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**AGUJEROS CUÁNTICOS: APROXIMACIONES
TEÓRICAS RESPECTO DE LOS AGUJEROS
NEGROS CUÁNTICOS, AGUJEROS BLANCOS
CUÁNTICOS Y AGUJEROS CUÁNTICOS DE
GUSANO EN RELACIÓN A CAMPOS CUÁNTICOS
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VOLUMEN I**

QUANTUM HOLES: THEORETICAL APPROACHES TO QUANTUM
BLACK HOLES, WHITE QUANTUM HOLES, AND QUANTUM
WORMHOLES IN RELATION TO RELATIVISTIC OR CURVED
QUANTUM FIELDS AND UNDER CONDITIONS OF QUANTUM
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AGUJEROS CUÁNTICOS: APROXIMACIONES TEÓRICAS RESPECTO DE LOS AGUJEROS NEGROS CUÁNTICOS, AGUJEROS BLANCOS CUÁNTICOS Y AGUJEROS CUÁNTICOS DE GUSANO EN RELACIÓN A CAMPOS CUÁNTICOS RELATIVISTAS O CURVOS Y EN CONDICIONES DE GRAVEDAD Y SUPERGRAVEDAD CUÁNTICAS. VOLUMEN I

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RESUMEN

Los agujeros negros cuánticos, son sin duda, un campo de estudio fascinante en la teoría cuántica de campos relativistas. Éstos, según la TCCR (Teoría Cuántica de Campos Relativistas), se forman por partículas supermasivas o en su defecto, por superpartículas, de tal suerte que, en escenarios de supergravedad o gravedad cuánticas, según sea el caso, su formación es inminente en la primera y potencial en la segunda. En este trabajo, abordaré más a detalle, la formación de agujeros negros cuánticos supermasivos, especialmente en escenarios de extrema gravitación e intentaremos descifrar la singularidad de los mismos, incluyendo la formación de agujeros cuánticos de gusano, que en sí, son disímiles a los agujeros negros cuánticos. Estos agujeros negros, de carácter subatómico, interactúan con campos cuánticos específicos, con la particularidad, de que devoran energía a escala microscópica, lo que explicaría, en forma exponencial, la formación de materia oscura.

Palabras Clave: materia oscura, agujero negro cuántico, relatividad cuántica, agujeros cuánticos de gusano, partícula oscura, partícula estrella.

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QUANTUM HOLES: THEORETICAL APPROACHES TO QUANTUM BLACK HOLES, WHITE QUANTUM HOLES, AND QUANTUM WORMHOLES IN RELATION TO RELATIVISTIC OR CURVED QUANTUM FIELDS AND UNDER CONDITIONS OF QUANTUM GRAVITY AND SUPERGRAVITY. VOLUME I

ABSTRACT

Quantum black holes are undoubtedly a fascinating field of study in the quantum theory of relativistic fields. These, according to the TCCR (Quantum Theory of Relativistic Fields), are formed by supermassive particles or, failing that, by superparticles, in such a way that, in scenarios of supergravity or quantum gravity, as the case may be, their formation is imminent in the former and potential in the latter. In this work, I will address in more detail the formation of supermassive quantum black holes, especially in extreme gravitation scenarios and we will try to decipher their singularity, including the formation of quantum wormholes, which in themselves, are dissimilar to quantum black holes. These black holes, of a subatomic nature, interact with specific quantum fields, with the particularity that they devour energy on a microscopic scale, which would explain, exponentially, the formation of dark matter.

Keywords: dark matter, quantum black hole, quantum relativity, wormholes, dark particle, star particle.

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INTRODUCCIÓN.

Los agujeros negros cuánticos, son aquellos fenómenos propios de la dualidad holográfica. Se forman principalmente por la deformación del espacio – tiempo cuántico, sea por una partícula supermasiva o en su defecto, por una superpartícula, aunque es inevitable en el segundo caso, sea por colapso o colisión. Estos agujeros negros cuánticos, a diferencia de los que ocurren a escala cosmológica, devoran energía para convertirla en materia oscura. En este trabajo se consolidará el modelo matemático de un agujero negro cuántico y de un agujero cuántico de gusano, éste último el mismo que se explica a través del puente de Einstein – Rosen, como ha quedado referido en trabajos anteriores, lo que además, explicaría la superposición y el entrelazamiento cuánticos. Se ha dicho que estos fenómenos microscópicos, ocurren por colapso o colisión de una partícula supermasiva o de una superpartícula, según sea el caso, sin embargo, los agujeros negros cuánticos también pueden colisionar, formando así, ondas cuánticas o denominadas también “ondas cuánticas gravitacionales”, como se ha teorizado en trabajos anteriores. En este artículo, abordaré los fenómenos antes referidos, con especial énfasis, en la descripción matemática de los mismos, en un espacio de Hilbert – Einstein, tanto en supergravedad como en gravedad cuánticas, según corresponda, incluyendo la morfología propiamente de un agujero negro cuántico y un agujero cuántico de gusano.

RESULTADOS Y DISCUSIÓN.

El desarrollo matemático aquí contenido, complementa el sistema de ecuaciones desplegado en el volumen II de este manuscrito. Dicho esto, prosigo:

Modelo de campo cuántico relativista en supergravedad, en el que se forman agujeros negros cuánticos, agujeros blancos cuánticos y agujeros cuánticos de gusano.

$$\begin{aligned}\delta e_\mu^a &= \bar{\epsilon} \gamma^a \chi_\mu, \delta \chi_\mu = \left(D_\mu - \frac{A}{2} \gamma_\mu \right) \epsilon, \delta A = -\frac{1}{2} \bar{\epsilon} \gamma^{\mu\nu} \left(D_\mu - \frac{A}{2} \gamma_\mu \right) \chi_\nu \\ E &= e \left[1 + \frac{1}{2} \bar{\theta} \gamma^\mu \chi_\mu + \frac{1}{2} \bar{\theta} \theta \left(-A + \frac{1}{4} \bar{\chi}_\mu \gamma^{\mu\nu} \chi_\nu \right) \right], \\ \mathcal{R} &= -A + \bar{\theta} \gamma^{\mu\nu} \left(D_\mu - \frac{A}{2} \gamma_\mu \right) \chi_\nu + \frac{1}{2} \bar{\theta} \theta \left(\frac{R}{2} - A^2 - \bar{\chi}^\mu \gamma^\nu D_{[\mu} \chi_{\nu]} - \frac{A}{4} \bar{\chi}^\mu \chi_\mu \right) \\ S_{\mathcal{N}=1}^{(h)} &= \int_{\Sigma_h} d^2x \, d^2\theta E \left(-\frac{\vartheta}{\pi} \mathcal{R} + 2i\mu \right)\end{aligned}$$



$$M = \begin{pmatrix} a & b & b\alpha - a\rho \\ c & d & d\alpha - c\rho \\ \alpha & \rho & 1 - \alpha\rho \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\mathrm{ber}(M) \equiv \mathrm{det}A \mathrm{det}(D - CA^{-1}B)^{-1} = 1$$

$$\begin{aligned} z(Z') &= \frac{\alpha z' + b}{cz' + d} + \theta' \frac{\alpha z' + \rho}{(cz' + d)^2} \\ \theta(Z') &= \frac{\alpha z' + \rho}{cz' + d} + \frac{\theta'}{cz' + d} \end{aligned}$$

$$\begin{aligned} \partial_{z'Z} &= \frac{1}{(cz' + d)^2} \left(1 + \frac{\theta'}{(cz' + d)} (\alpha(d - cz') - 2\rho c) \right) \\ \partial_{z'}\theta &= \frac{1}{(cz' + d)^2} (\alpha d - \rho c) - \theta' \frac{c}{(cz' + d)^2} \\ \partial_{\theta'Z} &= \frac{\alpha z' + \rho}{(cz' + d)^2}, \partial_{\theta'}\theta = \frac{1}{cz' + d} \end{aligned}$$

$$\sigma \equiv \mathrm{ber} \begin{pmatrix} \partial_{z'Z} & \partial_{z'}\theta \\ \partial_{\theta'Z} & \partial_{\theta'}\theta \end{pmatrix} = \frac{1}{cz' + d} + \frac{\theta'}{(cz' + d)^2} (\alpha d + \rho c)$$

$$\Phi'[Z'] = \Phi[z(Z') \mid \theta(Z')] + \frac{1}{\beta} \log \sigma^{-1},$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{grav},(0)} = e^{2\vartheta} \times \mathcal{A} \times \left(\frac{r}{\ell_{\mathrm{uv}}}\right)^{\frac{c_m+c_{\mathrm{bc}}+c_{\beta\gamma}}{3}} \times \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\mathrm{vol}_{\mathrm{OSp}(1|2;\mathbb{C})}} e^{-\mathcal{S}^{\mathcal{N}=1}_{\mathrm{L}}}$$

$$\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{L}}[\mu] \equiv \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\mathrm{vol}_{\mathrm{OSp}(1|2;\mathbb{C})}} e^{-\mathcal{S}^{\mathcal{N}=1}_{\mathrm{L}}}$$

$$\mathcal{S}^{\mathcal{N}=1}_{\mathrm{tL}} = \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2x \tilde{\epsilon} \left(-\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{\mathrm{i}}{2} \bar{\psi} \not{D} \psi + \frac{1}{2} F^2 - \frac{1}{2} q \tilde{R} \varphi + \mathrm{i} \mu e^{\beta \varphi} F + \frac{1}{2} \mu \beta e^{\beta \varphi} \bar{\psi} \psi \right)$$

$$q=\sqrt{\frac{c_m}{3}-\frac{9}{2}},\beta=\frac{1}{2}\left(\sqrt{\frac{c_m}{3}-\frac{1}{2}}-\sqrt{\frac{c_m}{3}-\frac{9}{2}}\right)$$

$$\mathcal{S}^{\mathcal{N}=1}_{\mathrm{tL}} = \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2x \tilde{\epsilon} \left[-\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{\mathrm{i}}{2} \bar{\psi} \not{D} \psi + \frac{1}{2} F^2 - \frac{1}{2} q \tilde{R} \varphi + \mathrm{i} \left(F W'(\varphi) - \frac{\mathrm{i}}{2} W''(\varphi) \bar{\psi} \psi \right) \right]$$

$$\delta\varphi=\bar{\epsilon}\psi,\delta\psi=\mathrm{i}\not{\partial}\varphi\epsilon+q\mathrm{i}\not{D}\epsilon-\mathrm{i}F\epsilon,\delta F=-\bar{\epsilon}\not{D}\psi$$

$$\begin{aligned} [\delta_{\epsilon_1},\delta_{\epsilon_2}]\varphi &= \mathcal{L}_\xi\varphi + \Delta q \\ [\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi &= \mathcal{L}_\xi\psi + \frac{1}{2}\Delta\psi \\ [\delta_{\epsilon_1},\delta_{\epsilon_2}]F &= \mathcal{L}_\xi F + \Delta F \end{aligned}$$

$$\xi^\mu \equiv 2\mathrm{i}\bar{\epsilon}_2\gamma^\mu\epsilon_1, \Delta \equiv \frac{1}{2}\nabla_\mu\xi^\mu$$

$$\delta\varphi=\bar{\epsilon}\psi,\delta\psi=\mathrm{i}\,\partial\varphi\epsilon-\frac{q}{r}\epsilon-\mathrm{i}F\epsilon,\delta F=-\bar{\epsilon}\not{D}\psi$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}}=1[\mu]=\int\frac{[\mathcal{D}\varphi][\mathcal{D}\psi]}{\mathrm{vol}_{\mathrm{OSp}(1|2;\mathbb{C})}}e^{-\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\bar{\epsilon}\Big(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi+\frac{\mathrm{i}}{2}\bar{\psi}\not{D}\psi-\frac{q}{r^2}\varphi+\frac{1}{2}\mu^2e^{2\beta\varphi}+\frac{1}{2}\mu\beta e^{\beta\varphi}\bar{\psi}\psi\Big)}$$

$$-\nabla^2\varphi=\mu^2\beta e^{2\beta\varphi}-\frac{q}{r^2}+\frac{1}{2}\mu\beta^2e^{\beta\varphi}\bar{\psi}\psi,\not{D}\psi=\mathrm{i}\mu\beta e^{\beta\varphi}\psi$$

$$\varphi_*=\frac{1}{2\beta}\log\frac{q}{r^2\mu^2\beta},\psi_*=0$$

$$\mathcal{Z}_{\mathrm{tL}}^{\mathcal{N}=1}[\mu]\approx\mathcal{Z}_{\mathrm{saddle}}^{\mathcal{N}=1}[\mu]=\Big(\frac{q}{e\mu^2r^2\beta}\Big)^{\frac{q}{2\beta}}$$

$$\varphi=\varphi_*+\delta\varphi,\psi=\psi_*+\delta\psi$$

$$\mathcal{L}_{\mathrm{bos}}^{(2)}[\delta\varphi]=-\frac{1}{8\pi}\delta\varphi\left(-\nabla^2-\frac{2q\beta}{r^2}\right)\delta\varphi,\mathcal{L}_{\mathrm{fer}}^{(2)}[\delta\psi]=\frac{\mathrm{i}}{8\pi}\delta\bar{\psi}\left(\not{D}-\mathrm{i}\frac{\sqrt{q\beta}}{r}\right)\delta\psi$$

$$\mathcal{Z}_{\mathrm{tL,pert}}^{\mathcal{N}=1}=\int\left[\mathcal{D}\delta\varphi\right]\left[\mathcal{D}\delta\psi\right]e^{-\int_{S^2}\mathrm{d}^2x\,\mathrm{e}\left(\mathcal{L}_{\mathrm{bos}}^{(2)}[\delta\varphi]+\mathcal{L}_{\mathrm{fer}}^{(2)}[\delta\psi]\right)}$$

$$\varphi(\Omega)=\sum_{l=0}^{\infty}\sum_{m=-l}^l\varphi_{lm}Y_{lm}(\Omega),-\nabla^2Y_{lm}(\Omega)=\frac{1}{r^2}(l+1)Y_{lm}(\Omega)$$

$$\int_{S^2}\mathrm{d}\Omega Y_{lm}(\Omega)Y_{l'm'}(\Omega)=\delta_{ll'}\delta_{mm'}$$

$$\psi(\Omega)=\frac{1}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^l\Big(\alpha_{+lm}\psi_{+lm}^{(+)}(\Omega)+\alpha_{-lm}\psi_{-lm}^{(+)}(\Omega)+\beta_{+lm}\psi_{+lm}^{(-)}(\Omega)+\beta_{-lm}\psi_{-lm}^{(-)}(\Omega)\Big)$$

$$\psi(\Omega)=\frac{1}{\sqrt{r}}\sum_{l=0}^{\infty}\sum_{m=0}^l\Big(\alpha_{+lm}\psi_{+lm}^{(+)}(\Omega)+\alpha_{-lm}\psi_{-lm}^{(+)}(\Omega)-\mathrm{i}\alpha_{-lm}^*\psi_{+lm}^{(-)}(\Omega)-\mathrm{i}\alpha_{+lm}^*\psi_{-lm}^{(-)}(\Omega)\Big)$$

$$\not{D}\psi_{\pm lm}^{(s)}(\Omega)=\pm\frac{\mathrm{i}}{r}(l+1)\psi_{\pm lm}^{(s)}(\Omega),s=\pm$$

$$\int_{S^2}\mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(+)}(\Omega)\psi_{\pm l'm'}^{(-)}(\Omega)=-\mathrm{i}\delta_{ll'}\delta_{m,m'},\int_{S^2}\mathrm{d}\Omega\bar{\psi}_{\pm lm}^{(-)}(\Omega)\psi_{\pm l'm'}^{(+)}(\Omega)=+\mathrm{i}\delta_{ll'}\delta_{m,m'}$$

$$[\mathcal{D}\psi]=\prod_{l=0}^{\infty}\prod_{m=0}^l\left(\frac{2\pi}{\Lambda_{\mathrm{uv}}r^2}\right)\prod_{\pm}\mathrm{d}\alpha_{\pm lm}\,\mathrm{d}\alpha_{\pm lm}^*,[\mathcal{D}\varphi]=\prod_{l=0}^{\infty}\prod_{m=-l}^l\left(\frac{\Lambda_{\mathrm{uv}}r^2}{\pi}\right)^{\frac{1}{2}}\mathrm{d}\varphi_{lm}$$

$$1=\int\left[\mathcal{D}\psi\right]e^{-\frac{1}{2\sqrt{2\pi}}\Lambda_{\mathrm{uv}}^{1/2}\int_{S^2}\mathrm{d}^2x\bar{\epsilon}\not{\psi}\psi},1=\int\left[\mathcal{D}\varphi\right]e^{-\Lambda_{\mathrm{uv}}\int_{S^2}\mathrm{d}^2x\bar{\epsilon}\varphi(x)^2}$$

$$S_{\text{tL,fer}}^{(2)}[\delta\psi] = \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=0}^l ((l+1-\sqrt{q\beta})\alpha_{+lm}\alpha_{-lm}^* - (l+1+\sqrt{q\beta})\alpha_{-lm}\alpha_{+lm}^*).$$

$$\mathcal{Z}_{\text{tL,fer},1\text{-loop}}^{\mathcal{N}=1} \equiv \frac{2\pi}{\Lambda_{\text{uv}} r^2} \times \int [\mathcal{D}'\delta\psi] e^{-S_{\text{tL,fer}}^{(2)}[\delta\psi]} = \frac{2\pi}{\Lambda_{\text{uv}} r^2} \times \prod_{l=1}^{\infty} \left(\frac{(1+l)^2 - q\beta}{8\pi\Lambda_{\text{uv}} r^2} \right)^{l+1}$$

$$\sum_{l=1}^{\infty} (l+1) \log \frac{(1+l)^2 - q\beta}{8\pi\Lambda_{\text{uv}} r^2} \\ = - \int_0^{\infty} \frac{dt}{2t} \left[\frac{2e^{-\frac{t}{2}}}{(1-e^{-t})} \left(\frac{2e^{-\frac{t}{2}+ivt}}{(1-e^{-t})} - 2e^{-\frac{t}{2}+ivt} - e^{-(\frac{5}{2}+iv)t} \frac{(1-2e^t + e^{2ivt})}{(1-e^{-t})} \right) \right]$$

$$\chi_{\Delta=3/2}(t) = \frac{e^{-\frac{3}{2}t}}{(1-e^{-t})}$$

$$-\frac{1}{2}\log\frac{\tilde{\lambda}}{8\pi r^2\Lambda_{\text{uv}}}=\int_0^\infty\frac{\text{d}\tau}{2\tau}e^{-\frac{\varepsilon^2}{4\tau}-\tilde{\lambda}\tau}=K_0(\sqrt{\tilde{\lambda}}\varepsilon)\approx-\frac{1}{2}\log\frac{\varepsilon^2e^{2\gamma_E\tilde{\lambda}}}{4}$$

$$\log \mathcal{Z}_{\text{tL,fer},1\text{-loop}}^{\mathcal{N}=1} = -\frac{2}{\varepsilon^2} + \left(\frac{5}{6} - \beta^2\right) \log \varepsilon + \mathfrak{q}$$

$$\mathcal{Z}_{\text{tL,bos},1\text{-loop}}^{\mathcal{N}=1} \equiv \pm \mathrm{i} \left(\frac{4\pi r^2 \Lambda_{\text{uv}}}{\beta q} \right)^{\frac{1}{2}} \left(\frac{r^2 \Lambda_{\text{uv}}}{\pi} \right)^{\frac{3}{2}} \times \int [\mathcal{D}'\delta\varphi] e^{-\frac{1}{4\pi} \int_{S^2} \mathrm{d}x^2 \mathrm{e}^{\frac{1}{2}\delta\varphi} \left(-\nabla^2 - \frac{2}{r^2} q\beta \right) \delta\varphi} \\ = \pm \mathrm{i} \left(\frac{4\pi r^2 \Lambda_{\text{uv}}}{\beta q} \right)^{\frac{1}{2}} \left(\frac{r^2 \Lambda_{\text{uv}}}{\pi} \right)^{\frac{3}{2}} \times \prod_{l=2}^{\infty} \left(\frac{8\pi r^2 \Lambda_{\text{uv}}}{l(l+1) - 2\beta q} \right)^{l+\frac{1}{2}}$$

$$-\frac{1}{2}\sum_{l=2}^{\infty}(2l+1)\log\left(\frac{l(l+1)-2\beta q}{8\pi\Lambda_{\text{uv}}r^2}\right)=\int_0^{\infty}\frac{\text{d}t}{2t}\left[\frac{1+e^{-t}}{(1-e^{-t})}\frac{2e^{-\frac{t}{2}+ivt}}{(1-e^{-t})}\right.\\ \left.+e^{-(\frac{5}{2}-iv)t}\left(\frac{3-2e^t+5e^{3t}}{1-e^{-t}}-e^{2t}\frac{(5e^t-3)(1-e^{-(3+2iv)t})}{(1-e^{-t})^2}\right)\right]$$

$$\chi_{\Delta=2}(t) = \frac{e^{-2t}}{(1-e^{-t})}$$

$$\log \mathcal{Z}_{\text{bos},1\text{-loop}}^{\mathcal{N}=1} = \frac{2}{\varepsilon^2} - \left(\frac{7}{3} - 2\beta^2\right) \log \varepsilon + \zeta_{\lambda}^{\mathfrak{c}}$$

$$\mathcal{Z}_{\text{tL}}^{\mathcal{N}=1} \approx \pm \mathrm{i} \times \text{const} \times \left(\frac{\mu}{\beta}\right)^{-\frac{1}{\beta^2}+1} \Lambda_{\text{uv}}^{\frac{5}{4}-\beta^2} r_{\text{tL}}^{c^{\mathcal{N}=1}/3} e^{-\frac{1}{2\beta^2} - \left(\frac{1}{\beta^2}-1\right) \log \beta^2} \times \beta^{-1} \times \left(1 + \mathcal{O}(\beta^2)\right)$$

$$4\pi\mathcal{L}_{\text{tL,int}}^{\mathcal{N}=1}[\delta\varphi,\delta\psi] = -\frac{2\mathrm{i}}{3r^2}q\beta^2\delta\varphi^3 + \frac{1}{3r^2}q\beta^3\delta\varphi^4 + \frac{\mathrm{i}}{2r}\beta\sqrt{q\beta}\delta\varphi\delta\bar{\psi}\delta\psi - \frac{1}{4r}\beta^2\sqrt{q\beta}\delta\varphi^2\delta\bar{\psi}\delta\psi$$

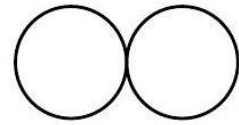
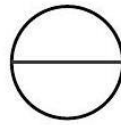
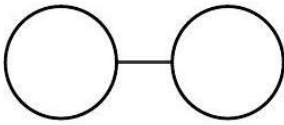
$$\begin{aligned}
H(\Omega, \Omega') &\equiv \frac{1}{\mathcal{Z}_{\text{fer}, 1-\text{loop}}^{\mathcal{N}=1}} \frac{2\pi}{\Lambda_{\text{uv}} r^2} \int [\mathcal{D}' \delta \psi] e^{-S_{\text{pert}}^{(2)}[\delta \psi]} \delta \bar{\psi}(\Omega) \delta \psi(\Omega') \\
&= \frac{4\pi}{r} \sum_{l \geq 1} \sum_{0 \leq m \leq l} \frac{1}{(l+1)^2 - q\beta} \left[i(l+1 + \sqrt{q\beta}) \left(\bar{\psi}_{+lm}^{(+)}(\Omega) \psi_{+lm}^{(-)}(\Omega') - \bar{\psi}_{+lm}^{(-)}(\Omega) \psi_{+lm}^{(+)}(\Omega') \right) \right. \\
&\quad \left. + i(l+1 - \sqrt{q\beta}) \left(\bar{\psi}_{-lm}^{(-)}(\Omega) \psi_{-lm}^{(+)}(\Omega') - \bar{\psi}_{-lm}^{(+)}(\Omega) \psi_{-lm}^{(-)}(\Omega') \right) \right],
\end{aligned}$$

$$H(\Omega_0, \Omega_0) = \frac{1}{4\pi} \int_{S^2} d\Omega H(\Omega, \Omega) = + \frac{4\sqrt{q\beta}}{r} \sum_{l \geq 1} \frac{l+1}{(l+1)^2 - q\beta}$$

$$G(\Omega, \Omega') = 4\pi \sum_{l \neq 1} \sum_{-l \leq m \leq l} \frac{Y_{lm}(\Omega) Y_{lm}(\Omega')}{l(l+1) - 2q\beta}.$$

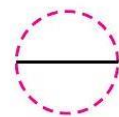
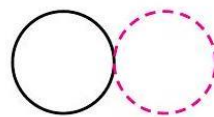
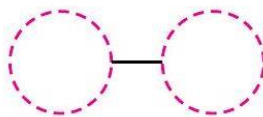
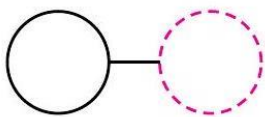
$$G(\Omega_0, \Omega_0) = \frac{1}{4\pi} \int_{S^2} d\Omega G(\Omega, \Omega) = \sum_{l \neq 1} \frac{2l+1}{l(l+1) - 2q\beta}$$

$$\begin{aligned}
\text{loops}_b = & -\frac{\beta^2}{8\pi^2} \int_{S^2} d\Omega d\Omega' G(\Omega, \Omega) G(\Omega', \Omega') G(\Omega, \Omega') - \frac{\beta^2}{12\pi^2} \int_{S^2} d\Omega d\Omega' G(\Omega, \Omega')^3 \\
& - \frac{\beta^2}{4\pi} \int_{S^2} d\Omega G(\Omega, \Omega)^2
\end{aligned}$$



$$\Theta \equiv -\frac{1}{6} \sum_{l_1, l_2, l_3 \neq 1} \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{(l_1(l_1+1)-2)(l_2(l_2+1)-2)(l_3(l_3+1)-2)} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2,$$

$$\begin{aligned}
\text{loops}_f = & \frac{r}{16\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega) G(\Omega', \Omega') G(\Omega, \Omega') - \frac{r^2}{128\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega) H(\Omega', \Omega') G(\Omega, \Omega') \\
& + \frac{r}{16\pi} \beta^2 \int_{S^2} d\Omega H(\Omega, \Omega) G(\Omega, \Omega) - \frac{r^2}{128\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega')^2 G(\Omega, \Omega')
\end{aligned}$$



$$\begin{aligned}
\frac{r}{16\pi^2} \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega) G(\Omega', \Omega') G(\Omega, \Omega') &= \frac{r}{(-2)} H(\Omega_0, \Omega_0) G(\Omega_0, \Omega_0) \approx -4 \log^2(\Lambda_{\text{uv}} r^2) + \dots, \\
-\frac{r^2}{128\pi^2} \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega) F(\Omega', \Omega') G(\Omega, \Omega') &= -\frac{r^2}{8} \frac{1}{(-2)} H(\Omega_0, \Omega_0)^2 \approx \log^2(\Lambda_{\text{uv}} r^2) + \dots, \\
\frac{r}{16\pi} \int_{S^2} d\Omega H(\Omega, \Omega) G(\Omega, \Omega) &= \frac{r}{4} H(\Omega_0, \Omega_0) G(\Omega_0, \Omega_0) \approx 2 \log^2(\Lambda_{\text{uv}} r^2) + \dots, \\
-\frac{r^2}{128\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' H(\Omega, \Omega')^2 G(\Omega, \Omega') &\approx \log^2(\Lambda_{\text{uv}} r^2) + \dots
\end{aligned}$$

$$Z_{\rm tL}^{\mathcal{N}=1} \approx \pm \mathrm{i} \, \mathrm{const} \times \left(\frac{\mu}{\beta}\right)^{-\frac{1}{\beta^2}+1} \Lambda_{\rm uv}^{\frac{5}{4}-\beta^2} r_{\rm tL}^{c_{\rm tL}^{\mathcal{N}=1}}/3 e^{-\frac{1}{2\beta^2}-\left(\frac{1}{\beta^2}-1\right)\log \beta^2} \times \beta^{-1} \times (1 + \mathrm{loops} \beta^2 + \cdots)$$

$$\mathcal{W}_\beta \equiv F e^{\beta \varphi} - \frac{\mathrm{i}}{2} \beta e^{\beta \varphi} \bar{\psi} \psi$$

$$\log Z_{\mathrm{grav},(0)}^{\mathcal{N}=1}=2\vartheta-\left(\frac{12}{\left(\sqrt{2c_m-27}-\sqrt{2c_m-3}\right)^2}-\frac{1}{2}\right)\log\frac{\mu^2}{\Lambda_{\mathrm{uv}}}+f_0(c_m)$$

$$\frac{1}{\pi}\int_{\Sigma_h}\mathrm{d}^2x\mathrm{e}\left(\vartheta\frac{R}{4}+\mathrm{i}\frac{v}{4}\epsilon^{\mu\nu}F_{\mu\nu}\right)=\vartheta\chi_h+\mathrm{i}vn_h$$

$$Z_{\mathrm{grav}}^{\mathcal{N}=2}=\sum_{h=0}^{\infty}\sum_{n_h\in\mathbb{Z}}e^{\vartheta\chi_h+\mathrm{i}vn_h}\int\left[\mathcal{D}\mathrm{e}_{\mu}^a\right]\left[\mathcal{D}\chi_{\mu}\right]\left[\mathcal{D}A_{\mu}\right]\left[\mathcal{D}B\right]e^{-\mathrm{i}\mu\int_{\Sigma_h}\mathrm{d}^2x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}\left(\mathrm{E}+\mathrm{h.c.}\right)}$$

$$\times Z_{\mathrm{SCFT}}^{(h,n_h)}\big[\mathrm{e}_{\mu}^a,\chi_{\mu},A_{\mu},B\big]$$

$$\mathrm{e}_{\mu}^a=e^{\frac{1}{2}b(\varphi+\tilde{\varphi})}\tilde{\mathrm{e}}_{\mu}^a,A^{\mu}=-\frac{\mathrm{i}}{2}\epsilon^{\mu\nu}\partial_{\nu}(\varphi-\tilde{\varphi})$$

$$\chi_{\mu}=e^{\frac{1}{2}b\varphi}\gamma_{\mu}\psi,\chi_{\mu}^*=e^{\frac{1}{2}b\tilde{\varphi}}\gamma_{\mu}\tilde{\psi},B=e^{-b\varphi}F,B^*=e^{-b\tilde{\varphi}}\tilde{F}$$

$$Z_{\mathrm{SCFT}}^{(0)}[\Phi,\tilde{\Phi}]=\mathcal{C}e^{-\mathcal{S}_{\mathrm{s-anomaly}}\left[\Phi,\tilde{\Phi}\right]}$$

$$S_{\mathrm{L}}^{\mathcal{N}=2}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathrm{e}}\left(\partial_{\mu}\tilde{\varphi}\partial^{\mu}\varphi-\mathrm{i}\bar{\tilde{\psi}}\not{\partial}\psi-F\tilde{F}+\frac{Q\tilde{R}}{2}(\varphi+\tilde{\varphi})\right.\\ \left.+\mu e^{b\varphi}F-\frac{\mathrm{i}}{2}\mu be^{b\varphi}\bar{\psi}\psi+\mu^*e^{b\tilde{\varphi}}\tilde{F}-\frac{\mathrm{i}}{2}\mu^*be^{b\tilde{\varphi}}\bar{\tilde{\psi}}\tilde{\psi}\right)$$

$$U(1)_V\colon \qquad \varphi\rightarrow\varphi+2\mathrm{i}Q\rho,\tilde{\varphi}\rightarrow\tilde{\varphi}-2\mathrm{i}Q\rho,\psi\rightarrow e^{-\mathrm{i}\rho}\psi,\tilde{\psi}\rightarrow e^{\mathrm{i}\rho}\tilde{\psi}\\ F\rightarrow e^{-2\mathrm{i}\rho}F,\tilde{F}\rightarrow e^{2\mathrm{i}\rho}\tilde{F}$$

$$U(1)_A\colon \qquad \psi\rightarrow e^{\mathrm{i}\alpha\gamma_*}\psi,\tilde{\psi}\rightarrow e^{-\mathrm{i}\alpha\gamma_*}\tilde{\psi}.$$

$$c_{\rm L}^{\mathcal{N}=2}+c_m+c_{\rm bc}+2\times c_{\beta\gamma}+c_{U(1)_V}=3+6Q^2+c_m-26+2\times 11-2=0.$$

$$Q=\frac{1}{b}=\sqrt{\frac{3-c_m}{6}}.$$

$$\begin{array}{c} c_m=3 \\ \hline \begin{array}{cc} \xleftarrow{\hspace{1.5cm}} & \xrightarrow{\hspace{1.5cm}} \\ \text{spacelike } \mathcal{N}=2 \text{ Liouville} & \text{timelike } \mathcal{N}=2 \text{ Liouville} \end{array} \end{array}$$

$$\psi\rightarrow e^{-\frac{\mathrm{i}}{2}b\varphi_2}\psi,\tilde{\psi}\rightarrow e^{\frac{\mathrm{i}}{2}b\varphi_2}\tilde{\psi},F\rightarrow e^{-\mathrm{i}b\varphi_2}F,\tilde{F}\rightarrow e^{\mathrm{i}b\varphi_2}\tilde{F}$$

$$M = \begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \zeta & \eta & e & f \\ \rho & \sigma & g & h \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix} \cdot \begin{pmatrix} \Omega_2 & 0 \\ 0 & \mathbb{S}_2 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Omega_2 & 0 \\ 0 & \mathbb{S}_2 \end{pmatrix}$$

$$\mathcal{Z}_{\text{grav},(0)}^{\mathcal{N}=2} = e^{2\vartheta} \times \mathcal{A} \times \left(\frac{r}{\ell_{\text{uv}}}\right)^{(c_m+c_{\text{bc}}+2c_{\beta\gamma}+c_{U(1)})/3} \times \int \frac{[\mathcal{D}\Phi][\mathcal{D}\tilde{\Phi}]}{\text{vol}_{\text{OSp}(2|2;\mathbb{C})}} e^{-\mathcal{S}_{\text{L}}^{\mathcal{N}=2}}$$

$$\mathcal{Z}_{\text{L}}^{\mathcal{N}=2}[\mu] \equiv \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}F]}{\text{vol}_{\text{OSp}(2|2;\mathbb{C})}} e^{-\mathcal{S}_{\text{L}}^{\mathcal{N}=2}}$$

$$\begin{aligned} \mathcal{S}_{\text{tL}}^{\mathcal{N}=2} = \frac{1}{4\pi} \int_{S^2} \text{d}^2x \tilde{\epsilon} & (-\partial_\mu \tilde{\varphi} \partial^\mu \varphi + \text{i} \bar{\tilde{\psi}} \not{\partial} \psi + F \tilde{F} - \frac{q \tilde{R}}{2} (\varphi + \tilde{\varphi}) \\ & + \text{i} \mu e^{\beta \varphi} F + \frac{1}{2} \mu \beta e^{\beta \varphi} \bar{\psi} \psi + \text{i} \mu^* e^{\beta \tilde{\varphi}} \tilde{F} + \frac{1}{2} \mu^* \beta e^{\beta \tilde{\varphi}} \bar{\tilde{\psi}} \tilde{\psi}) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{\text{tL}}^{\mathcal{N}=2} = \frac{1}{4\pi} \int_{S^2} \text{d}^2x \tilde{\epsilon} & [-\partial_\mu \tilde{\varphi} \partial^\mu \varphi + \text{i} \bar{\tilde{\psi}} \not{\partial} \psi + F \tilde{F} - \frac{q \tilde{R}}{2} (\varphi + \tilde{\varphi}) \\ & + \text{i} \left(F W'(\varphi) - \frac{\text{i}}{2} W''(\varphi) \bar{\psi} \psi \right) + \text{i} \left(\tilde{F} \tilde{W}'(\tilde{\varphi}) - \frac{\text{i}}{2} \tilde{W}''(\tilde{\varphi}) \bar{\tilde{\psi}} \tilde{\psi} \right) \Big] \end{aligned}$$

$$c_{\text{tL}}^{\mathcal{N}=2} = 3 - \frac{6}{\beta^2}$$

$$\begin{aligned} \delta\varphi &= \bar{\epsilon}\psi, & \delta\tilde{\varphi} &= \bar{\epsilon}\tilde{\psi} \\ \delta\psi &= \text{i}\not{\partial}\varphi\epsilon + \text{i}q\not{\partial}\epsilon - \text{i}F\tilde{\epsilon}, & \delta\tilde{\psi} &= \text{i}\not{\partial}\tilde{\varphi}\tilde{\epsilon} + \text{i}q\not{\partial}\tilde{\epsilon} - \text{i}\tilde{F}\epsilon \\ \delta F &= -\bar{\epsilon}\not{\partial}\psi, & \delta\tilde{F} &= \bar{\epsilon}\not{\partial}\tilde{\psi} \end{aligned}$$

$$\tilde{\varphi}=\varphi^*, \tilde{\psi}=\gamma_*\psi^C, \tilde{F}=F^*$$

$$\begin{aligned} [\delta_\epsilon,\delta_{\bar\epsilon}]\varphi &= \mathcal{L}_\xi\varphi + q\Delta + 2\text{i}q\rho \\ [\delta_\epsilon,\delta_{\bar\epsilon}]\psi &= \mathcal{L}_\xi\psi + \frac{1}{2}\Delta\psi - \text{i}\rho\psi + \text{i}\alpha\gamma_*\psi \\ [\delta_\epsilon,\delta_{\bar\epsilon}]F &= \mathcal{L}_\xi F + \Delta F - 2\text{i}\rho F \end{aligned}$$

$$\begin{aligned} \xi^\mu &= \text{i}\bar{\tilde{\epsilon}}\gamma^\mu\epsilon, & \Delta &= \frac{1}{2}\nabla_\mu\xi^\mu \\ \rho &= -\frac{1}{4}(D_\mu\bar{\tilde{\epsilon}}\gamma^\mu\epsilon - \bar{\tilde{\epsilon}}\gamma^\mu D_\mu\epsilon), & \alpha &= \frac{1}{4}(D_\mu\bar{\tilde{\epsilon}}\gamma_*\gamma^\mu\epsilon - \bar{\tilde{\epsilon}}\gamma_*\gamma^\mu D_\mu\epsilon) \end{aligned}$$

$$\begin{aligned} \varphi &\rightarrow \varphi + q\Delta, & \psi &\rightarrow e^{\frac{\Delta}{2}}\psi, & F &\rightarrow e^{\Delta}F \\ \tilde{\varphi} &\rightarrow \tilde{\varphi} - q\Delta, & \tilde{\psi} &\rightarrow e^{-\frac{\Delta}{2}}\tilde{\psi}, & \tilde{F} &\rightarrow e^{-\Delta}\tilde{F} \end{aligned}$$

$$0 = \nabla^2 \varphi - \frac{1}{\beta r^2} + i\mu\beta e^{\beta\varphi} F + \frac{1}{2}\mu\beta^2 \bar{\psi}\psi, 0 = \nabla^2 \tilde{\varphi} - \frac{1}{\beta r^2} + i\mu^*\beta e^{\beta\tilde{\varphi}} \tilde{F} + \frac{1}{2}\mu^*\beta^2 \bar{\tilde{\psi}}\tilde{\psi}$$

$$0 = -i\not{D}\bar{\tilde{\psi}} + \frac{1}{2}\mu\beta e^{\beta\varphi} \bar{\psi}, 0 = i\not{D}\psi + \frac{1}{2}\mu^*\beta e^{\beta\tilde{\varphi}} \tilde{\psi}$$

$$0 = \tilde{F} + i\mu e^{\beta\varphi}, 0 = F + i\mu^* e^{\beta\tilde{\varphi}}$$

$$(\varphi+\tilde{\varphi})_*=\frac{1}{\beta}\log\frac{1}{|\mu|^2\beta^2r^2},(\varphi-\tilde{\varphi})_*=c$$

$$\mathcal{S}_{\text{tL}}^{\mathcal{N}=2}=\frac{1}{4\pi}\int_{S^2}\text{d}^2x\sqrt{\tilde{g}}\left(-\partial_\mu\tilde{\varphi}\partial^\mu\varphi+i\bar{\tilde{\psi}}\not{D}\psi+|\mu|^2e^{\beta(\varphi+\tilde{\varphi})}\right.\\ \left.-\frac{1}{\beta r^2}(\varphi+\tilde{\varphi})+\frac{\beta}{2}\left(\mu e^{\beta\varphi}\bar{\psi}\psi+\mu^*e^{\beta\tilde{\varphi}}\bar{\tilde{\psi}}\tilde{\psi}\right)\right)$$

$$\mathcal{Z}_{\text{tL}}^{\mathcal{N}=2}[\mu] \approx \mathcal{Z}_{\text{saddle}}^{\mathcal{N}=2} = \left(\frac{1}{e|\mu|^2\beta^2r^2}\right)^{\frac{1}{\beta^2}}$$

$$\varphi_1\equiv\frac{1}{2}(\varphi+\tilde{\varphi}),\varphi_2\equiv-\frac{\mathrm{i}}{2}(\varphi-\tilde{\varphi})$$

$$\varphi_1=\varphi_{1,*}+\delta\varphi_1,\varphi_2=\varphi_{2,*}+\delta\varphi_2,\psi=\psi_*+\delta\psi,\tilde{\psi}=\tilde{\psi}_*+\delta\tilde{\psi}$$

$$S_{\text{pert}}^{(2)}=\frac{1}{4\pi}\int_{S^2}\text{d}^2x\tilde{\text{e}}\left[-\delta\varphi_1\left(-\nabla^2-\frac{2}{r^2}\right)\delta\varphi_1-\delta\varphi_2(-\nabla^2)\delta\varphi_2\right.\\ \left.+i\delta\bar{\tilde{\psi}}\not{D}\delta\psi+\frac{1}{2r}\frac{\mu}{|\mu|}\delta\bar{\psi}\delta\psi+\frac{1}{2r}\frac{\mu^*}{|\mu|}\delta\bar{\tilde{\psi}}\delta\tilde{\psi}\right]$$

$$1=\int\left[\mathcal{D}\varphi_1\right]\left[\mathcal{D}\varphi_2\right]e^{-\Lambda_{\text{uv}}\int_{S^2}\text{d}^2x\,\text{e}\,\varphi(x)\tilde{\varphi}(x)}$$

$$\mathcal{Z}_{\text{tL, bos}, 1\text{-loop}}^{\mathcal{N}=2}=\pm\mathrm{i}(2\pi r^2\Lambda_{\text{uv}})^{\frac{1}{2}}\Big(\frac{r^2\Lambda_{\text{uv}}}{\pi}\Big)^2\times\prod_{l=2}^{\infty}\left(\frac{4\pi r^2\Lambda_{\text{uv}}}{l(l+1)-2}\right)^{l+\frac{1}{2}}\prod_{l=1}^{\infty}\left(\frac{4\pi r^2\Lambda_{\text{uv}}}{l(l+1)}\right)^{l+\frac{1}{2}}$$

$$-\frac{1}{2}\sum_{l=2}^{\infty}(2l+1)\log\frac{l(l+1)-2}{4\pi r^2\Lambda_{\text{uv}}}=\int_0^{\infty}\frac{\text{d}t}{2t}\Big[\frac{1+e^{-t}}{(1-e^{-t})}\Big(\frac{2e^{-2t}}{(1-e^{-t})}+5e^{-t}-5e^{-2t}+3e^{-3t}\Big)\Big]$$

$$-\frac{1}{2}\sum_{l=1}^{\infty}(2l+1)\log\frac{l(l+1)}{4\pi r^2\Lambda_{\text{uv}}}=\int_0^{\infty}\frac{\text{d}t}{2t}\Big[\frac{1+e^{-t}}{(1-e^{-t})}\Big(\frac{2e^{-t}}{(1-e^{-t})}+e^{-t}\Big)\Big]$$

$$\chi_{\Delta=2}(t)=\frac{e^{-2t}}{(1-e^{-t})} \text{ and } \chi_{\Delta=1}(t)=\frac{e^{-t}}{(1-e^{-t})}$$

$$\log \mathcal{Z}_{\text{tL,bos},1\text{-loop}}^{\mathcal{N}=2} = 2 \times \frac{2}{\varepsilon^2} - \frac{8}{3} \log \varepsilon + \aleph$$

$$\psi(\Omega) = \frac{1}{\sqrt{r}} \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{\pm} \left(\alpha_{\pm lm} \psi_{\pm lm}^{(+)}(\Omega) + \beta_{\pm lm} \psi_{\pm lm}^{(-)}(\Omega) \right)$$

$$\tilde{\psi}(\Omega) = -\frac{i}{\sqrt{r}} \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{\pm} \left(\beta_{\mp lm}^* \psi_{\pm lm}^{(+)}(\Omega) + \alpha_{\mp lm}^* \psi_{\pm lm}^{(-)}(\Omega) \right)$$

$$[\mathcal{D}\psi] = \prod_{\substack{l \geq 0 \\ 0 \leq m \leq l}} \left(\frac{4\pi}{\Lambda_{uv} r^2} \right) \prod_{\pm} d\alpha_{\pm lm} d\beta_{\pm lm}, [\mathcal{D}\tilde{\psi}] = \prod_{\substack{l \geq 0 \\ 0 \leq m \leq l}} \prod_{\pm} \left(\frac{4\pi}{\Lambda_{uv} r^2} \right) d\alpha_{\pm lm}^* d\beta_{\pm lm}^*$$

$$\boldsymbol{x}_{lm}^T \equiv (\alpha_{+lm} \quad \alpha_{+lm}^* \quad \alpha_{-lm} \quad \alpha_{-lm}^* \quad \beta_{+lm} \quad \beta_{+lm}^* \quad \beta_{-lm} \quad \beta_{-lm}^*).$$

$$S_{\text{pert}}^{(2)}[\delta\psi,\delta\tilde{\psi}]\equiv\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\epsilon}\left(\mathrm{i}\delta\bar{\tilde{\psi}}\not{\partial}\delta\psi+\frac{1}{r}\frac{\mu}{2|\mu|}\delta\bar{\psi}\delta\psi+\frac{1}{r}\frac{\mu^*}{2|\mu|}\delta\bar{\tilde{\psi}}\delta\tilde{\psi}\right)=\frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^l\boldsymbol{x}_{lm}^TB_{lm}\boldsymbol{x}_{lm}$$

$$B_{lm}$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}(l+1) & -i\frac{\mu}{2|\mu|} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(l+1) & 0 & 0 & -i\frac{\mu^*}{2|\mu|} & 0 & 0 \\ 0 & -\frac{1}{2}(l+1) & 0 & 0 & 0 & 0 & -i\frac{\mu}{2|\mu|} & 0 \\ -\frac{1}{2}(l+1) & 0 & 0 & 0 & 0 & 0 & 0 & -i\frac{\mu^*}{2|\mu|} \\ i\frac{\mu}{2|\mu|} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(l+1) \\ 0 & i\frac{\mu^*}{2|\mu|} & 0 & 0 & 0 & 0 & -\frac{1}{2}(l+1) & 0 \\ 0 & 0 & i\frac{\mu}{2|\mu|} & 0 & 0 & \frac{1}{2}(l+1) & 0 & 0 \\ 0 & 0 & 0 & i\frac{\mu^*}{2|\mu|} & \frac{1}{2}(l+1) & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{Z}_{\text{fer},1\text{-loop}}^{\mathcal{N}=2} = \left(\frac{4\pi}{\Lambda_{uv}r^2}\right)^2 \prod_{l=1}^{\infty} \left(\frac{l(l+2)}{4\pi\Lambda_{uv}r^2}\right)^{2(l+1)}$$

$$\sum_{l=1}^{\infty} 2(l+1)\log\frac{l(l+2)}{4\pi\Lambda_{uv}r^2} = -\int_0^{\infty} \frac{dt}{2t} \left[\frac{2e^{-\frac{t}{2}}}{(1-e^{-t})} \left(4\chi_{\Delta=3/2}(t) + 4e^{-\frac{t}{2}} - 2e^{-\frac{3t}{2}} + 2e^{-\frac{5t}{2}} \right) \right]$$

$$\log \mathcal{Z}_{\text{fer},1\text{-loop}}^{\mathcal{N}=2} = -\frac{4}{\varepsilon^2} + \frac{5}{3} \log \varepsilon + \lambda$$

$$\mathcal{Z}_{\text{tL}}^{\mathcal{N}=2}[\mu] \approx \pm i \times \text{const} \times \left(\frac{\mu\mu^*}{\beta^2}\right)^{-\frac{1}{\beta^2}} \Lambda_{uv}^{\frac{1}{2}} r c_{\text{tL}}^{\mathcal{N}=2/3} \times e^{-\frac{1}{\beta^2}-\frac{2}{\beta^2}\log\beta^2} \times (1+\mathcal{O}(\beta^2))$$



$$S_{\text{L, int}}^{\mathcal{N}=2} = \frac{1}{4\pi} \int_{S^2} d^2x \tilde{e} \left[-\frac{4i}{3r^2} \beta \delta \varphi_1^3 + \frac{2}{3r^2} \beta^2 \delta \varphi_1^4 + \frac{\beta}{2r} \frac{\mu}{|\mu|} \delta \bar{\psi} \delta \psi (i\delta \varphi_1 - \delta \varphi_2) \right. \\ \left. + \frac{\beta}{2r} \frac{\mu^*}{|\mu|} \delta \bar{\tilde{\psi}} \delta \tilde{\psi} (i\delta \varphi_1 + \delta \varphi_2) - \frac{1}{4r} \beta^2 \frac{\mu}{|\mu|} \delta \bar{\psi} \delta \psi (\delta \varphi_1^2 - \delta \varphi_2^2) - \frac{1}{4r} \beta^2 \frac{\mu^*}{|\mu|} \delta \bar{\tilde{\psi}} \delta \tilde{\psi} (\delta \varphi_1^2 - \delta \varphi_2^2) \right]$$

$$G(\Omega; \Omega') = 2\pi \sum_{l \neq 1} \sum_{-l \leq m \leq l} \frac{Y_{lm}(\Omega) Y_{lm}(\Omega')}{l(l+1) - 2}$$

$$G(\Omega_0, \Omega_0) = \frac{1}{4\pi} \int_{S^2} d\Omega G(\Omega; \Omega) = \frac{1}{2} \sum_{l \neq 1} \frac{2l+1}{l(l+1) - 2}$$

$$\tilde{G}(\Omega; \Omega') = 2\pi \sum_{l=1}^{\infty} \sum_{-l \leq m \leq l} \frac{Y_{lm}(\Omega) Y_{lm}(\Omega')}{l(l+1)}$$

$$H(\Omega; \Omega') \equiv \frac{1}{\mathcal{Z}_{\text{fer}, 1\text{-loop}}^{\mathcal{N}=2}} \left(\frac{4\pi}{\Lambda_{\text{uv}} r^2} \right)^2 \int [\mathcal{D}' \delta \psi] [\mathcal{D}' \delta \tilde{\psi}] e^{-S_{\text{pert}}^{(2)}[\delta \psi, \delta \tilde{\psi}]} \delta \bar{\psi}(\Omega) \delta \psi(\Omega') \\ = \frac{4\pi i}{r} \frac{\mu^*}{|\mu|} \sum_{l \geq 1} \sum_{0 \leq m \leq l} \frac{1}{l(l+2)} \left(\bar{\psi}_{+lm}^{(+)}(\Omega) \psi_{+lm}^{(-)}(\Omega') + \bar{\psi}_{-lm}^{(+)}(\Omega) \psi_{-lm}^{(-)}(\Omega') \right. \\ \left. - \bar{\psi}_{+lm}^{(-)}(\Omega) \psi_{+lm}^{(+)}(\Omega') - \bar{\psi}_{-lm}^{(-)}(\Omega) \psi_{-lm}^{(+)}(\Omega') \right)$$

$$\tilde{H}(\Omega; \Omega') \equiv \frac{1}{\mathcal{Z}_{\text{fer}, 1\text{-loop}}^{\mathcal{N}=2}} \left(\frac{4\pi}{\Lambda_{\text{uv}} r^2} \right)^2 \int [\mathcal{D}' \delta \psi] [\mathcal{D}' \delta \tilde{\psi}] e^{-S_{\text{pert}}^{(2)}[\delta \psi, \delta \tilde{\psi}]} \delta \tilde{\bar{\psi}}(\Omega) \delta \tilde{\psi}(\Omega') \\ = \frac{4\pi i}{r} \frac{\mu}{|\mu|} \sum_{l \geq 1} \sum_{0 \leq m \leq l} \frac{1}{l(l+2)} \left(\bar{\tilde{\psi}}_{+lm}^{(+)}(\Omega) \tilde{\psi}_{+lm}^{(-)}(\Omega') + \bar{\tilde{\psi}}_{-lm}^{(+)}(\Omega) \tilde{\psi}_{-lm}^{(-)}(\Omega') \right. \\ \left. - \bar{\tilde{\psi}}_{+lm}^{(-)}(\Omega) \tilde{\psi}_{+lm}^{(+)}(\Omega') - \bar{\tilde{\psi}}_{-lm}^{(-)}(\Omega) \tilde{\psi}_{-lm}^{(+)}(\Omega') \right)$$

$$H(\Omega_0, \Omega_0) = \frac{1}{4\pi} \int_{S^2} d\Omega H(\Omega, \Omega) = \frac{4}{r} \frac{\mu^*}{|\mu|} \sum_{l=1}^{\infty} \frac{l+1}{l(l+2)}$$

$$\tilde{H}(\Omega_0, \Omega_0) = \frac{1}{4\pi} \int_{S^2} d\Omega \tilde{H}(\Omega, \Omega) = \frac{4}{r} \frac{\mu}{|\mu|} \sum_{l=1}^{\infty} \frac{l+1}{l(l+2)}$$

$$-\frac{1}{3\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' G(\Omega, \Omega')^3 - \frac{1}{2\pi^2} \beta^2 \int_{S^2} d\Omega d\Omega' G(\Omega, \Omega) G(\Omega', \Omega') G(\Omega, \Omega') - \frac{1}{2\pi} \beta^2 \int_{S^2} d\Omega G(\Omega, \Omega)^2$$

$$-\frac{r}{16\pi} \frac{\mu}{|\mu|} \beta^2 \int_{S^2} d\Omega H(\Omega, \Omega) G(\Omega, \Omega) - \frac{r}{16\pi} \frac{\mu^*}{|\mu|} \beta^2 \int_{S^2} d\Omega \tilde{H}(\Omega, \Omega) G(\Omega, \Omega) \\ + \frac{r}{16\pi} \frac{\mu}{|\mu|} \beta^2 \int_{S^2} d\Omega H(\Omega, \Omega) \tilde{G}(\Omega, \Omega) + \frac{r}{16\pi} \frac{\mu^*}{|\mu|} \beta^2 \int_{S^2} d\Omega \tilde{H}(\Omega, \Omega) \tilde{G}(\Omega, \Omega)$$



$$\begin{aligned}
& + \frac{r}{8\pi^2} \frac{\mu}{|\mu|} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega') G(\Omega, \Omega') G(\Omega', \Omega') + \frac{r}{8\pi^2} \frac{\mu^*}{|\mu|} \beta^2 \int_{S^2} d\Omega \, d\Omega' \tilde{H}(\Omega, \Omega') G(\Omega, \Omega') G(\Omega', \Omega') \\
& - \frac{r^2}{128\pi^2} \frac{\mu}{\mu^*} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega') G(\Omega, \Omega') F(\Omega', \Omega') - \frac{r^2}{128\pi^2} \frac{\mu^*}{\mu} \beta^2 \int_{S^2} d\Omega \, d\Omega' \tilde{H}(\Omega, \Omega') G(\Omega, \Omega') \tilde{H}(\Omega', \Omega') \\
& - \frac{r^2}{64\pi^2} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega') \tilde{H}(\Omega', \Omega) G(\Omega, \Omega') \\
& - \frac{r^2}{128\pi^2} \frac{\mu}{\mu^*} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega')^2 G(\Omega, \Omega') - \frac{r^2}{128\pi^2} \frac{\mu^*}{\mu} \beta^2 \int_{S^2} d\Omega \, d\Omega' \tilde{H}(\Omega, \Omega')^2 G(\Omega, \Omega') \\
& + \frac{r^2}{128\pi^2} \frac{\mu}{\mu^*} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega')^2 \tilde{G}(\Omega, \Omega') + \frac{r^2}{128\pi^2} \frac{\mu^*}{\mu} \beta^2 \int_{S^2} d\Omega \, d\Omega' \tilde{H}(\Omega, \Omega')^2 \tilde{G}(\Omega, \Omega') \\
& - \frac{r^2}{64\pi^2} \beta^2 \int_{S^2} d\Omega \, d\Omega' F(\Omega, \Omega') \tilde{H}(\Omega', \Omega) G(\Omega, \Omega') - \frac{r^2}{64\pi^2} \beta^2 \int_{S^2} d\Omega \, d\Omega' H(\Omega, \Omega') \tilde{H}(\Omega', \Omega) \tilde{G}(\Omega, \Omega').
\end{aligned}$$

$$Z_{\text{tL}}^{\mathcal{N}=2}[\mu] \approx \pm i \times \text{const} \times \left(\frac{\mu \mu^*}{\beta^2} \right)^{-\frac{1}{\beta^2}} \Lambda_{\text{uv}}^{\frac{1}{2}} r^{c_{\text{tL}}^{\mathcal{N}=2}/3} \times e^{-\frac{1}{\beta^2} - \frac{2}{\beta^2} \log \beta^2} \times (1 + \text{loops} \beta^2)$$

$$\log Z_{\text{grav},(0)}^{\mathcal{N}=2} = 2\vartheta - \left(\frac{c_m}{6} - \frac{3}{6}\right) \log \frac{|\mu|^2}{\Lambda_{\text{uv}}} + f_0(c_m)$$

$$D_\mu \epsilon = \frac{\mathrm{i}}{2r} \gamma_\mu \epsilon, D_\mu \tilde{\epsilon} = \frac{\mathrm{i}}{2r} \gamma_\mu \tilde{\epsilon}$$

$$\begin{aligned}
\delta\varphi &= \bar{\epsilon}\psi, & \delta\tilde{\varphi} &= \bar{\epsilon}\tilde{\psi}, \\
\delta\psi &= \mathrm{i}\not{\partial}\varphi\epsilon - \frac{1}{\beta r}\epsilon - \mathrm{i}F\tilde{\epsilon}, & \delta\tilde{\psi} &= \mathrm{i}\not{\partial}\tilde{\varphi}\tilde{\epsilon} - \frac{1}{\beta r}\tilde{\epsilon} - \mathrm{i}\tilde{F}\epsilon, \\
\delta F &= -\bar{\epsilon}\not{\partial}\psi, & \delta\tilde{F} &= \tilde{\epsilon}\not{\partial}\tilde{\psi}.
\end{aligned}$$

$$\begin{aligned}
[\delta_\epsilon,\delta_{\tilde\epsilon}]\varphi &= \mathcal{L}_\xi\varphi + \frac{2\mathrm{i}}{\beta}\rho \\
[\delta_\epsilon,\delta_{\tilde\epsilon}]\psi &= \mathcal{L}_\xi\psi - \mathrm{i}\rho\psi \\
[\delta_\epsilon,\delta_{\tilde\epsilon}]F &= \mathcal{L}_\xi F - 2\mathrm{i}\rho F
\end{aligned}$$

$$\int_{\mathcal{E}} [\mathcal{D}\Phi] e^{-S} \mathcal{O} = \text{vol}(F) \int_{\mathcal{E}/F} [\widetilde{\mathcal{D}\Phi}] e^{-S} \mathcal{O}$$

$$Z(t)=\int_{\mathcal{E}}[\mathcal{D}\Phi]e^{-S-tQV}$$

$$Z'(t)=-\int_{\mathcal{E}}[\mathcal{D}\Phi]\mathcal{Q}(Ve^{-S-tQV})$$

$$Q\equiv \epsilon_+^\alpha Q_\alpha, Q^\dagger\equiv (\epsilon_+^c)^\alpha \tilde{Q}_\alpha$$

$$\varphi^\dagger\equiv\tilde{\varphi},\psi^\dagger\equiv-\overline{\tilde{\psi}},F^\dagger=\tilde{F}$$

$$\begin{aligned}
Q\varphi &= 0, & Q^\dagger\varphi &= -\epsilon_+^\dagger\psi \\
Q\psi &= \left(\mathrm{i}\partial\varphi - \frac{q}{r}\right)\epsilon_+, & Q^\dagger\psi &= -\mathrm{i}F\epsilon_+^c \\
QF &= -\bar{\epsilon}_+\not{D}\psi, & Q^\dagger F &= 0 \\
Q\varphi^\dagger &= \psi^\dagger\epsilon_+, & Q^\dagger\varphi^\dagger &= 0 \\
Q\psi^\dagger &= \mathrm{i}F^\dagger\bar{\epsilon}_+, & Q^\dagger\psi^\dagger &= \epsilon_+^\dagger\left(-\mathrm{i}\partial\varphi^\dagger - \frac{q}{r}\right) \\
QF^\dagger &= 0, & Q^\dagger F^\dagger &= D_\mu\psi^\dagger\gamma^\mu\epsilon_+^c
\end{aligned}$$

$$\{Q, Q^\dagger\} = J + \frac{1}{2}R$$

$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= -\partial_\mu\varphi^\dagger\partial^\mu\varphi - \mathrm{i}\psi^\dagger\not{D}\psi + F^\dagger F - \frac{q}{r^2}(\varphi + \varphi^\dagger) - \frac{1}{\beta^2 r^2} \\
&= \frac{1}{s}\mathcal{Q}\left(-\mathrm{i}\epsilon_+^\dagger\partial\varphi^\dagger\psi - \frac{q}{r}\epsilon_+^\dagger\psi - \mathrm{i}F\psi^\dagger\epsilon_+^c - \frac{1}{r}(\varphi^\dagger - \varphi)\epsilon_+^\dagger\psi\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_W &\equiv FW'(\varphi) - \frac{\mathrm{i}}{2}W''(\varphi)\bar{\psi}\psi = \frac{\mathrm{i}}{s}\mathcal{Q}(W'(\varphi)\bar{\epsilon}_+\psi) \\
\mathcal{L}_{W^\dagger} &\equiv F^\dagger W'^\dagger(\varphi^\dagger) - \frac{\mathrm{i}}{2}W''^\dagger(\varphi^\dagger)\bar{\tilde{\psi}}\tilde{\psi} = -\frac{\mathrm{i}}{s}\mathcal{Q}(W'^\dagger(\varphi^\dagger)\psi^\dagger\epsilon_+^c)
\end{aligned}$$

$$\delta F_W = -\nabla_\mu(\bar{\epsilon}\gamma^\mu\psi_W)$$

$$V_{\text{can}} = (Q\psi)^\dagger\psi + \bar{\tilde{\psi}}(Q\tilde{\psi})^\dagger$$

$$\mathcal{Q}V_{\text{can}}|_{\text{bos}} = 2\left[\partial_\mu\varphi^\dagger\partial^\mu\varphi + \frac{1}{\beta^2 r^2} + \mathrm{i}\frac{1}{r}v^\mu\varphi^\dagger\partial_\mu\varphi\right] - 2F^\dagger F - 2\mathrm{i}(\mathrm{i}Fw^\mu\partial_\mu\varphi^\dagger + \text{c.c.}),$$

$$\epsilon_+ = e^{\mathrm{i}\frac{\phi}{2}}\begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\begin{aligned}
0 &= \sin\frac{\theta}{2}(2\partial_-\varphi - \mathrm{i}Fe^{-\mathrm{i}\phi}) + \cos\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \cos\frac{\theta}{2}(2\partial_+\varphi - \mathrm{i}Fe^{-\mathrm{i}\phi}) - \sin\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \cos\frac{\theta}{2}(2\partial_-\varphi^\dagger + \mathrm{i}F^\dagger e^{\mathrm{i}\phi}) - \sin\frac{\theta}{2}\frac{1}{\beta r} \\
0 &= \sin\frac{\theta}{2}(2\partial_+\varphi^\dagger + \mathrm{i}F^\dagger e^{\mathrm{i}\phi}) + \cos\frac{\theta}{2}\frac{1}{\beta r}
\end{aligned}$$

$$0 = F = \tilde{F}, 0 = \nabla^2\varphi - \frac{1}{\beta r^2} = \nabla^2\tilde{\varphi} - \frac{1}{\beta r^2}, 0 = \not{D}\psi = \not{D}\tilde{\psi}$$

$$\mathcal{L}_{\text{fermi}} = \mathrm{i}\tilde{\psi}_-(\partial_0 + \partial_1)\psi_- + \mathrm{i}\tilde{\psi}_+(\partial_0 - \partial_1)\psi_+ - \mathrm{i}(W''(\varphi)\psi_+\psi_- + \tilde{W}''(\tilde{\varphi})\tilde{\psi}_-\tilde{\psi}_+)$$

$$\mathcal{C} = -\mathcal{C}^T, \gamma_a^T = -\mathcal{C}\gamma_a\mathcal{C}^{-1}$$

$$\bar{\lambda} = \lambda^T \mathcal{C},$$

$$P_L=\frac{1}{2}(\mathbb{I}+\gamma_*), P_R=\frac{1}{2}(\mathbb{I}-\gamma_*)$$

$$P_{L,R}\lambda_{L,R}=\lambda_{L,R}$$

$$\lambda^C\equiv\mathcal{C}^{-1}\lambda^*$$

$$\lambda=\gamma_*\lambda^C$$

$$(\overline{\chi}M\lambda)^C=-\overline{\chi^C}M^C\lambda^C$$

$$(\overline{\chi_m}\lambda_m)^C=\overline{\chi_m}\lambda_m, (\overline{\chi_m}\gamma_a\lambda_m)^C=\overline{\chi_m}\gamma_a\lambda_m, (\overline{\chi_m}\partial\chi_m)^C=\overline{\chi_m}\partial\chi_m$$

$$\bar\lambda\eta=\lambda^\alpha\mathcal{C}_{\alpha\beta}\eta^\beta$$

$$\bar\lambda\eta=\bar\eta\lambda,\bar\lambda\gamma_a\eta=-\bar\eta\gamma_a\lambda,\bar\lambda\gamma_{ab}\eta=-\bar\eta\gamma_{ab}\lambda$$

$$\delta^\alpha{}_\delta\delta^\gamma{}_\beta=\frac{1}{2}\delta^\alpha{}_\beta\delta^\gamma{}_\delta+\frac{1}{2}(\gamma^a)^\alpha{}_\beta(\gamma_a)^\gamma{}_\delta+\frac{1}{2}(\gamma_*)^\alpha{}_\beta(\gamma_*)^\gamma{}_\delta.$$

$$\gamma_a=\sigma_a,\gamma_*= \sigma_3$$

$$\lambda^1=-i\lambda^{2*} \text{ and } \lambda^2=-i\lambda^{1*}.$$

$$D_\mu \epsilon = \frac{1}{d} \gamma_\mu \rlap{-}\!\!\!/\epsilon$$

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \epsilon$$

$$g_{\mu\nu}(x)\rightarrow\hat{g}_{\mu\nu}(x)=e^{2\sigma(x)}g_{\mu\nu}(x),\epsilon(x)\rightarrow\hat{\epsilon}(x)e^{\frac{1}{2}\sigma(x)}$$

$$D_\mu \epsilon = \kappa \gamma_\mu \epsilon$$

$$\kappa^2=-\frac{R}{4d(d-1)}$$

$$\mathrm{d} s^2=r^2(\,\mathrm{d} \theta^2+\sin^2\,\theta\,\mathrm{d} \phi^2)$$

$$D_\mu \epsilon_\pm = \pm \frac{\mathrm{i}}{2r} \gamma_\mu \epsilon_\pm$$

$$\mathbf{e}^1=r\,\mathrm{d}\theta,\mathbf{e}^2=r\sin\,\theta\,\mathrm{d}\phi,\omega^{12}=-\cos\,\theta\mathrm{d}\phi$$

$$\epsilon_+=C_1\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}\sin\frac{\theta}{2}\\-\mathrm{i}\cos\frac{\theta}{2}\end{pmatrix}+C_2\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\\mathrm{i}\sin\frac{\theta}{2}\end{pmatrix}$$

$$D_\mu=\partial_\mu+\mathrm{i}\hat{\mathbf{s}}\cos\,\theta(\,\mathrm{d}\phi)_\mu$$

$$D^2=\frac{1}{r^2}\Big[\frac{1}{\sin\,\theta}\partial_\theta(\sin\,\theta\partial_\theta)+\frac{1}{\sin^2\,\theta}(\partial_\phi+\mathrm{i}\hat{\mathbf{s}}\cos\,\theta)^2\Big]$$



$$\mathbf{L} = \mathbf{r} \wedge \mathbf{p} - \hat{s} \frac{\mathbf{r}}{r},$$

$$L_{\pm} = e^{\pm i\phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \hat{s} \frac{1}{\sin \theta} \right), L_z = -i \partial_{\phi}$$

$$[L_z, L_{\pm}] = \pm L_{\pm}, [L_+, L_-] = 2L_z$$

$$L^2 = \frac{1}{2} (L_+ L_- + L_- L_+) + L_z^2 = -r^2 D^2 + \hat{s}^2$$

$$L^2 \mathcal{Y}_{lm}(\Omega) = l(l+1) \mathcal{Y}_{lm}(\Omega), L_z \mathcal{Y}_{lm}(\Omega) = m \mathcal{Y}_{lm}(\Omega)$$

$$\mathcal{Y}_{lm}(\Omega) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\phi},$$

$$\int_{S^2} d\Omega (\mathcal{Y}_{lm}(\Omega))^* \mathcal{Y}_{l'm'}(\Omega) = \delta_{ll'} \delta_{mm'}$$

$$Y_{lm}(\Omega) \equiv \begin{cases} \frac{i}{\sqrt{2}} (\mathcal{Y}_{lm}(\Omega) - (-1)^m \mathcal{Y}_{l,-m}(\Omega)), & \text{if } m < 0 \\ \mathcal{Y}_{l0}(\Omega) \\ \frac{1}{\sqrt{2}} (\mathcal{Y}_{l,-m}(\Omega) + (-1)^m \mathcal{Y}_{lm}(\Omega)), & \text{if } m > 0 \end{cases}$$

$$\int_{S^2} d\Omega Y_{lm}(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'} \delta_{mm'}$$

$$\mathcal{D}^2 = D^2 - \frac{R}{4} = D^2 - \frac{1}{2r^2} = -\frac{1}{r^2} L^2 - \frac{1}{4r^2}$$

$$L^2 \psi_{\pm lm}^{(s)}(\Omega) = \left(l + \frac{1}{2}\right) \left(l + \frac{3}{2}\right) \psi_{\pm lm}^{(s)}(\Omega), L_z \psi_{\pm lm}^{(s)}(\Omega) = s \left(m + \frac{1}{2}\right) \psi_{\pm lm}^{(s)}(\Omega)$$

$$\mathcal{D} \psi_{\pm lm}^{(s)}(\Omega) = \pm \frac{i}{r} (l+1) \psi_{\pm lm}^{(s)}(\Omega), s = \pm$$

$$\psi_{\pm lm}^{(+)}(\Omega) = \frac{c_{lm}}{\sqrt{2}} e^{i(m+\frac{1}{2})\phi} \begin{pmatrix} \Phi_{lm}(\theta) \\ \pm i \Psi_{lm}(\theta) \end{pmatrix}, \psi_{\pm lm}^{(-)}(\Omega) = \frac{c_{lm}}{\sqrt{2}} e^{-i(m+\frac{1}{2})\phi} \begin{pmatrix} \pm i \Psi_{lm}(\theta) \\ \Phi_{lm}(\theta) \end{pmatrix}$$

$$\Phi_{lm}(\theta) = \left(\cos \frac{\theta}{2}\right)^{m+1} \left(\sin \frac{\theta}{2}\right)^m P_{l-m}^{(m,m+1)}(\cos \theta)$$

$$\Psi_{lm}(\theta) = \left(\cos \frac{\theta}{2}\right)^m \left(\sin \frac{\theta}{2}\right)^{m+1} P_{l-m}^{(m+1,m)}(\cos \theta) = (-1)^{l-m} \Phi_{lm}(\pi - \theta)$$

$$c_{lm}^2 = \frac{(l+m+1)!(l-m)!}{2\pi(l!)^2}$$

$$\int_{S^2} d\Omega \bar{\psi}_{\pm lm}^{(+)}(\Omega) \psi_{\pm l'm'}^{(-)}(\Omega) = -i \delta_{ll'} \delta_{m,m'}, \int_{S^2} d\Omega \bar{\psi}_{\pm lm}^{(-)}(\Omega) \psi_{\pm l'm'}^{(+)}(\Omega) = +i \delta_{ll'} \delta_{m,m'}$$



$$\eta = \frac{1}{\sqrt{r}} \sum_{l \geq 0} \sum_{0 \leq m \leq l} \left(\alpha_{+lm} \psi_{+lm}^{(+)} + \alpha_{-lm} \psi_{-lm}^{(+)} + \beta_{+lm} \psi_{+lm}^{(-)} + \beta_{-lm} \psi_{-lm}^{(-)} \right)$$

$$\tilde{\eta} = \frac{1}{\sqrt{r}} \sum_{l \geq 0} \sum_{0 \leq m \leq l} \left(\tilde{\alpha}_{+lm} \psi_{+lm}^{(+)} + \tilde{\alpha}_{-lm} \psi_{-lm}^{(+)} + \tilde{\beta}_{+lm} \psi_{+lm}^{(-)} + \tilde{\beta}_{-lm} \psi_{-lm}^{(-)} \right)$$

$$\int_{S^2} d\Omega i \tilde{\eta} \not{D} \eta = \frac{1}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (l+1) [\tilde{\alpha}_{+lm} \beta_{+lm} - \tilde{\alpha}_{-lm} \beta_{-lm} - \tilde{\beta}_{+lm} \alpha_{+lm} + \tilde{\beta}_{-lm} \alpha_{-lm}]$$

$$\int_{S^2} d\Omega \tilde{\eta} \eta = -\frac{i}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} [\tilde{\alpha}_{+lm} \beta_{+lm} + \tilde{\alpha}_{-lm} \beta_{-lm} - \tilde{\beta}_{+lm} \alpha_{+lm} - \tilde{\beta}_{-lm} \alpha_{-lm}]$$

$$\tilde{\alpha}_{\pm lm} = -i\beta_{\mp lm}^*, \tilde{\beta}_{\pm lm} = -i\alpha_{\mp lm}^*$$

$$\int_{S^2} d\Omega i \tilde{\eta} \not{D} \eta = \frac{1}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (l+1) [-\alpha_{-lm}^* \alpha_{+lm} - \alpha_{-lm} \alpha_{+lm}^* + \beta_{-lm}^* \beta_{+lm} + \beta_{-lm} \beta_{+lm}^*]$$

$$\int_{S^2} d\Omega \tilde{\eta} \eta = -\frac{2i}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (\alpha_{+lm} \beta_{+lm} + \alpha_{-lm} \beta_{-lm})$$

$$\int_{S^2} d\Omega \tilde{\eta} \tilde{\eta} = -\frac{2i}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (\alpha_{+lm}^* \beta_{+lm}^* + \alpha_{-lm}^* \beta_{-lm}^*)$$

$$\beta_{\pm lm} = -i\alpha_{\mp lm}^*$$

$$\int_{S^2} d\Omega i \tilde{\eta} \not{D} \eta = -\frac{2}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (l+1) (\alpha_{-lm}^* \alpha_{+lm} + \alpha_{-lm} \alpha_{+lm}^*)$$

$$\int_{S^2} d\Omega \tilde{\eta} \eta = \frac{2}{r} \sum_{l \geq 0} \sum_{0 \leq m \leq l} (\alpha_{-lm}^* \alpha_{+lm} - \alpha_{-lm} \alpha_{+lm}^*)$$

$$\mathcal{S}_L^{\mathcal{N}=1} = \frac{1}{4\pi} \int_{S^2} d^2x \tilde{e} \left(\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{i}{2} \bar{\psi} \not{D} \psi - \frac{1}{2} F^2 + \frac{1}{2} Q \tilde{R} \varphi + \mu e^{b\varphi} F - \frac{i}{2} \mu b e^{b\varphi} \bar{\psi} \psi \right),$$

$$\mathcal{Z}_L^{\mathcal{N}=1}[\mu] = \int \frac{[\mathcal{D}\varphi][\mathcal{D}\psi]}{\text{vol}_{\text{osp}(1|2;\mathbb{C})}} e^{-\frac{1}{4\pi} \int_{S^2} d^2x \tilde{e} \left(\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{i}{2} \bar{\psi} \not{D} \psi + \frac{1}{2} Q \tilde{R} \varphi + \frac{1}{2} \mu^2 e^{2b\varphi} - \frac{i}{2} \mu b e^{b\varphi} \bar{\psi} \psi \right)}$$

$$\varphi_*^{(n)} = \frac{1}{2b} \log \left(\frac{Q}{b\mu^2 r^2} \right) + \frac{\pi i}{2b} (2n+1), n \in \mathbb{Z}$$

$$\mathcal{Z}_{\text{L, saddle}}^{\mathcal{N}=1}[\mu] \approx -i \times \left(\frac{Q}{b\mu^2 r^2} \right)^{-\frac{Q}{2b}} \times e^{-\frac{Q}{2b}} e^{-\frac{i\pi}{2b^2}} \sum_{n \geq 0} (-1)^n e^{-\frac{i\pi}{b^2} n}$$

$$\mathcal{Z}_{\text{L, bos. fluct.}}^{\mathcal{N}=1} = \int [\mathcal{D}\delta\varphi] e^{-\frac{1}{8\pi} \int_{S^2} d^2x \tilde{e} \delta\varphi \left(-\nabla^2 - \frac{2Qb}{r^2} \right) \delta\varphi}$$

$$\mathcal{Z}_{\text{L, fer. fluct.}}^{\mathcal{N}=1} = \int [\mathcal{D}\delta\psi] e^{-\frac{i}{8\pi} \int_{S^2} d^2x \tilde{e} \delta\bar{\psi} \left(-\not{D} - i(-1)^n \frac{\sqrt{Qb}}{r} \right) \delta\psi}$$

$$Z_{\rm L}^{\mathcal{N}=1} \approx \text{const} \times \frac{1}{\cos \frac{\pi}{2b^2}} \times (\mu b)^{\frac{1}{b^2}+1} \Lambda_{\rm uv}^{\frac{5}{4}+b^2} r^{c_{\rm L}^{\mathcal{N}=1}/3} e^{-\frac{1}{2b^2}+\frac{1}{b^2}\log b^2} \times b^{-1} \times (1+\mathcal{O}(b^2))$$

$$\Delta_{\rm NS}=\frac{\alpha(Q-\alpha)}{2}=\frac{c_{\rm L}^{\mathcal{N}=1}}{24}-\frac{3}{48}+\frac{P^2}{2}, \Delta_{\rm R}=\Delta_{\rm NS}+\frac{1}{16},$$

$$\mathcal{W}_\alpha=\alpha^2\bar{\psi}\psi e^{\alpha\varphi}-2\pi\mathrm{i}\mu\alpha e^{(\alpha+b)\varphi}$$

$$C(b,b,b) \stackrel{b\rightarrow 0}{\approx} \mathrm{i}(\pi\mu b)^{\frac{1}{b^2}-2}e^{\frac{1}{b^2}(1+\log 2+\log b^2)}e^{-\frac{\mathrm{i}\pi}{2b^2}}\sum_{n=0}^{\infty}(-1)^ne^{-\frac{\mathrm{i}\pi}{b^2}n}b^{-\frac{9}{2}}\times\left(1-\frac{11}{6}b^2+\cdots\right)$$

$$S_{\rm L}^{\mathcal{N}=2}=\frac{1}{4\pi}\int_{S^2}\mathrm{d}^2x\tilde{\mathfrak{e}}\left(\partial_\mu\tilde{\varphi}\partial^\mu\varphi-\mathrm{i}\bar{\tilde{\psi}}\not{D}\psi-F\tilde{F}+\frac{Q\tilde{R}}{2}(\varphi+\tilde{\varphi})\right.\\ \left.+\mu e^{b\varphi}F-\frac{\mathrm{i}}{2}\mu b e^{b\varphi}\bar{\psi}\psi+\mu^*e^{b\tilde{\varphi}}\tilde{F}-\frac{\mathrm{i}}{2}\mu^*b e^{b\tilde{\varphi}}\bar{\tilde{\psi}}\tilde{\psi}\right)$$

$$\begin{array}{ll} 0=-\nabla^2\varphi+\frac{1}{br^2}+\mu be^{b\varphi}F-\frac{\mathrm{i}}{2}\mu b^2\bar{\psi}\psi, & 0=-\nabla^2\tilde{\varphi}+\frac{1}{br^2}+\mu^*be^{b\tilde{\varphi}}\tilde{F}-\frac{\mathrm{i}}{2}\mu^*b^2\bar{\tilde{\psi}}\tilde{\psi} \\ 0=\tilde{F}-\mu e^{b\varphi}, & 0=F-\mu^*e^{b\tilde{\varphi}} \end{array}$$

$$(\varphi+\tilde{\varphi})_*^{(n)}=\frac{1}{b}\log\left(\frac{1}{b^2|\mu|^2r^2}\right)+\frac{\pi\mathrm{i}}{b}(2n+1), n\in\mathbb{Z}$$

$$Z_{\rm L,\,saddle}^{\mathcal{N}=2}[\mu] = (|\mu^2|b^2r^2)^{\frac{1}{b^2}}e^{\frac{1}{b^2}-\frac{\mathrm{i}\pi}{b^2}}\sum_{n\geq 0}e^{-\frac{2\mathrm{i}\pi}{b^2}n} = -\frac{\mathrm{i}}{2}(e|\mu^2|b^2r^2)^{\frac{1}{b^2}}\times\frac{1}{\sin\left(\frac{\pi}{b^2}\right)}$$

$$\mathcal{L}_\xi\psi\equiv \xi^\mu D_\mu\psi+\frac{1}{4}\nabla_\mu\xi_\nu\gamma^{\mu\nu}\psi$$

$$\langle \mathcal{H}(\alpha_{\pm lm},\alpha_{\pm l'm'}^*)\rangle \equiv (Z_{\rm fer,1-loop}^{\mathcal{N}=1})^{-1}\int\left[\mathcal{D}\delta\psi\right]e^{-S_{\rm L,fer}^{(2)}[\delta\psi]}\mathcal{H}(\alpha_{\pm lm},\alpha_{\pm l'm'}^*)$$

$$\int \, \mathrm{d}\alpha_{+lm} \, \mathrm{d}\alpha_{+lm}^* \, \mathrm{d}\alpha_{-lm} \, \mathrm{d}\alpha_{-lm} \alpha_{+lm} \alpha_{+lm}^* \alpha_{-lm} \alpha_{-lm}^* = 1$$

$$\delta F=-\nabla_\mu(\bar{\epsilon}\gamma^\mu\psi)-\frac{r-2}{2}D_\mu\bar{\epsilon}\gamma^\mu\psi$$

$$L^2\psi_{\pm lm}^{(s)}=1(1+1)\psi_{\pm lm}^{(s)},L_z\psi_{\pm lm}^{(s)}=m\psi_{\pm lm}^{(s)},\not{D}\psi_{\pm lm}^{(s)}=\pm\mathrm{i}\frac{1+\frac{1}{2}}{r}\psi_{\pm lm}^{(s)}$$

$$C(b,b,b)=\frac{\mathrm{i}}{2}\Big(\frac{\pi\mu}{2b}\gamma\Big(\frac{1}{2}+\frac{b^2}{2}\Big)b^{1-b^2}\Big)^{\frac{1}{b^2}-2}\frac{\gamma_b\left(\frac{b}{2}\right)\gamma_b\left(\frac{1}{2b}\right)}{\gamma_b(b)}\times\frac{\gamma\left(\frac{1}{2}+\frac{b^2}{2}\right)^3\gamma\left(\frac{1}{2}-\frac{1}{2b^2}\right)}{\gamma\left(\frac{b^2}{2}-\frac{1}{2}\right)}b^{-\frac{1}{b^2}-2-2b^2},$$

$$n_H-n_V+29n_T=273$$



$$V = \frac{2g^2}{\kappa^4} U(q) e^\varphi$$

$$\mathcal{L}_6 = -\sqrt{-g} \left[\frac{1}{2\kappa^2} g^{MN} (R_{MN} + \partial_M \varphi \partial_N \varphi) + \frac{1}{4} e^{-\varphi} F_{MN} F^{MN} + \frac{1}{12} e^{-2\varphi} H_{MNP} H^{MNP} + \frac{2g^2}{\kappa^4} e^\varphi \right]$$

$$\square_6 \varphi + \frac{\kappa^2}{4} e^{-\varphi} F_{MN} F^{MN} + \frac{\kappa^2}{6} e^{-2\varphi} H_{MNP} H^{MNP} - \frac{2g^2}{\kappa^2} e^\varphi = 0$$

$$\nabla_M (e^{-\varphi} F^{MN}) + \kappa e^{-2\varphi} H^{PNQ} F_{PQ} = 0, \nabla_M (e^{-2\varphi} H^{MNP}) = 0$$

$$R_{MN} + \partial_M \varphi \partial_N \varphi + \kappa^2 e^{-\varphi} F_{MP} F_N^P + \frac{1}{2} (\square_6 \varphi) g_{MN} = 0$$

$$g_{MN} \rightarrow c g_{MN} \text{ and } e^{-\varphi} \rightarrow c e^{-\varphi} \text{ imply } \mathcal{L}_6 \rightarrow c^2 \mathcal{L}_6$$

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n$$

$$e^{-\varphi} f = \pm 2g/\kappa^2$$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \rho^2 (d\phi^2 + \sin^2 \phi d\theta^2)$$

$$\frac{1}{\rho^2} = \kappa^2 f^2 e^{-\varphi}, \rho^2 e^\varphi = \left(\frac{\kappa}{2g}\right)^2$$

$$V = \frac{2g^2 e^\varphi}{\rho^2} \left(1 - \frac{\kappa^4}{4g^2 e^\varphi \rho^2}\right)^2$$

$$ds^2 = \hat{g}_{MN} \, dx^M \, dx^N = W^2 g_{\mu\nu} dx^\mu dx^\nu + a^2 \, d\theta^2 + a^2 W^8 \, d\eta^2,$$

$$\hat{R}_{\mu\nu} + \frac{1}{2} (\square_2 \varphi) \hat{g}_{\mu\nu} = 0,$$

$$\int_{M_2} d^2y \sqrt{g_2} \hat{R}_4(x,y) = 2 \oint_{\partial M_2} dy n_m \sqrt{g_2} g^{mn} \partial_n \varphi$$

$$F_{\eta\theta} = Q a^2 e^\varphi$$

$$\varphi'' + \frac{\kappa^2}{2} Q^2 a^2 e^\varphi - \frac{2g^2}{\kappa^2} a^2 W^8 e^\varphi = 0$$

$$(\mu\nu): \left(\ln W + \frac{\varphi}{2}\right)'' = 3\zeta H^2 a^2 W^6$$

$$(\theta\theta): \left(\ln a + \frac{\varphi}{2}\right)'' = -\kappa^2 Q^2 a^2 e^\varphi$$

$$(\eta\eta): 6\zeta H^2 a^2 W^6 - \frac{4a'W'}{aW} - \frac{6(W')^2}{W^2} + \frac{1}{2} (\varphi')^2 + \frac{\kappa^2}{2} Q^2 a^2 e^\varphi - \frac{2g^2}{\kappa^2} a^2 W^8 e^\varphi = 0$$

$$\{\varphi,a,W,H\}\rightarrow\{\varphi+\varphi_0,ae^{-\varphi_0/2},W,He^{\varphi_0/2}\}$$



$$3\zeta H^2 \int_{\eta_1}^{\eta_2} d\eta a^2 W^6 = \left(\ln W + \frac{\varphi}{2} \right)'_{\eta=\eta_2} - \left(\ln W + \frac{\varphi}{2} \right)'_{\eta=\eta_1}$$

$$\frac{1}{\kappa_4^2} = \frac{1}{\kappa^2} \int d^2x \sqrt{g_2} W^2 = \frac{2\pi}{\kappa^2} \int_{-\infty}^{\infty} d\eta a^2 W^6$$

$$\zeta H^2 = \frac{8}{3} \pi G_N \left[\left(\ln W + \frac{\varphi}{2} \right)' \right]_{\eta=-\infty}^{\eta=+\infty}$$

$$W^2 e^{\varphi} = e^{-\lambda_3 \eta}, W^4 = \left(\frac{\kappa^2 Q \lambda_2}{2g \lambda_1} \right) \frac{\cosh [\lambda_1 (\eta - \eta_1)]}{\cosh [\lambda_2 (\eta - \eta_2)]}, F_{\eta\theta} = \left(\frac{Q a^2}{W^2} \right) e^{-\lambda_3 \eta}$$

$$\text{and } \frac{1}{a^4} = \left(\frac{4g\kappa^4 Q^3}{\lambda_1^3 \lambda_2} \right) e^{-2\lambda_3 \eta} \cosh^3 [\lambda_1 (\eta - \eta_1)] \cosh [\lambda_2 (\eta - \eta_2)]$$

$$\varphi'' \simeq (\ln W)'' \simeq (\ln a)'' \simeq 0$$

$$\varphi \simeq \mp \eta q_{\pm}, W \simeq W_{\pm} e^{\mp \eta \omega_{\pm}} \text{ and } a \simeq a_{\pm} e^{\mp \eta \alpha_{\pm}}$$

$$q_b^2=4\omega_b(2\alpha_b+3\omega_b)$$

$$2\alpha_b+6\omega_b>0, 2\alpha_b+q_b>0 \text{ and } 2\alpha_b+8\omega_b+q_b>0$$

$$\alpha_{\pm}=\frac{1}{4}(3\lambda_1+\lambda_2\mp2\lambda_3)\geq0, \omega_{\pm}=\frac{1}{4}(\lambda_2-\lambda_1)\geq0$$

$$q_{\pm}=\pm\lambda_3-\frac{1}{2}(\lambda_2-\lambda_1)$$

$$\omega_{\pm}+\frac{q_{\pm}}{2}=\pm\frac{\lambda_3}{2}$$

$$\zeta H^2 = -\frac{8}{3} \pi G_N \sum_{b=\pm} \left(\frac{q_b}{2} + \omega_b \right)$$

$$S_{Ren} = S_{QED} + S_{\Phi} = - \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} [\not{D} + m] \Psi + \bar{\Phi} [\not{D} + M] \Phi \right\}$$

$$S_{NRen} = - \int d^4x \left\{ \frac{\tilde{c}_d}{2} (\bar{\Phi} \gamma^{\mu\nu} \Phi) F_{\mu\nu} + \tilde{c}_s (\bar{\Psi} \Psi) (\bar{\Phi} \Phi) + \tilde{c}_v (\bar{\Psi} \gamma^{\mu} \Psi) (\bar{\Phi} \gamma_{\mu} \Phi) + \cdots \right\}$$

$$S_{Nucl} = - \int_{\mathcal{P}} ds \left\{ \sqrt{-\dot{y}^2} M - Ze \dot{y}^{\mu} A_{\mu} + c_s \sqrt{-\dot{y}^2} (\bar{\Psi} \Psi) + i c_v \dot{y}^{\mu} (\bar{\Psi} \gamma_{\mu} \Psi) + \cdots \right\}$$

$$S_b = \int d^Dx \int_W ds L_b[y(s)] \delta^D[x-y(s)] = \int d^Dx \frac{L_b}{\gamma} \delta^{D-1}[\mathbf{x}-\mathbf{y}(s)]$$

$$0 = \int_{P_\epsilon} d^{D-1}x \left\{ -\partial_M [\sqrt{-g} f(\phi) F^{MN}] + \sqrt{-g} \nabla_M [f(\phi) F^{MN}] + \frac{1}{\gamma} \frac{\partial L_b}{\partial A_N} \delta^{D-1}[\mathbf{x}-\mathbf{y}(s)] \right\}$$

$$\simeq - \oint_{\partial P_\epsilon} d^{D-2}x n_M [\sqrt{-g} f F^{MN}] + \frac{1}{\gamma} \left(\frac{\partial L_b}{\partial A_N} \right)_{\mathbf{x}=\mathbf{y}}$$



$$\oint_{C_\epsilon} d\theta [n_N \sqrt{-g} f(\phi) F^{NM}] = - \left(\frac{\delta S_b}{\delta A_M} \right)_\epsilon$$

$$\mathcal{L} = -\frac{1}{2} \kappa^{-2} \sqrt{-g} \mathcal{G}_{AB} \partial^M \phi^A \partial_M \phi^B$$

$$\oint_{C_\epsilon} d\theta \left[\frac{1}{\kappa^2} \sqrt{-g} \mathcal{G}_{AB} n_M \partial^M \phi^B \right] = - \left(\frac{\delta S_b}{\delta \phi^A} \right)_\epsilon$$

$$\lim_{r \rightarrow 0} \oint_{C_\epsilon} d\theta \left[\frac{1}{2\kappa^2} \sqrt{-g} (K^{ij} - K g^{ij}) - (\text{supercurvature}) \right] = - \left(\frac{\delta S_b}{\delta g_{ij}} \right)_\epsilon$$

$$S_b = \int_W d^4x \sqrt{-\gamma} T(\phi)$$

$$S_b = - \int d^4x \sqrt{-\gamma} L_b(\varphi) = - \int d^4x \sqrt{-g} W_b^4 L_b(\varphi) = - \int d^4x \sqrt{-g} T_b(\varphi)$$

$$\frac{2\pi}{\kappa^2} [aW^4 \partial_r \varphi]_{r=\epsilon} = \frac{\partial}{\partial \varphi} [W_b^4 L_b] \text{ and so } \lim_{\eta \rightarrow \pm \infty} [\mp \partial_\eta \varphi] = q_\pm = \frac{\kappa^2}{2\pi} \left(\frac{\partial T_\pm}{\partial \varphi} \right)$$

$$dr := \mp a W^4 \, d\eta,$$

$$-\frac{2\pi}{\kappa^2} \left\{ W^4 \left[a \left(3 \frac{\partial_r W}{W} + \frac{\partial_r a}{a} \right) - 1 \right] \right\}_{r=\epsilon} = W_b^4 L_b(\varphi)$$

$$\lim_{\eta \rightarrow \pm \infty} \left\{ \mp \left[3 \left(\frac{\partial_\eta W}{W} \right) + \left(\frac{\partial_\eta a}{a} \right) \right] - W^4 \right\} = 3\omega_\pm + \alpha_\pm - W_\pm^4 = -\frac{\kappa^2 T_\pm}{2\pi}$$

$$\alpha_b^2 e^{-2\alpha_b|\eta|} (W_b^8 \, d\eta^2 + d\theta^2) \simeq dr^2 + \left(\frac{r\alpha_b}{W_b^4} \right)^2 d\theta^2$$

$$\delta_b := 2\pi \left(1 - \frac{\alpha_b}{W_b^4} \right) = \frac{\kappa^2 T_b}{W_b^4} = \kappa^2 L_b$$

$$\frac{2\pi}{\kappa^2} [aW^4 \partial_r W]_{r=\epsilon} = W_b^4 \tilde{U}_b(\varphi) = U_b(\varphi) \text{ and so } \lim_{\eta \rightarrow \pm \infty} \left[\mp \left(\frac{\partial_\eta W}{W} \right) \right] = \omega_\pm = \frac{\kappa^2 U_\pm}{2\pi}$$

$$\begin{aligned} 0 &\simeq \left[-\frac{8a'W'}{aW} - \frac{12(W')^2}{W^2} + (\varphi')^2 \right]_b = -8\alpha_b\omega_b - 12\omega_b^2 + q_b^2 \\ &= -8\mathcal{U}_b(W_b^4 - \mathcal{T}_b - 3\mathcal{U}_b) - 12\mathcal{U}_b^2 + (\mathcal{T}_b')^2 \\ &= W_b^8 [-8\tilde{\mathcal{U}}_b(1 - \mathcal{L}_b - 3\tilde{\mathcal{U}}_b) - 12\tilde{\mathcal{U}}_b^2 + (\mathcal{L}_b')^2] \end{aligned}$$

$$\mathcal{U}_b = W_b^4 \tilde{\mathcal{U}}_b = \frac{W_b^4}{3} \left[(1 - \mathcal{L}_b) - \sqrt{(1 - \mathcal{L}_b)^2 - \frac{3}{4} (\mathcal{L}_b')^2} \right]$$

$$V_{\rm eff}(\varphi_0)=-\sum_b\left(U_b+\frac{T'_b}{2}\right)=-\sum_bW_b^4\left(\tilde{U}_b+\frac{L'_b}{2}\right),$$

$$\mathcal{L}_{10}=-\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_{10}^2}\hat{g}^{MN}(\hat{R}_{MN}+\partial_M\hat{\phi}\partial_N\hat{\phi})+\frac{1}{4}e^{-\hat{\phi}}\hat{F}_{MN}\hat{F}^{MN}+\frac{1}{12}e^{-2\varphi}\hat{H}_{MNP}\hat{H}^{MNP}\right]$$

$$y_1 + i y_2 = \cosh \rho e^{i \alpha}; \; y_3 + i y_4 = \sinh \rho e^{i \beta}$$

$$\mathrm{d}\hat{s}_{10}^2=(\cosh 2\rho)^{1/4}\left[e^{-\varphi/4}\,\mathrm{d}s_6^2+e^{\varphi/4}\,\mathrm{d}z^2+\frac{e^{\varphi/4}}{2\bar{g}^2}\left(\mathrm{d}\rho^2+\frac{\cosh^2\rho}{\cosh 2\rho}(D\alpha)^2+\frac{\sinh^2\rho}{\cosh 2\rho}(D\beta)^2\right)\right]$$

$$\hat{H}_{(3)}=H_{(3)}+\frac{\sinh 2\rho}{2\bar{g}(\cosh 2\rho)^2}\,\mathrm{d}\rho\wedge D\alpha\wedge D\beta+\frac{1}{2\bar{g}\cosh 2\rho}F_{(2)}\wedge(\cosh^2\rho D\alpha-\sinh^2\rho D\beta)$$

$$e^{\hat{\phi}} = (\cosh 2\rho)^{-1/2} e^{\varphi}$$

$$\mathcal{M}_{11}=\mathbb{R}^{1,4}\times Y_6; \, Y_6\simeq B_4\times T^2; \, T^2=S^1_A\times S^1_B$$

$$\mathcal{M}_{10}=\mathbb{R}^{1,4}\times \tilde{S}^1_B\times B_4\rightarrow \mathbb{R}^{1,5}\times B_4$$

$$S_{11}=-\frac{1}{2\kappa_{11}^2}\int\,\,\mathrm{d}^{11}x\left[\sqrt{-g}\left(R+\frac{1}{48}G_{MNPQ}G^{MNPQ}\right)+\frac{2}{3\cdot 4!^3}\epsilon^{M_1\cdots M_{11}}c_{M_1\cdots M_3}G_{M_4\cdots M_7}G_{M_8\cdots M_{11}}\right]$$

$$\mathrm{d}s^2=\mathrm{d}s_5^2+\mathrm{d}s_6^2$$

$$\int_{\gamma_\Lambda} \langle G_{(4)} \rangle = n_\Lambda; \; \langle G_{(4)} \rangle = n^\Lambda \omega_\Lambda$$

$$L^\Lambda=\frac{v^\Lambda}{\mathcal{V}^{1/3}}, \mathcal{N}\equiv \lambda_{\Lambda\Omega\Gamma}L^\Lambda L^\Omega L^\Gamma=1$$

$$\lambda_{\Lambda\Omega\Gamma}=\int_{Y_6}\tilde{\omega}_\Lambda\wedge\tilde{\omega}_\Omega\wedge\tilde{\omega}_\Gamma$$

$$D_\mu\Phi=\partial_\mu\Phi+2n_\Lambda A_\mu^\Lambda; \; V_{\rm flux}=\frac{1}{8\mathcal{V}^2}G^{\Lambda\Sigma}n_\Lambda n_\Sigma$$

$$G_{\Lambda\Sigma}=\frac{1}{2\mathcal{V}^{1/3}}\bigg(\int_{Y_6}\tilde{w}_\Lambda\wedge w_\Sigma\bigg)$$

$$n_H=h^{2,1}(Y_6)+1; \, n_T+1=h^{1,1}(B_4); \, \mathrm{rank} G=h^{1,1}(Y_6)-h^{1,1}(B_4)-1$$

$$\langle F_{(2)}\rangle=n^\Lambda\tilde{\omega}_\Lambda;\,\int\,\mathrm{d}^8x\sqrt{-g}F^{MN}F_{MN}\Rightarrow V_{\mathrm{D}7}\propto G^{\Lambda\Sigma}n_\Lambda n_\Sigma$$

$$-\frac{\mathcal{L}_6}{\sqrt{-g}}=\frac{1}{2\kappa^2}g^{MN}(R_{MN}+\partial_M\varphi\partial_N\varphi+h_{UV}D_Mq^UD_Nq^V)\\+\frac{1}{4}e^{-\varphi}F_{MN}F^{MN}+\frac{1}{12}e^{-2\varphi}H_{MNP}H^{MNP}+\frac{2\mathfrak{g}^2U(q)}{\kappa^4}e^{\varphi}$$



$$h(\mathcal{V}) := h_{\mathcal{V}\mathcal{V}} = 4h_{\Phi\Phi} = \frac{1}{2\mathcal{V}^2}; \quad h_{\tau\tau^*} = \frac{1}{2\tau_2^2}$$

$$V(\varphi,\chi)=\frac{2\mathfrak{g}^2}{\kappa^4\mathcal{V}^2}e^\varphi=\frac{2\mathfrak{g}^2}{\kappa^4}e^{\varphi-2\chi} \text{ where } \mathcal{V}:=e^\chi$$

$$R_{MN}=4e^{-\varphi}\kappa^2F_{MP}F_N^P+\partial_M\varphi\partial_N\varphi+\frac{1}{2}\partial_M\chi\partial_N\chi+\frac{1}{2}e^{-2\chi}D_M\Phi D_N\Phi+\frac{1}{2\tau_2^2}\partial_{(M}\tau\partial_{N)}\tau^*+\frac{1}{2}g_{MN}\Box\varphi$$

$$\nabla_R(e^{-\varphi}F^{LR})=\frac{k}{32}e^{-2\chi}D^L\Phi$$

$$\Box\chi=-4e^{\varphi-2\chi}\tilde{V}-e^{-2\chi}D_M\Phi D^M\Phi,\Box\Phi=2\partial_M\chi D^M\Phi,\\ \Box\varphi=\tilde{V}e^{\varphi-2\chi}-\kappa^2e^{-\varphi}F^{MN}F_{MN}\text{ and }\Box\tau=-\frac{i}{\tau_2}\partial^M\tau\partial_M\tau^*,$$

$$\nabla_M(e^{-\varphi}F^{MN})=0$$

$$\Box\chi+4\tilde{V}e^{\varphi-2\chi}=0\text{ and }\Box\varphi+\frac{1}{4}\kappa^2e^{-\varphi}F^{MN}F_{MN}-\tilde{V}e^{\varphi-2\chi}=0,$$

$$R_{MN}+\kappa^2e^{-\varphi}F_{MP}F_N^P+\partial_M\varphi\partial_N\varphi+\frac{1}{2}\partial_M\chi\partial_N\chi+\frac{1}{2}g_{MN}\Box_6\varphi=0$$

$$\mathrm{d}s^2=\hat{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+a^2\,\mathrm{d}\theta^2+a^2W^8\,\mathrm{d}\eta^2$$

$$F_{\eta\theta}=\hat{Q}a^2e^\varphi$$

$$\chi''=-4\tilde{V}a^2W^8e^{\varphi-2\chi}\text{ and }\varphi''=\tilde{V}a^2W^8e^{\varphi-2\chi}-\frac{1}{2}Q^2a^2e^\varphi$$

$$\left(\ln W+\frac{1}{2}\varphi\right)''=3\zeta H^2a^2W^6$$

$$\left(\ln a+\frac{1}{2}\varphi\right)''=-Q^2a^2e^\varphi$$

$$\varphi'^2+\frac{1}{2}\chi'^2-8(\ln W)'(\ln a)'-12[(\ln W)']^2+a^2e^\varphi(Q^2-2\tilde{V}W^8e^{-2\chi}+12\zeta H^2W^6e^{-\varphi})=0$$

$$L:=\left[\varphi'^2+\frac{1}{2}\chi'^2-8(\ln W)'(\ln a)'-12[(\ln W)']^2\right]N^{-1}-Na^2e^\varphi(Q^2-2\tilde{V}W^8e^{-2\chi}+12\zeta H^2W^6e^{-\varphi})$$

$$\varphi=\frac{1}{2}(\mathcal{X}-\mathcal{Y}-2\mathcal{Z}), \ln W=\frac{1}{4}(\mathcal{Y}-\mathcal{X}), \ln a=\frac{1}{4}(3\mathcal{X}+\mathcal{Y}+2\mathcal{Z})$$

$$X=\mathcal{X}+\ln Q, Y=\mathcal{Y}+\ln(2\tilde{V}) \text{ and } Z=\mathcal{Z}+\ln\left(\frac{6H^2}{\tilde{V}}\right)$$

$$L=\left[(X')^2-(Y')^2+(Z')^2+\frac{1}{2}(\chi')^2\right]N^{-1}-N[e^{2X}-e^{2Y-2\chi}+\zeta e^{2Y+Z}]$$



$$X'' + e^{2X} = 0, \chi'' + 2e^{2Y-2\chi} = 0, Z'' + \frac{1}{2}\zeta e^{2Y+Z} = 0 \text{ and } Y'' + e^{2Y-2\chi} - \zeta e^{2Y+Z} = 0.$$

$$X = \ln \left[\frac{\lambda_1}{\cosh [\lambda_1(\eta - \eta_1)]} \right],$$

$$\chi = 4Z + 2Y + b\eta + c$$

$$A:=\frac{1}{2}(2Y+Z) \text{ and } B:=Y-\chi=-(Y+4Z+b\eta+c)$$

$$B''=\zeta e^{2A}+e^{2B} \text{ and } A''=\frac{3}{4}\zeta e^{2A}-e^{2B}$$

$$B''=e^{2B} \text{ (4D curve solutions)}$$

$$B=\ln\left[\frac{\lambda_q}{\sinh\left[\lambda_q(\eta-\eta_0)\right]}\right],$$

$$\mathrm{d}s^2=e^{2\Gamma(r)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+e^{2\Omega(r)}\mathrm{d}\theta^2+\mathrm{d}r^2$$

$$J'=e^{\Omega+4\Gamma}j \text{ and } J''=e^{2\Omega+8\Gamma}[\ddot{j}+(\dot{\Omega}+4\dot{\Gamma})j],$$

$$\begin{aligned}\ddot{\varphi}+(\dot{\Omega}+4\dot{\Gamma})\dot{\varphi}&=\tilde{V}e^{\varphi-2\chi}-\frac{1}{2}Q^2e^{\varphi-8\Gamma}\\\ddot{\chi}+(\dot{\Omega}+4\dot{\Gamma})\dot{\chi}&=-4\tilde{V}e^{\varphi-2\chi}\\\ddot{\Gamma}+(\dot{\Omega}+4\dot{\Gamma})\dot{\Gamma}&=3\zeta H^2e^{-2\Gamma}-\frac{1}{2}[\dot{\varphi}+(\dot{\Omega}+4\dot{\Gamma})\dot{\varphi}]\\\ddot{\Omega}+(\dot{\Omega}+4\dot{\Gamma})\dot{\Omega}&=-Q^2e^{\varphi-8\Gamma}-\frac{1}{2}[\dot{\varphi}+(\dot{\Omega}+4\dot{\Gamma})\dot{\varphi}].\end{aligned}$$

$$\ddot{\chi}+3\ddot{\varphi}-2\ddot{\Omega}=-(\dot{\Omega}+4\dot{\Gamma})(\dot{\chi}+3\dot{\varphi}-2\dot{\Omega}),$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big[e^{\Omega+4\Gamma}\Big(\dot{\Gamma}+\frac{1}{2}\dot{\varphi}\Big)\Big]=3\zeta H^2e^{\Omega+2\Gamma} \text{ and } \frac{\mathrm{d}}{\mathrm{d}r}\Big[e^{\Omega+4\Gamma}\Big(\dot{\Omega}+\frac{1}{2}\dot{\varphi}\Big)\Big]=-Q^2e^{\varphi+\Omega-4\Gamma}$$

$$\dot{\varphi}^2+\frac{1}{2}\dot{\chi}^2-8\dot{\Omega}\dot{\Gamma}-12\dot{\Gamma}^2+12\zeta H^2e^{-2\Gamma}+Q^2e^{\varphi-8\Gamma}-2\tilde{V}e^{\varphi-2\chi}=0$$

$$\varphi=\varphi_0+q\ln\left(\frac{r}{r_0}\right), \chi=\chi_0+s\ln\left(\frac{r}{r_0}\right), \Gamma=\Gamma_0+w\ln\left(\frac{r}{r_0}\right) \text{ and } \Omega=\Omega_0+\alpha\ln\left(\frac{r}{r_0}\right),$$

$$a(r)=a_0\left(\frac{r}{r_0}\right)^\alpha\left[1+\mathcal{O}(r)\right], \text{ for } a(r)\propto e^\Omega$$

$$q>2(s-1) \text{ and } q>2(4w-1) \text{ and (if } H\neq 0) w<1$$

$$\alpha+4w=1$$

$$q^2+\frac{1}{2}s^2-8\alpha w-12w^2=q^2+\frac{1}{2}s^2-8w+20w^2=0$$



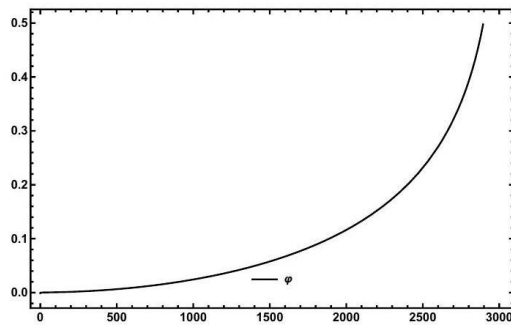
$$\begin{aligned}\frac{q}{r^2}(-1 + \alpha + 4w) &= \tilde{V} e^{\varphi_0 - 2\chi_0} \left(\frac{r_0}{r}\right)^{2s-q} - \frac{1}{2} Q^2 e^{\varphi_0 - 8\Gamma_0} \left(\frac{r_0}{r}\right)^{8w-q} \\ \frac{s}{r^2}(-1 + \alpha + 4w) &= -4\tilde{V} e^{\varphi_0 - 2\chi_0} \left(\frac{r_0}{r}\right)^{2s-q} \\ \frac{w + \frac{1}{2}q}{r^2}(-1 + \alpha + 4w) &= 3\zeta H^2 e^{-2\Gamma_0} \left(\frac{r_0}{r}\right)^{2w} \\ \frac{\alpha + \frac{1}{2}q}{r^2}(-1 + \alpha + 4w) &= -Q^2 e^{\varphi_0 - 8\Gamma_0} \left(\frac{r_0}{r}\right)^{8w-q}\end{aligned}$$

$$(s + 3q - 2\alpha)(-1 + \alpha + 4w) \frac{1}{r^2} = 0$$

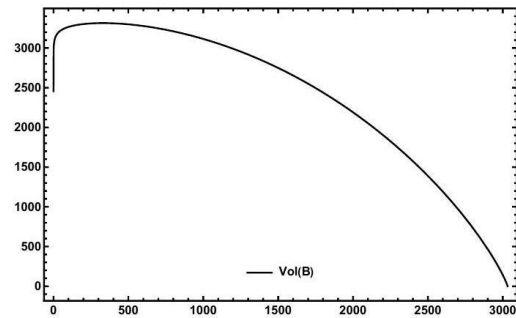
$$\frac{q^2 + \frac{1}{2}s^2 - 8\alpha w - 12w^2}{r^2} + 12\zeta H^2 e^{-2\Gamma_0} \left(\frac{r_0}{r}\right)^{2w} + Q^2 e^{\varphi_0 - 8\Gamma_0} \left(\frac{r_0}{r}\right)^{8w-q} - 2\tilde{V} e^{\varphi_0 - 2\chi_0} \left(\frac{r_0}{r}\right)^{2s-q} = 0$$

$$q = 2(s - 1), q = 2(4w - 1) \text{ and (if } H \neq 0) w = 1$$

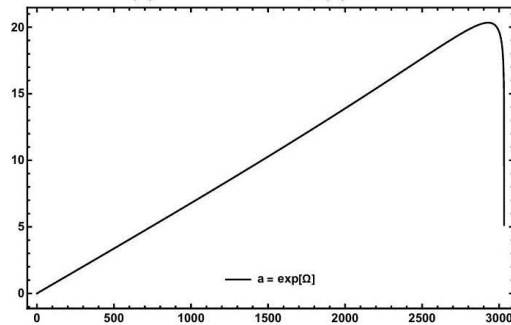
$$w = \frac{1}{9}, q = -\frac{2}{9}, s = \frac{8}{9}, \alpha = \frac{1}{9} \text{ and } \tilde{V} e^{\varphi_0 - 2\chi_0} r_0^2 = \frac{8}{81}$$



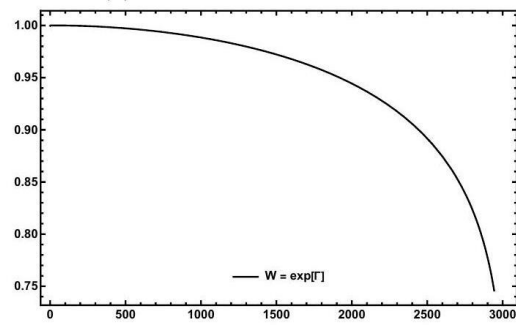
(a) The function $\varphi(r)$ vs r



(b) The function $\mathcal{V} = e^{\chi(r)}$ vs r .

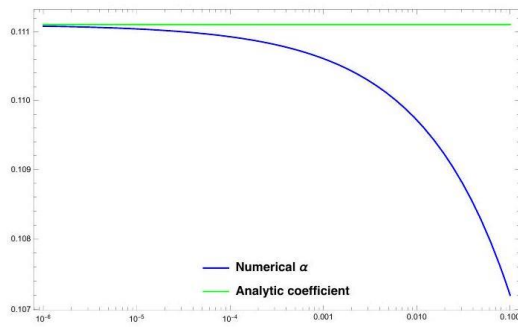


(c) The function $a(r) = e^{\Omega(r)}$ vs r .

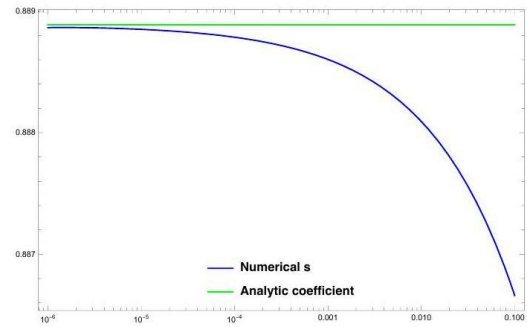


(d) The function $W(r) = e^{\Gamma(r)}$ vs r .

$$\alpha_0 = 0.9999, q_0 = 0.0001, s_0 \simeq 0.0200, w_0 = 0.000025$$



(a) $r \partial\Omega/\partial r$ vs $\alpha_e = \frac{1}{9}$



(b) $r \partial\chi/\partial r$ vs $s_e = \frac{8}{9}$

$$S_b = - \int d^4x \sqrt{-g} W_b^4 L_b(\varphi, \chi) = - \int d^4x \sqrt{-g} T_b(\varphi, \chi)$$

$$[a\partial_r\varphi]_{r=\epsilon} = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi}\right)_{r=\epsilon} \quad \text{and} \quad [a\partial_r\chi]_{r=\epsilon} = \left(\frac{\partial\mathcal{L}_b}{\partial\chi}\right)_{r=\epsilon}$$

$$1 - \left[a \left(3 \frac{\partial_r W}{W} + \frac{\partial_r a}{a} \right) \right]_{r=\epsilon} = (\mathcal{L}_b)_{r=\epsilon} \quad \text{and} \quad [a\partial_r W]_{r=\epsilon} = (\tilde{u}_b)_{r=\epsilon}$$

$$q_b \left(\frac{a_b}{r_b} \right) \left(\frac{\epsilon}{r_b} \right)^{\alpha_b-1} = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi} \right)_{r=\epsilon} \quad \text{and} \quad s_b \left(\frac{a_b}{r_b} \right) \left(\frac{\epsilon}{r_b} \right)^{\alpha_b-1} = \left(\frac{\partial\mathcal{L}_b}{\partial\chi} \right)_{r=\epsilon},$$

$$1 - (3w_b + \alpha_b) \left(\frac{a_b}{r_b} \right) \left(\frac{\epsilon}{r_b} \right)^{\alpha_b-1} = (\mathcal{L}_b)_{r=\epsilon}$$

$$w_b \left(\frac{a_b}{r_b} \right) \left(\frac{\epsilon}{r_b} \right)^{\alpha_b-1} = (\tilde{u}_b)_{r=\epsilon}$$

$$q_b \left(\frac{a_b}{r_b} \right) = \left(\frac{\partial\mathcal{L}_b}{\partial\varphi} \right)_{r=\epsilon}, \quad s_b \left(\frac{a_b}{r_b} \right) = \left(\frac{\partial\mathcal{L}_b}{\partial\chi} \right)_{r=\epsilon}, \quad w_b \left(\frac{a_b}{r_b} \right) = (\tilde{u}_b)_{r=\epsilon}$$

$$1 - (3w_b + \alpha_b) \left(\frac{a_b}{r_b} \right) = (\mathcal{L}_b)_{r=\epsilon}$$

$$\begin{aligned} 0 &\simeq \left[\dot{\varphi}^2 + \frac{1}{2} \dot{\chi}^2 - 8\dot{\Omega}\dot{\Gamma} - 12\dot{\Gamma}^2 \right]_b = q_b^2 + \frac{1}{2} s_b^2 - 8\alpha_b w_b - 12w_b^2 \\ &= W_b^4 \left[-8\tilde{u}_b (1 - \mathcal{L}_b - 3\tilde{u}_b) - 12\tilde{u}_b^2 + (\mathcal{L}_{b,\varphi})^2 + \frac{1}{2} (\mathcal{L}_{b,\chi})^2 \right] \end{aligned}$$

$$\tilde{u}_b = \frac{1}{3} \left[(1 - \mathcal{L}_b) - \sqrt{(1 - \mathcal{L}_b)^2 - \frac{3}{4} (\mathcal{L}_{b,\varphi})^2 - \frac{3}{8} (\mathcal{L}_{b,\chi})^2} \right] \simeq \frac{1}{8} (\mathcal{L}_{b,\varphi})^2 + \frac{1}{16} (\mathcal{L}_{b,\chi})^2 + \dots$$

$$w_b \left(\frac{a_b}{r_b} \right) \simeq \frac{1}{8} \left(\frac{\partial\mathcal{L}_b}{\partial\varphi} \right)_{r=\epsilon}^2 + \frac{1}{16} \left(\frac{\partial\mathcal{L}_b}{\partial\chi} \right)_{r=\epsilon}^2 + \dots$$

$$\frac{a_b}{r_b} \simeq 1 - (\mathcal{L}_b)_{r=\epsilon}$$

$$\delta_b=2\pi\left(1-\frac{a_b}{r_b}\right)\simeq\kappa^2L_b$$

$$\Omega_0=\log\left(\frac{a_b}{\ell}\right)+\alpha\log\left(\frac{\ell}{r_b}\right)=\log\left(\frac{a_b}{r_b}\right)-(1-\alpha)\log\left(\frac{\ell}{r_b}\right)$$

$$3\zeta H^2\int_0^{r_e}\mathrm{d}r aW^2=W_e^4\left(w_e+\frac{1}{2}q_e\right)\left(\frac{a_e}{r_e}\right)-W_0^4\left(w_0+\frac{1}{2}q_0\right)\left(\frac{a_0}{r_0}\right)=0$$

$$S=\frac{A}{4G}+C\log\lambda+\cdots, (\lambda\rightarrow+\infty)$$

$$C=C_{\mathrm{local}}+C_{\mathrm{global}}$$

$$C_{\mathrm{local}}\equiv\int\,d^dx\sqrt{g}a_4(x)$$

$$a_4(x)=-a_{\rm E}E_4+cW^2+b_1R^2+b_2RF_{\mu\nu}F^{\mu\nu}$$

Multiplet	$a_{\rm E}$	c	b_1	b_2
Free scalar ($m\ell=-2$)	$\frac{1}{360}$	$\frac{1}{120}$	$\frac{1}{288}(\Delta(\Delta-3)+2)^2$	0
Free Dirac spinor ($m=0$)	$-\frac{11}{360}$	$-\frac{1}{20}$	$\frac{1}{72}\left(\Delta-\frac{3}{2}\right)^2\left(\left(\Delta-\frac{3}{2}\right)^2-2\right)$	0
Free vector ($m=0$)	$\frac{31}{180}$	$\frac{1}{10}$	0	0
Free gravitino ($m=0$)	$-\frac{229}{720}$	$-\frac{77}{120}$	$-\frac{1}{9}$	0
Einstein-Maxwell	$\frac{53}{45}$	$\frac{137}{60}$	$-\frac{13}{36}$	0
$\mathcal{N}=2$ gravitini	$-\frac{589}{360}$	$-\frac{137}{60}$	0	$\frac{13}{18}$
$\mathcal{N}=2$ gravity multiplet	$-\frac{11}{24}$	0	$-\frac{13}{36}$	$\frac{13}{18}$

$$(4\pi)^2a_4(x)=\frac{11}{24}E_4-\frac{13}{36}R^2+\frac{13}{18}RF_{\mu\nu}F^{\mu\nu}.$$

$$C_{\rm local}=\frac{11}{6}-\frac{26}{3}\frac{a(\ell^2-4\ell-a^2)}{(\ell-a)(a^2+6a\ell+\ell^2)},$$

$$Z(\beta,\mu_\alpha)=\int\,D\Psi e^{-S_E(\Psi)}$$

$$S=\log Z+\beta M+\sum_\alpha \mu^\alpha q_\alpha$$

$$S=\frac{\text{Area}(q_\alpha)}{4G}+\cdots$$

$$Z(\beta,\mu_\alpha)\sim \frac{1}{\sqrt{\det Q}}e^{-S_E^{\rm classical}}$$

$$\mathcal{Q} = \frac{\delta^2 S_E}{\delta \Psi^2}$$

$$\delta S = -\frac{1}{2}\log \det \mathcal{Q}$$

$$\kappa_n=\lambda^2\kappa_n^{(0)}$$

$$S=\frac{A}{4G}+\sum_{i=1}^k C_i\log\ell_i+\cdots$$

$$C=\sum_{i=1}^k C_i$$

$$Z_{1-\text{loop}}(\beta,\mu_\alpha)=Z_{\text{nz}}Z_{\text{zm}}e^{-S_E^{\text{class.}}}$$

$$S=\frac{A}{4G}+(C_{\text{local}}+C_{\text{global}})\log\lambda+\cdots$$

$$\log Z_{\text{nz}}=-\frac{1}{2}\sum_n'\log\kappa_n$$

$$K(x,s)=\sum_n e^{-\kappa_n s} f_n^\ell(x) f_n^{\ell'}(x) G_{\ell\ell'}$$

$$\int_{\mathcal{M}} d^Dx \sqrt{g} K(x,s) = \sum_n e^{-s\kappa_n} = \sum_n' e^{-s\kappa_n} + N_{\text{zm}}$$

$$\log \kappa - \log \kappa^{(0)} = -\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{ds}{s} \Big(e^{-s\kappa} - e^{-s\kappa^{(0)}} \Big)$$

$$\log Z_{\text{nz}} - \log Z_{\text{nz}}^{(0)} = \frac{1}{2} \int_{\epsilon}^{\epsilon \lambda} \frac{ds}{s} \Big(\int_{\mathcal{M}} d^Dx \sqrt{g} K(x,s) - N_{\text{zm}} \Big)$$

$$K(x,s)=\sum_{n\geqslant 0} s^{n-D/2}a_{2n}(x)$$

$$K(x,s)=s^{-2}a_0(x)+s^{-1}a_2(x)+s^0a_4(x)+\mathcal{O}(s)$$

$$\log Z_{\text{nz}} = C_{\text{local}} \log \lambda + \cdots,$$

$$C_{\text{local}}\equiv\int d^4x\sqrt{g}a_4(x)$$

$$\mathcal{Q}_m^n=(\Box)I_m^n+2(\omega^\mu D_\mu)_m^n+P_m^n,$$

$$\mathcal{Q}_m^n=(\mathcal{D}^\mu \mathcal{D}_\mu)I_m^n+E_m^n,$$

$$E\equiv P-\omega^\mu\omega_\mu-(D^\mu\omega_\mu)$$

$$(4\pi)^2a_4(x)=\mathrm{Tr}\left[\frac{1}{2}E^2+\frac{1}{6}RE+\frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\right.\\ \left.+\frac{1}{360}I(5R^2+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu})\right]$$

$$\Omega_{\mu\nu} = [D_\mu + \omega_\mu, D_\nu + \omega_\nu]$$

$$\mathcal{L}=\bar{\psi}\mathcal{D}\psi \text{ where } \mathcal{D}=\mathbb{D}+L$$

$$\log \det \mathcal{D} = \frac{1}{2} \log \det \mathcal{D}^\dagger \mathcal{D}$$

$$\omega^\mu=\frac{1}{2}(\gamma^\mu L-L^\dagger\gamma^\mu), P=\mathcal{R}+(\mathbb{D}L)-L^\dagger L$$

$$\mathcal{R}=-\frac{1}{4}g_{\mu\nu}+\frac{1}{2}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma}$$

$$C_{\mathrm{global}}=C_{\mathrm{ens}}+C_{\mathrm{zm}}$$

$$\lambda^{\beta_\Psi}$$

$$C_{\mathrm{zm}}=\sum_{\Psi}(\beta_{\Psi}-1)n_{\Psi}^0$$

$$\beta_A=\frac{D}{2}-1, \beta_\psi=D-1, \beta_g=\frac{D}{2}$$

$$(4\pi)^2a_4(x)=-a_{\rm E}E_4+cW^2+b_1R^2+b_2RF_{\mu\nu}F^{\mu\nu}$$

$$E_4\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$$

$$W^2\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2$$

$$\chi=\frac{1}{32\pi^2}\int\,d^4x\sqrt{g}E_4,$$

$$ds^2=f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega^2, f(r)=1+\frac{r^2}{\ell^2}-\frac{2m}{r}$$

$$t\sim t+\beta, \beta=\frac{4\pi r_+}{1+\frac{3r_+^2}{\ell^2}},$$

$$E_4=\frac{24}{\ell^4}+\frac{48m^2}{r^6}, W^2=\frac{48m^2}{r^6}, R^2=\frac{144}{\ell^4}, RF_{\mu\nu}F^{\mu\nu}=0$$

$$a_4^{\rm CT}=\int_{\partial M}d^3y\sqrt{h}(c_1+c_2\mathcal{R})$$

$$\begin{aligned}\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} E_4 &= 4, & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} W^2 &= \frac{4(\ell^2 + r_+^2)^2}{\ell^2(\ell^2 + 3r_+^2)} \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R^2 &= \frac{24r_+^2(\ell^2 - r_+^2)}{\ell^2(\ell^2 + 3r_+^2)}, & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R F_{\mu\nu} F^{\mu\nu} &= 0\end{aligned}$$

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} E_4 = 2$$

$$C_{\text{local}} = \frac{4}{\ell^2(\ell^2 + 3r_+^2)} ((c - a_E)\ell^4 + (2c - 3a_E + 6b_1)\ell^2 r_+^2 + (c - 6b_1)r_+^4)$$

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + r^2 d\Omega^2$$

$$E_4 = \frac{24}{\ell^4}, W^2 = 0, R^2 = \frac{144}{\ell^4}, F_{\mu\nu} F^{\mu\nu} = 0$$

$$\begin{aligned}\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} E_4 &= 0, & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} W^2 &= 0 \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R^2 &= 0, & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R F_{\mu\nu} F^{\mu\nu} &= 0\end{aligned}$$

$$C_{\text{local}} = 0,$$

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, A = \frac{iq_e}{r} dt - q_m \cos \theta d\phi$$

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2m}{r} + \frac{q_e^2 + q_m^2}{r^2},$$

$$T_H = \beta^{-1} = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left(1 + \frac{3r_+^2}{\ell^2} - \frac{(q_e^2 + q_m^2)}{r_+^2} \right).$$

$$\begin{aligned}R^2 &= \frac{144}{\ell^4}, & E_4 &= \frac{24}{\ell^4} + \frac{8(6m^2r^2 - 12m(q_e^2 + q_m^2)r + 5(q_e^2 + q_m^2)^2)}{r^8} \\ F_{\mu\nu} F^{\mu\nu} &= -\frac{2(q_e^2 - q_m^2)}{r^4}, & W^2 &= \frac{48(mr - (q_e^2 + q_m^2))^2}{r^8}\end{aligned}$$

$$\begin{aligned}\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} E_4 &= 4 \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} W^2 &= \frac{2}{5} \left(2 - \frac{14r_+^2}{\ell^2} + \frac{16\pi r_+}{\beta} + \frac{(\ell^4 + \ell^2 r_+^2 + 4r_+^4)}{\pi \ell^4 r_+} \beta \right) \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R^2 &= \frac{12r_+(r_+^2 + \ell^2)}{\pi \ell^4} \beta - \frac{24r_+^2}{\ell^2} \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R F_{\mu\nu} F^{\mu\nu} &= \frac{6(3r_+^4 + \ell^2 r_+^2 - 2\ell^2 q_m^2)}{\pi \ell^4 r_+} \beta - \frac{24r_+^2}{\ell^2}\end{aligned}$$

$$C_{\text{local}} = \frac{2}{5} \left(2(c - 5a_E) - \frac{2r_+^2}{\ell^2} (7c + 30(b_1 + b_2)) + \frac{16\pi r_+}{\beta} c \right. \\ \left. + \frac{\beta}{\pi \ell^4 r_+} (c \ell^4 + (c + 30b_1 + 15b_2) \ell^2 r_+^2 + (4c + 30b_1 + 45b_2) r_+^4 - 30b_2 \ell^2 q_m^2) \right)$$

$$r_+ = r_0 + \frac{2\pi \ell_2^2}{\beta} + O(\beta^{-2})$$

$$r_0^2 = \frac{1}{6} \ell \left(\sqrt{\ell^2 + 12q^2} - \ell \right), \ell_2^2 = \frac{r_0^2}{1 + \frac{6r_0^2}{\ell^2}} = \frac{\ell^2}{6} \left(1 - \frac{\ell}{\sqrt{\ell^2 + 12q^2}} \right)$$

$$\int d^4x \sqrt{g} a_4(x) = C_1 \beta + C_0 + O(\beta^{-1})$$

$$\lim_{\beta \rightarrow +\infty} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} E_4 = 4 \\ \lim_{\beta \rightarrow +\infty} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} W^2 = -\frac{2(r_0^2 - \ell_2^2)^2}{3r_0^2 \ell_2^2} \\ \lim_{\beta \rightarrow +\infty} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R^2 = -\frac{2(r_0^2 - \ell_2^2)^2}{r_0^2 \ell_2^2} \\ \lim_{\beta \rightarrow +\infty} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R F_{\mu\nu} F^{\mu\nu} = -\frac{(r_0^2 - \ell_2^2)(r_0^4 + r_0^2 \ell_2^2 - 4q_m^2 \ell_2^2)}{r_0^4 \ell_2^2}$$

$$\lim_{\beta \rightarrow +\infty} C_{\text{local}} = -4a_E - \frac{r_0^2 - \ell_2^2}{r_0^2 \ell_2^2} \left(\left(\frac{2}{3}c + 2b_1 \right) (r_0^2 - \ell_2^2) + b_2 \left(r_0^2 + \ell_2^2 - \frac{4\ell_2^2 q_m^2}{r_0^2} \right) \right)$$

$$r \rightarrow r_0 + \epsilon \tilde{r}, t \rightarrow \ell_2^2 \frac{\tilde{t}}{\epsilon}$$

$$ds^2 = \ell_2^2 \left(\tilde{r}^2 d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} \right) + r_0^2 d\Omega_2^2, A = -\frac{i\ell_2^2 q_e}{r_0^2} \tilde{r} d\tilde{t} + q_m \cos \theta d\phi$$

$$\frac{6}{\ell^2} = \frac{1}{\ell_2^2} - \frac{1}{r_0^2}, q_e^2 + q_m^2 = \frac{r_0^2(r_0^2 + \ell_2^2)}{2\ell_2^2}$$

$$C_{\text{local}} = -4a_E - \frac{r_0^2 - \ell_2^2}{r_0^2 \ell_2^2} \left(\left(\frac{2}{3}c + 2b_1 \right) (r_0^2 - \ell_2^2) + b_2 \left(r_0^2 + \ell_2^2 - \frac{4\ell_2^2 q_m^2}{r_0^2} \right) \right)$$

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2$$

$$\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{\ell^2} \right) - 2mr + q_e^2 + q_m^2, \Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$



$$\Xi = 1 - \frac{a^2}{\ell^2}$$

$$A = -\frac{q_e r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right) - \frac{q_m \cos \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)$$

$$M = \frac{m}{\Xi^2}, J = \frac{am}{\Xi^2}, Q_e = \frac{q_e}{\Xi}, Q_m = \frac{q_m}{\Xi}$$

$$\beta = \frac{4\pi(r_+^2 + a^2)}{r_+ \left(1 + \frac{a^2}{\ell^2} + 3\frac{r_+^2}{\ell^2} - \frac{a^2 + q_\ell^2 + q_m^2}{r_+^2} \right)}$$

$$C_{\text{local}} = -4a_E + (6A_1 + cW_1)\beta + (24A_2 + cW_2) + \frac{cW_3}{\beta}$$

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} E_4 = 2$$

$$A_1 = \frac{(2b_1 + b_2)(a^2 + \ell^2)r_+^3 + (2b_1 + 3b_2)r_+^5 + ((2b_1 - b_2)a^2 - 2b_2q_m^2)r_+\ell^2}{\pi\ell^2(\ell^2 - a^2)(a^2 + r_+^2)}$$

$$A_2 = -\frac{b_1a^2 + (b_1 + b_2)r_+^2}{\ell^2 - a^2}$$

$$W_1 = \frac{1}{16\pi a^5 r_+^4 \ell^2 (\ell^2 - a^2) (a^2 + r_+^2)} [3ar_+ (a^8 (\ell^2 - r_+^2)^2 + r_+^8 (\ell^2 + 3r_+^2)^2) - 4a^3 r_+^3 (r_+^4 (\ell^4 - 9r_+^4) + a^4 (\ell^4 + 12\ell^2 r_+^2 + 3r_+^4) + 2a^5 r_+^5 (\ell^4 - 14\ell^2 r_+^2 + 5r_+^4) - 3(a^2 + r_+^2)(a^2 (\ell^2 - r_+^2) - r_+^2 (\ell^2 + 3r_+^2))^2 (r_+^4 - a^4) \arctan(a/r_+)]$$

$$W_2 = \frac{a^2 + r_+^2}{2a^5 r_+^3 (\ell^2 - a^2)} [4a^3 \ell^2 r_+^3 + 3ar_+ (a^4 (\ell^2 - r_+^2) - r_+^4 (\ell^2 + 3r_+^2)) - 3(a^2 (\ell^2 - r_+^2) - r_+^2 (\ell^2 + 3r_+^2)) (r_+^4 - a^4) \arctan(a/r_+)]$$

$$W_3 = \frac{\pi \ell^2 (a^2 + r_+^2)}{a^5 r_+^2 (\ell^2 - a^2)} [ar_+ (3a^4 + 2a^2 r_+^2 + 3r_+^4) - 3(r_+^2 + a^2) (r_+^4 - a^4) \arctan(a/r_+)]$$

$$r_+ = r_0 + \frac{2\pi\ell_2^2}{\beta} + O(\beta^{-2})$$

$$C_{\text{local}} = -4a_E + \frac{1}{2ar_0^5(\ell^2 - a^2)(a^2 + r_0^2)(a^2 + \ell^2 + 6r_0^2)} [-3a^7r_0(16b_1r_0^4 + c(\ell^2 - r_0^2)^2) + a^5r_0^3(c\ell^4 + 2(11c - 12b_2)\ell^2r_0^2 - 3(13c - 8b_2 + 80b_1)r_0^4) + a^3r_0^5(15c\ell^4 + 2(25c + 24b_2)\ell^2r_0^2 - (49c + 336b_1 - 48b_2)r_0^4 - 48b_2\ell^2q_m^2) + 3ar_0^7(c\ell^4 + 2(3c - 4b_2)\ell^2r_0^2 - (7c + 48b_1 + 24b_2)r_0^4 + 16b_2\ell^2q_m^2) - 3c(a^2 + r_0^2)(a^2(r_0^2 - \ell^2) + r_0^2(\ell^2 + 3r_0^2))^2 \arctan(a/r_0)].$$

$$r \rightarrow r_0 + \epsilon \tilde{r}, t \rightarrow \ell_2^2 \frac{\tilde{t}}{\epsilon}, \phi \rightarrow \phi - \frac{ia\ell_2^2(\ell^2 - a^2)}{\ell^2(a^2 + r_0^2)} \frac{t}{\epsilon}$$



$$d\tilde{s}^2 = \frac{\ell_2^2(r_0^2 + a^2 \cos^2 \theta)}{a^2 + r_0^2} \left(\tilde{r}^2 d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} \right) + \frac{\ell^2(r_0^2 + a^2 \cos^2 \theta)}{\ell^2 - a^2 \cos^2 \theta} d\theta^2 \\ + \frac{\ell^2(a^2 + r_0^2)^2(\ell^2 - a^2 \cos^2 \theta) \sin^2 \theta}{(\ell^2 - a^2)^2(r_0^2 + a^2 \cos^2 \theta)} \left(d\phi - \frac{2\ell_2^2 a r_0 (\ell^2 - a^2)}{\ell^2(a^2 + r_0^2)^2} i r dt \right)^2$$

$$\ell_2 = \ell \sqrt{\frac{a^2 + r_0^2}{a^2 + \ell^2 + 6r_0^2}}$$

$$\tilde{A} = \frac{1}{r_0^2 + a^2 \cos^2 \theta} \left[-\frac{i\ell_2^2}{a^2 + r_0^2} (q_e(r_0^2 - a^2 \cos^2 \theta) + 2q_m a r_0 \cos \theta) \tilde{r} d\tilde{t} \right. \\ \left. + \frac{\ell^2}{\ell^2 - a^2} (q_e a r_0 \sin^2 \theta + q_m (a^2 + r_0^2) \cos \theta) d\tilde{\phi} \right]$$

$$r_0 = \sqrt{a\ell}, q_m = 0$$

$$M = \frac{\sqrt{a\ell}}{\left(1 - \frac{a}{\ell}\right)^2}, Q_e = \frac{\sqrt{a\ell}}{1 - \frac{a}{\ell}}, Q_m = 0, J = \frac{a\sqrt{a\ell}}{\left(1 - \frac{a}{\ell}\right)^2}$$

$$M=Q_e+\frac{J}{\ell}$$

$$C_{\rm local} = -4a_{\rm E} + \frac{3\ell_2^2}{2a\ell^2(\ell^2 - a^2)} \left[(9c - 8b_2)a\ell^2 - (9c + 48b_1 - 8b_2)a^2\ell - (c + 16b_1)a^3 \right. \\ \left. + c\ell^3 - \frac{c(a + \ell)^4}{\sqrt{a\ell}} \arctan(\sqrt{a/\ell}) \right]$$

$$\ell_2 = \ell \sqrt{\frac{a(a + \ell)}{a^2 + 6a\ell + \ell^2}} \text{ (BPS case)}$$

$$S_{\rm EE} = -{\rm Tr} \rho_A \log \rho_A$$

$$\rho_A = {\rm Tr}_{\bar{A}}|0\rangle\langle 0|$$

$$S_{\rm EE} = \frac{\rm Area}{4G} + (C_{\rm local} + C_{\rm zm}) \log \ell + \cdots$$

$$ds^2 = \left(\frac{\rho^2}{\ell^2} - 1\right) dt^2 + \frac{d\rho^2}{\frac{\rho^2}{\ell^2} - 1} + \rho^2 ds_{H_2}^2, ds_{H_2}^2 = du^2 + \sinh^2 u d\phi^2$$

$$\beta=2\pi\ell$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}E_4=2\qquad\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}W^2=0\\ \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2=12,\qquad\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}RF_{\mu\nu}F^{\mu\nu}=0$$

$$C_{\rm local} = -2a_{\rm E} + 12b_1$$



$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} E_4 = 1$$

$$C_{\text{zm}} = 0 \text{ (full geometry)}$$

$$C_{\text{zm}} = -3 + 8\delta_{\text{BPS}} \text{ (near horizon geometry)},$$

$$C_{\text{ens}} = -K$$

Background spacetime	C_{zm}	C_{ens}	C_{global}
Schwarzschild	0	-3	-3
Reissner-Nordström	0	-3	-3
Kerr	0	-1	-1
Kerr-Newman	0	-1	-1
BPS Kerr-Newman	0	-1	-1
Reissner-Nordström near horizon	-3	-3	-6
Kerr-Newman near horizon	-3	-1	-4
BPS Kerr-Newman near horizon	5	-1	4
Thermal AdS ₄	0	-3	-3
AdS ₄ -Rindler	0	-3	-3

$$S = -\frac{1}{2} \int d^4x \sqrt{g} ((\partial\phi)^2 + m^2\phi^2)$$

$$P=E=m^2, \Omega=0,$$

$$\Delta=\frac{1}{2}\Big(3+\sqrt{9+4m^2\ell^2}\Big).$$

$$(4\pi)^2 a_4(x) = -\frac{1}{360} E_4 + \frac{1}{120} W^2 + \frac{1}{288} (\Delta(\Delta-3)-2)^2 R^2.$$



$$C_{\mathrm{local}}=-\frac{1}{90}-\frac{r_0^2}{20\ell^4}(24+5(\Delta+1)\Delta(\Delta-3)(\Delta-4))$$

$$S=\int\,d^{d+1}x\sqrt{g}\bar{\psi}(\gamma^\mu\nabla_\mu-m)\psi$$

$$\Delta=\frac{3}{2}+m\ell$$

$$(4\pi)^2a_4(x)=\frac{11}{360}E_4+\frac{1}{20}W^2+\frac{1}{72}\Big(\Delta-\frac{3}{2}\Big)^2\left(\Big(\Delta-\frac{3}{2}\Big)^2-2\right)R^2$$

$$\mathcal{L}=-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$$

$$\mathcal{L}_{\mathrm{g.f.}}=\frac{1}{2}(\nabla_\mu a^\mu)^2,$$

$$\mathcal{L}+\mathcal{L}_{\mathrm{g.f.}}=a^\mu\Box a_\mu-a^\mu R_{\mu\nu}a^\nu$$

$$(4\pi)^2a_4(x)=-\frac{31}{180}E_4+\frac{1}{10}W^2$$

$$\mathcal{L}_{3/2}=-\bar{\psi}_\mu\gamma^{\mu\rho\nu}\nabla_\rho\psi_\nu$$

$$\mathcal{L}_{\mathrm{g.f.}}=-\frac{1}{2}(\bar{\psi}_\mu\gamma^\mu)\gamma^\rho\nabla_\rho(\gamma^\nu\psi_\nu)$$

$$\mathcal{L}_{3/2}+\mathcal{L}_{\mathrm{g.f.}}=\bar{\chi}_\mu\gamma^\nu D_\nu\chi^\mu$$

$$(4\pi)^2a_4(x)=\frac{229}{720}E_4-\frac{77}{120}W^2-\frac{1}{9}R^2$$

$$a_{\rm E}=\frac{1}{720}(2n_S+22n_F+124n_V-229n_\psi)$$

$$c\,=\frac{1}{120}(n_S+6n_F+12n_V-77n_\psi)$$

$$b_1\,=\frac{1}{72}(n_S-8n_\psi)$$

$$C_{\mathrm{local}}=\frac{1}{180\ell^2(\ell^2+3r_+^2)}[\ell^4(4n_S+14n_F-52n_V-233n_\psi)\\+3r_+^2\ell^2(22n_S+2n_F-76n_V-239n_\psi)+18r_+^4(-3n_S+2n_F+4n_V+n_\psi)]$$

$$\lim_{\ell\rightarrow+\infty}C_{\mathrm{local}}=\frac{1}{180}(4n_S+14n_F-52n_V-233n_\psi)$$

$$S_{\mathrm{EE}}=\frac{\mathrm{Area}}{4G}+C\log\beta+\cdots$$

$$C=\frac{29}{180}+\frac{1}{24}(\Delta+1)\Delta(\Delta-3)(\Delta-4)$$



$$S = \int d^4x \sqrt{g} (R - 2\Lambda - F_{\mu\nu} F^{\mu\nu})$$

$$\delta g_{\mu\nu} = \sqrt{2} h_{\mu\nu}, \delta A_\mu = \frac{1}{2} a_\mu$$

$$S = - \int d^4x \sqrt{\det g} \left\{ \left(D^\mu h_{\mu\rho} - \frac{1}{2} D_\rho h \right) \left(D^\nu h^\rho_\nu - \frac{1}{2} D^\rho h \right) + \frac{1}{2} (D^\mu a_\mu) (D^\nu a_\nu) \right\}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 2 F_{\mu\rho} F^\rho_\nu - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$D^\mu F_{\mu\nu} = 0$$

$$D_{[\mu} F_{\nu\rho]} = 0$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$\phi_m I^{mn} \phi_n = h_{\mu\nu} \left(\frac{1}{2} g^{\mu\alpha} g^{\nu\beta} + \frac{1}{2} g^{\mu\beta} g^{\nu\alpha} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \right) h_{\alpha\beta} + a_\alpha g^{\alpha\beta} a_\beta$$

$$\phi_m E^{mn} \phi_n = h_{\mu\nu} (R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu} R^{\alpha\beta} - g^{\alpha\beta} R^{\mu\nu} + \Lambda g^{\mu\nu} g^{\alpha\beta}) h_{\alpha\beta}$$

$$+ a_\alpha \left(\frac{3}{2} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \Lambda g^{\alpha\beta} \right) a_\beta + \frac{\sqrt{2}}{2} h_{\mu\nu} (D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}) a_\alpha$$

$$+ \frac{\sqrt{2}}{2} a_\alpha (D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}) h_{\mu\nu}$$

$$\phi_m (\Omega^{\rho\sigma})^{mn} \phi_n = h_{\mu\nu} \left\{ \frac{1}{2} (g^{\nu\beta} R^{\mu\alpha\rho\sigma} + g^{\nu\alpha} R^{\mu\beta\rho\sigma} + g^{\mu\beta} R^{\nu\alpha\rho\sigma} + g^{\mu\alpha} R^{\nu\beta\rho\sigma}) \right.$$

$$+ [\omega^\rho, \omega^\sigma]^{\mu\nu\alpha\beta} \} h_{\alpha\beta} + a_\alpha \{ R^{\alpha\beta\rho\sigma} + [\omega^\rho, \omega^\sigma]^{\alpha\beta} \} a_\beta$$

$$+ h_{\mu\nu} (D^{[\rho} \omega^{\sigma]})^{\mu\nu\alpha} a_\alpha + a_\alpha (D^{[\rho} \omega^{\sigma]})^{\alpha\mu\nu} h_{\mu\nu}$$

$$\phi_m (\omega^\rho)^{mn} \phi_n = \frac{\sqrt{2}}{2} h_{\mu\nu} (g^{\alpha\mu} F^{\rho\nu} + g^{\alpha\nu} F^{\rho\mu} - g^{\mu\rho} F^{\alpha\nu} - g^{\nu\rho} F^{\alpha\mu} - g^{\mu\nu} F^{\rho\alpha}) a_\alpha$$

$$- \frac{\sqrt{2}}{2} a_\alpha (g^{\alpha\mu} F^{\rho\nu} + g^{\alpha\nu} F^{\rho\mu} - g^{\mu\rho} F^{\alpha\nu} - g^{\nu\rho} F^{\alpha\mu} - g^{\mu\nu} F^{\rho\alpha}) h_{\mu\nu}$$

$$(4\pi)^2 a_4^{\text{EM}}(x) = -\frac{277}{180} E_4 + \frac{38}{15} W^2 + \frac{7}{18} R^2.$$

$$\mathcal{S}_{\text{ghost},b} = \frac{1}{2} \int d^4x \sqrt{g} \{ 2b_\mu (g^{\mu\nu} \square + R^{\mu\nu}) c_\nu + 2b \square c - 4b F^{\rho\nu} D_\rho c_\nu \}$$

$$\phi_n E_m^n \phi^m = b_\mu (R^\mu_\nu) b^\nu + c_\mu (R^\mu_\nu) c^\nu$$

$$\phi_n (\Omega_{\alpha\beta})^n_m \phi^m = b_\mu (R^\mu_{\nu\alpha\beta}) b^\nu + c_\mu (R^\mu_{\nu\alpha\beta}) c^\nu - \frac{1}{2} (b_\mu - i c_\mu) (D^\mu F_{\alpha\beta}) (b + i c)$$

$$+ \frac{1}{2} (b + i c) (D_\nu F_{\alpha\beta}) (b^\nu - i c^\nu)$$

$$a_4^{\text{ghosts}}(x) = \frac{13}{36} E_4 - \frac{1}{4} W^2 - \frac{3}{4} R^2,$$

$$(4\pi)^2 a_4^{\text{B}}(x) = -\frac{53}{45} E_4 + \frac{137}{60} W^2 - \frac{13}{36} R^2.$$



$$a_{\mathrm{E}}=\frac{53}{45}, c=\frac{137}{60}, b_1=-\frac{13}{36}, b_2=0.$$

$$C_{\mathrm{local}}=-\frac{212}{45}+\frac{\ell_2^2}{120a\ell^2(\ell^2-a^2)}(629a^3-579a^2\ell+3699a\ell^2\\ +411\ell^3-411\frac{(a+\ell)^4}{\sqrt{a\ell}}\arctan(\sqrt{a/\ell}))$$

$$\mathcal{L}_f=\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho+\frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma-\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$$

$$D_\mu\psi_\nu=\nabla_\mu\psi_\nu-\frac{i}{\ell}A_\mu\psi_\nu$$

$$\mathcal{L}_{\rm g.f.}=-\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)\gamma^\nu D_\nu(\gamma^\rho\psi_\rho)$$

$$\psi_\mu=\sqrt{2}\left(\chi_\mu-\frac{1}{2}\gamma_\mu\gamma^\nu\chi_\nu\right)$$

$$\mathcal{L}_f=g^{\mu\nu}\bar{\chi}_\mu\gamma^\rho D_\rho\chi_\nu+\frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma-\frac{1}{\ell}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu$$

$$\chi^\mu=\chi_1^\mu+i\chi_2^\mu,$$

$$D_\mu\chi^\nu_A=\left(\delta_{AB}\nabla^\mu+\frac{1}{\ell}\varepsilon_{AB}A_\mu\right)\chi^\nu_B,$$

$$\mathcal{L}_f=\delta_{AB}g_{\mu\nu}\bar{\chi}^\mu_A\gamma^\rho D_\rho\chi^\nu_B-\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}^\rho_A\gamma^\mu\gamma_{\rho\sigma}\gamma^\nu\chi^\sigma_B-\frac{1}{\ell}\delta_{AB}\bar{\chi}^\mu_A\gamma_{\mu\nu}\chi^\nu_B$$

$$\chi_1^\mu=\frac{1}{2}(\lambda_1^\mu+\gamma^5\lambda_2^\mu)$$

$$\chi_2^\mu=\frac{i}{2}(\lambda_1^\mu-\gamma^5\lambda_2^\mu)$$

$$\mathcal{L}_f=\mathcal{L}_1+\mathcal{L}_2,$$

$$\mathcal{L}_1=g_{\mu\nu}\bar{\lambda}_1^\mu\gamma^\rho(\nabla_\rho+i\ell^{-1}A_\rho)\lambda_1^\nu-\frac{i}{2}F^{\mu\nu}\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma-\frac{1}{\ell}\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu$$

$$\mathcal{L}_2=g_{\mu\nu}\bar{\lambda}_2^\mu\gamma^\rho(\nabla_\rho-i\ell^{-1}A_\rho)\lambda_2^\nu-\frac{i}{2}F^{\mu\nu}\bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma+\frac{1}{\ell}\bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu$$

$$(4\pi)^2a_4^{\rm F,\,gravitini}(x)=\frac{139}{90}E_4-\frac{32}{15}W^2-\frac{2}{9}R^2+\frac{8}{9}RF_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}_{bc}=\delta_{AB}\bar{b}_A\gamma_\mu\delta_c\psi_B^\mu$$

$$\mathcal{L}_{bc}=\delta_{AB}\bar{b}_A\left(\gamma^\mu D_\mu+\frac{2}{\ell}\right)c_B$$

$$(4\pi)^2a_4^{\rm F,\,ghosts}(x)=\frac{11}{120}E_4-\frac{3}{20}W^2+\frac{2}{9}R^2-\frac{1}{6}RF_{\mu\nu}F^{\mu\nu},$$



$$(4\pi)^2 a_4^{\text{F}}(x) = \frac{589}{360} E_4 - \frac{137}{60} W^2 + \frac{13}{18} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{11}{24} E_4 - \frac{13}{36} R^2 + \frac{13}{18} R F_{\mu\nu} F^{\mu\nu}$$

$$C_{\text{local}} = \frac{11}{6} + \frac{26[r_+\beta(r_+^4 - \ell^2(a^2 + q_m^2)) - \pi\ell^2(r_+^4 - a^4)]}{3\pi\ell^2(\ell^2 - a^2)(a^2 + r_+^2)}.$$

$$C_{\text{local}} = \frac{11}{6} + \frac{26\ell_2^2[(a^2(a+\ell) + r_0^2(3a-\ell))(a^2(a-\ell) + r_0^2(3a+\ell)) + 2\ell^2q_m^2(r_0^2 - a^2)]}{3\ell^2(\ell^2 - a^2)(a^2 + r_0^2)^2},$$

$$\ell_2 = \ell \sqrt{\frac{a^2 + r_0^2}{a^2 + \ell^2 + 6r_0^2}}$$

$$C_{\text{local}} = \frac{11}{6} - \frac{26}{3} \frac{a(\ell^2 - 4a\ell - a^2)}{(\ell - a)(a^2 + 6a\ell + \ell^2)}$$

$$\frac{1}{(4\pi)^2} \int d^2x \sqrt{g} W^2 = \frac{3\ell_2^2}{2a\ell^2(\ell^2 - a^2)} \left((\ell - a)(\ell^2 + 10a\ell + a^2) - \frac{(a + \ell)^4}{\sqrt{a\ell}} \arctan(\sqrt{a/\ell}) \right)$$

$$(4\pi)^2 a_4(x) = \text{Tr} \left[\frac{1}{2} E^2 + \frac{1}{6} RE + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{360} (5R^2 + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu}) \right],$$

$$S = \int d^4x \sqrt{g} (R - 2\Lambda - F_{\mu\nu} F^{\mu\nu})$$

$$\delta g_{\mu\nu} = \sqrt{2} h_{\mu\nu}, \delta A_\mu = \frac{1}{2} a_\mu$$

$$S = - \int d^4x \sqrt{\det g} \left\{ \left(D^\mu h_{\mu\rho} - \frac{1}{2} D_\rho h \right) \left(D^\nu h^\rho_\nu - \frac{1}{2} D^\rho h \right) + \frac{1}{2} (D^\mu a_\mu) (D^\nu a_\nu) \right\}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 2 F_{\mu\rho} F_{\nu}{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\ D^\mu F_{\mu\nu} = 0$$

$$D_{[\mu} F_{\nu\rho]} = 0 \\ R_{\mu[\nu\rho\sigma]} = 0$$

$$\begin{aligned} \phi_m Q^{mn} \phi_n = & G^{\mu\nu\alpha\beta} h_{\mu\nu} \square h_{\alpha\beta} + g^{\alpha\beta} a_\alpha \square a_\beta - a_\alpha R^{\alpha\beta} a_\beta \\ & + h_{\mu\nu} \left\{ R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - \frac{1}{2} (g^{\mu\alpha} R^{\nu\beta} + g^{\mu\beta} R^{\nu\alpha} \right. \\ & + g^{\nu\alpha} R^{\mu\beta} + g^{\nu\beta} R^{\mu\alpha}) - 2(F^{\mu\alpha} F^{\nu\beta} + F^{\mu\beta} F^{\nu\alpha}) \\ & + \frac{1}{2} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) (F_{\theta\varphi} F^{\theta\varphi}) + 2G^{\alpha\beta\mu\nu} \Lambda \Big\} h_{\alpha\beta} \\ & - h_{\mu\nu} \left\{ \frac{1}{4} (D_\rho K^\rho)^{\mu\nu\alpha} - \frac{1}{2} (K^\rho)^{\mu\nu\alpha} D_\rho \right\} a_\alpha \\ & - a_\alpha \left\{ \frac{1}{4} (D_\rho K^\rho)^{\mu\nu\alpha} + \frac{1}{2} (K^\rho)^{\mu\nu\alpha} D_\rho \right\} h_{\mu\nu} \end{aligned}$$

$$(K^\rho)^{\mu\nu\alpha} = 2\sqrt{2}(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha})$$

$$G^{\mu\nu\alpha\beta} = \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})$$

$$h_{\mu\nu}(2g^{\mu\alpha}g^{\nu\beta}\Lambda - g^{\mu\nu}g^{\alpha\beta}\Lambda)h_{\alpha\beta} = h_{\mu\nu}(2G^{\alpha\beta\mu\nu}\Lambda)h_{\alpha\beta}$$

$$\phi_m I^{mn} \phi_n = h_{\mu\nu} G^{\mu\nu\alpha\beta} h_{\alpha\beta} + a_\alpha g^{\alpha\beta} a_\beta$$

$$\phi_m (\omega^\rho)^{mn} \phi_n = \frac{1}{4} h_{\mu\nu} (K^\rho)^{\mu\nu\alpha} a_\alpha - \frac{1}{4} a_\alpha (K^\rho)^{\mu\nu\alpha} h_{\mu\nu}$$

$$\begin{aligned} \phi_m P^{mn} \phi_n &= h_{\mu\nu} \left\{ R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - \frac{1}{2} (g^{\mu\alpha} R^{\nu\beta} + g^{\mu\beta} R^{\nu\alpha} \right. \\ &\quad \left. + g^{\nu\alpha} R^{\mu\beta} + g^{\nu\beta} R^{\mu\alpha}) - 2(F^{\mu\alpha} F^{\nu\beta} + F^{\mu\beta} F^{\nu\alpha}) \right. \\ &\quad \left. + \frac{1}{2} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) (F_{\theta\phi} F^{\theta\phi}) \right. \\ &\quad \left. + 2g^{\mu\alpha} g^{\nu\beta} \Lambda - g^{\mu\nu} g^{\alpha\beta} \Lambda \right\} h_{\alpha\beta} \\ &\quad - a_\alpha R^{\alpha\beta} a_\beta + \frac{\sqrt{2}}{2} h_{\mu\nu} \{ D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu} \} a_\alpha \\ &\quad + \frac{\sqrt{2}}{2} a_\alpha \{ D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu} \} h_{\mu\nu} \end{aligned}$$

$$\phi_m \Omega_{\mu\nu}^{mn} \phi_n = \phi_m [D_\mu + \omega_\mu, D_\nu + \omega_\nu] \phi_n = \phi_m \left\{ [D_\mu, D_\nu]^{mn} + D_{[\mu} \omega_{\nu]}^{mn} + [\omega_\mu, \omega_\nu]^{mn} \right\} \phi_n$$

$$[D_\mu \cdot D_\nu], (D^\rho \omega_\rho)^{mn} \text{ and } (\omega^\rho)^{mp} (\omega_\rho)_p{}^n$$

$$\begin{aligned} \phi_m [D^\rho, D^\sigma]^{mn} \phi_n &= h_{\mu\nu} [D^\rho, D^\sigma] h^{\mu\nu} + a_\alpha [D^\rho, D^\sigma] a^\alpha \\ &= h_{\mu\nu} \{ g^{\nu\beta} R^{\mu\alpha\rho\sigma} + g^{\mu\beta} R^{\nu\alpha\rho\sigma} \} h_{\alpha\beta} + a_\alpha R^{\alpha\beta\rho\sigma} a_\beta \\ &= \frac{1}{2} h_{\mu\nu} \{ g^{\nu\beta} R^{\mu\alpha\rho\sigma} + g^{\nu\alpha} R^{\mu\beta\rho\sigma} + g^{\mu\beta} R^{\nu\alpha\rho\sigma} + g^{\mu\alpha} R^{\nu\beta\rho\sigma} \} h_{\alpha\beta} \\ &\quad + a_\alpha R^{\alpha\beta\rho\sigma} a_\beta \end{aligned}$$

$$\phi_m (D^\rho \omega_\rho)^{mn} \phi_n = -\frac{\sqrt{2}}{2} h_{\mu\nu} (D^\nu F^{\alpha\mu} + D^\mu F^{\alpha\nu}) a_\alpha + \frac{\sqrt{2}}{2} a_\alpha (D^\nu F^{\alpha\mu} + D^\mu F^{\alpha\nu}) h_{\mu\nu}$$

$$\begin{aligned} \phi_m (\omega^\rho)^{mp} (\omega_\rho)_p{}^n \phi_n &= \frac{1}{16} h_{\mu\nu} (K^\rho)^{\mu\nu\alpha} (-K_\rho)^{\delta\gamma\beta} g_{\alpha\beta} h_{\delta\gamma} \\ &\quad + \frac{1}{16} a_\alpha (K^\rho)^{\mu\nu\alpha} (-K_\rho)^{\delta\gamma\beta} G_{\mu\nu\delta\gamma} a_\beta \end{aligned}$$

$$\begin{aligned} \phi_m (\omega^\rho)^{mp} (\omega_\rho)_p{}^n \phi_n &= h_{\mu\nu} (-2F^{\mu\delta} F^{\nu\gamma} - 2F^{\mu\gamma} F^{\nu\delta} + 2F^{\gamma\rho} F^{\delta}{}_\rho g^{\mu\nu} - F^{\nu\rho} F^{\delta}{}_\rho g^{\mu\gamma} \\ &\quad - F^{\nu\rho} F^{\gamma}{}_\rho g^{\mu\delta} - F^{\mu\rho} F^{\delta}{}_\rho g^{\nu\gamma} - F^{\mu\rho} F^{\gamma}{}_\rho g^{\nu\delta} + 2F^{\mu\rho} F^{\nu}{}_\rho g^{\gamma\delta} \\ &\quad - \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} g^{\gamma\delta}) h_{\gamma\delta} + a_\alpha (-2F^{\alpha\gamma} F^{\beta}{}_\gamma - F_{\theta\phi} F^{\theta\phi} g^{\alpha\beta}) a_\beta \end{aligned}$$



$$\begin{aligned}\phi_m(\omega^\rho)^{mp}(\omega_\rho)_p^n\phi_n=&h_{\mu\nu}\left(-2F^{\mu\alpha}F^{\nu\beta}-2F^{\mu\beta}F^{\nu\alpha}-\frac{1}{2}F_{ad}F^{ad}g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}F_{ad}F^{ad}g^{\mu\beta}g^{\nu\alpha}\right.\\&+\frac{1}{2}F_{ad}F^{ad}g^{\mu\nu}g^{\beta\alpha}+g^{\beta\alpha}R^{\mu\nu}-\frac{1}{2}g^{\nu\alpha}R^{\mu\beta}-\frac{1}{2}g^{\nu\beta}R^{\mu\alpha}-\frac{1}{2}g^{\mu\alpha}R^{\nu\beta}\\&-\frac{1}{2}g^{\mu\beta}R^{\nu\alpha}+g^{\mu\nu}R^{\beta\alpha}+\Lambda g^{\mu\alpha}g^{\nu\beta}+\Lambda g^{\mu\beta}g^{\nu\alpha}-2\Lambda g^{\mu\nu}g^{\beta\alpha})h_{\alpha\beta}\\&+\frac{1}{2}a_\alpha\left(-\frac{3}{2}F_{\theta\phi}F^{\theta\phi}g^{\alpha\beta}-R^{\alpha\beta}+\Lambda g^{\alpha\beta}\right)a_\beta\end{aligned}$$

$$\begin{aligned}\phi_mE^{mn}\phi_n=&h_{\mu\nu}(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta})h_{\alpha\beta}\\&+a_\alpha\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)a_\beta+\frac{\sqrt{2}}{2}h_{\mu\nu}(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})a_\alpha\\&+\frac{\sqrt{2}}{2}a_\alpha(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})h_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\phi_m(\Omega^{\rho\sigma})^{mn}\phi_n=&h_{\mu\nu}\left\{\frac{1}{2}(g^{\nu\beta}R^{\mu\alpha\rho\sigma}+g^{\nu\alpha}R^{\mu\beta\rho\sigma}+g^{\mu\beta}R^{\nu\alpha\rho\sigma}+g^{\mu\alpha}R^{\nu\beta\rho\sigma})\right.\\&+[\omega^\rho, \omega^\sigma]^{\mu\nu\alpha\beta}\}h_{\alpha\beta}+a_\alpha\{R^{\alpha\beta\rho\sigma}+[\omega^\rho, \omega^\sigma]^{\alpha\beta}\}a_\beta\\&+h_{\mu\nu}(D^{[\rho}\omega^{\sigma]})^{\mu\nu\alpha}a_\alpha+a_\alpha(D^{[\rho}\omega^{\sigma]})^{\alpha\mu\nu}h_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\text{Tr}(RE) =&\text{Tr}R[R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}]\\&+\text{Tr}\left(\frac{3}{2}g^{\alpha\beta}RF_{\mu\nu}F^{\mu\nu}-R\Lambda g^{\alpha\beta}\right)\\=&R(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda)G_{\mu\nu\alpha\beta}\\&+R\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)g_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}\text{Tr}(RE) =&R\left(-2R+(D-2)R+\left(D-\frac{D^2}{2}\right)\Lambda\right)+R\left(\frac{3D}{2}F_{\mu\nu}F^{\mu\nu}-d\Lambda\right)\\=&-32\Lambda^2+6RF_{\mu\nu}F^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\text{Tr}(E^2) =&\text{Tr}(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda)\\&\times(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda)\\&+\text{Tr}\left(\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)\right)\\&+\frac{1}{2}\text{Tr}\left((D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho})\right)\\&+\frac{1}{2}\text{Tr}\left((D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho})\right)\end{aligned}$$

$$\begin{aligned}&(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda)\\&\times(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda)G_{\mu\nu\rho\sigma}G_{\alpha\beta\tau\delta}\\&=16\Lambda^2-4R_{ab}R^{ab}+3R_{abcd}R^{abcd}\end{aligned}$$

$$\text{Tr}\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)=4\Lambda^2-12\Lambda F_{\alpha\beta}F^{\alpha\beta}+9(F_{\theta\phi}F^{\theta\phi})^2,$$



$$(D_\rho F_{\mu\nu})(D^\rho F^{\mu\nu}) = R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + \frac{1}{2} R^2 - \Lambda R - \frac{1}{2} R F_{\rho\sigma} F^{\rho\sigma},$$

$$(D_\mu F_\rho{}^\nu)(D_\nu F^{\rho\mu}) = \frac{1}{2} (D_\rho F_{\mu\nu})(D^\rho F^{\mu\nu}).$$

$$X \equiv \frac{1}{2} \text{Tr} \left((D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}) \right)$$

$$+ \frac{1}{2} \text{Tr} \left((D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}) \right)$$

$$= (D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu})(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}) g_{\alpha\beta} G_{\mu\nu\rho\sigma}$$

$$X = 2(D_\rho F_{\alpha\sigma})(D^\rho F^{\alpha\sigma}) + 2(D_\rho F_{\alpha\sigma})(D^\sigma F^{\alpha\rho}).$$

$$X = 2 \left(-R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} R^2 - \Lambda R - \frac{1}{2} R F_{\rho\sigma} F^{\rho\sigma} + R_{\mu\nu\rho\alpha} F^{\mu\nu} F^{\rho\alpha} \right)$$

$$+ \left(R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + \frac{1}{2} R^2 - \Lambda R - \frac{1}{2} R F_{\rho\sigma} F^{\rho\sigma} + R_{\mu\nu\rho\alpha} F^{\mu\nu} F^{\rho\alpha} \right)$$

$$= 3 \left(-R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} R^2 - \Lambda R - \frac{1}{2} R F_{\rho\sigma} F^{\rho\sigma} + R_{\mu\nu\rho\alpha} F^{\mu\nu} F^{\rho\alpha} \right)$$

$$\text{Tr} E^2 = (16\Lambda^2 - 4R_{ab} R^{ab} + 3R_{abcd} R^{abcd}) + (4\Lambda^2 - 12\Lambda F_{\alpha\beta} F^{\alpha\beta} + 9(F_{\theta\phi} F^{\theta\phi})^2)$$

$$+ 3 \left(-R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} R^2 - \Lambda R - \frac{1}{2} R F_{\rho\sigma} F^{\rho\sigma} + R_{\mu\nu\rho\alpha} F^{\mu\nu} F^{\rho\alpha} \right)$$

$$= 32\Lambda^2 - 7R_{\mu\nu} R^{\mu\nu} + 3R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma} F^{\rho\sigma} + 3R_{\mu\nu\rho\alpha} F^{\mu\nu} F^{\rho\alpha} + 9(F_{\theta\phi} F^{\theta\phi})^2.$$

$$\text{Tr}(I) = 14,$$

$$\text{Tr}(RE) = -32\Lambda^2 + 6R F_{\mu\nu} F^{\mu\nu}$$

$$\text{Tr}(E^2) = 32\Lambda^2 - 7R_{\mu\nu} R^{\mu\nu} + 3R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma} F^{\rho\sigma}$$

$$+ 3R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + 9(F_{\theta\phi} F^{\theta\phi})^2$$

$$\text{Tr}(\Omega^2) = -224\Lambda^2 + 60\Lambda F_{\mu\nu} F^{\mu\nu} - 54F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + 56R_{\mu\nu} R^{\mu\nu}$$

$$- 18F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma} - 7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$(4\pi)^2 a_4^{\text{EM}}(x) = -\frac{880}{180} \Lambda^2 + \frac{196}{180} R_{\mu\nu} R^{\mu\nu} + \frac{179}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$\mathcal{S}_{\text{ghost},b} = \frac{1}{2} \int d^4x \sqrt{g} \{ 2b_\mu (g^{\mu\nu} \square + R^{\mu\nu}) c_\nu + 2b \square c - 4b F^{\rho\nu} D_\rho c_\nu \}$$

$$\phi_n E_m^n \phi^m = b_\mu (R_\nu^\mu) b^\nu + c_\mu (R_\nu^\mu) c^\nu$$

$$\phi_n (\Omega_{\alpha\beta})_m^n \phi^m = b_\mu (R_{\nu\alpha\beta}^\mu) b^\nu + c_\mu (R_{\nu\alpha\beta}^\mu) c^\nu - \frac{1}{2} (b_\mu - ic_\mu) (D^\mu F_{\alpha\beta}) (b + ic)$$

$$+ \frac{1}{2} (b + ic) (D_\nu F_{\alpha\beta}) (b^\nu - ic^\nu)$$

$$a_4^{\text{ghosts, EM}}(x) = \frac{13}{36} E_4 - \frac{1}{4} W^2 - \frac{3}{4} R^2$$

$$(4\pi)^2 a_4^{\text{B}}(x) = -\frac{53}{45} E_4 + \frac{137}{60} W^2 - \frac{13}{36} R^2$$



$$a_E = \frac{53}{45}, c = \frac{137}{60}, b_1 = -\frac{13}{36}, b_2 = 0$$

$$\phi_m E^{mn} \phi_n = h_{\mu\nu} (R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu} R^{\alpha\beta} - g^{\alpha\beta} R^{\mu\nu} + \Lambda g^{\mu\nu} g^{\alpha\beta}) h_{\alpha\beta}$$

$$\mathrm{Tr}(RE)=-16\Lambda^2$$

$$\mathrm{Tr}(E^2)=16\Lambda^2-4R_{\mu\nu}R^{\mu\nu}+3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$\mathrm{Tr}(I)=10$$

$$\begin{aligned} 180(4\pi)^2 a_4^{\text{bulk},\Lambda} &= \frac{1}{2} \left[60(-16\Lambda^2) + 180(16\Lambda^2 - 4R_{\mu\nu}R^{\mu\nu} + 3R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}) + 30(-6)R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \right. \\ &\quad \left. + 10(5R^2 + 2R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu}) \right] \\ &= -120\Lambda^2 + 190R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \end{aligned}$$

$$\phi_n(\Omega_{\alpha\beta})_m^n \phi^m = b_\mu \left(R_{\nu\alpha\beta}^\mu\right) b^\nu + c_\mu \left(R_{\nu\alpha\beta}^\mu\right) c^\nu$$

$$\mathrm{Tr}(I) = \mathrm{Tr}(g^{\mu\nu}) + \mathrm{Tr}(g^{\mu\nu}) = 8$$

$$\begin{aligned} -(4\pi)^2 a_4^{\text{ghost},\text{B}}(x) &= \frac{1}{3}R^2 + R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \left(\frac{1}{9}R^2 + \frac{2}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{2}{45}R_{\mu\nu}R^{\mu\nu}\right) \\ &= \frac{4}{9}R^2 + \frac{43}{45}R_{\mu\nu}R^{\mu\nu} - \frac{43}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &= \frac{11}{90}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{164}{15}\Lambda^2 \end{aligned}$$

$$180(4\pi)^2 a_4^\Lambda(x) = 212 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2088 \Lambda^2$$

$$\mathcal{L}=\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho+\frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma-\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$$

$$\psi_\mu=\psi_\mu^1+i\psi_\mu^2$$

$$\gamma^0=\left(\begin{smallmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{smallmatrix}\right), \gamma^1=\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{smallmatrix}\right), \gamma^2=\left(\begin{smallmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{smallmatrix}\right), \gamma^3=\left(\begin{smallmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{smallmatrix}\right)$$

$$\psi^*=\psi$$

$$\mathcal{L}_{\text{g.f.}}=-\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)\gamma^\nu D_\nu(\gamma^\rho\psi_\rho)$$

$$\psi_\mu=\sqrt{2}\left(\chi_\mu-\frac{1}{2}\gamma_\mu\gamma^\nu\chi_\nu\right)$$

$$\mathcal{L}_{\text{Fermi}}+\mathcal{L}_{\text{g.f.}}=\mathcal{L}_{\text{kin}}+\mathcal{L}_{\bar{\psi}F\psi}+\mathcal{L}_{\bar{\psi}\psi}$$

$$\mathcal{L}_{\text{kin}}=\frac{1}{2}g^{\mu\nu}\bar{\psi}_\mu\gamma^\rho\nabla_\rho\psi_\nu$$

$$\mathcal{L}_{\bar{\psi}F\psi}=\frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma$$

$$\mathcal{L}_{\bar{\psi}\psi}=-\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$$

$$\bar{\chi}_\mu \gamma^\nu D_\nu \chi^\mu = \frac{1}{2} \bar{\psi}_\mu \left(\gamma^{\mu\nu\rho} D_\nu - \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho D_\nu \right) \psi_\rho$$

$$\begin{aligned} \mathcal{L}_{\bar{\psi}\psi} &= -\frac{1}{2\ell} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \\ &= -\frac{1}{\ell} \left(\bar{\chi}_\mu - \frac{1}{2} \bar{\chi}_\rho \gamma^\rho \gamma_\mu \right) \gamma^{\mu\nu} \left(\chi_\nu - \frac{1}{2} \gamma_\nu \gamma^\sigma \chi_\sigma \right) \\ &= -\frac{1}{\ell} \left(\bar{\chi}_\mu \gamma^{\mu\nu} \chi_\nu - \frac{1}{2} \bar{\chi}_\rho \gamma^\rho \gamma_\mu \gamma^{\mu\nu} \chi_\nu - \frac{1}{2} \bar{\chi}_\mu \gamma^{\mu\nu} \gamma_\nu \gamma^\sigma \chi_\sigma + \frac{1}{4} \bar{\chi}_\rho \gamma^\rho \gamma_\mu \gamma^{\mu\nu} \gamma_\nu \gamma^\sigma \chi_\sigma \right) \\ &= -\frac{1}{\ell} \left(\bar{\chi}_\mu \gamma^{\mu\nu} \chi_\nu - 3 \bar{\chi}_\mu \gamma^\mu \gamma^\nu \chi_\nu + 3 \bar{\chi}_\rho \gamma^\rho \gamma^\sigma \chi_\sigma \right) \\ &= -\frac{1}{\ell} \bar{\chi}_\mu \gamma^{\mu\nu} \chi_\nu \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{4} F^{\mu\nu} \bar{\psi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma \\ &= \frac{i}{2} F^{\mu\nu} \left(\bar{\chi}_\rho - \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \right) \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \left(\chi_\sigma - \frac{1}{2} \gamma_\sigma \gamma^\beta \chi_\beta \right) \\ &= \frac{i}{2} F^{\mu\nu} \left(\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma - \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma - \frac{1}{2} \bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \gamma_\sigma \gamma^\beta \chi_\beta \right. \\ &\quad \left. + \frac{1}{4} \bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \gamma_\sigma \gamma^\beta \chi_\beta \right) \end{aligned}$$

$$\begin{aligned} \gamma^\mu \gamma_\nu \gamma_\mu &= -2\gamma_\nu \\ \gamma_\rho \gamma_\mu \gamma^{\rho\sigma} &= \gamma_\rho \gamma_\mu (\gamma^\rho \gamma^\sigma - g^{\rho\sigma}) = -\gamma_\mu \gamma^\sigma - 2\delta_\mu^\sigma \\ \gamma_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \gamma_\sigma &= (-\gamma_\mu \gamma^\sigma - 2\delta_\mu^\sigma) \gamma_\nu \gamma_\sigma = 4\gamma_{\mu\nu} \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma &= \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha (\gamma_\mu \gamma^\sigma + 2\delta_\mu^\sigma) \gamma_\nu \chi_\sigma \\ &= \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma^\sigma \gamma_\nu \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu \\ &= \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu (2\delta_\nu^\sigma - \gamma_\nu \gamma^\sigma) \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu \\ &= \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \chi_\nu - \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma_\nu \gamma^\sigma \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_\nu \chi_\mu \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{2} F^{\mu\nu} (\bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma - \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \gamma_\nu \gamma^\sigma \chi_\sigma + \bar{\chi}_\alpha \gamma^\alpha \gamma_{\mu\nu} \gamma^\beta \chi_\beta) \\ &= \frac{i}{2} F^{\mu\nu} \bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma \end{aligned}$$

$$\mathcal{L}_f = g^{\mu\nu} \bar{\chi}_\mu \gamma^\rho D_\rho \chi_\nu + \frac{i}{2} F^{\mu\nu} \bar{\chi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma - \frac{1}{\ell} \bar{\chi}_\mu \gamma^{\mu\nu} \chi_\nu.$$

$$\chi_\mu = \chi_\mu^1 + i\chi_\mu^2,$$

$$\bar{\lambda} \gamma_{\mu_1 \dots \mu_r} \chi = t_r \bar{\chi} \gamma_{\mu_1 \dots \mu_r} \lambda,$$

$$t_r = \begin{cases} -1 & r = 1, 2 \bmod 4 \\ 1 & r = 3, 4 \bmod 4 \end{cases}.$$

$$\bar{\lambda} \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_p} \chi = (-1)^p \bar{\chi} \gamma^{\mu_p} \dots \gamma^{\mu_2} \gamma^{\mu_1} \lambda.$$



$$\begin{aligned}\bar{\chi}_\mu^1 \gamma^\rho D_\rho \chi_\nu^2 &= -D_\rho \bar{\chi}_\nu^2 \gamma^\rho \chi_\mu^1 \\ &= \bar{\chi}_\nu^2 \gamma^\rho D_\rho \chi_\mu^1\end{aligned}$$

$$\mathcal{L}_{\text{kin}} = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B$$

$$\begin{aligned}F^{\mu\nu} \bar{\psi}_\rho^1 \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma^1 &= F^{\mu\nu} (\gamma_\mu \psi_\rho^1)^\dagger \gamma^{\rho\sigma} (\gamma_\nu \psi_\sigma^1) \\ &= -F^{\mu\nu} (\gamma_\nu \psi_\sigma^1)^\dagger \gamma^{\rho\sigma} (\gamma_\mu \psi_\rho^1) \\ &= -F^{\mu\nu} \bar{\psi}_\rho^1 \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma^1 \\ &= 0\end{aligned}$$

$$\mathcal{L}_{\bar{\psi}F\psi} = -\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_\rho^A\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma^B$$

$$\begin{aligned}\bar{\chi}_\mu^1 \gamma^{\mu\nu} \chi_\nu^2 &= -\bar{\chi}_\nu^2 \gamma^{\mu\nu} \chi_\mu^1 \\ &= \bar{\chi}_\mu^2 \gamma^{\mu\nu} \chi_\nu^1\end{aligned}$$

$$\mathcal{L}_{\bar{\psi}\psi} = -\frac{1}{\ell}\delta_{AB}\bar{\chi}_\mu^A\gamma^{\mu\nu}\chi_\nu^B$$

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B - \frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_\rho^A \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma^B - \frac{1}{\ell} \delta_{AB} \bar{\chi}_\mu^A \gamma^{\mu\nu} \chi_\nu^B$$

$$(\gamma^\mu)^\dagger = (-1)^t A \gamma^\mu A^{-1}$$

$$(\gamma^\mu)^t = t_0 t_1 C \gamma^\mu C^{-1}$$

$$(\gamma^\mu)^* = -t_0 t_1 B \gamma^\mu B^{-1}$$

$$B=(CA^{-1})^t$$

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$(\gamma^5)^2=1, (\gamma^5)^\dagger=\gamma^5, (\gamma^5)^t=(\gamma^5)^*=-\gamma^5$$

$$\psi^*=B_+\psi=\psi$$

$$B_-(C_-A^{-1})^t=\gamma^5$$

$$(\lambda_A^\mu)^*=B_- \varepsilon_{AB} \lambda_B^\mu$$

$$\begin{aligned}\lambda_1^\mu &= \chi_1^\mu - i \chi_\mu^2 \\ \lambda_2^\mu &= \gamma^5 (\chi_1^\mu + i \chi_2^\mu)\end{aligned}$$

$$\begin{aligned}\chi_1^\mu &= \frac{1}{2}(\lambda_1^\mu + \gamma^5 \lambda_2^\mu) \\ \chi_2^\mu &= \frac{i}{2}(\lambda_1^\mu - \gamma^5 \lambda_2^\mu)\end{aligned}$$

$$\bar{\chi}_1^\mu = \frac{1}{2}(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu \gamma^5)$$

$$\bar{\chi}_2^\mu = -\frac{i}{2}(\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu \gamma^5)$$

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\chi^F \chi} + \mathcal{L}_{\chi \chi}$$

$$\mathcal{L}_{\text{kin}} = \delta_{AB} g_{\mu\nu} \bar{\chi}_A^\mu \gamma^\rho D_\rho \chi_B^\nu$$

$$\mathcal{L}_{\chi^F \chi} = -\frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_A^\rho \gamma_\mu \gamma_\rho \sigma \gamma_\nu \chi_B^\sigma$$

$$\mathcal{L}_{\chi \chi} = -\frac{1}{\ell} \delta_{AB} \bar{\chi}_A^\mu \gamma_{\mu\nu} \chi_B^\nu$$

$$\begin{aligned} g_{\mu\nu} \bar{\chi}_1^\mu \gamma^\rho \nabla_\rho \chi_1^\nu &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu \gamma^5) \gamma^\rho \nabla_\rho (\lambda_1^\mu + \gamma^5 \lambda_2^\mu) \\ &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu) \\ &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu + \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu) \end{aligned}$$

$$\begin{aligned} g_{\mu\nu} \bar{\chi}_2^\mu \gamma^\rho \nabla_\rho \chi_2^\nu &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu \gamma^5) \gamma^\rho \nabla_\rho (\lambda_1^\mu - \gamma^5 \lambda_2^\mu) \\ &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu - \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu - \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu) \\ &= \frac{1}{4} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu - \bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \gamma^5 \lambda_2^\mu + \bar{\lambda}_2^\mu \gamma^5 \gamma^\rho \nabla_\rho \lambda_1^\mu) \end{aligned}$$

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho \nabla_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho \nabla_\rho \lambda_2^\mu)$$

$$D_\mu \chi_A^\nu = \left(\delta_{AB} \nabla_\mu + \frac{1}{\ell} \varepsilon_{AB} A_\mu \right) \chi_B^\nu,$$

$$D_\mu \chi_1^\nu = \nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu$$

$$D_\mu \chi_2^\nu = \nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu$$

$$\begin{aligned} D_\mu \lambda_1^\nu &= D_\mu (\chi_1^\nu - i \chi_2^\nu) \\ &= \nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu - i \left(\nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu \right) \\ &= \nabla_\mu \lambda_1^\nu + \frac{i}{\ell} A_\mu \lambda_1^\nu \end{aligned}$$

$$\begin{aligned} D_\mu \lambda_2^\nu &= \gamma^5 D_\mu (\chi_1^\nu + i \chi_2^\nu) \\ &= \gamma^5 \left(\nabla_\mu \chi_1^\nu + \frac{1}{\ell} A_\mu \chi_2^\nu \right) + i \gamma^5 \left(\nabla_\mu \chi_2^\nu - \frac{1}{\ell} A_\mu \chi_1^\nu \right) \\ &= \nabla_\mu \lambda_2^\nu - \frac{i}{\ell} A_\mu \lambda_2^\nu \end{aligned}$$

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} g_{\mu\nu} (\bar{\lambda}_1^\mu \gamma^\rho D_\rho \lambda_1^\mu + \bar{\lambda}_2^\mu \gamma^\rho D_\rho \lambda_2^\mu)$$



$$\mathcal{L}_{\chi\chi} = -\frac{1}{\ell}\delta_{AB}\bar{\chi}_A^\mu\gamma_{\mu\nu}\chi_B^\nu$$

$$\begin{aligned}\bar{\chi}_1^\mu\gamma_{\mu\nu}\chi_1^\nu &= \frac{1}{4}(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu\gamma^5)\gamma_{\mu\nu}(\lambda_1^\nu + \gamma^5\lambda_2^\nu) \\ &= \frac{1}{4}(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma^5\gamma_{\mu\nu}\gamma^5\lambda_2^\nu + \bar{\lambda}_1^\mu\gamma_{\mu\nu}\gamma^5\lambda_2^\nu - \bar{\lambda}_2^\mu\gamma^5\gamma_{\mu\nu}\lambda_1^\nu) \\ &= \frac{1}{4}(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu + \bar{\lambda}_1^\mu\gamma_{\mu\nu}\gamma^5\lambda_2^\nu - \bar{\lambda}_2^\mu\gamma^5\gamma_{\mu\nu}\lambda_1^\nu)\end{aligned}$$

$$\begin{aligned}\bar{\chi}_2^\mu\gamma_{\mu\nu}\chi_2^\nu &= \frac{1}{4}(\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu\gamma^5)\gamma_{\mu\nu}(\lambda_1^\nu - \gamma^5\lambda_2^\nu) \\ &= \frac{1}{4}(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu - \bar{\lambda}_1^\mu\gamma_{\mu\nu}\gamma^5\lambda_2^\nu + \bar{\lambda}_2^\mu\gamma^5\gamma_{\mu\nu}\lambda_1^\nu)\end{aligned}$$

$$\mathcal{L}_{\chi\chi} = -\frac{1}{2\ell}(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu)$$

$$\begin{aligned}F^{\mu\nu}\bar{\chi}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_2^\sigma \\ &= \frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho - \bar{\lambda}_2^\rho\gamma^5)\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu(\lambda_1^\sigma - \gamma^5\lambda_2^\sigma) \\ &= \frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma - \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma - \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma) \\ &= \frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma - \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma - \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma)\end{aligned}$$

$$\begin{aligned}F^{\mu\nu}\bar{\chi}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_1^\sigma \\ &= -\frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho + \bar{\lambda}_2^\rho\gamma^5)\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu(\lambda_1^\sigma + \gamma^5\lambda_2^\sigma) \\ &= -\frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma) \\ &= -\frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma + \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma)\end{aligned}$$

$$\mathcal{L}_{\chi F\chi} = -\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_B^\sigma = -\frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma)$$

$$\mathcal{L}_f = \frac{1}{2}\delta_{AB}g_{\mu\nu}\bar{\lambda}_A^\mu\gamma^\rho D_\rho\lambda_B^\nu - \frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma) - \frac{1}{2\ell}(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu)$$

$$\mathcal{L}_f = \mathcal{L}_1 + \mathcal{L}_2,$$

$$\begin{aligned}\mathcal{L}_1 &= g_{\mu\nu}\bar{\lambda}_1^\mu\gamma^\rho(\nabla_\rho + i\ell^{-1}A_\rho)\lambda_1^\nu - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma - \frac{1}{\ell}\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu \\ \mathcal{L}_2 &= g_{\mu\nu}\bar{\lambda}_2^\mu\gamma^\rho(\nabla_\rho - i\ell^{-1}A_\rho)\lambda_2^\nu - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma + \frac{1}{\ell}\bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu\end{aligned}$$

$$\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^4$$

$$(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^\mu$$

$$\bar{\lambda}_A^\mu = (\lambda_A^\mu)^\dagger$$

$$Z = \int D\bar{\psi}_\mu D\psi_\nu e^{-S[\bar{\psi}_\mu, \psi_\nu]}$$

$$1 = \int D\epsilon D\bar{\epsilon} \delta\left(\xi - \gamma^\mu \psi_\mu^{(\epsilon)}\right) \delta\left(\bar{\xi} - \gamma^\mu \bar{\psi}_\mu^{(\epsilon)}\right) \Delta_{\text{FP}}^{-1}$$

$$\Delta_{\text{FP}} = \det\left(\frac{\delta\left(\gamma^\mu \psi_\mu^{(\epsilon)}\right)}{\delta\epsilon}\right) \det\left(\frac{\delta\left(\gamma^\mu \bar{\psi}_\mu^{(\epsilon)}\right)}{\delta\bar{\epsilon}}\right)$$

$$\psi_\mu^{(\epsilon)} = \psi_\mu + \mathcal{D}_\mu \epsilon$$

$$1 = \frac{1}{\det \mathbb{D}} \int D\xi D\bar{\xi} \exp(-\bar{\xi} \mathbb{D} \xi)$$

$$\begin{aligned} Z &= \frac{\Delta_{\text{FP}}^{-1}}{\det \mathbb{D}} \int D\bar{\psi}_\mu D\psi_\nu D\xi D\bar{\xi} D\epsilon D\bar{\epsilon} \exp(-\bar{\xi} \mathbb{D} \xi) \delta\left(\xi - \gamma^\mu \psi_\mu^{(\epsilon)}\right) \delta\left(\bar{\xi} - \gamma^\mu \bar{\psi}_\mu^{(\epsilon)}\right) e^{-S[\bar{\psi}_\mu, \psi_\nu]} \\ &= \frac{\Delta_{\text{FP}}^{-1}}{\det \mathbb{D}} \int D\bar{\psi}_\mu D\psi_\nu D\epsilon D\bar{\epsilon} \exp(-\bar{\psi}_\nu \gamma^\nu \mathbb{D} \gamma^\mu \psi_\mu) e^{-S[\bar{\psi}_\mu, \psi_\nu]} \end{aligned}$$

$$\psi_\mu \rightarrow \psi_\mu - \mathcal{D}_\mu \epsilon$$

$$\Delta_{\text{FP}}^{-1} = \int D b D c \exp \left(- b \gamma^\mu \mathcal{D}_\mu c \right)$$

$$\frac{1}{\det \mathbb{D}} = \int D d D \bar{d} \exp \left(- \bar{d} \gamma^\mu \mathcal{D}_\mu d \right)$$

$$\mathcal{L}_{\text{ghost}} = \bar{b}_A \left(\gamma^\mu D_\mu + \frac{2}{\ell} \right) c_A + \bar{c}_A \left(\gamma^\mu D_\mu + \frac{2}{\ell} \right) b_A + \bar{e}_A \gamma^\mu D_\mu e_A$$

$$\begin{aligned} b_1 &= \frac{1}{2}(\beta_1 + \gamma^5 \beta_2), & b_2 &= \frac{i}{2}(\beta_1 - \gamma^5 \beta_2) \;, \\ c_1 &= \frac{1}{2}(\eta_1 + \gamma^5 \eta_2), & c_2 &= \frac{i}{2}(\eta_1 - \gamma^5 \eta_2) \;, \\ e_1 &= \frac{1}{2}(\epsilon_1 + \gamma^5 \epsilon_2), & e_2 &= \frac{i}{2}(\epsilon_1 - \gamma^5 \epsilon_2) \;. \end{aligned}$$

$$\begin{aligned} \bar{b}_1 \gamma^\mu D_\mu c_1 &= \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2 \gamma^5) \gamma^\mu D_\mu (\eta_1 + \gamma^5 \eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1 \gamma^\mu D_\mu \eta_1 - \bar{\beta}_2 \gamma^5 \gamma^\mu D_\mu \gamma^5 \eta_2 + \bar{\beta}_1 \gamma^\mu D_\mu \gamma^5 \eta_2 - \bar{\beta}_2 \gamma^5 \gamma^\mu D_\mu \eta_1) \\ &= \frac{1}{4}(\bar{\beta}_1 \gamma^\mu D_\mu \eta_1 + \bar{\beta}_2 \gamma^\mu D_\mu \eta_2 + \bar{\beta}_1 \gamma^\mu D_\mu \gamma^5 \eta_2 - \bar{\beta}_2 \gamma^5 \gamma^\mu D_\mu \eta_1) \end{aligned}$$

$$\begin{aligned}\bar{b}_2 \gamma^\mu D_\mu c_2 &= \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2 \gamma^5) \gamma^\mu D_\mu (\eta_1 - \gamma^5 \eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1 \gamma^\mu D_\mu \eta_1 + \bar{\beta}_2 \gamma^\mu D_\mu \eta_2 - \bar{\beta}_1 \gamma^\mu D_\mu \gamma^5 \eta_2 + \bar{\beta}_2 \gamma^5 \gamma^\mu D_\mu \eta_1)\end{aligned}$$

$$\bar{b}_A \gamma^\mu D_\mu c_A = \frac{1}{2}(\bar{\beta}_1 \gamma^\mu D_\mu \eta_1 + \bar{\beta}_2 \gamma^\mu D_\mu \eta_2)$$

$$\bar{c}_A \gamma^\mu D_\mu b_A = \frac{1}{2}(\bar{\eta}_1 \gamma^\mu D_\mu \beta_1 + \bar{\eta}_2 \gamma^\mu D_\mu \beta_2)$$

$$\bar{e}_A \gamma^\mu D_\mu e_A = \frac{1}{2}(\bar{\epsilon}_1 \gamma^\mu D_\mu \epsilon_1 + \bar{\epsilon}_2 \gamma^\mu D_\mu \epsilon_2)$$

$$\begin{aligned}\bar{b}_1 c_1 &= \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2 \gamma^5)(\eta_1 + \gamma^5 \eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1 \eta_1 - \bar{\beta}_2 \eta_2 + \bar{\beta}_1 \gamma^5 \eta_2 - \bar{\beta}_2 \gamma^5 \eta_1)\end{aligned}$$

$$\begin{aligned}\bar{b}_2 c_2 &= \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2 \gamma^5)(\eta_1 - \gamma^5 \eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1 \eta_1 - \bar{\beta}_2 \eta_2 - \bar{\beta}_1 \gamma^5 \eta_2 + \bar{\beta}_2 \gamma^5 \eta_1)\end{aligned}$$

$$\bar{b}_A c_A = \frac{1}{2}(\bar{\beta}_1 \eta_1 - \bar{\beta}_2 \eta_2),$$

$$\bar{c}_A b_A = \frac{1}{2}(\bar{\eta}_1 \beta_1 - \bar{\eta}_2 \beta_2).$$

$$\begin{aligned}\mathcal{L}_{\text{ghosts}} &= \bar{\beta}_1 \left(\gamma^\mu D_\mu + \frac{2}{\ell} \right) \eta_1 + \bar{\eta}_1 \left(\gamma^\mu D_\mu + \frac{2}{\ell} \right) \beta_1 \\ &\quad + \bar{\beta}_2 \left(\gamma^\mu D_\mu - \frac{2}{\ell} \right) \eta_2 + \bar{\eta}_2 \left(\gamma^\mu D_\mu - \frac{2}{\ell} \right) \beta_2 \\ &\quad + \bar{\epsilon}_1 \gamma^\mu D_\mu \epsilon_1 + \bar{\epsilon}_2 \gamma^\mu D_\mu \epsilon_2\end{aligned}$$

$$\bar{\beta}_1 = \eta_1^\dagger, \bar{\beta}_2 = \eta_2^\dagger, \bar{\epsilon}_1 = \epsilon_1^\dagger$$

$$(4\pi)^2 a_4(x) = \frac{139}{90} E_4 - \frac{32}{15} W^2 - \frac{2}{9} R^2 + \frac{8}{9} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{11}{360} E_4 - \frac{1}{20} W^2 - \frac{1}{18} R F_{\mu\nu} F^{\mu\nu},$$

$$(4\pi)^2 a_4(x) = \frac{11}{360} E_4 - \frac{1}{20} W^2 + \frac{1}{9} R^2 - \frac{1}{18} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{589}{360} E_4 - \frac{137}{60} W^2 + \frac{13}{18} R F_{\mu\nu} F^{\mu\nu},$$

$$\begin{aligned}\mathcal{L}_{\text{g.f.}} &= -\frac{1}{4}(\bar{\psi}_\mu \gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho \psi_\rho) \\ &= -\frac{1}{4}\delta_{AB}(\bar{\psi}_\mu^A \gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho \psi_\rho^B)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\text{new}} &= \frac{1}{4} m \bar{\psi}_\mu \gamma^\mu \gamma^\nu \psi_\nu \\ &= \frac{1}{2} m \left(\bar{\chi}_\mu - \frac{1}{2} \bar{\chi}_\alpha \gamma^\alpha \gamma_\mu \right) \gamma^\mu \gamma^\nu \left(\chi_\nu - \frac{1}{2} \gamma_\nu \gamma^\beta \chi_\beta \right) \\ &= \frac{1}{2} m \delta_{AB} \bar{\chi}_\mu^A \gamma^\mu \gamma^\nu \chi_\nu^B\end{aligned}$$

$$g_{\mu\nu} = \gamma^\mu \gamma^\nu - \gamma^{\mu\nu}$$

$$\mathcal{L}_f = \delta_{AB} g^{\mu\nu} \bar{\chi}_\mu^A \gamma^\rho D_\rho \chi_\nu^B - \frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}_\rho^A \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi_\sigma^B + \frac{1}{\ell} \delta_{AB} \bar{\chi}_\mu^A g^{\mu\nu} \chi_\nu^B$$

$$(4\pi)^2 a_4(x) = \frac{139}{90} E_4 - \frac{32}{15} W^2 - \frac{1}{3} R^2 + \frac{8}{9} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = -\frac{11}{120} E_4 - \frac{3}{20} W^2 + \frac{1}{3} R^2 - \frac{1}{6} R F_{\mu\nu} F^{\mu\nu}$$

$$(4\pi)^2 a_4(x) = \frac{589}{360} E_4 - \frac{137}{60} W^2 + \frac{13}{18} R F_{\mu\nu} F^{\mu\nu}$$

$$S=S_{\text{classical}}+S_{1-\text{ loop}}+\cdots$$

$$S_{1\text{-loop}}=-\frac{1}{2}\int_{\epsilon}^{+\infty}\frac{ds}{s}\int d^4x\sqrt{g}K(x,s), K(x,s)=s^{-2}a_0(x)+s^{-1}a_2(x)+\cdots$$

$$S_{1-\text{ loop}} = \int d^4x \sqrt{g} \left(-\frac{1}{4\epsilon^2} a_0(x) - \frac{1}{2\epsilon} a_2(x) + \cdots \right)$$

$$(4\pi)^2 a_2(x) = d_1 R + d_2 F_{\mu\nu} F^{\mu\nu}$$

$$S_{1\text{-loop}} = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left(-\frac{1}{4\epsilon^2} a_0 - \frac{1}{2\epsilon} (d_1 R + d_2 F_{\mu\nu} F^{\mu\nu}) + \cdots \right)$$

$$(4\pi)^2 a_0 = \text{Tr}1$$

$$(4\pi)^2 a_2 = \text{Tr}\left(E+\frac{1}{6}R\right)$$

$$\frac{1}{32\pi^2}\int_{\mathcal{M}}d^dx\sqrt{g}E_4+\frac{1}{32\pi^2}B=\chi$$

Multiplet	a_0	d_1	d_2
Free scalar	1	$\frac{1}{12}(2 - \Delta(\Delta - 3))$	0
Free Dirac fermion	-4	$-\frac{1}{12}(5 + 4\Delta(\Delta - 3))$	0
Free vector	2	$-\frac{2}{3}$	0
Free gravitino	-2	$-\frac{1}{2}$	0
Einstein-Maxwell	4	$-\frac{10}{3}$	6
$\mathcal{N} = 2$ gravitini	-4	7	-8
$\mathcal{N} = 2$ gravity multiplet	0	$\frac{11}{3}$	-2

$$\lim_{r_c \rightarrow +\infty} \left[\frac{1}{32\pi^2} \int_{\mathcal{M}} d^4x \sqrt{g} E_4 + \int_{\partial\mathcal{M}} d^3y \sqrt{h} (c_1 + c_2 \mathcal{R}) \right] = \chi.$$

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} E_4 + \frac{1}{32\pi^2} B$$

$$B \equiv -2 \int \epsilon_{abcd} \theta_b^a \mathcal{R}_d^c + \frac{4}{3} \int \epsilon_{abcd} \theta_b^a \theta_e^c \theta_d^e$$

$$\begin{aligned} \frac{1}{32\pi^2} \int d^4x \sqrt{g} E_4 &= \int_{r_+}^{r_c} r^2 dr \int d\Omega^2 \int_0^\beta d\tau \left(\frac{24}{\ell^4} + \frac{48m^2}{r^6} \right) \\ &= \frac{\beta}{\pi} \left[2m^2 \left(\frac{1}{r_+^3} - \frac{1}{r_c^3} \right) + \frac{1}{\ell^4} (r_c^3 - r_+^3) \right] \end{aligned}$$

$$\theta_{01} = \frac{m\ell^2 + r_c^3}{r_c^2\ell^2} d\tau$$

$$\theta_{12} = -\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}} d\theta$$

$$\theta_{12} = -\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}} \sin\theta d\phi$$

$$\mathcal{R}_b^a = \mathcal{R}^a{}_{bcd} e^c \wedge e^d = \mathcal{R}^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$$

$$\begin{aligned}\mathcal{R}_{2\tau\theta}^0 &= -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2} \sqrt{\frac{r_c^3 + r_c \ell^2 - 2m\ell^2}{r_c \ell^2}} \\ \mathcal{R}_{3\tau\phi}^0 &= -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2} \sqrt{\frac{r_c^3 + r_c \ell^2 - 2m\ell^2}{r_c \ell^2}} \sin \theta \\ \mathcal{R}_{3\theta\phi}^0 &= \frac{2m\ell^2 - r_c^3}{r_c \ell^2}\end{aligned}$$

$$\begin{aligned}B &= 8(4\pi\beta) \frac{(m\ell^2 + r_c^3)(2m\ell^2 - r_c^3)}{\ell^4 r_c^3} \\ &= 4\pi\beta \left[-16m^2 \frac{1}{r_c^3} + 8m \frac{1}{\ell^2} - \frac{8}{\ell^4} r_c^3 \right] \\ &\quad 4\pi\beta \left[16m^2 \frac{1}{r_+^3} + 8m \frac{1}{\ell^2} - \frac{8}{\ell^4} r_+^3 \right].\end{aligned}$$

$$\chi = \frac{\beta}{\pi} \left[2m^2 \frac{1}{r_+^3} + m \frac{1}{\ell^2} - \frac{1}{\ell^3} r_+^3 \right] = 2,$$

$$ds^2 = \left(1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}\right) dt^2 + r_c^2 d\theta^2 + r_c^2 \sin^2 \theta d\phi^2$$

$$\begin{aligned}a^{\text{CT}} &= \int d^3y \sqrt{h} (c_1 + c_2 \mathcal{R}) \\ &= \int_0^\beta dt \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \sqrt{1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}} r_c^2 (c_1 + c_2 \mathcal{R}) \\ &= 4\pi\beta (c_1 r_c^2 + 2c_2) \sqrt{\frac{r_c^2}{\ell^2} - \frac{2m}{r_c} + 1} \\ &= 4\pi\beta \left[\frac{c_1}{\ell} r_c^3 + \frac{(c_1 \ell^2 + 4c_2)}{2\ell} r_c - c_1 m \ell \right] + \mathcal{O}(r_c^{-1})\end{aligned}$$

$$c_1 = -\frac{8}{\ell^3}, c_2 = \frac{2}{\ell}$$

$$a_4(x) = \cdots + \frac{1}{30} (5 \square E + \square R)$$

$$C_{\text{local}} = \cdots + \frac{1}{(4\pi)^2} \frac{1}{30} \int d^3y \sqrt{h} n^\mu \nabla_\mu (5 \text{Tr} E + R \text{Tr} 1)$$

$$\text{Tr} E = \alpha_1 R + \alpha_2 F_{\mu\nu} F^{\mu\nu},$$

$$C_{\text{bdy}} = \int d^3y \sqrt{h} a_4^\partial(y),$$

$$a_4^\partial = B_1 \text{Tr} 1 + B_2 \text{Tr} E,$$

$$\begin{aligned}
B_1 &= \frac{1}{360} [24K_{.bb} + 20RK + 4R_{anan}K - 12R_{anbn}K_{ab} + 4R_{abcb}K_{ac} + 480S^2K + 480S^3 \\
&\quad + \frac{1}{21} [(280\Pi_+ + 40\Pi_-)K^3 + (168\Pi_+ - 264\Pi_-)K_{ab}K_{ab}K_{cc}] + 120S_{.aa} \\
&\quad + \frac{1}{21} (224\Pi_+ + 320\Pi_-)K_{ab}K_{bc}K_{ac} + 120SR + 144SK^2 + 48SK_{ab}K_{ab}] \\
B_2 &= \frac{1}{3}(K + 6S) \\
R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &= \frac{24}{\ell^4} + \frac{8}{4(r^2 + a^2\cos^2\theta)^6} \left[-24m^2(a^2\cos^2\theta + r^2)^3 + 192r^4((q_e^2 + q_m^2) - 2mr)^2 \right. \\
&\quad \left. - 192r^2((q_e^2 + q_m^2) - 3mr)((q_e^2 + q_m^2) - 2mr)(a^2\cos^2\theta + r^2) \right. \\
&\quad \left. + 4((q_e^2 + q_m^2) - 6mr)(7(q_e^2 + q_m^2) - 18mr)(a^2\cos^2\theta + r^2)^2 \right] \\
&= \frac{24}{\ell^4} + \tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} \\
R_{\mu\nu}R^{\mu\nu} &= \frac{36}{\ell^4} + \frac{4(q_e^2 + q_m^2)^2}{(r^2 + a^2\cos^2\theta)^4} \\
&= \frac{36}{\ell^4} + \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} \\
R^2 &= \frac{144}{\ell^4}, \\
F_{\mu\nu}F^{\mu\nu} &= -\frac{2(q_e^2 - q_m^2)(r^4 - 6a^2r^2\cos^2\theta + a^4\cos^4\theta) + 16q_eq_mracos\theta(r^2 - a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^4}. \\
E_4 &= \frac{24}{\ell^4} + \frac{8}{(r^2 + a^2\cos^2\theta)^6} (6m^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta) \\
&\quad - 12mr(q_e^2 + q_m^2)(r^4 - 10a^2r^2\cos^2\theta + 5a^4\cos^4\theta) \\
&\quad + (q_e^2 + q_m^2)^2(5r^4 - 38a^2r^2\cos^2\theta + 5a^4\cos^4\theta))
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
&= - \frac{3(a^2 - r_+^2)(a^2 + r_+^2)^2(a^2(r_+^2 - \ell^2) + r_+^2(\ell^2 + 3r_+^2)) \arctan\left(\frac{a}{r_+}\right)}{a^5 \ell^2 \Xi r_+^3} \\
&+ \frac{(a^2(\ell^2 - r_+^2) - r_+^2(\ell^2 + 3r_+^2))^2}{\beta(3a^5 r_+ + 2a^3 r_+^3 + 3(a^2 - r_+^2)(a^2 + r_+^2)^2 \arctan\left(\frac{a}{r_+}\right) + 3ar_+^5)} \\
&\times \frac{8\pi a^5 \ell^4 \Xi r_+^4 (a^2 + r_+^2)}{2\pi(a^2 + r_+^2)(3a^5 r_+ + 2a^3 r_+^3 + 3(a^2 - r_+^2)(a^2 + r_+^2)^2 \arctan\left(\frac{a}{r_+}\right) + 3ar_+^5)} \\
&+ \frac{a^5 \beta \Xi r_+^2}{ar_+ \left(a^6(3\ell^2 - 7r_+^2) + a^4(3\ell^2 r_+^2 - 11r_+^4) + a^2 r_+^4(\ell^2 - 9r_+^2) - 3r_+^6(\ell^2 + 3r_+^2) \right)} \\
&+ \frac{a^5 \ell^2 \Xi r_+^3}{\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R_{\mu\nu} R^{\mu\nu}} \\
&= \frac{96\beta r_+^4(\beta r_+(a^2 + \ell^2 + 2r_+^2) - 2\pi\ell^2(a^2 + r_+^2)) - 96\beta^2 r_+^5(a^2 + r_+^2)}{32\pi\beta\ell^4\Xi r_+^4} \\
&+ \frac{(a^2(\beta r_+^2 - \ell^2(\beta + 4\pi r_+)) + \ell^2 r_+^2(\beta - 4\pi r_+) + 3\beta r_+^4)^2}{\left(3a^5 r_+ + 2a^3 r_+^3 + 3(a - r_+)(a + r_+)(a^2 + r_+^2)^2 \arctan\left(\frac{a}{r_+}\right) + 3ar_+^5\right)} \\
&\times \frac{32\pi\beta\ell^4\Xi r_+^4(a^5(a^2 + r_+^2))}{\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} R^2 = \frac{12(\beta r_+(\ell^2 + r_+^2) - 2\pi\ell^2(a^2 + r_+^2))}{\pi\ell^4\Xi}} \\
&\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} = 2r_+^2 - \frac{\beta r_+(3r_+^4 + (a^2 + \ell^2)r_+^2 - a^2\ell^2 - 2\ell^2 q_m^2)}{2\pi\ell^2(a^2 + r_+^2)}
\end{aligned}$$

Modelo de campo cuántico – relativista en el que se forman agujeros negros cuánticos en gravedad cuántica.

$$\psi_I^\alpha = ie^i(\gamma_i)^\alpha{}_\beta \chi_I^\beta$$

$$T^i = \mathcal{D}e^i \equiv de^i + \omega_j^i \wedge e^j = \tau \epsilon_{jk}^i e^j \wedge e^k$$

$$\mathcal{L}_{SUGRA} = \mathcal{L}_{CS}^{(G_+)} - \mathcal{L}_{CS}^{(G_-)}$$

$$\mathcal{A}_{(+)} = \frac{1}{2} \omega_{(+)}^{ij} \mathbb{J}_{ij} + A_{(+)}^{IJ} \mathbb{T}_{IJ} + \overline{\mathbb{Q}}^{\alpha I} \psi_{\alpha I}^{(+)}$$

$$\mathcal{A}_{(-)} = \frac{1}{2} \omega_{(-)}^{ij} \mathbb{J}_{ij} + A_{(-)}^{ij} \mathbb{T}_{ij} + \overline{\mathbb{Q}}^{\dot{\alpha} I} \psi_{\dot{\alpha} I}^{(-)}$$

$$\omega^{ij} = \frac{1}{2} \left(\omega_{(+)}^{ij} + \omega_{(-)}^{ij} \right)$$

$$E^k = \frac{\ell}{4} (\omega_{(+)}{}_{ij} - \omega_{(-)}{}_{ij}) \epsilon^{ijk}$$

$$\mathfrak{so}(1,2)_D \subset \mathfrak{so}(2,2) = \mathfrak{so}(1,2)_{(+)} \times \mathfrak{so}(1,2)_{(-)}$$



$$\mathcal{L}_{(3)} = \left(\mathcal{R}^{ij} - \frac{1}{3\ell^2} E^i E^j - \frac{1}{2\ell} \bar{\psi}_I \gamma^{ij} \psi_I \right) E^k \epsilon_{ijk} - \frac{1}{2\ell} A dA + 2\bar{\psi}_I \left(\mathcal{D}\psi_I - \frac{1}{2\ell} \epsilon_{IJ} A \psi_J \right)$$

$$\begin{aligned}\mathcal{R}_{(+)}^{IJ} &= \frac{i}{\ell} \bar{\psi}_I \wedge \gamma_k \psi_I \epsilon^{Ijk} \\ \mathcal{D}_{(+)} \psi_I &= \frac{1}{2\ell} A \wedge \epsilon_{IJ} \psi_J \\ dA &= \epsilon_{IJ} \bar{\psi}_I \wedge \psi_J \\ \mathcal{R}_{(-)}^{ij} &= 0\end{aligned}$$

$$R_{(+)}^{IJ} \equiv d\omega_{(+)}^{IJ} + \omega_{(+)}^{Ik} \wedge \omega_{(+)}^J{}_{k}, R_{(-)}^{ij} \equiv d\omega_{(-)}^{ij} + \omega_{(-)}^{ik} \wedge \omega_{(-)}^j{}_{k}, A_{(+)}^{IJ} = \epsilon^{IJ} A$$

$$\psi_I^\alpha = i(\gamma_i)^\alpha{}_\beta \chi_I^\beta e^i$$

$$\Gamma^{\hat{\mu}}\Psi_{\hat{\mu}I}=0\Rightarrow \gamma^\mu\psi_{\mu I}=-\gamma^r\psi_{rI}=3i\chi_I\neq 0$$

$$\psi_I^\alpha = i(\gamma_i)^\alpha{}_\beta \chi_I^{\beta'} e^i$$

$$\omega_{\pm}^{\alpha\beta}\equiv\frac{1}{2}\gamma_{ij}^{\alpha\beta}\left(\omega^{ij}\pm\frac{1}{\ell}E_k\epsilon^{ijk}\right)$$

$$\begin{aligned}\mathcal{R}_+^{\alpha\beta} &= -i\psi^{\alpha I}\wedge\psi_I^\beta, (\mathcal{D}\psi)_I^\alpha = \frac{1}{2}A\wedge\epsilon_{IJ}\psi_J^\alpha, dA = -i\frac{1}{2}\epsilon^{IJ}\epsilon_{\alpha\beta}\psi_I^\alpha\wedge\psi_J^\beta \\ \mathcal{R}_-^{\alpha\beta} &= 0\end{aligned}$$

$$\begin{aligned}\delta\omega_-^{\alpha\beta} &= 0 \\ \delta\omega_+^{\alpha\beta} &= -2i\epsilon^{(\alpha I}\psi_I^{\beta)} \\ \delta A &= -i\epsilon^{IJ}\epsilon_{\alpha\beta}\epsilon_I^\alpha\psi_J^\beta \\ \delta\psi_I^\alpha &= \mathcal{D}\epsilon_I^\alpha - \frac{1}{2}A\epsilon_{IJ}\epsilon_J^\alpha \equiv \nabla\epsilon_I^\alpha\end{aligned}$$

$$\begin{aligned}s\omega_-^{\alpha\beta} &= 0 \\ s\omega_+^{\alpha\beta} &= -2i\phi^{(\alpha I}\psi_I^{\beta)} \\ sA &= -i\epsilon^{IJ}\epsilon_{\alpha\beta}\phi_I^\alpha\psi_J^\beta \\ s\psi_I^\alpha &= \mathcal{D}\phi_I^\alpha - \frac{1}{2}A\epsilon_{IJ}\phi_J^\alpha \equiv \nabla\phi_I^\alpha \\ s\phi_I^\alpha &= 0\end{aligned}$$

$$\mu_+^{(\alpha\beta)}=-\phi_I^\alpha\phi_I^\beta, \mu_{+[IJ]}=-\epsilon_{\alpha\beta}\phi_I^\alpha\phi_J^\beta=\epsilon_{IJ}\mu_+,$$

$$\begin{aligned}s^2\omega_-^{\alpha\beta} &= 0 \\ s^2\omega_+^{\alpha\beta} &= i\nabla\mu_+^{(\alpha\beta)} \\ s^2A &= i\nabla\mu_+ \\ s^2\psi_I^\alpha &= \frac{i}{2}\mu_{IJ}^+\psi_J^\alpha + \frac{i}{2}\mu^{+l(\alpha\beta)}\epsilon_{\beta\gamma}\psi_I^\gamma \\ s^2\phi_I^\alpha &= 0\end{aligned}$$



$$s\bar{\phi}_\alpha^I = \bar{\eta}_\alpha^I, s\bar{\eta}_\alpha^I = \frac{i}{2}(\mu_{IJ}^+ \bar{\phi}_{\alpha J} + \mu^{+|(\delta\beta)} \epsilon_{\alpha\delta} \bar{\phi}_{\beta I}).$$

$$\mu_3^{\alpha\beta} = -\bar{\phi}_I^{(\alpha} \phi_I^{\beta)}, \mu_{3|IJ} = -\phi_{[I}^{\alpha} \bar{\phi}_{J]}^{\beta} \epsilon_{\alpha\beta}$$

$$\mu_-^{(\alpha\beta)} = -\bar{\phi}_I^{\alpha} \bar{\phi}_I^{\beta}, \mu_{-|IJ} = -\epsilon_{\alpha\beta} \bar{\phi}_I^{\alpha} \bar{\phi}_J^{\beta} = \epsilon_{IJ} \mu_-$$

$$\Phi_I^{\alpha|1} = \phi_I^{\alpha}, \Phi_I^{\alpha|2} = -\epsilon^{\alpha\beta} \bar{\phi}_{\beta I}$$

$$\Omega^{AB} = \epsilon_{\alpha\beta} d\Phi_I^{\alpha A} \wedge d\Phi_I^{\beta B}$$

$$\iota_{V^{(\alpha\beta)}}\Omega^{AB} = -d\mu^{AB|(\alpha\beta)}, \iota_{V_{IJ}}\Omega^{AB} = -d\mu_{IJ}^{AB}$$

$$\mu^{AB|(\alpha\beta)} = -\Phi_I^{(\alpha|A} \Phi_I^{\beta)B}, \mu_{IJ}^{AB} = -\epsilon_{\alpha\beta} \Phi_{[I}^{\alpha|A} \Phi_{J]}^{\beta|B} = \epsilon_{IJ} \mu^{AB}$$

$$\mu^{11|(\alpha\beta)} = \mu_+^{(\alpha\beta)}; \mu^{22|(\alpha\beta)} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \mu_{-(\gamma\delta)}; \mu^{12|(\alpha\beta)} = \mu_3^{(\alpha\beta)}; \\ \mu^{11} = \mu_+; \mu^{22} = \mu_-; \mu^{12} = \mu_3$$

$$\mu_+^{(\alpha\beta)} \mu_{+|(\alpha\beta)} + 2\mu_+^2 = 0$$

$$\mathcal{L}^{(3)} = \mathcal{L}_+^{(3)} - \mathcal{L}_-^{(3)}$$

$$\mathcal{L}_+^{(3)} = \frac{1}{2} \left(\omega_+^{\alpha\beta} \wedge d\omega_{+,\alpha\beta} - \frac{1}{3} \omega_+^{\alpha\alpha'} \wedge \omega_{+,\alpha'\beta'} \wedge \omega_+^{\beta'\beta} \right) - 2i\epsilon_{\alpha\beta} \psi^{\alpha I} \nabla \psi_I^\beta - A \wedge dA$$

$$\nabla \psi_I^\alpha = \left(\delta_\beta^\alpha d + \frac{1}{2} \omega_\beta^\alpha \right) \psi_I^\beta - \frac{1}{2} \epsilon_{IJ} A \wedge \psi_J^\alpha$$

$$\mathcal{L}_-^{(3)} = \frac{1}{2} \left(\omega_-^{\alpha\beta} \wedge d\omega_{-,\alpha\beta} - \frac{1}{3} \epsilon_{\alpha\beta} \omega_-^{\alpha\alpha'} \wedge \omega_{-,\alpha'\beta'} \wedge \omega_-^{\beta'\beta} \right)$$

$$\bar{s}\omega_-^{\alpha\beta} = 0$$

$$\bar{s}\omega_+^{\alpha\beta} = -2i\bar{\phi}_I^{(\alpha} \psi_I^{\beta)}$$

$$\bar{s}A = -i\epsilon_{\alpha\beta} \epsilon_{IJ} \bar{\phi}_I^\alpha \psi_J^\beta$$

$$\bar{s}\psi_I^\alpha = \mathcal{D}\bar{\phi}_I^\alpha - \frac{1}{2} A \epsilon_{IJ} \bar{\phi}_J^\alpha \equiv \nabla \bar{\phi}_I^\alpha$$

$$\bar{s}\bar{\phi}_\alpha^I = 0$$

$$\bar{s}\phi_I^\alpha = -\bar{\eta}_I^\alpha, \bar{s}\bar{\eta}_I^\alpha = -\frac{i}{2}(\mu_- \epsilon_{IJ} \phi_J^\alpha + \mu_-^{(\alpha\beta)} \epsilon_{\beta\gamma} \phi_I^\gamma).$$

$$\frac{1}{2}(s\bar{s} + \bar{s}s) = -\frac{i}{2}\mu_3^{\alpha\beta} \delta_{(\alpha\beta)} - i\mu_3 \delta$$

$$\frac{\delta S}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} = s\Gamma_{\mu\nu}$$

$$\partial^\mu \Gamma_{\mu\nu} = \delta_\nu A_\mu^{\alpha\beta} \frac{\delta S}{\delta A_\mu^{\alpha\beta}} + \delta_\nu A_{\mu,[IJ]} \frac{\delta S}{\delta A_{\mu,[IJ]}} + \delta_\nu \psi_{\mu,I}^\alpha \frac{\delta S}{\delta \psi_{\mu,I}^\alpha} + \delta_\nu \bar{\eta}_\alpha^I \frac{\delta S}{\delta \bar{\eta}_\alpha^I} + \delta_\nu \phi_I^\alpha \frac{\delta S}{\delta \phi_I^\alpha} + \delta_\nu \bar{\phi}_\alpha^I \frac{\delta S}{\delta \bar{\phi}_\alpha^I}$$



$$\Lambda_{I\alpha'\beta'}^\alpha=i\gamma_{\alpha'\beta'}^\mu\psi_{\mu,I}^\alpha-\frac{1}{2}\epsilon_{\alpha'\beta'}\bar\eta_I^\alpha$$

$$\Lambda_{I\alpha'\beta'}^\alpha=i(\gamma^\mu)^\alpha{}_{\alpha'}\chi_{\mu\beta',I}-2\delta^\alpha{}_{\alpha'}\chi_{I\beta'}$$

$$\Lambda_{I\alpha'\beta'}^\alpha\longrightarrow\Lambda_{I\alpha'\beta'}^\alpha$$

$$\epsilon^{\alpha'\dot\alpha'A}=i(\gamma^i)^{\alpha'\dot\alpha'}\epsilon^A_i+\epsilon^{\alpha'\dot\alpha'}\epsilon^A,$$

$$\mathcal{L}=\mathcal{L}^{(3)}+\mathcal{L}^{g.f.},\mathcal{L}^{g.f.}=s\left(\bar\phi^I_\alpha\mathcal{F}^\alpha_I(\omega_+,\psi^\alpha_I,A,\phi^\alpha_I)\right),S=\int_{\mathcal{M}_3}\mathcal{L}$$

$$S_A=2\int\,s(\bar\phi^I_\alpha\nabla\star\psi^\alpha_I)=-2\int\,(\nabla\phi^\alpha_I\wedge\star\nabla\bar\phi^I_\alpha+\psi^\alpha_I\wedge\star\nabla\bar\eta^I_\alpha)\\+2\int\,\left(-i\phi^{J(\alpha}\psi^\beta_{J)}\bar\phi^I_\alpha\star\psi_{\beta I}-\frac{1}{2}\epsilon^{KL}\phi^\gamma_K\psi^\delta_L\epsilon_{\gamma\delta}\bar\phi^I_\alpha\epsilon^J_I\star\psi^\alpha_J\right)$$

$$2\epsilon_{\alpha\beta}\nabla\psi_I^\beta+\star\nabla\bar\eta_{I\alpha}=0,\nabla\star\psi_I^\alpha=0$$

$$\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta}\nabla_\nu\psi_{\rho,I}^\alpha+\nabla^\mu\bar\eta_{I\alpha}=0,\nabla^\mu\psi_{\mu,I}^\alpha=0$$

$$2(\gamma_\mu)_{\alpha\beta}\nabla^\mu(\gamma^\nu\psi_{\nu,I})^\beta+(\gamma_\mu)_\alpha^\beta\nabla^\mu\bar\eta^I_\beta=\mathfrak{A}(2\epsilon_{\alpha\beta}(\psi_I^\beta)+\bar\eta^I_\alpha)=0$$

$$\bar\eta^I_\alpha=-2(\psi_I)_\alpha+\sigma^I_\alpha$$

$$S_B=2\int\,s(\beta\bar\eta^I_\alpha\epsilon^{\alpha\beta}\bar\phi_{\beta,I})V^{(3)}\\=2\int\,\beta\left(\bar\phi^I_\alpha\left(\mu_{+,IJ}(\phi)\epsilon^{\alpha\beta}+\mu_+^{\alpha\beta}(\phi)\delta_{IJ}\right)\bar\phi^J_\beta+\bar\eta^I_\alpha\epsilon^{\alpha\beta}\bar\eta_{\beta,I}\right)V^{(3)}\\=2\int\,\beta(\mu_{3,\alpha}{}^\beta\mu_{3,\gamma}{}^\delta\epsilon_{\alpha\gamma}\epsilon^{\beta\delta}+\mu_{3,K}^I\mu_{3,I}^K+\bar\eta^I_\alpha\epsilon^{\alpha\beta}\bar\eta_{\beta,I})V^{(3)}$$

$$S'_B=\int\,s\left(\bar\eta^I_\alpha V_{IJ}^{\alpha\beta}(\phi)\bar\phi^J_\beta\right)V^{(3)}\\=\int\,\left(\bar\phi^K_\alpha\bar\phi^J_\beta\left(\mu_{+,K}^I V_{IJ}^{\alpha\beta}+\mu_{+,\gamma}^\alpha V_{IJ}^{\gamma\beta}\right)+\bar\eta^I_\alpha V_{IJ}^{\alpha\beta}\bar\eta^J_\beta\right)V^{(3)}$$

$$\mathcal{L}=2\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho}\psi_{\mu,I}^\alpha\nabla_\nu\psi_{\rho,I}^\beta+\bar\eta^I_\alpha\left(\nabla^\mu\psi_{\mu,I}^\alpha-2i\beta(\gamma^i)^\alpha{}_\beta\psi_{i,I}^\beta\right)+2\beta\epsilon^{\alpha\beta}\bar\eta^I_\alpha\bar\eta_{\beta,I}$$

$$4\epsilon^{\mu\nu\rho}\nabla_\nu\psi_{\rho,I}^\alpha-2\nabla^\mu\bar\eta_I^\alpha+4i\beta\gamma^{\mu,\alpha}{}_\beta\bar\eta_I^\beta=0\\ \nabla^\mu\psi_{\mu,I}^\alpha-2i\beta(\gamma^\mu\psi_{\mu,I})^\alpha+2\beta\epsilon^{\alpha\beta}\bar\eta^I_\beta=0$$

$$S_A=\int\,s\left(\psi_I^\alpha\wedge\star\nabla\bar\phi^I_\alpha+\psi_I^\alpha\wedge\star e^a\gamma_{a,\alpha}^\beta\bar\phi^I_\beta\right)\\=\int\,\left[\nabla\phi_I^\alpha\wedge\star\nabla\bar\phi^I_\alpha+\psi_I^\alpha\wedge\star\nabla\bar\eta^I_\alpha+\psi_I^\alpha\wedge\star\left(\epsilon_{\alpha\beta}\psi_K^{(\beta}\eta^{KL}\phi_L^{\gamma)}\bar\phi^I_\gamma+\eta^{IJ}\psi_{[J}^\gamma\epsilon_{\gamma\beta}\phi_K^\beta\bar\phi^K_\alpha\right)\right.\\ \left.+\nabla\phi_I^\alpha\wedge\star e^i\gamma_{i,\alpha}^\beta\bar\phi^I_\beta+\psi_I^\alpha\wedge\star e^i\gamma_{i,\alpha}^\beta\bar\eta^I_\beta\right]$$

$$\mathcal{L}^{g.f.}=s\bar{S}\Sigma$$



$$\mathcal{L}^{g.f.} = s\bar{s}\left(\psi_I^\alpha \epsilon_{\alpha\beta} \eta^{IJ} \wedge \star \psi_J^\beta + \bar{\eta}_\alpha^I \epsilon^{\alpha\beta} \bar{\eta}_\beta^I \star 1\right)$$

$$\mathcal{L}^{g.f.} = s\bar{s}\Big(\psi_I^\alpha \epsilon_{\alpha\beta} \wedge \star \psi_I^\beta + \bar{\eta}_\alpha^I \epsilon^{\alpha\beta} \bar{\eta}_{\beta|I} \star 1 + \beta \psi_I^\alpha \wedge \star e^i \Gamma_{i\alpha}{}^\beta \bar{\eta}_\beta^I\Big)$$

$$s\bar{s}(A^{(\alpha\beta)}\gamma_{i\alpha\beta}\wedge\star e^i)$$

$$s(\psi^\alpha\wedge\bar{\phi}^\beta(\gamma_i)_{\alpha\beta}\star e^i)=\nabla\phi^\alpha\wedge\bar{\phi}^\beta(\gamma_i)_{\alpha\beta}\star e^i-\psi^\alpha\wedge\bar{\eta}^\beta(\gamma_i)_{\alpha\beta}\star e^i$$

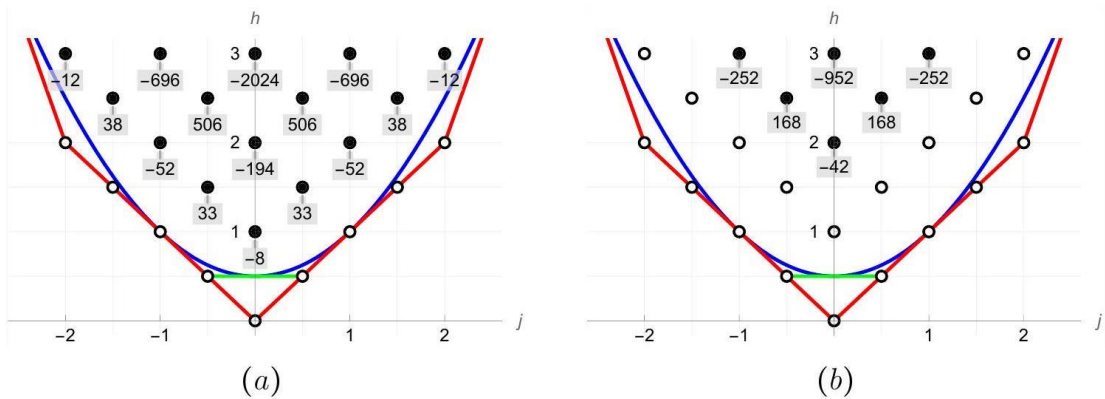
$$\delta_{\mathrm{Sp}(2)}\Phi_I^{\gamma A}=\frac{1}{2}\lambda_{(\alpha\beta)}(V^{(\alpha\beta)})_I^{\gamma A}=\frac{1}{2}\lambda_{(\alpha\beta)}\epsilon^{\gamma(\alpha}\Phi_I^{\beta)A},\delta_{\mathrm{SO}(2)}\Phi_I^{\alpha A}=\frac{1}{2}\lambda^{KL}(V_{KL})_I^{\alpha A}=\\-\frac{1}{2}\lambda^{KL}\delta_{I[K}\Phi_{L]}^{\alpha A}$$

$$|\Psi\rangle=|\psi_1\rangle_{a_1}|\psi_2\rangle_{a_2}|\psi_3\rangle_{a_3}\cdots$$

$$|\Psi\rangle = \mathcal{O}_1|\phi_1\rangle_{a_1}\mathcal{O}_2|\phi_2\rangle_{a_2}\cdots$$

$$\mathcal{G}|\phi\rangle_a\,\underbrace{|0\rangle_1|0\rangle_1\cdots}_{N-a},$$

$$\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots$$



$$\begin{aligned} &\mathcal{A}^{(\mathrm{T})}(\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots) \\ &= (\mathcal{A}\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots) + (\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{A}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1|0\rangle_1\cdots) + \cdots \\ &+ (\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots\mathcal{A}|0\rangle_1|0\rangle_1\cdots) + (\mathcal{G}_1|\phi_1\rangle_{a_1}\mathcal{G}_2|\phi_2\rangle_{a_2}\cdots|0\rangle_1\mathcal{A}|0\rangle_1\cdots) + \cdots \end{aligned}$$

$$X^{\dot{A}A}(t,\sigma),\psi^{\alpha\dot{A}}(t,\sigma),\tilde{\psi}^{\dot{\alpha}\dot{A}}(t,\sigma)$$

$$\alpha\in\{+,-\},\dot{\alpha}\in\{+,\dot{-}\},A\in\{1,2\},\dot{A}\in\{\dot{1},\dot{2}\}$$

$$SO(4)_E\cong SU(2)_L\times SU(2)_R$$

$$SO(4)_I\cong SU(2)_1\times SU(2)_2$$

$$\partial X^{\dot{A}A}(t,\sigma+2\pi R_\sigma)=\partial X^{\dot{A}A}(t,\sigma)\,,\psi^{\alpha\dot{A}}(t,\sigma+2\pi R_\sigma)=-\psi^{\alpha\dot{A}}(t,\sigma)$$

$$\partial X^{\dot{A}A}(z)=\sum_{n\in\mathbb{Z}}\alpha_{-n}^{\dot{A}A}z^{n-1},\psi^{\alpha\dot{A}}(z)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\psi_{-r}^{\alpha\dot{A}}z^{r-\frac{1}{2}},\tilde{\psi}^{\dot{\alpha}\dot{A}}(\tilde{z})=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\tilde{\psi}_{-r}^{\dot{\alpha}\dot{A}}\tilde{z}^{r-\frac{1}{2}}$$

$$\{L_n,J_n^{\pm,3},G_r^{\alpha A}\},\{\tilde{L}_n,\tilde{J}_n^{\pm,3},\tilde{G}_r^{\dot{\alpha}A}\}$$

$$\mathcal{H}\big(\mathrm{Sym}^N(\mathcal{M}_4)\big)\equiv \mathcal{H}_{\mathrm{seed}}^{\otimes N}/S_N=\bigoplus_{[g]}\mathcal{H}_{[g]}$$

$$[g]\equiv\{\tilde{g}\in S_N\mid\exists h\in S_N\text{ s.t. }h\tilde{g}h^{-1}=g\}$$

$$\sum_{a=1}^N an_a=N$$

$$\mathcal{H}_{\{n_a\}}=\bigotimes_{a=1}^N \underbrace{(\mathcal{H}_a\otimes\cdots\otimes\mathcal{H}_a)}_{n_a}^{S_{n_a}}$$

$$N=\{\underbrace{1,\ldots,1}_{n_1},\underbrace{2,\ldots,2}_{n_2},\ldots\}$$

$$\{\partial X^{\dot{A}A(i)},\psi^{\alpha\dot{A}(i)},\tilde{\psi}^{\dot{\alpha}\dot{A}(i)}\}$$

$$L_n^{(\mathrm{T})}\equiv\sum_{i=1}^NL_n^{(i)}$$

$$\begin{aligned}\partial X^{\dot{A}A(i)}(t,\sigma+2\pi R_\sigma)&=\partial X^{\dot{A}A(i+1)}(t,\sigma) \text{ for } i=1,\ldots,a-1, \\ \partial X^{\dot{A}A(a)}(t,\sigma+2\pi R_\sigma)&=\partial X^{\dot{A}A(1)}(t,\sigma),\end{aligned}$$

$$\mathcal{C}_{\{n_a\}}=\prod_{a=1}^N\mathcal{C}_{n_a}=\prod_{a=1}^N\left(S_{n_a}\ltimes\mathbb{Z}_a^{n_a}\right)$$

$$L_n(\eta)\!=\!L_n+\eta J_n^3+\eta^2\frac{c}{24}\delta_{n,0}$$

$$J_n^3(\eta)=J_n^3+\eta\frac{c}{12}\delta_{n,0}$$

$$J_n^{\pm}(\eta)=J_{n\mp\eta}^{\pm}$$

$$G_r^{\pm A}(\eta)=G_{r\mp\frac{\eta}{2}}^{\pm A}$$

$$\left\{L_{\pm 1},L_0,G_{\pm\frac{1}{2}}^{\alpha A},J_0^a\right\}$$

$$\begin{aligned}L_n|\phi\rangle &=J_n^a|\phi\rangle=G_r^{\alpha A}|\phi\rangle=0\text{ for }n>0,r\geq\frac{1}{2},\\ L_0|\phi\rangle &=h|\phi\rangle,J_0^3|\phi\rangle=j|\phi\rangle,\end{aligned}$$

$$h\geq |j|, h\geq J$$

$$G_{-\frac{1}{2}}^{+A}|\phi\rangle=0$$

$$h=j=\frac{a-1}{2},\frac{a}{2},\frac{a+1}{2}.$$

$$|-\rangle_a\colon h=j=\frac{a-1}{2},$$

$$|\dot{A}\rangle_a\colon h=j=\frac{a}{2}$$

$$|+\rangle_a\colon h=j=\frac{a+1}{2},$$

$$|\dot{-}\rangle_a\colon \tilde{h}=\tilde{j}=\frac{a-1}{2},$$

$$|\dot{A}\rangle_a\colon \tilde{h}=\tilde{j}=\frac{a}{2},$$

$$|\dot{+}\rangle_a\colon \tilde{h}=\tilde{j}=\frac{a+1}{2}.$$

$$\frac{1}{2}-\text{BPS: } \mid \text{chiral primary} \rangle \overline{\mid \text{chiral primary} \rangle}$$

$$\frac{1}{4}-\text{BPS: } \mid \text{anything} \rangle \overline{\mid \text{chiral primary} \rangle}$$

$$(1) \; |\phi\rangle, \quad (4) \; G|\phi\rangle, \quad (6) \; G^2|\phi\rangle, \quad (4) \; G^3|\phi\rangle, \quad (1) \; G^4|\phi\rangle \; ,$$

$$(1) \; |\phi\rangle, \quad (2) \; G|\phi\rangle, \quad (1) \; G^2|\phi\rangle \; .$$

$$h(\eta)=h+\eta j+\frac{c}{24}\eta^2, j(\eta)=j+\frac{c}{12}\eta$$

$$\frac{1}{2}-\text{BPS: } \mid \text{R ground state} \rangle \overline{\mid \text{R ground state} \rangle},$$

$$\frac{1}{4}-\text{BPS: } \mid \text{anything} \rangle \mid \overline{\text{R ground state}} \rangle.$$

$$X^{\dot{A}A}\rightarrow -X^{\dot{A}A}, \psi^{\alpha\dot{A}}\rightarrow -\psi^{\alpha\dot{A}}, \tilde{\psi}^{\dot{\alpha}\dot{A}}\rightarrow -\tilde{\psi}^{\dot{\alpha}\dot{A}}$$

$$|-\dot{\pm}\rangle_a\colon h=j=\frac{a-1}{2},\tilde{h}=\tilde{j}=\frac{a\pm1}{2}$$

$$|I\rangle_a\colon h=j=\tilde{h}=\tilde{j}=\frac{a}{2}$$

$$|+\dot{\pm}\rangle_a\colon h=j=\frac{a+1}{2},\tilde{h}=\tilde{j}=\frac{a\pm1}{2}$$

$$\Pi_{\psi} \left(|\psi\rangle \right)^{N_{\psi}}$$

$$\Pi_{\psi,n,m} \left[(L_{-1})^n (J_0^-)^m |\psi\rangle \right]^{N_{\psi,n,m}}$$

$$\Pi_{\psi,n,m} \left[(L_{-1})^n (J_0^-)^m G_{-\frac{1}{2}}^{-A} |\psi\rangle \right]^{N'_{\psi,n,m}}$$

$$\Pi_{\psi,n,m} \left[(L_{-1})^n (J_0^-)^m \left(G_{-\frac{1}{2}}^{-1} G_{-\frac{1}{2}}^{-2} - \frac{1}{2h} L_{-1} J_0^- \right) |\psi\rangle \right]^{N''_{\psi,n,m}}$$

$$\mathcal{H}_N^{\text{graviton}}=\bigoplus_{\{j,\tilde{j};d\}}^{\blacksquare}\oplus'\mathfrak{K}\otimes_i\left(j_i,\tilde{j}_i;d_i\right)_s$$



$$Z_R^{\text{CFT}}(q, y, \tilde{q}, \tilde{y}) \equiv \text{Tr}_{\mathcal{H}_R^{\text{CFT}}} \left[(-1)^F q^{L_0 - \frac{c}{24}} y^{2J_0^3} \tilde{q}^{\tilde{L}_0 - \frac{c}{24}} \tilde{y}^{2\tilde{J}_0^3} \right]$$

$$q = e^{2\pi i \tau}, y = e^{2\pi i \nu}, \tilde{q} = e^{2\pi i \tilde{\tau}}, \tilde{y} = e^{2\pi i \tilde{\nu}}$$

$$\mathcal{E}_R^{\text{CFT}}(q, y) \equiv Z_R^{\text{CFT}}(q, y, 1, 1) = \text{Tr}_{\mathcal{H}_R^{\text{CFT}}} \left[(-1)^F q^{L_0 - \frac{c}{24}} y^{2J_0^3} \right]$$

$$Z_{\text{NS}}^{\text{CFT}}(q, y, \tilde{q}, \tilde{y}) \equiv (y\tilde{y})^{\frac{c}{6}} (q\tilde{q})^{\frac{c}{12}} Z_R^{\text{CFT}} \left(q, yq^{\frac{1}{2}}, \tilde{q}, \tilde{y}\tilde{q}^{\frac{1}{2}} \right) = \text{Tr}_{\mathcal{H}(\text{NS})} \left[(-1)^F q^{L_0} y^{2J_0^3} \tilde{q}^{\tilde{L}_0} \tilde{y}^{2\tilde{J}_0^3} \right]$$

$$\mathcal{E}_{\text{NS}}^{\text{CFT}}(q, y) \equiv Z_{\text{NS}}^{\text{CFT}}(q, y, 1, 1) = \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{CFT}}} \left[(-1)^F q^{L_0} y^{2J_0^3} \right]$$

$$\hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q, y) \equiv \frac{1}{2} (\tilde{y} \partial_{\tilde{y}})^2 Z_{\text{NS}}^{\text{CFT}}(q, y, \tilde{q}, \tilde{y}) \Big|_{\tilde{q}=\tilde{y}=1} = \frac{1}{2} \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{CFT}}} \left[(-1)^F q^{L_0} y^{2J_0^3} (2\tilde{J}_0^3)^2 \right]$$

$$\begin{aligned} \hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q, y) &\supset \frac{1}{2} (-1)^{2j} q^h y^{2j} \left[(-1)^{2\tilde{j}} (2\tilde{j})^2 + 2(-1)^{2\tilde{j}+1} (2\tilde{j}+1)^2 + (-1)^{\tilde{j}+2} (2\tilde{j}+2)^2 \right] \\ &= (-1)^{2j-2\tilde{j}} q^h y^{2j} \end{aligned}$$

$$\mathcal{E}_{R,K3}^{\text{CFT}}(q, y) = 8 \sum_{i=2}^4 \frac{\vartheta_i(\nu, \tau)}{\vartheta_i(0, \tau)} = \sum_{m, \ell} c_R^{K3}(m, \ell) q^m y^\ell$$

$$c_R^{K3}(m, \ell) = c(4m - \ell^2)$$

$$Z_{R,K3}^{\text{CFT}}(p, q, y) \equiv \sum_{N=1}^{\infty} p^N \mathcal{E}_R^{\text{CFT}}(q, y; N) = \prod_{n, m, \ell} (1 - p^n q^m y^\ell)^{-c_R^{K3}(nm, \ell)}$$

$$Z_{\text{NS},K3}^{\text{CFT}}(p, q, y) \equiv \sum_{N=1}^{\infty} p^N \mathcal{E}_{\text{NS}}^{\text{CFT}}(q, y; N) = \prod_{n, m, \ell} (1 - p^n q^m y^\ell)^{-c_{\text{NS}}^{K3}(n, m, \ell)}$$

$$c_{\text{NS}}^{K3}(n, m, \ell) = c(4nm - n^2 - \ell^2)$$

$$\begin{aligned} Z_{K3}^{\text{graviton}}(p, q, y, \tilde{y}) &= \sum_{k=1}^{\infty} p^k \left[\phi_{\frac{k-1}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20 \phi_{\frac{k}{2}}^{(s)}(q, y) \tilde{\chi}_0(\tilde{y}) + \phi_{\frac{k+1}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}}(\tilde{y}) \right] \\ &= \frac{p}{(1-q)(1-y^2)} \left[\frac{(1-\sqrt{q}y)^2 \left(\tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20\sqrt{q}y^{-1} \tilde{\chi}_0(\tilde{y}) + qy^{-2} \tilde{\chi}_{\frac{1}{2}}(\tilde{y}) \right)}{1-p\sqrt{q}y^{-1}} \right] \\ &\quad - \frac{y^2(1-\sqrt{q}y^{-1})^2 \left(\tilde{\chi}_{\frac{1}{2}}(\tilde{y}) + 20\sqrt{q}y \tilde{\chi}_0(\tilde{y}) + qy^2 \tilde{\chi}_{\frac{1}{2}}(\tilde{y}) \right)}{1-p\sqrt{q}y} \Bigg] \\ &= \sum_{n, m, \ell} c_{\text{graviton}}^{K3}(n, m, \ell, \tilde{\ell}) p^n q^m y^\ell \tilde{y}^{\tilde{\ell}} \end{aligned}$$

$$X_k^{K3}(q, y, \tilde{y}) \equiv \phi_{\frac{k-1}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, k}(\tilde{y}) + 20 \phi_{\frac{k}{2}}^{(s)}(q, y) \tilde{\chi}_{0, k}(\tilde{y}) + \phi_{\frac{k+1}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, k}(\tilde{y})$$

$$\begin{aligned} Z_{K3}^{\text{graviton}}(p, q, y, \tilde{y}) &\equiv \sum_{N=1}^{\infty} p^N Z_{K3}^{\text{graviton}}(q, y, \tilde{y}; N) \\ &= \text{PE} \left[Z_{K3}^{\text{graviton}} \right] (p, q, y, \tilde{y}) \\ &\equiv \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} p^{nk} X_k^{K3}(q^n, y^n, \tilde{y}^n) \right) \\ &= \prod_{n, m, \ell, \tilde{\ell}} (1 - p^n q^m y^\ell \tilde{y}^{\tilde{\ell}})^{-c_{\text{graviton}}^{K3}(n, m, \ell, \tilde{\ell})} \end{aligned}$$



$$\mathcal{E}_{K3}^{\text{graviton}}(q,y;N)=Z_{K3}^{\text{graviton}}(q,y,1;N)$$

$$z_{R,T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y})=\left(\frac{\vartheta_1(\nu,\tau)}{\eta(q)^3}\right)^2\left(\frac{\vartheta_1(\tilde{\nu},\tilde{\tau})}{\eta(\tilde{q})^3}\right)^2=\sum_{m,\ell,\tilde{m},\tilde{\ell}}c_R^{T^4}(m,\ell,\tilde{m},\tilde{\ell})q^my^\ell\tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}},$$

$$\mathcal{Z}_{R,T^4}^{\text{CFT}}(p,q,y,\tilde{q},\tilde{y})\equiv \sum_{N=1}^{\infty}p^NZ_{R,T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y};N)=\prod_{n,m,\ell,\tilde{m},\tilde{\ell}}\left(1-p^nq^my^\ell\tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}}\right)^{-c_R^{T^4}(nm,\ell,n\tilde{m},\tilde{\ell})}$$

$$Z_{\text{NS},T^4}^{\text{CFT}}(q,y,\tilde{q},\tilde{y};N)=(y\tilde{y})^{\frac{c}{6}}(q\tilde{q})^{\frac{c}{12}}Z_{R,T^4}^{\text{CFT}}\left(q,yq^{\frac{1}{2}},\tilde{q},\tilde{y}\tilde{q}^{\frac{1}{2}};N\right),$$

$$\begin{aligned} z_{T^4}^{\text{graviton}}(p,q,y,\tilde{q},\tilde{y})&= \sum_{k=1}^{\infty}p^k\left[\phi_{\frac{k-1}{2}}^{(s)}(q,y)-2\phi_{\frac{k}{2}}^{(s)}(q,y)+\phi_{\frac{k+1}{2}}^{(s)}(q,y)\right]\omega_k(\tilde{q},\tilde{y})\\ &= \sum_{n,m,l,\tilde{m},\tilde{\ell}}c_{T^4}^{\text{graviton}}(n,m,l,\tilde{m},\tilde{\ell})p^nq^my^l\tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}} \end{aligned}$$

$$\omega_k(\tilde{q}^n,\tilde{y}^n)\equiv \tilde{q}^{n\frac{k-1}{2}}\tilde{y}^{n(k-1)}\left(1-\tilde{q}^{\frac{n}{2}}\tilde{y}^n\right)^2.$$

$$X_k^{T^4}(q,y)\equiv \phi_{\frac{k-1}{2}}^{(s)}(q,y)-2\phi_{\frac{k}{2}}^{(s)}(q,y)+\phi_{\frac{k+1}{2}}^{(s)}(q,y),$$

$$\begin{aligned} z_{T^4}^{\text{graviton}}(p,q,y,\tilde{q},\tilde{y})&\equiv \sum_{N=1}^{\infty}p^N z_{T^4}^{\text{graviton}}(q,y,\tilde{q},\tilde{y};N)\\ &= \text{PE}\left[z_{T^4}^{\text{graviton}}\right](p,q,y,\tilde{q},\tilde{y})\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}p^{nk}X_k^{T^4}(q^n,y^n)\omega_k(\tilde{q}^n,\tilde{y}^n)\right)\\ &= \prod_{n,m,l,\tilde{m},\tilde{\ell}}\left(1-p^nq^my^l\tilde{q}^{\tilde{m}}\tilde{y}^{\tilde{\ell}}\right)^{-c_{T^4}^{\text{graviton}}(n,m,l,\tilde{m},\tilde{\ell})} \end{aligned}$$

$$h\leq \frac{N+1}{4}~(K3)$$

$$h<\frac{N}{4}~(T^4)$$

$$J_{-(n+1)}^+|-\rangle_1=\frac{1}{n!}L_{-1}^n|+\rangle_1,n\geq 0$$

$$\{|h\rangle,L_{-1}|h\rangle,L_{-1}^2|h\rangle,\ldots\}=\{L_{-1}^n|h\rangle\}_{n=0}^\infty$$

$$z(q)=q^h(1+q+q^2+\cdots)=\frac{q^h}{1-q}=\chi_h^{\text{global}}(q)=\sum_m c(m)q^m$$

$$\prod_{n\geq 0} [(L_{-1})^n|h\rangle]^{N_n}$$

$$Z(q,p)=\sum_N Z_N(q)p^N$$

$$Z(q,p)=\prod_m \frac{1}{(1-q^mp)^{c(m)}}=\prod_{n\geq 0} \frac{1}{1-q^{h+np}}$$

$$h^{(\mathrm{T})}=2h\colon (|h\rangle)^2$$

$$h^{(\mathrm{T})}=2h+1\colon (L_{-1}|h\rangle)|h\rangle$$

$$h^{(\mathrm{T})}=2h+2\colon (L_{-1}^2|h\rangle)|h\rangle,(L_{-1}|h\rangle)^2$$

$$h^{(\mathrm{T})}=2h+3\colon (L_{-1}^3|h\rangle)|h\rangle,(L_{-1}^2|h\rangle)(L_{-1}|h\rangle)$$

$$\ldots$$

$$|\phi\rangle = a(L_{-1}^2|h\rangle)|h\rangle + 2b(L_1L_{-1}|h\rangle)(L_{-1}|h\rangle) + d(L_{-2}|h\rangle)|h\rangle$$

$$L_1^{(T)}|\phi\rangle = ((4h+1)a + 4hb + 3d)(L_{-1}|h\rangle)|h\rangle = 0$$

$$L_2^{(T)}|\phi\rangle = \left(6ha + \left(\frac{c}{2} + 4h\right)d\right)(|h\rangle)^2 = 0$$

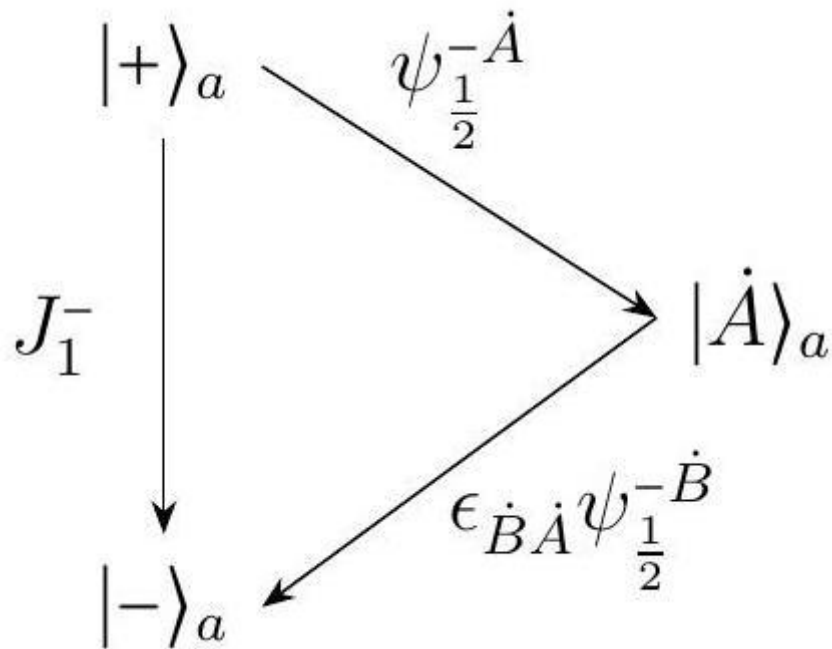
$$Z_N(q) = \frac{1}{1-q} \cdot (1-q)Z_N(q) = \sum_{h'} c_{h'} \chi_{h'}^{\text{global}}(q)$$

$$(1-q)Z_N(q) \equiv \sum_{h'} c_{h'} q^{h'}$$

$$Z_k(q) = \sum_{h'} c_{h'} \chi_{h'}^{\text{global}}(q) \rightarrow \sum_{h'} c_{h'} \chi_{h'}^{\text{Vir}}(q), \chi_{h'}^{\text{Vir}}(q) = \frac{q^{h'}}{\prod_{n \geq 1} (1-q^n)}$$

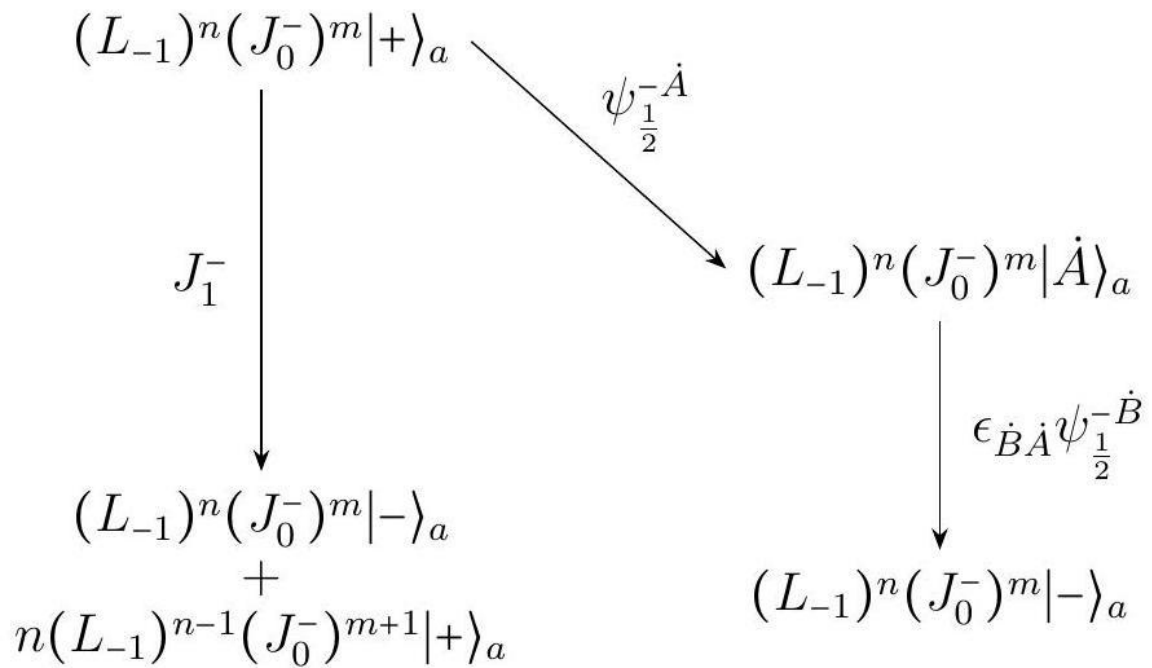
$$Z_N(q) \rightarrow \frac{Z_N(q)}{\prod_{n \geq 2} (1-q^n)}$$

$$|-\rangle_a, \psi_{\frac{1}{2}}^{+\dot{A}}|-\rangle_a = |\dot{A}\rangle_a, \psi_{\frac{1}{2}}^{+\dot{1}}\psi_{\frac{1}{2}}^{+\dot{2}}|-\rangle_a = J_1^+|-\rangle_a = |+\rangle_a$$

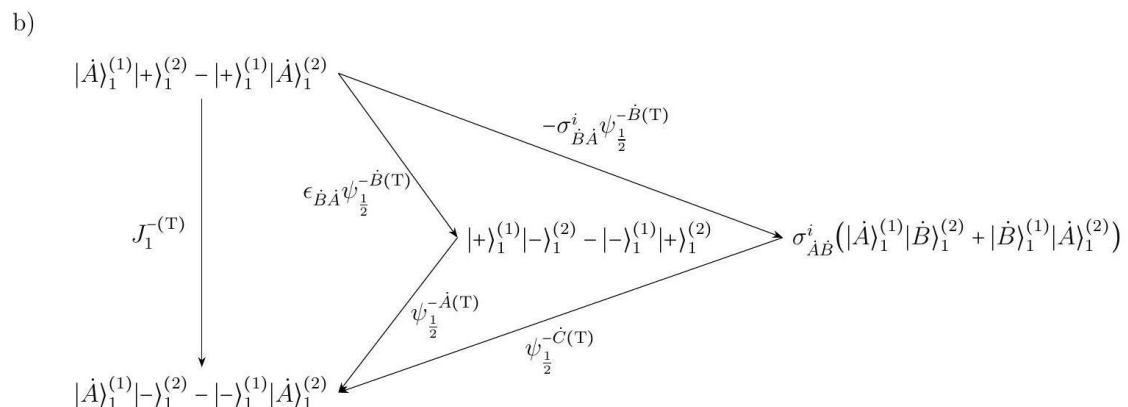
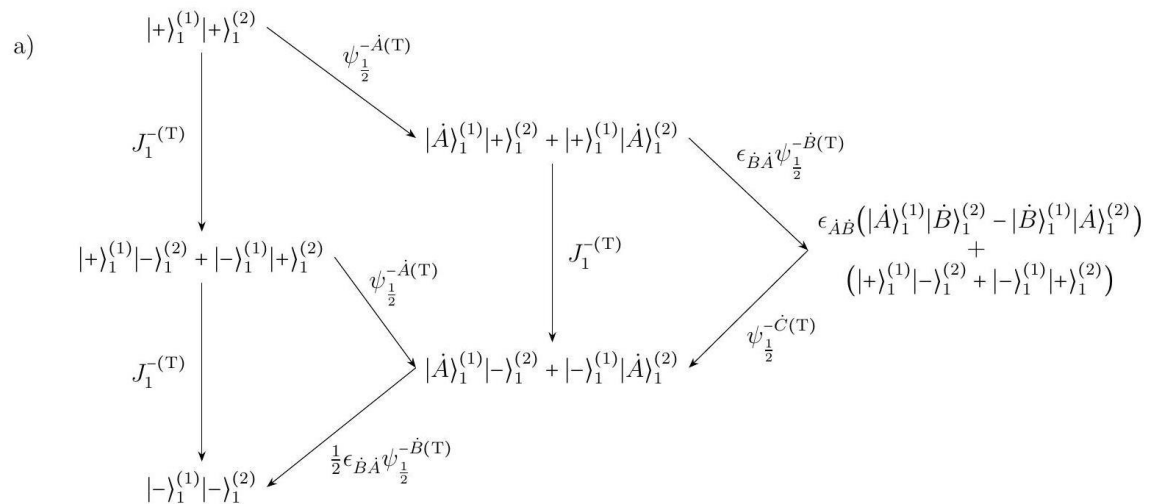


$$J_1^- \mathcal{O} |+\rangle_a = \mathcal{O}' |+\rangle_a + \mathcal{O} |-\rangle_a$$

$$\psi_{\frac{1}{2}}^{-\dot{A}} \mathcal{O} |+\rangle_a = \mathcal{O}' |+\rangle_a \pm \mathcal{O} |\dot{A}\rangle_a$$



$$\begin{aligned}
 J_1^{-(T)} \left(\mathcal{O}_1^{(1)} |+\rangle_1^{(1)} \mathcal{O}_2^{(2)} |+\rangle_1^{(2)} \right) = & \mathcal{O}_1^{(1)} |-\rangle_1^{(1)} \mathcal{O}_2^{(2)} |+\rangle_1^{(2)} + \mathcal{O}_1^{(1)} |+\rangle_1^{(1)} \mathcal{O}_2^{(2)} |-\rangle_1^{(2)} \\
 & + \mathcal{O}_1'^{(1)} |+\rangle_1^{(1)} \mathcal{O}_2^{(2)} |+\rangle_1^{(2)} + \mathcal{O}_1^{(1)} |+\rangle_1^{(1)} \mathcal{O}_2'^{(2)} |+\rangle_1^{(2)}
 \end{aligned}$$



$$Z_{T^4}^{\text{graviton}}(q, y, \tilde{y}; \{2\}) \equiv \Pi_{m, l, \tilde{m}, \tilde{l}} \left(1 - p^2 q^m y^l \tilde{q}^{\tilde{m}} \tilde{y}^{\tilde{l}} \right)^{-c_{T^4}^{\text{graviton}}(2, m, l, \tilde{m}, \tilde{l})} \Big|_{\tilde{p}^2}$$

$$= X_2^{T^4}(q, y) \omega_2(\tilde{y})$$

$$\omega_k(\tilde{y}) \equiv \omega_k(1, \tilde{y}) = \tilde{\chi}_{\frac{1}{2}; k}(\tilde{y}) - 2\tilde{\chi}_{0; k}(\tilde{y})$$

$$Z_{T^4}^{\text{graviton}}(q, y, \tilde{y}, \xi; \{2\}) = \left(\xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) - 2\xi_0 \phi_1^{(s)}(q, y) + \xi_+ \phi_{\frac{3}{2}}^{(s)}(q, y) \right) \omega_2(\tilde{y})$$

$$Z_{T^4}^{\text{red}}(q, y, \tilde{y}, \xi; \{2\}) = \xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) \omega_2(\tilde{y})$$

$$Z_{T^4}^{\text{gen}}(q, y, \tilde{y}; \{2\}) = \Phi_{\frac{1}{2}; 2}^{(s)}(q, y) \omega_2(\tilde{y})$$

$$\begin{array}{ccc} \xi_+ \phi_{\frac{3}{2}}^{(s)}(q, y) \omega_2(\tilde{y}) & \xrightarrow{\psi_{\frac{1}{2}}^{-\dot{A}(T)}} & -2\xi_0 \phi_1^{(s)}(q, y) \omega_2(\tilde{y}) \\ \downarrow J_1^{-(T)} & & \uparrow \epsilon_{\dot{A}\dot{B}} \psi_{\frac{1}{2}}^{-\dot{B}(T)} \\ \xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) \omega_2(\tilde{y}) & & \end{array}$$

$$Z_{T^4}^{\text{gen}}(q, y, \tilde{y}; \{2\}) = \left[\phi_{\frac{1}{2}}^{(s)} - 2\phi_1^{(s)} + \phi_{\frac{3}{2}}^{(s)} - 2\phi_{0,1}^{(\ell)} - 12\phi_{0,2}^{(\ell)} + 5\phi_{\frac{1}{2},2}^{(\ell)} - 4\phi_{1,2}^{(\ell)} + \dots \right] \omega_2(\tilde{y})$$

$$= Z_{T^4}^{\text{graviton}}(q, y, \tilde{y}; \{2\}) + \left[-2\phi_{0,1}^{(\ell)} - 12\phi_{0,2}^{(\ell)} + 5\phi_{\frac{1}{2},2}^{(\ell)} - 4\phi_{1,2}^{(\ell)} + \dots \right] \omega_2(\tilde{y})$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q, y; \{2\}) \equiv \frac{1}{2} (\tilde{y} \partial_{\tilde{y}})^2 Z_{T^4}^{\text{gen}}(q, y, \tilde{y}; \{2\}) \Big|_{\tilde{y}=1} = \Phi_{\frac{1}{2}; 2}^{(s)}(q, y).$$

$$Z_{T^4}^{\text{graviton}}(q, y, \tilde{y}; \{1, 1\}) \equiv \Pi_{m, l, \tilde{m}, \tilde{l}} \left(1 - p q^m y^l \tilde{q}^{\tilde{m}} \tilde{y}^{\tilde{l}} \right)^{-c_{T^4}^{\text{graviton}}(1, m, l, \tilde{m}, \tilde{l})} \Big|_{\substack{p^2 \\ \tilde{q}=1}}$$

$$= \frac{1}{2} (X_1^{T^4}(q, y) \omega_1(\bar{q}, \bar{y}))^2 + \frac{1}{2} X_1^{T^4}(q^2, y^2) \omega_1(\bar{q}^2, \bar{y}^2)$$

$$Z_{T^4\{1,1\}}^{\text{graviton}} = \frac{1}{2}\xi_-^2 \left[\left(\phi_0^{(s)}(q, y) \omega_1(\bar{y}) \right)^2 + \phi_0^{(s)}(q^2, y^2) \omega_1(\bar{y}^2) \right] - 2\xi_- \xi_0 \phi_0^{(s)}(q, y) \phi_{\frac{1}{2}}^{(s)}(q, y) (\omega_1(\bar{y}))^2 \\ + \xi_- \xi_+ \phi_0^{(s)}(q, y) \phi_1^{(s)}(q, y) (\omega_1(\bar{y}))^2 + \xi_0^2 \left[2 \left(\phi_{\frac{1}{2}}^{(s)}(q, y) \omega_1(\bar{y}) \right)^2 - \phi_{\frac{1}{2}}^{(s)}(q^2, y^2) \omega_1(\bar{y}^2) \right] \\ - 2\xi_0 \xi_+ \phi_{\frac{1}{2}}^{(s)}(q \cdot y) \phi_1^{(s)}(q, y) (\omega_1(\bar{y}))^2 + \frac{1}{2} \xi_+^2 \left[\left(\phi_1^{(s)}(q, y) \omega_1(\bar{y}) \right)^2 + \phi_1^{(s)}(q^2, y^2) \omega_1(\bar{y}^2) \right]$$

$$Z_{T^4\{1,1\}}^{\text{graviton}} \Big|_{\xi_-^2} = \frac{1}{2} \left[\left(\phi_0^{(s)}(q, y) \omega_1(\bar{y}) \right)^2 + \phi_0^{(s)}(q^2, y^2) \omega_1(\bar{y}^2) \right] \\ = \phi_0^{(s)}(q, y) \frac{1}{2} [(\omega_1(\bar{y}))^2 + \omega_1(\bar{y}^2)]$$

$$\omega_{\text{sym}}(\tilde{y}) \equiv \frac{1}{2} [(\omega_1(\bar{y}))^2 + \omega_1(\bar{y}^2)] = \tilde{\chi}_{1;2}(\tilde{y}) - 2\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + \tilde{\chi}_{0;2}(\tilde{y})$$

$$\omega_{\text{asym}}(\tilde{y}) \equiv \frac{1}{2} [(\omega_1(\bar{y}))^2 - \omega_1(\bar{y}^2)] = -2\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) + 4\tilde{\chi}_{0;2}(\tilde{y})$$

$$\phi_0^{(s)} \tilde{\chi}_{1;2} : | - \rangle_1^{(1)} | - \rangle_1^{(2)} \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} + |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right) \\ - 2\phi_0^{(s)} \tilde{\chi}_{\frac{1}{2};2} : | - \rangle_1^{(1)} | - \rangle_1^{(2)} \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} + |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right) \\ \phi_0^{(s)} \tilde{\chi}_{0;2} : | - \rangle_1^{(1)} | - \rangle_1^{(2)} \left(|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} - |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} \right)$$

$$Z_{T^4\{1,1\}}^{\text{graviton}} \Big|_{\xi_- \xi_0} = -2\phi_0^{(s)}(q, y) \phi_{\frac{1}{2}}^{(s)}(q, y) (\omega_1(\bar{y}))^2 \\ = -2\phi_{\frac{1}{2}}^{(s)}(q, y) \left(\tilde{\chi}_{1;2} - 4\tilde{\chi}_{\frac{1}{2};2} + 5\tilde{\chi}_{0;2} \right) \\ - 2\phi_{\frac{1}{2}}^{(s)}(\tilde{\chi}_{1;2} + \tilde{\chi}_{0;2}) : \left(| - \rangle_1^{(1)} |\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} \pm |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right), \\ 8\phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{\frac{1}{2};2} : \left(| - \rangle_1^{(1)} |\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} \pm |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right), \\ - 8\phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{0;2} : \left(| - \rangle_1^{(1)} |\dot{A}\rangle_1^{(2)} \pm |\dot{A}\rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} \mp |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} \right).$$

$$-2\xi_- \xi_0 \phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{1;2} \rightarrow \xi_-^2 \phi_0^{(s)} \tilde{\chi}_{1;2}$$

$$4\xi_- \xi_0 \phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{\frac{1}{2};2} \rightarrow -2\xi_-^2 \phi_0^{(s)} \tilde{\chi}_{\frac{1}{2};2}$$

$$-2\xi_- \xi_0 \phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{0;2} \rightarrow \xi_-^2 \phi_0^{(s)} \tilde{\chi}_{0;2}$$

$$Z_{T^4\{1,1\}}^{\text{graviton}} \Big|_{\xi_- \xi_+} = \phi_0^{(s)}(q, y) \phi_1^{(s)}(q, y) (\omega_1(\bar{y}))^2 \\ = \phi_1^{(s)}(q, y) \left(\tilde{\chi}_{1;2} - 4\tilde{\chi}_{\frac{1}{2};2} + 5\tilde{\chi}_{0;2} \right)$$

$$\phi_1^{(s)}(\tilde{\chi}_{1;2} + \tilde{\chi}_{0;2}) : \left(| - \rangle_1^{(1)} | + \rangle_1^{(2)} \pm | + \rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} \pm |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right), \\ - 4\phi_1^{(s)} \tilde{\chi}_{\frac{1}{2};2} : \left(| - \rangle_1^{(1)} | + \rangle_1^{(2)} \pm | + \rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} \pm |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)} \right), \\ 4\phi_1^{(s)} \tilde{\chi}_{0;2} : \left(| - \rangle_1^{(1)} | + \rangle_1^{(2)} \pm | + \rangle_1^{(1)} | - \rangle_1^{(2)} \right) \left(|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} \mp |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} \right).$$



$$\begin{aligned}
Z_{T^4\{1,1\}}^{\text{graviton}} \Big|_{\xi_0^2} &= 2 \left(\phi_{\frac{1}{2}}^{(s)}(q, y) \omega_1(\tilde{y}) \right)^2 - \phi_{\frac{1}{2}}^{(s)}(q^2, y^2) \omega_1(\tilde{y}^2) \\
&= \phi_1^{(s)}(q, y) [\omega_{\text{sym}}(\tilde{y}) + 3\omega_{\text{asym}}(\tilde{y})] + \sum_{\substack{h \geq 1 \\ \text{odd}}} \phi_{0,h}^{(\ell)}(q, y) [3\omega_{\text{sym}}(\tilde{y}) + \omega_{\text{asym}}(\tilde{y})] \\
&\quad + \sum_{\substack{h \geq 2 \\ \text{even}}} \phi_{0,h}^{(\ell)}(q, y) [\omega_{\text{sym}}(\tilde{y}) + 3\omega_{\text{asym}}(\tilde{y})] \\
&\quad \phi_1^{(s)} \tilde{\chi}_{1,2}: \left(|A\rangle_1^{(1)} |B\rangle_1^{(2)} - |B\rangle_1^{(1)} |A\rangle_1^{(2)} \right) \left(|\dot{a}\rangle_1^{(1)} |\dot{b}\rangle_1^{(2)} + |\dot{b}\rangle_1^{(1)} |\dot{a}\rangle_1^{(2)} \right), \\
&\quad 3\phi_1^{(s)} \tilde{\chi}_{0,2}: \left(|A\rangle_1^{(1)} |B\rangle_1^{(2)} + |B\rangle_1^{(1)} |A\rangle_1^{(2)} \right) \left(|\dot{a}\rangle_1^{(1)} |\dot{b}\rangle_1^{(2)} - |\dot{b}\rangle_1^{(1)} |\dot{a}\rangle_1^{(2)} \right), \\
&\quad -8\phi_1^{(s)} \tilde{\chi}_{\frac{1}{2},2}: \left(|A\dot{a}\rangle_1^{(1)} |B\dot{c}\rangle_1^{(2)} \pm |B\dot{c}\rangle_1^{(1)} |A\dot{a}\rangle_1^{(2)} \right), \\
&\quad 10\phi_1^{(s)} \tilde{\chi}_{0,2}: \left(|A\rangle_1^{(1)} |B\rangle_1^{(2)} \pm |B\rangle_1^{(1)} |A\rangle_1^{(2)} \right) \left(|\dot{c}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} \pm |\dot{D}\rangle_1^{(1)} |\dot{c}\rangle_1^{(2)} \right),
\end{aligned}$$

$$Z_{T^4}^{\text{red}}(q, y, \tilde{y}; \{1,1\}) = \phi_0^{(s)}(q, y) \omega_{\text{sym}}(\tilde{y}) - 2\phi_{\frac{1}{2}}^{(s)}(q, y) \omega_{\text{asym}}(\tilde{y})$$

$$Z_{T^4}^{\text{gen}}(q, y, \tilde{y}; \{1,1\}) = \Phi_{0,2}^{(s)}(q, y) \omega_{\text{sym}}(\tilde{y}) - 2\Phi_{\frac{1}{2},2}^{(s)}(q, y) \omega_{\text{asym}}(\tilde{y})$$

$$\begin{aligned}
Z_{T^4\{1,1\}}^{\text{gen}} &\approx Z_{T^4\{1,1\}}^{\text{graviton}} + 6\phi_{0,1}^{(\ell)} \left(2\tilde{\chi}_{0,2} - \tilde{\chi}_{\frac{1}{2},2} \right) - 2\phi_{\frac{1}{2},2}^{(\ell)} \left(17\tilde{\chi}_{0,2} - 10\tilde{\chi}_{\frac{1}{2},2} + \tilde{\chi}_{1,2} \right) \\
&\quad + 2\phi_{0,2}^{(\ell)} \left(47\tilde{\chi}_{0,2} - 31\tilde{\chi}_{\frac{1}{2},2} + 5\tilde{\chi}_{1,2} \right) + 2\phi_{1,2}^{(\ell)} \left(16\tilde{\chi}_{0,2} - 11\tilde{\chi}_{\frac{1}{2},2} + 2\tilde{\chi}_{1,2} \right) + \dots
\end{aligned}$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q, y; \{1,1\}) \equiv \frac{1}{2} (\tilde{y} \partial_{\tilde{y}})^2 Z_{T^4}^{\text{gen}}(q, y, \tilde{y}; \{1,1\}) \Big|_{\tilde{y}=1} = 2\Phi_{0,2}^{(s)}(q, y) + 4\Phi_{\frac{1}{2},2}^{(s)}(q, y)$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q, y; 2) = 2\Phi_{0,2}^{(s)}(q, y) + 5\Phi_{\frac{1}{2},2}^{(s)}(q, y)$$

$$\hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q, y; 2) = 2 + \sqrt{q}(y^{-1} + y) + q(-12 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 39y^{-1} + 39y + y^3) + q^2(-192 + 2y^{-4} - 56y^2 - 56y^{-2} + 2y^4) + \dots$$

$$\hat{\mathcal{E}}_{T^4}^{\text{graviton}}(q, y; 2) = 2 + \sqrt{q}(y^{-1} + y) + q(-4 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 6y^{-1} + 6y + y^3) + q^2(2 + 2y^{-4} - 4y^2 - 4y^{-2} + 2y^4) + \dots$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q, y; 2) = 2 + \sqrt{q}(y^{-1} + y) + q(-12 - 6y^{-2} - 6y^2) + q^{\frac{3}{2}}(y^{-3} + 39y^{-1} + 39y + y^3) + q^2(-150 + 2y^{-4} - 56y^2 - 56y^{-2} + 2y^4) + \dots$$

$$\hat{\mathcal{E}}_{T^4}^{\text{gen}}(q, y; 2) - \hat{\mathcal{E}}_{T^4}^{\text{graviton}}(q, y; 2) \approx -8q + 33q^{\frac{3}{2}}(y^{-1} + y) + q^2(-52y^{-2} - 152 - 52y^2).$$

$$\left(Z_{T^4\{1,1\}}^{\text{gen}} - Z_{T^4\{1,1\}}^{\text{graviton}} \right) \Big|_q = 3\phi_{0,1}^{(\ell)}(q, y) \omega_{\text{asym}}(\tilde{y}) \Big|_q = 3\omega_{\text{asym}}(\tilde{y}) \xrightarrow{\text{MEG}} -6,$$

$$\left(Z_{T^4\{2\}}^{\text{gen}} - Z_{T^4\{2\}}^{\text{graviton}} \right) \Big|_q = -2\phi_{0,1}^{(\ell)}(q, y) \omega_2(\tilde{y}) \Big|_q = -2\omega_2(\tilde{y}) \xrightarrow{\text{MEG}} -2,$$

$$\psi_{\frac{1}{2}}^{-A} | - \rangle_2 \otimes \begin{Bmatrix} |\dot{a}\rangle_2 \\ |\dot{B}\rangle_2 \end{Bmatrix},$$

$$\sigma_{\dot{A}\dot{B}}^i \psi_{-\frac{1}{2}}^{-\dot{A}(\text{T})} \left(|-\rangle_1^{(1)} |\dot{B}\rangle_1^{(2)} - |\dot{B}\rangle_1^{(1)} |-\rangle_1^{(2)} \right) \otimes \begin{cases} (|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} - |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} - |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} + |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)}) \end{cases}$$

$$\psi_{-\frac{1}{2}}^{+\dot{A}} \psi_{-\frac{1}{2}}^{-\dot{B}} |-\rangle_2 \otimes \begin{Bmatrix} |\dot{\alpha}\rangle_2 \\ |\dot{B}\rangle_2 \end{Bmatrix}, J_{-1}^3 |-\rangle_2 \otimes \begin{Bmatrix} |\dot{\alpha}\rangle_2 \\ |\dot{B}\rangle_2 \end{Bmatrix},$$

$$G_{-\frac{3}{2}}^{+\dot{A}(\text{T})} |-\rangle_1^{(1)} |-\rangle_1^{(2)} \otimes \begin{cases} (|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} + |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} + |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} - |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)}) \end{cases},$$

$$\psi_{-\frac{1}{2}}^{-\dot{A}(\text{T})} \psi_{-\frac{1}{2}}^{+\dot{B}(\text{T})} \left(|-\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} - |\dot{C}\rangle_1^{(1)} |-\rangle_1^{(2)} \right) \otimes \begin{cases} (|\dot{\alpha}\rangle_1^{(1)} |\dot{\beta}\rangle_1^{(2)} - |\dot{\beta}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{\alpha}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)} - |\dot{C}\rangle_1^{(1)} |\dot{\alpha}\rangle_1^{(2)}) \\ (|\dot{C}\rangle_1^{(1)} |\dot{D}\rangle_1^{(2)} + |\dot{D}\rangle_1^{(1)} |\dot{C}\rangle_1^{(2)}) \end{cases}$$

$$\hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q,y;2)=2\Phi_{0;2}^{(s)}+5\Phi_{\frac{1}{2};2}^{(s)}-42\Phi_{0,2;2}^{(\ell)}-70\Phi_{0,3;2}^{(\ell)}-324\Phi_{0,4;2}^{(\ell)}+\cdots$$

$$\hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q,y;\{1,1\})=2\Phi_{0;2}^{(s)}+4\Phi_{\frac{1}{2};2}^{(s)}+6\Phi_{0,1;2}^{(\ell)}-14\Phi_{0,2;2}^{(\ell)}+28\Phi_{0,3;2}^{(\ell)}-42\Phi_{0,4;2}^{(\ell)}+\cdots$$

$$\hat{\mathcal{E}}_{\text{NS}}^{\text{CFT}}(q,y;\{2\})=\Phi_{\frac{1}{2};2}^{(s)}-6\Phi_{0,1;2}^{(\ell)}-28\Phi_{0,2;2}^{(\ell)}-98\Phi_{0,3;2}^{(\ell)}-282\Phi_{0,4;2}^{(\ell)}+\cdots$$

$$\begin{array}{c} |+\dot{\alpha}\rangle_a \\ \downarrow J_1^- \\ |-\dot{\alpha}\rangle_a \end{array}$$

$$Z_{K3}^{\text{graviton}}(q, y, \tilde{y}; \{2\}) \equiv \Pi_{m, \ell, \tilde{\ell}} \left(1 - p^2 q^m y^\ell \tilde{y}^{\tilde{\ell}} \right)^{-c_{\text{graviton}}^{K3}(2, m, \ell, \tilde{\ell})}} \Big|_{p^2} = X_2^{K3}(q, y, \tilde{y})$$

$$\begin{aligned} Z_{K3}^{\text{graviton}}(q, y, \tilde{y}; \{1, 1\}) &\equiv \Pi_{m, \ell, \tilde{\ell}} \left(1 - p q^m y^\ell \tilde{y}^{\tilde{\ell}} \right)^{-c_{\text{graviton}}^{K3}(1, m, \ell, \tilde{\ell})}} \Big|_{p^2} \\ &= \frac{1}{2} (X_1^{K3}(q, y, \tilde{y}))^2 + \frac{1}{2} X_1^{K3}(q^2, y^2, \tilde{y}^2) \end{aligned}$$

$$Z_{K3}^{\text{graviton}}(q, y, \tilde{y}, \xi; \{2\}) = \xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, 2}(\tilde{y}) + 20 \xi_0 \phi_1^{(s)}(q, y) \tilde{\chi}_{0, 2}(\tilde{y}) + \xi_+ \phi_{\frac{3}{2}}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, 2}(\tilde{y})$$

$$Z_{K3}^{\text{gen}}(q, y, \tilde{y}; \{2\}) = \Phi_{\frac{1}{2}, 2}^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, 2}(\tilde{y}) + 20 \Phi_{1, 2}^{(s)}(q, y) \tilde{\chi}_{0, 2}(\tilde{y})$$

$$Z_{K3\{1, 1\}}^{\text{graviton}} \equiv Z_{K3}^{\text{graviton}}(q, y, \tilde{y}, \xi; \{1, 1\})$$

$$\begin{aligned} Z_{K3\{1, 1\}}^{\text{graviton}} &= \xi_-^2 \frac{1}{2} \left[\left(\phi_0^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) \right)^2 + \phi_0^{(s)}(q^2, y^2) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}^2) \right] + \xi_- \xi_+ \phi_0^{(s)}(q, y) \phi_1^{(s)}(q, y) \left(\tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) \right)^2 \\ &\quad + \xi_+^2 \frac{1}{2} \left[\left(\phi_1^{(s)}(q, y) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) \right)^2 + \phi_1^{(s)}(q^2, y^2) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}^2) \right] \\ &\quad + 20 \xi_0 \xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) \phi_0^{(s)}(q, y) \tilde{\chi}_{0, 1}(\tilde{y}) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) + 20 \xi_0 \xi_+ \phi_{\frac{1}{2}}^{(s)}(q, y) \phi_1^{(s)}(q, y) \tilde{\chi}_{0, 1}(\tilde{y}) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) \\ &\quad + \xi_0^2 \frac{1}{2} \left[\left(20 \phi_{\frac{1}{2}}^{(s)}(q, y) \tilde{\chi}_{0, 1}(\tilde{y}) \right)^2 + 20 \phi_{\frac{1}{2}}^{(s)}(q^2, y^2) \tilde{\chi}_{0, 1}(\tilde{y}^2) \right] \end{aligned}$$

$$\begin{aligned} Z_{K3\{1, 1\}}^{\text{graviton}} \Big|_{\xi_0} &= 20 \left[\xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) \phi_0^{(s)}(q, y) + \xi_+ \phi_{\frac{1}{2}}^{(s)}(q, y) \phi_1^{(s)}(q, y) \right] \tilde{\chi}_{0, 1}(\tilde{y}) \tilde{\chi}_{\frac{1}{2}, 1}(\tilde{y}) \\ &= 20 \left[\xi_- \phi_{\frac{1}{2}}^{(s)}(q, y) + \xi_+ \left(\phi_{\frac{3}{2}}^{(s)}(q, y) + \sum_{h \geq 1} \phi_{\frac{1}{2}h + \frac{1}{2}}^{(\ell)}(q, y) \right) \right] \tilde{\chi}_{\frac{1}{2}, 2}(\tilde{y}) \end{aligned}$$

$$20 \xi_0 \xi_{\pm} \phi_{\frac{1}{2}}^{(s)} \tilde{\chi}_{\frac{1}{2}, 2} : | \pm \dot{\alpha} \rangle_1^{(1)} | I \rangle_1^{(2)} + | I \rangle_1^{(1)} | \pm \dot{\alpha} \rangle_1^{(2)}.$$

$$\begin{aligned} Z_{K3\{1, 1\}}^{\text{graviton}} \Big|_{(\xi_0)^0} &\supset \xi_-^2 \phi_0^{(s)}(q, y) \tilde{\chi}_{1, 2}(\tilde{y}) + \xi_- \xi_+ \phi_1^{(s)}(q, y) [\tilde{\chi}_{1, 2}(\tilde{y}) + \tilde{\chi}_{0, 2}(\tilde{y})] \\ &\quad + \xi_+^2 \phi_2^{(s)}(q, y) \tilde{\chi}_{1, 2}(\tilde{y}) \end{aligned}$$

$$\begin{aligned} &\xi_-^2 : | - \dot{\alpha} \rangle_1^{(1)} | - \dot{\beta} \rangle_1^{(2)} + | - \dot{\beta} \rangle_1^{(1)} | - \dot{\alpha} \rangle_1^{(2)} \\ \xi_- \xi_+ : &\left(| + \rangle_1^{(1)} | - \rangle_1^{(2)} \pm | - \rangle_1^{(1)} | + \rangle_1^{(2)} \right) \left(| \dot{\alpha} \rangle_1^{(1)} | \dot{\beta} \rangle_1^{(2)} \pm | \dot{\beta} \rangle_1^{(1)} | \dot{\alpha} \rangle_1^{(2)} \right) \\ &\xi_+^2 : | + \dot{\alpha} \rangle_1^{(1)} | + \dot{\beta} \rangle_1^{(2)} + | + \dot{\beta} \rangle_1^{(1)} | + \dot{\alpha} \rangle_1^{(2)} \end{aligned}$$

$$\begin{array}{ccc}
|+\dot{\alpha}\rangle_1^{(1)}|+\dot{\beta}\rangle_1^{(2)}+|+\dot{\beta}\rangle_1^{(1)}|+\dot{\alpha}\rangle_1^{(2)} & & |+\dot{\alpha}\rangle_1^{(1)}|I\rangle_1^{(2)}+|I\rangle_1^{(1)}|+\dot{\alpha}\rangle_1^{(2)} \\
\downarrow J_1^{-(T)} & & \downarrow J_1^{-(T)} \\
(|+\rangle_1^{(1)}|-\rangle_1^{(2)}+|-\rangle_1^{(1)}|+\rangle_1^{(2)}) (|\dot{\alpha}\rangle_1^{(1)}|\dot{\beta}\rangle_1^{(2)}+|\dot{\beta}\rangle_1^{(1)}|\dot{\alpha}\rangle_1^{(2)}) & & \\
\downarrow J_1^{-(T)} & & \\
|-\dot{\alpha}\rangle_1^{(1)}|-\dot{\beta}\rangle_1^{(2)}+|-\dot{\beta}\rangle_1^{(1)}|-\dot{\alpha}\rangle_1^{(2)} & & |-\dot{\alpha}\rangle_1^{(1)}|I\rangle_1^{(2)}+|I\rangle_1^{(1)}|-\dot{\alpha}\rangle_1^{(2)}
\end{array}$$

$$\begin{aligned}
Z_{K3}^{\text{red}}(q,y,\tilde{y},\xi;\{1,1\})= & \xi_-^2\phi_0^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y})+\xi_-\xi_+\phi_1^{(s)}(q,y)\tilde{\chi}_{0;2}(\tilde{y})+20\xi_0\xi_-\phi_{\frac{1}{2}}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) \\
& +\xi_0^2\left[210\phi_1^{(s)}(q,y)+190\sum_{\text{odd}}_{h\geq 1}\phi_{0,h}^{(\ell)}(q,y)+210\sum_{\text{even}}_{h\geq 2}\phi_{0,h}^{(\ell)}(q,y)\right]\tilde{\chi}_{0;2}(\tilde{y})
\end{aligned}$$

$$\begin{aligned}
Z_{K3}^{\text{gen}}(q,y,\tilde{y};\{1,1\})= & \Phi_{0;2}^{(s)}(q,y)\tilde{\chi}_{1;2}(\tilde{y})+\Phi_{1;2}^{(s)}(q,y)\tilde{\chi}_{0;2}(\tilde{y})+20\Phi_{\frac{1}{2};2}^{(s)}(q,y)\tilde{\chi}_{\frac{1}{2};2}(\tilde{y}) \\
& +\left[210\Phi_{1;2}^{(s)}(q,y)+190\sum_{\text{odd}}_{h\geq 1}\Phi_{0,h;2}^{(\ell)}(q,y)+210\sum_{\text{even}}_{h\geq 2}\Phi_{0,h;2}^{(\ell)}(q,y)\right]\tilde{\chi}_{0;2}(\tilde{y})
\end{aligned}$$

$$\mathcal{E}_{K3}^{\text{gen}}(q,y;2)=3\Phi_{0;2}^{(s)}(q,y)+42\Phi_{\frac{1}{2};2}^{(s)}(q,y)+231\Phi_{1;2}^{(s)}(q,y)+\sum_{h\geq 1}n_h^{(\ell)}\Phi_{0,h;2}^{(\ell)}(q,y)$$

$$\mathcal{E}_{K3}^{\text{gen}}(q,y;2)-\mathcal{E}_{K3}^{\text{graviton}}(q,y;2)=2\phi_{\frac{1}{2};2}^{(\ell)}+231\phi_{0,2}^{(\ell)}+420\phi_{1,2}^{(\ell)}+\cdots$$

$$\mathcal{E}_{K3}^{\text{gen}}(q,y;\{1,1\})-\mathcal{E}_{K3}^{\text{graviton}}(q,y;\{1,1\})=211\phi_{0,2}^{(\ell)}+400\phi_{1,2}^{(\ell)}+\cdots$$

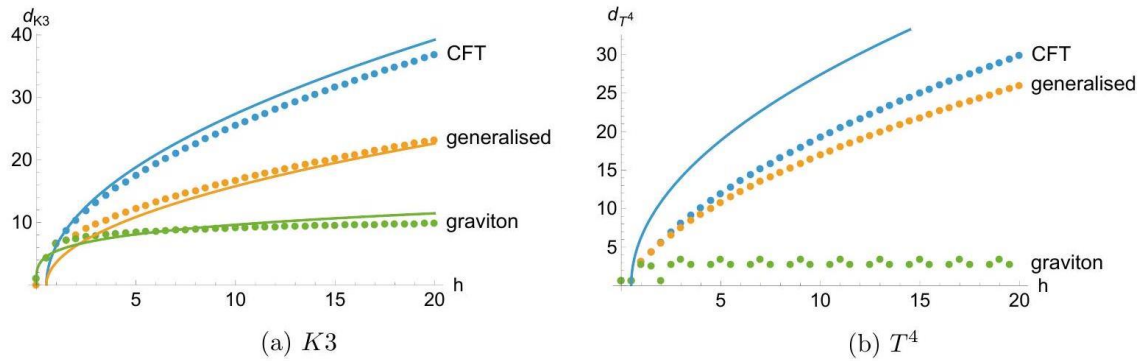
$$\mathcal{E}_{K3}^{\text{gen}}(q,y;\{2\})-\mathcal{E}_{K3}^{\text{graviton}}(q,y;\{2\})=2\phi_{\frac{1}{2};2}^{(\ell)}+20\phi_{0,2}^{(\ell)}+20\phi_{1,2}^{(\ell)}+\cdots$$

$$J_{-1}^3|-\dot{\alpha}\rangle_2$$

$$\begin{aligned}
\mathcal{E}_{\text{NS}}^{\text{CFT}}(q,y;2)\approx & 3+\sqrt{q}(42y^{-1}+42y)+q(234y^{-2}+360+234y^2) \\
& +q^{\frac{3}{2}}(42y^{-3}-3114y^{-1}-3114y+42y^3)+\cdots
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{K3}^{\text{graviton}}(q,y;2)\approx & 3+\sqrt{q}(42y^{-1}+42y)+q(234y^{-2}+340+234y^2) \\
& +q^{\frac{3}{2}}(42y^{-3}-724y^{-1}-724y+42y^3)+\cdots
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{K3}^{\text{gen}}(q,y;2)\approx & 3+\sqrt{q}(42y^{-1}+42y)+q(234y^{-2}+340+234y^2) \\
& +q^{\frac{3}{2}}(42y^{-3}-722y^{-1}-722y+42y^3)+\cdots
\end{aligned}$$



$$T(z) = \frac{1}{2} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \partial X^{\dot{A}\dot{A}}(z) \partial X^{\dot{B}\dot{B}}(z) + \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\dot{A}\dot{B}} \psi^{\alpha\dot{A}}(z) \psi^{\beta\dot{B}}(z)$$

$$J^a(z) = \frac{1}{4} \epsilon_{\alpha\beta} \epsilon_{\dot{A}\dot{B}} (\sigma^{aT})^\beta{}_\gamma \psi^{\alpha\dot{A}}(z) \psi^{\gamma\dot{B}}(z)$$

$$G^{\alpha A}(z) = \epsilon_{\dot{A}\dot{B}} \psi^{\alpha\dot{A}}(z) \partial X^{\dot{B}A}(z)$$

$$(\sigma^1)^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (\sigma^2)^\alpha{}_\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, (\sigma^3)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{O}(z) = \sum_n \mathcal{O}_n z^{-h-n}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0},$$

$$[J_m^a, J_n^b] = \frac{c}{12} m \delta^{ab} \delta_{m+n,0} + i \epsilon_c^{ab} J_{m+n}^c,$$

$$\{G_r^{\alpha A}, G_s^{\beta B}\} = \epsilon^{AB} \left[\epsilon^{\alpha\beta} \frac{c}{6} \left(\frac{1}{4} - r^2 \right) \delta_{r+s,0} + (\sigma^{aT})^\alpha{}_\gamma \epsilon^{\beta\gamma} (r-s) J_{r+s}^a - \epsilon^{\alpha\beta} L_{r+s} \right],$$

$$[J_m^a, G_r^{\alpha A}] = \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta G_{m+r}^{\beta A},$$

$$[L_m, G_r^{\alpha A}] = \left(\frac{m}{2} - r \right) G_{m+r}^{\alpha A},$$

$$[L_m, J_n^a] = -n J_{m+n}^a.$$

$$[\alpha_n^{\dot{A}\dot{A}}, \alpha_m^{\dot{B}\dot{B}}] = n \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} \delta_{n+m,0}$$

$$\{\psi_r^{\alpha\dot{A}}, \psi_s^{\beta\dot{B}}\} = -\epsilon^{\alpha\beta} \epsilon^{\dot{A}\dot{B}} \delta_{r+s,0}$$

$$[L_n, \alpha_m^{\dot{A}\dot{A}}] = -m \alpha_{n+m}^{\dot{A}\dot{A}}$$

$$[L_n, \psi_r^{\alpha\dot{A}}] = -\left(\frac{n}{2} + r\right) \psi_{n+r}^{\alpha\dot{A}}$$

$$[J_n^a, \alpha_m^{\dot{A}\dot{A}}] = 0$$

$$[J_n^a, \psi_r^{\alpha\dot{A}}] = \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta \psi_{n+r}^{\beta\dot{A}}$$

$$[G_r^{\alpha A}, \alpha_m^{\dot{B}\dot{B}}] = -m \epsilon^{AB} \psi_{r+m}^{\alpha\dot{B}}$$

$$\{G_r^{\alpha A}, \psi_s^{\beta\dot{A}}\} = \epsilon^{\alpha\beta} \alpha_{r+s}^{\dot{A}\dot{A}}$$

$$J_n^\pm = J_n^1 \pm i J_n^2$$

$$[J_n^+, J_m^-] = n \frac{c}{6} \delta_{n+m,0} + 2 J_{n+m}^3$$

$$[J_n^3, J_m^\pm] = \pm J_{n+m}^\pm$$

$$[L_n, J_m^\pm] = -m J_{n+m}^\pm$$

$$[J_n^\pm, G_r^{\mp A}] = G_{n+r}^{\pm A}$$

$$[J_n^\pm, \psi_r^{\mp \dot{A}}] = \psi_{n+r}^{\pm \dot{A}}$$



$$\vartheta(\nu,\tau)=\sum_{n=-\infty}^{\infty}e^{2\pi i\nu\nu}e^{\pi i n^2\tau}$$

$$\vartheta_{\alpha,\beta}(\nu,\tau)\equiv e^{\pi i\alpha^2\tau}e^{2\pi i\alpha(\nu+\beta)}\vartheta(\nu+\alpha\tau+\beta,\tau)$$

$$\vartheta_1(\nu,\tau)\equiv -\vartheta_{\frac{1}{2},\frac{1}{2}}(\nu,\tau)$$

$$\vartheta_2(\nu,\tau)\equiv \vartheta_{\frac{1}{2},0}(\nu,\tau)$$

$$\vartheta_3(\nu,\tau)\equiv \vartheta_{0,0}(\nu,\tau)$$

$$\vartheta_4(\nu,\tau)\equiv \vartheta_{0,\frac{1}{2}}(\nu,\tau)$$

$$\eta(q)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\tilde{\chi}_{\tilde{J};N}(\tilde{y})=\tilde{y}^N\frac{\tilde{y}^{2\tilde{J}+1}-\tilde{y}^{-2\tilde{J}-1}}{\tilde{y}-\tilde{y}^{-1}},$$

$$\Phi_{j;N}^{(s)}(q,y) \, = F(q,y)\varphi_{j;N}^{(s)}(q,y)$$

$$\Phi_{j,h;N}^{(\ell)}(q,y) \, = F(q,y)\varphi_{j,h;N}^{(\ell)}(q,y)$$

$$F(q,y)=\prod_{k=1}^{\infty}\frac{(1-yq^{k-1/2})^2(1-y^{-1}q^{k-1/2})^2}{(1-q^k)^2(1-y^2q^k)(1-y^{-2}q^{k-1})}=\frac{-iq^{1/4}\vartheta_4(\nu,\tau)^2}{\vartheta_1(2\nu,\tau)\eta(q)^3}$$

$$\varphi_{j;N}^{(s)}(q,y) \, = q^j \sum_{k \in \mathbb{Z}} \left[\frac{y^{2(N+1)k+2j}}{(1-yq^{k+1/2})^2} - \frac{y^{-2(N+1)k-2(j+1)}}{(1-y^{-1}q^{k+1/2})^2} \right] q^{(N+1)k^2+(2j+1)k}$$

$$\varphi_{j,h;N}^{(\ell)}(q,y) \, = q^h \sum_{k \in \mathbb{Z}} \left(y^{2(N+1)k+2j} - y^{-2(N+1)k-2(j+1)} \right) q^{(N+1)k^2+(2j+1)k}$$

$$\Phi_{j;N}^{(s)}(q,y)=\frac{q^{\frac{1}{8}}}{\eta(q)^3}\Phi_{0,0;1}^{(\ell)}(q,y)\Phi_{j;N-1}^{(s)}(q,y)$$

$$\Phi_{j,h;N}^{(\ell)}(q,y)=\frac{q^{\frac{1}{8}}}{\eta(q)^3}\Phi_{0,0;1}^{(\ell)}(q,y)\Phi_{j,h;N-1}^{(\ell)}(q,y)$$

$$\phi_j^{(s)}(q,y)=\frac{q^j}{1-q}\frac{y^{2j}\Big(y-2q^{\frac{1}{2}}+y^{-1}q\Big)-y^{-2j}\Big(y^{-1}-2q^{\frac{1}{2}}+yq\Big)}{y-y^{-1}},$$

$$\phi_{j,h}^{(\ell)}(q,y)=\phi_{j,h}^{(s)}(q,y)-2\phi_{j+\frac{1}{2},h+\frac{1}{2}}^{(s)}(q,y)+\phi_{j+1,h+1}^{(s)}(q,y)$$

$$=\frac{q^h}{1-q}\frac{(y^{2j+1}-y^{-2j-1})\Big(y^{-1}-2q^{\frac{1}{2}}+qy\Big)\Big(y-2q^{\frac{1}{2}}+qy^{-1}\Big)}{y-y^{-1}}$$

$$\phi_{j_1}^{(s)}(q,y)\times\phi_{j_2}^{(s)}(q,y)=\phi_{j_1+j_2}^{(s)}(q,y)+\sum_{h\geq j_1+j_2}\sum_{j=|j_1-j_2|}^{j_1+j_2-1}\phi_{j,h}^{(\ell)}(q,y).$$

$$\phi_j^{(s)}(q^2,y^2)=\phi_{2j}^{(s)}(q,y)+\sum_{h\geq 2j}\sum_{j'=0}^{2j-1}(-1)^{h+j'}\phi_{j',h}^{(\ell)}(q,y).$$

$$\tilde{\chi}_{j_1;N_1}(\bar{y})\times\tilde{\chi}_{j_2;N_2}(\bar{y})=\sum_{j=|j_1-j_2|}^{j_1+j_2}\tilde{\chi}_{j;N_1+N_2}(\bar{y}),$$

$$\tilde{\chi}_{j_1;N}(\bar{y}^2)=\sum_{j=0}^{2j_1}(-1)^{2j_1-j}\tilde{\chi}_{j;2N}(\bar{y})$$



$$\begin{aligned}
&+12q: \sigma_{\dot{A}\dot{B}}^i|-\dot{\alpha}\rangle_1\psi_{-\frac{1}{2}}^{-\dot{A}}|\dot{B}\dot{\beta}\rangle_1 \quad (0; \mathbf{1}, \mathbf{3}) + 2(1; \mathbf{1}, \mathbf{3}) + (2; \mathbf{1}, \mathbf{3}) \\
&-192q: \sigma_{\dot{A}\dot{B}}^i|-\dot{\alpha}\rangle_1\alpha_{-\frac{1}{2}}^{\dot{A}\dot{A}}\psi_0^{-\dot{B}}|r\rangle_1 \quad \left(\frac{1}{2}; \mathbf{2}, \mathbf{3}\right)_r + \left(\frac{3}{2}; \mathbf{2}, \mathbf{3}\right)_r \\
&+4q: \epsilon_{\dot{C}\dot{D}}\psi_0^{-\dot{C}}\psi_0^{-\dot{D}}|\dot{A}\dot{B}\rangle_2 + f\psi_{-\frac{1}{2}}^{-\dot{A}}|-\dot{B}\rangle_2 \quad \left(\frac{1}{2}; \mathbf{1}, \mathbf{1}\right) + \left(\frac{1}{2}; \mathbf{1}, \mathbf{3}\right) \\
&-12q: \sigma_{\dot{A}\dot{B}}^i\alpha_{-\frac{1}{2}}^{\dot{A}\dot{A}}\psi_0^{-\dot{B}}|-\dot{\alpha}\rangle_2 \quad (0; \mathbf{2}, \mathbf{3}) + (1; \mathbf{2}, \mathbf{3}) \\
&+160q: \alpha_{-\frac{1}{4}}^{\dot{A}\dot{A}}\alpha_{-\frac{1}{4}}^{\dot{B}\dot{B}}J_{\frac{1}{2}}^-|r\rangle_2 \quad (1, \mathbf{3}, \mathbf{3})_r + \left(\frac{1}{2}; \mathbf{1}, \mathbf{1}\right)_r \\
&+48q: \sigma_{\dot{A}\dot{B}}^i\psi_{-\frac{1}{4}}^{-\dot{A}}\psi_{\frac{1}{4}}^{-\dot{B}}|r\rangle_2 \quad \left(\frac{1}{2}; \mathbf{1}, \mathbf{3}\right)_r
\end{aligned}$$

$$(\tilde{J}; \mathbf{d}_1, \mathbf{d}_2), \left(\tilde{J} + \frac{1}{2}; \mathbf{d}_1 \otimes \mathbf{2}, \mathbf{d}_2\right), (\tilde{J} + 1; \mathbf{d}_1, \mathbf{d}_2).$$

$$(a\cdot\rho)R\wedge R+\sum_A\,(b_A\cdot\rho)F^A\wedge F^A$$

$$H-V=273-29T$$

$$y^2=x^3+fx+g$$

$$-12[K] = [\Delta] = \sum_A \, v_A [S_A] + [Y]$$

$$T=h^{1,1}(B)-1$$

$$r=h^{1,1}(\hat{X})-h^{1,1}(B)-1$$

$$H_{\mathrm{neutral}}=h^{2,1}(\hat{X})+1$$

$$\chi(\hat{X})=2(h^{1,1}(X)-h^{2,1}(X))=2(r+T-H_{\mathrm{neutral}}+3)$$

$$(K\cdot B)\wedge R\wedge R, (S_A\cdot B)\wedge F^A\wedge F^A$$

$$\begin{aligned}
S^{(6)} = & -\int \frac{1}{2} \left(j^\alpha \Omega_{\alpha\beta} a^\beta \right) \mathrm{tr} R \wedge * R + \frac{2}{\lambda_A} \left(j^\alpha \Omega_{\alpha\beta} b_A^\beta \right) \mathrm{tr} F^A \wedge * F^A \\
& + \frac{1}{4} \left(B^\alpha \Omega_{\alpha\beta} a^\beta \right) \wedge \mathrm{tr} R \wedge R + \frac{1}{\lambda_A} \left(B^\alpha \Omega_{\alpha\beta} b_A^\beta \right) \wedge \mathrm{tr} F^A \wedge F^A \\
& + \frac{1}{4} G_{\alpha\beta} H_3^\alpha \wedge * H_3^\beta + \frac{1}{2} G_{\alpha\beta} d j^\alpha \wedge * d j^\beta
\end{aligned}$$

$$H_3^\alpha=dB_2^\alpha+\frac{1}{2}a^\alpha w_{\mathrm{CS}}(R)+2\frac{b_A^\alpha}{\lambda_A}w_{\mathrm{CS}}^A(F)$$

$$\begin{aligned}
w_{\mathrm{CS}}(R) &= \mathrm{tr}\left(\hat{\omega}\wedge d\hat{\omega}+\frac{3}{2}\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega}\right) \\
w_{\mathrm{CS}}^B(F) &= \mathrm{tr}\left(A^B\wedge dA^B+\frac{3}{2}A^B\wedge A^B\wedge A^B\right)
\end{aligned}$$

$$\Omega_{\alpha\beta} * H_3^\beta = G_{\alpha\beta} H_3^\beta$$

$$S_{\mathrm{GS}}^{(6)}=-\frac{1}{2}\int \Omega_{\alpha\beta}B_2^\alpha\wedge X_4^\beta$$

$$X_4^\alpha=\frac{1}{2}a^\alpha\mathrm{tr}R\wedge R+\sum_A\,b_A^\alpha\left(\frac{2}{\lambda_A}\mathrm{tr}F^A\wedge F^A\right)$$



$$a^\alpha = K^\alpha, b_A^\alpha = C_A^\alpha$$

$$[K]=K^\alpha\omega_\alpha, [S_A]=C_A^\alpha\omega_\alpha$$

$$\int_{M_6\times S_A} C_4\wedge\Big(\mathrm{tr}(\hat{F}^A)^2-\frac{1}{48}\mathrm{tr}\hat{R}^2\Big),$$

$$x\cdot y=x^\alpha\Omega_{\alpha\beta}y^\beta$$

$$\Gamma=H_2(B,\mathbb{Z})$$

$$\int_{M_6\times S_A} \mathrm{Tr}\big(\hat{F}^A\wedge*_8\hat{F}^A\big)=\int_{M_6} \mathrm{Tr}(F^A\wedge*F^A)\cdot\frac{\int_{S_A}J_{\mathrm{b}}}{\mathcal{V}_{\mathrm{b}}^{1/2}}$$

$$j^\alpha=\frac{v_{\rm b}^\alpha}{(2\mathcal{V}_{\rm b})^{1/2}}, j^\alpha\Omega_{\alpha\beta}j^\beta=1$$

$$G_{\alpha\beta}=-\Omega_{\alpha\beta}+\mathcal{O}(j^2)$$

$$\omega_f=[\Sigma+(m/2)F], \omega_b=[F]$$

$$\Omega_{\alpha\beta}=\int_{B_2}\omega_\alpha\wedge\omega_\beta, \Omega=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$-[K]=[2\Sigma+(2+m)F]=2\omega_f+2\omega_b, a=(K^\alpha)=(-2,-2)$$

$$[S]=p\omega_f+(q-pm/2)\omega_b, b=(C^\alpha)=(p,q-pm/2)$$

$$S_{\rm GS}^{(6)}=\frac{1}{2}\int\,\left(B_f+B_b\right)\wedge\mathrm{tr}R^2-\left(pB_b+(q-pm/2)B_f\right)\wedge\frac{2}{\lambda}\mathrm{tr}F^2$$

$$d\hat{H}=\frac{2}{\lambda}\mathrm{Tr}\hat{F}^2-\mathrm{tr}\hat{R}^2, \hat{H}=d\hat{B}+\frac{2}{\lambda}w_{\mathrm{CS}}^B(F)-w_{\mathrm{CS}}(R)$$

$$S_B^{(10)}=\int\,\,-\frac{1}{4}e^{-2\phi}\hat{H}\wedge*\hat{H}-\frac{1}{4}e^{2\phi}\hat{H}_7\wedge*\hat{H}_7-\frac{1}{3}\hat{B}\wedge\hat{X}_8-\frac{1}{2}\hat{B}_6\wedge\hat{X}_4$$

$$\hat{X}_8=\frac{1}{24}\mathrm{Tr}\hat{F}^4-\frac{1}{7200}\big(\mathrm{Tr}\hat{F}^2\big)^2-\frac{1}{240}\mathrm{Tr}\hat{F}^2\mathrm{tr}\hat{R}^2+\frac{1}{8}\big(\mathrm{tr}\hat{R}^4\big)+\frac{1}{32}\big(\mathrm{tr}\hat{R}^2\big)^2\\ \hat{X}_4=\frac{2}{\lambda}\mathrm{tr}\hat{F}^2-\mathrm{tr}\hat{R}^2$$

$$S_{\rm GS}^{(6)}(B_1)=\frac{1}{2}\int\,B_1\wedge\Big(\mathrm{tr}R^2-\frac{2}{\lambda}\mathrm{tr}F^2\Big)$$

$$\frac{2}{\lambda}\int_{\mathrm{K}3}\,\,\mathrm{tr}\mathcal{F}^2=\int_{\mathrm{K}3}\,\,\mathrm{tr}\mathcal{R}^2=24$$

$$\mathrm{Tr}\hat{F}^2=30\mathrm{tr}\hat{F}^2,\mathrm{Tr}\hat{F}^4=24\mathrm{tr}\hat{F}^4+3\big(\mathrm{tr}\hat{F}^2\big)^2$$

$$\hat{X}_8^{SO(32)}=\int_{\mathrm{K}3}\,\,\Big(-\frac{1}{8}\mathrm{tr}F^2\mathrm{tr}\mathcal{R}^2-\frac{1}{8}\mathrm{tr}R^2\mathrm{tr}\mathcal{F}^2+\frac{1}{16}\mathrm{tr}R^2\mathrm{tr}\mathcal{R}^2\Big).$$

$$S_{SO(32)}^{(6)}=\frac{1}{2}\int\,\left(B_1+B_0\right)\wedge\mathrm{tr}R^2-\left(B_1-2B_0\right)\wedge\mathrm{tr}F^2$$



$$a=(-2,-2), b=(1,-2)$$

$$(B_f,B_b)\leftrightarrow (B_0,B_1).$$

$$\mathrm{Tr}\hat{F}^2=\mathrm{tr}\hat{F}^2,\mathrm{Tr}\hat{F}^4=\frac{1}{100}\mathrm{tr}\hat{F}^4$$

$$\begin{aligned}\hat{X}_8^{E_8} = \int_{K_3} \Big(& \frac{1}{3600} [2\mathrm{tr} F_1^2 \mathrm{tr} \mathcal{F}_1^2 + 2\mathrm{tr} F_2^2 \mathrm{tr} \mathcal{F}_2^2 - \mathrm{tr} F_2^2 \mathrm{tr} \mathcal{F}_1^2 - \mathrm{tr} F_1^2 \mathrm{tr} \mathcal{F}_2^2] \\ & - \frac{1}{240} [\mathrm{tr} F_1^2 \mathrm{tr} \mathcal{R}^2 + \mathrm{tr} R^2 \mathrm{tr} \mathcal{F}_1^2 + \mathrm{tr} F_2^2 \mathrm{tr} \mathcal{R}^2 + \mathrm{tr} R^2 \mathrm{tr} \mathcal{F}_2^2] \\ & + \frac{1}{16} [\mathrm{tr} R^2 \mathrm{tr} \mathcal{R}^2] \Big)\end{aligned}$$

$$S_{E_8}^{(6)}=\frac{1}{2}\int\;(B_1+B_0)\wedge\mathrm{tr}R^2-\frac{1}{30}\Big(B_1-\frac{k}{2}B_0\Big)\wedge\mathrm{tr}F_1^2-\frac{1}{30}\Big(B_1+\frac{k}{2}B_0\Big)\wedge\mathrm{tr}F_2^2$$

$$a=(-2,-2), b_1=(1,-k/2), b_2=(1,k/2)$$

$$J_{\rm b}=v_{\rm b}^{\alpha}\omega_{\alpha}, C_4=B_2^{\alpha}\wedge\omega_{\alpha}$$

$$\begin{aligned}C_{\rm cs} &= h^{3,1}(\hat{X}) - 1 \\ C_{\rm sa} &= h^{1,1}(\mathcal{B}_3) + 1 \\ C_{21} &= h^{2,1}(\hat{X}) - h^{2,1}(\mathcal{B}_3)\end{aligned}$$

$$T_\alpha=\frac{1}{2}\kappa_{\alpha\beta\gamma}v_{\rm b}^\beta v_{\rm b}^\gamma+\frac{1}{4}d_{\alpha ab}(N+\bar{N})^a(N+\bar{N})^b+i\rho_\alpha$$

$$N^a\rightarrow N^a+i\Lambda^a, T_\alpha\rightarrow T_\alpha+i\Lambda_\alpha.$$

$$r_{\rm v}=h^{1,1}(\hat{X})-h^{1,1}(\mathcal{B}_3)-1$$

$$f_A=\frac{1}{2}C_A^\alpha T_\alpha, [S_A]=C_A^\alpha \omega_\alpha$$

$$r_{21}=h^{2,1}(\mathcal{B}_3)$$

$$\tau_{\kappa\lambda}(z)=\frac{i}{2}\big(\int_B\beta^\mu\wedge\bar\psi^\kappa\big)^{-1}\int_B\alpha_\lambda\wedge\bar\psi^\mu$$

$$\Theta_{\Sigma\Lambda}=\int_{\hat{X}}\omega_{\Sigma}\wedge\omega_{\Lambda}\wedge G_4$$

$$DT_\alpha=dT_\alpha+i\Theta_{\alpha i_B}A^{i_B}$$

$$DT_\alpha=dT_\alpha+i\mathrm{Tr}(\underline{\Theta}_{\alpha B}A^B)$$

$$\Theta_{i_Aj_B}=\int_{\hat{X}}\omega_{i_A}\wedge\omega_{j_B}\wedge G_4$$

$$\frac{\chi(\hat{X})}{24}=\frac{1}{2}\int\;G_4\wedge G_4+N_3$$

$$h^{2,2}(X)=44+4h^{1,1}(X)+2h^{2,1}(X)-4h^{3,1}(X)$$

$$\begin{aligned}\chi(\hat{X})&=6(8+h^{1,1}(\hat{X})+h^{3,1}(\hat{X})-h^{2,1}(\hat{X}))\\&=6(9+C_{\rm sa}+C_{\rm cs}+r_{\rm v}-(C_{21}+r_{21}))\end{aligned}$$



$$\Delta C_{\rm cs}=3N_3, \Delta r_{\rm v}=N_3$$

$$S_{ax}^{(4)}=\frac{1}{8}\int\;\frac{1}{2}a^{\mathcal{A}}\rho_{\mathcal{A}}\mathrm{tr}R\wedge R+\frac{2}{\lambda_A}b_A^{\mathcal{A}}\rho_{\mathcal{A}}\mathrm{tr}F^A\wedge F^A$$

$$b_A^\alpha = C_A^\alpha, [S_A] = C_A^\alpha \omega_\alpha$$

$$a^\alpha = K^\alpha, c_1(\mathcal{B}_3) = -K^\alpha \omega_\alpha$$

$$S_{\mathrm{D}7}^{CS}(R,F) \!=\! -\frac{1}{2}\!\int_{M^{3,1}} C_{\hat{A}}^{\alpha}\rho_{\alpha}\!\left(\frac{1}{96}\mathrm{tr}(R\wedge R)N_{\mathrm{D}7}^{\hat{A}}\!+\!\frac{1}{2}\mathrm{tr}(F^{\hat{A}}\wedge F^{\hat{A}})\right)\!+\!\cdots$$

$$S_{07}^{CS}(R)=+2\int_{M^{3,1}}\tilde{C}^{\alpha}\rho_{\alpha}\left(-\frac{1}{192}\mathrm{tr}(R\wedge R)\right)+\cdots$$

$$-12[K] = [\Delta] = 2[S_{07}] + N_{\mathrm{D}7}^{\hat{A}}[S_{\hat{A}}] = \left(2\tilde{C}^\alpha + N_{\mathrm{D}7}^{\hat{A}}C_{\hat{A}}^\alpha\right)\omega_\alpha$$

$$T_0=-i\tau_{\mathrm{IIB}}=e^{-\phi}-iC_0$$

$$\frac{1}{2}\tilde{a}(z)\mathrm{tr}R\wedge R+\frac{2}{\lambda_A}\tilde{b}_A(z)\mathrm{tr}F^A\wedge F^A$$

$$H_3^\alpha=dB_2^\alpha+\frac{1}{2}a^\alpha w_{\rm CS}(R)+2\frac{b_A^\alpha}{\lambda_A}w_{\rm CS}^A(F)$$

$$\tilde{G}_{\alpha\beta}\propto\kappa_{\alpha\beta\gamma}L^\gamma+\cdots,$$

$$r=c_1(\mathcal{O}(1))=[\Sigma], t=c_1(\mathcal{L})$$

$$r(r+t)=0$$

$$\begin{aligned} c_1(\mathcal{B}_3) &= c_1(\mathcal{B}_2) + (2r+t) \\ c_2(\mathcal{B}_3) &= c_2(\mathcal{B}_2) + c_1(\mathcal{B}_2)(2r+t) \end{aligned}$$

$$\omega_f=r+\frac{t}{2},\omega_i$$

$$\kappa_{ijk}=0, \kappa_{fij}=\kappa_{ij}, \kappa_{ffi}=0, \kappa_{fff}=\frac{1}{4}t^it^j\kappa_{ij}$$

$$(a_F^\alpha)\equiv (K^\alpha)=(-2,K^i)$$

$$[S]=p\omega_f+\sum_i\;q_i\omega_i$$

$$t=k[H]$$

$$c_1(\tilde{\mathbb{F}}_k)=2[\Sigma]+(3+k)[H], c_2(\tilde{\mathbb{F}}_k)=6[\Sigma\cdot H]+(3+3k)[H]^2$$

$$\Sigma H^2=1, H^3=0, \Sigma^2 H=-k, \Sigma^3=k^2,$$

$$\hat{B}=b_2+\chi^l\omega_l=b_2+\chi_f\omega_B+\sum_i\;\chi^i\omega_i$$

$$c_l-\lambda_l=0$$



$$c_I = \int_{Z_3} \omega_I \wedge \text{tr} \mathcal{R}^2 = \int_{Z_3} \omega_I \wedge c_2(Z_3), \lambda_I = \frac{2}{\lambda} \int_{Z_3} \omega_I \wedge \text{Tr} \mathcal{F}^2$$

$$c_2(Z_3) = 11c_1(\mathcal{B}_2)^2 + c_2(\mathcal{B}_2) + 12c_1(\mathcal{B}_2) \wedge \omega_{\mathcal{B}},$$

$$c_i=-12\kappa_{ij}K^j$$

$$c_f=\int_{\mathcal{B}_2}c_2(Z_3)=\int_{\mathcal{B}_2}c_2(\mathcal{B}_2)-c_1(\mathcal{B}_2)^2=-8+2h^{1,1}(\mathcal{B}_2)$$

$$\chi_0\left(\text{tr}R^2-\frac{2}{\lambda}\text{tr}F^2\right)$$

$$\begin{aligned} S_{SO(32)}^{(4)} &= \frac{1}{2} \int \left(\chi_0 + \frac{1}{24} \chi^I c_I \right) \text{tr} R^2 - \left(\chi_0 - \frac{1}{12} \chi^I c_I \right) \text{tr} F^2 \\ &= \frac{1}{2} \int \left(\chi_0 - \frac{1}{2} \chi^i \kappa_{ij} K^j + \frac{1}{24} \chi_f c_f \right) \text{tr} R^2 \\ &\quad - \left(\chi_0 + \chi^i \kappa_{ij} K^j - \frac{1}{12} \chi_f c_f \right) \text{tr} F^2 \end{aligned}$$

$$\tilde{\chi}_0=8\chi_0,\tilde{\chi}_i=8\kappa_{ij}\chi^j,\tilde{\chi}_f=8\chi_f$$

$$a=\left(-2,K^i,-\frac{1}{12}c_f\right),b=\left(1,K^i,-\frac{1}{12}c_f\right)$$

$$\frac{1}{30}\text{tr}\mathcal{F}_i^2=\eta_i\wedge\omega_{\mathcal{B}}+\zeta_i$$

$$\text{tr}\mathcal{F}_1^2+\text{tr}\mathcal{F}_2^2=30\text{tr}\mathcal{R}^2$$

$$\eta_1+\eta_2\,=\,12c_1(\mathcal{B}_2)$$

$$\zeta_1+\zeta_2\,=\,11c_1(\mathcal{B}_2)^2+c_2(\mathcal{B}_2)\equiv\mathcal{C}_2$$

$$\eta_1\,=\,6c_1(\mathcal{B}_2)+\tilde{t},\eta_2\,=\,6c_1(\mathcal{B}_2)-\tilde{t}$$

$$\zeta_1\,=\,\frac{1}{2}\mathcal{C}_2+\Phi,\zeta_2\,=\,\frac{1}{2}\mathcal{C}_2-\Phi$$

$$S_{E_8}^{(4)}=\frac{1}{2}\int\left(\chi_0+\frac{1}{24}\chi^Ic_I\right)\text{tr}R^2-\left(\chi_0-\frac{1}{12}\chi^I\tilde{c}_I\right)\text{tr}F_1^2-\left(\chi_0+\frac{1}{12}\chi^I\tilde{c}_I\right)\text{tr}F_2^2$$

$$\tilde{c}_I=\int_{Z_3}\omega_I\wedge(\tilde{t}\wedge\omega_{\mathcal{B}}+\Phi)=(-1)^i\int_{Z_3}\omega_I\wedge\left(\frac{1}{2}\text{tr}\mathcal{R}^2-\frac{1}{30}\text{tr}\mathcal{F}_i\right)$$

$$\tilde{c}_i=\kappa_{ij}\tilde{t}^j,\tilde{c}_f=\int_{\mathcal{B}_2}\Phi-\tilde{t}^2$$

$$a=\left(-2,K^i,-\frac{1}{12}c_f\right),\qquad\begin{aligned}b_1&=\left(1,-\frac{1}{2}\tilde{t}^i,-\frac{1}{12}\tilde{c}_f\right)\\b_2&=\left(1,\frac{1}{2}\tilde{t}^i,\frac{1}{12}\tilde{c}_f\right)\end{aligned}$$

$$\rho_0\leftrightarrow\tilde{\chi}_f$$

$$\rho_b\leftrightarrow\tilde{\chi}_0$$

$$\rho_i\leftrightarrow\tilde{\chi}_i$$

$$t^i\leftrightarrow\tilde{t}^i,$$

$$\eta_1=8c_1(\mathcal{B}_2),\eta_2=4c_1(\mathcal{B}_2)$$



$$\begin{array}{l}a\cdot a=9-T\\273=29T+H\end{array}$$

$$K\cdot K=10-h^{1,1}(B)$$

$$\chi_0(B)=\frac{1}{12}\int_B\;(c_1^2+c_2)$$

$$\chi(B)=\int_B\;c_2=\sum_i\;(-1)^ib^i=2+h^{1,1}$$

$$n_{U(1)}=h^{1,1}(X)-h^{1,1}(B)-1,$$

$$\chi(X)=-60\int_B\;c_1(B)^2=-60\Omega_{\alpha\beta}K^\alpha K^\beta=2(T-H+3),$$

$$30(9-T)=H-T-3\Rightarrow 273=H+29T$$

$$\pi\colon B'\rightarrow B$$

$$c_1(B')=\pi^*c_1(B)-[E]$$

$$\chi(X')=\chi(X)-60$$

$$\Delta T=1, \Delta H_{\mathrm{neutral}}=-29$$

$$\begin{array}{l}(-12a-v_Ab_A)\cdot b_A=0\\b_A\cdot b_B=0, A\neq B\\29T-273=V-H\end{array}$$

$$\chi(\hat{X})=-60\int_B\;c_1^2(B)-\sum_A\;r_{G_A}c_{G_A}(2-2g_A)$$

$$\chi(X)=60T-540+2\sum_A\;(r_{G_A}-\dim_{G_A})$$

$$29T-273=\sum_A\;\dim_{G_A}-H$$

$$\begin{array}{l}j\cdot(-a)>0\\j\cdot b>0\\j\cdot(-12a-\sum_A\;v_Ab_A)>0\end{array}$$

$$-12K^\alpha-\sum_A\;v_AC_A^\alpha-Y^\alpha=0$$

$$\chi(X)=288+360\int\;c_1^3(\mathcal{B}_3)$$

$$39-60\kappa_{\alpha\beta\gamma}K^\alpha K^\beta K^\gamma=39-60\langle\langle a,a,a\rangle\rangle=C_{\mathrm{sa}}+C_{\mathrm{cs}}-C_{21},$$

$$\langle\langle x,y,z\rangle\rangle\cong\kappa_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma$$

$$C_{\mathrm{sa}}+C_{\mathrm{cs}}-C_{21}\equiv 39(\mathrm{mod}60)$$

$$\kappa=\kappa_{HHH}=1$$

$$39-60\kappa a^3=3879=2+C_{\mathrm{cs}}-(C_{21}+r_{21}).$$

$$k=2: 60\langle\langle -K,-K,-K\rangle\rangle=3720$$



$$C_{\text{sa}}+C_{\text{cs}}-(C_{21}+r_{21})=3+3756-(0+0)=3759=39-60\langle\langle a,a,a\rangle\rangle.$$

$$C_{\text{sa}}+C_{\text{cs}}-C_{21}=4359$$

$$\chi(X')=\chi(X)-2880$$

$$\Delta C_{\text{cs}}=-481,\Delta C_{\text{sa}}=+1,\Delta C_{21}=\Delta r_{21}=0$$

$$(x_1,x_2,x_3,x_4,x_5)\sim(\lambda x_1,\lambda\mu x_2,\lambda x_3,\mu x_4,\lambda x_5).$$

$$\chi(X')=\chi(X)-1440\left(2-2g_{\mathcal{C}}+\int_{\mathcal{C}}c_1(\mathcal{B}_3)\right)$$

$$\Delta C_{\text{cs}}=-481(1-g_{\mathcal{C}})-240\int_{\mathcal{C}}c_1(\mathcal{B}_3),\Delta C_{\text{sa}}=+1\\ \Delta r_{21}=g_{\mathcal{C}},\Delta C_{21}=0$$

$$39-60\kappa_{\alpha\beta\gamma}K^{\alpha}K^{\beta}K^{\gamma}=39-60\langle\langle a,a,a\rangle\rangle=C_{\text{sa}}+C_{\text{cs}}-C_{21}-r_{21}$$

$$C_{\text{sa}}+C_{\text{cs}}-C_{21}-r_{21}\equiv 39(\text{mod}60)$$

$$\kappa_{\alpha\beta\gamma}S^{\alpha}_AS^{\beta}_B=0,\forall\gamma$$

$$\Theta_{i_Ai_B}=\int_{\hat{X}}\omega_{i_A}\wedge\omega_{i_B}\wedge G_4=0$$

$$\chi(X)=288+360\int_Bc_1(B)^3-\sum_Ar_{G_A}c_{G_A}(c_{G_A}+1)\int_{S_A}c_1(S_A)^2$$

$$39-60\langle\langle a,a,a\rangle\rangle$$

$$=C_{\text{sa}}+C_{\text{cs}}+r_v-(C_{21}+r_{21})+\frac{1}{6}\sum_Ar_{G_A}c_{G_A}(c_{G_A}+1)\langle\langle a+b_A,a+b_A,b_A\rangle\rangle.$$

$$39-60\langle\langle a,a,a\rangle\rangle=5199=3+5186+1+\frac{1}{6}(1\cdot2\cdot3\cdot9),$$

$$\langle\langle -K,v,v\rangle\rangle=\kappa_{\alpha\beta\gamma}(-a^{\alpha})v^{\beta}v^{\gamma}>0$$

$$\langle\langle S_A,v,v\rangle\rangle=\kappa_{\alpha\beta\gamma}b^{\alpha}_Av^{\beta}v^{\gamma}>0$$

$$12\langle\langle -K,v,v\rangle\rangle\geq\sum_Av_A\langle\langle S_A,v,v\rangle\rangle>0$$

$$48\geq mN$$

$$\rho_{\alpha}\rightarrow\rho_{\alpha}+l_{\alpha},l\in L$$

$$a,b_A\in L^*$$

$$C_{\text{sa}}+C_{\text{cs}}-C_{21}\equiv 39(\text{mod}60)$$

$$c_1(B')=\pi^*c_1(B)-2[E],c_2(B')=\pi^*(c_2(B))$$

$$E^2=f,E\cdot f=-1,E\cdot\pi^*D=E\cdot\pi^*\tilde{C}=f\cdot\pi^*D=0$$



$$\begin{aligned}c_1(B') &= \pi^*c_1(B) - [E] \\ c_2(B') &= \pi^*(c_2(B) + [\mathcal{C}]) - [E] \wedge \pi^*c_1(B)\end{aligned}$$

$$\begin{aligned}E^2 &= -\pi^*\mathcal{C} + \deg(N_B\mathcal{C})f, E\cdot f = -1 \\ E\cdot \pi^*D &= (C\cdot D)f, f\cdot \pi^*D = 0, E\cdot \pi^*\tilde{\mathcal{C}} = 0\end{aligned}$$

$$\deg(N_B\mathcal{C})=-\chi(\mathcal{C})+\int_{\mathcal{C}}c_1(B)$$

$$I_8(R,F)=\frac{1}{2}\Omega_{\alpha\beta}X_4^\alpha X_4^\beta$$

$$X_4^\alpha=\frac{1}{2}a^\alpha\mathrm{tr}R^2+\sum_ib_i^\alpha\left(\frac{2}{\lambda_i}\mathrm{tr}F_i^2\right)$$

$$\begin{aligned}H-V &= 273-29T \\ 0 &= B_{\mathrm{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i B_{\mathbf{R}}^i \\ a\cdot a &= 9-T \\ -a\cdot b_i &= \frac{1}{6}\lambda_i(\sum_{\mathbf{R}} x_{\mathbf{R}}^i A_{\mathbf{R}}^i - A_{\mathrm{adj}}^i) \\ b_i\cdot b_i &= \frac{1}{3}\lambda_i^2(\sum_{\mathbf{R}} x_{\mathbf{R}}^i C_{\mathbf{R}}^i - C_{\mathrm{adj}}^i) \\ b_i\cdot b_j &= \lambda_i\lambda_j\sum_{\mathbf{RS}} x_{\mathbf{RS}}^{ij} A_{\mathbf{R}}^i A_{\mathbf{S}}^j\end{aligned}$$

$$\mathrm{tr}_{\mathbf{R}}F^2=A_{\mathbf{R}}\mathrm{tr}F^2,\mathrm{tr}_{\mathbf{R}}F^4=B_{\mathbf{R}}\mathrm{tr}F^4+C_{\mathbf{R}}(\mathrm{tr}F^2)^2$$

$$B\cdot X_4=B^\alpha\Omega_{\alpha\beta}\left[\frac{1}{2}a^\beta\mathrm{tr}R^2+\sum_ib_i^\beta\left(\frac{2}{\lambda_i}\mathrm{tr}F_i^2\right)\right]$$

$$r\rightarrow r'N^{1/3},x^\mu\rightarrow\frac{x'^\mu}{N^{1/3}}$$

$$T_{\mu\nu}=\frac{2h_{\phi}(\beta)}{\beta\pi^2}\gamma^4(\theta)(4u_{\mu}u_{\nu}+g_{\mu\nu})$$

$$\begin{aligned}ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \\ u^\mu\partial_\mu &= \gamma\big(\partial_t - \omega_1\partial_{\phi_1} - \omega_2\partial_{\phi_2}\big)\end{aligned}$$

$$\gamma(\theta)=\frac{1}{\sqrt{1-\omega_1^2\sin^2\theta-\omega_2^2\cos^2\theta}}$$

$$Z_{\rm gas}(\beta,\omega_i)=\frac{4h_{\phi}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$T_{\mu\nu}=\frac{2h_{\rm YM}(\beta)}{\beta\pi^2}\gamma^4(\theta)(4u_{\mu}u_{\nu}+g_{\mu\nu})$$

$$ds^2=-\Delta_{\tilde{\theta}}(1+r^2)\rho^2dt^2+\frac{f}{\rho^4}\Big(\frac{\Delta_{\tilde{\theta}}dt}{\Xi_a\Xi_b}-\omega\Big)^2+\frac{\rho^2dr^2}{\Delta_r}+\frac{\rho^2d\tilde{\theta}^2}{\Delta_{\tilde{\theta}}}+\frac{r^2+a^2}{\Xi_a}\sin^2\tilde{\theta}d\phi_1^2+\frac{r^2+b^2}{\Xi_b}\cos^2\tilde{\theta}d\phi_2^2$$



$$\begin{aligned}\Xi_a &= 1 - a^2 \\ \Xi_b &= 1 - b^2 \\ \Delta_{\tilde{\theta}} &= 1 - a^2 \cos^2 \tilde{\theta} - b^2 \sin^2 \tilde{\theta} \\ \Delta_r &= \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2)}{r^2} - 2m \\ \rho^2 &= r^2 + a^2 \cos^2 \tilde{\theta} + b^2 \sin^2 \tilde{\theta} \\ f &= 2m\rho^2 \\ \omega &= a \sin^2 \tilde{\theta} \frac{d\phi_1}{\Xi_a} + b \cos^2 \tilde{\theta} \frac{d\phi_2}{\Xi_b}\end{aligned}$$

$$a,b\in [-1,1]$$

$$(r_+^2+a^2)(r_+^2+b^2)(r_+^2+1)-2mr_+^2=0$$

$$x^3+x^2(a^2+b^2+1)+x(a^2b^2+a^2+b^2-2m)+a^2b^2=0$$

$$2m>a^2+b^2+a^2b^2$$

$$m_{ext}=\frac{1}{2}\Big(\frac{1}{6}\sqrt{(a^2+b^2+1)^4+216a^2b^2(a^2+b^2+1)}\cos\left(\frac{\alpha}{3}\right)+a^2(b^2+1)-\frac{1}{12}(a^2+b^2+1)^2+b^2\Big)$$

$$\begin{aligned}\cos\left(\alpha\right) &= \left(\frac{1}{(a^2+b^2+1)^4+216a^2b^2(a^2+b^2+1)}\right)^{3/2} \\ &\times (5832a^4b^4-(a^2+b^2+1)^6+540a^2b^2(a^2+b^2+1)^3)\end{aligned}$$

$$x_{\rm ext}=\frac{1}{6}(a^2+b^2+1)\Big(2\cos\left(\frac{1}{3}\xi\right)-1\Big)$$

$$\cos\left(\xi\right)=\left(\frac{54a^2b^2}{(a^2+b^2+1)^3}-1\right)$$

$$x\approx\sqrt{2m}$$

$$ds^2=-\Xi_a(1+r^2)(r^2+a^2)dt^2+\frac{2m}{(r^2+a^2)\Xi_a^2}(dt-a\sigma_3)^2+\frac{(r^2+a^2)dr^2}{\Delta_r}+\frac{(r^2+a^2)}{\Xi_a}(\sigma_1^2+\sigma_2^2+\sigma_3^2)$$

$$\begin{aligned}\sigma_1&=\frac{1}{2}(\sin{(\phi)}d(2\tilde{\theta})-\cos{(\phi)}\sin{(2\tilde{\theta})}d\psi)\\ \sigma_2&=\frac{1}{2}(\cos{(\phi)}d(2\tilde{\theta})+\sin{(\phi)}\sin{(2\tilde{\theta})}d\psi)\\ \sigma_3&=\frac{1}{2}(d\phi-\cos{(2\tilde{\theta})}d\psi)\end{aligned}$$

$$\phi=\phi_1+\phi_2,\psi=\phi_1-\phi_2$$

$$\begin{aligned}m_{\rm ext}&=\frac{1}{128}(\sqrt{8a^2+1}-1)(\sqrt{8a^2+1}+3)^3\\ (r_+)_{\rm ext}^2&=\frac{1}{4}(\sqrt{1+8a^2}-1)\end{aligned}$$

$$\begin{aligned}y^2\sin^2\theta&=\frac{(r^2+a^2)\sin^2\tilde{\theta}}{\Sigma_a}\\ y^2\cos^2\theta&=\frac{(r^2+b^2)\cos^2\tilde{\theta}}{\Sigma_b}\end{aligned}$$



$$\begin{aligned}
ds^2 = & -(1+y^2)dt^2 + \frac{dy^2}{1+y^2} + y^2(d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2) \\
& + \frac{2m}{y^6 \Delta_\theta^3} dy^2 + \frac{2m}{y^2 \Delta_\theta^3} dt^2 \\
& - \frac{4am \sin^2 \theta}{y^2 \Delta_\theta^3} dt d\phi - \frac{4bm \cos^2 \theta}{y^2 \Delta_\theta^3} dt d\psi \\
& - \frac{2ma^2 \sin^4 \theta}{y^2 \Delta_\theta^3} d\phi^2 + \frac{2mb^2 \cos^4 \theta}{y^2 \Delta_\theta^3} d\psi^2 \\
& + \frac{2abm \sin^2 \theta \cos^2 \theta}{y^2 \Delta_\theta^3} d\psi d\phi
\end{aligned}$$

$$ds^2 = -y^2 dt^2 + \frac{dy^2}{y^2} + y^2(d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2)$$

$$E = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b)}{4\Xi_a^2 \Xi_b^2}$$

$$J_a = \frac{2\pi am}{4\Xi_a^2 \Xi_b}$$

$$J_b = \frac{2\pi bm}{4\Xi_a \Xi_b^2}$$

$$S = \frac{\pi^2(r_+^2 + a^2)(r_+^2 + b^2)}{2\Xi_a \Xi_b r_+}$$

$$T = \frac{r_+}{2\pi} (1 + r_+^2) \left(\frac{1}{r_+^2 + a^2} + \frac{1}{r_+^2 + b^2} \right) - \frac{1}{2\pi r_+}$$

$$\omega_a = \frac{a(r_+^2 + 1)}{(r_+^2 + a^2)}$$

$$\omega_b = \frac{b(r_+^2 + 1)}{(r_+^2 + b^2)}$$

$$\Xi_a = 1 - a^2$$

$$\Xi_b = 1 - b^2$$

$$(r_+)_{ext} = \frac{\sqrt{ab}}{(1+b^2)^{\frac{1}{4}}}$$

$$m_{ext} = \frac{b^2}{2} + ab\sqrt{1+b^2}$$

$$E = -\frac{b^2(b^2-3\pi)}{8(b^2-1)^2}$$

$$J_a = \frac{ab^2\pi}{4(1-b^2)}$$

$$J_b = \frac{b^3\pi}{4(b^2-1)^2}$$

$$S = \frac{\sqrt{ab}b^2\pi^2}{2(1+b^2)^{\frac{1}{4}}(1-b^2)}$$

$$m \leq \frac{(1+a)^2(a+b^2)}{2}$$

$$\Delta m = a\left(\frac{1}{2} + b^2 - b\sqrt{1+b^2}\right),$$

$$m = \frac{a(1+a)^3}{2}.$$



$$T = \frac{1}{2\pi\sqrt{a}}$$

$$E = \frac{a(3+a^2)\pi}{8(1-a)^3}$$

$$J = J_a = J_b = \frac{\pi a^2}{(1-a)^3}$$

$$S = \frac{\pi^2 a^{3/2}}{2(a-1)^2}$$

$$E = -\frac{\pi+48\pi^5T^4}{8(1-4\pi^2T^2)^3}, \frac{1}{2\pi} \leq T < \infty$$

$$E \approx \frac{3}{32\pi T^2}$$

$$c = \left(\frac{dE}{dT}\right)\Big|_{\omega_a=\omega_b=1} = -\frac{3\pi^3T(4\pi^2T^2+1)^2}{(1-4\pi^2T^2)^4}$$

$$c \approx -\frac{3}{128\pi^2\left(T-\frac{1}{2\pi}\right)^4}$$

$$c \approx -\frac{3}{16\pi T^3}$$

$$\frac{dE_{\omega}(J)}{dJ}=\frac{3(a+1)^2}{2a(a+2)}$$

$$E_{\omega}(J)=\int_0^J\frac{dE_{\omega}(J)}{dJ}>2J$$

$$T^{tt}=\frac{m}{8\pi G_5}(4\gamma^6-\gamma^4)$$

$$T\phi_1\phi_1=\frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2a^2+\frac{1}{\sin^2(\theta)}\right)$$

$$T\phi_2\phi_2=\frac{m}{8\pi G_5}\gamma^4\left(4\gamma^2b^2+\frac{1}{\cos^2(\theta)}\right)$$

$$T^t\phi_1=\frac{4m}{8\pi G_5}a\gamma^6,T^t\phi_2=\frac{4m}{8\pi G_5}b\gamma^6$$

$$T\phi_1\phi_2=\frac{4m}{8\pi G_5}ab\gamma^6,T^{\theta\theta}=\frac{m}{8\pi G_5}\gamma^4$$

$$T^{\mu\nu}=\frac{N^2m}{4\pi^2}\gamma^4(\theta)(4u_{bh}^{\mu}u_{bh}^{\nu}+g^{\mu\nu})$$

$$u_{bh}^{\mu}=\gamma(\partial_t-a\partial_{\phi_1}-b\partial_{\phi_2}),\gamma^{-2}=1-a^2\sin^2\theta-b^2\cos^2\theta$$

$$u_{bh}^{\mu}=\gamma(\partial_t-\omega_a\partial_{\phi_1}-\omega_b\partial_{\phi_2}),\gamma^{-2}=1-\omega_a^2\sin^2\theta-\omega_b^2\cos^2\theta$$

$$E_5 = \int d\Omega_3 T_0^0$$

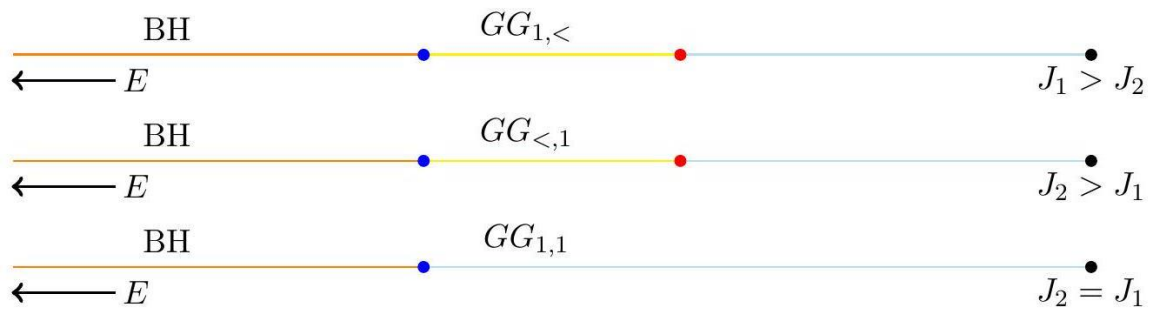
$$= \frac{m}{8\pi G_5} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{\frac{\pi}{2}} d\theta \sin(\theta) \cos(\theta) (4\gamma^6 - \gamma^4)$$

$$\equiv \int \rho(\theta) d\theta$$

$$= \frac{m\pi(2-2a^2+2-2b^2-(1-a^2)(1-b^2))}{4G_5(1-a^2)^2(1-b^2)^2}$$

$$\rho(\theta)=\frac{m\pi\sin(\theta)\cos(\theta)(3+a^2\sin^2(\theta)+b^2\cos^2(\theta))}{2G_5(1-a^2\sin^2(\theta)-b^2\cos^2(\theta))^3}$$





$$(J_1,J_2,E) = (\alpha,\beta,E_{B_1}(\alpha,\beta))$$

$$(\beta,\beta,E_{\omega}(\beta))$$

$$E_{\omega}(\beta)-E_{B_1}(\alpha,\beta)=\beta-\alpha$$

$$(\alpha,\beta,E_{\omega}(\beta-x)+(\alpha-(\beta-x))+(\beta-(\beta-x)))=(\alpha,\beta,E_{\omega}(\beta-x)+\alpha-\beta+2x)$$

$$(\alpha,\beta,E_{B_1}(\alpha,\beta)+E_{\omega}(\beta-x)-E_{\omega}(\beta)+2x)$$

$$(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_J}O$$

$$\mathrm{Tr} e^{-\beta H+\beta(\omega_1+\omega_2)J_L^Z+\beta(\omega_1-\omega_2)J_R^Z}$$

$$\ln Z(\beta,\omega_1,\omega_2)=-\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\sum_{m_L=-J/2}^{J/2}\sum_{m_R=-J/2}^{J/2}\ln\big(1-e^{-\beta(\Delta+2n+J)+\beta\omega_1(m_L+m_R)+\beta\omega_2(m_L-m_R)}\big)$$

$$\ln Z(\beta,\omega_1,\omega_2)\!\approx\!\sum_{q=1}^{\infty}\frac{1}{q}\Bigg[-\frac{e^{-q\beta(\Delta-6)}}{(e^{2\beta q}-1)^4}+\frac{e^{-q\beta(\Delta-2)}}{4\beta^2q^2(1-\omega_1)(1-\omega_2)\sinh^2(\beta q)}\Bigg]$$

$$\approx \frac{h_{\Delta}(\beta)}{(1-\omega_1)(1-\omega_2)}$$

$$h_{\Delta}(\beta)=\sum_{q=1}^{\infty}\frac{e^{-q\beta(\Delta-2)}}{4\beta^2q^3\sinh^2(\beta q)}$$

$$\ln Z \approx \frac{4h_{\Delta}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$J_L=\frac{J_1+J_2}{2},J_R=\frac{J_1-J_2}{2}$$

$$\ln Z=-\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\sum_{|m_1|+|m_2|\leq J}\ln\big(1-e^{-\beta(\Delta+2n+J)+\beta\omega_1m_1+\beta\omega_2m_2}\big)$$

$$E=\frac{8h_{\Delta}(\beta)}{\beta}\bigg(\frac{1}{(1-\omega_2^2)^2(1-\omega_1^2)}+\frac{1}{(1-\omega_1^2)^2(1-\omega_2^2)}\bigg)$$

$$J_1=\frac{1}{\beta}\frac{8h_{\Delta}(\beta)}{(1-\omega_1^2)^2(1-\omega_2^2)}$$

$$J_2=\frac{1}{\beta}\frac{8h_{\Delta}(\beta)}{(1-\omega_2^2)^2(1-\omega_1^2)}$$

$$S=\frac{(\beta\partial_{\beta}-1)4h_{\Delta}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$d_n=\frac{(2n+4)(n+1)(n+2)(n+3)}{24}.$$



$$\ln Z \approx \sum_{n=0}^{\infty} d_n \frac{4h_{\Delta}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$h_{\Delta}(\beta) = \sum_{q=1}^{\infty} \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^3 \sinh^2(\beta q)}, \Delta = n+4,$$

$$\ln Z = \frac{1}{(1-\omega_1^2)(1-\omega_2^2)} \sum_{q=1}^{\infty} \frac{4\operatorname{csch}^6\left(\frac{\beta q}{2}\right)\operatorname{csch}(\beta q)}{128\beta^2 q^3}$$

$$= \frac{4h_{\phi}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$\ln Z = \frac{4h_{YM}(\beta)}{(1-\omega_1^2)(1-\omega_2^2)}$$

$$h_{YM}(\beta) = h_{YM}^B(\beta) + h_{YM}^F(\beta)$$

$$h_{YM}^B(\beta)$$

$$= \sum_{n=1}^{\infty} \frac{(22\cosh(\beta n) + 17\cosh(2\beta n) + 6\cosh(3\beta n) + \cosh(4\beta n) + 18)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)\operatorname{sech}\left(\frac{\beta n}{2}\right)}{128\beta^2 n^3}$$

$$h_{YM}^F(\beta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cosh(\beta n)(\cosh(\beta n) + \cosh(2\beta n) + 2)\operatorname{csch}^7\left(\frac{\beta n}{2}\right)}{8\beta^2 n^3}$$

$$1-\omega_1=\frac{\frac{\alpha_1}{2}}{N^{\frac{3}{2}}}$$

$$1-\omega_2=\frac{\frac{\alpha_2}{2}}{N^{\frac{3}{2}}}$$

$$\mathcal{Z} = \operatorname{Tr}\big[\mathrm{e}^{-\beta(\mathrm{H}-\omega_1\,\mathrm{J}_1-\omega_2\,\mathrm{J}_2)}\big]$$

$$(\tau,\phi_1,\phi_2,\Omega_i)\sim(\tau+\beta,\phi_1-i\beta\omega_1,\phi_2-i\beta\omega_2,\Omega_i)$$

$$T_{\mu\nu}=\partial_\mu\Phi\partial_\nu\Phi-\frac{g_{\mu\nu}}{2}(\partial\Phi)^2$$

$$\langle T_{\mu\nu}\rangle = \lim_{x_1\rightarrow x_2}\left[\partial_\mu^{x_1}\partial_\nu^{x_2}\tilde{G}(x_1-x_2)-\frac{g_{\mu\nu}}{2}\left(\partial^{\alpha,x_1}\partial_\alpha^{x_2}\tilde{G}(x_1-x_2)\right)\right]$$

$$\tilde{G}(x_1,x_2)=\sum_{q=-\infty}^{\infty}G(x_1,R^q(x_2))$$

$$G(x_1,x_2)=\frac{\Gamma[4]}{4\pi^5}\Big(\frac{1}{u+v}\Big)^4$$

$$u=-2+2\left[\sqrt{(1+r^2)(1+r'^2)}\cosh\left(\tau^E-\tau'^E\right)\right.\\ \left.-rr'\left(\sin\theta\sin\theta'\cos\left(\phi_1-\phi_1'\right)+\cos\theta\cos\theta'\cos\left(\phi_2-\phi_2'\right)\right)\right]$$

$$u_q=-2+2[(1+r^2)\cosh(q\beta+\tau'-\tau)\\ -r^2(\sin^2\theta\cos(\phi_1-\phi_1'-iq\beta\omega_1)+\cos^2\theta\cos(\phi_2-\phi_2'-iq\beta\omega_2))]$$

$$\tau'=\tau,\phi_1'=\phi_1,\phi_2'=\phi_2$$



$$r \sim \frac{1}{\sqrt{(1-\omega_1 \sin^2(\theta) - \omega_2 \cos^2(\theta))}}$$

$$r^2(1 - \omega_1 \sin^2(\theta) - \omega_2 \cos^2(\theta)) = x^2$$

$$2(1 - \omega_1 \sin^2(\theta) - \omega_2 \cos^2(\theta)) = 1 - \omega_1^2 \sin^2(\theta) - \omega_2^2 \cos^2(\theta) = \frac{1}{r^2(\theta)}$$

$$x^2 = \frac{r^2}{2\gamma^2(\theta)}$$

$$\begin{aligned} u_q &= 2(\cosh(\beta q) + \beta q r^2 \sinh(\beta q)(1 - \omega_1 \sin^2(\theta) - \omega_2 \cos^2(\theta)) - 1) \\ &= 2(\cosh(\beta q) + \beta q x^2 \sinh(\beta q) - 1) \end{aligned}$$

$$\begin{aligned} \partial_\tau u_q &= -\partial_{\tau'} u_q = -2((r^2 + 1) \sinh(\beta q)) \\ \partial_{\phi_1} u_q &= -\partial_{\phi_1'} u_q = 2ir^2 \sin^2(\theta) \sinh(\beta q \omega_1) \\ \partial_{\phi_2} u_q &= -\partial_{\phi_2'} u_q = 2ir^2 \sin^2(\theta) \sinh(\beta q \omega_2) \end{aligned}$$

$$w^\mu \partial_\mu = (\partial_t - \partial_{\phi_1} - \partial_{\phi_2})$$

$$g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \partial_{\mu_1} \partial_{\mu_2} \langle \Phi(x_1) \Phi(x_2) \rangle_q \propto u^{\nu_1} u^{\nu_2}$$

$$\begin{aligned} T_{\mu_1 \mu_2} &= -2 \sum_{q=1}^{\infty} K_{\mu_1 \mu_2}^q \\ K_{\mu_1 \mu_2}^q &= \partial_{\mu_1} \partial_{\mu_2} \langle \Phi(x_1) \Phi(x_2) \rangle_q \\ &= \partial_u^2 G(u, v) \frac{\partial u_q}{\partial \mu_1} \frac{\partial u_q}{\partial \mu_2} + \partial_u G(u, v) \frac{\partial^2 u_q}{\partial \mu_1 \partial \mu_2} \end{aligned}$$

$$K_{\tau\tau}^q = \partial_u^2 G(u, v) \frac{\partial u_q}{\partial \tau} \frac{\partial u_q}{\partial \tau'} \Big|_{u=u_q}$$

$$\begin{aligned} K_{\tau\tau}^q &= -16\gamma^4(\theta)x^4\sinh^2(q\beta)\partial_u^2G(u,v)|_{v\rightarrow 0} \\ &= -\frac{4\gamma^4(\theta)x^4\sinh^2(\beta q)\Gamma[6]}{\pi^5u_q^6} \end{aligned}$$

$$\langle T_{00} \rangle = \sum_{q=1}^{\infty} \frac{8\gamma^4(\theta)x^4\sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5u_q^6}$$

$$\begin{aligned} \langle T_{0\phi_1} \rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4\sinh^2(q\beta)\sin^2\theta\Gamma[6]}{\pi^5u_q^6} \\ \langle T_{0\phi_2} \rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4\sinh^2(q\beta)\cos^2\theta\Gamma[6]}{\pi^5u_q^6} \\ \langle T_{\phi_1\phi_1} \rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4\sinh^2(q\beta)\sin^4\theta\Gamma[6]}{\pi^5u_q^6} \\ \langle T_{\phi_2\phi_2} \rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4\sinh^2(q\beta)\cos^4\theta\Gamma[6]}{\pi^5u_q^6} \\ \langle T_{\phi_1\phi_2} \rangle &= \sum_{q=1}^{\infty} \frac{4\gamma^4(\theta)x^4\sinh^2(q\beta)\sin^2\theta\cos^2\theta\Gamma[6]}{\pi^5u_q^6} \\ \langle T_{\theta i} \rangle &= 0, i \in \{\tau_E, \theta, \phi_1, \phi_2\} \\ \langle T_{ri} \rangle &= 0 \\ \langle T_{\alpha\beta} \rangle &= 0 \\ \langle T_{\mu\alpha} \rangle &= 0, y_\alpha, y_\beta \in S^5, x_\mu \in AdS_5 \end{aligned}$$



$$\langle T_{\mu\nu} \rangle = 4\mathcal{F}(x)\gamma^4(\theta) \begin{pmatrix} 1 & 0 & 0 & \sin^2(\theta) & \cos^2(\theta) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sin^2(\theta) & 0 & 0 & \sin^4(\theta) & \sin^2(\theta)\cos^2(\theta) \\ \cos^2(\theta) & 0 & 0 & \sin^2(\theta)\cos^2(\theta) & \cos^4(\theta) \end{pmatrix}$$

$$\begin{aligned}\mathcal{F}(x) &= \sum_{q=1}^{\infty} \frac{2x^4\Gamma[6]\sinh^2\beta q}{\pi^5 u_q^6} \\ &= \sum_{q=1}^{\infty} \frac{x^4\Gamma[6]\sinh^2\beta q}{2^5\pi^5((\cosh(\beta q)+\beta qx^2\sinh(\beta q)-1))^6}\end{aligned}$$

$$T_{\mu\nu}=4\mathcal{F}(x)\gamma^4(\theta)w_\mu w_\nu$$

$$\begin{aligned}\mathcal{F}(x) &= x^4 D + \mathcal{O}(x^6) \\ D &= \left(\frac{\Gamma[6]}{2^6\pi^5}\sum_{q=1}^{\infty}\frac{1}{2\sinh^2\beta q(\cosh(\beta q)-1)^6}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{F}(x) &= \frac{B}{x^8} + \mathcal{O}\left(\frac{1}{x^{10}}\right) \\ B &= \frac{15\Gamma[6]}{4\pi^5\beta^6}\sum_{q=1}^{\infty}\frac{\operatorname{csch}^4(\beta q)}{q^6}\end{aligned}$$

$$\begin{aligned}E &= \int d^{10}x \sqrt{-g} g^{00} T_{00} \\ &= \Omega_5 \int d\Omega_3 dr r \times 4\gamma^4(\theta) \mathcal{F}(x) \\ &= \Omega_5 \int_{S_3} \gamma^6(\theta) \int_0^\infty dx x \times 8\mathcal{F}(x)\end{aligned}$$

$$\begin{aligned}\int_{S_3} \gamma^6(\theta) &= 4\pi^2 \int_0^\pi \frac{d\theta \sin\theta \cos\theta}{(1-\omega_1^2\sin^2\theta-\omega_2^2\cos^2\theta)^3} \\ &= \pi^2 \left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} \right)\end{aligned}$$

$$\begin{aligned}&\int_0^\infty dx x \mathcal{F}(x) \\ &= \sum_{q=1}^\infty \int_0^\infty \frac{2\Gamma(6)\sinh^2\beta q dx x^5}{\pi^5(2(\cosh(\beta q)+\beta qx^2\sinh(\beta q)-1))^6} \\ &= \sum_{q=1}^\infty \frac{\operatorname{csch}^6(\frac{\beta q}{2})\operatorname{csch}(\beta q)}{128\pi^5\beta^3q^3} \\ &= \frac{h_\phi(\beta)}{\pi^5\beta}\end{aligned}$$

$$\begin{aligned}E &= \left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right) \sum_{q=1}^\infty \frac{\operatorname{csch}^6(\frac{\beta q}{2})\operatorname{csch}(\beta q)}{16\beta^3q^3} \\ &= \frac{8h_\phi(\beta)}{\beta} \left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2}\right)\end{aligned}$$

$$ds^2=d\theta^2+\sin{(\theta)}^2d\phi_1^2+\cos{(\theta)}^2d\phi_2^2$$

$$ds^2=(\sigma_1)^2+(\sigma_2)^2+(\sigma_3)^2$$

$$ds_{AdS_5}^2=-(1+r^2)dt^2+\frac{dr^2}{1+r^2}+r^2((\sigma_1)^2+(\sigma_2)^2+(\sigma_3)^2)$$

$$T=4\mathcal{F}(x)\gamma^4(\theta)(dt-\sigma_3)(dt-\sigma_3)$$



$$x = \frac{r}{\sqrt{2}\gamma(\theta)}$$

$$t = t_0 + \frac{\tau}{\sqrt{2}\gamma(\theta_0)}$$

$$\phi = \phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)}$$

$$\psi = \psi_0 + \frac{\psi'}{\sqrt{2}\gamma(\theta_0)}$$

$$\theta = \theta_0 + \frac{\theta'}{\sqrt{2}\gamma(\theta_0)}$$

$$ds^2 = -x^2 d\tau^2 + \frac{dx^2}{x^2} + x^2((\sigma'_1)^2 + (\sigma'_2)^2 + (\sigma'_3)^2) + d\Omega_5^2$$

$$\sigma'_1 = \frac{1}{2} \left(\sin \left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)} \right) d(2\theta') - \cos \left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)} \right) \sin \left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)} \right) d\psi' \right)$$

$$\sigma'_2 = \frac{1}{2} \left(\cos \left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)} \right) d(2\theta') + \sin \left(\phi_0 + \frac{\phi'}{\sqrt{2}\gamma(\theta_0)} \right) \sin \left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)} \right) d\psi' \right)$$

$$\sigma'_3 = \frac{1}{2} \left(d\phi' - \cos \left(2\theta_0 + \frac{2\theta'}{\sqrt{2}\gamma(\theta_0)} \right) d\psi' \right)$$

$$dw = \sigma'_1, dy = \sigma'_2, dz = \sigma'_3$$

$$ds^2 = \frac{dx^2}{x^2} + x^2(-d\tau^2 + dw^2 + dy^2 + dz^2)$$

$$T = \mathcal{F}(x)2\gamma^2(\theta_0)(d\tau - dz)(d\tau - dz)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 6g_{\mu\nu} = 8\pi G_N^{(10)}T_{\mu\nu}$$

$$G^{(10)} = \frac{\pi^4 R_{AdS}^8}{2N^2}$$

$$G^{(5)} = \frac{G^{(10)}}{\Omega_5 R_{AdS}^5} = \frac{\pi^4 R_{AdS}^3}{2N^2 \pi^3} = \frac{\pi R_{AdS}^3}{2N^2}$$

$$\begin{aligned} \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 6g_{\mu\nu}\right)dx^\mu dx^\nu &= 8\pi \left(\frac{\pi^4}{2N^2}\right)\mathcal{F}(x)(2\gamma^2(\theta_0))(d\tau - dz)(d\tau - dz) \\ &= \left(\frac{8\pi^5\gamma^2(\theta_0)}{N^2}\right)\mathcal{F}(x)(d\tau - dz)(d\tau - dz) \end{aligned}$$

$$\delta ds^2 = \frac{8\pi^5\gamma^2(\theta_0)}{N^2}x^2f(x)(d\tau - dz)(d\tau - dz)$$

$$\frac{1}{x^3}\partial_x(x^5\partial_x f) = -2\frac{x\mathcal{F}(x)}{x^3}$$

$$f(x) = 2\int_x^\infty dy \left(\frac{1}{y^5} \int_0^y dz z \mathcal{F}(z) \right)$$

$$ds_{AdS_5}^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2((\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2) + \frac{8\pi^2\gamma^2(\theta)}{N^2}r^2f\left(\frac{r}{\gamma(\theta)}\right)(dt - \sigma_3)(dt - \sigma_3)$$

$$\begin{aligned} \int_0^y dz z \mathcal{F}(z) &= \int_0^\infty dz z \mathcal{F}(z) - \int_y^\infty dz z \mathcal{F}(z) \\ &= C - \frac{B}{6y^6} + \mathcal{O}(1/y^8) \end{aligned}$$



$$\begin{aligned} C &= \int_0^\infty dz z \mathcal{F}(z) \\ &= \sum_{q=1}^\infty \frac{\text{csch}^6\left(\frac{\beta q}{2}\right) \text{csch}(\beta q)}{128 \pi^5 \beta^3 q^3} \end{aligned}$$

$$C=\frac{h_\phi(\beta)}{\beta\pi^5}$$

$$f(x)=\frac{h_\phi(\beta)}{2\pi^5x^4\beta}-\frac{B}{60x^{10}}+\mathcal{O}\left(\frac{1}{x^{12}}\right)$$

$$\begin{aligned} f(x) &= \frac{1}{2} \int_x^\infty dy \frac{\mathcal{F}(y)}{y^3} + \frac{1}{2x^4} \int_0^x dy y \mathcal{F}(y) \\ &= \frac{1}{2} \int_0^\infty dy \frac{\mathcal{F}(y)}{y^3} - \frac{1}{2} \int_0^x dy \frac{\mathcal{F}(y)}{y^3} + \frac{1}{2x^4} \int_0^x dy y \mathcal{F}(y) \\ &= \frac{1}{2} \int_0^\infty dy \frac{\mathcal{F}(y)}{y^3} - \frac{D}{6} x^2 + \mathcal{O}(x^4) \end{aligned}$$

$$ds_{AdS_5}^2=-(1+r^2)dt^2+\frac{dr^2}{1+r^2}+r^2((\sigma_1)^2+(\sigma_2)^2+(\sigma_3)^2)+r^2(dt-\sigma_3)(dt-\sigma_3)\frac{16h_\phi(\beta)\gamma^6(\theta)}{N^2r^4\beta}$$

$$\begin{aligned} T_{\text{bdry}} &= \left(\frac{4}{16\pi G_5}\right) \frac{16h_\phi(\beta)\gamma^6(\theta)}{\beta N^2} (dt-\sigma_3)(dt-\sigma_3) \\ &= \frac{8h_\phi(\beta)\gamma^6(\theta)}{\pi^2\beta} (dt-\sigma_3)(dt-\sigma_3) \end{aligned}$$

$$u^\mu\partial_\mu=\gamma(\theta)(\partial_t-\omega_1\partial_{\phi_1}-\omega_2\partial_{\phi_2})$$

$$u^\mu\partial_\mu=\gamma(\theta)\left(\partial_t-\partial_{\phi_1}-\partial_{\phi_2}+\mathcal{O}\left(\frac{1}{\gamma(\theta)}\right)\right)$$

$$u_u dx^\mu = \gamma(\theta)(-dt + \sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2) = \gamma(\theta)(dt - \sigma_3)$$

$$\left(T_{\text{bdry}}\right)_{\mu\nu}=\frac{2h_\phi(\beta)\gamma^4(\theta)}{\beta\pi^2}\left(4u_\mu u_\nu+g_{\mu\nu}\right)$$

$$\begin{aligned} E &= \frac{h_\phi(\beta)}{\beta\pi^5} \pi^3 \int_{S^3} 8\gamma^6(\theta) \\ &= \frac{8h_\phi(\beta)}{\beta} \left(\frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} + \frac{1}{(1-\omega_1^2)(1-\omega_2^2)^2} \right) \end{aligned}$$

$$\int_0^\infty dx x \mathcal{F}_{YM}(x) = \frac{h_{YM}(\beta)}{\pi^5 \beta}$$

$$f_{YM}(x)=2\int_x^\infty dy\left(\frac{1}{y^5}\int_0^y dz z \mathcal{F}_{YM}(z)\right)$$

$$f_{YM}(x)=\frac{h_{YM}(\beta)}{2\pi^5x^4\beta}-\frac{B_{YM}}{60x^{10}}+\mathcal{O}\left(\frac{1}{x^{12}}\right)$$

$$\left(T_{\text{bdry}}^{YM}\right)_{\mu\nu}=\frac{2h_{YM}(\beta)\gamma^4(\theta)}{\beta\pi^2}\left(4u_\mu u_\nu+g_{\mu\nu}\right)$$

$$\omega J^{\frac{1}{D-2}}=g\left(\frac{M}{J^{\frac{D-3}{D-2}}}\right)$$



$$\omega \sim \frac{M}{J}$$

$$\Delta = a^4(r_1-r_2)^2(r_1-r_3)^2(r_2-r_3)^2.$$

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

$$(r^2+a^2)(r^2+b^2)(r^2+1)-2mr^2=0$$

$$\Delta = -27a^4b^4 - 18a^2b^2(a^2+b^2+1)(a^2(-b^2) - a^2 - b^2 + 2m) + 4(a^2(-b^2) - a^2 - b^2 + 2m)^3 \\ + (a^2+b^2+1)^2(a^2(-b^2) - a^2 - b^2 + 2m)^2 - 4a^2b^2(a^2+b^2+1)^3$$

$$\frac{1}{16}a^2b^2(a^6+3a^4(b^2+1)+3a^2(b^4-7b^2+1)+(b^2+1)^3)^3\geq 0$$

$$m_{\rm ext} = \frac{1}{2} \Big(\frac{1}{6} \sqrt{(a^2+b^2+1)^4 + 216a^2b^2(a^2+b^2+1)} \cos\left(\frac{\theta}{3}\right) + a^2(b^2+1) - \frac{1}{12}(a^2+b^2+1)^2 + b^2 \Big)$$

$$\cos{(\theta)} = \left(\frac{1}{(a^2+b^2+1)^4+216a^2b^2(a^2+b^2+1)}\right)^{3/2} \\ \times (5832a^4b^4-(a^2+b^2+1)^6+540a^2b^2(a^2+b^2+1)^3)$$

$$\Delta_{a=b}=4m(2a^2(-1+a^2)^3+m-4a^2(5+2a^2)m+8m^2)$$

$$m_{\rm ext} = \frac{1}{128}(\sqrt{8a^2+1}-1)(\sqrt{8a^2+1}+3)^3$$

$$(r_+)_{\rm ext}^2 = \frac{1}{4}(\sqrt{1+8a^2}-1)$$

$$\ln Z = -\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m_L, m_R = -\frac{J}{2}}^{\frac{J}{2}} \ln \left(1 - e^{-\beta(\Delta + 2n + J) + \beta(\omega_1 + \omega_2)m_L + \beta(\omega_1 - \omega_2)m_R} \right) \\ = -\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{a,b=0}^J \ln \left(1 - e^{-\beta(\Delta + 2n + a(\omega_1 + \omega_2) + b(\omega_1 - \omega_2)) + \beta(1 - \omega_1)J} \right) \\ = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{a,b=0}^J \sum_{q=1}^{\infty} \frac{e^{-q\beta(\Delta + 2n + a(\omega_1 + \omega_2) + b(\omega_1 - \omega_2)) + q\beta(1 - \omega_1)J}}{q}$$

$$\ln Z = \sum_{n,J=0}^{\infty} \sum_{q=1}^{\infty} \frac{1}{q} \left[-\frac{\exp\left(\beta(J+1)q\omega_1 - \beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n)\right)}{(e^{\beta q\omega_1} - e^{\beta q\omega_2})(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right. \\ \left. + \frac{\exp\left(\beta(J+1)q\omega_2 - \beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n)\right)}{(e^{\beta q\omega_1} - e^{\beta q\omega_2})(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right. \\ \left. + \frac{\exp\left(-\beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n) + \beta(J+1)q\omega_1 + \beta(J+1)q(\omega_1 + \omega_2)\right)}{(e^{\beta q\omega_1} - e^{\beta q\omega_2})(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right. \\ \left. - \frac{\exp\left(-\beta q(\Delta + J\omega_1 + J\omega_2 + J + 2n) + \beta(J+1)q\omega_2 + \beta(J+1)q(\omega_1 + \omega_2)\right)}{(e^{\beta q\omega_1} - e^{\beta q\omega_2})(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right]$$

$$\ln Z = \sum_{q=1}^{\infty} \frac{1}{q} \left[-\frac{(\coth(\beta q) - 1)e^{\beta q(-\Delta + \omega_1 + \omega_2 + 4)}}{2(e^{\beta q(\omega_1 + 1)} - 1)(e^{\beta q(\omega_2 + 1)} - 1)(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right. \\ \left. + \frac{(\coth(\beta q) - 1)e^{\beta q(-\Delta + \omega_1 + \omega_2 + 4)}}{2(e^{\beta q} - e^{\beta q\omega_1})(e^{\beta q} - e^{\beta q\omega_2})(e^{\beta q(\omega_1 + \omega_2)} - 1)} \right]$$



$$\ln \mathcal{Z} = \sum_{q=1}^{\infty} \frac{1}{q} \left[-\frac{e^{\beta(\Delta-6)(-q)}}{(e^{2\beta q}-1)^4} + \frac{e^{\beta(\Delta-4)(-q)}}{\beta^2 q^2 (\omega_1-1)(\omega_2-1)(e^{2\beta q}-1)^2} + O\left(\frac{1}{1-\omega_1}\right) + O\left(\frac{1}{1-\omega_2}\right) \right] \\ \approx \frac{e^{-q\beta(\Delta-2)}}{4\beta^2 q^3 (1-\omega_1)(1-\omega_2) \sinh^2(\beta q)}$$

$$\Delta > j_1+j_2+2-\delta_{j_1,0}-\delta_{j_2,0}+\delta_{j_1,0}\delta_{j_2,0}$$

$$\mathrm{Tr}(s^H x^{j_1^Z} y^{j_2^Z})$$

$$z_{B/F}^{\Delta j_1 j_2}(s,x,y)=\frac{s^{\Delta}\chi_{j_1}^L(x)\chi_{j_2}^R(y)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{100}(s,x,y)=\frac{s(1-s^2)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{j_1+1,j_10}(s,x,y)=\frac{s^{j_1+1}\Big(\chi_{j_1}^L(x)-s\chi_{j_1-1/2}^L(x)\chi_{1/2}^R(y)+s^2\chi_{j_1-1}^L(x)\Big)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$z_{B/F}^{j_1+j_2+2,j_1j_2}(s,x,y)=\frac{s^{j_1+j_2+2}\Big(\chi_{j_1}^L(x)\chi_{j_2}^R(y)-s\chi_{j_1-1/2}^L(x)\chi_{j_2-1/2}^R(y)\Big)}{\left(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\left(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(1-sx^{\frac{1}{2}}y^{\frac{1}{2}}\right)}$$

$$\mathrm{Tre}^{\mathrm{i}(\theta_1\mathrm{R}_1+\theta_2\mathrm{R}_2+\theta_3\mathrm{R}_3)}=\mathrm{Tr}(\mathrm{p}^{\mathrm{R}_1}\mathrm{q}^{\mathrm{R}_2}\mathrm{r}^{\mathrm{R}_3})$$

$$[1,0,0], [-1,1,0], [0,-1,1], \text{ and } [0,0,-1]$$

$$v_1=e^{i\theta_1}, v_2=e^{i(\theta_2-\theta_1)}, v_3=e^{i(\theta_3-\theta_2)}, v_4=e^{-i\theta_3}.$$

$$\chi=\frac{\det\big[v_i^{l_j+n-j}\big]}{\det\big[v_i^{n-j}\big]}$$

$$\begin{array}{l} l_1=r_1+r_2+r_3\\ l_2=r_2+r_3\\ l_3=r_3\\ l_4=0 \end{array}$$

$$\chi_{(r_1,r_2,r_3)}(p,q,r) = \frac{\det \begin{pmatrix} v_1^{r_1+r_2+r_3+3} & v_2^{r_1+r_2+r_3+3} & v_3^{r_1+r_2+r_3+3} & v_4^{r_1+r_2+r_3+3} \\ v_1^{r_2+r_3+2} & v_2^{r_2+r_3+2} & v_3^{r_2+r_3+2} & v_4^{r_2+r_3+2} \\ v_1^{r_3+1} & v_2^{r_3+1} & v_3^{r_3+1} & v_4^{r_3+1} \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det \begin{pmatrix} v_1^3 & v_2^3 & v_3^3 & v_4^3 \\ v_1^2 & v_2^2 & v_3^2 & v_4^2 \\ v_1 & v_2 & v_3 & v_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

(j_1,j_2)	2Δ	Fields	$SU(4)$ Dynkin labels
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$(0,0)$	2	$\varphi^{(1)}$	$(0,1,0)$
$\left(\frac{1}{2}, 0\right)$	3	$\lambda^{(1)}$	$(0,0,1)$
$\left(0, \frac{1}{2}\right)$	3	$\lambda^{(1)}$	$(1,0,0)$
$(1,0)$	4	$A_{\mu\nu}^{(1)}$	$(0,0,0)$
$(0,1)$	4	$\tilde{A}_{\mu\nu}^{(1)}$	$(0,0,0)$

$SO(4)$ Labels	2Δ	Fields	$SU(4)$ Dynkin Labels
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$2p + 2 = 6$	$A_{\mu}^{(1)}$	$(1, p - 2, 1) = (1, 0, 1)$
$\left(1, \frac{1}{2}\right)$	$2p + 3 = 7$	$\psi_{+\mu}^{(1)}$	$(1, p - 2, 0) = (1, 0, 0)$
$\left(\frac{1}{2}, 1\right)$	$2p + 3 = 7$	$\psi_{-\mu}^{(1)}$	$(0, p - 2, 1) = (0, 0, 1)$
$(1, 1)$	$2p + 4 = 8$	$h_{\mu\nu}$	$(0, p - 2, 0) = (0, 0, 0)$

$$\begin{aligned}
z_B = & \mathcal{D}s(s(-s^2(s^2-1)xyr^4 + (s(s-q)(qs-1)(x+1)\sqrt{x}y^{3/2} + s^3(1-qs)xy^2 - (q(s-q)x^2s^3 \\
& + q(s-q)s^3 + (s^2+1)((q^2+1)s^3 + qs^2 - 2(q^2+1)s + q)x)y + s^3(1-qs)x \\
& + s(s-q)(qs-1)(x+1)\sqrt{x}\sqrt{y})r^2 - q^2s^2(s^2-1)xy)p^4 + (s^2(s-q)(qs-1)(r^4 + q(s^2+1)^2r^2 \\
& + q^2)(x+1)\sqrt{x}y^{3/2} + qs(q^3xs^3 + r^2(-sr^2 + s^2+1)xs^3 + q^2(r^2(s^2+1) - s)xs^3 \\
& + qr^2((-2xs^4 - (x(x+3) + 1)s^2 + r^2xs + x^2 + 1)s^2 + x))y^2 + (s^2(2 - s(r^2 + s)(s^2-1))xq^4 \\
& + (r^2(s^2+1) - s)((x^2 + x + 1)s^4 + 2xs^2 + x)q^3 + s(s(-s^4 + s^2 + 2)xr^4 - (xs^8 + (x(2x+3) + 2)s^6 \\
& + (x(3x+7) + 3)s^4 + 3xs^2 - (x-1)^2)r^2 + s^3 + sx(-s^4 + (x+1)s^2 + 2))q^2 \\
& + r^2(-sr^2 + s^2+1)((x^2 + x + 1)s^4 + 2xs^2 + x)q + r^4s^4 + r^4s^4x^2 + r^2s^2(-s^3 + s + \\
& r^2(-s^4 + s^2+2))x)y + qs(q^3xs^3 + r^2(-sr^2 + s^2+1)xs^3 + q^2(r^2(s^2+1) - s)xs^3 \\
& + qr^2((-2xs^4 - (x(x+3) + 1)s^2 + r^2xs + x^2 + 1)s^2 + x)) + s^2(s-q)(qs-1)(r^4 + q(s^2+1)^2r^2 \\
& + q^2)(x+1)\sqrt{x}\sqrt{y})p^2 - qr(r^2 + q)(s^2(s^2+1)(q(s^3 + s - q) - 1)(x+1)\sqrt{x}y^{3/2} \\
& - s^4(s-q)(qs-1)xy^2 + (q-s)(qs-1)((x^2 + x + 1)s^4 + 2xs^2 + x)y - s^4(s-q)(qs-1)x \\
& + s^2(s^2+1)(q(s^3 + s - q) - 1)(x+1)\sqrt{x}\sqrt{y})p - p^3r(r^2 + q)(s^2(s^2+1)(q(s^3 + s - q) \\
& - 1)(x+1)\sqrt{x}y^{3/2} - s^4(s-q)(qs-1)xy^2 + (q-s)(qs-1)((x^2 + x + 1)s^4 + 2xs^2 + x)y \\
& - s^4(s-q)(qs-1)x + s^2(s^2+1)(q(s^3 + s - q) - 1)(x+1)\sqrt{x}\sqrt{y}) + q^2s(-s^2(s^2-1)xyr^4 \\
& + (s(s-q)(qs-1)(x+1)\sqrt{x}y^{3/2} + s^3(1-qs)xy^2 - (q(s-q)x^2s^3 + q(s-q)s^3 + (s^2+1)((q^2+1)s^3 \\
& + qs^2 - 2(q^2+1)s + q)x)y + s^3(1-qs)x + s(s-q)(qs-1)(x+1)\sqrt{x}\sqrt{y})r^2 - q^2s^2(s^2-1)xy))
\end{aligned}$$

$$\begin{aligned}
z_F = & -\mathcal{D}(s^{3/2}(s^2+1)(p^4rs^2\sqrt{x}\sqrt{y}(q+r^2)(\sqrt{x}(y+1)(qs-1) + x\sqrt{y}(s-q) + \sqrt{y}(s-q)) \\
& + p^3(y^{3/2}(q-s)(q^2s^2x(r^2s-1) - qr^2(s^2(x(s^2+x+4)+1) + x) + r^2sx(1-r^2s)) \\
& + \sqrt{y}(q-s)(q^2s^2x(r^2s-1) - qr^2(s^2(xs^2+x+4)+1) + x) + r^2sx(1-r^2s)) \\
& + (x+1)\sqrt{xy}(qs-1)(q^2s(s-r^2) + qr^2(s^4+4s^2+1) + r^2s^2(r^2-s)) + qr^2s^2(x+1)\sqrt{xy}^2(qs-1) \\
& + qr^2s^2(x+1)\sqrt{x}(qs-1) + p^2r(q+r^2)(-y^{3/2}(qs-1)(q^2sx - q(s^2(x(s^2+x+4)+1) + x) \\
& + s^3x)) - \sqrt{y}(qs-1)(q^2sx - q(s^2(x(s^2+x+4)+1) + x) + s^3x) + (x+1)\sqrt{xy}(q-s)(q^2s^3 \\
& - q(s^4+4s^2+1) + s) - qs^2\sqrt{x}(x+1)y^2(q-s) - qs^2(x+1)\sqrt{x}(q-s) + pq(y^{3/2}(q-s)(q^2s^2x(r^2s-1) \\
& - qr^2(s^2(x(s^2+x+4)+1) + x) + r^2sx(1-r^2s)) + \sqrt{y}(q-s)(q^2s^2x(r^2s-1) \\
& - qr^2(s^2(x(s^2+x+4)+1) + x) + r^2sx(1-r^2s)) + (x+1)\sqrt{xy}(qs-1)(q^2s(s-r^2) + qr^2(s^4+4s^2 \\
& + 1) + r^2s^2(r^2-s)) + qr^2s^2(x+1)\sqrt{xy}^2(qs-1) + qr^2s^2(x+1)\sqrt{x}(qs-1) \\
& + q^2rs^2\sqrt{x}\sqrt{y}(q+r^2)(\sqrt{x}(y+1)(qs-1) + x\sqrt{y}(s-q) + \sqrt{y}(s-q)))
\end{aligned}$$

$$\frac{1}{\mathcal{D}} = (q-s)(qs-1)(ps-r)(rs-p)(s\sqrt{x} - \sqrt{y})(s\sqrt{y} - \sqrt{x})(s - \sqrt{x}\sqrt{y})(s\sqrt{x}\sqrt{y} - 1)(qs - pr)(q - prs)$$

$$i_{s,p} = \text{Tr}_{s,p} [(-1)^F u^\delta t^{2(E+j_1)} \bar{y}^{2j_2} v^{R_2} w^{R_3}]$$

$$s \rightarrow ut^2, x \rightarrow u^{-2}t^2, y \rightarrow \bar{y}^2, p \rightarrow u^{-3/2}, q \rightarrow u^{-1}v, r \rightarrow u^{-1/2}w$$

$$i_{s,p} = -\frac{t^3}{\bar{y}\left(1-\frac{t^3}{\bar{y}}\right)} - \frac{t^3\bar{y}}{1-t^3\bar{y}} + \frac{t^2w}{v\left(1-\frac{t^2w}{v}\right)} + \frac{t^2v}{1-t^2v} + \frac{t^2}{w\left(1-\frac{t^2}{w}\right)}$$



$$\begin{aligned}
z_B(s, x, y, 1, 1, 1) &= \mathcal{N} s \left(-s^3(s+1)x^2 - s^3(s+1) + s^2((s((s-5)s-2) - 14)s^2 + s - 13)(x+1)\sqrt{x}y^{3/2} \right. \\
&\quad + s^2((s((s-5)s-2) - 14)s^2 + s - 13)(x+1)\sqrt{x}\sqrt{y} - y(s(s(s(s(s^2+s+4)+6) \\
&\quad + 31) + 5) + 26)x + s((2(s-2)s+11)s^2 + s+1) + ((2(s-2)s+11)s^3 + s^2 + s+1)x^2 \\
&\quad - 8x+1) + 6x) - s((2(s-2)s+11)s^3 + s^2 + s+1)x \\
&\quad \left. - sy^2(s(s((s+1)x^2 + s(2(s-2)s+11)x + s+x+1) + x) + x) \right) \\
z_F(s, x, y, 1, 1, 1) &= -4\mathcal{N}s^{3/2}(s^2+1)(\sqrt{x}(y+1) + x\sqrt{y} + \sqrt{y})(s^2(x+1)y + s^2(x+1) \\
&\quad + (s(s((s-1)s+4) - 1) + 1)\sqrt{x}\sqrt{y})
\end{aligned}$$

$$\mathcal{N} = \frac{1}{(s-1)^5(s\sqrt{x}-\sqrt{y})(s\sqrt{y}-\sqrt{x})(s-\sqrt{x}\sqrt{y})(s\sqrt{x}\sqrt{y}-1)}$$

$$\ln Z_B = \sum_{n=1}^{\infty} \frac{1}{n} z_B(s^n, x^n, y^n, p^n, q^n, r^n)$$

$$\ln Z_F = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z_F(s^n, x^n, y^n, p^n, q^n, r^n)$$

$$\ln Z_B = \sum_{n=1}^{\infty} \frac{(22 \cosh(\beta n) + 17 \cosh(2\beta n) + 6 \cosh(3\beta n) + \cosh(4\beta n) + 18) \operatorname{csch}^7\left(\frac{\beta n}{2}\right) \operatorname{sech}\left(\frac{\beta n}{2}\right)}{32\beta^2 n^3 (1-\omega_1^2)(1-\omega_2^2)}$$

$$\ln Z_F = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh(\beta n) (\cosh(\beta n) + \cosh(2\beta n) + 2) \operatorname{csch}^7\left(\frac{\beta n}{2}\right)}{2\beta^2 n^3 (1-\omega_1^2)(1-\omega_2^2)}$$

$$\begin{aligned}
ds^2 &= R_{AdS_5}^2 \left(\frac{dz^2}{z^2} + \frac{dx_i^2}{z^2} + d\Omega_5^2 \right) \\
&= \frac{R_{AdS_5}^2}{z^2} (dx_i^2 + dz^2 + z^2 d\Omega_5^2) \\
&= \frac{R_{AdS_5}^2}{z^2} (dx_i^2 + dy_j^2)
\end{aligned}$$

$$\nabla_{\text{conf}, g_{\mu\nu}}^2 \equiv \left(g^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{D-2}{4(D-1)} R \right)$$

$$\nabla_{x,\text{conf}}^2 e^{2\chi} g_{\mu\nu} \left(e^{-\chi \frac{D-2}{2}} \phi \right) = e^{-\frac{(D+2)\chi}{2}} \nabla_{\text{conf}, g_{\mu\nu}}^2 (\phi)$$

$$\nabla_{\text{conf}, g_{\mu\nu}}^2 G(x,y) = \frac{1}{\sqrt{-g}} \delta^D(x-y)$$

$$\begin{aligned}
\nabla_{\text{conf}, e^2\chi}^2 g_{\mu\nu} \left(e^{-\chi(x) \frac{D-2}{2}} G(x,y) \right) &= \frac{e^{-\frac{(D+2)\chi(y)}{2}} \times e^{\chi(y)D}}{\sqrt{-g e^{2\chi(x)}}} \delta^D(x-y) \\
&= \frac{e^{\frac{(D-2)\chi(y)}{2}}}{\sqrt{-g e^{2\chi(x)}}} \delta^D(x-y)
\end{aligned}$$

$$\nabla_{\text{conf}, e^2\chi}^2 g_{\mu\nu} \left(e^{-\chi(x) \frac{D-2}{2}} G(x,y) e^{-\chi(y) \frac{D-2}{2}} \right) = \frac{1}{\sqrt{-g e^{2\chi(x)}}} \delta^D(x-y)$$

$$G_{ge^{2\chi}}(x,y) = e^{-\chi(x) \frac{D-2}{2}} G_{g_{\mu\nu}}(x,y) e^{-\chi(y) \frac{D-2}{2}}$$

$$g_{\mu\nu} \rightarrow e^{2\chi} g_{\mu\nu} \Rightarrow G(x,y) \rightarrow e^{-\chi(x) \frac{D-2}{2}} G(x,y) e^{-\chi(y) \frac{D-2}{2}}$$

$$\langle \phi(x) \phi(y) \rangle = z_x^4 G(x,y) z_y^4$$



$$G(x,x')=\frac{\Gamma[4]}{4\pi^5}\left(\frac{1}{(x_i-x'_j)^2}\right)^4$$

$$\begin{aligned}\langle\phi(\tilde{x}_1)\phi(\tilde{x}_2)\rangle &= z_1^4G(\tilde{x}_1,\tilde{x}_2)z_2^4\\ &= \frac{\Gamma[4]}{4\pi^5}z_1^4\left(\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}\right)^4z_2^4\\ &= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(y_1-y_2)^2}{2z_1z_2}}\right)^4\\ &= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1y_1'-z_2y_2')^2}{2z_1z_2}}\right)^4\\ &= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2+2z_1z_2-2z_1z_2y_1'y_2'}{2z_1z_2}}\right)^4\\ &= \frac{\Gamma[4]}{2^6\pi^5}\left(\frac{1}{\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2}{2z_1z_2}+1-y_1'y_2'}\right)^4\\ &= \frac{\Gamma[4]}{4\pi^5}\left(\frac{1}{u+v}\right)^4\end{aligned}$$

$$v=2(1-y_1'\cdot y_2'), u=2\left(\frac{(x_1-x_2)^2+(z_1-z_2)^2}{2z_1z_2}\right)$$

$$(y_1'-y_2')^2=2(1-y_1'\cdot y_2')=v$$

$$-X_{-1}^2-X_0^2+\sum_1^{d-1}X_i^2=-1$$

$$\begin{aligned}X_i&=\frac{x_i}{z}, i=0,\ldots d-2\\ X_{-1}&=\frac{1}{2}\left(\frac{(z^2+x^2)}{z}+\frac{1}{z}\right)\\ X_{d-1}&=\frac{1}{2}\left(\frac{(z^2+x^2)}{z}-\frac{1}{z}\right)\end{aligned}$$

$$(X_1-X_2)^2=-2-2X_1\cdot X_2=-2+2\left(1+\frac{(x_1-x_2)^2}{2z_1z_2}+\frac{(z_1-z_2)^2}{2z_1z_2}\right)=u$$

$$ds^2=-(1+r^2)dt^2+\frac{dr^2}{1+r^2}+r^2(d\theta^2+\sin^2\theta d\phi_1^2+\cos^2\theta d\phi_2^2)+d\Omega_5^2$$

$$G^{(10)}((x,\theta),(x',\theta'))=\sum_{l=0}^{\infty}\sum_{\alpha=1}^{d_l}G_l^{(5)}(x,x')Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta')$$

$$\begin{aligned}-\nabla_{10}^2G^{(10)}((x,\theta),(x',\theta'))&=\delta^{(10)}(x-x',\theta-\theta')\\ -\nabla_5^2G_l^{(5)}(x,x')&=\delta^{(5)}(x-x')\end{aligned}$$

$$\int_{S^5}Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta)=1$$

$$T_{\mu\nu}^{(10)}=\sum_{l=0}^{\infty}\sum_{\alpha=1}^{d_l}(T_l)_{\mu\nu}^{(5)}Y_l^{\alpha}(\theta)Y_l^{*\alpha}(\theta)$$



$$\sum_{\alpha} Y_l^{\alpha}(\hat{n}) Y_l^{*\alpha}(\hat{n}') = \frac{d_l}{\Omega_5 P^l(\hat{n}, \hat{n}'=1)} P^l(\hat{n}, \hat{n}')$$

$$T_{\mu\nu}^{(10)} = \sum_{l=0}^{\infty} d_l \frac{(T_l)_{\mu\nu}^{(5)}}{\Omega_5}$$

$$\int_{S^5} T_{\mu\nu}^{(10)} = \sum_{l=0}^{\infty} d_l (T_l)_{\mu\nu}^{(5)}$$

$$M^2 = \Delta(\Delta - 4) = m_{KK}^2 = l(l + 4)$$

$$\begin{aligned} G_l^{(5)}(y, y') &= \frac{\Gamma(\Delta)}{2\pi^2 \Gamma(\Delta-1)} \frac{1}{u^\Delta} {}_2F_1\left(\Delta, \Delta - \frac{3}{2}, 2\Delta - 3, -\frac{4}{u}\right) \\ &= \frac{\Gamma(l+4)}{2\pi^2 \Gamma(l+3)} \frac{1}{u^{l+4}} {}_2F_1\left(l+4, l+5/2, 2l+5, -\frac{4}{u}\right) \end{aligned}$$

$$G^{(10)}(x, x') = \sum_{l=0}^{\infty} \sum_{\alpha=1}^{d_l} \frac{\Gamma(l+4)}{2\pi^2 \Gamma(l+3)} \frac{1}{u^{l+4}} {}_2F_1\left(l+4, l+5/2, 2l+5, -\frac{4}{u}\right) Y_l^{\alpha}(\Omega) Y_l^{*\alpha}(\Omega')$$

$$\left\langle T_{\mu\nu}^{(10)} \right\rangle = -2 \sum_{q=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\alpha=1}^{d_l} K_{l,\mu\nu}^q Y_l^{\alpha}(\Omega) Y_l^{*\alpha}(\Omega)$$

$$\begin{aligned} -K_{l,\tau\tau}^q &= -\frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)}{2\pi^2} \frac{\Gammal+4}{\Gamma[l+3](4+u_q)u_q^{l+6}} \left[5(u_q+2) {}_2F_1\left(l+\frac{5}{2}, l+5; 2l+5; -\frac{4}{u_q}\right) \right. \\ &\quad \left. + (lu_q) {}_2F_1\left(l+\frac{5}{2}, l+4; 2l+5; -\frac{4}{u_q}\right) \right] \\ &= -\frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)}{2\pi^2} \frac{\Gamma[l+5]}{\Gamma[l+3]u_q^{l+6}} \left[(l+5) {}_2F_1\left(l+\frac{5}{2}, l+6; 2l+5; -\frac{4}{u_q}\right) \right] \end{aligned}$$

$$\sum_{l=0}^{\infty} \sum_{\alpha=1}^{d_l} Y_l^{\alpha}(\Omega) Y_l^{*\alpha}(\Omega') = \sum_{l=0}^{\infty} \frac{(2l+4)\Gamma(2)}{4\Omega_5} C_l^{(2)}(\cos \Theta), \cos \Theta = 1 - \frac{v}{2}$$

$$\begin{aligned} \left\langle T_{00}^{(10)} \right\rangle &= \lim_{\Theta \rightarrow 0} \sum_{q=1}^{\infty} \sum_{l=0}^{\infty} \frac{4x^4\gamma^4(\theta)\sinh^2(q\beta)(2l+4)}{8\pi^5} \frac{\Gamma[(l+6)]}{\Gamma[l+3]u_q^{l+6}} \\ &\quad \times \left[{}_2F_1\left(l+\frac{5}{2}, l+6; 2l+5; -\frac{4}{u_q}\right) C_l^{(2)}(\cos \Theta) \right] \\ &= \lim_{\Theta \rightarrow 0} \sum_{q=1}^{\infty} \sum_{l=0}^{\infty} \frac{x^4\gamma^4(\theta)\sinh^2(q\beta)}{\pi^5} \frac{\Gamma[(l+6)]}{\Gamma[l+2](u_q+2)^{l+6}} \\ &\quad \times \left[{}_2F_1\left(\frac{l}{2}+3, \frac{l}{2}+\frac{7}{2}; l+3; \frac{4}{(u_q+2)^2}\right) C_l^{(2)}(\cos \Theta) \right] \end{aligned}$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(l+\alpha)}{\Gamma(l+\beta)} \left(\frac{z}{2}\right)^l F\left(\frac{l}{2}+\frac{\alpha}{2}, \frac{l}{2}+\frac{\alpha}{2}+\frac{1}{2}; l+\beta+1; z^2\right) C_l^{(\beta)}(\cos \Theta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1-z\cos \Theta)^\alpha}$$

$$\left\langle T_{00}^{(10)} \right\rangle = \lim_{\Theta \rightarrow 0} 2 \sum_{q=1}^{\infty} \frac{x^4\gamma^4(\theta)\sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5(u_q+2)^6 \left(1-\frac{2}{u_q+2}\cos \Theta\right)^6}$$

$$\left\langle T_{00}^{(10)} \right\rangle = \sum_{q=1}^{\infty} \frac{2x^4\gamma^4(\theta)\sinh^2(q\beta)\Gamma[6]}{\Gamma[2]\pi^5u_q^6}$$



$$\begin{aligned}
ds^2 &= dt^2 + d\Omega_3^2 \\
&= \frac{dr^2}{r^2} + d\Omega_3^2 \\
&= \frac{1}{r^2} (dr^2 + r^2 d\Omega_3^2)
\end{aligned}$$

$$\vec{r} = e^t \hat{n}$$

$$\begin{aligned}
e^{-t} G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') e^{-t'} &= \frac{\mathcal{N}}{|e^t \hat{n}_1 - e^{t'} \hat{n}'|^2} \\
\Rightarrow G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') &= \frac{\mathcal{N} e^t e^{t'}}{|e^t \hat{n} - e^{t'} \hat{n}'|^2} = \frac{\mathcal{N}}{\left| e^{\frac{t-t'}{2}} \hat{n} - e^{\frac{t'-t}{2}} \hat{n}' \right|^2}
\end{aligned}$$

$$\mathcal{N} = \frac{1}{4\pi^2}$$

$$\hat{n} = (\sin \theta \cos \phi_1, \sin \theta \sin \phi_1, \cos \theta \cos \phi_2, \cos \theta \sin \phi_2)$$

$$G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = \frac{\mathcal{N}}{e^{t-t'} + e^{t'-t-2} (\cos \theta \cos \theta' \cos (\phi_1 - \phi'_1) + \sin \theta \sin \theta' \cos (\phi_2 - \phi'_2))}$$

$$\begin{aligned}
G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') &= G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') \\
&+ \sum_{q=-\infty; q \neq 0}^{\infty} \frac{\mathcal{N}}{\left(e^{t-t'+q\beta} + e^{t'-t-q\beta-2} (\cos \theta \cos \theta' \cos (\phi_1 - \phi'_1 - iq\beta\omega_1) + \sin \theta \sin \theta' \cos (\phi_2 - \phi'_2 - iq\beta\omega_2)) \right)}
\end{aligned}$$

$$G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') - G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') = \gamma^2(\theta) \sum_{q=1}^{\infty} \frac{2\mathcal{N}}{\beta q \sinh(\beta q)}$$

$$\begin{aligned}
\lim_{\beta \rightarrow 0} G_{S^3 \times S^1}(t, \hat{n}; t', \hat{n}') - G_{S^3 \times R}(t, \hat{n}; t', \hat{n}') &= 2 \sum_{q=1}^{\infty} \frac{\mathcal{N}}{q^2 \beta^2 (1 - \omega_2^2 \sin^2(\theta) - \omega_1^2 \cos^2(\theta))} \\
&= 2 \sum_{q=1}^{\infty} \frac{\mathcal{N} \gamma^2(\theta)}{q^2 \beta^2}
\end{aligned}$$

$$\begin{aligned}
&\partial_t \partial_{t'} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}') \Big|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} \\
&\approx \sum_{q=1}^{\infty} - \frac{8\gamma^6(\theta) \operatorname{csch}(\beta q) \left(2 - \beta q \coth(\beta q) \left(\frac{1}{2\gamma^2(\theta)} \right) \right)}{4\pi^2 \beta^3 q^3} \\
&\approx \sum_{q=1}^{\infty} - \frac{4\gamma^6(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3}
\end{aligned}$$



$$\begin{aligned}
\partial_{\phi_1} \partial_{\phi'_1} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t', \hat{n}') \rightarrow (t, \hat{n})} &\approx \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_1^2 \cos^4(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\partial_{\phi_2} \partial_{\phi'_2} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &\approx \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_2^2 \sin^4(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\partial_{\phi_1} \partial_{\phi'_2} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &= \partial_{\phi_2} \partial_{\phi'_1} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')}^{\infty} \\
&\approx \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_1 \omega_2 \sin^2(2\theta) \operatorname{csch}(\beta q)}{\pi^2 (\beta q)^3} \\
\partial_t \partial_{\phi'_2} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &= \partial_{\phi_2} \partial_{t'} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} \\
&\approx \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{i4\omega_2 \sin^2(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\partial_t \partial_{\phi'_1} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &= \partial_{\phi_1} \partial_{t'} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} \\
&\approx \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{i4\omega_1 \cos^2(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\partial_{\theta} \partial_{\phi'_1} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &= \partial_{\theta} \partial_{\phi'_2} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} \approx O\left(\frac{1}{\gamma^2(\theta)}\right)
\end{aligned}$$

$$\begin{aligned}
\partial_{\theta} \partial_{\theta'} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &\approx \gamma^4(\theta) \sum_{q=1}^{\infty} \frac{\operatorname{csch}^2(\beta q)}{\pi^2 \beta^2 q^2} \\
\partial_t \partial_{\theta'} G_{S^3 \times S^1}(t, \hat{n}, t', \hat{n}')|_{(t, \hat{n}) \rightarrow (t', \hat{n}')} &\approx O\left(\frac{1}{\gamma^2(\theta)}\right)
\end{aligned}$$

$$S = \frac{1}{2} \int \sqrt{-g} d^4x (-g^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - \xi R \phi^2), \xi = \frac{1}{6}$$

$$T_{\mu\nu} = \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \nabla_{\alpha} \phi \nabla^{\alpha} \phi + \xi G_{\mu\nu} \phi^2 + \xi (g_{\mu\nu} \nabla^{\alpha} \nabla_{\alpha} - \nabla_{\mu} \nabla_{\nu}) \phi^2$$

$$\begin{aligned}
T_{\mu}^{\mu} &= \left(1 - \frac{D}{2}\right) \nabla_{\alpha} \phi \nabla^{\alpha} \phi + \xi R \left(1 - \frac{D}{2}\right) \phi^2 + \xi (2(D-1) \nabla_{\alpha} \phi \nabla^{\alpha} \phi + 2(D-1) \phi \nabla^2 \phi) \\
&= \left(1 - \frac{D}{2}\right) \phi (-\nabla^2 + \xi R) \phi = 0
\end{aligned}$$

$$\begin{aligned}
\langle T_{00} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4 \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\langle T_{\phi_1 \phi_1} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_1^2 \cos^4(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\langle T_{\phi_2 \phi_2} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_2^2 \sin^4(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\langle T_{\theta\theta} \rangle &= \gamma^4(\theta) \sum_{q=1}^{\infty} \frac{\operatorname{csch}^2(\beta q)}{\pi^2 \beta^2 q^2} \\
\langle T_{0\phi_1} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_1 \cos^2(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\langle T_{0\phi_2} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_2 \sin^2(\theta) \operatorname{csch}(\beta q)}{\pi^2 \beta^3 q^3} \\
\langle T_{\phi_1 \phi_2} \rangle &= \gamma^6(\theta) \sum_{q=1}^{\infty} \frac{4\omega_1 \omega_2 \sin^2(2\theta) \operatorname{csch}(\beta q)}{\pi^2 (\beta q)^3}
\end{aligned}$$

$$T_{\mu\nu} = \gamma^4(\theta) (4u_{\mu} u_{\nu} + g_{\mu\nu}) \sum_{q=1}^{\infty} \frac{1}{\pi^2 \beta^3 \sinh(q\beta) q^3}$$

$$\lim_{\beta \rightarrow 0} \langle T_{\mu\nu} \rangle = \frac{\pi^2}{90\beta^4} \gamma^4(\theta) (g_{\mu\nu} + 4u_{\mu} u_{\nu}), u_{\mu} = \gamma(-1, 0, \omega_1 \cos^2 \theta, \omega_2 \sin^2 \theta).$$



$$T_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + M^2 A_\mu A_\nu - g_{\mu\nu} \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{M^2}{2} A_\alpha A^\alpha \right)$$

$$\begin{aligned} \langle T_{\mu\nu} \rangle &= \lim_{x_1 \rightarrow x_2} \left[\langle F_{\mu\alpha}(x_1) F_\nu{}^\alpha(x_2) \rangle + M^2 \langle A_\mu(x_1) A_\nu(x_2) \rangle - \frac{g_{\mu\nu}}{4} \left(\langle F_{\alpha\beta}(x_1) F^{\alpha\beta}(x_2) \rangle + \langle A_\alpha(x_1) A^\alpha(x_2) \rangle \right) \right] \\ &= \lim_{x_1 \rightarrow x_2} \left[\partial_\mu^{x_1} \partial_\nu^{x_2} \tilde{\Pi}_\alpha{}^\alpha(x_1, x_2) - \partial_\mu^{x_1} \partial^{x_2, \alpha} \tilde{\Pi}_{\alpha, \nu}(x_1, x_2) - \partial_\alpha^{x_1} \partial_\nu^{x_2} \tilde{\Pi}_\mu{}^\alpha(x_1, x_2) + \partial_\alpha^{x_1} \partial^{x_2, \alpha} \tilde{\Pi}_{\mu, \nu}(x_1, x_2) \right. \\ &\quad \left. - \frac{g_{\mu\nu}}{4} \left(\partial_\alpha^{x_1} \partial^{x_2, \alpha} \tilde{\Pi}_\beta{}^\beta(x_1, x_2) + \partial_\beta^{x_1} \partial^{x_2, \beta} \tilde{\Pi}_\alpha{}^\alpha(x_1, x_2) - \partial_\alpha^{x_1} \partial^{x_2, \beta} \tilde{\Pi}_\beta{}^\alpha(x_1, x_2) - \partial_\beta^{x_1} \partial^{x_2, \alpha} \tilde{\Pi}_\alpha{}^\beta(x_1, x_2) \right) \right. \\ &\quad \left. + M^2 \tilde{\Pi}_{\mu, \nu}(x_1, x_2) - \frac{g_{\mu\nu}}{4} M^2 \tilde{\Pi}_\alpha{}^\alpha(x_1, x_2) \right] \end{aligned}$$

$$\tilde{\Pi}_{\mu, \nu}(x_1, x_2) = \sum_{q=-\infty}^{\infty} \Pi_{\mu, \nu}(x_1, R^q(x_2))$$

$$\langle A_\mu(x_1) A_\nu(x_2) \rangle = \Pi_{\mu, \nu}(x_1, x_2) = -\frac{\partial^2 u}{\partial x_1^\mu \partial x_2^\nu} g_0(u) + \frac{\partial u}{\partial x_1^\mu} \frac{\partial u}{\partial x_2^\nu} g_1(u)$$

$$\begin{aligned} g_0(u) &= \frac{\mathcal{N}}{(u)^\Delta} \left[(4-\Delta) {}_2F_1\left(\Delta, \frac{1}{2}(-3+2\Delta); -3+2\Delta; -\frac{1}{u}\right) \right. \\ &\quad \left. - \frac{2+u}{u} {}_2F_1\left(\Delta+1, \frac{1}{2}(-3+2\Delta); -3+2\Delta; -\frac{1}{u}\right) \right] \\ g_1(u) &= \frac{\mathcal{N}}{(u)^\Delta} \left[\frac{2(2+u)(4-\Delta)}{u(u+4)} {}_2F_1\left(\Delta, \frac{1}{2}(-3+2\Delta); -3+2\Delta; -\frac{1}{u}\right) \right. \\ &\quad \left. - 2 \frac{4+(u+2)^2}{u^2(u+4)} {}_2F_1\left(\Delta+1, \frac{1}{2}(-3+2\Delta); -3+2\Delta; -\frac{1}{u}\right) \right] \end{aligned}$$

$$\mathcal{N} = \frac{\Gamma(\Delta+1)}{2\pi^2(3-\Delta)(\Delta-1)\Gamma(\Delta-1)}$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{0\lambda}(x_1) F_0^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) f^{(1)}(x)$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{0\lambda}(x_1) F_{\phi_1}^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) \sin^2(\theta) f^{(1)}(x)$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{0\lambda}(x_1) F_{\phi_2}^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) \cos^2(\theta) f^{(1)}(x)$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{\phi_1\lambda}(x_1) F_{\phi_1}^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) \sin^4(\theta) f^{(1)}(x)$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{\phi_2\lambda}(x_1) F_{\phi_2}^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) \cos^4(\theta) f^{(1)}(x)$$

$$\lim_{x_1 \rightarrow x_2} \langle F_{\phi_1\lambda}(x_1) F_{\phi_2}^\lambda(x_2) \rangle = 4x^4 \gamma^4(\theta) \sin^2(\theta) \cos^2(\theta) f^{(1)}(x)$$

$$f^{(1)}(x) = \sum_{q=1}^{\infty} \sinh^2(\beta q) \left(g_0''(u_q) + g_1'(u_q) \right) (3 \cosh(\beta q) + \beta q x^2 \sinh(\beta q)) + 2g_0'(u_q) + 2g_1(u_q)$$

$$g_{\mu\nu} \lim_{x_1 \rightarrow x_2} \langle F_{\alpha\beta}(x_1) F^{\alpha\beta}(x_2) \rangle = O(\gamma^2(\theta))$$

$$\langle T_{\mu\nu} \rangle = 4\mathcal{F}_\Delta^{(1)}(x) \gamma^4(\theta) \begin{pmatrix} 1 & 0 & 0 & \sin^2(\theta) & \cos^2(\theta) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sin^2(\theta) & 0 & 0 & \sin^4(\theta) & \sin^2(\theta) \cos^2(\theta) \\ \cos^2(\theta) & 0 & 0 & \sin^2(\theta) \cos^2(\theta) & \cos^4(\theta) \end{pmatrix}$$



$$\begin{aligned}\mathcal{F}_{\Delta}^{(1)}(x) &= \sum_{q=1}^{\infty} \left[\sinh^2(\beta q) \left(g_0''(u_q) + g_1'(u_q) \right) (3 \cosh(\beta q) + \beta q x^2 \sinh(\beta q)) \right. \\ &\quad \left. + 2g_0'(u_q) + 2g_1(u_q) + M^2 g_1(u_q) \right] \\ &= \sum_{q=1}^{\infty} \left[\sinh^2(\beta q) \left(g_0''(u_q) + g_1'(u_q) \right) (3 \cosh(\beta q) + \beta q x^2 \sinh(\beta q)) \right. \\ &\quad \left. + 2g_0'(u_q) + 2g_1(u_q) + (\Delta - 1)(\Delta - 3)g_1(u_q) \right]\end{aligned}$$

$$T_{\mu\nu} = \mathcal{F}_{\Delta}^{(1)}(x) \gamma^4(\theta) 4w_{\mu}w_{\nu}$$

$$y^2 = \frac{r^2(1-a^2\cos^2\theta - b^2\sin^2\theta) + a^2\sin^2\theta + b^2\cos^2\theta - a^2b^2}{\Sigma_a\Sigma_b}$$

$$\gamma(x,\theta)(\partial_t-\omega_1\partial_{\phi_1}-\omega_2\partial_{\phi_2}),\gamma(x,\theta)=\frac{1}{\sqrt{1+\frac{r^2}{\gamma^2(\theta)}}}=\frac{1}{\sqrt{1+2x^2}}$$

$$\begin{aligned}\frac{\partial u}{\partial \tau^E} \frac{\partial u}{\partial \tau'^E} &= -4r^4 \sinh^2(q\beta) \\ &= -16\gamma^4(\theta) x^4 \sinh^2(q\beta) \\ \frac{\partial^2 u}{\partial \tau^E \partial \tau'^E} &= -2r^2 \cosh(q\beta) \\ &= -4\gamma^2(\theta) x^2 \cosh(q\beta)\end{aligned}$$

$$\gamma=\frac{1}{\sqrt{1-\omega^2\sin^2\theta}}\approx\frac{\sqrt{2}}{\sqrt{\delta^2+\delta\theta^2}},1-\omega^2\approx\frac{\delta^2}{2}$$

$$TJ^{\frac{1}{D-2}}=f\left(\frac{M}{j^{\frac{D-3}{D-2}}}\right)$$

$$\sum y_i^2 = z^2$$

$$\Sigma_{l,\alpha} \, Y_l^\alpha(\theta) Y_l^{*\alpha}(\theta') = \delta^{(5)}(\theta - \theta')$$

$${}_2F_1(a,b,2b,z)=\left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(a/2,a/2+1/2,b+1/2,\frac{z^2}{(2-z)^2}\right)$$

Formación de un agujero negro cuántico por colapso interior de una partícula oscura o blanca.

Modelo Unificado.

$$R^a_{bcd}=\Gamma^a_{bd,c}-\Gamma^a_{bc,d}+\Gamma^e_{bd}\Gamma^a_{ce}-\Gamma^e_{bc}\Gamma^a_{de},$$

$$\Gamma^a_{bd}=\frac{1}{2}g^{ae}(g_{be,d}+g_{ed,b}-g_{bd,e})$$

$$R_{ab}=g^{cd}R_{acbd}$$

$$T_{(ab)}=\frac{1}{2}(T_{ab}+T_{ba}),T_{[ab]}=\frac{1}{2}(T_{ab}-T_{ba}).$$

$$\mathcal{A}=\frac{1}{2}\int d^4x\sqrt{-g}[R+2\mathcal{L}_m]$$

$$N^b_a\equiv h^b_a-e_ae^b=g^b_a+u_au^b-e_ae^b,N^a{}_a=2$$



$$\varepsilon_{ab} \equiv \varepsilon_{abc} e^c = \eta_{dabc} e^c u^d$$

$$\begin{aligned} \dot{X}^{a..b}{}_{c..d} &\equiv u^e \nabla_e X^{a..b}{}_{c..d}, \\ D_e X^{a..b}{}_{c..d} &\equiv h^a{}_f h^p{}_{c...} h^b{}_g h^q{}_d h^r{}_e \nabla_r X^{f..g}{}_{p..q}, \\ \hat{X}^{a..b}{}_{c..d} &\equiv e^f D_f X^{a..b}{}_{c..d}, \\ \delta_f X^{a..b}{}_{c..d} &\equiv N_a{}^f \dots N_b{}^g N_h{}^c \dots N_i{}^d N_f{}^j D_j X_{f..g}{}^{i..j} \end{aligned}$$

$$\{\phi,\xi,\Theta,\mathcal{A},\Omega,\Sigma,\mathcal{E},\mathcal{H}\}$$

$$\{a^b,\mathcal{A}^a,\Omega^a,\Sigma^a,\mathcal{E}^a,\mathcal{H}^a\}$$

$$\{\zeta^{ab},\Sigma^{ab},\mathcal{E}^{ab},\mathcal{H}^{ab}\}$$

$$\begin{aligned} \mathcal{A} &= e_a \dot{u}^a, \mathcal{A}^a = N_{ab} \dot{e}^b, \\ \tilde{\theta} &= \delta_a u^a, \bar{\theta} = a_b u^b, \tilde{\theta} + \bar{\theta} = D_a u^a = \Theta, \\ \Omega &= \frac{1}{2} \varepsilon^{abc} D_{[a} u_{b]} e_a, \Omega^a = \frac{1}{2} \varepsilon^{abd} D_{[a} u_{b]} N_d^a, \\ \Sigma &= \sigma^{ab} \left(e_a e_b - \frac{1}{2} N_{ab} \right), \Sigma_a = \sigma_{cd} e^c N_a^d, \\ \Sigma_{ab} &= \left(N_{(a}^c N_{b)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \sigma_{cd}, \\ \sigma_{ab} &= \left(n_{(a}^c h_{b)}^d - \frac{1}{3} h_{ab} h^{cd} \right) D_c u_d, \\ a_b &= e^c D_c e_b = \hat{e}_b, \phi = \delta_a e^a, \\ \zeta_{ab} &= \left(N_{(a}^c N_{b)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \delta_c e_d, \\ \xi &= \frac{1}{2} \varepsilon^{ab} \delta_a e_b, \end{aligned}$$

$$\begin{aligned} \mathcal{E} &= C^{ab}{}_{cd} u^c u^d \left(e_a e_b - \frac{1}{2} N_{ab} \right) \\ \mathcal{E}_a &= C_{cdef} u^e u^f e^c N^d{}_a, \mathcal{E}_{ab} = C_{\{ab\}cd} u^c u^d \\ \mathcal{H} &= \frac{1}{2} \varepsilon^a{}_{de} C^{deb}{}_c u^c \left(e_a e_b - \frac{1}{2} N_{ab} \right) \\ \mathcal{H}_a &= \frac{1}{2} \varepsilon_{cfe} C^{fe}{}_{dh} u^h e^c N^d{}_a \\ \mathcal{H}_{ab} &= \frac{1}{2} \varepsilon_{\{a}{}^{de} C_{b\}cde} u^c \end{aligned}$$

$$\{\mu,p,Q,\Pi\},$$

$$\{Q^a,\Pi^a\},$$



$$\begin{aligned}\mu &= T_{ab}u^au^b, \\ p &= \frac{1}{3}T_{ab}(e^ae^b + N^{ab}), \\ \Pi &= \frac{1}{3}T_{ab}(2e^ae^b - N^{ab}), \\ Q &= \frac{1}{2}T_{ab}e^au^b, \\ Q_a &= T_{cd}N^a{}_cu^d, \\ \Pi_a &= T_{cd}N^a{}_ce^d, \\ \Pi_{ab} &= T_{\{ab\}}.\end{aligned}$$

$$\begin{aligned}p_r &= p + \Pi \\ p_\perp &= p - \frac{1}{2}\Pi\end{aligned}$$

$$\begin{aligned}\hat{\phi} &= -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \varepsilon, \\ \hat{\varepsilon} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} &= -\frac{3}{2}\phi\left(\varepsilon + \frac{1}{2}\Pi\right), \\ \hat{p} + \hat{\Pi} &= -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\mathcal{A}, \\ \hat{\mathcal{A}} &= -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{2}(\mu + 3p), \\ \hat{K} &= -\phi K,\end{aligned}$$

$$\begin{aligned}0 &= -\mathcal{A}\phi + \frac{1}{3}(\mu + 3p) - \varepsilon + \frac{1}{2}\Pi, \\ K &= \frac{1}{3}\mu - \varepsilon - \frac{1}{2}\Pi + \frac{1}{4}\phi^2.\end{aligned}$$

$$K=K_0e^{-\rho}$$

$$\rho=2\ln{(r/r_0)},$$

$$\begin{aligned}X &= \frac{\phi_{,\rho}}{\phi}, \quad Y = \frac{\mathcal{A}}{\phi}, \quad \mathcal{K} = \frac{K}{\phi^2}, \quad E = \frac{\varepsilon}{\phi^2} \\ \mathbb{M} &= \frac{\mu}{\phi^2}, \quad \mathbb{P} = \frac{\Pi}{\phi^2}, \quad P = \frac{p}{\phi^2},\end{aligned}$$

$$\begin{aligned}Y_{,\rho} &= M + 3P - 2Y(X + Y + 1) \\ \mathcal{K}_{,\rho} &= -\mathcal{K}(1 + 2X) \\ P_{,\rho} + \mathbb{P}_{,\rho} &= -2Y(M + \mathbb{P}) - 2P(2X + Y) \\ &\quad - \mathbb{P}(4X + 3)\end{aligned}$$

$$\begin{aligned}2\mathbb{M} + 2P + 2\mathbb{P} + 2X - 2Y + 1 &= 0 \\ 1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P} &= 0 \\ 2\mathbb{M} + 6P + 3\mathbb{P} - 6Y - 6E &= 0\end{aligned}$$

$$\mathcal{A} = \frac{1}{2A\sqrt{B}} \frac{dA}{d\rho}, \phi = \frac{1}{\sqrt{B}}$$

$$K = \frac{1}{C_0} e^{-\rho} = K_0 e^{-\rho}$$

$$Y = \frac{1}{2} \frac{A_{,\rho}}{A}, \mathcal{K} = K_0 B(\rho) e^{-\rho}$$

$$\mathcal{A} = \frac{1}{2\sqrt{B}} \frac{dA}{dr}, \phi = \frac{2}{r\sqrt{B}}$$

$$K = \frac{\bar{K}_0}{r^2}$$

$$Y = \frac{1}{4} \frac{A_{,r}}{A}, \mathcal{K} = \bar{K}_0 B(r).$$

$$[g_{ab}^S] = [N_{ab} + u_a u_b] = 0,$$

$$[\bar{K}_{cd}] = \left[\frac{1}{2} \phi N_{ab} - u_a u_b \mathcal{A} \right] = 0,$$

$$[u_a] = 0, [N_{ab}] = 0$$

$$[\mathcal{A}] = 0, [\phi] = 0$$

$$T_{ab}^S = (N_{ab} + u_a u_b) [\bar{K}] - [\bar{K}_{cd}] =$$

$$= u_a u_b [\phi + 2\mathcal{A}] + N_{ab} \left[\frac{\phi}{2} + \mathcal{A} \right],$$

$$0 = [\mathcal{K}] = [P + \mathbb{P} - Y]$$

$$[P + \mathbb{P}] = 0$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = P \left(\mathbb{M} - 2\mathbb{P} - 3\mathcal{K} + \frac{7}{4} \right)$$

$$+ \mathbb{P} \left(\mathbb{M} - 3\mathcal{K} + \frac{1}{4} \right)$$

$$+ \left(\frac{1}{4} - \mathcal{K} \right) \mathbb{M} - P^2 - \mathbb{P}^2$$

$$\mathcal{K}_{,\rho} = -2\mathcal{K} \left(\mathcal{K} - \frac{1}{4} - \mathbb{M} \right)$$

$$\mathbb{P} = \frac{\mu_0^2 P_0 (1-h) e^{2\rho} \left(1 - \frac{\mu_0 e^\rho}{3}\right)^{h/2}}{18 \left(1 - \frac{\mu_0 e^\rho}{3}\right)^2 \left[\left(1 - \frac{\mu_0 e^\rho}{3}\right)^{h/2} + 3P_0 \right]^2},$$



$$P = \frac{\mu_0 e^\rho}{\left[\left(1 - \frac{\mu_0 e^\rho}{3}\right)^{h/2} + 3P_0 \right]^2} \left[\frac{3P_0^2 + \left(1 - \frac{\mu_0 e^\rho}{3}\right)^h}{4 \left(\frac{\mu_0 e^\rho}{3} - 1\right)} + \frac{2P_0[4(h+5)\mu_0 e^\rho - 72] \left(1 - \frac{\mu_0 e^\rho}{3}\right)^{h/2}}{(4\mu_0 e^\rho - 12)^2} \right],$$

$$\mathcal{K} = -\frac{3K_0 e^{\rho/2}}{16\mu_0 e^{3\rho/2} - 12K_0 e^{\rho/2}}$$

$$Y = \frac{\mu_0 e^\rho (3K_0 - \mu_0 e^\rho)^{h/2}}{2(\mu_0 e^\rho - 3K_0)[(3K_0 - \mu_0 e^\rho)^{h/2} + 9K_0 P_0]}$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$A = A_0 \left[\left(1 - \frac{\mu_0}{3} r^2\right)^{h/2} + 3P_0 \right]^{2/h}$$

$$B = \frac{3}{3 - \mu_0 r^2}$$

$$p_r = -\frac{\mu_0 \left[\left(1 - \frac{\mu_0 r^2}{3}\right)^{h/2} + P_0 \right]}{\left(1 - \frac{\mu_0 r^2}{3}\right)^{h/2} + 3P_0},$$

$$p_\perp = p_r - \frac{\mu_0^2 P_0 r^2 (1-h) \left(1 - \frac{\mu_0 r^2}{3}\right)^{\frac{h}{2}-1}}{3 \left[\left(1 - \frac{\mu_0 r^2}{3}\right)^{h/2} + 3P_0 \right]^2},$$

$$\mathbb{P} = \frac{1}{6} \mathbb{M} (1 - 4\mathcal{K}),$$

$$\mathcal{P}_{,\rho} + \mathcal{P}^2 + \mathcal{P} \left(3\mathcal{K} - \mathbb{M} - \frac{7}{4} \right) = 0$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$A = \frac{A_0 (e^{\mathcal{P}_0} r_0^2 + r^2)^2}{\sqrt{3 - \mu_0 r^2}}$$

$$B = \frac{3}{3 - \mu_0 r^2}$$

$$p_r = \frac{3 - \mu_0 r^2}{3(r^2 + e^{\mathcal{P}_0} r_0^2)}$$

$$p_\perp = \frac{\mu_0^2 r^2}{4(3 - \mu_0 r^2)} + p_r$$

$$\begin{aligned}
ds^2 &= -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\
A &= A_0 \psi (12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4)^{\frac{4b-5}{8b-4}} \\
\psi &= \exp \left(\frac{\sqrt{3}(4b+1) \tan^{-1} \left(\frac{a(3-12b) + 2b(4b-2)r^2}{a\sqrt{48b^2-24b-9}} \right)}{(2-4b)\sqrt{(4b-3)(4b+1)}} \right), \\
B &= \frac{12(br^2 - a)^2}{12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4}, \\
\mu &= \frac{(4b+1)(9a^2 - 7abr^2 + 2b^2r^4)}{12(br^2 - a)^3}, \\
p_r &= \frac{1}{a - br^2}, \\
p_{\perp} &= \frac{[4b^2r^4 + 3a(4a + r^2) - b(12ar^2 + 2r^4)]^{-1}}{48(br^2 - a)^3} \times \\
&\quad \{9a^3(64a + 19r^2) \\
&\quad - b^3(624a^2r^4 + 800ar^6 + 100r^8) \\
&\quad + ab^2r^2(432a^2 + 1608ar^2 + 356r^4) \\
&\quad - 9a^2br^2(168a + 47r^2) \\
&\quad + 80b^4r^6(4a + 2r^2) - 64b^5r^8\}.
\end{aligned}$$

$$\begin{aligned}
P_{,\rho} + P^2 - P \left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4} \right) \\
+ \left(\frac{1}{4} - \mathcal{K} \right) \mathbb{M} = \\
\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P} \left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4} \right), \\
\mathcal{K}_{,\rho} = -2\mathcal{K} \left(\mathcal{K} - \frac{1}{4} - \mathbb{M} \right).
\end{aligned}$$

$$\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P} \left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4} \right) = 0$$

$$P = P_0 - \mathbb{P}, \mathbb{M} = \mathbb{M}_0 + \alpha \mathbb{M}_1$$

$$\begin{aligned}
\mathbb{P} = \frac{1}{6} [&P_0(4\alpha \mathbb{M}_1 + 4\mathbb{M}_0 - 12\mathcal{K} + 7) - 4P_{0,\rho} \\
&- 4P_0^2 - (4\mathcal{K} - 1)(\alpha \mathbb{M}_1 + \mathbb{M}_0)].
\end{aligned}$$

$$\begin{aligned}
\mathcal{K} &= \frac{e^F}{\mathcal{K}_* - 2 \int e^F d\rho} \\
F &= \int \frac{1}{2} (4\alpha \mathbb{M}_1 + 4\mathbb{M}_0 + 1) d\rho
\end{aligned}$$



$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$A = A_0 \left(\sqrt{3 - \mu_0 r^2} + P_0 \right)^2$$

$$B = \frac{3}{3 - \mu_0 r^2}$$

$$p = -\frac{\mu_0(P_0 + 3\sqrt{3 - \mu_0 r^2})}{3(P_0 + \sqrt{3 - \mu_0 r^2})}$$

$$\mathbb{M}_1 = \frac{\mu_1 e^\rho}{K_0} \mathcal{K}$$

$$\mathcal{K} = \frac{3K_0}{4(3K_0 - 4e^\rho(\alpha\mu_1 + \mu_0))}.$$

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A = \frac{A_0 \sqrt{3 - \mu_0 r^2} (3 - P_0^2 r_0^2 - \mu_0 r^2)}{\sqrt{3 - r^2} (\alpha \mu_1 + \mu_0)} \times$$

$$\exp \left(2 \tanh^{-1} \left(\frac{\sqrt{3 - \mu_0 r^2}}{P_0 r_0} \right) \right),$$

$$B = \frac{3}{3 - r^2 (\alpha \mu_1 + \mu_0)},$$

$$p_r = \frac{\mu_0 [3 - r^2 (\alpha \mu_1 + \mu_0)] (3 \sqrt{3 - \mu_0 r^2} + P_0 r_0)}{3 (\mu_0 r^2 - 3) (\sqrt{3 - \mu_0 r^2} + P_0 r_0)}$$

$$p_\perp = -192(\mu_0 r^2 - 3)^2 (r^2 (\beta \mu_1 + \mu_0) - 3) p_r \\ - \frac{36 \beta \mu_1 r^2 (8 \mu_0 (r^2 (\beta \mu_1 + \mu_0) - 3) - 12 \beta \mu_1)}{48 (\mu_0 r^2 - 3)^2 (12 - 4 r^2 (\beta \mu_1 + \mu_0))}$$

$$P = P_0 + P_1, Y = Y_0 + Y_1$$

$$A \rightarrow A_0(\rho) \exp \left(\int Y_1 d\rho \right)$$

$$B \rightarrow B_0(\rho)$$

$$C \rightarrow C_0(\rho)$$

$$P_{1,\rho} + P_1^2 + P_1 \left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4} \right) = 0,$$

$$Y_1 = P_1$$

$$P_1 = \frac{e^F}{P_* + \int e^F d\rho}$$

$$F = \int \left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4} \right) d\rho$$



$$P=P_0+P_1, \mathbb{M}=\mathbb{M}_0+\mathbb{M}_1,$$

$$\mathcal{K}=\frac{1}{\mathcal{K}_0+\mathcal{K}_1},$$

$$A\rightarrow A_0(\rho)$$

$$B^{-1}\rightarrow B_0(\rho)+\frac{e^{\rho}}{K_0}\mathcal{K}_1$$

$$C\rightarrow C_0(\rho)$$

$$\mathcal{K}_{1,\rho}=-\mathcal{K}_1\Phi+\Gamma$$

$$\Phi=\frac{12\mathcal{K}_0-4\mathbb{M}_0-1+Y_0(8\mathcal{K}_0-8\mathbb{M}_0-2)}{2(1+2Y_0)}$$

$$\Gamma=\frac{8\mathcal{K}_0^2(Y_0+1)+\mathcal{K}_0(4\mathbb{M}_0+1)(2Y_0+1)-4}{1+2Y_0}$$

$$P_1=\frac{\mathcal{K}_0^2+\mathcal{K}_1\mathcal{K}_0-1}{\mathcal{K}_0+\mathcal{K}_1}$$

$$\mathbb{M}_1=-\frac{[\mathcal{K}_0(\mathcal{K}_0+\mathcal{K}_1)-1](2Y_0+3)}{(\mathcal{K}_0+\mathcal{K}_1)(2Y_0+1)}$$

$$\mathcal{K}_1=e^{-F}\Big(\mathcal{K}_*-\int e^F\Gamma d\rho\Big)$$

$$F=\int\Phi d\rho$$

$$\mathbb{P}=\mathbb{P}_0+\mathbb{P}_1, Y=Y_0+Y_1$$

$$Y_1=\mathbb{P}_1$$

$$\mathbb{P}_1=\frac{e^F}{\mathbb{P}_*+\int e^F d\rho}$$

$$F=\int\left(3\mathcal{K}_0-\mathbb{M}_0+2P_0+2\mathbb{P}_0-\frac{1}{4}\right)d\rho$$

$$\mathbb{P}=\mathbb{P}_0+\mathbb{P}_1, \mathbb{M}=\mathbb{M}_0+\mathbb{M}_1$$

$$\mathcal{K}=\frac{1}{\mathcal{K}_0+\mathcal{K}_1}$$

$$\mathcal{K}_{1,\rho}=\mathcal{K}_1\Phi+\Gamma$$

$$\Phi=\frac{Y_0(8\mathcal{K}_0-8\mathbb{M}_0-2)-4\mathbb{M}_0-1}{2(1+2Y_0)}$$

$$\Gamma=\frac{\mathcal{K}_0^2(8Y_0+2)-\mathcal{K}_0(4\mathbb{M}_0+1)(2Y_0+1)+2}{1+2Y_0}$$

$$\mathbb{P}_1=\frac{\mathcal{K}_0^2+\mathcal{K}_1\mathcal{K}_0-1}{\mathcal{K}_0+\mathcal{K}_1}$$

$$\mathbb{M}_1=-\frac{2[\mathcal{K}_0(\mathcal{K}_0+\mathcal{K}_1)-1]Y_0}{(\mathcal{K}_0+\mathcal{K}_1)(2Y_0+1)}$$



$$\mathcal{K}_1=e^F\left(\mathcal{K}_*-\int e^{-F}\Gamma d\rho\right)$$

$$F=\int\Phi d\rho$$

$$\mathbb{P}=\mathbb{P}_0+\mathbb{P}_1,P=P_0+P_1$$

$$\mathcal{K}=\frac{1}{\mathcal{K}_0+\mathcal{K}_1}$$

$$\mathcal{K}_{1,\rho}=-\mathcal{K}_1\Phi+\Gamma$$

$$\Phi=\frac{1}{2}(4\mathbb{M}_0+1)$$

$$\Gamma=\frac{1}{2}[4\mathcal{K}_0^2-2\mathcal{K}_0(4\mathbb{M}_0+1)+4]$$

$$\mathbb{P}_1=\frac{(\mathcal{K}_0(\mathcal{K}_0+\mathcal{K}_1)-1)(2Y_0+3)}{3(\mathcal{K}_0+\mathcal{K}_1)}$$

$$P_1=\frac{2}{3}\Big(\frac{1}{\mathcal{K}_0+\mathcal{K}_1}-\mathcal{K}_0\Big)Y_0$$

$$\mathcal{K}_1=e^{-F}\left(\mathcal{K}_*-\int e^F\Gamma d\rho\right)$$

$$F=\int\Phi d\rho$$

$$\mathbb{P}=\mathbb{P}_0+\mathbb{P}_1,P=P_0+P_1,$$

$$\mathcal{K}=\mathcal{K}_1,$$

$$\mathcal{K}_{1,\rho}=-\mathcal{K}_1\left(-2\mathbb{M}_0-\frac{1}{2}\right)-2\mathcal{K}_1^2$$

$$\mathbb{P}_1=(2\mathcal{K}_0-1)P_0-\mathbb{P}_0+Y_0+\frac{1}{4}$$

$$\frac{1}{6}\mathcal{K}_1(-4P_0-4\mathbb{P}_0-5)+\frac{2}{3}\mathcal{K}_0(\mathbb{M}_0+3\mathbb{P}_0)$$

$$P_1=\frac{1}{6}\mathcal{K}_1(4P_0+4\mathbb{P}_0-1)$$

$$-\frac{2}{3}\mathcal{K}_0(\mathbb{M}_0+3P_0+3\mathbb{P}_0)$$

$$\mathbb{M}=\frac{\mathcal{K}_\rho}{2\mathcal{K}}+\mathcal{K}-\frac{1}{4}$$

$$P=\frac{1}{3}(2Y_\rho+2Y^2+Y)$$

$$-\frac{2Y+1}{6}\frac{\mathcal{K}_\rho}{\mathcal{K}}-\frac{1}{3}\mathcal{K}+\frac{1}{12}$$

$$\mathbb{P}=(2Y+1)\mathcal{K}_\rho-4\mathcal{K}^2$$

$$-\mathcal{K}[4Y_\rho+4(Y-1)Y-1]$$

$$w_r = \frac{\mathcal{K}^2(-4\mathcal{K} - 4Y_{,\rho} + 4Y + 1) + (4Y + 1)\mathcal{K}\mathcal{K}_{,\rho}}{4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}) + 4\mathcal{K}_{,\rho}^2}$$

$$w_{\perp} = -\frac{1 + 16\mathcal{K}Y^2(\mathcal{K}_{,\rho} + \mathcal{K}) + \mathcal{K}\mathcal{K}_{,\rho}(1 - 12Y_{,\rho})}{[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}) + 4\mathcal{K}_{,\rho}^2]}$$

$$+ \frac{2\mathcal{K}^2(-8Y_{,\rho} + 8Y_{,\rho\rho} + 1) - 8\mathcal{K}^3}{[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}) + 4\mathcal{K}_{,\rho}^2]}$$

$$+ \frac{Y(4\mathcal{K}(\mathcal{K}_{,\rho} + \mathcal{K}(-4\mathcal{K} + 8Y_{,\rho} + 1)) - 2)}{[4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}) + 4\mathcal{K}_{,\rho}^2]}$$

$$A = (a + br^{2\rho})^{\delta}$$

$$B = \frac{3\mu_0}{3\mu_0 - r^2(\mu_0 - \mu_1 r^{2\alpha})^{\beta+1}\mathcal{F}}$$

$$\mathcal{F} = {}_2F_1\left(1, \beta + \frac{3}{2\alpha} + 1; 1 + \frac{3}{2\alpha}; \frac{r^{2\alpha\rho}\mu_1}{\mu_0}\right)$$

$$\mu = (\mu_0 - \mu_1 r^{2\alpha})^{\beta}$$

$$p_r = \frac{2b\delta\mu_0}{\mu_0(a + br^2)}$$

$$- \frac{[a + b(2\delta + 1)r^2](\mu_0 - \mu_1 r^{2\alpha})^{\beta+1}}{3\mu_0(a + br^2)}\mathcal{F}$$

$$p_{\perp} = \frac{2[a^2 + ab(2 - 3\delta)r^2 + b^2(-2\delta^2 + \delta + 1)r^4]\mathcal{F}}{\mu_0 12(a + br^2)^2(\mu_0 - \mu_1 r^{2\alpha})^{-(\beta+1)}}$$

$$+ \frac{12b\delta(2a + b\delta r^2) - 6(a + br^2)(a + b(\delta + 1)r^2)}{12(a + br^2)^2(\mu_0 - \mu_1 r^{2\alpha})^{-\beta}}$$

$$w_r = \frac{(\mu_0 - \mu_1 r^{2\alpha})}{3\alpha\beta\mu_0\mu_1 r^{2\alpha}(a + br^2)^2} \times$$

$$\left\{ \frac{3}{2}\mu_0(a + br^2)[a + b(2\delta + 1)r^2] \right.$$

$$\frac{\mathcal{F}}{2}(\mu_0 - \mu_1 r^{2\alpha})[2b\delta r^2(a + 3br^2) + 3(a + br^2)^2]$$

$$\left. + 6b^2\delta\mu_0 r^2(\mu_0 - \mu_1 r^{2\alpha})^{-\beta} \right\}$$

$$w_{\perp} = \frac{r^{-2\alpha}}{12\alpha\beta\mu_1(a + br^2)^3} \times$$

$$\left\{ \frac{\mathcal{F}}{\mu_0}(\mu_0 - \mu_1 r^{2\alpha})^2[3a^3 - 3a^2b(\delta - 3)r^2 \right.$$

$$+ ab^2(2(\delta - 8)\delta + 9)r^4 + 3b^3(-2\delta^2 + \delta + 1)r^6]$$

$$+ 12b^2\delta r^2[b\delta r^2 - a(\delta - 4)](\mu_0 - \mu_1 r^{2\alpha})^{1-\beta}$$

$$- 3\mu_0(a + br^2)[a^2 + ab(2 - 5\delta)r^2$$

$$+ b^2(-2\delta^2 + \delta + 1)r^4]$$

$$\left. - 3\mu_1 r^{2\alpha}(a + br^2)[ab\delta r^2(2\alpha\beta + 5) + 2b^2\delta^2 r^4 \right.$$

$$\left. + b^2\delta r^4(2\alpha\beta - 1) + (2\alpha\beta - 1)(a + br^2)^2 \right\}$$



$$A = (a + (c - br^2)^\gamma)^\delta$$

$$B = \frac{3\mu_0}{3\mu_0 - r^2(\mu_0 - \mu_1 r^{2\alpha})^{\beta+1} \mathcal{F}}$$

$$\mathcal{F} = {}_2F_1\left(1, \beta + \frac{3}{2\alpha} + 1; 1 + \frac{3}{2\alpha}; \frac{r^{2\alpha\beta}\mu_1}{\mu_0}\right)$$

$$\mu = (\mu_0 - \mu_1 r^{2\alpha})^\beta$$

$$p_r = \frac{1}{6(br^2 - c)(a + (c - br^2)^\gamma)} \times$$

$$\left\{ 12b\gamma\delta(c - br^2)^\gamma - 2\frac{\mathcal{F}}{\mu_0} [a(br^2 - c) + (c - br^2)^\gamma (br^2(2\gamma\delta + 1) - c)] \right\}$$

$$p_\perp = \frac{\mu_0^\beta (\mu_0 - \mu_1 r^{2\alpha})^{-\beta}}{12(br^2 - c)^2 (a + (c - br^2)^\gamma)^2}$$

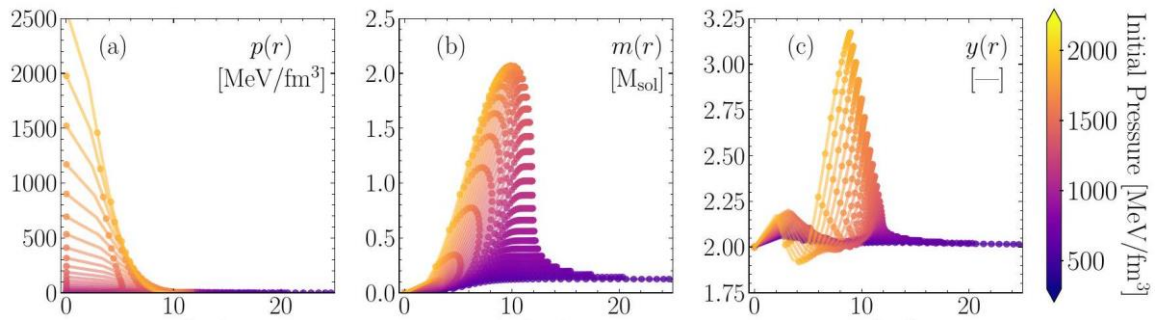
$$\left\{ 2\mathcal{F}(\mu_0 - \mu_1 r^{2\alpha})^\beta [a^2 (br^2 - c)^2 - a(c - br^2)^\gamma [b^2 r^4 (4\gamma^2 \delta - \gamma\delta - 2) + bcr^2 (4 - 3\gamma\delta) - 2c^2] + (c - br^2)^{2\gamma} \times \right.$$

$$(b^2 r^4 (-2\gamma^2 \delta^2 + \gamma\delta + 1) + bcr^2 (3\gamma\delta - 2) + c^2)] - 3\mu_0^{-\beta} (\mu_0 - \mu_1 r^{2\alpha})^\beta [2a^2 (br^2 - c)^2 (\mu_0 - \mu_1 r^{2\alpha})^\beta$$

$$+ 2a(c - br^2)^\gamma (b^2 r^4 (\gamma\delta + 2) (\mu_0 - \mu_1 r^{2\alpha})^\beta - 4b^2 \gamma^2 \delta r^2 + bc(4\gamma\delta - r^2 (\gamma\delta + 4) (\mu_0 - \mu_1 r^{2\alpha})^\beta)$$

$$+ 2c^2 (\mu_0 - \mu_1 r^{2\alpha})^\beta] + (c - br^2)^{2\gamma} \times (b^2 r^2 (2r^2 (\gamma\delta + 1) (\mu_0 - \mu_1 r^{2\alpha})^\beta - 4\gamma^2 \delta^2)$$

$$+ 2bc(4\gamma\delta - r^2 (\gamma\delta + 2) (\mu_0 - \mu_1 r^{2\alpha})^\beta) + 2c^2 (\mu_0 - \mu_1 r^{2\alpha})^\beta \left. \right\}$$



$$\frac{dp}{dx} = -\frac{1}{2} \frac{(\varepsilon(x) + p(x))(m(x) + 3x^3 p(x))}{x^2(1 - m(x)/x)}$$

$$\frac{dm}{dx} = 3x^2 \varepsilon(x)$$

$$\frac{dy}{dx} = -\frac{y(x)^2}{x} - \frac{F(x)y(x)}{x} - \frac{Q(x)}{x}$$

$$F(x) = \left[1 - \frac{3}{2}(\varepsilon(x) - p(x))\right] [1 - m(x)/x]^{-1},$$

$$Q(x) = \frac{\frac{3}{2}x^2}{\left(1 - \frac{m(x)}{x}\right)} \left[5\varepsilon(x) + 9p(x) + \frac{\varepsilon(x) + p(x)}{c_s^2(x)} - \frac{4}{x^2} \right] - \left(\frac{m(x)/x + 3x^2p(x)}{\left(1 - \frac{m(x)}{x}\right)} \right)^2,$$

$$\frac{\partial y}{\partial r} = \mathcal{F}(y(x, r)),$$

$$\frac{dy}{dr} = \hat{A}y,$$

	HF		SLM		Error (%)		Time (s)	
EOS	Max. Mass [$M_{\odot}$]	Radius [nm]	Max. Mass [$M_{\odot}$]	Radius [nm]	Max. Mass	Radius	HF	SLM
Tabular EOS								
SLy4	2.067	10.033	2.068	10.033	0.02	0.00	9.218	2.135 $\times 10^{-4}$
APR	2.193	9.979	2.194	9.982	0.05	0.03	9.237	2.149 $\times 10^{-4}$
FSU Garnet	2.054	10.998	2.068	11.000	0.66	0.02	9.226	2.319 $\times 10^{-4}$
BL	2.083	10.194	2.085	10.204	0.10	0.11	9.246	2.099 $\times 10^{-4}$

DS-CMF-5	2.023	11.896	2.026	11.647	0.17	2.09	9.228	2.215 $\times 10^{-4}$
Quarkyonic EOS								
Λ = 300.00, κ = 0.19	2.947	14.763	2.917	14.666	1.00	0.66	9.286	3.822 $\times 10^{-4}$
Λ = 352.63, κ = 0.14	2.653	12.512	2.675	12.561	0.83	0.39	9.299	4.051 $\times 10^{-4}$
Λ = 405.26, κ = 0.10	2.441	11.199	2.441	11.063	0.40	1.21	9.234	4.361 $\times 10^{-4}$
Λ = 447.37, κ = 0.28	2.294	10.462	2.294	10.463	0.01	0.01	9.292	4.261 $\times 10^{-4}$
Λ = 500.00, κ = 0.23	2.226	10.180	2.221	10.177	0.22	0.03	9.156	3.898 $\times 10^{-4}$

$$\frac{d\mathbf{y}^\theta}{dr} = \hat{\mathbf{A}}(\boldsymbol{\theta})\mathbf{y}^\theta,$$

$$\frac{dp}{dr} = -\frac{G}{c^2}(\epsilon(r) + P(r))\frac{M(r) + 4\pi r^3 P(r)/c^2}{r(r - 2GM(r)/c^2)}$$

$$\frac{dm}{dr} = 4\pi r^2 \frac{\epsilon(r)}{c^2}$$

$$\frac{dy}{dr} = -\frac{y(r)^2}{r} - \frac{F(r)y(r)}{r} - \frac{Q(r)}{r}$$



$$F(r)=\frac{1-4\pi Gr^2(\epsilon(r)-P(r))/c^4}{1-2\frac{GM}{rc^2}}$$

$$Q(r)=\frac{4\pi Gr^2/c^4}{1-\frac{2GM}{rc^2}}\Big(5\epsilon(r)+9p(r)+\frac{\epsilon(r)+p(r)}{c_s(r)^2}c^2-\frac{6c^4}{4\pi r^2G}\Big)-4\left(\frac{G(M(r)/(rc^2)+4\pi r^2p(r)/c^4)}{1-2GM/(rc^2)}\right)^2$$

$$y(r)=\frac{R\beta(R)}{H(R)}\text{ and }\beta=dH/dr^1$$

$$r=R_0x, M=M_0m(x), P=P_0p(x), \epsilon=\varepsilon_0\varepsilon(x)$$

$$\varepsilon_0=P_0\equiv\frac{1}{8\pi^2}\frac{(m_n c^2)^4}{(\hbar c)^3}\approx 1.285\mathrm{GeV}/\mathrm{fm}^3$$

$$\left[\frac{2GM_0}{c^2R_0}\right]=\left[\frac{4\pi R_0^3\varepsilon_0}{3M_0c^2}\right]=1.$$

$$R_0=\sqrt{\frac{3\pi}{\alpha_G}}\lambda_n\approx 8.378\text{ nm},$$

$$M_0=\left(\frac{R_0}{R_s^\odot}\right)M_\odot\approx 2.837M_\odot,$$

$$\alpha_G=\frac{Gm_n^2}{\hbar c}\approx 5.922\times 10^{-39}$$

$$\lambda_n=\frac{\hbar c}{m_n c^2}\approx 0.210\times 10^{-18}\text{ nm}$$

$$R_s^\odot=\frac{2GM_\odot}{c^2}\approx 2.953\text{ nm}$$

$$\frac{dp}{dx}=-\frac{1}{2}\frac{(\varepsilon(x)+p(x))(m(x)+3x^3p(x))}{x^2(1-m(x)/x)}$$

$$\frac{dm}{dx}=3x^2\varepsilon(x)$$

$$\frac{dy}{dx}=-\frac{y(x)^2}{x}-\frac{F(x)y(x)}{x}-\frac{Q(x)}{x}$$

$$F(x)=\frac{1-\frac{3}{2}(\varepsilon(x)-p(x))}{1-\frac{m(x)}{x}},$$

$$Q(x)=\frac{\frac{3}{2}x^2}{\left(1-\frac{m(x)}{x}\right)}\left[5\varepsilon(x)+9p(x)+\frac{\varepsilon(x)+p(x)}{c_s^2(x)}-\frac{4}{x^2}\right] \\ -\left(\frac{m(x)/x+3x^2p(x)}{\left(1-\frac{m(x)}{x}\right)}\right)^2$$

$$k_2\!=\!\frac{8\beta^5}{5}(1-2\beta)^2[2+2\beta(y_R-1)-y_R] \\ \times\{2\beta(6-3y_R+3\beta(5y_R-8)) \\ +4\beta^3[13-11y_R+\beta(3y_R-2)+2\beta^2(1+y_R)] \\ +3(1-2\beta)^2[2-y_R+2\beta(y_R-1)]\log{(1-2\beta)}\}^{-1}.$$

$$\frac{\partial y}{\partial r} = \mathcal{F}(y(x,r))$$

$$\frac{d\boldsymbol{y}}{dr}=\boldsymbol{F}(\boldsymbol{y}(r),r)\Big),$$

$$\frac{d\boldsymbol{y}}{dr}=\hat{\boldsymbol{A}}\boldsymbol{y}$$

$$\log \, \boldsymbol{y}_{r+\delta r} = \hat{\boldsymbol{A}} \log \, \boldsymbol{y}_r$$

$$X_j(y)=\{y,y^2,y^3,\ldots,y^n\}$$

$$X_j(\boldsymbol{z})=\{\boldsymbol{z},\boldsymbol{z}^2\}$$

$$\begin{array}{l} \boldsymbol{X}^- = \left[\boldsymbol{z}_0 \quad \boldsymbol{z}_1 \quad \ldots \quad \boldsymbol{z}_{m-1} \quad \boldsymbol{z}_0^2 \quad \ldots \quad \boldsymbol{z}_i \boldsymbol{z}_j \quad \ldots \quad \boldsymbol{z}_{m-1}^2 \right] \\ \boldsymbol{X}^+ = \left[\boldsymbol{z}_1 \quad \boldsymbol{z}_2 \quad \ldots \quad \boldsymbol{z}_m \quad \boldsymbol{z}_1^2 \quad \ldots \quad \boldsymbol{z}_i \boldsymbol{z}_j \quad \ldots \boldsymbol{z}_m^2 \right] \end{array}$$

$$s=\operatorname{argmin}_j\frac{\sum_{i=1}^{i=j}\sigma_i}{\sum_{i=1}^{i=n}\sigma_i},\kappa$$

$$\boldsymbol{X}^- \approx \boldsymbol{U}_s \boldsymbol{\Sigma}_s \boldsymbol{V}_s^T$$

$$\boldsymbol{A}_s \equiv \boldsymbol{U}_s^T \boldsymbol{A} \boldsymbol{U}_s = \boldsymbol{U}_s^T \boldsymbol{X}^+ \boldsymbol{V}_s \boldsymbol{\Sigma}_s^{-1}.$$

$$A_{\rm s}W=\Lambda W.$$

$$\boldsymbol{y}(l)=\boldsymbol{\Phi}\boldsymbol{\Lambda}^{l/\Delta l}\boldsymbol{b}_0=\Sigma_ib_{0i}\phi_i(\lambda_i)^{l/\Delta l},$$

$$\frac{\partial y^\mu}{\partial r} = \mathcal{F}(y^\mu(x,r;\mu),r;\mu).$$

$$\frac{d\boldsymbol{y}^\mu}{dr}=\boldsymbol{F}(\boldsymbol{y}^\mu(r;\mu),r;\mu).$$

$$\frac{d\boldsymbol{y}^\theta}{dt}=\hat{\boldsymbol{A}}(\boldsymbol{\theta})\boldsymbol{y}^\theta\,,$$

$$\mathbf{A}_{js} = \mathbf{U}_{js}^T \mathbf{X}_j^+ \mathbf{V}_{js} \mathbf{\Sigma}_{js}^{-1}$$

$$\mathbf{A}_{js}\mathbf{W}_{js} = \mathbf{\Lambda}_{js}\mathbf{W}_{js}.$$

$$u\colon=\operatorname{argmin} E[w]\colon w\in\{u_1,\ldots,u_s\}$$

$$ds^2=r^2(d\theta^2+\sin^2\theta d\phi^2)+..$$

$$\begin{aligned}\frac{dm(r)}{dr}&=4\pi r^2\rho(r)=12\pi r^2p(r)\\\frac{dp(r)}{dr}&=-\frac{\rho(r)+p(r)}{2}\frac{dv(r)}{dr}=-2p(r)\frac{dv(r)}{dr}\\\frac{dv(r)}{dr}&=\frac{N(r)}{1-2m(r)/r}\\N(r)&=(2/r^2)(m(r)+4\pi r^3p(r))\end{aligned}$$

$$\begin{array}{l}r\rightarrow r/L\\m(r)\rightarrow m(r/L)/L\\v(r)\rightarrow v(r/L)\\p(r)\rightarrow p(r/L)L^2\\\rho(r)\rightarrow \rho(r/L)L^2\end{array}$$

$$\begin{array}{l}t=\log r\\dt=dr/r\\\alpha(t)=m(r)/r\\\delta(t)=4\pi r^2p(r)\end{array}$$

$$\begin{array}{l}\frac{d\alpha(t)}{dt}=3\delta(t)-\alpha(t)\\\frac{d\delta(t)}{dt}=-4\delta(t)\frac{\delta(t)+2\alpha(t)-1/2}{1-2\alpha(t)}\end{array}$$

$$D(r)\equiv 1-2m(r)/r$$

$$\begin{array}{l}r=f(r')\\t=t'\end{array}$$

$$f'\equiv dr/dr'=df(r')/dr'$$

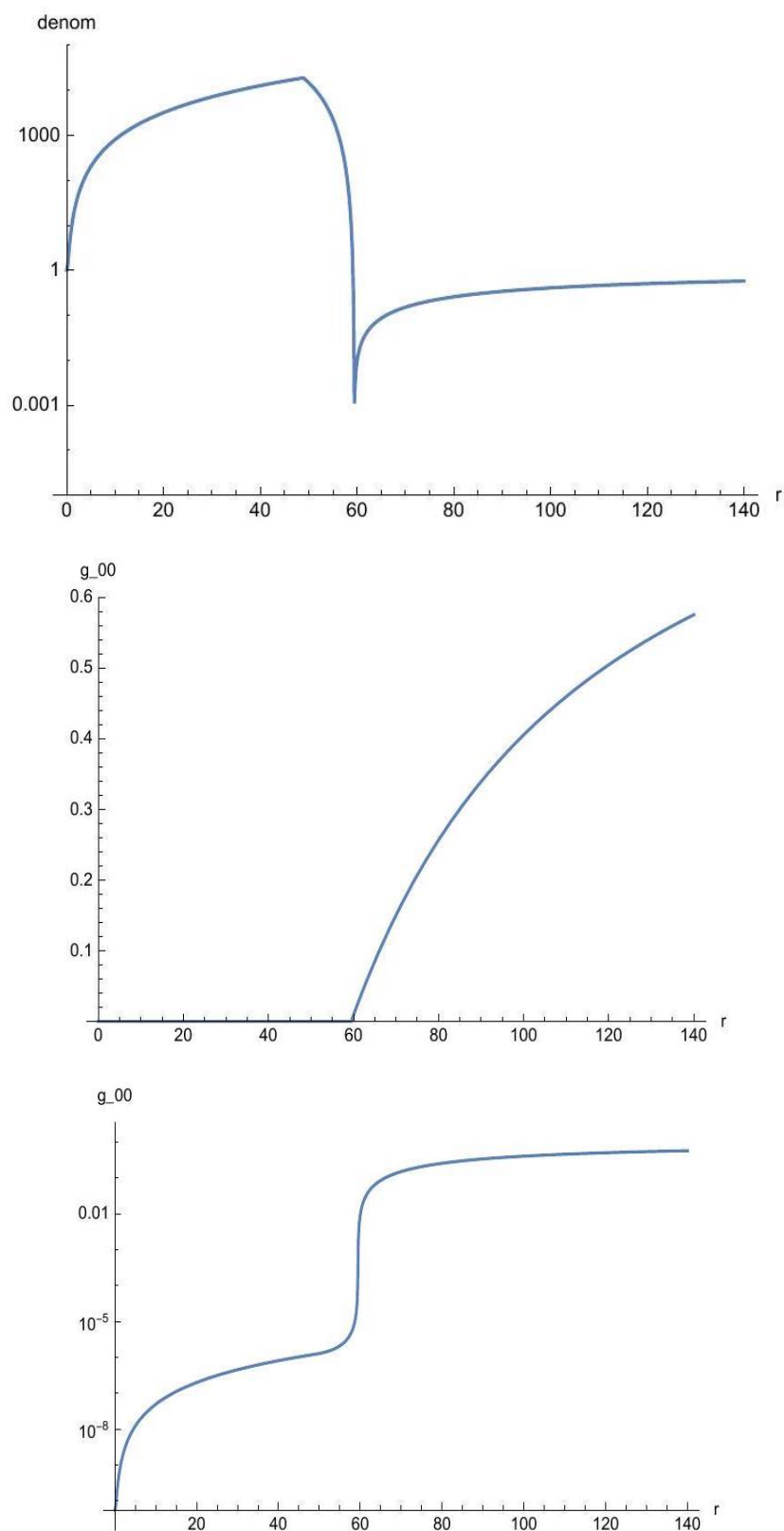
$$\begin{array}{l}\frac{dm(r)}{dr'}=12\pi r^2f'p(r)\\\frac{dp(r)}{dr'}=-2p(r)f'\frac{N(r)}{D(r)}\\N(r)=(2/r^2)(m(r)+4\pi r^3p(r))\\D(r)=D(f(r'))=1-2m(r)/r\end{array}$$

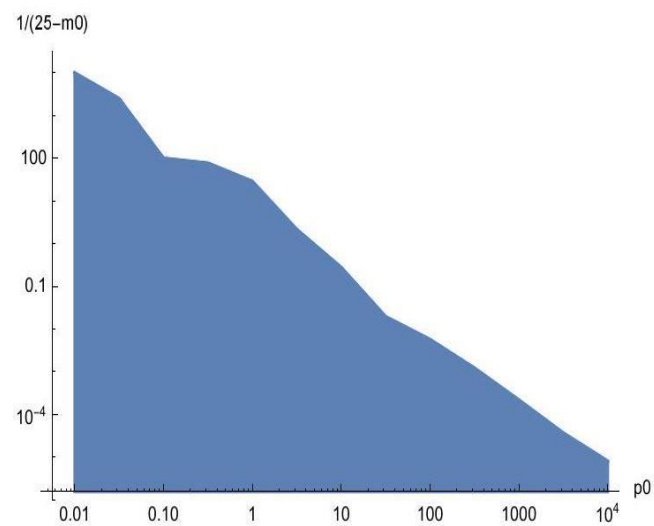
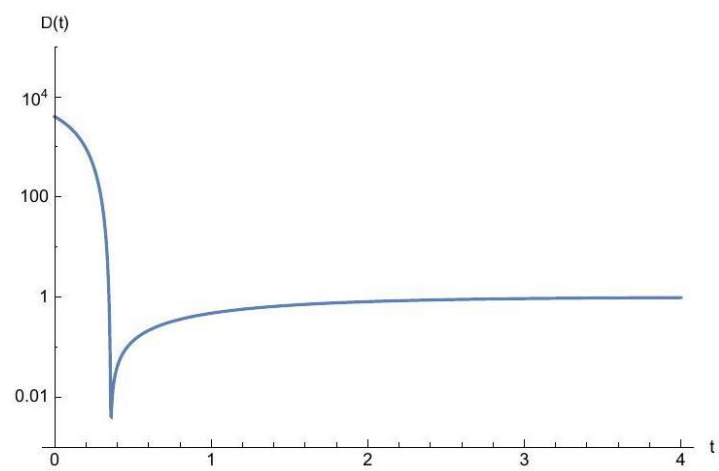
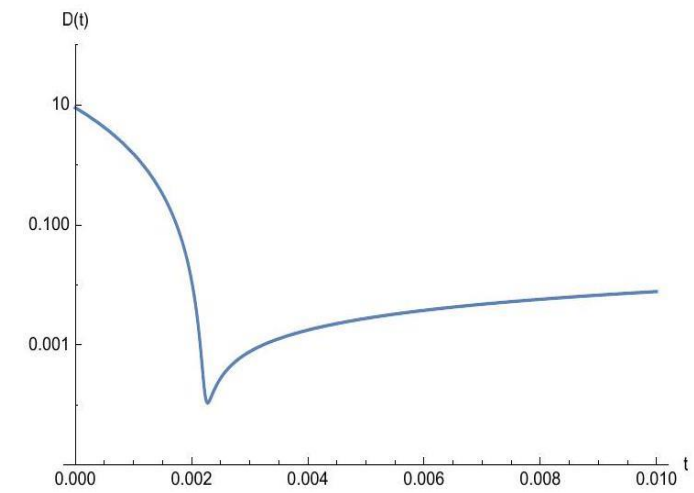
$$d\hat{D}(r')/dr'|_{r'=r'_{\min}}=f'dD/dr|_{r=r_{\min}}=0$$

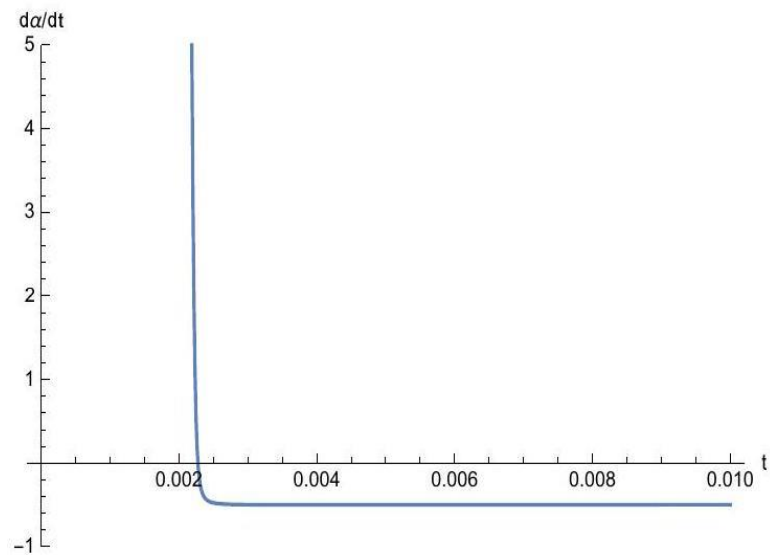
$$\begin{array}{l}d^2\hat{D}(r')/(dr')^2|_{r'=r'_{\min}}=d^2D(r)/(dr)^2|_{r=r_{\min}}(f')^2|_{r'=r'_{\min}}+dD(r)/dr|_{r=r_{\min}}df'/dr'|_{r'=r'_{\min}}\\=d^2D(r)/(dr)^2|_{r=r_{\min}}(f')^2|_{r'=r'_{\min}}>0\end{array}$$



Figuras. Densidad de energía y masa de una partícula oscura y blanca para formar un agujero negro cuántico.







$$\alpha(t) \simeq 1/2 - \theta_0 - \theta_2 t^2 + O(t^3)$$

$$\delta(t) \simeq 1/6 - (2/9)\text{ArcTan}((\theta_2/\theta_1)^{1/2}t)/(\theta_2\theta_1)^{1/2} + O(\theta_1, \theta_2, t^2)$$

p_0	m_0	α_0	δ_0
0.01	24.9998999	. 499998	314.159
. 03162	24.9996004	. 499992	993.37
0.1	24.9899999	. 499800	3141.59
. 3162	24.9869469	. 499739	9933.71
1	24.9651	. 499302	31415.9
3.162	24.5310	. 49012	99337.1
10	21.366	. 42732	314159
31.62	-26.75	-. 535	993371
100	-153	-3.06	3141590
316.2	-806	-16.12	9933708
1000	-4,640	-92.8	31415900

3162	-27,600	-63.24	99337080
10000	-126,000	-2520	314159000

$$ds^2 = -L^2(x)dt^2 + h_{ij}(x)dx^i dx^j$$

$$T_{\mu\nu} = \rho n_\mu n_\nu + P(g_{\mu\nu} + n_\mu n_\nu)$$

$$\frac{\nabla_i P}{\rho + P} = -\frac{\nabla_i L}{L}$$

$$Tds = d\rho - \sum_a \mu_a dn_a$$

$$\mu_a = -T\frac{\partial s}{\partial n_a}$$

$$T=\left(\frac{\partial s}{\partial \rho}\right)^{-1}_{n_a}$$

$$\rho + P - Ts - \sum_a \mu_a n_a = 0$$

$$N_a=\int_C d^3x\sqrt{h}n_a$$

$$\Omega = S + \sum_a b_a N_a$$

$$\delta\Omega=\sum_a\int_Cd^3x\sqrt{h}\Big(-\frac{\mu_a}{T}+b_a\Big)\delta n_a=0$$

$$\omega(\rho,b_a)\!:=\!s-\sum_a\frac{\partial s}{\partial n_a}n_a=s+\sum_ab_an_a=\frac{\rho+P}{T}.$$

$$\Omega=\int_C d^3x\sqrt{h}\omega(\rho,b_a)$$

$$d\omega=\frac{d\rho}{T}+\sum_an_adb_a$$

$$T^{-1}=(\partial\omega/\partial\rho)_{b_a},\,n_a=(\partial\omega/\partial b_a)_\rho$$

$$dP=\omega dT+T\sum_an_adb_a$$

$$\frac{dP}{dT} = \omega = \frac{P + \rho}{T}$$

$$\frac{\nabla_i T}{T} = -\frac{\nabla_i L}{L}$$

$$LT=T_\infty$$

$$n(T,b) = \frac{g}{\pi^2} \int_0^\infty \frac{dpp^2}{e^{-b+\epsilon_p/T} \pm 1}$$

$$P(T,b) = \frac{gT}{\pi^2} \int_0^\infty dpp^2 \log [1 \pm e^{b-\epsilon_p/T}]$$

$$\rho = T(\partial P/\partial T)_b - P$$

$$n(T,b) \, = -\frac{2}{\pi^2} f_1^\pm(b) T^3$$

$$P(T,b) \, = -\frac{2}{\pi^2} f_2^\pm(b) T^4$$

$$f_1^+=-\mathrm{Li}_3(-e^b), f_1^-=\mathrm{Li}_3(e^b), f_2^+=-\mathrm{Li}_4(-e^b), f_2^-=\mathrm{Li}_4(e^b); \mathrm{Li}_k(x)\!:=\!\sum_{n=1} \frac{x^n}{n^k}$$

$$\omega(\rho,b)=\frac{8}{6^{3/4}\sqrt{\pi}}\rho^{3/4}[f_2^\pm(b)]^{1/4}$$

$$\epsilon_p \simeq m + \frac{p^2}{2m}$$

$$n(T,b)=\frac{1}{2}\Big(\frac{2m}{\pi}\Big)^{3/2}e^{b-\frac{m}{T}}T^{3/2}$$

$$P(T,b)=\frac{1}{2}\Big(\frac{2m}{\pi}\Big)^{3/2}e^{b-\frac{m}{T}}T^{5/2}$$

$$\rho=\left(m+\frac{3}{2}T\right)n$$

$$(\partial P/\partial T)_{b_a}=\omega\geq 0$$

$$\left(\frac{\partial^2\rho}{\partial\omega^2}\right)_{b_a}\geq 0$$

$$\left(\frac{\partial^2\rho}{\partial\omega^2}\right)_{b_a}=\left(\frac{\partial T}{\partial\rho}\right)_{b_a}$$

$$(\partial\rho/\partial T)_{b_a}\geq 0$$

$$2P\leq \omega T=T\left(\frac{\partial P}{\partial T}\right)_{b_a}$$

$$\rho=f(b_a)T,\omega=f(b_a)$$



$$\kappa\!:=\!\left(\frac{\partial\log P}{\partial\log T}\right)_{b_a}\geq 2$$

$$\lim_{T\rightarrow\infty}P(T,b_a)=\lim_{T\rightarrow\infty}\rho(T,b_a)=\infty$$

$$\lim_{T\rightarrow 0}P(T,b_a)=\lim_{T\rightarrow 0}\rho(T,b_a)=0.$$

$$\frac{d\rho}{dt}=\frac{\rho+f(\rho)}{f'(\rho)}$$

$$X_f\!:=\!\frac{x\!+\!f(x)}{f'(x)}\frac{\partial}{\partial x}\,\boldsymbol{R}_*^+$$

$$f(x)=kx^a \text{ as } x\rightarrow \infty, k,a>0$$

$$\frac{d\rho}{dt}=\frac{\rho^{2-a}}{ka}$$

$$\frac{1}{\rho^{1-a}}=\frac{1}{\rho_0^{1-a}}-\frac{1-a}{ak}\log{(T/T_0)}$$

$$T=T_0\mathrm{exp}\left(\frac{ak}{(1-a)\rho_0^{1-a}}\right)$$

$$\frac{d\rho}{dt}=\left(k'a'\rho^{a'-2}\right)^{-1}$$

$$\rho^{a'-1}=\rho_0^{a'-1}+\frac{a'-1}{a'k'}\log{(T/T_0)}.$$

$$T=T_0\mathrm{exp}\left(-\frac{a'k'\rho_0'-1}{a'-1}\right)$$

$$ds^2=-L^2(r)dt^2+\frac{dr^2}{1-\frac{2m(r)}{r}}+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

$$\frac{dP}{dr}=-\frac{(\rho+P)(m+4\pi r^3P)}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\frac{dm}{dr}=4\pi r^2\rho$$

$$\frac{1}{L}\frac{dL}{dr}=-\frac{1}{\rho+P}\frac{dP}{dr}$$

$$\frac{d\log T}{dr}=-\frac{m+4\pi r^3P(T)}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\frac{dm}{dr}=4\pi r^2\rho(T)$$



$$\begin{aligned}u&:=\frac{2m}{r}\\v&:=4\pi r^2\rho\\w&:=4\pi r^2P\leq v\\\xi&:=-\log{(r/r_B)}\\t&:=\log{(T/T_B)}.\end{aligned}$$

$$\begin{aligned}\frac{dt}{d\xi}&=\frac{\frac{1}{2}u+w}{1-u}\\\frac{du}{d\xi}&=u-2v\end{aligned}$$

$$u(0)=u_B=\frac{2M}{r_B},\,t(0)=0$$

$$\lim_{T\rightarrow\infty}P/\rho=\lambda\leq 1$$

$$\begin{aligned}\frac{dt}{dx}&=-\frac{\frac{1}{2}+w}{\epsilon}\\\frac{d\epsilon}{dx}&=1-2v.\end{aligned}$$

$$-\frac{dt}{dx}<\frac{1}{\epsilon}<\frac{1}{Cx}$$

$$-\log\frac{T(x)}{T(x_r)}<C^{-1}\log{(x/x_r)}$$

$$\frac{d\epsilon}{dx}=f(x)$$

$$\epsilon=F(x)=\int_0^x dx' f(x')$$

$$-\log\frac{T(x)}{T(x_r)}<\int_{x_r}^x\frac{dx'}{F(x')}$$

$$(\, P = \tfrac{1}{3} \rho \sim T^4 \,) \, u_{\max} = 1 - \epsilon, \, u_B \sqrt{v(0)}$$

$$u_m<1$$

$$m(r)<\frac{1}{2}r \text{ for all } r<r_B$$

$$\lim_{r\rightarrow 0}m(r)=0 \text{ or } \lim_{r\rightarrow 0}m(r)<0$$

$$\frac{dt}{dr}=f_1(r,m,t)\,\,\frac{dm}{dr}=f_2(r,m,t)$$

$$\xi=-\log{(r_1/r_B)}$$



$$du/d\xi < 0 \text{ for } u < 0, u(\xi) < 0 \text{ for all } \xi > \xi_1$$

$$\frac{dt}{d\xi}(\xi_1) = w(\xi_1) > 0, \xi > \xi_1$$

$$\frac{dt}{d\xi}(\xi_2) = 0$$

$$\frac{dt}{d\xi} > 0 \text{ for all } \xi > \xi_1, w > -\frac{1}{2}u \text{ for all } \xi > \xi_1.$$

$$v > -\frac{1}{2}u, u-2v < 2u.$$

$$\frac{du}{d\xi} < 2u,$$

$$u < -ke^{2\xi}, k > 0.$$

$$r^{-1} \text{ as } r \rightarrow 0$$

$$\lim_{T \rightarrow \infty} \frac{P}{\rho} = \lambda \leq 1.$$

$$P=\lambda \rho \sim T^{1+\frac{1}{\lambda}}$$

$$\frac{d\log T}{dr}=\frac{m+4\pi r^3P}{2mr}=\frac{1}{2r}+\frac{\lambda}{2mr}\frac{dm}{dr}=\frac{1}{2r}+\frac{\lambda}{2r}\frac{d\log |m|}{dr}.$$

$$T|m|^{-\lambda/2}\sim \sqrt{r}$$

$$r^{-2}dm/dr$$

$$d|m|/dr=-Kr^2\big(r^{1/2}|m|^{\lambda/2}\big)^{1+\frac{1}{\lambda}}=-Kr^{\frac{5}{2}+\frac{1}{2\lambda}}|m|^{\frac{1}{2}+\frac{\lambda}{2}},$$

$$|m|^{(1-\lambda)/2}=c_1-c_2r^{\frac{7}{2}+\frac{1}{2\lambda}},\,c_1,c_2\geq 0.$$

$$|m|=Ce^{\frac{Kr^5}{5}}$$

$$\frac{d^2t}{d\xi^2}=\frac{\left(\frac{1}{2}+w\right)}{(1-u)^2}\frac{du}{d\xi}-\frac{2w}{1-u}+\kappa\frac{w}{1-u}\frac{dt}{d\xi}$$

$$\frac{d^2t}{d\xi^2}(\xi_2)<0$$

$$\bar{\xi}_2 > \xi_2, \frac{dt}{d\xi}(\bar{\xi}_2) < 0 \text{ for all } \xi \in (\xi_2, \bar{\xi}_2].$$

$$\frac{dy}{d\xi} > g(\xi)y,$$



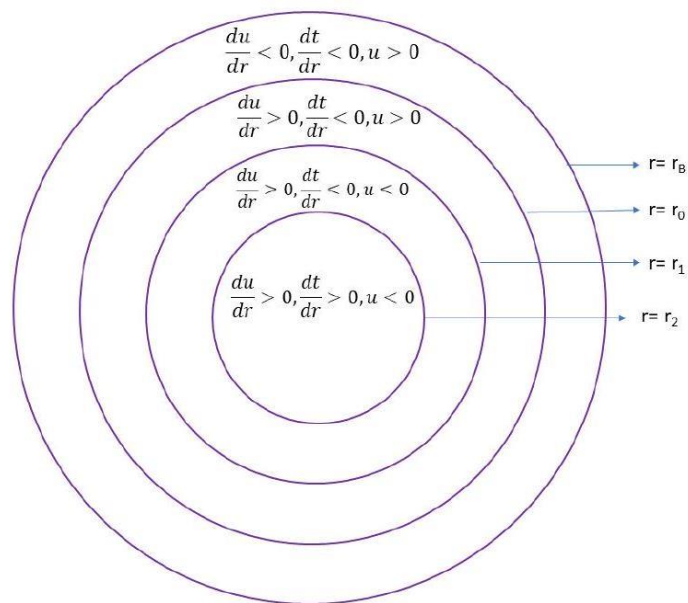
$$y = -\frac{dt}{d\xi}$$

$$y(\xi)/y(\bar{\xi}_2) = \exp \left[\int_{\bar{\xi}_2}^{\xi} d\xi' g(\xi') \right] > 0$$

$$\frac{dt}{d\xi}(\xi) < 0$$

$$\xi \rightarrow \infty, t(\xi) \sim -\frac{1}{2}\xi \text{ and } u(\xi) \sim e^{\xi}$$

$$0 < \frac{dt}{d\xi} + \frac{1}{2} = \frac{\frac{1}{2} + w}{1 - u} < \frac{\frac{1}{2} + w(\xi_2)}{1 - u} < \frac{\frac{1}{2} + w(\xi_2)}{-u} = \frac{1}{2}(1 + |u_2|^{-1})e^{-\xi}.$$

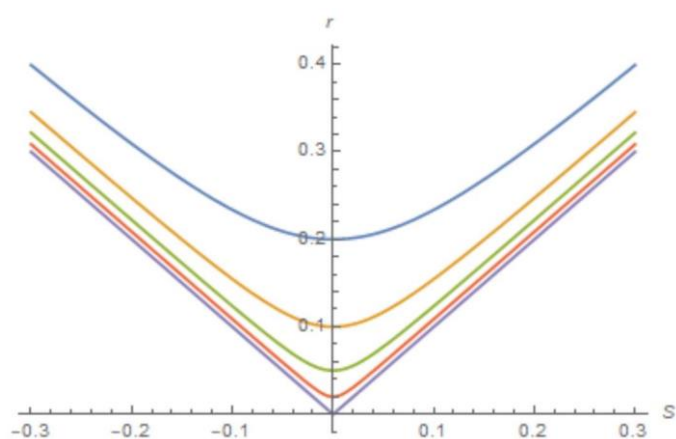
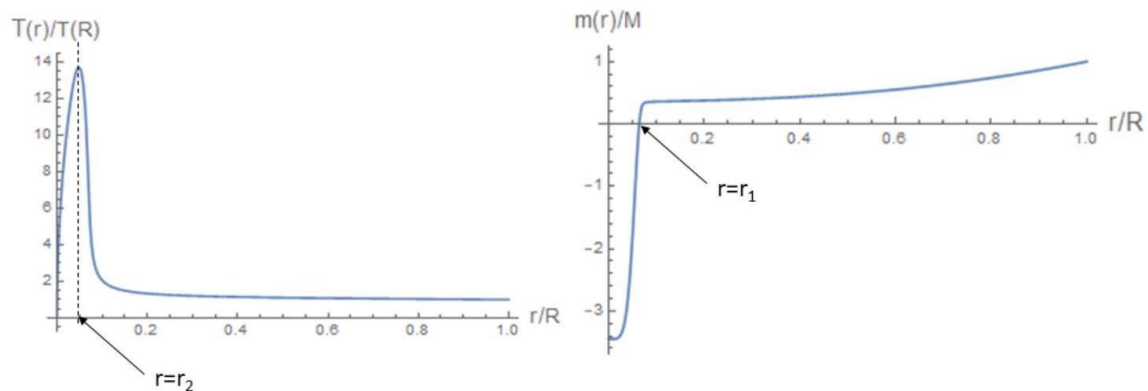


$$\frac{du}{dr}(r_B) > 0, r_0$$

$$u = 2m/r$$

$$r = r_2$$

Figuras. Termodinámica de una partícula oscura y blanca.



$$L(r) = \frac{\eta}{\sqrt{r}},$$

$$ds^2 = -\frac{\eta^2}{r} dt^2 + \frac{r dr^2}{2M_0} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$ds^2 = -\frac{2M_0}{r} d\tilde{t}^2 + \frac{r dr^2}{2M_0} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\tilde{t} = \frac{\eta}{\sqrt{2M_0}} t$$

$$g_{rr} = \frac{r}{2M_0} \left[1 - \frac{r}{2M_0} + \frac{r^2}{4M_0^2} - \frac{r^3}{M_0^3} + O[(r/M_0)^4] \right]$$

$$r \frac{d \log T}{dr} = - \frac{-M_0 + k r^\delta + 4 \pi r^3 P}{2 M_0 + r - k r^\delta}.$$

$$r \frac{d \log T}{dr} = \frac{1}{2} \left(1 + \frac{r}{2 M_0} \right)^{-1}$$

$$g_{tt} = \frac{\eta^2}{2M_0} \left(1 + \frac{2M_0}{r} \right) + O[(r/M_0)^4]$$

$$K := \frac{48M_0^2}{r^6}$$

$$\dot{r}^2 = \frac{2M_0\epsilon^2}{\eta^2} - \frac{2M_0\sigma}{r} - \frac{2M_0\ell^2}{r^3}$$

$$t\eta^2 = \epsilon r$$

$$r_{\min} = (\eta\ell/\epsilon)^{2/3}$$

$$\dot{r} = \pm\sqrt{2M_0}\epsilon/\eta$$

$$r = \pm(\sqrt{2M_0}\epsilon/\eta)\lambda$$

$$t = t_0 + \frac{1}{2}(\sqrt{2M_0}\epsilon^2\eta^3)\lambda^2$$

$$S = \sqrt{2M_0}\epsilon\lambda/\eta$$

$$\delta\tau = \frac{2\eta}{\sqrt{2M_0}}r_0$$

$$a = \frac{d\log L}{dr}dr = -\frac{1}{2r}dr$$

$$t = \frac{\sqrt{r}}{\eta} \sqrt{1 + \frac{r\dot{r}^2}{2M_0} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2}$$

$$\dot{t} \simeq \frac{\sqrt{r}}{\eta}$$

$$\dot{t} = \frac{r|\dot{r}|}{\eta\sqrt{2M_0}}$$

$$a^2 := a_\mu a^\mu \sim r^{-(4s+1)}$$

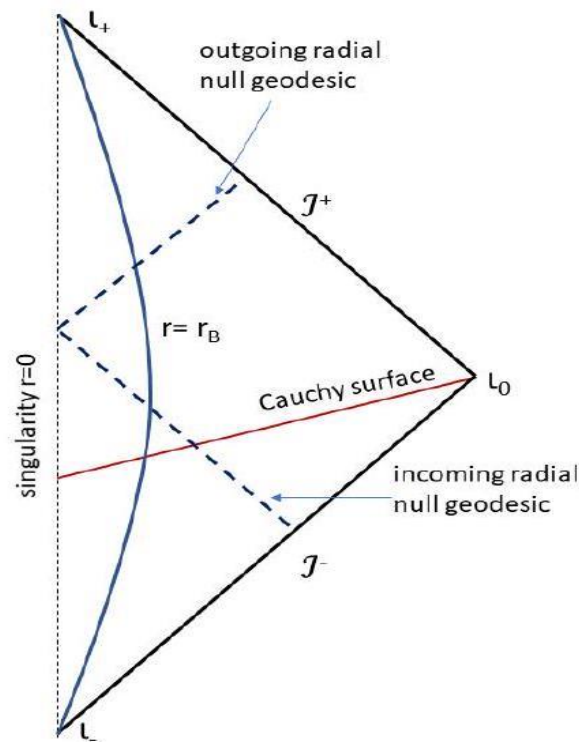
$$d\tilde{s}^2 = -d\tilde{t}^2 + dx^2 + (2x)^{3/2}(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\tilde{t} = \eta^2/(8M_0^3)t \text{ and } x = \frac{r^2}{8M_0^2}$$

$$d\sigma^2 = dx^2 + (2x)^{3/2}(d\theta^2 + \sin^2\theta d\phi^2)$$

Figura. Diagrama Penrose aplicado a una partícula blanca y oscura.





$$n = \frac{8D}{m_f} \left[\theta^{3/2} F_{\frac{1}{2}}(\theta, b - \theta^{-1}) + \theta^{5/2} F_{\frac{3}{2}}(\theta, b - \theta^{-1}) \right]$$

$$P = \frac{16D}{3} \left[\theta^{5/2} F_{\frac{3}{2}}(\theta, b - \theta^{-1}) + \frac{1}{2} \theta^{7/2} F_{\frac{5}{2}}(\theta, b - \theta^{-1}) \right]$$

$$\rho = m_f n + 8D \left[\theta^{5/2} F_{\frac{3}{2}}(\theta, b - \theta^{-1}) + \theta^{7/2} F_{\frac{5}{2}}(\theta, b - \theta^{-1}) \right]$$

$$D = \frac{m_f^4}{8\pi^2 \hbar^3} \cdot F_\alpha$$

$$F_\alpha(\theta, s) = \int_0^\infty dx \frac{x^\alpha \sqrt{2 + \theta x}}{e^{x-s} + 1}$$

$$n = \frac{D}{m_f} v(Y, b), P = D\sigma(Y, b), \rho = D\psi(Y, b)$$

$$F_\alpha(\theta, s) \simeq \int_0^{b-\theta^{-1}} x^\alpha \sqrt{2 + \theta x}$$

$$v(Y, b) = v_0(Y) := \frac{8}{3} (Y^2 - 1)^{3/2}$$

$$\sigma(Y, b) = \sigma_0(Y) := \frac{1}{3} Y \sqrt{Y^2 - 1} (2Y^2 - 5) + \sinh^{-1} \sqrt{Y^2 - 1}$$

$$\psi(Y, b) = \psi_0(Y) := Y \sqrt{Y^2 - 1} (2Y^2 - 1) - \sinh^{-1} \sqrt{Y^2 - 1}$$

$$\frac{dz}{dx} = x^2 \psi(Y, b)$$

$$\frac{dY}{dx} = - \frac{Y[z + x^3 \sigma(Y, b)]}{x^2 \left(1 - \frac{2z}{x}\right)}$$

$$x = x_B := \sqrt{4\pi D} r_B$$

$$z_B := \sqrt{4\pi D} M$$

$$T_\infty = \frac{m_f}{b} \sqrt{1 - \frac{2z_B}{x_B}}$$

$$z(0) - z(x_c) = \int_0^{x_c} dx x^2 \psi(Y(x), b)$$

$$\psi(1, b) = c_1 b^{-3/2} + O(b^{-5/2})$$

$$c_1 = 8 \int_0^\infty dx \sqrt{2x} (e^x + 1)^{-1} \simeq 7.67$$

$$|z(0) - z(x_c)| < \frac{c_1 x_c^3}{3b^{3/2}}$$

$$N_0 = \frac{1}{\mu \sqrt{4\pi D}} \int_0^{x_c} dx \frac{x^2 v[Y(x), b]}{\sqrt{1 - \frac{2w(x)}{x}}}$$

$$N_0 < \frac{c_1 x_c^3}{6\mu \sqrt{\pi D} b^{3/2}}$$

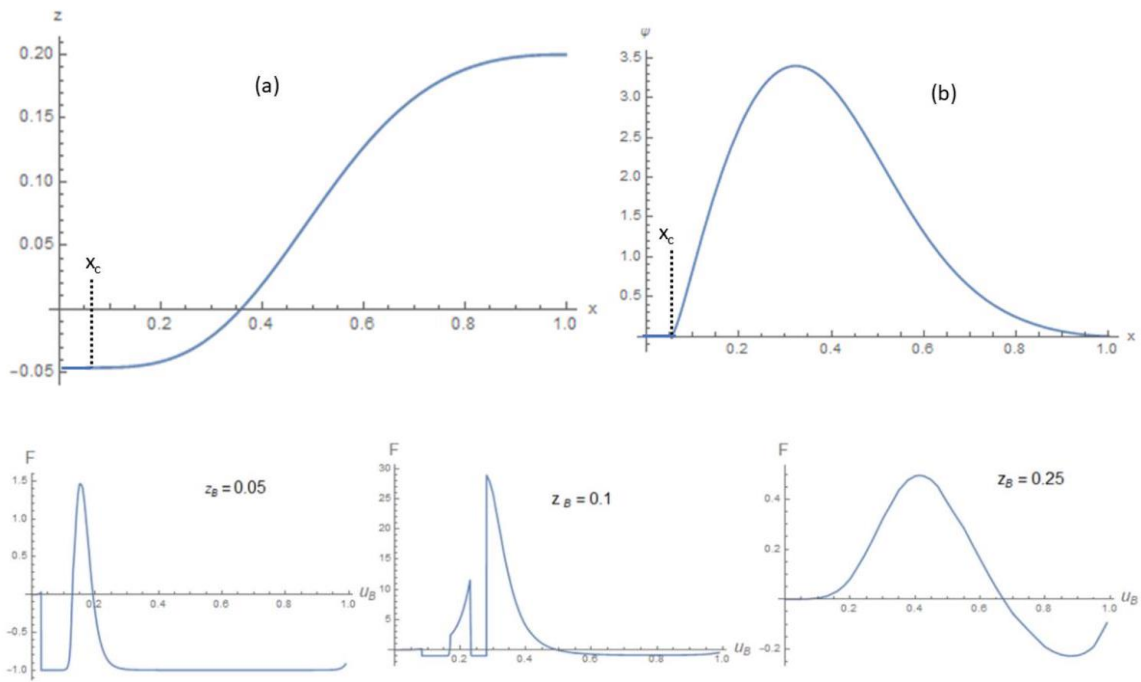
$$M_0 = -\sqrt{4\pi D} z_0, \text{ where } z_0 = -z(0) > 0.$$

$$Y = \sqrt{\frac{1 + 2z_0/x_c}{1 + 2z_0/x}}$$

$$\eta = \frac{T_\infty b}{m_f (4\pi D)^{1/4}} \sqrt{\frac{2z_0}{1 + \frac{2z_0}{x_c}}}$$

Figuras. Masa y energía en dimensiones.





$$\rho = \rho_0 \left(\frac{P}{\rho_0} \right)^s + 3P$$

$$P = \rho_0 \left[\frac{(T/T_0)^{4(1-s)} - 1}{4} \right]^{\frac{1}{1-s}}$$

$$\rho = \frac{1}{4} \rho_0 [3(T/T_0)^{4(1-s)} + 1] \left[\frac{(T/T_0)^{4(1-s)} - 1}{4} \right]^{\frac{s}{1-s}}$$

$$P=Ce^{-m_f/T}T^{5/2}, \rho=Ce^{-m_f/T}T^{3/2}\left(m_f+\frac{3}{2}T\right)$$

$$ds^2=-L^2(r,t)dt^2+\frac{dr^2}{1-\frac{2m(r,t)}{r}}+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

$$W\ddot{f}=(Sf')'+Qf$$

$$W(\xi)=\frac{1}{4\pi r_B^3}e^{3\xi}\frac{L(v+w)}{(1-u)^{3/2}}>0$$

$$S(\xi)=\frac{1}{4\pi r_B^5}e^{5\xi}\frac{L^3\Gamma_1w}{\sqrt{1-u}}>0$$

$$Q(\xi)=\frac{1}{4\pi r_B^5}e^{5\xi}\frac{L^3(v+w)}{\sqrt{1-u}}\left[(t')^2+4t'-\frac{2w}{1-u}\right]$$

$$f(r)=f_{\omega}(r)e^{-i\omega t}$$

$$(Sf_{\omega}')'+Qf_{\omega}+\omega^2Wf_{\omega}=0$$

$$Lf\colon=\frac{1}{W}(Sf')'+Qf$$



$$\omega_0^2=\min_{f\in\mathcal{K}}R[f]$$

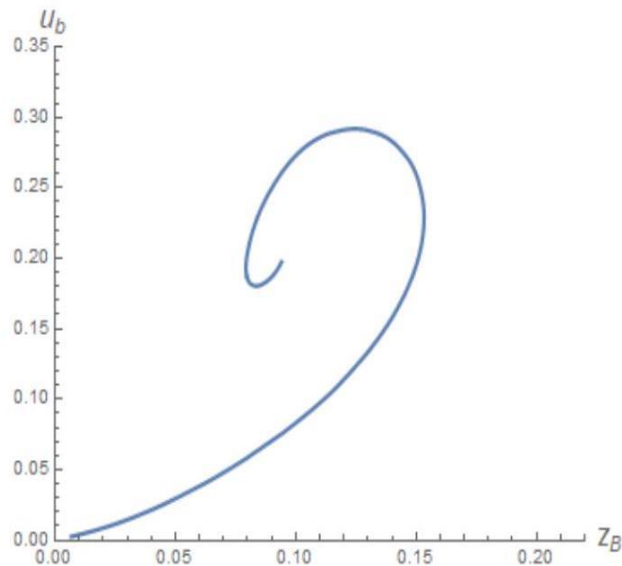
$$R[f]\colon=\frac{\int_0^\infty d\xi(Sf'^2-Qf^2)}{\int_0^\infty d\xi Wf^2}$$

$$\int_0^{r_B} dr (S\zeta'^2-Q\zeta^2)>0$$

$$\frac{Q}{W}\rightarrow 8\pi\bar{L}\bar{P}$$

$$-\frac{7}{4}u^2+2u+w^2+w-uw=\left(-\frac{7}{4}\delta^2+2\delta+1\right)w^2+2(1-\delta)w$$

Figura. Curvatura previa a la formación de un agujero negro cuántico, provocada por la gravedad, sea por la partícula blanca y/o por la partícula oscura, según sea el caso.



$$F(z_B,u_B)=\exp\left[\frac{\omega_0^2(z_B,u_B)}{4\pi D^2}\right]-1$$

$$R_{MN}-\frac{1}{2}g_{MN}R=\kappa^2T_{MN},\sum_M\nabla_MT_N^M=0$$

$$ds^2=-e^{2\lambda^0}dt^2+\sum_ie^{2\lambda^i}(dx^i)^2+e^{2\lambda}dr^2+e^{2\sigma}d\Omega_n^2$$

$$(T_0^0,T_i^i,T_r^r,T_a^a)=(p_0,p_i,\Pi,p_a)$$

$$\begin{aligned}\Pi_r &= \sum_{\alpha} (-\Pi + p_{\alpha}) \lambda_r^{\alpha} \\ \sum_{\alpha\beta} G_{\alpha\beta} \lambda_r^{\alpha} \lambda_r^{\beta} &= 2\kappa^2 \Pi e^{2\lambda} + n(n-1) e^{2\lambda-2\sigma} \\ \lambda_{rr}^{\alpha} + (\Lambda_r - \lambda_r) \lambda_r^{\alpha} &= \kappa^2 \sum_{\beta} G^{\alpha\beta} (\Pi + p_{\beta}) e^{2\lambda} + \delta^{\alpha a} (n-1) e^{2\lambda-2\sigma}\end{aligned}$$

$$\alpha=(0,i,a),\lambda^{\alpha}=(\lambda^0,\lambda^i,\lambda^a),p_{\alpha}=(p_0,p_i,p_a)$$

$$G_{\alpha\beta}=1-\delta_{\alpha\beta},G^{\alpha\beta}=\frac{1}{D-2}-\delta^{\alpha\beta},\Lambda=\sum_{\alpha}\lambda^{\alpha}$$

$$\sum_{\beta}G^{\alpha\beta}G_{\beta\gamma}=\delta^{\alpha}_{\gamma},\sum_{\beta}G^{\alpha\beta}=\frac{1}{D-2}$$

$$\sum_{\beta}G^{\alpha\beta}(\Pi+p_{\beta})=-p_{\alpha}+\frac{T}{D-2}$$

$$T_{MN}=\sum_I T_{MN(I)};\sum_M \nabla_M T^M_{N(I)}=0$$

$$(T^0_{0(I)},T^i_{i(I)},T^r_{r(I)},T^a_{a(I)})=(p_{0I},p_{iI},\Pi_I,p_{aI})$$

$$\Pi=\sum_I\Pi_I,p_{\alpha}=\sum_Ip_{\alpha I};\left(\Pi_I\right)_r=\sum_{\alpha}\left(-\Pi_I+p_{\alpha I}\right)\lambda_r^{\alpha}$$

$$\Pi_I=\Pi_{I0}e^{\phi^I},\phi^I=\sum_{\alpha}(c_{\alpha I}-1)\lambda^{\alpha}$$

$$p_{0*}=p_{i*}=\Pi_*,p_*=\frac{n-2}{n}\Pi_*$$

$$T_*=\Pi_*+\sum_{\beta}p_{\beta*}=(n_c+n)\Pi_*$$

$$\sum_{\beta}G^{\alpha\beta}(\Pi_*+p_{\beta*})=\Pi_*-p_{\alpha*}=\delta^{\alpha a}\frac{2}{n}\Pi_*$$

$$\tilde{\Pi}=\sum_{\bar{I}}\Pi_{\bar{I}},\tilde{p}_{\alpha}=\sum_{\bar{I}}p_{\alpha\bar{I}};\left(\Pi_{\bar{I}}\right)_r=\sum_{\alpha}\left(-\Pi_{\bar{I}}+p_{\alpha\bar{I}}\right)\lambda_r^{\alpha}$$

$$e^{\lambda}dr=dl,$$

$$ds^2=-e^{2\lambda^0}dt^2+\sum_ie^{2\lambda^i}(dx^i)^2+dl^2+e^{2\sigma}d\Omega_n^2$$

$$\psi_l=e^{-\lambda}\psi_r,\psi_u=e^{-2\lambda}(\psi_{rr}-\lambda_r\psi_r)$$

$$\sum_{\alpha\beta} G_{\alpha\beta} \lambda_l^\alpha \lambda_l^\beta = 2\kappa^2 \tilde{\Pi}$$

$$\lambda_{ll}^\alpha + \Lambda_l \lambda_l^\alpha = \kappa^2 \sum_{\beta} G^{\alpha\beta} (\tilde{\Pi} + \tilde{p}_\beta)$$

$$\tilde{\Pi} = \sum_{\tilde{I}} \Pi_{\tilde{I}}, \tilde{p}_\alpha = \sum_{\tilde{I}} p_{\alpha\tilde{I}}, \tilde{I} = *, 1, 2, \dots, \mathcal{N}$$

$$(\Pi_{\tilde{I}})_l = \sum_{\alpha} (-\Pi_{\tilde{I}} + p_{\alpha\tilde{I}}) \lambda_l^\alpha$$

$$dl^2+e^{2\sigma}d\Omega_n^2=e^{2\lambda}dr^2+r^2d\Omega_n^2.$$

$$e^{-\lambda}=\frac{dr}{dl}=e^{\sigma}\sigma_l.$$

$$Y_\alpha=\sum_\beta G_{\alpha\beta}\lambda_l^\beta=\Lambda_l-\lambda_l^\alpha\Rightarrow\lambda_l^\alpha=\sum_\beta G^{\alpha\beta}Y_\beta$$

$$\sum_\alpha Y_\alpha=(D-2)\Lambda_l,\sum_\alpha\lambda_l^\alpha Y_\alpha=2\kappa^2\tilde{\Pi}.$$

$$\lambda_{ll}^\alpha+\sum_\beta\frac{(\lambda_l^\alpha-\lambda_l^\beta)Y_\beta}{D-2}=\kappa^2\sum_\beta G^{\alpha\beta}(-\tilde{\Pi}+\tilde{p}_\beta).$$

$$\tilde{\Pi}_l=\sum_{\alpha}(-\tilde{\Pi}+\tilde{p}_{\alpha})\lambda_l^{\alpha}.$$

$$\alpha=(0,i,a), m^{\alpha}=(m^0,m^i,m^a)$$

$$f^{\alpha}=f(m^{\alpha}),g_{\alpha}=\frac{df^{\alpha}}{dm^{\alpha}},X_{\alpha}=g_{\alpha}\sum_{\beta}G_{\alpha\beta}f^{\beta}$$

$$\sum_{\alpha\beta} G_{\alpha\beta} f^{\alpha} f^{\beta} = 2 l_{qm}^2 \kappa^2 \tilde{\Pi}$$

$$(m^{\alpha})_l + \sum_{\beta} \frac{(m^{\alpha} - m^{\beta}) X_{\beta}}{(D-2) l_{qm}} = l_{qm} \kappa^2 \sum_{\beta} G^{\alpha\beta} (-\tilde{\Pi} + \tilde{p}_{\beta})$$

$$l_{qm} \lambda_l^{\alpha} = \sum_{\beta} G^{\alpha\beta} X_{\beta}$$

$$\leftrightarrow X_{\alpha} = l_{qm} Y_{\alpha} = l_{qm} \sum_{\beta} G_{\alpha\beta} \lambda_l^{\beta}$$

$$l_{qm} (\Pi_{\tilde{I}})_l = \sum_{\alpha\beta} G^{\alpha\beta} (-\Pi_{\tilde{I}} + p_{\alpha\tilde{I}}) X_{\beta}$$

$$l_{qm} \tilde{\Pi}_l = \sum_{\alpha\beta} G^{\alpha\beta} (-\tilde{\Pi} + \tilde{p}_{\alpha}) X_{\beta}$$

$$f^\alpha = cm^\alpha + c_0, g_\alpha = c, X_\alpha = c \sum_\beta G_{\alpha\beta} f^\beta, l_{qm} \lambda_l^\alpha = c f^\alpha$$

$$(m^0, f^0, g_0, X_0) = (m, f, g, X) \\ (m^a, f^a, g_a, X_a) = (\hat{m}, \hat{f}, \hat{g}, \hat{X})$$

$$\sum_{\beta} G^{\alpha\beta} X_{\beta} = \hat{X} - X_{\alpha} + \frac{X}{n}$$

$$\sum_{\beta} G^{\alpha\beta}(-\tilde{\Pi}+\tilde{p}_{\beta})=\tilde{p}-\tilde{p}_{\alpha}-\frac{\tilde{\Pi}+\tilde{\rho}}{n}$$

$$2nf\hat{f}+n(n-1)\hat{f}^2=2l_{qm}^2\kappa^2\tilde{\Pi} \\ nl_{qm}\hat{m}_l+(\hat{m}-m)X=-l_{qm}^2\kappa^2(\tilde{\Pi}+\tilde{\rho}) \\ l_{qm}m_l+(m-\hat{m})\hat{X}=l_{qm}^2\kappa^2\left(\tilde{p}+\tilde{\rho}-\frac{\tilde{\Pi}+\tilde{\rho}}{n}\right) \\ l_{qm}\lambda_l^0=\hat{g}f+(n-1)(\hat{g}-g)\hat{f},l_{qm}\sigma_l=g\hat{f}$$

$$(\Pi_l)_l=-(\Pi_l+\rho_l)\lambda_l^0-n(\Pi_l-p_l)\sigma_l$$

$$2f\hat{f}+(n-1)\hat{f}^2=l_{qm}^2\left(\frac{2\kappa^2}{n}\Pi+(n-1)e^{-2\sigma}\right)$$

$$2l_{qm}\hat{m}_l+2(\hat{m}-m)g\hat{f}+2f\hat{f}+(n-1)\hat{f}^2 \\ =l_{qm}^2\left(-\frac{2\kappa^2}{n}\rho+(n-1)e^{-2\sigma}\right).$$

$$\tilde{\Pi} \leq f_{\max}^2 \Pi_{qm}, \Pi_{qm} = \frac{(D-1)(D-2)}{2l_{qm}^2\kappa^2}$$

$$g_\alpha=\frac{df^\alpha}{dm^\alpha}$$

$$\Pi=p, \tilde{\Pi}=p+\Pi_*, \tilde{\rho}=\rho+\rho_*, \tilde{p}=p+p_*.$$

$$p+\Pi_*\leq \Pi_{qm}$$

$$p_c<\Pi_{qm}$$

$$2\kappa^2\Pi_*\propto e^{-2\sigma}$$

$$e^\sigma > \mathcal{O}(1) l_{qm}$$

$$\rho \ll \Pi_{qm}$$

$$R_{\rm star} \propto \rho^{-\frac{1}{2}}$$



$$M_{star} \propto \rho^{\frac{n-1}{2}}, K_{star} \simeq \frac{4n}{(n+1)^2} \left(1 - \frac{(n-1)^2 \rho}{n(n+1)\Pi_{qm}} \right).$$

$$R_{star} \simeq (\kappa^2 \Pi_{qm})^{-\frac{1}{2}} \simeq l_{qm}, M_{star} \simeq l_{qm}^{n-1}, K_{star} \simeq \frac{2n-1}{n^2}.$$

$$\alpha=(0,a), \lambda^\alpha=(\lambda^0,\sigma), p_\alpha=(-\rho,p)$$

$$\Lambda=\lambda^0+n\sigma,\sum_\beta G^{\alpha\beta}(\Pi+p_\beta)=-p_\alpha+\frac{T}{n}$$

$$\begin{aligned} 2n\lambda_r^0\sigma_r+n(n-1)(\sigma_r)^2&=2\kappa^2\Pi e^{2\lambda}+n(n-1)e^{2\lambda-2\sigma}\\ n\sigma_{rr}+n(\Lambda_r-\lambda_r)\sigma_r&=\kappa^2(\Pi-\rho)e^{2\lambda}+n(n-1)e^{2\lambda-2\sigma}\\ \lambda_{rr}^0+(\Lambda_r-\lambda_r)\lambda_r^0&=\kappa^2\left(\rho+\frac{T}{n}\right)e^{2\lambda} \end{aligned}$$

$$K(r)=\frac{M(r)}{r^{n-1}}=1-e^{-2\lambda(r)}$$

$$ds^2=-e^{2\lambda^0}dt^2+\frac{dr^2}{1-K}+r^2d\Omega_n^2$$

$$\begin{aligned} 2nr\lambda_r^0e^{-2\lambda}&=2\kappa^2r^2\Pi+n(n-1)K\\ 2nr\lambda_re^{-2\lambda}&=nrK_r=2\kappa^2r^2\rho-n(n-1)K \end{aligned}$$

$$nM_r=2\kappa^2r^n\rho\rightarrow M=\frac{2\kappa^2}{n}\int_0^rdr r^n\rho$$

$$\begin{aligned}\Pi_r&=-(\Pi+\rho)\lambda_r^0-n(\Pi-p)\sigma_r\\ \lambda_r^0&=\frac{2\kappa^2r^{n+1}\Pi+n(n-1)M}{2nr(r^{n-1}-M)}\end{aligned}$$

$$M(r)=\frac{2\kappa^2\rho r^{n+1}}{n(n+1)}, K(r)=\frac{M(r)}{r^{n-1}}=\frac{2\kappa^2\rho r^2}{n(n+1)}.$$

$$p_r=-\kappa^2r\frac{(p+\rho)((n+1)p+(n-1)\rho)}{n(n+1)-2\kappa^2\rho r^2}$$

$$\frac{(n+1)p+(n-1)\rho}{(n-1)(p+\rho)}=\sqrt{\frac{1-K(r)}{1-K(R)}}$$

$$\frac{p}{\rho}=\frac{(n-1)(\sqrt{1-K(r)}-\sqrt{1-K(R)})}{(n+1)\sqrt{1-K(R)}-(n-1)\sqrt{1-K(r)}}.$$

$$e^{2\lambda^0}=\frac{\text{const}}{(\rho+p)^2}=\left(\frac{(n+1)\sqrt{1-K(R)}-(n-1)\sqrt{1-K(r)}}{2}\right)^2$$

$$e^{2\lambda^0}=e^{-2\lambda}=1-\frac{M(R)}{r^{n-1}}$$



$$K_{\text{star}} = \frac{M_{\text{star}}}{R_{\text{star}}^{n-1}} = \frac{2\kappa^2 \rho R_{\text{star}}^2}{n(n+1)} = \frac{4p_c(np_c + (n-1)\rho)}{((n+1)p_c + (n-1)\rho)^2}$$

$$M_{star} \propto \rho^{-\frac{n-1}{2}}, K_{star} \simeq \frac{4n}{(n+1)^2} \left(1 - \frac{(n-1)^2 \rho}{n(n+1)p_c} \right)$$

$$R_{star} \simeq (\kappa^2 p_c)^{-\frac{1}{2}}, M_{star} \simeq (\kappa^2 p_c)^{-\frac{n-1}{2}}, K_{star} \simeq \frac{2n-1}{n^2}.$$

$$ds^2 = -dt^2 + \sum_i e^{2\lambda^i} (dx^i)^2$$

$$\sum_{ij} G_{ij} \lambda_t^i \lambda_t^j = 2\kappa^2 \rho$$

$$\lambda_{tt}^i + \Lambda_t \lambda_t^i = \kappa^2 \sum_j G^{ij} (\rho - p_j)$$

$$\rho_t + \sum_i (\rho + p_i) \lambda_t^i = 0$$

$$G_{ij} = 1 - \delta_{ij}, G^{ij} = \frac{1}{D-2} - \delta^{ij}, \Lambda = \sum_i \lambda^i$$

$$Y_i = \sum_j G_{ij} \lambda_t^j \Leftrightarrow \lambda_t^i = \sum_j G^{ij} Y_j$$

$$\lambda_{tt}^i + \sum_j \frac{(\lambda_t^i - \lambda_t^j) Y_j}{D-2} = -\kappa^2 \sum_j G^{ij} (\rho + p_j)$$

$$\rho_t = - \sum_i (\rho + p_i) \lambda_t^i$$

$$f^i = f(m^i), g_i = \frac{df^i}{dm^i}, X_i = g_i \sum_j G_{ij} f^j$$

$$\sum_{ij} G_{ij} f^i f^j = 2\gamma^2 \lambda_{qm}^2 \kappa^2 \rho$$

$$(m^i)_t + \sum_j \frac{(m^i - m^j) X_j}{(D-2) \gamma \lambda_{qm}} = -\gamma \lambda_{qm} \kappa^2 \sum_j G^{ij} (\rho + p_j)$$

$$\gamma \lambda_{qm} \lambda_t^i = \sum_j G^{ij} X_j$$

$$\leftrightarrow X_i = \gamma \lambda_{qm} Y_i = \gamma \lambda_{qm} \sum_j G_{ij} \lambda_t^j$$

$$\lambda_{qm}^{D-2} = \mathcal{O}(1) \gamma \kappa^2$$

$$(\gamma\lambda_{qm})\rho_t = - \sum_{ij} G^{ij}(\rho + p_i)X_j$$

$$f^i = cm^i + c_0, g_i = c, (\gamma\lambda_{qm})\lambda_t^i = cf^i$$

Modelo matemático para describir el comportamiento fenomenológico y morfología de un Agujero Negro Cuántico y su conexión sustancial tanto con el Agujero Cuántico de Gusano como con el Agujero Blanco Cuántico (Modelo Unificado), todo esto, en escenarios de gravedad y supergravedad cuánticas respectivamente o teorizados como campos cuánticos relativistas.

Incluye cálculos termodinámicos.

$$(16\pi G_5)e^{-1}\mathcal{L}=R+\frac{i}{2}\bar{\psi}_{\mu}^a\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho a}-\frac{1}{8}G_{AB,CD}F_{\mu\nu}^{AB}F^{\mu\nu CD}-\frac{i}{12}\bar{\lambda}^{abc}\gamma^{\mu}D_{\mu}\lambda_{abc}-\frac{1}{6}P_{\mu abcd}P^{\mu abcd} \\ -\frac{i}{3\sqrt{2}}P_{\mu abcd}\bar{\psi}_{\mu}^a\gamma^{\rho}\gamma^{\mu}\lambda^{bcd}-\frac{i}{8}F_{\mu\nu}^{ab}\left(\bar{\psi}_{\mu}^{\rho}\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_b^{\sigma}+\frac{1}{\sqrt{2}}\bar{\psi}_{\rho}^c\gamma^{\mu\nu}\gamma^{\rho}\lambda_{abc} \right. \\ \left. +\frac{1}{2}\bar{\lambda}_{acd}\gamma^{\mu\nu}\lambda_b^{cd}\right)+\frac{1}{24}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu})_b^a(F_{\rho\sigma})_c^b(A_{\lambda})_a^c$$

$$G_{AB,CD}=V_{ABab}\Omega^{ac}\Omega^{bd}V_{CDcd}$$

$$V_{ij}^{ab}=\frac{1}{8}(\Gamma_{kl})^{ab}S_i^kS_j^l+\frac{1}{4}(\Gamma^{k\alpha})^{ab}\varphi_{ijk\alpha}, \\ V_{i\alpha}^{ab}=\frac{1}{4\sqrt{2}}(\Gamma^{kl})^{ab}\varphi_{ikl\alpha}+\frac{1}{2\sqrt{2}}(\Gamma_{k\beta})^{ab}S_i^k\Lambda_{\alpha}^{\beta}.$$

$$G_{ij,kl}=\frac{1}{4}(\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np})S_i^mS_j^nS_k^pS_l^q+\varphi_{ijm\alpha}\varphi_{klm\alpha}=\frac{1}{2}T_{ik}^{-1}T_{jl}^{-1}+\varphi_{ijm\alpha}\varphi_{klm\alpha}$$

$$W_{abcd}=\epsilon^{\alpha\beta}\delta^{IJ}V_{I\alpha ab}V_{J\beta cd}$$

$$\mathcal{T}_{ab}=\frac{15}{4}W_{acb}^c,\mathcal{A}_{abcd}=-3W_{a[bcd]}$$

$$(16\pi G_5)e^{-1}\mathcal{L}=\mathcal{L}_{g+v}+\mathcal{L}_s+\mathcal{L}_f$$

$$\mathcal{L}_{g+v}=R-\frac{1}{8}G_{ij,kl}F_{\mu\nu}^{ij}F^{\mu\nu kl}-\frac{1}{96}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\epsilon_{ijklmn}(F_{\mu\nu}^{ij}F_{\rho\sigma}^{kl}A_{\tau}^{mn}+gF_{\mu\nu}^{ij}A_{\rho}^{kl}A_{\sigma}^{mn}A_{\tau}^{pn} \\ +\frac{2}{5}g^2A_{\mu}^{ij}A_{\nu}^{kp}A_{\rho}^{pl}A_{\sigma}^{mr}A_{\tau}^{rn})+\frac{1}{8g}e^{-1}\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}^{i\alpha}D_{\rho}B_{\sigma\tau}^{j\beta}-\frac{1}{4}G_{i\alpha j\beta}B_{\mu\nu i\alpha}B^{\mu\nu j\beta}$$

$$\mathcal{L}_{g+v}=R\star 1-\frac{1}{8}T_{ik}^{-1}T_{jl}^{-1}F^{ij}\wedge\star F^{kl}-\frac{1}{4}\varphi_{ijm\alpha}\varphi_{klm\alpha}F^{ij}\wedge\star F^{kl}-\frac{1}{48}\epsilon_{i_1\dots i_6}(F^{i_1i_2}\wedge F^{i_3i_4}\wedge A^{i_5i_6} \\ -gF^{i_1i_2}\wedge A^{i_3i_4}\wedge A^{i_5j}\wedge A^{ji_6}+\frac{2}{5}g^2A^{i_1i_2}\wedge A^{i_3j}\wedge A^{ji_4}\wedge A^{i_5k}\wedge A^{ki_6}) \\ +\frac{1}{8g}e^{-1}\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}^{i\alpha}D_{\rho}B_{\sigma\tau}^{j\beta}-\frac{1}{4}G_{i\alpha j\beta}B_{\mu\nu i\alpha}B^{\mu\nu j\beta}$$



$$\mathcal{L}_s = -\frac{1}{6}P_{\mu abcd}P^{\mu abcd} + \frac{1}{6}g^2 \left(\frac{64}{225}(\mathcal{T}_{ab})^2 - (\mathcal{A}_{abcd})^2 \right)$$

$$\mathcal{L}_s = -\frac{1}{4}T_{ij}^{-1}T_{kl}^{-1}DT_{li} \wedge \star DT_{jk} - \frac{1}{2}D\Lambda_\alpha^\beta \wedge \star D\Lambda_\alpha^\beta - \frac{1}{2}D\varphi_{ijk\alpha} \wedge \star D\varphi_{ijk\alpha} - V \star 1$$

$$V = \frac{1}{2}g^2 \left(2T_{ij}T^{ij} - (T_i^i)^2 \right) - \frac{1}{4}g^2\varphi_{ijk\alpha}\varphi^{ijk\alpha}$$

$$\begin{aligned} \mathcal{L}_f = & \frac{i}{2}\bar{\psi}_\mu^a\gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{i}{12}\bar{\lambda}^{abc}\gamma^\mu D_\mu\lambda_{abc} - \frac{i}{3\sqrt{2}}P_{\mu abcd}\bar{\psi}_\mu^a\gamma^\rho\gamma^\mu\lambda^{bcd} \\ & - \frac{i}{8}F_{\mu\nu}^{ab}\left(\bar{\psi}_a^\rho\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_b^\sigma + \frac{1}{\sqrt{2}}\bar{\psi}_\rho^c\gamma^{\mu\nu}\gamma^\rho\lambda_{abc} + \frac{1}{2}\bar{\lambda}_{acd}\gamma^{\mu\nu}\lambda_b^{cd}\right) \\ & + \frac{2i}{15}g\mathcal{T}_{ab}\bar{\psi}_\mu^a\gamma^{\mu\nu}\psi_\nu^b - \frac{i}{6\sqrt{2}}g\mathcal{A}_{dabc}\bar{\lambda}^{abc}\gamma^\mu\psi_\mu^d - ig\bar{\lambda}^{abc}\left(\frac{1}{2}\mathcal{A}_{bcde} - \frac{1}{45}\Omega_{bd}\mathcal{T}_{ce}\right)\lambda_a^{de} \end{aligned}$$

$$DT_{ij} = dT_{ij} + g(A^{ik}T_{kj} + A^{jk}T_{ki})$$

$$D\Lambda_\alpha^\beta = d\Lambda_\alpha^\beta$$

$$D\varphi_{ijk\alpha} = d\varphi_{ijk\alpha} + gA_{l[i}\varphi_{jk]l\alpha}$$

$$F^{ij} = dA^{ij} + gA^{ik}\wedge A^{kj}$$

$$D_\mu B_{\rho\sigma}^{i\alpha} = \partial_\mu B_{\rho\sigma}^{i\alpha} + gA_\mu^{ij}B_{\rho\sigma}^{ja}$$

$$D(T_{ik}^{-1}T_{jl}^{-1}\star F^{kl})=-2gT_{k[i}^{-1}\star DT_{j]k}-\frac{1}{8}\epsilon_{ijk_1\dots k_4}F^{k_1k_2}\wedge F^{k_3k_4},$$

$$D_{[\mu}B_{\nu\rho]}^{i\alpha}-\frac{1}{12}g\epsilon^{\alpha\beta}\epsilon_{\mu\nu\rho\sigma\tau}V_{i\beta}^{ab}V_{ab}^{j\gamma}B_{j\gamma}^{\sigma\tau}=0$$

$$\begin{aligned} D(T_{ik}^{-1}\star DT_{kj}) = & -2g^2(2T_{ik}T_{jk}-T_{ij}T_{kk})\star 1+T_{ik}^{-1}T_{lm}^{-1}F^{lk}\wedge\star F^{mj} \\ & -\frac{1}{6}\delta_{ij}[-2g^2(T_{kl}T_{kl}-T_{kk}^2)\star 1+T_{pk}^{-1}T_{lm}^{-1}F^{lk}\wedge\star F^{mp}]. \end{aligned}$$

$$\Box\,\Lambda_\alpha^\beta=0$$

$$D_\mu D^\mu \varphi_{ijk\alpha} - 3g^2 \varphi_{ijk\alpha} - \frac{1}{2}F_{\mu\nu}^{lm}F^{\mu\nu [ij}\varphi_{k]lm\alpha} = 0$$

$$\begin{aligned} & \gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{1}{3\sqrt{2}}P_{vabcd}\gamma^v\gamma^\mu\lambda^{bcd} - \frac{1}{4}F_{ab}^{\rho\sigma}\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu^b - \frac{1}{8\sqrt{2}}F_{\rho\sigma}^{bc}\gamma^{\rho\sigma}\gamma^\mu\lambda_{abc} \\ & + \frac{4}{15}g\mathcal{T}_{ab}\gamma^{\mu\nu}\psi_\nu^b - \frac{1}{6\sqrt{2}}g\mathcal{A}_{abcd}\gamma^\mu\lambda^{bcd} = 0 \\ & \frac{1}{6}\gamma^\mu D_\mu\lambda_{abc} - \frac{1}{3\sqrt{2}}P_{vabcd}\gamma^\mu\gamma^\nu\psi_\mu^d - \frac{1}{8\sqrt{2}}\gamma^\rho\gamma_{\mu\nu}\psi_{\rho[a}F_{bc]}^{\mu\nu} \\ & + \frac{1}{8}\gamma_{\mu\nu}\lambda_{[ab}^dF_{c]d}^{\mu\nu} + \frac{1}{6\sqrt{2}}g\mathcal{A}_{dabc}\gamma^\mu\psi_\mu^d + g\left(\frac{1}{2}\mathcal{A}_{[ab}^{ef} + \frac{1}{45}\delta_{[a}^e\mathcal{T}_{b]}^f\right)\lambda_{c]ef} = 0 \end{aligned}$$

$$\mathcal{L}_{\text{EMAdS}} = \frac{1}{16\pi G_5} \left((R + 12g^2) \star 1 - \frac{3}{2} F \wedge \star F - F \wedge F \wedge A \right)$$

$$d\star F + F\wedge F = 0$$

$$R_{\mu\nu} + 4g^2g_{\mu\nu} = \frac{3}{2}F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}$$



$$M = \frac{\pi}{4G_5} \frac{m(3 + a^2 g^2) + 4qa^2 g^2}{\Xi^3}$$

$$Q = \frac{\pi}{4G_5} \frac{q}{\Xi^2}$$

$$J = \frac{\pi}{4G_5} \frac{2ma + qa(1 + a^2 g^2)}{\Xi^3}$$

$$ds^2 = -\frac{[(1 + g^2 r^2)\rho^2 dt + 2qv]dt}{\Xi\rho^2} + \frac{2qv^2}{\Xi\rho^2} + \frac{f}{\Xi^2\rho^4}(dt - v)^2$$

$$+ \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Xi} + \frac{r^2 + a^2}{\Xi}(\sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2)$$

$$A = \frac{3q}{\Xi\rho^2}(dt - v)$$

$$\Delta_r = \frac{(r^2 + a^2)^2(1 + g^2 r^2) + q^2 + 2a^2 q}{r^2} - 2m, \rho^2 = r^2 + a^2$$

$$v = a(\sin^2 \theta d\phi + \cos^2 \theta d\psi), f = 2m\rho^2 - q^2 + 2a^2 g^2 q\rho^2$$

$$S = \frac{\pi^2}{2G_5} \frac{(r_+^2 + a^2)^2 + a^2 q}{\Xi^2 r_+}$$

$$T = \frac{r_+^4(1 + 2g^2(r_+^2 + a^2)) - (q + a^2)^2}{2\pi r_+((r_+^2 + a^2)^2 + a^2 q)}$$

$$\Omega = a \frac{(r_+^2 + a^2)(1 + g^2 r_+^2) + q}{(r_+^2 + a^2)^2 + a^2 q}$$

$$\Phi = \frac{qr_+^2}{(r_+^2 + a^2)^2 + a^2 q}$$

$$(1 + g^2 r_+^2)(r_+^2 + a^2)^2 + q^2 + 2a^2 q - 2mr_+^2 = 0$$

$$\Lambda^\beta_\alpha = \varphi_{ijk\alpha} = B^{i\alpha}_{\mu\nu} = \psi^a_\mu = \lambda_{abc} = 0$$

$$T_{ij} = \delta_{ij}, A_{\mu ij} = A_\mu \Omega_{ij}$$

$$\Omega^{ij} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

$$\left((3_{-2} \oplus \overline{3}_2) \otimes (3_{-2} \oplus \overline{3}_2) \right)_{\text{sym, tr}} = (3_{-2} \otimes 3_{-2})_{\text{sym}} \oplus (\overline{3}_2 \otimes \overline{3}_2)_{\text{sym}} \oplus (\overline{3} \otimes 3)_{\text{tr}} = \mathbf{6}_{-4} \oplus \overline{\mathbf{6}}_4 \oplus \mathbf{8}_0.$$

$$T_{ij} = \delta_{ij} + t_{ij}, A^{ij} = A\Omega^{ij} + a^{ij}$$

$$DT_{ij} = \underbrace{dt_{ij} + gA(\Omega^{ik}t_{kj} + \Omega^{jk}t_{ki})}_{Dt_{ij}} + \underbrace{g(a^{ik}t_{kj} + a^{jk}t_{ki})}_{f^{ij} = D a^{ij}}$$

$$F^{ij} = F\Omega^{ij} + \underbrace{da^{ij} + gA\wedge(\Omega^{ik}a^{kj} - a^{ik}\Omega^{kj})}_{f^{ij} = D a^{ij}} + \underbrace{ga^{ik}\wedge a^{kj}}_{f^{ij} = D a^{ij}}$$



$$D \star Dt_{ij} - 4g^2 t_{ij} \star 1 + 2\Omega^{il} t_{lm} \Omega^{mj} (F \wedge \star F) = -F \wedge \star (f^{kj} \Omega^{ki} + f^{ki} \Omega^{kj}),$$

$$D \star f^{ij} + F \wedge \star (\Omega^{ik} f^{kl} \Omega^{lj}) = D \left((t_{ik} \Omega^{kj} + t_{jk} \Omega^{ki}) \star F \right).$$

$$t_{ij} = t_{ij}^+ + t_{ij}^-$$

$$\Omega^{im} t_{mn}^{\pm} \Omega^{nj} = \pm t_{ij}^{\pm}$$

$$t_{ij}^- = \frac{1}{2} \begin{pmatrix} t_{11} + t_{22} & t_{13} + t_{24} \\ t_{33} + t_{44} & t_{35} + t_{26} \\ & t_{55} + t_{66} \end{pmatrix} \otimes I + \frac{1}{2} \begin{pmatrix} 0 & t_{14} - t_{23} & t_{16} - t_{25} \\ & 0 & t_{36} - t_{45} \\ & & 0 \end{pmatrix} \otimes (i\sigma_2)$$

$$t_{ij}^+ = \frac{1}{2} \begin{pmatrix} t_{11} - t_{22} & t_{13} - t_{24} \\ t_{33} - t_{44} & t_{35} - t_{26} \\ & t_{55} - t_{66} \end{pmatrix} \otimes \sigma_3 + \frac{1}{2} \begin{pmatrix} 2t_{12} & t_{14} + t_{23} \\ & t_{16} + t_{25} \\ & 2t_{34} \\ & & t_{36} + t_{45} \end{pmatrix} \otimes \sigma_1.$$

$$a_{ij} = a_{ij}^+ + a_{ij}^-$$

$$\Omega^{im} a_{mn}^{\pm} \Omega^{nj} = \pm a_{ij}^{\pm}.$$

$$a_{ij}^- = \frac{1}{2} \begin{pmatrix} 0 & a^{13} + a^{24} & a^{15} + a^{26} \\ & 0 & a^{35} + a^{46} \\ & & 0 \end{pmatrix} \otimes I + \frac{1}{2} \begin{pmatrix} 2a^{12} & a^{14} - a^{23} \\ & a^{16} - a^{25} \\ & 2a^{34} \\ & & a^{36} - a^{45} \\ & & & 2a^{56} \end{pmatrix} \otimes (i\sigma_2)$$

$$a_{ij}^+ = \frac{1}{2} \begin{pmatrix} 0 & a^{13} - a^{24} & a^{15} - a^{26} \\ & 0 & a^{35} - a^{46} \\ & & 0 \end{pmatrix} \otimes \sigma_3 + \frac{1}{2} \begin{pmatrix} 0 & a^{14} + a^{23} & a^{16} + a^{25} \\ & 0 & a^{36} + a^{45} \\ & & 0 \end{pmatrix} \otimes \sigma_1,$$

$$D \star Dt_{ij}^+ - 4g^2 t_{ij}^+ \star 1 - 2t_{ij}^+ (F \wedge \star F) = 0,$$

$$D \star f_+^{ij} + F \wedge f_+^{ij} = 0$$

$$Dt_{ij}^+ = dt_{ij}^+ + 2gA\Omega^{ik} t_{kj}^+$$

$$f_+^{ij} = Da_+^{ij} = da_+^{ij} + 2g\Omega^{ik} A \wedge a_+^{kj}$$

$$D \star Dt_{ij}^- - 4g^2 t_{ij}^- \star 1 + 2t_{ij}^- (F \wedge \star F) = 2F \wedge \star (f_-^{ki} \Omega^{kj})$$

$$D \star (f_-^{ki} \Omega^{kj}) - F \wedge (f_-^{ki} \Omega^{kj}) = -2D(t_{ij}^- \star F)$$

$$Dt_{ij}^- = dt_{ij}^-, f_-^{ij} = Da_-^{ij} = da_-^{ij}$$

$$D^\mu D_\mu \varphi_{ijk\alpha} - 3g^2 \varphi_{ijk\alpha} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Omega_{[ij} \varphi_{k]l m \alpha} \Omega^{lm} = 0$$

$$\Omega_{[ij} \varphi_{k]l m \alpha}^{(1)} \Omega^{lm} = 0$$

$$\Omega_{[ij} \varphi_{k]l m \alpha}^{(3)} \Omega^{lm} = 0$$



$$+\frac{1}{3}F_{\mu\nu}F^{\mu\nu}\left(\varphi_{ijm\alpha}^{(6)}\right)^2,$$

$$D_{\mu}\varphi_{ijk\alpha}^{(1)}=\partial_{\mu}\varphi_{ijk\alpha}^{(1)}\pm3gA_{\mu}\varphi_{ijk\alpha}^{(1)}$$

$$D_{\mu}\varphi_{ijk\alpha}^{(3)}=\partial_{\mu}\varphi_{ijk\alpha}^{(3)}\pm gA_{\mu}\varphi_{ijk\alpha}^{(3)}$$

$$D_{\mu}\varphi_{ijk\alpha}^{(6)}=\partial_{\mu}\varphi_{ijk\alpha}^{(6)}\pm gA_{\mu}\varphi_{ijk\alpha}^{(6)}$$

$$\square\,\Lambda_{\alpha}^{\beta}=0$$

$$\epsilon^{\mu\nu\rho\sigma\tau}D_{\rho}B_{\sigma\tau}^{i\pm}+iB_{\mu\nu}^{i\pm}=0$$

$$B^{\mu\nu i\pm}=\frac{1}{2}(B^{\mu\nu i1}+\Omega^{ij}B^{\mu\nu j2})\pm\frac{i}{2}(B^{\mu\nu i2}-\Omega^{ij}B^{\mu\nu j1})$$

$$D_{\rho}B_{\sigma\tau}^{i\pm}=\partial_{\rho}B_{\sigma\tau}^{i\pm}\mp gA_{\mu}B_{\sigma\tau}^{j\pm}$$

$$\sigma_1=-\sin\,\psi d\theta+\cos\,\psi\sin\,\theta d\phi$$

$$\sigma_2=\cos\,\psi d\theta+\sin\,\psi\sin\,\theta d\phi$$

$$\sigma_3=d\psi+\cos\,\theta d\phi$$

$$ds^2=-\frac{r^2}{(r^2+a^2)^2(1-a^2)}\Delta_r e^{U_2-U_1}dt^2+\frac{(r^2+a^2)dr^2}{\Delta_r}+e^{-U_1}(\sigma_1^2+\sigma_2^2)+e^{-U_2}(\sigma_3+\omega dt)^2,$$

$$e^{-U_1}=\frac{r^2+a^2}{4(1-a^2)}$$

$$e^{-U_2}=\frac{r^2+a^2}{4(1-a^2)}\bigg(1+\frac{2(m+q)a^2}{(1-a^2)(r^2+a^2)^2}-\frac{a^2q^2}{(1-a^2)(r^2+a^2)^3}\bigg)$$

$$\omega=\frac{a}{2(1-a^2)}\bigg((r^2+a^2)-\frac{2m+q}{r^2+a^2}+\frac{q^2}{(r^2+a^2)^2}\bigg)e^{U_2}-2a$$

$$A=\frac{q}{(1-a^2)(r^2+a^2)}\bigg(dt-\frac{1}{2}a\sigma_3\bigg).$$

$$q=r_+^2\sqrt{2r_+^2+1+2a^2}-a^2$$

$$2m=(r_+^2+a^2)(3r_+^2+2+a^2)$$

$$Q=\frac{\pi}{4G_5}\frac{1}{(1-a^2)^2}\bigg(r_+^2\sqrt{2r_+^2+1+2a^2}-a^2\bigg)$$

$$J=\frac{\pi}{4G_5}\frac{a}{(1-a^2)^3}\bigg(3r_+^4+2r_+^2(1+2a^2)+a^2+r_+^2(1+a^2)\sqrt{2r_+^2+1+2a^2}\bigg)$$

$$\Phi = \frac{-a^2 + r_+^2 \sqrt{2r_+^2 + 2a^2 + 1}}{r_+^2 + 2a^2 + a^2 \sqrt{2r_+^2 + 2a^2 + 1}}$$

$$\Omega = a \frac{r_+^2 + a^2 + 1 + \sqrt{2r_+^2 + 1 + 2a^2}}{r_+^2 + 2a^2 + a^2 \sqrt{2r_+^2 + 1 + 2a^2}}$$

$$S = \frac{\pi^2}{2G_5} \frac{r_+ \left(r_+^2 + 2a^2 + a^2 \sqrt{2r_+^2 + 2a^2 + 1} \right)}{(1 - a^2)^2}$$

$$r_+^2 \geq \frac{1}{4} \left(\sqrt{1 + 8a^2} - 1 \right)$$

$$\left(Q^3 + \frac{\pi}{4G_5} J^2 \right) - \left(3Q + \frac{\pi}{4G_5} \right) \left(3Q^2 - \frac{\pi}{2G_5} J \right) = 0$$

$$M - 3Q - J = \frac{\pi}{4G_5} \frac{3 - a}{(1 - a)^2 (1 + a)^3} \left(\frac{1}{2} (r_+^2 + a^2) (3r_+^2 + 2 + a^2) - (1 \right.$$

$$\left. + 2a) \left(r_+^2 \sqrt{2r_+^2 + 1 + 2a^2} - a^2 \right) \right) \geq 0$$

$$r_*^2 = a(2 + a)$$

$$\frac{r^2}{(1-a^2)(r^2+a^2)^2} \Delta_r e^{U_2-U_1} = e^{-U_2} \omega^2.$$

$$ds^2 = -A^2 (r-r_+)^2 dt^2 + \frac{B^2}{(r-r_+)^2} dr^2 + \frac{1}{4} C^2 (\sigma_1^2 + \sigma_2^2)$$

$$+ \frac{1}{4} D^2 (\sigma_3 + E(r-r_+) dt)^2$$

$$\sigma_{3\mathrm{as}} = \sigma_{3\mathrm{hor}} - \omega|_{r=r_+} dt = \sigma_{3\mathrm{hor}} + 2\Omega dt$$

$$A = \frac{2\sqrt{(r_+^2 + a^2)(3r_+^2 + 2a^2 + 1)}}{r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2}}$$

$$B = \sqrt{\frac{r_+^2 + a^2}{4(3r_+^2 + 2a^2 + 1)}}$$

$$C = \sqrt{\frac{r_+^2 + a^2}{1 - a^2}}$$

$$D = \frac{r_+}{(1 - a^2)(r_+^2 + a^2)} \left(r_+^2 + 2a^2 + a^2 \sqrt{2r_+^2 + 1 + 2a^2} \right)$$

$$E = \frac{4a(1 - a^2)}{r_+} \frac{r_+^2 + (2r_+^2 + a^2)\sqrt{2r_+^2 + 1 + 2a^2}}{\left(r_+^2 + 2a^2 + a^2\sqrt{2r_+^2 + 1 + 2a^2} \right)^2}$$

$$\mathcal{A}_t|_{r=r_+} = 0$$



$$\mathcal{A} = -F(r - r_+)dt - G\sigma_3$$

$$\mathcal{F} = d\mathcal{A} = -Fdrdt - G\sigma_1\sigma_2$$

$$\mathcal{A} = \frac{q}{(1-a^2)}(1-a\Omega)\left(\frac{1}{r^2+a^2} - \frac{1}{r_+^2+a^2}\right)dt - \frac{qa}{2(1-a^2)(r^2+a^2)}\sigma_3$$

$$\mathcal{A} = -\frac{2qr_+}{(r_+^2+a^2)\left(r_+^2+2a^2+a^2\sqrt{2r_+^2+1+2a^2}\right)}(r-r_+)dt - \frac{qa}{2(1-a^2)(r_+^2+a^2)}\sigma_3$$

$$F = \frac{2\left(r_+^2\sqrt{1+2a^2+2r_+^2}-a^2\right)r_+}{(r_+^2+a^2)\left(r_+^2+2a^2+a^2\sqrt{2r_+^2+1+2a^2}\right)}$$

$$G = \frac{\left(r_+^2\sqrt{1+2a^2+2r_+^2}-a^2\right)a}{2(1-a^2)(r^2+a^2)}$$

$$\begin{aligned}\mathcal{A} \wedge \star \mathcal{A} &= \mathcal{A}_\mu \mathcal{A}^\mu \star 1 = \left(\frac{4G^2}{D^2} - \frac{(F-EG)^2}{A^2} \right) \star 1 \\ &= \frac{\left(a^2 - r_+^2 \sqrt{1+2a^2+2r_+^2} \right)^2 \left(-r_+^2 + a^2 \left(1 + 3r_+^2 + 3a^2 + 2\sqrt{1+2a^2+2r_+^2} \right) \right)}{(r_+^2+a^2)(1+2a^2+3r_+^2)\left(r_+^2+a^2\left(2+\sqrt{1+2a^2+2r_+^2}\right)\right)^2} \star 1\end{aligned}$$

$$\begin{aligned}\mathcal{F} \wedge \star \mathcal{F} &= \frac{1}{2} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \star 1 = \left(\frac{16G^2}{C^4} - \frac{F^2}{A^2 B^2} \right) \star 1 \\ &= \frac{4(a^2 - r_+^2) \left(a^2 - r_+^2 \sqrt{1+2a^2+2r_+^2} \right)^2}{(r_+^2+a^2)^4} \star 1\end{aligned}$$

$$E = \frac{16AB(ABD + C^2F)}{C^4DF}G$$

$$R_{\mu\nu} + 4g_{\mu\nu} - \left(\frac{3}{2} \mathcal{F}_{\mu\lambda} \mathcal{F}_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} \mathcal{F}_{\lambda\rho} \mathcal{F}^{\lambda\rho} \right) = 0$$

$$B = \frac{C}{2\sqrt{1+3C^2}}$$

$$E = \frac{1}{CD} \sqrt{\frac{2A^2D^2}{1+3C^2} + 8A^2C^2 - 6C^2F^2}$$

$$G = \frac{1}{2\sqrt{2}A} \sqrt{A^2(2C^2+2C^4-D^2) - C^2(1+3C^2)F^2}$$

$$ds_2^2 = -A^2(r-r_+)^2 dt^2 + \frac{B^2}{(r-r_+)^2} dr^2$$

$$z = \frac{1}{r-r_+}, \tau = \frac{B}{A}t$$

$$ds_2^2 = \frac{B^2}{z^2} (-d\tau^2 + dz^2)$$

$$\begin{aligned} (\square - m^2)\Phi &= \left(\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu) - m^2 \right) \Phi \\ &= \left(\frac{1}{B^2} \partial_r ((r - r_+)^2 \partial_r) + \frac{\omega^2}{A^2 (r - r_+)^2} - m^2 \right) \Phi = 0 \end{aligned}$$

$$\left(-\frac{\partial^2}{\partial z^2} + \frac{m^2 B^2}{z^2}\right)\Phi = \frac{B^2}{A^2} \omega^2 \Phi$$

$$m^2 \ell_2^2 \geq -\frac{1}{4}$$

$$\Delta_0 = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2 \ell_2^2}$$

$$\ell_2 = B = \sqrt{\frac{r_+^2 + a^2}{4(3r_+^2 + 2a^2 + 1)}},$$

$$\mathcal{L} = d\Phi_p \wedge \star d\Phi - m^2 \Phi_p \wedge \star \Phi \leftrightarrow d \star d\Phi - (-1)^p m^2 \star \Phi_p = 0$$

$$d\star\Phi_p=0\,,(m^2\neq0)$$

$$D\phi = d\phi + ie\mathcal{A}\phi$$

$$\mathcal{L} = -\frac{1}{2}d\phi\wedge d\phi^* - \frac{1}{2}(m^2\star 1 + e^2\mathcal{A}\wedge\star\mathcal{A} - p\mathcal{F}\wedge\star\mathcal{F})\phi\phi^*$$

$$m_{\rm eff}^2 = m^2 + e^2 \mathcal{A}_\mu \mathcal{A}^\mu - \frac{1}{2} p \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$$

$$\begin{aligned} m_{t^+}^2 &= -4 + 4\mathcal{A}_\mu \mathcal{A}^\mu - \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \\ &= -4 + \frac{4\left(a^2\left(2\sqrt{2a^2+2r_+^2+1}+3a^2+3r_+^2+1\right)-r_+^2\right)\left(a^2-r_+^2\sqrt{2a^2+2r_+^2+1}\right)^2}{\left(a^2+r_+^2\right)\left(2a^2+3r_+^2+1\right)\left(a^2\left(\sqrt{2a^2+2r_+^2+1}+2\right)+r_+^2\right)^2} \\ &\quad + \frac{8\left(r_+^2-a^2\right)\left(a^2-r_+^2\sqrt{2a^2+2r_+^2+1}\right)^2}{\left(a^2+r_+^2\right)^4} \end{aligned}$$

$$m_{t^+}^2\ell_2^2=\frac{1-\sqrt{1+8a^2}}{2\sqrt{1+8a^2}}\in\left(-\frac{1}{3},0\right)$$

$$m_{t^+}^2\ell_2^2=\frac{2+6a-4a^2}{(1+5a)^2}\in\left(\frac{1}{9},2\right)$$

$$m_{t^+}^2\ell_2^2=\frac{2+8r_+^2+7r_+^4}{(1+3r_+^2)^2}\in\left(\frac{7}{9},2\right)$$

$$\varphi^{(1)}\colon m_{\varphi(1)}^2=-3+9\mathcal{A}_\mu\mathcal{A}^\mu$$



$$\varphi^{(3)}: m_{\varphi(3)}^2 = -3 + \mathcal{A}_\mu \mathcal{A}^\mu$$

$$\varphi^{(6)}: m_{\varphi(6)}^2 = -3 + \mathcal{A}_\mu \mathcal{A}^\mu - 2\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$$

$$m^2\ell_2^2=\frac{3(1-\sqrt{1+8a^2})}{8\sqrt{1+8a^2}}\in\left(-\frac{1}{4},0\right)$$

$$m_{\varphi(1)}^2\ell_2^2=-\frac{6a(1-a)}{(1+5a)^2}\in\left(-\frac{1}{4},0\right)$$

$$m_{\varphi(3)}^2\ell_2^2=-\frac{2a(1+3a)}{(1+5a)^2}\in\left(-\frac{2}{9},0\right)$$

$$m_{\varphi(6)}^2\ell_2^2=\frac{2(2+9a-3a^2)}{(1+5a)^2}\in\left(\frac{4}{9},4\right)$$

$$m_{\varphi(1)}^2\ell_2^2=\frac{-3r_+^2(4+9r_+^2)}{4(1+3r_+^2)^2}\in\left(-\frac{3}{4},0\right)$$

$$m_{\varphi(3)}^2\ell_2^2=\frac{-r_+^2(4+11r_+^2)}{4(1+3r_+^2)^2}\in\left(-\frac{11}{36},0\right)$$

$$m_{\varphi(6)}^2\ell_2^2=\frac{16+76r_+^2+85r_+^2}{4(1+3r_+^2)^2}\in\left(\frac{85}{36},4\right)$$

$$\begin{aligned}d\star dt_- + 4g^2 t_- \star 1 - t_-(\mathcal{F} \wedge \star \mathcal{F}) &= 2\mathcal{F} \wedge \star f_- \\ d\star f_- - \mathcal{F} \wedge f_- &= 2d(t_- \star \mathcal{F})\end{aligned}$$

$$a_- = a^-_t dt + a^-_r dr + \sum_{i=1}^3 a^-_i \sigma_i$$

$$\begin{aligned}\left[\frac{1}{A^2(r-r_+)^2}\partial_t^2-\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)+\left(\frac{4}{D^2}-\frac{E^2}{A^2}-\frac{2F}{ABD}\right)\right]a_1^-&=\frac{2E}{A^2(r-r_+)}\partial_ta_2^-\\ \left[\frac{1}{A^2(r-r_+)^2}\partial_t^2-\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)+\left(\frac{4}{D^2}-\frac{E^2}{A^2}-\frac{2F}{ABD}\right)\right]a_2^-&=-\frac{2E}{A^2(r-r_+)}\partial_ta_1^-\end{aligned}$$

$$a_{\perp}^{-}=a_1^{-}+ia_2^{-}$$

$$\left[\frac{1}{A^2(r-r_+)^2}\partial_t^2-\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)+\left(\frac{4}{D^2}-\frac{E^2}{A^2}-\frac{2F}{ABD}\right)\right]a_{\perp}^{-}=-\frac{2iE}{A^2(r-r_+)}\partial_ta_{\perp}^{-}$$

$$(\Box_2-m_{\rm eff}^2)\chi=0$$

$$\Box_2=-\frac{1}{A^2(r-r_+)^2}\partial_t^2+\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)$$

$$m_{1-}^2=\frac{4}{D^2}-\frac{E^2}{A^2}-\frac{2F}{ABD}$$

$$0=d\left(\partial_{{\scriptscriptstyle{r}}}a_{{\scriptscriptstyle{t}}}^- - \partial_{{\scriptscriptstyle{t}}}a_{{\scriptscriptstyle{r}}}^- - Er\partial_{{\scriptscriptstyle{r}}}a_3^- + 2Ft_- + \frac{8ABG}{DC^2}a_3^- \right)$$

$$\partial_r a_t^- - \partial_t a_r^- - Er \partial_r a_3^- + 2Ft_- + \frac{8ABG}{DC^2} a_3^- = 0$$

$$\Box_2 \left(\begin{smallmatrix} a_3^- \\ t^- \end{smallmatrix} \right) - M_{\text{eff}}^2 \left(\begin{smallmatrix} a_3^- \\ t^- \end{smallmatrix} \right) = 0$$

$$M_{\text{eff}}^2 = \left(\begin{array}{cc} \frac{4D^2}{C^4} - \frac{2D(F-EG)}{ABC^2} + \frac{16G^2}{C^4} & \frac{16DG}{C^4} + \frac{8FG}{ABC^2} \\ \frac{16DG}{C^4} + \frac{8FG}{ABC^2} & -4g^2 + \frac{16G^2}{C^4} + \frac{3F^2}{A^2B^2} \end{array} \right)$$

$$m_{1-}^2\ell_2^2 = -\frac{8(1-a^2)^2}{(8a^2+1)(3+\sqrt{8a^2+1})^2} \in \left(-\frac{1}{2},0\right),$$

$$m_{1-}^2\ell_2^2 = -\frac{6a(1-a)}{(1+5a)^2} \in \left(-\frac{1}{4},0\right),$$

$$m_{1-}^2\ell_2^2 = \frac{1-\sqrt{1+2r_+^2}}{1+3r_+^2} \in \left(\sqrt{\frac{2}{3}}-1,0\right).$$

$$\text{Kerr: } \{m_{3<}^2, m_{3>}^2\} \ell_2^2 = \left\{ \frac{\sqrt{1+8a^2}-1}{2\sqrt{1+8a^2}}, \frac{2}{\sqrt{1+8a^2}} \right\} \in \left\{ \left(-\frac{1}{3}, 0\right), \left(\frac{2}{3}, 2\right) \right\},$$

$$\text{BPS: } \{m_{3<}^2, m_{3>}^2\} \ell_2^2 = \left\{ \frac{3+2a \pm \sqrt{9+20a+20a^2}}{2(1+5a)} \right\} \in \left\{ \left(-\frac{1}{6}, 0\right), (1,3) \right\},$$

$$\text{RN: } \{m_{3<}^2, m_{3>}^2\} \ell_2^2 = \left\{ \frac{4+5r_+^2+\sqrt{1+2r_+^2} \pm \sqrt{5+22r_+^2+25r_+^4+2(1+5r_+^2)\sqrt{1+2r_+^2}}}{2(1+3r_+^2)} \right\} \\ \in \left\{ \left(-1+\sqrt{\frac{2}{3}}, 0\right), \left(\frac{5}{3}, 3\right) \right\}.$$

$$D\star f^+ + \mathcal{F} \wedge f^+ = 0$$

$$DB=i\star B$$

$$a_+=a_t^+dt+a_r^+dr+\sum_{i=1}^3a_i^+\sigma_i$$

$$a_{\perp}^{\dagger}=a_1^{\dagger}+ia_2^{\dagger}$$

$$\left[\frac{1}{A^2(r-r_+)^2}(\partial_t+2iE(r-r_+))\partial_t-\frac{1}{B^2}\partial_r((r-r_+)^2\partial_r)+m_{\perp}^2\right]a_{\perp}^{\dagger}=0,$$

$$m_{1+}^2=\frac{4}{D^2}+\frac{2F}{ABD}-\frac{E^2}{A^2}+\frac{16G^2}{D^2}-\frac{4(F-EG)^2}{A^2}$$



$$m_{1+}^2 \ell_2^2 = -\frac{8(1-a^2)^2}{(8a^2+1)(3+\sqrt{8a^2+1})^2} \in \left(-\frac{1}{2}, 0\right)$$

$$m_{1+}^2 \ell_2^2 = \frac{2-10a^2}{(1+5a)^2} \in \left(-\frac{2}{9}, 2\right)$$

$$m_{1+}^2 \ell_2^2 = \frac{1+2r_+^2-2r_+^4+(1+3r_+^2)\sqrt{1+2r_+^2}}{(1+3r_+^2)^2} \in \left(-\frac{2}{9}, 2\right)$$

$$\partial_r a_t^+ - \partial_t a_r^+ - E(r-r_+) \partial_r a_3^+ + 2(F-EG)(r-r_+) r a_r^+ - \frac{8ABG}{C^2 D} a_3^+ = 0$$

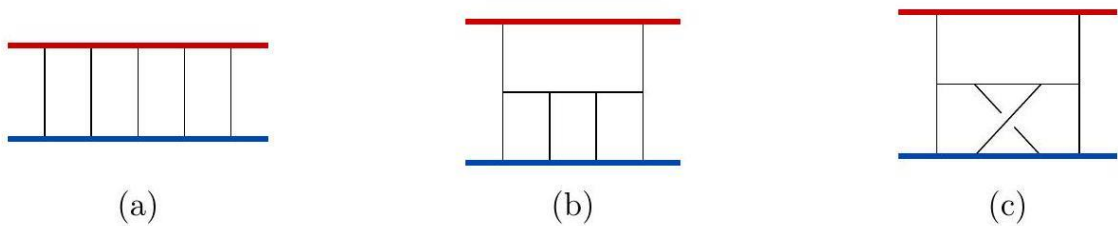
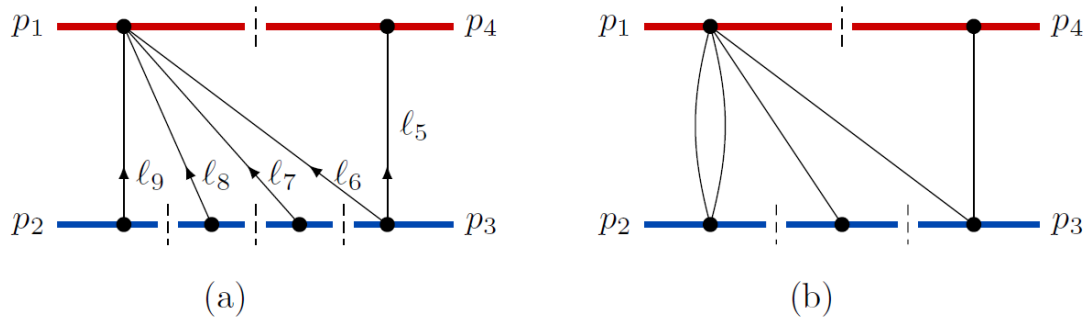
$$m_{3+}^2 = \frac{4D^2}{C^4} + \frac{2D(F-EG)}{ABC^2} - \frac{4(F-EG)^2}{A^2} + \frac{16G^2}{C^4}$$

$$m_{3+}^2 \ell_2^2 = \frac{2}{\sqrt{1+8a^2}} \in \left(\frac{2}{3}, 2\right)$$

$$m_{3+}^2 \ell_2^2 = \frac{4+24a+27a^2-a^3}{(2+a)(1+5a)^2} \in \left(\frac{1}{2}, 2\right)$$

$$m_{3+}^2 \ell_2^2 = \frac{1-2r_+^4+\sqrt{1+2r_+^2}+r_+^2(2+3\sqrt{1+2r_+^2})}{1+3r_+^2} \in \left(-\frac{2}{9}, 2\right)$$

$$m_B^2 = 1 + \mathcal{A}^\mu \mathcal{A}_\mu = 1 + \frac{4G^2}{D^2} - \frac{(F-EG)^2}{A^2}$$



$$\mathcal{C}_{(a)} = \delta((p_2 + \ell_9)^2 - m_2^2) \delta((p_2 + \ell_{89})^2 - m_2^2) \delta((p_2 + \ell_{789})^2 - m_2^2)$$

$$\times \delta((p_4 - \ell_5)^2 - m_1^2) \prod_{i=5}^9 \delta(\ell_i^2) M_0(p_1, -\ell_9, -\ell_8, -\ell_7, -\ell_6) M_0(p_4, -\ell_5)$$

$$\times M_0(p_2, \ell_9) M_0(p_2 + \ell_9, \ell_8) M_0(p_2 + \ell_{89}, \ell_7) M_0(p_2 + \ell_{789}, \ell_6, \ell_5),$$

$$M_L = G^{L+1} \sum_i \int \prod_{j=1}^L \frac{d^D \ell_j}{(2\pi)^D} \frac{N_i(p_e, \ell_j)}{\prod_{\alpha_i} (k_{\alpha_i}^2 - m_{\alpha_i}^2 + i\varepsilon)},$$

$$k_1=\begin{pmatrix} p_1 \\ 0 \\ m_1 \end{pmatrix}, k_2=\begin{pmatrix} p_2 \\ m_2 \sin \phi \\ m_2 \cos \phi \end{pmatrix}, k_3=\begin{pmatrix} p_3 \\ -m_2 \sin \phi \\ -m_2 \cos \phi \end{pmatrix}, k_4=\begin{pmatrix} p_4 \\ 0 \\ -m_1 \end{pmatrix},$$

$$(k_1+k_2)^2\rightarrow s-|m_1+m_2e^{i\phi}|^2, (k_1+k_4)^2\rightarrow t, (k_1+k_3)^2\rightarrow u-|m_1+m_2e^{i\phi}|^2$$

$$s,u,m_1^2,m_2^2\sim J^2|t|\gg|t|=|q^2|$$

$$\text{hard: } \ell \gg |q|, \text{ soft: } \ell \sim |q|,$$

$$\text{potential: } \ell \sim (v,\mathbf{1})|q|, \text{ radiation: } \ell \sim (v,\boldsymbol{v})|q|.$$

$$p_1=-\bar{m}_1u_1+\frac{q}{2}, p_2=-\bar{m}_2u_2-\frac{q}{2}, p_3=\bar{m}_2u_2-\frac{q}{2}, p_4=\bar{m}_1u_1+\frac{q}{2},$$

$$u_1^2=u_2^2=1, u_1\cdot q=u_2\cdot q=0, \bar{m}_i^2=m_i^2\left(1-\frac{q^2}{4m_i^2}\right)$$

$$y=u_1\cdot u_2$$

$$y=\sigma+\mathcal{O}(q^2), \text{ where } \sigma=\frac{p_1\cdot p_2}{m_1m_2}$$

$$\delta((p_2+\ell_9)^2-m_2^2)=\delta(2p_2\cdot\ell_9)=\delta(-2\bar{m}_2u_2\cdot\ell_9+q\cdot\ell_9)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^{(n)}(-2\bar{m}_2u_2\cdot\ell_9)(q\cdot\ell_9)^n$$

$$\frac{1}{(p_2+\ell_9)^2-m_2^2+i\varepsilon}=\frac{1}{-2\bar{m}_2u_2\cdot\ell_9+q\cdot\ell_9+i\varepsilon}=\sum_{n=0}^{\infty}\frac{(-q\cdot\ell_9)^n}{(-2\bar{m}_2u_2\cdot\ell_9+i\varepsilon)^{n+1}},$$

$$\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}=-2\pi i\delta(x),\frac{1}{(x+i\varepsilon)^{n+1}}-\frac{1}{(x-i\varepsilon)^{n+1}}=-2\pi i\frac{(-1)^n}{n!}\delta^{(n)}(x)$$

$$\delta(\omega_1+\cdots\omega_n)\sum_{\text{perms}\,\omega_i}\frac{1}{\omega_1+i\varepsilon}\cdots\frac{1}{\omega_1+\cdots+\omega_{n-1}+i\varepsilon}=(-2\pi i)^n\delta(\omega_1)\cdots\delta(\omega_n),$$

$$\mathrm{i}\mathcal{M}(\boldsymbol{q})=\int_J\left(e^{\mathrm{i}I_r(J)}-1\right)$$

$$\tilde{I}_r(\boldsymbol{q})=\int_J I_r(J)\colon=\mu^{-2\epsilon}4E|\boldsymbol{p}|\int\,\mathrm{d}^{D-2}\boldsymbol{b}e^{\mathrm{i}q\cdot\boldsymbol{b}}I_r(J)$$

$$E=\sqrt{s}=\sqrt{2m_1m_2\sigma+m_1^2+m_2^2}, |\boldsymbol{p}|=\frac{m_1m_2\sqrt{\sigma^2-1}}{E}$$



$$\int_J \frac{(\mathrm{i}I_r(J))^n}{n!} = \mathrm{i} \int_{\boldsymbol{\ell}} \frac{\tilde{I}_r(\boldsymbol{\ell}_1) \dots \tilde{I}_r(\boldsymbol{\ell}_n)}{Z_1 \dots Z_{n-1}}$$

$$Z_j = -4E|\boldsymbol{p}| \left((\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \dots + \boldsymbol{\ell}_j) \cdot \hat{\boldsymbol{z}} + \mathrm{i}\varepsilon \right)$$

$$\int_{\boldsymbol{\ell}} := \mu^{2(n-1)\epsilon} \int \left(\prod_{i=1}^n \frac{\mathrm{d}^{D-1}\boldsymbol{\ell}_i}{(2\pi)^{D-1}} \right) (2\pi)^{D-1} \delta^{(D-1)} \left(\sum_{j=1}^n \boldsymbol{\ell}_j - \boldsymbol{q} \right)$$

$$\mathcal{M}_1=\tilde{I}_{r,1}, \mathcal{M}_2=\tilde{I}_{r,2}+\int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}}{Z_1}$$

$$\mathcal{M}=\sum_{k=1}\mathcal{M}_k, \tilde{I}_r=\sum_{k=1}\tilde{I}_{r,k}$$

$$\mathcal{M}_k=\mathcal{O}(G^k), \tilde{I}_{r,k}=\mathcal{O}(G^k)$$

$$\mathcal{M}_5=\tilde{I}_{r,5}+\mathcal{M}_5^{\mathrm{it}}$$

$$\mathcal{M}_5^{\mathrm{it}}=\int_{\ell}\frac{\tilde{I}_{r,1}^5}{Z_1Z_2Z_3Z_4}+\int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,2}}{Z_1Z_2Z_3}+\int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,2}\tilde{I}_{r,2}}{Z_1Z_2}+\int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1Z_2}$$

$$+\int_{\ell}\frac{\tilde{I}_{r,1}\tilde{I}_{r,4}}{Z_1}+\int_{\ell}\frac{\tilde{I}_{r,2}\tilde{I}_{r,3}}{Z_1}$$

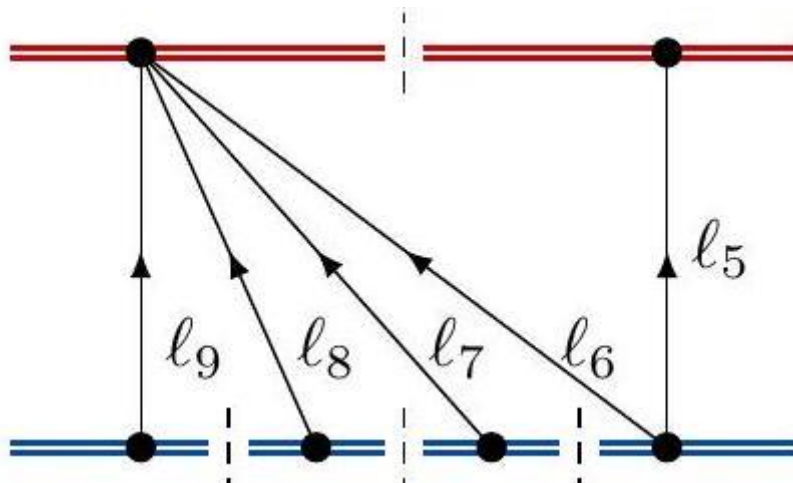
$$\tilde{I}_{r,1}\tilde{I}_{r,4}\!:=\!\tilde{I}_{r,1}(\boldsymbol{\ell}_1)\tilde{I}_{r,4}(\boldsymbol{\ell}_2)+\tilde{I}_{r,4}(\boldsymbol{\ell}_1)\tilde{I}_{r,1}(\boldsymbol{\ell}_2),$$

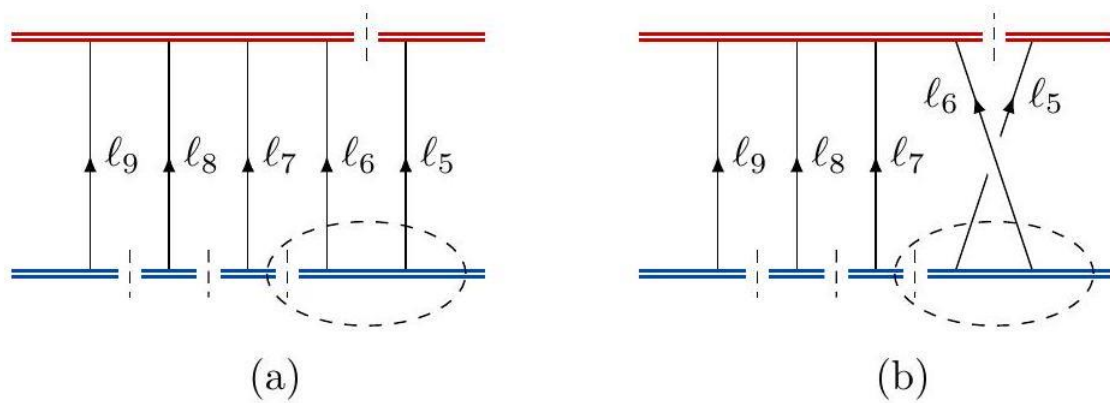
$$\chi=-\frac{\partial I_r(J)}{\partial J}$$

$$\chi=\sum_{k=1}\chi_k, \chi_k=\mathcal{O}(G^k)$$

$$\begin{aligned}\tilde{I}_{r,5} &= m_1^2 m_2^2 (m_1^4 + m_2^4) \tilde{I}_{r,5}^{\mathrm{0SF}} + m_1^3 m_2^3 (m_1^2 + m_2^2) \tilde{I}_{r,5}^{\mathrm{1SF}} + m_1^4 m_2^4 \tilde{I}_{r,5}^{\mathrm{2SF}} \\ &= M^8 v^2 [\tilde{I}_{r,5}^{\mathrm{0SF}} + v (\tilde{I}_{r,5}^{\mathrm{1SF}} - 4 \tilde{I}_{r,5}^{\mathrm{0SF}}) + v^2 (\tilde{I}_{r,5}^{\mathrm{2SF}} - 2 \tilde{I}_{r,5}^{\mathrm{1SF}} + 2 \tilde{I}_{r,5}^{\mathrm{0SF}})]\end{aligned}$$

$$M=m_1+m_2, \text{ and } v=\frac{m_1m_2}{M^2},$$





$$\delta(\ell_{56} \cdot u_2) \frac{1}{2\ell_6 \cdot u_2 + i\varepsilon} = -\delta(\ell_{56} \cdot u_2) \frac{1}{2\ell_5 \cdot u_2 - i\varepsilon},$$

$$\delta^{(n)}(\ell_{56} \cdot u_2) \frac{1}{2\ell_6 \cdot u_2 + i\varepsilon} \neq -\delta^{(n)}(\ell_{56} \cdot u_2) \frac{1}{2\ell_5 \cdot u_2 - i\varepsilon}, n \neq 0$$

$$\check{u}_2 \cdot \partial_{\ell_5}, \text{ with } \check{u}_2^\mu = \frac{yu_1^\mu - u_2^\mu}{y^2 - 1}$$

$$\check{u}_2 \cdot u_2 = 1, \check{u}_2 \cdot u_1 = 0, \check{u}_2 \cdot q = 0$$

$$\partial_x \vec{l} = A(x,\epsilon) \vec{l}, y = \frac{1+x^2}{2x},$$

$$\sqrt{y^2-1}=\frac{1-x^2}{2x}$$

$$\partial_x \vec{J} = \epsilon B(x) \vec{J}, \vec{J} = T(\epsilon, x) \vec{l}.$$

$$\Psi(x)\!:=\!\frac{K(1-x^2)}{\pi},$$

$$K(z)\!:=\!\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-zt^2}}.$$

$$B(x)=\sum_{i=1}^{10}b_ik_i(x)$$

$$\mathbb{W}=\left\{\frac{1}{x},\frac{1}{x+1},\frac{1}{x-1},\frac{x}{x^2+1},\frac{1}{x(1-x^2)\Psi(x)^2},\frac{\Psi(x)^2}{x},x\Psi(x)^2,\right.\\ \left.\frac{x\Psi(x)^2}{1-x^2},\frac{(1+x^2)\Psi(x)^2}{1-x^2},\frac{(1+x^2)(x^2-2x-1)(x^2+2x-1)\Psi(x)^4}{x(1-x^2)}\right\}$$

$$\Gamma_{i_1,\ldots i_n}(x)=\int_1^x\mathrm{d}x_nk_{i_n}(x_n)\cdots\int_1^{x_2}\mathrm{d}x_1k_{i_1}(x_1)$$

$$\begin{pmatrix} D_{1,1}(\epsilon, x) & 0 & 0 \\ D_{2,1}(\epsilon, x) & D_{2,2}(\epsilon, x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \epsilon D'_{1,1}(x) & 0 & 0 \\ D'_{2,1}(\epsilon, x) & \epsilon D'_{2,2}(x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{T'} \begin{pmatrix} \epsilon D''_{1,1}(x) & 0 & 0 \\ \epsilon D''_{2,1}(x) & \epsilon D''_{2,2}(x) & 0 \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{aligned} \text{Diagram} &= \int \prod_{i=1}^4 \frac{d^D \ell_i}{(2\pi)^D} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 (\ell_{12} - \ell_4)^2 \ell_4^2 (\ell_{123} + \hat{q})^2} \\ &\times \frac{1}{(\ell_4 + \hat{q})^2 (-2u_1 \cdot \ell_4) (2u_2 \cdot \ell_1) (2u_2 \cdot \ell_{12}) (2u_2 \cdot \ell_{123})} \end{aligned}$$

$$\text{Diagram} = c x \Psi^2 + \mathcal{O}(\epsilon)$$

$$u_1=(y,0,0,v), u_2=(1,0,0,0), q=(0,\boldsymbol{q}_x,\boldsymbol{q}_y,0), y=\frac{1+x^2}{2x}, v=\frac{1-x^2}{2x}.$$

$$\frac{1}{\ell^2}=\frac{1}{\omega^2-\ell^2}=-\frac{1}{\ell^2}\sum_{k=0}^{\infty}\frac{\omega^{2k}}{(\ell^2)^k}$$

$$\text{Diagram} = \text{Diagram} + \mathcal{O}(v^2).$$

$$\begin{aligned} \text{Diagram} &= \int \prod_{i=1}^4 \frac{d\omega_i}{2\pi} \frac{d^{D-1} \ell_i}{(2\pi)^{D-1}} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 (\ell_{12} - \ell_4)^2 \ell_4^2 (\ell_{123} + \hat{\boldsymbol{q}})^2} \\ &\times \frac{1}{(\ell_4 + \hat{\boldsymbol{q}})^2 (-2(\omega_4 - \hat{\boldsymbol{z}} \cdot \ell_4)) 2\omega_1 2\omega_{12} 2\omega_{123}}, \end{aligned}$$

$$\text{Diagram} = \text{Diagram} \times \left[\frac{1-10\epsilon}{2\epsilon} + \mathcal{O}(v^2) \right] + \text{Diagram} \times \left[\frac{(4\epsilon-1)(6\epsilon-1)}{8\epsilon^2} + \mathcal{O}(v^2) \right].$$

$$B(x)\vec{J}^{(0)} = \frac{1}{x-1} \left(b_3 - 2b_5 - \frac{1}{8}b_8 - \frac{1}{4}b_9 + \frac{1}{4}b_{10} \right) \vec{J}^{(0)} + \mathcal{O}((1-x)^0) \\ = \mathcal{O}((1-x)^0),$$

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \times I_1(x, \epsilon) + \begin{array}{c} \text{Diagram 3} \end{array} \times I_2(x, \epsilon)$$

$$\vec{J}^{(0)} = r_1(\epsilon)\vec{c}_1 \begin{array}{c} \text{Diagram 2} \end{array} + r_2(\epsilon)\vec{c}_2 \begin{array}{c} \text{Diagram 3} \end{array} + \dots$$

$$r_1(\epsilon) = \frac{1-10\epsilon}{2\epsilon}, r_2(\epsilon) = \frac{(4\epsilon-1)(6\epsilon-1)}{8\epsilon^2}$$

$$\mathcal{M}_5 = \left(\frac{|\boldsymbol{q}|^2}{\mu^2}\right)^{-4\epsilon} \left[\sum_{k=-2}^2 |\boldsymbol{q}|^k \sum_{n=1}^{N_k} f_{k,n}(\sigma, \epsilon) \mathcal{E}_{k,n}(\epsilon) + \mathcal{O}(|\boldsymbol{q}|^3) \right]$$

$$\begin{aligned}
\mathcal{E}_{-2,1} = & 1 \times \text{diag}_1 + 1 \times \text{diag}_2 + 1 \times \text{diag}_3 + 1 \times \text{diag}_4 + 1 \times \text{diag}_5 + \\
& 1 \times \text{diag}_6 + 2 \times \text{diag}_7 + 2 \times \text{diag}_8 + 2 \times \text{diag}_9 + 2 \times \text{diag}_{10} + \\
& 2 \times \text{diag}_{11} + 2 \times \text{diag}_{12} + 2 \times \text{diag}_{13} + 2 \times \text{diag}_{14} + 2 \times \text{diag}_{15} + \\
& 2 \times \text{diag}_{16} + 2 \times \text{diag}_{17} + 2 \times \text{diag}_{18} + 2 \times \text{diag}_{19} + 2 \times \text{diag}_{20} + \\
& 2 \times \text{diag}_{21} + 2 \times \text{diag}_{22} + 4 \times \text{diag}_{23} + 4 \times \text{diag}_{24} + 4 \times \text{diag}_{25} + \\
& 4 \times \text{diag}_{26} + 4 \times \text{diag}_{27} + 4 \times \text{diag}_{28} + 4 \times \text{diag}_{29} + 4 \times \text{diag}_{30} + \\
& 4 \times \text{diag}_{31} + 4 \times \text{diag}_{32} + 4 \times \text{diag}_{33} + 4 \times \text{diag}_{34} + 4 \times \text{diag}_{35} .
\end{aligned}$$

$$f_{-2,1} = \frac{(8\pi G 4m_1^2 m_2^2 c_\phi^2)^5}{(m_1 m_2 \sqrt{\sigma^2 - 1})^4} \propto \tilde{I}_{r,0}^5$$

$$\frac{5!}{120}\mathcal{E}_{-2,1}=\left(\frac{\mathbf{i}}{2}\right)^4\text{diag}_1=\frac{1}{16}\int\prod_{i=1}^4\frac{\mathrm{d}^{3-2\epsilon}\ell_i}{(2\pi)^{3-2\epsilon}}\frac{1}{\ell_1^2\ell_2^2\ell_3^2\ell_4^2(\ell_{1234}+\hat{q})^2}\frac{1}{(2\hat{\mathbf{z}}\cdot\ell_1)(2\hat{\mathbf{z}}\cdot\ell_{12})(2\hat{\mathbf{z}}\cdot\ell_{123})(2\hat{\mathbf{z}}\cdot\ell_{1234})}$$

$$\begin{aligned}
\mathcal{E}_{0,1} &= \text{diag}_1 + \dots, & \mathcal{E}_{0,2} &= \text{diag}_2 + \dots, \\
\mathcal{E}_{0,3} &= \text{diag}_3 + \dots, & \mathcal{E}_{0,4} &= \text{diag}_4 + \dots, \\
\mathcal{E}_{0,5} &= \text{diag}_5 + \dots.
\end{aligned}$$

$$\frac{4!}{60}\mathcal{E}_{0,1} = \frac{4!}{144}\mathcal{E}_{0,4} = \left(\frac{i}{2}\right)^4 \quad \text{Diagram: A triangle with a vertical line from the top vertex to the base, and two lines from the top vertex to the base forming a smaller triangle inside. The base is double-lined.$$

$$\frac{4!}{72}\mathcal{E}_{0,5} = \left(\frac{i}{2}\right)^4 \quad \text{Diagram: A triangle with a vertical line from the top vertex to the base, and two lines from the top vertex to the base forming a figure-eight shape inside. The base is double-lined.$$

$$\frac{4!}{90}\mathcal{E}_{0,2} = \frac{4!}{72}\mathcal{E}_{0,3} = \left(\frac{i}{2}\right)^4 \quad \text{Diagram: A triangle with a vertical line from the top vertex to the base, and two lines from the top vertex to the base forming a lens shape inside. The base is double-lined.$$

$$\text{Diagram: Triangle with internal vertical line and two lines to base} = \left(\frac{1}{16\pi^2\epsilon} + \mathcal{O}(\epsilon^0)\right) \int \prod_{i=1}^2 \frac{d^{3-2\epsilon}\ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2(\ell_2^2)^{2\epsilon}(\ell_{12} + \hat{q})^2} \frac{1}{(2\hat{z} \cdot \ell_1)(2\hat{z} \cdot \ell_{12})}$$

$$\text{Diagram: Triangle with internal figure-eight shape} = \left(\frac{1}{16} + \mathcal{O}(\epsilon)\right) \int \prod_{i=1}^2 \frac{d^{3-2\epsilon}\ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2(\ell_2^2)^{2\epsilon}(\ell_{12} + \hat{q})^2} \frac{1}{(2\hat{z} \cdot \ell_1)(2\hat{z} \cdot \ell_{12})}$$

$$\text{Diagram: Triangle with internal lens shape} = \left(\frac{1}{16} + \mathcal{O}(\epsilon)\right) \int \prod_{i=1}^2 \frac{d^{3-2\epsilon}\ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{(\ell_1^2)^{1/2+2\epsilon}\ell_2^2[(\ell_{12} + \hat{q})^2]^{1/2+2\epsilon}} \frac{1}{(2\hat{z} \cdot \ell_1)(2\hat{z} \cdot \ell_{12})}$$

$$f_{0,1} = -(8\pi G)^5 \frac{(4m_1 m_2 c_\phi^2)^5 (m_1^2 + m_2^2)}{4m_1 m_2 (\sigma^2 - 1)^3} \propto \tilde{I}_{r,0} \tilde{I}_{r,2}^{\text{0SF}}$$

$$f_{0,4} = -(8\pi G)^5 \frac{(m_1 m_2)^5 (4c_\phi^2)^4}{(\sigma^2 - 1)^3} \left[\frac{2}{3} c_\phi^2 \sigma - (\sigma^2 - 1)^{\frac{3}{2}} \text{arccosh}(\sigma) \right] \propto \tilde{I}_{r,0}^2 \tilde{I}_{r,2}^{\text{1SF}}$$

$$\tilde{I}_{r,2} = m_1^2 m_2^2 [(m_1^2 + m_2^2) \tilde{I}_{r,2}^{\text{0SF}} + m_1 m_2 \tilde{I}_{r,2}^{\text{1SF}}]$$

$$\mathcal{E}_{2,1} = 1 \times \quad \text{Diagram: A triangle with dashed lines from the top vertex to the base, and a solid line from the top vertex to the base. The base is double-lined, with a red line above it and a blue line below it.$$

$$\mathcal{E}_{2,2} = 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4$$

$$\begin{aligned} \mathcal{E}_{2,3} = & 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4 + 1 \times \text{diagram}_5 + \\ & 1 \times \text{diagram}_6 + 1 \times \text{diagram}_7 + 1 \times \text{diagram}_8 + 1 \times \text{diagram}_9 + \\ & 1 \times \text{diagram}_{10} + 1 \times \text{diagram}_{11} . \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{2,4} = & 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4 + 1 \times \text{diagram}_5 + \\ & 1 \times \text{diagram}_6 , \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{2,5} = & 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4 + 2 \times \text{diagram}_5 + \\ & 2 \times \text{diagram}_6 + 2 \times \text{diagram}_7 + 2 \times \text{diagram}_8 , \end{aligned}$$

$$\mathcal{E}_{2,6} = 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4$$

$$\mathcal{E}_{2,7} = \text{diagram}_1$$

$$5! \times \mathcal{E}_{1,2} = \frac{4! \times 2!}{4} \times \mathcal{E}_{2,2} = \frac{4! \times 2!}{12} \times \mathcal{E}_{2,3} = \left(\frac{i}{2}\right)^4 \text{ (diagram: a circle with three vertical lines) },$$

$$\frac{2! \times 4!}{6} \times \mathcal{E}_{2,4} = \frac{2! \times 4!}{12} \times \mathcal{E}_{2,5} = \left(\frac{i}{2}\right)^4 \text{ (diagram: a circle with two vertical lines) },$$

$$\frac{2! \times 4!}{4} \times \mathcal{E}_{2,6} = \left(\frac{i}{2}\right)^4 \text{ (diagram: a circle with one vertical line) },$$

$$2! \times 4! \times \mathcal{E}_{2,7} = \left(\frac{i}{2}\right)^4 \text{ (diagram: a circle with no vertical lines) }.$$

$$\text{(diagram: circle with 3 vertical lines)} = \int \prod_{i=1}^4 \frac{d^{3-2\epsilon} \ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2 (\ell_{1234} + \hat{q})^2} = \frac{1}{(16\pi^2)^4} \left[-\frac{16\pi^4}{3\epsilon} + \mathcal{O}(\epsilon^0) \right],$$

$$\text{(diagram: circle with 2 vertical lines)} = \int \prod_{i=1}^4 \frac{d^{3-2\epsilon} \ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2 (\ell_{123} + \hat{q})^2 \ell_{234}^2} = \frac{1}{(16\pi^2)^4} \left[\frac{8\pi^6}{\epsilon} + \mathcal{O}(\epsilon^0) \right],$$

$$\begin{aligned} \text{(diagram: circle with 1 vertical line)} &= \int \prod_{i=1}^4 \frac{d^{3-2\epsilon} \ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2 (\ell_{123} + \hat{q})^2 \ell_{1234}^2} \\ &= \frac{1}{(16\pi^2)^4} \left[\frac{8\pi^4}{\epsilon^2} + \frac{32\pi^4}{\epsilon} (4 - \gamma_E + \ln(4\pi)) + \mathcal{O}(\epsilon^0) \right], \end{aligned}$$

$$\text{(diagram: circle with 0 vertical lines)} = \int \prod_{i=1}^4 \frac{d^{3-2\epsilon} \ell_i}{(2\pi)^{3-2\epsilon}} \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2 (\ell_{123} + \hat{q})^2 (\ell_4 + \hat{q})^2} = \mathcal{O}(\epsilon^0),$$

$$\begin{aligned} \tilde{I}_{r,5}^{\text{OSF}} &= G^5 \pi c_\phi^4 |\mathbf{q}|^2 \left(\frac{|\mathbf{q}|^2}{\bar{\mu}^2} \right)^{-4\epsilon} \left[-\frac{4c_\phi^6}{5\epsilon(\sigma^2 - 1)^4} + \mathcal{O}(\epsilon^0) \right] \\ \tilde{I}_{r,5}^{\text{1SF}} &= G^5 \pi c_\phi^4 |\mathbf{q}|^2 \left(\frac{|\mathbf{q}|^2}{\bar{\mu}^2} \right)^{-4\epsilon} \left[\frac{I_{r,5}^{\text{1SF,div.}}}{\epsilon^2} + \frac{I_{r,5}^{\text{1SF,fin.}}}{\epsilon} + \mathcal{O}(\epsilon^0) \right] \end{aligned}$$

$$\begin{aligned}
\tilde{I}_{r,5}^{1\text{SF, div.}} &= 16c_\phi^2 \left\{ F_0 \left[\frac{\sigma^2 c_\phi^2}{(\sigma^2 - 1)^2} + \frac{4\sigma c_\phi}{\sigma^2 - 1} + 4 \right] + \frac{\sigma c_\phi^2}{(\sigma^2 - 1)^2} + \frac{2c_\phi}{\sigma^2 - 1} \right\} \\
\tilde{I}_{r,5}^{1\text{SF, fin.}} &= r_1 + r_2 F_0 + r_3 F_0^2 + r_4 F_1 + r_5 F_2 \\
F_0 &= \frac{2x}{1-x^2} \ln(x) \\
F_1 &= \frac{2x}{1-x^2} \left[-\text{Li}_2(1-x) - \text{Li}_2(-x) - \ln(x) \ln(x+1) - \frac{1}{2} \zeta_2 \right] \\
F_2 &= \frac{2x}{1-x^2} \left[-\text{Li}_2(1-x) + \text{Li}_2(-x) - \frac{1}{2} \ln^2(x) + \ln(x) \ln(x+1) + \frac{1}{2} \zeta_2 \right] \\
r_1 &= \frac{16c_\phi^3}{\sigma^2 - 1} \left[-\frac{\sigma(5\sigma^2 - 4)c_\phi^3}{5(\sigma^2 - 1)^3} - \frac{2c_\phi^2}{\sigma^2 - 1} + 8 \right] \\
r_2 &= 32c_\phi^2 \left[-\frac{\sigma^2 c_\phi^4}{(\sigma^2 - 1)^3} - \frac{4\sigma c_\phi^3}{(\sigma^2 - 1)^2} - \frac{9(2\sigma^2 - 1)c_\phi^2}{2(\sigma^2 - 1)^2} - \frac{8\sigma c_\phi}{\sigma^2 - 1} + 1 \right] \\
r_3 &= 16c_\phi \left[-\frac{\sigma^3 c_\phi^5}{(\sigma^2 - 1)^3} - \frac{6\sigma^2 c_\phi^4}{(\sigma^2 - 1)^2} + \frac{6\sigma(2 - 3\sigma^2)c_\phi^3}{(\sigma^2 - 1)^2} + \frac{(8 - 32\sigma^2)c_\phi^2}{\sigma^2 - 1} \right. \\
&\quad \left. - \frac{92\sigma c_\phi}{3} - \frac{40}{3}(\sigma^2 - 1) \right] \\
r_4 &= 32c_\phi \left[-\frac{\sigma c_\phi^3}{(\sigma^2 - 1)^2} - \frac{2c_\phi^2}{\sigma^2 - 1} - \frac{4\sigma c_\phi}{3(\sigma^2 - 1)} - \frac{8}{3} \right] \\
r_5 &= 64c_\phi \left[\frac{\sigma^2 c_\phi^3}{(\sigma^2 - 1)^2} + \frac{4\sigma c_\phi^2}{\sigma^2 - 1} + \frac{2(7\sigma^2 - 6)c_\phi}{3(\sigma^2 - 1)} + \frac{4\sigma}{3} \right] \\
16c_\phi^2 &\left[F_0 \left(\frac{\sigma^2 c_\phi^2}{(\sigma^2 - 1)^2} + \frac{4\sigma c_\phi}{\sigma^2 - 1} + 4 \right) + \frac{\sigma c_\phi^2}{(\sigma^2 - 1)^2} + \frac{2c_\phi}{\sigma^2 - 1} \right] \\
\chi_5 &= (m_1^4 + m_2^4) \chi_5^{\text{OSF}} + m_1 m_2 (m_1^2 + m_2^2) \chi_5^{1\text{SF}} + m_1^2 m_2^2 \chi_5^{2\text{SF}} \\
|q|^2 \left(\frac{|q|^2}{\bar{\mu}^2} \right)^{-4\epsilon} &\mapsto (\mu^2 \tilde{b}^2)^{5\epsilon} \frac{1}{b^5} \frac{\sqrt{2m_1 m_2 \sigma + m_1^2 + m_2^2}}{\pi m_1^2 m_2^2 (\sigma^2 - 1)} \frac{2(e^{-9\epsilon \gamma_E}) \Gamma(3 - 5\epsilon)}{\Gamma(4\epsilon - 1)}, \\
&\quad = (-16\epsilon + 184\epsilon^2 + \mathcal{O}(\epsilon^3)) \\
\chi_5^{\text{OSF}} &= \frac{G^5}{b^5} (\mu^2 \tilde{b}^2)^{5\epsilon} c_\phi^4 \frac{\sqrt{2m_1 m_2 \sigma + m_1^2 + m_2^2}}{(\sigma^2 - 1)} \left[\frac{64c_\phi^6}{5(\sigma^2 - 1)^4} \right] + \mathcal{O}(\epsilon) \\
\chi_5^{1\text{SF}} &= \frac{G^5}{b^5} (\mu^2 \tilde{b}^2)^{5\epsilon} c_\phi^4 \frac{\sqrt{2m_1 m_2 \sigma + m_1^2 + m_2^2}}{(\sigma^2 - 1)} \\
&\times \left[-\frac{16}{\epsilon} I_{r,5}^{1\text{SF, div.}} + (184 I_{r,5}^{1\text{SF, div.}} - 16 I_{r,5}^{1\text{SF, fin.}}) \right] + \mathcal{O}(\epsilon) \\
\chi^{\text{OSF}} &= 2 \arctan \left(\frac{G m_1 m_2}{J} \frac{2c_\phi^2}{\sqrt{\sigma^2 - 1}} \right) \\
\chi &= \pi - 2J \int_{r_{\min}}^{\infty} \frac{dr}{r^2 \sqrt{p_r^2(r)}}
\end{aligned}$$

$$p_r^2(r) = \mathbf{p}^2 - \frac{J^2}{r^2} + \sum_{k=1} \left(\frac{G}{r}\right)^k (\mu r)^{2k\epsilon} P_k(\sigma)$$

$$\chi_5 = \frac{16\chi_1\chi_4\left[1+\epsilon\left(\frac{55}{6}-16\ln 2\right)\right]}{3\pi}-\frac{9\chi_1^5}{80}+\frac{3\chi_2\chi_1^3}{\pi}-\frac{3}{2}\chi_3\chi_1^2-\frac{12\chi_2^2\chi_1}{\pi^2}+\frac{4\chi_2\chi_3}{\pi}+\mathcal{O}(v^{-2},\epsilon).$$

$$\begin{aligned} \mathcal{E}_{0,1} = & 1\times \text{diagram}_1 + 1\times \text{diagram}_2 + 1\times \text{diagram}_3 + 1\times \text{diagram}_4 + 2\times \text{diagram}_5 + \\ & 2\times \text{diagram}_6 + 2\times \text{diagram}_7 + 2\times \text{diagram}_8 + 2\times \text{diagram}_9 + 2\times \text{diagram}_{10} + \\ & 2\times \text{diagram}_{11} + 2\times \text{diagram}_{12} + 2\times \text{diagram}_{13} + 2\times \text{diagram}_{14} + 2\times \text{diagram}_{15} + \\ & 2\times \text{diagram}_{16} + 2\times \text{diagram}_{17} + 2\times \text{diagram}_{18} + 2\times \text{diagram}_{19} + 2\times \text{diagram}_{20} + \\ & 2\times \text{diagram}_{21} + 2\times \text{diagram}_{22} . \end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{0,2} = & 1 \times \text{diagram} + 1 \times \text{diagram} + 1 \times \text{diagram} + 1 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + 2 \times \text{diagram} + \\
& 2 \times \text{diagram} + 2 \times \text{diagram} .
\end{aligned}$$

$$\begin{aligned}
 \mathcal{E}_{0,3} = & 1 \times \text{diagram}_1 + 1 \times \text{diagram}_2 + 1 \times \text{diagram}_3 + 1 \times \text{diagram}_4 + 1 \times \text{diagram}_5 + \\
 & 1 \times \text{diagram}_6 + 1 \times \text{diagram}_7 + 1 \times \text{diagram}_8 + 1 \times \text{diagram}_9 + 1 \times \text{diagram}_{10} + \\
 & 1 \times \text{diagram}_{11} + 1 \times \text{diagram}_{12} + 2 \times \text{diagram}_{13} + 2 \times \text{diagram}_{14} + 2 \times \text{diagram}_{15} + \\
 & 2 \times \text{diagram}_{16} + 2 \times \text{diagram}_{17} + 2 \times \text{diagram}_{18} + 2 \times \text{diagram}_{19} + 2 \times \text{diagram}_{20} + \\
 & 2 \times \text{diagram}_{21} + 2 \times \text{diagram}_{22} + 2 \times \text{diagram}_{23} + 2 \times \text{diagram}_{24} + 2 \times \text{diagram}_{25} + \\
 & 2 \times \text{diagram}_{26} + 2 \times \text{diagram}_{27} + 2 \times \text{diagram}_{28} + 2 \times \text{diagram}_{29} + 2 \times \text{diagram}_{30} + \\
 & 2 \times \text{diagram}_{31} + 2 \times \text{diagram}_{32} .
 \end{aligned}$$



$$\mathcal{E}_{0,5} = 1 \times \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \\ \text{diagram 5} \\ \text{diagram 6} \end{array} + 1 \times \begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \\ \text{diagram 9} \\ \text{diagram 10} \\ \text{diagram 11} \\ \text{diagram 12} \end{array} + 1 \times \begin{array}{c} \text{diagram 13} \\ \text{diagram 14} \\ \text{diagram 15} \\ \text{diagram 16} \\ \text{diagram 17} \\ \text{diagram 18} \end{array} + 1 \times \begin{array}{c} \text{diagram 19} \\ \text{diagram 20} \\ \text{diagram 21} \\ \text{diagram 22} \\ \text{diagram 23} \\ \text{diagram 24} \end{array} + 2 \times \begin{array}{c} \text{diagram 25} \\ \text{diagram 26} \\ \text{diagram 27} \\ \text{diagram 28} \\ \text{diagram 29} \\ \text{diagram 30} \end{array} +$$

$$2 \times \begin{array}{c} \text{diagram 31} \\ \text{diagram 32} \\ \text{diagram 33} \\ \text{diagram 34} \\ \text{diagram 35} \\ \text{diagram 36} \end{array} + 2 \times \begin{array}{c} \text{diagram 37} \\ \text{diagram 38} \\ \text{diagram 39} \\ \text{diagram 40} \\ \text{diagram 41} \\ \text{diagram 42} \end{array} + 2 \times \begin{array}{c} \text{diagram 43} \\ \text{diagram 44} \\ \text{diagram 45} \\ \text{diagram 46} \\ \text{diagram 47} \\ \text{diagram 48} \end{array} + 2 \times \begin{array}{c} \text{diagram 49} \\ \text{diagram 50} \\ \text{diagram 51} \\ \text{diagram 52} \\ \text{diagram 53} \\ \text{diagram 54} \end{array} + 2 \times \begin{array}{c} \text{diagram 55} \\ \text{diagram 56} \\ \text{diagram 57} \\ \text{diagram 58} \\ \text{diagram 59} \\ \text{diagram 60} \end{array} +$$

$$2 \times \begin{array}{c} \text{diagram 61} \\ \text{diagram 62} \\ \text{diagram 63} \\ \text{diagram 64} \\ \text{diagram 65} \\ \text{diagram 66} \end{array} + 2 \times \begin{array}{c} \text{diagram 67} \\ \text{diagram 68} \\ \text{diagram 69} \\ \text{diagram 70} \\ \text{diagram 71} \\ \text{diagram 72} \end{array} + 2 \times \begin{array}{c} \text{diagram 73} \\ \text{diagram 74} \\ \text{diagram 75} \\ \text{diagram 76} \\ \text{diagram 77} \\ \text{diagram 78} \end{array} + 4 \times \begin{array}{c} \text{diagram 79} \\ \text{diagram 80} \\ \text{diagram 81} \\ \text{diagram 82} \\ \text{diagram 83} \\ \text{diagram 84} \end{array} +$$

$$4 \times \begin{array}{c} \text{diagram 85} \\ \text{diagram 86} \\ \text{diagram 87} \\ \text{diagram 88} \\ \text{diagram 89} \\ \text{diagram 90} \end{array} + 4 \times \begin{array}{c} \text{diagram 91} \\ \text{diagram 92} \\ \text{diagram 93} \\ \text{diagram 94} \\ \text{diagram 95} \\ \text{diagram 96} \end{array} + 4 \times \begin{array}{c} \text{diagram 97} \\ \text{diagram 98} \\ \text{diagram 99} \\ \text{diagram 100} \\ \text{diagram 101} \\ \text{diagram 102} \end{array} + 4 \times \begin{array}{c} \text{diagram 103} \\ \text{diagram 104} \\ \text{diagram 105} \\ \text{diagram 106} \\ \text{diagram 107} \\ \text{diagram 108} \end{array} +$$

$$4 \times \begin{array}{c} \text{diagram 109} \\ \text{diagram 110} \\ \text{diagram 111} \\ \text{diagram 112} \\ \text{diagram 113} \\ \text{diagram 114} \end{array} + 4 \times \begin{array}{c} \text{diagram 115} \\ \text{diagram 116} \\ \text{diagram 117} \\ \text{diagram 118} \\ \text{diagram 119} \\ \text{diagram 120} \end{array} +$$

$$4 \times \begin{array}{c} \text{diagram 121} \\ \text{diagram 122} \\ \text{diagram 123} \\ \text{diagram 124} \\ \text{diagram 125} \\ \text{diagram 126} \end{array} .$$

$$I^\sigma(\varphi, \epsilon; n_\pm) = \frac{i\pi}{8G_N^{(4)}} \frac{1}{\epsilon} (F(\varphi^I + \epsilon n^I) - \sigma F(\varphi^I - \epsilon n^I))$$

$$\mathfrak{n}^R = \xi_I \mathfrak{n}^I = \frac{n_+ + \sigma n_-}{n_+ n_-}, \mathfrak{n}_\alpha^m = \zeta_{\alpha, I} \mathfrak{n}^I = 0$$

$$S^\sigma(q,J,\varphi,\epsilon;n_\pm)=\frac{i\pi}{2G_N^{(4)}}(\varphi^Iq_I+\epsilon J)-I^\sigma(\varphi,\epsilon;n_\pm)+\lambda\left(\xi_I\varphi^I-\frac{n_+-\sigma n_-}{n_+n_-}\epsilon-2\right)+\mu^\alpha\zeta_{\alpha,I}\varphi^I$$

$$S_{\rm on-shell}^\sigma(q,J;n_\pm)=S^\sigma(q_I,J,\bar\varphi^I,\bar\epsilon;n_\pm),\;\partial_{\varphi^I,\epsilon}S^\sigma\big|_{\bar\varphi^I,\bar\epsilon}=\partial_{\lambda,\mu^\alpha}S^\sigma\big|_{\bar\varphi^I,\bar\epsilon}=0$$

$$F=-2i\sqrt{X^0X^1X^2X^3}$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R-\sum_{i=1}^3\partial_\mu u_i\partial^\mu u_i-\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{4\phi}D_\mu\sigma D^\mu\sigma-g^2\mathcal{V}\\ -\big[e^{2u_1+2u_2+2u_3}F_{\mu\nu}^0F^{0\mu\nu}+e^{2u_1-2u_2-2u_3}F_{\mu\nu}^1F^{1\mu\nu}\\ +e^{-2u_1+2u_2-2u_3}F_{\mu\nu}^2F^{2\mu\nu}+e^{-2u_1-2u_2+2u_3}F_{\mu\nu}^3F^{3\mu\nu}\big]$$

$$D\sigma = d\sigma + g(e_0A^0 - 2A^1 - 2A^2 - 2A^3)$$

$$\mathcal{V}=\sum_{i=1}^3\left(\frac{\partial W}{\partial u_i}\right)^2+\left(\frac{\partial W}{\partial \phi}\right)^2-3W^2$$

$$W=\frac{1}{4}e^{-u_1-u_2-u_3}\left(8-e_0e^{2\phi}+2e^{2\phi}(e^{2u_1+2u_2}+e^{2u_2+2u_3}+e^{2u_3+2u_1})\right)$$

$$\begin{aligned} A^{\rm R} &= 4\sqrt{2}A^0 \\ A^m &= \sqrt{2}(e_0A^0 - 2A^1 - 2A^2 - 2A^3) \\ A^{\mathcal{B}_1} &= 4\sqrt{2}(A^1 - 2A^2 + A^3) \\ A^{\mathcal{B}_2} &= 4\sqrt{2}(A^1 - A^3) \end{aligned}$$

$$\begin{aligned} [2\nabla_\mu - iB_\mu - gW\gamma_\mu + 4iH_{\mu\nu}\gamma^\nu]\epsilon &= 0 \\ [\partial_\mu u_i\gamma^\mu + g\partial_{u_i}W + i\partial_{u_i}H_{\mu\nu}\gamma^{\mu\nu}]\epsilon &= 0 \\ \left[\partial_\mu\phi\gamma^\mu + g\partial_\phi W + \frac{i}{2}\partial_\phi B_\mu\gamma^\mu\right]\epsilon &= 0 \end{aligned}$$

$$\begin{aligned} H_{\mu\nu} &= -\frac{1}{2\sqrt{2}}\big(e^{u_1+u_2+u_3}F_{\mu\nu}^{-0}+e^{u_1-u_2-u_3}F_{\mu\nu}^{-1}+e^{-u_1+u_2-u_3}F_{\mu\nu}^{-2}+e^{-u_1-u_2+u_3}F_{\mu\nu}^{-3}\big) \\ B_\mu &= -4\sqrt{2}gA_\mu^0+\frac{1}{2}e^{2\phi}D_\mu\sigma \end{aligned}$$

$$e^{2u_i}=\sqrt{\frac{e_0}{6}}, e^{-2\phi}=\frac{e_0}{6}, L_{AdS_4}=\frac{1}{2}\left(\frac{e_0}{6}\right)^{3/4}$$

$$L_{AdS_4}^2=-\frac{3}{g^2\mathcal{V}_*}$$

$$u_3=u_2, A^3=A^2$$

$$\mathcal{F}_{S^3}=\frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$

$$\mathcal{F}_{S^3}=N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{vol}(Y)}},$$

$$\begin{aligned}\mathcal{F}_{S^3}^{\mathrm{ABJM}} &= \frac{\sqrt{2}\pi}{3}N^{3/2}, & \mathrm{vol}(S^7) &= \frac{\pi^4}{3} \\ \mathcal{F}_{S^3}^{Q^{1,1,1}/\mathbb{Z}_n} &= \frac{4\pi}{3\sqrt{3}}n^{1/2}N^{3/2}, & \mathrm{vol}(Q^{1,1,1}/\mathbb{Z}_n) &= \frac{\pi^4}{8n} \\ \mathcal{F}_{S^3}^{M^{1,1,1}/\mathbb{Z}_n} &= \frac{16\pi}{9\sqrt{3}}n^{1/2}N^{3/2}, & \mathrm{vol}(M^{1,1,1}/\mathbb{Z}_n) &= \frac{9\pi^4}{128n}\end{aligned}$$

$$\begin{aligned} ds^2 &= e^{2V}ds_{AdS_2}^2 + f^2 dy^2 + h^2 dz^2 \\ A^I &= a^I dz \end{aligned}$$

$$B_\mu dx^\mu \equiv B_z dz$$

$$e^a=e^V\bar{e}^a,e^2=f dy,e^3=h dz$$

$$F_{23}^I=f^{-1}h^{-1}(a^I)'$$

$$\begin{aligned} &-\frac{1}{\sqrt{2}}e^{2V}(e^{2u_1-2u_2-2u_3}F_{23}^1-e^{-2u_1+2u_2-2u_3}F_{23}^2)=\mathcal{E}_{F_1}\\ &-\frac{1}{\sqrt{2}}e^{2V}(e^{-2u_1+2u_2-2u_3}F_{23}^2-e^{-2u_1-2u_2+2u_3}F_{23}^3)=\mathcal{E}_{F_2}\\ &-\frac{1}{\sqrt{2}}e^{2V}(e^{-2u_1-2u_2+2u_3}F_{23}^3-e^{2u_1-2u_2-2u_3}F_{23}^1)=\mathcal{E}_{F_3} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{2u_1-2u_2-2u_3}F_{23}^1\right)=\mathcal{E}_{R_1}, \\
& -\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{-2u_1+2u_2-2u_3}F_{23}^2\right)=\mathcal{E}_{R_2}, \\
& -\frac{1}{\sqrt{2}}e^{2V}\left(e^{2u_1+2u_2+2u_3}F_{23}^0+\frac{e_0}{2}e^{-2u_1-2u_2+2u_3}F_{23}^3\right)=\mathcal{E}_{R_3},
\end{aligned}$$

$$(e^{2V-2u_1-2u_2+2u_3}F_{23}^3)'=\sqrt{2}ge^{2V}fh^{-1}e^{4\phi}D_z\sigma,$$

$$\gamma^m=\Gamma^m\otimes\sigma^3,\gamma^2=I_2\otimes\sigma^1,\gamma^3=I_2\otimes\sigma^2,$$

$$\epsilon=\psi\otimes\chi$$

$$D_m\psi=\frac{1}{2}\kappa\Gamma_m\psi$$

$$\begin{aligned}
f^{-1}\xi' &= 2gW\cos\xi + \kappa e^{-V} \\
f^{-1}V' &= gW\sin\xi \\
f^{-1}u_i' &= -g\partial_{u_i}W\sin\xi \\
f^{-1}\phi' &= -\frac{g\partial_\phi W}{\sin\xi} \\
f^{-1}\frac{h'}{h} &= \frac{1}{\sin\xi}(\kappa e^{-V}\cos\xi + gW(1+\cos^2\xi))
\end{aligned}$$

$$\begin{aligned}
(s-B_z)\sin\xi &= -2gWh\cos\xi - \kappa h e^{-V} \\
g\partial_\phi W\cos\xi &= \frac{1}{2}\partial_\phi B_z\sin\xi h^{-1}
\end{aligned}$$

$$\begin{aligned}
\partial_{u_i}H_{23} &= -\frac{1}{4}g\partial_{u_i}W\cos\xi \\
H_{23} &= -\frac{1}{4}gW\cos\xi - \frac{1}{4}\kappa e^{-V}
\end{aligned}$$

$$he^{-V}=k\sin\xi$$

$$\xi'=-k^{-1}(s-B_z)(e^{-V}f),$$

$$\begin{aligned}
(s-B_z) &= -k[2ge^V\cos\xi + \kappa] \\
g\partial_\phi W\cos\xi &= \frac{1}{2}k^{-1}e^{-V}\partial_\phi B_z
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{R_1} &= e^V[4ge^V\cos\xi + \kappa e^{u_1}\cosh(u_2+u_3)] - \frac{2-e_0}{4}\kappa e^Ve^{2u_1}e^{-u_1-u_2-u_3}, \\
\mathcal{E}_{R_2} &= e^V[4ge^V\cos\xi + \kappa e^{u_2}\cosh(u_3+u_1)] - \frac{2-e_0}{4}\kappa e^Ve^{2u_2}e^{-u_1-u_2-u_3}, \\
\mathcal{E}_{R_3} &= e^V[4ge^V\cos\xi + \kappa e^{u_3}\cosh(u_1+u_2)] - \frac{2-e_0}{4}\kappa e^Ve^{2u_3}e^{-u_1-u_2-u_3},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{F_1} &= \kappa e^Ve^{u_3}\sinh(u_1-u_2) \\
\mathcal{E}_{F_2} &= \kappa e^Ve^{u_1}\sinh(u_2-u_3) \\
\mathcal{E}_{F_3} &= \kappa e^Ve^{u_2}\sinh(u_3-u_1)
\end{aligned}$$

$$f=-e^V,$$

$$ds^2=e^{2V}\big[ds_{AdS_2}^2+ds_{\Sigma}^2\big],$$

$$ds_{\Sigma}^2=dy^2+k^2\sin^2\xi dz^2.$$

$$\Delta z=2\pi$$

$$h\leq 0, \Leftrightarrow k\sin \xi \leq 0$$

$$(k\sin \xi)'|_{N,S}=-\frac{(-1)^{l_{N,S}}}{n_{N,S}}, l_N=0, l_S=1$$

$$\cos \xi|_{N,S}=(-1)^{t_{N,S}}; \qquad (t_N,t_S)=(1,1) \text{ or } (0,0) \\ (t_N,t_S)=(1,0) \text{ or } (0,1)$$

$$(s-B_z)|_{N,S}=\frac{1}{n_{N,S}}(-1)^{l_{N,S}+t_{N,S}+1}$$

$$-gF^R=dB+d\left(\frac{1}{2}e^{2\phi}D\sigma\right)$$

$$\frac{1}{2\pi}\int_{\Sigma}gF^R\equiv\frac{1}{2\pi}\int_{\Sigma}(-dB)=\frac{n_N(-1)^{t_S}+n_S(-1)^{t_N}}{n_Nn_S}$$

$$\partial_{\phi}B_z|_{N,S}=\partial_{\phi}W|_{N,S}=0$$

$$\phi|_N,\phi|_S\neq 0,\Rightarrow D_z\sigma|_N=D_z\sigma|_S=0$$

$$\frac{1}{2\pi}\int_{\Sigma}\sqrt{2}g(e_0F^0-2F^1-2F^2-2F^3)=(D_z\sigma)|_{y_N}^{y_S}=0$$

$$\left(e^{-2u_1}+e^{-2u_2}+e^{-2u_3}-\frac{e_0}{2}e^{-2u_1-2u_2-2u_3}\right)\Big|_{N,S}=0,\Rightarrow W|_{N,S}=2e^{-u_1-u_2-u_3}|_{N,S}$$

$$M_{(1)}\equiv -2ge^Ve^{-u_1-u_2-u_3}, M_{(2)}\equiv -\frac{\kappa}{4}M_{(1)}+M_{(1)}^2\cos\xi$$

$$\mathcal{E}_{Ri}=-\frac{e_0}{2}\frac{\kappa}{4g}M_{(1)}e^{2u_i}+\frac{1}{g}M_{(2)}e^{2u_1+2u_2+2u_3}$$

$$M_{(1)}=\frac{1}{2}(-1)^{t_{N,S}}\kappa-\frac{1}{2kn_{N,S}}(-1)^{l_{N,S}},\\ M_{(2)}=\frac{1}{8}(-1)^{t_{N,S}}+\frac{1}{4k^2n_{N,S}^2}(-1)^{t_{N,S}}-\frac{3\kappa}{8kn_{N,S}}(-1)^{l_{N,S}}.$$

$$\mathcal{E}_{R_1}(u_{1N},u_{2N})=\mathcal{E}_{R_1}(u_{1S},u_{2S}),\\ \mathcal{E}_{R_2}(u_{1N},u_{2N})=\mathcal{E}_{R_2}(u_{1S},u_{2S}),\\ \mathcal{E}_{R_3}(u_{1N},u_{2N})=\mathcal{E}_{R_3}(u_{1S},u_{2S}).$$

$$F^I_{yz}=(a^I)'=(\mathcal{I}^{(I)})'$$



$$\begin{aligned} \mathcal{J}^{(0)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{-u_1-u_2-u_3} \\ \mathcal{J}^{(1)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{-u_1+u_2+u_3} \\ \mathcal{J}^{(2)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{u_1-u_2+u_3} \\ \mathcal{J}^{(3)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{u_1+u_2-u_3} \end{aligned}$$

$$\frac{p_I}{n_N n_S} \equiv \frac{1}{2\pi} \int_{\Sigma} g F^I = g \mathcal{J}_I|_N^S$$

$$\begin{aligned} \mathcal{J}_R|_{N,S} &\equiv 4\sqrt{2}\mathcal{J}^{(0)}|_{N,S} \\ \mathcal{J}_m|_{N,S} &\equiv \sqrt{2}(e_0\mathcal{J}^{(0)} - 2\mathcal{J}^{(1)} - 2\mathcal{J}^{(2)} - 2\mathcal{J}^{(3)})|_{N,S} \\ \mathcal{J}_{B_1}|_{N,S} &\equiv 4\sqrt{2}(\mathcal{J}^{(1)} - 2\mathcal{J}^{(2)} + \mathcal{J}^{(3)})|_{N,S} \\ \mathcal{J}_{B_2}|_{N,S} &\equiv 4\sqrt{2}(\mathcal{J}^{(1)} - 2\mathcal{J}^{(3)})|_{N,S} \end{aligned}$$

$$\begin{aligned} g\mathcal{J}_R|_{N,S} &= 4\sqrt{2}g\mathcal{J}^{(0)}|_{N,S} \\ &= 4gk e^V \cos \xi e^{-u_1-u_2-u_3}|_{N,S} \\ &= -2kM_{(1)}(-1)^{t_{N,S}+1}|_{N,S} \end{aligned}$$

$$\begin{aligned} g\mathcal{J}_m|_{N,S} &= \sqrt{2}g(e_0\mathcal{J}^{(0)} - 2\mathcal{J}^{(1)} - 2\mathcal{J}^{(2)} - 2\mathcal{J}^{(3)})|_{N,S} \\ &= gk e^V \cos \xi (e_0 e^{-u_1-u_2-u_3} - 2e^{-u_1+u_2+u_3} - 2e^{u_1-u_2+u_3} - 2e^{u_1+u_2-u_3})|_{N,S} \\ &= gk e^V \cos \xi e^{u_1+u_2+u_3} (e_0 e^{-2u_1-2u_2-2u_3} - 2e^{-2u_1} - 2e^{-2u_2} - 2e^{-2u_3})|_{N,S} = 0 \end{aligned}$$

$$\begin{aligned} g\mathcal{J}_R|_N^S &= \frac{n_N(-1)^{t_S} + n_S(-1)^{t_N}}{n_N n_S} \\ g\mathcal{J}_m|_N^S &= 0 \end{aligned}$$

$$\begin{aligned} \frac{p_{B_1}}{n_N n_S} &\equiv g\mathcal{J}_{B_1}|_N^S = 4\sqrt{2}g(\mathcal{J}^{(1)} - 2\mathcal{J}^{(2)} + \mathcal{J}^{(3)})|_N^S \\ &= 4gk e^V \cos \xi (e^{-u_1+u_2+u_3} - 2e^{u_1-u_2+u_3} + e^{u_1+u_2-u_3})|_N^S \\ &= 2kM_{(1)}(-1)^{t_{N,S}}(e^{2u_2+2u_3} - 2e^{2u_1+2u_3} + e^{2u_1+2u_2})|_N^S, \end{aligned}$$

$$\begin{aligned} \frac{p_{B_2}}{n_N n_S} &\equiv g\mathcal{J}_{B_2}|_N^S = 4\sqrt{2}g(\mathcal{J}^{(1)} - \mathcal{J}^{(3)})|_N^S \\ &= 4gk e^V \cos \xi (e^{-u_1+u_2+u_3} - e^{u_1+u_2-u_3})|_N^S \\ &= 2kM_{(1)}(-1)^{t_{N,S}}(e^{2u_2+2u_3} - e^{2u_1+2u_2})|_N^S, \end{aligned}$$

$$\begin{aligned} n_N &= 4, & n_S &= 1, & p_{B_1} &= 1, & p_{B_2} &= 2, \\ g &= 1, & \kappa &= +1, & e_0 &= 2, \end{aligned}$$



$$\begin{aligned}e^{2u_{1N}} &\approx 0.174581, & e^{2u_{1S}} &\approx 0.0768503, \\e^{2u_{2N}} &\approx 0.418164, & e^{2u_{2S}} &\approx 0.411234, \\e^{2u_{3N}} &\approx 1.5639, & e^{2u_{3S}} &\approx 1.98408, \\k &\approx -0.683914.\end{aligned}$$

$$\frac{1}{2G_N^{(4)}}=\left(\frac{6}{e_0}\right)^{3/2}\frac{16}{3\sqrt{3}}N^{3/2}$$

$$\left(G_N^{(2)}\right)^{-1}=\left(G_N^{(4)}\right)^{-1}\Delta z\int_{y_N}^{y_S}|fh|dy$$

$$fh=ke^Vf\sin\,\xi=+\frac{k}{\kappa}(e^{2V}\cos\,\xi)'$$

$$\begin{aligned}S_{\rm BH} &= \frac{1}{4G_N^{(2)}} = \left(\frac{6}{e_0}\right)^{3/2} \frac{16\pi}{3\sqrt{3}} N^{3/2} g^2 \left(-\frac{k}{\kappa}\right) [e^{2V} \cos \xi]_N^S \\ &= -\left(\frac{6}{e_0}\right)^{3/2} \frac{16N^{3/2}k}{3\sqrt{3}\kappa} \left(\frac{1}{4}M_{(1)}^2\right)\Big|_S e^{2u_{1S}+2u_{2S}+2u_{3S}}(-1)^{t_S} - \frac{1}{4}M_{(1)}^2\Big|_N e^{2u_{1N}+2u_{2N}+2u_{3N}}(-1)^{t_N}\end{aligned}$$

$$e^{2u_i}=\sqrt{\frac{e_0}{6}},e^{-2\phi}=\frac{e_0}{6},L_{AdS_4}=\frac{1}{2}\Big(\frac{e_0}{6}\Big)^{3/4}$$

$$ds^2\!=\!\frac{1}{2g^2}\!\left[\!\frac{y^2}{4}ds_{AdS_2}^2\!+\!\frac{y^2}{q(y)}dy^2\!+\!\frac{q(y)}{4y^2}c_0^2dz^2\!\right]$$

$$A^0=A^1=A^2=A^3=-\left[\frac{c_0\kappa}{2g}\left(1-\frac{a}{y}\right)+\frac{S}{g}\right]dz$$

$$\sin\,\xi=-\frac{\sqrt{q(y)}}{y^2},\cos\,\xi=\kappa\frac{2y-a}{y^2}.$$

$$q(y)=y^4-4y^2+4ay-a^2,$$

$$a\!=\!\frac{n_S^2-n_N^2}{n_S^2+n_N^2}$$

$$c_0=\frac{\sqrt{n_S^2+n_N^2}}{\sqrt{2}n_Sn_N}$$

$$y_N=-1+\sqrt{1+a},y_S=1-\sqrt{1-a}.$$

$$\begin{aligned}S_{\rm BH} &= \frac{\sqrt{2}\sqrt{n_S^2+n_N^2-(n_S+n_N)}\pi L_{AdS_4}^2}{n_Sn_N4G_N^{(4)}} \\ &= \frac{\sqrt{2}\sqrt{n_S^2+n_N^2-(n_S+n_N)}}{n_Sn_N}\frac{1}{2}\mathcal{F}_{S^3}^{Q^{1,1,1}},\end{aligned}$$

$$\begin{aligned}n_N &= 4, & n_S &= 1, & p_{B_1} &= 1 \\g &= 1, & \kappa &= +1, & e_0 &= 2\end{aligned}$$

$$\begin{aligned}e^{2u_{1N}} &\approx 0.49043, & e^{2u_{1S}} &\approx 0.436592 \\e^{2u_{2N}} &\approx 0.774299, & e^{2u_{2S}} &\approx 0.926937, \\k &\approx -0.725009.\end{aligned}$$

$$\mathcal{B}(X^I,\epsilon)=\frac{i\pi}{8G_N^{(4)}}\frac{F(X^I)}{\epsilon},$$

$$\mathcal{F}_{\partial M_4}(\varphi,\epsilon)=\sum_{s=1}^{\chi_E(M_4)}\sigma_{(s)}\mathcal{B}(X^I_{(s)},\epsilon_{(s)}),$$

$$\mathcal{F}_{S^3}(\varphi)=\mathcal{B}(X^I=2\varphi^I,\epsilon=1)=\frac{i\pi}{2G_N^{(4)}}F(\varphi^I)$$

$$S^\sigma(\varphi,\epsilon;\mathfrak{n}_+,\mathfrak{n}_-)= -\frac{i\pi}{8G_N^{(4)}\epsilon}\big(F(\varphi^I+\epsilon\mathfrak{n}^I)-\sigma F(\varphi^I-\epsilon\mathfrak{n}^I)\big)$$

$$\mathfrak{n}^R=\frac{1}{2\pi}\int_{\Sigma}dA^R=\frac{n_++\sigma n_-}{n_+n_-}$$

$$\varphi^R-\frac{n_+-\sigma n_-}{n_+n_-}\epsilon=2.$$

$$F=-2i\sqrt{X^0X^1X^2X^3},$$

$$A^R=A^0+A^1+A^2+A^3.$$

$$\mathcal{F}_{S^3}^{\text{ABJM}}(\varphi)=\frac{\pi}{G_N^{(4)}}\sqrt{\varphi^0\varphi^1\varphi^2\varphi^3}$$

$$\mathcal{F}_{S^3}^{\text{ABJM}}:=\mathcal{F}_{S^3}^{\text{ABJM}}(\bar{\varphi})=\frac{\pi}{4G_N^{(4)}}=\frac{\pi L_{AdS_4}^2}{2G_N^{(4)}},$$

$$\mathfrak{n}=\frac{n_++\sigma n_-}{4n_+n_-},$$

$$\varphi-\frac{n_+-\sigma n_-}{4n_+n_-}\epsilon=\frac{1}{2}$$

$$S^-(\epsilon;n_+,n_-)=-\frac{\pi}{4G_N^{(4)}\epsilon}((\varphi+\epsilon\mathfrak{n})^2+(\varphi-\epsilon\mathfrak{n})^2)=-\frac{\pi}{2G_N^{(4)}\epsilon}(\varphi^2+\epsilon^2\mathfrak{n}^2)$$



$$\bar{\epsilon} = \pm \frac{\sqrt{2}n_+n_-}{\sqrt{n_+^2 + n_-^2}},$$

$$S^-(\bar{\epsilon};n_+,n_-)=\frac{\sqrt{2}\sqrt{n_+^2+n_-^2}-(n_++n_-)}{2n_+n_-}\mathcal{F}_{S^3}^{\text{ABJM}}$$

$$F=-2i\sqrt{X^0X^1X^2X^3}$$

$$A^R=A^0+A^1+A^2+A^3, A^m=A^0-A^1-A^2-A^3$$

$$F=-2i\sqrt{X^0X^1X^2X^3}\Rightarrow F^{\text{eff}}=-2i\sqrt{(X^1+X^2+X^3)X^1X^2X^3}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}}(\varphi)=\frac{\pi}{G_N^{(4)}}\sqrt{(\varphi^1+\varphi^2+\varphi^3)\varphi^1\varphi^2\varphi^3}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}}:=\mathcal{F}_{S^3}^{\text{mABJM}}(\bar{\varphi})=\frac{\pi}{3\sqrt{3}G_N^{(4)}}=\frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}}=\frac{4}{3\sqrt{3}}\mathcal{F}_{S^3}^{\text{ABJM}}$$

$$\mathfrak{n}^0=\sum_{i=1}^3\mathfrak{n}^i=\frac{n_++\sigma n_-}{2n_+n_-},$$

$$\sum_{i=1}^3\varphi^i-\frac{n_+-\sigma n_-}{2n_+n_-}\epsilon=1$$

$$S^-(\varphi,\epsilon;\mathfrak{n},n_+,n_-)=-\frac{i\pi}{8G_N^{(4)}\epsilon}\Big(F^{\text{eff}}(\varphi^i+\epsilon\mathfrak{n}^I)+F^{\text{eff}}(\varphi^i-\epsilon\mathfrak{n}^I)\Big)$$

$$n_N=n_+,n_S=n_-,\frac{p_{F_1}}{n_+n_-}=-\frac{1}{2}(\mathfrak{n}^1-\mathfrak{n}^2),\frac{p_{F_2}}{n_+n_-}=-\frac{1}{\sqrt{3}}(\mathfrak{n}^1+\mathfrak{n}^2-2\mathfrak{n}^3)$$

$$\frac{\bar{\varphi}^1+\mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_+}=e^{-\lambda_2-\lambda_3}\Big|_N,$$

$$\frac{\bar{\varphi}^1-\mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_-}=e^{-\lambda_2-\lambda_3}\Big|_{S'},$$

$$F=-2i\sqrt{X^0X^1X^2X^3}$$

$$A^R=4\sqrt{2}A^0,A^m=\sqrt{2}(e_0A^0-2A^1-2A^2-2A^3),$$

$$\xi_0=4\sqrt{2},\xi_1=\xi_2=\xi_3=0$$

$$\zeta_0=\sqrt{2}e_0,\zeta_1=\zeta_2=\zeta_3=-2\sqrt{2}$$

$$L_{AdS_4}^2=\sqrt{e_0^3/4\sqrt{6^3}}$$



$$F = -2i\sqrt{X^0X^1X^2X^3} \Rightarrow F^{\text{eff}} = -2i\sqrt{\frac{2}{e_0}(X^1 + X^2 + X^3)X^1X^2X^3}$$

$$\mathcal{F}_{S^3}^{Q^{1,1,1}}(\varphi) = \frac{\pi}{G_N^{(4)}}\sqrt{\frac{2}{e_0}(\varphi^1 + \varphi^2 + \varphi^3)\varphi^1\varphi^2\varphi^3}$$

$$\varphi^0 = 2/e_0 \sum_{i=1}^3 \varphi^i = 1/(2\sqrt{2})$$

$$\bar{\varphi}^i = e_0/(12\sqrt{2})$$

$$\mathcal{F}_{S^3}^{Q^{1,1,1}} := \mathcal{F}_{S^3}^{Q^{1,1,1}}(\bar{\varphi}) = \frac{\pi\sqrt{e_0^3}}{8\sqrt{6^3}G_N^{(4)}} = \frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$

$$\mathfrak{n}^0 = \frac{1}{4\sqrt{2}}\frac{n_+ + \sigma n_-}{n_+ n_-}, \sum_{i=1}^3 \mathfrak{n}^i = \frac{e_0}{2}\frac{1}{4\sqrt{2}}\frac{n_+ + \sigma n_-}{n_+ n_-}.$$

$$\varphi^0 = \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\frac{n_+ - \sigma n_-}{n_+ n_-}\epsilon, \sum_{i=1}^3 \varphi^i = \frac{e_0}{2}\Big(\frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\frac{n_+ - \sigma n_-}{n_+ n_-}\epsilon\Big)$$

$$S^-(\varphi,\epsilon;\mathfrak{n},n_+,n_-)=-\frac{i\pi}{8G_N^{(4)}\epsilon}\Big(F^{\text{eff}}(\varphi^i+\epsilon\mathfrak{n}^i)+F^{\text{eff}}(\varphi^i-\epsilon\mathfrak{n}^i)\Big)$$

$$n_N=n_-,n_S=n_+, \frac{p_{B_1}}{n_+n_-}=4\sqrt{2}(\mathfrak{n}^1-2\mathfrak{n}^2+\mathfrak{n}^3), \frac{p_{B_2}}{n_+n_-}=4\sqrt{2}(\mathfrak{n}^1-\mathfrak{n}^3),$$

$$\frac{4\sqrt{2}}{e_0}\frac{\bar{\varphi}^1+\mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_-}=e^{u_2+u_3}|_N$$

$$\frac{4\sqrt{2}}{e_0}\frac{\bar{\varphi}^1-\mathfrak{n}^1\bar{\epsilon}}{1+\bar{\epsilon}/n_+}=e^{u_2+u_3}|_S$$

$$X^3=X^2,\mathfrak{n}^3=\mathfrak{n}^2,\varphi^3=\varphi^2$$

$$\mathcal{M}_v\times\mathcal{M}_h=\left(\frac{SU(1,1)}{U(1)}\right)^3\times\frac{SU(2,1)}{S(U(2)\times U(1))}$$

$$h=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}e^{4\phi} & -\frac{1}{8}e^{4\phi}\tilde{\xi}_0 & \frac{1}{8}e^{4\phi}\xi^0 \\ 0 & -\frac{1}{8}e^{4\phi}\tilde{\xi}_0 & \frac{1}{4}e^{2\phi}\left(1+\frac{1}{4}e^{2\phi}(\tilde{\xi}_0)^2\right) & -\frac{1}{16}e^{4\phi}\xi^0\tilde{\xi}_0 \\ 0 & \frac{1}{8}e^{4\phi}\xi^0 & -\frac{1}{16}e^{4\phi}\xi^0\tilde{\xi}_0 & \frac{1}{4}e^{2\phi}\left(1+\frac{1}{4}e^{2\phi}(\xi^0)^2\right) \end{pmatrix}.$$

$$k^{\mathbb{R}}=\partial_\sigma,k^{U(1)}=-\tilde{\xi}_0\partial_{\xi^0}+\xi^0\partial_{\tilde{\xi}_0}$$



$$P^{\mathbb{R}} = \left(0, 0, -\frac{1}{2}e^{2\phi}\right), P^{U(1)} = \left(\xi_0 e^\phi, -\xi^0 e^\phi, 1 - \frac{1}{4}\left((\xi^0)^2 + (\xi_0)^2\right)e^{2\phi}\right)$$

$$P^{\mathbb{R}} = \left(0, 0, -\frac{1}{2}e^{2\phi}\right), P^{U(1)}(\xi^0 = \tilde{\xi}_0 = 0) = (0, 0, 1)$$

$$\omega^x = e^\phi d\xi^0 = 0, \omega^y = e^\phi d\tilde{\xi}_0 = 0, \omega^z = -\frac{1}{2}e^{2\phi} (d\sigma + \xi^0 d\tilde{\xi}_0) = -\frac{1}{2}e^{2\phi} d\sigma$$

$$\Omega^x := d\omega^x - \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z$$

$$\Omega^z_{\sigma\phi} = -\Omega^z_{\phi\sigma} = \frac{1}{2}e^{2\phi}$$

$$X^0=1, X^1=z^2z^3, X^2=z^3z^1, X^3=z^1z^2$$

$$e^{-K}=8\mathrm{Re}[z^1]\mathrm{Re}[z^2]\mathrm{Re}[z^3]=8z^1z^2z^3$$

$$K_i=\partial_{z^i}K=-(2z^i)^{-1}, g_{i\bar{j}}=K_{i\bar{j}}=\delta_{i\bar{j}}(2z^i)^{-2}$$

$$\mathcal{N}=-i\mathrm{diag}\left(z^1z^2z^3,\frac{z^1}{z^2z^3},\frac{z^2}{z^3z^1},\frac{z^3}{z^1z^2}\right)$$

$$k_I = (\sqrt{2}(4k^{U(1)} + e_0 k^{\mathbb{R}}), -2\sqrt{2}k^{\mathbb{R}}, -2\sqrt{2}k^{\mathbb{R}}, -2\sqrt{2}k^{\mathbb{R}})$$

$$P_I^Z = \left(\sqrt{2}\left(4-\frac{1}{2}e^{2\phi}e_0\right), e^{2\phi}, e^{2\phi}, e^{2\phi}\right)$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R+\sum_i\frac{(\partial z^i)^2}{4(z^i)^2}+(\partial\phi)^2+\frac{1}{4}e^{4\phi}(D\sigma)^2+\mathrm{Im}\mathcal{N}_{IJ}F_{\mu\nu}^IF^{I\mu\nu}-g^2\mathcal{V}$$

$$D_\mu\sigma=\partial_\mu\sigma+gA_\mu^m$$

$$A^m=\sqrt{2}(e_0A^0-2A^1-2A^2-2A^3),$$

$$A^R=4\sqrt{2}A^0$$

$$\begin{aligned}\delta\psi_{\mu A}=&\nabla_\mu\varepsilon_A+\frac{i}{2}\Big(4\sqrt{2}gA_\mu^0-\frac{1}{2}e^{2\phi}\nabla_\mu\sigma\Big)\sigma_A^3{}^B\varepsilon_B-\frac{g}{2}e^{K/2}P_I^3X^I\gamma_\mu\sigma^3{}_{AB}\varepsilon^B\\&+2ie^{K/2}X^I\mathrm{Im}\mathcal{N}_{IJ}F_{\mu\nu}^{-J}\gamma^\nu\varepsilon_{AB}\varepsilon^B\\ \delta\lambda^{iA}=&i\partial_\mu z^i\gamma^\mu\varepsilon^A+ig e^{K/2}g^{ij}(\partial_j+K_j)X^IP_I^3\sigma^{3AB}\varepsilon_B\\&-e^{K/2}g^{ij}(\partial_j+K_j)X^I\mathrm{Im}\mathcal{N}_{IJ}F_{\mu\nu}^{-J}\gamma^{\mu\nu}\varepsilon^{AB}\varepsilon_B\\ \delta\zeta_\alpha=&U_{u\alpha A}(i\nabla_\mu q^u\gamma^\mu\varepsilon^A+2ge^{K/2}k_I^uX^I\varepsilon^{AB}\varepsilon_B)\end{aligned}$$

$$W\equiv e^{K/2}P_I^3X^I=\frac{1}{4\sqrt{z^1z^2z^3}}\Big(8-e_0e^{2\phi}+2e^{2\phi}(z^1z^2+z^3z^1+z^2z^3)\Big)$$

$$H_{\mu\nu}\equiv e^{K/2}X^I\mathrm{Im}\mathcal{N}_{IJ}F_{\mu\nu}^{-J}=-\frac{1}{2\sqrt{2z^1z^2z^3}}\Big(z^1z^2z^3F_{\mu\nu}^{-0}+z^1F_{\mu\nu}^{-1}+z^2F_{\mu\nu}^{-2}+z^3F_{\mu\nu}^{-3}\Big)$$



$$z^i = ib^i + v^i$$

$$v^i = e^{2u_i}.$$

$$\begin{aligned} e^{-1}\mathcal{L} = & \frac{1}{2}R - \sum_{i=1}^3 \partial_\mu u_i \partial^\mu u_i - \partial_\mu \phi \partial^\mu \phi - \frac{1}{4}e^{4\phi}D_\mu \sigma D^\mu \sigma - g^2\mathcal{V} \\ & - [e^{2u_1+2u_2+2u_3}F_{\mu\nu}^0 F^{0\mu\nu} + e^{2u_1-2u_2-2u_3}F_{\mu\nu}^1 F^{1\mu\nu} \\ & + e^{-2u_1+2u_2-2u_3}F_{\mu\nu}^2 F^{2\mu\nu} + e^{-2u_1-2u_2+2u_3}F_{\mu\nu}^3 F^{3\mu\nu}] \end{aligned}$$

$$\begin{aligned} \mathcal{V} = & -8e^{2\phi}(e^{-2u_1}+e^{-2u_2}+e^{-2u_3})+e^{4\phi}(e^{-2u_1+2u_2+2u_3}+e^{2u_1-2u_2+2u_3}+e^{2u_1+2u_2-2u_3}) \\ & +\frac{1}{4}e_0^2e^{4\phi}e^{-2u_1-2u_2-2u_3}, \end{aligned}$$

$$\mathcal{V}=\sum_{i=1}^3\left(\frac{\partial W}{\partial u_i}\right)^2+\left(\frac{\partial W}{\partial \phi}\right)^2-3W^2$$

$$W=\frac{1}{4}e^{-u_1-u_2-u_3}\left(8-e_0e^{2\phi}+2e^{2\phi}(e^{2u_1+2u_2}+e^{2u_2+2u_3}+e^{2u_3+2u_1})\right)$$

$$\begin{aligned} A^{\mathcal{R}} &= 4\sqrt{2}A^0 \\ A^m &= \sqrt{2}(e_0A^0-2A^1-2A^2-2A^3) \\ A^{\mathcal{B}_1} &= 4\sqrt{2}(A^1-2A^2+A^3) \\ A^{\mathcal{B}_2} &= 4\sqrt{2}(A^1-A^3) \end{aligned}$$

$$\begin{aligned} \delta\psi_{\mu A} &\sim 2\nabla_\mu \varepsilon_A - iB_\mu \sigma^3{}_A{}^B \varepsilon_B - gW\gamma_\mu \sigma^3{}_{AB} \varepsilon^B + 4iH_{\mu\nu} \gamma^\nu \epsilon_{AB} \varepsilon^B \\ \delta\lambda^{iA} &\sim \partial_\mu u_i \gamma^\mu \varepsilon^A + g\partial_{u_i} W \sigma^{3AB} \varepsilon_B - i\partial_{u_i} H_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \varepsilon_B \\ U_\phi^{\alpha A} \delta\zeta_\alpha &\sim U_\sigma^{\alpha A} \delta\zeta_\alpha \sim \partial_\mu \phi \gamma^\mu \varepsilon^A + \frac{i}{2} \partial_\phi B_\mu \gamma^\mu \sigma^3{}_B{}^A \varepsilon^B + g\partial_\phi W \sigma^{3AB} \varepsilon_B \end{aligned}$$

$$\begin{aligned} H_{\mu\nu} = & -\frac{1}{2\sqrt{2}}\big(e^{u_1+u_2+u_3}F_{\mu\nu}^{-0}+e^{u_1-u_2-u_3}F_{\mu\nu}^{-1}+e^{-u_1+u_2-u_3}F_{\mu\nu}^{-2}+e^{-u_1-u_2+u_3}F_{\mu\nu}^{-3}\big) \\ B_\mu = & -4\sqrt{2}gA_\mu^0+\frac{1}{2}e^{2\phi}D_\mu\sigma \end{aligned}$$

$$F^{-I}=\frac{1}{2}(F^I-i\ast F^I)$$

$$\begin{aligned}
F^{-0} &= \frac{1}{\sqrt{2}} e^{-u_1-u_2-u_3} (\bar{F}^{-12} + \bar{F}^{-34} + \bar{F}^{-56} + \bar{F}^{-78}) \\
F^{-1} &= \frac{1}{\sqrt{2}} e^{-u_1+u_2+u_3} (\bar{F}^{-12} - \bar{F}^{-34} - \bar{F}^{-56} + \bar{F}^{-78}) \\
F^{-2} &= \frac{1}{\sqrt{2}} e^{u_1-u_2+u_3} (-\bar{F}^{-12} + \bar{F}^{-34} - \bar{F}^{-56} + \bar{F}^{-78}) \\
F^{-3} &= \frac{1}{\sqrt{2}} e^{u_1+u_2-u_3} (-\bar{F}^{-12} - \bar{F}^{-34} + \bar{F}^{-56} + \bar{F}^{-78})
\end{aligned}$$

$$\bar{F}_{\mu\nu}^{-12} = -4\partial_{u_1} H_{\mu\nu}, \bar{F}_{\mu\nu}^{-34} = -4\partial_{u_2} H_{\mu\nu}, \bar{F}_{\mu\nu}^{-56} = -4\partial_{u_3} H_{\mu\nu}, \bar{F}_{\mu\nu}^{-78} = 4H_{\mu\nu}.$$

$$\epsilon = \epsilon_1 + \epsilon^2.$$

$$\begin{aligned}
[2\nabla_\mu - iB_\mu - gW\gamma_\mu + 4iH_{\mu\nu}\gamma^\nu]\epsilon &= 0 \\
[\partial_\mu u_i \gamma^\mu + g\partial_{u_i} W + i\partial_{u_i} H_{\mu\nu} \gamma^{\mu\nu}]\epsilon &= 0 \\
\left[\partial_\mu \phi \gamma^\mu + g\partial_\phi W + \frac{i}{2}\partial_\phi B_\mu \gamma^\mu\right]\epsilon &= 0
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + g^2\mathcal{V}g_{\mu\nu} - 2\left(T_{\mu\nu}^\phi + T_{\mu\nu}^{u_1} + T_{\mu\nu}^{u_2} + T_{\mu\nu}^{u_3}\right) - \frac{1}{2}e^{4\phi}T_{\mu\nu}^\sigma \\
-e^{2(u_1+u_2+u_3)}T_{\mu\nu}^{A^0} - e^{2(u_1-u_2-u_3)}T_{\mu\nu}^{A^1} - e^{-2(u_1-u_2+u_3)}T_{\mu\nu}^{A^2} - e^{-2(u_1+u_2-u_3)}T_{\mu\nu}^{A^3} = 0,
\end{aligned}$$

$$\begin{aligned}
T_{\mu\nu}^X &= \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\rho X \partial^\rho X, \\
T_{\mu\nu}^{A^I} &= g^{\rho\sigma} F_{\mu\rho}^I F_{\nu\sigma}^I - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^I F^{I\rho\sigma},
\end{aligned}$$

$$\begin{aligned}
\partial_\nu(\sqrt{-g}e^{2(u_1+u_2+u_3)}F^{0\mu\nu}) + \frac{e_0}{2}\sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma &= 0, \\
\partial_\nu(\sqrt{-g}e^{2(u_1-u_2-u_3)}F^{1\mu\nu}) - \sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma &= 0, \\
\partial_\nu(\sqrt{-g}e^{-2(u_1-u_2+u_3)}F^{2\mu\nu}) - \sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma &= 0, \\
\partial_\nu(\sqrt{-g}e^{-2(u_1+u_2-u_3)}F^{3\mu\nu}) - \sqrt{2}\sqrt{-g}e^{4\phi}g^{\mu\nu}D_\nu\sigma &= 0.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu u_1) - \frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_1} \\
&+ e^{2(u_1+u_2+u_3)}F_{\mu\nu}^0F^{0\mu\nu} + e^{2(u_1-u_2-u_3)}F_{\mu\nu}^1F^{1\mu\nu} \\
&- e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^2F^{2\mu\nu} - e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^3F^{3\mu\nu} = 0 \\
&\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu u_2) - \frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_2} \\
&+ e^{2(u_1+u_2+u_3)}F_{\mu\nu}^0F^{0\mu\nu} - e^{2(u_1-u_2-u_3)}F_{\mu\nu}^1F^{1\mu\nu} \\
&+ e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^2F^{2\mu\nu} - e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^3F^{3\mu\nu} = 0 \\
&\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu u_3) - \frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial u_3} \\
&+ e^{2(u_1+u_2+u_3)}F_{\mu\nu}^0F^{0\mu\nu} - e^{2(u_1-u_2-u_3)}F_{\mu\nu}^1F^{1\mu\nu} \\
&- e^{-2(u_1-u_2+u_3)}F_{\mu\nu}^2F^{2\mu\nu} + e^{-2(u_1+u_2-u_3)}F_{\mu\nu}^3F^{3\mu\nu} = 0
\end{aligned}$$

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi)-\frac{g^2}{2}\frac{\partial\mathcal{V}}{\partial\phi}-\frac{1}{2}e^{4\phi}D_{\mu}\sigma D^{\mu}\sigma=0$$

$$e^{2u_i}=\sqrt{\frac{e_0}{6}},e^{-2\phi}=\frac{e_0}{6},L_{AdS_4}=\frac{1}{2}\left(\frac{e_0}{6}\right)^{3/4}$$

$$A\equiv A^0=A^1=A^2=A^3$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R+12g^2-4F_{\mu\nu}F^{\mu\nu}$$

$$ds^2=e^{2V}ds_{AdS_2}^2+f^2dy^2+h^2dz^2$$

$$A^I=a^I dz(\text{B.1})$$

$$\gamma^m=\Gamma^m\otimes\sigma^3,\gamma^2=1_2\otimes\sigma^1,\gamma^3=1_2\otimes\sigma^2$$

$$\epsilon=\psi\otimes\chi$$

$$D_m\psi=\frac{1}{2}\kappa\Gamma_m\psi$$

$$[-i(\kappa e^{-V}+4H_{23})\gamma^{23}+V'f^{-1}\gamma^2]\epsilon=gW\epsilon.$$

$$[i\cos\xi\gamma^{23}+\sin\xi\gamma^2]\epsilon=\epsilon$$

$$-\kappa e^{-V}-4H_{23}=gW\cos\xi,V'f^{-1}=gW\cos\xi$$

$$\epsilon=e^{i\frac{\xi}{2}\gamma^3}\eta,\gamma^2\eta=i\gamma^3\eta$$

$$\xi=0\pi,\gamma^{23}\epsilon=\pm i\epsilon$$

$$\left[\partial_y\eta-\frac{1}{2}V'\eta+\frac{i}{2}(\partial_y\xi+8fH_{23}+\kappa fe^{-V})\gamma^3\right]\eta=0$$

$$[2\partial_z-iB_z+if^{-1}h'\cos\xi-4iH_{23}h\sin\xi$$

$$+(f^{-1}h'\sin\xi-gWh+4H_{23}h\cos\xi)\gamma^3]\eta=0$$

$$\eta=e^{V/2}e^{isz}\eta_0$$

$$\partial_y\xi+8fH_{23}+\kappa fe^{-V}=0$$

$$(s-B_z)+f^{-1}h'\cos\xi-4H_{23}h\sin\xi=0$$

$$f^{-1}h'\sin\xi-gWh+4H_{23}h\cos\xi=0$$

$$f^{-1}h'=gWh\sin\xi-(s-B_z)\cos\xi$$

$$hH_{23}=\frac{1}{4}gWh\cos\xi+\frac{1}{4}(s-B_z)\sin\xi$$

$$(s-B_z)\sin\xi=-2gWh\cos\xi-\kappa he^{-V},$$

$$H_{23}=-\frac{1}{4}gW\cos\xi-\frac{1}{4}\kappa e^{-V}$$

$$f^{-1}\partial_y\xi=2gW\cos\xi+\kappa e^{-V}$$



$$f^{-1} \frac{h'}{h} \sin \xi = \kappa e^{-V} \cos \xi + gW(1 + \cos^2 \xi).$$

$$f^{-1} u'_i + g \partial_{u_i} W \sin \xi = 0,$$

$$g \partial_{u_i} W \cos \xi + 4 \partial_{u_i} H_{23} = 0.$$

$$f^{-1} \phi' \sin \xi + g \partial_\phi W = 0$$

$$g \partial_\phi W \cos \xi - \frac{1}{2} \partial_\phi B_z \sin \xi h^{-1} = 0$$

$$f^{-1} \xi' = 2gW \cos \xi + \kappa e^{-V}$$

$$f^{-1} V' = gW \sin \xi$$

$$f^{-1} u'_i = -g \partial_{u_i} W \sin \xi$$

$$f^{-1} \phi' = -\frac{g \partial_\phi W}{\sin \xi}$$

$$f^{-1} \frac{h'}{h} \sin \xi = \kappa e^{-V} \cos \xi + gW(1 + \cos^2 \xi)$$

$$(s - B_z) \sin \xi = -2gWh \cos \xi - \kappa h e^{-V}$$

$$g \partial_\phi W \cos \xi = \frac{1}{2} \partial_\phi B_z \sin \xi h^{-1}$$

$$\partial_{u_i} H_{23} = -\frac{1}{4} g \partial_{u_i} W \cos \xi$$

$$H_{23} = -\frac{1}{4} gW \cos \xi - \frac{1}{4} \kappa e^{-V}$$

$$\partial_y W = -gf \sin \xi \left[\sum_{i=1}^3 (\partial_{u_i} W)^2 + \frac{1}{\sin^2 \xi} (\partial_\phi W)^2 \right]$$

$$h e^{-V} = k \sin \xi$$

$$f^{-1} \xi' = -k^{-1} (s - B_z) e^{-V}$$

$$f^{-1} V' = gW \sin \xi$$

$$f^{-1} u'_i = -g \partial_{u_i} W \sin \xi$$

$$f^{-1} \phi' = -\frac{g \partial_\phi W}{\sin \xi}$$

$$(s - B_z) = -k(2gW e^V \cos \xi + \kappa)$$

$$g \partial_\phi W \cos \xi = \frac{1}{2} k^{-1} e^{-V} \partial_\phi B_z$$

$$\partial_\phi B_z = e^{2\phi} D_z \sigma.$$

$$D_z \sigma = \frac{2gk e^V \partial_\phi W \cos \xi}{e^{2\phi}}$$

$$F_{yz}^I = (a^I)' = (\mathcal{I}^{(I)})'$$



$$\begin{aligned} \mathcal{I}^{(0)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{-u_1-u_2-u_3} \\ \mathcal{I}^{(1)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{-u_1+u_2+u_3} \\ \mathcal{I}^{(2)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{u_1-u_2+u_3} \\ \mathcal{I}^{(3)} &\equiv \frac{1}{\sqrt{2}} g k e^V \cos \xi e^{u_1+u_2-u_3} \end{aligned}$$

$$h \rightarrow -h, z \rightarrow -z,$$

$$U_{u\alpha A}U_v^{\alpha B}=\frac{1}{2}h_{uv}\delta_A{}^B+\frac{i}{2}\Omega_{uv}^x\sigma^3{}_A{}^B$$

$$\delta\lambda^{iA}\sim\frac{1}{4(z^i)^2}\partial_\mu z^i\gamma^\mu\varepsilon^A+g\partial_{zi}W\sigma^{3AB}\varepsilon_B-i\partial_{zi}H_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B$$

$$\begin{aligned} S = \frac{1}{16\pi G_N} \int \big[& R \star 1 + 2g^2 \left(8X_1X_2 + \frac{4(X_1+X_2)}{X_1^2X_2^2} - \frac{1}{X_1^4X_2^4} \right) \star 1 - \frac{1}{2} \sum_{I=1}^2 d\varphi_I \wedge \star d\varphi_I \\ & - \frac{1}{2} \sum_{I=1}^2 \frac{1}{X_I^2} F_{(2)}^I \wedge \star F_{(2)}^I - \frac{1}{2} X_1^2 X_2^2 F_{(4)} \wedge \star F_{(4)} \\ & + g F_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)} \big] \end{aligned}$$

$$X_1=e^{-\frac{1}{\sqrt{10}}\varphi_1-\frac{1}{\sqrt{2}}\varphi_2},X_2=e^{-\frac{1}{\sqrt{10}}\varphi_1+\frac{1}{\sqrt{2}}\varphi_2},F_{(2)}^I=dA_{(1)}^I,F_{(4)}=dA_{(3)}$$

$$\begin{aligned} 0=&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\left[R+2g^2\left(8X_1X_2+\frac{4(X_1+X_2)}{X_1^2X_2^2}-\frac{1}{X_1^4X_2^4}\right)\right] \\ &-\sum_{I=1}^2\left[\frac{1}{2}\partial_\mu\varphi_I\partial_\nu\varphi_I-\frac{1}{4}g_{\mu\nu}\partial^\rho\varphi_I\partial_\rho\varphi_I\right]-\sum_{I=1}^2\frac{1}{X_I^2}\left[\frac{1}{2}F_{\mu\rho}^IF_\nu^{I\rho}-\frac{1}{8}g_{\mu\nu}F_{\rho\sigma}^IF^{I\rho\sigma}\right] \\ &-X_1^2X_2^2\left[\frac{1}{12}F_{\mu\rho\sigma\lambda}F_\nu^{\rho\sigma\lambda}-\frac{1}{96}g_{\mu\nu}F_{\rho\sigma\lambda\delta}F^{\rho\sigma\lambda\delta}\right]. \end{aligned}$$

$$\begin{aligned} \square \varphi_1 &= \frac{1}{2\sqrt{10}} \sum_{I=1}^2 \frac{1}{X_I^2} F_{\mu\nu}^I F^{I\mu\nu} - \frac{1}{12\sqrt{10}} X_1^2 X_2^2 F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \\ &\quad + \frac{8g^2}{\sqrt{10}} \left(4X_1X_2 - \frac{3(X_1+X_2)}{X_1^2X_2^2} + \frac{2}{X_1^4X_2^4} \right) \\ \square \varphi_2 &= \frac{1}{2\sqrt{2}} \left[\frac{1}{X_1^2} F_{\mu\nu}^1 F^{1\mu\nu} - \frac{1}{X_2^2} F_{\mu\nu}^2 F^{2\mu\nu} \right] + 4\sqrt{2}g^2 \frac{X_1-X_2}{X_1^2X_2^2} \end{aligned}$$

$$\begin{aligned} d(X_1^{-2}\star F_{(2)}^1)&=F_{(2)}^2\wedge F_{(4)}\\ d(X_2^{-2}\star F_{(2)}^2)&=F_{(2)}^1\wedge F_{(4)}\\ d(X_1^2X_2^2\star F_{(4)})&=2gF_{(4)}+F_{(2)}^1\wedge F_{(2)}^2 \end{aligned}$$

$$X_1^2 X_2^2 \star F_{(4)} = 2gA_{(3)} + \frac{1}{2}(A_{(1)}^1 \wedge F_{(2)}^2 + A_{(1)}^2 \wedge F_{(2)}^1) - dA_{(2)}$$

$$ds^2 = (H_1 H_2)^{\frac{1}{5}} \left(- \frac{(1+g^2 r^2)(1-g^2 y^2)(1-g^2 z^2)}{\Xi_1 \Xi_2 \Xi_3} dt^2 + \frac{(r^2+y^2)(r^2+z^2)}{U} dr^2 \right. \\ + \frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2 y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)} dy^2 \\ + \frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2 z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)} dz^2 \\ + \frac{(a_1^2+r^2)(a_1^2-y^2)(a_1^2-z^2)}{\Xi_1(a_1^2-a_2^2)(a_1^2-a_3^2)} d\phi_1^2 + \frac{(a_2^2+r^2)(a_2^2-y^2)(a_2^2-z^2)}{\Xi_2(a_2^2-a_1^2)(a_2^2-a_3^2)} d\phi_2^2 \\ \left. + \frac{(a_3^2+r^2)(a_3^2-y^2)(a_3^2-z^2)}{\Xi_3(a_3^2-a_1^2)(a_3^2-a_2^2)} d\phi_3^2 + \frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2} K_1^2 + \frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2} K_2^2 \right)$$

$$s_I = \sinh \delta_I, c_I = \cosh \delta_I$$

$$\Xi_i = 1 - a_i^2 g^2$$

$$H_I(r,y,z) = 1 + \frac{2ms_I^2}{(r^2+y^2)(r^2+z^2)}$$

$$U(r) = \frac{(1+g^2r^2)\prod_{i=1}^3(r^2+a_i^2)}{r^2} - 2m + mg^2(s_1^2+s_2^2)\left(2r^2 + \sum_{i=1}^3 a_i^2\right) + \frac{4m^2g^2s_1^2s_2^2}{r^2}$$

$$- \frac{2mg(s_1^2+s_2^2)a_1a_2a_3}{r^2} + \frac{2mg(c_1-c_2)^2}{r^2}(a_1+a_2a_3g)(a_2+a_3a_1g)(a_3+a_1a_2g)$$

$$-\frac{1}{2}mg^2(c_1-c_2)^2(-2(a_1^2+a_2^2+a_3^2)-8a_1a_2a_3g$$

$$-(a_1+a_2+a_3)(a_2+a_3-a_1)(a_3+a_1-a_2)(a_1+a_2-a_3)g^2)$$

$$K_1 = \frac{c_1+c_2}{2s_1}\mathcal{A}[y^2,z^2,0] + \frac{c_1-c_2}{2s_1}y$$

$$K_2 = \frac{c_1+c_2}{2s_2}\mathcal{A}[y^2,z^2,0] - \frac{c_1-c_2}{2s_2}y$$

$$\mathcal{A}[y^2,z^2,0] = \frac{(1-g^2y^2)(1-g^2z^2)}{\Xi_1\Xi_2\Xi_3}dt - \sum_{i=1}^3 \frac{a_i(a_i^2-y^2)(a_i^2-z^2)}{\Xi_i(a_i^2-a_j^2)(a_i^2-a_k^2)}d\phi_i$$

$$y = \frac{(1-g^2y^2)(1-g^2z^2)(1-(a_1^2+a_2^2+a_3^2)g^2-2a_1a_2a_3g^3)}{\Xi_1\Xi_2\Xi_3}dt$$

$$+ \sum_{i=1}^3 \frac{a_i(a_i^2-y^2)(a_i^2-z^2)\left(1-(a_i^2-a_j^2-a_k^2)g^2+\frac{2a_ja_k}{a_i}g\right)}{\Xi_i(a_i^2-a_j^2)(a_i^2-a_k^2)}d\phi_i$$

$$A_{(1)}^I = \left(1 - \frac{1}{H_I}\right) K_I + \alpha_I dt$$

$$A_{(3)} = 2ms_1s_2g^4a_1a_2a_3[\mathcal{A}[y^2,z^2,0] - \mathcal{A}[y^2,z^2,g^{-2}]] \\ \wedge \left[\frac{dz \wedge (\mathcal{A}[y^2,0,0] - \mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)z} + \frac{dy \wedge (\mathcal{A}[z^2,0,0] - \mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)y} \right] \\ + 2ms_1s_2g^3\mathcal{A}[y^2,z^2,0] \\ \wedge \left[\frac{zdz \wedge (\mathcal{A}[y^2,0,0] - \mathcal{A}[y^2,0,g^{-2}])}{(r^2+y^2)} + \frac{ydy \wedge (\mathcal{A}[z^2,0,0] - \mathcal{A}[z^2,0,g^{-2}])}{(r^2+z^2)} \right]$$

$$\mathcal{A}[v_1,v_2,v_3]=\left[\prod_{i=1}^3\frac{1-g^2v_i}{\Xi_i}\right]dt-\sum_{i=1}^3\left[\frac{(a_i^2-v_1)(a_i^2-v_2)(a_i^2-v_3)}{a_i\Xi_i\prod_{j=1(\neq i)}^3(a_i^2-a_j^2)}\right]d\phi_i$$

$$X_I=\frac{(H_1H_2)^{\frac{2}{5}}}{H_I}$$

$$A_{(2)}\!=\!\left(\frac{1}{H_1}\!+\!\frac{1}{H_2}\right)\!\frac{ms_1s_2(a_1\!+\!a_2a_3g)}{(r^2\!+\!y^2)(r^2\!+\!z^2)}\!\left(\frac{(1\!-\!g^2y^2)(1\!-\!g^2z^2)\mu_1^2}{\Xi_1(1\!-\!a_2^2g^2)(1\!-\!a_3^2g^2)}dt\wedge d\phi_1\right.\\ \left.+\frac{g(a_3^2\!-\!a_2^2)\mu_2^2\mu_3^2}{\Xi_2\Xi_3}d\phi_2\wedge d\phi_3\right)\\ \!+\!(\text{cyclic-permutations})\\ \!-\!\frac{1}{2}dt\wedge(\alpha_1A_{(1)}^2\!+\!\alpha_2A_{(1)}^1)$$

$$\mu_i^2=\frac{(a_i^2-y^2)(a_i^2-z^2)}{\prod_{j=1(\neq i)}^3(a_i^2-a_j^2)},$$

$$(1-g^2a_i^2)\tilde{r}^2\tilde{\mu}_i^2=(r^2+a_i^2)\mu_i^2$$

$$ds^2\sim \frac{d\tilde{r}^2}{g^2\tilde{r}^2}+\tilde{r}^2\left(-g^2dt^2+\sum_i\left(d\tilde{\mu}_i^2+\tilde{\mu}_i^2d\phi_i^2\right)\right)+\cdots$$

$$0\leq a_1\leq z=\sqrt{a_1^2\cos^2\theta_2+a_2^2\sin^2\theta_2}\leq a_2\leq y=\sqrt{a_2^2\cos^2\theta_1+a_3^2\sin^2\theta_1}\leq a_3\leq g^{-1},$$

$$S=\frac{1}{4G_N}\int_{S_5}d^5x\sqrt{h}\Big|_{r=r_+}=\frac{1}{4G_N}\int_{a_1}^{a_2}dz\int_{a_2}^{a_3}dy\int_0^{2\pi}d\phi_1d\phi_2d\phi_3\sqrt{h}\Big|_{r=r_+}\\ =\frac{\pi^3}{4G_N\Xi_1\Xi_2\Xi_3}\frac{\sqrt{\mathcal{S}(r_+)}}{r_+}$$

$$h_{ij}=g_{\mu\nu}\frac{\partial x^\mu}{\partial x^i}\frac{dx^\nu}{\partial x^j}\Big|_{t,r=\aleph}$$

$$h=\frac{y^2z^2(y^2-z^2)^2}{\Xi_1^2\Xi_2^2\Xi_3^2(a_1^2-a_2^2)^2(a_2^2-a_3^2)^2(a_3^2-a_1^2)^2}\\ \times\left[\frac{\Xi_1\Xi_2\Xi_3r^2((r^2+y^2)(r^2+z^2)-r^2(r^2+g^2y^2z^2))}{(1-g^2y^2)(1-g^2z^2)}U(r)+\frac{\mathcal{S}(r)}{r^2}\right.\\ \left.-\left((a_1^2+a_2^2+a_3^2+a_1^2a_2^2a_3^2g^4)r^4+(1-g^2r^2)\left((a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2)r^2+a_1^2a_2^2a_3^2\right)\right)U(r)\right]$$



$$\mathcal{S}(r) \equiv \prod_{I=1}^2 \left((r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2) + 2ms_I^2(r^2 - a_1a_2a_3g) \right) \\ + 2mg(c_1 - c_2)^2(r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2)(a_1 + a_2a_3g)(a_2 + a_3a_1g)(a_3 + a_1a_2g)$$

$$J_1 = -\frac{1}{16\pi G_N} \int_{S^5} \star dK_1 \\ = \frac{\pi^2 m}{16G_N \Xi_1 \prod_{j=1}^3 \Xi_j} [4a_1c_1c_2 + 4g(1 - c_1c_2)(a_2 + a_1a_3g)(a_3 + a_1a_2g) \\ + (c_1 - c_2)^2 \left(2a_2a_3g + a_1 \left(1 + 2\Xi_1 - \sum_{j=1}^3 \Xi_j \right) \right) \left(1 + 2a_1a_2a_3g^3 - \sum_{j=1}^3 \Xi_j \right)]$$

$$K_i^\mu \partial_\mu = -\partial_{\phi_i} \Leftrightarrow K_i = -g_{\mu\phi_i} dx^\mu$$

$$Q_1 = -\frac{1}{16\pi G_N} \int_{S^5} (X_1^{-2} \star F_{(2)}^1 - F_{(2)}^1 \wedge A_{(3)}) \\ = \frac{\pi^2 m s_1}{4G_N \Xi_1 \Xi_2 \Xi_3} [2c_1 - (c_1 - c_2)g^2(a_1^2 + a_2^2 + a_3^2 + 2a_1a_2a_3g)]$$

$$ds^2|_{\rho^2 \ll r_+} = (H_1 H_2)^{\frac{1}{5}} \frac{(r_+^2 + y^2)(y^2 - z^2)y^2}{(1 - g^2 y^2)(a_1^2 - y^2)(a_2^2 - y^2)(a_3^2 - y^2)} dy^2 \\ + (H_1 H_2)^{\frac{1}{5}} \frac{(r_+^2 + z^2)(z^2 - y^2)z^2}{(1 - g^2 z^2)(a_1^2 - z^2)(a_2^2 - z^2)(a_3^2 - z^2)} dz^2 \\ + (H_1 H_2)^{\frac{1}{5}} (r_+^2 + y^2)(r_+^2 + z^2) \left[\frac{r_+^4}{\mathcal{S}(r_+)} U'(r_+) \rho^2 d\tau^2 + \frac{4d\rho^2}{U'(r_+)} \right] \\ + \sum_{i,j=1}^3 g_{ij}(r_+, y, z) (d\phi_i + i\Omega_i d\tau) (d\phi_j + i\Omega_j d\tau)$$

$$(\tau, \phi_i) \sim (\tau + \beta, \phi_i - i\Omega_i \beta) \text{ where } \beta = 2\pi \frac{2\sqrt{\mathcal{S}(r_+)}}{r_+^2 U'(r_+)}.$$

$$T = \beta^{-1} = \frac{r_+^2 U'(r_+)}{4\pi \sqrt{\mathcal{S}(r_+)}}$$

$$\ell = \partial_t + \sum_{i=1}^3 \Omega_i \partial_{\phi_i}$$

$$\begin{aligned}
w_1(r) = & \frac{1}{\mathcal{S}(r)} \left[\frac{1}{2} \left(\prod_{i=1}^3 (r^2 + a_i^2) + 2ms_1^2(r^2 - a_1a_2a_3g) \right) \right. \\
& \times \left(a_1(1 + g^2r^2)(r^2 + a_2^2)(r^2 + a_3^2) + 2ms_2^2(a_1g^2r^2 - a_2a_3g) \right) \\
& + \frac{1}{2} \left(\prod_{i=1}^3 (r^2 + a_i^2) + 2ms_2^2(r^2 - a_1a_2a_3g) \right) \\
& \times \left(a_1(1 + g^2r^2)(r^2 + a_2^2)(r^2 + a_3^2) + 2ms_1^2(a_1g^2r^2 - a_2a_3g) \right) \\
& - m(c_1 - c_2)^2(r^2 + a_2^2)(r^2 + a_3^2)\{2g(gr^2 - a_1)(a_1 + a_2a_3g)(a_2 + a_3a_1g)(a_3 + a_1a_2g) \\
& + (1 + a_1g)^2(a_1 + a_2a_3g)(1 - a_1g - a_2g - a_3g)(1 - a_1g + a_2g + a_3g)r^2\}
\end{aligned}$$

$$\Phi_I = \ell^\mu A_{(1)\mu}^I \Big|_{r=r_+} - \ell^\mu A_{(1)\mu}^I \Big|_{r \rightarrow \infty}$$

$$\begin{aligned}
\Phi_1 = & \frac{2mr_+^2s_1c_1\left(\prod_i (r_+^2 + a_i^2) + 2ms_2^2(r_+^2 - a_1a_2a_3g)\right)}{\mathcal{S}(r_+)} \\
- & \frac{mgr_+^2s_1(c_1 - c_2)}{\mathcal{S}(r_+)} \left\{ g(a_1^2 + a_2^2 + a_3^2 + 2a_1a_2a_3g) \left(\prod_i (r_+^2 + a_i^2) + 2ms_2^2(r_+^2 - a_1a_2a_3g) \right) \right. \\
& \left. + 4ms_2^2(a_1 + a_2a_3g)(a_2 + a_3a_1g)(a_3 + a_1a_2g) \right\}
\end{aligned}$$

$$E = \frac{1}{8(D-3)\pi g^3} \int_B d\mathcal{B}_a \tilde{\Omega}^{-(D-3)} \bar{n}^c \bar{n}^d \bar{C}^a{}_{cbd} \xi^b$$

$$\tilde{\Omega}=\frac{1}{gr}$$

$$g_{rr}=\frac{1}{g^2r^2}, g^{rr}=g^2r^2, \bar{C}_{cbd}^a=\mathcal{O}(r^{-8})$$

$$\bar{n}^r=\bar{g}^{ra}\bar{n}_a=\bar{g}^{rr}\left(-\frac{1}{gr^2}\right)=\tilde{\Omega}^{-2}g^{rr}\left(-\frac{1}{gr^2}\right)=-\tilde{\Omega}^{-2}g$$

$$\text{vol}_6=\frac{yz|y^2-z^2|dt\wedge dy\wedge dz\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3}{g^5(a_1^2-a_2^2)(a_3^2-a_1^2)(a_2^2-a_3^2)\Xi_1\Xi_2\Xi_3}+\mathcal{O}(r^{-1})$$

$$d\mathcal{B}_t=\frac{yz|y^2-z^2|dy\wedge dz\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3}{g^5(a_1^2-a_2^2)(a_3^2-a_1^2)(a_2^2-a_3^2)\Xi_1\Xi_2\Xi_3}$$

$$E=\frac{1}{32\pi g}\int_B d\mathcal{B}_t\tilde{\Omega}^{-8}\bar{C}^t{}_{rtr}$$

$$\begin{aligned}
E = & \frac{m\pi^2}{8G_N\Xi_1\Xi_2\Xi_3} \left[\sum_{i=1}^3 \frac{2}{\Xi_i} - 1 + \frac{5(s_1^2 + s_2^2)}{2} + \frac{s_1^2 + s_2^2}{2} \sum_{i=1}^3 \left(\frac{2(1 + a_i^2g^2 - \Sigma_2 - 2\Pi_1)}{\Xi_i} - \Xi_i \right) \right] \\
& + \frac{m\pi^2(c_1 - c_2)^2}{32G_N\Xi_1^2\Xi_2^2\Xi_3^2} [-10\Sigma_2 - 16\Pi_1 + 11\Sigma_4 + 13\Pi_{22} + 32\Pi_1\Sigma_2 - 3(\Sigma_6 + 5\Pi_{42} + 4\Pi_1^2) \\
& - 16\Pi_1\Sigma_2^2 + \Pi_{62} + 3\Pi_{44} - 5\Pi_1^2\Sigma_2 + 8\Pi_1(2\Pi_2 + \Pi_{42}) + \Pi_1^2(\Sigma_4 + 3\Pi_{22})]
\end{aligned}$$



$$\begin{aligned}\Sigma_n &\equiv (a_1 g)^n + (a_2 g)^n + (a_3 g)^n \\ \Pi_n &\equiv (a_1 g)^n (a_2 g)^n (a_3 g)^n \\ \Pi_{nm} &\equiv (a_1 g)^n ((a_2 g)^m + (a_3 g)^m) + (a_2 g)^n ((a_3 g)^m + (a_1 g)^m) + (a_3 g)^n ((a_1 g)^m + (a_2 g)^m)\end{aligned}$$

$$dE = TdS + \sum_{i=1}^3 \Omega_i dJ_i + \sum_{I=1}^2 \Phi_I dQ_I$$

$$\begin{aligned}I &= -S + \beta \left[E - \sum_{i=1}^3 \Omega_i J_i - \sum_{I=1}^2 \Phi_I Q_I \right] \\ &= \frac{\pi^2 \beta}{16 G_N \Xi_1 \Xi_2 \Xi_3 r_+^2} \left[(1 - g^2 r_+^2) \prod_{i=1}^3 (r_+^2 + a_i^2) - 2m(s_1^2 + s_2^2)(g^2 r_+^4 + a_1 a_2 a_3 g) \right. \\ &\quad \left. - \frac{2m^2 s_1^2 s_2^2}{g^4 \mathcal{S}(r_+)} \left(\prod_{i=1}^3 (r_+^2 + a_i^2) + 2ms_1^2(r_+^2 - a_1 a_2 a_3 g) \right) \right. \\ &\quad \times \left(g^6 r_+^6 + (\Sigma_2 + 2\Pi_1)g^4 r_+^4 - \left(2\Pi_1 + \frac{1}{2}\Pi_{22} - 2g^4 m s_2^2 \right) g^2 r_+^2 + \Pi_1(2g^4 m s_2^2 - \Pi_1) \right) \\ &\quad \left. - \frac{2m^2 s_1^2 s_2^2}{g^4 \mathcal{S}(r_+)} \left(\Pi_{i=1}^3 (r_+^2 + a_i^2) + 2ms_2^2(r_+^2 - a_1 a_2 a_3 g) \right) \right. \\ &\quad \times \left(g^6 r_+^6 + (\Sigma_2 + 2\Pi_1)g^4 r_+^4 - \left(2\Pi_1 + \frac{1}{2}\Pi_{22} - 2g^4 m s_1^2 \right) g^2 r_+^2 + \Pi_1(2g^4 m s_1^2 - \Pi_1) \right) \\ &\quad \left. + \frac{m(c_1 - c_2)^2 (a_1 + a_2 a_3 g)(a_2 + a_3 a_1 g)(a_3 + a_1 a_2 g)}{g^{11} \mathcal{S}(r_+)} \right. \\ &\quad \times \left(-2\Xi_1 \Xi_2 \Xi_3 \left(g^6 r_+^6 + \Sigma_2 g^4 r_+^4 + \left(\frac{1}{2}\Pi_{22} + 2mg^4(s_1^2 + s_2^2) \right) g^2 r_+^2 + \Pi_2 \right) \right. \\ &\quad \left. - 2g^4 m(1 + s_1^2 + s_2^2) - \frac{1}{2}(c_1 - c_2)^2 g^4 m(-2\Sigma_2 - 8\Pi_1 + \Sigma_4 - \Pi_{22}) + (s_1^2 + s_2^2)g^4 m \Sigma_2 \right) \\ &\quad \left. + g^4 m \left(4 + 8g^4 m s_1^2 s_2^2 + (2 - \Sigma_2 - 2\Pi_1) \left(2(s_1^2 + s_2^2) - (c_1 - c_2)^2 (\Sigma_2 + 2\Pi_1) \right) \right) \right. \\ &\quad \left. \times \frac{g^8 r_+^2 \Pi_{i=1}^3 (r_+^2 + a_i^2) - \Xi_1 \Xi_2 \Xi_3}{1 + g^2 r_+^2} - 8g^8 m^2 s_1^2 s_2^2 (2g^6 r_+^6 + \Sigma_2 g^4 r_+^4 - \Pi_2) \right) \Bigg].\end{aligned}$$

$$(a_1 + a_2 + a_3)g = \frac{2}{1 - e^{\delta_1 + \delta_2}}$$

$$E + g \sum_{i=1}^3 J_i - \sum_{I=1}^2 Q_I = 0$$

$$\beta \left[g - \sum_{i=1}^3 \Omega_i - 2g \sum_{I=1}^2 \Phi_I \right] = \pm 2\pi i$$

$$I = -S + \beta \left[E - \sum_{i=1}^3 \Omega_i J_i - \sum_{I=1}^2 \Phi_I Q_I \right] = \frac{\pi^2}{8G_N g} \frac{\varphi_1^2 \varphi_2^2}{\omega_1 \omega_2 \omega_3}$$

$$\varphi_I \equiv \beta(\Phi_I - 1), \omega_i \equiv \beta(\Omega_i + g)$$



$$T = 0 \Leftrightarrow U'(r_+) = 0$$

$$m^{\star}=\frac{M_1}{g^4M_2},$$

$$M_1=2(2\Sigma_1-2\Sigma_2-\Pi_{11}+2\Pi_{21}+2\Pi_1)(2\Pi_1+\Pi_{21})\prod_{i=1}^3(1-a_i g),$$

$$M_2=2(1-\Sigma_1)^3\Big(2-(\Sigma_2+2\Pi_{11})(s_1^2+s_2^2)\Big)-8(s_1^2+s_2^2)(1-\Sigma_1)^2(2\Pi_1+\Pi_{21})\\+(c_1-c_2)^2(1-\Sigma_1)^2(-2\Sigma_2+2\Sigma_1\Sigma_2-8\Pi_1+\Sigma_4-\Pi_{22}+8\Sigma_1\Pi_1\\-\Sigma_1^2(\Sigma_3-\Pi_{21}+2\Pi_1)),$$

$$r_+^{\star}=\sqrt{\frac{a_1a_2+a_2a_3+a_3a_1-a_1a_2a_3g}{1-(a_1+a_2+a_3)g}}$$

$$\Omega_i^{\star}=-g, \Phi_I^{\star}=1$$

$$\frac{\frac{2}{g^3}Q_1^{\star}Q_2^{\star}(Q_1^{\star}+Q_2^{\star})-\frac{\pi^2}{G_Ng^5}(J_1^{\star}J_2^{\star}+J_2^{\star}J_3^{\star}+J_3^{\star}J_1^{\star})}{\frac{8}{g}(Q_1^{\star}+Q_2^{\star})-\frac{\pi^2}{G_Ng^5}}\\=\left(-\frac{\pi^2}{16G_Ng^5}(J_1^{\star}+J_2^{\star}+J_3^{\star})+\frac{1}{8g^2}((Q_1^{\star})^2+(Q_2^{\star})^2)+\frac{1}{2g^2}Q_1^{\star}Q_2^{\star}\right)\\ \times \left(1-\sqrt{1-\frac{\pi^2}{\left(-\frac{\pi^2}{16G_Ng^5}(J_1^{\star}+J_2^{\star}+J_3^{\star})+\frac{1}{8g^2}((Q_1^{\star})^2+(Q_2^{\star})^2)+\frac{1}{2g^2}Q_1^{\star}Q_2^{\star}\right)^2}}\right)$$

$$S^{\star}=2\pi\sqrt{\frac{\frac{2}{g^3}Q_1^{\star}Q_2^{\star}(Q_1^{\star}+Q_2^{\star})-\frac{\pi^2}{G_Ng^5}(J_1^{\star}J_2^{\star}+J_2^{\star}J_3^{\star}+J_3^{\star}J_1^{\star})}{\frac{8}{g}(Q_1^{\star}+Q_2^{\star})-\frac{\pi^2}{G_Ng^5}}}$$

$$-\log Z_{S^1\times S^5}=\frac{N^3}{24}\frac{\Delta_1^2\Delta_2^2}{\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3},$$

$$\Delta_1+\Delta_2-\hat{\omega}_1-\hat{\omega}_2-\hat{\omega}_3=\pm 2\pi\mathrm{i}$$

$$N^3=\frac{3\pi^2}{16G_Ng^5}$$

$$\hat{\omega}_i=\beta(\Omega_i+g)=\omega_i,$$

$$\Delta_I=-2g\beta(\Phi_I-1)=-2g\varphi_I$$

$$-\log Z_{S^1\times S^5}=\frac{N^3}{24}\frac{\Delta_1^2\Delta_2^2}{\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3}=\frac{\pi^2}{8G_Ng}\frac{\varphi_1^2\varphi_2^2}{\omega_1\omega_2\omega_3}.$$



$$J_i \sim N^3, \frac{Q_I}{g} \sim N^3.$$

$$N^3=\frac{3\pi^2}{64G_Ng^5}$$

$$-\log Z_{S^1\times S^5}=\frac{N^3}{6}\frac{\Delta_1^2\Delta_2^2}{\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3}$$

$$X_{(p)}=\frac{1}{p!}X_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots dx^{\mu_p}\;(0\leq p\leq d)$$

$$\star\left(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\right)=\frac{\sqrt{|g|}}{(d-p)!}\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{d-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{d-p}}$$

$$\varepsilon_{\mu_1\cdots\mu_d}=\begin{cases}+1 & \text{if } \mu_1\cdots\mu_d \text{ even permutation} \\ -1 & \text{if } \mu_1\cdots\mu_d \text{ odd permutation} \\ 0 & \text{otherwise}\end{cases}$$

$$\begin{aligned}\star\left(\star X_{(p)}\right)&=(-1)^{s+p(d-p)}X_{(p)}\\X_{(p)}\wedge\star Y_{(p)}&=Y_{(p)}\wedge\star X_{(p)}\\&=\frac{1}{p!}X_{\mu_1\cdots\mu_p}Y^{\mu_1\cdots\mu_p}\sqrt{|g|}dx^1\wedge\cdots\wedge dx^d\end{aligned}$$

$$\begin{aligned}ds^2=&H^{\frac{2}{5}}\bigg(-\frac{(1+g^2r^2)(1-g^2y^2)(1-g^2z^2)}{(1-a_1^2g^2)(1-a_2^2g^2)(1-a_3^2g^2)}dt^2+\frac{(r^2+y^2)(r^2+z^2)}{R}dr^2\\&+\frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)}dy^2\\&+\frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)}dz^2+\frac{(a_1^2+r^2)(a_1^2-y^2)(a_1^2-z^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)(1-a_1^2g^2)}d\phi_1^2\\&+\frac{(a_2^2+r^2)(a_2^2-y^2)(a_2^2-z^2)}{(a_2^2-a_1^2)(a_2^2-a_3^2)(1-a_2^2g^2)}d\phi_2^2+\frac{(a_3^2+r^2)(a_3^2-y^2)(a_3^2-z^2)}{(a_3^2-a_1^2)(a_3^2-a_2^2)(1-a_3^2g^2)}d\phi_3^2\\&+\frac{1-\frac{1}{H}}{H}\mathcal{A}[y^2,z^2,0](C_0dt+C_1d\phi_1+C_2d\phi_2+C_3d\phi_3)\bigg)\end{aligned}$$

$$A_{(1)}^l=\frac{2msc}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}[y^2,z^2,0]=\left(1-\frac{1}{H}\right)\frac{c}{s}\mathcal{A}[y^2,z^2,0]$$

$$\begin{aligned}
A_{(3)} &= 2ms^2 g^4 a_1 a_2 a_3 [\mathcal{A}[y^2, z^2, 0] - \mathcal{A}[y^2, z^2, g^{-2}]] \\
&\quad \wedge \left[\frac{dz \wedge (\mathcal{A}[y^2, 0, 0] - \mathcal{A}[y^2, 0, g^{-2}])}{(r^2 + y^2)z} + \frac{dy \wedge (\mathcal{A}[z^2, 0, 0] - \mathcal{A}[z^2, 0, g^{-2}])}{(r^2 + z^2)y} \right] \\
&\quad + 2ms^2 g^3 \mathcal{A}[y^2, z^2, 0] \\
&\quad \wedge \left[\frac{zdz \wedge (\mathcal{A}[y^2, 0, 0] - \mathcal{A}[y^2, 0, g^{-2}])}{(r^2 + y^2)} + \frac{ydy \wedge (\mathcal{A}[z^2, 0, 0] - \mathcal{A}[z^2, 0, g^{-2}])}{(r^2 + z^2)} \right] \\
A_{(2)} &= \frac{2ms^2(a_1 + a_2 a_3 g)}{H(r^2 + y^2)(r^2 + z^2)} \left(\frac{(1 - g^2 y^2)(1 - g^2 z^2)\mu_1^2}{(1 - a_1^2 g^2)(1 - a_2^2 g^2)(1 - a_3^2 g^2)} dt \wedge d\phi_1 \right. \\
&\quad \left. + \frac{g(a_3^2 - a_2^2)\mu_2^2 \mu_3^2}{(1 - a_2^2 g^2)(1 - a_3^2 g^2)} d\phi_2 \wedge d\phi_3 \right) \\
&\quad + (\text{cyclic-permutations})
\end{aligned}$$

$$s = \sinh \delta, c = \cosh \delta$$

$$\begin{aligned}
R(r) &= \frac{1 + g^2 r^2}{r^2} (r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2) - 2m + g^2 r^2 \left(r^2 + \frac{2ms^2}{r^2} \right)^2 - g^2 r^6 \\
&\quad + 2ms^2 g^2 (a_1^2 + a_2^2 + a_3^2) - \frac{4ms^2 g a_1 a_2 a_3}{r^2} \\
C_0 &= \frac{(1 - g^2 y^2)(1 - g^2 z^2)(c^2/s^2 + (1 - (a_1^2 + a_2^2 + a_3^2)g^2 - 2a_1 a_2 a_3 g^3)H)}{(1 - a_1^2 g^2)(1 - a_2^2 g^2)(1 - a_3^2 g^2)} \\
C_i &= - \frac{a_i(a_i^2 - y^2)(a_i^2 - z^2) \left(c^2/s^2 - \left(1 - (a_i^2 - a_j^2 - a_k^2)g^2 + \frac{2a_j a_k}{a_i} g \right) H \right)}{(a_i^2 - a_j^2)(a_i^2 - a_k^2)(1 - a_i^2 g^2)}
\end{aligned}$$

$$\delta = \delta_1 = \delta_2$$

$$\begin{aligned}
(t, r, y = a \sin \theta_1, \theta_2, \phi_1, \phi_2, \phi_3) &= (t, r, y = a \cos \theta, \psi, \xi, \zeta, \phi)^\blacksquare \\
&= (t, r, y, \vartheta, \phi_2, \phi_3, \phi)^\#
\end{aligned}$$

$$\begin{aligned}
ds^2 &= (H_1 H_2)^{\frac{1}{5}} \left(\frac{r^2 + y^2}{\Delta_r} dr^2 + \frac{r^2 + y^2}{\Delta_y} dy^2 + \frac{r^2 y^2}{a^2} (d\theta_2^2 + \sin^2 \theta_2 d\phi_1^2 + \cos^2 \theta_2 d\phi_2^2) \right. \\
&\quad \left. + \frac{1}{H_1 H_2 (1 - a^2 g^2)^2 (r^2 + y^2)} [-(V_y^2 \Delta_r - \mathcal{V}_1 \mathcal{V}_2 \Delta_y) dt^2 + (\mathcal{W}_1 \mathcal{W}_2 \Delta_y - \tilde{V}_y^2 \Delta_r) a^2 d\phi_3^2 \right. \\
&\quad \left. - \frac{4m' c'_1 c'_2 \tilde{c}'_1 \tilde{c}'_2 \Delta_y}{ar^2} dt d\phi_3 \right] \\
&= (H_1 H_2)^{\frac{1}{5}} \left(- \frac{(1 + g^2 r^2)(1 - g^2 y^2)}{1 - a^2 g^2} dt^2 + \frac{r^2 + y^2}{\Delta_r} dr^2 + \frac{r^2 + y^2}{1 - g^2 y^2} d\theta_1^2 \right. \\
&\quad \left. + r^2 \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 d\phi_1^2 + \cos^2 \theta_2 d\phi_2^2) + \frac{r^2 + a^2}{1 - a^2 g^2} \cos^2 \theta_1 d\phi_3^2 \right. \\
&\quad \left. + \frac{2m' s'_1 k_1^2}{r^2 (r^2 + y^2) H_1 (1 - a^2 g^2)^2 (s_1'^2 - s_2'^2)} + \frac{2m' s_2'^2 k_2^2}{r^2 (r^2 + y^2) H_2 (1 - a^2 g^2)^2 (s_2'^2 - s_1'^2)} \right) \\
A_{\blacksquare}^1 &= \frac{2m' s'_1}{H_1 (r^2 + y^2) r^2 (1 - a^2 g^2)} k_1 \\
A_{\blacksquare}^2 &= \frac{2m' s'_2}{H_2 (r^2 + y^2) r^2 (1 - a^2 g^2)} k_2
\end{aligned}$$



$$A_{\#} = -\frac{2m's'_1s'_2a}{r^2+y^2}\sin^4\theta_1\text{vol}_{S^3} - \frac{2m's'_1s'_2ag}{(1-a^2g^2)r^2}\sin\theta_1\cos\theta_1dt\wedge d\theta_1\wedge d\phi_3,$$

$$\text{vol}_{S^3} = \sin\theta_2\cos\theta_2d\theta_2\wedge d\phi_1\wedge d\phi_2$$

$$A_{\Delta} = \left(\frac{1}{H_1} + \frac{1}{H_2}\right)\frac{m's'_1s'_2(1-g^2y^2)a}{(r^2+y^2)r^2(1-a^2g^2)}\cos^2\theta_1dt\wedge d\phi_3$$

$$s'_I = \sinh\delta'_I, c'_I = \cosh\delta'_I$$

$$\tilde{c}'_I = \sqrt{1+a^2g^2s'^2_I}$$

$$H_I(r,y) = 1 + \frac{2m's'^2_I}{(r^2+y^2)r^2}$$

$$\Delta_r(r) = \left(r^2 + a^2 - \frac{2m'}{r^2}\right)\left(1 + g^2r^2 - \frac{2m'g^2s'^2_1s'^2_2}{r^2}\right) + 2m'g^2c'^2_1c'^2_2$$

$$\Delta_y(y) = (1-g^2y^2)(a^2-y^2)$$

$$\mathcal{V}_I(r) = 1 + g^2r^2 + \frac{2m's'^2_Ig^2}{r^2}$$

$$\mathcal{V}_y(y) = 1 - g^2y^2$$

$$\mathcal{W}_I(r) = 1 + \frac{r^2}{a^2} + \frac{2m's'^2_I}{r^2a^2}$$

$$\tilde{\mathcal{V}}_y(y) = 1 - \frac{y^2}{a^2}$$

$$k_1 = c'_1\tilde{c}'_2(1-g^2y^2)dt - c'_2\tilde{c}'_1a\cos^2\theta_1d\phi_3$$

$$k_2 = c'_2\tilde{c}'_1(1-g^2y^2)dt - c'_1\tilde{c}'_2a\cos^2\theta_1d\phi_3$$

$$\frac{c_1 - \frac{1}{2}a^2g^2(c_1 - c_2)}{s_1} = \frac{c'_1\tilde{c}'_2}{s'_1}, \frac{c_2 + \frac{1}{2}a^2g^2(c_1 - c_2)}{s_2} = \frac{c'_2\tilde{c}'_1}{s'_2}, ms'^2_I = m's'^2_I$$

$$ds^2 = (H_1H_2)^{\frac{1}{5}} \left(-\frac{(1+g^2r^2)(1-g^2y^2)}{1-a^2g^2}dt^2 + \frac{r^2+y^2}{\Delta_r}dr^2 + \frac{r^2+y^2}{1-g^2y^2}d\theta_1^2 \right. \\ \left. + r^2\sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2d\phi_1^2 + \cos^2\theta_2d\phi_2^2) + \frac{r^2+a^2}{1-a^2g^2}\cos^2\theta_1d\phi_3^2 \right. \\ \left. + \frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2}K_1^2 + \frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2}K_2^2 \right)$$

$$A^I_{(1)} = \left(1 - \frac{1}{H_I}\right)K_I$$

$$A_{(2)} = \left(\frac{1}{H_1} + \frac{1}{H_2}\right)\frac{ms_1s_2(1-g^2y^2)a}{(r^2+y^2)r^2(1-a^2g^2)}\cos^2\theta_1dt\wedge d\phi_3$$

$$A_{(3)} = -\frac{2ms_1s_2a}{r^2+y^2}\sin^4\theta_1\text{vol}_{S^3} - \frac{2ms_1s_2ag}{(1-a^2g^2)r^2}\sin\theta_1\cos\theta_1dt\wedge d\theta_1\wedge d\phi_3$$

$$K_1 = \frac{\left(c_1 - \frac{1}{2}a^2g^2(c_1 - c_2)\right)(1 - g^2y^2)dt - \left(c_2 + \frac{1}{2}a^2g^2(c_1 - c_2)\right)a\cos^2\theta_1d\phi_3}{s_1(1 - a^2g^2)}$$

$$K_2 = \frac{\left(c_2 + \frac{1}{2}a^2g^2(c_1 - c_2)\right)(1 - g^2y^2)dt - \left(c_1 - \frac{1}{2}a^2g^2(c_1 - c_2)\right)a\cos^2\theta_1d\phi_3}{s_2(1 - a^2g^2)}$$

$$\lim_{a_2\rightarrow 0}\lim_{a_1\rightarrow 0}\lim_{a_3\rightarrow a}.$$

$$\begin{aligned} T &= T^\Delta = \frac{1}{2}T^\odot, \quad \Omega_i = (0,0,\Omega)^\square, \quad \Phi_I = \Phi_I^\oplus \\ S &= \frac{1}{G_N}S^\Delta, \quad J_i = \frac{1}{G_N}(0,0,J)^\odot, \quad Q_I = \frac{1}{G_N}Q_I^\boxplus \\ E &= \frac{1}{G_N}E^\otimes, \end{aligned}$$

$$t\rightarrow (1-ag)t, \text{ ii) } \tau\rightarrow \tau-gt,$$

$$\Delta=[1,13,30]|_{g\rightarrow -g}$$

$$g\rightarrow -g.$$

$$(t,r,\psi,\xi,\varphi_1,\varphi_2,\varphi_3)=\left(t,\sqrt{\rho^2-a^2},\tau,\xi,\varphi_1,\varphi_2,\varphi_3\right)^\boxtimes$$

$$\begin{aligned}\sigma_1 &= \cos\varphi_3 d\varphi_1 + \sin\varphi_1 \sin\varphi_3 d\varphi_2 \\ \sigma_2 &= \sin\varphi_3 d\varphi_1 - \sin\varphi_1 \cos\varphi_3 d\varphi_2 \\ \sigma_3 &= d\varphi_3 + \cos\varphi_1 d\varphi_2\end{aligned}$$

$$\begin{aligned} ds^2 =& (H_1H_2)^{\frac{1}{5}}\bigg(-\frac{W}{f_1}dt^2+\frac{r^2(r^2+a^2)^2}{W}dr^2+\frac{r^2+a^2}{1-a^2g^2}ds_{\mathbb{CP}^2}^2 \\ & +\frac{f_1}{(r^2+a^2)^2H_1H_2(1-a^2g^2)^2}\Big(\sigma+gdt-\frac{2f_2}{f_1}(1+ag)dt\Big)^2\bigg) \\ =& (H_1H_2)^{\frac{1}{5}}\bigg(-\frac{1+g^2r^2}{1-a^2g^2}dt^2+\frac{r^2(r^2+a^2)^2}{W}dr^2+\frac{r^2+a^2}{1-a^2g^2}(\sigma^2+ds_{\mathbb{CP}^2}^2) \\ & +\frac{2ms_1^2}{(s_1^2-s_2^2)(r^2+a^2)^2H_1(1-a^2g^2)^2}(\alpha_1(1+ag)dt-a\alpha_2(\sigma+gdt))^2 \\ & +\frac{2ms_2^2}{(s_2^2-s_1^2)(r^2+a^2)^2H_2(1-a^2g^2)^2}(\alpha_2(1+ag)dt-a\alpha_1(\sigma+gdt))^2\bigg) \end{aligned}$$

$$\sigma\!=\!d\psi+\frac{1}{2}\sin^2\xi\sigma_3$$

$$ds_{\mathbb{CP}^2}^2=d\xi^2+\frac{1}{4}\sin^2\xi(\sigma_1^2+\sigma_2^2)+\frac{1}{4}\sin^2\xi\cos^2\xi\sigma_3^2$$

$$A^1_{(1)}\!=\!\frac{2ms_1}{(1-a^2g^2)H_1(r^2+a^2)^2}(\alpha_1(1+ag)dt-a\alpha_2(\sigma+gdt))$$

$$A^2_{(1)}\!=\!\frac{2ms_2}{(1-a^2g^2)H_2(r^2+a^2)^2}(\alpha_2(1+ag)dt-a\alpha_1(\sigma+gdt))$$



$$A_{(3)} = \frac{mas_1s_2}{(1-a^2g^2)(1+ag)(r^2+a^2)}(\sigma + gdt) \wedge d\sigma$$

$$A_{(2)} = \left(\frac{1}{H_1} + \frac{1}{H_2}\right) \frac{mas_1s_2}{(1+ag)(r^2+a^2)^2} dt \wedge \sigma$$

$$s_I = \sinh \delta_I, c_I = \cosh \delta_I$$

$$\alpha_1 = c_1 - \frac{1}{2}(1 - (1 - ag)^2)(c_1 - c_2)$$

$$\alpha_2 = c_2 + \frac{1}{2}(1 - (1 - ag)^2)(c_1 - c_2)$$

$$H_I(r) = 1 + \frac{2ms_I^2}{(r^2 + a^2)^2}$$

$$f_1(r) = (1 - a^2g^2)H_1H_2(r^2 + a^2)^3 - \frac{4(1 - ag)^2m^2a^2s_1^2s_2^2}{(r^2 + a^2)^2} \\ + \frac{1}{2}ma^2\left(4(1 - ag)^2 + 2(1 - (1 - ag)^4)c_1c_2 + (1 - (1 - ag)^2)^2(c_1^2 + c_2^2)\right)$$

$$f_2(r) = \frac{1}{2}g(1 - ag)H_1H_2(r^2 + a^2)^3 \\ + \frac{1}{4}ma\left(2(1 + (1 - ag)^4)c_1c_2 + (1 - (1 - ag)^4)(c_1^2 + c_2^2)\right)$$

$$W(r) = g^2H_1H_2(r^2 + a^2)^4 + (1 - a^2g^2)(r^2 + a^2)^3 + \frac{1}{2}ma^2(4(1 - ag)^2 \\ + 2(1 - (1 - ag)^4)c_1c_2 + (1 - (1 - ag)^2)^2(c_1^2 + c_2^2)) \\ - \frac{1}{2}m(r^2 + a^2)(4(1 - a^2g^2) + 2a^2g^2(6 - 8ag + 3a^2g^2)c_1c_2 \\ - a^2g^2(2 - ag)(2 - 3ag)(c_1^2 + c_2^2))$$

$$(\psi, \xi, \varphi_1, \varphi_2, \varphi_3) = (-\phi_3, \theta_1, 2\theta_2, \phi_1 - \phi_2, 2\phi_3 - \phi_1 - \phi_2)$$

$$\sigma = -\cos^2 \theta_1 d\phi_3 - \sin^2 \theta_1 (\sin^2 \theta_2 d\phi_1 + \cos^2 \theta_2 d\phi_2) \\ ds_{\mathbb{CP}^2}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_2 (d\phi_1 - d\phi_2)^2 \\ + \sin^2 \theta_1 \cos^2 \theta_1 (d\phi_3 - \sin^2 \theta_2 d\phi_1 - \cos^2 \theta_2 d\phi_2)^2$$

$$ds^2 = (H_1H_2)^{\frac{1}{5}} \left(-\frac{1+g^2r^2}{1-a^2g^2} dt^2 + \frac{r^2(r^2+a^2)^2}{W} dr^2 + \frac{1-\frac{1}{H_1}}{1-(s_2/s_1)^2} K_1^2 + \frac{1-\frac{1}{H_2}}{1-(s_1/s_2)^2} K_2^2 \right. \\ \left. + \frac{r^2+a^2}{1-a^2g^2} (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\phi_1^2 + \sin^2 \theta_1 \cos^2 \theta_2 d\phi_2^2 + \cos^2 \theta_1 d\phi_3^2) \right)$$

$$A_{(1)}^I = \left(1 - \frac{1}{H_I}\right) K_I$$

$$A_{(3)} = \frac{mas_1s_2}{(1-a^2g^2)(1+ag)(r^2+a^2)}(\sigma + gdt) \wedge d\sigma$$

$$A_{(2)} = \left(\frac{1}{H_1} + \frac{1}{H_2}\right) \frac{mas_1s_2}{(1+ag)(r^2+a^2)^2} dt \wedge \sigma$$



$$K_1 = \frac{1}{s_1(1-a^2g^2)}(\alpha_1(1+ag)dt - a\alpha_2(\sigma + gdt))$$

$$K_2 = \frac{1}{s_2(1-a^2g^2)}(\alpha_2(1+ag)dt - a\alpha_1(\sigma + gdt))$$

$$\lim_{a_3 \rightarrow -a} \lim_{a_2 \rightarrow -a} \lim_{a_1 \rightarrow -a}$$

$$T = [gT^{\oplus}]_{g \rightarrow -g}, \quad \Omega_i = [g\Omega^{\square}]_{g \rightarrow -g}, \Phi_I = \left[\frac{1}{2}\Phi^{\dagger}\right]_{g \rightarrow -g}$$

$$S = [S^{\boxtimes}]_{g \rightarrow -g}, \quad J_i = [J^{\boxplus}]_{g \rightarrow -g}, Q_I = [2gQ_I^*]_{g \rightarrow -g}$$

$$E = [gE^{\boxminus}]_{g \rightarrow -g},$$

$$\omega_i = \frac{\Omega_i + g}{T} = \frac{[g\Omega^{\circ}]_{g \rightarrow -g} + g}{[gT^{\ominus}]_{g \rightarrow -g}} = \frac{\Omega^V - 1}{T^{\Delta}} \Big|_{g \rightarrow -g} = -\zeta^{\ominus} \Big|_{g \rightarrow -g}$$

$$\varphi_I = \frac{\Phi_I - 1}{T} = \frac{\left[\frac{1}{2}\Phi_I^{\ominus}\right]_{g \rightarrow -g} - 1}{[gT^{\odot}]_{g \rightarrow -g}} = \frac{\Phi_I^{\ominus} - 2}{2gT^{[13]}} \Big|_{g \rightarrow -g} = -\frac{1}{2g}\xi_I^{\wedge} \Big|_{g \rightarrow -g}$$

$$ds^2=ds_5^2+(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+y^2)(y^2-z^2)y^2}{(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)(a_3^2-y^2)}dy^2$$

$$+(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+z^2)(z^2-y^2)z^2}{(1-g^2z^2)(a_1^2-z^2)(a_2^2-z^2)(a_3^2-z^2)}dz^2$$

$$ds_5^2=-g_{00}(r,y,z)dt^2+\sum_{i,j=1}^3g_{ij}(r,y,z)(d\phi_i-W_i(r,y,z)dt)(d\phi_j-W_j(r,y,z)dt)$$

$$+(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}\frac{(r^2+y^2)(r^2+z^2)}{U}dr^2$$

$$g_{00}(r,y,z)=\frac{(H_1(r,y,z)H_2(r,y,z))^{\frac{1}{5}}r^4(r^2+y^2)(r^2+z^2)U(r)}{\mathcal{S}(r)}\Big[1+\frac{Q(r,y,z)}{P(r,y,z)}U(r)\Big]$$

$$W_i(r,y,z)=\frac{\frac{1+g^2r^2}{r^2}(1-g^2y^2)(1-g^2z^2)\mathcal{S}(r)}{P(r,y,z)}\Big[w_i(r)-a_ir^2(r^2+a_j^2)(r^2+a_k^2)\frac{U(r)}{\mathcal{S}(r)}\Big]$$

$$Q(r,y,z)=(1+g^2r^2)[a_1^2a_2^2a_3^2(1-g^2(r^2+y^2+z^2)+g^4(r^2+y^2)(r^2+z^2))$$

$$+(a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2)r^2(1-g^2(r^2+y^2+z^2))$$

$$+(a_1^2+a_2^2+a_3^2)r^2(r^2+g^2y^2z^2)+r^2(r^2(r^2+g^2y^2z^2)-(r^2+y^2)(r^2+z^2))]$$

$$P(r,y,z)=\frac{1+g^2r^2}{r^2}(1-g^2y^2)(1-g^2z^2)\mathcal{S}(r)$$

$$+\left[r^2(r^2+y^2)(r^2+z^2)\Xi_1\Xi_2\Xi_3-(1-g^2y^2)(1-g^2z^2)\prod_{i=1}^3(r^2+a_i^2)\right]U(r)$$

$$g_{00}(r,y,z)=\frac{(H_1(r_+,y,z)H_2(r_+,y,z))^{\frac{1}{5}}r_+^4(r_+^2+y^2)(r_+^2+z^2)}{\mathcal{S}(r_+)}U'(r_+)\rho^2+\mathcal{O}(\rho^4)$$

$$W_i(r_+,y,z)=\Omega_i+\mathcal{O}(\rho^2)$$



$$r = r_+ + \rho^2 \ (\rho^2 \ll r_+)$$

$$\begin{aligned} & ds_5^2|_{\rho^2 \ll r_+} \\ &= (H_1(r_+, y, z)H_2(r_+, y, z))^{\frac{1}{5}}(r_+^2 + y^2)(r_+^2 + z^2) \left[-\frac{r_+^4}{\mathcal{S}(r_+)}U'(r_+)\rho^2 dt^2 + \frac{4d\rho^2}{U'(r_+)} \right] \\ &+ \sum_{i,j=1}^3 g_{ij}(r_+, y, z)(d\phi_i - \Omega_i dt)(d\phi_j - \Omega_j dt), \end{aligned}$$

$$I_{11}=\frac{1}{16\pi G_{11}}\int\left(\star_{11}R-\frac{1}{2}\mathcal{F}\wedge\star_{11}\mathcal{F}+\frac{1}{6}\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{A}\right),$$

$$\mathrm{d}s_4^2=\frac{1}{\mu^2}\big(e^{-2\lambda_1}[\,\mathrm{d}v_1^2+v_1^2(D\phi_1)^2]+e^{-2\lambda_2}[\,\mathrm{d}v_2^2+v_2^2(D\phi_2)^2]+e^{4\lambda_1+4\lambda_2}\,\mathrm{d}v_3^2\big)$$

$$v_1=\sin\alpha\cos\beta, v_2=\sin\alpha\sin\beta, v_3=\cos\alpha; \ v_1^2+v_2^2+v_3^2=1$$

$$\begin{aligned} \mathrm{d}s_{11}^2=&\Delta^{1/3}\,\mathrm{d}s_7^2+\Delta^{-2/3}\,\mathrm{d}s_4^2\\ \mathcal{F}= &-e^{-4\lambda_1-4\lambda_2}v_3\star_7S+\frac{1}{\mu}S\wedge\,\mathrm{d}v_3\\ &+\frac{1}{\Delta\mu^2}\big[-e^{2\lambda_1}v_1^2D\phi_1\wedge F_2\wedge\,\mathrm{d}v_3-e^{2\lambda_2}v_2^2D\phi_2\wedge F_1\wedge\,\mathrm{d}v_3\\ &+e^{-4\lambda_1-4\lambda_2}v_1v_3D\phi_1\wedge F_2\wedge\,\mathrm{d}v_1+e^{-4\lambda_1-4\lambda_2}v_2v_3D\phi_2\wedge F_1\wedge\,\mathrm{d}v_2\big]\\ &+\frac{Uv_1v_2}{\Delta^2\mu^3}\big[v_1D\phi_1\wedge D\phi_2\wedge\,\mathrm{d}v_2\wedge\,\mathrm{d}v_3+v_2D\phi_2\wedge D\phi_1\wedge\,\mathrm{d}v_1\wedge\,\mathrm{d}v_3\\ &+v_3D\phi_1\wedge D\phi_2\wedge\,\mathrm{d}v_1\wedge\,\mathrm{d}v_2\big]\\ &+\frac{v_1v_2}{\Delta^2\mu^3}\big[2e^{2\lambda_1+2\lambda_2}v_1v_2D\phi_1\wedge D\phi_2\wedge\,\mathrm{d}(\lambda_1-\lambda_2)\wedge\mathrm{d}v_3\\ &+2e^{-2\lambda_1-4\lambda_2}v_1v_3D\phi_1\wedge D\phi_2\wedge\,\mathrm{d}(3\lambda_1-2\lambda_2)\wedge\mathrm{d}v_2\\ &+2e^{-4\lambda_1-2\lambda_2}v_2v_3D\phi_1\wedge D\phi_2\wedge\,\mathrm{d}(3\lambda_1+2\lambda_2)\wedge\mathrm{d}v_1\big] \\ \Delta=&e^{2\lambda_1}v_1^2+e^{2\lambda_2}v_2^2+e^{-4\lambda_1-4\lambda_2}v_3^2\\ U=&-(2e^{2\lambda_1+2\lambda_2}+e^{-2\lambda_1-4\lambda_2})v_1^2-(2e^{2\lambda_1+2\lambda_2}+e^{-4\lambda_1-2\lambda_2})v_2^2\\ &+(e^{-8\lambda_1-8\lambda_2}-2e^{-2\lambda_1-4\lambda_2}-2e^{-4\lambda_1-2\lambda_2})v_3^2 \end{aligned}$$

$$\begin{aligned} I_7 = \frac{1}{16\pi G_7} \int \big[& \star_7 (R - V) - 5 \, \mathrm{d}\lambda_+ \wedge \star_7 \, \mathrm{d}\lambda_+ - \mathrm{d}\lambda_- \wedge \star_7 \, \mathrm{d}\lambda_- - \frac{1}{2} e^{-4\lambda_1} F_1 \wedge \star_7 F_1 \\ & - \frac{1}{2} e^{-4\lambda_2} F_2 \wedge \star_7 F_2 - \frac{1}{2} e^{-4\lambda_+} S \wedge \star_7 S + \frac{1}{2\mu} S \wedge \mathrm{d}S \\ & - \frac{1}{\mu} F_1 \wedge F_2 \wedge S + \frac{1}{4\mu} (A_1 \wedge F_1 \wedge F_2 \wedge F_2 + A_2 \wedge F_2 \wedge F_1 \wedge F_1) \big] \end{aligned}$$

$$V=\frac{\mu^2}{2}\big(e^{-8\lambda_1-8\lambda_2}-4e^{-2\lambda_1-4\lambda_2}-4e^{-4\lambda_1-2\lambda_2}-8e^{2\lambda_1+2\lambda_2}\big)$$



$$\begin{aligned}
dS &= \mu e^{-4\lambda_+} S + F_1 \wedge F_2 \\
d(e^{-4\lambda_{1,2}} \star_7 F_{1,2}) &= \frac{1}{\mu} F_{1,2} \wedge F_{2,1} \wedge F_{2,1} - \frac{1}{\mu} F_{2,1} \wedge dS \\
d \star_7 d(3\lambda_{1,2} + 2\lambda_{2,1}) &= -\frac{1}{2} e^{-4\lambda_{1,2}} F_{1,2} \wedge \star_7 F_{1,2} - \frac{1}{2} e^{-4\lambda_+} S \wedge \star_7 S + \frac{1}{4} \star_7 \frac{\partial V}{\partial \lambda_{1,2}} \\
R_{MN} &= 5\partial_M \lambda_+ \partial_N \lambda_- + \partial_M \lambda_- \partial_N \lambda_+ + \frac{1}{2} e^{-4\lambda_1} F_{1MN}^2 + \frac{1}{2} e^{-4\lambda_2} F_{2MN}^2 \\
&+ \frac{1}{4} e^{-4\lambda_+} S_{MN}^2 + \frac{1}{5} g_{MN} \left[V - \frac{1}{4} e^{-4\lambda_1} F_1^2 - \frac{1}{4} e^{-4\lambda_2} F_2^2 - \frac{1}{6} e^{-4\lambda_+} S^2 \right]
\end{aligned}$$

$$\begin{aligned}
\delta \tilde{\psi}_M &= [\partial_M + \frac{1}{4} \omega_{\hat{M}\hat{A}\hat{B}} \tilde{\gamma}^{\hat{A}\hat{B}} \otimes 1_4 + \frac{\mu}{2} 1_8 \otimes (A_{1M} \Gamma^{12} + A_{2M} \Gamma^{34}) \\
&+ \left(\frac{\mu}{4} e^{-4\lambda_+} \tilde{\gamma}_M + \frac{1}{2} \tilde{\gamma}_M \tilde{\gamma}^N \partial_N \lambda_+ \right) \otimes 1_4 \\
&+ \frac{1}{4} (e^{-2\lambda_1} F_{1MN} \tilde{\gamma}^N \otimes \Gamma^{12} + e^{-2\lambda_2} F_{2MN} \tilde{\gamma}^N \otimes \Gamma^{34}) \\
&- \frac{1}{8} e^{-2\lambda_+} S_{MNP} \tilde{\gamma}^{NP} \otimes \Gamma^5] \tilde{\epsilon} \otimes \mathcal{E} \\
\delta \tilde{\zeta}_{1,2} &= \left[\frac{\mu}{4} (e^{2\lambda_{1,2}} - e^{-4\lambda_+}) - \frac{1}{4} \tilde{\gamma}^M \partial_M (3\lambda_{1,2} + 2\lambda_{2,1}) \otimes 1_4 \right. \\
&- \left. \frac{1}{16} e^{-2\lambda_{1,2}} F_{1,2MN} \tilde{\gamma}^{MN} \otimes \Gamma^{12} + \frac{1}{48} e^{-2\lambda_+} S_{MNP} \tilde{\gamma}^{MNP} \otimes \Gamma^5 \right] \tilde{\epsilon} \otimes \mathcal{E}.
\end{aligned}$$

$$ds_2^2 = (dx_1^2 + dx_2^2)/x_2^2$$

$$\begin{aligned}
\Lambda_1 &= \frac{1}{10} \ln \left(\frac{(1+7z+7z^2+33z^3-(1+4z+19z^2)\sqrt{1+3z^2})}{4z(1-z)^2} \right) \\
\Lambda_2 &= \Lambda_1 - \frac{1}{2} \ln \left(\frac{1+z}{2z+\sqrt{1+3z^2}} \right) \\
f_0 &= 4\Lambda_+ \\
g_0 &= \frac{1}{2} \ln \left(\frac{1}{8} e^{2\Lambda_+} [(1-z)e^{2\Lambda_1} + (1+z)e^{2\Lambda_2}] \right)
\end{aligned}$$

$$\begin{aligned}
ds_7^2 &= e^{2f_0} ds_5^2 + e^{2g_0} L^2 ds_2^2 \\
S &= c e^{f_0+2g_0+2\Lambda_+} L^2 \star_7 (F \wedge d\omega) \\
A_{1,2} &= a(1 \pm z) L \omega + b e^{f_0+2\Lambda_{1,2}} A
\end{aligned}$$

$$I_5 = \frac{1}{16\pi G_5} \int \left[\star_5 \left(R + \frac{12}{L^2} \right) - \frac{\chi^2}{2} F \wedge \star_5 F + \frac{\chi^3}{3\sqrt{3}} A \wedge F \wedge F \right]$$

$$G_7 = e^{3f_0+2g_0} L^2 4\pi (g-1) G_5$$

$$\begin{aligned}
d \star_5 F &= \frac{\chi}{\sqrt{3}} F \wedge F \\
R_{\mu\nu} &= -\frac{4}{L^2} g_{\mu\nu} + \frac{\chi^2}{2} F_\mu{}^\alpha F_{\nu\alpha} - \frac{\chi^2}{12} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}
\end{aligned}$$

$$\delta\psi_\mu=\left[\partial_\mu+\frac{1}{4}\omega_{\mu\hat\alpha\hat\beta}\gamma^{\hat\alpha\hat\beta}-\frac{\mathrm{i}\chi}{8\sqrt{3}}(\gamma_{\mu\alpha\beta}F^{\alpha\beta}-4F_{\mu\alpha}\gamma^\alpha)-\frac{1}{2L}(\gamma_\mu+\mathrm{i}\sqrt{3}\chi A_\mu)\right]\epsilon$$



$$\tilde{\gamma}_\mu = -\chi_7 \gamma_\mu \otimes \sigma_1, \tilde{\gamma}_{\hat{x}_1} = \chi_7 1_4 \otimes \sigma_2, \tilde{\gamma}_{\hat{x}_2} = \chi_7 1_4 \otimes \sigma_3, \tilde{\epsilon} = \epsilon \otimes \begin{pmatrix} 1 \\ \chi_7 \end{pmatrix},$$

$$(\tilde{\gamma}_{\hat{x}_1\hat{x}_2} \otimes \Gamma^{12})(\tilde{\epsilon} \otimes \mathcal{E}) = \chi_7 \tilde{\epsilon} \otimes \mathcal{E}, (1_8 \otimes \Gamma^5)(\tilde{\epsilon} \otimes \mathcal{E}) = \tilde{\epsilon} \otimes \mathcal{E}$$

$$a=-\frac{\chi_7}{4}, c=b=\frac{\chi}{\sqrt{3}}, \partial_{x_1}(\tilde{\epsilon} \otimes \mathcal{E})=\partial_{x_2}(\tilde{\epsilon} \otimes \mathcal{E})=0$$

$$-1 < ag < 1, -1 < bg < 1, 0 \leq q \leq \frac{m}{1+ag+bg}$$

$$\begin{aligned} ds_{\text{CCLP}}^2 = & -\frac{\Delta_\eta[(1+g^2r^2)\rho^2\,dt+2qv]dt}{\Xi_a\Xi_b\rho^2}+\frac{2qv\omega}{\rho^2}+\frac{f}{\rho^4}\Big(\frac{\Delta_\eta dt}{\Xi_a\Xi_b}-\omega\Big)^2 \\ & +\frac{\rho^2\,dr^2}{\Delta_r}+\frac{\rho^2\,d\eta^2}{\Delta_\eta}+\frac{r^2+a^2}{\Xi_a}\sin^2\eta\,d\xi_1^2+\frac{r^2+b^2}{\Xi_b}\cos^2\eta\,d\xi_2^2 \\ A_{\text{CCLP}} = & \frac{\sqrt{3}q}{\chi\rho^2}\Big(\frac{\Delta_\eta dt}{\Xi_a\Xi_b}-\omega\Big)+\alpha dt \end{aligned}$$

$$v=b\sin^2\eta\,d\xi_1+a\cos^2\eta\,d\xi_2,\omega=\frac{a\sin^2\eta}{\Xi_a}\,d\xi_1+\frac{b\cos^2\eta}{\Xi_b}\,d\xi_2$$

$$\begin{aligned}\Delta_r(r) &= \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2}-2m \\ \Delta_\eta(\eta) &= 1-a^2g^2\cos^2\eta-b^2g^2\sin^2\eta \\ \rho^2(r,\eta) &= r^2+a^2\cos^2\eta+b^2\sin^2\eta \\ f(r,\eta) &= 2(m+abqg^2)\rho^2(r,\eta)-q^2\end{aligned}$$

$$\Xi_a=1-a^2g^2,\Xi_b=1-b^2g^2$$

$$-\infty < t < \infty, r_+ < r < \infty, 0 < \eta < \frac{\pi}{2}, \xi_1 \sim \xi_1 + 2\pi, \xi_2 \sim \xi_2 + 2\pi$$

$$m=\frac{(r_+^2+a^2)(r_+^2+b^2)(1+g^2r_+^2)+q^2+2abq}{2r_+^2}$$

$$ds_{\text{CCLP},\mathfrak{E}}^2=ds_{\text{CCLP}}^2|_{t\rightarrow -i\tau}, A_{\text{CCLP},\mathfrak{E}}=A_{\text{CCLP}}|_{t\rightarrow -i\tau}.$$

$$\begin{aligned} ds_{\text{CCLP},\mathcal{B}}^2 &= \lim_{r\rightarrow\infty} g^{-2} r^{-2} \, ds_{\text{CCLP}}^2 \\ &= -\frac{\Delta_\eta}{\Xi_a\Xi_b} \, dt^2 + \frac{1}{g^2\Delta_\eta} d\eta^2 + \frac{\sin^2\eta}{g^2\Xi_a} \, d\xi_1^2 + \frac{\cos^2\eta}{g^2\Xi_b} \, d\xi_2^2 \end{aligned}$$



$$\beta = \frac{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{r_+^4[1 + g^2(2r_+^2 + a^2 + b^2)] - (ab + q)^2}$$

$$\Omega_1 = \frac{a(r_+^2 + b^2)(1 + g^2 r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \Omega_2 = \frac{b(r_+^2 + a^2)(1 + g^2 r_+^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}$$

$$\Phi = \frac{\sqrt{3}qr_+^2}{\chi[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}$$

$$ds_{\mathcal{H}}^2 = ds_{\text{CCLP}}^2|_{dt \rightarrow 0, dr \rightarrow 0, r \rightarrow r_+}$$

$$S = \frac{\pi^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2\Xi_a\Xi_b r_+ G_5}$$

$$E = (E_{\text{AdS}} - \tilde{\zeta}') + \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2abq^2\pi(\Xi_a + \Xi_b)}{4\Xi_a^2\Xi_b^2 G_5}$$

$$J_1 = \frac{\pi[2am + bq(1 + a^2g^2)]}{4\Xi_a^2\Xi_b G_5}, J_2 = \frac{\pi[2bm + aq(1 + b^2g^2)]}{4\Xi_a\Xi_b^2 G_5}$$

$$Q = \frac{\chi\sqrt{3}\pi q}{4\Xi_a\Xi_b G_5}$$

$$E_{\text{AdS}} = \frac{3\pi}{32g^2 G_5} \left(1 + \frac{(\Xi_a - \Xi_b)^2}{9\Xi_a\Xi_b} \right), \tilde{\zeta}' = \frac{9g\pi\beta(\Xi_a^2 - \Xi_a\Xi_b + \Xi_b^2)}{\Xi_a\Xi_b G_5} \zeta'$$

$$J_{\text{bs}} = \frac{\pi\beta}{4\Xi_a\Xi_b G_5} \left[m - g^2(a^2 + r_+^2)(b^2 + r_+^2) - \frac{q^2 r_+^2}{(a^2 + r_+^2)(b^2 + r_+^2) + abq} \right]$$

$$\mathcal{I} = \beta(E_{\text{AdS}} - \tilde{\zeta}') + J_{\text{bs}}$$

$$\mathcal{I} = \beta E - S - \beta\Omega_1 J_1 - \beta\Omega_2 J_2 - \beta\Phi Q.$$

$$dE = \frac{1}{\beta} dS + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ$$

$$dE = \delta_\sigma E + \frac{1}{\beta} dS + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ$$

$$\delta_\sigma \mathcal{I} = \beta d(E_{\text{AdS}} - \tilde{\zeta}') = \beta \delta_\sigma E$$

$$d\left(E - (E_{\text{AdS}} - \tilde{\zeta}')\right) = \frac{1}{\beta} dS + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ$$

$$\mathcal{I}(\beta, \Omega_1, \Omega_2, \Phi) = -\log Z(\beta, \Omega_1, \Omega_2, \Phi)$$

$$P_i = \frac{1}{Z} e^{-\beta(E_i - \Omega_1 J_{1,i} - \Omega_2 J_{2,i} - \Phi Q_i)}, Z = \sum_i e^{-\beta(E_i - \Omega_1 J_{1,i} - \Omega_2 J_{2,i} - \Phi Q_i)}$$

$$E = \sum_i P_i E_i = \frac{\partial \mathcal{I}}{\partial \beta} + \Omega_1 J_1 + \Omega_2 J_2 + \Phi Q$$

$$J_1 = \sum_i P_i J_{1,i} = -\frac{1}{\beta} \frac{\partial \mathcal{I}}{\partial \Omega_1}, J_2 = \sum_i P_i J_{2,i} = -\frac{1}{\beta} \frac{\partial \mathcal{I}}{\partial \Omega_2}$$

$$Q = \sum_i P_i Q_i = -\frac{1}{\beta} \frac{\partial \mathcal{I}}{\partial \Phi}$$

$$\frac{\partial \mathcal{I}}{\partial (a, b, r_+, q)} = \frac{\partial \mathcal{I}}{\partial \beta} \frac{\partial \beta}{\partial (a, b, r_+, q)} + \frac{\partial \mathcal{I}}{\partial \Omega_1} \frac{\partial \Omega_1}{\partial (a, b, r_+, q)} + \frac{\partial \mathcal{I}}{\partial \Omega_2} \frac{\partial \Omega_2}{\partial (a, b, r_+, q)} + \frac{\partial \mathcal{I}}{\partial \Phi} \frac{\partial \Phi}{\partial (a, b, r_+, q)}$$

$$\left(\frac{\partial \mathcal{I}}{\partial \beta}, \frac{\partial \mathcal{I}}{\partial \Omega_1}, \frac{\partial \mathcal{I}}{\partial \Omega_2}, \frac{\partial \mathcal{I}}{\partial \Phi} \right)$$

$$S(E, J_1, J_2, Q) = \beta E - \beta \Omega_1 J_1 - \beta \Omega_2 J_2 - \beta \Phi Q - \mathcal{I}(\beta, \Omega_1, \Omega_2, \Phi)$$

$$\beta = \frac{\partial S}{\partial E}, \Omega_1 = -\frac{1}{\beta} \frac{\partial S}{\partial J_1}, \Omega_2 = -\frac{1}{\beta} \frac{\partial S}{\partial J_2}, \Phi = -\frac{1}{\beta} \frac{\partial S}{\partial Q}.$$

$$\frac{\partial S}{\partial (a, b, r_+, q)} = \frac{\partial S}{\partial E} \frac{\partial E}{\partial (a, b, r_+, q)} + \frac{\partial S}{\partial J_1} \frac{\partial J_1}{\partial (a, b, r_+, q)} + \frac{\partial S}{\partial J_2} \frac{\partial J_2}{\partial (a, b, r_+, q)} + \frac{\partial S}{\partial Q} \frac{\partial Q}{\partial (a, b, r_+, q)}$$

$$\left(\frac{\partial S}{\partial E}, \frac{\partial S}{\partial J_1}, \frac{\partial S}{\partial J_2}, \frac{\partial S}{\partial Q} \right)$$

$$q = \frac{m}{1 + ag + bg}$$

$$\tilde{m} = \frac{mg}{(a+b)(1+ag)(1+bg)(1+ag+bg)} - 1$$

$$r_*^2 = \frac{1}{g}(a+b+abg),$$

$$q = -ab + r_+^2(1 + ag + bg) \pm \sqrt{-r_+^2(r_*^2 - r_+^2)}$$

$$\begin{aligned}
\beta &= -\frac{2\pi(a - ir_+)(b - ir_+)(gr_*^2 + ir_+)}{g(r_*^2 - r_+^2)[2r_+(1 + ag + bg) + ig(r_*^2 - 3r_+^2)]} \\
\Omega_1 &= \frac{g(r_*^2 + iar_+)(1 - igr_+)}{(gr_*^2 + ir_+)(a - ir_+)}, \Omega_2 = \Omega_1|_{a \leftrightarrow b} \\
\Phi &= \frac{\sqrt{3}ir_+(1 - igr_+)}{\chi(gr_*^2 + ir_+)}, \\
\tilde{E} &= \frac{\pi[3 - g^2(a^2(1 + bg) + b^2(1 + ag) - ab)][-ab + r_+^2(1 + ag + bg) + igr_+(r_*^2 - r_+^2)]}{4\Xi_a\Xi_b(1 - ag)(1 - bg)G_5} \\
J_1 &= \frac{\pi[2a + b(1 + ag)][-ab + r_+^2(1 + ag + bg) + igr_+(r_*^2 - r_+^2)]}{4\Xi_a\Xi_b(1 - ag)G_5}, J_2 = J_1 \Big|_{a \leftrightarrow b}, \\
Q &= \frac{\chi\sqrt{3}\pi[-ab + r_+^2(1 + ag + bg) + igr_+(r_*^2 - r_+^2)]}{4\Xi_a\Xi_bG_5}, \\
S &= \frac{\pi^2[r_+(r_+^2 + a^2(1 + bg) + b^2(1 + ag) + ab) + iabg(r_*^2 - r_+^2)]}{2\Xi_a\Xi_bG_5} \\
\tilde{J} &= \frac{\pi^2(a - ir_+)^2(b - ir_+)^2(1 + g^2r_*^2)}{2\Xi_a\Xi_b[2r_+(1 + ag + bg) + ig(r_*^2 - 3r_+^2)]G_5},
\end{aligned}$$

$$E - gJ_1 - gJ_2 - \frac{\sqrt{3}}{\chi}Q = E_{\text{AdS}} - \zeta'$$

$$\beta(g + \Omega_1 + \Omega_2 - g\chi\sqrt{3}\Phi) = 2\pi i$$

$$\omega_1 = \beta(\Omega_1 - g), \omega_2 = \beta(\Omega_2 - g), \varphi = \beta\left(\Phi - \frac{\sqrt{3}}{\chi}\right)$$

$$\mathcal{J} = \frac{\pi\chi^3}{12\sqrt{3}G_5} \frac{\varphi^3}{\omega_1\omega_2}$$

$$\mathcal{J} = -S - \omega_1J_1 - \omega_2J_2 - \varphi Q; \omega_1 + \omega_2 - g\chi\sqrt{3}\varphi = 2\pi i$$

$$\omega_1 = -\frac{\pi(1 - ag)(b - ir_*)}{r_*[1 + g(a + b - ir_*)]}, \omega_2 = \omega_1|_{a \leftrightarrow b}, \varphi = \frac{\sqrt{3}\pi(a - ir_*)(b - ir_*)}{\chi r_*[1 + g(a + b - ir_*)]},$$

$$\begin{aligned}
E &= \frac{\pi(a + b)[3 - g^2(a^2(1 + bg) + b^2(1 + ag) - ab)]}{4g(1 - ag)^2(1 - bg)^2G_5}, Q = \frac{\chi\sqrt{3}\pi(a + b)}{4(1 - ag)(1 - bg)G_5}, \\
J_1 &= \frac{\pi(a + b)[2a + b(1 + ag)]}{4g(1 - ag)^2(1 - bg)G_5}, J_2 = J_1|_{a \leftrightarrow b}, S = \frac{\pi^2(a + b)r_*}{2g(1 - ag)(1 - bg)G_5},
\end{aligned}$$

$$\mathcal{J} = \mu \frac{\varphi^3}{\omega_1\omega_2}$$

$$\mathcal{J} = -S - \omega_1J_1 - \omega_2J_2 - \varphi Q; \omega_1 + \omega_2 - \nu\varphi = 2\pi i$$

$$\mathcal{S}(J_1, J_2, Q) = -\mathcal{J}(\omega_1, \omega_2, \varphi) - \omega_1J_1 - \omega_2J_2 - \varphi Q - \lambda(\omega_1 + \omega_2 - \nu\varphi - 2\pi i)$$



$$-\frac{\partial \mathcal{I}}{\partial \omega_1} = J_1 + \lambda, \quad -\frac{\partial \mathcal{I}}{\partial \omega_2} = J_2 + \lambda, \quad -\frac{\partial \mathcal{I}}{\partial \varphi} = Q - \nu \lambda$$

$$\Downarrow$$

$$\mu \frac{\varphi^3}{\omega_1^2 \omega_2} = J_1 + \lambda, \quad \mu \frac{\varphi^3}{\omega_1 \omega_2^2} = J_2 + \lambda, \quad -\mu \frac{3\varphi^2}{\omega_1 \omega_2^2} = Q - \nu \lambda$$

$$S = 2\pi i \lambda$$

$$0 = \frac{1}{\nu^3} \left[\left(\frac{\partial \mathcal{I}}{\partial \varphi} \right)^3 - 27\mu \frac{\partial \mathcal{I}}{\partial \omega_1} \frac{\partial \mathcal{I}}{\partial \omega_2} \right] \Rightarrow 0 = p_0 + p_1 \lambda + p_2 \lambda^2 + \lambda^3,$$

$$p_0 = -\frac{1}{\nu^3} Q^3 - \frac{27\mu}{\nu^3} J_1 J_2, p_1 = \frac{3}{\nu^2} Q^2 - \frac{27\mu}{\nu^3} (J_1 + J_2), p_2 = -\frac{3}{\nu} Q - \frac{27\mu}{\nu^3}.$$

$$\left(\frac{2}{g\chi\sqrt{3}} Q \right)^3 + \frac{2\pi}{g^3 G_5} J_1 J_2 = \left(\frac{2\sqrt{3}}{g\chi} Q + \frac{\pi}{2g^3 G_5} \right) \left[\left(\frac{2}{g\chi} Q \right)^2 - \frac{\pi}{g^3 G_5} (J_1 + J_2) \right].$$

$$S = \frac{\pi}{g} \sqrt{\frac{4}{\chi^2} Q^2 - \frac{\pi}{g G_5} (J_1 + J_2)}.$$

$$\log J_{\text{QFT}} = \frac{\varphi_R^3}{48\omega_1\omega_2} k_{RRR} - \frac{\varphi_R(\omega_1^2 + \omega_2^2 - 4\pi^2)}{48\omega_1\omega_2} k_R + \log |G|,$$

$$\varphi_R = (\omega_1 + \omega_2 + 2\pi i n_0)$$

$$k_{RRR} = \frac{2(g-1)}{27Z^2} \left[9Z^2 - 1 + (3Z^2 + 1)^{\frac{3}{2}} \right] N^3, \quad k_{RRF} = \frac{(g-1)}{9} ZN$$

$$k_{RFF} = -\frac{(g-1)}{3} \sqrt{3Z^2 + 1} N^3, \quad k_{FFF} = (g-1) ZN^3$$

$$k_R = \frac{g-1}{3} \left[4 - \sqrt{3Z^2 + 1} \right] N, \quad k_F = (g-1) ZN$$

$$a = \frac{1}{32} (9k_{RRR} - 3k_R) = (g-1) \frac{\left(9Z^2 - 1 + (1 + 3Z^2)^{\frac{3}{2}} \right)}{48Z^2} N^3$$

$$c = \frac{1}{32} (9k_{RRR} - 5k_R) = (g-1) \frac{\left(9Z^2 - 1 + (1 + 3Z^2)^{\frac{3}{2}} \right)}{48Z^2} N^3$$

$$k_{RRR} = \frac{4\pi L^3}{9G_5}$$

$$\omega_{1,2} \leftrightarrow \omega_{1,2}, \varphi_R \leftrightarrow \frac{\chi\sqrt{3}}{L} \varphi$$

$$\log J_{\text{QFT}} = \frac{k_{RRR}\varphi_R^3 + k_{RRF}\varphi_R^2\varphi_F + k_{RFF}\varphi_R\varphi_F^2 + k_{FFF}\varphi_F^3}{48\omega_1\omega_2},$$

$$\omega_{1,2} \leftrightarrow \omega_{1,2}, \varphi_R \leftrightarrow \nu \varphi_R, \varphi_F \leftrightarrow \rho \varphi_F,$$



$$\mathcal{J} = \frac{\mu_1 \varphi_R^3 + \mu_2 \varphi_R^2 \varphi_F + \mu_3 \varphi_R \varphi_F^2 + \mu_4 \varphi_F^3}{\omega_1 \omega_2},$$

$$\mu_1 = \frac{k_{RRR} v^3}{48}, \mu_2 = \frac{k_{RRF} v^2 \rho}{48}, \mu_3 = \frac{k_{RFF} v \rho^2}{48}, \mu_4 = \frac{k_{FFF} \rho^3}{48},$$

$$\mathcal{J} = \frac{\pi}{24 \omega_1 \omega_2} \kappa_{IJK} \varphi^I \varphi^J \varphi^K$$

$$\kappa_{111} = \frac{24\mu_1}{\pi}, \kappa_{112} = \frac{8\mu_2}{\pi}, \kappa_{122} = \frac{8\mu_3}{\pi}, \kappa_{222} = \frac{24\mu_4}{\pi},$$

$$\boldsymbol{S} = -I - \varphi^I \boldsymbol{Q}_I - \omega_1 \boldsymbol{J}_1 - \omega_2 \boldsymbol{J}_2 - \lambda (\omega_1 + \omega_2 - \nu \varphi_R - 2\pi i),$$

$$-\frac{\partial \mathcal{J}}{\partial \omega_1} = \boldsymbol{J}_1 + \lambda, -\frac{\partial \mathcal{J}}{\partial \omega_2} = \boldsymbol{J}_2 + \lambda, -\frac{\partial \mathcal{J}}{\partial \varphi_R} = \boldsymbol{Q}_R - \nu \lambda, -\frac{\partial \mathcal{J}}{\partial \varphi_F} = \boldsymbol{Q}_F.$$

$$0=\frac{1}{6}\kappa^{IJK}\frac{\partial \mathcal{J}}{\partial \varphi^I}\frac{\partial \mathcal{J}}{\partial \varphi^J}\frac{\partial \mathcal{J}}{\partial \varphi^K}-\frac{\pi}{4}\frac{\partial \mathcal{J}}{\partial \omega_1}\frac{\partial \mathcal{J}}{\partial \omega_2},$$

$$\kappa_{IJK}\kappa_{J'(LM}\kappa_{PQ)K'}g^{JJ'}g^{KK'}=\frac{4}{3}g_{I(L}\kappa_{MPQ)}$$

$$g_{IJ}=6\left(\frac{\pi}{2\mu_1}\right)^{-2/3}\delta_{IJ}, g^{IJ}=\frac{1}{6}\left(\frac{\pi}{2\mu_1}\right)^{2/3}\delta^{IJ}$$

$$\kappa^{111}=\frac{\pi}{18\mu_1}, \kappa^{112}=\frac{\pi\mu_2}{54\mu_1^2}, \kappa^{122}=\frac{\pi\mu_3}{54\mu_1^2}, \kappa^{222}=\frac{\pi\mu_4}{18\mu_1^2}$$

$$0\!=\!p_0+p_1\lambda+p_2\lambda^2+\lambda^3$$

$$p_0\!=\!-\frac{1}{v^3}\boldsymbol{Q}_R^3\!-\!\frac{27\mu_1}{v^3}\boldsymbol{J}_1\boldsymbol{J}_2\!-\!\frac{\mu_2}{\mu_1v^3}\boldsymbol{Q}_R^2\boldsymbol{Q}_F\!-\!\frac{\mu_4}{\mu_1v^3}\boldsymbol{Q}_F^3$$

$$p_1\!=\!-\frac{27\mu_1}{v^3}(\boldsymbol{J}_1+\boldsymbol{J}_2)\!+\!\frac{3}{v^2}\boldsymbol{Q}_R^2\!+\!\frac{2\mu_2}{\mu_1v^2}\boldsymbol{Q}_R\boldsymbol{Q}_F\!+\!\frac{\mu_3}{\mu_1v^2}\boldsymbol{Q}_F^2$$

$$p_2\!=\!-\frac{27\mu_1}{v^3}\!-\!\frac{3}{v}\boldsymbol{Q}_R\!-\!\frac{\mu_2}{\mu_1v}\boldsymbol{Q}_F$$

$$p_0=p_1p_2, \boldsymbol{S}=2\pi\sqrt{p_1}$$

$$\mu_1=\frac{k_{RRR}v^3}{48}=\frac{1}{48}\frac{4\pi L^3}{9G_5}\left(\frac{\chi\sqrt{3}}{L}\right)^3=\frac{\pi\chi^3}{12\sqrt{3}G_5}=\mu$$

$$\mu_3=-\frac{3}{2}\mu_1, \mu_4=\frac{1}{\sqrt{2}}\mu_1$$

$$k_{RFF}\rho^2=-\frac{3}{2}k_{RRR}v^2,k_{FFF}\rho^3=\frac{1}{\sqrt{2}}k_{RRR}v^3$$

$$\varepsilon_{(7)}=\star_7 1, \varepsilon_{(5)}=\star_5 1, \varepsilon_{(2)}=\star_2 1=\mathrm{d}\omega$$

$$\varepsilon_{(7)}=e^{5f_0+2g_0}L^2\varepsilon_{(5)}\wedge\mathrm{d}\omega$$



$$\begin{aligned}
S &= ce^{2f_0+2\Lambda_+} \star_5 F \\
\star_7 S &= -ce^{f_0+2g_0+2\Lambda_+} L^2 F \wedge d\omega \\
F_{1,2} &= a(1 \pm z)L d\omega + be^{f_0+2\Lambda_{1,2}} F \\
\star_7 F_{1,2} &= ae^{5f_0-2g_0}(1 \pm z)L^{-1}\varepsilon_{(5)} + be^{2f_0+2g_0+2\Lambda_{1,2}} L^2 \star_5 F \wedge d\omega
\end{aligned}$$

$$R_{(7)} = e^{-2f_0} R_{(5)} + e^{-2g_0} L^{-2} R_{(2)}, R_{(2)} = -2$$

$$\begin{aligned}
\star_7 R &= -2e^{5f_0}\varepsilon_{(5)} \wedge d\omega + L_0^2 \star_5 R \wedge d\omega \\
-\star_7 V &= -e^{5f_0+2g_0} L^2 V \varepsilon_{(5)} \wedge d\omega \\
-\frac{1}{2}e^{-4\Lambda_{1,2}} F_{1,2} \wedge \star_7 F_{1,2} &= -\frac{a^2}{2}e^{5f_0-2g_0-4\Lambda_{1,2}}(1 \pm z)^2 \varepsilon_{(5)} \wedge d\omega - \frac{b^2}{2}L_0^2 F \wedge \star_5 F \wedge d\omega, \\
-\frac{1}{2}e^{-4\Lambda_+} S \wedge \star_7 S &= \frac{c^2}{2}L_0^2 F \wedge \star_5 F \wedge d\omega, \\
\frac{L}{4}S \wedge dS &= 0, \\
-\frac{L}{2}F_1 \wedge F_2 \wedge S &= -4abcL_0^2 F \wedge \star_5 F \wedge d\omega,
\end{aligned}$$

$$\frac{L}{8}[A_1 \wedge F_1 \wedge F_2 \wedge F_2 + (1 \leftrightarrow 2)] = 4ab^3 L_0^2 A \wedge F \wedge F \wedge d\omega.$$

$$-\frac{1}{2L^2}e^{2f_0-2g_0-2\Lambda_+}\Big(2e^{2g_0+4\Lambda_+}(2+e^{2g_0}L^2V)+a^2(e^{4\Lambda_1}(1-z)^2+e^{4\Lambda_2}(1+z)^2)\Big)$$

$$a=\pm\frac{1}{4}$$

$$I_7=\frac{L_0^2}{16\pi G_7}\int\left[\star_5\left(R+\frac{12}{L^2}\right)-\frac{1}{2}(2b^2+2bc-c^2)F\wedge\star_5F+b^3A\wedge F\wedge F\right]\wedge d\omega$$

$$\Sigma_{\rm g}\colon\int\,\,\mathrm{d}\omega=4\pi(\mathrm{g}-1)$$

$$g_{MN}=e^{2f_0}g_{\mu\nu}\oplus e^{2g_0}L^{-2}g_{mn}$$

$$e_M^{\hat{M}}=e^{f_0}e_{\mu}^{\hat{\mu}}\oplus e^{g_0}Le_m^{\hat{m}}$$

$$\begin{aligned}
S_{MNR} &= ce^{2f_0+2\Lambda_+}\frac{1}{2!}F^{\alpha\beta}\varepsilon_{\alpha\beta\mu\nu\rho} \\
F_{1,2MN} &= be^{f_0+2\Lambda_{1,2}}F_{\mu\nu}\oplus a(1\pm z)L(d\omega)_{mn} \\
A_{1,2M} &= be^{f_0+2\Lambda_{1,2}}A_{\mu}\oplus a(1\pm z)L\omega_m
\end{aligned}$$

$$\tilde{\gamma}^{MN}=e^{-2f_0}\tilde{\gamma}^{\mu\nu}\oplus e^{-2g_0}L^{-2}\tilde{\gamma}^{mn},\tilde{\gamma}^M=e^{-f_0}\tilde{\gamma}^{\mu}\oplus e^{-g_0}L^{-1}\tilde{\gamma}^m$$

$$\omega_{M\hat{A}\hat{B}}=\omega_{\mu\hat{\alpha}\hat{\beta}}\oplus\omega_{m\hat{a}\hat{b}}$$

$$\omega_{x_1\hat{x}_1\hat{x}_2}=-x_2^{-1}$$

$$\tilde{\gamma}^{MNR}=\chi_7\frac{1}{4!}\varepsilon^{MNRABCD}\tilde{\gamma}_{ABCD},\tilde{\gamma}^{MN}=\chi_7\frac{1}{5!}\varepsilon^{MNABCDE}\tilde{\gamma}_{ABCDE}$$



$$\tilde{\gamma}^{\mu\nu\rho}=\chi_7\frac{1}{2!}e^{-3f_0}\varepsilon^{\mu\nu\rho\alpha\beta}\tilde{\gamma}_{\alpha\beta}\tilde{\gamma}_{\hat{x}_1\hat{x}_2},\tilde{\gamma}^{\mu\nu}=\chi_7\frac{1}{3!}e^{-2f_0}\varepsilon^{\mu\nu\alpha\beta\gamma}\tilde{\gamma}_{\alpha\beta\gamma}\tilde{\gamma}_{\hat{x}_1\hat{x}_2}$$

$$\tilde{\gamma}^m(\text{d}\omega)_{mn}=-\tilde{\gamma}_n\tilde{\gamma}_{x_1x_2}$$

$$\delta\tilde{\zeta}_1=\frac{L^{-1}}{2}\Big[(e^{2\Lambda_1}-e^{-4\Lambda_+})-\frac{a}{4}(1+z)e^{-2g_0-2\Lambda_1}\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\otimes\Gamma^{12}\Big]\tilde{\epsilon}\otimes\mathcal{E}\\-\frac{e^{-f_0}}{16}\tilde{\gamma}^{\mu\nu}F_{\mu\nu}\big[61_8\otimes\Gamma^{12}+\chi_7c\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\otimes\Gamma^5\big]\tilde{\epsilon}\otimes\mathcal{E}$$

$$(\tilde{\gamma}_{\widehat{x_1\widehat{x_2}}}\otimes 1_4)(\tilde{\epsilon}\otimes E)=p_2(1_8\otimes\Gamma^{12})(\tilde{\epsilon}\otimes E), (1_8\otimes\Gamma^5)(\tilde{\epsilon}\otimes E)=p_5\tilde{\epsilon}\otimes E$$

$$p_2=s_a,p_5=-\chi_7s_a;\,a=\frac{s_a}{4},c=b=\frac{\chi}{\sqrt{3}},$$

$$(1_8\otimes\Gamma^5)(\tilde{\epsilon}\otimes\mathcal{E})=p_5\tilde{\epsilon}\otimes\mathcal{E}\Rightarrow(1_8\otimes\Gamma^{34})(\tilde{\epsilon}\otimes\mathcal{E})=p_5(1_8\otimes\Gamma^{12})(\tilde{\epsilon}\otimes\mathcal{E})$$

$$\delta\tilde{\psi}_m\stackrel{p_5=1}{=} \left[\partial_m-\frac{1}{2}x_2^{-1}\delta_{mx_1}(\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\otimes 1_4-4a1_8\otimes\Gamma^{12})\right.\\ \left.+\frac{1}{2}e^{-f_0+g_0}\tilde{\gamma}_m(1_8\otimes 1_4+4a\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\otimes\Gamma^{12})\right]\tilde{\epsilon}\otimes\mathcal{E}$$

$$p_2=-\chi_7,p_5=1;\,a=-\frac{\chi_7}{4},c=b=\frac{\chi}{\sqrt{3}};\,\partial_m(\tilde{\epsilon}\otimes\mathcal{E})=0$$

$$\delta\tilde{\psi}_\mu=\Big([\partial_\mu+\frac{1}{4}\omega_{\mu\hat{\alpha}\hat{\beta}}\tilde{\gamma}^{\hat{\alpha}\hat{\beta}}+\chi_7\frac{\chi}{8\sqrt{3}}(\tilde{\gamma}_{\mu\alpha\beta}F^{\alpha\beta}-4F_{\mu\alpha}\tilde{\gamma}^\alpha)\tilde{\gamma}_{\hat{x}_1\hat{x}_2}\\+\frac{1}{2L}(\tilde{\gamma}_\mu-\chi_7\sqrt{3}\chi A_\mu\tilde{\gamma}_{\hat{x}_1\hat{x}_2})\Big]\tilde{\epsilon}\Big)\otimes E$$

$$\tilde{\gamma}_\mu=\gamma_\mu\otimes g_1,\tilde{\gamma}_{\hat{x}_1\hat{x}_2}=1_4\otimes g_2,\tilde{\epsilon}=\epsilon\otimes\eta$$

$$\Big[\partial_\mu+\frac{1}{4}\omega_{\mu\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}\hat{\beta}}\otimes g_1^2+\chi_7\frac{\chi}{8\sqrt{3}}(\gamma_{\mu\alpha\beta}F^{\alpha\beta}\otimes g_1^3g_2-4F_{\mu\alpha}\gamma^\alpha\otimes g_1g_2)\\-\frac{1}{2L}(-\gamma_\mu\otimes g_1+\chi_7\sqrt{3}\chi A_\mu1_4\otimes g_2)\Big]\epsilon\otimes\eta$$

$$g_1\eta=-\eta,g_2\eta=\mathrm{i}\chi_7\eta$$

$$\Gamma^{12}\mathcal{E}=-\mathrm{i}\mathfrak{E},\Gamma^5\mathcal{E}=\mathcal{E}:\Rightarrow\mathcal{E}_1=\mathcal{E}_2=\mathcal{E}_3=0$$

$$g_1=-\chi_7\sigma_1,g_2=\mathrm{i}\sigma_1$$

$$\varepsilon_{01234}=\sqrt{-g},\varepsilon^{01234}=\frac{s}{\sqrt{-g}},$$

$$I=\frac{1}{16\pi G_5}\int\,\,\,\mathrm{d}^5x\sqrt{-g}\Big(R+\frac{12}{L^2}-\frac{\chi^2}{4}F^{\mu\nu}F_{\mu\nu}+\frac{s\chi^3}{12\sqrt{3}}\varepsilon^{\nu\alpha\beta\gamma\delta}A_\nu F_{\alpha\beta}F_{\gamma\delta}\Big)$$

$$t\rightarrow -\mathrm{i}\tau$$

$$T_{0ij\dots}\rightarrow \mathrm{i}T_{0ij\dots},T^{0ij\dots}\rightarrow -\mathrm{i}T^{0ij\dots},$$



$$d^5x \rightarrow -id^5x, \sqrt{-g} \rightarrow \sqrt{g}, R \rightarrow R, F^{\mu\nu}F_{\mu\nu} \rightarrow F^{\mu\nu}F_{\mu\nu}$$

$$\varepsilon_{0ijkl} \rightarrow (-i) \cdot i\varepsilon_{0ijkl} = \varepsilon_{0ijkl}, \varepsilon^{0ijkl} \rightarrow i \cdot (-i) \cdot s\varepsilon^{0ijkl} = s\varepsilon^{0ijkl},$$

$$\varepsilon_{01234} = \sqrt{g}, \varepsilon^{01234} = \frac{1}{\sqrt{g}}$$

$$\begin{aligned}\varepsilon^{\nu\alpha\beta\gamma\delta}A_{\nu}F_{\alpha\beta}F_{\gamma\delta} &= (\varepsilon^{0ijkl}A_0F_{ij}F_{kl} - \varepsilon^{0ijkl}A_iF_{0j}F_{kl} + \cdots) \\ &\rightarrow i s (\varepsilon^{0ijkl}A_0F_{ij}F_{kl} - \varepsilon^{0ijkl}A_iF_{0j}F_{kl} + \cdots) \\ &= i s \varepsilon^{\nu\alpha\beta\gamma\delta}A_{\nu}F_{\alpha\beta}F_{\gamma\delta}\end{aligned}$$

$$I_E = -\frac{1}{16\pi G_5} \int d^5x \sqrt{g} \left(R + \frac{12}{L^2} - \frac{\chi^2}{4} F^{\mu\nu}F_{\mu\nu} + \frac{i\chi^3}{12\sqrt{3}} \varepsilon^{\nu\alpha\beta\gamma\delta} A_{\nu}F_{\alpha\beta}F_{\gamma\delta} \right)$$

$$\begin{aligned}R_{\mu\nu} + \frac{4}{L^2}g_{\mu\nu} - \frac{\chi^2}{2}F_{\mu}^{\alpha}F_{\nu\alpha} + \frac{\chi^2}{12}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} &= 0 \\ \nabla_{\mu}F^{\mu\nu} + \frac{i\chi}{4\sqrt{3}}\varepsilon^{\nu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} &= 0\end{aligned}$$

$$r=r_+ + \varrho^2,$$

$$\begin{aligned}ds^2_{\text{CCLP},\mathcal{H}} &= \tilde{g}_{\varrho\varrho} \Big(d\varrho^2 - \Big(\frac{2\pi}{\beta^{\mathcal{H}}}\Big)^2 \varrho^2 dt^2 \Big) + \tilde{g}_{\eta\eta} d\eta^2 + 2\tilde{g}_{12} \big(d\xi_1 - \Omega_1^{\mathcal{H}} dt \big) \big(d\xi_2 - \Omega_2^{\mathcal{H}} dt \big) \\ &\quad + \tilde{g}_{11} \big(d\xi_1 - \Omega_1^{\mathcal{H}} dt \big)^2 + \tilde{g}_{22} \big(d\xi_2 - \Omega_2^{\mathcal{H}} dt \big)^2\end{aligned}$$

$$\begin{aligned}\beta^{\mathcal{H}} &= \frac{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(ab+q)^2} \\ \Omega_1^{\mathcal{H}} &= \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}, \Omega_2^{\mathcal{H}} = \Omega_1^{\mathcal{H}}|_{a\leftrightarrow b},\end{aligned}$$

$$\tilde{g}_{\varrho\varrho}=\frac{4\rho_+^2}{\Delta_{r,+}'},\tilde{g}_{\eta\eta}=\frac{\rho_+^2}{\Delta_{\eta}},$$

$$\begin{aligned}\frac{\tilde{g}_{11}}{\mathcal{S}_{\eta}^2} = & -\frac{(\Delta_{\eta}-\Xi_a)(a^2f_++2abq\Xi_a\rho_+^2)-(a^2+r_+^2)\Xi_a(\Xi_a-\Xi_b)\rho_+^4}{\Xi_a^2(\Xi_a-\Xi_b)\rho_+^4} \\ & -\frac{(\Delta_{\eta}-\Xi_a)[a^2f_+' \rho_+^2-2a(af_++bq\Xi_a\rho_+^2)(\rho_+^2)']-2r_+\Xi_a(\Xi_a-\Xi_b)\rho_+^6}{\Xi_a^2(\Xi_a-\Xi_b)\rho_+^6}\varrho^2,\end{aligned}$$

$$\begin{aligned}\frac{\tilde{g}_{12}}{\mathcal{S}_{\eta}^2\mathcal{C}_{\eta}^2} = & \frac{abf_++(b^2\Xi_a+a^2\Xi_b)q\rho_+^2}{\Xi_a\Xi_b\rho_+^4} \\ & +\frac{ab[f_+' \rho_+^2-2f_+(\rho_+^2)']-(b^2\Xi_a+a^2\Xi_b)\rho_+^2(\rho_+^2)'}{\Xi_a\Xi_b\rho_+^6}\varrho^2,\end{aligned}$$

$$\begin{aligned}\frac{\tilde{g}_{22}}{\mathcal{C}_{\eta}^2} = & \frac{(\Delta_{\eta}-\Xi_b)(b^2f_++2abq\Xi_b\rho_+^2)+(b^2+r_+^2)\Xi_b(\Xi_a-\Xi_b)\rho_+^4}{\Xi_b^2(\Xi_a-\Xi_b)\rho_+^4} \\ & +\frac{(\Delta_{\eta}-\Xi_b)[b^2f_+' \rho_+^2-2b(bf_++aq\Xi_b\rho_+^2)(\rho_+^2)'] + 2r_+\Xi_b(\Xi_a-\Xi_b)\rho_+^6}{\Xi_b^2(\Xi_a-\Xi_b)\rho_+^6}\varrho^2,\end{aligned}$$



$$f_+ = f(r_+, \eta), \rho_+^2 = \rho^2(r_+, \eta), S_\eta = \sin \eta, C_\eta = \cos \eta$$

$$f'_+ = \frac{\partial f}{\partial r}(r_+, \eta), (\rho_+^2)' = \frac{\partial \rho^2}{\partial r}(r_+, \eta), \Delta'_{r,+} = \frac{\partial \Delta_r}{\partial r}(r_+)$$

$$r = \hat{r} \left(1 + \frac{1}{\hat{r}^2} f_1(\hat{\eta}) + \frac{1}{\hat{r}^4} f_2(\hat{\eta}) \right), \eta = \hat{\eta} + \frac{1}{\hat{r}^4} h_1(\hat{\eta}) + \frac{1}{\hat{r}^6} h_2(\hat{\eta})$$

$$\begin{matrix} \hat{g}_{tt} & \hat{g}_{t1} & \hat{g}_{t2} & \hat{g}_{\hat{r}\hat{r}} & \hat{g}_{\hat{r}\hat{\eta}} & \hat{g}_{\hat{\eta}\hat{\eta}} & \hat{g}_{11} & \hat{g}_{12} \\ O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-6}) & O(\hat{r}^{-5}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2}) & O(\hat{r}^{-2}) \end{matrix} O(\hat{r}^{-2}).$$

$$f_1(\hat{\eta}) = -\frac{1}{4g^2} [2 - \Xi_a - \Xi_b + \Delta_\eta(\hat{\eta})]$$

$$f_2(\hat{\eta}) = \frac{1}{8g^4} [1 - 2g^2m - \Xi_a \sin^2 \hat{\eta} - \Xi_b \cos^2 \hat{\eta} - \cos^2 \hat{\eta} \sin^2 \hat{\eta} (\Xi_a - \Xi_b)^2]$$

$$h_1(\hat{\eta}) = -\frac{\Xi_a - \Xi_b}{8g^4} \cos \hat{\eta} \sin^2 \hat{\eta} \Delta_\eta(\hat{\eta})$$

$$h_2(\hat{\eta}) = -\frac{\Xi_a - \Xi_b}{16g^6} \cos \hat{\eta} \sin^2 \hat{\eta} [2\Xi_a^2 \cos^2 \hat{\eta} - \Xi_a \Xi_b + 2\Xi_b^2 \sin^2 \hat{\eta} - 3\cos^2 \hat{\eta} \sin^2 \hat{\eta} (\Xi_a - \Xi_b)^2]$$

$$\begin{aligned} ds_{\text{CCLP},\mathcal{A}}^2 = & \frac{1}{g^2 \hat{r}^2} d\hat{r}^2 + \hat{g}_{tt} dt^2 + 2\hat{g}_{t1} dt d\xi_1 + 2\hat{g}_{t2} dt d\xi_2 + \hat{g}_{\hat{\eta}\hat{\eta}} d\hat{\eta}^2 \\ & + \hat{g}_{11} d\xi_1^2 + 2\hat{g}_{12} d\xi_1 d\xi_2 + \hat{g}_{22} d\xi_2^2 \end{aligned}$$

$$\begin{aligned} \hat{g}_{tt} = & -\frac{g^2 \hat{\Delta}_\eta \hat{r}^2}{\Xi_a \Xi_b} - \frac{\hat{\Delta}_\eta (\Xi_a + \Xi_b - \hat{\Delta}_\eta)}{2\Xi_a \Xi_b} \\ & + \frac{2\hat{\Delta}_\eta^2 (abg^2q + m) - \frac{\Xi_a \Xi_b \hat{\Delta}_\eta (-2\hat{\Delta}_\eta (\Xi_a + \Xi_b) + (\Xi_a + \Xi_b)^2 + \hat{\Delta}_\eta^2 + 8g^2m)}{16g^2}}{\Xi_a^2 \Xi_b^2 \hat{r}^2} \end{aligned}$$

$$\frac{\hat{g}_{t1}}{\hat{S}_\eta^2} = -\frac{\hat{\Delta}_\eta (a^2bg^2q + 2am + bq)}{\Xi_a^2 \Xi_b \hat{r}^2}$$

$$\frac{\hat{g}_{t2}}{\hat{C}_\eta^2} = -\frac{\hat{\Delta}_\eta (ab^2g^2q + aq + 2bm)}{\Xi_a \Xi_b^2 \hat{r}^2},$$

$$\hat{g}_{\hat{\eta}\hat{\eta}} = \frac{\hat{r}^2}{\hat{\Delta}_\eta} + \frac{\Xi_a + \Xi_b - 3\hat{\Delta}_\eta}{2g^2 \hat{\Delta}_\eta} + \frac{-6\hat{\Delta}_\eta (\Xi_a + \Xi_b) + (\Xi_a + \Xi_b)^2 + 9\hat{\Delta}_\eta^2 + 8g^2m}{16g^4 \hat{\Delta}_\eta \hat{r}^2}$$

$$\begin{aligned} \frac{\hat{g}_{11}}{\hat{S}_\eta^2} = & \frac{\hat{r}^2}{\Xi_a} - \frac{\Xi_a - \Xi_b + \hat{\Delta}_\eta}{2g^2 \Xi_a} \\ & - \frac{8g^2 \left(4a^2g^2m\hat{\Delta}_\eta + 4abg^2q(\hat{\Delta}_\eta - \Xi_a) + m\Xi_a(3\Xi_a + \Xi_b - 4) \right)}{\Xi_a - \Xi_b} - \frac{\Xi_a(\Xi_a - \Xi_b + \hat{\Delta}_\eta)^2}{16g^4 \Xi_a^2 \hat{r}^2} \end{aligned}$$

$$\frac{\hat{g}_{12}}{\hat{S}_\eta^2 \hat{C}_\eta^2} = \frac{a^2q + 2abm + b^2q}{\Xi_a \Xi_b \hat{r}^2},$$

$$\begin{aligned} \frac{\hat{g}_{22}}{\hat{C}_\eta^2} = & \frac{\hat{r}^2}{\Xi_b} + \frac{\Xi_a - \Xi_b - \hat{\Delta}_\eta}{2g^2 \Xi_b} \\ & + \frac{8g^2 \left(4b^2g^2m\hat{\Delta}_\eta + 4abg^2q(\hat{\Delta}_\eta - \Xi_b) + m\Xi_b(\Xi_a + 3\Xi_b - 4) \right)}{\Xi_a - \Xi_b} + \frac{\Xi_b(-\Xi_a + \Xi_b + \hat{\Delta}_\eta)^2}{16g^4 \Xi_b^2 \hat{r}^2} \end{aligned}$$



$$\widehat{\Delta}_\eta \equiv \Delta_\eta(\hat{\eta}), \hat{S}_\eta = \sin \hat{\eta}, \hat{C}_\eta = \cos \hat{\eta}$$

$$\Omega_i=\Omega_i^{\mathcal{H}}-\Omega_i^{\mathcal{B}},\Omega_i^{\mathcal{H}}=\left.\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right|_{r\rightarrow r_+},\Omega_i^{\mathcal{B}}=\left.\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right|_{r\rightarrow\infty},i=1,2$$

$$\Omega_1=\frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq},\Omega_2=\frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}$$

$$\mathrm{d} s_{\mathrm{CCLP},\mathcal{H},\mathcal{E}}^2=\tilde{g}_{\varrho\varrho}\left(\mathrm{d} \varrho^2+\left(\frac{2\pi}{\beta^{\mathcal{H}}}\right)^2\varrho^2\mathrm{d} \tau^2\right)+\tilde{g}_{\eta\eta}\mathrm{d} \eta^2+2\tilde{g}_{12}(\mathrm{d} \xi_1+\mathrm{i}\Omega_1\mathrm{d} t)(\mathrm{d} \xi_2+\mathrm{i}\Omega_2\mathrm{d} t)\\ +\tilde{g}_{11}(\mathrm{d} \xi_1+\mathrm{i}\Omega_1\mathrm{d} t)^2+\tilde{g}_{22}(\mathrm{d} \xi_2+\mathrm{i}\Omega_2\mathrm{d} t)^2$$

$$\tau\sim\tau+\beta^{\mathcal{H}}.$$

$$(\tau,\xi_1,\xi_2)\sim(\tau+\beta^{\mathcal{H}},\xi_1-\mathrm{i}\Omega_1\beta^{\mathcal{H}},\xi_2-\mathrm{i}\Omega_2\beta^{\mathcal{H}}).$$

$$\beta=\frac{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(ab+q)^2}$$

$$V=V^\mu\partial_\mu=\frac{\partial}{\partial t}+\Omega_1\frac{\partial}{\partial\xi_1}+\Omega_2\frac{\partial}{\partial\xi_2};\;\nabla_\mu V_\nu+\nabla_\nu V_\mu=0$$

$$V_\mu V^\mu|_{r=r_+}=0$$

$$\kappa_{\mathcal{H}}=\sqrt{-\frac{1}{2}\nabla^\mu V^\nu\nabla_\mu V_\nu}\Big|_{r\rightarrow r_+}$$

$$\Phi=V^\mu A_\mu|_{r=r_+}-V^\mu A_\mu|_{r\rightarrow\infty}$$

$$V^\mu A_\mu|_{r=r_+}=\frac{\sqrt{3}qr_+^2}{\chi[(r_+^2+a^2)(r_+^2+b^2)+abq]}+\alpha,\,V^\mu A_\mu|_{r\rightarrow\infty}=\alpha$$

$$\Phi=\frac{\sqrt{3}qr_+^2}{\chi[(r_+^2+a^2)(r_+^2+b^2)+abq]}$$

$$\alpha=-\Phi$$

$$A^\mu A_\mu|_{r=r_+}=0,\,V^\mu A_\mu|_{r=r_+}=0$$

$$K_{(t)}=\frac{\partial}{\partial t},K_{(\xi_1)}=\frac{\partial}{\partial \xi_1},K_{(\xi_2)}=\frac{\partial}{\partial \xi_2}.$$

$$\tilde{K}_\odot=K_\odot^\mu g_{\mu\nu}\mathrm{d} x^\nu$$

$$J_i=\lim_{r\rightarrow\infty}\frac{1}{16\pi G_5}\int_{S^3}\star_5\,\,\mathrm{d}\tilde{K}_{(\xi_i)},i=1,2$$

$$J_1 = \frac{\pi[2am + bq(1 + a^2g^2)]}{4\Xi_a^2\Xi_bG_5}, J_2 = \frac{\pi[2bm + aq(1 + b^2g^2)]}{4\Xi_a\Xi_b^2G_5}.$$

$$0=\sigma\left[\mathrm{d}\star_5F-\frac{1}{\sqrt{3}}F\wedge F\right]=\sigma\left[\mathrm{d}\star_5F-\frac{1}{\sqrt{3}}\mathrm{d}(F\wedge A)\right]=\mathrm{d}\left[\sigma\left(\star_5F-\frac{1}{\sqrt{3}}F\wedge A\right)\right]$$

$$Q=\lim_{r\rightarrow\infty}\frac{\sigma}{16\pi G_5}\int_{S^3}\left(\star_5F-\frac{1}{\sqrt{3}}F\wedge A\right)=-\frac{\sigma\sqrt{3}\pi q}{4\chi\Xi_a\Xi_bG_5}$$

$$Q=\frac{\chi\sqrt{3}\pi q}{4\Xi_a\Xi_bG_5}$$

$$\begin{aligned} I_{\rm GH} &= \frac{1}{8\pi G_5} \int \mathrm{d}^4x \sqrt{-\gamma} K \\ I_{\rm ct} &= \frac{1}{8\pi G_5} \int \mathrm{d}^4x \sqrt{-\gamma} \left(-3g - \frac{1}{4g} \mathcal{R}\right) \\ I_{\rm fct} &= \frac{1}{8\pi G_5} \int \mathrm{d}^4x \sqrt{-\gamma} (\zeta' \mathcal{R}^2 - \zeta'' F_{ij} F^{ij}) \end{aligned}$$

$$K_{ij}=\frac{g\hat{r}}{2}\frac{\partial}{\partial\hat{r}}\gamma_{ij}, K=\gamma^{ij}K_{ij}$$

$$I_{\rm an}=\frac{1}{8\pi G_5}\int\mathrm{d}^4x\sqrt{-\gamma}\frac{1}{16g^3}\Big(\mathcal{R}_{ij}\mathcal{R}^{ij}-\frac{1}{3}\mathcal{R}^2-4F_{ij}F^{ij}\Big)\log e^{-2\hat{R}_0}$$

$$\mathcal{I}_{\rm hr} = I_{\mathcal{E}} + I_{\rm GH}^{\mathcal{E}} + I_{\rm ct}^{\mathcal{E}} + I_{\rm fct}^{\mathcal{E}},$$

$$\begin{aligned} T_{ij} &= -\frac{2}{\sqrt{-\gamma}\delta\gamma^{ij}}(\mathcal{L}_{\rm GH}+\mathcal{L}_{\rm ct}+\mathcal{L}_{\rm fct}) \\ &= \frac{1}{8\pi G_5}\Big[K_{ij}-\gamma_{ij}K+3g\gamma_{ij}-\frac{1}{2g}\mathcal{R}_{ij}+\frac{1}{4g}\mathcal{R}\gamma_{ij} \\ &\quad +4\zeta'\Big(\mathcal{R}_{ij}\mathcal{R}-\frac{1}{4}\gamma_{ij}\mathcal{R}^2-\nabla_i\nabla_j\mathcal{R}+\gamma_{ij}\nabla_k\nabla^k\mathcal{R}\Big)\Big] \end{aligned}$$

$$\mathcal{Q}[K]=\int_0^{2\pi}\mathrm{d}\xi_1\int_0^{2\pi}\mathrm{d}\xi_2\int_0^{\pi/2}\mathrm{d}\hat{\eta}\sqrt{-\gamma}T_j^iK^j$$

$$J_1=\mathcal{Q}[\partial_{\xi_1}]=\frac{\pi(2am+bq(1+a^2g^2))}{4\Xi_a^2\Xi_bG_5}, J_2=\mathcal{Q}[\partial_{\xi_2}]=J_1|_{b\leftrightarrow a}$$

$$E=\mathcal{Q}[\partial_t]=(E_{\rm AdS}-\tilde{\zeta}')+\frac{m\pi(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)+2abq^2\pi(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2G_5},$$

$$\epsilon=\exp\left\{\frac{\mathrm{i}}{2}\left[g(1+\sqrt{3}\chi\alpha)t+\xi_1+\xi_2\right]\right\}\frac{1}{\rho}[\tilde{m}\rho^2-(1+\tilde{m})(r^2-r_*^2)]^{1/2}\epsilon_0$$

$$\frac{\mathrm{i}}{2}(\gamma^{12}-\gamma^{34})\epsilon_0=\epsilon_0$$

$$\begin{aligned}
e^0 &= -\frac{g^2 \Delta_\eta}{\Xi_a \Xi_b} \left[\frac{(a+b)(1+\tilde{m})(1+g^2 r_*^2)}{g^3 \rho^2} - (g^{-2} + r_*^2) - \frac{\rho^2}{1+\tilde{m}} \left(1 - \frac{\tilde{m} \rho}{h} \right) \right] dt \\
&\quad + \frac{g \sin^2 \eta}{\Xi_a} \left[\frac{a(a+b)(1+\tilde{m})(1+g^2 r_*^2)}{g^2 \rho^2} - (a^2 + r_*^2) - \frac{\rho^2}{1+\tilde{m}} \left(1 - \frac{\tilde{m} \rho}{h} \right) \right] d\xi_1 \\
&\quad + \frac{g \cos^2 \eta}{\Xi_b} \left[\frac{b(a+b)(1+\tilde{m})(1+g^2 r_*^2)}{g^2 \rho^2} - (b^2 + r_*^2) - \frac{\rho^2}{1+\tilde{m}} \left(1 - \frac{\tilde{m} \rho}{h} \right) \right] d\xi_1 \\
e^1 &= \sqrt{\frac{\rho^2}{\Delta_r}} dr \\
e^2 &= \frac{r \sqrt{\Delta_r \rho^2}}{h} \left(-\frac{g \Delta_\eta}{\Xi_a \Xi_b} dt + \frac{\sin^2 \eta}{\Xi_a} d\xi_2 + \frac{\cos^2 \eta}{\Xi_b} d\xi_1 \right) \\
e^3 &= -\sqrt{\frac{\rho^2}{\Delta_\eta}} d\eta \\
e^4 &= \frac{\sin \eta \cos \eta \sqrt{\Delta_\eta \rho^2}}{h} \left\{ -\frac{g(\Xi_a - \Xi_b)}{\Xi_a \Xi_b} (g^{-2} + (1+\tilde{m})r_*^2 - r^2) dt \right. \\
&\quad \left. + \frac{1}{\Xi_a} (a^2 + (1+\tilde{m})r_*^2 - r^2) d\xi_1 + \frac{1}{\Xi_b} (b^2 + (1+\tilde{m})r_*^2 - r^2) d\xi_2 \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_V \epsilon &= V^\mu (\partial_\mu \epsilon + \omega_{\mu ab} \gamma^{ab} \epsilon) - \frac{1}{4} \nabla_\mu V_\nu \gamma^{\mu\nu} \epsilon \\
&= \frac{i}{2} \left(g + \Omega_1 + \Omega_2 + g\chi\sqrt{3}\alpha + 2V^\mu A_\mu \Big|_{r=r_+} \right) \epsilon
\end{aligned}$$

$$\mathcal{L}_V \epsilon = \frac{\pi}{\beta} \epsilon \Rightarrow e^{t\mathcal{L}_V} \epsilon \Big|_{t=t_0}^{t=t_0-i\beta} = e^{-i\beta\mathcal{L}_V} \epsilon = -\epsilon$$

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} \, \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}$$

$$A=A_\mu\mathrm{d}x^\mu, F=\frac{1}{2!}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, F=\mathrm{d}A=\partial_{[\mu}A_{\nu]}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$

$$T_{[\mu_1 \dots \mu_p]} = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \text{sign}(\sigma) T_{\mu_{\sigma(1)} \dots \mu_{\sigma(p)}}$$

$$F_{\mu\nu} = 2! \, \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\star \omega = \frac{1}{p! \, (D-p)!} \omega_{\mu_1 \dots \mu_p} \varepsilon^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_{D-p}} \, \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_{D-p}}$$

$$\varepsilon_{\mu_1 \dots \mu_D} = \sqrt{|g|} \eta_{\mu_1 \dots \mu_D}, \varepsilon^{\mu_1 \dots \mu_D} = \frac{\text{sgn}(g)}{\sqrt{|g|}} \eta^{\mu_1 \dots \mu_D}$$

$$R^\rho_{\mu\nu\,\sigma} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma},$$

$$R_{\mu\nu} = R^\rho_{\mu\nu\rho} = R^\rho_{\mu\rho\nu}, R = R^\mu_\mu$$



$$e^{-1}\mathcal{L} = \frac{1}{2}R - \partial_\mu\chi\partial^\mu\chi - \frac{1}{4}\sinh^2(2\chi)D_\mu\psi D^\mu\psi - \sum_{i=1}^3\partial_\mu\lambda_i\partial^\mu\lambda_i - g^2\mathcal{P} \\ - \frac{1}{4}\left[e^{-2(\lambda_1+\lambda_2+\lambda_3)}F_{\mu\nu}^0F^{0\mu\nu} + e^{-2(\lambda_1-\lambda_2-\lambda_3)}F_{\mu\nu}^1F^{1\mu\nu} + e^{2(\lambda_1-\lambda_2+\lambda_3)}F_{\mu\nu}^2F^{2\mu\nu} + e^{2(\lambda_1+\lambda_2-\lambda_3)}F_{\mu\nu}^3F^{3\mu\nu}\right]$$

$$D\psi \equiv d\psi - g(A^0 - A^1 - A^2 - A^3).$$

$$\mathcal{P} = \frac{1}{2}\left(\frac{\partial W}{\partial \chi}\right)^2 + \frac{1}{2}\sum_{i=1}^3\left(\frac{\partial W}{\partial \lambda_i}\right)^2 - \frac{3}{2}W^2$$

$$W = \frac{2e^{2(\lambda_1+\lambda_2+\lambda_3)}\sinh^2\chi - \cosh^2\chi(e^{2(\lambda_1+\lambda_2+\lambda_2)} + e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})}{2e^{\lambda_1+\lambda_2+\lambda_3}}.$$

$$A_R = -g(A^0 + A^1 + A^2 + A^3)$$

$$A_m = g(A^0 - A^1 - A^2 - A^3)$$

$$A_{F_1} = \frac{1}{2}g(A^1 - A^2)$$

$$A_{F_2} = \frac{1}{\sqrt{3}}g(A^1 + A^2 - 2A^3)$$

$$\left[2\nabla_\mu-iB_\mu-\frac{g}{\sqrt{2}}W\gamma_\mu-i\frac{1}{2\sqrt{2}}H_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\right]\epsilon=0$$

$$H_{\mu\nu}\equiv\bar{F}_{\mu\nu}^{78}=\frac{1}{2}\big(e^{-\lambda_1-\lambda_2-\lambda_3}F_{\mu\nu}^0+e^{-\lambda_1+\lambda_2+\lambda_3}F_{\mu\nu}^1+e^{\lambda_1-\lambda_2+\lambda_3}F_{\mu\nu}^2+e^{\lambda_1+\lambda_2-\lambda_3}F_{\mu\nu}^3\big)$$

$$B_\mu\equiv -g(A_\mu^0+A_\mu^1+A_\mu^2+A_\mu^3)-\frac{1}{2}(\cosh(2\chi)-1)D_\mu\psi$$

$$\left[\gamma^\mu\partial_\mu\lambda_1+\frac{g}{\sqrt{2}}\partial_{\lambda_1}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{12}\right]\epsilon=0$$

$$\left[\gamma^\mu\partial_\mu\lambda_2+\frac{g}{\sqrt{2}}\partial_{\lambda_2}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{34}\right]\epsilon=0$$

$$\left[\gamma^\mu\partial_\mu\lambda_3+\frac{g}{\sqrt{2}}\partial_{\lambda_3}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{56}\right]\epsilon=0$$

$$\left[\gamma^\mu\partial_\mu\chi+\frac{g}{\sqrt{2}}\partial_\chi W+i\frac{1}{2}\partial_\chi B_\mu\gamma^\mu\right]\epsilon=0$$

$$\tanh\chi=\frac{1}{\sqrt{3}},\psi=\frac{\pi}{2},\tanh\lambda_i=2-\sqrt{3},\varphi_i=\pi$$

$$L_{AdS_4}^2 = -\frac{3}{g^2\mathcal{P}_*} = \begin{cases} \frac{1}{2g^2}, & SO(8) \\ \frac{2}{3\sqrt{3}g^2}, & SU(3)\times U(1)_R \end{cases}$$

$$\mathcal{F}_{S^3} = \frac{\pi L_{AdS_4}^2}{2G_N^{(4)}}$$



$$\mathcal{F}_{S^3}^{\text{ABJM}} = \frac{\sqrt{2}\pi}{3} N^{3/2}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}} = \frac{4\sqrt{2}\pi}{9\sqrt{3}} N^{3/2}$$

$$\frac{\mathcal{F}_{S^3}}{L_{AdS_4}^2} = \frac{2\sqrt{2}\pi}{3} g^2 N^{3/2}$$

$$ds^2 = e^{2V} ds_{AdS_2}^2 + f^2 dy^2 + h^2 dz^2$$

$$A^\alpha = a^\alpha dz$$

$$B_\mu dx^\mu \equiv B_z dz$$

$$e^a = e^V \bar{e}^a, e^2 = f dy, e^3 = h dz$$

$$F_{23}^\alpha = f^{-1} h^{-1} (a^\alpha)'.$$

$$e^{2V}\big(e^{-2\lambda_1+2\lambda_2+2\lambda_3}F_{23}^1-e^{2\lambda_1-2\lambda_2+2\lambda_3}F_{23}^2\big)=\mathcal{E}_{F_1}$$

$$e^{2V}\big(e^{2\lambda_1-2\lambda_2+2\lambda_3}F_{23}^2-e^{2\lambda_1+2\lambda_2-2\lambda_3}F_{23}^3\big)=\mathcal{E}_{F_2}$$

$$e^{2V}\big(e^{2\lambda_1+2\lambda_2-2\lambda_3}F_{23}^3-e^{-2\lambda_1+2\lambda_2+2\lambda_3}F_{23}^1\big)=\mathcal{E}_{F_3}$$

$$e^{2V}\big(e^{-2\lambda_1-2\lambda_2-2\lambda_3}F_{23}^0+e^{-2\lambda_1+2\lambda_2+2\lambda_3}F_{23}^1\big)=\mathcal{E}_{R_1}$$

$$e^{2V}\big(e^{-2\lambda_1-2\lambda_2-2\lambda_3}F_{23}^0+e^{2\lambda_1-2\lambda_2+2\lambda_3}F_{23}^2\big)=\mathcal{E}_{R_2}$$

$$e^{2V}\big(e^{-2\lambda_1-2\lambda_2-2\lambda_3}F_{23}^0+e^{2\lambda_1+2\lambda_2-2\lambda_3}F_{23}^3\big)=\mathcal{E}_{R_3}$$

$$\left(e^{2V+2\lambda_1+2\lambda_2-2\lambda_3}F_{23}^3\right)'=-e^{2V}fh^{-1}\frac{1}{2}\sinh^2\left(2\chi\right)D_z\psi,$$

$$\gamma^m = \Gamma^m \otimes \sigma^3, \gamma^2 =]_2 \otimes \sigma^1, \gamma^3 =]_2 \otimes \sigma^2$$

$$\epsilon = \psi \otimes \chi$$

$$D_m\psi = \frac{1}{2}\kappa\Gamma_m\psi$$

$$\chi = e^{V/2}e^{isz/2}\begin{pmatrix}\sin\frac{\xi}{2} \\ \cos\frac{\xi}{2}\end{pmatrix}$$

$$f^{-1}\xi'=\sqrt{2}gW\cos\xi+\kappa e^{-V}$$

$$f^{-1}V'=\frac{g}{\sqrt{2}}W\sin\xi$$

$$f^{-1}\lambda_i'=-\frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin\xi$$

$$f^{-1}\chi'=-\frac{g}{\sqrt{2}}\frac{\partial_\chi W}{\sin\xi}$$

$$f^{-1}\frac{h'}{h}=\frac{1}{\sin\xi}\left(\kappa e^{-V}\cos\xi+\frac{gW}{\sqrt{2}}(1+\cos^2\xi)\right)$$

$$(s - B_z)\sin \xi = -\sqrt{2}gWh\cos \xi - \kappa h e^{-V}$$

$$\sqrt{2}g\partial_\chi W\cos \xi = \partial_\chi B_z\sin \xi h^{-1}$$

$$\bar{F}_{23}^{12} = -g\partial_{\lambda_1} W\cos \xi$$

$$\bar{F}_{23}^{34} = -g\partial_{\lambda_2} W\cos \xi$$

$$\bar{F}_{23}^{56} = -g\partial_{\lambda_3} W\cos \xi$$

$$H_{23} = -gW\cos \xi - \sqrt{2}\kappa e^{-V}$$

$$he^{-V} = k\sin \xi$$

$$\xi' = -k^{-1}(s - B_z)(e^{-V}f)$$

$$(s - B_z) = -k[\sqrt{2}gWe^V\cos \xi + \kappa]$$

$$\sqrt{2}g\partial_\chi W\cos \xi = k^{-1}e^{-V}\partial_\chi B_z$$

$$\mathcal{E}_{R_1} = e^V[2ge^V\cos \xi - \sqrt{2}\kappa e^{-\lambda_1}\cosh(\lambda_2 + \lambda_3)],$$

$$\mathcal{E}_{R_2} = e^V[2ge^V\cos \xi - \sqrt{2}\kappa e^{-\lambda_2}\cosh(\lambda_3 + \lambda_1)],$$

$$\mathcal{E}_{R_3} = e^V[2ge^V\cos \xi - \sqrt{2}\kappa e^{-\lambda_3}\cosh(\lambda_1 + \lambda_2)],$$

$$\mathcal{E}_{F_1} = \sqrt{2}\kappa e^V e^{\lambda_3}\sinh(\lambda_1 - \lambda_2)$$

$$\mathcal{E}_{F_2} = \sqrt{2}\kappa e^V e^{\lambda_1}\sinh(\lambda_2 - \lambda_3)$$

$$\mathcal{E}_{F_3} = \sqrt{2}\kappa e^V e^{\lambda_2}\sinh(\lambda_3 - \lambda_1)$$

$$f = e^V$$

$$ds^2 = e^{2V}[ds_{AdS_2}^2 + ds_\Sigma^2]$$

$$ds_\Sigma^2 = dy^2 + k^2\sin^2 \xi dz^2$$

$$\Delta z = 2\pi$$

$$\cos \xi_{N,S} = (-1)^{t_{N,S}} 2\pi \left(1 - \frac{1}{n_{N,S}}\right) |(k\sin \xi)'|_{N,S} = \frac{1}{n_{N,S}}$$

$$h \geq 0, \Leftrightarrow k\sin \xi \geq 0$$

$$(k\sin \xi)'|_{N,S} = \frac{(-1)^{l_{N,S}}}{n_{N,S}}, l_N = 0, l_S = 1$$

$$\cos \xi|_{N,S} = (-1)^{t_{N,S}}; \quad (t_N, t_S) = (1,1) \quad \text{or} \quad (0,0)$$

$$(t_N, t_S) = (1,0) \quad \text{or} \quad (0,1)$$

$$(s - B_z)|_{N,S} = \frac{1}{n_{N,S}} (-1)^{l_{N,S} + t_{N,S} + 1}$$

$$F^R = dB + d\left(\frac{1}{2}(\cosh(2\chi) - 1)D\psi\right)$$



$$\frac{1}{2\pi} \int_{\bar{\Sigma}} F^R \equiv \frac{1}{2\pi} \int_{\bar{\Sigma}} -g(F^0 + F^1 + F^2 + F^3) = \frac{n_N(-1)^{t_S+1} + n_S(-1)^{t_N+1}}{n_N n_S}$$

$$\partial_z B = -\frac{1}{2} \sinh(2\chi) D_z \psi$$

$$\partial_\chi B_z|_{N,S} = \partial_\chi W|_{N,S} = 0.$$

$$\chi|_N, \chi|_S \neq 0, \Rightarrow D_z \psi|_N = D_z \psi|_S = 0$$

$$\frac{1}{2\pi} \int_{\Sigma} g(F^0 - F^1 - F^2 - F^3) = (D_z \psi)|_{y_N^S} = 0$$

$$(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3} - e^{2\lambda_1+2\lambda_2+2\lambda_3})|_{N,S} = 0, \Rightarrow W|_{N,S} = -e^{\lambda_1+\lambda_2+\lambda_3}|_{N,S}.$$

$$M_{(1)} \equiv g e^{\lambda_1+\lambda_2+\lambda_3} e^V, M_{(2)} \equiv -\frac{\kappa}{\sqrt{2}} M_{(1)} + 2M_{(1)}^2 \cos \xi$$

$$\mathcal{E}_{R_i} = -\frac{\kappa}{\sqrt{2}g} M_{(1)} e^{-2\lambda_i} + \frac{1}{g} M_{(2)} e^{-2\lambda_1-2\lambda_2-2\lambda_3},$$

$$M_{(1)|N,S} = \frac{1}{\sqrt{2}} (-1)^{t_{N,S} \kappa} - \frac{1}{\sqrt{2} \kappa n_{N,S}} (-1)^{l_{N,S}},$$

$$M_{(2)|N,S} = \frac{1}{2} (-1)^{t_{N,S}} + \frac{1}{k^2 n_{N,S}^2} (-1)^{t_{N,S}} - \frac{3\kappa}{2\kappa n_{N,S}} (-1)^{l_{N,S}}.$$

$$\mathcal{E}_{R_1}(\lambda_{1N},\lambda_{2N})=\mathcal{E}_{R_1}(\lambda_{1S},\lambda_{2S}),$$

$$\mathcal{E}_{R_2}(\lambda_{1N},\lambda_{2N})=\mathcal{E}_{R_2}(\lambda_{1S},\lambda_{2S}),$$

$$\mathcal{E}_{R_3}(\lambda_{1N},\lambda_{2N})=\mathcal{E}_{R_3}(\lambda_{1S},\lambda_{2S}).$$

$$F_{yz}^\alpha=(a^\alpha)'=(\mathcal{I}^\alpha)'$$

$$\mathcal{I}^{(0)}\!\!\equiv\!\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{\lambda_1+\lambda_2+\lambda_3}$$

$$\mathcal{I}^{(1)}\!\!\equiv\!\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{\lambda_1-\lambda_2-\lambda_3}$$

$$\mathcal{I}^{(2)}\!\!\equiv\!\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{-\lambda_1+\lambda_2-\lambda_3}$$

$$\mathcal{I}^{(3)}\!\!\equiv\!\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$\frac{p_\alpha}{n_N n_S} \equiv \frac{1}{2\pi} \int_{\Sigma} g F^\alpha = g \mathcal{I}^\alpha|_N^S$$

$$\mathcal{I}_0|_{N,S} \equiv -(\mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)})|_{N,S}$$

$$\mathcal{I}_{\Delta_1}|_{N,S} \equiv \frac{1}{2}(\mathcal{I}^{(1)} - \mathcal{I}^{(2)})|_{N,S}$$

$$\mathcal{I}_{\Delta_2}|_{N,S} \equiv \frac{1}{\sqrt{3}}(\mathcal{I}^{(1)} + \mathcal{I}^{(2)} - 2\mathcal{I}^{(3)})|_{N,S}$$



$$\begin{aligned}
g\mathcal{J}_0|_{N,S} &= -g(\mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)})|_{N,S} \\
&= -\frac{1}{\sqrt{2}}gke^V \cos \xi (e^{\lambda_1+\lambda_2+\lambda_3} + e^{\lambda_1-\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{-\lambda_1-\lambda_2+\lambda_3})|_{N,S} \\
&= -\frac{1}{\sqrt{2}}gke^V \cos \xi e^{-\lambda_1-\lambda_2-\lambda_3} (e^{2\lambda_1+2\lambda_2+2\lambda_3} + e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})|_{N,S} \\
&= -\frac{1}{\sqrt{2}}gke^V \cos \xi 2e^{\lambda_1+\lambda_2+\lambda_3}|_{N,S} \\
&= \sqrt{2}kM_{(1)}|_{N,S}(-1)^{t_{N,S}+1}
\end{aligned}$$

$$\begin{aligned}
&g(\mathcal{J}^{(0)} - \mathcal{J}^{(1)} - \mathcal{J}^{(2)} - \mathcal{J}^{(3)})|_{N,S} \\
&= \frac{1}{\sqrt{2}}gke^V \cos \xi (e^{\lambda_1+\lambda_2+\lambda_3} - e^{\lambda_1-\lambda_2-\lambda_3} - e^{-\lambda_1+\lambda_2-\lambda_3} - e^{-\lambda_1-\lambda_2+\lambda_3})|_{N,S} \\
&= \frac{1}{\sqrt{2}}gke^V \cos \xi e^{-\lambda_1-\lambda_2-\lambda_3} (-e^{2\lambda_1+2\lambda_2+2\lambda_3} + e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})|_{N,S} = 0,
\end{aligned}$$

$$\begin{aligned}
g\mathcal{J}_0|_N^S &= -g(\mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)})|_N^S = \frac{n_N(-1)^{t_S+1} + n_S(-1)^{t_N+1}}{n_N n_S} \\
g(\mathcal{J}^{(0)} - \mathcal{J}^{(1)} - \mathcal{J}^{(2)} - \mathcal{J}^{(3)})|_N^S &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{p_{F_1}}{n_N n_S} &\equiv g\mathcal{J}_{\Delta_1}|_N^S = \frac{1}{2}g(\mathcal{J}^{(1)} - \mathcal{J}^{(2)})|_N^S \\
&= \frac{1}{2\sqrt{2}}gke^V \cos \xi (e^{\lambda_1-\lambda_2-\lambda_3} - e^{-\lambda_1+\lambda_2-\lambda_3})|_N^S \\
&= \frac{1}{2\sqrt{2}}kM_{(1)}(-1)^{t_{N,S}}(e^{-2\lambda_2-2\lambda_3} - e^{-2\lambda_3-2\lambda_1})|_N^S \\
\frac{p_{F_2}}{n_N n_S} &\equiv g\mathcal{J}_{\Delta_2}|_N^S = \frac{1}{\sqrt{3}}g(\mathcal{J}^{(1)} + \mathcal{J}^{(2)} - 2\mathcal{J}^{(3)})|_N^S \\
&= \frac{1}{\sqrt{6}}gke^V \cos \xi (e^{\lambda_1-\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} - 2e^{-\lambda_1-\lambda_2+\lambda_3})|_N^S \\
&= \frac{1}{\sqrt{6}}kM_{(1)}(-1)^{t_{N,S}}(e^{-2\lambda_2-2\lambda_3} + e^{-2\lambda_3-2\lambda_1} - 2e^{-2\lambda_1-2\lambda_2})|_N^S
\end{aligned}$$

$$\begin{aligned}
n_N &= 8, n_S = 1, p_{F_1} = 1, p_{F_2} = 2 \\
g &= 1, \kappa = -1,
\end{aligned}$$

$$\begin{aligned}
e^{-2\lambda_{1N}} &\approx 0.0452746, & e^{-2\lambda_{1S}} &\approx 0.012202, \\
e^{-2\lambda_{2N}} &\approx 0.110402, & e^{-2\lambda_{2S}} &\approx 0.101903, \\
e^{-2\lambda_{3N}} &\approx 6.39148, & e^{-2\lambda_{3S}} &\approx 8.75297, \\
k &\approx 0.631939.
\end{aligned}$$

$$\frac{1}{2G_N^{(4)}} = \frac{2\sqrt{2}}{3}g^2N^{3/2}.$$



$$\left(G_N^{(2)}\right)^{-1}=\left(G_N^{(4)}\right)^{-1} \Delta z \int_{y_N}^{y_S}|f h| d y$$

$$fh=ke^Vf\sin \xi=-\frac{k}{\kappa}(e^{2V}\cos \xi) '.$$

$$\begin{aligned} S_{\text{BH}} &= \frac{1}{4G_N^{(2)}} = \frac{2\sqrt{2}\pi}{3} N^{3/2} g^2 \left(-\frac{k}{\kappa}\right) [e^{2V} \cos \xi]_N^S \\ &= -\frac{2\sqrt{2}N^{3/2}k}{3\kappa} \left(M_{(1)}^2 \Big|_S e^{-2(\lambda_{1S}+\lambda_{2S}+\lambda_{3S})} (-1)^{t_S} - M_{(1)}^2 \Big|_N e^{-2(\lambda_{1N}+\lambda_{2N}+\lambda_{3N})} (-1)^{t_N} \right), \end{aligned}$$

$$\tanh \chi = \frac{1}{\sqrt{3}}, \psi = \frac{\pi}{2}, \tanh \lambda_i = 2 - \sqrt{3}, \varphi_i = \pi$$

$$\begin{aligned} ds^2 &= \frac{2}{3\sqrt{3}g^2} \left[\frac{y^2}{4} ds_{AdS_2}^2 + \frac{y^2}{q(y)} dy^2 + \frac{q(y)}{4y^2} c_0^2 dz^2 \right] \\ \frac{1}{3} A^0 &= A^1 = A^2 = A^3 = - \left[\frac{2c_0\kappa}{3g} \left(1 - \frac{a}{y} \right) + \frac{s}{g} \right] dz \end{aligned}$$

$$\sin \xi=-\frac{\sqrt{q(y)}}{y^2}, \cos \xi=\kappa \frac{2 y-a}{y^2}$$

$$L_{AdS_4}^2=\frac{2}{3\sqrt{3}g^2}$$

$$q(y)=y^4-4y^2+4ay-a^2$$

$$\begin{aligned} a &= \frac{n_S^2-n_N^2}{n_S^2+n_N^2} \\ c_0 &= \frac{\sqrt{n_S^2+n_N^2}}{\sqrt{2}n_Sn_N} \end{aligned}$$

$$y_N=-1+\sqrt{1+a}, y_S=1-\sqrt{1-a}$$

$$\begin{aligned} S_{\text{BH}} &= \frac{\sqrt{2}\sqrt{n_S^2+n_N^2}-(n_S+n_N)}{n_Sn_N}\frac{\pi L_{AdS_4}^2}{4G_N^{(4)}} \\ &= \frac{\sqrt{2}\sqrt{n_S^2+n_N^2}-(n_S+n_N)}{n_Sn_N}\frac{1}{2}\mathcal{F}_{S^3}^{\text{mABJM}}, \end{aligned}$$

$$\mathcal{F}_{S^3}^{\text{mABJM}}=\frac{4\sqrt{2}\pi}{9\sqrt{3}}N^{3/2}$$

$$\frac{1}{2}(T_{12}-T_{34}),\frac{1}{\sqrt{3}}(T_{12}+T_{34}-2T_{56}),$$

$$A^{12} = \frac{1}{2}(A^0 + A^1 - A^2 - A^3), \quad A^{34} = \frac{1}{2}(A^0 - A^1 + A^2 - A^3), \\ A^{56} = \frac{1}{2}(A^0 - A^1 - A^2 + A^3), \quad A^{78} = \frac{1}{2}(A^0 + A^1 + A^2 + A^3).$$

$$\mathcal{V} \equiv \begin{pmatrix} u_{ij}^{IJ} & v_{ijKL} \\ v^{klIJ} & u^{klKL} \end{pmatrix} = \exp \begin{pmatrix} 0 & \phi_{ijkl} \\ \phi^{ijkl} & 0 \end{pmatrix} \in E_{7(7)}/SU(8),$$

$$\phi_{ijkl} = \frac{1}{24}\epsilon_{ijklmnpq}\phi^{mnpq}, \phi^{ijkl} = (\phi_{ijkl})^*$$

$$\phi_{1278} = -\frac{1}{2}\lambda_1 e^{i\varphi_1}, \phi_{3478} = -\frac{1}{2}\lambda_2 e^{i\varphi_2}, \phi_{5678} = -\frac{1}{2}\lambda_3 e^{i\varphi_3} \\ \phi_{1234} = -\frac{1}{2}\lambda_3 e^{-i\varphi_3}, \phi_{1256} = -\frac{1}{2}\lambda_2 e^{-i\varphi_2}, \phi_{3456} = -\frac{1}{2}\lambda_1 e^{-i\varphi_1} \\ \phi_{1357} = -\phi_{1467} = -\phi_{2367} = -\phi_{2457} = \frac{1}{4}(\chi_1 \cos \psi_1 + i\chi_2 \sin \psi_2) \\ \phi_{1367} = \phi_{1457} = \phi_{2357} = -\phi_{2467} = -\frac{1}{4}(\chi_1 \sin \psi_1 - i\chi_2 \cos \psi_2) \\ \phi_{1368} = \phi_{1458} = \phi_{2358} = -\phi_{2468} = -\frac{1}{4}(\chi_1 \cos \psi_1 - i\chi_2 \sin \psi_2) \\ \phi_{1358} = -\phi_{1468} = -\phi_{2368} = -\phi_{2458} = -\frac{1}{4}(\chi_1 \sin \psi_1 + i\chi_2 \cos \psi_2)$$

$$\mathcal{M}_V \times \mathcal{M}_H = \left[\frac{SU(1,1)}{U(1)} \right]^3 \times \frac{SU(2,1)}{SU(2) \times U(1)}.$$

$$\chi \equiv \chi_2, \psi \equiv \psi_2, \chi_1 = 0, \psi_1 = \frac{\pi}{2}, \varphi_1 = \varphi_2 = \varphi_3 = \pi$$

$$e^{-1}\mathcal{L}=\frac{1}{2}R-\partial_\mu\chi\partial^\mu\chi-\frac{1}{4}\sinh^2\left(2\chi\right)D_\mu\psi D^\mu\psi-\sum_{i=1}^3\partial_\mu\lambda_i\partial^\mu\lambda_i-g^2\mathcal{P} \\ -\frac{1}{4}\big[e^{-2(\lambda_1+\lambda_2+\lambda_3)}F_{\mu\nu}^0F^{0\mu\nu}+e^{-2(\lambda_1-\lambda_2-\lambda_3)}F_{\mu\nu}^1F^{1\mu\nu}+e^{2(\lambda_1-\lambda_2+\lambda_3)}F_{\mu\nu}^2F^{2\mu\nu}+e^{2(\lambda_1+\lambda_2-\lambda_3)}F_{\mu\nu}^3F^{3\mu\nu}\big]$$

$$D\psi\equiv d\psi-g(A^0-A^1-A^2-A^3)$$

$$\mathcal{P} = \frac{1}{2} \Big(\frac{\partial W}{\partial \chi} \Big)^2 + \frac{1}{2} \sum_{i=1}^3 \Big(\frac{\partial W}{\partial \lambda_i} \Big)^2 - \frac{3}{2} W^2$$

$$W=\frac{2e^{2(\lambda_1+\lambda_2+\lambda_3)}\sinh^2\chi-\cosh^2\chi(e^{2(\lambda_1+\lambda_2+\lambda_2)}+e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3})}{2e^{\lambda_1+\lambda_2+\lambda_3}}.$$

$$A_R = -g(A^0 + A^1 + A^2 + A^3) \\ A_m = g(A^0 - A^1 - A^2 - A^3) \\ A_{F_1} = \frac{1}{2}g(A^1 - A^2) \\ A_{F_2} = \frac{1}{\sqrt{3}}g(A^1 + A^2 - 2A^3)$$



$$\delta\chi^{ijk} = -\mathcal{A}_\mu{}^{ijkl}\gamma^\mu\epsilon_l + \frac{3}{2}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{-[ij}\epsilon^{k]} - 2gA_{2l}{}^{ijk}\epsilon^l$$

$$\delta\psi_\mu{}^i = 2D_\mu\epsilon^i + \frac{\sqrt{2}}{4}\bar{F}_{\nu\rho}^{-ij}\gamma^{\nu\rho}\gamma_\mu\epsilon_j + \sqrt{2}gA_1{}^{ij}\gamma_\mu\epsilon_j$$

$$F^{-IJ} = (u_{ij}^{IJ} + v_{ijIJ})\bar{F}^{-ij}, F^{-IJ} = \frac{1}{2}(F^{IJ} - i * F^{IJ})$$

$$\epsilon^1=\ldots=\epsilon^6=0$$

$$A_1{}^{77}\epsilon_7=-\frac{1}{2}W\epsilon_7, A_1{}^{88}\epsilon_8=-\frac{1}{2}W\epsilon_8$$

$$B_\mu\equiv B_\mu{}^7{}_8=-B_\mu{}^8{}_7,\\ H_{\mu\nu}\equiv\bar{F}_{\mu\nu}^{78},$$

$$H_{\mu\nu}\equiv\bar{F}_{\mu\nu}^{78}=\frac{1}{2}\big(e^{-\lambda_1-\lambda_2-\lambda_3}F_{\mu\nu}^0+e^{-\lambda_1+\lambda_2+\lambda_3}F_{\mu\nu}^1+e^{\lambda_1-\lambda_2+\lambda_3}F_{\mu\nu}^2+e^{\lambda_1+\lambda_2-\lambda_3}F_{\mu\nu}^3\big)$$

$$B_\mu\equiv -g\big(A_\mu^0+A_\mu^1+A_\mu^2+A_\mu^3\big)-\frac{1}{2}(\cosh{(2\chi)}-1)D_\mu\psi$$

$$\epsilon=(\epsilon_7+\epsilon^7)+i(\epsilon_8+\epsilon^8)$$

$$\left[2\nabla_\mu-iB_\mu-\frac{g}{\sqrt{2}}W\gamma_\mu-i\frac{1}{2\sqrt{2}}H_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\right]\epsilon=0$$

$$\left[\gamma^\mu\partial_\mu\lambda_1+\frac{g}{\sqrt{2}}\partial_{\lambda_1}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{12}\right]\epsilon=0,$$

$$\left[\gamma^\mu\partial_\mu\lambda_2+\frac{g}{\sqrt{2}}\partial_{\lambda_2}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{34}\right]\epsilon=0,$$

$$\left[\gamma^\mu\partial_\mu\lambda_3+\frac{g}{\sqrt{2}}\partial_{\lambda_3}W+i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}_{\mu\nu}^{56}\right]\epsilon=0,$$

$$\left[\gamma^\mu\partial_\mu\chi+\frac{g}{\sqrt{2}}\partial_\chi W+i\frac{1}{2}\partial_\chi B_\mu\gamma^\mu\right]\epsilon=0$$

$$F^0=\frac{1}{2}e^{\lambda_1+\lambda_2+\lambda_3}(\bar{F}^{12}+\bar{F}^{34}+\bar{F}^{56}+\bar{F}^{78})$$

$$F^1=\frac{1}{2}e^{\lambda_1-\lambda_2-\lambda_3}(\bar{F}^{12}-\bar{F}^{34}-\bar{F}^{56}+\bar{F}^{78})$$

$$F^2=-\frac{1}{2}e^{-\lambda_1+\lambda_2-\lambda_3}(\bar{F}^{12}-\bar{F}^{34}+\bar{F}^{56}-\bar{F}^{78})$$

$$F^3=-\frac{1}{2}e^{-\lambda_1-\lambda_2+\lambda_3}(\bar{F}^{12}+\bar{F}^{34}-\bar{F}^{56}-\bar{F}^{78})$$

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+g^2\mathcal{P}g_{\mu\nu}-2\left(T_{\mu\nu}^\chi+T_{\mu\nu}^{\lambda_1}+T_{\mu\nu}^{\lambda_2}+T_{\mu\nu}^{\lambda_3}\right)-\frac{1}{2}\sinh^2{(2\chi)}T_{\mu\nu}^\psi\\ -e^{-2(\lambda_1+\lambda_2+\lambda_3)}T_{\mu\nu}^{A^0}-e^{-2(\lambda_1-\lambda_2-\lambda_3)}T_{\mu\nu}^{A^1}-e^{2(\lambda_1-\lambda_2+\lambda_3)}T_{\mu\nu}^{A^2}-e^{2(\lambda_1+\lambda_2-\lambda_3)}T_{\mu\nu}^{A^3}=0$$

$$T_{\mu\nu}^X = \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} \partial_\rho X \partial^\rho X,$$

$$T_{\mu\nu}^{A^\alpha} = g^{\rho\sigma} F_{\mu\rho}^\alpha F_{\nu\sigma}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^\alpha F^{\alpha\rho\sigma},$$

$$\begin{aligned} \partial_\nu (\sqrt{-g} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F^{0\mu\nu}) - \frac{1}{2} \sqrt{-g} \sinh^2(2\chi) g^{\mu\nu} D_\nu \psi &= 0 \\ \partial_\nu (\sqrt{-g} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F^{1\mu\nu}) + \frac{1}{2} \sqrt{-g} \sinh^2(2\chi) g^{\mu\nu} D_\nu \psi &= 0 \\ \partial_\nu (\sqrt{-g} e^{2(\lambda_1-\lambda_2+\lambda_3)} F^{2\mu\nu}) + \frac{1}{2} \sqrt{-g} \sinh^2(2\chi) g^{\mu\nu} D_\nu \psi &= 0 \\ \partial_\nu (\sqrt{-g} e^{2(\lambda_1+\lambda_2-\lambda_3)} F^{3\mu\nu}) + \frac{1}{2} \sqrt{-g} \sinh^2(2\chi) g^{\mu\nu} D_\nu \psi &= 0 \end{aligned}$$

$$\begin{aligned} & \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_1) - \frac{g^2}{2} \frac{\partial \mathcal{P}}{\partial \lambda_1} \\ & + \frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^0 F^{0\mu\nu} + \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^1 F^{1\mu\nu} \\ & - \frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^2 F^{2\mu\nu} - \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^3 F^{3\mu\nu} = 0 \\ & \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_2) - \frac{g^2}{2} \frac{\partial \mathcal{P}}{\partial \lambda_2} \\ & + \frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^1 F^{1\mu\nu} \\ & + \frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^2 F^{2\mu\nu} - \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^3 F^{3\mu\nu} = 0 \\ & \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \lambda_3) - \frac{g^2}{2} \frac{\partial \mathcal{P}}{\partial \lambda_3} \\ & + \frac{1}{4} e^{-2(\lambda_1+\lambda_2+\lambda_3)} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{4} e^{-2(\lambda_1-\lambda_2-\lambda_3)} F_{\mu\nu}^1 F^{1\mu\nu} \\ & - \frac{1}{4} e^{2(\lambda_1-\lambda_2+\lambda_3)} F_{\mu\nu}^2 F^{2\mu\nu} + \frac{1}{4} e^{2(\lambda_1+\lambda_2-\lambda_3)} F_{\mu\nu}^3 F^{3\mu\nu} = 0 \\ & \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \chi) - \frac{g^2}{2} \frac{\partial \mathcal{P}}{\partial \chi} + \frac{1}{8} \sinh(4\chi) D_\mu \psi D^\mu \psi = 0 \end{aligned}$$

$$\chi = 0, \psi = \frac{\pi}{2}$$

$$\mathcal{P} = -2[\cosh(2\lambda_1) + \cosh(2\lambda_2) + \cosh(2\lambda_3)]$$

$$\lambda_i = 0, \varphi_i = \pi, A \equiv A^0 = A^1 = A^2 = A^3$$

$$e^{-1} \mathcal{L} = \frac{1}{2} R - 6g^2 - F_{\mu\nu} F^{\mu\nu}$$

$$\tanh \chi = \frac{1}{\sqrt{3}}, \psi = \frac{\pi}{2}, \tanh \lambda_i = 2 - \sqrt{3}, \varphi_i = \pi$$

$$\frac{1}{3} A \equiv \frac{1}{3} A^0 = A^1 = A^2 = A^3$$



$$e^{-1}\mathcal{L}=\frac{1}{2}R-\frac{9\sqrt{3}}{2}g^2-\frac{1}{12\sqrt{3}}F_{\mu\nu}F^{\mu\nu}$$

$$ds^2=e^{2V}ds_{AdS_2}^2+f^2dy^2+h^2dz^2$$

$$A^\alpha=a^\alpha dz$$

$$\gamma^m=\Gamma^m\otimes\sigma^3,\gamma^2=]\,]_2\otimes\sigma^1,\gamma^3=]\,]_2\otimes\sigma^2$$

$$\epsilon=\psi\otimes\chi,$$

$$D_m\psi=\frac{1}{2}\kappa\Gamma_m\psi$$

$$[-i(\sqrt{2}\kappa e^{-V}+H_{23})\gamma^{23}+\sqrt{2}V'f^{-1}\gamma^2]\epsilon=gW\epsilon.$$

$$[i\cos\xi\gamma^{23}+\sin\xi\gamma^2]\epsilon=+\epsilon$$

$$-\sqrt{2}\kappa e^{-V}-H_{23}=gW\cos\xi,\sqrt{2}V'f^{-1}=gW\sin\xi$$

$$\epsilon=e^{i\frac{\xi}{2}\gamma^3}\eta,\gamma^2\eta=+i\gamma^3\eta$$

$$\xi=0,\pi,\gamma^{23}\epsilon=\pm i\epsilon$$

$$\left[\partial_y-\frac{1}{2}V'+\frac{i}{2}(\partial_y\xi+\sqrt{2}fH_{23}+\kappa fe^{-V})\gamma^3\right]\eta=0$$

$$[2\partial_z-iB_z+if^{-1}h'\cos\xi-i\frac{1}{\sqrt{2}}H_{23}h\sin\xi$$

$$+\left(f^{-1}h'\sin\xi-\frac{1}{\sqrt{2}}gWh+\frac{1}{\sqrt{2}}H_{23}h\cos\xi\right)\gamma^3]\eta=0$$

$$\eta=e^{V/2}e^{isz}\eta_0$$

$$\partial_y\xi+\sqrt{2}fH_{23}+\kappa fe^{-V}=0$$

$$(s-B_z)+f^{-1}h'\cos\xi-\frac{1}{\sqrt{2}}H_{23}h\sin\xi=0$$

$$f^{-1}h'\sin\xi-\frac{gWh}{\sqrt{2}}+\frac{1}{\sqrt{2}}H_{23}h\cos\xi=0$$

$$f^{-1}h'=\frac{gWh}{\sqrt{2}}\sin\xi-(s-B_z)\cos\xi$$

$$hH_{23}=gWh\cos\xi+\sqrt{2}(s-B_z)\sin\xi$$

$$(s-B_z)\sin\xi=-\sqrt{2}gWh\cos\xi-\kappa he^{-V}$$

$$H_{23}=-gW\cos\xi-\sqrt{2}\kappa e^{-V}$$

$$f^{-1}\partial_y\xi=\sqrt{2}gW\cos\xi+\kappa e^{-V}$$

$$f^{-1}\frac{h'}{h}\sin\xi=\kappa e^{-V}\cos\xi+\frac{gW}{\sqrt{2}}(1+\cos^2\xi)$$



$$f^{-1}\lambda'_i + \frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin \xi = 0$$

$$g\partial_{\lambda_1}W\cos \xi + \bar{F}_{23}^{12} = 0$$

$$g\partial_{\lambda_2}W\cos \xi + \bar{F}_{23}^{34} = 0$$

$$g\partial_{\lambda_3}W\cos \xi + \bar{F}_{23}^{56} = 0$$

$$f^{-1}\chi'\sin \xi + \frac{g}{\sqrt{2}}\partial_{\chi}W = 0$$

$$\sqrt{2}g\partial_{\chi}W\cos \xi - \partial_{\chi}B_z\sin \xi h^{-1} = 0$$

$$f^{-1}\xi' = \sqrt{2}gW\cos \xi + \kappa e^{-V}$$

$$f^{-1}V' = \frac{g}{\sqrt{2}}W\sin \xi$$

$$f^{-1}\lambda'_i = -\frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin \xi$$

$$f^{-1}\chi' = -\frac{g}{\sqrt{2}}\frac{\partial_{\chi}W}{\sin \xi}$$

$$f^{-1}\frac{h'}{h}\sin \xi = \kappa e^{-V}\cos \xi + \frac{gW}{\sqrt{2}}(1 + \cos^2 \xi)$$

$$(s - B_z)\sin \xi = -\sqrt{2}gWh\cos \xi - \kappa h e^{-V}$$

$$\sqrt{2}g\partial_{\chi}W\cos \xi = \partial_{\chi}B_z\sin \xi h^{-1}$$

$$\bar{F}_{23}^{12} = -g\partial_{\lambda_1}W\cos \xi$$

$$\bar{F}_{23}^{34} = -g\partial_{\lambda_2}W\cos \xi$$

$$\bar{F}_{23}^{56} = -g\partial_{\lambda_3}W\cos \xi$$

$$H_{23} = -gW\cos \xi - \sqrt{2}\kappa e^{-V}$$

$$\partial_{\chi}W = -\frac{g}{\sqrt{2}}f\sin \xi \left[\sum_{i=1}^3 (\partial_{\lambda_i}W)^2 + \frac{1}{\sin^2 \xi} (\partial_{\chi}W)^2 \right]$$

$$h e^{-V} = k \sin \xi$$

$$f^{-1}\xi' = -k^{-1}(s - B_z)e^{-V}$$

$$f^{-1}V' = \frac{g}{\sqrt{2}}W\sin \xi$$

$$f^{-1}\lambda'_i = -\frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin \xi$$

$$f^{-1}\chi' = -\frac{g}{\sqrt{2}}\frac{\partial_{\chi}W}{\sin \xi}$$

$$(s - B_z) = -k(\sqrt{2}gWe^V\cos \xi + \kappa)$$

$$\sqrt{2}g\partial_{\chi}W\cos \xi = k^{-1}e^{-V}\partial_{\chi}B_z$$

$$\partial_{\chi}B_z = -\sinh(2\chi)D_z\psi$$



$$D_z\psi=-\frac{\sqrt{2}gke^V\partial_\chi W\cos\xi}{\sinh\left(2\chi\right)}$$

$$F_{yz}^\alpha=(a^\alpha)'=(\mathcal{I}^{(\alpha)})'$$

$$\mathcal{I}^{(0)}\!\equiv\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{\lambda_1+\lambda_2+\lambda_3}$$

$$\mathcal{I}^{(1)}\!\equiv\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{\lambda_1-\lambda_2-\lambda_3}$$

$$\mathcal{I}^{(2)}\!\equiv\!\frac{1}{\sqrt{2}}ke^V\cos\xi e^{-\lambda_1+\lambda_2-\lambda_3}$$

$$\mathcal{I}^{(3)}\equiv\frac{1}{\sqrt{2}}ke^V\cos\xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$C(\Delta)=\sum_{c\geq 1}K_c(\Delta)\int\overset{\text{integration measure}}{\text{on localization locus}}\frac{\overbrace{d\phi^0}^{\text{Bulk modes}}}{(\phi^0)^{3/2}}\underbrace{(\phi^0)^5}_{\text{one-loop determinant}}\overset{\text{Induced}}{\text{one-loop determinant}}\frac{\overbrace{1}^{\text{Supergravity action}}}{c\phi^0}\exp\left[\underbrace{-\frac{\pi\Delta\phi^0}{4c}-\frac{\pi}{c\phi^0}}_{\text{along locus}}\right],$$

$$\Delta(\mathcal{Q})=C^{abcd}\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c\mathcal{Q}_d$$

$$S_{\rm BH}(\mathcal{Q})=\pi\sqrt{\Delta(\mathcal{Q})}$$

$$\Delta=4n-\ell^2$$

$$\varphi_{-2,1}(\tau,z)=\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}$$

$$\varphi_{-2,1}(\tau,z)=\sum_{n,\ell\in\mathbb{Z}}c_{-2,1}(n,\ell)q^n\zeta^\ell, q=\mathrm{e}^{2\pi\mathrm{i}\tau}, \zeta=\mathrm{e}^{2\pi\mathrm{i}z}$$

$$c_{-2,1}(n,\ell)=C(\Delta), \Delta=4n-\ell^2$$

$$C(\Delta)=2\pi\left(\frac{\pi}{2}\right)^{7/2}\sum_{c=1}^\infty c^{-9/2}K_c(\Delta)\tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$\tilde{I}_\rho(z)=\left(\frac{z}{2}\right)^{-\rho}I_\rho(z)=\frac{1}{2\pi\mathrm{i}}\int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty}\frac{d\sigma}{\sigma^{\rho+1}}\exp\left(\sigma+\frac{z^2}{4\sigma}\right)$$

$$K_c(\Delta)\colon=\mathrm{e}^{5\pi\mathrm{i}/4}\sum_{\substack{-c\leq d<0\\(d,c)=1}}\mathrm{e}^{2\pi\mathrm{i}\frac{d}{c}(\Delta/4)}M(\gamma_{c,d})_{\nu 1}^{-1}\mathrm{e}^{2\pi\mathrm{i}\frac{a}{c}(-1/4)},\nu=\Delta\bmod 2$$

$$M^{-1}(\gamma_{c,d})_{\nu 1}=\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\frac{1}{\sqrt{6c}}\exp\left(-\frac{\mathrm{i}\pi}{6}\Phi(\gamma)\right)\times\\ \sum_{\varepsilon=\pm 1}\sum_{n=0}^{c-1}\varepsilon\exp\left(\frac{\mathrm{i}\pi}{6c}(d(\nu+1)^2-4(\nu+1)(3n+\varepsilon)+4a(3n+\varepsilon)^2)\right)$$

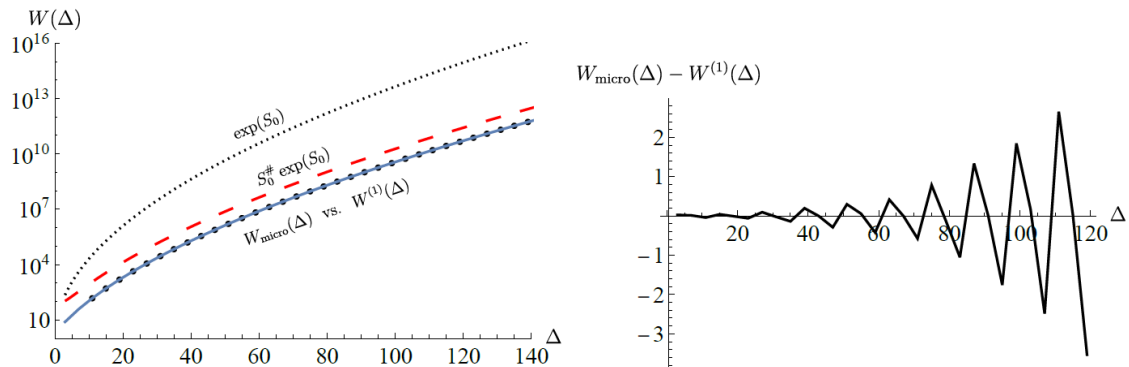


$$W_{\text{micro}}(\Delta) = (-1)^{\Delta+1} C(\Delta)$$

$$\log W_{\text{micro}}(\Delta) \sim \pi\sqrt{\Delta} + \cdots, \Delta \rightarrow \infty$$

$$\log W_{\text{micro}}(\Delta) = \pi\sqrt{\Delta} - 2\log \Delta + O(1/\sqrt{\Delta})$$

$$W_{\text{micro}}(\Delta) = \underbrace{\frac{\pi^{9/2}}{8} \tilde{I}_{7/2}(\pi\sqrt{\Delta})}_{\equiv W^{(1)}(\Delta)} + O\left(e^{\sqrt{\Delta}/2}\right)$$



$$\frac{1}{2\pi}\left(\frac{2}{\pi}\right)^{7/2} W_{\text{micro}}(\Delta) = \sum_{c=1}^{\infty} A_c(\Delta), A_c(\Delta) = c^{-9/2} K_c(\Delta) \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)$$

$$W(q,p)=\left\langle \exp\left(-iq_{\Lambda}\oint A^{\Lambda}\right)\right\rangle_{\text{AdS}_2\times\text{S}^2\text{ with T}\rightarrow 0}^{\text{reg}}\\ =\int [DgD\Psi DAD\Phi]e^{-S_{\text{suggra}}^{\text{bulk}}-S_{\text{sugra}}^{\text{bdry}}}$$

$$S_{\text{sugra}}^{\text{bdry}}\supset\int_{\partial(\text{AdS}_2\times\text{S}^2)}d^3x\sqrt{h}A^{\mu}F_{\mu\nu}n^{\nu}$$

$$S_{\text{sugra}}=\int\,\text{d}^4x\left(\mathcal{L}_{\text{sugra}}^{\text{phys}}-Q_{\text{eq}}(b_{\alpha}F^{\alpha})\right)$$

$$S_{\text{sugra}}=S(0)\rightarrow S(t)=S(0)+\lambda Q_{\text{eq}}\mathcal{V}$$

$$\mathcal{V}=\int\,d^4x\sqrt{\overset{\circ}{g}}\sum_{\psi}\bar{\psi}Q_{\text{eq}}\psi$$

$$ds^2=d\rho^2+\sinh^2\rho d\tau^2+d\theta^2+\sin^2\theta d\psi^2$$

$$(\tau,\psi)\sim(\tau+2\pi,\psi)\sim(\tau,\psi+2\pi)$$

$$F_{\rho\tau}^{\Lambda}=-ie_{\star}^{\Lambda}\sinh\rho, F_{\theta\psi}^{\Lambda}=p^{\Lambda}\sin\theta$$

$$X^\Lambda=\frac{1}{2}(e^\Lambda_\star+ip^\Lambda)+\frac{\phi^\Lambda-e^\Lambda_\star}{2\cosh\rho},\bar{X}^\Lambda=\frac{1}{2}(e^\Lambda_\star-ip^\Lambda)+\frac{\phi^\Lambda-e^\Lambda_\star}{2\cosh\rho}.$$

$$Y_1^{\Lambda,1}=-Y_2^{\Lambda,2}=\frac{\phi^\Lambda-e^\Lambda_\star}{\cosh^2\rho}$$

$$\ell^2 = \mathrm{e}^{-\mathcal{K}} \equiv -\mathrm{i}(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda)\big|_{\rho=0}$$

$$(\tau,\phi)\sim\left(\tau+\frac{2\pi}{c},\psi+\frac{2\pi}{c}\right)\sim(\tau,\psi+2\pi).$$

$$A^0_{(c,d)}=A^0+\frac{d}{c}d\tau, A^0=-ie^\circ_\star(\cosh\rho-1)d\tau$$

$$F(X^I)=C_{IJK}\frac{X^IX^JX^K}{X^0}+\cdots$$

$$W^{(1)}(q,p)=\int_{\mathcal{M}_Q}[\,\mathrm{d}^{n_\nu+1}\phi]\exp\left(-\pi q_\Lambda\phi^\Lambda+4\pi\mathrm{Im}F\left(\frac{\phi^\Lambda+\mathrm{i}p^\Lambda}{2}\right)\right)Z^{Q_{\mathrm{eq}}\nu}_{1-\mathrm{loop}}(\phi^\Lambda)$$

$$W^{(\mathrm{c})}(q,p)=\int_{\mathcal{M}_Q}[\,\mathrm{d}^{n_\nu+1}\phi]_c\exp\left(-\frac{\pi q_\Lambda\phi^\Lambda}{c}+\frac{4\pi}{c}\mathrm{Im}F\left(\frac{\phi^\Lambda+\mathrm{i}p^\Lambda}{2}\right)\right)Z^{Q_{\mathrm{eq}}\nu}_{1-\mathrm{loop},c}(\phi^\Lambda)Z_c^{\mathrm{top}}(q,p)$$

$$F^{(0)}(X)=C_{IJK}\frac{X^IX^JX^K}{X^0}, C_{IJK}\equiv\frac{1}{6}\int_{T^6}\alpha_I\wedge\alpha_J\wedge\alpha_K$$

$$\Delta=4C_{IJK}p^Ip^Jp^K\left(q_0+\frac{1}{12}q_I C^{IJ}q_J\right)$$

$$\int\,[D\Phi^m] \mathrm{e}^{-\int\,\sqrt{g}G_{mn}(\Phi)\delta\Phi^m\delta\Phi^n}$$

$$[\mathrm{d}^{16}\phi]_c\equiv\,\mathrm{d}^{16}\phi\sqrt{2\mathrm{det}\Big(\frac{1}{2c}\mathrm{Im}F_{\Lambda\Sigma}\Big)},$$

$$\mathrm{d}^{16}\phi=\prod_{\Lambda=0}^{16}\mathrm{d}\phi^\Lambda$$

$$\mathrm{Im}(F_{IJ})=\frac{6C_{IJ}}{\phi^0}$$

$$\mathrm{Im}(F_{0I})=\mathrm{Im}(F_{I0})=-\frac{6C_{IJ}\phi^J}{(\phi^0)^2}$$

$$\mathrm{Im}(F_{00})=-\frac{2C_{IJ}p^Ip^J}{(\phi^0)^3}+\frac{6C_{IJ}\phi^I\phi^J}{(\phi^0)^3}$$

$$[d^{16}\phi]_c = d^{16}\phi \frac{1}{c^8} \sqrt{\frac{-2C_{IJK}p^I p^J p^K \det(3C_{IJ})}{(\phi^0)^{18}}}$$

$$Z_c^{\rm top} = \sqrt{c} K_c(\Delta).$$

$$\mathrm{e}^{-\mathcal{K}(\phi^\Lambda)} = \frac{-2C_{IJK}p^Ip^Jp^K}{\phi^0}$$

$$Z_{\rm index}=\sqrt{\frac{\det_{\Psi}(H/\ell)}{\det_{\Phi}(H/\ell)}}$$

$$\ell^2=e^{-\mathcal{K}(\phi^\Lambda+\mathrm{i}p^\Lambda)}$$

$$\mathrm{ind}(D_{10})(t) \mathrel{\mathop:}= \mathrm{Tr}_{\mathrm{Ker}\,D_{10}} \mathrm{e}^{\frac{tH}{\ell}} - \mathrm{Tr}_{\mathrm{Coker}\,D_{10}} \mathrm{e}^{\frac{tH}{\ell}}$$

$$\mathrm{ind}(D_{10})(t)=\sum_n\alpha(n)\mathrm{e}^{\mathrm{i}\lambda_nt}$$

$$Z_{\mathrm{index}}=\prod_n\left(\lambda_n\right)^{-\frac{1}{2}\alpha(n)},$$

$$\mathrm{ind}(D_{10})(t)=\sum_{\{x|\tilde{x}=x\}}\frac{\mathrm{Tr}_{\Phi}e^{\frac{tH}{\ell}}-\mathrm{Tr}_{\Psi}e^{\frac{tH}{\ell}}}{\det(1-\partial\tilde{x}/\partial x)}$$

$$ds^2=d\rho^2+\sinh^2\rho d\tau^2+d\theta^2+\sin^2\theta d\psi^2=\frac{4dwd\bar{w}}{(1-w\bar{w})^2}+\frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

$$H\equiv L_0-J_0=(\partial_\tau-\partial_\psi)$$

$$(w,z)\rightarrow (e^{\mathrm{i}t/\ell}w,e^{-\mathrm{i}t/\ell}z)$$

$$\det(1-\partial\tilde{x}/\partial x)=(1-q)^2(1-q^{-1})^2$$

$$NP\colon \varepsilon^i_{+\alpha}=0, \varepsilon^i_{-\dot{\alpha}}=\left(\sigma_3\exp\left[\mathrm{i}\frac{(\tau+\psi)}{2}\sigma_3\right]\right)^i_{\dot{\alpha}}, H=L_0-J_0=2J_0^{(+)}$$

$$SP\colon \varepsilon^i_{+\alpha}=\left(-\mathrm{i}\sigma_3\exp\left[\mathrm{i}\frac{(\tau+\psi)}{2}\sigma_3\right]\right)^i_{\alpha}, \varepsilon^i_{-\dot{\alpha}}=0, H=L_0-J_0=2J_0^{(-)}$$

$$NP\colon \mathrm{Tr}_{(m,n)}e^{\frac{tH}{\ell}}=n\big(q^{-|m-1|}+q^{-|m-1|+2}\ldots+q^{|m-1|-2}+q^{|m-1|}\big)$$

$$SP\colon \mathrm{Tr}_{(m,n)}e^{\frac{tH}{\ell}}=m\big(q^{-|n-1|}+q^{-|n-1|+2}\ldots+q^{|n-1|-2}+q^{|n-1|}\big)$$

$$\log Z_{\rm index} = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{dt}{t} \mathrm{ind}(D_{10})(t)$$



$$\frac{1}{4}\mathrm{ind}(D_{10})(t)=\cdots+\frac{a_{-2}}{t^2}+a_0+a_2t^2+\cdots$$

$$\{Q^i,\bar{Q}_j\}=(\gamma^aP_a+\mathrm{i}Z)\delta_j^i$$

$$\big[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)\big]=\delta_{\mathrm{cgct}}(\xi)+\delta_M(\varepsilon)+\delta_K(\Lambda_K)+\delta_S(\eta)+\delta_{\mathrm{gauge}}$$

$$\varepsilon^{ab}=\mathrm{i}\frac{1}{2}\bar{\varepsilon}_{2i+}\varepsilon_{1+}^iT^{ab+}+\mathrm{i}\frac{1}{2}\bar{\varepsilon}_{2i-}\varepsilon_{1-}^iT^{ab-}$$

$$\{Q^i,\bar{S}_j\}=-\delta_j^i(\gamma^{ab}M_{ab}+Dil-\gamma_5A)+2V_j^i$$

$$(\tau,\psi)\sim\left(\tau+\frac{2\pi}{c},\psi+\frac{2\pi}{c}\right)\sim(\tau,\psi+2\pi).$$

$$\frac{1}{|\mathbb{Z}_c|}\sum_{\gamma\in\mathbb{Z}_c}\delta^4(\tilde{x}(\gamma,t)-x)$$

$$\tilde{x}(\gamma,t)=\left(\gamma\cdot\mathrm{e}^{\frac{tH}{\ell}}\right)x=\mathrm{e}^{\left(\frac{t}{\ell}+\frac{2\pi}{c}\right)H}x$$

$$\mathrm{ind}_c(D_{10})(t)=\frac{1}{c}\sum_{\gamma\in\mathbb{Z}_c}\sum_{\{x|\tilde{x}(\gamma,t)=x\}}\frac{\mathrm{Tr}_{\Phi}\left(\gamma\cdot\mathrm{e}^{\frac{tH}{\ell}}\right)-\mathrm{Tr}_{\Psi}\left(\gamma\cdot\mathrm{e}^{\frac{tH}{\ell}}\right)}{\det(1-\partial\tilde{x}/\partial x)},$$

$$q=\mathrm{e}^{\frac{t}{\ell}+\frac{2\pi}{c}}$$

$$\lambda_{(2,2)}\equiv\lambda_\mu\equiv\bar{\varepsilon}_i\gamma_\mu\lambda^i,\lambda_{(1,1)}\equiv\lambda\equiv\bar{\varepsilon}_i\gamma_5\lambda^i,\lambda_{(1,3)}\equiv\lambda^{ij}\equiv-2\varepsilon^{(i}C\lambda^{j)},$$

$$Q_{\mathrm{eq}}A_\mu=\lambda_\mu+\partial_\mu c.$$

$$Q_{\mathrm{eq}}\mathrm{Im}X=\lambda,$$

$$Q_{\mathrm{eq}}\lambda^{ij}\supset Y^{ij},$$

$$Q_{\mathrm{eq}}c\supset i(\bar{\epsilon}_i\epsilon^i)\mathrm{Re}X,$$

$$Q_{\mathrm{eq}}b=B,$$

$$\begin{aligned}\mathrm{Tr}_{\Phi}^{NP}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)&=\mathrm{Tr}_{\Phi}^{SP}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)=\mathrm{Tr}_{(2,2)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)+\mathrm{Tr}_{(1,1)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)\\ \mathrm{Tr}_{\Psi}^{NP}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)&=\mathrm{Tr}_{\Psi}^{SP}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)=\mathrm{Tr}_{(1,3)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)+2\mathrm{Tr}_{(1,1)}\left(\mathrm{e}^{\frac{tH}{\ell}}\right)\end{aligned}$$

$$\mathrm{ind}_{\mathrm{vect.}}^{\mathcal{N}=2}(D_{10})(t)=\frac{4(q+q^{-1})-8}{(1-q)^2(1-q^{-1})^2}=\frac{4q}{(1-q)^2}\quad\mathrm{with}\;q=e^{\frac{it}{\ell}}$$

$$\mathrm{ind}_{\mathrm{vect.,}c}^{\mathcal{N}=2\;\mathrm{s.c.}}(D_{10})(t)\Big|_{\mathrm{NP/SP}}=\frac{2cq^c}{(1-q^c)^2}\Big|_{\mathrm{NP/SP}},\;\mathrm{with}\;q=e^{\frac{it}{\ell}},$$

$$\frac{2cq^c}{(1-q^c)^2} = \sum_{n \geq 1} 2cne^{-it\frac{n}{\ell}c}$$

$$\langle e^{\partial_s(\sum_{n \geq 1} 2cn(nc/\ell)^{-s})} \big|_{s \rightarrow 0}$$

$$\log \left(Z_{1-\text{loop}}^{\text{vec}} \right) = -\frac{i\pi}{12}c + \frac{c}{6} \log c + 2c\zeta'(-1) - \frac{c}{12} \log \ell^2.$$

$$\delta e_{\mu}{}^a = \bar{\epsilon}^i \gamma^a \psi_{\mu i} + \text{h.c.} \, ,$$

$$\delta \psi_{\mu}{}^i \supset -b_{\mu} \epsilon^i - 2V_{\mu}{}^i{}_j \epsilon^j - \frac{1}{2} \gamma^{ab} T_{ab}{}^{ij} \gamma_{\mu} \epsilon_j - \gamma_{\mu} \eta^i,$$

$$\delta b_{\mu} \supset -\frac{1}{2} \bar{\psi}_{\mu}{}^i \eta_i + \text{h.c.} \, ,$$

$$\delta V_{\mu}{}^i{}_j \supset \bar{\epsilon}^k \gamma_{\mu} \chi^i{}_{kj} - \bar{\psi}_{\mu}{}^i \eta_j + \frac{1}{4} \delta^i{}_j \bar{\psi}_{\mu}{}^k \eta_k,$$

$$\delta \Lambda_i \supset E_{ij} \epsilon^j + \frac{1}{2} \varepsilon_{ijkl} T_{bc}{}^{kl} \gamma^{bc} \epsilon^j,$$

$$\delta E_{ij} \supset -2\bar{\epsilon}^k \chi_{(i}^{mn} \varepsilon_{j)kmn} + 2\bar{\eta}_{(i} \Lambda_{j)},$$

$$\delta T_{ab}{}^{ij} \supset -2\bar{\epsilon}^{[i} \left(\gamma_{[a} \phi_{b]}{}^{j]l} + \frac{1}{2} \gamma \cdot T^{j]lk} \gamma_{[a} \psi_{b]k} \right) + \frac{1}{2} \bar{\epsilon}^k \gamma_{ab} \chi^{ij}{}_k - \frac{1}{4} \varepsilon^{ijkl} \bar{\eta}_k \gamma_{ab} \Lambda_l,$$

$$\delta \chi^{ij}{}_k \supset -\frac{1}{2} \not{D} T_{ab}^{ij} \epsilon_k + D^{ij}{}_{kl} \epsilon^l + \frac{1}{2} T_{ab}^{ij} \gamma^{ab} \eta_k + \frac{2}{3} \delta_k^{[i} T_{ab}{}^{j]l} \gamma^{ab} \eta_l - \frac{1}{2} \varepsilon^{ijlm} E_{kl} \eta_m,$$

$$\delta D^{ij}{}_{kl} \supset \varepsilon^{ijmn} \bar{\epsilon}^p T^{ab}{}_{kl} [2T_{abmp} \lambda_m + T_{abmn} \Lambda_p],$$

$$e_{\mu}^a = e_{(3,3,1) \oplus (1,3,1) \oplus (3,1,1) \oplus (1,1,1)} \in \Phi$$

$$\psi_{\mu}^a = \psi_{(2,2,1) \oplus (2,2,1)} = \psi_{(3,3,1) \oplus (1,3,1) \oplus (3,1,1) \oplus (1,1,1)} \notin \Psi$$

$$\psi_{(2,4,1)} \in \Psi, \psi_{(2,2,1) \otimes (1,1,1)} \in \Psi$$

$$V_{(2,2,1)} \notin \Phi, V_{(2,4,1)} \notin \Phi.$$

$$\psi_{(2,2,1)'} \in \Psi, (c_S)_{(2,2,1)} \notin \Phi$$

$$\left. \text{ind}_{\text{vect.}}^{\mathcal{N}=2 \text{ s.c.}} (D_{10})(t) \right|_{\text{NP/SP}} = \left. \frac{2(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2} \right|_{\text{NP/SP}},$$

$$\left. \text{ind}_{\text{hyper}}^{\mathcal{N}=2 \text{ s.c.}} (D_{10})(t) \right|_{\text{NP/SP}} = - \left. \frac{2(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2} \right|_{\text{NP/SP}},$$

$$\left. \text{ind}_{\text{Weyl}}^{\mathcal{N}=2 \text{ s.c.}} (D_{10})(t) \right|_{\text{NP/SP}} = \left. \frac{2(q^2+q^{-2})-6(q+q^{-1})+8}{(1-q)^2(1-q^{-1})^2} \right|_{\text{NP/SP}},$$

$$\left. \text{ind}_{\text{spin } 3/2}^{\mathcal{N}=2 \text{ s.c.}} (D_{10})(t) \right|_{\text{NP/SP}} = \left. \frac{-2(q^2+q^{-2})+4(q+q^{-1})-4}{(1-q)^2(1-q^{-1})^2} \right|_{\text{NP/SP}},$$

$$\left. \text{ind}_{\text{chiral}}^{\mathcal{N}=2 \text{ s.c.}} (D_{10})(t) \right|_{\text{NP/SP}} = 0|_{\text{NP}},$$

$$\begin{aligned}\text{ind}_{\text{vect., } c}^{\mathcal{N}=2 \text{ s.P.}} (D_{10})(t)|_{\text{NP}} &= \frac{2cq^c}{(1-q^c)^2} \Big|_{\text{NP}}, \\ \text{ind}_{\text{hyper. } c}^{\mathcal{N}=2 \text{ s.P.}} (D_{10})(t)|_{\text{NP}} &= \frac{-2cq^c}{(1-q^c)^2} \Big|_{\text{NP}}, \\ \text{ind}_{\text{Graviton, } c}^{\mathcal{N}=2 \text{ s.P.}} (D_{10})(t)|_{\text{NP}} &= \frac{2q^{2c} + 2(c-2)q^c + 2}{(1-q^c)^2} \Big|_{\text{NP}}, \\ \text{ind}_{\text{Gravitino, } c}^{\mathcal{N}=2 \text{ s.P.}} (D_{10})(t)|_{\text{NP}} &= -\frac{2q^{2c} + 2(c-2)q^c + 2}{(1-q^c)^2} \Big|_{\text{NP}},\end{aligned}$$

$$\begin{aligned}\log (Z_{\text{index}}^{\mathcal{N}=2 \text{ vector}}) &= -\frac{i\pi}{12}c + \frac{c}{6}\log c + 2c\zeta'(-1) - \frac{c}{12}\log \ell^2, \\ \log (Z_{\text{index}}^{\mathcal{N}=2 \text{ hyper}}) &= \frac{i\pi}{12}c - \frac{c}{6}\log c - 2c\zeta'(-1) + \frac{c}{12}\log \ell^2, \\ \log (Z_{\text{index}}^{\mathcal{N}=2 \text{ graviton}}) &= -\frac{i\pi}{12}c + \frac{c}{6}\log c + 2c\zeta'(-1) + \left(1 - \frac{c}{12}\right)\log \ell^2 \\ \log (Z_{\text{index}}^{\mathcal{N}=2 \text{ gravitino}}) &= \frac{i\pi}{12}c - \frac{c}{6}\log c - 2c\zeta'(-1) - \left(1 - \frac{c}{12}\right)\log \ell^2.\end{aligned}$$

$$\begin{aligned}Z_{\text{index}}^{\mathcal{N}=4 \text{ vector}} &= 1 \\ \log (Z_{\text{index}}^{\mathcal{N}=4 \text{ graviton}}) &= -\log \ell^2.\end{aligned}$$

$$\log (Z_{\text{index}}^{\mathcal{N}=8 \text{ graviton}}) = -5\log \ell^2.$$

$$Z_{1\text{-loop}}^{Q_{\text{eq}}\mathcal{V}} \equiv Z_{\text{zero-modes}}^{Q_{\text{eq}}\mathcal{V}} Z_{\text{non-zero}}^{Q_{\text{eq}}\mathcal{V}}$$

$$Q_{\text{eq}}^2 A_{\mu}^{\text{bdry}} = H A_{\mu}^{\text{bdry}} = n^{\nu} F_{\nu\mu}^{\text{bdry}} = 0$$

$$Q_{\text{eq}}^2 \Phi^{\text{bdry}} = H \Phi^{\text{bdry}} = 0$$

$$Q_{\text{eq}} \mathcal{V}|_{A_{\mu}^{\text{bdry}}} = 0$$

$$Q_{\text{eq}} \mathcal{V}|_{\Phi^{\text{bdry}}} = 0$$

$$Q_{\text{eq}} \mathcal{V}|_{Q_{\text{eq}} \Phi^{\text{bdry}}} = -(Q_{\text{eq}} \Phi^{\text{bdry}}) H (Q_{\text{eq}} \Phi^{\text{bdry}}) = -Q_{\text{eq}} [\Phi^{\text{bdry}} H (Q_{\text{eq}} \Phi^{\text{bdry}})]$$

$$Q_{\text{eq}} \mathcal{V}|_{\Psi^{\text{bdry}}} = Q_{\text{eq}} \mathcal{V}|_{Q_{\text{eq}} \Phi^{\text{pre-bdry}}} = 0$$

$$Q_{\text{eq}} \mathcal{V}|_{Q_{\text{eq}} \Psi^{\text{bdry}}} = -(Q_{\text{eq}} \Psi^{\text{bdry}}) 1 (Q_{\text{eq}} \Psi^{\text{bdry}}), \quad Q_{\text{eq}} \mathcal{V}|_{\Phi^{\text{pre-bdry}}} = \Phi^{\text{pre-bdry}} H^2 \Phi^{\text{pre-bdry}}$$

$$Z_{\text{non-zero}}^{Q_{\text{eq}}\mathcal{V}} = \sqrt{\det^{*f} K_f / \det^{*b} K_b}$$

$$Z_{\text{non-zero}}^{Q_{\text{eq}} \mathcal{V}} = \sqrt{\frac{\det^{*f} K_f}{\det^{*b} K_b}} = \sqrt{\frac{\det_{Q_{\text{eq}} \Phi^{\text{bdry}}} (H) \det' K_f}{\det_{\Phi^{\text{pre-bdy}}} (H^2) \det' K_b}} = \sqrt{\frac{\det_{\Phi^{\text{bdry}}} (H) \det'_{\Psi} H}{\det_{\Phi^{\text{pre-bdy}}} (H^2) \det'_{\Phi} H}}$$

$$\begin{aligned} \log Z_{\text{non-zero}}^{Q_{\text{eq}} \mathcal{V}} &= -\frac{1}{2} \int \frac{dt}{t} \left(\text{Tr}_{\Phi^{\text{bdry}}} e^{\frac{tH}{\ell}} - 2\text{Tr}_{\Phi^{\text{pre-bdy}}} e^{\frac{tH}{\ell}} + \text{Tr}_{\Psi'} e^{\frac{tH}{\ell}} - \text{Tr}_{\Phi'} e^{\frac{tH}{\ell}} \right) \\ &= \int \frac{dt}{t} \left[-\text{Tr}_{\Phi^{\text{bdry}}} e^{\frac{tH}{\ell}} + \frac{1}{2} \left(\text{Tr}_{\Psi^{\text{bdry}}} e^{\frac{tH}{\ell}} + \text{Tr}_{Q_{\text{eq}} \Phi^{\text{pre-bdy}}} e^{\frac{tH}{\ell}} \right) \right] \\ &\quad - \frac{1}{2} \int \frac{dt}{t} \left(\text{Tr}_{\Psi} e^{\frac{tH}{\ell}} - \text{Tr}_{\Phi} e^{\frac{tH}{\ell}} \right) \end{aligned}$$

$$\text{ind}_{\text{Weyl}}^{\mathcal{N}=8 \text{ s.P.}} (D_{10})(t) = \text{Tr}_{\Phi} e^{\frac{tH}{\ell}} - \text{Tr}_{\Psi} e^{\frac{tH}{\ell}}$$

$$n_{\text{bdry}}^{\text{bos}} \equiv \text{Tr}_{\Phi^{\text{bdry}}} e^{\tilde{t}H}, n_{\text{bdry}}^{\text{ferm}} \equiv \text{Tr}_{\Psi^{\text{bdry}}} e^{\tilde{t}H}, \text{ with } \tilde{t} = \frac{t}{\ell}$$

$$n_{\text{bdry}}^{\text{bos}} \equiv n_{\text{bdry}}^{1\text{-form}} + n_{\text{bdry}}^{\text{grav}}, n_{\text{bdry}}^{\text{grav}} = -6, n_{\text{bdry}}^{1\text{-form}} = -n_v, n_{\text{bdry}}^{\text{ferm}} = -8, \text{ for } c = 1.$$

$$n_{\text{bdry},c}^{\text{grav}} = -2, n_{\text{bdry},c}^{1\text{-form}} = -n_v, n_{\text{bdry},c}^{\text{ferm}} = -4, \text{ for } c > 1.$$

$$\log Z_{\text{non-zero},c}^{Q_{\text{eq}} \mathcal{V}} = \left(-n_{\text{bdry},c}^{\text{bos}} + \frac{1}{2} n_{\text{bdry},c}^{\text{ferm}} \right) \log(\ell) + \log Z_{\text{index},c}^{Q_{\text{eq}} \mathcal{V}}$$

$$\log Z_{\text{index},c}^{Q_{\text{eq}} \mathcal{V}} = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{dt}{t} \text{ind}_{\text{Weyl},c}^{\mathcal{N}=8 \text{ s.P.}} (D_{10})(t)$$

$$-n_{\text{bdry},c}^{\text{bos}} + \frac{1}{2} n_{\text{bdry},c}^{\text{ferm}} = 2 + \frac{n_v}{2}$$

$$-n_{\text{bdry},c}^{\text{bos}} + \frac{1}{2} n_{\text{bdry},c}^{\text{ferm}} = \frac{n_v}{2}$$

$$Z_{\text{zero-modes},c}^{\text{bdy.}} = \int_{\frac{\text{Large-diffeomorphisms}}{\text{Super-isometries}}} \left[d(A_{\mu})^{\text{Large gauge transf.}} \right] \left[dh_{\mu\nu}^{\text{Large super-diffs.}} \right] \left[d\psi_{\mu}^{\text{Large super-diffs.}} \right],$$

$$\int [d(A_{\mu})] e^{-\int d^4x \sqrt{g} g^{\mu\nu} A_{\mu} A_{\nu}} = 1$$

$$Z_{\text{zero-modes},c=1}^{\text{bdy.}} = (\ell)^{c_{\text{zero-modes}}} \int_{\frac{\text{Diff}(S^{1|4})}{\text{PSU}(1,1|2)}} \mathcal{D}_{\text{large diffs}} \int_{[\text{Loop } U(1)]^{nv}} \mathcal{D}_{\text{large gauge}}$$

$$Z_{\text{zero-modes},c}^{\text{bdy.}} = (\ell)^{c_{\text{zero-modes},c}} \int_{\frac{\text{Diff}(S^{1|4})}{H_{\text{orb.}}}} \mathcal{D}_{\text{large diffs}} \int_{[\text{Loop } U(1)]^{nv}} \mathcal{D}_{\text{large gauge}}$$

$$c_{\text{zero-modes},c} = \beta^{1\text{-form}} n_{\text{bdry},c}^{1\text{-form}} + \beta^{\text{grav}} n_{\text{bdry},c}^{\text{grav}} - \frac{1}{2} \beta^{\text{ferm}} n_{\text{bdry},c}^{\text{ferm}}.$$

$$\beta^{1\text{-form}} = 1, \beta^{\text{grav}} = 2, \beta^{\text{ferm}} = 3$$



$$c_{\rm non-index} = (\beta^{1\text{-form}} - 1)n_{\rm bdry,c}^{1\text{-form}} + (\beta^{\rm grav} - 1)n_{\rm bdry,c}^{\rm grav} - \frac{1}{2}(\beta^{\rm ferm} - 1)n_{\rm bdry,c}^{\rm ferm}$$

$$\begin{aligned}\mathrm{Vol}\left[\frac{\mathrm{Diff}(S^{1|4})}{H_{\mathrm{Disk}}}\right]&=\int_{\frac{\mathrm{Diff}(S^{1|4})}{H_{\mathrm{Disk}}}}\mathbb{D}_{zero\;modes}\mathrm{Vol}\left[\frac{\mathrm{Diff}(S^{1|4})}{H_{\mathrm{Orb}}}\right]=\int_{\frac{\mathrm{Diff}(S^{1|4})}{H_{\mathrm{Orb}}}}\mathbb{D}_{zero\;modes} \\ \mathrm{Vol}[\mathrm{Loop}(U(1))]&=\int_{\mathrm{Loop}(U(1))}\mathfrak{D}_{large\;gauge}\end{aligned}$$

$$S_{\mathrm{sugra,\;bdy}} \supset \int_{\partial(\mathrm{AdS}_2 \times S^2)} d^3x \sqrt{h} A^\mu F_{\mu\nu} n^\nu$$

$$I_{\mathrm{Particle\;on\;}U(1)}=-\frac{T}{E_{U(1)}}\int_0^{2\pi}d\tau(\partial_\tau\Lambda(\tau))^2$$

$$\Lambda(\tau+2\pi)=\Lambda(\tau)+\alpha_{U(1)}$$

$$\begin{aligned}Z^{\mathrm{Particle\;on\;}U(1)}(\beta,\tilde{q}=0)&=\int_0^{2\pi}d\alpha_{U(1)}Z(\beta,\alpha_{U(1)})\\&=\int_0^{2\pi}d\alpha_{U(1)}\int_{\frac{\mathrm{Loop}(U(1))}{U(1)}}D\Lambda e^{-I_{\mathrm{Particle\;on\;}U(1)}[\Lambda,\alpha_{U(1)}]}\end{aligned}$$

$$\mathrm{Vol}[\mathrm{Loop}(U(1))]=\lim_{T\rightarrow 0}\big[Z^{\mathrm{Particle\;on\;}U(1)}(\beta,\tilde{q}=0)\big]=1.$$

$$I_{\mathcal{N}=4\;\mathrm{Schw.}}=-\frac{T}{E_{\mathrm{gap}}}\int_0^{2\pi}d\tau[\mathrm{Sch}(f,\tau)+\mathrm{Tr}(g^{-1}\partial_\tau g)^2+(\;\mathrm{white\;particles})]$$

$$E_{\mathrm{gap}}\;\sim\;\frac{1}{\ell_{\mathrm{P1}}\Delta^{3/2}}$$

$$\mathrm{Vol}\left[\frac{\mathrm{Diff}(S^{1|4})}{H_{\mathrm{Disk}}}\right]=\lim_{T\rightarrow 0}\int_{\frac{\mathrm{Diff}(S^{1|4})}{\mathrm{PSU}(1,1|2)}}DfDgD\eta D\bar\eta e^{-I_{\mathcal{N}=4\;\mathrm{super-Schw.}}},$$

$$f(\tau+2\pi)=f(\tau),g(\tau+2\pi)=\mathrm{e}^{\mathrm{i}2\pi\alpha\sigma_3}g(\tau),\eta(\tau+2\pi)=-\mathrm{e}^{\mathrm{i}2\pi\alpha\sigma_3}\eta(\tau)$$

$$\alpha=\frac{\beta\Omega_E}{4\pi},$$

$$f(\tau)=\tan\left(\frac{\tau}{2}\right),g(\tau)=\mathrm{e}^{i\tau[n+\alpha]\sigma_3},\eta(\tau)=\bar{\eta}(\tau)=0$$

$$Z_{\mathrm{Disk}}^{\mathcal{N}=4\;\mathrm{Schw.}}(\beta,\alpha)=\sum_{n\in\mathbb{Z}}\frac{E_{\mathrm{gap}}}{T}\frac{2\cot{(\pi\alpha)}(\alpha+n)}{\pi^3(1-4(n+\alpha)^2)^2}\mathrm{e}^{2\pi^2\frac{T}{E_{\mathrm{gap}}}(1-4(n+\alpha)^2)}$$

$$Z_{\mathrm{Disk}}^{\mathcal{N}=4\;\mathrm{Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)=1$$



$$\mathrm{Vol}\left[\frac{\mathrm{Diff}(S^{14})}{H_{\mathrm{Disk}}}\right]=\lim_{T\rightarrow 0}\left[Z_{\mathrm{Disk}}^{\mathcal{N}=4\text{ Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)\right]=1$$

$$(\tau,\psi)\sim\left(\tau+\frac{2\pi}{c},\psi+\frac{2\pi}{c}\right)\sim(\tau,\psi+2\pi).$$

$$f(\tau+2\pi)=\frac{f(\tau)+\tan{(\pi/c)}}{1-f(\tau)\tan{(\pi/c)}},g(\tau+2\pi)=\mathrm{e}^{2\pi i\alpha\sigma_3}g(\tau)\mathrm{e}^{2\pi i\frac{2\pi}{c}\sigma_3},\eta(\tau+2\pi)=-\mathrm{e}^{2\pi i\alpha\sigma_3}\eta(\tau)$$

$$f(\tau)=\tan\left(\frac{\tau}{2c}\right),g(\tau)=\mathrm{e}^{i\tau\left[n+\alpha\pm\frac{1}{2}\left(1-\frac{1}{c}\right)\right]\sigma_3},\eta(\tau)=\bar{\eta}(\tau)=0$$

$$Z_{(c)}^{\mathcal{N}=4\text{ Schw.}}(\beta,\alpha)=\sum_{n\in\mathbb{Z}}\frac{E_{\mathrm{gap}}}{T}\frac{\cot{(\pi\alpha)}}{4\pi^3}\left(\frac{e^{\frac{2\pi^2}{E_{\mathrm{gap}}}\left[\frac{1}{c^2}-4\left(n+\frac{1}{2}-\alpha-\frac{1}{2c}\right)^2\right]}}{(1-2\alpha+2n)^2}-\frac{e^{\frac{2\pi^2}{E_{\mathrm{gap}}}\left[\frac{1}{c^2}-4\left(n+\frac{1}{2}+\alpha-\frac{1}{2c}\right)^2\right]}}{(1+2\alpha+2n)^2}\right)$$

$$Z_{(c)}^{\mathcal{N}=4\text{ Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)=\frac{1}{c}$$

$$\mathrm{Vol}\left[\frac{\mathrm{Diff}(S^{14})}{H_{\mathrm{Orb}}}\right]=\lim_{T\rightarrow 0}\left[Z_{(c)}^{\mathcal{N}=4\text{ Schw.}}\left(\beta,\alpha=\frac{1}{2}\right)\right]=\frac{1}{c}$$

$$\begin{aligned} I[\phi]&=\frac{\pi q_{\Lambda}\phi^{\Lambda}}{c}-\frac{4\pi}{c}\mathrm{Im}F\left(\frac{\phi^{\Lambda}+\mathrm{i}p^{\Lambda}}{2}\right)\\ &=\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}-\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}+\frac{\pi q_0\phi^0}{c}+\frac{\pi q_I\phi^I}{c} \end{aligned}$$

$$Z_{\mathrm{non-zero},c}^{Q_{\mathrm{eq}}\mathcal{V}}=\begin{cases} \mathrm{e}^{5\mathcal{K}}\mathrm{e}^{-\frac{17}{2}\mathcal{K}}, & c=1, \\ \mathrm{e}^{5\mathcal{K}}\mathrm{e}^{-\frac{15}{2}\mathcal{K}}, & c>1, \end{cases} Z_{\mathrm{zero-mode},c}^{Q_{\mathrm{eq}}\mathcal{V}}=\begin{cases} \mathrm{e}^{\frac{15}{2}\mathcal{K}}, & c=1, \\ -\frac{1}{c}\mathrm{e}^{\frac{13}{2}\mathcal{K}}, & c>1. \end{cases}$$

$$\begin{aligned} Z_{1\text{-loop},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)&=Z_{\mathrm{non-zero},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)Z_{\mathrm{zero-modes},c}^{Q_{\mathrm{eq}}\mathcal{V}}(\phi)=\frac{e^{4\mathcal{K}}}{c}\\ &=\frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4 \end{aligned}$$

$$\begin{aligned} W^{(c)}(q,p)=&\sqrt{c}K_c(\Delta)\int\mathrm{d}^{16}\phi\sqrt{\frac{-2C_{IJK}p^Ip^Jp^K\det(3C_{IJ})}{c^{16}(\phi^0)^{18}}}\frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4\\ &\times\exp\left(-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}+\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_0\phi^0}{c}-\frac{\pi q_I\phi^I}{c}\right) \end{aligned}$$

$$\int\mathrm{d}^{15}\phi\exp\left(\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_I\phi^I}{c}\right)=\sqrt{-\frac{c^{15}(\phi^0)^{15}}{\det(3C_{IJ})}}\exp\left(-\frac{\pi}{12c}\phi^0q_I C^{IJ}q_J\right)$$

$$W^{(c)}(q,p)=K_c(\Delta)\int\mathrm{d}\phi^0\frac{1}{c}\frac{(\phi^0)^{5/2}}{\mathrm{i}(-2C_{IJK}p^Ip^Jp^K)^{7/2}}\exp\left(-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}-\frac{\pi\phi^0}{c}\left(q^0+\frac{1}{12}q_I C^{IJ}q_J\right)\right)$$



$$\Delta = 4C_{IJK}p^I p^J p^K \left(q^0 + \frac{1}{12} q_I C^{IJ} q_J \right)$$

$$\sigma = \frac{-\pi C_{IJK}p^I p^J p^K}{c\phi^0}$$

$$W^{(c)}(q,p)=K_c(\Delta)\frac{2\pi}{c^{9/2}}\left(\frac{\pi}{2}\right)^{7/2}\frac{1}{2\pi i}\int\frac{d\sigma}{\sigma^{9/2}}\exp\left(\sigma+\frac{\pi^2\Delta}{4c^2\sigma}\right)$$

$$W(q,p)=\sum_c W^{(c)}(q,p)=\sum_{c=1}^{\infty} \frac{2\pi}{c^{9/2}}\left(\frac{\pi}{2}\right)^{7/2} K_c(\Delta) \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right)=W_{\rm micro}\left(\Delta\right)$$

$$Z_{\frac{1}{8}\text{-BPSBH},\Sigma}^{T=0}\left(\Delta,\Omega_E=\frac{2\pi}{\beta}\right)=Z_{\frac{1}{8}\text{-BPSBH},\Sigma}^{T=0}(\Delta,J=0),$$

$$\lim_{T\rightarrow 0} Z_{\text{Disk}}^{\mathcal{N}=4\text{ Schw.}}(\beta,\alpha)=1,\forall \alpha$$

$$\lim_{T\rightarrow 0} Z_{\text{Orb}}^{\mathcal{N}=4\text{ Schw.}}(\beta,\alpha)=\frac{1}{c},\forall \alpha$$

$$\lim_{T\rightarrow 0} Z_{\text{Disk}}^{\mathcal{N}=4\text{ Schw.}}(\beta,J)=\delta_{J,0},\lim_{T\rightarrow 0} Z_{\text{Orb}}^{\mathcal{N}=4\text{ Schw.}}(\beta,J)=\frac{\delta_{J,0}}{c}$$

$$Z_{\frac{1}{8}\text{-BPSBH,disk/orb}}^{T=0}\left(\Delta,\Omega_E=\frac{2\pi}{\beta}\right)=Z_{\frac{1}{8}\text{-BPSBH,disk/orb}}^{T=0}(\Delta,J=0).$$

$$(\tau,\psi)\sim\left(\tau+\frac{2\pi}{c},\psi-4\pi\varphi\right)\sim(\tau,\psi+2\pi)$$

$$g(\tau+\beta)=\mathrm{e}^{i2\pi\alpha\sigma_3}g(\tau)\mathrm{e}^{-i2\pi\varphi\sigma_3}$$

$$Z_{(c,\varphi)}^{\mathcal{N}=4\text{ Schw.}}(\beta,\alpha)=\sum_{n\in\mathbb{Z}}\frac{T}{E_{\text{gap}}}\frac{\cot{(\pi\alpha)}}{2\pi}\left(\mathrm{e}^{2\pi^2\frac{T}{E_{\text{gap}}}\left(\frac{1}{c^2}-4(n-\alpha+\varphi)^2\right)}-\mathrm{e}^{2\pi^2\frac{T}{E_{\text{gap}}}\left(\frac{1}{c^2}-4(n+\alpha+\varphi)^2\right)}\right)$$

$$\lim_{T\rightarrow 0} Z_{\frac{1}{8}\text{-BPSBH},(b,\varphi)}^{T=0}(\Delta,J)=0$$

$$f(\tau+2\pi)=\mathrm{e}^{\frac{b}{2}}f(\tau),g(\tau+2\pi)=\mathrm{e}^{2\pi i\alpha\sigma_3}g(\tau)\mathrm{e}^{2\pi i\varphi\sigma_3},\eta(\tau+2\pi)=-\mathrm{e}^{i2\pi i\alpha\sigma_3}\eta(\tau)$$

$$\lim_{T\rightarrow 0} Z_{\text{Trumpet } (b,\varphi)}^{\mathcal{N}=4\text{ Schw.}}(\beta,\alpha)=0,\forall \alpha$$

$$E_{\frac{1}{8}\text{-BPS BH}} \sim \Delta^{1/4}$$

$$\rho_{\text{trumpet } ,(b,\varphi)}\sim \Theta\left(E-E_{\frac{1}{8}\text{-BPS BH}}-E_{\text{gap}}\right).$$

$$\lim_{T\rightarrow 0}\int d\mu[b]d\mu[\varphi]Z_{\text{Trumpet } (b,\varphi)}^{\mathcal{N}=4\text{ Schw.}}(\beta,J)Z_{(b,\varphi)}^{\text{matter}}\text{Vol}_{g,m}=0$$

$$W(p,q)=Z_{\frac{1}{8}\text{-BPSBH}}^{T=0}(\Delta,\Omega_E\beta=2\pi)=Z_{\frac{1}{8}\text{-BPSBH}}^{T=0}(\Delta,\Omega_E=0)=d_{\frac{1}{8}\text{-BPSBH}}.$$

$$E=E_{\frac{1}{8}\text{-BPS BH}}+E_{\text{gap}}$$

$$\rho_{\text{Disk},(c,\varphi)}^{\text{cont.}}\sim\Theta\left(E-E_{\frac{1}{8}\text{-BPS BH}}-E_{\text{gap}}\right),\rho_{\text{Orbifold},c}^{\text{cont.}}\sim\Theta\left(E-E_{\frac{1}{8}\text{-BPS BH}}-E_{\text{gap}}\right)$$

$$E_{\text{gap}}\sim\frac{1}{\Delta^{3/2}},\Delta\rightarrow\infty$$

$$\rho_{\text{orbifold},(c,\varphi)}\sim\Theta\left(E-E_{\frac{1}{8}\text{-BPS BH}}-E_{\text{gap}}\right),\rho_{\text{trumpet},(b,\varphi)}\sim\Theta\left(E-E_{\frac{1}{8}\text{-BPS BH}}-E_{\text{gap}}\right),$$

$$E_{\frac{1}{8}\text{-BPS BH}}+E_{\text{gap}}\left(1+O(1/\Delta)\right)$$

$$\int_{E_{\frac{1}{8}\text{-BPSBH}}}^{E_{\frac{1}{8}\text{-BPSBH}}+\#E_{\text{gap}}}dE\rho(E)=d_{\frac{1}{8}\text{-BPSBH}}$$

$$\rho_{\text{TFD}}=|\text{TFD}(T=0)\rangle\langle\text{TFD}(T=0)|,\rho_L=\text{Tr}_R(\rho_{\text{TFD}})$$

$$S_\alpha=\frac{1}{1-\alpha}\frac{\text{Tr}\left(\rho_L\right)^\alpha}{\left(\text{Tr}\,\rho_L\right)^\alpha}=\frac{1}{1-\alpha}\left[\frac{\overbrace{\left(\text{circle with red dots}\right)\left(\text{oval with green dots}\right)}^{\alpha\text{ copies of segments of proper length }\beta}}{\left(\text{circle with red dots}\right)\left(\text{oval with green dots}\right)^\alpha}\right]$$

$$=\frac{\left(\text{circle with red dots}\right)\left(\text{oval with green dots}\right)+\left(\text{circle with red dots}\right)\left(\text{trapezoid with green dots}\right)^{1-\alpha}}{1-\alpha}=\frac{\left(d_{\frac{1}{8}\text{-BPS BH}}\right)^{1-\alpha}}{1-\alpha}.$$

$$\vartheta_1(\tau,z)=\sum_{n\in\mathbb{Z}+\frac{1}{2}}(-1)^nq^{n^2/2}\zeta^n$$

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^ke^{\frac{2\pi imcz^2}{c\tau+d}}\varphi(\tau,z)\,\forall\,\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2;\mathbb{Z})$$

$$\varphi(\tau,z+\lambda\tau+\mu)=\mathrm{e}^{-2\pi im(\lambda^2\tau+2\lambda z)}\varphi(\tau,z)\,\forall\,\lambda,\mu\in\mathbb{Z}$$

$$\varphi(\tau,z)=\sum_{n,r\in\mathbb{Z}}c(n,r)q^n\zeta^r$$

$$c(n,r)=C_r(4nm-r^2)\because C_r(\Delta)\parallel r\mathrm{mod}2m.$$

$$\varphi(\tau,z)=\sum_{n,\ell\in\mathbb{Z}}c(n,\ell)q^n\zeta^\ell$$

$$c(n,\ell)=C_v(\Delta), \Delta=4mn-\ell^2, v=\ell\mathrm{mod}2m$$

$$C_v(\Delta)=\sum_{c=1}^{\infty}\left(\frac{c}{2\pi}\right)^{k-\frac{5}{2}}\sum_{\widetilde{\Delta}<0}C_{\mu}(\widetilde{\Delta})\mathrm{Kl}(\Delta,v,\widetilde{\Delta},\widetilde{v};c)\left|\frac{\widetilde{\Delta}}{4k}\right|^{\frac{3}{2}-k}I_{\frac{3}{2}-k}\left(\frac{\pi}{c}\sqrt{|\widetilde{\Delta}|\Delta}\right)$$

$$\tilde{I}_\rho(z)=\frac{1}{2\pi\mathrm{i}}\int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty}\frac{d\sigma}{\sigma^{\rho+1}}\exp\left(\sigma+\frac{z^2}{4\sigma}\right)$$

$$\tilde{I}_\rho(z)=\left(\frac{z}{2}\right)^{-\rho}I_\rho(z)$$

$$I_\rho(z)\sim \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\bigg(1-\frac{(\mu-1)}{8z}+\frac{(\mu-1)(\mu-3^2)}{2!(8z)^3}-\frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!(8z)^5}+\cdots\bigg).$$

$$\mathrm{Kl}(\Delta,v;\widetilde{\Delta},\widetilde{v};c):=\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(\frac{1}{2}-k)}\sum_{\substack{-c\leq d<0\\(d,c)=1}}\mathrm{e}^{2\pi\mathrm{i}\frac{d}{c}(\Delta/4m)}M(\gamma_{c,d})_{v\widetilde{v}}^{-1}\mathrm{e}^{2\pi\mathrm{i}\frac{a}{c}(\widetilde{\Delta}/4m)}$$

$$\vartheta_{m,\mu}(\tau,z)=\sum_{\ell=\mu\mathrm{mod}2m}q^{\ell^2/4m}\zeta^\ell$$

$$\vartheta_{m,\mu}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^{\frac{1}{2}}\mathrm{e}^{2\pi\mathrm{i}m\frac{cz^2}{c\tau+d}}M(\gamma)_{v\mu}^{-1}\vartheta_{\mu,\ell}(\tau,z)$$

$$M^{-1}(\gamma)_{v\mu}=\frac{\frac{\pi\mathrm{i}}{4}\operatorname{sign}(c)}{\sqrt{2r|c|}}\exp\left(-\frac{\mathrm{i}\pi}{6}\Phi(\gamma)\right)\\ \sum_{\varepsilon=\pm}\sum_{n=0}^{c-1}\varepsilon\exp\left(\frac{\mathrm{i}\pi}{2rc}(d(v+1)^2-2(v+1)(2rn+\varepsilon(\mu+1))+a(2rn+\varepsilon(\mu+1))^2)\right)$$

$$\Phi(\gamma)=\frac{a+d}{c}-12\operatorname{sign}(c)s(a,|c|)$$

$$s(a,c)=\frac{1}{4c}\sum_{j=1}^{c-1}\cot\left(\frac{\pi j}{c}\right)\cot\left(\frac{\pi ja}{c}\right).$$

$$\delta_\xi\phi^i=R(\phi)^i_\alpha\xi^\alpha,$$

$$\delta_{\xi_1}\delta_{\xi_2}-\delta_{\xi_2}\delta_{\xi_1}=\delta_{\xi_3}$$

$$\xi_3^\alpha=f(\phi)_{\beta\gamma}{}^\alpha\xi_1^\beta\xi_2^\gamma$$



$$\delta_{\mathrm{brst}}\phi^i=R(\phi)^i_{\alpha}\Lambda c^{\alpha},\delta_{\mathrm{brst}}c^{\alpha}=\frac{1}{2}f(\phi)_{\beta\gamma}{}^{\alpha}c^{\beta}\Lambda c^{\alpha}$$

$$\delta_{\mathrm{brst}}^2=0$$

$$\delta_{\mathrm{brst}}\,\tilde{\phi}^i=\delta_{\mathrm{brst}}\,\phi^i-\delta_{\mathrm{brst}}\,\overline{\phi^i},\delta_{\mathrm{brst}}\,\tilde{c}^{\alpha}=\delta_{\mathrm{brst}}\,c^{\alpha}-\delta_{\mathrm{brst}}\,\bar{c}^{\alpha}$$

$$\delta_{\mathrm{brst}}\overline{\phi^i}=\Lambda\bar{c}^{\alpha}R(\bar{\phi})^i_{\alpha}=0$$

$$\delta_{\mathrm{brst}}\Rightarrow\delta_{\mathrm{eq}}$$

$$\delta_{\mathrm{eq}}\bar{\phi}^i=0,\quad\delta_{\mathrm{eq}}\tilde{\phi}^i=R(\phi)^i_{\alpha}\Lambda(\bar{c}+\tilde{c})^{\alpha}\\ \delta_{\mathrm{eq}}\bar{c}^{\alpha}=0,\quad\delta_{\mathrm{eq}}\tilde{c}^{\alpha}=\frac{1}{2}f(\phi)^{\alpha}_{\beta\gamma}(\bar{c}+\tilde{c})^{\beta}\Lambda(\bar{c}+\tilde{c})^{\alpha}-\frac{1}{2}f(\bar{\phi})^{\alpha}_{\beta\gamma}\bar{c}^{\beta}\Lambda\bar{c}^{\alpha}$$

$$\delta_{\mathrm{eq}}^2=\delta_{\bar{\xi}}\neq 0, \left[\delta_{\mathrm{eq}},\delta_{\bar{\xi}}\right]=0$$

$$\delta_{\bar{\xi}}\bar{\phi}=0,\delta_{\bar{\xi}}\tilde{\phi}^i=\bar{\xi}^{\alpha}R(\phi^i)_{\alpha}^i,\delta_{\bar{\xi}}\bar{c}^{\alpha}=0,\delta_{\bar{\xi}}\tilde{c}^{\alpha}=f(\phi)^{\alpha}_{\beta\gamma}(\bar{c}+\tilde{c})^{\beta}\bar{\xi}^{\gamma}$$

$$\bar{\xi}=\Lambda_{[2}f(\bar{\phi})_{\beta\gamma}{}^{\alpha}\bar{c}^{\beta}\Lambda_{1]}\bar{c}^{\gamma}$$

$$\delta_{\mathrm{eq}}\tilde{b}_{\alpha}=\Lambda B_{\alpha},\delta_{\mathrm{eq}}B^{\alpha}=\frac{1}{2}f(\bar{\phi})_{\delta\epsilon}{}^{\beta}\bar{c}^{\delta}\Lambda\bar{c}^{\epsilon}f(\bar{\phi})^{\gamma}_{\alpha\beta}$$

$$\delta_{\mathrm{eq}}^2\,b_{\alpha}=\delta_{\bar{\xi}}\,b_{\alpha}=\bar{\xi}f(\phi)_{\beta\alpha}{}^{\gamma}b_{\gamma},\delta_{\mathrm{eq}}^2\,B_{\alpha}=\delta_{\bar{\xi}}\,B_{\alpha}=\bar{\xi}f(\phi)_{\beta\alpha}{}^{\gamma}B_{\gamma}$$

$$\begin{array}{l} \Sigma^{45}\rightarrow X^1,\Sigma^{67}\rightarrow X^2,\Sigma^{68}\rightarrow X^3,\Sigma^{69}\rightarrow X^4,\Sigma^{78}\rightarrow X^5,\Sigma^{79}\rightarrow X^6,\Sigma^{89}\rightarrow X^7\\ \Sigma^{46}\rightarrow X^8,\Sigma^{47}\rightarrow X^9,\Sigma^{48}\rightarrow X^{10},\Sigma^{49}\rightarrow X^{11},\Sigma^{56}\rightarrow X^{12}\\ \Sigma^{57}\rightarrow X^{13},\Sigma^{58}\rightarrow X^{14},\Sigma^{59}\rightarrow X^{15} \end{array}$$

$$F_{\mathrm{trunc.}}\left(X\right)=-\frac{1}{2}\frac{X^1C_{ab}X^aX^b}{X^0},a=2,...,7$$

$$\int \prod_{I=8}^{15} \mathrm{d}\phi^I \mathrm{exp}\left(\sum_{I,J=8}^{15} \frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}\right)\Bigg|_{p^8=\ldots=p^{15}=0}=\frac{4c^4(\phi^0)^4}{(P^2)^2}, P^2=p^aC_{ab}p^b$$

$$[{\rm d}^8\phi]_c\equiv\,{\rm d}^8\phi\,\sqrt{2{\rm det}'\left(\frac{1}{2c}{\rm Im}F_{\Lambda\Sigma}\right)}$$

$$W^{(c)}(q,p)=\sqrt{c}K_c(\Delta)\int\,[{\rm d}^8\phi]_c c^{-1} {\rm e}^{4{\mathcal K}(\phi)} e^{\frac{4\pi}{c}{\rm Im}F_{\rm trunc}\left(\frac{\phi^\Lambda+i p^\Lambda}{2}\right)}\frac{\pi}{c}q_\Lambda\phi^\Lambda$$

$$A^{\pm}=\left(\omega^a\pm\frac{e^a}{l}\right)T_a$$

$$\omega_a\equiv\frac{1}{2}\epsilon_{abc}\omega^{bc}$$

$$I=I_{CS}(A^+)-I_{CS}(A^-)$$

$$I_{CS}(A^\pm)=\frac{\mathrm{k}}{4\pi}\int\mathrm{Tr}\left[A^\pm\wedge dA^\pm+\frac{2}{3}(A^\pm)^3\right]+\mathcal{B}_\infty(A^\pm)$$

$$\mathrm{k}=\frac{l}{4G}$$

$$\mathrm{Tr}(\mathrm{L}_0\,\mathrm{L}_0)=\frac{1}{2},\mathrm{Tr}(\,\mathrm{L}_1\,\mathrm{L}_{-1})=-1$$

$$g_{\mu\nu}=\frac{l^2}{2}\mathrm{Tr}((A^+-A^-)_{\mu}(A^+-A^-)_{\nu})$$

$$A^\pm=b_\pm^{-1}(d+a^\pm)b_\pm$$

$$a^\pm(t,\theta)=(\xi_\pm(t,\theta)dt\pm p_\pm(t,\theta)d\theta)L_0$$

$$dA^\pm + A^{\pm 2} = 0$$

$$\dot{p}_\pm(t,\theta)=\pm\xi'_\pm(t,\theta)$$

$$\delta\mathcal{B}_\infty^\pm=-\frac{\mathrm{k}}{2\pi}\int\,dtd\theta\mathrm{Tr}\big(A_t^\pm\delta A_\theta^\pm\big)=\mp\frac{\mathrm{k}}{4\pi}\int\,dtd\theta\xi_\pm\delta p_\pm$$

$$\xi_\pm=-\frac{4\pi}{\mathrm{k}}\frac{\delta H^\pm}{\delta p_\pm}$$

$$H^\pm=\int\,d\theta\mathcal{H}^\pm(p_\pm,p'_\pm,\cdots)$$

$$\mathcal{B}_\infty^\pm=\pm\int\,dtd\theta\mathcal{H}^\pm=\pm\int\,dtH^\pm$$

$$\dot{p}_\pm(t,\theta)=\mp\frac{4\pi}{\mathrm{k}}\frac{\partial}{\partial\theta}\Big(\frac{\delta H^\pm}{\delta p_\pm}\Big)$$

$$\dot{p}_\pm(t,\theta)=\{p_\pm(t,\theta),H^\pm\}_{PB}$$

$$\{p_\pm(t,\theta),p_\pm(t,\theta')\}_{PB}=\mp\frac{4\pi}{\mathrm{k}}\frac{\partial}{\partial\theta}\delta(\theta-\theta')$$

$$H_{CFT}=\int\,d\theta\sigma(t,\theta)\left[\frac{1}{2}\Big(\frac{\partial\Pi(t,\theta)}{\partial\theta}\Big)^2+\frac{\pi^2}{6}\sigma^2(t,\theta)+W(\theta)\right]$$

$$\partial_t\sigma(t,\theta)+\partial_\theta(\sigma(t,\theta)v(t,\theta))=0$$

$$\partial_tv(t,\theta)+v(t,\theta)\partial_\theta v(t,\theta)=-\partial_\theta\left(\frac{\pi^2}{2}\sigma^2(t,\theta)+W(\theta)\right)$$

$$v(t,\theta)=\partial_\theta\Pi(t,\theta)$$

$$p_\pm(t,\theta)=v(t,\theta)\pm\pi\sigma(t,\theta)$$



$$\partial_t p_{\pm}(t,\theta) + p_{\pm}(t,\theta) \partial_{\theta} p_{\pm}(t,\theta) + W'(\theta) = 0$$

$$H_{CFT} = H_{CFT}^{+} + H_{CFT}^{-}, \text{ where } H_{CFT}^{\pm} = \pm \frac{1}{2\pi} \int d\theta \left(\frac{p_{\pm}^3(t,\theta)}{6} + W(\theta) p_{\pm}(t,\theta) \right)$$

$$\delta a^{\pm} = d\lambda^{\pm} + [a^{\pm}, \lambda^{\pm}]$$

$$\delta p_{\pm} = \eta'_{\pm}, \text{ and } \delta \xi_{\pm} = \dot{\eta}_{\pm}$$

$$Q_n^{\pm} = \pm \frac{k}{4\pi} \int d\theta \sum_{k=0}^n \frac{{}^nC_k}{2^k(2k+1)} p_{\pm}^{2k+1}(t,\theta) W^{n-k}(\theta)$$

$$\{Q_m^{\pm}, Q_n^{\pm}\}_{PB} = 0, \forall m, n \geq 1.$$

$$H_{(1)}^{\pm} = \frac{C_{\pm}k}{2} H_{CFT}^{\pm} = \pm \frac{C_{\pm}k}{4\pi} \int d\theta \left(\frac{p_{\pm}^3(t,\theta)}{6} + W(\theta) p_{\pm}(t,\theta) \right)$$

$$\xi^{\pm} = \mp C_{\pm} \left(\frac{p_{\pm}^2(t,\theta)}{2} + W(\theta) \right)$$

$$p_{\pm}(\theta)\partial_{\theta}p_{\pm}(\theta)+W'(\theta)=0$$

$$\frac{\partial}{\partial \theta}(v(\theta)\sigma(\theta))=0, \text{ and } \frac{\partial}{\partial \theta}\left(\frac{\sigma^2(\theta)+v^2(\theta)}{2}\right)+W'(\theta)=0.$$

$$\mathrm{Hol}_{\tau}(A^{\pm})\sim \exp{(\beta_{\pm}\xi^{\pm})}\sim \mathbb{I}\Rightarrow \beta_{\pm}\xi^{\pm}=\mp 2\pi$$

$$\sigma=\frac{(\beta_{-}+\beta_{+})l}{\beta_{-}\beta_{+}}, v=\frac{\pi l(\beta_{-}-\beta_{+})}{\beta_{-}\beta_{+}}$$

$$b_{\pm}(r)=\exp\left(\pm\frac{r}{2l}(L_{+}-L_{-})\right)$$

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2\left(d\phi+\frac{A_\phi}{r^2}dt\right)^2$$

$$f(r)=\frac{r^2}{l^2}-\frac{2\pi^2l^2(\beta_-^2+\beta_+^2)}{\beta_-^2\beta_+^2}+\frac{A_\phi^2}{r^2}, \text{ and } A_\phi=\pi^2l^3\left(\frac{1}{\beta_+^2}-\frac{1}{\beta_-^2}\right)$$

$$p_{\pm}=\pm\frac{2\pi l}{\beta_{\pm}}$$

$$C_{\pm}=\frac{\beta_{\pm}}{\pi l^2}$$

$$r_{\pm}=\pi l^2\left(\frac{1}{\beta_{+}}\pm\frac{1}{\beta_{-}}\right)$$

$$M=\frac{\pi^2\sigma^2+v^2}{8G}, \text{ and } J=\frac{\pi l\sigma v}{4G}$$



$$|v| \leq \pi \sigma.$$

$$v\sigma = \frac{4GJ}{\pi l} \text{ and } \sigma^2 = \frac{4G\left(\sqrt{l^2M^2 - J^2} + lM\right)}{\pi^2 l}.$$

$$r_+ = l\pi\sigma \text{ and } r_- = |lv|.$$

$$\beta = \frac{\beta_+ + \beta_-}{2},$$

$$T = \frac{1}{\beta} = \frac{\pi^2 \sigma^2 - v^2}{2\pi^2 l \sigma}$$

$$\frac{\beta_- - \beta_+}{2} = \frac{2\pi l v}{\pi^2 \sigma^2 - v^2} = \frac{v}{\pi \sigma} \beta$$

$$H_{(2)}^\pm = \pm \frac{\mathcal{C}_\pm k}{4\pi} \int d\theta \left(\frac{p_\pm^2(t,\theta)}{2} + W(\theta) p_\pm(t,\theta) \right)$$

$$S=\frac{2\pi r_+}{4G}=\frac{\pi^2 l^2}{2G}\Big(\frac{1}{\beta_+}+\frac{1}{\beta_-}\Big)$$

$$\mathcal{H} = \mathcal{H}^+ \otimes \mathcal{H}^-.$$

$$p_{\pm}(t,\theta)=2\pi\sqrt{\frac{2}{\mathbf{k}}}\tilde{p}_{\pm}(t,\theta)$$

$$\{\tilde{p}_{\pm}(t,\theta),\tilde{p}_{\pm}(t,\theta')\}_{PB}=\mp\frac{1}{2\pi}\frac{\partial}{\partial\theta}\delta(\theta-\theta')$$

$$c=\frac{3l}{2G},$$

$$[\tilde{p}_{\pm}(t,\theta),\tilde{p}_{\pm}(t,\theta')]=\mp\frac{i}{2\pi c}\frac{\partial}{\partial\theta}\delta(\theta-\theta')$$

$$\tilde{p}_{\pm}(t,\theta)=\frac{1}{2\pi\sqrt{c}}\sum_n\alpha_{\pm n}^{\pm}e^{in\theta}$$

$$[\alpha_m^\pm,\alpha_n^\pm]=m\delta_{m+n}$$

$$\psi(t,\theta)=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi_{m-1/2}e^{im\theta},\psi^\dagger(t,\theta)=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\psi_{m-1/2}^\dagger e^{-im\theta}$$

$$\{\psi_q,\psi_r\}=0,\{\psi_q^\dagger,\psi_r^\dagger\}=0,\{\psi_q,\psi_r^\dagger\}=\delta_{q,r}$$

$$\begin{aligned}\{\psi(t,\theta),\psi(t,\phi)\}&=0,\{\psi^\dagger(t,\theta),\psi^\dagger(t,\phi)\}=0\\ \{\psi(t,\theta),\psi^\dagger(t,\phi)\}&=\delta(\theta-\phi)\end{aligned}$$

$$:\psi_q^\dagger\psi_r:=\begin{cases}\psi_q^\dagger\psi_r & r>0\\-\psi_r\psi_q^\dagger & r<0\end{cases}$$

$$:\psi_q^\dagger\psi_r:=\psi_q^\dagger\psi_r-\langle\psi_q^\dagger\psi_r\rangle$$

$$B_n^K=\sum_{m\in\mathbb{Z}}m^K:\psi_{m-n-1/2}^\dagger\psi_{m-1/2}:$$

$$[B_m^{K_1},B_n^{K_2}]=\sum_{q\in\mathbb{Z}}((q-n)^{K_1}q^{K_2}-(q-m)^{K_2}q^{K_1}): \psi_{q-m-n-1/2}^\dagger\psi_{q-1/2}:\\ +\delta_{m+n}\sum_{l=0}^{m-1}l^{K_1}(l-m)^{K_2}$$

$$:\psi^\dagger(t,\theta)\psi(t,\theta):=\frac{1}{2\pi}\sum_{m,n\in\mathbb{Z}}:\psi_{m-n-1/2}^\dagger\psi_{m-1/2}:e^{in\theta}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}B_n^0e^{in\theta}.$$

$$[B_m^0,B_n^0]=m\delta_{m+n}.$$

$$:\psi^\dagger(t,\theta)\psi(t,\theta):\equiv\sqrt{c}\tilde{p}_\pm(t,\theta).$$

$$\psi(\theta)=\frac{1}{\sqrt{2\pi}}:e^{-2\pi i\varphi(\theta)}$$

$$\tilde{p}(\theta)=\frac{\varphi'(\theta)}{\sqrt{c}}.$$

$$2\pi c:\frac{\tilde{p}^2(\theta)}{2}:= -\frac{i}{2}:(\psi^\dagger\partial_\theta\psi-\partial_\theta\psi^\dagger\psi):\\ =\frac{1}{2\pi}\sum_p\left(B_p^1-\frac{p}{2}B_p^0\right)e^{ip\theta}$$

$$(2\pi)^2c^{3/2}:\frac{\tilde{p}^3(\theta)}{3}:= -\frac{1}{6}:\partial_\theta^2\psi^\dagger\psi+\psi^\dagger\partial_\theta^2\psi-4\partial_\theta\psi^\dagger\partial_\theta\psi:\\ =\frac{1}{2\pi}\sum_p\left(B_p^2-pB_p^1+\frac{p^2}{6}B_p^0\right)e^{ip\theta}$$

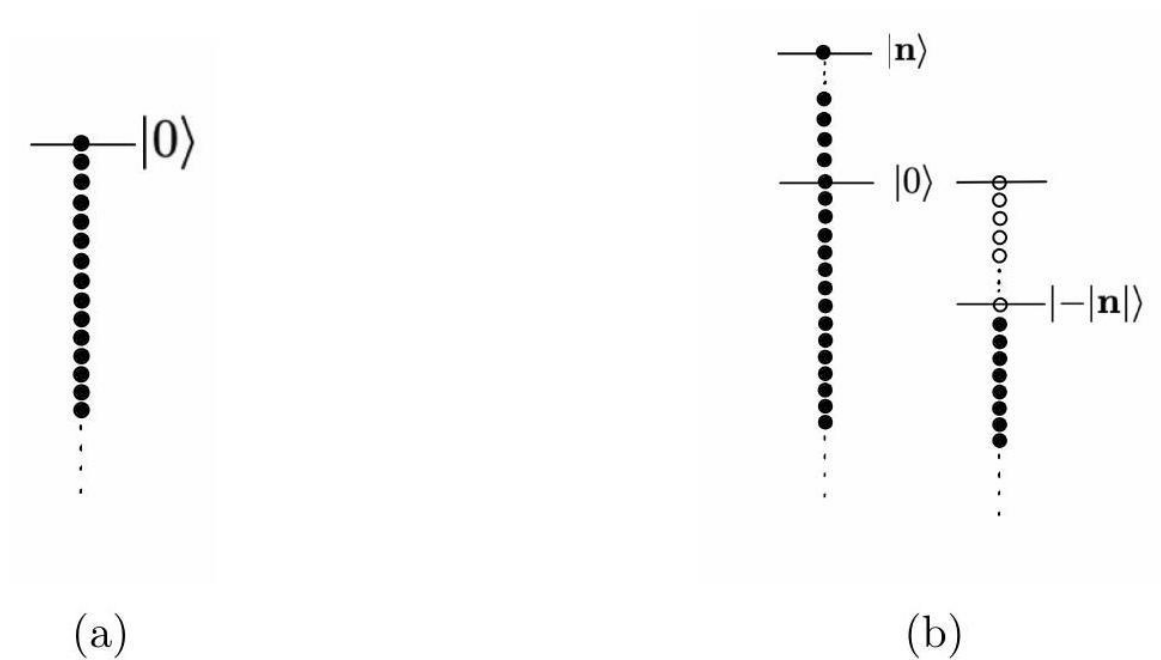
$$\hat{p}_\pm^2(t,\theta):=\frac{24}{c^2}\sum_p\left(B_p^1-\frac{p}{2}B_p^0\right)e^{ip\theta}$$

$$H^\pm=\pm\mathcal{C}_\pm\frac{\sqrt{3}}{c^2}B_0^2$$

$$Q_n^\pm=\pm\frac{1}{2\sqrt{3}}\Big(\frac{6}{c^2}\Big)^nB_0^{2n}$$

$$\psi_{k+\frac{1}{2}}|0\rangle=0,\forall k\geq 0$$

$$\psi_{k-\frac{1}{2}}^{\dagger}|0\rangle=0,\forall k\leq 0$$



$$\prod_{i=1}^{n_p}\prod_{j=1}^{n_h}\psi_{n_i+\frac{1}{2}}^{\dagger}\psi_{-m_j-\frac{1}{2}}|0\rangle.$$

$$|\mathbf{n}\rangle=\begin{cases}\psi_{\mathbf{n}-\frac{1}{2}}^{\dagger}\cdots\psi_{\frac{1}{2}}^{\dagger}|0\rangle & \mathbf{n}>0,\\ \psi_{\mathbf{n}+\frac{1}{2}}\cdots\psi_{-\frac{1}{2}}|0\rangle & \mathbf{n}<0.\end{cases}$$

$$\left[B_m^K,\psi_{k-\frac{1}{2}}\right]=-(k+m)^K\psi_{k+m-\frac{1}{2}},\left[B_m^K,\psi_{k-\frac{1}{2}}^{\dagger}\right]=(k-m)^K\psi_{k-m-\frac{1}{2}}^{\dagger},$$

$$B_0^0|\mathbf{n}\rangle=\mathbf{n}|\mathbf{n}\rangle,B_m^0|\mathbf{n}\rangle=0,\forall m>0.$$

$$|\vec{k};\mathbf{n}\rangle=\prod_{m>0}(B_{-m}^0)^{k_m}|\mathbf{n}\rangle$$

$$\langle \vec{k};\mathbf{n}\mid \vec{k}';\mathbf{n}\rangle=z_{\vec{k}}^{-1}\delta_{\vec{k},\vec{k}'},z_{\vec{k}}=\prod_mk_m!m^{k_m}$$

$$\mathcal{H}=\bigoplus_{\mathbf{n}\in\mathbb{Z}}\mathcal{H}_{\mathbf{n}}.$$

$$|\vec{m};\vec{n};\mathbf{n}\rangle=\prod_i\psi_{\mathbf{n}+n_i+\frac{1}{2}}^{\dagger}\psi_{\mathbf{n}-m_i-\frac{1}{2}}|\mathbf{n}\rangle,n_i,m_i\geq 0$$

$$\langle R';\mathbf{n}\mid R;\mathbf{n}\rangle=\delta_{RR'}$$

$$|\vec{m}; \vec{n}; \mathbf{n}\rangle = |R; \mathbf{n}\rangle = \prod_i \psi_{\mathbf{n}+n_i+\frac{1}{2}}^\dagger \psi_{\mathbf{n}-m_i-\frac{1}{2}} |\mathbf{n}\rangle, n_i, m_i \geq 0$$

$$\mathcal{E}^K = \mathcal{E}_R^K(n_i, m_i) + g_K(\mathbf{n}),$$

$$\mathcal{E}_R^K(n_i, m_i) = \sum_i ((\mathbf{n} + n_i + 1)^K - (\mathbf{n} - m_i)^K),$$

$$\mathcal{E}_R^0 = 0, g_0(\mathbf{n}) = \mathbf{n}$$

$$\mathcal{E}_R^1 = \sum_i (n_i + m_i + 1), g_1(\mathbf{n}) = \frac{\mathbf{n}(\mathbf{n} + 1)}{2}$$

$$\mathcal{E}_R^2 = \sum_i \left(\left(n_i + \frac{1}{2} \right)^2 - \left(m_i + \frac{1}{2} \right)^2 \right) + (2\mathbf{n} + 1) \sum_i (n_i + m_i + 1),$$

$$g_2(\mathbf{n}) = \frac{\mathbf{n}(\mathbf{n} + 1)(2\mathbf{n} + 1)}{3}$$

$$\sum_i (n_i + m_i + 1) = \sum_{i=1}^{v_1} l_i = |R| \equiv \mathfrak{R}$$

$$\mathcal{E}_R^1 = |R|.$$

$$\sum_i \left(\left(n_i + \frac{1}{2} \right)^2 - \left(m_i + \frac{1}{2} \right)^2 \right) = \sum_{i=1}^{v_1} l_i(l_i - 2i) + |R| \equiv \kappa_R$$

$$\mathcal{E}_R^2 = \kappa_R + (2\mathbf{n} + 1)|R|.$$

$$\mathcal{E}_R^2 = -\kappa_R - (2|\mathbf{n}| - 1)|R|, \text{ for } \mathbf{n} < 0.$$

$$\langle \Psi | : \hat{v}(t, \theta) \hat{\sigma}(t, \theta) : | \Psi \rangle = \frac{4GJ}{\pi l} \text{ and } \langle \Psi | : \pi^2 \hat{\sigma}^2(t, \theta) + \hat{v}^2(t, \theta) : | \Psi \rangle = 8GM$$

$$\langle \Psi^\pm | : \hat{p}_\pm^2(t, \theta) : | \Psi^\pm \rangle = \frac{12}{c} (Ml \pm J) = \frac{4\pi^2 l^2}{\beta_\pm^2}$$

$$\langle \mathbf{n}_\pm | \hat{p}_\pm | \mathbf{n}_\pm \rangle = \frac{2\sqrt{3}}{c} \mathbf{n}_\pm, \langle \mathbf{n}_\pm | \hat{p}_\pm^2 | \mathbf{n}_\pm \rangle = \frac{12}{c^2} (\mathbf{n}_\pm)^2$$

$$\langle R^\pm; \mathbf{n}_\pm | B_0^0 | R^\pm; \mathbf{n}_\pm \rangle = \mathbf{n}_\pm, \langle R^\pm; \mathbf{n}_\pm | B_0^1 | R^\pm; \mathbf{n}_\pm \rangle = \frac{\mathbf{n}_\pm(\mathbf{n}_\pm + 1)}{2} + |R^\pm|,$$

$$\langle R^\pm; \mathbf{n}_\pm | \hat{p}_\pm | R^\pm; \mathbf{n}_\pm \rangle = \frac{2\sqrt{3}}{c} \mathbf{n}_\pm, \langle R^\pm; \mathbf{n}_\pm | \hat{p}_\pm^2 | R^\pm; \mathbf{n}_\pm \rangle = \frac{24}{c^2} \left(\frac{\mathbf{n}_\pm(\mathbf{n}_\pm + 1)}{2} + |R^\pm| \right).$$

$$\langle R^\pm | : \hat{p}_\pm^2(t, \theta) : | R^\pm \rangle = \frac{24}{c^2} |R^\pm|$$

$$|R^\pm| = \frac{c}{2} (Ml \pm J) = \frac{c^2 \pi^2 l^2}{6\beta_\pm^2}$$



$$d(p_+,p_-)=e^{2\pi\sqrt{\frac{|R^+|}{6}}+2\pi\sqrt{\frac{|R^-|}{6}}}$$

$$S_{\rm stat}=\ln\,d(p_+,p_-)=2\pi\sqrt{\frac{|R^+|}{6}}+2\pi\sqrt{\frac{|R^-|}{6}}=\frac{\pi^2l^2}{2G}\Big(\frac{1}{\beta_+}+\frac{1}{\beta_-}\Big)$$

$$S=\log\, \mathcal{Z}+\beta \bar{E}$$

$$\mathcal{Z}=\mathrm{Tr}\mathrm{exp}\left(-\beta_+H_{(1)}^+\right)\mathrm{Tr}\mathrm{exp}\left(-\beta_-H_{(1)}^-\right)$$

$$\mathbf{n}_{\pm}=\pm\frac{\pi cl}{\sqrt{3}\beta_{\pm}}$$

$$\mathcal{Z}=\sum_{R^+}\exp\left[-\frac{\sqrt{3}}{\pi c^2}\Big(\frac{\beta_+}{l}\Big)^2\left\langle R^+;\mathbf{n}_+|B_0^2|R^+;\mathbf{n}_+\right\rangle\right]\cdot\sum_{R^-}\exp\left[\frac{\sqrt{3}}{\pi c^2}\Big(\frac{\beta_-}{l}\Big)^2\left\langle R^-;\mathbf{n}_-|B_0^2|R^-;\mathbf{n}_-\right\rangle\right].$$

$$\begin{aligned}\langle R^\pm;\mathbf{n}_\pm|B_0^2|R^\pm;\mathbf{n}_\pm\rangle&=\pm\big(\kappa_{R^\pm}+2|\mathbf{n}_\pm||R^\pm|\big)+g_2(\mathbf{n}_\pm)\\&=\pm\bigg(\sum_i l_i^\pm(l_i^\pm-2i)+|R^\pm|+(2|\mathbf{n}_\pm|\pm1)|R^\pm|\bigg)+g_2(\mathbf{n}_\pm)\end{aligned}$$

$$\mathcal{Z}_\pm=\sum_{R^\pm}\exp\left[-\frac{\sqrt{3}}{\pi c^2}\Big(\frac{\beta_\pm}{l}\Big)^2\bigg(\sum_i l_i^\pm(l_i^\pm+1-2i)+(2|\mathbf{n}_\pm|\pm1)|R^\pm|\bigg)\right].$$

$$\mathcal{Z}_\pm=\sum_{R^\pm}\exp\left[-\frac{\tilde{A}_\pm}{2|\mathbf{n}_\pm|}\sum_i l_i^\pm(l_i^\pm+1-2i)-\tilde{A}_\pm\sum_i l_i^\pm\right]$$

$$\tilde{A}_\pm=2\left(\frac{\beta_\pm}{cl}\right)$$

$$\mathcal{F}^\pm=\ln\,\mathcal{Z}_\pm$$

$$\mathcal{F}^\pm=\sum_{g=0}^\infty N_\pm^{2-2g}F_g^\pm(\tilde{A}_\pm)$$

$$F_0^\pm(\tilde{A}_\pm)=-\frac{\tilde{A}_\pm}{48}, F_1^\pm(\tilde{A}_\pm)=-\log\,\eta(Q_\pm)$$

$$\eta(Q_\pm)\colon=Q_\pm^{1/24}\prod_{n=1}^\infty(1-Q_\pm^n)$$

$$F_g(\tilde{A}_\pm)\sim (\tilde{A}_\pm)^{-2g+1}\pi^{2g}\mathbf{c}_g, g\geq 2$$

$$F_1(\tilde{A}_{\pm})=\frac{\pi^2}{3\tilde{A}_{\pm}}+\frac{1}{2}\ln\tilde{A}_{\pm}$$

$$\mathcal{F}^{\pm}=\mathcal{F}_0^{\pm}+\frac{1}{2}\ln\tilde{A}_{\pm}$$

$$\mathcal{F}_0^{\pm}=\frac{\pi^2}{3\tilde{A}_{\pm}}\sum_{g\geq 0}\frac{3^g\mathbf{c}_g}{16^{g-1}},\text{ with }\mathbf{c}_0=-\frac{1}{192},\text{ and }\mathbf{c}_1=\frac{1}{3}$$

$$\mathcal{F}_0^{\pm}=\frac{\pi^2}{3\tilde{A}_{\pm}}-\frac{\pi^2}{3\tilde{A}_{\pm}}\bigg(\frac{1}{12}-16\sum_{g\geq 2}\left(\frac{3}{16}\right)^g\mathbf{c}_g\bigg)$$

$$\Delta S = \frac{1}{2} \ln \left(\frac{\beta_+ \beta_-}{c^2 l^2} \right) = - \frac{1}{2} \ln \left(\frac{9 l^4}{4 G^2 \beta^2 (1 - l^2 \Omega^2)} \right)$$

$$Z_{\pm}=\mathrm{Tr}\left[\exp\left(-\frac{k}{4\pi}\frac{\beta_{\pm}}{l}\int d\theta\frac{p_{\pm}^2(t,\theta)}{2}\right)\right]=\mathrm{Tr}\exp\left(-\frac{\beta_{\pm}}{cl}B_0^1(p_{\pm})\right)$$

$$B_0^1|R^{\pm},\mathbf{n}\rangle=\left(|R^{\pm}|+\frac{1}{2}\mathbf{n}(\mathbf{n}+1)\right)|R^{\pm},\mathbf{n}\rangle.$$

$$\mathcal{Z}_{\pm}=\sum_{R^{\pm}}\exp\left(-\frac{\beta_{\pm}}{cl}|R^{\pm}|\right)$$

$$\mathcal{Z}_{\pm}=\sum_{n=0}^{\infty}p(n)e^{-\mu_{\pm}n}$$

$$\mathcal{Z}_{\pm}=\prod_{k=1}^{\infty}\frac{1}{1-e^{-\mu_{\pm}k}},\mu_{\pm}>0$$

$$\log\mathcal{Z}_{\pm}=\frac{\pi^2}{6\mu_{\pm}}+\frac{1}{2}\log\mu_{\pm}+\cdots$$

$$\mathcal{F}_0=\frac{l^2\pi^2}{2\beta G(1-l^2\Omega^2)}$$

$$\Delta S = -\frac{1}{2}\log\left(\frac{9l^4}{4G^2\beta^2(1-l^2\Omega^2)}\right)$$

$$\xi^{\pm}=\mp \mathcal{C}_{\pm}\frac{p_{\pm}^2}{2}$$



$$ds^2 = \frac{\pi^4 l^6}{\beta_-^4 \beta_+^4} \left(\beta_-^4 C_+^2 + 2\beta_-^2 \beta_+^2 C_+ C_- \cosh \left(\frac{2\rho}{l} \right) + \beta_+^4 C_-^2 \right) dt^2 + d\rho^2 \\ - \frac{2\pi^3 l^5}{\beta_-^3 \beta_+^3} \left(\beta_-^3 C_+ - \beta_- \beta_+ \cosh \left(\frac{2\rho}{l} \right) (\beta_- C_+ - \beta_+ C_-) - \beta_+^3 C_- \right) dt d\theta \\ + \frac{\pi^2 l^4}{\beta_-^2 \beta_+^2} \left(\beta_-^2 + \beta_+^2 - 2\beta_- \beta_+ \cosh \left(\frac{2\rho}{l} \right) \right) d\theta^2$$

$$p_{\pm}(t, \theta) = v \pm \pi \sigma = \pm \frac{2\pi l}{\beta_{\pm}}$$

$$\rho = \frac{l}{2} \cosh^{-1} \left(\frac{\pi^2 l^4 (\beta_-^2 + \beta_+^2) - \beta_-^2 \beta_+^2 r^2}{2\pi^2 \beta_- \beta_+ l^4} \right).$$

$$ds^2 = \frac{\pi^2 l^2}{\beta_-^4 \beta_+^4} (\pi^2 \beta_-^4 C_+^2 l^4 + \beta_- \beta_+ C_+ C_- (\pi^2 l^4 (\beta_-^2 + \beta_+^2) - \beta_-^2 \beta_+^2 r^2) + \pi^2 \beta_+^4 C_-^2 l^4) dt^2 \\ + r^2 d\theta^2 + \left(\frac{l^2 r^2 \beta_-^4 \beta_+^4}{\pi^4 l^8 (\beta_-^2 - \beta_+^2)^2 - 2\pi^2 \beta_-^2 \beta_+^2 l^4 r^2 (\beta_-^2 + \beta_+^2) + \beta_-^4 \beta_+^4 r^4} \right) dr^2 \\ - \frac{\pi l}{\beta_-^3 \beta_+^3} (C_+ (\pi^2 \beta_- l^4 (\beta_-^2 - \beta_+^2) + \beta_-^3 \beta_+^2 r^2) - \beta_+ C_- (\pi^2 l^4 (\beta_+^2 - \beta_-^2) + \beta_-^2 \beta_+^2 r^2)) dt d\theta$$

$$t \rightarrow t, \theta \rightarrow \theta + \left(\frac{2A\phi}{r^2} \right) t, C_{\pm} = \pm \frac{2}{l(v \pm \pi \sigma)} = \pm \frac{2}{lp_{\pm}} = \frac{\beta_{\pm}}{\pi l^2}$$

$$\xi_{\pm} = \mp C_{\pm} p_{\pm}$$

$$ds^2 = \frac{\pi^2 l^4}{\beta_-^2 \beta_+^2} \left(\beta_-^2 C_+^2 - 2\beta_- \beta_+ C_+ C_- \cosh \left(\frac{2\rho}{l} \right) + \beta_+^2 C_-^2 \right) dt^2 + d\rho^2 \\ - \frac{2\pi^2 l^4}{\beta_-^2 \beta_+^2} \left(\beta_-^2 C_+ - \beta_- \beta_+ (C_+ + C_-) \cosh \left(\frac{2\rho}{l} \right) + \beta_+^2 C_- \right) dt d\theta \\ + \frac{\pi^2 l^4}{\beta_-^2 \beta_+^2} \left(\beta_-^2 + \beta_+^2 - 2\beta_- \beta_+ \cosh \left(\frac{2\rho}{l} \right) \right) d\theta^2$$

$$\rho = \rho = \frac{l}{2} \cosh^{-1} \left(\frac{\pi^2 l^4 (\beta_-^2 + \beta_+^2) - \beta_-^2 \beta_+^2 r^2}{2\pi^2 \beta_- \beta_+ l^4} \right).$$

$$ds^2 = \frac{1}{\beta_-^2 \beta_+^2} (\pi^2 \beta_-^2 C_+^2 l^4 + C_+ C_- (\beta_-^2 \beta_+^2 r^2 - \pi^2 l^4 (\beta_-^2 + \beta_+^2)) + \pi^2 \beta_+^2 C_-^2 l^4) dt^2 \\ + r^2 d\theta^2 + \frac{l^2 r^2 \beta_-^4 \beta_+^4}{\pi^4 l^8 (\beta_-^2 - \beta_+^2)^2 - 2\pi^2 \beta_-^2 \beta_+^2 l^4 r^2 (\beta_-^2 + \beta_+^2) + \beta_-^4 \beta_+^4 r^4} dr^2 \\ - \frac{1}{\beta_-^2 \beta_+^2} (\pi^2 C_+ l^4 (\beta_-^2 - \beta_+^2) + \beta_-^2 \beta_+^2 C_+ r^2 + \pi^2 C_- l^4 (\beta_+^2 - \beta_-^2) + \beta_-^2 \beta_+^2 C_- r^2) dt d\theta$$

$$C_{\pm} = \pm \frac{1}{l}.$$

$$\prod_{k=1}^{\infty} \frac{1}{1 - e^{-\mu k}}$$



$$q=e^{-\mu}, \text{ and } \tau=\frac{i\mu}{2\pi}$$

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k)$$

$$\prod_{k=1}^{\infty}\frac{1}{1-q^k}=e^{-\mu/24}\eta\left(\frac{i\mu}{2\pi}\right)^{-1}$$

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\eta(\tau)$$

$$\eta\left(\frac{i\mu}{2\pi}\right)=\left(\frac{\mu}{2\pi}\right)^{-1/2}\eta\left(\frac{2\pi i}{\mu}\right).$$

$$\eta\left(\frac{2\pi i}{\mu}\right)$$

$$\begin{aligned}\eta\left(\frac{2\pi i}{\mu}\right)&=q'^{1/24}\prod_{k=1}^{\infty}(1-q'^k)\\&=e^{\frac{2\pi i(2\pi i/\mu)}{24}}=e^{\pi^2/6\mu}\end{aligned}$$

$$\prod_{k=1}^{\infty}\frac{1}{(1-e^{-\mu k})}=\sqrt{\frac{\mu}{2\pi}}e^{-\frac{t}{24}}e^{\frac{\pi^2}{6\mu}}$$

$$\mathcal{S}=\frac{1}{2}\int_{\mathcal{M}}\mathrm{d}^n x\sqrt{-g}\left[\mathcal{R}-\frac{(n-2)(n-1)}{3}\Lambda+\alpha_2\mathcal{L}_{\mathrm{GB}}-\mathcal{F}^q\right],$$

$$\mathcal{L}_{\mathrm{GB}}=\mathcal{R}_{\gamma\zeta\varpi\delta}\mathcal{R}^{\gamma\zeta\varpi\delta}-4\mathcal{R}_{\gamma\zeta}\mathcal{R}^{\gamma\zeta}+\mathcal{R}^2$$

$$\mathcal{F}=\sum_{a=1}^{\frac{(n-1)(n-2)}{2}}\mathrm{Tr}\left(F_{\gamma\zeta}^{(a)}F^{\gamma\zeta(a)}\right)$$

$$\mathbf{F}^{(a)}=\mathbf{d}\mathbf{A}^{(a)}+\frac{1}{2\sigma}C_{(b)(c)}^{(a)}\mathbf{A}^{(b)}\wedge\mathbf{A}^{(c)},$$

$$F_{\gamma\zeta}^{(a)}=\partial_{\gamma}A_{\zeta}^{(a)}-\partial_{\zeta}A_{\gamma}^{(a)}+\frac{1}{2\sigma}C_{(b)(c)}^{(a)}A_{\gamma}^{(b)}A_{\zeta}^{(c)}$$

$$\mathbf{d}\big(\,{}^*\mathbf{F}^{(a)}\mathcal{F}^{q-1}\big)+\frac{1}{\sigma}C_{(b)(c)}^{(a)}\mathcal{F}^{q-1}\mathbf{A}^{(b)}\wedge\,{}^*\mathbf{F}^{(c)}\!=\!0$$

$$\mathcal{G}_{\mu\nu}+\alpha_2\mathcal{H}_{\mu\nu}+\frac{(n-2)(n-1)}{6}\Lambda g_{\mu\nu}=\mathcal{T}_{\mu\nu},$$

$$\mathcal{G}_{\mu\nu} \, = \, \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$$

$$\mathcal{H}_{\mu\nu} \, = \, 2\big(-\mathcal{R}_{\mu\sigma\kappa\tau}\mathcal{R}^{\kappa\tau\sigma}{}_{\nu} - 2\mathcal{R}_{\mu\rho\nu\sigma}\mathcal{R}^{\rho\sigma}{}_{\nu} - 2\mathcal{R}_{\mu\sigma}\mathcal{R}^{\sigma}{}_{\nu} + \mathcal{R}\mathcal{R}_{\mu\nu}\big) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{\mathrm{GB}}$$



$$\mathcal{F}^{\mu}{}_{\nu} = -\frac{1}{2}\delta^{\mu}{}_{\nu}\mathcal{F}^q + 2q\mathrm{Tr}(F_{\nu\lambda}{}^{(a)}F^{\mu\lambda(a)})\mathcal{F}^{q-1}$$

$$\mathbf{A}^{(a)}=\frac{\mathcal{Q}}{r^2}\mathcal{C}_{(i)(j)}^{(a)}\chi^i\,\mathrm{d}\chi^j$$

$$\begin{array}{l} \chi_1=r\cos\theta_{n-3}\sin\theta_{n-4}\cdots\sin\theta_1\\ \chi_2=r\sin\theta_{n-3}\sin\theta_{n-4}\cdots\sin\theta_1\\ \chi_3=r\cos\theta_{n-4}\sin\theta_{n-5}\cdots\sin\theta_1\\ \chi_4=r\sin\theta_{n-4}\sin\theta_{n-5}\cdots\sin\theta_1\\ \vdots\\ \chi_{n-2}=r\cos\theta_1 \end{array}$$

$$\mathcal{F}=\frac{(n-2)(n-3)\mathcal{Q}^2}{r^4}$$

$$\mathrm{Tr}\left(F_{\theta i\lambda}^{(a)}F^{\theta i\lambda(a)}\right)=\frac{1}{(n-2)}\mathcal{F}=\frac{(n-3)\mathcal{Q}^2}{r^4}$$

$$\mathcal{F}^a{}_b=-\frac{1}{2}\mathcal{F}^q\mathrm{diag}\left[1,1,\left(1-\frac{4q}{n-2}\right),\left(1-\frac{4q}{n-2}\right),\cdots,\left(1-\frac{4q}{n-2}\right)\right]$$

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}=-f(r)\mathrm{d}t^2+\frac{1}{f(r)}\mathrm{d}r^2+r^2\sum_{i=2}^{n-2}\mathfrak{h}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

$$f_{\pm}(r)=\begin{cases} 1+\frac{r^2}{2\tilde{\alpha}_2}\left(1\pm\sqrt{1+\frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}}+\frac{4\tilde{\alpha}_2Q_1}{r^{4q}}}\right), & q\neq\frac{n-1}{4};\\ 1+\frac{r^2}{2\tilde{\alpha}_2}\left(1\pm\sqrt{1+\frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}}+\frac{4\tilde{\alpha}_2Q_2\ln{(r)}}{r^{n-1}}}\right), & q=\frac{n-1}{4}, \end{cases}$$

$$Q_1\!=\!\frac{[(n-3)(n-2)\mathcal{Q}^2]^q}{(n-2)(n-1-4q)}$$

$$Q_2=\frac{[(n-3)(n-2)\mathcal{Q}^2]^{\frac{(n-1)}{4}}}{(n-2)}$$

$$\mathfrak{M}=\frac{M}{2\Sigma_{(n-2)}}$$

$$\Sigma_{(n-2)}=\frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$f_{-}(r)=\begin{cases} 1+\frac{r^2}{2\tilde{\alpha}_2}\left(1-\sqrt{1+\frac{64\pi\mathcal{G}_M\tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)}r^{n-1}}+\frac{4\tilde{\alpha}_2Q_1}{r^{4q}}}\right), & \frac{n-1}{4}<q<\frac{n-1}{2};\\ 1+\frac{r^2}{2\tilde{\alpha}_2}\left(1-\sqrt{1+\frac{64\pi\mathcal{G}_M\tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)}r^{n-1}}+\frac{4\tilde{\alpha}_2Q_2\ln{(r)}}{r^{n-1}}}\right), & q=\frac{n-1}{4}. \end{cases}$$

$$f(r) = 1 - \frac{16\pi\mathcal{G}M}{(n-2)\Sigma_{(n-2)}r^{n-3}}.$$

$$f(r) = 1 + \frac{r^2}{2\tilde{\alpha}_2} \left(1 - \sqrt{1 + \frac{64\pi\mathcal{G}M\tilde{\alpha}_2}{(n-2)\Sigma_{(n-2)}r^{n-1}}} \right).$$

$$f(r) = \begin{cases} 1 - \frac{8\mathcal{G}M}{3\pi r^2} - \frac{2Q^2 \ln(r)}{r^2}, & (n=5); \\ 1 - \frac{16\pi\mathcal{G}M}{(n-2)\Sigma_{(n-2)}r^{n-3}} - \frac{(n-3)Q^2}{(n-5)r^2}, & (n>5). \end{cases}$$

$$1 + \frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}} + \frac{4\tilde{\alpha}_2Q_1}{r^{4q}} > 0,$$

$$1 + \frac{16\mathfrak{M}\tilde{\alpha}_2}{(n-2)r^{n-1}} + \frac{4\tilde{\alpha}_2Q_2 \ln(r)}{r^{n-1}} > 0.$$

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + k(r)\mathrm{d}\Omega_{n-2}^2$$

$$\begin{aligned} \mathrm{d}\Omega_{n-2}^2 &= \mathrm{d}\theta_1^2 + \sin^2 \theta_1 \, \mathrm{d}\theta_2^2 + \cdots + (\sin^2 \theta_1 \cdots \sin^2 \theta_{n-3}) \mathrm{d}\theta_{n-2}^2 \\ &= \mathrm{d}\theta_1^2 + \sin^2 \theta_1 \, \mathrm{d}\theta_2^2 + \cdots + \left(\prod_{i=1}^{n-3} \sin^2 \theta_i \right) \mathrm{d}\theta_{n-2}^2, \end{aligned}$$

$$g_{a\,b} = \begin{pmatrix} g_{0\,0} & & & & & \\ & g_{1\,1} & & & & \\ & & g_{2\,2} & & & \\ & & & g_{3\,3} & & \\ & & & & \ddots & \\ & & & & & g_{\chi+1\,\chi+1} \\ & & & & & & \ddots \\ & & & & & & & g_{n-1\,n-1} \end{pmatrix}_{n \times n}$$

$$= \begin{pmatrix} -h(r) & & & & & \\ & \frac{1}{f(r)} & & & & \\ & & k(r) & & & \\ & & & k(r) \sin^2 \theta_1 & & \\ & & & & \ddots & \\ & & & & & k(r) \prod_{i=1}^{\chi-1} \sin^2 \theta_i \\ & & & & & & \ddots \\ & & & & & & & k(r) \prod_{i=1}^{n-3} \sin^2 \theta_i \end{pmatrix}_{n \times n}$$

$$\tilde{\mathcal{L}} = \frac{1}{2}(g_{ab})\dot{x}^a\dot{x}^b = \frac{1}{2}(g_{ab})\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$$

$$\begin{aligned}\tilde{\mathcal{L}} = & \frac{1}{2}[(g_{00})\dot{t}^2 + (g_{11})\dot{r}^2 + (g_{22})\dot{\theta}_1^2 + (g_{33})\dot{\theta}_2^2 \\ & + \cdots + (g_{\chi+1\chi+1})\dot{\theta}_\chi^2 + \cdots + (g_{n-1n-1})\dot{\theta}_{n-2}^2] \\ = & \frac{1}{2}\left[(-h(r))\dot{t}^2 + \left(\frac{1}{f(r)}\right)\dot{r}^2 + (k(r))\dot{\theta}_1^2 + (k(r)\sin^2 \theta_1)\dot{\theta}_2^2 \right. \\ & \left. + \cdots + \left(k(r)\prod_{i=1}^{\chi-1}\sin^2 \theta_i\right)\dot{\theta}_\chi^2 + \cdots + \left(k(r)\prod_{i=1}^{n-3}\sin^2 \theta_i\right)\dot{\theta}_{n-2}^2\right].\end{aligned}$$

$$P_a = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^a} = (g_{ab})\dot{x}^b = \dot{x}_a$$

$$P_0 = P_t = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^0} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{t}} = (-h(r))\dot{t} \equiv -E,$$

$$P_1 = P_r = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^1} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{r}} = \left(\frac{1}{f(r)}\right)\dot{r},$$

$$P_2 = P_{\theta_1} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^2} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}_1} = (k(r))\dot{\theta}_1,$$

$$P_3 = P_{\theta_2} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^3} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}_2} = (k(r)\sin^2 \theta_1)\dot{\theta}_2,$$

⋮

$$P_{\chi+1} = P_{\theta_\chi} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^{\chi+1}} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}_\chi} = \left(k(r)\prod_{i=1}^{\chi-1}\sin^2 \theta_i\right)\dot{\theta}_\chi,$$

⋮

$$P_{n-1} = P_{\theta_{n-2}} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^{n-1}} = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}_{n-2}} = \left(k(r)\prod_{i=1}^{n-3}\sin^2 \theta_i\right)\dot{\theta}_{n-2} \equiv \mathbb{L},$$

$$\frac{\partial S}{\partial \tau} = -\mathcal{H} = -\frac{1}{2}(g^{ab})\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}$$

$$\mathcal{H} = P_a\dot{x}^a - \tilde{\mathcal{L}} = \frac{1}{2}(g_{ab})\dot{x}^a\dot{x}^b = -\frac{1}{2}m^2$$

$$\begin{aligned}-2\frac{\partial S}{\partial \tau} = & -\frac{1}{h(r)}\left(\frac{\partial S_t}{\partial t}\right)^2 + f(r)\left(\frac{\partial S_r}{\partial r}\right)^2 \\ & + \sum_{\chi=1}^{n-3}\frac{1}{(k(r)\prod_{i=1}^{\chi-1}\sin^2 \theta_i)}\left(\frac{\partial S_{\theta_\chi}}{\partial \theta_\chi}\right)^2 \\ & + \frac{1}{(k(r)\prod_{i=1}^{n-3}\sin^2 \theta_i)}\left(\frac{\partial S_{\theta_{n-2}}}{\partial \theta_{n-2}}\right)^2\end{aligned}$$

$$S = \frac{1}{2}m^2\tau - Et + S_r(r) + \sum_{\chi=1}^{n-3}S_{\theta_\chi}(\theta_\chi) + \mathbb{L}\theta_{n-2}$$



$$\frac{\partial S_\tau}{\partial \tau} = -\frac{1}{2}(g^{ab})P_a P_b = \frac{1}{2}m^2,$$

$$\frac{\partial S_t}{\partial t} = P_t = -h(r)\dot{t} = -E,$$

$$\frac{\partial S_r}{\partial r} = P_r = \frac{1}{f(r)}\dot{r},$$

$$\frac{\partial S_{\theta_\chi}}{\partial \theta_\chi} = P_{\theta_\chi} = k(r) \left(\prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) \dot{\theta}_\chi,$$

$$\frac{\partial S_{\theta_{n-2}}}{\partial \theta_{n-2}} = P_{\theta_{n-2}} = k(r) \left(\prod_{i=1}^{n-3} \sin^2 \theta_i \right) \dot{\theta}_{n-2} = \mathbb{L}.$$

$$k(r)m^2 = \frac{k(r)}{h(r)}(-E)^2 - \frac{k(r)}{f(r)}(\dot{r})^2 \\ - k^2(r) \sum_{\chi=1}^{n-3} \left[\left(\prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) (\dot{\theta}_\chi)^2 \right] - \left[\prod_{i=1}^{n-3} (1 + \cot^2 \theta_i) \right] \mathbb{L}^2$$

$$\prod_{i=1}^{n-3} (1 + \cot^2 \theta_i) = 1 + \prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{D}(\theta_i)$$

$$0 = \left\{ \frac{k(r)}{h(r)} E^2 - \frac{k(r)}{f(r)} (\dot{r})^2 - \mathbb{L}^2 - k(r)m^2 \right\} \\ - \left\{ k^2(r) \sum_{\chi=1}^{n-3} \left[\left(\prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) (\dot{\theta}_\chi)^2 \right] + \left[\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{D}(\theta_i) \right] \mathbb{L}^2 \right\}$$

$$\mathcal{K} = \left\{ \frac{k(r)}{h(r)} E^2 - \frac{k(r)}{f(r)} (\dot{r})^2 - \mathbb{L}^2 - k(r)m^2 \right\}$$

$$\mathcal{K} = \left\{ k^2(r) \sum_{\chi=1}^{n-3} \left[\left(\prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) (\dot{\theta}_\chi)^2 \right] + \left[\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{D}(\theta_i) \right] \mathbb{L}^2 \right\}$$

$$\dot{t} = \frac{1}{h(r)} E$$

$$\dot{r} = \pm \sqrt{\frac{f(r)}{h(r)} E^2 - \frac{f(r)}{k(r)} (\mathcal{K} + \mathbb{L}^2) - f(r)m^2}$$

$$\sqrt{\sum_{\chi=1}^{n-3} \prod_{i=1}^{\chi-1} \sin^2 \theta_i (\dot{\theta}_\chi)^2} = \pm \sqrt{\frac{\mathcal{K} - [\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{D}(\theta_i)] \mathbb{L}^2}{k^2(r)}}$$

$$\dot{\theta}_{n-2} = \frac{1}{k(r) (\prod_{i=1}^{n-3} \sin^2 \theta_i)} \mathbb{L}$$

$$\dot{t}=\frac{1}{h(r)}\mathcal{E}$$

$$\dot{r}=\pm\frac{1}{k(r)}\sqrt{\mathfrak{R}}$$

$$\sqrt{\sum_{\chi=1}^{n-3}\prod_{i=1}^{\chi-1}\sin^2\theta_i(\dot{\theta}_{\chi})^2}=\pm\frac{1}{k(r)}\sqrt{\Theta_{\chi}}$$

$$\dot{\theta}_{n-2}=\frac{1}{k(r)(\prod_{i=1}^{n-3}\sin^2\theta_i)}\mathfrak{L}$$

$$\mathfrak{R}=k^2(r)\frac{f(r)}{h(r)}\mathcal{E}^2-k(r)f(r)(\mathcal{K}+\mathfrak{L}^2)-k^2(r)f(r)m^2$$

$$\Theta_{\chi}=\mathcal{K}-\left[\prod_{i=1}^{n-3}\cot^2\theta_i+\mathfrak{D}(\theta_i)\right]\mathfrak{L}^2$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+V_{\mathrm{eff}}=0$$

$$V_{\mathrm{eff}}=\frac{f(r)}{k(r)}(\mathcal{K}+\mathfrak{L}^2)-\frac{f(r)}{h(r)}\mathcal{E}^2$$

$$\mathfrak{R}=k^2(r)\frac{f(r)}{h(r)}\mathcal{E}^2-k(r)f(r)(\mathcal{K}+\mathfrak{L}^2).$$

$$\xi=\frac{\mathfrak{L}}{\mathcal{E}}$$

$$\eta=\frac{\mathcal{K}}{\mathcal{E}^2}$$

$$V_{\mathrm{eff}}=\mathcal{E}^2\left\{\frac{f(r)}{k(r)}(\eta+\xi^2)-\frac{f(r)}{h(r)}\right\}$$

$$\mathfrak{R}=\mathcal{E}^2\left\{k^2(r)\frac{f(r)}{h(r)}\mathcal{E}^2-k(r)f(r)(\eta+\xi^2)\right\}$$

$$V_{\mathrm{eff}}=0,\qquad\qquad\mathfrak{R}=0$$

$$\frac{\partial V_{\mathrm{eff}}}{\partial r}=0,\qquad\text{or}\qquad\frac{\partial \mathfrak{R}}{\partial r}=0$$

$$\frac{\partial^2 V_{\mathrm{eff}}}{\partial r^2}<0,\qquad\qquad\frac{\partial^2 \mathfrak{R}}{\partial r^2}>0$$

$$k(r_p)h'(r_p)-h(r_p)k'(r_p)=0$$

$$(\eta+\xi^2)=\frac{h(r)k^2(r)f'(r)-f(r)k^2(r)h'(r)+2f(r)h(r)k(r)k'(r)}{h^2(r)[k(r)f'(r)+f(r)k'(r)]}$$



$$(\eta + \xi^2) = \frac{k^2(r_p)}{h(r_p)[k(r_p)f'(r_p) + f(r_p)k'(r_p)]} \left\{ f'(r_p) - f(r_p) \left[\frac{h'(r_p)}{h(r_p)} - \frac{2k'(r_p)}{k(r_p)} \right] \right\}$$

$$e_{\hat{\alpha}} = \{e_{\hat{t}}, e_{\hat{r}}, \cdots, e_{\hat{\theta}_{\chi}}, \cdots, e_{\hat{\theta}_{n-2}}\}$$

$$e_{\mu} = \{e_t, e_r, \cdots, e_{\theta_{\chi}}, \cdots, e_{\theta_{n-2}}\}$$

$$e_{\hat{\alpha}} = e_{\hat{\alpha}}^{\mu} e_{\mu}$$

$$e_{\hat{\alpha}}^{\mu} e_{\hat{\beta}}^{\nu} g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}$$

$$-1=e_{\hat{t}}\cdot e_{\hat{t}}$$

$$1=e_{\hat{r}}\cdot e_{\hat{r}}$$

$$1=e_{\hat{r}}\cdot e_{\hat{r}}$$

$$1=e_{\hat{\theta}_{n-2}}\cdot e_{\hat{\theta}_{n-2}},$$

$$e_{\hat{t}}=\zeta e_t+\gamma e_{\theta_{n-2}},$$

$$e_{\hat{r}}=\iota e_r,$$

$$\vdots$$

$$e_{\hat{\theta}_{\chi}}=\varsigma e_{\theta_{\chi}},$$

$$\vdots$$

$$e_{\hat{\theta}_{n-2}}=q e_{\theta_{n-2}},$$

$$\zeta=e_{\hat{t}}^t=\sqrt{\frac{g_{n-1n-1}}{(g_{0n-1})^2-g_{00}g_{n-1n-1}}},$$

$$\gamma=e_{\hat{t}}^{\theta_{n-2}}=-\frac{g_{0n-1}}{g_{n-1n-1}}\sqrt{\frac{g_{n-1n-1}}{(g_{0n-1})^2-g_{00}g_{n-1n-1}}},$$

$$\iota=e_{\hat{r}}^r=\frac{1}{\sqrt{g_{11}}},$$

$$\varsigma=e_{\hat{\theta}_{\chi}}^{\theta_{\chi}}=\frac{1}{\sqrt{g_{\chi+1\chi+1}}},$$

$$q=e_{\hat{\theta}_{n-2}}^{\theta_{n-2}}=\frac{1}{\sqrt{g_{n-1n-1}}}.$$

$$P^{\hat{t}}=-(\zeta P_t+\gamma P_{\theta_{n-2}})$$

$$P^{\hat{r}}=\iota P_r$$

$$P^{\hat{\theta}_{\chi}}=\varsigma P_{\theta_{\chi}}$$

$$P^{\hat{\theta}_{n-2}}=q P_{\theta_{n-2}}$$

$$P^{\hat{t}} = \zeta E - \gamma E = \left(\frac{1}{h(r)} \right)^{\frac{1}{2}} E,$$

$$P^{\hat{r}} = \frac{1}{\sqrt{g_{11}}} \left(\frac{1}{f(r)} \right) \dot{r} = \left(\frac{1}{f(r)} \right)^{\frac{1}{2}} |\dot{r}|,$$

$$P^{\hat{\theta}_{\chi}} = \frac{1}{\sqrt{g_{\chi+1\chi+1}}} \left(k(r) \prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) \dot{\theta}_{\chi} = \left(k(r) \prod_{i=1}^{\chi-1} \sin^2 \theta_i \right)^{\frac{1}{2}} \dot{\theta}_{\chi},$$

$$P^{\hat{\theta}_{n-2}} = \frac{1}{\sqrt{g_{n-1n-1}}} L = \left(\frac{1}{k(r) \prod_{i=1}^{n-3} \sin^2 \theta_i} \right)^{\frac{1}{2}} L.$$

$$\mathbf{P} = P^{\hat{r}} e_{\hat{r}} + \sum_{\chi=1}^{n-3} P^{\hat{\theta}_{\chi}} e_{\hat{\theta}_{\chi}} + P^{\hat{\theta}_{n-2}} e_{\hat{\theta}_{n-2}}$$

$$\left(P^{\hat{t}}\right)^2 = \mathbf{P}^2 = \left(P^{\hat{r}}\right)^2 + \sum_{\chi=1}^{n-3} \left(P^{\hat{\theta}_{\chi}}\right)^2 + \left(P^{\hat{\theta}_{n-2}}\right)^2$$

$$P^{\hat{r}} = |\mathbf{P}| \cos \alpha \cos \beta,$$

$$P^{\hat{\theta}_{\chi}} = |\mathbf{P}| \sin \alpha,$$

$$P^{\hat{\theta}_{n-2}} = |\mathbf{P}| \cos \alpha \sin \beta,$$

$$\tan \beta = \frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{r}}},$$

$$\sin \alpha = \frac{P^{\hat{\theta}_{\chi}}}{P^{\hat{t}}}.$$

$$\begin{aligned} X &\equiv -r_o \beta \\ Y &\equiv r_o \alpha \end{aligned}$$

$$X \equiv \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{r}}} \right) \right]_{(r=r_o)},$$

$$Y \equiv \lim_{r_o \rightarrow \infty} \left[r_o \left(\frac{\|\mathbf{P}^{\hat{\theta}_{\chi}}\|}{P^{\hat{t}}} \right) \right]_{(r=r_o)},$$

$$\|\mathbf{P}^{\hat{\theta}_{\chi}}\| \equiv \sqrt{\sum_{\chi=1}^{n-3} |P^{\hat{\theta}_{\chi}}|^2} = \sqrt{\sum_{\chi=1}^{n-3} \left(k(r) \prod_{i=1}^{\chi-1} \sin^2 \theta_i \right) (\dot{\theta}_{\chi})^2}.$$

$$\begin{aligned}
X &= \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{\left(\frac{1}{k(r) \prod_{i=1}^{n-3} \sin^2 \theta_i} \right)^{\frac{1}{2}} \mathfrak{L}}{\left(\frac{1}{f(r)} \right)^{\frac{1}{2}} |\dot{r}|} \right) \right]_{(r=r_o)} \\
&= \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{\left(\frac{1}{\sqrt{\prod_{i=1}^{n-3} \sin^2 \theta_i}} \right) \mathfrak{L}}{\sqrt{\frac{k(r)}{h(r)} E^2 - (\mathcal{K} + \mathfrak{L}^2)}} \right) \right]_{(r=r_o)} \\
&= \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{1}{\sqrt{\frac{k(r)}{h(r)} - (\eta + \xi^2)}} \cdot \frac{\xi}{\prod_{i=1}^{n-3} \sin \theta_i} \right) \right]_{(r=r_o)} \\
&= -\frac{1}{\sqrt{E}} \left(\frac{\xi}{\prod_{i=1}^{n-3} \sin \theta_i} \right), \\
Y &= \lim_{r_o \rightarrow \infty} \left[r_o \left(\frac{\sqrt{k(r) \sum_{\chi=1}^{n-3} \prod_{i=1}^{\chi-1} \sin^2 \theta_i (\dot{\theta}_\chi)^2}}{E \sqrt{\frac{1}{h(r)}}} \right) \right]_{(r=r_o)} \\
&= \lim_{r_o \rightarrow \infty} \left[r_o \left(\pm \sqrt{\frac{\mathcal{K} - [\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{O}(\theta_i)] E^2}{E^2}} \cdot \sqrt{\frac{h(r)}{k(r)}} \right) \right]_{(r=r_o)} \\
&= \lim_{r_o \rightarrow \infty} \left[r_o \left(\pm \sqrt{\eta - \xi^2 \left[\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{O}(\theta_i) \right]} \cdot \sqrt{\frac{h(r)}{k(r)}} \right) \right]_{(r=r_o)} \\
&= \pm \frac{1}{\sqrt{E}} \sqrt{\eta - \xi^2 \left[\prod_{i=1}^{n-3} \cot^2 \theta_i + \mathfrak{O}(\theta_i) \right]}
\end{aligned}$$

$$X = -\frac{\xi}{\sqrt{E}}$$

$$Y = \pm \sqrt{\frac{\eta}{E}}$$

$$(R_s)^2 \equiv X^2 + Y^2 = \frac{(\xi^2 + \eta)}{E}$$

$$R_s = \sqrt{\frac{(r_p)^4 \mathcal{E}^2}{\mathcal{E} h(r_p) [(r_p)^2 \mathcal{E} f'(r_p) + 2r_p \mathcal{E} f(r_p)]} \left\{ f'(r_p) - f(r_p) \left[\frac{h'(r_p)}{h(r_p)} - \frac{4r_p \mathcal{E}}{(r_p)^2 \mathcal{E}} \right] \right\}}$$

$$= \sqrt{\frac{(r_p)^3}{h(r_p) [r_p f'(r_p) + 2f(r_p)]} \left\{ f'(r_p) - f(r_p) \left[\frac{h'(r_p)}{h(r_p)} - \frac{4}{r_p} \right] \right\}}$$

$$X = \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{P^{\hat{\theta}_{n-2}}}{P^{\hat{t}}} \right) \right]_{(r=r_o)}$$

$$= \lim_{r_o \rightarrow \infty} \left[-r_o \left(\frac{\xi}{\prod_{i=1}^{n-3} \sin \theta_i} \cdot \sqrt{\frac{h(r)}{k(r)}} \right) \right]_{(r=r_o)}$$

$$= -\frac{1}{\sqrt{\mathcal{E}}} \left(\frac{\xi}{\prod_{i=1}^{n-3} \sin \theta_i} \right).$$

$$\theta_{\rm g} = \frac{GM}{c^2 D}$$

$$\delta \equiv \frac{\theta_{\rm dyn}}{\theta_{\rm g}} - 1$$

$$\theta_{\rm dyn} = \frac{GM_{\rm dyn}}{c^2 D_{\rm dyn}}$$

$$\theta_{\rm sh} = \frac{r_{\rm sh}}{D},$$

$$r_{\rm sh} = \frac{3\sqrt{3}GM_{\rm dyn}}{c^2},$$

$$\theta_{\rm sh} = \frac{r_{\rm sh}}{D_{\rm dyn}} = \frac{\left(\frac{3\sqrt{3}GM_{\rm dyn}}{c^2}\right)}{D_{\rm dyn}} = 3\sqrt{3} \left(\frac{GM_{\rm dyn}}{c^2 D_{\rm dyn}}\right)$$

$$\theta_{\rm sh} = 3\sqrt{3}\theta_{\rm dyn}$$

$$\delta = \frac{\left(\frac{\theta_{\rm sh}}{3\sqrt{3}}\right)}{\theta_{\rm g}} - 1$$

$$\theta_{\rm sh} = 3\sqrt{3}(\delta + 1)\theta_{\rm g}$$

$$\theta_{\rm sh} = \frac{r_{\rm sh}}{D} = \frac{r_{\rm sh}}{\left(\frac{GM}{c^2 \theta_{\rm g}}\right)} = 3\sqrt{3}(\delta + 1)\theta_{\rm g}$$



$$r_{\text{sh}} = 3\sqrt{3}(\delta + 1)GM$$

$$r_{\text{sh}} = \sqrt{\frac{n-1}{n-3} \left[\frac{16\pi \mathcal{G} M_{\text{dyn}} (n-1)}{2(n-2)\Sigma_{(n-2)}} \right]^{\frac{2}{n-3}}}$$

$$\begin{aligned} \theta_{\text{sh}} &= \frac{r_{\text{sh}}}{D_{\text{dyn}}} \\ &= \left\{ \left(\frac{n-1}{n-3} \right)^{\frac{1}{2}} \left[\frac{16\pi(n-1)}{2(n-2)\Sigma_{(n-2)}} \right]^{\frac{1}{n-3}} \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}} \right\} \cdot \frac{1}{D_{\text{dyn}}} \end{aligned}$$

$$\mathfrak{C}(n) \equiv \left(\frac{n-1}{n-3} \right)^{\frac{1}{2}} \left[\frac{16\pi(n-1)}{2(n-2)\Sigma_{(n-2)}} \right]^{\frac{1}{n-3}}$$

$$\begin{aligned} \theta_{\text{sh}} &= \mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}} \cdot \frac{1}{D_{\text{dyn}}} \\ &= \mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}-1} \cdot \left(\frac{\mathcal{G} M_{\text{dyn}}}{D_{\text{dyn}}} \right) \\ &= \mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}-1} \cdot \theta_{\text{dyn}} \end{aligned}$$

$$\delta = \frac{\left[\frac{\theta_{\text{sh}}}{\mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}-1}} \right]}{\theta_{\text{g}}} - 1$$

$$\theta_{\text{sh}} = \left[\mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}-1} \right] (\delta + 1) \theta_{\text{g}}$$

$$\theta_{\text{sh}} = \frac{r_{\text{sh}}}{D} = \frac{r_{\text{sh}}}{\left(\frac{\mathcal{G} M}{\theta_{\text{g}}} \right)} = \left[\mathfrak{C}(n) \cdot (\mathcal{G} M_{\text{dyn}})^{\frac{1}{n-3}-1} \right] (\delta + 1) \theta_{\text{g}}$$

$$r_{\text{sh}} = \left[\mathfrak{C}(n) \cdot (\mathcal{G})^{\frac{1}{n-3}} \cdot (M_{\text{dyn}})^{\frac{1}{n-3}-1} \right] (\delta + 1) M$$

$$G = \frac{\mathcal{G}}{\mathcal{V}_{(n-4)}}$$

$$\mathcal{G} = G(2\pi\mathcal{R})^{n-4},$$

$$r_{\text{sh}} = \left[\mathfrak{C}(n) \cdot (G)^{\frac{1}{n-3}} \cdot \left(\frac{M_{\text{dyn}}}{2\pi\mathcal{R}} \right)^{\frac{1}{n-3}-1} \right] (\delta + 1) M$$

$$r_{\text{sh}} = \left[\mathfrak{C}(n) \cdot (G)^{\frac{1}{n-3}} \right] (\delta + 1) M$$



$$\mathcal{S}^{(0)} = \int \mathrm{d}^n x \sqrt{-g} \left[-\frac{1}{2} (\nabla \Phi)^2 - \frac{1}{2} m^2 \Phi^2 \right]$$

$$\Phi(x)=\sum_{\ell,\mathbf{m}}\frac{\Psi_{\ell,\mathbf{m}}^{(0)}(t,r)}{r^{(n-2)/2}}Y_{\ell,\mathbf{m}}(\theta)$$

$$\mathcal{S}^{(0)}=\sum_{\ell,\mathbf{m}}\mathcal{S}_{\ell,\mathbf{m}}^{(0)}$$

$$\mathcal{S}_{\ell,\mathbf{m}}^{(0)} = \int \mathrm{d}^2 x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell,\mathbf{m}}^{(0)} D^a \Psi_{\ell,\mathbf{m}}^{(0)} - \frac{1}{2 f_t} V_\ell^{(0)}(r) \left| \Psi_{\ell,\mathbf{m}}^{(0)} \right|^2 \right]$$

$$V_\ell^{(0)}(r) \equiv f_t \left\{ \frac{\ell(\ell+n-3)}{r^2} + \frac{(n-2)(n-4)}{4r^2} f_r + \frac{n-2}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] + m^2 \right\}.$$

$$\mathrm{d}r_* = \mathrm{d}r/\sqrt{f_t \bar{f}_r}$$

$$\mathcal{S}_{\ell,\mathbf{m}}^{(0)} = \int \mathrm{d}t \, \mathrm{d}r_* \left[\frac{1}{2} \left| \partial_t \Psi_{\ell,\mathbf{m}}^{(0)} \right|^2 - \frac{1}{2} \left| \partial_{r_*} \Psi_{\ell,\mathbf{m}}^{(0)} \right|^2 - \frac{1}{2} V_\ell^{(0)}(r) \left| \Psi_{\ell,\mathbf{m}}^{(0)} \right|^2 \right],$$

$$\left[\partial_{r_*}^2-\partial_t^2-V_\ell^{(0)}(r)\right]\Psi_{\ell,\mathbf{m}}^{(0)}(r_*)=0$$

$$\begin{aligned}\mathcal{S}^{(1)} &= \int \mathrm{d}^n x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned}$$

$$A_\mu(x)=\left(\begin{array}{c} A_a(x) \\ D_A A^{(\mathrm{L})}(x)+A_A^{(\mathrm{T})}(x)\end{array}\right).$$

$$\begin{aligned}\delta_\Lambda A_a &= \partial_a \Lambda, \\ \delta_\Lambda A^{(\mathrm{L})} &= \Lambda, \\ \delta_\Lambda A_A^{(\mathrm{T})} &= 0.\end{aligned}$$

$$\mathcal{A}_a=A_a-D_aA^{(\mathrm{L})}$$

$$\begin{aligned}F_{ab}&=D_a\mathcal{A}_b-D_b\mathcal{A}_a\\F_{aA}&=D_aA^{(\mathrm{T})}_A-D_A\mathcal{A}_a\\F_{AB}&=D_AA^{(\mathrm{T})}_B-D_BA^{(\mathrm{T})}_A\end{aligned}$$

$$\mathcal{A}^a(x) \, = \, \sum_{\ell,\mathbf{m}} \mathcal{A}^a_{\ell,\mathbf{m}}(t,r) \mathcal{Y}_{\ell,\mathbf{m}}(\theta)$$

$$A^{(\mathrm{T})}_A(x) \, = \, \sum_{\ell,\mathbf{m}} A^{(\mathrm{V})}_{\ell,\mathbf{m}}(t,r) \mathcal{Y}^{(\mathrm{T})}_{A;\ell,\mathbf{m}}(\theta)$$

$$\mathcal{S}^{(1)} = \sum_{\ell, \mathbf{m}} \left(\mathcal{S}_{\ell, \mathbf{m}}^{(V)} + \mathcal{S}_{\ell, \mathbf{m}}^{(S)} \right)$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(V)} = \int d^2x \sqrt{-g^{(2)}} r^{n-4} \left[-\frac{1}{2} D_a \bar{A}_{\ell, \mathbf{m}}^{(V)} D^a A_{\ell, \mathbf{m}}^{(V)} - \frac{1}{2} \frac{(\ell+1)(\ell+n-4)}{r^2} |A_{\ell, \mathbf{m}}^{(V)}|^2 \right]$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(S)} = \int d^2x \sqrt{-g^{(2)}} r^{n-2} \left[-\frac{1}{4} \bar{f}_{ab; \ell, \mathbf{m}} f_{\ell, \mathbf{m}}^{ab} - \frac{1}{2} \frac{\ell(\ell+n-3)}{r^2} \bar{\mathcal{A}}_{a; \ell, \mathbf{m}} \mathcal{A}_{\ell, \mathbf{m}}^a \right]$$

$$f_{\ell, \mathbf{m}}^{ab} \equiv D^a \mathcal{A}_{\ell, \mathbf{m}}^b - D^b \mathcal{A}_{\ell, \mathbf{m}}^a$$

$$A_{\ell, \mathbf{m}}^{(V)}(t, r) = \frac{\Psi_{\ell, \mathbf{m}}^{(V)}(t, r)}{r^{(n-4)/2}}$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(V)} = \int d^2x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell, \mathbf{m}}^{(V)} D^a \Psi_{\ell, \mathbf{m}}^{(V)} - \frac{1}{2 f_t} V_{\ell}^{(V)}(r) |\Psi_{\ell, \mathbf{m}}^{(V)}|^2 \right]$$

$$V_{\ell}^{(V)}(r) \equiv f_t \left\{ \frac{(\ell+1)(\ell+n-4)}{r^2} + \frac{(n-4)(n-6)}{4r^2} f_r + \frac{n-4}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(V)} = \int dt dr_* \left[\frac{1}{2} |\partial_t \Psi_{\ell, \mathbf{m}}^{(V)}|^2 - \frac{1}{2} |\partial_{r_*} \Psi_{\ell, \mathbf{m}}^{(V)}|^2 - \frac{1}{2} V_{\ell}^{(V)}(r) |\Psi_{\ell, \mathbf{m}}^{(V)}|^2 \right]$$

$$\left[\partial_{r_*}^2 - \partial_t^2 - V_{\ell}^{(V)}(r) \right] \Psi_{\ell, \mathbf{m}}^{(V)}(r_*) = 0$$

$$\tilde{\mathcal{S}}_{\ell, \mathbf{m}}^{(S)} = \int d^2x \sqrt{-g^{(2)}} \left[\frac{1}{2} \sqrt{\ell(\ell+n-3)} r^{\frac{(n-4)}{2}} \text{Re} \left\{ \bar{\Psi}_{\ell, \mathbf{m}}^{(S)} \varepsilon_{ab} F_{\ell, \mathbf{m}}^{ab} \right\} - \frac{1}{2} \frac{\ell(\ell+n-3)}{r^2} \left(|\Psi_{\ell, \mathbf{m}}^{(S)}|^2 + r^{n-2} \bar{\mathcal{A}}_{a; \ell, \mathbf{m}} \mathcal{A}_{\ell, \mathbf{m}}^a \right) \right].$$

$$\Psi_{\ell, \mathbf{m}}^{(S)} = \frac{1}{2} \frac{r^{\frac{n}{2}}}{\sqrt{\ell(\ell+n-3)}} \varepsilon_{ab} f_{\ell, \mathbf{m}}^{ab}$$

$$\mathcal{A}_{\ell, \mathbf{m}}^a = \frac{r^{\frac{(n-4)}{2}}}{\sqrt{\ell(\ell+n-3)}} \left\{ \varepsilon^{ab} D_b - t^a \frac{n-4}{2r} \sqrt{\frac{f_r}{f_t}} \right\} \Psi_{\ell, \mathbf{m}}^{(S)}$$

$$\begin{aligned} \tilde{\mathcal{S}}_{\ell, \mathbf{m}}^{(S)} &= \int d^2x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell, \mathbf{m}}^{(S)} D^a \Psi_{\ell, \mathbf{m}}^{(S)} - \frac{1}{2 f_t} V_{\ell}^{(S)}(r) |\Psi_{\ell, \mathbf{m}}^{(S)}|^2 \right] \\ &= \int dt dr_* \left[\frac{1}{2} |\partial_t \Psi_{\ell, \mathbf{m}}^{(S)}|^2 - \frac{1}{2} |\partial_{r_*} \Psi_{\ell, \mathbf{m}}^{(S)}|^2 - \frac{1}{2} V_{\ell}^{(S)}(r) |\Psi_{\ell, \mathbf{m}}^{(S)}|^2 \right] \end{aligned}$$

$$\left[\partial_{r_*}^2 - \partial_t^2 - V_{\ell}^{(S)}(r) \right] \Psi_{\ell, \mathbf{m}}^{(S)}(r_*) = 0,$$

$$V_{\ell}^{(S)}(r) \equiv f_t \left\{ \frac{\ell(\ell+n-3)}{r^2} + \frac{(n-2)(n-4)}{4r^2} f_r - \frac{n-4}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}$$

$$\mathbb{A}^{(p)} = \frac{1}{p!} A_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$



$$\mathcal{S}^{(p)} = -\frac{1}{2} \int \mathbb{F}^{(p+1)} \wedge^* \mathbb{F}^{(p+1)}$$

$$F_{\mu_1\mu_2\cdots\mu_{p+1}}=(p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]}$$

$$\mathcal{S}^{(p)}=-\frac{1}{2(p+1)!}\int\,\mathrm{d}^n x\sqrt{-g}F_{\mu_1\mu_2\cdots\mu_{p+1}}F^{\mu_1\mu_2\cdots\mu_{p+1}}$$

$$\hat{\mathbb{A}}^{(p)}=\frac{1}{p!}A_{A_1A_2\cdots A_p}\mathcal{D}^{A_1A_2\cdots A_p},$$

$$\mathcal{D}^{A_1A_2\cdots A_p}\equiv \mathrm{d}x^{A_1}\wedge \mathrm{d}x^{A_2}\wedge \cdots \wedge \mathrm{d}x^{A_p}$$

$$\begin{aligned}\mathbb{A}^{(p)}=&\frac{1}{2}\frac{1}{(p-2)!}A_{abA_1\cdots A_{p-2}}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\mathcal{D}^{A_1\cdots A_{p-2}}\\&+\frac{1}{(p-1)!}A_{aA_1\cdots A_{p-1}}\,\mathrm{d}x^a\wedge\mathcal{D}^{A_1\cdots A_{p-1}}\\&+\frac{1}{p!}A_{A_1\cdots A_p}\mathcal{D}^{A_1\cdots A_p}\end{aligned}$$

$$(T_{ab})_{A_1\cdots A_{p-2}}\equiv A_{abA_1\cdots A_{p-2}}$$

$$(V_a)_{A_1\cdots A_{p-1}}\equiv A_{aA_1\cdots A_{p-1}}$$

$$X_{A_1\cdots A_p}\equiv A_{A_1\cdots A_p}$$

$$\mathbb{A}^{(p)}=\frac{1}{2}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\hat{\mathbb{T}}_{ab}^{(p-2)}+\mathrm{d}x^a\wedge\hat{\mathbb{V}}_a^{(p-1)}+\hat{\mathbb{X}}^{(p)}$$

$$\begin{aligned}\hat{\mathbb{A}}^{(p)}&= \hat{\mathrm{d}}\hat{\mathbb{A}}^{(p-1)} + \hat{\mathcal{A}}^{(p)} \\ &= \hat{\mathrm{d}}\big(\hat{\mathrm{d}}\hat{\mathbb{A}}^{(p-2)} + \hat{\mathcal{A}}^{(p-1)}\big) + \hat{\mathcal{A}}^{(p)} \\ &= \hat{\mathrm{d}}\hat{\mathcal{A}}^{(p-1)} + \hat{\mathcal{A}}^{(p)},\end{aligned}$$

$$D^{A_1}\mathcal{A}_{A_1A_2\cdots A_{p+1}}=0$$

$$\begin{aligned}\mathbb{A}^{(p)}=&\frac{1}{2}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\Big(\hat{\mathrm{d}}\hat{\mathcal{T}}_{ab}^{(p-3)}+\hat{\mathcal{T}}_{ab}^{(p-2)}\Big)\\&+\mathrm{d}x^a\wedge\Big(\hat{\mathrm{d}}\hat{\mathcal{V}}_a^{(p-2)}+\hat{\mathcal{V}}_a^{(p-1)}\Big)\\&+\big(\hat{\mathrm{d}}\hat{\mathcal{X}}^{(p-1)}+\hat{\mathcal{X}}^{(p)}\big),\end{aligned}$$

$$\begin{aligned}\mathrm{d}\Lambda^{(p-1)}&=\delta_\Lambda\mathbb{A}^{(p)}\\&=\frac{1}{2}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\Big(\hat{\mathrm{d}}\hat{\Lambda}_{ab}^{(p-3)}+2D_{[a}\hat{\Lambda}_{b]}^{(p-2)}\Big)\\&\quad+\mathrm{d}x^a\wedge\Big(-\hat{\mathrm{d}}\hat{\Lambda}_a^{(p-2)}+D_a\hat{\Lambda}^{(p-1)}\Big)\\&\quad+\hat{\mathrm{d}}\hat{\Lambda}^{(p-1)}\end{aligned}$$

$$\begin{aligned}\delta_\Lambda\hat{\mathcal{T}}_{ab}^{(p-3)}&=\hat{\Lambda}_{ab}^{(p-3)},\quad\delta_\Lambda\hat{\mathcal{T}}_{ab}^{(p-2)}=2D_{[a}\hat{\Lambda}_{b]}^{(p-2)}\\ \delta_\Lambda\hat{\mathcal{V}}_a^{(p-2)}&=-\hat{\Lambda}_a^{(p-2)},\quad\delta_\Lambda\hat{\mathcal{V}}_a^{(p-1)}=D_a\hat{\Lambda}^{(p-1)}\\ \delta_\Lambda\hat{\mathcal{X}}^{(p-1)}&=\hat{\Lambda}^{(p-1)},\quad\delta_\Lambda\hat{\mathcal{X}}^{(p)}=0\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{H}}_{ab}^{(p-2)} &= \hat{\mathcal{J}}_{ab}^{(p-2)} + 2D_{[a}\hat{\mathcal{V}}_{b]}^{(p-2)} \\ \hat{\mathcal{A}}_a^{(p-1)} &= \hat{\mathcal{V}}_a^{(p-1)} - D_a\hat{\mathcal{X}}^{(p-1)}\end{aligned}$$

$$\begin{aligned}\mathbb{F}^{(p+1)} = & \frac{1}{2} \, \mathrm{d} x^a \wedge \, \mathrm{d} x^b \wedge \Big(2 D_{[a} \hat{\mathcal{A}}_{b]}^{(p-1)} + (p-2)! \, \hat{\mathrm{d}} \hat{\mathcal{H}}_{ab}^{(p-2)} \Big) \\ & + \mathrm{d} x^a \wedge \Big(D_a \hat{\mathcal{X}}^{(p)} - \hat{\mathrm{d}} \hat{\mathcal{A}}_a^{(p-1)} \Big) \\ & + \hat{\mathrm{d}} \hat{\mathcal{X}}^{(p)}\end{aligned}$$

$$\begin{aligned}(\mathcal{H}^{ab})^{A_1\cdots A_{p-2}}(x) &= \sum_{\ell,\mathbf{m}} \mathcal{H}_{\ell,\mathbf{m}}^{ab}(t,r) \mathfrak{Y}_{\ell,\mathbf{m}}^{(\mathrm{T})A_1\cdots A_{p-2}}(\theta) \\ (\mathcal{A}^a)^{A_1\cdots A_{p-1}}(x) &= \sum_{\ell,\mathbf{m}} \mathcal{A}_{\ell,\mathbf{m}}^a(t,r) \mathfrak{Y}_{\ell,\mathbf{m}}^{(\mathrm{T})A_1\cdots A_{p-1}}(\theta) \\ \mathcal{X}^{A_1\cdots A_p}(x) &= \sum_{\ell,\mathbf{m}} \mathcal{X}_{\ell,\mathbf{m}}(t,r) \mathfrak{Y}_{\ell,\mathbf{m}}^{(\mathrm{T})A_1\cdots A_p}(\theta)\end{aligned}$$

$$\mathfrak{Y}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)}=\frac{1}{p!}\mathfrak{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}\mathcal{D}^{A_1\cdots A_p},$$

$$\begin{aligned}\hat{\Delta}\hat{\mathfrak{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} &= \lambda_p\hat{\mathfrak{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} \\ \hat{\delta}_{\Lambda}\hat{\mathfrak{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)} &= 0\end{aligned}$$

$$\hat{\Delta}\equiv\hat{\delta}\hat{\mathrm{d}}+\hat{\mathrm{d}}\hat{\delta}$$

$$\lambda_p\equiv(\ell+p)(\ell+n-p-3)$$

$$\hat{\delta}\hat{\mathrm{d}}\hat{\mathfrak{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)}=\lambda_p\hat{\mathfrak{Y}}_{\ell,\mathbf{m}}^{(\mathrm{T})(p)}$$

$$\tilde{\Delta}\mathfrak{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}=-\gamma_p\mathfrak{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}$$

$$\tilde{\Delta}\mathfrak{Y}_{A_1\cdots A_p;\ell,\mathbf{m}}^{(\mathrm{T})}=\mathfrak{Y}^{\star}_{A_1\cdots A_p;\ell,\mathbf{m}}\mathrel{\mathop:}\mathrel{\mathop:}\blacksquare^{\eta}$$

$$\gamma_p=\ell(\ell+n-3)-p$$

$$\begin{aligned}\mathcal{S}^{(p)} &= \sum_{\ell,\mathbf{m}} \left(\mathcal{S}_{\ell,\mathbf{m}}^{(p)} + \mathcal{S}_{\ell,\mathbf{m}}^{(p-1)} + \mathcal{S}_{\ell,\mathbf{m}}^{(p-2)} \right), \\ \mathcal{S}_{\ell,\mathbf{m}}^{(p)} &= \int \, \mathrm{d}^2x \sqrt{-g^{(2)}} r^{n-2p-2} \left[-\frac{1}{2p!} D_a \overline{\mathcal{X}}_{\ell,\mathbf{m}} D^a \mathcal{X}_{\ell,\mathbf{m}} - \frac{1}{2p!} \frac{(\ell+p)(\ell+n-p-3)}{r^2} |\mathcal{X}_{\ell,\mathbf{m}}|^2 \right], \\ \mathcal{S}_{\ell,\mathbf{m}}^{(p-1)} &= \int \, \mathrm{d}^2x \sqrt{-g^{(2)}} r^{n-2p} \left[-\frac{1}{4(p-1)!} \overline{\mathfrak{F}}_{ab;\ell,\mathbf{m}} \mathfrak{F}_{\ell,\mathbf{m}}^{ab} \right. \\ &\quad \left. - \frac{1}{2(p-1)!} \frac{(\ell+p-1)(\ell+n-p-2)}{r^2} \overline{\mathcal{A}}_{a;\ell,\mathbf{m}} \mathcal{A}_{\ell,\mathbf{m}}^a \right], \\ \mathcal{S}_{\ell,\mathbf{m}}^{(p-2)} &= \int \, \mathrm{d}^2x \sqrt{-g^{(2)}} r^{n-2p+2} \left[-\frac{(p-2)! (\ell+p-2)(\ell+n-p-1)}{4} \frac{\overline{\mathcal{H}}_{ab;\ell,\mathbf{m}} \mathcal{H}_{\ell,\mathbf{m}}^{ab}}{r^2} \right], \\ \mathfrak{F}_{\ell,\mathbf{m}}^{ab} &\equiv D^a \mathcal{A}_{\ell,\mathbf{m}}^b - D^b \mathcal{A}_{\ell,\mathbf{m}}^a.\end{aligned}$$



$$\mathcal{X}_{\ell,\mathbf{m}}(t,r) = \sqrt{p!} \frac{\Psi_{\ell,\mathbf{m}}^{(p)}(t,r)}{r^{(n-2p-2)/2}},$$

$$\mathcal{S}_{\ell,\mathbf{m}}^{(p)} = \int d^2x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell,\mathbf{m}}^{(p)} D^a \Psi_{\ell,\mathbf{m}}^{(p)} - \frac{1}{2f_t} V_\ell^{(p)}(r) |\Psi_{\ell,\mathbf{m}}^{(p)}|^2 \right]$$

$$V_\ell^{(p)}(r) \equiv f_t \left\{ \frac{(\ell+p)(\ell+n-p-3)}{r^2} + \frac{(n-2p-2)(n-2p-4)}{4r^2} f_r \right. \\ \left. + \frac{n-2p-2}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}.$$

$$\tilde{\mathcal{S}}_{\ell,\mathbf{m}}^{(p-1)} = \int d^2x \sqrt{-g^{(2)}} \left[\frac{1}{2} \sqrt{\frac{(\ell+p-1)(\ell+n-p-2)}{(p-1)!}} r^{\frac{(n-2p-2)}{2}} \text{Re} \left\{ \bar{\Psi}_{\ell,\mathbf{m}}^{(\tilde{p})} \varepsilon_{ab} \tilde{\mathcal{S}}_{\ell,\mathbf{m}}^{ab} \right\} \right. \\ \left. - \frac{1}{2} \frac{(\ell+p-1)(\ell+n-p-2)}{r^2} \left(|\Psi_{\ell,\mathbf{m}}^{(\tilde{p})}|^2 + \frac{r^{n-2p}}{(p-1)!} \bar{\mathcal{A}}_{a;\ell,\mathbf{m}} \mathcal{A}_{\ell,\mathbf{m}}^a \right) \right],$$

$$\Psi_{\ell,\mathbf{m}}^{(\tilde{p})} = \frac{1}{2} \frac{r^{\frac{(n-2p+2)}{2}}}{\sqrt{(p-1)!(\ell+p-1)(\ell+n-p-2)}} \varepsilon_{ab} \tilde{\mathcal{S}}_{\ell,\mathbf{m}}^{ab}$$

$$\mathcal{A}_{\ell,\mathbf{m}}^a = \sqrt{\frac{(p-1)!}{(\ell+p-1)(\ell+n-p-2)}} r^{-\frac{(n-2p-2)}{2}} \left\{ \varepsilon^{ab} D_b - t^a \frac{n-2p-2}{2r} \sqrt{\frac{f_r}{f_t}} \right\} \Psi_{\ell,\mathbf{m}}^{(\tilde{p})}$$

$$\tilde{\mathcal{S}}_{\ell,\mathbf{m}}^{(p-1)} = \int d^2x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell,\mathbf{m}}^{(\tilde{p})} D^a \Psi_{\ell,\mathbf{m}}^{(\tilde{p})} - \frac{1}{2f_t} V_\ell^{(\tilde{p})}(r) |\Psi_{\ell,\mathbf{m}}^{(\tilde{p})}|^2 \right]$$

$$V_\ell^{(\tilde{p})}(r) \equiv f_t \left\{ \frac{(\ell+p-1)(\ell+n-p-2)}{r^2} + \frac{(n-2p)(n-2p-2)}{4r^2} f_r \right. \\ \left. - \frac{(n-2p-2)}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}.$$

$$V_\ell^{(\kappa)}(r) \equiv f_t \left\{ \frac{(\ell+\kappa)(\ell+n-\kappa-3)}{r^2} + \frac{(n-2\kappa-2)(n-2\kappa-4)}{4r^2} f_r \right. \\ \left. + \frac{n-2\kappa-2}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}.$$

$$V_\ell^{(\kappa)}(r) \Rightarrow \begin{cases} V_\ell^{(0)}(r), & \kappa = 0; \\ V_\ell^{(\text{V})}(r), & \kappa = 1; \\ V_\ell^{(\text{S})}(r), & \kappa = n-3; \\ V_\ell^{(p)}(r), & \kappa = p; \\ V_\ell^{(\tilde{p})}(r), & \kappa = \tilde{p} = n-p-2. \end{cases}$$

$$h_{ab}(x) = H_{ab}(x)$$

$$h_{aA}(x) = D_A H_a^{(\text{S})}(x) + h_{aA}^{(\text{V})}(x)$$

$$h_{AB}(x) = r^2 \left[K(x) \Omega_{AB} + D_{\langle A} D_{B \rangle} G(x) + D_{(A} h_{B)}^{(\text{V})}(x) + h_{AB}^{(\text{TT})}(x) \right]$$



$$H_{ab}(x), H_a^{(\text{L})}(x), K(x), G(x).$$

$$h_{aA}^{(\text{V})}(x), h_A^{(\text{V})}(x);$$

$$D^A h_{aA}^{(\text{V})}(x) = 0$$

$$D^A h_A^{(\text{V})}(x) = 0$$

$$h_{AB}^{(\text{TT})}(x);$$

$$D^A h_{AB}^{(\text{TT})}(x) = 0,$$

$$\Omega^{AB} h_{AB}^{(\text{TT})}(x) = 0.$$

$$\delta_{\xi} h_{\mu\nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}$$

$$\xi_a(x) = D_a \xi^{(\text{S})}(x) + \xi_a^{(\text{V})}(x)$$

$$\xi_A(x) = D_A \xi^{(\text{S})}(x) + \xi_A^{(\text{V})}(x)$$

$$\delta_{\xi} H_{ab} = D_a \xi_b + D_b \xi_a$$

$$\delta_{\xi} H_a^{(\text{S})} = \xi_a + D_a \xi^{(\text{S})} - \frac{2}{r} r_a \xi^{(\text{S})}$$

$$\delta_{\xi} K = \frac{2}{r} r^a \xi_a + \frac{2}{n-2} \frac{1}{r^2} D_A D^A \xi^{(\text{S})}$$

$$\delta_{\xi} G = \frac{2}{r^2} \xi^{(\text{S})}$$

$$\delta_{\xi} h_{aA}^{(\text{V})} = D_a \xi_A^{(\text{V})} - \frac{2}{r} r_a \xi_A^{(\text{V})}$$

$$\delta_{\xi} h_A^{(\text{V})} = \frac{2}{r^2} \xi_A^{(\text{V})}$$

$$\delta_{\xi} h_{AB}^{(\text{TT})} = 0$$

$$h_{AB}^{(\text{TT})} \otimes \oint \mathfrak{d}^4 \mathfrak{S}_{gravity}$$

$$\mathcal{H}_{aA}^{(\text{V})} = h_{aA}^{(\text{V})} - \frac{1}{2} r^2 D_a h_A^{(\text{V})}$$

$$\mathcal{H}_{ab} = H_{ab} - 2D_{(a} H_{b)}^{(\text{S})} + D_{(a} (r^2 D_{b)} G)$$

$$\mathcal{K} = K - \frac{1}{n-2} D_A D^A G + r r^a D_a G - \frac{2}{r} r^a H_a^{(\text{S})}$$

$$\mathcal{H}_{ab}(x) = \sum_{\ell,\mathbf{m}} \mathcal{H}_{ab;\ell,\mathbf{m}}(t,r) \mathcal{Y}_{\ell,\mathbf{m}}(\theta)$$

$$\mathcal{K}(x) = \sum_{\ell,\mathbf{m}} \mathcal{K}_{\ell,\mathbf{m}}(t,r) \mathcal{Y}_{\ell,\mathbf{m}}(\theta)$$

$$\mathcal{H}_{aA}^{(\text{V})}(x) = \sum_{\ell,\mathbf{m}} \mathcal{H}_{a;\ell,\mathbf{m}}(t,r) \mathcal{Y}_{A;\ell,\mathbf{m}}^{(\text{T})}(\theta)$$

$$h_{AB}^{(\text{TT})}(x) = \sum_{\ell,\mathbf{m}} h_{\ell,\mathbf{m}}^{(\text{T})}(t,r) \mathcal{Y}_{AB;\ell,\mathbf{m}}^{(\text{TT})}(\theta)$$

$$\mathcal{S}^{(\text{gr})} = \int d^n x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho h_{\mu\nu} \nabla^\rho h^{\mu\nu} + \nabla_\rho h_{\mu\nu} \nabla^\nu h^{\mu\rho} - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla_\mu h \nabla^\mu h \right]$$

$$g_{\mu\nu}^{\text{full}} = g_{\mu\nu} + \sqrt{32\pi\mathcal{G}} h_{\mu\nu}$$

$$\mathcal{S}^{(\text{gr})} = \sum_{\ell, \mathbf{m}} \left(\mathcal{S}_{\ell, \mathbf{m}}^{(\text{T})} + \mathcal{S}_{\ell, \mathbf{m}}^{(\text{RW})} + \mathcal{S}_{\ell, \mathbf{m}}^{(\text{Z})} \right)$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(\text{T})} = \int d^2 x \sqrt{-g^{(2)}} r^{n-2} \left[-\frac{1}{2} D_a \bar{h}_{\ell, \mathbf{m}}^{(\text{T})} D^a h_{\ell, \mathbf{m}}^{(\text{T})} - \frac{1}{r} r^a D_a |h_{\ell, \mathbf{m}}^{(\text{T})}|^2 \right. \\ \left. - \frac{1}{2} \frac{\ell(\ell+n-3) + 2(n-3)}{r^2} |h_{\ell, \mathbf{m}}^{(\text{T})}|^2 \right]$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(\text{RW})} = \int d^2 x \sqrt{-g^{(2)}} 2r^{n-4} \left[-\frac{1}{4} \bar{\mathcal{F}}_{ab; \ell, \mathbf{m}} \mathcal{F}_{\ell, \mathbf{m}}^{ab} - \frac{2}{r} r_a \text{Re} \left\{ \bar{\mathcal{H}}_{\ell, \mathbf{m}}^b D_b \mathcal{H}_{\ell, \mathbf{m}}^a \right\} \right. \\ \left. - \frac{1}{2} \left(\frac{(\ell+1)(\ell+n-4)}{r^2} \bar{\mathcal{H}}_{a; \ell, \mathbf{m}} \mathcal{H}_{\ell, \mathbf{m}}^a - 4 |r_a \mathcal{H}_{\ell, \mathbf{m}}^a|^2 \right) \right]$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(\text{Z})} = \int d^2 x \sqrt{-g^{(2)}} r^{n-2} \left[-\frac{1}{2} D_c \bar{\mathcal{H}}_{ab; \ell, \mathbf{m}} D^c \mathcal{H}_{\ell, \mathbf{m}}^{ab} + D_c \bar{\mathcal{H}}_{ab; \ell, \mathbf{m}} D^b \mathcal{H}_{\ell, \mathbf{m}}^{ac} \right. \\ \left. - \text{Re} \{ D_a \bar{\mathcal{H}}_{\ell, \mathbf{m}} D_b \mathcal{H}_{\ell, \mathbf{m}}^{ab} \} \right. \\ \left. + \frac{1}{2} D_a \bar{\mathcal{H}}_{\ell, \mathbf{m}} D^a \mathcal{H}_{\ell, \mathbf{m}} + \frac{(n-2)(n-3)}{2} D_a \bar{\mathcal{K}}_{\ell, \mathbf{m}} D^a \mathcal{K}_{\ell, \mathbf{m}} \right. \\ \left. - (n-2) \text{Re} \{ D_a \bar{\mathcal{K}}_{\ell, \mathbf{m}} (D_b \mathcal{H}_{\ell, \mathbf{m}}^{ab} - D^a \mathcal{H}_{\ell, \mathbf{m}}) \} \right. \\ \left. - \frac{n-2}{r} \text{Re} \{ (D_a \bar{\mathcal{H}}_{\ell, \mathbf{m}} + (n-4) D_a \bar{\mathcal{K}}_{\ell, \mathbf{m}}) (r_b \mathcal{H}_{\ell, \mathbf{m}}^{ab} - r^a \mathcal{K}_{\ell, \mathbf{m}}) \} \right. \\ \left. - \frac{1}{2} \frac{\ell(\ell+n-3)}{r^2} \left(\bar{\mathcal{H}}_{ab; \ell, \mathbf{m}} \mathcal{H}_{\ell, \mathbf{m}}^{ab} - |\mathcal{H}_{\ell, \mathbf{m}}|^2 - (n-3)(n-4) |\mathcal{K}_{\ell, \mathbf{m}}|^2 \right. \right. \\ \left. \left. - 2(n-3) \text{Re} \{ \bar{\mathcal{H}}_{\ell, \mathbf{m}} \mathcal{K}_{\ell, \mathbf{m}} \} \right) \right],$$

$$\mathcal{F}_{\ell, \mathbf{m}}^{ab} \equiv D^a \mathcal{H}_{\ell, \mathbf{m}}^b - D^b \mathcal{H}_{\ell, \mathbf{m}}^a$$

$$\mathcal{H}_{\ell, \mathbf{m}} \equiv g_{ab} \mathcal{H}_{\ell, \mathbf{m}}^{ab}$$

$$h_{\ell, \mathbf{m}}^{(\text{T})} = \frac{\Psi_{\ell, \mathbf{m}}^{(\text{T})}}{r^{(n-2)/2}}$$

$$\mathcal{S}_{\ell, \mathbf{m}}^{(\text{T})} = \int d^2 x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell, \mathbf{m}}^{(\text{T})} D^a \Psi_{\ell, \mathbf{m}}^{(\text{T})} - \frac{1}{2 f_t} V_\ell^{(\text{T})}(r) |\Psi_{\ell, \mathbf{m}}^{(\text{T})}|^2 \right] \\ = \int dt dr_* \left[\frac{1}{2} |\partial_t \Psi_{\ell, \mathbf{m}}^{(\text{T})}|^2 - \frac{1}{2} |\partial_{r_*} \Psi_{\ell, \mathbf{m}}^{(\text{T})}|^2 - \frac{1}{2} V_\ell^{(\text{T})}(r) |\Psi_{\ell, \mathbf{m}}^{(\text{T})}|^2 \right]$$

$$V_\ell^{(\text{T})}(r) \equiv f_t \left\{ \frac{\ell(\ell+n-3) + 2(n-3)}{r^2} + \frac{n^2 - 14n + 32}{4r^2} f_r + \frac{n-6}{2r} \left[\frac{(f_t f_r)'}{2f_t} \right] \right\}$$

$$\tilde{\mathcal{S}}_{\ell, \mathbf{m}}^{(\text{RW})} = \int d^2 x \sqrt{-g^{(2)}} \left[\sqrt{\frac{\mathbb{F}_\ell(r)}{2}} r^{\frac{(n-6)}{2}} \text{Re} \left\{ \bar{\Psi}_{\ell, \mathbf{m}}^{(\text{RW})} \left(\varepsilon_{ab} \mathcal{F}_{\ell, \mathbf{m}}^{ab} - \frac{4}{r} \sqrt{\frac{f_r}{f_t}} t_a \mathcal{H}_{\ell, \mathbf{m}}^a \right) \right\} \right. \\ \left. - \frac{1}{2} \frac{\mathbb{F}_\ell(r)}{r^2} \left(|\Psi_{\ell, \mathbf{m}}^{(\text{RW})}|^2 + 2r^{n-4} \bar{\mathcal{H}}_{a; \ell, \mathbf{m}} \mathcal{H}_{\ell, \mathbf{m}}^a \right) \right]$$

$$\mathbb{F}_\ell(r) \equiv (\ell+1)(\ell+n-4) - 2(n-3)r_a r^a - 2r D_a r^a$$



$$\Psi_{\ell,\mathbf{m}}^{(\text{RW})}=\frac{r^{\frac{(n-2)}{2}}}{\sqrt{2\mathbb{F}_{\ell}(r)}}\left(\varepsilon_{ab}\mathcal{F}_{\ell,\mathbf{m}}^{ab}-\frac{4}{r}\sqrt{\frac{f_r}{f_t}}t_a\mathcal{H}_{\ell,\mathbf{m}}^a\right)$$

$$\mathcal{H}_{\ell,\mathbf{m}}^a=\frac{r^{\frac{(n-6)}{2}}}{\sqrt{2\mathbb{F}_{\ell}(r)}}\left\{\varepsilon^{ab}D_b-t^a\left[\frac{n-2}{2r}+\frac{\mathbb{F}'_{\ell}(r)}{2\mathbb{F}_{\ell}(r)}\right]\sqrt{\frac{f_r}{f_t}}\right\}\Psi_{\ell,\mathbf{m}}^{(\text{RW})}$$

$$\begin{aligned}\tilde{\mathcal{S}}_{\ell,\mathbf{m}}^{(\text{RW})} &= \int \mathrm{d}^2x \sqrt{-g^{(2)}} \left[-\frac{1}{2} D_a \bar{\Psi}_{\ell,\mathbf{m}}^{(\text{RW})} D^a \Psi_{\ell,\mathbf{m}}^{(\text{RW})} - \frac{1}{2 f_t} V_{\ell}^{(\text{RW})}(r) \left| \Psi_{\ell,\mathbf{m}}^{(\text{RW})} \right|^2 \right] \\ &= \int \mathrm{d}t \, \mathrm{d}r_* \left[\frac{1}{2} \left| \partial_t \Psi_{\ell,\mathbf{m}}^{(\text{RW})} \right|^2 - \frac{1}{2} \left| \partial_{r_*} \Psi_{\ell,\mathbf{m}}^{(\text{RW})} \right|^2 - \frac{1}{2} V_{\ell}^{(\text{RW})}(r) \left| \Psi_{\ell,\mathbf{m}}^{(\text{RW})} \right|^2 \right]\end{aligned}$$

$$\begin{aligned}V_{\ell}^{(\text{RW})}(r)\equiv&f_t\left\{\frac{(\ell+1)(\ell+n-4)}{r^2}+\frac{(n-4)(n-6)}{4r^2}f_r-\frac{n+2}{2r}\left[\frac{(f_tf_r)'}{2f_t}\right]\right.\\&\left.+\left[\frac{\mathbb{F}'_{\ell}}{2\mathbb{F}_{\ell}}\left(\frac{n-4}{r}+\frac{\mathbb{F}'_{\ell}}{2\mathbb{F}_{\ell}}\right)-\left(\frac{\mathbb{F}'_{\ell}}{2\mathbb{F}_{\ell}}\right)'\right]f_r-\frac{\mathbb{F}'_{\ell}}{2\mathbb{F}_{\ell}}\left[\frac{(f_tf_r)'}{2f_t}\right]\right\}\end{aligned}$$

$$\mathbb{F}_{\ell}(r)\equiv(\ell+1)(\ell+n-4)-2(n-3)f_r-2r\left[\frac{(f_tf_r)'}{2f_t}\right]$$

$$\begin{aligned}V_{\ell}^{(\text{Z})}(r)\equiv&\left\{\frac{\ell(\ell+n-3)}{r^2}+\frac{(n-2)(n-4)}{4r^2}\text{f}(r)+\frac{n-2}{2r}\text{f}'(r)\right\}\text{f}(r)\\&-\left\{\frac{2\text{f}'(r)}{r}\frac{[2\lambda_{\ell}+(n-1)(n-2)][\text{H}_{\ell}(r)+2(n-3)\lambda_{\ell}]}{\text{H}_{\ell}^2(r)}\right\}\text{f}(r)\end{aligned}$$

$$\text{H}_{\ell}(r)=2\ell(\ell+n-3)-2(n-2)\text{f}(r)+(n-2)r\text{f}'(r)$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+[\omega^2-V(x)]\Psi(x)=0$$

$$\Psi(x\rightarrow\pm\infty)\propto e^{\pm i\omega x}$$

$$\begin{aligned}\omega^2=&V_0+A_2(\mathcal{K}^2)+A_4(\mathcal{K}^2)+A_6(\mathcal{K}^2)+\cdots\\&-i\mathcal{K}\sqrt{-2V_2}[1+A_3(\mathcal{K}^2)+A_5(\mathcal{K}^2)+A_7(\mathcal{K}^2)+\cdots],\end{aligned}$$

$$\mathcal{K}=\begin{cases}+n+\frac{1}{2},\text{Re}(\omega)>0\\-n-\frac{1}{2},\text{Re}(\omega)<0\end{cases}$$

$$\begin{aligned}P_k(\epsilon)=&V_0+A_2(\mathcal{K}^2)\epsilon^2+A_4(\mathcal{K}^2)\epsilon^4+A_6(\mathcal{K}^2)\epsilon^6+\cdots\\&-i\mathcal{K}\sqrt{-2V_2}[\epsilon+A_3(\mathcal{K}^2)\epsilon^3+A_5(\mathcal{K}^2)\epsilon^5+A_7(\mathcal{K}^2)\epsilon^7+\cdots],\end{aligned}$$

$$\omega^2=P_k(1)$$

$$P_{\tilde{n}/\tilde{m}}(\epsilon)=\frac{Q_0+Q_1\epsilon+\cdots+Q_{\tilde{n}}\epsilon^{\tilde{n}}}{R_0+R_1\epsilon+\cdots+R_{\tilde{m}}\epsilon^{\tilde{m}}}$$

$$P_{\tilde{n}/\tilde{m}}(\epsilon)-P_k(\epsilon)=\mathcal{O}(\epsilon^{k+1})$$



$$\chi''=\lambda_0(x)\chi'+s_0(x)\chi$$

$$\chi'''=\lambda_1(x)\chi'+s_1(x)\chi,$$

$$\chi^{(n+2)}=\lambda_n(x)\chi'+s_n(x)\chi$$

$$\begin{aligned}\lambda_n(x) &= \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \\ s_n(x) &= s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x)\end{aligned}$$

$$\frac{s_n(x)}{\lambda_n(x)}=\frac{s_{n-1}(x)}{\lambda_{n-1}(x)}=\mathfrak{B}\mathfrak{h}$$

$$\begin{aligned}\lambda_n(x') &= \sum_{i=0}^{\infty} c_n^i(x-x')^i \\ s_n(x') &= \sum_{i=0}^{\infty} d_n^i(x-x')^i\end{aligned}$$

$$\begin{aligned}c_n^i &= \sum_{k=0}^i c_0^k c_{n-1}^{i-k} + (i+1)c_{n-1}^{i+1} + d_{n-1}^i \\ d_n^i &= \sum_{k=0}^i d_0^k c_{n-1}^{i-k} + (i+1)d_{n-1}^{i+1}\end{aligned}$$

$$d_n^0c_{n-1}^0-c_n^0d_{n-1}^0=0$$

$$\frac{\partial^2\Psi}{\partial t^2}-\frac{\partial^2\Psi}{\partial x^2}+V(t,x)\Psi=0$$

$$\left[4\frac{\partial^2}{\partial u\partial v}+V(u,v)\right]\Psi(u,v)=0$$

$$\Psi(u=u_0,v)=\exp\left[-\frac{(v-v_c)^2}{2\sigma^2}\right], \Psi(u,v=v_0)=\Psi(u=u_0,0)$$

$$\Psi(N)=\Psi(W)+\Psi(E)-\Psi(S)-\frac{\Delta^2}{8}[\Psi(W)+\Psi(E)]V(S)+\mathcal{O}(\Delta^4)$$

$$N=(u+\Delta,v+\Delta), W=(u+\Delta,v), E=(u,v+\Delta) \text{ and } S=(u,v)$$

$$\Psi\Big(t=\frac{v+u}{2},x=\frac{v-u}{2}\Big)$$

$$\{t_0,t_0+\Delta,t_0+2\Delta,\cdots,t_0+\infty\Delta\}$$

$$\hat{g}\colon=-\left(1-\frac{M}{r}\right)^2 dt^2+\left(1-\frac{M}{r}\right)^{-2} dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

$$\tau=\frac{t}{4M}\in\mathbb{R}, x=\log\left(\frac{r}{M}-1\right)\in\mathbb{R}, \hat{g}=16M^2(1+e^{-x})^{-2}g$$



$$g = -dt^2 + \cosh^4\left(\frac{x}{2}\right)(dx^2 + d\theta^2 + \sin^2\theta d\varphi^2)$$

$$A(t,x)=\varphi(t,x)\omega(\tau_1,\tau_2)+\tau_3\cos\theta d\varphi$$

$$\omega(\tau_1,\tau_2)=\tau_1d\theta+\tau_2\sin\theta d\varphi,\varphi=\varphi(t,x)$$

$$\mathcal{L}[x,\varphi,\partial_x\varphi,\partial_t\varphi]=-\frac{1}{2}\cosh^2\left(\frac{x}{2}\right)(\partial_t\varphi)^2+\frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\left((\partial_x\varphi)^2+\frac{1}{2}(1-\varphi^2)^2\right)$$

$$\partial_t^2\varphi-Q\partial_x(Q\partial_x\varphi)+Q^2(\varphi^2-1)\varphi=0$$

$$\varphi(t,x)\mapsto\varphi\left(\frac{3}{2}t,x\right)$$

$$Q(x)\!:=\!\frac{3}{2}\operatorname{sech}^2\!\left(\frac{x}{2}\right)$$

$$\mathcal{E}[\varphi,\varphi_t]=\int\left(\frac{1}{2}Q^{-1}(\partial_t\varphi)^2+\frac{1}{2}Q\left((\partial_x\varphi)^2+\frac{1}{2}(1-\varphi^2)^2\right)\right)dx$$

$$Q^{-1}\partial_t(\partial_t\varphi)^2+Q\partial_t\left((\partial_x\varphi)^2+\frac{1}{2}(1-\varphi^2)^2\right)+\partial_x(Q\partial_t\varphi\partial_x\varphi)=0$$

$$\mathcal{P}[\varphi,\partial_t\varphi]=\int\;Q^{-1}\partial_t\varphi\partial_x\varphi dx$$

$$H''-\tanh\left(\frac{x}{2}\right)H'+H(1-H^2)=0,x\in\mathbb{R}$$

$$H(x)=\tanh\left(\frac{x}{2}\right)$$

$$\alpha(x)=\frac{1}{3}(\sinh\,x+x),$$

$$\tilde{Q}(x)=Q(\alpha^{-1}(x)),\tilde{H}(x)=H(\alpha^{-1}(x)),$$

$$0\leq \tilde{Q}(x)\lesssim \frac{1}{|x|}, |\tilde{H}(x)\mp 1|\lesssim \frac{1}{|x|}, \text{ as } x\rightarrow \pm\infty.$$

$$\partial_t^2\phi-\partial_x^2\phi-\tilde{Q}^2(\phi-\phi^3)=0$$

$$\begin{cases} \partial_t\phi_1=\phi_2\\ \partial_t\phi_2=\partial_x^2\phi_1+\tilde{Q}^2(x)(1-\phi_1^2)\phi_1 \end{cases}$$

$$E[\phi_1,\phi_2]=\frac{1}{2}\int\left(\phi_2^2+(\partial_x\phi_1)^2+\frac{1}{2}\tilde{Q}^2(1-\phi_1^2)^2\right)dx$$

$$E[\phi_1,\phi_2]:0=E[\pm1,0]<E[\tilde{H},0]=\frac{1}{2}\int\;QH'^2+\frac{1}{4}\int\;Q(1-H^2)^2=\frac{6}{5}$$

$$\mathbf{E}:=\left\{\phi=(\phi_1,\phi_2)\in\left(L^1_{\text{loc}}(\mathbb{R})\right)^2\colon\partial_x\phi_1\in L^2(\mathbb{R}),\tilde{Q}(1-\phi_1^2)\in L^2(\mathbb{R}),\phi_2\in L^2(\mathbb{R})\right\}$$

$$H_0(\mathbb{R})\colon=\left\{\phi\in L^1_{\text{loc}}(\mathbb{R})\colon\partial_x\phi\in L^2(\mathbb{R}),\tilde{Q}\phi\in L^2(\mathbb{R})\right\}$$

$$\|\phi\|^2_{H_0(\mathbb{R})}\colon=\|\partial_x\phi\|^2_{L^2(\mathbb{R})}+\|\tilde{Q}\phi\|^2_{L^2(\mathbb{R})}$$

$$|\phi(x)|\leq |\phi(0)|+\|\partial_x\phi\|_{L^2}|x|^{\frac{1}{2}}$$

$$H_0(\mathbb{R})\times L^2(\mathbb{R})$$

$$\mathbf{E}=\left\{\phi=(\phi_1,\phi_2)\in H_0(\mathbb{R})\times L^2(\mathbb{R})\colon\tilde{Q}(1-\phi_1^2)\in L^2(\mathbb{R})\right\}$$

$$\|\phi\|^2_{(H_0\times L^2)(\mathbb{R})}\colon=\|\phi_1\|^2_{H_0(\mathbb{R})}+\|\phi_2\|^2_{L^2(\mathbb{R})}$$

$$\sup_{t\geq 0}\|\phi(t)-\tilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}<\delta$$

$$\lim_{t\rightarrow\infty}\|\phi(t)-\tilde{\mathbf{H}}\|_{(H^1\times L^2)(I)}=0$$

$$\partial_t^2\phi-\partial_r^2\phi-\frac{1}{r}\partial_r\phi-\frac{2}{r^2}(\phi-\phi^3)=0, t\in\mathbb{R}, r>0$$

$$H(r)=\frac{1-r^2}{1+r^2}$$

$$\phi_1(t,x)=\tilde{H}(x)+w_1(t,x),\phi_2(t,x)=w_2(t,x)$$

$$\begin{cases} \partial_t w_1 = w_2 \\ \partial_t w_2 = -Lw_1 - \tilde{Q}^2(3\tilde{H}w_1^2 + w_1^3) \end{cases}$$

$$Lw=-\partial_x^2w+V(x)w, \text{ with } V(x)=2\tilde{Q}^2(1-\tilde{Q})$$

$$L\phi_0=-\mu_0^2\phi_0, \left|\partial_x^k\phi_0(x)\right|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x}, k=0,1,2.$$

$$\mathbf{Y}_{\pm}=\begin{pmatrix}\phi_0\\ \pm\mu_0\phi_0\end{pmatrix},\mathbf{Z}_{\pm}=\begin{pmatrix}\phi_0\\ \pm\mu_0^{-1}\phi_0\end{pmatrix}$$

$$\mathcal{A}_0=\left\{\boldsymbol{\varepsilon}\in H_0(\mathbb{R})\times L^2(\mathbb{R})\text{ such that }\|\boldsymbol{\varepsilon}\|_{(H_0\times L^2)(\mathbb{R})}<\delta_0\text{ and }\langle\boldsymbol{\varepsilon},\mathbf{Z}_+\rangle=0\right\}$$

$$|h(\varepsilon)|\leq C\|\varepsilon\|_{H_0\times L^2}^{3/2}$$

$$\mathcal{M}=\{\tilde{\mathbf{H}}+\varepsilon+h(\varepsilon)\mathbf{Y}_+\text{ with }\varepsilon\in\mathcal{A}_0\}$$

$$\|\boldsymbol{\phi}(t)-\tilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}\leq C\|\boldsymbol{\phi}(0)-\tilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}$$

$$\|\phi(t)-\tilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}\leq \frac{\delta_0}{2},$$

$$(\phi,\partial_t\phi)(0)=(1+a(0))\tilde{\mathbf{H}}+(\tilde{u}_1,u_2)(0)+b_-(0)\mathbf{Y}_-+h(\boldsymbol{\varepsilon})\mathbf{Y}_+,$$



$$\|u_2\|_2^2 + \|\tilde{u}_1\|_{H_0}^2 + a^2 \lesssim \|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1 \rangle + \int \tilde{Q}^2(u_1^2 + 2\tilde{H}u_1)^2 \lesssim \delta_0^2$$

$$\partial_x\alpha(x)=Q^{-1}(x),\partial_x\alpha^{-1}(x)=\tilde{Q}(x)$$

$$\partial_x\tilde{Q}(x)=-\tilde{Q}^2(x)\tilde{H}(x),\partial_x^2\tilde{Q}(x)=2\tilde{Q}^3(x)-\frac{5}{3}\tilde{Q}^4(x)$$

$$\alpha'(x)=\frac{1}{3}(\cosh x+1)=\frac{2}{3}\cosh^2\left(\frac{x}{2}\right)=\frac{1}{Q(x)}$$

$$(\alpha^{-1})'(x)=\frac{1}{\alpha'(\alpha^{-1}(x))}=Q(\alpha^{-1}(x))=\tilde{Q}(x)$$

$$Q'(x)=-\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)=-Q(x)H(x)$$

$$\partial_x\tilde{Q}(x)=Q'(\alpha^{-1}(x))(\alpha^{-1})'(x)=-\tilde{Q}^2(x)\tilde{H}(x).$$

$$\tilde{H}'(x)=\frac{1}{3}\tilde{Q}^2(x)\text{ and }\tilde{H}^2=1-\frac{2}{3}\tilde{Q}$$

$$\begin{aligned}\partial_x^2\tilde{Q}(x) &= -2\tilde{Q}(x)\tilde{Q}'(x)\tilde{H}(x)-\tilde{Q}^2(x)\tilde{H}'(x) \\ &= 2\tilde{Q}^3(x)\tilde{H}^2(x)-\frac{1}{3}\tilde{Q}^4(x)=2\tilde{Q}^3(x)-\frac{5}{3}\tilde{Q}^4(x)\end{aligned}$$

$$\alpha^{-1}(x)=\frac{3}{2}x+\mathcal{O}(x^2),\tilde{Q}(x)=\frac{3}{2}-\frac{27}{32}x^2+\mathcal{O}(x^4),\tilde{H}(x)=\frac{3}{4}x-\frac{27}{16}x^3+\mathcal{O}(x^5).$$

$$\lim_{x\rightarrow\pm\infty}\frac{|\alpha^{-1}(x)|}{\ln(|x|)}=1,\lim_{x\rightarrow\pm\infty}(1+|x|)\tilde{Q}(x)=1,\lim_{x\rightarrow\pm\infty}(1+|x|)|\tilde{H}(x)\mp1|=-\frac{1}{3}$$

$$\int \tilde{Q}^{1+\varepsilon}dx$$

$$\lim_{x\rightarrow\pm\infty}x\tilde{Q}(x)=\lim_{y\rightarrow\pm\infty}\alpha(y)Q(y)=\lim_{y\rightarrow\pm\infty}\frac{1}{2}\frac{\sinh y+y}{\cosh^2\left(\frac{y}{2}\right)}$$

$$=\lim_{y\rightarrow\pm\infty}\frac{\cosh y+1}{\sinh y}=\pm1$$

$$\lim_{x\rightarrow\pm\infty}\frac{|\alpha^{-1}(x)|}{\ln(|x|)}=\lim_{x\rightarrow\pm\infty}|x|\tilde{Q}(x)=1$$

$$\operatorname{sech}^2\left(\frac{x}{2}\right)=\frac{1}{\cosh^2\left(\frac{x}{2}\right)}=\frac{4}{e^x+2+e^{-x}}=\frac{4}{3e^{-x}+2-2x+6\alpha(x)}$$

$$\tilde{Q}(x)=\frac{6}{3e^{-\alpha^{-1}(x)}+2-2\alpha^{-1}(x)+6x}\leq\frac{3}{1-\alpha^{-1}(x)+3x}$$

$$\tilde{Q}(x)\geq\frac{3}{1-\alpha^{-1}(x)+3x}\geq\frac{3}{1+3x}$$



$$\lim_{x \rightarrow +\infty} (1+x)\tilde{Q}(x) = 1$$

$$\lim_{x \rightarrow \pm\infty} x(\tilde{H}(x) - 1) = \lim_{y \rightarrow \pm\infty} \alpha(y) \left(\tanh\left(\frac{y}{2}\right) \mp 1 \right) = -\frac{1}{3}$$

$$\alpha^{-1}(x) = \alpha^{-1}(0) + \partial_x \alpha^{-1}(0)x + \mathcal{O}(x^2) = \frac{3}{2}x + \mathcal{O}(x^2)$$

$$\begin{aligned}\tilde{Q}(x) &= \tilde{Q}(0) + \tilde{Q}'(0)x + \frac{1}{2}\tilde{Q}''(0)x^2 + \frac{1}{6}\tilde{Q}'''(0)x^3 + \mathcal{O}(x^4) \\ &= \frac{3}{2} - \frac{27}{32}x^2 + \mathcal{O}(x^4)\end{aligned}$$

$$\begin{aligned}\tilde{H}(x) &= \tilde{H}(0) + \tilde{H}'(0)x + \frac{1}{2}\tilde{H}''(0)x^2 + \frac{1}{6}\tilde{H}'''(0)x^3 + \frac{1}{24}\tilde{H}''''(0)x^4 + \mathcal{O}(x^5) \\ &= \frac{3}{4}x - \frac{27}{16}x^3 + \mathcal{O}(x^5)\end{aligned}$$

$$\tilde{Q}'(x) = -\tilde{Q}^2(x)\tilde{H}(x), \tilde{Q}''(x) = \tilde{Q}^2(x)(1 - \tilde{Q}(x)), \tilde{H}'(x) = \frac{1}{3}\tilde{Q}^2(x), \tilde{H}''(x) = -\frac{2}{3}\tilde{Q}^3(x)\tilde{H}(x)$$

$$\tilde{H}'''(x) = 2\tilde{Q}^4\tilde{H}^2(x) - \frac{2}{9}\tilde{Q}^5$$

$$\int \tilde{Q}^{1+\varepsilon}(x)dx = \int Q^\varepsilon(s)ds < +\infty$$

$$\phi_1(t,x) = \tilde{H} + \bar{w}_1(t,x), \phi_2(t,x) = \bar{w}_2(t,x)$$

$$(1-\phi_1^2)^2 = (1-\tilde{H}^2)^2 - 2(1-\tilde{H}^2)(2\tilde{H}\bar{w}_1 + \bar{w}_1^2) + (2\tilde{H}\bar{w}_1 + \bar{w}_1^2)^2.$$

$$\begin{aligned}E[\phi_1, \phi_2] &= \int \left(\frac{1}{2}\phi_2^2 + \frac{1}{2}(\partial_x \phi_1)^2 + \frac{1}{4}\tilde{Q}^2(1-\phi_1^2)^2 \right) dx \\ &= E[\tilde{H}, 0] + \frac{1}{2} \int \bar{w}_2^2 + \frac{1}{2} \int (\partial_x \bar{w}_1)^2 - \int \tilde{H}'' \bar{w}_1 \\ &\quad - \frac{1}{2} \int \tilde{Q}^2(1-\tilde{H}^2)(2\tilde{H}\bar{w}_1 + \bar{w}_1^2) + \frac{1}{4} \int \tilde{Q}^2(2\tilde{H}\bar{w}_1 + \bar{w}_1^2)^2 \\ &= E[\tilde{H}, 0] + \frac{1}{2} \int \bar{w}_2^2 + \frac{1}{2} \int \bar{w}_1(-\partial_x^2 \bar{w}_1 + 2\tilde{Q}^2(1-\tilde{Q})\bar{w}_1) \\ &\quad + \frac{1}{4} \int \tilde{Q}^2(4\tilde{H}\bar{w}_1^3 + \bar{w}_1^4)\end{aligned}$$

$$2\{E(\phi_1, \phi_2) - E(\tilde{H}, 0)\} = \int \bar{w}_2^2 + \langle L\bar{w}_1, \bar{w}_1 \rangle + \frac{1}{2} \int \tilde{Q}^2(4\tilde{H}\bar{w}_1^3 + \bar{w}_1^4)$$

$$\begin{aligned}\langle \phi(t,x) - \tilde{H} &= a_1(t)\phi_0(x) + u_1(t,x) \\ \partial_t \phi(t,x) &= \mu_0 a_2(t)\phi_0(x) + u_2(t,x) \end{aligned} \rangle$$

$$a_1(t) = \langle \phi(t) - \tilde{H}, \phi_0 \rangle = -\frac{1}{\mu_0^2} \langle \phi(t) - \tilde{H}, L[\phi_0] \rangle,$$

$$a_2(t) = \frac{1}{\mu_0} \langle \partial_t \phi(t), \phi_0 \rangle = -\frac{1}{\mu_0^3} \langle \partial_t \phi(t), L[\phi_0] \rangle,$$

$$\langle u_1(t), \phi_0 \rangle = 0 = \langle u_2(t), \phi_0 \rangle.$$

$$b_+ = \frac{1}{2}(a_1 + a_2), b_- = \frac{1}{2}(a_1 - a_2).$$

$$\|u_1(t)\|_{H_0} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| + |b_+(t)| + |b_-(t)| \leq C\delta.$$

$$\begin{aligned} \|\phi_2\|_{L^2}^2 &= \mu_0^2 |a_2|^2 \|\phi_0\|_{L^2}^2 + \mu_0 a_2 \langle u_2(t), \phi_0 \rangle + \|u_2(t)\|_{L^2}^2 \\ &= \mu_0^2 |a_2|^2 \|\phi_0\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 \leq \delta^2. \end{aligned}$$

$$|a_2|, \|u_2(t)\|_{L^2} \lesssim \delta$$

$$\|\partial_x(\phi_1 - \tilde{H})\|_{L^2(\mathbb{R})}^2 + \|\tilde{Q}(\phi_1 - \tilde{H})\|_{L^2(\mathbb{R})}^2 \leq \delta^2$$

$$\int_{-R}^R (a_1 \phi_0 + u_1)^2 \lesssim R^2 \delta^2$$

$$a_1^2 + \int_{-R}^R u_1^2 \leq CR^2 \delta^2 + C|a_1| \left| \int_{-R}^R \phi_0 u_1 \right|$$

$$\begin{aligned} & C|a_1| \left| \int_{-R}^R \phi_0 u_1 \right| \\ & \leq C|a_1| \|\tilde{Q}^{-1} \phi_0\|_{L^2(|x| \geq R)} \|\tilde{Q}(a_1 \phi_0 + u_1)\|_{L^2(|x| \geq R)} + C|a_1|^2 \|\phi_0\|_{L^2(|x| \geq R)}^2 \\ & \leq \delta^2 + C e^{-2c_0 R} a_1^2 \end{aligned}$$

$$|a_1(t)| \leq C\delta, |b_+(t)| + |b_-(t)| \leq C\delta$$

$$\|u_1(t)\|_{L^2(-R,R)} \leq CR^2\delta$$

$$\|\tilde{Q}(a_1 \phi_0 + u_1)\|_{L^2(\mathbb{R})} \leq \delta$$

$$\|\tilde{Q}u_1\|_{L^2(\mathbb{R})} \leq C\delta$$

$$\|a_1 \phi'_0 + \partial_x u_1\|_{L^2(\mathbb{R})} \leq \delta$$

$$\|\partial_x u_1\|_{L^2(\mathbb{R})} \leq C\delta$$

$$\left\| \tilde{Q}^{\frac{p+2}{2p}} u \right\|_{L^p} \leq \sqrt{2} \|u\|_{H_0}$$

$$\|u\|^2 := \|\partial_x u\|_{L^2(\mathbb{R})}^2 + \left\| \tilde{Q}^{\frac{3}{2}} u \right\|_{L^2(\mathbb{R})}^2$$



$$u(x) = v(\alpha^{-1}(x))$$

$$\partial_x u(x) = \partial_y v(\alpha^{-1}(x))Q(\alpha^{-1}(x))$$

$$\int (\partial_x u)^2 = \int Q(\partial_y v)^2, \int \tilde{Q}^k u^2 = \int Q^{k-1} v^2, \text{ for } k = 2, 3$$

$$\|u\|_{H_0}^2 = \int \left((\partial_y g)^2 + Hg\partial_y g + \frac{1}{4}H^2g^2 + g^2 \right) = \int (\partial_y g)^2 + \left(\frac{5}{4} - \frac{1}{3}Q \right) g^2 \geq \frac{1}{2}\|g\|_{H^1}^2$$

$$\left\| \tilde{Q}^{\frac{p+2}{2p}} u \right\|_{L^p} = \|g\|_{L^p} \leq \|g\|_{H^1}$$

$$\frac{1}{2}\int (\partial_x u)^2 + \int \tilde{Q}^3 u^2 = \frac{1}{2}\int (\partial_y g)^2 + \left(\frac{1}{8} + \frac{5}{6}Q\right) g^2 \geq \frac{1}{8}\|g\|_{H^1}^2$$

$$\|\tilde{Q}u\|_{L^2(\mathbb{R})} \leq \|g\|_{H^1}$$

$$\|u\|_{H_0} \lesssim \|u\|$$

$$\begin{cases} \dot{a}_1(t) = \mu_0 a_2(t) \\ \dot{a}_2(t) = \mu_0 a_1(t) - \frac{N_0}{\mu_0}, \end{cases} \text{ or equivalently } \begin{cases} \dot{b}_+(t) = \mu_0 b_+(t) - \frac{N_0}{2\mu_0} \\ \dot{b}_-(t) = -\mu_0 b_-(t) + \frac{N_0}{2\mu_0} \end{cases}$$

$$N=\tilde{Q}^2\big(3\tilde{H}(a_1\phi_0+u_1)^2+(a_1\phi_0+u_1)^3\big)$$

$$N^\perp = N - N_0\phi_0, \text{ and } N_0 = \langle N, \phi_0 \rangle.$$

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = -Lu_1 - N^\perp \end{cases}$$

$$\|\phi(0)-\tilde{\mathbf{H}}\|_{(H_0\times L^2)(\mathbb{R})}\leq \delta$$

$$\phi_1(t,x)=\tilde{H}(x)+w_1(t,x), \phi_2(t,x)=w_2(t,x)$$

$$\begin{cases} \partial_t w_1 = w_2 \\ \partial_t w_2 = \partial_x^2 w_1 - F(t,x,w_1) \end{cases}$$

$$F(t,x,w_1)=2\tilde{Q}^2(1-\tilde{Q})w_1+\tilde{Q}^2(3\tilde{H}w_1^2+w_1^3)$$

$$w_1, \tilde{w}_1, \text{ if } \|\tilde{Q}^{1/2}w_1\|_{L^\infty} \leq 1 \text{ and } \|\tilde{Q}^{1/2}\tilde{w}_1\|_{L^\infty} \leq 1$$

$$|F(t,x,w_1)-F(t,x,\tilde{w}_1)|\leq C\tilde{Q}|w_1-\tilde{w}_1|$$

$$|f^{(k)}(x)|\leq C_k\tilde{Q}(x)^3 \text{ for all } x\in\mathbb{R}$$

$$\chi(x)=1 \text{ for } |x|\leq 1, \chi(x)=0 \text{ for } |x|\geq 2, \chi'(x)\leq 0 \text{ for } x\geq 0.$$

$$\zeta_A^2(x)=\exp\left(-\frac{1}{A}|\alpha^{-1}(x)|(1-\chi(x))\right), \varphi_A(x)=\int_0^x \tilde{Q}\zeta_A^2(y)dy, x\in\mathbb{R}$$

$$\sigma_A(x) = \operatorname{sech}\left(\frac{1}{A}\alpha^{-1}(x)\right)$$

$$\zeta_B^2(x) = \exp\left(-\frac{1}{B}|\alpha^{-1}(x)|(1-\chi(x))\right), \varphi_B(x) = \int_0^x \tilde{Q}\zeta_B^2(y)dy, x \in \mathbb{R}$$

$$\psi_{A,B}(x) = \tilde{\chi}_A^2(x)\varphi_B(x), \tilde{\chi}_A(x) = \chi\left(\frac{\alpha^{-1}(x)}{A}\right), \tilde{\chi}_B(x) = \chi\left(\frac{\alpha^{-1}(x)}{B^2}\right)$$

$$1\ll B\ll B^2\ll A.$$

$$\mathcal{I} = \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi_A' u_1 \right) u_2$$

$$w_i = \zeta_A u_i, i=1,2$$

$$A=\delta^{-1/4}$$

$$\frac{d}{dt}\mathcal{I} \leq -\frac{1}{2}C_0 \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \int \tilde{Q}^7 u_1^2 + C|a_1|^4$$

$$\frac{d}{dt}\mathcal{I} = - \int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4} \int \varphi_A''' u_1^2 + \frac{1}{2} \int \varphi_A V' u_1^2 - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi_A' u_1 \right) N^\perp$$

$$\mathcal{I}_1 = \int \varphi_A u_2 \partial_x u_1, \mathcal{I}_2(t) = \frac{1}{2} \int \varphi_A' u_1 u_2$$

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_1(t) &= \int \varphi_A(\dot{u}_2\partial_x u_1 + u_2\partial_x\dot{u}_1) = - \int \varphi_A(L[u_1] + N^\perp)\partial_x u_1 + \int \varphi_A u_2 \partial_x u_2 \\ &= - \int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^\perp - \frac{1}{2} \int \varphi_A' u_2^2 = - \int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^\perp - \frac{1}{2} \int \varphi_A' u_2^2 \end{aligned}$$

$$\begin{aligned} \int \varphi_A L[u_1]\partial_x u_1 &= \int \varphi_A(-\partial_x^2 u_1 + V u_1)\partial_x u_1 = -\frac{1}{2} \int \varphi_A \partial_x (\partial_x u_1)^2 + \frac{1}{2} \int \varphi_A V \partial_x u_1^2 \\ &= \frac{1}{2} \int \varphi_A' (\partial_x u_1)^2 - \frac{1}{2} \int \varphi_A' V u_1^2 - \frac{1}{2} \int \varphi_A V' u_1^2 \end{aligned}$$

$$\frac{d}{dt}\mathcal{I}_1 = -\frac{1}{2} \int \varphi_A' (\partial_x u_1)^2 + \frac{1}{2} \int \varphi_A' V u_1^2 + \frac{1}{2} \int \varphi_A V' u_1^2 - \frac{1}{2} \int \varphi_A' u_2^2 - \int \varphi_A \partial_x u_1 N^\perp$$

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_2 &= \frac{1}{2} \int \varphi_A'(\dot{u}_1 u_2 + u_1 \dot{u}_2) = \frac{1}{2} \int \varphi_A' u_2^2 - \frac{1}{2} \int \varphi_A' u_1 (L[u_1] + N^\perp) \\ &= \frac{1}{2} \int \varphi_A' u_2^2 - \frac{1}{2} \int \varphi_A' u_1 L[u_1] - \frac{1}{2} \int \varphi_A' u_1 N^\perp \end{aligned}$$

$$\begin{aligned} \int \varphi_A' u_1 L[u_1] &= \int \varphi_A' u_1 (-\partial_x^2 u_1 + V u_1) = \int (\varphi_A' u_1)_x \partial_x u_1 + \int \varphi_A' V u_1^2 \\ &= \int \varphi_A'' u_1 \partial_x u_1 + \int \varphi_A' (\partial_x u_1)^2 + \int \varphi_A' V u_1^2 = -\frac{1}{2} \int \varphi_A''' u_1^2 + \int \varphi_A' (\partial_x u_1)^2 + \int \varphi_A' V u_1^2 \end{aligned}$$

$$\frac{d}{dt}\mathcal{I}_2(t) = \frac{1}{2} \int \varphi_A' u_2^2 + \frac{1}{4} \int \varphi_A''' u_1^2 - \frac{1}{2} \int \varphi_A' (\partial_x u_1)^2 - \frac{1}{2} \int \varphi_A' V u_1^2 - \frac{1}{2} \int \varphi_A' u_1 N^\perp$$



$$\begin{aligned}
\frac{d}{dt} \mathcal{J}(t) &= - \int \varphi'_A (\partial_x u_1)^2 + \frac{1}{4} \int \varphi_A''' u_1^2 + \frac{1}{2} \int \varphi_A V' u_1^2 - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) N^\perp \\
&\quad - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) N^\perp \\
&\quad - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \left(\tilde{Q}^2 (3\tilde{H} (a_1 \phi_0 + u_1)^2 + (a_1 \phi_0 + u_1)^3) - N_0 \phi_0 \right) \\
&\leq C a_1^4 + C \int \tilde{Q}^7 u_1^2 + C A \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^3 w_1^2 + \frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 + \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 \\
&\quad \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \tilde{Q}^2 (3\tilde{H} (a_1 \phi_0 + u_1)^2 + (a_1 \phi_0 + u_1)^3) \\
&\quad = a_1^2 \int \tilde{Q}^2 (3\tilde{H} + a_1 \phi_0) \phi_0^2 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) + 3 a_1 \int \tilde{Q}^2 (2\tilde{H} + a_1 \phi_0) \phi_0 u_1 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \\
&\quad + 3 \int \tilde{Q}^2 (\tilde{H} + a_1 \phi_0) u_1^2 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) + \int \tilde{Q}^2 u_1^3 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \\
&\quad = I_1 + I_2 + I_3 + I_4 \\
|I_1| &\leq a_1^2 \int |\partial_x (\tilde{Q}^2 (3\tilde{H} + a_1 \phi_0) \phi_0^2) \varphi_A u_1| + \frac{1}{2} a_1^2 \int |\tilde{Q}^2 (3\tilde{H} + a_1 \phi_0) \phi_0^2 \varphi'_A u_1| \\
&\lesssim a_1^2 \left[\left(\int \tilde{Q} \phi_0^4 |\alpha^{-1}|^2 \right)^{\frac{1}{2}} + \left(\int \tilde{Q}^3 |\phi_0 \phi_0' \alpha^{-1}|^2 \right)^{\frac{1}{2}} \right] \left(\int \tilde{Q}^7 u_1^2 \right)^{\frac{1}{2}} \lesssim a_1^4 + \int \tilde{Q}^7 u_1^2 \\
|I_2| &= \frac{3}{2} |a_1| \left| \int \partial_x (\tilde{Q}^2 (2\tilde{H} + a_1 \phi_0) \phi_0) \varphi_A u_1^2 \right| \lesssim |a_1| \int |\alpha^{-1}(x)| (\tilde{Q} \phi_0 + \phi_0') \tilde{Q}^2 u_1^2 \lesssim \int \tilde{Q}^7 u_1^2 \\
I_4 &= \int \tilde{Q}^2 u_1^3 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) = \frac{1}{4} \int \tilde{Q}^2 \varphi'_A u_1^4 + \frac{1}{2} \int \tilde{Q}^3 \tilde{H} \varphi_A u_1^4 = \frac{1}{4} \int \tilde{Q}^2 (2\varphi'_A + \tilde{Q} \tilde{H} \varphi_A) u_1^4 \\
I_{3,1} &:= \int \tilde{Q}^2 \tilde{H} u_1^2 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right), I_{3,2} := a_1 \int \tilde{Q}^2 \phi_0 u_1^2 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \\
|I_{3,2}| &\lesssim A |a_1| \int \tilde{Q}^2 (|\phi_0| + |\phi_0'|) |u_1|^3 \lesssim A \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^3 w_1^2 \\
I_{3,1} &= \int \left(\frac{1}{6} \varphi'_A \tilde{Q}^2 \tilde{H} - \frac{1}{3} \varphi_A (\tilde{Q}^2 \tilde{H})' \right) u_1^3 = \int \left(\frac{1}{6} \varphi'_A \tilde{Q}^2 \tilde{H} - \frac{1}{3} \varphi_A \tilde{Q}^3 \left(\frac{1}{3} \tilde{Q} - 2\tilde{H}^2 \right) \right) u_1^3 \\
3I_{3,1} + I_4 &= \frac{1}{6} \int \varphi'_A \tilde{Q}^2 \tilde{H} u_1^3 + \frac{1}{2} \int \tilde{Q}^2 \varphi'_A u_1^4 - \frac{1}{9} \int \varphi_A \tilde{Q}^4 u_1^3 + \frac{2}{3} \int \tilde{Q}^3 \tilde{H} |\varphi_A \tilde{H}| u_1^3 + \frac{1}{4} \int \tilde{Q}^3 |\varphi_A \tilde{H}| u_1^4 \\
&= \frac{1}{2} \int \varphi'_A \tilde{Q}^2 u_1^2 \left(u_1 + \frac{1}{6} \tilde{H} \right)^2 - \frac{1}{72} \int \varphi'_A \tilde{Q}^2 \tilde{H}^2 u_1^2 \\
&\quad - \frac{1}{9} \int \varphi_A \tilde{Q}^4 u_1^3 + \frac{1}{4} \int \tilde{Q}^3 |\varphi_A \tilde{H}| u_1^2 \left(u_1 + \frac{4}{3} \tilde{H} \right)^2 - \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 \\
&\quad \int \varphi_A \tilde{Q}^4 u_1^3 \lesssim A \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^{7/2} u_1^2
\end{aligned}$$

$$\begin{aligned}
3I_{3,1} + I_4 &\geq -\frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 - \frac{1}{9} \int \varphi_A \tilde{Q}^4 u_1^3 - \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 \\
&\geq -\frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 - \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 - CA \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^{7/2} \zeta_A^{-2} w_1^2
\end{aligned}$$

$$N_0 \int \phi_0 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right)$$

$$N = \tilde{Q}^2 (3\tilde{H}(a_1^2 \phi_0^2 + 2a_1 \phi_0 u_1 + u_1^2) + a_1^3 \phi_0^3 + 3a_1^2 \phi_0^2 u_1 + 3a_1 \phi_0 u_1^2 + u_1^3)$$

$$|N| \lesssim \tilde{Q}^2 (a_1^2 \phi_0^2 + u_1^2 + |u_1|^3 + u_1^4)$$

$$\phi_0, \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \lesssim \|\tilde{Q}^{1/2} u_1\|_{H^1} \lesssim 1, A \geq 2$$

$$|N_0| = |\langle \phi_0, N \rangle| \lesssim a_1^2 + \int \tilde{Q}^2 \phi_0 u_1^2 \lesssim a_1^2 + \int \tilde{Q}^7 u_1^2$$

$$-\int \phi_0 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) = \int u_1 \left(\varphi_A \phi'_0 + \frac{1}{2} \varphi'_A \phi_0 \right)$$

$$|\varphi_A \phi'_0 + \varphi'_A \phi_0| \lesssim \alpha^{-1}(x) \phi'_0 + \tilde{Q} \zeta_A^2 \phi_0 \lesssim \tilde{Q}^7$$

$$\begin{aligned}
\left| N_0 \int \phi_0 \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) \right| &\lesssim \left(a_1^2 + \int \tilde{Q}^7 u_1^2 \right) \int \tilde{Q}^7 |u_1| \\
&\lesssim \left(a_1^2 + \int \tilde{Q}^7 u_1^2 \right) \left(\int \tilde{Q}^7 u_1^2 \right)^{\frac{1}{2}} \left(\int \tilde{Q}^7 \right)^{\frac{1}{2}} \lesssim a_1^4 + \int \tilde{Q}^7 u_1^2
\end{aligned}$$

$$\begin{aligned}
& - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi'_A u_1 \right) (\tilde{Q}^2 (3\tilde{H}(a_1 \phi_0 + u_1)^2 + (a_1 \phi_0 + u_1)^3) - N_0 \phi_0) \\
& \leq C a_1^4 + C \int \tilde{Q}^7 u_1^2 + \frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 + \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 \\
& + CA \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^3 w_1^2 + CA \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^{7/2} \zeta_A^{-2} w_1^2 \\
& \leq C a_1^4 + C \int \tilde{Q}^7 u_1^2 + \frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 + \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 + CA \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \int \tilde{Q}^3 w_1^2 \\
& - \int \varphi'_A (\partial_x u_1)^2 + \frac{1}{4} \int \varphi_A''' u_1^2 + \frac{1}{2} \int \varphi_A V' u_1^2 + \frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 + \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2 \\
& = - \int \tilde{Q} (\partial_x w_1)^2 + \frac{1}{2} \int \left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right] \tilde{Q} w_1^2 + \frac{1}{72} \int \tilde{Q}^3 \tilde{H}^2 w_1^2 \\
& + \frac{1}{4} \int \tilde{Q}'' w_1^2 + \frac{1}{2} \int \varphi_A V' u_1^2 + \frac{4}{9} \int \tilde{Q}^3 |\varphi_A \tilde{H}| \tilde{H}^2 u_1^2
\end{aligned}$$

$$\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 = \frac{1}{A} [\chi'' |\alpha^{-1}| + 2\chi' \tilde{Q} \operatorname{sgn}(\alpha^{-1}) + (1 - \chi) \tilde{Q}^2 \tilde{H} \operatorname{sgn}(\alpha^{-1})]$$

$$\left| \frac{\zeta_A'}{\zeta_A} \right| \lesssim \frac{1}{A} \tilde{Q} \mathbf{1}_{\{|x| \geq 1\}}, \left| \frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right| \lesssim \frac{1}{A} \tilde{Q}^2 \mathbf{1}_{\{|x| \geq 1\}}$$



$$\frac{1}{4} \left(\tilde{Q}''(x) + \frac{1}{18} \tilde{Q}^3(x) \tilde{H}^2(x) \right) \tilde{Q}^{-1}(x) \varphi'_A(x) + \frac{1}{2} \varphi_A(x) V'(x) + \frac{4}{9} |\varphi_A(x) \tilde{H}(x)| \tilde{H}^2 \tilde{Q}^3(x) \leq -C \tilde{Q}^3(x)$$

$$\frac{1}{4} \int \tilde{Q}'' w_1^2$$

$$\begin{aligned} \int \varphi'_A (\partial_x u_1)^2 &= \int \tilde{Q} \left(\partial_x w_1 - \frac{\zeta'_A}{\zeta_A} w_1 \right)^2 = \int \tilde{Q} (\partial_x w_1)^2 - 2 \int \tilde{Q} \frac{\zeta'_A}{\zeta_A} w_1 \partial_x w_1 + \int \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \tilde{Q} w_1^2 \\ &= \int \tilde{Q} (\partial_x w_1)^2 + \int \left[\left(\frac{\tilde{Q} \zeta'_A}{\zeta_A} \right)' + \tilde{Q} \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right] w_1^2 = \int \tilde{Q} (\partial_x w_1)^2 + \int \frac{(\tilde{Q} \zeta'_A)'}{\zeta_A} w_1^2 \end{aligned}$$

$$\int \varphi_A''' u_1^2 = \int \left[\tilde{Q}'' + 2\tilde{Q}' \left(\frac{\zeta'_A}{\zeta_A} \right) + 2 \frac{(\tilde{Q} \zeta'_A)'}{\zeta_A} + 2\tilde{Q} \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right] w_1^2$$

$$\begin{aligned} - \int \varphi'_A (\partial_x u_1)^2 + \frac{1}{4} \int \varphi_A''' u_1^2 &= - \int \tilde{Q} (\partial_x w_1)^2 + \frac{1}{4} \int \tilde{Q}'' w_1^2 + \frac{1}{2} \int \left[\tilde{Q}' \left(\frac{\zeta'_A}{\zeta_A} \right) - \frac{(\tilde{Q} \zeta'_A)'}{\zeta_A} + \tilde{Q} \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right] w_1^2 \\ &= - \int \tilde{Q} (\partial_x w_1)^2 + \frac{1}{4} \int \tilde{Q}'' w_1^2 + \frac{1}{2} \int \tilde{Q} \left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right] w_1^2 \end{aligned}$$

$$\frac{\zeta'_A}{\zeta_A} = \frac{1}{A} [\chi' |\alpha^{-1}| - \text{sgn}(\alpha^{-1}) (\alpha^{-1})' (1 - \chi)]$$

$$\frac{\zeta_A''}{\zeta_A} = \left(\frac{\zeta'_A}{\zeta_A} \right)^2 + \frac{1}{A} [\chi'' |\alpha^{-1}| + 2\chi' (\alpha^{-1})' \text{sgn}(\alpha^{-1}) - (1 - \chi) (\alpha^{-1})'' \text{sgn}(\alpha^{-1})]$$

$$\left| \frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right| \lesssim \frac{1}{A}$$

$$\left| \frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right| = \frac{1}{A} \tilde{Q}^2 |\tilde{H}| \leq \frac{1}{A} \tilde{Q}^2(x)$$

$$\left| \frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta'_A}{\zeta_A} \right)^2 \right| \lesssim \frac{\tilde{Q}^2 \mathbf{1}_{\{|x| \geq 1\}}}{A}$$

$$\begin{aligned} &\frac{1}{4} \left(\tilde{Q}'' + \frac{1}{18} \tilde{H}^2 \tilde{Q}^3 \right) \varphi'_A + \frac{1}{2} \varphi_A V' + \frac{4}{9} (\varphi_A \tilde{H}) \tilde{H}^2 \tilde{Q}^3 \\ &= \left[\left(\frac{1}{2} - \frac{5}{12} \tilde{Q} + \frac{1}{72} \tilde{H}^2 \right) \varphi'_A - \left(2 - 3\tilde{Q} - \frac{4}{9} \tilde{H}^2 \right) \varphi_A \tilde{H} \right] \tilde{Q}^3 \\ &= \left[\frac{1}{18} \left(\frac{37}{4} - \frac{23}{3} \tilde{Q} \right) \zeta_A^2 - \frac{1}{9} \left(14 - \frac{73}{3} \tilde{Q} \right) \varphi_A \tilde{H} \right] \tilde{Q}^3 \end{aligned}$$

$$14 - \frac{73}{3} \tilde{Q} \geq \frac{53}{9} > \frac{47}{8}, \left(\frac{37}{8} - \frac{23}{6} \tilde{Q} \right) \zeta_A^2 \leq \frac{37}{8}$$

$$\varphi_A = \int_0^{\alpha^{-1}(x)} e^{-\frac{2}{A}s(1-\chi(\alpha(s)))} ds \geq \int_0^1 ds + \int_1^{\alpha^{-1}(x)} e^{-\frac{2}{A}s} ds \geq 1$$



$$\frac{1}{4}\left(\tilde{Q}''+\frac{1}{18}\tilde{H}^2\tilde{Q}^3\right)\varphi_A'+\frac{1}{2}\varphi_AV'+\frac{4}{9}(\varphi_A\tilde{H})\tilde{H}^2\tilde{Q}^3\leq\frac{37}{72}\left(1-\frac{47}{37}\varphi_A\tilde{H}\right)\tilde{Q}^3\leq\frac{37}{72}R(\alpha^{-1}(x))\tilde{Q}^3$$

$$R(s)\colon=1-\frac{6}{5}H(s)$$

$$\begin{aligned} & -\int \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int \varphi_A'''u_1^2+\frac{1}{2}\int \varphi_AV'u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2 \\ & \leq -C_0\int \tilde{Q}[(\partial_x w_1)^2+\tilde{Q}^2w_1^2]+C'\int \tilde{Q}^7u_1^2 \end{aligned}$$

$$\int \tilde{Q}^7u_1^2$$

$$\frac{1}{4}\int \tilde{Q}''u_1^2+\frac{1}{2}\int \varphi_AV'u_1^2$$

$$\begin{aligned} & -\int \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int \varphi_A'''u_1^2+\frac{1}{2}\int \varphi_AV'u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2 \\ & \leq -\int \tilde{Q}(\partial_x w_1)^2+\frac{C'}{A}\int_{|x|\geq 1}\tilde{Q}^3w_1^2-C\int_{|x|\geq \tilde{x}}\tilde{Q}^3u_1^2+C\int_{|x|\leq \tilde{x}}\tilde{Q}^2u_1^2 \end{aligned}$$

$$\left|\frac{1}{4}\left(\tilde{Q}''+\frac{1}{18}\tilde{H}^2\tilde{Q}^3\right)\tilde{Q}^{-1}\varphi_A'+\frac{1}{2}\varphi_AV'+\frac{4}{9}(\varphi_A\tilde{H})\tilde{H}^2\tilde{Q}^3\right|\lesssim\tilde{Q}^2$$

$$\begin{aligned} & -\int \varphi_A'(\partial_x u_1)^2+\frac{1}{4}\int \varphi_A'''u_1^2+\frac{1}{2}\int \varphi_AV'u_1^2+\frac{1}{72}\int \tilde{Q}^3\tilde{H}^2w_1^2+\frac{4}{9}\int \tilde{Q}^3|\varphi_A\tilde{H}|\tilde{H}^2u_1^2 \\ & \leq -C_0\int \tilde{Q}[(\partial_x w_1)^2+\tilde{Q}^2w_1^2]+C\int \tilde{Q}^7u_1^2 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}J &= -\int \tilde{Q}(\partial_x w_1)^2-\frac{1}{2}\int \left[\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]\tilde{Q}w_1^2+\frac{1}{4}\int \tilde{Q}''u_1^2+\int \varphi_AV'u_1^2 \\ & \quad -\int \left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A'u_1\right)N^\perp \\ & \leq -C_0\int \tilde{Q}[(\partial_x w_1)^2+\tilde{Q}^2w_1^2]+C\int \tilde{Q}^7u_1^2+C|a_1|^4+CA\|\tilde{Q}^{1/2}u_1\|_{L^\infty}\int \tilde{Q}^3w_1^2 \end{aligned}$$

$$L_0=-\partial_x^2+V_0, \text{ with } V_0\colon=2\left(\frac{\partial_x\phi_0}{\phi_0}\right)^2-2\mu_0^2-V,$$

$$U=\phi_0\cdot\partial_x\cdot\phi_0^{-1}, U^*=-\phi_0^{-1}\cdot\partial_x\cdot\phi_0$$

$$L=U^*U-\mu_0^2, L_0=UU^*-\mu_0^2$$

$$UL=L_0U$$

$$\begin{cases} \dot{v}_1=v_2 \\ \dot{v}_2=-L_0[v_1] \end{cases}$$

$$X_\gamma\colon L^2(\mathbb{R})\rightarrow H^2(\mathbb{R}), X_\gamma=(1-\gamma\partial_x^2)^{-1}$$

$$\widehat{X_\gamma h}(\xi) = \frac{\hat{h}(\xi)}{1 + \gamma \xi^2}.$$

$$X_\gamma^{-1}[fX_\gamma g] = fg + \gamma X_\gamma^{-1}[\partial_x((\partial_x f)g)] + \gamma X_\gamma^{-1}[(\partial_x f)(\partial_x g)]$$

$$X_\gamma^{-1}[fX_\gamma g] = fg + h$$

$$X_\gamma h = fX_\gamma g - X_\gamma(fg) = \gamma(\partial_x^2 fg + 2\partial_x f \partial_x g + f\partial_x^2 g - f\partial_x^2 g) = \gamma(\partial_x((\partial_x f)g) + \partial_x f \partial_x g)$$

$$\begin{cases} v_1 = (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B u_1) \\ v_2 = (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B u_2) \end{cases}$$

$$\begin{cases} \dot{v}_1 = v_2 \\ \dot{v}_2 = -(1 - \gamma \partial_x^2)^{-1} ULu_1 - (1 - \gamma \partial_x^2)^{-1} U(N^\perp) \end{cases}$$

$$\tilde{\chi}_B Lu_1 = L(\tilde{\chi}_B u_1) + 2\tilde{\chi}'_B \partial_x u_1 + \tilde{\chi}''_B u_1.$$

$$\begin{aligned} -(1 - \gamma \partial_x^2)^{-1} UL(\tilde{\chi}_B u_1) &= -(1 - \gamma \partial_x^2)^{-1} L_0 U(\tilde{\chi}_B u_1) = -(1 - \gamma \partial_x^2)^{-1} L_0 [(1 - \gamma \partial_x^2) v_1] \\ &= -(1 - \gamma \partial_x^2)^{-1} (-\partial_x^2 + V_0)(1 - \gamma \partial_x^2) v_1 = \partial_x^2 v_1 - (1 - \gamma \partial_x^2)^{-1} [V_0(1 - \gamma \partial_x^2) v_1] \end{aligned}$$

$$\begin{aligned} (1 - \gamma \partial_x^2)[V_0 v_1] &= V_0 v_1 - \gamma(V_0'' v_1 + 2V_0' \partial_x v_1 + V_0 \partial_x^2 v_1) \\ &= V_0(1 - \gamma \partial_x^2) v_1 - \gamma(V_0'' v_1 + 2V_0' \partial_x v_1) \end{aligned}$$

$$-(1 - \gamma \partial_x^2)^{-1} UL(\tilde{\chi}_B u_1) = -L_0 v_1 - \gamma(1 - \gamma \partial_x^2)^{-1} (V_0'' v_1 + 2V_0' \partial_x v_1)$$

$$\left\| \begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -L_0 v_1 - \gamma(1 - \gamma \partial_x^2)^{-1} (V_0'' v_1 + 2V_0' \partial_x v_1) \\ &\quad -(1 - \gamma \partial_x^2)^{-1} U[2\tilde{\chi}'_B \partial_x u_1 + \tilde{\chi}''_B u_1] - (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N^\perp) \end{aligned} \right\|$$

$$\mathcal{J}(t) = \int \left(\psi_{A,B}(x) \partial_x v_1(t,x) + \frac{1}{2} \psi'_{A,B}(x) v_1(t,x) \right) v_2(t,x) dx$$

$$z\!:=\!\tilde{\chi}_A\zeta_Bv_1$$

$$B=\alpha^{-1}(\delta^{-1/8})$$

$$\frac{d}{dt}\mathcal{J}\leq -C_2\int\tilde{Q}[(\partial_xz)^2+\tilde{Q}^2z^2]+C\ln\left(\delta^{-\frac{1}{8}}\right)^{-1}\int\tilde{Q}[(\partial_xw_1)^2+\tilde{Q}^2w_1^2]+C\delta^{\frac{1}{2}}|a_1|^3$$



$$\begin{aligned}
\frac{d}{dt} \mathcal{J} &= \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) \dot{v}_2 \\
&= - \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) L_0 v_1 \\
&\quad - \gamma \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) (1 - \gamma \partial_x^2)^{-1} (V_0'' v_1 + 2V_0' \partial_x v_1) \\
&\quad - \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) (1 - \gamma \partial_x^2)^{-1} U [2\tilde{\chi}'_B \partial_x u_1 + \tilde{\chi}''_B u_1] \\
&\quad - \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}_B N^\perp) \\
&=: J_1 + J_2 + J_3 + J_4
\end{aligned}$$

$$J_1 = - \int \psi'_{A,B} (\partial_x v_1)^2 + \frac{1}{4} \int \psi'''_{A,B} v_1^2 - \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) V_0 v_1$$

$$\begin{aligned}
\psi'_{A,B} &= \tilde{Q} \tilde{\chi}_A^2 \zeta_B^2 + (\tilde{\chi}_A^2)' \varphi_B \\
\psi''_{A,B} &= \tilde{Q}' \tilde{\chi}_A^2 (\zeta_B^2) + \tilde{Q} \tilde{\chi}_A^2 (\zeta_B^2)' + 2\tilde{Q} (\tilde{\chi}_A^2)' \zeta_B^2 + (\tilde{\chi}_A^2)'' \varphi_B \\
\psi'''_{A,B} &= \tilde{Q}'' \tilde{\chi}_A^2 \zeta_B^2 + 3\tilde{Q}' (\tilde{\chi}_A^2)' \zeta_B^2 + 2\tilde{Q}' \tilde{\chi}_A^2 (\zeta_B^2)' + 3\tilde{Q} (\tilde{\chi}_A^2)' (\zeta_B^2)' \\
&\quad + 3\tilde{Q} (\tilde{\chi}_A^2)'' \zeta_B^2 + \tilde{Q} \tilde{\chi}_A^2 (\zeta_B^2)'' + (\tilde{\chi}_A^2)''' \varphi_B
\end{aligned}$$

$$\begin{aligned}
&- \int \psi'_{A,B} (\partial_x v_1)^2 + \frac{1}{4} \int \psi'''_{A,B} v_1^2 \\
&= - \int \tilde{Q}^2 \zeta_B^2 (\partial_x v_1)^2 + \frac{1}{4} \int \tilde{Q}'' \tilde{\chi}_A^2 \zeta_B^2 v_1^2 + \frac{1}{4} \int \tilde{Q} \tilde{\chi}_A^2 (\zeta_B^2)'' v_1^2 \\
&\quad + \frac{3}{4} \int \tilde{Q}' (\tilde{\chi}_A^2)' \zeta_B^2 v_1^2 + \frac{3}{4} \int \tilde{Q} (\tilde{\chi}_A^2)' (\zeta_B^2)' v_1^2 + \frac{3}{4} \int \tilde{Q} (\tilde{\chi}_A^2)'' \zeta_B^2 v_1^2 \\
&\quad + \frac{1}{2} \int \tilde{Q}' \tilde{\chi}_A^2 (\zeta_B^2)' v_1^2 - \int (\tilde{\chi}_A^2)' \varphi_B (\partial_x v_1)^2 + \frac{1}{4} \int (\tilde{\chi}_A^2)''' \varphi_B v_1^2
\end{aligned}$$

$$\begin{aligned}
\int \tilde{Q} \tilde{\chi}_A^2 \zeta_B^2 (\partial_x v_1)^2 &= \int \tilde{Q} (\partial_x z)^2 + \int (\tilde{Q} (\tilde{\chi}_A \zeta_B)')' \tilde{\chi}_A \zeta_B v_1^2 \\
&= \int \tilde{Q} (\partial_x z)^2 + \int \tilde{Q} \frac{\zeta_B''}{\zeta_B} z^2 + \int \tilde{Q} \tilde{\chi}_A'' \tilde{\chi}_A \zeta_B^2 v_1^2 + \frac{1}{2} \int \tilde{Q} (\tilde{\chi}_A^2)' (\zeta_B^2)' v_1^2 \\
&\quad + \frac{1}{2} \int \tilde{Q}' (\tilde{\chi}_A^2)' \zeta_B^2 v_1^2 + \frac{1}{2} \int \tilde{Q}' \tilde{\chi}_A^2 (\zeta_B^2)' v_1^2 \\
&\quad \frac{1}{4} \int \tilde{Q} \tilde{\chi}_A^2 (\zeta_B^2)'' v_1^2 = \frac{1}{2} \int \tilde{Q} \left(\frac{\zeta_B''}{\zeta_B} + \frac{(\zeta_B')^2}{\zeta_B^2} \right) z^2
\end{aligned}$$

$$- \int \psi'_{A,B} (\partial_x v_1)^2 + \frac{1}{4} \int \psi'''_{A,B} v_1^2 = - \left\{ \int \tilde{Q} (\partial_x z)^2 - \frac{1}{4} \int \tilde{Q}'' z^2 + \frac{1}{2} \int \tilde{Q} \left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2} \right) z^2 \right\} + \tilde{J}_1$$

$$\begin{aligned}
\tilde{J}_1 &= \frac{1}{4} \int \tilde{Q} (\tilde{\chi}_A^2)' (\zeta_B^2)' v_1^2 + \frac{1}{4} \int \tilde{Q}' (\tilde{\chi}_A^2)' \zeta_B^2 v_1^2 + \frac{1}{2} \int \tilde{Q} [3(\tilde{\chi}_A')^2 + \tilde{\chi}_A'' \tilde{\chi}_A] \zeta_B^2 v_1^2 \\
&\quad - \int (\tilde{\chi}_A^2)' \varphi_B (\partial_x v_1)^2 + \frac{1}{4} \int (\tilde{\chi}_A^2)''' \varphi_B v_1^2
\end{aligned}$$

$$\int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) V_0 v_1 = \frac{1}{2} \int V_0 \partial_x (\psi_{A,B} v_1^2) = -\frac{1}{2} \int \frac{\varphi_B}{\zeta_B^2} V_0' z^2$$



$$V_B = -\frac{1}{4}\tilde{Q}'' + \frac{1}{2}\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{2}\frac{\varphi_B}{\zeta_B^2}V_0'$$

$$V_B = \left[\frac{1}{2}\tilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{10}\frac{\varphi_B}{\zeta_B^2}V_0'\right] + \left[-\frac{1}{4}\tilde{Q}'' - \frac{2}{5}\frac{\varphi_B}{\zeta_B^2}V_0'\right] =: V_{\otimes}^{\dagger} + V_{\boxtimes}^{\triangle}$$

$$J_1=-\int\left[\tilde{Q}(\partial_xz)^2+V_{\boxplus}^{\star}z^2+V_{\boxcup}^{\odot}z^2\right]+\tilde{J}_1$$

$$V_B^I \geq V_1 \text{ where } V_1 = C_1' \tilde{Q}^3(x) \mathbf{1}_{\{|x| \geq 1\}}(x),$$

$$\left|\frac{\zeta_B''}{\zeta_B}-\left(\frac{\zeta_B'}{\zeta_B}\right)^2\right|\leq \frac{C}{B}\tilde{Q}^2(x)\mathbf{1}_{\{|x|\geq 1\}}(x)$$

$$\frac{\varphi_B}{\zeta_B^2}=\frac{1}{\zeta_B^2}\int_0^{\alpha^{-1}(x)}\zeta_B^2(\alpha(s))ds\geq \alpha^{-1}(x)$$

$$\begin{aligned} V_B^{\mathrm{I}}(x) &\geq \frac{1}{10}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3(x)\mathbf{1}_{\{|x|\geq 1\}}(x) \\ &\geq \Big(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3(x)\Big)\mathbf{1}_{\{1\leq x\leq x_{2,2}\}}(x)+\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| \\ &\quad +\Big(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3(x)\Big)\mathbf{1}_{\{x\geq x_{2,2}\}}(x) \end{aligned}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|\geq \tilde{C}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3\geq \tilde{C}-\frac{27}{8}\frac{C}{B}\geq \frac{1}{2}\tilde{C}>0$$

$$\tilde{Q}^3\lesssim |V_0'|\lesssim \tilde{Q}^3$$

$$C'\tilde{Q}^3\leq |V_0'(x)|$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3\geq \left(\frac{C'}{20}\alpha^{-1}(x)-\frac{C}{B}\right)\tilde{Q}^3$$

$$x\in [x_{2,2},+\infty)\mapsto \alpha^{-1}(x)$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3\geq \left(\frac{C'}{20}\alpha^{-1}(x_{2,2})-\frac{C}{B}\right)\tilde{Q}^3$$

$$B_2=\frac{10}{\alpha^{-1}(x_{2,2})}\frac{C}{C'}$$

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)|-\frac{C}{B}\tilde{Q}^3\geq \frac{1}{2}C'\tilde{Q}^3$$

$$\alpha^{-1}\colon \mathbb{R}_+\mapsto \mathbb{R}_+$$



$$\begin{aligned} V_{\odot}^+(x) &\geq \frac{1}{2} \tilde{C} \mathbf{1}_{\{1 \leq x \leq \bar{x}\}}(x) + \frac{1}{2} C' \tilde{Q}^3 \mathbf{1}_{\{x \geq \bar{x}\}}(x) + \frac{1}{20} \alpha^{-1}(x) |V_0'(x)| \\ &\geq \frac{1}{2} \tilde{C} \mathbf{1}_{\{1 \leq x \leq \bar{x}\}}(x) + \frac{1}{2} C' \tilde{Q}^3 \mathbf{1}_{\{x \geq \bar{x}\}}(x) \end{aligned}$$

$$V_B^{\mathrm{I}}(x) \geq C_1' \tilde{Q}^3 \mathbf{1}_{\{x \geq 1\}}(x),$$

$$V_{\underline{\Delta}}^{\star} \geq V_2 \text{ where } V_2 = C_1'' \tilde{Q}^3(x),$$

$$-\frac{1}{4}\tilde{Q}''=\frac{1}{2}\tilde{Q}^3\left(\frac{5}{6}\tilde{Q}-1\right)$$

$$\bar{x}=\alpha\left(2\mathrm{arccosh}\left(\sqrt{\frac{4}{5}}\right)\right)\sim0.576$$

$$V_{\wedge}^{\square}(x)>0$$

$$\begin{aligned} V_{\infty}^{\vee}(x) &\geq -\frac{1}{4}\tilde{Q}''-\frac{1}{5}\alpha^{-1}(x)V'(x)=\frac{1}{2}\tilde{Q}^3\left(\frac{5}{6}\tilde{Q}-1\right)+\frac{2}{5}\alpha^{-1}(x)(2-3\tilde{Q})\tilde{Q}^3\tilde{H} \\ &=\left(\frac{4}{5}\alpha^{-1}(x)\tilde{H}-\frac{1}{2}\right)\tilde{Q}^3+\left(\frac{5}{12}-\frac{6}{5}\alpha^{-1}(x)\tilde{H}\right)\tilde{Q}^4=k(\alpha^{-1}(x))\tilde{Q}^3 \end{aligned}$$

$$k\colon \mathbb{R}_+ \mapsto \mathbb{R}$$

$$k(s)\!:=\!\frac{4}{5}sH(s)-\frac{1}{2}+\left(\frac{5}{12}-\frac{6}{5}sH(s)\right)Q(s).$$

$$V_{\sqcap}^{\circ}(x) \geq C \mathbf{1}_{|x| \leq x_{2,2}}(x) + \tilde{C} \tilde{Q}^3 \mathbf{1}_{|x| \geq x_{2,2}}(x)$$

$$V_B^{\sqcup}(x) \geq C_1'' \tilde{Q}^3(x)$$

$$\frac{d}{dt}\mathcal{J}\leq -\int\tilde{Q}[(\partial_xz)^2+c_1\tilde{Q}^2z^2]+\tilde{J}_1+J_2+J_3+J_4$$

$$\begin{aligned}\|(1-\gamma\partial_x^2)^{-1}f\|_{L^2}&\leq\|f\|_{L^2},\|(1-\gamma\partial_x^2)^{-1}\partial_xf\|_{L^2}\leq\gamma^{-\frac{1}{2}}\|f\|_{L^2}\\\|(1-\gamma\partial_x^2)^{-1}\partial_x^2f\|_{L^2}&\leq\gamma^{-1}\|f\|_{L^2}\end{aligned}$$

$$\|\mathrm{sech}(K\alpha^{-1}(x))(1-\gamma\partial_x^2)^{-1}f\|_{L^2}\leq (1+m_0)\|(1-\gamma\partial_x^2)^{-1}[\mathrm{sech}(K\alpha^{-1}(x))f]\|_{L^2},$$

$$\|\cosh\left(K\alpha^{-1}(x)\right)(1-\gamma\partial_x^2)^{-1}f\|_{L^2}\lesssim\|(1-\gamma\partial_x^2)^{-1}[\cosh\left(K\alpha^{-1}(x)\right)f]\|_{L^2}$$

$$g=\mathrm{sech}(K\alpha^{-1})(1-\gamma\partial_x^2)^{-1}f \text{ and } k=(1-\gamma\partial_x^2)^{-1}[\mathrm{sech}(K\alpha^{-1})f]$$

$$\begin{aligned} f &= \cosh\left(K\alpha^{-1}\right)(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)[\cosh\left(K\alpha^{-1}\right)g] \\ &= \cosh\left(K\alpha^{-1}\right)g - \gamma[\cosh\left(K\alpha^{-1}\right)''g + 2\cosh\left(K\alpha^{-1}\right)'\partial_xg + \cosh\left(K\alpha^{-1}\right)\partial_x^2g] \\ &= \cosh\left(K\alpha^{-1}\right)(1-\gamma\partial_x^2)g - \gamma K\cosh\left(K\alpha^{-1}\right)\tilde{Q}^2\big[K-\tilde{H}\tanh\left(K\alpha^{-1}\right)\big]g \\ &\quad - 2\gamma K\cosh\left(K\alpha^{-1}\right)\tilde{Q}\tanh\left(K\alpha^{-1}\right)\partial_xg \end{aligned}$$

$$(1-\gamma\partial_x^2)k=(1-\gamma\partial_x^2)g-\gamma K\tilde{Q}^2\big[K-\tilde{H}\tanh\left(K\alpha^{-1}\right)\big]g-2\gamma K\tilde{Q}\tanh\left(K\alpha^{-1}\right)\partial_xg$$

$$g = k + \gamma K(1 - \gamma \partial_x^2)^{-1} \{ \tilde{Q}^2 [K - \tilde{H} \tanh(K\alpha^{-1})] g \} \\ + 2\gamma K(1 - \gamma \partial_x^2)^{-1} [\tilde{Q} \tanh(K\alpha^{-1}) \partial_x g]$$

$$\|(1 - \gamma \partial_x^2)^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq 1, \|(1 - \gamma \partial_x^2)^{-1} \partial_x\|_{\mathcal{L}(L^2, L^2)} \leq \gamma^{-\frac{1}{2}}$$

$$\gamma K \|(1 - \gamma \partial_x^2)^{-1} \{ \tilde{Q}^2 [K - \tilde{H} \tanh(K\alpha^{-1})] g \}\|_{L^2} \\ \leq \gamma K \|\tilde{Q}^2 [K - \tilde{H} \tanh(K\alpha^{-1})] g\|_{L^2} \leq (1 + K) \gamma K \|\tilde{Q}^2 g\|_{L^2}$$

$$\|(1 - \gamma \partial_x^2)^{-1} [\tilde{Q} \tanh(K\alpha^{-1}) \partial_x g]\|_{L^2} \\ \leq \|(1 - \gamma \partial_x^2)^{-1} \partial_x [\tilde{Q} \tanh(K\alpha^{-1}) g]\|_{L^2} + \|(1 - \gamma \partial_x^2)^{-1} [\partial_x (\tilde{Q} \tanh(K\alpha^{-1})) g]\|_{L^2} \\ \leq \gamma^{-\frac{1}{2}} \|\tilde{Q} \tanh(K\alpha^{-1}) g\|_{L^2} + \|\tilde{Q}^2 (K \operatorname{sech}^2(K\alpha^{-1}) - \tilde{H} \tanh(K\alpha^{-1})) g\|_{L^2} \\ \leq \gamma^{-\frac{1}{2}} K \|\tilde{Q}^2 g\|_{L^2} + \|\tilde{Q}^2 g\|_{L^2} \leq 3\gamma^{-\frac{1}{2}} \|\tilde{Q}^2 g\|_{L^2}$$

$$\|g\|_{L^2} \leq \|k\|_{L^2} + (1 + K) \gamma K \|\tilde{Q}^2 g\|_{L^2} + 6K\gamma^{\frac{1}{2}} \|\tilde{Q}^2 g\|_{L^2}.$$

$$\|g\|_{L^2} \leq (1 + m_0) \|k\|_{L^2},$$

$$g = \cosh(K\alpha^{-1})(1 - \gamma \partial_x^2)^{-1} f \text{ and } K = (1 - \gamma \partial_x^2)^{-1} [\cosh(K\alpha^{-1}) f]$$

$$f = \operatorname{sech}(K\alpha^{-1})(1 - \gamma \partial_x^2) k = (1 - \gamma \partial_x^2) [\operatorname{sech}(K\alpha^{-1}) g] \\ = \operatorname{sech}(K\alpha^{-1}) g - \gamma [\operatorname{sech}(K\alpha^{-1})'' g + 2\operatorname{sech}(K\alpha^{-1})' \partial_x g + \operatorname{sech}(K\alpha^{-1}) \partial_x^2 g] \\ = \operatorname{sech}(K\alpha^{-1})(1 - \gamma \partial_x^2) g - \gamma K \tilde{Q} \operatorname{sech}(K\alpha^{-1}) [K \tilde{Q} (1 - 2\operatorname{sech}^2(K\alpha^{-1})) g - 2 \tanh(K\alpha^{-1}) \partial_x g]$$

$$g = k + \gamma K^2 (1 - \gamma \partial_x^2)^{-1} [\tilde{Q}^2 (1 - 2\operatorname{sech}^2(K\alpha^{-1})) g] \\ - 2\gamma K (1 - \gamma \partial_x^2)^{-1} [\tilde{Q} \tanh(K\alpha^{-1}) \partial_x g]$$

$$\|(1 - \gamma \partial_x^2)^{-1} [\tilde{Q}^2 (1 - 2\operatorname{sech}^2(K\alpha^{-1})) g]\|_{L^2} \lesssim \|\tilde{Q}^2 (1 - 2\operatorname{sech}^2(K\alpha^{-1})) g\|_{L^2} \lesssim \|g\|_{L^2}$$

$$\|(1 - \gamma \partial_x^2)^{-1} [\tilde{Q} \tanh(K\alpha^{-1}) \partial_x g]\|_{L^2} \\ \lesssim \|(1 - \gamma \partial_x^2)^{-1} \partial_x [\tilde{Q} \tanh(K\alpha^{-1}) g]\|_{L^2} + \|(1 - \gamma \partial_x^2)^{-1} [\partial_x (\tilde{Q} \tanh(K\alpha^{-1})) g]\|_{L^2} \\ \lesssim \gamma^{-\frac{1}{2}} \|\tilde{Q} \tanh(K\alpha^{-1}) g\|_{L^2} + \|\tilde{Q}^2 [K \operatorname{sech}^2(K\alpha^{-1}) - \tilde{H} \tanh(K\alpha^{-1})] g\|_{L^2} \\ \lesssim \gamma^{-\frac{1}{2}} \|g\|_{L^2}.$$

$$\|g\|_{L^2} \leq \|k\|_{L^2} + \tilde{C} \gamma^{\frac{1}{2}} \|g\|_{L^2}$$

$$\left\| \operatorname{sech} \left(\left(\frac{3}{2} + \frac{1}{A} \right) \alpha^{-1} \right) (1 - \gamma \partial_x^2)^{-1} f \right\| \lesssim \left\| (1 - \gamma \partial_x^2)^{-1} \left[\operatorname{sech} \left(\left(\frac{3}{2} + \frac{1}{A} \right) \alpha^{-1} \right) f \right] \right\|,$$

$$\left\| \operatorname{sech} \left(\left(\frac{1}{2} + \frac{1}{A} \right) \alpha^{-1} \right) (1 - \gamma \partial_x^2)^{-1} f \right\| \lesssim \left\| (1 - \gamma \partial_x^2)^{-1} \left[\operatorname{sech} \left(\left(\frac{1}{2} + \frac{1}{A} \right) \alpha^{-1} \right) f \right] \right\|$$

$$\sigma_A \tilde{Q}^{-\frac{1}{2}} \lesssim \cosh \left(\frac{2-A}{2A} \alpha^{-1} \right) \lesssim \sigma_A \tilde{Q}^{-\frac{n}{2}}$$



$$\begin{aligned}\left\|\sigma_A \tilde{Q}^{-\frac{1}{2}}(1-\gamma \partial_x^2)^{-1} f\right\| &\lesssim\left\|\sigma_A \tilde{Q}^{-\frac{1}{2}} f\right\|. \\ \left\|\sigma_A \tilde{Q}^{-\frac{1}{2}}(1-\gamma \partial_x^2)^{-1} \partial_x f\right\| &\lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{-\frac{1}{2}} f\right\|.\end{aligned}$$

$$v=(1-\gamma \partial_x^2)^{-1} U u$$

$$\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} u\right\|$$

$$\left\|\sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x u\right\|+\left\|\sigma_A \tilde{Q}^{\frac{5}{2}} u\right\|$$

$$\sigma_A \tilde{Q}^{\frac{n}{2}} \lesssim \operatorname{sech}\left(\left(\frac{n}{2}+\frac{1}{A}\right) \alpha^{-1}\right) \lesssim \sigma_A \tilde{Q}^{\frac{n}{2}}$$

$$\begin{aligned}\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} v\right\| &\lesssim\left\|\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) v\right\| \\ &\lesssim\left\|\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right)\left(1-\gamma \partial_x^2\right)^{-1} \partial_x u\right\|+\left\|\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right)\left(1-\gamma \partial_x^2\right)^{-1}\left[h_0 u\right]\right\| \\ &\lesssim\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) \partial_x u\right]\right\|+\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) h_0 u\right]\right\| \\ &\lesssim\left\|\left(1-\gamma \partial_x^2\right)^{-1} \partial_x\left[\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) u\right]\right\| \\ &\quad +\frac{3 A+2}{2 A}\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\tilde{Q} \operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) u\right]\right\|+\left\|\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) h_0 u\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) u\right\|+\left\|\tilde{Q} \operatorname{sech}\left(\frac{3 A+2}{2 A} \alpha^{-1}\right) u\right\|+\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} h_0 u\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} u\right\|+\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} u\right\|+\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} h_0 u\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{\frac{3}{2}} u\right\|.\end{aligned}$$

$$\partial_x U=\partial_x^2-h_0 \partial_x-h_0'$$

$$\left|h_0'\right| \lesssim|V| \lesssim \tilde{Q}^2$$

$$\begin{aligned}\left\|\sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x v\right\| &\lesssim\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) \partial_x v\right\| \\ &\lesssim\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right)\left(1-\gamma \partial_x^2\right)^{-1} \partial_x^2 u\right\|+\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right)\left(1-\gamma \partial_x^2\right)^{-1}\left[h_0 \partial_x u\right]\right\| \\ &\quad +\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right)\left(1-\gamma \partial_x^2\right)^{-1}\left[h_0' u\right]\right\| \\ &\lesssim\left\|\left(1-\gamma \partial_x^2\right)^{-1} \partial_x\left[\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) \partial_x u\right]\right\|+\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right)' \partial_x u\right]\right\| \\ &\quad +\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) h_0 \partial_x u\right]\right\|+\left\|\left(1-\gamma \partial_x^2\right)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) h_0' u\right]\right\| \\ &\lesssim \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) \partial_x u\right\|+\left\|\operatorname{sech}\left(\frac{A+2}{2 A} \alpha^{-1}\right) \tilde{Q}^2 u\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x u\right\|+\left\|\sigma_A \tilde{Q}^{\frac{5}{2}} u\right\|\end{aligned}$$

$$\left\|\sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x\left(\tilde{\chi}_B u_1\right)\right\| \lesssim\left\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\right\|+\left\|\tilde{Q}^{\frac{3}{2}} w_1\right\|.$$



$$\begin{aligned}
& \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} v_1 \right\|_{L^2} \lesssim \gamma^{-\frac{1}{2}} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \\
& \left\| \sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x v_1 \right\| \lesssim \gamma^{-\frac{1}{2}} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) \\
& \left\| \sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x (\tilde{\chi}_B u_1) \right\| \lesssim \left\| \zeta_A \tilde{Q}^{\frac{1}{2}} \partial_x u_1 \right\| + \left\| \zeta_A \tilde{Q}^{\frac{3}{2}} u_1 \right\| \\
& \lesssim \left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| + \left\| \tilde{Q}^{\frac{1}{2}} \zeta'_A u_1 \right\| \lesssim \left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|. \\
& \left\| \sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x v_1 \right\| \lesssim \gamma^{-\frac{1}{2}} \left\| \sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x (\tilde{\chi}_B u_1) \right\| + \left\| \sigma_A \tilde{Q}^{\frac{5}{2}} \tilde{\chi}_B u_1 \right\| \\
& \lesssim \gamma^{-\frac{1}{2}} \left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \gamma^{-\frac{1}{2}} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| + \left\| \sigma_A \tilde{Q}^{\frac{5}{2}} u_1 \right\| \lesssim \gamma^{-\frac{1}{2}} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right),
\end{aligned}$$

$$\begin{aligned}
& \zeta_B(x) \lesssim e^{-\frac{1}{B}|\alpha^{-1}(x)|}, |\zeta'_B(x)| \lesssim \frac{1}{B} \tilde{Q} e^{-\frac{1}{B}|\alpha^{-1}(x)|}, |\varphi_B| \lesssim B \\
& |\tilde{\chi}'_A| \lesssim \frac{1}{A} \tilde{Q}, |(\tilde{\chi}^2_A)'| \lesssim \frac{1}{A} \tilde{Q}, |\tilde{\chi}''_A| \lesssim \frac{1}{A} \tilde{Q}^2, |(\tilde{\chi}^2_A)'''| \lesssim \frac{1}{A} \tilde{Q}^3
\end{aligned}$$

$$\tilde{\chi}'_A(x) = \tilde{\chi}''_A(x) = \tilde{\chi}'''_A(x) = 0$$

$$\begin{aligned}
\tilde{J}_1 &= \frac{1}{4} \int \tilde{Q} (\tilde{\chi}^2_A)' (\zeta^2_B)' v_1^2 + \frac{1}{4} \int \tilde{Q}' (\tilde{\chi}^2_A)' \zeta^2_B v_1^2 + \frac{1}{2} \int \tilde{Q} [3(\tilde{\chi}'_A)^2 + \tilde{\chi}''_A \tilde{\chi}_A] \zeta^2_B v_1^2 \\
&+ \frac{1}{4} \int (\tilde{\chi}^2_A)''' \varphi_B v_1^2 - \int (\tilde{\chi}^2_A)' \varphi_B (\partial_x v_1)^2 \\
&= J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}
\end{aligned}$$

$$\begin{aligned}
& |(\tilde{\chi}^2_A)' (\zeta^2_B)'| \lesssim \frac{1}{AB} e^{-2\frac{A}{B}} \sigma_A^2 \tilde{Q}^2, |(\tilde{\chi}^2_A)' \zeta^2_B| \lesssim \frac{1}{A} e^{-2\frac{A}{B}} \sigma_A^2 \tilde{Q} \\
& (\tilde{\chi}'_A)^2 \zeta^2_B + |\tilde{\chi}''_A \tilde{\chi}_A| \zeta^2_B \lesssim \frac{1}{A} e^{-2\frac{A}{B}} \sigma_A^2 \tilde{Q}^2, |(\tilde{\chi}^2_A)''' \varphi_B| \lesssim \frac{B}{A} \sigma_A^2 \tilde{Q}^3, |(\tilde{\chi}^2_A)' \varphi_B| \lesssim \frac{B}{A} \sigma_A^2 \tilde{Q}
\end{aligned}$$

$$|J_{1,1}| + |J_{1,2}| + |J_{1,3}| + |J_{1,4}| \lesssim \frac{B}{A} \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} v_1 \right\|^2 \lesssim \gamma^{-1} \frac{B}{A} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|^2$$

$$|J_{1,5}| \lesssim \frac{B}{A} \left\| \sigma_A \tilde{Q}^{\frac{1}{2}} \partial_x v_1 \right\|^2 \lesssim \gamma^{-1} \frac{B}{A} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\|^2 + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|^2 \right)$$

$$|\tilde{J}_1| \lesssim \gamma^{-1} \frac{B}{A} \left(\int \tilde{Q} (\partial_x w_1)^2 + \int \tilde{Q}^3 w_1^2 \right)$$

$$|J_2| \lesssim \gamma \left\| \tilde{Q} (1 - \gamma \partial_x^2)^{-1} \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) \right\| \left\| \tilde{Q}^{-1} (V_0'' v_1 + V_0' \partial_x v_1) \right\|$$

$$\begin{aligned}
\left\| \tilde{Q} (1 - \gamma \partial_x^2)^{-1} (\psi_{A,B} \partial_x v_1) \right\| &\lesssim \left\| \operatorname{sech}(\alpha^{-1}) (1 - \gamma \partial_x^2)^{-1} (\psi_{A,B} \partial_x v_1) \right\| \lesssim \left\| (1 - \gamma \partial_x^2)^{-1} (\operatorname{sech}(\alpha^{-1}) \psi_{A,B} \partial_x v_1) \right\| \\
&\lesssim \left\| \operatorname{sech}(\alpha^{-1}) \psi_{A,B} \partial_x v_1 \right\| \lesssim \left\| \tilde{Q} \psi_{A,B} \partial_x v_1 \right\|.
\end{aligned}$$

$$\partial_x z = \tilde{\chi}_A \zeta_B \partial_x v_1 + (\tilde{\chi}_A \zeta_B)' v_1 \Rightarrow \tilde{\chi}_A^2 \zeta_B^2 (\partial_x v_1)^2 \lesssim (\partial_x z)^2 + |(\tilde{\chi}_A \zeta_B)' v_1|^2$$

$$|(\tilde{\chi}_A \zeta_B)' v_1|^2 \tilde{\chi}_A^2 \lesssim \left(\frac{1}{A} + \frac{1}{B} \right)^2 \tilde{Q}^2 \tilde{\chi}_A^2 \zeta_B^2 v_1^2 \lesssim \frac{1}{B^2} \tilde{Q}^2 z^2$$



$$\tilde{\chi}_A^4 \zeta_B^2 (\partial_x v_1)^2 \lesssim \tilde{\chi}_A^2 (\partial_x z)^2 + \frac{1}{B^2} \tilde{Q}^2 z^2$$

$$|\psi_{A,B}| \leq |\alpha^{-1}(x)| \tilde{\chi}_A^2$$

$$\tilde{Q}^2 |\psi_{A,B} \partial_x v_1|^2 \lesssim |\alpha^{-1}(x)|^2 \tilde{Q}^2 \tilde{\chi}_A^4 (\partial_x v_1)^2 \lesssim \tilde{Q} \tilde{\chi}_A^4 \zeta_B^2 (\partial_x v_1)^2 \lesssim \tilde{Q} (\partial_x z)^2 + \frac{1}{B^2} \tilde{Q}^3 z^2$$

$$\|\tilde{Q} \psi_{A,B} \partial_x v_1\| \lesssim \left(\tilde{Q} (\partial_x z)^2 + \frac{1}{B^2} \tilde{Q}^3 z^2 \right)^{\frac{1}{2}}$$

$$\|\tilde{Q} (1 - \gamma \partial_x^2)^{-1} (\psi'_{A,B} v_1)\| \lesssim \|\tilde{Q} \psi'_{A,B} v_1\|$$

$$(\psi'_{A,B})^2 \leq \frac{21}{10} \tilde{Q}^2 \tilde{\chi}_A^2$$

$$|\alpha^{-1}(x)| \geq 2A$$

$$(\psi'_{A,B})^2 = [(\tilde{\chi}_A^2)' \varphi_B + \tilde{\chi}_A^2 \tilde{Q} \zeta_B^2]^2 \leq 8(\tilde{\chi}_A \tilde{\chi}'_A \varphi_B)^2 + 2\tilde{Q}^2 \tilde{\chi}_A^4 \zeta_B^4 \leq \tilde{Q}^2 \tilde{\chi}_A^2 \left(\frac{C}{A^2} B^2 + 2 \right) \leq \frac{21}{10} \tilde{Q}^2 \tilde{\chi}_A^2$$

$$|\tilde{Q}^2 (\psi'_{A,B})^2 v_1^2| \leq \frac{21}{10} \tilde{Q}^4 \tilde{\chi}_A^2 v_1^2 \leq \frac{63}{20} \tilde{Q}^3 z^2$$

$$\|\tilde{Q} \psi_{A,B} v_1\|^2 \leq 4 \int \tilde{Q}^3 z^2$$

$$\left\| \tilde{Q} (1 - \gamma \partial_x^2)^{-1} \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) \right\| \lesssim \left(\int \tilde{Q} (\partial_x z)^2 + \tilde{Q}^3 z^2 \right)^{\frac{1}{2}}$$

$$|h_0| \lesssim 1, h'_0 = \frac{1}{4h_0} (V'_0 + V'), h''_0 = V' - 2h_0 h'_0$$

$$V''_0 = 4(h'_0)^2 + 4h_0 h''_0 - V''$$

$$|V''_0 v_1| + |V'_0 \partial_x v_1| \lesssim \tilde{Q}^3 |v_1| + \tilde{Q}^3 |\partial_x v_1|$$

$$\|\tilde{Q}^{-1} (V''_0 v_1 + V'_0 \partial_x v_1)\| \lesssim \|\tilde{Q}^2 v_1\| + \|\tilde{Q}^2 \partial_x v_1\|$$

$$\tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^2 v_1^2 \lesssim \tilde{\chi}_A^2 \zeta_B^2 v_1^2 = z^2$$

$$\tilde{Q}^4 v_1^2 = \tilde{Q}^4 v_1^2 \tilde{\chi}_A^2 + \tilde{Q}^4 v_1^2 (1 - \tilde{\chi}_A^2) \lesssim \tilde{Q}^{\frac{7}{2}} z^2 + e^{-\frac{A}{2}} \tilde{Q}^{\frac{7}{2}} v_1^2.$$

$$\|\tilde{Q}^2 v_1\| \lesssim \left\| \tilde{Q}^{\frac{7}{2}} z \right\| + e^{-\frac{A}{2}} \left\| \tilde{Q}^{\frac{7}{4}} v_1 \right\| \lesssim \left\| \tilde{Q}^{\frac{3}{2}} z \right\| + e^{-\frac{A}{4}} \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} v_1 \right\|.$$

$$\|\tilde{Q}^2 v_1\| \lesssim \left\| \tilde{Q}^{\frac{3}{2}} z \right\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{4}} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|.$$

$$\tilde{\chi}_A \zeta_B \partial_x v_1 = \partial_x z - \frac{\zeta'_B}{\zeta_B} z - \tilde{\chi}'_A \zeta_B v_1$$



$$|\tilde{\chi}_A \zeta_B \partial_x v_1| \lesssim \partial_x z + \frac{1}{B} \tilde{Q} z$$

$$\tilde{Q}^4 (\partial_x v_1)^2 = \tilde{Q}^4 (\partial_x v_1)^2 \tilde{\chi}_A^2 + \tilde{Q}^4 (\partial_x v_1)^2 (1 - \tilde{\chi}_A^2) \lesssim \tilde{Q}^{\frac{7}{2}} (\partial_x z)^2 + \frac{1}{B} \tilde{Q}^{\frac{11}{2}} z^2 + e^{-A} \tilde{Q}^3 (\partial_x v_1)^2$$

$$\begin{aligned} \|\tilde{Q}^2 \partial_x v_1\| &\lesssim \left\| \tilde{Q}^{\frac{7}{4}} \partial_x z \right\| + \frac{1}{\sqrt{B}} \left\| \tilde{Q}^{\frac{11}{4}} z \right\| + e^{-\frac{A}{2}} \left\| \tilde{Q}^{\frac{7}{4}} \partial_x v_1 \right\| \\ &\lesssim \left\| \tilde{Q}^{\frac{1}{2}} \partial_x z \right\| + \frac{1}{\sqrt{B}} \left\| \tilde{Q}^{\frac{3}{2}} z \right\| + e^{-\frac{A}{2}} \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} \partial_x v_1 \right\| \\ &\lesssim \left\| \tilde{Q}^{\frac{1}{2}} \partial_x z \right\| + \frac{1}{\sqrt{B}} \left\| \tilde{Q}^{\frac{3}{2}} z \right\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{2}} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) \end{aligned}$$

$$\|\tilde{Q}^2 v_1\| + \|\tilde{Q}^2 \partial_x v_1\| \lesssim \left\| \tilde{Q}^{\frac{1}{2}} \partial_x z \right\| + \left\| \tilde{Q}^{\frac{3}{2}} z \right\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{4}} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right)$$

$$|U_2| \lesssim \gamma \left(\int \tilde{Q} (\partial_x z)^2 + \tilde{Q}^3 z^2 \right) + e^{-\frac{A}{4}} \left(\int \tilde{Q} (\partial_x w_1)^2 + \tilde{Q}^3 w_1^2 \right)$$

$$|U_3| \lesssim \left(\left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi_{A,B} \partial_x v_1 \right\| + \left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi'_{A,B} v_1 \right\| \right) \left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}'_B \partial_x u_1 + \tilde{\chi}''_B u_1) \right\|$$

$$\left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi_{A,B} \partial_x v_1 \right\| \lesssim B \left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A \partial_x v_1 \right\| \lesssim \gamma^{-\frac{1}{2}} B \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right)$$

$$|\psi'_{A,B}| \leq |(\tilde{\chi}_A^2)' \varphi_B| + \tilde{Q} \tilde{\chi}_A^2 \zeta_B^2 \lesssim \frac{B}{A} \tilde{Q} \tilde{\chi}_A 1_{\{A \leq |\alpha^{-1}(x)| \leq 2A\}} + \tilde{Q} \tilde{\chi}_A^2 \zeta_B^2$$

$$\left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi'_{A,B} v_1 \right\| \lesssim \frac{B}{A} \left\| \tilde{Q}^{\frac{3}{2}} 1_{\{A \leq |x| \leq 2A\}} v_1 \right\| + \|\tilde{Q}^{3/2} z\| \lesssim \gamma^{-\frac{1}{2}} \frac{B}{A} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| + \|\tilde{Q}^{3/2} z\|$$

$$\left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi_{A,B} \partial_x v_1 \right\| + \left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi'_{A,B} v_1 \right\| \lesssim \gamma^{-\frac{1}{2}} B \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) + \|\tilde{Q}^{3/2} z\|$$

$$1 \lesssim \sigma_A \lesssim \zeta_A[-2\alpha(A), 2\alpha(A)], U = \partial_x - h_0$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}''_B u_1) \right\| \lesssim \gamma^{-\frac{1}{2}} \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}''_B u_1 \right\|$$

$$\tilde{\chi}''_B \lesssim B^{-4} \tilde{Q}^2 [-2\alpha(B^2), 2\alpha(B^2)]$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}''_B u_1) \right\| \lesssim \gamma^{-\frac{1}{2}} B^{-4} \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} u_1 \right\| \lesssim \gamma^{-\frac{1}{2}} B^{-4} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}'_B \partial_x u_1) \right\| \lesssim \gamma^{-\frac{1}{2}} \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}'_B \partial_x u_1 \right\| \lesssim \gamma^{-\frac{1}{2}} B^{-2} \left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\|$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U (\tilde{\chi}'_B \partial_x u_1 + \tilde{\chi}''_B u_1) \right\| \lesssim \gamma^{-\frac{1}{2}} B^{-2} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right)$$

$$|U_3| \lesssim \gamma^{-1} B^{-1} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right)^2 + \gamma^{-\frac{1}{2}} B^{-2} \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) \|\tilde{Q}^{3/2} z\|$$

$$N = N_g + N_b$$



$$N_g := \tilde{Q}^2(3\tilde{H}(a_1^2\phi_0^2 + 2a_1\phi_0u_1) + a_1^3\phi_0^3 + 3a_1^2\phi_0^2u_1 + 3a_1\phi_0u_1^2) \\ N_b := \tilde{Q}^2u_1^2(3\tilde{H} + u_1)$$

$$N^\perp = N - N_0\phi_0 = (N_g - N_0\phi_0) + N_b := N_g^\perp + N_b$$

$$J_{4,g} = - \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g^\perp)$$

$$|J_{4,g}| \lesssim \left(\left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi_{A,B} \partial_x v_1 \right\| + \left\| \tilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^{-1} \psi'_{A,B} v_1 \right\| \right) \left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g^\perp) \right\|$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g^\perp) \right\| \\ \leq \left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g) \right\| + |N_0| \left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B \phi_0) \right\|.$$

$$|N_g| \lesssim \tilde{Q}^2 \phi_0 (a_1^2 + u_1^2) \\ |N_0| \lesssim a_1^2 + \int \tilde{Q}^2 \phi_0 u_1^2 \lesssim a_1^2 + \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\|.$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g) \right\| \lesssim \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g) \right\| \lesssim \gamma^{-\frac{1}{2}} \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} N_g \right\|$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_g) \right\| \lesssim \gamma^{-\frac{1}{2}} \left(a_1^2 \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} \phi_0 \right\| + \left\| \sigma_A \tilde{Q}^{\frac{3}{2}} \phi_0 u_1^2 \right\| \right) \\ \lesssim \gamma^{-\frac{1}{2}} \left(a_1^2 + \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right).$$

$$\left\| \tilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U \phi_0 \right\| \lesssim \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} (1 - \gamma \partial_x^2)^{-1} U \phi_0 \right\| \lesssim \gamma^{-\frac{1}{2}} \left\| \sigma_A \tilde{Q}^{-\frac{1}{2}} \phi_0 \right\| \lesssim \gamma^{-\frac{1}{2}}$$

$$|J_{4,g}| \lesssim \gamma^{-1} B \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) \left(a_1^2 + \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) \\ + \gamma^{-\frac{1}{2}} \left\| \tilde{Q}^{\frac{3}{2}} z \right\| \left(a_1^2 + \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right)$$

$$J_{4,b} = \int \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_b)$$

$$\left\| \tilde{Q}^{1/2} \tilde{\chi}_A^{-1} \left(\psi_{A,B} \partial_x v_1 + \frac{1}{2} \psi'_{A,B} v_1 \right) \right\| \lesssim \gamma^{-\frac{1}{2}} B \left(\left\| \tilde{Q}^{\frac{1}{2}} \partial_x w_1 \right\| + \left\| \tilde{Q}^{\frac{3}{2}} w_1 \right\| \right) + \|\tilde{Q}^{3/2} z\|$$

$$\left\| \tilde{Q}^{-1/2} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_b) \right\| \lesssim \left\| \sigma_A \tilde{Q}^{-1/2} (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_b) \right\| \\ \lesssim \gamma^{-\frac{1}{2}} (\|\sigma_A \tilde{\chi}_B \tilde{Q}^{3/2} u_1^2\| + \|\sigma_A \tilde{\chi}_B \tilde{Q}^{3/2} u_1^3\|).$$

$$\left\| \tilde{Q}^{-1/2} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U(\tilde{\chi}_B N_b) \right\| \lesssim \gamma^{-\frac{1}{2}} \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \|\sigma_A \tilde{Q}^{3/2} u_1\| + \gamma^{-\frac{1}{2}} \alpha(B)^2 \|\tilde{Q}^{1/2} u_1\|_{L^\infty}^2 \|\sigma_A \tilde{Q}^{3/2} u_1\| \\ \lesssim \gamma^{-\frac{1}{2}} \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \right) \|\tilde{Q}^{3/2} w_1\|.$$



$$|J_{4,b}| \lesssim \gamma^{-1} B \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \left(\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \|\tilde{Q}^{3/2} w_1\| \\ + \gamma^{-\frac{1}{2}} B \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|\tilde{Q}^{3/2} z\| \|\tilde{Q}^{3/2} w_1\|$$

$$|J_4| \lesssim \gamma^{-1} B \left(\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \left(a_1^2 + \alpha(B) \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \\ + \gamma^{-\frac{1}{2}} \|\tilde{Q}^{\frac{3}{2}} z\| \left(a_1^2 + B \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right)$$

$$\frac{d}{dt} J \leq -4C_2 \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] + \gamma^{-1} \frac{CB}{A} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] \\ + \gamma C \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] + C e^{-\frac{A}{4}} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] \\ + \gamma^{-1} B^{-1} \left(\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\tilde{Q}^{\frac{3}{2}} w_1\|\right)^2 + \gamma^{-\frac{1}{2}} B^{-2} \left(\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \|\tilde{Q}^{3/2} z\| \\ + \gamma^{-1} CB \left(\|\tilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \left(a_1^2 + \alpha(B) \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right) \\ + \gamma^{-\frac{1}{2}} C \|\tilde{Q}^{\frac{3}{2}} z\| \left(a_1^2 + B \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right)$$

$$\frac{d}{dt} J \leq -C_2 \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] + C \left(\frac{1}{B} + \frac{B}{A} + e^{-\frac{A}{4}}\right) \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] \\ + CB \alpha(B) \left(a_1^2 + B \left(1 + \alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty}\right) \|u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right)^2$$

$$\alpha(B) \|\tilde{Q}^{1/2} u_1\|_{L^\infty} \lesssim \delta^{\frac{7}{8}} \lesssim 1, B^{-1} + A^{-1} B + e^{-\frac{A}{4}} \lesssim \ln \left(\delta^{-\frac{1}{8}}\right)^{-1} \\ B \alpha(B) \left(\alpha(B) \|u_1\|_{L^\infty} \|\tilde{Q}^{\frac{3}{2}} w_1\|\right)^2 \lesssim \delta^{-\frac{1}{2}} \|u_1\|_{L^\infty}^2 \|\tilde{Q}^{\frac{3}{2}} w_1\|^2 \lesssim \delta^{\frac{3}{2}} \|\tilde{Q}^{\frac{3}{2}} w_1\|^2.$$

$$\frac{d}{dt} J \leq -C_2 \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] + C \ln \left(\delta^{-\frac{1}{8}}\right)^{-1} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \delta^{\frac{1}{2}} |a_1|^3$$

$$\int \tilde{Q}^7 u_1^2$$

$$v = (1 - \gamma \partial_x^2)^{-1} U u$$

$$\int \tilde{Q}^7 u^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2]$$

$$v - \gamma \partial_x^2 v = \phi_0 \partial_x \left(\frac{u}{\phi_0} \right)$$

$$\partial_x \left(\frac{u}{\phi_0} + \gamma \frac{\partial_x v}{\phi_0} \right) = \frac{1}{\phi_0} (v - \gamma h_0 \partial_x v)$$

$$\frac{u}{\phi_0} + \gamma \frac{\partial_x v}{\phi_0} = a + \int_0^x \frac{1}{\phi_0} (v - \gamma h_0 \partial_x v)$$

$$u = a \phi_0 - \gamma \partial_x v + \tilde{u}$$



$$\tilde{u} = \phi_0 \int_0^x \frac{1}{\phi_0} (v - \gamma h_0 \partial_x v)$$

$$\phi_0 \int_0^x \frac{|v|}{\phi_0} \lesssim \phi_0 \left(\int \tilde{Q}^{\frac{9}{2}} v^2 \right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\tilde{Q}^{\frac{9}{2}} \phi_0^2} \right)^{\frac{1}{2}} \lesssim \|\tilde{Q}^{\frac{9}{4}} v\| \left(\int_0^{\alpha^{-1}(x)} \frac{1}{Q^{\frac{11}{2}}} \right)^{\frac{1}{2}} \lesssim \tilde{Q}^{-\frac{11}{4}} \|\tilde{Q}^{\frac{9}{4}} v\|$$

$$\phi_0 \int_0^x \frac{|h_0 \partial_x v|}{\phi_0} \lesssim \phi_0 \left(\int \tilde{Q}^{\frac{9}{2}} (h_0 \partial_x v)^2 \right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\tilde{Q}^{\frac{9}{2}} \phi_0^2} \right)^{\frac{1}{2}} \lesssim \tilde{Q}^{-\frac{11}{4}} \|\tilde{Q}^{\frac{9}{4}} \partial_x v\|$$

$$\tilde{Q}^{\frac{11}{2}} \tilde{u}^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2]$$

$$\int \tilde{Q}^7 \tilde{u}^2 \lesssim \left(\int \tilde{Q}^{\frac{3}{2}} \right) \left(\int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2] \right) \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v)^2 + v^2]$$

$$a = \gamma \langle \partial_x v, \phi_0 \rangle - \langle \tilde{u}, \phi_0 \rangle.$$

$$a^2 \lesssim \left(\int \phi_0 \partial_x v \right)^2 + \left(\int \phi_0 \tilde{u} \right)^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} (\partial_x v)^2 + \int \tilde{Q}^7 \tilde{u}^2 \lesssim \int \tilde{Q}^{\frac{7}{2}} [(\partial_x v)^2 + v^2]$$

$$\int \tilde{Q}^7 u_1^2 \lesssim \int \tilde{Q}^2 [(\partial_x z)^2 + \tilde{Q}^2 z^2] + e^{-\frac{A}{4}} \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2]$$

$$\int \tilde{Q}^7 u_1^2 \lesssim \int \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2]$$

$$\bar{Q} \lesssim e^{-|\alpha^{-1}(x)|}, \bar{Q}^{\frac{1}{4}} \lesssim \zeta_B^2$$

$$\begin{aligned} \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2] &\lesssim \int \tilde{Q}^4 \zeta_B^2 (\partial_x v_1)^2 + \int \tilde{Q}^4 \zeta_B^2 v_1^2 \\ &\lesssim \int \tilde{Q}^4 [(\partial_x z)^2 + \tilde{Q}^2 z^2] + \int \tilde{Q}^4 z^2 + e^{-3A} \int \tilde{Q} \zeta_B^2 (1 - \tilde{\chi}_A^2) (\partial_x v_1)^2 + e^{-A} \int \tilde{Q}^3 \zeta_B^2 (1 - \tilde{\chi}_A^2) v_1^2 \end{aligned}$$

$$\begin{aligned} \tilde{Q}^{\frac{9}{2}} [(\partial_x v_1)^2 + v_1^2] &\lesssim \int \tilde{Q}^2 [(\partial_x z)^2 + \tilde{Q}^2 z^2] + e^{-3A} \int \sigma_A^2 \tilde{Q} (\partial_x v_1)^2 + e^{-A} \int \sigma_A^2 \tilde{Q}^3 v_1^2 \\ &\lesssim \int \tilde{Q}^2 [(\partial_x z)^2 + \tilde{Q}^2 z^2] + e^{-A} \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] \end{aligned}$$

$$\mathcal{H} = \mathcal{J} + 8C C_0^{-1} \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \mathcal{J}$$

$$\frac{d}{dt} \mathcal{H} \leq -C_3 \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + |a_1|^3$$

$$\begin{aligned} \frac{d}{dt} \mathcal{J} &\leq -\frac{1}{2} C_0 \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \int \tilde{Q} [(\partial_x z)^2 + \tilde{Q}^2 z^2] + C e^{-\frac{A}{4}} \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \delta_3 |a_1|^3 \\ &\leq -\frac{1}{4} C_0 \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \int \tilde{Q} [(\partial_x z)^2 + \tilde{Q}^2 z^2] + |a_1|^3 \end{aligned}$$



$$\frac{d}{dt}\mathcal{J} \leq -C_2 \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] + C \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \delta^{\frac{1}{2}} |a_1|^3$$

$$\begin{aligned} \frac{d}{dt}\mathcal{H} \leq & \left(-C_2 + 8C^2 C_0^{-1} \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \right) \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] \\ & - C \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + C \left(\delta^{\frac{1}{2}} + 8C_0^{-1} \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \right) |a_1|^3 \end{aligned}$$

$$\frac{d}{dt}\mathcal{H} \leq -\frac{C_2}{2} \int \tilde{Q}[(\partial_x z)^2 + \tilde{Q}^2 z^2] - C \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] + |a_1|^3$$

$$C_3 = C \ln \left(\delta_3^{-\frac{1}{8}} \right)^{-1} > 0$$

$$\mathcal{B} = b_+^2 - b_-^2$$

$$|\dot{b}_+ - \mu_0 b_+| + |\dot{b}_- + \mu_0 b_-| \leq C_4 \left(b_+^2 + b_-^2 + \int \tilde{Q}^3 w_1^2 \right)$$

$$\left| \frac{d}{dt}(b_+^2) - 2\mu_0 b_+^2 \right| + \left| \frac{d}{dt}(b_-^2) + 2\mu_0 b_-^2 \right| \leq C_4 \left(b_+^2 + b_-^2 + \int \tilde{Q}^3 w_1^2 \right)^{\frac{3}{2}}$$

$$\frac{d}{dt}\mathcal{B} \geq \mu_0(b_+^2 + b_-^2) - C_4 \int \tilde{Q}^3 w_1^2 = \frac{\mu_0}{2}(a_1^2 + a_2^2) - C_4 \int \tilde{Q}^3 w_1^2$$

$$|N_0| \lesssim a_1^2 + \int \tilde{Q}^3 w_1^2 \lesssim b_+^2 + b_-^2 + \int \tilde{Q}^3 w_1^2$$

$$\frac{d}{dt}(\mathcal{B} - 2C_4 C_3^{-1} \mathcal{H}) \geq \frac{\mu_0}{2}(a_1^2 + a_2^2) + C_4 \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] - 2C_4 |a_1|^3$$

$$\frac{d}{dt}(\mathcal{B} - 2C_4 C_3^{-1} \mathcal{H}) \geq \frac{\mu_0}{4}(a_1^2 + a_2^2) + C_4 \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2]$$

$$|\mathcal{J}| \leq \left| \int \left(\varphi_A \partial_x u_1 + \frac{1}{2} \varphi_A' u_1 \right) u_2 \right| \lesssim A \left(\|\partial_x u_1\|_{L^2} + \|\tilde{Q} u_1\|_{L^2} \right) \|u_2\|_{L^2} \lesssim \delta$$

$$U = \partial_x - h_0, \psi'_{A,B} = \tilde{Q} \tilde{\chi}_A^2 \tilde{\zeta}_B^2 + (\tilde{\chi}_A^2)' \varphi_B$$

$$|\mathcal{J}| = \left| \int \left(\psi_{A,B}(x) \partial_x v_1(t,x) + \frac{1}{2} \psi'_{A,B}(x) v_1(t,x) \right) v_2(t,x) dx \right| \lesssim B \left(\|\partial_x v_1\|_{L^2} + \|\tilde{Q} v_1\|_{L^2} \right) \|v_2\|_{L^2}$$

$$\lesssim \delta$$

$$|\mathcal{H}| \lesssim \delta$$

$$\int_0^\infty \left\{ a_1^2 + a_2^2 + \int \tilde{Q}[(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2] \right\} dt \lesssim \delta$$



$$\int \tilde{Q}^2 \left[(\partial_x u_1)^2 + \tilde{Q}^{\frac{3}{2}} u_1^2 \right] \lesssim \int \tilde{Q} [(\partial_x w_1)^2 + \tilde{Q}^2 w_1^2]$$

$$\int_0^\infty \left\{ a_1^2 + a_2^2 + \int \tilde{Q}^2 \left[(\partial_x u_1)^2 + \tilde{Q}^{\frac{3}{2}} u_1^2 \right] \right\} dt \lesssim \delta$$

$$\mathcal{K} = \int u_1 u_2 \tilde{Q}^2 \text{ and } \mathcal{G} = \frac{1}{2} \int \left[(\partial_x u_1)^2 + \tilde{Q}^{\frac{3}{2}} u_1^2 + u_2^2 \right] \tilde{Q}^2$$

$$\begin{aligned} \frac{d}{dt} \mathcal{K} &= \int [\dot{u}_1 u_2 + u_1 \dot{u}_2] \tilde{Q}^2 = \int [u_2^2 - u_1 (Lu_1 + N^\perp)] \tilde{Q}^2 \\ &= \int [u_2^2 - (\partial_x u_1)^2 - 2\tilde{Q}^2(1 - \tilde{Q})u_1^2] \tilde{Q}^2 + \frac{1}{2} \int (\tilde{Q}^2)'' u_1^2 - \int N^\perp \tilde{Q}^2 u_1 \end{aligned}$$

$$\int N^\perp \tilde{Q}^2 u_1 \lesssim a_1^2 + \int \tilde{Q}^{\frac{7}{2}} u_1^2$$

$$\int \tilde{Q}^2 u_2^2 \leq \frac{d}{dt} \mathcal{K} + C a_1^2 + C \int \tilde{Q}^2 \left[(\partial_x u_1)^2 + \tilde{Q}^{\frac{3}{2}} u_1^2 \right]$$

$$\int_0^\infty [a_1^2 + a_2^2 + \mathcal{G}] dt \lesssim \delta$$

$$\begin{aligned} \frac{d}{dt} \mathcal{G} &= \int \left[(\partial_x \dot{u}_1)(\partial_x u_1) + \tilde{Q}^{\frac{3}{2}} \dot{u}_1 u_1 + \dot{u}_2 u_2 \right] \tilde{Q}^2 = \int \left[(\partial_x u_2)(\partial_x u_1) + \tilde{Q}^{\frac{3}{2}} u_2 u_1 - (Lu_1 + N^\perp) u_2 \right] \tilde{Q}^2 \\ &= -2 \int \tilde{Q} \tilde{Q}' u_2 \partial_x u_1 + \int \left(2\tilde{Q}^{\frac{3}{2}} - 2\tilde{Q}^{\frac{1}{2}} + 1 \right) \tilde{Q}^{\frac{7}{2}} u_1 u_2 - \int \tilde{Q}^2 N^\perp u_2 \end{aligned}$$

$$\left| \frac{d}{dt} \mathcal{G} \right| \lesssim a_1^2 + \mathcal{G}$$

$$\lim_{n \rightarrow \infty} [a_1^2(t_n) + a_2^2(t_n) + \mathcal{G}(t_n)] = 0$$

$$\mathcal{G}(t) \lesssim \int_t^\infty [a_1^2 + \mathcal{G}] dt$$

$$\left| \frac{d}{dt} (a_1^2) \right| + \left| \frac{d}{dt} (a_2^2) \right| \lesssim a_1^2 + a_2^2 + \int \tilde{Q}^{\frac{7}{2}} u_1^2$$

$$a_1^2(t) + a_2^2(t) \lesssim \int_t^\infty [a_1^2 + a_2^2 + \mathcal{G}] dt$$

$$\lim_{t \rightarrow \infty} (|a_1(t)| + |a_2(t)|) = 0$$

$$\begin{aligned} \widehat{H}(x) &:= (\sinh(\alpha^{-1}(x)) + 3\alpha^{-1}(x))\widetilde{H}(x) - 4 =: \beta(x)\widetilde{H}(x) - 4 \\ \beta(x) &= 3x + 2\alpha^{-1}(x) \end{aligned}$$

$$\langle \tilde{L}u_1, u_1 \rangle \geq 0$$

$$\phi_1 = \alpha_{00} \widetilde{H} + \beta_{00} \widehat{H} + \frac{1}{3} \left(\widehat{H} \int_x^\infty \phi_0 \widetilde{H} + \widetilde{H} \int_0^x \phi_0 \widehat{H} \right)$$



$$\phi_1 = \frac{1}{3} \left(\hat{H} \int_x^\infty \phi_0 \tilde{H} + \tilde{H} \int_0^x \phi_0 \hat{H} \right)$$

$$\phi_{1,x} = \frac{1}{3} \left(\hat{H}' \int_x^\infty \phi_0 \tilde{H} + \frac{1}{3} \tilde{Q}^2 \int_0^x \phi_0 \hat{H} \right) \in L^2(\mathbb{R})$$

$$\lim_{\infty} \phi_{1,x} \hat{H} = \lim_{\infty} \phi_1 \hat{H}_x = 0$$

$$\langle \phi_0, \hat{H} \rangle = \langle \tilde{L} \phi_1, \hat{H} \rangle = \langle \phi_1, \tilde{L} \hat{H} \rangle = 0$$

$$\int_0^\infty \phi_0 \hat{H} = 0$$

$$[0, \infty) \ni x \mapsto g(x) := \int_0^x \phi_0 \hat{H}$$

$$\langle \phi_0, \phi_1 \rangle = \frac{2}{3} \int_0^\infty \phi_0 \left(\hat{H} \int_x^\infty \phi_0 \tilde{H} + \tilde{H} \int_0^x \phi_0 \hat{H} \right) = \frac{4}{3} \int_0^\infty \phi_0 \tilde{H} \int_0^x \phi_0 \hat{H} = \frac{4}{3} \int_0^\infty \phi_0 \tilde{H} g < 0$$

$$\langle \phi_0, u \rangle = \langle \tilde{Q}^2 \tilde{H}^3, u \rangle = 0$$

$$\langle \tilde{L}u, u \rangle \geq c_0 \|u\|_{H_0}^2$$

$$\tau = \inf\{\langle \tilde{L}u, u \rangle: \|u\|_{H_0} = 1, \langle \phi_0, u \rangle = \langle \tilde{Q}^2 \tilde{H}^3, u \rangle = 0\}.$$

$$0 < \int (\partial_x u_n)^2 + \tilde{Q}^3 u_n^2 \leq \frac{5}{3} \tilde{Q}^3 u_n^2 + \eta$$

$$\int \tilde{Q}^3 (u_n - u_\infty)^2 \rightarrow 0, \text{ as } n \rightarrow +\infty$$

$$v_\infty = u_\infty / \|u_\infty\|_{H_0}$$

$$\langle \tilde{L}u_\infty, u_\infty \rangle \leq \liminf_{n \rightarrow \infty} \langle \tilde{L}u_n, u_n \rangle = 0$$

$$\tilde{L}u = \alpha(-\partial_x^2 u + \tilde{Q}^2 u) + \beta \phi_0 + \gamma \tilde{Q}^2 \tilde{H}^3$$

$$\|u\|_{H_0} = 1, \langle \phi_0, u \rangle = \langle \tilde{Q}^2 \tilde{H}^3, u \rangle = 0$$

$$\alpha = \langle \tilde{L}u, u \rangle, \text{ so } \alpha = \tau = 0$$

$$\tilde{L}u_\infty = \beta \phi_0 + \gamma \tilde{Q}^2 \tilde{H}^3$$

$$u_\infty = \frac{\beta}{3} \left(\hat{H} \int_x^\infty \phi_0 \tilde{H} + \tilde{H} \int_0^x \phi_0 \hat{H} \right)$$

$$\langle Lu, u \rangle \geq C_5 \left(\int (\partial_x u)^2 + \int \tilde{Q}^2 u^2 \right)$$

$$L = \varepsilon_0(-\partial_x^2 + \tilde{Q}^2) + L_{\varepsilon_0}$$



$$L_{\varepsilon_0} := -(1 - \varepsilon_0)\partial_x^2 + 2\tilde{Q}^2\left(1 - \frac{1}{2}\varepsilon_0 - \tilde{Q}\right).$$

$$\begin{aligned}\langle L_{\varepsilon_0} f, f \rangle &= (1 - \varepsilon_0) \int f'^2 + 2 \int f^2 \tilde{Q}^2 \left(1 - \frac{1}{2}\varepsilon_0 - \tilde{Q}\right) \\ &= (1 - \varepsilon_0) \int f'^2(x) dx + 2 \int f^2(\alpha(y)) Q(1 - Q)(y) dy - \varepsilon_0 \int f^2(\alpha(y)) Q(y) dy \\ &\sim -0.6564\end{aligned}$$

$$L \geq L_{\varepsilon_0} \geq (1 - \varepsilon_0) \left(-\partial_x^2 - 0.845 Q_p^{7/2} \right)$$

$$0.808 \leq \mu_{\varepsilon_0} \leq 0.882$$

$$L_{\varepsilon_0}\phi_0=-(1-\varepsilon_0)\mu_0^2\phi_0-\varepsilon_0\tilde{Q}^3\phi_0$$

$$L_{\varepsilon_0}^{-1}\phi_0=-\frac{\phi_0+\varepsilon_0L_{\varepsilon_0}^{-1}(\tilde{Q}^3\phi_0)}{\mu_0^2(1-\varepsilon_0)}$$

$$\langle L_{\varepsilon_0}^{-1}\phi_0,\phi_0\rangle=-\frac{1+\varepsilon_0\langle L_{\varepsilon_0}^{-1}(\tilde{Q}^3\phi_0),\phi_0\rangle}{\mu_0^2(1-\varepsilon_0)}$$

$$u_\varepsilon=L_\varepsilon^{-1}(\tilde{Q}^3\phi_0)\in L^\infty(\mathbb{R})$$

$$-(1-\varepsilon)\partial_x^2u+2\tilde{Q}^2\left(1-\frac{\varepsilon}{2}-\tilde{Q}\right)u=\tilde{Q}^3\phi_0$$

$$-(1-\varepsilon)\langle \partial_x^2 u_\varepsilon, \partial_x^{-2} \phi_0 \rangle + 2 \left\langle \tilde{Q}^2 \left(1 - \frac{\varepsilon}{2} - \tilde{Q} \right) u_\varepsilon, \partial_x^{-2} \phi_0 \right\rangle = \langle \tilde{Q}^3 \phi_0, \partial_x^{-2} \phi_0 \rangle$$

$$\langle L_{\varepsilon_0}^{-1}\phi_0,\phi_0\rangle\leq -\frac{1-\varepsilon_0C_{\varepsilon_1}}{\mu_0^2(1-\varepsilon_0)}<0$$

$$\begin{aligned}&2\{E(\phi_1,\phi_2)-E(\tilde{H},0)\}\\&=\mu_0^2(a_2^2-a_1^2)+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle+2\int\tilde{Q}^2\tilde{H}(a_1\phi_0+u_1)^3+\frac{1}{2}\int\tilde{Q}^2(a_1\phi_0+u_1)^4\\&=\mu_0^2(a_2^2-a_1^2)+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle+\frac{1}{2}\int\tilde{Q}^2(u_1^2+4\tilde{H}u_1)u_1^2\\&\quad +2a_1\int\tilde{Q}^2\phi_0u_1^3+3a_1\int\tilde{Q}^2(2\tilde{H}+a_1\phi_0)\phi_0u_1^2+2a_1^2\int\tilde{Q}^2(3\tilde{H}+a_1\phi_0)\phi_0^2u_1\\&\quad +\frac{1}{2}a_1^3\int\tilde{Q}^2(4\tilde{H}+a_1\phi_0)\phi_0^3\\&\doteq I_1+I_2+I_3\\&\frac{1}{2}\int\tilde{Q}^2(u_1^2+4\tilde{H}u_1+(2\tilde{H})^2-(2\tilde{H})^2)u_1^2=\frac{1}{2}\int\tilde{Q}^2(u_1+2\tilde{H})^2u_1^2-2\int\tilde{Q}^2\tilde{H}^2u_1^2\\&V-2\tilde{Q}^2\tilde{H}^2=2\tilde{Q}^2(1-\tilde{Q}-\tilde{H}^2)=2\tilde{Q}^2\left(\frac{2}{3}\tilde{Q}-\tilde{Q}\right)=-\frac{2}{3}\tilde{Q}^3,\end{aligned}$$

$$I_1 = \mu_0^2(a_2^2 - a_1^2) + \|u_2\|_{L^2}^2 + \int \underbrace{\left(-\partial_x^2 u_1 - \frac{2}{3}\tilde{Q}^3 u_1\right)}_{\tilde{L}u_1} u_1 + \frac{1}{2} \int \tilde{Q}^2(u_1 + 2\tilde{H})^2 u_1^2$$

$$\begin{aligned} \left| 2a_1 \int \tilde{Q}^2 \phi_0 u_1^3 \right| &\lesssim |a_1| \|\tilde{Q}u_1\|_{L^2}^3, \\ \left| 3a_1 \int \tilde{Q}^2(2\tilde{H} + a_1\phi_0)\phi_0 u_1^2 \right| &\lesssim |a_1| \|\tilde{Q}u_1\|_{L^2}^2, \end{aligned}$$

$$\left| 2a_1^2 \int \tilde{Q}^2(3\tilde{H} + a_1\phi_0)\phi_0^2 u_1 \right| \lesssim C_\epsilon a_1^4 + \epsilon \|\tilde{Q}u_1\|_{L^2}^2$$

$$|I_2| \leq C|a_1|^3 + \epsilon \|\tilde{Q}u_1\|_{L^2}^2$$

$$|I_3| = \left| \frac{1}{2} a_1^3 \int \tilde{Q}^2(4\tilde{H} + a_1\phi_0)\phi_0^3 \right| \lesssim |a_1|^3.$$

$$-4\mu_0^2 b_+ b_- + \|u_2\|_{L^2}^2 + \langle \tilde{L}u_1, u_1 \rangle + \int \tilde{Q}^2(u_1 + 2\tilde{H})^2 u_1^2 \leq C\{E(\phi_1, \phi_2) - E(\tilde{H}, 0)\} + C|a_1|^3 + \epsilon \|u_1\|_{H_0}^2$$

$$u_1 = a(t)\tilde{H} + \tilde{u}_1, \langle \phi_0, \tilde{u}_1 \rangle = \langle \tilde{Q}^2 \tilde{H}^3, \tilde{u}_1 \rangle = 0$$

$$a(t) = \frac{\langle u_1, \tilde{Q}^2 \tilde{H}^3 \rangle}{\langle \tilde{H}, \tilde{Q}^2 \tilde{H}^3 \rangle}$$

$$\langle \tilde{L}u_1, u_1 \rangle = \langle \tilde{L}\tilde{u}_1, \tilde{u}_1 \rangle$$

$$a^2 \|\tilde{H}\|_{H_0}^2 + \|\tilde{u}_1\|_{H_0}^2 = \|u_1\|_{H_0}^2$$

$$\delta_0^2 = b_+^2(0) + b_-^2(0) + \|u_2(0)\|_{L^2}^2 + \|u_1(0)\|_{H_0}^2 + \|\tilde{Q}(u_1(0) + 2\tilde{H})u_1(0)\|_{L^2}^2$$

$$|E(\phi_1, \phi_2) - E(\tilde{H}, 0)| \lesssim \delta_0^2$$

$$\|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1 \rangle + \int \tilde{Q}^2(u_1 + 2\tilde{H})^2 u_1^2 \lesssim \delta_0^2 + b_+^2 + b_-^2 + |a_1|^3 + \epsilon \|u_1\|_{H_0}^2$$

$$\begin{aligned} \int \tilde{Q}^2(u_1 + 2\tilde{H})^2 u_1^2 &= \int \tilde{Q}^2(a(2+a)\tilde{H}^2 + 2(1+a)\tilde{H}\tilde{u}_1 + \tilde{u}_1^2)^2 \\ &\geq a^2(2+a)^2 \int \tilde{Q}^2\tilde{H}^4 + 4a(1+a)(2+a) \int \tilde{Q}^2\tilde{H}^3\tilde{u}_1 + 2a(2+a) \int \tilde{Q}^2\tilde{H}^2\tilde{u}_1^2 \\ &\geq a^2(2+a)^2 \int \tilde{Q}^2\tilde{H}^4 - C|a| \|u_1\|_{H_0}^2 \end{aligned}$$

$$\|u_2\|_2^2 + \langle \tilde{L}\tilde{u}_1, \tilde{u}_1 \rangle + a^2(2+a)^2 \lesssim \delta_0^2 + b_+^2 + b_-^2 + |a_1|^3 + \epsilon a^2 + \epsilon \|\tilde{u}_1\|_{H_0}^2 + |a| \|\tilde{u}_1\|_{H_0}^2.$$

$$a^2 + \|u_2\|_{L^2}^2 + \|\tilde{u}_1\|_{H_0}^2 \lesssim |b_+|^2 + |b_-|^2 + \delta_0^2 + O(|b_+|^3, |b_-|^3, |a|^3, \|\tilde{u}_1\|_{H_0}^3)$$

$$\langle \varepsilon_1, \phi_0 \rangle + \langle \varepsilon_2, \mu_0^{-1} \phi_0 \rangle = 0$$

$$b_-(0) = \langle \varepsilon_1, \phi_0 \rangle = -\langle \varepsilon_2, \mu_0^{-1} \phi_0 \rangle$$



$$\varepsilon_1 = b_-(0)\phi_0 + a(0)\tilde{H} + \tilde{u}_1(0), \varepsilon_2 = -b_-(0)\mu_0\phi_0 + u_2(0).$$

$$\langle \tilde{u}_1(0), \phi_0 \rangle = \langle \tilde{u}_1(0), \tilde{Q}^3 \tilde{H} \rangle = \langle u_2(0), \phi_0 \rangle = 0$$

$$(\phi, \partial_t \phi)(0) = (1 + a(0))\tilde{\mathbf{H}} + (\tilde{u}_1, u_2)(0) + b_-(0)\mathbf{Y}_- + h(\varepsilon)\mathbf{Y}_+.$$

$$h(\varepsilon) := b_+(0)$$

$$\|u_1\|_{H_0}^2 = \|\partial_x u_1\|_{L^2}^2 + \|\tilde{Q}u_1\|_{L^2}^2.$$

$$|a| \leq K^2 \delta_0, \|\tilde{u}_1\|_{H_0} \leq K^2 \delta_0 \text{ and } \|u_2\|_{L^2} \leq K^2 \delta_0$$

$$|b_-| \leq K \delta_0$$

$$|b_+| \leq K^5 \delta_0^2$$

$$(\tilde{u}_1(0), u_2(0)), b_+(0), b_-(0)$$

$$|a(0)| \leq \delta_0, \|\tilde{u}_1(0)\|_{H_0} \leq \delta_0, \|\tilde{Q}(u_1(0) + 2\tilde{H})u_1(0)\|_{L^2} \leq \delta_0, \|u_2(0)\|_{L^2} \leq \delta_0, |b_-(0)| \leq \delta_0,$$

$$|b_+(0)| \leq K^5 \delta_0$$

$$b_+(0) \in [-K^5 \delta_0^2, K^5 \delta_0^2]$$

$$b_+(0) \in [-K^5 \delta_0^2, K^5 \delta_0^2]$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \leq 3K^4 \delta_0^2$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \leq C_6(K^2 \delta_0^2 + K^{10} \delta_0^4 + \delta_0^2)$$

$$C_6 \leq \frac{1}{4} K^2, K^4 \delta_0 \leq 1$$

$$a^2 + \|\tilde{u}_1\|_{H_0}^2 + \|u_2\|_{L^2}^2 \leq \frac{3}{4} K^4 \delta_0^2$$

$$\left| \frac{d}{dt} (e^{2\mu_0 t} b_-^2) \right| \leq C_7 (K^{15} \delta_0^6 + K^6 \delta_0^3) e^{2\mu_0 t}$$

$$b_-^2 \leq \frac{C_7}{2\mu_0} (K^{15} \delta_0^6 + K^6 \delta_0^3) + \delta_0^2$$

$$\frac{C_7}{2\mu_0} K^{15} \delta_0^4 \leq \frac{1}{4} K^2, \frac{C_7}{2\mu_0} K^6 \delta_0 \leq \frac{1}{4} K^2, 1 \leq \frac{1}{4} K^4$$

$$\begin{aligned} \frac{d}{dt} (b_+^2) &\geq 2\mu_0 b_+^2 - 2C_4 |b_+| \left(b_+^2 + b_-^2 + \int \tilde{Q}^3 w_1^2 \right) \geq 2\mu_0 b_+^2 - 2C_4 |b_+| (b_+^2 + K^2 \delta_0^2 + K^4 \delta_0^2) \\ &\geq 2\mu_0 K^{10} \delta_0^4 - C_8 K^5 \delta_0 (K^{10} \delta_0^4 + K^4 \delta_0^2) \end{aligned}$$

$$C_8 K^{15} \delta_0^2 \leq \frac{1}{2} \mu_0 K^{10}, C_8 K^9 \leq \frac{1}{2} \mu_0 K^{10}$$

$$\frac{d}{dt} (b_+^2) \geq \mu_0 K^{10} \delta_0^4 > 0$$



$$b_+(0) \in [-K^5 \delta_0^2, K^5 \delta_0^2] \mapsto b_+(T) \in \{-K^5 \delta_0^2, K^5 \delta_0^2\}$$

$$\|(\phi, \partial_t \phi)(t) - (\tilde{H}, 0)\|_{H_0 \times L^2} < \delta, \|(\tilde{\phi}, \partial_t \tilde{\phi})(t) - (\tilde{H}, 0)\|_{H_0 \times L^2} < \delta$$

$$(\phi, \partial_t \phi) = (\tilde{H}, 0) + \varepsilon + b_+(0) \mathbf{Y}_+, (\tilde{\phi}, \partial_t \tilde{\phi}) = (\tilde{H}, 0) + \tilde{\varepsilon} + \tilde{b}_+(0) \mathbf{Y}_+$$

$$|b_+(0) - \tilde{b}_+(0)| \leq C \delta^{\frac{1}{2}} \|\varepsilon - \tilde{\varepsilon}\|_{H_0 \times L^2}$$

$$\|\partial_x u_1(t)\|_{L^2} + \|\partial_x \tilde{u}_1(t)\|_{L^2} + \|\tilde{Q}u_1\|_{L^2} + \|\tilde{Q}\tilde{u}_1\|_{L^2} \\ + \|u_2(t)\|_{L^2} + \|\tilde{u}_2(t)\|_{L^2} + |b_{\pm}(t)| + |\tilde{b}_{\pm}(t)| \leq C_0 \delta.$$

$$\check{a}_1 = a_1 - \tilde{a}_1, \check{a}_2 = a_2 - \tilde{a}_2, \check{b}_+ = b_+ - \tilde{b}_+, \check{b}_- = b_- - \tilde{b}_-, \\ \check{u}_1 = u_1 - \tilde{u}_1, \check{u}_2 = u_2 - \tilde{u}_2, \check{N} = N - \tilde{N}, \check{N}^\perp = N^\perp - \tilde{N}^\perp, \check{N}_0^\perp = N_0^\perp - \tilde{N}_0^\perp.$$

$$\begin{cases} \dot{\check{b}}_+(t) = \mu_0 \check{b}_+(t) - \frac{\check{N}_0}{2\mu_0} \\ \dot{\check{b}}_-(t) = -\mu_0 \check{b}_-(t) + \frac{\check{N}_0}{2\mu_0}, \end{cases} \text{ and } \begin{cases} \dot{\check{u}}_1 = \check{u}_2 \\ \dot{\check{u}}_2 = -L\check{u}_1 - \check{N}^\perp \end{cases}$$

$$|\check{N}_0| + \|\check{N}^\perp\|_{L^2} \leq C\delta \left(|\check{b}_+| + |\check{b}_-| + \|\tilde{Q}u_1\|_{L^2} \right).$$

$$|\check{N}| \lesssim \tilde{Q}^2 (|\check{a}_1|\phi_0 + |\check{u}_1|)(|a_1|\phi_0 + |\tilde{a}_1|\phi_0 + |u_1| + |\tilde{u}_1|).$$

$$\|\check{N}\|_{L^2} \leq \delta \left(|\check{a}_1| + \|\tilde{Q}u_1\|_{L^2} \right)$$

$$\beta_+ = \check{b}_+^2, \beta_- = \check{b}_-^2, \beta_c = \langle L\check{u}_1, \check{u}_1 \rangle + \langle \check{u}_2, \check{u}_2 \rangle.$$

$$\dot{\beta}_c = -2\langle \check{N}^\perp, \check{u}_2 \rangle, \dot{\beta}_+ - 2\mu_0\beta_+ = -\frac{1}{\mu_0}\check{b}_+\check{N}_0, \dot{\beta}_- + 2\mu_0\beta_- = \frac{1}{\mu_0}\check{b}_-\check{N}_0$$

$$|\dot{\beta}_c| + |\dot{\beta}_+ - 2\mu_0\beta_+| + |\dot{\beta}_- + 2\mu_0\beta_-| \leq K\delta(\beta_c + \beta_+ + \beta_-),$$

$$0 \leq K\delta(\beta_c(0) + \beta_+(0) + \beta_-(0)) < \frac{\mu_0}{10}\beta_+(0).$$

$$K\delta(\beta_c + \beta_+ + \beta_-) \leq \mu_0\beta_+.$$

$$\mu_0\beta_+ \leq 2\mu_0\beta_+ - K\delta(\beta_c + \beta_+ + \beta_-) \leq \dot{\beta}_+$$

$$\dot{\beta}_c \leq \mu_0\beta_+ \leq \dot{\beta}_+,$$

$$\beta_c(t) \leq \beta_c(0) + \beta_+(t) - \beta_+(0) \leq \beta_c(0) + \beta_+(t)$$

$$K\delta\beta_c(t) \leq K\delta(\beta_c(0) + \beta_+(t)) \leq \frac{\mu_0}{10}\beta_+(0) + K\delta\beta_+(t) \leq \frac{\mu_0}{5}\beta_+(t)$$

$$\dot{\beta}_- \leq -2\mu_0\beta_- + \mu_0\beta_+,$$

$$\beta_{-}(t) \leq e^{-2\mu_0 t} \beta_{-}(0) + \mu_0 \beta_{+}(t) e^{-2\mu_0 t} \int_0^t e^{2\mu_0 s} ds \leq \beta_{-}(0) + \frac{1}{2} \beta_{+}(t)$$

$$K\delta\beta_{-}(t) \leq K\delta(\beta_{-}(0) + \beta_{+}(t)) \leq \frac{\mu_0}{10} \beta_{+}(0) + K\delta\beta_{+}(t) \leq \frac{\mu_0}{5} \beta_{+}(t)$$

$$K\delta(\beta_c(t) + \beta_+(t) + \beta_-(t)) \leq \frac{3}{5} \mu_0 \beta_+(t).$$

$$\varepsilon = \mathbf{u}(0) + b_-(0) \mathbf{Y}_-, \tilde{\varepsilon} = \tilde{\mathbf{u}}(0) + \tilde{b}_-(0) \mathbf{Y}_-,$$

$$V(x)=2\tilde{Q}^2(x)(1-\tilde{Q}(x))$$

$$\tilde{Q}(x)=Q(\alpha^{-1}(x)) \text{ with } \alpha(x)=\frac{1}{3}(\sinh x+x).$$

$$L\phi\colon=-\partial_x^2\phi+V(x)\phi, \text{ with } V(x)=2\tilde{Q}^2(x)(1-\tilde{Q}(x)),$$

$$L\phi_0=\lambda_0\phi_0, \phi_0\in H^2(\mathbb{R}).$$

$$L\tilde{Q}=-\partial_x^2\tilde{Q}+2\tilde{Q}^3(1-\tilde{Q})=\partial_x(\tilde{Q}^2\tilde{H})+2\tilde{Q}^3(1-\tilde{Q})=-2\tilde{Q}^3\tilde{H}^2+\frac{1}{3}\tilde{Q}^4+2\tilde{Q}^3(1-\tilde{Q})=-\frac{5}{3}\tilde{Q}^4,$$

$$\langle L\tilde{Q},\tilde{Q}\rangle=-\frac{5}{3}\int\tilde{Q}^5(x)dx=-\frac{5}{3}\int Q^4(y)dy<0$$

$$\left\{ \begin{array}{l} -L_c v(y)=f(y), y\in \mathbb{R} \\ v\in H^2(\mathbb{R}), \end{array} \right.$$

$$L_c^{-1}\colon \mathcal{C}^1(\mathbb{R})\rightarrow \mathcal{C}^1(\mathbb{R})$$

$$-L\phi_0(x)=\lambda_0\phi_0(x), x\in \mathbb{R}$$

$$\int_0^\infty |V(x)|dx=2\int_0^\infty \tilde{Q}^2(x)|1-\tilde{Q}(x)|dx=2\int_0^\infty Q(s)|1-Q(s)|ds\leq \int_0^\infty Q(s)ds<+\infty$$

$$0.808\leq \mu_0\leq 0.883$$

$$\lambda_0=\inf_{\|f\|_{L^2}=1}(Lf,f)$$

$$f(x)\colon= c_0e^{-\frac{1}{2}x^2}(a_4x^4+a_2x^2+a_0),$$

$$a_4\colon=-0.0574167, a_2\colon=0.115416, a_0\colon=-0.761391.$$

$$1=\int f^2=c_0^2\int e^{-x^2}(a_4^2x^8+2a_4a_2x^6+(a_2^2+2a_4a_0)x^4+2a_2a_0x^2+a_0^2)$$

$$\int e^{-x^2}=\sqrt{\pi}, \int x^2e^{-x^2}=\frac{\sqrt{\pi}}{2}, \int x^4e^{-x^2}=\frac{3\sqrt{\pi}}{4}$$

$$\int x^6e^{-x^2}=\frac{15\sqrt{\pi}}{8}, \int x^8e^{-x^2}=\frac{105\sqrt{\pi}}{16}$$



$$(Lf, f) = \int f'^2 + 2 \int f^2 \tilde{Q}^2(1 - \tilde{Q}) = \int f'^2(x) dx + 2 \int f^2(\alpha(y)) Q(1 - Q)(y) dy \sim -0.652$$

$$Q_p=\left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)}, p=9/2$$

$$2\tilde{Q}^2(x)(1-\tilde{Q}(x))\geq -0.845Q_p^{7/2}(x)=-\frac{2.32375}{\cosh^2\left(\frac{7}{4}x\right)}$$

$$|\phi_0(x)|, |\partial_x \phi_0(x)|, |\partial_x^2 \phi_0(x)| \lesssim e^{-\frac{\sqrt{2}}{2} \mu_0 x}$$

$$\partial_x^2\phi_0=q(x)\phi_0$$

$$q(x)\geq \mu_0^2,$$

$$\frac{1}{2}\partial_x^2v(x)=(\partial_xu)^2(x)+q(x)u^2(x)\geq\mu_0^2v^2(x),$$

$$z=e^{-\sqrt{2}\mu_0x}(\partial_xv+\sqrt{2}\mu_0v)$$

$$\partial_xz=e^{-\sqrt{2}\mu_0x}(\partial_x^2v-2\mu_0^2v)\geq 0$$

$$z(x)\geq z(x_0)>0,$$

$$\partial_xv+\sqrt{2}\mu_0v\geq z(x_0)e^{\sqrt{2}\mu_0x},$$

$$\partial_x\left(e^{\sqrt{2}\mu_0x}v\right)=e^{2\sqrt{2}\mu_0x}z\leq 0\text{ for }x\geq x_r,$$

$$|\phi_0(x)|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x}.$$

$$\mu_0^2\leq q(x)\leq \mu_0^2+1,$$

$$\mu_0^2\int_{x_1}^{x_2}\phi_0\leq\partial_x\phi_0(x_2)-\partial_x\phi_0(x_1)\leq(\mu_0^2+1)\int_{x_1}^{x_2}\phi_0,$$

$$|\partial_x\phi_0(x)|\leq(\mu_0^2+1)\int_x^\infty|\phi_0|\lesssim e^{-\frac{\sqrt{2}}{2}\mu_0x}$$

$$\phi'(x)=\int_0^x(\mu_0^2+V(y))\phi_0(y)dy<0.$$

$$\phi_0'(x)=-\int_x^\infty(\mu_0^2+V(y))\phi_0(y)dy<0$$

$$h_0(x)=\frac{\phi_0'(x)}{\phi_0(x)}.$$



$$\phi_0(x) = \phi_0(0) \exp \left(\int_0^x h_0(y) dy \right)$$

$$\begin{cases} h'_0(x) + h_0^2(x) = \mu_0^2 + V(x), & \text{for } x \geq 0 \\ h_0(0) = 0 \end{cases}$$

$$h'_0(x) = -\frac{1}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy$$

$$\phi_0''(x) = (\mu_0^2 + V(x)) \phi_0$$

$$\phi_0(x) = \phi_0(0) \exp \left(\int_0^x h_0(y) dy \right)$$

$$\begin{aligned} (\phi'_0(x))^2 &= - \int_x^\infty (\mu_0^2 + V(y)) (\phi_0^2)'(y) dy \\ &= (\mu_0^2 + V(x)) \phi_0^2(x) + \int_x^\infty V'(y) \phi_0^2(y) dy \end{aligned}$$

$$h_0^2(x) = \mu_0^2 + V(x) + \frac{1}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy$$

$$\begin{aligned} L_0 &= -\partial_x^2 + 2(h_0^2 - \mu_0^2) - V, \\ U &= \partial_x - h_0, U^* = -\partial_x - h_0. \end{aligned}$$

$$\begin{aligned} &\begin{cases} V(x) \leq 0 & \text{for } 0 \leq x \leq x_0, \\ V(x) \geq 0 & \text{for } x \geq x_0 \end{cases} \quad \begin{cases} V'(x) \geq 0 & \text{for } 0 \leq x \leq x_1 \\ V'(x) \leq 0 & \text{for } x \geq x_1 \end{cases} \\ &\begin{cases} V''(x) \geq 0 & \text{for } 0 \leq x \leq x_{2,1} \\ V''(x) \leq 0 & \text{for } x_{2,1} \leq x \leq x_{2,2} \\ V''(x) \geq 0 & \text{for } x \geq x_{2,2} \end{cases} \end{aligned}$$

$$V(x) = 2\tilde{Q}^2(x)(1 - \tilde{Q}(x))$$

$$\begin{aligned} V'(x) &= 4\tilde{Q}(x)\tilde{Q}'(x) - 6\tilde{Q}^2(x)\tilde{Q}'(x) \\ &= 2\tilde{Q}^2(x)Q'(\alpha^{-1}(x))(2 - 3\tilde{Q}(x)) \end{aligned}$$

$$V(x_0) = 2\tilde{Q}^2(x_0)(1 - \tilde{Q}(x_0)) = 0$$

$$V'(x_0) = 2\tilde{Q}^2(x_0)Q'(\alpha^{-1}(x_0))(2 - 3\tilde{Q}(x_0)) = -2\tilde{Q}^3(x_0)Q'(\alpha^{-1}(x_0)) > 0$$

$$\begin{aligned} V''(x) &= 8\tilde{Q}^2(x)Q'^2(\alpha^{-1}(x)) + 4\tilde{Q}^3(x)Q''(\alpha^{-1}(x)) - 18\tilde{Q}^3(x)Q'^2(\alpha^{-1}(x)) \\ &\quad - 6\tilde{Q}^4(x)Q''(\alpha^{-1}(x)) \end{aligned}$$

$$V''(x) = 2\tilde{Q}^4(x) \left(6 - \frac{50}{3}\tilde{Q}(x) + 9\tilde{Q}^2(x) \right)$$

$$V''(0) = 2\left(\frac{3}{2}\right)^4 \left(6 - \frac{50}{3}\left(\frac{3}{2}\right) + 9\left(\frac{3}{2}\right)^2 \right) \sim 12.65, V''(x_0) = -\frac{5}{3} < 0$$



$$V''(x_1) = 2\left(\frac{2}{3}\right)^4 \left(6 - \frac{50}{3}\left(\frac{2}{3}\right) + 9\left(\frac{2}{3}\right)^2\right) \sim -0.44$$

$$\tilde{\mu}_0 := \sqrt{\mu_0^2 + \max_{y>0} V(y)}, \quad \max_{y>0} V(y) = \frac{8}{27},$$

$$-\tilde{\mu}_0 \leq h_0(x)$$

$$h_0(x) \leq -\mu_0$$

$$\lim_{x \rightarrow +\infty} h_0(x) = -\mu_0$$

$$h'_0(0) = \mu_0^2 + V(0) = \mu_0^2 - \frac{9}{4} \sim -1.59.$$

$$h_0^2(x) \leq h_0^2(\bar{x}) = \mu_0^2 + V(\bar{x}) \leq \mu_0^2 + \max_{y>0} V(y) = \tilde{\mu}_0^2$$

$$\begin{aligned} \mu_0^2 - h_0^2(x) &= -\frac{1}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy - V(x) \\ &\leq -\int_x^\infty V'(y) dy - V(x) = 0 \end{aligned}$$

$$\begin{aligned} \mu_0^2 - h_0^2(x) &= -\frac{1}{\phi_0^2(x)} \int_x^{x_1} V'(y) \phi_0^2(y) dy + \frac{1}{\phi_0^2(x)} \int_{x_1}^\infty |V'(y)| \phi_0^2(y) dy - V(x) \\ &\leq -\frac{\phi_0^2(x_1)}{\phi_0^2(x)} \int_x^{x_1} V'(y) dy - \frac{\phi_0^2(x_1)}{\phi_0^2(x)} \int_{x_1}^\infty V'(y) dy - V(x) \\ &= -\left(1 - \frac{\phi_0^2(x_1)}{\phi_0^2(x)}\right) V(x) \leq 0 \end{aligned}$$

$$|h_0^2(x) - \mu_0^2| \leq \frac{1}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy + |V(x)| \leq 2|V(x)|$$

$$(\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x) \leq h_0(x) \leq \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x),$$

$$\mu_0^2 x - R(x) \leq h_0 \quad \text{for all } x > 0$$

$$R(x) := 2\ln\left(\frac{3}{2}\right) - 2\ln(\tilde{Q}) + 2\tilde{Q}\tilde{H} + \frac{\mu_0^2 - \tilde{\mu}_0^2}{2}x^2$$

$$\begin{cases} h'_1 = \mu_0^2 + V \\ h_1(0) = 0 \end{cases}$$

$$\begin{aligned} \int_0^x V(y) dy &= 2 \int_0^x \tilde{Q}^2(y)(1 - \tilde{Q}(y)) dy = 2 \int_0^{\alpha^{-1}(x)} Q(s)(1 - Q(s)) ds \\ &= 2 \int_0^{\alpha^{-1}(x)} Q''(s) ds = 2Q'(\alpha^{-1}(x)) = -2\tilde{Q}(x)\tilde{H}(x) \end{aligned}$$

$$h_1(x) = \int_0^x (\mu_0^2 + V(y)) dy = \mu_0^2 x + \int_0^x V(y) dy = \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x)$$



$$h_0(x) \leq \mu_0^2 x - 2\tilde{Q}(x)\tilde{H}(x)$$

$$\begin{cases} h_2' = \mu_0^2 - \tilde{\mu}_0^2 + V \\ h_2(0) = 0 \end{cases}$$

$$\begin{aligned} h_2(x) &= \int_0^x (\mu_0^2 - \tilde{\mu}_0^2 + V(y)) dy = (\mu_0^2 - \tilde{\mu}_0^2)x + \int_0^x V(y) dy \\ &= (\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x) \end{aligned}$$

$$h_2'(x) \leq \mu_0^2 + V(x) - h_0^2(x) = h_0'(x)$$

$$(\mu_0^2 - \tilde{\mu}_0^2)x - 2\tilde{Q}(x)\tilde{H}(x) \leq h_0(x)$$

$$\begin{cases} h_3' = \mu_0^2 - h_2^2 + V \\ h_3(0) = 0 \end{cases}$$

$$h_3(x) = \mu_0^2 x - 2\ln\left(\frac{3}{2}\right) + 2\ln(\tilde{Q}) - 2\tilde{Q}\tilde{H} - \frac{\mu_0^2 - \tilde{\mu}_0^2}{2}x^2$$

$$\frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{H}(x) \leq h_0(x)$$

$$(\mu_0^2 - \tilde{\mu}_0^2)(x - x_0) - \tilde{\mu}_0 \leq h_0(x) \leq -\frac{\mu_0}{x_0}x.$$

$$g(x) = h_0(x) - \frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{H}(x).$$

$$g(x) = g'(\xi)x$$

$$h_0^2 \leq \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{Q}^2$$

$$h_0^2 \leq \mu_0^4 x^2 - 2\mu_0^2 x R(x) + R^2(x)$$

$$\mu_0^4 x^2 - 2\mu_0^2 x R(x) + R^2(x) \leq \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\tilde{Q}^2$$

$$\mu_0^4 x^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \leq V + \tilde{Q}^2 - R^2$$

$$\mu_0^4 x^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \leq G(\alpha^{-1}(x))$$

$$G(s) := (0.883)^4 \alpha(s)^2 + \left(\frac{4}{9}\tilde{Q}^2 - 2\alpha(s)R(\alpha(s)) - 1\right)(0.808)^2$$

$$G(s) \leq V(\alpha(s)) + Q^2(s) - R^2(\alpha(s))$$

$$h_0'(x_0) = \mu_0^2 - h_0^2(x_0) \leq \mu_0^2 - \mu_0^2 = 0.$$

$$\mu_0^2 - \tilde{\mu}_0^2 + V(x) \leq \mu_0^2 - h_0^2 + V = h_0'(x) \leq 0.$$



$$\begin{aligned}
h_0'' &= V' - 2h_0 h_0' \geq V' - 2(\mu_0^2 x - R(x))h_0' \\
&\geq V' - 2(\mu_0^2 x - R(x))(\mu_0^2 - \tilde{\mu}_0^2 + V) \geq V' - 2(0.808^2 x - R(x))\left(-\frac{8}{27} + V\right) =: j_1(\alpha^{-1}(x))
\end{aligned}$$

$$h_0''(x) \geq j_1(\alpha^{-1}(x)) > 0$$

$$\begin{aligned}
h_0(x) &\geq h_0'(x_0)(x - x_1) + h_0(x_0) \\
&= (\mu_0^2 - h_0^2(x_0))(x - x_0) + h_0(x_0) \geq (\mu_0^2 - \tilde{\mu}_0^2)(x - x_0) - \tilde{\mu}_0
\end{aligned}$$

$$\begin{aligned}
h_0'' &= V' - 2h_0 h_0' \\
&\geq V' - 2\left(\frac{4\mu_0^2 - 9}{3}\right)\left(\mu_0^2 + V - \left(\frac{4\mu_0^2 - 9}{3}\right)^2 \tilde{H}^2\right)\tilde{H} \\
&\geq V' - 2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 + V - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 \tilde{H}^2\right)\tilde{H} =: j_2(\alpha^{-1}(x))
\end{aligned}$$

$$\begin{aligned}
j_2(s) &= 2Q^3 H(3Q - 2) \\
&\quad - 2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 + \frac{2}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)^2 Q + 2Q^2(1 - Q)\right) \\
&= -2\left(\frac{4(0.808)^2 - 9}{3}\right)\left(\mu_0^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 - \frac{4}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)\right)^3 Q \\
&\quad - 4\left(\frac{4(0.808)^2 - 9}{3}\right)Q^2 + 4\left(\frac{4(0.808)^2 - 9}{3} - H\right)Q^3 + 6HQ^4.
\end{aligned}$$

$$h_0(x) \leq \frac{h_0(x_0)}{x_0}x \leq -\frac{\mu_0}{x_0}x.$$

$$V_0(x) = V(x) + \frac{2}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy$$

$$V_0(x) = \frac{1}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy - \frac{1}{\phi_0^2(x)} \int_x^\infty V(y) (\phi_0^2(y))' dy$$

$$= \frac{1}{\phi_0^2(x)} \int_x^\infty [V'(y) - 2h_0(y)V(y)] \phi_0^2(y) dy$$

$$V_0(x) = \frac{1}{\phi_0^2(x)} \int_x^\infty K(y) \phi_0^2(y) dy$$

$$K(y) := V'(y) - 2h_0(y)V(y)$$

$$\begin{aligned}
K(y) &= V' - 2h_0 V \geq V' - 2(\mu_0^2 x - R)V \geq V' - 2(0.808^2 x - R)V \\
&= 2\tilde{Q}^2 \left[2(0.808^2 x - R) - 2\left((0.808^2 x - R) + \tilde{H}\right)\tilde{Q} + 3\tilde{H}\tilde{Q}^2 \right] =: 2\tilde{Q}^2 k_1(\alpha^{-1}(y)).
\end{aligned}$$



$$\begin{aligned}
K(y) &\geq V' + \frac{8}{3} \left(\mu_0^2 - \frac{9}{4} \right) \tilde{H} V \\
&= 2\tilde{Q}^2 \tilde{H} \left[\frac{2}{3} (4\mu_0^2 - 9) + 2 \left(1 - \frac{4}{3} \mu_0^2 \right) \tilde{Q} + 3\tilde{Q}^2 \right] \\
&\geq 4\tilde{Q}^2 \tilde{H} \left[\frac{1}{3} (4(0.808)^2 - 9) + \left(1 - \frac{4}{3} (0.883)^2 \right) \tilde{Q} + \frac{3}{2} \tilde{Q}^2 \right] =: 4\tilde{Q}^2 \tilde{H} k_2(\alpha^{-1}(y)),
\end{aligned}$$

$$L = (-\partial_x - h_0)(\partial_x + h_0) - \mu_0^2 = U^*U - \mu_0^2$$

$$L_0 = (\partial_x + h_0)(-\partial_x - h_0) - \mu_0^2 = UU^* - \mu_0^2$$

$$UL = L_0U$$

$$\sigma_p(L_0) = \sigma_p(L) \setminus \{-\mu_0^2\}$$

$$\begin{aligned}
V'_0(x) &= 4h_0(x)h'_0(x) - V'(x) \\
&= -\frac{2h_0(x)}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy - \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \frac{V'(y)}{h_0(y)} (\phi_0^2(y))' dy - V'(x) \\
&= -\frac{2h_0(x)}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy + \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(\frac{V'(y)}{h_0(y)} \right)' \phi_0^2(y)dy \\
&\quad - \frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h_0(y)} \Big|_x^\infty - V'(x) \\
&= \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(\frac{V'(y)}{h_0(y)} - 2V(y) \right)' \phi_0^2(y)dy = \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(\frac{V''(y)}{h_0(y)} - \frac{V'(y)h'_0(y)}{h_0^2(y)} - 2V'(y) \right) \phi_0^2(y)dy \\
&= \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty (V''(y)h_0(y) - V'(y)h'_0(y) - 2V'(y)h_0^2(y)) \left(\frac{\phi_0}{h_0} \right)^2(y)dy
\end{aligned}$$

$$V'_0(x) = \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty I(y) \left(\frac{\phi_0}{h_0} \right)^2(y)dy$$

$$I(y) = V''(y)h_0(y) - V'(y)(h_0^2(y) + \mu_0^2 + V(y)).$$

$$\begin{aligned}
I(y) &= -V''(y)|h_0(y)| + |V'(y)|(h_0^2(y) + \mu_0^2 + V(y)) \\
&\geq -\tilde{\mu}_0 V''(y) + |V'(y)|(2\mu_0^2 + V(y)) \geq -1.038V''(y) + (2 \cdot 0.808^2 + V(y))|V'(y)|
\end{aligned}$$

$$\begin{aligned}
I(\alpha(s)) &\geq -2.075Q^4 \left(6 - \frac{50}{3}Q + 9Q^2 \right) + 4(2 - 3Q)(0.652 + Q^2 - Q^3)Q^3H \\
&= 2Q^3 \left[2.611 - 6(1.038 + 0.652H)Q + \left(\frac{50}{3}1.038 + 4H \right) Q^2 - (9.342 + 20H)Q^3 + 6Q^4H \right] =: 2Q^3 i_1(s)
\end{aligned}$$

$$\begin{aligned}
I(y) &= |V''(y)h_0| + |V'(y)|(h_0^2(y) + \mu_0^2 + V(y)) \geq \mu_0|V''(y)| + |V'(y)|(2\mu_0^2 + V(y)) \\
&\geq 0.808|V''(y)| + |V'(y)|(2 \cdot 0.808^2 + V(y))
\end{aligned}$$

$$\begin{aligned}
I(\alpha(s)) &= -2\mu_0Q^4 \left(6 - \frac{50}{3}Q + 9Q^2 \right) + 4(2 - 3Q)(0.808^2 + Q^2 - Q^3)Q^3H \\
&= 2Q^3H \left[4 \cdot 0.808^2H - 6 \cdot 0.808(1 + 0.808H)Q + \left(\frac{50}{3} \cdot 0.808 + 4H \right) Q^2 \right. \\
&\quad \left. - (10H + 9 \cdot 0.808)Q^3 + 6HQ^4 \right] =: 2Q^3H i_2(s)
\end{aligned}$$



$$I(y) = |V''(y)h_0(y)| - V'(y)(h_0^2(y) + \mu_0^2 + V(y)) \\ \geq \mu_0|V''(y)| - V'(y)(\tilde{\mu}_0^2 + \mu_0^2 + V(y)) \geq 0.808|V''(y)| - V'(y)(1.959 + V(y))$$

$$I(\alpha(s)) \geq -2 \cdot 0.808Q^4 \left(6 - \frac{50}{3}Q + 9Q^2\right) + 2(2 - 3Q)(1.959 + 2Q^2 - 2Q^3)Q^3H \\ = 2Q^3 \left[3.842H - 3(1.959 + 1.616H)Q + \left(\frac{50}{3} \cdot 0.808 + 4H\right)Q^2 - (7.272 + 10H)Q^3 + 6HQ^4\right] =: 2Q^3i_3(s)$$

$$(\alpha^{-1}(x_0), \alpha^{-1}(x_1)), (\alpha^{-1}(x_1), \alpha^{-1}(x_{2,2}))$$

$$(\alpha^{-1}(x_{2,2}), \infty)$$

$$(0, \alpha^{-1}(x_{2,1})) \text{ and } (\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$$

$$I(y) = |V''(y)h_0(y)| + |V'(y)(h_0^2(y) + \mu_0^2 + V(y))| \geq 0.$$

$$h_0^2(y) + \mu_0^2 + V(y) \geq 0$$

$$h_0^2(y) + \mu_0^2 + V(y) \geq \left(\frac{8}{27}(x - x_0) + \tilde{\mu}_0\right)^2 + \mu_0^2 + V(y) \\ \geq \left(\frac{8}{27}(x - x_0) + 0.974\right)^2 + 0.652 + V(y).$$

$$I(y) \geq -\frac{\mu_0}{x_0}yV''(y) - V'(y)\left(\left(\frac{8}{27}(y - x_0) + 0.974\right)^2 + 0.652 + V(y)\right)$$

$$I(\alpha(s)) \geq -2\frac{0.808}{x_0}\alpha(s)Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) \\ + 2(2 - 3Q)Q^3H\left(\left(\frac{8}{27}(\alpha(s) - x_0) + 0.974\right)^2 + 0.652 + 2Q^2(1 - Q)\right) =: m(s),$$

$$s \in (\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$$

$$h_0^2(y) + \mu_0^2 + V(y) \leq \left(\mu_0^2 - \frac{9}{4}\right)y^2 + \mu_0^2 + V(y) \leq \left(0.652 - \frac{9}{4}\right)y^2 + 0.78 + V(y) \leq 0,$$

$$I(y) = V''(y)|h_0(y)| + V'(y)|h_0^2(y) + \mu_0^2 + V(y)|.$$

$$I(y) \geq \left(0.652 - \frac{9}{4}\right)yV''(y) - V'(y)\left(\left(0.652 - \frac{9}{4}\right)y^2 + 0.652 + V(y)\right)$$

$$I(\alpha(s)) \geq 2\left(0.652 - \frac{9}{4}\right)\alpha(s)Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) \\ + 2Q^3H(2 - 3Q)\left(\left(0.652 - \frac{9}{4}\right)\alpha(s)^2 + 0.652 + 2Q^2(1 - Q)\right) =: \hat{m}(s)$$

$$V'_0(x) = \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty (V''(y)h_0(y) - V'(y)h'_0(y) - 2V'(y)h_0^2(y)) \left(\frac{\phi_0}{h_0}\right)^2(y) dy \leq 0$$



$$3V'(x) \leq V'_0(x) \leq \frac{1}{2}V'(x),$$

$$\begin{aligned} |V'_0(x)| &\leq \left| \frac{4h_0(x)}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy \right| + |V'(x)| \\ &= \left| \frac{4h_0(x)}{\phi_0^2(x)} \int_x^\infty \frac{V'(y)}{2h_0(y)} (\phi_0^2(y))' dy \right| + |V'(x)| \leq \left| \frac{2\mu_0^{-1}h_0(x)V'(x)}{\phi_0^2(x)} \int_x^\infty (\phi_0^2(y))' dy \right| + |V'(x)| \leq 3|V'(x)| \end{aligned}$$

$$\begin{aligned} V'_0(x) &= 4h_0(x)h'_0(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\ &= h_0(x)h'_0(x) + 3h_0(x)h'_0(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\ &= -\frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy - \frac{3}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \frac{V'(y)}{h_0(y)} (\phi_0^2(y))' dy - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\ &= -\frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty V'(y) \phi_0^2(y) dy + \frac{3}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(\frac{V'(y)}{h_0(y)} \right)' \phi_0^2(y) dy \\ &\quad - \frac{3}{2} \frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h(y)} \Big|_x^\infty - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\ &= \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(\frac{3}{2} \frac{V'(y)}{h_0(y)} - V(y) \right)' \phi_0^2(y) dy + \frac{1}{2}V'(x) \\ &= \frac{1}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(3 \frac{V''(y)}{h_0(y)} - 3 \frac{V'(y)h'_0(y)}{h_0^2(y)} - 2V'(y) \right) \phi_0^2(y) dy + \frac{1}{2}V'(x) \\ &= \frac{1}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty (3V''(y)h_0(y) - 3V'(y)h'_0(y) - 2V'(y)h_0^2(y)) \left(\frac{\phi_0}{h_0} \right)^2(y) dy + \frac{1}{2}V'(x) \end{aligned}$$

$$V'_0(x) = \frac{1}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty J(y) \left(\frac{\phi_0}{h_0} \right)^2(y) dy + \frac{1}{2}V'(x)$$

$$J(y) = 3V''(y)h_0(y) - V'(y)(3\mu_0^2 - h_0^2(y) + 3V(y)).$$

$$3\mu_0^2 - h_0^2(y) + 3V(y) \geq 3\mu_0^2 - \tilde{\mu}_0^2 + 3V(y) = 2\mu_0^2 - \frac{8}{27} > 0.$$

$$J(y) \geq -3V''(y) - (3\mu_0^2 - \tilde{\mu}_0^2 + 3V(y))V'(y) \geq -3V''(y) - (1.3 + 3V(y))V'(y) \geq 0$$

$$V'_0(x) = \underbrace{\frac{1}{2} \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty J(y) \left(\frac{\phi_0}{h_0} \right)^2(y) dy}_{\leq 0} + \underbrace{\frac{1}{2}V'(x)}_{\geq 0} \leq \frac{1}{2}V'(x) \leq 0$$

$$E \geq R_1 + R_2 + R_3 + J_1 + J_2,$$

$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2).$$

$$Q\phi^m=0, Q\psi_m=-\frac{i}{2}\epsilon_{mnp}[\phi^n,\phi^p], Qf=-i[\phi^m,\psi_m].$$

$$\mathcal{I} = \mathcal{I}_{\text{s.c.d.}} \cdot \mathcal{I}_{w_2} \cdot \sum_i (-1)^{F_i} \chi(\mathbf{R}_i) t^{j_i}.$$



$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2) = 24 + 2n$$

$$\mathcal{O}_{24}^2 = \epsilon^{pqr} v^m{}_q v^n{}_r \text{tr}(\psi_{(m} \psi_n \psi_{p)})$$

$$v^m{}_n = \text{tr}(\phi^m \psi_n) - \frac{1}{3} \delta_n^m \text{tr}(\phi^p \psi_p)$$

$$u_2{}^{ij} \equiv \text{tr}(\phi^{(i} \phi^{j)}), \quad u_3{}^{ijk} \equiv \text{tr}(\phi^{(i} \phi^j \phi^{k)})$$

$$v_2{}^i{}_j \equiv \text{tr}(\phi^i \psi_j) - \frac{1}{3} \delta_j^i \text{tr}(\phi^a \psi_a), \quad v_3{}^{ij}{}_k \equiv \text{tr}(\phi^{(i} \phi^{j)} \psi_k) - \frac{1}{4} \delta_k^i \text{tr}(\phi^{(j} \phi^{a)} \psi_a) - \frac{1}{4} \delta_k^j \text{tr}(\phi^{(i} \phi^{a)} \psi_a)$$

$$w_2{}^i \equiv \text{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), \quad w_3{}^{ij} \equiv \text{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$\sum_a f_a(g_I) R_a(g_I) = 0$$

$$\sum_a f_a(g_I) r_a$$

$$\begin{aligned}\phi^m &= \phi^m{}_1 T^1 + \phi^m{}_2 T^2 + \phi^m{}_3 T^3 + \sqrt{3} \hat{\phi}^m T^8 \\ \psi_m &= \psi_{m_1} T^1 + \psi_{m_2} T^2 + \psi_{m_3} T^3 \\ f &= f_1 T^1 + f_2 T^2 + f_3 T^3\end{aligned}$$

$$\phi^m = \begin{pmatrix} \phi^m|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix} + \hat{\phi}^m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \psi_m = \begin{pmatrix} \psi_m|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} f|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{tr}[\phi^{(i}\phi^j\phi^{k)}] \stackrel{SU(2)\times U(1)}{\rightarrow} \text{tr}[\phi^{(i}\phi^j\phi^{k)}]_{SU(2)} + 3\hat{\phi}^{(i}\text{tr}[\phi^{j}\phi^{k)}]_{SU(2)} - 6\hat{\phi}^{(i}\hat{\phi}^j\hat{\phi}^{k)}$$

$$\mathcal{O}^3[R] \stackrel{SU(2)\times U(1)}{\rightarrow} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2]$$

$$Q\Psi = \begin{pmatrix} Q\Psi|_{SU(2)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q\Psi = \begin{pmatrix} Q\Psi|_{SU(2)} & 0 \\ 0 & Q\Psi|_{U(1)} = 0 \end{pmatrix}$$

$$\begin{aligned} & \mathcal{QO}^3[R] \stackrel{SU(2)\times U(1)}{\rightarrow} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2] \\ \Rightarrow \mathcal{O}^3[R] & \stackrel{SU(2)\times U(1)}{\rightarrow} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2], \text{ where } \mathcal{O}^2[R_2] = Q\mathcal{O}^2[R_2]. \end{aligned}$$

$$([1,1] \otimes [1,1])_A = [3,0] + [0,3] + [1,1].$$

$$[2,0] \otimes ([1,1] \otimes [1,1])_A \otimes [1,3] = \cdots + (7 \times [3,0]) + \cdots$$

$$([2,1] \otimes [2,1])_A \otimes [1,3] = \cdots + (4 \times [3,0]) + \cdots$$



$$0 = 24s_1 - \frac{16}{15}s_3 + \frac{4}{15}s_4 - \frac{32}{15}s_5 + 16s_9 + 85s_{10} + 5s_{11} - 40s_{12} - \frac{5}{3}s_{13} + \frac{5}{3}s_{14} - 40s_{15} \\ + \frac{125}{7}s_{17} + 31s_{18} + \frac{24}{7}s_{19} + \frac{543}{7}s_{20} + \frac{8}{5}s_{22} + \frac{16}{15}s_{23} + \frac{251}{7}s_{30} - s_{31} - 9s_{32} + \frac{174}{7}s_{33} - \frac{17}{2}s_{34}$$

$$\mathcal{O}^{ijk} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \tilde{\mathcal{F}}_{(abc)}^{(ijk)}$$

$$\tilde{\mathcal{F}}_{(abc)}^{(ijk)} = \mathcal{T}_{(abc)}^{(ijk)} + \delta_a^i \mathcal{T}_{(bc)}^{(jk)} + \delta_{(ab)}^{(ij)} \mathcal{T}_c^k + \delta_{(abc)}^{(ijk)} \mathcal{T}$$

$$[3,0] \otimes [0,3] = [3,3] \oplus [2,2] \oplus [1,1] \oplus [0,0]$$

$$\mathcal{O}^{ijk} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \mathcal{T} + \hat{\phi}^a \hat{\phi}^i (\hat{\phi}^j \mathcal{T}_a^k) + \hat{\phi}^a \hat{\phi}^b \hat{\phi}^i (\mathcal{T}_{ab}^{jk}) + \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \mathcal{T}_{abc}^{ijk}$$

$$(\mathcal{O}_{30}^3)^{ijk} = 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c_dv^{lk)}_{a_1}v^d_{b_1}(r_{12}^{(0,2)})_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_bv^b{}_cv^c_{a_1}(r_{12}^{(0,2)})_{a_2}^{lk)} \\ + \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(r_{12}^{(0,2)})_b^k) - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}{}_bv^k)_{a_1}v^b{}_c(r_{12}^{(0,2)})_{a_2}^c + 16\epsilon^{a_1a_2(i}v^{j|}{}_dv^b{}_{a_1}v^d{}_c(r_{12}^{(0,2)})_{a_2b}^{lk)} \\ + 85\epsilon^{a_1a_2(i}u^{bc}v_{a_1}v^{j|}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + 5\epsilon^{a_1a_2(i}u^{bc}v^d{}_bv^{j|}{}_a(r_{14}^{(0,3)})_{a_2cd}^k) - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^d{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} \\ - \frac{5}{3}\epsilon^{a_1a_2(i}u^{cd}v^b_{a_1}v^{j|}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b_{a_1}v^d{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^{lk)}{}_d(r_{14}^{(0,3)})_{a_2bc}^d \\ + \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_bv^{cd}{}_{a_1}(r_{14}^{(0,3)})_{a_2cd}^b \\ + \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(r_{14}^{(0,3)})_{a_3bc}^{lk)} + \frac{8}{5}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{j|k)}{}_b(r_{16}^{(0,2)})_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^k{}^b{}_{a_2}(r_{16}^{(0,3)})_b \\ + \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(r_{16}^{(0,3)})_{a_3b_1b_2}^{jk)} - \epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{b_1b_2}{}_{a_2}(r_{16}^{(0,3)})_{a_3b_1b_2}^{j|k)} - 9\epsilon^{a_1a_2(i}v^{j|}{}_av^{bc}{}_d(r_{16}^{(0,3)})_{a_2bc}^{d|k)} \\ + \frac{174}{7}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{c|j}{}_d(r_{16}^{(0,3)})_{a_2bc}^k) - \frac{17}{2}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{cd}{}_{a_2}(r_{16}^{(0,3)})_{bcd}^{j|k)}$$

$$(\mathcal{O}_{30}^3)^{ijk} = 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c_dv^{lk)}_{a_1}v^d_{b_1}(r_{12}^{(0,2)})_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_bv^b{}_cv^c_{a_1}(r_{12}^{(0,2)})_{a_2}^{lk)} \\ + \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(r_{12}^{(0,2)})_b^k) - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}{}_bv^k)_{a_1}v^b{}_c(r_{12}^{(0,2)})_{a_2}^c + 16\epsilon^{a_1a_2(i}v^{j|}{}_dv^b{}_{a_1}v^d{}_c(r_{12}^{(0,2)})_{a_2b}^{lk)} \\ + 85\epsilon^{a_1a_2(i}u^{bc}v_{a_1}v^{j|}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + 5\epsilon^{a_1a_2(i}u^{bc}v^d{}_bv^{j|}{}_a(r_{14}^{(0,3)})_{a_2cd}^k) - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^d{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} \\ - \frac{5}{3}\epsilon^{a_1a_2(i}u^{cd}v^b_{a_1}v^{j|}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b_{a_1}v^d{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c_{a_1}v^{lk)}{}_d(r_{14}^{(0,3)})_{a_2bc}^d \\ + \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(r_{14}^{(0,3)})_{a_2cd}^{lk)} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(r_{14}^{(0,3)})_{a_2cd}^k) + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_bv^{cd}{}_{a_1}(r_{14}^{(0,3)})_{a_2cd}^b \\ + \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(r_{14}^{(0,3)})_{a_3bc}^{lk)} + \frac{8}{5}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{j|k)}{}_b(r_{16}^{(0,2)})_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^k{}^b{}_{a_2}(r_{16}^{(0,3)})_b \\ + \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(r_{16}^{(0,3)})_{a_3b_1b_2}^{jk)} - \epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{b_1b_2}{}_{a_2}(r_{16}^{(0,3)})_{a_3b_1b_2}^{j|k)} - 9\epsilon^{a_1a_2(i}v^{j|}{}_av^{bc}{}_d(r_{16}^{(0,3)})_{a_2bc}^{d|k)} \\ + \frac{174}{7}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{c|j}{}_d(r_{16}^{(0,3)})_{a_2bc}^k) - \frac{17}{2}\epsilon^{a_1a_2(i}v^b{}_{a_1}v^{cd}{}_{a_2}(r_{16}^{(0,3)})_{bcd}^{j|k)}$$

$$\mathcal{T} = -\frac{648}{5}\mathcal{O}_{24}^2$$





$$\begin{aligned}
(\mathcal{O}_{32}^3)^{ijkl} = & 162576\epsilon^{a_1a_2}(i\epsilon^{j|a_3a_4}v^{[k}_{a_1}v^{l]}_{a_5}v^{a_5a_6}_{a_3}(r_{12}^{(0,1)})_{a_2a_4a_6} - 3240\epsilon^{a_1a_2}(i\epsilon^{j|a_3a_4}v^{[k}_{a_1}v^{a_5}_{a_2}v^{l]a_6}_{a_3}(r_{12}^{(0,1)})_{a_4a_5a_6} \\
& + 19080\epsilon^{a_1a_2}(i\epsilon^{j|a_3a_4}v^{[k}_{a_1}v^{a_5}_{a_3}v^{l]a_6}_{a_2}(r_{12}^{(0,1)})_{a_4a_5a_6} - 183096\epsilon^{a_1a_2}(i\epsilon^{j|a_3a_4}v^{[k}_{a_1}v^{a_5}_{a_6}v^{l]a_6}_{a_3}(r_{12}^{(0,1)})_{a_2a_4a_5} \\
& - 283500\epsilon^{a_1a_2}(i\epsilon^{j|a_3a_4}v^{a_5}_{a_1}v^{a_6}_{a_2}v^{[kl]}_{a_3}(r_{12}^{(0,1)})_{a_4a_5a_6} + 126000\epsilon^{a_1a_2}(i v^j_{a_1}v^k_{a_2}v^{l]a_3}_{a_4}(r_{12}^{(0,1)})_{a_3} \\
& - 16800\epsilon^{a_1a_2}(i v^j_{a_1}v^k_{a_3}v^{l]a_3}_{a_4}(r_{12}^{(0,1)})_{a_2} + 75600\epsilon^{a_1a_2}(i v^j_{a_1}v^k_{a_4}v^{l]a_3}_{a_2}(r_{12}^{(0,1)})_{a_3} \\
& - 46200\epsilon^{a_1a_2}(i v^j_{a_1}v^{a_3}_{a_4}v^{[kl]}_{a_2}(r_{12}^{(0,1)})_{a_3}^{a_4} - 21000\epsilon^{a_1a_2}(i v^j_{a_1}v^{a_3}_{a_4}v^{[kl]}_{a_3}(r_{12}^{(0,1)})_{a_2}^{a_4} \\
& + 75600\epsilon^{a_1a_2}(i v^j_{a_4}v^{a_3}_{a_1}v^{[kl]}_{a_2}(r_{12}^{(0,1)})_{a_3}^{a_4} + 75600\epsilon^{a_1a_2}(i v^j_{a_1}v^{a_3}_{a_4}v^{[k|a_4}_{a_2}(r_{12}^{(0,1)})_{a_3}^{l]} \\
& - 37800\epsilon^{a_1a_2}(i|v^{a_3}_{a_1}v^{a_4}_{a_2}v^{[jk}_{a_3}(r_{12}^{(0,1)})_{a_4}^l - 14000\epsilon^{a_1a_2}(i v^j_{a_1}v^k_{a_2}v^{a_3a_4}_{a_5}(r_{12}^{(0,1)})_{a_3a_4}^{l]a_5} \\
& + 14000\epsilon^{a_1a_2}(i v^j_{a_1}v^{a_3}_{a_4}v^{[kl]}_{a_5}(r_{12}^{(0,1)})_{a_2a_3}^{a_4a_5} - 28000\epsilon^{a_1a_2}(i v^j_{a_1}v^{a_3}_{a_5}v^{a_4|k}_{a_2}(r_{12}^{(0,1)})_{a_3a_4}^{l]a_5} \\
& + 14000\epsilon^{a_1a_2}(i|v^{a_3}_{a_1}v^{a_4}_{a_5}v^{[jk}_{a_2}(r_{12}^{(0,1)})_{a_3a_4}^{l]a_5} + 31500\epsilon^{a_1a_2a_3}v^{(i}_{a_1}v^j_{a_2}v^k_{a_3}(r_{14}^{(0,1)})_{a_4}^{l]} \\
& - 31500\epsilon^{a_1a_2}(i\epsilon^{a_3a_4}|j v^k_{a_1}v^l_{a_2}v^{a_5}_{a_3}(r_{14}^{(0,1)})_{a_4a_5} + 315000\epsilon^{a_1a_2}(i|u^{a_3a_4a_5}v^{[j}_{a_1}v^k_{a_2}(r_{14}^{(0,2)})_{a_3a_4a_5}^l] \\
& - 1055700\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_1}v^{a_4a_5}_{a_2}(r_{14}^{(0,2)})_{a_3a_4a_5}^l] + 363660\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_1}v^{a_4a_5}_{a_3}(r_{14}^{(0,2)})_{a_2a_4a_5}^l] \\
& - 596400\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_4}v^{a_4a_5}_{a_1}(r_{14}^{(0,2)})_{a_2a_3a_5}^l] + 1905300\epsilon^{a_1a_2}(i u^j|a_3v^{a_4}_{a_1}v^{a_5|k}_{a_2}(r_{14}^{(0,2)})_{a_3a_4a_5}^l] \\
& + 890400\epsilon^{a_1a_2}(i u^j|a_3v^{a_4}_{a_1}v^{a_5|k}_{a_3}(r_{14}^{(0,2)})_{a_2a_4a_5}^l] + 82500\epsilon^{a_1a_2}(i u^j|a_3v^{a_4}_{a_5}v^{a_5|k}_{a_1}(r_{14}^{(0,2)})_{a_2a_3a_4}^l] \\
& + 2288100\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^k_{a_2}(r_{14}^{(0,2)})_{a_3a_4a_5}^l] - 247560\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^k_{a_3}(r_{14}^{(0,2)})_{a_2a_4a_5}^l] \\
& + 1506900\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_5}v^k_{a_1}(r_{14}^{(0,2)})_{a_2a_3a_4}^l] - 3441300\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{a_5}_{a_1}v^{[jk}_{a_2}(r_{14}^{(0,2)})_{a_3a_4a_5}^l] \\
& - 4275300\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{a_5}_{a_1}v^{[jk}_{a_3}(r_{14}^{(0,2)})_{a_2a_4a_5}^l] + 30360\epsilon^{a_1a_2}(i u^{jk}v^l_{a_1}v^{a_3a_4}_{a_5}(r_{14}^{(0,2)})_{a_2a_3a_4}^{a_5} \\
& - 134400\epsilon^{a_1a_2}(i u^{jk}|v^{a_3}_{a_1}v^{l]a_4}_{a_5}(r_{14}^{(0,2)})_{a_2a_3a_4}^{a_5} + 340680\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_1}v^{l]a_4}_{a_5}(r_{14}^{(0,2)})_{a_2a_3a_4}^{a_5} \\
& + 1010400\epsilon^{a_1a_2}(i u^j|a_3v^{a_4}_{a_1}v^{[kl]}_{a_5}(r_{14}^{(0,2)})_{a_2a_3a_4}^{a_5} - 414240\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^{kl]}_{a_5}(r_{14}^{(0,2)})_{a_2a_3a_4}^{a_5} \\
& + 48000\epsilon^{a_1a_2}(i v^{jk}|_{a_1}v^{a_3a_4}_{a_5}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} - 58800\epsilon^{a_1a_2}(i v^{jk}|_{a_5}v^{a_3a_4}_{a_1}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} \\
& - 369600\epsilon^{a_1a_2}(i v^j|a_3v^{a_4|k}_{a_1}v^{a_5}_{a_2}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} + 33600\epsilon^{a_1a_2}(i v^j|a_3v^{a_4a_5}_{a_2}(r_{16}^{(0,2)})_{a_3a_4a_5}^{[kl]} \\
& + 12250\epsilon^{a_1a_2}(i u^{jk}|v^{a_3}_{a_1}v^{a_4}_{a_5}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} - 615850\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_1}v^{a_4}_{a_5}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} \\
& + 369250\epsilon^{a_1a_2}(i u^j|a_3v^{[k}_{a_5}v^{a_4}_{a_1}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} + 184625\epsilon^{a_1a_2}(i u^j|a_3v^{a_4}_{a_1}v^{a_5}_{a_2}(r_{16}^{(0,2)})_{a_3a_4a_5}^{[kl]} \\
& - 504800\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^k_{a_5}(r_{16}^{(0,2)})_{a_2a_3a_4}^{l]a_5} - 1085950\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^{a_5}_{a_2}(r_{16}^{(0,2)})_{a_3a_4a_5}^{[kl]} \\
& + 701100\epsilon^{a_1a_2}(i|u^{a_3a_4}v^{[j}_{a_1}v^{a_5}_{a_3}(r_{16}^{(0,2)})_{a_2a_4a_5}^{[kl]}
\end{aligned}$$

$$\mathcal{O}^{ijkl} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \hat{\phi}^l \mathcal{T} + \hat{\phi}^a \hat{\phi}^i (\hat{\phi}^j \hat{\phi}^k \mathcal{T}_a^l) + \hat{\phi}^a \hat{\phi}^b \hat{\phi}^i (\hat{\phi}^j \mathcal{T}_{ab}^{kl}) + \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \hat{\phi}^i (\mathcal{T}_{abc}^{jkl}) + \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \hat{\phi}^d \mathcal{T}_{abcd}^{ijkl}.$$

$$\mathcal{T} = -1020600 \mathcal{O}_{24}^2.$$

$$u_2^{ij} \equiv \text{tr}(\phi^{(i} \phi^{j)}), u_3^{ijk} \equiv \text{tr}(\phi^{(i} \phi^j \phi^{k)})$$

$$(u_2 \mathcal{O}_{30}^3)^{ijklm} = u_2^{(ij} (\mathcal{O}_{30}^3)^{klm)}$$

$$(u_2 \mathcal{O}_{30}^3)_l^{ijk} = \epsilon_{lab} u_2^{a(i} (\mathcal{O}_{30}^3)^{jk)b}$$

$$(u_2 \mathcal{O}_{30}^3)_{jk}^i = \epsilon_{jab} \epsilon_{kcd} u_2^{ac} (\mathcal{O}_{30}^3)^{ibd}$$

$$\text{tr}[\phi^{(i} \phi^{j)}] \xrightarrow{SU(2) \times U(1)} \text{tr}[\phi^{(i} \phi^{j)}]_{SU(2)} + 6 \hat{\phi}^i \hat{\phi}^j.$$



$$\begin{aligned}
(u_2 \mathcal{O}_{30}^3)^{ijklm} &\xrightarrow{SU(2) \times U(1)} \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \hat{\phi}^l \hat{\phi}^m \cdot 6\mathcal{T} + (\mathfrak{Q}) \\
(u_2 \mathcal{O}_{30}^3)_l^{ijk} &\xrightarrow{SU(2) \times U(1)} \epsilon_{lab} \hat{\phi}^a \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \hat{\phi}^b \cdot 6\mathcal{T} + (\mathfrak{Q}) = 0 + (\mathfrak{Q}) \\
(u_2 \mathcal{O}_{30}^3)_{jk}^i &\xrightarrow{SU(2) \times U(1)} \epsilon_{jab} \epsilon_{kcd} \hat{\phi}^a \hat{\phi}^c \hat{\phi}^i \hat{\phi}^b \hat{\phi}^d \cdot 6\mathcal{T} + (\mathfrak{Q}) = 0 + (\mathfrak{Q})
\end{aligned}$$

$$(u \cdots u \mathcal{O}_{24+2k}^3)^{i_1 \cdots i_n} = u^{(i_1 \cdots \cdots u^{\cdots i_{n-k}} (\mathcal{O}_{24+2k}^3)^{i_{n-k+1} \cdots i_n})}$$

$$u^{(i_1 \cdots \cdots u^{\cdots i_{n-k}} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_{n-k}} C + \cdots$$

$$(u \cdots u \mathcal{O}_{24+2k}^3)^{i_1 \cdots i_n} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_n} C \mathcal{T} + (\mathfrak{Q}),$$

$$(s_{26,1})^i = \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{a_5}{}_{a_2} \left(R_{14}^{(0,2)} \right)_{a_3 a_4 a_5}^i$$

$$(s_{26,2})^i = v^i{}_a \left(R_{20}^{(0,3)} \right)^a$$

$$(s_{26,3})^i = \left(R_{26}^{(0,4)} \right)^i$$

$$(s_{28,1})^{ij} = \epsilon^{a_1 a_2 (i l} v^{a_3}{}_{a_5} v^{a_4 a_5}{}_{a_1} \left(R_{14}^{(0,2)} \right)_{a_2 a_3 a_4}^{j) \quad ij},$$

$$(s_{28,2})^{ij} = \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{a_5 (i}{}_{a_2} \left(R_{14}^{(0,2)} \right)_{a_3 a_4 a_5}^{j) \quad ij},$$

$$(s_{28,3})^{ij} = \epsilon^{a_1 a_2 (i l} v^{a_3}{}_{a_5} v^{j) a_4}{}_{a_1} \left(R_{14}^{(0,2)} \right)_{a_2 a_3 a_4}^{a_5 \quad ij},$$

$$(s_{28,4})^{ij} = \epsilon^{a_1 a_2 (i} v^{j) a_3}{}_{a_1} v^{a_4 a_5}{}_{a_2} \left(R_{14}^{(0,2)} \right)_{a_2 a_3 a_4}^{a_5 \quad ij},$$

$$(s_{28,5})^{ij} = \epsilon^{a_1 a_2 (i} v^{j) a_3}{}_{a_5} v^{a_4 a_5}{}_{a_1} \left(R_{14}^{(0,2)} \right)_{a_2 a_3 a_4}^{a_5 \quad ij},$$

$$(s_{28,6})^{ij} = \epsilon^{a_1 a_2 (i} v^{j) a_3}{}_{a_3} v^{a_4}{}_{a_1} \left(R_{16}^{(0,2)} \right)_{a_2}{}_{a_2},$$

$$(s_{28,7})^{ij} = \epsilon^{a_1 a_2 (i} v^{j) a_3}{}_{a_1} v^{a_4}{}_{a_2} \left(R_{16}^{(0,2)} \right)_{a_3}{}_{a_3},$$

$$(s_{28,8})^{ij} = \epsilon^{a_1 a_2 (i l} v^{a_3}{}_{a_4} v^{a_4}{}_{a_1} \left(R_{16}^{(0,2)} \right)_{a_2 a_3}^{j) \quad ij},$$

$$(s_{28,9})^{ij} = \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{(i}{}_{a_2} \left(R_{16}^{(0,2)} \right)_{a_3 a_4}^{j) \quad ij},$$

$$(s_{28,10})^{ij} = \epsilon^{a_1 a_2 (i} v^{j) a_3}{}_{a_1} v^{a_4}{}_{a_4} \left(R_{16}^{(0,2)} \right)_{a_2 a_3}^{a_4 \quad ij},$$

$$(s_{28,11})^{ij} = \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{a_5}{}_{a_2} \left(R_{16}^{(0,2)} \right)_{a_3 a_4 a_5}^{ij \quad ij},$$

$$(s_{28,12})^{ij} = v^{ij}{}_a \left(R_{20}^{(0,3)} \right)^a,$$

$$(s_{28,13})^{ij} = v^{(i}{}_a \left(R_{22}^{(0,3)} \right)^{j)a}.$$

$$\begin{aligned}
-30s_{28,1} + 30s_{28,3} + \frac{15}{4}s_{28,4} - 15s_{28,5} - \frac{7}{20}s_{28,6} + \frac{21}{80}s_{28,7} \\
+ \frac{49}{36}s_{28,8} - \frac{7}{12}s_{28,9} + \frac{7}{24}s_{28,10} + s_{28,11} = 0
\end{aligned}$$



$$\mathcal{O}^{ij} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^i \hat{\phi}^j \mathcal{T} + \hat{\phi}^a \hat{\phi}^{(i} \mathcal{T}_a^{j)} + \hat{\phi}^a \hat{\phi}^b \mathcal{T}_{ab}^{ij}$$

$$(\mathcal{O}_{24+2n}^3)^{i_1 \cdots i_n} \xrightarrow{SU(2) \times U(1)} \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_n} C_n \mathcal{O}_{24}^2 + (\mathfrak{Q}). (n \geq 3, C_n \neq 0)$$

$$w_2{}^i \equiv \text{tr} \left(f \phi^i + \frac{1}{2} \epsilon^{ia_1 a_2} \psi_{a_1} \psi_{a_2} \right)$$

$$[1,0] \otimes [n,0] = [n+1,0] \oplus [n-1,1]$$

$$\begin{aligned} (w_2 \mathcal{O}_{24+2n}^3)^{i_1 \cdots i_{n+1}} &= w^{(i_1} (\mathcal{O}_{24+2n}^3)^{i_2 \cdots i_{n+1})} \\ (w_2 \mathcal{O}_{24+2n}^3)_j^{i_1 \cdots i_{n-1}} &= \epsilon_{jab} w^a (\mathcal{O}_{24+2n}^3)^{bi_1 \cdots i_{n-1}} \end{aligned}$$

$$w_2{}^i \equiv \text{tr} \left(f \phi^i + \frac{1}{2} \epsilon^{ia_1 a_2} \psi_{a_1} \psi_{a_2} \right)$$

$$\begin{aligned} (w_2 \mathcal{O}_{24+2n}^3)^{i_1 \cdots i_{n+1}} &\xrightarrow{SU(2) \times U(1)} \hat{\phi}^{(i_2} \cdots \hat{\phi}^{i_{n+1}} C_n w^{i_1)} \mathcal{O}_{24}^2 + (\mathfrak{Q}), \\ (w_2 \mathcal{O}_{24+2n}^3)_j^{i_1 \cdots i_{n-1}} &\xrightarrow{SU(2) \times U(1)} \epsilon_{jab} \hat{\phi}^b \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_{n-1}} C_n w^a \mathcal{O}_{24}^2 + (\mathfrak{Q}). \end{aligned}$$

$$[1,0]_S^{\otimes k} \otimes [n,0] = [k,0] \otimes [n,0]. [1,0]_S^{\otimes k}$$

$$(w_2)^k \mathcal{O}_{24+2n}^3$$

$$(w_2)^k \mathcal{O}_{24+2n}^3 \xrightarrow{SU(2) \times U(1)} \hat{\phi}^n C_n (w_2)^k \mathcal{O}_{24}^2 + (\mathfrak{Q})$$

$$[2,0] \otimes [n,0] = [n+2,0] \oplus [n,1] \oplus [n-2,2].$$

$$\begin{aligned} (w_3 \mathcal{O}_{24+2n}^3)^{i_1 \cdots i_{n+2}} &= w^{(i_1 i_2} (\mathcal{O}_{24+2n}^3)^{i_3 \cdots i_{n+2})} \\ (w_3 \mathcal{O}_{24+2n}^3)_j^{i_1 \cdots i_n} &= \epsilon_{ja_1 a_2} w^{a_1 (i_1} (\mathcal{O}_{24+2n}^3)^{i_2 \cdots i_n) a_2} \\ (w_3 \mathcal{O}_{24+2n}^3)_{jk}^{i_1 \cdots i_{n-2}} &= \epsilon_{a_1 a_2 (j} \epsilon_{k) b_1 b_2} w^{a_1 b_1} (\mathcal{O}_{24+2n}^3)^{a_2 b_2 i_1 \cdots i_{n-2}} \end{aligned}$$

$$w_3 = \text{tr} (f \phi^{(i} \phi^{j)} + \epsilon^{a_1 a_2 (i} \phi^{j)} \psi_{a_1} \psi_{a_2})$$

$$w_3^{ij} \rightarrow 2 \hat{\phi}^{(i} w_2^{j)}.$$

$$\begin{aligned} w^{(i_1 i_2} (\mathcal{O}_{24+2n}^3)^{i_3 \cdots i_{n+2}}) &\xrightarrow{SU(2) \times U(1)} 2 C_n \hat{\phi}^{(i_1 \cdots i_{n+1}} w^{i_{n+2})} \mathcal{O}_{24}^2 + (\mathfrak{Q}), \\ \epsilon_{ja_1 a_2} w^{a_1 (i_1} (\mathcal{O}_{24+2n}^3)^{i_2 \cdots i_n) a_2} &\xrightarrow{SU(2) \times U(1)} - C_n \epsilon_{ja_1 a_2} \hat{\phi}^{(i_1 \cdots i_n) a_1} w^{a_2} \mathcal{O}_{24}^2 + (\mathfrak{Q}), \\ \epsilon_{a_1 a_2 (j} \epsilon_{k) b_1 b_2} w^{a_1 b_1} (\mathcal{O}_{24+2n}^3)^{a_2 b_2 i_1 \cdots i_{n-2}} &\xrightarrow{SU(2) \times U(1)} (\mathfrak{Q}). \end{aligned}$$

$$(w_3)^k \mathcal{O}_{24+2n}^3$$

$$(w_3)^k \mathcal{O}_{24+2n}^3 \xrightarrow{SU(2) \times U(1)} \hat{\phi}^{n+k} C'_n (w_2)^k \mathcal{O}_{24}^2 + (\mathfrak{Q}),$$

$$[2,0]_S^{\otimes k} \otimes [n,0] = \left([1,0]_{\hat{\phi}} \otimes_S [1,0]_{w_2} \right)_S^{\otimes k} \otimes [n,0]$$

$$(w_2)^k \mathcal{O}_{24+2(n+k)}^3 [1,0]_S^{\otimes k} \otimes [n+k,0] = [k,0] \otimes [n+k,0]$$

$$[2,0] \otimes [n,0] = [n+2,0] \oplus [n,1] \oplus [n-2,2]$$

$$[1,0] \otimes [n+1,0] = [n+2,0] \oplus [n,1]$$

$$[2,0] \otimes [n+2,0] = [n+4,0] \oplus [n+2,1] \oplus [n,2]$$

$$(w_2)^2 \mathcal{O}_{24+2(n+2)}^3$$

$$\mathcal{O} \cdot \partial_1^{j_1} \partial_2^{j_2} \mathrm{tr}(X^2)$$

$$\mathcal{O} \cdot \mathrm{tr}(X(z)X(z))$$

$$X(z) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 D_1 + z_2 D_2)^n X(0) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_{\alpha} D_{\alpha})^n X(0)$$

$$\begin{aligned} Q\phi^m(z) &= [\lambda(z), \phi^m(z)], & Q\psi_m(z) &= \epsilon_{mnp} [\phi^n(z), \phi^p(z)] + \{\lambda(z), \psi_m(z)\} \\ Qf(z) &= [\phi^m(z), \psi_m(z)] + [\lambda(z), f(z)], & Q\lambda(z) &= \{\lambda(z), \lambda(z)\} \end{aligned}$$

$$\begin{aligned} \phi^m(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_{\alpha} D_{\alpha})^n \phi^m(0), & \psi_m(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_{\alpha} D_{\alpha})^n \psi_m(0) \\ f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_{\alpha} D_{\alpha})^n f(0), & \lambda(z) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (z_{\alpha} D_{\alpha})^n (z_{\beta} \lambda_{\beta}(0)). \end{aligned}$$

$$\mathcal{O} \cdot \left[\mathrm{tr}(X(z)^2) + \sum_{i=1}^n a_i \mathrm{tr}(X(z_i)^2) \right]$$

$$R=\frac{N^2}{2}\left(\frac{q}{\ell^2}+\frac{q^2}{2\ell^4}\right), J=\frac{N^2}{2}\left(\frac{3q^2}{2\ell^4}+\frac{q^3}{\ell^6}\right)$$

$$J(R)=-\frac{N^2}{2}-3R+\frac{(N^2+4R)^{\frac{3}{2}}}{2N}$$

$$j\equiv j_1+j_2>2\left(\frac{2J}{N^2}\right)^{\frac{1}{3}}$$

$$J(R)\sim \frac{4R^{\frac{3}{2}}}{N}$$

$$J_1\approx J_2=J\sim \frac{4R^{\frac{3}{2}}}{N}$$

$$\frac{J}{R}\sim \frac{J^{\frac{1}{3}}}{N^{\frac{2}{3}}}$$



$$E - E_{\text{BPS}} = cj^{-\tau} + \dots,$$

$$u_2^{ij} \equiv \text{tr}(\phi^{(i}\phi^{j)}), \quad u_3^{ijk} \equiv \text{tr}(\phi^{(i}\phi^j\phi^{k)})$$

$$v_2^i{}_j \equiv \text{tr}(\phi^i\psi_j) - \frac{1}{3}\delta_j^i \text{tr}(\phi^a\psi_a), \quad v_3^{ij}{}_k \equiv \text{tr}(\phi^{(i}\phi^j)\psi_k) - \frac{1}{4}\delta_k^i \text{tr}(\phi^j\phi^a\psi_a) - \frac{1}{4}\delta_k^j \text{tr}(\phi^i\phi^a\psi_a)$$

$$\begin{aligned} t^{10}[1,2](u_2u_3) &: (R_{10}^{(0,0)})_{jk}^i = \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}u^{a_1b_1}u^{ia_2b_2} \\ t^{12}[0,0](u_2u_2u_2) &: (R_{12}^{(0,0)}) = \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3} \\ t^{12}[2,2](u_2u_2u_2,u_3u_3) &: (R_{12}^{(0,0)})_{kl}^{ij} = \epsilon_{a_1a_2(k}\epsilon_{l)b_1b_2}(u^{a_1b_1}u^{a_2b_2}u^{ij} + 6u^{a_1b_1(i}u^{j)a_2b_2}) \\ t^{12}[0,3](u_2v_3) &: (R_{12}^{(0,1)})_{ijk} = \epsilon_{(i|a_1a_2}\epsilon_{j|b_1b_2}u^{a_1b_1}v^{a_2b_2|k)} \\ t^{12}[1,1](u_2v_3,u_3v_2) &: (R_{12}^{(0,1)})_j^i = \epsilon_{ja_1a_2}(4u^{a_1b}v^{ia_2}{}_b + 3u^{ia_1b}v^{a_2}{}_b) \\ t^{12}[2,2](u_2v_3,u_3v_2) &: (R_{12}^{(0,1)})_{kl}^{ij} = \epsilon_{a_1a_2(k}(u^{a_1(i}v^{j)a_2}{}_l) + u^{ija_1}v^{a_2}{}_l)) \\ t^{14}[1,0](u_2u_2v_2) &: (R_{14}^{(0,1)})^i = \epsilon_{a_1a_2a_3}u^{ia_1}u^{ba_2}v^{a_3}{}_b \\ t^{14}[0,2](u_2u_2v_2,u_3v_3) &: (R_{14}^{(0,1)})_{ij} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_j) - 2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_c) \\ t^{14}[2,1](u_2u_2v_2,u_3v_3) &: (R_{14}^{(0,1)})_{jk}^{ij} = \epsilon_{ka_1a_2}(3u^{(a_1b}u^{ij)}v^{a_2}{}_b + 4u^{a_1b}u^{a_2(i}v^{j)}{}_b + 24u^{a_1b(i}v^{j)a_2}{}_b) \\ t^{14}[1,3](u_2u_2v_2,u_3v_3) &: (R_{14}^{(0,1)})_{jkl}^i = \epsilon_{(j|a_1a_2}\epsilon_{k|b_1b_2}(u^{a_1b_1}u^{a_2b_2}v^i{}_{|l}) + 6u^{ia_1b_1}v^{a_2b_2}{}_l)) \\ t^{14}[3,2](u_2u_2v_2,u_3v_3) &: (R_{14}^{(0,1)})_{lm}^{ijk} = \epsilon_{a_1a_2(l}(u^{(a_1i}u^{jk)}v^{a_2}{}_m) + 6u^{a_1(ij}v^{k)a_2}{}_m)) \\ t^{14}[1,3](v_2v_3) &: (R_{14}^{(0,2)})_{jkl}^i = \epsilon_{a_1a_2(j}v^{a_1}{}_kv^{ia_2}{}_l) \\ t^{16}[0,1](u_2v_2v_2,v_3v_3) &: (R_{16}^{(0,2)})_i = \epsilon_{ia_1a_2}(12u^{bc}v^{a_1}{}_bv^{a_2}{}_c + 13u^{a_1b}v^{a_2}{}_cv^c{}_b + 12v^{a_1b}{}_cv^{a_2c}{}_b) \\ t^{16}[1,2](u_2v_2v_2,v_3v_3) &: (R_{16}^{(0,2)})_{jk}^i = \epsilon_{a_1a_2(j}(3u^{ib}v^{a_1}{}_k)v^{a_2}{}_b - 7u^{ia_1}v^b{}_kv^{a_2}{}_b + 6u^{a_1b}v^i{}_kv^{a_2}{}_b + 24v^{a_1b}{}_kv^{ia_2}{}_b) \\ t^{16}[2,3](u_2v_2v_2,v_3v_3) &: (R_{16}^{(0,2)})_{klm}^{ij} = \epsilon_{a_1a_2(k}(u^{a_1(i}v^{j)}{}_lv^{a_2}{}_m) + 3v^{a_1(i}{}_lv^{j)a_2}{}_m)) \\ t^{18}[0,0](u_3v_2v_2) &: (R_{18}^{(0,2)}) = \epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_bv^{a_3}{}_c \\ t^{20}[1,0](v_2v_2v_3) &: (R_{20}^{(0,3)})^i = 2v^a{}_cv^b{}_av^{ic}{}_b - 3v^i{}_av^c{}_bv^{ab}{}_c \\ t^{22}[2,0](u_2v_2v_2v_2) &: (R_{22}^{(0,3)})^{ij} = u^{ij}v^a{}_bv^b{}_cv^c{}_a - 3u^{a(i}v^{j)}{}_bv^b{}_cv^c{}_a + 3u^{ab}v^{(i}{}_av^{j)}{}_cv^c{}_b \\ t^{24}[0,0](u_2v_2v_2v_3) &: (R_{24}^{(0,3)}) = \epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_bv^{a_3c}{}_dv^d{}_c \\ t^{26}[1,0](v_2v_2v_2v_3) &: (R_{26}^{(0,4)})^i = v^i{}_av^a{}_bv^d{}_cv^{bc}{}_d \\ t^{30}[0,0](v_2v_2v_2v_2v_2) &: (R_{30}^{(0,5)}) = v^a{}_bv^b{}_cv^c{}_dv^d{}_ev^e{}_a \end{aligned}$$

$$t^{30}[3,0](v_2v_2v_2v_2v_2) : (R_{30}^{(0,5)})^{ijk} = \epsilon^{a_1a_2(i}v^j{}_{a_1}v^k{}_{a_2}v^b{}_cv^c{}_dv^d{}_b$$

$$\left(r_{10}^{(0,1)}\right)_{jk}^i = -2\epsilon_{a_1a_2(j}\text{tr}(\phi^{a_1}\phi^{a_2}\phi^i\psi_k)),$$

$$r_{12}^{(0,1)} = \epsilon_{a_1a_2a_3}[6\text{tr}(\psi_b\phi^{a_1})\text{tr}(\phi^b\phi^{a_2}\phi^{a_3}) - \text{tr}(\psi_b\phi^{a_1}\phi^{a_2})\text{tr}(\phi^b\phi^{a_3})]$$



$$\begin{aligned}
& -3\epsilon_{a_1 a_2 a_3} [\text{tr}(\psi_b \phi^b \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^b \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^b \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^{a_3} \phi^b)], \\
& (r_{12}^{(0,1)})_{kl}^{ij} = -2\epsilon_{a_1 a_2 k} [\text{tr}(\psi_l \phi^{(i} \phi^{j)}) \phi^{a_1} \phi^{a_2}) + 7\text{tr}(\psi_l \phi^{(il} \phi^{a_1} \phi^{lj)}) \phi^{a_2}), \\
& (r_{12}^{(0,2)})_{ijk} = \frac{1}{2} \epsilon_{a_1 a_2 (i} \text{tr}(\phi^{a_1} \psi_j \phi^{a_2} \psi_k), \\
& (r_{12}^{(0,2)})_j^i = 6\text{tr}(\phi^{(i} \phi^{a)} \psi_{(a} \psi_{j)}) - 5\text{tr}(\phi^{[i} \psi_a \phi^{a]} \psi_j), \\
& (r_{12}^{(0,2)})_{kl}^{ij} = \text{tr}(\phi^{(i} \phi^{j)}) \psi_{(k} \psi_{l)}), \\
& (r_{14}^{(0,2)})^i = 3\text{tr}(\phi^i \psi_{a_1} \phi^{a_1} \phi^{a_2} \psi_{a_2}) + 2\text{tr}(\phi^i \phi^{a_1}) \text{tr}(\phi^{a_2} \psi_{(a_1} \psi_{a_2)}) \\
& - 6\text{tr}(\phi^i \psi_{a_1}) \text{tr}(\phi^{[a_1} \phi^{a_2]} \psi_{a_2}) - \text{tr}(\phi^i \psi_{a_1} \psi_{a_2}) \text{tr}(\phi^{a_1} \phi^{a_2}), \\
& (r_{14}^{(0,2)})_{ij} = \frac{5}{9} \epsilon_{a_1 a_2 a_3} [2\text{tr}(\psi_{(i} \psi_{j)}) \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_{(i} \phi^{a_1} \psi_{j)}) \phi^{a_2} \phi^{a_3})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_{j)}) \psi_{a_3} \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j \psi_{a_3} \phi^{a_1} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_{j)}) \psi_{a_3} \phi^{a_3} \phi^{a_1} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_{j)}) \phi^{a_1} \psi_{a_3} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j \phi^{a_1} \psi_{a_3} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_{j)}) \phi^{a_3} \psi_{a_3} \phi^{a_1} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_{j)}) \phi^{a_1} \phi^{a_2} \psi_{a_3} \phi^{a_3}) + \text{tr}(\psi_j \phi^{a_1} \phi^{a_3} \psi_{a_3} \phi^{a_2}) + \text{tr}(\psi_{j)}) \phi^{a_3} \phi^{a_1} \psi_{a_3} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_{j)}) \phi^{a_1} \phi^{a_2} \phi^{a_3} \psi_{a_3}) + \text{tr}(\psi_j \phi^{a_1} \phi^{a_3} \phi^{a_2} \psi_{a_3}) + \text{tr}(\psi_j \phi^{a_3} \phi^{a_1} \phi^{a_2} \psi_{a_3})] \\
& - \frac{1}{3} \epsilon_{a_1 a_2 (i} [5\text{tr}(\psi_{j)}) \phi^{a_1} \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{a_3}) + 2\text{tr}(\psi_j \phi^{(a_1} \phi^{a_3)}) \text{tr}(\psi_{a_3} \phi^{a_2}) \\
& - 2\text{tr}(\psi_j \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{(a_1} \phi^{a_3)})], \\
& (r_{14}^{(0,2)})_k^{ij} = 12\text{tr}(\phi^{(i} \phi^a \phi^{j)}) \psi_{(a} \psi_{k)}) + 12\text{tr}(\phi^{(il} \phi^a \phi^{lj)}) \psi_{(a} \psi_{k)}) + 54\text{tr}(\phi^{(i} \phi^j \psi_{(a} \phi^a) \psi_{k)}) - 36\text{tr}(\phi^{(i} \phi^j) \psi_{(a} \phi^a \psi_{k)}), \\
& (r_{14}^{(0,2)})_{jkl}^i = 2\epsilon_{a_1 a_2 j} [\text{tr}(\phi^i \phi^{a_1} \phi^{a_2} \psi_k \psi_l) + 3\text{tr}(\phi^i \phi^{a_1} \psi_k \phi^{a_2} \psi_l) - 2\text{tr}(\phi^i \psi_k \phi^{a_1} \phi^{a_2} \psi_l)], \\
& (r_{14}^{(0,2)})_{lm}^{ijk} = 4\text{tr}(\phi^{(i} \phi^j \phi^k) \psi_{(l} \psi_{m)}) + 3\text{tr}(\phi^{(i} \phi^j \psi_{(l} \phi^k) \psi_{m)}) \\
& (r_{14}^{(0,3)})_{jkl}^i = -\frac{1}{2} \text{tr}(\phi^i \psi_{(j} \psi_k \psi_{l)}), \\
& (r_{16}^{(0,3)})_i = \frac{39}{4} \text{tr}(\psi_i \{\psi_{b_1} \psi_{b_2}, \phi^{b_1} \phi^{b_2}\}) + 2\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \psi_{b_2} \phi^{b_2}) - \frac{61}{4} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \psi_{b_2} \phi^{b_1}) \\
& + \frac{97}{4} \text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \psi_{b_2} \phi^{b_2}) - \frac{41}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \psi_{b_2} \phi^{b_1}) - 5\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \phi^{b_2} \psi_{b_2}) \\
& - \frac{25}{2} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \phi^{b_1} \psi_{b_2}) + 2\text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \phi^{b_2} \psi_{b_2}) - \frac{61}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \phi^{b_1} \psi_{b_2}) \\
& - \frac{11}{4} \text{tr}(\phi^{b_1} \phi^{b_2}) \text{tr}(\psi_i \psi_{b_1} \psi_{b_2}) - \frac{27}{2} \text{tr}(\psi_{b_1} \psi_{b_2}) \text{tr}(\psi_i \phi^{b_1} \phi^{b_2}) + \frac{29}{4} \text{tr}(\phi^{b_2} \psi_{b_2}) \text{tr}(\psi_i [\psi_{b_1}, \phi^{b_1}]), \\
& (r_{16}^{(0,3)})_{jk}^i = 2\text{tr}(\psi_{(j} \psi_{k)} \psi_b \phi^b \phi^i) - 4\text{tr}(\psi_{(j} \psi_{k)} \psi_b \phi^i \phi^b) - \text{tr}(\psi_{(j} \psi_b \psi_{|k)} \{\phi^b, \phi^i\}) - 4\text{tr}(\psi_{(j} \psi_{k)} \phi^{(b} \psi_b \phi^{i)}) \\
& + 7\text{tr}(\psi_{(j} \{\psi_b, \phi^b\} \psi_{|k)} \phi^i) - 11\text{tr}(\psi_{(j} \{\psi_b, \phi^i\} \psi_{|k)} \phi^b) - 4\text{tr}(\psi_{(j} \psi_{k)} \phi^b \phi^i \psi_b) \\
& + 2\text{tr}(\psi_{(j} \psi_{k)} \phi^i \phi^b \psi_b) + 3\text{tr}(\psi_{(j} \psi_b) \text{tr}(\psi_{|k)} [\phi^b, \phi^i]) + 6\text{tr}(\psi_{(j} \phi^{[b} \text{tr}(\{\psi_{k)}, \psi_b\} \phi^{i]}), \\
& (r_{16}^{(0,3)})_{klm}^{ij} = -\text{tr}(\phi^{(i} \phi^{j)}) \psi_{(k} \psi_l \psi_{m)}) - 2\text{tr}(\phi^{(i} \psi_{(k} \phi^{j)}) \psi_l \psi_{m)}).
\end{aligned}$$

$$(r_{10}^{(0,1)})_{jk}^i = -2 \epsilon_{a_1 a_2 (j} \text{tr}(\phi^{a_1} \phi^{a_2} \phi^i \psi_{k)}) ,$$

$$r_{12}^{(0,1)} = \epsilon_{a_1 a_2 a_3} [6\text{tr}(\psi_b \phi^{a_1}) \text{tr}(\phi^b \phi^{a_2} \phi^{a_3}) - \text{tr}(\psi_b \phi^{a_1} \phi^{a_2}) \text{tr}(\phi^b \phi^{a_3})]$$



$$\begin{aligned}
& -3\epsilon_{a_1 a_2 a_3} [\text{tr}(\psi_b \phi^b \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^b \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^b \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^{a_3} \phi^b)] , \\
(r_{12}^{(0,1)})_{kl}^{ij} &= -2\epsilon_{a_1 a_2 (k} [\text{tr}(\psi_l) \phi^{(i} \phi^{j)} \phi^{a_1} \phi^{a_2}) + 7\text{tr}(\psi_l) \phi^{(i|} \phi^{a_1} \phi^{j|} \phi^{a_2})] , \\
(r_{12}^{(0,2)})_{ijk} &= \frac{1}{2} \epsilon_{a_1 a_2 (i} \text{tr}(\phi^{a_1} \psi_j \phi^{a_2} \psi_k) , \\
(r_{12}^{(0,2)})_j^i &= 6\text{tr}(\phi^{(i} \phi^{a)} \psi_{(a} \psi_{j)}) - 5\text{tr}(\phi^{[i} \psi_a \phi^{a]} \psi_j) , \\
(r_{12}^{(0,2)})_{kl}^{ij} &= \text{tr}(\phi^{(i} \phi^{j)} \psi_{(k} \psi_{l)}) , \\
(r_{14}^{(0,2)})^i &= 3\text{tr}(\phi^i \psi_{a_1} \phi^{a_1} \phi^{a_2} \psi_{a_2}) + 2\text{tr}(\phi^i \phi^{a_1}) \text{tr}(\phi^{a_2} \psi_{(a_1} \psi_{a_2)}) \\
& - 6\text{tr}(\phi^i \psi_{a_1}) \text{tr}(\phi^{[a_1} \phi^{a_2]} \psi_{a_2}) - \text{tr}(\phi^i \psi_{a_1} \psi_{a_2}) \text{tr}(\phi^{a_1} \phi^{a_2}) , \\
(r_{14}^{(0,2)})_{ij} &= \frac{5}{9} \epsilon_{a_1 a_2 a_3} [2\text{tr}(\psi_{(i} \psi_{j)} \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_{(i} \phi^{a_1} \psi_{j)} \phi^{a_2} \phi^{a_3})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_j) \psi_{a_3} \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j) \psi_{a_3} \phi^{a_1} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_j) \psi_{a_3} \phi^{a_3} \phi^{a_1} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_j) \phi^{a_1} \psi_{a_3} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j) \phi^{a_1} \psi_{a_3} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_j) \phi^{a_3} \psi_{a_3} \phi^{a_1} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_j) \phi^{a_1} \phi^{a_2} \psi_{a_3} \phi^{a_3}) + \text{tr}(\psi_j) \phi^{a_1} \phi^{a_3} \psi_{a_3} \phi^{a_2}) + \text{tr}(\psi_j) \phi^{a_3} \phi^{a_1} \psi_{a_3} \phi^{a_2})] \\
& + \epsilon_{a_1 a_2 (i} [\text{tr}(\psi_j) \phi^{a_1} \phi^{a_2} \phi^{a_3} \psi_{a_3}) + \text{tr}(\psi_j) \phi^{a_1} \phi^{a_3} \phi^{a_2} \psi_{a_3}) + \text{tr}(\psi_j) \phi^{a_3} \phi^{a_1} \phi^{a_2} \psi_{a_3})] \\
& - \frac{1}{3} \epsilon_{a_1 a_2 (i} [5\text{tr}(\psi_j) \phi^{a_1} \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{a_3}) + 2\text{tr}(\psi_j) \phi^{(a_1} \phi^{a_3)}) \text{tr}(\psi_{a_3} \phi^{a_2}) \\
& - 2\text{tr}(\psi_j) \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{(a_1} \phi^{a_3)})] , \\
(r_{14}^{(0,2)})_k^{ij} &= 12\text{tr}(\phi^{(i} \phi^a \phi^j) \psi_{(a} \psi_k) + 12\text{tr}(\phi^{(i|} \phi^a \phi^{j|}) \psi_{(a} \psi_k) + 54\text{tr}(\phi^{(i} \phi^j \psi_{(a} \phi^a) \psi_k) - 36\text{tr}(\phi^{(i} \phi^j) \psi_{(a} \phi^a \psi_k) \\
(r_{14}^{(0,2)})_{jkl}^i &= 2\epsilon_{a_1 a_2 (j} [\text{tr}(\phi^i \phi^{a_1} \phi^{a_2} \psi_k \psi_l) + 3\text{tr}(\phi^i \phi^{a_1} \psi_k \phi^{a_2} \psi_l) - 2\text{tr}(\phi^i \psi_k \phi^{a_1} \phi^{a_2} \psi_l)] , \\
(r_{14}^{(0,2)})_{lm}^{ijk} &= 4\text{tr}(\phi^{(i} \phi^j \phi^k) \psi_l \psi_m) + 3\text{tr}(\phi^{(i} \phi^j \psi_l \phi^k) \psi_m) \\
(r_{14}^{(0,3)})_{jkl}^i &= -\frac{1}{2} \text{tr}(\phi^i \psi_{(j} \psi_k \psi_{l)}) , \\
(r_{16}^{(0,3)})_i &= \frac{39}{4} \text{tr}(\psi_i \{\psi_{b_1} \psi_{b_2}, \phi^{b_1} \phi^{b_2}\}) + 2\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \psi_{b_2} \phi^{b_2}) - \frac{61}{4} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \psi_{b_2} \phi^{b_1}) \\
& + \frac{97}{4} \text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \psi_{b_2} \phi^{b_2}) - \frac{41}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \psi_{b_2} \phi^{b_1}) - 5\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \phi^{b_2} \psi_{b_2}) \\
& - \frac{25}{2} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \phi^{b_1} \psi_{b_2}) + 2\text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \phi^{b_2} \psi_{b_2}) - \frac{61}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \phi^{b_1} \psi_{b_2}) \\
& - \frac{11}{4} \text{tr}(\phi^{b_1} \phi^{b_2}) \text{tr}(\psi_i \psi_{b_1} \psi_{b_2}) - \frac{27}{2} \text{tr}(\psi_{b_1} \psi_{b_2}) \text{tr}(\psi_i \phi^{b_1} \phi^{b_2}) + \frac{29}{4} \text{tr}(\phi^{b_2} \psi_{b_2}) \text{tr}(\psi_i [\psi_{b_1}, \phi^{b_1}]) , \\
(r_{16}^{(0,3)})_{jk}^i &= 2\text{tr}(\psi_{(j} \psi_k) \psi_b \phi^b \phi^i) - 4\text{tr}(\psi_{(j} \psi_k) \psi_b \phi^i \phi^b) - \text{tr}(\psi_{(j|} \psi_b \psi_{|k)} \{\phi^b, \phi^i\}) - 4\text{tr}(\psi_{(j} \psi_k) \phi^{(b} \psi_b \phi^{i)}) \\
& + 7\text{tr}(\psi_{(j|} \{\psi_b, \phi^b\} \psi_{|k)} \phi^i) - 11\text{tr}(\psi_{(j|} \{\psi_b, \phi^i\} \psi_{|k)} \phi^b) - 4\text{tr}(\psi_{(j} \psi_k) \phi^b \phi^i \psi_b) \\
& + 2\text{tr}(\psi_{(j} \psi_k) \phi^i \phi^b \psi_b) + 3\text{tr}(\psi_{(j|} \psi_b) \text{tr}(\psi_{|k)} [\phi^b, \phi^i]) + 6\text{tr}(\psi_{(j} \phi^{[b} \text{tr}(\{\psi_k, \psi_b\} \phi^{i]}) , \\
(r_{16}^{(0,3)})_{klm}^{ij} &= -\text{tr}(\phi^{(i} \phi^{j)} \psi_{(k} \psi_l \psi_m) - 2\text{tr}(\phi^{(i} \psi_{(k} \phi^{j)} \psi_l \psi_m)
\end{aligned}$$

$$\begin{aligned}
s_1 &= \epsilon^{a_1 a_2 (i} \epsilon^{j|b_1 b_2} v^c{}_d v^{(k)}{}_{a_1} v^d{}_{b_1} (R_{12}^{(0,1)})_{a_2 b_2 c} , \\
s_3 &= \epsilon^{a_1 a_2 (i} v^j|_b v^b{}_c v^c{}_{a_1} (R_{12}^{(0,1)})^{(k)}{}_{a_2} , \\
s_5 &= \epsilon^{a_1 a_2 (i} v^j{}_b v^k{}_{a_1} v^b{}_c (R_{12}^{(0,1)})^c{}_{a_2} , \\
s_7 &= \epsilon^{a_1 a_2 a_3} v^b{}_{a_1} v^c{}_{a_2} v^{(i}{}_{a_3} (R_{12}^{(0,1)})^{jk)}{}_{bc} , \\
s_9 &= \epsilon^{a_1 a_2 (i} v^j|_d v^b{}_a v^d{}_c (R_{12}^{(0,1)})^{(k)c}{}_{a_2 b} , \\
s_{11} &= \epsilon^{a_1 a_2 (i|} u^{bc} v^d{}_b v^{(j}{}_{a_1} (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{13} &= \epsilon^{a_1 a_2 (i|} u^{cd} v^b{}_a v^{(j}{}_{b} (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{15} &= \epsilon^{a_1 a_2 (i} v^j|_b v^c{}_{a_1} v^{(k)}{}_d (R_{14}^{(0,2)})^d{}_{a_2 bc} , \\
s_{17} &= \epsilon^{a_1 a_2 (i} v^j|_b v^{cd}{}_a (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{19} &= \epsilon^{a_1 a_2 (i} v^{jk)}{}_b v^{cd}{}_{a_1} (R_{14}^{(0,2)})^b{}_{a_2 cd} , \\
s_{21} &= \epsilon^{a_1 a_2 (i} v^j{}_b v^{(k)b}{}_{a_1} (R_{16}^{(0,2)})_{a_2} , \\
s_{23} &= \epsilon^{a_1 a_2 (i} v^j{}_a v^{(k)b}{}_{a_2} (R_{16}^{(0,2)})_b , \\
s_{25} &= \epsilon^{a_1 a_2 (i|} v^c{}_b v^{(j|}{}_b (R_{16}^{(0,2)})^{(k)}{}_{a_2 c} , \\
s_{27} &= \epsilon^{a_1 a_2 b} v^c{}_{a_1} v^{(ij}{}_{a_2} (R_{16}^{(0,2)})^{(k)}{}_{bc} , \\
s_{29} &= \epsilon^{a_1 a_2 (i|} v^b{}_{a_1} v^{(jk)}{}_c (R_{16}^{(0,2)})^c{}_{a_2 b} , \\
s_{31} &= \epsilon^{a_1 a_2 a_3} v^{(i|}{}_{a_1} v^{b_1 b_2}{}_{a_2} (R_{16}^{(0,2)})^{(jk)}{}_{a_3 b_1 b_2} , \\
s_{33} &= \epsilon^{a_1 a_2 (i|} v^b{}_{a_1} v^{c|}{}_d (R_{16}^{(0,2)})^{(k)d}{}_{a_2 bc} , \\
s_{35} &= \epsilon^{ab(i} v^j{}_a v^k{}_{b} R_{18}^{(0,2)} , \\
s_{37} &= u^{(ij} v^k{}_a (R_{20}^{(0,3)})^a , \\
s_{39} &= u^{(ij} (R_{26}^{(0,4)})^k) . \\
s_2 &= \epsilon^{a_1 a_2 (i} \epsilon^{j|b_1 b_2} v^{(k)}{}_d v^c{}_{a_1} v^d{}_{b_1} (R_{12}^{(0,1)})_{a_2 b_2 c} , \\
s_4 &= \epsilon^{a_1 a_2 a_3} v^b{}_{a_1} v^{(i}{}_{a_2} v^j{}_{a_3} (R_{12}^{(0,1)})^{(k)}{}_b , \\
s_6 &= \epsilon^{a_1 a_2 (i|} v^b{}_c v^c{}_d v^d{}_{a_1} (R_{12}^{(0,1)})^{(jk)}{}_{a_2 b} , \\
s_8 &= \epsilon^{a_1 a_2 (i} v^j|_a v^b{}_d v^d{}_c (R_{12}^{(0,1)})^{(k)c}{}_{a_2 b} , \\
s_{10} &= \epsilon^{a_1 a_2 (i|} u^{bc} v^d{}_a v^{(j}{}_{b} (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{12} &= \epsilon^{a_1 a_2 (i} u^{j|}{}_b v^c{}_{a_1} v^d{}_b (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{14} &= \epsilon^{a_1 a_2 (i} u^{j|}{}_c v^b{}_{a_1} v^d{}_b (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{16} &= \epsilon^{a_1 a_2 a_3} u^{(ij} v^c{}_{a_1} v^d{}_{a_2} (R_{14}^{(0,2)})^{(k)}{}_{a_3 cd} , \\
s_{18} &= \epsilon^{a_1 a_2 (i} v^{bc}{}_{a_1} v^{d|}{}_b (R_{14}^{(0,2)})^{(k)}{}_{a_2 cd} , \\
s_{20} &= \epsilon^{a_1 a_2 a_3} v^{b(i}{}_{a_1} v^{j|}{}_c v^{(k)}{}_{a_2} (R_{14}^{(0,2)})^{(k)}{}_{a_3 bc} , \\
s_{22} &= \epsilon^{a_1 a_2 (i|} v^b{}_{a_1} v^{(jk)}{}_b (R_{16}^{(0,2)})_{a_2} , \\
s_{24} &= \epsilon^{a_1 a_2 (i} v^j|_b v^{bc}{}_{a_1} (R_{16}^{(0,2)})^{(k)}{}_{a_2 c} , \\
s_{26} &= \epsilon^{a_1 a_2 (i|} v^b{}_a v^{(j|}{}_b (R_{16}^{(0,2)})^{(k)}{}_{a_2 c} , \\
s_{28} &= \epsilon^{a_1 a_2 (i} v^j{}_a v^{(k)b}{}_c (R_{16}^{(0,2)})^c{}_{a_2 b} , \\
s_{30} &= \epsilon^{a_1 a_2 a_3} v^{b_1}{}_{a_1} v^{b_2(i}{}_{a_2} (R_{16}^{(0,2)})^{(jk)}{}_{a_3 b_1 b_2} , \\
s_{32} &= \epsilon^{a_1 a_2 (i} v^j|_a v^{bc}{}_d (R_{16}^{(0,2)})^{(k)}{}_{a_2 bc} , \\
s_{34} &= \epsilon^{a_1 a_2 (i|} v^b{}_{a_1} v^{cd}{}_{a_2} (R_{16}^{(0,2)})^{(jk)}{}_{bcd} , \\
s_{36} &= u^{a(i} v^j{}_a (R_{20}^{(0,3)})^k) , \\
s_{38} &= v^{(ij}{}_a (R_{22}^{(0,3)})^k)^a ,
\end{aligned}$$

$$\begin{aligned}
s_1 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_1} v^l |_{a_2} v^{a_5 a_6} |_{a_3} (R_{12}^{(0,1)})_{a_4 a_5 a_6} , \\
s_3 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_1} v^{a_5} |_{a_2} v^l |_{a_3} (R_{12}^{(0,1)})_{a_4 a_5 a_6} , \\
s_5 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_1} v^{a_5} |_{a_6} v^l |_{a_3} (R_{12}^{(0,1)})_{a_2 a_4 a_5} , \\
s_7 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v^{a_5} |_{a_1} v^{a_6} |_{a_2} v^l |_{a_3} (R_{12}^{(0,1)})_{a_4 a_5 a_6} , \\
s_9 &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_3} v^l |_{a_4} (R_{12}^{(0,1)})_{a_2}^{a_4} , \\
s_{11} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^l |_{a_4} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{13} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^l |_{a_3} (R_{12}^{(0,1)})_{a_2}^{a_4} , \\
s_{15} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^l |_{a_4} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{17} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^l |_{a_4} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{19} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_2} v^{a_3 a_4} |_{a_5} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{21} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^l |_{a_5} (R_{12}^{(0,1)})_{a_2 a_3}^{a_4 a_5} , \\
s_{23} &= \epsilon^{a_1 a_2} (i v^j |_{a_4} v^{a_3} |_{a_5} v^l |_{a_1} (R_{12}^{(0,1)})_{a_2 a_3}^{a_4 a_5} , \\
s_{25} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^{a_4} |_{a_5} (R_{12}^{(0,1)})_{a_2 a_3}^{a_5} , \\
s_{27} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_5} v^{a_4} |_{a_3} (R_{12}^{(0,1)})_{a_2 a_4}^{a_5} , \\
s_{29} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4 a_5} |_{a_3} (R_{12}^{(0,1)})_{a_4 a_5}^{kl} , \\
s_{31} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4} |_{a_5} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{33} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4} |_{a_5} (R_{12}^{(0,1)})_{a_2 a_3}^{a_5} , \\
s_{35} &= \epsilon^{a_1 a_2 a_3} v^{(i} |_{a_1} v^j |_{a_2} v^k |_{a_3} (R_{14}^{(0,1)})^l , \\
s_{37} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4} |_{a_3} (R_{14}^{(0,1)})_{a_4}^{kl} , \\
s_{39} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_2} v^{a_3} |_{a_4} (R_{14}^{(0,1)})_{a_3}^{l a_4} , \\
s_{41} &= \epsilon^{a_1 a_2} (i |_{a_3} \epsilon^{a_3 a_4} |_{a_1} v^k |_{a_2} v^{a_5} |_{a_3} v^{a_6} |_{a_5} (R_{14}^{(0,1)})_{a_2 a_4 a_6}^{l} , \\
s_{43} &= \epsilon^{a_1 a_2} (i |_{a_3} v^{a_3} |_{a_1} v^{a_4} |_{a_3} v^{a_5} |_{a_4} (R_{14}^{(0,1)})_{a_2 a_5}^{jkl} , \\
s_{45} &= \epsilon^{a_1 a_2} (i v^j |_{a_5} v^{a_3} |_{a_1} v^{a_4} |_{a_3} (R_{14}^{(0,1)})_{a_2 a_4}^{kl a_5} , \\
s_{47} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_4} |_{a_1} v^{a_5} |_{a_3} (R_{14}^{(0,2)})_{a_2 a_4 a_5}^{l} , \\
s_{49} &= \epsilon^{a_1 a_2} (i u^j |_{a_3} a_3 a_4 v |^k |_{a_1} v^{a_5} |_{a_2} (R_{14}^{(0,2)})_{a_3 a_4 a_5}^{l} , \\
s_{51} &= \epsilon^{a_1 a_2} (i u^j |_{a_3} a_3 a_4 v |^k |_{a_5} v^{a_5} |_{a_1} (R_{14}^{(0,2)})_{a_2 a_3 a_4}^{l} , \\
s_{53} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^l |_{a_5} v^{a_4} |_{a_1} (R_{14}^{(0,2)})_{a_2 a_3 a_4}^{a_5} , \\
s_{55} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_3} |_{a_1} v^{a_4 a_5} |_{a_3} (R_{14}^{(0,2)})_{a_2 a_4 a_5}^{l} , \\
s_{57} &= \epsilon^{a_1 a_2} (i u^j |_{a_3} v^l |_{a_1} v^{a_4 a_5} |_{a_2} (R_{14}^{(0,2)})_{a_3 a_4 a_5}^{l} , \\
s_2 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_1} v^l |_{a_5} v^{a_5 a_6} |_{a_3} (R_{12}^{(0,1)})_{a_2 a_4 a_6} , \\
s_4 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_1} v^{a_5} |_{a_3} v^l |_{a_2} (R_{12}^{(0,1)})_{a_4 a_5 a_6} , \\
s_6 &= \epsilon^{a_1 a_2} (i \epsilon^j | a_3 a_4 v |^k |_{a_6} v^{a_5} |_{a_1} v^l |_{a_3} (R_{12}^{(0,1)})_{a_2 a_4 a_5} , \\
s_8 &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_2} v^l |_{a_3} (R_{12}^{(0,1)})_{a_4}^{a_4} , \\
s_{10} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_4} v^l |_{a_2} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{12} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^l |_{a_2} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{14} &= \epsilon^{a_1 a_2} (i v^j |_{a_4} v^{a_3} |_{a_1} v^l |_{a_2} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{16} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^l |_{a_2} (R_{12}^{(0,1)})_{a_3}^{a_4} , \\
s_{18} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_4} v^l |_{a_5} (R_{12}^{(0,1)})_{a_2 a_3}^{a_4 a_5} , \\
s_{20} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^k |_{a_5} v^{a_3 a_4} |_{a_2} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{22} &= \epsilon^{a_1 a_2} (i v^j |_{a_4} v^{a_3} |_{a_1} v^l |_{a_2} (R_{12}^{(0,1)})_{a_2 a_3}^{a_4 a_5} , \\
s_{24} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4} |_{a_5} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{26} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_5} v^{a_4} |_{a_2} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{28} &= \epsilon^{a_1 a_2} (i v^j |_{a_5} v^{a_3} |_{a_1} v^{a_4} |_{a_2} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{30} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_4} v^{a_4 a_5} |_{a_2} (R_{12}^{(0,1)})_{a_3 a_5}^{kl} , \\
s_{32} &= \epsilon^{a_1 a_2} (i |_{a_3} v^{a_3} |_{a_1} v^{a_4} |_{a_5} v^l |_{a_2} (R_{12}^{(0,1)})_{a_3 a_4}^{a_5} , \\
s_{34} &= \epsilon^{a_1 a_2} (i |_{a_3} v^{a_3} |_{a_1} v^{a_4} |_{a_2} v^{a_5} |_{a_3} (R_{12}^{(0,1)})_{a_4 a_5}^{kl} , \\
s_{36} &= \epsilon^{a_1 a_2} (i |_{a_3} \epsilon^{a_3 a_4} |_{a_1} v^k |_{a_2} v^{a_5} |_{a_3} (R_{14}^{(0,1)})_{a_4 a_5} , \\
s_{38} &= \epsilon^{a_1 a_2} (i v^j |_{a_3} v^{a_3} |_{a_1} v^{a_4} |_{a_2} (R_{14}^{(0,1)})_{a_4}^{kl} , \\
s_{40} &= \epsilon^{a_1 a_2} (i |_{a_3} \epsilon^{a_3 a_4} |_{a_1} v^k |_{a_2} v^{a_5} |_{a_2} v^{a_6} |_{a_3} (R_{14}^{(0,1)})_{a_4 a_5 a_6}^{l} , \\
s_{42} &= \epsilon^{a_1 a_2} (i |_{a_3} v^{a_3} |_{a_1} v^{a_4} |_{a_2} v^{a_5} |_{a_3} (R_{14}^{(0,1)})_{a_4 a_5}^{jkl} , \\
s_{44} &= \epsilon^{a_1 a_2} (i v^j |_{a_1} v^{a_3} |_{a_2} v^{a_4} |_{a_5} (R_{14}^{(0,1)})_{a_3 a_4}^{kl a_5} , \\
s_{46} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_4} |_{a_1} v^{a_5} |_{a_2} (R_{14}^{(0,2)})_{a_3 a_4 a_5}^{l} , \\
s_{48} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_4} |_{a_1} v^{a_5} |_{a_4} (R_{14}^{(0,2)})_{a_2 a_3 a_5}^{l} , \\
s_{50} &= \epsilon^{a_1 a_2} (i u^j |_{a_3} a_3 a_4 v |^k |_{a_1} v^{a_5} |_{a_3} (R_{14}^{(0,2)})_{a_2 a_4 a_5}^{l} , \\
s_{52} &= \epsilon^{a_1 a_2} (i |_{a_3} u^{a_3 a_4 a_5} v^j |_{a_1} v^k |_{a_2} (R_{14}^{(0,2)})_{a_3 a_4 a_5}^{l} , \\
s_{54} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_3} |_{a_1} v^{a_4 a_5} |_{a_2} (R_{14}^{(0,2)})_{a_3 a_4 a_5}^{l} , \\
s_{56} &= \epsilon^{a_1 a_2} (i u^{jk} |_{a_3} v^{a_3} |_{a_4} v^{a_4 a_5} |_{a_1} (R_{14}^{(0,2)})_{a_2 a_3 a_5}^{l} , \\
s_{58} &= \epsilon^{a_1 a_2} (i u^j |_{a_3} v^l |_{a_1} v^{a_4 a_5} |_{a_3} (R_{14}^{(0,2)})_{a_2 a_4 a_5}^{l} ,
\end{aligned}$$



$$\begin{aligned}
s_{59} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} v^{a_4 a_5} a_1 (R_{14}^{(0,2)})^{|l|})_{a_2 a_3 a_5}}, \\
s_{61} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{a_4} a_1 v^{a_5|k} (R_{14}^{(0,2)})^l)_{a_2 a_4 a_5}}, \\
s_{63} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^{k|a_5} a_2 (R_{14}^{(0,2)})^{|l|})_{a_3 a_4 a_5}}, \\
s_{65} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_5 v^{k|a_5} a_1 (R_{14}^{(0,2)})^{|l|})_{a_2 a_3 a_4}}, \\
s_{67} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^{a_5} a_1 v^{jk} a_3 (R_{14}^{(0,2)})^l)_{a_2 a_4 a_5}}, \\
s_{69} &= \epsilon^{a_1 a_2 (i|u^{jk} v^{a_3} a_1 v^l a_4 a_5 (R_{14}^{(0,2)})^{a_5})_{a_2 a_3 a_4}}, \\
s_{71} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{a_4} a_1 v^{kl})_{a_5} (R_{14}^{(0,2)})^{a_5}}_{a_2 a_3 a_4}, \\
s_{73} &= \epsilon^{a_1 a_2 (i|v^{jk} a_1 v^{a_3 a_4} a_5 (R_{16}^{(0,2)})^{|l|})^{a_5}}_{a_2 a_3 a_4}, \\
s_{75} &= \epsilon^{a_1 a_2 (i|v^j|a_3 a_1 v^{a_4|k} a_5 (R_{16}^{(0,2)})^{a_5})_{a_2 a_3 a_4}}, \\
s_{77} &= \epsilon^{a_1 a_2 a_3 v^{(ij} a_1 v^{k|b} a_2 (R_{16}^{(0,2)})^{|l|})_{a_3 b}}, \\
s_{79} &= \epsilon^{a_1 a_2 (i|v^{jk} a_1 v^{bc} a_2 (R_{16}^{(0,2)})^{|l|})_{bc}}, \\
s_{81} &= \epsilon^{a_1 a_2 a_3 v^{(ij} a_1 v^{kl})_{a_2} (R_{16}^{(0,2)})_{a_3}}, \\
s_{83} &= \epsilon^{a_1 a_2 (i|u^{jk} v^{a_3} a_1 v^{a_4} a_5 (R_{16}^{(0,2)})^{|l|})^{a_5}}_{a_2 a_3 a_4}, \\
s_{85} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} a_5 v^{a_4} a_1 (R_{16}^{(0,2)})^{|l|})^{a_5}}_{a_2 a_3 a_4}, \\
s_{87} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^k a_5 (R_{16}^{(0,2)})^l)_{a_2 a_3 a_4}}, \\
s_{89} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^{a_5} a_3 (R_{16}^{(0,2)})^{|kl|})_{a_2 a_4 a_5}}, \\
s_{91} &= \epsilon^{a_1 a_2 (i|u^{jk} v^l)_{a_4} v^{a_3} a_1 (R_{16}^{(0,2)})^{a_4}}_{a_2 a_3}, \\
s_{93} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} a_1 v^l)_{a_4} (R_{16}^{(0,2)})^{a_4}}_{a_2 a_3}, \\
s_{95} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} a_1 v^{a_4} a_3 (R_{16}^{(0,2)})^{|l|})_{a_2 a_4}}, \\
s_{97} &= \epsilon^{a_1 a_2 a_3 u^{(ij} v^k a_1 v^l)_{a_2} (R_{16}^{(0,2)})_{a_3}}, \\
s_{99} &= \epsilon^{a_1 a_2 (i|u^j|b v^{|k} a_1 v^l)_{a_2} (R_{16}^{(0,2)})_b}, \\
s_{101} &= u^{a(i} v^{jk} a (R_{20}^{(0,3)})^l), \\
s_{103} &= u^{a(ij} v^k a (R_{20}^{(0,3)})^l), \\
s_{105} &= u^{a(i} v^j a (R_{22}^{(0,3)})^{kl}), \\
s_{107} &= u^{(ijk} (R_{26}^{(0,4)})^l), \\
s_{60} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{a_4} a_1 v^{a_5|k} a_2 (R_{14}^{(0,2)})^l)_{a_3 a_4 a_5}}, \\
s_{62} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{a_4} a_5 v^{a_5|k} a_1 (R_{14}^{(0,2)})^l)_{a_2 a_3 a_4}}, \\
s_{64} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^{k|a_5} a_3 (R_{14}^{(0,2)})^{|l|})_{a_2 a_4 a_5}}, \\
s_{66} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^{a_5} a_1 v^{jk} a_2 (R_{14}^{(0,2)})^l)_{a_3 a_4 a_5}}, \\
s_{68} &= \epsilon^{a_1 a_2 (i|u^{jk} v^l)_{a_1} v^{a_3 a_4} a_5 (R_{14}^{(0,2)})^{a_5}}_{a_2 a_3 a_4}, \\
s_{70} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k} a_1 v^l a_4 a_5 (R_{14}^{(0,2)})^{a_5})_{a_2 a_3 a_4}}, \\
s_{72} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^{kl})_{a_5} (R_{14}^{(0,2)})^{a_5}}_{a_2 a_3 a_4}, \\
s_{74} &= \epsilon^{a_1 a_2 (i|v^{jk} a_5 v^{a_3 a_4} a_1 (R_{16}^{(0,2)})^{|l|})^{a_5}}_{a_2 a_3 a_4}, \\
s_{76} &= \epsilon^{a_1 a_2 (i|v^j|a_3 a_1 v^{a_4 a_5} a_2 (R_{16}^{(0,2)})^{|kl|})_{a_3 a_4 a_5}}, \\
s_{78} &= \epsilon^{a_1 a_2 (i|v^{jk} a_1 v^l c_b (R_{16}^{(0,2)})^b)_{a_2 c}}, \\
s_{80} &= \epsilon^{a_1 a_2 (i|v^j|b a_1 v^{c|k} a_2 (R_{16}^{(0,2)})^{|l|})_{bc}}, \\
s_{82} &= \epsilon^{a_1 a_2 (i|v^{jk} a_1 v^l b_{a_2} (R_{16}^{(0,2)})^b)}, \\
s_{84} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} a_1 v^{a_4} a_5 (R_{16}^{(0,2)})^{|l|})^{a_5}}_{a_2 a_3 a_4}, \\
s_{86} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{a_4} a_1 v^{a_5} a_2 (R_{16}^{(0,2)})^{|kl|})_{a_3 a_4 a_5}}, \\
s_{88} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^{a_5} a_2 (R_{16}^{(0,2)})^{|kl|})_{a_3 a_4 a_5}}, \\
s_{90} &= \epsilon^{a_1 a_2 (i|u^{jk} v^l)_{a_1} v^{a_3} a_4 (R_{16}^{(0,2)})^{a_4}}_{a_2 a_3}, \\
s_{92} &= \epsilon^{a_1 a_2 (i|u^{jk} v^{a_3} a_1 v^{a_4} a_2 (R_{16}^{(0,2)})^l)_{a_3 a_4}}, \\
s_{94} &= \epsilon^{a_1 a_2 (i|u^j|a_3 v^{|k|} a_1 v^{a_4} a_2 (R_{16}^{(0,2)})^l)_{a_3 a_4}}, \\
s_{96} &= \epsilon^{a_1 a_2 (i|u^{a_3 a_4} v^j a_1 v^k a_2 (R_{16}^{(0,2)})^l)_{a_3 a_4}}, \\
s_{98} &= \epsilon^{a_1 a_2 (i|u^{jk} v^l)_{a_1} v^b a_2 (R_{16}^{(0,2)})_b}, \\
s_{100} &= \epsilon^{a_1 a_2 (i|v^j a_1 v^{kl})_{a_2} (R_{18}^{(0,2)})}, \\
s_{102} &= u^{(ij} v^{kl} a (R_{20}^{(0,3)})^a), \\
s_{104} &= u^{(ijk} v^l a (R_{20}^{(0,3)})^a), \\
s_{106} &= u^{(ij} v^k a (R_{22}^{(0,3)})^l a),
\end{aligned}$$

$$\frac{ds_{10}^2}{R_{AdS}^2} = g_{\mu\nu} dx^\mu dx^\nu = ds_5^2 + ds^2(\mathbb{CP}^2) + (d\Psi + \Theta - A)^2$$

$$R_{AdS}^4 = 4\pi g_s N(\alpha')^2$$

$$ds^2(\mathbb{CP}^2) + (d\Psi + \Theta - A)^2 = \sum_{i=1}^3 dl_i^2 + l_i^2 (d\phi_i - A)^2$$

$$\sum_{i=1}^3 l_i^2 = 1$$

$$\Psi = \frac{\phi_1 + \phi_2 + \phi_3}{3}$$



$$2J = d\Theta = \sum_i d(l_i^2) \wedge d\phi_i$$

$$\Theta = \sum_i l_i^2 d\phi_i - d\Psi = \sum_i \left(l_i^2 - \frac{1}{3} \right) d\phi_i$$

$$\epsilon^{(10)} = \epsilon^{(5)} \wedge \epsilon^{(4)}(\mathbb{CP}^2) \wedge (d\Psi + \Theta - A), \epsilon^{(4)}(\mathbb{CP}^2) = \frac{1}{2} J \wedge J$$

$$ds_5^2 = -\frac{r^2 W(r)}{4b(r)^2} dt^2 + \frac{dr^2}{W(r)} + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2) + b(r)^2 (\sigma_3 + f(r) dt)^2$$

$$A = \frac{q}{r^2} \left(dt - \frac{j}{2} \sigma_3 \right)$$

$$\sigma_1 = \cos \psi d\theta_a + \sin \psi \sin \theta_a d\phi_a$$

$$\sigma_2 = -\sin \psi d\theta_a + \cos \psi \sin \theta_a d\phi_a$$

$$\sigma_3 = d\psi + \cos \theta_a d\phi_a$$

$$\sum_i \frac{\sigma_i^2}{4}$$

$$\Omega = f(R)$$

$$\mu = A_t(R) - \Omega A_{\sigma_3}(R)$$

$$A = \left(\frac{q}{r^2} - \mu\right) dt - \frac{j q}{2 r^2} \sigma_3$$

$$b(r)^2 = \frac{r^2}{4} \left(1 - \frac{j^2 q^2}{r^6} + \frac{2 j^2 p}{r^4} \right)$$

$$f(r) = -\frac{j}{2b^2} \left(\frac{2p-q}{r^2} - \frac{q^2}{r^4} \right)$$

$$W(r) = 1 + 4b^2 - \frac{1}{r^2} (2p - 2q) + \frac{1}{r^4} (q^2 + 2pj^2)$$

$$M = \frac{1}{2} (3p - 3q + pj^2)$$

$$Q = \frac{q}{2}$$

$$J = \frac{1}{2} j (2p - q)$$

$$T = \frac{RW'(R)}{8\pi b(R)}$$

$$\frac{F^{(5)}}{R_{AdS_5}^4} = G^{(5)} + *_{{10}} G^{(5)}$$

$$G^{(5)} = -4\epsilon^{(5)} + J \wedge *_5 F^{(2)}$$

$$dC = F^{(5)}$$



$$M-2J-3Q=0$$

$$\frac{C}{R_{AdS_5}^4}=C_V+\Theta\wedge(-*_5F+A\wedge F)+l_2^2d(l_3^2)\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3-A\wedge(d\Psi+\Theta)\wedge J$$

$$C_V=\frac{r^4-R^4}{8}dt\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3$$

$$\begin{aligned} ds^2 &= -\frac{Y}{R^2}dt^2 + \frac{R\rho^2}{Y}d\rho^2 + Rd\Omega_3^2 \\ A_i &= \frac{r_jr_k - \Gamma r_i}{H_i}dt \\ X_i &= \frac{R}{H_i} \end{aligned}$$

$$\begin{aligned} R &= (H_1H_2H_3)^{\frac{1}{3}} \\ Y &= R^3 + \rho^4 + \rho^2 \left(\frac{1}{3}(r_1^2 + r_2^2 + r_3^2) - \Gamma^2 \right) + \frac{1}{3}\Gamma^2(r_1^2 + r_2^2 + r_3^2) \\ &\quad - 2\Gamma r_1r_2r_3 + \left[\frac{5}{18}(r_1^2 + r_2^2 + r_3^2)^2 - \frac{1}{2}(r_1^4 + r_2^4 + r_3^4) \right] \\ H_i &= \rho^2 + \frac{\mu(s_i^2 - s_j^2) + \mu(s_i^2 - s_k^2)}{3} \\ \Gamma &= \sqrt{\mu}[c_1c_2c_3 - s_1s_2s_3] \\ r_i &= \sqrt{\mu}(c_is_js_k - s_ic_jc_k) \\ c_i &= \cosh \delta_i \\ s_i &= \sinh \delta_i \end{aligned}$$

$$E=\frac{\mu}{4}[\cosh 2\delta_1+\cosh 2\delta_2+\cosh 2\delta_3]$$

$$Q_i=\frac{\mu}{4}\sinh 2\delta_i$$

$$\begin{aligned} S &= \pi (H_1 H_2 H_3)^{\frac{1}{2}} \\ T &= \frac{Y'(\rho_0)}{4\pi \rho_0 (H_1 H_2 H_3)^{\frac{1}{2}}} \end{aligned}$$

$$\mu_i=\frac{3\mu\sinh 2\delta_i}{6\rho_0^2+\mu(2\cosh 2\delta_i-\cosh 2\delta_j-\cosh 2\delta_k)}$$

$$\begin{aligned} N_{\rm eff} &= N - m \\ E^{BH} &= E^{\rm tot} - \sum_{i=1}^3 Q_i^D \\ Q_i^{BH} &= Q_i^{\rm tot} - Q_i^D \\ J_i^{BH} &= J_i^{\rm tot} \end{aligned}$$



$$S_{BH} = N_{\text{eff}}^2 s_{BH} \left(\frac{E^{BH}}{N_{\text{eff}}^2}, \frac{Q_i^{BH}}{N_{\text{eff}}^2}, \frac{J_j^{BH}}{N_{\text{eff}}^2} \right)$$

$$N_{\text{eff}} = N \left(1 - \frac{m}{N} \right)$$

$$S_{BH} = N^2 s_{BH} \left(\frac{E^{BH}}{N^2}, \frac{Q_i^{BH}}{N^2}, \frac{J_j^{BH}}{N^2} \right) + 2Nm\beta \left(\frac{E^{BH}}{N^2} - \mu_i \frac{Q_i^{BH}}{N^2} - \omega_j \frac{J_j^{BH}}{N^2} - s_{BH} T \right)$$

$$\beta = N^2 \frac{\partial S_{BH}}{\partial E^{BH}}$$

$$\beta \mu_i = -N^2 \frac{\partial S_{BH}}{\partial Q_i^{BH}}$$

$$\beta \omega_j = -N^2 \frac{\partial S_{BH}}{\partial J_j^{BH}}$$

$$\frac{\partial S_{BH}}{\partial Q_i^p} = \beta(\mu_i - 1), Q^{BH} < Q^{\text{tot}}$$

$$\mu_i(Q_i^{BH} = Q_i^{\text{tot}}) < 1$$

$$2Nm\beta \left(\frac{E^{BH}}{N^2} - \mu_i \frac{Q_i^{BH}}{N^2} - \omega_j \frac{J_j^{BH}}{N^2} - s_{BH} T \right)$$

$$e^{-\beta q_i(1-\mu_i)}$$

$$Z_{\text{Dual}}^i(\beta, \mu_i) = \sum_{n=1}^{\infty} \left(Z_{\text{excit}}^{(n)}(\beta) e^{-\beta(1-\mu)n} \right) \approx \frac{Z_{\text{excit}}^{(\infty)}(\beta)}{\beta(1-\mu_i)}$$

$$\ln Z_{\text{Dual}} = -\ln(\beta(1-\mu_i))$$

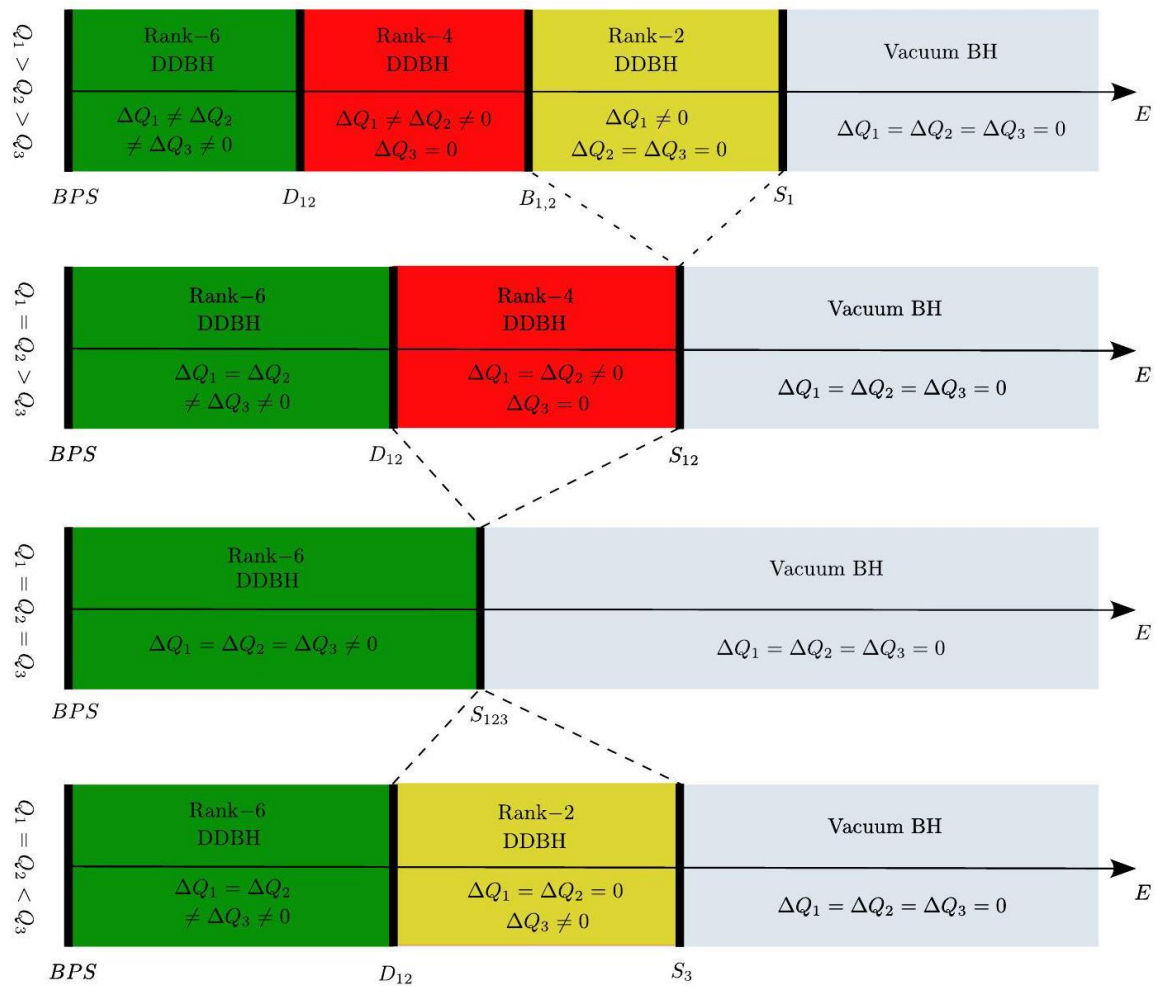
$$\ln Z_{\text{Tot}} = \ln Z_{\text{BH}} - \sum_{i=1}^3 \ln \beta(1-\mu_i)$$

$$Q_{\text{Tot}}^i(\beta, \mu_i) = Q_{BH}^i(\beta, \mu_i) + \frac{1}{\beta(1-\mu_i)}$$

$$\mu_i = 1 - \frac{\chi_i}{N^2}$$

$$\Delta E = \Delta Q_1 + \Delta Q_2 + \Delta Q_3$$





$$E = 3Q^2 + 3Q \approx 3Q^2$$

$$T = \frac{\tau}{\epsilon}, \mu = \frac{\nu}{\epsilon}$$

$$E = \frac{3(1+x^2)}{4\epsilon^4}$$

$$Q = \frac{x}{2\epsilon^3}$$

$$E = cQ^{\frac{4}{3}}$$

$$c = \frac{3}{4}(1+x^2)\left(\frac{2}{x}\right)^{4/3}$$

$$\frac{\nu}{\tau} = \frac{2\pi x}{2-x^2}$$

$$r' = \epsilon r$$

$$t' = \frac{t}{\epsilon}$$

$$x'_i = \frac{\theta_i}{\epsilon}$$



$$Q_{gg}=\frac{x}{2\epsilon^3}-\mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

$$Q_{BH}=\frac{\sqrt{1+x^2}}{2\epsilon^2}+\cdots$$

$$E_{BH}=\frac{3(1+x^2)}{4\epsilon^4}-\mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$r=\sqrt{\frac{Nx}{2}}\frac{1}{\epsilon^{3/2}},\text{ i.e }r'=\sqrt{\frac{Nx}{2}}\frac{1}{\epsilon^{1/2}}$$

$$ds^2=R_{AdS_5}^2g_{\mu\nu}dx^\mu dx^\nu$$

$$ds^2=R_{AdS_5}^2h_{ab}dx^adx^b$$

$$S=\frac{R_{AdS_5}^4}{(2\pi)^3\alpha'^2g_s}\Big(-\int d^4y\sqrt{-\text{deth}_{ab}}+\frac{1}{4!}\int \epsilon^{abcd}C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx^{\mu_1}}{dy^a}\frac{dx^{\mu_2}}{dy^b}\frac{dx^{\mu_3}}{dy^c}\frac{dx^{\mu_4}}{dy^d}d^4y\Big)$$

$$=\frac{N}{2\pi^2}\Big(-\int d^4y\sqrt{-\text{deth}_{ab}}+\frac{1}{4!}\int \epsilon^{abcd}C_{\mu_1\mu_2\mu_3\mu_4}\frac{dx^{\mu_1}}{dy^a}\frac{dx^{\mu_2}}{dy^b}\frac{dx^{\mu_3}}{dy^c}\frac{dx^{\mu_4}}{dy^d}d^4y\Big)$$

$$R_{AdS_5}^4=4\pi gN\alpha'^2$$

$$S=S_{DBI}+S_{WZ}$$

$$=2N\int r^2\sqrt{b^2+A_{\sigma_3}^2}\sqrt{-\tilde{g}_{\mu\nu}dx^\mu dx^\nu}+8N\int\left[\left(\frac{r^4-R^4}{8}\right)dt\right]$$

$$ds_7^2=\tilde{g}_{\mu\nu}dx^\mu dx^\nu=-\frac{r^2W}{4b^2}dt^2+\frac{dr^2}{W}+ds^2(\mathbb{CP}^2)+\frac{b^2}{b^2+A_{\sigma_3}^2}(d\Psi+\Theta+(fA_{\sigma_3}-A_t)dt)^2$$

$$\Delta=\tilde{g}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$$

$$m=2r^2\sqrt{b^2+A_{\sigma_3}^2}$$

$$\begin{pmatrix}q_1&0&0\\0&q_2&0\\0&0&0\end{pmatrix}$$

$$Q_{ij}=\frac{i}{2}\Big(z_ip_{z_j}-\bar{z}_jp_{\bar{z}_i}\Big)$$

$$z_i=l_ie^{i\phi_i}$$

$$p_{z_i}=\frac{\partial L}{\partial \dot{z}_i}=-\frac{i}{z_i}\left(\frac{\partial L}{\partial \phi_i}\right)+\frac{l_i}{z_i}\frac{\partial L}{\partial \dot{l}_i}$$

$$p_{\bar{z}_i}=\frac{\partial L}{\partial \dot{\bar{z}}_i}=\frac{i}{\bar{z}_i}\left(\frac{\partial L}{\partial \phi_i}\right)+\frac{l_i}{\bar{z}_i}\frac{\partial L}{\partial \dot{l}_i}$$

$$Q_{ii}\equiv Nq_i=\frac{\partial L}{\partial \dot{\phi}_i}$$

$$N \begin{pmatrix} q_1 & \frac{e^{i(\phi_1-\phi_2)}(l_2^2 q_1 + l_1^2 q_2 - 2il_1^2 l_2^2(i_1 - i_2))}{2l_1 l_2} & \frac{e^{i(\phi_1-\phi_3)}(l_3^2 q_1 + l_1^2 q_2 - 2il_1^2 l_3^2(i_1 - i_3))}{2l_1 l_3} \\ -\frac{e^{-i(\phi_1-\phi_2)}(l_2^2 q_1 + l_1^2 q_2 - 2il_1^2 l_2^2(i_1 - i_2))}{2l_1 l_2} & q_2 & \frac{e^{i(\phi_2-\phi_3)}(l_3^2 q_2 + l_2^2 q_3 - 2il_2^2 l_3^2(i_2 - i_3))}{2l_2 l_3} \\ -\frac{e^{-i(\phi_1-\phi_3)}(l_3^2 q_1 + l_1^2 q_3 - 2il_1^2 l_3^2(i_1 - i_3))}{2l_1 l_3} & -\frac{e^{-i(\phi_2-\phi_3)}(l_3^2 q_2 + l_2^2 q_3 - 2il_2^2 l_3^2(i_2 - i_3))}{2l_2 l_3} & \end{pmatrix}$$

$$l_1 = l_2 = 0, l_3 = 0, \frac{l_1^2}{q_1} = -\frac{l_2^2}{q_2}$$

$$l_1 = \sqrt{\frac{q_1}{q_1 - q_2}}, l_2 = \sqrt{\frac{q_2}{q_2 - q_1}}, l_3 = 0$$

$$q_1 = m \frac{l_1^2 \dot{\phi}_1 - l_1^2 \left(A_t - \frac{A_t A_{\sigma_3} + b^2 f - \sum_i (l_i^2 A_{\sigma_3} \dot{\phi}_i)}{b^2 + A_{\sigma_3}^2} A_{\sigma_3} \right)}{\sqrt{-\Delta}}$$

$$q_2 = m \frac{l_2^2 \dot{\phi}_2 - l_2^2 \left(A_t - \frac{A_t A_{\sigma_3} + b^2 f - \sum_i (l_i^2 A_{\sigma_3} \dot{\phi}_i)}{b^2 + A_{\sigma_3}^2} A_{\sigma_3} \right)}{\sqrt{-\Delta}}$$

$$L_{\text{eff}} = L - p_i \dot{\phi}_i$$

$$L_{\text{eff}} = N \left(-\frac{1}{2b^2} \sqrt{\frac{(r^2 W^2 - 4b^2 \dot{r}^2) \left((q_1 + q_2)^2 A_{\sigma_3}^2 + b^2 (m^2 + (q_1 - q_2)^2) \right)}{W}} \right. \\ \left. + (q_1 + q_2)(f A_{\sigma_3} - A_t) + r^4 - R^4 \right)$$

$$H = N \left(\frac{r\sqrt{W}}{2b^2} \sqrt{(q_1 + q_2)^2 A_{\sigma_3}^2 + b^2 (m^2 + W q_r^2 + (q_1 - q_2)^2)} + (q_1 + q_2)(A_t - f A_{\sigma_3}) - r^4 + R^4 \right)$$

$$V_{\text{cl}}(r) = N \left(\frac{r\sqrt{W}}{2b^2} \sqrt{(q_1 + q_2)^2 A_{\sigma_3}^2 + b^2 (m^2 + (q_1 - q_2)^2)} + (q_1 + q_2)(A_t - f A_{\sigma_3}) - r^4 + R^4 \right)$$

$$L_{\text{eff}} = N \left(-(q_1 + q_2) A_t(r) - \sqrt{\frac{(r^6 + (q_1 - q_2)^2)(W(r)^2 - \dot{r}^2)}{W(r)}} + r^4 - R^4 \right)$$

$$H = N \left((q_1 + q_2) A_t + \sqrt{W(r)((q_1 - q_2)^2 + q_r^2 W(r) + r^6)} - r^4 + R^4 \right)$$

$$V_{\text{cl}}(r) = N \left((q_1 + q_2) A_t + \sqrt{W(r)((q_1 - q_2)^2 + r^6)} - r^4 + R^4 \right)$$

$$W(r) = \frac{(r - R)(r + R)(r^4 + r^2(R^2 + 1) - \mu^2 R^2)}{r^4}$$

$$A_t(r) = \mu \left(\frac{R^2}{r^2} - 1 \right)$$



$$V_{\text{cl}}(r) = \frac{Nr\sqrt{W}}{2b^2} \sqrt{q_c^2 A_{\sigma_3}^2 + b^2(m^2 + q_n^2) + q_c(A_t - fA_{\sigma_3}) - r^4 + R^4}$$

$$V_{\text{cl}}(R) = N\mu q_c$$

$$V_{\text{cl}}(r) = N\frac{r^2}{2} + \mathcal{O}(1)$$

$$V_{\text{cl}}(r) = N\left(\sqrt{(1+r^2)(q_n^2+r^6)} - r^4\right)$$

$$V_{\text{cl}}(r) = N\left(\frac{q_n^2+r^4}{2r^2} + \left((j^2-1)p + \frac{qq_c}{r^2} + q - \frac{(q_n^2-r^4)^2}{8r^8} + R^4\right)\right)$$

$$r_{\min} = \sqrt{|q_n|} + \frac{qq_c}{2q_n^{3/2}}$$

$$E = V_{\text{cl}}(r_{\min}) = N\left(q_n + \left((j^2-1)p + \frac{qq_c}{q_n} + q + R^4\right)\right)$$

$$\mu_c > \frac{q_n}{q_c}$$

$$E = V_{\text{cl}}(r_{\min}) = N\left(q_n + ((j^2-1)p + 2q + R^4)\right)$$

$$(j^2-1)p + 2q + R^4$$

$$V_{\text{cl}}(r) = N\left(\frac{q_n^2+r^4}{2r^2} + \frac{-q_n^4 + 2q_n^2r^4 + 8\mu q_cr^6R^2 + r^8(4R^4 - 4(\mu^2+1)R^2 - 1)}{8r^8}\right)$$

$$r_{\min} = \sqrt{q_n} + \frac{(\mu q_c R^2)}{2q_n^{3/2}}$$

$$E = V_{\text{cl}}(r_{\min}) = q_n + \frac{R^2(q_n(-\mu^2 + R^2 - 1) + 2\mu q_c)}{2q_n}$$

$$q = R^2\left(1 + \frac{R^2}{2}\right)$$

$$p = 2R^2\left(1 + \frac{R^2}{2}\right)^2$$

$$j = \frac{R^2}{2l}\left(1 + \frac{R^2}{2}\right)^{-1}$$

$$V_{\text{cl}}(r) = q_c + \sqrt{A^2 + B} - A$$



$$A = q_c \left(\frac{2(r^2 - R^2)(2r^4 - r^2 R^4 + R^6)}{4r^6 + 4r^2 R^6 - R^8} \right) + r^4 - R^4$$

$$B = \frac{(r^2 - R^2)^2 (-2q_c(r^2 + R^2) + 2r^4 - r^2 R^4 + R^6)^2}{4r^6 + 4r^2 R^6 - R^8}$$

$$\sqrt{A^2 + B} - A \geq 0$$

$$q_c = f(r); f(r) = \frac{2r^4 - r^2 R^4 + R^6}{2(r^2 + R^2)}$$

$$r^2 = \frac{R^4 + 2q_c \pm \sqrt{(R^4 + 2q_c)^2 - 8(R^4 - 2q_c)R^2}}{4}$$

$$f(r) = \frac{R^2}{2} \left(4\sqrt{R^2 + 1} - R^2 - 4 \right)$$

$$\frac{R^2}{2} \left(4\sqrt{R^2 + 1} - R^2 - 4 \right) \leq q_c \leq \frac{R^2}{2}$$

$$h_{\mu\nu} = m(r)^2 \tilde{g}_{\mu\nu}$$

$$C_t = (r^4 - R^4).$$

$$\left(-\frac{1}{N^2}h^{\mu\nu}D_\mu D_\nu +1\right)\phi=0$$

$$(-h^{\mu\nu}D_\mu D_\nu + N^2)\phi=0$$

$$t=v-\int\frac{dr}{W(r)}$$

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -W(r)dv^2 + 2drdv + ds^2(\mathbb{CP}^2) + \frac{b^2}{b^2 + A_{\sigma_3}^2} \left(d\Psi + \Theta + (fA_{\sigma_3} - A_t) \left(dv - \frac{dr}{W} \right) \right)^2$$

$$C_v=C_t=r^4-R^4, C_r=-\frac{C_t}{W(r)}=\frac{R^4-r^4}{W(r)}$$

$$\left(-\Theta+(A_t-fA_{\sigma_3})dv-\frac{A_t-fA_{\sigma_3}}{W}dr\right).$$

$$-e^{-ik\Psi}\nabla^2\phi=-\frac{1}{m^2}(D_vD_r+D_rD_v)\chi-\frac{1}{m^7}D_r(m^5W(r)D^r\chi)$$

$$+\frac{k^2}{m^2}\left(\frac{b^2+A_{\sigma_3}^2}{b^2}\right)\chi-\frac{1}{m^2}\nabla_{CP^2,k}^2\chi$$



$$D_r = \partial_r + i \frac{k(A_t - fA_{\sigma_3}) - NC_t}{W}, D_v = (\partial_v + i(k(A_t - fA_{\sigma_3}) - iNC_t))$$

$$\chi = e^{-i\omega v} \phi_{n_Z, n_{\bar{Z}}} h(r)$$

$$m^2 \frac{e^{i\omega v}}{\phi_{n_Z, n_{\bar{Z}}}} (-\nabla^2 + N^2) \phi = -(D_v D_r + D_r D_v) h(r) - \frac{1}{m^5} D_r (m^5 W(r) D^r h(r)) \\ + \left(\left(\frac{b^2 + A_{\sigma_3}^2}{b^2} \right) (n_Z - n_{\bar{Z}})^2 + 4(n_Z n_{\bar{Z}} + n_Z + n_{\bar{Z}}) + N^2 m^2 \right) h(r)$$

$$D_v = -i\omega + i \left((n_Z - n_{\bar{Z}}) (A_t(r) - fA_{\sigma_3}(r)) - NC_t(r) \right)$$

$$h(r) = \frac{e^{i\omega r_s}}{\sqrt{W}} g(r)$$

$$r_s = \int_R^r \frac{dr}{W(r)}$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2}+V(r)\right)g(r)=0$$

$$V(r) = \frac{(m^5 W)''}{4N^2(m^5 W)} - \frac{((m^5 W)')^2}{8N^2(m^5 W)^2} - \frac{\left(\frac{\omega}{N} + r^4 - R^4 - (A_t - fA_{\sigma_3})\left(\frac{n_Z - n_{\bar{Z}}}{N}\right)\right)^2}{2W^2} + \frac{m^2}{2W} \\ + \frac{1}{2W} \left(\left(\frac{b^2 + A_{\sigma_3}^2}{b^2} \right) \left(\frac{n_Z - n_{\bar{Z}}}{N} \right)^2 + 4 \left(\frac{n_Z n_{\bar{Z}} + n_Z + n_{\bar{Z}}}{N^2} \right) \right)$$

$$g(r) \sim \sqrt{W(r)} e^{-i\omega r_s} \sim (r-R)^{-\frac{i\omega}{W'(R)} + \frac{1}{2}}$$

$$\frac{n_Z+n_{\bar{Z}}}{N}=q_n\\ \frac{n_Z-n_{\bar{Z}}}{N}=q_c$$

$$r\rightarrow\frac{r}{\epsilon}, \omega\rightarrow\frac{\omega}{\epsilon^2} \; n_Z\rightarrow\frac{n_Z}{\epsilon^2} \; n_{\bar{Z}}\rightarrow\frac{n_{\bar{Z}}}{\epsilon^2}$$

$$V(r)=\frac{r^2}{2}+\frac{(n_Z+n_{\bar{Z}})^2}{2N^2r^2}-\frac{\omega}{N}+\mu\frac{(n_{\bar{Z}}-n_Z)}{N}$$

$$\omega=N\left(\frac{r^2}{2}+\frac{(n_Z+n_{\bar{Z}})^2}{2N^2r^2}-\mu\frac{n_Z-n_{\bar{Z}}}{N}\right)$$

$$r_*^2=\frac{n_Z+n_{\bar{Z}}}{N}=q_n$$



$$\omega_* = (n_Z + n_{\bar{Z}}) - \mu(n_Z - n_{\bar{Z}}) = N(q_n - \mu q_c)$$

$$V''(r_*)|_{\omega=\omega_*}=4$$

$$V(r)=-\frac{(W'(R))^2+4\omega^2}{8N^2(W'(R)(r-R))^2}$$

$$\omega_\star\sim\mathcal{O}(Nq_n),$$

$$V(r)\sim -\frac{q_n^2}{(r-R)^2}$$

$$V(r)=q_n^2\sum_{n=-2}^{\infty}A_n(r-R)^n$$

$$V(r_*)\sim q_n-\mu q_c-\frac{\omega}{N}$$

$$r_*\sim\sqrt{q_n}$$

$$\frac{\max(\frac{\omega}{N},\frac{1}{N})^2}{q_n^2})\omega\sim\omega_*\sim\mathcal{O}(1)$$

$$V(r)=-\frac{\frac{1}{4}+\frac{\omega_q^{*2}}{W'(R)^2}}{2N^2(r-R)^2}+\frac{b}{N^2(r-R)}$$

$$\int \sqrt{V} dr \sim q_n$$

$$b=\frac{1}{8W'(R)}\bigg(4\bigg(\frac{j^2\mu^2R^4(n_Z-n_{\bar{Z}})^2}{R^6-j^2(q^2-2pR^2)}+n_Z^2+2n_Zn_{\bar{Z}}+2\bigg)+n_{\bar{Z}}^2+4n_{\bar{Z}}\bigg)+4m^2N^2\\ +W''(R)+4\frac{W''(R)}{W'^2(R)}\omega^2\bigg)$$

$$r_1-R=\frac{\frac{1}{4}+\frac{\omega^2}{W'(R)^2}}{2b}$$

$$r_1-R\sim\mathcal{O}\left(\max\left(\frac{\omega}{N},\frac{1}{N}\right)\right)$$

$$g(r)=\frac{1}{\sqrt{N}(2V(r))^{\frac{1}{4}}}\Big(a_1e^{N\int dr\sqrt{2V(r)}}+a_2e^{-N\int dr\sqrt{2V(r)}}\Big)$$

$$\frac{V'}{NV^{\frac{3}{2}}}\ll 1$$

$$V(r)\sim q_n^2(r-r_1)$$



$$\frac{1}{Nq_n(r-r_1)^{\frac{3}{2}}}$$

$$|r-r_1|\gg\frac{1}{(Nq_n)^{\frac{2}{3}}}$$

$$|r-r_1|\ll 1$$

$$\frac{1}{(Nq_n)^{\frac{2}{3}}}\ll 1$$

$$V=\frac{\alpha^2}{(r-R)^2}+\frac{q_n^2}{(r-R)}$$

$$\alpha^2=\max\left(\frac{\omega^2}{N^2},\frac{1}{N^2}\right)$$

$$r_1-R\sim\frac{\alpha^2}{q_n^2}$$

$$|r_1-r|\ll r_1-R\sim\frac{\alpha^2}{q_n^2}$$

$$V(r)\sim\frac{q_n^6}{\alpha^4}(r-r_1)$$

$$|r-r_1|\gg\left(\frac{\alpha^2}{Nq_n^3}\right)^{\frac{2}{3}}$$

$$\alpha\gg\frac{1}{N}$$

$$|\omega|\gg 1$$

$$\omega_*=N(q_n-\mu q_c)=-Nq_c\left(\mu-\frac{\mu q_n}{q_c}\right)$$

$$\left|\mu-\frac{q_n}{q_c}\right|\gg\frac{1}{Nq_c}$$

$$\omega=\omega_*^q+\epsilon$$

$$\omega_*^q=\omega_*+1+\cdots$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2}+\frac{V''(r_*)}{2}(r-r_*)^2\right)g(r)=\left(\frac{\sqrt{V''(r_*)}}{2N}+\epsilon\right)g(r)$$

$$r_*=\frac{\sqrt{n_Z+n_{\bar{Z}}}}{N}V''(r_*)=4$$

$$\left(-\frac{1}{2N^2}\frac{\partial^2}{\partial r^2}+2(r-r_*)^2\right)g(r)=\left(\frac{1}{N}+\epsilon\right)g(r)$$

$$g(r)=c_1D_{\frac{N\epsilon}{2}}\Big(2\sqrt{N}(r-r_*)\Big)+c_2D_{\frac{N\epsilon}{2}-1}\Big(2i\sqrt{N}(r-r_*)\Big)$$

$$g(r)=c_1e^{-N(r-r_*)^2}-ic_2\frac{e^{N(r-r_*)^2}}{2\sqrt{N}(r-r_*)}$$

$$g(r)=c_1D_{\frac{N\epsilon}{2}}\Big(2\sqrt{N}(r-r_*)\Big)$$

$$r_*-r\gg\frac{1}{\sqrt{N}},g(r)$$

$$\begin{aligned} g(r) &= c_1 \left(\frac{\sqrt{\pi} e^{N(r-r_*)^2}}{\sqrt{2N}(r_*-r)\Gamma\left(-\frac{1}{2}(N\epsilon)\right)} + e^{-N(r-r_*)^2} \right) \\ &= c_1 \left(\frac{-\epsilon\sqrt{N\pi}}{2\sqrt{2}(r_*-r)} e^{N(r-r_*)^2} + e^{-N(r-r_*)^2} \right) \end{aligned}$$

$$g(r)=\frac{1}{(2V(r))^{1/4}}\bigg(A\exp\left(-\int_r^{r^*}dr'\sqrt{2V(r')}\right)+B\exp\left(\int_r^{r^*}dr'\sqrt{2V(r')}\right)\bigg)$$

$$A=\frac{i}{\sqrt[4]{e}},B=-i\sqrt{\frac{\pi}{2}}\sqrt[4]{e}N^{3/2}\epsilon$$

$$g(r)=\frac{1}{(2V(r))^{1/4}}\Big(A\exp\left(\int_{r_1}^rN\sqrt{2V}\right)e^{-\beta}+B\exp\left(-\int_{r_1}^rN\sqrt{2V}\right)e^{\beta}\Big)$$

$$\beta=\int_{r_1}^{r^*}N\sqrt{2V}$$

$$g(r)\sim\frac{Ae^{-\beta}}{(2V)^{-\frac{1}{4}}}\left(\sin\left(\int_{r_1}^rN\sqrt{-2V}+\frac{\pi}{4}\right)+\frac{2B}{A}e^{2\beta}\cos\left(\int_{r_1}^rN\sqrt{-2V}+\frac{\pi}{4}\right)\right)$$

$$V(r)=-\frac{\left(\omega_*^q\right)^2}{2N^2W'(R)^2(r-R)^2}$$

$$\sqrt{-2V}=\frac{\left|\omega_*^q\right|}{N\sqrt{W'(R)}(r-R)}$$

$$g(r) = (-1)^{1/4} C (r - R)^{1/2} \left(\left(-i + \frac{2B}{A} e^{2\beta} \right) e^{\frac{i|\omega_q^*|}{\sqrt{W'(R)}} \log(r-R)} + \left(\frac{1}{2} - i \frac{B}{A} e^{2\beta} \right) e^{-i \frac{|\omega_q^*|}{\sqrt{W'(R)}} \log(r-R)} \right) \\ = (-1)^{1/4} C (r - R)^{1/2} \left(\left(-i + \frac{2B}{A} e^{2\beta} \right) e^{i|\omega_q^*| r_s} + \left(\frac{1}{2} - i \frac{B}{A} e^{2\beta} \right) e^{-i|\omega_q^*| r_s} \right)$$

$$C = A e^{-\beta} \left(\frac{|\omega_*^q|}{N \sqrt{W'(R)}} \right)^{-\frac{1}{2}}$$

$$\frac{2B}{A} e^{2\beta} = i \\ \epsilon = -\frac{i}{N^{3/2} \sqrt{2e\pi}} \exp(-2\beta)$$

$$\omega = \omega_*^q - \frac{i}{N^{3/2} \sqrt{2e\pi}} \exp(-2\beta)$$

$$\frac{2B}{A} e^{2\beta} = -i \\ \epsilon = \frac{i}{N^{3/2} \sqrt{2e\pi}} \exp(-2\beta)$$

$$\frac{\omega}{N} \ll q_n.$$

$$1 \ll \omega \ll Nq_n$$

$$g(r) = 2\sqrt{2b(r-R)}f(y)$$

$$y=2\sqrt{2b(r-R)}$$

$$y^2 f''(y) + y f'(y) + \left(\frac{4(\omega_*^q)^2}{W'(R)^2} - y^2 \right) f(y) = 0$$

$$f=c_1\tilde{I}_\nu(y)+c_2\tilde{K}_\nu(y)$$

$$\tilde{I}_\nu(y)=\mathrm{Re}(I_{i\nu}(y)), \tilde{K}_\nu(y)=K_{i\nu}(y)$$

$$\nu=\frac{2\omega_*^q}{W'}$$

$$f^{y\rightarrow 0}c_1\left(\frac{\sinh{(\pi\nu)}}{\pi\nu}\right)^{\frac{1}{2}}\cos{(\nu\log{(y/2)}-\gamma_\nu)}-c_2\left(\frac{\pi}{\nu\sinh{(\pi\nu)}}\right)^{\frac{1}{2}}\sin{(\nu\log{(y/2)}-\gamma_\nu)}$$

$$\frac{c_2}{c_1}=\frac{i\sinh{\pi\nu}}{\pi}$$

$$\tilde{I}_\nu(y) \approx \frac{1}{\sqrt{2\pi y}} e^y$$

$$\tilde{K}_\nu(y) \approx \sqrt{\frac{\pi}{2y}} e^{-y}$$

$$f^{y \rightarrow \infty} \frac{c_1}{\sqrt{2\pi y}} (e^y + i \sinh(\pi \nu) e^{-y})$$

$$g(r) = \frac{c_1 \sqrt{2} b^{1/4} (r-R)^{1/4}}{\sqrt{2\pi}} \left(e^{2\sqrt{2b(r-R)}} + i \sinh \left(\frac{2\pi \omega_*^q}{W'(R)} \right) e^{-2\sqrt{2b(r-R)}} \right)$$

$$g(r) = \frac{\sqrt{N^4} \sqrt{(r-R)}}{\sqrt[4]{2} \sqrt{b}} \left(A \exp(-\beta) e^{2\sqrt{2b(r-R)}-\alpha} + B \exp(\beta) e^{\alpha-2\sqrt{2b(r-R)}} \right)$$

$$\alpha = \pi \sqrt{\frac{1}{4} + \frac{(\omega_*^q)^2}{W'(R)^2}}$$

$$\frac{B}{A} e^{2\beta} e^{2\alpha} = i \sinh \left(\frac{2\pi \omega_*^q}{W'(R)} \right)$$

$$\epsilon = -\frac{1}{N^{3/2}} \sqrt{\frac{2}{e\pi}} \left(i \sinh \left(\frac{2\pi \omega_*^q}{W'(R)} \right) \right) e^{-2\beta} e^{-2\alpha}$$

$$\omega \gg 1, \alpha \rightarrow \frac{\pi |\omega_*^q|}{W'(r)}$$

$$\epsilon = -\frac{i}{N^{3/2} \sqrt{2e\pi}} e^{-2\beta}$$

$$\epsilon = \frac{i}{N^{3/2} \sqrt{2e\pi}} e^{-2\beta}$$

$$J_\mu = -i(\phi^* D_\mu \phi - \phi D_\mu \phi^*)$$

$$\sqrt{g}J^v = 2 \left(\frac{(A_t - f A_{\sigma_3})(n_Z - n_{\bar{Z}}) + r^4 - R^4}{W} \right) |h|^2 e^{2\text{Im}(\omega)v} - i(h^* \partial_r h - h \partial_r h^*) e^{2\text{Im}(\omega)v}$$

$$\sqrt{g}J^r = -2(\text{Re}(\omega)) |h|^2 e^{2\text{Im}(\omega)v} - iW(h^* \partial_r h - h \partial_r h^*) e^{2\text{Im}(\omega)v}$$

$$h^* \partial_r h - h \partial_r h^* = (g^* \partial_r g - g \partial_r g^*) e^{-2\text{Im}(\omega)r_s} + \frac{2i\text{Re}(\omega)}{W} |g|^2 e^{-2\text{Im}(\omega)r_s}$$

$$\sqrt{g}J^v = \frac{1}{W} \left(\left(\frac{2(A_t - f A_{\sigma_3})}{W} \right) |g|^2 - i(g^* \partial_r g - g \partial_r g^*) \right) e^{2\text{Im}(\omega)v} e^{-2\text{Im}(\omega)r_s}$$

$$\sqrt{g}J^r = -i(g^* \partial_r g - g \partial_r g^*) e^{2\text{Im}(\omega)v} e^{-2\text{Im}(\omega)r_s}$$

$$J^r(R)=\int_r^{\infty}\partial_vJ^v$$

$$J^r(R)=2\omega_i\int_r^{\infty}J^v$$

$$J^r(R)=\begin{cases} 0, & r-r_1\gg\frac{1}{\alpha}\\ 2\omega_i, & r-r_1\ll\frac{1}{\alpha} \end{cases}$$

$$\sqrt{g}J^r(R)=-8N^2\mathrm{sgn}(\omega_*^q)|A|^2e^{-2\beta}W'(R)e^{2\omega_iv}$$

$$\omega_*^q\left(\frac{\sinh\left(\frac{2\pi|\omega_*^q|}{W'(R)}\right)}{\frac{2|\omega_q^*|}{W'(R)}}\right)\approx\pi\frac{\omega_q^*}{W'(R)}$$

$$q_c^{\mathrm{crit}}(\delta)$$

$$\omega=\omega_*^q-\frac{i}{N^{3/2}}\sqrt{\frac{2}{e\pi}}\left(\sinh\left(\frac{2\pi\omega_*^q}{W'(r)}\right)\right)e^{-2\beta}e^{-2\alpha}$$

$$\sum_{i=0}^{n_z}\sum_{j=0}^{n_z-i}\sum_{k=0}^{2i}(2n_z+i-j-k,n_z-i+k,j)$$

$$\sum_{i=0}^{n_z}\sum_{j=0}^{n_z-i}\sum_{k=0}^{2i}(Q+2n_z+i-j-k,Q+n_z-i+k,Q+j)$$

$$(Q+n_Z,Q+n_Z,Q+n_Z)$$

$$\sum_{i=0}^{n_Z}(Q+n_Z+i,Q+n_Z-i,Q_1)$$

$$(Q_1+n_Z,Q_2,Q_3)$$

$$\sum_{i=0}^{n_Z}(n_Z+i,n_Z-i,0).$$

$$ds^2=l_{N-1}^2(ds_{(0)}^2+ds_{(1)}^2)$$

$$F^{(5)}=F_{(0)}^{(5)}+F_{(1)}^{(5)}$$

$$ds_{(0)}^2=(ds_{AdS_5}^2+ds_{S^5}^2)$$

$$(l_{N-1})^4=4\pi g_s(N-1)(\alpha')^2$$



$$ds_{AdS_5}^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_3^2$$

$$ds_{S^5}^2 = \sin^2 \theta d\tilde{\Omega}_3^2 + \cos^2 \theta d\phi^2 + d\theta^2$$

$$\frac{F_{(5)}^{(0)}}{(l_{N-1})^4} = -4\epsilon_5 + 4\tilde{\epsilon}_5$$

$$ds_{(1)}^2 = \rho_1^2 \left[-\frac{(r^2 + \sin^2 \theta)^2}{2x^4} r^2 d\Omega_3^2 + \frac{(r^2 + \sin^2 \theta)^2}{2x^4} \sin^2 \theta d\tilde{\Omega}_3^2 \right. \\ + \left(\frac{(r^2 + \cos^2 \theta)^2 - 4(1+r^2)\cos^2 \theta - 1 + 4d\sqrt{1+r^2}\cos \theta \cos(\phi-t)}{2x^4} \right) (1+r^2)dt^2 \\ + \left(\frac{r^4 - \sin^4 \theta}{2x^4} \right) \frac{dr^2}{1+r^2} + \left(\frac{r^4 - \sin^4 \theta}{2x^4} \right) d\theta^2 \\ + \left(\frac{(r^2 + \cos^2 \theta)^2 - 4(1+r^2)\cos^2 \theta - 1 + 4d\sqrt{1+r^2}\cos \theta \cos(\phi-t)}{2x^4} \right) \cos^2 \theta d\phi^2 \\ - \frac{2d\sin(\phi-t)}{x^4} (dt(1+r^2) - \cos^2 \theta d\phi) d(\sqrt{1+r^2}\cos \theta) \\ \left. + 2d\phi dt \frac{\cos \theta (2(1+r^2)\cos \theta - d\sqrt{1+r^2}(1+r^2 + \cos^2 \theta)\cos(\phi-t))}{x^4} \right] + O(\rho_1^4)$$

$$x^2 = \rho_0^2(r^2 + \cos^2 \theta) + d^2 - 2d\rho_0\sqrt{1+r^2}\cos \theta \cos(\phi-t)$$

$$F_{(1)}^{(5)} = 4(F \wedge d\Omega_3 + \tilde{F} \wedge d\tilde{\Omega}_3)$$

$$F_{tr} = \frac{2(d^2-1)r\sin^4 \theta(r^2 + \sin^2 \theta)}{2Nx^6}, F_{t\phi} = \frac{d\sqrt{r^2+1}\sin^3 \theta \sin 2\theta(\sin^2 \theta + r^2)^2 \sin(t-\phi)}{2Nx^6}$$

$$F_{t\theta} = \frac{\sqrt{1+r^2}\sin^3 \theta}{8Nx^6} \left[8\sqrt{r^2+1}\cos \theta ((d^2(r^2+2) + 2r^2 + 1)\cos \theta + d^2\cos^2 \theta \cos(2t-2\phi)) \right. \\ \left. - 4d\cos(t-\phi)((d^2+3r^2+2)\cos 2\theta + d^2 + 2r^4 + 7r^2 + 4) \right]$$

$$F_{r\theta} = \frac{dr\sin^3 \theta \sin(t-\phi)((d^2+3r^2+2)\cos 2\theta - d^2 + 4d\sqrt{r^2+1}\cos^3 \theta \cos(t-\phi) + 2r^4 + r^2)}{2N\sqrt{r^2+1}x^6}$$

$$F_{r\phi} = \frac{2r\sin^4 \theta \cos \theta(r^2 + \sin^2 \theta)(d(\cos 2\theta + 2r^2 + 3)\cos(t-\phi) - 4\sqrt{r^2+1}\cos \theta)}{4N\sqrt{r^2+1}x^6}$$

$$F_{\theta\phi} = \frac{\sin^3 \theta \cos \theta}{4Nx^6} [2\cos^2 \theta (2d^2(2(r^2+1)\cos^2(t-\phi) + 2r^2 + 1) + \sin^2 \theta (4d^2 + 3r^4 + 2r^2 - 1) \\ + 2(2r^2 + 1)\sin^4 \theta) \\ + 2d\sqrt{r^2+1}\cos(t-\phi)\cos \theta (2\cos^2 \theta (2\cos 2\theta - 5(r^2+1)) - (2(d^2-r^4) + \sin^2 \theta (-\cos 2\theta + 2r^2 + 1))) \\ - \frac{1}{4}(r^2+1)(16r^4 - 8r^2\cos 2\theta + (r^2+1)\cos 4\theta + 7r^2 - 1 + 4\cos^4 \theta (-\cos 2\theta + 2r^2 + 2))]]$$



$$\begin{aligned}\tilde{F}_{tr} &= \frac{r^3}{32Nx^6} [(48d^2(r^2+1) - 16r^4 + 17)\cos 2\theta - \cos 6\theta + 2(6r^2+5)\cos 4\theta \\ &\quad + 2(-8(4d^2+3)r^4 - 6(4d^2+1)r^2 + 2d\cos \theta (\sqrt{r^2+1}\cos(t-\phi)(-8d^2 + \cos 4\theta - 8(2r^2+3)\cos 2\theta \\ &\quad + 24(r^4+r^2) - 1) + 8d(r^2+1)\cos \theta \cos(2t-2\phi)) + 8d^2 - 16r^6 + 3)] \\ \tilde{F}_{t\theta} &= -\frac{4r^4\sin \theta(r^2+\sin^2 \theta)(4(r^2+1)\cos \theta - d\sqrt{r^2+1}(\cos 2\theta + 2r^2+3)\cos(t-\phi))}{8Nx^6} \\ \tilde{F}_{t\phi} &= -\frac{dr^4\sqrt{r^2+1}\cos \theta(\sin^2 \theta + r^2)^2\sin(t-\phi)}{Nx^6} \\ \tilde{F}_{r\theta} &= \frac{-dr^3\sin \theta\sin(t-\phi)}{8N\sqrt{r^2+1}x^6} [8d(r^2+1)(d-2(r^2+1)^{1/2}\cos \theta\cos(t-\phi)) - \cos 4\theta + 4(3r^2+2)\cos 2\theta \\ &\quad + 4r^2+1]\end{aligned}$$

$$\begin{aligned}\tilde{F}_{r\phi} &= -\frac{r^3\cos \theta}{8Nx^6} \left[\frac{\cos(t-\phi)(d(8d^2(r^2+1) + \cos 4\theta + 4(3r^2+4)\cos 2\theta + 4r^2+7))}{\sqrt{r^2+1}} \right. \\ &\quad \left. - 4\cos \theta((d^2+2)\cos 2\theta - 2d^2(r^2+1)\cos(2t-2\phi) + 3d^2) \right] \\ \tilde{F}_{\theta\phi} &= \frac{2(d^2-1)r^4\sin(2\theta)(-\cos(2\theta) + 2r^2+1)}{8Nx^6}\end{aligned}$$

$$\begin{aligned}& -\frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h^\alpha_\alpha + \frac{1}{2}\nabla_\alpha\nabla_\nu h^\alpha_\mu + \frac{1}{2}\nabla_\alpha\nabla_\mu h^\alpha_\nu - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \frac{1}{2}R_{\nu\alpha}h^\alpha_\mu + \frac{1}{2}R_{\mu\alpha}h^\alpha_\nu \\ &= \left(2\left(F_{(1)}^{(5)}\right)_{\mu\rho\sigma\delta\kappa}\left(F_{(0)}^{(5)}\right)^\rho{}_{\sigma\delta\kappa} - 4h^{\alpha\rho}\left(F_{(0)}^{(5)}\right)_{\mu\rho\sigma\delta\kappa}\left(F_{(0)}^{(5)}\right)^\sigma{}_{\delta\kappa} \right) \\ F_{(1)}^{(5)} &= \star F_{(1)}^{(5)} + \frac{1}{2}h^\alpha_\alpha \star F_{(0)}^{(5)} - 5\sqrt{-g}\epsilon_{\mu_1\mu_2\cdots\mu_{10}}h^{\mu_6\nu_6}g^{\mu_7\nu_7}\cdots g^{\mu_{10}\nu_{10}}\left(F_{(0)}^{(5)}\right)_{\nu_6\cdots\nu_{10}}dx^{\mu_1}\wedge\cdots dx^{\mu_5}\end{aligned}$$

$$ds^2=l_{N-1}^2\left(\frac{dr^2}{d^2}-dt^2\times d^2+(d^2-1)d\alpha_i^2+dy_j^2+d\phi^2\right)$$

$$\theta \hat{n}^j = y^j, j = 1 \ldots 4$$

$$\begin{aligned}l_{N-1}\theta\hat{n}^j &= y^j, & j &= 1 \ldots 4 \\ l_{N-1}\left(r-\sqrt{d^2-1}\right) &= d\times y^5 & i &= 1 \ldots 3 \\ l_{N-1}\alpha_i &= \frac{x_i}{\sqrt{d^2-1}}, \\ l_{N-1}(\phi-t) &= \frac{\sqrt{d^2-1}}{d}y^6 \\ l_{N-1}x^0 &= \frac{d^2t-\phi}{\sqrt{d^2-1}}\end{aligned}$$

$$ds^2=dx_\mu dx^\mu+dy_idy^i, \mu=0\ldots 3, i=1\ldots 6$$

$$x^2=(d^2-1)\frac{(y_iy^i)}{l_{N-1}^2}$$

$$F_{y^i x^0 x^1 x^2 x^3} = -16\pi\alpha' g_s^2 \frac{y^i}{(y^j y_j)^3}$$



$$h_{\mu\nu} = -2\pi\alpha' g_s^2 \frac{1}{(y_j y^j)^2} \eta_{\mu\nu}$$

$$h_{ij} = 2\pi\alpha' g_s^2 \frac{1}{(y_j y^j)^2} \delta_{ij}$$

$$G_{AB} = \frac{1}{2} (\partial^C \partial_A h_{BC} + \partial^C \partial_B h_{AC} - \square h_{AB} - \partial_A \partial_B h - \partial^C \partial^D h_{CD} \eta_{AB} + \square h \eta_{AB}) = 8\pi G_N T_{AB} \nabla_i F^{i\mu\nu\rho\sigma}$$

$$= -16\pi^4 g_s \alpha^2 \epsilon^{\mu\nu\rho\sigma} \delta^6(\vec{y})$$

$$\frac{ds_{(1)}^2}{l_{N-1}^2} \approx \left(\frac{r^2}{2}\right) \frac{dt^2}{N} + \left(\frac{1}{2r^2}\right) \frac{dr^2}{N} - \frac{r^2}{2N} d\Omega_3^2 + \frac{1}{2N} d\Omega_5^2$$

$$r=r'+\frac{r'}{2N}$$

$$\frac{ds_{(1)}^2}{l_{N-1}^2} \approx \frac{1}{2N} \left(-r^2 dt^2 + \left(\frac{dr^2}{r^2} \right) + r^2 d\Omega_3^2 + d\Omega_5^2 \right)$$

$$= \frac{1}{2N} (ds_{AdS_5}^2 + d\Omega_5^2)$$

$$ds^2=ds_{(0)}^2+ds_{(1)}^2=l_{N-1}^2\left(1+\frac{1}{2N}\right)(ds_{AdS_5}^2+d\Omega_5^2)=l_N^2(ds_{AdS_5}^2+d\Omega_5^2)$$

$$\frac{l_N^2}{l_{N-1}^2}=1+\frac{1}{2N}$$

$$r=r'+\frac{1}{N}\left(d\cos\theta'\cos(\phi'-t')-\frac{2\cos^2\theta(1-3d^2\cos^2(\phi'-t'))+(d^2-1)}{2r'}\right)$$

$$\phi=\phi'-\frac{d\sin(\phi'-t')}{r'\cos\theta'N}$$

$$\theta=\theta'-\frac{d\sin\theta'\cos(\phi'-t')}{Nr'}$$

$$\frac{ds^2}{l_N^2} \approx ds_{AdS_5 \times S^5}^2 + \frac{1}{N} \Big[-(2\cos^2\theta(1-3d^2\cos^2(\phi-t))+(d^2-1))dt^2$$

$$+ \frac{4\cos^2\theta(1-2d^2\cos^2(\phi-t))}{r^2}d\phi dt + \left(\frac{\sin^2\theta(\cos^2\theta(6d^2\cos^2(\phi-t)-1)-d^2+\sin^2\theta)}{r^2} \right) d\tilde{\Omega}_3^2$$

$$+ \left(\frac{\cos^2\theta(6d^2\cos^2(\phi-t)-1)-d^2}{r^4} \right) dr^2 + (\cos^2\theta(1-6d^2\cos^2(\phi-t))+d^2-\sin^2\theta) d\Omega_3^2$$

$$+ \left(\frac{2d\cos\theta\cos(\phi-t)}{r} + \frac{\cos^2\theta(6d^2\cos^2(\phi-t)-1)-d^2}{r^2} \right) d\theta^2 - \left(\frac{d^2\sin(2\theta)\sin(2t-2\phi)}{r^2} \right) d\theta dt \Big]$$

$$A_{\mu}^{\zeta}=-3\frac{\int d\Omega_5\zeta^{\alpha}g_{\alpha\mu}}{\int d\Omega_5}$$

$$A=\frac{2(d^2-1)}{N}\frac{dt}{r^2}$$



$$Q = N(d^2 - 1)$$

$$ds^2 = \left(-(1+r^2) + \frac{6d^2 + 6r^4 + 2r^2 - 7}{6Nr^2} \right) dt^2 + \left(\frac{r^4 - r^2 + 1}{r^6} - \frac{r^2 - 1}{3Nr^6} \right) dr^2 \\ + \left(r^2 - \frac{-2d^2 + 6r^4 + 2r^2 + 1}{6Nr^2} \right) d\Omega_3^2$$

$$n^\mu = \left(0, \frac{1}{\sqrt{g_{rr}}}, 0, 0, 0 \right)$$

$$\Theta^\mu_\nu = -\frac{1}{2}(\nabla^\mu n_\nu + \nabla_\nu n^\mu)$$

$$T^\mu_\nu = \frac{N^2 r^4}{4\pi^2} \left(\Theta^\mu_\nu - \Theta \gamma^\mu_\nu - 3 \gamma^\mu_\nu + \frac{1}{2} G^\mu_\nu \right)$$

$$T^0_0 = \frac{(N-1)^2}{4\pi^2} \left(\frac{3}{8} \left(1 + \frac{2}{N} \right) + \frac{2(d^2-1)}{N} \right) \approx \frac{N^2}{4\pi^2} \left(\frac{3}{8} + \frac{2(d^2-1)}{N} \right) \\ T^i_j = -\frac{\delta^j_i}{3} \frac{(N-1)^2}{4\pi^2} \left(\frac{3}{8} \left(1 + \frac{2}{N} \right) + \frac{2(d^2-1)}{N} \right) \approx -\frac{\delta^j_i}{3} \frac{N^2}{4\pi^2} \left(\frac{3}{8} + \frac{2(d^2-1)}{N} \right)$$

$$E = \int d^3x \sqrt{\gamma} T^0_0 = \frac{N^2}{2} \left(\frac{3}{8} \left(1 + \frac{2}{N} \right) + \frac{2(d^2-1)}{N} \right)$$

$$\delta \tilde{F} = -\frac{\sin^3\theta \cos\theta}{N} d\theta \wedge d\phi$$

$$\delta F = \frac{r'^3}{N} dt \wedge dr'$$

$$F_{(5)} = l_{N-1}^4 \left(1 + \frac{1}{N} \right) (-4\epsilon_5 + 4\tilde{\epsilon}_5) \approx l_N^4 (-4\epsilon_5 + 4\tilde{\epsilon}_5)$$

$$F_5 = \sqrt{g'} \tilde{\epsilon}_5 \chi$$

$$dx^\mu = dx'^\mu + \frac{1}{N} \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

$$\chi = \sum_k (-k)(k+4)(-s^I+t^I)Y^I$$

$$s^I = -\frac{\alpha^I}{r^k}$$

$$\alpha^I = \frac{4i^k}{N(k+4)} \left(\frac{(k^2-1)}{\sqrt{2(k+1)(k+2)}} \right) (d)^k$$



$$ds_{(1)}^2 = \frac{1}{d^3 N} \left[(2(1+r^2)^{1/2} \cos \theta \cos(t-\phi))((1+r^2)dt^2 + \cos^2 \theta d\phi^2) \right. \\ \left. + \left(\frac{2r \cos \theta \sin(t-\phi)}{\sqrt{r^2+1}} dr - 2((r^2+1)^{1/2} \sin \theta \sin(t-\phi))d\theta \right) ((1+r^2)dt - \cos^2 \theta d\phi) \right. \\ \left. - 2(\sqrt{r^2+1} \cos \theta (\cos^2 \theta + r^2 + 1) \cos(t-\phi)) dt d\phi \right] + \mathcal{O}\left(\frac{1}{d^4 N}\right)$$

$$F_{t\theta} = F_{\theta\phi} = -\frac{\sqrt{r^2+1} \sin^3 \theta \cos^2 \theta \cos(t-\phi)}{d^3 N}, \quad F_{r\theta} = -\frac{r \sin^3 \theta \cos^2 \theta \sin(t-\phi)}{d^3 N \sqrt{r^2+1}} \\ \tilde{F}_{tr} = \tilde{F}_{r\phi} = -\frac{r^3 \sqrt{r^2+1} \cos \theta \cos(t-\phi)}{d^3 N}, \quad \tilde{F}_{r\theta} = -\frac{r^3 \sqrt{r^2+1} \sin \theta \sin(t-\phi)}{d^3 N}$$

$$\phi' = \phi - \frac{\sqrt{r^2+1} \cos \theta \sin(t-\phi)}{x^3 N} \\ t' = t - \frac{\sqrt{r^2+1} \cos \theta \sin(t-\phi)}{x^3 N}$$

$$ds_{(1)}^2 = \left(\frac{1}{2Nx^4} \right) \left[(r^2 + \sin^2 \theta)^2 (r^2 d\Omega_3^2 + \sin^2 \theta d\tilde{\Omega}_3^2) + (r^4 - \sin^4 \theta) \left(\frac{dr^2}{1+r^2} + d\theta^2 \right) \right. \\ \left. + \left((r^2 + \cos^2 \theta)^2 - 4(1+r^2) \cos^2 \theta - 1 + 4d\sqrt{1+r^2} \cos \theta \cos(\phi-t) \right) \right. \\ \left. + \frac{4\sqrt{1+r^2} \cos \theta}{x} \left(3d\sqrt{1+r^2} \cos \theta \sin^2(t-\phi) - x^2 \cos(t-\phi) \right) \right] ((1+r^2)dt^2 + \cos^2 \theta d\phi^2) \\ + \left(4 \left(2(1+r^2) \cos \theta - d\sqrt{1+r^2} (1+r^2 + \cos^2 \theta) \cos(\phi-t) \right) + \right. \\ \left. + \frac{2\sqrt{1+r^2} (3+2r^2 + \cos 2\theta)}{x} \left(-3d\sqrt{1+r^2} \cos \theta \sin^2(t-\phi) + x^2 \cos(t-\phi) \right) \right) \cos \theta d\phi dt \\ + \frac{12}{x} \sqrt{1+r^2} \cos \theta \sin(\phi-t) (dt(1+r^2) - \cos^2 \theta d\phi) (rdr - \sin \theta \cos \theta d\theta) \\ + \sin(\phi-t) (dt(1+r^2) - \cos^2 \theta d\phi) \left(\frac{r \cos \theta dr}{\sqrt{1+r^2}} - \sqrt{1+r^2} \sin \theta d\theta \right) (4(x-d) \\ + 12 \frac{d}{x} \sqrt{1+r^2} \cos \theta \cos(t-\phi)) \Bigg]$$

$$F_{tr} = \frac{2(d^2-1)r \sin^4 \theta (r^2 + \sin^2 \theta)}{2Nx^6}, F_{t\phi} = \frac{d\sqrt{r^2+1} \sin^3 \theta \sin 2\theta (\sin^2 \theta + r^2)^2 \sin(t-\phi)}{2Nx^6} \\ F_{t\theta} = \frac{\sin^3 \theta}{2Nx^6} \left[2(r^2+1)(d^2(r^2+2) + 2r^2+1) \cos \theta + 2 \cos^2 \theta \sqrt{r^2+1} x^3 \cos(t-\phi) \right. \\ \left. + d(r^2+1) \cos^3 \theta ((2d+3x) \cos(2t-2\phi) - 3x) - d\sqrt{r^2+1} \cos(t-\phi) ((d^2+3r^2+2) \cos(2\theta) \right. \\ \left. + d^2 + 2r^4 + 7r^2 + 4) \right] \\ F_{r\theta} = \frac{r \sin^3 \theta \sin(t-\phi)}{2N\sqrt{r^2+1} x^6} [d(\sin^2 \theta + r^2)^2 + 6dx \cos^3 \theta] \sqrt{r^2+1} \cos(t-\phi) \\ + 2x \cos^2 \theta (-dx - 3r^2 + x^2 - 3) \\ F_{r\phi} = \frac{2r \sin^4 \theta \cos \theta (r^2 + \sin^2 \theta) (d(\cos 2\theta + 2r^2 + 3) \cos(t-\phi) - 4\sqrt{r^2+1} \cos \theta)}{4N\sqrt{r^2+1} x^6}$$



$$\begin{aligned}
F_{\theta\phi} &= \frac{\sin^3 \theta \cos \theta}{8Nx^6} \left[8\cos^2 \theta ((2r^2 + 1)(d^2 + \sin^4 \theta) + 2d^2(r^2 + 1)\cos^2(t - \phi) - 3d(r^2 + 1)x\sin^2(t - \phi)) \right. \\
&\quad + \sqrt{r^2 + 1}\cos \theta \cos(t - \phi)(-d(8d^2 - 8r^4 + 4r^2 + 3) - d\cos 4\theta + 4d(r^2 + 1)\cos(2\theta) + 8x^3) \\
&\quad - 2((d^2 + r^4 + r^2)\cos(4\theta) - d^2 + 4r^6 + 5r^4 + r^2 - 2(r^4 + r^2)\cos(2\theta)) \\
&\quad \left. + 8d\sqrt{r^2 + 1}\cos^3 \theta (2\cos 2\theta - 5(r^2 + 1))\cos(t - \phi) - 8(r^2 + 1)\cos^4 \theta (\cos 2\theta - 2(r^2 + 1)) \right] \\
\tilde{F}_{tr} &= \frac{r^3}{8Nx^6} \left[-2(\cos 2\theta - 2r^2 - 1)((d^2 + r^4)\cos 2\theta - d^2(4r^2 + 1) - r^4) \right. \\
&\quad + \cos \theta \left(\sqrt{r^2 + 1}\cos(t - \phi)(d(-8d^2 + 24(r^4 + r^2) - 1) + d(\cos 4\theta - 8(2r^2 + 3)\cos 2\theta) + 8x^3) \right. \\
&\quad + r^2\cos 3\theta) + \cos^2 \theta (2(2d^2 + (r^2 + 2)^2)\cos 2\theta + 4d(r^2 + 1)(2d + 3x)\cos(2t - 2\phi) \\
&\quad \left. + 12d(d - (r^2 + 1)x) - 2\cos 4\theta - 8r^6 - 22r^4 - 13r^2 - 6) + 4\cos^4 \theta (\cos 2\theta + r^4 + r^2 + 1) \right] \\
\tilde{F}_{t\theta} &= -\frac{4r^4\sin \theta (r^2 + \sin^2 \theta)(4(r^2 + 1)\cos \theta - d\sqrt{r^2 + 1}(\cos 2\theta + 2r^2 + 3)\cos(t - \phi))}{8Nx^6} \\
\tilde{F}_{t\phi} &= -\frac{dr^4\sqrt{r^2 + 1}\cos \theta (\sin^2 \theta + r^2)^2\sin(t - \phi)}{Nx^6} \\
\tilde{F}_{r\theta} &= -\frac{r^3\sin \theta \sin(t - \phi)}{8N\sqrt{r^2 + 1}x^6} \left[8(r^2 + 1)(d^3 - x(d^2 + r^2 - 1)) + 4dr^2 + d \right. \\
&\quad \left. - d(8(r^2 + 1)^{3/2}\cos \theta (2d + x)\cos(t - \phi) + \cos 4\theta) + 4\cos 2\theta (d(3r^2 + 2) + 2(r^2 + 1)x) \right] \\
\tilde{F}_{r\phi} &= \frac{r^3\cos \theta [2(7d^2 + 2)\cos \theta + 2(d^2 + 2)\cos 3\theta + 8d^2(r^2 + 1)\cos \theta \cos(2t - 2\phi)]}{8Nx^6} \\
&\quad - \frac{d\cos(t - \phi)(8d^2(r^2 + 1) + \cos 4\theta + 4(3r^2 + 4)\cos 2\theta + 4r^2 + 7)}{\sqrt{r^2 + 1}} \\
&\quad - 24d(r^2 + 1)x\cos \theta \sin^2(t - \phi) + 8\sqrt{r^2 + 1}x^3\cos(t - \phi) \\
\tilde{F}_{\theta\phi} &= \frac{2(d^2 - 1)r^4\sin 2\theta(-\cos 2\theta + 2r^2 + 1)}{8Nx^6}
\end{aligned}$$

$$\epsilon^{\text{red}} = \frac{1}{N^2}$$

$$\epsilon^{\text{irred}} = \frac{1}{N^3}$$

$$Q > Nq_c(\delta)$$

$$S = \frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4} \sqrt{-g} \left(R - \frac{1}{2(5!)} F_5^2 - \frac{H^2}{2(3!)} - \partial_\mu \phi \partial^\mu \phi \right)$$

$$(F_5)_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \partial_{\mu_1} C_{\mu_2\mu_3\mu_4\mu_5} + \partial_{\mu_2} C_{\mu_3\mu_4\mu_5\mu_1} + 3 \text{ other}$$

$$\int_{S^5} * F_5 = (2\pi)^4 N g_s \alpha'^2$$

$$S = \frac{1}{(2\pi)^3 \alpha'^2 g_s} \left(- \int d^4 y \sqrt{-\text{deth}_{ab}} + \frac{1}{4!} \int \epsilon^{abcd} C_{\mu_1\mu_2\mu_3\mu_4} \frac{dx^{\mu_1}}{dy^a} \frac{dx^{\mu_2}}{dy^b} \frac{dx^{\mu_3}}{dy^c} \frac{dx^{\mu_4}}{dy^d} d^4 y \right)$$



$$\frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4} \nabla_\mu F^{\mu\mu_1\mu_2\mu_3\mu_4} = -\frac{N}{(2\pi)^3 \alpha'^2 g_s} \delta^6(\vec{r}) \epsilon^{\mu_1 \dots \mu_4}$$

$$\Rightarrow \frac{1}{16\pi^4 g_s \alpha'^2} \nabla_\mu F^{\mu\mu_1\mu_2\mu_3\mu_4} = -N \delta^6(\vec{r}) \epsilon^{\mu_1 \dots \mu_4}$$

$$F_{\mu_1\mu_2\mu_3\mu_4 r} = A(r) \epsilon_{\mu_1\mu_2\mu_3\mu_4 r}$$

$$\frac{1}{(2\pi)^4 g_s \alpha'^2} \frac{1}{r^5} \partial_r (r^5 F^r \mu_1 \mu_2 \mu_3 \mu_4) = -N \frac{\delta(r)}{\pi^3 r^5} \epsilon^{\mu_1 \dots \mu_4}$$

$$A = -N \frac{(2\pi)^4 g_s \alpha'^2}{\pi^3}, F = -A \frac{\epsilon_{\mu_1\mu_2\mu_3\mu_4 r}}{r^5}$$

$$T^{\mu\nu} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \eta^{\mu\nu} \delta^6(\vec{r})$$

$$ds_{(1)}^2 = \phi(r) (dx^\mu dx_\mu - dy^i dy_i)$$

$$4\partial_i \partial^i \phi = 8\pi G_N \frac{1}{(2\pi)^3 \alpha'^2 g_s} \delta^6(\vec{r})$$

$$\phi = \frac{2\pi \alpha'^2 g_s}{r^4}$$

$$\frac{1}{\alpha'^4 g_s^2} \int F^2 \sim N^2 \int \frac{dr r^5}{r^{10}} \sim N^2 \frac{1}{r_0^4} \sim \frac{N}{g_s \alpha'^2}$$

$$T_D^{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$= \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \int d^4 y \sqrt{-h} h^{ab} \frac{\partial x_0^\mu(y)}{\partial y^a} \frac{\partial x_0^\nu(y)}{\partial y^b} \frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g(x)}}$$

$$\delta h_{ab} = \partial_a X^\mu \partial_b X^\nu \delta g_{\mu\nu}$$

$$\int \delta = \frac{\delta^{10}}{\sqrt{-g}}$$

$$n_\mu = k(x) \frac{\partial f(x)}{\partial x^\mu}$$

$$n_\mu T_D^{\mu\nu} = \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \int d^4 y \sqrt{-h} h^{\alpha\beta} k(x) \frac{\partial f(x)}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g}}$$

$$= \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \int d^4 y \sqrt{-h} h^{\alpha\beta} k(x) \frac{\partial f(x(y))}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g}} = 0$$

$$\begin{aligned}
\frac{-\delta \int d^4 y \sqrt{-h}}{\delta x_0^\alpha(y')} &= - \int d^4 y \frac{\delta \sqrt{-h}}{\delta h_{ab}} \frac{\delta h_{ab}}{\delta x_0^\alpha(y')} \\
&= - \frac{1}{2} \int d^4 y \sqrt{-h} h^{ab} \frac{\delta \left(\frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} g_{\mu\nu} \right)}{\delta x_0^\alpha(y')} \\
&= - \frac{\sqrt{-h} h^{ab}}{2} \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \frac{\partial g_{\mu\nu}}{\partial x_0^\alpha} + \frac{\partial}{\partial y^b} \left(\sqrt{-h} h^{ab} g_{\alpha\nu} \frac{\partial x_0^\nu}{\partial y^a} \right) \\
&= - \frac{\sqrt{-h} h^{ab}}{2} \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \frac{\partial g_{\mu\nu}}{\partial x_0^\alpha} + \sqrt{-h} h^{ab} (\hat{\nabla}_b g_{\alpha\nu}) \left(\frac{\partial x_0^\nu}{\partial y^a} \right) + \sqrt{-h} h^{ab} (g_{\alpha\nu}) \hat{\nabla}_b \left(\frac{\partial x_0^\nu}{\partial y^a} \right) \\
&= - \frac{\sqrt{-h} h^{ab}}{2} \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \frac{\partial g_{\mu\nu}}{\partial x_0^\alpha} + \sqrt{-h} h^{ab} \frac{\partial g_{\alpha\nu}}{\partial x_0^\mu} \frac{\partial x_0^\mu}{\partial y^b} \frac{\partial x_0^\nu}{\partial y^a} + \sqrt{-h} h^{ab} g_{\alpha\nu} \hat{\nabla}_b \left(\frac{\partial x_0^\nu}{\partial y^a} \right) \\
&= \sqrt{-h} h^{ab} g_{\alpha\nu} \left(\hat{\nabla}_b \left(\frac{\partial x_0^\nu}{\partial y^a} \right) + \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\beta}{\partial y^b} \Gamma_{\mu\beta}^\nu \right)
\end{aligned}$$

$$\partial_a (\sqrt{-h} h^{ab} A_b) = \sqrt{-h} \nabla_b A^b$$

$$\begin{aligned}
&\frac{1}{4!} \frac{\delta}{\delta x_0^\alpha(y')} \int d^4 y \epsilon^{a_1 a_2 a_3 a_4} C_{\mu_1 \mu_2 \mu_3 \mu_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} \\
&= \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} \left(\frac{\partial C_{\mu_1 \mu_2 \mu_3 \mu_4}}{\partial x_0^\alpha} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} - 4 \frac{\partial}{\partial y_{a_1}} \left(C_{\alpha \mu_2 \mu_3 \mu_4} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} \right) \right) \\
&= \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} \left(\frac{\partial C_{\mu_1 \mu_2 \mu_3 \mu_4}}{\partial x_0^\alpha} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} + 4 \frac{\partial}{\partial y_{a_1}} \left(C_{\alpha \mu_2 \mu_3 \mu_4} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} \right) \right) \\
&= \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} \left(\frac{\partial C_{\mu_1 \mu_2 \mu_3 \mu_4}}{\partial x_0^\alpha} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} - 4 \frac{\partial C_{\alpha \mu_2 \mu_3 \mu_4}}{\partial x_0^{\mu_1}} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} \right) \\
&= \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} F_{\alpha \mu_1 \mu_2 \mu_3 \mu_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}}
\end{aligned}$$

$$\sqrt{-h} h^{ab} g_{\alpha\nu} \left(\hat{\nabla}_b \left(\frac{\partial x_0^\nu}{\partial y^a} \right) + \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\beta}{\partial y^b} \Gamma_{\mu\beta}^\nu \right) + \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} F_{\alpha \mu_1 \mu_2 \mu_3 \mu_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} = 0$$

$$\begin{aligned}
\nabla_\mu T_D^{\mu\nu}(x) &= \partial_\mu T_D^{\mu\nu}(x) + \Gamma_{\mu\alpha}^\nu T^{\mu\alpha} + \Gamma_{\mu\alpha}^\mu T^{\alpha\nu} \\
&= \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \left[\int d^4 y \sqrt{-h} h^{ab} \frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \frac{\partial}{\partial x^\mu} \left(\frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g(x)}} \right) \right. \\
&\quad \left. + \sqrt{-h} h^{ab} \left(\frac{\partial x_0^\alpha}{\partial y^a} \frac{\partial x_0^\mu}{\partial y^b} \Gamma_{\mu\alpha}^\nu + \frac{\partial x_0^\alpha}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \Gamma_{\mu\alpha}^\mu \right) \left(\frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g(x)}} \right) \right] \\
&= \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \left[\int d^4 y \frac{1}{\sqrt{-g(x)}} \frac{\partial}{\partial y^a} \left(\sqrt{-h} h^{ab} \frac{\partial x_0^\nu}{\partial y^b} \right) \delta^{10}(x^\mu - x_0^\mu(y)) \right. \\
&\quad \left. + \sqrt{-h} h^{ab} \delta^{10}(x^\mu - x_0^\mu(y)) \left(\frac{\partial x_0^\mu}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \frac{\partial}{\partial x^\mu} \left(\frac{1}{\sqrt{-g(x)}} \right) + \left(\frac{\partial x_0^\alpha}{\partial y^a} \frac{\partial x_0^\mu}{\partial y^b} \Gamma_{\mu\alpha}^\nu + \frac{\partial x_0^\alpha}{\partial y^a} \frac{\partial x_0^\nu}{\partial y^b} \Gamma_{\mu\alpha}^\mu \right) \right) \right] \\
&= \frac{-1}{(2\pi)^3 \alpha'^2 g_s} \frac{1}{\sqrt{-g(x)}} \left[\int d^4 y \delta^{10}(x^\mu - x_0^\mu(y)) \sqrt{-h(y)} h^{ab} \left(\hat{\nabla}_a \frac{\partial x_0^\nu}{\partial y^b} + \frac{\partial x_0^\beta}{\partial y^a} \frac{\partial x_0^\mu}{\partial y^b} \Gamma_{\mu\beta}^\nu \right) \right]
\end{aligned}$$



$$\frac{\partial x_0^\mu}{\partial y^a} - \frac{\partial}{\partial y^a} (\delta^{10}(x^\mu - x_0^\mu(y)))$$

$$\frac{\partial g}{\partial x^\mu} = 2g\Gamma_{\alpha\mu}^\alpha$$

$$T_{\rm int}^{\mu\nu} = \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha' 45!} (g^{\mu\nu} F \cdot \tilde{F} - 5 F^{\mu\nu_1\nu_2\nu_3\nu_4} \tilde{F}_{\nu_1\nu_2\nu_3\nu_4}^\nu - 5 \tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4}^\nu)$$

$$\begin{aligned} \nabla_\mu T_{int}^{\mu\nu}(x) &= \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha' 5!} (g^{\mu\nu} \partial_\mu (F \cdot \tilde{F}) - 5 F^{\mu\nu_1\nu_2\nu_3\nu_4} \nabla_\mu \tilde{F}_{\nu_1\nu_2\nu_3\nu_4}^\nu \\ &\quad - 5 \nabla_\mu \tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4}^\nu - 5 \tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4} \nabla_\mu F_{\nu_1\nu_2\nu_3\nu_4}^\nu) \\ &= \frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha' 45!} 5 \nabla_\mu \tilde{F}^{\mu\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4}^\nu \\ &= \frac{-1}{(2\pi)^3 \alpha'^2 g_s 4!} \frac{1}{\sqrt{-g}} \int d^4 y \delta^{10}(x^\mu - x_0^\mu(y)) \epsilon^{a_1 a_2 a_3 a_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} F_{\mu_1 \mu_2 \mu_3 \mu_4}^\nu \end{aligned}$$

$$\nabla_\mu F^{\mu\nu_1\nu_2\nu_3\nu_4} = 0$$

$$\frac{1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4} \nabla_\mu \tilde{F}^{\mu\mu_1\mu_2\mu_3\mu_4} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \int d^4 y \frac{\delta^{10}(x^\mu - x_0^\mu(y))}{\sqrt{-g}} \epsilon^{a_1 a_2 a_3 a_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}}$$

$$\nabla_\mu (T_D^{\mu\nu} + T_F^{\mu\nu}) = 0$$

$$\Rightarrow \sqrt{-h(y)} h^{ab} \left(\hat{\nabla}_a \frac{\partial x_0^\nu}{\partial y^b} + \frac{\partial x_0^\beta}{\partial y^a} \frac{\partial x_0^\mu}{\partial y^b} \Gamma_{\mu\beta}^\nu \right) + \frac{1}{4!} \epsilon^{a_1 a_2 a_3 a_4} \frac{dx_0^{\mu_1}}{dy^{a_1}} \frac{dx_0^{\mu_2}}{dy^{a_2}} \frac{dx_0^{\mu_3}}{dy^{a_3}} \frac{dx_0^{\mu_4}}{dy^{a_4}} F_{\mu_1 \mu_2 \mu_3 \mu_4}^\nu = 0$$

$$dC_V=-4\epsilon^{(5)}$$

$$d(\Theta\wedge(-*_5F+A\wedge F))=-2J\wedge*_5F+2J\wedge A\wedge F$$

$$\begin{aligned} d(l_2^2 d(l_3^2) \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) &= -2J \wedge J \wedge (d\Psi + \Theta) = -4 *_5 \epsilon^{(5)} - 2J \wedge J \wedge A \\ d(-A \wedge (d\Psi + \Theta) \wedge J) &= -F \wedge (d\Psi + \Theta) \wedge J + 2A \wedge J \wedge J \end{aligned}$$

$$d(-*_5F+A\wedge F)=0$$

$$(-*_5F+A\wedge F)=\frac{2\pi^2Q}{4}\sigma_1\wedge\sigma_2\wedge\sigma_3$$

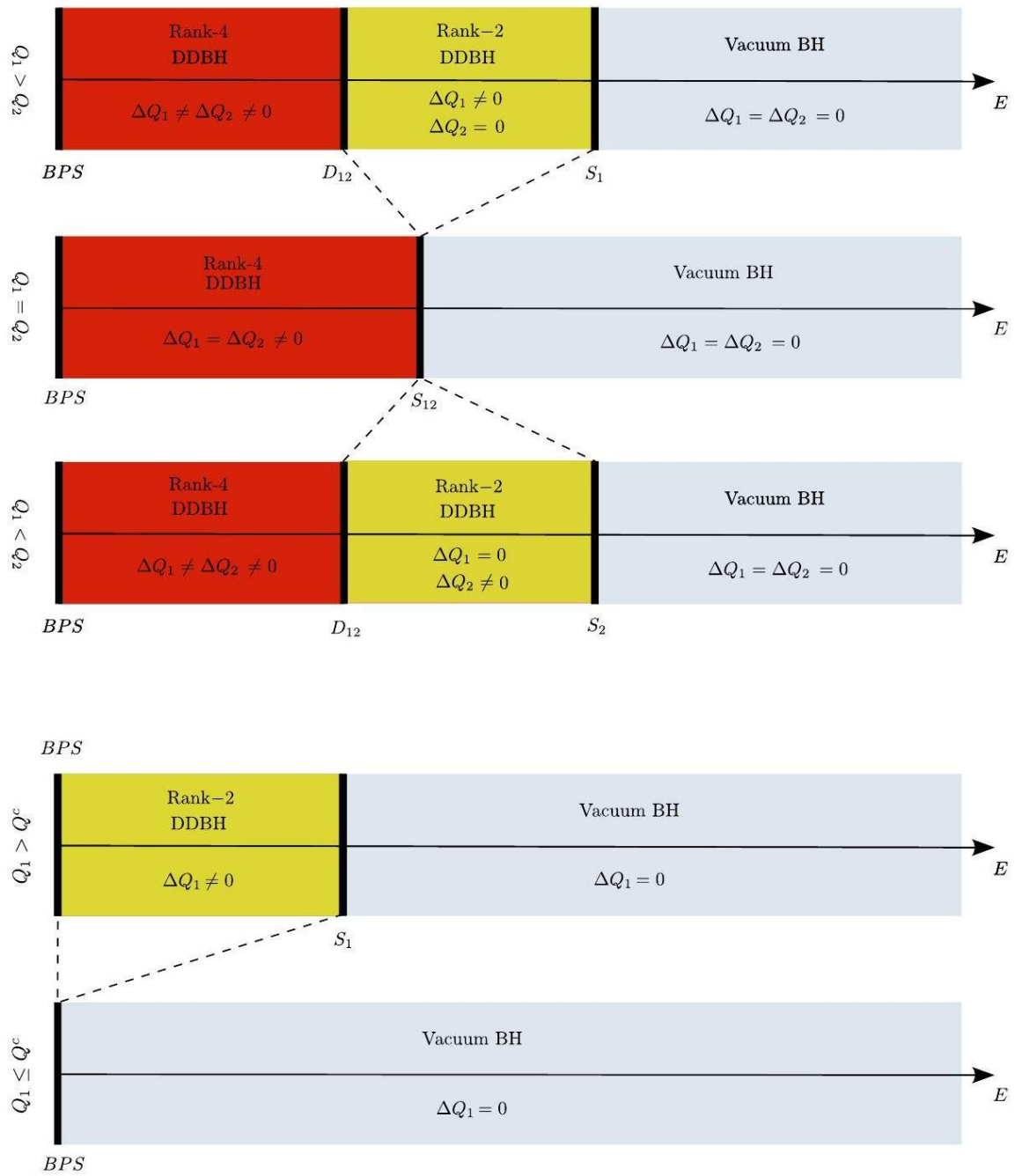
$$\begin{aligned} \frac{C}{R_{AdS_5^4}} &= C_V + \left(\frac{\pi^2 Q}{4}\right) \Theta \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ + l_2^2 d(l_3^2) \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 &+ A \wedge (d\Psi + \Theta) \wedge J \end{aligned}$$

$$\sigma_1\wedge\sigma_2\wedge\sigma_3=d(\cos\theta_ad\psi\wedge d\phi)$$

$$\Theta\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3=-d\left((\Theta\wedge(\cos\theta_ad\psi\wedge d\phi))+J\wedge(\cos\theta_ad\psi\wedge d\phi)\right)$$

$$\begin{aligned} \frac{C}{R_{AdS_5^4}} &= C_V + \left(\frac{\pi^2 Q}{4}\right) \cos\theta_a J \wedge d\psi \wedge d\phi \\ + l_2^2 d(l_3^2) \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 &+ A \wedge (d\Psi + \Theta) \wedge J \end{aligned}$$





$$\frac{NV_3}{2\pi^2} V_3 = 2\pi^2 \left(2r^2 \sqrt{b^2 + A_3^2} \right)$$

$$2Nr^2 \sqrt{b^2 + A_3^2}$$

$$S_{\text{DBI}} = 2N \int r^2 \sqrt{b^2 + A_3^2} \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ab} dy^a dy^b + 2g_{a\mu} dy^a dx^\mu$$

$$ds_7^2 = (g_{\mu\nu} - g_{\mu a} g_{\nu b} g_3^{ab}) dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

$$\begin{aligned} ds_7^2 &= \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \left(-\frac{r^2 W}{4b^2} + A_t^2 + b^2 f^2 \right) dt^2 + \frac{dr^2}{W} + d\alpha^2 + \cos^2 \alpha d\beta^2 + \sum_{i=1}^3 (l_i^2 d\phi_i^2 - 2A_t l_i^2 d\phi_i dt) \\ &\quad - \frac{(dt(A_t A_{\sigma_3} + b^2 f) - A_{\sigma_3} \sum_i l_i^2 d\phi_i)^2}{b^2 + A_{\sigma_3}^2} \\ &= -\frac{r^2 W}{4b^2} dt^2 + \frac{dr^2}{W} + ds^2(\mathbb{CP}^2) + \frac{b^2}{b^2 + A_{\sigma_3}^2} (d\Psi + \Theta + (f A_{\sigma_3} - A_t) dt)^2 \end{aligned}$$

$$l_i(\tau), r(\tau), \phi_{1,2,3}(\tau)$$

$$\begin{aligned} ds_{\text{ind}}^2 &= \left(g_{tt} + g_{rr} \dot{r}^2 + \sum_{i=1}^2 (2l_i \dot{l}_i) + \sum_{i=1}^3 (g_{\phi_i \phi_i} \dot{\phi}_i^2 - 2g_{t\phi_i} \dot{\phi}_i) \right) dt^2 \\ &\quad + \left(g_{t\sigma_3} + \sum_{i=1}^3 g_{\sigma_3 \phi_i} \dot{\phi}_i \right) dt d\sigma_3 + \sum_{i=1}^3 g_{\sigma_i \sigma_i} d\sigma_i^2 \end{aligned}$$

$$ds_{\text{ind}}^2 = \Delta dt^2 + (b^2 + A_{\sigma_3}^2) \left(\sigma_3 + \frac{\gamma}{b^2 + A_{\sigma_3}^2} dt \right)^2 + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2)$$

$$\Delta = \tilde{g}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = -\frac{r^2 W}{4b^2} + A_t^2 + b^2 f^2 + \frac{\dot{r}^2}{W} + \sum_{i=1}^2 (2l_i \dot{l}_i) + \sum_{i=1}^3 (l_i^2 \dot{\phi}_i^2 - 2A_t l_i^2 \dot{\phi}_i) - \frac{\gamma^2}{q^2}$$

$$q^2 = b^2 + A_{\sigma_3}^2, \text{ and } \gamma = A_t A_{\sigma_3} + b^2 f - l_i^2 A_{\sigma_3} \dot{\phi}_i$$

$$C_V = \left(\frac{r^4 - R^4}{8} \right) dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

$$S_{WZ} = 8N \int \left(\frac{r^4 - R^4}{8} \right) dt$$

$$\begin{aligned} S &= S_{DBI} + S_{WZ} \\ &= 2N \int r^2 \sqrt{b^2 + A_3^2} \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu} + 8N \int \left[\left(\frac{r^4 - R^4}{8} \right) dt \right] \end{aligned}$$

$$z_1 \rightarrow \cos \zeta z_1 + i \sin \zeta \bar{z}_2, z_2 \rightarrow \cos \zeta z_2 - i \sin \zeta \bar{z}_1$$

$$dz_1 \wedge d\bar{z}_1 \rightarrow \cos^2 \zeta dz_1 \wedge d\bar{z}_1 - \sin^2 \zeta dz_2 \wedge d\bar{z}_2 - i \sin \zeta \cos \zeta (dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2)$$

$$\cos^2 \zeta dz_1 \wedge d\bar{z}_1 - \sin^2 \zeta dz_2 \wedge d\bar{z}_2$$

$$\begin{pmatrix} \cos^2 \zeta & 0 & 0 \\ 0 & -\sin^2 \zeta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$dz_1 \wedge d\bar{z}_1 \rightarrow \cos^2 \zeta dz_1 \wedge d\bar{z}_1 - \sin^2 \zeta dz_2 \wedge d\bar{z}_2 - i \sin \zeta \cos \zeta (e^{2i\alpha} dz_1 \wedge dz_2 + e^{-2i\alpha} d\bar{z}_1 \wedge d\bar{z}_2)$$

$$\begin{aligned} x_1 &\rightarrow \cos \zeta x_1 + \sin \zeta y_2, & y_1 &\rightarrow \cos \zeta y_1 + \sin \zeta x_2 \\ x_2 &\rightarrow \cos \zeta x_2 - \sin \zeta y_1, & y_2 &\rightarrow \cos \zeta y_2 - \sin \zeta x_1 \end{aligned}$$

$$ds^2_{(5)} = -f_b^2(dt_b - \omega)^2 + \frac{1}{f_b} \left[\frac{\xi}{F(\xi)} d\xi^2 + \frac{F}{\xi} (d\Phi + \eta \Psi_b)^2 + \xi \left(\frac{d\eta^2}{1-\eta^2} + (1-\eta^2) d\Psi_b^2 \right) \right]$$

$$\omega = \frac{1}{3f_b} \left(\frac{F'(\xi)}{2\xi} - 1 \right) \blacksquare$$

$$A=-f_b dt_b-\frac{3a^2f_b}{8(1-a)^2\xi}(d\Phi+\eta\Psi_b)$$

$$t_b=t, \eta=\cos\theta_a, \Phi=\phi_a-2t, \Psi_b=\psi, \xi=\frac{r^2}{4}\left(1-\frac{R_0^2}{r^2}\right)$$

$$f_b=1-\frac{R_0^2}{r^2}, F=\frac{1}{16}(r^2-R_0^2)^2(1+2R_0^2+r^2), a=j$$

$$e^0=f_b(dt-\omega), e^1=-\frac{1}{\sqrt{f_b}}\sqrt{\frac{\xi}{F(\xi)}}d\xi, e^2=\frac{1}{\sqrt{f_b}}\sqrt{\frac{F}{\xi}}(d\Phi+\eta d\Psi_b), e^3=\sqrt{\frac{\xi}{f_b(1-\eta^2)}}d\eta$$

$$e^4=\sqrt{\frac{\xi}{f_b}}(1-\eta^2)d\Psi_b, e^5=d\alpha, e^6=\sin\alpha\cos\alpha(\cos^2\beta(d\phi_1-d\phi_2)+d\phi_2-d\phi_3)$$

$$e^7=\sin\alpha d\beta, e^8=\sin\alpha\sin\beta\cos\beta d(\phi_2-\phi_1), e^9=\sum_i l_i^2 d\phi_i-A$$

$$\Gamma \epsilon = \epsilon$$

$$\Gamma=\frac{1}{4!}\frac{\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}}{\sqrt{-h}}\frac{\partial X^{\mu_1}}{\partial\sigma^{\alpha_1}}\frac{\partial X^{\mu_2}}{\partial\sigma^{\alpha_2}}\frac{\partial X^{\mu_3}}{\partial\sigma^{\alpha_3}}\frac{\partial X^{\mu_4}}{\partial\sigma^{\alpha_4}}e_{\mu_1}^{M_1}e_{\mu_2}^{M_2}e_{\mu_3}^{M_3}e_{\mu_4}^{M_4}\Gamma_{M_1M_2M_3M_4}$$

$$\sigma^0=t, \sigma^i=y^i$$

$$(r,\phi_1,\phi_2,\phi_3,l_1,l_2,l_3)=(r_0,t,t,t,1,0,0)$$

$$\epsilon=e^{\frac{i}{2}(\phi-2t+\phi_1+\phi_2+\phi_3)}\sqrt{f}\epsilon_0$$

$$\Gamma^{09}\epsilon_0=\epsilon_0, \Gamma^{12}\epsilon_0=-i\epsilon_0, \Gamma^{34}\epsilon_0=\Gamma^{56}\epsilon_0=\Gamma^{78}\epsilon_0=i\epsilon_0$$

$$\Gamma^{11}\epsilon=-\epsilon$$

$$\left(\Gamma^{0349}+\frac{R\sqrt{1+R^2+2R_0^2}}{R^2+R_0^2}(\Gamma^{0234}-\Gamma^{2349})\right)\epsilon_0=\epsilon_0$$

$$(\Gamma^0+\Gamma^9)\epsilon_0=0$$



$$\Gamma^{0349}\epsilon_0=\epsilon_0$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ (a_{12})^* & a_{22} & a_{23} \\ (a_{13})^* & (a_{23})^* & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11}(i\sigma_2) & \operatorname{Re}(a_{12})(i\sigma_2)-\operatorname{Im}(a_{12})\mathbb{I}_2 & \operatorname{Re}(a_{13})(i\sigma_2)-\operatorname{Im}(a_{13})\mathbb{I}_2 \\ \operatorname{Re}(a_{12})(i\sigma_2)+\operatorname{Im}(a_{12})\mathbb{I}_2 & a_{22}(i\sigma_2) & \operatorname{Re}(a_{23})(i\sigma_2)-\operatorname{Im}(a_{23})\mathbb{I}_2 \\ \operatorname{Re}(a_{13})(i\sigma_2)+\operatorname{Im}(a_{13})\mathbb{I}_2 & \operatorname{Re}(a_{23})(i\sigma_2)+\operatorname{Im}(a_{23})\mathbb{I}_2 & a_{33}(i\sigma_2) \end{pmatrix}$$

$$\mathbf{z}'=U\mathbf{z}$$

$$V_B=\partial_{z_i}B_i{}^jz_j-\partial_{z_i}^*(B^*)_i{}^jz_j^*=B_i{}^j\left(z_i\partial_{z_j}-z^*{}_j\partial_{z_i}^*\right)$$

$$z_i=l_ie^{i\phi_i}$$

$$\partial_{z_i}=-\frac{i}{z_i}\partial_{\phi_i}+\frac{l_i}{z_i}\partial_{l_i}$$

$$B_i^je^{i(\phi_i-\phi_j)}\left(l_i\frac{\partial}{\partial l_j}-l_j\frac{\partial}{\partial l_i}-i\frac{l_i}{l_j}\frac{\partial}{\partial \phi_j}-i\frac{l_j}{l_i}\frac{\partial}{\partial \phi_i}\right)$$

$$n_{SO_6}=\sum_{n_Z=0}^n(n_Z,n-n_Z)$$

$$-\nabla^2\phi_n=n(n+4)\phi_n$$

$$ds_{S^5}^2=ds_{CP^2}^2+(d\Psi+\Theta)^2$$

$$\phi_n=e^{i(n_Z-n_{\bar{Z}})\Psi}\phi_{n_Z,n_{\bar{Z}}}$$

$$-\nabla_{S^5}^2=\left(-\nabla_{CP^2,(n_Z-n_{\bar{Z}})}^2+(n_Z-n_{\bar{Z}})^2\right)\phi_{n_Z,n_{\bar{Z}}}$$

$$\begin{aligned} \text{As } -\nabla_{S^5}^2 &= (n_Z+n_{\bar{Z}})(n_Z+n_{\bar{Z}}+4), \\ -\nabla_{CP^2,(n_Z-n_{\bar{Z}})}^2\phi_{n_Z,n_{\bar{Z}}} &= (n_Z+n_{\bar{Z}})(n_Z+n_{\bar{Z}}+4)-(n_Z-n_{\bar{Z}})^2 \\ &= 4(n_Zn_{\bar{Z}}+n_Z+n_{\bar{Z}}) \end{aligned}$$

$$\begin{aligned} g(r) &= \frac{1}{(2V(r))^{1/4}} \left(A \exp \left(\int_{r_1}^r N \sqrt{2V} \right) e^{-\beta} + B \exp \left(- \int_{r_1}^r N \sqrt{2V} \right) e^{\beta} \right) \\ &= \frac{A e^{-\beta}}{(2V(r))^{1/4}} \left(\exp \left(\int_{r_1}^r N \sqrt{2V} \right) + \frac{B}{A} e^{2\beta} \exp \left(- \int_{r_1}^r N \sqrt{2V} \right) \right) \end{aligned}$$

$$\beta=\int_{r_1}^{r^*}N\sqrt{2V}$$

$$\left(-\frac{1}{2N^2}\frac{\partial}{\partial r}+V'(r_1)(r-r_1)\right)g(r)=0$$



$$g(r)=c_1\mathrm{Bi}\left(N^{2/3}V'(r_1)^{1/3}(r-r_1)\right)+c_2\mathrm{Ai}\left(N^{2/3}V'(r_1)^{1/3}(r-r_1)\right)$$

$$g(r)\approx c_1\frac{e^{\frac{2}{3}N\sqrt{V'(r_1)}r^{3/2}}}{\sqrt{\pi}^{12}\sqrt{V'(r_1)}^4\sqrt[4]{r}}+c_2\frac{e^{-\frac{2}{3}N\sqrt{V'(r_1)}r^{3/2}}}{2\sqrt{\pi}^{12}\sqrt{V'(r_1)}^4\sqrt[4]{r}}$$

$$c_1=\frac{\sqrt{\pi}}{V'(r_1)^{1/6}}Ae^{-\beta},c_2=\frac{2\sqrt{\pi}}{V'(r_1)^{1/6}}Be^{\beta}$$

$$g(r)\sim \frac{Ae^{-\beta}}{(2V)^{-\frac{1}{4}}}\bigg(\sin\left(\int_{r_1}^rN\sqrt{-2V}+\frac{\pi}{4}\right)+\frac{2B}{A}e^{2\beta}\cos\left(\int_{r_1}^rN\sqrt{-2V}+\frac{\pi}{4}\right)\bigg)$$

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu+e^{H+G}d\Omega_3^2+e^{H-G}d\tilde{\Omega}_3^2\\ F_{(5)}=4(F\wedge d\Omega_3+\tilde{F}\wedge d\tilde{\Omega}_3)$$

$$F=dB_t\wedge (dt+V)+B_t dV+d\hat{B}\\ \tilde{F}=d\tilde{B}_t\wedge (dt+V)+\tilde{B}_tdV+d\hat{\tilde{B}}$$

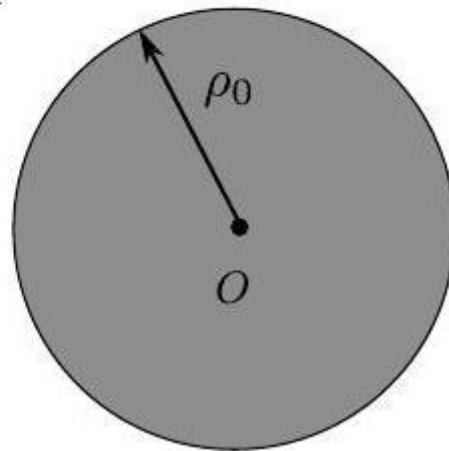
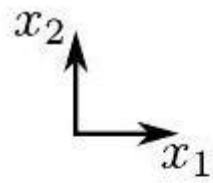
$$F=e^{3G}\star_4\tilde{F}$$

$$ds^2=-h^{-2}(dt+V_id x^i)^2+h^2(dy^2+d x^i d x^i)+ye^Gd\Omega_3^2+ye^{-G}d\tilde{\Omega}_3^2\\ B_t=-\frac{1}{4}y^2e^{2G},\tilde{B}_t=-\frac{1}{4}y^2e^{-2G}\\ d\hat{B}=-\frac{1}{4}y^3\star_3\left(\frac{z+\frac{1}{2}}{y^2}\right),d\hat{\tilde{B}}=-\frac{1}{4}y^3\star_3\left(\frac{z-\frac{1}{2}}{y^2}\right)$$

$$z=\frac{1}{2}\tanh G,\quad h^{-2}=2y\cosh G\\ y\partial_yV_i=\epsilon_{ij}\partial_jz,\quad y(\partial_iV_j-\partial_jV_i)=\epsilon_{ij}\partial_yz$$

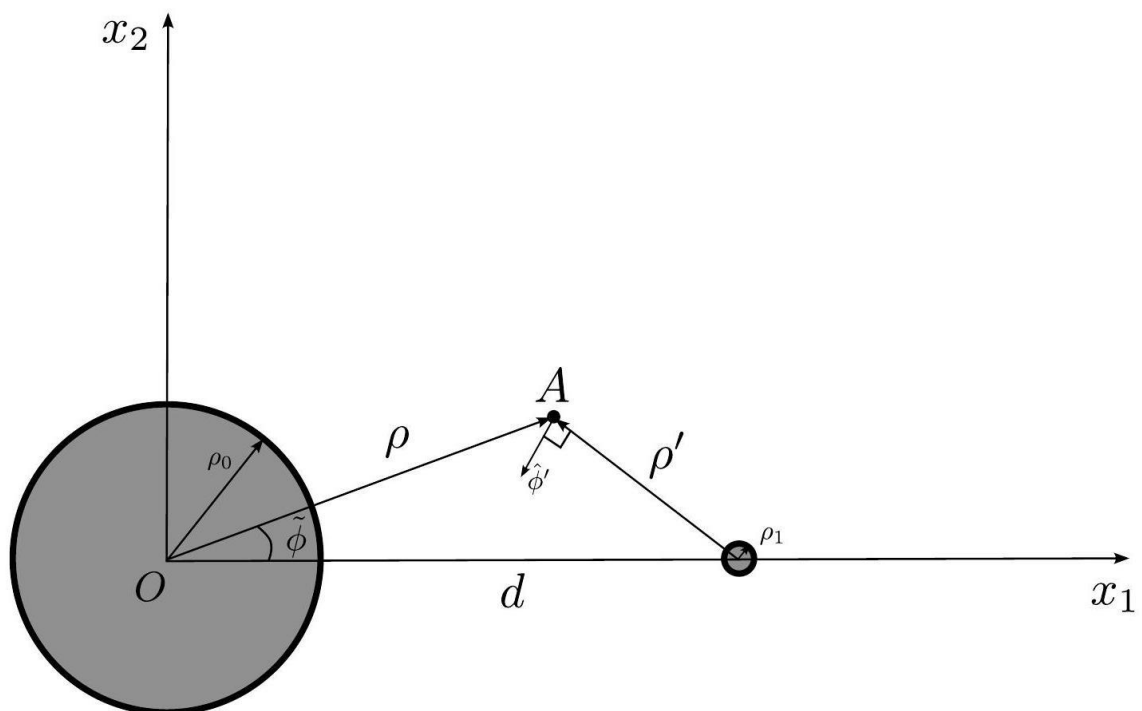
$$z(x_1,x_2,y)=\frac{y^2}{\pi}\int_{R^2}\frac{z(x'_1,x'_2,0)dx'_1dx'_2}{((\mathbf{x}-\mathbf{x}')^2+y^2)^2}\\ V_i=\frac{\epsilon_{ij}}{\pi}\int_{R^2}\frac{z(x'_1,x'_2,0)(x^j-x'^j)dx'_1dx'_2}{((\mathbf{x}-\mathbf{x}')^2+y^2)^2}$$

$$\frac{ds^2}{l_N^2}=\rho_0\left(-(1+r^2)dt^2+\frac{dr^2}{1+r^2}+r^2d\Omega_3^2+d\theta^2+\cos^2\theta d\phi^2+\sin^2\theta d\tilde{\Omega}_3^2\right)$$



$$\begin{aligned}\rho &= \rho_0 \sqrt{1 + r^2} \cos \theta \\ y &= \rho_0 r \sin \theta \\ \tilde{\phi} &= \phi - t\end{aligned}$$

$$\begin{aligned}z &= \tilde{z}_1 + \tilde{z}_2 + \frac{1}{2} \\ \tilde{z}_1 &= \frac{(\rho^2 - \rho_0^2 + y^2)}{2\sqrt{(\rho^2 + \rho_0^2 + y^2)^2 - 4\rho^2\rho_0^2}} - \frac{1}{2} = z_1 - \frac{1}{2} \\ \tilde{z}_2 &= \frac{(\rho'^2 - \rho_1^2 + y^2)}{2\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2\rho_1^2}} - \frac{1}{2} = z_2 - \frac{1}{2} \approx -\frac{(\rho_1)^2 y^2}{(\rho'^2 + y^2)^2}\end{aligned}$$



$$\begin{aligned}
V_1 &= V_\phi \hat{\phi} = -\frac{1}{2} \left(\frac{\rho^2 + y^2 + \rho_0^2}{\sqrt{(\rho^2 + \rho_0^2 + y^2)^2 - 4\rho^2 \rho_0^2}} - 1 \right) (-\sin \tilde{\phi} \hat{x}_1 + \cos \tilde{\phi} \hat{x}_2) \\
V_2 &= V'_\phi \hat{\phi}' = -\frac{1}{2} \left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2 \rho_1^2}} - 1 \right) \frac{1}{\rho'} (-\rho \sin \tilde{\phi} \hat{x}_1 + (\rho \cos \tilde{\phi} - d) \hat{x}_2) \\
&= V_2^\rho \hat{\rho} + V_2^\phi \hat{\phi}
\end{aligned}$$

$$\vec{\rho}' = \vec{\rho} - \vec{d} = (\rho \cos \tilde{\phi} - d, \rho \sin \tilde{\phi})$$

$$\rho' = \sqrt{\rho^2 + d^2 - 2\rho d \cos \tilde{\phi}}, \hat{\phi}' = \frac{(-\rho \sin \tilde{\phi}, \rho \cos \tilde{\phi} - d)}{\rho'}$$

$$\begin{aligned}
V_2^\rho &= \frac{d \sin \tilde{\phi}}{2} \left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2 \rho_1^2}} - 1 \right) \frac{1}{\rho'^2} \approx \frac{(\rho_1)^2 d \sin \tilde{\phi}}{(\rho'^2 + y^2)^2} \\
V_2^\phi &= -\frac{\rho^2 - d \rho \cos \tilde{\phi}}{2} \left(\frac{\rho'^2 + y^2 + \rho_1^2}{\sqrt{(\rho'^2 + \rho_1^2 + y^2)^2 - 4\rho'^2 \rho_1^2}} - 1 \right) \frac{1}{\rho'^2} \approx -\frac{(\rho_1)^2 (\rho^2 - d \rho \cos \tilde{\phi})}{(\rho'^2 + y^2)^2}
\end{aligned}$$

$$z = z_1 = \frac{1}{2} \frac{r^2 - \sin^2 \theta}{r^2 + \sin^2 \theta}$$

$$V = V_1 = \frac{-\cos^2 \theta}{r^2 + \sin^2 \theta} \hat{\phi}$$

$$h^{-2} = \frac{2y}{\sqrt{1 - 4z^2}} = \rho_0(r^2 + \sin^2 \theta)$$

$$\begin{aligned}
\cosh G &= \frac{1}{\sqrt{4z^2 - 1}} = \frac{r^2 + \sin^2 \theta}{2r \sin \theta} \\
e^G &= \frac{r}{\sin \theta}
\end{aligned}$$

$$h^{-2} = \frac{2y}{\sqrt{1 - 4(z_1 + \tilde{z}_2)^2}} \approx \rho_0(r^2 + \sin^2 \theta) \left(1 - \frac{\rho_1^2 (r^4 - \sin^4 \theta)}{2x^4} \right)$$

$$h^2 = \frac{\sqrt{1 - 4(z_1 + \tilde{z}_2)^2}}{2y} \approx \frac{1}{\rho_0(r^2 + \sin^2 \theta)} \left(1 + \frac{\rho_1^2 (r^4 - \sin^4 \theta)}{2x^4} \right)$$

$$e^G = \frac{\sqrt{1 + 2(z_1 + \tilde{z}_2)}}{\sqrt{1 - 2(z_1 + \tilde{z}_2)}} \approx \frac{r}{\sin \theta} \left(1 - \frac{\rho_1^2 (r^2 + \sin^2 \theta)^2}{2x^4} \right)$$

$$e^{-G} = \frac{\sqrt{1 - 2(z_1 + \tilde{z}_2)}}{\sqrt{1 + 2(z_1 + \tilde{z}_2)}} \approx \frac{\sin \theta}{r} \left(1 + \frac{\rho_1^2 (r^2 + \sin^2 \theta)^2}{2x^4} \right)$$

$$V^\rho \approx \rho_1^2 \frac{d \sin(\phi - t)}{x^4}$$

$$V^\phi \approx \frac{-\cos^2 \theta}{r^2 + \sin^2 \theta} - \rho_1^2 \frac{\sqrt{1 + r^2} \cos \theta (\sqrt{1 + r^2} \cos \theta - d \cos(\phi - t))}{x^4}$$

$$x^2 = r^2 + \cos^2 \theta + d^2 - 2d\sqrt{1 + r^2} \cos \theta \cos(\phi - t)$$



$$\begin{aligned}
\frac{ds^2}{l_{N-1}^2} = & r^2 d\Omega_3^2 \left(1 - \frac{\rho_1^2 (r^2 + \sin^2 \theta)^2}{x^4} \right) + \sin^2 \theta d\tilde{\Omega}_3^2 \left(1 + \frac{\rho_1^2 (r^2 + \sin^2 \theta)^2}{x^4} \right) \\
& - (1 + r^2) dt^2 \left(1 - \frac{\rho_1^2 (r^2 + \cos^2 \theta)^2 - 4(1 + r^2) \cos^2 \theta - 1 + 4d\sqrt{1 + r^2} \cos \theta \cos(\phi - t)}{x^4} \right) \\
& + \frac{dr^2}{1 + r^2} \left(1 + \frac{\rho_1^2 r^4 - \sin^4 \theta}{x^4} \right) + d\theta^2 \left(1 + \frac{\rho_1^2 r^4 - \sin^4 \theta}{x^4} \right) \\
& + \cos^2 \theta d\phi^2 \left(1 + \frac{\rho_1^2 (r^2 + \cos^2 \theta)^2 - 4(1 + r^2) \cos^2 \theta - 1 + 4d\sqrt{1 + r^2} \cos \theta \cos(\phi - t)}{x^4} \right) \\
& - \rho_1^2 \frac{2d\sin(\phi - t)}{x^4} (dt(1 + r^2) - \cos^2 \theta d\phi) \left(\frac{r \cos \theta dr}{\sqrt{1 + r^2}} - \sqrt{1 + r^2} \sin \theta d\theta \right) \\
& + 2d\phi dt \rho_1^2 \frac{\cos \theta (2(1 + r^2) \cos \theta - d\sqrt{1 + r^2} (1 + r^2 + \cos^2 \theta) \cos(\phi - t))}{x^4} + O(\rho_1^4) \\
F_{tr} = & \frac{2(d^2 - 1)r \sin^4 \theta (r^2 + \sin^2 \theta)}{2Nx^6}, F_{t\phi} = \frac{d\sqrt{r^2 + 1} \sin^3 \theta \sin 2\theta (\sin^2 \theta + r^2)^2 \sin(t - \phi)}{2Nx^6} \\
F_{t\theta} = & \frac{\sqrt{1 + r^2} \sin^3 \theta}{8Nx^6} \left[8\sqrt{r^2 + 1} \cos \theta ((d^2(r^2 + 2) + 2r^2 + 1) \cos \theta + d^2 \cos^2 \theta \cos(2t - 2\phi)) \right. \\
& \left. - 4d \cos(t - \phi) ((d^2 + 3r^2 + 2) \cos 2\theta + d^2 + 2r^4 + 7r^2 + 4) \right] \\
F_{r\theta} = & \frac{dr \sin^3 \theta \sin(t - \phi) ((d^2 + 3r^2 + 2) \cos 2\theta - d^2 + 4d\sqrt{r^2 + 1} \cos^3 \theta \cos(t - \phi) + 2r^4 + r^2)}{2N\sqrt{r^2 + 1}x^6} \\
F_{r\phi} = & \frac{2r \sin^4 \theta \cos \theta (r^2 + \sin^2 \theta) (d(\cos 2\theta + 2r^2 + 3) \cos(t - \phi) - 4\sqrt{r^2 + 1} \cos \theta)}{4N\sqrt{r^2 + 1}x^6} \\
F_{\theta\phi} = & \frac{\sin^3 \theta \cos \theta}{4Nx^6} [2\cos^2 \theta (2d^2(2(r^2 + 1) \cos^2(t - \phi) + 2r^2 + 1) + \sin^2 \theta (4d^2 + 3r^4 + 2r^2 - 1) \\
& + 2(2r^2 + 1) \sin^4 \theta) \\
& + 2d\sqrt{r^2 + 1} \cos(t - \phi) \cos \theta (2\cos^2 \theta (2\cos 2\theta - 5(r^2 + 1)) - (2(d^2 - r^4) + \sin^2 \theta (-\cos 2\theta + 2r^2 + 1))) \\
& - \frac{1}{4} (r^2 + 1) (16r^4 - 8r^2 \cos 2\theta + (r^2 + 1) \cos 4\theta + 7r^2 - 1 + 4\cos^4 \theta (-\cos 2\theta + 2r^2 + 2))] \\
\tilde{F}_{tr} = & \frac{r^3}{32Nx^6} [(48d^2(r^2 + 1) - 16r^4 + 17) \cos 2\theta - \cos 6\theta + 2(6r^2 + 5) \cos 4\theta \\
& + 2(-8(4d^2 + 3)r^4 - 6(4d^2 + 1)r^2 + 2d \cos \theta (\sqrt{r^2 + 1} \cos(t - \phi) (-8d^2 + \cos 4\theta - 8(2r^2 + 3) \cos 2\theta \\
& + 24(r^4 + r^2) - 1) + 8d(r^2 + 1) \cos \theta \cos(2t - 2\phi)) + 8d^2 - 16r^6 + 3)] \\
\tilde{F}_{t\theta} = & - \frac{4r^4 \sin \theta (r^2 + \sin^2 \theta) (4(r^2 + 1) \cos \theta - d\sqrt{r^2 + 1} (\cos 2\theta + 2r^2 + 3) \cos(t - \phi))}{8Nx^6} \\
\tilde{F}_{t\phi} = & - \frac{dr^4 \sqrt{r^2 + 1} \cos \theta (\sin^2 \theta + r^2)^2 \sin(t - \phi)}{Nx^6} \\
\tilde{F}_{r\theta} = & \frac{-dr^3 \sin \theta \sin(t - \phi)}{8N\sqrt{r^2 + 1}x^6} [8d(r^2 + 1) (d - 2(r^2 + 1)^{1/2} \cos \theta \cos(t - \phi)) - \cos 4\theta + 4(3r^2 + 2) \cos 2\theta \\
& + 4r^2 + 1] \\
\tilde{F}_{r\phi} = & - \frac{r^3 \cos \theta}{8Nx^6} \left[\frac{\cos(t - \phi) (d(8d^2(r^2 + 1) + \cos 4\theta + 4(3r^2 + 4) \cos 2\theta + 4r^2 + 7))}{\sqrt{r^2 + 1}} \right. \\
& \left. - 4\cos \theta ((d^2 + 2) \cos 2\theta - 2d^2(r^2 + 1) \cos(2t - 2\phi) + 3d^2) \right] \\
\tilde{F}_{\theta\phi} = & \frac{2(d^2 - 1)r^4 \sin(2\theta) (-\cos(2\theta) + 2r^2 + 1)}{8Nx^6}
\end{aligned}$$



$$ds^2 = d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_3^2$$

$$\left(\frac{1}{\sin^3 \theta \cos \theta} \frac{\partial}{\partial \theta} \left(\sin^3 \theta \cos \theta \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial \phi^2} \right) Y(\theta, \phi) = -k(k+4)$$

$$Y(\theta, \phi) = Y_{k,m}(\theta) e^{im\phi}$$

$$\frac{\partial^2 Y_{k,m}(\theta)}{\partial \theta^2} - \frac{m^2 Y_{k,m}(\theta)}{\cos^2 \theta} + \left(\frac{3 \cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} \right) \frac{\partial Y_{k,m}(\theta)}{\partial \theta} = -k(k+4)$$

$$Y_{k,m}(\theta) = a(\cos \theta)^{-m} {}_2F_1 \left(-\frac{k}{2} - \frac{m}{2}, \frac{k}{2} - \frac{m}{2} + 2; 1 - m; \cos^2 \theta \right) \\ + b(\cos \theta)^m {}_2F_1 \left(\frac{m}{2} - \frac{k}{2}, \frac{k}{2} + \frac{m}{2} + 2; m + 1; \cos^2 \theta \right)$$

$$Y_{k,m}(\theta) = b(\cos \theta)^m {}_2F_1 \left(\frac{m}{2} - \frac{k}{2}, \frac{k}{2} + \frac{m}{2} + 2; m + 1; \cos^2 \theta \right)$$

$$\lim_{\theta \rightarrow 0} Y_{k,m}(\theta) = b \frac{i^m \Gamma(m+1)}{\theta^2 \Gamma\left(\frac{m-k}{2}\right) \Gamma\left(\frac{1}{2}(k+m+4)\right)}$$

$$\frac{2}{\pi} \int d\theta d\phi \sin^3 \theta \cos \theta Y(\theta, \phi) Y^*(\theta, \phi) = \delta_{k,k'} \delta_{m,m'}$$

$$Y(\theta, \phi) = b(\cos \theta)^{|m|} {}_2F_1 \left(\frac{|m|}{2} - \frac{k}{2}, \frac{k}{2} + \frac{|m|}{2} + 2; |m| + 1; \cos^2 \theta \right) e^{im\phi}$$

$$Y_{k,k}(\theta, \phi) = \sqrt{\frac{(k+1)(k+2)}{2}} (\cos \theta)^k e^{ik\phi}$$

$$\chi = 4 \left(-1 - \frac{(\cos 2\theta - 2r^2 - 1)}{64N\sqrt{r^2 + 1}x^6} \left[4\cos 2\theta \left(\sqrt{r^2 + 1}(5d^2 + 10r^2 + 6) - 6d(r^2 + 1)\cos \theta \cos(t - \phi) \right) \right. \right. \\ \left. \left. + \sqrt{r^2 + 1}(8(5d^2 + 2)r^2 + 28d^2 + 8r^4 + 23) - 24d(2r^4 + 5r^2 + 3)\cos \theta \cos(t - \phi) + \sqrt{r^2 + 1}\cos 4\theta \right] \right)$$

$$\chi(\theta, \phi) \approx 4 \left(\frac{-6\cos^2 \theta (8d^2 \cos^2(t - \phi) + 1) + 8d^2 + 15\cos 2\theta + 7}{16Nr^2} \right) + O\left(\frac{1}{r^3}\right) \\ \approx \frac{1}{r^2} \left(\frac{d^2 \sqrt{6}}{2} (Y_{2,2}(\theta, \tilde{\phi}) + Y_{2,-2}(\theta, \tilde{\phi})) + \sqrt{2}(d^2 - 1)Y_{2,0}(\theta, \tilde{\phi}) \right) + O\left(\frac{1}{r^3}\right)$$

$$s^I = \frac{1}{k(k+4)} \left(\frac{2}{\pi} \int d\theta d\phi \sin^3 \theta \cos \theta Y_{k,k}^*(\theta, \phi) \chi(\theta, \phi) \right)$$

$$z_{\pm} = \frac{\sec \theta \left(2d^2 + \cos 2\theta + 2r^2 + 1 \pm \sqrt{(2d^2 + \cos 2\theta + 2r^2 + 1)^2 - 16d^2(r^2 + 1)\cos^2 \theta} \right)}{4d\sqrt{r^2 + 1}}$$

$$s^I = -\frac{4i^k}{Nk(k+4)} \left(\int d\theta (\sqrt{2(k+1)(k+2)}(k^2 - 1) \sin^3 \theta \cos^{k+1} \theta) \left(\frac{d \cos \theta}{r} \right)^k \right)$$



$$s^I=-\frac{4i^k}{Nk(k+4)}\Bigg(\frac{(k^2-1)}{\sqrt{2(k+1)(k+2)}}\Bigg)\Bigg(\frac{d}{r}\Bigg)^k$$

$$l_1=\cos\,\alpha, l_2=\sin\,\alpha\cos\,\beta, l_3=\sin\,\alpha\sin\,\beta$$

$$Q=\frac{1}{16\pi G}\int_{S^3}{}^*F$$

$$\partial_t - \Omega (\partial_\psi + \cos \theta \partial_\phi)$$

$$M_{\rm there}=\frac{\pi}{2}M_{\rm here}\,,J_{\rm there}=\frac{\pi}{2}J_{\rm here}\,,Q_{\rm there}=2Q_{\rm here}$$

$$q_c<\frac{R^2}{2}\Big(4\sqrt{R^2+1}-R^2-4\Big)$$

$$l_i = \sqrt{\frac{q_i}{\sum_j q_j}}$$

$$-e^{-ik\psi}\nabla^2\big(B(x_i)e^{ik\psi}\big)=-H^{ij}D_iD_jB+e^{-2\sigma}k^2B$$

$$\int \sqrt{V} dr \sim q_n$$

$$V(r)=-\frac{\frac{R^2\omega^2}{4(\mu^2-2R^2-1)^2}+\frac{1}{4}}{2N^2(r-R)^2}$$

$$\omega=\omega_*+1+\mathcal{O}\left(\frac{1}{q_nN}\right)$$

$$V(r)=2(r-r_*)^2-\frac{1}{N}-\epsilon$$

$$r-r_1\gg\frac{1}{N^{\frac{2}{3}}}$$

$$g(r)=2c_1\sqrt{2b(r-R)}e^{-\frac{i\omega_*^q}{w^l(r)}\log b(r-R)}$$

$$\tilde{\epsilon}_5=\epsilon^{(4)}(\mathbb{CP}^2)\wedge (d\Psi+\Theta-A)$$

$$\delta F_{(5)}=\frac{l_{N-1}^4}{N}4(F\wedge d\Omega_3+\tilde{F}\wedge d\tilde{\Omega}_3)=\frac{l_{N-1}^4}{N}(-4\epsilon_5+4\tilde{\epsilon}_5)$$



$$l_p^8=g_s^2\alpha'^4$$

$$\begin{aligned} T_F^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S_F}{\delta g_{\mu\nu}} \\ &= \frac{-1}{16\pi(2\pi^2)^3 g_s^2 \alpha'^4 5!} \Big(\frac{1}{2} g^{\mu\nu} F_5^2 - 5 F^{\mu\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4}^\nu \Big) \end{aligned}$$

$$\int \vec{E}_1\cdot\vec{E}_2=\int \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2$$

$$-\int \phi_2 \vec{\nabla}^2 \phi_1 + \int_B \phi_2 \vec{\nabla} \phi_1$$

$$ds_3^2=\frac{r^2}{4}(\sigma_1^2+\sigma_2^2)+(b^2+A_{\sigma_3}^2)\sigma_3^2\otimes\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{4}$$

$$l_1=\cos\alpha, l_2=\sin\alpha\cos\beta, l_3=\sin\alpha\sin\beta$$

$$\langle A_2 \mid A_1 \rangle = \mathrm{Tr}(A_2^\dagger A_1)$$

$$u_{11}=\frac{1}{\sqrt{2}}\begin{pmatrix}i\sigma_2&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$u_{12,r}=\frac{1}{2}\begin{pmatrix}0&-\mathbb{I}_2&0\\ \mathbb{I}_2&0&0\\ 0&0&0\end{pmatrix}, u_{12,c}=\frac{1}{2}\begin{pmatrix}0&i\sigma_2&0\\ i\sigma_2&0&0\\ 0&0&0\end{pmatrix}\mathrm{Tr}(OB^\dagger)$$

$$\frac{\partial M_{ext}(\vec{Q},\epsilon)}{\partial \epsilon}=\lim_{M\rightarrow M_{ext}(\vec{Q},\epsilon)}-T\left(\frac{\partial S(M,\vec{Q},\epsilon)}{\partial \epsilon}\right)_{M,\vec{Q}}$$

$$\Delta \mathcal{I} = \mathcal{K} \int \, d^4x \sqrt{-g} \epsilon \Lambda$$

$$\mathcal{I}_{\mathrm{Total}} \, = \mathcal{I} + \Delta \mathcal{I}.$$

$$g_{\mu\nu}=\hat{g}_{\mu\nu}+\epsilon\Delta g_{\mu\nu}=\hat{g}_{\mu\nu}+\epsilon\mathsf{h}_{\mu\nu}$$

$$G(T,\vec{\mu})=M(T,\vec{\mu})-TS-\vec{\mu}\cdot\vec{Q}$$

$$G(T,\vec{\mu})\rightarrow G(T,\vec{\mu})+\epsilon\Delta G(T,\vec{\mu}).$$

$$S(T,\vec{\mu},\epsilon)=-\left(\frac{\partial G}{\partial T}\right)_{\vec{\mu},\epsilon}\;;\;Q_i(T,\vec{\mu},\epsilon)=-\left(\frac{\partial G}{\partial \mu_i}\right)_{T,\mu_{j\neq i},\epsilon}$$

$$\lim_{T\rightarrow 0}T\left(\frac{\partial S(T,\vec{Q},\epsilon)}{\partial \epsilon}\right)_{T,\vec{Q}}=0$$

$$M_{\mathrm{ext}}(\vec{Q},\epsilon)\equiv\lim_{T\rightarrow 0}M(T,\vec{Q},\epsilon)$$

$$\begin{aligned}\left(\frac{\partial M}{\partial \epsilon}\right)_{T,\vec{Q}} &= \left(\frac{\partial}{\partial \epsilon}(G + TS + \vec{\mu} \cdot \vec{Q})\right)_{T,\vec{Q}} \\ &= \left(\frac{\partial G}{\partial \vec{\mu}}\right)_{T,\epsilon} \cdot \left(\frac{\partial \vec{\mu}}{\partial \epsilon}\right)_{T,\vec{Q}} + \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}} + T \left(\frac{\partial S}{\partial \epsilon}\right)_{T,\vec{Q}} + \vec{Q} \cdot \left(\frac{\partial \vec{\mu}}{\partial \epsilon}\right)_{T,\vec{Q}} \\ &= \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}} + T \left(\frac{\partial S}{\partial \epsilon}\right)_{T,\vec{Q}}.\end{aligned}$$

$$\lim_{T \rightarrow 0} \left(\frac{\partial M}{\partial \epsilon}\right)_{T,\vec{Q}} = \lim_{T \rightarrow 0} \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}}$$

$$-T \left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}} = \left(\frac{\partial G}{\partial \epsilon}\right)_{T,\vec{\mu}}.$$

$$\Delta M_{ext}(\vec{Q}) \approx -T_0(M, \vec{Q}) \Delta S(M, \vec{Q})|_{M \approx M_{ext}^0(\vec{Q})}$$

$$\mathcal{I}_{\text{AdS}} = \frac{1}{2} \int d^4x \sqrt{-g} (R - 2\Lambda + \mathcal{I}_{\text{interaction}})$$

$$\mathcal{I}_{\text{PM}} = \frac{1}{2} \int_M d^4x \sqrt{-g} [R - 2\Lambda - \mathcal{F}^\gamma]$$

$$\begin{aligned}G_{\mu\nu} + \Lambda g_{\mu\nu} - \gamma(\mathcal{F})^{\gamma-1} \left(F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) &= 0 \\ \frac{1}{\sqrt{-g}} \nabla_{\nu} (\sqrt{-g} (\mathcal{F})^{\gamma-1} F^{\mu\nu}) &= 0\end{aligned}$$

$$ds^2 = -f_{\text{PM}}(r) dt^2 + \frac{dr^2}{f_{\text{PM}}(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

$$f_{\text{PM}}(r) = 1 + \frac{r^2}{\ell^2} - \frac{2M}{r} - \frac{(2\gamma-1)^{2-2\gamma} q^{2\gamma}}{(2^{1-\gamma}(2\gamma-3)^{1-2\gamma}) r^{\frac{2}{2\gamma-1}}}.$$

$$A = -qr^{(2\gamma-3)/(2\gamma-1)}dt\,; \, F = dA,$$

$$Q = 2^{\gamma-1} \frac{(3-2\gamma)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} q^{2\gamma-1}.$$

$$T = \frac{f'_{\text{PM}}(r_+)}{4\pi} = \frac{1}{4\pi} \left[\frac{2r_+}{\ell^2} + \frac{2M}{r_+^2} + 2^\gamma \left(\frac{2}{1-2\gamma} + 1 \right)^{2\gamma-1} \frac{q^{2\gamma}}{r_+^{\frac{2}{2\gamma-1}+1}} \right].$$

$$M = -\frac{1}{2}r_+ \left(-\frac{r_+^2}{l^2} + 2^{\gamma-1} \frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} \frac{q^{2\gamma}}{r_+^{\frac{2}{2\gamma-1}}} - 1 \right)$$

$$\Phi = \frac{q}{r_+^{(3-2\gamma)/(2\gamma-1)}}\,;\, S = \pi r_+^2$$



$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi\ell^2} ; V = \frac{4\pi r_+^3}{3}$$

$$M = 2TS + \frac{\Phi Q}{\gamma(2\gamma - 1)} - VP$$

$$dM = TdS + \Phi dQ + VdP$$

$$J_{\text{PYM}} = \frac{1}{2} \int d^4x \sqrt{-g} [R - 2\Lambda - \mathcal{F}^\xi]$$

$$\mathcal{F} = \text{Tr} \left(F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} \right) ; \text{Tr}(\cdot) = \sum_{a=1}^3 (\cdot)$$

$$F_{\lambda\sigma}^{(a)} = \partial_\lambda A_\sigma^{(a)} - \partial_\sigma A_\lambda^{(a)} + \frac{1}{2\zeta} \Xi_{(b)(c)}^{(a)} A_\lambda^{(b)} A_\sigma^{(c)}$$

$$F^{(a)} = dA^{(a)} + \frac{1}{2\sigma} \Xi_{(b)(c)}^{(a)} A^{(b)} \wedge A^{(c)}.$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{1}{2} \left(\delta_\nu^\mu \mathcal{F}^\xi - 4\xi \text{Tr} \left[F_{\nu\lambda}^{(a)} F^{(a)\mu\lambda} \right] \mathcal{F}^{\xi-1} \right),$$

$$d\left({}^*F^{(a)}\mathcal{F}^{\xi-1} \right) + \frac{1}{\zeta} \Xi_{(b)(c)}^{(a)} \mathcal{F}^{\xi-1} A^{(b)} \wedge {}^*F^{(c)} = 0$$

$$f_{\text{PYM}}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{(2Q^2)^\xi}{2(4\xi - 3)r^{4\xi-2}}.$$

$$M = \frac{1}{4}r_h \left(\frac{2r_h^2}{\ell^2} + \frac{2^\xi r_h^{2-4\xi}}{4\xi - 3} Q^{2\xi} + 2 \right)$$

$$T = \frac{1}{4\pi} \left[\frac{2r_h}{l^2} + \frac{2M}{r_h^2} + \frac{2^\xi (1-2\xi) r_h^{1-4\xi}}{4\xi - 3} Q^{2\xi} \right].$$

$$\Phi_Q = \frac{\xi(2Q^2)^\xi}{2(4\xi - 3)Q} r_h^{3-4\xi} ; S = \pi r_h^2$$

$$M = 2TS + \frac{2\xi - 1}{\xi} \Phi_Q Q - 2VP$$

$$dM = TdS + \Phi_Q dQ + VdP$$

$$J_{\text{MPYM}} = \frac{1}{2} \int d^4x \sqrt{-g} [(R - 2\Lambda) - F_{\mu\nu} F^{\mu\nu} - \mathcal{F}^\xi],$$

$$f_{\text{MPYM}}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{Q_M}{r^2} + \frac{(2Q_{YM})^\xi}{2(4\xi - 3)r^{4\xi-2}},$$



$$M = \frac{1}{4r_h} \left[2Q_M^2 + \frac{2r_h^4}{\ell^2} + \frac{2^\xi (Q_{YM})^{2\xi} r_h^{4-4\xi}}{4\xi - 3} + 2r_h^2 \right].$$

$$T = \frac{1}{4\pi} \left[-\frac{2Q_M^2}{r_h^3} + \frac{2r_h}{\ell^2} + \frac{2M}{r_h^2} + \frac{2^\xi (1-2\xi)(Q_{YM})^{2\xi} r_h^{1-4\xi}}{4\xi - 3} \right].$$

$$\Phi_{Q_M} = \frac{Q_M}{r_h} \quad \text{and} \quad \Phi_{Q_{YM}} = \frac{\xi(2Q^2)^\xi}{2(4\xi - 3)Q} r_h^{3-4\xi}.$$

$$M = 2TS + \Phi_{Q_M} Q_M + \frac{2\xi - 1}{\xi} \Phi_{Q_{YM}} Q_{YM} - 2VP.$$

$$dM = TdS + \Phi_{Q_M} dQ_M + \Phi_{Q_{YM}} dQ_{YM} + VdP$$

$$ds_{\text{Perturbed}}^2 = -\frac{r^2}{\ell^2} dt^2 - \frac{r^2}{\ell^2 f_{\text{PM}}(r)} dr^2.$$

$$M_{\text{ext}}(\vec{Q}, \epsilon) = \frac{\sqrt{S}}{2\sqrt{\pi}} \left(1 + \frac{S(\epsilon + 1)}{\pi \ell^2} - 2^{\gamma-1} \pi^{\frac{1}{2\gamma-1}} \frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} \frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}}} \right).$$

$$T(\vec{Q}, \epsilon) = \frac{\epsilon + 1}{8\pi^{3/2} \ell^2 \sqrt{S}} \left[\frac{\pi \ell^2}{\epsilon + 1} \left(2 - 2^\gamma \pi^{\frac{1}{2\gamma-1}} \frac{(2\gamma-3)^{2\gamma}}{(2\gamma-1)^{2\gamma-1}} \frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}}} \right) + 6S \right].$$

$$\epsilon = -\frac{\ell^2}{2S^{3/2}} \left[\frac{2S^{3/2}}{\ell^2} - 2^\gamma \pi^{\frac{1}{2\gamma-1}+1} \frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} \frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1} - \frac{1}{2}}} + 2\pi\sqrt{S} - 4M\pi^{3/2} \right]$$

$$-T \left(\frac{\partial S}{\partial \epsilon} \right)_{M,Q} = \frac{S^2 \left[\pi \ell^2 \left(2 - 2^\gamma \pi^{\frac{1}{2\gamma-1}} \frac{(2\gamma-3)^{2\gamma}}{(2\gamma-1)^{2\gamma-1}} \frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1}}} \right) + 6S(\epsilon + 1) \right]}{8\pi^{5/2} \ell^4 \left[-3\sqrt{\pi}M - 2^\gamma \pi^{\frac{1}{2\gamma-1}} \left(\frac{2}{1-2\gamma} + 1 \right)^{2\gamma-1} \gamma \frac{q^{2\gamma}}{S^{2\gamma-1} + \frac{1}{2}} + \sqrt{S} \right]}$$

$$-T \left(\frac{\partial S}{\partial \epsilon} \right)_{M,\vec{Q}} = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon} \right)_{\vec{Q}} = \frac{r_+^3}{2\ell^2}.$$

$$\epsilon(P) = -\frac{1}{16PS^{3/2}} \left[16PS^{3/2} - 32^\gamma \pi^{\frac{1}{2\gamma-1}} \frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} \frac{q^{2\gamma}}{S^{\frac{1}{2\gamma-1} - \frac{1}{2}}} + 6\sqrt{S} - 12M\sqrt{\pi} \right]$$

$$-V \left(\frac{\partial P}{\partial \epsilon} \right) = \frac{64P^2 S^3}{3\sqrt{\pi} \left(-12\sqrt{\pi}M - 32^\gamma \pi^{\frac{1}{2\gamma-1}} \frac{(2\gamma-3)^{2\gamma-1}}{(2\gamma-1)^{2\gamma-2}} \frac{q^{2\gamma}}{S^{2\gamma-1} - \frac{1}{2}} + 6\sqrt{S} \right)}$$



$$-V\left(\frac{\partial P}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{r_+^3}{2\ell^2}$$

$$ds_{\text{Perturbed}}^2 = -\frac{r^2}{\ell^2} dt^2 - \frac{r^2}{\ell^2 f_{\text{PYM}}(r)} dr^2.$$

$$M_{\text{ext}}(\vec{Q}, \epsilon) = \frac{1}{4\pi^{3/2}} \left[\frac{2\sqrt{S}(\pi l^2 + S\epsilon + S)}{l^2} + \frac{2^\xi \pi^{2\xi} (Q^2)^\xi S^{\frac{3}{2}-2\xi}}{4\xi - 3} \right],$$

$$T(\vec{Q}, \epsilon) = \frac{S^{-2\xi-\frac{1}{2}}}{8\pi^{3/2}\ell^2} [2(\epsilon + 1)(\pi\ell^2 + 3S)S^{2\xi} - 2^\xi \pi^{2\xi} \ell^2 S(Q^2)^\xi].$$

$$\epsilon = \ell^2 \left(\frac{2\pi^{3/2}M}{S^{3/2}} + \frac{2^{\xi-1}\pi^{2\xi}(Q^2)^\xi S^{-2\xi}}{3-4\xi} - \frac{\pi}{S} \right) - 1.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,Q} = \frac{(4\xi - 3)S^2(2(\epsilon + 1)(\pi l^2 + 3S)S^{2\xi} - 2^\xi \pi^{2\xi} l^2 S(Q^2)^\xi)}{8\pi^{3/2}l^4 \left(-3\pi^{3/2}(4\xi - 3)MS^{2\xi} + 2^\xi \pi^{2\xi} \xi S^{3/2}(Q^2)^\xi + \pi(4\xi - 3)S^{2\xi+\frac{1}{2}} \right)}.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}} = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}.$$

$$\epsilon(P) = \frac{3}{16P} \left[\frac{4\sqrt{\pi}M}{S^{3/2}} + \frac{2^\xi \pi^{2\xi-1} (Q^2)^\xi S^{-2\xi}}{3-4\xi} - \frac{2}{S} \right] - 1$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right) = \frac{64P^2 S^{3/2}}{9\sqrt{\pi}} \left[-\frac{4\sqrt{\pi}M}{S^{3/2}} + \frac{2^\xi \pi^{2\xi-1} (Q^2)^\xi S^{-2\xi}}{4\xi - 3} + \frac{2}{S} \right]^{-1}.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}.$$

$$-\Phi_Q\left(\frac{\partial Q}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}.$$

$$ds_{\text{Perturbed}}^2 = -\frac{r^2}{\ell^2} dt^2 + \frac{r^2}{\ell^2 f_{\text{MPYM}}(r)} dr^2$$

$$M_{\text{ext}}(\vec{Q}, \epsilon) = \frac{\sqrt{S}}{2\sqrt{\pi}} \left[\frac{1}{2\pi} S \left(\frac{2(\epsilon + 1)}{\ell^2} + \frac{2^\xi \pi^{2\xi} S^{-2\xi} (Q_{\text{YM}}^2)^\xi}{4\xi - 3} \right) + \frac{\pi Q_M^2}{S} + 1 \right]$$

$$T(\vec{Q}, \epsilon) = \frac{S^{-2\xi-\frac{3}{2}} \left(2S^{2\xi} (S(\pi\ell^2 + 3S(\epsilon + 1)) - \pi^2 \ell^2 Q_M^2) - 2^\xi \pi^{2\xi} \ell^2 S^2 (Q_{\text{YM}}^2)^\xi \right)}{8\pi^{3/2}\ell^2}.$$

$$\epsilon = \frac{1}{2} \ell^2 \left[\frac{2^\xi \pi^{2\xi} S^{-2\xi} (Q_{\text{YM}}^2)^\xi}{3-4\xi} - \frac{2\pi(\pi Q_M^2 - 2\sqrt{\pi}M\sqrt{S} + S)}{S^2} \right] - 1.$$



$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,Q} = \frac{(\epsilon+1)S^{\frac{3}{2}-2\xi} \left(\frac{l^2 \left(-2\pi S^{2\xi} (\pi Q_M^2 - S) - 2^\xi \pi^{2\xi} S^2 (Q_{YM}^2)^\xi \right)}{\epsilon+1} + 6S^{2\xi+2} \right)}{8\pi^{3/2} l^4 \left(\pi(2\pi Q_M^2 - 3\sqrt{\pi} M \sqrt{S} + S) + \frac{2^\xi \pi^{2\xi} \xi S^{2-2\xi} (Q_{YM}^2)^\xi}{4\xi-3} \right)}.$$

$$-T\left(\frac{\partial S}{\partial \epsilon}\right)_{M,\vec{Q}} = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}.$$

$$\epsilon(P) = \frac{3}{16\pi P} \left[\frac{2^\xi \pi^{2\xi} (Q_{YM}^2)^\xi S^{-2\xi}}{3-4\xi} - \frac{2\pi(\pi Q_M^2 - 2\sqrt{\pi} M \sqrt{S} + S)}{S^2} \right] - 1.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right) = -\frac{64\sqrt{\pi} P^2 S^{3/2}}{9 \left(\frac{2^\xi \pi^{2\xi} S^{-2\xi} (Q_{YM}^2)^\xi}{3-4\xi} - \frac{2\pi(\pi Q_M^2 - 2\sqrt{\pi} M \sqrt{S} + S)}{S^2} \right)}.$$

$$-V\left(\frac{\partial P}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}.$$

$$-\Phi_{Q_M}\left(\frac{\partial Q_M}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}$$

$$-\Phi_{Q_{YM}}\left(\frac{\partial Q_{YM}}{\partial \epsilon}\right) = \left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{\vec{Q}} = \frac{1}{2\ell^2} \left(\frac{S}{\pi}\right)^{3/2}$$

$$dM = TdS + \vec{\mu} \cdot d\vec{Q}$$

$$L_0 - \frac{k_R}{4} = \frac{\epsilon + j}{2}, \tilde{L}_0 - \frac{k_L}{4} = \frac{\epsilon - j}{2}.$$

$$6k=c=\frac{3R}{2G_3}.$$

$$T=\frac{\hbar c^3}{8\pi G_N k_B M'}$$

$$S=2\pi\sqrt{\frac{1}{2}k_R(E+J)-\frac{1}{4}Q_R^2}+2\pi\sqrt{\frac{1}{2}k_L(E-J)-\frac{1}{4}Q_L^2}.$$

$$TdS = dE - \mu dJ - \omega_L dQ_L - \omega_R dQ_R$$

$$\begin{aligned}
\beta &= \frac{\partial S}{\partial E} = \frac{1}{2} \sqrt{\frac{2\pi^2 k_L}{E - J - \frac{Q_L^2}{2k_L}}} + \frac{1}{2} \sqrt{\frac{2\pi^2 k_R}{E + J - \frac{Q_R^2}{2k_R}}} \\
\mu &= -\frac{1}{\beta} \frac{\partial S}{\partial J} = \frac{-\sqrt{\frac{E - J - \frac{Q_L^2}{2k_L}}{k_L}} + \sqrt{\frac{E + J - \frac{Q_R^2}{2k_R}}{k_R}}}{\sqrt{\frac{E - J - \frac{Q_L^2}{2k_L}}{k_L}} + \sqrt{\frac{E + J - \frac{Q_R^2}{2k_R}}{k_R}}} \\
\omega_R &= -\frac{1}{\beta} \frac{\partial S}{\partial Q_R} = \frac{\sqrt{\frac{E + J - \frac{Q_R^2}{2k_R}}{k_R}}}{\sqrt{\frac{E - J - \frac{Q_L^2}{2k_L}}{k_L}} + \sqrt{\frac{E + J - \frac{Q_R^2}{2k_R}}{k_R}}} \cdot \frac{Q_R}{k_R} \\
\omega_L &= -\frac{1}{\beta} \frac{\partial S}{\partial Q_L} = \frac{\sqrt{\frac{E - J - \frac{Q_L^2}{2k_L}}{k_L}}}{\sqrt{\frac{E - J - \frac{Q_L^2}{2k_L}}{k_L}} + \sqrt{\frac{E + J - \frac{Q_R^2}{2k_R}}{k_R}}} \cdot \frac{Q_L}{k_L}. \quad (2.5)
\end{aligned}$$

$$E_R \equiv E + J = \frac{2k_R}{\beta^2(1 - \mu)^2} (\pi^2 + \beta^2 \omega_R^2)$$

$$E_L \equiv E - J = \frac{2k_L}{\beta^2(1 + \mu)^2} (\pi^2 + \beta^2 \omega_L^2)$$

$$Q_R = \frac{2k_R}{1 - \mu} \omega_R$$

$$Q_L = \frac{2k_L}{1 + \mu} \omega_L$$

$$Z(\beta, \mu, \omega_R, \omega_L) = \text{Tr}[e^{-\beta(E - \mu J - \omega_R Q_R - \omega_L Q_L)}]$$

$$\begin{aligned}
\log Z &= S - \beta(E - \mu J - \omega_R Q_R - \omega_L Q_L) \\
&= \frac{k_R}{\beta(1 - \mu)} (\pi^2 + \beta^2 \omega_R^2) + \frac{k_L}{\beta(1 + \mu)} (\pi^2 + \beta^2 \omega_L^2).
\end{aligned}$$

$$\begin{aligned}
E - \mu J - \omega_R Q_R - \omega_L Q_L &= -\frac{\partial \log Z}{\partial \beta} \\
&= \frac{k_R}{\beta^2(1 - \mu)} (\pi^2 - \beta^2 \omega_R^2) + \frac{k_L}{\beta^2(1 + \mu)} (\pi^2 - \beta^2 \omega_L^2) \\
J &= \frac{1}{\beta} \frac{\partial \log Z}{\partial \mu} \\
&= \frac{k_R}{\beta^2(1 - \mu)^2} (\pi^2 + \beta^2 \omega_R^2) - \frac{k_L}{\beta^2(1 + \mu)^2} (\pi^2 + \beta^2 \omega_L^2) \\
Q_{L,R} &= \frac{1}{\beta} \frac{\partial \log Z}{\partial \omega_{L,R}} = \frac{2k_{L,R}}{1 \pm \mu} \omega_{L,R}
\end{aligned}$$



$$E - J - \frac{1}{2k_L} Q_L^2 \geq 0$$

$$E + J - \frac{1}{2k_R} Q_R^2 \geq 0$$

$$\epsilon - j + \frac{1}{2} k_L \geq q_L$$

$$E - J + \frac{1}{2} k_L \geq Q_L$$

$$Q_L = k_L$$

$$E = \frac{k_R}{\beta^2(1-\mu)^2} (\pi^2 + \beta^2 \omega_R^2) + \frac{k_L}{\beta^2(1+\mu)^2} (\pi^2 + \beta^2 \omega_L^2),$$

$$J = \frac{k_R}{\beta^2(1-\mu)^2} (\pi^2 + \beta^2 \omega_R^2) - \frac{k_L}{\beta^2(1+\mu)^2} (\pi^2 + \beta^2 \omega_L^2),$$

$$Q_{L,R} = \frac{2k_{L,R}}{1 \pm \mu} \omega_{L,R}$$

$$E + J - \frac{Q_R^2}{2k_R} = \frac{2k_R \pi^2}{\beta^2(1-\mu)^2} \geq 0$$

$$E - J - \frac{Q_L^2}{2k_L} = \frac{2k_L \pi^2}{\beta^2(1+\mu)^2} \geq 0$$

$$\text{Extremal limit: } \begin{cases} \beta \rightarrow \infty, \\ \mu \rightarrow 1 \\ \omega_R \rightarrow 0 \\ \omega_L \text{ finite.} \end{cases} \quad \begin{cases} \text{with } \tilde{\mu} \equiv \beta(\mu - 1) \text{ finite,} \\ \text{with } \tilde{\omega}_R \equiv \beta \omega_R \text{ finite,} \end{cases}$$

$$E = \frac{k_R}{\tilde{\mu}^2} (\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{4} \omega_L^2,$$

$$J = \frac{k_R}{\tilde{\mu}^2} (\pi^2 + \tilde{\omega}_R^2) - \frac{k_L}{4} \omega_L^2,$$

$$Q_R = -\frac{2k_R}{\tilde{\mu}} \tilde{\omega}_R,$$

$$Q_L = k_L \omega_L.$$

$$E_{\text{ext}} = J + \frac{1}{2k_L} Q_L^2$$

$$S_{\text{ext}} = -\frac{2k_R \pi^2}{\tilde{\mu}} = 2\pi \sqrt{\frac{1}{2} k_R (E_{\text{ext}} + J) - \frac{1}{4} Q_R^2}$$

$$= 2\pi \sqrt{k_R J + \frac{k_R}{4k_L} Q_L^2 - \frac{1}{4} Q_R^2}$$

$$E - J - Q_L + \frac{1}{2} k_L \geq 0$$



$$E - J - Q_L + \frac{1}{2}k_L = \frac{2k_L\pi^2}{\beta^2(1+\mu)^2} + \frac{k_L}{2}\left(1 - \frac{2\omega_L}{1+\mu}\right)^2$$

$$\varphi \equiv 1 + \mu - 2\omega_L = 0$$

$$T = \beta^{-1} \rightarrow 0$$

$$\tilde{\mu} = \beta(\mu - 1), \tilde{\omega}_R = \beta\omega_R, \tilde{\omega}_L = \beta(\omega_L - 1)$$

$$\begin{aligned} Z(\beta, \mu, \omega_R, \omega_L) &= \text{Tr}[e^{-\beta(E - \mu J - \omega_R Q_R - \omega_L Q_L)}] \\ &= e^{\frac{1}{2}\beta k_L} \text{Tr}\left[e^{-\beta(E - J - Q_L + \frac{k_L}{2}) + \tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \end{aligned}$$

$$E_{\text{SUSY}} = -\frac{1}{2}k_L,$$

$$\begin{aligned} Z_{\text{BPS}}(\beta, \tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) &= \lim_{\beta \rightarrow \infty} e^{\frac{1}{2}\beta k_L} \text{Tr}\left[e^{-\beta(E - J - Q_L + \frac{k_L}{2}) + \tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \\ &= \left(\lim_{\beta \rightarrow \infty} e^{\frac{1}{2}\beta k_L}\right) \text{Tr}_{\text{BPS}}[e^{\tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}] \end{aligned}$$

$$\log Z_{\text{BPS}} = \frac{1}{2}\beta k_L - \frac{k_R}{\tilde{\mu}}(\pi^2 + \tilde{\omega}_R^2) + k_L\left(\tilde{\omega}_L - \frac{1}{4}\tilde{\mu}\right).$$

$$E = \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{4}$$

$$J = \frac{k_R}{\tilde{\mu}^2}(\pi^2 + \tilde{\omega}_R^2) - \frac{k_L}{4}$$

$$Q_R = -\frac{2k_R}{\tilde{\mu}}\tilde{\omega}_R$$

$$Q_L = k_L$$

$$S_{\text{BPS}} = 2\pi \sqrt{k_R\left(J + \frac{1}{4}k_L\right) - \frac{1}{4}Q_R^2}$$

$$TdS = dE - \mu dJ - \omega_R dQ_R - \omega_L dQ_L$$

$$TdS = -\epsilon + \mu j + \omega_R q_R + \omega_L q_L$$

$$\omega_R \leq 1, \omega_L \leq 1$$

$$0 \geq \omega_R - \frac{1-\mu}{2} \Leftrightarrow Q_R \leq k_R$$

$$0 \geq \omega_L - \frac{1+\mu}{2} \Leftrightarrow Q_L \leq k_L$$

$$E - J + \frac{1}{2}k_L \geq Q_L, E + J + \frac{1}{2}k_R \geq Q_R$$



$$S_{\text{bh}} = 2\pi \sqrt{\frac{1}{2}k_R E_{R,\text{bh}} - \frac{1}{4}Q_{R,\text{bh}}^2} + 2\pi \sqrt{\frac{1}{2}k_L E_{L,\text{bh}} - \frac{1}{4}Q_{L,\text{bh}}^2}.$$

$$\begin{aligned} S_{R,\text{bh}} &= 2\pi \sqrt{\frac{1}{2}k_R E_{R,\text{bh}} - \frac{1}{4}Q_{R,\text{bh}}^2} \\ &= 2\pi \sqrt{\frac{1}{2}k_R (E_{R,\text{tot}} - E_{R,\text{gp}}) - \frac{1}{4}(Q_{R,\text{tot}} - E_{R,\text{gp}})^2} \\ &= \pi \sqrt{2k_L \left(E_{R,\text{tot}} - Q_{R,\text{tot}} + \frac{k_R}{2}\right) - (Q_{R,\text{tot}} - k_R - E_{R,\text{gp}})^2} \end{aligned}$$

$$E_{R,\text{gp}} = \begin{cases} Q_{R,\text{tot}} - k_R \Rightarrow S_{R,\text{bh}} = \begin{cases} 2\pi \sqrt{\frac{1}{2}k_L \left(E_{R,\text{tot}} - Q_{R,\text{tot}} + \frac{k_R}{2}\right)} & (Q_{R,\text{tot}} > k_R) \\ 2\pi \sqrt{\frac{1}{2}k_R E_{R,\text{tot}} - \frac{1}{4}Q_{R,\text{tot}}^2} & (Q_{R,\text{tot}} \leq k_R) \end{cases} \\ 0 \end{cases}$$

$$E = \frac{\pi}{4G_5} \frac{m(2(1-a^2) + 2(1-b^2) - (1-a^2)(1-b^2)) + 2qab((1-a^2) + (1-b^2))}{(1-a^2)^2(1-b^2)^2}$$

$$Q = \frac{\pi}{4G_5} \frac{q}{(1-a^2)(1-b^2)}$$

$$J_1 = \frac{\pi}{4G_5} \frac{2ma + qb(1+a^2g^2)}{(1-a^2)^2(1-b^2)}$$

$$J_2 = \frac{\pi}{4G_5} \frac{2mb + qa(1+b^2g^2)}{(1-a^2)(1-b^2)^2}$$

$$2m = \frac{(r_+^2 + a^2)(r_+^2 + b^2)(1 + g^2 r_+^2) + q^2 + 2abq}{r_+^2}.$$

$$S = 2\pi \cdot \frac{\pi}{4G_5} \frac{(r_+^2 + a^2)(r_+^2 + b^2) + abq}{(1-a^2)(1-b^2)r_+}.$$

$$TdS = dE - \Phi dQ - \Omega_1 dJ_1 - \Omega_2 dJ_2.$$

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1} = \frac{r_+^4[1 + (2r_+^2 + a^2 + b^2)] - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}$$

$$E - (3Q + J_1 + J_2) \geq 0$$

$$E^* - 3Q^* - J_1^* - J_2^* = 0$$

$$E - (3Q + J_1 + J_2) = \frac{\pi}{4G_5} \frac{3 + (a+b) - ab}{(1-a)(1+a)^2(1-b)(1+b)^2} [m - q(1+a+b)],$$

$$q^* = \frac{m^*}{1+a+b}$$



$$m - q(1 + a + b) = \frac{r_+^2(q - q^*)^2 + (((1 + a + b)^2 + r_+^2)(r_+^2 - r^{*2}) - (1 + a + b)(q - q^*))^2}{2r_+^2((1 + a + b)^2 + r_+^2)}$$

$$q^* = (a + b)(1 + a)(1 + b), r^* \equiv r_+ = \sqrt{a + b + ab}.$$

$$E^* = \frac{\pi}{4G_5} \frac{(3(a + b) - (a^3 + b^3) - ab(a + b)^2)}{(1 - a)^2(1 - b)^2},$$

$$Q^* = \frac{\pi}{4G_5} \frac{a + b}{(1 - a)(1 - b)},$$

$$J_1^* = \frac{\pi}{4G_5} \frac{(a + b)(2a + b + ab)}{(1 - a)^2(1 - b)},$$

$$J_2^* = \frac{\pi}{4G_5} \frac{(a + b)(a + 2b + ab)}{(1 - a)(1 - b)^2}.$$

$$T = \frac{[1 + 3(a + b) + (a^2 + b^2 + 3ab)](r_+^2 - r^{*2}) - (1 + a + b)(q - q^*)}{\pi r^* q^*}.$$

$$\begin{aligned} & \left(Q_1 Q_2 Q_3 + \frac{\pi}{4G_5} J_1 J_2 \right) \\ &= \left(Q_1 + Q_2 + Q_3 + \frac{\pi}{4G_5} \right) \left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{\pi}{4G_5} (J_1 + J_2) \right). \end{aligned}$$

$$S^* = 2\pi \cdot \frac{\pi}{4G_5} \frac{a + b}{(1 - a)(1 - b)} \sqrt{a + b + ab}$$

$$S^* = 2\pi \sqrt{Q_1^* Q_2^* + Q_2^* Q_3^* + Q_3^* Q_1^* - \frac{\pi}{4G_5} (J_1^* + J_2^*)}.$$

$$\begin{aligned} J_1 &= \frac{j + \bar{j}}{2}, & J_2 &= \frac{j - \bar{j}}{2}, & Q_3 &= \frac{R_1 - R_3}{2}. \\ Q_1 &= R_2 + \frac{R_1 + R_3}{2}, & Q_2 &= \frac{R_1 + R_3}{2}, \end{aligned}$$

$$Z(\beta, \Delta_I, \omega_i) \equiv \text{Tr}[e^{-\beta E} e^{\Delta_I Q_I + \omega_i J_i}]$$

$$2\{Q, Q^\dagger\} = E - (Q_1 + Q_2 + Q_3 + J_1 + J_2)$$

$$E \geq Q_1 + Q_2 + Q_3 + J_1 + J_2$$

$$E = Q_1 + Q_2 + Q_3 + J_1 + J_2$$

$$Z(\beta, \Delta_I, \omega_i) = \text{Tr}[e^{-\beta(E - Q_1 - Q_2 - Q_3 - J_1 - J_2)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}]$$

$$\begin{aligned} Z_{\text{BPS}}(\tilde{\Delta}_I, \tilde{\omega}_i) &= \lim_{\beta \rightarrow \infty} \text{Tr}[e^{-\beta(E - Q_1 - Q_2 - Q_3 - J_1 - J_2)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}] \\ &= \text{Tr}_{\text{BPS}}[e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}] \end{aligned}$$

$$e^{\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2}{2}} = -1$$



$$\mathcal{I}(\tilde{\Delta}_I, \tilde{\omega}_i) = \text{Tr}[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}] \\ = \text{Tr}_{\text{BPS}}[e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}], \text{ where } e^{\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2}{2}} = -1.$$

$$\mathcal{I}(\tilde{\Delta}_I, \tilde{\omega}_i) = \text{Tr}[(-1)^F e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}], \text{ where } e^{\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2}{2}} = 1$$

$$A_\mu \sim A_{\alpha\dot\beta}, \qquad (\mu=1,2,3,4, \alpha=\pm, \dot\beta=\pm)$$

$$\Phi_{ij}(=-\Phi_{ji}), \bar{\Phi}^{ij} \sim \frac{1}{2} \epsilon^{ijkl} \Phi_{kl}, \quad (i,j,k,l=1,2,3,4)$$

$$:\Psi_{i\alpha},\bar{\Psi}^i_{\dot{\alpha}}$$

$$\partial_{+\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}=0\Leftrightarrow\partial_{+[\dot{\alpha}}\bar{\lambda}_{\dot{\beta}]}=0$$

$$J_1=\frac{j+\bar J}{2},\;\;J_2=\frac{j-\bar J}{2},\\ Q_1=R_2+\frac{R_1+R_3}{2},\;\;Q_2=\frac{R_1+R_3}{2},\;\;Q_3=\frac{R_1-R_3}{2},$$

$$x_B \equiv e^{\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2}{2}}$$

$$\begin{aligned} 1+x_B+x_B^2+\cdots &= \frac{1}{1-x_B} \\ &= \exp\left[-\log\left(1-x_B\right)\right] \\ &= \exp\left[\sum_{n=1}^{\infty}\frac{x_B^n}{n}\right] \\ &\equiv \text{PE}[x_B] \end{aligned}$$

$$\begin{aligned} 1-x_F &= \exp\left[\log\left(1-x_F\right)\right] \\ &= \exp\left[-\sum_{n=1}^{\infty}\frac{x_F^n}{n}\right] \\ &\equiv \text{PE}[-x_F] \end{aligned}$$

$$\mathcal{I}_{\text{Fock}}=\text{PE}\left[\sum_{\text{dark particle letters}}x_B-\sum_{\text{white particle letters}}x_F\right]$$

$$x^2=e^{\tilde{\Delta}_1}, y^2=e^{\tilde{\Delta}_2}, z^2=e^{\tilde{\Delta}_3}, p^2=e^{\tilde{\omega}_1}, q^2=e^{\tilde{\omega}_2}$$

$$\mathcal{I}_{\text{Fock}}\left(x,y,z,p,q;\zeta_a\right)=\text{Tr}_{\text{Fock}}\left[(-1)^F x^{2Q_1}y^{2Q_2}z^{2Q_3}p^{2J_1}q^{2J_2}\prod_a e^{i\alpha_a\zeta_a}\right]$$

$$\mathcal{I}_{\text{Fock}}\left(x,y,z,p,q;\zeta_a\right)=\text{PE}\big[f(x,y,z,p,q)\cdot\chi^{\text{adj.}}\left(\alpha_a\right)\big]$$

$$\begin{aligned} f(x,y,z,p,q) &= \frac{x^2+y^2+z^2-xyzpq\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)+p^2q^2}{(1-p^2)(1-q^2)} \\ &= 1-\frac{(1-x^2)(1-y^2)(1-z^2)}{(1-p^2)(1-q^2)} \end{aligned}$$



$$\chi^{adj} \cdot (\alpha_a) = \sum_{a,b=1}^N e^{i(\alpha_a - \alpha_b)}$$

$$\mathcal{I}(x,y,z,p,q)=\oint d\mu[\alpha_a]\mathcal{I}_{\text{Fock}}(x,y,z,p,q;\alpha_a)$$

$$\begin{aligned}\oint d\mu[\alpha_a]&=\frac{1}{N!}\cdot\int_0^{2\pi}\prod_{a=1}^N\frac{d\alpha_a}{2\pi}\cdot\prod_{a,b=1}^N(1-e^{i(\alpha_a-\alpha_b)})\\&=\frac{1}{N!}\cdot\int_0^{2\pi}\prod_{a=1}^N\frac{d\alpha_a}{2\pi}\cdot\text{PE}\left[-\sum_{a,b=1}^Ne^{i(\alpha_a-\alpha_b)}\right]\end{aligned}$$

$$\begin{aligned}\mathcal{I}(x,y,z,p,q)&=\frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^N\frac{d\alpha_a}{2\pi}\cdot\text{PE}\left[-(1-f(x,y,z,p,q))\cdot\sum_{a,b=1}^Ne^{i(\alpha_a-\alpha_b)}\right] \\&=\frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^N\frac{d\alpha_a}{2\pi}\cdot\exp\left[-\sum_{n=1}^{\infty}\sum_{a,b=1}^N\frac{1}{n}(1-f(x^n,y^n,z^n,p^n,q^n))\cdot e^{in(\alpha_a-\alpha_b)}\right]\end{aligned}$$

$$\int_0^{2\pi}d\theta\rho(\alpha)=1$$

$$\mathcal{I}(x,y,z,p,q)=\frac{1}{N!}\int\,[d\rho]e^{-S[\rho(\alpha)]},$$

$$\begin{aligned}S[\rho(\alpha)]&=N^2\sum_{n=1}^{\infty}\int_0^{2\pi}d\alpha_1d\alpha_2\rho(\alpha_1)\rho(\alpha_2)e^{in(\alpha_1-\alpha_2)}\cdot\frac{1}{n}(1-f(x^n,y^n,z^n,p^n,q^n))\\&=N^2\sum_{n=1}^{\infty}\frac{1}{n}(1-f(x^n,y^n,z^n,p^n,q^n))\cdot|\rho_n|^2\end{aligned}$$

$$\rho_n\equiv\int_0^{2\pi}d\alpha\rho(\alpha)e^{in\alpha}$$

$$1-f(x^n,y^n,z^n,p^n,q^n)=\frac{(1-x^{2n})(1-y^{2n})(1-z^{2n})}{(1-p^{2n})(1-q^{2n})}>0,$$

$$\mathcal{I}(x,y,z,p,q)=e^{\mathcal{O}(N^0)}$$

$$\frac{\pi}{4G_5}=\frac{N^2}{2}$$

$$e^{\frac{\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3-\bar{\omega}_1-\bar{\omega}_2}{2}}=-1$$

$$\mathcal{I}(x)=\text{Tr}[(-1)^F x^{2Q_1+2Q_2+2Q_3+3J_1+3J_2}]\equiv\text{Tr}[(-1)^F x^{\mathcal{J}}]$$

$$\mathcal{J}\equiv 2Q_1+2Q_2+2Q_3+3J_1+3J_2$$

$$x^2=e^{\Delta_1}=e^{\Delta_2}=e^{\Delta_3}$$



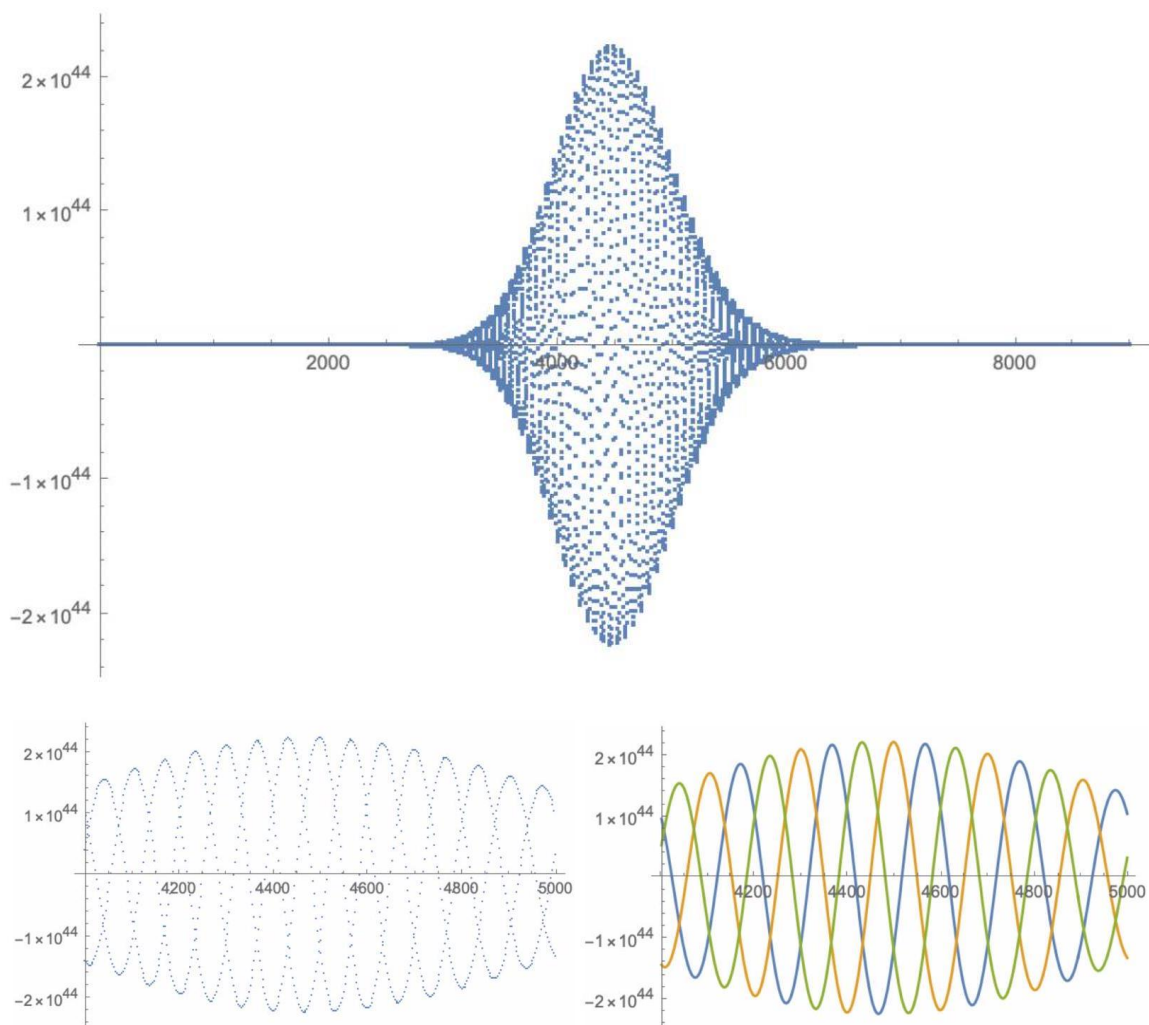
$$x^3=e^{\omega_1}=e^{\omega_2}$$

$$e^{\frac{\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3-\bar{\omega}_1-\bar{\omega}_2}{2}}=1$$

$$\mathcal{I}(x)=\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}(1-f(x^n))\right],$$

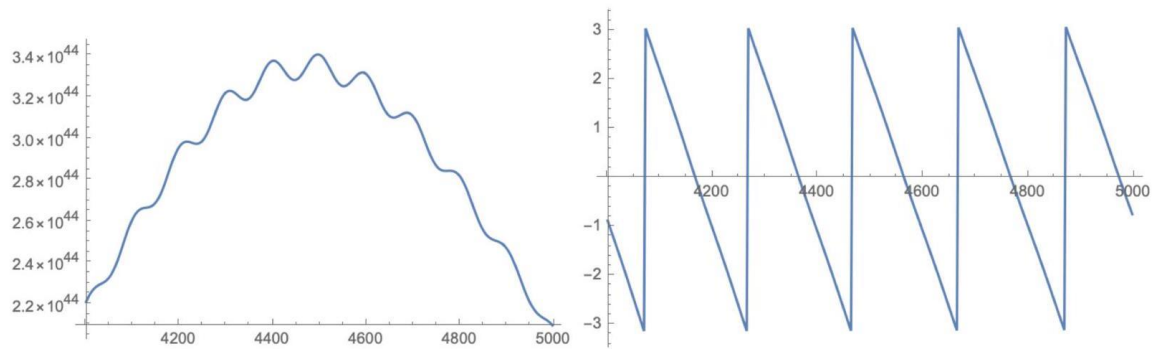
$$\begin{aligned} f(x) &= \frac{3x^2-2x^3-3x^4+2x^6}{(1-x^3)^2} \\ &= 1-\frac{(1-x^2)^3}{(1-x^3)^2} \end{aligned}$$

$$\mathcal{I}(x)=\sum_{j=0}^{\infty}\Omega_jx^j$$



$$\Omega_jx^j+\Omega_{j+1}x^{j+1}+\Omega_{j+2}x^{j+2}$$

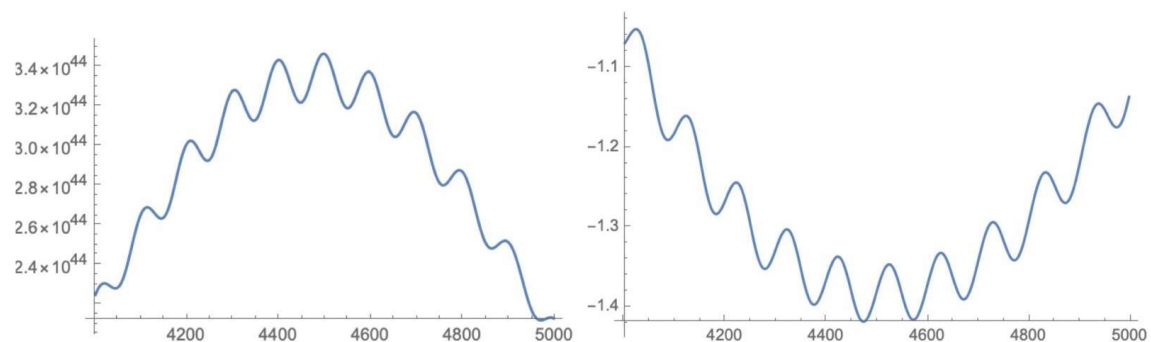




$$\Omega_J x^J + \Omega_{J+1} x^{J+1} + \Omega_{J+2} x^{J+2}$$

$$\Omega_J x^J + \Omega_{J+1} x^{J+1} + \Omega_{J+2} x^{J+2}, \text{ where } x = 0.95 \cdot \exp\left(\frac{2\pi i}{3}\right),$$

$$x = 0.95 \cdot \exp\left(i\left(\frac{2\pi}{3} + 0.0313458\right)\right).$$



$$\Omega_J x^J + \Omega_{J+1} x^{J+1} + \Omega_{J+2} x^{J+2} \text{ where } x = 0.95 \cdot e^{i\left(\frac{2\pi}{3} + 0.0313458\right)}$$

$$J\left(0.95 \cdot e^{i\left(\frac{2\pi}{3} + 0.0313458\right)}\right) = 1.33 \times 10^{47}$$

$$J(0.95) = 2.1 \times 10^{31}.$$

$$\log J = \frac{8(5a - 3c)}{27\omega^2}(-\pi i + \omega)^3 + \frac{8\pi^2(a - c)}{3\omega^2}(-\pi i + \omega),$$

$$\operatorname{Re} \omega = \sqrt{3} \cdot \operatorname{Im} \omega \leftrightarrow -\log |x| = \sqrt{3} \cdot \left(\arg(x) - \frac{2\pi}{3}\right)$$

$$\operatorname{Re}(\log J) = \frac{2\pi^2(3c - 2a)}{9\sqrt{3}(\operatorname{Im} \omega)^2}$$

$$\begin{aligned} Z(\beta, \mu, \omega_R, \omega_L) &= \operatorname{Tr}[e^{-\beta(E - \mu J - \omega_R Q_R - \omega_L Q_L)}] \\ &= e^{\frac{1}{2}\beta k_L} \operatorname{Tr}\left[e^{-\beta(E - J - Q_L + \frac{k_L}{2}) + \tilde{\mu} J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right] \end{aligned}$$

$$Z_{\text{BPS}}(\beta, \tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) = \left(\lim_{\beta \rightarrow \infty} e^{\frac{1}{2}\beta k_L}\right) \operatorname{Tr}_{\text{BPS}}[e^{\tilde{\mu} J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}]$$

$$\tilde{\mu} - 2\tilde{\omega}_L = 2\pi i$$

$$\begin{aligned} \mathcal{I} &\equiv e^{\beta E_{\text{SUSY}} Z} \Big|_{\tilde{\omega}_L = \frac{\tilde{\mu}}{2} - i\pi} \\ &= \text{Tr}_{\text{BPS}} \left[e^{\tilde{\mu} J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L} \right] \Big|_{\tilde{\omega}_L = \frac{\tilde{\mu}}{2} - i\pi}, \end{aligned}$$

$$e^{-\beta E_{\text{SUSY}}} \Big|_{\beta \rightarrow \infty}$$

$$\begin{aligned} \log \mathcal{I} &= -\frac{k_L}{2} \beta + \frac{k_R}{\beta(1-\mu)} (\pi^2 + \beta^2 \omega_R^2) + \frac{k_L}{\beta(1+\mu)} (\pi^2 + \beta^2 \omega_L^2) \\ &= -\frac{k_L}{2} \beta - \frac{k_R}{\tilde{\mu}} (\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{\tilde{\mu} + 2\beta} (\pi^2 + (\tilde{\omega}_L + \beta)^2) \\ &= -\frac{k_R}{\tilde{\mu}} (\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{4} (\tilde{\mu} - 4\pi i) \\ &= -\frac{k_R}{\tilde{\mu}} (\pi^2 + \tilde{\omega}_R^2) + \frac{k_L}{\tilde{\mu}} (\pi^2 + \tilde{\omega}_L^2) \end{aligned}$$

$$\log \mathcal{I} = k \frac{\tilde{\omega}_1 \tilde{\omega}_2}{\tilde{\mu}},$$

$$\tilde{\omega}_{L,R} = \frac{1}{2} (\tilde{\omega}_1 \pm \tilde{\omega}_2)$$

$$S(\tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) \equiv \log \mathcal{I} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu} J,$$

$$\begin{aligned} S &= S(\tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) - \Lambda(\tilde{\mu} - 2\tilde{\omega}_L - 2\pi i) \\ &= \frac{k_L(\tilde{\omega}_L^2 + \pi^2) - k_R(\tilde{\omega}_R^2 + \pi^2)}{\tilde{\mu}} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu} J - \Lambda(\tilde{\mu} - 2\tilde{\omega}_L - 2\pi i) \end{aligned}$$

$$0 = (\tilde{\omega}_L \partial_{\tilde{\omega}_L} + \tilde{\omega}_R \partial_{\tilde{\omega}_R} + \tilde{\mu} \partial_{\tilde{\mu}}) S = S - 2\pi i \Lambda + \frac{2\pi^2(k_R - k_L)}{\tilde{\mu}},$$

$$S = 2\pi i \Lambda - \frac{2\pi^2(k_R - k_L)}{\tilde{\mu}}.$$

$$\partial_{\tilde{\omega}_L} S = k_L \frac{2\tilde{\omega}_L}{\tilde{\mu}} + (2\Lambda - Q_L) = 0,$$

$$\partial_{\tilde{\omega}_R} S = -k_R \frac{2\tilde{\omega}_R}{\tilde{\mu}} - Q_R = 0,$$

$$\partial_{\tilde{\mu}} S = -\frac{k_L(\tilde{\omega}_L^2 + \pi^2) - k_R(\tilde{\omega}_R^2 + \pi^2)}{\tilde{\mu}^2} - (\Lambda + J) = 0.$$

$$k_L \frac{\tilde{\mu} - 2\pi i}{\tilde{\mu}} = Q_L - 2\Lambda \Rightarrow \frac{\pi i k_L}{\tilde{\mu}} = \Lambda - \frac{1}{2} (Q_L - k_L).$$

$$S = 2\pi i \left[\Lambda + \frac{i\pi}{\tilde{\mu}} (k_R - k_L) \right] = 2\pi i \left[\frac{k_R}{k_L} \Lambda - \frac{1}{2k_L} (k_R - k_L)(Q_L - k_L) \right] \equiv 2\pi i \Lambda_{\text{eff}}$$

$$\Lambda_{\text{eff}} = \frac{k_R}{k_L} \Lambda - \frac{1}{2k_L} (k_R - k_L)(Q_L - k_L).$$



$$-\frac{1}{k_L}\left(\Lambda-\frac{1}{2}Q_L\right)^2+\frac{1}{4k_R}Q_R^2-(\Lambda+J)-\frac{1}{k_L^2}(k_R-k_L)\left(\Lambda-\frac{1}{2}(Q_L-k_L)\right)^2=0,$$

$$\Lambda_{\rm eff}^2-(Q_L-k_L)\Lambda_{\rm eff}+\frac{1}{4}(Q_L-k_L)^2+k_R\left(J+\frac{Q_L}{2}-\frac{k_L}{4}\right)-\frac{1}{4}Q_R^2=0.$$

$$S=2\pi i\Lambda_{\rm eff}=2\pi\sqrt{k_R\left(J+\frac{Q_L}{2}-\frac{k_L}{4}\right)-\frac{Q_R^2}{4}}+\pi i(Q_L-k_L).$$

$$Q_L=k_L,$$

$$S_{\rm BPS}=2\pi\sqrt{k_R\left(J+\frac{1}{4}k_L\right)-\frac{1}{4}Q_R^2},$$

$$\begin{aligned}\mathcal{I}(\tilde{\Delta}_I,\tilde{\omega}_i) &= \text{Tr}[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_IQ_I+\tilde{\omega}_iJ_i}] \\ &= \text{Tr}_{\text{BPS}}[e^{\tilde{\Delta}_IQ_I+\tilde{\omega}_iJ_i}](3.61)\end{aligned}$$

$$\tilde{\Delta}_I\equiv\Delta_I-\beta\otimes\tilde{\omega}_i\equiv\omega_i-\beta\iiint e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}=-1$$

$$\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2=2\pi i$$

$$\log \mathcal{I}=-\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}$$

$$S(\tilde{\Delta}_I,\tilde{\omega}_i)\equiv\log \mathcal{I}-\sum_I\tilde{\Delta}_IQ_I-\sum_i\tilde{\omega}_iJ_i$$

$$\begin{aligned}S&=S(\tilde{\Delta}_I,\tilde{\omega}_i)-\Lambda\left(\sum_I\tilde{\Delta}_I-\sum_i\tilde{\omega}_i-2\pi i\right)\\&=-\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}-\sum_I\tilde{\Delta}_IQ_I-\sum_i\tilde{\omega}_iJ_i-\Lambda\left(\sum_I\tilde{\Delta}_I-\sum_i\tilde{\omega}_i-2\pi i\right)\end{aligned}$$

$$0=\left(\sum_I\tilde{\Delta}_I\partial_{\tilde{\Delta}_I}+\sum_i\tilde{\omega}_i\partial_{\tilde{\omega}_i}\right)S=S-2\pi i\Lambda$$

$$S=2\pi i\Lambda$$

$$-\tilde{\Delta}_I\partial_{\tilde{\Delta}_I}S=\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}+\tilde{\Delta}_I(Q_I+\Lambda)=0$$

$$\tilde{\omega}_i\partial_{\tilde{\omega}_i}S=\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}+\tilde{\omega}_i(-J_i+\Lambda)=0$$

$$\tilde{\Delta}_I = -\frac{1}{Q_I + \Lambda} \cdot \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2}$$

$$\tilde{\omega}_i = -\frac{1}{-J_i + \Lambda} \cdot \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2}$$

$$\frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} = \frac{N^2}{2} \cdot \left(-\frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} \right)^{3-2} \cdot \frac{(-J_1 + \Lambda)(-J_2 + \Lambda)}{(Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda)}$$

$$\Rightarrow 0 = (Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda) + \frac{N^2}{2}(-J_1 + \Lambda)(-J_2 + \Lambda)$$

$$\Lambda^3 + \left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{N^2}{2} (J_1 + J_2) \right) \Lambda = 0$$

$$\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2} \right) \Lambda^2 + \left(Q_1 Q_2 Q_3 + \frac{N^2}{2} J_1 J_2 \right) = 0$$

$$Q_1 Q_2 Q_3 + \frac{N^2}{2} J_1 J_2$$

$$= \left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{N^2}{2} (J_1 + J_2) \right) \left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2} \right),$$

$$S = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{N^2}{2} (J_1 + J_2)}$$

$$N^2 = \frac{\pi}{2G_5}$$

$$E \geq J + Q_L - \frac{k_L}{2}$$

$$E \geq J - \frac{Q_L^2}{2k_L},$$

$$Q_L = k_L$$

$$\text{ch}_{h,q_L}(q,y)=q^hF^{NS}\sum_{m=-\infty}^{\infty}\left(y^{2(k_L+1)m+q_L+1}-y^{-2(k_L+1)m-q_L-1}\right)\frac{q^{(k_L+1)m^2+(q_L+1)m}}{y-y^{-1}}$$

$$\chi_{q_L}(q,y)=q^{\frac{q_L}{2}}F^{NS}\sum_{m=-\infty}^{\infty}\left(\frac{y^{2(k_L+1)m+q_L+1}}{\left(1+yq^{m+\frac{1}{2}}\right)^2}-\frac{y^{-2(k_L+1)m-q_L-1}}{\left(1+y^{-1}q^{m+\frac{1}{2}}\right)^2}\right)\frac{q^{(k_L+1)m^2+(q_L+1)m}}{y-y^{-1}}$$

$$F^{NS}=\prod_{n\geq 1}\frac{\left(1+yq^{n-\frac{1}{2}}\right)^2\left(1+y^{-1}q^{n-\frac{1}{2}}\right)^2}{(1-y^2q^n)(1-q^n)^2(1-y^{-2}q^n)},$$



$$\begin{aligned}\chi_{q_L}(q,y) &= \chi_{q_L}(q,q^{-1}y^{-1}) \cdot q^{k_L}y^{2k_L} \\ \text{ch}_{h,q_L}(q,y) &= \text{ch}_{h,q_L}(q,q^{-1}y^{-1}) \cdot q^{k_L}y^{2k_L}.\end{aligned}$$

$$q^h(q^{-1}y^{-1})^{q_L} \cdot q^k y^{2k_L} = q^{h+k_L-q_L} y^{2k_L-q_L}.$$

$$h-\frac{1}{2}q_L=(h+k_L-q_L)-\frac{1}{2}(2k_L-q_L).$$

$$\langle Q_L\rangle = y\frac{\partial \log Z}{\partial y}\Big|_{y=q^{-\frac{1}{2}}} = k_L,$$

$$\chi_{h,q_L-1}(q,y)+2\chi_{h,q_L}(q,y)+\chi_{h,q_L+1}(q,y)=\text{ch}_{h,q_L}(q,y)\Big|_{h=\frac{1}{2}q_L}$$

$$\text{BPS: } \{Q,Q^\dagger\}=E-\sum_i J_i-\sum_I Q_I=0$$

$$\begin{aligned}Z&\equiv \text{Tr}_{\text{BPS}}\Big[e^{-\beta\{Q,Q^\dagger\}}e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}\Big] \\ &= \text{Tr}_{\text{BPS}}\big[e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}\big]\end{aligned}$$

$$1+x_B+x_B^2+\cdots=\frac{1}{1-x_B}$$

$$Z=\frac{\prod_{j=1}^{N_F}\left(1+x_{F,j}\right)}{\prod_{i=1}^{N_B}\left(1-x_{B,i}\right)}.$$

$$Z=e^{Z_{\rm sp}}=\exp\left(\sum_{i=1}^{N_B}x_{B,i}+\sum_{j=1}^{N_F}x_{F,j}\right).$$

$$\begin{aligned}Q_J&=\frac{\text{Tr}_{\text{BPS}}[Q_J\cdot e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}]}{\text{Tr}_{\text{BPS}}[e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}]}=\frac{\partial}{\partial\Delta_J}\log Z,\\ J_j&=\frac{\text{Tr}_{\text{BPS}}[J_j\cdot e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}]}{\text{Tr}_{\text{BPS}}[e^{\sum_i\omega_iJ_i+\sum_I\Delta_IQ_I}]}=\frac{\partial}{\partial\omega_j}\log Z.\end{aligned}$$

$$E\geq J_1+J_2+Q_1+Q_2+Q_3$$

$$\begin{aligned}&\left(Q_1Q_2Q_3+\frac{N^2}{2}J_1J_2\right)\\ &=\left(Q_1+Q_2+Q_3+\frac{N^2}{2}\right)\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}(J_1+J_2)\right).\end{aligned}$$

$$\frac{1}{2}N^2=\frac{\pi\ell_5^3}{4G_5}$$

$$\left(Q^3+\frac{N^2}{2}J^2\right)-\left(3Q+\frac{N^2}{2}\right)(3Q^2-N^2J)=0$$



$$[j;\bar{j}]_E^{[R_1,R_2,R_3]}$$

$$J_1=\frac{j+\bar{j}}{2}, \qquad J_2=\frac{j-\bar{j}}{2}, \\ Q_1=R_2+\frac{R_1+R_3}{2}, \quad Q_2=\frac{R_1+R_3}{2}, \quad Q_3=\frac{R_1-R_3}{2}$$

$$2\{Q,Q^\dagger\}=E-\left(j+\frac{3}{2}R_1+R_2+\frac{1}{2}R_3\right) \\ =E-(Q_1+Q_2+Q_3+J_1+J_2)\geq 0$$

$$e^{\omega_1}=p^2,e^{\omega_2}=q^2,e^{\Delta_1}=x^2,e^{\Delta_2}=y^2,e^{\Delta_3}=z^2$$

$$Z_{\rm sp}\equiv {\rm Tr}_{\rm BPS}[p^{2J_1}q^{2J_2}x^{2Q_1}y^{2Q_2}z^{2Q_3}]$$

$$x^2+y^2+z^2+xyzpq\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)+p^2q^2.$$

$$Z_{\rm sp}=\frac{x^2+y^2+z^2+xyzpq\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)+p^2q^2}{(1-p^2)(1-q^2)}$$

$$Z\equiv \exp\left[Z_{\rm sp}\right]$$

$$2Q_1=x\frac{\partial}{\partial x}\log Z, 2J_1=p\frac{\partial}{\partial p}\log Z$$

$$Q_1=\frac{2x^2+xyzpq\left(-\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)}{2(1-p^2)(1-q^2)} \\ J_1=\frac{2p^2(q^2+x^2+y^2+z^2)+xyzpq(1+p^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{1}{p^2}+\frac{1}{q^2}-1\right)+4xyzpq}{2(1-p^2)^2(1-q^2)}$$

$$(E,J_1,J_2,Q_1,Q_2,Q_3)=(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$$

$$\frac{xyz}{pq}=1,$$

$$(Q_1Q_2Q_3+J_1J_2)-(Q_1+Q_2+Q_3+1)(Q_1Q_2+Q_2Q_3+Q_3Q_1-J_1-J_2)=0$$

$$\frac{1}{2}N^2=1\leftrightarrow \frac{\pi \ell_5^3}{4G_5}=1$$

$$Q\!=\!\frac{2x^2+2x^3+xp^2-x^3p^2}{2(1-p^2)^2} \\ J\!=\!\frac{3x(1+x)^2p^2+(2+3x-x^3)p^4}{2(1-p^2)^3}$$

$$E\geq J+\frac{1}{2}(Q_1+Q_2+Q_3+Q_4)$$



$$(\mathbb{Q}_3)^2 - (\mathbb{Q}_1)(\mathbb{Q}_2)(\mathbb{Q}_3) + (\mathbb{Q}_1)^2(\mathbb{Q}_4) = 0,$$

$$(\mathbb{Q}_1) \equiv Q_1 + Q_2 + Q_3 + Q_4$$

$$(\mathbb{Q}_2) \equiv Q_1 Q_2 + Q_1 Q_3 + Q_1 Q_4 + Q_2 Q_3 + Q_2 Q_4 + Q_3 Q_4 + \frac{2N^3}{9}$$

$$(\mathbb{Q}_3) \equiv Q_1 Q_2 Q_3 + Q_1 Q_2 Q_4 + Q_1 Q_3 Q_4 + Q_2 Q_3 Q_4 - \frac{4N^3}{9} J$$

$$(\mathbb{Q}_4) \equiv Q_1 Q_2 Q_3 Q_4 + \frac{2N^3}{9} J^2.$$

$$N^{\frac{3}{2}} = \frac{3}{2\sqrt{2}G_4},$$

$$Q_1 Q_2 (Q_1 + Q_2)^2 - (Q_1 + Q_2) \cdot \frac{2N^3}{9} J - \frac{2N^3}{9} J^2 = 0.$$

$$4Q^4 - 2Q \cdot \frac{2N^3}{9} J - \frac{2N^3}{9} J^2 = 0.$$

$$[j]_E^{[R_1, R_2, R_3, R_4]}$$

$$J=\frac{j}{2}$$

$$Q_1=R_3+R_2+\frac{R_1+R_4}{2}, Q_2=R_2+\frac{R_1+R_4}{2}, Q_3=\frac{R_1+R_4}{2}, Q_4=\frac{R_1-R_4}{2}.$$

$$\begin{aligned} E &\geq \frac{1}{2}j + R_1 + R_2 + \frac{1}{2}R_3 + \frac{1}{2}R_4 \\ &= J + \frac{1}{2}(Q_1 + Q_2 + Q_3 + Q_4) \end{aligned}$$

$$\begin{aligned} Z_{\text{sp}} &\equiv \text{Tr}_{\text{BPS}}[p^{2J}x^{2Q_1}y^{2Q_2}z^{2Q_3}w^{2Q_4}] \\ &= \frac{x^2+y^2+z^2+w^2+pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)}{1-p^2} \end{aligned}$$

$$Q_1=\frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)+2x^2-\frac{2pxyzw}{x^2}}{2(1-p^2)}$$

$$J=\frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)}{2(1-p^2)}+\frac{p^2\left(x^2+y^2+z^2+w^2+pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)\right)}{(1-p^2)^2}$$

$$\frac{xyzw}{p}=1$$

$$(\mathbb{Q}_3)_{\frac{9}{2}}^2 - (\mathbb{Q}_1)_{\frac{9}{2}}(\mathbb{Q}_2)_{\frac{9}{2}}(\mathbb{Q}_3)_{\frac{9}{2}} + (\mathbb{Q}_1)_{\frac{9}{2}}^2(\mathbb{Q}_4)_{\frac{9}{2}} = 0$$



$$\begin{aligned}
(\mathbb{Q}_1)_{\frac{9}{2}} &\equiv Q_1 + Q_2 + Q_3 + Q_4 \\
(\mathbb{Q}_2)_{\frac{9}{2}} &\equiv Q_1 Q_2 + Q_1 Q_3 + Q_1 Q_4 + Q_2 Q_3 + Q_2 Q_4 + Q_3 Q_4 + 1 \\
(\mathbb{Q}_3)_{\frac{9}{2}} &\equiv Q_1 Q_2 Q_3 + Q_1 Q_2 Q_4 + Q_1 Q_3 Q_4 + Q_2 Q_3 Q_4 - 2J \\
(\mathbb{Q}_4)_{\frac{9}{2}} &\equiv Q_1 Q_2 Q_3 Q_4 + J^2
\end{aligned}$$

$$\frac{\sqrt{2}}{3} N^{\frac{3}{2}} = 1 \leftrightarrow G_4 = \frac{1}{2}$$

$$\begin{aligned}
Q_1 = Q_3 &= \frac{x^2}{1-p} \\
Q_2 = Q_4 &= \frac{y^2}{1-p} \\
J &= \frac{p(x^2 + y^2)}{(1-p)^2}
\end{aligned}$$

$$Q_1 Q_2 (Q_1 + Q_2)^2 - (Q_1 + Q_2) J - J^2 = 0.$$

$$\begin{aligned}
Q &\equiv Q_{1,2,3,4} = \frac{x^2}{1-p} \\
J &= \frac{2px^2}{(1-p)^2}
\end{aligned}$$

$$4Q^4 - 2QJ - J^2 = 0$$

$$E \geq J_1 + J_2 + J_3 + Q_1 + Q_2$$

$$\begin{aligned}
&\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3) - \frac{Q_1Q_2(Q_1 + Q_2) - \frac{N^3}{3}(J_1J_2 + J_2J_3 + J_3J_1)}{Q_1 + Q_2 - \frac{N^3}{3}} \\
&= \sqrt{\left(\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3)\right)^2 - \left(Q_1^2Q_2^2 + \frac{2N^3}{3}J_1J_2J_3\right)}
\end{aligned}$$

$$N^3 = \frac{3\pi^2}{16G_7}$$

$$\begin{aligned}
&\left(Q^4 + \frac{2N^3}{3}J^3\right)\left(Q - \frac{N^3}{6}\right)^2 \\
&= 2(3Q^2 + N^3J)\left(Q^3 - \frac{N^3}{2}J^2\right)\left(Q - \frac{N^3}{6}\right) - \left(Q^3 - \frac{N^3}{2}J^2\right)^2.
\end{aligned}$$

$$J_1 = j_1 + \frac{j_2 + j_3}{2}, \quad J_2 = \frac{j_2 + j_3}{2}, \quad J_3 = \frac{-j_2 + j_3}{2},$$

$$Q_1 = R_1 + \frac{R_2}{2}, \quad Q_2 = \frac{R_2}{2}.$$

$$E \geq j_1 + \frac{1}{2}j_2 + \frac{3}{2}j_3 + 2R_1 + 2R_2$$

$$= J_1 + J_2 + J_3 + 2Q_1 + 2Q_2$$

$$Z_{\text{sp}} \equiv \text{Tr}_{\text{BPS}}[p^{2J_1} q^{2J_2} r^{2J_3} x^{2Q_1} y^{2Q_2}]$$

$$= \frac{x^2 + y^2 + xypqr \left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1 \right)}{(1-p^2)(1-q^2)(1-r^2)}$$

$$Q_1 = \frac{2x^2 + xypqr \left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1 \right)}{2(1-p^2)(1-q^2)(1-r^2)}$$

$$J_1 = \frac{2p^2(x^2 + y^2) + xypqr(1+p^2) \left(-\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} - 1 \right) + 4xypqr}{2(1-p^2)^2(1-q^2)(1-r^2)}$$

$$\frac{1}{2}(J_1 + J_2 + J_3) + \frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) - \frac{Q_1Q_2(Q_1 + Q_2) - \frac{1}{2}(J_1J_2 + J_2J_3 + J_3J_1)}{Q_1 + Q_2 - \frac{1}{2}}$$

$$= \sqrt{\left(\frac{1}{2}(J_1 + J_2 + J_3) + \frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) \right)^2 - (J_1J_2J_3 + Q_1^2Q_2^2)}$$

$$\frac{2}{3}N^3 = 1 \leftrightarrow \frac{\pi^2}{8G_7} = 1.$$

$$Q = \frac{2x^2 + 3x^2p - x^2p^3}{2(1-p^2)^3},$$

$$J = \frac{x^2p + 4x^2p^2 + 4x^2p^3 - x^2p^5}{2(1-p^2)^4}.$$

$$A_\mu \sim A_{\alpha\dot\beta}, \qquad (\mu=1,2,3,4, \alpha=\pm, \dot\beta=\pm)$$

$$\Phi_{ij}(=-\Phi_{ji}), \bar{\Phi}^{ij} \sim \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}, \quad (i,j,k,l=1,2,3,4)$$

$$\Psi_{i\alpha}, \bar{\Psi}^i_{\dot{\alpha}}.$$

$$A_{\alpha\dot\beta}, \lambda_\alpha = \Psi_{4\alpha}, \bar\lambda_{\dot\alpha} = \bar\Psi^4_{\dot\alpha}$$

$$\phi_m = \Phi_{4m}, \bar\phi^m = \bar\Phi^{4m}, \psi_{m\alpha} = -i\Psi_{m\alpha}, \bar\psi^m_{\dot\alpha} = i\bar\Psi^m_{\dot\alpha}$$

$$\left\{Q^i_\alpha,S^\beta_j\right\}=\frac{1}{2}E\delta^i_j\delta^\beta_\alpha+R^i{}_j\delta^\beta_\alpha+J_\alpha{}^\beta\delta^i_j$$

$$2\{Q,Q^{\dagger}\}=E-(Q_1+Q_2+Q_3+J_1+J_2).$$

$$[Q,O]=0, [Q^\dagger,O]=0$$

$$E=Q_1+Q_2+Q_3+J_1+J_2$$

$$\phi^m,\psi_m,f,\lambda_{\dot\alpha},\partial_{\dot\alpha}$$

$$\partial_{\dot\alpha}\lambda^{\dot\alpha}=0\Leftrightarrow\partial_{[\dot\alpha}\lambda_{\dot\beta]}=0$$

$$\begin{array}{l} \partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}\phi^m,\quad \partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}\psi_m,\partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}f\\ \partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_{n-1}}\lambda_{\dot\alpha_n)} \end{array}$$

$$g^2_{\rm YM}, H(g_{\rm YM}) = \sum_{L=0}^\infty g^{2L}_{\rm YM} H_{(L)}$$

$$\{Q(g_{\rm YM}),Q^+(g_{\rm YM})\}=H(g_{\rm YM})-\sum_I R_I-\sum_i J_i=\sum_{L=1}^\infty g^{2L}_{\rm YM} H_{(L)}.$$

$$\begin{array}{l} Q\phi^m=0, Q\lambda_{\dot\alpha}=0, Q\psi_m=-\frac{i}{2}\epsilon_{mnp}[\phi^n,\phi^p] \\ Qf=-i[\phi^m,\psi_m],[Q,D_{\dot\alpha}]=-i[\lambda_{\dot\alpha}\} \end{array}$$

$$\phi^m,\psi_m,f$$

$$Q\colon [Q,\phi^m]=0,\{Q,\psi_m\}=-i\epsilon_{mnp}[\phi^m,\phi^n],[Q,f]=-i[\psi_m,\phi^m]$$

$$Z(\Delta_I,\omega_i)=\text{Tr}[(-1)^Fe^{\Delta_IQ_I+\omega_iJ_i}],\text{ where }e^{\frac{\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2}{2}}=1$$

$$\begin{array}{l} e^{\Delta_1}=e^{\Delta_2}=e^{\Delta_3}=t^2,e^{\omega_1}=e^{\omega_2}=t^3\\ \Rightarrow Z(t)=\text{Tr}[(-1)^Ft^{2(Q_1+Q_2+Q_3)+3(J_1+J_2)}]\equiv\text{Tr}[(-1)^F x^J] \end{array}$$

$$\mathcal{J}\equiv 2Q_1+2Q_2+2Q_3+3J_1+3J_2$$

$$\omega=\frac{\Delta_1+\Delta_2+\Delta_3}{2}$$

$$Z_{\rm BMN}(\Delta_I)=\text{Tr}_{\rm BMN}[(-1)^Fe^{\Delta_I(Q_I+J)}]$$

$$q_I\equiv Q_I+J$$

$$Z_{\rm BMN}(t)=\text{Tr}_{\rm BMN}[(-1)^Ft^{2(Q_1+Q_2+Q_3)+6J}]$$

$$S(j,N)=N^2f\left(\frac{j}{N^2}\right)$$



$$Z_{\text{BMN}}(\Delta_I) = \frac{1}{N!} \int_0^{2\pi} \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \frac{\prod_{a \neq b} (1 - e^{i\alpha_{ab}}) \prod_{a,b=1}^N \prod_{I < J} (1 - e^{\Delta_I + \Delta_J} e^{i\alpha_{ab}})}{\prod_{a,b=1}^N [(1 - e^{\Delta_1 + \Delta_2 + \Delta_3} e^{i\alpha_{ab}}) \prod_{I=1}^3 (1 - e^{\Delta_I} e^{i\alpha_{ab}})]} \\ \times \frac{(1 - e^{\Delta_1 + \Delta_2 + \Delta_3}) \prod_{I=1}^3 (1 - e^{\Delta_I})}{\prod_{I < J} (1 - e^{\Delta_I + \Delta_J})}$$

$$(e^{\Delta_1}, e^{\Delta_2}, e^{\Delta_3}) = t^2(x, y^{-1}, x^{-1}y)$$

$$\rho(\alpha) = \frac{3}{4\pi^3}(\pi^2 - \alpha^2).$$

$$\log Z = \frac{3N^2}{\pi^2} \Delta_1 \Delta_2 \Delta_3.$$

$$S_{\text{BMN}}(q_I;\Delta_I)=\log Z-\sum_I q_I\Delta_I,$$

$$S_{\text{BMN}}(q_I) = 2\pi \sqrt{\frac{q_1 q_2 q_3}{3N^2}}$$

$$q_I = N^2 \epsilon_I \sim N^2 \sqrt{\epsilon_1 \epsilon_2 \epsilon_3} \propto N^2$$

$$S(q_I) = 2\pi \sqrt{\frac{2q_1q_2q_3}{N^2}}$$

$$u^{i_1i_2\cdots i_n}=\text{tr}(\phi^{(i_1}\phi^{i_2}\cdots\phi^{i_n)})$$

$$(u_n)^{i_1\cdots i_n}=\text{tr}(\phi^{(i_1}\cdots\phi^{i_n)})\\ (v_n)^{i_1\cdots i_{n-1}}{}_j=\text{tr}(\phi^{(i_1}\cdots\phi^{i_{n-1}})\psi_j)-\text{trace}\\ (w_n)^{i_1\cdots i_{n-1}}=\text{tr}\left(\phi^{(i_1\cdots i_{n-1})}f+\frac{1}{2}\epsilon^{jk(i_p}\sum_{p=1}^{n-1}\phi^{i_1}\cdots\phi^{i_{p-1}}\psi_j\phi^{i_{p+1}}\cdots\phi^{i_{n-1}})\psi_k\right).$$

$$R_{ij} \equiv \epsilon_{ikm} \epsilon_{jln} (u_2)^{kl} (u_2)^{mn} = Q[-i \epsilon_{a_1 a_2 (i} \text{tr}(\psi_j) \phi^{a_1} \phi^{a_2})]$$

$$\text{tr}(X^2)\text{tr}(Y^2)-[\text{tr}(XY)]^2\sim 0, \text{tr}(XY)\text{tr}(XZ)-\text{tr}(X^2)\text{tr}(YZ)\sim 0,$$

$$\sum_a f_a(\{u_n,v_n,w_n\})R_a(\{u_n,v_n,w_n\}).$$



$(-1)^F E'$	J'	R'_1	R'_2	construction
n	0	n	0	$ n\rangle$
$-(n + \frac{1}{2})$	$\frac{1}{2}$	$n - 1$	0	$\overline{Q}_{m\dot{\alpha}} n\rangle$
$n + 1$	0	$n - 2$	0	$\overline{Q}_{m\dot{+}}\overline{Q}_{n\dot{-}} n\rangle$
$-(n + 1)$	0	$n - 1$	1	$Q^m_+ n\rangle$
$n + \frac{3}{2}$	$\frac{1}{2}$	$n - 2$	1	$Q^m_+\overline{Q}_{n\dot{\alpha}} n\rangle$
$-(n + 2)$	0	$n - 3$	1	$Q^m_+\overline{Q}_{n\dot{+}}\overline{Q}_{p\dot{-}} n\rangle$
$n + 2$	0	$n - 1$	0	$Q^m_+Q^n_+ n\rangle$
$-(n + \frac{5}{2})$	$\frac{1}{2}$	$n - 2$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{\alpha}} n\rangle$
$n + 3$	0	$n - 3$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{+}}\overline{Q}_{q\dot{-}} n\rangle$

$$u^{ik}R_{jk}(u_2)-\frac{1}{3}\delta^i_ju^{kl}R_{kl}(u_2)=0,$$

$$u^{1i}R_{2i}=u^{11}[u^{23}u^{13}-u^{12}u^{33}]+u^{12}[u^{33}u^{11}-(u^{13})^2]+u^{13}[u^{12}u^{13}-u^{11}u^{23}]=0.$$

$$G_i(g_i,\lambda_I)\equiv g_i-g_i(\lambda_I)=0$$

$$\sum_i q_i(g_i,\lambda_I)G_i(g_i,\lambda_I)$$

$$p(g_i,\lambda_I)=\sum_a q_a(g_i,\lambda_I)G_a(g_i,\lambda_I)+r(g_i,\lambda_I)$$

$$f=\begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & -f_1-f_2 \end{pmatrix}$$

$$f=\begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3-f_1 & 0 \\ 0 & 0 & 0 & -f_2-f_3 \end{pmatrix}$$

$$f=\mathrm{diag}(f_1,f_2,f_3,-f_1-f_2-f_3)$$

$$\mathbf{6}\colon x^2,y^2,z^2,xy,yz,zx$$

$$\mathbf{8}\colon \psi_1\cdot (y,z),\psi_2\cdot (z,x),\psi_3\cdot (x,y),\psi_1x-\psi_2y,\psi_2y-\psi_3z$$

$$\mathbf{3}\colon xf-\frac{1}{2}\psi_2\psi_3,yf-\frac{1}{2}\psi_3\psi_1,zf-\frac{1}{2}\psi_1\psi_2$$

$$V_k = \text{span}\{\mathfrak{v} \times (xf)^a(yf)^b(zf)^c \mid \mathfrak{v} \in \mathfrak{B}, a+b+c \leq k\}$$

$$\tilde{V}_k = \text{span}\left\{\mathfrak{v} \times \left(xf - \frac{1}{2}\psi_2\psi_3\right)^a \left(yf - \frac{1}{2}\psi_3\psi_1\right)^b \left(zf - \frac{1}{2}\psi_1\psi_2\right)^c \mid \mathfrak{v} \in \mathfrak{B}, a+b+c \leq k\right\}$$

$$v = \sum_{i=1}^n r_i (xf)^{a_i} (yf)^{b_i} (zf)^{c_i}$$

$$\tilde{v} = \sum_{i=1}^n r_i \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i}$$

$$\begin{aligned} \tilde{v} = & -\frac{1}{2} \sum_i^n r_i [a_i \psi_2 \psi_3 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i-1} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + b_i \psi_3 \psi_1 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i-1} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + c_i \psi_1 \psi_2 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i-1}]. \end{aligned}$$

$$r_1=\alpha_1\psi_1y,r_2=\alpha_2\psi_1z,\cdots,r_8=\alpha_8(\psi_2y-\psi_3z)$$

$$G_0=\{x^ay^bz^cf^d\mid a,b,c,d\in\mathbb{Z}^{\geq 0},a+b+c\geq d,a+b+c+d=0(\text{mod}2)\}.$$

$$Z_0(x,y,z,f)=\sum_{g\in G_0}g$$

$$\frac{1}{(1-x)(1-y)(1-z)(1-f)}$$

$$\frac{1}{(1-xf)(1-yf)(1-zf)}\cdot\frac{f}{1-f}.$$

$$\begin{aligned} Z_0 &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f} \right]_{\text{even}} \\ &= \left[\frac{1-f(xy+yz+zx-xyz)+f^2xyz}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)} \right]_{\text{even}} \\ &= \frac{1+\chi_2+f(\chi_3-\chi_1\chi_2)+f^2(\chi_3^2+\chi_1\chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \end{aligned}$$

$$\begin{array}{l} \chi_1=x+y+z\\ \chi_2=xy+yz+zx\\ \chi_3=xyz \end{array}$$

$$\{x^ay^bz^cf^d\psi_1\mid a,b,c,d\in\mathbb{Z}^{\geq 0},b+c\geq 1,a+b+c-1\geq d,a+b+c+d=1(\text{mod}2)\}.$$

$$G_{\psi_1}=\{x^ay^bz^cf^d\psi_1\mid a,b,c,d\in\mathbb{Z}^{\geq 0},a+b+c-1\geq d,a+b+c+d=1(\text{mod}2)\}\setminus\{x\psi_1\}.$$

$$G_1=G_{\psi_1}\cup G_{\psi_2}\cup G_{\psi_3}\cup\{x\psi_1-y\psi_2,y\psi_2-z\psi_3\}.$$

$$x\rightarrow [x],y\rightarrow [y],z\rightarrow [z],f\rightarrow [f],\psi_1\rightarrow \frac{[\psi]}{[x]},\psi_2\rightarrow \frac{[\psi]}{[y]},\psi_3\rightarrow \frac{[\psi]}{[z]}.$$



$$\frac{1}{(1-x)(1-y)(1-z)(1-f)} \cdot \frac{\psi}{x}$$

$$(x, y, z, f) \rightarrow (-x, -y, -z, -f)x\psi_1 \rightarrow \psi$$

$$\begin{aligned} Z_{\psi_1} &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{1}{1-f} \right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \left[\frac{x+y+z-(xy+yz+zx)(1+f)+xyz(1+f+f^2)}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)} \right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \frac{\chi_1 + \chi_3 - f(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^2\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi}{x} - \psi \end{aligned}$$

$$Z_1 = \frac{\chi_1 + \chi_3 - f(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^2\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_2}{\chi_3} \cdot \psi - \psi$$

$$\{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+c \geq 1, a+b+c-2 \geq d, a+b+c+d = 0(\text{mod}2)\} \setminus \{x^2 \psi_1 \psi_2\}.$$

$$\{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, b+c \geq 1, a+b+c-2 \geq d, a+b+c+d = 0(\text{mod}2)\} \setminus \{y^2 \psi_1 \psi_2\}.$$

$$G_{\psi_1 \psi_2} = \{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+b+c-2 \geq d, a+b+c+d = 0(\text{mod}2)\}$$

$$\begin{aligned} (x\psi_1 - y\psi_2)(x\psi_1 - y\psi_2) &\sim xy\psi_1\psi_2 \\ (y\psi_2 - z\psi_3)(y\psi_2 - z\psi_3) &\sim yz\psi_2\psi_3 \\ (x\psi_1 - y\psi_2)(y\psi_2 - z\psi_3) &\sim xy\psi_1\psi_2 + yz\psi_2\psi_3 + zx\psi_3\psi_1 \end{aligned}$$

$$G_2 = G_{\psi_1 \psi_2} \cup G_{\psi_2 \psi_3} \cup G_{\psi_3 \psi_1}$$

$$Z_{\psi_1 \psi_2} = \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi^2}{xy}$$

$$\begin{aligned} Z_2 &= Z_{\psi_1 \psi_2} + Z_{\psi_2 \psi_3} + Z_{\psi_3 \psi_1} \\ &= \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_1}{\chi_3} \cdot \psi^2 \end{aligned}$$

$$\{x^a y^b z^c f^d \psi_1 \psi_2 \psi_3 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+b \geq 1, a+b+c-3 \geq d, a+b+c+d = 1(\text{mod}2)\}.$$

$$G_{\psi_1 \psi_2 \psi_3} = \{x^a y^b z^c f^d \psi_1 \psi_2 \psi_3 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+b+c-3 \geq d, a+b+c+d = 1(\text{mod}2)\}.$$

$$Z_3 = \left[\frac{-1 + \chi_1^2 - 2\chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_1 + \chi_3) - f^2(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^3\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} + 1 \right] \cdot \frac{\psi^3}{f\chi_3}.$$

$$(x, y, z) = (e^{\Delta_1}, e^{\Delta_2}, e^{\Delta_3})$$

$$x, y, z \rightarrow t^2, f \rightarrow t^6, \psi \rightarrow -t^6.$$



$$Z_{\text{grav}} = \frac{1 + 3t^4 - 8t^6 - 6t^{10} + 10t^{12} + 9t^{14} - 9t^{16} + 16t^{18} - 18t^{20} - 3t^{22} + t^{24} - 3t^{26} + 9t^{28} - 2t^{30} + 3t^{32} - 3t^{34}}{(1-t^4)^3(1-t^8)^3}.$$

$$(e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3} \equiv t^2)$$

$$Z = [1 + 3t^2 + 12t^4 + 20t^6 + 42t^8 + 48t^{10} + 75t^{12} + 66t^{14} + 81t^{16} + 55t^{18} + 54t^{20} + 27t^{22} + 19t^{24} + 6t^{26} + 3t^{28}] \frac{(1-t^2)^3}{(1-t^{12})(1-t^8)^3}$$

$$Z - Z_{\text{grav}} = \left[-\frac{e^{4(\Delta_1+\Delta_2+\Delta_3)}}{1-e^{2(\Delta_1+\Delta_2+\Delta_3)}} \right] \cdot \left[\prod_{l=1}^3 (1-e^{\Delta_l}) \right] \cdot \left[\prod_{l=1}^3 \frac{1}{1-e^{\Delta_l}e^{\Delta_1+\Delta_2+\Delta_3}} \right]$$

$$Z - Z_{\text{grav}} = -\frac{t^{24}}{1-t^{12}} \cdot \frac{(1-t^2)^3}{(1-t^8)^3}$$

$$\begin{aligned} Z - Z_{\text{grav}} = & \left[-t^{24} - \chi_{(1,3)}t^{32} - (\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)} + \chi_{(3,8)})t^{36} \right. \\ & - (\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37} + \chi_{(5,3)}t^{38} + (\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39} \\ & \left. - (2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)})t^{40} \right] \chi_D + \mathcal{O}(t^{41}) \end{aligned}$$

$$\begin{aligned} \chi_{(2J'+1,R)} &\equiv \chi_{J'}^{SU(2)_R}(p) \chi_R^{SU(3)}(x,y) \\ \chi_D &\equiv \frac{(1-t^2z_1)\left(1-\frac{t^2}{z_2}\right)\left(1-\frac{t^2z_2}{z_1}\right)\left(1-\frac{tp}{z_1}\right)\left(1-\frac{t}{pz_1}\right)(1-tz_2p)\left(1-\frac{tz_2}{p}\right)\left(1-\frac{tz_1p}{z_2}\right)\left(1-\frac{tz_1}{z_2p}\right)}{(1-t^3p)\left(1-\frac{t^3}{p}\right)} \end{aligned}$$

$$t^6 = e^{\Delta_1+\Delta_2+\Delta_3} = e^{\omega_1+\omega_2}, z_1 = e^{\frac{2\Delta_1-\Delta_2-\Delta_3}{3}}, z_2^{-1} = e^{\frac{-\Delta_1+2\Delta_2-\Delta_3}{3}}, p = e^{\frac{\omega_1-\omega_2}{2}}$$

$$Z - Z_{\text{grav}} = Z_{\text{core}}(\Delta_l) \cdot \prod_{l=1}^3 \frac{1}{1 - e^{\Delta_l} e^{\Delta_1 + \Delta_2 + \Delta_3}} \cdot \prod_{l < j} (1 - e^{\Delta_l + \Delta_j}).$$

$$Z_{\text{core}}(\Delta_l) \equiv f(t,x,y) \text{ with } e^{\Delta_1} = t^2x, e^{\Delta_2} = t^2y^{-1}, e^{\Delta_3} = t^2x^{-1}y$$

$$f(t,x,y) = \sum_{j=0}^{54} \sum_{\mathbf{R}_j} (-1)^{F(\mathbf{R}_j)} \chi_{\mathbf{R}_j}(x,y) t^j + \mathcal{O}(t^{56}),$$

$$\prod_{l<J} (1-e^{\Delta_1+\Delta_2})$$

$$Z - Z_{\text{grav}} = [-\chi_{[2,0]}(x,y)t^{28} - \chi_{[3,0]}(x,y)t^{30} + \mathcal{O}(t^{32})] \cdot \prod_{l < j} (1 - e^{\Delta_l + \Delta_j})$$

$$-\frac{t^{24}}{1-t^{12}} = -t^{24} - t^{36} - t^{48} - t^{60} - \dots$$



$$Q\phi^m = 0, Q\psi_m = \frac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p, Qf = \phi^m \times \psi_m$$

$$E = \frac{19}{2}, Q_1 = Q_2 = Q_3 = \frac{3}{2}, J_1 = J_2 = \frac{5}{2}$$

$$O'_0 = (X \cdot \psi_1 - Y \cdot \psi_2)(X \cdot \psi_3)(\psi_2 \cdot \psi_1 \times \psi_1) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Y \cdot \psi_1)(\psi_3 \cdot \psi_2 \times \psi_2) \\ + (Z \cdot \psi_3 - X \cdot \psi_1)(Z \cdot \psi_2)(\psi_1 \cdot \psi_3 \times \psi_3)$$

$$O''_0 = (X \cdot \psi_1 - Y \cdot \psi_2)(Y \cdot \psi_3)(\psi_1 \cdot \psi_2 \times \psi_2) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Z \cdot \psi_1)(\psi_2 \cdot \psi_3 \times \psi_3) \\ + (Z \cdot \psi_3 - X \cdot \psi_1)(X \cdot \psi_2)(\psi_3 \cdot \psi_1 \times \psi_1)$$

$$O'_0 - O''_0 = -2Q[(\psi_1 \cdot \psi_2)(\psi_2 \cdot \psi_3)(\psi_3 \cdot \psi_1)] \\ O_0 \equiv -5(O'_0 + O''_0) = \epsilon^{p_1 p_2 p_3} v_{p_1}^m v_{p_2}^n (\psi_m \cdot \psi_n \times \psi_{p_3})$$

$$v_n^m \equiv (\phi^m \cdot \psi_n) - \frac{1}{3} \delta_n^m (\phi^p \cdot \psi_p)$$

$$O_0 = \epsilon^{p_1 p_2 p_3} (\phi^m \cdot \psi_{p_1}) (\phi^n \cdot \psi_{p_2}) (\psi_m \cdot \psi_n \times \psi_{p_3}).$$

$$\psi_m \cdot \psi_n \times \psi_{p_3}$$

$$Q(\psi_m \cdot \psi_n \times \psi_p) = \frac{3}{2} \epsilon_{(m|qr} (\phi^q \times \phi^r) \cdot (\psi_{|n} \times \psi_p) = 3 \epsilon_{(m|qr} (\phi^q \cdot \psi_{|n}) (\phi^r \cdot \psi_p) = 3 \epsilon_{(m|qr} v_{|n}^q v_p^r$$

$$\phi^2\psi^5=Q_f[f\phi\psi^4]+Q[\psi^6].$$

$$\phi^{(m}_i\phi^{n)}_j):(\mathbf{6},\mathbf{5}+\mathbf{1})\in SU(3)\times SU(2)$$

$$\psi_{p_1(i)}\psi_{p_2(j)}(\psi_m\cdot\psi_n\times\psi_{p_3}):(\mathbf{3},\mathbf{5}+\mathbf{1})\otimes(\overline{\mathbf{10}},\mathbf{1})$$

$$(f\cdot\phi^m)(\psi_{[n_1}\cdot\psi_{n_2]})(\psi_{[p_1}\cdot\psi_{p_2]}),(\phi^m\cdot\psi_{n_1})(f\cdot\psi_{n_2})(\psi_{[p_1}\cdot\psi_{p_2]}).$$

$$Q[f\cdot\phi^m]=(\phi^n\times\psi_n)\cdot\phi^m=(\phi^m\times\phi^n)\cdot\psi_n$$

$$(\mathbf{8}\oplus\mathbf{1})\otimes\mathbf{3}\otimes\overline{\mathbf{3}}\rightarrow\mathbf{27}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}\oplus\mathbf{1}$$

$$(\phi^m\cdot\psi_m)\epsilon^{npq}(f\cdot\psi_n)(\psi_p\cdot\psi_q),(\phi^m\cdot\psi_n)(f\cdot\psi_m)\epsilon^{npq}(\psi_p\cdot\psi_q)$$

$$\psi^{(i}_m(\psi_r)\times\psi_n)^j)\epsilon^{npq}(\psi_p\cdot\psi_q),\psi^{(i}_n(\psi_r\times\psi_m)^j)\epsilon^{npq}(\psi_p\cdot\psi_q)$$

$$\epsilon^{pqr}\psi^{(i}_p\psi^{j)}_q(\psi_m\cdot\psi_n\times\psi_r)=A\psi^{(i}_m(\psi_n)\times\psi_r)^j)\epsilon^{pqr}(\psi_p\cdot\psi_q)+B\psi^{(i}_r(\psi_n\times\psi_m)^j)\epsilon^{pqr}(\psi_p\cdot\psi_q)$$

$$Q_+^2Q_+^1O'_0=\\ -(Y\cdot f+\psi_3\cdot\psi_1)^2\psi_3\cdot(\psi_2\times\psi_2)-(X\cdot f+\psi_2\cdot\psi_3)(Z\cdot f+\psi_1\cdot\psi_2)\psi_1\cdot(\psi_3\times\psi_3)\\ -(X\cdot f+\psi_2\cdot\psi_3)(X\cdot\psi_3)f\cdot(\psi_1\times\psi_1)+2(Y\cdot\psi_2-Z\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)\psi_3\cdot(\psi_2\times f)\\ -2(Y\cdot f+\psi_3\cdot\psi_1)(X\cdot\psi_3)\psi_2\cdot(\psi_1\times f)-(Z\cdot\psi_3-X\cdot\psi_1)(Z\cdot f+\psi_1\cdot\psi_2)f\cdot(\psi_3\times\psi_3)$$

$$Q_+^1Q_+^2Q_+^3O'_0=Q_+^1Q_+^2Q_+^3O''_0=\\ (X\cdot f+\psi_2\cdot\psi_3)^2f\cdot(\psi_1\times\psi_1)+2(X\cdot f+\psi_2\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)f\cdot(\psi_1\times\psi_2)\\ +(1,2,3\rightarrow2,3,1)+(1,2,3\rightarrow3,1,2)=G^mG^nf\cdot(\psi_m\times\psi_n)$$



$$G^m \equiv \phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p$$

$$Q_+^2 Q_+^1 [(Z \cdot f + \psi_1 \cdot \psi_2) O'_0] = (Z \cdot f + \psi_1 \cdot \psi_2) Q_+^2 Q_+^1 O'_0$$

$$\begin{aligned} O_1 = & (f \cdot f) \epsilon^{c_1 c_2 c_3} (\phi^a \cdot \psi_{c_1}) (\phi^b \cdot \psi_{c_2}) (\psi_a \cdot \psi_b \times \psi_{c_3}) \\ & + \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (f \cdot \psi_{b_1}) (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ & - \frac{1}{72} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \end{aligned}$$

$$O_1 = (f \cdot f) O_0 + f \cdot \xi + \chi$$

$$\begin{aligned} \vec{\xi} = & \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} \vec{\psi}_{b_1} (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ \chi = & -\frac{1}{72} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \\ = & -120 \psi_1^1 \psi_1^2 \psi_1^3 \psi_2^1 \psi_2^2 \psi_2^3 \psi_3^1 \psi_3^2 \psi_3^3 \end{aligned}$$

$$2(\vec{\phi}^m \times \vec{\psi}_m) O_0 + Q_\psi \vec{\xi} = 0, \vec{\phi}^m \cdot (\overrightarrow{\psi_m} \times \vec{\xi}) + Q_\psi \chi = 0$$

$$\vec{\xi} = -\frac{1}{2} \epsilon^{mnp} \vec{\psi}_m \psi_n \cdot \frac{\partial}{\partial \phi^p} O_0$$

$$\begin{aligned} & 4(\vec{\phi}^m \times \vec{\psi}_m) O_0 \\ = & \left[(\vec{\phi}^a \times \vec{\phi}^b) \psi_a \cdot \frac{\partial}{\partial \phi^b} + \vec{\psi}_a (\phi^a \times \phi^b) \cdot \frac{\partial}{\partial \phi^b} - \vec{\psi}_a (\psi_b \times \phi^a) \cdot \frac{\partial}{\partial \psi_b} + \vec{\psi}_b (\psi_a \times \phi^a) \cdot \frac{\partial}{\partial \psi_b} \right] O_0. \end{aligned}$$

$$(\phi^b \times \frac{\partial}{\partial \phi^b} + \psi_b \times \frac{\partial}{\partial \psi_b}) O_0$$

$$\phi^m \cdot \frac{\partial}{\partial \phi^m} O_0 = n_{\text{B}} O_0, \psi_m \cdot \frac{\partial}{\partial \psi_m} O_0 = n_{\text{F}} O_0 \quad (n_{\text{B}}, n_{\text{F}}) = \mathfrak{G}$$

$$\vec{\varepsilon}_a^b \left[\phi^a \cdot \frac{\partial}{\partial \phi^b} - \psi_b \cdot \frac{\partial}{\partial \psi_a} \right] O_0 = 0 \quad (\vec{\varepsilon}_a^a = 0).$$

$$\left[\phi^a \cdot \frac{\partial}{\partial \phi^b} - \psi_b \cdot \frac{\partial}{\partial \psi_a} \right] O_0 = \frac{1}{3} (n_{\text{B}} - n_{\text{F}}) \delta_b^a O_0.$$

$$\left[\phi_i^a \cdot \frac{\partial}{\partial \phi_j^a} - \psi_a^j \cdot \frac{\partial}{\partial \psi_i^j} \right] O_0 = \frac{1}{3} (n_{\text{B}} - n_{\text{F}}) \delta_i^j O_0.$$

$$(\vec{\psi}_a \times \vec{\phi}^b) \left(\phi^a \cdot \frac{\partial}{\partial \phi^b} \right) O_0 = \left(4 - \frac{n_{\text{F}} + 2n_{\text{B}}}{3} \right) (\vec{\phi}^m \times \vec{\psi}_m) O_0 = (\vec{\phi}^m \times \vec{\psi}_m) O_0$$

$$\epsilon_{\text{man}} \epsilon_{ijk} \frac{\partial}{\partial \psi_n^k} \chi$$

$$\phi^m \cdot (\psi_m \times \xi) = -\frac{1}{2} \phi_i^m \phi_j^a \epsilon_{\text{man}} \epsilon_{ijk} \frac{\partial \chi}{\partial \psi_n^k} = -\frac{1}{2} \epsilon_{\text{man}} (\phi^m \times \phi^a) \cdot \frac{\partial \chi}{\partial \psi_n} = -Q_\psi \chi,$$

$$O_0 \text{ is } (Q_+^m Q_+^n O_0) \left(\phi^p \cdot f + \frac{1}{2} \epsilon^{pqr} \psi_q \cdot \psi_r \right)$$



$$O_n \equiv (f \cdot f)^n O_0 + n(f \cdot f)^{n-1} f \cdot \xi + \frac{2n^2 + n}{3} (f \cdot f)^{n-1} \chi$$

$$QO_n = (f \cdot f)^{n-1} \left[\vec{f} \cdot (2n(\vec{\phi}^m \times \vec{\psi}_m) O_0 + nQ_\psi \vec{\xi}) + n(\vec{\phi}^m \times \vec{\psi}_m) \cdot \vec{\xi} + \frac{2n^2 + n}{3} Q_\psi \chi \right] \\ + 2(n^2 - n)(f \cdot f)^{n-2} \vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m)(f \cdot \xi) + \frac{2n(n-1)(2n+1)}{3} (f \cdot f)^{n-2} \vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m) \chi$$

$$QO_n = \frac{2(n^2 - n)}{3} (f \cdot f)^{n-2} [-(f \cdot f)(\phi^m \times \psi_m) \cdot \xi + 3(f \times \phi^m) \cdot \psi_m(f \cdot \xi)]$$

$$\epsilon_{\text{man}} \delta_{ij} \frac{\partial}{\partial \psi_n^k} \chi, \epsilon_{\text{man}} \delta_{k(i} \frac{\partial}{\partial \psi_n^{j)}} \chi$$

$$\epsilon_{\text{man}} \left[\delta_{ij} \frac{\partial}{\partial \psi_n^k} - \delta_{k(i} \frac{\partial}{\partial \psi_n^{j)}} \right] \chi$$

$$\epsilon_{\text{man}} \left[(f \cdot \phi^a)(f \times \phi^m) \cdot \frac{\partial}{\partial \psi_n} - \frac{1}{2} [(f \times \phi^m) \cdot \phi^a] f \cdot \frac{\partial}{\partial \psi_n} \right] \chi \\ = \frac{1}{2} \epsilon_{\text{man}} \left[[f \times (f \times (\phi^m \times \phi^a))] \cdot \frac{\partial}{\partial \psi_n} - [(f \times \phi^m) \cdot \phi^a] f \cdot \frac{\partial}{\partial \psi_n} \right] \chi \\ = -\frac{1}{2} \epsilon_{\text{man}} (f \cdot f)(\phi^m \times \phi^a) \cdot \frac{\partial}{\partial \psi_n} \chi = -(f \cdot f) Q_\psi \chi = (f \cdot f)(\phi^m \times \psi_m) \cdot \xi$$

$$O_n = (f \cdot f)^n \epsilon^{c_1 c_2 c_3} (\phi^a \cdot \psi_{c_1}) (\phi^b \cdot \psi_{c_2}) (\psi_a \cdot \psi_b \times \psi_{c_3}) \\ + n(f \cdot f)^{n-1} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (f \cdot \psi_{b_1}) (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ - \left(\frac{n}{72} + \frac{n^2 - n}{108} \right) (f \cdot f)^{n-1} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} \\ \cdot (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3})$$

$$2p+8q=12(n-n'), p=0,1,2,3, q\geq 0$$

$$(p,q,n-n')=(2,1,1),(0,3,2),(2,4,3),(0,6,4),(2,7,5),(0,9,6),\cdots$$

$$\phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \psi_p$$

$$\mathcal{O}(f^{2n-2}\phi^0\psi^9)(Q_+^aQ_+^bO_{n'}) (\phi\cdot f+\psi\cdot\psi)^q\boxtimes f^{2n-2}\phi^0\psi^9$$

$$f^{2n'}\phi^2\psi^5\rightarrow f^{2n'}\psi^7,f^{2n'-1}\phi\psi^7\rightarrow f^{2n'}\psi^7,f^{2n'-2}\psi^9\rightarrow f^{2n'}\psi^7.$$

$$O_n \stackrel{\mathfrak{m}}{\sim} \epsilon_{abc} (Q_+^a Q_+^b O_{n-1}) \left(\phi^c \cdot f + \frac{1}{2} \epsilon^{cde} \psi_d \cdot \psi_e \right)$$

$$O_n=(f\cdot f)^nO_0+\frac{20n}{3}(f\cdot f)^{n-1}\sum_{\text{cyclic}}(f\cdot\psi_3)(\psi_3\cdot\psi_2)(X\cdot\psi_2)(\psi_1\cdot\psi_1\times\psi_1)\\ -\frac{10}{3}\left(\frac{n}{6}+\frac{n^2-n}{9}\right)(f\cdot f)^{n-1}(\psi_1\cdot\psi_1\times\psi_1)(\psi_2\cdot\psi_2\times\psi_2)(\psi_3\cdot\psi_3\times\psi_3)$$



$$\begin{aligned}
& Q_1^+ Q_2^+ Q_3^+ O_0 \\
& = -10(X \cdot f + \psi_2 \cdot \psi_3)^2 f \cdot (\psi_1 \times \psi_1) \\
& - 20(X \cdot f + \psi_2 \cdot \psi_3)(Y \cdot f + \psi_3 \cdot \psi_1) f \cdot (\psi_1 \times \psi_2) + \mathfrak{C}_{\text{cyclic}} \\
& \rightarrow -20(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1) + \mathfrak{C}_{\text{cyclic}} \\
& Q_1^+ Q_2^+ Q_3^+ (f \cdot \psi_3)(\psi_3 \cdot \psi_2)(X \cdot \psi_2)(\psi_1 \cdot \psi_1 \times \psi_1) + \mathfrak{C}_{\text{cyclic}} \\
& \rightarrow -3(f \cdot f)(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1) + 6(f \cdot \psi_2)(f \cdot \psi_3)(\psi_2 \cdot \psi_3)(f \cdot \psi_1 \times \psi_1) + \mathfrak{C}_{\text{cyclic}} \\
& Q_1^+ Q_2^+ Q_3^+ (\psi_1 \cdot \psi_1 \times \psi_1)(\psi_2 \cdot \psi_2 \times \psi_2)(\psi_3 \cdot \psi_3 \times \psi_3) \\
& = -27(f \cdot \psi_1 \times \psi_1)(f \cdot \psi_2 \times \psi_2)(f \cdot \psi_3 \times \psi_3) \\
& = 18((f \cdot f)(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1)) - 2(f \cdot \psi_2)(f \cdot \psi_3)(\psi_2 \cdot \psi_3)(f \cdot \psi_1 \times \psi_1) + \mathfrak{C}_{\text{cyclic}}
\end{aligned}$$

$$\chi_{24} \equiv -t^{24} \chi_D(t, x, y, p)$$

$$\begin{aligned}
Z - Z_{\text{grav}} - \chi_{24} = & \left[-\chi_{(1,3)} t^{32} - \left(\chi_{(1,\bar{3})} + \chi_{(3,6)} \right) t^{34} - \chi_{(2,3)} t^{35} + \left(\chi_{(3,1)} + \chi_{(3,8)} \right) t^{36} \right. \\
& - \left(\chi_{(2,\bar{3})} + \chi_{(4,6)} \right) t^{37} + \chi_{(5,3)} t^{38} + \left(\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)} \right) t^{39} \\
& \left. - \left(2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)} \right) t^{40} \right] \chi_D + \mathcal{O}(t^{41}).
\end{aligned}$$

$$\begin{aligned}
& O_0 \cdot (\phi^{(m)} \cdot \phi^n), O_0 \cdot (\phi^m \cdot \lambda_{\dot{\alpha}}), O_0 \cdot (\lambda_+ \cdot \lambda_-) \\
& O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right), O_0 \cdot \left(\lambda_{\dot{\alpha}} \cdot \psi_m - \frac{1}{2} \epsilon_{mnp} \phi^n \cdot D_{\dot{\alpha}} \phi^p \right) \\
& O_0 \cdot \partial_{\dot{\alpha}} (\phi^{(m)} \cdot \phi^n)
\end{aligned}$$

$$\begin{aligned}
& \bar{Q} O_0 \times \left(\phi^m \cdot \phi^n, \phi^m \cdot \lambda_{\dot{\alpha}}, \lambda_+ \cdot \lambda_-, \phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right) \\
& (Q, \overline{Q\bar{Q}}) O_0 \times (\phi^m \cdot \phi^n, \phi^m \cdot \lambda_{\dot{\alpha}}) \\
& (Q\bar{Q}, \overline{Q\bar{Q}Q}, \partial) O_0 \times (\phi^m \cdot \phi^n)
\end{aligned}$$

$$O_0 \cdot (\phi^m \cdot \phi^n), O_0 \cdot (\phi^m \cdot \lambda_{\dot{\alpha}}), O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right)$$

$$\begin{aligned}
O_0 \cdot (\phi^{(m)} \cdot \phi^n) = & -\frac{1}{14} Q [20 \epsilon^{rs(m} (\phi^n) \cdot \psi_p) (\phi^p \cdot \psi_r) (\phi^q \cdot \psi_q) (f \cdot \psi_s) \\
& - 20 \epsilon^{prs} (\phi^{(m)} \cdot \psi_p) (\phi^n) \cdot \psi_r) (\phi^q \cdot \psi_q) (f \cdot \psi_s) \\
& + 30 \epsilon^{prs} (\phi^{(m)} \cdot \psi_p) (\phi^n) \cdot \psi_r) (\phi^q \cdot \psi_s) (f \cdot \psi_q) \\
& - 7 \epsilon^{a_1 a_2 p} \epsilon^{b_1 b_2 (m} (\phi^n) \cdot \psi_p) (\phi^q \cdot \psi_q) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2}) \\
& + 18 \epsilon^{a_1 a_2 p} \epsilon^{b_1 b_2 (m} (\phi^n) \cdot \psi_q) (\phi^q \cdot \psi_p) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2})].
\end{aligned}$$

$$O_0 \cdot (\phi^m \cdot \lambda_{\dot{\alpha}})$$

$$\begin{aligned}
O_0 \cdot (\phi^m \cdot \lambda_{\dot{\alpha}}) = & \frac{1}{8} Q [40 \epsilon^{mnp} (f \cdot \psi_q) (\lambda_{\dot{\alpha}} \cdot \psi_r) (\phi^q \cdot \psi_n) (\phi^r \cdot \psi_p) \\
& - 4 \epsilon^{ma_1 a_2} \epsilon^{nb_1 b_2} (\lambda_{\dot{\alpha}} \cdot \psi_n) (\phi^p \cdot \psi_p) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2}) \\
& + 6 \epsilon^{ma_1 a_2} \epsilon^{nb_1 b_2} (\lambda_{\dot{\alpha}} \cdot \psi_p) (\phi^p \cdot \psi_n) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2}) \\
& + \epsilon^{na_1 a_2} \epsilon^{pb_1 b_2} (\lambda_{\dot{\alpha}} \cdot \psi_n) (\phi^m \cdot \psi_p) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2})].
\end{aligned}$$



$$\begin{aligned}
& O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right) \\
& O_0 \cdot \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right) \\
& = \frac{1}{4} Q [\epsilon_{npq} \epsilon^{ra_1 a_2} \epsilon^{qb_1 b_2} \epsilon^{mc_1 c_2} (\phi^p \cdot \psi_r) (\psi_{a_1} \cdot \psi_{a_2}) (\psi_{b_1} \cdot \psi_{b_2}) (\psi_{c_1} \cdot \psi_{c_2})] \\
& O_0 \cdot \left(\phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p \right) \\
& f \cdot \lambda_{\dot{\alpha}} + \frac{2}{3} \psi_m \cdot D_{\dot{\alpha}} \phi^m - \frac{1}{3} \phi^m \cdot D_{\dot{\alpha}} \psi_m \text{ at } \mathcal{O}(t^9) \\
& \phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p \\
& \phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p \\
& O_0 \cdot \partial_{\dot{\alpha}} (\lambda_{\dot{\beta}} \cdot \lambda^{\dot{\beta}}), O_0 \cdot \left(f \cdot \lambda_{\dot{\alpha}} + \frac{2}{3} \psi_m \cdot D_{\dot{\alpha}} \phi^m - \frac{1}{3} \phi^m \cdot D_{\dot{\alpha}} \psi_m \right) \\
& Z - Z_{\text{grav}} - \chi_{24} - \chi_{32} \text{ is } - \left(\chi_{(1, \bar{3})} + \chi_{(3, 6)} \right) t^{34} \\
& O_0 \cdot \partial^{\dot{\alpha}} \left(\lambda_{\dot{\alpha}} \cdot \psi_m - \frac{1}{2} \epsilon_{mnp} \phi^n \cdot D_{\dot{\alpha}} \phi^p \right), O_0 \cdot \partial_{\dot{\alpha}} \partial_{\dot{\beta}} (\phi^m \cdot \phi^n) \\
& O_0 \cdot \partial_{\dot{\alpha}} \left(f \cdot \phi^m + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p \right) \\
& O_0 \cdot \partial_{\dot{\alpha}} \left(f \cdot \lambda_{\dot{\beta}} + \frac{2}{3} \psi_m \cdot D_{\dot{\beta}} \phi^m - \frac{1}{3} \phi^m \cdot D_{\dot{\beta}} \psi_m \right) \\
& O_0 \cdot \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right) \\
& O_0 \cdot \partial_{\dot{\alpha}} \partial^{\dot{\beta}} \left(\lambda_{\dot{\beta}} \cdot \psi_m - \frac{1}{2} \epsilon_{mnp} \phi^n \cdot D_{\dot{\beta}} \phi^p \right), O_0 \cdot \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \partial_{\dot{\gamma}} (\phi^m \cdot \phi^n). \\
& O_0 \cdot \partial_{(\dot{\alpha}} \partial_{\dot{\beta}} \partial_{\dot{\gamma}} (\lambda_{\dot{\delta}} \cdot \phi^m) \\
& O_0 \cdot \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \partial^{\dot{\gamma}} (\lambda_{\dot{\gamma}} \cdot \phi^m), O_0 \cdot \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \left(f \cdot \phi^m + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p \right) \\
& O_0 \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \partial_{\dot{\gamma}} (\lambda_{\dot{\delta}} \cdot \lambda^{\dot{\delta}}) \\
& O_0 \partial_{\dot{\alpha}} \partial^{\dot{\beta}} \left(f \cdot \lambda_{\dot{\beta}} + \frac{2}{3} \psi_m \cdot D_{\dot{\beta}} \phi^m - \frac{1}{3} \phi^m \cdot D_{\dot{\beta}} \psi_m \right) \\
& O_0 \partial_{(\dot{\alpha}} \partial_{\dot{\beta}} (f \cdot \lambda_{\dot{\gamma}}) + \frac{2}{3} \psi_m \cdot D_{\dot{\gamma}} \phi^m - \frac{1}{3} \phi^m \cdot D_{\dot{\gamma}} \psi_m) \\
& O_0 \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \partial_{\dot{\gamma}} \left(\phi^m \cdot \psi_n - \frac{1}{3} \delta_n^m \phi^p \cdot \psi_p \right) \\
& O_0 \equiv \epsilon^{abc} (v_2)^m{}_a (v_2)^n{}_b \text{tr}(\psi_{(c} \psi_m \psi_{n)})
\end{aligned}$$

$$Q\text{tr}(\psi_{(c}\psi_m\psi_n)) \propto \epsilon_{ab(c}(v_2)^a{}_m(v_2)^b{}_n) \equiv R(v_2)_{cmn}$$

$$QO_0 \propto \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn} = 0.$$

$$\mathcal{J} \equiv 2(Q_1 + Q_2 + Q_3) + 6J = 24$$

$$R \equiv \frac{Q_1 + Q_2 + Q_3}{3}$$

$$SU(3) \subset SU(4)_R$$

$$(R,J)=\left(\frac{n}{2},\frac{8-n}{2}\right)$$

$$u^{ij}\equiv\text{tr}(\phi^{(i}\phi^{j)}), u^{ijk}\equiv\text{tr}(\phi^{(i}\phi^j\phi^{k)})$$

$$v_j^i\equiv\text{tr}(\phi^i\psi_j)-\frac{1}{3}\delta_j^i\text{tr}(\phi^a\psi_a), v^{ij}{}_k\equiv\text{tr}(\phi^{(i}\phi^{j)}\psi_k)-\frac{1}{4}\delta_k^i\text{tr}(\phi^{(j}\phi^a)\psi_a)-\frac{1}{4}\delta_k^j\text{tr}(\phi^{(i}\phi^a)\psi_a)$$

$$w^i\equiv\text{tr}\left(f\phi^i+\frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), w^{ij}\equiv\text{tr}(f\phi^{(i}\phi^{j)}+\epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2})$$

$$\begin{aligned} O^{(2,1)} \equiv & 65u^{ij}\left(r_{20}^{(2,1)}\right)_{ij} - 39w^{ij}\left(r_{14}^{(1,1)}\right)_{ij} + 5w^i\left(r_{16}^{(1,1)}\right)_i \\ & + 312v_i^{jk}\left(r_{16}^{(1,2)}\right)_{jk}^i + 26v_i^j\left(r_{18}^{(1,2)}\right)_j^i + 6w^i\left(r_{16}^{(0,3)}\right)_i \end{aligned}$$

$$O_1^{(1,2)} \equiv -3v^{(j}{}_i w^{k)}\left(r_{10}^{(0,1)}\right)_{jk}^i - 3u^{(ij}w^{k)}\left(r_{12}^{(0,2)}\right)_{ijk} + \epsilon_{a_1a_2i}u^{a_1j}w^{a_2}\left(r_{12}^{(0,2)}\right)_j^i$$

$$\begin{aligned} O_2^{(1,2)} \equiv & -9u^{a(i}v^{j)}{}_a\left(r_{14}^{(1,1)}\right)_{ij} + 10\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_j)\left(r_{14}^{(1,1)}\right)_k^{ij} \\ & + 30v^{(j}{}_i w^{k)}\left(r_{10}^{(0,1)}\right)_{jk}^i + 60u^{(jk}v^{l)}{}_i\left(r_{14}^{(0,3)}\right)_{jkl}^i, \end{aligned}$$

$$\begin{aligned} O_3^{(1,2)} \equiv & -3u^{a(i}v^{j)}{}_a\left(r_{14}^{(1,1)}\right)_{ij} + 6\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_j)\left(r_{14}^{(1,1)}\right)_k^{ij} + 4u^{ijk}\left(r_{18}^{(1,2)}\right)_{ijk} + 14v^{(j}{}_i w^{k)}\left(r_{10}^{(0,1)}\right)_{jk}^i \\ & - 6w^{ij}\left(r_{14}^{(0,2)}\right)_{ij} - 12\epsilon^{a_1a_2(i}v^{j}{}_a v^{k)}{}_a\left(r_{12}^{(0,2)}\right)_{ijk} - 4v^j{}_a v^a{}_i\left(r_{12}^{(0,2)}\right)_j^i \end{aligned}$$

$$\begin{aligned} O_4^{(1,2)} \equiv & -3u^{a(i}v^{j)}{}_a\left(r_{14}^{(1,1)}\right)_{ij} + 14\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_j)\left(r_{14}^{(1,1)}\right)_k^{ij} + 8v^{jk}{}_i\left(r_{16}^{(1,1)}\right)_{jk}^i + 42v^{(j}{}_i w^{k)}\left(r_{10}^{(0,1)}\right)_{jk}^i \\ & + 12u^{(ij}w^{k)}\left(r_{12}^{(0,2)}\right)_{ijk} - 24w^{ij}\left(r_{14}^{(0,2)}\right)_{ij} - 36\epsilon^{a_1a_2(i}v^{j}{}_a v^{k)}{}_a\left(r_{12}^{(0,2)}\right)_{ijk} - 8v^{jk}{}_i\left(r_{16}^{(0,3)}\right)_{jk}^i. \end{aligned}$$

$$\begin{aligned}
O_1^{(1,1)} &\equiv \epsilon_{a_1 a_2 i} u^{a_1(j w^k) a_2} \left(r_{10}^{(0,1)} \right)_{jk}^i \\
O_2^{(1,1)} &\equiv \epsilon_{a_1 a_2 i} u^{a_1 j k} w^{a_2} \left(r_{10}^{(0,1)} \right)_{jk}^i \\
O_3^{(1,1)} &\equiv \epsilon_{a_1 a_2 i} \epsilon_{b_1 b_2 j} u^{a_1 b_1} u^{a_2 b_2 k} \left(r_{14}^{(1,1)} \right)_k^{ij} + 5 v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i - 2 v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i \\
O_1^{(0,3)} &= -\epsilon_{i a_1 a_2} (4 u^{a_1 b} v^{j a_2}{}_b + 3 u^{j a_1 b} v^{a_2}{}_b) \left(r_{12}^{(0,2)} \right)_j^i = \frac{1}{2} i Q \left(\left(r_{12}^{(0,2)} \right)_j^i \left(r_{12}^{(0,2)} \right)_i^j \right) \\
O_2^{(0,3)} &= -\epsilon_{a_1 a_2 (i} (u^{a_1(k} v^{l) a_2}{}_j) + u^{k l a_1} v^{a_2}{}_j) \left(r_{12}^{(0,2)} \right)_{kl}^{ij} = \frac{1}{2} i Q \left(\left(r_{12}^{(0,2)} \right)_{ij}^{kl} \left(r_{12}^{(0,2)} \right)_{kl}^{ij} \right) \\
O_3^{(0,3)} &\equiv -u^{a(i} v^{jk)}{}_a \left(r_{12}^{(0,2)} \right)_{ijk}, \\
O_4^{(0,3)} &\equiv -\epsilon_{a_1 a_2 i} u^{a_1 b} v^{a_2}{}_b \left(r_{14}^{(0,2)} \right)^i, \\
O_5^{(0,3)} &\equiv 6 v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i + 6 u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)} \right)_{ijk} + \epsilon_{a_1 a_2 i} u^{a_1 b j} v^{a_2}{}_b \left(r_{12}^{(0,2)} \right)_j^i, \\
O_6^{(0,3)} &\equiv 24 v^{(j}{}_a v^{k)}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i + 6 u^{a(i} v^{j)}{}_a \left(r_{14}^{(0,2)} \right)_{ij} - \epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_j) \left(r_{14}^{(0,2)} \right)_k^{ij}, \\
O_7^{(0,3)} &\equiv v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i - 10 v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i + 6 u^{a(ij} v^{k)} \left(r_{12}^{(0,2)} \right)_{ijk} + 10 \epsilon_{a_1 a_2 (i} u^{a_1 k l} v^{a_2)} \left(r_{12}^{(0,2)} \right)_{kl}^{ij}, \\
O_8^{(0,3)} &\equiv 5 v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i - 2 v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i + 9 u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)} \right)_{ijk} + 6 \epsilon_{a_1 a_2 i} u^{a_1(j} u^{kl) a_2} \left(r_{14}^{(0,3)} \right)_{jkl}^i, \\
O_9^{(0,3)} &\equiv 6 v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i + 12 v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i + 18 u^{a(ij} v^{k)} \left(r_{12}^{(0,2)} \right)_{ijk} - \epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_j) \left(r_{14}^{(0,2)} \right)_k^{ij}, \\
O_{10}^{(0,3)} &\equiv 38 v^a{}_i v^{jk}{}_a \left(r_{10}^{(0,1)} \right)_{jk}^i + 4 v^{(j}{}_a v^{k)a}{}_i \left(r_{10}^{(0,1)} \right)_{jk}^i + 24 u^{a(ij} v^{k)}{}_a \left(r_{12}^{(0,2)} \right)_{ijk} + 5 u^{(jk} v^{l)}{}_i \left(r_{14}^{(0,2)} \right)_{jkl}^i.
\end{aligned}$$

$$\begin{aligned}
O &\equiv -6 O_6^{(0,3)} \\
&= 288 v^j{}_a v^{ka}{}_i \epsilon_{c_1 c_2 (j} \text{tr}(\phi^{c_1} \phi^{c_2} \phi^i \psi_k) - 72 v^a{}_b v^{bk}{}_a \epsilon_{c_1 c_2 (k} \text{tr}(\phi^{c_1} \phi^{c_2} \phi^d \psi_d) \\
&\quad + 36 \epsilon_{a_1 a_2} u^{a_1 k} v^{a_2}{}_j [2 \text{tr}(\phi^{(i} \phi^c \phi^j) \psi_{(c} \psi_k) + 2 \text{tr}(\phi^{(i} \phi^c \phi^{j)} \psi_{(c} \psi_k) \\
&\quad + 9 \text{tr}(\phi^{(i} \phi^j \psi_{(c} \phi^c) \psi_k) - 6 \text{tr}(\phi^{(i} \phi^j) \psi_{(c} \phi^c \psi_k)] \\
&\quad - 9 \epsilon_{a_1 a_2 j} u^{a_1 b} v^{a_2}{}_b [2 \text{tr}(\phi^{(j} \phi^c \phi^d) \psi_{(c} \psi_d) + 2 \text{tr}(\phi^{(j} \phi^c \phi^{ld)} \psi_{(c} \psi_d) \\
&\quad + 9 \text{tr}(\phi^{(j} \phi^d \psi_{(c} \phi^c) \psi_d) - 6 \text{tr}(\phi^{(j} \phi^d) \psi_{(c} \phi^c \psi_d)] \\
&\quad - 20 u^{ai} v^j{}_a \epsilon_{b_1 b_2 b_3} [2 \text{tr}(\psi_{(i} \psi_j) \phi^{b_1} \phi^{b_2} \phi^{b_3}) + \text{tr}(\psi_{(i} \phi^{b_1} \psi_j) \phi^{b_2} \phi^{b_3})] \\
&\quad - 36 u^{ai} v^j{}_a \epsilon_{b_1 b_2 (i} [\text{tr}(\psi_j) \psi_c \phi^{b_1} \phi^{b_2} \phi^c) + \text{tr}(\psi_j) \psi_c \phi^{b_1} \phi^c \phi^{b_2}) + \text{tr}(\psi_j) \psi_c \phi^c \phi^{b_1} \phi^{b_2})] \\
&\quad - 36 u^{ai} v^j{}_a \epsilon_{b_1 b_2 (i} [\text{tr}(\psi_j) \phi^{b_1} \psi_c \phi^{b_2} \phi^c) + \text{tr}(\psi_j) \phi^{b_1} \psi_c \phi^c \phi^{b_2}) + \text{tr}(\psi_j \phi^c \psi_c \phi^{b_1} \phi^{b_2})] \\
&\quad - 36 u^{ai} v^j{}_a \epsilon_{b_1 b_2 (i} [\text{tr}(\psi_j) \phi^{b_1} \phi^{b_2} \psi_c \phi^c) + \text{tr}(\psi_j) \phi^{b_1} \phi^c \psi_c \phi^{b_2}) + \text{tr}(\psi_j) \phi^c \phi^{b_1} \psi_c \phi^{b_2})] \\
&\quad - 36 u^{ai} v^j{}_a \epsilon_{b_1 b_2 (i} [\text{tr}(\psi_j) \phi^{b_1} \phi^{b_2} \phi^c \psi_c) + \text{tr}(\psi_j) \phi^{b_1} \phi^c \phi^{b_2} \psi_c) + \text{tr}(\psi_j) \phi^c \phi^{b_1} \phi^{b_2} \psi_c) \\
&\quad + 12 u^{ai} v^j{}_a \epsilon_{b_1 b_2 (i} [5 \text{tr}(\psi_j) \phi^{b_1} \phi^{b_2}) \text{tr}(\psi_c \phi^c) + 2 \text{tr}(\psi_j \phi^{(b_1} \phi^{c)}) \text{tr}(\psi_c \phi^{b_2}) \\
&\quad - 2 \text{tr}(\psi_j) \phi^{b_2}) \text{tr}(\psi_c \phi^{(b_1} \phi^c)])].
\end{aligned}$$



$$\begin{aligned}
O_1^{(0,2)} &\equiv -\epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i + 2\epsilon_{a_1 a_2} u^{a_1 b} u^{a_2(j} v^{k)}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i \\
O_2^{(0,2)} &\equiv -6\epsilon_{a_1 a_2} i u^{a_1 b(j} v^{k)a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i - \epsilon_{a_1 a_2} i u^{a_1(k} v^{l)a_2}{}_j \left(r_{12}^{(0,1)}\right)_{kl}^{ij} \\
O_3^{(0,2)} &\equiv -\epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i - \epsilon_{a_1 a_2} i u^{a_1 kl} v^{a_2}{}_j \left(r_{12}^{(0,1)}\right)_{kl}^{ij} \\
O_4^{(0,2)} &\equiv -\epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i + \epsilon_{a_1 a_2(i} \epsilon_{j)b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} u^{kl} \left(r_{12}^{(0,2)}\right)_{kl}^{ij} \\
O_5^{(0,2)} &\equiv -4\epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i - 24\epsilon_{a_1 a_2} i u^{a_1 b(j} v^{k)a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i \\
&\quad - \epsilon_{a_1 a_2(i} \epsilon_{j)b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} k \left(r_{14}^{(0,2)}\right)_k^{ij} \\
O_6^{(0,2)} &\equiv -\epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i + 12\epsilon_{a_1 a_2} i u^{a_1 b(j} v^{k)a_2}{}_b \left(r_{10}^{(0,1)}\right)_{jk}^i + 3\epsilon_{a_1 a_2} i u^{a_1(j} u^{kl)a_2} \left(r_{14}^{(0,2)}\right)_{jkl}^i.
\end{aligned}$$

$$\begin{aligned}
O^{(0,1)} &\equiv 36\epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} u^{a_3 jk} \left(r_{10}^{(0,1)}\right)_{jk}^i + 5\epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} u^{a_3 b_3} r_{12}^{(0,1)} \\
&\quad - 6\epsilon_{a_1 a_2(i} \epsilon_{j)b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} u^{kl} \left(r_{12}^{(0,1)}\right)_{kl}^{ij}
\end{aligned}$$

$$\begin{aligned}
&\text{tr}(\phi^1 \phi^1 \psi_1 \phi^1 \psi_1 \psi_1 \phi^1 \psi_1) + \text{tr}(\phi^2 \phi^2 \psi_2 \phi^2 \psi_2 \psi_2 \phi^2 \psi_2) + \text{tr}(\phi^3 \phi^3 \psi_3 \phi^3 \psi_3 \psi_3 \phi^3 \psi_3) \\
&\text{tr}(\phi^1 \phi^1 \phi^2 \psi_2) \text{tr}(\psi_1 \psi_2) \text{tr}(\phi^2 \psi_1) + \text{tr}(\phi^2 \phi^2 \phi^3 \psi_3) \text{tr}(\psi_2 \psi_3) \text{tr}(\phi^3 \psi_2) \\
&+ \text{tr}(\phi^3 \phi^3 \phi^1 \psi_1) \text{tr}(\psi_3 \psi_1) \text{tr}(\phi^1 \psi_3) + \text{tr}(\phi^3 \phi^3 \phi^2 \psi_2) \text{tr}(\psi_3 \psi_2) \text{tr}(\phi^2 \psi_3) \\
&+ \text{tr}(\phi^1 \phi^1 \phi^3 \psi_3) \text{tr}(\psi_1 \psi_3) \text{tr}(\phi^3 \psi_1) + \text{tr}(\phi^2 \phi^2 \phi^1 \psi_1) \text{tr}(\psi_2 \psi_1) \text{tr}(\phi^1 \psi_2) \\
&\text{tr}(\phi^2 \phi^2 \psi_1 \psi_2 \phi^3) \text{tr}(f \phi^1 \phi^1) + \text{tr}(\phi^3 \phi^3 \psi_2 \psi_3 \phi^1) \text{tr}(f \phi^2 \phi^2) \\
&+ \text{tr}(\phi^1 \phi^1 \psi_3 \psi_1 \phi^2) \text{tr}(f \phi^3 \phi^3) - \text{tr}(\phi^3 \phi^3 \psi_1 \psi_3 \phi^2) \text{tr}(f \phi^1 \phi^1) \\
&- \text{tr}(\phi^1 \phi^1 \psi_2 \psi_1 \phi^3) \text{tr}(f \phi^2 \phi^2) - \text{tr}(\phi^2 \phi^2 \psi_3 \psi_2 \phi^1) \text{tr}(f \phi^3 \phi^3)
\end{aligned}$$

$$f = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & -f_1 - f_5 \end{pmatrix},$$

$$\begin{aligned}
u^{ij} &\equiv \text{tr}(\phi^{(i} \phi^{j)}), u^{ijk} \equiv \text{tr}(\phi^{(i} \phi^j \phi^{k)}) \\
v_j^i &\equiv \text{tr}(\phi^i \psi_j) - \frac{1}{3} \delta_j^i \text{tr}(\phi^a \psi_a), v_k^{ij} \equiv \text{tr}(\phi^{(i} \phi^{j)} \psi_k) - \frac{1}{4} \delta_k^i \text{tr}(\phi^{(j} \phi^a) \psi_a) - \frac{1}{4} \delta_k^j \text{tr}(\phi^{(i} \phi^a) \psi_a) \\
w^i &\equiv \text{tr}\left(f \phi^i + \frac{1}{2} \epsilon^{ia_1 a_2} \psi_{a_1} \psi_{a_2}\right), w^{ij} \equiv \text{tr}(f \phi^{(i} \phi^{j)} + \epsilon^{a_1 a_2(i} \phi^{j)} \psi_{a_1} \psi_{a_2})
\end{aligned}$$

$$Q\phi^m = 0, Q\psi_m = -\frac{i}{2} \epsilon_{mnp} [\phi^n, \phi^p], Qf = -i[\phi^m, \psi_m]$$

$$\begin{aligned}
[n, 0]: O^{i_1 i_2 i_3 \cdots i_n} &\rightarrow O^{i_1 i_2 i_3 \cdots i_n}, [0, n]: O_{i_1 i_2 i_3 \cdots i_n} \rightarrow O_{i_1 i_2 i_3 \cdots i_n} \\
[1, 1]: O_j^i &\rightarrow O_j^i - \frac{1}{3} \delta_j^i O_a^a \\
[2, 1]: O_k^{ij} &\rightarrow O_k^{ij} - \frac{1}{2} \delta_k^{(i} O_a^{j)a}, [1, 2]: O_{jk}^i \rightarrow O_{jk}^i - \frac{1}{2} \delta_{(j}^i O_{k)a}^a \\
[3, 1]: O_l^{ijk} &\rightarrow O_l^{ijk} - \frac{3}{5} \delta_l^{(i} O_a^{jk)a}, [1, 3]: O_{jkl}^i \rightarrow O_{jkl}^i - \frac{3}{5} \delta_{(j}^i O_{kl)a}^a \\
[2, 2]: O_{kl}^{ij} &\rightarrow O_{kl}^{ij} - \frac{4}{5} \delta_{(k}^{(i} O_{l)a}^{j)a} + \frac{1}{10} \delta_{(k}^{(i} \delta_{l)}^{j)} O_{a_1 a_2}^{a_1 a_2}
\end{aligned}$$



$$\begin{aligned}
t^{10}[1,2](u_2 u_3): \left(R_{10}^{(0,0)}\right)_{jk}^i &= \epsilon_{a_1 a_2(j} \epsilon_{k)b_1 b_2} u^{a_1 b_1} u^{i a_2 b_2} \\
t^{12}[0,0](u_2 u_2 u_2): R_{12}^{(0,0)} &= \epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} u^{a_3 b_3} \\
t^{12}[2,2](u_2 u_2 u_2, u_3 u_3): \left(R_{12}^{(0,0)}\right)_{kl}^{ij} &= \epsilon_{a_1 a_2(k} \epsilon_{l)b_1 b_2} (u^{a_1 b_1} u^{a_2 b_2} u^{ij} + 6u^{a_1 b_1(i} u^{j)a_2 b_2}) \\
t^{12}[0,3](u_2 v_3): \left(R_{12}^{(0,1)}\right)_{ijk} &= \epsilon_{(i|a_1 a_2} \epsilon_{|j|b_1 b_2} u^{a_1 b_1} v^{a_2 b_2}{}_{|k)} \\
t^{12}[1,1](u_2 v_3, u_3 v_2): \left(R_{12}^{(0,1)}\right)_j^i &= \epsilon_{j a_1 a_2} (4u^{a_1 b} v^{i a_2}{}_b + 3u^{i a_1 b} v^{a_2}{}_b) \\
t^{12}[2,2](u_2 v_3, u_3 v_2): \left(R_{12}^{(0,1)}\right)_{kl}^{ij} &= \epsilon_{a_1 a_2(k} (u^{a_1(i} v^{j)a_2}{}_{l)} + u^{ij a_1} v^{a_2}{}_{l)}) \\
t^{14}[1,0](u_2 u_2 v_2): \left(R_{14}^{(0,1)}\right)^i &= \epsilon_{a_1 a_2 a_3} u^{i a_1} u^{b a_2} v^{a_3}{}_b \\
t^{14}[0,2](u_2 u_2 v_2, u_3 v_3): \left(R_{14}^{(0,1)}\right)_{ij}^i &= \epsilon_{a_1 a_2(i} (\epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} v^{b_3}{}_{|j)} - 2\epsilon_{|j)b_1 b_2} u^{a_1 b_1 c} v^{a_2 b_2}{}_c) \\
t^{14}[2,1](u_2 u_2 v_2, u_3 v_3): \left(R_{14}^{(0,1)}\right)_k^{ij} &= \epsilon_{k a_1 a_2} (3u^{(a_1 b} u^{ij)} v^{a_2}{}_b + 4u^{a_1 b} u^{a_2(i} v^{j)}{}_b + 24u^{a_1 b(i} v^{j)a_2}{}_b) \\
t^{14}[1,3](u_2 u_2 v_2, u_3 v_3): \left(R_{14}^{(0,1)}\right)_{jkl}^i &= \epsilon_{(j|a_1 a_2} \epsilon_{|k|b_1 b_2} (u^{a_1 b_1} u^{a_2 b_2} v^i{}_{|l)} + 6u^{i a_1 b_1} v^{a_2 b_2}{}_{|l)}) \\
t^{14}[3,2](u_2 u_2 v_2, u_3 v_3): \left(R_{14}^{(0,1)}\right)_{lm}^{ijk} &= \epsilon_{a_1 a_2(l} (u^{(a_1 i} u^{jk)} v^{a_2}{}_{m)}) + 6u^{a_1(ij} v^{k)a_2}{}_{m)}) \\
t^{14}[1,3](v_2 v_3): \left(R_{14}^{(0,2)}\right)_{jkl}^i &= \epsilon_{a_1 a_2(j} v^{a_1}{}_k v^{i a_2}{}_{l)} \\
t^{16}[0,1](u_2 v_2 v_2, v_3 v_3): \left(R_{16}^{(0,2)}\right)_i^i &= \epsilon_{i a_1 a_2} (12u^{bc} v^{a_1}{}_b v^{a_2}{}_c + 13u^{a_1 b} v^{a_2}{}_c v^c{}_b + 12v^{a_1 b}{}_c v^{a_2 c}{}_b) \\
t^{16}[1,2](u_2 v_2 v_2, v_3 v_3): \left(R_{16}^{(0,2)}\right)_{jk}^i &= \epsilon_{a_1 a_2(j} (3u^{ib} v^{a_1}{}_k) v^{a_2}{}_b - 7u^{i a_1} v^b{}_k) v^{a_2}{}_b \\
&+ 6u^{a_1 b} v^i{}_k v^{a_2}{}_b + 24v^{a_1 b}{}_k v^{i a_2}{}_b) \\
t^{16}[2,3](u_2 v_2 v_2, v_3 v_3): \left(R_{16}^{(0,2)}\right)_{klm}^{ij} &= \epsilon_{a_1 a_2(k} (u^{a_1(i} j} v^j{}_l v^{a_2}{}_{m)}) + 3v^{a_1(i}{}_l v^{j)a_2}{}_{m)}) \\
t^{18}[0,0](u_3 v_2 v_2): R_{18}^{(0,2)} &= \epsilon_{a_1 a_2 a_3} u^{a_1 b c} v^{a_2}{}_b v^{a_3}{}_c \\
t^{20}[1,0](v_2 v_2 v_3): \left(R_{20}^{(0,3)}\right)^i &= 2v^a{}_c v^b{}_a v^{ic}{}_b - 3v^i{}_a v^c{}_b v^{ab}{}_c \\
t^{22}[2,0](u_2 v_2 v_2 v_2): \left(R_{22}^{(0,3)}\right)^{ij} &= u^{ij} v^a{}_b v^b{}_c v^c{}_a - 3u^{a(i} v^{j)}{}_b v^b{}_c v^c{}_a + 3u^{ab} v^{(i}{}_a v^{j)}{}_c v^c{}_b \\
t^{24}[0,0](u_2 v_2 v_2 v_3): R_{24}^{(0,3)} &= \epsilon_{a_1 a_2 a_3} u^{a_1 b} v^{a_2}{}_b v^{a_3 c}{}_d v^d{}_c \\
t^{26}[1,0](v_2 v_2 v_2 v_3): \left(R_{26}^{(0,4)}\right)^i &= v^i{}_a v^a{}_b v^b{}_c v^{bc}{}_d \\
t^{30}[0,0](v_2 v_2 v_2 v_2 v_2): R_{30}^{(0,5)} &= v^a{}_b v^b{}_c v^c{}_d v^d{}_e v^e{}_a \\
t^{30}[3,0](v_2 v_2 v_2 v_2 v_2): \left(R_{30}^{(0,5)}\right)^{ijk} &= \epsilon^{a_1 a_2(i} v^{j}{}_{a_1} v^{k)}{}_{a_2} v^b{}_c v^c{}_d v^d{}_b.
\end{aligned}$$

$$\begin{aligned}
& t^{14}[0,2](v_2 v_3, u_2 w_3): \left(R_{14}^{(1,0)}\right)_{ij} = \epsilon_{a_1 a_2}(i(8 v^{a_1 b}{}_j) v^{a_2}{}_b + 5 \epsilon_{j) b_1 b_2} u^{a_1 b_1} w^{a_2 b_2}) \\
& t^{14}[2,1](v_2 v_3, u_2 w_3, u_3 w_2): \\
& \left(R_{14}^{(1,0)}\right)_k^{ij} = 2 v^i{}_a v^{j a}{}_k - 5 v^a{}_k v^{ij}{}_a + 3 \epsilon_{k a_1 a_2} u^{a_1(i} w^{j) a_2} + 3 \epsilon_{k a_1 a_2} u^{ij a_1} w^{a_2} \\
& t^{16}[0,1](v_3 v_3, u_2 v_2 v_2, u_2 u_2 w_2): \\
& \left(R_{16}^{(1,0)}\right)_i = \epsilon_{i a_1 a_2} (48 v^{a_1 b_1}{}_b v^{a_2 b_2}{}_{b_1} + 9 u^{b_1 b_2} v^{a_1}{}_{b_1} v^{a_2}{}_{b_2} - 13 \epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} w^{b_3}) \\
& t^{16}[1,2](v_3 v_3, u_2 v_2 v_2, u_3 w_3, u_2 u_2 w_2): \\
& \left(R_{16}^{(1,0)}\right)_{jk}^i = \epsilon_{a_1 a_2}(j(24 v^{i a_1}{}_b v^{b a_2}{}_{|k}) + 2 u^{i a_1} v^{a_2}{}_b v^b{}_{|k}) - 6 u^{a_1 b} v^{a_2}{}_b v^i{}_{|k}) \\
& + 6 \epsilon_{|k) b_1 b_2} u^{i a_1 b_1} w^{a_2 b_2} + \epsilon_{|k) b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} w^i) \\
& t^{16}[3,1](v_3 v_3, u_2 v_2 v_2, u_3 w_3, u_2 u_2 w_2): \\
& \left(R_{16}^{(1,0)}\right)_l^{ijk} = 24 v^{(ij}{}_a v^{k) a}{}_l + 7 u^{(ij} v^k){}_a v^a{}_l - 6 u^{a(i} v^j{}_a v^k){}_l \\
& + 18 \epsilon_{l a_1 a_2} u^{a_1(ij} w^{k) a_2} + 3 \epsilon_{l a_1 a_2} u^{(ij} u^{k) a_1} w^{a_2} \\
& t^{16}[1,2](v_2 w_3, v_3 w_2): \left(R_{16}^{(1,1)}\right)_{jk}^i = \epsilon_{a_1 a_2}(j(v^{a_1}{}_k) w^{a_2 i} + v^{a_1 i}{}_k) w^{a_2}) \\
& t^{18}[0,0](v_2 v_2 v_2, u_2 v_2 w_2): R_{18}^{(1,1)} = v^{a_1}{}_{a_2} v^{a_2}{}_{a_3} v^{a_3}{}_{a_1} - 3 \epsilon_{a_1 a_2 a_3} u^{a_1 b} v^{a_2}{}_b w^{a_3} \\
& t^{18}[1,1](v_2 v_2 v_2, v_3 w_3, u_2 v_2 w_2): \\
& \left(R_{18}^{(1,1)}\right)_j^i = 9 v^i{}_{a_1} v^{a_1}{}_{a_2} v^{a_2}{}_j - 24 \epsilon_{j a_1 a_2} v^{i a_1}{}_b w^{b a_2} \\
& - 13 \epsilon_{j a_1 a_2} u^{i a_1} v^{a_2}{}_b w^b - 16 \epsilon_{j a_1 a_2} u^{i b} v^{a_1}{}_b w^{a_2} + 5 \epsilon_{j a_1 a_2} u^{a_1 b} v^i{}_b w^{a_2} \\
& t^{18}[0,3](v_2 v_2 v_2, v_3 w_3, u_2 v_2 w_2): \\
& \left(R_{18}^{(1,1)}\right)_{ijk} = \epsilon_{a_1 a_2}(i(3 v^{a_1}{}_{|j} v^{a_2}{}_b v^b{}_{|k}) - 3 \epsilon_{b_1 b_2|j} v^{a_1 b_1}{}_k) w^{a_2 b_2} - \epsilon_{b_1 b_2|j} u^{a_1 b_1} v^{a_2}{}_k) w^{b_2}) \\
& t^{18}[2,2](v_2 v_2 v_2, v_3 w_3, u_2 v_2 w_2): \\
& \left(R_{18}^{(1,1)}\right)_{ij}^{kl} = 2 v^{(i}{}_a v^{j) a}{}_k (v^l{}_a) - 6 \epsilon_{a_1 a_2}(k v^{a_1(i}{}_l) w^{j) a_2} - \epsilon_{a_1 a_2}(k u^{ij} v^{a_1}{}_l) w^{a_2} \\
& t^{20}[0,2](v_2 v_2 w_2, u_2 w_2 w_2, w_3 w_3): \\
& \left(R_{20}^{(2,0)}\right)_{ij} = 2 \epsilon_{a_1 a_2}(i v^{a_1}{}_b v^b{}_{|j}) w^{a_2} - 3 \epsilon_{a_1 a_2 a_3} v^{a_1}{}_i v^{a_2}{}_j w^{a_3} \\
& + \epsilon_{i a_1 a_2} \epsilon_{j b_1 b_2} u^{a_1 b_1} w^{a_2} w^{b_2} + 3 \epsilon_{i a_1 a_2} \epsilon_{j b_1 b_2} w^{a_1 b_1} w^{a_2 b_2} .
\end{aligned}$$

$$i Q \tau_j^{(n_f, n_\psi)} = R_j^{(n_f, n_\psi - 1)}$$



$$\begin{aligned}
(r_{10}^{(0,1)})_{jk}^i &= -2\epsilon_{a_1 a_2(j)} \text{tr}(\phi^{a_1} \phi^{a_2} \phi^i \psi_k), \\
r_{12}^{(0,1)} &= \epsilon_{a_1 a_2 a_3} [6\text{tr}(\psi_b \phi^{a_1}) \text{tr}(\phi^b \phi^{a_2} \phi^{a_3}) - \text{tr}(\psi_b \phi^{a_1} \phi^{a_2}) \text{tr}(\phi^b \phi^{a_3})] \\
&\quad - 3\epsilon_{a_1 a_2 a_3} [\text{tr}(\psi_b \phi^b \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^b \phi^{a_2} \phi^{a_3}) \\
&\quad + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^b \phi^{a_3}) + \text{tr}(\psi_b \phi^{a_1} \phi^{a_2} \phi^{a_3} \phi^b)], \\
(r_{12}^{(0,1)})_{kl}^{ij} &= -2\epsilon_{a_1 a_2(k)} [\text{tr}(\psi_l) \phi^{(i} \phi^{j)} \phi^{a_1} \phi^{a_2}) + 7\text{tr}(\psi_l) \phi^{(il} \phi^{a_1} \phi^{lj)} \phi^{a_2})], \\
(r_{12}^{(0,2)})_{ijk} &= \frac{1}{2} \epsilon_{a_1 a_2(i)} \text{tr}(\phi^{a_1} \psi_j \phi^{a_2} \psi_k), \\
(r_{12}^{(0,2)})_j^i &= 6\text{tr}(\phi^{(i} \phi^{a)} \psi_{(a} \psi_{j)}) - 5\text{tr}(\phi^{[i} \psi_a \phi^{a]} \psi_j), \\
(r_{12}^{(0,2)})_{kl}^{ij} &= \text{tr}(\phi^{(i} \phi^{j)} \psi_{(k} \psi_{l)}), \\
(r_{14}^{(0,2)})^i &= 3\text{tr}(\phi^i \psi_{a_1} \phi^{a_1} \phi^{a_2} \psi_{a_2}) + 2\text{tr}(\phi^i \phi^{a_1}) \text{tr}(\phi^{a_2} \psi_{(a_1} \psi_{a_2)}) \\
&\quad - 6\text{tr}(\phi^i \psi_{a_1}) \text{tr}(\phi^{[a_1} \phi^{a_2]} \psi_{a_2}) - \text{tr}(\phi^i \psi_{a_1} \psi_{a_2}) \text{tr}(\phi^{a_1} \phi^{a_2}), \\
(r_{14}^{(0,2)})_{ij} &= \frac{5}{9} \epsilon_{a_1 a_2 a_3} [2\text{tr}(\psi_{(i} \psi_{j)} \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_{(i} \phi^{a_1} \psi_{j)} \phi^{a_2} \phi^{a_3})] \\
&\quad + \epsilon_{a_1 a_2(i)} [\text{tr}(\psi_{j)} \psi_{a_3} \phi^{a_1} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j \psi_{a_3} \phi^{a_1} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_{j)} \psi_{a_3} \phi^{a_3} \phi^{a_1} \phi^{a_2})] \\
&\quad + \epsilon_{a_1 a_2(i)} [\text{tr}(\psi_j)^{a_1} \psi_{a_3} \phi^{a_2} \phi^{a_3}) + \text{tr}(\psi_j \phi^{a_1} \psi_{a_3} \phi^{a_3} \phi^{a_2}) + \text{tr}(\psi_j \phi^{a_3} \psi_{a_3} \phi^{a_1} \phi^{a_2})] \\
&\quad + \epsilon_{a_1 a_2(i)} [\text{tr}(\psi_j) \phi^{a_1} \phi^{a_2} \psi_{a_3} \phi^{a_3}) + \text{tr}(\psi_j \phi^{a_1} \phi^{a_3} \psi_{a_3} \phi^{a_2}) + \text{tr}(\psi_j \phi^{a_3} \phi^{a_1} \psi_{a_3} \phi^{a_2})] \\
&\quad + \epsilon_{a_1 a_2(i)} [\text{tr}(\psi_j) \phi^{a_1} \phi^{a_2} \phi^{a_3} \psi_{a_3}) + \text{tr}(\psi_j \phi^{a_1} \phi^{a_3} \phi^{a_2} \psi_{a_3}) + \text{tr}(\psi_j \phi^{a_3} \phi^{a_1} \phi^{a_2} \psi_{a_3})] \\
&\quad - \frac{1}{3} \epsilon_{a_1 a_2(i)} [5\text{tr}(\psi_{j)} \phi^{a_1} \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{a_3}) + 2\text{tr}(\psi_j \phi^{(a_1} \phi^{a_3)}) \text{tr}(\psi_{a_3} \phi^{a_2}) \\
&\quad - 2\text{tr}(\psi_j \phi^{a_2}) \text{tr}(\psi_{a_3} \phi^{(a_1} \phi^{a_3)})], \\
(r_{14}^{(0,2)})_k^{ij} &= 12\text{tr}(\phi^{(i} \phi^{a} \phi^{j)} \psi_{(a} \psi_{k)}) + 12\text{tr}(\phi^{(il} \phi^{a} \phi^{lj)} \psi_{(a} \psi_{k)}) \\
&\quad + 54\text{tr}(\phi^{(i} \phi^{j} \psi_{(a} \phi^{a)} \psi_{k)}) - 36\text{tr}(\phi^{(i} \phi^{j)} \psi_{(a} \phi^{a} \psi_{k)}), \\
(r_{14}^{(0,2)})_{jkl}^i &= 2\epsilon_{a_1 a_2(j)} [\text{tr}(\phi^i \phi^{a_1} \phi^{a_2} \psi_k \psi_l) + 3\text{tr}(\phi^i \phi^{a_1} \psi_k \phi^{a_2} \psi_l) - 2\text{tr}(\phi^i \psi_k \phi^{a_1} \phi^{a_2} \psi_l)], \\
(r_{14}^{(0,3)})_{jkl}^i &= -\frac{1}{2} \text{tr}(\phi^i \psi_{(j} \psi_k \psi_{l)}), \\
(r_{16}^{(0,3)})_i &= \frac{39}{4} \text{tr}(\psi_i \{\psi_{b_1} \psi_{b_2}, \phi^{b_1} \phi^{b_2}\}) + 2\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \psi_{b_2} \phi^{b_2}) - \frac{61}{4} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \psi_{b_2} \phi^{b_1}) \\
&\quad + \frac{97}{4} \text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \psi_{b_2} \phi^{b_2}) - \frac{41}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \psi_{b_2} \phi^{b_1}) - 5\text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \phi^{b_2} \psi_{b_2}) \\
&\quad - \frac{25}{2} \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \phi^{b_1} \psi_{b_2}) + 2\text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \phi^{b_2} \psi_{b_2}) - \frac{61}{4} \text{tr}(\psi_i \phi^{b_2} \psi_{b_1} \phi^{b_1} \psi_{b_2}) \\
&\quad - \frac{11}{4} \text{tr}(\phi^{b_1} \phi^{b_2}) \text{tr}(\psi_i \psi_{b_1} \psi_{b_2}) - \frac{27}{2} \text{tr}(\psi_{b_1} \psi_{b_2}) \text{tr}(\psi_i \phi^{b_1} \phi^{b_2})
\end{aligned}$$

$$\begin{aligned}
& + \frac{29}{4} \text{tr}(\phi^{b_2} \psi_{b_2}) \text{tr}(\psi_i [\psi_{b_1}, \phi^{b_1}]) \\
(r_{16}^{(0,3)})^i_{jk} &= 2 \text{tr}(\psi_{(j} \psi_k) \psi_b \phi^b \phi^i) - 4 \text{tr}(\psi_{(j} \psi_k) \psi_b \phi^i \phi^b) - \text{tr}(\psi_{(j} \psi_b \psi_{|k)} \{\phi^b, \phi^i\}) \\
& - 4 \text{tr}(\psi_{(j} \psi_k) \phi^{(b} \psi_b \phi^{i)}) + 7 \text{tr}(\psi_{(j} \{\psi_b, \phi^b\} \psi_{|k)} \phi^i) - 11 \text{tr}(\psi_{(j} \{\psi_b, \phi^i\} \psi_{|k)} \phi^b) \\
& - 4 \text{tr}(\psi_{(j} \psi_k) \phi^b \phi^i \psi_b) + 2 \text{tr}(\psi_{(j} \psi_k) \phi^i \phi^b \psi_b) \\
& + 3 \text{tr}(\psi_{(j} \psi_b) \text{tr}(\psi_{|k)} [\phi^b, \phi^i]) + 6 \text{tr}(\psi_{(j} \phi^{[b} \text{tr}(\{\psi_k, \psi_b\} \phi^{i]}) \\
(r_{14}^{(1,1)})_{ij} &= 5 \epsilon_{a_1 a_2} (i \text{tr}(f \phi^{a_1} \psi_j) \phi^{a_2}) + \text{tr}(\phi^a \{\psi_a, \psi_{(i} \psi_{j)}\}) - 4 \text{tr}(\phi^a \psi_{(i} \psi_a \psi_{j)}), \\
(r_{14}^{(1,1)})^{ij}_k &= 3 \text{tr}(f \phi^{(i} \phi^{j)} \psi_k) - 3 \text{tr}(f \psi_k \phi^{(i} \phi^{j)}) \\
& + \epsilon^{a_1 a_2} (i \text{tr}(\phi^{j)} \psi_k \psi_{a_1} \psi_{a_2}) - \epsilon^{a_1 a_2} (i \text{tr}(\phi^{j)} \psi_{a_1} \psi_{a_2} \psi_k), \\
(r_{16}^{(1,1)})_i &= 13 \epsilon_{a_1 a_2 a_3} \text{tr}(f \psi_i) \text{tr}(\phi^{a_1} \phi^{a_2} \phi^{a_3}) + \frac{10}{3} \epsilon_{a_1 a_2 a_3} \text{tr}(f \phi^{a_1}) \text{tr}(\psi_i \phi^{a_2} \phi^{a_3}) \\
& + \frac{10}{3} \epsilon_{a_1 a_2 a_3} \text{tr}(f \phi^{a_1} \phi^{a_2}) \text{tr}(\psi_i \phi^{a_3}) + 46 \epsilon_{ia_1 a_2} \text{tr}(f \phi^b) \text{tr}(\psi_b \phi^{a_1} \phi^{a_2}) \\
& - 7 \epsilon_{ia_1 a_2} \text{tr}(f \phi^{a_1}) \text{tr}(\psi_b \phi^{a_2} \phi^b) - 7 \epsilon_{ia_1 a_2} \text{tr}(f \phi^b \phi^{a_1}) \text{tr}(\psi_b \phi^{a_2}) \\
& + 6 \epsilon_{ia_1 a_2} \text{tr}(f \phi^{a_1} \phi^{a_2}) \text{tr}(\psi_b \phi^b) - \frac{115}{3} \epsilon_{a_1 a_2 a_3} \text{tr}(f \psi_i \phi^{a_1} \phi^{a_2} \phi^{a_3}) \\
& - \frac{95}{3} \epsilon_{a_1 a_2 a_3} \text{tr}(f \phi^{a_1} \psi_i \phi^{a_2} \phi^{a_3}) + 5 \epsilon_{a_1 a_2 a_3} \text{tr}(f \phi^{a_1} \phi^{a_2} \psi_i \phi^{a_3}) \\
& + 36 \epsilon_{ia_1 a_2} \text{tr}(f \psi_b \phi^{a_1} \phi^{a_2} \phi^b) - 43 \epsilon_{ia_1 a_2} \text{tr}(f \psi_b \phi^{a_1} \phi^b \phi^{a_2}) \\
& + 39 \epsilon_{ia_1 a_2} \text{tr}(f \phi^{a_1} \psi_b \phi^{a_2} \phi^b) - 68 \epsilon_{ia_1 a_2} \text{tr}(f \phi^{a_1} \phi^{a_2} \psi_b \phi^b) \\
& + 39 \epsilon_{ia_1 a_2} \text{tr}(f \phi^{a_1} \phi^b \psi_b \phi^{a_2}) + 13 \text{tr}(\psi_i \{\psi_{b_1} \psi_{b_2}, \phi^{b_1} \phi^{b_2}\}) \\
& - 31 \text{tr}(\psi_i \{\psi_{b_1} \psi_{b_2}, \phi^{b_2} \phi^{b_1}\}) + 14 \text{tr}(\psi_i \psi_{b_1} \phi^{b_1} \psi_{b_2} \phi^{b_2}) \\
& - 22 \text{tr}(\psi_i \psi_{b_1} \phi^{b_2} \phi^{b_1} \psi_{b_2}) + 14 \text{tr}(\psi_i \phi^{b_1} \psi_{b_1} \phi^{b_2} \psi_{b_2}), \\
(r_{16}^{(1,1)})^i_{jk} &= \epsilon_{a_1 a_2} (j [-4 \text{tr}(f \phi^i) \text{tr}(\psi_k) \phi^{a_1} \phi^{a_2}) - \text{tr}(\phi^i \phi^{a_2}) \text{tr}(f [\psi_k], \phi^{a_1})]) \\
& + \epsilon_{a_1 a_2} (j [3 \text{tr}(f \phi^{a_1} \{\psi_k, \phi^i\} \phi^{a_2}) + 5 \text{tr}(f \{\psi_k, \phi^{a_1} \phi^i \phi^{a_2}\}) \\
& - 4 \text{tr}(f \psi_k) \phi^i \phi^{a_1} \phi^{a_2}) - 4 \text{tr}(f \phi^{a_1} \phi^{a_2} \phi^i \psi_k)]) \\
& + 2 \text{tr}(\psi_{(j} \psi_k) \psi_b [\phi^b, \phi^i]) - 3 \text{tr}(\psi_{(j} \psi_b \psi_{|k)} \{\phi^b, \phi^i\}) + 6 \text{tr}(\psi_{(j} \{\psi_b, \phi^b\} \psi_{|k)} \phi^i) \\
& - 9 \text{tr}(\psi_{(j} \{\psi_b, \phi^i\} \psi_{|k)} \phi^b) - 2 \text{tr}(\psi_{(j} \psi_k) [\phi^b, \phi^i] \psi_b) \\
& + \text{tr}(\psi_{(j} \psi_b) \text{tr}(\psi_{|k)} [\phi^b, \phi^i]) + \text{tr}(\psi_{(j} \phi^b) \text{tr}(\{\psi_{|k}, \psi_b\} \phi^i), \\
(r_{16}^{(1,2)})^i_{jk} &= -\frac{1}{2} \text{tr}(f \phi^i \psi_{(j} \psi_{k)}) - \frac{1}{2} \text{tr}(f \psi_{(j} \phi^i \psi_{k)}) \\
& - \frac{1}{2} \text{tr}(f \psi_{(j} \psi_{k)} \phi^i) - \frac{1}{4} \epsilon^{ia_1 a_2} \text{tr}(\psi_{a_1} \psi_{a_2} \psi_{(j} \psi_{k)}),
\end{aligned}$$

$$\begin{aligned}
(r_{18}^{(1,2)})_j^i &= -4\text{tr}(f\phi^i\phi^a)\text{tr}(\psi_j\psi_a) - 5\text{tr}(f\phi^a\phi^i)\text{tr}(\psi_j\psi_a) - \frac{53}{2}\text{tr}(f\phi^i\psi_j)\text{tr}(\phi^a\psi_a) \\
&+ 7\text{tr}(f\phi^i\psi_a)\text{tr}(\phi^a\psi_j) + \frac{15}{2}\text{tr}(f\phi^a\psi_j)\text{tr}(\phi^i\psi_a) + 12\text{tr}(f\phi^a\psi_a)\text{tr}(\phi^i\psi_j) \\
&+ 2\text{tr}(f\psi_j\phi^i)\text{tr}(\phi^a\psi_a) - 13\text{tr}(f\psi_a\phi^i)\text{tr}(\phi^a\psi_j) + 4\text{tr}(f\psi_j\phi^a)\text{tr}(\phi^i\psi_a) \\
&+ 6\text{tr}(f\psi_j\psi_a)\text{tr}(\phi^i\phi^a) + \frac{13}{2}\text{tr}(f\psi_a\psi_j)\text{tr}(\phi^i\phi^a) - 4\text{tr}(f\phi^i)\text{tr}(\phi^a\psi_j\psi_a) \\
&+ 14\text{tr}(f\phi^i)\text{tr}(\phi^a\psi_a\psi_j) - 8\text{tr}(f\phi^a)\text{tr}(\phi^i\psi_j\psi_a) - 8\text{tr}(f\phi^a)\text{tr}(\phi^i\psi_a\psi_j) \\
&- 4\text{tr}(f\psi_j)\text{tr}(\psi_a\phi^i\phi^a) - 9\text{tr}(f\psi_a)\text{tr}(\psi_j\phi^i\phi^a) + 6\text{tr}(f\psi_a)\text{tr}(\psi_j\phi^a\phi^i) \\
&+ 3\text{tr}(f\phi^i\phi^a\psi_j\psi_a) - \frac{31}{2}\text{tr}(f\phi^i\phi^a\psi_a\psi_j) + 3\text{tr}(f\phi^a\phi^i\psi_j\psi_a) \\
&+ \frac{5}{2}\text{tr}(f\phi^a\phi^i\psi_a\psi_j) + 12\text{tr}(f\phi^i\psi_j\phi^a\psi_a) - \frac{13}{2}\text{tr}(f\phi^i\psi_a\phi^a\psi_j) \\
&- 6\text{tr}(f\phi^a\psi_j\phi^i\psi_a) - \frac{13}{2}\text{tr}(f\phi^a\psi_a\phi^i\psi_j) + 18\text{tr}(f\phi^i\psi_j\psi_a\phi^a) \\
&- 12\text{tr}(f\psi_j\phi^i\phi^a\psi_a) + \frac{17}{2}\text{tr}(f\psi_a\phi^i\phi^a\psi_j) - \frac{43}{2}\text{tr}(f\psi_a\phi^a\phi^i\psi_j) \\
&+ \frac{1}{3}\epsilon^{a_1a_2a_3}\text{tr}(\phi^i\psi_j)\text{tr}(\psi_{a_1}\psi_{a_2}\psi_{a_3}) - 2\epsilon^{a_1a_2i}\text{tr}(\phi^b\psi_{a_1})\text{tr}(\psi_b\psi_j\psi_{a_2}) \\
&- 10\epsilon^{a_1a_2a_3}\text{tr}(\phi^i\psi_j\psi_{a_1}\psi_{a_2}\psi_{a_3}) + 8\epsilon^{a_1a_2a_3}\text{tr}(\phi^i\psi_{a_1}\psi_j\psi_{a_2}\psi_{a_3}) \\
&- 2\epsilon^{a_1a_2a_3}\text{tr}(\phi^i\psi_{a_1}\psi_{a_2}\psi_j\psi_{a_3}), \\
(r_{18}^{(1,2)})_{ijk} &= -\epsilon_{a_1a_2i}\left[\text{tr}(f\phi^{a_1})\text{tr}(\phi^{a_2}\psi_j\psi_k) - \frac{3}{2}\text{tr}(f\psi_j)\text{tr}(\psi_k)\phi^{a_1}\phi^{a_2}\right] \\
&+ 3\text{tr}(f\phi^{a_1}\psi_j\phi^{a_2}\psi_k) - 3\text{tr}(f\psi_j\phi^{a_1}\psi_k\phi^{a_2}) \\
&- \frac{1}{2}\text{tr}(\phi^a\psi_a)\text{tr}(\psi_{(i}\psi_j\psi_{k)}) + \frac{3}{2}\text{tr}(\phi^a\psi_{(i})\text{tr}(\psi_a\psi_{|j}\psi_{k)}) \\
&+ \frac{1}{2}\text{tr}(\phi^a\psi_{(i}\psi_{j|})\text{tr}(\psi_a\psi_{|k)}) + \frac{3}{2}\text{tr}(\phi^a\psi_{(i|}\psi_a\psi_{|j}\psi_{k)}) - \frac{3}{2}\text{tr}(\phi^a\psi_{(i}\psi_{j|}\psi_a\psi_{|k)}), \\
(r_{20}^{(2,1)})_{ij} &= -\epsilon_{a_1a_2i}\left[\text{tr}(ff)\text{tr}(\phi^{a_1}\phi^{a_2}\psi_j) + \frac{1}{2}\text{tr}(f\psi_j)\right]\text{tr}(f\phi^{a_1}\phi^{a_2}) \\
&+ 2\text{tr}(f\phi^{a_1})\text{tr}(f[\phi^{a_2}, \psi_j]) \\
&+ \epsilon_{a_1a_2i}\left[4\text{tr}(ff\phi^{a_1}\phi^{a_2}\psi_j) - \text{tr}(f\phi^{a_1}\phi^{a_2}f\psi_j)\right] + 2\text{tr}(f\phi^a\psi_{(i})\text{tr}(\psi_{j)})\psi_a) \\
&- 4\text{tr}(f\psi_{(i}\phi^a)\text{tr}(\psi_{j)})\psi_a) - \frac{1}{2}\text{tr}(f\psi_{(i})(\phi^a\psi_{j)})\psi_a) - \frac{5}{2}\text{tr}(f\psi_{(i|})(\phi^a\psi_{a|j|)}) \\
&+ 2\text{tr}(f\psi_a)(\phi^a\psi_{(i}\psi_{j)}) - 4\text{tr}(f\psi_{(i|}\psi_a)(\phi^a\psi_{|j|)}) + 2\text{tr}(f\phi^a\psi_{(i}[\psi_{j|}, \psi_a]) \\
&+ 4\text{tr}(f\phi^a\psi_a\phi^a\psi_{(i}\psi_{j)}) + 4\text{tr}(f\psi_{(i}\phi^a\psi_{j)})\psi_a) - 3\text{tr}(f\psi_{(i|}\phi^a\psi_a\psi_{|j|)}) \\
&- 2\text{tr}(f\psi_a\phi^a\psi_{(i}\psi_{j)}) - \text{tr}(f\psi_{(i}\psi_a\phi^a\psi_{|j|)}) + 4\text{tr}(f\psi_a\psi_{(i}\phi^a\psi_{j)}) \\
&+ \frac{2}{5}\epsilon^{a_1a_2a_3}\left[2\text{tr}(\psi_{a_1}\psi_{a_2})\text{tr}(\psi_{a_3}\psi_{(i}\psi_{j)}) - 3\text{tr}(\psi_{(i|}\psi_{a_1}\psi_{|j|})\psi_{a_2}\psi_{a_3})\right] \\
s_1^{(2,0)} &= u^{ij}\left(R_{20}^{(2,0)}\right)_{ij}, \quad s_2^{(2,0)} = w^{ij}\left(R_{14}^{(1,0)}\right)_{ij}, \quad s_3^{(2,0)} = w^i\left(R_{16}^{(1,0)}\right)_i \\
s_1^{(1,2)} &= v_i^{jk}\left(R_{16}^{(1,1)}\right)_{jk}^i, \quad s_2^{(1,2)} = v_i^j\left(R_{18}^{(1,1)}\right)_j^i, \quad s_3^{(1,2)} = w^i\left(R_{16}^{(0,2)}\right)_i. \\
iQO^{(2,1)} &\equiv 65s_1^{(2,0)} - 39s_2^{(2,0)} + 5s_3^{(2,0)} - 312s_1^{(1,2)} - 26s_2^{(1,2)} + 6s_3^{(1,2)} = 0
\end{aligned}$$



$$\begin{aligned}
s_1^{(1,1)} &= u^{a(i} v^{j)}{}_a \left(R_{14}^{(1,0)} \right)_{ij}, s_2^{(1,1)} = \epsilon_{a_1 a_2} (i u^{a_1 k} v^{a_2}{}_j) \left(R_{14}^{(1,0)} \right)_k^{ij}, s_3^{(1,1)} = v^{jk}{}_i \left(R_{16}^{(1,0)} \right)_{jk}^i, \\
s_4^{(1,1)} &= u^{ijk} \left(R_{18}^{(1,1)} \right)_{ijk}, s_5^{(1,1)} = v^{(j}{}_i w^k) \left(R_{10}^{(0,0)} \right)_{jk}^i, s_6^{(1,1)} = u^{(ij} w^k) \left(R_{12}^{(0,1)} \right)_{ijk}, \\
s_7^{(1,1)} &= \epsilon_{a_1 a_2} i u^{a_1 j} w^{a_2} \left(R_{12}^{(0,1)} \right)_j^i, s_8^{(1,1)} = w^{ij} \left(R_{14}^{(0,1)} \right)_{ij}, \\
s_1^{(0,3)} &= \epsilon^{a_1 a_2} (i v^j{}_a v^k)_{a_2} \left(R_{12}^{(0,1)} \right)_{ijk}, s_2^{(0,3)} = v^j{}_a v^a{}_i \left(R_{12}^{(0,1)} \right)_j^i, \\
s_3^{(0,3)} &= u^{(jk} v^k){}_i \left(R_{14}^{(0,2)} \right)_{jkl}^i, s_4^{(0,3)} = v^{jk}{}_i \left(R_{16}^{(0,2)} \right)_{jk}^i.
\end{aligned}$$

$$\begin{aligned}
iQO_1^{(1,2)} &\equiv 3s_5^{(1,1)} - 3s_6^{(1,1)} + s_7^{(1,1)} = 0 \\
iQO_2^{(1,2)} &\equiv 9s_1^{(1,1)} - 10s_2^{(1,1)} - 30s_5^{(1,1)} - 60s_3^{(0,3)} = 0 \\
iQO_3^{(1,2)} &\equiv 3s_1^{(1,1)} - 6s_2^{(1,1)} + 4s_4^{(1,1)} - 14s_5^{(1,1)} - 6s_8^{(1,1)} - 12s_1^{(0,3)} - 4s_2^{(0,3)} = 0 \\
iQO_4^{(1,2)} &\equiv 3s_1^{(1,1)} - 14s_2^{(1,1)} - 8s_3^{(1,1)} - 42s_5^{(1,1)} + 12s_6^{(1,1)} - 24s_8^{(1,1)} - 36s_1^{(0,3)} + 8s_4^{(0,3)} = 0
\end{aligned}$$

$$\begin{aligned}
s_1^{(1,0)} &= \epsilon_{a_1 a_2} i \epsilon_{b_1 b_2} j u^{a_1 b_1} u^{a_2 b_2 k} \left(R_{14}^{(1,0)} \right)_k^{ij}, \\
s_2^{(1,0)} &= \epsilon_{a_1 a_2} i u^{a_1(j} w^{k)a_2} \left(R_{10}^{(0,0)} \right)_{jk}^i, \\
s_3^{(1,0)} &= \epsilon_{a_1 a_2} i u^{a_1 j k} w^{a_2} \left(R_{10}^{(0,0)} \right)_{jk}^i, \\
s_1^{(0,2)} &= v^a{}_i v^{jk}{}_a \left(R_{10}^{(0,0)} \right)_{jk}^i, \\
s_2^{(0,2)} &= v^{(j}{}_a v^{k)a}{}_i \left(R_{10}^{(0,0)} \right)_{jk}^i, \\
s_3^{(0,2)} &= u^{a(i} v^{jk)}{}_a \left(R_{12}^{(0,1)} \right)_{ijk}, \\
s_4^{(0,2)} &= u^{a(ij} v^{k)}{}_a \left(R_{12}^{(0,1)} \right)_{ijk}, \\
s_5^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1 b} v^{a_2 j}{}_b \left(R_{12}^{(0,1)} \right)_j^i, \\
s_6^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1 b j} v^{a_2}{}_b \left(R_{12}^{(0,1)} \right)_j^i, \\
s_7^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1(k} v^{l)a_2}{}_j \left(R_{12}^{(0,1)} \right)_{kl}^{ij}, \\
s_8^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1 k l} v^{a_2}{}_j \left(R_{12}^{(0,1)} \right)_{kl}^{ij}, \\
s_9^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1(j} u^{kl)a_2} \left(R_{14}^{(0,2)} \right)_{jkl}^i, \\
s_{10}^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1 b} v^{a_2}{}_b \left(R_{14}^{(0,1)} \right)_j^i, \\
s_{11}^{(0,2)} &= u^{a(i} v^{j)}{}_a \left(R_{14}^{(0,1)} \right)_{ij}, \\
s_{12}^{(0,2)} &= \epsilon_{a_1 a_2} i u^{a_1 k} v^{a_2}{}_j \left(R_{14}^{(0,1)} \right)_k^{ij}, \\
s_{13}^{(0,2)} &= u^{(jk} v^{l)}{}_i \left(R_{14}^{(0,1)} \right)_{jkl}^i.
\end{aligned}$$



$$\begin{aligned}
iQO_1^{(1,1)} &\equiv s_2^{(1,0)} = 0 \\
iQO_2^{(1,1)} &\equiv s_3^{(1,0)} = 0 \\
iQO_3^{(1,1)} &\equiv s_1^{(1,0)} + 5s_1^{(0,2)} - 2s_2^{(0,2)} = 0 \\
iQO_1^{(0,3)} &\equiv 4s_5^{(0,2)} + 3s_6^{(0,2)} = \left(R_{12}^{(0,1)}\right)_j^i \left(R_{12}^{(0,1)}\right)_i^j = iQ \left[\frac{1}{2} iQ \left(\left(r_{12}^{(0,2)}\right)_j^i \left(r_{12}^{(0,2)}\right)_i^j \right) \right] = 0 \\
iQO_2^{(0,3)} &\equiv s_7^{(0,2)} + s_8^{(0,2)} = \left(R_{12}^{(0,1)}\right)_{kl}^{ij} \left(R_{12}^{(0,1)}\right)_{ij}^{kl} = iQ \left[\frac{1}{2} iQ \left(\left(r_{12}^{(0,2)}\right)_{kl}^{ij} \left(r_{12}^{(0,2)}\right)_{ij}^{kl} \right) \right] = 0 \\
iQO_3^{(0,3)} &\equiv s_3^{(0,2)} = 0 \\
iQO_4^{(0,3)} &\equiv s_{10}^{(0,2)} = 0 \\
iQO_5^{(0,3)} &\equiv 6s_1^{(0,2)} - 6s_4^{(0,2)} - s_6^{(0,2)} = 0 \\
iQO_6^{(0,3)} &\equiv 24s_2^{(0,2)} - 6s_{11}^{(0,2)} + s_{12}^{(0,2)} = 0 \\
iQO_7^{(0,3)} &\equiv s_1^{(0,2)} - 10s_2^{(0,2)} - 6s_4^{(0,2)} - 10s_8^{(0,2)} = 0 \\
iQO_8^{(0,3)} &\equiv 5s_1^{(0,2)} - 2s_2^{(0,2)} - 9s_4^{(0,2)} + 6s_9^{(0,2)} = 0 \\
iQO_9^{(0,3)} &\equiv 6s_1^{(0,2)} + 12s_2^{(0,2)} - 18s_4^{(0,2)} + s_{12}^{(0,2)} = 0 \\
iQO_{10}^{(0,3)} &\equiv 38s_1^{(0,2)} + 4s_2^{(0,2)} - 24s_4^{(0,2)} - 5s_{13}^{(0,2)} = 0
\end{aligned}$$

$$\begin{aligned}
s_1^{(0,1)} &= \epsilon_{a_1 a_2} i u^{a_1 b} u^{jk} v^{a_2}{}_b \left(R_{10}^{(0,0)}\right)_{jk}^i, s_2^{(0,1)} = \epsilon_{a_1 a_2} i u^{a_1 b} u^{a_2(j} v^{k)}{}_b \left(R_{10}^{(0,0)}\right)_{jk}^i, \\
s_3^{(0,1)} &= \epsilon_{a_1 a_2} i u^{a_1 b(j} v^{k) a_2}{}_b \left(R_{10}^{(0,0)}\right)_{jk}^i, s_4^{(0,1)} = \epsilon_{a_1 a_2} (i u^{a_1(k} v^{l) a_2}{}_j) \left(R_{12}^{(0,0)}\right)_{kl}^{ij}, \\
s_5^{(0,1)} &= \epsilon_{a_1 a_2} (i u^{a_1 kl} v^{a_2}{}_j) \left(R_{12}^{(0,0)}\right)_{kl}^{ij}, s_6^{(0,1)} = \epsilon_{a_1 a_2} (i \epsilon_j)_{b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} u^{kl} \left(R_{12}^{(0,1)}\right)_{kl}^{ij}, \\
s_7^{(0,1)} &= \epsilon_{a_1 a_2} (i \epsilon_j)_{b_1 b_2} u^{a_1 b_1} u^{a_2 b_2 k} \left(R_{14}^{(0,1)}\right)_k^{ij}, s_8^{(0,1)} = \epsilon_{a_1 a_2} i u^{a_1(j} u^{kl) a_2} \left(R_{14}^{(0,1)}\right)_{jkl}^i.
\end{aligned}$$

$$\begin{aligned}
iQO_1^{(0,2)} &\equiv s_1^{(0,1)} - 2s_2^{(0,1)} = 0 \\
iQO_2^{(0,2)} &\equiv 6s_3^{(0,1)} + s_4^{(0,1)} = 0 \\
iQO_3^{(0,2)} &\equiv s_1^{(0,1)} + s_5^{(0,1)} = 0 \\
iQO_4^{(0,2)} &\equiv s_1^{(0,1)} + s_6^{(0,1)} = 0 \\
iQO_5^{(0,2)} &\equiv 4s_1^{(0,1)} + 24s_3^{(0,1)} - s_7^{(0,1)} = 0 \\
iQO_6^{(0,2)} &\equiv s_1^{(0,1)} - 12s_3^{(0,1)} + 3s_8^{(0,1)} = 0
\end{aligned}$$

$$\begin{aligned}
s_1^{(0,0)} &= \epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2} i u^{a_1 b_1} u^{a_2 b_2} u^{a_3 jk} \left(R_{10}^{(0,0)}\right)_{jk}^i, \\
s_2^{(0,0)} &= R_{12}^{(0,0)} R_{12}^{(0,0)} \\
s_3^{(0,0)} &= \epsilon_{a_1 a_2} (i \epsilon_j)_{b_1 b_2} u^{a_1 b_1} u^{a_2 b_2} u^{kl} \left(R_{12}^{(0,0)}\right)_{kl}^{ij}, \\
s_4^{(0,0)} &= \epsilon_{a_1 a_2} (i \epsilon_j)_{b_1 b_2} u^{a_1 b_1(k} u^{l) a_2 b_2} \left(R_{12}^{(0,0)}\right)_{kl}^{ij}.
\end{aligned}$$

$$iQO^{(0,1)} \equiv 36s_1^{(0,0)} + 5s_2^{(0,0)} - 6s_3^{(0,0)} = 0$$

$$r_c \ll \min(\mathcal{R}, \mathcal{L}).$$



$$I = \frac{1}{16\pi G} \int_{\mathcal{M}_D} \left(\star (R - 2\Lambda) - \frac{1}{2} d\phi \wedge \star d\phi - \frac{1}{2} e^{a_q \phi} F_{q+2} \wedge \star F_{q+2} \right)$$

$$ds^2_{AdS_D} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, f(r) = 1 + \frac{r^2}{L^2}$$

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}_{10}} \left(\star R - \frac{1}{2} d\phi \wedge \star d\phi - \frac{1}{2} e^{-\phi} H_3 \wedge \star H_3 - \frac{1}{4} \sum_q e^{a_q \phi} F_{q+2} \wedge \star F_{q+2} \right)$$

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}_{11}} \left(\star R - \frac{1}{2} F_4 \wedge \star F_4 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge \star F_4$$

$$ds^2 = ds^2_{AdS_l} + ds^2_{S^m}$$

$$\begin{aligned} ds^2_{AdS_l} &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{l-2}^2 \\ ds^2_{S^m} &= L^2(d\zeta^2 + \cos^2 \zeta d\phi^2 + \sin^2 \zeta d\Omega_{m-2}^2) \end{aligned}$$

$$f(r) \equiv 1 + \frac{r^2}{\tilde{L}^2}$$

$$\tilde{L} = 2L/(l-3), d\Omega_{l-2}^2 \otimes d\Omega_{m-2}^2$$

$$\begin{aligned} A_{l-1} &= -\frac{r^{l-1}}{\tilde{L}} dt \wedge d\Omega_{l-2} \\ A_{m-1} &= \beta_m (L \sin \zeta)^{m-1} d\phi \wedge d\Omega_{m-2} \\ \beta_m &= (-1)^{D-m-1} \end{aligned}$$

$$\tilde{T}^{\mu\nu}(x) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \tilde{\delta}(x) T^{\mu\nu}(\sigma)$$

$$\tilde{\delta}(x) = \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta^{(D)}(x-X(\sigma))$$

$$\tilde{\mathcal{J}}_{q+1}(x) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \tilde{\delta}(x) \mathcal{J}_{q+1}(\sigma)$$

$$\begin{aligned} \nabla_\mu \tilde{T}^{\mu\nu} &= \frac{1}{(q+1)!} F^{\nu\alpha_1 \dots \alpha_{q+1}} \tilde{\mathcal{J}}_{\alpha_1 \dots \alpha_{q+1}} \\ \nabla_\mu \tilde{\mathcal{J}}^{\mu\alpha_1 \dots \alpha_q} &= 0 \end{aligned}$$

$$D_a T^{ab} = 0, D_a \mathcal{J}^{ab_1 \dots b_q} = 0$$

$$T^{ab}K_{ab}{}^\mu=\perp_\nu{}^\mu F^{\nu a_1\dots a_{q+1}}\mathcal{J}_{a_1\dots a_{q+1}},$$

$$K_{ab}^\mu = \perp_\nu^\mu \left(\partial_a \partial_b X^\nu + \Gamma_{\alpha\beta}^\nu \partial_a X^\alpha \partial_b X^\beta \right),$$



$$SO(p) \rightarrow SO\left(p - \sum_l^m q_l\right) \times SO(q_l) \times \dots \times SO(q_l)$$

$$\mathcal{I}_{q+1} = \mathcal{Q}_q \mathcal{V}_{q+1}$$

$$\mathcal{C}_{q+1} \subset \mathcal{W}_{p+1}$$

$$\mathcal{V}_{q+1} = u \wedge v^{(1)} \wedge \dots \wedge v^{(q)}$$

$$\partial_a \mathcal{Q}_p = 0$$

$$\begin{aligned} D_a(\mathcal{Q}_q u^a) + \mathcal{Q}_q u^a \dot{v}_a &= 0, \\ D_a(\mathcal{Q}_q v^{a,i}) + \mathcal{Q}_q v^{a,i} \dot{v}_a - \mathcal{Q}_q v^{a,i} \dot{u}_a &= 0, \end{aligned}$$

$$\dot{v}_a \equiv v^{b,j} D_b v_{a,j}$$

$$P_{\perp} - P_{\parallel} = \Phi_q \mathcal{Q}_q$$

$$\hat{h}_{ab}^{(q)} = -u_a u_b + (1 - \delta_{q,0}) v_a^i v_{b,i},$$

$$\hat{h}_{ab}^{(0)} = -u_a u_b, \hat{h}_{ab}^{(1)} = -u_a u_b + v_a v_b$$

$$\hat{h}_{ab}^{(p)} = \gamma_{ab}$$

$$\hat{\mathbb{L}}^a_b \left(v^c_{[i} D_c v^b_{j]} \right) = \hat{\mathbb{L}}^a_b \left(v^c_i D_c u^b - u^c D_c v^b_i \right) = 0.$$

$$T_{ab} = (\varepsilon + P_{\parallel}) u_a u_b + (P_{\parallel} - P_{\perp}) \hat{h}_{ab}^{(q)} + P_{\perp} \gamma_{ab},$$

$$\varepsilon + P_{\perp} = \mathcal{T} s + \Phi_q \mathcal{Q}_q$$

$$d\varepsilon = \mathcal{T} ds + \Phi_q d\mathcal{Q}_q, dP_{\perp} = s d\mathcal{T} + \mathcal{Q}_q d\Phi_q, dP_{\parallel} = s d\mathcal{T} - \mathcal{Q}_q d\Phi_q$$

$$\begin{aligned} D_a(su^a) &= 0, \\ (\hat{h}^{ab} + u^a u^b)(\dot{u}_b + D_b \ln \mathcal{T}) &= 0, \\ \mathcal{T} s \, \hat{\mathbb{L}}^{ab} (\dot{u}_b + D_b \ln \mathcal{T}) - \Phi_q \mathcal{Q}_q (\hat{K}_{(q)}^a - \hat{\mathbb{L}}^{ab} D_b \ln \Phi_q) &= 0, \end{aligned}$$

$$u^a \partial_a f|_{\partial \mathcal{W}_{p+1}} = 0,$$

$$\prod_{j=1}^q \mathcal{Q}_q u^{[a} v_j^{b]} \partial_a f \Big|_{\partial \mathcal{W}_{p+1}} = 0$$

$$\mathcal{Q}_q v_j^a \partial_a f|_{\partial \mathcal{W}_{p+1}} = 0$$

$$P_{\perp}|_{\partial\mathcal{W}_{p+1}}=0$$

$$r_0|_{\partial\mathcal{W}_{p+1}}=0,$$

$$k=\xi+\Omega\chi$$

$$u^a\partial_a=\frac{k^a\partial_a}{|k|},|k|=\sqrt{-\gamma_{ab}k^ak^b}$$

$$v_i^a\partial_a=\mathbb{M}_i{}^j\zeta_j^a\partial_a,\mathbb{M}_i{}^k\mathbb{M}_j{}^l\zeta_k^a\zeta_{a,l}=\delta_{ij},\zeta_j^a\partial_a\equiv\left(\psi_j^a\partial_a-\frac{k_b\psi_j^b}{k^2}k^a\partial_a\right)$$

$$\begin{aligned} (\hat{h}^{ab}+u^au^b)D_b\ln\left(|k|\mathcal{T}\right)&=0,\\ \mathcal{T}S\,\hat{1}^{ab}\,D_b\ln\left(|k|\mathcal{T}\right)-\Phi_q\mathcal{Q}_q\,\hat{1}^{ab}\left(\dot{v}_b-D_b\ln\left(|k|\Phi_q\right)\right)&=0, \end{aligned}$$

$$\mathcal{T}(\sigma)=\frac{T}{|k|},$$

$$\Phi_q(\sigma)=\frac{\phi_q(\sigma)}{\left|\hat{h}^{(q)}\right|^{1/2}}$$

$$\left|\hat{h}^{(q)}\right|^{1/2}=|k||\mathbb{M}|$$

$$\Phi_H^{(q)}=\int_{\mathcal{C}_q}d^q\sigma|\hat{h}^{(q)}|^{1/2}\Phi_q(\sigma)$$

$$v^{a,i}\dot{v}_a=v^{a,i}\dot{u}_a=u^a\dot{v}_a=0$$

$$v_i^aD_a\mathcal{Q}=u^aD_a\mathcal{Q}=0$$

$$-\xi^2|\mathcal{W}_{p+1}=R_0^2(\sigma),\chi^2|\mathcal{W}_{p+1}=R^2(\sigma),$$

$$n^a\partial_a=\frac{1}{R_0}\xi^a\partial_a$$

$$n^a u_a = -\cosh \eta$$

$$\mathcal{Q}_k=\int_{\mathcal{B}_p}dV_p(T^{\mu\nu}+\mathcal{V}^{\mu\nu})n_\mu k_\nu\Big|_{x^\mu=X^\mu}$$

$$\mathcal{V}^{\mu\nu}=\frac{1}{q!}A_{\alpha_1\ldots\alpha_q}^\nu\mathcal{J}^{\mu\alpha_1\ldots\alpha_q}$$

$$M=\int_{\mathcal{B}_p}dV_p(T^{\mu\nu}+\mathcal{V}^{\mu\nu})n_\mu\xi_\nu\Big|_{x^\mu=X^\mu},J=-\int_{\mathcal{B}_p}dV_p(T^{\mu\nu}+\mathcal{V}^{\mu\nu})n_\mu\chi_\nu\Big|_{x^\mu=X^\mu}$$



$$\omega_q = \frac{-\mathcal{V}_{q+1} \cdot n}{\sqrt{-\hat{h}_{ab}^{(q)} n^a n^b}},$$

$$Q_q = - \int_{\mathcal{B}_{p-q}} dV_{p-q} \mathcal{I}_{q+1} \cdot (n \wedge \omega_q) = \int_{\mathcal{B}_{p-q}} dV_{p-q} \sqrt{-\hat{h}_{ab}^{(q)} n^a n^b} \mathcal{Q}_q$$

$$\Phi_H^{(q)} = \int_{\mathcal{C}_q} dV_q \frac{R_0}{\sqrt{-\hat{h}_{ab}^{(q)} n^a n^b}} \Phi_q$$

$$I = - \int_{\mathcal{W}_{p+1}}^{\star} \, {}_{(p+1)}\mathcal{G}$$

$$\mathcal{G} = \varepsilon - \mathcal{T} s - \sum_q^p \Phi_q \mathcal{Q}_q$$

$$\bar{\Phi}_H^{(p)} = R_0 \int_{\mathcal{B}_p} \star(p) \Phi_p - \int_{\mathcal{B}_p} \mathbb{P}[A_{q+1}]$$

$$\tilde{I} = - \int_{\mathcal{B}_p} \star(p) R_0 \left(\mathcal{G} + \sum_{q \neq p} \Phi_q \mathcal{Q}_q \right) - \bar{\Phi}_H^{(p)} \mathcal{Q}_p = \int_{\mathcal{B}_p} \star(p) (R_0 P_{\parallel} - \mathbb{P}[A_{q+1}])$$

$$I = \int_{\mathcal{W}_{p+1}} \left(\star(p+1) P_{\parallel} + Q_p \mathbb{P}[A_{p+1}] \right)$$

$$\tilde{I}_E = M - \Omega J - TS,$$

$$S = - \int_{\mathcal{B}_p} dV_p s u^a n_a$$

$$\varepsilon = \frac{\Omega_{n+1}}{16\pi G} r_0^n (1+n+nN\sinh^2\alpha)$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \cosh^N \alpha}, s = \frac{\Omega_{n+1}}{4G} r_0^{n+1} \cosh^N \alpha$$

$$\mathcal{Q}_q = \frac{\Omega_{n+1}}{16\pi G} n \sqrt{N} r_0^n \sinh \alpha \cosh \alpha, \Phi_q = \sqrt{N} \tanh \alpha$$

$$P_{\parallel} = -\frac{\Omega_{n+1}}{16\pi G} r_0^n (1+nN\sinh^2\alpha), P_{\perp} = -\frac{\Omega_{n+1}}{16\pi G} r_0^n$$

$$a_q^2 = \frac{4}{N} - \frac{2(p+1)n}{D-2} \Rightarrow N \leq 2 \left(\frac{n+p+1}{n(p+1)} \right)$$

$$\mathcal{G} = \frac{\Omega_{n+1}}{16\pi G} r_0^n$$



$$\begin{aligned}\varepsilon &= -(n+1)P_{\perp} + \Phi_q \mathcal{Q}_q \\ \varepsilon &= -(n+1)P - n\Phi_p \mathcal{Q}_p\end{aligned}$$

$$T_{ab} = \mathcal{T}s\left(u_a u_b - \frac{1}{n} \gamma_{ab}\right) - \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)}$$

$$r_0=\frac{n|k|}{4\pi T}\left(1-\frac{\Phi_H^2}{c^2N|\hat{h}^{(q)}|}\right)^{N/2}, \tanh\alpha=\frac{\Phi_H}{\sqrt{N}c|\hat{h}^{(q)}|^{1/2}}$$

$$c|\hat{h}^{(q)}|^{1/2}=\begin{cases} |k| \text{ for } q=0 \\ \Omega_q|k||\mathbb{M}| \text{ for } 0<q<p \\ \Omega_p|\gamma| \text{ for } q=p \end{cases}$$

$$(D-3)M-(D-2)(TS+\Omega J)-(D-q-3)\Phi_H^{(q)}Q_q=\mathcal{T}_{tot}$$

$$\mathcal{T}_{\mathrm{tot}}=-\int_{\mathcal{B}_p}dV_pR_0(\gamma^{ab}+n^an^b)T_{ab}$$

$$F=M-TS-\Omega J-\Phi_H^{(q)}Q_q$$

$$T_{ab}=-\sqrt{N}\mathcal{Q}_q\hat{h}_{ab}^{(q)}$$

$$-\sqrt{N}\mathcal{Q}_p\hat{h}_{ab}^{(p)}=P\gamma_{ab}$$

$$M=(q+1)\Phi_H^{(q)}Q_q-\mathcal{T}_{tot},\Omega J=q\Phi_H^{(q)}Q_q-\mathcal{T}_{tot}.$$

$$M=\Phi_H^{(q)}Q_q+\Omega J$$

$$\sqrt{\mathcal{T}}su^a=\mathcal{K}^{1/2}l^a$$

$$T_{ab}=\mathcal{K}l_al_b-\sqrt{N}\mathcal{Q}_q\hat{h}_{ab}^{(q)}$$

$$\mathcal{T}_{tot}=-\int_{\mathcal{B}_p}dV_pR_0(\mathcal{K}-q\sqrt{N}\mathcal{Q}_q)$$

$$M=\int_{\mathcal{B}_p}dV_pR_0(\mathcal{K}+\sqrt{N}\mathcal{Q}_q)$$

$$J=\int_{\mathcal{B}_p}dV_p\mathcal{R}\mathcal{K},\mathcal{R}\equiv l_a\chi^a$$

$$M=(q+1)\Phi_H^{(q)}Q_q-\mathcal{T}_{tot}=\frac{1}{q}(V_pR_0(q+1)\mathcal{K}+\mathcal{T}_{tot})$$

$$\frac{J}{\mathcal{R}}=\frac{1}{R_0}\Big(q\Phi_H^{(q)}Q_q-\mathcal{T}_{tot}\Big)=V_p\mathcal{K}$$

$$M=V_pR_0\sqrt{N}Q_p+R_0\frac{J}{\mathcal{R}},$$

$$\sigma^0=t=\tau, r=R, \sigma_i=\phi_i, \sigma_j=\mu_i$$



$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R)d\tau^2+R^2\sum_{i=1}^{k+1}(d\mu_i^2+\mu_i^2d\phi_i^2),\sum_{i=1}^{k+1}\mu_i^2=1$$

$$k^a\partial_a=\partial_\tau+\Omega\partial_\Phi,\partial_\Phi\equiv\sum_{i=1}^{k+1}\partial_{\phi_i}$$

$$\zeta=\frac{1}{|k|^2}(\Omega^2R^2\partial_\tau+f(R)\Omega\partial_\Phi)$$

$$\tilde{I}_E=\frac{\Omega_{n+1}V_p}{16\pi G}\sqrt{f(R)}\Big(\frac{n}{4\pi T}\Big)^n\left(f(R)-\Omega^2R^2\right)^{n/2}\Bigg(1-\frac{\Phi_H^2}{c^2N|\hat{h}(q)|}\Bigg)^{nN/2}$$

$$\Omega^2=\frac{f(R)}{R^2}\frac{p\mathcal{T}s+qn\Phi_q\mathcal{Q}_q+\frac{R^2}{L^2}\Xi_q}{(n+p)\mathcal{T}s+(\delta_{q,0}+q)n\Phi_q\mathcal{Q}_q+\frac{R^2}{L^2}\Xi_q},$$

$$\Xi_q=\mathcal{T}s(1+n+p)+n(1+q)\Phi_q\mathcal{Q}_q$$

$$M\,=\,V_pR_0\left(\cosh^2\,\eta\big(\mathcal{T}s+\delta_{q,0}\Phi_q\mathcal{Q}_q\big)+\frac{\mathcal{T}s}{n}+(1-\delta_{q,0})\Phi_q\mathcal{Q}_q\right)$$

$$J\,=\,V_pR\big(\mathcal{T}s+\delta_{q,0}\Phi_q\mathcal{Q}_q\big)\cosh\,\eta\sinh\,\eta$$

$$S\,=\,V_p s\cosh\,\eta,T=\mathcal{T}R_0\operatorname{sech}\eta$$

$$\Phi_H^{(0)}=R_0\operatorname{sech}\eta\Phi_0,Q_0=V_p\cosh\,\eta\mathcal{Q}_0$$

$$\Phi_H^{(1)}=2\pi RR_0\Phi_1,Q_1=\frac{V_p}{2\pi R}\mathcal{Q}_1$$

$$\Phi_H^{(p)}=V_pR_0\Phi_p,Q_p=\mathcal{Q}_p$$

$$\mathcal{T}_{tot}=-\frac{V_p}{n}\sqrt{f(R)}\frac{R^2}{L^2}\Xi_q$$

$$\Omega^2=\frac{f'(R)}{R^2}\frac{p\mathcal{T}s+qn\Phi_q\mathcal{Q}_q-\frac{R^2}{L^2}\Xi_q}{(n+p)\mathcal{T}s+(\delta_{q,0}+q)n\Phi_q\mathcal{Q}_q-\frac{R^2}{L^2}\Xi_q},f'(R)=1-\frac{R^2}{L^2}.$$

$$\left(\frac{(n+p)\mathcal{T}s+(\delta_{q,0}+q)n\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2}<\frac{R}{L}\leq 1\,\vee\,\frac{R}{L}\leq \left(\frac{p\mathcal{T}s+qn\Phi_q\mathcal{Q}_q}{\Xi_q}\right)^{1/2}$$

$$\mathcal{T}_{tot}=\frac{V_p}{n}\sqrt{f(R)}\frac{R^2}{L^2}\Xi_q,$$

$$\Omega^2=\pm\frac{1}{L^2}$$

$$M=\frac{J}{L}+\sqrt{Nf(R)}Q_0$$



$$l=\frac{1}{\sqrt{f(R)}}\partial_\tau+\frac{1}{R}\partial_\Phi$$

$$R=\frac{L(q+1)J}{\sqrt{L^2(qM-\mathcal{T}_{tot})^2-(q+1)^2J^2}}$$

$$Q_q=\frac{(M+\mathcal{T}_{tot})(L^2(qM-\mathcal{T}_{tot})^2-(q+1)^2J^2)^{\frac{2q+1}{2}}}{\sqrt{N}\Omega_q(L(q+1))^{q+1}J^q(qM-\mathcal{T}_{tot})}$$

$$M=V_q\sqrt{Nf(R)}Q_q+\sqrt{f(R)}\frac{J}{R}$$

$$R=\frac{L(q+1)J}{\sqrt{L^2(qM-\mathcal{T}_{tot})^2+(q+1)^2J^2}}$$

$$\lim_{\alpha\rightarrow\infty}\big(\lim_{\Omega\rightarrow 0}\Omega^2\big)\Rightarrow \frac{R^2}{L^2}=\frac{p}{1+p}\vee\frac{R^2}{L^2}=1$$

$$\mathcal{B}_p=\prod_{a=1}^bS^{p_a}$$

$$p_a=2k_a+1$$

$$\sum_{a=1}^bp_a=p$$

$$\prod_a \mathbb{S}^{p_a} \times \mathbb{S}^{n+1}$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R)d\tau+\sum_{a=1}^b\sum_{i=1}^{k+1}R_a^2(d\mu_{a,i}^2+\mu_{a,i}^2d\phi_{a,i}^2),R^2\equiv\sum_{a=1}^bR_a^2,\sum_{i=1}^{k_a+1}\mu_i^2=1$$

$$k=\partial_\tau+\sum_{a=1}^b\Omega_a\partial_{\Phi_a},\partial_{\Phi_a}\equiv\sum_{i=1}^{k+1}\partial_{\phi_{a,i}}$$

$$\psi^\mu\partial_\mu=\sum_a^m\Omega_a\partial_{\Phi_a}$$

$$\zeta=\frac{1}{|k|^2}\sum_{a=1}^b\left(\Omega_a^2R_a^2\partial_\tau+f(R)\Omega_a\partial_{\Phi_a}\right)$$

$$\tilde{I}_E=\frac{\Omega_{n+1}V_p}{16\pi G}\Big(\frac{n}{4\pi T}\Big)^n\Bigg(f(R)-\sum_{a=1}^b\Omega_a^2R_a^2\Bigg)^{n/2}\left(1-\frac{\Phi_H^2}{c^2N|\hat{h}(q)|}\right)^{nN/2},$$



$$V_p = \prod_{a=1}^m V_{p_a} \text{ and } c^2 |\hat{h}^{(q)}|$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2} \frac{p_a \mathcal{T}s + q_a n \Phi_q \mathcal{Q}_q + \frac{R_a^2}{L^2} \Xi_q}{(n+p) \mathcal{T}s + (\delta_{q,0} + q) n \Phi_q \mathcal{Q}_q + \frac{R^2}{L^2} \Xi_q}, q_a = (0, p_a),$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2} \frac{\left(p_a \mathcal{T}s + \frac{R_a^2}{L^2} \Xi_1\right) \left(p \mathcal{T}s + n \Phi_1 \mathcal{Q}_1 + \frac{R^2}{L^2} \Xi_1\right)}{\left(p \mathcal{T}s + \frac{R^2}{L^2} \Xi_1\right) \left((n+p) \mathcal{T}s + n \Phi_1 \mathcal{Q}_1 + \frac{R^2}{L^2} \Xi_1\right)}, q = 1.$$

$$\tanh \eta = \frac{\sum_{a=1}^b \Omega_a R_a}{\sqrt{f(R)}}$$

$$J_a = V_p \frac{\Omega_a R_a^2}{|k|} \mathcal{T} \text{scosh } \eta$$

$$\frac{R_a}{L} \leq \left(\frac{p_a \mathcal{T}s + q_a n \Phi_q \mathcal{Q}_q}{\Xi_q}\right)^{1/2} \vee \\ \frac{R_a}{L} \geq \left(\frac{p_a \mathcal{T}s + q_a n \Phi_q \mathcal{Q}_q}{\Xi_q}\right)^{1/2} \wedge \left(\frac{(n+p) \mathcal{T}s + (\delta_{q,0} + q) n \Phi_q \mathcal{Q}_q}{\Xi_q}\right)^{1/2} < \frac{R}{L} < 1$$

$$\frac{R_a}{L} \leq \left(\frac{p_a \mathcal{T}s}{\Xi_1}\right)^{1/2} \vee \frac{R_a}{L} \geq \left(\frac{p_a \mathcal{T}s}{\Xi_1}\right)^{1/2} \wedge \\ \left(\frac{R}{L} \leq \left(\frac{p \mathcal{T}s + n \Phi_1 \mathcal{Q}_1}{\Xi_1}\right)^{1/2} \vee \left(\frac{(n+p) \mathcal{T}s}{\Xi_1}\right)^{1/2} < \frac{R}{L} < 1\right)$$

$$M = \sum_a^b \Omega_a J_a + \Phi_H^{(q)} Q_q \\ \Rightarrow M = \sqrt{f(R)} \frac{J}{L} + \sqrt{Nf(R)} Q_0$$

$$J \equiv \sum_a^b J_a$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2} \frac{p_a + (1+p) \frac{R_a^2}{L^2}}{p + (1+p) \frac{R^2}{L^2}} \equiv \frac{f(R)}{R_a^2} \rho_a$$

$$\sum_{a=1}^b \Omega_a^2 R_a^2 = f(R)$$

$$l = \frac{1}{\sqrt{f(R)}} \partial_t + \sum_{a=1}^b \left(\frac{\rho_a}{R_a^2}\right)^{1/2} \partial_{\Phi_a}$$

$$l_a\chi^a=\sum_{a=1}^bR_a\sqrt{\rho_a}$$

$$J_a=V_p\mathcal{K}\sum_{a=1}^bR_a\sqrt{\rho_a}$$

$$M=\frac{q+1}{q}\left(\sum_{a=1}^b\frac{\sqrt{f(R)}J_a}{R_a\sqrt{\rho_a}}+\frac{\mathcal{T}_{tot}}{q+1}\right)$$

$$\lim_{\alpha\rightarrow\infty}\left(\lim_{\Omega\rightarrow 0}\Omega^2\right)\Rightarrow\frac{R_a^2}{L^2}=\frac{p_a}{1+p}\vee\frac{R^2}{L^2}=1$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(r)d\tau^2+f(r)^{-1}dr^2+r^2d\phi^2$$

$$k^a\partial_a=\partial_\tau+\Omega\partial_\phi$$

$$\zeta^a\partial_a=\frac{1}{|k|^2}(\Omega r^2\partial_\tau+f(r)\partial_\phi)$$

$$r_0(r)\,=\,\frac{n}{4\pi T}\,(f(r)-\Omega^2r^2)^{\frac{1-N}{2}}\!\left(f(r)-\Omega^2r^2-\frac{\Phi_H^2}{N}\right)^{\frac{N}{2}}$$

$$\tanh \alpha(r) = \frac{\Phi_H}{\sqrt{N}\sqrt{f(r)-\Omega^2r^2}}$$

$$r=r_{\max}=\sqrt{\frac{1-\frac{\Phi_H^2}{N}}{\Omega^2-\frac{1}{L^2}}},$$

$$\begin{aligned}\tilde{I}_E&=\frac{\Omega_{(n+1)}}{8G}\Big(\frac{n}{4\pi T}\Big)^n\int_0^{r_{\max}}drr(f(r)-\Omega^2r^2)^{\frac{n-nN}{2}}\Big(f(r)-\Omega^2r^2-\frac{\Phi_H^2}{N}\Big)^{\frac{nN}{2}}\\&=\frac{\Omega_{(n+1)}}{8G}\Big(\frac{n}{4\pi T}\Big)^n\frac{x^{2+nN}}{(2+nN)\Big(\Omega^2-\frac{1}{L^2}\Big)}{}_2F_1\Big(1,\frac{1}{2}(N-1)n;\frac{Nn}{2}+2;x^2\Big)\end{aligned}$$

$$x\equiv\left(1-\frac{\Phi_H^2}{N}\right)^{1/2}$$

$$r_0(r)\,=\,\frac{n}{4\pi T}\,(f(r)-\Omega^2r^2)^{\frac{1}{2}}\!\left(1-\frac{\Phi_H^2}{Nf(r)(2\pi r)^2}\right)^{\frac{N}{2}}$$

$$\tanh \alpha(r) = \frac{\Phi_H}{2\pi\sqrt{N}\sqrt{f(r)}r}$$

$$r_{\min} = \frac{L}{\sqrt{2}} \sqrt{\sqrt{1 + \frac{\Phi_H^2}{NL^2\pi^2}} - 1} \leq r \leq r_{\max} = \sqrt{\frac{1}{\Omega^2 - \frac{1}{L^2}}},$$

$$\tilde{I}_E = \frac{\Omega_{(n+1)}}{8G} \left(\frac{n}{4\pi T}\right)^n \int_{r_{\min}}^{r_{\max}} dr r (f(r) - \Omega^2 r^2)^{\frac{n}{2}} \left(1 - \frac{\Phi_H^2}{Nf(r)(2\pi r)^2}\right)^{\frac{nN}{2}}$$

$$\Phi_H \rightarrow \sqrt{N} \text{ as } \alpha \rightarrow \infty$$

$$r_{\max}\sqrt{\Omega^2-1/L^2}\rightarrow 0$$

$$\Phi_H \leq \frac{2\pi\Omega\sqrt{N}}{\Omega^2-1/L^2}$$

$$r_{\max}\sqrt{\Omega^2+1/L^2}\rightarrow 0$$

$$r_{\max}=L\sqrt{1-\Phi_H^2/N}$$

$$r_{\min} = \frac{L}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \frac{\Phi_H^2}{NL^2\pi^2}}} \leq r \leq r'_{\max} = \frac{L}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{\Phi_H^2}{NL^2\pi^2}}},$$

$$T_{ab} = \mathcal{T} s u_a u_b - \mathcal{G} \gamma_{ab} - \sum_q^p \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)},$$

$$\varepsilon = \frac{n+1}{n} \mathcal{T} s + \sum_q \Phi_q \mathcal{Q}_q \Rightarrow \mathcal{G} = \frac{1}{n} \mathcal{T} s$$

$$T_{ab} = \mathcal{T} s \left(u_a u_b + \frac{1}{n} \gamma_{ab}\right) - \sum_q \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)},$$

$$(D-3)M-(D-2)(TS+\Omega J)-\sum_q^p (D-q-3)\Phi_H^{(q)}Q_q=\mathcal{T}_{tot},$$

$$\mathcal{T}_{\rm tot} = - \int_{\mathcal{B}_p} dV_p \left(R_0 T + (D-2) \mathcal{L}_p + (T^{\mu\nu} + \mathcal{V}^{\mu\nu}) n_\mu \xi_\nu \Big|_{x^\mu = X^\mu} \right)$$

$$\mathcal{L}_p \equiv \mathcal{Q}_p \mathbb{P}[A_{l-1}]$$

$$\mathcal{T} s \left(\dot{u}^\mu - \frac{1}{n} K^\mu \right) - \sum_q \Phi_q \mathcal{Q}_q \hat{K}_{(q)}^\mu = \mathcal{F}^\mu$$

$$\mathcal{F}^\mu = \perp_\nu{}^\mu F^{\nu a_1 \dots a_{q+1}} \mathcal{J}_{a_1 \dots a_{q+1}}/(q+1)!$$

$$\hat{K}_{(0)}^\mu = -\perp^\mu_\nu \dot{u}^\nu, \hat{K}_{(q)}^\mu = -\perp^\mu_\nu (\dot{u}^\nu - \dot{v}^\nu), \hat{K}_{(p)}^\mu = K^\mu,$$

$$I = - \int_{\mathcal{W}_{p+1}} \mathcal{Q}_p(\star_{(p+1)} \mathbf{1} - \mathbb{P}[A_{l-1}])$$

$$M=\Omega J+\Phi_H^{(p)}Q_p(1-\tilde{A}), \tilde{A}\equiv \frac{A_{\sigma^0}\dots\sigma^p}{R_0R^p}$$

$$R_0V_pQ_p=\Phi_H^{(p)}Q_p$$

$$\begin{array}{l} M \, = \, \Phi_H Q_p ((n+p+1)\tilde{A} + p + 1) - \mathcal{T}_{tot} \\ \Omega J \, = \, \Phi_H Q_p ((p+n)\tilde{A} + p) - \mathcal{T}_{tot} \end{array}$$

$$M=\Omega J+\sum_q^p\Phi_H^{(q)}Q_q-\Phi_H^{(p)}Q_p\tilde{A}$$

$$T_{ab} = \mathcal{K} l_a l_b - \sum_q \Phi_q \mathcal{Q}_q \hat{h}_{ab}^{(q)}$$

$$\mathcal{T}_{tot} \, = \, - \int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} - \sum_q q \Phi_q \mathcal{Q}_q + \tilde{\mathcal{V}}_{\mathcal{T}} \right)$$

$$M \, = \, \int_{\mathcal{B}_p} dV_p R_0 \left(\mathcal{K} + \sum_q \Phi_q \mathcal{Q}_q + \tilde{\mathcal{V}}_M \right)$$

$$J \, = \, \int_{\mathcal{B}_p} dV_p (\mathcal{R}\mathcal{K} + \tilde{\mathcal{V}}_J)$$

$$\tilde{\mathcal{V}}_M = \mathcal{V}^{\mu\nu} n_\mu \xi_\nu / R_0$$

$$\tilde{\mathcal{V}}_{\mathcal{T}} = \tilde{\mathcal{V}}_M + (D-2)\mathcal{L}_p$$

$$M=V_pR_0\left(\tilde{\mathcal{V}}_M-\frac{\tilde{\mathcal{V}}_J}{\mathcal{R}}+\sum_q^p\Phi_q\mathcal{Q}_q\right)+R_0\frac{J}{\mathcal{R}}$$

$$\sigma^0=t=\tau, r=R, \sigma_i=\theta_i, \sigma_j=\mu_i, \zeta=\rho_i=0$$

$$\dot{u}^r \, = \, \cosh^2 \, \eta \left(\frac{R}{\tilde{L}^2} - \Omega^2 R \right), \dot{v}^r = \frac{R}{\tilde{L}^2} \sinh^2 \, \eta - \frac{f(R)}{R} \cosh^2 \, \eta$$

$$K^r \, = \, - \left(\frac{R}{\tilde{L}^2} + \frac{p}{R} f(R) \right), \mathcal{F}_{p+2}^r = \frac{l-1}{\tilde{L}} \sqrt{f(R)} \mathcal{Q}_p$$

$$\Omega^2 = \frac{f(R)}{R^2} \frac{p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p} - \frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}\mathcal{Q}_p}{(n+p)\mathcal{T}s + pn\Phi_p\mathcal{Q}_p + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p} - \frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}\mathcal{Q}_p},$$

$$\Xi_{1,p} \equiv \mathcal{T}s(1+n+p) + n(1+p)\Phi_p\mathcal{Q}_p + 2n\Phi_1\mathcal{Q}_1$$

$$\mathcal{V}^{\mu\nu}n_{\mu}\chi_{\nu}\big|_{x^{\mu}=X^{\mu}}=0$$

$$M=V_p\left(R_0\left(\cosh^2\eta(\mathcal{T}s+\delta_{\bar{q},0}\Phi_{\bar{q}}\mathcal{Q}_{\bar{q}})+\frac{\mathcal{T}s}{n}+\Phi_p\mathcal{Q}_p+(1-\delta_{\bar{q},0})\Phi_{\bar{q}}\mathcal{Q}_{\bar{q}}\right)-\frac{R\mathcal{Q}_p}{\tilde{L}}\right)\\ \mathcal{T}_{\text{tot}}=-\frac{V_p}{n}\left(R_0\frac{R^2}{L^2}\Xi_{\bar{q},p}+n\mathcal{Q}_p\frac{R}{L}(R_0^2(l-1)+1)\right)$$

$$\sigma^0=t=\tau, r=\theta_i=0, \sigma_i=\rho_i, \sigma_j=\mu_i, \zeta=\zeta_0$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-d\tau^2+R^2d\Omega_{m-2}$$

$$\mathcal{V}^{\mu\nu}n_{\mu}\xi_{\nu}=0 \text{ since } A^t\cdot\xi_t=0$$

$$\mathcal{V}^{\mu\nu}n_{\mu}\chi_{\nu}=0 \text{ since } A^{\chi_i}\cdot J^t=0$$

$$R_0\equiv\sqrt{-\xi^2}\Big|_{\mathcal{W}_{p+1}}=1$$

$$\Omega^2=\frac{1}{R^2}\frac{p(\mathcal{T}s+n\Phi_p\mathcal{Q}_p)+n\Phi_1\mathcal{Q}_1}{(n+p)\mathcal{T}s+pn\Phi_p\mathcal{Q}_p+n\Phi_1\mathcal{Q}_1},$$

$$d\Omega_{m-2}^2=d\delta_1^2+\cos^2\delta_1d\delta_2^2+\sin^2\delta_1d\Omega_{m-4}^2$$

$$\sigma^0=t=\tau, r=R_1, \sigma_1=\theta_1, \theta_{2,3}=\pi/2, \zeta=\zeta_0, \sigma_2=\phi, \delta_1=\delta_0, \sigma_3=\delta_2, \rho_1=0$$

$$\sigma^0=t=\tau, r=R_1, \sigma_i=\theta_i, \sigma_j=\mu_i, \zeta=\zeta_0, \sigma_5=\phi, \delta_1=\delta_0, \sigma_6=\delta_2, \rho_1=0$$

$$d\Omega_5^2=d\omega_1^2+\cos^2\omega_1d\omega_2^2+\sin^2\omega_1d\Omega_3^2$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R_1)d\tau^2+R_1^2d\Omega_{p_a}^2+R_2^2d\phi^2+R_3^2d\delta_2^2$$

$$k^a\partial_a=\partial_\tau+\Omega_1\partial_\Theta+\Omega_2\partial_\phi+\Omega_3\partial_{\delta_2}, \zeta^a\partial_a=\frac{1}{|k|^2}\left(\sum_{a=1}^3\Omega_a^2R_a^2\partial_\tau+f(R_1)(\Omega_1\partial_\Theta+\Omega_2\partial_\phi+\Omega_3\partial_{\delta_2})\right),$$

$$\dot{u}^r=\frac{f(R_1)}{|k|^2}\left(\frac{R_1}{L^2}-\Omega_1^2R_1\right), \dot{u}^\zeta=\frac{R_2}{L^2|k|^2}\left(\Omega_2^2c_\zeta+\frac{\Omega_3^2R_3^2}{c_\zeta}\right), \dot{u}^{\delta_1}=\frac{\Omega_3^2R_3c_{\delta_1}}{|k|^2},\\ \dot{v}^r=\frac{1}{|k|^2}\left(\frac{R_1}{L^2}\sum_{a=1}^2\Omega_a^2R_a^2-\frac{\Omega_1^2R_1f(R_1)}{\sum_{a=1}^2\Omega_a^2R_a^2}\right), \dot{v}^\zeta=\frac{f(R_1)}{\sum_{a=1}^3\Omega_a^2R_a^2}\dot{u}^\zeta, \dot{v}^{\delta_1}=\frac{f(R_1)\dot{u}^{\delta_1}}{\sum_{a=1}^3\Omega_a^2R_a^2},\\ K^r=-\left(\frac{R_1}{L^2}+\frac{p_a}{R_1}f(R_1)\right), K^\zeta=\frac{c_\zeta}{R_2}+\frac{R_3^2}{R_2L^2c_\zeta}, K^{\delta_1}=\frac{c_{\delta_1}}{R_3},$$



$$c_{\delta_1} = \sqrt{L^2 - R_2^2 - R_3^2/c_\zeta^2}$$

$$c_\zeta = \sqrt{L^2 - R_2^2}$$

$$\Omega_1^2 = \frac{f(R_1)}{R_1^2} \frac{\left(p_a(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + n(p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_a^2 = \frac{f(R_1)}{R_a^2} \frac{(\mathcal{T}s + n\Phi_p\mathcal{Q}_p)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p\mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\sigma^0 = t = \tau, r = R, \omega_1 = \omega_0, \sigma_1 = \omega_2, \sigma_2 = \theta, \zeta = \zeta_0, \phi = \phi_0, \sigma_3 = \rho_1, \rho_{2,3} = \pi/2$$

$$\sigma^0 = t = \tau, r = R, \omega_1 = \omega_0, \sigma_1 = \omega_2, \sigma_2 = \theta, \zeta = \zeta_0, \phi = \phi_0, \sigma_i = \rho_i, \sigma_j = \mu_i,$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R)d\tau^2+R_1^2d\theta^2+R_2^2d\omega^2+R_3^2d\Omega_{p_a}^2,$$

$$k^a\partial_a=\partial_\tau+\Omega_1\partial_{\theta_1}+\Omega_2\partial_{\omega_2}+\Omega_3\partial_{\mathcal{P}},\zeta^a\partial_a=\frac{1}{|k|^2}\left(\sum_{a=1}^3\Omega_a^2R_a^2\partial_\tau+f(R)(\Omega_1\partial_{\omega_2}+\Omega_2\partial_\phi+\Omega_3\partial_{\mathcal{P}})\right),$$

$$\dot{u}^r=\frac{f(R)}{|k|^2}\left(\frac{R}{L^2}-\frac{\sum_a^2\Omega_a^2R_a^2}{R}\right),\dot{u}^{\omega_1}=\frac{R_1R_2}{R^2|k|^2}(\Omega_1^2-\Omega_2^2),\dot{u}^\zeta=-\frac{\Omega_3^2R_3\sqrt{L^2-R_3^2}}{L^2|k|^2}$$

$$\dot{v}^r=\frac{1}{|k|^2}\left(\frac{R}{L^2}\sum_a^3\Omega_a^2R_a^2-\frac{\sum_a^2\Omega_a^2R_af(R)}{\sum_a^3\Omega_a^2R_a^2}\right),\dot{v}^{\omega_1}=\frac{f(R)\dot{u}^{\omega_1}}{\sum_a^2\Omega_a^2R_a^2},\dot{v}^\zeta=\frac{f(R)\dot{u}^\zeta}{\sum_a^2\Omega_a^2R_a^2}$$

$$K^r=-\left(\frac{R_1}{L^2}+2\frac{f(R)}{R_1}\right),K^{\omega_1}=\frac{1}{R^2}\left(\frac{R_2}{R_1}-\frac{R_1}{R_2}\right),K^\zeta=\frac{p_a\dot{u}^\zeta}{\Omega_3^2R_3^2}$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2} \frac{\left(\mathcal{T}s + n\Phi_p\mathcal{Q}_p + \frac{R_a^2}{\tilde{L}^2}\Xi_{1,p}\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + n(p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\Omega_3^2 = \frac{f(R)}{R_3^2} \frac{\left(p_a(\mathcal{T}s + n\Phi_p\mathcal{Q}_p)\right)\left(p(\mathcal{T}s + n\Phi_p\mathcal{Q}_p) + n\Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p\mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)\left((n+p)\mathcal{T}s + p\Phi_p\mathcal{Q}_p + \Phi_1\mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2}\Xi_{1,p}\right)}$$

$$\sigma^0 = t = \tau, r = R_1, \sigma_1 = \theta_1, \theta_2 = \pi/2, \zeta = \zeta_0, \sigma_2 = \phi, \delta_1 = \delta_0, \sigma_i = \rho_i, \sigma_j = \mu_i$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R_1)d\tau^2+R_1^2d\theta_1^2+R_2^2d\phi^2+R_3^2d\Omega_3^2$$

$$k^a\partial_a=\partial_\tau+\Omega_1\partial_{\theta_1}+\Omega_2\partial_\phi+\Omega_3\partial_{\mathcal{P}},\zeta^a\partial_a=\frac{1}{|k|^2}\left(\sum_{a=1}^3\Omega_a^2R_a^2\partial_\tau+f(R)(\Omega_1\partial_{\theta_1}+\Omega_2\partial_\phi+\Omega_3\partial_{\mathcal{P}})\right)$$



$$K^\zeta = \frac{c_\zeta}{R_2} + \frac{3R_3^2}{R_2 L^2 c_\zeta}$$

$$\begin{aligned}\Omega_1^2 &= \frac{f(R_1)}{R_1^2} \frac{\left(\mathcal{T}s + n\Phi_p \mathcal{Q}_p + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + n(p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1) + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right)} \\ \Omega_2^2 &= \frac{f(R_1)}{R_2^2} \frac{\left((\mathcal{T}s + n\Phi_p \mathcal{Q}_p)\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + n(p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1) + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right)} \\ \Omega_3^2 &= \frac{f(R_1)}{R_3^2} \frac{\left(3(\mathcal{T}s + n\Phi_p \mathcal{Q}_p)\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p \mathcal{Q}_p) + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}\end{aligned}$$

$$\sigma^0 = t = \tau, r = R_1, \omega_1 = \omega_0, \sigma_1 = \omega_2, \sigma_{2,3} = \theta_{2,3}, \sigma_{4,5} = \mu_{2,3}, \zeta = \zeta_0, \sigma_6 = \phi$$

$$\delta_1 = \pi/2$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R)d\tau^2+R_1^2d\omega_2^2+R_2^2d\Omega_3^2+R_3^2d\phi^2$$

$$k^a\partial_a=\partial_\tau+\Omega_1\partial_{\omega_2}+\Omega_2\partial_\Theta+\Omega_3\partial_\phi,\zeta^a\partial_a=\frac{1}{|k|^2}\left(\sum_{a=1}^3\Omega_a^2R_a^2\partial_\tau+f(R)(\Omega_1\partial_{\omega_2}+\Omega_2\partial_\Theta+\Omega_3\partial_\phi)\right),$$

$$K^{\omega_1}=\frac{1}{R^2}\Big(\frac{R_2}{R_1}-3\frac{R_1}{R_2}\Big)$$

$$\begin{aligned}\Omega_1^2 &= \frac{f(R)}{R_1^2} \frac{\left(\mathcal{T}s + n\Phi_p \mathcal{Q}_p + \frac{R_1^2}{\tilde{L}^2} \Xi_{1,p}\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + n(p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1) + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)} \\ \Omega_2^2 &= \frac{f(R)}{R_2^2} \frac{\left(3(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + \frac{R_2^2}{\tilde{L}^2} \Xi_{1,p}\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + n(p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1) + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)} \\ \Omega_3^2 &= \frac{f(R)}{R_3^2} \frac{\left((\mathcal{T}s + n\Phi_p \mathcal{Q}_p)\right) \left(p(\mathcal{T}s + n\Phi_p \mathcal{Q}_p) + n\Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}{\left(p(\mathcal{T}s + \Phi_p \mathcal{Q}_p) + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right) \left((n+p)\mathcal{T}s + p\Phi_p \mathcal{Q}_p + \Phi_1 \mathcal{Q}_1 + \frac{R^2}{\tilde{L}^2} \Xi_{1,p}\right)}\end{aligned}$$

$$\gamma_{ab}d\sigma^ad\sigma^b=-f(R_1)d\tau^2+R_2^2d\phi^2+R_3^2d\delta_2^2$$

$$M\,=\,V_pR_0\left(\mathcal{T}s\mathrm{cosh}^2\,\eta+\frac{\mathcal{T}s}{n}+\Phi_1Q_1+\Phi_pQ_p\right)$$

$$J_a\,=\,V_p\,\frac{\Omega_aR_a^2}{|k|}\mathcal{T}s\mathrm{cosh}\,\eta$$

$$\mathcal{T}_{\mathrm{tot}}\,=\,-\frac{V_pR_0}{n}\frac{R^2}{L^2}\Xi_{1,p}$$



$$\Omega^2=\frac{f(R)}{R^2}\frac{p(1+nsh_\alpha^2)+\frac{R^2}{\tilde{L}^2}(1+n+p+n(1+p)sh_\alpha^2)-\frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}sh_\alpha ch_\alpha}{n+p(1+nsh_\alpha^2)+\frac{R^2}{\tilde{L}^2}(1+n+p+n(1+p)sh_\alpha^2)-\frac{Rn(l-1)}{\tilde{L}}\sqrt{f(R)}sh_\alpha ch_\alpha}.$$

$$r_c \ll r_{\rm int}, r_c \ll r_{\rm ext}, r_c \ll \mathcal{L}$$

$$r_{\rm int}=\left| \mathcal{R}_\gamma \right|^{-1/2}$$

$$r_{\rm ext}=\left| K^\mu K_\mu \right|^{-1/2}$$

$$\frac{r_0}{R} \ll (\cosh \alpha)^{-2/n} \text{ and } \frac{r_0}{L} \ll (\cosh \alpha)^{-2/n}.$$

$$r_c \sim N_{l-2}^{\frac{1}{l-1}} T_{l-2}^{-\frac{1}{m-1}}$$

$$1 \ll N_{D_p} \ll N, 1 \ll N_2 \ll N^2, 1 \ll N_5^2 \ll N$$

$$\prod_a^b \mathbb{S}^{p_a} \times \mathbb{S}^{n+1}$$

$$u^b\partial_b=\frac{1}{\sqrt{f(R)}}\partial_\tau, v^b_a\partial_b=\frac{1}{R_a}\partial_{\Phi_a}$$

$$\tilde{I}_E=\frac{\Omega_{n+1}V_p}{16\pi G}\Big(\frac{n}{4\pi T}\Big)^n\,f'(R)^{\frac{n+1}{2}}\Big(1-\frac{\Phi_H^2}{4\pi^4NV_q^2}\Big)^{nN/2}$$

$$c|\hat{h}^{(q)}|^{1/2}=V_q=(2\pi)^q\prod_a^qR_a\sqrt{f'(R)}$$

$$\frac{R}{L}=\left(\frac{p+qnN\sinh^2\alpha}{1+n+p+(q+1)nN\sinh^2\alpha}\right)^{1/2}$$

$$\begin{aligned}
M &= \frac{\Omega_{(n+1)} x^{nN}}{8\pi G \left(\Omega^2 - \frac{1}{L^2}\right)^2} \left(\frac{n}{4\pi T}\right)^n \left[\left(\Omega^2 - \frac{1}{L^2}\right) \frac{\Phi_H^2}{N} {}_2F_1\left(1, 1 + \frac{n}{2}(N-1); 1 + \frac{nN}{2}; x^2\right) \right. \\
&\quad \left. + x^2 \left(\frac{1}{L^2} + \frac{\frac{-1-2n}{L^2} + (1+n)\Omega^2 - \frac{n(N-1)\Phi_H^2}{L^2 N}}{2+nN} {}_2F_1\left(1, 1 + \frac{n}{2}(N-1); 2 + \frac{nN}{2}; x^2\right) \right) \right. \\
&\quad \left. + \frac{2\left(\frac{n+1}{L^2} - \Omega^2\right) x^4}{(2+nN)(4+nN)} {}_2F_1\left(2, 1 + \frac{n}{2}; 3 + \frac{nN}{2}; x^2\right) \right] \\
J &= \frac{\Omega_{(n+1)} \Omega}{4G \left(\Omega^2 - \frac{1}{L^2}\right)^2} \left(\frac{n}{4\pi T}\right)^n \left[\frac{\Phi_H^2 x^{nN+2}}{N(nN+2)} {}_2F_1\left(2, 1 + \frac{n}{2}(N-1); \frac{nN}{2} + 2; x^2\right) \right. \\
&\quad \left. + \frac{nx^{nN+4}}{(nN+2)(nN+4)} {}_2F_1\left(2, 1 + \frac{n}{2}(N-1); \frac{nN}{2} + 3; x^2\right) \right], \\
S &= \frac{n\Omega_{(n+1)}}{8GT} \left(\frac{n}{4\pi T}\right)^n \frac{x^{2+nN}}{(2+nN) \left(\Omega^2 - \frac{1}{L^2}\right)} {}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2} + 2; x^2\right) \\
Q &= \frac{\Phi_H \Omega_{(n+1)}}{8GN} \left(\frac{n}{4\pi T}\right)^n \frac{x^{nN}}{\left(\Omega^2 - \frac{1}{L^2}\right)} {}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2} + 1; x^2\right), \\
\mathcal{T}_{\text{tot}} &= \frac{\Omega_{(n+1)} x^{nN+2}}{8G \left(\Omega^2 - \frac{1}{L^2}\right)^2} \left[\left(\Omega^2 - \frac{1}{L^2}\right) \frac{\Phi_H^2}{N} {}_2F_1\left(1, 1 + \frac{n}{2}(N-1); 1 + \frac{nN}{2}; x^2\right) \right. \\
&\quad \left. + \frac{\frac{4+n}{L^2} - (2+n)\Omega^2}{(2+n)} + \frac{n}{(2+n)(2+nN)} \left((2+n) \left(\Omega^2 - \frac{1}{L^2}\right) \right. \right. \\
&\quad \left. \left. + \left(\frac{4+n-2N}{L^2} - \Omega^2(2+n) \right) \frac{\Phi_H^2}{N} \right) \times {}_2F_1\left(1, 1 + \frac{n}{2}(N-1); 2 + \frac{nN}{2}; x^2\right) \right] \\
x &\equiv \left(1 - \frac{\Phi_H^2}{N}\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
M &= \frac{\Omega_{(n+1)}}{8G} \left(\frac{n}{4\pi T} \right)^n \int_{r_{\min}}^{r_{\max}} dr I_r \left(f(r) - \Omega^2 r^2 + n + n \frac{\Phi_H^2}{4\pi^2 r^2} \frac{f(r) - \Omega^2 r^2}{f(r) - \frac{\Phi_H^2}{N(2\pi r)^2}} \right) \\
J &= \frac{n\Omega_{(n+1)}}{8G} \left(\frac{n}{4\pi T} \right)^n \int_{r_{\min}}^{r_{\max}} dr r^2 I_r \\
S &= \frac{n\Omega_{(n+1)}}{8GT} \left(\frac{n}{4\pi T} \right)^n \int_{r_{\min}}^{r_{\max}} dr I_r \\
Q &= \frac{n\Omega_{(n+1)}\Phi_H}{16\pi G} \left(\frac{n}{4\pi T} \right)^n \int_{r_{\min}}^{r_{\max}} dr r^{-2} I_r \\
\mathcal{T}_{\text{tot}} &= -\frac{\Omega(n+1)}{16\pi G} \int_{r_{\min}}^{r_{\max}} dr I_r [(n+p)\Omega^2 r^2 - pf(r) \\
&\quad + nN \left(-f(r) + 2\Omega^2 r^2 - f(r)\Omega^2 r^2 \frac{\frac{\Phi_H^2}{N(2\pi r)^2}}{f(r) - \frac{\Phi_H^2}{N(2\pi r)^2}} \right)] \\
I_r &= r(f(r) - \Omega^2 r^2)^{\frac{n}{2}} \left(1 - \frac{\Phi_H^2}{Nf(r)(2\pi r)^2} \right)^{\frac{nN}{2}}
\end{aligned}$$

$$\sinh \alpha_i \equiv \sin \theta \sinh \alpha, \sinh \alpha_p \equiv \cos \theta \sinh \alpha$$

$$s_i \equiv \sinh \alpha_i, c \equiv \cosh \alpha$$

$$\begin{aligned}
\Omega^2 &= \frac{f(R)p(1+(7-p)s_p^2)+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}{R^2\frac{7+p(7-p)s_p^2+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}{7+p(7-p)s_p^2+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}} \\
M &= \frac{\Omega_{8-p}V_p}{16\pi G}r_0^{7-p}\left(f(R)^{3/2}\Xi_{1,p}-(7-p)\frac{R}{L}(4f(R)+1)s_p c\right) \\
J &= \frac{\Omega_{8-p}V_p R}{16\pi G}r_0^{7-p}\sqrt{p(1+(7-p)s_p^2)+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}\times \\
&\quad \times \sqrt{7+p(7-p)s_p^2+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}, \\
S &= \frac{\Omega_{8-p}V_p c}{4G\sqrt{7-p}}r^{8-p}\sqrt{7+p(7-p)s_p^2+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c} \\
T &= \frac{1}{4\pi r_0 c}\sqrt{\frac{f(R)(7-p)^3}{7+p(7-p)s_p^2+(7-p)s_1^2+\frac{R^2}{L^2}\Xi_{1,p}-4\frac{R(7-p)}{L}\sqrt{f(R)}s_p c}} \\
\mathcal{T}_{\text{tot}} &= -\frac{\Omega_{8-p}V_p}{16\pi G}\left(\sqrt{f(R)}\frac{R^2}{L^2}\Xi_{1,p}+(7-p)\frac{R}{L}(4f(R)+1)s_p c\right) \\
\Xi_{1,p} &= 8+(7-p)((1+p)s_p^2+2s_1^2)
\end{aligned}$$

$$\sin \theta = 0, \tilde{L} = \frac{L}{2}, n = 3$$

$$\Omega_a^2 = \frac{f(R)}{R_a^2} \frac{1 + 2s_p^2 + \frac{R_a^2}{L^2} \Xi_5}{7 + 2(5s_p^2 + s_1^2) + \frac{R^2}{L^2} \Xi_5}$$

$$\Omega_3^2 = \frac{f(R)}{R_3^2} \frac{3(1 + 2s_p^2)}{7 + 2(5s_p^2 + s_1^2) + \frac{R^2}{L^2} \Xi_5}$$

$$M = \frac{\Omega_3 V_5}{16\pi G} r_0^2 (f(R)^{3/2} \Xi_{1,5})$$

$$J_a = \frac{\Omega_3 V_5 R_a}{16\pi G} r_0^2 \sqrt{1 + 2s_p^2 + \frac{R_a^2}{L^2} \Xi_5} \sqrt{7 + 2(5s_p^2 + s_1^2) + \frac{R^2}{L^2} \Xi_5}$$

$$J_3 = \frac{\Omega_3 V_5 R_3}{16\pi G} r_0^2 \sqrt{3(1 + 2s_p^2)} \sqrt{7 + 2(5s_p^2 + s_1^2) + \frac{R^2}{L^2} \Xi_5}$$

$$S = \frac{\Omega_3 V_5 c r_0^3}{4G \sqrt{f(R) - \sum_a^3 \Omega_a^2 R_a^2}}$$

$$T = \frac{1}{4\pi r_0 c} \sqrt{f(R) - \sum_a^3 \Omega_a^2 R_a^2}$$

$$\mathcal{T}_{tot} = -\frac{\Omega_3 V_3}{16\pi G} \sqrt{f(R)} \frac{R^2}{L^2} \Xi_5$$

$$\Omega_a^2 = \frac{f(R_1)}{R_a^2} \frac{1 + 6s_2^2}{7 + 2s_p^2 + \frac{R_1^2}{\tilde{L}^2} \Xi_2}$$

$$M = \frac{\Omega_7 V_2}{16\pi G} r_0^7 (f(R)^{3/2} \Xi_2)$$

$$J_a = \frac{\Omega_7 V_2 R_a}{16\pi G} r_0^7 \sqrt{1 + 6s_2^2} \sqrt{7 + 2s_p^2 + \frac{R_1^2}{\tilde{L}^2} \Xi_2}$$

$$S = \frac{\Omega_7 V_2 c r_0^7}{4G \sqrt{f(R) - \sum_a^2 \Omega_a^2 R_a^2}}$$

$$T = \frac{1}{4\pi r_0 c} \sqrt{f(R) - \sum_a^2 \Omega_a^2 R_a^2}$$

$$\mathcal{T}_{tot} = -\frac{\Omega_7 V_2}{16\pi G} \sqrt{f(R)} \frac{R^2}{L^2} \Xi_2$$

$$d\varepsilon = \mathcal{T}ds + \sum_q^{p-1} \Phi_q dQ_q$$

$$Q_2 u^{[a} v^b w^{c]} \partial_a f|_{\partial \mathcal{W}_{p+1}} = 0$$



$$\mathcal{Q}_2 v^a \partial_a f|_{\partial \mathcal{W}_{p+1}} = 0$$

$$v^{ia}D_{(a}v_{b)i}=\dot{v}_b=\delta^{ij}v_i^a\zeta_{ka}D_b\mathbb{M}_j{}^k$$

$$v_i^a\zeta_{ka}=(\mathbb{M}^{-1})_k{}^j\delta_{ij}$$

$$\dot{v}_b=(\mathbb{M}^{-1})_k{}^jD_b\mathbb{M}_j{}^k=D_b\ln|\mathbb{M}|$$

$$I=\frac{1}{16\pi}\int\,d^Dx\sqrt{-g}\Big[R-2\Lambda+\frac{g}{D-4}\mathcal{G}+F_{\mu\alpha}^{(a)}F^{(a)\mu\alpha}\Big]+\int_{\Sigma}\mu\sqrt{-\gamma}d\tau d\sigma$$

$$\Lambda=\frac{-(D-1)(D-2)}{2l^2}$$

$$\mathcal{G}=R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$F_{\mu\nu}^{(a)}=\partial_\mu A_\nu^{(a)}-\partial_\nu A_\mu^{(a)}-i\left[A_\mu^{(a)},A_\nu^{(a)}\right]$$

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega_{2,\kappa}^2,d\Omega_{2,\kappa}^2=\frac{dx^2}{1-\kappa x^2}+x^2d\varphi^2$$

$$f(r)=1-r^2\psi(r)$$

$$\nabla_\mu F^{(a)\mu\nu}=0$$

$$\mathbf{A}^{(a)}=\frac{i}{2}h(r)dt\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$h(r)=C_2+Q\int^r\frac{1}{u^{D-2}}\frac{1}{u^{D-2}}du$$

$$h'(r)=\frac{Q}{r^{D-2}}$$

$$I=\frac{\Omega_{D-2}(D-2)}{16\pi}$$

$$\int\,dtdr\left[r^{D-1}\psi(1+g(D-3)\psi)+\frac{r^{D-1}}{l^2}+\frac{2Q^2r^{3-D}}{(D-3)(D-2)}-\frac{2ar}{(D-2)}\right]'$$

$$\psi+g(D-3)\psi^2=\frac{16\psi M}{(D-2)r^{D-1}\Omega_{D-2}}-\frac{1}{l^2}-\frac{2Q^2r^{4-2D}}{(D-3)(D-2)}-\frac{2a}{(D-2)r^{D-2}},$$

$$f(r)=1-\frac{r^2}{2g(D-3)}[-1\pm\sqrt{1-4(D-3)g\left(\frac{1}{l^2}+\frac{2Q^2r^{4-2D}}{(D-3)(D-2)}-\frac{16\pi Mr^{1-D}}{(D-2)\Omega_{D-2}}+\frac{2ar^{2-D}}{D-2}\right)}]$$

$$M=\frac{(D-2)\Omega_{D-2}}{16\pi}m$$

$$Q^2 = \frac{(D-2)(D-3)}{2} q^2$$

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}$$

$$\frac{(D-3)g}{l^2} = \lambda_{gb}$$

$$\frac{2a}{D-2} = A,$$

$$f(r) = 1 + \frac{r^2}{2\lambda_{gb}l^2} \left[1 \mp \sqrt{1 - 4\lambda_{gb} \left(1 + \frac{q^2l^2}{r^{2D-4}} - \frac{ml^2}{r^{D-1}} + \frac{Al^2}{r^{D-2}} \right)} \right].$$

$$f(r) = 1 + \frac{r^2}{2g} \left(1 - \sqrt{1 - 4g \left(-\frac{\Lambda}{3} + \frac{Q^2}{r^4} - \frac{2m}{r^3} + \frac{a}{r^2} \right)} \right)$$

$$m = -\frac{r_+^3\Lambda}{6} + \frac{ar_+}{2} + \frac{Q^2}{2r_+} + \frac{r_+}{2} + \frac{g}{2r_+},$$

$$K=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$K=\frac{4}{r^4}(f(r)-1)^2+\frac{4f'(r)^2}{r^2}+f''(r)^2$$

$$K=\frac{(1-\sqrt{\chi})^2}{g^2}+\frac{4\left(\frac{r(1-\sqrt{\chi})}{g}+\frac{2r^2\mathcal{Y}}{\sqrt{\chi}}\right)^2}{r^2}+ \left(\frac{1-\sqrt{\chi}}{g}+\frac{8r\mathcal{Y}}{\sqrt{\chi}}+\frac{8r^2\mathcal{Y}^2g}{\chi^{\frac{3}{2}}}-\frac{2r\mathcal{C}}{\sqrt{\chi}}\right)^2$$

$$\chi=1-4g\left(-\frac{\Lambda}{3}+\frac{Q^2}{r^4}-\frac{2m}{r^3}+\frac{a}{r^2}\right),$$

$$\mathcal{Y}=-\frac{2Q^2}{r^5}+\frac{3m}{r^4}-\frac{a}{r^3}$$

$$\mathcal{C}=-\frac{10Q^2}{r^5}+\frac{12m}{r^4}-\frac{3a}{r^3}$$

$$H=-\frac{r_h^3\Lambda}{6}+\frac{r_ha}{2}+\frac{g}{2r_h}+\frac{r_h}{2}+\frac{Q^2}{2r_h}=\frac{4r_h^3\pi P}{3}+\frac{r_ha}{2}+\frac{g}{2r_h}+\frac{r_h}{2}+\frac{Q^2}{2r_h}$$

$$T = \frac{\mathcal{K}(r_h)}{2\pi} = \frac{1}{4\pi} \left(\frac{\partial f(r)}{\partial r} \right)_{r=r_h} = \frac{-r_h^4 \Lambda + (a+1)r_h^2 - Q^2 - g}{4\pi r_h(r_h^2 + 2g)} = \frac{8r_h^4 \pi P + (a+1)r_h^2 - Q^2 - g}{4\pi r_h(r_h^2 + 2g)}$$

$$S = \int_0^{r_h} \frac{1}{T} \frac{\partial H}{\partial r_h} dr_h$$

$$S = \pi r_h^2 + 4\pi g \ln(r_h)$$

$$dH(S, P, Q, a, g) = TdS + VdP + \phi dQ + \mathcal{A}da + \mathcal{B}dg$$

$$V = \left(\frac{\partial H}{\partial P} \right)_{S, Q, a, g} = \frac{4\pi}{3} r_h^3$$

$$\phi = \left(\frac{\partial H}{\partial Q} \right)_{S, P, a, g} = \frac{Q}{r_h}$$

$$\mathcal{A} = \left(\frac{\partial H}{\partial a} \right)_{S, P, Q, g} = \frac{1}{2} r_h$$

$$\mathcal{B} = \left(\frac{\partial H}{\partial g} \right)_{S, P, Q, a} = \frac{1}{2r_h}$$

$$H = 2TS - 2PV + Q\phi + a\mathcal{A} + g\mathcal{B}$$

$$P = \frac{T}{2r_h} + \frac{gT}{r_h^3} - \frac{a+1}{8\pi r_h^2} + \frac{Q^2}{8\pi r_h^4} + \frac{g}{8\pi r_h^4}$$

$$\left. \frac{\partial P}{\partial r_h} \right|_{T=T_c, r_h=r_c} = 0, \quad \left. \frac{\partial^2 P}{\partial^2 r_h} \right|_{T=T_c, r_h=r_c} = 0$$

$$r_c = \sqrt{\frac{3Q^2 + 3ag + Y + 6g}{a+1}}$$

$$P_c = \frac{(Q^2 ag + 3a^2 g^2 + 3YQ^2 + 4Yag + Q^2 g + 7ag^2 + Y^2 + 7Yg + 4g^2)(a+1)^2}{8\pi(3Q^2 + 3ag + Y + 6g)^2(3Q^2 + 9ag + Y + 12g)}$$

$$T_c = \frac{(Q^2 + 3ag + Y + 4g)(a+1)}{2\pi \sqrt{\frac{3Q^2 + 3ag + Y + 6g}{a+1}} (3Q^2 + 9ag + Y + 12g)}$$

$$Y = \sqrt{9Q^4 + 30Q^2 ag + 9a^2 g^2 + 48Q^2 g + 48ag^2 + 48g^2}$$



$$Z=\frac{PV}{T}.$$

$$c_P\!=\!\left(\frac{\partial H}{\partial r_h}\right)\!\left(\frac{\partial r_h}{\partial T}\right)\!=\!\frac{2\!\left(r_h^2+2g\right)^2\!\left(8r_h^4\pi P+r_h^2a-Q^2+r_h^2-g\right)\pi}{8\pi Pr_h^6+48\pi gPr_h^4-ar_h^4+3Q^2r_h^2+2agr_h^2-r_h^4+2Q^2g+5r_h^2g+2g^2}$$

$$\delta = |r_c - r_d| \simeq 0.01$$

$$G=\frac{\left(-8r_h^4\pi P-(a+1)r_h^2+Q^2+g\right)\left(4\ln\left(r_h\right)g+r_h^2\right)}{4r_h\left(r_h^2+2g\right)}+\frac{4r_h^3\pi P}{3}+\frac{r_ha}{2}+\frac{g}{2r_h}+\frac{r_h}{2}+\frac{Q^2}{2r_h}$$

$$\begin{aligned}C_v&=\left(T\frac{\partial S}{\partial T}\right)_v\propto |t|^{-\alpha},\\ \eta&=v_l-v_s\propto |t|^{\beta},\\ \kappa_T&=-\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T\propto |t|^{-\gamma}\end{aligned}$$

$$|P-P_c|\propto |v-v_c|^{-\delta}.$$

$$p=\frac{T_c}{v_cp_c}\frac{\tau}{v}+\frac{8gT_c}{v_c^3p_c}\frac{\tau}{v^3}-\frac{a+1}{2\pi v_c^2p_c}\frac{1}{v^2}+\frac{2Q^2+2g}{\pi v_c^4p_c}\frac{1}{v^4}.$$

$$p=1+\Theta t+\Phi t\omega+\Omega\omega^3+\cdots$$

$$\begin{aligned}\Theta&=\frac{T_c(v_c^2+8g)}{v_c^3p_c}\\ \Phi&=-\frac{T_c(v_c^2+24g)}{v_c^3p_c}\end{aligned}$$

$$\Omega=-\frac{\pi T_c v_c^3+80\pi g T_c v_c-2a v_c^2+40Q^2-2v_c^2+40g}{\pi v_c^4 p_c}$$

$$1+\Theta t+\Phi t\omega_l+\Omega\omega_l^3=1+\Theta t+\Phi t\omega_s+\Omega\omega_s^3$$

$$\int_{\omega_s}^{\omega_l}\omega\frac{dp}{d\omega}d\omega=0$$

$$\Phi t(\omega_l^2-\omega_s^2)=-\frac{3}{2}\Omega(\omega_l^4-\omega_s^4)$$

$$\omega_l=-\omega_s=\sqrt{-\frac{\Phi t}{\Omega}}$$

$$\eta=v_l-v_s=v_c(\omega_l-\omega_s)=2v_c\omega_l\propto\sqrt{-t}$$



$$\kappa_T = -\frac{1}{v_c(1+\omega)}\frac{\partial v}{\partial \omega}\left(\frac{\partial \omega}{\partial P}\right)_T \propto \left(-\frac{1}{\frac{\partial p}{\partial \omega}}\right)_{\omega=0} = -\frac{t^{-1}}{\Phi}$$

$$p=1+\Omega\omega^3$$

$$\alpha+2\beta+\gamma=2, \alpha+\beta(\delta+1)=2, (2-\alpha)=\delta(1-\alpha)-1\\ \gamma=\beta(\delta-1), \gamma(\delta+1)=(\delta-1)(2-\alpha)$$

$$\frac{dP}{dT}=\frac{T\Delta S}{\Delta V}=\frac{(\Delta H)_P}{\Delta V}$$

$$\mu=\left(\frac{\partial T}{\partial P}\right)_H$$

$$\mu=\frac{1}{C_P}\Big[T\left(\frac{\partial V}{\partial T}\right)_P-V\Big]$$

$$T_i=V\left(\frac{\partial T}{\partial V}\right)_P.$$

$$I_{\rm QT}=\frac{1}{16\pi G}\int\;{\rm d}^Dx\sqrt{-g}\left(R-2\Lambda+\sum_n^{n_{\rm max}}\alpha_n{\cal Z}_n\right)$$

$$\Lambda = -(D-1)(D-2)/(2L^2)$$

$$\mathrm{d} s^2=-f(r) \mathrm{d} t^2+\frac{\mathrm{d} r^2}{f(r)}+\frac{r^2}{L^2} \mathrm{~d} \Sigma_{D-2, k}^2, \mathrm{~d} \Sigma_{D-2, k}^2=\left\{\begin{array}{ll} L^2 \mathrm{~d} \Omega_{D-2}^2, & \text { for } k=1 \\ \mathrm{d} \mathbf{x}_{D-2}^2, & \text { for } k=0 \\ L^2 \mathrm{~d} \Xi_{D-2}^2, & \text { for } k=-1 \end{array}\right.$$

$$\sum_{n=0}^{n_{\max}}\alpha_n r^{D-2n-3}(k-f(r))^n=m$$

$$f(r)\propto \frac{1}{r^{\frac{D-2n_{\max}-1}{n_{\max}}}}+\cdots$$

$$h(\psi)=\frac{1}{L^2}+\psi+\sum_{n=2}^{\infty}\frac{D-2n}{D-2}\alpha_n\psi^n=\frac{m}{r^{D-1}},\text{ with }\psi\colon=\frac{k-f(r)}{r^2}$$

$$h_{\rm H}(\psi)=\frac{1}{L^2}+\frac{\psi}{1-\alpha\psi},\qquad\text{if }\alpha_n=\frac{D-2}{D-2n}\alpha^{n-1}\\ h_{\rm D}(\psi)=\frac{1}{L^2}-\frac{\log{(1-\alpha\psi)}}{\alpha},\qquad\text{if }\alpha_n=\frac{D-2}{n(D-2n)}\alpha^{n-1}$$

$$h_{\rm B}(\psi)=\frac{1}{L^2}+\frac{\psi}{\sqrt{1-\alpha^2\psi^2}},\text{ if }\alpha_n=\frac{(1-(-1)^n)(D-2)\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}(D-2n)\Gamma\left(\frac{D+1}{2}\right)}\alpha^{n-1}.$$



$$f_{\text{H}}(r) = k - \frac{r^{D+1} - L^2 m r^2}{\alpha r^{D-1} - L^2 (r^{D-1} + \alpha m)}$$

$$f_{\text{D}}(r) = k + \frac{r^2}{\alpha} \left(e^{\alpha(L^{-2} - m r^{1-D})} - 1 \right)$$

$$f_{\text{B}}(r) = k + \frac{r^2 (r^{D-1} - L^2 m)}{\sqrt{L^4 r^{2D-2} + \alpha^2 (r^{D-1} - L^2 m)^2}}$$

$$M = \frac{(D-2)r_+^{D-1}\Sigma_{D-2,k}}{16\pi G}h(\psi_+)$$

$$T = \frac{1}{4\pi r_+} \left(\frac{(D-1)r_+^2 h(\psi_+)}{h'(\psi_+)} - 2k \right)$$

$$S = -\frac{(D-2)\Sigma_{D-2,k}k^{D/2-1}}{8G} \int \frac{h'(\psi_+)}{\psi_+^{D/2}} d\psi_+$$

$$\mathrm{d}M = T \, \mathrm{d}S$$

$$f_{\text{H}}(r) \stackrel{m \rightarrow 0}{=} k + \frac{r^2}{L^2 - \alpha} = k + \frac{r^2}{L_\star^2}$$

$$\frac{m}{L_\star^{D-3}} = \frac{y_+^D (1-\beta)^{\frac{D-3}{2}} (y_+^2 + k(1-\beta))}{y_+^2 - k\beta}$$

$$\beta = \alpha/L^2 = \alpha/(\alpha + L_\star^2)$$

$$y_+ = r_+/L = r_+ \sqrt{1 - \beta}/L_\star$$

$$\frac{m_{\text{ext}}}{L_\star^{D-3}} = \frac{k^{\frac{D-1}{2}}}{2^{\frac{D+3}{2}} \beta (1-\beta)^{\frac{D-3}{2}} (D-1)^{\frac{D-1}{2}}} \left((D-3 + 4\beta + \sqrt{(D-3)^2 + 8(D-1)\beta}) \right. \\ \left. \times \left(3 - D + 2(D-1)\beta + \sqrt{(D-3)^2 + 8(D-1)\beta} \right)^{\frac{D-1}{2}}, \right.$$

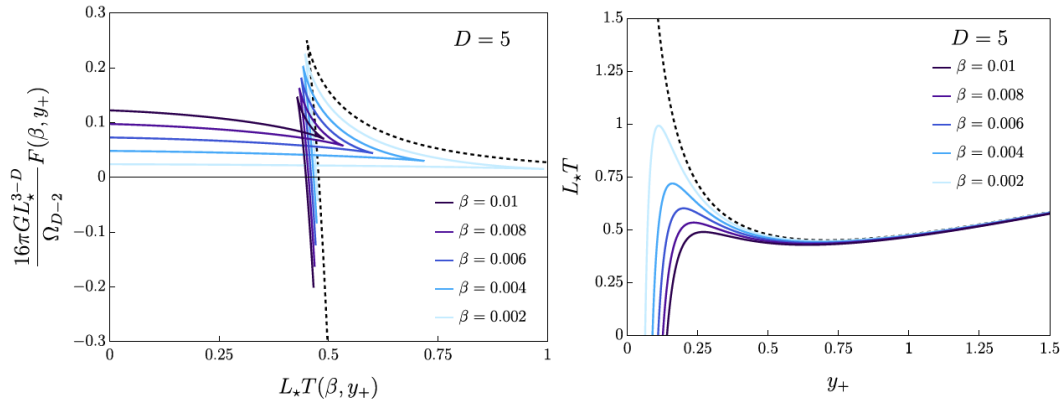
$$L_\star T = \frac{\sqrt{1-\beta} \left(k(D-3)y_+^2 + (D-1)(y_+^4 - k\beta(k(1-\beta) + 2y_+^2)) \right)}{4\pi y_+^3}.$$

$$\frac{M}{L_\star^{D-3}} = \frac{(D-2)\Sigma_{D-2,k}y_+^{D-1}}{16\pi G(1-\beta)^{\frac{D-3}{2}}} \frac{(k(\beta-1) - y_+^2)}{(k\beta - y_+^2)}$$

$$\frac{S}{L_\star^{D-2}} = \frac{\Sigma_{D-2,k}y_+^{D-2}}{4G(1-\beta)^{\frac{D-2}{2}}} {}_2F_1\left(2, 1-\frac{D}{2}, 2-\frac{D}{2}, \frac{k\beta}{y_+^2}\right).$$



$$\frac{F}{L_\star^{D-3}} = \frac{y_+^{D-5} \Sigma_{D-2}^k}{16\pi G(k\beta - y_+^2)(1 - \beta)^{\frac{D-3}{2}}} ((D-2)y_+^4(k(\beta-1) - y_+^2) - (k\beta - y_+^2)) \\ \times \left(k(D-3)y_+^2 + (D-1)(y_+^4 + k\beta(k(\beta-1) - 2y_+^2)) \right) {}_2F_1\left(2, 1 - \frac{D}{2}, 2 - \frac{D}{2}, \frac{k\beta}{y_+^2}\right)$$



$$f_D(r) \stackrel{m \rightarrow 0}{=} k + \frac{r^2}{\alpha} (e^{\alpha/L^2} - 1) = 1 + \frac{r^2}{L_\star^2},$$

$$L_\star^2 = \alpha / (e^{\alpha/L^2} - 1)$$

$$\frac{m}{L_\star^{D-3}} = \left(\frac{e^\beta - 1}{\beta} \right)^{\frac{D-3}{2}} \left[1 - \frac{1}{\beta} \log \left(1 - k \frac{\beta}{y_+^2} \right) \right] y_+^{D-1}$$

$$L_\star T = \sqrt{\frac{\beta}{e^\beta - 1}} \frac{(D-1)y_+^2 + k[(D-1)\beta - 2] + (D-1)\left(k - \frac{y_+^2}{\beta}\right) \log\left(1 - k \frac{\beta}{y_+^2}\right)}{4\pi y_+}.$$

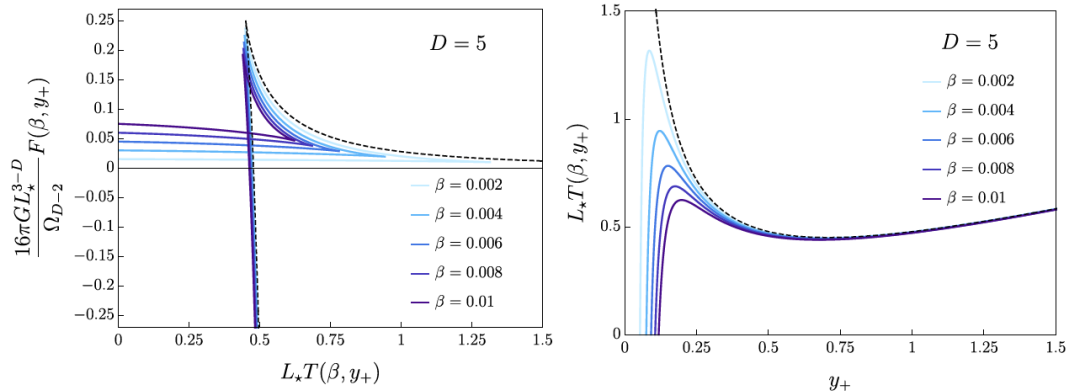
$$(D-1)y_{+, \text{ext}}^2 + k[(D-1)\beta - 2] + (D-1)\left(k - \frac{y_{+, \text{ext}}^2}{\beta}\right) \log\left(1 - k \frac{\beta}{y_{+, \text{ext}}^2}\right) = 0$$

$$\frac{M}{L_\star^{D-3}} = \frac{(D-2)\Sigma_{D-2,k}y_+^{D-1}}{16\pi G} \left(\frac{e^\beta - 1}{\beta} \right)^{\frac{D-3}{2}} \left[1 - \frac{1}{\beta} \log \left(1 - k \frac{\beta}{y_+^2} \right) \right] \\ = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G} m$$

$$\frac{S}{L_\star^{D-2}} = \frac{\Sigma_{D-2,k}y_+^{D-2}}{4G} \left(\frac{e^\beta - 1}{\beta} \right)^{\frac{D-2}{2}} {}_2F_1\left(1, 1 - \frac{D}{2}, 2 - \frac{D}{2}, \frac{k\beta}{y_+^2}\right)$$

$$\frac{F}{L_\star^{D-3}} = \frac{\Sigma_{D-2,k}y_+^{D-1}}{16\pi G} \left(\frac{e^\beta - 1}{\beta} \right)^{\frac{D-3}{2}} \left\{ (D-2) \left[1 - \frac{1}{\beta} \log \left(1 - k \frac{\beta}{y_+^2} \right) \right] \right\}$$

$$- \left[(D-1)y_+^2 + k[(D-1)\beta - 2] + (D-1) \left(k - \frac{y_+^2}{\beta} \right) \ln \left(1 - k \frac{\beta}{y_+^2} \right) \right] \\ \times {}_2F_1 \left(1, 1 - \frac{D}{2}, 2 - \frac{D}{2}, \frac{k\beta}{y_+^2} \right) \Bigg\}$$



$$f_B(r) \stackrel{m \rightarrow 0}{=} 1 + \frac{r^2}{\sqrt{L^4 + \alpha^2}} = 1 + \frac{r^2}{L_*^2},$$

$$L_*^2 = \sqrt{L^4 + \alpha^2}$$

$$\frac{m}{L_*^{D-3}} = \frac{y_+^{D-1}}{(1+\beta)^{\frac{D-3}{4}}} \left(1 + \frac{k}{\sqrt{y_+^4 - k^2\beta^2}} \right)$$

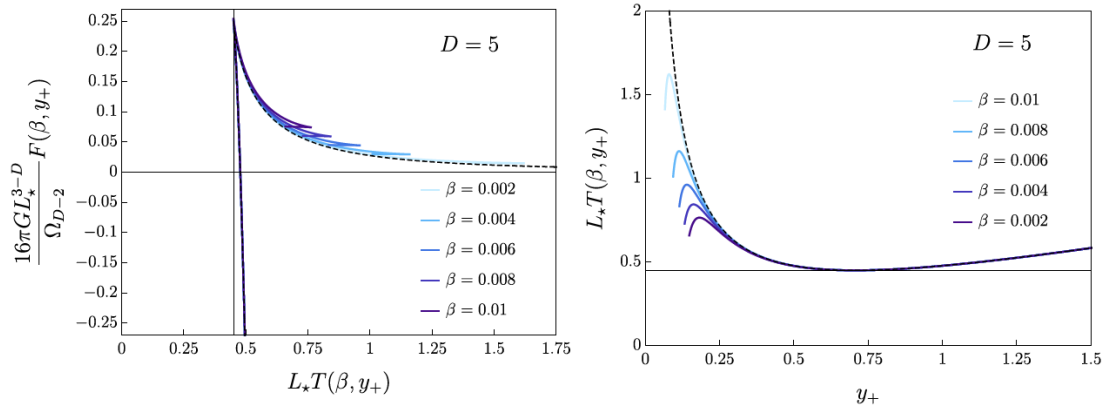
$$L_*T = (1+\beta^2)^{1/4} \frac{(D-1)((y_+^4 - k^2\beta^2)^{3/2} - k^3\beta^2) + (D-3)ky_+^4}{4\pi y_+^5}$$

$$(D-1)((y_+^4 - k^2\beta^2)^{3/2} - k^3\beta^2) + (D-3)ky_+^4 = 0$$

$$\frac{M}{L_*^{D-3}} = \frac{(D-2)\Sigma_{D-2,k}y_+^{D-1}}{16\pi G} \frac{1}{(1+\beta)^{\frac{D-3}{4}}} \left(1 + \frac{k}{\sqrt{y_+^4 - k^2\beta^2}} \right) \\ = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G} m$$

$$\frac{S}{L_*^{D-2}} = \frac{\Sigma_{D-2,k}y_+^{D-2}}{4G} \frac{1}{(1+\beta)^{\frac{D-2}{4}}} {}_2F_1 \left(\frac{3}{2}, \frac{1}{2} - \frac{D}{4}, \frac{3}{2} - \frac{D}{4}, \frac{k^2\beta^2}{y_+^4} \right)$$

$$\frac{F}{L_*^{D-3}} = \frac{\Sigma_{D-2,k}y_+^{D-7}}{16\pi G} \frac{1}{(1+\beta^2)^{\frac{D-3}{4}}} \left\{ (D-2)y_+^6 \left(1 + \frac{k}{\sqrt{y_+^4 - k^2\beta^2}} \right) - \right. \\ \left. [(D-1)((y_+^4 - k^2\beta^2)^{3/2} - k^3\beta^2) + (D-3)ky_+^4] \times \right. \\ \left. {}_2F_1 \left(\frac{3}{2}, \frac{1}{2} - \frac{D}{4}, \frac{3}{2} - \frac{D}{4}, \frac{k^2\beta^2}{y_+^4} \right) \right\}$$



$$\square \Phi = 0,$$

$$ds^2 = -f(r)dv^2 + 2\,dv\,dr + r^2\,d\Omega_{D-2}^2$$

$$\Phi = e^{-i\omega v}R(r)Y(\Omega_{D-2}),$$

$$\nabla_{\Omega_{D-2}}Y = -l(l+D-3)Y$$

$$fR'' + (Dr^{-1}f - 2i\omega - 2r^{-1}f + f')R' + (-i\omega r^{-1}D + 2i\omega r^{-1} - l(l+D-3)r^{-2})R(r) = 0$$

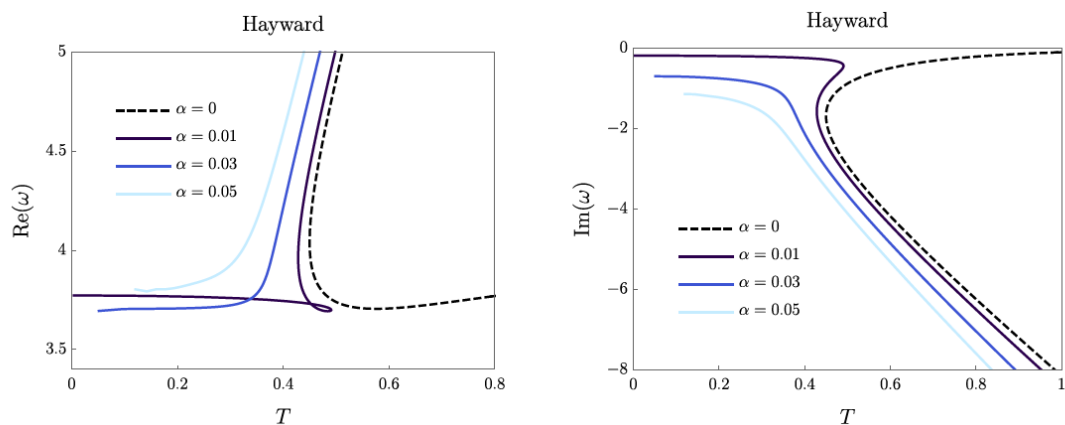
$$R(r) = C_1(1 + \mathcal{O}(r-r_+)) + C_2(r-r_+)^{2i\omega/4\pi T}(1 + \mathcal{O}(r-r_+))$$

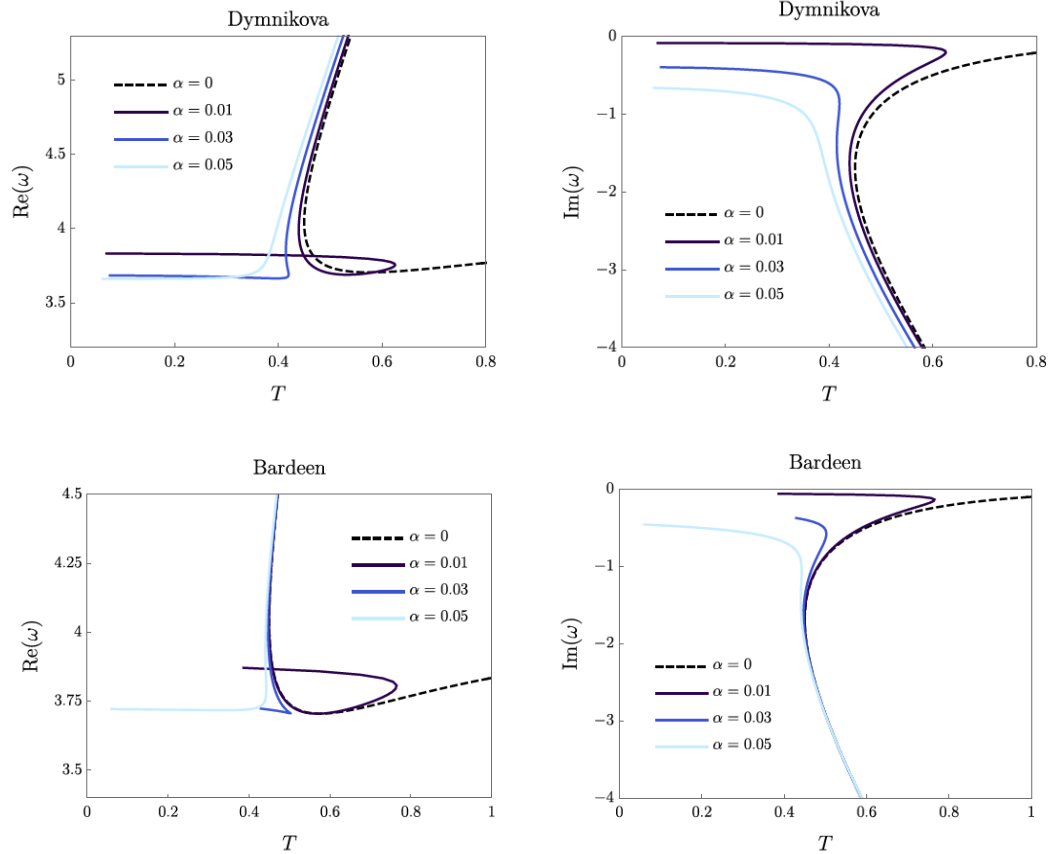
$$T=f'(r_+)/4\pi$$

$$\Phi \sim \exp\left[-i\omega\big(t+(4\pi T)^{-1}\log{(r-r_+)}\big)\right]$$

$$R(r)=\frac{D_1}{r^{\Delta_+}}\big(1+\mathcal{O}(r^{-1})\big)+\frac{D_2}{r^{\Delta_-}}\big(1+\mathcal{O}(r^{-1})\big)$$

$$\Phi \sim r^{-\Delta_+} \sim r^{-4}, \Delta_+$$





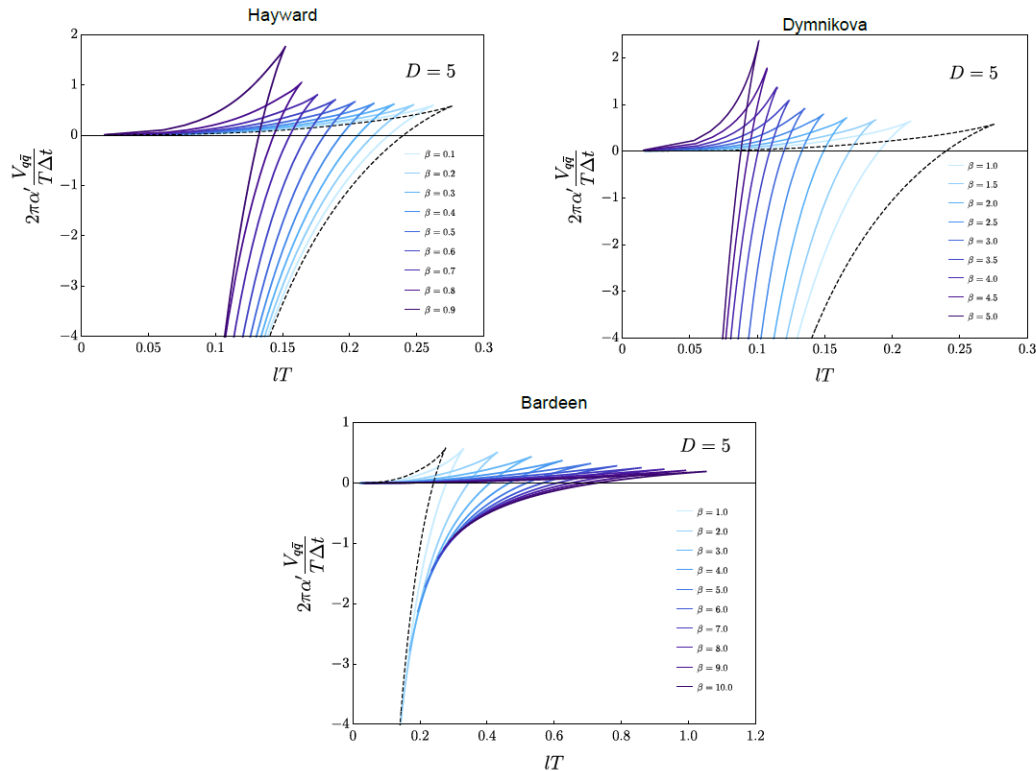
$$\langle W(\mathcal{C}) \rangle \sim e^{-\Delta t V_{q\bar{q}},}$$

$$\langle W(\mathcal{C}) \rangle \sim e^{-I_{\text{NG}}}.$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2 + dz^2)$$

$$\frac{I_{\text{reg}}(r_0)}{2\Delta t} = \int_{r_0}^{\infty} dr \left[\frac{f(r)^{1/2}r}{\sqrt{f(r)r^2 - f(r_0)r_0^2}} - 1 \right] - \int_{r_{\min}}^{r_0} dr$$

$$l(r_0) = 2 \int_{r_0}^{\infty} \frac{r_0}{r} \left(\frac{f(r_0)}{f(r)} \right)^{1/2} \frac{dr}{\sqrt{f(r)r^2 - f(r_0)r_0^2}}$$



$$I_{\text{hyperspace}_{\mathcal{A}_5 \times \mathcal{S}^5}} = \frac{1}{16\pi G} \int d^5x \sqrt{|g|} \left[R + \frac{12}{L^2} + \frac{\zeta(3)}{8} \alpha'^3 W^4 \right]$$

$$W^4 = \left(W_{abcd} W^{ebcf} + \frac{1}{2} W_{adbc} W^{efbc} \right) W^{ag}{}_{he} W_{fg}{}^{hd}$$

$$g_{ab} \rightarrow g_{ab} - \frac{\zeta(3)}{8} \alpha'^3 \left[\frac{1}{L^4} \left(\hat{\mathcal{C}}_{ab}^{(1)} - \frac{g_{ab}}{3} \hat{\mathcal{C}}^{(1)} \right) + \frac{1}{L^2} \left(\hat{\mathcal{C}}_{ab}^{(2)} - \frac{g_{ab}}{3} \hat{\mathcal{C}}^{(2)} \right) + \hat{\mathcal{C}}_{ab}^{(3)} - \frac{g_{ab}}{3} \hat{\mathcal{C}}^{(3)} \right]$$

$$\begin{aligned} \hat{\mathcal{C}}_{ab}^{(3)} = & -\frac{6935}{1584} R^{cdef} R_c{}^g{}_{ea} R_{dgfb} - \frac{18625}{19008} R^{cdef} R_{cd}{}^g R_{efgb} + \frac{7}{44} R_a{}^c{}_{b}{}^d R_{efgc} R^{efg}{}_d \\ & + \frac{2053}{1584} g_{ab} R_g{}^c{}_{h}{}^d R_c{}^e{}_{d}{}^f R_e{}^g{}_{f}{}^h + \frac{1241}{19008} g_{ab} R_{gh}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{gh} + \frac{6035}{3168} R^{cd} R^e{}_{a}{}^f{}_{c} R_{ebfd} \\ & + \frac{24215}{6336} R^{cd} R^e{}_{a}{}^f{}_{b} R_{ecfd} - \frac{1291}{12672} R^{cd} R^{ef}{}_{ac} R_{efbd} - \frac{767}{1584} R_{ab} R_{cdef} R^{cdef} \\ & + \frac{2155}{12672} g_{ab} R_{ghcd} R^{ghc}{}_e R^{de} - \frac{521}{1408} R R_{ghca} R^{ghc} + \frac{1679}{12672} g_{ab} R R_{ghcd} R^{ghcd} \\ & - \frac{25}{44} R^{gh} R_{ghad} R_b{}^d - \frac{3569}{3168} R R_{gacb} R^{gc} + \frac{7}{704} g_{ab} R^3 \\ \hat{\mathcal{C}}_{ab}^{(2)} = & \frac{15}{44} R_{acde} R_b{}^{cde} - \frac{285}{176} R_{cd} R_a{}^c{}_{b}{}^d + \frac{15}{44} R R_{ab}, \hat{\mathcal{C}}_{ab}^{(1)} = \frac{45}{44} R_{ab} \end{aligned}$$

$$\begin{aligned} \tilde{I} = & \frac{1}{16\pi G} \int d^5x \sqrt{|g|} \left(\frac{12}{L^2} + R + \frac{15\zeta(3)}{88} \alpha'^3 \left[\frac{3}{L^6} R + \frac{1}{L^4} (R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2) \right. \right. \\ & \left. \left. + \frac{1}{L^2} Q^{(3)}(\mathcal{R}^3) + Q^{(4)}(\mathcal{R}^4) \right] \right) \end{aligned}$$

$$Q^{(3)}(\mathcal{R}^3) = \frac{11}{15} \left(4\hat{\mathcal{C}}^{(3)} + R^{ab} \hat{\mathcal{C}}_{ab}^{(2)} \right)$$

$$Q^{(4)}(\mathcal{R}^4) = \frac{11}{15} \left(W^4 + R^{ab} \hat{\mathcal{C}}_{ab}^{(3)} \right)$$

$$\frac{r^2}{L^2} + (k-f) + \frac{15\zeta(3)\alpha^3}{88} \left(\frac{3(k-f)}{L^6} + \frac{2(k-f)^2}{r^2 L^4} - \frac{4(k-f)^3}{3r^4 L^2} - \frac{(k-f)^4}{r^6} \right) = \frac{m}{r^2}$$

$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G} \left(\frac{h(\psi_+)}{\psi_+^{D/2}} - \int h(\psi_+) d\left(\frac{1}{\psi_+^{D/2}}\right) \right)$$

$$t=1/\psi^{D/2}=r_+^D$$

$$h'=-\frac{D}{2}\frac{\dot{h}}{\psi^{D/2+1}}=-\frac{D}{2}\dot{h}t^{1+\frac{2}{D}}$$

$$M = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G} t^{1-\frac{1}{D}} h(t)$$

$$T = \frac{1}{4\pi} \frac{1}{t^{\frac{1}{D}}} \left(-\frac{2(D-1)}{D} \frac{h(t)}{\dot{h}(t)t} - 2 \right)$$

$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G} \left(h(t)t - \int h(t) dt \right)$$

$$M = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G} t^{1-\frac{1}{D}} \dot{H}(t)$$

$$T = -\frac{1}{2\pi} \frac{1}{t^{\frac{1}{D}}} \left(\frac{(D-1)}{D} \frac{\dot{H}(t)}{t\ddot{H}(t)} + 1 \right)$$

$$S = -\frac{(D-2)\Sigma_{D-2,k}}{8G} (\dot{H}(t)t - H(t))$$

$$Dt^2\ddot{H}(t)\dot{H}(t)+(D-2)((D-1)\dot{H}(t)(\dot{H}(t)t-H(t))-Dt\ddot{H}(t)H(t))=0$$

$${\rm d}M = T\,{\rm d}S + V\,{\rm d}P + \mu_\alpha {\rm d}\alpha$$

$$P = -\frac{\Lambda}{8\pi G} = \frac{(D-1)(D-2)}{16\pi GL^2}$$

$$V = \frac{\Sigma_{D-2,k}}{(D-1)} r_+^{D-1}$$

$$\mu_\alpha = \frac{(D-2)\Sigma_{D-2,k}}{16\pi G} \left[r_+^{D-1} \frac{\partial h(\psi_+)}{\partial \alpha} \right]_{L,r_+}^+$$

$$\frac{k^{D/2-1}}{2\pi r_+} \left(\frac{(D-1)r_+^2 h(\psi_+)}{h'(\psi_+)} - 2k \right) \int \frac{d\psi_+}{\psi_+^{D/2}} \frac{\partial h'(\psi_+)}{\partial \alpha} \bigg|_{L,r_+}$$

$$(D-3)M = (D-2)TS - 2VP + 2\mu_\alpha \alpha$$

$$\begin{aligned} I &= \int \sqrt{-g} d^D x \left(R - 2\Lambda + \sum_{n=2}^{\infty} \alpha_n \mathcal{Z}_n \right) \\ &= \int \sqrt{-g} d^D x (R - 2\Lambda + F(R, \mathcal{Z}_2, \dots, \mathcal{Z}_5)) \end{aligned}$$

$$y_{+, \text{ext}} = \frac{\sqrt{k(3-2\beta + D(-1+2\beta)) + \sqrt{(D-3)^2 + 8(D-1)\beta}}}{\sqrt{2(D-1)}}, \text{ such that } T=0$$

$$\begin{aligned} S = & \frac{1}{2\kappa_4^2} \int \mathcal{R} \star 1 + \frac{1}{2} \text{Re} \mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} \text{Im} \mathcal{N}_{AB} F^A \wedge \star F^B \\ & - \frac{1}{\kappa_4^2} \int G_{a\bar{b}} dz^a \wedge \star d\bar{z}^b + h_{pq} d\xi^p \wedge \star d\xi^q \end{aligned}$$

$$K=-\log\, i\big(\bar{X}^A\mathcal{F}_A-X^A\overline{\mathcal{F}}_A\big)$$

$$\mathcal{N}_{AB}=\overline{\mathcal{F}}_{AB}+2i\frac{(\text{Im}\mathcal{F})_{AC}X^C(\text{Im}\mathcal{F})_{BD}X^D}{X^C(\text{Im}\mathcal{F})_{CD}X^D},$$

$$\mathcal{L}_{\text{h.d.}} \supset -\frac{i}{2} \int \text{d}^4\theta \sum_{g \geq 1} \mathcal{F}_g(\mathcal{X}^A) (\mathcal{W}^{ij} \mathcal{W}_{ij})^g + \text{h.c.}$$

$$\begin{aligned} \mathcal{W}^{ij}_{\mu\nu} &= W^{ij,-}_{\mu\nu} - \mathcal{R}^{-}_{\mu\nu\rho\sigma}\theta^i\sigma^{\rho\sigma}\theta^j + \dots \\ \mathcal{X}^A &= X^A + \frac{1}{2}\epsilon_{ij}\theta^i\sigma^{\mu\nu}\theta^j\big(F^{A,-}_{\mu\nu} - ie^{K/2}\bar{X}^AW^{-}_{\mu\nu}\big) + \dots \end{aligned}$$

$$W_{\mu\nu}^{-}=2ie^{K/2}\text{Im}\mathcal{N}_{AB}X^AF_{\mu\nu}^{B,-}, W_{\mu\nu}^{ij,-}=\frac{\epsilon^{ij}}{2}W_{\mu\nu}^{-}$$

$$\mathcal{L}_{\text{h.d.}} \supset -\frac{i}{2} \sum_{g \geq 1} \mathcal{F}_g(X^A) \mathcal{R}^2_- W_-^{2g-2} + \text{h.c.}$$

$$\sum_{g\geq 0}\delta\mathcal{F}_g^{(\text{hyp})}(X^A)W_-^{2g-2}=-\frac{1}{4}\int_{i0^+}^{i\infty}\frac{\mathrm{d}\tau}{\tau}\frac{1}{\sin^2\frac{\tau W_-e^{K/2}\bar{Z}_p}{2}}e^{-\tau|Z_p|^2}$$

$$Y^A=CX^A, \Upsilon=C^2W^2, \text{ with } C^2=e^{\mathcal{K}}\bar{Z}^2,$$

$$\begin{aligned} |\mathcal{Z}|^2 &= e^{\mathcal{K}}|p^AF_A(X,W^2)-q_AX^A|^2 \\ e^{-\mathcal{K}} &= i\bar{X}^AF_A(X,W^2)-iX^A\bar{F}_A(\bar{X},\bar{W}^2) \end{aligned}$$

$$F_A(X,W^2)=\partial_{X^A}F(X,W^2)$$

$$F(X,W^2)=\sum_{g=0}^{\infty}F_g(X^A)W^{2g}$$

$$F(Y,\Upsilon)\colon=C^2F(X,W^2)=\sum_{g=0}^{\infty}F_g(Y^A)\Upsilon^g$$

$$2F(\Upsilon,Y^A)=2\Upsilon F_\Upsilon+Y^AF_A$$

$$F_g(Y^A)=(-1)^g2^{-6g}\mathcal{F}_g(Y^A)$$

$$Z=\overline{Z}^{-1}(p^AF_A(Y,\Upsilon)-q_AY^A),$$

$$ds^2=-e^{2U(y)}dt^2+e^{-2U(y)}(dy^2+y^2d\Omega_2^2),\text{ with }e^{-2U(y)}=\frac{|Z|^2\kappa_4^2}{8\pi y^2}$$

$$ip^A=Y^A-\bar{Y}^A$$

$$iq_A=F_A(Y,\Upsilon)-\bar{F}_A(\bar{Y},\bar{\Upsilon})$$

$$\mathcal{S}_{\mathrm{BH}}=\pi[|Z|^2+4\mathrm{Im}(\Upsilon\partial_\Upsilon F(Y,\Upsilon))],$$

$$\mathcal{S}_{\mathrm{BH}}=4\pi\left[\mathrm{Im}F-\frac{1}{2}q_A\mathrm{Re}Y^A\right],$$

$$F(Y,\Upsilon)=\frac{D_{abc}Y^aY^bY^c}{Y^0}+d_a\frac{Y^a}{Y^0}\Upsilon+G(Y^0,\Upsilon)$$

$$\mathcal{S}_{\mathrm{BH}}\!=\!\frac{\pi}{3p^0}\sqrt{\frac{4}{3}\big(\tilde{\Delta}_a\tilde{x}^a\big)^2-9\big(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c\big)^2}\\+4\pi\left[\mathrm{Im}G-\mathrm{Re}Y^0\mathrm{Im}G_0-\frac{1}{\sqrt{3}}\frac{(\mathrm{Re}Y^0)^2}{|Y^0|^3}\tilde{x}^ad_ad_aY\right]$$

$$\mathcal{F}_0(X^A) \!=\! -\frac{1}{6}\mathcal{K}_{abc}\frac{X^aX^bX^c}{X^0} + K_{ab}^{(1)}X^aX^b + K_a^{(2)}X^0X^a + K^{(3)}(X^0)^2 \\ - \frac{(X^0)^2}{(2\pi i)^3}\sum_{k>0}n_k^{(0)}\sum_{m\geq 1}\frac{1}{m^3}e^{2\pi i m k_a z^a}$$

$$\mathcal{K}_{abc}=\omega_a\cdot\omega_b\cdot\omega_c, K_a^{(2)}=\frac{1}{24}c_2(TX_3)\cdot\omega_a, K^{(3)}=\frac{i\zeta(3)}{2(2\pi)^3}\chi_E(X_3)$$

$$\chi_E(X_3)=2h^{1,1}(X_3)-2h^{2,1}(X_3)$$

$$\boldsymbol{k}=k_a\gamma^a\in H_2^+(X_3,\mathbb{Z})$$

$$\mathcal{F}_1(X^A)=\frac{1}{24}\int_{X_3}J_{\mathbb{C}}\wedge c_2(TX_3)+\mathcal{O}(\log\,z^a)=\frac{1}{24}c_{2,a}z^a+\mathcal{O}(e^{2\pi iz^a})$$

$$J_{\mathbb{C}}=z^a\omega_a=(b^a+it^a)\omega_a$$

$$\mathcal{F}_{g>1}(X^A)\supset\frac{\chi(X_3)}{2}(-1)^{g-1}2(2g-1)\frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}(X^0)^{2-2g},$$

$$F(Y, Y) = \frac{D_{abc} Y^a Y^b Y^c}{Y^0} + d_a \frac{Y^a}{Y^0} Y + G(Y^0, Y) + \mathcal{O}(e^{2\pi i z^a})$$

$$G(Y^0, Y) = -\frac{i}{2(2\pi)^3} \chi_E(X_3) (Y^0)^2 \sum_{g=0,2,3,\dots} c_{g-1}^3 \alpha^{2g} + \dots$$

$$c_{g-1}^3 = (-1)^{g-1} 2(2g-1) \frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}, \alpha^2 = -\frac{1}{64} \frac{Y}{(Y^0)^2}$$

$$\begin{aligned} q_a &= 2\mathrm{Im}F_a \\ q_0 &= 2\mathrm{Im}F_0 \\ p^a &= 2\mathrm{Im}Y^a \\ p^0 &= 2\mathrm{Im}Y^0 \end{aligned}$$

$$D_{abc}\left(\mathrm{Re}Y^b-\frac{\mathrm{Re}Y^0}{p^0}p^b\right)\left(\mathrm{Re}Y^c-\frac{\mathrm{Re}Y^0}{p^0}p^c\right)=\Delta_a$$

$$\Delta_a=|Y^0|^2\left(-\frac{\tilde{q}_a}{3p^0}+\frac{1}{(p^0)^2}D_{abc}p^bp^c\right),\tilde{q}_a\equiv q_a+p^0\frac{d_aY}{|Y^0|^2}$$

$$\mathrm{Re}Y^a=x^a+\frac{\mathrm{Re}Y^0}{p^0}p^a$$

$$\tilde{q}_0=2\mathrm{Im}\left[-\frac{1}{(Y^0)^2}D_{abc}Y^aY^bY^c\right]$$

$$\tilde{q}_0\equiv q_0+2\mathrm{Im}\left[\frac{1}{(Y^0)^2}d_aY^aY\right]-2\mathrm{Im}G_0$$

$$(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)|Y^0|^4=2(p^0)^3\mathrm{Re}Y^0x^a\Delta_a$$

$$\begin{aligned} \frac{\mathcal{S}_{\mathrm{BH}}}{4\pi} = & -2\mathrm{Im}\left(\frac{1}{Y^0}\right)\Delta_a\mathrm{Re}Y^a+\mathrm{Im}G-\frac{1}{2}q_0\mathrm{Re}Y^0 \\ & +\frac{1}{2}\mathrm{Re}\left(\frac{1}{Y^0}\right)p^a\left(\frac{q_a}{6\mathrm{Im}\left(\frac{1}{Y^0}\right)}+\frac{2}{3}d_aY\right) \end{aligned}$$

$$\frac{\mathcal{S}_{\mathrm{BH}}}{4\pi}=\frac{1}{8\mathrm{Re}Y^0}(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)+\mathrm{Im}G+\frac{\mathrm{Re}Y^0}{2p^0}p^A\delta q_A$$

$$x^a=\tilde{x}^a\frac{|Y^0|}{\sqrt{3}p^0},\tilde{\Delta}_a=D_{abc}\tilde{x}^b\tilde{x}^c=\Delta_a\frac{3(p^0)^2}{|Y^0|^2},$$

$$(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)|Y^0|=\frac{2}{3\sqrt{3}}\mathrm{Re}Y^0\tilde{x}^a\tilde{\Delta}_a$$

$$|Y^0|^2=\frac{(p^0)^2}{3}\frac{(\tilde{\Delta}_a\tilde{x}^a)^2}{\frac{4}{3}(\tilde{\Delta}_a\tilde{x}^a)^2-9(p^0p^A\tilde{q}_A-2D_{abc}p^ap^bp^c)^2}$$



$$\mathcal{S}_{\text{BH}} = \frac{\pi}{3p^0} \sqrt{\frac{4}{3}(\tilde{\Delta}_a \tilde{x}^a)^2 - 9(p^0 p^A \tilde{q}_A - 2D_{abc} p^a p^b p^c)^2} \\ + 4\pi \left[\text{Im} G - \text{Re} Y^0 \text{Im} G_0 - \frac{1}{\sqrt{3}} \frac{(\text{Re} Y^0)^2}{|Y^0|^3} \tilde{x}^a d_a Y \right]$$

$$\mathcal{S}_{\text{BH}} = \frac{\pi}{3p^0} \sqrt{\frac{4}{3}(\tilde{\Delta}_a \tilde{x}^a)^2 - 9(p^0 p^A q_A - 2D_{abc} p^a p^b p^c)^2},$$

$$\mathcal{S}_{\text{BH}} = -\frac{4\pi}{3} Y^a \left(q_a + \frac{4d_a Y}{p^0} \right) + 4\pi \text{Im} G,$$

$$d_a = -\frac{1}{24} \frac{1}{64} c_a$$

$$\tilde{x}^a = -\sqrt{3} p^a + p^0 \left[\frac{\sqrt{3}}{6} D^{ab} q_b \right] + (p^0)^2 \frac{\sqrt{3}}{2} \left[\frac{p^a}{4(\text{Re} Y^0)^2} - \frac{1}{(\text{Re} Y^0)^3} \Gamma^a \right] + \dots$$

$$\tilde{\Delta}_a = 3D_{abc} p^b p^c - p^0 q_a - (p^0)^2 \frac{d_a Y}{(\text{Re} Y^0)^2} + \dots$$

$$\tilde{q}_a = q_a + p^0 \frac{d_a Y}{(\text{Re} Y^0)^2} + \dots$$

$$\tilde{q}_0 = q_0 - 2\text{Im} G_0 + \frac{p^a d_a Y}{(\text{Re} Y^0)^2} + \dots$$

$$D_{abc} p^a p^b \Gamma^c = \text{Re} Y^0 \left[-\frac{1}{36} (\text{Re} Y^0)^2 + \frac{1}{4} D_{abc} p^a p^b p^c - \frac{1}{3} p^a d_a Y \right] + \mathcal{O}(p^0)$$

$$\frac{\mathcal{S}_{\text{BH}}}{4\pi} = \frac{1}{2} \sqrt{D_{abc} p^a p^b p^c \left[\hat{q}_0 - 2\text{Im} G_0 + \frac{p^a d_a Y}{(Y^0)^2} \right]} + \text{Im} G - Y^0 \text{Im} G_0 + \frac{p^a d_a Y}{Y^0}$$

$$\hat{q}_0 = q_0 + \frac{1}{12} D^{ab} q_a q_b$$

$$(Y^0)^2 = \frac{\frac{1}{4} D_{abc} p^a p^b p^c - d_a p^a Y}{\hat{q}_0 + i(G_0 - \bar{G}_0)}$$

$$\frac{\mathcal{S}_{\text{BH}}}{4\pi} = -\frac{1}{4} \frac{D_{abc} p^a p^b p^c}{Y^0} + \text{Im} G - Y^0 \text{Im} G_0 + \frac{p^a d_a Y}{Y^0} \\ = -\hat{q}_0 Y^0 + \text{Im} G + Y^0 \text{Im} G_0$$

$$F(Y^A) = D_{abc} \frac{Y^a Y^b Y^c}{Y^0} + K_a^{(2)} Y^0 Y^a$$

$$F(Y^A, Y) = \frac{D_{abc} Y^a Y^b Y^c}{Y^0} + d_a \frac{Y^a}{Y^0} Y + G(Y^0, Y)$$

$$|\mathcal{Z}|^2 = -\frac{D_{abc} p^a p^b p^c - 2d_a p^a Y}{Y^0} - 2Y^0 \text{Im} G_0$$



$$\frac{\mathcal{S}_{\text{BH}}^{\text{shift}}}{\pi} = 2 \sqrt{D_{abc} p^a p^b p^c \left[\hat{q}_0 - 2 \text{Im} G_0 + \frac{p^a d_a Y}{(Y^0)^2} \right]} = - \frac{D_{abc} p^a p^b p^c}{Y^0} \neq |Z|^2$$

$$\frac{\mathcal{S}_{\text{BH}}}{\pi} = \left[-\frac{D_{abc} p^a p^b p^c}{Y^0} + 2 \frac{p^a d_a Y}{Y^0} - 2 Y^0 \text{Im} G_0 \right] + \left[4 \text{Im} G - 2 Y^0 \text{Im} G_0 + 2 \frac{p^a d_a Y}{Y^0} \right]$$

$$F_{\mu\nu}^{A,-} = \frac{1}{2} (F_{\mu\nu}^A + i \star F_{\mu\nu}^A), G_A^- = \overline{\mathcal{N}}_{AB} F^{B,-}$$

$$S \supset \frac{1}{4\kappa_4^2} \int F^{A,-} \wedge G_A^- + \text{h.c.}$$

$$\begin{pmatrix} F^{A,-} \\ G_A^- \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} F^{A,-} \\ G_A^- \end{pmatrix}$$

$$\mathcal{A}^{\text{T}}\mathcal{D}-\mathcal{C}^{\text{T}}\mathcal{B}=\mathbb{1}_{n_{\text{v}}+1}, \mathcal{A}^{\text{T}}\mathcal{C}-\mathcal{C}^{\text{T}}\mathcal{A}=\mathcal{B}^{\text{T}}\mathcal{D}-\mathcal{D}^{\text{T}}\mathcal{B}=0$$

$$\mathcal{N}_{AB} \rightarrow (\mathcal{D}_A{}^C \mathcal{N}_{CD} + \mathcal{C}_{AD}) [(\mathcal{C} + \mathcal{B} \mathcal{N})^{-1}]_B^D$$

$$\begin{pmatrix} X^A \\ \mathcal{F}_A \end{pmatrix} \rightarrow \begin{pmatrix} X^{A'} \\ \mathcal{F}'_A \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} X^A \\ \mathcal{F}_A \end{pmatrix}$$

$$\begin{aligned} F'(X^{A'}) &= F(X^A) - \frac{1}{2} X^A \mathcal{F}_A + \frac{1}{2} X^A (\mathcal{C}^{\text{T}} \mathcal{B} + \mathcal{A}^{\text{T}} \mathcal{D})_A{}^B \mathcal{F}_B \\ &\quad + \frac{1}{2} X^A (\mathcal{C}^{\text{T}} \mathcal{A})_{AB} X^B + \frac{1}{2} \mathcal{F}_A (\mathcal{D}^{\text{T}} \mathcal{B})^{AB} \mathcal{F}_B \end{aligned}$$

$$\begin{aligned} F'(Y^{A'}) &= F(Y^A) - \frac{1}{2} Y^A F_A + \frac{1}{2} Y^A (\mathcal{C}^{\text{T}} \mathcal{B} + \mathcal{A}^{\text{T}} \mathcal{D})_A{}^B F_B \\ &\quad + \frac{1}{2} Y^A (\mathcal{C}^{\text{T}} \mathcal{A})_{AB} Y^B + \frac{1}{2} F_A (\mathcal{D}^{\text{T}} \mathcal{B})^{AB} F_B \end{aligned}$$

$$Y^{0'}=Y^0, Y^{a'}=Y^a+\frac{\gamma^a}{p^0}Y^0$$

$$\begin{aligned} F'_0 &= F_0 - \frac{\gamma^a F_a}{p^0} - 3 D_{abc} \frac{\gamma^a \gamma^b Y^c}{(p^0)^2} - D_{abc} \frac{\gamma^a \gamma^b \gamma^c Y^0}{(p^0)^3} \\ F'_a &= F_a + 6 D_{abc} \frac{Y^b \gamma^c}{p^0} + 3 D_{abc} \frac{\gamma^b \gamma^c Y^0}{(p^0)^2} \end{aligned}$$

$$\begin{aligned} p^{0'} &= p^0, p^{a'} = p^a + \gamma^a, q'_a = q_a + 6 D_{abc} \frac{p^b \gamma^c}{p^0} + 3 D_{abc} \frac{\gamma^b \gamma^c}{p^0} \\ q'_0 &= q_0 - \frac{\gamma^a q_a}{p^0} - 3 D_{abc} \frac{\gamma^a \gamma^b p^c}{(p^0)^2} - D_{abc} \frac{\gamma^a \gamma^b \gamma^c}{(p^0)^2} \end{aligned}$$

$$p^A F_A \rightarrow p^A F_A + \Delta, q_A Y^A \rightarrow q_A Y^A + \Delta, \text{Im}(\Upsilon \partial_\Upsilon F) \rightarrow \text{Im}(\Upsilon \partial_\Upsilon F)$$

$$\Delta = 6 D_{abc} \frac{p^a Y^b \gamma^c}{p^0} + 3 D_{abc} \frac{\gamma^a Y^b \gamma^c}{p^0} + 3 D_{abc} \frac{p^a \gamma^b \gamma^c Y^0}{(p^0)^2} + 2 D_{abc} \frac{\gamma^a \gamma^b \gamma^c Y^0}{(p^0)^2},$$



$$\mathcal{S}'_{\text{BH}}[p^{0'}, p^{a'}, q'_a, q'_0] = \mathcal{S}_{\text{BH}}[p^{0'}, p^{a'}, q'_a, q'_0] = \mathcal{S}_{\text{BH}}[p^0, p^a, q_a, q_0],$$

$$\begin{pmatrix} Y^{0'} \\ Y^{a'} \\ F'_0 \\ F'_a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\gamma^a}{p^0} & \delta^a_b & 0 & 0 \\ -D_{abc} \frac{\gamma^a \gamma^b \gamma^c}{(p^0)^3} & -3D_{bcd} \frac{\gamma^c \gamma^d}{(p^0)^2} & 1 & -\frac{\gamma^a}{p^0} \\ 3D_{abc} \frac{\gamma^c \gamma^d}{(p^0)^2} & 6D_{abc} \frac{\gamma^c}{(p^0)^2} & 0 & \delta^b_a \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^b \\ F_0 \\ F_b \end{pmatrix}$$

$$\mathbf{M}^{\text{T}}\boldsymbol{\eta}\mathbf{M}=\boldsymbol{\eta}\equiv\begin{pmatrix}0&\mathbb{1}_{h^{1,1}+1}\\-\mathbb{1}_{h^{1,1}+1}&0\end{pmatrix}$$

$$|Z|^2 = \mathbf{q}^{\text{T}} \cdot \boldsymbol{\eta} \cdot C \boldsymbol{\Pi}, \text{Im}(\Upsilon \partial_Y F(Y, \Upsilon))$$

$$\begin{aligned} F'(Y^{A'},Y')&=F(Y^A,Y)+3\frac{\gamma^aD_{abc}Y^bY^c}{p^0}+3\frac{\gamma^a\gamma^bD_{abc}Y^0Y^c}{(p^0)^2}+\frac{D_{abc}\gamma^a\gamma^b\gamma^c(Y^0)^2}{(p^0)^3}\\ &=\frac{D_{abc}Y^{a'}Y^{b'}Y^{c'}}{Y^{0'}}+d_a\frac{Y^{a'}}{Y^{0'}}Y'+G(Y^{0'},Y')-d_a\frac{\gamma^a}{p^0}Y'=F(Y^{A'}) \end{aligned}$$

$$\begin{aligned} q_a &= -\frac{1}{p^0}3D_{abc}(2\text{Re}Y^b)(2\text{Re}Y^c)+\frac{1}{p^0}3D_{abc}p^bp^c-\frac{4}{p^0}d_aY \\ q_0 &= \frac{1}{(p^0)^2}3D_{abc}p^a(2\text{Re}Y^b)(2\text{Re}Y^c)-\frac{1}{(p^0)^2}D_{abc}p^ap^bp^c+\frac{4}{(p^0)^2}d_ap^aY \end{aligned}$$

$$q_0=-\frac{p^aq_a}{p^0}+\frac{2}{(p^0)^2}D_{abc}p^ap^bp^c$$

$$p^{0'}=p^0, p^{a'}=\gamma^a, q'_a=q_a+3D_{abc}\frac{\gamma^b\gamma^c}{p^0}, q'_0=-\frac{\gamma^aq_a}{p^0}-D_{abc}\frac{\gamma^a\gamma^b\gamma^c}{(p^0)^2}$$

$$q'_0=-\frac{p^{a'}q'_a}{p^{0'}}+\frac{2}{(p^{0'})^2}D_{abc}p^{a'}p^{b'}p^{c'}$$

$$p^{0'}=p^0, q'_a=q_a-3D_{abc}\frac{p^bp^c}{p^0}$$

$$D_{abc}x^ax^b=\Delta_a=-|Y^0|^2\frac{\tilde{q}_a}{3p^0}, \tilde{q}_a\equiv q_a+p^0\frac{d_aY}{|Y^0|^2},$$

$$q_0-2\text{Im}G_0=\left[\frac{2}{3p^0}|Y^0|^2q_ax^a-\frac{4}{3}d_ax^aY\right]\text{Im}((Y^0)^{-2})=\left[-\frac{2q_ax^a}{3|Y^0|^2}+\frac{4p^0d_ax^aY}{3|Y^0|^4}\right]\text{Re}Y^0$$

$$\begin{aligned} \frac{\mathcal{S}_{\text{BH}}}{4\pi} &= 2\text{Im}\left(\frac{1}{Y^0}\right)\left[\frac{1}{3p^0}|Y^0|^2q_ax^a+\frac{1}{3}d_ax^aY\right]+\text{Im}G-\frac{1}{2}q_0\text{Re}Y^0 \\ &= -\frac{\tilde{q}_ax^a}{3}+\text{Im}G-\frac{1}{2}q_0\text{Re}Y^0 \end{aligned}$$



$$p^0 = p^{0'}, q_a = q'_a - 3D_{abc} \frac{p^{b'} p^{c'}}{p^{0'}}, q_0 = q'_0 + \frac{p^{a'} q'_a}{p^{0'}} - 2D_{abc} \frac{p^{a'} p^{b'} p^{c'}}{(p^{0'})^2}$$

$$\tilde{q}_a = \tilde{q}'_a - 3D_{abc} \frac{p^{b'} p^{c'}}{p^{0'}} = -3p^{0'} \frac{\Delta'_a}{|Y^{0'}|^2}, x^a = x^{a'} = \text{Re} Y^{a'} - \frac{\text{Re} Y^{0'}}{p^{0'}} p^{a'}, Y^0 = Y^{0'}$$

$$\frac{\mathcal{S}_{\text{BH}}}{4\pi} = p^{0'} \frac{\Delta'_a}{|Y^{0'}|^2} \text{Re} Y^{a'} + \text{Im} G' - \frac{1}{2} q'_0 \text{Re} Y^{0'} + \frac{1}{2} \frac{\text{Re} Y^{0'}}{|Y^{0'}|^2} p^{a'} \left(-\frac{q'_a |Y^{0'}|^2}{6p^{0'}} + \frac{2}{3} d_a Y' \right)$$

$$G(Y^0, Y) = \frac{i}{4(2\pi)^3} \chi_E(X_3) (Y^0)^2 \alpha^2 \oint \frac{ds}{s} \frac{1}{1 - e^{-2\pi i s}} \frac{1}{\sinh^2 \left(\frac{\pi \alpha s}{2} \right)}$$

$$s = \frac{2\pi n}{|\alpha|} e^{i(\frac{\pi}{2} - \theta_\alpha)}, \text{ with } n \in \mathbb{Z}$$

$$G^{(p)}(Y^0, Y) = -\frac{i}{2(2\pi)^3} \chi_E(X_3) (Y^0)^2 \alpha^2 \sum_{n=1}^{\infty} n \log(1 - e^{-\alpha n})$$

$$\begin{aligned} G^{(np)}(Y^0, Y = -64) &= \frac{\chi_E}{8\pi^2} Y^0 \sum_{n,k=1}^{\infty} \frac{n}{k} e^{-4\pi^2 kn Y^0} \left(1 + \frac{1}{4\pi^2 kn Y^0} \right) \\ &= \frac{\chi_E}{8\pi^2} Y^0 \sum_n \left(n \text{Li}_1(e^{-4\pi^2 n Y^0}) + \frac{1}{4\pi^2 Y^0} \text{Li}_2(e^{-4\pi^2 n Y^0}) \right) \end{aligned}$$

$$\begin{aligned} G_0^{(np)}(Y^0, Y = -64) &= -\frac{\chi_E}{2} Y^0 \sum_{n,k=1}^{\infty} n^2 e^{-4\pi^2 kn Y^0} = \frac{\chi_E}{2} Y^0 \sum_{n=1}^{\infty} n^2 \frac{1}{1 - e^{4\pi^2 n Y^0}} \\ &= -\frac{\chi_E}{2} Y^0 \sum_{n=1}^{\infty} n^2 \text{Li}_0(e^{-4\pi^2 n Y^0}) \end{aligned}$$

$$\frac{\mathcal{S}_{\text{BH}}}{4\pi} = -\frac{\tilde{q}_a x^a}{3} + \text{Im} G - \frac{1}{2} q_0 \text{Re} Y^0,$$

$$\text{Im} G^{(np)} = \frac{\chi_E}{8\pi^2} \sum_{n,k=1}^{\infty} \frac{n}{k} e^{-4\pi^2 kn \text{Re} Y^0} \left[\frac{p^0}{2} \cos(2\pi^2 kn p^0) - \left(\text{Re} Y^0 + \frac{1}{4\pi^2 kn} \right) \sin(2\pi^2 kn p^0) \right]$$

$$\text{Im} G_0^{(np)} = -\frac{\chi_E}{2} \sum_{n,k=1}^{\infty} n^2 e^{-4\pi^2 kn \text{Re} Y^0} \left[\frac{p^0}{2} \cos(2\pi^2 kn p^0) - \text{Re} Y^0 \sin(2\pi^2 kn p^0) \right],$$

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega_2^2, \text{ with } f(r) = 1 - \frac{r_h}{r}.$$

$$f(y) = \left(1 + \frac{r_h}{y}\right)^{-1}$$

$$ds^2 = -\frac{y^2}{r_h^2} dt^2 + \frac{r_h^2}{y^2} dy^2 + r_h^2 d\Omega_2^2$$

$$ds^2 = \frac{r_h^2}{\rho^2} (-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2)$$

$$-(X^0)^2 - (X^2)^2 + (X^1)^2 = -r_h^2 := -R_{\text{AdS}}^2$$

$$X^0 = R_{\text{AdS}} \cosh(\xi) \sin(\tau/R_{\text{AdS}}), X^2 = R_{\text{AdS}} \cosh(\xi) \cos(\tau/R_{\text{AdS}})$$

$$X^1 = R_{\text{AdS}} \sinh(\xi).$$

$$ds^2 = -\cosh^2(\xi) d\tau^2 + R_{\text{AdS}}^2 (d\xi^2 + d\Omega_2^2)$$

$$\psi = \sin^{-1} \left(\frac{1}{\cosh \xi} \right), \tilde{\tau} = \frac{\tau}{R_{\text{AdS}}}, \text{ with } \psi \in [0, \pi], \tilde{\tau} \in (-\infty, \infty)$$

$$ds^2 = \frac{R_{\text{AdS}}^2}{\sin^2(\psi)} (-d\tilde{\tau}^2 + d\psi^2) + R_{\text{AdS}}^2 d\Omega_2^2$$

$$r_h^2 = R_{\text{AdS}}^2 = |C|^2 e^{-K} = |Z_{\text{BH}}|^2.$$

$$m = |Z|, Z = e^{K/2} (q_A X^A - p^A \mathcal{F}_A),$$

$$S_{\text{wt}} = -2|Z|R_{\text{AdS}} \int_{\gamma} d\sigma \sqrt{\rho^{-2}(t^2 - \dot{\rho}^2) - \dot{\theta}^2 - \sin^2 \theta \dot{\phi}^2} + \int_{\Sigma} q_A F^A - p^A G_A,$$

$$p^{A'} = \frac{1}{4\pi} \int_{\mathbb{S}^2} F^A, q'_A = \frac{1}{4\pi} \int_{\mathbb{S}^2} G_A, G_A^- = \overline{\mathcal{N}}_{AB} F^{B,-},$$

$$F^A = R_{\text{AdS}}^{-2} (p^{A'} \omega_{\mathbb{S}^2} + 2 \text{Re} C X^A \omega_{\text{AdS}_2}), G_A = R_{\text{AdS}}^{-2} (q'_A \omega_{\mathbb{S}^2} + 2 \text{Re} C \mathcal{F}_A \omega_{\text{AdS}_2}),$$

$$C X^A = \text{Re} C X^A + \frac{i}{2} p^{A'}, C \mathcal{F}_A = \text{Re} C \mathcal{F}_A + \frac{i}{2} q'_A,$$

$$\omega_{\mathbb{S}^2} = R_{\text{AdS}}^2 \sin \theta d\theta \wedge d\phi, \omega_{\text{AdS}_2} = \frac{R_{\text{AdS}}^2}{\rho^2} dt \wedge d\rho$$

$$\star_4 \omega_{\mathbb{S}^2} = -\omega_{\text{AdS}_2}, \star_4 \omega_{\text{AdS}_2} = \omega_{\mathbb{S}^2},$$

$$R_{\text{AdS}}^2 F^{A,-} = \overline{C X}^A (\omega_{\text{AdS}_2} + i \omega_{\mathbb{S}^2}), R_{\text{AdS}}^2 G_A^- = \overline{C \mathcal{F}_A} (\omega_{\text{AdS}_2} + i \omega_{\mathbb{S}^2})$$

$$\begin{aligned} W^- &= e^{K/2} \mathcal{F}_A F^{A,-} - e^{K/2} X^A G_A^- = \frac{e^{K/2}}{R_{\text{AdS}}^2} \left[\mathcal{F}_A \overline{C X}^A - X^A \overline{C \mathcal{F}_A} \right] (\omega_{\text{AdS}_2} + i \omega_{\mathbb{S}^2}) \\ &= -i e^{-K/2} \bar{C} R_{\text{AdS}}^{-2} (\omega_{\text{AdS}_2} + i \omega_{\mathbb{S}^2}) = Z_{\text{BH}} R_{\text{AdS}}^{-2} (\omega_{\mathbb{S}^2} - i \omega_{\text{AdS}_2}) \end{aligned}$$



$$ie^{K/2}\bar{X}^AW^- = \overline{CX}^AR_{\text{AdS}}^{-2}(\omega_{\text{AdS}_2} + i\omega_{\text{S}^2}) = F^{A,-}$$

$$S_{wl} \supset \int_{\Sigma} q_A F^A - p^A G_A = q_e \int_{\gamma} \frac{dt}{\rho} + q_m \int_{\gamma} \cos \theta d\phi$$

$$q_e = 2e^{-K/2}\text{Re} CZ, q_m = -2e^{-K/2}\text{Im} CZ = p^A q_A' - q_A p^{A'}$$

$$\tilde{m}:=2|Z|R_{\text{AdS}}=2e^{-K/2}|CZ|,$$

$$q_e^2+q_m^2=\tilde{m}^2\Rightarrow |q_e|=2e^{-K/2}|\text{Re} CZ|\leq 2e^{-K/2}|CZ|=\tilde{m},$$

$$(2mR_{\text{AdS}})^2\geq \left|2e^{-K/2}CZ\right|^2=q_e^2+q_m^2$$

$$S_{wl}=-\tilde{m}\int_{\gamma}d\sigma\sqrt{\rho^{-2}(t^2-\dot{\rho}^2)-\dot{\theta}^2-\sin^2\theta\dot{\phi}^2}+q_e\int_{\gamma}\frac{dt}{\rho}+q_m\int_{\gamma}\cos\theta d\phi$$

$$S_{wl}=\frac{1}{2}\int_{\gamma}d\sigma[h^{-1}(\rho^{-2}(-t^2+\dot{\rho}^2)+\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)-h\tilde{m}^2]+\int_{\gamma}d\sigma(q_e\rho^{-1}t+q_m\cos\theta\dot{\phi})$$

$$\rho^{-2}(-t^2+\dot{\rho}^2)+\dot{\theta}^2+\sin^2\theta\dot{\phi}^2+h^2\tilde{m}^2=0$$

$$H=\frac{\rho^2}{2}\left[p_\rho^2-\left(p_t-\frac{q_e}{\rho}\right)^2\right]+\frac{1}{2}p_\theta^2+\frac{1}{2}\csc^2\theta(p_\phi-q_m\cos\theta)^2\stackrel{!}{=}-\frac{\tilde{m}^2}{2}.$$

$$p_\rho=\frac{\dot{\rho}}{\rho^2}, p_t=-\frac{\dot{t}}{\rho^2}+\frac{q_e}{\rho}, p_\theta=\dot{\theta}, p_\phi=\sin^2\theta\dot{\phi}+q_m\cos\theta$$

$$\frac{dp_\phi}{d\sigma}=\frac{dp_t}{d\sigma}=0$$

$$j=p_\phi=\sin^2\theta\dot{\phi}+q_m\cos\theta, E=-p_t=\frac{\dot{t}}{\rho^2}-\frac{q_e}{\rho}.$$

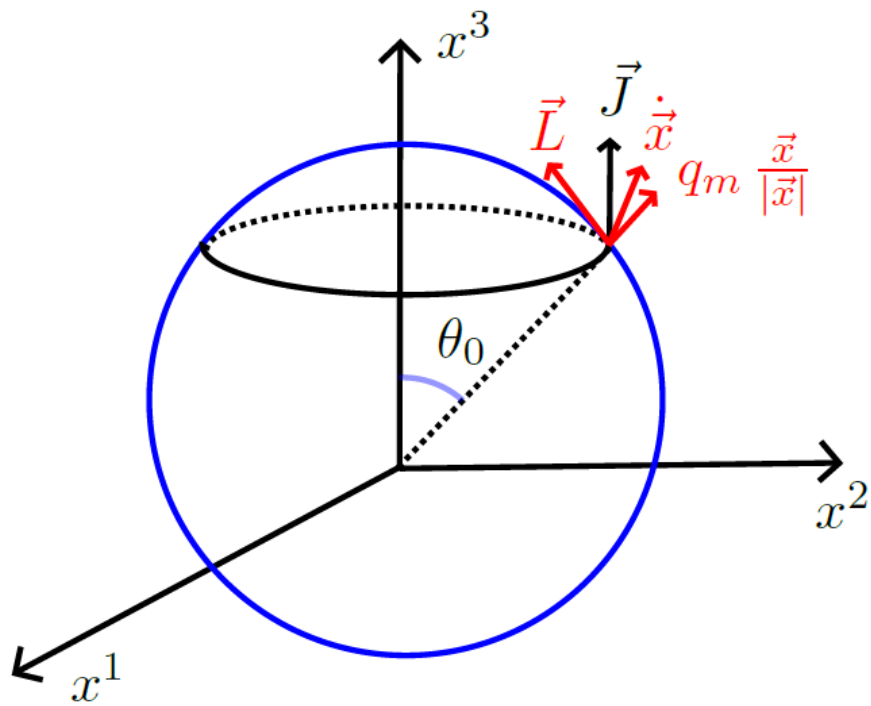
$$\dot{p}_\rho=\frac{d}{d\sigma}\left(\frac{\dot{\rho}}{\rho^2}\right)=E\frac{\dot{t}}{\rho}-\frac{\dot{\rho}^2}{\rho^3}$$

$$\dot{p}_\theta=\ddot{\theta}=\sin\theta(\cos\theta\dot{\phi}^2-q_m\dot{\phi})=\dot{\phi}\tan\theta(\dot{\phi}-j).$$

$$p_\rho^2+\frac{p_\theta^2}{\rho^2}+V(\rho,\theta)=0, V(\rho,\theta)=\frac{\tilde{m}^2}{\rho^2}-\left(E+\frac{q_e}{\rho}\right)^2+\frac{1}{\rho^2\sin^2\theta}(j-q_m\cos\theta)^2.$$

$$\dot{p}_\theta=0\Rightarrow\cos\theta_0=\frac{q_m}{j},\text{ with }j=\dot{\phi}$$

$$|L|=\ell=\sin\theta_0\dot{\phi}=\sqrt{j^2-q_m^2}$$



$$p_\rho^2 + V(\rho) = 0, V(\rho) = \frac{\tilde{m}^2 + \ell^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2$$

$$\tilde{m}^2 \rightarrow m_{\text{eff}}^2 = \tilde{m}^2 + \ell^2$$

$$p_\rho^2 + V(\rho) = 0, V(\rho) = \frac{\tilde{m}^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2$$

$$p_\theta = \pm i(\cot \theta p_\phi - q_m \csc \theta) = 0 \Leftrightarrow \cos \theta_0 = \frac{q_m}{j}$$

$$p_\rho^2 + V_{\text{ext}}(\rho) = 0, V_{\text{ext}}(\rho) = \frac{\tilde{m}^2 + \ell^2}{\rho^2} - \left(E + \frac{\tilde{m}}{\rho}\right)^2.$$

$$p_\rho^2 + V_{\text{mag}}(\rho) = 0, V_{\text{mag}}(\rho) = \frac{j^2}{\rho^2} - E^2,$$

$$\ell = \sqrt{j^2 - \tilde{m}^2}$$

$$V_{\text{mag}}(\rho) = \frac{\tilde{m}^2}{\rho^2} - E^2$$

$$H = \frac{1}{2}\rho^2 \left[p_\rho^2 - \left(p_t - \frac{q_e}{\rho} \right)^2 \right] + \frac{1}{2}p_\theta^2 + \frac{1}{2}\csc^2 \theta (p_\phi - q_m \cos \theta)^2.$$

$$\{q^j, p_k\}_{\text{PB}} = \delta_k^j$$

$$\{A(q,p),B(q,p)\}_{\text{PB}}=\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}-\frac{\partial B}{\partial q^i}\frac{\partial A}{\partial p_i}$$

$$\dot{q}^i=\frac{\partial H}{\partial p_i},\dot{p}_i=-\frac{\partial H}{\partial q^i}$$

$$\frac{dA(q,p)}{d\sigma}=\{A(q,p),H(q,p)\}_{\text{PB}}$$

$$\begin{aligned} K_+ &= p_t \\ K_- &= (t^2 + \rho^2)p_t + 2t\rho p_\rho - 2q_e\rho \\ K_0 &= tp_t + \rho p_\rho \end{aligned}$$

$$\begin{aligned} J_1 &= -\sin\phi p_\theta - \cot\theta \cos\phi p_\phi + q_m \csc\theta \cos\phi \\ J_2 &= \cos\phi p_\theta - \cot\theta \sin\phi p_\phi + q_m \csc\theta \sin\phi \\ J_3 &= p_\phi \end{aligned}$$

$$\begin{aligned}\{J_i,J_j\}_{\text{PB}} &= \epsilon_{ijk}J_k \\ \{K_+,K_-\}_{\text{PB}} &= -2K_0,\{K_0,K_\pm\}_{\text{PB}} = \pm K_\pm \\ \{J_i,K_j\}_{\text{PB}} &= 0\end{aligned}$$

$$2H=\delta^{ik}J_iJ_k+K_0^2-\frac{1}{2}(K_+K_-+K_-K_+)-q_e^2-q_m^2=C_2^{\mathcal{S}^2}+C_2^{\text{AdS}_2}-\tilde{m}^2$$

$$H\stackrel{!}{=}-\frac{1}{2}\tilde{m}^2\Leftrightarrow C_2^{\mathcal{S}^2}+C_2^{\text{AdS}_2}=0$$

$$C_2^{\mathcal{S}^2}+C_2^{\text{AdS}_2}=-\Delta^2$$

$$\Delta^2=\tilde{m}^2-q_e^2-q_m^2>0$$

$$\left(\rho-\frac{q_e}{K_+}\right)^2-\left(t-\frac{K_0}{K_+}\right)^2=\frac{\tilde{m}^2+\ell^2}{K_+^2}.$$

$$\text{sgn}(q_e/K_+)=\text{sgn}(q_ep_t)$$

$$S_E=\frac{1}{2}\int d\sigma\left[\left(\frac{1}{h\rho^2}(\dot{t}^2+\dot{\rho}^2)\right)+hm_{\text{eff}}^2\right]-q_e\int d\sigma\frac{\dot{t}}{\rho}$$

$$m_{\text{eff}}=\sqrt{\tilde{m}^2+\ell(\ell+1)}$$

$$H_E=\frac{\rho^2}{2}\left[p_\rho^2+\left(p_\tau-\frac{q_e}{\rho}\right)^2\right]-\frac{m_{\text{eff}}^2}{2}\stackrel{!}{=}0$$

$$p_\rho^2-V(\rho)=0,V(\rho)=\frac{m_{\text{eff}}^2}{\rho^2}-\left(p_\tau-\frac{q_e}{\rho}\right)^2$$

$$V(\rho)=0\Leftrightarrow \rho_\pm=\frac{q_e\pm m_{\text{eff}}}{p_\tau}$$

$$\mathcal{L}_E = p_i \dot{q}^i - H_E = p_i \dot{q}^i$$

$$S_E = \oint p_i dq^i$$

$$S_E = 2 \int_{\rho_-}^{\rho_+} d\rho p_\rho = 2 \int_{\rho_-}^{\rho_+} d\rho \sqrt{V(\rho)}$$

$$S_E = 2 m_{\rm eff} \int_{-1}^1 dy \frac{\sqrt{1-y^2}}{y+q_e/m_{\rm eff}}$$

$$S_E = 2\pi \left(q_e - \sqrt{q_e^2 - \tilde{m}^2 - \ell(\ell+1)} \right)$$

$$\frac{d\tau}{d\rho} = \pm \frac{q_e}{m_{\rm eff}} \frac{\frac{p_\tau}{q_e} - 1}{\sqrt{1 - \frac{q_e^2}{m_{\rm eff}^2} \left(1 - \frac{p_\tau}{q_e} \rho\right)^2}}$$

$$\dot{\tau} = q_e \rho \left(\frac{p_\tau}{q_e} \rho - 1 \right), \dot{\rho} = \pm m_{\rm eff} \rho \sqrt{1 - \frac{q_e^2}{m_{\rm eff}^2} \left(1 - \frac{p_\tau}{q_e} \rho\right)^2}.$$

$$\tau = \tau_0 \mp m_{\rm eff}^2 \frac{\sqrt{1 - \frac{q_e^2}{m_{\rm eff}^2} \left(1 - \frac{p_\tau}{q_e} \rho\right)^2}}{p_\tau q_e} \Rightarrow (\tau - \tau_0)^2 + \left(\rho - \frac{q_e}{p_\tau}\right)^2 = \frac{m_{\rm eff}^2}{p_\tau^2},$$

$$S_{\rm eff}^{\rm 2d}[A] = - \int_0^\infty \frac{dh}{h} \int \mathcal{D}\tau(h) \mathcal{D}\rho(h) e^{-S_{\rm E}[x^i,A]}$$

$$S_{\rm eff}^{\rm 2d}[A] \sim i \int d^2x \sqrt{g} \sum_{k=1}^\infty \sum_{\ell=0}^{\ell_{\rm max}} f(q_e^2 - \tilde{m}^2 - \ell(\ell+1)) e^{-2\pi k \left(q_e - \sqrt{q_e^2 - \tilde{m}^2 - \ell(\ell+1)} \right)} + \dots$$

$$e^{-K/2} CZ_{\rm D0} = nCX^0 = \frac{1}{2} \Big(q_e^{(n)} - i q_m^{(n)} \Big)$$

$$\mathrm{Im} G^{(np)} \supset -\frac{\chi_E}{16\pi^2} \sum_{n,k=1}^\infty \frac{q_m^{(n)}}{k} \mathrm{Re} \Big[e^{-2\pi^2 k (q_e^{(n)} - i q_m^{(n)})} \Big]$$

$$S_{wl} = -\tilde{m} \int_{\gamma} d\sigma \sqrt{\csc^2 \psi (\dot{\tilde{\tau}}^2 - \dot{\psi}^2) - \dot{\theta}^2 - \sin^2 \theta \dot{\phi}^2} + \int_{\Sigma} q_A F^A - p^A G_A$$

$$\omega_{\mathrm{AdS}_2} = \frac{R_{\mathrm{AdS}}^2}{\sin^2 \psi} d\tilde{\tau} \wedge d\psi$$



$$S_{wl} \supset \int_{\Sigma} q_A F^A - p^A G_A = \int_{\gamma} d\sigma \left(q_e \frac{\dot{\tau}}{\tan \psi} + q_m \cos \theta \dot{\phi} \right)$$

$$S_{wl} = \frac{1}{2} \int_{\gamma} d\sigma [h^{-1}(\csc^2 \psi (-\dot{\tau}^2 + \dot{\psi}^2) + \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - h\tilde{m}^2] + \int_{\gamma} d\sigma \left(q_e \frac{\dot{\tau}}{\tan \psi} + q_m \cos \theta \dot{\phi} \right)$$

$$H = \frac{\sin^2 \psi}{2} \left[p_{\psi}^2 - \left(p_{\tilde{\tau}} - \frac{q_e}{\tan \psi} \right)^2 \right] + \frac{1}{2} p_{\theta}^2 + \frac{1}{2} \csc^2 \theta (p_{\phi} - q_m \cos \theta)^2 \stackrel{!}{=} -\frac{\tilde{m}^2}{2},$$

$$p_{\psi} = \frac{\dot{\psi}}{\sin^2 \psi}, p_{\tilde{\tau}} = -\frac{\dot{\tau}}{\sin^2 \psi} + \frac{q_e}{\tan \psi}, p_{\theta} = \dot{\theta}, p_{\phi} = \sin^2 \theta \dot{\phi} + q_m \cos \theta$$

$$j = p_{\phi}, \tilde{E} = -p_{\tilde{\tau}},$$

$$\dot{p}_{\psi} = \frac{d}{d\sigma} \left(\frac{\dot{\psi}}{\sin^2 \psi} \right) = \frac{\dot{\tau}}{\cos \psi \sin \psi} (\tilde{E} - \dot{\tau}) - \frac{\dot{\psi}^2}{\sin^3 \psi} \cos \psi$$

$$\dot{p}_{\theta} = \ddot{\theta} = \sin \theta (\cos \theta \dot{\phi}^2 - q_m \dot{\phi}) = \dot{\phi} \tan \theta (\dot{\phi} - j)$$

$$p_{\psi}^2 + V(\psi) = 0, V(\psi) = \frac{\tilde{m}^2 + \ell^2}{\sin^2 \psi} - \left(\tilde{E} + \frac{q_e}{\tan \psi} \right)^2$$

$$K_0 = \cos \psi \cos \tilde{\tau} p_{\psi} + \cos \psi \sin \tilde{\tau} p_{\tilde{\tau}} + q_e \sin \psi \sin \tilde{\tau} \\ K_{\pm} = \mp \sin \psi \sin \tilde{\tau} p_{\psi} + (1 \pm \cos \psi \cos \tilde{\tau}) p_{\tilde{\tau}} \pm q_e \sin \psi \cos \tilde{\tau}$$

$$\{K_+, K_-\}_{\text{PB}} = -2K_0, \{K_0, K_{\pm}\}_{\text{PB}} = \pm K_{\pm}$$

$$(K_+ - K_-) \cos \tilde{\tau} + 2K_0 \sin \tilde{\tau} = 2q_e \sin \psi + (K_+ + K_-) \cos \psi$$

$$\frac{1}{2} (K_+ K_- + K_- K_+) - K_0^2 = \Delta^2 + j^2,$$

$$\Delta^2 = \tilde{m}^2 - q_e^2 - q_m^2$$

$$\tanh \xi = \frac{\text{Re} CZ}{|CZ|} = \frac{q_e}{\tilde{m}},$$

$$K_+ = K_-, K_0 = 0, \ell = 0$$

$$K_+^2 = q_m^2$$

$$\tan \psi = -\frac{q_m}{q_e} = \tan [\arg (CZ)]$$

$$\sin \psi = \sqrt{1 - \tanh^2 \xi} = \sin [\arg (CZ)].$$

$$\psi = \arg (CZ), \theta, \phi = \lambda,$$

$$K_+^2 = j^2 = \ell^2 + q_m^2.$$



$$\tan \psi = -\frac{j}{q_e},$$

$$S_{wl} = -\tilde{m} \int d\tilde{\tau} \left[\sqrt{\csc^2 \psi - \csc^2 \psi \left(\frac{d\psi}{d\tilde{\tau}} \right)^2 - \left(\frac{d\theta}{d\tilde{\tau}} \right)^2 - \sin^2 \theta \left(\frac{d\phi}{d\tilde{\tau}} \right)^2} + \frac{q_e}{\tan \psi} + q_m \cos \theta \frac{d\phi}{d\tilde{\tau}} \right].$$

$$\mathcal{H} = P_i \frac{d}{d\tilde{\tau}} q^i - \mathcal{L} = \csc \psi \sqrt{\tilde{m}^2 + \sin^2 \psi P_\psi^2 + P_\theta^2 + \csc^2 \theta (P_\phi - q_m \cos \theta)^2} - \frac{q_e}{\tan \psi},$$

$$P_\psi = \frac{\tilde{m}}{\sqrt{h}} \csc^2 \psi \frac{d\psi}{d\tilde{\tau}}, P_\theta = \frac{\tilde{m}}{\sqrt{h}} \frac{d\theta}{d\tilde{\tau}}, P_\phi = \frac{\tilde{m}}{\sqrt{h}} \sin^2 \theta \frac{d\phi}{d\tilde{\tau}} + q_m \cos \theta$$

$$h = g_{\mu\nu} \frac{dx^\mu}{d\tilde{\tau}} \frac{dx^\nu}{d\tilde{\tau}} = \frac{\tilde{m}^2 \csc^2 \psi}{\tilde{m}^2 + \sin^2 \psi P_\psi^2 + P_\theta^2 + \csc^2 \theta (P_\phi - q_m \cos \theta)^2}$$

$$J_1 = -\sin \phi P_\theta - \cot \theta \cos \phi P_\phi + q_m \csc \theta \cos \phi$$

$$J_2 = \cos \phi P_\theta - \cot \theta \sin \phi P_\phi + q_m \csc \theta \sin \phi$$

$$J_3 = P_\phi$$

$$\mathcal{H} = \csc \psi \sqrt{q_e^2 + \sin^2 \psi P_\psi^2 + J^2} - \frac{q_e}{\tan \psi}.$$

$$\cos \psi = q_e / \sqrt{q_e^2 + J^2}$$

$$\mathcal{H} \geq \sqrt{J^2}$$

$$Z_{1-\text{loop}}[A] = - \int_0^\infty \frac{dh}{h} \int \mathcal{D}\tau(h) \mathcal{D}\rho(h) e^{-S_E[x^i, A]}$$

$$\tau(\sigma) = \tau_0 - R_0 \sin \theta(\sigma), \rho(\sigma) = \rho_0 \mp R_0 \cos \theta(\sigma),$$

$$\dot{\theta} = \pm p_\tau \rho \Rightarrow \theta(\sigma) = 2 \tan^{-1} \left(\sqrt{\frac{1 \mp m_{\text{eff}}/q_e}{1 \pm m_{\text{eff}}/q_e}} \tan \left(\frac{\sigma}{2} \sqrt{q_e^2 - m_{\text{eff}}^2} + c_1 \right) \right)$$

$$\tau(\sigma) = \tau_0 - R \sin \theta(\sigma), \rho(\sigma) = \rho_0 - R \cos \theta(\sigma),$$

$$\theta(\sigma) = 2 \tan^{-1} \left(\sqrt{\frac{\rho_0/R - 1}{\rho_0/R + 1}} \tan \left(\frac{q_e \sigma}{2 \rho_0} \sqrt{\rho_0^2 - R^2} \right) \right)$$

$$R = \frac{m_{\text{eff}}}{q_e} \rho_0, \rho_0 = \frac{q_e}{p_\tau}$$

$$2\pi\rho_0/\left(q_e\sqrt{\rho_0^2-R^2}\right)$$

$$Z_{\text{mini}}[A] = - \int \frac{d\tau_0 d\rho_0}{\rho_0^2} \int \frac{RdR}{\rho_0^2} \int_0^\infty \frac{dh}{h} e^{-S_E}$$

$$\begin{aligned} S_E &= \int_0^T d\sigma \left[\frac{(q_e R)^2}{2h\rho_0^2} + \frac{hm_{\text{eff}}^2}{2} - q_e^2 \frac{\rho - \rho_0}{\rho_0} \right] \\ &= \pi \left[\frac{q_e R^2}{h\rho_0\sqrt{\rho_0^2 - R^2}} + \frac{hm_{\text{eff}}^2\rho_0}{q_e\sqrt{\rho_0^2 - R^2}} \right] - \frac{q_e^2 R}{\rho_0} \int_0^T d\sigma \cos \theta(\sigma) \end{aligned}$$

$$\dot{t}^2 + \dot{\rho}^2 = R^2 \dot{\theta}^2 = (q_e R)^2 \frac{\rho^2}{\rho_0^2}$$

$$\dot{t} = \dot{\theta}(\rho - \rho_0) = \frac{q_e \rho}{\rho_0} (\rho - \rho_0)$$

$$\int_0^T d\sigma \cos \theta(\sigma) = \frac{2\pi\rho_0}{q_e R} \left(\frac{\rho_0}{\sqrt{\rho_0^2 - R^2}} - 1 \right)$$

$$S_E = \pi \left[\frac{q_e R^2}{h\rho_0\sqrt{\rho_0^2 - R^2}} + \frac{hm_{\text{eff}}^2\rho_0}{q_e\sqrt{\rho_0^2 - R^2}} - 2q_e \left(\frac{\rho_0}{\sqrt{\rho_0^2 - R^2}} - 1 \right) \right]$$

$$\frac{\partial S_E}{\partial h} = 0 \Rightarrow h_{\text{min}} = \frac{q_e}{m_{\text{eff}}} \frac{R}{\rho_0},$$

$$S_E = 2\pi \left[\frac{m_{\text{eff}}}{\sqrt{\rho_0^2/R^2 - 1}} - q_e \frac{\rho_0/R}{\sqrt{\rho_0^2/R^2 - 1}} + q_e \right]$$

$$R_{\text{min}} = \frac{m_{\text{eff}}}{q_e} \rho_0 \Rightarrow S_{E,\text{min}} = 2\pi \left(q_e - \sqrt{q_e^2 - m_{\text{eff}}^2} \right)$$

$$\begin{aligned} \delta S_E &= \pi \left[-\frac{q_e R}{h^2 \rho_0 \sqrt{\rho_0^2/R^2 - 1}} + \frac{m_{\text{eff}}^2 \rho_0/R}{q_e \sqrt{\rho_0^2/R^2 - 1}} \right] \delta h \\ &+ \pi \left[\frac{q_e}{h\rho_0\sqrt{\rho_0^2/R^2 - 1}} + \frac{q_e \rho_0}{R^2 (\rho_0^2/R^2 - 1)^{3/2}} \left(h^{-1} - 2 + h \frac{m_{\text{eff}}^2}{q_e^2} \right) \right] \delta R \end{aligned}$$

$$h_{\text{min}} = \frac{m_{\text{eff}}}{q_e} \rho_0, \frac{R_{\text{min}}}{\rho_0} = \sqrt{2 - 2h + \frac{h^2 m_{\text{eff}}^2}{q_e^2}}.$$

$$\delta^2 S_E = \frac{\delta^2 S_E}{\delta h^2} (\delta h)^2 + 2 \frac{\delta^2 S_E}{\delta h \delta R} \delta h \delta R + \frac{\delta^2 S_E}{\delta R^2} (\delta R)^2$$



$$\frac{\delta^2 S_E}{\delta h^2} = \frac{2\pi q_e R}{h^3 \rho_0 \sqrt{\rho_0^2/R^2 - 1}} \triangleq \frac{2\pi m_{\text{eff}}}{\sqrt{q_e^2/m_{\text{eff}}^2 - 1}}$$

$$\frac{\delta^2 S_E}{\delta h \delta R} = -\frac{\pi q_e}{h^2 \rho_0 \sqrt{\rho_0^2/R^2 - 1}} - \frac{\pi q_e \rho_0}{h^2 R^2 (\rho_0^2/R^2 - 1)^{3/2}} + \frac{\pi h m_{\text{eff}}^2 R}{q_e \rho_0^2 (1 - R^2/\rho_0^2)^{3/2}} \stackrel{*}{=} -\frac{2\pi q_e}{\rho_0 \sqrt{q_e^2/m_{\text{eff}}^2 - 1}}$$

$$\frac{\delta^2 S_E}{\delta R^2} = -\frac{2\pi q_e \rho_0 \left((2h)^{-1} - 2 + h \frac{m_{\text{eff}}^2}{q_e^2} \right)}{R^3 (\rho_0^2/R^2 - 1)^{3/2}} + \frac{3\pi q_e \rho_0^3 \left(h^{-1} - 2 + h \frac{m_{\text{eff}}^2}{q_e^2} \right)}{R^5 (\rho_0^2/R^2 - 1)^{5/2}} \stackrel{\dagger}{=} -\frac{2\pi q_e^2}{m_{\text{eff}} \rho_0^2 (q_e^2/m_{\text{eff}}^2 - 1)^{3/2}}$$

$$Z_{\text{mini}}[A] = - \sum_{\text{saddles}} \int \frac{d\tau_0 d\rho_0}{\rho_0^2} \frac{R_{\text{min}}}{h_{\text{min}} \rho_0^2} \mathcal{A}_{1-\text{loop}} e^{-S_{E, \text{min}}} + (\S)$$

$$= i \int \frac{d\tau_0 d\rho_0}{\rho_0^2} \sum_{k=1}^{\infty} \sum_{\ell=0}^{\ell_{\text{max}}} \frac{(q_e^2 - \tilde{m}^2 - \ell(\ell+1))}{2\pi k q_e^3} e^{-2\pi k \left(q_e - \sqrt{q_e^2 - \tilde{m}^2 - \ell(\ell+1)} \right)} + (\S)$$

$$\mathcal{A}_{1-\text{loop}} = \left(\det \frac{\delta^2 S_E}{\delta x^i \delta x^j} \right)^{-1/2} = \frac{m_{\text{eff}} \rho_0 (q_e^2/m_{\text{eff}}^2 - 1)}{2\pi i q_e^2}$$

$$s = \frac{\tilde{\Delta}_a \tilde{x}^a}{\sqrt{(\tilde{\Delta}_a \tilde{x}^a)^2}}$$

$$X^0 = \frac{y}{r_h} t = \frac{r_h}{\rho} t, X^2 - X^1 = y = \frac{r_h^2}{\rho}, X^2 + X^1 = \frac{r_h^2}{y} - \frac{y t^2}{r_h^2} = \rho - \frac{t^2}{\rho}$$

$$\tilde{\sigma} = \tilde{\rho}_+ + \tilde{\rho}_-, \tilde{t} = \tilde{\rho}_+ - \tilde{\rho}_-, \text{ with } \tilde{\rho}_{\pm} = \tan^{-1}(\rho \pm t),$$

$$J_{\pm} = J_1 \pm iJ_2 = \pm i e^{\pm i\phi} [p_{\theta} \pm i(\cot \theta p_{\phi} - q_m \csc \theta)]$$

$$d\rho/dt = \sqrt{1 - \frac{\tilde{m}^2}{(E\rho + q_e)^2}}$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2$$

$$t \rightarrow t + F(r); \; dt \rightarrow dt + f(r)dr$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) (dt + f(r)dr)^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2(1 - 2m/r)f(r)drdt \\ + \left[\frac{1}{1 - 2m/r} - (1 - 2m/r)f(r)^2\right] dr^2 + r^2 d\Omega^2$$

$$f(r) = \frac{h(r)}{1 - 2m/r}$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 - 2h(r)drdt + \left[\frac{1 - h(r)^2}{1 - 2m/r}\right]dr^2 + r^2d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 \mp 2\sqrt{2m/r}drdt + dr^2 + r^2d\Omega^2$$

$$-dt^2 + (dr \mp \sqrt{2m/r}dt)^2 = 0$$

$$\frac{dr}{dt} = \pm 1 \pm \sqrt{2m/r}$$

$$\frac{dr}{dt} = \pm 1 - \sqrt{2m/r}$$

$$\frac{dr}{dt} = \pm 1 + \sqrt{2m/r}$$

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega^2 + \frac{2m}{r}(dt \pm dr)^2$$

$$-dt^2 + dr^2 + (2m/r)(dt \pm dr)^2 = 0$$

$$\frac{dr}{dt} = \mp 1, \text{ or } \frac{dr}{dt} = \pm 1 \mp \frac{4m}{r + 2m},$$

$$\frac{dr}{dt} = -1 \text{ (ingoing), or } \frac{dr}{dt} = 1 - \frac{4m}{r + 2m} \text{ ("outgoing")},$$

$$\frac{dr}{dt} = 1 \text{ (outgoing); or } \frac{dr}{dt} = -1 + \frac{4m}{r + 2m} \text{ ("ingoing")},$$

$$ds^2 = -(1 - 2m/r)dt^2 \mp 2drdt + r^2d\Omega^2$$

$$ds^2 = -(1 - 2m/r)dv^2 + 2dvdr + r^2d\Omega^2$$

$$[-(1 - 2m/r)dv + 2dr]dv = 0$$

$$\frac{dr}{dv} = -\infty; \frac{dr}{dv} = \frac{1 - 2m/r}{2}$$

$$ds^2 = -(1 - 2m/r)du^2 - 2dudr + r^2d\Omega^2$$

$$[-(1 - 2m/r)du - 2dr]du = 0$$

$$\frac{dr}{du} = +\infty; \frac{dr}{du} = -\frac{1 - 2m/r}{2}$$

$$F(r) = \int f(r)dr = \int \frac{h(r)}{1 - 2m/r}dr$$



$$F_{PG}(r) = \pm \int \frac{\sqrt{2m/r}}{1 - 2m/r} dr = \pm 2\sqrt{2mr} \pm 2m \ln \left(\frac{1 - \sqrt{2m/r}}{1 + \sqrt{2m/r}} \right)$$

$$F_{KS}(r) = \pm \int \frac{2m/r}{1 - 2m/r} dr = \pm 2m \ln (r - 2m)$$

$$F_{EF}(r) = \pm \int \frac{1}{1 - 2m/r} dr = \pm r \pm 2m \ln (r - 2m)$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r) dr dt + \left[\frac{1 - h(r)^2}{1 - 2m/r} \right] dr^2 + r^2 d\Omega^2$$

$$-\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r) dr dt + \left[\frac{1 - h(r)^2}{1 - 2m/r} \right] dr^2 = 0$$

$$-\left(1 - \frac{2m}{r}\right)^2 - 2\left(1 - \frac{2m}{r}\right) h(r) \dot{r} + [1 - h(r)^2] \dot{r}^2 = 0$$

$$\dot{r} = \mp \frac{1 - 2m/r}{1 \pm h(r)}.$$

$$\dot{r}_H = \pm \frac{1}{2mh'(2m)}$$

$$\Delta F = F(\infty) - F(0) = F(r_>) - F(r_<) = \int_{r_<}^{r_>} \frac{h(r)}{1 - 2m/r} dr = \int_{r_<}^{r_>} \frac{rh(r)}{r - 2m} dr$$

$$h(r) \rightarrow s(t)h(r).$$

$$ds^2 = -(1 - 2m/r) dt^2 - 2s(t)h(r) dr dt + \left[\frac{1 - s(t)^2 h(r)^2}{1 - 2m/r} \right] dr^2 + r^2 d\Omega^2$$

$$G_{tt} = 0; \quad G_{rr} = -\frac{2\dot{s}(t)h(r)}{r(1 - 2m/r)}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{d^2[s^2(t)]/dt^2 h(r)^2}{2(1 - 2m/r)} + h'(r)\dot{s}(t) - \frac{(1 - m/r)h(r)\dot{s}(t)}{r(1 - 2m/r)}$$

$$R = -\frac{d^2[s^2(t)]/dt^2 h(r)^2}{(1 - 2m/r)} - 2h'(r)\dot{s}(t) + \frac{2(2 - 3m/r)h(r)\dot{s}(t)}{r(1 - 2m/r)}.$$

$$\sqrt{-g} = r^2 \sin \theta$$

$$S = \int \sqrt{-g} R d^4 x = \int \sqrt{-g} R d^4 x = 4\pi \int r^2 R dt dr$$

$$\int_{-\infty}^{+\infty} \left(\frac{d^2[s^2(t)]}{dt^2} \right) dt = \left[\frac{d[s^2(t)]}{dt} \right]_{-\infty}^{+\infty} = 0 - 0 = 0$$

$$\int_{-\infty}^{+\infty} \left(\frac{ds(t)}{dt} \right) dt = [s(t)]_{-\infty}^{+\infty} = \pm 1 - (\mp 1) = \pm 2$$



$$S = \pm 4\pi \int r^2 \left[4h'(r) + \frac{4(2 - 3m/r)h(r)}{r(1 - 2m/r)} \right] dr$$

$$\int_{-\infty}^{+\infty} r^2 h'(r) dr = [r^2 h(r)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2rh(r) dr = - \int_{-\infty}^{+\infty} 2rh(r) dr$$

$$S = \pm 4\pi \int r \left[-8h(r) + \frac{4(2 - 3m/r)h(r)}{(1 - 2m/r)} \right] dr$$

$$S = \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1 - 2m/r)} dr = \pm 16\pi m \int_{r_<}^{r_>} \frac{rh(r)}{(r - 2m)} dr$$

$$S = \pm 16\pi m \Delta F$$

$$S = \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1 - 2m/r)} dr = \pm 16\pi m \int_{2m-\Delta}^{2m+\Delta} \frac{(2m/r)B(|r - 2m|)}{(1 - 2m/r)} dr$$

$$= \pm 32\pi m^2 \int_{2m-\Delta}^{2m+\Delta} \frac{B(|r - 2m|)}{(r - 2m)} dr = \pm 32\pi m^2 \int_{-\Delta}^{+\Delta} \frac{B(|z|)}{z} dz$$

$$S \propto \lim_{\epsilon \rightarrow 0} \left(\int_{-\Delta}^{-\epsilon} \frac{B(|z|)}{z} dz + \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz \right)$$

$$S \propto \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz - \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z} dz \right) = 0$$

$$-(1 - 2m/r)dt^2 - 2s(t)h(r)drdt + \left[\frac{1 - s(t)^2 h(r)^2}{1 - 2m/r} \right] dr^2 = 0$$

$$-(1 - 2m/r)^2 - 2s(t)h(r)(1 - 2m/r)\dot{r} + [1 - s(t)^2 h(r)^2]\dot{r}^2 = 0$$

$$\frac{dr}{dt} = \pm \frac{(1 - 2m/r)}{[1 \mp s(t)h(r)]}$$

$$k^a \propto (1, \dot{r}; 0, 0) = \left(1, \pm \frac{(1 - 2m/r)}{[1 \mp s(t)h(r)]}; 0, 0 \right)$$

$$G_{ab}k^a k^b \propto G_{rr} \left(\frac{(1 - 2m/r)^2}{[1 \mp s(t)h(r)]^2} \right) = - \frac{2\dot{s}(t)h(r)(1 - 2m/r)}{[1 \mp s(t)h(r)]^2}$$

$$G_{ab}k^a k^b \propto -\dot{s}(t)h(r)(1 - 2m/r)$$

$$g_{ab}T^{ab} = -(\rho - 3p) \leq 0$$

$$ds^2 = - \left(1 - \frac{2m}{r} \right) dt^2 - 2h(r)drdt + \left[\frac{1 - h(r)^2}{1 - 2m/r} \right] dr^2 + r^2 d\Omega^2$$

$$\dot{r}_H = \pm \frac{1}{2mh'(2m)}$$



$$ds^2 = -(1 - 2m/r)dt^2 - 2s(t)h(r)drdt + \left[\frac{1 - s(t)^2 h(r)^2}{1 - 2m/r} \right] dr^2 + r^2 d\Omega^2$$

$$\dot{r}_H = \pm \frac{1}{2mh'(2m)}$$

$$S = 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1 - 2m/r)} dr$$

$$g^{ab} = \Sigma^{-1} \begin{pmatrix} A_3 - B_3 & A_4 - B_4 \\ A_4 - B_4 & A_5 - B_5 \end{pmatrix}, g^{ij} = -\Sigma^{-1} \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$$

$$\Sigma = A_1 + B_1.$$

$$\nabla_{(\alpha} K_{\mu\nu)} = 0$$

$$\Sigma K^{ab} = \begin{pmatrix} A_1 B_3 + A_3 B_1 & A_1 B_4 + A_4 B_1 \\ A_1 B_4 + A_4 B_1 & A_1 B_5 + A_5 B_1 \end{pmatrix}, \Sigma K^{ij} = \begin{pmatrix} -A_2 B_1 & 0 \\ 0 & A_1 B_2 \end{pmatrix}$$

$$A_1 \leftrightarrow -B_1, A_i \leftrightarrow B_i, (i = 2..4), g^{rr} \leftrightarrow -g^{yy}, K^{rr} \leftrightarrow -K^{yy}.$$

$$g_{ab} = \frac{\Sigma}{\mathcal{P}} \begin{pmatrix} A_5 - B_5 & -A_4 + B_4 \\ -A_4 + B_4 & A_3 - B_3 \end{pmatrix}, g_{ij} = -\Sigma \begin{pmatrix} A_2^{-1} & 0 \\ 0 & B_2^{-1} \end{pmatrix}$$

$$\mathcal{P} = (A_3 - B_3)(A_5 - B_5) - (A_4 - B_4)^2.$$

$$A_3 - B_3 > 0, A_5 - B_5 < 0, \mathcal{P} < 0.$$

$$A_4 \equiv 0, B_4 \equiv 0.$$

$$r \rightarrow \tilde{r}(r), y \rightarrow \tilde{y}(y),$$

$$A_2 A_5 = a^2 = \text{const}, B_2 B_5 = b^2 = \Re.$$

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu.$$

$$l = 1/\sqrt{2\Sigma}(\sqrt{A_3}\partial_t + C\partial_\varphi - \sqrt{A_2}\partial_r), \quad m = 1/\sqrt{2\Sigma}(\sqrt{B_3}\partial_t + D\partial_\varphi - i\sqrt{B_2}\partial_y), \\ n = 1/\sqrt{2\Sigma}(\sqrt{A_3}\partial_t + C\partial_\varphi + \sqrt{A_2}\partial_r), \quad \bar{m} = 1/\sqrt{2\Sigma}(\sqrt{B_3}\partial_t + D\partial_\varphi + i\sqrt{B_2}\partial_y).$$

$$(A_3 - B_3)C = \sqrt{A_3}(A_4 - B_4) + \sqrt{B_3}\sqrt{-\mathcal{P}}, (A_3 - B_3)D = \sqrt{B_3}(A_4 - B_4) + \sqrt{A_3}\sqrt{-\mathcal{P}}.$$

$$A_4 = \sqrt{A_3 A_5}, B_4 = \sqrt{B_3 B_5}.$$

$$\sqrt{-\mathcal{P}} = \sqrt{A_3 B_5} - \sqrt{A_5 B_3}, \Rightarrow C = \sqrt{A_5}, D = \sqrt{B_5}.$$

$$l = 1/\sqrt{2\Sigma}(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi - \sqrt{A_2}\partial_r), \quad m = 1/\sqrt{2\Sigma}(\sqrt{B_3}\partial_t + \sqrt{B_5}\partial_\varphi - i\sqrt{B_2}\partial_y), \\ n = 1/\sqrt{2\Sigma}(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi + \sqrt{A_2}\partial_r), \quad \bar{m} = 1/\sqrt{2\Sigma}(\sqrt{B_3}\partial_t + \sqrt{B_5}\partial_\varphi + i\sqrt{B_2}\partial_y).$$

$$A_5 \equiv 0, B_3 \equiv 0, A_3 \neq 0, B_5 \neq 0$$

$$D = l^\mu \nabla_\mu, \Delta = n^\mu \nabla_\mu, \delta = m^\mu \nabla_\mu, \bar{\delta} = \bar{m}^\mu \nabla_\mu$$



$$Dl^\mu = (\epsilon + \bar{\epsilon})l^\mu - \bar{\kappa}m^\mu - \kappa\bar{m}^\mu$$

$$\Delta n^\mu = -(\gamma + \bar{\gamma})n^\mu + \nu m^\mu + \bar{\nu}\bar{m}^\mu$$

$$\sigma=-m^\mu\delta l_\mu,\bar{\lambda}=m^\mu\delta n_\mu$$

$$\kappa=\nu=0,\sigma=\lambda=0$$

$$\mu=\rho,\tau=\pi,\epsilon=\gamma,\alpha=\beta.$$

$$\begin{aligned}\Psi_0 &= -C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta \\ \Psi_1 &= -C_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta \\ \Psi_2 &= -C_{\alpha\beta\gamma\delta}(l^\alpha n^\beta l^\gamma n^\delta - l^\alpha n^\beta m^\gamma \bar{m}^\delta)/2 \\ \Psi_3 &= -C_{\alpha\beta\gamma\delta}n^\alpha l^\beta n^\gamma m^\delta \\ \Psi_4 &= -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta\end{aligned}$$

$$\Psi_0=0=\Psi_4$$

$$\Psi_1=\Psi_3$$

$$I=\Psi_0\Psi_4-4\Psi_1\Psi_3+3\Psi_2^2,J=\begin{vmatrix}\Psi_4&\Psi_3&\Psi_2\\ \Psi_3&\Psi_2&\Psi_1\\ \Psi_2&\Psi_1&\Psi_0\end{vmatrix}$$

$$I^3=27J^2$$

$$\Psi_1^2=k\Psi_2^2, \text{ with } k=9/16, \text{ or } 0.$$

$$l^\mu\rightarrow f^{-1}(r,y)l^\mu,n^\mu\rightarrow f(r,y)n^\mu,f=\sqrt{\Sigma/A_2}$$

$$\mathbf{Q}=\left(\begin{array}{ccc}\Psi_2-\frac{1}{2}(\Psi_0+\Psi_4) & \frac{1}{2}i(\Psi_4-\Psi_0) & \Psi_1-\Psi_3 \\ \frac{1}{2}i(\Psi_4-\Psi_0) & \Psi_2+\frac{1}{2}(\Psi_0+\Psi_4) & i(\Psi_1+\Psi_3) \\ \Psi_1-\Psi_3 & i(\Psi_1+\Psi_2) & -2\Psi_2\end{array}\right)$$

$$\mathbf{Q}=\left(\begin{array}{ccc}\Psi_2&0&0\\ 0&\Psi_2&2i\Psi_1\\ 0&2i\Psi_1&-2\Psi_2\end{array}\right).$$

$$K_{ln}=B_1,K_{m\bar{m}}=A_1.$$

$$\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}g^{\mu\nu}=\mu^2$$

$$S=-Et+L\varphi+S_r(r)+S_y(y),$$

$$\begin{aligned}A_2p_r^2+U_r&=0, & B_2p_y^2+U_y&=0, \\ U_r&=\mathcal{C}-\varepsilon_a^2+A_1\mu^2, & U_y&=-\mathcal{C}+\varepsilon_b^2+B_1\mu^2, \\ \varepsilon_a&=E\sqrt{A_3}-L\sqrt{A_5}, & \varepsilon_b&=E\sqrt{B_3}-L\sqrt{B_5},\end{aligned}$$

$$\mathcal{C}=p_\mu p_\nu K^{\mu\nu}=\Sigma^{-1}\big[A_1\big(\varepsilon_b^2+B_2p_y^2\big)+B_1\big(\varepsilon_a^2-A_2p_r^2\big)\big].$$



$$U_r=0=U_r'$$

$$\mu^2 \Rightarrow \mathcal{M}^2(x)$$

$$\mathcal{M}^2(x)=\frac{\mathcal{M}_r^2(r)+\mathcal{M}_y^2(y)}{A_1+B_1}.$$

$$\square\,\phi=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)=-\mu^2\phi$$

$$\sqrt{-g}=\frac{\Sigma^2}{\sqrt{A_2B_2}\sqrt{-\mathcal{P}}}$$

$$\sqrt{-g}=\frac{\Sigma^2}{\sqrt{A_2B_2}(\sqrt{A_3B_5}-\sqrt{A_5B_3})}$$

$$\sqrt{-g}=\Sigma$$

$$\Sigma=\sqrt{A_2B_2}(\sqrt{A_3B_5}-\sqrt{B_3A_5})$$

$$A_1+B_1=bA_{23}-aB_{23}$$

$$A_{23}=\sqrt{A_2A_3},B_{23}=\sqrt{B_2B_3},$$

$$A_1'=bA_{23}',B_1'=aB_{23}'$$

$$\phi(x^\mu) = \mathrm{e}^{-i\omega t + im\varphi} R(r) Y(y),$$

$$\frac{((A_2)'R)'}{R}+\frac{((B_2)'Y)'}{Y}+U(r)-V(y)=0,$$

$$U(r)=\left(\omega\sqrt{A_3}-m\sqrt{A_5}\right)^2-\mu^2A_1,$$

$$V(y)=\left(\omega\sqrt{B_3}-m\sqrt{B_5}\right)^2+\mu^2B_1.$$

$$\hat{K}=\nabla_\mu K^{\mu\nu}\nabla_\nu$$

$$[\square,\hat{K}]\phi=\frac{4}{3}\nabla_\alpha(K_\sigma\,^{[\alpha}R^{\beta]\sigma})\nabla_\beta\phi$$

$$K_\sigma\,^{[\alpha}R^{\beta]\sigma}=2(K_{ln}+K_{m\bar{m}})(n^\beta(\bar{m}^\alpha\Phi_{01}+m^\alpha\Phi_{10})-n^\alpha(\bar{m}^\beta\Phi_{01}+m^\beta\Phi_{10})\\ + (l^\beta\bar{m}^\alpha-l^\alpha\bar{m}^\beta)\Phi_{12}+(l^\beta m^\alpha-l^\alpha m^\beta)\Phi_{21})$$

$$\Phi_{01}=R_{\mu\nu}l^\mu m^\nu/2=\overline{\Phi_{10}},\Phi_{12}=R_{\mu\nu}n^\mu m^\nu/2=\overline{\Phi_{21}}$$

$$aA_{23}''-bB_{23}''=0$$

$$a^2A_1''-b^2B_1''=0$$



$$\begin{aligned}\square &= (D + \varepsilon + \bar{\varepsilon} - \rho - \bar{\rho})\Delta + (\Delta + \mu + \bar{\mu} - \gamma - \bar{\gamma})D \\ &\quad - (\delta - \tau - \bar{\alpha} + \beta + \bar{\pi})\bar{\delta} - (\bar{\delta} - \alpha - \bar{\tau} + \pi + \bar{\beta})\delta \\ \hat{K} &= (D + \varepsilon + \bar{\varepsilon} - \rho - \bar{\rho})K_{ln}\Delta + (\Delta + \mu + \bar{\mu} - \gamma - \bar{\gamma})K_{ln}D \\ &\quad + (\delta - \tau - \bar{\alpha} + \beta + \bar{\pi})K_{m\bar{m}}\bar{\delta} + (\bar{\delta} - \alpha - \bar{\tau} + \pi + \bar{\beta})K_{m\bar{m}}\delta\end{aligned}$$

$$\begin{aligned}D_n^\pm &= \sqrt{A_3/A_2}\partial_t \pm \partial_r + \sqrt{A_5/A_2}\partial_\varphi \pm n\partial_r(\ln A_2) \\ L_s^\pm &= \sqrt{B_3/B_2}\partial_t \pm i\partial_y + \sqrt{B_5/B_2}\partial_\varphi \pm is\partial_y(\ln B_2)\end{aligned}$$

$$D_n^\pm A_2{}^k = A_2{}^k D_{n+k}^\pm, L_s^\pm B_2{}^l = B_2{}^l L_{s+l}^\pm$$

$$D=\sqrt{A_2/2\Sigma}D_0^-, \Delta=\sqrt{A_2/2\Sigma}D_0^+, \delta=\sqrt{B_2/2\Sigma}L_0^-, \bar{\delta}=\sqrt{B_2/2\Sigma}L_0^+$$

$$\begin{aligned}\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho}&=\sqrt{A_2/8\Sigma^3}\big(2\Sigma\partial_r\ln\big[\sqrt{A_3B_5}-\sqrt{A_5B_3}\big]-3\partial_r\Sigma\big) \\ \bar{\pi}-\tau+\beta-\bar{\alpha}&=-i\sqrt{B_2/8\Sigma^3}\big(2\Sigma\partial_y\ln\big[\sqrt{A_3B_5}-\sqrt{A_5B_3}\big]-3\partial_y\Sigma\big)\end{aligned}$$

$$\begin{aligned}\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho}&=\sqrt{A_2}\partial_r(2\Sigma)^{-1/2}-(2\Sigma)^{-1/2}\partial_r\sqrt{A_2} \\ \bar{\pi}-\tau+\beta-\bar{\alpha}&=i\sqrt{B_2}\partial_y(2\Sigma)^{-1/2}-i(2\Sigma)^{-1/2}\partial_r\sqrt{B_2}\end{aligned}$$

$$\square=\frac{1}{2\Sigma}\left[A_2(D_1^-D_0^++D_1^+D_0^-)-B_2(L_1^-L_0^++L_1^+L_0^-)\right],$$

$$\hat{K}=\frac{1}{2\Sigma}\left[A_2B_1(D_1^-D_0^++D_1^+D_0^-)+B_2A_1(L_1^-L_0^++L_1^+L_0^-)\right]=\frac{B_2}{2}(L_1^-L_0^++L_1^+L_0^-)+B_1\square.$$

$$ds^2=\frac{A_2B_2}{\Sigma}(\sqrt{B_5}dt-\sqrt{B_3}d\varphi)^2-\frac{A_2B_2}{\Sigma}(\sqrt{A_5}dt-\sqrt{A_3}d\varphi)^2-\frac{\Sigma}{A_2}dr^2-\frac{\Sigma}{B_2}dy^2$$

$$\Sigma=\sqrt{A_2B_2}(\sqrt{A_3B_5}-\sqrt{A_5B_3}).$$

$$S=\int\,d^4x\left[\left(R-\frac{1}{2}f_{AB}\partial_\mu\Psi^A\partial^\mu\Psi^B-\frac{1}{2}K_{IJ}F_{\mu\nu}^IF^{J\mu\nu}\right)\sqrt{-g}-\frac{1}{2}H_{IJ}F_{\mu\nu}^IF_{\lambda\tau}^J\epsilon^{\mu\nu\lambda\tau}\right]$$

$$R_{\mu\nu}=\frac{1}{2}f_{AB}\Psi_{,\mu}^A\Psi_{,\nu}^B-K_{IJ}\left(F_{\mu\lambda}^IF_{\nu}^{J\lambda}+\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}^IF^{J\alpha\beta}\right)$$

$$\nabla_{(\alpha}Y_{\mu)\nu}=0$$

$$Y_\mu{}^\alpha Y_{\alpha\nu}=K_{\mu\nu}$$

$$Y_{ln}^2=K_{ln}, Y_{m\bar{m}}^2=-K_{m\bar{m}}$$

$$Y_{ln}=p\sqrt{B_1}, Y_{m\bar{m}}=iq\sqrt{A_1}, p=\pm1, q=\pm1$$

$$\begin{aligned}(\tau+\bar{\pi})Y_{ln}-(\tau-\bar{\pi})Y_{m\bar{m}}&=0, & (\rho+\bar{\rho})Y_{ln}-(\rho-\bar{\rho})Y_{m\bar{m}}&=0 \\ DY_{m\bar{m}}-\rho(Y_{ln}-Y_{m\bar{m}})&=0, & \delta Y_{ln}-\bar{\pi}(Y_{ln}+Y_{m\bar{m}})&=0 \\ \Delta Y_{m\bar{m}}+\mu(Y_{ln}-Y_{m\bar{m}})&=0, & \delta Y_{ln}+\tau(Y_{ln}-Y_{m\bar{m}})&=0\end{aligned}$$

$$\mu=\rho=\frac{b}{2\Sigma}\sqrt{A_2/2\Sigma}(A'_{23}-iB'_{23}), \tau=\pi=\frac{a}{2\Sigma}\sqrt{B_2/2\Sigma}(A'_{23}-iB'_{23})$$



$$pA'_{23}\sqrt{B_1}=qB'_{23}\sqrt{A_1}$$

$$pA'_{23}=2\sqrt{A_1},\,qB'_{23}=2\sqrt{B_1}$$

$$(\sqrt{A_1})'=b/p,(\sqrt{B_1})'=-a/q$$

$$A_1+B_1=bA_{23}-aB_{23}$$

$$\begin{array}{l} A_1' = 2b\sqrt{A_1}, \quad B_1' = 2a\sqrt{B_1} \\ A_{23}' = 2\sqrt{A_1}, \quad B_{23}' = -2\sqrt{B_1} \end{array}$$

$$\begin{array}{ll} A_1=(br+c_1)^2, & B_1=(ay+d_1)^2 \\ A_{23}=br^2+2c_1r+c_2, & B_{23}=-(ay^2+2d_1y+d_2) \end{array}$$

$$bc_2+ad_2=c_1^2+d_1^2$$

$$Y_{ln}=\sqrt{B_1}, Y_{m\bar{m}}=-i\sqrt{A_1}$$

$$8\Sigma^3\Psi_{1,3}=-\sqrt{A_2B_2}\{\Sigma(aA_{23}''-bB_{23}'')-ab(A_{23}'^2+B_{23}'^2)\},$$

$$\nabla_{(\alpha}\mathcal{K}_{\mu\nu)}=\Omega_{(\alpha}g_{\mu\nu)},6\Omega_{\alpha}=(2\nabla_{\sigma}\mathcal{K}_{\alpha}{}^{\sigma}+\nabla_{\alpha}\mathcal{K}),\mathcal{K}=g^{\mu\nu}\mathcal{K}_{\mu\nu}$$

$$\begin{array}{ll} (D+\rho+\bar{\rho})(\mathcal{K}_{ln}+\mathcal{K}_{m\bar{m}})=0, & (\delta+\tau-\bar{\pi})(\mathcal{K}_{ln}+\mathcal{K}_{m\bar{m}})=0 \\ (\Delta-\mu-\bar{\mu})(\mathcal{K}_{ln}+\mathcal{K}_{m\bar{m}})=0, & (\bar{\delta}+\bar{\tau}-\pi)(\mathcal{K}_{ln}+\mathcal{K}_{m\bar{m}})=0. \end{array}$$

$$\mathcal{K}_{ln}=\Sigma-S(r,y),\mathcal{K}_{m\bar{m}}=S(r,y)$$

$$K^{\mu\nu}=-A_1g^{\mu\nu}-A_2\delta_r^\mu\delta_r^\nu+\tilde{K}_r^{\mu\nu}$$

$$\tilde{K}_r^{\mu\nu}=A_3\delta_t^\mu\delta_t^\nu+2A_4\delta_t^{(\mu}\delta_\varphi^{\nu)}+A_5\delta_\varphi^\mu\delta_\varphi^\nu$$

$$K^{\mu\nu}=B_1g^{\mu\nu}+B_2\delta_y^\mu\delta_y^\nu+\tilde{K}_y^{\mu\nu}$$

$$\tilde{K}_y^{\mu\nu}=B_3\delta_t^\mu\delta_t^\nu+2B_4\delta_t^{(\mu}\delta_\varphi^{\nu)}+B_5\delta_\varphi^\mu\delta_\varphi^\nu$$

$$A_2(r_h)=0, A_2>0 \text{ for } r>r_h$$

$$n_\mu=-\sqrt{\Sigma/A_2}\delta_\mu^r$$

$$h_{\mu\nu}=g_{\mu\nu}+n_\mu n_\nu, \chi_{\mu\nu}=h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta=h_\mu^\alpha h_\nu^\beta (n_{\beta,\alpha}-\Gamma_{\alpha\beta}^\lambda n_\lambda)$$

$$\chi_{\mu\nu}dx^\mu dx^\nu=\frac{n_r}{2}\,g^{rr}\big(g_{ab,r}dx^a dx^b+g_{yy,r}dy^2\big)$$

$$\chi_{\mu\nu}=\frac{\chi^\alpha_\alpha}{3}\,h_{\mu\nu},\mu,\nu=t,\varphi,y$$

$$(\ln\;g_{tt})_{,r}=(\ln\;|g_{\varphi\varphi}|)_{,r}=(\ln\;|g_{yy}|)_{,r}$$

$$(\ln\;A_2)'=2(\ln\;A_{23})', \text{ or simply } A_3'=0,$$



$$A_2 = r^2 - 2Mr + Q^2, A_{23} = r^2$$

$$r^2 - 3Mr + 2Q^2 = 0 \Rightarrow r = \frac{3}{2} \left(M \pm \sqrt{M^2 - 8Q^2/9} \right)$$

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{A_2}{\Sigma} (bdt - B_{23} d\varphi)^2 - \frac{B_2}{\Sigma} (adt - A_{23} d\varphi)^2 - \frac{\Sigma}{B_2} dy^2$$

$$\chi_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2} \sqrt{\frac{A_2}{\Sigma}} \partial_r h_{\mu\nu} dx^\mu dx^\nu$$

$$\frac{1}{2} h_{yy} \sqrt{|g^{rr}|} \partial_r p^2 = \chi_{yy} (p^2 - \mu^2), p^2 = g^{ab} p_a p_b, p_a = (-E, L)$$

$$(\sqrt{A_3}E - \sqrt{A_5}L)'(\sqrt{A_3}E - \sqrt{A_5}L) = \frac{1}{2} \mu^2 A_1'$$

$$ds^2 = \Delta_r/\Sigma (dt - a \sin^2 \theta d\varphi)^2 - 1/\Sigma \sin^2 \theta (adt - (r^2 + a^2) d\varphi)^2 - \Sigma/\Delta_r dr^2 - \Sigma d\theta^2$$

$$\Delta_r = r^2 - 2Mr + a^2 + Q^2, \Sigma = r^2 + a^2 \cos \theta$$

$$A_1 = r^2, \quad B_1 = a^2 y^2$$

$$A_3 = (r^2 + a^2)^2 / \Delta_r, \quad B_3 = a^2 (1 - y^2)$$

$$A_5 = a^2 / A_2 = a^2 / \Delta_r, \quad B_5 = 1 / B_2 = 1 / (1 - y^2)$$

$$A_4^2 = A_3 A_5, B_4^2 = B_3 B_5, b = 1, A_1 + B_1 = \sqrt{A_2 B_2} (\sqrt{A_3 B_5} - \sqrt{A_5 B_3}).$$

$$\Psi_2 = \frac{Q^2 - M(r - iay)}{(r - iay)^3 (r + iay)}, \Phi_{11} = \frac{Q^2}{2(r^2 + a^2)^2}$$

$$(E(r^2 + a^2) - aL)[Er(\Delta_r - Mr + Q^2) + aL(r - m) + a^2 EM] - \mu^2 r \Delta_r^2 = 0,$$

$$ds^2 = \frac{\Delta_r - a^2 \sin^2 \theta}{\Sigma} (dt - w d\varphi)^2 - \Sigma \left(\frac{dr^2}{\Delta_r} + d\theta^2 + \frac{\Delta_r \sin^2 \theta}{\Delta_r - a^2 \sin^2 \theta} d\varphi^2 \right)$$

$$\Delta_r = (r - r_-)(r - 2M) + a^2 - (N - N_-)^2$$

$$\Sigma = r(r - r_-) + (a \cos \theta + N)^2 - N_-^2$$

$$w = \frac{2}{a^2 \sin^2 \theta - \Delta_r} [N \Delta_r \cos \theta + a \sin^2 \theta (M(r - r_-) + N(N - N_-))]$$

$$r_- = \frac{M|Q - iP|^2}{|M + iN|^2}, N_- = \frac{N|Q - iP|^2}{2|M + iN|^2}$$

$$A_1 = r(r - r_-), \quad B_1 = (ay + N)^2 - N_-^2$$

$$A_3 = (r(r - r_-) + a^2 + N^2 - N_-^2)^2 / \Delta_r, \quad B_3 = [a(1 - y^2) - 2Ny]^2 / 1 - y^2$$

$$A_5 = a^2 / A_2 = a^2 / \Delta_r, \quad B_5 = 1 / B_2 = 1 / 1 - y^2$$

$$A_4^2 = A_3 A_5, B_4^2 = B_3 B_5, a = a, b = 1, \quad A_1 + B_1 = \sqrt{A_2 B_2} (\sqrt{A_3 B_5} - \sqrt{A_5 B_3}),$$



$$\Psi_1 = \Psi_3 = \frac{a(4N_-^2 + r_-^2)\sin \theta \sqrt{\Delta_r}}{8\Sigma^3}$$

$$12\Sigma^3\Psi_2 = -12(M - iN)(r + i(\cos \theta + N))^3 + 6NN_-(2(r + i(N + \cos \theta)) - r_-)^2$$

$$+ r_-^3(8M - r) + 8NN_-^3 - 4N_-^4 - 6ar_- \cos \theta (5M - 3iN)(\cos \theta + 2N - 2ir)$$

$$+ 2r_-(M(15(r + iN)^2 + 7N_-^2) + 18N^2r + 9iN^3 + 3iN(N_-^2 - 3r^2) - 2N_-^2r)$$

$$+ 4N_-^2(2a^2 - \cos \theta (3i(M + iN) + \cos \theta) - 5Mr + 2N^2 - 3iN(M + r))$$

$$+ 4N_-^2r^2 + r_-^2(2a^2 - 24iMN - 7N^2 + 2NN_- - N_-^2)$$

$$- r_-^2(\cos \theta (6i(4M - iN) + \cos \theta) + 26Mr - 6iNr - r^2)$$

$$16\Sigma^3\Phi_{11} = a^2\cos^2(\theta)(8Mr_- + 16NN_- - 4N_-^2 - r_-^2) + 16MN_-^2r_- - 24MN_-^2r$$

$$+ 8aN\cos \theta (2Mr_- + 4NN_- - 4N_-^2 - r_-^2) + N^2(8Mr_- - 28N_-^2 - 7r_-^2)$$

$$+ 8Mr_-r^2 - 14Mr_-^2r + 16N^3N_- + 2NN_-(4N_-^2 + 3r_-^2 + 8r^2 - 8r_-r)$$

$$+ N_-^2r_-^2 - 4N_-^2r^2 + 4N_-^2r_-r + 4N_-^4 - r_-^2r^2 + r_-^3r + 6Mr_-^3$$

$$\Lambda = \frac{R}{24} = -\frac{(4N_-^2 + r_-^2)(a^2\sin^2 \theta + \Delta_r)}{48\Sigma^3}$$

$$E(r(r - r_-) + N^2 - N_-^2)[(2r - r_-)(r^2 - 2rr_- + a^2 + N^2 - 2NN_- + 3N_-^2) -$$

$$- 2M(3r^2 - rr_- + 2r_-^2 + N^2 - N_-^2)] - 4a^2ELM + a^4E^2(2r + 2M - r_-) +$$

$$+ a^2[2E^2(2(2r - r_-)(N_-^2 - NN_-) - 2M((r - r_-)^2) - N^2 + N_-^2) - L^2(2r - r_- - 2M)] +$$

$$+ 4aEL[M(r - r_-) - N^2 + N_-^2 + N_-(N - N_-)(2r - r_-)] - \mu^2\Delta^2(2r - r_-) = 0.$$

$$w = \frac{2}{\Delta_r - a^2\sin^2 \theta} \left(N\Delta_r \cos \theta + a\sin^2 \theta \left[m(r - (m + |Y|)) + \frac{1}{2}(|\mathcal{M}|^2 - |Y|^2) \right] \right)$$

$$\Delta_r = r[r - 2(m + |Y|)] + a^2 + (m + |Y|)^2$$

$$\Sigma = r(r - 2|Y|) + (\cos \theta + N)^2$$

$$\Upsilon = -\frac{2}{\mathcal{M}} \sum_n [\bar{\Gamma}_{(n)}]^2$$

$$\Gamma_{(n)} = \frac{1}{2}(Q_{(n)} + iP_{(n)}), n = 1, 2$$

$$|\mathcal{M}|^2 + |Y|^2 - 4 \sum_n |\Gamma_{(n)}|^2 = 0$$

$$\begin{aligned} A_1 &= r(r - 2|Y|), & B_1 &= N + ay \\ A_3 &= (r(r - 2|Y|) + N^2 + a^2)^2/\Delta_r, & B_3 &= (a(1 - y^2) - 2Ny)^2/1 - y^2 \\ A_5 &= a^2/A_2 = a^2/\Delta_r, & B_5 &= 1/B_2 = 1/(1 - y^2) \end{aligned}$$

$$\begin{aligned}\Psi_1 = \Psi_3 &= \frac{a \sin \theta |Y|^2 \sqrt{\Delta_r}}{2\Sigma^3} \\ 3\Sigma^3\Psi_2 &= 3(m - iN)(m - r - i a \cos \theta)(r + i(N + a \cos \theta))^2 + \\ &\quad + 3(m - iN)(3(r + i a \cos \theta) - 2m + iN)(r + i(N + a \cos \theta))|Y| - \\ &\quad - 2|Y|^2(3N^2 - 2(m^2 + r^2) + 7mr + 6iN(m - r) + a^2 \cos 2\theta) - \\ &\quad - 2|Y|^3(4(r - m) + 3i(N + a \cos \theta)) + 3ia|Y|^2 \cos \theta(2(m - iN) - r) + 4|Y|^4 \\ 4\Sigma^3\Phi_{11} &= 3|Y|^2(a^2 \cos^2 \theta + (m - r + |Y|)^2) + \\ &\quad + 2\Sigma(m^2 - a \cos \theta(2N + a \cos \theta) - (r - |Y|)^2 + \Sigma) \\ \Phi_{00} &= \\ \Lambda = \Phi_{22} &= \frac{|Y|^2 \Delta_r}{2\Sigma^3}, \Phi_{02} = -\frac{a^2 |Y|^2 \sin^2 \theta}{2\Sigma^3}\end{aligned}$$

$$\begin{aligned}\Psi_1 = \Psi_3 &= \frac{a \sin \theta |\Upsilon|^2 \sqrt{\Delta_r}}{2\Sigma^3}, \\ 3\Sigma^3\Psi_2 &= 3(m - iN)(m - r - i a \cos \theta)(r + i(N + a \cos \theta))^2 + \\ &\quad + 3(m - iN)(3(r + i a \cos \theta) - 2m + iN)(r + i(N + a \cos \theta))|\Upsilon| - \\ &\quad - 2|\Upsilon|^2(3N^2 - 2(m^2 + r^2) + 7mr + 6iN(m - r) + a^2 \cos 2\theta) - \\ &\quad - 2|\Upsilon|^3(4(r - m) + 3i(N + a \cos \theta)) + 3ia|\Upsilon|^2 \cos \theta(2(m - iN) - r) + 4|\Upsilon|^4, \\ 4\Sigma^3\Phi_{11} &= 3|\Upsilon|^2(a^2 \cos^2 \theta + (m - r + |\Upsilon|)^2) + \\ &\quad + 2\Sigma(m^2 - a \cos \theta(2N + a \cos \theta) - (r - |\Upsilon|)^2 + \Sigma), \\ \Phi_{00} = \Phi_{22} &= \frac{|\Upsilon|^2 \Delta_r}{2\Sigma^3}, \quad \Phi_{02} = -\frac{a^2 |\Upsilon|^2 \sin^2 \theta}{2\Sigma^3}, \\ \Lambda &= -\frac{|\Upsilon|(a^2(1 + \sin^2 \theta) + (m - r + |\Upsilon|)^2)}{12\Sigma^3}.\end{aligned}$$

$$E(r(r - 2|Y|) + N^2 + a^2)[\{E(\Delta_r + |Y|^2 - M^2 - N^2) + aL\}(r - M - |Y|) + 2a^2EM] - \mu^2 \Delta_r^2(r - |Y|) = 0$$

$$\begin{aligned}\Delta_r &= r^2 - 2Mr + a^2 \\ w &= -\frac{2Ma\omega \sin^2 \theta}{\Delta_r - a^2 \sin^2 \theta}, \omega = ((\Pi_c - \Pi_s)r + 2M\Pi_s) \\ \Sigma^2 &= \prod_{l=0}^4 (r + 2Ms_l^2) + a^4 \cos^4 \theta + \\ &\quad + 2a^2 \cos^2 \theta \left(r^2 + Mr \sum_{l=0}^3 s_l^2 + 4M^2(\Pi_c - \Pi_s)\Pi_s - 2M^2 \sum_{l < j < k}^3 s_l^2 s_j^2 s_k^2 \right) \\ \Pi_s &= \prod_{i=1}^4 s_i = \prod_{l=1}^4 \sinh \delta_l, \Pi_c = \prod_{l=1}^4 \sqrt{1 + s_l^2} = \prod_{l=1}^4 \cosh \delta_l, s_l^2 = \sinh^2 \delta_l\end{aligned}$$

$$\Sigma_{2ch} = r^2 + a^2 \cos^2 \theta + 2Mr(\mathcal{S}_1^2 + \mathcal{S}_2^2) + 4M^2 \mathcal{S}_1^2 \mathcal{S}_2^2 = r^2 - 2Mr + a^2 \cos^2 \theta + 2Mw_{2ch}$$



$$\begin{aligned}
A_1 &= r^2 - 2Mr + 2Mw_{2ch}, & B_1 &= a^2 y^2 \\
A_3 &= (r^2 - 2Mr + a^2 + 2Mw_{2ch})^2 / \Delta_r, & B_3 &= a^2 (1 - y^2) \\
A_5 &= a^2 / A_2 = a^2 / \Delta_r, & B_5 &= 1 / B_2 = 1 / (1 - y^2)
\end{aligned}$$

$$\Psi_1 = \Psi_3 = \frac{aM^2 \sin \theta (S_1^2 - S_2^2)^2 \sqrt{\Delta_r}}{2\Sigma_{2ch}^3}$$

$$\begin{aligned}
\frac{3\Sigma_{2ch}^3}{M} \Psi_2 &= -3(r + i \cos \theta)^3 - 3(r + i \cos \theta)^2 (M + r + i \cos \theta) (S_1^2 + S_2^2) \\
&\quad - 2M(r^2 + 2Mr - a^2 + 3i \cos \theta + a^2 y^2) (S_1^4 + S_2^4) \\
&\quad + 12M^2 S_1^2 S_2^2 (M - r - i \cos \theta) (S_1^2 + S_2^2) \\
&\quad + 4M S_1^2 S_2^2 (5a^2 \cos^2 \theta - a^2 + 4Mr - 5r^2 + 3iaM \cos \theta - 9i \cos \theta) \\
\Phi_{00} = \Phi_{22} &= \frac{M^2 \Delta_r (S_1^2 - S_2^2)^2}{2\Sigma_{2ch}^3}, \quad \Phi_{02} = \overline{\Phi_{20}} = -\frac{a^2 M^2 \sin^2 \theta (S_1^2 - S_2^2)^2}{2\Sigma_{2ch}^3}
\end{aligned}$$

$$\begin{aligned}
\frac{4\Sigma_{2ch}^3}{M^2} \Phi_{11} &= 4(r^2 + a^2 \cos^2 \theta) (S_1^2 + S_2^2) + (2Mr + 3r^2 + 3a^2 \cos^2 \theta) (S_1^4 + S_2^4) \\
&\quad + 16M(r + M) S_1^2 S_2^2 (S_1^2 + S_2^2) + 32M^2 S_1^4 S_2^4 + 2S_1^2 S_2^2 (r^2 + 14Mr + a^2 \cos^2 \theta) \\
\Lambda &= \frac{1}{24R} = -\frac{M^2 (S_1^2 - S_2^2)^2 (\Delta_r + a^2 \sin^2 \theta)}{12\Sigma_{2ch}^3}
\end{aligned}$$

$$\begin{aligned}
E^2 [\Delta^2 (r - M + 2M\Sigma_{s^2}) - 4M^2 r (Mr - a^2) \Sigma_{s^2}^2 - 16M^4 (r - M) \Pi_{s^2}^2 - 8M^3 (r^2 - a^2) \Pi_{s^2} \Sigma_{s^2}] \\
+ 2aELM [4M(r - M) \Pi_{s^2} - (r^2 - a^2) \Sigma_{s^2}] - a^2 L(r - M) - \mu^2 \Delta^2 (r - M + M\Sigma_{s^2}) = 0,
\end{aligned}$$

$$\Sigma_{s^2} = S_1^2 + S_2^2, \Pi_{s^2} = S_1^2 S_2^2$$

$$\begin{aligned}
\mu = \rho &= -\sqrt{A_2} \frac{\partial}{\partial r} \frac{1}{\sqrt{2\Sigma}} - \frac{iB_5 \sqrt{B_2}}{2\sqrt{2\Sigma}} \frac{\partial_y \sqrt{B_3/B_5}}{\sqrt{A_3 B_5} - \sqrt{A_5 B_3}} \\
\tau = \pi &= -i\sqrt{B_2} \frac{\partial}{\partial y} \frac{1}{\sqrt{2\Sigma}} + \frac{A_5 \sqrt{A_2}}{2\sqrt{2\Sigma}} \frac{\partial_r \sqrt{A_3/A_5}}{\sqrt{A_3 B_5} - \sqrt{A_5 B_3}} \\
\epsilon = \gamma &= \frac{1}{2} \rho + \frac{\sqrt{A_2}}{2\sqrt{2\Sigma}^{3/2}} (\Sigma \partial_r \ln [\sqrt{A_3 B_5} - \sqrt{A_5 B_3}] - \partial_r \Sigma) \\
\alpha = \beta &= \frac{1}{2} \tau + i \frac{\sqrt{B_2}}{2\sqrt{2\Sigma}^{3/2}} (\Sigma \partial_y \ln [\sqrt{A_3 B_5} - \sqrt{A_5 B_3}] - \partial_y \Sigma)
\end{aligned}$$

$$\begin{aligned}
\Phi_{00} &= \Phi_{22} = \frac{bA_2}{8\Sigma^3} (bA_{23}'^2 + bB_{23}'^2 - 2\Sigma A_{23}''), \\
\Phi_{01} &= \overline{\Phi_{10}} = \Phi_{12} = \overline{\Phi_{21}} = -\frac{\sqrt{A_2 B_2}}{8\Sigma^3} (aA_{23}'' - bB_{23}''), \\
\Phi_{02} &= \overline{\Phi_{20}} = -\frac{aB_2}{8\Sigma^3} (aA_{23}'^2 + aB_{23}'^2 + 2\Sigma B_{23}''), \\
64\Sigma^3\Phi_{11} &= \Sigma^2\{A_2[(\ln A_5)_{,r}^2 - 4(\ln A_5)_{,rr}] - B_2[(\ln B_5)_{,r}^2 - 4(\ln B_5)_{,rr}]\} \\
&+ 3\{4b^2A_2B_{23}'^2 - 4a^2B_2A_{23}'^2 + A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}']^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}']^2\} \\
&+ 4\Sigma\{A_2((\ln A_5)_{,r})[\Sigma(\ln A_5)_{,r} + 2bA_{23}'] - A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}'], r\} \\
&- 4\Sigma\{B_2((\ln B_5)_{,y})[\Sigma(\ln B_5)_{,y} - 2aB_{23}'] - B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}'], y\}, \\
192\Sigma^3\Lambda &= 8\Sigma^3R = 3\Sigma^2\{A_2[3(\ln A_5)_{,r}^2 - 4(\ln A_5)_{,rr}] + B_2[3(\ln B_5)_{,y}^2 - 4(\ln B_5)_{,yy}]\} \\
&+ 4A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}'], r + 4B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}'], y - 4a^2B_2A_{23}'^2 \\
&- A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}']^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}']^2 - 4b^2A_2B_{23}'^2 \\
-12\Sigma^3\Psi_2 &= \Sigma\{A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}'], r + B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}'], y\} + 2a^2B_2A_{23}'^2 \\
&- A_2[\Sigma(\ln A_5)_{,r} + 2bA_{23}']^2 - B_2[\Sigma(\ln B_5)_{,y} - 2aB_{23}']^2 + 2b^2A_2B_{23}'^2 \\
&+ 3i\{aB_2A_{23}'[\Sigma(\ln B_5)_{,y} - 2aB_{23}'] + bA_2B_{23}'[\Sigma(\ln A_5)_{,r} + 2bA_{23}']\}
\end{aligned}$$

$$S = \frac{1}{16\pi G} \int d^5x e \left[R + 12g^2 - \frac{1}{4}F^2 - \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda \right]$$

$$\begin{aligned}
E &= \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a + \Xi_b)}{4G\Xi_a^2\Xi_b^2}, Q = \frac{\sqrt{3}\pi q}{4G\Xi_a\Xi_b} \\
J_1 &= \frac{\pi[2am + qb(1 + a^2g^2)]}{4G\Xi_a^2\Xi_b}, J_2 = \frac{\pi[2bm + qa(1 + b^2g^2)]}{4G\Xi_b^2\Xi_a}
\end{aligned}$$

$$\frac{\pi\ell_5^3}{4G}=\frac{1}{2}N^2$$

$$\Delta_r(r)=\frac{(r^2+a^2)(r^2+b^2)(1+r^2)+q^2+2abq}{r^2}-2m=0,$$

$$\begin{aligned}
T &\equiv \beta^{-1} = \frac{r_+^4[(1+g^2(2r_+^2+a^2+b^2)) - (ab+q)^2]}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2) + abq]} \\
\Omega_1 &= \frac{a(r_+^2+b^2)(1+g^2r_+^2) + bq}{(r_+^2+a^2)(r_+^2+b^2) + abq}, \Omega_2 = \frac{b(r_+^2+a^2)(1+g^2r_+^2) + aq}{(r_+^2+a^2)(r_+^2+b^2) + abq} \\
\Phi &= \frac{\sqrt{3}qr_+^2}{(r_+^2+a^2)(r_+^2+b^2) + abq}
\end{aligned}$$

$$\mathcal{S} = \frac{\text{Area}}{4G} = \frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2) + abq]}{2G\Xi_a\Xi_br_+}$$

$$dE=Td\mathcal{S}+\Omega_1dJ_1+\Omega_2dJ_2+\Phi dQ$$

$$I=\beta E-\mathcal{S}-\beta\Omega_1J_1-\beta\Omega_2J_2-\beta\Phi Q,$$

$$I=\frac{\pi\beta}{4G\Xi_a\Xi_b}\left[m-g^2(r_+^2+a^2)(r_+^2+b^2)-\frac{q^2r_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq}\right]$$



$$E = \frac{\partial I}{\partial \beta}, J_1 = -\frac{1}{\beta} \frac{\partial I}{\partial \Omega_1}, J_2 = -\frac{1}{\beta} \frac{\partial I}{\partial \Omega_2}, Q = -\frac{1}{\beta} \frac{\partial I}{\partial \Phi}$$

$$q=\frac{m}{1+ag+bg}$$

$$q=\frac{1}{g}(a+b)(1+ag)(1+bg)$$

$$r_*=\sqrt{\frac{1}{g}(a+b+abg)}$$

$$\beta\rightarrow\infty,\Omega_1\rightarrow\Omega_1^*=g,\Omega_2\rightarrow\Omega_2^*=g,\Phi\rightarrow\Phi^*=\sqrt{3}$$

$$\{Q,\overline{Q}\}\propto E-gJ_1-gJ_2-\sqrt{3}Q.$$

$$E^*-gJ_1^*-gJ_2^*-\sqrt{3}Q^*=0$$

$$G(T,\Omega_a,\Omega_b,\Phi)=M-TS-\Phi Q-\Omega_aJ_a-\Omega_bJ_b$$

$$F=E-T\mathcal{S}-\Phi Q-2\Omega J$$

$$m=\frac{(r_+^2+a^2)^2(1+r_+^2)+q^2+2a^2q}{2r_+^2}$$

$$F(r_+,T,\Phi,\Omega)=\left(\frac{\pi}{8(a^2-1)^3Gr_+^2}\right)\Big(-\big(a^6(r_+^2+4\pi r_+T+1)\big)-3q^2+r_+^4(-3r_+^2+4\pi r_+T-3)\\+4a^5(r_+^2+1)\Omega+4a^3\Omega((q+2)r_+^2+2q+2r_+^4)+4a\Omega(q^2+qr_+^2+r_+^6+r_+^4)\\+2\sqrt{3}qr_+^2\Phi-a^2(q^2+2q(r_+(\sqrt{3}r_+\Phi+4r_+-2\pi T)+3)\\+r_+^2(r_+^4+4\pi(r_+^2-2)r_+T+7r_+^2+6))\\-a^4(4\pi r_+T(q+2r_+^2-1)+2q+2r_+^4+5r_+^2+3))\Big)$$

$$q=\frac{\Phi(r_+^2+a^2)^2}{\sqrt{3}r_+^2-a^2\Phi}, 1-\Omega=\frac{1-a}{r_+^2+a^2}\Big[r_+^2-r_*^2+a(1+a)\Big(1-\frac{\Phi}{\sqrt{3}}\Big)\Big]$$

$$F(r_+,T,\Phi)=\frac{1}{4}r_+^2(3+3r_+^2-4\pi r_+T-\Phi^2)$$

$$T=\frac{6r_+^2-\Phi^2+3}{6\pi r_+},$$

$$F(r_+,\Phi)=-\frac{1}{12}r_+^2(3r_+^2+\Phi^2-3),$$

$$\begin{array}{l} F=0,\\ \frac{\partial F}{\partial r_+}=0, \end{array}$$

$$r_+=\frac{\sqrt{3-\Phi^2}}{\sqrt{3}}, T_{\rm HP}=\frac{\sqrt{3}\sqrt{3-\Phi^2}}{2\pi}.$$



$$a = \frac{1 + r_+^2 - f_\Omega}{2\Omega}, \text{ where } f_\Omega = \sqrt{(2 - 4\Omega^2)r_+^2 + r_+^4 + 1}.$$

$$F = \left(\frac{1}{8r_+^2(\Omega^2 - 1)^2} \right) (-4\pi r_+^3 T(f_\Omega - 2\Omega^2) + r_+^4(3f_\Omega - 10\Omega^2 + 3) + r_+^2((6 - 4f_\Omega)\Omega^2 - 1) + 4\pi(f_\Omega - 1)r_+T + f_\Omega - 4\pi r_+^5 T + 3r_+^6 - 1)$$

$$T = \frac{f_\Omega + r_+^2}{2\pi r_+}$$

$$T_{\text{HP}} = \frac{2\sqrt{1 - \Omega^2} + 1}{2\pi},$$

$$F = \frac{1}{4}r_+^2(3r_+^2 - 4\pi r_+T + 3), T = \frac{(2r_+^2 + 1)}{2\pi r_+}$$

$$S = \frac{1}{16\pi G} \int d^5x e \left\{ \tilde{c}_0 R + 12\tilde{c}_1 g^2 - \frac{\tilde{c}_2}{4} F^2 - \frac{\tilde{c}_3}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda + \lambda_1 \alpha \left[\mathcal{X}_{\text{GB}} - \frac{1}{2} C_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + \frac{1}{8} F^4 - \frac{1}{2\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_\lambda \right] \right\}$$

$$\mathcal{X}_{\text{GB}} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{3} (R_{\mu[\rho} g_{\sigma]\nu} + R_{\nu[\sigma} g_{\rho]\mu}) + \frac{1}{6} R g_{\mu[\rho} g_{\sigma]\nu}$$

$$\tilde{c}_i = 1 + \alpha g^2 \delta \tilde{c}_i$$

$$\delta \tilde{c}_0 = 4\lambda_2, \delta \tilde{c}_1 = -10\lambda_1 + 4\lambda_2, \delta \tilde{c}_2 = 4\lambda_1 + 4\lambda_2, \delta \tilde{c}_3 = -12\lambda_1 + 4\lambda_2.$$

$$Q = \frac{\sqrt{3}\pi q}{4G(1-a^2)^2} (1 + 4\lambda_2\alpha) + \lambda_1\alpha\Delta Q(q,a,r_+)$$

$$J = \frac{a\pi(1+4\lambda_2\alpha)}{4(1-a^2)^3Gr_+^2} [(a^2+q)^2 + (a^4+q+a^2(2+q))r_+^2 + (1+2a^2)r_+^4 + r_+^6]$$

$$+ \lambda_1\alpha\Delta J(q,a,r_+) \\ \mathcal{S} = \frac{\pi^2[r_+^4 + a^4 + a^2(q + 2r_+^2)]}{2Gr_+(1-a^2)^2} (1 + 4\lambda_2\alpha) + \lambda_1\alpha\Delta\mathcal{S}(q,a,r_+)$$

$$F(r_+,T,\Phi,\Omega,\alpha,\lambda_1,\lambda_2)=E-T\mathcal{S}-\Phi Q-2\Omega J$$

$$F(r_+,T,\Phi,\alpha,\lambda_1,\lambda_2)=F_0(r_+,T,\Phi)+4\pi\alpha r_+^3T\left(-\lambda_2-\frac{3\lambda_1(9r_+^2+\Phi^2+9)}{6r_+^2+\Phi^2-3}\right)\\ +\frac{\alpha}{6(6r_+^2+\Phi^2-3)}[6\lambda_2r_+^2(-3r_+^2(\Phi^2-3)+18r_+^4-(\Phi^2-3)^2)\\ +\lambda_1\left(r_+^4(729-99\Phi^2)-12r_+^2(\Phi^4+6\Phi^2-27)+486r_+^6+(\Phi^2-3)^2(\Phi^2+9)\right)]$$

$$M=3Q+2J$$



$$M - 3Q - 2J = \frac{1}{2}N^2 \frac{3 + 2a - a^2}{(1 - a)^2(1 + a)^4} (m - q(1 + 2a)),$$

$$W(\Phi', \Omega') = \frac{G}{T} = -S - \Phi'Q - \Omega'J$$

$$\Phi' = (\Phi - \Phi^*)/T, \Omega' = (\Omega - \Omega^*)/T$$

$$\begin{aligned}\Phi' &= -\frac{2}{\tau} + \varphi', \\ \Omega' &= -\frac{1}{\tau}.\end{aligned}$$

$$\begin{aligned}W(a) &= \frac{2a\left(\frac{\pi(-2a^2-5a+1)}{\sqrt{\frac{2}{a}+1}} - \frac{1}{2}(3-a)a\varphi'\right)}{(1-a)^2(5a+1)} \\ \tau_{\rm H} &= \frac{5a+1}{(1-a)\left(\frac{3\pi}{\sqrt{\frac{2}{a}+1}} + \frac{\varphi'}{2}\right)}\end{aligned}$$

$$\begin{aligned}F(u) &\approx \sqrt{2}\pi u^3 - \frac{33\pi u^5}{2\sqrt{2}} + \mathcal{O}(u^7) \\ \tau_{\rm H} &\approx \frac{25u}{6\sqrt{2}\pi} + \frac{\sqrt{2}}{3\pi u} + \mathcal{O}(u^3)\end{aligned}$$

$$F(u,\tau)=\frac{\pi}{2\sqrt{2}}(c_3u^3-c_4\tau u^4+c_5u^5)$$

$$F(u)=\frac{1}{12}u^3(3\sqrt{2}\pi c_3-2c_4)+\frac{1}{24}u^5(6\sqrt{2}\pi c_5-25c_4).$$

$$c_5=-\frac{198\sqrt{2}\pi-25c_4}{6\sqrt{2}\pi}, c_3=\frac{\sqrt{2}(c_4+6\sqrt{2}\pi)}{3\pi}$$

$$F(u,\tau)=-\frac{\pi c_4\tau u^4}{2\sqrt{2}}+\frac{1}{24}(25c_4-198\sqrt{2}\pi)u^5+\frac{1}{6}(c_4+6\sqrt{2}\pi)u^3$$

$$\partial_u F(u,\tau)=0$$

$$\frac{u(25u^2+4)c_4'(u)+(125u^2+12)c_4(u)+18\sqrt{2}\pi(4-55u^2)}{6\sqrt{2}\pi u(uc_4'(u)+4c_4(u))}=\frac{25u}{6\sqrt{2}\pi}+\frac{\sqrt{2}}{3\pi u},$$

$$c_4(u)=18\sqrt{2}\pi-135(\sqrt{2}\pi)u^2+O(u^4)$$



$$F(u, \tau) = 4\sqrt{2}\pi u^3 - 9(\pi^2 \tau)u^4 - 12(\sqrt{2}\pi)u^5 + \frac{135}{2}\pi^2 \tau u^6 + O(u^7)$$

$$F(r_+, T) = G(r_+) + g(r_+) \left(1 - \frac{T}{T_H}\right).$$

$$F'(r_+, T) = \left[G'(r_+) + g(r_+) \left(1 - \frac{T}{T_H(r_+)}\right)' + g'(r_+) \left(1 - \frac{T}{T_H}\right) \right] = 0$$

$$G = \frac{r_+^2(1-r_+^2)}{4}$$

$$T_H = \frac{1+2r_+^2}{2\pi r_+}$$

$$F(r_+, T) = \frac{r_+^2(1-r_+^2)}{4} + g(r_+) \left[1 - \left(\frac{2\pi r_+}{1+2r_+^2} \right) T \right]$$

$$g(r_+) = r^4 + \frac{r^2}{2}$$

$$F(r_+, T) = \frac{1}{4}r^2(3r^2 - 4\pi rT + 3)$$

$$F(a, \tau, \phi') = W(a) + g(a, \phi') \left(1 - \frac{\tau}{\tau_H}\right)$$

$$g(a, \phi') = \frac{a(a+1)^2 \left(\pi \sqrt{\frac{a+2}{a}} (17a^2 + 20a - 1) \phi + 6\pi^2 (11a^2 + 8a - 1) + (a+2)^2 \phi^2 \right)}{(a-1)^2 (5a+1) \left(\pi \sqrt{\frac{a+2}{a}} (11a^2 + 8a - 1) + (a+2)^2 \phi \right)}.$$

$$F = \frac{a}{(a-1)^2 (5a+1)^2} \left(2(5a+1) \left(\frac{\pi(-2a^2 - 5a + 1)}{\sqrt{\frac{a+2}{a}}} + \frac{1}{2}(a-3)a\phi' \right) \right. \\ \left. - \frac{(a+1)^2 \left((a+2)\phi' + 6\pi a \sqrt{\frac{a+2}{a}} \right) \left((1-a)\tau \left(\frac{3\pi}{\sqrt{\frac{a+2}{a}}} + \frac{\phi'}{2} \right) - 5a - 1 \right)}{a+2} \right)$$

$$F(a, \tau) = W(a) + 2 \left(\frac{J^* + Q^*}{\tau_H} \right) \left(1 - \frac{\tau}{\tau_H} \right)$$

$$W(a) = -S^* + 2 \left(\frac{J^* + Q^*}{\tau_H} \right)$$

$$M^* = N^2 \frac{a(3 - a^2 - 2a^3)}{(1 - a)^4}, Q^* = N^2 \frac{a}{(1 - a)^2}$$

$$J^* = N^2 \frac{a^2(a + 3)}{(1 - a)^3}, S^* = N^2 \frac{2\pi a \sqrt{a(a + 2)}}{(1 - a)^2}$$

$$\left(Q^{*3} + \frac{1}{2}N^2J^{*2}\right) - \left(3Q^* + \frac{1}{2}N^2\right)(3Q^{*2} - N^2J^*) = 0$$

$$F(a, \tau) = 4\sqrt{2}\pi a^{3/2} - 12\sqrt{2}\pi a^{5/2} - 9\pi^2 a^2 \tau + \varphi' \left(\frac{87\pi a^{5/2} \tau}{2\sqrt{2}} - 3\sqrt{2}\pi a^{3/2} \tau - 4a^2 + a \right) \\ + \mathcal{O}(a^{5/2})$$

$$F = \mathcal{E} - \tau \mathcal{S}$$

$$\mathcal{S} = -\frac{\partial F(a, \tau)}{\partial \tau} = 2 \left(\frac{J^* + Q^*}{\tau_H^2} \right) = \frac{18\pi^2 a^2 (a + 1)^2}{(5a + 1)^2 (2 - a - a^2)},$$

$$\mathcal{E} = -S^* + 4 \left(\frac{J^* + Q^*}{\tau_H} \right) = \frac{2\pi a^{3/2} (a^2 + a + 4)}{(a - 1)^2 \sqrt{a + 2} (5a + 1)}$$

$$Q^* = \frac{\pi a}{2Gg(1-ag)^2} \left[1 + 4\lambda_2 \alpha g^2 + 4\lambda_1 \alpha g^2 \frac{1 + 8ag + 36a^2 g^2 + 44a^3 g^3 + 19a^4 g^4}{ag(-1 + 8ag + 11a^2 g^2)} \right],$$

$$J^* = \frac{\pi a^2(3+ag)}{2Gg(1-ag)^3} \left[1 + 4\lambda_2 \alpha g^2 + 24\lambda_1 \alpha g^2 \frac{1 + 9ag + 29a^2 g^2 + 25a^3 g^3 + 8a^4 g^4}{ag(3+ag)(-1 + 8ag + 11a^2 g^2)} \right],$$

$$S^* = \frac{\pi^2 a \sqrt{ag(ag+2)}}{g^2 G(1-ag)^2} \left[1 + 4\lambda_2 \alpha g^2 + 48\lambda_1 \alpha g^2 \frac{2a^2 g^2 + 5ag + 2}{11a^2 g^2 + 8ag - 1} \right].$$

$$F(a, \tau, \alpha, \lambda_1, \lambda_2) = F_0(a, \tau) + \frac{(8\sqrt{a}\pi\alpha(f_{\lambda_1} + f_{\lambda_2}))}{(a-1)^2(a+2)^{3/2}(5a+1)^2(a(11a+8)-1)},$$

$$f_{\lambda_1} = 3\lambda_1(3\pi\sqrt{a(a+2)}(29a^6 + 96a^5 + 57a^4 - 100a^3 - 69a^2 - 12a - 1)\tau \\ + 90a^7 + 508a^6 + 1158a^5 + 1272a^4 + 622a^3 + 192a^2 + 42a + 4),$$

$$f_{\lambda_2} = \lambda_2 \left(9\pi\sqrt{a^3(a+2)}(11a^3 - 3a^2 - 9a + 1)(a+1)^2\tau \right. \\ \left. + a(a+2)(5a+1)(a^2 + a + 4)(a(11a+8)-1) \right),$$

$$G(r_+, \alpha, \lambda_1, \lambda_2) = -\frac{1}{4}(r_+^2 - 1)r_+^2(4\alpha\lambda_2 + 1) - \frac{9}{2}\alpha\lambda_1(r_+^2 + 1)^2,$$

$$W = -\frac{\pi r_+((r_+^2 - 1)r_+^2(4\alpha\lambda_2 + 1) + 18\alpha\lambda_1(r_+^2 + 1)^2)}{4r_+^2 + 2}$$

$$W \sim -9r_+(\pi\alpha\lambda_1) + \frac{1}{2}\pi r_+^3(4\alpha\lambda_2 + 1) + O(r_+^4)$$

$$W \sim -T^{-1} \text{ as } T \rightarrow \infty$$

$$W \sim -\frac{1}{4}\pi r_+^3(18\alpha\lambda_1 + 4\alpha\lambda_2 + 1) - \frac{3}{8}\pi r_+(18\alpha\lambda_1 - 4\alpha\lambda_2 - 1) + \cdots$$



$$W \sim -T^3 \text{ as } T \rightarrow \infty$$

$$W(a, \tau, \alpha) = W(a, \tau) - \frac{8\pi \sqrt{\frac{a}{a+2}} \alpha (3(a^3 + 5a^2 + 5a + 1)\lambda_1 + a(2a^2 + 5a - 1)\lambda_2)}{(a-1)^2(5a+1)}$$

$$W(a, \alpha) = -12\sqrt{2}\pi\alpha\sqrt{a}\lambda_1 - \sqrt{2}\pi a^{3/2}(21\alpha\lambda_1 - 4\alpha\lambda_2 - 1) + O(a^{5/2})$$

$$W \sim -\tau^{-1} \text{ as } \tau \rightarrow \infty,$$

$$W(a, \alpha) = -\frac{4\pi(4\alpha(3\lambda_1 + \lambda_2) + 1)}{\sqrt{3}(1-a)} - \frac{2\pi(4\alpha(6\lambda_1 + \lambda_2) + 1)}{\sqrt{3}(1-a)^2} - \frac{\pi(4\alpha(6\lambda_1 + 5\lambda_2) + 5)}{3\sqrt{3}} + \dots,$$

$$W \sim -\tau^2 \text{ as } \tau \rightarrow \infty,$$

$$W(Q, \Phi, T) = \frac{Q \left(6Q - \Phi \left(4\pi\sqrt{2}T\sqrt{\frac{Q}{\Phi}} + \Phi^2 - 3 \right) \right)}{2\Phi^2}$$

$$T = \frac{\sqrt{\frac{Q}{\Phi}}(12Q - \Phi^3 + 3\Phi)}{6\sqrt{2}\pi Q}$$

$$T = \frac{\sqrt{\frac{Q}{\Phi}}(6Q - \Phi^3 + 3\Phi)}{4\sqrt{2}\pi Q}$$

$$Q_c = -\frac{1}{6}\Phi(\Phi^2 - 3)$$

$$T_c = \frac{\sqrt{9 - 3\Phi^2}}{2\pi}$$

CONCLUSIONES.

Los agujeros negros cuánticos se forman en gravedad cuántica o en su defecto en supergravedad cuántica, aunque es inevitable en el segundo caso. La singularidad de un agujero negro cuántico, es en sí, la masa compacta de la partícula colapsada o colisionada, cuyo punto concéntrico es extremadamente denso, capaz de devorar materia y energía, transformándola en materia y energía oscuras, es decir, materia y energía inertes atravesadas por gravedad en condiciones perturbativas o entrópicas. El agujero cuántico de gusano, es distinto al agujero negro cuántico, es decir, no se forma por el colapso o colisión de una partícula oscura o de una partícula blanca, sino por la distorsión del espacio – tiempo cuántico, en condiciones de gravedad exógena, esto es, cuando cualquiera de las partículas antes referidas, interactúan con el gravitón o su supercompañera, el supergravitón o gravitino, formándose así, lo que se



conoce también como puente de Einstein – Rosen, lo que explicaría la superposición y entrelazamiento cuánticos desde la perspectiva de la relatividad general. El radio de un agujero negro cuántico, es infinito en proporción a su disco de acreción. El horizonte de eventos de un agujero negro cuántico, inicialmente es débil, más su acaparamiento es mayor en tanto la absorción de materia y energía se vuelve más agresiva. El agujero negro cuántico, sea en gravedad o supergravedad cuánticas, produce materia y energía oscuras, es decir, tejido cuánticamente muerto, sin interacción más que la gravedad en sí misma, sin que sea posible la formación de otro fenómeno subatómico que requiera energía o materia. Un agujero negro cuántico, no puede crecer más allá del límite del campo en el que se ha formado, además que, es capaz de producir radiación bajo parámetros termodinámicos hostiles. Finalmente, es preciso señalar que un agujero blanco cuántico, se forma por el colapso o colisión de una partícula blanca en sentido estricto, más no por una partícula oscura, aunque puede formarse en condiciones de antimateria, lo que causa una supercurvatura y en consecuencia, la formación de un agujero blanco cuántico cuya principal característica es, expulsar materia y energía, sin capacidad de absorción o atracción, por lo que, todo escapa a su horizonte de eventos. La singularidad de un agujero blanco cuántico, es el centro de masa y energía de la partícula colapsada o colisionada, cuya densidad es extremadamente ligera, la misma que expulsa paquetes de energía de manera infinita al igual que repele materia aunque la produce. Por tanto, la gravedad no interactúa en un agujero blanco cuántico, aunque éstos pueden formarse a partir de la distorsión del espacio – tiempo cuántico. El agujero blanco cuántico, es vital para la creación de multidimensiones cuánticas y por ende, de hiperespacios paralelos al espacio – tiempo cuántico de origen, pues se constituye como un punto de salida de un puente de Einstein – Rosen propiamente dicho, arrojando energía y materia, bien sea, por atrapamiento desde el otro extremo del puente, por un agujero negro cuántico, aunque el referido puente, no necesariamente se forma a partir de un agujero negro cuántico, es decir, que su opuesto puede producir un puente ER de manera directa, cuando una partícula blanca o estrella colapsa o colisiona, en cuyo caso, la formación de materia y energía y la repulsión de la misma, se origina desde la singularidad del agujero blanco cuántico, aunque en condiciones extremas y para estos efectos, el agujero blanco cuántico pueda apalancarse de un agujero negro cuántico para efectos de desplegar su fenomenología, es decir, se forma de manera ulterior al agujero blanco cuántico. En todos los casos, el puente ER, es el ducto de comunicación entre ambos agujeros, pero esta



configuración no es absoluta, pues un agujero blanco cuántico o su opuesto, pueden formarse a partir de una partícula blanca u oscura, según sea el caso, expulsando o en su defecto, absorbiendo masa y energía, que no se transporta a través del puente ER, sino que interactúa con la singularidad del agujero cuántico de que se trate, sea formando gravedad o reduciéndola a cero.

ACLARACIONES FINALES:

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo $\dagger$ será reemplazado por este símbolo $\dagger^\circ$ o por este símbolo $\ddagger$, equivaliendo lo mismo.

Símbolo a ser reemplazado.	Símbolos de reemplazo.
$\dagger$	$\dagger^\circ$
	$\ddagger$

2. En todos los casos, este símbolo $\ddagger$, será reemplazado por este símbolo $\ddagger^\circ$ o por este símbolo $\S$.

Símbolo a ser reemplazado.	Símbolos de reemplazo.
$\ddagger$	$\ddagger^\circ$
	$\S$

3. En todos los casos, se añadirá y por ende, se calculará la magnitud $\S$ que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo $\bullet$ podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

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