4\bar{w}\lambda^2 z^2 + 4\bar{z}\lambda^2 z^2 + 4\bar{w}\bar{z}\lambda^2 z^2 - 4\bar{z}^2\lambda^2 z^2,$
$\bar{z}^2 w^2 - \bar{z}^3 w^2 + 2\bar{w}\bar{z}wz - 4\bar{w}^2\bar{z}wz - 4\bar{z}^2 wz + 6\bar{w}\bar{z}^2 wz - 4\bar{w}\bar{z}w^2 z + 4\bar{w}^2\bar{z}w^2 z + 3\bar{z}^2 w^2 z - 4\bar{w}\bar{z}^2 w^2 z + \bar{z}^3 w^2 z +$
$4\bar{w}^2\bar{z}\lambda^2 w^2 z - 8\bar{w}\bar{z}^2\lambda^2 w^2 z + 4\bar{z}^3\lambda^2 w^2 z + \bar{w}^2 z^2 - 4\bar{w}\bar{z}z^2 + 3\bar{w}^2\bar{z}z^2 + 4\bar{z}^2 z^2 - 4\bar{w}\bar{z}^2 z^2 + 6\bar{w}\bar{z}wz^2 - 4\bar{w}^2\bar{z}wz^2 -$
$4\bar{z}^2 wz^2 + 2\bar{w}\bar{z}^2 wz^2 - 8\bar{w}^2\bar{z}\lambda^2 wz^2 + 16\bar{w}\bar{z}^2\lambda^2 wz^2 - 8\bar{z}^3\lambda^2 wz^2 - \bar{w}^2 z^3 + \bar{w}^2\bar{z}z^3 + 4\bar{w}^2\bar{z}\lambda^2 z^3 - 8\bar{w}\bar{z}^2\lambda^2 z^3 + 4\bar{z}^3\lambda^2 z^3,$
$-\bar{w} + \bar{w}^2 + \bar{z} - \bar{w}\bar{z} - w + 3\bar{w}w - 2\bar{w}^2 w - \bar{z}w + \bar{w}\bar{z}w + 2\bar{w}\lambda^2 w - 2\bar{w}^2\lambda^2 w - 2\bar{z}\lambda^2 w + 2\bar{w}\bar{z}\lambda^2 w + w^2 - 2\bar{w}w^2 +$
$\bar{w}^2 w^2 - 2\bar{w}\lambda^2 w^2 + 2\bar{w}^2\lambda^2 w^2 + 2\bar{z}\lambda^2 w^2 - 2\bar{w}\bar{z}\lambda^2 w^2 + \bar{w}^2\lambda^4 w^2 - 2\bar{w}\bar{z}\lambda^4 w^2 + \bar{z}^2\lambda^4 w^2 + z - \bar{w}z - \bar{z}z + \bar{w}\bar{z}z -$
$2\bar{w}\lambda^2 z + 2\bar{w}^2\lambda^2 z + 2\bar{z}\lambda^2 z - 2\bar{w}\bar{z}\lambda^2 z - wz + \bar{w}wz + \bar{z}wz - \bar{w}\bar{z}wz + 2\bar{w}\lambda^2 wz - 2\bar{w}^2\lambda^2 wz - 2\bar{z}\lambda^2 wz + 2\bar{w}\bar{z}\lambda^2 wz -$
$2\bar{w}^2\lambda^4 wz + 4\bar{w}\bar{z}\lambda^4 wz - 2\bar{z}^2\lambda^4 wz + \bar{w}^2\lambda^4 z^2 - 2\bar{w}\bar{z}\lambda^4 z^2 + \bar{z}^2\lambda^4 z^2, \bar{z}^2 w^2 - 2\bar{w}\bar{z}^2 w^2 + \bar{w}^2\bar{z}^2 w^2 - 2\bar{w}\bar{z}wz + 2\bar{w}^2\bar{z}wz +$
$2\bar{w}\bar{z}^2 wz - 2\bar{w}^2\bar{z}^2 wz + 2\bar{w}\bar{z}w^2 z - 2\bar{w}^2\bar{z}w^2 z - 2\bar{z}^2 w^2 z + 2\bar{w}\bar{z}^2 w^2 z - 4\bar{w}^2\bar{z}\lambda^2 w^2 z + 4\bar{w}\bar{z}^2\lambda^2 w^2 z + 4\bar{w}^2\bar{z}^2\lambda^2 w^2 z -$
$4\bar{w}\bar{z}^3\lambda^2 w^2 z + \bar{w}^2 z^2 - 2\bar{w}^2\bar{z}z^2 + \bar{w}^2\bar{z}^2 z^2 - 2\bar{w}^2 wz^2 + 2\bar{w}\bar{z}wz^2 + 2\bar{w}^2\bar{z}wz^2 - 2\bar{w}\bar{z}^2 wz^2 + 4\bar{w}^2\bar{z}\lambda^2 wz^2 -$
$4\bar{w}\bar{z}^2\lambda^2 wz^2 - 4\bar{w}^2\bar{z}^2\lambda^2 wz^2 + 4\bar{w}\bar{z}^3\lambda^2 wz^2 + \bar{w}^2 w^2 z^2 - 2\bar{w}\bar{z}w^2 z^2 + \bar{z}^2 w^2 z^2 + 4\bar{w}^2\bar{z}\lambda^2 w^2 z^2 - 4\bar{w}\bar{z}^2\lambda^2 w^2 z^2 -$
$4\bar{w}^2\bar{z}^2\lambda^2 w^2 z^2 + 4\bar{w}\bar{z}^3\lambda^2 w^2 z^2 - 4\bar{w}^2\bar{z}\lambda^2 wz^3 + 4\bar{w}\bar{z}^2\lambda^2 wz^3 + 4\bar{w}^2\bar{z}^2\lambda^2 wz^3 - 4\bar{w}\bar{z}^3\lambda^2 wz^3,$
$-\bar{w}w + \bar{w}^2 w + \bar{w}\bar{z}w - \bar{w}^2\bar{z}w + \bar{w}w^2 - \bar{w}^2 w^2 - \bar{w}\bar{z}w^2 + \bar{w}^2\bar{z}w^2 + \bar{z}z - 2\bar{w}\bar{z}z + \bar{w}^2\bar{z}z + \bar{w}wz - \bar{w}^2 wz - 2\bar{z}wz +$
$3\bar{w}\bar{z}wz - \bar{w}^2\bar{z}wz + 2\bar{w}\bar{z}\lambda^2 wz - 2\bar{w}^2\bar{z}\lambda^2 wz - 2\bar{z}^2\lambda^2 wz + 2\bar{w}\bar{z}^2\lambda^2 wz - \bar{w}w^2 z + \bar{w}^2 w^2 z + \bar{z}w^2 z - \bar{w}\bar{z}w^2 z -$
$2\bar{w}\bar{z}\lambda^2 w^2 z + 2\bar{w}^2\bar{z}\lambda^2 w^2 z + 2\bar{z}^2\lambda^2 w^2 z - 2\bar{w}\bar{z}^2\lambda^2 w^2 z + \bar{w}^2\bar{z}\lambda^4 w^2 z - 2\bar{w}\bar{z}^2\lambda^4 w^2 z + \bar{z}^3\lambda^4 w^2 z - 2\bar{w}\bar{z}\lambda^2 z^2 +$
$2\bar{w}^2\bar{z}\lambda^2 z^2 + 2\bar{z}^2\lambda^2 z^2 - 2\bar{w}\bar{z}^2\lambda^2 z^2 + 2\bar{w}\bar{z}\lambda^2 wz^2 - 2\bar{w}^2\bar{z}\lambda^2 wz^2 - 2\bar{z}^2\lambda^2 wz^2 + 2\bar{w}\bar{z}^2\lambda^2 wz^2 - 2\bar{w}^2\bar{z}\lambda^4 wz^2 +$
$4\bar{w}\bar{z}^2\lambda^4 wz^2 - 2\bar{z}^3\lambda^4 wz^2 + \bar{w}^2\bar{z}\lambda^4 z^3 - 2\bar{w}\bar{z}^2\lambda^4 z^3 + \bar{z}^3\lambda^4 z^3,$
$4\bar{w}^2 w^2 - 4\bar{w}^3 w^2 - 4\bar{w}\bar{z}w^2 + 4\bar{w}^2\bar{z}w^2 + \bar{z}^2 w^2 - 2\bar{w}\bar{z}^2 w^2 + \bar{w}^2\bar{z}^2 w^2 - 4\bar{w}^2 w^3 + 4\bar{w}^3 w^3 + 4\bar{w}\bar{z}w^3 - 4\bar{w}^2\bar{z}w^3$
$- 4\bar{w}^2\bar{z}\lambda^2 w^3 + 4\bar{w}^3\bar{z}\lambda^2 w^3 + 4\bar{w}^2\lambda^2 w^3 - 4\bar{w}^2\bar{z}^2\lambda^2 w^3 - 4\bar{w}^2 wz + 4\bar{w}^3 wz + 2\bar{w}\bar{z}wz - 2\bar{w}^2\bar{z}wz$
$+ 2\bar{w}\bar{z}^2 wz - 2\bar{w}^2\bar{z}^2 wz + 4\bar{w}^2 w^2 z - 4\bar{w}^3 w^2 z - 2\bar{w}\bar{z}w^2 z + 2\bar{w}^2\bar{z}w^2 z - 2\bar{z}^2 w^2 z + 2\bar{w}\bar{z}^2 w^2 z$
$- 4\bar{w}^3\lambda^2 w^2 z + 16\bar{w}^2\bar{z}\lambda^2 w^2 z - 8\bar{w}^3\bar{z}\lambda^2 w^2 z - 12\bar{w}^2\lambda^2 w^2 z + 8\bar{w}^2\bar{z}^2\lambda^2 w^2 z + 4\bar{w}^3\lambda^2 w^3 z -$
$8\bar{w}^2\bar{z}\lambda^2 w^3 z + 4\bar{w}\bar{z}^2\lambda^2 w^3 z + 4\bar{w}^3\bar{z}\lambda^4 w^3 z - 8\bar{w}^2\bar{z}^2\lambda^4 w^3 z + 4\bar{w}\bar{z}^3\lambda^4 w^3 z + \bar{w}^2 z^2 - 2\bar{w}^2\bar{z}z^2 + \bar{w}^2\bar{z}^2 z^2 - 2\bar{w}^2 wz^2 +$
$2\bar{w}\bar{z}wz^2 + 2\bar{w}^2\bar{z}wz^2 - 2\bar{w}\bar{z}^2 wz^2 + 4\bar{w}^3\lambda^2 wz^2 - 12\bar{w}^2\bar{z}\lambda^2 wz^2 + 4\bar{w}^3\bar{z}\lambda^2 wz^2 + 8\bar{w}\bar{z}^2\lambda^2 wz^2 - 4\bar{w}^2\bar{z}^2\lambda^2 wz^2 +$
$\bar{w}^2 w^2 z^2 - 2\bar{w}\bar{z}w^2 z^2 + \bar{z}^2 w^2 z^2 - 4\bar{w}^3\lambda^2 w^2 z^2 + 8\bar{w}^2\bar{z}\lambda^2 w^2 z^2 - 4\bar{w}\bar{z}^2\lambda^2 w^2 z^2 - 8\bar{w}^3\bar{z}\lambda^4 w^2 z^2 + 16\bar{w}^2\bar{z}^2\lambda^4 w^2 z^2 -$
$8\bar{w}\bar{z}^3\lambda^4 w^2 z^2 + 4\bar{w}^3\bar{z}\lambda^4 wz^3 - 8\bar{w}^2\bar{z}^2\lambda^4 wz^3 + 4\bar{w}\bar{z}^3\lambda^4 wz^3,$
$-\bar{z}^2 w^2 + 2\bar{w}\bar{z}^2 w^2 - \bar{w}^2\bar{z}^2 w^2 + 2\bar{w}\bar{z}\lambda^2 w^2 - 2\bar{w}^2\bar{z}\lambda^2 w^2 - 2\bar{z}^2\lambda^2 w^2 + 2\bar{w}\bar{z}^2\lambda^2 w^2 - \bar{w}^2\lambda^4 w^2 + 2\bar{w}\bar{z}\lambda^4 w^2 - \bar{z}^2\lambda^4 w^2 +$
$2\bar{w}\bar{z}wz - 2\bar{w}^2\bar{z}wz - 2\bar{w}\bar{z}^2 wz + 2\bar{w}^2\bar{z}^2 wz + 2\bar{w}^2\lambda^2 wz - 4\bar{w}^2\bar{z}\lambda^2 wz + 2\bar{w}^2\bar{z}^2\lambda^2 wz + 2\bar{w}^2\lambda^4 wz - 2\bar{w}\bar{z}\lambda^4 wz -$
$2\bar{w}^2\bar{z}\lambda^4 wz + 2\bar{w}\bar{z}^2\lambda^4 wz - 2\bar{w}\bar{z}w^2 z + 2\bar{w}^2\bar{z}w^2 z + 2\bar{z}^2 w^2 z - 2\bar{w}\bar{z}^2 w^2 z - 2\bar{w}^2\lambda^2 w^2 z - 4\bar{w}\bar{z}\lambda^2 w^2 z + 8\bar{w}^2\bar{z}\lambda^2 w^2 z +$
$4\bar{z}^2\lambda^2 w^2 z - 4\bar{w}\bar{z}^2\lambda^2 w^2 z - 2\bar{w}^2\bar{z}^2\lambda^2 w^2 z - 2\bar{w}\bar{z}\lambda^4 w^2 z - 2\bar{w}^2\bar{z}\lambda^4 w^2 z + 2\bar{z}^2\lambda^4 w^2 z + 2\bar{w}\bar{z}^2\lambda^4 w^2 z + 4\bar{w}^2\bar{z}^2\lambda^4 w^2 z -$
$4\bar{w}^3\lambda^4 w^2 z - 4\bar{w}^2\bar{z}\lambda^6 w^2 z + 4\bar{w}\bar{z}^2\lambda^6 w^2 z + 4\bar{w}^2\bar{z}^2\lambda^6 w^2 z - 4\bar{w}\bar{z}^3\lambda^6 w^2 z - \bar{w}^2 z^2 + 2\bar{w}^2\bar{z}z^2 - \bar{w}^2\bar{z}^2 z^2 - 2\bar{w}^2\lambda^2 z^2 +$
$4\bar{w}^2\bar{z}\lambda^2 z^2 - 2\bar{w}^2\bar{z}^2\lambda^2 z^2 - \bar{w}^2\lambda^4 z^2 + 2\bar{w}^2\bar{z}\lambda^4 z^2 - \bar{w}^2\bar{z}^2\lambda^4 z^2 + 2\bar{w}^2 wz^2 - 2\bar{w}\bar{z}wz^2 - 2\bar{w}^2\bar{z}wz^2 + 2\bar{w}\bar{z}^2 wz^2 +$
$2\bar{w}^2\lambda^2 wz^2 - 4\bar{w}^2\bar{z}\lambda^2 wz^2 + 2\bar{w}^2\bar{z}^2\lambda^2 wz^2 + 2\bar{w}\bar{z}\lambda^4 wz^2 + 2\bar{w}^2\bar{z}\lambda^4 wz^2 - 6\bar{w}\bar{z}^2\lambda^4 wz^2 - 2\bar{w}^2\bar{z}^2\lambda^4 wz^2 + 4\bar{w}\bar{z}^3\lambda^4 wz^2 +$
$4\bar{w}^2\bar{z}\lambda^6 wz^2 - 4\bar{w}\bar{z}^2\lambda^6 wz^2 - 4\bar{w}^2\bar{z}^2\lambda^6 wz^2 + 4\bar{w}\bar{z}^3\lambda^6 wz^2 - \bar{w}^2 w^2 z^2 + 2\bar{w}\bar{z}w^2 z^2 - \bar{z}^2 w^2 z^2 + 2\bar{w}\bar{z}\lambda^2 w^2 z^2 -$
$2\bar{w}^2\bar{z}\lambda^2 w^2 z^2 - 2\bar{z}^2\lambda^2 w^2 z^2 + 2\bar{w}\bar{z}^2\lambda^2 w^2 z^2 + 4\bar{w}^2\bar{z}\lambda^4 w^2 z^2 - \bar{z}^2\lambda^4 w^2 z^2 - 2\bar{w}\bar{z}^2\lambda^4 w^2 z^2 - 5\bar{w}^2\bar{z}^2\lambda^4 w^2 z^2 +$
$4\bar{w}\bar{z}^3\lambda^4 w^2 z^2 + 4\bar{w}^2\bar{z}\lambda^6 w^2 z^2 - 4\bar{w}\bar{z}^2\lambda^6 w^2 z^2 - 4\bar{w}^2\bar{z}^2\lambda^6 w^2 z^2 + 4\bar{w}\bar{z}^3\lambda^6 w^2 z^2 - 4\bar{w}^2\bar{z}\lambda^4 wz^3 + 4\bar{w}\bar{z}^2\lambda^4 wz^3 +$
$4\bar{w}^2\bar{z}^2\lambda^4 wz^3 - 4\bar{w}\bar{z}^3\lambda^4 wz^3 - 4\bar{w}^2\bar{z}\lambda^6 wz^3 + 4\bar{w}\bar{z}^2\lambda^6 wz^3 + 4\bar{w}^2\bar{z}^2\lambda^6 wz^3 - 4\bar{w}\bar{z}^3\lambda^6 wz^3,$
$1 - 2\bar{w} + \bar{w}^2 - 2w + 4\bar{w}w - 2\bar{w}^2 w + 2\bar{w}\lambda^2 w - 2\bar{w}^2\lambda^2 w - 4\bar{z}\lambda^2 w + 4\bar{w}\bar{z}\lambda^2 w + 2\bar{z}^2\lambda^2 w - 2\bar{w}\bar{z}^2\lambda^2 w + w^2 - 2\bar{w}w^2 +$
$\bar{w}^2 w^2 - 2\bar{w}\lambda^2 w^2 + 2\bar{w}^2\lambda^2 w^2 + 4\bar{z}\lambda^2 w^2 - 4\bar{w}\bar{z}\lambda^2 w^2 - 2\bar{z}^2\lambda^2 w^2 + 2\bar{w}\bar{z}^2\lambda^2 w^2 + \bar{w}^2\lambda^4 w^2 - 4\bar{w}\bar{z}\lambda^4 w^2 + 4\bar{z}^2\lambda^4 w^2 +$
$2\bar{w}\bar{z}^2\lambda^4 w^2 - 4\bar{z}^3\lambda^4 w^2 + \bar{z}^4\lambda^4 w^2 - 4\bar{w}\lambda^2 z + 4\bar{w}^2\lambda^2 z + 8\bar{z}\lambda^2 z - 8\bar{w}\bar{z}\lambda^2 z - 4\bar{z}^2\lambda^2 z + 4\bar{w}^2\lambda^2 z + 4\bar{w}\lambda^2 wz - 4\bar{w}^2\lambda^2 wz -$
$8\bar{z}\lambda^2 wz + 10\bar{w}\bar{z}\lambda^2 wz - 2\bar{w}^2\bar{z}\lambda^2 wz + 2\bar{z}^2\lambda^2 wz - 2\bar{w}\bar{z}^2\lambda^2 wz - 4\bar{w}^2\lambda^4 wz + 14\bar{w}\bar{z}\lambda^4 wz - 2\bar{w}^2\bar{z}\lambda^4 wz - 14\bar{z}^2\lambda^4 wz +$
$2\bar{w}\bar{z}^2\lambda^4 wz + 8\bar{z}^3\lambda^4 wz - 4\bar{w}\bar{z}^3\lambda^4 wz - 2\bar{w}\bar{z}\lambda^2 w^2 z + 2\bar{w}^2\bar{z}\lambda^2 w^2 z + 2\bar{z}^2\lambda^2 w^2 z - 2\bar{w}\bar{z}^2\lambda^2 w^2 z - 2\bar{w}\bar{z}\lambda^4 w^2 z +$
$4\bar{w}^2\bar{z}\lambda^4 w^2 z + 2\bar{z}^2\lambda^4 w^2 z - 8\bar{w}\bar{z}^2\lambda^4 w^2 z + 4\bar{z}^3\lambda^4 w^2 z + 2\bar{w}\bar{z}^3\lambda^4 w^2 z - 2\bar{z}^4\lambda^4 w^2 z + 2\bar{w}^2\bar{z}\lambda^6 w^2 z - 6\bar{w}\bar{z}^2\lambda^6 w^2 z +$
$4\bar{z}^3\lambda^6 w^2 z + 2\bar{w}\bar{z}^3\lambda^6 w^2 z - 2\bar{z}^4\lambda^6 w^2 z + 2\bar{w}\lambda^2 z^2 - 2\bar{w}^2\lambda^2 z^2 - 4\bar{z}\lambda^2 z^2 + 2\bar{w}\bar{z}\lambda^2 z^2 + 2\bar{w}^2\bar{z}\lambda^2 z^2 + 4\bar{z}^2\lambda^2 z^2 -$
$4\bar{w}\bar{z}^2\lambda^2 z^2 + 4\bar{w}^2\lambda^4 z^2 - 14\bar{w}\bar{z}\lambda^4 z^2 + 2\bar{w}^2\bar{z}\lambda^4 z^2 + 14\bar{z}^2\lambda^4 z^2 - 2\bar{w}\bar{z}^2\lambda^4 z^2 - 8\bar{z}^3\lambda^4 z^2 + 4\bar{w}^3\lambda^4 z^2 - 2\bar{w}\lambda^2 wz^2 +$
$2\bar{w}^2\lambda^2 wz^2 + 4\bar{z}\lambda^2 wz^2 - 2\bar{w}\bar{z}\lambda^2 wz^2 - 2\bar{w}^2\bar{z}\lambda^2 wz^2 - 4\bar{z}^2\lambda^2 wz^2 + 4\bar{w}\bar{z}^2\lambda^2 wz^2 + 2\bar{w}^2\lambda^4 wz^2 + 2\bar{w}\bar{z}\lambda^4 wz^2 -$
$8\bar{w}^2\bar{z}\lambda^4 wz^2 - 2\bar{z}^2\lambda^4 wz^2 + 8\bar{w}\bar{z}^2\lambda^4 wz^2 - 4\bar{z}^3\lambda^4 wz^2 + 2\bar{w}\bar{z}^3\lambda^4 wz^2 - 6\bar{w}^2\bar{z}\lambda^6 wz^2 + 18\bar{w}\bar{z}^2\lambda^6 wz^2 - 12\bar{z}^3\lambda^6 wz^2 -$
$6\bar{w}\bar{z}^3\lambda^6 wz^2 + 6\bar{z}^4\lambda^6 wz^2 + \bar{w}^2\bar{z}^2\lambda^4 w^2 z^2 - 2\bar{w}\bar{z}^3\lambda^4 w^2 z^2 + \bar{z}^4\lambda^4 w^2 z^2 + 2\bar{w}^2\bar{z}^2\lambda^6 w^2 z^2 - 4\bar{w}\bar{z}^3\lambda^6 w^2 z^2 + 2\bar{z}^4\lambda^6 w^2 z^2 +$
$\bar{w}^2\bar{z}^2\lambda^8 w^2 z^2 - 2\bar{w}\bar{z}^3\lambda^8 w^2 z^2 + \bar{z}^4\lambda^8 w^2 z^2 - 4\bar{w}^2\lambda^4 z^3 + 8\bar{w}\bar{z}\lambda^4 z^3 + 4\bar{w}^2\bar{z}\lambda^4 z^3 - 8\bar{z}^2\lambda^4 z^3 - 4\bar{w}\bar{z}^2\lambda^4 z^3 + 8\bar{z}^3\lambda^4 z^3 -$
$4\bar{w}\bar{z}^3\lambda^4 z^3 + 4\bar{w}^2\bar{z}\lambda^6 z^3 - 12\bar{w}\bar{z}^2\lambda^6 z^3 + 8\bar{z}^3\lambda^6 z^3 + 4\bar{w}\bar{z}^3\lambda^6 z^3 - 4\bar{z}^4\lambda^6 z^3 - 4\bar{w}\bar{z}\lambda^4 wz^3 + 2\bar{w}^2\bar{z}\lambda^4 wz^3 + 4\bar{z}^2\lambda^4 wz^3 +$
$2\bar{w}\bar{z}^2\lambda^4 wz^3 - 2\bar{w}^2\bar{z}^2\lambda^4 wz^3 - 4\bar{z}^3\lambda^4 wz^3 + 2\bar{w}\bar{z}^3\lambda^4 wz^3 + 2\bar{w}^2\bar{z}\lambda^6 wz^3 - 6\bar{w}\bar{z}^2\lambda^6 wz^3 - 4\bar{w}^2\bar{z}^2\lambda^6 wz^3 + 4\bar{z}^3\lambda^6 wz^3 +$

$10\bar{w}\bar{z}^3\lambda^6wz^3-6\bar{z}^4\lambda^6wz^3-2\bar{w}^2\bar{z}^2\lambda^8wz^3+4\bar{w}\bar{z}^3\lambda^8wz^3-2\bar{z}^4\lambda^8wz^3+\bar{w}^2\lambda^4z^4-2\bar{w}^2\bar{z}\lambda^4z^4+\bar{w}^2\bar{z}^2\lambda^4z^4-$
$2\bar{w}^2\bar{z}\lambda^6z^4+6\bar{w}\bar{z}^2\lambda^6z^4+2\bar{w}^2\bar{z}^2\lambda^6z^4-4\bar{z}^3\lambda^6z^4-6\bar{w}\bar{z}^3\lambda^6z^4+4\bar{z}^4\lambda^6z^4+\bar{w}^2\bar{z}^2\lambda^8z^4-2\bar{w}\bar{z}^3\lambda^8z^4+\bar{z}^4\lambda^8z^4,$
$\bar{z}^2\lambda^4w^2-2\bar{z}^3\lambda^4w^2+\bar{z}^4\lambda^4w^2+4\bar{w}\bar{z}\lambda^2wz-4\bar{w}^2\bar{z}\lambda^2wz-2\bar{z}^2\lambda^2wz-2\bar{w}\bar{z}^2\lambda^2wz+4\bar{w}^2\bar{z}^2\lambda^2wz+2\bar{z}^3\lambda^2wz$
$-2\bar{w}\bar{z}^3\lambda^2wz+2\bar{w}\bar{z}\lambda^4wz-4\bar{w}^2\bar{z}\lambda^4wz-2\bar{z}^2\lambda^4wz+2\bar{w}\bar{z}^2\lambda^4wz+4\bar{w}^2\bar{z}^2\lambda^4wz+2\bar{z}^3\lambda^4wz$
$-4\bar{w}\bar{z}^3\lambda^4wz-4\bar{w}\bar{z}\lambda^2w^2z+4\bar{w}^2\bar{z}\lambda^2w^2z+2\bar{z}^2\lambda^2w^2z+2\bar{w}\bar{z}^2\lambda^2w^2z-4\bar{w}^2\bar{z}^2\lambda^2w^2z-2\bar{z}^3\lambda^2w^2z$
$+2\bar{w}\bar{z}^3\lambda^2w^2z-4\bar{w}\bar{z}\lambda^4w^2z+8\bar{w}^2\bar{z}\lambda^4w^2z-4\bar{w}\bar{z}^2\lambda^4w^2z-8\bar{w}^2\bar{z}^2\lambda^4w^2z+4\bar{z}^3\lambda^4w^2z$
$+8\bar{w}\bar{z}^3\lambda^4w^2z-4\bar{z}^4\lambda^4w^2z+4\bar{w}^2\bar{z}\lambda^6w^2z-6\bar{w}\bar{z}^2\lambda^6w^2z-4\bar{w}^2\bar{z}^2\lambda^6w^2z+2\bar{z}^3\lambda^6w^2z+6\bar{w}\bar{z}^3\lambda^6w^2z$
$-$
$2\bar{z}^4\lambda^6w^2z+\bar{z}^2z^2-2\bar{w}\bar{z}^2z^2+\bar{w}^2\bar{z}^2z^2-2\bar{w}\bar{z}\lambda^2z^2+2\bar{w}^2\bar{z}\lambda^2z^2+2\bar{z}^2\lambda^2z^2-2\bar{w}\bar{z}^2\lambda^2z^2+\bar{w}^2\lambda^4z^2-2\bar{w}\bar{z}\lambda^4z^2+$
$\bar{z}^2\lambda^4z^2-2\bar{z}^2wz^2+4\bar{w}\bar{z}^2wz^2-2\bar{w}^2\bar{z}^2wz^2-2\bar{w}\bar{z}\lambda^2wz^2+2\bar{w}^2\bar{z}\lambda^2wz^2-2\bar{z}^2\lambda^2wz^2+10\bar{w}\bar{z}^2\lambda^2wz^2-8\bar{w}^2\bar{z}^2\lambda^2wz^2-$
$4\bar{z}^3\lambda^2wz^2+4\bar{w}\bar{z}^3\lambda^2wz^2+2\bar{w}\bar{z}\lambda^4wz^2-4\bar{w}^2\bar{z}\lambda^4wz^2+8\bar{w}\bar{z}^2\lambda^4wz^2-2\bar{w}^2\bar{z}^2\lambda^4wz^2-8\bar{z}^3\lambda^4wz^2+2\bar{w}\bar{z}^3\lambda^4wz^2+$
$2\bar{z}^4\lambda^4wz^2-6\bar{w}^2\bar{z}\lambda^6wz^2+10\bar{w}\bar{z}^2\lambda^6wz^2+4\bar{w}^2\bar{z}^2\lambda^6wz^2-4\bar{z}^3\lambda^6wz^2-6\bar{w}\bar{z}^3\lambda^6wz^2+2\bar{z}^4\lambda^6wz^2+\bar{z}^2w^2z^2-$
$2\bar{w}\bar{z}^2w^2z^2+\bar{w}^2\bar{z}^2w^2z^2+4\bar{w}\bar{z}\lambda^2w^2z^2-4\bar{w}^2\bar{z}\lambda^2w^2z^2-8\bar{w}\bar{z}^2\lambda^2w^2z^2+8\bar{w}^2\bar{z}^2\lambda^2w^2z^2+4\bar{z}^3\lambda^2w^2z^2-$
$4\bar{w}\bar{z}^3\lambda^2w^2z^2+4\bar{w}\bar{z}\lambda^4w^2z^2-8\bar{w}^2\bar{z}\lambda^4w^2z^2-2\bar{w}\bar{z}^2\lambda^4w^2z^2+14\bar{w}^2\bar{z}^2\lambda^4w^2z^2+2\bar{z}^3\lambda^4w^2z^2-14\bar{w}\bar{z}^3\lambda^4w^2z^2+$
$4\bar{z}^4\lambda^4w^2z^2-4\bar{w}^2\bar{z}\lambda^6w^2z^2+4\bar{w}\bar{z}^2\lambda^6w^2z^2+8\bar{w}^2\bar{z}^2\lambda^6w^2z^2-12\bar{w}\bar{z}^3\lambda^6w^2z^2+4\bar{z}^4\lambda^6w^2z^2+\bar{w}^2\bar{z}^2\lambda^8w^2z^2-$
$2\bar{w}\bar{z}^3\lambda^8w^2z^2+\bar{z}^4\lambda^8w^2z^2+2\bar{w}\bar{z}\lambda^2z^3-2\bar{w}^2\bar{z}\lambda^2z^3-4\bar{w}\bar{z}^2\lambda^2z^3+4\bar{w}^2\bar{z}^2\lambda^2z^3+2\bar{z}^3\lambda^2z^3-2\bar{w}\bar{z}^3\lambda^2z^3-2\bar{w}^2\lambda^4z^3+$
$2\bar{w}\bar{z}\lambda^4z^3+4\bar{w}^2\bar{z}\lambda^4z^3-8\bar{w}\bar{z}^2\lambda^4z^3+2\bar{w}^2\bar{z}^2\lambda^4z^3+4\bar{z}^3\lambda^4z^3-2\bar{w}\bar{z}^3\lambda^4z^3+2\bar{w}^2\bar{z}\lambda^6z^3-4\bar{w}\bar{z}^2\lambda^6z^3+2\bar{z}^3\lambda^6z^3-$
$2\bar{w}\bar{z}\lambda^2wz^3+2\bar{w}^2\bar{z}\lambda^2wz^3+4\bar{w}\bar{z}^2\lambda^2wz^3-4\bar{w}^2\bar{z}^2\lambda^2wz^3-2\bar{z}^3\lambda^2wz^3+2\bar{w}\bar{z}^3\lambda^2wz^3-4\bar{w}\bar{z}\lambda^4wz^3+8\bar{w}^2\bar{z}\lambda^4wz^3+$
$2\bar{w}\bar{z}^2\lambda^4wz^3-14\bar{w}^2\bar{z}^2\lambda^4wz^3-2\bar{z}^3\lambda^4wz^3+14\bar{w}\bar{z}^3\lambda^4wz^3-4\bar{z}^4\lambda^4wz^3+6\bar{w}^2\bar{z}\lambda^6wz^3-6\bar{w}\bar{z}^2\lambda^6wz^3-$
$12\bar{w}^2\bar{z}^2\lambda^6wz^3+18\bar{w}\bar{z}^3\lambda^6wz^3-6\bar{z}^4\lambda^6wz^3-2\bar{w}^2\bar{z}^2\lambda^8wz^3+4\bar{w}\bar{z}^3\lambda^8wz^3-2\bar{z}^4\lambda^8wz^3+\bar{w}^2\lambda^4z^4-4\bar{w}^2\bar{z}\lambda^4z^4+$
$2\bar{w}\bar{z}^2\lambda^4z^4+4\bar{w}^2\bar{z}^2\lambda^4z^4-4\bar{w}\bar{z}^3\lambda^4z^4+\bar{z}^4\lambda^4z^4-2\bar{w}^2\bar{z}\lambda^6z^4+2\bar{w}\bar{z}^2\lambda^6z^4+4\bar{w}^2\bar{z}^2\lambda^6z^4-6\bar{w}\bar{z}^3\lambda^6z^4+2\bar{z}^4\lambda^6z^4+$
$\bar{w}^2\bar{z}^2\lambda^8z^4-2\bar{w}\bar{z}^3\lambda^8z^4+\bar{z}^4\lambda^8z^4,$
$\bar{z}^2-2\bar{w}\bar{z}^2+\bar{w}^2\bar{z}^2-2\bar{z}^2w+4\bar{w}\bar{z}^2w-2\bar{w}^2\bar{z}^2w+2\bar{w}\bar{z}\lambda^2w-2\bar{w}^2\bar{z}\lambda^2w-2\bar{z}^2\lambda^2w+2\bar{w}\bar{z}^2\lambda^2w+\bar{z}^2w^2-2\bar{w}\bar{z}^2w^2$
$+\bar{w}^2\bar{z}^2w^2-2\bar{w}\bar{z}\lambda^2w^2+2\bar{w}^2\bar{z}\lambda^2w^2+2\bar{z}^2\lambda^2w^2-2\bar{w}\bar{z}^2\lambda^2w^2+\bar{w}^2\lambda^4w^2-2\bar{w}\bar{z}\lambda^4w^2+\bar{z}^2\lambda^4w^2$
$-2\bar{z}z+4\bar{w}\bar{z}z-2\bar{w}^2\bar{z}z-2\bar{w}\bar{z}\lambda^2z+2\bar{w}^2\bar{z}\lambda^2z+2\bar{z}^2\lambda^2z-2\bar{w}^2\lambda^2z+4\bar{z}wz-8\bar{w}\bar{z}wz+4\bar{w}^2\bar{z}wz$
$+2\bar{w}\lambda^2wz-2\bar{w}^2\lambda^2wz-2\bar{z}\lambda^2wz-4\bar{w}\bar{z}\lambda^2wz+6\bar{w}^2\bar{z}\lambda^2wz+6\bar{z}^2\lambda^2wz-4\bar{w}\bar{z}^2\lambda^2wz-2\bar{w}^2\bar{z}^2\lambda^2wz$
$-2\bar{z}^3\lambda^2wz+2\bar{w}\bar{z}^3\lambda^2wz-2\bar{w}^2\lambda^4wz+8\bar{w}\bar{z}\lambda^4wz-4\bar{w}^2\bar{z}\lambda^4wz-6\bar{z}^2\lambda^4wz+4\bar{w}^2\bar{z}^2\lambda^4wz$
$+4\bar{z}^3\lambda^4wz-4\bar{w}\bar{z}^3\lambda^4wz-2\bar{z}w^2z+4\bar{w}\bar{z}w^2z-2\bar{w}^2\bar{z}w^2z-2\bar{w}\lambda^2w^2z+2\bar{w}^2\lambda^2w^2z+2\bar{z}\lambda^2w^2z$
$+6\bar{w}\bar{z}\lambda^2w^2z-8\bar{w}^2\bar{z}\lambda^2w^2z-8\bar{z}^2\lambda^2w^2z+6\bar{w}\bar{z}^2\lambda^2w^2z+2\bar{w}^2\bar{z}^2\lambda^2w^2z+2\bar{z}^3\lambda^2w^2z-2\bar{w}\bar{z}^3\lambda^2w^2z$
$-4\bar{w}\bar{z}\lambda^4w^2z+2\bar{w}^2\bar{z}\lambda^4w^2z+4\bar{z}^2\lambda^4w^2z+4\bar{w}\bar{z}^2\lambda^4w^2z-4\bar{w}^2\bar{z}^2\lambda^4w^2z-6\bar{z}^3\lambda^4w^2z+4\bar{w}\bar{z}^3\lambda^4w^2z$
$+4\bar{w}^2\bar{z}\lambda^6w^2z-8\bar{w}\bar{z}^2\lambda^6w^2z-4\bar{w}^2\bar{z}^2\lambda^6w^2z+4\bar{z}^3\lambda^6w^2z+8\bar{w}\bar{z}^3\lambda^6w^2z-4\bar{z}^4\lambda^6w^2z+z^2-2\bar{w}z^2$
$+\bar{w}^2z^2-2\bar{w}\lambda^2z^2+2\bar{w}^2\lambda^2z^2+2\bar{z}\lambda^2z^2+6\bar{w}\bar{z}\lambda^2z^2-8\bar{w}^2\bar{z}\lambda^2z^2-8\bar{z}^2\lambda^2z^2+6\bar{w}\bar{z}^2\lambda^2z^2$
$+2\bar{w}^2\bar{z}^2\lambda^2z^2+2\bar{z}^3\lambda^2z^2-2\bar{w}\bar{z}^3\lambda^2z^2+\bar{w}^2\lambda^4z^2-6\bar{w}\bar{z}\lambda^4z^2+4\bar{w}^2\bar{z}\lambda^4z^2+5\bar{z}^2\lambda^4z^2-4\bar{w}^2\bar{z}^2\lambda^4z^2$
$-4\bar{z}^3\lambda^4z^2+4\bar{w}\bar{z}^3\lambda^4z^2-2wz^2+4\bar{w}wz^2-2\bar{w}^2wz^2+2\bar{w}\lambda^2wz^2-2\bar{w}^2\lambda^2wz^2-2\bar{z}\lambda^2wz^2$
$-4\bar{w}\bar{z}\lambda^2wz^2+6\bar{w}^2\bar{z}\lambda^2wz^2+6\bar{z}^2\lambda^2wz^2-4\bar{w}\bar{z}^2\lambda^2wz^2-2\bar{w}^2\bar{z}^2\lambda^2wz^2-2\bar{z}^3\lambda^2wz^2+2\bar{w}\bar{z}^3\lambda^2wz^2$
$+4\bar{w}^2\bar{z}\lambda^4wz^2-8\bar{w}^2\lambda^4wz^2+4\bar{z}^3\lambda^4wz^2-8\bar{w}^2\bar{z}\lambda^6wz^2+16\bar{w}\bar{z}^2\lambda^6wz^2+8\bar{w}^2\bar{z}^2\lambda^6wz^2-8\bar{z}^3\lambda^6wz^2$
$-16\bar{w}\bar{z}^3\lambda^6wz^2+8\bar{z}^4\lambda^6wz^2+w^2z^2-2\bar{w}w^2z^2+\bar{w}^2w^2z^2-2\bar{w}\bar{z}\lambda^2w^2z^2+2\bar{w}^2\bar{z}\lambda^2w^2z^2$
$+2\bar{z}^2\lambda^2w^2z^2-2\bar{w}\bar{z}^2\lambda^2w^2z^2+4\bar{w}\bar{z}\lambda^4w^2z^2-4\bar{w}^2\bar{z}\lambda^4w^2z^2-4\bar{z}^2\lambda^4w^2z^2+5\bar{w}^2\bar{z}^2\lambda^4w^2z^2$
$+4\bar{z}^3\lambda^4w^2z^2-6\bar{w}^3\lambda^4w^2z^2+\bar{z}^4\lambda^4w^2z^2-4\bar{w}^2\bar{z}\lambda^6w^2z^2+8\bar{w}\bar{z}^2\lambda^6w^2z^2+4\bar{w}^2\bar{z}^2\lambda^6w^2z^2$
$-4\bar{z}^3\lambda^6w^2z^2-8\bar{w}^3\lambda^6w^2z^2+4\bar{z}^4\lambda^6w^2z^2-2\bar{w}\bar{z}\lambda^2z^3+2\bar{w}^2\bar{z}\lambda^2z^3+2\bar{z}^2\lambda^2z^3-2\bar{w}\bar{z}^2\lambda^2z^3$
$+4\bar{w}\bar{z}\lambda^4z^3-6\bar{w}^2\bar{z}\lambda^4z^3-4\bar{z}^2\lambda^4z^3+4\bar{w}\bar{z}^2\lambda^4z^3+4\bar{w}^2\bar{z}^2\lambda^4z^3+2\bar{z}^3\lambda^4z^3-4\bar{w}\bar{z}^3\lambda^4z^3+4\bar{w}^2\bar{z}\lambda^6z^3$
$-8\bar{w}\bar{z}^2\lambda^6z^3-4\bar{w}^2\bar{z}^2\lambda^6z^3+4\bar{z}^3\lambda^6z^3+8\bar{w}\bar{z}^3\lambda^6z^3-4\bar{z}^4\lambda^6z^3+2\bar{w}\bar{z}\lambda^2wz^3-2\bar{w}^2\bar{z}\lambda^2wz^3$
$-2\bar{z}^2\lambda^2wz^3+2\bar{w}\bar{z}^2\lambda^2wz^3-4\bar{w}\bar{z}\lambda^4wz^3+4\bar{w}^2\bar{z}\lambda^4wz^3+4\bar{z}^2\lambda^4wz^3-6\bar{w}^2\bar{z}^2\lambda^4wz^3-4\bar{z}^3\lambda^4wz^3$
$+8\bar{w}\bar{z}^3\lambda^4wz^3-2\bar{z}^4\lambda^4wz^3+8\bar{w}^2\bar{z}\lambda^6wz^3-16\bar{w}\bar{z}^2\lambda^6wz^3-8\bar{w}^2\bar{z}^2\lambda^6wz^3+8\bar{z}^3\lambda^6wz^3$
$+16\bar{w}\bar{z}^3\lambda^6wz^3-8\bar{z}^4\lambda^6wz^3+\bar{w}^2\bar{z}^2\lambda^4z^4-2\bar{w}\bar{z}^3\lambda^4z^4+\bar{z}^4\lambda^4z^4-4\bar{w}^2\bar{z}\lambda^6z^4+8\bar{w}^2\lambda^6z^4$
$+4\bar{w}^2\bar{z}^2\lambda^6z^4-4\bar{z}^3\lambda^6z^4-8\bar{w}\bar{z}^3\lambda^6z^4+4\bar{z}^4\lambda^6z^4$

$\lambda, z, \bar{z}, w-z, \bar{z}-z, -1+z, \bar{w}-\bar{z}, -1+\bar{z}, 1+\lambda^2,\ w-\bar{w}z, \bar{w}-\bar{z}w, -1+\bar{z}z, 1+\lambda^2 z, 1+\bar{z}\lambda^2, \bar{z}w-\bar{w}z, \bar{w}w-\bar{z}z, 1+\bar{z}\lambda^2 z, -1+\bar{z}\lambda^4 z, \bar{w}w-\bar{z}w-\bar{w}z, -1+w+\lambda^2 w-\lambda^2 z, -1+\bar{z}+z+\bar{z}\lambda^2 z, \bar{z}+z-\bar{z}z+\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2-\bar{z}\lambda^2, -1+w+\lambda^2 w-\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2-\bar{z}\lambda^2 z, \bar{w}-w-\bar{z}\lambda^2 w+\bar{w}\lambda^2 z, -1+w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, \bar{w}-z+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z, -\bar{z}+w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z, -\bar{w}+\bar{z}+w-\bar{z}w-z+\bar{w}z, 1-\bar{w}w-\bar{w}\lambda^2 w+\bar{z}\lambda^2 z, 1-\bar{w}z-\bar{w}\lambda^2 z+\bar{z}\lambda^2 z, 1-\bar{w}-w+\bar{z}w+\bar{w}z-\bar{z}z, -1+\bar{z}w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, -1+w+\lambda^2 w-2\lambda^2 z+\lambda^2 z^2, -1+\bar{w}+\bar{w}\lambda^2-2\bar{z}\lambda^2+\bar{z}^2\lambda^2, -\lambda^2 w-z+wz+2\lambda^2 wz-\lambda^2 z^2, \bar{z}-\bar{w}\bar{z}+\bar{w}\lambda^2-2\bar{w}\bar{z}\lambda^2+\bar{z}^2\lambda^2, -w+\bar{w}w-\bar{w}z+wz+\bar{w}\lambda^2 wz-\bar{w}\lambda^2 z^2, -\bar{z}\lambda^2 w-z+wz+2\bar{z}\lambda^2 wz-\bar{z}\lambda^2 z^2, \bar{z}-\bar{w}\bar{z}+\bar{w}\lambda^2 z-2\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z, -\bar{w}+\bar{w}\bar{z}+\bar{w}w-\bar{z}w+\bar{w}\bar{z}\lambda^2 w-\bar{z}^2\lambda^2 w, -\bar{z}w+\bar{w}\bar{z}w+\bar{w}z-\bar{w}\bar{z}z-\bar{w}wz+\bar{z}wz, -\bar{w}w+\bar{z}w+\bar{w}z-\bar{w}\bar{z}z-\bar{z}wz+\bar{w}\bar{z}wz,$

$1-\bar{w}-w+\bar{w}w+\bar{w}\lambda^2 w-\bar{z}\lambda^2 w-\bar{w}\lambda^2 z+\bar{z}\lambda^2 z,$

$1-\bar{w}-w+\bar{w}w-\bar{w}\lambda^4 w+\bar{z}\lambda^4 w+\bar{w}\lambda^4 z-\bar{z}\lambda^4 z,$

$\bar{z}-\bar{z}w-\bar{z}\lambda^2 w-z+\bar{z}\lambda^2 z+wz+\bar{z}\lambda^2 wz-\bar{z}\lambda^2 z^2,$

$-\bar{z}w+\bar{z}w^2+\bar{z}\lambda^2 w^2+z-wz-2\bar{z}\lambda^2 wz+\bar{z}\lambda^2 z^2,$

$1-w-\lambda^2 w-\bar{z}\lambda^2 w-\bar{z}\lambda^4 w+\lambda^2 z+\bar{z}\lambda^2 z+\bar{z}\lambda^4 z^2,$

$\bar{z}-\bar{w}\bar{z}-\bar{w}z+\bar{w}^2 z+\bar{w}^2\lambda^2 z-2\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z,$

$\bar{z}-\bar{w}\bar{z}-z+\bar{w}z+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z-\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z,$

$-1+\bar{w}+\bar{w}\lambda^2-\bar{z}\lambda^2+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z+\bar{w}\lambda^4 z-\bar{z}^2\lambda^4 z,$

$-\bar{w}+\bar{z}-w+\bar{w}w+\bar{w}\lambda^2 w-\bar{z}\lambda^2 w+z-\bar{z}z-\bar{w}\lambda^2 z+\bar{z}\lambda^2 z,$

$1-\bar{z}-\bar{w}w+\bar{z}w-\bar{w}\lambda^2 w+\bar{z}\lambda^2 w-z+\bar{w}z+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z,$

$\bar{z}w-\bar{w}\bar{z}w+\bar{w}z-\bar{w}wz-2\bar{w}\bar{z}\lambda^2 wz+\bar{z}^2\lambda^2 wz+\bar{w}\bar{z}\lambda^2 z^2,$

$-\bar{z}^2+2\bar{z}z+4\bar{z}\lambda^2 z-4\bar{z}^2\lambda^2 z-z^2-4\bar{z}\lambda^2 z^2+4\bar{z}^2\lambda^2 z^2,$

$-1+\bar{w}+w-\bar{w}w+\bar{w}\bar{z}\lambda^4 wz-\bar{z}^2\lambda^4 wz-\bar{w}\bar{z}\lambda^4 z^2+\bar{z}^2\lambda^4 z^2,$

$\bar{z}w-\bar{w}\bar{z}w-\bar{w}z+2\bar{w}\bar{z}z-\bar{z}^2 z+\bar{w}wz-2\bar{z}wz+\bar{z}^2 wz+\bar{z}z^2-\bar{w}\bar{z}z^2,$

$-\bar{w}+\bar{z}+w-\bar{z}w+\bar{z}\lambda^2 w-\bar{z}^2\lambda^2 w-z+\bar{w}z-\bar{w}\lambda^2 z+\bar{z}^2\lambda^2 z+\bar{w}\lambda^2 z^2-\bar{z}\lambda^2 z^2,$

$\bar{z}w-\bar{w}\bar{z}w+\bar{w}z-\bar{z}z-\bar{w}wz+\bar{w}\bar{z}wz-\bar{w}\bar{z}\lambda^2 wz+\bar{z}^2\lambda^2 wz+\bar{w}\bar{z}\lambda^2 z^2-\bar{z}^2\lambda^2 z^2,$

$-\bar{w}w+\bar{w}\bar{z}w+\bar{z}z-\bar{w}\bar{z}z+\bar{w}wz-\bar{z}wz+\bar{w}\bar{z}\lambda^2 wz-\bar{z}^2\lambda^2 wz-\bar{w}\bar{z}\lambda^2 z^2+\bar{z}^2\lambda^2 z^2,$

$-w+\bar{w}w+\bar{w}\lambda^2 w-\bar{z}\lambda^2 w-\bar{w}z-\bar{w}\lambda^2 z+wz+\bar{w}\lambda^2 wz+\bar{z}\lambda^2 wz+\bar{w}\lambda^4 wz-\bar{w}\lambda^2 z^2-\bar{w}\lambda^4 z^2,$

$-\bar{w}+\bar{w}\bar{z}+\bar{w}w-\bar{z}w+\bar{w}\lambda^2 w-\bar{z}\lambda^2 w+\bar{w}\bar{z}\lambda^2 w-\bar{z}^2\lambda^2 w+\bar{w}\bar{z}\lambda^4 w-\bar{z}^2\lambda^4 w-\bar{w}\lambda^2 z+\bar{w}\bar{z}\lambda^2 z,$

$-\bar{z}w+\bar{w}\bar{z}w+\bar{w}\bar{z}\lambda^2 w-\bar{z}^2\lambda^2 w+\bar{w}z-\bar{w}\bar{z}z-\bar{w}wz+\bar{z}wz-\bar{w}\lambda^2 wz+\bar{z}^2\lambda^2 wz+\bar{w}\lambda^2 z^2-\bar{w}\bar{z}\lambda^2 z^2,$

$\bar{w}^2 w-\bar{z}w-\bar{w}w^2+\bar{z}w^2-\bar{w}\bar{z}\lambda^2 w^2+\bar{z}^2\lambda^2 w^2+\bar{w}z-\bar{w}^2 z+\bar{w}^2\lambda^2 wz-\bar{z}^2\lambda^2 wz-\bar{w}^2\lambda^2 z^2+\bar{w}\bar{z}\lambda^2 z^2,$

$-\bar{w}w+\bar{z}w+\bar{w}w^2-\bar{z}w^2+\bar{w}\lambda^2 w^2-\bar{z}\lambda^2 w^2-z+\bar{w}z+wz-\bar{w}wz-2\bar{w}\lambda^2 wz+2\bar{z}\lambda^2 wz+\bar{w}\lambda^2 z^2-\bar{z}\lambda^2 z^2,$

$$-\bar{z} + \bar{w}\bar{z} - \bar{w}w + \bar{w}^2w + \bar{z}w - \bar{w}\bar{z}w + \bar{w}^2\lambda^2w - 2\bar{w}\bar{z}\lambda^2w + \bar{z}^2\lambda^2w + \bar{w}z - \bar{w}^2z - \bar{w}^2\lambda^2z + 2\bar{w}\bar{z}\lambda^2z - \bar{z}^2\lambda^2z,$$

$$\bar{w}^2 - 2\bar{w}w - 4\bar{w}\lambda^2w + 2\bar{w}\bar{z}\lambda^2w + w^2 + 2\bar{z}\lambda^2w^2 + \bar{z}^2\lambda^4w^2 + 2\bar{w}^2\lambda^2z + 2\bar{w}\lambda^2wz - 4\bar{w}\bar{z}\lambda^2wz - 2\bar{w}\bar{z}\lambda^4wz + \bar{w}^2\lambda^4z^2,$$

$$-w + \bar{w}w + \bar{w}\lambda^2w - \bar{z}\lambda^2w + z - \bar{w}z + \lambda^2z - \bar{w}\lambda^2z - \lambda^2wz + \bar{w}\lambda^2wz + \bar{w}\lambda^4wz - \bar{z}\lambda^4wz - \bar{w}\lambda^2z^2 + \bar{z}\lambda^2z^2 - \bar{w}\lambda^4z^2 + \bar{z}\lambda^4z^2,$$

$$1 - \bar{w} - w + \bar{w}w - \bar{w}\lambda^4w + \bar{z}\lambda^4w + \lambda^2z - 2\bar{w}\lambda^2z + \bar{z}\lambda^2z - \lambda^2wz + 2\bar{w}\lambda^2wz - \bar{z}\lambda^2wz + 2\bar{w}\lambda^4wz - 2\bar{z}\lambda^4wz - \bar{w}\lambda^4z^2 + \bar{z}\lambda^4z^2,$$

$$-\bar{w} + \bar{z} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 + \bar{w}w - \bar{z}w + \bar{w}\lambda^2w - \bar{z}\lambda^2w + \bar{w}\bar{z}\lambda^2w - \bar{z}^2\lambda^2w + \bar{w}\bar{z}\lambda^4w - \bar{z}^2\lambda^4w - \bar{w}\lambda^2z + \bar{z}^2\lambda^2z - \bar{w}\bar{z}\lambda^4z + \bar{z}^2\lambda^4z,$$

$$1 - \bar{w} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 - w + \bar{w}w - 2\bar{z}\lambda^2w + 2\bar{w}\bar{z}\lambda^2w - \bar{w}\lambda^4w + 2\bar{w}\bar{z}\lambda^4w - \bar{z}^2\lambda^4w + \bar{z}\lambda^2z - \bar{w}\bar{z}\lambda^2z + \bar{w}\lambda^4z - 2\bar{w}\bar{z}\lambda^4z + \bar{z}^2\lambda^4z,$$

$$-\bar{w} + \bar{z} - w + \bar{w}w - \bar{z}\lambda^2w + \bar{w}\bar{z}\lambda^2w + z - \bar{z}z - \bar{w}\lambda^2z + 2\bar{z}\lambda^2z - \bar{w}\bar{z}\lambda^2z + \bar{w}\lambda^2wz - \bar{z}\lambda^2wz + \bar{w}\bar{z}\lambda^4wz - \bar{z}^2\lambda^4wz - \bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2,$$

$$-\bar{z} + \bar{w}\bar{z} + \bar{z}w - \bar{w}\bar{z}w + \bar{z}\lambda^2w - \bar{w}\bar{z}\lambda^2w + z - \bar{w}z - \bar{w}\lambda^2z + 2\bar{w}\bar{z}\lambda^2z - \bar{z}^2\lambda^2z - wz + \bar{w}wz + \bar{w}\lambda^2wz - 2\bar{z}\lambda^2wz + \bar{z}^2\lambda^2wz + \bar{z}\lambda^2z^2 - \bar{w}\bar{z}\lambda^2z^2,$$

$$-\bar{w}w + \bar{w}\bar{z}w + \bar{z}z - \bar{w}\bar{z}z - \bar{w}\bar{z}\lambda^2z + \bar{z}^2\lambda^2z + \bar{w}wz - \bar{z}wz - \bar{z}\lambda^2wz + 2\bar{w}\bar{z}\lambda^2wz - \bar{z}^2\lambda^2wz + \bar{w}\bar{z}\lambda^4wz - \bar{z}^2\lambda^4wz + \bar{z}\lambda^2z^2 - \bar{w}\bar{z}\lambda^2z^2 - \bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2,$$

$$-1 + \bar{w} + \bar{z} - \bar{w}\bar{z} + w - \bar{w}w - \bar{z}w + \bar{w}\bar{z}w + z - \bar{w}z - \bar{w}\bar{z}\lambda^2z + \bar{z}^2\lambda^2z - wz + \bar{w}wz - \bar{z}\lambda^2wz + 2\bar{w}\bar{z}\lambda^2wz - \bar{z}^2\lambda^2wz + \bar{w}\bar{z}\lambda^4wz - \bar{z}^2\lambda^4wz + \bar{z}\lambda^2z^2 - \bar{w}\bar{z}\lambda^2z^2 - \bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2,$$

$$1 - 2\bar{w} + \bar{w}^2 - 2w + 4\bar{w}w - 2\bar{w}^2w + 2\bar{w}\lambda^2w - 2\bar{w}^2\lambda^2w + w^2 - 2\bar{w}w^2 + \bar{w}^2w^2 - 2\bar{w}\lambda^2w^2 + 2\bar{w}^2\lambda^2w^2 + \bar{w}^2\lambda^4w^2 + 2\bar{z}\lambda^2z - 2\bar{w}\bar{z}\lambda^2z - 2\bar{z}\lambda^2wz + 2\bar{w}\bar{z}\lambda^2wz - 2\bar{w}\bar{z}\lambda^4wz + \bar{z}^2\lambda^4z^2,$$

$$\bar{w} - \bar{w}^2 - \bar{w}w + \bar{w}^2w - \bar{w}\lambda^2w + \bar{w}^2\lambda^2w - z + \bar{w}z + 2\bar{w}\lambda^2z - 2\bar{w}^2\lambda^2z - \bar{z}\lambda^2z + \bar{w}\bar{z}\lambda^2z + wz - \bar{w}wz - \bar{w}\lambda^2wz + \bar{w}^2\lambda^2wz + 2\bar{z}\lambda^2wz - 2\bar{w}\bar{z}\lambda^2wz + \bar{w}^2\lambda^4wz - 2\bar{w}\bar{z}\lambda^4wz + \bar{z}^2\lambda^4wz - \bar{z}\lambda^2z^2 + \bar{w}\bar{z}\lambda^2z^2 - \bar{w}^2\lambda^4z^2 + 2\bar{w}\bar{z}\lambda^4z^2 - \bar{z}^2\lambda^4z^2,$$

$$\bar{z} - \bar{w}\bar{z} - w + \bar{w}w - \bar{z}w + \bar{w}\bar{z}w + \bar{w}\lambda^2w - 2\bar{z}\lambda^2w + \bar{w}\bar{z}\lambda^2w + w^2 - \bar{w}w^2 - \bar{w}\lambda^2w^2 + 2\bar{z}\lambda^2w^2 - \bar{w}\bar{z}\lambda^2w^2 - \bar{w}\bar{z}\lambda^4w^2 + \bar{z}^2\lambda^4w^2 + \bar{z}\lambda^2z - 2\bar{w}\bar{z}\lambda^2z + \bar{z}^2\lambda^2z - \bar{z}\lambda^2wz + 2\bar{w}\bar{z}\lambda^2wz - \bar{z}^2\lambda^2wz + 2\bar{w}\bar{z}\lambda^4wz - 2\bar{z}^2\lambda^4wz - \bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2,$$

$$1 - \bar{w} + \bar{z}\lambda^2 - \bar{w}\bar{z}\lambda^2 - w + \bar{w}w - 2\bar{z}\lambda^2w + 2\bar{w}\bar{z}\lambda^2w - \bar{w}\lambda^4w + 2\bar{w}\bar{z}\lambda^4w - \bar{z}^2\lambda^4w + \lambda^2z - 2\bar{w}\lambda^2z + \bar{z}\lambda^2z + \bar{z}\lambda^4z - 2\bar{w}\bar{z}\lambda^4z + \bar{z}^2\lambda^4z - \lambda^2wz + 2\bar{w}\lambda^2wz - \bar{w}\bar{z}\lambda^2wz + 2\bar{w}\lambda^4wz - 2\bar{z}\lambda^4wz + \bar{w}\bar{z}\lambda^6wz - \bar{z}^2\lambda^6wz - \bar{w}\lambda^4z^2 + \bar{z}\lambda^4z^2 - \bar{w}\bar{z}\lambda^6z^2 + \bar{z}^2\lambda^6z^2,$$

$$1 - \bar{w} - \bar{w}\lambda^2 + \bar{z}\lambda^2 - w + \bar{w}w - \lambda^2w + \bar{w}\lambda^2w - \bar{z}\lambda^2w + \bar{w}\bar{z}\lambda^2w - \bar{z}\lambda^4w + \bar{w}\bar{z}\lambda^4w + \lambda^2z - \bar{w}\lambda^2z + \bar{z}\lambda^2z - \bar{w}\bar{z}\lambda^2z - \bar{w}\lambda^4z + \bar{z}\lambda^4z - \bar{w}\bar{z}\lambda^4z + \bar{z}^2\lambda^4z + \bar{w}\lambda^2wz - \bar{z}\lambda^2wz + \bar{w}\lambda^4wz - \bar{z}\lambda^4wz + \bar{w}\bar{z}\lambda^4wz - \bar{z}^2\lambda^4wz + \bar{w}\bar{z}\lambda^6wz - \bar{z}^2\lambda^6wz + \bar{z}\lambda^4z^2 - \bar{w}\bar{z}\lambda^4z^2 - \bar{w}\bar{z}\lambda^6z^2 + \bar{z}^2\lambda^6z^2,$$

$$\bar{w}^2 - 2\bar{w}\bar{z} + \bar{z}^2 - 2\bar{w}w + 2\bar{z}w + 2\bar{w}\bar{z}w - 2\bar{z}^2w - 4\bar{w}\lambda^2w + 4\bar{z}\lambda^2w + 4\bar{w}\bar{z}\lambda^2w - 4\bar{z}^2\lambda^2w + w^2 - 2\bar{z}w^2 + \bar{z}^2w^2 + 2\bar{w}z - 2\bar{w}^2z - 2\bar{z}z + 2\bar{w}\bar{z}z + 4\bar{w}\lambda^2z - 4\bar{z}\lambda^2z - 4\bar{w}\bar{z}\lambda^2z + 4\bar{z}^2\lambda^2z - 2wz + 2\bar{w}wz + 2\bar{z}wz - 2\bar{w}\bar{z}wz + 4\bar{w}\lambda^2wz - 4\bar{z}\lambda^2wz - 4\bar{w}\bar{z}\lambda^2wz + 4\bar{z}^2\lambda^2wz + z^2 - 2\bar{w}z^2 + \bar{w}^2z^2 - 4\bar{w}\lambda^2z^2 + 4\bar{z}\lambda^2z^2 + 4\bar{w}\bar{z}\lambda^2z^2 - 4\bar{z}^2\lambda^2z^2,$$

$$\bar{z}^2w^2 - \bar{z}^3w^2 + 2\bar{w}\bar{z}wz - 4\bar{w}^2\bar{z}wz - 4\bar{z}^2wz + 6\bar{w}\bar{z}^2wz - 4\bar{w}\bar{z}w^2z + 4\bar{w}^2\bar{z}w^2z + 3\bar{z}^2w^2z - 4\bar{w}\bar{z}^2w^2z + \bar{z}^3w^2z + 4\bar{w}^2\bar{z}\lambda^2w^2z - 8\bar{w}\bar{z}^2\lambda^2w^2z + 4\bar{z}^3\lambda^2w^2z + \bar{w}^2z^2 - 4\bar{w}\bar{z}z^2 + 3\bar{w}^2\bar{z}z^2 + 4\bar{z}^2z^2 - 4\bar{w}\bar{z}^2z^2 + 6\bar{w}\bar{z}wz^2 - 4\bar{w}^2\bar{z}wz^2 - 4\bar{z}^2wz^2 + 2\bar{w}\bar{z}^2wz^2 - 8\bar{w}^2\bar{z}\lambda^2wz^2 + 16\bar{w}\bar{z}^2\lambda^2wz^2 - 8\bar{z}^3\lambda^2wz^2 - \bar{w}^2z^3 + \bar{w}^2\bar{z}z^3 + 4\bar{w}^2\bar{z}\lambda^2z^3 - 8\bar{w}\bar{z}^2\lambda^2z^3 + 4\bar{z}^3\lambda^2z^3,$$

$$-\bar{w} + \bar{w}^2 + \bar{z} - \bar{w}\bar{z} - w + 3\bar{w}w - 2\bar{w}^2w - \bar{z}w + \bar{w}\bar{z}w + 2\bar{w}\lambda^2w - 2\bar{w}^2\lambda^2w - 2\bar{z}\lambda^2w + 2\bar{w}\bar{z}\lambda^2w + w^2 - 2\bar{w}w^2 + \bar{w}^2w^2 - 2\bar{w}\lambda^2w^2 + 2\bar{w}^2\lambda^2w^2 + 2\bar{z}\lambda^2w^2 - 2\bar{w}\bar{z}\lambda^2w^2 + \bar{w}^2\lambda^4w^2 - 2\bar{w}\bar{z}\lambda^4w^2 + \bar{z}^2\lambda^4w^2 + z - \bar{w}z - \bar{z}z + \bar{w}\bar{z}z - 2\bar{w}\lambda^2z + 2\bar{w}^2\lambda^2z + 2\bar{z}\lambda^2z - 2\bar{w}\bar{z}\lambda^2z - wz + \bar{w}wz + \bar{z}wz - \bar{w}\bar{z}wz + 2\bar{w}\lambda^2wz - 2\bar{w}^2\lambda^2wz - 2\bar{z}\lambda^2wz + 2\bar{w}\bar{z}\lambda^2wz - 2\bar{w}^2\lambda^4wz + 4\bar{w}\bar{z}\lambda^4wz - 2\bar{z}^2\lambda^4wz + \bar{w}^2\lambda^4z^2 - 2\bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2,$$

$$\bar{z}^2w^2 - 2\bar{w}\bar{z}^2w^2 + \bar{w}^2\bar{z}^2w^2 - 2\bar{w}\bar{z}wz + 2\bar{w}^2\bar{z}wz + 2\bar{w}\bar{z}^2wz - 2\bar{w}^2\bar{z}^2wz + 2\bar{w}\bar{z}w^2z - 2\bar{w}^2\bar{z}w^2z - 2\bar{z}^2w^2z + 2\bar{w}\bar{z}^2w^2z - 4\bar{w}^2\bar{z}\lambda^2w^2z + 4\bar{w}\bar{z}^2\lambda^2w^2z + 4\bar{w}^2\bar{z}^2\lambda^2w^2z - 4\bar{w}\bar{z}^3\lambda^2w^2z + \bar{w}^2z^2 - 2\bar{w}^2\bar{z}z^2 + \bar{w}^2\bar{z}^2z^2 - 2\bar{w}^2wz^2 + 2\bar{w}\bar{z}wz^2 + 2\bar{w}^2\bar{z}wz^2 - 2\bar{w}\bar{z}^2wz^2 + 4\bar{w}^2\bar{z}\lambda^2wz^2 - 4\bar{w}\bar{z}^2\lambda^2wz^2 - 4\bar{w}^2\bar{z}^2\lambda^2wz^2 + 4\bar{w}\bar{z}^3\lambda^2wz^2 + \bar{w}^2w^2z^2 - 2\bar{w}\bar{z}w^2z^2 + \bar{z}^2w^2z^2 + 4\bar{w}^2\bar{z}\lambda^2w^2z^2 - 4\bar{w}\bar{z}^2\lambda^2w^2z^2 - 4\bar{w}^2\bar{z}^2\lambda^2w^2z^2 + 4\bar{w}\bar{z}^3\lambda^2w^2z^2 - 4\bar{w}^2\bar{z}\lambda^2wz^3 + 4\bar{w}\bar{z}^2\lambda^2wz^3 + 4\bar{w}^2\bar{z}^2\lambda^2wz^3 - 4\bar{w}\bar{z}^3\lambda^2wz^3,$$

$$-\bar{w}w + \bar{w}^2w + \bar{w}\bar{z}w - \bar{w}^2\bar{z}w + \bar{w}w^2 - \bar{w}^2w^2 - \bar{w}\bar{z}w^2 + \bar{w}^2\bar{z}w^2 + \bar{z}z - 2\bar{w}\bar{z}z + \bar{w}^2\bar{z}z + \bar{w}wz - \bar{w}^2wz - 2\bar{z}wz + 3\bar{w}\bar{z}wz - \bar{w}^2\bar{z}wz + 2\bar{w}\bar{z}\lambda^2wz - 2\bar{w}^2\bar{z}\lambda^2wz - 2\bar{z}^2\lambda^2wz + 2\bar{w}\bar{z}^2\lambda^2wz - \bar{w}w^2z + \bar{w}^2w^2z + \bar{z}w^2z - \bar{w}\bar{z}w^2z - 2\bar{w}\bar{z}\lambda^2w^2z + 2\bar{w}^2\bar{z}\lambda^2w^2z + 2\bar{z}^2\lambda^2w^2z - 2\bar{w}\bar{z}^2\lambda^2w^2z + \bar{w}^2\bar{z}\lambda^4w^2z - 2\bar{w}\bar{z}^2\lambda^4w^2z + \bar{z}^3\lambda^4w^2z - 2\bar{w}\bar{z}\lambda^2z^2 + 2\bar{w}^2\bar{z}\lambda^2z^2 + 2\bar{z}^2\lambda^2z^2 - 2\bar{w}\bar{z}^2\lambda^2z^2 + 2\bar{w}\bar{z}\lambda^2wz^2 - 2\bar{w}^2\bar{z}\lambda^2wz^2 - 2\bar{z}^2\lambda^2wz^2 + 2\bar{w}\bar{z}^2\lambda^2wz^2 - 2\bar{w}^2\bar{z}\lambda^4wz^2 + 4\bar{w}\bar{z}^2\lambda^4wz^2 - 2\bar{z}^3\lambda^4wz^2 + \bar{w}^2\bar{z}\lambda^4z^3 - 2\bar{w}\bar{z}^2\lambda^4z^3 + \bar{z}^3\lambda^4z^3,$$

$$4\bar{w}^2w^2 - 4\bar{w}^3w^2 - 4\bar{w}\bar{z}w^2 + 4\bar{w}^2\bar{z}w^2 + \bar{z}^2w^2 - 2\bar{w}\bar{z}^2w^2 + \bar{w}^2\bar{z}^2w^2 - 4\bar{w}^2w^3 + 4\bar{w}^3w^3 + 4\bar{w}\bar{z}w^3 - 4\bar{w}^2\bar{z}w^3 - 4\bar{w}^2\bar{z}\lambda^2w^3 + 4\bar{w}^3\bar{z}\lambda^2w^3 + 4\bar{w}\bar{z}^2\lambda^2w^3 - 4\bar{w}^2\bar{z}^2\lambda^2w^3 - 4\bar{w}^2wz + 4\bar{w}^3wz + 2\bar{w}\bar{z}wz - 2\bar{w}^2\bar{z}wz + 2\bar{w}\bar{z}^2wz - 2\bar{w}^2\bar{z}^2wz + 4\bar{w}^2w^2z - 4\bar{w}^3w^2z - 2\bar{w}\bar{z}w^2z + 2\bar{w}^2\bar{z}w^2z - 2\bar{z}^2w^2z + 2\bar{w}\bar{z}^2w^2z - 4\bar{w}^3\lambda^2w^2z + 16\bar{w}^2\bar{z}\lambda^2w^2z - 8\bar{w}^3\bar{z}\lambda^2w^2z - 12\bar{w}\bar{z}^2\lambda^2w^2z + 8\bar{w}^2\bar{z}^2\lambda^2w^2z + 4\bar{w}^3\lambda^2w^3z -$$

$$8\bar{w}^2\bar{z}\lambda^2w^3z + 4\bar{w}\bar{z}^2\lambda^2w^3z + 4\bar{w}^3\bar{z}\lambda^4w^3z - 8\bar{w}^2\bar{z}^2\lambda^4w^3z + 4\bar{w}\bar{z}^3\lambda^4w^3z + \bar{w}^2z^2 - 2\bar{w}^2\bar{z}z^2 + \bar{w}^2\bar{z}^2z^2 - 2\bar{w}^2wz^2 + 2\bar{w}\bar{z}wz^2 + 2\bar{w}^2\bar{z}wz^2 - 2\bar{w}\bar{z}^2wz^2 + 4\bar{w}^3\lambda^2wz^2 - 12\bar{w}^2\bar{z}\lambda^2wz^2 + 4\bar{w}^3\bar{z}\lambda^2wz^2 + 8\bar{w}\bar{z}^2\lambda^2wz^2 - 4\bar{w}^2\bar{z}^2\lambda^2wz^2 + \bar{w}^2w^2z^2 - 2\bar{w}\bar{z}w^2z^2 + \bar{z}^2w^2z^2 - 4\bar{w}^3\lambda^2w^2z^2 + 8\bar{w}^2\bar{z}\lambda^2w^2z^2 - 4\bar{w}\bar{z}^2\lambda^2w^2z^2 - 8\bar{w}^3\bar{z}\lambda^4w^2z^2 + 16\bar{w}^2\bar{z}^2\lambda^4w^2z^2 - 8\bar{w}\bar{z}^3\lambda^4w^2z^2 + 4\bar{w}^3\bar{z}\lambda^4wz^3 - 8\bar{w}^2\bar{z}^2\lambda^4wz^3 + 4\bar{w}\bar{z}^3\lambda^4wz^3,$$

$$-\bar{z}^2w^2 + 2\bar{w}\bar{z}^2w^2 - \bar{w}^2\bar{z}^2w^2 + 2\bar{w}\bar{z}\lambda^2w^2 - 2\bar{w}^2\bar{z}\lambda^2w^2 - 2\bar{z}^2\lambda^2w^2 + 2\bar{w}\bar{z}^2\lambda^2w^2 - \bar{w}^2\lambda^4w^2 + 2\bar{w}\bar{z}\lambda^4w^2 - \bar{z}^2\lambda^4w^2 + 2\bar{w}\bar{z}wz - 2\bar{w}^2\bar{z}wz - 2\bar{w}\bar{z}^2wz + 2\bar{w}^2\bar{z}^2wz + 2\bar{w}^2\lambda^2wz - 4\bar{w}^2\bar{z}\lambda^2wz + 2\bar{w}^2\bar{z}^2\lambda^2wz + 2\bar{w}^2\lambda^4wz - 2\bar{w}\bar{z}\lambda^4wz - 2\bar{w}^2\bar{z}\lambda^4wz + 2\bar{w}\bar{z}^2\lambda^4wz - 2\bar{w}\bar{z}w^2z + 2\bar{w}^2\bar{z}w^2z + 2\bar{z}^2w^2z - 2\bar{w}\bar{z}^2w^2z - 2\bar{w}^2\lambda^2w^2z - 4\bar{w}\bar{z}\lambda^2w^2z + 8\bar{w}^2\bar{z}\lambda^2w^2z + 4\bar{z}^2\lambda^2w^2z - 4\bar{w}\bar{z}^2\lambda^2w^2z - 2\bar{w}^2\bar{z}^2\lambda^2w^2z - 2\bar{w}\bar{z}\lambda^4w^2z - 2\bar{w}^2\bar{z}\lambda^4w^2z + 2\bar{z}^2\lambda^4w^2z + 2\bar{w}\bar{z}^2\lambda^4w^2z + 4\bar{w}^2\bar{z}^2\lambda^4w^2z - 4\bar{w}\bar{z}^3\lambda^4w^2z - 4\bar{w}^2\bar{z}\lambda^6w^2z + 4\bar{w}\bar{z}^2\lambda^6w^2z + 4\bar{w}^2\bar{z}^2\lambda^6w^2z - 4\bar{w}\bar{z}^3\lambda^6w^2z - \bar{w}^2z^2 + 2\bar{w}^2\bar{z}z^2 - \bar{w}^2\bar{z}^2z^2 - 2\bar{w}^2\lambda^2z^2 + 4\bar{w}^2\bar{z}\lambda^2z^2 - 2\bar{w}^2\bar{z}^2\lambda^2z^2 - \bar{w}^2\lambda^4z^2 + 2\bar{w}^2\bar{z}\lambda^4z^2 - \bar{w}^2\bar{z}^2\lambda^4z^2 + 2\bar{w}^2wz^2 - 2\bar{w}\bar{z}wz^2 - 2\bar{w}^2\bar{z}wz^2 + 2\bar{w}\bar{z}^2wz^2 + 2\bar{w}^2\lambda^2wz^2 - 4\bar{w}^2\bar{z}\lambda^2wz^2 + 2\bar{w}^2\bar{z}^2\lambda^2wz^2 + 2\bar{w}\bar{z}\lambda^4wz^2 + 2\bar{w}^2\bar{z}\lambda^4wz^2 - 6\bar{w}\bar{z}^2\lambda^4wz^2 - 2\bar{w}^2\bar{z}^2\lambda^4wz^2 + 4\bar{w}\bar{z}^3\lambda^4wz^2 + 4\bar{w}^2\bar{z}\lambda^6wz^2 - 4\bar{w}\bar{z}^2\lambda^6wz^2 - 4\bar{w}^2\bar{z}^2\lambda^6wz^2 + 4\bar{w}\bar{z}^3\lambda^6wz^2 - \bar{w}^2w^2z^2 + 2\bar{w}\bar{z}w^2z^2 - \bar{z}^2w^2z^2 + 2\bar{w}\bar{z}\lambda^2w^2z^2 - 2\bar{w}^2\bar{z}\lambda^2w^2z^2 - 2\bar{z}^2\lambda^2w^2z^2 + 2\bar{w}\bar{z}^2\lambda^2w^2z^2 + 4\bar{w}^2\bar{z}\lambda^4w^2z^2 - \bar{z}^2\lambda^4w^2z^2 - 2\bar{w}\bar{z}^2\lambda^4w^2z^2 - 5\bar{w}^2\bar{z}^2\lambda^4w^2z^2 + 4\bar{w}\bar{z}^3\lambda^4w^2z^2 + 4\bar{w}^2\bar{z}\lambda^6w^2z^2 - 4\bar{w}\bar{z}^2\lambda^6w^2z^2 - 4\bar{w}^2\bar{z}^2\lambda^6w^2z^2 + 4\bar{w}\bar{z}^3\lambda^6w^2z^2 - 4\bar{w}^2\bar{z}\lambda^4wz^3 + 4\bar{w}\bar{z}^2\lambda^4wz^3 + 4\bar{w}^2\bar{z}^2\lambda^4wz^3 - 4\bar{w}\bar{z}^3\lambda^4wz^3 - 4\bar{w}^2\bar{z}\lambda^6wz^3 + 4\bar{w}\bar{z}^2\lambda^6wz^3 + 4\bar{w}^2\bar{z}^2\lambda^6wz^3 - 4\bar{w}\bar{z}^3\lambda^6wz^3,$$

$$1 - 2\bar{w} + \bar{w}^2 - 2w + 4\bar{w}w - 2\bar{w}^2w + 2\bar{w}\lambda^2w - 2\bar{w}^2\lambda^2w - 4\bar{z}\lambda^2w + 4\bar{w}\bar{z}\lambda^2w + 2\bar{z}^2\lambda^2w -$$
$$2\bar{w}\bar{z}^2\lambda^2w + w^2 - 2\bar{w}w^2 + \bar{w}^2w^2 - 2\bar{w}\lambda^2w^2 + 2\bar{w}^2\lambda^2w^2 + 4\bar{z}\lambda^2w^2 - 4\bar{w}\bar{z}\lambda^2w^2 - 2\bar{z}^2\lambda^2w^2 +$$
$$2\bar{w}\bar{z}^2\lambda^2w^2 + \bar{w}^2\lambda^4w^2 - 4\bar{w}\bar{z}\lambda^4w^2 + 4\bar{z}^2\lambda^4w^2 + 2\bar{w}\bar{z}^2\lambda^4w^2 - 4\bar{z}^3\lambda^4w^2 + \bar{z}^4\lambda^4w^2 - 4\bar{w}\lambda^2z +$$
$$4\bar{w}^2\lambda^2z + 8\bar{z}\lambda^2z - 8\bar{w}\bar{z}\lambda^2z - 4\bar{z}^2\lambda^2z + 4\bar{w}\bar{z}^2\lambda^2z + 4\bar{w}\lambda^2wz - 4\bar{w}^2\lambda^2wz - 8\bar{z}\lambda^2wz + 10\bar{w}\bar{z}\lambda^2wz -$$
$$2\bar{w}^2\bar{z}\lambda^2wz + 2\bar{z}^2\lambda^2wz - 2\bar{w}\bar{z}^2\lambda^2wz - 4\bar{w}^2\lambda^4wz + 14\bar{w}\bar{z}\lambda^4wz - 2\bar{w}^2\bar{z}\lambda^4wz - 14\bar{z}^2\lambda^4wz +$$
$$2\bar{w}\bar{z}^2\lambda^4wz + 8\bar{z}^3\lambda^4wz - 4\bar{w}\bar{z}^3\lambda^4wz - 2\bar{w}\bar{z}\lambda^2w^2z + 2\bar{w}^2\bar{z}\lambda^2w^2z + 2\bar{z}^2\lambda^2w^2z - 2\bar{w}\bar{z}^2\lambda^2w^2z -$$
$$2\bar{w}\bar{z}\lambda^4w^2z + 4\bar{w}^2\bar{z}\lambda^4w^2z + 2\bar{z}^2\lambda^4w^2z - 8\bar{w}\bar{z}^2\lambda^4w^2z + 4\bar{z}^3\lambda^4w^2z + 2\bar{w}\bar{z}^3\lambda^4w^2z - 2\bar{z}^4\lambda^4w^2z +$$
$$2\bar{w}^2\bar{z}\lambda^6w^2z - 6\bar{w}\bar{z}^2\lambda^6w^2z + 4\bar{z}^3\lambda^6w^2z + 2\bar{w}\bar{z}^3\lambda^6w^2z - 2\bar{z}^4\lambda^6w^2z + 2\bar{w}\lambda^2z^2 - 2\bar{w}^2\lambda^2z^2 -$$
$$4\bar{z}\lambda^2z^2 + 2\bar{w}\bar{z}\lambda^2z^2 + 2\bar{w}^2\bar{z}\lambda^2z^2 + 4\bar{z}^2\lambda^2z^2 - 4\bar{w}\bar{z}^2\lambda^2z^2 + 4\bar{w}^2\lambda^4z^2 - 14\bar{w}\bar{z}\lambda^4z^2 + 2\bar{w}^2\bar{z}\lambda^4z^2 +$$
$$14\bar{z}^2\lambda^4z^2 - 2\bar{w}\bar{z}^2\lambda^4z^2 - 8\bar{z}^3\lambda^4z^2 + 4\bar{w}\bar{z}^3\lambda^4z^2 - 2\bar{w}\lambda^2wz^2 + 2\bar{w}^2\lambda^2wz^2 + 4\bar{z}\lambda^2wz^2 - 2\bar{w}\bar{z}\lambda^2wz^2 -$$
$$2\bar{w}^2\bar{z}\lambda^2wz^2 - 4\bar{z}^2\lambda^2wz^2 + 4\bar{w}\bar{z}^2\lambda^2wz^2 + 2\bar{w}^2\lambda^4wz^2 + 2\bar{w}\bar{z}\lambda^4wz^2 - 8\bar{w}^2\bar{z}\lambda^4wz^2 - 2\bar{z}^2\lambda^4wz^2 +$$
$$8\bar{w}\bar{z}^2\lambda^4wz^2 - 4\bar{z}^3\lambda^4wz^2 + 2\bar{w}\bar{z}^3\lambda^4wz^2 - 6\bar{w}^2\bar{z}\lambda^6wz^2 + 18\bar{w}\bar{z}^2\lambda^6wz^2 - 12\bar{z}^3\lambda^6wz^2 - 6\bar{w}\bar{z}^3\lambda^6wz^2 +$$
$$6\bar{z}^4\lambda^6wz^2 + \bar{w}^2\bar{z}^2\lambda^4w^2z^2 - 2\bar{w}\bar{z}^3\lambda^4w^2z^2 + \bar{z}^4\lambda^4w^2z^2 + 2\bar{w}^2\bar{z}^2\lambda^6w^2z^2 - 4\bar{w}\bar{z}^3\lambda^6w^2z^2 + 2\bar{z}^4\lambda^6w^2z^2 +$$
$$\bar{w}^2\bar{z}^2\lambda^8w^2z^2 - 2\bar{w}\bar{z}^3\lambda^8w^2z^2 + \bar{z}^4\lambda^8w^2z^2 - 4\bar{w}^2\lambda^4z^3 + 8\bar{w}\bar{z}\lambda^4z^3 + 4\bar{w}^2\bar{z}\lambda^4z^3 - 8\bar{z}^2\lambda^4z^3 -$$
$$4\bar{w}\bar{z}^2\lambda^4z^3 + 8\bar{z}^3\lambda^4z^3 - 4\bar{w}\bar{z}^3\lambda^4z^3 + 4\bar{w}^2\bar{z}\lambda^6z^3 - 12\bar{w}\bar{z}^2\lambda^6z^3 + 8\bar{z}^3\lambda^6z^3 + 4\bar{w}\bar{z}^3\lambda^6z^3 - 4\bar{z}^4\lambda^6z^3 -$$
$$4\bar{w}\bar{z}\lambda^4wz^3 + 2\bar{w}^2\bar{z}\lambda^4wz^3 + 4\bar{z}^2\lambda^4wz^3 + 2\bar{w}\bar{z}^2\lambda^4wz^3 - 2\bar{w}^2\bar{z}^2\lambda^4wz^3 - 4\bar{z}^3\lambda^4wz^3 + 2\bar{w}\bar{z}^3\lambda^4wz^3 +$$
$$2\bar{w}^2\bar{z}\lambda^6wz^3 - 6\bar{w}\bar{z}^2\lambda^6wz^3 - 4\bar{w}^2\bar{z}^2\lambda^6wz^3 + 4\bar{z}^3\lambda^6wz^3 + 10\bar{w}\bar{z}^3\lambda^6wz^3 - 6\bar{z}^4\lambda^6wz^3 - 2\bar{w}^2\bar{z}^2\lambda^8wz^3 +$$
$$4\bar{w}\bar{z}^3\lambda^8wz^3 - 2\bar{z}^4\lambda^8wz^3 + \bar{w}^2\lambda^4z^4 - 2\bar{w}^2\bar{z}\lambda^4z^4 + \bar{w}^2\bar{z}^2\lambda^4z^4 - 2\bar{w}^2\bar{z}\lambda^6z^4 + 6\bar{w}\bar{z}^2\lambda^6z^4 + 2\bar{w}^2\bar{z}^2\lambda^6z^4 -$$
$$4\bar{z}^3\lambda^6z^4 - 6\bar{w}\bar{z}^3\lambda^6z^4 + 4\bar{z}^4\lambda^6z^4 + \bar{w}^2\bar{z}^2\lambda^8z^4 - 2\bar{w}\bar{z}^3\lambda^8z^4 + \bar{z}^4\lambda^8z^4,$$

$$\bar{z}^2\lambda^4w^2 - 2\bar{z}^3\lambda^4w^2 + \bar{z}^4\lambda^4w^2 + 4\bar{w}\bar{z}\lambda^2wz - 4\bar{w}^2\bar{z}\lambda^2wz - 2\bar{z}^2\lambda^2wz - 2\bar{w}\bar{z}^2\lambda^2wz + 4\bar{w}^2\bar{z}^2\lambda^2wz +$$
$$2\bar{z}^3\lambda^2wz - 2\bar{w}\bar{z}^3\lambda^2wz + 2\bar{w}\bar{z}\lambda^4wz - 4\bar{w}^2\bar{z}\lambda^4wz - 2\bar{z}^2\lambda^4wz + 2\bar{w}\bar{z}^2\lambda^4wz + 4\bar{w}^2\bar{z}^2\lambda^4wz +$$
$$2\bar{z}^3\lambda^4wz - 4\bar{w}\bar{z}^3\lambda^4wz - 4\bar{w}\bar{z}\lambda^2w^2z + 4\bar{w}^2\bar{z}\lambda^2w^2z + 2\bar{z}^2\lambda^2w^2z + 2\bar{w}\bar{z}^2\lambda^2w^2z - 4\bar{w}^2\bar{z}^2\lambda^2w^2z -$$
$$2\bar{z}^3\lambda^2w^2z + 2\bar{w}\bar{z}^3\lambda^2w^2z - 4\bar{w}\bar{z}\lambda^4w^2z + 8\bar{w}^2\bar{z}\lambda^4w^2z - 4\bar{w}\bar{z}^2\lambda^4w^2z - 8\bar{w}^2\bar{z}^2\lambda^4w^2z + 4\bar{z}^3\lambda^4w^2z +$$
$$8\bar{w}\bar{z}^3\lambda^4w^2z - 4\bar{z}^4\lambda^4w^2z + 4\bar{w}^2\bar{z}\lambda^6w^2z - 6\bar{w}\bar{z}^2\lambda^6w^2z - 4\bar{w}^2\bar{z}^2\lambda^6w^2z + 2\bar{z}^3\lambda^6w^2z + 6\bar{w}\bar{z}^3\lambda^6w^2z -$$

$$2\bar{z}^4\lambda^6w^2z + \bar{z}^2z^2 - 2\bar{w}\bar{z}^2z^2 + \bar{w}^2\bar{z}^2z^2 - 2\bar{w}\bar{z}\lambda^2z^2 + 2\bar{w}^2\bar{z}\lambda^2z^2 + 2\bar{z}^2\lambda^2z^2 - 2\bar{w}\bar{z}^2\lambda^2z^2 + \bar{w}^2\lambda^4z^2 -$$
$$2\bar{w}\bar{z}\lambda^4z^2 + \bar{z}^2\lambda^4z^2 - 2\bar{z}^2wz^2 + 4\bar{w}\bar{z}^2wz^2 - 2\bar{w}^2\bar{z}^2wz^2 - 2\bar{w}\bar{z}\lambda^2wz^2 + 2\bar{w}^2\bar{z}\lambda^2wz^2 - 2\bar{z}^2\lambda^2wz^2 +$$
$$10\bar{w}\bar{z}^2\lambda^2wz^2 - 8\bar{w}^2\bar{z}^2\lambda^2wz^2 - 4\bar{z}^3\lambda^2wz^2 + 4\bar{w}\bar{z}^3\lambda^2wz^2 + 2\bar{w}\bar{z}\lambda^4wz^2 - 4\bar{w}^2\bar{z}\lambda^4wz^2 + 8\bar{w}\bar{z}^2\lambda^4wz^2 -$$
$$2\bar{w}^2\bar{z}^2\lambda^4wz^2 - 8\bar{z}^3\lambda^4wz^2 + 2\bar{w}\bar{z}^3\lambda^4wz^2 + 2\bar{z}^4\lambda^4wz^2 - 6\bar{w}^2\bar{z}\lambda^6wz^2 + 10\bar{w}\bar{z}^2\lambda^6wz^2 + 4\bar{w}^2\bar{z}^2\lambda^6wz^2 -$$
$$4\bar{z}^3\lambda^6wz^2 - 6\bar{w}\bar{z}^3\lambda^6wz^2 + 2\bar{z}^4\lambda^6wz^2 + \bar{z}^2w^2z^2 - 2\bar{w}\bar{z}^2w^2z^2 + \bar{w}^2\bar{z}^2w^2z^2 + 4\bar{w}\bar{z}\lambda^2w^2z^2 -$$
$$4\bar{w}^2\bar{z}\lambda^2w^2z^2 - 8\bar{w}\bar{z}^2\lambda^2w^2z^2 + 8\bar{w}^2\bar{z}^2\lambda^2w^2z^2 + 4\bar{z}^3\lambda^2w^2z^2 - 4\bar{w}\bar{z}^3\lambda^2w^2z^2 + 4\bar{w}\bar{z}\lambda^4w^2z^2 -$$
$$8\bar{w}^2\bar{z}\lambda^4w^2z^2 - 2\bar{w}\bar{z}^2\lambda^4w^2z^2 + 14\bar{w}^2\bar{z}^2\lambda^4w^2z^2 + 2\bar{z}^3\lambda^4w^2z^2 - 14\bar{w}\bar{z}^3\lambda^4w^2z^2 + 4\bar{z}^4\lambda^4w^2z^2 -$$
$$4\bar{w}^2\bar{z}\lambda^6w^2z^2 + 4\bar{w}\bar{z}^2\lambda^6w^2z^2 + 8\bar{w}^2\bar{z}^2\lambda^6w^2z^2 - 12\bar{w}\bar{z}^3\lambda^6w^2z^2 + 4\bar{z}^4\lambda^6w^2z^2 + \bar{w}^2\bar{z}^2\lambda^8w^2z^2 -$$
$$2\bar{w}\bar{z}^3\lambda^8w^2z^2 + \bar{z}^4\lambda^8w^2z^2 + 2\bar{w}\bar{z}\lambda^2z^3 - 2\bar{w}^2\bar{z}\lambda^2z^3 - 4\bar{w}\bar{z}^2\lambda^2z^3 + 4\bar{w}^2\bar{z}^2\lambda^2z^3 + 2\bar{z}^3\lambda^2z^3 -$$
$$2\bar{w}\bar{z}^3\lambda^2z^3 - 2\bar{w}^2\lambda^4z^3 + 2\bar{w}\bar{z}\lambda^4z^3 + 4\bar{w}^2\bar{z}\lambda^4z^3 - 8\bar{w}\bar{z}^2\lambda^4z^3 + 2\bar{w}^2\bar{z}^2\lambda^4z^3 + 4\bar{z}^3\lambda^4z^3 - 2\bar{w}\bar{z}^3\lambda^4z^3 +$$
$$2\bar{w}^2\bar{z}\lambda^6z^3 - 4\bar{w}\bar{z}^2\lambda^6z^3 + 2\bar{z}^3\lambda^6z^3 - 2\bar{w}\bar{z}\lambda^2wz^3 + 2\bar{w}^2\bar{z}\lambda^2wz^3 + 4\bar{w}\bar{z}^2\lambda^2wz^3 - 4\bar{w}^2\bar{z}^2\lambda^2wz^3 -$$
$$2\bar{z}^3\lambda^2wz^3 + 2\bar{w}\bar{z}^3\lambda^2wz^3 - 4\bar{w}\bar{z}\lambda^4wz^3 + 8\bar{w}^2\bar{z}\lambda^4wz^3 + 2\bar{w}\bar{z}^2\lambda^4wz^3 - 14\bar{w}^2\bar{z}^2\lambda^4wz^3 - 2\bar{z}^3\lambda^4wz^3 +$$
$$14\bar{w}\bar{z}^3\lambda^4wz^3 - 4\bar{z}^4\lambda^4wz^3 + 6\bar{w}^2\bar{z}\lambda^6wz^3 - 6\bar{w}\bar{z}^2\lambda^6wz^3 - 12\bar{w}^2\bar{z}^2\lambda^6wz^3 + 18\bar{w}\bar{z}^3\lambda^6wz^3 - 6\bar{z}^4\lambda^6wz^3 -$$
$$2\bar{w}^2\bar{z}^2\lambda^8wz^3 + 4\bar{w}\bar{z}^3\lambda^8wz^3 - 2\bar{z}^4\lambda^8wz^3 + \bar{w}^2\lambda^4z^4 - 4\bar{w}^2\bar{z}\lambda^4z^4 + 2\bar{w}\bar{z}^2\lambda^4z^4 + 4\bar{w}^2\bar{z}^2\lambda^4z^4 -$$
$$4\bar{w}\bar{z}^3\lambda^4z^4 + \bar{z}^4\lambda^4z^4 - 2\bar{w}^2\bar{z}\lambda^6z^4 + 2\bar{w}\bar{z}^2\lambda^6z^4 + 4\bar{w}^2\bar{z}^2\lambda^6z^4 - 6\bar{w}\bar{z}^3\lambda^6z^4 + 2\bar{z}^4\lambda^6z^4 + \bar{w}^2\bar{z}^2\lambda^8z^4 -$$
$$2\bar{w}\bar{z}^3\lambda^8z^4 + \bar{z}^4\lambda^8z^4,$$

$$\bar{z}^2-2\bar{w}\bar{z}^2+\bar{w}^2\bar{z}^2-2\bar{z}^2w+4\bar{w}\bar{z}^2w-2\bar{w}^2\bar{z}^2w+2\bar{w}\bar{z}\lambda^2w-2\bar{w}^2\bar{z}\lambda^2w-2\bar{z}^2\lambda^2w+2\bar{w}\bar{z}^2\lambda^2w+$$
$$\bar{z}^2w^2-2\bar{w}\bar{z}^2w^2+\bar{w}^2\bar{z}^2w^2-2\bar{w}\bar{z}\lambda^2w^2+2\bar{w}^2\bar{z}\lambda^2w^2+2\bar{z}^2\lambda^2w^2-2\bar{w}\bar{z}^2\lambda^2w^2+\bar{w}^2\lambda^4w^2-$$
$$2\bar{w}\bar{z}\lambda^4w^2+\bar{z}^2\lambda^4w^2-2\bar{z}z+4\bar{w}\bar{z}z-2\bar{w}^2\bar{z}z-2\bar{w}\bar{z}\lambda^2z+2\bar{w}^2\bar{z}\lambda^2z+2\bar{z}^2\lambda^2z-2\bar{w}\bar{z}^2\lambda^2z+$$
$$4\bar{z}wz-8\bar{w}\bar{z}wz+4\bar{w}^2\bar{z}wz+2\bar{w}\lambda^2wz-2\bar{w}^2\lambda^2wz-2\bar{z}\lambda^2wz-4\bar{w}\bar{z}\lambda^2wz+6\bar{w}^2\bar{z}\lambda^2wz+$$
$$6\bar{z}^2\lambda^2wz-4\bar{w}\bar{z}^2\lambda^2wz-2\bar{w}^2\bar{z}^2\lambda^2wz-2\bar{z}^3\lambda^2wz+2\bar{w}\bar{z}^3\lambda^2wz-2\bar{w}^2\lambda^4wz+8\bar{w}\bar{z}\lambda^4wz-$$
$$4\bar{w}^2\bar{z}\lambda^4wz-6\bar{z}^2\lambda^4wz+4\bar{w}^2\bar{z}^2\lambda^4wz+4\bar{z}^3\lambda^4wz-4\bar{w}\bar{z}^3\lambda^4wz-2\bar{z}w^2z+4\bar{w}\bar{z}w^2z-2\bar{w}^2\bar{z}w^2z-$$
$$2\bar{w}\lambda^2w^2z+2\bar{w}^2\lambda^2w^2z+2\bar{z}\lambda^2w^2z+6\bar{w}\bar{z}\lambda^2w^2z-8\bar{w}^2\bar{z}\lambda^2w^2z-8\bar{z}^2\lambda^2w^2z+6\bar{w}\bar{z}^2\lambda^2w^2z+$$
$$2\bar{w}^2\bar{z}^2\lambda^2w^2z+2\bar{z}^3\lambda^2w^2z-2\bar{w}\bar{z}^3\lambda^2w^2z-4\bar{w}\bar{z}\lambda^4w^2z+2\bar{w}^2\bar{z}\lambda^4w^2z+4\bar{z}^2\lambda^4w^2z+4\bar{w}\bar{z}^2\lambda^4w^2z-$$
$$4\bar{w}^2\bar{z}^2\lambda^4w^2z-6\bar{z}^3\lambda^4w^2z+4\bar{w}\bar{z}^3\lambda^4w^2z+4\bar{w}^2\bar{z}\lambda^6w^2z-8\bar{w}\bar{z}^2\lambda^6w^2z-4\bar{w}^2\bar{z}^2\lambda^6w^2z+4\bar{z}^3\lambda^6w^2z+$$
$$8\bar{w}\bar{z}^3\lambda^6w^2z-4\bar{z}^4\lambda^6w^2z+z^2-2\bar{w}z^2+\bar{w}^2z^2-2\bar{w}\lambda^2z^2+2\bar{w}^2\lambda^2z^2+2\bar{z}\lambda^2z^2+6\bar{w}\bar{z}\lambda^2z^2-$$
$$8\bar{w}^2\bar{z}\lambda^2z^2-8\bar{z}^2\lambda^2z^2+6\bar{w}\bar{z}^2\lambda^2z^2+2\bar{w}^2\bar{z}^2\lambda^2z^2+2\bar{z}^3\lambda^2z^2-2\bar{w}\bar{z}^3\lambda^2z^2+\bar{w}^2\lambda^4z^2-6\bar{w}\bar{z}\lambda^4z^2+$$
$$4\bar{w}^2\bar{z}\lambda^4z^2+5\bar{z}^2\lambda^4z^2-4\bar{w}^2\bar{z}^2\lambda^4z^2-4\bar{z}^3\lambda^4z^2+4\bar{w}\bar{z}^3\lambda^4z^2-2wz^2+4\bar{w}wz^2-2\bar{w}^2wz^2+$$
$$2\bar{w}\lambda^2wz^2-2\bar{w}^2\lambda^2wz^2-2\bar{z}\lambda^2wz^2-4\bar{w}\bar{z}\lambda^2wz^2+6\bar{w}^2\bar{z}\lambda^2wz^2+6\bar{z}^2\lambda^2wz^2-4\bar{w}\bar{z}^2\lambda^2wz^2-$$
$$2\bar{w}^2\bar{z}^2\lambda^2wz^2-2\bar{z}^3\lambda^2wz^2+2\bar{w}\bar{z}^3\lambda^2wz^2+4\bar{w}^2\bar{z}\lambda^4wz^2-8\bar{w}\bar{z}^2\lambda^4wz^2+4\bar{z}^3\lambda^4wz^2-8\bar{w}^2\bar{z}\lambda^6wz^2+$$
$$16\bar{w}\bar{z}^2\lambda^6wz^2+8\bar{w}^2\bar{z}^2\lambda^6wz^2-8\bar{z}^3\lambda^6wz^2-16\bar{w}\bar{z}^3\lambda^6wz^2+8\bar{z}^4\lambda^6wz^2+w^2z^2-2\bar{w}w^2z^2+$$
$$\bar{w}^2w^2z^2-2\bar{w}\bar{z}\lambda^2w^2z^2+2\bar{w}^2\bar{z}\lambda^2w^2z^2+2\bar{z}^2\lambda^2w^2z^2-2\bar{w}\bar{z}^2\lambda^2w^2z^2+4\bar{w}\bar{z}\lambda^4w^2z^2-4\bar{w}^2\bar{z}\lambda^4w^2z^2-$$
$$4\bar{z}^2\lambda^4w^2z^2+5\bar{w}^2\bar{z}^2\lambda^4w^2z^2+4\bar{z}^3\lambda^4w^2z^2-6\bar{w}\bar{z}^3\lambda^4w^2z^2+\bar{z}^4\lambda^4w^2z^2-4\bar{w}^2\bar{z}\lambda^6w^2z^2+8\bar{w}\bar{z}^2\lambda^6w^2z^2+$$
$$4\bar{w}^2\bar{z}^2\lambda^6w^2z^2-4\bar{z}^3\lambda^6w^2z^2-8\bar{w}\bar{z}^3\lambda^6w^2z^2+4\bar{z}^4\lambda^6w^2z^2-2\bar{w}\bar{z}\lambda^2z^3+2\bar{w}^2\bar{z}\lambda^2z^3+2\bar{z}^2\lambda^2z^3-$$
$$2\bar{w}\bar{z}^2\lambda^2z^3+4\bar{w}\bar{z}\lambda^4z^3-6\bar{w}^2\bar{z}\lambda^4z^3-4\bar{z}^2\lambda^4z^3+4\bar{w}\bar{z}^2\lambda^4z^3+4\bar{w}^2\bar{z}^2\lambda^4z^3+2\bar{z}^3\lambda^4z^3-4\bar{w}\bar{z}^3\lambda^4z^3+$$
$$4\bar{w}^2\bar{z}\lambda^6z^3-8\bar{w}\bar{z}^2\lambda^6z^3-4\bar{w}^2\bar{z}^2\lambda^6z^3+4\bar{z}^3\lambda^6z^3+8\bar{w}\bar{z}^3\lambda^6z^3-4\bar{z}^4\lambda^6z^3+2\bar{w}\bar{z}\lambda^2wz^3-2\bar{w}^2\bar{z}\lambda^2wz^3-$$
$$2\bar{z}^2\lambda^2wz^3+2\bar{w}\bar{z}^2\lambda^2wz^3-4\bar{w}\bar{z}\lambda^4wz^3+4\bar{w}^2\bar{z}\lambda^4wz^3+4\bar{z}^2\lambda^4wz^3-6\bar{w}^2\bar{z}^2\lambda^4wz^3-4\bar{z}^3\lambda^4wz^3+$$
$$8\bar{w}\bar{z}^3\lambda^4wz^3-2\bar{z}^4\lambda^4wz^3+8\bar{w}^2\bar{z}\lambda^6wz^3-16\bar{w}\bar{z}^2\lambda^6wz^3-8\bar{w}^2\bar{z}^2\lambda^6wz^3+8\bar{z}^3\lambda^6wz^3+16\bar{w}\bar{z}^3\lambda^6wz^3-$$
$$8\bar{z}^4\lambda^6wz^3+\bar{w}^2\bar{z}^2\lambda^4z^4-2\bar{w}\bar{z}^3\lambda^4z^4+\bar{z}^4\lambda^4z^4-4\bar{w}^2\bar{z}\lambda^6z^4+8\bar{w}\bar{z}^2\lambda^6z^4+4\bar{w}^2\bar{z}^2\lambda^6z^4-4\bar{z}^3\lambda^6z^4-$$
$$8\bar{w}\bar{z}^3\lambda^6z^4+4\bar{z}^4\lambda^6z^4$$

CONCLUSIONES

En mérito a los resultados obtenidos, se concluye lo que sigue: 1) Las superpartículas, llamadas también partículas estrella o blancas, son aquellas cuyo centro de masa – energía es extremadamente denso, lo que le permite deformar el espacio – tiempo cuántico; 2) La deformación del espacio – tiempo cuántico, causada en supergravedad, puede ser por intervención gravitónica o no; de tal suerte que, es endógena, cuando la superpartícula, por su propia masa y energía, deforma el espacio – tiempo cuántico en tanto que, es exógena, cuando la superpartícula interactúa con el supergravitón o gravitino, lo que ocurre cuando el campo supergravitónico, permea el espacio – tiempo cuántico repercutido; 3) la deformación del espacio – tiempo cuántico en supergravedad, es drástica e intensa, en la medida en que, cuando la superpartícula interactúa, deforma el espacio – tiempo cuántico hasta formar un supercurvatura, en la que, el campo cuántico se vuelve perturbativo, y ésta supercurvatura deviene en la formación de un agujero negro cuántico, por la colisión o colapso inminente y previo de la partícula estrella o blanca, formándose consecuentemente, un agujero cuántico de gusano y finalmente, un agujero blanco cuántico, por la implosión de la singularidad. Téngase en cuenta, la definición concebida por este autor, respecto de la singularidad de un agujero negro cuántico. Véanse mis trabajos paralelos en este punto; 4) En supergravedad, el espacio – tiempo cuántico, se deforma en dimensiones altas, lo que da lugar a la creación de superespacios y por ende, supermembranas, entendidas éstas últimas, como regiones de dimensión disociada; 5) En supergravedad cuántica relativista, la supersimetría se vuelve esencial, incluso como contrapeso de la antisimetría; 6) En supergravedad, el entorno es eminentemente entrópico, por lo que, el principio de incertidumbre queda ratificado; 7) Las partículas supermasivas, son aquellas, cuya masa es extremadamente densa, con capacidad por tanto, de deformar el espacio – tiempo cuántico en términos racionales; 8) La colisión de una partícula supermasiva, produce agujeros negros cuánticos o en su defecto, su colapso en entornos cuánticos entrópicos; 9) En gravedad cuántica, es decir, la deformación del espacio – tiempo cuántico, al igual que en supergravedad, se produce por intervención gravitónica o sin intervención gravitónica, por lo que, ésta se tiene por exógena, cuando la partícula supermasiva interactúa con el gravitón en tanto que, se tiene por endógena, cuando la propia partícula supermasiva, a razón de su interacción, colisión o colapso, deforma el espacio – tiempo cuántico, en ambos casos, pudiendo provocar agujeros negros cuánticos; 9) La simetría se vuelve esencial en gravedad cuántica, como contrapeso de la simetría; y, 10) Las partículas blancas u oscuras, para efectos de determinar sus interacciones, matemáticamente es posible, a través de la identificación de sus osciladores y propagadores armónicos.

Aclaraciones Finales

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo † será reemplazado por este símbolo†o por este símbolo †, equivaliendo lo mismo.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
†	†
	†

2. En todos los casos, este símbolo ‡, será reemplazado por este símbolo‡o por este símbolo ‡.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
‡	‡
	‡

3. En todos los casos, se añadirá y por ende, se calculará la magnitud§que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo • podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

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