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RELATIVISTIC QUANTUM GRAVITY. AdS/CFT CALIBER SYMMETRIES IN HILBERT–EINSTEIN–RIEMANN SPACES. VOLUME I.

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RESUMEN

En este trabajo, abordaré la gravedad cuántica relativista desde la TCCR, esto es, desde la Teoría Cuántica de Campos Relativistas. Es aquí donde se establece una clara distinción entre superpartícula y partícula supermasiva respectivamente, en la medida en que, la primera, refiere a aquellas partículas cuyo centro de masa y energía es extremadamente denso, lo que provoca la deformación intensa del espacio – tiempo cuántico, todo esto, en dimensiones mas altas y en condiciones de supersimetría, causando incluso la formación de agujeros negros cuánticos, esto, sea por razones endógenas o en su defecto, por razones exógenas, esto último, cuando la superpartícula interactúa con el supergravitón, es decir, por permeabilización del espacio – tiempo cuántico a través del campo supergravitónico, en tanto que, la segunda, refiere a las partículas cuyo centro de masa, es extremadamente denso, causando así, la deformación del espacio – tiempo cuántico, en simetrías de calibre compactas, pudiendo provocar o no la formación de agujeros negros, salvo en circunstancias de colisión, en cuyo caso, es inevitable. La partícula supermasiva, por tanto, es la responsable de la gravedad cuántica relativista, sea por razones endógenas, es decir, cuando la partícula supermasiva, por sí misma, distorsiona el espacio – tiempo cuántico o en su defecto, por interacción con el gravitón, esto es, la permeabilización del plano cuántico por intrusión del campo gravitónico.

Palabras Clave: Simetrías de gauge, gravedad cuántica, relatividad general, partícula supermasiva, gravitón, campo gravitónico.

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RELATIVISTIC QUANTUM GRAVITY. AdS/CFT CALIBER SYMMETRIES IN HILBERT–EINSTEIN–RIEMANN SPACES. VOLUME I.

ABSTRACT

In this work, I will address relativistic quantum gravity from the TCCR, that is, from the Quantum Theory of Relativistic Fields. It is here that a clear distinction is established between superparticle and supermassive particle respectively, insofar as the former refers to those particles whose center of mass and energy is extremely dense, which causes the intense deformation of quantum space-time, all this, in higher dimensions and under conditions of supersymmetry, even causing the formation of quantum black holes. This, either for endogenous reasons or, failing that, for exogenous reasons, the latter, when the superparticle interacts with the supergraviton, that is, by permeabilization of quantum space-time through the supergravitonic field, while the second refers to particles whose center of mass is extremely dense, thus causing the deformation of quantum space-time. in compact gauge symmetries, which may or may not cause the formation of black holes, except in collision circumstances, in which case, it is inevitable. The supermassive particle, therefore, is responsible for relativistic quantum gravity, either for endogenous reasons, that is, when the supermassive particle, by itself, distorts quantum space-time or, failing that, by interaction with the graviton, that is, the permeabilization of the quantum plane by intrusion of the gravitonic field.

Keywords: Gauge symmetries, quantum gravity, general relativity, supermassive particle, graviton, gravitonic field.

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INTRODUCCIÓN.

Las partículas supermasivas, han sido teorizadas en trabajos anteriores, por este autor, sin embargo, existen aclaraciones que precisar. Las partículas supermasivas, o llamadas también “partículas oscuras” o “partículas cosmológicas”, son aquellas, que a propósito de su masa en extremo pesada, son capaces de deformar el espacio – tiempo cuántico, con posibilidad de provocar agujeros negros cuánticos, en casos específicos de colapso o colisión, en tanto que, las superpartículas o llamadas también “partículas estrella” o “partículas blancas”, son aquellas cuyo centro de masa y energía es tan extremo, que no solamente deforma el espacio – tiempo cuántico, sin que además, provoca agujeros negros cuánticos e incluso, agujeros de gusano. La otra gran diferencia, está en cuanto a la gravedad exógena se refiere, esto es, que la partícula supermasiva, distorsiona el espacio – tiempo cuántico en el que interactúa, cuando empata o ancla con el gravitón, lo que supone por tanto, la intrusión y permeo del campo gravitónico en el plano repercutido, en tanto que, cuando se trata de la superpartícula, ésta distorsiona el espacio – tiempo cuántico en el que interactúa, cuando empata o ancla con el supergravitón, lo que supone por tanto, la intrusión y permeo del campo supergravitónico en el plano repercutido. Cabe aclarar que los campos de permeabilización antes referidos, pueden ser fantasmas aunque no por regla general. En cuanto a la gravedad cuántica relativista, no existe la formación de supersimetrías, sino de simetrías de calibre, con propagadores y osciladores armónicos masivos pero no caóticos, es decir, existe un mínimo de compactación. Trabajaremos en un espacio de Hilbert – Einstein para campos cuánticos relativistas, demostrando la acción gravitatoria de las partículas supermasivas en entornos de entropía sin que esto, comporte dimensiones más altas, salvo en casos de colapso o colisión de estas partículas pesadas.

RESULTADOS Y DISCUSIÓN:

A continuación, se expresa el Modelo de Gravedad Cuántica Relativista, en dimensión $\mathbb{R}^4$ para simetrías de gauge puras, con osciladores armónicos y propagadores $\left\langle \psi_{\blacksquare}^{\square} \right\| :: \|\aleph^{\mu\nu}\rangle$, en un espacio de Hilbert – Einstein, tanto con interferencia gravitónica como sin interferencia gravitónica.

Modelo de Gravedad Cuántica Relativista con intervención gravitónica (gravedad cuántica exógena).

$$I_D = I - nI_0$$

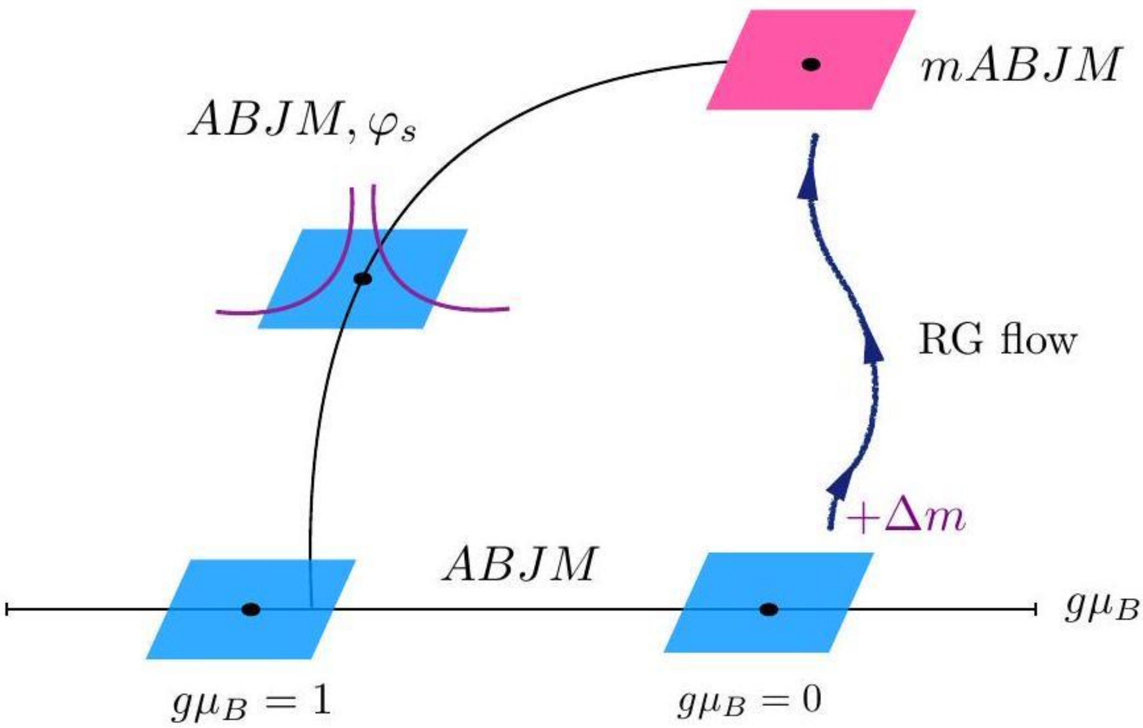


Figura 1. Plano cuántico – relativista en condiciones de entropía.

$$\mathcal{L} = \frac{1}{2}R - \partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\sinh^2(2\varphi)D_\mu\theta D^\mu\theta - \sum_{i=1}^{3}\partial_\mu\lambda_i\partial^\mu\lambda_i - g^2\mathcal{P}$$
$$-\frac{1}{4}\left[e^{-2(\lambda_1+\lambda_2+\lambda_3)}F_{\mu\nu}^0F^{0\mu\nu} + e^{-2(\lambda_1-\lambda_2-\lambda_3)}F_{\mu\nu}^1F^{1\mu\nu}\right.$$
$$\left.+e^{-2(-\lambda_1+\lambda_2-\lambda_3)}F_{\mu\nu}^2F^{2\mu\nu} + e^{-2(-\lambda_1-\lambda_2+\lambda_3)}F_{\mu\nu}^3F^{3\mu\nu}\right]$$

$$S_{\text{bulk}} = \frac{1}{8\pi G}\int \sqrt{-g}\mathcal{L}, \frac{1}{G} = \frac{4\sqrt{2}g^2}{3}N^{3/2}$$

$$D\theta \equiv d\theta - g(A^0 - A^1 - A^2 - A^3) \equiv d\theta - gA_B$$

$$\mathcal{P} = \frac{1}{2}\left(\frac{\partial W}{\partial\varphi}\right)^2 + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial W}{\partial\lambda_i}\right)^2 - \frac{3}{2}W^2$$

$$W = e^{\lambda_1+\lambda_2+\lambda_3}\sinh^2\varphi - \frac{1}{2}\left(e^{\lambda_1+\lambda_2+\lambda_3} + e^{\lambda_1-\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{-\lambda_1-\lambda_2+\lambda_3}\right)\cosh^2\varphi$$

$$\left[\nabla_\mu - \frac{i}{2}Q_\mu - \frac{g}{2\sqrt{2}}W\gamma_\mu - i\frac{1}{4\sqrt{2}}H_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\right]\epsilon = 0$$

$$H_{\mu\nu} \equiv \bar{F}^{78} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} + e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} + e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} + e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right)$$

$$Q_\mu \equiv -g\left(A^0_\mu + A^1_\mu + A^2_\mu + A^3_\mu\right) - \frac{1}{2}(\cosh\, 2\varphi - 1)D_\mu\theta$$

$$\left[\gamma^\mu\partial_\mu\lambda_1 + \frac{g}{\sqrt{2}}\partial_{\lambda_1}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{12}_{\mu\nu}\right]\epsilon = 0,$$
$$\left[\gamma^\mu\partial_\mu\lambda_2 + \frac{g}{\sqrt{2}}\partial_{\lambda_2}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{34}_{\mu\nu}\right]\epsilon = 0,$$
$$\left[\gamma^\mu\partial_\mu\lambda_3 + \frac{g}{\sqrt{2}}\partial_{\lambda_3}W + i\frac{1}{2\sqrt{2}}\gamma^{\mu\nu}\bar{F}^{56}_{\mu\nu}\right]\epsilon = 0,$$
$$\left[\gamma^\mu\partial_\mu\varphi + \frac{g}{\sqrt{2}}\partial_\varphi W + i\frac{1}{2}\partial_\varphi Q_\mu\gamma^\mu\right]\epsilon = 0,$$

$$\bar{F}^{12}_{\mu\nu} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} + e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} - e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} - e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right),$$
$$\bar{F}^{34}_{\mu\nu} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} - e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} + e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} - e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right),$$
$$\bar{F}^{56}_{\mu\nu} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} - e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} - e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} + e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right),$$
$$\bar{F}^{78}_{\mu\nu} = \frac{1}{2}\left(e^{-\lambda_1-\lambda_2-\lambda_3}F^0_{\mu\nu} + e^{-\lambda_1+\lambda_2+\lambda_3}F^1_{\mu\nu} + e^{\lambda_1-\lambda_2+\lambda_3}F^2_{\mu\nu} + e^{\lambda_1+\lambda_2-\lambda_3}F^3_{\mu\nu}\right),$$

$$L^2 \equiv \frac{1}{2g^2}$$

$$A_R = A^0 + A^1 + A^2 + A^3, \qquad A_{F_1} = A^1 - A^2$$
$$A_{F_2} = A^2 - A^3, \qquad A_{F'} = \frac{1}{2}(3A^0 - A^1 - A^2 - A^3)$$

$$J_R^{ABJM} = \frac{1}{4}(J_0 + J_1 + J_2 + J_3), \qquad J_{F_1} = \frac{1}{3}(2J_1 - J_2 - J_3)$$
$$J_{F_2} = \frac{1}{3}(J_1 + J_2 - 2J_3) \qquad J_{F'} = \frac{1}{6}(3J_0 - J_1 - J_2 - J_3),$$

$$A_R = A^0 + A^1 + A^2 + A^3, \qquad A_{F_1} = A^1 - A^2$$
$$A_{F_2} = A^2 - A^3, \qquad A_B = A^0 - A^1 - A^2 - A^3$$

$$J_R^\varphi = \frac{1}{6}(3J_0 + J_1 + J_2 + J_3), \quad J_{F_1} = \frac{1}{3}(2J_1 - J_2 - J_3),$$
$$J_{F_2} = \frac{1}{3}(J_1 + J_2 - 2J_3), \qquad J_B = \frac{1}{6}(3J_0 - J_1 - J_2 - J_3),$$

$$A^\alpha J_\alpha = A_R J_R^\varphi + A_{F_1}J_{F_1} + A_{F_2}J_{F_2} + A_B J_B$$

$$J_R^{ABJM} = J_R^\varphi - \tfrac{1}{2}J_B, J_B = J_{F'}$$

$$\tilde{L}^2 \equiv \frac{2}{3\sqrt{3}g^2}, e^{\lambda_i} = 3^{1/4}, \tanh\, \varphi = \frac{1}{\sqrt{3}}$$

$$A_R^{mABJM} = \frac{1}{2}(A^0 + 3A^1 + 3A^2 + 3A^3), A_{F_1} = A^1 - A^2$$
$$A_{F_2} = A^2 - A^3, A_B = A^0 - A^1 - A^2 - A^3$$

$$ds^2 = e^{2V} ds^2(AdS_2) + f^2 dy^2 + h^2 dz^2$$
$$A^\alpha = a^\alpha dz$$

$$ds^2 = R^2 \left(\frac{dy^2}{y^2} + e^{2V_0} y^2 [ds^2(AdS_2) + n^2 dz^2] \right) + \cdots$$

$$e^{2V_0}(ds^2(AdS_2) + n^2 dz^2) = e^{2V_0} \left(\frac{1}{\rho^2} [-dt^2 + d\rho^2 + n^2 \rho^2 dz^2] \right)$$

$$a^\alpha = \mu^\alpha + \frac{j^\alpha}{y} + \cdots$$

$$g\mu_R \equiv g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -\kappa n - s$$

$$g\mu_B \equiv g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3 = \kappa n$$

$$g\mu_B = 0$$

$$g\mu_R \equiv g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -\kappa n - s$$

$$g\mu_R^{mABJM} \equiv \frac{1}{2}(g\mu^0 + 3g\mu^1 + 3g\mu^2 + 3g\mu^3) = 2g\mu^0 = g\mu^R$$

$$\varphi \neq 0,$$

$$\mathcal{E}_{R_1} \equiv e^{2V}\left(e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3} F_{23}^0 + e^{-2\lambda_1 + 2\lambda_2 + 2\lambda_3} F_{23}^1\right),$$
$$\mathcal{E}_{R_2} \equiv e^{2V}\left(e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3} F_{23}^0 + e^{2\lambda_1 - 2\lambda_2 + 2\lambda_3} F_{23}^2\right),$$
$$\mathcal{E}_{R_3} \equiv e^{2V}\left(e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3} F_{23}^0 + e^{2\lambda_1 + 2\lambda_2 - 2\lambda_3} F_{23}^3\right),$$

$$F_{23}^\alpha = f^{-1} h^{-1} (a^\alpha)'.$$

$$\mathcal{E}_{F_1} \equiv \mathcal{E}_{R_1} - \mathcal{E}_{R_2} = e^{2V}\left(e^{-2\lambda_1 + 2\lambda_2 + 2\lambda_3} F_{23}^1 - e^{2\lambda_1 - 2\lambda_2 + 2\lambda_3} F_{23}^2\right)$$
$$\mathcal{E}_{F_2} \equiv \mathcal{E}_{R_2} - \mathcal{E}_{R_3} = e^{2V}\left(e^{2\lambda_1 - 2\lambda_2 + 2\lambda_3} F_{23}^2 - e^{2\lambda_1 + 2\lambda_2 - 2\lambda_3} F_{23}^3\right)$$
$$\mathcal{E}_{F_3} \equiv \mathcal{E}_{R_3} - \mathcal{E}_{R_1} = e^{2V}\left(e^{2\lambda_1 + 2\lambda_2 - 2\lambda_3} F_{23}^3 - e^{-2\lambda_1 + 2\lambda_2 + 2\lambda_3} F_{23}^1\right).$$

$$\mathcal{E}_B \equiv e^{2V}\left(3e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3} F_{23}^0 - e^{-2\lambda_1 + 2\lambda_2 + 2\lambda_3} F_{23}^1 - e^{2\lambda_1 - 2\lambda_2 + 2\lambda_3} F_{23}^2 - e^{2\lambda_1 + 2\lambda_2 - 2\lambda_3} F_{23}^3\right)$$

$$\mathcal{E}_B' = -3g e^{2V} f h^{-1} \frac{1}{2} \sinh^2(2\varphi) D_z \theta$$

$$\chi = e^{V/2} e^{\frac{isz}{2}} \begin{pmatrix} \sin \frac{\xi}{2} \\ \cos \frac{\xi}{2} \end{pmatrix}$$

$$h e^{-V} = -n \sin \xi$$

$$\mathcal{E}_{R_1} = 2ge^{2V}\cos\,\xi - \sqrt{2}\kappa e^V e^{-\lambda_1}\cosh\,(\lambda_2+\lambda_3),$$
$$\mathcal{E}_{R_2} = 2ge^{2V}\cos\,\xi - \sqrt{2}\kappa e^V e^{-\lambda_2}\cosh\,(\lambda_3+\lambda_1),$$
$$\mathcal{E}_{R_3} = 2ge^{2V}\cos\,\xi - \sqrt{2}\kappa e^V e^{-\lambda_3}\cosh\,(\lambda_1+\lambda_2),$$

$$\mathcal{E}_{F_1} = \sqrt{2}\kappa e^V e^{\lambda_3}\sinh\,(\lambda_1-\lambda_2),$$
$$\mathcal{E}_{F_2} = \sqrt{2}\kappa e^V e^{\lambda_1}\sinh\,(\lambda_2-\lambda_3),$$
$$\mathcal{E}_{F_3} = \sqrt{2}\kappa e^V e^{\lambda_2}\sinh\,(\lambda_3-\lambda_1).$$

$$\mathcal{E}_B = \frac{\kappa}{\sqrt{2}}e^V e^{-\lambda_1-\lambda_2-\lambda_3}\left(-3+e^{2\lambda_1+2\lambda_2}+e^{2\lambda_2+2\lambda_3}+e^{2\lambda_1+2\lambda_3}\right)$$

$$F^{\alpha}_{yz} = (a^{\alpha})' = (\mathcal{J}^{\alpha})'$$

$$\mathcal{J}^0 \equiv -\frac{1}{\sqrt{2}}ne^V\cos\,\xi e^{\lambda_1+\lambda_2+\lambda_3}, \mathcal{J}^1 \equiv -\frac{1}{\sqrt{2}}ne^V\cos\,\xi e^{\lambda_1-\lambda_2-\lambda_3}$$
$$\mathcal{J}^2 \equiv -\frac{1}{\sqrt{2}}ne^V\cos\,\xi e^{-\lambda_1+\lambda_2-\lambda_3}, \mathcal{J}^3 \equiv -\frac{1}{\sqrt{2}}ne^V\cos\,\xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$\mathcal{J}^0\mathcal{J}^1 = \frac{1}{2}n^2\cos^2\,\xi e^{2V}e^{2\lambda_1}, \mathcal{J}^0\mathcal{J}^2 = \frac{1}{2}n^2\cos^2\,\xi e^{2V}e^{2\lambda_2}, \mathcal{J}^0\mathcal{J}^3 = \frac{1}{2}n^2\cos^2\,\xi e^{2V}e^{2\lambda_2}$$
$$\mathcal{J}^0\mathcal{J}^1\mathcal{J}^2\mathcal{J}^3 = \frac{1}{4}n^4\cos^4\,\xi e^{4V}$$

$$n > 0.$$

$$\bar{\theta} = 0.$$

$$D_z\theta = 0, Q_z = 0.$$

$$f = e^V.$$

$$(n\sin\,\xi)' = -1, \cos\,\xi = (-1)^t,$$

$$s = (-1)^{t+1}.$$

$$\partial_{\varphi}Q_z = \partial_{\varphi}W = 0.$$

$$e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3}-e^{2\lambda_1+2\lambda_2+2\lambda_3} = 0, W = -e^{\lambda_1+\lambda_2+\lambda_3}.$$

$$M \equiv \sqrt{2}ge^{\lambda_1+\lambda_2+\lambda_3}e^V, M > 0.$$

$$\mathcal{E}_{R_i} = \frac{M^2}{g}\cos\,\xi e^{-2(\lambda_1+\lambda_2+\lambda_3)} - \frac{\kappa M}{2g}\left(e^{-2\lambda_i}+e^{-2(\lambda_1+\lambda_2+\lambda_3)}\right),$$

$$\mathcal{E}_{F_1} = -\frac{\kappa M}{2g}\left(e^{-2\lambda_1}-e^{-2\lambda_2}\right), \mathcal{E}_{F_2} = -\frac{\kappa M}{2g}\left(e^{-2\lambda_2}-e^{-2\lambda_3}\right).$$

$$M = -s\kappa + \frac{1}{n}.$$

$$\varphi \to 0: \mathcal{E}_B = \frac{\kappa M}{2g} e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3}\left(-3 + e^{2\lambda_1 + 2\lambda_2} + e^{2\lambda_2 + 2\lambda_3} + e^{2\lambda_1 + 2\lambda_3}\right).$$

$$s = -\kappa: g\mu_R = \kappa(1-n);\ g\mu_R^{mABJM} = \kappa(1-n)$$

$$n = 1,\ \text{Core:}\ \mathcal{E}_{R_i} = \frac{\kappa}{g}\left(3e^{-2\lambda_1 - 2\lambda_2 - 2\lambda_3} - e^{-2\lambda_i}\right),$$

$$n = 1,\ \mathcal{E}_{F_1} = -\frac{\kappa}{g}\left(e^{-2\lambda_1} - e^{-2\lambda_2}\right), \mathcal{E}_{F_2} = -\frac{\kappa}{g}\left(e^{-2\lambda_2} - e^{-2\lambda_3}\right).$$

$$(\mathcal{J}^0 - \mathcal{J}^1 - \mathcal{J}^2 - \mathcal{J}^3)|_{\text{core}} = 0,\ (\mathcal{J}^0)|_{\text{core}} = \frac{1}{2g}(-\kappa n + s).$$

$$\begin{aligned}
\mathcal{E}_{R_1} &= c(\langle J^0\rangle + \langle J^1\rangle) = c\left(2\langle J_R^\varphi\rangle + \langle J_{F_1}\rangle\right) \\
\mathcal{E}_{R_2} &= c(\langle J^0\rangle + \langle J^2\rangle) = c\left(2\langle J_R^\varphi\rangle - \langle J_{F_1}\rangle + \langle J_{F_2}\rangle\right) \\
\mathcal{E}_{R_3} &= c(\langle J^0\rangle + \langle J^3\rangle) = c\left(2\langle J_R^\varphi\rangle - \langle J_{F_2}\rangle\right)
\end{aligned}$$

$$\mathcal{E}_{R_1} + \mathcal{E}_{R_2} + \mathcal{E}_{R_3} = 6c\langle J_R^\varphi\rangle$$

$$\begin{aligned}
\mathcal{E}_{F_1} &= c(\langle J^1\rangle - \langle J^2\rangle) = c\left(2\langle J_{F_1}\rangle - \langle J_{F_2}\rangle\right) \\
\mathcal{E}_{F_2} &= c(\langle J^2\rangle - \langle J^3\rangle) = c\left(-\langle J_{F_1}\rangle + 2\langle J_{F_2}\rangle\right)
\end{aligned}$$

$$c = -3\sqrt{2}\pi e^{V_0}/\left(ngN^{3/2}\right),$$

$$\mathcal{E}_B = c(3\langle J^0\rangle - \langle J^1\rangle - \langle J^2\rangle - \langle J\rangle) = 6c\langle J_B\rangle,$$

$$F_{yz}^\alpha = (A^\alpha)' = (\mathcal{J}^\alpha)'$$

$$g\mu^\alpha = g\mathcal{J}^\alpha|_{\text{bdry}} - g\mathcal{J}^\alpha|_{\text{core}},$$

$$\begin{aligned}
e^{4V}|_{\text{core}} &= \frac{4}{n^4}(\mathcal{J}^0|_{\text{bdry}} - \mu^0)(\mathcal{J}^1|_{\text{bdry}} - \mu^1)(\mathcal{J}^2|_{\text{bdry}} - \mu^2)(\mathcal{J}^3|_{\text{bdry}} - \mu^3) \\
&= R^4\left(1 - \frac{\mu^0}{\mathcal{J}^0|_{\text{bdry}}}\right)\left(1 - \frac{\mu^1}{\mathcal{J}^1|_{\text{bdry}}}\right)\left(1 - \frac{\mu^2}{\mathcal{J}^2|_{\text{bdry}}}\right)\left(1 - \frac{\mu^3}{\mathcal{J}^3|_{\text{bdry}}}\right).
\end{aligned}$$

$$e^{2\lambda_1}\big|_{\text{core}} = (e^{-2V}|_{\text{core}})R^2 e^{2(\lambda_1)_{fp}}\left(1 - \frac{\mu^0}{\mathcal{J}^0|_{\text{bdry}}}\right)\left(1 - \frac{\mu^1}{\mathcal{J}^1|_{\text{bdry}}}\right),$$

$$\left(1 - \frac{\mu^\alpha}{\mathcal{J}^\alpha|_{\text{bdry}}}\right) > 0\ \left(1 - \frac{\mu^\alpha}{\mathcal{J}^\alpha|_{\text{bdry}}}\right) < 0\ \left(1 - \frac{\mu^\alpha}{\mathcal{J}^\alpha|_{\text{bdry}}}\right) = \mathcal{J}^\alpha|_{\text{core}} / \mathcal{J}^\alpha|_{\text{bdry}}$$

$$\begin{aligned}
s &= -\kappa, \left(1 - \frac{\mu^\alpha}{\mathcal{J}^\alpha|_{\text{bdry}}}\right) > 0, n > 0 \\
s &= +\kappa, \left(1 - \frac{\mu^\alpha}{\mathcal{J}^\alpha|_{\text{bdry}}}\right) < 0, 0 < n < 1
\end{aligned}$$

$$L^{-2}e^{2V}|_{\text{core}} = \mathcal{F}^{ABJM} \equiv \left[\left(1+\frac{2g\mu^0}{\kappa n}\right)\left(1+\frac{2g\mu^1}{\kappa n}\right)\left(1+\frac{2g\mu^2}{\kappa n}\right)\left(1+\frac{2g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}}$$
$$e^{2\lambda_1}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1+\frac{2g\mu^2}{\kappa n}\right)^{-1}\left(1+\frac{2g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_2}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1+\frac{2g\mu^1}{\kappa n}\right)^{-1}\left(1+\frac{2g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_3}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1+\frac{2g\mu^1}{\kappa n}\right)^{-1}\left(1+\frac{2g\mu^2}{\kappa n}\right)^{-1}$$

$$g\mu_R \equiv [g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3] = -\kappa n - s$$
$$g\mu_B \equiv [g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3] = \kappa n$$

$$g\mu_R = 0, g\mu_B = \kappa$$

$$g\mu_{F_1} \equiv g\mu^1 - g\mu^2, g\mu_{F_2} \equiv g\mu^2 - g\mu^3$$

$$g\mu^0 = -\frac{s}{2}, g\mu^1 = \frac{1}{6}\left(4g\mu_{F_1} + 2g\mu_{F_2} - s - 2\kappa n\right)$$
$$g\mu^2 = \frac{1}{6}\left(-2g\mu_{F_1} + 2g\mu_{F_2} - s - 2\kappa n\right), g\mu^3 = \frac{1}{6}\left(-2g\mu_{F_1} - 4g\mu_{F_2} - s - 2\kappa n\right)$$

$$e^{2V}|_{\text{core}} = L^2\mathcal{F}^{ABJM} \equiv \frac{L^2}{3^{3/2}n^2}\left[(s-\kappa n)\left(-4g\mu_{F_1} - 2g\mu_{F_2} + s - \kappa n\right)\right.$$
$$\left.\left(2g\mu_{F_1} - 2g\mu_{F_2} + s - \kappa n\right)\left(2g\mu_{F_1} + 4g\mu_{F_2} + s - \kappa n\right)\right]^{\frac{1}{2}}$$
$$e^{2\lambda_1}|_{\text{core}=}\mathcal{F}^{ABJM}9n^2\left(2g\mu_{F_1} - 2g\mu_{F_2} + s - \kappa n\right)^{-1}\left(2g\mu_{F_1} + 4g\mu_{F_2} + s - \kappa n\right)^{-1}$$
$$e^{2\lambda_2}|_{\text{core}=}\mathcal{F}^{ABJM}9n^2\left(-4g\mu_{F_1} - 2g\mu_{F_2} + s - \kappa n\right)^{-1}\left(2g\mu_{F_1} + 4g\mu_{F_2} + s - \kappa n\right)^{-1}$$
$$e^{2\lambda_3}|_{\text{core}=}\mathcal{F}^{ABJM}9n^2\left(-4g\mu_{F_1} - 2g\mu_{F_2} + s - \kappa n\right)^{-1}\left(2g\mu_{F_1} + 4g\mu_{F_2} + s - \kappa n\right)^{-1}$$

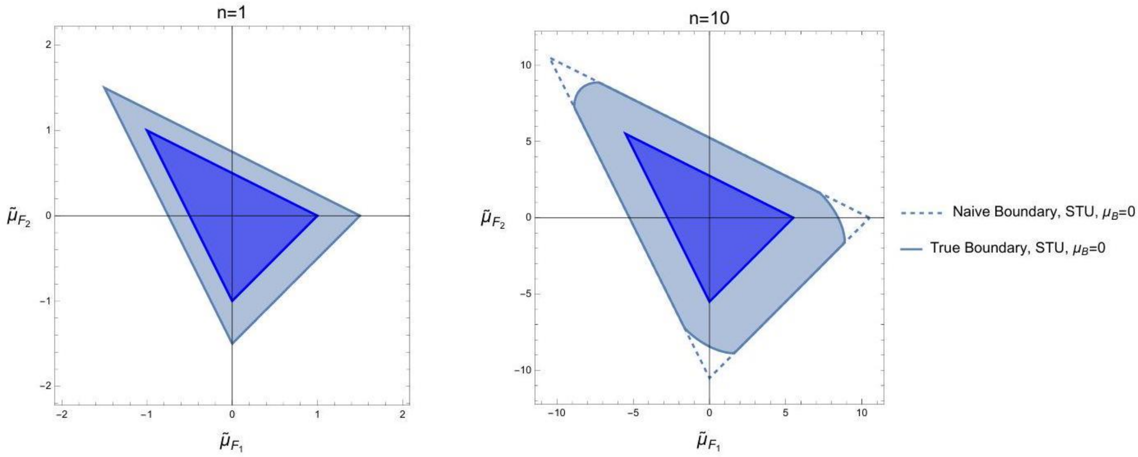


Figura 2. Sector de materia oscura.

$$\langle J^1\rangle - \langle J^2\rangle = 2\langle J_{F_1}\rangle - \langle J_{F_2}\rangle = \frac{\kappa(1-s\kappa n)N^{3/2}}{6\sqrt{2}\pi e^{V_0}}\left(e^{-2\lambda_1} - e^{-2\lambda_2}\right)\Big|_{\text{core}}$$
$$\langle J^2\rangle - \langle J^3\rangle = -\langle J_{F_1}\rangle + 2\langle J_{F_2}\rangle = \frac{\kappa(1-s\kappa n)N^{3/2}}{6\sqrt{2}\pi e^{V_0}}\left(e^{-2\lambda_2} - e^{-2\lambda_3}\right)\Big|_{\text{core}}$$

$$3\langle J^0\rangle + \langle J^1\rangle + \langle J^2\rangle + \langle J^3\rangle = 6\langle J_R^{\varphi}\rangle$$
$$= \frac{\kappa(1-s\kappa n)N^{3/2}}{6\sqrt{2}\pi e^{V_0}}\left(-\left(3 - \frac{6s\kappa}{n}\right)e^{-2\lambda_1-2\lambda_2-2\lambda_3} + e^{-2\lambda_1} + e^{-2\lambda_2} + e^{-2\lambda_3}\right)\Big|_{\text{core}}$$

$$3\langle J^0\rangle - \langle J^1\rangle - \langle J^2\rangle - \langle J^3\rangle = 6\langle J_B\rangle$$
$$= \frac{\kappa(1-s\kappa n)N^{3/2}}{6\sqrt{2}\pi e^{V_0}}\left(3e^{-2\lambda_1-2\lambda_2-2\lambda_3} - e^{-2\lambda_1} - e^{-2\lambda_2} - e^{-2\lambda_3}\right)\Big|_{\text{core}}$$

$$\langle T_{ab}\rangle dx^a dx^b = -\frac{h_D}{2\pi}\left[ds^2(AdS_2) - 2n^2 dz^2\right],$$

$$\frac{h_D}{2\pi} = \frac{1}{4\kappa n}\sum_{\alpha}\langle J^{\alpha}\rangle = \frac{1}{\kappa n}\langle J_R^{ABJM}\rangle$$
$$= \frac{1}{\kappa n}\left(\langle J_R^{\varphi}\rangle - \frac{1}{2}\langle J_B\rangle\right).$$

$$\langle \mathcal{O}_{\varphi}^{\Delta=2}\rangle = 0$$
$$\langle \mathcal{O}_{\Pi_1}^{\Delta=1}\rangle = -\frac{g}{2\kappa n}\left(\langle J^0\rangle + \langle J^1\rangle - \langle J^2\rangle - \langle J^3\rangle\right) = -\frac{g}{\kappa n}\left(\langle J_B\rangle + \langle J_{F_1}\rangle\right)$$
$$\langle \mathcal{O}_{\Pi_2}^{\Delta=1}\rangle = -\frac{g}{2\kappa n}\left(\langle J^0\rangle - \langle J^1\rangle + \langle J^2\rangle - \langle J^3\rangle\right) = -\frac{g}{\kappa n}\left(\langle J_B\rangle - \langle J_{F_1}\rangle + \langle J_{F_2}\rangle\right)$$
$$\langle \mathcal{O}_{\Pi_3}^{\Delta=1}\rangle = -\frac{g}{2\kappa n}\left(\langle J^0\rangle - \langle J^1\rangle - \langle J^2\rangle + \langle J^3\rangle\right) = -\frac{g}{\kappa n}\left(\langle J_B\rangle - \langle J_{F_2}\rangle\right)$$

$$\mathcal{J}^0\big|_{\text{bdry}} = -\frac{\kappa n}{g},\ \mathcal{J}^i\big|_{\text{bdry}} = -\frac{\kappa n}{3g},\ \text{for } i = 1,2,3.$$

$$\tilde{L}^{-2}e^{2V}\big|_{\text{core}} = \mathcal{F}^{mABJM} \equiv \left[\left(1+\frac{g\mu^0}{\kappa n}\right)\left(1+\frac{3g\mu^1}{\kappa n}\right)\left(1+\frac{3g\mu^2}{\kappa n}\right)\left(1+\frac{3g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}}$$
$$e^{2\lambda_1}\big|_{\text{core}} = \sqrt{3}\mathcal{F}^{mABJM}\left(1+\frac{3g\mu^2}{\kappa n}\right)^{-1}\left(1+\frac{3g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_2}\big|_{\text{core}} = \sqrt{3}\mathcal{F}^{mABJM}\left(1+\frac{3g\mu^1}{\kappa n}\right)^{-1}\left(1+\frac{3g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_3}\big|_{\text{core}} = \sqrt{3}\mathcal{F}^{mABJM}\left(1+\frac{3g\mu^1}{\kappa n}\right)^{-1}\left(1+\frac{3g\mu^2}{\kappa n}\right)^{-1}$$

$$g\mu_R \equiv g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -\kappa n - s$$
$$g\mu_B \equiv g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3 = 0$$

$$g\mu_B = 0, g\mu_R = 0$$

$$g\mu_{F_1} \equiv g\mu^1 - g\mu^2, g\mu_{F_2} \equiv g\mu^2 - g\mu^3$$

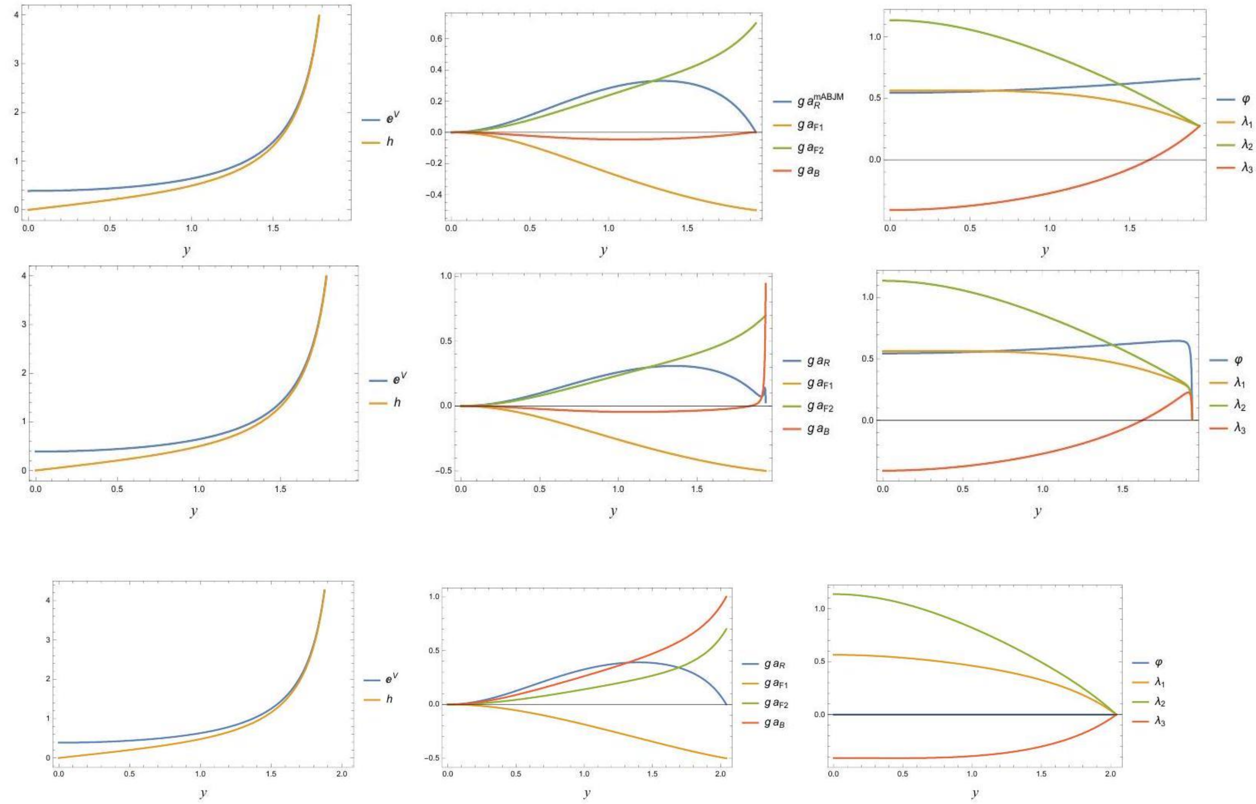
$$g\mu^0 = \frac{1}{2}(-s-\kappa n), \qquad g\mu^1 = \frac{1}{6}\left(4g\mu_{F_1} + 2g\mu_{F_2} - s - \kappa n\right),$$
$$g\mu^2 = \frac{1}{6}\left(-2g\mu_{F_1} + 2g\mu_{F_2} - s - \kappa n\right), \qquad g\mu^3 = \frac{1}{6}\left(-2g\mu_{F_1} - 4g\mu_{F_2} - s - \kappa n\right).$$

$$\tilde{L}^2\mathcal{F}^{mABJM}\left(g\mu_{F_i}, n\right) = L^2\mathcal{F}^{ABJM}\left(g\mu_{F_i}, n\right)$$

$$\langle T_{ab}\rangle dx^a dx^b = -\frac{h_D}{2\pi}\left[ds^2(AdS_2) - 2n^2dz^2\right],$$

$$\frac{h_D}{2\pi} = \frac{1}{\kappa n}\langle J_R^{\varphi}\rangle.$$

$$\langle \mathcal{O}_1^{\Delta=1}\rangle = \frac{3^{1/4}g}{2\kappa n}\langle J_{F_1}\rangle, \langle \mathcal{O}_2^{\Delta=1}\rangle = -\frac{g}{2\times 3^{1/4}\kappa n}\left(\langle J_{F_1}\rangle - 2\langle J_{F_2}\rangle\right)$$
$$\left\langle \mathcal{O}^{\Delta=\frac{1}{2}(1+\sqrt{17})}\right\rangle = 0$$



Figuras 3 y 4. Fluctuaciones gravitónicas.

$$g\mu^{\alpha} :: g\mu_R = g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -\kappa n - s.$$

$$\mathcal{E}^0 = e^{2V}e^{-2\lambda_1-2\lambda_2-2\lambda_3}F_{23}^0, \quad \mathcal{E}^1 = e^{2V}e^{-2\lambda_1+2\lambda_2+2\lambda_3}F_{23}^1$$
$$\mathcal{E}^2 = e^{2V}e^{2\lambda_1-2\lambda_2+2\lambda_3}F_{23}^2, \quad \mathcal{E}^3 = e^{2V}e^{2\lambda_1+2\lambda_2-2\lambda_3}F_{23}^3$$

$$\mathcal{E}^{\alpha} = c\langle J^{\alpha}\rangle.$$

$$\mathcal{E}^0 = e^{2V} g\cos\,\xi - \frac{\kappa}{\sqrt{2}} e^{V-\lambda_1-\lambda_2-\lambda_3}, \quad \mathcal{E}^1 = e^{2V} g\cos\,\xi - \frac{\kappa}{\sqrt{2}} e^{V-\lambda_1+\lambda_2+\lambda_3},$$
$$\mathcal{E}^2 = e^{2V} g\cos\,\xi - \frac{\kappa}{\sqrt{2}} e^{V+\lambda_1-\lambda_2+\lambda_3}, \quad \mathcal{E}^3 = e^{2V} g\cos\,\xi - \frac{\kappa}{\sqrt{2}} e^{V+\lambda_1+\lambda_2-\lambda_3},$$

$$\mathcal{E}_\alpha = g\cos\,\xi e^{2V}\left(1 + \frac{\kappa n}{2g\mathcal{J}^\alpha}\right),$$

$$W = \frac{1}{\sqrt{2} n e^V \cos\,\xi} \sum_\alpha \mathcal{J}^\alpha$$

$$g\mu^\alpha = g\mathcal{J}^\alpha|_{\text{bdry}} - g\mathcal{J}^\alpha|_{\text{core}},$$

$$g\mathcal{J}_\alpha|_{\text{core}} = -\left(\frac{\kappa n}{2} + g\mu^\alpha\right).$$

$$L^{-2} e^{2V}|_{\text{core}} = \mathcal{F}^{ABJM} \equiv \left[\left(1 + \frac{2g\mu^0}{\kappa n}\right)\left(1 + \frac{2g\mu^1}{\kappa n}\right)\left(1 + \frac{2g\mu^2}{\kappa n}\right)\left(1 + \frac{2g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}}$$
$$e^{2\lambda_1}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1 + \frac{2g\mu^2}{\kappa n}\right)^{-1}\left(1 + \frac{2g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_2}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1 + \frac{2g\mu^1}{\kappa n}\right)^{-1}\left(1 + \frac{2g\mu^3}{\kappa n}\right)^{-1}$$
$$e^{2\lambda_3}|_{\text{core}} = \mathcal{F}^{ABJM}\left(1 + \frac{2g\mu^1}{\kappa n}\right)^{-1}\left(1 + \frac{2g\mu^2}{\kappa n}\right)^{-1}$$

$$s = -\kappa, \left(1 + \frac{2g\mu^\alpha}{\kappa n}\right) > 0, n > 0$$
$$s = +\kappa, \left(1 + \frac{2g\mu^\alpha}{\kappa n}\right) < 0, 0 < n < 1$$

$$\langle J^\alpha \rangle = \frac{N^{3/2}}{6\sqrt{2}\pi} s\kappa e^{-V_0} \frac{(2g\mu^\alpha)\mathcal{F}^{ABJM}}{\left(1 + \frac{2g\mu^\alpha}{\kappa n}\right)}$$

$$\langle T_{ab} \rangle dx^a dx^b = -\frac{h_D}{2\pi}[ds^2(AdS_2) - 2n^2 dz^2]$$

$$h_D = \frac{2\pi}{4\kappa n} \sum_\alpha \langle J^\alpha \rangle.$$

$$h_D = -\frac{N^{3/2}}{12\sqrt{2}} e^{-V_0} \sum_\alpha \frac{(2\kappa g\mu^\alpha)\mathcal{F}^{ABJM}}{(1 + 2g\kappa\mu^\alpha)}.$$

$$h_D = \frac{s\kappa(1 - n^2)}{n^2} \frac{N^{3/2}}{12\sqrt{2} e^{V_0}}$$

$$e^{V_0} = 1$$

$$S = \frac{L^2}{4G}\mathrm{Vol}(AdS_2)(-s\kappa n)\mathcal{F}^{ABJM}$$

$$\mathcal{F}^{ABJM} \equiv \left[\left(1+\frac{2g\mu^0}{\kappa n}\right)\left(1+\frac{2g\mu^1}{\kappa n}\right)\left(1+\frac{2g\mu^2}{\kappa n}\right)\left(1+\frac{2g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}},$$

$$S_0 = \mathrm{Vol}(AdS_2)\frac{L^2}{4G}$$

$$I_0 = F_{S^3}^{ABJM} \equiv \frac{\sqrt{2}\pi}{3}N^{3/2}$$

$$I = (-s\kappa n)\mathcal{F}^{ABJM}F_{S^3}^{ABJM}$$

$$I_D^{ABJM} \equiv I^{ABJM} - nI_0^{ABJM} = n(-s\kappa\mathcal{F}^{ABJM} - 1)F_{S^3}^{ABJM}$$

$$\begin{aligned}\frac{1}{(2\pi)^2}dI_D^{ABJM} &= \frac{1}{n}\left(\sum_\alpha \langle J^\alpha\rangle d[g\mu^\alpha]\right) + \left(2\frac{h_D}{2\pi} - \frac{F_{S^3}^{ABJM}}{(2\pi)^2}\right)dn \\ &= \frac{1}{n}\left(\sum_\alpha \langle J^\alpha\rangle d[g\mu^\alpha] + \frac{\kappa}{2}\sum_\alpha \langle J^\alpha\rangle dn\right) - \frac{F_{S^3}^{ABJM}}{(2\pi)^2}dn\end{aligned}$$

$$\sum_\alpha \langle J^\alpha\rangle d[g\mu^\alpha] = \langle J_{F_1}\rangle d[g\mu^{F_1}] + \langle J_{F_2}\rangle d[g\mu^{F_2}] + \langle J_{F'}\rangle d[g\mu_{F'}] + \langle J_R^{ABJM}\rangle d[g\mu_R]$$

$$S = -\frac{\tilde{L}^2}{4G}(s\kappa n)\mathrm{Vol}(AdS_2)\mathcal{F}^{mABJM}$$

$$\mathcal{F}^{mABJM} \equiv \left[\left(1+\frac{g\mu^0}{\kappa n}\right)\left(1+\frac{3g\mu^1}{\kappa n}\right)\left(1+\frac{3g\mu^2}{\kappa n}\right)\left(1+\frac{3g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}}$$

$$S_0 = \mathrm{Vol}(AdS_2)\frac{\tilde{L}^2}{4G}$$

$$I_0 = F_{S^3}^{mABJM}$$

$$F_{S^3}^{mABJM} \equiv \frac{4\sqrt{2}\pi}{9\sqrt{3}}N^{3/2} = \frac{4}{3\sqrt{3}}F_{S^3}^{ABJM}$$

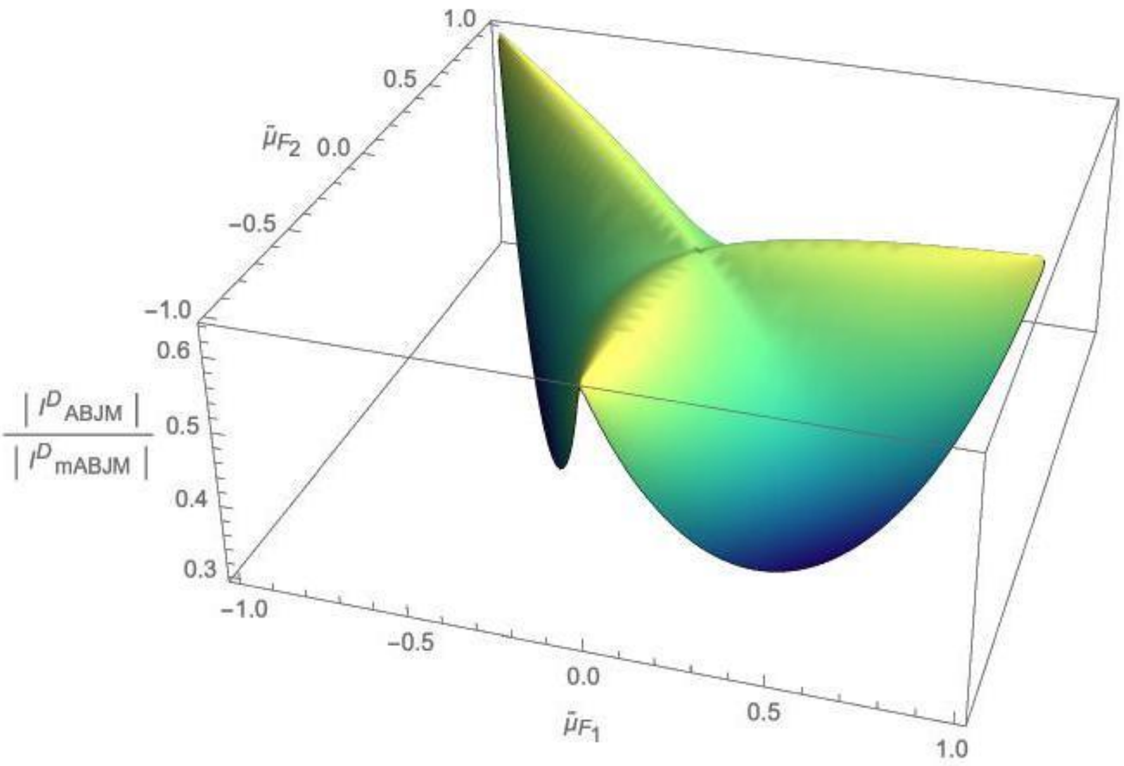
$$I_D^{mABJM} = n(-s\kappa\mathcal{F}^{mABJM} - 1)F_{S^3}^{mABJM}$$

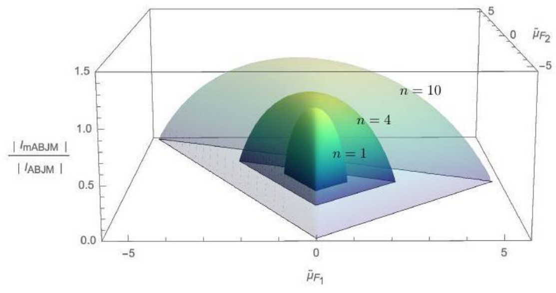
$$\begin{aligned}\frac{1}{(2\pi)^2}dI_D^{mABJM} &= \frac{1}{n}\langle J_{F_1}\rangle d[g\mu^{F_1}] + \frac{1}{n}\langle J_{F_2}\rangle d[g\mu^{F_2}] + \frac{1}{(2\pi)^2}(2\pi h_D - F_{S^3}^{mABJM})dn \\ &= \frac{1}{n}(\langle J_{F_1}\rangle d[g\mu^{F_1}] + \langle J_{F_2}\rangle d[g\mu^{F_2}] + \kappa\langle J_R^{\varphi}\rangle dn) - \frac{F_{S^3}^{mABJM}}{(2\pi)^2}dn\end{aligned}$$

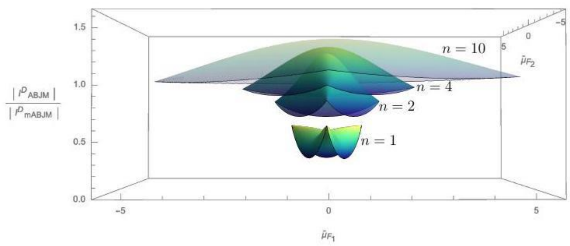
$$I^{ABJM,\varphi\neq 0} = (-s\kappa n)\mathcal{F}^{ABJM}F_{S^3}^{ABJM}\Big|_{g\mu_B=\kappa n}$$
$$= \frac{(-s\kappa)}{4n}\left[(s-n\kappa)\left(s-n\kappa-4g\mu_{F_1}-2g\mu_{F_2}\right)\right.$$
$$\left.\left(s-n\kappa+2g\mu_{F_1}-2g\mu_{F_2}\right)\left(s-n\kappa+2g\mu_{F_1}+4g\mu_{F_2}\right)\right]^{1/2}\frac{4F_{S^3}^{ABJM}}{3\sqrt{3}}$$

$$I^{ABJM,\varphi\neq 0} = I^{ABJM}\Big|_{g\mu_B=\kappa n}$$

$$I^{ABJM,\varphi\neq 0} = I^{mABJM}$$







Figuras 5 y 6. Deformación del espacio – tiempo cuántico por intervención gravitónica.

$$s = -\kappa$$

$$S_n^{\mathrm{SRE}} = -\frac{I_n(g\mu^\alpha) - nI_{n=1}(g\mu^\alpha)}{1-n}$$

$$I_n(g\mu^i) = n\mathcal{F}^{ABJM}F_{S^3}^{ABJM}$$

$$\mathcal{F}^{ABJM} = \left[\left(1+\frac{2g\mu^0}{\kappa n}\right)\left(1+\frac{2g\mu^1}{\kappa n}\right)\left(1+\frac{2g\mu^2}{\kappa n}\right)\left(1+\frac{2g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}}$$

$$S_n^{\mathrm{SRE}} = -\frac{1+3n}{4n} F_{S^3}^{ABJM}$$

$$S_n^{\mathrm{SRE}} = -\frac{9n^2-\sqrt{3}(1+2n)^{3/2}}{9n(n-1)} F_{S^3}^{ABJM}$$

$$g\mu^\alpha = \kappa\left(\Delta^\alpha - \frac{1}{2}\right) + \frac{\kappa}{2}(\Delta^\alpha - 1)(n-1), \sum_\alpha \Delta^\alpha = \mathfrak{I}$$

$$S_n^{SRE} = -\frac{1+3n}{n}\sqrt{\Delta^0\Delta^1\Delta^2\Delta^3} F_{S^3}^{ABJM}$$

$$g\mu^\alpha = \frac{\kappa}{2}(1-n)\Delta^\alpha, \sum_\alpha \Delta^\alpha = 2$$

$$S_n^{SRE} = -\frac{\sqrt{\prod_{\alpha=0}^{3}\left(n(\Delta^\alpha-1)-\Delta^\alpha\right)} - n^2}{n(1-n)} F_{S^3}^{ABJM}$$

$$\Delta^\alpha : \lim_{n\to 1} S_n^{SRE} = -F_{S^3}^{ABJM}$$

$$I_n = n\mathcal{F}^{mABJM} F_{S^3}^{mABJM}$$

$$\mathcal{F}^{mABJM} \equiv \left[\left(1+\frac{g\mu^0}{\kappa n}\right)\left(1+\frac{3g\mu^1}{\kappa n}\right)\left(1+\frac{3g\mu^2}{\kappa n}\right)\left(1+\frac{3g\mu^3}{\kappa n}\right)\right]^{\frac{1}{2}},$$

$$S_n^{\mathrm{SRE}} = -\frac{1+3n}{4n} F_{S^3}^{mABJM}$$

$$g\mu^0 = \kappa(\Delta^0 - 1) + \frac{\kappa}{2}(\Delta^0 - 2)(n-1),$$
$$g\mu^a = \kappa\left(\Delta^a - \frac{1}{3}\right) + \frac{\kappa}{2}\left(\Delta^a - \frac{2}{3}\right)(n-1), a = 1{,}2{,}3,$$

$$\sum_\alpha \Delta^\alpha = 2, \Delta^0 - \Delta^1 - \Delta^2 - \Delta^3 = 0$$

$$S_n^{SRE} = -3\sqrt{3}\frac{1+3n}{4n}\sqrt{\Delta^0\Delta^1\Delta^2\Delta^3} F_{S^3}^{mABJM}.$$

$$g\mu^\alpha = \frac{\kappa}{2}(1-n)\Delta^\alpha, \sum_\alpha \Delta^\alpha = 2, \Delta^0 - \Delta^1 - \Delta^2 - \Delta^3 = 0.$$

$$S_n^{SRE} = -\frac{\sqrt{\left(\Delta^0(n-1)-2n\right)\prod_{\alpha=1}^{3}\left(3\Delta^\alpha(n-1)-2n\right)} - 4n^2}{4n(1-n)} F_{S^3}^{mABJM}.$$

$$f^{-1}\xi' = \sqrt{2}gW\cos\,\xi + \kappa e^{-V}$$
$$f^{-1}V' = \frac{g}{\sqrt{2}}W\sin\,\xi$$
$$f^{-1}\lambda_i' = -\frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin\,\xi$$
$$f^{-1}\varphi' = -\frac{g}{\sqrt{2}}\frac{\partial_\varphi W}{\sin\,\xi}$$
$$f^{-1}\frac{h'}{h}\sin\,\xi = \kappa e^{-V}\cos\,\xi + \frac{gW}{\sqrt{2}}(1+\cos^2\,\xi)$$

$$(s-Q_z)\sin\,\xi = -\sqrt{2}gWh\cos\,\xi - \kappa h e^{-V}$$
$$\sqrt{2}g\partial_\varphi W\cos\,\xi = \partial_\varphi Q_z \sin\,\xi h^{-1}$$

$$\bar{F}^{12}_{23} = -g\partial_{\lambda_1}W\cos\,\xi$$
$$\bar{F}^{34}_{23} = -g\partial_{\lambda_2}W\cos\,\xi$$
$$\bar{F}^{56}_{23} = -g\partial_{\lambda_3}W\cos\,\xi$$
$$H_{23} = -gW\cos\,\xi - \sqrt{2}\kappa e^{-V}$$

$$F^0 = \frac{1}{2}e^{\lambda_1+\lambda_2+\lambda_3}(\bar{F}^{12}+\bar{F}^{34}+\bar{F}^{56}+\bar{F}^{78})$$
$$F^1 = \frac{1}{2}e^{\lambda_1-\lambda_2-\lambda_3}(\bar{F}^{12}-\bar{F}^{34}-\bar{F}^{56}+\bar{F}^{78})$$
$$F^2 = -\frac{1}{2}e^{-\lambda_1+\lambda_2-\lambda_3}(\bar{F}^{12}-\bar{F}^{34}+\bar{F}^{56}-\bar{F}^{78})$$
$$F^3 = -\frac{1}{2}e^{-\lambda_1-\lambda_2+\lambda_3}(\bar{F}^{12}+\bar{F}^{34}-\bar{F}^{56}-\bar{F}^{78})$$

$$he^{-V} = -n\sin\,\xi$$

$$f^{-1}\xi' = n^{-1}(s-Q_z)e^{-V}$$
$$f^{-1}V' = \frac{g}{\sqrt{2}}W\sin\,\xi$$
$$f^{-1}\lambda_i' = -\frac{g}{\sqrt{2}}\partial_{\lambda_i}W\sin\,\xi$$
$$f^{-1}\varphi' = -\frac{g}{\sqrt{2}}\frac{\partial_\varphi W}{\sin\,\xi}$$

$$(s-Q_z) = n\left(\sqrt{2}gWe^V\cos\,\xi + \kappa\right)$$
$$\sqrt{2}g\partial_\varphi W\cos\,\xi = -n^{-1}e^{-V}\partial_\varphi Q_z$$

$$\partial_\varphi Q_z = -\sinh\,2\varphi D_z\theta.$$

$$D_z\theta = \frac{\sqrt{2}gne^V\partial_\varphi W\cos\,\xi}{\sinh\,2\varphi}$$

$$F^\alpha_{yz} = (a^\alpha)' = (\mathcal{J}^\alpha)'$$

$$\mathcal{J}^0 \equiv -\frac{1}{\sqrt{2}} n e^V \cos \xi e^{\lambda_1+\lambda_2+\lambda_3}, \qquad \mathcal{J}^1 \equiv -\frac{1}{\sqrt{2}} n e^V \cos \xi e^{\lambda_1-\lambda_2-\lambda_3}$$

$$\mathcal{J}^2 \equiv -\frac{1}{\sqrt{2}} n e^V \cos \xi e^{-\lambda_1+\lambda_2-\lambda_3}, \qquad \mathcal{J}^3 \equiv -\frac{1}{\sqrt{2}} n e^V \cos \xi e^{-\lambda_1-\lambda_2+\lambda_3}$$

$$h \to -h, z \to -z,$$

$$f = \frac{L}{y}$$

$$\lambda_i = \frac{\lambda_i^{(1)}}{y} + \frac{\lambda_i^{(2)}}{y^2} + \cdots$$

$$\varphi = \frac{\varphi^{(1)}}{y} + \frac{\varphi^{(2)}}{y^2} + \cdots$$

$$a^\alpha = \mu^\alpha + \frac{j^\alpha}{y} + \cdots$$

$$\varphi_s \equiv \varphi^{(1)}.$$

$$e^{2V} = e^{2V_0} y^2 + \frac{1}{2}\left(L^2 - e^{2V_0}\left[\sum_i \left(\lambda_i^{(1)}\right)^2 + \left(\varphi^{(1)}\right)^2\right]\right) + \frac{V_{(2)}}{y} + \cdots,$$

$$\frac{h^2}{h_0^2} = y^2 - \frac{1}{2}\left(\frac{L^2}{e^{2V_0}} + \sum_i \left(\lambda_i^{(1)}\right)^2 + \left(\varphi^{(1)}\right)^2\right)$$

$$-2\left(\frac{V_{(2)}}{e^{2V_0}} + \frac{4}{3}\left[\sum_i \lambda_i^{(1)}\lambda_i^{(2)} + \varphi^{(1)}\varphi^{(2)}\right]\right)\frac{1}{y} + \cdots$$

$$ds^2 = \gamma_{ab} dx^a dx^b + \frac{L^2}{y^2} dy^2 + \cdots, \gamma_{ab} = y^2 h_{ab}$$

$$h_{ab} dx^a dx^b = e^{2V_0}\left(ds^2(AdS_2) + \frac{h_0^2}{e^{2V_0}} dz^2\right).$$

$$\xi = -\frac{\pi}{2} + \frac{\kappa L}{e^{V_0} y} + \cdots,$$

$$n = \frac{h_0}{e^{V_0}}, \sum_\alpha g\mu^\alpha = -\kappa n - s$$

$$\bar{\theta} - (g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3) = -\frac{\kappa h_0}{e^{V_0}}$$

$$\lambda_1^{(2)} = \lambda_2^{(1)}\lambda_3^{(1)} - \frac{1}{2}\left(\varphi^{(1)}\right)^2$$

$$\lambda_2^{(2)} = \lambda_1^{(1)}\lambda_3^{(1)} - \frac{1}{2}\left(\varphi^{(1)}\right)^2$$

$$\lambda_3^{(2)} = \lambda_1^{(1)}\lambda_2^{(1)} - \frac{1}{2}\left(\varphi^{(1)}\right)^2$$

$$\varphi^{(2)} = -\left(\lambda_1^{(1)} + \lambda_2^{(1)} + \lambda_3^{(1)}\right)\varphi^{(1)}$$

$$V_{(2)} = \frac{4}{3}e^{2V_0}\left(\left(\lambda_1^{(1)} + \lambda_2^{(1)} + \lambda_3^{(1)}\right)\left(\varphi^{(1)}\right)^2 - 2\lambda_1^{(1)}\lambda_2^{(1)}\lambda_3^{(1)}\right) - \frac{\kappa e^{V_0}}{6gh_0}(j^0 + j^1 + j^2 + j^3)$$

$$\lambda_1^{(1)} = -\frac{g}{2\kappa n}(j^0 + j^1 - j^2 - j^3)$$
$$\lambda_2^{(1)} = -\frac{g}{2\kappa n}(j^0 - j^1 + j^2 - j^3)$$
$$\lambda_3^{(1)} = -\frac{g}{2\kappa n}(j^0 - j^1 - j^2 + j^3)$$

$$\mathcal{E}_{R_1} = -\frac{e^{2V_0}}{h_0 L}(j^0 + j^1)$$
$$\mathcal{E}_{R_2} = -\frac{e^{2V_0}}{h_0 L}(j^0 + j^2)$$
$$\mathcal{E}_{R_3} = -\frac{e^{2V_0}}{h_0 L}(j^0 + j^3)\text{(B.12)}$$

$$\mathcal{E}_B = -\frac{e^{2V_0}}{h_0 L}(3j^0 - j^1 - j^2 - j^3)$$

$$g\mu_R \equiv g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -s - \kappa n$$
$$g\mu_B \equiv g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3 = \kappa n$$

$$\sqrt{-g}\mathcal{L} = \partial_y\left(-\frac{e^{2V}hV'}{\rho^2 f}\right) + \partial_\rho\left(\frac{fh}{\rho}\right)$$

$$\sqrt{-g}\mathcal{L} =-\partial_y\left(\frac{e^{2V}h'}{\rho^2 f} + \frac{e^{2V}}{\rho^2 fh}\left[e^{-2(\lambda_1+\lambda_2+\lambda_3)}a_0a_0' + e^{2(-\lambda_1+\lambda_2+\lambda_3)}a_1a_1'\right.\right.$$
$$\left.\left. + e^{2(\lambda_1-\lambda_2+\lambda_3)}a_2a_2' + e^{2(\lambda_1+\lambda_2-\lambda_3)}a_3a_3'\right]\right)$$
$$-\frac{e^{2V}f}{2\rho^2 h}\sinh^2\ 2\varphi\bar{\theta}D_z\theta$$

$$S = S_{bulk} + S_{bdy}$$

$$S_{bdy} = \frac{1}{8\pi G}\int\ d^3x\sqrt{-\gamma}\left(\mathrm{Tr}K + \frac{1}{L}W + LR(\gamma)\left[-\frac{1}{2} + \sum_i\ b_i\lambda_i + b_4\varphi\right]\right)$$

$$\left[S + S_{bdy}\right] = \left[S + S_{bdy}\right]\left(g\mu^\alpha, \lambda_i^{(1)}, \varphi^{(1)}, h_0, e^{2V_0}\right)$$

$$\Pi_i \equiv \left(\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\partial_y\lambda_i} + \frac{\delta\left(\sqrt{-\gamma}\left[\frac{1}{L}W + LR(\gamma)(\sum_i\ b_i\lambda_i + b_4\varphi)\right]\right)}{\delta\lambda_i}\right)$$

$$\Pi_i = \frac{h_0 e^{2V_0}}{\rho^2}\Pi_i^{(s)}\frac{y}{L} + \cdots$$

$$\Pi_1^{(s)} \equiv 2\lambda_1^{(2)} - 2\lambda_2^{(1)}\lambda_3^{(1)} + \left(\varphi^{(1)}\right)^2 - 2e^{-2V_0}b_1L^2$$

$$S_{Tot} = S_{bulk} + S_{bdy} - \frac{1}{8\pi G}\int d^3x \sum_i \Pi_i \lambda_i$$

$$\delta S_{Tot} = \int d^3x\sqrt{-h}\left[\langle \mathcal{O}_\varphi\rangle \delta\varphi^{(1)} - \langle \mathcal{O}_{\Pi_i}\rangle \delta \Pi_i^{(s)}\right]$$

$$\langle \mathcal{O}_\varphi\rangle = \frac{1}{8\pi G}\frac{1}{L}\left(2\sum_i \lambda_i^{(1)}\varphi^{(1)} + 2\varphi^{(2)} - 2b_4L^2e^{-2V_0}\right)$$
$$\langle \mathcal{O}_{\Pi_i}\rangle = \frac{1}{8\pi G}\frac{1}{L}\lambda_i^{(1)}$$

$$\delta S_{Tot} = \int d^3x\sqrt{-h}\langle J^{\alpha z}\rangle \delta(g\mu^\alpha),$$

$$\langle J^{\alpha z}\rangle = \frac{1}{8\pi G}\frac{1}{gLh_0^2}j^\alpha, \Rightarrow \langle J_z^\alpha\rangle = \frac{1}{8\pi G}\frac{1}{gL}j^\alpha \equiv \langle J^\alpha\rangle,$$

$$\delta S_{Tot} = \int d^3x\sqrt{-h}\left[\langle T^{uu}\rangle \delta\left(\frac{e^{2V_0}}{\rho^2}\right) + \frac{1}{2}\langle T^{zz}\rangle \delta h_0^2\right]$$

$$\langle T^{uu}\rangle = \frac{1}{8\pi G}\rho^2\frac{e^{-2V_0}}{L}\left(4\lambda_1^{(1)}\lambda_2^{(1)}\lambda_3^{(1)} + 2\varphi^{(1)}\varphi^{(2)} + \frac{3}{2}e^{-2V_0}V_{(2)} + \sum_i 2b_iL^2e^{-2V_0}\lambda_i^{(1)}\right)$$
$$\langle T^{zz}\rangle = \frac{1}{8\pi G}\frac{h_0^{-2}}{L}\left(-4\sum_i \lambda_i^{(1)}\lambda_i^{(2)} + 4\lambda_1^{(1)}\lambda_2^{(1)}\lambda_3^{(1)} - 2\varphi^{(1)}\varphi^{(2)} - 3e^{-2V_0}V_{(2)} - 2b_4L^2e^{-2V_0}\varphi^{(1)}\right)$$

$$\langle T_a^a\rangle = \left(3 - \Delta_{\mathcal{O}_\varphi}\right)\langle \mathcal{O}_\varphi\rangle \varphi^{(1)} - \sum_i \left(3 - \Delta_{\mathcal{O}_{\lambda_i}}\right)\langle \mathcal{O}_{\Pi_i}\rangle \Pi_i^{(s)} = 0$$

$$\langle T_u^u\rangle = -\frac{\kappa n}{4h_0^2}\sum_\alpha \langle J^{(\alpha)}\rangle + \frac{1}{8\pi G}\frac{1}{L}\left(\sum_i 2b_iL^2e^{-2V_0}\lambda_i^{(1)}\right)$$
$$\langle T_z^z\rangle = \frac{\kappa n}{2h_0^2}\sum_\alpha \langle J^{(\alpha)}\rangle + \frac{1}{8\pi G}\frac{1}{L}\left(-2b_4L^2e^{-2V_0}\varphi^{(1)}\right)$$
$$\langle \mathcal{O}_\varphi\rangle = \frac{1}{8\pi G}\frac{1}{L}\left(-2b_4L^2e^{-2V_0}\right)$$
$$\langle \mathcal{O}_{\Pi_i}\rangle = \frac{1}{8\pi G}\frac{1}{L}\lambda_i^{(1)}$$

$$\frac{1}{8\pi GL}\lambda_1^{(1)} = -\frac{g}{2\kappa n}\left(\langle J^0\rangle + \langle J^1\rangle - \langle J^2\rangle - \langle J^3\rangle\right)$$
$$\frac{1}{8\pi GL}\lambda_2^{(1)} = -\frac{g}{2\kappa n}\left(\langle J^0\rangle - \langle J^1\rangle + \langle J^2\rangle - \langle J^3\rangle\right)$$
$$\frac{1}{8\pi GL}\lambda_3^{(1)} = -\frac{g}{2\kappa n}\left(\langle J^0\rangle - \langle J^1\rangle - \langle J^2\rangle + \langle J^3\rangle\right)$$

$$b_i = b_4 = 0.$$

$$\langle T_{ab}\rangle dx^a dx^b = -\frac{1}{4\kappa n}\sum_{\alpha} \langle J^{(\alpha)}\rangle [ds^2(AdS_2) - 2n^2 dz^2],$$

$$\langle \mathcal{O}_\varphi\rangle = 0,$$

$$\langle \mathcal{O}_{\Pi_1}\rangle = -\frac{g}{2\kappa n}(\langle J^0\rangle + \langle J^1\rangle - \langle J^2\rangle - \langle J^3\rangle),$$

$$\langle \mathcal{O}_{\Pi_2}\rangle = -\frac{g}{2\kappa n}(\langle J^0\rangle - \langle J^1\rangle + \langle J^2\rangle - \langle J^3\rangle),$$

$$\langle \mathcal{O}_{\Pi_3}\rangle = -\frac{g}{2\kappa n}(\langle J^0\rangle - \langle J^1\rangle - \langle J^2\rangle + \langle J^3\rangle),$$

$$\lambda_i = \frac{1}{4}\ln\ 3, \varphi = \frac{1}{2}\text{arccosh}2.$$

$$f = \frac{\tilde{L}}{y}, \tilde{L} = \frac{\sqrt{2}}{3^{3/4} g}$$

$$V = V_0 + \ln\ y + y^{-2\delta}(v_s + \cdots) + \frac{v_1}{y} + \frac{v_2}{y^2} + \frac{v_3}{y^3} + y^{-\delta}\left(\frac{v_m}{y} + \cdots\right) + \cdots$$

$$h = h_0 y\left[1 + y^{-2\delta}(\eta_s + \cdots) + \frac{h_1}{y} + \frac{h_2}{y^2} + \frac{h_3}{y^3} + y^{-\delta}\left(\frac{\eta_m}{y} + \cdots\right) + \cdots\right]$$

$$A^\alpha = \mu^\alpha + \frac{j^\alpha}{y} + \cdots + y^{-3+\delta}(m_v^\alpha + \cdots) + \cdots$$

$$\lambda_i = \frac{1}{4}\ln\ 3 + y^{-\delta}(\zeta_i^s + \cdots) + \frac{l_i^1}{y} + \frac{l_i^2}{y^2} + \cdots + y^{-3+\delta}(\zeta_i^v + \cdots) + y^{-5+\delta}(\zeta_i^{Iv} + \cdots) + \cdots$$

$$\varphi = \frac{1}{2}\text{arccosh}2 + y^{-\delta}(Z^s + \cdots) + \frac{f^1}{y} + \frac{f^2}{y^2} + \cdots + y^{-3+\delta}(Z^v + \cdots) + y^{-5+\delta}(Z^{Iv} + \cdots) + \cdots$$

$$g\mu_B \equiv g\mu^0 - g\mu^1 - g\mu^2 - g\mu^3 = 0$$

$$g\mu_R^{mABJM} \equiv \frac{1}{2}(g\mu^0 + 3g\mu^1 + 3g\mu^2 + 3g\mu^3) = 2g\mu^0 = g\mu^R$$

$$0 = j^0 - j^1 - j^2 - j^3$$

$$\xi = -\frac{\pi}{2} + \frac{x_1}{y} + \frac{x_2}{y^2} + \cdots$$

$$x_1 = \frac{\sqrt{2}}{3^{3/4}}\frac{\kappa}{g}e^{-V_0}, x_2 = -\frac{\sqrt{2}}{3^{3/4}}\frac{e^{-V_0}}{n}j^0$$

$$\zeta_i^s = Z^s = v_s = \eta_s = f_1 = v_1 = 0$$

$$h_0 = ne^{V_0}, f_2 = \frac{2}{\sqrt{3}}(l_1^{12} + l_1^1 l_2^1 + (l_2^1)^2)$$
$$l_1^2 = \frac{1}{3}(4l_1^{12} + l_1^1 l_2^1 + (l_2^1)^2), l_2^2 = \frac{1}{3}(l_1^{12} + l_1^1 l_2^1 + 4(l_2^1)^2)$$
$$l_3^2 = \frac{1}{3}(4(l_1^1)^2 + 7l_1^1 l_2^1 + 4(l_2^1)^2), \zeta_i^v = -\frac{1+\sqrt{17}}{4\sqrt{3}} Z^v$$
$$v_2 = \frac{1}{18}\left(-9((l_1^1)^2 + l_1^1 l_2^1 + (l_2^1)^2) + \frac{\sqrt{3}}{g^2} e^{-2V_0}\right)$$
$$v_3 = \frac{1}{9}\left(12 l_1^1 l_2^1 (l_1^1 + l_2^1) + 3^{1/4}\sqrt{2} x_2 \frac{\kappa}{g} e^{-V_0}\right)$$

$$l_1^1 = \frac{g\kappa}{2n}(j^0 - 3j^1), l_2^1 = \frac{g\kappa}{2n}(j^0 - 3j^2), l_3^1 = -\frac{g\kappa}{2n}\left(2j^0 - 3(j^1 + j^2)\right)$$

$$g\mu_R \equiv g\mu^0 + g\mu^1 + g\mu^2 + g\mu^3 = -\kappa n - s$$

$$S = S_{bulk} + S_{bdy}$$

$$S_{bdy} = \frac{1}{8\pi G}\int d^3x\sqrt{-\gamma}\left(\mathrm{Tr}K + \frac{1}{L}W - \frac{\tilde{L}}{2}R(\gamma)\right)$$

$$\begin{aligned}\delta[S_{bulk} + S_{bdy}] &= \frac{1}{8\pi G}\int d^3x\left(\Pi^{(A)}\right)_\alpha^\mu \delta A_\mu^\alpha \\ &= \frac{1}{8\pi G}\int d^3x\left(\Pi^{(A)}\right)_\alpha^\mu M^\alpha{}_\beta \delta\tilde{A}_\mu^\beta\end{aligned}$$

$$M_\beta^\alpha = \frac{1}{12}\begin{pmatrix} 6 & 0 & 0 & 9 \\ 2 & 8 & 4 & -1 \\ 2 & -4 & 4 & -1 \\ 2 & -4 & -8 & -1 \end{pmatrix}.$$

$$\begin{aligned}\langle J_R^\varphi\rangle &= \frac{1}{8\pi G}\frac{3^{1/4}}{\sqrt{2}}\frac{1}{6}(3j^0 + j^1 + j^2 + j^3), \\ \langle J_{F_1}\rangle &= \frac{1}{8\pi G}\frac{3^{1/4}}{\sqrt{2}}(2j^1 - j^2 - j^3), \\ \langle J_{F_2}\rangle &= \frac{1}{8\pi G}\frac{3^{1/4}}{\sqrt{2}}(j^1 + j^2 - 2j^3).\end{aligned}$$

$$S_{ct}^{mv} = -\frac{1}{8\pi G}\frac{g3^{1/4}}{2\sqrt{2}}\int d^3x\sqrt{-\gamma}A_B^2$$

$$\langle J_B^{mABJM}\rangle = \frac{1}{8\pi G}\frac{\kappa n}{g2^{5/2}3^{1/4}}(5 - 3\sqrt{17})Z^v.$$

$$\langle T_{ab}\rangle dx^a dx^b = -\frac{h_D}{2\pi}[ds^2(AdS_2) - 2n^2 dz^2],$$

$$h_D = \frac{2\pi}{\kappa n}J_R^{mABJM}.$$

$$\delta\left[S_{bulk}+S_{bdy}+S_{ct}^{mv}\right]=\frac{1}{8\pi G}\int\ d^3x(\Pi_a\delta\Phi^a+\cdots)=\frac{1}{8\pi G}\int\ d^3x\left(\Pi_a\mathbb{S}^a{}_b\delta\tilde{\Phi}^b+\cdots\right).$$

$$S^{\delta\tilde{\Phi}^{1,2}}=-\frac{1}{8\pi G}\int\ d^3x\sum_{a=1}^{4}\sum_{b=1}^{2}\Pi_a\mathbb{S}_b^a\tilde{\Phi}^b,$$

$$\begin{pmatrix} -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}}\sqrt{1-\frac{1}{\sqrt{17}}} & 2\frac{\sqrt{17}-3}{\sqrt{17}-5}\sqrt{\frac{2}{3(17+\sqrt{17})}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\sqrt{1-\frac{1}{\sqrt{17}}} & 2\frac{\sqrt{17}-3}{\sqrt{17}-5}\sqrt{\frac{2}{3(17+\sqrt{17})}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\sqrt{1-\frac{1}{\sqrt{17}}} & 2\frac{\sqrt{17}-3}{\sqrt{17}-5}\sqrt{\frac{2}{3(17+\sqrt{17})}} \\ 0 & 0 & \sqrt{\frac{1}{34}(17+\sqrt{17})} & 2\sqrt{\frac{2}{17+\sqrt{17}}} \end{pmatrix}$$

$$\tilde{\Phi}^1=\frac{1}{\sqrt{6}}(\lambda^3+\lambda^2-2\lambda^1)$$
$$\tilde{\Phi}^2=\frac{1}{\sqrt{2}}(\lambda^3-\lambda^2)$$

$$\langle\mathcal{O}_1^{\Delta=1}\rangle=\frac{3^{1/4}g}{2\kappa n}\langle J_{F_1}\rangle,\langle\mathcal{O}_2^{\Delta=1}\rangle=-\frac{g}{2\times3^{1/4}\kappa n}\left(\langle J_{F_1}\rangle-2\langle J_{F_2}\rangle\right).$$

$$\left\langle\mathcal{O}^{\Delta=\frac{1}{2}(1+\sqrt{17})}\right\rangle=0$$

$$\delta S^{\delta\tilde{\Phi}^4}=\frac{1}{8\pi G}\frac{\sqrt{13+\frac{43}{\sqrt{17}}}}{2\times3^{3/4}g}\int\ d^3x\sqrt{-\gamma}(\mathbb{S}^{-1})^4{}_a\delta\Phi^a\left(\varphi-\frac{1}{2}\text{arccosh}2\right)R(\gamma)$$

$$\left\langle\mathcal{O}^{\Delta=\frac{1}{2}(5+\sqrt{17})}\right\rangle=\frac{2}{3^{3/4}g}\sqrt{5+\frac{13}{\sqrt{17}}}e^{-2V_0}Z^v$$

$$\begin{aligned} f^{-1}\xi'+2^{3/2}g\cos\,\xi-\kappa e^{-V}&=0\\ f^{-1}V'+2^{1/2}g\sin\,\xi&=0\\ h&=-ne^V\sin\,\xi\\ a_R&=-2^{3/2}ne^V\cos\,\xi+c_0 \end{aligned}$$

$$gc_0-\kappa n+s=0$$

$$e^V=e^{V_0}y,f=\frac{h_0}{h}$$

$$(h^2)'|_{y=y_*}=2h_0$$

$$\sqrt{2}gh_0 = ne^{V_0}$$

$$\xi' = \frac{2}{y\tan\,\xi} - \frac{\kappa e^{-V_0}}{\sqrt{2}gy^2\sin\,\xi}$$

$$\tilde{y} = 2^{3/2}ge^{V_0}\kappa y,$$

$$\xi = \pm\arccos\left[\frac{2\tilde{y}+a}{\tilde{y}^2}\right]$$

$$ds^2 = \frac{1}{2g^2}\left[\frac{\tilde{y}^2}{4}ds^2(AdS_2) + \frac{\tilde{y}^2}{q}d\tilde{y}^2 + \frac{q}{4\tilde{y}^2}n^2dz^2\right]$$

$$ga_R = -s - n\kappa\left(1+\frac{a}{\tilde{y}}\right)$$

$$q = (\tilde{y}^2 - [2\tilde{y}+a])(\tilde{y}^2 + [2\tilde{y}+a]),$$

$$\tilde{y}_*(\sigma_i) = \sigma_2 + \sigma_1\sqrt{1+\sigma_2 a}$$

$$\kappa = +1$$

$$a = \frac{1-n^2}{n^2},$$

$$h_D = \frac{n^2-1}{n^2}\frac{N^{3/2}}{12\sqrt{2}}, -I_D = \frac{(n-1)(3n+1)}{4n}F_{S^3}^{ABJM}$$

$$a = \frac{n^2-1}{n^2}$$

$$0 < n < 1/\sqrt{2}$$

$$h_D = \frac{1-n^2}{n^2}\frac{N^{3/2}}{12\sqrt{2}}, -I_D = \frac{(5n^2-2n+1)}{4n}F_{S^3}^{ABJM}$$

$$ds^2 = \frac{1}{2g^2}\left[\frac{y^2}{4}ds^2(AdS_2) + \frac{y^2}{(y-1)^2(y^2+2y-1)}dy^2 + \frac{(y-1)^2(y^2+2y-1)}{4y^2}n^2dz^2\right]$$

$$F^R = -\frac{n}{gy^2}dy\wedge dz$$

$$\frac{1}{k}\sum_{i=1}^{k} f(x_i) \le f\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right),$$

$$\frac{1}{k}\sum_{i=1}^{k} f(x_i) \ge f\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right).$$

$$\log\,\mathcal{F}^{ABJM} = \frac{1}{2}\sum_{\alpha=0}^{3} f\left(\frac{2g\mu^\alpha}{\kappa n}\right),$$

$$\log\,\mathcal{F}^{ABJM} \leq 2\log\left(1+\frac{1}{4}\sum_{\alpha=0}^{3}\frac{2g\mu^{\alpha}}{\kappa n}\right) = 2\log\left(1-\frac{n-1}{2n}\right),$$

$$\mathcal{F}^{ABJM} \leq \left(1-\frac{n-1}{2n}\right)^{2} = \mathcal{F}^{ABJM}|_{\mu_0=\mu_1=\mu_2=\mu_3},$$

$$I_D^{ABJM} = n(\mathcal{F}^{ABJM}-1)F_{S^3}^{ABJM} \leq 0.$$

$$-c\frac{h_D}{\mathcal{F}^{ABJM}} = \frac{1}{4}\sum_{\alpha=0}^{3} f\left(\frac{2g\mu^{\alpha}}{\kappa n}\right),$$

$$-c\frac{h_D}{\mathcal{F}^{ABJM}} \leq f\left(\frac{1}{4}\sum_{\alpha=0}^{3}\frac{2g\mu^{\alpha}}{\kappa n}\right) = \frac{1-n}{1+n} \leq 0$$

$$\begin{aligned}\log\,\mathcal{F}^{mABJM} &= \frac{1}{2}f\left(\frac{g\mu^{0}}{\kappa n}\right)+\frac{1}{2}\sum_{i=1}^{3} f\left(\frac{3g\mu^{i}}{\kappa n}\right) \leq \frac{1}{2}f\left(\frac{g\mu^{0}}{\kappa n}\right)+\frac{3}{2}f\left(\sum_{i=1}^{3}\frac{g\mu^{i}}{\kappa n}\right)\\ &= 2f\left(\frac{g\mu^{0}}{\kappa n}\right) = 2\log\left(1-\frac{n-1}{2n}\right)\end{aligned}$$

$$\mathcal{F}^{mABJM} \leq \left(1-\frac{n-1}{2n}\right)^{2} = \mathcal{F}^{mABJM}|_{\mu_0/3=\mu_1=\mu_2=\mu_3} \leq 1$$

$$\begin{aligned}ch_D\mathcal{F}^{mABJM}\prod_{i=1}^{3}\left(1+\frac{3g\mu_i}{\kappa n}\right)^{-1} &= \sum_{i=1}^{3} f\left(\frac{3g\mu^{i}}{\kappa n}\right)-\left(1+\frac{2}{n}\right)f\left(\frac{g\mu^{0}}{\kappa n}\right)\\ \geq 3f\left(\sum_{i=1}^{3}\frac{g\mu^{i}}{\kappa n}\right)-\left(1+\frac{2}{n}\right)f\left(\frac{g\mu^{0}}{\kappa n}\right) &= f\left(\frac{g\mu^{0}}{\kappa n}\right)\frac{2(n-1)}{n} \geq 0,\end{aligned}$$

$$\mathcal{L} = \gamma R - V(\varphi) - \frac{Z(\varphi)}{4}F^2 - \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - X(\varphi)\big(D_\mu\theta D^\mu\theta\big),$$

$$(z,\bar{z}) \mapsto \left(\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right), \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

$$p^\mu = \omega q^\mu, q^\mu = \frac{1}{2}(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2)$$

$$\omega \mapsto (cz+d)(\bar{c}\bar{z}+\bar{d})\omega, q^\mu \mapsto q'^\mu = (cz+d)^{-1}(\bar{c}\bar{z}+\bar{d})^{-1}\Lambda^\mu{}_\nu q^\nu.$$

$$q^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}}q^\mu = \begin{pmatrix} 1 & \bar{z}\\ z & z\bar{z}\end{pmatrix} = \begin{pmatrix} 1\\ z\end{pmatrix}\begin{pmatrix} 1 & \bar{z}\end{pmatrix}$$

$$h^\alpha \equiv \langle p|^\alpha = \sqrt{\omega}\begin{pmatrix} 1\\ z\end{pmatrix} = \sqrt{\omega}\langle q|^\alpha, \tilde{h}^{\dot{\alpha}} \equiv |p]^{\dot{\alpha}} = \sqrt{\omega}\begin{pmatrix} 1\\ \bar{z}\end{pmatrix} = \sqrt{\omega}\Big|\,q\Big]^{\dot{\alpha}},$$

$$\langle q|^\alpha = \begin{pmatrix} 1\\ z\end{pmatrix}, |q]^{\dot{\alpha}} = \begin{pmatrix} 1\\ \bar{z}\end{pmatrix}$$

$$\langle ij\rangle = -\sqrt{\omega_i\omega_j}z_{ij}, [ij] = \sqrt{\omega_i\omega_j}\bar{z}_{ij}$$

$$\Psi(p,\eta) = G^{+}(p) + \eta_a\Gamma_{+}^{a}(p) + \frac{1}{2!}\eta_a\eta_b\Phi^{ab}(p) + \frac{1}{3!}\epsilon^{abcd}\eta_a\eta_b\eta_c\Gamma_{d}^{-}(p) + \frac{1}{4!}\epsilon^{abcd}\eta_a\eta_b\eta_c\eta_d G^{-}(p)$$

$$\mathcal{A}_n(\{p_1,\eta^1\},\ldots\{p_n,\eta^n\}) \equiv \langle\Psi_1(p_1,\eta^1)\ldots\Psi_n(p_n,\eta^n)\rangle.$$

$$\mathcal{A}_n(\cdots,a,s,b,\cdots) \overset{p_s\to 0}{\to} \mathrm{Soft}^{\mathrm{SYM}}(a,s,b)\mathcal{A}_{n-1}(\cdots,a,b,\cdots),$$

$$\mathrm{Soft}_{\mathrm{hol}}^{\mathrm{SYM}}(a,s,b) = \frac{1}{\varepsilon^2}\mathrm{Soft}(0)_{\mathrm{hol}}^{\mathrm{SYM}}(a,s,b) + \frac{1}{\varepsilon}\mathrm{Soft}(1)_{\mathrm{hol}}^{\mathrm{SYM}}(a,s,b).$$

$$\mathrm{Soft}(k)_{\mathrm{hol}}^{\mathrm{SYM}}(a,s,b) = \frac{1}{k!}\frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle}\left[\frac{\langle sa\rangle}{\langle ba\rangle}\left(\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_b^{\dot{\alpha}}} + (\eta^s)_c\frac{\partial}{\partial(\eta^b)_c}\right) + \frac{\langle sb\rangle}{\langle ab\rangle}\left(\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_a^{\dot{\alpha}}} + (\eta^s)_c\frac{\partial}{\partial(\eta^a)_c}\right)\right]^k$$

$$\mathrm{Soft}(k)_{\text{anti-hol}}^{\mathrm{SYM}}(a,s,b) = \frac{1}{k!}\frac{[ab]}{[as][sb]}\delta^4\left(\eta^s + \frac{[as]}{[ab]}\eta^b + \frac{[sb]}{[ab]}\eta^a\right)\left[\frac{[sb]}{[ab]}h_s^{\alpha}\frac{\partial}{\partial h_a^{\alpha}} + \frac{[as]}{[ab]}h_s^{\alpha}\frac{\partial}{\partial h_b^{\alpha}}\right]^k$$

$$p_1 = zp_{12}, p_2 = (1-z)p_{12}$$

$$\mathcal{A}_n(1,2,3,\cdots,n) \overset{p_{12}^2\to 0}{\to} \sum_{l=1}^{2}\int d^4\eta^{p_{12}}\mathrm{Split}_{1-l}(1,2,p_{12})\mathcal{A}_{n-1}(p_{12},3,\cdots,n)$$

$$\mathrm{Split}_0(z;\eta^1,\eta^2,\eta^{p_{12}}) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}\prod_{a=1}^{4}\left(\eta_a^{p_{12}} - \sqrt{z}\eta_a^1 - \sqrt{1-z}\eta_a^2\right)$$

$$\mathrm{Split}_{-1}(z;\eta^1,\eta^2,\eta^{p_{12}}) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{[12]}\prod_{a=1}^{4}\left(\eta_a^1\eta_a^2 - \sqrt{1-z}\eta_a^1\eta_a^{p_{12}} + \sqrt{z}\eta_a^2\eta_a^{p_{12}}\right)$$

$$\int d\eta_a^{p_{12}}\prod_{a=1}^{4}\left(\eta_a^1\eta_a^2 - \sqrt{1-z}\eta_a^1\eta_a^{p_{12}} + \sqrt{z}\eta_a^2\eta_a^{p_{12}}\right)f(\eta_a^{p_{12}}) = \delta^{(4)}\left(\sqrt{1-z}\eta_a^1 - \sqrt{z}\eta_a^2\right)f\left(\frac{\eta_a^2}{\sqrt{1-z}}\right)$$

$$\mathcal{A}_n(1,2,3,\cdots,n) \overset{p_{12}^2\to 0}{\to} \frac{1}{\sqrt{z(1-z)}}\frac{1}{[12]}\delta^{(4)}\left(\sqrt{1-z}\eta_a^1 - \sqrt{z}\eta_a^2\right)\mathcal{A}_{n-1}\left(\left\{p_{12},\frac{\eta_a^2}{\sqrt{1-z}}\right\},3,\cdots,n\right) + \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}\mathcal{A}_{n-1}\left(\{p_{12},\sqrt{z}\eta_a^1 + \sqrt{1-z}\eta_a^2\},3,\cdots,n\right)$$

$$A_n(1^{h_1},2^{h_2},\ldots,n) \overset{1\|2}{\to} \sum_h \mathrm{Split}_{-h}^{\mathrm{SYM}}(z,1^{h_1},2^{h_2})A_{n-1}(p^h,\ldots,n),$$

$$\mathrm{Split}_{-h}\left(z;a^{h_1},b^{h_2}\right)=\mathrm{Split}_{+h}\left(z;a^{-h_1},b^{-h_2}\right)\Big|_{[ab]\leftrightarrow\langle ab\rangle}$$

$$\mathrm{Split}_{+1}^{\mathrm{SYM}}(z,a^{+1},b^{+1})=0,\mathrm{Split}_{-1}^{\mathrm{SYM}}(z,a^{+1},b^{+1})=\frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle ab\rangle}$$

$$\mathrm{Split}_{+1}^{\mathrm{SYM}}(z,a^{-1},b^{+1})=\sqrt{\frac{z^3}{1-z}}\frac{1}{\langle ab\rangle},\mathrm{Split}_{+1}^{\mathrm{SYM}}(z,a^{+1},b^{-1})=\frac{(1-z)^2}{\sqrt{z(1-z)}}\frac{1}{\langle ab\rangle}$$

$$\mathrm{Split}_{0}^{\mathrm{SYM}}\left(z,a^{+\frac{1}{2}},b^{+\frac{1}{2}}\right)=\frac{1}{\langle ab\rangle},\mathrm{Split}_{+1}^{\mathrm{SYM}}\left(z,a^{+\frac{1}{2}},b^{-\frac{1}{2}}\right)=\frac{(1-z)}{\langle ab\rangle}$$

$$\mathrm{Split}_{+1}^{\mathrm{SYM}}\left(z,a^{-\frac{1}{2}},b^{+\frac{1}{2}}\right)=\frac{z}{\langle ab\rangle},\mathrm{Split}_{1}^{\mathrm{SYM}}\left(z,a^{0},b^{0}\right)=\sqrt{z(1-z)}\frac{1}{\langle ab\rangle}.$$

$$\mathrm{Split}_{+\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{-\frac{1}{2}},b^{+1}\right)=\frac{z}{\sqrt{(1-z)}}\frac{1}{\langle ab\rangle},\mathrm{Split}_{+\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{+1},b^{-\frac{1}{2}}\right)=\frac{1-z}{\sqrt{z}}\frac{1}{\langle ab\rangle},$$

$$\mathrm{Split}_{-\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{+\frac{1}{2}},b^{+1}\right)=\frac{1}{\sqrt{(1-z)}}\frac{1}{\langle ab\rangle},\mathrm{Split}_{-\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{+1},b^{+\frac{1}{2}}\right)=\frac{1}{\sqrt{z}}\frac{1}{\langle ab\rangle},$$

$$\mathrm{Split}_{0}^{\mathrm{SYM}}(z,a^{0},b^{+1})=\sqrt{\frac{z}{1-z}}\frac{1}{\langle ab\rangle},\mathrm{Split}_{0}^{\mathrm{SYM}}(z,a^{+1},b^{0})=\sqrt{\frac{1-z}{z}}\frac{1}{\langle ab\rangle},$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{0},b^{+\frac{1}{2}}\right)=\sqrt{z}\frac{1}{\langle ab\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SYM}}\left(z,a^{+\frac{1}{2}},b^{0}\right)=\sqrt{(1-z)}\frac{1}{\langle ab\rangle}.$$

Collinear fields	Resulting index structure
Γ_+^a,Γ_+^b	Φ^{ab}
Γ_a^-,Γ_b^-	$\frac{1}{2!}\epsilon_{abcd}\Phi^{cd}$
Γ_+^a,Γ_b^-	$\delta_b^a G^\pm$
Γ_+^a,Φ^{bc}	$\epsilon^{abcd}\Gamma_d^-$
Γ_a^-,Φ^{bc}	$2!\,\delta_a^b\Gamma_+^c$
Φ^{ab},Φ^{cd}	$\epsilon^{abcd}G^\pm$

Collinear fields	Resulting index structure
Γ^a_+, Γ^b_+	Φ^{ab}
Γ^-_a, Γ^-_b	$\frac{1}{2!}\epsilon_{abcd}\Phi^{cd}$
Γ^a_+, Γ^-_b	$\delta^a_b G^\pm$
Γ^a_+, Φ^{bc}	$\epsilon^{abcd}\Gamma^-_d$
Γ^-_a, Φ^{bc}	$2!\delta^{[b}_a\Gamma^{c]}_+$
Φ^{ab}, Φ^{cd}	$\epsilon^{abcd}G^\pm$

$$
\begin{aligned}
M_n^{\text{tree}}(1,2,\dots,n) = {} & i(-1)^{n+1} A_n^{\text{tree}}(1,2,\dots,n) \\
& \times \sum_{\substack{\sigma\in S_{n/2-1} \\ \tau\in S_{n/2-2}}} f(\sigma(1),\dots,\sigma(n/2-1))\bar{f}(\tau(n/2+1),\dots,\tau(n-2)) \\
& \times A_n^{\text{tree}}(\sigma(1),\dots,\sigma(n/2-1),1,n-1,\tau(n/2+1),\dots,\tau(n-2),n) \\
& + \text{Permutations of } (2,\dots,n-2)
\end{aligned}
$$

$$
f(i_1,\dots,i_j) = s(1,i_j)\prod_{m=1}^{j-1}\left(s(1,i_m) + \sum_{k=m+1}^{j} g(i_m,i_k)\right)
$$

$$
\bar{f}(l_1,\dots,l_{j'}) = s(l_1,n-1)\prod_{m=2}^{j'}\left(s(l_m,n-1) + \sum_{k=1}^{m-1} g(l_k,l_m)\right)
$$

$$
g(i,j) = \begin{cases} s(i,j) := s_{ij} := \langle ij\rangle[ji], & i > j \\ 0, & \text{otherwise.} \end{cases}
$$

$$
M_n^{\text{tree}}(1^{h_1},2^{h_2},\dots,n) \overset{1\|2}{\to} \sum_{h=\pm} \text{Split}_{-h}^{\text{gravity}}(z,1^{h_1},2^{h_2}) \times M_{n-1}^{\text{tree}}(P^h,3,\dots,n).
$$

$$
\begin{aligned}
\text{Split}^{\text{gravity}}_{-(h+\tilde{h})}(z,1^{h_1+\tilde{h}_1},2^{h_2+\tilde{h}_2}) = -s_{12} \times \text{Split}_{-h}^{\text{gauge}}(z,1^{h_1},2^{h_2}) \\
\times \text{Split}_{-\tilde{h}}^{\text{gauge}}(z,2^{\tilde{h}_2},1^{\tilde{h}_1})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_n^{\text{tree}}(\dots,a,\varepsilon i^\pm,b,\dots) \overset{\varepsilon\to 0}{\to} \left(\frac{1}{\varepsilon^2}\mathcal{S}^{(0)}_{\text{Gauge}}(i,a,b) + \frac{1}{\varepsilon}\mathcal{S}^{(1)}_{\text{Gauge}}(i,a,b) + \mathcal{O}(1)\right) \\
\times \mathcal{A}_{n-1}^{\text{tree}}(\dots,a,b,\dots)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_n^{\text{tree}}(\dots,a,\varepsilon i^\pm,b,\dots) \overset{\varepsilon\to 0}{\to} & \left(\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}_{\text{Gravity}}(i,a,b) + \frac{1}{\varepsilon^2}\mathcal{S}^{(1)}_{\text{Gravity}}(i,a,b)\right. \\
& \left. + \frac{1}{\varepsilon}\mathcal{S}^{(2)}_{\text{Gravity}}(i,a,b) + \mathcal{O}(1)\right)\mathcal{M}_{n-1}^{\text{tree}}(\dots,a,b,\dots)
\end{aligned}
$$

$$\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}_{\text{Gravity}}(s,n,1)+\frac{1}{\varepsilon^2}\mathcal{S}^{(1)}_{\text{Gravity}}(s,n,1)+\frac{1}{\varepsilon}\mathcal{S}^{(2)}_{\text{Gravity}}(s,n,1)$$
$$=\sum_{j=1}^{n}K_{sj}^2\left(\frac{1}{\varepsilon^2}\mathcal{S}^{(0)}_{\text{Gauge}}(j,s,n)+\frac{1}{2\varepsilon}\mathcal{S}^{(1)}_{\text{Gauge}}(j,s,n)\right)^2$$

$$\Psi(p,\eta)=H^+(p)+\eta_A\psi_+^A(p)+\eta_{AB}G_+^{AB}(p)+\eta_{ABC}\chi_+^{ABC}(p)$$
$$+\eta_{ABCD}\Phi^{ABCD}(p)+\tilde{\eta}^{ABC}\chi^-_{ABC}(p)+\tilde{\eta}^{AB}G^-_{AB}(p)+\tilde{\eta}^{A}\psi^-_{A}(p)+\tilde{\eta}H^-(p),$$

$$\eta_{A_1\ldots A_n}\equiv\frac{1}{n!}\eta_{A_1}\ldots\eta_{A_2}$$
$$\tilde{\eta}^{A_1\ldots A_n}\equiv\epsilon^{A_1\ldots A_nB_1\ldots B_{8-n}}\eta_{B^1\ldots B^{8-n}}$$
$$\tilde{\eta}\equiv\prod_{A=1}^{8}\eta_A$$

$$\mathcal{M}_n(\{p_1,\eta^1\},\ldots\{p_n,\eta^n\})=\langle\Psi_1(p_1,\eta^1)\ldots\Psi_n(p_n,\eta^n)\rangle.$$

$(\mathcal{N}=4\text{SYM})\otimes(\mathcal{N}=4\text{SYM})$
$36(0\otimes 0);1(-1\otimes +1);1(+1\otimes -1);16\left(+\frac{1}{2}\otimes -\frac{1}{2}\right);16\left(-\frac{1}{2}\otimes +\frac{1}{2}\right)$
$48\left(\pm\frac{1}{2}\otimes 0\right);48\left(0\otimes \pm\frac{1}{2}\right);8\left(\pm\frac{1}{2}\otimes \mp 1\right);8\left(\pm 1\otimes \mp\frac{1}{2}\right)$
$12(\pm 1\otimes 0);16\left(+\frac{1}{2}\otimes +\frac{1}{2}\right);16\left(-\frac{1}{2}\otimes -\frac{1}{2}\right)$
$8\left(\pm\frac{1}{2}\otimes \pm 1\right);8\left(\pm 1\otimes \pm\frac{1}{2}\right)$
$2(\pm 1\otimes \pm 1)$

$\mathcal{N}=8$ Supergravity	$(\mathcal{N}=4\text{ SYM})\otimes(\mathcal{N}=4\text{ SYM})$
70 Scalars	$36(0\otimes 0);1(-1\otimes +1);1(+1\otimes -1);16(+\frac{1}{2}\otimes -\frac{1}{2});16(-\frac{1}{2}\otimes +\frac{1}{2})$
112 Gravi-photinos $(\pm)$	$48(\pm\frac{1}{2}\otimes 0);48(0\otimes \pm\frac{1}{2});8(\pm\frac{1}{2}\otimes \mp 1);8(\pm 1\otimes \mp\frac{1}{2})$
56 Graviphotons $(\pm)$	$12(\pm 1\otimes 0);16(+\frac{1}{2}\otimes +\frac{1}{2});16(-\frac{1}{2}\otimes -\frac{1}{2})$
16 Gravitinos $(\pm)$	$8(\pm\frac{1}{2}\otimes \pm 1);8(\pm 1\otimes \pm\frac{1}{2})$
2 Gravitons $(\pm)$	$2(\pm 1\otimes \pm 1)$

$H^{+} = G^{+}\tilde{G}^{+}$	$H^{-} = G^{-}\tilde{G}^{-}$
$S_{+}^{a} = \Gamma_{+}^{a}\tilde{G}^{+}$	$f_{a}^{-} = \Gamma_{a}^{-}\tilde{G}^{-}$
$S_{+}^{r} = G^{+}\tilde{\Gamma}_{+}^{r}$	$f_{r}^{-} = G^{-}\tilde{\Gamma}_{r}^{-}$
$G_{ab}^{+} = \Phi^{ab}\tilde{G}^{+}$	$G_{ab}^{-} = \Phi_{ab}\tilde{G}^{-}$
$G_{ar}^{+} = \Gamma_{+}^{a}\tilde{\Gamma}_{+}^{r}$	$G_{ar}^{-} = -\Gamma_{a}^{-}\tilde{\Gamma}_{r}^{-}$
$G_{rs}^{+} = G^{+}\tilde{\Phi}^{rs}$	$\bar{G}_{rs}^{-} = G^{-}\tilde{\Phi}_{rs}$
$\chi_{+}^{abc} = \alpha_{4}\epsilon^{abcd}\Gamma_{d}^{-}\tilde{G}^{+}$	χ_{abc}^{-} $= -\alpha_{4}\epsilon_{abcd}\Gamma_{+}^{d}\tilde{G}^{-}$
$\chi_{+}^{abr} = \Phi^{ab}\tilde{\Gamma}_{+}^{r}\chi_{+}^{ars}$ $= \Gamma_{+}^{a}\tilde{\Phi}^{rs}$	$\chi_{abr}^{-} = \Phi_{ab}\tilde{\Gamma}_{r}^{-}$ $\chi_{ars}^{-} = \Gamma_{a}^{-}\tilde{\Phi}_{rs}$ $\chi_{rst}^{-} = -\tilde{\alpha}_{4}\epsilon_{rstu}G^{-}\tilde{\Gamma}_{+}^{u}$
$\Phi^{abcd} = \alpha_{4}\epsilon^{abcd}G^{-}\tilde{G}^{+}$	Φ_{abcd} $= \alpha_{4}\epsilon_{abcd}G^{+}\tilde{G}^{-}$
$\Phi^{abcr} = \alpha_{4}\epsilon^{abcd}\Gamma_{d}^{-}\tilde{\Gamma}_{+}^{r}$	$\Phi_{abcr} = \alpha_{4}\epsilon_{abcd}\Gamma_{+}^{d}\tilde{\Gamma}_{r}^{-}$
$\Phi^{abrs} = \Phi^{ab}\tilde{\Phi}^{rs}$	$\Phi_{abrs} = \Phi_{ab}\Phi_{rs}$
$\Phi^{\mathrm{arst}} = \tilde{\alpha}_{4}\epsilon^{rstu}\Gamma_{+}^{a}\tilde{\Gamma}_{u}^{-}$	$\Phi_{\mathrm{arst}} = \tilde{\alpha}_{4}\epsilon_{rstu}\Gamma_{a}^{-}\tilde{\Gamma}_{+}^{u}$
$\Phi^{rstu} = \tilde{\alpha}_{4}\epsilon^{rstu}G^{+}\tilde{G}^{-}$	$\Phi_{\mathrm{rstu}} = \tilde{\alpha}_{4}\epsilon_{\mathrm{rstu}}G^{-}\tilde{G}^{+}$

$H^+ = G^+\tilde{G}^+$	$H^- = G^-\tilde{G}^-$
$S^a_+ = \Gamma^a_+\tilde{G}^+$	$f^-_a = \Gamma^-_a\tilde{G}^-$
$S^r_+ = G^+\tilde{\Gamma}^r_+$	$f^-_r = G^-\tilde{\Gamma}^-_r$
$G^+_{ab} = \Phi^{ab}\tilde{G}^+$	$G^-_{ab} = \Phi_{ab}\tilde{G}^-$
$G^+_{ar} = \Gamma^a_+\tilde{\Gamma}^r_+$	$G^-_{ar} = -\Gamma^-_a\tilde{\Gamma}^-_r$
$G^+_{rs} = G^+\tilde{\Phi}^{rs}$	$G^-_{rs} = G^-\tilde{\Phi}_{rs}$
$\chi^{abc}_+ = \alpha_4\epsilon^{abcd}\Gamma^-_d\tilde{G}^+$	$\chi^-_{abc} = -\alpha_4\epsilon_{abcd}\Gamma^d_+\tilde{G}^-$
$\chi^{abr}_+ = \Phi^{ab}\tilde{\Gamma}^r_+$	$\chi^-_{abr} = \Phi_{ab}\tilde{\Gamma}^-_r$
$\chi^{ars}_+ = \Gamma^a_+\tilde{\Phi}^{rs}$	$\chi^-_{ars} = \Gamma^-_a\tilde{\Phi}_{rs}$
$\chi^{rst}_+ = \tilde{\alpha}_4\epsilon^{rstu}G^+\tilde{\Gamma}^-_u$	$\chi^-_{rst} = -\tilde{\alpha}_4\epsilon_{rstu}G^-\tilde{\Gamma}^u_+$
$\Phi^{abcd} = \alpha_4\epsilon^{abcd}G^-\tilde{G}^+$	$\Phi_{abcd} = \alpha_4\epsilon_{abcd}G^+\tilde{G}^-$
$\Phi^{abcr} = \alpha_4\epsilon^{abcd}\Gamma^-_d\tilde{\Gamma}^r_+$	$\Phi_{abcr} = \alpha_4\epsilon_{abcd}\Gamma^d_+\tilde{\Gamma}^-_r$
$\Phi^{abrs} = \Phi^{ab}\tilde{\Phi}^{rs}$	$\Phi_{abrs} = \Phi_{ab}\tilde{\Phi}_{rs}$
$\Phi^{arst} = \tilde{\alpha}_4\epsilon^{rstu}\Gamma^a_+\tilde{\Gamma}^-_u$	$\Phi_{arst} = \tilde{\alpha}_4\epsilon_{rstu}\Gamma^-_a\tilde{\Gamma}^u_+$
$\Phi^{rstu} = \tilde{\alpha}_4\epsilon^{rstu}G^+\tilde{G}^-$	$\Phi_{rstu} = \tilde{\alpha}_4\epsilon_{rstu}G^-\tilde{G}^+$

$$\Phi_{ABCD} = \frac{1}{4!}\alpha_8\epsilon_{ABCDEFGH}\Phi^{EFGH}$$
$$\Phi_{ab} = \frac{1}{2!}\alpha_4\epsilon_{abcd}\Phi^{cd}$$
$$\tilde{\Phi}_{rs} = \frac{1}{2!}\tilde{\alpha}_4\epsilon_{rstu}\Phi^{tu}$$

$$\alpha_4\tilde{\alpha}_4 = \alpha_8.$$

$$M_n\left(1^{h_1}, 2^{h_2}, \dots, n\right) \overset{1\|2}{\rightarrow} \sum_h \text{Split}^{\text{SG}}_{-h}\left(z, 1^{h_1}, 2^{h_2}\right) M_{n-1}\left(p^h, \dots, n\right),$$

$$\text{Split}^{\text{SG}}_{-(h+\tilde{h})}\left(z, 1^{h_1+\tilde{h}_1}, 2^{h_2+\tilde{h}_2}\right) = -s_{12}\times \text{Split}^{\text{SYM}}_{-h}\left(z, 1^{h_1}, 2^{h_2}\right) \times \text{Split}^{\text{SYM}}_{-\tilde{h}}\left(z, 2^{\tilde{h}_2}, 1^{\tilde{h}_1}\right)$$

$$\text{Split}_-\left(z; a^{h_1}, b^{h_2}\right) = \text{Split}_+\left(z; a^{-h_1}, b^{-h_2}\right)\Big|_{[ab]\leftrightarrow\langle ab\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-(h+\tilde{h})}(z,1^{\pm 2},2^{\pm 2}) = -s_{12}\times \mathrm{Split}^{\mathrm{SYM}}_{-h}(z,1^{\pm 1},2^{\pm 1}) \\ \times \mathrm{Split}^{\mathrm{SM}}_{-\tilde{h}}(z,2^{\pm 1},1^{\pm 1})$$

$$\mathrm{Split}^{\mathrm{SG}}_{+2}(z,a^{+2},b^{+2}) = 0 = \mathrm{Split}^{\mathrm{SG}}_{-2}(z,a^{-2},b^{-2})$$

$$\mathrm{Split}^{\mathrm{SG}}_{-2}(z,a^{+2},b^{+2}) = -\frac{1}{z(1-z)}\frac{[ab]}{\langle ab\rangle}, \mathrm{Split}^{\mathrm{SG}}_{+2}(z,a^{-2},b^{-2}) = -\frac{1}{z(1-z)}\frac{\langle ab\rangle}{[ab]}$$

$$\mathrm{Split}^{\mathrm{SG}}_{+2}(z,a^{-2},b^{+2}) = -\frac{z^3}{(1-z)}\frac{[ab]}{\langle ab\rangle}, \mathrm{Split}^{\mathrm{SG}}_{-2}(z,a^{-2},b^{+2}) = -\frac{(1-z)^3}{z}\frac{\langle ab\rangle}{[ab]}.$$

$$p_1 = zp, p_2 = (1-z)p.$$

$$z = \frac{\omega_1}{\omega_p}, (1-z) = \frac{\omega_2}{\omega_p}$$

$M_n(1^{+2},2^{+2},\cdots,n)$	$\frac{\omega_p^2}{\omega_1 2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}(p^{+2},\dots,n)$
$M_n(1^{-2},2^{-2},\cdots,n)$	$\frac{\omega_p^2}{\omega_1\omega_2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}(p^{-2},\dots,n)$
$M_n(1^{+2},2^{-2},\dots,n)$	$\frac{\omega_1^3}{\omega_p^2\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}(p^{-2},3,\dots,n) + \frac{\omega_2^3}{\omega_p^2\omega_1}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}(p^{+2},3,\dots,n)$

$M_n\left(1^{+2},2^{+2},\cdots,n\right)$	$\frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{+2},\dots,n\right)$
$M_n\left(1^{-2},2^{-2},\cdots,n\right)$	$\frac{\omega_p^2}{\omega_1\omega_2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{-2},\dots,n\right)$
$M_n\left(1^{+2},2^{-2},\dots,n\right)$	$\frac{\omega_1^3}{\omega_p^2\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},3,\dots,n\right) + \frac{\omega_2^3}{\omega_p^2\omega_1}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},3,\dots,n\right)$

$$\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+1}\right) = -\frac{1}{\sqrt{z(1-z)}}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{1+\frac{1}{2}}\right) = -\frac{1}{\sqrt{z(1-z)}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right) = -\sqrt{\frac{z^5}{(1-z)}\frac{\langle 12\rangle}{[12]}}, \mathrm{Split}^{\mathrm{SG}}_{+2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right) = -\sqrt{\frac{(1-z)^5}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\begin{cases}\left(a;\frac{3}{2}\right) = \left(a;\frac{1}{2}\right)\otimes 1 \\ \left(r;\frac{3}{2}\right) = 1\otimes\left(r;\frac{1}{2}\right)\end{cases}$$

$M_n\left(1^{A;+\frac{3}{2}},2^{B;+\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}(p^{AB;+1},\cdots,n)$

$M_n\left(1^{A;+\frac{3}{2}}, 2_B^{-\frac{3}{2}}, \cdots, n\right)$	$\delta_B^A \frac{\omega_1^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2} \frac{\bar{z}_{12}}{z_{12}} M_{n-1}(p^{-2}, \cdots, n) + \delta_B^A \frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2} \frac{z_{12}}{\bar{z}_{12}} M_{n-1}(p^{+2}, \cdots, n)$

$M_n\left(1^{A;+\frac{3}{2}}, 2^{B;+\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_p}{\sqrt{\omega_1\omega_2}} \frac{\bar{z}_{12}}{z_{12}} M_{n-1}\left(p^{AB;+1}, \cdots, n\right)$
$M_n\left(1^{A;+\frac{3}{2}}, 2_B^{-\frac{3}{2}}, \cdots, n\right)$	$\delta_B^A \frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2} \frac{\bar{z}_{12}}{z_{12}} M_{n-1}\left(p^{-2}, \cdots, n\right) + \delta_B^A \frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2} \frac{z_{12}}{\bar{z}_{12}} M_{n-1}\left(p^{+2}, \cdots, n\right)$

$$\begin{cases} (ab;1) = (ab;0) \otimes 1 \\ (ar;1) = \left(a, \frac{1}{2}\right) \otimes \left(r; \frac{1}{2}\right) \\ (rs;1) = 1 \otimes (rs;0) \end{cases}$$

$M_n(1^{AB;+1}, 2^{CD;+1}, \cdots)$	$\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{ABCD;0}, \cdots)$
$M_n(1^{AB;+1}, 2_{CD}^{-1}, \cdots)$	$-\delta_{CD}^{AB}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-2}, \cdots) + \frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{+2}, \cdots)\right]$

$M_n\left(1^{AB;+1}, 2^{CD;+1}, \cdots\right)$	$\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{ABCD;0}, \cdots\right)$
$M_n\left(1^{AB;+1}, 2_{CD}^{-1}, \cdots\right)$	$-\delta_{CD}^{AB}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots\right) + \frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots\right)\right]$

$$\begin{cases} \left(abr; \frac{1}{2}\right) = (ab;0) \otimes \left(r; \frac{1}{2}\right) \\ \left(ars; \frac{1}{2}\right) = \left(a; \frac{1}{2}\right) \otimes (rs;0) \end{cases}$$

$$\begin{cases} \left(rst; \frac{1}{2}\right) = -\epsilon^{rstu}\left(1 \otimes \left(u; -\frac{1}{2}\right)\right) \\ \left(abc; \frac{1}{2}\right) = -\epsilon^{abcd}\left(\left(d; -\frac{1}{2}\right) \otimes 1\right) \end{cases} \text{(sum over } u, d)$$

$M_n\left(1^{\mathrm{ars};+\frac{1}{2}}, 2^{\mathrm{btu};+\frac{1}{2}}, \cdots\right)$	$\epsilon^{\mathrm{rstu}}\epsilon^{abcd} \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{cd}^{-1}, \cdots\right)$
$M_n\left(1^{\mathrm{ars};+\frac{1}{2}}, 2^{\mathrm{bct};+\frac{1}{2}}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu} \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{du}^{-1}, \cdots\right)$
$M_n\left(1^{rst;+\frac{1}{2}}, 2^{abc;+\frac{1}{2}}, \cdots\right)$	$\epsilon^{rstu}\epsilon^{abcd} \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{ud}^{-1}, \cdots\right)$

$M_n\left(1^{ars;+\frac{1}{2}}, 2_{btu}^{-\frac{1}{2}}, \cdots\right)$	$\epsilon_{tuvw}\epsilon^{rsvw}\delta_b^a\left[\frac{\omega_1^{\frac{3}{2}}\omega_2^{\frac{1}{2}}}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}(p^{+2},\cdots) + \frac{\omega_2^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}(p^{-2},\cdots)\right]$

$M_n\left(1^{ars;+\frac{1}{2}}, 2^{btu;+\frac{1}{2}}, \cdots\right)$	$\epsilon^{rstu}\epsilon^{abcd}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{cd}^{-1}, \cdots\right)$
$M_n\left(1^{ars;+\frac{1}{2}}, 2^{bct;+\frac{1}{2}}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{du}^{-1}, \cdots\right)$
$M_n\left(1^{rst;+\frac{1}{2}}, 2^{abc;+\frac{1}{2}}, \cdots\right)$	$\epsilon^{rstu}\epsilon^{abcd}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{ud}^{-1}, \cdots\right)$
$M_n\left(1^{ars;+\frac{1}{2}}, 2_{btu}^{-\frac{1}{2}}, \cdots\right)$	$\epsilon_{tuvw}\epsilon^{rsvw}\delta_b^a\left[\frac{\omega_1^{\frac{3}{2}}\omega_2^{\frac{1}{2}}}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2}, \cdots\right) + \frac{\omega_2^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2}, \cdots\right)\right]$

$$(abrs;0) = (ab;0) \otimes (rs;0)$$

$$\begin{cases}(abcd;0) = -\epsilon^{abcd}(-1 \otimes 1)\\(rstu;0) = -\epsilon^{rstu}(1 \otimes -1)\end{cases}$$

$$\begin{cases}(\text{ abcr };0) = -\epsilon^{abcd}\left(d;-\frac{1}{2}\right) \otimes \left(r;\frac{1}{2}\right)\\(\text{ arst };0) = -\epsilon^{\text{rstu}}\left(a;\frac{1}{2}\right) \otimes \left(u;-\frac{1}{2}\right)\end{cases}$$

$M_n\left(1^{abrs;0}, 2^{cdtu;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_1\omega_2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{+2},\cdots) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-2},\cdots)\right]$
$M_n\left(1^{abcd;0}, 2^{rstu;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{+2},\cdots)\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-2},\cdots)\right.$
$M_n\left(1^{abcu;0}, 2^{drst;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{+2},\cdots) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-2},\cdots)\right.$
$M_n\left(1^{\text{arst };0}, 2^{\text{bcdu; }0}, \cdots\right)$	$\epsilon^{\text{rstu }}\epsilon^{abcd}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{+2},\cdots) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-2},\cdots)\right.$

$M_n\left(1^{abrs;0}, 2^{cdtu;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_1\omega_2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots\right)\right]$
$M_n\left(1^{abcd;0}, 2^{rstu;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots\right) \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots\right)\right]$
$M_n\left(1^{abcu;0}, 2^{drst;0}, \cdots\right)$	$\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots\right)\right]$
$M_n\left(1^{arst;0}, 2^{bcdu;0}, \cdots\right)$	$\epsilon^{rstu}\epsilon^{abcd}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots\right)\right]$

$M_n\left(1^{+2}, 2^{r;+\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{r;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_r^{-\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{z_{12}}{z_{12}} \times M_{n-1}\left(p_r^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2^{a;+\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_a^{-\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_a^{-\frac{3}{2}}, \cdots, n\right)$

$M_n\left(1^{+2}, 2^{r;+\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{r;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_r^{-\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_r^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2^{a;+\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_a^{-\frac{3}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_a^{-\frac{3}{2}}, \cdots, n\right)$

$$M_n(1^{+2}, 2_{AB}^{-1}, \cdots, n) = \frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p_{AB}^{-1}, \cdots, n)$$

$M_n\left(1^{+2}, 2^{abr;+\frac{1}{2}}, \cdots, n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{abr;+\frac{1}{2}}, \cdots, n\right)$

$M_n\left(1^{+2}, 2^{\text{ars};+\frac{1}{2}}, \cdots, n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{z_{12}}{z_{12}} \times M_{n-1}\left(p^{\text{ars};+\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2^{abc;+\frac{1}{2}}, \cdots, n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{z_{12}}{z_{12}} \times M_{n-1}\left(p^{abc;+\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2^{rst;+\frac{1}{2}}, \cdots, n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{z_{12}}{z_{12}} \times M_{n-1}\left(p^{rst;+\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_{abr}^{-\frac{1}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{abr}^{-\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_{\text{ars}}^{-\frac{1}{2}}, \cdots, n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{\text{ars}}^{-\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_{abc}^{-\frac{1}{2}}, \cdots, n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{abc}^{-\frac{1}{2}}, \cdots, n\right)$
$M_n\left(1^{+2}, 2_{rst}^{-\frac{1}{2}}, \cdots, n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{rst}^{-\frac{1}{2}}, \cdots, n\right)$

$M_n\left(1^{+2},2^{abr;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{ars;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ars;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{abc;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abc;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{rst;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rst;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{-\frac{1}{2}}_{abr},\cdots,n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{abr},\cdots,n\right)$
$M_n\left(1^{+2},2^{-\frac{1}{2}}_{ars},\cdots,n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{ars},\cdots,n\right)$
$M_n\left(1^{+2},2^{-\frac{1}{2}}_{abc},\cdots,n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{abc},\cdots,n\right)$
$M_n\left(1^{+2},2^{-\frac{1}{2}}_{rst},\cdots,n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{rst},\cdots,n\right)$

$$\boxed{M_n\left(1^{+2},2^{-1}_{AB},\cdots,n\right)=\frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{AB},\cdots,n\right)}$$

$$\boxed{M_n\left(1^{+2},2^{ABCD;0},\cdots,n\right)=\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ABCD;0},\cdots,n\right)}$$

$$\boxed{M_n\left(1^{A;+\frac{3}{2}},2^{-1}_{BC},\cdots,n\right)=\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}2!\delta^A_{[B}\times M_{n-1}\left(p^{-\frac{3}{2}}_{C]},\cdots,n\right)}$$

$M_n\left(1^{A;+\frac{3}{2}},2^{BCD;+\frac{1}{2}},\cdots\right)$	$\sqrt{\frac{\omega_2}{\omega_1}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ABCD;0},\cdots\right)$
$M_n\left(1^{A;+\frac{3}{2}},2^{-\frac{1}{2}}_{BCD},\cdots\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times 3\delta^A_{(B}M_{n-1}\left(p^{-1}_{CD)},\cdots\right)$

$$M_n(1^{+2},2^{ABCD;0},\cdots,n)=\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{ABCD;0},\cdots,n)$$

$$M_n\left(1^{A;+\frac{3}{2}}, 2_{BC}^{-1}, \cdots, n\right) = \frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} 2!\,\delta_{[B}^{A} \times M_{n-1}\left(p_{C]}^{-\frac{3}{2}}, \cdots, n\right)$$

$$p_{[A_1 \ldots A_n]} := \frac{1}{n!}\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) p_{A_{\sigma(1)} \ldots A_{\sigma(n)}}.$$

$M_n\left(1^{A;+\frac{3}{2}}, 2^{BCD;+\frac{1}{2}}, \cdots\right)$	$\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}(p^{ABCD;0}, \cdots)$
$M_n\left(1^{A;+\frac{3}{2}}, 2_{BCD}^{-\frac{1}{2}}, \cdots\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times 3\delta_{(B}^{A} M_{n-1}(p_{CD)}^{-1}, \cdots)$

$$p_{(A_1 \ldots A_n)} := \frac{1}{n!}\sum_{\sigma \in S_n} p_{A_{\sigma(1)} \ldots A_{\sigma(n)}}.$$

$$M_n\left(1^{A;+\frac{3}{2}}, 2^{BCDE;0}, \cdots, n\right) = -\frac{1}{3!}\epsilon^{ABCDEFGH}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{FGH}^{-\frac{1}{2}}, \cdots, n\right)$$

$M_n\left(1^{A;+\frac{3}{2}}, 2^{BCDE;0}, \cdots, n\right) = -\frac{1}{3!}\epsilon^{ABCDEFGH}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{FGH}^{-\frac{1}{2}}, \cdots, n\right)$

$M_n\left(1^{ab;+1}, 2^{cdr;+\frac{1}{2}}, \cdots, n\right)$	$\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}(p^{ABCD;0}, \cdots)$
$M_n\left(1^{ab;+1}, 2_{cdr;-\frac{1}{2}}, \cdots, n\right)$	$-\delta_{cd}^{ab}\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_r^{-\frac{3}{2}}, \cdots, n\right)$

$M_n\left(1^{ab;+1}, 2^{cdr;+\frac{1}{2}}, \cdots, n\right)$	$\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{ABCD;0}, \cdots\right)$
$M_n\left(1^{ab;+1}, 2_{cdr;-\frac{1}{2}}, \cdots, n\right)$	$-\delta_{cd}^{ab}\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_r^{-\frac{3}{2}}, \cdots, n\right)$

$M_n(1^{ab;+1}, 2^{cdrs;0}, \cdots, n)$	$\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p_{tu}^{-1}, \cdots, n)$

$M_n(1^{rs;+1}, 2^{abtu;0}, \cdots, n)$	$\epsilon^{rstu}\,\epsilon^{abcd}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-1}_{cd}, \cdots, n)$
$M_n(1^{ar;\,+1}, 2^{bcst\,;0}, \cdots, n)$	$\epsilon^{abcd}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-1}_{du}, \cdots, n)$
$M_n(1^{ab;+1}, 2^{cdef;0}, \cdots, n)$	$-\epsilon^{cdef}\,\epsilon^{abgh}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-1}_{gh}, \cdots, n)$
$M_n(1^{rs;\,+1}, 2^{cdef\,;0}, \cdots, n)$	$-\epsilon^{cdef}\,\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-1}_{tu}, \cdots, n)$
$M_n(1^{ar;+1}, 2^{bcds;0}, \cdots, n)$	$-\epsilon^{abcd}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p^{-1}_{tu}, \cdots, n)$
$M_n(1^{-1}_{ab}, 2^{cdef\,;0}, \cdots, n)$	$-\epsilon^{cdef}\,\epsilon_{abgh}\,\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{gh;+1}, \cdots, n)$
$M_n(1^{-1}_{ar}, 2^{bcds;0}, \cdots, n)$	$-\epsilon^{bcde}\,\delta^s_r\epsilon_{aefg}\,\frac{\omega_1}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}(p^{fg;+1}, \cdots, n)$
$M_n(1^{-1}_{ar}, 2^{bcst;0}, \cdots, n)$	$-\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}\,4!\,\delta^{[b}_a\,\delta^s_r M_{n-1}(p^{tc];+1}, \cdots, n)$

$M_n\left(1^{ab;+1}, 2^{cdrs;0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{tu}, \cdots, n\right)$
$M_n\left(1^{rs;+1}, 2^{abtu;0}, \cdots, n\right)$	$\epsilon^{rstu}\epsilon^{abcd}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{cd}, \cdots, n\right)$
$M_n\left(1^{ar;+1}, 2^{bcst;0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{du}, \cdots, n\right)$
$M_n\left(1^{ab;+1}, 2^{cdef;0}, \cdots, n\right)$	$-\epsilon^{cdef}\epsilon^{abgh}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{gh}, \cdots, n\right)$
$M_n\left(1^{rs;+1}, 2^{cdef;0}, \cdots, n\right)$	$-\epsilon^{cdef}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{tu}, \cdots, n\right)$
$M_n\left(1^{ar;+1}, 2^{bcds;0}, \cdots, n\right)$	$-\epsilon^{abcd}\epsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-1}_{tu}, \cdots, n\right)$
$M_n\left(1^{-1}_{ab}, 2^{cdef;0}, \cdots, n\right)$	$-\epsilon^{cdef}\epsilon_{abgh}\,\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{gh;+1}, \cdots, n\right)$
$M_n\left(1^{-1}_{ar}, 2^{bcds;0}, \cdots, n\right)$	$-\epsilon^{bcde}\delta^s_r\epsilon_{aefg}\,\frac{\omega_1}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{fg;+1}, \cdots, n\right)$
$M_n\left(1^{-1}_{ar}, 2^{bcst;0}, \cdots, n\right)$	$-\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}4!\delta^{[b}_a\delta^s_r\; M_{n-1}\left(p^{tc];+1}, \cdots, n\right)$

$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cdst;0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_u^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cstu;0}, \cdots, n\right)$	$-\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{\mathrm{ars};+\frac{1}{2}}, 2^{\mathrm{bctu};\,0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{bctu;0}, \cdots, n\right)$	$-2!\,\delta_{rs}^{tu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\delta_a^{[b} \times M_{n-1}\left(p^{c];+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{\mathrm{ars}}^{-\frac{1}{2}}, 2^{\mathrm{btuv}\ 0}, \cdots, n\right)$	$-\delta_a^b\epsilon^{tuvw}\epsilon_{wrsx}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{x;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{\mathrm{avwx}\ ;0}, \cdots, n\right)$	$-\epsilon_{\mathrm{rstu}}\,\epsilon^{\mathrm{vwxu}}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{uvwx;0}, \cdots, n\right)$	$-\epsilon_{rsty}\epsilon^{uvwx}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)$

$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cdst;0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_u^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cstu;0}, \cdots, n\right)$	$-\epsilon^{abcd}\epsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{ars;+\frac{1}{2}}, 2^{bctu;0}, \cdots, n\right)$	$\epsilon^{abcd}\epsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{bctu;0}, \cdots, n\right)$	$-2!\delta_{rs}^{tu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\delta_a^{[b} \times M_{n-1}\left(p^{c];+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{btuv\ 0}, \cdots, n\right)$	$-\delta_a^b\epsilon^{tuvw}\epsilon_{wrsx}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{x;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{avwx;0}, \cdots, n\right)$	$-\epsilon_{rstu}\epsilon^{vwxu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{uvwx;0}, \cdots, n\right)$	$-\epsilon_{rsty}\epsilon^{uvwx}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)$

$$\mathcal{M}_{n+1}(\Psi_s, \Psi_1, \dots, \Psi_n) \overset{\varepsilon\to 0}{=} \textstyle\sum_{k=0}^{2} \frac{1}{\varepsilon^{3-k}} \mathcal{S}^{(k)} \mathcal{M}_n(\Psi_1, \dots, \Psi_n)$$

$$\mathcal{M}_{n+1}(\Psi_s, \Psi_1, \dots, \Psi_n) \overset{\varepsilon\to 0}{=} \textstyle\sum_{k=0}^{2} \frac{1}{\varepsilon^{3-k}} \overline{\mathcal{S}}^{(k)} \mathcal{M}_n(\Psi_1, \dots, \Psi_n)$$

$$\mathrm{Soft}_{\mathrm{leading}}^{\mathrm{SYM}}(a, s, b) = \frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle} + \frac{[ab]}{[as][sb]}\delta^4(\eta^s)$$

$$\begin{aligned}\frac{1}{\varepsilon^3}\mathrm{Soft}(0)^{\mathrm{SG}} &+ \frac{1}{\varepsilon^2}\mathrm{Soft}(1)^{\mathrm{SG}} + \frac{1}{\varepsilon}\mathrm{Soft}(2)^{\mathrm{SG}} \\ &= \sum_{i=1}^{n} \varepsilon\langle si\rangle[is]\left(\frac{1}{\varepsilon^2}\mathrm{Soft}(0)^{\mathrm{SYM}}(i, s, a) + \frac{1}{2\varepsilon}\mathrm{Soft}(1)^{\mathrm{SYM}}(i, s, a)\right)^2\end{aligned}$$

$$\mathrm{Soft}(0)^{\mathrm{SG}} = \sum_{i=1}^{n} \langle si\rangle[is]\left[\mathrm{Soft}(0)^{\mathrm{SYM}}(i, s, a)\right]^2$$

$$\mathrm{Soft}(1)^{\mathrm{SG}} = \sum_{i=1}^{n} \langle si\rangle[is]\left[\mathrm{Soft}(0)^{\mathrm{SYM}}(i, s, a) \times \mathrm{Soft}(1)^{\mathrm{SYM}}(i, s, a)\right]$$

$$\mathrm{Soft}(2)^{\mathrm{SG}} = \frac{1}{4}\sum_{i=1}^{n} \langle si\rangle[is]\left[\mathrm{Soft}(1)^{\mathrm{SYM}}(i, s, a)\right]^2$$

$$\mathrm{Soft}_{\mathrm{leading}}^{\mathrm{SG}} = \sum_{i=1}^{n} \langle si\rangle[is]\left(\left[\mathrm{Soft}(0)_{\mathrm{hol}}^{\mathrm{SYM}}(i, s, a)\right]^2 + \left[\mathrm{Soft}(0)_{\text{anti-hol}}^{\mathrm{SYM}}(i, s, a)\right]^2\right)$$

$$\begin{aligned}\mathrm{Soft}_{\mathrm{subleading}}^{\mathrm{SG}} &= \sum_{i=1}^{n} \langle si\rangle[is]\left[\mathrm{Soft}(0)_{\mathrm{hol}}^{\mathrm{SYM}}(i, s, a) \times \mathrm{Soft}(1)_{\mathrm{hol}}^{\mathrm{SYM}}(i, s, a)\right. \\ &\quad \left. + \mathrm{Soft}(0)_{\text{anti-hol}}^{\mathrm{SYM}}(i, s, a) \times \mathrm{Soft}(1)_{\text{anti-hol}}^{\mathrm{SYM}}(i, s, a)\right]\end{aligned}$$

$$\left(\delta^4(\eta_a)\right)^2 = \delta^8(\eta_A)$$

$$\mathrm{Soft}^{\mathrm{SG}}_{\mathrm{leading}}(a,s,b) = \sum_{i=1}^{n}\left(\frac{[si]}{\langle si\rangle}\frac{\langle ai\rangle^2}{\langle as\rangle^2} + \frac{\langle si\rangle}{[si]}\frac{[ai]^2}{[as]^2}\delta^8(\eta_A)\right)$$

$$= \sum_{i=1}^{n}\langle si\rangle[is]\left[\frac{\langle ia\rangle}{\langle is\rangle\langle sa\rangle}\left\{\frac{1}{\langle sa\rangle}\left(\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_a^{\dot{\alpha}}} + \eta_A^s\frac{\partial}{\partial\eta_A^a}\right) + \frac{1}{\langle is\rangle}\left(\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_i^{\dot{\alpha}}} + \eta_s^A\frac{\partial}{\partial\eta_i^A}\right)\right\}\right.$$
$$\left.+\frac{[ia]}{[is][sa]}\delta^8\left(\eta^s + \frac{[as]}{[ab]}\eta^b + \frac{[sb]}{[ab]}\eta^a\right)\left(\frac{1}{[is]}h_s^{\alpha}\frac{\partial}{\partial h_i^{\alpha}} + \frac{1}{[sa]}h_s^{\alpha}\frac{\partial}{\partial h_a^{\alpha}}\right)\right]$$
$$= \sum_{i=1}^{n}\langle si\rangle[is]\left[\frac{\langle ia\rangle}{\langle is\rangle^2\langle sa\rangle}\left(\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_i^{\dot{\alpha}}} + \eta_A^s\frac{\partial}{\partial\eta_A^i}\right) + \frac{[ia]}{[is]^2[sa]}\delta^8\left(\eta^s + \frac{[as]}{[ab]}\eta^b + \frac{[sb]}{[ab]}\eta^a\right)\left(h_s^{\alpha}\frac{\partial}{\partial h_i^{\alpha}}\right)\right]$$

$$\sum_i \langle si\rangle[ia] = \sum_i [si]\langle ia\rangle = 0.$$

$$\mathrm{Soft}^{\mathrm{SG}}_{\mathrm{subleading}}(a,s,b) = \sum_{i=1}^{n}\frac{[is]\langle ia\rangle}{\langle si\rangle\langle sa\rangle}\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_i^{\dot{\alpha}}} + \frac{\langle si\rangle[ia]}{[is][sa]}\delta^8(\eta^s)h_s^{\alpha}\frac{\partial}{\partial h_i^{\alpha}}$$

$$\mathrm{Soft}^{\mathrm{SG}}_{\mathrm{subleading}}(a,s,b) = \sum_{i=1}^{n}\frac{[is]\langle ir\rangle}{\langle si\rangle\langle sr\rangle}\tilde{h}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{h}_i^{\dot{\alpha}}} + \frac{\langle si\rangle[ir]}{[is][sr]}\delta^8(\eta^s)h_s^{\alpha}\frac{\partial}{\partial h_i^{\alpha}}$$

$$\mathcal{M}_{n+1}(\Psi_s,\Psi_1,\dots,\Psi_n) = \left(\frac{1}{\varepsilon^3}\mathcal{S}^{(0)} + \frac{1}{\varepsilon^2}\mathcal{S}^{(1)} + \frac{1}{\varepsilon}\mathcal{S}^{(2)}\right)\mathcal{M}_n(\Psi_1,\dots,\Psi_n) + O(\varepsilon^0)$$

$$\mathcal{S}^{(0)} = \sum_{i=1}^{n}\frac{[si]\langle ri\rangle^2}{\langle si\rangle\langle rs\rangle^2} = S^{(0)}$$

$$\mathcal{S}^{(1)} = \sum_{i=1}^{n}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\left(\tilde{\lambda}_{s\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_{i\dot{\alpha}}} + \eta_{sA}\frac{\partial}{\partial\eta_{iA}}\right) = S^{(1)} + \eta_{sA}\mathcal{S}^{A(1)}.$$

$$\mathcal{S}^{A(1)} = \sum_{i=1}^{n}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\frac{\partial}{\partial\eta_{iA}}$$

$$\mathcal{S}^{(2)} = S^{(2)} + \eta_{sA}\mathcal{S}^{A(2)} + \frac{1}{2}\eta_{sA}\eta_{sB}\mathcal{S}^{AB(2)}$$

$$S^{(2)} = \frac{1}{2}\sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\tilde{\lambda}_{s\dot{\alpha}}\tilde{\lambda}_{s\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_{i\dot{\alpha}}\partial\tilde{\lambda}_{i\dot{\beta}}}$$
$$\mathcal{S}^{A(2)} = \sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\tilde{\lambda}_{s\dot{\alpha}}\frac{\partial^2}{\partial\tilde{\lambda}_{i\dot{\alpha}}\partial\eta_{aA}}$$
$$\mathcal{S}^{AB(2)} = \sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\frac{\partial^2}{\partial\eta_{aB}\partial\eta_{aA}}$$

$$\mathcal{M}_{n+1}(\Psi_s,\Psi_1,\dots,\Psi_n)=\mathcal{M}_{n+1}(H_{s+},\Psi_1,\dots,\Psi_n)+\eta_{sA}\mathcal{M}_{n+1}(S^A_{s+},\Psi_1,\dots,\Psi_n)+\frac{1}{2}\eta_{sA}\eta_{sB}\mathcal{M}_{n+1}(G^{AB}_{s+},\Psi_1,\dots,\Psi_n)+\cdots$$

Superamplitude expansion on $\varepsilon \to 0$
$\mathcal{M}_{n+1}(H_{s+},\dots)=\left(\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}+\frac{1}{\varepsilon^2}\mathcal{S}^{(1)}+\frac{1}{\varepsilon}\mathcal{S}^{(2)}\right)\mathcal{M}_n+O(\varepsilon^0)$
$\mathcal{M}_{n+1}(S^A_{s+},\dots)=\left(\frac{1}{\varepsilon^2}\mathcal{S}^{A(1)}+\frac{1}{\varepsilon}\mathcal{S}^{A(2)}\right)\mathcal{M}_n+O(\varepsilon^0)$
$\mathcal{M}_{n+1}(G^{AB}_{s+}\dots)=\frac{1}{\varepsilon}\mathcal{S}^{AB(2)}\mathcal{M}_n+\mathcal{O}(\varepsilon^0)$
$\mathcal{M}_{n+1}(\chi^{ABC}_s\dots)=\frac{0}{\varepsilon}+\mathcal{O}(\varepsilon^0)$
$\mathcal{M}_{n+1}(\Phi^{ABCD}_s,\dots)=\frac{0}{\varepsilon}+\mathcal{O}(\varepsilon^0)$

Soft Superfields	**Superamplitude expansion on $\varepsilon \to 0$**
Soft graviton	$\mathcal{M}_{n+1}\left(H_{s+},\dots\right)=\left(\frac{1}{\varepsilon^3}S^{(0)}+\frac{1}{\varepsilon^2}S^{(1)}+\frac{1}{\varepsilon}S^{(2)}\right)\mathcal{M}_n+O\left(\varepsilon^0\right)$
Soft gravitino	$\mathcal{M}_{n+1}\left(S^A_{s+},\dots\right)=\left(\frac{1}{\varepsilon^2}\mathcal{S}^{A(1)}+\frac{1}{\varepsilon}\mathcal{S}^{A(2)}\right)\mathcal{M}_n+O\left(\varepsilon^0\right)$
Soft graviphoton	$\mathcal{M}_{n+1}\left(G^{AB}_{s+}\dots\right)=\frac{1}{\varepsilon}\mathcal{S}^{AB(2)}\mathcal{M}_n+\mathcal{O}\left(\varepsilon^0\right)$
Soft graviphotino	$\mathcal{M}_{n+1}\left(\chi^{ABC}_s\dots\right)=\frac{0}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right)$
Soft scalar	$\mathcal{M}_{n+1}\left(\Phi^{ABCD}_s,\dots\right)=\frac{0}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right)$

$$\epsilon^{\alpha\beta}=-\epsilon_{\alpha\beta}=\epsilon^{\dot\alpha\dot\beta}=-\epsilon_{\dot\alpha\dot\beta}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

$$\langle\lambda\chi\rangle\equiv\epsilon^{\alpha\beta}\lambda_\alpha\chi_\beta=\lambda_\alpha\chi^\alpha=-\lambda^\alpha\chi_\alpha=-\langle\chi\lambda\rangle$$
$$[\lambda\chi]\equiv\epsilon_{\dot\alpha\dot\beta}\tilde\lambda^{\tilde\lambda^2}\tilde\chi^{\dot\beta}=\tilde\lambda^{\dot\alpha}\tilde\chi_{\dot\alpha}=-\tilde\lambda_{\dot\alpha}\tilde\chi^{\dot\alpha}=-[\chi\lambda].$$

$$p^{\alpha\dot\alpha}=\sigma^{\alpha\dot\alpha}_\mu p^\mu=\begin{pmatrix}p^0+p^3 & p^1-ip^2\\ p^1+ip^2 & p^0-p^3\end{pmatrix}\equiv|p\rangle[p|$$

$$p^\mu=\frac{1}{2}\sigma^{\mu\alpha\dot\alpha}p_{\dot\alpha\alpha}=\frac{1}{2}\bar\sigma^\mu_{\dot\alpha\alpha}p^{\alpha\dot\alpha},$$

$$p\cdot q=\frac{1}{2}[pq]\langle qp\rangle.$$

$$\sum_{j=1}^{n} \langle ij\rangle[ji] = 0$$

$$\mathrm{Split}_0^{\mathrm{SG}}(z, 1^{1+0}, 2^{1+0}) = -\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_0^{\mathrm{SG}}(z, 1^{1+0}, 2^{0+1}) = -\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_0^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{2}}\right) = -\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{+2}^{\mathrm{SG}}(z, 1^{1+0}, 2^{-1+0}) = -(1-z)^2\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-2}^{\mathrm{SG}}(z, 1^{1+0}, 2^{-1+0}) = -z^2\frac{\langle 12\rangle}{[12]}, \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z, 1^{-\frac{1}{2}-\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{2}}\right) = -(1-z)^2\frac{\langle 12\rangle}{[12]}$$

$$\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z, 1^{-\frac{1}{2}-\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{2}}\right) = -z^2\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_1^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+0}, 2^{\frac{1}{2}+0}\right) = -\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_1^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+0}, 2^{0+\frac{1}{2}}\right) = -\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+0}, 2^{-\frac{1}{2}+0}\right) = -\sqrt{z(1-z)^3}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+0}, 2^{-\frac{1}{2}+0}\right) = -\sqrt{z^3(1-z)}\frac{\langle 12\rangle}{[12]}$$

$$\mathrm{Split}_1^{\mathrm{SG}}\left(z, 1^{1-\frac{1}{2}}, 2^{-\frac{1}{2}+1}\right) = -\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}-1}, 2^{-\frac{1}{2}+1}\right) = -\sqrt{z(1-z)^3}\frac{\langle 12\rangle}{[12]}$$

$$\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}-1}, 2^{-\frac{1}{2}+1}\right) = -\sqrt{z^3(1-z)}\frac{[12]}{\langle 12\rangle}.$$

$$\mathrm{Split}_{-2}^{\mathrm{SG}}(z, 1^{0+0}, 2^{0+0}) - z(1-z)\frac{\langle 12\rangle}{[12]}, \mathrm{Split}_{+2}^{\mathrm{SG}}(z, 1^{0+0}, 2^{0+0}) = -z(1-z)\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-2}^{\mathrm{SG}}(z, 1^{-1+1}, 2^{+1-1}) = -z(1-z)\frac{\langle 12\rangle}{[12]}, \mathrm{Split}_{+2}^{\mathrm{SG}}(z, 1^{-1+1}, 2^{+1-1}) = -z(1-z)\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z, 1^{-\frac{1}{2}+\frac{1}{2}}, 2^{+\frac{1}{2}-\frac{1}{2}}\right) = -z(1-z)\frac{\langle 12\rangle}{[12]}, \mathrm{Split}_{+2}^{\mathrm{SG}}\left(z, 1^{-\frac{1}{2}+\frac{1}{2}}, 2^{+\frac{1}{2}-\frac{1}{2}}\right) = -z(1-z)\frac{[12]}{\langle 12\rangle}.$$

$$\mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{1+\frac{1}{2}}\right) = -\frac{1}{z\sqrt{1-z}}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{\frac{1}{2}+1}\right) = -\frac{1}{z\sqrt{(1-z)}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{+\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{-1-\frac{1}{2}}\right) = -\frac{\sqrt{(1-z)^5}}{z}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{+\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{-\frac{1}{2}-1}\right) = -\frac{\sqrt{(1-z)^5}}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-1}^{\mathrm{SG}}(z, 1^{1+1}, 2^{1+0}) = -\frac{1}{z}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{-1}^{\mathrm{SG}}(z, 1^{1+1}, 2^{0+1}) = -\frac{1}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{+1}^{\mathrm{SG}}(z, 1^{1+1}, 2^{0-1}) = -\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{+1}^{\mathrm{SG}}(z, 1^{1+1}, 2^{-1+0}) = -\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{\frac{1}{2}+\frac{1}{2}}\right) = -\frac{1}{z}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{+1}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{-\frac{1}{2}-\frac{1}{2}}\right) = -\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}_{-\frac{1}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{\frac{1}{2}+0}\right) = -\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle}, \mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z, 1^{1+1}, 2^{-\frac{1}{2}+0}\right) = -\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{0+\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{0-\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{1-\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{-1+\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{-\frac{1}{2}+1}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{\frac{1}{2}-1}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}(z,1^{1+1},2^{0+0})=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}(z,1^{1+1},2^{1-1})=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+1},2^{\frac{1}{2}-\frac{1}{2}}\right)=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{0+1}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{1+0}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{0+1}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{1+0}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{+\frac{3}{2}}\left(z,1^{\frac{1}{2}+1},2^{0-1}\right)=-\frac{(1-z)^2}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{+\frac{3}{2}}\left(z,1^{1+\frac{1}{2}},2^{-1+0}\right)=-\frac{(1-z)^2}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{0+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{1-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}+1}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{0-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)^3}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}-1}\right)=-\sqrt{\frac{(1-z)^3}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{0+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{1-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{-\frac{1}{2}+1}\right)=-\sqrt{\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}},\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+\frac{1}{2}},2^{0-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}\frac{[12]}{\langle 12\rangle}}$$

$$\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{1+\frac{1}{2}},2^{-\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{1+\frac{1}{2}},2^{-1+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{0+0}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{1-1}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}-\frac{1}{2}}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+\frac{1}{2}},2^{0+0}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{0+1},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{0+1},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{0+1},2^{1-\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{1-\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+1}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+0},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+0},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+0},2^{-\frac{1}{2}+1}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{0+1},2^{0-\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{0+1},2^{\frac{1}{2}-1}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0-\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+0}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+0},2^{-\frac{1}{2}+0}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle},\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+0},2^{-1+\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{1}^{\mathrm{SG}}(z,1^{0+1},2^{0+0})=-(1-z)\frac{[12]}{\langle12\rangle},\mathrm{Split}_{1}^{\mathrm{SG}}(z,1^{0+1},2^{1-1})=-(1-z)\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{1}^{\mathrm{SG}}\left(z,1^{0+1},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{[12]}{\langle12\rangle},\mathrm{Split}_{1}^{\mathrm{SG}}(z,1^{1+0},2^{0+0})=-(1-z)\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{1}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+0}\right)=-(1-z)\frac{[12]}{\langle12\rangle},\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{[12]}{\langle12\rangle}$$

$$\mathrm{Split}_{-1}^{\mathrm{SG}}(z,1^{-1+0},2^{1-1})=-(1-z)\frac{\langle12\rangle}{[12]},\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-1+0},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{\langle12\rangle}{[12]}$$

$$\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{\langle12\rangle}{[12]}$$

$$\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{0+0}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle}$$

$$\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+0},2^{0+0}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle},\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-\frac{1}{2}+0},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]}$$

$$\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-1+\frac{1}{2}},2^{1-1}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]},\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-1+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]}$$

$$\begin{cases}\left(a;\frac{3}{2}\right)=\left(a;\frac{1}{2}\right)\otimes 1\\ \left(r;\frac{3}{2}\right)=1\otimes\left(r;\frac{1}{2}\right)\end{cases}$$

$$\begin{aligned}M_n\left(1^{a;+\frac{3}{2}},2^{b;+\frac{3}{2}},\cdots,n\right)&=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{\left(b;\frac{1}{2}\right)\otimes 1},\cdots,n\right)\\&=\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+1}\right)\times M_{n-1}(p^{ab;+1},\cdots,n)\\&=\frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{ab;+1},\cdots,n)\\M_n\left(1^{a;+\frac{3}{2}},2^{r;+\frac{3}{2}},\cdots,n\right)&=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{1\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)\\&=\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{1+\frac{1}{2}}\right)\times M_{n-1}(p^{ar;+1},\cdots,n)\\&=\frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{ar;+1},\cdots,n)\end{aligned}$$

$$M_n\left(1^{A;+\frac{3}{2}},2^{B;+\frac{3}{2}},\cdots,n\right)=\frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{AB;+1},\cdots,n)$$

$$\begin{aligned}M_n\left(1^{a;+\frac{3}{2}},2_b^{-\frac{3}{2}},\cdots,n\right)&=\delta^a_b\mathrm{Split}^{\mathrm{SG}}_{-2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)M_{n-1}\left(p^{+2},\cdots,n\right)\\&\quad+\delta^a_b\mathrm{Split}^{\mathrm{SG}}_{+2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)M_{n-1}(p^{-2},\cdots,n)\\&=\delta^a_b\frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{-2},\cdots,n)+\delta^a_b\frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2}\frac{z_{12}}{\bar z_{12}}M_{n-1}(p^{+2},\cdots,n).\end{aligned}$$

$$M_n\left(1^{A;+\frac{3}{2}},2_B^{-\frac{3}{2}},\cdots,n\right)=\delta^A_B\frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{-2},\cdots,n)+\delta^A_B\frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2}\frac{z_{12}}{\bar z_{12}}M_{n-1}(p^{+2},\cdots,n).$$

$$\begin{cases}(ab;1)=(ab;0)\otimes 1\\(ar;1)=\left(a,\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)\\(rs;1)=1\otimes(rs;0)\end{cases}$$

$$
\begin{aligned}
M_n\left(1^{ab;+1},2^{cd;+1},\cdots,n\right) &= M_n\left(1^{ab;(1\otimes 0)},2^{cd;(1\otimes 0)},\cdots,n\right)\\
&= \mathrm{Split}_0^{\mathrm{SG}}(z,1^{1+0},2^{1+0})\times M_{n-1}\left(p^{abcd;0},\cdots,n\right)\\
&= \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abcd;0},\cdots,n\right)
\end{aligned}
$$

$$
M_n(1^{rs;+1},2^{tu;+1},\cdots,n) = \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{rstu;0},\cdots,n)
$$

$$
\begin{aligned}
M_n\left(1^{rs;+1},2^{ab;+1},\cdots,n\right) &= M_n\left(1^{rs;(1\otimes 0)},2^{ab;(0\otimes 1)},\cdots,n\right)\\
&= \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rsab;0},\cdots,n\right)
\end{aligned}
$$

$$
M_n\left(1^{ab;+1},2^{rs;+1},\cdots,n\right) = \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abrs;0},\cdots,n\right)
$$

$$
\begin{aligned}
M_n\left(1^{ar;+1},2^{bs;+1},\cdots,n\right) &= M_n\left(1^{ar;\left(\frac{1}{2}\otimes\frac{1}{2}\right)},2^{bs;\left(\frac{1}{2}\otimes\frac{1}{2}\right)},\cdots,n\right)\\
&= \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{arbs;0},\cdots,n\right)
\end{aligned}
$$

$$
M_n(1^{AB;+1},2^{CD;+1},\cdots,n) = \frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{ABCD;0},\cdots,n)
$$

$$
\begin{aligned}
M_n\left(1^{ar;+1},2_{bs}^{-1},\cdots,n\right) &= M_n\left(1^{ar;\left(\frac{1}{2}\otimes\frac{1}{2}\right)},2_{bs}^{\left(-\frac{1}{2}\otimes-\frac{1}{2}\right)},\cdots,n\right)\\
&= -\delta_b^a\delta_s^r\left[\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\quad \left.+\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}(p^{-2},\cdots,n)\right]\\
&= -\delta_b^a\delta_s^r\left[\frac{\omega_2^2\bar{z}_{12}}{\omega_p^2}\frac{1}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)+\frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right]
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{ab;+1},2_{cd}^{-1},\cdots,n\right) &= M_n\left(1^{ab;(1\otimes 0)},2_{cd}^{(-1\otimes 0)},\cdots,n\right)\\
&= \frac{1}{2!}\alpha_4\epsilon_{cdef}\epsilon^{abef}\left[\mathrm{Split}_{-2}^{\mathrm{SG}}(z,1^{1+0},2^{-1+0})\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\quad \left.+\mathrm{Split}_{+2}^{\mathrm{SG}}(z,1^{1+0},2^{-1+0})\times M_{n-1}(p^{-2},\cdots,n)\right]\\
&= \alpha_4\delta_{cd}^{ab}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)+\frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right]
\end{aligned}
$$

$$
\delta_{b_1\ldots b_n}^{a_1\ldots a_n} = \sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\delta_{b_1}^{a_{\sigma(1)}}\ldots\delta_{b_n}^{a_{\sigma(n)}}
$$

$$
\begin{aligned}
M_n(1^{AB;+1},2_{CD}^{-1},\cdots,n) &= -\delta_{CD}^{AB}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\right.\\
&\quad \left.+\frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right].
\end{aligned}
$$

$$\begin{cases} \left(abr;\frac{1}{2}\right) = (ab;0)\otimes\left(r;\frac{1}{2}\right) \\ \left(ars;\frac{1}{2}\right) = \left(a;\frac{1}{2}\right)\otimes(rs;0) \end{cases}$$

$$\begin{cases} \left(rst;\frac{1}{2}\right) = -\epsilon^{rstu}\left(1\otimes\left(u;-\frac{1}{2}\right)\right) \\ \left(abc;\frac{1}{2}\right) = -\epsilon^{abcd}\left(\left(d;-\frac{1}{2}\right)\otimes 1\right) \end{cases} \text{(sum over } u,d)$$

$$\begin{aligned} M_n\left(1^{ars;+\frac{1}{2}},2^{btu;\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;\left(\frac{1}{2}\otimes 0\right)},2^{btu;\left(\frac{1}{2}\otimes 0\right)},\cdots,n\right) \\ &= \epsilon^{rstu}\epsilon^{abcd}\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{\frac{1}{2}+0}\right)\times M_{n-1}(p_{cd}^{-1},\cdots,n) \\ &= \epsilon^{rstu}\epsilon^{abcd}\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{z_{12}}{z_{12}}\times M_{n-1}(p_{cd}^{-1},\cdots,n) \end{aligned}$$

$$\begin{aligned} M_n\left(1^{ars;+\frac{1}{2}},2^{bct;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;\left(\frac{1}{2}\otimes 0\right)},2^{bct;\left(0\otimes\frac{1}{2}\right)},\cdots,n\right) \\ &= \epsilon^{abcd}\epsilon^{rstu}\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{du}^{-1},\cdots,n) \\ M_n\left(1^{rst;+\frac{1}{2}},2^{abc;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{rst;\left(1\otimes-\frac{1}{2}\right)},2^{abc;\left(-\frac{1}{2}\otimes 1\right)},\cdots,n\right) \\ &= \epsilon^{rstu}\epsilon^{abcd}\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{ud}^{-1},\cdots,n) \\ M_n\left(1^{ars;+\frac{1}{2}},2_{btu}^{-\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;\left(\frac{1}{2}\otimes 0\right)},2_{btu}^{\left(-\frac{1}{2}\otimes 0\right)},\cdots,n\right) \\ &= \epsilon_{tuvw}\epsilon^{rsvw}\delta_b^a\Big[\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)\times M_{n-1}(p^{+2},\cdots,n) \\ &\quad +\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)\times M_{n-1}(p^{-2},\cdots,n)\Big] \\ &= \epsilon_{tuvw}\epsilon^{rsvw}\delta_b^a\Bigg[\frac{\omega_1^{\frac{3}{2}}\omega_2^{\frac{1}{2}}}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n) \\ &\quad +\frac{\omega_2^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\Bigg] \end{aligned}$$

$$(abrs;0) = (ab;0)\otimes(rs;0)$$

$$\begin{cases} (abcd;0) = -\epsilon^{abcd}(-1\otimes 1) \\ (rstu;0) = -\epsilon^{rstu}(1\otimes -1) \end{cases}$$

$$\begin{cases} (abcr;0) = -\epsilon^{abcd}\left(d;-\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right) \\ (arst;0) = -\epsilon^{rstu}\left(a;\frac{1}{2}\right)\otimes\left(u;-\frac{1}{2}\right) \end{cases}$$

$$
\begin{aligned}
M_n\left(1^{abrs;0},2^{cdtu;0},\cdots,n\right) =& M_n\left(1^{abrs;(0\otimes 0)},2^{cdtu;(0\otimes 0)},\cdots,n\right)\\
=&\epsilon^{abcd}\epsilon^{rstu}\big[\mathrm{Split}_{-2}^{\mathrm{SG}}(z,1^{0+0},2^{0+0})\times M_{n-1}(p^{+2},\cdots,n)\\
&+\mathrm{Split}_{+2}^{\mathrm{SG}}(z,1^{0+0},2^{0+0})\times M_{n-1}(p^{-2},\cdots,n)\big]\\
=&\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_1\omega_2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\left.+\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\right]
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{abcd;0},2^{rstu;0},\cdots,n\right) =& M_n\left(1^{abcd;(-1\otimes 1)},2^{rstu;(+1\otimes -1)},\cdots,n\right)\\
=&\epsilon^{abcd}\epsilon^{rstu}\big[\mathrm{Split}_{-2}^{\mathrm{SG}}(z,1^{-1+1},2^{+1-1})\times M_{n-1}(p^{+2},\cdots,n)\\
&+\mathrm{Split}_{+2}^{\mathrm{SG}}(z,1^{-1+1},2^{+1-1})\times M_{n-1}(p^{-2},\cdots,n)\big]\\
=&\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\left.+\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\right]
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{abcu;0},2^{drst;0},\cdots,n\right) =&\epsilon^{abce}\epsilon^{rstv}M_n\left(1^{\left(e,-\frac{1}{2}\right)\otimes\left(u,\frac{1}{2}\right)},2^{\left(d,+\frac{1}{2}\right)\otimes\left(v,-\frac{1}{2}\right)},\cdots,n\right)\\
=&\epsilon^{abce}\epsilon^{rstv}\delta_e^d\delta_v^u\left[\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\left.+\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}+\frac{1}{2}},2^{+\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}(p^{-2},\cdots,n)\right]\\
=&\epsilon^{abcd}\epsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\left.+\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\right]
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{\mathrm{arst};0},2^{\mathrm{bcdu};0},\cdots,n\right) =&\epsilon^{\mathrm{rstu}}\epsilon^{abcd}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}(p^{+2},\cdots,n)\right.\\
&\left.+\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-2},\cdots,n)\right]
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{+2},2^{r;+\frac{3}{2}},\cdots,n\right) &= M_n\left(1^{(1\otimes1)},2^{1\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{1+1},2^{1+\frac{1}{2}}\right)\times M_{n-1}\left(p^{r;+\frac{3}{2}},\cdots,n\right)\\
&= \frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{r;+\frac{3}{2}},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{+2},2_r^{-\frac{3}{2}},\cdots,n\right) &= M_n\left(1^{(1\otimes1)},2^{-1\otimes\left(r;-\frac{1}{2}\right)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{+\frac{3}{2}}\left(z,1^{1+1},2^{-1-\frac{1}{2}}\right)\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)\\
&= \frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)
\end{aligned}
$$

$$
M_n\left(1^{+2},2^{a;+\frac{3}{2}},\cdots,n\right) = \frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{a;+\frac{3}{2}},\cdots,n\right)
$$

$$
M_n\left(1^{+2},2_a^{-\frac{3}{2}},\cdots,n\right) = \frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_a^{-\frac{3}{2}},\cdots,n\right)
$$

$$
\begin{aligned}
M_n\left(1^{+2},2^{ab;+1},\cdots,n\right) &= M_n\left(1^{(1\otimes1)},2^{(ab;0)\otimes1},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{1+1},2^{0+1}\right)\times M_{n-1}\left(p^{ab;+1},\cdots,n\right)\\
&= \frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ab;+1},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n(1^{+2},2^{rs;+1},\cdots,n) &= M_n\left(1^{(1\otimes 1)},2^{1\otimes(rs;0)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{-1}(z,1^{1+1},2^{1+0})\times M_{n-1}(p^{rs;+1},\cdots,n)\\
&= \frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{rs;+1},\cdots,n)
\end{aligned}
$$

$$
\begin{aligned}
M_n(1^{+2},2^{ar;+1},\cdots,n) &= M_n\left(1^{(1\otimes 1)},2^{\left(a;\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{1+1},2^{\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}(p^{ar;+1},\cdots,n)\\
&= \frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{ar;+1},\cdots,n)
\end{aligned}
$$

$$
\begin{aligned}
M_n(1^{+2},2^{-1}_{ab},\cdots,n) &= M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes -1},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{+1}(z,1^{1+1},2^{0-1})\times M_{n-1}\left(p^{-1}_{ab},\cdots,n\right)\\
&= \frac{\omega_2^2}{\omega_p\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{ab},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n(1^{+2},2^{-1}_{rs},\cdots,n) &= M_n\left(1^{(1\otimes 1)},2^{-1\otimes(rs;0)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{+1}(z,1^{1+1},2^{-1+0})\times M_{n-1}(p^{-1}_{rs},\cdots,n)\\
&= \frac{\omega_2^2}{\omega_p\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-1}_{rs},\cdots,n)
\end{aligned}
$$

$$
\begin{aligned}
M_n(1^{+2},2^{-1}_{ar},\cdots,n) &= M_n\left(1^{(1\otimes 1)},2^{\left(a;-\frac{1}{2}\right)\otimes\left(r;-\frac{1}{2}\right)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+1},2^{-\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}(p^{-1}_{ar},\cdots,n)\\
&= \frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-1}_{ar},\cdots,n)
\end{aligned}
$$

$$
M_n(1^{+2},2^{-1}_{AB},\cdots,n) = \frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{-1}_{AB},\cdots,n)
$$

$$
\begin{aligned}
M_n\left(1^{+2},2^{abr;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)\\
&= \mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{0+\frac{1}{2}}\right)\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right)\\
&= \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{z_{12}}{z_{12}}\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right)
\end{aligned}
$$

$$
M_n\left(1^{+2},2^{\mathrm{ars};+\frac{1}{2}},\cdots,n\right) = \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{\mathrm{ars};+\frac{1}{2}},\cdots,n\right)
$$

$$
\begin{aligned}
M_n\left(1^{+2},2^{abc;+\frac{1}{2}},\cdots,n\right) &= -\epsilon^{abcd}M_n\left(1^{(1\otimes 1)},2^{\left(d;-\frac{1}{2}\right)\otimes 1},\cdots,n\right)\\
&= -\frac{1}{3!}\epsilon^{abcd}\epsilon_{defg}\mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{-\frac{1}{2}+1}\right)\times M_{n-1}\left(p^{efg;+\frac{1}{2}},\cdots,n\right)\\
&= \frac{1}{3!}\delta^{abc}_{efg}\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{efg;+\frac{1}{2}},\cdots,n\right)\\
&= \frac{\sqrt{\omega_2\omega_p}\bar{z}_{12}}{\omega_1}\times M_{n-1}\left(p^{abc;+\frac{1}{2}},\cdots,n\right)
\end{aligned}
$$

$$\Gamma_a^- \equiv \frac{1}{3!}\epsilon_{abcd}\Gamma^{-bcd}$$

$$M_n\left(1^{+2},2^{rst;+\frac{1}{2}},\cdots,n\right)=\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rst;+\frac{1}{2}},\cdots,n\right)$$

$$\begin{aligned}M_n\left(1^{+2},2_{abr}^{-\frac{1}{2}},\cdots,n\right)&=M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes\left(r;-\frac{1}{2}\right)},\cdots,n\right)\\&=\mathrm{Split}_{+\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}\left(p_{abr}^{-\frac{1}{2}},\cdots,n\right)\\&=\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{abr}^{-\frac{1}{2}},\cdots,n\right)\end{aligned}$$

$$M_n\left(1^{+2},2_{ars}^{-\frac{1}{2}},\cdots,n\right)=\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{ars}^{-\frac{1}{2}},\cdots,n\right)$$

$$\begin{aligned}M_n\left(1^{+2},2_{abc}^{-\frac{1}{2}},\cdots,n\right)&=\epsilon_{abcd}M_n\left(1^{(1\otimes 1)},2^{\left(d;\frac{1}{2}\right)\otimes -1},\cdots,n\right)\\&=\frac{1}{3!}\epsilon_{abcd}\epsilon^{defg}\mathrm{Split}_{\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{\frac{1}{2}-1}\right)\times M_{n-1}\left(p_{efg;}^{-\frac{1}{2}},\cdots,n\right)\\&=-\frac{1}{3!}\delta_{abc}^{efg}\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{efg;}^{-\frac{1}{2}},\cdots,n\right)\\&=-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{abc}^{-\frac{1}{2}},\cdots,n\right)\end{aligned}$$

$$M_n\left(1^{+2},2_{rst}^{-\frac{1}{2}},\cdots,n\right)=-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{rst}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n(1^{+2},2^{ABCD;0},\cdots,n)=\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p^{ABCD;0},\cdots,n)$$

$$\begin{aligned}M_n\left(1^{a;+\frac{3}{2}},2^{bc;+1},\cdots,n\right)&=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{(bc;0)\otimes 1},\cdots,n\right)\\&=\mathrm{Split}_{-\frac{1}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{0+1}\right)\times M_{n-1}\left(p^{abc;\frac{1}{2}},\cdots,n\right)\\&=\sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{abc;+\frac{1}{2}},\cdots,n\right)\end{aligned}$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{st;+1},\cdots,n\right)=\sqrt{\frac{\omega_p}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}\left(p^{rst;+\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{rs;+1},\cdots,n\right)=\sqrt{\frac{\omega_p}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}\left(p^{ars;+\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{ab;+1},\cdots,n\right)=\sqrt{\frac{\omega_p}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}\left(p^{rab;+\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{A;+\frac{3}{2}},2^{BC;+1},\cdots,n\right)=\sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{ABC;+\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{-1}_{bc},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{(bc;0)\otimes 1},\cdots,n\right)$$

$$=\frac{1}{2!}\epsilon_{bcde}\epsilon^{adef}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+1},2^{0-1}\right)\times M_{n-1}\left(p_f^{-\frac{3}{2}},\cdots,n\right)$$

$$=\delta^{af}_{bc}\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_f^{-\frac{3}{2}},\cdots,n\right)$$

$$=\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}2!\,\delta^{a}_{[b}\times M_{n-1}\left(p_{c]}^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{-1}_{st},\cdots,n\right)=\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}2!\,\delta^{r}_{[s}\times M_{n-1}\left(p_{t]}^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{A;+\frac{3}{2}},2^{-1}_{BC},\cdots,n\right)=\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}2!\,\delta^{A}_{[B}\times M_{n-1}\left(p_{C]}^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{brs;+\frac{1}{2}},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{\left(b;\frac{1}{2}\right)\otimes(rs;0)},\cdots,n\right)$$

$$=\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+0}\right)\times M_{n-1}(p^{abrs;0},\cdots,n)$$

$$=\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}(p^{abrs;0},\cdots,n)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{bcr;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}(p^{abcr;0},\cdots,n)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{sta;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}(p^{rsta;0},\cdots,n)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{abs;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}(p^{abrs;0},\cdots,n)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{bcd;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}(p^{abcd;0},\cdots,n)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{rst;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{z_{12}}{z_{12}}}\times M_{n-1}(p^{arst;0},\cdots,n)$$

$$M_n\left(1^{A;+\frac{3}{2}},2^{BCD;+\frac{1}{2}},\cdots,n\right)=\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}(p^{ABCD;0},\cdots,n)$$

$$M_n\left(1^{a;+\frac{3}{2}},2_{bcr}^{-\frac{1}{2}},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{(bc;0)\otimes\left(r;-\frac{1}{2}\right)},\cdots,n\right)$$

$$=-\frac{1}{2!}\epsilon_{bcde}\epsilon^{deaf}\,\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}(p_{fr}^{-1},\cdots,n)$$

$$=-\delta_{bc}^{af}\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{fr}^{-1},\cdots,n)$$

$$=-2!\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[b}^{a}\times M_{n-1}(p_{c]r}^{-1},\cdots,n)$$

$$=-3!\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[b}^{a}\times M_{n-1}(p_{cr]}^{-1},\cdots,n)$$

$$2!\,\delta_{[b]r}^{a}p_{c]r}^{-1}=\delta_b^a p_{cr}^{-1}-\delta_b^a p_{rc}^{-1}+\delta_c^a p_{rb}^{-1}-\delta_c^a p_{br}^{-1}+\delta_r^a p_{bc}^{-1}-\delta_r^a p_{cb}^{-1}$$

$$=3!\,\delta_{[b}^{a}p_{cr]}^{-1}$$

$$M_n\left(1^{a;+\frac{3}{2}},2_{brs}^{-\frac{1}{2}},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{\left(b;-\frac{1}{2}\right)\otimes(rs;0)},\cdots,n\right)$$

$$=\delta_b^a\,\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}+0}\right)\times M_{n-1}(p_{rs}^{-1},\cdots,n)$$

$$=\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_b^a\times M_{n-1}(p_{rs}^{-1},\cdots,n)$$

$$M_n\left(1^{r;+\frac{3}{2}}, 2_{ast}^{-\frac{1}{2}}, \cdots, n\right) = -\frac{1}{2!}\epsilon_{stuv}\epsilon^{uvrw}\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p_{wa}^{-1}, \cdots, n)$$

$$= -3!\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[s}^{r} \times M_{n-1}(p_{ta]}^{-1}, \cdots, n)$$

$$M_n\left(1^{t;+\frac{3}{2}}, 2_{rab}^{-\frac{1}{2}}, \cdots, n\right) = \frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_r^t \times M_{n-1}(p_{ab}^{-1}, \cdots, n)$$

$$M_n\left(1^{a;+\frac{3}{2}}, 2_{bcd}^{-\frac{1}{2}}, \cdots, n\right) = \epsilon_{bcde}M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1}, 2^{\left(e;+\frac{1}{2}\right)\otimes -1}, \cdots, n\right)$$

$$= -\frac{1}{2!}\epsilon_{bcde}\epsilon^{aefg}\mathrm{Split}_{+1}^{\mathrm{SG}}\left(z, 1^{\frac{1}{2}+1}, 2^{\frac{1}{2}-1}\right) \times M_{n-1}(p_{fg}^{-1}, \cdots, n)$$

$$= -\frac{1}{2!}\epsilon_{bcde}\epsilon^{afge}\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p_{fg}^{-1}, \cdots, n)$$

$$= -\frac{1}{2!}\delta_{bcd}^{afg}\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}(p_{fg}^{-1}, \cdots, n)$$

$$= 3\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[b}^{a} \times M_{n-1}(p_{cd]}^{-1}, \cdots, n)$$

$$G_{fg}^{-1} = \Phi_{fg} \otimes G^{-1} = -\frac{1}{2!}\epsilon_{aefg}\Phi^{ae} \otimes G^{-1} =: -\frac{1}{2!}\epsilon_{aefg}G^{ae-1}$$

$$M_n\left(1^{d;+\frac{3}{2}}, 2_{abc}^{-\frac{1}{2}}, \cdots, n\right) = -3\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[a}^{d} \times M_{n-1}(p_{bc]}^{-1}, \cdots, n)$$

$$M_n\left(1^{u;+\frac{3}{2}}, 2_{rst}^{-\frac{1}{2}}, \cdots, n\right) = -3\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta_{[r}^{u} \times M_{n-1}(p_{st]}^{-1}, \cdots, n)$$

$$M_n\left(1^{A;+\frac{3}{2}}, 2_{BCD}^{-\frac{1}{2}}, \cdots, n\right) = -\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}[\delta_B^A M_{n-1}(p_{CD}^{-1}, \cdots, n)$$
$$+\delta_C^A M_{n-1}(p_{DB}^{-1}, \cdots, n) + \delta_D^A M_{n-1}(p_{BC}^{-1}, \cdots, n)]$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{bcrs;0},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{(bc;0)\otimes(rs;0)},\cdots,n\right)$$
$$=\epsilon^{rstu}\epsilon^{abcd}\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{0+0}\right)\times M_{n-1}\left(p_{dtu}^{-\frac{1}{2}},\cdots,n\right)$$
$$=\epsilon^{rstu}\epsilon^{abcd}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{drs}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{rstu;0},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{rstu;(1\otimes 1)},\cdots,n\right)$$
$$=-\epsilon^{rstu}\epsilon^{abcd}\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{1-1}\right)\times M_{n-1}\left(p_{bcd}^{-\frac{1}{2}},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{bcd}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{abcd;0},\cdots,n\right)=M_n\left(1^{\left(r;\frac{1}{2}\right)\otimes 1},2^{abcd;(1\otimes 1)},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{-1+1}\right)\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{A;+\frac{3}{2}},2^{BCDE;0},\cdots,n\right)=-\frac{1}{3!}\epsilon^{ABCDEFGH}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{FGH}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{ab;+1},2^{cdr;+\frac{1}{2}},\cdots,n\right)=M_n\left(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)$$
$$=\frac{1}{3!}\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{0+1},2^{0+\frac{1}{2}}\right)\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)$$
$$=\frac{1}{3!}\epsilon^{abcd}\epsilon^{rstu}\sqrt{\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)$$

$$M_n\left(1^{ab;+1},2_{cdr;-\frac{1}{2}},\cdots,n\right)=M_n\left(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right)$$
$$=-\frac{1}{2!}\epsilon^{abef}\epsilon_{cdef}\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)$$
$$=-\delta^{ab}_{cd}\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{z_{12}}\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n(1^{ab;+1},2^{cdrs;0},\cdots,n)=M_n(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes(rs;0)},\cdots,n)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}^{\mathrm{SG}}_{1}(z,1^{0+1},2^{0+0})\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$

$$M_n\left(1^{rs;+1},2^{abtu;0},\cdots,n\right)=M_n\left(1^{1\otimes(rs;0)},2^{(ab;0)\otimes(tu;0)},\cdots,n\right)$$
$$=\epsilon^{rstu}\epsilon^{abcd}\mathrm{Split}_1^{\mathrm{SG}}(z,1^{1+0},2^{0+0})\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right)$$
$$=\epsilon^{rstu}\epsilon^{abcd}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right)$$

$$M_n\left(1^{ar;+1},2^{bcst;0},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)},2^{(bc;0)\otimes(st;0)},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p_{du}^{-1},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{du}^{-1},\cdots,n\right)$$

$$M_n\left(1^{ab;+1},2^{cdef;0},\cdots,n\right)=-M_n\left(1^{(ab;0)\otimes 1},2^{cdef;(-1\otimes 1)},\cdots,n\right)$$
$$=-\epsilon^{cdef}\epsilon^{abgh}\mathrm{Split}_1^{\mathrm{SG}}(z,1^{0+1},2^{-1+1})\times M_{n-1}\left(p_{gh}^{-1},\cdots,n\right)$$
$$=-\epsilon^{cdef}\epsilon^{abgh}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{gh}^{-1},\cdots,n\right)$$

$$M_n\left(1^{rs;+1},2^{cdef;0},\cdots,n\right)=-M_n\left(1^{1\otimes(rs;0)},2^{cdef;(-1\otimes 1)},\cdots,n\right)$$
$$=-\epsilon^{cdef}\epsilon^{rstu}\mathrm{Split}_1^{\mathrm{SG}}(z,1^{1+0},2^{-1+1})\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$
$$=-\epsilon^{cdef}\epsilon^{rstu}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$

$$M_n\left(1^{ar;+1},2^{bcds;0},\cdots,n\right)=-M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)},2^{bcds;\left(-\frac{1}{2}+\frac{1}{2}\right)},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}(p_{tu}^{-1},\cdots,n)$$

$$M_n\left(1_{ab}^{-1},2^{\mathrm{cdef};0},\cdots,n\right)=-M_n\left(1^{(ab;0)\otimes -1},2^{\mathrm{cdef};(-1\otimes 1)},\cdots,n\right)$$
$$=-\epsilon^{\mathrm{cdef}}\epsilon_{\mathrm{abgh}}\,\mathrm{Split}_{-1}^{\mathrm{SG}}(z,1^{0-1},2^{-1+1})\times M_{n-1}\left(p^{\mathrm{gh};+1},\cdots,n\right)$$
$$=-\epsilon^{\mathrm{cdef}}\epsilon_{\mathrm{abgh}}\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{\mathrm{gh};+1},\cdots,n\right)$$

$$M_n\left(1_{ar}^{-1},2^{bcds;0},\cdots,n\right)=-M_n\left(1^{\left(a;-\frac{1}{2}\right)\otimes\left(r;-\frac{1}{2}\right)},2^{bcds;\left(-\frac{1}{2}\otimes\frac{1}{2}\right)},\cdots,n\right)$$
$$=-\epsilon^{bcde}\delta_r^s\epsilon_{aefg}\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{-\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}\left(p^{fg;+1},\cdots,n\right)$$
$$=-\epsilon^{bcde}\delta_r^s\epsilon_{aefg}\frac{\omega_1}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{fg;+1},\cdots,n\right)$$

$$M_n\left(1_{ar}^{-1},2^{bcst;0},\cdots,n\right)=-M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)},2^{(bc;0)\otimes(st;0)},\cdots,n\right)$$
$$=\Big[-\delta_a^b\delta_r^s\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}(p^{ct;+1},\cdots,n)$$
$$+\delta_a^c\delta_r^t\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}(p^{bs;+1},\cdots,n)$$
$$+\delta_a^b\delta_r^t\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}(p^{cs;+1},\cdots,n)$$
$$-\delta_a^c\delta_r^s\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}(p^{bt;+1},\cdots,n)\Big]$$
$$=-\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}4!\,\delta_a^{[b}\delta_r^s M_{n-1}(p^{tc];+1},\cdots,n)$$

$$M_n\left(1^{abr;+\frac{1}{2}},2^{cdst;0},\cdots,n\right)=M_n\left(1^{(ab;0)\otimes\left(r;\frac{1}{2}\right)},2^{(cd;0)\otimes(st;0)},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{0+\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p_u^{-\frac{3}{2}},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_u^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{abr;+\frac{1}{2}},2^{cstu;0},\cdots,n\right)=-M_n\left(1^{(ab;0)\otimes\left(r;\frac{1}{2}\right)},2^{cstu;\left(\frac{1}{2}\otimes-\frac{1}{2}\right)},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{0+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)$$
$$=-\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{ars;+\frac{1}{2}},2^{bctu;0},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes(rs;0)},2^{(bc;0)\otimes(tu;0)},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\mathrm{Split}_{\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{0+0}\right)\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)$$
$$=\epsilon^{abcd}\epsilon^{rstu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1_{ars}^{-\frac{1}{2}},2^{bctu;0},\cdots,n\right)=M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes(rs;0)},2^{(bc;0)\otimes(tu;0)},\cdots,n\right)$$
$$=-\frac{1}{2!}\epsilon^{tuvw}\epsilon_{vwrs}2!\,\delta_a^{[b}\mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}+0},2^{0+0}\right)\times M_{n-1}\left(p^{c];+\frac{3}{2}},\cdots,n\right)$$
$$=-2!\,\delta_{rs}^{tu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\delta_a^{[b}\times M_{n-1}\left(p^{c];+\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{btuv}0, \cdots, n\right) = -M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes(rs;0)}, 2^{btuv;\left(\frac{1}{2}\otimes-\frac{1}{2}\right)}, \cdots, n\right)$$
$$= -\delta_a^b \epsilon^{tuvw} \epsilon_{wrsx} \mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{-\frac{1}{2}+0}, 2^{-\frac{1}{2}+\frac{1}{2}}\right) \times M_{n-1}\left(p^{x;+\frac{3}{2}}, \cdots, n\right)$$
$$= -\delta_a^b \epsilon^{tuvw} \epsilon_{wrsx} \frac{\omega_1^{\frac{1}{2}} \omega_2}{\omega_p^{\frac{3}{2}}} \frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{x;+\frac{3}{2}}, \cdots, n\right)$$

$$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{avwx;0}, \cdots, n\right) = -M_n\left(1^{rst;\left(-1\otimes\frac{1}{2}\right)}, 2^{avwx;\left(\frac{1}{2}\otimes-\frac{1}{2}\right)}, \cdots, n\right)$$
$$= -\epsilon_{rstu} \epsilon^{vwxy} \delta_y^u \mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z, 1^{-1+\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$$
$$= -\epsilon_{rstu} \epsilon^{vwxu} \frac{\omega_1^{\frac{1}{2}} \omega_2}{\omega_p^{\frac{3}{2}}} \frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$$

$$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{\mathrm{uvw}\,x;0}, \cdots, n\right) = -M_n\left(1^{\mathrm{rst}\,;\left(-1\otimes\frac{1}{2}\right)}, 2^{\mathrm{uvw}\,x;(1\otimes-1)}, \cdots, n\right)$$
$$= -\epsilon_{\mathrm{rsty}}\, \epsilon^{\mathrm{uvwx}}\, \mathrm{SplitSGG}_{-\frac{3}{2}}^{\mathrm{Sin}}\left(z, 1^{-1+\frac{1}{2}}, 2^{1-1}\right) \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)$$
$$= -\epsilon_{\mathrm{rsty}}\, \epsilon^{\mathrm{uvwx}} \frac{\omega_1^{\frac{1}{2}} \omega_2}{\omega_p^{\frac{3}{2}}} \frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)$$

$$\int d^4\eta^p \delta^{(4)}(\eta^p - \eta) f(\eta^p) = \int d^4\eta^p \prod_{a=1}^{4} (\eta_a^p - \eta_a) f(\eta^p) = f(\eta)$$

$$\delta^{(4)}(\eta^p - \eta) = \prod_{a=1}^{4} (\eta_a^p - \eta_a)$$

$$\begin{aligned}
R_{mn} - \frac{1}{4} H_{mpq} H_n^{pq} + 2\nabla_{(m} Z_{n)} &= T_{mn}, \\
-\frac{1}{2} \nabla^k H_{kmn} + Z^k H_{kmn} + 2\nabla_{[m} I_{n]} &= K_{mn}, \\
R - \frac{1}{2} |H_3|^2 + 4(\nabla^m Z_m - I^m I_m - Z^m Z_m) &= 0, \\
d * F_p - H_3 \wedge * F_{p+2} - \iota_I B_2 \wedge * F_p - \iota_I * F_{p-2} &= 0,
\end{aligned}$$

$$\begin{aligned}
T_{mn} &= \frac{1}{4} e^{2\Phi} \sum_p \left[\frac{1}{p!} F_m^{\ k_1 \dots k_p} F_{n k_1 \dots k_p} - \frac{1}{2} g_{mn} \left|F_{p+1}\right|^2\right] \\
K_{mn} &= \frac{1}{4} e^{2\Phi} \sum_p \frac{1}{p!} F_{k_1 \dots k_p} F_{mn}^{\ k_1 \dots k_p}
\end{aligned}$$

$$r^{i_1 [j_1} r^{j_2 | i_2 |} f_{i_1 i_2}^{\ \ j_3]} = 0$$

$$I^m = r^{i_1 i_2} f_{i_1 i_2}{}^j k_j{}^m$$

$$6\rho^{[i_2|i_7 j_1}\rho^{|i_3 i_4|j_2} f_{j_1 j_2}{}^{|i_5]} + \rho^{j_1 j_2 [i_2}\rho^{i_3 i_4 i_5]} f_{j_1 j_2}{}^{i_7} = 0$$

$$\rho^{i_1 i_2 i_3} f_{i_2 i_3}{}^{i_4} = 0$$

$$J^{mn} = \frac{1}{4}\rho^{i_1 i_2 i_3} f^{i_4}_{i_2 i_3} k_{i_1}{}^m k_{i_4}{}^n$$

$$\begin{aligned}
0 =& \mathcal{R}_{mn}[h_{(4)}] - 7\tilde{\nabla}_{(m} Z_{n)} + T_{mn} \\
&+8(1+V^2)\left(J_{(mn)} J^k{}_k - 2J_{mk} J^k{}_n\right) + 4V_m V_n\left(J^{kl} J_{kl} - 2J^{kl} J_{lk}\right) \\
&+4V_k V_l\left(4J_m{}^k J_n{}^l - J^k{}_m J^l{}_n - 2J^{kl} J_{(mn)}\right) + 8V_k V_{(m}\left(2J^l{}_{n)} J^k{}_l - JJ_{n)}{}^k - JJ^k{}_{n)} + J^{kl} J_{n)l}\right), \\
0 =& \frac{1}{7} e^{2\phi}\mathcal{R}[\bar{g}_{(7)}] + \frac{1}{6}(\nabla V)^2 + \tilde{\nabla}^m Z_m - 6Z_m Z^m - 2J^{mn} J_{mn} + \frac{4}{3} J_{mn} J^{nm}, \\
0 =& \tilde{\nabla}^m F_{mnkl} - 6\left(Z^m F_{mnkl} + 2J^{pm} C_{m[nk} J_{l]p} - J^{pm} J_{p[n} C_{kl]m}\right),
\end{aligned}$$

$$\begin{aligned}
Z_m &= \partial_m \phi - \frac{2}{3}\epsilon_{mnkl} J^{nk} V^l \\
\tilde{\nabla}_m &= \nabla_m - \partial_m \phi
\end{aligned}$$

$$\begin{aligned}
0 &= J^{m[n} J^{kl]} \\
0 &= \partial_m J^{kl} + J^{kn}\partial_n e^a_m e^l_a + J^{nl}\partial_n e^a_m e^k_a + J^{nl}\delta^k_m \partial_n \phi \\
0 &= \nabla_m\left(e^{-\phi} J^{[mn]}\right) \\
0 &= J^{mn}\partial_n \phi \\
0 &= \nabla_{[m} Z_{n]} - \frac{1}{3} J^{kl} F_{mnkl} \\
0 &= \nabla_k\left(e^{-\phi} J^{k[l} V^{p]}\right) \\
0 &= \nabla_k\left(J^{(pl)} V^k\right) - \nabla_k\left(V^{(p} J^{l)k}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\Lambda V^M &= \Lambda^N \partial_N V^M - V^N \partial_N \Lambda^M + \eta^{MN}\eta_{KL}\partial_N \Lambda^K V^L \\
\mathcal{L}_\Lambda d &= \Lambda^M \partial_M d - \frac{1}{2}\partial_M \Lambda^M
\end{aligned}$$

$$\eta^{MN}\partial_M f \partial_N g = 0, \eta^{MN}\partial_M \partial_N f = 0$$

$$\eta^{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix},$$

$$\begin{aligned}
S_{DFT} = \int d^{10}\Re e^{-2d}\Big[&\mathcal{H}^{AB}\mathcal{F}_A\mathcal{F}_B + \mathcal{F}_{ABC}\mathcal{F}_{DEF}\left(\frac{1}{4}\mathcal{H}^{AD}\eta^{BE}\eta^{CF} - \frac{1}{12}\mathcal{H}^{AD}\mathcal{H}^{BE}\mathcal{H}^{CF}\right) \\
&-\mathcal{F}_A\mathcal{F}^A - \frac{1}{6}\mathcal{F}_{ABC}\mathcal{F}^{ABC}\Big]
\end{aligned}$$

$$\mathcal{F}_{ABC} = 3E_{N[C}\partial_A E^N_{B]}, \mathcal{F}_A = 2\partial_A d - \partial_M E^M_A$$

$$\eta^{AB} = \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}, \mathcal{H}^{AB} = \begin{pmatrix} h^{ab} & 0 \\ 0 & h_{ab} \end{pmatrix}$$

$$d = \phi - \frac{1}{2}\log\ e, e = \det e_k^a,$$
$$E_M^A = \begin{bmatrix} e_m^a & 0 \\ -e^k{}_a b_{km} & e^m{}_a \end{bmatrix}, E^M{}_A = \begin{bmatrix} e^m{}_a & -e^k{}_a b_{km} \\ 0 & e_m{}^a \end{bmatrix}$$

$$u_A^B = E_M^B \delta E_A^M = \begin{bmatrix} e_m{}^b \delta e_a^m & 0 \\ e^p{}_a e^m{}_b \delta b_{mp} & -e_m{}^a \delta e^m{}_b \end{bmatrix}, \delta d = \delta\phi + \frac{1}{2} e_m{}^a \delta e^m{}_a$$

$$\delta\mathcal{F}_A = \mathcal{F}_B u_A^B - E_B^M \partial_M u_A^B + 2E_A^M \partial_M \delta d,$$
$$\delta\mathcal{F}_{ABC} = 3u_{[A}^D \mathcal{F}_{BC]D} + 3E_{[A}^M \eta_{C|D} \partial_M u_{|B]}^D.$$

$$e^{2d}\delta\mathcal{L}_{DFT} = \delta d\Big(2\mathcal{F}_A\mathcal{F}_B\mathcal{H}^{AB} - 4\partial_B\mathcal{F}_A\mathcal{H}^{AB} + \frac{1}{6}\mathcal{F}_{ABC}\mathcal{F}_{DEF}\mathcal{H}^{AD}\mathcal{H}^{BE}\mathcal{H}^{CF}$$
$$-\frac{1}{2}\mathcal{F}_{ABC}\mathcal{F}^{AB}{}_D\mathcal{H}^{CD} - 2\mathcal{F}_A\mathcal{F}^A + 4\partial_A\mathcal{F}^A + \frac{1}{3}\mathcal{F}_{ABC}\mathcal{F}^{ABC}\Big)$$
$$+u^A{}_B\Big(2\partial_M\mathcal{F}_C\mathcal{H}^{BC}E_A{}^M + \frac{1}{2}\mathcal{F}_A{}^{CD}\mathcal{F}_{CDG}\mathcal{H}^{BG} + \mathcal{F}_{ACD}\mathcal{F}^{BC}{}_G\mathcal{H}^{DG} + \frac{1}{2}\mathcal{F}^B{}_{AC}\mathcal{F}_D\mathcal{H}^{CD}$$
$$-\frac{1}{2}\partial_M\mathcal{F}^B{}_{AC}\mathcal{H}^{CD}E_D{}^M + \frac{1}{2}\mathcal{F}_{ACD}\mathcal{F}^C\mathcal{H}^{BD} + \frac{1}{2}\partial_D\mathcal{F}_{AE}{}^D\mathcal{H}^{BE} + \frac{1}{2}\mathcal{F}^{BC}{}_D\mathcal{F}^D\mathcal{H}_{CA}$$
$$+\frac{1}{2}\partial_C\mathcal{F}^{BCD}\mathcal{H}_{DA} - \frac{1}{2}\mathcal{F}_{ACD}\mathcal{F}_{EFG}\mathcal{H}^{BE}\mathcal{H}^{CF}\mathcal{H}^{DG} + \frac{1}{2}\mathcal{F}_{CD}{}^E\mathcal{F}_F\mathcal{H}^{BC}\mathcal{H}^{DF}\mathcal{H}_{AE}$$
$$-\frac{1}{2}\partial_G\mathcal{F}_{CE}{}^D\mathcal{H}^{BC}\mathcal{H}^{EG}H_{DA} - 2\partial_A\mathcal{F}^B - \mathcal{F}_{ACD}\mathcal{F}^{BCD} - \mathcal{F}^B{}_{AC}\mathcal{F}^C - \partial_C\mathcal{F}^{BC}{}_A\Big)$$

$$\mathcal{F}_{abc} = -H_{abc}, \mathcal{F}_a = 2e_a^m\nabla_m\phi + f_a,$$
$$\mathcal{F}_{ab}^c = f_{ab}^c,$$
$$f_{ab}^c = -2e^m{}_a e^n{}_b \partial_{[m}e_{n]}^c, f_a = f_{ab}^b.$$

$$\delta d: R - \frac{1}{12}H^2 + 4\nabla^m\nabla_m\phi - 4(\nabla\phi)^2 = 0,$$
$$\delta e_a^m: R_{mn} - \frac{1}{4}H_{mkl}H_n^{kl} + \nabla_m\nabla_n\phi + \nabla_n\nabla_m\phi = 0,$$
$$\delta b_{mn}: \frac{1}{2}\nabla_k H^{kmn} - H^{kmn}\nabla_k\phi = 0,$$

$$E'_M{}^A = O_M{}^N E_N{}^A$$

$$O_K^M = \exp(\beta^{mn}T_{mn}) = \begin{bmatrix} \delta^m{}_k & 0 \\ \beta^{mk} & \delta_m{}^k \end{bmatrix}$$

$$[T_{MN}, T_{KL}] = 2\eta_{K[M}T_{N]L} - 2\eta_{L[M}T_{N]K}$$

$$\begin{cases} f_{j_1 j_2}{}^{[i_1} r^{i_2|j_1|} r^{i_3]j_2} = 0, \\ f_{i_1 i_2}{}^j r^{i_1 i_2} = 0, \end{cases}$$

$$\delta_I\mathcal{F}_{ABC} = 0, \delta_I\mathcal{F}_A = 2E_A^M \begin{bmatrix} 0 \\ I^m \end{bmatrix}$$

$$\mathcal{F}'_{ABC} = \mathcal{F}_{ABC}$$
$$\mathcal{F}'_A - X'_A = \mathcal{F}_A$$

$$
\begin{aligned}
0 &= \partial_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}^{E}_{[AB}\mathcal{F}_{CD]E} \\
0 &= 2\partial_{[A}\mathcal{F}_{B]} + \partial^{C}\mathcal{F}_{CAB} - \mathcal{F}^{C}\mathcal{F}_{CAB} \\
0 &= \partial^{A}\mathcal{F}_{A} - \frac{1}{2}\mathcal{F}^{A}\mathcal{F}_{A} + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC}
\end{aligned}
$$

$$
\begin{aligned}
\partial'_A &= E'^{m}_{A}\partial_m = E_A^N O_N^m \partial_m \\
&= E_A{}^{m}\partial_m + E_{An}\beta^{mn}\partial_m = E_A{}^{M}\partial_M = \partial_A
\end{aligned}
$$

$$
\begin{aligned}
0 &= 2\partial_{[A}\mathcal{F}_{B]} + \partial^{C}\mathcal{F}_{CAB} - \mathcal{F}^{C}\mathcal{F}_{CAB} = 2\partial'_{[A}\mathcal{F}_{B]} + \partial'^{C}\mathcal{F}_{CAB} - \mathcal{F}^{C}\mathcal{F}_{CAB} \\
&= 2\partial'_{[A}\left(\mathcal{F}'_{B]} - X_{B]}\right) + \partial'^{C}\mathcal{F}'_{CAB} - (\mathcal{F}'^{C} - X^{C})\mathcal{F}'_{CAB} \\
&= 2\partial'_{[A}\mathcal{F}'_{B]} - 2\partial'_{[A}X_{B]} + \partial'^{C}\mathcal{F}'_{CAB} - \mathcal{F}'^{C}\mathcal{F}'_{CAB} + X^{C}\mathcal{F}'_{CAB} \\
&= -2\partial'_{[A}X_{B]} + X^{C}\mathcal{F}'_{CAB}
\end{aligned}
$$

$$
\begin{aligned}
0 &= 2\partial'_{[A}X_{B]} - X^{C}\mathcal{F}'_{CAB} \\
&= 2E'_{[A}{}^{M}\partial'_{M}\left(E'_{B]}{}^{N}X_N\right) - X^{M}E'_{M}{}^{C}\left(2\partial'_{[A}E'_{B]}{}^{N}E'_{CN} + \partial'_{C}E'_{A}{}^{N}E'_{BN}\right) \\
&= 2E'_{[A}{}^{M}E'_{B]}{}^{N}\partial_M X_N + 2E'_{[A}{}^{M}\partial_M E'_{B]}{}^{N}X_N - 2X_N E'_{[A}{}^{M}\partial_M E'_{B]}{}^{N} - X^{M}\partial_M E'_{A}{}^{N}E'_{BN} \\
&= E'_{A}{}^{M}E'_{B}{}^{N}\partial_M X_N - E'_{B}{}^{N}E'_{A}{}^{M}\partial_N X_M - X^{M}\partial_M E'_{A}{}^{N}E'_{BN} \\
&= -E'_{BN}(X^{M}\partial_M E'_{A}{}^{N} - E'_{A}{}^{M}\partial_M X^{N} + E'_{A}{}^{M}\partial^{N}X_M) = -E'_{BM}\mathcal{L}_X E'_{A}{}^{M}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \partial'^{A}\mathcal{F}_A - \frac{1}{2}\mathcal{F}^{A}\mathcal{F}_A + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} \\
&= \partial'^{A}(\mathcal{F}'_A - X_A) - \frac{1}{2}(\mathcal{F}'^{A} - X^{A})(\mathcal{F}'_A - X_A) + \frac{1}{12}\mathcal{F}'^{ABC}\mathcal{F}'_{ABC} \\
&= -\partial'^{A}X_A + \mathcal{F}'_A X^{A} - \frac{1}{2}X_A X^{A} \\
&= -E'_{A}{}^{M}\partial_M(E'_{N}{}^{A}X^{N}) + (2\partial_M d' - \partial_N E'_{A}{}^{N}E'_{M}{}^{A})X^{M} - X_M X^{M} \\
&= 2X^{M}\partial_M d' - \partial_M X^{M} - X_M X^{M} = 2\mathcal{L}_X d' - X_M X^{M}
\end{aligned}
$$

$$
\begin{tikzcd}
E_M{}^{A} \arrow[r] \arrow[d, dashed, leftrightarrow] & E'_M{}^{A} = (O_M{}^{N}E_N{}^{A})|_{b-frame} \arrow[d, dashed, leftrightarrow] \\
\mathcal{F}_{ABC},\, \mathcal{F}_A \arrow[r] \arrow[d] & \mathcal{F}'_{ABC} = \mathcal{F}_{ABC},\, \mathcal{F}'_A = \mathcal{F}_A + X_A \arrow[d] \\
\mathrm{EoMs}(\mathcal{F}_{ABC},\, \mathcal{F}_A) = 0 \arrow[d, dash] & \mathrm{EoMs}(\mathcal{F}'_{ABC},\, \mathcal{F}'_A - X_A) = 0 \arrow[d, dash] \\
\mathrm{EoMs}_{SUGRA}(g, b, \phi) = 0 & \mathrm{EoMs}_{I}(g', b', \phi') = 0
\end{tikzcd}
$$

$$\nabla_M d \longrightarrow \nabla_M d + X_M.$$

$$\mathcal{L}_\Lambda \longrightarrow \mathcal{L}_\Lambda + \Lambda^{M}X_M$$

$$\{g_{\mu\nu}, A_{\mu}^{MN}, M_{MN}, B_{\mu\nu M}\}$$

$$\epsilon^{MNKLP}\partial_{MN}\cdot\otimes\partial_{KL}\cdot = 0$$

$$\delta_{\Lambda}V^{M} = \mathcal{L}_{\Lambda}V^{M} = \frac{1}{2}\Lambda^{KL}\partial_{KL}V^{M} - V^{L}\partial_{LK}\Lambda^{MK} + \left(\frac{1}{4}+\lambda\right)V^{M}\partial_{KL}\Lambda^{KL}$$

$$e^{-1}\mathcal{L} = \hat{R}[g_{(7)}] \mp \frac{1}{8}m_{MN}m_{KL}\mathcal{F}_{\mu\nu}{}^{MK}\mathcal{F}^{\mu\nu NL} + \frac{1}{48}g^{\mu\nu}\mathcal{D}_{\mu}m_{MN}\mathcal{D}_{\nu}m^{MN} + e^{-1}\mathcal{L}_{sc}$$
$$+\frac{1}{3\cdot(16)^{2}}m^{MN}\mathcal{F}_{\mu\nu\rho M}\mathcal{F}^{\mu\nu\rho}{}_{N} + e\mathcal{L}_{\text{top}}$$

$$\mathcal{D}_{\mu} = \partial_{\mu} - \mathcal{L}_{A_{\mu}}$$

$$e^{-1}\mathcal{L}_{sc} = \pm\left(\frac{1}{8}\partial_{MN}m_{PQ}\partial_{KL}m^{PQ}m^{MK}m^{NL} + \frac{1}{2}\partial_{MN}m_{PQ}\partial_{KL}m^{MP}m^{NK}m^{LQ}\right.$$
$$+\frac{1}{2}\partial_{MN}m^{LN}\partial_{KL}m^{MK} + \frac{1}{2}m^{MK}\partial_{MN}m^{NL}(g^{-1}\partial_{KL}g) + \frac{1}{8}m^{MK}m^{NL}(g^{-1}\partial_{MN}g)(g^{-1}\partial_{KL}g)$$
$$\left.+\frac{1}{8}m^{MK}m^{NL}\partial_{MN}g^{\mu\nu}\partial_{KL}g_{\mu\nu}\right)$$

$$m_{MN} = h^{\frac{1}{10}}\begin{bmatrix} h^{-\frac{1}{2}}h_{mn} & -V_{m} \\ -V_{n} & \pm h^{\frac{1}{2}}(1\pm V_{k}V^{k}) \end{bmatrix}, m^{MN} = h^{-\frac{1}{10}}\begin{bmatrix} h^{\frac{1}{2}}(h^{mn}\pm V^{m}V^{n}) & \pm V^{m} \\ \pm V_{n} & \pm h^{-\frac{1}{2}} \end{bmatrix}$$

$$E_{\hat{\mu}}^{\hat{a}} = \begin{bmatrix} h^{-\frac{1}{5}}e_{\mu}^{\alpha} & A_{\mu}{}^{m}h_{m}{}^{a} \\ 0 & h_{m}{}^{a} \end{bmatrix}$$

$$\begin{array}{ll} g_{\mu\nu} = g_{\mu\nu}(x^{\mu}, x^{m}), & m_{MN} = m_{MN}(x^{m}) \\ A_{\mu}{}^{MN} = 0, & B_{\mu\nu M} = 0 \end{array}$$

$$h_{\mu\nu} = e^{-2\phi}h^{\frac{1}{5}}\bar{h}_{\mu\nu}$$
$$m_{MN} = e^{-\phi}h^{\frac{1}{10}}M_{MN}$$

$$\mathcal{L} = \bar{e}M^{-1}\left(\mathcal{R}[\bar{h}_{(7)}] - \frac{1}{8}M^{KL}M^{MN}\partial_{KM}M_{PQ}\partial_{LN}M^{PQ} - \frac{1}{2}\partial_{NK}M^{MN}\partial_{ML}M^{KL}\right.$$
$$+\frac{1}{2}M^{KL}M^{MN}\partial_{MK}M^{PQ}\partial_{PL}M_{NQ} + M^{KL}M^{MN}\partial_{KP}M_{MN}\partial_{LQ}M^{PQ}$$
$$\left.-\frac{15}{24}M^{KL}M^{MN}M^{PQ}M^{RS}\partial_{MP}M_{KL}\partial_{NQ}M_{RS}\right),$$

$$\mathcal{R}_{\mu\nu}[\bar{h}_{(7)}] - \frac{1}{7}\bar{h}_{\mu\nu}\mathcal{R}[\bar{h}_{(7)}] = 0$$

$$E_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} e^{-\phi}\bar{e}_{\mu}{}^{a} & A_{\mu}{}^{m}h_{m}{}^{\alpha} \\ 0 & h_{m}{}^{\alpha} \end{pmatrix}$$

$$M_{MN} = e^{\phi}\begin{bmatrix} |h|^{-\frac{1}{2}}h_{mn} & -V_n \\ -V_m & \pm|h|^{\frac{1}{2}}(1 \pm V_kV^k) \end{bmatrix}, M^{MN} = e^{-\phi}\begin{bmatrix} |h|^{\frac{1}{2}}(h^{mn} \pm V^mV^n) & \pm V_n \\ \pm V_m & \pm|h|^{-\frac{1}{2}} \end{bmatrix},$$

$$\bar{e}^{-1}h^{-\frac{1}{2}}\mathcal{L} = e^{-5\phi}\mathcal{R}[\bar{h}_{(7)}] + e^{-7\phi}\left(\mathcal{R}[h_{(4)}] + 42h^{mn}\partial_m\phi\partial_n\phi \mp \frac{1}{2}\nabla_mV^m\nabla_nV^n\right).$$

$$\mathcal{L}_{E_{AB}}E_C{}^M = \mathcal{F}_{ABC}{}^DE_D{}^M$$

$$\mathcal{F}_{ABC}{}^D = \frac{3}{2}E_N{}^D\partial_{[AB}E^N{}_{C]} - E^M{}_C\partial_{MN}E^N{}_{[B}\delta^D{}_{A]} - \frac{1}{2}E^M{}_{[B|}\partial_{MN}E^N{}_{|A]}\delta^D{}_C.$$

$$E_M{}^A = e^{\frac{\phi}{2}}\begin{bmatrix} e^{-1/2}e_m{}^a & e^{1/2}V^a \\ 0 & e^{1/2} \end{bmatrix}, E^M{}_A = e^{-\frac{\phi}{2}}\begin{bmatrix} e^{1/2}e^m{}_a & 0 \\ -e^{1/2}V^m & e^{-1/2} \end{bmatrix}.$$

$$\mathcal{F}^D_{ABC} = \frac{3}{2}Z^D_{ABC} + 5\theta_{[AB}\delta^D_{C]} + \delta^D_{[A}Y_{B]C}$$

$$\theta_{AB} = \frac{1}{10}E^M_{[A}\partial_{MN}E^N_{B]} - \frac{1}{10}E^{-1}E^{MN}_{AB}{}_{AB}\partial_{MN}E,$$

$$Y_{AB} = -E^M_{(A}\partial_{MN}E^N_{B)},$$

$$Z_{ABC}{}^D = E^M_{[A}E^N_{B|}E^D_K\partial_{MN}E^K_{|C]} + \frac{1}{3}(2E^M_{[A|}\partial_{MN}E^N_{|B|} + E^M_{[A}E^N_{B|}E^{-1}\partial_{MN}E)\delta^D_{|C]}.$$

$$m\Delta\mathcal{L} = 3\delta_{M1}{}^{N1}{}_{M2}{}^{N2}{}_{M3}{}^{N3}{}_{M4}{}^{N4}\partial_{N1N2}E_A{}^{M1}\partial_{N3N4}E_B{}^{M2}E_C{}^{M3}E_D{}^{M4}m^{AC}m^{BD}$$

$$m\mathcal{L} = -\frac{700}{3}\theta_{AB}\theta_{CD}m^{AC}m^{BD} + Y_{AB}Y_{CD}m^{AC}m^{BD} - \frac{1}{2}Y_{AB}Y_{CD}m^{AB}m^{CD}$$
$$+\frac{9}{4}Z_{ABC}{}^DZ_{DEF}{}^Am^{BE}m^{CF} + \frac{3}{4}Z_{AA1B}{}^CZ_{DEF}{}^Gm_{CG}m^{AD}m^{A1E}m^{BF}$$

$$\delta E^M{}_AE^B{}_M \equiv u^B{}_A \equiv -\delta E^B{}_ME^M{}_A.$$

$$u^5{}_5 = \frac{1}{2}de_a{}^me^a{}_m - \frac{1}{2}\delta\phi$$

$$u^a{}_5 = -\delta V^me^a{}_m + V^me^a{}_me^b{}_n\delta e_b{}^n$$

$$u^a{}_b = \delta e_b{}^me^a{}_m - \frac{1}{2}\delta e_c{}^me^c{}_m\delta^a{}_b - \frac{1}{2}\delta\phi\delta^a{}_b$$

$$\delta\theta_{AB} = -\frac{1}{10}\partial_{C[A}u^C_{B]} + \frac{1}{10}\partial_{AB}u^C_C - 2u^C_{[A}\theta_{B]C},$$

$$dY_{AB} = \partial_{C(A}u^C_{B)} + 2u^C_{(A}Y_{B)C},$$

$$\delta Z^{EFD} = -\frac{1}{16}\partial_{AB}u^D_C\epsilon^{EFABC} - 2Z^{EF,[A}u^{D]}_A{}_A - 2u^{[F}{}_AZ^{E]A,D} + \frac{1}{48}\partial_{AB}u^C{}_C\epsilon^{DEFAB}$$
$$+\frac{1}{24}\partial_{AB}u^A{}_C\epsilon^{DEFBC}.$$

$$
\begin{aligned}
&\frac{14}{3}\partial_{AC}\theta_{DE}m^{BD}m^{CE}+\frac{14}{3}\delta_A{}^B\theta_{CD}\theta_{EF}m^{CE}m^{DF}+\frac{14}{3}\delta_A{}^B\partial_{CD}\theta_{EF}m^{CE}m^{DF}-4Y_{CD}\theta_{AE}m^{BC}m^{DE}\\
&+2\delta_A{}^BY_{CD}Y_{EF}m^{CE}m^{DF}+4Y_{AC}Y_{DE}m^{BD}m^{CE}-2\partial_{AC}Y_{DE}m^{BD}m^{CE}-\delta_A{}^BY_{CD}Y_{EF}m^{CD}m^{EF}\\
&+2Y_{CD}\theta_{AE}m^{BE}m^{CD}-64Z^{CDB}Z^{EFG}m_{AG}m_{CE}m_{DF}+128\delta_A{}^BZ^{CDE}Z^{FGH}m_{CF}m_{DG}m_{EH}\\
&+\partial_{AC}Y_{DE}m^{BC}m^{DE}+8\theta_{CD}Z^{EFG}\epsilon^{BCDHA1}m_{AG}m_{EH}m_{FA1}+4\partial_{CD}Z^{EFG}\epsilon^{BCDHA1}m_{AG}m_{EH}m_{FA1}\\
&+64Z^{BCD}Z^{EFG}m_{AE}m_{CD}m_{FG}-128Z^{BcD}Z^{EFG}m_{AE}m_{CF}m_{DG}+128\delta_A{}^BZ^{CDE}Z^{FGH}m_{CE}m_{DF}m_{GH}\\
&-2Y_{AC}Y_{DE}m^{BC}m^{DE}-8\theta_{CD}Z^{EFG}\epsilon^{BCDHA1}m_{AH}m_{EG}m_{FA1}-64Z^{CDB}Z^{EFG}m_{AC}m_{DE}m_{FG}\\
&-4\partial_{CD}Z^{EFG}\epsilon^{BCDHA1}m_{AH}m_{EG}m_{FA1}-64Z^{BCD}Z^{EFG}m_{AD}m_{CE}m_{FG}=0
\end{aligned}
$$

$$
\begin{aligned}
&12\nabla^m\nabla_m\phi-42\nabla^m\phi\nabla_m\phi-\frac{1}{2}\nabla_mV^m\nabla_nV^n+\left(2e^{am}\partial_mf_a-f^af_a-\frac{1}{4}f^{ab}{}_cf_{ab}{}^c-\frac{1}{2}f^a{}_b{}^cf_{ac}{}^b\right)=0,\\
&e_{bm}e^{cn}\partial_nf^a{}_c{}^b+e^b{}_me^{cn}\partial_nf_{bc}{}^a-e_{bm}f^{acb}f_c-2\partial_mf^a-e^b{}_mf^{ac}{}_df_{cb}{}^d-\frac{1}{2}e_{bm}f^{cda}f_{cd}{}^b+e^b{}_mf^c{}_b{}^af_c\\
&-e^b{}_mf^a{}_b{}^cf_c+e_b{}^ne^c{}_m\partial_nf^a{}_c{}^b+e^b{}_mf^a{}_c{}^df_{bd}{}^c+14e^{an}\nabla_m\phi\nabla_n\phi-14e^{an}\nabla_m\nabla_n\phi\\
&+e^a{}_m\left(\nabla_mV^m\nabla_nV^n+V^n\nabla_n\nabla_kV^k-2\nabla^n\phi\nabla_n\phi+14\nabla^n\nabla_n\phi-7V^n\nabla_n\phi\nabla_kV^k\right)=0,\\
&7\nabla_m\phi\nabla_nV^n-\nabla_m\nabla_nV^n=0,
\end{aligned}
$$

$$
f_{ab}^c=-2e_a^me_b^n\partial_{[m}e_{n]}^c,f_a=f_{ab}^b
$$

$$
\begin{aligned}
\delta\phi:&\quad \frac{5}{7}e^{2\phi}\mathcal{R}[\bar{g}_{(7)}]+\mathcal{R}[h_{(4)}]+12\nabla_m\nabla_n\phi h^{mn}-42\nabla_m\phi\nabla_n\phi h^{mn}+\frac{1}{2}(\nabla V)^2=0,\\
\delta V^m:&\quad \partial_m(\nabla V)-7(\nabla V)\partial_m\phi=0,\\
\delta h^{mn}:&\quad \mathcal{R}_{mn}[h_{(4)}]-7\partial_m\phi\partial_n\phi+7\nabla_m\nabla_n\phi\\
&\quad +h_{mn}\left(-\frac{1}{2}e^{2\phi}\mathcal{R}[\bar{g}_{(7)}]-\frac{1}{2}\mathcal{R}[h_{(4)}]+28\partial_k\phi\partial_l\phi h^{kl}-7\nabla_k\nabla_l\phi h^{kl}+\frac{1}{4}(\nabla V)^2\right)=0,
\end{aligned}
$$

$$
E_A'^M=O^M{}_NE_A{}^N,O_M{}^N=\begin{bmatrix}\delta_m{}^n & 0\\ \frac{1}{3!}\epsilon_{mpqr}\Omega^{pqr} & 1\end{bmatrix},
$$

$$
\begin{aligned}
\delta_\rho\mathcal{F}_{ABC}{}^D=&E^m{}_AE^n{}_BE^k{}_CE_l{}^DJ^{lp}\epsilon_{mnkp}\\
&-\frac{1}{2}e_{(4)}^{-2}E_5{}^EE^{mnk}{}_{[ABC]}k_{i7p}k_{i2n}k_{i3m}k_{i4k}k_{i5}{}^p\left(6\rho^{[i2|i7j1}\rho^{|i3i4|j2}f_{j1j2}{}^{|i5]}+\rho^{j1j2[i2}\rho^{i3i4i5]}f_{j1j2}{}^{i7}\right),
\end{aligned}
$$

$$
J^{mn}=\frac{1}{4}k_{i_1}{}^mk_{i_4}{}^n\rho^{i_1i_2i_3}f_{i_2i_3}{}^{i_4}=S^{(mn)}+I^{[mn]}
$$

$$
\delta_\rho\mathcal{F}_{ABC}{}^D=X_{ABC}{}^D
$$

$$
X_{mnk}^l=\epsilon_{mnkp}J^{lp}
$$

$$
X_{ABC}'{}^D=X_{ABC}{}^D-E_A{}^ME_B{}^NE_C{}^KE_5{}^DW_lX_{MNK}{}^l
$$

$$
W_l\epsilon_{mnkp}J^{lp}=0
$$

$$\mathcal{F}^{D}_{ABC} = \mathcal{F}'_{ABC}{}^{D} - X^{D}_{ABC}$$

$$\delta_\Lambda \mathcal{F}_{ABC}{}^{D} = \frac{1}{2}\Lambda^{MN}\partial_{MN}\mathcal{F}_{ABC}{}^{D}$$

$$\delta_\Lambda E^{M}_{C} = \frac{1}{2}\Lambda^{AB}\partial_{AB}E^{M}_{C} - E_{C}{}^{L}\partial_{LK}\Lambda^{MK} + \frac{1}{4}E_{C}{}^{M}\partial_{KL}\Lambda^{KL}$$

$$\delta_\Lambda E^{M}_{C} = \mathcal{F}^{E}_{ABC}E^{M}_{E}\Lambda^{AB} - E^{M}_{A}\partial_{CB}\Lambda^{AB} + \frac{1}{4}E^{M}_{C}\partial_{AB}\Lambda^{AB}$$

$$\begin{aligned}\mathcal{Z}^{E}_{DF,ABC} =& \frac{1}{2}\partial_{AB}\mathcal{F}^{E}_{DFC} + \frac{1}{2}\partial_{BC}\mathcal{F}^{E}_{DFA} - \frac{1}{2}\delta^{E}_{A}\partial_{CG}\mathcal{F}^{G}_{DFB}\\ &-\frac{1}{4}\delta^{E}_{C}\partial_{BG}\mathcal{F}_{DFA}{}^{G} + \frac{1}{4}\delta^{E}_{C}\partial_{AG}\mathcal{F}_{DFB}{}^{G} + \frac{1}{2}\delta^{E}_{B}\partial_{CG}\mathcal{F}_{DFA}{}^{G}\\ &-\frac{1}{2}\partial_{AC}\mathcal{F}_{DFB}{}^{E} - \mathcal{F}_{BGC}{}^{E}\mathcal{F}_{DFA}{}^{G} + \mathcal{F}_{AGC}{}^{E}\mathcal{F}_{DFB}{}^{G}\\ &+\mathcal{F}_{ABG}{}^{E}\mathcal{F}_{DFC}{}^{G} - \mathcal{F}_{ABC}{}^{G}\mathcal{F}_{DFG}{}^{E} - \frac{1}{2}\partial_{DF}\mathcal{F}_{ABC}{}^{E} = 0\end{aligned}$$

$$\delta_\xi A_\mu = \xi^\nu\partial_\nu A_\mu + A_\nu\partial_\mu\xi^\nu = \xi^\nu F_{\nu\mu} + \partial_\mu(A_\nu\xi^\nu)$$

$$\delta_\xi F_{\mu\nu} = 2\partial_{[\mu}\delta_\xi A_{\nu]} = L_\xi F_{\mu\nu} - 3\xi^\rho\partial_{[\mu}F_{\nu\rho]}$$

$$\begin{aligned}0=& \frac{1}{2}\partial'_{CD}X^{F}_{ABE} - \partial'_{CE}X^{F}_{ABD} - \frac{1}{2}\partial'_{AB}X^{F}_{CDE} + \frac{1}{2}\delta^{F}_{E}\partial'_{CG}X^{G}_{ABD} - \delta^{F}_{C}\partial'_{EG}X^{G}_{ABD}\\ &-2X^{G}_{ABC}\mathcal{F}'^{F}_{DGE} + \mathcal{F}'_{ABE}{}^{G}X_{CDG}{}^{F} + X_{ABE}{}^{G}\mathcal{F}'^{F}_{CDG} - \mathcal{F}'_{ABG}{}^{F}X_{CDE}{}^{G}\\ &-X_{ABG}{}^{F}\mathcal{F}'_{CDE}{}^{G} + 2\mathcal{F}'_{ABC}{}^{G}X_{DEG}{}^{F} - X_{BGC}{}^{E}X_{DFA}{}^{G} + X_{AGC}{}^{E}X_{DFB}{}^{G} + X_{ABG}{}^{E}X_{DFC}{}^{G}\end{aligned}$$

$$\begin{aligned}\partial'_{AB} =& E'_{A}{}^{M}E'_{B}{}^{N}\partial_{MN} = E'_{A}{}^{K}E'_{B}{}^{L}O^{M}{}_{K}O^{N}{}_{L}\partial_{MN} = 2E'_{[A}{}^{K}E'_{B]}{}^{L}O^{5}{}_{K}O^{m}{}_{L}\partial_{5m}\\ &= 2E'_{[A}{}^{5}E'_{B]}{}^{l}O^{5}{}_{5}O^{m}{}_{l}\partial_{5m} + 2E'_{[A}{}^{k}E'_{B]}{}^{l}O^{5}{}_{k}O^{m}{}_{l}\partial_{5m}\\ &= E_{A}{}^{M}E_{B}{}^{N}\partial_{MN} + 2E'_{[A}{}^{k}E'_{B]}{}^{m}W_{k}\partial_{5m}\end{aligned}$$

$$W_{[m}\partial_{n]} = \epsilon_{mnkl}\epsilon^{klpq}W_p\partial_q = \epsilon_{mnkl}\Omega^{klq}\partial_q \simeq 0$$

$$-\frac{1}{16}E^{k}{}_{A}E^{l}{}_{B}E^{m}{}_{C}E^{n}{}_{D}E^{p}{}_{F}E_{q}{}^{E}J^{q[r}J^{st]}\left(\epsilon_{klmn}\epsilon_{prst} - \epsilon_{klmp}\epsilon_{nrst}\right) = 0$$

$$J^{m[n}J^{kl]} = 0$$

$$L_{e_a}J^{kl} + J^{nl}\partial_n\phi e^{k}_{a} = 0$$

$$[e_a, J]^{kl} + J^{nl}\partial_n\phi e^{k}_{a} = 0$$

$$[A,B]^{m_1\dots m_{p+q-1}} = pA^{n[m_1\dots m_{p-1}}\partial_n B^{m_p\dots m_{p+q-1}]} + q(-1)^{pq}B^{n[m_1\dots m_{q-1}}\partial_n A^{m_q\dots m_{p+q-1}}$$

$$\begin{aligned}P_a &= \partial_a, K_a = x^2\partial_a + 2x_a D\\ D &= -x^m\partial_m, M_{ab} = x_a\partial_b - x_b\partial_a\end{aligned}$$

$$\Omega = \frac{1}{8}\rho^{ab,cd}P_a \wedge P_b \wedge M_{cd} = \frac{1}{2}[(\alpha-\alpha')x_0 + (\beta-\beta')x_1 + (\gamma-\gamma')x_2]\partial_0 \wedge \partial_1 \wedge \partial_2$$

$$\alpha \quad = \rho^{01,02}, \quad \beta = \rho^{01,12}, \gamma = \rho^{02,12},$$

$$ds^2 = \frac{R^2}{4z^2} K^{-\frac{2}{3}} [-(dx^0)^2 + (dx^1)^2 + (dx^2)^2] + R^2 K^{\frac{1}{3}} \left[\frac{dz^2}{4z^2} + d\Omega_{(7)}^2\right]$$

$$F = -\frac{3R^3}{8z^4} K^{-2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dz$$

$$K = 1 + \frac{\rho_a x^a}{z^3}$$

$$\alpha = -\alpha', \beta = -\beta', \gamma = -\gamma'$$

$$\alpha^2 = \beta^2 + \gamma^2$$

$$J = \frac{1}{4}(\alpha \partial_2 \wedge \partial_1 + \beta \partial_2 \wedge \partial_0 + \gamma \partial_0 \wedge \partial_1)$$

$$\Omega = \frac{2}{R^3} D \wedge \left(\rho_a \epsilon^{abc} P_b \wedge P_c\right)$$

$$ds^2 = \frac{R^2}{4z^2} K^{-\frac{2}{3}} \left[-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + \left(1 + \frac{\rho_a x^a}{z^3}\right) dz^2 - \frac{1}{z^2} \rho_a dx^a dz\right] + R^2 K^{\frac{1}{3}} d\Omega_{(7)}^2$$

$$F = -\frac{3R^3}{8z^4} \left(1 + \frac{\rho^2}{12z^4}\right) K^{-2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dz$$

$$J^{ab} = -\frac{4}{3R^3} \epsilon^{abc} \rho_c$$

$$K = 1 + \frac{\rho_a x^a}{z^3} - \frac{\rho^2}{4z^4}$$

$$\Omega = \frac{4}{R^3} \rho_a \epsilon^{abc} D \wedge M_{bd} \wedge M_c^d = \frac{4}{R^3} \rho_a x^a \left(x^b x_b \partial_0 \wedge \partial_1 \wedge \partial_2 - \frac{1}{2} z \epsilon^{bcd} x_b \partial_c \wedge \partial_d \wedge \partial_z\right)$$

$$ds^2 = \frac{R^2}{4z^2} K^{-\frac{2}{3}} \left[dx_a dx^a + \frac{1}{z^2} \rho_a x^a x^b dx_b dz + \left(1 - \frac{x_a x^a \rho_b x^b}{z^3}\right) dz^2\right] + R^2 K^{\frac{1}{3}} d\Omega_{(7)}^2$$

$$F_{012z} = -\frac{3}{8} \frac{R^3}{z^4} K^{-2} \left(1 + \frac{1}{12} \frac{x_a x^a \rho_b \rho_c x^b x^c}{z^4}\right)$$

$$J^{ma} = \frac{32}{R^3} \rho_b \epsilon^{abc} x_c x^m$$

$$K = 1 + \frac{x_a x^a}{z^3} \rho_b x^b \left(1 - \frac{\rho_c x^c}{4z}\right)$$

$$J^{mn} \rightarrow J^{**}, J^{*a}, J^{a*}, J^{ab}$$

$$J^{*a} = \rho^{*\alpha_1 \alpha_2} f_{\alpha_1 \alpha_2}^{\alpha_3} k_*^* k_{\alpha_3}^a = I^a$$

$$f_{ab}^c = -2 e_a^m e_b^n \partial_{[m} e_{n]}^c, f_a = f_{ab}^b$$

$$\Gamma_{mn}{}^{k} = \frac{1}{2} h^{kl}(\partial_m h_{ln} + \partial_n h_{lm} - \partial_l h_{mn})$$
$$= -\frac{1}{2} e^a{}_m e^b{}_n e^c{}_l f_{ac}{}^d h_{bd} h^{kl} - \frac{1}{2} f_{ab}{}^c e_c{}^k e^a{}_m e^b{}_n - \frac{1}{2} e^a{}_m e^b{}_n e^c{}_l f_{bc}{}^d h_{ad} h^{kl}$$

$$\begin{aligned}
R_{nl} =& \partial_k \Gamma_{nl}{}^k - \partial_n \Gamma_{kl}{}^k + \Gamma_{nl}{}^p \Gamma_{kp}{}^k - \Gamma_{kl}{}^p \Gamma_{np}{}^k \\
=& -\frac{1}{2} e^a{}_l e^b{}_n f_{bc}{}^d f_a{}^c{}_d - \frac{1}{2}\partial_m f_a{}^b{}_c e_b{}^m e^c{}_l e^a{}_n + \frac{1}{2} e_{al} e^b{}_n f^c f_{bc}{}^a - \frac{1}{2} e^a{}_l e^b{}_n f_{ac}{}^d f_{bd}{}^c \\
& -\frac{1}{2}\partial_m f_{ab}{}^c e_c{}^m e^a{}_l e^b{}_n - \frac{1}{2}\partial_m f_a{}^b{}_c e_b{}^m e^a{}_l e^c{}_n + \frac{1}{2} e_{an} e^b{}_l f^c f_{bc}{}^a + \partial_n f_a e^a{}_l \\
& +\frac{1}{2} e^a{}_l e^b{}_n f_c f_{ab}{}^c + \frac{1}{4} e_{al} e_{bn} f_{cd}{}^b f^{cda},
\end{aligned}$$

$$\begin{aligned}
R =& h_{nl}\left(\partial_k \Gamma^k_{nl} - \partial_n \Gamma^k_{kl} + \Gamma^p_{nl}\Gamma^k_{kp} - \Gamma^p_{kl}\Gamma^k_{np}\right) \\
=& -\frac{1}{2} f^c_{ab} f^a_{cd} h^{bd} - \frac{1}{4} f^c_{ab} f^g_{df} h_{cg} h^{ad} h^{bf} + \partial_m f_a e^m_b h^{ab} - f_a f_b h^{ab} + \partial_m f_a e^a_n h^{mn}
\end{aligned}$$

$$\begin{aligned}
Z^l_{[12,34,5]} =& \partial_p J^{pl} + J^{pq}\partial_p e_q{}^a e_a{}^l - J^{pl}\partial_p \phi + J^{pl}\partial_p e_q{}^a e_a{}^q \\
Z_{[mn],[kl]}{}^p =& \delta_m{}^p\left(-\epsilon_{nklq}\partial_r J^{rq} + 2J^{qr}\epsilon_{nrkl}\partial_q \phi - J^{qr}\epsilon_{nkls}\partial_q e_r{}^a e_a{}^s - J^{qr}\epsilon_{nrkl}\partial_q e_s{}^a e_a{}^s\right) \\
& -\epsilon_{mklq}\partial_n J^{pq} - J^{pq}\epsilon_{mklr}\partial_n e_q{}^a e_a{}^r + J^{pq}\epsilon_{mqkl}\partial_n \phi + J^{pq}\epsilon_{mnqr}\partial_k e_l{}^a e_a{}^r + J^{pq}\epsilon_{mnqk}\partial_l e_r{}^a e_a{}^r \\
& -J^{pq}\epsilon_{mnqk}\partial_r e_l{}^a e_a{}^r - J^{pq}\epsilon_{mnqk}\partial_l \phi - J^{pq}\epsilon_{qklr}\partial_m e_n{}^a e_a{}^r - J^{qr}\epsilon_{mrkl}\partial_q e_n{}^a e_a{}^p \\
Z_{m,l,[nk]}{}^p =& \delta_l{}^p\left(\frac{1}{2}\epsilon_{mnkq}\partial_r J^{rq} - \frac{3}{2}J^{qr}\epsilon_{mrnk}\partial_q \phi + \frac{1}{2}J^{qr}\epsilon_{mnks}\partial_q e_r{}^a e_a{}^s + \frac{1}{2}J^{qr}\epsilon_{mrnk}\partial_q e_s{}^a e_a{}^s\right) \\
& -\delta_m{}^p J^{qr}\epsilon_{rlnk}\partial_q \phi + \epsilon_{lnkq}\partial_m J^{pq} - \epsilon_{mnkq}\partial_l J^{pq} + J^{pq}\epsilon_{lnkr}\partial_m e_q{}^a e_a{}^r + J^{pq}\epsilon_{qlnk}\partial_m \phi \\
& -J^{pq}\epsilon_{mnkr}\partial_l e_q{}^a e_a{}^r + J^{pq}\epsilon_{mqnk}\partial_l \phi - 2J^{pq}\epsilon_{mqlr}\partial_n e_k{}^a e_a{}^r - 2J^{pq}\epsilon_{mqln}\partial_k e_r{}^a e_a{}^r \\
& +2J^{pq}\epsilon_{mqln}\partial_r e_k{}^a e_a{}^r + 2J^{pq}\epsilon_{mqln}\partial_k \phi - J^{pq}\epsilon_{qnkr}\partial_m e_l{}^a e_a{}^r + J^{pq}\epsilon_{qnkr}\partial_l e_m{}^a e_a{}^r \\
& -J^{qr}\epsilon_{mrnk}\partial_q e_l{}^a e_a{}^p - J^{qr}\epsilon_{rlnk}\partial_q e_m{}^a e_a{}^p; \\
Z_{[mn],k,l}{}^p =& \delta_k{}^p J^{qr}\epsilon_{mnrl}\partial_q \phi - \epsilon_{mnlq}\partial_k J^{pq} - 2J^{pq}\epsilon_{mqlr}\partial_n e_k{}^a e_a{}^r + 2J^{pq}\epsilon_{mqlr}\partial_k e_n{}^a e_a{}^r \\
& -J^{pq}\epsilon_{mnql}\partial_k e_r{}^a e_a{}^r + J^{pq}\epsilon_{mnql}\partial_r e_k{}^a e_a{}^r + J^{pq}\epsilon_{mnql}\partial_k \phi - 2J^{pq}\epsilon_{mqlk}\partial_n \phi \\
& -J^{pq}\epsilon_{mnqr}\partial_l e_k{}^a e_a{}^r + J^{pq}\epsilon_{mnqr}\partial_k e_l{}^a e_a{}^r - J^{pq}\epsilon_{mnqk}\partial_l \phi - J^{pq}\epsilon_{mnlr}\partial_k e_q{}^a e_a{}^r \\
& +J^{qr}\epsilon_{mnrl}\partial_q e_k{}^a e_a{}^p.
\end{aligned}$$

$$Z^p_{[mn],[kl]^p} \in \mathbf{6}\times\mathbf{6}\times\overline{\mathbf{4}} \to (\mathbf{1}+\mathbf{15}+\mathbf{20})')\times\overline{\mathbf{4}}$$

$$Z^p_{[mn],[kl]^p} \in (\mathbf{1}+\mathbf{15})\times\overline{\mathbf{4}} = \overline{\mathbf{4}}+\overline{\mathbf{4}}+\overline{\mathbf{20}}+\overline{\mathbf{36}}$$

$$\begin{aligned}
\overline{\mathbf{4}}:&\quad \epsilon^{mnkl} Z^p_{mnkl}, \\
\overline{\mathbf{4}}:&\quad \epsilon^{mnkl} Z^p_{mnkp} - \frac{1}{4}\epsilon^{mnkp} Z^l_{mnkp}, \\
\overline{\mathbf{20}}:&\quad \epsilon^{mnk[p} Z_{mnkl}{}^{q]} - tr, \\
\overline{\mathbf{36}}:&\quad \epsilon^{mnk(p} Z_{mnkl}{}^{q)} - tr.
\end{aligned}$$

$\overline{\mathbf{4}}$: $\partial_n I^{nm} - 2I^{nm}\partial_n\phi + I^{nm}e^{-1}\partial_n e = 0$

$\overline{\mathbf{4}}$: $-4U_m\partial_n J^{nm} + 12U_m J^{nm}d_n - 8U_m J^{nk}\partial_n e_k{}^a e_a{}^m - 4U_m J^{nm}\partial_n e_k{}^a e_a{}^k - 2U_m\partial_n J^{mn}$
$+3U_m J^{mn}d_n + 2U_m J^{nk}\partial_k e_n{}^a e_a{}^m - 2U_m J^{mn}\partial_n e_k{}^a e_a{}^k$

$\overline{\mathbf{20}}$: $A_{mn}U^k\left(-J^{ml}\partial_l e_k{}^a e_a{}^n + J^{lm}\partial_l e_k{}^a e_a{}^n - \partial_k J^{mn}\right)$
$+A_{mn}U^m\left(J^{kl}\partial_k e_l{}^a e_a{}^n - J^{kl}\partial_l e_k{}^a e_a{}^n\right) + 6A_{mn}U^n J^{km}d_k = 0$

$\overline{\mathbf{36}}$: $S_{mn}U^k\left(2J^{ml}\partial_l e_k{}^a e_a{}^n + 2\partial_k J^{mn} + 2J^{lm}\partial_l e_k{}^a e_a{}^n\right) + S_{mn}U^n\left(-J^{mk}d_k + 2J^{km}d_k\right); = 0$

$$Z_{[mn],k,l} \in \mathbf{6}\times\mathbf{4}\times\mathbf{4} \to \mathbf{6}\times(\mathbf{6}+\mathbf{10}) = \left(\mathbf{1}+\mathbf{15}+\mathbf{20})'\right) + (\mathbf{15}+\mathbf{45}).$$

$$\partial_m J^{kl} + J^{kn}\partial_n e_m{}^a e_a{}^l + J^{nl}\partial_n e_m{}^a e_a{}^k + J^{kn}\delta_m{}^l\partial_n\phi + J^{nl}\delta_m{}^k\partial_n\phi = 0$$

$$J^{mn}\partial_n\phi = 0$$

$$J^{mn}\partial_n\phi = 0,$$

$$\partial_p J^{pl} + J^{pq}\partial_p e_q{}^a e_a{}^l - J^{pl}\partial_p\phi + J^{pl}\partial_p e_q{}^a e_a{}^q = 0,$$

$$\partial_n I^{nm} - 2I^{nm}\partial_n\phi + I^{nm}e^{-1}\partial_n e = 0,$$

$$-2U_m\partial_n J^{nm} + 6U_m J^{nm}d_n - 4U_m J^{nk}\partial_n e_k{}^a e_a{}^m - 2U_m J^{nm}\partial_n e_k{}^a e_a{}^k - U_m\partial_n J^{mn}$$

$$+U_m J^{nk}\partial_k e_n{}^a e_a{}^m - U_m J^{mn}\partial_n e_k{}^a e_a{}^k = 0,$$

$$\partial_m J^{kl} + J^{kn}\partial_n e_m{}^a e_a{}^l + J^{nl}\partial_n e_m{}^a e_a{}^k + J^{nl}\delta_m{}^k\partial_n\phi = 0,$$

$$A_{mn}U^k\left(-J^{ml}\partial_l e_k{}^a e_a{}^n + J^{lm}\partial_l e_k{}^a e_a{}^n - \partial_k J^{mn}\right)$$

$$+A_{mn}U^m\left(J^{kl}\partial_k e_l{}^a e_a{}^n - J^{kl}\partial_l e_k{}^a e_a{}^n\right) + 6A_{mn}U^n J^{km}d_k = 0,$$

$$S_{mn}U^k\left(J^{ml}\partial_l e_k{}^a e_a{}^n + \partial_k J^{mn} + J^{lm}\partial_l e_k{}^a e_a{}^n\right) + S_{mn}U^n J^{km}d_k = 0.$$

$$-U_k J^{mn}\partial_m e_n{}^a e_a{}^k + 5U_m J^{nm}d_n + U_k J^{mn}\partial_n e_m{}^a e_a{}^k = 0.$$

$$\begin{aligned}
0 &= \partial_m J^{kl} + J^{kn}\partial_n e^a_m e^l_a + J^{nl}\partial_n e^a_m e^k_a + J^{nl}\delta^k_m\partial_n\phi,\\
0 &= I^{mn}\partial_m e^a_n e^k_a - \frac{5}{2}J^{lk}\partial_l\phi \longrightarrow \nabla_m\left(e^{-\phi}I^{mn}\right) = 0,\\
0 &= J^{mn}\partial_n\phi.
\end{aligned}$$

$$\begin{aligned}
\nabla_{[m}Z_{n]} - \frac{1}{3}J^{kl}F_{mnkl} &= 0,\\
\nabla_k\left(e^{-\phi}J^{k[l}V^{p]}\right) &= 0,\\
\nabla_k\left(J^{(pl)}V^k\right) - \nabla_k\left(V^{(p}J^{l)k}\right) &= 0.
\end{aligned}$$

$$\left[\nabla_\alpha, \nabla_\beta\right] = T^\gamma_{\alpha\beta}\nabla_\gamma + T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} + \frac{1}{2}R_{\alpha\beta\underline{c}}\mathcal{M}_{\underline{\underline{d}}}{}^{\underline{c}}.$$

$$\left(\gamma_{\underline{d}}\right)^{\alpha\beta}T_{\alpha\beta\underline{c}} \propto \delta^{\underline{\underline{c}}}_{\underline{\underline{d}}},$$

$$T_{\alpha\beta\underline{c}} \propto (\gamma^{\underline{c}})_{\alpha\beta}.$$

$$\mathrm{E}_{\alpha} = \Psi^{1/2}\left[\exp\left(\frac{1}{2}\mathcal{A}^{\underline{a}}\gamma_{\underline{ab}}\right)\right]_{\alpha}{}^{\beta}\left[\mathcal{N}_{\beta}^{\gamma}\right]\left[\mathrm{D}_{\gamma} + \widehat{\mathrm{H}}_{\gamma}^{\underline{b}}\partial_{\underline{b}}\right]$$

$$\mathcal{N}_{\alpha}{}^{\beta} \equiv \left[\mathbf{I} + \mathcal{A}^{\underline{a}}\gamma_{\underline{a}} + \frac{1}{3!}\mathcal{A}^{[3]}\gamma_{[3]} + \frac{1}{4!}\mathcal{A}^{[4]}\gamma_{[4]} + \frac{1}{5!}\mathcal{A}^{[5]}\gamma_{[5]}\right]_{\alpha}{}^{\beta}$$

$$\widehat{\mathrm{H}}_{\beta\underline{b}} = \mathrm{H}_{\beta\underline{b}} - \frac{1}{\mathrm{D}}\left(\gamma^{\underline{b}}\gamma_{\underline{d}}\right)_{\beta}{}^{\delta}\mathrm{H}_{\delta}{}^{\underline{d}},$$

$$\mathrm{H}_{\beta\underline{b}} \to \mathrm{H}_{\beta\underline{b}} + \left(\gamma^{\underline{b}}\right)_{\beta}{}^{\alpha}\Lambda_{\alpha}$$

$$\nabla_{\alpha} = \mathrm{E}_{\alpha} + \frac{1}{2}\omega_{\alpha\underline{d}^{\underline{e}}}\mathcal{M}_{\underline{\underline{e}}}{}^{\underline{\underline{d}}}$$

$$\nabla_{\underline{a}} = \mathrm{E}_{\underline{a}} + \frac{1}{2}\omega_{\underline{a}\underline{\underline{d}}^{\underline{\underline{e}}}}\mathcal{M}_{\underline{\underline{e}}}{}^{\underline{\underline{d}}}$$

$$\left[\nabla_{\alpha},\nabla_{\beta}\right\} = T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\beta}^{\gamma}\nabla_{\gamma} + \frac{1}{2}R_{\alpha\beta\underline{\underline{d}}^{\underline{e}}}\mathcal{M}_{\underline{\underline{e}}}{}^{\underline{d}}$$

$$\left[\nabla_{\alpha},\nabla_{\underline{\underline{b}}}\right\} = T_{\alpha\underline{\underline{b}}}{}^{\underline{c}}\nabla_{\underline{\underline{c}}} + T_{\alpha\underline{\underline{b}}}^{\gamma}\nabla_{\gamma} + \frac{1}{2}R_{\alpha\underline{\underline{b}}}{}^{\underline{\underline{e}}}\mathcal{M}_{\underline{\underline{e}}}{}^{\underline{d}}$$

$$\left[\nabla_{\underline{a}},\nabla_{\underline{\underline{b}}}\right\} = T_{\underline{a}\underline{\underline{b}}}{}^{\underline{c}}\nabla_{\underline{\underline{c}}} + T_{\underline{a}\underline{\underline{b}}}^{\gamma}\nabla_{\gamma} + \frac{1}{2}R_{\underline{a}\underline{b}\underline{\underline{d}}^{\underline{e}}}\mathcal{M}_{\underline{\underline{e}}}{}^{\underline{d}}$$

$$i\frac{1}{32}\left(\gamma_{\underline{a}}\right)^{\alpha\beta}T_{\alpha\beta\underline{\underline{b}}} = \delta_{\underline{a}}^{\underline{b}} \qquad \left(\gamma_{\underline{a}}\right)^{\alpha\beta}T_{\alpha\beta}{}^{\gamma} = 0$$

$$T_{\alpha[\underline{de}]} - \frac{2}{55}\left(\gamma_{\underline{de}}\right)_{\alpha}^{\gamma}T_{\gamma\underline{\underline{b}}} = 0 \qquad \left(\gamma_{\underline{a}}\right)^{\alpha\beta}R_{\alpha\beta\underline{\underline{de}}} = 0$$

$$\left(\gamma_{\underline{abcde}}\right)^{\alpha\beta}T_{\alpha\beta\underline{\underline{e}}} = 0 \qquad \left(\gamma_{[\underline{abcde}]}\right)^{\alpha\beta}T_{\alpha\beta|\underline{f}]} = 0$$

$$\left(\gamma_{\underline{ab}}\right)^{\alpha\beta}T_{\alpha\beta\underline{\underline{b}}} = 0 \qquad \left(\gamma_{[\underline{ab}]}\right)^{\alpha\beta}T_{\alpha\beta|\underline{c}]} = 0$$

$$i\frac{1}{32}\left(\gamma_{\underline{a}}\right)^{\alpha\beta}T_{\alpha\beta^{\underline{b}}} = \delta_{\underline{a}}^{\underline{b}} \qquad \left(\gamma_{\underline{a}}\right)^{\alpha\beta}T_{\alpha\beta^{\underline{b}}} = 0$$

$$T_{\alpha[\underline{de}]} - \frac{2}{55}\left(\gamma_{\underline{de}}\right)_{\alpha}^{\gamma}T_{\gamma\underline{b}} = 0 \qquad \left(\gamma_{\underline{a}}\right)^{\alpha\beta}R_{\alpha\beta^{\underline{de}}} = 0$$

$$\left(\gamma_{\underline{abcde}}\right)^{\alpha\beta}T_{\alpha\beta^{\underline{e}}} = 0 \qquad \left(\gamma_{[\underline{abcde}]}\right)^{\alpha\beta}T_{\alpha\beta|\underline{f}]} = 0$$

$$\nabla_{\alpha} = \mathrm{D}_{\alpha} + \frac{1}{2}\Psi\mathrm{D}_{\alpha} + \frac{1}{10}\left(\mathrm{D}_{\beta}\Psi\right)\left(\gamma^{\underline{de}}\right)_{\alpha}^{\beta}\mathcal{M}_{\underline{de}}$$

$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} + i\frac{1}{4}\left(\gamma_{\underline{a}}\right)^{\alpha\beta}\left(\mathrm{D}_{\alpha}\Psi\right)\mathrm{D}_{\beta} + \frac{1}{5}\left(\partial_{\underline{c}}\Psi\right)\mathcal{M}_{\underline{a}}^{\underline{c}}$$

$$+i\frac{1}{160}\left(\gamma_{\underline{a}}^{\underline{de}}\right)^{\alpha\beta}\left(\mathrm{D}_{\alpha}\mathrm{D}_{\beta}\Psi\right)\mathcal{M}_{\underline{de}}$$

$$\delta_{S}\nabla_{\alpha} = \frac{1}{2}L\nabla_{\alpha} + \frac{1}{10}\left(\nabla_{\beta}L\right)\left(\gamma^{\underline{de}}\right)_{\alpha}^{\beta}\mathcal{M}_{\underline{de}}$$

$$\delta_{S}\nabla_{\underline{a}} = L\nabla_{\underline{a}} + i\frac{1}{4}\left(\gamma_{\underline{a}}\right)^{\alpha\beta}\left(\nabla_{\alpha}L\right)\nabla_{\beta} + \frac{1}{5}\left(\nabla_{\underline{c}}L\right)\mathcal{M}_{\underline{a}}^{\underline{c}}$$

$$+i\frac{1}{160}\left(\gamma_{\underline{a}}^{\underline{de}}\right)^{\alpha\beta}\left(\nabla_{\alpha}\nabla_{\beta}L\right)\mathcal{M}_{\underline{de}}$$

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0$$

$$\delta_S T_{\alpha\beta}{}^{\gamma} = \frac{1}{2} L T_{\alpha\beta}{}^{\gamma} - i\frac{1}{4} T_{\alpha\beta}{}^{\underline{c}} (\gamma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L) + \frac{1}{2} (\nabla_{(\alpha} L) \delta_{\beta)}{}^{\gamma}$$

$$+ \frac{1}{20} (\nabla_\delta L) (\gamma^{[2]})^{\delta}_{(\alpha} (\gamma_{[2]})^{\gamma}_{\beta)}$$

$$\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} = \frac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{c}} - i\frac{1}{4} (\gamma_{\underline{b}})^{\gamma\delta} (\nabla_\gamma L) T_{\alpha\delta}{}^{\underline{c}} + (\nabla_\alpha L) \delta_{\underline{b}}{}^{\underline{c}}$$

$$+ \frac{1}{5} (\nabla_\gamma L) (\gamma_{\underline{b}})_{\alpha}{}^{\gamma}$$

$$\delta_S T_{\alpha\underline{b}}{}^{\gamma} = L T_{\alpha\underline{b}}{}^{\gamma} - i\frac{1}{4} \left(\gamma_{\underline{b}}\right)^{\delta\epsilon} (\nabla_\delta L) T_{\alpha\epsilon}{}^{\gamma} - i\frac{1}{4} T_{\alpha\underline{b}}{}^{\underline{c}} (\gamma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L)$$

$$- \frac{1}{2} (\nabla_{\underline{b}} L) \delta_{\alpha}{}^{\gamma} - \frac{1}{10} (\nabla_{\underline{d}} L) (\gamma_{\underline{b}}{}^{\underline{d}})_{\alpha}{}^{\gamma}$$

$$+ i\frac{1}{4} (\gamma_{\underline{b}})^{\delta\gamma} (\nabla_\alpha \nabla_\delta L) - i\frac{1}{320} \left(\gamma_{\underline{b}\underline{\underline{de}}}\right)^{\delta\epsilon} (\nabla_\delta \nabla_\epsilon L) (\gamma_{\underline{de}})_{\alpha}{}^{\gamma},$$

$$\delta_S T_{\underline{ab}}{}^{\underline{c}} = L T_{\underline{ab}}{}^{\underline{c}} + i\frac{1}{4} (\gamma_{[\underline{a}|})^{\delta\epsilon} (\nabla_\delta L) T_{\epsilon|\underline{b}]}{}^{\underline{c}} + \frac{6}{5} (\nabla_{[\underline{a}|} L) \delta_{|\underline{b}]}{}^{\underline{c}}$$

$$+ i\frac{1}{40} (\gamma_{\underline{ab}}{}^{\underline{c}})^{\delta\epsilon} (\nabla_\delta \nabla_\epsilon L),$$

$$\delta_S R_{\alpha\beta} \xrightarrow{\mathrm{de}} = L R_{\alpha\beta}{}^{\mathrm{de}} + \frac{1}{5} T_{\alpha\beta}{}^{[\mathrm{d}} (\nabla^{\mathrm{e}]} L) + \frac{1}{5} T_{\alpha\beta}{}^{\delta} (\nabla_\gamma L) (\gamma^{\mathrm{de}})_{\delta}{}^{\gamma}$$

$$- \frac{1}{5} (\nabla_{(\alpha|} \nabla_\gamma L) (\gamma^{\underline{de}})_{|\beta)}{}^{\gamma} + i\frac{1}{80} T_{\alpha\beta}{}^{\underline{c}} \left(\gamma_{\underline{\underline{d}}}{}^{\underline{de}}\right)^{\delta\epsilon} (\nabla_\delta \nabla_\epsilon L).$$

$$T_{\alpha\beta}{}^{\underline{c}} = i(\gamma^{\underline{c}})_{\alpha\beta} + i\frac{1}{32 \cdot 5!} (\gamma^{\underline{b_1 \cdots b_5}})_{\alpha\beta} X_{\underline{b_1} \cdots \underline{b_5}}^{\underline{\underline{c}}}, X_{\underline{abcde}\underline{\underline{e}}} = X_{[\underline{abcde\underline{f}}]} = 0$$

$$X_{\underline{a_1} \cdots \underline{a_5} \underline{\underline{b}}} \equiv i(\gamma_{\underline{a_1} \cdots \underline{a_5}})^{\alpha\beta} T_{\alpha\beta\underline{\underline{b}}}$$

$$\delta_S T_{\alpha\beta}^{\beta} = \frac{1}{2} L T_{\alpha\beta}^{\beta} - i\frac{1}{4} \left[i(\gamma^{\underline{c}})_{\alpha\beta} + i\frac{1}{32 \cdot 5!} (\gamma^{\underline{b_1 \cdots b_5}})_{\alpha\beta} X_{\underline{b_1} \cdots \underline{b_5}}{}^{\underline{c}} \right] (\gamma_{\underline{c}})^{\delta\beta} (\nabla_\delta L)$$

$$+ \frac{1}{2} \left(\nabla_{(\alpha} L) \delta_{\beta}\right)^{\beta} + \frac{1}{20} (\nabla_\delta L) (\gamma^{[2]})^{\delta}_{(\alpha} (\gamma_{[2]})_{\beta}\Big)^{\beta}$$

$$= \frac{1}{2} L T_{\alpha\beta}^{\beta} + \frac{1}{4} (\gamma^{\underline{c}})_{\alpha\beta} (\gamma_{\underline{c}})^{\delta\beta} (\nabla_\delta L) + \frac{1}{2} \left(\nabla_{(\alpha} L) \delta_{\beta}\right)^{\beta}$$

$$+ \frac{1}{20} (\nabla_\delta L) (\gamma^{[2]})^{\delta}_{(\alpha} (\gamma_{[2]})^{\beta}_{\beta)}$$

$$= \frac{1}{2} L T_{\alpha\beta}^{\beta} + \frac{33}{4} (\nabla_\alpha L)$$

$$\delta_S T_{\alpha\underline{b}}{}^{\underline{b}} = \frac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{b}} + \frac{1}{4} (\gamma_{\underline{b}})^{\gamma\delta} (\nabla_\gamma L) \left[(\gamma^{\underline{b}})_{\alpha\delta} + \frac{1}{32 \cdot 5!} (\gamma^{\underline{b_1 \cdots b_5}})_{\alpha\delta} X_{\underline{b_1} \cdots \underline{b_5}}{}^{\underline{b}} \right] + (\nabla_\alpha L) \delta_{\underline{b}}{}^{\underline{b}}$$

$$= \frac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{b}} + \frac{1}{4} (\gamma_{\underline{b}})^{\gamma\delta} (\nabla_\gamma L) (\gamma^{\underline{b}})_{\alpha\delta} + (\nabla_\alpha L) \delta_{\underline{b}}{}^{\underline{b}}$$

$$= \frac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{b}} + \frac{33}{4} (\nabla_\alpha L)$$

$$\mathcal{J}_\alpha{}^{(1)} = \frac{4}{33} T_{\alpha\beta}{}^{\beta}, \mathcal{J}_\alpha{}^{(2)} = \frac{4}{33} T_{\alpha\underline{b}}{}^{\underline{b}}$$

$$\delta_S \mathcal{J}_\alpha{}^{(1)} = \frac{1}{2} L \mathcal{J}_\alpha{}^{(1)} + (\nabla_\alpha L)\,, \delta_S \mathcal{J}_\alpha{}^{(2)} = \frac{1}{2} L \mathcal{J}_\alpha{}^{(2)} + (\nabla_\alpha L)$$

$$\mathcal{J}_\alpha{}^{(\pm)} = \frac{1}{2}\left[\mathcal{J}_\alpha{}^{(1)} \pm \mathcal{J}_\alpha{}^{(2)}\right]$$

$$\delta_S \mathcal{J}_\alpha{}^{(+)} = \frac{1}{2} L \mathcal{J}_\alpha{}^{(+)} + (\nabla_\alpha L), \delta_S \mathcal{J}_\alpha{}^{(-)} = \frac{1}{2} L \mathcal{J}_\alpha{}^{(-)}$$

$$\mathcal{W}_{\underline{abcd}} \equiv \frac{1}{32}\left[\left(\gamma^{\underline{e}}\gamma_{\underline{abcd}}\right)_\gamma{}^\alpha T_{\alpha\underline{e}}{}^\gamma + i\frac{11}{4}\left(\gamma_{\underline{abcd}}\right)^{\alpha\beta}\left(\nabla_\alpha \mathcal{J}_\beta{}^{(+)} - \frac{23}{220}\mathcal{J}_\alpha{}^{(+)}\mathcal{J}_\beta{}^{(+)}\right)\right]$$

$$\nabla_\alpha = \mathrm{D}_\alpha + \frac{1}{2}\Psi \mathrm{D}_\alpha + \frac{1}{10}\left(\sigma^{\underline{ab}}\right)_\alpha{}^\beta (\mathrm{D}_\beta \Psi)\mathcal{M}_{\underline{ab}}$$
$$\nabla_{\dot\alpha} = \mathrm{D}_{\dot\alpha} + \frac{1}{2}\Psi \mathrm{D}_{\dot\alpha} + \frac{1}{10}\left(\sigma^{\underline{ab}}\right)_{\dot\alpha}^{\dot\beta} (\mathrm{D}_{\dot\beta} \Psi)\mathcal{M}_{\underline{ab}}$$
$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\frac{1}{5}\left(\sigma_{\underline{a}}\right)^{\delta\gamma}(\mathrm{D}_\delta\Psi)\mathrm{D}_\gamma - i\frac{1}{5}\left(\sigma_{\underline{a}}\right)^{\dot\delta\dot\gamma}(\mathrm{D}_{\dot\delta}\Psi)\mathrm{D}_{\dot\gamma} - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}}$$

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}}\,, \{\mathrm{D}_{\dot\alpha}, \mathrm{D}_{\dot\beta}\} = i(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}\partial_{\underline{a}}\,, \{\mathrm{D}_\alpha, \mathrm{D}_{\dot\beta}\} = 0$$

$$T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta}\,, T_{\dot\alpha\dot\beta} = i(\sigma^{\underline{c}})_{\dot\alpha\dot\beta}\,, T_{\alpha\dot\beta}{}^{\underline{c}} = 0$$

$$\delta_S \nabla_\alpha = \frac{1}{2} L \nabla_\alpha + \frac{1}{10}\left(\sigma^{ab}\right)_\alpha^\beta (\nabla_\beta L)\mathcal{M}_{\underline{ab}}$$
$$\delta_S \nabla_{\dot\alpha} = \frac{1}{2} L \nabla_{\dot\alpha} + \frac{1}{10}\left(\sigma^{ab}\right)_{\dot\alpha}^{\dot\beta} (\nabla_{\dot\beta} L)\mathcal{M}_{\underline{ab}}$$
$$\delta_S \nabla_{\underline{a}} = L\nabla_{\underline{a}} - i\frac{1}{5}\left(\sigma_{\underline{a}}\right)^{\delta\gamma}(\nabla_\delta L)\nabla_\gamma - i\frac{1}{5}\left(\sigma_{\underline{a}}\right)^{\dot\delta\dot\gamma}(\nabla_{\dot\delta} L)\nabla_{\dot\gamma} - (\nabla_{\underline{c}} L)\mathcal{M}_{\underline{a}}^{\underline{c}}$$

$$\begin{aligned}
[\nabla_\alpha\,,\ \nabla_\beta\} &= T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\beta}{}^\gamma\nabla_\gamma + T_{\alpha\beta}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\alpha\beta\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\nabla_\alpha\,,\ \nabla_{\dot\beta}\} &= T_{\alpha\dot\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\dot\beta}{}^\gamma\nabla_\gamma + T_{\alpha\dot\beta}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\alpha\dot\beta\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\nabla_{\dot\alpha}\,,\ \nabla_{\dot\beta}\} &= T_{\dot\alpha\dot\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\dot\alpha\dot\beta}{}^\gamma\nabla_\gamma + T_{\dot\alpha\dot\beta}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\dot\alpha\dot\beta\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\nabla_\alpha\,,\ \nabla_{\underline{b}}\} &= T_{\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\underline{b}}{}^\gamma\nabla_\gamma + T_{\alpha\underline{b}}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\nabla_{\dot\alpha}\,,\ \nabla_{\underline{b}}\} &= T_{\dot\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\dot\alpha\underline{b}}{}^\gamma\nabla_\gamma + T_{\dot\alpha\underline{b}}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\dot\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\nabla_{\underline{a}}\,,\ \nabla_{\underline{b}}\} &= T_{\underline{ab}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\underline{ab}}{}^\gamma\nabla_\gamma + T_{\underline{ab}}{}^{\dot\gamma}\nabla_{\dot\gamma} + \tfrac{1}{2}R_{\underline{ab}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,
\end{aligned}$$

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0$$
$$\delta_S T_{\alpha\dot{\beta}}{}^{\underline{c}} = 0$$
$$\delta_S T_{\dot{\alpha}\dot{\beta}} = 0$$
$$\delta_S T_{\alpha\beta}{}^{\gamma} = \frac{1}{2} L T_{\alpha\beta}{}^{\gamma} + i\frac{1}{5} T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) + \frac{1}{2} (\nabla_{(\alpha} L)\delta_{\beta)}\gamma + \frac{1}{20} (\sigma^{[2]})_{(\alpha}{}^{\delta} (\sigma_{[2]})_{\beta)}{}^{\gamma} (\nabla_\delta L)$$
$$\delta_S T_{\alpha\beta}{}^{\dot{\gamma}} = \frac{1}{2} L T_{\alpha\beta}{}^{\dot{\gamma}} + i\frac{1}{5} T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot{\delta}\dot{\gamma}} (\nabla_{\dot{\delta}} L)$$
$$\delta_S T_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \frac{1}{2} L T_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + i\frac{1}{5} T_{\dot{\alpha}\dot{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot{\gamma}\dot{\delta}} (\nabla_{\dot{\delta}} L) + \frac{1}{2} (\nabla_{(\dot{\alpha}} L)\delta_{\dot{\beta}}\dot{\gamma} + \frac{1}{20} (\sigma^{[2]})_{(\dot{\alpha}}\dot{\delta} (\sigma_{[2]})_{\dot{\beta})}\dot{\gamma}^{\dot{\gamma}} (\nabla_{\dot{\delta}} L)$$
$$\delta_S T_{\dot{\alpha}\dot{\beta}}{}^{\gamma} = \frac{1}{2} L T_{\dot{\alpha}\dot{\beta}}{}^{\gamma} + i\frac{1}{5} T_{\dot{\alpha}\dot{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L)$$
$$\delta_S T_{\alpha\dot{\beta}}{}^{\gamma} = \frac{1}{2} L T_{\alpha\dot{\beta}}{}^{\gamma} + i\frac{1}{5} T_{\alpha\dot{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L) + \frac{1}{2} (\nabla_{\dot{\beta}} L)\delta_{\alpha}{}^{\gamma} + \frac{1}{20} (\sigma^{[2]})_{\dot{\beta}}^{\dot{\delta}} (\sigma_{[2]})_{\alpha}{}^{\gamma} (\nabla_{\dot{\delta}} L)$$
$$\delta_S T^{\dot{\gamma}}_{\alpha\dot{\beta}} = \frac{1}{2} L T^{\dot{\gamma}}_{\alpha\dot{\beta}} + i\frac{1}{5} T_{\alpha\dot{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot{\delta}\dot{\gamma}} (\nabla_{\dot{\delta}} L) + \frac{1}{2} (\nabla_\alpha L)\delta_{\dot{\beta}}^{\dot{\gamma}} + \frac{1}{20} (\sigma^{[2]})_{\dot{\beta}}^{\dot{\gamma}} (\sigma_{[2]})_{\alpha}{}^{\delta} (\nabla_\delta L)$$

$$\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} = \frac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{c}} + (\nabla_\alpha L)\delta_{\underline{b}}{}^{\underline{c}} + \frac{1}{5} (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\beta} (\nabla_\beta L) + i\frac{1}{5} (\sigma_{\underline{b}})^{\beta\delta} (\nabla_\delta L) T_{\alpha\beta}{}^{\underline{c}}$$
$$+ i\frac{1}{5} (\sigma_{\underline{b}})^{\dot{\beta}\dot{\delta}} (\nabla_{\dot{\delta}} L) T_{\alpha\dot{\beta}}{}^{\underline{c}}$$
$$\delta_S T_{\dot{\alpha}\underline{\underline{\beta}}}{}^{\underline{c}} = \frac{1}{2} L T_{\dot{\alpha}\underline{\underline{c}}}{}^{\underline{c}} + (\nabla_{\dot{\alpha}} L)\delta_{\underline{\underline{b}}\underline{\underline{c}}} + \frac{1}{5} (\sigma_{\underline{\underline{b}}})_{\dot{\alpha}}{}^{\dot{\beta}\dot{\beta}} (\nabla_{\dot{\beta}} L) + i\frac{1}{5} (\sigma_{\underline{b}})^{\beta\delta} (\nabla_\delta L) T_{\dot{\alpha}\beta}{}^{\underline{c}}$$
$$+ i\frac{1}{5} (\sigma_{\underline{b}})^{\dot{\beta}\dot{\delta}} (\nabla_{\dot{\delta}} L) T_{\dot{\alpha}\dot{\beta}}{}^{\underline{c}}$$

$$\delta_S T^{\gamma}_{\alpha\underline{b}} = L T^{\gamma}_{\alpha\underline{b}} + i\frac{1}{5} T_{\alpha\underline{b}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) - \frac{1}{2} (\nabla_{\underline{b}} L)\delta^{\gamma}_{\alpha} - i\frac{1}{5} (\sigma_{\underline{b}})^{\gamma\beta} (\nabla_\alpha \nabla_\beta L)$$
$$+ i\frac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\alpha\beta}{}^{\gamma} + i\frac{1}{5} (\sigma_{\underline{b}})^{\dot{\delta}\dot{\beta}} (\nabla_{\dot{\delta}} L) T^{\gamma}_{\alpha\dot{\beta}} + \frac{1}{2} (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\gamma} (\nabla_{\underline{c}} L),$$
$$\delta_S T_{\alpha\underline{\underline{b}}}{}^{\dot{\gamma}} = L T_{\alpha\underline{\underline{b}}}{}^{\dot{\gamma}} - i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\dot{\gamma}\dot{\beta}} (\nabla_\alpha \nabla_{\dot{\beta}} L) + i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\delta\beta} (\nabla_\delta L) T_{\alpha\beta}{}^{\dot{\gamma}} + i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\dot{\delta}\dot{\beta}} (\nabla_{\dot{\delta}} L) T_{\alpha\dot{\beta}}\dot{\gamma}^{\dot{\gamma}}$$
$$- i\frac{1}{5} (\sigma_{\underline{c}})^{\dot{\delta}\dot{\gamma}} (\nabla_{\dot{\delta}} L) T_{\alpha\underline{\underline{b}}}{}^{\underline{c}},$$
$$\delta_S T^{\gamma}_{\dot{\alpha}\underline{\underline{b}}} = L T^{\gamma}_{\dot{\alpha}\underline{\underline{b}}} - i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\gamma\beta} (\nabla_{\dot{\alpha}} \nabla_\beta L) + i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\delta\beta} (\nabla_\delta L) T^{\gamma}_{\dot{\alpha}\beta} + i\frac{1}{5} (\sigma_{\underline{\underline{b}}})^{\dot{\delta}\dot{\beta}} (\nabla_{\dot{\delta}} L) T^{\gamma}_{\dot{\alpha}\dot{\beta}}$$
$$- i\frac{1}{5} (\sigma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L) T_{\dot{\alpha}\underline{\underline{b}}}{}^{\underline{c}},$$
$$\delta_S T_{\dot{\alpha}\underline{\underline{b}}}{}^{\dot{\gamma}} = L T_{\dot{\alpha}\underline{\underline{b}}}{}^{\dot{\gamma}} + i\frac{1}{5} T_{\dot{\alpha}\underline{\underline{b}}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot{\gamma}\dot{\delta}} (\nabla_{\dot{\delta}} L) - \frac{1}{2} (\nabla_{\underline{b}} L) \delta_{\dot{\alpha}}\dot{\gamma} - i\frac{1}{5} (\sigma_{\underline{b}})^{\dot{\gamma}\dot{\beta}} (\nabla_{\dot{\alpha}} \nabla_{\dot{\beta}} L)$$
$$+ i\frac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\dot{\alpha}\beta}{}^{\dot{\gamma}} + i\frac{1}{5} (\sigma_{\underline{b}})^{\dot{\delta}\dot{\beta}} (\nabla_{\dot{\delta}} L) T_{\dot{\alpha}\dot{\beta}}\dot{\gamma} + \frac{1}{2} (\sigma_{\underline{b}}{}^{\underline{c}})_{\dot{\alpha}}\dot{\gamma} (\nabla_{\underline{c}} L),$$
$$\delta_S T_{\underline{ab}}{}^{\underline{c}} = L T_{\underline{ab}}{}^{\underline{c}} - i\frac{1}{5} (\sigma_{[\underline{a}})^{\alpha\beta} (\nabla_\alpha L) T_{\beta|\underline{b}]}{}^{\underline{c}} - i\frac{1}{5} (\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\beta}} (\nabla_{\dot{\alpha}} L) T_{\dot{\beta}|\underline{b}]}{}^{\underline{c}},$$

$$
\begin{aligned}
\delta_S R_{\alpha\beta}{}^{\underline{de}} &= L R_{\alpha\beta}{}^{\underline{de}} - T_{\alpha\beta}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \tfrac{1}{5} T_{\alpha\beta}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) \ + \tfrac{1}{5} T_{\alpha\beta}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\
&\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\nabla_{\beta)}\nabla_{\delta} L) \quad , \\
\delta_S R_{\alpha\dot{\beta}}{}^{\underline{de}} &= L R_{\alpha\dot{\beta}}{}^{\underline{de}} - T_{\alpha\dot{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \tfrac{1}{5} T_{\alpha\dot{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) \ + \tfrac{1}{5} T_{\alpha\dot{\beta}}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\
&\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\delta}(\nabla_{\dot{\beta}}\nabla_{\delta} L) \ - \tfrac{1}{5}(\sigma^{\underline{de}})_{\dot{\beta}}{}^{\dot{\gamma}}(\nabla_{\alpha}\nabla_{\dot{\gamma}} L) \quad , \\
\delta_S R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} &= L R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} - T_{\dot{\alpha}\dot{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \tfrac{1}{5} T_{\dot{\alpha}\dot{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) \ + \tfrac{1}{5} T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\
&\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\dot{\alpha}}{}^{\dot{\delta}}(\nabla_{\dot{\beta})}\nabla_{\dot{\delta}} L) \quad .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{\alpha}^{(1)} &= \frac{1}{2} T_{\alpha\beta}{}^{\beta} \, , \mathcal{J}_{\dot{\alpha}}^{(1)} = \frac{1}{2} T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\beta}} \\
\mathcal{J}_{\alpha}^{(2)} &= \frac{1}{8} T_{\alpha\dot{\beta}}{}^{\dot{\beta}} \, , \mathcal{J}_{\dot{\alpha}}^{(2)} = \frac{1}{8} T_{\dot{\alpha}\beta}{}^{\beta} \\
\mathcal{J}_{\alpha}^{(3)} &= \frac{1}{8} T_{\alpha\underline{b}} \, , \mathcal{J}_{\dot{\alpha}}^{(3)} = \frac{1}{8} T_{\dot{\alpha}\underline{b}}{}^{\underline{b}}
\end{aligned}
$$

$$
\begin{aligned}
\delta_S \mathcal{J}_{\alpha}^{(1)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(1)} + (\nabla_{\alpha} L), \delta_S \mathcal{J}_{\dot{\alpha}}^{(1)} = \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(1)} + (\nabla_{\dot{\alpha}} L), \\
\delta_S \mathcal{J}_{\alpha}^{(2)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(2)} + (\nabla_{\alpha} L), \delta_S \mathcal{J}_{\dot{\alpha}}^{(2)} = \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(2)} + (\nabla_{\dot{\alpha}} L), \\
\delta_S \mathcal{J}_{\alpha}^{(3)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(3)} + (\nabla_{\alpha} L), \delta_S \mathcal{J}_{\dot{\alpha}}^{(3)} = \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(3)} + (\nabla_{\dot{\alpha}} L),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{\alpha}^{(+)} &= \frac{1}{3}\left(\mathcal{J}_{\alpha}^{(3)} + \mathcal{J}_{\alpha}^{(1)} + \mathcal{J}_{\alpha}^{(2)}\right), & \mathcal{J}_{\dot{\alpha}}^{(+)} &= \frac{1}{3}\left(\mathcal{J}_{\dot{\alpha}}^{(3)} + \mathcal{J}_{\dot{\alpha}}^{(1)} + \mathcal{J}_{\dot{\alpha}}^{(2)}\right) \\
\mathcal{J}_{\alpha}^{(-1)} &= \frac{1}{2}\left(\mathcal{J}_{\alpha}^{(3)} - \mathcal{J}_{\alpha}^{(1)}\right), & \mathcal{J}_{\dot{\alpha}}^{(-1)} &= \frac{1}{2}\left(\mathcal{J}_{\dot{\alpha}}^{(3)} - \mathcal{J}_{\dot{\alpha}}^{(1)}\right) \\
\mathcal{J}_{\alpha}^{(-2)} &= \frac{1}{6}\left(\mathcal{J}_{\alpha}^{(3)} + \mathcal{J}_{\alpha}^{(1)} - 2\mathcal{J}_{\alpha}^{(2)}\right), & \mathcal{J}_{\dot{\alpha}}^{(-2)} &= \frac{1}{6}\left(\mathcal{J}_{\dot{\alpha}}^{(3)} + \mathcal{J}_{\dot{\alpha}}^{(1)} - 2\mathcal{J}_{\dot{\alpha}}^{(2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta_S \mathcal{J}_{\alpha}^{(+)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(+)} + (\nabla_{\alpha} L), & \delta_S \mathcal{J}_{\dot{\alpha}}^{(+)} &= \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(+)} + (\nabla_{\dot{\alpha}} L) \\
\delta_S \mathcal{J}_{\alpha}^{(-1)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(-1)}, & \delta_S \mathcal{J}_{\dot{\alpha}}^{(-1)} &= \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(-1)} \\
\delta_S \mathcal{J}_{\alpha}^{(-2)} &= \frac{1}{2} L \mathcal{J}_{\alpha}^{(-2)}, & \delta_S \mathcal{J}_{\dot{\alpha}}^{(-2)} &= \frac{1}{2} L \mathcal{J}_{\dot{\alpha}}^{(-2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{\underline{abc}} = \frac{1}{32}\{ & (\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha} T_{\alpha\underline{d}}{}^{\gamma} - i2(\sigma_{\underline{abc}})^{\alpha\beta}\left[\nabla_{\alpha}\mathcal{J}_{\beta}^{(+)} - \frac{6}{5}\mathcal{J}_{\alpha}^{(+)}\mathcal{J}_{\beta}^{(+)}\right] \\
& + (\sigma^{\underline{d}})_{\dot{\gamma}\dot{\beta}}(\sigma_{\underline{abc}})^{\dot{\beta}\dot{\alpha}} T_{\dot{\alpha}\underline{d}}^{\dot{\gamma}} - i2(\sigma_{\underline{abc}})^{\dot{\alpha}\dot{\beta}}\left[\nabla_{\dot{\alpha}}\mathcal{J}_{\dot{\beta}}^{(+)} - \frac{6}{5}\mathcal{J}_{\dot{\alpha}}^{(+)}\mathcal{J}_{\dot{\beta}}^{(+)}\right]\}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\alpha} &= \mathrm{D}_{\alpha} + \frac{1}{2}\Psi \mathrm{D}_{\alpha} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{ab}} \\
\bar{\nabla}_{\alpha} &= \bar{\mathrm{D}}_{\alpha} + \frac{1}{2}\overline{\Psi \mathrm{D}}_{\alpha} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\bar{\mathrm{D}}_{\beta}\bar{\Psi})\mathcal{M}_{\underline{ab}} \\
\nabla_{\underline{a}} &= \partial_{\underline{a}} + \frac{1}{2}(\Psi + \bar{\Psi})\partial_{\underline{a}} - i\frac{1}{32}(\sigma_{\underline{a}})^{\alpha\beta}\left[\bar{\mathrm{D}}_{\alpha}\left(\Psi + \frac{27}{5}\bar{\Psi}\right)\right]\mathrm{D}_{\beta} \\
& - i\frac{1}{32}(\sigma_{\underline{a}})^{\alpha\beta}\left[\mathrm{D}_{\alpha}\left(\bar{\Psi} + \frac{27}{5}\Psi\right)\right]\bar{\mathrm{D}}_{\beta} - \frac{1}{2}[\partial_{\underline{c}}(\Psi + \bar{\Psi})]\mathcal{M}_{\underline{a}}{}^{\underline{c}}
\end{aligned}
$$

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = 0\,, \{\overline{\mathrm{D}}_\alpha, \overline{\mathrm{D}}_\beta\} = 0\,, \{\mathrm{D}_\alpha, \overline{\mathrm{D}}_\beta\} = i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}}$$

$$T_{\alpha\beta}{}^{\underline{c}} = 0\,, T_{\bar{\alpha}\beta}{}^{\underline{c}} = 0\,, T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta}$$

$$\delta_S\nabla_\alpha = \frac{1}{2}L\nabla_\alpha + \frac{1}{10}\left(\sigma^{\underline{ab}}\right)_\alpha{}^\beta(\nabla_\beta L)\mathcal{M}_{\underline{ab}}$$
$$\delta_S\bar{\nabla}_\alpha = \frac{1}{2}\overline{L\nabla}_\alpha + \frac{1}{10}\left(\sigma^{\underline{ab}}\right)_\alpha{}^\beta(\bar{\nabla}_\beta\bar{L})\mathcal{M}_{\underline{ab}}$$
$$\delta_S\nabla_{\underline{a}} = \frac{1}{2}(L+\bar{L})\nabla_{\underline{a}} - i\frac{1}{32}\left(\sigma_{\underline{a}}\right)^{\alpha\beta}\left[\bar{\nabla}_\alpha\left(L + \frac{27}{5}\bar{L}\right)\right]\nabla_\beta$$
$$-i\frac{1}{32}\left(\sigma_{\underline{a}}\right)^{\alpha\beta}\left[\nabla_\alpha\left(\bar{L} + \frac{27}{5}L\right)\right]\bar{\nabla}_\beta - \frac{1}{2}[\nabla_{\underline{c}}(L+\bar{L})]\mathcal{M}_{\underline{a}}{}^{\underline{c}}$$

$$[\nabla_\alpha\,,\ \nabla_\beta\} = T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\beta}{}^\gamma\nabla_\gamma + T_{\alpha\beta}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\alpha\beta\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$
$$[\nabla_\alpha\,,\ \overline{\nabla}_\beta\} = T_{\alpha\overline{\beta}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\overline{\beta}}{}^\gamma\nabla_\gamma + T_{\alpha\overline{\beta}}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\alpha\overline{\beta}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$
$$[\overline{\nabla}_\alpha\,,\ \overline{\nabla}_\beta\} = T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\overline{\alpha}\overline{\beta}}{}^\gamma\nabla_\gamma + T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\overline{\alpha}\overline{\beta}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$
$$[\nabla_\alpha\,,\ \nabla_{\underline{b}}\} = T_{\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\underline{b}}{}^\gamma\nabla_\gamma + T_{\alpha\underline{b}}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$
$$[\overline{\nabla}_\alpha\,,\ \nabla_{\underline{b}}\} = T_{\overline{\alpha}\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\overline{\alpha}\underline{b}}{}^\gamma\nabla_\gamma + T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\overline{\alpha}\underline{b}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$
$$[\nabla_{\underline{a}}\,,\ \nabla_{\underline{b}}\} = T_{\underline{ab}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\underline{ab}}{}^\gamma\nabla_\gamma + T_{\underline{ab}}{}^{\overline{\gamma}}\overline{\nabla}_\gamma + \tfrac{1}{2}R_{\underline{ab}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}}\quad,$$

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = \frac{1}{2}(L-\bar{L})T_{\alpha\beta}{}^{\underline{c}}$$
$$\delta_S T_{\alpha\bar{\beta}}{}^{\underline{c}} = 0$$
$$\delta_S T_{\bar{\alpha}\bar{\beta}} = -\frac{1}{2}(L-\bar{L})T_{\bar{\alpha}\bar{\beta}}{}^{\underline{c}}$$
$$\delta_S T_{\alpha\beta}{}^{\bar{\gamma}} = \left(L - \frac{1}{2}\bar{L}\right)T_{\alpha\beta}{}^{\bar{\gamma}} + iT_{\alpha\beta}{}^{\underline{c}}\left(\sigma_{\underline{c}}\right)^{\delta\gamma}\left(\nabla_\delta\left(\frac{1}{32}\bar{L} + \frac{27}{160}L\right)\right)$$

$$
\begin{aligned}
\delta_S T_{\alpha\beta}{}^{\gamma} &= \tfrac{1}{2} L T_{\alpha\beta}{}^{\gamma} + i T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\overline{\nabla}_{\delta} (\tfrac{1}{32} L + \tfrac{27}{160} \overline{L})) \\
&\quad + \tfrac{1}{2} (\nabla_{(\alpha} L) \delta_{\beta)}{}^{\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{(\alpha}{}^{\delta} (\sigma_{[2]})_{\beta)}{}^{\gamma} (\nabla_{\delta} L) \quad , \\
\delta_S T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} &= \tfrac{1}{2} L T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} + i T_{\alpha\overline{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\delta\gamma} (\nabla_{\delta} (\tfrac{1}{32} \overline{L} + \tfrac{27}{160} L)) + \tfrac{1}{2} (\nabla_{\alpha} \overline{L}) \delta_{\beta}{}^{\gamma} \\
&\quad + \tfrac{1}{20} (\sigma^{[2]})_{\beta}{}^{\gamma} (\sigma_{[2]})_{\alpha}{}^{\delta} (\nabla_{\delta} L) \quad , \\
\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} &= \tfrac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{c}} + \tfrac{1}{2} (\nabla_{\alpha} (L + \overline{L})) \delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5} (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\beta} (\nabla_{\beta} L) \\
&\quad + i (\sigma_{\underline{b}})^{\delta\beta} (\nabla_{\delta} (\tfrac{1}{32} \overline{L} + \tfrac{27}{160} L)) T_{\alpha\overline{\beta}}{}^{\underline{c}} + i (\sigma_{\underline{b}})^{\delta\beta} (\overline{\nabla}_{\delta} (\tfrac{1}{32} L + \tfrac{27}{160} \overline{L})) T_{\alpha\beta}{}^{\underline{c}} \quad , \\
\delta_S T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} &= \tfrac{1}{2} \overline{L} T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} + \tfrac{1}{2} (\overline{\nabla}_{\alpha} (L + \overline{L})) \delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5} (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\beta} (\overline{\nabla}_{\beta} \overline{L}) \\
&\quad + i (\sigma_{\underline{b}})^{\beta\delta} (\overline{\nabla}_{\delta} (\tfrac{1}{32} L + \tfrac{27}{160} \overline{L})) T_{\overline{\alpha}\beta}{}^{\underline{c}} + i (\sigma_{\underline{b}})^{\beta\delta} (\nabla_{\delta} (\tfrac{1}{32} \overline{L} + \tfrac{27}{160} L)) T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} \quad , \\
\delta_S T_{\alpha\overline{\beta}}{}^{\gamma} &= \tfrac{1}{2} \overline{L} T_{\alpha\overline{\beta}}{}^{\gamma} + i T_{\alpha\overline{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\delta\gamma} (\overline{\nabla}_{\delta} (\tfrac{1}{32} L + \tfrac{27}{160} \overline{L})) + \tfrac{1}{2} (\overline{\nabla}_{\beta} L) \delta_{\alpha}{}^{\gamma} \\
&\quad + \tfrac{1}{20} (\sigma^{[2]})_{\beta}{}^{\delta} (\sigma_{[2]})_{\alpha}{}^{\gamma} (\overline{\nabla}_{\delta} \overline{L}) \quad , \\
\delta_S T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} &= \tfrac{1}{2} \overline{L} T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + i T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_{\delta} (\tfrac{1}{32} \overline{L} + \tfrac{27}{160} L)) \\
&\quad + \tfrac{1}{2} (\overline{\nabla}_{(\alpha} \overline{L}) \delta_{\beta)}{}^{\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{(\alpha}{}^{\delta} (\sigma_{[2]})_{\beta)}{}^{\gamma} (\overline{\nabla}_{\delta} \overline{L}) \quad , \\
\delta_S T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} &= (\overline{L} - \tfrac{1}{2} L) T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} + i T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\overline{\nabla}_{\delta} (\tfrac{1}{32} L + \tfrac{27}{160} \overline{L})) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_S T_{\alpha\underline{b}}{}^{\overline{\gamma}} &= L T_{\alpha\underline{b}}{}^{\overline{\gamma}} - i(\sigma_{\underline{b}})^{\gamma\beta}(\nabla_\alpha\nabla_\beta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\alpha\beta}{}^{\overline{\gamma}} \\
&\quad + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} - i(\sigma_{\underline{c}})^{\delta\gamma}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\alpha\underline{b}}{}^{\underline{c}} \quad , \\
\delta_S T_{\alpha\underline{b}}{}^{\gamma} &= \tfrac{1}{2}(L+\overline{L})T_{\alpha\underline{b}}{}^{\gamma} + iT_{\alpha\underline{b}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) - \tfrac{1}{2}(\nabla_{\underline{b}}L)\delta_\alpha{}^{\gamma} \\
&\quad - i(\sigma_{\underline{b}})^{\gamma\beta}(\nabla_\alpha\overline{\nabla}_\beta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\alpha\overline{\beta}}{}^{\gamma} \\
&\quad + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\alpha\beta}{}^{\gamma} + \tfrac{1}{4}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma}(\nabla_{\underline{c}}(L+\overline{L})) \quad , \\
\delta_S T_{\underline{ab}}{}^{\underline{c}} &= \tfrac{1}{2}(L+\overline{L})T_{\underline{ab}}{}^{\underline{c}} - i(\sigma_{[\underline{a}})^{\alpha\beta}\left[(\overline{\nabla}_\alpha(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\beta|\underline{b}]}{}^{\underline{c}}(\nabla_\alpha(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\beta}|\underline{b}]}{}^{\underline{c}}\right] \\
\delta_S T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}} &= \tfrac{1}{2}(L+\overline{L})T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}} + iT_{\overline{\alpha}\underline{b}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) - \tfrac{1}{2}(\nabla_{\underline{b}}\overline{L})\delta_\alpha{}^{\gamma} \\
&\quad - i(\sigma_{\underline{b}})^{\gamma\beta}(\overline{\nabla}_\alpha\nabla_\beta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\beta}{}^{\overline{\gamma}} \\
&\quad + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + \tfrac{1}{4}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma}(\nabla_{\underline{c}}(L+\overline{L})) \quad , \\
\delta_S T_{\overline{\alpha}\underline{b}}{}^{\gamma} &= \overline{L}T_{\overline{\alpha}\underline{b}}{}^{\gamma} - i(\sigma_{\underline{b}})^{\gamma\beta}(\overline{\nabla}_\alpha\overline{\nabla}_\beta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\beta}{}^{\gamma} \\
&\quad + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} - i(\sigma_{\underline{c}})^{\delta\gamma}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_S R_{\alpha\beta}{}^{\underline{de}} &= L R_{\alpha\beta}{}^{\underline{de}} - \tfrac{1}{2}T_{\alpha\beta}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L+\overline{L})) + \tfrac{1}{5}T_{\alpha\beta}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L) \\
&\quad + \tfrac{1}{5}T_{\alpha\beta}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\nabla_{\beta)}\nabla_\delta L) \quad , \\
\delta_S R_{\alpha\overline{\beta}}{}^{\underline{de}} &= \tfrac{1}{2}(L+\overline{L})R_{\alpha\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}T_{\alpha\overline{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L+\overline{L})) + \tfrac{1}{5}T_{\alpha\overline{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \tfrac{1}{5}T_{\alpha\overline{\beta}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_\alpha{}^{\delta}(\overline{\nabla}_\beta\nabla_\delta L) - \tfrac{1}{5}(\sigma^{\underline{de}})_\beta{}^{\gamma}(\nabla_\alpha\overline{\nabla}_\gamma\overline{L}) \quad , \\
\delta_S R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}} &= \overline{L}R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}T_{\overline{\alpha}\overline{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L+\overline{L})) + \tfrac{1}{5}T_{\overline{\alpha}\overline{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L) \\
&\quad + \tfrac{1}{5}T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\overline{\nabla}_{\beta)}\overline{\nabla}_\delta\overline{L}) \quad .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_\alpha^{(1)} &= \frac{1}{4}T_{\alpha\beta}{}^{\beta}, & \overline{\mathcal{J}}_\alpha^{(1)} &= \frac{1}{4}T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\beta}} \\
\mathcal{J}_\alpha^{(2)} &= \frac{5}{12}T_{\alpha\beta}{}^{\beta} + \frac{1}{3}T_{\alpha\bar{\beta}}{}^{\bar{\beta}} - \frac{1}{3}T_{\alpha\underline{\underline{b}}}{}^{\underline{b}} & \overline{\mathcal{J}}_\alpha^{(2)} &= \frac{5}{12}T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\beta}} + \frac{1}{3}T_{\bar{\alpha}\beta}{}^{\beta} - \frac{1}{3}T_{\bar{\alpha}\underline{\underline{b}}}{}^{\underline{b}}.
\end{aligned}
$$

$$
\begin{aligned}
\delta_S\mathcal{J}_\alpha^{(1)} &= \frac{1}{2}L\mathcal{J}_\alpha^{(1)} + (\nabla_\alpha L), & \delta_S\overline{\mathcal{J}}_\alpha^{(1)} &= \frac{1}{2}\bar{L}\overline{\mathcal{J}}_\alpha^{(1)} + (\bar{\nabla}_\alpha\bar{L}), \\
\delta_S\mathcal{J}_\alpha^{(2)} &= \frac{1}{2}L\mathcal{J}_\alpha^{(2)} + (\nabla_\alpha\bar{L}), & \delta_S\overline{\mathcal{J}}_\alpha^{(2)} &= \frac{1}{2}\bar{L}\overline{\mathcal{J}}_\alpha^{(2)} + (\bar{\nabla}_\alpha L).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{\underline{abc}} = \frac{1}{32}\Big\{ & (\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha} T_{\alpha\underline{d}}^{\gamma} + (\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha} T_{\bar{\alpha}\underline{d}}^{\bar{\gamma}} - 4T_{[\underline{abc}]} \\
& -i(\sigma_{\underline{abc}})^{\alpha\beta}\Big[\frac{27}{16}\left(\nabla_\alpha \bar{\mathcal{J}}_\beta^{(1)} + \bar{\nabla}_\alpha \mathcal{J}_\beta^{(1)}\right) + \frac{5}{16}\left(\nabla_\alpha \bar{\mathcal{J}}_\beta^{(2)} + \bar{\nabla}_\alpha \mathcal{J}_\beta^{(2)}\right) \\
& + \frac{1{,}863}{3{,}200}\mathcal{J}_\alpha^{(1)}\bar{\mathcal{J}}_\beta^{(1)} - \frac{33}{128}\mathcal{J}_\alpha^{(2)}\bar{\mathcal{J}}_\beta^{(2)} - \frac{411}{640}\left(\mathcal{J}_\alpha^{(1)}\bar{\mathcal{J}}_\beta^{(2)} + \mathcal{J}_\alpha^{(2)}\bar{\mathcal{J}}_\beta^{(1)}\right)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\alpha &= \mathrm{D}_\alpha + \frac{1}{2}\Psi\mathrm{D}_\alpha + \frac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{ab}} \\
\nabla_{\underline{a}} &= \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\frac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\mathrm{D}_\beta - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}}
\end{aligned}
$$

$$
\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}}
$$

$$
\begin{aligned}
\delta_S\nabla_\alpha &= \frac{1}{2}L\nabla_\alpha + \frac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\nabla_\beta L)\mathcal{M}_{\underline{ab}} \\
\delta_S\nabla_{\underline{a}} &= L\nabla_{\underline{a}} - i\frac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\nabla_\alpha L)\nabla_\beta - (\nabla_{\underline{c}} L)\mathcal{M}_{\underline{a}}{}^{\underline{c}}
\end{aligned}
$$

$$
\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0 \quad ,
$$

$$
\begin{aligned}
\delta_S T_{\alpha\beta}{}^\gamma &= \tfrac{1}{2}L\, T_{\alpha\beta}{}^\gamma + i\tfrac{2}{5}T_{\alpha\beta}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta L) + \tfrac{1}{2}(\nabla_{(\alpha}L)\delta_{\beta)}{}^\gamma \\
&\quad + \tfrac{1}{20}(\sigma^{[2]})_{(\alpha}{}^\delta(\sigma_{[2]})_{\beta)}{}^\gamma(\nabla_\delta L) \quad , \\
\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} &= \tfrac{1}{2}L T_{\alpha\underline{b}}{}^{\underline{c}} + (\nabla_\alpha L)\delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\beta(\nabla_\beta L) + i\tfrac{2}{5}(\sigma_{\underline{b}})^{\beta\delta}(\nabla_\delta L)T_{\alpha\beta}{}^{\underline{c}} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_S T_{\alpha\underline{b}}{}^\gamma &= L\, T_{\alpha\underline{b}}{}^\gamma + i\tfrac{2}{5}T_{\alpha\underline{b}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta L) - \tfrac{1}{2}(\nabla_{\underline{b}}L)\delta_\alpha{}^\gamma \\
&\quad - i\tfrac{2}{5}(\sigma_{\underline{b}})^{\gamma\beta}(\nabla_\alpha\nabla_\beta L) + i\tfrac{2}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta L)T_{\alpha\beta}{}^\gamma + \tfrac{1}{2}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\nabla_{\underline{c}}L) \quad , \\
\delta_S T_{\underline{ab}}{}^{\underline{c}} &= L\, T_{\underline{ab}}{}^{\underline{c}} - i\tfrac{2}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_\alpha L)\, T_{\beta|\underline{b}]}{}^{\underline{c}} \quad , \\
\delta_S R_{\alpha\beta}{}^{\underline{de}} &= L\, R_{\alpha\beta}{}^{\underline{de}} - T_{\alpha\beta}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) + \tfrac{1}{5}T_{\alpha\beta}{}^\gamma(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) \\
&\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^\delta(\nabla_{\beta)}\nabla_\delta L) \quad .
\end{aligned}
$$

$$
\mathcal{J}_\alpha^{(+)} = \frac{1}{6}T_{\alpha\underline{b}}{}^{\underline{b}}\, ,\mathcal{J}_\alpha^{(-)} = T_{\alpha\beta}{}^{\beta}
$$

$$
\delta_S\mathcal{J}_\alpha^{(+)} = \frac{1}{2}L\mathcal{J}_\alpha^{(+)} + (\nabla_\alpha L)\, ,\delta_S\mathcal{J}_\alpha^{(-)} = \frac{1}{2}L\mathcal{J}_\alpha^{(-)}
$$

$$
\mathcal{W}_{\underline{abc}} = \frac{1}{16}\Big\{(\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha}T_{\alpha\underline{d}}^{\gamma} - i4(\sigma_{\underline{abc}})^{\alpha\beta}\Big[\nabla_\alpha\mathcal{J}_\beta^{(+)} - \frac{22}{25}\mathcal{J}_\alpha^{(+)}\mathcal{J}_\beta^{(+)}\Big]\Big\}
$$

$$
\mathcal{G} \;=\; 1 \,\oplus\, \ell\,\{(\square) \times [a_1, b_1, c_1, d_1, e_1]\} \,\oplus\, \bigoplus_{p=2}^{16} \tfrac{1}{p!}\,(\ell)^p\,\{(\square\,(\wedge\square)^{p-2}\wedge\square) \times [a_p, b_p, c_p, d_p, e_p]\}
$$

$$
\begin{aligned}
\mathcal{G} \;=\; & \ell \,\{(\square) \times [a_1, b_1, c_1, d_1, e_1]\} \\
& \oplus \bigoplus_{p=1}^{7} \frac{1}{p!} (\ell)^{2p+1} \{(\square\,(\wedge\square)^{2p-1} \wedge \square) \times [a_{2p+1}, b_{2p+1}, c_{2p+1}, d_{2p+1}, e_{2p+1}]\} \\
& \oplus 1 \oplus \bigoplus_{p=1}^{8} \frac{1}{(2p)!} (\ell)^{2p} \{(\square\,(\wedge\square)^{2(p-1)} \wedge \square) \times [a_{2p}, b_{2p}, c_{2p}, d_{2p}, e_{2p}]\} \quad ,
\end{aligned}
$$

$$
\square \wedge \square \;=\; \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \;=\; \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}
$$

$$
\begin{aligned}
\mathcal{V}_{[0,0,1,0,0]} &= [0,0,1,0,0] \otimes \mathcal{V} \quad , \quad & \mathcal{V}_{[1,0,1,0,1]} &= [1,0,1,0,1] \otimes \mathcal{V} \quad , \\
\mathcal{V}_{[3,0,0,0,1]} &= [3,0,0,0,1] \otimes \mathcal{V} \quad , \quad & \mathcal{V}_{[4,0,0,0,0]} &= [4,0,0,0,0] \otimes \mathcal{V} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\{\mathcal{F}\}_{\rm SGI} &= (\,\chi_{\alpha},\, \psi_{\underline{a}}{}^{\alpha}\,) &&= ([0,0,0,1,0],\, [1,0,0,0,1]) \\
\{\mathcal{B}\}_{\rm SGI} &= (\,\varphi,\, B_{\underline{a}\,\underline{b}},\, {\rm e}_{\underline{a}}{}^{\underline{m}}\,) &&= ([0,0,0,0,0],\, [0,1,0,0,0],\, [2,0,0,0,0])
\end{aligned}
$$

$$
\begin{aligned}
\{\mathcal{F}\}_{\rm IIAMGM} &= (\,\chi_{\dot{\alpha}},\, \psi_{\underline{a}}{}^{\dot{\alpha}}\,) &&= ([0,0,0,0,1],\, [1,0,0,1,0]) \\
\{\mathcal{B}\}_{\rm IIAMGM} &= (\,B_{\underline{a}},\, A_{\underline{a}\,\underline{b}\,\underline{c}}\,) &&= ([1,0,0,0,0],\, [0,0,1,0,0])
\end{aligned}
$$

$$
\begin{aligned}
\{\mathcal{F}\}_{\rm IIBMGM} &= (\,\chi'_{\alpha},\, \psi'_{\underline{a}}{}^{\alpha}\,) &&= ([0,0,0,1,0],\, [1,0,0,0,1]) \\
\{\mathcal{B}\}_{\rm IIBMGM} &= (\,A,\, B'_{\underline{a}\,\underline{b}},\, A_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,) &&= ([0,0,0,0,0],\, [0,1,0,0,0],\, [0,0,0,1,1])
\end{aligned}
$$

$$
\mathcal{V}_{IIA}(x^{\underline{a}},\, \theta^{\alpha},\, \theta^{\dot{\alpha}}) \;=\; \mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\, \mathcal{V}^{(1)}_{\dot{\alpha}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \mathcal{V}^{(2)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \dots
$$

$$
\mathcal{V}_{IIB}(x^{\underline{a}},\, \theta^{\alpha},\, \theta^{\alpha}) \;=\; \mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\alpha}\, \mathcal{V}^{(1)}_{\alpha}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\alpha}\theta^{\beta}\, \mathcal{V}^{(2)}_{\alpha\beta}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \dots
$$

$$
\mathcal{V}_{[4,0,0,0,0]}(x^{\underline{a}},\, \theta^{\alpha}) \;=\; \mathcal{V}_{\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4\}}(x^{\underline{a}},\, \theta^{\alpha})
$$

$$
\begin{aligned}
\delta_S T_{\underline{ab}}{}^{\gamma} \ &= \frac{3}{2} L T_{\underline{ab}}{}^{\gamma} + i\frac{1}{5}(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\nabla_{\alpha} L) - i\frac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha} L) T_{\beta|\underline{b}]}{}^{\gamma} \\
&\quad - i\frac{1}{5}(\sigma_{[\underline{a}})^{\dot\alpha\dot\beta}(\nabla_{\dot\alpha} L) T_{\dot\beta|\underline{b}]}{}^{\gamma} + i\frac{1}{5}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_{\delta} L) T_{\underline{ab}}{}^{\underline{c}} \quad , \\
\delta_S T_{\underline{ab}}{}^{\dot\gamma} \ &= \frac{3}{2} L T_{\underline{ab}}{}^{\dot\gamma} + i\frac{1}{5}(\sigma_{[\underline{a}})^{\dot\alpha\dot\gamma}(\nabla_{\underline{b}]}\nabla_{\dot\alpha} L) - i\frac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha} L) T_{\beta|\underline{b}]}{}^{\dot\gamma} \\
&\quad - i\frac{1}{5}(\sigma_{[\underline{a}})^{\dot\alpha\dot\beta}(\nabla_{\dot\alpha} L) T_{\dot\beta|\underline{b}]}{}^{\dot\gamma} + i\frac{1}{5}(\sigma_{\underline{c}})^{\dot\gamma\dot\delta}(\nabla_{\dot\delta} L) T_{\underline{ab}}{}^{\underline{c}} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_S R_{\alpha\underline{b}}{}^{\underline{de}} \ &= \frac{3}{2} L R_{\alpha\underline{b}}{}^{\underline{de}} - T_{\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \frac{1}{5} T_{\alpha\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \frac{1}{5} T_{\alpha\underline{b}}{}^{\dot\gamma}(\sigma^{\underline{de}})_{\dot\gamma}{}^{\dot\delta}(\nabla_{\dot\delta} L) \\
&\quad + \frac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\nabla_{\beta} L) + i\frac{1}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta} L) R_{\alpha\beta}{}^{\underline{de}} + i\frac{1}{5}(\sigma_{\underline{b}})^{\dot\delta\dot\beta}(\nabla_{\dot\delta} L) R_{\alpha\dot\beta}{}^{\underline{de}} \\
&\quad - (\nabla_{\alpha}\nabla^{[\underline{d}} L)\delta_{\underline{b}}{}^{\underline{e}]} \quad , \\
\delta_S R_{\dot\alpha\underline{b}}{}^{\underline{de}} \ &= \frac{3}{2} L R_{\dot\alpha\underline{b}}{}^{\underline{de}} - T_{\dot\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \frac{1}{5} T_{\dot\alpha\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \frac{1}{5} T_{\dot\alpha\underline{b}}{}^{\dot\gamma}(\sigma^{\underline{de}})_{\dot\gamma}{}^{\dot\delta}(\nabla_{\dot\delta} L) \\
&\quad + \frac{1}{5}(\sigma^{\underline{de}})_{\dot\alpha}{}^{\dot\beta}(\nabla_{\underline{b}}\nabla_{\dot\beta} L) + i\frac{1}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta} L) R_{\dot\alpha\beta}{}^{\underline{de}} + i\frac{1}{5}(\sigma_{\underline{b}})^{\dot\delta\dot\beta}(\nabla_{\dot\delta} L) R_{\dot\alpha\dot\beta}{}^{\underline{de}} \\
&\quad - (\nabla_{\dot\alpha}\nabla^{[\underline{d}} L)\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_S R_{\underline{ab}}{}^{\underline{de}} \ &= 2 L R_{\underline{ab}}{}^{\underline{de}} - T_{\underline{ab}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \frac{1}{5} T_{\underline{ab}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \frac{1}{5} T_{\underline{ab}}{}^{\dot\gamma}(\sigma^{\underline{de}})_{\dot\gamma}{}^{\dot\delta}(\nabla_{\dot\delta} L) \\
&\quad - i\frac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha} L) R_{\beta|\underline{b}]}{}^{\underline{de}} - i\frac{1}{5}(\sigma_{[\underline{a}})^{\dot\alpha\dot\beta}(\nabla_{\dot\alpha} L) R_{\dot\beta|\underline{b}]}{}^{\underline{de}} - (\nabla_{[\underline{a}}\nabla^{[\underline{d}} L)\delta_{\underline{b}]}{}^{\underline{e}]}
\end{aligned}
$$

$$
\delta_S T_{\underline{ab}}{}^{\gamma} \ = (\tfrac{1}{2} L + \overline{L}) T_{\underline{ab}}{}^{\gamma} + i(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\overline{\nabla}_{\alpha}(\tfrac{1}{32} L + \tfrac{27}{160}\overline{L})) - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_{\alpha}(\tfrac{1}{32} L + \tfrac{27}{160}\overline{L})) T_{\beta|\underline{b}]}{}^{\gamma}
$$

$$
- i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}(\frac{1}{32}\overline{L} + \frac{27}{160} L)) T_{\overline{\beta}|\underline{b}]}{}^{\gamma} + i(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_{\delta}(\frac{1}{32} L + \frac{27}{160}\overline{L})) T_{\underline{ab}}{}^{\underline{c}}
$$

$$
\delta_S T_{\underline{ab}}{}^{\overline{\gamma}} \ = (\tfrac{1}{2}\overline{L} + L) T_{\underline{ab}}{}^{\overline{\gamma}} + i(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\nabla_{\alpha}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160} L)) - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_{\alpha}(\tfrac{1}{32} L + \tfrac{27}{160}\overline{L})) T_{\beta|\underline{b}]}{}^{\overline{\gamma}}
$$

$$
- i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}(\frac{1}{32}\overline{L} + \frac{27}{160} L)) T_{\overline{\beta}|\underline{b}]}{}^{\overline{\gamma}} + i(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_{\delta}(\frac{1}{32}\overline{L} + \frac{27}{160} L)) T_{\underline{ab}}{}^{\underline{c}}
$$

$$
\delta_S R_{\alpha\underline{b}}{}^{\underline{de}} \ = (\tfrac{1}{2}\overline{L} + L) R_{\alpha\underline{b}}{}^{\underline{de}} - \tfrac{1}{2} T_{\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5} T_{\alpha\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \tfrac{1}{5} T_{\alpha\underline{b}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\overline{\nabla}_{\delta}\overline{L})
$$

$$
\begin{aligned}
&+ \tfrac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\nabla_{\beta} L) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_{\delta}(\tfrac{1}{32} L + \tfrac{27}{160}\overline{L})) R_{\alpha\beta}{}^{\underline{de}} \\
&+ i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160} L)) R_{\alpha\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}(\nabla_{\alpha}\nabla^{[\underline{d}}(L + \overline{L}))\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned}
$$

$$
\delta_S R_{\overline{\alpha}\underline{b}}{}^{\underline{de}} \ = (\tfrac{1}{2} L + \overline{L}) R_{\overline{\alpha}\underline{b}}{}^{\underline{de}} - \tfrac{1}{2} T_{\overline{\alpha}\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5} T_{\overline{\alpha}\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \tfrac{1}{5} T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\overline{\nabla}_{\delta}\overline{L})
$$

$$
+ \frac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\overline{\nabla}_{\beta}\overline{L}) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_{\delta}(\frac{1}{32} L + \frac{27}{160}\overline{L})) R_{\overline{\alpha}\beta}{}^{\underline{de}} + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta}(\frac{1}{32}\overline{L} + \frac{27}{160} L)) R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}}
$$

$$- \tfrac{1}{2}(\overline{\nabla}_\alpha \nabla^{[\underline{d}}(L+\overline{L}))\delta_{\underline{b}}{}^{\underline{e}]} \quad ,$$

$$\begin{aligned}\delta_S R_{\underline{ab}}{}^{\underline{de}} &= (L+\overline{L})R_{\underline{ab}}{}^{\underline{de}} - \tfrac{1}{2}T_{\underline{ab}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L+\overline{L})) + \tfrac{1}{5}T_{\underline{ab}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) + \tfrac{1}{5}T_{\underline{ab}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^\delta(\overline{\nabla}_\delta \overline{L}) \\ &\quad - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_\alpha(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))R_{\beta|\underline{b}]}{}^{\underline{de}} - i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_\alpha(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))R_{\overline{\beta}|\underline{b}]}{}^{\underline{de}} \\ &\quad - \tfrac{1}{2}(\nabla_{[\underline{a}}\nabla^{[\underline{d}}(L+\overline{L}))\delta_{\underline{b}]}{}^{\underline{e}]} \quad .\end{aligned}$$

$$\begin{aligned}\delta_S T_{\underline{ab}}{}^{\gamma} &= \tfrac{3}{2}L\, T_{\underline{ab}}{}^{\gamma} \;+\; i\,\tfrac{2}{5}(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\nabla_\alpha L) \;-\; i\,\tfrac{2}{5}\,(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_\alpha L)\, T_{\beta|\underline{b}]}{}^{\gamma} \\ &\quad +\; i\,\tfrac{2}{5}\,(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta L)\, T_{\underline{ab}}{}^{\underline{c}} \quad , \\ \delta_S R_{\alpha\underline{b}}{}^{\underline{de}} &= \tfrac{3}{2}L\, R_{\alpha\underline{b}}{}^{\underline{de}} \;-\; T_{\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) \;+\; \tfrac{1}{5}\, T_{\alpha\underline{b}}{}^{\gamma}\,(\sigma^{\underline{de}})_\gamma{}^\delta\,(\nabla_\delta L) \\ &\quad +\; \tfrac{1}{5}\,(\sigma^{\underline{de}})_\alpha{}^\beta(\nabla_{\underline{b}}\nabla_\beta L) \;+\; i\,\tfrac{2}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta L)R_{\alpha\beta}{}^{\underline{de}} \;-\; (\nabla_\alpha\nabla^{[\underline{d}}L)\delta_{\underline{b}}{}^{\underline{e}]} \quad ,\end{aligned}$$

$$\begin{aligned}\delta_S R_{\underline{ab}}{}^{\underline{de}} &= 2L\, R_{\underline{ab}}{}^{\underline{de}} \;-\; T_{\underline{ab}}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) \;+\; \tfrac{1}{5}T_{\underline{ab}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) \\ &\quad -\; i\,\tfrac{2}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_\alpha L)R_{\beta|\underline{b}]}{}^{\underline{de}} \;-\; (\nabla_{[\underline{a}}\nabla^{[\underline{d}}L)\delta_{\underline{b}]}{}^{\underline{e}]} \quad .\end{aligned}$$

Dynkin $\mathcal{V}_{[0,0,1,0,0]}$

- Nivel-0: [0, 0, 1, 0, 0]
- Nivel-1: [0,0,0,0,1] ⊕ [1,0,0,1,0] ⊕ [0,1,0,0,1] ⊕ [0,0,1,1,0]
- Nivel-2: [0,0,0,0,0] ⊕ [0,1,0,0,0] ⊕ [2,0,0,0,0] ⊕ (2)[0,0,0,1,1] ⊕ [0,2,0,0,0] ⊕ [1,0,1,0,0] ⊕ [1,0,0,2,0] ⊕ [1,0,0,0,2] ⊕ [0,0,2,0,0] ⊕ [0,1,0,1,1]
- Nivel-3: [0,0,0,1,0] ⊕ (2)[1,0,0,0,1] ⊕ (2)[0,1,0,1,0] ⊕ [0,0,0,0,3] ⊕ [2,0,0,1,0] ⊕ (2)[0,0,1,0,1] ⊕ [0,0,0,2,1] ⊕ (2)[1,1,0,0,1] ⊕ [0,2,0,1,0] ⊕ [1,0,1,1,0] ⊕ [1,0,0,1,2] ⊕ [0,1,1,0,1]
- Nivel-4: (2)[0,0,1,0,0] ⊕ [0,0,0,0,2] ⊕ (2)[1,1,0,0,0] ⊕ (3)[1,0,0,1,1] ⊕ (2)[0,1,1,0,0] ⊕ [0,1,0,2,0] ⊕ (3)[0,1,0,0,2] ⊕ (2)[2,0,1,0,0] ⊕ [1,2,0,0,0] ⊕ [2,0,0,0,2] ⊕ [0,0,0,1,3] ⊕ [0,0,1,1,1] ⊕ [0,2,1,0,0] ⊕ (2)[1,1,0,1,1] ⊕ [1,0,1,0,2]
- Nivel-5: [1,0,0,1,0] ⊕ (3)[0,1,0,0,1] ⊕ (2)[2,0,0,0,1] ⊕ [0,0,1,1,0] ⊕ (2)[0,0,0,1,2] ⊕ [3,0,0,1,0] ⊕ (3)[1,1,0,1,0] ⊕ (2)[1,0,0,0,3] ⊕ (2)[0,2,0,0,1] ⊕ (4)[1,0,1,0,1] ⊕ [1,0,0,2,1] ⊕ (2)[2,1,0,0,1] ⊕ [0,1,1,1,0] ⊕ [0,0,1,0,3] ⊕ (2)[0,1,0,1,2] ⊕ [2,0,1,1,0] ⊕ [1,2,0,1,0] ⊕ [2,0,0,1,2] ⊕ [1,1,1,0,1]

- Nivel-6: [2,0,0,0,0] ⊕ [0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ (2)[0,2,0,0,0] ⊕ (2)[1,0,1,0,0] ⊕ [1,0,0,2,0] ⊕ (3)[1,0,0,0,2] ⊕ (2)[2,1,0,0,0] ⊕ [0,0,0,0,4] ⊕ (2)[0,0,2,0,0] ⊕ (3)[0,1,0,1,1] ⊕ (2)[0,0,1,0,2] ⊕ (4)[2,0,0,1,1] ⊕ [0,0,0,2,2] ⊕ [3,0,1,0,0] ⊕ [2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ [3,0,0,0,2] ⊕ (3)[1,1,1,0,0] ⊕ [1,1,0,2,0] ⊕ (3)[1,1,0,0,2] ⊕ [1,0,0,1,3] ⊕ (2)[1,0,1,1,1] ⊕ [0,2,0,1,1] ⊕ [2,0,2,0,0] ⊕ [0,1,1,0,2] ⊕ [2,1,0,1,1]
- Nivel-7: [1,0,0,0,1] ⊕ [0,1,0,1,0] ⊕ [0,0,0,0,3] ⊕ (2)[2,0,0,1,0] ⊕ (2)[0,0,1,0,1] ⊕ [0,0,0,2,1] ⊕ (3)[3,0,0,0,1] ⊕ (4)[1,1,0,0,1] ⊕ [4,0,0,1,0] ⊕ [0,2,0,1,0] ⊕ (4)[1,0,1,1,0] ⊕ (3)[1,0,0,1,2] ⊕ (3)[2,1,0,1,0] ⊕ [0,1,0,0,3] ⊕ [2,0,0,0,3] ⊕ (3)[0,1,1,0,1] ⊕ [0,0,2,1,0] ⊕ [0,1,0,2,1] ⊕ (3)[2,0,1,0,1] ⊕ [1,2,0,0,1] ⊕ [3,1,0,0,1] ⊕ (2)[2,0,0,2,1] ⊕ [0,0,1,1,2] ⊕ [3,0,1,1,0] ⊕ [1,1,1,1,0] ⊕ [1,0,2,0,1] ⊕ [1,1,0,1,2]
- Nivel-8: [0,0,1,0,0] ⊕ [3,0,0,0,0] ⊕ [1,1,0,0,0] ⊕ (3)[1,0,0,1,1] ⊕ (2)[0,1,1,0,0] ⊕ (2)[0,1,0,2,0] ⊕ (2)[0,1,0,0,2] ⊕ (5)[2,0,1,0,0] ⊕ [1,2,0,0,0] ⊕ (2)[3,1,0,0,0] ⊕ (2)[2,0,0,2,0] ⊕ (2)[2,0,0,0,2] ⊕ (3)[0,0,1,1,1] ⊕ (2)[1,0,2,0,0] ⊕ (3)[3,0,0,1,1] ⊕ [0,2,1,0,0] ⊕ (4)[1,1,0,1,1] ⊕ [4,0,1,0,0] ⊕ (2)[1,0,1,2,0] ⊕ (2)[1,0,1,0,2] ⊕ [1,0,0,2,2] ⊕ [2,1,1,0,0] ⊕ [0,0,3,0,0] ⊕ [2,1,0,2,0] ⊕ [2,1,0,0,2] ⊕ [0,1,1,1,1] ⊕ [2,0,1,1,1]
- Nivel-9: [1,0,0,1,0] ⊕ [0,1,0,0,1] ⊕ [0,0,0,3,0] ⊕ (2)[2,0,0,0,1] ⊕ (2)[0,0,1,1,0] ⊕ [0,0,0,1,2] ⊕ (3)[3,0,0,1,0] ⊕ (4)[1,1,0,1,0] ⊕ [4,0,0,0,1] ⊕ [0,2,0,0,1] ⊕ (4)[1,0,1,0,1] ⊕ (3)[1,0,0,2,1] ⊕ (3)[2,1,0,0,1] ⊕ [0,1,0,3,0] ⊕ [2,0,0,3,0] ⊕ (3)[0,1,1,1,0] ⊕ [0,0,2,0,1] ⊕ [0,1,0,1,2] ⊕ (3)[2,0,1,1,0] ⊕ [1,2,0,1,0] ⊕ [3,1,0,1,0] ⊕ (2)[2,0,0,1,2] ⊕ [0,0,1,2,1] ⊕ [3,0,1,0,1] ⊕ [1,1,1,0,1] ⊕ [1,0,2,1,0] ⊕ [1,1,0,2,1]
- Nivel-10: [2,0,0,0,0] ⊕ [0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ (2)[0,2,0,0,0] ⊕ (2)[1,0,1,0,0] ⊕ (3)[1,0,0,2,0] ⊕ [1,0,0,0,2] ⊕ (2)[2,1,0,0,0] ⊕ [0,0,0,4,0] ⊕ (2)[0,0,2,0,0] ⊕ (3)[0,1,0,1,1] ⊕ (2)[0,0,1,2,0] ⊕ (4)[2,0,0,1,1] ⊕ [0,0,0,2,2] ⊕ [3,0,1,0,0] ⊕ [2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ [3,0,0,0,2] ⊕ (3)[1,1,1,0,0] ⊕ (3)[1,1,0,2,0] ⊕ [1,1,0,0,2] ⊕ [1,0,0,3,1] ⊕ (2)[1,0,1,1,1] ⊕ [0,2,0,1,1] ⊕ [2,0,2,0,0] ⊕ [0,1,1,2,0] ⊕ [2,1,0,1,1]

- Nivel-11: $[1,0,0,0,1] \oplus (3)[0,1,0,1,0] \oplus (2)[2,0,0,1,0] \oplus [0,0,1,0,1] \oplus (2)[0,0,0,2,1] \oplus [3,0,0,0,1] \oplus (3)[1,1,0,0,1] \oplus (2)[1,0,0,3,0] \oplus (2)[0,2,0,1,0] \oplus (4)[1,0,1,1,0] \oplus [1,0,0,1,2] \oplus (2)[2,1,0,1,0] \oplus [0,1,1,0,1] \oplus [0,0,1,3,0] \oplus (2)[0,1,0,2,1] \oplus [2,0,1,0,1] \oplus [1,2,0,0,1] \oplus [2,0,0,2,1] \oplus [1,1,1,1,0]$
- Nivel-12: $(2)[0,0,1,0,0] \oplus [0,0,0,2,0] \oplus (2)[1,1,0,0,0] \oplus (3)[1,0,0,1,1] \oplus (2)[0,1,1,0,0] \oplus (3)[0,1,0,2,0] \oplus [0,1,0,0,2] \oplus (2)[2,0,1,0,0] \oplus [1,2,0,0,0] \oplus [2,0,0,2,0] \oplus [0,0,0,3,1] \oplus [0,0,1,1,1] \oplus [0,2,1,0,0] \oplus (2)[1,1,0,1,1] \oplus [1,0,1,2,0]$
- Nivel-13: $[0,0,0,0,1] \oplus (2)[1,0,0,1,0] \oplus (2)[0,1,0,0,1] \oplus [0,0,0,3,0] \oplus [2,0,0,0,1] \oplus (2)[0,0,1,1,0] \oplus [0,0,0,1,2] \oplus (2)[1,1,0,1,0] \oplus [0,2,0,0,1] \oplus [1,0,1,0,1] \oplus [1,0,0,2,1] \oplus [0, 1, 1, 1, 0]$
- Nivel-14: $[0,0,0,0,0] \oplus [0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (2)[0,0,0,1,1] \oplus [0,2,0,0,0] \oplus [1,0,1,0,0] \oplus [1,0,0,2,0] \oplus [1,0,0,0,2] \oplus [0,0,2,0,0] \oplus [0,1,0,1,1]$
- Nivel-15: $[0,0,0,1,0] \oplus [1,0,0,0,1] \oplus [0,1,0,1,0] \oplus [0,0,1,0,1]$
- Nivel-16: $[0, 0, 1, 0, 0]$

Dynkin $\mathcal{V}_{[1,0,1,0,1]}$

- Nivel-0: $[1, 0, 1, 0, 1]$
- Nivel-1: $[1,0,1,0,0] \oplus [1,0,0,0,2] \oplus [0,0,2,0,0] \oplus [0,1,0,1,1] \oplus [0,0,1,0,2] \oplus [2,0,0,1,1] \oplus [1,1,1,0,0] \oplus [1,1,0,0,2] \oplus [1,0,1,1,1]$
- Nivel-2: $[1,0,0,0,1] \oplus [0,1,0,1,0] \oplus [0,0,0,0,3] \oplus [2,0,0,1,0] \oplus (2)[0,0,1,0,1] \oplus [0,0,0,2,1] \oplus [3,0,0,0,1] \oplus (3)[1,1,0,0,1] \oplus [0,2,0,1,0] \oplus (3)[1,0,1,1,0] \oplus (3)[1,0,0,1,2] \oplus [2,1,0,1,0] \oplus [0,1,0,0,3] \oplus [2,0,0,0,3] \oplus (3)[0,1,1,0,1] \oplus [0,0,2,1,0] \oplus [0,1,0,2,1] \oplus (2)[2,0,1,0,1] \oplus [1,2,0,0,1] \oplus [2,0,0,2,1] \oplus [0,0,1,1,2] \oplus [1,1,1,1,0] \oplus [1,0,2,0,1] \oplus [1,1,0,1,2]$
- Nivel-3: $[0,0,1,0,0] \oplus [0,0,0,2,0] \oplus [0,0,0,0,2] \oplus [3,0,0,0,0] \oplus (2)[1,1,0,0,0] \oplus (5)[1,0,0,1,1] \oplus (4)[0,1,1,0,0] \oplus (3)[0,1,0,2,0] \oplus (4)[0,1,0,0,2] \oplus (4)[2,0,1,0,0] \oplus (2)[1,2,0,0,0] \oplus [3,1,0,0,0] \oplus (2)[2,0,0,2,0] \oplus (4)[2,0,0,0,2] \oplus [0,0,0,3,1] \oplus$

(2)[0,0,0,1,3] ⊕ (5)[0,0,1,1,1] ⊕ [1,0,0,0,4] ⊕ (4)[1,0,2,0,0] ⊕ (2)[3,0,0,1,1] ⊕ (2)[0,2,1,0,0] ⊕ (7)[1,1,0,1,1] ⊕ [0,2,0,2,0] ⊕ (2)[0,2,0,0,2] ⊕ (2)[1,0,1,2,0] ⊕ (5)[1,0,1,0,2] ⊕ (2)[1,0,0,2,2] ⊕ (2)[2,1,1,0,0] ⊕ [0,0,3,0,0] ⊕ [2,1,0,2,0] ⊕ (2)[2,1,0,0,2] ⊕ [0,0,2,0,2] ⊕ [0,1,0,1,3] ⊕ (3)[0,1,1,1,1] ⊕ [2,0,0,1,3] ⊕ (2)[2,0,1,1,1] ⊕ [1,1,2,0,0] ⊕ [1,2,0,1,1] ⊕ [1,1,1,0,2]

- Nivel-4: (2)[1,0,0,1,0] ⊕ (4)[0,1,0,0,1] ⊕ (2)[0,0,0,3,0] ⊕ (4)[2,0,0,0,1] ⊕ (5)[0,0,1,1,0] ⊕ (4)[0,0,0,1,2] ⊕ (3)[3,0,0,1,0] ⊕ (8)[1,1,0,1,0] ⊕ (4)[1,0,0,0,3] ⊕ [4,0,0,0,1] ⊕ (5)[0,2,0,0,1] ⊕ (11)[1,0,1,0,1] ⊕ (7)[1,0,0,2,1] ⊕ (7)[2,1,0,0,1] ⊕ (2)[0,1,0,3,0] ⊕ [2,0,0,3,0] ⊕ (8)[0,1,1,1,0] ⊕ (3)[0,0,1,0,3] ⊕ (5)[0,0,2,0,1] ⊕ (8)[0,1,0,1,2] ⊕ (7)[2,0,1,1,0] ⊕ [0,0,0,2,3] ⊕ (4)[1,2,0,1,0] ⊕ (2)[3,1,0,1,0] ⊕ (7)[2,0,0,1,2] ⊕ (3)[0,0,1,2,1] ⊕ [0,3,0,0,1] ⊕ [3,0,0,0,3] ⊕ (4)[1,1,0,0,3] ⊕ (3)[3,0,1,0,1] ⊕ (9)[1,1,1,0,1] ⊕ [2,2,0,0,1] ⊕ [3,0,0,2,1] ⊕ [1,0,0,1,4] ⊕ (3)[1,0,2,1,0] ⊕ (4)[1,1,0,2,1] ⊕ (2)[0,2,1,1,0] ⊕ (4)[1,0,1,1,2] ⊕ (2)[0,2,0,1,2] ⊕ (2)[0,1,2,0,1] ⊕ [0,1,1,0,3] ⊕ (2)[2,1,1,1,0] ⊕ [2,0,2,0,1] ⊕ (2)[2,1,0,1,2] ⊕ [2,0,1,0,3] ⊕ [1, 2, 1, 0, 1]
- Nivel-5: [0,1,0,0,0] ⊕ [2,0,0,0,0] ⊕ (3)[0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ (4)[0,2,0,0,0] ⊕ (7)[1,0,1,0,0] ⊕ (6)[1,0,0,2,0] ⊕ (5)[1,0,0,0,2] ⊕ (5)[2,1,0,0,0] ⊕ [0,0,0,4,0] ⊕ [0,0,0,0,4] ⊕ (6)[0,0,2,0,0] ⊕ (13)[0,1,0,1,1] ⊕ (6)[0,0,1,2,0] ⊕ (7)[0,0,1,0,2] ⊕ (2)[0,3,0,0,0] ⊕ (12)[2,0,0,1,1] ⊕ (5)[0,0,0,2,2] ⊕ [4,1,0,0,0] ⊕ (6)[3,0,1,0,0] ⊕ (3)[2,2,0,0,0] ⊕ (3)[3,0,0,2,0] ⊕ (5)[3,0,0,0,2] ⊕ (13)[1,1,1,0,0] ⊕ (9)[1,1,0,2,0] ⊕ (12)[1,1,0,0,2] ⊕ (3)[1,0,0,3,1] ⊕ (7)[1,0,0,1,3] ⊕ (2)[0,1,0,0,4] ⊕ (6)[0,1,2,0,0] ⊕ (16)[1,0,1,1,1] ⊕ (9)[0,2,0,1,1] ⊕ (2)[4,0,0,1,1] ⊕ (2)[2,0,0,0,4] ⊕ (6)[2,0,2,0,0] ⊕ (4)[0,1,1,2,0] ⊕ (8)[0,1,1,0,2] ⊕ (11)[2,1,0,1,1] ⊕ (4)[1,2,1,0,0] ⊕ (4)[0,1,0,2,2] ⊕ (3)[2,0,1,2,0] ⊕ (8)[2,0,1,0,2] ⊕ (2)[3,1,1,0,0] ⊕ (3)[0,0,2,1,1] ⊕ (2)[0,0,1,1,3] ⊕ (2)[1,2,0,2,0] ⊕ (4)[1,2,0,0,2] ⊕ (3)[2,0,0,2,2] ⊕ [3,1,0,2,0] ⊕ (2)[3,1,0,0,2] ⊕ [1,0,3,0,0] ⊕ [0,3,0,1,1] ⊕ [0,2,2,0,0] ⊕ [1,0,1,0,4] ⊕ [3,0,0,1,3] ⊕ (3)[1,1,0,1,3] ⊕

(2)[1,0,2,0,2] ⊕ (2)[3,0,1,1,1] ⊕ (6)[1,1,1,1,1] ⊕ [0,2,1,0,2] ⊕ [2,2,0,1,1] ⊕ [2,1,2,0,0] ⊕ [2,1,1,0,2]

- Nivel-6: [0,0,0,1,0] ⊕ (3)[1,0,0,0,1] ⊕ (7)[0,1,0,1,0] ⊕ (2)[0,0,0,0,3] ⊕ (6)[2,0,0,1,0] ⊕ (7)[0,0,1,0,1] ⊕ (7)[0,0,0,2,1] ⊕ (6)[3,0,0,0,1] ⊕ (13)[1,1,0,0,1] ⊕ (5)[1,0,0,3,0] ⊕ (3)[4,0,0,1,0] ⊕ (10)[0,2,0,1,0] ⊕ (17)[1,0,1,1,0] ⊕ (14)[1,0,0,1,2] ⊕ (12)[2,1,0,1,0] ⊕ (6)[0,1,0,0,3] ⊕ [5,0,0,0,1] ⊕ (7)[2,0,0,0,3] ⊕ (16)[0,1,1,0,1] ⊕ (2)[0,0,0,1,4] ⊕ (2)[0,0,1,3,0] ⊕ (7)[0,0,2,1,0] ⊕ (13)[0,1,0,2,1] ⊕ (17)[2,0,1,0,1] ⊕ (2)[0,0,0,3,2] ⊕ (11)[1,2,0,0,1] ⊕ (7)[3,1,0,0,1] ⊕ (11)[2,0,0,2,1] ⊕ (9)[0,0,1,1,2] ⊕ [1,0,0,0,5] ⊕ (3)[0,3,0,1,0] ⊕ [3,0,0,3,0] ⊕ (3)[1,1,0,3,0] ⊕ (7)[3,0,1,1,0] ⊕ [4,1,0,1,0] ⊕ (16)[1,1,1,1,0] ⊕ (4)[2,2,0,1,0] ⊕ (7)[3,0,0,1,2] ⊕ (10)[1,0,2,0,1] ⊕ (6)[1,0,1,0,3] ⊕ (3)[0,2,0,0,3] ⊕ (16)[1,1,0,1,2] ⊕ [4,0,0,0,3] ⊕ (6)[0,2,1,0,1] ⊕ (3)[1,0,0,2,3] ⊕ [1,3,0,0,1] ⊕ (2)[4,0,1,0,1] ⊕ (6)[1,0,1,2,1] ⊕ (4)[0,2,0,2,1] ⊕ (4)[2,1,0,0,3] ⊕ [0,0,3,0,1] ⊕ [3,2,0,0,1] ⊕ [4,0,0,2,1] ⊕ (3)[0,1,2,1,0] ⊕ [0,1,0,1,4] ⊕ [0,0,2,0,3] ⊕ (9)[2,1,1,0,1] ⊕ (3)[2,0,2,1,0] ⊕ [2,0,0,1,4] ⊕ (4)[2,1,0,2,1] ⊕ (4)[0,1,1,1,2] ⊕ (2)[1,2,1,1,0] ⊕ (4)[2,0,1,1,2] ⊕ [3,1,1,1,0] ⊕ (2)[1,2,0,1,2] ⊕ [3,1,0,1,2] ⊕ [3,0,2,0,1] ⊕ (2)[1,1,2,0,1] ⊕ [1,1,1,0,3]
- Nivel-7: [1,0,0,0,0] ⊕ (4)[0,0,1,0,0] ⊕ (3)[0,0,0,2,0] ⊕ (2)[0,0,0,0,2] ⊕ (2)[3,0,0,0,0] ⊕ (5)[1,1,0,0,0] ⊕ (13)[1,0,0,1,1] ⊕ [5,0,0,0,0] ⊕ (12)[0,1,1,0,0] ⊕ (11)[0,1,0,2,0] ⊕ (9)[0,1,0,0,2] ⊕ (12)[2,0,1,0,0] ⊕ (8)[1,2,0,0,0] ⊕ (5)[3,1,0,0,0] ⊕ (9)[2,0,0,2,0] ⊕ (9)[2,0,0,0,2] ⊕ (5)[0,0,0,3,1] ⊕ (5)[0,0,0,1,3] ⊕ (15)[0,0,1,1,1] ⊕ [1,0,0,4,0] ⊕ (3)[1,0,0,0,4] ⊕ (13)[1,0,2,0,0] ⊕ (12)[3,0,0,1,1] ⊕ (10)[0,2,1,0,0] ⊕ (28)[1,1,0,1,1] ⊕ (3)[1,3,0,0,0] ⊕ (4)[4,0,1,0,0] ⊕ (8)[0,2,0,2,0] ⊕ (9)[0,2,0,0,2] ⊕ (13)[1,0,1,2,0] ⊕ (15)[1,0,1,0,2] ⊕ (2)[3,2,0,0,0] ⊕ (2)[4,0,0,2,0] ⊕ (4)[4,0,0,0,2] ⊕ (12)[1,0,0,2,2] ⊕ (14)[2,1,1,0,0] ⊕ (3)[0,0,3,0,0] ⊕ (9)[2,1,0,2,0] ⊕ (13)[2,1,0,0,2] ⊕ [0,0,1,0,4] ⊕ [0,0,0,2,4] ⊕ (2)[0,0,2,2,0] ⊕ (4)[0,0,2,0,2] ⊕ (4)[0,1,0,3,1] ⊕ (7)[0,1,0,1,3] ⊕ (16)[0,1,1,1,1] ⊕ [5,0,0,1,1] ⊕ (3)[2,0,0,3,1] ⊕ (7)[2,0,0,1,3] ⊕ [0,3,1,0,0] ⊕ (3)[0,0,1,2,2] ⊕ [3,0,0,0,4] ⊕ (17)[2,0,1,1,1] ⊕ (4)[3,0,2,0,0] ⊕ (2)[1,1,0,0,4] ⊕

$[0,3,0,2,0] \oplus [0,3,0,0,2] \oplus (7)[1,1,2,0,0] \oplus (11)[1,2,0,1,1] \oplus (7)[3,1,0,1,1] \oplus [4,1,1,0,0] \oplus (2)[2,2,1,0,0] \oplus (2)[3,0,1,2,0] \oplus (5)[3,0,1,0,2] \oplus (5)[1,1,1,2,0] \oplus (9)[1,1,1,0,2] \oplus [4,1,0,0,2] \oplus [2,2,0,2,0] \oplus (2)[2,2,0,0,2] \oplus (2)[3,0,0,2,2] \oplus [0,1,3,0,0] \oplus (5)[1,1,0,2,2] \oplus (4)[1,0,2,1,1] \oplus (2)[1,0,1,1,3] \oplus [0,2,0,1,3] \oplus [2,0,3,0,0] \oplus (2)[0,2,1,1,1] \oplus [0,1,2,0,2] \oplus [4,0,1,1,1] \oplus [2,1,0,1,3] \oplus [2,0,2,0,2] \oplus (3)[2,1,1,1,1]$

- Nivel-8: $(2)[0,0,0,0,1] \oplus (5)[1,0,0,1,0] \oplus (8)[0,1,0,0,1] \oplus (3)[0,0,0,3,0] \oplus (6)[2,0,0,0,1] \oplus (10)[0,0,1,1,0] \oplus (7)[0,0,0,1,2] \oplus (6)[3,0,0,1,0] \oplus (16)[1,1,0,1,0] \oplus (5)[1,0,0,0,3] \oplus (4)[4,0,0,0,1] \oplus (12)[0,2,0,0,1] \oplus (19)[1,0,1,0,1] \oplus (17)[1,0,0,2,1] \oplus (15)[2,1,0,0,1] \oplus (6)[0,1,0,3,0] \oplus [5,0,0,1,0] \oplus (5)[2,0,0,3,0] \oplus (19)[0,1,1,1,0] \oplus [0,0,0,4,1] \oplus (4)[0,0,1,0,3] \oplus (8)[0,0,2,0,1] \oplus (17)[0,1,0,1,2] \oplus (19)[2,0,1,1,0] \oplus (4)[0,0,0,2,3] \oplus (14)[1,2,0,1,0] \oplus (8)[3,1,0,1,0] \oplus (16)[2,0,0,1,2] \oplus (10)[0,0,1,2,1] \oplus (5)[0,3,0,0,1] \oplus (4)[3,0,0,0,3] \oplus (8)[1,1,0,0,3] \oplus (12)[3,0,1,0,1] \oplus (3)[4,1,0,0,1] \oplus (22)[1,1,1,0,1] \oplus (7)[2,2,0,0,1] \oplus (7)[3,0,0,2,1] \oplus (2)[1,0,0,1,4] \oplus (10)[1,0,2,1,0] \oplus (3)[1,0,1,3,0] \oplus (2)[0,2,0,3,0] \oplus (18)[1,1,0,2,1] \oplus (7)[0,2,1,1,0] \oplus (3)[1,0,0,3,2] \oplus (2)[1,3,0,1,0] \oplus (3)[4,0,1,1,0] \oplus (12)[1,0,1,1,2] \oplus (7)[0,2,0,1,2] \oplus (2)[2,1,0,3,0] \oplus [0,0,3,1,0] \oplus [3,2,0,1,0] \oplus (3)[4,0,0,1,2] \oplus (5)[0,1,2,0,1] \oplus (2)[0,1,1,0,3] \oplus (10)[2,1,1,1,0] \oplus (2)[0,1,0,2,3] \oplus (6)[2,0,2,0,1] \oplus (10)[2,1,0,1,2] \oplus (3)[2,0,1,0,3] \oplus [5,0,1,0,1] \oplus (4)[0,1,1,2,1] \oplus [0,0,2,1,2] \oplus [1,2,0,0,3] \oplus [2,0,0,2,3] \oplus [3,1,0,0,3] \oplus (3)[1,2,1,0,1] \oplus (4)[2,0,1,2,1] \oplus (3)[3,1,1,0,1] \oplus (2)[1,2,0,2,1] \oplus [3,1,0,2,1] \oplus [3,0,2,1,0] \oplus (2)[1,1,2,1,0] \oplus [1,0,3,0,1] \oplus [3,0,1,1,2] \oplus (2)[1,1,1,1,2]$
- Nivel-9: $[0,0,0,0,0] \oplus (3)[0,1,0,0,0] \oplus (2)[2,0,0,0,0] \oplus (6)[0,0,0,1,1] \oplus [4,0,0,0,0] \oplus (6)[0,2,0,0,0] \oplus (10)[1,0,1,0,0] \oplus (8)[1,0,0,2,0] \oplus (7)[1,0,0,0,2] \oplus (6)[2,1,0,0,0] \oplus [0,0,0,4,0] \oplus [0,0,0,0,4] \oplus (8)[0,0,2,0,0] \oplus (19)[0,1,0,1,1] \oplus (9)[0,0,1,2,0] \oplus (8)[0,0,1,0,2] \oplus (5)[0,3,0,0,0] \oplus (16)[2,0,0,1,1] \oplus (8)[0,0,0,2,2] \oplus (2)[4,1,0,0,0] \oplus (8)[3,0,1,0,0] \oplus (6)[2,2,0,0,0] \oplus (6)[3,0,0,2,0] \oplus (6)[3,0,0,0,2] \oplus (19)[1,1,1,0,0] \oplus (16)[1,1,0,2,0] \oplus (15)[1,1,0,0,2] \oplus (8)[1,0,0,3,1] \oplus (8)[1,0,0,1,3] \oplus [0,4,0,0,0] \oplus$

[0,1,0,4,0] ⊕ [0,1,0,0,4] ⊕ (8)[0,1,2,0,0] ⊕ (24)[1,0,1,1,1] ⊕ (16)[0,2,0,1,1] ⊕ (5)[4,0,0,1,1] ⊕ [2,0,0,4,0] ⊕ [2,0,0,0,4] ⊕ [5,0,1,0,0] ⊕ (9)[2,0,2,0,0] ⊕ [2,3,0,0,0] ⊕ [5,0,0,0,2] ⊕ (9)[0,1,1,2,0] ⊕ (9)[0,1,1,0,2] ⊕ (20)[2,1,0,1,1] ⊕ [0,0,0,3,3] ⊕ (8)[1,2,1,0,0] ⊕ (9)[0,1,0,2,2] ⊕ (9)[2,0,1,2,0] ⊕ (10)[2,0,1,0,2] ⊕ (5)[3,1,1,0,0] ⊕ (5)[0,0,2,1,1] ⊕ (2)[0,0,1,3,1] ⊕ (2)[0,0,1,1,3] ⊕ (6)[1,2,0,2,0] ⊕ (7)[1,2,0,0,2] ⊕ (8)[2,0,0,2,2] ⊕ (3)[3,1,0,2,0] ⊕ (5)[3,1,0,0,2] ⊕ (2)[1,0,3,0,0] ⊕ (2)[0,3,0,1,1] ⊕ [0,2,2,0,0] ⊕ [3,0,0,3,1] ⊕ (2)[3,0,0,1,3] ⊕ (3) [1,1,0,3,1] ⊕ (4)[1,1,0,1,3] ⊕ (2)[1,0,2,2,0] ⊕ (2)[1,0,2,0,2] ⊕ [4,0,2,0,0] ⊕ (6)[3,0,1,1,1] ⊕ (12)[1,1,1,1,1] ⊕ [4,1,0,1,1] ⊕ [0,2,1,2,0] ⊕ [0,2,1,0,2] ⊕ (3)[2,2,0,1,1] ⊕ (2)[2,1,2,0,0] ⊕ [4,0,1,0,2] ⊕ (2)[1,0,1,2,2] ⊕ [0,2,0,2,2] ⊕ [2,1,1,2,0] ⊕ (2)[2,1,1,0,2] ⊕ [0,1,2,1,1] ⊕ [2,1,0,2,2] ⊕ [2,0,2,1,1]

- Nivel-10: (2)[0,0,0,1,0] ⊕ (5)[1,0,0,0,1] ⊕ (8)[0,1,0,1,0] ⊕ (3)[0,0,0,0,3] ⊕ (6)[2,0,0,1,0] ⊕ (9)[0,0,1,0,1] ⊕ (7)[0,0,0,2,1] ⊕ (4)[3,0,0,0,1] ⊕ (14)[1,1,0,0,1] ⊕ (5)[1,0,0,3,0] ⊕ (2)[4,0,0,1,0] ⊕ (11)[0,2,0,1,0] ⊕ (17)[1,0,1,1,0] ⊕ (14)[1,0,0,1,2] ⊕ (12)[2,1,0,1,0] ⊕ (5)[0,1,0,0,3] ⊕ [5,0,0,0,1] ⊕ (4)[2,0,0,0,3] ⊕ (16)[0,1,1,0,1] ⊕ [0,0,0,1,4] ⊕ (3)[0,0,1,3,0] ⊕ (7)[0,0,2,1,0] ⊕ (14)[0,1,0,2,1] ⊕ (14)[2,0,1,0,1] ⊕ (3)[0,0,0,3,2] ⊕ (12)[1,2,0,0,1] ⊕ (6)[3,1,0,0,1] ⊕ (13)[2,0,0,2,1] ⊕ (8)[0,0,1,1,2] ⊕ (4)[0,3,0,1,0] ⊕ (2)[3,0,0,3,0] ⊕ (5)[1,1,0,3,0] ⊕ (7)[3,0,1,1,0] ⊕ [4,1,0,1,0] ⊕ (17)[1,1,1,1,0] ⊕ (5)[2,2,0,1,0] ⊕ (6)[3,0,0,1,2] ⊕ [1,0,0,4,1] ⊕ (7)[1,0,2,0,1] ⊕ (2)[1,0,1,0,3] ⊕ (2)[0,2,0,0,3] ⊕ (15)[1,1,0,1,2] ⊕ [4,0,0,0,3] ⊕ (6)[0,2,1,0,1] ⊕ (3)[1,0,0,2,3] ⊕ (2)[1,3,0,0,1] ⊕ (2)[4,0,1,0,1] ⊕ (9)[1,0,1,2,1] ⊕ (5)[0,2,0,2,1] ⊕ (2)[2,1,0,0,3] ⊕ [3,2,0,0,1] ⊕ [4,0,0,2,1] ⊕ (3)[0,1,2,1,0] ⊕ [0,1,1,3,0] ⊕ (8)[2,1,1,0,1] ⊕ [0,1,0,3,2] ⊕ (3)[2,0,2,1,0] ⊕ (6)[2,1,0,2,1] ⊕ [2,0,1,3,0] ⊕ (3)[0,1,1,1,2] ⊕ [0,0,2,2,1] ⊕ [2,0,0,3,2] ⊕ (2)[1,2,1,1,0] ⊕ (3)[2,0,1,1,2] ⊕ [3,1,1,1,0] ⊕ (2)[1,2,0,1,2] ⊕ [3,1,0,1,2] ⊕ [3,0,2,0,1] ⊕ [1,1,2,0,1] ⊕ [1,1,1,2,1]
- Nivel-11: [1,0,0,0,0] ⊕ (4)[0,0,1,0,0] ⊕ (2)[0,0,0,2,0] ⊕ (3)[0,0,0,0,2] ⊕ [3,0,0,0,0] ⊕ (5)[1,1,0,0,0] ⊕ (11)[1,0,0,1,1] ⊕ (10)[0,1,1,0,0] ⊕ (7)[0,1,0,2,0] ⊕ (9)[0,1,0,0,2] ⊕

$(8)[2,0,1,0,0] \oplus (6)[1,2,0,0,0] \oplus (2)[3,1,0,0,0] \oplus (6)[2,0,0,2,0] \oplus (6)[2,0,0,0,2] \oplus$
$(3)[0,0,0,3,1] \oplus (4)[0,0,0,1,3] \oplus (11)[0,0,1,1,1] \oplus [1,0,0,4,0] \oplus [1,0,0,0,4] \oplus$
$(8)[1,0,2,0,0] \oplus (6)[3,0,0,1,1] \oplus (7)[0,2,1,0,0] \oplus (19)[1,1,0,1,1] \oplus (2)[1,3,0,0,0] \oplus$
$[4,0,1,0,0] \oplus (5)[0,2,0,2,0] \oplus (6)[0,2,0,0,2] \oplus (9)[1,0,1,2,0] \oplus (9)[1,0,1,0,2] \oplus$
$[3,2,0,0,0] \oplus [4,0,0,2,0] \oplus [4,0,0,0,2] \oplus (8)[1,0,0,2,2] \oplus (8)[2,1,1,0,0] \oplus [0,0,3,0,0] \oplus$
$(6)[2,1,0,2,0] \oplus (6)[2,1,0,0,2] \oplus (2)[0,0,2,2,0] \oplus [0,0,2,0,2] \oplus (3)[0,1,0,3,1] \oplus$
$(3)[0,1,0,1,3] \oplus (10)[0,1,1,1,1] \oplus (3)[2,0,0,3,1] \oplus (2)[2,0,0,1,3] \oplus [0,3,1,0,0] \oplus$
$(2)[0,0,1,2,2] \oplus (9)[2,0,1,1,1] \oplus [3,0,2,0,0] \oplus [0,3,0,0,2] \oplus (3)[1,1,2,0,0] \oplus$
$(7)[1,2,0,1,1] \oplus (3)[3,1,0,1,1] \oplus [2,2,1,0,0] \oplus [3,0,1,2,0] \oplus [3,0,1,0,2] \oplus$
$(3)[1,1,1,2,0] \oplus (3)[1,1,1,0,2] \oplus [2,2,0,0,2] \oplus [3,0,0,2,2] \oplus (3)[1,1,0,2,2] \oplus$
$[1,0,2,1,1] \oplus [1,0,1,3,1] \oplus [0,2,1,1,1] \oplus [2,1,1,1,1]$

- Nivel-12: $[0,0,0,0,1] \oplus (3)[1,0,0,1,0] \oplus (6)[0,1,0,0,1] \oplus [0,0,0,3,0] \oplus (4)[2,0,0,0,1] \oplus$ $(5)[0,0,1,1,0] \oplus (5)[0,0,0,1,2] \oplus (2)[3,0,0,1,0] \oplus (8)[1,1,0,1,0] \oplus (4)[1,0,0,0,3]$
$\oplus (6)[0,2,0,0,1] \oplus (11)[1,0,1,0,1] \oplus (7)[1,0,0,2,1] \oplus (6)[2,1,0,0,1]$
$\oplus (2)[0,1,0,3,0] \oplus (2)[2,0,0,3,0] \oplus (8)[0,1,1,1,0] \oplus (2)[0,0,1,0,3]$
$\oplus (4)[0,0,2,0,1] \oplus (8)[0,1,0,1,2] \oplus (7)[2,0,1,1,0] \oplus [0,0,0,2,3]$
$\oplus (5)[1,2,0,1,0] \oplus (2)[3,1,0,1,0] \oplus (6)[2,0,0,1,2] \oplus (4)[0,0,1,2,1]$
$\oplus (2)[0,3,0,0,1] \oplus (2)[1,1,0,0,3] \oplus (2)[3,0,1,0,1] \oplus (8)[1,1,1,0,1]$
$\oplus (2)[2,2,0,0,1] \oplus (2)[3,0,0,2,1] \oplus (3)[1,0,2,1,0] \oplus [1,0,1,3,0]$
$\oplus (6)[1,1,0,2,1] \oplus (2)[0,2,1,1,0] \oplus [1,0,0,3,2] \oplus (3)[1,0,1,1,2]$
$\oplus (2)[0,2,0,1,2] \oplus [0,1,2,0,1] \oplus (2)[2,1,1,1,0] \oplus (2)[2,1,0,1,2]$
$\oplus [0,1,1,2,1] \oplus [1,2,1,0,1] \oplus [2,0,1,2,1]$
- Nivel-13: $[0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (2)[0,0,0,1,1] \oplus (2)[0,2,0,0,0] \oplus (4)[1,0,1,0,0] \oplus$ $(2)[1,0,0,2,0] \oplus (4)[1,0,0,0,2] \oplus (2)[2,1,0,0,0] \oplus [0,0,0,0,4] \oplus (3)[0,0,2,0,0] \oplus$ $(6)[0,1,0,1,1] \oplus (2)[0,0,1,2,0] \oplus (4)[0,0,1,0,2] \oplus [0,3,0,0,0] \oplus (5)[2,0,0,1,1] \oplus$ $(2)[0,0,0,2,2] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus (5)[1,1,1,0,0] \oplus$ $(3)[1,1,0,2,0] \oplus (5)[1,1,0,0,2] \oplus [1,0,0,3,1] \oplus (2)[1,0,0,1,3] \oplus (2)[0,1,2,0,0] \oplus$

(6)[1,0,1,1,1] ⊕ (3)[0,2,0,1,1] ⊕ [2,0,2,0,0] ⊕ [0,1,1,2,0] ⊕ (2)[0,1,1,0,2] ⊕ (3)[2,1,0,1,1] ⊕ [1,2,1,0,0] ⊕ [0,1,0,2,2] ⊕ [2,0,1,2,0] ⊕ [2,0,1,0,2] ⊕ [0,0,2,1,1] ⊕ [1,2,0,0,2] ⊕ [2,0,0,2,2] ⊕ [1,1,1,1,1]

- Nivel-14: [1,0,0,0,1] ⊕ [0,1,0,1,0] ⊕ [0,0,0,0,3] ⊕ [2,0,0,1,0] ⊕ (2)[0,0,1,0,1] ⊕ [0,0,0,2,1] ⊕ [3,0,0,0,1] ⊕ (3)[1,1,0,0,1] ⊕ [0,2,0,1,0] ⊕ (3)[1,0,1,1,0] ⊕ (3)[1,0,0,1,2] ⊕ [2,1,0,1,0] ⊕ [0,1,0,0,3] ⊕ [2,0,0,0,3] ⊕ (3)[0,1,1,0,1] ⊕ [0,0,2,1,0] ⊕ [0,1,0,2,1] ⊕ (2)[2,0,1,0,1] ⊕ [1,2,0,0,1] ⊕ [2,0,0,2,1] ⊕ [0,0,1,1,2] ⊕ [1,1,1,1,0] ⊕ [1,0,2,0,1] ⊕ [1,1,0,1,2]
- Nivel-15: [1,0,0,1,1] ⊕ [0,1,1,0,0] ⊕ [0,1,0,0,2] ⊕ [2,0,1,0,0] ⊕ [2,0,0,0,2] ⊕ [0,0,1,1,1] ⊕ [1,0,2,0,0] ⊕ [1,1,0,1,1] ⊕ [1,0,1,0,2]
- Nivel-16: [1, 0, 1, 0, 1]

Dynkin $\mathcal{V}_{[3,0,0,0,1]}$

- Nivel-0: [3, 0, 0, 0, 1]
- Nivel-1: [3,0,0,0,0] ⊕ [2,0,1,0,0] ⊕ [3,1,0,0,0] ⊕ [2,0,0,0,2] ⊕ [3,0,0,1,1]
- Nivel-2: [2,0,0,0,1] ⊕ (2)[3,0,0,1,0] ⊕ [1,1,0,1,0] ⊕ [4,0,0,0,1] ⊕ [1,0,1,0,1] ⊕ (2)[2,1,0,0,1] ⊕ [2,0,1,1,0] ⊕ [3,1,0,1,0] ⊕ [2,0,0,1,2] ⊕ [3,0,1,0,1]
- Nivel-3: [4,0,0,0,0] ⊕ [0,2,0,0,0] ⊕ [1,0,1,0,0] ⊕ [1,0,0,2,0] ⊕ (2)[2,1,0,0,0] ⊕ [0,1,0,1,1] ⊕ (3)[2,0,0,1,1] ⊕ [4,1,0,0,0] ⊕ (3)[3,0,1,0,0] ⊕ [2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ (2)[3,0,0,0,2] ⊕ (2)[1,1,1,0,0] ⊕ [1,1,0,2,0] ⊕ [1,1,0,0,2] ⊕ [1,0,1,1,1] ⊕ [4,0,0,1,1] ⊕ [2,0,2,0,0] ⊕ (2)[2,1,0,1,1] ⊕ [2,0,1,0,2] ⊕ [3,1,1,0,0] ⊕ [3,1,0,0,2]
- Nivel-4: (2)[0,1,0,1,0] ⊕ (2)[2,0,0,1,0] ⊕ [0,0,0,2,1] ⊕ (3)[3,0,0,0,1] ⊕ (3)[1,1,0,0,1] ⊕ [1,0,0,3,0] ⊕ (2)[4,0,0,1,0] ⊕ (2)[0,2,0,1,0] ⊕ (3)[1,0,1,1,0] ⊕ [1,0,0,1,2] ⊕ (4)[2,1,0,1,0] ⊕ [5,0,0,0,1] ⊕ [2,0,0,0,3] ⊕ [0,1,1,0,1] ⊕ [0,1,0,2,1] ⊕ (4)[2,0,1,0,1] ⊕ (2)[1,2,0,0,1] ⊕ (4)[3,1,0,0,1] ⊕ (2)[2,0,0,2,1] ⊕ (2)[3,0,1,1,0] ⊕ [4,1,0,1,0] ⊕ (2)[1,1,1,1,0] ⊕ [2,2,0,1,0] ⊕ (2)[3,0,0,1,2] ⊕ [1,0,2,0,1] ⊕ [1,1,0,1,2] ⊕ [4,0,0,0,3] ⊕ [4,0,1,0,1] ⊕ [2,1,0,0,3] ⊕ [3,2,0,0,1] ⊕ (2)[2,1,1,0,1]

- Nivel-5: [0,0,1,0,0] ⊕ [0,0,0,2,0] ⊕ [3,0,0,0,0] ⊕ (2)[1,1,0,0,0] ⊕ (3)[1,0,0,1,1] ⊕ [5,0,0,0,0] ⊕ (3)[0,1,1,0,0] ⊕ (3)[0,1,0,2,0] ⊕ [0,1,0,0,2] ⊕ (4)[2,0,1,0,0] ⊕ (3)[1,2,0,0,0] ⊕ (3)[3,1,0,0,0] ⊕ (3)[2,0,0,2,0] ⊕ (2)[2,0,0,0,2] ⊕ [0,0,0,3,1] ⊕ [0,0,1,1,1] ⊕ (2)[1,0,2,0,0] ⊕ (5)[3,0,0,1,1] ⊕ [5,1,0,0,0] ⊕ (2)[0,2,1,0,0] ⊕ (6)[1,1,0,1,1] ⊕ [1,3,0,0,0] ⊕ (3)[4,0,1,0,0] ⊕ [0,2,0,2,0] ⊕ [0,2,0,0,2] ⊕ (2)[1,0,1,2,0] ⊕ [1,0,1,0,2] ⊕ (2)[3,2,0,0,0] ⊕ [4,0,0,2,0] ⊕ (3)[4,0,0,0,2] ⊕ [1,0,0,2,2] ⊕ (5)[2,1,1,0,0] ⊕ (2)[2,1,0,2,0] ⊕ (4)[2,1,0,0,2] ⊕ [0,1,1,1,1] ⊕ [5,0,0,1,1] ⊕ [2,0,0,1,3] ⊕ [3,0,0,0,4] ⊕ (3)[2,0,1,1,1] ⊕ [3,0,2,0,0] ⊕ [1,1,2,0,0] ⊕ (2)[1,2,0,1,1] ⊕ (3)[3,1,0,1,1] ⊕ [4,1,1,0,0] ⊕ [2,2,1,0,0] ⊕ (2)[3,0,1,0,2] ⊕ [1,1,1,0,2] ⊕ [4,1,0,0,2] ⊕ [2,2,0,0,2]
- Nivel-6: [0,0,0,0,1] ⊕ (2)[1,0,0,1,0] ⊕ (3)[0,1,0,0,1] ⊕ [0,0,0,3,0] ⊕ (2)[2,0,0,0,1] ⊕ (3)[0,0,1,1,0] ⊕ [0,0,0,1,2] ⊕ (3)[3,0,0,1,0] ⊕ (6)[1,1,0,1,0] ⊕ (3)[4,0,0,0,1] ⊕ (4)[0,2,0,0,1] ⊕ (4)[1,0,1,0,1] ⊕ (4)[1,0,0,2,1] ⊕ (6)[2,1,0,0,1] ⊕ [0,1,0,3,0] ⊕ (2)[5,0,0,1,0] ⊕ [2,0,0,3,0] ⊕ (4)[0,1,1,1,0] ⊕ (2)[0,1,0,1,2] ⊕ (6)[2,0,1,1,0] ⊕ (5)[1,2,0,1,0] ⊕ (5)[3,1,0,1,0] ⊕ [6,0,0,0,1] ⊕ (4)[2,0,0,1,2] ⊕ [0,0,1,2,1] ⊕ (2)[0,3,0,0,1] ⊕ (2)[3,0,0,0,3] ⊕ [1,1,0,0,3] ⊕ (6)[3,0,1,0,1] ⊕ (4)[4,1,0,0,1] ⊕ (5)[1,1,1,0,1] ⊕ (4)[2,2,0,0,1] ⊕ (2)[3,0,0,2,1] ⊕ [1,0,2,1,0] ⊕ (3)[1,1,0,2,1] ⊕ [0,2,1,1,0] ⊕ [5,1,0,1,0] ⊕ [1,3,0,1,0] ⊕ (2)[4,0,1,1,0] ⊕ [1,0,1,1,2] ⊕ [0,2,0,1,2] ⊕ [3,2,0,1,0] ⊕ (2)[4,0,0,1,2] ⊕ (3)[2,1,1,1,0] ⊕ [2,0,2,0,1] ⊕ (3)[2,1,0,1,2] ⊕ [2,0,1,0,3] ⊕ [5,0,1,0,1] ⊕ [3,1,0,0,3] ⊕ [1,2,1,0,1] ⊕ (2)[3,1,1,0,1]
- Nivel-7: [0,0,0,0,0] ⊕ (2)[0,1,0,0,0] ⊕ [2,0,0,0,0] ⊕ (3)[0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ (3)[0,2,0,0,0] ⊕ (4)[1,0,1,0,0] ⊕ (3)[1,0,0,2,0] ⊕ (2)[1,0,0,0,2] ⊕ (3)[2,1,0,0,0] ⊕ (2)[0,0,2,0,0] ⊕ [6,0,0,0,0] ⊕ (6)[0,1,0,1,1] ⊕ (2)[0,0,1,2,0] ⊕ [0,0,1,0,2] ⊕ (3)[0,3,0,0,0] ⊕ (6)[2,0,0,1,1] ⊕ [0,0,0,2,2] ⊕ (3)[4,1,0,0,0] ⊕ (5)[3,0,1,0,0] ⊕ (4)[2,2,0,0,0] ⊕ (3)[3,0,0,2,0] ⊕ (3)[3,0,0,0,2] ⊕ (7)[1,1,1,0,0] ⊕ (5)[1,1,0,2,0] ⊕ (4)[1,1,0,0,2] ⊕ [1,0,0,3,1] ⊕ [1,0,0,1,3] ⊕

  [0,4,0,0,0] ⊕ [0,1,2,0,0] ⊕ [6,1,0,0,0] ⊕ (5)[1,0,1,1,1] ⊕ (5)[0,2,0,1,1] ⊕

$(5)[4,0,0,1,1] \oplus (3)[5,0,1,0,0] \oplus (3)[2,0,2,0,0] \oplus [4,2,0,0,0] \oplus [2,3,0,0,0] \oplus$ $[5,0,0,2,0] \oplus (2)[5,0,0,0,2] \oplus [0,1,1,2,0] \oplus [0,1,1,0,2] \oplus (9)[2,1,0,1,1] \oplus$ $(4)[1,2,1,0,0] \oplus [0,1,0,2,2] \oplus (2)[2,0,1,2,0] \oplus (3)[2,0,1,0,2] \oplus (5)[3,1,1,0,0] \oplus$ $(2)[1,2,0,2,0] \oplus (3)[1,2,0,0,2] \oplus (2)[2,0,0,2,2] \oplus (2)[3,1,0,2,0] \oplus (4)[3,1,0,0,2] \oplus$ $[6,0,0,1,1] \oplus [0,3,0,1,1] \oplus [3,0,0,1,3] \oplus [1,1,0,1,3] \oplus [4,0,2,0,0] \oplus (3)[3,0,1,1,1] \oplus$ $(2)[1,1,1,1,1] \oplus (2)[4,1,0,1,1] \oplus (2)[2,2,0,1,1] \oplus [2,1,2,0,0] \oplus [4,0,1,0,2] \oplus [2,1,1,0,2]$

- Nivel-8: $(2)[0,0,0,1,0] \oplus (3)[1,0,0,0,1] \oplus (4)[0,1,0,1,0] \oplus [0,0,0,0,3] \oplus (3)[2,0,0,1,0] \oplus$ $(4)[0,0,1,0,1] \oplus (2)[0,0,0,2,1] \oplus (3)[3,0,0,0,1] \oplus (6)[1,1,0,0,1] \oplus [1,0,0,3,0] \oplus$ $(3)[4,0,0,1,0] \oplus (5)[0,2,0,1,0] \oplus (6)[1,0,1,1,0] \oplus (4)[1,0,0,1,2] \oplus (7)[2,1,0,1,0] \oplus$ $[0,1,0,0,3] \oplus (3)[5,0,0,0,1] \oplus [2,0,0,0,3] \oplus (5)[0,1,1,0,1] \oplus [0,0,2,1,0] \oplus$ $(3)[0,1,0,2,1] \oplus (6)[2,0,1,0,1] \oplus (7)[1,2,0,0,1] \oplus (6)[3,1,0,0,1] \oplus (2)[6,0,0,1,0] \oplus$ $(5)[2,0,0,2,1] \oplus [0,0,1,1,2] \oplus (3)[0,3,0,1,0] \oplus [3,0,0,3,0] \oplus [1,1,0,3,0] \oplus [7,0,0,0,1] \oplus$ $(6)[3,0,1,1,0] \oplus (4)[4,1,0,1,0] \oplus (6)[1,1,1,1,0] \oplus (5)[2,2,0,1,0] \oplus (4)[3,0,0,1,2] \oplus$ $[1,0,2,0,1] \oplus [0,2,0,0,3] \oplus (5)[1,1,0,1,2] \oplus [4,0,0,0,3] \oplus (2)[0,2,1,0,1] \oplus [1,0,0,2,3] \oplus$ $(2)[5,1,0,0,1] \oplus (2)[1,3,0,0,1] \oplus (4)[4,0,1,0,1] \oplus [1,0,1,2,1] \oplus [0,2,0,2,1] \oplus$ $[2,1,0,0,3] \oplus (2)[3,2,0,0,1] \oplus (2)[4,0,0,2,1] \oplus (5)[2,1,1,0,1] \oplus [2,0,2,1,0] \oplus$ $(3)[2,1,0,2,1] \oplus [5,0,1,1,0] \oplus [5,0,0,1,2] \oplus [1,2,1,1,0] \oplus [2,0,1,1,2] \oplus (2)[3,1,1,1,0] \oplus$ $[1,2,0,1,2] \oplus [3,1,0,1,2] \oplus [3,0,2,0,1]$
- Nivel-9: $[1,0,0,0,0] \oplus (3)[0,0,1,0,0] \oplus [0,0,0,2,0] \oplus (2)[0,0,0,0,2] \oplus [3,0,0,0,0] \oplus (3)$ $[1,1,0,0,0] \oplus (5)[1,0,0,1,1] \oplus [5,0,0,0,0] \oplus (5)[0,1,1,0,0] \oplus (2)[0,1,0,2,0] \oplus$ $(4)[0,1,0,0,2] \oplus (5)[2,0,1,0,0] \oplus (4)[1,2,0,0,0] \oplus (3)[3,1,0,0,0] \oplus (3)[2,0,0,2,0] \oplus$ $(3)[2,0,0,0,2] \oplus [0,0,0,1,3] \oplus [7,0,0,0,0] \oplus (3)[0,0,1,1,1] \oplus (3)[1,0,2,0,0] \oplus$ $(6)[3,0,0,1,1] \oplus (2)[5,1,0,0,0] \oplus (4)[0,2,1,0,0] \oplus (9)[1,1,0,1,1] \oplus (3)[1,3,0,0,0] \oplus$ $(4)[4,0,1,0,0] \oplus (2)[0,2,0,2,0] \oplus (3)[0,2,0,0,2] \oplus (2)[1,0,1,2,0] \oplus (3)[1,0,1,0,2] \oplus$ $(3)[3,2,0,0,0] \oplus (3)[4,0,0,2,0] \oplus (2)[4,0,0,0,2] \oplus (2)[1,0,0,2,2] \oplus (7)[2,1,1,0,0] \oplus$ $(5)[2,1,0,2,0] \oplus (4)[2,1,0,0,2] \oplus [0,1,0,1,3] \oplus (2)[0,1,1,1,1] \oplus (3)[5,0,0,1,1] \oplus$ $[2,0,0,3,1] \oplus [2,0,0,1,3] \oplus [6,0,1,0,0] \oplus [0,3,1,0,0] \oplus [6,0,0,0,2] \oplus (5)[2,0,1,1,1] \oplus$

(2)[3,0,2,0,0] ⊕ [0,3,0,0,2] ⊕ [1,1,2,0,0] ⊕ (5)[1,2,0,1,1] ⊕ (6)[3,1,0,1,1] ⊕ (2)[4,1,1,0,0] ⊕ (2)[2,2,1,0,0] ⊕ (2)[3,0,1,2,0] ⊕ [3,0,1,0,2] ⊕ [1,1,1,2,0] ⊕ [1,1,1,0,2] ⊕ [4,1,0,2,0] ⊕ [4,1,0,0,2] ⊕ [2,2,0,2,0] ⊕ [2,2,0,0,2] ⊕ [3,0,0,2,2] ⊕ [1,1,0,2,2] ⊕ [4,0,1,1,1] ⊕ [2, 1, 1, 1, 1]

- Nivel-10: [0,0,0,0,1] ⊕ (2)[1,0,0,1,0] ⊕ (4)[0,1,0,0,1] ⊕ (3)[2,0,0,0,1] ⊕ (2)[0,0,1,1,0] ⊕ (2)[0,0,0,1,2] ⊕ (3)[3,0,0,1,0] ⊕ (5)[1,1,0,1,0] ⊕ (2)[1,0,0,0,3] ⊕ (2)[4,0,0,0,1] ⊕ (4) [0,2,0,0,1] ⊕ (6)[1,0,1,0,1] ⊕ (2)[1,0,0,2,1] ⊕ (6)[2,1,0,0,1] ⊕ (2)[5,0,0,1,0] ⊕ [2,0,0,3,0] ⊕ (3) [0,1,1,1,0] ⊕ [0,0,1,0,3] ⊕ [0,0,2,0,1] ⊕ (3)[0,1,0,1,2] ⊕ (6)[2,0,1,1,0] ⊕ (5)[1,2,0,1,0] ⊕ (6)[3,1,0,1,0] ⊕ [6,0,0,0,1] ⊕ (4)[2,0,0,1,2] ⊕ (2)[0,3,0,0,1] ⊕ [1,1,0,0,3] ⊕ (4)[3,0,1,0,1] ⊕ (3)[4,1,0,0,1] ⊕ (5)[1,1,1,0,1] ⊕ (4)[2,2,0,0,1] ⊕ (4)[3,0,0,2,1] ⊕ [1,0,2,1,0] ⊕ (3)[1,1,0,2,1] ⊕ [4,0,0,3,0] ⊕ [0,2,1,1,0] ⊕ [5,1,0,1,0] ⊕ [1,3,0,1,0] ⊕ (3)[4,0,1,1,0] ⊕ [1,0,1,1,2] ⊕ [0,2,0,1,2] ⊕ [2,1,0,3,0] ⊕ (2)[3,2,0,1,0] ⊕ [4,0,0,1,2] ⊕ (4)[2,1,1,1,0] ⊕ (2)[2,1,0,1,2] ⊕ [5,0,1,0,1] ⊕ [1,2,1,0,1] ⊕ [2,0,1,2,1] ⊕ [3,1,1,0,1] ⊕ [3,1,0,2,1]
- Nivel-11: [0,1,0,0,0] ⊕ [2,0,0,0,0] ⊕ [0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ (2)[0,2,0,0,0] ⊕ (3) [1,0,1,0,0] ⊕ [1,0,0,2,0] ⊕ (3)[1,0,0,0,2] ⊕ (3)[2,1,0,0,0] ⊕ [0,0,0,0,4] ⊕ [0,0,2,0,0] ⊕ (3)[0,1,0,1,1] ⊕ (2)[0,0,1,0,2] ⊕ [0,3,0,0,0] ⊕ (5)[2,0,0,1,1] ⊕ (2)[4,1,0,0,0] ⊕ (4)[3,0,1,0,0] ⊕ (3)[2,2,0,0,0] ⊕ (3)[3,0,0,2,0] ⊕ (2)[3,0,0,0,2] ⊕ (5)[1,1,1,0,0] ⊕ (2)[1,1,0,2,0] ⊕ (4) [1,1,0,0,2] ⊕ [1,0,0,1,3] ⊕ [0,1,2,0,0] ⊕ (3)[1,0,1,1,1] ⊕ (2)[0,2,0,1,1] ⊕ (3)[4,0,0,1,1] ⊕ [5,0,1,0,0] ⊕ (2)[2,0,2,0,0] ⊕ [4,2,0,0,0] ⊕ [2,3,0,0,0] ⊕ [5,0,0,2,0] ⊕ [0,1,1,0,2] ⊕ (6)[2,1,0,1,1] ⊕ (2)[1,2,1,0,0] ⊕ (2)[2,0,1,2,0] ⊕ [2,0,1,0,2] ⊕ (3)[3,1,1,0,0] ⊕ [1,2,0,2,0] ⊕ [1,2,0,0,2] ⊕ [2,0,0,2,2] ⊕ (3)[3,1,0,2,0] ⊕ [3,1,0,0,2] ⊕ [3,0,0,3,1] ⊕ [3,0,1,1,1] ⊕ [1,1,1,1,1] ⊕ [4,1,0,1,1] ⊕ [2,2,0,1,1]
- Nivel-12: [1,0,0,0,1] ⊕ [0,1,0,1,0] ⊕ [0,0,0,0,3] ⊕ (2)[2,0,0,1,0] ⊕ [0,0,1,0,1] ⊕ (3)[3,0,0,0,1] ⊕ (4)[1,1,0,0,1] ⊕ (2)[4,0,0,1,0] ⊕ [0,2,0,1,0] ⊕ (2)[1,0,1,1,0] ⊕ (2)[1,0,0,1,2] ⊕ (4) [2,1,0,1,0] ⊕ [0,1,0,0,3] ⊕ [2,0,0,0,3] ⊕ (2)[0,1,1,0,1] ⊕

(4)[2,0,1,0,1] ⊕ (2)[1,2,0,0,1] ⊕ (3)[3,1,0,0,1] ⊕ (2)[2,0,0,2,1] ⊕ [3,0,0,3,0] ⊕ (3)[3,0,1,1,0] ⊕ (2)[4,1,0,1,0] ⊕ (2)[1,1,1,1,0] ⊕ (2)[2,2,0,1,0] ⊕ [3,0,0,1,2] ⊕ [1,0,2,0,1] ⊕ [1,1,0,1,2] ⊕ [3,2,0,0,1] ⊕ [4,0,0,2,1] ⊕ [2,1,1,0,1] ⊕ [2,1,0,2,1]

- Nivel-13: [3,0,0,0,0] ⊕ [1,1,0,0,0] ⊕ [1,0,0,1,1] ⊕ [0,1,1,0,0] ⊕ [0,1,0,0,2] ⊕ (3)[2,0,1,0,0] ⊕ [1,2,0,0,0] ⊕ (2)[3,1,0,0,0] ⊕ [2,0,0,2,0] ⊕ (2)[2,0,0,0,2] ⊕ [1,0,2,0,0] ⊕ (3)[3,0,0,1,1] ⊕ (2)[1,1,0,1,1] ⊕ [4,0,1,0,0] ⊕ [1,0,1,0,2] ⊕ [3,2,0,0,0] ⊕ [4,0,0,2,0] ⊕ (2)[2,1,1,0,0] ⊕ [2,1,0,2,0] ⊕ [2,1,0,0,2] ⊕ [2,0,1,1,1] ⊕ [3,1,0,1,1]
- Nivel-14: [2,0,0,0,1] ⊕ (2)[3,0,0,1,0] ⊕ [1,1,0,1,0] ⊕ [4,0,0,0,1] ⊕ [1,0,1,0,1] ⊕ (2)[2,1,0,0,1] ⊕ [2,0,1,1,0] ⊕ [3,1,0,1,0] ⊕ [2,0,0,1,2] ⊕ [3,0,1,0,1]
- Nivel-15: [4,0,0,0,0] ⊕ [2,1,0,0,0] ⊕ [2,0,0,1,1] ⊕ [3,0,1,0,0] ⊕ [3,0,0,0,2]
- Nivel-16: [3, 0, 0, 0, 1]

Dynkin $\mathcal{V}_{[4,0,0,0,0]}$

- Nivel-0: [4, 0, 0, 0, 0]
- Nivel-1: [3,0,0,0,1] ⊕ [4,0,0,1,0]
- Nivel-2: [2,0,1,0,0] ⊕ [3,1,0,0,0] ⊕ [3,0,0,1,1] ⊕ [4,0,1,0,0]
- Nivel-3: [3,0,0,1,0] ⊕ [1,1,0,1,0] ⊕ [4,0,0,0,1] ⊕ [2,1,0,0,1] ⊕ [2,0,1,1,0] ⊕ [3,1,0,1,0] ⊕ [3,0,1,0,1] ⊕ [4,1,0,0,1]
- Nivel-4: [4,0,0,0,0] ⊕ [0,2,0,0,0] ⊕ [1,0,0,2,0] ⊕ [2,1,0,0,0] ⊕ [2,0,0,1,1] ⊕ [4,1,0,0,0] ⊕ [3,0,1,0,0] ⊕ [2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ [3,0,0,0,2] ⊕ [1,1,1,0,0] ⊕ [1,1,0,2,0] ⊕ [4,0,0,1,1] ⊕ [2,0,2,0,0] ⊕ [4,2,0,0,0] ⊕ [5,0,0,0,2] ⊕ [2,1,0,1,1] ⊕ [3,1,1,0,0] ⊕ [3,1,0,0,2]
- Nivel-5: [0,1,0,1,0] ⊕ [2,0,0,1,0] ⊕ [3,0,0,0,1] ⊕ [1,1,0,0,1] ⊕ [1,0,0,3,0] ⊕ [4,0,0,1,0] ⊕ [0,2,0,1,0] ⊕ [1,0,1,1,0] ⊕ (2)[2,1,0,1,0] ⊕ [5,0,0,0,1] ⊕ [2,0,1,0,1] ⊕ [1,2,0,0,1] ⊕ (2)[3,1,0,0,1] ⊕ [2,0,0,2,1] ⊕ [3,0,1,1,0] ⊕ [4,1,0,1,0] ⊕ [1,1,1,1,0] ⊕ [2,2,0,1,0] ⊕ [3,0,0,1,2] ⊕ [4,0,0,0,3] ⊕ [5,1,0,0,1] ⊕ [4,0,1,0,1] ⊕ [3,2,0,0,1] ⊕ [2,1,1,0,1]

- Nivel-6: [0,0,1,0,0] ⊕ [1,1,0,0,0] ⊕ [1,0,0,1,1] ⊕ [0,1,1,0,0] ⊕ [0,1,0,2,0] ⊕ (2)[2,0,1,0,0] ⊕ [1,2,0,0,0] ⊕ [3,1,0,0,0] ⊕ [2,0,0,2,0] ⊕ (2)[3,0,0,1,1] ⊕ [5,1,0,0,0] ⊕ [0,2,1,0,0] ⊕ (2)[1,1,0,1,1] ⊕ [1,3,0,0,0] ⊕ (2)[4,0,1,0,0] ⊕ [1,0,1,2,0] ⊕ [3,2,0,0,0] ⊕ [4,0,0,0,2] ⊕ (2)[2,1,1,0,0] ⊕ [2,1,0,2,0] ⊕ [2,1,0,0,2] ⊕ [5,0,0,1,1] ⊕ [6,0,1,0,0] ⊕ [2,0,1,1,1] ⊕ [1,2,0,1,1] ⊕ (2)[3,1,0,1,1] ⊕ [4,1,1,0,0] ⊕ [2,2,1,0,0] ⊕ [3,0,1,0,2] ⊕ [4,1,0,0,2]
- Nivel-7: [0,0,0,0,1] ⊕ [1,0,0,1,0] ⊕ [0,1,0,0,1] ⊕ [2,0,0,0,1] ⊕ [0,0,1,1,0] ⊕ [3,0,0,1,0] ⊕ (2)[1,1,0,1,0] ⊕ [4,0,0,0,1] ⊕ [0,2,0,0,1] ⊕ [1,0,1,0,1] ⊕ [1,0,0,2,1] ⊕ (2)[2,1,0,0,1] ⊕ [5,0,0,1,0] ⊕ [0,1,1,1,0] ⊕ (2)[2,0,1,1,0] ⊕ (2)[1,2,0,1,0] ⊕ (2)[3,1,0,1,0] ⊕ [6,0,0,0,1] ⊕ [2,0,0,1,2] ⊕ [0,3,0,0,1] ⊕ [7,0,0,1,0] ⊕ (2)[3,0,1,0,1] ⊕ (2)[4,1,0,0,1] ⊕ [1,1,1,0,1] ⊕ (2)[2,2,0,0,1] ⊕ [3,0,0,2,1] ⊕ [1,1,0,2,1] ⊕ [5,1,0,1,0] ⊕ [1,3,0,1,0] ⊕ [4,0,1,1,0] ⊕ [3,2,0,1,0] ⊕ [4,0,0,1,2] ⊕ [2,1,1,1,0] ⊕ [2,1,0,1,2] ⊕ [5,0,1,0,1] ⊕ [3,1,1,0,1]
- Nivel-8: [0,0,0,0,0] ⊕ [0,1,0,0,0] ⊕ [2,0,0,0,0] ⊕ [0,0,0,1,1] ⊕ [4,0,0,0,0] ⊕ [0,2,0,0,0] ⊕ [1,0,1,0,0] ⊕ [1,0,0,2,0] ⊕ [1,0,0,0,2] ⊕ [2,1,0,0,0] ⊕ [0,0,2,0,0] ⊕ [6,0,0,0,0] ⊕ [0,1,0,1,1] ⊕ [0,3,0,0,0] ⊕ (2)[2,0,0,1,1] ⊕ [4,1,0,0,0] ⊕ [3,0,1,0,0] ⊕ (2)[2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ [3,0,0,0,2] ⊕ (2)[1,1,1,0,0] ⊕ [8,0,0,0,0] ⊕ [1,1,0,2,0] ⊕ [1,1,0,0,2] ⊕ [0,4,0,0,0] ⊕ [6,1,0,0,0] ⊕ [1,0,1,1,1] ⊕ [0,2,0,1,1] ⊕ (2)[4,0,0,1,1] ⊕ [5,0,1,0,0] ⊕ [2,0,2,0,0] ⊕ [4,2,0,0,0] ⊕ [2,3,0,0,0] ⊕ [5,0,0,2,0] ⊕ [5,0,0,0,2] ⊕ (3)[2,1,0,1,1] ⊕ [1,2,1,0,0] ⊕ (2)[3,1,1,0,0] ⊕ [1,2,0,2,0] ⊕ [1,2,0,0,2] ⊕ [2,0,0,2,2] ⊕ [3,1,0,2,0] ⊕ [3,1,0,0,2] ⊕ [6,0,0,1,1] ⊕ [4,0,2,0,0] ⊕ [3,0,1,1,1] ⊕ [4,1,0,1,1] ⊕ [2,2,0,1,1]
- Nivel-9: [0,0,0,1,0] ⊕ [1,0,0,0,1] ⊕ [0,1,0,1,0] ⊕ [2,0,0,1,0] ⊕ [0,0,1,0,1] ⊕ [3,0,0,0,1] ⊕ (2)[1,1,0,0,1] ⊕ [4,0,0,1,0] ⊕ [0,2,0,1,0] ⊕ [1,0,1,1,0] ⊕ [1,0,0,1,2] ⊕ (2)[2,1,0,1,0] ⊕ [5,0,0,0,1] ⊕ [0,1,1,0,1] ⊕ (2)[2,0,1,0,1] ⊕ (2)[1,2,0,0,1] ⊕ (2)[3,1,0,0,1] ⊕ [6,0,0,1,0] ⊕ [2,0,0,2,1] ⊕ [0,3,0,1,0] ⊕ [7,0,0,0,1] ⊕ (2)[3,0,1,1,0] ⊕ (2)[4,1,0,1,0] ⊕ [1,1,1,1,0] ⊕ (2)[2,2,0,1,0] ⊕ [3,0,0,1,2] ⊕ [1,1,0,1,2] ⊕ [5,1,0,0,1] ⊕ [1,3,0,0,1] ⊕

[4,0,1,0,1] ⊕ [3,2,0,0,1] ⊕ [4,0,0,2,1] ⊕ [2,1,1,0,1] ⊕ [2,1,0,2,1] ⊕ [5,0,1,1,0] ⊕ [3,1,1,1,0]

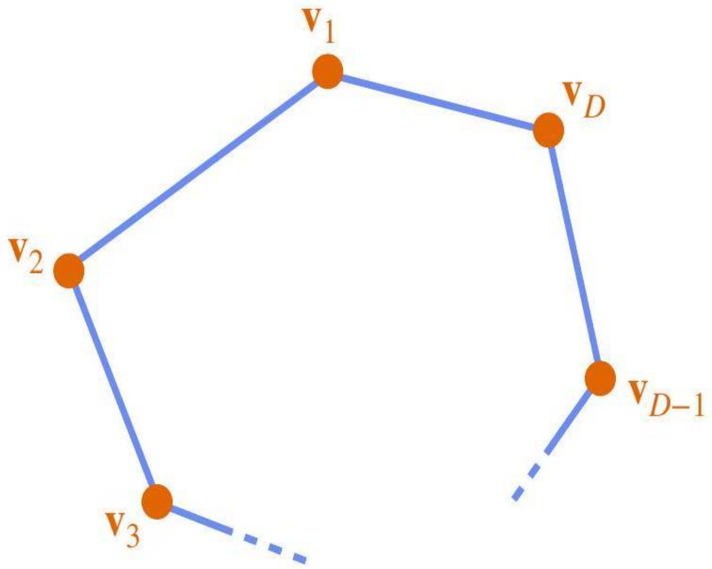
- Nivel-10: [0,0,1,0,0] ⊕ [1,1,0,0,0] ⊕ [1,0,0,1,1] ⊕ [0,1,1,0,0] ⊕ [0,1,0,0,2] ⊕ (2)[2,0,1,0,0] ⊕ [1,2,0,0,0] ⊕ [3,1,0,0,0] ⊕ [2,0,0,0,2] ⊕ (2)[3,0,0,1,1] ⊕ [5,1,0,0,0] ⊕ [0,2,1,0,0] ⊕ (2)[1,1,0,1,1] ⊕ [1,3,0,0,0] ⊕ (2)[4,0,1,0,0] ⊕ [1,0,1,0,2] ⊕ [3,2,0,0,0] ⊕ [4,0,0,2,0] ⊕ (2)[2,1,1,0,0] ⊕ [2,1,0,2,0] ⊕ [2,1,0,0,2] ⊕ [5,0,0,1,1] ⊕ [6,0,1,0,0] ⊕ [2,0,1,1,1] ⊕ [1,2,0,1,1] ⊕ (2)[3,1,0,1,1] ⊕ [4,1,1,0,0] ⊕ [2,2,1,0,0] ⊕ [3,0,1,2,0] ⊕ [4,1,0,2,0]
- Nivel-11: [0,1,0,0,1] ⊕ [2,0,0,0,1] ⊕ [3,0,0,1,0] ⊕ [1,1,0,1,0] ⊕ [1,0,0,0,3] ⊕ [4,0,0,0,1] ⊕ [0,2,0,0,1] ⊕ [1,0,1,0,1] ⊕ (2)[2,1,0,0,1] ⊕ [5,0,0,1,0] ⊕ [2,0,1,1,0] ⊕ [1,2,0,1,0] ⊕ (2)[3,1,0,1,0] ⊕ [2,0,0,1,2] ⊕ [3,0,1,0,1] ⊕ [4,1,0,0,1] ⊕ [1,1,1,0,1] ⊕ [2,2,0,0,1] ⊕ [3,0,0,2,1] ⊕ [4,0,0,3,0] ⊕ [5,1,0,1,0] ⊕ [4,0,1,1,0] ⊕ [3,2,0,1,0] ⊕ [2,1,1,1,0]
- Nivel-12: [4,0,0,0,0] ⊕ [0,2,0,0,0] ⊕ [1,0,0,0,2] ⊕ [2,1,0,0,0] ⊕ [2,0,0,1,1] ⊕ [4,1,0,0,0] ⊕ [3,0,1,0,0] ⊕ [2,2,0,0,0] ⊕ [3,0,0,2,0] ⊕ [3,0,0,0,2] ⊕ [1,1,1,0,0] ⊕ [1,1,0,0,2] ⊕ [4,0,0,1,1] ⊕ [2,0,2,0,0] ⊕ [4,2,0,0,0] ⊕ [5,0,0,2,0] ⊕ [2,1,0,1,1] ⊕ [3,1,1,0,0] ⊕ [3,1,0,2,0]
- Nivel-13: [3,0,0,0,1] ⊕ [1,1,0,0,1] ⊕ [4,0,0,1,0] ⊕ [2,1,0,1,0] ⊕ [2,0,1,0,1] ⊕ [3,1,0,0,1] ⊕ [3,0,1,1,0] ⊕ [4,1,0,1,0]
- Nivel-14: [2,0,1,0,0] ⊕ [3,1,0,0,0] ⊕ [3,0,0,1,1] ⊕ [4,0,1,0,0]
- Nivel-15: [3,0,0,1,0] ⊕ [4,0,0,0,1]
- Nivel-16: [4, 0, 0, 0, 0]

$$\mathcal{I}(\omega_i,\Delta_I)=\mathcal{I}_\infty(\omega_i,\Delta_I)\sum_{n_1,\dots,n_D=0}^{\infty}e^{-Nn_I\Delta_I}\mathcal{I}_{(n_1,\dots,n_D)}(\omega_i,\Delta_I)$$

$$\log\,\mathcal{I}_{(n_1,\dots,n_D)}(\omega_i,\Delta_I)\sim-\frac{\omega_1\omega_2}{(2\pi)^3}\mathcal{V}(\mathbf{b},\lambda_I)\ \text{ with }\ \mathbf{b}=\sum_I\Delta_I\mathbf{v}_I,\lambda_I=n_I$$

$$\log \mathcal{I}(\omega_i, \Delta_I) \sim \frac{N^2}{12\omega_1\omega_2} \sum_{I,J,K} C_{I,J,K}\Delta_I\Delta_J\Delta_K, C_{I,J,K} := |\mathbf{v}_I \cdot (\mathbf{v}_J \times \mathbf{v}_K)|.$$

$$ds_6^2 = dr^2 + r^2 ds_5^2$$



$$\mathcal{C} = \{\mathbf{u} \in \mathbb{R}^3 : \mathbf{u} \cdot \mathbf{v}_I \geq 0, I = 1, \ldots, D\}$$

$$\mathcal{P} = \mathcal{C} \cap \mathcal{H}(\mathbf{b})$$

$$\mathcal{H}(\mathbf{b}) := \left\{\mathbf{u} \in \mathbb{R}^3 : \mathbf{u} \cdot \mathbf{b} = \frac{1}{2}\right\}$$

$$\mathcal{V}(\mathbf{b}, \lambda_I) = \frac{(2\pi)^3}{2} \sum_{I=1}^{D} \left(\frac{\lambda_I(\lambda_I \mathbf{b} \cdot (\mathbf{v}_{I+1} \times \mathbf{v}_{I-1}) + \lambda_{I+1}\mathbf{b} \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_I) + \lambda_{I-1}\mathbf{b} \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1}))}{\mathbf{b} \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_I)\mathbf{b} \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1})} \right)$$

$$\mathcal{V}(\mathbf{b}, \lambda_I) = \mathcal{V}(\mathbf{b}, \lambda_I')$$

$$\lambda_I' = \lambda_I + \mathbf{u} \cdot (b^3 \mathbf{v}_I - \mathbf{b})$$

$$\left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right) \text{ under } (\Delta, J_i, Q_I).$$

$$\mathcal{I}(\omega_i, \Delta_I) = \mathrm{Tr}\left[e^{-\omega_i J_i - \Delta_I Q_I}\right]$$

$$\omega_1 + \omega_2 - \sum_{I=1}^{D} \Delta_I = 2\pi i \ (\mathrm{mod} 4\pi i)$$

$$\log \mathcal{I}(\omega_i, \Delta_I) \sim \frac{N^2}{12\omega_1\omega_2} \sum_{I,J,K} C_{I,J,K}\Delta_I\Delta_J\Delta_K, C_{I,J,K} := |\mathbf{v}_I \cdot (\mathbf{v}_J \times \mathbf{v}_K)|$$

$$S_I \cong S^3/\mathbb{Z}_{k_I}, k_I = \mathbf{v}_{I-1} \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1}) > 0$$

$$S_I \cap S_{I+1} \cong S^1$$

$$\mathcal{I}(\omega_i, \Delta_I) = \mathcal{I}_\infty(\omega_i, \Delta_I) \sum_{n_I \in \mathbb{Z}_{\geq 0}} e^{-N n_I \Delta_I} \mathcal{I}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I),$$

$$\mathcal{I}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I) = \int \left(\prod_{I=1}^{D} d^{n_I} u_I\right)\left(\prod_{I=1}^{D} F_I^{(4d)}(\omega_i, \Delta_I; u_I)\right)\left(\prod_{I=1}^{D} F_{I,I+1}^{(2d)}(\omega_i, \Delta_I; u_I, u_{I+1})\right)$$

$$F_I^{(4d)}(\omega_i, \Delta_I; u_I) = \Upsilon_I^{(4d)}(\omega_i, \Delta_I) \tilde{F}_I^{(4d)}(\omega_i, \Delta_I; u_I),$$

$$F_{I,I+1}^{(2d)}(\omega_i, \Delta_I; u_I, u_{I+1}) = \Upsilon_{I,I+1}^{(2d)}(\omega_i, \Delta_I) \tilde{F}_I^{(2d)}(\omega_i, \Delta_I; u_I, u_{I+1}),$$

$$\mathcal{I}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I) = \left(\prod_{I=1}^{D} \Upsilon_I^{(4d)}(\omega_i, \Delta_I) \Upsilon_{I,I+1}^{(2d)}(\omega_i, \Delta_I)\right) \tilde{\mathcal{I}}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I)$$

$$\tilde{\mathcal{I}}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I) = \int \left(\prod_{I=1}^{D} d^{n_I} u_I\right)\left(\prod_{I=1}^{D} \tilde{F}_I^{(4d)}(\omega_i, \Delta_I; u_I)\right)\left(\prod_{I=1}^{D} \tilde{F}_{I,I+1}^{(2d)}(\omega_i, \Delta_I; u_I, u_{I+1})\right)$$

$$\log\, \tilde{\mathcal{I}}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I) \sim \sum_I n_I \log\, k_I$$

$$\log\, \mathcal{I}_{(n_1,\dots,n_D)}(\omega_i, \Delta_I) \sim \sum_{I=1}^{D} \log\, \Upsilon_I^{(4d)}(\omega_i, \Delta_I) + \sum_{I=1}^{D} \log\, \Upsilon_{I,I+1}^{(2d)}(\omega_i, \Delta_I)$$

$$\varepsilon = \sum_{I=1}^{D} \Delta_I \mathbf{v}_I$$

$$\log\, \Upsilon_I^{(4d)}(\omega_i, \Delta_I) = \frac{n_I^2 \omega_1 \omega_2}{2}\left(\frac{\varepsilon \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_{I+1})}{\varepsilon \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_I) \varepsilon \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1})}\right)$$

$$\log\, \Upsilon_I^{(4d)}(\omega_i, \Delta_I) = \frac{n_I^2}{2k_I} \frac{\Delta_1^{(I)} \Delta_2^{(I)} \Delta_3^{(I)}}{\omega_1^{(I)} \omega_2^{(I)}}$$

$$\omega_1^{(I)} + \omega_2^{(I)} - \Delta_1^{(I)} - \Delta_2^{(I)} - \Delta_3^{(I)} = 2\pi i \ (\mathrm{mod} 4\pi i)$$

$$\omega_1^{(I)} = \frac{1}{k_I} \varepsilon \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_I), \quad \omega_2^{(I)} = \frac{1}{k_I} \varepsilon \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1}), \quad \Delta_3^{(I)} = \frac{1}{k_I} \varepsilon \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_{I+1}),$$
$$\Delta_1^{(I)} = \omega_1, \quad \Delta_2^{(I)} = \omega_2,$$

$$\mathbf{v}_J \cdot (\mathbf{v}_{I-1} \times \mathbf{v}_I + \mathbf{v}_I \times \mathbf{v}_{I+1} + \mathbf{v}_{I+1} \times \mathbf{v}_{I-1}) = k_I$$

$$\log\, \Upsilon_{I,I+1}^{(2d)}(\omega_i, \Delta_I) = -\frac{n_I n_{I+1} \omega_1 \omega_2}{\varepsilon \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1})}$$

$$\log \Upsilon^{(2d)}_{I,I+1}(\omega_i,\Delta_I) = -n_I n_{I+1}\frac{\Delta_1^{(I,I+1)}\Delta_2^{(I,I+1)}}{\omega^{(I,I+1)}}$$

$$\Delta_1^{(I,I+1)} = \omega_1, \Delta_2^{(I,I+1)} = \omega_2, \omega^{(I,I+1)} = \boldsymbol{\varepsilon}\cdot(\mathbf{v}_I \times \mathbf{v}_{I+1})$$

$$\log \mathcal{I}_{(n_1,\dots,n_D)}(\omega_i,\Delta_I) \sim \frac{\omega_1\omega_2}{2}\sum_{I=1}^{D}\left(\frac{n_I^2\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_{I+1})}{\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_I)\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})} - \frac{2n_I n_{I+1}}{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}\right)$$
$$= -\frac{\omega_1\omega_2}{2}\sum_{I=1}^{D}\left(\frac{n_I\big(n_I\varepsilon\cdot(\mathbf{v}_{I+1}\times\mathbf{v}_{I-1}) + n_{I+1}\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_I) + n_{I-1}\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})\big)}{\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_I)\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}\right)$$

$$\log \mathcal{I}_{(n_1,\dots,n_D)}(\omega_i,\Delta_I) \sim -\frac{\omega_1\omega_2}{(2\pi)^3}\mathcal{V}(\mathbf{b},\lambda_I)$$

$$\mathbf{b} = \varepsilon = \sum_I \Delta_I\mathbf{v}_I, \lambda_I = n_I$$

$$\frac{\mathcal{I}(\omega_i,\Delta_I)}{\mathcal{I}_\infty(\omega_i\Delta_I)} \sim \sum_{n_I\in\mathbb{Z}_{\geq 0}} e^{-\mathcal{A}(n_I)}$$

$$\mathcal{A}(n_I) = N\sum_I n_I\Delta_I - \frac{\omega_1\omega_2}{2}\sum_I\left(\frac{n_I^2\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_{I+1})}{\varepsilon\cdot(\mathbf{v}_{I-1}\times\mathbf{v}_I)\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})} - \frac{2n_I n_{I+1}}{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}\right),$$

$$\frac{\mathcal{I}(\omega_i,\Delta_I)}{\mathcal{I}_\infty(\omega_i,\Delta_I)} \sim \int_0^\infty \prod_I dn_I e^{-\mathcal{A}(n_I)}$$

$$\mathcal{A}(n_I') = \mathcal{A}(n_I)$$

$$n_I' = n_I + \sum_{J=1}^{D}\Delta_J\mathbf{u}\cdot(\mathbf{v}_I - \mathbf{v}_J),\ \text{for any } \mathbf{u}\in\mathbb{R}^3$$

$$\sum_I (n_I' - n_I)\Delta_I = \sum_{I,J}\Delta_I\Delta_J\mathbf{u}\cdot(\mathbf{v}_I - \mathbf{v}_J) = 0$$

$$n_I = \begin{cases} m_I + \sum_J \Delta_J\mathbf{u}\cdot(\mathbf{v}_I - \mathbf{v}_J), & I = 1,\dots,D-2 \\ \sum_J \Delta_J\mathbf{u}\cdot(\mathbf{v}_I - \mathbf{v}_J), & I = D-1, D\end{cases} \quad \text{where } \mathbf{u} = \begin{pmatrix}\sigma_1\\ \sigma_2\\ 0\end{pmatrix}$$

$$\int_0^\infty \prod_I dn_I e^{-\mathcal{A}(n_I)} = \int_0^\infty \prod_A dm_A V(m_A) e^{-\mathcal{B}(m_A)}$$

$$\mathcal{B}(m_A) = N\sum_{A=1}^{D-2} m_A\Delta_A - \frac{\omega_1\omega_2}{2}\sum_{A=1}^{D-2}\left(\frac{m_A^2\boldsymbol{\varepsilon}\cdot(\mathbf{v}_{A-1}\times\mathbf{v}_{A+1})}{\boldsymbol{\varepsilon}\cdot(\mathbf{v}_{A-1}\times\mathbf{v}_A)\boldsymbol{\varepsilon}\cdot(\mathbf{v}_A\times\mathbf{v}_{A+1})} - \frac{2m_A m_{A+1}}{\boldsymbol{\varepsilon}\cdot(\mathbf{v}_A\times\mathbf{v}_{A+1})}\right)$$

$$V(m_A) = J\iint_{P(m_A)\subset\mathbb{R}^2} d\sigma_1 d\sigma_2 = J\times \text{Area}[P(m_A)]$$

$$\int_0^\infty \prod_I dn_I e^{-\mathcal{A}(n_I)} \sim V(m_A^*)e^{-\mathcal{B}(m_A^*)}$$

$$\frac{\partial \mathcal{B}}{\partial m_A}(m_A^*) = 0$$

$$m_A^* = -\frac{N}{\omega_1\omega_2}(\varepsilon \times \mathbf{v}_A)\cdot\left(\sum_{I=A+1}^{D-1} \Delta_I \mathbf{v}_I\right)$$

$$\mathcal{B}(m_A^*) = -\frac{N^2}{12\omega_1\omega_2}\sum_{I,J,K} C_{I,J,K}\Delta_I\Delta_J\Delta_K,$$

$$\log \mathcal{I}(\omega_i,\Delta_I) \sim \frac{N^2}{12\omega_1\omega_2}\sum_{I,J,K} C_{I,J,K}\Delta_I\Delta_J\Delta_K + \log V(m_A^*) + \cdots$$

$$\log V(m_A^*) = 2\log N + \cdots,$$

$$n_3 = n_3^* = -\frac{\Delta_1\Delta_2}{\omega_1\omega_2}N$$

$$\frac{1}{2}N^2\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} = S - \omega_i J_I - \Delta_I Q_I$$

$$\begin{aligned} J_1 + Q_3 &= \frac{1}{2}N^2\frac{\Delta_1\Delta_2}{\omega_1\omega_2}\left(\frac{\Delta_3}{\omega_1} - 1\right) \\ J_2 + Q_3 &= \frac{1}{2}N^2\frac{\Delta_1\Delta_2}{\omega_1\omega_2}\left(\frac{\Delta_3}{\omega_2} - 1\right) \\ Q_3 - Q_1 &= \frac{1}{2}N^2\frac{\Delta_2}{\omega_1\omega_2}(\Delta_3 - \Delta_1) \\ Q_3 - Q_2 &= \frac{1}{2}N^2\frac{\Delta_1}{\omega_1\omega_2}(\Delta_3 - \Delta_2) \end{aligned}$$

$$n_3^* = -\frac{\Delta_1\Delta_2}{\omega_1\omega_2}N = \frac{2}{N}Q_3 + \frac{i}{\pi N}S$$

$$\mathcal{V}(\mathbf{b},\lambda_I) = -\frac{(2\pi)^4}{3!}\sum_{I=1}^{D}\lambda_I\sum_{k=2}^{l_I-1}\frac{X_{I,k}\tilde{X}_{I,k}}{(\mathbf{v}_I,\mathbf{v}_{I,k-1},\mathbf{v}_{I,k},\mathbf{b})(\mathbf{v}_{I,l_I},\mathbf{v}_I,\mathbf{v}_{I,1},\mathbf{b})(\mathbf{v}_{I,k},\mathbf{v}_{I,k+1},\mathbf{v}_I,\mathbf{b})},$$

$$\begin{aligned} X_{I,k} =& -\lambda_I(\mathbf{v}_{I,k-1},\mathbf{v}_{I,k},\mathbf{v}_{I,k+1},\mathbf{b}) + \lambda_{I,k-1}(\mathbf{v}_{I,k},\mathbf{v}_{I,k+1},\mathbf{v}_I,\mathbf{b}) \\ & -\lambda_{I,k}(\mathbf{v}_{I,k+1},\mathbf{v}_I,\mathbf{v}_{I,k-1},\mathbf{b}) + \lambda_{I,k+1}(\mathbf{v}_I,\mathbf{v}_{I,k-1},\mathbf{v}_{I,k},\mathbf{b}), \\ \tilde{X}_{I,k} =& -\lambda_I(\mathbf{v}_{I,1},\mathbf{v}_{I,k},\mathbf{v}_{I,l_I},\mathbf{b}) + \lambda_{I,1}(\mathbf{v}_{I,k},\mathbf{v}_{I,l_I},\mathbf{v}_I,\mathbf{b}) \\ & -\lambda_{I,k}(\mathbf{v}_{I,l_I},\mathbf{v}_I,\mathbf{v}_{I,1},\mathbf{b}) + \lambda_{I,l_I}(\mathbf{v}_I,\mathbf{v}_{I,1},\mathbf{v}_{I,k},\mathbf{b}). \end{aligned}$$

$$\left(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right) \text{ under } (\Delta, J, Q_I).$$

$$\mathcal{I}(\omega,\Delta_I) = \mathrm{Tr}\left[e^{-\omega J-\Delta_I Q_I}\right]$$

$$\omega - \sum_{I=1}^{D} \Delta_I = 2\pi i \ (\mathrm{mod} 4\pi i)$$

$$\mathcal{I}(\omega, \Delta_I) = \mathcal{I}_\infty(\omega, \Delta_I) \sum_{n_I \in \mathbb{Z}_{\geq 0}} e^{-N n_I \Delta_I} \mathcal{I}_{(n_1, \ldots, n_D)}(\omega, \Delta_I)$$

$$\log \mathcal{I}_{(n_1, \ldots, n_D)}(\omega, \Delta_I) \sim -\frac{\omega^2}{4(2\pi)^4} \mathcal{V}(\mathbf{b}, \lambda_I)$$

$$\mathbf{b} = \varepsilon = \sum_I \Delta_I \mathbf{v}_I, \lambda_I = n_I$$

$$\log \mathcal{I}_{(n_1, n_2, n_3, n_4)}(\omega, \Delta_I) \sim \sum_{I=1}^{4} \log \mathcal{I}_I^{(6d)}(\omega, \Delta_I) + \sum_{I<J} \log \mathcal{I}_{I,J}^{(4d)}(\omega, \Delta_I) + \sum_{I<J<K} \mathcal{I}_{I,J,K}^{(2d)}(\omega, \Delta_I)$$

$$\log \mathcal{I}_{(n_1, n_2, n_3, n_4)}(\omega, \Delta_I) \sim -\frac{\omega^2 (\sum_{I=1}^{4} n_I \Delta_I)^3}{24 \Delta_1 \Delta_2 \Delta_3 \Delta_4}$$

$$\mathcal{V}(\mathbf{b}, \lambda_I) = \frac{(2\pi)^4}{6} \frac{\left(-\lambda_1(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{b}) + \lambda_2(\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1, \mathbf{b}) - \lambda_3(\mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{b}) + \lambda_4(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b})\right)^3}{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b})(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{b})(\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1, \mathbf{b})(\mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{b})}$$

$$\mathcal{V}(\mathbf{b}, \lambda_I) = \frac{(2\pi)^4}{6} \frac{(\sum_{I=1}^{4} n_I \Delta_I)^3}{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$$

$$\log \mathcal{I}(\omega, \Delta_I) \sim \pm \frac{4\sqrt{2} i}{3} N^{3/2} \frac{\sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}}{\omega},$$

$$\omega_1^{(I)} + \omega_2^{(I)} - \Delta_1^{(I)} - \Delta_2^{(I)} - \Delta_3^{(I)} = 2\pi i \ (\mathrm{mod} 4\pi i).$$

$$\exp\left(\frac{2\pi i}{\mathbf{v}_{I-1} \cdot (\mathbf{v}_I \times \mathbf{v}_{I+1})} (J_1 + \alpha_I J_2 + (1 + \alpha_I) Q_3)\right).$$

$$p_{1,2} = e^{2\pi i \sigma_{1,2}} = e^{-\omega_{1,2}^{(I)}}, x_{1,2,3} = e^{2\pi i \nu_{1,2,3}} = e^{-\Delta_{1,2,3}^{(I)}}$$

$$\sigma_1 + \sigma_2 - \nu_1 - \nu_2 - \nu_3 = 1 \ (\mathrm{mod} 2)$$

$$\sigma_1 + \sigma_2 - \nu_1 - \nu_2 - \nu_3 = -1$$

$$z_{I,a} = e^{2\pi i u_{I,a}}$$

$$F^{(4d)} = \sum_{h=(h_1, \ldots, h_n)} F_h^{(4d)},$$

$$F_h^{(4d)} = \mu_h(z) \mathrm{Pexp}\left[\frac{1}{k} \sum_{l=0}^{k-1} \left(1 - \frac{(1 - x_1)(1 - x_2)\left(1 - \omega^{(1+\alpha)l} x_3\right)}{(1 - \omega^l p_1)(1 - \omega^{\alpha l} p_2)}\right) \sum_{a,b=1}^{n} \omega^{(h_a - h_b)l} \frac{z_a}{z_b}\right],$$

$$\mu_h(z)=\prod_{\substack{(a,b)\in K_h\\ a\neq b}}\left(1-\frac{z_a}{z_b}\right)$$

$$\mathrm{Pexp}\left[\frac{1}{k}\sum_{l=0}^{k-1}\sum_{a,b=1}^{n}\omega^{(h_a-h_b)l}\frac{z_a}{z_b}\right]=\mathrm{Pexp}\left[\sum_{(a,b)\in K_h}\frac{z_a}{z_b}\right]=\mu_h(z)^{-1}$$

$$f(x)=\sum_{n=0}^{\infty}a_nx^n\ \Longrightarrow\ \mathrm{Pexp}[f(x)]=\exp\left(\sum_{n=1}^{\infty}\frac{f(x^n)-f(0)}{n}\right)=\prod_{n=1}^{\infty}(1-x^n)^{-a_n}.$$

$$\begin{aligned}
&\mathrm{Pexp}\left[-\frac{1}{k}\sum_{l=0}^{k-1}\left(\frac{(1-x_1)(1-x_2)\left(1-\omega^{(1+\alpha)l}x_3\right)}{(1-\omega^l p_1)(1-\omega^{\alpha l}p_2)}\right)\sum_{a,b=1}^{n}\omega^{(h_a-h_b)l}\frac{z_a}{z_b}\right]\\
&=\mathrm{Pexp}\left[-\frac{1}{k}\sum_{a,b=1}^{n}\sum_{l,m_1,m_2=0}^{k-1}\left(\frac{(1-x_1)(1-x_2)\left(1-\omega^{(1+\alpha)l}x_3\right)}{(1-p_1^k)(1-p_2^k)}\right)p_1^{m_1}p_2^{m_2}\omega^{(h_a-h_b+m_1+\alpha m_2)l}\frac{z_a}{z_b}\right]\\
&=\mathrm{Pexp}\left[-\frac{(1-x_1)(1-x_2)}{(1-p_1^k)(1-p_2^k)}\times\sum_{a,b=1}^{n}\sum_{m_1,m_2=0}^{k-1}\left(\delta_{h_a-h_b+m_1+\alpha m_2,0}-\delta_{h_a-h_b+(m_1+1)+\alpha(m_2+1),0}x_3\right)\ p_1^{m_1}p_2^{m_2}\frac{z_a}{z_b}\right]\\
&=\mathrm{Pexp}\left[-\frac{(1-x_1)(1-x_2)}{(1-p_1^k)(1-p_2^k)}\right.\\
&\left.\times\sum_{a,b=1}^{n}\sum_{m_1,m_2=0}^{k-1}\delta_{h_a-h_b+m_1+\alpha m_2,0}\left(p_1^{m_1}p_2^{m_2}\frac{z_a}{z_b}-x_3p_1^{k-1-m_1}p_2^{k-1-m_2}\frac{z_b}{z_a}\right)\right]
\end{aligned}$$

$$F_h^{(4d)}=\prod_{a\neq b}\prod_{m=0}^{k-1}\frac{\Gamma(u_{ab}+[\![h_b-h_a-\alpha m]\!]\sigma_1+m\sigma_2+\nu_1,k\sigma_1,k\sigma_2)\Gamma(u_{ab}+[\![h_b-h_a-\alpha m]\!]\sigma_1+m\sigma_2+\nu_2,k\sigma_1,k\sigma_2)}{\Gamma(u_{ab}+[\![h_b-h_a-\alpha m]\!]\sigma_1+m\sigma_2+\nu_1+\nu_2,k\sigma_1,k\sigma_2)\Gamma(u_{ab}+[\![h_b-h_a-\alpha m]\!]\sigma_1+m\sigma_2,k\sigma_1,k\sigma_2)},$$

$$\Gamma(x,\sigma,\tau)=\mathrm{Pexp}\left[\frac{e^{2\pi ix}-e^{2\pi i(\sigma+\tau-x)}}{(1-e^{2\pi i\sigma})(1-e^{2\pi i\tau})}\right]=\prod_{n,m\geq 0}\frac{1-e^{2\pi i(-x+(n+1)\sigma+(m+1)\tau)}}{1-e^{2\pi i(x+n\sigma+m\tau)}}$$

$$\begin{aligned}
\Gamma(x,\sigma,\tau)&=e^{-i\pi P_+(x,\sigma,\tau)}\Gamma\left(-\frac{x+1}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(\frac{x}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\\
&=e^{-i\pi P_-(x,\sigma,\tau)}\Gamma\left(-\frac{x}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(\frac{x-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)
\end{aligned}$$

$$\begin{aligned}
P_\pm(x,\sigma,\tau)&=\frac{1}{3\sigma\tau}x^3-\frac{\sigma+\tau\mp 1}{2\sigma\tau}x^2+\frac{\sigma^2+\tau^2+3\sigma\tau\mp 3\sigma\mp 3\tau+1}{6\sigma\tau}x\\
&\pm\frac{1}{12}(\sigma+\tau\mp 1)\left(\frac{1}{\sigma}+\frac{1}{\tau}\mp 1\right)
\end{aligned}$$

$$\frac{\Gamma(x+\gamma+\alpha,\sigma,\tau)\Gamma(x+\gamma+\beta,\sigma,\tau)\Gamma(-x+\gamma+\alpha,\sigma,\tau)\Gamma(-x+\gamma+\beta,\sigma,\tau)}{\Gamma(x+\gamma+\alpha+\beta,\sigma,\tau)\Gamma(x+\gamma,\sigma,\tau)\Gamma(-x+\gamma+\alpha+\beta,\sigma,\tau)\Gamma(-x+\gamma,\sigma,\tau)} = \exp\left(\frac{2\pi i\alpha\beta(\alpha+\beta-\sigma-\tau+2\gamma-1)}{\sigma\tau}\right)$$
$$\times\left(\frac{\Gamma\left(-\frac{x+\gamma+\alpha}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{x+\gamma+\beta}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{-x+\gamma+\alpha}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{-x+\gamma+\beta}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)}{\Gamma\left(-\frac{x+\gamma+\alpha+\beta}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{x+\gamma}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{-x+\gamma+\alpha+\beta}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)\Gamma\left(-\frac{-x+\gamma}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)}\right)$$
$$\times\left(\frac{\Gamma\left(\frac{x+\gamma+\alpha-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{x+\gamma+\beta-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{-x+\gamma+\alpha-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{-x+\gamma+\beta-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{x+\gamma+\alpha+\beta-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{x+\gamma-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{-x+\gamma+\alpha+\beta-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{-x+\gamma-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)}\right)$$

$$F_h^{(4d)} = \Upsilon_h^{(4d)}\tilde{F}_h^{(4d)}$$

$$\Upsilon_h^{(4d)} = \exp\left(\frac{2\pi i\nu_1\nu_2}{k^2\sigma_1\sigma_2}\sum_{a<b}\left(-k\sigma_1-k\sigma_2+\sum_{m=0}^{k-1}(\nu_1+\nu_2-1)\right)\right)$$
$$= \exp\left(\frac{\pi i}{k}n(n-1)\frac{\nu_1\nu_2}{\sigma_1\sigma_2}(\nu_1+\nu_2-\sigma_1-\sigma_2-1)\right)$$
$$= \exp\left(-\frac{\pi i}{k}n(n-1)\frac{\nu_1\nu_2\nu_3}{\sigma_1\sigma_2}\right)$$

$$F_I^{(4d)} = \Upsilon_I^{(4d)}\tilde{F}_I^{(4d)}$$

$$\tilde{F}_I^{(4d)} = \sum_h \tilde{F}_{I,h}^{(4d)}$$

$$\Upsilon_I^{(4d)} = \exp\left(\frac{n_I(n_I-1)}{2k_I}\frac{\Delta_1^{(I)}\Delta_2^{(I)}\Delta_3^{(I)}}{\omega_1^{(I)}\omega_2^{(I)}}\right)$$

$$\log\,\Upsilon_I^{(4d)} \sim \frac{n_I^2}{2k_I}\frac{\Delta_1^{(I)}\Delta_2^{(I)}\Delta_3^{(I)}}{\omega_1^{(I)}\omega_2^{(I)}}$$

$$F_{I,I+1}^{(2d)} = \prod_{a=1}^{n_I}\prod_{b=1}^{n_{I+1}}\left[\frac{\vartheta\left(u_{I,a}-u_{I+1,b}-\frac{1}{2}-\frac{1}{4\pi i}(\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})+\omega_1-\omega_2),-\frac{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}{2\pi i}\right)}{\vartheta\left(u_{I,a}-u_{I+1,b}-\frac{1}{2}-\frac{1}{4\pi i}(\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})-\omega_1-\omega_2),-\frac{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}{2\pi i}\right)}\right.$$
$$\left.\times\frac{\vartheta\left(-u_{I,a}+u_{I+1,b}-\frac{1}{2}-\frac{1}{4\pi i}(\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})+\omega_1-\omega_2),-\frac{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}{2\pi i}\right)}{\vartheta\left(-u_a^I+u_b^{I+1}-\frac{1}{2}-\frac{1}{4\pi i}(\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})-\omega_1-\omega_2),-\frac{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}{2\pi i}\right)}\right]$$

$$\vartheta(x,\tau) = \prod_{n=0}^{\infty}\left(1-e^{2\pi i(-x+(n+1)\tau)}\right)\left(1-e^{2\pi i(x+n\tau)}\right)$$

$$\vartheta(x,\tau) = e^{-i\pi B(x,\tau)}\vartheta\left(\frac{x}{\tau},-\frac{1}{\tau}\right),$$

$$B(x,\tau)=\frac{1}{\tau}x^2+\left(\frac{1}{\tau}-1\right)x+\frac{1}{6}\left(\tau+\frac{1}{\tau}\right)-\frac{1}{2}$$

$$\frac{\vartheta(x+\alpha+\gamma,\beta)\vartheta(-x+\alpha+\gamma,\beta)}{\vartheta(x-\alpha+\gamma,\beta)\vartheta(-x-\alpha+\gamma,\beta)}=\exp\left(\frac{4\pi i\alpha(\beta-2\gamma-1)}{\beta}\right)\frac{\vartheta\left(\frac{x+\alpha+\gamma}{\beta},-\frac{1}{\beta}\right)\vartheta\left(\frac{-x+\alpha+\gamma}{\beta},-\frac{1}{\beta}\right)}{\vartheta\left(\frac{x-\alpha+\gamma}{\beta},-\frac{1}{\beta}\right)\vartheta\left(\frac{-x-\alpha+\gamma}{\beta},-\frac{1}{\beta}\right)}$$

$$F_{I,I+1}^{(2d)}=\Upsilon_{I,I+1}^{(2d)}\tilde{F}_{I,I+1}^{(2d)}$$

$$\Upsilon_{I,I+1}^{(2d)}=\exp\left(-n_I n_{I+1}\frac{\omega_1\omega_2}{\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1})}\right)$$

$$\tilde{\mathcal{I}}_{(n_1,\ldots,n_D)}(\omega_i,\Delta_I)=\sum_{h^{(1)},\ldots,h^{(D)}}\tilde{\mathcal{I}}_{(n_1,\ldots,n_D)}\left(\omega_i,\Delta_I;h^{(I)}\right).$$

$$\tilde{\mathcal{I}}_{(n_1,\ldots,n_D)}\left(\omega_i,\Delta_I;h^{(I)}\right)\sim 1$$

$$\log\,\tilde{\mathcal{I}}_{(n_1,\ldots,n_D)}(\omega_i,\Delta_I)\sim\sum_I n_I\log\,k_I,$$

$$\begin{aligned}&\tilde{\mathcal{I}}_{(n_1,\ldots,n_D)}\left(\omega_i,\Delta_I;h^{(I)}\right)\\&=\int\left(\prod_{I=1}^{D}d^{n_I}u_I\right)\left(\prod_{I=1}^{D}\tilde{F}_{I,h^{(I)}}^{(4d)}(\omega_i,\Delta_I;u_I)\right)\left(\prod_{I=1}^{D}\tilde{F}_{I,I+1}^{(2d)}(\omega_i,\Delta_I;u_I,u_{I+1})\right)\end{aligned}$$

$$\rho_{I,I+1}:=-\frac{1}{2\pi i}\varepsilon\cdot(\mathbf{v}_I\times\mathbf{v}_{I+1}).$$

$$\omega_1+\omega_2=-2\pi i\left(1+\frac{1}{\kappa}\sum_{I=1}^{D}\rho_{I,I+1}\right)$$

$$\sum_{I=1}^{D}(\mathbf{v}_I\times\mathbf{v}_{I+1})=(0,0,\kappa)$$

$$\tilde{F}^{(4d)}_{I,h^{(I)}}(\omega_i, \Delta_I; u_I)$$

$$= \prod_{\substack{a,b=1\\ a\neq b}}^{n_I} \prod_{m=0}^{k_I-1} \Gamma\Gamma\left(-\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_1}{2\pi i}}{\rho_{I-1,I}}, -\frac{1}{\rho_{I-1,I}}, -\frac{\rho_{I,I+1}}{\rho_{I-1,I}}\right)$$

$$\times \Gamma\left(\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_1}{2\pi i} - 1}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}, \frac{\rho_{I-1,I}}{\rho_{I,I+1}}\right)$$

$$\times \Gamma\left(-\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_2}{2\pi i}}{\rho_{I-1,I}}, -\frac{1}{\rho_{I-1,I}}, -\frac{\rho_{I,I+1}}{\rho_{I-1,I}}\right)$$

$$\times \Gamma\left(\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_2}{2\pi i} - 1}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}, \frac{\rho_{I-1,I}}{\rho_{I,I+1}}\right)$$

$$\times \Gamma\left(-\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I}}{\rho_{I-1,I}}, -\frac{1}{\rho_{I-1,I}}, -\frac{\rho_{I,I+1}}{\rho_{I-1,I}}\right)^{-1}$$

$$\times \Gamma\left(\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - 1}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}, \frac{\rho_{I-1,I}}{\rho_{I,I+1}}\right)^{-1}$$

$$\times \Gamma\left(-\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_1+\omega_2}{2\pi i}}{\rho_{I-1,I}}, -\frac{1}{\rho_{I-1,I}}, -\frac{\rho_{I,I+1}}{\rho_{I-1,I}}\right)^{-1}$$

$$\times \Gamma\left(\frac{u_{I,ab} + [\![h_{I,b} - h_{I,a} - \alpha_I m]\!]\frac{\rho_{I-1,I}}{k_I} + m\frac{\rho_{I,I+1}}{k_I} - \frac{\omega_1+\omega_2}{2\pi i} - 1}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}, \frac{\rho_{I-1,I}}{\rho_{I,I+1}}\right)^{-1}$$

$$\tilde{F}^{(2d)}_{I,I+1}(\omega_i, \Delta_I; u_I, u_{I+1}) = \prod_{a=1}^{n_I} \prod_{b=1}^{n_I+1} \vartheta\left(\frac{u_a^I - u_b^{I+1} + \frac{1}{2}(\rho_{I,I+1} - 1) - \frac{\omega_1 - \omega_2}{4\pi i}}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}\right)$$

$$\times \vartheta\left(\frac{-u_a^I + u_b^{I+1} + \frac{1}{2}(\rho_{I,I+1} - 1) - \frac{\omega_1 - \omega_2}{4\pi i}}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}\right)$$

$$\times \vartheta\left(\frac{u_a^I - u_b^{I+1} + \frac{1}{2}(\rho_{I,I+1} - 1) - \frac{\omega_1 + \omega_2}{4\pi i}}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}\right)^{-1}$$

$$\times \vartheta\left(\frac{-u_a^I + u_b^{I+1} + \frac{1}{2}(\rho_{I,I+1} - 1) - \frac{\omega_1 + \omega_2}{4\pi i}}{\rho_{I,I+1}}, -\frac{1}{\rho_{I,I+1}}\right)^{-1}$$

$$\frac{\partial V(u_I)}{\partial u_{I,a_I}} = 0$$

$$K = \int d^{N_1}v_1 \dots d^{N_p}v_p e^{-V(v_1,\dots,v_k)}$$

$$\frac{\partial V(v_1, \dots, v_p)}{\partial v_{i,a_i}} = 0, a_i = 1, \dots, N_i, i = 1, \dots, p$$

$$V(v_1, \dots, v_p) = \sum_{i=1}^{p} \sum_{a_i \neq b_i} A_i(v_{i,a_i} - v_{i,b_i}) + \sum_{i<j} \sum_{a_i=1}^{N_i} \sum_{a_j=1}^{N_j} B_{ij}\left(v_{i,a_i} - v_{j,a_j}\right),$$

$$A_i(z) = A_i^{s_i}(z) + A_i^{t_i}(z), A_i^{s_i}(z + s_i) = A_i^{s_i}(z), A_i^{t_i}(z + t_i) = A_i^{t_i}(z)$$

$$\frac{\partial V(v_1, \dots, v_p)}{\partial v_{i,a_i}} = -\sum_{\substack{b_i=1 \\ b_i \neq a_i}}^{N_i} \frac{\partial}{\partial v_{i,b_i}}\left(A_i^{s_i}(v_{i,a_i} - v_{i,b_i}) + A_i^{t_i}(v_{i,a_i} - v_{i,b_i})\right)$$

$$v_{i,a_i} \to v_i(x_i, y_i) = x_i s_i + y_i t_i$$

$$\frac{\delta V(v_1, \dots, v_p)}{\delta v_i(\tilde{x}, \tilde{y})} = -N_i \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \left(\frac{1}{s_i}\frac{\partial}{\partial x} A_i^{s_i}(v_i(\tilde{x}, \tilde{y}) - xs_i - yt_i) + \frac{1}{t_i}\frac{\partial}{\partial y} A_i^{t_i}(v_i(\tilde{x}, \tilde{y}) - xs_i - yt_i)\right)$$

$$= -\frac{N_i}{s_i} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \left(A_i^{s_i}\left(v_i(\tilde{x}, \tilde{y}) - \frac{1}{2}s_i - yt_i\right) - A_i^{s_i}\left(v_i(\tilde{x}, \tilde{y}) + \frac{1}{2}s_i - yt_i\right)\right)$$

$$-\frac{N_i}{t_i} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \left(A_i^{t_i}\left(v_i(\tilde{x}, \tilde{y}) - xs_i - \frac{1}{2}t_i\right) - A_i^{t_i}\left(v_i(\tilde{x}, \tilde{y}) - xs_i + \frac{1}{2}t_i\right)\right) = 0$$

$$s_1 = s_2$$

$$B_{12}(z + s_1) = B_{12}(z)$$

$$\frac{\partial V(v_1, \dots, v_p)}{\partial v_{1,a_1}} = -\sum_{a_2=1}^{N_2} \frac{\partial}{\partial v_{2,a_2}} B_{12}(v_{1,a_1} - v_{2,a_2}).$$

$$\frac{\delta V(v_1, \dots, v_p)}{\delta v_1(\tilde{x}, \tilde{y})} = -\frac{N_2}{s_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \frac{\partial}{\partial x} B_{12}(v_1(\tilde{x}, \tilde{y}) - xs_2 - yt_2)$$

$$= -\frac{N_2}{s_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \left(B_{12}\left(v_1(\tilde{x}, \tilde{y}) - \frac{1}{2}s_1 - yt_2\right) - B_{12}\left(v_1(\tilde{x}, \tilde{y}) + \frac{1}{2}s_1 - yt_2\right)\right) = 0$$

$$s_1 = s_2, t_1 = t_2$$

$$B_{12}(z) = B_{12}^{s_1}(z) + B_{12}^{t_1}(z), B_{12}^{s_1}(z + s_1) = B_{12}^{s_1}(z), B_{12}^{t_1}(z + t_1) = B_{12}^{t_1}(z),$$

$$A_i^{s_i}(z+s_i)=A_i^{s_i}(z), A_i^{t_i}(z+t_i)=A_i^{t_i}(z)$$

$$B_{ij}(z+\gamma)=B_{ij}(z)$$

$$B_{ij}^{\gamma_1}(z+\gamma_1)=B_{ij}^{\gamma_1}(z), B_{ij}^{\gamma_2}(z+\gamma_2)=B_{ij}^{\gamma_2}(z)$$

$$V|_{\text{saddle}} = \cdots + N_i^2 \int_{-1/2}^{1/2} dx_1 dy_1 dx_2 dy_2 A_i^{s_i}(x_1 s_i + y_1 t_i - x_2 s_i - y_2 t_i) + \cdots$$

$$\log\left(1-b(y_1-y_2)e^{2\pi i c(x_1-x_2)}\right)$$

$$|b(y_1-y_2)|<1$$

$$\int_{-1/2}^{1/2} dx_1 \int_{-1/2}^{1/2} dx_2 \log\left(1-b(y_1-y_2)e^{2\pi i c(x_1-x_2)}\right)$$
$$=\begin{cases} 0 & \text{if } |b(y_1-y_2)|<1 \\ 2\log|b(y_1-y_2)| & \text{if } |b(y_1-y_2)|>1 \end{cases}$$

$$V|_{\text{saddle}} = 0$$

$$V|_{\text{saddle}} = \mathcal{O}\left(N_i^2\right),$$

$$\rho_{I-1,I}, \rho_{I,I+1}$$

$$\Gamma(x+1,\sigma,\tau)=\Gamma(x,\sigma,\tau)$$

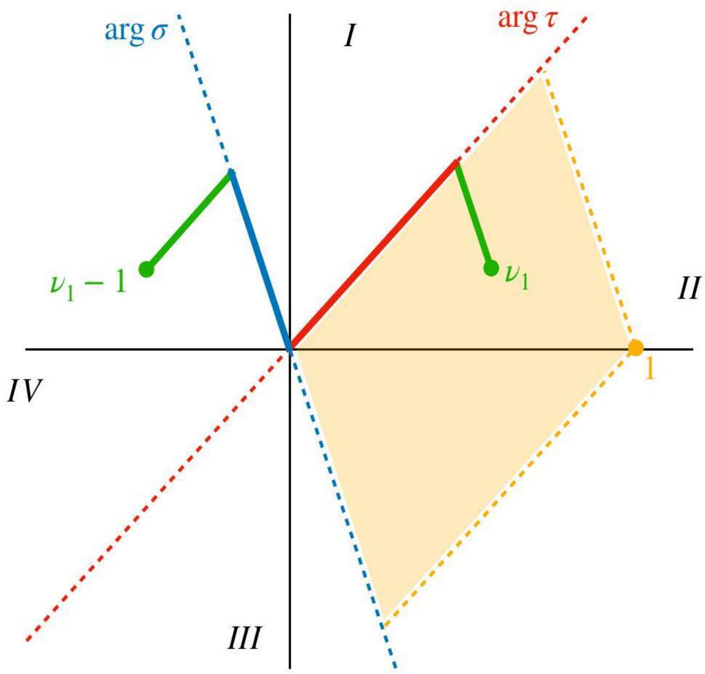
$$\vartheta(x+1,\tau)=\vartheta(x,\tau)$$

$$\mathcal{I}_N(\omega_1,\omega_2,\Delta_1,\Delta_2,\Delta_3)=\mathcal{I}_\infty(\omega_i,\Delta_I)\sum_{n=0}^{\infty} e^{-Nn\Delta_3}\tilde{\mathcal{I}}_n(\Delta_1,\Delta_2,\omega_1,\omega_2,-\Delta_3)$$

$$\mathcal{I}_m(-2\pi i\sigma,-2\pi i\tau,-2\pi i\nu_1,-2\pi i\nu_2,-2\pi i\nu_3)=\int d^m u \prod_{a\neq b}\frac{\Gamma(u_{ab}+\nu_1,\sigma,\tau)\Gamma(u_{ab}+\nu_2,\sigma,\tau)}{\Gamma(u_{ab}+\nu_1+\nu_2,\sigma,\tau)\Gamma(u_{ab},\sigma,\tau)}$$

$$\log \mathcal{I}_m \sim -i\pi m^2\frac{\nu_1\nu_2\nu_3}{\sigma\tau}$$

$$\operatorname{Im}\left(-\frac{\nu_I}{\tau}\right)>0, \operatorname{Im}\left(\frac{\nu_I-1}{\sigma}\right)>0, \operatorname{Im}\left(\frac{\tau-\nu_I}{\sigma}\right)>0, \operatorname{Im}\left(\frac{(\nu_I-1)-\sigma}{\tau}\right)>0,$$



$$\mathbf{w}_I := \sum_{J=1}^{D} \Delta_J (\mathbf{v}_I - \mathbf{v}_J),$$

$$\mathbf{w}_I \times \mathbf{w}_J = \left(\sum_{K,L} \Delta_K \Delta_L\right) \mathbf{v}_I \times \mathbf{v}_J + \left(\sum_{K} \Delta_K\right) \varepsilon \times (\mathbf{v}_I - \mathbf{v}_J)$$

$$J = \det\left(\frac{\partial(n_1, \dots, n_A)}{\partial(m_1, \dots, m_{D-2}, \sigma_1, \sigma_2)}\right) = \hat{\mathbf{z}} \cdot (\mathbf{w}_{D-1} \times \mathbf{w}_D)$$

$$\iint_{P(m_A)\subset \mathbb{R}^2} d\sigma_1 d\sigma_2 = \mathrm{Area}[P(m_A)]$$

$$\mathbf{u} \cdot \mathbf{w}_A \geq -m_A, A = 1, \dots, D-2, \mathbf{u} \cdot \mathbf{w}_{D-1} \geq 0, \mathbf{u} \cdot \mathbf{w}_D \geq 0, \mathbf{u} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 0 \end{pmatrix}$$

$$\begin{aligned} V(m_A) &= J \times \mathrm{Area}[P(m_A)] \\ &= -\frac{1}{2} \varepsilon \cdot (\mathbf{w}_{D-1} \times \mathbf{w}_D) \sum_{A=1}^{D-2} \left(\frac{m_A^2 \varepsilon \cdot (\mathbf{w}_{A-1} \times \mathbf{w}_{A+1})}{\varepsilon \cdot (\mathbf{w}_{A-1} \times \mathbf{w}_A) \varepsilon \cdot (\mathbf{w}_A \times \mathbf{w}_{A+1})} - \frac{2 m_A m_{A+1}}{\varepsilon \cdot (\mathbf{w}_A \times \mathbf{w}_{A+1})} \right) \end{aligned}$$

$$V(m_A) = -\frac{1}{2} \varepsilon \cdot (\mathbf{v}_{D-1} \times \mathbf{v}_D) \sum_{A=1}^{D-2} \left(\frac{m_A^2 \varepsilon \cdot (\mathbf{v}_{A-1} \times \mathbf{v}_{A+1})}{\varepsilon \cdot (\mathbf{v}_{A-1} \times \mathbf{v}_A) \varepsilon \cdot (\mathbf{v}_A \times \mathbf{v}_{A+1})} - \frac{2 m_A m_{A+1}}{\varepsilon \cdot (\mathbf{v}_A \times \mathbf{v}_{A+1})} \right)$$

$$V(m_A^*) = -\frac{\varepsilon \cdot (\mathbf{v}_{D-1} \times \mathbf{v}_D)}{\omega_1 \omega_2} \mathcal{B}(m_A^*) = \frac{1}{12} \left(\frac{N}{\omega_1 \omega_2}\right)^2 \varepsilon \cdot (\mathbf{v}_{D-1} \times \mathbf{v}_D) \sum_{I,J,K} C_{I,J,K} \Delta_I \Delta_J \Delta_K$$

$$\partial_{\mu_1} \cdots \partial_{\mu_{s-t}} X^{\mu_1 \cdots \mu_s} = 0$$

$$\sum_{i=1}^{n} \langle O_1 \ldots [Q, O_i] \ldots O_n \rangle = 0$$

$$\sum_{j} \left[a_{1j}\partial^{n_{1j}}\langle O_j O_2 O_3\rangle + a_{2j}\partial^{n_{2j}}\langle O_1 O_j O_3\rangle + a_{3j}\partial^{n_{3j}}\langle O_1 O_2 O_j\rangle\right] = 0.$$

$$O^{(s)}(x,z) \equiv O_{\mu_1 \ldots \mu_s}(x) z^{\mu_1} \cdots z^{\mu_s},$$

$$\mathrm{d}s^2 = \frac{L^2}{\eta^2}(-\mathrm{d}\eta^2 + \mathrm{d}x^2)$$

$$(\Box - [d - (s-1)(s+d-4)]L^{-2} - m^2)\Phi_{M_1 \cdots M_s} = 0$$

$$m^2 L^2 = (s-t-1)(s+t+d-3)$$

$$\delta_\xi \Phi_{M_1 \cdots M_s} = \nabla_{(M_{t+1}} \cdots \nabla_{M_s} \xi_{M_1 \cdots M_t)} + g$$

$$\Phi_{\mu_1 \cdots \mu_s}(\eta, x) \overset{\eta \to 0}{\rightarrow} \phi_{\mu_1 \cdots \mu_s}(x)\eta^{\Delta_- - s} + j_{\mu_1 \cdots \mu_s}(x)\eta^{\Delta_+ - s}$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2} + s - 2\right)^2 - m^2 L^2}$$

$$\phi^{(s)}(x,z) \equiv \phi_{\mu_1 \ldots \mu_s}(x) z^{\mu_1} \cdots z^{\mu_s}$$

$$D_z^\mu \equiv \left(\frac{d}{2} - 1 + z \cdot \frac{\partial}{\partial z}\right)\frac{\partial}{\partial z_\mu} - \frac{1}{2} z^\mu \frac{\partial^2}{\partial z \cdot \partial z}$$

$$\delta_\xi \phi^{(s)} = (z \cdot \partial)^{s-t} \xi^{(t)}$$

$$(\partial \cdot D_z)^{s-t} J_{(s,t)} = 0$$

$$\Delta = d - 1 + t$$

$$\Psi[\phi] = \int^{\Phi = \phi} \mathcal{D}\,\mathcal{D}\Phi e^{iS[\Phi]},$$

$$\Phi(\eta, k) \sim \frac{1}{\sqrt{2|k|}} e^{+i|k|\eta}, \text{ as } \eta \to -\infty$$

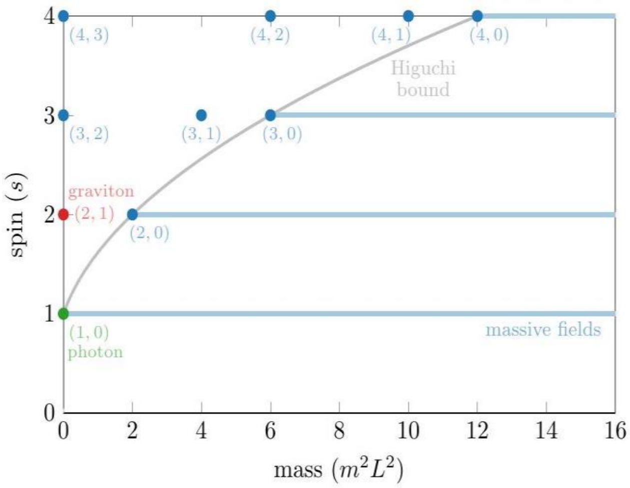
$$\Psi[\phi(x)] \approx \exp\left(-\sum_{n=2}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \langle O(x_1) \ldots O(x_n)\rangle \phi(x_1) \cdots \phi(x_n)\right)$$

$$m^2 L^2 = \begin{cases} \Delta(d - \Delta), & s = 0 \\ (\Delta + s - 2)(d - \Delta + s - 2), & s \geq 1 \end{cases}$$

$$m^2L^2 \geq \left(\frac{d}{2} + s - 2\right)^2$$

$$(s-1)(d+s-3) < m^2L^2 < \left(\frac{d}{2} + s - 2\right)^2$$

$$m^2L^2 = (s-t-1)(s+t+d-3)$$



$$P_AP^A = 0, P^A \sim \lambda P^A,$$

$$(P^+, P^-, x^\mu) = (1, |x|^2, x^\mu),$$

$$\begin{aligned} O_{A_1 \cdots A_s}(\lambda P) &= \lambda^{-\Delta} O_{A_1 \cdots A_s}(P) \\ P^{A_1} O_{A_1 \cdots A_s}(P) &= 0 \end{aligned}$$

$$\begin{aligned} P_{ij} &\equiv P_i \cdot P_j \\ H_{ij} &\equiv -2\left[(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(Z_j \cdot P_i)\right] \\ V_{i,jk} &\equiv \frac{(Z_i \cdot P_j)(P_i \cdot P_k) - (Z_i \cdot P_k)(P_i \cdot P_j)}{P_j \cdot P_k} \end{aligned}$$

$$\begin{aligned} P_{ij} &\rightarrow -\frac{1}{2} x_{ij}^2 \\ H_{ij} &\rightarrow (z_i \cdot z_j) x_{ij}^2 - 2(z_i \cdot x_{ij})(z_j \cdot x_{ij}) \\ V_{i,jk} &\rightarrow \frac{(z_i \cdot x_{ik}) x_{ij}^2 - (z_i \cdot x_{ij}) x_{ik}^2}{x_{jk}^2} \end{aligned}$$

$$\left\langle O_1^{(s_1)}(P_1, Z_1) O_2^{(s_2)}(P_2, Z_2) O_3^{(s_3)}(P_3, Z_3) \right\rangle = f(P_{ij}, H_{ij}, V_k),$$

$$-2H_{12}H_{23}H_{31} = (V_1H_{23} + V_2H_{31} + V_3H_{12} + 2V_1V_2V_3)^2$$

$$(\partial \cdot D_z)^{s-t} O_{(s,t)} = 0$$

$$\langle T_{(2,1)}X_{(2,0)}X_{(2,0)}\rangle = \frac{1}{P_{12}^{\frac{d+2}{2}}P_{23}^{\frac{d}{2}}P_{31}^{\frac{d+2}{2}}}\sum_{n=1}^{8} c_n G_n, \text{ where } G_n \equiv \begin{pmatrix} V_1^2V_2^2V_3^2 \\ V_1V_2V_3^2H_{12} + (2 \leftrightarrow 3) \\ V_1^2V_2V_3H_{23} \\ V_3^2H_{12}^2 + (2 \leftrightarrow 3) \\ V_1^2H_{23}^2 \\ V_1V_3H_{12}H_{23} + (2 \leftrightarrow 3) \\ V_2V_3H_{12}H_{31} \\ H_{12}H_{23}H_{31} \end{pmatrix}$$

$$c_6 = \frac{4}{(d-2)(d+4)}c_1 + \frac{(d-4)}{(d-2)(d+4)}c_2 + \frac{1}{2-d}c_3 + \frac{d^2}{(d-2)(d+4)}c_4 + \frac{d}{d-2}c_5,$$

$$c_7 = \frac{4}{(d-2)(d+4)}c_1 - \frac{8}{(d-2)(d+4)}c_2 + \frac{2d(d+2)}{(d-2)(d+4)}c_4,$$

$$c_8 = -\frac{4}{(d-2)^2(d+4)}c_1 + \frac{4d}{(d-2)^2(d+2)(d+4)}c_2 + \frac{2d}{(d-2)^2(d+2)}c_3 + \frac{2d(d^2+2d-12)}{(d-2)^2(d+2)(d+4)}c_4 - \frac{4d}{(d-2)^2(d+2)}c_5.$$

$$S = \int d^dx\phi_a^\dagger \,\square^k\, \phi_a$$

$$\square^k\, \phi_a = 0$$

$$X_{(s,s-1)} \sim \phi_a^\dagger(z\cdot\partial)^s \,\square\, \phi_a + \cdots,$$

$$s \geq 1: (\partial\cdot D_z)X_{(s,s-1)} = 0$$

$$X_{(s,s-3)} \sim \phi_a^\dagger(z\cdot\partial)^s\phi_a + \cdots,$$

$$s \geq 3: (\partial\cdot D_z)^3X_{(s,s-3)} = 0$$

$$S = -\frac{\lambda^2}{8}\int d^4x\sqrt{-g}C^{MNLK}C_{MNLK}$$

$$S = -\frac{\lambda^2}{4}\int d^4x\sqrt{-g}\left(R_{MN}R^{MN} - \frac{1}{3}R^2\right)$$

$$S = \lambda^2\int d^4x\sqrt{-g}\left(\frac{\Lambda}{6}(R-2\Lambda) + (G_{MN} - \Lambda g_{MN})f^{MN} + f_{MN}f^{MN} - f^2\right)$$

$$f_{MN} = -\frac{1}{2}\left(R_{MN} - \frac{1}{6}Rg_{MN}\right) + \frac{\Lambda}{6}g_{MN}$$

$$g_{MN} \mapsto g_{MN} - \frac{6}{\Lambda}f_{MN}$$

$$S = \lambda^2\int d^4x\sqrt{-g}\left[\frac{\Lambda}{6}(R-2\Lambda) - \frac{3}{\Lambda}\left(-\frac{1}{2}(\nabla f)^2 + \frac{2\Lambda}{3}\left(f^{MN}f_{MN} - \frac{1}{4}f^2\right) + (G_{MN} + \Lambda g_{MN})F^{MN}\right) + \mathcal{O}(f^3)\right]$$

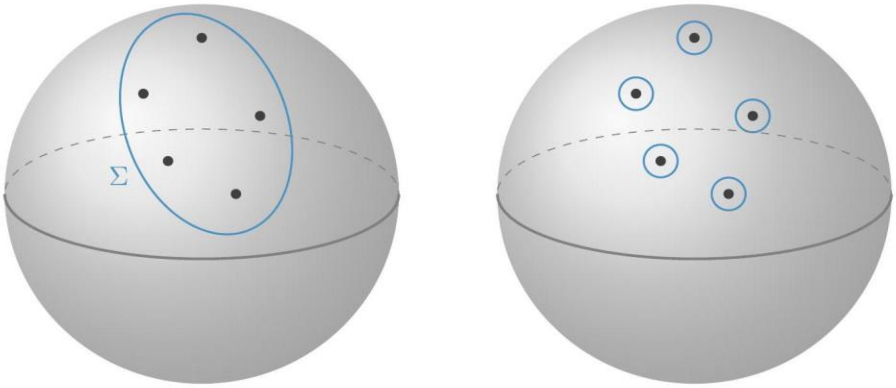
$$-\frac{1}{2}(\nabla f)^2 \equiv -\frac{1}{2}\nabla_K f_{MN}\nabla^K f^{MN} + \nabla_K f_{MN}\nabla^N f^{MK} - \nabla_M f\nabla_N f^{MN} + \frac{1}{2}\nabla_M f\nabla^M f$$

$$S = \lambda^2 \int \mathrm{d}^4x \left(\frac{1}{4L^2}\mathcal{L}_{\mathrm{FP},0}(h) - L^2\mathcal{L}_{\mathrm{FP},2L^{-2}}(f)\right)$$

$$\frac{\mathcal{L}_{\mathrm{FP},m^2}(X)}{\sqrt{-g}} = -\frac{1}{2}\nabla_K X_{MN}\nabla^K X^{MN} + \nabla_K X_{MN}\nabla^N X^{MK} - \nabla_M X\nabla_N X^{MN} + \frac{1}{2}\nabla_M X\nabla^M X$$

$$+3L^{-2}\left(X^{MN}X_{MN} - \frac{1}{2}X^2\right) - \frac{1}{2}m^2(X_{MN}X^{MN} - X^2)$$

$$Q_{(s,t)}^{\mu_1\dots\mu_t}[\Sigma] \equiv \oint_\Sigma \mathrm{d}\Sigma_{\nu_1}\partial_{\nu_2}\cdots\partial_{\nu_{s-t}}X_{(s,t)}^{\mu_1\cdots\mu_t\nu_1\nu_2\cdots\nu_{s-t}}$$



$$\langle Q[\Sigma]O_1\dots O_n\rangle \equiv \langle [Q, O_1\dots O_n]\rangle = \sum_{i=1}^{n}\langle O_1\dots[Q,O_i]\dots O_n\rangle = 0$$

$$[Q, O_i(x)] \sim \sum_j a_{ij}\partial^{n_{ij}}O_j(x)$$

$$\langle [Q, O_1O_2O_3]\rangle = \langle [Q,O_1]O_2O_3\rangle + \langle O_1[Q,O_2]O_3\rangle + \langle O_1O_2[Q,O_3]\rangle = 0$$

$$\sum_j \left[a_{1j}\partial^{n_{1j}}\langle O_jO_2O_3\rangle + a_{2j}\partial^{n_{2j}}\langle O_1O_jO_3\rangle + a_{3j}\partial^{n_{3j}}\langle O_1O_2O_j\rangle\right] = 0$$

$$\oint_\Sigma \mathrm{d}\Sigma_{\nu_1}\partial_{\nu_2}\cdots\partial_{\nu_{s-t}}\left\langle X_{(s,t)}^{\mu_1\cdots\mu_t\nu_1\nu_2\cdots\nu_{s-t}}(x,z)O_1(x_1)O_2(x_2)O_3(x_3)\right\rangle = 0$$

$$\square\, O_\Delta^{(s)} \sim O_{\Delta+2}^{(s)}, (z\cdot\partial)O_\Delta^{(s)} \sim O_{\Delta+1}^{(s+1)}, (\partial\cdot D_z)O_\Delta^{(s)} \sim O_{\Delta+1}^{(s-1)}$$

$$\left[Q_{(s,0)}, X_{(s',t')}(x,z)\right] \supset \mathcal{P}_k\left(\square, (z\cdot\partial)(\partial\cdot D_z)\right) \times \begin{cases}(\partial\cdot D_z)^{s''-s'}X_{(s'',t'')}(x,z) & \text{if } s'' \geq s' \\ (z\cdot\partial)^{s'-s''}X_{(s'',t'')}(x,z) & \text{if } s' > s''\end{cases}$$

$$k \equiv \frac{1}{2}(s - 1 - |s'' - s'| + t' - t'')$$

$$t' - t'' \geq |s'' - s'| - s + 1$$

$$\left\langle\left[Q_{(s,t)}, X_{(s',t')}X_{(s'',t'')}\right]\right\rangle = \left\langle\left[Q_{(s,t)}, X_{(s',t')}\right]X_{(s'',t'')}\right\rangle + \left\langle X_{(s',t')}\left[Q_{(s,t)}, X_{(s'',t'')}\right]\right\rangle = 0$$

$$t'' - t' \geq |s'' - s'| - s + 1$$

$$s'' \in \mathbb{Z}_{\geq 0}, t'' \in \mathbb{Z}, s - 1 - |s'' - s'| - |t'' - t'| \in 2\mathbb{Z}_{\geq 0}$$
$$t'' \leq s'' - 1, |s'' - s'| \leq s - 1$$

$$(s'', t'') \in \{(1,0), (2,-1), (2,1), (3,0)\}$$

$$\left[Q_{(2,0)}, X_{(2,0)}(x,z)\right] = a_1(z\cdot\partial)X_{(1,0)}(x,z) + [a_2\,\Box + a_3(z\cdot\partial)(\partial\cdot D_z)]X_{(2,-1)}(x,z)$$
$$+ a_4 T_{(2,1)}(x,z) + a_5(\partial\cdot D_z)X_{(3,0)}(x,z)$$

$$\left[Q_{(1,0)}, O_i(x)\right] = q_i O_i(x)$$

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \frac{\lambda_{ijk}}{P_{12}^{\Delta_i+\Delta_j-\Delta_k} P_{23}^{\Delta_j+\Delta_k-\Delta_i} P_{31}^{\Delta_k+\Delta_i-\Delta_j}}$$

$$\langle[Q_{(1,0)}, O_1(x_1)O_2(x_2)O_3(x_3)]\rangle = (q_1 + q_2 + q_3)\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = 0$$

$$\left[Q^{\mu}_{(2,1)}, O_i(x)\right] = \kappa_i \partial^{\mu} O_i(x)$$

$$\left\langle\left[Q^{\mu}_{(2,1)}, O_1(x_1)O_2(x_2)O_3(x_3)\right]\right\rangle = \left(\kappa_1\partial_1^{\mu} + \kappa_2\partial_2^{\mu} + \kappa_3\partial_3^{\mu}\right)\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = 0$$

$$\left[Q_{(3,2)}(z), O_i(x)\right] = \tau_i(z\cdot\partial)^2 O_i(x)$$

$$\left\langle\left[Q^{\mu\nu}_{(3,2)}, O_1(x_1)O_2(x_2)O_3(x_3)\right]\right\rangle$$
$$= \left[\tau_1\left(\partial_1^{\mu}\partial_1^{\nu} - \frac{1}{d}g^{\mu\nu}\,\Box_1\right) + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right]\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = 0.$$

$$z_{\mu_1}\cdots z_{\mu_t}\partial_{\nu_1}\cdots\partial_{\nu_{s-t}}\left\langle X_{(s,t)}^{\nu_1\cdots\nu_{s-t}\mu_1\cdots\mu_t}(x)O_1(x_1)\cdots O_n(x_n)\right\rangle$$
$$= \sum_{i=1}^{n} \delta^{(d)}(x - x_i)\langle O_1(x_1)\cdots\left[Q_{(s,t)}(z), O_i(x_i)\right]\cdots O_n(x_n)\rangle$$

$$\lim_{k\to 0}\int \mathrm{d}^d x e^{ik\cdot x}\partial_{\nu_1}\ldots\partial_{\nu_{s-t}}\left\langle X_{(s,t)}^{\nu_1\ldots\nu_{s-t}\mu_1\ldots\mu_t}(x)O_1(x_1)\ldots O_n(x_n)\right\rangle$$
$$= (-i)^{s-t}\lim_{k\to 0} k_{\nu_1}\ldots k_{\nu_{s-t}}\left\langle X_{(s,t)}^{\nu_1\ldots\nu_{s-t}\mu_1\ldots\mu_t}(k)O_1(x_1)\ldots O_n(x_n)\right\rangle$$

$$\lim_{k\to 0} k_{\nu_1}\cdots k_{\nu_{s-t}} z_{\mu_1}\cdots z_{\mu_t}\left\langle X_{(s,t)}^{\nu_1\cdots\nu_{s-t}\mu_1\cdots\mu_t}(k)O_1(k_1)\cdots O_n(k_n)\right\rangle = 0$$

$$\lim_{k\to 0} k_{\nu_1}\cdots k_{\nu_{s-t}} z_{\mu_1}\cdots z_{\mu_t}\mathcal{A}_{n+1}^{\nu_1\cdots\nu_{s-t}\mu_1\cdots\mu_t}\left(\Phi_{(s,t)}(k), \Phi(k_1), \cdots, \Phi(k_n)\right) = 0$$

$$k_{\mu}\left\langle J^{\mu}_{(1,0)}(k)O_1(k_1)O_2(k_2)O_3(k_3)\right\rangle = \lambda_{123}\left[q_1 \log\left(\frac{|k + k_1| + |k_2| + |k_3|}{E_0}\right)\right.$$
$$\left. + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right]$$

$$\lim_{k\to 0} k_\mu \left\langle J^\mu_{(1,0)}(k) O_1(k_1) O_2(k_2) O_3(k_3) \right\rangle = \lambda_{123}(q_1+q_2+q_3)\log\left(\frac{E}{E_0}\right).$$

$$\lim_{k\to 0} k_\mu \mathcal{A}_4^\mu\big(\gamma(k), \Phi_1(k_1), \Phi_2(k_2), \Phi_3(k_3)\big) = (q_1+q_2+q_3)\mathcal{A}_3\big(\Phi_1(k_1), \Phi_2(k_2), \Phi_3(k_3)\big) = 0$$

$$\left[Q_{(s,t)}, X_{(s,t)}\right] = 0$$

$$\left[Q_{(s,t)}(z), O_i(x)\right] = \sum_j a_{ij} \,\Box^{n_{ij}}\, (z\cdot\partial)^t O_j(x)$$

$$2n_{ij} = \Delta_i - \Delta_j + s - 1 - t$$

$$\Delta_i - \Delta_j \in 2\mathbb{Z} + r$$
$$\left|\Delta_i - \Delta_j\right| \leq r$$

$$\left\langle X_{(s,t)}(x_1,z_1) O_i(x_2) O_j(x_3)\right\rangle = c_{ij} \frac{V_1^s}{P_{12}^{(d-1+t-s)/2} P_{13}^{(d-1+t-s)/2} P_{23}^{(2\Delta-d+1-t-s)/2}}$$

$$\left\langle O_i(x_2) O_j(x_3)\right\rangle = \frac{\delta_{ij}}{P_{23}^{\Delta}}$$

$$\left[Q_{(s,t)}(z), O_i(x)\right] = g_i \,\Box^{(s-t-1)/2}\, (z\cdot\partial)^t O_i(x), \text{ if } s \in 2\mathbb{Z}_{\geq 0}$$
$$\left[Q_{(s,t)}(z), O_i(x)\right] = i\lambda_i \,\Box^{(s-t-1)/2}\, (z\cdot\partial)^t O_i(x), \text{ if } s \in 2\mathbb{Z}_{\geq 0}+1$$

$$\left\langle\left[Q_{(s,t)}(z), X_{(s,t)}(x_1,z_1) O_i(x_2) O_i(x_3)\right]\right\rangle = 0$$

$$g_i\left(\Box_2^{(s-t-1)/2}\,(z\cdot\partial_2)^t + \Box_3^{(s-t-1)/2}\,(z\cdot\partial_3)^t\right)\left\langle X_{(s,t)}(x_1,z_1) O_i(x_2) O_i(x_3)\right\rangle = 0$$

$$i\lambda_i\left(\Box_2^{(s-t-1)/2}\,(z\cdot\partial_2)^t - \Box_3^{(s-t-1)/2}\,(z\cdot\partial_3)^t\right)\left\langle X_{(s,t)}(x_1,z_1) O_i(x_2) O_i^\dagger(x_3)\right\rangle = 0$$

$$\left[Q_{(s,s-2)}, O_\Delta\right] \sim O_{\Delta+1} + O_{\Delta-1}$$

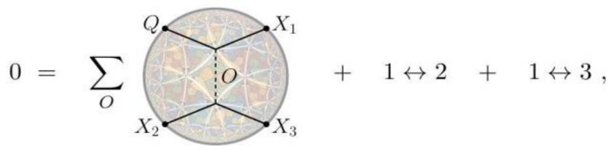
$$\left[Q_{(s,s-2)}(z), O_\Delta(x)\right] = a(z\cdot\partial)^{s-2} O_{\Delta+1}(x).$$

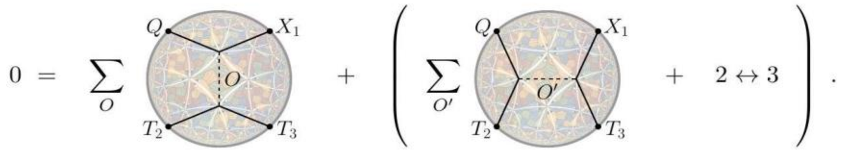
$$\left\langle\left[Q_{(s,s-2)}(z), X_{(s,s-2)}(x_1,z_1) O_\Delta(x_2) O_\Delta(x_3)\right]\right\rangle = 0$$

$$a(z\cdot\partial_2)^{s-2}\left\langle X_{(s,s-2)}(x_1,z_1) O_{\Delta+1}(x_2) O_\Delta(x_3)\right\rangle + (2\leftrightarrow 3) = 0.$$

$$\left[Q_{(s,s-5)}(z), O_\Delta(x)\right] = a_1 \,\Box\, (z\cdot\partial)^{s-5} O_{\Delta+1}(x) + a_2 (z\cdot\partial)^{s-5} O_{\Delta+3}(x)$$

$$\left\langle\left[Q_{(s,t)}, X_{(s,t)} X_{(s,t)} X_{(s,t)}\right]\right\rangle = 0$$
$$\left\langle\left[Q_{(s,t)}, X_{(s,t)} T_{(2,1)} T_{(2,1)}\right]\right\rangle = 0$$





$$\left[\hat{D}, Q_{(s,t)}\right] = (s-1)Q_{(s,t)}$$

$$\left\langle T_{(2,1)}X_{(s,t)}X_{(s,t)}\right\rangle \neq 0$$
$$\left\langle T_{(2,1)}T_{(2,1)}T_{(2,1)}\right\rangle \neq 0$$

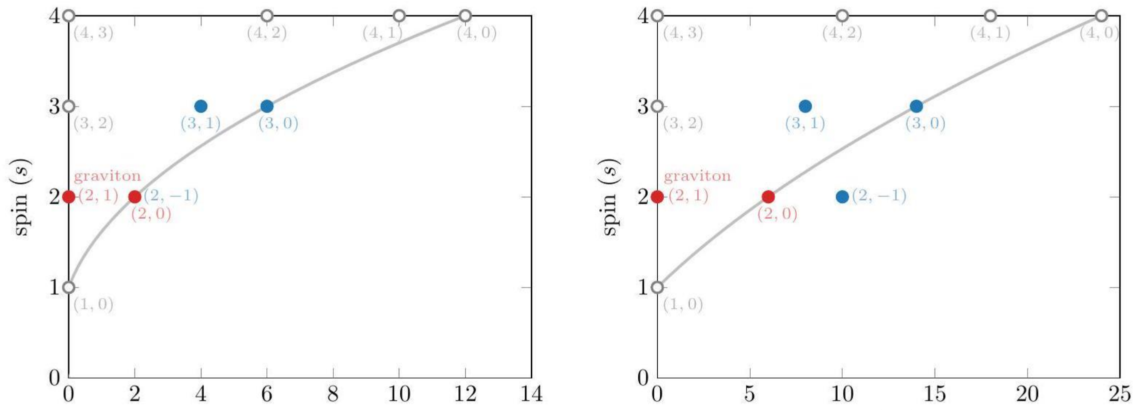
$$\left[Q_{(2,0)}, X_{(2,0)}\right] = a_1 T_{(2,1)}$$
$$\left[Q_{(2,0)}, T_{(2,1)}\right] = b_1\left(\Box - \frac{2}{d}(z\cdot\partial)(\partial\cdot D_z)\right)X_{(2,0)}$$

$$\left\langle\left[Q_{(2,0)}, X_{(2,0)}X_{(2,0)}X_{(2,0)}\right]\right\rangle = 0$$
$$\left\langle\left[Q_{(2,0)}, X_{(2,0)}T_{(2,1)}T_{(2,1)}\right]\right\rangle = 0$$

$$0 = a_1\left\langle T_{(2,1)}X_{(2,0)}X_{(2,0)}\right\rangle + (1\leftrightarrow 2) + (1\leftrightarrow 3),$$
$$0 = a_1\left\langle T_{(2,1)}T_{(2,1)}T_{(2,1)}\right\rangle$$
$$+\left\{b_1\left(\Box_2 - \frac{2}{d}(z_2\cdot\partial_2)(\partial_2\cdot D_{z_2})\right)\left\langle X_{(2,0)}X_{(2,0)}T_{(2,1)}\right\rangle + (2\leftrightarrow 3)\right\}.$$

$$\left[Q_{(2,0)}, X_{(2,0)}\right] = a_1 T_{(2,1)} + a_2(\partial\cdot D_z)X_{(3,0)}$$
$$+a_3\left(\Box - \frac{1}{d-1}(z\cdot\partial)(\partial\cdot D_z)\right)X_{(2,-1)}$$
$$\left[Q_{(2,0)}, T_{(2,1)}\right] = b_1\left(\Box - \frac{2}{d}(z\cdot\partial)(\partial\cdot D_z)\right)X_{(2,0)} + b_2(\partial\cdot D_z)X_{(3,1)}$$

$$0 = a_1\left\langle T_{(2,1)}X_{(2,0)}X_{(2,0)}\right\rangle + a_3\left(\Box_1 - \frac{1}{d-1}(z_1\cdot\partial_1)(\partial_1\cdot D_{z_1})\right)\left\langle X_{(2,-1)}X_{(2,0)}X_{(2,0)}\right\rangle$$



$$+a_2(\partial_1 \cdot D_{z_1})\langle X_{(3,0)}X_{(2,0)}X_{(2,0)}\rangle + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)$$

$$0 = a_1\langle T_{(2,1)}T_{(2,1)}T_{(2,1)}\rangle + a_3\left(\square_1 - \frac{1}{d-1}(z_1 \cdot \partial_1)(\partial_1 \cdot D_{z_1})\right)\langle X_{(2,-1)}T_{(2,1)}T_{(2,1)}\rangle$$

$$+\left\{b_1\left(\square_2 - \frac{2}{d}(z_2 \cdot \partial_2)(\partial_2 \cdot D_{z_2})\right)\langle X_{(2,0)}X_{(2,0)}T_{(2,1)}\rangle\right.$$

$$\left.+b_2(z_2 \cdot \partial_2)(\partial_2 \cdot D_{z_2})\langle X_{(2,0)}X_{(3,1)}T_{(2,1)}\rangle + (2 \leftrightarrow 3)\right\},$$

$$\begin{aligned} \text{S1:}\quad & \{T_{(2,1)}, X_{(2,0)}, X_{(2,-1)}, X_{(3,0)}\}, \\ \text{S2:}\quad & \{T_{(2,1)}, X_{(2,0)}, X_{(2,-1)}, X_{(3,1)}\}, \\ \text{S3:}\quad & \{T_{(2,1)}, X_{(2,0)}, X_{(3,0)}, X_{(3,1)}\}. \end{aligned}$$

$$\text{S1:}\{T_{(2,1)}, X_{(2,0)}, X_{(2,-1)}\},$$
$$\text{S2:}\{T_{(2,1)}, X_{(3,0)}, X_{(3,0)}, X_{(3,1)}\}.$$

$$\begin{aligned} [Q_{(3,0)}, X_{(3,0)}] &= a_1(z \cdot \partial)T_{(2,1)} \\ [Q_{(3,0)}, T_{(2,1)}] &= b_1\left(\square\,(\partial \cdot D_z) - \frac{2}{d}(z \cdot \partial)(\partial \cdot D_z)^2\right)X_{(3,0)} \end{aligned}$$

$$\begin{aligned} \langle[Q_{(3,0)}, X_{(3,0)}X_{(3,0)}X_{(3,0)}]\rangle &= 0 \\ \langle[Q_{(3,0)}, X_{(3,0)}T_{(2,1)}T_{(2,1)}]\rangle &= 0 \end{aligned}$$

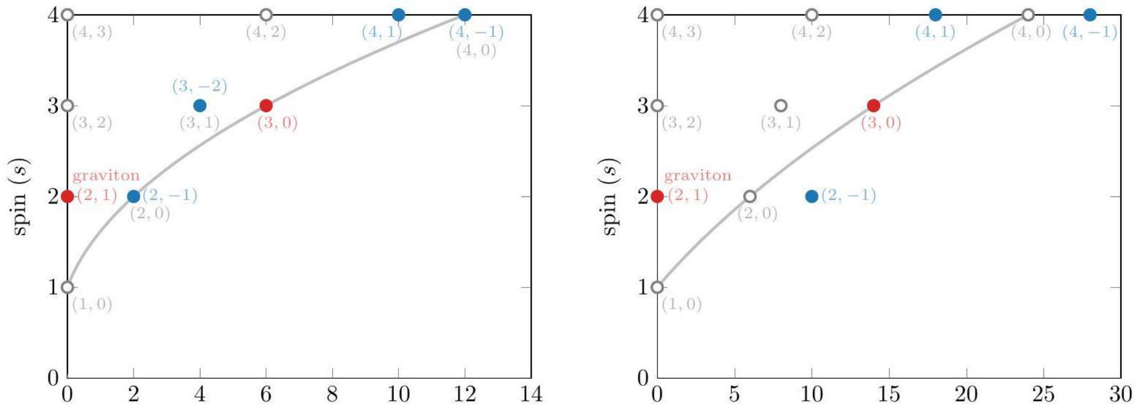
$$0 = a_1(z_1 \cdot \partial_1)\langle T_{(2,1)}X_{(3,0)}X_{(3,0)}\rangle + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)$$
$$0 = a_1(z_1 \cdot \partial_1)\langle T_{(2,1)}T_{(2,1)}T_{(2,1)}\rangle$$
$$+\left\{b_1\left(\square_2\,(\partial_2 \cdot D_{z_2}) - \frac{2}{d}(z_2 \cdot \partial_2)(\partial_2 \cdot D_{z_2})^2\right)\langle X_{(3,0)}X_{(3,0)}T_{(2,1)}\rangle + (2 \leftrightarrow 3)\right\}.$$

$$
\begin{aligned}
\left[Q_{(3,0)}, X_{(3,0)}\right] &= a_1 (z\cdot\partial) T_{(2,1)} \\
&+ \left[a_2\left(\Box^2 - \frac{2}{3d(d-1)}(z\cdot\partial)^2(\partial\cdot D_z)^2\right)\right. \\
&\left. + a_3\left(\Box\,(z\cdot\partial)(\partial\cdot D_z) - \frac{1}{d-1}(z\cdot\partial)^2(\partial\cdot D_z)^2\right)\right] X_{(3,-2)} \\
&+ a_4 (\partial\cdot D_z) X_{(4,1)} + a_5\left(\Box\,(z\cdot\partial) - \frac{2}{3d}(z\cdot\partial)^2(\partial\cdot D_z)\right) X_{(4,-1)} \\
&+ a_6\left(\Box\,(z\cdot\partial) - \frac{1}{d-1}(z\cdot\partial)^2(\partial\cdot D_z)\right) X_{(2,-1)} \\
&+ a_7 (\partial\cdot D_z)^2 X_{(5,0)}
\end{aligned}
$$

$$
\begin{aligned}
\left[Q_{(3,0)}, T_{(2,1)}\right] &= b_1\left(\Box\,(\partial\cdot D_z) - \frac{2}{d}(z\cdot\partial)(\partial\cdot D_z)^2\right) X_{(3,0)} \\
&+ \left[b_2\left(\Box^2 - \frac{2}{d}\Box\,(z\cdot\partial)(\partial\cdot D_z)\right)\right. \\
&\left. + b_3\left(\Box^2 + \frac{2}{d(d-1)}(z\cdot\partial)^2(\partial\cdot D_z)^2\right)\right] \tilde{X}_{(2,-1)} \\
&+ b_4 (\partial\cdot D_z)^2 \tilde{X}_{(4,1)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= a_1 (z_1\cdot\partial_1)\langle T_{(2,1)} X_{(3,0)} X_{(3,0)}\rangle \\
&+ \left[a_2\left(\Box_1^2 - \frac{2}{3d(d-1)}(z_1\cdot\partial_1)^2\left(\partial_1\cdot D_{z_1}\right)^2\right)\right. \\
&\left. + a_3\left(\Box_1\,(z_1\cdot\partial_1)(\partial_1\cdot D_1) - \frac{1}{d-1}(z_1\cdot\partial_1)^2\left(\partial_1\cdot D_{z_1}\right)^2\right)\right] \langle X_{(3,-2)} X_{(3,0)} X_{(3,0)}\rangle \\
&+ a_4\left(\partial_1\cdot D_{z_1}\right)\langle X_{(4,1)} X_{(3,0)} X_{(3,0)}\rangle \\
&+ a_5\left(\Box_1\,(z_1\cdot\partial_1) - \frac{2}{3d}(z_1\cdot\partial_1)^2\left(\partial_1\cdot D_{z_1}\right)\right)\langle X_{(4,-1)} X_{(3,0)} X_{(3,0)}\rangle \\
&+ a_6\left(\Box_1\,(z_1\cdot\partial_1) - \frac{1}{d-1}(z_1\cdot\partial_1)^2\left(\partial_1\cdot D_{z_1}\right)\right)\langle X_{(2,-1)} X_{(3,0)} X_{(3,0)}\rangle \\
&+ a_7\left(\partial_1\cdot D_{z_1}\right)^2\langle X_{(5,0)} X_{(3,0)} X_{(3,0)}\rangle + (1\leftrightarrow 2) + (1\leftrightarrow 3)
\end{aligned}
$$

$$
\begin{aligned}
0 &= a_1(z_1\cdot\partial_1)\langle T_{(2,1)}T_{(2,1)}T_{(2,1)}\rangle \\
&+\Big[a_2\Big(\Box_1^2-\frac{2}{3d(d-1)}(z_1\cdot\partial_1)^2\big(\partial_1\cdot D_{z_1}\big)^2\Big) \\
&+a_3\Big(\Box_1\,(z_1\cdot\partial_1)\big(\partial_1\cdot D_{z_1}\big)-\frac{1}{d-1}(z_1\cdot\partial_1)^2\big(\partial_1\cdot D_{z_1}\big)^2\Big)\Big]\langle X_{(3,-2)}T_{(2,1)}T_{(2,1)}\rangle \\
&+a_4\big(\partial_1\cdot D_{z_1}\big)\langle X_{(4,1)}T_{(2,1)}T_{(2,1)}\rangle \\
&+a_5\Big(\Box_1\,(z_1\cdot\partial_1)-\frac{2}{3d}(z_1\cdot\partial_1)^2\big(\partial_1\cdot D_{z_1}\big)\Big)\langle X_{(4,-1)}T_{(2,1)}T_{(2,1)}\rangle \\
&+a_6\Big(\Box_1\,(z_1\cdot\partial_1)-\frac{1}{d-1}(z_1\cdot\partial_1)^2\big(\partial_1\cdot D_{z_1}\big)\Big)\langle X_{(2,-1)}T_{(2,1)}T_{(2,1)}\rangle \\
&+a_7\big(\partial_1\cdot D_{z_1}\big)^2\langle X_{(5,0)}T_{(2,1)}T_{(2,1)}\rangle \\
&+\Big\{b_1\Big(\Box_2\,\big(\partial_2\cdot D_{z_2}\big)-\frac{2}{d}(z_2\cdot\partial_2)\big(\partial_2\cdot D_{z_2}\big)^2\Big)\langle X_{(3,0)}X_{(3,0)}T_{(2,1)}\rangle \\
&+\Big[b_2\Big(\Box_2^2-\frac{2}{d}\Box_2\,(z_2\cdot\partial_2)\big(\partial_2\cdot D_{z_2}\big)\Big) \\
&+b_3\Big(\Box_2^2+\frac{2}{d(d-1)}(z_2\cdot\partial_2)^2\big(\partial_2\cdot D_{z_2}\big)^2\Big)\Big]\langle X_{(3,0)}\tilde{X}_{(2,-1)}T_{(2,1)}\rangle
\end{aligned}
$$



$$
+b_4\big(\partial_2\cdot D_{z_2}\big)^2\langle X_{(3,0)}\tilde{X}_{(4,1)}T_{(2,1)}\rangle+(2\leftrightarrow 3)\Big\}
$$

$$
\begin{aligned}
&\mathrm{S1}\colon\{T_{(2,1)},X_{(3,0)},X_{(4,-1)},X_{(4,1)}\},\\
&\mathrm{S2}\colon\{T_{(2,1)},X_{(3,0)},X_{(2,-1)},\tilde{X}_{(2,-1)},X_{(3,-2)},X_{(4,1)}\}.
\end{aligned}
$$

$$
\mathrm{S1}\colon\{T_{(2,1)},X_{(3,0)},X_{(4,-1)},X_{(4,1)}\},
$$

$$
\begin{aligned}
&\mathrm{S2}\colon\quad\{T_{(2,1)},X_{(3,0)},X_{(2,-1)},X_{(4,1)}\}\\
&\mathrm{S3}\colon\quad\{T_{(2,1)},X_{(3,0)},X_{(2,-1)},X_{(4,-1)}\}
\end{aligned}
$$

$$
\{X_{(2,-1)},T_{(2,1)},X_{(3,0)},X_{(4,1)}\}
$$

$$
S_{(s,t)}=\lambda\int\ \mathrm{d}^{d+1}y\sqrt{-g}\,\Phi^{M_1\ldots M_s}\phi_1\nabla_{M_1}\ldots\nabla_{M_s}\phi_2
$$

$$
\delta_\xi S_{(s,t)}=\lambda\mathcal{P}_{(s,t)}(m_1^2,m_2^2)\int\ \mathrm{d}^{d+1}y\sqrt{-g}\,\xi^{M_1\ldots M_t}\phi_1\nabla_{M_1}\ldots\nabla_{M_t}\phi_2
$$

$$\mathcal{P}_{(s,t)}(m_1^2, m_2^2) = 0.$$

$$\mathcal{P}_{(2,0)} = [(\Delta_1 - \Delta_2)^2 - 1][(\Delta_1 + \Delta_2 - d)^2 - 1].$$

$$\langle O_1 O_2 X_{(2,0)} \rangle = \lambda \int \mathrm{d}^{d+1} y \sqrt{-g} K_{\Delta_1}(x_1; y) \nabla_{M_1} \nabla_{M_2} K_{\Delta_2}(x_2; y) K_{d-1}^{M_1 M_2}(z_3, x_3; y),$$

$$(\partial_x \cdot D_z)^{s-t} K_{d-1+t}^{M_1 \dots M_s}(z, x; y) \propto \nabla^{(M_{t+1}} \dots \nabla^{M_s} K_{d-1+s}^{M_1 \dots M_t)}(x, y) - \aleph$$

$$\begin{aligned}(\partial_3 \cdot D_{z_3})^2 \langle O_1 O_2 X_{(2,0)} \rangle \propto & [(\Delta_1 - \Delta_2)^2 - 1][(\Delta_1 + \Delta_2 - d)^2 - 1] \\ & \times \int \mathrm{d}^{d+1} y \sqrt{-g} K_{\Delta_1}(x_1; y) K_{\Delta_2}(x_2; y) K_{d+1}(x_3, y)\end{aligned}$$

$$\int \mathrm{d}^{d+1} y \sqrt{-g} K_{\Delta_1}(x_1; y) K_{\Delta_2}(x_2; y) K_{d+1}(x_3, y) \propto \Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_2 - d - 1)\right)$$

$$(\partial_3 \cdot D_{z_3})^2 \langle O_1 O_2 X_{(2,0)} \rangle \propto \langle O_1(x_1) O_1(x_3) \rangle \langle O_2(x_2) O_2(x_3) \rangle = \frac{1}{P_{23}^{\Delta_2} P_{31}^{\Delta_1}}$$

$$(\partial \cdot D_z)^2 X_{(2,0)}(z, x) \propto O_1(x) O_2(x)$$

$$\langle O_1(x_1) O_2(x_2) O_1(x) O_2(y) \rangle = \langle O_1(x_1) O_1(x) \rangle \langle O_2(x_2) O_2(y) \rangle + \mathfrak{H}^{\dagger}$$

$$\partial_\mu \partial_\nu \langle X^{\mu\nu} X X \rangle = \frac{9}{64\pi^4 M_{\mathrm{Pl}}} \frac{(H_{23} + 2V_2 V_3)^2}{P_{12}^2 P_{23}^2 P_{31}^2}$$

$$\partial_\mu \partial_\nu X^{\mu\nu} = \frac{1}{2M_{\mathrm{Pl}}} : X^{\mu\nu} X_{\mu\nu} :$$

$$\langle XX \rangle = -\frac{3}{\pi^2} \frac{H_{12}^2}{(-2P_{12})^{4+\gamma}}$$

$$\gamma = \frac{3}{16\pi^2 M_{\mathrm{Pl}}^2}$$

$$\sum_{i,j} a_i c_j F_{ij}(x_1, x_2, x_3) = 0$$

$$X_{(0,-3)} = \sum_{a=1}^{N} \phi_a^\dagger \phi_a,$$

$$X_{(0,-1)} = \sum_{a=1}^{N} \left[\phi_a^\dagger \Box \phi_a + \frac{2}{d-4}\partial_z\phi_a^\dagger \partial_z \phi_a\right] + \text{ h.c.},$$

$$X_{(1,-2)} = i\sum_{a=1}^{N} \phi_a^\dagger \partial_z \phi_a + \text{ h.c.},$$

$$X_{(1,0)} = i\sum_{a=1}^{N} \left[\phi_a^\dagger \Box \partial_z \phi_a + \frac{d}{d-4}\partial_z\phi_a^\dagger \Box \partial_z \phi_a + \frac{4}{d-4}\partial_\mu \phi_a^\dagger \partial^\mu \partial_z \phi_a\right] + \text{ h.c.}$$

$$X_{(2,-1)} = \sum_{a=1}^{N} \left[\phi_a^\dagger \partial_z^2 \phi_a - \frac{d-2}{d-4}\partial_z\phi_a^\dagger \partial_z \phi_a\right] + \text{ h.c.}$$

$$\begin{aligned} T_{(2,1)} = \sum_{a=1}^{N} \Big[&\phi_a^\dagger \Box \partial_z^2 \phi_a - \frac{2(d+2)}{d-4}\partial_z\phi_a^\dagger \Box \partial_z \phi_a + \frac{d(d+2)}{(d-2)(d-4)}\partial_z^2\phi_a^\dagger \Box \phi_a \\ &+ \frac{4}{d-4}\partial_z^2\partial_\mu \phi_a^\dagger \partial^\mu \phi_a - \frac{4d}{(d-2)(d-4)}\partial_\mu\partial_z\phi_a^\dagger \partial^\mu \partial_z \phi_a\Big] + \text{ h.c.} \end{aligned}$$

$$X_{(3,0)} = i\sum_{a=1}^{N} \left[\phi_a^\dagger \partial_z^3 \phi_a - \frac{3d}{d-4}\partial_z\phi_a^\dagger \partial_z^2 \phi_a\right] + \text{ h.c.}$$

$$\begin{aligned} X_{(3,2)} = i\sum_{a=1}^{N} \Big[&\phi_a^\dagger \Box \partial_z^3 \phi_a - \frac{3(d+4)}{d-4}\partial_z^2\phi_a^\dagger \Box \partial_z \phi_a + \frac{3(d+2)(d+4)}{(d-4)(d-2)}\Box \partial_z^2\phi_a^\dagger \partial_z \phi_a \\ &- \frac{(d+2)(d+4)}{(d-4)(d-2)}\partial_z^3\phi_a^\dagger \Box \phi_a + \frac{4}{d-4}\partial_\mu \phi_a^\dagger \partial^\mu \partial_z^3 \phi_a \end{aligned}$$

$$X_{(4,1)} = \sum_{a=1}^{N} \left[\phi_a^\dagger \partial_z^4 \phi_a - \frac{4(d+2)}{d-4}\partial_z\phi_a^\dagger \partial_z^3 \phi_a + \frac{3d(d+2)}{(d-2)(d-4)}\partial_z^2\phi_a^\dagger \partial_z^2 \phi_a\right] + \text{ h.c.}$$

$$\begin{aligned} X_{(4,3)} = \sum_{a=1}^{N} \Big[&\phi_a^\dagger \Box \partial_z^4 \phi_a - \frac{4(d+6)}{d-4}\partial_z\phi_a^\dagger \Box \partial_z \phi_a - \frac{4(d+2)(d+4)(d+6)}{(d-4)(d-2)d}\Box \partial_z^2\phi_a^\dagger \partial_z^2 \phi_a \\ &+ \frac{(d+4)(d+6)}{(d-4)(d-2)}\partial_z^4\phi_a^\dagger \Box \phi_a + \frac{6(d+4)(d+6)}{(d-4)(d-2)}\partial_z\phi_a^\dagger \Box \partial_z \phi_a + \frac{4}{d-4}\partial_\mu \phi_a^\dagger \partial^\mu \partial_z^4 \phi_a \\ &- \frac{16(d+4)}{(d-4)(d-2)}\partial_\mu \partial_z\phi_a^\dagger \partial^\mu \partial_z^3 \phi_a + \frac{12(d+2)(d+4)}{(d-4)(d-2)d}\partial_\mu \partial_z^2\phi_a^\dagger \partial^\mu \partial_z^2 \phi_a\Big] + \text{ h.c.} \end{aligned}$$

$$\begin{aligned} [Q_{(3,0)}, \phi_a] &= i\delta\phi_a = i \Box \phi_a, \\ [Q_{(3,0)}, \phi_a^\dagger] &= -i\delta\phi_a^\dagger = -i \Box \phi_a^\dagger. \end{aligned}$$

$$[Q_{(3,0)},X_{(0,-3)}]=\frac{2}{d-2}(\partial\cdot D_z)X_{(1,-2)}$$
$$[Q_{(3,0)},X_{(0,-1)}]=\frac{4}{(d-2)(d-4)}\square(\partial\cdot D_z)X_{(1,-2)}$$
$$[Q_{(3,0)},X_{(1,-2)}]=-\frac{2}{(d-3)(d-6)}\square\,\partial_z X_{(0,-3)}+\frac{d-4}{d(d-3)}(\partial\cdot D_z)X_{(2,-1)}+\frac{2(d-4)}{d(d-6)}\partial_z X_{(0,-1)}$$
$$[Q_{(3,0)},X_{(1,0)}]=\frac{2}{d(d-2)(d-3)}\big((d-2)\square-2\partial_z(\partial\cdot D_z)\big)(\partial\cdot D_z)X_{(2,-1)}$$
$$[Q_{(3,0)},X_{(2,-1)}]=-\frac{4}{(d-1)(d-4)^2}\partial_z\big((d-3)\square-\partial_z(\partial\cdot D_z)\big)X_{(1,-2)}+\frac{2(d-3)}{3(d-1)(d+2)}(\partial\cdot D_z)X_{(3,0)}+\frac{4(d-3)}{(d-4)(d+2)}\partial_z X_{(1,0)}$$
$$[Q_{(3,0)},T_{(2,1)}]=\frac{4}{3(d-2)d(d+2)}\big(d\square-2\partial_z(\partial\cdot D_z)\big)(\partial\cdot D_z)X_{(3,0)}$$
$$[Q_{(3,0)},X_{(3,0)}]=\frac{6}{d+4}\partial_z T_{(2,1)}+\frac{d-2}{2(d+1)(d+4)}(\partial\cdot D_z)X_{(4,1)}-\frac{6}{(d-3)(d-2)(d+1)}\partial_z\big((d-1)\square-\partial_z(\partial\cdot D_z)\big)X_{(2,-1)}$$
$$[Q_{(3,0)},X_{(3,2)}]=\frac{1}{(d-1)(d+2)(d+4)}\big((d+2)\square-2\partial_z(\partial\cdot D_z)\big)(\partial\cdot D_z)X_{(4,1)}$$

$$X_{(2,-1)}\overset{d\to 3}{\propto}\partial_z^2 X_{(0,-3)}$$

$$\langle\phi_a^\dagger(x)\phi_b(0)\rangle\overset{d\to 4}{=}1.$$

$$X_{(0,-1)}\overset{d\to 6}{\propto}\square\,X_{(0,-3)}.$$

$$h_{MN}=\frac{2L}{|\lambda|}\hat{h}_{MN},f_{MN}=\frac{1}{|\lambda|L}\hat{f}_{MN}$$

$$\mathcal{L}=\mathcal{L}_{\mathrm{FP},2L^{-2}}(\hat{h})-\mathcal{L}_{\mathrm{FP},0}(\hat{f})+\mathcal{L}_{\hat{h}\hat{h}\hat{h}}+\mathcal{L}_{\hat{h}\hat{f}\hat{f}}+\mathcal{L}_{\hat{f}\hat{f}\hat{f}}$$

$$\frac{\mathcal{L}_{hhh}}{\sqrt{-g}}=-\frac{1}{M_{\mathrm{Pl}}}(2h^{MN}\nabla_N h_{KL}\nabla^L h_M^K+h_{MN}h^{KL}\nabla_K\nabla_L h^{MN}+2L^{-2}h_N^M h_K^N h_M^K)$$
$$\frac{\mathcal{L}_{hff}}{\sqrt{-g}}=\frac{1}{M_{\mathrm{Pl}}}(2h_{MN}f^{KL}\nabla_K\nabla_L f^{MN}+f_{MN}h^{KL}\nabla_K\nabla_L f^{MN}+2f^{MK}\nabla_M f^{NL}\nabla_L h_{KN}+4h^{MK}\nabla_K f_{NL}\nabla^L f_M^N+6L^{-2}h_N^M f_K^N f_M^K)$$
$$\frac{\mathcal{L}_{fff}}{\sqrt{-g}}=-\frac{2}{M_{\mathrm{Pl}}}(2f^{MN}\nabla_N f_{KL}\nabla^L f_M^K+f_{MN}f^{KL}\nabla_K\nabla_L f^{MN}+2L^{-2}f_N^M f_K^N f_M^K)$$

$$\Pi_\Delta=\frac{\mathcal{C}_\Delta}{(-2P\cdot Y)^\Delta},\mathcal{C}_\Delta\equiv\frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}$$

$$Y^A=(Y^+,Y^-,Y^\mu)=\frac{1}{z}(1+z^2+|x|^2,x^\mu)$$

$$\begin{aligned}\mathcal{E}^A &= N_{\mathcal{E}}\big((\Delta+\ell)\delta_B^A + P^A\partial_{P^B}\big)Z^B \\ \mathcal{P}^A &= N_{\mathcal{P}}\big(c_1\delta_B^A + P^A\partial_{P^B}\big)\big(c_2\delta_C^B + Z^B\partial_{Z^C}\big)\big(c_3\delta_D^C - \partial_Z^C Z_D\big)\partial_{P_D}\end{aligned}$$

$$N_{\mathcal{E}} \equiv \frac{\Delta-\ell}{\Delta(\Delta-1)}, N_{\mathcal{P}} \equiv \frac{2i}{(\ell+1)(1-\Delta)(d-\Delta-2)(d-2\Delta-2)}$$

$$c_1 \equiv 2-d+2\Delta, c_2 \equiv 2-d+\Delta-\ell, c_3 \equiv \Delta+\ell$$

$$\begin{aligned}(Y\cdot\mathcal{E})\Pi_\Delta &= 0 \\ (Y\cdot\mathcal{P})\Pi_{\Delta-1} &= i(d-\Delta)\Pi_\Delta \\ (Y\cdot\mathcal{E})\mathcal{P}^A\Pi_{\Delta-1} &= i\mathcal{E}^A\Pi_\Delta\end{aligned}$$

$$\begin{aligned}\Pi_\Delta^{AB} &= \mathcal{E}^A\mathcal{E}^B\Pi_\Delta \\ &= \mathcal{C}_{\Delta,2}\frac{4(P^AY\cdot Z - Z^AY\cdot P)(P^BY\cdot Z - Z^BY\cdot P)}{(-2Y\cdot P)^{\Delta+2}}\end{aligned}$$

$$\mathcal{C}_{\Delta,J} = \frac{(J+\Delta-1)\Gamma(\Delta)}{2\pi^{\frac{d}{2}}(\Delta-1)\Gamma\left(\Delta+1-\frac{d}{2}\right)}$$

$$\nabla_C\Pi_\Delta^{AB} = i\mathcal{E}^A\mathcal{E}^B\mathcal{P}_C\Pi_{\Delta-1} - \big[(d-\Delta)Y_C\mathcal{E}^A\mathcal{E}^B + Y^{(A}\mathcal{E}^{B)}\mathcal{E}_C\big]\Pi_\Delta$$

$$\begin{aligned}\langle XXX\rangle = -\frac{2}{M_{\mathrm{Pl}}}\int_{\mathrm{AdS}_{d+1}} &\big(2\Pi^{CD}\nabla_D\Pi_{AB}\nabla^B\Pi_C^A - \Pi^{AB}\nabla_A\Pi_{CD}\nabla_B\Pi^{CD} \\ &-2\Pi_B^A\Pi_C^B\Pi^C{}_A\big)\end{aligned}$$

$$\langle XXX\rangle = -\frac{4}{M_{\mathrm{Pl}}}\mathcal{D}_{XXX}\langle\phi\phi\phi\rangle,$$

$$\begin{aligned}\mathcal{D}_{XXX} &= (\mathcal{E}_1\cdot\mathcal{E}_2)^2(\mathcal{E}_3\cdot\mathcal{P}_1)^2 + 2\mathcal{E}_1\cdot\mathcal{E}_2\mathcal{E}_1\cdot\mathcal{E}_3\mathcal{E}_3\cdot\mathcal{P}_1\mathcal{E}_2\cdot\mathcal{P}_3 + \text{ cyc.} \\ &=: (\mathcal{E}_1\cdot\mathcal{E}_2\mathcal{E}_3\cdot\mathcal{P}_1 + \text{ cyc.})^2:\end{aligned}$$

$$\langle\phi_{\Delta_1}\phi_{\Delta_2}\phi_{\Delta_3}\rangle = \frac{c_{\Delta_1\Delta_2\Delta_3}}{(-2P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(-2P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(-2P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}},$$

$$c_{\Delta_1\Delta_2\Delta_3} \equiv \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}.$$

$$\langle XXX\rangle = \frac{3}{256\pi^4 M_{\mathrm{Pl}}}\frac{12Q_1 - 2Q_2 + Q_3 + 8Q_4}{P_{12}^2P_{13}^2P_{23}^2}, Q_n = \begin{pmatrix} V_1^2V_2^2V_3^2 \\ H_{13}H_{23}V_1V_2 + \text{ cyc.} \\ H_{12}^2V_3^2 + \text{ cyc.} \\ H_{12}V_1V_2V_3^2 + \text{ cyc.}\end{pmatrix}$$

$$\partial_\mu\partial_\nu X^{\mu\nu} = \frac{g}{\sqrt{N}}:X_{\mu\nu}X^{\mu\nu}:$$

$$\langle XX\rangle = -c_X \frac{H_{12}^2}{(-2P_{12})^{4+\gamma}}, \text{ with } c_X = \mathcal{C}_{2,2} = \frac{3}{\pi^2}$$

$$\partial_1^\mu \partial_1^\nu \langle X_{\mu\nu}(x_1) X(x_2,z_2) X(x_3,z_3)\rangle = \frac{9}{64\pi^4 M_{\mathrm{Pl}}} \frac{(H_{12}+2V_1V_2)^2}{P_{12}^2 P_{13}^2 P_{23}^2},$$

$$X^{\mu\nu}(x) = \frac{2}{3} D_z^\mu D_z^\nu X(x,z)$$

$$\begin{aligned}
\partial_1^\mu \partial_1^\nu \langle X_{\mu\nu}(x_1) X(x_2,z_2) X(x_3,z_3)\rangle &= \frac{g}{\sqrt{N}} \langle : X^{\mu\nu}(x_1) X_{\mu\nu}(x_1) : X(x_2) X(x_3)\rangle \\
&= \frac{2g}{\sqrt{N}} \langle X^{\mu\nu}(x_1) X(x_2)\rangle \langle X_{\mu\nu}(x_1) X(x_3)\rangle \\
&= \frac{c_X^2 g}{32\sqrt{N}} \frac{(H_{12}+2V_1V_2)^2}{P_{12}^2 P_{13}^2 P_{23}^2}
\end{aligned}$$

$$\frac{g}{\sqrt{N}} = \frac{1}{2M_{\mathrm{Pl}}}$$

$$\partial_1^\mu \partial_1^\nu \partial_2^\rho \partial_2^\sigma \langle X_{\mu\nu}(x_1) X_{\rho\sigma}(x_2)\rangle = \frac{5c_X}{2} \frac{\gamma}{P_{12}^4} + \mathcal{O}(\gamma^2)$$

$$\begin{aligned}
\partial_1^\mu \partial_1^\nu \partial_2^\rho \partial_2^\sigma \langle X_{\mu\nu}(x_1) X_{\rho\sigma}(x_2)\rangle &= \frac{g^2}{N} \langle : X^{\mu\nu}(x_1) X_{\mu\nu}(x_1) :: X^{\rho\sigma}(x_2) X_{\rho\sigma}(x_2) :\rangle \\
&= \frac{2g^2}{N} \langle X^{\mu\nu}(x_1) X^{\rho\sigma}(x_2)\rangle \langle X_{\mu\nu}(x_1) X_{\rho\sigma}(x_2)\rangle \\
&= \frac{5c_X^2}{8} \frac{g^2}{N} \frac{1}{P_{12}^4} + \mathcal{O}(\gamma^2)
\end{aligned}$$

$$\gamma = \frac{c_X}{4} \frac{g^2}{N} = \frac{3}{16\pi^2 M_{\mathrm{Pl}}^2}$$

$$Q_{(s,t)}^\zeta[\Sigma] = \oiiint_\Sigma \mathrm{d}\Sigma_\mu J_\zeta^\mu$$

$$\begin{aligned}
J_\zeta^\mu =& \zeta_{\mu_1\cdots\mu_t} \partial_{\nu_2} \cdots \partial_{\nu_{s-t}} X_{(s,t)}^{\mu_1\cdots\mu_t\mu\nu_2\cdots\nu_{s-t}} - \partial_{\nu_2} \zeta_{\mu_1\cdots\mu_t} \partial_{\nu_3} \cdots \partial_{\nu_{s-t}} X_{(s,t)}^{\mu_1\cdots\mu_t\mu\nu_2\cdots\nu_{s-t}} \\
&+ \cdots + (-1)^{s-t-1} \partial_{\nu_2} \cdots \partial_{\nu_{s-t}} \zeta_{\mu_1\cdots\mu_t} X_{(s,t)}^{\mu_1\cdots\mu_t\mu\nu_2\cdots\nu_{s-t}}
\end{aligned}$$

$$(z\cdot\partial)^{s-t}\zeta = 0, \zeta(x,z) \equiv z_{\nu_1} \cdots z_{\nu_t} \zeta^{\nu_1\cdots\nu_t}(x)$$

$$\mathcal{D}_{XXX} - \mathcal{D}_{TTT} \propto \mathcal{E}_1 \cdot \mathcal{E}_2 \mathcal{E}_1 \cdot \mathcal{E}_3 \mathcal{E}_2 \cdot \mathcal{E}_3$$

$$\langle XX\rangle = -i^{d+1} \tilde{c}_X \frac{H_{12}^2}{(-2P_{12})^{d+1}}, \tilde{c}_X = \left(\frac{M_{\mathrm{Pl}}}{H}\right)^{d-1} \frac{\Gamma(d+1)}{4\pi^{\frac{d}{2}}(d-1)\Gamma\left(\frac{d}{2}-1\right)}$$

$$\tilde{c}_X = \frac{3}{4\pi^2}\left(\frac{M_{\mathrm{Pl}}}{H}\right)^{d-1}$$

Modelo de Gravedad Cuántica Relativista sin intervención gravitónica (gravedad cuántica endógena).

$$R_{MN} - \frac{1}{12}\left[F_{MPQR}F_N^{PQR} - \frac{1}{12}g_{MN}F^2\right] = 0$$
$$\mathrm{d}\star_{11} F + \frac{1}{2}F \wedge F = 0, F^2 \equiv F_{MNPQ}F^{MNPQ}$$

$$\nabla_M F^{MNPQ} + \frac{1}{2^7 3^2}\epsilon^{NPQM_1M_2M_3M_4M_5M_6M_7M_8}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8} = 0$$

$$ds^2 = e^{2f_0(\theta)}ds^2_{\mathrm{AdS}_4} + e^{2f_1(\theta)}d\theta^2 + e^{2f_2(\theta)}d\Omega_6^2$$

$$A_{\theta mn} = g_1(\theta)J_{mn}, A_{mnp} = g_2(\theta)T_{mnp} + g_3(\theta)S_{mnp},$$

$$F^{(4)} = g_0 \mathrm{vol}_{\mathrm{AdS}_4} + \mathrm{d}A^{(3)}$$

$$F_{\mu\nu\rho\sigma} = g_0\epsilon_{\mu\nu\rho\sigma}, F_{\theta mnp} = (g_2' - 3g_1)T_{mnp} + g_3' S_{mnp}, F_{mnpq} = 2g_3\epsilon_{mnpqrs}J^{rs},$$

$$\begin{aligned}
&-f_0'' + f_0'(-4f_0' + f_1' - 6f_2') + \frac{1}{3}g_0^2\left(e^{6f_2} + 2g_3^2\right)e^{-8f_0+2f_1-6f_2} \\
&-3e^{2f_1-2f_0} + 8g_3^2e^{2f_1-8f_2} + \frac{2}{3}e^{-6f_2}g_3'^2 = 0 \\
&-f_2'' + f_2'(-4f_0' + f_1' - 6f_2') - \frac{1}{6}g_0^2\left(e^{6f_2} + 2g_3^2\right)e^{-8f_0+2f_1-6f_2} \\
&-8g_3^2e^{2f_1-8f_2} + 5e^{2f_1-2f_2} - \frac{1}{3}e^{-6f_2}g_3'^2 = 0 \\
&8f_0'f_2' + 2f_0'^2 - \frac{1}{12}g_0^2\left(e^{6f_2} + 4g_3^2\right)e^{-8f_0+2f_1-6f_2} \\
&+e^{2f_1}\left(2e^{-2f_0} - 5e^{-2f_2}\right) + 4g_3^2e^{2f_1-8f_2} + 5f_2'^2 - \frac{1}{3}e^{-6f_2}g_3'^2 = 0
\end{aligned}$$

$$g_3'' + g_3'(4f_0' - f_1') + e^{2f_1}\left(-12e^{-2f_2} + e^{-8f_0}g_0^2\right)g_3 = 0$$
$$(g_2' - 3g_1) = e^{f_1-4f_0}g_0g_3$$

$$e^{2f_0} = \sigma^2(\theta), e^{2f_1} = \rho_1^2(\theta), e^{2f_2} = \rho_2^2\sin^2\theta$$
$$g_3 = \sqrt{2}f_1, g_2' - 3g_1 = -\sqrt{2}f_2, g_3' = -\sqrt{2}f_3, g_0 = \sqrt{2}f$$

$$\mathrm{SO}(8): e^{f_0} = \frac{g_0^{1/3}}{3^{1/3}}, e^{f_1} = \frac{2g_0^{1/3}}{3^{1/3}}, e^{f_2} = \frac{2g_0^{1/3}}{3^{1/3}}\sin\theta, (g_2' - 3g_1) = g_3 = 0$$

$$G_2 : e^{f_0} = \frac{2^{\frac{1}{6}}}{3^{\frac{1}{6}}5^{\frac{1}{6}}} g_0^{\frac{1}{3}} (2 + \cos\ 2\theta)^{\frac{1}{3}}, e^{f_1} = \frac{2^{\frac{5}{3}}}{3^{\frac{1}{6}}5^{\frac{2}{3}}} g_0^{\frac{1}{3}} (2 + \cos\ 2\theta)^{\frac{1}{3}}$$

$$e^{f_2} = \frac{2^{\frac{5}{3}}3^{\frac{1}{3}}}{5^{\frac{2}{3}}} g_0^{\frac{1}{3}} \sin\ \theta (2 + \cos\ 2\theta)^{-\frac{1}{6}}$$

$$g_2' - 3g_1 = \frac{2^6 3^{\frac{3}{2}}}{5^{\frac{5}{2}}} g_0 \sin^4\ \theta (2 + \cos\ 2\theta)^{-2}, g_3 = \frac{2^5 3^1}{5^{\frac{5}{2}}} g_0 \sin^4\ \theta (2 + \cos\ 2\theta)^{-1}$$

$$e^{f_i} \to \lambda e^{f_i};\ g_j \to \lambda^3 g_j . i = 1,2,3;\ j = 0,1,2,3.$$

$$\delta\Psi_M \ = \nabla_M \epsilon + \frac{1}{288} \left(\Gamma_M^{N_1 N_2 N_3 N_4} - 8\delta_M^{N_1} \Gamma^{N_2 N_3 N_4}\right) F_{N_1 N_2 N_3 N_4} \epsilon = 0$$

$$\nabla_M \epsilon \ \equiv \partial_M \epsilon + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} \epsilon, \omega_M^{AB} \ \equiv E_N^A \bar{E}^{BL} \Gamma_{ML}^N - \bar{E}^{LB} \partial_M E_L^A$$

$$E_1^1 \delta\Psi_1 = \left(\partial_1 + \frac{1}{2} e^{-f_1} f_0' \Gamma^{15} - \frac{1}{6} e^{-4f_0} g_0 \Gamma^{234} + \frac{1}{72} e^{-f_1} \Gamma^{15mnp} F_{\theta mnp} + \frac{1}{288} \Gamma^{1mnpq} F_{mnpq}\right) \epsilon,$$
$$E_2^2 \delta\Psi_2 = \left(\frac{1}{2} e^{-f_0} \Gamma^{12} - \frac{1}{2} f_0' e^{-f_1} \Gamma^{25} + \frac{1}{6} g_0 e^{-4f_0} \Gamma^{134} + \frac{1}{72} e^{-f_1} \Gamma^{25mnp} F_{\theta mnp} + \frac{1}{288} \Gamma^{2mnpq} F_{mnpq}\right) \epsilon,$$
$$E_3^3 \delta\Psi_3 = \left(-\frac{1}{2} e^{-f_0} \Gamma^{13} + \frac{1}{2} f_0' e^{-f_1} \Gamma^{35} - \frac{1}{6} e^{-4f_0} g_0 \Gamma^{124} + \frac{1}{72} e^{-f_1} \Gamma^{35mnp} F_{\theta mnp} + \frac{1}{288} \Gamma^{3mnpq} F_{mnpq}\right) \epsilon,$$
$$E_4^4 \delta\Psi_4 = \left(-\frac{1}{2} e^{-f_0} \Gamma^{14} + \frac{1}{2} e^{-f_1} f_0' \Gamma^{45} + \frac{1}{6} e^{-4f_0} g_0 \Gamma^{123} + \frac{1}{72} e^{-f_1} \Gamma^{45mnp} F_{\theta mnp} + \frac{1}{288} \Gamma^{4mnpq} F_{mnpq}\right) \epsilon,$$
$$E_5^5 \delta\Psi_5 = \left(\partial_5 - \frac{1}{36} e^{-f_1} \Gamma^{mnp} F_{\theta mnp} + \frac{1}{12} g_0 \Gamma^{12345} e^{-4f_0} + \frac{1}{288} \Gamma_5^{mnpq} F_{mnpq}\right) \epsilon,$$
$$E_6^6 \delta\Psi_6 = \left(\partial_6 - \frac{1}{2} f_2' e^{-f_1} \Gamma^{56} + \frac{1}{12} e^{-4f_0} g_0 \Gamma^{12346} + \frac{1}{12} e^{-f_1} \Gamma^{5mn} F_{\theta 6mn} - \frac{1}{72} e^{-f_1} \Gamma^{56mnp} F_{\theta mnp}\right.$$
$$\left. + \frac{1}{288} \Gamma^{6mnpq} F_{mnpq} - \frac{1}{36} \Gamma^{mnp} F_{6mnp}\right) \epsilon,$$
….

$$E_M^M \delta\Psi_M = (\partial_M + P_M)\epsilon$$

$$\Gamma^{12} P_2 \epsilon = \Gamma^{13} P_3 \epsilon = \Gamma^{14} P_4 \epsilon = \left(\frac{1}{2} e^{-f_0} + P_1\right) \epsilon = 0$$

$$P_1 \epsilon = -\frac{1}{2} e^{-f_0} \epsilon$$

$$\delta\Psi_1 = (E_1^1 \partial_r + P_1)\epsilon = \left(e^{-f_0} \partial_r - \frac{1}{2} e^{-f_0}\right) \epsilon = 0 \ \Rightarrow \ \epsilon = e^{r/2} \epsilon_{\hat{r}}$$

$$\left(\frac{1}{3} g_0 e^{-3f_0} \Gamma^{234} - f_0' e^{f_0 - f_1} \Gamma^{15} - \frac{1}{36} e^{f_0 - f_1} \Gamma^{15mnp} F_{\theta mnp} - \frac{1}{144} e^{f_0} \Gamma^{1mnpq} F_{mnpq}\right) \epsilon_{\hat{r}} = \epsilon_{\hat{r}}$$

$$\epsilon = \epsilon_{(4)} \otimes \theta \otimes \epsilon_{(6)} + \epsilon_{(4)}^c \otimes \theta \otimes \epsilon_{(6)}^c$$

$$\gamma^1 = \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1}, \gamma^2 = \sigma_2 \otimes \mathbf{1} \otimes \mathbf{1}, \gamma^3 = \sigma_3 \otimes \sigma_1 \otimes \mathbf{1}, \gamma^4 = \sigma_3 \otimes \sigma_2 \otimes \mathbf{1}$$
$$\gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \mathbf{1}, \gamma^6 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2, \gamma^7 = -i\sigma_3 \otimes \sigma_3 \otimes \sigma_3$$

$$(\gamma^m)^\dagger = \gamma^m;\ \{\gamma^m, \gamma^n\} = 2\delta^{mn}, m, n = 1,2,\cdots,6;$$
$$(\gamma^7)^2 = -\mathbf{1}, \gamma^7 = \gamma^1 \cdots \gamma^6.$$

$$\rho^1 = i\sigma_1 \otimes 1, \rho^2 = -\sigma_2 \otimes 1, \rho^3 = i\sigma_3 \otimes \sigma_1, \rho^4 = i\sigma_3 \otimes \sigma_2, \rho^5 = i\sigma_3 \otimes \sigma_3$$

$$\{\rho^a, \rho^b\} = -2\eta^{ab}, a, b = 1,2,\cdots,5, \eta^{ab} = \mathrm{diag}(1,-1,1,1,1), \rho^1 \cdots \rho^5 = +\mathbf{1}$$

$$\Gamma^a = \rho^a \otimes \gamma^7, a = 1,2,\cdots,5;$$
$$\Gamma^{5+m} = \mathbf{1}_{4\times4} \otimes \gamma^m, m = 1,2,\cdots,6,$$

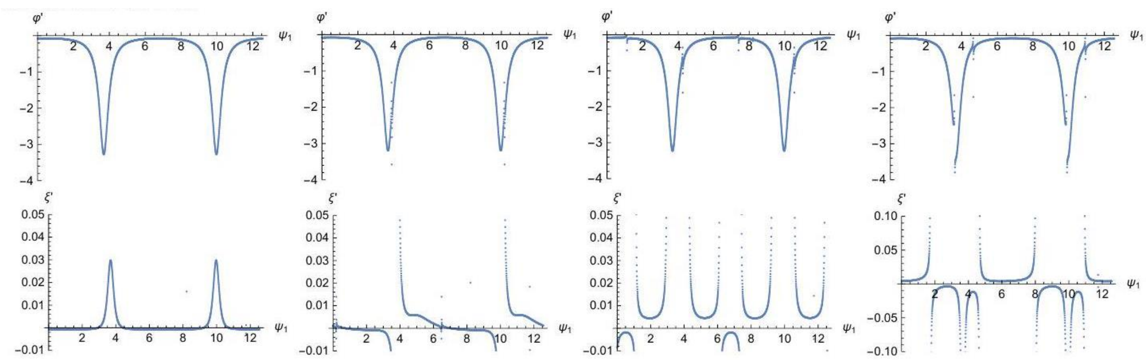
$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \Gamma_* = \Gamma^1 \cdots \Gamma^{11} = -\mathbf{1}$$

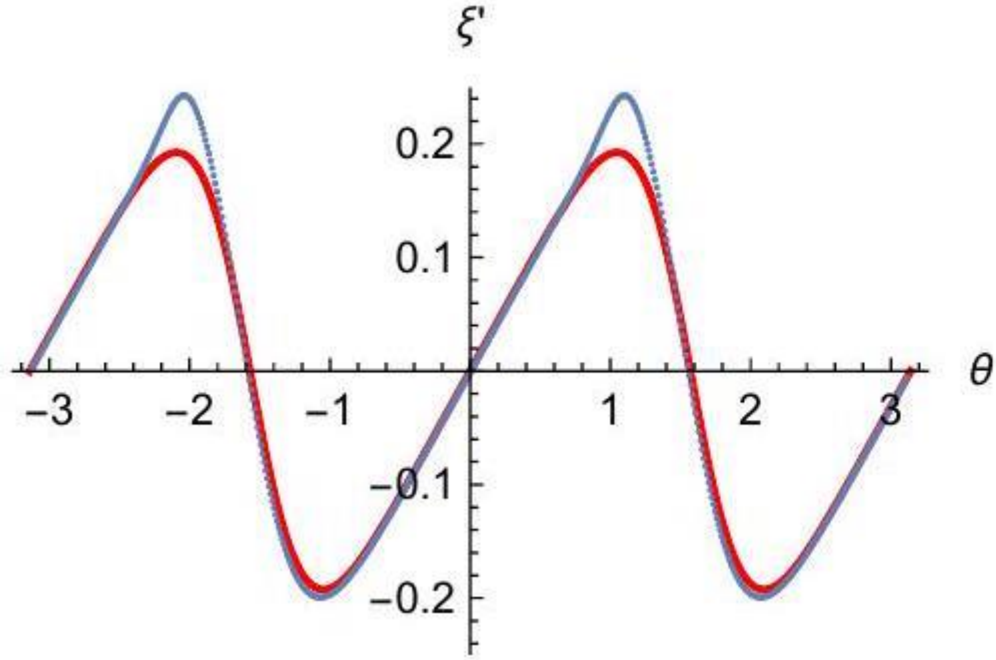
$$\epsilon_{\mathrm{num}} = e^{\xi} e^{i\varphi} \epsilon_{\hat{r}}$$

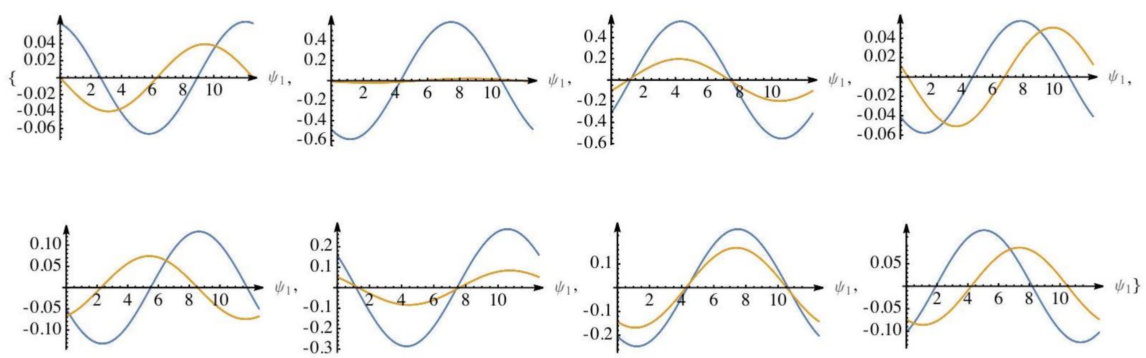
$$(D_6 + P_6)\epsilon_{\mathrm{num}} = 0, D_6 \equiv \partial_6 - iE_6^6 \varphi'(\psi_1) - E_6^6 \xi'(\psi_1)$$

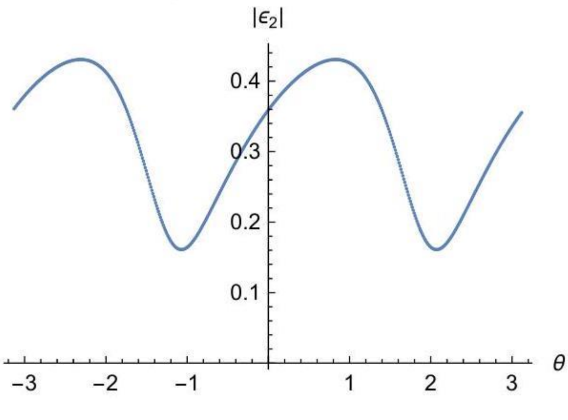
$$\partial_6 \epsilon^R + E_6^6 \varphi' \epsilon^I - E_6^6 \xi' \epsilon^R = -(P_6\epsilon)^R, \partial_6 \epsilon^I - E_6^6 \varphi' \epsilon^R - E_6^6 \xi' \epsilon^I = -(P_6\epsilon)^I$$

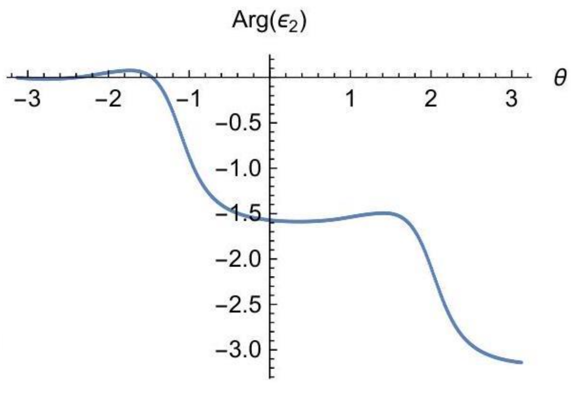
$$E_6^6 \varphi' = \frac{-(P_6\epsilon)^R + (P_6\epsilon)^I + \partial_6(\epsilon^I - \epsilon^R)}{\epsilon^R + \epsilon^I}, E_6^6 \xi' = \frac{\epsilon^I(\partial_6\epsilon^I + (P_6\epsilon)^I) + \epsilon^R(\partial_6\epsilon^R + (P_6\epsilon)^R)}{\epsilon^R(\epsilon^R + \epsilon^I)}.$$











Figuras 7, 8, 9 y 10. Fluctuaciones de masa y radiación de una partícula supermasiva.

$$\xi'(\theta) = -\frac{1}{2} f_0'(\theta) = \frac{\sin(2\theta)}{3(2+\cos(2\theta))} \Rightarrow \xi(\theta) = -\frac{1}{2} f_0(\theta)$$

$$\epsilon = e^{r/2}\epsilon_{\hat{r}} = e^{r/2} e^{f_0/2} \epsilon_{\hat{r}}^{\mathrm{nor}},$$

$$\epsilon = e^{r/2} e^{f_0/2} \mathcal{R}(\theta, \psi_i, \Gamma) \epsilon_{\mathrm{const}}^{\mathrm{nor}},$$

$$\eta_{\hat{r}} = e^{i\delta}\Gamma^{23}\epsilon_{\hat{r}}$$

$$\epsilon_{\hat{r}} = e^{i\delta'}\Gamma^{156810}\epsilon_{\hat{r}}^{*}$$

$$\epsilon_{\hat{r},\alpha} = c_{\hat{r},\alpha}\cos\frac{\psi_i}{2} + s_{\hat{r},\alpha}\sin\frac{\psi_i}{2}, \alpha = 1,2,\cdots,32.$$

$$\epsilon^{(6)} = \left(\mathbf{1}\cos\frac{\psi_1}{2} + i\gamma_1\sin\frac{\psi_1}{2}\right)\prod_{i=2}^{6}\left(\mathbf{1}\cos\frac{\psi_i}{2} + \gamma_{i-1,i}\sin\frac{\psi_i}{2}\right)\epsilon^{(6)}_{\hat{\psi}_1,\cdots,\hat{\psi}_6},$$

$$\epsilon_{\hat{r},\alpha} = \sum_{\sigma\in\mathbb{Z}_2^6} C_{\sigma,\alpha} f_{\sigma_1}\left(\frac{\psi_1}{2}\right) f_{\sigma_2}\left(\frac{\psi_2}{2}\right) f_{\sigma_3}\left(\frac{\psi_3}{2}\right) f_{\sigma_4}\left(\frac{\psi_4}{2}\right) f_{\sigma_5}\left(\frac{\psi_5}{2}\right) f_{\sigma_6}\left(\frac{\psi_6}{2}\right)$$

$$f_\sigma(x) \equiv \begin{cases}\cos x, & \sigma = 0\\ \sin x, & \sigma = 1\end{cases}$$

$$\epsilon_{\hat{r}} = \mathcal{R}_{S^6}(\psi_i,\Gamma)\epsilon_{\hat{r},\hat{\psi}_1,\cdots,\hat{\psi}_6}, \mathcal{R}_{S^6} = \left(\cos\frac{\psi_4}{2}\mathbf{1} - \Gamma^{10,11}\sin\frac{\psi_4}{2}\right)\left(\mathcal{A}\cos\frac{\psi_3}{2} + \mathcal{B}\sin\frac{\psi_3}{2}\right),$$

$$\begin{aligned}\mathcal{A} =& (c_1c_6s_2s_5 - c_2c_5s_1s_6 + c_1c_2c_5c_6 + s_1s_2s_5s_6)\mathbf{1} + (c_1c_5c_6s_2 + c_5s_1s_6s_2 - c_1c_2c_6s_5 + c_2s_1s_5s_6)\Gamma^{67}\\ &+(-c_2c_5c_6s_1 - c_6s_2s_5s_1 - c_1c_2c_5s_6 + c_1s_2s_5s_6)\Gamma^{89} + (-c_5c_6s_1s_2 + c_1c_5s_6s_2 + c_2c_6s_1s_5 + c_1c_2s_5s_6)\Gamma^{810}\\ \mathcal{B} =& (c_1c_5c_6s_2 + c_5s_1s_6s_2 + c_1c_2c_6s_5 - c_2s_1s_5s_6)\Gamma^{68} + (c_5c_6s_1s_2 - c_1c_5s_6s_2 + c_2c_6s_1s_5 + c_1c_2s_5s_6)\Gamma^{69}\\ &+(c_2c_5c_6s_1 - c_6s_2s_5s_1 + c_1c_2c_5s_6 + c_1s_2s_5s_6)\Gamma^{610} + (-c_1c_6s_2s_5 - c_2c_5s_1s_6 + c_1c_2c_5c_6 - s_1s_2s_5s_6)\Gamma^{611}\\ c_i \equiv& \cos\frac{\psi_i}{2}, s_i \equiv \sin\frac{\psi_i}{2}\end{aligned}$$

$$\epsilon = e^{r/2}e^{f_0/2}\mathcal{R}_{S^6}(\psi_i,\Gamma)\mathcal{R}(\theta,\Gamma)\epsilon_0(\theta), \eta = \Gamma^{23}\epsilon$$

$$\epsilon|_{r=\psi_i=0} = e^{r/2}e^{f_0/2}\mathcal{R}_{S^6}(\psi_i,\Gamma)\mathcal{R}(\theta,\Gamma)\epsilon_0(\theta)\big|_{r=\psi_i=0} = e^{f_0/2}\mathcal{R}(\theta,\Gamma)\epsilon_0(\theta)$$

$$(\mathbf{1} - \Gamma^{891011})\epsilon = 0, (\mathbf{1} + \Gamma^{67811})\epsilon = 0, (\mathbf{1} - \Gamma^{67910})\epsilon = 0$$

$$\epsilon = \Gamma^{156810}\epsilon^{*} \text{ or } \epsilon = \Gamma^{2347911}\epsilon^{*}$$

$$\epsilon = \Gamma^{57810}\epsilon(-\theta)$$

$$(\partial_{\psi_i} + P_{\psi_i})\epsilon = \left[\partial_{\psi_i}\mathcal{R}_{S^6}(\psi_i,\Gamma) + P_{\psi_i}\mathcal{R}_{S^6}(\psi_i,\Gamma)\right]\epsilon_{\hat{\psi}}.$$

$$e^{-2f_1}(f_0' + 2f_2')^2 - 4g_3^2e^{-8f_2} - 4e^{-2f_2} + e^{-2f_0} = 0.$$

$$4e^{-2f_1}f_0'f_2' + e^{-2f_1}f_0'^2\left(3 + g_3^{-2}e^{6f_2}\right) + \frac{1}{9}g_0^2e^{-8f_0} + 4g_3^2e^{-8f_2} - e^{-2f_0} = 0,$$

$$2g_3g_3' + 12g_3^2f_2' + \left(12g_3^2 + 3e^{6f_2}\right)f_0' = 0$$

$$
\begin{aligned}
&4e^{-2f_1}f_0'f_2' + e^{-2f_1}f_0'^2\left(3+g_3^{-2}e^{6f_2}\right) + \frac{1}{9}g_0^2e^{-8f_0} + 4g_3^2e^{-8f_2} - e^{-2f_0}\\
&e^{-2f_1}(f_0'+2f_2')^2 - 4g_3^2e^{-8f_2} - 4e^{-2f_2} + e^{-2f_0},\\
&2g_3g_3' + 12g_3^2f_2' + \left(12g_3^2+3e^{6f_2}\right)f_0',\\
&(g_2'-3g_1) = e^{f_1-4f_0}g_0g_3,
\end{aligned}
$$

$$f_0' = \mathcal{F}_0^{(1),(2)}\left(e^{f_0},e^{f_2},g_3\right), f_2' = \mathcal{F}_2^{(1),(2)}\left(e^{f_0},e^{f_2},g_3\right), g_3' = \mathcal{G}_3^{(1),(2)}\left(e^{f_0},e^{f_2},g_3\right)$$

$$4e^{2f_0}\left(e^{6f_2}+g_3^2\right) \geq e^{8f_2}, \left(9e^{6f_0}-1\right)e^{6f_2} \geq g_3^2$$

$$g_3(\theta_*) = (6\alpha^4)^3\sqrt{9\alpha^6-1}, \beta = 6\alpha^4, \beta \equiv e^{f_2}(\theta_*)$$

$$g_3(\theta_*) = 0, \beta = 2\alpha$$

$$\alpha \equiv e^{f_0}(\theta_*), \beta \equiv e^{f_2}(\theta_*),$$

$$g_3(\theta_*)^2 = \frac{\beta^6}{288\alpha^8}\left[(36\alpha^6-1)\beta^2 - 144\alpha^8 + \beta\sqrt{(36\alpha^6-1)^2\beta^2 - 288\alpha^8(18\alpha^6-1)}\right].$$

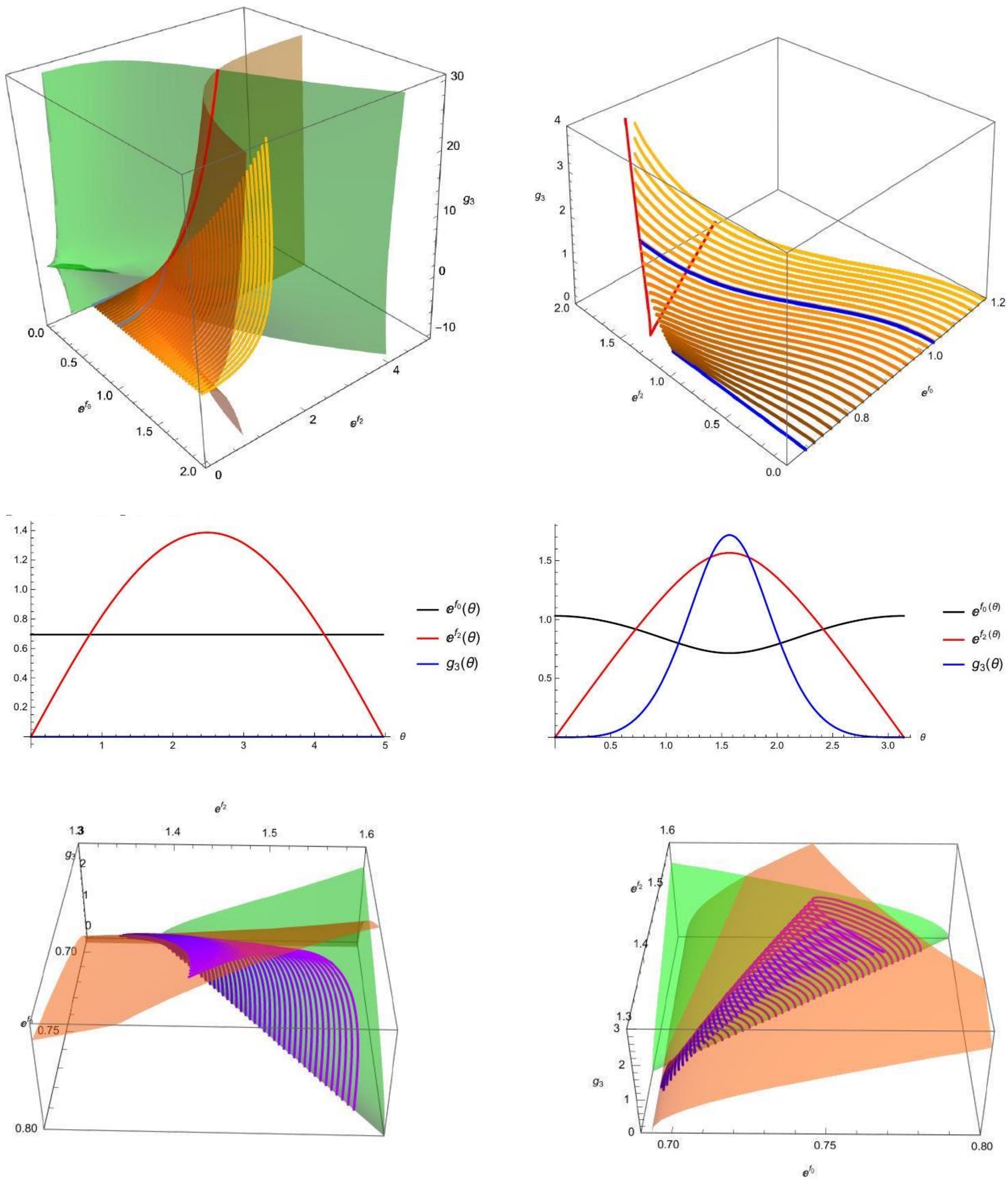
$$e^{f_0} = \sum_{i=0}^{\infty}\tilde{f}_{0,i}\theta^i, e^{f_1} = \sum_{i=0}^{\infty}\tilde{f}_{1,i}\theta^i, e^{f_2} = \sum_{i=1}^{\infty}\tilde{f}_{2,i}\theta^i, \tilde{g}_3 \equiv g_3^2 = \sum_{i=0}^{\infty}\tilde{g}_{3,i}\theta^i$$

$$\mathrm{SO}(8): e^{f_1} = 2e^{f_0}, G_2: e^{f_1} = \frac{2^{3/2}}{5^{1/2}}e^{f_0}$$

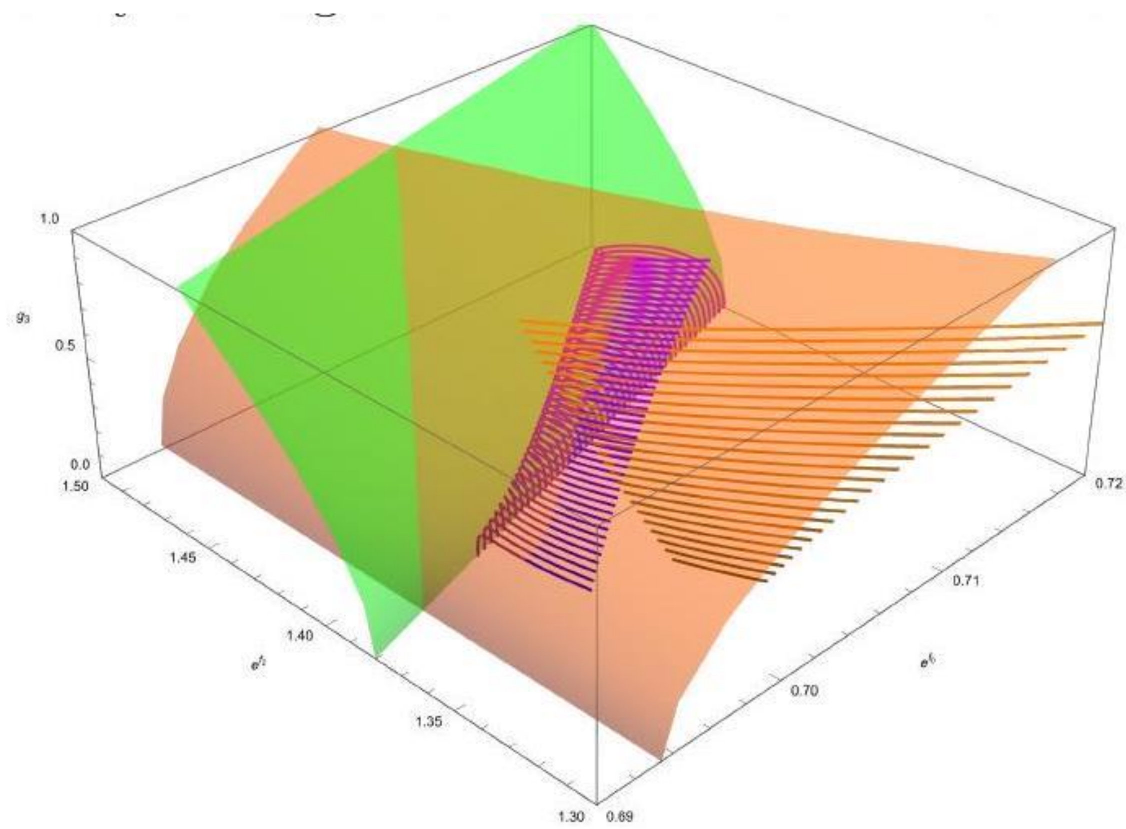
$$
\begin{aligned}
e^{f_0} &= \tilde{f}_{0,0} - \frac{4\left(9\tilde{f}_{0,0}^6-1\right)}{3\cdot 7^2\tilde{f}_{0,0}^5}\theta^2 + \frac{8\left(9\tilde{f}_{0,0}^6-1\right)\left(65\tilde{f}_{0,0}^6+69\right)\theta^4}{324135\tilde{f}_{0,0}^{11}} + O(\theta)^6\\
e^{f_2} &= \frac{2^{3/2}}{5^{1/2}}\theta\left[\tilde{f}_{0,0} - \frac{\left(40\tilde{f}_{0,0}^6+1\right)\theta^2}{735\tilde{f}_{0,0}^5} + \frac{\left(201-1760\tilde{f}_{0,0}^6\right)\theta^4}{330750\tilde{f}_{0,0}^{11}} + O(\theta)^6\right]\\
\tilde{g}_3 \equiv g_3^2 &= \theta^8\left[\frac{1024\left(9\tilde{f}_{0,0}^6-1\right)}{5^4\cdot 7^2} + \frac{65536\left(9\tilde{f}_{0,0}^6-1\right)\left(5\tilde{f}_{0,0}^6-6\right)\theta^2}{67528125\tilde{f}_{0,0}^6} + O(\theta)^4\right].
\end{aligned}
$$

$$R = -2e^{-2f_1}[4f_0'' + 6f_2'' + 10(f_0')^2 + 21(f_2')^2 - 4f_0'f_1' + 24f_0'f_2' - 6f_1'f_2'] - 12e^{-2f_0} + 30e^{-2f_2}$$

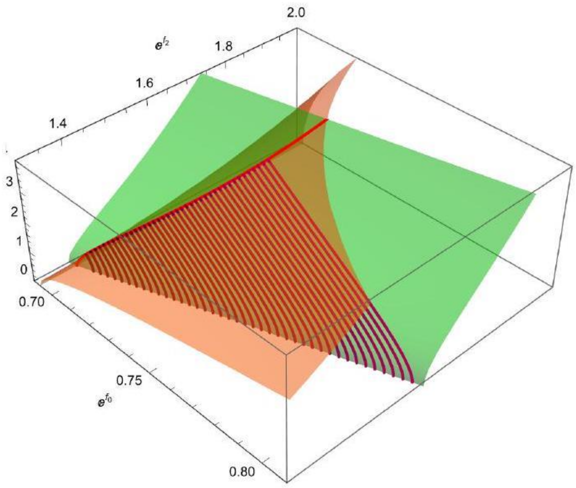
$$\tilde{f}_{2,1} = \tilde{f}_{1,0}, \tilde{f}_{2,2} = \frac{1}{2}\tilde{f}_{1,1} - \frac{\tilde{f}_{0,1}\tilde{f}_{1,0}}{3\tilde{f}_{0,0}}$$



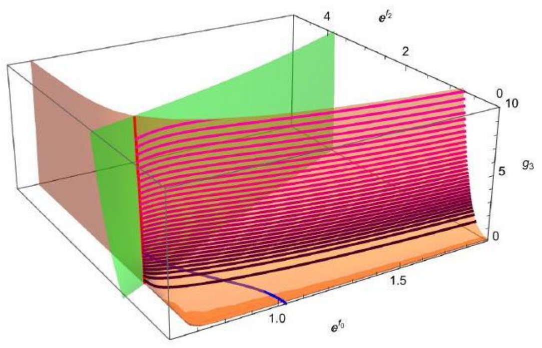
e^{f_0}
e^{f_2}
g_3
$e^{f_0}(\theta)$
$e^{f_2}(\theta)$
$g_3(\theta)$
θ
$e^{f_0}(\theta)$
$e^{f_2}(\theta)$
$g_3(\theta)$
θ



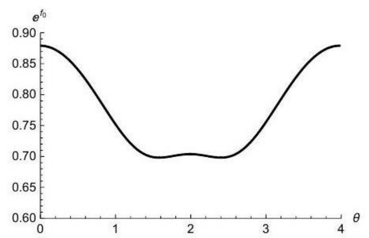
1.0
g_3
0.5
0.0
1.50
1.45
1.40
e^{f_2}
1.35
1.30
0.69
0.70
0.71
0.72
e^{f_0}



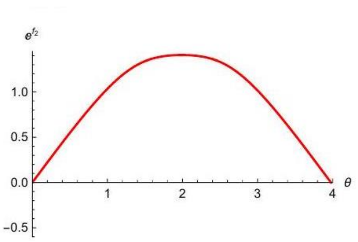
2.0
e^{f_2}
1.8
1.6
1.4
3
2
1
0
0.70
0.75
e^{f_0}
0.80



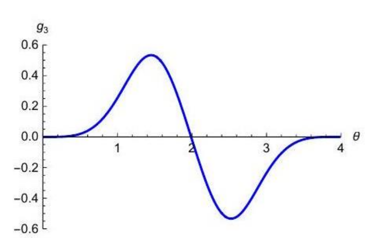
4
e^{f_2}
2
0
10
5
g_3
0
1.5
1.0
e^{f_0}



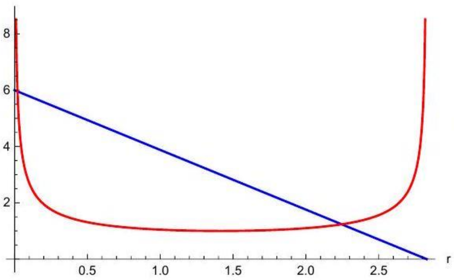
e^{f_0}
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0
1
2
3
4
θ

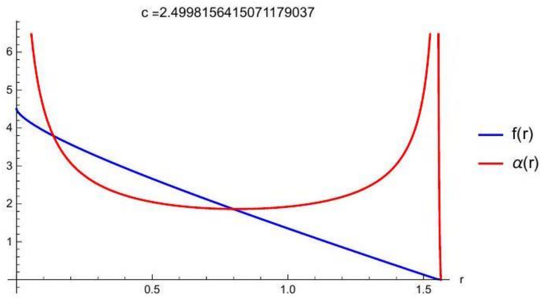


e^{f_2}
1.0
0.5
0.0
-0.5
1
2
3
4
θ



g_3
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
1
2
3
4
θ





Figuras 11, 12, 13, 14, 15, 16 y 17. Deformación del espacio – tiempo cuántico sin permeabilización gravitónica e isometrías.

$$e^{f_0} = \sum_{i=0}^{\infty} \hat{f}_{0,i}(\theta-\theta_*)^i, e^{f_1} = \frac{2^{3/2}}{5^{1/2}} e^{f_0}, e^{f_2} = \sum_{i=0, i\neq 1}^{\infty} \hat{f}_{2,i}(\theta-\theta_*)^i, \tilde{g}_3 = \sum_{i=0}^{\infty} \hat{g}_{3,i}(\theta-\theta_*)^i$$

$$\hat{f}_{2,0} = 6\hat{f}_{0,0}^4, \quad \hat{f}_{2,0} = 2\hat{f}_{0,0}$$

$$e^{f_0} = \hat{f}_{0,0} - \frac{2(9\hat{f}_{0,0}^6-1)\rho}{15\hat{f}_{0,0}^5} + \frac{2(9\hat{f}_{0,0}^6-1)(39\hat{f}_{0,0}^6+2)\rho^2}{225\hat{f}_{0,0}^{11}} + O(\rho)^3, \rho \equiv (\theta-\theta_*)^2$$

$$e^{f_2} = 2\hat{f}_{0,0} + \frac{2(6\hat{f}_{0,0}^6-1)\rho}{15\hat{f}_{0,0}^5} - \frac{(996\hat{f}_{0,0}^{12}-48\hat{f}_{0,0}^6-7)\rho^2}{225\hat{f}_{0,0}^{11}} + O(\rho)^3$$

$$\tilde{g}_3 = \frac{128}{5}(9\hat{f}_{0,0}^6-1)\left[\rho + \frac{4(5\hat{f}_{0,0}^6-1)\rho^2}{5\hat{f}_{0,0}^6} + \frac{4(152\hat{f}_{0,0}^{12}-137\hat{f}_{0,0}^6+15)\rho^3}{125\hat{f}_{0,0}^{12}} + O(\rho)^4\right]$$

$$g_3 = \frac{1}{6} e^{-4f_0+4f_2}\sqrt{9e^{6f_0}-1}.$$

$$\tilde{f}_{0,0} = 0.879040\cdots, \hat{f}_{0,0} = 0.703761\cdots$$

$$\frac{F^{\mathrm{IR}}}{F^{\mathrm{UV}}} = \frac{16(p-1)^{3/2}}{3\sqrt{3}p^2} = \begin{cases} \frac{2^2}{3^{3/2}} = 0.7698\cdots, & p=2 \\ \frac{2^{11/2}}{3^{7/2}} = 0.9677\cdots, & p=3 \end{cases}$$

$$\begin{aligned} ds_{11}^2 &= \frac{1}{4} e^{2\Delta} ds_4^2 + e^{2\Delta} ds_7^2 \\ ds_7^2 &= \frac{f\alpha}{4\sqrt{1+(1+r^2)\alpha^2}} ds_{\mathbb{CP}^2}^2 + \frac{\alpha^2}{16}\left[dr^2 + \frac{r^2 f^2}{1+r^2}(d\tau+\mathcal{A})^2 \right.\\ &\left. + \frac{1+r^2}{1+(1+r^2)\alpha^2}\left(d\psi + \frac{f}{1+r^2}(d\tau+\mathcal{A})\right)^2\right] \\ e^{6\Delta} &= \left(\frac{m}{6}\right)^2 \frac{1+(1+r^2)\alpha^2}{\alpha^2}, 0<\tau<2\pi, 0<\psi<4\pi \\ F &= \frac{m}{16}\mathrm{vol}_4 + F_{\mathrm{internal}} \end{aligned}$$

$$F=\sqrt{\frac{m^3\pi^6}{2^2 3^6 \int e^{9\Delta}\mathrm{vol}_7}}N^{3/2}=\frac{2^{\frac{9}{2}}\pi}{3^{\frac{3}{2}}\sqrt{\int f^3\alpha^2 r dr}}N^{3/2},$$

$$\alpha(r)=\sqrt{\frac{2}{r(2\sqrt{2}-r)}}, f(r)=\frac{3}{\sqrt{2}}(2\sqrt{2}-r), r\in[0,2\sqrt{2}].$$

$$F_{\mathrm{CPW}}=\left(\frac{2}{3}\right)^{5/2}\pi N^{3/2}\Rightarrow\frac{F_{SU(3)\times U(1)}}{F_{SO(8)}}=\frac{2^2}{3^{\frac{3}{2}}}=0.7698\cdots,$$

$$\frac{F^{p=3}_{SU(3)\times U(1)}}{N^{3/2}}=1.433135\cdots,\Rightarrow\frac{F^{p=3}_{SU(3)\times U(1)}}{F_{SO(8)}}=0.9677\cdots,$$

$$F_{\mathrm{SO(8)}}>F_{\mathrm{G_2}}>F_{\mathrm{SU(3)\times U(1)}}.$$

$$F_{G_2}=\frac{2^2 5^{\frac{1}{2}}\pi^{\frac{3}{2}}}{3^{\frac{5}{2}}}\sqrt{\frac{g_0^3}{\int e^{2f_0+f_1+6f_2}d\theta}}=\frac{2^{\frac{5}{4}}5^{\frac{3}{4}}\pi^{\frac{3}{2}}}{3^{\frac{5}{2}}}\sqrt{\frac{g_0^3}{\int e^{3f_0+6f_2}d\theta}},$$

$$I_1\equiv\int_0^{\pi}e^{3f_0+6f_2}\Big|_{\mathrm{dWNW}}d\theta=\frac{2^{\frac{13}{2}}3^{\frac{3}{2}}}{5^{\frac{7}{2}}}\pi g_0^3\approx 5.286 g_0^3$$

$$\frac{F_{G_2'}}{N^{3/2}}\approx 1.45669$$

$\frac{F}{N^{3/2}}$	global symmetry	supercharges
$\frac{\sqrt{2}\pi}{3}\approx 1.4810$	SO(8)	32
$\frac{5^{5/2}\pi}{2^2 3^{13/4}}\approx 1.2356$	G_2	4
$\left(\frac{2}{3}\right)^{5/2}\pi\approx 1.1400$	$\mathrm{SU(3)\times U(1)}$	8
≈ 1.4567	G_2	4
$\frac{2^6\pi}{3^{9/2}}\approx 1.4331$	$\mathrm{SU(3)\times U(1)}$	8

$$\frac{\pi}{3^{1/3}5^{1/4}} \approx 1.456689$$

$$\Delta W = \mathrm{Tr}XX$$

$$\begin{aligned}&\mathbf{35}_{\mathrm{v}} \to \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1}\\&\mathbf{56}_{\mathrm{s}} \to \mathbf{27} \oplus \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{7} \oplus \mathbf{1}\\&\mathbf{355}_{\mathrm{c}} \to \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1}\end{aligned}$$

$$\Delta W = \mathrm{Tr}XXX$$

$$\begin{aligned}&\mathbf{112}' \to \mathbf{77} \oplus \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1},\\&\mathbf{224}_{\mathrm{vc}} \to \mathbf{77} \oplus \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{27} \oplus \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{7} \oplus \mathbf{1},\\&\mathbf{224}_{\mathrm{cv}} \to \mathbf{77} \oplus \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{27} \oplus \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{7} \oplus \mathbf{1}.\end{aligned}$$

$$r = \left(\frac{1}{6} - \frac{2}{3p}\right) r_1 + \frac{1}{2} r_2 = \begin{cases} -\frac{1}{6} r_1 + \frac{1}{2} r_2, & p = 2 \\ -\frac{1}{18} r_1 + \frac{1}{2} r_2, & p = 3 \end{cases}$$

$$\begin{aligned}&X^2: \quad (2000) = \mathbf{35_v} \quad \to \quad \mathbf{1}_{-2} + \mathbf{1}_0 + \mathbf{1}_2 + \cdots\\&\lambda\lambda: \quad (0020) = \mathbf{35_c} \quad \to \quad \mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + \cdots\\&\lambda X: \quad (1010) = \mathbf{56_s} \quad \to \quad \mathbf{1}_{-1} + \mathbf{1}_{-1} + \mathbf{1}_1 + \mathbf{1}_1 + \cdots\end{aligned}$$

$$\begin{aligned}&X^3: (3000) = \mathbf{112_v} \to \mathbf{1}_{-2} + \mathbf{1}_{-\frac{2}{3}} + \mathbf{1}_{\frac{2}{3}} + \mathbf{1}_2 + \cdots\\&\lambda\lambda X: (1020) = \mathbf{224_{cv}} \to \mathbf{1}_{-\frac{4}{3}} + \mathbf{1}_{-\frac{2}{3}} + \mathbf{1}_0 + \mathbf{1}_{\frac{2}{3}} + \mathbf{1}_{\frac{4}{3}} + \cdots\\&\lambda X^2: (2010) = \mathbf{224_{vc}} \to \mathbf{1}_{-\frac{5}{3}} + \mathbf{1}_{-1} + \mathbf{1}_{-\frac{1}{3}} + \mathbf{1}_{\frac{1}{3}} + \mathbf{1}_1 + \mathbf{1}_{\frac{5}{3}} + \cdots\end{aligned}$$

$$J_p(v) = J(p, v) \equiv p \times v, v \in T_p S^6, p \in \mathrm{Im}\mathbb{O}$$

$$J_\nu^\rho v^\nu e_\rho \equiv J_p v = \frac{1}{2}(pv - vp) = p^\mu v^\nu \tilde{\eta}_{\mu\nu}^\rho e_\rho$$

$$e_\mu e_\nu = -\delta_{\mu\nu} e_0 + \tilde{\eta}_{[\mu\nu}{}^{\rho]} e_\rho, \tilde{\eta}_{[\mu\nu}{}^{\rho]} = 1 \text{ when } \mu\nu\rho = 123,471,572,673,354,246,165$$

$$J_{\rho\nu}(p) = p^\mu \tilde{\eta}_{\mu\nu\rho}$$

$$dx^\mu = \frac{\partial x^\mu}{\partial x^m} dx^m \equiv \Lambda_m^\mu dx^m$$

$$J_{mn}(p) = J_{\rho\nu}(p) \Lambda^\rho{}_m \Lambda_n^\nu = p^\mu \tilde{\eta}_{\mu\nu\rho} \Lambda^\rho{}_m \Lambda_n^\nu,$$

$$J_\mu{}^\nu e_\nu = e_\mu \times p$$

$$\begin{aligned}&x^1 = \cos\psi^1\\&x^2 = \sin\psi^1 \cos\psi^2\\&x^3 = \sin\psi^1 \sin\psi^2 \cos\psi^3\\&\dots\dots\\&x^6 = \sin\psi^1 \sin\psi^2 \sin\psi^3 \sin\psi^4 \sin\psi^5 \cos\psi^6\\&x^7 = \sin\psi^1 \sin\psi^2 \sin\psi^3 \sin\psi^4 \sin\psi^5 \sin\psi^6\end{aligned}$$

$$T_{\mu\nu\rho} = \left(e_\mu \times e_\nu, e_\rho\right) = \tilde{\eta}_{\mu\nu\rho}, \Rightarrow T_{mnp} = \Lambda_m^\mu \Lambda_n^\nu \Lambda_p^\rho \tilde{\eta}_{\mu\nu\rho},$$

$$\left(e_\mu, e_\nu\right) \equiv e_\mu^\alpha e_{\nu\alpha}$$

$$S_{mnp} \equiv \frac{1}{3!}\epsilon_{mnpqrs}T^{qrs}$$

$$e^{f_0} = \hat{f}_{0,0} + \frac{4\left(9\hat{f}_{0,0}^6 - 1\right)(\theta-\theta_*)^2}{9\hat{f}_{0,0}^5} + \frac{8\left(9\hat{f}_{0,0}^6 - 1\right)\left(195\hat{f}_{0,0}^6 - 29\right)(\theta-\theta_*)^4}{405\hat{f}_{0,0}^{11}} + O(\theta-\theta_*)^6$$

$$e^{f_2} = 6\hat{f}_{0,0}^4 - \frac{2\left(180\hat{f}_{0,0}^6 - 19\right)(\theta-\theta_*)^2}{15\hat{f}_{0,0}^2} - \frac{\left(75600\hat{f}_{0,0}^{12} - 20760\hat{f}_{0,0}^6 + 1373\right)(\theta-\theta_*)^4}{675\hat{f}_{0,0}^8} + O(\theta-\theta_*)^6$$

$$\tilde{g}_3 = 6^6\hat{f}_{0,0}^{24}\left(9\hat{f}_{0,0}^6 - 1\right)\left[1 - \frac{16(\theta-\theta_*)^2}{15\hat{f}_{0,0}^6} + \frac{8\left(340\hat{f}_{0,0}^6 - 27\right)(\theta-\theta_*)^4}{225\hat{f}_{0,0}^{12}} + O(\theta-\theta_*)^6\right]$$

$$e^{f_0} = \hat{f}_{0,0} + \frac{2\left(9\hat{f}_{0,0}^6 - 1\right)\rho}{45\hat{f}_{0,0}^{11}} - \frac{4\left(9\hat{f}_{0,0}^6 - 1\right)\rho^2}{2025\hat{f}_{0,0}^{23}} + O(\rho)^3, \rho \equiv (\theta-\theta_*)^2$$

$$e^{f_2} = 6\hat{f}_{0,0}^4 - \frac{2\left(9\hat{f}_{0,0}^6 - 1\right)\rho}{15\hat{f}_{0,0}^8} + \frac{\left(243\hat{f}_{0,0}^{12} - 18\hat{f}_{0,0}^6 - 1\right)\rho^2}{675\hat{f}_{0,0}^{20}} + O(\rho)^3$$

$$\tilde{g}_3 = 6^6\hat{f}_{0,0}^{24}\left(9\hat{f}_{0,0}^6 - 1\right)\left[1 - \frac{2\left(18\hat{f}_{0,0}^6 - 1\right)\rho}{15\hat{f}_{0,0}^{12}} + \frac{4\left(567\hat{f}_{0,0}^{12} - 63\hat{f}_{0,0}^6 + 1\right)\rho^2}{675\hat{f}_{0,0}^{24}} + O(\rho)^3\right]$$

$$g_3 = e^{3f_2}\sqrt{\frac{1}{4}e^{-2f_0+2f_2} - 1}$$

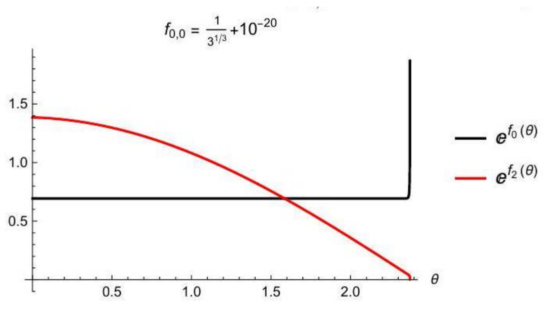
$$f_0'(\theta) + 2f_2'(\theta) = 0 \;\Rightarrow\; e^{f_2} = Ce^{-\frac{1}{2}f_0}$$

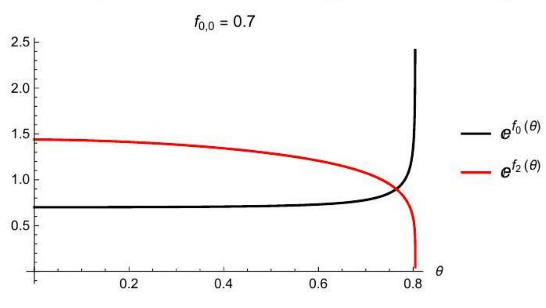
$$e^{f_0} = \hat{f}_{0,0} + \frac{2\left(9\hat{f}_{0,0}^6 - 1\right)\left(10\hat{f}_{0,0}^6 - 1\right)\rho}{45\hat{f}_{0,0}^{11}} - \frac{2\left(9\hat{f}_{0,0}^6 - 1\right)\left(100\hat{f}_{0,0}^{12} - 1\right)\left(39\hat{f}_{0,0}^6 - 4\right)\rho^2}{2025\hat{f}_{0,0}^{23}} + O(\rho)^3$$

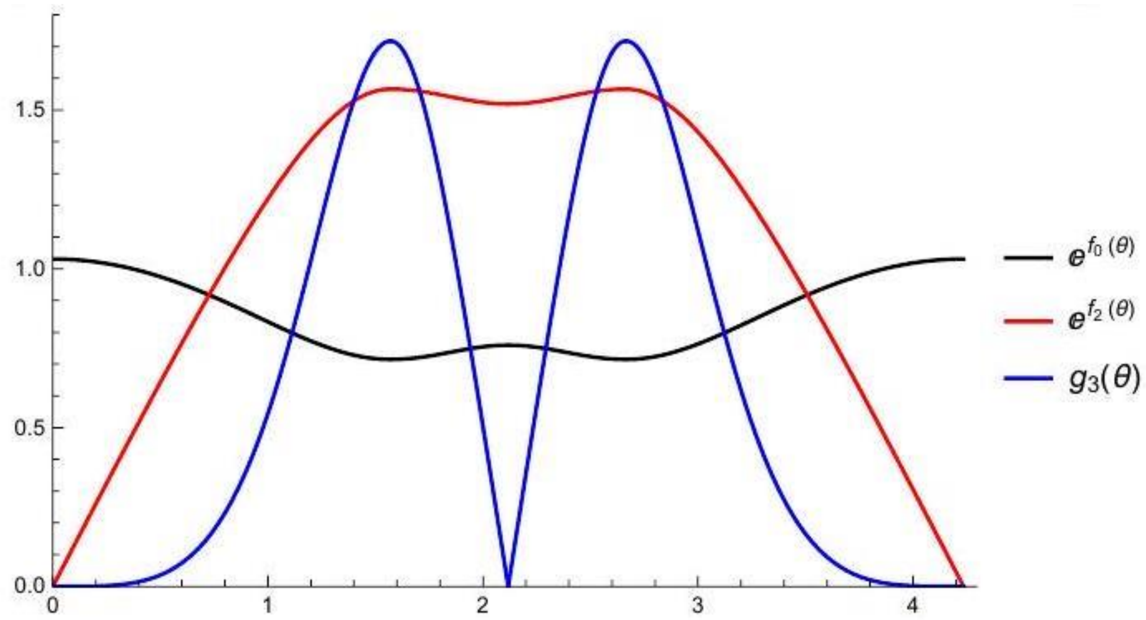
$$e^{f_2} = 6\hat{f}_{0,0}^4 - \frac{2\left(10\hat{f}_{0,0}^6 - 1\right)\left(18\hat{f}_{0,0}^6 - 1\right)\rho}{15\hat{f}_{0,0}^8} - \frac{\left(10\hat{f}_{0,0}^6 - 1\right)\left(7560\hat{f}_{0,0}^{18} - 348\hat{f}_{0,0}^{12} - 154\hat{f}_{0,0}^6 + 11\right)\rho^2}{675\hat{f}_{0,0}^{20}} + O(\rho)^3$$

$$\tilde{g}_3 = 6^6\hat{f}_{0,0}^{24}\left(9\hat{f}_{0,0}^6 - 1\right)\left[1 + \frac{2\left(10\hat{f}_{0,0}^6 - 1\right)\rho}{15\hat{f}_{0,0}^{12}} - \frac{4\left(10\hat{f}_{0,0}^6 - 1\right)\left(120\hat{f}_{0,0}^{12} - 51\hat{f}_{0,0}^6 + 4\right)\rho^2}{675\hat{f}_{0,0}^{24}} + O(\rho)^3\right]$$

$$g_3 = e^{3f_2}\sqrt{9e^{6f_0} - 1}.$$







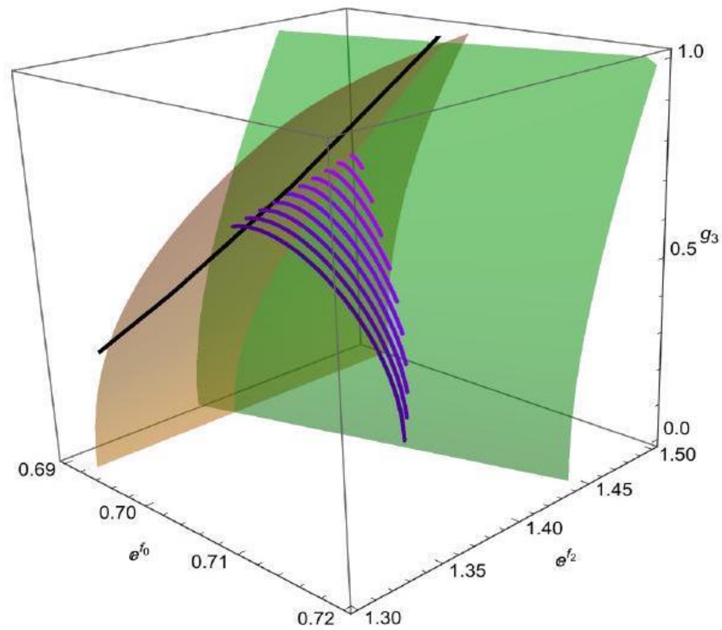
Figuras 18 y 19. Densidad de masa de una partícula supermasiva y registros de ondas cuánticas gravitacionales.

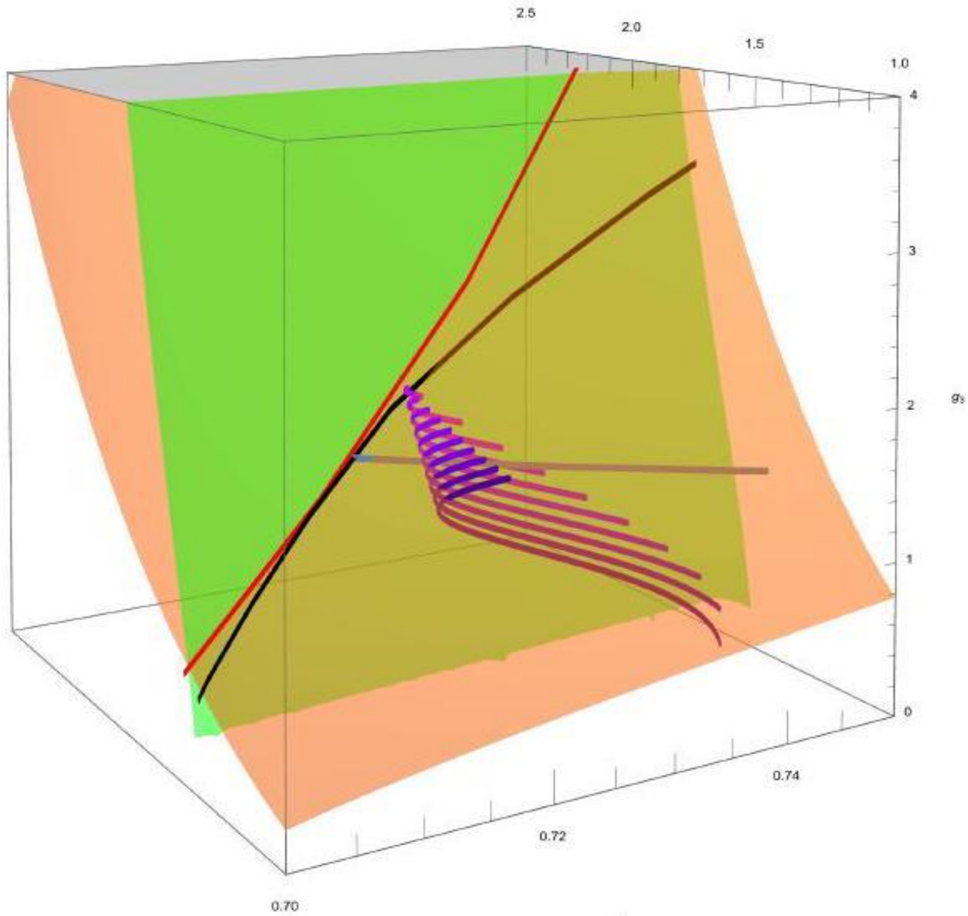
$$e^{f_2} = 2\cos\left(\sqrt{\frac{2}{5}}(\theta-\theta_*)\right)e^{f_0}, e^{f_0} \equiv \hat{f}_{0,0}, g_3 = 0$$

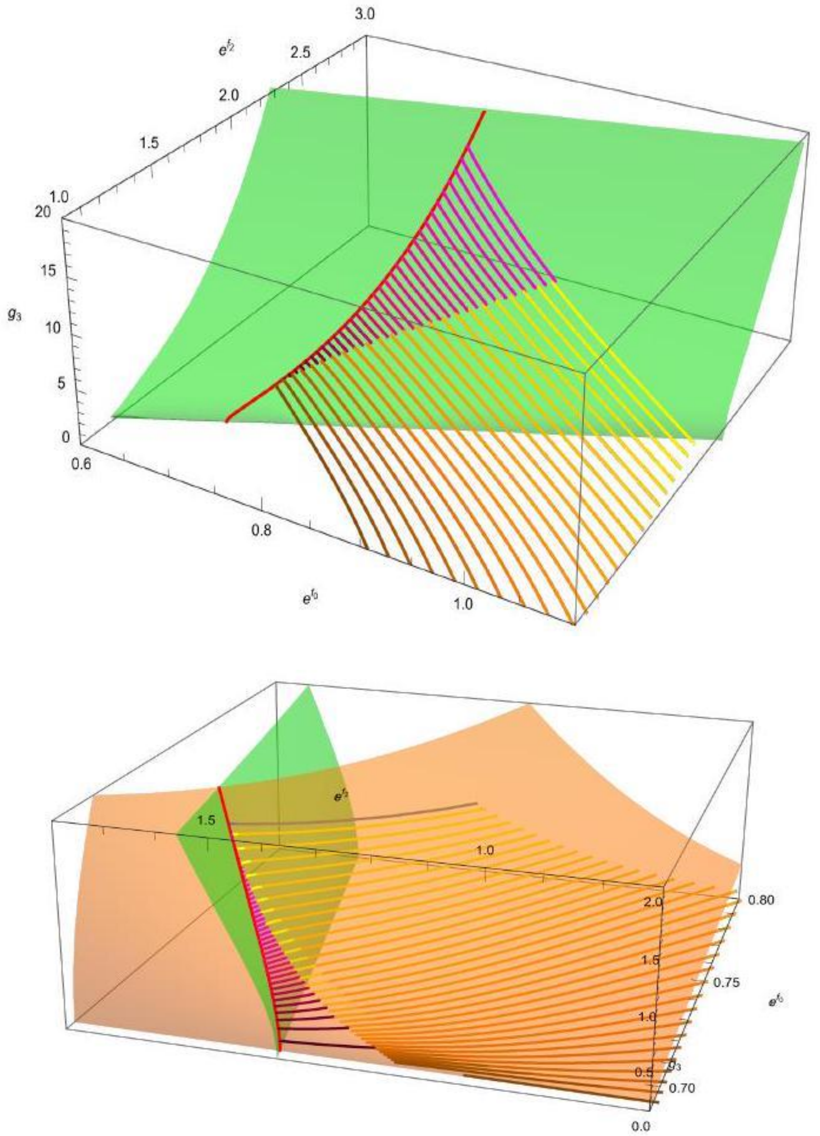
$$e^{f_0} = \hat{f}_{0,0} - \frac{2\left(9\hat{f}_{0,0}^6-1\right)\rho}{15\hat{f}_{0,0}^5} + \frac{4\left(9\hat{f}_{0,0}^6-1\right)\left(18\hat{f}_{0,0}^6+1\right)\rho^2}{225\hat{f}_{0,0}^{11}} + O(\rho)^3, \rho \equiv (\theta-\theta_*)^2$$

$$e^{f_2} = 2\hat{f}_{0,0} + \frac{2\left(9\hat{f}_{0,0}^6-1\right)\rho}{15\hat{f}_{0,0}^5} - \frac{\left(9\hat{f}_{0,0}^6-1\right)\left(45\hat{f}_{0,0}^6+7\right)\rho^2}{225\hat{f}_{0,0}^{11}} + O(\rho)^3$$

$$\tilde{g}_3 = \frac{128}{5}\left(9\hat{f}_{0,0}^6-1\right)\left[\rho + \frac{2\left(9\hat{f}_{0,0}^6-2\right)\rho^2}{5\hat{f}_{0,0}^6} - \frac{12\left(39\hat{f}_{0,0}^6-5\right)\rho^3}{125\hat{f}_{0,0}^{12}} + O(\rho)^4\right]$$


1.0
0.5
g_3
0.0
1.50
1.45
1.40
1.35
1.30
e^{f_2}
0.69
0.70
0.71
0.72
e^{f_0}


2.5
2.0
1.5
1.0
4
3
2
1
0
g_3
0.74
0.72
0.70



Figuras 20, 21, 22 y 23. Distorsión del espacio – tiempo cuántico por una partícula supermasiva, en dimensión D=4.

$$g_0 = \frac{1}{4} F \left(\frac{\ell_4}{\ell_4^{(0)}} \right)^4, e^{f_0} = \frac{1}{2} \Delta^{-\frac{1}{2}} \frac{\ell_4}{\ell_4^{(0)}}, e^{f_1} = e^{\frac{3}{2}\phi} \Delta^{-\frac{1}{2}}, e^{f_2} = e^{-\phi/4} \Delta^{1/4} \sin\, \theta$$

$$e^{\phi} \to \lambda^3 e^{\phi}, \Delta \to \lambda^7 \Delta, A \to \lambda^3 A, F \to \lambda^{-15} F, \ell_4 \to \lambda^{7/2} \ell_4.$$

$$\frac{F_{\text{holo}}}{N^{3/2}} = \frac{\sqrt{2}}{3} \pi \left(\frac{\ell_4}{\ell_4^{(0)}} \right)^5$$

	$\frac{F_{\text{holo}}}{N^{3/2}}$	ℓ_4	$\ell_4^{(0)}$	$\ell_4/\ell_4^{(0)}$
SO(8)	$\frac{\sqrt{2}\pi}{3} \approx 1.4810$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\mathrm{SO}(7)_+$	$\frac{2^{1/2}}{5^{3/4}}\pi \approx 1.3287$	0.489270	$\frac{1}{2}$	$\frac{3^{1/5}}{5^{3/20}} \approx 0.97854$
$\mathrm{SO}(7)_-$	$\frac{2^{9/2}}{5^{5/2}}\pi \approx 1.2716$	0.497590	0.512989	$\frac{2^{4/5}3^{1/5}}{5^{1/2}}$ ≈ 0.96998
G_2	$\frac{5^{5/2}}{2^2 3^{13/4}}\pi$ ≈ 1.2356	0.489049	0.507092	$\frac{5^{1/2}}{2^{1/2}3^{9/20}}$ ≈ 0.9644
G_2'	1.45669	0.504244	0.505913	0.996701

$$d\varphi = e^{f_1} d\theta$$

$$e^{f_0} = \frac{1}{3^{1/3}}, e^{f_1} = \frac{2^{3/2}}{5^{1/2}} e^{f_0}, e^{f_2} = \frac{2}{3^{1/3}} \sin\left(\sqrt{\frac{2}{5}}\theta\right)$$

$$F_{\mathrm{SO}(8)} = \frac{\sqrt{2}\pi}{3} N^{3/2}$$

$$\frac{\langle V \rangle}{m_{\mathrm{KK}}^2} \sim \frac{V_{\mathrm{O}p}}{m_{\mathrm{KK}}^2} \sim \rho^{\frac{p-7}{2}} e^{\phi}.$$

$$\mathrm{d}s_{10}^2 = \tau^{-2}\, \mathrm{d}s_D^2 + \rho \mathrm{d}\tilde{s}_{10-D}^2$$

$$\tau^{D-2} = e^{-2\phi} \rho^{\frac{10-D}{2}}, \text{ for } 2 < D < 10$$

$$S_D = M_{Pl}^{D-2} \int d^D x \sqrt{-g_D} \left(R_D - (D-2)\tau^{-2} (\partial_\mu \tau)^2 - \frac{10-D}{4} \rho^{-2} (\partial_\mu \rho)^2 - V \right)$$

$$M_{Pl}^{D-2} = \frac{1}{2\kappa_{10}^2} \int d^{10-D} y \sqrt{\tilde{g}} = \frac{l_s^{10-D}}{2\kappa_{10}^2}$$

$$V = V_R + V_H + \sum_q V_{F_q} + \sum_p V_{Dp/Op} + V_{NS5/ONS5},$$

$$q = 0,2,4,6,8,10, \quad p = 0,2,4,6,8$$
$$q = 1,3,5,7,9, \quad p = 1,3,5,7,9$$

$$\begin{aligned} dF_q &= H_3 \wedge F_{q-2} + \mu_{8-q} J_{q+1}, \\ d(\star F_q) &= -H_3 \wedge \star F_{q+2} + (-1)^{\frac{q(q-1)}{2}} \mu_{q-2} J_{11-q}, \\ dH_3 &= \mu_{NS5} J_4^{NS}, \\ d(e^{-2\phi} \star H_3) &= -\frac{1}{2} \sum_q \star F_q \wedge F_{q-2}. \end{aligned}$$

$$\tau \to e^{\sqrt{\frac{1}{2(D-2)}}\tilde{\tau}}, \rho \to e^{\sqrt{\frac{2}{10-D}}\tilde{\rho}}$$

$$M^2 = \begin{pmatrix} \partial_{\tilde{\rho}}^2 V & \partial_{\tilde{\rho}} \partial_{\tilde{\tau}} V \\ \partial_{\tilde{\rho}} \partial_{\tilde{\tau}} V & \partial_{\tilde{\tau}}^2 V \end{pmatrix} = \begin{pmatrix} \frac{2}{10-D} \rho^2 \partial_\rho^2 V & \frac{1}{\sqrt{(D-2)(10-D)}} \rho \tau \partial_\rho \partial_\tau V \\ \frac{1}{\sqrt{(D-2)(10-D)}} \rho \tau \partial_\rho \partial_\tau V & \frac{1}{2(D-2)} \tau^2 \partial_\tau^2 V \end{pmatrix}$$

$$\lambda_\pm = \frac{1}{2} \left(\partial_{\tilde{\rho}}^2 V + \partial_{\tilde{\tau}}^2 V \pm \sqrt{\left(\partial_{\tilde{\rho}}^2 V - \partial_{\tilde{\tau}}^2 V\right)^2 + 4\left(\partial_{\tilde{\rho}} \partial_{\tilde{\tau}} V\right)^2} \right).$$

$$\begin{aligned} V_R &= -\tilde{R}_{10-D} \rho^{-1} \tau^{-2}, \\ V_H &= |\tilde{H}_3|^2 \rho^{-3} \tau^{-2}, \\ V_{RR}^q &= |\tilde{F}_q|^2 \rho^{\frac{10-D-2q}{2}} \tau^{-D}, \\ V_{Dp/Op} &= T_p \rho^{\frac{2p-D-8}{4}} \tau^{-\frac{D+2}{2}}, \\ V_{NS5/ONS5} &= T_{NS5} \rho^{-2} \tau^{-2}. \end{aligned}$$

$$\begin{aligned} V_R &= -\tilde{R}_{10-D} \rho^{-1} \tau^{-2} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}} (-\tilde{R}_{10-D}), \\ V_H &= |\tilde{H}_3|^2 \rho^{-3} \tau^{-2} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}} \left(|\tilde{H}_3|^2 \rho^{-2}\right), \\ V_{RR}^q &= |\tilde{F}_q|^2 \rho^{\frac{10-D-2q}{2}} \tau^{-D} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}} \left(|\tilde{F}_q|^2 e^{2\phi} \rho^{1-q}\right), \\ V_{Dp/Op} &= T_p \rho^{\frac{2p-D-8}{4}} \tau^{-\frac{D+2}{2}} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}} \left(T_p e^{\phi} \rho^{\frac{p-7}{2}}\right), \\ V_{NS5/ONS5} &= T_{NS5} \rho^{-2} \tau^{-2} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}} (T_{NS5} \rho^{-1}). \end{aligned}$$

$$e^{-\frac{4}{D-2}\phi} \rho^{\frac{8}{D-2}} \partial_{e^\phi} V = e^{-\frac{4}{D-2}\phi} \rho^{\frac{8}{D-2}} \partial_\rho V = 0$$

$$m_{\mathrm{KK}}^2 \sim \rho^{-1} \tau^{-2} = e^{\frac{4}{D-2}\phi} \rho^{-\frac{8}{D-2}}$$

$$\begin{aligned} &g_{mn} \sim \rho \sim N^r, \quad &&\tau \sim N^t, \quad &&e^\phi \sim N^d, \quad &&\tilde{R}_{10-D} \sim N^c, \\ &H_3 \sim N^h, \quad &&F_q \sim N^{f_q}, \quad &&T_p \sim N^0, \quad &&T_{NS5} \sim N^n, \end{aligned}$$

$$
\begin{aligned}
V_R &\sim N^{c-r-2t},\\
V_H &\sim N^{2h-3r-2t},\\
V_{RR}^q &\sim N^{2f_q+\left(\frac{10-D-2q}{2}\right)r-Dt} \sim N^{2f_q+2d-qr-2t},\\
V_{Dp/Op} &\sim N^{\frac{2p-D-8}{4}r-\frac{D+2}{2}t} \sim N^{d+\frac{p-9}{2}r-2t},\\
V_{NS5/ONS5} &\sim N^{n-2r-2t},
\end{aligned}
$$

$$0 = -V_R - 3V_H + \frac{1}{2}(10-D-2q)\sum_q V_{RR}^q + \frac{1}{4}(2p-D-8)V_{Dp/Op} - 2V_{NS5/ONS5}$$

$$0 = 2V_R + 2V_H + D\sum_q V_{RR}^q + \frac{1}{2}(D+2)V_{Dp/Op} + 2V_{NS5/ONS5}$$

$$\frac{V_R}{m_{\text{KK}}^2} \sim \frac{-\tilde{R}_{10-D}\rho^{-1}\tau^{-2}}{\rho^{-1}\tau^{-2}} \sim -\tilde{R}_{10-D}$$

$$\frac{V_H}{m_{\text{KK}}^2} \sim \frac{\left|\tilde{H}_3\right|^2\rho^{-3}\tau^{-2}}{\rho^{-1}\tau^{-2}} \sim \left|\tilde{H}_3\right|^2\rho^{-2}$$

$$\frac{V_{RR}^q}{m_{\text{KK}}^2} \sim \frac{\rho^{-1}\tau^{-2}\left(\left|\tilde{F}_q\right|^2 e^{2\phi}\rho^{1-q}\right)}{\rho^{-1}\tau^{-2}} \sim \left|\tilde{F}_q\right|^2 e^{2\phi}\rho^{1-q}$$

$$\frac{V_{Dp/Op}}{m_{\text{KK}}^2} \sim \frac{\rho^{-1}\tau^{-2}\left(T_p e^{\phi}\rho^{\frac{p-7}{2}}\right)}{\rho^{-1}\tau^{-2}} = T_p e^{\phi}\rho^{\frac{p-7}{2}}.$$

$$\frac{V_{NS5/ONS5}}{m_{\text{KK}}^2} \sim \frac{T_{NS5}\rho^{-2}\tau^{-2}}{\rho^{-1}\tau^{-2}} = T_{NS5}\rho^{-1}$$

$$V = V_{LO}N^a + V_{NLO}N^b + \cdots, \text{ for } a > b$$

$$H_3 \wedge \star_6 F_4 = 0, H_3 \wedge F_2 = 0$$

$$\sum_s \star F_s \wedge F_{s-2} = \cdots + \star F_{q+2} \wedge A_q + \star A_q \wedge F_{q-2} + \cdots$$

$$\sum_i h_i f_i = 0$$

$$0 = H_3 \wedge F_q + \mu_{D(6-q)}J_{q+3} + \mu_{O(6-q)}J_{q+3} = \mu_{D(6-q)}J_{q+3} + \mu_{O(6-q)}J_{q+3}$$

$$H_3 \wedge F_q = H_3 \wedge F_{q,A} + H_3 \wedge F_{q,B}$$

$$
\begin{aligned}
0 &= H_3 \wedge F_{q,B} + \mu_{6-q}J_{q+1}\\
0 &= H_3 \wedge F_{q,A}
\end{aligned}
$$

$$V_{RR,AA} \sim N^{f_{q,A}}V_{RR,AB} \sim N^{2f_{q,A}}V_{RR,BB}$$

$$V_{RR,AA} > V_{RR,AB} > V_{RR,BB}, \text{ for } N \gg 1$$

$$dF_q = dF_q^{(\text{closed})} + dF_q^{(\text{non-closed})} = dF_q^{(\text{non-closed})} = \mu_{8-q} J_{q+1}$$

$$T_p = -\frac{4\left|\tilde{H}_3\right| e^{-\phi} \rho^{\frac{3-p}{2}} + 2D \sum_q \left|\tilde{F}_q\right|^2 e^{\phi} \rho^{\frac{9-p-2q}{2}}}{D+2}$$

$$\left|\tilde{H}_3\right| e^{-\phi} \rho^{\frac{3-p}{2}} \sim N^{2h-d+\frac{3-p}{2}r} \sim N^0$$
$$\left|\tilde{F}_{6-p}\right|^2 e^{\phi} \rho^{\frac{9-p-2(6-p)}{2}} \sim N^{2f_{6-p}+d-\frac{3-p}{2}r} \sim N^0$$

$$d = \frac{3-p}{2} r$$

$$t = \frac{2(p+2)-D}{2(D-2)} r > 0$$

$$V \sim V_{Op} \sim V_H \sim V_{RR}^{6-p} \sim \rho^{-\frac{2(D+p-1)}{D-2}}$$

$$\frac{V_{RR}^q}{V_{RR}^{6-p}} \sim N^{2f_q - 2f_{6-p} + (6-p-q)r}$$

$$f_q = \frac{q-(6-p)}{2} r$$

$$V = V_H + V_{Op} + V_{RR}^{6-p},$$

$$V_{RR}^{6-p} = V_H, V_{Op} = -2V_H$$

$$V = V_H + V_{Op} + V_{RR}^{6-p} + \sum_{q \neq 6-p} V_{RR}^q$$

$$\langle V \rangle = -\frac{1}{2} \sum_{q \neq 6-p} \frac{(D-2)(q-(6-p))}{D+p-1} V_{RR}^q$$

Op	F_{6-p}	other RR fluxes	EFT dimensions	Scalar Potential
O2	F_4	$\{F_0, F_2, F_6\}$	$D = 3$	$\langle V \rangle \sim 2V_{RR}^0 + V_{RR}^2 - V_{RR}^6$
O3	F_3	$\{F_1, F_5, F_7\}$	$D = 3$	$\langle V \rangle \sim V_{RR}^1 - V_{RR}^5 - 2V_{RR}^7$
O3	F_3	$\{F_1, F_5\}$	$D = 4$	$\langle V \rangle \sim V_{RR}^1 - V_{RR}^5$
O4	F_2	$\{F_0, F_4, F_6\}$	$D = 3$	$\langle V \rangle \sim V_{RR}^0 - V_{RR}^4 - 2V_{RR}^6$
O4	F_2	$\{F_0, F_4, F_6\}$	$D = 4$	$\langle V \rangle \sim V_{RR}^0 - V_{RR}^4 - 2V_{RR}^6$

O4	F_2	$\{F_0, F_4\}$	$D = 5$	$\langle V \rangle \sim V_{RR}^0 - V_{RR}^4$
O5	F_1	$\{F_3, F_5, F_7\}$	$D = 3$	$\langle V \rangle < 0$, AdS
O5	F_1	$\{F_3, F_5\}$	$D = 4$	$\langle V \rangle < 0$, AdS
O5	F_1	$\{F_3, F_5\}$	$D = 5$	$\langle V \rangle < 0$, AdS
O5	F_1	$\{F_3\}$	$D = 6$	$\langle V \rangle < 0$, AdS
O6	F_0	$\{F_2, F_4, F_6\}$	$D = 3$	$\langle V \rangle < 0$, AdS
O6	F_0	$\{F_2, F_4, F_6\}$	$D = 4$	$\langle V \rangle < 0$, AdS
O6	F_0	$\{F_2, F_4\}$	$D = 5$	$\langle V \rangle < 0$, AdS
O6	F_0	$\{F_2, F_4\}$	$D = 6$	$\langle V \rangle < 0$, AdS
O6	F_0	$\{F_2\}$	$D = 7$	$\langle V \rangle < 0$, AdS

$$\langle V \rangle = -\frac{1}{4} V_{RR}^6$$

$$\rho \sim N^r \sim N^{f_6}, e^{\phi} \sim N^{\frac{1}{2}f_6}, \tau \sim N^{\frac{5}{2}f_6}, \frac{L_{KK}^2}{L_{AdS}^2} \sim N^{-2f_6}$$

$$\langle V \rangle = -\frac{D-2}{D+3}(V_{RR}^4 + 2V_{RR}^6),$$

$$\begin{gathered} 0 = H_3 \wedge F_2 + \mu_{O4} J_5 \\ 0 = H_3 \wedge F_4 \\ 0 = H_3 \wedge\star F_4 \\ 0 = H_3 \wedge\star F_6 \\ 0 = F_2 \wedge\star F_4 + F_4 \wedge\star F_6 \end{gathered}$$

$$F_4 \sim N^{f_4}, F_6 \sim N^{f_6} \sim N^{2f_4},$$

$$\rho \sim N^r \sim N^{f_4}, e^{\phi} \sim N^{-\frac{1}{2}f_4}, \tau \sim N^{\frac{112-D}{2\,D-2}f_4}, \frac{L_{KK}^2}{L_{AdS}^2} \sim N^{-2f_4}$$

$$V_{RR}^4 \leq 2(D+1)V_{RR}^6.$$

$$V_H > \frac{D}{3+D}\frac{(V_{RR}^4 + 2V_{RR}^6)(V_{RR}^4 - 2(D+1)V_{RR}^6)}{(2D+1)V_{RR}^4 + 2(3D-1)V_{RR}^6}.$$

$$\langle V\rangle = -\frac{D-2}{D+5}(V_{RR}^2 + 2V_{RR}^4 + 3V_{RR}^6),$$

$$F_2 \sim N^{f_2} \sim N^{\frac{1}{2}f_4}, F_4 \sim N^{f_4}, F_6 \sim N^{f_6} \sim N^{\frac{3}{2}f_4},$$

$$\rho \sim N^r \sim N^{\frac{1}{2}f_4}, e^{\phi} \sim N^{-\frac{3}{4}f_4}, \tau \sim N^{\frac{1}{4}\frac{16-D}{D-2}f_4}, \frac{L_{KK}^2}{L_{AdS}^2} \sim N^{-f_4}$$

$$V_{RR}^2 \leq \frac{2}{3}(D-1)V_{RR}^4 + (2D+1)V_{RR}^6$$

$$V_H > \frac{D}{5+D}\frac{(V_{RR}^2 + 2V_{RR}^4 + 3V_{RR}^6)\left(3V_{RR}^2 + 2V_{RR}^4 - 3V_{RR}^6 - 2D(V_{RR}^4 + 3V_{RR}^6)\right)}{3V_{RR}^2 + 2V_{RR}^4 - 3V_{RR}^6 + 2D(V_{RR}^2 + 3V_{RR}^4 + 6V_{RR}^6)}$$

$$\langle V\rangle = -\frac{D-2}{D+2}V_{RR}^5$$

$$\begin{gathered} 0 = H_3 \wedge \star F_3 + \mu_{O3}J_6 \\ 0 = H_3 \wedge \star F_3 \\ 0 = H_3 \wedge \star F_5 \\ 0 = F_3 \wedge \star F_5 \end{gathered}$$

$$F_5 \sim N^{f_5}$$

$$\rho \sim N^r \sim N^{f_5}, e^{\phi} \sim N^0, \tau \sim N^{\frac{1}{2}\frac{D-10}{2-D}f_5}, \frac{L_{KK}^2}{L_{AdS}^2} \sim N^{-2f_5}$$

$$\lambda_+ = \frac{2D}{10-D}V_{RR}^5, \lambda_- = 2V_H$$

$$\langle V\rangle = -\frac{D-2}{D+4}(V_{RR}^3 + 2V_{RR}^5)$$

$$\begin{gathered} 0 = H_3 \wedge F_1 + \mu_5 J_4 \\ 0 = H_3 \wedge \star F_3 \\ 0 = H_3 \wedge \star F_5 \\ 0 = \star F_3 \wedge F_1 + \star F_5 \wedge F_3 \end{gathered}$$

$$F_3 \sim N^{f_3} \sim N^{\frac{1}{2}f_5}, F_5 \sim N^{f_5} \sim N^{2f_3},$$

$$\rho \sim N^r \sim N^{\frac{1}{2}f_5}, e^{\phi} \sim N^{-\frac{1}{2}f_5}, \tau \sim N^{\frac{1}{4}\frac{14-D}{D-2}f_5}, \frac{L_{KK}^2}{L_{AdS}^2} \sim N^{-f_5}$$

$$V_{RR}^3 \leq DV_{RR}^5$$

$$V_H > \frac{D}{D+4}\frac{(V_{RR}^3 + 2V_{RR}^5)(V_{RR}^3 - DV_{RR}^5)}{V_{RR}^3(D+1) + 3DV_{RR}^5}$$

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}X\sqrt{-G}e^{-2\phi}\left(R_{10} + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2 - \frac{1}{4}e^{2\phi}\sum_q |F_q|^2\right)$$

$$S_{loc} = -T_p \int \mathrm{d}^{p+1}X e^{-\phi}\sqrt{-P[G]} + T_p \int C_{p+1}$$

$$G_{MN} = e^{\phi/2} G_{MN}$$

$$S_E = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}X\sqrt{-G}\left(R_{10} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi}|H_3|^2 - \frac{1}{4}e^{\frac{5-q}{2}\phi}\sum_q |F_q|^2\right)$$
$$-T_p\int \mathrm{d}^{10}X\sqrt{-G}e^{\frac{p-3}{4}\phi}\delta(\Sigma_i)$$

$$R_{10} = R_D + R_{(10-D)} + \cdots,$$

$$\left(R_{MN} - \frac{1}{2}G_{MN}R_{10}\right) - \frac{1}{2}\left(\partial_M\phi\partial_N\phi - \frac{1}{2}G_{MN}(\partial\phi)^2\right)$$
$$-\frac{1}{2}e^{-\phi}\left(|H_3|^2_{MN} - \frac{1}{2}G_{MN}|H_3|^2\right) - \frac{1}{4}e^{\frac{5-q}{2}\phi}\sum_q\left(|F_q|^2_{MN} - \frac{1}{2}G_{MN}|F_q|^2\right)$$
$$+\frac{1}{2}e^{\frac{p-3}{4}\phi}T^{loc}_{MN} = 0$$

$$T_{MN} = 2\kappa_{10}^2 T_p P[G]_{MN}\delta_{\Sigma_i}$$

$$T^{loc} = G^{MN}T^{loc}_{MN} = 2\kappa_{10}^2 T_p(p+1)\delta_{\Sigma_i},$$
$$G^{\mu\nu}T^{loc}_{\mu\nu} = 2\kappa_{10}^2 T_p D\delta_{\Sigma_i},$$
$$G^{mn}T^{loc}_{mn} = 2\kappa_{10}^2 T_p(p+1-D)\delta_{\Sigma_i}.$$

$$R_{10} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{4}e^{-\phi}|H_3|^2 + \frac{5-q}{16}\sum_q e^{\frac{5-q}{2}\phi}|F_q|^2 + \frac{1}{8}e^{\frac{p-3}{4}\phi}T^{loc}$$

$$R_{MN} - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{2}e^{-\phi}\left(|H_3|^2_{MN} - \frac{1}{4}G_{MN}|H_3|^2\right)$$
$$-\frac{1}{4}\sum_q e^{\frac{5-q}{2}\phi}\left(|F_q|^2_{MN} - \frac{q-1}{8}G_{MN}|F_q|^2\right) + \frac{1}{2}e^{\frac{p-3}{4}\phi}\left(T^{loc}_{MN} - \frac{1}{8}G_{MN}T^{loc}\right)$$

$$R_D = -\frac{D}{8}e^{-\phi}|H_3|^2 - \sum_p \frac{D(q-1)}{32}e^{\frac{5-q}{2}\phi}|F_q|^2 - \kappa_{10}^2 T_p D\left(\frac{7-p}{8}\right)e^{\frac{p-3}{4}\phi}\delta_{\Sigma_i}$$

$$R_{mn} - \frac{1}{2}\partial_m\phi\partial_n\phi - \frac{1}{2}e^{-\phi}\left(|H_3|^2_{mn} - \frac{1}{4}G_{mn}|H_3|^2\right)$$
$$-\frac{1}{4}\sum_q e^{\frac{5-q}{2}\phi}\left(|F_q|^2_{mn} - \frac{q-1}{8}G_{mn}|F_q|^2\right) + \frac{1}{2}e^{\frac{p-3}{4}\phi}\left(T^{loc}_{mn} - \frac{1}{8}G_{mn}T^{loc}\right)$$

$$R_{(10-D)} = \frac{1}{2}(\partial_m\phi)^2 + \frac{2+D}{8}e^{-\phi}|H_3|^2 + \sum_q\frac{8q-(10-D)(q-1)}{32}e^{\frac{5-q}{2}\phi}|F_q|^2$$
$$+\kappa_{10}^2 T_p\left(\frac{8D-(D-2)(p+1)}{8}\right)e^{\frac{p-3}{4}\phi}\delta_{\Sigma_i}$$

$$\left|\frac{\int R_D}{\int R_{(10-D)}}\right| = \frac{\frac{D}{8}k + \sum_p \frac{D(q-1)}{16} f_q + T_p D\left(\frac{7-p}{8}\right) s_p}{\frac{1}{2}\sigma + \frac{2+D}{8}k + \sum_q \frac{8q-(10-D)(q-1)}{16} f_q + T_p\left(\frac{8D-(D-2)(p+1)}{8}\right) s_p},$$

$$k = \int e^{-\phi}|H_3|^2, f_q = \int e^{\frac{5-q}{2}\phi}\left|F_q\right|^2, s_p = \kappa_{10}^2 \int e^{\frac{p-3}{4}\phi}\delta_{\Sigma_i}, \sigma = \int (\partial_m\phi)^2$$

$$M_d = \mathbb{R}^d/\Gamma, \Gamma \subset \mathrm{SO}(d) \ltimes \mathbb{R}^d$$

$$\Gamma \ni \gamma = (R, \vec{b}), R \in \mathrm{SO}(d), \vec{b} \in \mathbb{R}^d$$

$$\gamma = \begin{pmatrix} R & \vec{b} \\ 0 & 1 \end{pmatrix}$$

$$\Lambda = \Gamma \cap \mathbb{R}^d$$

$$r: \mathrm{O}(d) \ltimes \mathbb{R}^d \to \mathrm{O}(d)$$

$$\Lambda^* \equiv \left\{\vec{k}^* \in \mathbb{R}^d \mid \vec{v} \cdot \vec{k}^* \in \mathbb{Z}, \forall \vec{v} \in \Lambda\right\}$$

$$\Lambda^* \ni \vec{k}^* = n_a \vec{e}_a^*, n_a \in \mathbb{Z}$$

$$\left\|\vec{k}^*\right\|^2 = n_a n_b G_{ab}$$

$$G = A^{-1}(A^{-1})^T$$

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}$$

$$\mu: \mathrm{Spin}(d) \to \mathrm{SO}(d),$$

$$\varepsilon: \Gamma \to \mathrm{Spin}(d)$$

$$r = \mu \circ \varepsilon,$$

$$
\begin{tikzcd}
\Gamma \arrow[r, "\varepsilon"] \arrow[dr, "r"'] & \mathrm{Spin(d)} \arrow[d, "\mu"] \\
& \mathrm{SO(d)}
\end{tikzcd}
$$

$$\varepsilon(e_a) = \delta_a \mathbb{1}, \delta_a = \pm 1.$$

$$\psi(\vec{y} + \vec{e}_a) = \delta_a \psi(\vec{y}),$$

$$\psi(\gamma(y)) = \varepsilon(\gamma)\psi(y), \forall \gamma \in \Gamma.$$

$$\alpha = \begin{pmatrix} R_\alpha & \vec{b}_\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\dfrac{2\pi}{3} & \sin\dfrac{2\pi}{3} & 0 & 0 \\ -\sin\dfrac{2\pi}{3} & \cos\dfrac{2\pi}{3} & 0 & 0 \\ 0 & 0 & 1 & \dfrac{L_3}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & -\dfrac{L}{2} \\ 0 & 1 & 0 & \dfrac{\sqrt{3}L}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma = \langle e_1, e_2, \alpha \mid e_1 e_2 = e_2 e_1, \alpha e_1 \alpha^{-1} = e_2^{-1} e_1^{-1}, \alpha e_2 \alpha^{-1} = e_1 \rangle$$

$$A = \begin{pmatrix} L & -\dfrac{L}{2} & 0 \\ 0 & \dfrac{\sqrt{3}L}{2} & 0 \\ 0 & 0 & L_3 \end{pmatrix}$$

$$e_a^* = \begin{pmatrix} \mathbb{1} & \vec{e}_a^{\,*} \\ 0 & 1 \end{pmatrix}$$

$$\vec{e}_1^{\,*} = \begin{pmatrix} \dfrac{1}{L} \\ \dfrac{1}{\sqrt{3}L} \\ 0 \end{pmatrix}, \vec{e}_2^{\,*} = \begin{pmatrix} 0 \\ \dfrac{2}{\sqrt{3}L} \\ 0 \end{pmatrix}, \vec{e}_3^{\,*} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_3} \end{pmatrix}$$

$$R_\alpha \vec{e}_1 = -\vec{e}_1 - \vec{e}_2, R_\alpha \vec{e}_2 = \vec{e}_1, R_\alpha \vec{e}_3 = \vec{e}_3$$

$$R_\alpha \vec{e}_1^* = -\vec{e}_2^*, R_\alpha \vec{e}_2^* = \vec{e}_1^* - \vec{e}_2^*, R_\alpha \vec{e}_3^* = \vec{e}_3^*$$

$$\varepsilon(\alpha) = -\delta_3 \exp\left(\frac{2\pi}{3}\frac{1}{2}\sigma_1\sigma_2\right) = -\delta_3 \begin{pmatrix} e^{\frac{i\pi}{3}} & 0 \\ 0 & e^{-\frac{i\pi}{3}} \end{pmatrix},$$

$$\varepsilon(e_1) = \mathbb{1}, \varepsilon(e_2) = \mathbb{1}, \varepsilon(e_3) = \delta_3 \mathbb{1},$$

$$\varepsilon(\alpha)^3 = \varepsilon(e_3)$$

$$\varepsilon(\alpha)\varepsilon(e_1)\varepsilon(\alpha)^{-1} = \varepsilon(e_1)^{-1}\varepsilon(e_2)^{-1},$$
$$\varepsilon(\alpha)\varepsilon(e_2)\varepsilon(\alpha)^{-1} = \varepsilon(e_1).$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & \frac{L_1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} -1 & 0 & 0 & \frac{L_1}{2} \\ 0 & 1 & 0 & \frac{L_2}{2} \\ 0 & 0 & -1 & \frac{L_3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma = \langle \alpha, \beta \mid \alpha^2\beta^2 = \beta^2\alpha^2, \beta\alpha\beta\alpha^2 = \alpha^2\beta\alpha\beta, \alpha\beta\alpha\beta^2 = \beta^2\alpha\beta\alpha \rangle,$$

$$e_1 = (\mathbb{1}_3, \vec{e}_1) = \alpha^2, e_2 = (\mathbb{1}_3, \vec{e}_2) = \beta^2, e_3 = (\mathbb{1}_3, \vec{e}_3) = (\beta\alpha)^2,$$

$$A = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix}$$

$$R_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_{\alpha\beta} = R_{\beta\alpha} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{cases} y^1 \sim y^1 + \frac{L_1}{2}, \\ y^2 \sim -y^2, \\ y^3 \sim -y^3, \end{cases} \begin{cases} y^1 \sim -y^1 + \frac{L_1}{2}, \\ y^2 \sim y^2 + \frac{L_2}{2}, \\ y^3 \sim -y^3 + \frac{L_3}{2}, \end{cases} \begin{cases} y^1 \sim -y^1, \\ y^2 \sim -y^2 + \frac{L_2}{2}, \\ y^3 \sim y^3 + \frac{L_3}{2}, \end{cases}$$

$$D =]0, \frac{L_1}{2}[\times]0, \frac{L_2}{2}[\times]0, \frac{L_3}{2}[\cup] -\frac{L_1}{2}, 0[\times] -\frac{L_2}{2}, 0[\times] -\frac{L_3}{2}, 0[$$

$$\varepsilon(\alpha) = s_1 \exp\left(\pi \frac{1}{2}\sigma_2\sigma_3\right) = i s_1 \sigma_1$$
$$\varepsilon(\beta) = s_2 \exp\left(\pi \frac{1}{2}\sigma_3\sigma_1\right) = i s_2 \sigma_2$$

$$
\begin{aligned}
\psi\left(y^1+\frac{L_1}{2},-y^2,-y^3\right) &= is_1\sigma_1\psi(y^1,y^2,y^3),\\
\psi\left(-y^1+\frac{L_1}{2},y^2+\frac{L_2}{2},-y^3+\frac{L_3}{2}\right) &= is_2\sigma_2\psi(y^1,y^2,y^3),\\
\psi\left(-y^1,-y^2+\frac{L_2}{2},y^3+\frac{L_3}{2}\right) &= -is_1s_2\sigma_3\psi(y^1,y^2,y^3),\\
\psi(y^1+L_1,y^2,y^3) &= -\psi(y^1,y^2,y^3),\\
\psi(y^1,y^2+L_2,y^3) &= -\psi(y^1,y^2,y^3),\\
\psi(y^1,y^2,y^3+L_3) &= -\psi(y^1,y^2,y^3),
\end{aligned}
$$

$$\boxtimes\, Y_0 = \Box\, Y_0 = \partial^i\partial_i Y_0 = \delta^{ij}\partial_i\partial_j Y_0$$

$$\boxtimes\, Y_{i_1\ldots i_p} = (p+1)\partial^j\partial_{[j}Y_{i_1\ldots i_p]}$$

$$\boxtimes\, Y_{i_1\ldots i_p} = \epsilon_{i_1\ldots i_p}{}^{j_1\ldots j_{p+1}}\partial_{j_1}Y_{j_2\ldots j_{p+1}}$$

$$\boxtimes\, Y_{ij} = 3\partial^k\partial_{(k}Y_{ij)}$$

$$\boxtimes\, \Xi = \gamma^i\partial_i\Xi$$

$$\boxtimes\, \Xi_i = \gamma_i{}^{jk}\partial_j\Xi_k,$$

$$Y_{\vec{k}^*} = \frac{1}{|r(\Gamma)|}\sum_{\gamma\in\Gamma/\Lambda}\exp\left[2\pi i\vec{k}^*\cdot\left(R_\gamma\vec{y}+\vec{b}_\gamma\right)\right], \vec{k}^* = n_a\vec{e}_a^{\,*}\in\Lambda^*$$

$$\Box\, Y_{\vec{k}^*} = -4\pi^2\left\|\vec{k}^*\right\|^2 Y_{\vec{k}^*} \equiv -M_{\vec{k}^*}^2 Y_{\vec{k}^*}$$

$$
\begin{aligned}
Y_{n_1,n_2,n_3} &= \frac{1}{|r(\Gamma)|}\sum_{\gamma\in\Gamma/\Lambda}\exp\left[2\pi i\vec{k}^*\cdot\left(R_\gamma\vec{y}+\vec{b}_\gamma\right)\right]\\
&= \frac{1}{3}\sum_{\gamma\in\{e_3,\alpha,\alpha^2\}}\exp\left[2\pi i\vec{k}^*\cdot\left(R_\gamma\vec{y}+\vec{b}_\gamma\right)\right]\\
&= \frac{1}{3}\left[e^{2\pi i\left(n_1\frac{y^1}{L}+\frac{n_1+2n_2}{\sqrt{3}}\frac{y^2}{L}+n_3\frac{y^3}{L_3}\right)}+e^{2\pi i\left(-(n_1+n_2)\frac{y^1}{L}+\frac{n_1-n_2}{\sqrt{3}}\frac{y^2}{L}+n_3\left(\frac{y^3}{L_3}+\frac{1}{3}\right)\right)}\right.\\
&\quad\left.+e^{2\pi i\left(n_2\frac{y^1}{L}-\frac{2n_1+n_2}{\sqrt{3}}\frac{y^2}{L}+n_3\left(\frac{y^3}{L_3}+\frac{2}{3}\right)\right)}\right]
\end{aligned}
$$

$$\vec{k}^* = n_1\vec{e}_1^{\,*}+n_2\vec{e}_2^{\,*}+n_3\vec{e}_3^{\,*} = \begin{pmatrix}\frac{n_1}{L}\\ \frac{n_1+2n_2}{\sqrt{3}L}\\ \frac{n_3}{L_3}\end{pmatrix}$$

$$n_1,n_2,n_3\in\mathbb{Z}, n_1\geq 1, n_2 > -n_1.$$

$$Y_{0,0,n_3} = \frac{1}{3}e^{2\pi i n_3\frac{y^3}{L_3}}\left(1+e^{2\pi i\frac{n_3}{3}}+e^{2\pi i\frac{2n_3}{3}}\right)$$

$$Y_{(p)\vec{k}^*} = \sum_{\gamma\in\Gamma/\Lambda} \left(R_\gamma d\vec{y}\right)^{i_1} \otimes \dots \left(R_\gamma d\vec{y}\right)^{i_p} c_{i_1\dots i_p} \exp\left[2\pi i\vec{k}^* \cdot \left(R_\gamma \vec{y} + \vec{b}_\gamma\right)\right]$$

$$\boxtimes\, Y_{(p)\vec{k}^*} = -4\pi^2\left\|\vec{k}^*\right\|^2 Y_{(p)\vec{k}^*} \equiv -M_{\vec{k}^*}^2 Y_{(p)\vec{k}^*}$$

$$d_{M_{\vec{k}^*}^2} = \frac{1}{|r(\Gamma)|} \sum_{\gamma\in\Gamma/\Lambda} \mathrm{tr}_p\left(R_\gamma\right) \sum_{\vec{k}^*\in\Lambda^*|R_\gamma\vec{k}^*=\vec{k}^*} e^{2\pi i\vec{k}^*\cdot\vec{b}_\gamma}$$

$$\exp\left(2\pi i n_3 \frac{y^3}{L_3}\right) dy^3,\ \text{ for }\ n_3 \in 3\mathbb{Z}$$

$$\exp\left(2\pi i n_3 \frac{y^3}{L_3}\right) \left(dy^1 + i dy^2\right),\ \text{ for }\ n_3 \in 3\mathbb{Z}+1, \text{ or } n_3 \in 3\mathbb{Z}+2$$

$$\Lambda_\varepsilon^* = \Lambda^* + \vec{a}_\varepsilon^*$$

$$e^{2\pi i\vec{a}_\varepsilon^*\cdot\vec{e}_a} = \delta_a$$

$$\Xi_{\vec{k}_\varepsilon^*} = \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right][\varepsilon(\gamma)]^{-1}\psi_0, \vec{k}_\varepsilon^* \in \Lambda_\varepsilon^*$$

$$\vec{k}_\varepsilon^* = \vec{k}^* + \vec{a}_\varepsilon^* = n_{\varepsilon a}\vec{e}_a^* = \left(n_a + \frac{1}{4}(1-\delta_a)\right)\vec{e}_a^*$$

$$\partial\Xi_{\vec{k}_\varepsilon^*} = \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right]\left(2\pi i n_{\varepsilon a}\vec{e}_{ai}^* R_j^i \gamma^j\right)[\varepsilon(\gamma)]^{-1}\psi_0$$

$$[\varepsilon(\gamma)]^{-1}\gamma^i[\varepsilon(\gamma)] = R^i{}_j\gamma^j,$$

$$\partial\Xi_{\vec{k}_\varepsilon^*} = \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right][\varepsilon(\gamma)]^{-1}\left(2\pi i n_{\varepsilon a}\vec{e}_{ai}^*\gamma^i\right)\psi_0$$

$$\Xi_{\vec{k}_\varepsilon^*,\pm} = \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right][\varepsilon(\gamma)]^{-1}\left(1 \pm \frac{n_{\varepsilon a}\vec{e}_{ai}^*}{\left\|\vec{k}_\varepsilon^*\right\|}\gamma^i\right)\psi_0$$

$$\begin{aligned}\partial\Xi_{\vec{k}_\varepsilon^*,\pm} &= \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right]\left(2\pi i n_{\varepsilon b}\vec{e}_{bj}^* R_k^j\gamma^k\right)[\varepsilon(\gamma)]^{-1}\left(1 \pm \frac{n_{\varepsilon a}\vec{e}_{ai}^*}{\left\|\vec{k}_\varepsilon^*\right\|}\gamma^i\right)\psi_0 \\ &= \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right][\varepsilon(\gamma)]^{-1}\left(2\pi i n_{\varepsilon b}\vec{e}_{bj}^*\gamma^j\right)\left(1 \pm \frac{n_{\varepsilon a}\vec{e}_{ai}^*}{\left\|\vec{k}_\varepsilon^*\right\|}\gamma^i\right)\psi_0 \\ &= \pm 2\pi i\left\|\vec{k}_\varepsilon^*\right\|\Xi_{\vec{k}^*,\pm}\end{aligned}$$

$$\Xi_{\vec{k}_\varepsilon^*}^{3/2} = \sum_{\gamma\in\Gamma/\Lambda} R_\gamma{}^i{}_j dx^j c_i \exp\left[2\pi i\vec{k}_\varepsilon^* \cdot \left(R_\gamma\vec{y} + \vec{b}_\gamma\right)\right][\varepsilon(\gamma)]^{-1}\psi_0, \vec{k}_\varepsilon^* \in \Lambda_\varepsilon^*$$

$$\dim(\ker\partial\!\!\!/) = \frac{1}{|r(\Gamma)|}\sum_{R\in r(\Gamma)} \chi(R),$$

$$V_1 = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\sum_I (-1)^{F_I} N_I \sum_{\vec{k}_I^*} \log\left(p^2 + M_{\vec{k}_I^*}^2\right)$$

$$V_1 = -\frac{d}{ds}\left[\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\sum_I\sum_{\vec{k}_I^*} (-1)^{F_I} N_I \left(p^2 + M_{\vec{k}_I^*}^2\right)^{-s}\right]_{s=0}$$

$$V_1 = -\frac{1}{2^{D+1}\pi^{D/2}}\sum_I\sum_{\vec{k}_I^*} (-1)^{F_I} N_I \frac{d}{ds}\left[\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)} M_{\vec{k}_I^*}^{D-2s}\right]_{s=0}$$

$$Z_\Lambda[s,\vec{a},\vec{b}^*] = \sum_{\vec{z}\in\Lambda} \frac{e^{-2\pi i\vec{z}\cdot\vec{b}^*}}{\|\vec{z}+\vec{a}\|^s}$$

$$Z_{\mathbb{Z}}[s,a,0] = 2\zeta(s,a)$$

$$V_1 = -\frac{\pi^{D/2}}{2}\sum_I (-1)^{F_I} N_I \frac{d}{ds}\left[\frac{\Gamma\left(s-\frac{D}{2}\right)}{(2\pi)^{2s}\Gamma(s)|r(\Gamma_I)|} Z_{\Lambda_I^*}(-D+2s,\vec{a}_I^*,0)\right]_{s=0}$$

$$e^{-2\pi i\vec{a}^*\cdot\vec{b}} \frac{\Gamma\left(\frac{d-x}{2}\right)}{\pi^{\frac{d-x}{2}}} Z_{\Lambda^*}[d-x,\vec{a}^*,-\vec{b}] = \sqrt{\Delta}\frac{\Gamma(x/2)}{\pi^{x/2}} Z_\Lambda[x,\vec{b},\vec{a}^*]$$

$$V_1 = -\frac{\sqrt{\Delta}}{2\pi^{\frac{D+d}{2}}}\sum_I (-1)^{F_I} N_I \frac{d}{ds}\left[\frac{\Gamma\left(\frac{D+d}{2}-s\right)}{2^{2s}\Gamma(s)|r(\Gamma_I)|} Z_{\Lambda_I}(D+d-2s,0,\vec{a}_I^*)\right]_{s=0}$$

$$\begin{aligned} V_1 &= -\frac{\sqrt{\Delta}\,\Gamma\left(\frac{D+d}{2}\right)}{2\pi^{\frac{D+d}{2}}}\sum_I (-1)^{F_I}\frac{N_I}{|r(\Gamma_I)|} Z_{\Lambda_I}(D+d,0,\vec{a}_I^*) \\ &= -\frac{\pi^{D/2}}{2}\Gamma(-D/2)\sum_I (-1)^{F_I}\frac{N_I}{|r(\Gamma_I)|} Z_{\Lambda_I^*}(-D,\vec{a}_I^*,0) \end{aligned}$$

$$\mathrm{Str}\mathcal{M}^{2p} \equiv \sum_I (-1)^{F_I} N_I M_{\vec{k}_I^*}^{2p}$$

$$V_1 = -\frac{1}{2^{D+1}\pi^{D/2}}\sum_{\vec{k}^*}\frac{d}{ds}\left[\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)}\mathrm{Str}^{D-2s}\right]_{s=0}$$

$$V_{\mathrm{red}} = \frac{1}{32\pi^2}\mathrm{Str}\mathcal{M}_0^2\mu^2 + \frac{1}{64\pi^2}\mathrm{Str}\left[\mathcal{M}_0^4\log\frac{\mathcal{M}_0^2}{\mu^2}\right]$$

$$V_1 = -L\frac{\Gamma\left(\frac{11}{2}\right)}{2\pi^{\frac{11}{2}}}\frac{128}{2}\left[Z_{L\mathbb{Z}}(11{,}0{,}0) - Z_{L\mathbb{Z}}(11{,}0{,}1/(2L))\right]$$
$$= -\frac{\pi^5}{2}64\Gamma(-5)\left[Z_{\mathbb{Z}/L}(-10{,}0{,}0) - Z_{\mathbb{Z}/L}(-10{,}1/(2L){,}0)\right]$$
$$= -\frac{\pi^5}{2L^{10}}64\Gamma(-5)(2\zeta(-10) - 2\zeta(-10{,}1/2))$$
$$= -\frac{2^{-4}\pi^5}{L^{10}}(2^{11}-1)\Gamma(-5)\zeta(-10) = -\frac{\Gamma\left(\frac{11}{2}\right)}{2^4 L^{10}\pi^{\frac{11}{2}}}(2^{11}-1)\zeta(11),$$

$$V_1 = -L\frac{\Gamma\left(\frac{5}{2}\right)}{2\pi^{\frac{5}{2}}}\frac{8}{2}\left[Z_{L\mathbb{Z}}(5{,}0{,}0) - Z_{L\mathbb{Z}}(5{,}0{,}1/(2L))\right]$$
$$= -\frac{\pi^2}{2}4\Gamma(-2)\left[Z_{\mathbb{Z}/L}(-4{,}0{,}0) - Z_{\mathbb{Z}/L}(-4{,}1/(2L){,}0)\right]$$
$$= -\frac{\pi^2}{2L^4}4\Gamma(-2)(2\zeta(-4) - 2\zeta(-4{,}1/2)) = -\frac{\pi^2}{4L^4}(2^5-1)\Gamma(-2)\zeta(-4)$$
$$= -\frac{\Gamma\left(\frac{5}{2}\right)}{4L^4\pi^{\frac{5}{2}}}(2^5-1)\zeta(5) = -\frac{3}{\pi^2 L^4}\frac{31}{16}\zeta(5),$$

$$\zeta(5) - \mathrm{Li}_5(-1) = \frac{31}{16}\zeta(5).$$

$$\partial^i\left(g_{ij} - \frac{1}{3}\delta_{ij}g_k^k\right) = 0 = \partial^i g_{\mu i} = \partial^i A_i = \partial^i B_{\mu i} = \partial^i B_{ij} = \partial^i C_{\mu ij} = \partial^i C_{ijk},$$

$$\Gamma^i\psi_i(x,y) = 0,$$

$$C_{ijk}(x,y) = \epsilon_{ijk}c^I(x)Y^I(y).$$

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}_2 \otimes \sigma_1, \Gamma^i = \mathbb{1}_8 \otimes \sigma^i \otimes \sigma^2.$$

$$\lambda_{10} = \begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix}.$$

state	$M \neq 0$	$M = 0$	state	$M \neq 0$	$M = 0$
$g_{\mu\nu}$	20	14	$c_{\mu\nu\rho}$	20	10
ψ_μ	20	16	χ	4	4
a_μ	6	5	ϕ	1	1
$b_{\mu\nu}$	15	10			

field	spin	origin field	harmonic	field	spin	origin field	harmonic
$h_{\mu\nu}$	2	$g_{\mu\nu}$	Y_0	ψ^A	1/2	ψ_i	$\Xi_{3/2}$
ψ_μ^A	3/2	ψ_μ	$\Xi_{1/2}$	η^A	1/2	ψ_i	$\Xi_{1/2}$
a_μ	1	A_μ	Y_0	χ^A	1/2	χ	$\Xi_{1/2}$
h_μ	1	$g_{\mu i}$	Y_1	ϕ	0	ϕ	Y_0
b_μ	1	$B_{\mu i}$	Y_1	g	0	g_{ij}	Y_{sym}
c_μ	1	$C_{\mu ij}$	$\star\, dY_0$	h	0	g_{ij}	Y_0
$b_{\mu\nu}$	tensor	$B_{\mu\nu}$	Y_0	a	0	A_i	Y_1
$c_{\mu\nu}$	tensor	$C_{\mu\nu i}$	Y_1	b	0	B_{ij}	$\star\, dY_0$
$c_{\mu\nu\rho}$	tensor	$C_{\mu\nu\rho}$	Y_0	c	0	C_{ijk}	$\mathfrak{I}$

$$\left\|\vec{k}^*\right\|^2 = \left[\frac{n_1^2}{L^2} + \frac{(n_1+2n_2)^2}{3L^2} + \frac{n_3^2}{L_3^2}\right], (n_1, n_2, n_3 \in \mathbb{Z}, n_1 \geq 1, n_2 > -n_1),$$

harmonic	degeneracy	$L_3^2\left\|\vec{k}^*\right\|^2$
Y_0	1	$9n_3^2$
$\Xi_{1/2}$	2	$(3n_3+1)^2$
Y_1	2	$(3n_3+1)^2$
$\Xi_{3/2}$	2	$9n_3^2$
Y_{sym}	2	$(3n_3+1)^2$

state	dofs	$[n_3]-$ level
$g_{\mu\nu}$	20	$\lvert 3[n_3]\rvert$
ψ_μ	20	$2\times\lvert 3[n_3]+1\rvert, 2\times\lvert 3[-n_3]+1\rvert,$
a_μ	6	$2\times\lvert 3[n_3]\rvert\oplus 2\times\lvert 3[n_3-1]+1\rvert\oplus 2\times\lvert 3[-n_3-1]+1\rvert$
$b_{\mu\nu}$	15	$\lvert 3[n_3]\rvert\oplus\lvert 3[n_3-1]+1\rvert\oplus\lvert 3[-n_3-1]+1\rvert$
$c_{\mu\nu\rho}$	20	$\lvert 3[n_3]\rvert$
χ	4	$2\times\lvert 3[n_3+1]\rvert\oplus 2\times\lvert 3[-n_3+1]\rvert\oplus 4\times\lvert 3[n_3]+1\rvert\oplus 4\times\lvert 3[-n_3]+1\rvert$
ϕ	1	$3\times\lvert 3[n_3]\rvert\oplus\lvert 3[-n_3+1]+1\rvert\oplus\lvert 3[n_3-1]+1\rvert\oplus\lvert 3[-n_3-1]+1\rvert$ $\oplus\lvert 3[n_3+1]+1\rvert$

$$\mathrm{Str}(\mathcal{M}^2)=\mathrm{Str}(\mathcal{M}^4)=\mathrm{Str}(\mathcal{M}^6)=0,$$

state	dofs	(n_1', n_2', n_3')
$g_{\mu\nu}$	20	(n_1, n_2, n_3)
ψ_μ	20	$2\times\left(n_1,n_2,n_3+\frac{1}{2}\right), 2\times\left(n_1,n_2,n_3-\frac{1}{2}\right),$
a_μ	6	$2\times(n_1,n_2,n_3)\oplus 2\times(n_1,n_2,n_3+1)\oplus 2\times(n_1,n_2,n_3-1)$
$b_{\mu\nu}$	15	$(n_1,n_2,n_3)\oplus(n_1,n_2,n_3+1)\oplus(n_1,n_2,n_3-1)$
$c_{\mu\nu\rho}$	20	(n_1, n_2, n_3)
χ	4	$2\times\left(n_1,n_2,n_3+\frac{1}{2}\right)\oplus 2\times\left(n_1,n_2,n_3-\frac{1}{2}\right)\oplus 4\times\left(n_1,n_2,n_3+\frac{3}{2}\right)$ $\oplus 4\times\left(n_1,n_2,n_3-\frac{3}{2}\right)$

ϕ	1	$3\times(n_1,n_2,n_3)\oplus(n_1,n_2,n_3+1)\oplus(n_1,n_2,n_3-1)\oplus(n_1,n_2,n_3+2)$ $\oplus(n_1,n_2,n_3-2)$

$$\mathrm{Str}(\mathcal{M}^8)=40320\frac{\pi^8}{L_3^8},$$

$$V_1=-\frac{54\cdot 3^7\pi^{\frac{7}{2}}}{2L_3^7}\Gamma\left(-\frac{7}{2}\right)[2\zeta(-7)-(\zeta(-7,1/3)+\zeta(-7,2/3))]$$
$$=-\frac{162}{\pi^4L_3^7}[\zeta(8,1/3)+\zeta(8,2/3)]=-\frac{162}{\pi^4L_3^7}(3^8-1)\zeta(8)=-\frac{3936}{35}\frac{\pi^4}{L_3^7}$$

$$\mathrm{Str}\mathcal{M}^8=40320\frac{\pi^8}{L_3^8},$$

$$V_1^{(I)}=-\frac{\sqrt{3}}{2}L^2L_3\frac{\Gamma(5)}{2\pi^5}\frac{128}{3}\Big[Z_\Lambda(10,\vec{0},\vec{0})-Z_\Lambda\left(10,\vec{0},(0,0,1/(2L_3))\right)$$
$$-\frac{2}{L_3^{10}}(\zeta(10)+\eta(10))\Big]$$

$$V_1^{(II)}=-\frac{3^7\pi^{\frac{7}{2}}}{2L_3^7}\Gamma(-7/2)(140\zeta(-7)+58(\zeta(-7,1/3)+\zeta(-7,2/3))$$
$$-112(\zeta(-7,1/6)+\zeta(-7,5/6))-32\zeta(-7,1/2))=-\frac{197}{105}\frac{\pi^4}{L_3^7}$$

$$\left\|\vec{k}^*\right\|^2=\left[\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right]$$

state	dofs	(n_1',n_2',n_3')
$g_{\mu\nu}$	20	(n_1,n_2,n_3)
ψ_μ	20	$2\times\left(n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}\right),2\times\left(n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2}\right),$
a_μ	6	$2\times(n_1,n_2,n_3),2\times(n_1+1,n_2+1,n_3+1),2\times(n_1-1,n_2-1,n_3-1)$
$b_{\mu\nu}$	15	$(n_1,n_2,n_3),(n_1+1,n_2+1,n_3+1),(n_1-1,n_2-1,n_3-1)$
$c_{\mu\nu\rho}$	20	(n_1,n_2,n_3)

χ	4	$2\times\left(n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}\right),2\times\left(n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2}\right),$ $4\times\left(n_1+\frac{3}{2},n_2+\frac{3}{2},n_3+\frac{3}{2}\right),4\times\left(n_1-\frac{3}{2},n_2-\frac{3}{2},n_3-\frac{3}{2}\right)$
ϕ	1	$3\times(n_1,n_2,n_3),(n_1+1,n_2+1,n_3+1),(n_1-1,n_2-1,n_3-1),$ $(n_1+2,n_2+2,n_3+2),(n_1-2,n_2-2,n_3-2)$

$$\mathrm{Str}\mathcal{M}^8 = 40320\frac{\pi^8(L_1^2L_2^2+L_2^2L_3^2+L_1^2L_3^2)^4}{L_1^8L_2^8L_3^8}$$

$$V_1 = -\frac{384}{\pi^5L^7}\left(Z_{\mathbb{Z}^3}(10,0,0,0)-Z_{\mathbb{Z}^3}\left(10,0,\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\right)-6(\zeta(10)+\eta(10))\right)$$

$$\begin{aligned} |+2\rangle&:\vec{0},\\ |+3/2,i\rangle = Q_i|+2\rangle&:\vec{q_i},\\ |+1,[ij]\rangle = Q_iQ_j|+2\rangle&:\vec{q_i}+\vec{q_j},\\ |+1/2,[ijk]\rangle = Q_iQ_jQ_k|+2\rangle&:\vec{q_i}+\vec{q_j}+\vec{q_k},\\ |0,[ijkl]\rangle = Q_iQ_jQ_kQ_l|+2\rangle&:\vec{q_i}+\vec{q_j}+\vec{q_k}+\vec{q_l},\end{aligned}$$

$$\sum_{i=1}^{8}\vec{q}_i=\vec{0}$$

$$\begin{aligned} |2\rangle&:M^2=\vec{n}^2,\\ |3/2,i\rangle&:M_i^2=(\vec{n}+\vec{q}_i)^2,\\ |1,[ij]\rangle&:M_{ij}^2=\left(\vec{n}+\vec{q}_i+\vec{q}_j\right)^2,\\ |1/2,[ijk]\rangle&:M_{ijk}^2=\left(\vec{n}+\vec{q}_i+\vec{q}_j+\vec{q}_k\right)^2,\\ |0,[ijkl]\rangle&:M_{ijkl}^2=\left(\vec{n}+\vec{q}_i+\vec{q}_j+\vec{q}_k+\vec{q}_l\right)^2,\end{aligned}$$

$$\vec{M}_a=\{\vec{n},\vec{n}+\vec{q}_i,\vec{n}+\vec{q}_i+\vec{q}_j,\vec{n}+\vec{q}_i+\vec{q}_j+\vec{q}_k,\vec{n}+\vec{q}_i+\vec{q}_j+\vec{q}_k+\vec{q}_l\}$$

$$\left(\vec{M}_a\right)^2=\sum_{A=1}^{n}(M_a^A)^2\mu_A^2$$

$$\mathrm{Str}\mathcal{M}^{2p}=\mathrm{Tr}\left[\mathcal{M}_{(0)}^2\right]^p-2\mathrm{Tr}\left[\mathcal{M}_{(1/2)}^2\right]^p+2\mathrm{Tr}\left[\mathcal{M}_{(1)}^2\right]^p-2\mathrm{Tr}\left[\mathcal{M}_{(3/2)}^2\right]^p+\mathrm{Tr}\left[\mathcal{M}_{(2)}^2\right]^p.$$

$$\mathrm{Str}\mathcal{M}^2=\mathrm{Str}\mathcal{M}^4=\mathrm{Str}\mathcal{M}^6=0,$$

$$\mathrm{Str}\mathcal{M}^8=\sum_a\epsilon_a\left(\vec{M}_a^2\right)^k=\sum_a\epsilon_a\left[\sum_A(M_a^A)^2\mu_A^2\right]^k.$$

$$\mathrm{Str}\mathcal{M}^8=40320\left(\prod_{i=1}^{8}q_i\right)\mu^8.$$

$$\mathrm{Str}\mathcal{M}^8 = 40320\left(\prod_{i=1}^{8} q_i\right)\left(\sum_A \mu_A^2\right)^4.$$

$$e^a = dy^i e_i^a = \begin{pmatrix} \cos(qz)dy^1 - \sin(qz)dy^2 \\ \sin(qz)dy^1 + \cos(qz)dy^2 \\ dy^3 \end{pmatrix}$$

$$e^1 + ie^2 = \exp(iqz)(dy^1 + idy^2), e^3 = dy^3$$

$$A(x,y) = dy^i a_i(x) + e^a(y)a_a(x) = a_7(x) + e^a(y)a_a(x)$$

$$de^1 = qe^2 \wedge e^3, de^2 = -qe^1 \wedge e^3, de^3 = 0$$

$$e^a(y)a_a(x) = dy^i e_i^a(y)a_a(x) = dy^i Y_i^I(y)a_I(x)$$

$$ds_3^2 = e^a(y) \otimes e^b(y) g_{ab}(x)$$

$$ds_3^2 = dy^i \otimes dy^j S_I(x) Y_{ij}^I(y)$$

$$D_i g_{ij} = \partial_i g_{ij} - g_{ik}\tau_{jk}^k g_i{}^k - g_{jk}\tau_{ik}^k g_i{}^k,$$

$$\begin{aligned} D_i g_{31} &= \partial_i g_{31} + q g_i^2 \\ D_i g_{32} &= \partial_i g_{32} - q g_i^1 \end{aligned}$$

$$\mathcal{L}_{1/2} = \bar{\psi}\sigma^a e_a^i \left(\partial_i + \frac{1}{4}\omega_i^{bc}\sigma_b\sigma_c\right)\psi,$$

$$de^a + \omega_b^a e^b = 0$$

$$\omega_3^{12} = q$$

$$\mathcal{L}_{1/2} = \bar{\psi}\left[\sigma^3\partial_3\psi + (\cos(qy^3)\sigma^1 + \sin(qy^3)\sigma^2)\partial_1\psi + (\cos(qy^3)\sigma^2 - \sin(qy^3)\sigma^1)\partial_2\psi + \frac{i}{2}q\psi\right]$$

$$\begin{aligned} \mathcal{L}_{1/2} &= \bar{\psi}\sigma^3\partial_3\psi + e^{\left(-\frac{i}{2}qy^3\sigma^3\right)}\sigma^1 e^{\left(\frac{i}{2}qy^3\sigma^3\right)}\partial_1\psi + e^{\left(-\frac{i}{2}qy^3\sigma^3\right)}\sigma^2 e^{\left(\frac{i}{2}qy^3\sigma^3\right)}\partial_2\psi + \frac{i}{2}q\psi \\ &= \bar{\psi}e^{\left(-\frac{i}{2}qy^3\sigma^3\right)}\sigma^i\partial_i\left[e^{\left(\frac{i}{2}qy^3\sigma^3\right)}\psi\right] \end{aligned}$$

$$\Xi = e^{\left(\frac{i}{2}qy^3\sigma^3\right)}\psi$$

$$\Xi \to e^{i\pi(kp+1)}\Xi$$

$$p = 2\pi\left(k\mathbb{Z} + \frac{k+1}{2}\right)$$

$$p = 2\pi\left(k\mathbb{Z} + \frac{1}{2}\right)$$

$$\mathcal{L}_{3/2} = \epsilon^{abc} e_b{}^i \bar{\psi}_a \left(\partial_i + \frac{1}{4}\omega_i{}^{ef}\sigma_e\sigma_f\right)\psi_c + \epsilon^{abc} e_b{}^i \omega_{ic}{}^d \bar{\psi}_a\psi_d$$

$$\partial_3\psi_w + iq\left(\mathbb{1}+\frac{\sigma_3}{2}\right)\psi_w = 0$$

$$\partial_3\left(e^{iq\left(\mathbb{1}+\frac{\sigma_3}{2}\right)y^3}\psi_w\right) = 0$$

$$\Xi_w = e^{iq\left(\mathbb{1}+\frac{\sigma_3}{2}\right)y^3}\psi_w,$$

$$Z_\Lambda[s,\vec{a},\vec{b}^*] = \sum_{\vec{z}\in\Lambda}\frac{e^{-2\pi i\vec{b}^*\cdot\vec{z}}}{\|\vec{z}+\vec{a}\|^s}$$

$$Z_\Lambda[s,-\vec{a},\vec{b}^*] = Z_\Lambda[s,\vec{a},-\vec{b}^*]$$

$$Z_\Lambda[s,-\vec{a},-\vec{b}^*] = Z_\Lambda[s,\vec{a},\vec{b}^*]$$

$$e^{-2\pi i\vec{a}^*\cdot\vec{b}}\frac{\Gamma\left(\frac{d-x}{2}\right)}{\pi^{\frac{d-x}{2}}}Z_{\Lambda^*}[d-x,\vec{a}^*,-\vec{b}] = \sqrt{\Delta}\frac{\Gamma(x/2)}{\pi^{x/2}}Z_\Lambda[x,\vec{b},\vec{a}^*]$$

$$Z_\Lambda[-2n,\vec{a},\vec{b}^*] = 0, \forall n\in\mathbb{N}$$

$$Z_\Lambda[0,\vec{a},\vec{b}^*] = 0$$

$$Z_\Lambda[0,\vec{a},\vec{b}^*] = -\exp\left(-2\pi i\vec{a}\cdot\vec{b}^*\right)$$

$$Z_{\mathbb{Z}}[s,a,0] = 2\zeta(s,a),$$

$$\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s)$$

$$\sum_{p=1}^{n}\zeta(s,p/n) = (n^s-1)\zeta(s)$$

$$S = \int d\tau\left[P^a\partial_\tau X_a + p_\alpha\partial_\tau\theta^\alpha + w_\alpha\partial_\tau\lambda^\alpha - \frac{1}{2}P^2\right]$$

$$Q\Psi = 0 \to D_{(\alpha}A_{\beta\delta\epsilon)} = (\gamma^a)_{(\alpha\beta}A_{a\delta\epsilon)}$$

$$\delta\Psi = Q\Lambda \to \delta A_{\alpha\beta\delta} = D_{(\alpha}\Lambda_{\beta\delta)}$$

$$\begin{aligned}U^{(3)} =& -\frac{3}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)\epsilon^{b_2b_3} - \frac{1}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)c^{b_1b_2b_3}\\ &+\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3}\Psi^{b_2}) - \frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3})\\ &+O(\theta^5)\end{aligned}$$

$$\partial^d\partial_{[d}c_{abc]} = 0, \Box\,\epsilon_{bc} - 2\partial^a\partial_{(b}\epsilon_{c)a} + \partial_b\partial_c\left(\eta^{ad}\epsilon_{ad}\right) = 0, \left(\gamma^{abc}\right)_{\alpha\beta}\partial_b\Psi_c^\beta = 0,$$

$$\delta c_{abc} = \partial_{[a}s_{bc]}, \delta\epsilon_{ab} = \partial_{(a}t_{b)}, \delta\Psi_a^\alpha = \partial_a\kappa^\beta$$

$$\mathcal{D}T^A = E^B R_B^A \,, \mathcal{D}R_A^B = 0$$

$$\mathcal{D}\mathcal{F}_{A_1\dots A_m}{}^{B_1\dots B_n} = d\mathcal{F}_{A_1\dots A_m}{}^{B_1\dots B_n} + \Omega_{A_1}^C \mathcal{F}_{CA_2\dots A_m}^{B_1\dots B_n} + \cdots$$
$$-\mathcal{F}_{A_1\dots A_m}{}^{C\dots B_n}\Omega_C^{B_1} - \cdots$$

$$\Omega_{A\beta}^{\delta} = \frac{1}{4}\left(\gamma^{bc}\right)_{\alpha}^{\beta}\Omega_{Abc}$$
$$R_{AB,\alpha}^{\beta} = \frac{1}{4}\left(\gamma^{cd}\right)_{\alpha}^{\beta}R_{AB,cd}$$

$$U^{(1)} = \lambda^\alpha\left(P_a h_\alpha{}^a + d_\beta h_\alpha{}^\beta - \Omega_{\alpha\beta}{}^\delta \lambda^\beta w_\delta\right),$$

$$\lambda^\alpha\lambda^\beta P_a\left[D_\alpha h_\beta^a - h_\alpha^\delta(\gamma^a)_{\beta\delta}\right] = 0$$
$$\lambda^\alpha\lambda^\beta d_\delta\left[D_\alpha h_\beta^\delta - \Omega_{\alpha\beta}^\delta\right] = 0$$
$$\lambda^\alpha\lambda^\beta\lambda^\delta w_\epsilon R_{\alpha\beta,\delta}^\epsilon = 0$$

$$[\nabla_A, \nabla_B\} = -T_{AB}^C\nabla_C - 2\Omega_{[AB\}}^C\nabla_C$$
$$R_{AB,C}{}^D = 2\nabla_{[A}\Omega_{B\}C}{}^D + T_{AB}{}^F\Omega_{FC}{}^D - \Omega_{[A|C|}{}^F\Omega_{B\}F}{}^D$$

$$\nabla_A = D_A - h_A^B D_B$$

$$2D_{(\alpha}h_{\beta)}^a - 2h_{(\alpha}^\delta(\gamma^a)_{\beta)\delta} + h_b^a\left(\gamma^b\right)_{\alpha\beta} = 0$$
$$2D_{(\alpha}h_{\beta)}^\delta - 2\Omega_{(\alpha\beta)}^\delta + (\gamma^a)_{\alpha\beta}h_a^\delta = 0$$
$$\partial_a h_\alpha^\beta - D_\alpha h_a^\beta - T_{a\alpha}^\beta - \Omega_{a\alpha}^\beta = 0$$
$$\partial_a h_\alpha^b - D_\alpha h_a^b - h_a^\beta\left(\gamma^b\right)_{\beta\alpha} + \Omega_{\alpha a}^b = 0$$
$$\partial_a h_b^\alpha - \partial_b h_a^\alpha - T_{ab}^\alpha = 0$$
$$\partial_a h_b^c - \partial_b h_a^c - 2\Omega_{[ab]}^c = 0$$

$$R_{(\alpha\beta,\delta)}{}^\epsilon + (\gamma^a)_{(\alpha\beta}T_{a\delta)}{}^\epsilon = 0,$$
$$R_{(\alpha\beta),b}{}^c + 2(\gamma^c)_{\gamma(\beta}T_{|b|\alpha)}{}^\gamma = 0,$$
$$R_{(\alpha\beta),c}{}^d - 2D_{(\alpha}\Omega_{\beta)c}{}^d - (\gamma^a)_{\alpha\beta}\Omega_{ac}{}^d = 0,$$

$$T_{a\alpha}{}^\beta = \left(\mathcal{T}_a{}^{bcde}\right)_\alpha{}^\beta H_{bcde},$$

$$\left(\mathcal{T}_a{}^{bcde}\right)_\alpha{}^\beta = \frac{1}{36}\left[\delta_a^{[b}\left(\gamma^{cde]}\right)_\alpha{}^\beta + \frac{1}{8}\left(\gamma_a{}^{bcde}\right)_\alpha{}^\beta\right]$$

$$R_{\alpha\beta,b}^c = \left(\mathcal{R}_b{}^{cdefg}\right)_{\alpha\beta}H_{defg},$$

$$\left(\mathcal{R}_{bc}{}^{defg}\right)_{\alpha\beta} = \frac{1}{6}\left[\delta_b^{[d}\delta_c^e\left(\gamma^{fg]}\right)_{\alpha\beta} + \frac{1}{24}\left(\gamma_{bc}{}^{defg}\right)_{\alpha\beta}\right].$$

$$\Omega_{abc} = \partial_{[a}h_{b]c} - \partial_{[a}h_{c]b} + \partial_{[c}h_{b]a}$$

$$H_{ABCD} = \hat{E}_{[D}{}^Q\hat{E}_C{}^P\hat{E}_B{}^N\hat{E}_{A\}}{}^M G_{MNPQ}$$

$$
\begin{aligned}
4D_{(\alpha}C_{\beta\delta\epsilon)} + 6(\gamma^{a})_{(\alpha\beta}C_{a\delta\epsilon)} &= 0 \\
\partial_a C_{\alpha\beta\delta} - 3D_{(\alpha}C_{a\beta\delta)} + 3\left(\gamma^{b}\right)_{(\alpha\beta}C_{ba\delta)} &= -3(\gamma_{ab})_{(\alpha\beta}h^{b}_{\delta)} \\
2\partial_{[a}C_{b]\alpha\beta} + 2D_{(\alpha}C_{\beta)ab} + (\gamma^{c})_{\alpha\beta}C_{cab} &= 2\left(\gamma^{c}_{[b}\right)_{\alpha\beta}h_{a]c} - 2(\gamma_{ab})_{\delta(\alpha}h^{\delta}_{\beta)} \\
3\partial_{[a}C_{bc]\alpha} - D_\alpha C_{abc} &= -3\left(\gamma_{[ab}\right)_{\alpha\beta}h^{\beta}_{c]}
\end{aligned}
$$

$$
\delta h_\alpha{}^{a} = D_\alpha \Lambda^{a} + (\gamma^{a})_{\alpha\beta}\Lambda^{\beta}\,, \delta h_\alpha{}^{\beta} = D_\alpha\Lambda^{\beta} + \Lambda_\alpha{}^{\beta}, \delta\Omega_{\alpha\beta}{}^{\epsilon} = D_\alpha\Lambda_\beta{}^{\epsilon}
$$
$$
\delta h_a{}^{b} = \partial_a\Lambda^{b} + \Lambda_a{}^{b}\,, \delta h_a{}^{\beta} = \partial_a\Lambda^{\beta}, \delta\Omega_{a\alpha}{}^{\beta} = \partial_a\Lambda_\alpha{}^{\beta},
$$

$$
\begin{aligned}
\delta C_{\alpha\beta\epsilon} &= D_{(\alpha}\Lambda_{\beta\epsilon)} + (\gamma^{a})_{(\alpha\beta}\Lambda_{a\epsilon)} \\
\delta C_{a\alpha\epsilon} &= \frac{1}{3}\partial_a\Lambda_{\alpha\epsilon} + \frac{2}{3}D_{(\alpha}\Lambda_{\epsilon)a} + \frac{1}{3}\left(\gamma^{b}\right)_{\alpha\epsilon}\Lambda_{ba} + (\gamma_{ab})_{\alpha\epsilon}\Lambda^{b} \\
\delta C_{ab\alpha} &= \frac{2}{3}\partial_{[a}\Lambda_{b]\alpha} + \frac{1}{3}D_\alpha\Lambda_{ab} - (\gamma_{ab})_{\alpha\beta}\Lambda^{\beta} \\
\delta C_{abc} &= \partial_{[a}\Lambda_{bc]}
\end{aligned}
$$

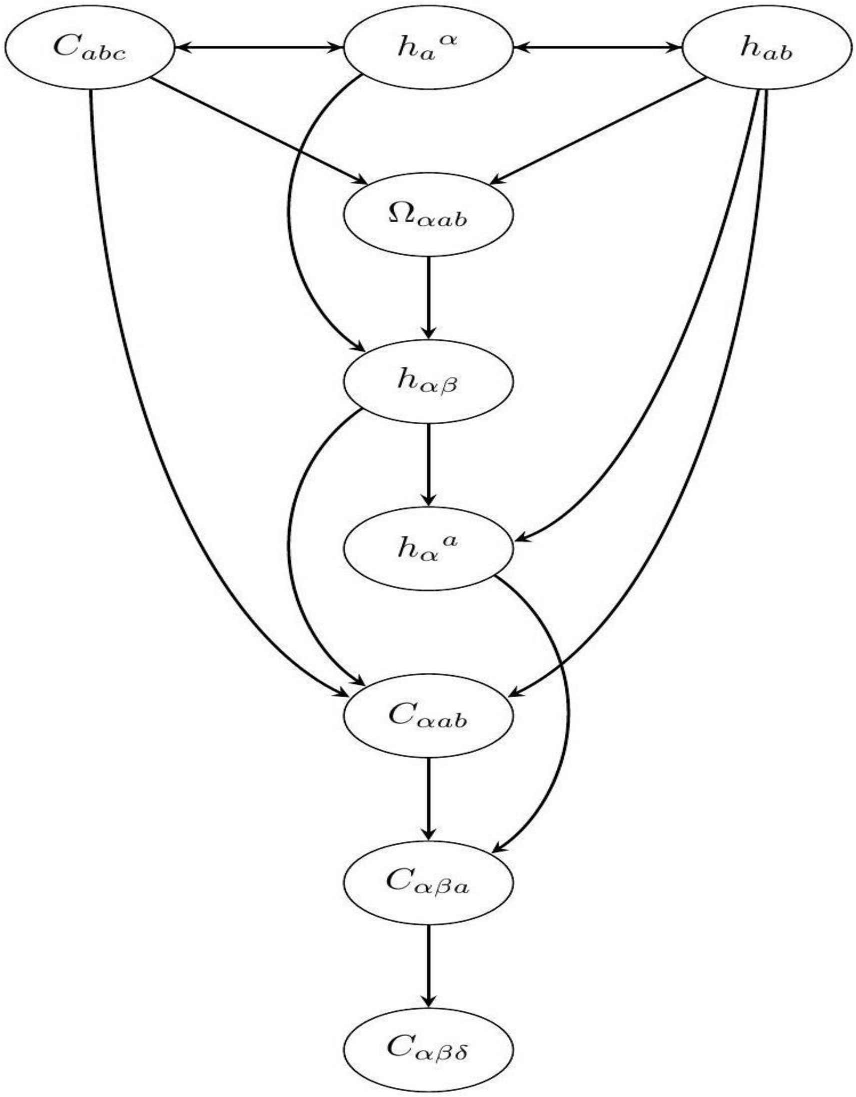
$$
\theta^{\alpha}h_\alpha{}^{A} = 0\,, \theta^{\alpha}\Omega_{\alpha A}{}^{B} = 0\,, \theta^{\alpha}C_{\alpha AB} = 0
$$

$$
\begin{aligned}
(D+3)C_{\beta\delta\epsilon} + 3\theta^{\alpha}(\gamma^{a})_{\alpha(\beta}C_{a\delta\epsilon)} &= 0 \\
(D+2)C_{a\beta\delta} - 2\theta^{\alpha}\left(\gamma^{b}\right)_{\alpha(\beta}C_{ba\delta)} &= 2\theta^{\alpha}(\gamma_{ab})_{\alpha(\beta}h^{b}_{\delta)} \\
(D+1)C_{\beta ab} + \theta^{\alpha}(\gamma^{c})_{\alpha\beta}C_{cab} &= 2\theta^{\alpha}\left(\gamma^{c}_{[b}\right)_{\alpha\beta}h_{a]c} - \theta^{\alpha}(\gamma_{ab})_{\alpha\delta}h^{\delta}_{\beta} \\
DC_{abc} &= 3\left(\theta\gamma_{[ab}\right)_{\beta}h^{\beta}_{c]}
\end{aligned}
$$

$$
\begin{aligned}
(D+1)h^{a}_{\beta} - h^{\delta}_{\beta}(\theta\gamma^{a})_{\delta} + h^{a}_{b}\left(\theta\gamma^{b}\right)_{\beta} &= 0, \\
(D+1)h^{\delta}_{\beta} - \frac{1}{4}\left(\theta\gamma^{bc}\right)^{\delta}\Omega_{\beta bc} + (\theta\gamma^{a})_{\beta}h^{\delta}_{a} &= 0, \\
Dh_a{}^{\beta} + 4\theta^{\alpha}\left(\mathcal{T}^{bcde}_{a}\right)_{\alpha}{}^{\beta}\partial_b C_{cde} + \frac{1}{4}\left(\theta\gamma^{bc}\right)^{\beta}\Omega_{abc} &= 0, \\
Dh_a{}^{b} + \left(\theta\gamma^{b}\right)_{\beta}h_a{}^{\beta} &= 0,
\end{aligned}
$$

$$
(1+D)\Omega_{\beta c}{}^{d} = \left(\theta\mathcal{R}_c{}^{defgl}\right)_{\beta}H_{efgl} - (\theta\gamma^{a})_{\beta}\Omega_{ac}{}^{d} = 0.
$$

$$
C_{abc} = c_{abc} + \mathcal{O}(\theta), h^{\alpha}_{a} = -\Psi^{\alpha}_{a} + \mathcal{O}(\theta), h_{ab} = -\epsilon_{ab} + \mathcal{O}(\theta)
$$



$$
\begin{aligned}
h_a^{\beta}\big|_{\theta^n} &= -\frac{4}{n}\theta^{\alpha}\left(\mathcal{T}_a^{bcde}\right)_{\alpha}^{\beta}\partial_b C_{cde}\Big|_{\theta^{n-1}} - \frac{1}{4n}\left(\theta\gamma^{bc}\right)^{\beta}\Omega_{abc}\Big|_{\theta^{n-1}}, \\
h_a^{b}\big|_{\theta^n} &= -\frac{1}{n}\left(\theta\gamma^{b}\right)_{\beta} h_a^{\beta}\Big|_{\theta^{n-1}}, \\
C_{abc}\big|_{\theta^n} &= \frac{3}{n}\theta^{\alpha}\left(\gamma_{[ab}\right)_{\alpha\beta} h_{c]}^{\beta}\Big|_{\theta^{n-1}}.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{\alpha c}{}^{d}\big|_{\theta^n} &= \frac{4}{n+1}\left(\theta\mathcal{R}_c{}^{defgl}\right)_{\beta}\partial_e C_{fgl}\Big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^{a})_{\alpha}\Omega_{ac}{}^{d}\Big|_{\theta^{n-1}}, \\
h_{\beta}{}^{\delta}\big|_{\theta^n} &= \frac{1}{4(n+1)}\left(\theta\gamma^{bc}\right)^{\delta}\Omega_{\beta bc}\Big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^{a})_{\beta} h_a{}^{\delta}\Big|_{\theta^{n-1}}, \\
h_{\beta}{}^{a}\big|_{\theta^n} &= \frac{1}{n+1}(\theta\gamma^{a})_{\delta} h_{\beta}{}^{\delta}\Big|_{\theta^{n-1}} - \frac{1}{n+1}\left(\theta\gamma^{b}\right)_{\beta} h_b{}^{a}\Big|_{\theta^{n-1}}, \\
C_{\beta ab}\big|_{\theta^n} &= \frac{2}{n+1}\left(\theta\gamma_{[b}{}^{c}\right)_{\beta} h_{a]c}\Big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^{c})_{\beta} C_{cab}\Big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma_{ab})_{\delta} h_{\beta}{}^{\delta}\Big|_{\theta^{n-1}} \\
C_{\alpha\beta\delta}\big|_{\theta^n} &= \frac{2}{n+2}\left(\theta\gamma^{b}\right)_{(\beta} C_{\delta)ba}\Big|_{\theta^{n-1}} + \frac{2}{n+2}(\theta\gamma_{ab})_{(\beta} h_{\delta)}{}^{b}\Big|_{\theta^{n-1}}, \\
C_{\beta\delta\epsilon}\big|_{\theta^n} &= -\frac{3}{n+3}(\theta\gamma^{a})_{(\beta} C_{\delta\epsilon)}\Big|_{\theta^{n-1}},
\end{aligned}
$$

$$h^{a_1a_2}|_{\theta^0} = -\epsilon^{a_1a_2}$$
$$h^{a_1a_2}|_{\theta^1} = +(\theta\gamma^{a_2}\Psi^{a_1})$$
$$h^{a_1a_2}|_{\theta^2} = +\frac{1}{4}(\theta\gamma^{a_2b_1k}\theta)\epsilon^{a_1b_1}$$
$$-\frac{1}{2}(\theta\mathcal{T}^{a_1b_1b_2b_3b_4}\gamma^{a_2}\theta)h^{b_1b_2b_3b_4}$$
$$h^{a_1a_2}|_{\theta^3} = -\frac{1}{24}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{a_1}\Psi^{b_1})$$
$$-\frac{1}{24}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{b_1}\Psi^{a_1})$$
$$+\frac{1}{24}(\theta\gamma^{a_2b_1b_2}\theta)(\theta\gamma^{b_1}\Psi^{b_2})k^{a_1}$$
$$-2(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3})$$
$$h^{a_1a_2}|_{\theta^4} = -\frac{1}{192}(\theta\gamma^{a_1b_1k}\theta)(\theta\gamma^{a_2b_2k}\theta)\epsilon^{b_1b_2}$$
$$-\frac{1}{192}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{b_1b_2k}\theta)\epsilon^{a_1b_2}$$
$$+\frac{1}{192}(\theta\gamma^{a_2b_1b_2}\theta)(\theta\gamma^{b_1b_3k}\theta)\epsilon^{b_2b_3}k^{a_1}$$
$$-\frac{1}{4}(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\gamma^{b_1b_2b_4k}\theta)\epsilon^{b_3b_4}$$
$$-\frac{1}{2}(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta)h^{b_4b_5b_6b_7}$$
$$+\frac{1}{96}(\theta\mathcal{T}^{a_1b_1b_2b_3b_4}\gamma^{b_5}\theta)(\theta\gamma^{a_2b_5k}\theta)h^{b_1b_2b_3b_4}$$
$$+\frac{1}{96}(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{a_1}\theta)(\theta\gamma^{a_2b_1k}\theta)h^{b_2b_3b_4b_5}$$
$$+\frac{1}{96}(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{b_6}\theta)(\theta\gamma^{a_2b_1b_6}\theta)h^{b_2b_3b_4b_5}k^{a_1}$$
$$+\mathcal{O}(\theta^5)$$

$$h^{\alpha}_{a_1}\big|_{\theta^0} = -\psi^{\alpha}_{a}$$

$$h^{\alpha}_{a_1}\big|_{\theta^1} = +\frac{1}{2}\left(\gamma^{b_1k}\theta\right)^{\alpha}\epsilon^{a_1b_1}$$
$$-\left(\theta\mathcal{T}^{a_1b_1b_2b_3b_4}\right)^{\alpha}h^{b_1b_2b_3b_4}$$

$$h^{\alpha}_{a_1}\big|_{\theta^2} = -\frac{1}{8}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\gamma^{a_1}\Psi^{b_1}\right)$$
$$-\frac{1}{8}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\gamma^{b_1}\Psi^{a_1}\right)$$
$$+\frac{1}{8}\left(\gamma^{b_1b_2}\theta\right)^{\alpha}\left(\theta\gamma^{b_1}\Psi^{b_2}\right)k^{a_1}$$
$$-6\left(\theta\mathcal{T}^{a_1kb_1b_2b_3}\right)^{\alpha}\left(\theta\gamma^{b_1b_2}\Psi^{b_3}\right)$$

$$h^{\alpha}_{a_1}\big|_{\theta^3} = -\frac{1}{48}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\gamma^{a_1b_2k}\theta\right)\epsilon^{b_1b_2}$$
$$-\frac{1}{48}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\gamma^{b_1b_2k}\theta\right)\epsilon^{a_1b_2}$$
$$+\frac{1}{48}\left(\gamma^{b_1b_2}\theta\right)^{\alpha}\left(\theta\gamma^{b_1b_3k}\theta\right)\epsilon^{b_2b_3}k^{a_1}$$
$$+\frac{1}{24}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\mathcal{T}^{a_1b_2b_3b_4b_5}\gamma^{b_1}\theta\right)h^{b_2b_3b_4b_5}$$
$$+\frac{1}{24}\left(\gamma^{b_1k}\theta\right)^{\alpha}\left(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{a_1}\theta\right)h^{b_2b_3b_4b_5}$$
$$+\frac{1}{24}\left(\gamma^{b_1b_2}\theta\right)^{\alpha}\left(\theta\mathcal{T}^{b_1b_3b_4b_5b_6}\gamma^{b_2}\theta\right)h^{b_3b_4b_5b_6}k^{a_1}$$
$$-\left(\theta\mathcal{T}^{a_1kb_1b_2b_3}\right)^{\alpha}\left(\theta\gamma^{b_1b_2b_4k}\theta\right)\epsilon^{b_3b_4}$$
$$-2\left(\theta\mathcal{T}^{a_1kb_1b_2b_3}\right)^{\alpha}\left(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta\right)h^{b_4b_5b_6b_7}$$
$$+\mathcal{O}(\theta^4)$$

$$C_{\lambda\lambda\lambda}\big|_{\theta^3} = -\frac{3}{8}\left(\lambda\gamma^{b_1}\theta\right)\left(\lambda\gamma^{b_2}\theta\right)\left(\lambda\gamma^{b_1b_3}\theta\right)\epsilon^{b_2b_3}$$

$$-\frac{1}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)c^{b_1b_2b_3}$$

$$C_{\lambda\lambda\lambda}|_{\theta^4} = +\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3}\Psi^{b_2})$$

$$-\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3})$$

$$C_{\lambda\lambda\lambda}|_{\theta^5} = +\frac{1}{32}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3b_4k}\theta)\epsilon^{b_2b_4}$$

$$-\frac{1}{32}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2b_4k}\theta)\epsilon^{b_3b_4}$$

$$-\frac{11}{192}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\mathcal{T}^{b_2b_4b_5b_6b_7}\gamma^{b_3}\theta)h^{b_4b_5b_6b_7}$$

$$-\frac{11}{192}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta)h^{b_4b_5b_6b_7}$$

$$-\frac{1}{1536}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\mathcal{R}^{b_1b_2b_3b_4b_5b_6}\theta)(\theta\theta)h^{b_3b_4b_5b_6}$$

$$+\frac{1}{3072}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_1b_2}\theta)(\lambda\mathcal{R}^{b_3b_4b_5b_6b_7b_8}\theta)(\theta\gamma^{b_2b_3b_4}\theta)h^{b_5b_6b_7b_8}$$

$$+\frac{1}{3072}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\mathcal{R}^{b_3b_4b_5b_6b_7b_8}\theta)(\theta\gamma^{b_1b_2b_3b_4}\theta)h^{b_5b_6b_7b_8}$$

$$+\mathcal{O}(\theta^6)$$

$$h_{abcd} = 4\partial_{[a}c_{bcd]}$$

$$S = \int d\tau[P^a\partial_\tau X_a + p_\alpha\partial_\tau\theta^\alpha + \rho\mathcal{A} + \xi_\alpha\mathcal{B}^\alpha + \iota^{\alpha\beta}\mathcal{C}_{\alpha\beta}]$$

$$\mathcal{A} = P^aP_a\,,\mathcal{B}^\alpha = (\gamma^a d)^\alpha P_a\,,\mathcal{C}_{\alpha\beta} = d_{[\alpha}d_{\beta]}$$

$$\{\mathcal{B}^\alpha,\mathcal{B}^\beta\} = -(\gamma^a)^{\alpha\beta}P_a\mathcal{A},[\mathcal{C}_{\alpha\beta},\mathcal{C}_{\delta\epsilon}] = -4(\gamma^a)_{\underline{\beta}\bar{\delta}}P_a\mathcal{C}_{\underline{\alpha}\bar{\epsilon}}\,,[\mathcal{B}^\alpha,\mathcal{C}_{\beta\delta}] = 2\delta^\alpha_{[\delta}d_{\beta]}\mathcal{A}$$

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$$

$$\gamma^i_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i_{A\dot{A}} \\ -\sigma^i_{\dot{B}B} & 0 \end{pmatrix},\gamma^+_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -i\sqrt{2}I_{\dot{B}\dot{A}} \end{pmatrix},\gamma^-_{\alpha\beta} = \begin{pmatrix} -i\sqrt{2}I_{AB} & 0 \\ 0 & 0 \end{pmatrix}$$

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -I_{\dot{A}A} \\ I_{B\dot{B}} & 0 \end{pmatrix}$$

$$X^+ = x^+ + \tau P^+\,,P^- = \frac{P^iP^i}{2P^+}$$

$$(\gamma^+\theta)_\alpha = 0\,,(\gamma^-d)^\alpha = \frac{1}{P^+}\left[-(\gamma^+d)^\alpha P^- + \left(\gamma^i d\right)^\alpha P_i\right]$$

$$\{d_\alpha,d_\beta\} = -(\gamma^a)_{\alpha\beta}P_a\,,\{d_\alpha,q_\beta\} = 0\,,\{q_\alpha,q_\beta\} = (\gamma^a)_{\alpha\beta}P_a$$

$$\{d_A,d_B\} = -\delta_{AB}\,,\{d_A,q_B\} = 0\,,\{q_A,q_B\} = \delta_{AB}$$

$$d_A \to \sqrt{\sqrt{2}iP^+}d_A$$

$$\delta d_A = m_A^B d_B$$

$$V = P^a P^b h_{ab} + P^a h_a^\alpha d_\alpha$$

$$\epsilon_a^+ = 0, \Psi^{\alpha +} = 0, c_{ab}^+ = 0$$

$$\epsilon_{ij} k^i = 0, \epsilon_i{}^i = 0$$

$$c_{iab} k^i = 0.$$

$$(\theta \gamma^{a+b} \theta) = \frac{1}{3}(\theta \gamma^a \gamma^+ \gamma^b \theta) = 0$$

$$(\theta \gamma^{a+bc} \theta) = 0$$

$$V|_{\epsilon^{--}} = -P^+ P^+ \epsilon^{--}$$

$$(\theta \gamma^{i_1 \dots i_n} \theta) \propto (\theta \gamma^{i_1 \dots i_n} \gamma^+ \gamma^- \theta) = 0$$

$$\theta^\alpha = \frac{1}{2P^+} (\gamma^+)^{\alpha\beta} (q_\beta - d_\beta)$$

$$\begin{aligned} V|_{\epsilon^{i-}} &= \epsilon^{i-} P_i P^+ - \frac{1}{16}(q\gamma^{+ik} q)\epsilon^{i-} + \frac{1}{16}(d\gamma^{+ik_1} d)\epsilon^{i-} \\ &= \epsilon^{i-} P_i P^+ - \frac{1}{16}(q\gamma^{+ik} q)\epsilon^{i-} \end{aligned}$$

$$R^{ij} = \frac{1}{16P^+}(q\gamma^{+ij} q)$$

$$[R^{ij}, R_{lk}] = \delta_{[k}^{[i} \delta_{l]}^{j]}$$

$$V|_{\epsilon^{i-}} = \epsilon^{i-} P_i P^+ - R^{ik} P^+ \epsilon^{i-}$$

$$\begin{aligned} V|_{\epsilon^{ij}} = &-\epsilon_{ij} P^i P^j + \frac{1}{8}(q\gamma^{+ik} q)\epsilon_{ij} P^j (P^+)^{-1} \\ &+ \frac{1}{768}(q\gamma^{+aik} q)(q\gamma^{+ajk} q)\epsilon_{ij}(P^+)^{-2} - \frac{1}{768}(q\gamma^{+ik} q)(q\gamma^{+jk} q)\epsilon_{ij}(P^+)^{-2} \end{aligned}$$

$$q_A q_B = q^2 \delta_{AB} + \frac{1}{32}(\sigma_{ij})_{AB}(q\sigma^{ij} q) + \frac{1}{96}(\sigma_{ijl})_{AB}(q\sigma^{ijl} q)$$

$$(q\gamma^{+(i|ja} q)(q\gamma^{+|l)jb} q) k_a k_b h_{il} = 5(q\gamma^{+(i|a} q)(q\gamma^{+|l)b} q) k_a k_b h_{il},$$

$$\begin{aligned} V|_{\epsilon^{ij}} &= -\epsilon_{ij} P^i P^j + \frac{1}{8}(q\gamma^{+ik} q)\epsilon_{ij} P^j (P^+)^{-1} - \frac{1}{128}(q\gamma^{+ik} q)(q\gamma^{+jk} q)\epsilon_{ij}(P^+)^{-2} \\ &= -\epsilon_{ij} P^i P^j + 2R^{ik}\epsilon_{ij} P^j - 2R^{ik} R^{jk} \epsilon_{ij} \end{aligned}$$

$$V|_{c_{ij-}} = \frac{1}{96}(q\gamma^{+ijl} q) H_{ijl}^-$$

$$V|_{c_{ijl}} = H_{i_1 i_2 i_3 i_4}\left(P^{i_1} - \frac{1}{24}(q\gamma^{+i_1 k} q)\right)\frac{1}{96}(q\gamma^{+i_2 i_3 i_4} q)$$

$$V|_{c_{ij-}} = R^{ijl}H^{-}_{ijl}$$
$$V|_{c_{ijl}} = H_{i_1i_2i_3i_4}\left(P^{i_1} - \frac{2}{3}R^{i_1k}\right)R^{i_2i_3i_4}$$

$$\gamma^0_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\gamma^9_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\gamma^{i=1,\dots,8}_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i \\ \left(\sigma^i\right)^T & 0 \end{pmatrix},$$

$$\sigma^1 = \tau^2 \otimes \tau^2 \otimes \tau^2$$
$$\sigma^2 = 1 \otimes \tau^1 \otimes \tau^2$$
$$\sigma^3 = 1 \otimes \tau^3 \otimes \tau^2$$
$$\sigma^4 = \tau^1 \otimes \tau^2 \otimes 1$$
$$\sigma^5 = \tau^3 \otimes \tau^2 \otimes 1$$
$$\sigma^6 = \tau^2 \otimes 1 \otimes \tau^1$$
$$\sigma^7 = \tau^2 \otimes 1 \otimes \tau^3$$
$$\sigma^8 = 1 \otimes 1 \otimes 1$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\gamma^{10}_{\alpha\beta} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\gamma^{i=0,\dots,9}_{\alpha\beta} = \begin{pmatrix} i\gamma^i & 0 \\ 0 & -i\gamma^i \end{pmatrix}.$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left(\gamma^{ab}\right)^{\alpha\beta} = C^{\beta\gamma}\left(\gamma^{ab}\right)^{\alpha}{}_{\gamma} = -\left(\gamma^{ab}C\right)^{\alpha\beta},$$

$$\left(\lambda\gamma^{ab}\lambda\right) = \left(\lambda\gamma^{ab}C\lambda\right)$$

$$\mathcal{A} = P^m P_m, \mathcal{B} = (\gamma^m d)_\alpha P_m, \mathcal{C}^{mnp} = (d\gamma^{mnp}d)$$

$$V = P^m A_m + d_\alpha W^\alpha$$

$$D_\alpha A_\beta + D_\beta A_\alpha = (\gamma^m)_{\alpha\beta}A_m, D_\alpha A_m = \partial_m A_\alpha + (\gamma_m W)_\alpha$$
$$D_\alpha W^\beta = -\frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}, D_\alpha F_{mn} = 2\left(\gamma_{[m}\partial_{n]}W\right)_\alpha$$

$$X^+ = x^+_0 + P^+\tau$$
$$(\gamma^+\theta)_\alpha = 0$$

$$\begin{aligned} d_\alpha &= p_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha P_m \\ q_\alpha &= p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha P_m \end{aligned}$$

$$\bar{d}_{\dot{a}} = \bar{p}_{\dot{a}} + \frac{1}{2}\left(\sigma^i\right)_{\dot{a}a}\theta^a P_i, \quad d_a = p_a - \frac{\sqrt{2}}{2}\theta_a P^+$$
$$\bar{q}_{\dot{a}} = \bar{p}_{\dot{a}} - \frac{1}{2}\left(\sigma^i\right)_{\dot{a}a}\theta^a P_i, \quad q_a = p_a + \frac{\sqrt{2}}{2}\theta_a P^+$$

$$\mathcal{C}^{+ij} = \sqrt{2}d_a\left(\sigma^{ij}\right)^{ab}d_b\,,\mathcal{C}^{ijk} = 2d_a\left(\sigma^{ijk}\right)^{a\dot{a}}\bar{d}_{\dot{a}}$$

$$S^a = \frac{q^a}{\sqrt{\sqrt{2}P^+}}$$

$$k^+ = 0, k^- = \frac{k^ik^i}{2k^+}, \epsilon^+ = 0, \epsilon^- = \frac{\epsilon^ik^i}{k^+}, \chi_a = -\frac{1}{\sqrt{2}}\left(\sigma^i\right)_{a\dot{a}}k_i\bar{\xi}^{\dot{a}}, \bar{\xi}^{\dot{a}} = \frac{\bar{\chi}^{\dot{a}}}{k^+},$$

$$\begin{aligned} A_m =& \epsilon_m - (\chi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)f_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)\left(\partial_p\chi\gamma_q\theta\right) \\ &+ \frac{1}{192}(\theta\gamma_{mrs}\theta)(\theta\gamma^{spq}\theta)\partial^r f_{pq} + O(\theta^5) \\ W^\alpha =& \chi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha f_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\chi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m f_{pq} \\ &- \frac{1}{96}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\left(\partial_m\partial_p\chi\gamma_q\theta\right) + O(\theta^5) \end{aligned}$$

$$V = -P^+A^- + P^iA^i + d_aW^a + \bar{d}_{\dot{a}}\bar{W}^{\dot{a}}$$

$$\begin{aligned} V =& -P^+\left[\epsilon^- + \frac{\sqrt{2}}{4}\left(\theta\sigma^{ij}\theta\right)k_i\epsilon_j\right] + P^i\epsilon^i + d_a\left[-\frac{1}{2}\left(\sigma^{ij}\theta\right)^a k_i\epsilon_j\right] \\ =& -P^+\epsilon^- + P^i\epsilon^i - \frac{\sqrt{2}}{4}\left(\theta\sigma^{ij}\theta\right)k_i\epsilon_jP^+ - \frac{1}{2}\left(d\sigma^{ij}\theta\right)k_i\epsilon_j \end{aligned}$$

$$\theta^a = \frac{1}{\sqrt{2}P^+}(q^a - d^a)$$

$$\begin{aligned} V =& -P^+\epsilon^- + P^i\epsilon^i - \frac{\sqrt{2}}{8P^+}\left(q\sigma^{ij}q\right)k_i\epsilon_j + \frac{\sqrt{2}}{4P^+}\left(q\sigma^{ij}d\right)k_i\epsilon_j - \frac{1}{2\sqrt{2}P^+}\left(d\sigma^{ij}q\right)k_i\epsilon_j \\ =& -P^+\epsilon^- + P^i\epsilon^i - \frac{\sqrt{2}}{8P^+}\left(q\sigma^{ij}q\right)k_i\epsilon_j \end{aligned}$$

$$V = -P^+\epsilon^- + P^i\epsilon^i - \frac{1}{4}\left(S\sigma^{ij}S\right)k_i\epsilon_j,$$

$$\hat{\mathcal{L}}_{E_A}E_B = -X_{AB}^C E_C$$

$$\hat{\mathcal{L}}_U V^I = U^J\partial_J V^I - \left(\partial_J U^I - \partial^I U_J\right)V^J$$

$$[T_A, T_B] = F_{AB}^C T_C,$$

$$U_I V^I = \eta_{IJ} U^I V^J = u_i v^i + u^i v_i$$

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \eta_{AB} = E_A{}^I E_B{}^J \eta_{IJ}$$

$$F_{abc} = H_{abc}, F_{ab}^c = f_{ab}^c, F_a^{bc} = Q_a^{bc}, \text{ and } F^{abc} = R^{abc}.$$

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix} \text{ and } R^{abc} = 0$$

$$E_A{}^I = M_A{}^B \begin{pmatrix} v_b{}^j & 0 \\ 0 & v^b{}_j \end{pmatrix} \begin{pmatrix} \delta_j^i & -B_{ji} \\ 0 & \delta_i^j \end{pmatrix}$$

$$M_A{}^B T_B = m^{-1} T_A m$$

$$T_A v^A = m^{-1} \,\mathrm{d}m = T_a v_i^a \,\mathrm{d}x^i + T^a A_{ia} \,\mathrm{d}x^i$$

$$B = \frac{1}{2!} B_{ij} \,\mathrm{d}x^i \wedge \,\mathrm{d}x^j$$

$$\hat{\mathcal{L}}_{E_A} E_B{}^I + \hat{\mathcal{L}}_{E_B} E_A{}^I = -2X_{(AB)}{}^C E_C{}^I = \partial_I (E_{AJ} E_B{}^J) = \partial_I \eta_{AB} = 0.$$

$$X_{ABC} = 3\Omega_{[ABC]}$$

$$\Omega_{AB}{}^C = E_A{}^I E_B{}^J \partial_I E^C{}_J.$$

$$E_A{}^I = M_A{}^B V_B{}^I,$$

$$\Omega_{AB}{}^C = M_A{}^D M_B{}^E (M^{-1})_F{}^C (\hat{\Omega}_{DE}{}^F - V_D{}^I (M^{-1} \partial_I M)_E{}^F),$$

$$\hat{\Omega}_{AB}{}^C := V_A{}^I V_B{}^J \partial_I V^C{}_J.$$

$$-V_A^I (M^{-1} \partial_I M)_B^C = \begin{cases} A_a^D F_{DB}^C \\ 0, \end{cases}$$

$$A_a{}^B = \begin{pmatrix} \delta_a^b \\ v_a{}^i A_{ib} \end{pmatrix}.$$

$$X_{aB}^C = \begin{cases} Q_a^{bc} \\ \hat{X}_{ab}^c + 2A_{id} v_{[a}^i Q_{b]}^{cd} + 2f_{ab}^c \\ \hat{X}_{abc} + 3A_{id} v_{[a}^i{}^i f_{bc]}^d + 3H_{abc}, \end{cases}$$

$$\hat{X}_{ABC} = 3\hat{\Omega}_{[ABC]}.$$

$$\hat{X}_{ab}^c = 2v_a^i v_b^j \partial_{[i} v_{j]}^c = -\iota_{v_a} \iota_{v_b} \,\mathrm{d}v^c$$

$$\hat{X}_{abc} = 3v_a^i v_b^j v_c^k \partial_{[i} B_{jk]}.$$

$$\mathrm{d}v^A = -\frac{1}{2} F_{BC}{}^A v^B \wedge v^C,$$

$$\hat{X}_{ab}^{c} = -f_{ab}^{c} - 2A_{id}v_{[a}^{i}Q_{b]}^{cd}.$$

$$\begin{aligned} \mathrm{d}B = H &= \frac{1}{6}(v^a H_{abc} - 3v^A F_{Abc}) \wedge v^b \wedge v^c \\ &= -\frac{1}{3}H_{abc}v^a \wedge v^b \wedge v^c - \frac{1}{2}f_{ab}^{c}v^a \wedge v^b \wedge A_c \end{aligned}$$

$$\mathrm{d}H = 0.$$

$$\frac{3}{4}F_{[AB}^{E}F_{CD]E} = J_{ABCD} = 0$$

$$\begin{aligned} J_{abcd} &= \frac{3}{2}f_{[ab}^{e}H_{cd]e} = 0, \\ J_{abc}^{d} &= \frac{3}{4}\left(H_{e[ab}Q_{c]}^{de} - f_{[ab}^{e}f_{c]e}^{d}\right) = 0, \\ J_{ab}^{cd} &= \frac{1}{4}\left(f_{ab}^{e}Q_{e}^{cd} - 4f_{e[a}^{[c}Q_{b]}^{d]e}\right) = 0, \\ J_{a}^{bcd} &= \frac{3}{4}Q_{a}^{e[b}Q_{e}^{cd]} = 0. \end{aligned}$$

$$\mathrm{d}H = -\frac{1}{2}J_{abcd}v^a \wedge v^b \wedge v^c \wedge v^d - \frac{4}{3}J_{abc}{}^{d}v^a \wedge v^b \wedge v^c \wedge A_d - J_{ab}{}^{cd}v^a \wedge v^b \wedge A_c \wedge A_d = 0$$

$$\begin{aligned} &\mathrm{E}_{n(n)} \to \mathrm{GL}(n) \\ &\mathrm{E}_{n(n)} \to \mathrm{GL}(n-1) \times \mathrm{SL}(2) \end{aligned}$$

$$\hat{\mathcal{L}}_U V^I = U^J \partial_J V^I - V^J \partial_J U^I + Y_{KL}^{IJ} \partial_J U^K V^L$$

$$Y_{KL}^{IJ} = \eta^{IJ}\eta_{KL}$$

$$Y_{KL}^{IJ} = (t^{\alpha})_L^I (t_{\alpha})_K^J + \beta \delta_L^I \delta_K^J + \delta_K^I \delta_L^J$$

	$O(D,D)$	$\mathrm{E}_{4(4)}$	$\mathrm{E}_{5(5)}$	$\mathrm{E}_{6(6)}$	$\mathrm{E}_{7(7)}$
α	2	3	4	6	12
β	0	1/5	1/4	1/3	1/2
R_1	$\mathbf{2D}$	$\overline{\mathbf{10}}$	$\mathbf{16}$	$\mathbf{27}$	$\mathbf{56}$
R_2	$\mathbf{1}$	$\mathbf{5}$	$\mathbf{10}$	$\overline{\mathbf{27}}$	$\mathbf{133}$

$$\kappa_{\alpha\beta} = -\frac{1}{\alpha}\mathrm{tr}_{R_1}\left(t_{\alpha}t_{\beta}\right).$$

$$Y^{IJ}_{KL}\partial_I \cdot \partial_J \cdot = 0$$

$$X_{AB}{}^{C} = W_{AB}{}^{C} + (t^{\boldsymbol{\alpha}})_D{}^{E}(t_{\boldsymbol{\alpha}})_B{}^{C}W_{EA}{}^{D} - \beta W_{DA}{}^{D}(t_0)_B{}^{C},$$

$$W_{AB}{}^{C} := W_A{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_B{}^{C} + W_A{}^{0}(t_0)_B{}^{C}.$$

$$Y^{AB}_{CD}W^{\dot{\boldsymbol{\alpha}}}_{A}W^{\dot{\boldsymbol{\beta}}}_{B} = 0,$$

$$\Omega_{AB}{}^{C} := E_A{}^{I}E_B{}^{J}\partial_I E_J{}^{C} \text{ where } E_I{}^{A} := (E^{-1})_I{}^{A},$$

$$W^{C}_{AB} = \Omega^{C}_{AB}\big|_{x^i=0}.$$

$$Y^{IJ}_{KL}\partial_I E_A{}^{M}\partial_J E_B{}^{N}\big|_{x^i=0} = 0.$$

$$\mathbf{56} \to \mathbf{7} \oplus \overline{\mathbf{21}} \oplus \mathbf{21} \oplus \overline{\mathbf{7}}$$

$$W^{\dot{\boldsymbol{\alpha}}}_{A} = \delta^{b}_{A}W^{\dot{\boldsymbol{\alpha}}}_{b}\ (a, b = 1, \dots, n)$$

$$\mathbf{27} \to \mathbf{6} \oplus \overline{\mathbf{15}} \oplus \overline{\mathbf{6}}$$

$$\mathbf{56} \to (\mathbf{6}, \mathbf{1}) \oplus (\overline{\mathbf{6}}, \mathbf{2}) \oplus (\mathbf{20}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2}) \oplus (\overline{\mathbf{6}}, \mathbf{1})$$

$$\partial_I = \left(\partial_{\mathrm{m}} \quad \tilde{\partial}^{\mathrm{m}}_{\mu} \quad \tilde{\partial}^{\mathrm{m}_1 \dots \mathrm{m}_3} \quad \tilde{\partial}^{\mathrm{m}_1 \dots \mathrm{m}_5}_{\mu} \quad \tilde{\partial}^{\mathrm{m}_1 \dots \mathrm{m}_6, \mathrm{m}'}\right)$$

$$W^{\dot{\alpha}}_{A} = \delta^{\mathrm{b}}_{A}W^{\dot{\alpha}}_{\mathrm{b}}\ (\mathrm{a}, \mathrm{b} = 1, \dots, n-1).$$

$$133 \to 7 \oplus \overline{35} \oplus (48+1) \oplus 35 \oplus \overline{7}$$

$$t_{\boldsymbol{\alpha}} = (R^{a_1 \dots a_6} \quad R^{a_1 \dots a_3} \quad K^{a}{}_{b} \quad R_{a_1 \dots a_3} \quad R_{a_1 \dots a_6}).$$

$$f_{a,b_1 \dots b_6}, f_{a,b_1 \dots b_3}, f_{a_1 a_2}{}^{b}, f_a{}^{b_1 \dots b_3}, f_a{}^{b_1 \dots b_6}, \text{ and } Z_a.$$

$$f_{a_1 \dots a_7}, f_{a_1 \dots a_4}, f^{b}_{a_1 a_2}, f^{b_1 \dots b_3}_{a}, f^{b_1 \dots b_6}_{a}, \text{ and } Z_a.$$

$$T_A = \left(T_a\, T^{a_1 a_2}\, T^{a_1 \dots a_5}\, T^{a_1 \dots a_7, a'}\right)$$

$$
\begin{aligned}
T_a \circ T_b &= f_{ab}{}^{c} T_c + f_{abc_2} T^{c_2} \\
T_a \circ T^{b_2} &= -f_a{}^{b_2 c} T_c + \delta^{b_2}_{de} f_{ac}{}^{d} T^{ec} + 3 Z_a T^{b_2} - f_{ac_3} T^{b_2 c_3} \\
T_a \circ T^{b_5} &= f_a{}^{b_5 c} T_c + \delta^{b_5}_{c_3 d_2} f_a{}^{c_3} T^{d_2} - \delta^{b_5}_{de_4} f_{ac}{}^{d} T^{e_4 c} + 6 Z_a T^{b_5} \\
T^{a_2} \circ T_b &= f_b{}^{a_2 c} T_c + \delta^{a_2}_{de} \delta^{f_2 e}_{c_2 b} f_{f_2}{}^{d} T^{c_2} - 3 Z_c \delta^{c a_2}_{b e_2} T^{e_2} + \delta^{a_2 c_4}_{b d_5} f_{c_4} T^{d_5} \\
T^{a_2} \circ T^{b_2} &= \delta^{b_2}_{de} f_c{}^{a_2 d} T^{ec} - \delta^{a_2}_{de} f_{c_2}{}^{d} T^{e b_2 c_2} + 3 Z_c T^{a_2 b_2 c} \\
T^{a_2} \circ T^{b_5} &= -\delta^{b_5}_{de_4} f_c{}^{a_2 d} T^{e_4 c} \\
T^{a_5} \circ T_b &= -f_b{}^{a_5 c} T_c - \delta^{a_5}_{c_3 d_2} f_b{}^{c_3} T^{d_2} - \delta^{a_5}_{d_3 b f} f_c^{d_3} T^{fc} \\
&\quad + \delta^{a_5}_{de_4} f_{bc}{}^{d} T^{e_4 c} + \delta^{a_5}_{dbf_3} f_{c2}{}^{d} T^{f_3 c_2} - 6 \delta^{c a_5}_{b d_5} Z_c T^{d_5} \\
T^{a_5} \circ T^{b_2} &= -\delta^{b_2}_{de} f_c{}^{a_5 d} T^{ec} + \delta^{a_5}_{d_3 e_2} f_c{}^{d_3} T^{e_2 b_2 c} \\
T^{a_5} \circ T^{b_5} &= \delta^{b_5}_{de_4} f_c{}^{a_5 d} T^{e_4 c}
\end{aligned}
$$

$$
\mathbf{1} \oplus \overline{\mathbf{35}} \oplus (\mathbf{140} \oplus \mathbf{7}) \oplus (\overline{\mathbf{224}} \oplus \overline{\mathbf{21}}) \oplus (\mathbf{28} \oplus \mathbf{21}) \oplus 7
$$

$$
\mathbf{133} \rightarrow 2(\mathbf{1},\mathbf{2}) \oplus 2(\mathbf{15},\mathbf{1}) \oplus 2(\overline{\mathbf{15}},\mathbf{2}) \oplus (\mathbf{35}+\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}).
$$

$$
t_{\boldsymbol{\alpha}} = \left(R_\alpha^{\mathrm{a}_1\ldots\mathrm{a}_6} \quad R^{\mathrm{a}_1\ldots\mathrm{a}_4} \quad R_\alpha^{\mathrm{a}_1\mathrm{a}_2} \quad K^{\mathrm{ab}} \quad R^{\alpha}{}_{\beta} \quad R^{\alpha}_{\mathrm{a}_1\mathrm{a}_2} \quad R_{\mathrm{a}_1\ldots\mathrm{a}_4} \quad R^{\alpha}_{\mathrm{a}_1\ldots\mathrm{a}_6}\right),
$$

$$
f^{\beta}_{\mathrm{a},\,\mathrm{b}_1\ldots\,\mathrm{b}_6}, f_{\mathrm{a},\,\mathrm{b}_1\ldots\,\mathrm{b}_4}, f^{\beta}_{\mathrm{a},\,\mathrm{b}_1\,\mathrm{b}_2}, f^{\mathrm{c}}_{\mathrm{a},\,\mathrm{b}}, f^{\gamma}_{\mathrm{a},\beta}, f_{\mathrm{a}\beta}^{\mathrm{b}_1\,\mathrm{b}_2}, f_{\mathrm{a}}^{\mathrm{b}_1\ldots\,\mathrm{b}_4}, f_{\mathrm{a}\beta}^{\mathrm{b}_1\ldots\,\mathrm{b}_6}, \text{ and } Z_{\mathrm{a}}.
$$

$$
f_{\mathrm{a}_1\ldots\mathrm{a}_5}, f^{\alpha}_{\mathrm{a}_1\ldots\mathrm{a}_3}, f^{\mathrm{b}}_{\mathrm{a}_1\mathrm{a}_2}, f^{\gamma}_{\mathrm{a}\beta}, f_{\mathrm{a}_\beta}^{\mathrm{b}_1\,\mathrm{b}_2}, f_{\mathrm{a}}^{\mathrm{b}_1\ldots\,\mathrm{b}_4}, f_{\mathrm{a}_\beta}^{\mathrm{b}_1\ldots\,\mathrm{b}_6}, \text{ and } Z_{\mathrm{a}}
$$

$$
T_A = \left(T_{\mathrm{a}} \quad T^{\mathrm{a}}_{\alpha} \quad T^{\mathrm{a}_1\ldots\mathrm{a}_3} \quad T_\alpha^{\mathrm{a}_1\ldots\mathrm{a}_5} \quad T^{\mathrm{a}_1\ldots\mathrm{a}_6,\mathrm{a}'}\right)
$$

$$T_a \circ T_b = f_{ab}{}^{c} T_c + f^{\gamma}_{abc} T^{c}_{\gamma} + f_{abc_3} T^{c_3},$$
$$T_a \circ T^{b}_{\beta} = f^{bc}_{a\beta} T_c - f^{\gamma}_{a\beta} T^{b}_{\gamma} - f_{ac}{}^{b} T^{c}_{\beta} + 2Z_a T^{b}_{\beta} - \epsilon_{\beta\gamma} f^{\gamma}_{ac_2} T^{bc_2} + f_{ac_4} T^{bc4}_{\beta},$$
$$T_a \circ T^{b_3} = f_a{}^{b_3 c} T_c - \delta^{b_3}_{c_2\, d} \epsilon^{\gamma\delta} f^{c_2}_{a\gamma} T^{d}_{\delta} - \delta^{b_3}_{de_2} f_{ac}{}^{d} T^{e_2 c} + 4Z_a T^{b_3} - f^{\gamma}_{ac_2} T^{b_3 c_2}_{\gamma},$$
$$T_a \circ T^{b_5}_{\beta} = f^{b_5 c}_{a\beta} T_c - \delta^{b_5}_{c_4\, d} f_a^{c_4} T^{d}_{\beta} + \delta^{b_5}_{c_2\, d_3} f_a^{c_2} T^{d_3}$$
$$-f_{a\beta}{}^{\gamma} T^{b_5}_{\gamma} - \delta^{b_5}_{de_4} f_{ac}{}^{d} T^{e_4 c}_{\beta} + 6Z_a T^{b_5}_{\beta},$$
$$T^{a}_{\alpha} \circ T_b = -f_b{}^{ac}_{\alpha} T_c - \delta^{ad}_{bc} f_{d\alpha}{}^{\gamma} T^{c}_{\gamma} + f_{bc}{}^{a} T^{c}_{\alpha} + 2\delta^{ac}_{bd} Z_c T^{d}_{\alpha} - \delta^{ad_3}_{bc_3} \epsilon_{\alpha\gamma} f^{\gamma}_{d_3} T^{c_3},$$
$$T^{a}_{\alpha} \circ T^{b}_{\beta} = f^{ab}_{c_\alpha} T^{c}_{\beta} + f_{c\alpha}{}^{\gamma} \epsilon_{\gamma\beta} T^{cab} + \epsilon_{\alpha\beta} f_{c_2}{}^{a} T^{c_2\, b} - 2\epsilon_{\alpha\beta} Z_c T^{abc} + \epsilon_{\alpha\gamma} f^{\gamma}_{c_3} T^{abcc_3}_{\beta},$$
$$T^{a}_{\alpha} \circ T^{b_3} = \delta^{b_3}_{de_2} f^{ad}_{c\alpha} T^{e_2 c} + f^{\gamma}_{c\alpha} T^{acb_3}_{\gamma} - f_{c_2}{}^{a} T^{c_2\, b_3}_{\alpha} + 2Z_c T^{ab_3 c}_{\alpha},$$
$$T^{a}_{\alpha} \circ T^{b_5}_{\beta} = \delta^{b_5}_{de_4} f^{ad}_{c\alpha} T^{e_4 c}_{\beta},$$
$$T^{a_3} \circ T_b = -f_b{}^{a_3 c} T_c - \delta^{a_3}_{d_2 e} \delta^{ef}_{bc} \epsilon^{\gamma\delta} f_f^{d_2} T^{c}_{\delta} + \delta^{a_3}_{de_2} f_{bc}{}^{d} T^{e_2 c}$$
$$-\delta^{a_3}_{bde} f_{c_2}{}^{d} T^{ec_2} + 4\delta^{a_3 c}_{bd_3} Z_c T^{d_3},$$
$$T^{a_3} \circ T^{b}_{\beta} = f_c^{a_3\, b} T^{c}_{\beta} - \delta^{a_3}_{d_2 e} f^{d_2}_{c\beta} T^{ebc} + \delta^{a_3}_{de_2} f_{c_2}{}^{d} T^{e_2\, bc_2}_{\beta} - 4Z_c T^{a_3 bc}_{\beta},$$
$$T^{a_3} \circ T^{b_3} = \delta^{b_3}_{de_2} f_c{}^{a_3\, d} T^{e_2 c} - \delta^{a_3}_{d_2 e} \epsilon^{\gamma\delta} f^{d_2}_{c\gamma} T^{eb_3 c}_{\delta},$$
$$T^{a_3} \circ T^{b_5}_{\beta} = \delta^{b_5}_{de_4} f_c{}^{a_3\, d} T^{e_4 c}_{\beta},$$
$$T^{a_5}_{\alpha} \circ T_b = -f^{a_5 c}_{b_\alpha} T_c - \delta^{a_5}_{d_4 e} \delta^{ef}_{bc} f_f^{d_4} T^{c}_{\alpha} + \delta^{a_5}_{bd_2 e_2} f^{d_2}_{c\alpha} T^{e_2 c} - \delta^{a_5}_{c_2\, d_3} f^{c_2}_{b_\alpha} T^{d_3}$$
$$-\delta^{a_5}_{bd_4} f_{c\alpha}{}^{\gamma} T^{d_4 c}_{\gamma} + f_{b\alpha}{}^{\gamma} T^{a_5}_{\gamma} + \delta^{a_5}_{de_4} f_{bc}{}^{d} T^{e_4 c}_{\alpha} + \delta^{a_5}_{dbe_3} f_{c_2}{}^{d} T^{e_3 c_2}_{\alpha},$$
$$T^{a_5}_{\alpha} \circ T^{b}_{\beta} = f^{a_5\, b}_{c_\alpha} T^{c}_{\beta} + \delta^{a_5}_{d_4 e} \epsilon_{\alpha\beta} f_c^{d_4} T^{ebc} - \delta^{a_5}_{d_2 e_3} f_c{}^{d_2} T^{e_3 bc}_{\beta},$$
$$T^{a_5}_{\alpha} \circ T^{b_3} = \delta^{b_3}_{de_2} f^{a_5\, d}_{c\alpha} T^{e_2 c} - \delta^{a_5}_{d_4 e} f_c^{d_4} T^{eb_3 c}_{\alpha},$$
$$T^{a_5}_{\alpha} \circ T^{b_5}_{\beta} = \delta^{b_5}_{de_4} f^{a_5\, d}_{c\alpha} T^{e_4 c}_{\beta}.$$

$$(\overline{\mathbf{6}},\mathbf{1}) \oplus (\mathbf{20},\mathbf{2}) \oplus (\mathbf{84}+\mathbf{6},\mathbf{1}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\overline{\mathbf{84}} \oplus \overline{\mathbf{6}},\mathbf{2}) \oplus (\mathbf{70} \oplus \mathbf{20},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1}).$$

$$2D \to D \oplus \bar{D}$$

$$W^{\dot{\alpha}}_{A} = \delta^{b}_{A} W^{\dot{\alpha}}_{b}\ (a,b = 1,\dots,D)$$

$$D(2D-1) \to \overline{\frac{D(D-1)}{2}} \oplus (\bar{D} \otimes D) \oplus \frac{D(D-1)}{2}$$

$$t_{\boldsymbol{\alpha}} = (R^{a_1 a_2} \quad K^{a}{}_{b} \quad R_{a_1 a_2})$$

$$(K^{c}_{d})^{B}_{A} = \begin{pmatrix} \delta^{c}_{a}\delta^{b}_{d} & 0 \\ 0 & -\delta^{a}_{d}\delta^{c}_{b} \end{pmatrix}, (R^{c_1 c_2})^{B}_{A} = \begin{pmatrix} 0 & -2\delta^{c_1 c_2}_{ab} \\ 0 & 0 \end{pmatrix}$$
$$\left(R_{c_1 c_2}\right)^{B}_{A} = \begin{pmatrix} 0 & 0 \\ -2\delta^{ab}_{c_1 c_2} & 0 \end{pmatrix}$$

$$[K^{a}_{b}, R^{c_1 c_2}] = 2\delta^{c_1 c_2}_{bd} R^{ad}, \left[K^{a}_{b}, R_{c_1 c_2}\right] = -2\delta^{ad}_{c_1 c_2} R_{bd}$$
$$\left[R_{a_1 a_2}, R^{b_1 b_2}\right] = 4\delta^{[b_1}_{[a_1} K^{b_2]}{}_{a_2]}.$$

$$f_{a,b_1 b_2}, f^{c}_{a,b}, f^{b_1 b_2}_{a}, \text{ and } Z_a.$$

$$W_a = -\frac{1}{2!} f_{abc} R^{bc} + \frac{1}{2} f_{ab}{}^{c} K^{b}{}_{c} - \frac{1}{3!} f_a{}^{d_1 d_2} R_{d_1 d_2}, W^a = 0$$

$$f_{a_1 \ldots a_3}, f^{b}_{a_1 a_2}, \text{ and } f_a^{b_1 b_2}$$

$$\begin{aligned} X_a &= -\frac{1}{2} f_{a b_1 b_2} R^{b_1 b_2} + f^{c}_{ab} K^{b}_{c} - \frac{1}{2} f_a^{b_1 b_2} R_{b_1 b_2} \\ X^a &= -f^{ac}_{b} K^{b}_{c} - \frac{1}{2} f_{b_1 b_2}{}^{a} R^{b_1 b_2} \end{aligned}$$

$$X_{ABC} = \begin{cases} f_{abc} = H_{abc} \\ f_{ab}{}^{c} \\ f_a{}^{bc} = Q_a{}^{bc} \end{cases}$$

$$N_{\text{geo}} = \frac{D}{6}(D-1)(7D-2).$$

$$N = \frac{2D}{3}(D-1)(2D-1)$$

$$\frac{N_{\text{geo}}}{N} = \frac{7}{8} + \frac{3}{16D-8}$$

$$X^{E}_{AC} X^{D}_{BE} - X^{E}_{BC} X^{D}_{AE} = -X^{E}_{AB} X^{D}_{EC}$$

$$X^{E}_{[AB]} X^{D}_{[EC]} + X^{E}_{[BC]} X^{D}_{[EA]} + X^{E}_{[CA]} X^{D}_{[EB]} = X^{E}_{[AB} Z^{D}_{C]E}$$

$$Z_{AB}{}^{C} := X_{(AB)}{}^{C},$$

$$[T_A, T_B]_- := \frac{1}{2}(T_A \circ T_B - T_B \circ T_A) \text{ with } [T_A, T_B]_- = X^{C}_{[AB]} T_C$$

$$Z_{AB}{}^{E} X_{EC}{}^{D} = 0$$

$$[T_A, T_B]_+ = \frac{1}{2}(T_A \circ T_B + T_B \circ T_A) = Z^{C}_{AB} T_C$$

$$T_A = (T_{\grave{a}} \quad T_{\dot{\alpha}}) = (T_a \quad T_{\grave{\alpha}} \quad T_{\dot{\alpha}}) = (T_a \quad T_{\check{\alpha}}).$$

$a = 1, \ldots, n$ (a $= 1, \ldots, n-1$),
$\grave{\alpha} = 1, \ldots, \dim R_1 - \dim \mathcal{J} - n$ ($\grave{\alpha} = 1, \ldots, \dim R_1 - \dim \mathcal{J} - n + 1$) and
$\dot{\alpha} = 1, \ldots, \dim \mathcal{J}$

$$T_{\grave{a}} = (T_a \quad T_{\grave{\alpha}}) \text{ and } T_{\check{\alpha}} = (T_{\grave{\alpha}} \quad T_{\dot{\alpha}}).$$

$$Z_{AB}{}^{\grave{c}} = 0 \text{ and } X^{C}_{\dot{\alpha} B} = 0$$

$$3 X^{\grave{e}}_{[\grave{a}\grave{b}} X^{\grave{d}}_{\grave{c}]\grave{e}} = 0$$

$$T_{\dot{\alpha}} \circ T_B = X_{\dot{\alpha} B}{}^{C} T_C = 0 \subset \mathcal{J} \text{ and } T_B \circ T_{\dot{\alpha}} = 2 Z_{\dot{\alpha} B}{}^{C} T_C = 2 Z_{\dot{\alpha} B}{}^{\dot{\gamma}} T_{\dot{\gamma}} \subset \mathcal{J}.$$

$$X^{c}_{\check{\alpha}\check{\beta}} = 0 \left(X^{c}_{\check{\alpha}\check{\beta}} = 0\right)$$

$$\begin{aligned} T_{\breve{\alpha}} &= (T^{a_1a_2} \quad T^{a_1\ldots a_5} \quad T^{a_1\ldots a_7,b}) \\ (T_{\breve{\alpha}} &= \left(T^{\mathrm{a}}_{\alpha} \quad T^{\mathrm{a}_1\ldots\mathrm{a}_3} \quad T^{\mathrm{a}_1\ldots\mathrm{a}_5}_{\alpha} \quad T^{\mathrm{a}_1\ldots\mathrm{a}_6,\,\mathrm{b}}\right)\Big). \end{aligned}$$

$$G \ni g = \exp\left(x^{\grave{a}}T_{\grave{a}}\right) = \sum_{n=0}^{\infty} \frac{\left(x^{\grave{a}}T_{\grave{a}}\right)^n}{n!}$$

$$(T_{\grave{a}})_A^B = -X_{\grave{a}B}^C = \begin{pmatrix} -X_{\grave{a}\grave{b}}^{\grave{c}} & -X'_{\grave{a}\grave{b}}{}^{\hat{\gamma}} \\ 0 & -X'_{\grave{a}\hat{\beta}} \end{pmatrix}.$$

$$\left[T_{\grave{a}}, T_{\grave{b}}\right] = X_{\grave{a}\grave{b}}^{\grave{c}} T_{\grave{c}}$$

$$\begin{aligned} 2X_{[\grave{a}|\grave{c}}^{\grave{e}} X_{|\grave{b}]\grave{e}}^{\grave{d}} &= -X_{\grave{a}\grave{b}}^{\grave{e}} X_{\grave{e}\grave{c}}^{\grave{d}} \\ 2X_{[\grave{a}|\dot{\gamma}}{}^{\hat{\epsilon}} X_{|\grave{b}]\dot{\epsilon}}{}^{\hat{\delta}} &= -X_{\grave{a}\grave{b}}{}^{\grave{e}} X_{\grave{e}\dot{\gamma}}{}^{\delta} \\ 2X_{[\grave{a}|\grave{c}}{}^{\grave{e}} X_{|\grave{b}]\grave{e}}{}^{\hat{\delta}} + 2X_{[\grave{a}|\grave{c}}{}^{\hat{\epsilon}} X_{|\grave{b}]\dot{\epsilon}}{}^{\hat{\delta}} &= -X_{\grave{a}\grave{b}}{}^{\grave{e}} X_{\grave{e}\grave{c}}{}^{\hat{\delta}} \end{aligned}$$

$$(M^{-1})_A{}^C \, \mathrm{d}M_C{}^B = -v^C X_{CA}{}^B = v^C (T_C)_A{}^B$$

$$\mathrm{d}v^A T_A = -\frac{1}{2} X_{BC}{}^A v^B \wedge v^C T_A$$

$$\mathrm{d}v^A = -\frac{1}{2} X_{BC}{}^A v^B \wedge v^C$$

$$M_A{}^B := \exp\left(x^i T_i\right)_A{}^B$$

$$M_A^B = \begin{pmatrix} M_{\grave{a}}^{\grave{b}} & I_{\grave{a}}{}^{\dot{\gamma}} M_{\gamma}{}^{\dot{\beta}} \\ 0 & M_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix}, (M^{-1})_A^B = \begin{pmatrix} (M^{-1})_{\grave{a}}{}^{\grave{b}} & -(M^{-1})_{\grave{a}}{}^{\grave{c}} I_{\grave{c}}{}^{\dot{\beta}} \\ 0 & (M^{-1})_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix}.$$

$$(M^{-1})_A{}^C \, \mathrm{d}M_C{}^B = \begin{pmatrix} \mathrm{d}M_{\grave{c}}^{\grave{b}} (M^{-1})_{\grave{a}}^{\grave{c}} & (M^{-1})_{\grave{a}}^{\grave{c}} \, \mathrm{d}I_{\grave{c}}^{\hat{\delta}} M_{\hat{\delta}}^{\hat{\beta}} \\ 0 & \mathrm{d}M_{\dot{\gamma}}^{\hat{\beta}} (M^{-1})_{\dot{\alpha}}^{\hat{\gamma}} \end{pmatrix} = -v^C X_{CA}{}^B = -v^{\grave{c}} X_{\grave{c}A}{}^B$$

$$(M^{-1})_{\grave{a}}^{\grave{c}} \, \mathrm{d}M_{\grave{c}}^{\grave{b}} = -v^{\grave{c}} X_{\grave{c}\grave{a}}^{\grave{b}}$$

$$\mathrm{d}v^{\grave{a}} = -\frac{1}{2} X_{\grave{b}\grave{c}}^{\grave{a}} v^{\grave{b}} \wedge v^{\grave{c}}$$

$$\mathrm{d}v^A + \frac{1}{2} X_{BC}{}^A v^B \wedge v^C + w^A = 0,$$

$$w^{\grave{a}} = 0 \;\text{ and }\; w^{\dot{\alpha}} = -\frac{1}{2} X_{\grave{b}\grave{c}}^{\dot{\alpha}} v^{\grave{b}} \wedge v^{\grave{c}}.$$

$$
\begin{aligned}
\mathrm{d}w^A &= X^A_{[BC]} w^B \wedge v^C + \frac{1}{2} X^E_{[BC]} X^A_{[ED]} v^B \wedge v^C \wedge v^D \\
&= -Z^A_{BC} w^B \wedge v^C + \frac{1}{3!} X^E_{[BC} Z^A_{D]E} v^B \wedge v^C \wedge v^D .
\end{aligned}
$$

$$
w^A X_{AB}{}^C = 0,
$$

$$
\begin{aligned}
(M^{-1})_A{}^C \, \mathrm{d}M_C{}^B &= -v^C X_{CA}{}^B, \text{ and} \\
\mathrm{d}v^A &= -\frac{1}{2} X_{[BC]}{}^A v^B \wedge v^C - w^A \text{ with } w^A X_{AB}{}^C = 0
\end{aligned}
$$

$$
E_A{}^I = M_A{}^B V_B{}^I,
$$

$$
\boldsymbol{X}_{AB}{}^C = \Omega_{AB}{}^C + (t^{\boldsymbol{\alpha}})_D{}^E (t_{\boldsymbol{\alpha}})_B{}^C \Omega_{EA}{}^D - \beta \Omega_{DA}{}^D (t_0)_B{}^C,
$$

$$
\Omega_{AB}{}^C := E_A{}^I E_B{}^J \partial_I E_J{}^C
$$

$$
\begin{aligned}
\Omega^C_{AB} &= M^D_A M^E_B (M^{-1})^C_F \left[\hat{\Omega}^C_{DE} - V^I_D (M^{-1} \partial_I M)^F_E \right] \\
&= M^D_A M^E_B (M^{-1})^C_F \delta^d_D \left(\hat{\Omega}^C_{dE} + A^G_d X^C_{GE} \right).
\end{aligned}
$$

$$
\hat{\Omega}_{aB}{}^C := v^i_a V_B{}^J \partial_i V_J{}^C \text{ and } A_a{}^B := v^i_a v^B_i .
$$

$$
v^A_i = \begin{pmatrix} v^a_i & A_{ia_1a_2} & A_{ia_1\dots a_5} & A_{ia_1\dots a_7,a'} \end{pmatrix},
$$

$$
A_a{}^B = v^i_a v^B_i = \begin{pmatrix} \delta^b_a & A_{ab_1b_2} & A_{ab_1\dots b_5} & A_{ab_1\dots b_7,b'} \end{pmatrix}.
$$

$$
\hat{V}^J_A := \begin{pmatrix}
v^j_a & 0 & 0 & 0 \\
0 & 2!\, v^{[a_1}_{[j_1} v^{a_2]}_{j_2]} & 0 & 0 \\
0 & 0 & 5!\, v^{[a_1}_{[j_1} \dots v^{a_5]}_{j_5]} & 0 \\
0 & 0 & 0 & 7!\, v^{[a_1}_{[j_1} \dots v^{a_7]}_{j_7]} v^{a'}_{j'}
\end{pmatrix}.
$$

$$
N_I{}^J = \left[\exp\left(\frac{1}{3!} C_{i_1\dots i_3} R^{i_1\dots i_3} \right) \exp\left(\frac{1}{6!} C_{i_1\dots i_6} R^{i_1\dots i_6} \right) \right]_I{}^J,
$$

$$
N^J_I = \begin{pmatrix}
\delta^j_i & -C_{ij_1j_2} & -\left(C_{ij_1\dots j_5} + 5 C_{i[j_1j_2} C_{j_3j_4j_5]} \right) \\
0 & 2!\, \delta^{i_1i_2}_{j_1j_2} & 20 \delta^{i_1i_2}_{[j_1j_2} C_{j_3j_4j_5]} \\
0 & 0 & 5!\, \delta^{i_1\dots i_5}_{j_1\dots j_5}
\end{pmatrix}.
$$

$$
V_A{}^I = \hat{V}_A{}^J N_J{}^I .
$$

$$
\begin{aligned}
\hat{\Omega}^C_{aB} &= -D_a \hat{V}^J_B \hat{V}^C_J - \hat{V}^J_B \hat{V}^C_K D_a N^L_J (N^{-1})^K_L \\
&= -\left(D_{(a} v^j_{b)} v^c_j + \frac{1}{2} [v_a, v_b]^j v^c_j \right) \left(\widetilde{K}^b_c \right)_B{}^C - \hat{V}_B{}^J \hat{V}_K{}^C D_a N_J{}^L (N^{-1})_L{}^K
\end{aligned}
$$

$$v^c([v_a, v_b]) = f_{ab}^c - \frac{2}{2!} f_{[a}^{cd_1d_2} A_{b]d_1d_2} - \frac{2}{5!} f_{[a}^{cd_1\dots d_5} A_{b]d_1\dots d_5}, \text{ and}$$

$$\mathrm{d}NN^{-1} = \frac{1}{3!}\mathrm{d}C_{i_1\dots i_3} R^{i_1\dots i_3} + \frac{1}{6!}\left(\mathrm{d}C_{i_1\dots i_6} - 10\,\mathrm{d}C_{i_1i_2i_3} C_{i_4i_5i_6}\right) R^{i_1\dots i_6}$$

$$\begin{aligned}\Omega_a \sim & \left(-\frac{1}{2} f_{ab}^c + \frac{1}{2!} f_{[a}^{cd_1d_2} A_{b]d_1d_2} + \frac{1}{5!} f_{[a}^{cd_1\dots d_5} A_{b]d_1\dots d_5}\right) \tilde{K}_c^b \\ & - \frac{1}{4!} F_{abcd} R^{bcd} - \frac{1}{7!} F_{ab_1\dots b_6} R^{b_1\dots b_6}\end{aligned}$$

$$F_{ijkl} := 4\partial_{[i} C_{jkl]} \text{ and } F_{i_1\dots i_7} := 7\partial_{[i_1} C_{i_2\dots i_7]} - \frac{35}{2} F_{[i_1\dots i_4} C_{i_5i_6i_7]}$$

$$\begin{aligned}&\hat{\Omega}_a + A_a{}^B X_B \sim X_a - \frac{1}{2} f_{ab}{}^c \tilde{K}^b{}_c \\ &- \frac{1}{4!}\left(F_{abcd} - 12 f_{[ab}{}^e A_{cd]e} + 36 Z_{[a} A_{bcd]} + 2 f_{[a}{}^{efg} A_{bcd]efg} + \frac{1}{30} f_d{}^{e_1\dots e_6} A_{abe_1\dots e_6,c}\right) R^{bcd} \\ &- \frac{1}{7!}\left(F_{ab_1\dots b_6} + 105 f_{[ab_1b_2b_3} A_{b_4b_5b_6]} + 105 f_{[ab_1}{}^e A_{b_2\dots b_6]e}\right. \\ &\left. + 252 Z_{[a} A_{b_1\dots b_6]} + 21 f_{[a}{}^{c_1c_2d} A_{b_1\dots b_6]c_1c_2,d}\right) R^{b_1\dots b_6}\end{aligned}$$

$$A_a{}^B X_B = X_a + \frac{1}{2!} A_{ab_1b_2} X^{b_1b_2} + \frac{1}{5!} A_{ab_1\dots b_5} X^{b_1\dots b_5} + \frac{1}{7!} A_{ab_1\dots b_7,b'} X^{b_1\dots b_7,b'}$$

$$\begin{aligned}X_a =& -\frac{1}{6!} f_{ab_1\dots b_6} R^{b_1\dots b_6} - \frac{1}{3!} f_{ab_1b_2b_3} R^{b_1b_2b_3} \\ & + f_{ab}{}^c \tilde{K}^b{}_c - Z_a\left(\tilde{K}^b{}_b + t_0\right) - \frac{1}{3!} f_a{}^{b_1b_2b_3} R_{b_1b_2b_3} - \frac{1}{6!} f_a{}^{b_1\dots b_6} R_{b_1\dots b_6} \\ X^{a_1a_2} =& f_b{}^{ca_1a_2} \tilde{K}^b{}_c - f_{c_1c_2}{}^{[a_1} R^{a_2]c_1c_2} + 3 Z_b R^{ba_1a_2} - \frac{1}{4!} f_{b_1\dots b_4} R^{a_1a_2b_1\dots b_4}, \\ X^{a_1\dots a_5} =& f_b{}^{ca_1\dots a_5} \tilde{K}^b{}_c + \frac{5!}{3!\,2!} f_b{}^{[a_1a_2a_3} R^{a_4a_5]b} - \frac{5!}{2!\,4!} f_{bc}{}^{[a_1} R^{a_2\dots a_5]bc} + 6 Z_b R^{ba_1\dots a_5}, \\ X^{a_1\dots a_7,a'} =& -7 f_b{}^{[a_1\dots a_6} R^{a_7]a'b} - 21 f_b{}^{a'[a_1a_2} R^{a_3\dots a_7]b}.\end{aligned}$$

$$W_a = W_a{}^{\dot{\alpha}} t_{\dot{\alpha}} = X_a - \frac{1}{2} f_{ab}{}^c \tilde{K}^b{}_c + \frac{3}{4!} f_{ab_1b_2b_3} R^{b_1b_2b_3} + \frac{6}{7!} f_{ab_1\dots b_6} R^{b_1\dots b_6},$$

$$\begin{aligned}F_{abcd} =& -3 f_{abcd} + 12 f_{[ab}^e A_{cd]e} - 36 Z_{[a} A_{bcd]} \\ & -2 f_{[a}^{efg} A_{bcd]efg} - \frac{1}{30} f_d^{e_1\dots e_6} A_{abe_1\dots e_6,c} \text{ and} \\ F_{a_1\dots b_7} =& -6 f_{a_1\dots a_7} - 105 f_{[a_1\dots a_4} A_{a_5a_6a_7]} - 105 f_{[a_1a_2}^b A_{a_3\dots a_7]b} \\ & -252 Z_{[a_1} A_{a_2\dots a_7]} - 21 f_{[a_1}^{b_1b_2c} A_{a_2\dots a_7]b_1b_2,c}\end{aligned}$$

$$F_4 = -\frac{3}{4!} f_{abcd} v^a \wedge \cdots \wedge v^d$$

$$+\frac{1}{2}\left(f_{ab}{}^{d} A_{cd} + 3Z_a A_{bc} + \frac{1}{3!} f_a{}^{e_1e_2e_3} A_{bce_1e_2e_3} - \frac{2}{6!} f_a{}^{e_1\dots e_6} A_{be_1\dots e_6,c}\right) \wedge v^a \wedge v^b \wedge v^c$$

$$F_7 = -\frac{6}{7!} f_{a_1\dots a_7} v^{a_1} \wedge \cdots \wedge v^{a_7}$$

$$-\frac{1}{2}\left(\frac{1}{4!} f_{a_1\dots a_4} A_{a_5a_6} + \frac{1}{4!} f_{a_1a_2}{}^{b} A_{a_3\dots a_6 b}\right.$$

$$\left.-\frac{1}{10} Z_{a_1} A_{a_2\dots a_6} - \frac{1}{5!} f_{a_1}{}^{b_1b_2c} A_{a_2\dots a_6 b_1 b_2, c}\right) \wedge v^{a_1} \wedge \cdots \wedge v^{a_6}.$$

$$\mathrm{d}F_4 = 0 \ \text{ and } \ \mathrm{d}F_7 + \frac{1}{2} F_4 \wedge F_4 = 0.$$

$$F_4 = \frac{1}{4!} f_{abcd} v^a \wedge \cdots \wedge v^d - \frac{1}{3!} X_{A,b,c_1c_2} v^A \wedge v^b \wedge v^{c_1} \wedge v^{c_2}$$

$$F_7 = \frac{1}{7!} f_{a_1\dots a_7} v^{a_1} \wedge \cdots \wedge v^{a_7} - \frac{1}{6!} X_{A,b,c_1\dots c_5} v^A \wedge v^b \wedge v^{c_1} \wedge \cdots \wedge v^{c_5}.$$

$$X_{AB}^{C} = -X_{BA}^{C}, X_{\breve{\alpha}\breve{\beta}}^{C} = 0, \ \text{ and } \ X_{AB}^{C} X_C = -X_{BA}^{C} X_C,$$

$$\mathrm{d}F_4 = -\frac{1}{12}\left(X_{ab}^{G} X_{G,c,de} + 2 f_{ab}^{g} f_{gcde}\right) v^a \wedge v^b \wedge v^c \wedge v^d \wedge v^e$$

$$-\frac{1}{12}\left(4 f_{gbcd} X_{e\breve{\alpha}}^{g} + 2 X_{e\breve{\alpha}}^{G} X_{G,b,cd} - 3 f_{bc}^{g} X_{\breve{\alpha},g,de}\right) v^{\breve{\alpha}} \wedge v^b \wedge v^c \wedge v^d \wedge v^e$$

$$-\frac{1}{2\cdot 3!}\left(X_{\breve{\alpha}\breve{\beta}}^{G} X_{G,c,de} + 6 X_{\breve{\alpha}c}^{f} X_{\breve{\beta},f,de}\right) v^{\breve{\alpha}} \wedge v^{\breve{\beta}} \wedge v^c \wedge v^d \wedge v^e.$$

$$L_{ABC}{}^{D} := X_{AC}{}^{E} X_{BE}{}^{D} - X_{BC}{}^{E} X_{AE}{}^{D} + X_{AB}{}^{E} X_{EC}{}^{D} = 0$$

$$\mathrm{d}F_4 = -\frac{1}{20} L_{a,b,c,de} v^a \wedge v^b \wedge v^c \wedge v^d \wedge v^e$$

$$+\frac{1}{24}\left(L_{\breve{\alpha},b,c,de} + 4 L_{b,\breve{\alpha},c,de}\right) v^{\breve{\alpha}} \wedge v^b \wedge v^c \wedge v^d \wedge v^e$$

$$-\frac{1}{2\cdot 3!} L_{\breve{\alpha},\breve{\beta},c,de} v^{\breve{\alpha}} \wedge v^{\breve{\beta}} \wedge v^c \wedge v^d \wedge v^e = 0$$

$$v_{\mathrm{m}}^{A} = \begin{pmatrix} v_{\mathrm{m}}^{\mathrm{a}} & A_{\mathrm{ma}_1}^{\alpha} & A_{\mathrm{ma}_1\mathrm{a}_2\mathrm{a}_3} & A_{\mathrm{ma}_1\dots\mathrm{a}_5}^{\alpha} & A_{\mathrm{ma}_1\dots\mathrm{a}_6,\mathrm{a}'} \end{pmatrix}$$

$$A_{\mathrm{a}}{}^{B} = \begin{pmatrix} \delta_{\mathrm{a}}^{\mathrm{b}} & A_{\mathrm{ab}}^{\beta} & A_{\mathrm{ab}_1\mathrm{b}_2\mathrm{b}_3} & A_{\mathrm{ab}_1\dots\mathrm{b}_5}^{\beta} & A_{\mathrm{ab}_1\dots\mathrm{b}_6,\mathrm{b}'} \end{pmatrix}$$

$$\hat{V}_A^{J} = \begin{pmatrix} v_{\mathrm{a}}^{\mathrm{n}} & 0 & 0 & 0 & 0 \\ 0 & \delta_{\alpha}^{\nu} v_{\mathrm{n}}^{\mathrm{a}} & 0 & 0 & 0 \\ 0 & 0 & 3!\, v_{[\mathrm{n}_1}^{[\mathrm{a}_1} v_{\mathrm{n}_2}^{\mathrm{a}_2} v_{\mathrm{n}_3]}^{\mathrm{a}_3]} & 0 & 0 \\ 0 & 0 & 0 & 5!\, \delta_{\alpha}^{\nu} v_{[\mathrm{n}_1}^{[\mathrm{a}_1} \dots v_{\mathrm{n}_5]}^{\mathrm{a}_5]} & 0 \\ 0 & 0 & 0 & 0 & 6!\, v_{[\mathrm{n}_1}^{[\mathrm{a}_1} \dots v_{\mathrm{n}_6]}^{\mathrm{a}_6]} v_{\mathrm{n}'}^{\mathrm{a}'} \end{pmatrix}$$

$$N_I{}^{J} = \left[\exp\left(\frac{1}{2!} B_{\mathrm{mn}}^{\mu} R_{\mu}^{\mathrm{mn}}\right) \exp\left(\frac{1}{4!} D_{\mathrm{m}_1\dots\mathrm{m}_4} R^{\mathrm{m}_1\dots\mathrm{m}_4}\right) \exp\left(\frac{1}{6!} B_{\mathrm{m}_1\dots\mathrm{m}_6}^{\mu} R_{\mu}^{\mathrm{m}_1\dots\mathrm{m}_6}\right)\right]_I{}^{J}.$$

$$N_I^{\,J} = \begin{pmatrix} \delta_{\mathrm{m}}^{\mathrm{n}} & -B_{\mathrm{mn}}^{\nu} & -D_{\mathrm{mn_1n_2n_3}} - \frac{3}{2}\epsilon_{\rho\sigma}B^{\rho}_{\mathrm{m[n_1}}B^{\sigma}_{\mathrm{n_2n_3]}} & -5\epsilon_{\rho\sigma}B^{\rho}_{\mathrm{m[n_1}}B^{\sigma}_{\mathrm{n_2n_3}}B^{\nu}_{\mathrm{n_4n_5]}} \\ 0 & \delta_\mu^\nu\delta_{\mathrm{n}}^{\mathrm{m}} & 3\epsilon_{\mu\rho}\delta^{\mathrm{m}}_{\mathrm{[n_1}}B^{\rho}_{\mathrm{n_2n_3]}} & -5\delta_\mu^\nu\delta^{\mathrm{m}}_{\mathrm{[n_1}}D_{\mathrm{n_2\ldots n_5]}} + 15\epsilon_{\mu\rho}\delta^{\mathrm{m}}_{\mathrm{[n_1}}B^{\rho}_{\mathrm{n_2n_3}}B^{\nu}_{\mathrm{n_4n_5]}} \\ 0 & 0 & 3!\,\delta^{\mathrm{m_3}}_{\mathrm{n_1n_2n_3}} & 60\delta^{\mathrm{m_3}}_{\mathrm{[n_1n_2n_3}}B^{\nu}_{\mathrm{n_4n_5]}} \\ 0 & 0 & 0 & 5!\,\delta_\mu^\nu\delta^{\mathrm{m_1\ldots m_5}}_{\mathrm{n_1\ldots n_5}} \end{pmatrix}$$

$$v^{\mathrm{c}}([v_{\mathrm{a}}, v_{\mathrm{b}}]) = f_{\mathrm{ab}}{}^{\mathrm{c}} - 2f^{\mathrm{cd}}_{\mathrm{[a|\alpha}}A^{\alpha}_{\mathrm{|b]d}} - \frac{2}{3!}f_{\mathrm{[a}}{}^{\mathrm{cd_1\,d_2\,d_3}}A_{\mathrm{b]d_1\,d_2\,d_3}} - \frac{2}{5!}f_{\mathrm{[a}}{}^{\mathrm{cd_1\ldots\,d_5}}A_{\mathrm{b]d_1\ldots\,d_5}},$$

$$\begin{aligned} \mathrm{d}NN^{-1} =& \frac{1}{2!}\mathrm{d}B^{\mu}_{\mathrm{mn}}R^{\mathrm{mn}}_{\mu} \\ &+ \frac{1}{4!}\left(\mathrm{d}D_{\mathrm{m_1\ldots\,m_4}} - 3\epsilon_{\mu\nu}\mathrm{d}B^{\mu}_{\mathrm{m_1\,m_2}}B^{\nu}_{\mathrm{m_3\,m_4}}\right)R^{\mathrm{m_1\ldots\,m_4}} \\ &+ \frac{1}{6!}(\ldots)^{\mu}_{\mathrm{m_1\ldots\,m_6}}R_{\nu}^{\mathrm{m_1\ldots\,m_6}}. \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_{\mathrm{a}B}{}^{C} =& -D_{\mathrm{a}}\hat{V}_B{}^J\hat{V}_J{}^C - \hat{V}_B{}^J\hat{V}_K{}^C D_a N_J{}^L(N^{-1})_L{}^K \\ &\sim \left[\left(-\frac{1}{2}f_{\mathrm{ab}}{}^{\mathrm{c}} + f^{\mathrm{cd}}_{\mathrm{[a|\alpha}}A^{\alpha}_{\mathrm{|b]d}} + \frac{1}{3!}f_{\mathrm{[a}}{}^{\mathrm{cd_1\,d_2\,d_3}}A_{\mathrm{b]d_1\,d_2\,d_3}} + \frac{1}{5!}f_{\mathrm{[a}}{}^{\mathrm{cd_1\ldots\,d_5}}A_{\mathrm{b]d_1\ldots\,d_5}}\right)\widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}}\right]_B{}^{C} \\ &- \left[\frac{1}{3!}F^{\alpha}_{\mathrm{abc}}R^{\mathrm{bc}}_{\alpha} + \frac{1}{5!}F_{\mathrm{ab_1\ldots\,b_4}}R^{\mathrm{b_1\ldots\,b_4}}\right]_B{}^{C}. \end{aligned}$$

$$F^{\alpha}_{\mathrm{abc}} := 3v_{\mathrm{a}}^{\mathrm{m}}v_{\mathrm{b}}^{\mathrm{n}}v_{\mathrm{c}}^{\mathrm{p}}\delta^{\alpha}_{\mu}\partial_{\mathrm{[m}}B^{\mu}_{\mathrm{np]}} \text{ and}$$

$$F_{\mathrm{a_1\ldots a_5}} := v_{\mathrm{a_1}}^{\mathrm{m_1}}\ldots v_{\mathrm{a_5}}^{\mathrm{m_5}}\left(5\partial_{\mathrm{[m_1}}D_{\mathrm{m_2\,m_3\,m_4\,m_5]}} - 5\epsilon_{\mu\nu}F^{\mu}_{\mathrm{[m_1\,m_2\,m_3}}B^{\nu}_{\mathrm{m_4\,m_5]}}\right)$$

$$\begin{aligned} &\hat{\Omega}_{\mathrm{a}} + A_{\mathrm{a}}{}^B X_B \\ &\sim X_{\mathrm{a}} - \frac{1}{2}f_{\mathrm{ab}}{}^{\mathrm{c}}\widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} \\ &- \frac{1}{3!}\left(F^{\alpha}_{\mathrm{abc}} + 3f_{\mathrm{[ab}}{}^{\mathrm{d}}A^{\alpha}_{\mathrm{c]d}} - 6f^{\alpha}_{\mathrm{[a|\beta}}A^{\beta}_{\mathrm{|bc]}} + 12Z_{\mathrm{[a}}A^{\alpha}_{\mathrm{bc]}} + 3\epsilon^{\alpha\beta}f^{\mathrm{d_1\,d_2}}_{\mathrm{[a|\beta}}A_{\mathrm{|bc]d_1\,d_2}}\right. \\ &\left.- \frac{1}{4}f_{\mathrm{[a}}{}^{\mathrm{d_1\ldots\,d_4}}A^{\alpha}_{\mathrm{bc]d_1\ldots\,d_4}} + \frac{1}{5!}\epsilon^{\alpha\beta}f_{\mathrm{b}\beta}^{\mathrm{d_1\ldots\,d_6}}A_{\mathrm{ad_1\ldots\,d_6,c}}\right)R^{\mathrm{bc}}_{\alpha} \\ &- \frac{1}{5!}\left(F_{\mathrm{ab_1\ldots\,b_4}} + 30f_{\mathrm{[ab_1}}{}^{\mathrm{c}}A_{\mathrm{b_2\,b_3\,b_4]c}} + 80Z_{\mathrm{[a}}A_{\mathrm{b_1\ldots\,b_4]}} + 10f^{\mathrm{c_1c_2}}_{\mathrm{[a|\alpha}}A^{\alpha}_{\mathrm{|b_1\ldots\,b_4]c_1c_2}}\right. \\ &\left.-20\epsilon_{\alpha\beta}A^{\alpha}_{\mathrm{a[\,b_1}}f^{\beta}_{\mathrm{b_2\,b_3\,b_4]}} + \frac{10}{3}f_{\mathrm{[b_1}}{}^{\mathrm{c_{c_2}c_3\,d}}A_{\mathrm{b_2\,b_3\,b_4]c_1c_2c_3,\,d}}\right)R^{\mathrm{b_1\ldots\,b_4}} \end{aligned}$$

$$
\begin{aligned}
X_{\mathrm{a}} =& -\frac{1}{4!} f_{\mathrm{ab}_1\ldots\mathrm{b}_4} R^{\mathrm{b}_1\ldots\mathrm{b}_4} - \frac{1}{2!} f^{\beta}_{\mathrm{ab}_1\mathrm{b}_2} R^{\mathrm{b}_1\mathrm{b}_2}_{\beta} \\
& + f_{\mathrm{ab}}{}^{\mathrm{c}} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} - Z_{\mathrm{a}}\left(\widetilde{K}^{\mathrm{b}}_{\mathrm{b}} + t_0\right) - f_{\mathrm{a}\alpha}{}^{\beta} R^{\alpha}{}_{\beta} \\
& - \frac{1}{2!} f_{\mathrm{a}_\beta}^{\mathrm{b}_1\mathrm{b}_2} R^{\beta}_{\mathrm{b}_1\mathrm{b}_2} - \frac{1}{4!} f_{\mathrm{a}}^{\mathrm{b}_1\ldots\mathrm{b}_4} R_{\mathrm{b}_1\ldots\mathrm{b}_4} - \frac{1}{6!} f_{\mathrm{a}_\beta}^{\mathrm{b}_1\ldots\mathrm{b}_6} R^{\beta}_{\mathrm{b}_1\ldots\mathrm{b}_6} \\
X^{\mathrm{a}}_{\alpha} =& f_{\mathrm{b}_\alpha}^{\mathrm{ca}} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} - \frac{1}{2!}\left(\delta^{\beta}_{\alpha} f_{\mathrm{b}_1\mathrm{b}_2}{}^{\mathrm{a}} - 2\delta^{\mathrm{a}}_{[\mathrm{b}_1} f_{\mathrm{b}_2]\alpha}{}^{\beta}\right) R^{\mathrm{b}_1\mathrm{b}_2}_{\beta} - 2Z_{\mathrm{b}} R^{\mathrm{ab}}_{\alpha} \\
& + \frac{1}{3!} \epsilon_{\alpha\beta} f^{\beta}_{\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3} R^{\mathrm{ab}_1\mathrm{b}_2\mathrm{b}_3} - \frac{1}{5!} f_{\mathrm{b}_1\ldots\mathrm{b}_5} R^{\mathrm{ab}_1\ldots\mathrm{b}_5}_{\alpha} \\
X^{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3} =& f_{\mathrm{b}}{}^{\mathrm{ca}_1\mathrm{a}_2\mathrm{a}_3} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} + 3\epsilon^{\beta\gamma} f_{\mathrm{b}}{}^{[\mathrm{a}_1\mathrm{a}_2} R^{\mathrm{a}_3]\mathrm{b}}_{\gamma} - \frac{3}{2} f_{\mathrm{b}_1\mathrm{b}_2}{}^{[\mathrm{a}_1} R^{\mathrm{a}_2\mathrm{a}_3]\mathrm{b}_1\mathrm{b}_2} - 4Z_{\mathrm{b}} R^{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\mathrm{b}} \\
& + \frac{1}{3!} f^{\alpha}_{\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3} R^{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3}_{\alpha}, \\
X^{\mathrm{a}_1\ldots\mathrm{a}_5}_{\alpha} =& f_{\mathrm{b}_\alpha}^{\mathrm{ca}_1\ldots\mathrm{a}_5} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} + 5 f_{\mathrm{b}}{}^{[\mathrm{a}_1\ldots\mathrm{a}_4} R^{\mathrm{a}_5]\mathrm{b}}_{\alpha} - 10 f_{\mathrm{b}_\alpha}^{[\mathrm{a}_1\mathrm{a}_2} R^{\mathrm{a}_3\mathrm{a}_4\mathrm{a}_5]\mathrm{b}} \\
& - \frac{5}{2} f_{\mathrm{b}_1\mathrm{b}_2}{}^{[\mathrm{a}_1} R^{\mathrm{a}_2\ldots\mathrm{a}_5]\mathrm{b}_1\mathrm{b}_2}_{\alpha} + f_{\mathrm{b}_\alpha}{}^{\beta} R^{\mathrm{a}_1\ldots\mathrm{a}_5\mathrm{b}}_{\beta} - 6Z_{\mathrm{b}} R^{\mathrm{a}_1\ldots\mathrm{a}_5\mathrm{b}}_{\alpha} \\
X^{\mathrm{a}_1\ldots\mathrm{a}_6,\mathrm{a}'} =& -\epsilon^{\beta\gamma} f_{\mathrm{b}_\beta}^{\mathrm{a}_1\ldots\mathrm{a}_6} R^{\mathrm{a}'\mathrm{b}}_{\gamma} + 20 f_{\mathrm{b}}{}^{\mathrm{a}'[\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3} R^{\mathrm{a}_4\mathrm{a}_5\mathrm{a}_6]\mathrm{b}} - 6\epsilon^{\beta\gamma} f_{\mathrm{b}_\beta}^{\mathrm{a}'[\mathrm{a}_1} R^{\mathrm{a}_2\ldots\mathrm{a}_6]\mathrm{b}}_{\gamma}
\end{aligned}
$$

$$
W_{\mathrm{a}} = X_{\mathrm{a}} - \frac{1}{2} f_{\mathrm{ab}^{\mathrm{c}}} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} + \frac{2}{3!} f^{\alpha}_{\mathrm{ab}_1\mathrm{b}_2} R^{\mathrm{b}_1\mathrm{b}_2}_{\alpha} + \frac{4}{5!} f_{\mathrm{ab}_1\ldots\mathrm{b}_4} R^{\mathrm{b}_1\ldots\mathrm{b}_4},
$$

$$
\begin{aligned}
F^{\mu}_3 =& -\frac{2}{3!} f^{\mu}_{\mathrm{abc}} v^{\mathrm{a}} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}} - \left(\frac{1}{2} f_{\mathrm{ab}}{}^{\mathrm{d}} A^{\mu}_{\mathrm{d}} + f_{\mathrm{a}\beta}{}^{\mu} A^{\beta}_{\mathrm{b}} - 2Z_{\mathrm{a}} A^{\mu}_{\mathrm{b}}\right) \wedge v^{\mathrm{a}} \wedge v^{\mathrm{b}} \quad \text{and} \\
& + \left(\frac{1}{2!} \epsilon^{\mu\beta} f_{\mathrm{a}\beta}^{\mathrm{c}_1\mathrm{c}_2} A_{\mathrm{bc}_1\mathrm{c}_2} - \frac{1}{4!} f_{\mathrm{a}}{}^{\mathrm{c}_1\ldots\mathrm{c}_4} A^{\mu}_{\mathrm{bc}_1\ldots\mathrm{c}_4} - \frac{1}{6!} \epsilon^{\mu\beta} f_{\mathrm{a}\beta}^{\mathrm{c}_1\ldots\mathrm{c}_6} A_{\mathrm{c}_1\ldots\mathrm{c}_6,\,\mathrm{b}}\right) \wedge v^{\mathrm{a}} \wedge v^{\mathrm{b}} \\
F_5 =& -\frac{4}{5!} f_{\mathrm{a}_1\ldots\mathrm{a}_5} v^{\mathrm{a}_1} \wedge \cdots \wedge v^{\mathrm{a}_5} \\
& - \left(\frac{1}{2!\,2!} f_{\mathrm{a}_1\mathrm{a}_2}{}^{\mathrm{b}} A_{\mathrm{a}_3\mathrm{a}_4\mathrm{b}} - \frac{4}{3!} Z_{\mathrm{a}_1} A_{\mathrm{a}_2\ldots\mathrm{a}_4} - \frac{1}{3!\,2!} f_{\mathrm{a}_1\alpha}^{\mathrm{b}_1\mathrm{b}_2} A^{\alpha}_{\mathrm{a}_2\ldots\mathrm{a}_4\mathrm{b}_1\mathrm{b}_2} \right. \\
& \left. - \frac{1}{3!} \epsilon_{\alpha\beta} f^{\alpha}_{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3} A^{\beta}_{\mathrm{a}_4} + \frac{1}{36} f_{\mathrm{a}_1}^{\mathrm{b}_1\ldots\mathrm{b}_4} A_{\mathrm{a}_2\mathrm{a}_3\mathrm{a}_4\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3,\,\mathrm{b}_4}\right) \wedge v^{\mathrm{a}_1} \wedge \cdots \wedge v^{\mathrm{a}_4}
\end{aligned}
$$

$$
\mathrm{d}F^{\mu}_3 = 0 \quad \text{and} \quad \mathrm{d}F_5 - \frac{1}{2} \epsilon_{\mu\nu} F^{\mu}_3 \wedge F^{\nu}_3 = 0
$$

$$
\begin{aligned}
F^{\mu}_3 &= \frac{1}{3!} f^{\mu}_{\mathrm{abc}} v^{\mathrm{a}} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}} - \frac{1}{2!} X^{\mu}_{A\mathrm{bc}} v^{A} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}} \text{ and} \\
F_5 &= \frac{1}{5!} f_{\mathrm{a}_1\ldots\mathrm{a}_5} v^{\mathrm{a}_1} \wedge \cdots \wedge v^{\mathrm{a}_5} - \frac{1}{4!} X_{A\mathrm{bc}_1\ldots\mathrm{c}_3} v^{A} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}_1} \wedge \cdots \wedge v^{\mathrm{c}_3}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}F^{\mu}_3 =& -\frac{1}{4}\left(f^{\mathrm{e}}_{\mathrm{ab}} f^{\mu}_{\mathrm{ecd}} + X^{E}_{\mathrm{a},\,\mathrm{b}} X^{\mu}_{E,c,d}\right) v^{\mathrm{a}} \wedge \cdots \wedge v^{\mathrm{d}} \\
& - \frac{1}{2}\left(f^{\mu}_{\mathrm{gbc}} X^{\mathrm{g}}_{\check{\alpha}\mathrm{d}} + X^{G}_{\check{\alpha}\mathrm{b}} X^{\mu}_{G,\mathrm{c},\,\mathrm{d}} - f^{\mathrm{g}}_{\mathrm{bc}} X^{\mu}_{\check{\alpha},\mathrm{g},\,\mathrm{d}}\right) v^{\check{\alpha}} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}} \wedge v^{\mathrm{d}} \\
& - \frac{1}{2\cdot 2!}\left(X^{E}_{\check{\alpha}\check{\beta}} X^{\mu}_{E,\mathrm{c},\,\mathrm{d}} + 4X^{\mu}_{\check{\alpha},\mathrm{e},\mathrm{c}} X^{\mathrm{e}}_{\check{\beta},\mathrm{d}}\right) v^{\check{\alpha}} \wedge v^{\check{\beta}} \wedge v^{\mathrm{c}} \wedge v^{\mathrm{d}},
\end{aligned}
$$

$$\mathrm{d}F_5-\frac{1}{2}\epsilon_{\mu\nu}F_3^{\mu}\wedge F_3^{\nu}$$
$$=-\left(\frac{1}{16}f_{\mathrm{a}_1\mathrm{a}_2}{}^{\mathrm{b}}f_{\mathrm{ba}_3\ldots\mathrm{a}_6}+\frac{1}{48}X_{\mathrm{a}_1\mathrm{a}_2}{}^{B}X_{B,\mathrm{a}_3,\mathrm{a}_4\mathrm{a}_5\mathrm{a}_6}-\frac{1}{18}\epsilon_{\mu\nu}f^{\mu}_{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3}f^{\nu}_{\mathrm{a}_4\mathrm{a}_5\mathrm{a}_6}\right)v^{\mathrm{a}_1}\wedge\cdots\wedge v^{\mathrm{a}_6}$$
$$-\left(\frac{1}{8}X_{\check{\alpha}\mathrm{b}_1}{}^{\mathrm{c}}f_{\mathrm{cb}_2\ldots\,\mathrm{b}_5}+\frac{1}{24}X_{\check{\alpha}\mathrm{b}_1}{}^{C}X_{C,\,\mathrm{b}_2,\,\mathrm{b}_3\,\mathrm{b}_4\,\mathrm{b}_5}\right.$$
$$\left.-\frac{1}{12}f_{\mathrm{b}_1\,\mathrm{b}_2}{}^{\mathrm{c}}X_{\check{\alpha},\mathrm{c},\,\mathrm{b}_3\,\mathrm{b}_4\,\mathrm{b}_5}+\frac{1}{6}\epsilon_{\mu\nu}f^{\mu}_{\mathrm{b}_1\,\mathrm{b}_2\,\mathrm{b}_3}X^{\nu}_{\check{\alpha},\mathrm{b}_4,\,\mathrm{b}_5}\right)v^{\check{\alpha}}\wedge v^{\mathrm{b}_1}\wedge\cdots\wedge v^{\mathrm{b}_5}$$
$$-\frac{1}{2\cdot 4!}\left(X_{\check{\alpha}\check{\beta}}{}^{G}X_{G,\mathrm{c},\mathrm{def}}+8X_{\check{\alpha}c}{}^{\mathrm{g}}X_{\check{\beta},\mathrm{g},\mathrm{def}}-6\epsilon_{\mu\nu}X^{\mu}_{\check{\alpha},\mathrm{c},\,\mathrm{d}}X^{\nu}_{\check{\beta},\mathrm{e},\mathrm{f}}\right)v^{\check{\alpha}}\wedge v^{\check{\beta}}\wedge v^{\mathrm{c}}\wedge\cdots\wedge v^{\mathrm{f}}$$

$$\mathrm{d}F_3^{\mu}=-\frac{1}{8}L_{\mathrm{a},\,\mathrm{b},\mathrm{c},\,\mathrm{d}}\mu v^{\mathrm{a}}\wedge\cdots\wedge v^{\mathrm{d}}-\frac{1}{3}L^{\mu}_{\check{\alpha},\mathrm{b},\mathrm{c},\,\mathrm{d}}v^{\check{\alpha}}\wedge v^{\mathrm{b}}\wedge v^{\mathrm{c}}\wedge v^{\mathrm{d}}$$
$$-\frac{1}{2\cdot 2!}L^{\mu}_{\check{\alpha},\check{\beta},\mathrm{c},\,\mathrm{d}}v^{\check{\alpha}}\wedge v^{\check{\beta}}\wedge v^{\mathrm{c}}\wedge v^{\mathrm{d}}$$

$$\mathrm{d}F_5-\frac{1}{2}\epsilon_{\mu\nu}F_3^{\mu}\wedge F_3^{\nu}=-\frac{1}{72}L_{\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3,\mathrm{a}_4\mathrm{a}_5\mathrm{a}_6}v^{\mathrm{a}_1}\wedge\cdots\wedge v^{\mathrm{a}_6}-\frac{1}{30}L_{\check{\alpha},\mathrm{b}_1,\,\mathrm{b}_2,\,\mathrm{b}_3\,\mathrm{b}_4\,\mathrm{b}_5}v^{\check{\alpha}}\wedge v^{\mathrm{b}_1}\wedge\cdots\wedge v^{\mathrm{b}_5}$$
$$-\frac{1}{2\cdot 4!}L_{\check{\alpha},\check{\beta},\mathrm{c},\mathrm{def}}v^{\check{\alpha}}\wedge v^{\check{\beta}}\wedge v^{\mathrm{c}}\wedge\cdots\wedge v^{\mathrm{f}}$$

$$v_i^{A}=\begin{pmatrix}v_i^{a} & A_{ia}\end{pmatrix}\ \text{and}\ A_a^{B}=v_a^{i}v_i^{B}=\begin{pmatrix}\delta_a^{b} & A_{ab}\end{pmatrix}$$

$$\hat{V}_A^{J}=\begin{pmatrix}v_a^{j} & 0\\ 0 & v_j^{a}\end{pmatrix}$$

$$N_I^{J}=\exp\left(\frac{1}{2!}B_{ij}R^{ij}\right)I^{J}=\begin{pmatrix}\delta_i^{j} & -B_{ij}\\ 0 & \delta_j^{i}\end{pmatrix}.$$

$$v^{c}([v_a,v_b])=f_{ab}{}^{c}-2f_{[a}{}^{cd}A_{b]d}\ \text{and}\ \mathrm{d}NN^{-1}=\frac{1}{2!}\mathrm{d}B_{ij}R^{ij}$$

$$\hat{\Omega}_{aB}{}^{C}=D_a\hat{V}_B{}^{J}\hat{V}_J{}^{C}+\hat{V}_B{}^{J}\hat{V}_K{}^{C}D_aN_J{}^{L}(N^{-1})_L{}^{K}$$
$$=-\left(D_{(a}v_{b)}^{j}v_j^{c}+\frac{1}{2}[v_a,v_b]^{j}v_j^{c}\right)\left(K^{b}{}_{c}\right)_B{}^{C}+\hat{V}_B{}^{J}\hat{V}_K{}^{C}D_aN_J{}^{L}(N^{-1})_L{}^{K}$$
$$\sim\left[\left(-\frac{1}{2}f_{ab}{}^{c}+f_{[a}{}^{cd}A_{b]d}\right)K^{b}{}_{c}-\frac{1}{3!}F_{abc}R^{bc}\right]_B{}^{C},$$

$$F_{ijk}:=3\partial_{[i}B_{jk]}.$$

$$\hat{\Omega}_a+A_a{}^{B}X_B\sim\left(-\frac{1}{2}f_{ab}{}^{c}+f_{[a}{}^{cd}A_{b]d}\right)K^{b}{}_{c}-\frac{1}{3!}F_{abc}R^{bc}+X_a+A_{ab}X^{b}$$
$$\sim X_a-\frac{1}{2}f_{ab}{}^{c}K^{b}{}_{c}-\frac{1}{3!}(F_{abc}+3f_{[ab}{}^{d}A_{c]d})R^{bc}$$

$$F_{abc}=-2f_{abc}-3f_{[ab}{}^{d}A_{c]d}$$

$$F_3=-\frac{2}{3!}f_{abc}v^{a}\wedge v^{b}\wedge v^{c}-\frac{1}{2}f_{ab}{}^{c}A_c\wedge v^{a}\wedge v^{b}=H.$$

$$E_A{}^{I}=M_A{}^{B}\hat{V}_B{}^{J}N_J{}^{I}$$

$$T_a \circ T_b = f_{ab}^{c} T_c + f_{abc_2} T^{c_2} + f_{abc_5} T^{c_5},$$
$$T_a \circ T^{b_2} = -f_a{}^{b_2 c} T_c + \delta_{de}^{b_2} f_{ac}{}^{d} T^{ec} + 3Z_a T^{b_2} - f_{ac_3} T^{b_2 c_3} + \delta_{ad}^{b_2} f_{c7} T^{c_7,d},$$
$$T_a \circ T^{b_5} = f_a{}^{b_5 c} T_c + \delta_{c_3 d_2}^{b_5} f_a{}^{c_3} T^{d_2} - \delta_{de_4}^{b_5} f_{ac}{}^{d} T^{e_4 c} + 6Z_a T^{b_5} + \delta_{d_4 e}^{b_5} f_{ac_3} T^{c_3 d_4,e},$$
$$T_a \circ T^{b_7,b'} = -\delta_{c_6 d}^{b_7} f_a{}^{c_6} T^{db'} + \delta_{c_2 d_5}^{b_7} f_a{}^{b' c_2} T^{d_5}$$
$$-\delta_{de_6}^{b_7} f_{ac}{}^{d} T^{e_6 c,b'} - f_{ac}{}^{b'} T^{b_7,c} + 9Z_a T^{b_7,b'},$$
$$T^{a_2} \circ T_b = f_b{}^{a_2 c} T_c + \delta_{de}^{a_2} \delta_{c_2 b}^{f_2 e} f_{f_2}{}^{d} T^{c_2} - 3Z_c \delta_{be_2}^{ca_2} T^{e_2} + \delta_{bd_5}^{a_2 c_4} f_{c_4} T^{d_5},$$
$$T^{a_2} \circ T^{b_2} = \delta_{de}^{b_2} f_c{}^{a_2 d} T^{ec} - \delta_{de}^{a_2} f_{c_2}{}^{d} T^{eb_2 c_2} + 3Z_c T^{a_2 b_2 c} + \delta_{de}^{b_2} f_{c_4} T^{a_2 c_4 d,e},$$
$$T^{a_2} \circ T^{b_5} = -\delta_{de_4}^{b_5} f_c{}^{a_2 d} T^{e_4 c} - \delta_{de}^{a_2} \delta_{f_2 g}^{c_2 e} f_{c_2}{}^{d} T^{b_5 f_2,g} + 3\delta_{d_2 e}^{a_2 c} Z_c T^{b_5 d_2,e},$$
$$T^{a_2} \circ T^{b_7,b'} = -\delta_{de_6}^{b_7} f_c{}^{a_2 d} T^{e_6 c,b'} - f_c{}^{a_2 b'} T^{b_7,c},$$
$$T^{a_5} \circ T_b = -f_b{}^{a_5 c} T_c - \delta_{c_3 d_2}^{a_5} f_b{}^{c_3} T^{d_2} - \delta_{d_3 bf}^{a_5} f_c{}^{d_3} T^{fc}$$
$$+\delta_{de_4}^{a_5} f_{bc}{}^{d} T^{e_4 c} + \delta_{dbf_3}^{a_5} f_{c_2}{}^{d} T^{f_3 c_2} - 6\delta_{bd_5}^{ca_5} Z_c T^{d_5},$$
$$T^{a_5} \circ T^{b_2} = -\delta_{de}^{b_2} f_c{}^{a_5 d} T^{ec} + \delta_{d_3 e_2}^{a_5} f_c{}^{d_3} T^{e_2 b_2 c}$$
$$+\delta_{de_4}^{a_5} \delta_{fg}^{b_2} f_{c_2}{}^{d} T^{e_4 c_2 f,g} - 6\delta_{de}^{b_2} Z_c T^{ca_5 d,e},$$
$$T^{a_5} \circ T^{b_5} = \delta_{de_4}^{b_5} f_c{}^{a_5 d} T^{e_4 c} + \delta_{f_3 g_2}^{a_5} \delta_{d_2 e}^{g_2 c} f_c{}^{f_3} T^{b_5 d_2,e},$$
$$T^{a_5} \circ T^{b_7,b'} = \delta_{de6}^{b_7} f_c{}^{a_5 d} T^{e_6 c,b'} + f_c{}^{a_5 b'} T^{b_7,c},$$
$$T^{a_7,a'} \circ T_b = \delta_{e_6 f}^{a_7} \delta_{bd_2}^{f' c} f_c{}^{e_6} T^{d_2} + \delta_{e_2 f_5}^{a_7} f_c{}^{a' e_2} \delta_{bd_5}^{f_5 c} T^{d_5},$$
$$T^{a_7,a'} \circ T^{b_2} = -\delta_{d_6 e}^{a_7} f_c{}^{d_6} T^{ea' cb_2} + \delta_{d_2 e_5}^{a_7} \delta_{fg}^{b_2} f_c{}^{a' d_2} T^{e_5 cf,g},$$
$$T^{a_7,a'} \circ T^{b_5} = -\delta_{f_6 g}^{a_7} \delta_{d_2 e}^{ga' c} f_c{}^{f_6} T^{b_5 d_2,e},$$
$$T^{a_7,a'} \circ T^{b_7,b'} = 0.$$

$$T_A = \left(T_a \; T^{a_2} \; T^{a_5} \; T^{a_7,a'} \; T^{a_8,a_3'} \; T^{a_8,a_6'} \; T^{a_8,a_8',a''}\right)$$

$$f_{a_7}, f_{a_4}, f_{a_2}{}^{b}, f_a{}^{b_3}, f_a{}^{b_6}, f_a{}^{b_8,b'}, \text{ and } Z_a.$$

$$X_a = -f_{ab_6} R^{b_6} - f_{ab_3} R^{b_3} + f_{ab}{}^{c} \widetilde{K}_c^b - Z_a\left(\widetilde{K}_b^b + t_0\right)$$
$$-f_a^{b_3} R_{b_3} - f_a^{b_6} R_{b_6} - f_a^{b_8,c} R_{b_8,c}$$
$$X^{a_2} = f_c^{da_2} \widetilde{K}_d^c - \delta_{de}^{a_2} f_{c_2}{}^{d} R^{ec_2} + 3Z_d R^{da_2} - f_{b_4} R^{a_2 b_4} + \delta_{cd}^{a_2} f_{b_7} R^{b_7 c,d}$$
$$X^{a_5} = f_c^{da_5} \widetilde{K}_d^c + \delta_{c_3 d_2}^{a_5} f_b{}^{c_3} R^{d_2 b} - \delta_{de_4}^{a_4} f_{c_2}{}^{d} R^{e_4 c_2} + 6Z_b R^{ba_5} + \delta_{c_3 d}^{b_4} f_{b_4} R^{a_5 c_3,d}$$
$$X^{a_7,a'} = \left(f_c^{da_7,a'} - \frac{1}{4} f_c^{a_7 a',d}\right) \widetilde{K}_d^c - \delta_{c_6 d}^{a_7} f_b{}^{c_6} R^{da' b} + \frac{1}{2} \delta_{c_6 d_2}^{a_7 a'} f_b{}^{c_6} R^{d_2 b}$$
$$-\delta_{c_2 d_5}^{a_7} f_b{}^{a' c_2} R^{d_5 b} - \frac{1}{4} \delta_{c_3 d_5}^{a_7 a'} f_b{}^{c_3} R^{d_5 b} - \delta_{cd_6}^{a_7} f_{b_2}{}^{c} R^{d_6 b_2,a'} + f_{bc}{}^{a\prime} R^{a_7 b,c}$$
$$+Z_b \left(9R^{ba_7,a'} + \frac{3}{4} R^{a_7 a',b}\right)$$
$$X^{a_8,a_3'} = -\delta_{cd_2}^{a_3'} f_b{}^{a_8,c} R^{d_2 b} + \delta_{c_3 d_5}^{a_8} f_b{}^{a_3' c_3} R^{d_5 b} - \delta_{c_2 d}^{a_3'} f_b{}^{c_2 b} R^{a_8,d} - f_b{}^{a_3} R^{a_8,b}$$
$$X^{a_8,a_6'} = \delta_{cd_5}^{a_6'} f_b{}^{a_8,c} R^{d_5 b} - \delta_{c_5 d}^{a_6'} f_b{}^{c_5 b} R^{8,d} - f_b{}^{a_6'} R^{a_8,b}$$
$$X^{a_8,a_8',a''} = -2f_d{}^{a_8,a''} R^{a_8',d} - f_d{}^{a_8,d} R^{a_8',a''}$$

$$T_a \circ T_b = f_{ab}{}^{c}T_c + f_{abc}^{\gamma}T_{\gamma}^{c} + f_{abc_3}T^{c_3},$$

$$T_a \circ T_{\beta}^{b} = f_{a\beta}^{bc}T_c - f_{a\beta}{}^{\gamma}T_{\gamma}^{b} - f_{ac}{}^{b}T_{\beta}^{c} + 2Z_a T_{\beta}^{b} - \epsilon_{\beta\gamma}f_{ac_2}^{\gamma}T^{bc_2} + f_{ac_4}T_{\beta}^{bc_4},$$

$$T_a \circ T^{b_3} = f_a^{b_3 c}T_c - \delta_{c_2\, d}^{b_3}\epsilon^{\gamma\delta}f_{a\gamma}^{c_2}T_{\delta}^{d} - \delta_{de_2}^{b_3}f_{ac}{}^{d}T^{e_2 c} + 4Z_a T^{b_3} - f_{ac_2}^{\gamma}T_{\gamma}^{b_3 c_2} - \delta_{c_2\, d}^{b_3}f_{ae_4}T^{c_2 e_4,\, d},$$

$$T_a \circ T_{\beta}^{b_5} = f_{a\beta}^{b_5 c}T_c - \delta_{c_4\, d}^{b_5}f_a{}^{c_4}T_{\beta}^{d} + \delta_{c_2\, d_3}^{b_5}f_a{}_{\beta}^{c_2}T^{d_3}$$

$$-f_{a\beta}{}^{\gamma}T_{\gamma}^{b_5} - \delta_{de_4}^{b_5}f_{ac}{}^{d}T_{\beta}^{e_4 c} + 6Z_a T_{\beta}^{b_5} + \epsilon_{\beta\gamma}f_{acd}^{\gamma}T^{b_5 c,\, d},$$

$$T_a \circ T^{b_6,\, b'} = \epsilon^{\gamma\delta}f_{a\gamma}^{b_6}T_{\delta}^{b'} - \delta_{c_3\, d_3}^{b_6}f_a^{b' c_3}T^{d_3} + \delta_{cd_5}^{b_6}\epsilon^{\gamma\delta}f_{a\gamma}^{b' c}T_{\delta}^{d_5}$$

$$-\delta_{de_5}^{b_6}f_{ac}{}^{d}T^{ce_5,\, b'} - f_{ac}{}^{b'}T^{b_6,c} + 8Z_a T^{b_6,\, b'},$$

$$T_{\alpha}^{a} \circ T_b = -f_b{}_{\alpha}^{ac}T_c - \delta_{bc}^{ad}f_{d\alpha}{}^{\gamma}T_{\gamma}^{c} + f_{bc}{}^{a}T_{\alpha}^{c} + 2\delta_{bd}^{ac}Z_c T_{\alpha}^{d} - \delta_{b_3}^{ad_3}\epsilon_{\alpha\gamma}f_{d_3}^{\gamma}T^{c_3} + \delta_{bd_5}^{ac_5}f_{c_5}T_{\alpha}^{d_5},$$

$$T_{\alpha}^{a} \circ T_{\beta}^{b} = f_{c_{\alpha}}^{ab}T_{\beta}^{c} + f_{c\alpha}{}^{\gamma}\epsilon_{\gamma\beta}T^{cab} + \epsilon_{\alpha\beta}f_{c_2}{}^{a}T^{c_2\, b} - 2\epsilon_{\alpha\beta}Z_c T^{abc} + \epsilon_{\alpha\gamma}f_{c_3}^{\gamma}T_{\beta}^{abc_3} - \epsilon_{\alpha\beta}f_{c_5}T^{ac_5,\, b},$$

$$T_{\alpha}^{a} \circ T^{b_3} = \delta_{de_2}^{b_3}f_{c\alpha}^{ad}T^{e_2 c} + f_{c\alpha}{}^{\gamma}T_{\gamma}^{acb_3} - f_{c_2}{}^{a}T_{\alpha}^{c_2\, b_3} + 2Z_c T_{\alpha}^{ab_3 c} + \delta_{ec_3}^{ad_3}\epsilon_{\alpha\gamma}f_{d_3}^{\gamma}T^{b_3 c_3,e},$$

$$T_{\alpha}^{a} \circ T_{\beta}^{b_5} = \delta_{de_4}^{b_5}f_{c\alpha}^{ad}T_{\beta}^{e_4 c} + \delta_{de}^{ac}f_{c\alpha}^{\gamma}\epsilon_{\gamma\beta}T^{b_5\, d,e} - \epsilon_{\alpha\beta}f_{cd}{}^{a}T^{b_5 c,\, d} + 4\epsilon_{\alpha\beta}Z_c T^{ab_5,c},$$

$$T_{\alpha}^{a} \circ T^{b_6,\, b'} = \delta_{de_5}^{b_6}f_{c\alpha}^{ad^{ad}}T^{ce_5,\, b'} + f_{c\alpha}^{ab'}T^{b_6,c},$$

$$T^{a_3} \circ T_b = -f_b{}^{a_3 c}T_c - \delta_{d_2 e}^{a_3}\delta_{bc}^{ef}\epsilon^{\gamma\delta}f_f^{d_2}T_{\delta}^{c} + \delta_{de_2}^{a_3}f_{bc}{}^{d}T^{e_2 c} - \delta_{bde}^{a_3}f_{c_2}{}^{d}T^{ec_2}$$

$$+4\delta_{bd_3}^{a_3 c}Z_c T^{d_3} - \delta_{bc_5}^{a_3\, d_3}f_{d_3}^{\gamma}T_{\gamma}^{c_5},$$

$$T^{a_3} \circ T_{\beta}^{b} = f_c^{a_3\, b}T_{\beta}^{c} - \delta_{d_2 e}^{a_3}f_{c\beta}^{d_2}T^{ebc} + \delta_{de_2}^{a_3}f_{c_2}{}^{d}T_{\beta}^{e_2 bc_2} - 4Z_c T_{\beta}^{a_3 bc} - \delta_{c_6}^{a_3\, d_3}\epsilon_{\beta\gamma}f_{d_3}^{\gamma}T^{c_6,\, b},$$

$$T^{a_3} \circ T^{b_3} = \delta_{de_2}^{b_3}f_c{}^{a_3\, d}T^{e_2 c} - \delta_{d_2 e}^{a_3}\epsilon^{\gamma\delta}f_{c\gamma}^{d_2}T_{\delta}^{eb_3 c}$$

$$+\delta_{def}^{a_3}f_{c_2}{}^{d}T^{b_3 c_2 e,f} + \delta_{ef_2}^{a_3}f_{cd}{}^{e}T^{f_2\, b_3 c,\, d} + 4\delta_{d_3 e}^{a_3 c}Z_c T^{b_3\, d_3,e},$$

$$T^{a_3} \circ T_{\beta}^{b_5} = \delta_{de_4}^{b_5}f_c^{a_3\, d}T_{\beta}^{e_4 c} - \delta_{f_2\, g}^{a_3}\delta_{de}^{gc}f_{c\beta}^{f_2}T^{b_5\, d,e},$$

$$T^{a_3} \circ T^{b_6,\, b'} = \delta_{de_5}^{b_6}f_c{}^{a_3\, d}T^{c_5,\, b'} + f_c{}^{a_3\, b'}T^{b_6,c},$$

$$T_{\alpha}^{a_5} \circ T_b = -f_{b_{\alpha}}^{a_5 c}T_c - \delta_{d_4 e}^{a_5}\delta_{bc}^{ef}f_f{}^{d_4}T_{\alpha}^{c} + \delta_{bd_2 e_2}^{a_5}f_{c\alpha}^{d_2}T^{e_2 c} - \delta_{c_2\, d_3}^{a_5}f_b{}^{c_2}T^{d_3} - \delta_{bd_4}^{a_5}f_{c\alpha}^{\gamma}T_{\gamma}^{d_4 c}$$

$$+f_{b\alpha}{}^{\gamma}T_{\gamma}^{a_5} + \delta_{de_4}^{a_5}f_{bc}{}^{d}T_{\alpha}^{e_4 c} + \delta_{dbe_3}^{a_5}f_{c_2}{}^{d}T_{\alpha}^{e_3 c_2} + 6\delta_{bd_5}^{a_5 c}Z_c T_{\alpha}^{d_5},$$

$$T_{\alpha}^{a_5} \circ T_{\beta}^{b} = f_{c\alpha}^{a_5\, b}T_{\beta}^{c} + \delta_{d_4 e}^{a_5}\epsilon_{\alpha\beta}f_c^{d_4}T^{ebc} - \delta_{d_2 e_3}^{a_5}f_{c\alpha}^{d_2}T_{\beta}^{e_3 bc}$$

$$-f_{c\alpha}{}^{\gamma}\epsilon_{\gamma\beta}T^{ca_6,\, b} - \delta_{de_4}^{a_5}\epsilon_{\alpha\beta}f_{c_2}{}^{d}T^{e_4 c_2,\, b} - 6\epsilon_{\alpha\beta}Z_c T^{a_5 c,\, b},$$

$$T_{\alpha}^{a_5} \circ T^{b_3} = \delta_{de_2}^{b_3}f_{c\alpha}^{a_5\, d}T^{e_2 c} - \delta_{d_4 e}^{a_5}f_c^{d_4}T_{\alpha}^{eb_3 c} + \delta_{f_2\, g_3}^{a_5}\delta_{d_3 e}^{g_3 c}f_{c_{\alpha}}^{f_2}T^{b_3\, d_3,e},$$

$$T_{\alpha}^{a_5} \circ T_{\beta}^{b_5} = \delta_{de_4}^{b_5}f_{c\alpha}^{a_5\, d}T_{\beta}^{e_4 c} + \delta_{f_4\, g}^{a_5}\delta_{de}^{gc}\epsilon_{\alpha\beta}f_c^{f_4}T^{b_5\, d,e},$$

$$T^{a_5} \circ T^{b_6,\, b'} = \delta_{de_5}^{b_6}f_{c\alpha}^{a_5\, d}T^{ce_5,\, b'} + f_{c\alpha}^{a_5\, b'}T^{b_6,c},$$

$$T^{a_6,a'} \circ T_b = \delta_{bd}^{a' c}\epsilon^{\gamma\delta}f_{c\gamma}^{a_6}T_{\delta}^{d} + \delta_{c_3\, d_3}^{a_6}f_b{}^{a' c_3}T^{d_3} + \delta_{bd_3 e_2}^{a_6}f_c{}^{a' d_3}T^{e_2 c}$$

$$-\delta_{cd_5}^{a_6}\epsilon^{\gamma\delta}f_{b_{\gamma}}{}^{a' c}T_{\delta}^{d_5} - \delta_{bde_4}^{a_6}\epsilon^{\gamma\delta}f_{c\gamma}^{a' d}T_{\delta}^{e_4 c},$$

$$T^{a_6,a'} \circ T_{\beta}^{b} = f_{c\beta}^{a_6}T^{a' bc} + \delta_{d_3 e_3}^{a_6}f_c^{a' d_3}T_{\beta}^{e_3 bc} + \delta_{de_5}^{a_6}f_{c\beta}^{a' d}T^{e_5 c,\, b},$$

$$T^{a_6,a'} \circ T^{b_3} = \epsilon^{\gamma\delta}f_{c\gamma}^{a_6}T_{\delta}^{a' b_3 c} - \delta_{f_3\, g_3}^{a_6}\delta_{d_3 e}^{g_3 c}f_c^{a' f_3}T^{b_3\, d_3,e},$$

$$T^{a_6,a'} \circ T_{\beta}^{b_5} = \delta_{de}^{a' c}f_{c\beta}^{a_6}T^{b_5\, d,e},$$

$$T^{a_6,a'} \circ T^{b_6,\, b'} = 0.$$

$$T_A = \left(T_a \quad T_{\alpha}^{a} \quad T^{a_3} \quad T_{\alpha}^{a_5} \quad T^{a_6,\, b} \quad T_{(\alpha_1\alpha_2)}^{a_7} \quad T_{\alpha}^{a_7,a_2'} \quad T^{a_7,a_4'} \quad T_{\alpha}^{a_7,a_6'} \quad T^{a_7,a_7',a''}\right).$$

$$
\begin{aligned}
&X_{\mathrm{a}} = -f_{\mathrm{ab}_6}^{\beta} R_{\beta}^{\mathrm{b}_6} - f_{\mathrm{ab}_4} R^{\mathrm{b}_4} - f_{\mathrm{ab}_2}^{\beta} R_{\beta}^{\mathrm{b}_2} + f_{\mathrm{ab}}{}^{\mathrm{c}} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} - Z_{\mathrm{a}}\left(\widetilde{K}^{\mathrm{b}}{}_{\mathrm{b}} + t_0\right)\\
&-f_{\mathrm{a}\beta}{}^{\gamma} R^{\beta}{}_{\gamma} - f_{\mathrm{a}\beta}^{\mathrm{b}_2} R_{\mathrm{b}_2}^{\beta} - f_{\mathrm{a}}{}^{\mathrm{b}_4} R_{\mathrm{b}_4} - f_{\mathrm{a}}{}^{\mathrm{b}_6}_{\beta} R_{\mathrm{b}_6}^{\beta} - f_{\mathrm{a}}{}^{\mathrm{b}_7,\,\mathrm{b}'} R_{\mathrm{b}_7,\,\mathrm{b}'},\\
&X_{\alpha}^{\mathrm{a}} = f_{\mathrm{b}}{}^{\mathrm{ca}}_{\alpha} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} - \left(\delta_{\alpha}^{\beta} f_{\mathrm{b}_2}{}^{\mathrm{a}} - \delta_{\mathrm{b}_2}^{\mathrm{ac}} f_{\mathrm{c}\alpha}{}^{\beta}\right) R_{\beta}^{\mathrm{b}_2} - 2Z_{\mathrm{b}} R_{\alpha}^{\mathrm{ab}}\\
&+\epsilon_{\alpha\beta} f_{\mathrm{b}_3}^{\beta} R^{\mathrm{ab}_3} - f_{\mathrm{b}_5} R_{\alpha}^{\mathrm{ab}_5} + \epsilon_{\alpha\beta} f_{\mathrm{b}_7}^{\beta} R^{\mathrm{b}_7,\mathrm{a}},\\
&X^{\mathrm{a}_3} = f_{\mathrm{b}}{}^{\mathrm{ca}_3} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} + \epsilon^{\beta\gamma} \delta_{\mathrm{c}_2\,\mathrm{d}}^{\mathrm{a}_3} f_{\mathrm{b}}{}^{\mathrm{c}_2}_{\beta} R_{\gamma}^{\mathrm{db}} - \delta_{\mathrm{cd}_2}^{\mathrm{a}_3} f_{\mathrm{b}_2}{}^{\mathrm{c}} R^{\mathrm{d}_2\,\mathrm{b}_2} - 4Z_{\mathrm{b}} R^{\mathrm{a}_3\,\mathrm{b}}\\
&+f_{\mathrm{b}_3}^{\alpha} R_{\alpha}^{\mathrm{a}_3\,\mathrm{b}_3} + \frac{1}{2} \delta_{\mathrm{c}_2\,\mathrm{d}}^{\mathrm{a}_3} f_{\mathrm{b}_5} R^{\mathrm{b}_5\mathrm{c}_2,\,\mathrm{d}},\\
&X_{\alpha}^{\mathrm{a}_5} = f_{\mathrm{b}}{}^{\mathrm{ca}_5}_{\alpha} \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} + \delta_{\mathrm{c}_4\,\mathrm{d}}^{\mathrm{a}_5} f_{\mathrm{b}}{}^{\mathrm{c}_4} R_{\alpha}^{\mathrm{db}} - \delta_{\mathrm{c}_2\,\mathrm{d}_3}^{\mathrm{a}_5} f_{\mathrm{b}}{}^{\mathrm{c}_2} R^{\mathrm{d}_3\,\mathrm{b}}\\
&-\delta_{\mathrm{cd}_4}^{\mathrm{a}_5} f_{\mathrm{b}_2}{}^{\mathrm{c}} R_{\alpha}^{\mathrm{d}_4\,\mathrm{b}_2} + f_{\mathrm{b}\alpha}{}^{\beta} R_{\beta}^{\mathrm{a}_5\,\mathrm{b}} - 6Z_{\mathrm{b}} R_{\alpha}^{\mathrm{a}_5\,\mathrm{b}} + \epsilon_{\alpha\beta} \delta_{\mathrm{c}_2\,\mathrm{d}}^{\mathrm{b}_3} f_{\mathrm{b}_3}^{\beta} R^{\mathrm{a}_5\mathrm{c}_2,\,\mathrm{d}},\\
&X^{\mathrm{a}_6,\mathrm{a}'} = \left(f_{\mathrm{b}}{}^{\mathrm{ca}_6,\mathrm{a}'} - c_{7,1} f_{\mathrm{b}}{}^{\mathrm{a}_6\mathrm{a}',\mathrm{c}}\right) \widetilde{K}^{\mathrm{b}}{}_{\mathrm{c}} - \epsilon^{\beta\gamma}\left(f_{\mathrm{b}}{}^{\mathrm{a}_6}_{\beta} R_{\gamma}^{\mathrm{a}'\mathrm{b}} - c_6 \delta_{\mathrm{c}_6\,\mathrm{d}}^{\mathrm{a}_6\mathrm{a}'} f_{\mathrm{b}_{\beta}}^{\mathrm{c}_6} R_{\gamma}^{\mathrm{db}}\right)\\
&+\delta_{\mathrm{c}_3\,\mathrm{d}_3}^{\mathrm{a}_6} f_{\mathrm{b}}{}^{\mathrm{a}'\mathrm{c}_3} R^{\mathrm{d}_3\,\mathrm{b}} - c_4 \delta_{\mathrm{c}_4\,\mathrm{d}_3}^{\mathrm{a}_6\mathrm{a}'} f_{\mathrm{b}}{}^{\mathrm{c}_4} R^{\mathrm{d}_3\,\mathrm{b}} - \epsilon^{\beta\gamma}\left(\delta_{\mathrm{cd}_5}^{\mathrm{a}_6} f_{\mathrm{b}}{}^{\mathrm{a}'\,\mathrm{c}} R_{\gamma}^{\mathrm{d}_5\,\mathrm{b}} - c_2 \delta_{\mathrm{c}_2\,\mathrm{d}_5}^{\mathrm{a}_6\mathrm{a}'} f_{\mathrm{b}_{\beta}}^{\mathrm{c}_2} R_{\gamma}^{\mathrm{d}_5\,\mathrm{b}}\right)\\
&-\delta_{\mathrm{cd}_5}^{\mathrm{a}_6} f_{\mathrm{b}_2}{}^{\mathrm{c}} R^{\mathrm{d}_5\,\mathrm{b}_2,\mathrm{a}'} - f_{\mathrm{bc}}{}^{\mathrm{a}'} R^{\mathrm{a}_6\,\mathrm{b},\mathrm{c}} + 8Z_{\mathrm{b}} R^{\mathrm{a}_6\,\mathrm{b},\mathrm{a}'} - \left(1 + c_{7,1}\right) Z_{\mathrm{b}} R^{\mathrm{a}_6\mathrm{a}',\mathrm{b}},\\
&X_{(\alpha_1\alpha_2)}^{\mathrm{a}_7} = \delta_{\mathrm{c}_6\,\mathrm{d}}^{\mathrm{a}_7} f_{\mathrm{b}}{}^{\mathrm{c}_6}_{(\alpha_1} R_{\alpha_2)}^{\mathrm{db}} - \delta_{\mathrm{c}_2\,\mathrm{d}_5}^{\mathrm{a}_7} f_{\mathrm{b}}{}^{\mathrm{c}_2}_{(\alpha_1} R_{\alpha_2)}^{\mathrm{d}_5\,\mathrm{b}} - f_{\mathrm{b}(\alpha_1}{}^{\beta} \epsilon_{\alpha_2)\beta} R^{\mathrm{a}_7,\,\mathrm{b}},\\
&X_{\alpha}^{\mathrm{a}_7,\mathrm{a}_2'} = \delta_{\mathrm{cd}}^{\mathrm{a}_2'} f_{\mathrm{b}}{}^{\mathrm{a}_7,\mathrm{c}} R_{\alpha}^{\mathrm{db}} + \delta_{\mathrm{c}_6\,\mathrm{d}}^{\mathrm{a}_7} f_{\mathrm{b}}{}^{\mathrm{c}_6} R^{\mathrm{da}_2'\mathrm{b}} - \delta_{\mathrm{c}_2\,\mathrm{d}_5}^{\mathrm{a}_7} f_{\mathrm{b}}{}^{\mathrm{a}_2'\mathrm{c}_2} R_{\alpha}^{\mathrm{d}_5\,\mathrm{b}} - \delta_{\mathrm{cd}}^{\mathrm{a}_2'} f_{\mathrm{b}}{}^{\mathrm{cb}} R^{\mathrm{a}_7,\,\mathrm{d}} - f_{\mathrm{b}}{}^{\mathrm{a}_2'} R^{\mathrm{a}_7,\,\mathrm{b}},\\
&X^{\mathrm{a}_7,\mathrm{a}_4'} = \delta_{\mathrm{cd}_3}^{\mathrm{a}_4'} f_{\mathrm{b}}{}^{\mathrm{a}_7,\mathrm{c}} R^{\mathrm{d}_3\,\mathrm{b}} + \epsilon^{\beta\gamma} \delta_{\mathrm{c}_6\,\mathrm{d}}^{\mathrm{a}_7} f_{\mathrm{b}}{}^{\mathrm{c}_6}_{\beta} R_{\gamma}^{\mathrm{da}_4'\,'\mathrm{b}} - \delta_{\mathrm{c}_3\,\mathrm{d}}^{\mathrm{a}_4'} f_{\mathrm{b}}{}^{\mathrm{c}_3\,\mathrm{b}} R^{\mathrm{a}_7,\,\mathrm{d}} - f_{\mathrm{b}}{}^{\mathrm{a}_4'} R^{\mathrm{a}_7,\,\mathrm{b}},\\
&X_{\alpha}^{\mathrm{a}_7,\mathrm{a}_6'} = \delta_{\mathrm{cd}_5}^{\mathrm{a}_6'} f_{\mathrm{b}}{}^{\mathrm{a}_7,\mathrm{c}} R_{\alpha}^{\mathrm{d}_5\,\mathrm{b}} - \delta_{\mathrm{c}_5\,\mathrm{d}}^{\mathrm{a}_6'} f_{\mathrm{b}}{}^{\mathrm{c}_5\,\mathrm{b}} R^{\mathrm{a}_7,\,\mathrm{d}} - f_{\mathrm{b}\alpha}{}^{\mathrm{a}_6'} R^{\mathrm{a}_7,\,\mathrm{b}},\\
&X^{\mathrm{a}_7,\mathrm{a}_7',\mathrm{a}''} = -f_{\mathrm{b}}{}^{\mathrm{a}_7,\,\mathrm{b}} R^{\mathrm{a}_7',\mathrm{a}''} - 2f_{\mathrm{b}}{}^{\mathrm{a}_7,\mathrm{a}''} R^{\mathrm{a}_7',\mathrm{b}},
\end{aligned}
$$

$$F_7^{\mu} := \mathrm{d}B_6^{\mu} - B_2^{\mu} \wedge \mathrm{d}D_4 - \frac{1}{3!} \epsilon_{\nu\rho} B_2^{\mu} \wedge B_2^{\nu} \wedge F_3^{\rho}$$

$$F_7^{\alpha} = \frac{1}{7!} f_{\mathrm{a}_1\ldots\mathrm{a}_7}^{\alpha} v^{\mathrm{a}_1} \wedge \cdots \wedge v^{\mathrm{a}_7} - \frac{1}{6!} X_{A,\,\mathrm{b},\mathrm{c}_1\ldots\mathrm{c}_5}^{\gamma} v^{A} \wedge v^{\mathrm{b}} \wedge v^{\mathrm{c}_1} \wedge \cdots \wedge v^{\mathrm{c}_5}.$$

$$\hat{\mathcal{L}}_V W^I = \begin{pmatrix} \mathcal{L}_v w^i \\ (\mathcal{L}_v w_2 - \iota_w\, \mathrm{d}v_2)_{i_1 i_2} \\ (\mathcal{L}_v w_5 + \mathrm{d}v_2 \wedge w_2 - \iota_w\, \mathrm{d}v_5)_{i_1\ldots i_5} \\ \left[\mathcal{L}_v w_{7,i'} + (\iota_{i'} \mathrm{d}v_2) \wedge w_5 + (\iota_{i'} w_2) \wedge \mathrm{d}v_5\right]_{i_1\ldots i_7} \end{pmatrix},$$

$$\hat{\mathcal{L}}_V W^I = \begin{pmatrix} \mathcal{L}_v w^{\mathrm{m}} \\ \left(\mathcal{L}_v w_1^{\mu} - \iota_w\, \mathrm{d}v_1^{\mu}\right)_{\mathrm{m}} \\ \left(\mathcal{L}_v w_3 - \epsilon_{\mu\nu} \mathrm{d}v_1^{\mu} \wedge w_1^{\nu} - \iota_w\, \mathrm{d}v_3\right)_{\mathrm{m}_1\,\mathrm{m}_2\,\mathrm{m}_3} \\ \left(\mathcal{L}_v w_5^{\mu} + \mathrm{d}v_1^{\mu} \wedge w_3 - \mathrm{d}v_3 \wedge w_1^{\mu} - \iota_w\, \mathrm{d}v_5^{\mu}\right)_{\mathrm{m}_1\ldots\,\mathrm{m}_5} \\ \left[\mathcal{L}_v w_{6,\,\mathrm{m}'} + \epsilon_{\mu\nu}\left(\iota_{\mathrm{m}'} \mathrm{d}v_1^{\mu}\right) \wedge w_5^{\nu} - (\iota_{\mathrm{m}'} \mathrm{d}v_3) \wedge w_3 + \epsilon_{\mu\nu} \mathrm{d}v_5^{\mu} w_{\mathrm{m}'}^{\nu}\right]_{\mathrm{m}_1\ldots\,\mathrm{m}_6} \end{pmatrix},$$

$$[V,W]^I + [W,V]^I = Y_{KL}^{IJ}\left(\partial_J V^K W^L + \partial_J W^K V^L\right)$$

$$[V,W]^I + [W,V]^I = \mathcal{D}^{I;\mathcal{J}} (V \times_N W)_{\mathcal{J}}$$

$$\mathcal{D}^{I;\mathcal{J}} := 2\eta^{IK;\mathcal{J}} \partial_K,$$

$$(V \times_N W)_{\mathcal{J}} = \eta_{JK;\mathcal{J}} V^J W^K$$

$$\mathcal{D}^{I;\mathcal{J}}\left(f N_{\mathcal{J}}\right) = f\mathcal{D}^{I;\mathcal{J}} N_{\mathcal{J}} + 2\eta^{IK;\mathcal{J}} \partial_K f N_{\mathcal{J}} = f\mathcal{D}^{I;\mathcal{J}} N_{\mathcal{J}} + \partial_K f N^{KI} + \partial_K f N^{IK}$$

$$Y^{IJ}_{KL} = \eta^{IJ;\mathcal{J}} \eta_{KL;\mathcal{J}} - \frac{1}{2}\Omega^{IJ}\Omega_{KL}$$

$$\begin{aligned}[V, W]^I + [W, V]^I &= \mathfrak{D}^I(V \otimes W) \\ &:= \mathcal{D}^{I;\mathcal{J}}(V \times_N W)_{\mathcal{J}} - \frac{1}{2}\Omega_{JK}\left(\tilde{\mathcal{D}}^I V^J W^K - V^J \tilde{\mathcal{D}}^I W^K\right),\end{aligned}$$

$$[E_A, E_B] = -X^C_{AB} E_C$$

$$\left[V_1, [V_2, V_3]\right] = \left[[V_1, V_2], V_3\right] + \left[V_2, [V_1, V_3]\right]$$

$$[t(w), V] = 0, t(w) \circ t(w) = 0, U \overset{s}{\circ} [V, V] = V \circ [U, V], \text{ and } [V, V] = t(V \circ V),$$

$$t\left(v^{\dot{a}}\right) = V^A = \left(0, v^{\dot{a}}\right),$$

$$U \overset{s}{\circ} V = U^A V^B Z_{AB}{}^{\dot{c}} T_{\hat{c}}.$$

$$x \cdot m := [x + \mathcal{I}, m] = [x, m].$$

$$(T_{\dot{\alpha}})_{\dot{\beta}}{}^{\dot{\gamma}} = -X_{\dot{\alpha}\dot{\beta}}{}^{\hat{\gamma}}.$$

$$[(x, m), (y, n)] = ([x, y], x \cdot n + \alpha(x, y)).$$

$$\alpha\left(T_{\dot{a}}, T_{\dot{b}}\right) = X_{\dot{a}\dot{b}}{}^{\hat{\gamma}} T_{\hat{\gamma}},$$

$$x \cdot \alpha(y, z) - y \cdot \alpha(x, z) = \alpha([x, y], z) + \alpha(y, [x, z]) - \alpha(x, [y, z]).$$

$$F = \mathrm{d}A + \frac{1}{2}[A \wedge A] = 0$$

$$\begin{aligned}F &= \mathrm{d}A + \frac{1}{2}[A \wedge A]_- - t(B), \text{ and} \\ G &= \mathrm{d}B + t(B) \circ A - \frac{1}{3!}\gamma(A \wedge, A \wedge, A) - A \circ F,\end{aligned}$$

$$\gamma(V_1, V_2, V_3) := -X^E_{[AB} Z^D_{C]E} V_1^A V_2^B V_3^C.$$

$$F_{p+2} = \frac{1}{(p+2)!} f_{a_1 \cdots a_{p+2}} v^{a_1} \wedge \cdots \wedge v^{a_{p+2}} + \frac{1}{(p+1)!} v_{a, b_1 \cdots b_p} \wedge v^a \wedge v^{b_1} \wedge \cdots \wedge v^{b_p}$$

$$g_{ab}(x) dx^a dx^b = e^{2\omega}(d\phi + v dt)(d\phi + \bar{v} dt) = e^{2\sigma} dx^- dx^+$$

$$T^{\mathrm{F}}_{--}(x^-,x^+) = T_{--}(F_t(\phi)) - \frac{c}{24\pi}\frac{1}{F_t'(\phi)^2}\{F_t(\phi),\phi\}$$
$$T^{\mathrm{F}}_{++}(x^-,x^+) = T_{++}(\bar{F}_t(\phi)) - \frac{c}{24\pi}\frac{1}{\bar{F}_t'(\phi)^2}\{\bar{F}_t(\phi),\phi\}$$
$$T^{\mathrm{F}}_{-+}(x^-,x^+) = 0$$

$$F_t'(\phi)^2 T^{\mathrm{F}}_{--}(\phi,t) \equiv \lim_{\epsilon\to 0}\left[\frac{1}{2}F_t'(\phi+\epsilon)F_t'(\phi)[\Pi_-(F_t(\phi+\epsilon)),\Pi_-(F_t(\phi))]_+ + \frac{1}{4\pi\epsilon^2}\right]$$

$$H_{\mathrm{S}}(t) = -\int_0^{2\pi} d\phi\, v(\phi,t)T_{--}(\phi) + \int_0^{2\pi} d\phi\, \bar{v}(\phi,t)T_{++}(\phi)$$

$$U(t) = \mathcal{T}\exp\left(-i\int_0^t ds H_{\mathrm{S}}(s)\right) = V_{f_t}\otimes \bar{V}_{\bar{f}_t}$$

$$T^{\mathrm{H}}_{\pm\pm}(\phi,t) = U(t)^\dagger T^{\mathrm{S}}_{\pm\pm}(\phi)U(t)$$

$$T^{\mathrm{H}}_{--}(\phi,t) = F_t'(\phi)^2 T^{\mathrm{F}}_{--}(x^-,x^+), T^{\mathrm{H}}_{++}(\phi,t) = \bar{F}_t'(\phi)^2 T^{\mathrm{F}}_{++}(x^-,x^+).$$

$$\langle A(\phi_1,t_1)\cdots A(\phi_n,t_n)\rangle = \langle 0|A_{\mathrm{H}}(\phi_1,t_1)\cdots A_{\mathrm{H}}(\phi_n,t_n)|0\rangle.$$

$$\langle T_{--}(\phi,t)\rangle = -\frac{c}{48\pi} - \frac{c}{24\pi}\frac{1}{F_t'(\phi)^2}\{F_t(\phi),\phi\}, \langle T_{++}(\phi,t)\rangle = -\frac{c}{48\pi} - \frac{c}{24\pi}\frac{1}{\bar{F}_t'(\phi)^2}\{\bar{F}_t(\phi),\phi\}$$

$$\langle \mathcal{O}(\phi,t)\mathcal{O}(\theta,s)\rangle = \frac{b_{\mathcal{O}}}{(2\pi)^4}\left[\frac{e^{-2(\omega(\phi,t)+\omega(\theta,s))}F_t'(\phi)\bar{F}_t'(\phi)F_s'(\theta)\bar{F}_s'(\theta)}{16\sin^2\left(\frac{F_t(\phi)-F_s(\theta)}{2}\right)\sin^2\left(\frac{\bar{F}_t(\phi)-\bar{F}_s(\theta)}{2}\right)}\right]^{\Delta/2}.$$

$$S_{\mathrm{D}}(t) = \frac{c}{6}\big(\omega(\phi_1,t)+\omega(\phi_2,t)\big) + \frac{c}{12}\left[\log\left(\frac{4\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2 F_t'(\phi_1)F_t'(\phi_2)}\right) + (F_t\leftrightarrow \bar{F}_t)\right]$$

$$S(t) = \frac{c}{12}\log\left(\frac{4\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2 F_t'(\phi_1)F_t'(\phi_2)}\right) + (F_t\leftrightarrow \bar{F}_t)$$

$$\tilde{G}_{\mu\nu}(X)dX^\mu dX^\nu = \ell^2\left[\frac{1}{z^2}(dz^2 + e^{2\sigma}dx^-dx^+) + g_{(2)ab}dx^a dx^b + z^2 g_{(4)ab}dx^a dx^b\right]$$

$$g_{(2)\pm\pm} = \partial_\pm^2\sigma - (\partial_\pm\sigma)^2 - \frac{1}{4}, g_{(2)-+} = \partial_-\partial_+\sigma$$
$$g_{(4)\pm\pm} = e^{-2\sigma}g_{(2)-+}g_{(2)\pm\pm}, g_{(4)-+} = \frac{1}{2}e^{-2\sigma}\big(g_{(2)--}g_{(2)++} + g_{(2)-+}^2\big)$$

$$g_{ab}(x)dx^a dx^b = e^{2\omega}(d\phi + v dt)(d\phi + \bar{v} dt)$$

$$\widetilde{\mathrm{Diff}}_+ S^1 \equiv \{F:\mathbb{R}\to\mathbb{R} \mid F(\phi+2\pi) = F(\phi)+2\pi, F'(\phi)>0\}$$

$$v(\phi,t)=\frac{\dot{F}_t(\phi)}{F_t'(\phi)},\bar{v}(\phi,t)=\frac{\dot{\bar{F}}_t(\phi)}{\bar{F}_t'(\phi)}$$

$$x^-=F_t(\phi),x^+=\bar{F}_t(\phi)$$

$$g_{ab}(x)dx^a dx^b=e^{2\omega+2\varphi}dx^- dx^+,e^{-2\varphi}=F_t'(\phi)\bar{F}_t'(\phi)$$

$$v(\phi,t)=-\frac{1}{p'(\phi)},\bar{v}(\phi,t)=\frac{1}{\bar{p}'(\phi)}.$$

$$F_t(\phi)=(h\circ p^{-1})(p(\phi)-t),\bar{F}_t(\phi)=(\bar{h}\circ\bar{p}^{-1})(\bar{p}(\phi)+t),$$

$$g_{ab}(x)dx^a dx^b=e^{2\omega}(-dt^2+d\phi^2)$$

$$g_{ab}(x)dx^a dx^b=-\frac{1}{p'(\phi)^2}dt^2+d\phi^2$$

$$\begin{aligned}(\psi g)_{ab}(x)&=e^{2\chi(D(x))}\frac{\partial D^c}{\partial x^a}\frac{\partial D^d}{\partial x^b}g_{cd}(D(x)),\\(\psi\Phi)_{a_1\ldots a_n}(x)&=e^{-\Delta_\Phi\chi(D(x))}\frac{\partial D^{b_1}}{\partial x^{a_1}}\cdots\frac{\partial D^{b_n}}{\partial x^{a_n}}\Phi_{b_1\ldots b_n}(D(x)),\end{aligned}$$

$$T_{ab}^{\mathrm{cl}}(x)\equiv-\frac{2}{\sqrt{-g}}\frac{\delta I[\Phi,g]}{\delta g^{ab}(x)}$$

$$\nabla^b T_{ab}^{\mathrm{cl}}=E_\alpha(\Phi)\pounds_a\Phi^\alpha,g^{ab}T_{ab}^{\mathrm{cl}}=\Delta_\Phi E_\alpha(\Phi)\Phi^\alpha,$$

$$\nabla^b T_{ab}^{\mathrm{D}}=0,g^{ab}T_{ab}^{\mathrm{D}}=-\frac{c}{24\pi}R$$

$$T_{ab}^{\mathrm{D}}=T_{ab}+C_{ab}^{\mathrm{D}},$$

$$\nabla^b T_{ab}=0,g^{ab}T_{ab}=0$$

$$\nabla^b C_{ab}^{\mathrm{D}}=0,g^{ab}C_{ab}^{\mathrm{D}}=-\frac{c}{24\pi}R$$

$$C_{ab}^{\mathrm{D}}=-\frac{2}{\sqrt{-g}}\frac{\delta A_{\mathrm{D}}[g]}{\delta g^{ab}}$$

$$A_{\mathrm{D}}[g]=-\frac{c}{96\pi}\int d^2x\sqrt{-g}R\frac{1}{\nabla^2}R$$

$$I_{\mathrm{Lio}}[\chi,g]\equiv\frac{c}{24\pi}\int d^2x\sqrt{-g}\left(g^{ab}\partial_a\chi\partial_b\chi+\chi R\right)$$

$$C_{ab}^{\mathrm{D}}=\frac{c}{12\pi}\left[\nabla_a\sigma\nabla_b\sigma+\nabla_a\nabla_b\sigma-g_{ab}\left(\nabla^2\sigma+\frac{1}{2}\nabla^c\sigma\nabla_c\sigma\right)\right].$$

$$I_{\mathrm{D}}[\Phi_{\mathrm{H}},g]=\mathcal{N}_{\mathrm{D}}\{I[\Phi_{\mathrm{H}},g]\}+A_{\mathrm{D}}[g]$$

$$T_{\pm\pm}(x^-,x^+)=T_{\pm\pm}(x^\pm),T_{-+}(x^-,x^+)=0$$

$$T^{\mathrm{F}}_{ab} = T_{ab} + C^{\mathrm{F}}_{ab},$$

$$A_{\mathrm{D}}[g] = I_{\mathrm{Lio}}[\omega,\hat{g}] + A_{\mathrm{D}}[\hat{g}] = I_{\mathrm{Lio}}[\varphi,\eta] + I_{\mathrm{Lio}}[\omega,\hat{g}] + A_{\mathrm{D}}[\eta],$$

$$I_{\mathrm{Lio}}[\varphi,\eta] = \Gamma[\nu] + \bar{\Gamma}[\bar{\nu}] + K[\nu,\bar{\nu}]$$

$$\Gamma[\nu] = \frac{c}{48\pi}\int d\phi dt \nu \partial_\phi^2 \log\, F_t'(\phi), \bar{\Gamma}[\bar{\nu}] = -\frac{c}{48\pi}\int d\phi dt \bar{\nu}\partial_\phi^2 \log \bar{F}_t'(\phi)$$

$$K[\nu,\bar{\nu}] = \frac{c}{48\pi}\int d\phi dt \frac{1}{\bar{\nu}-\nu}\left[(\partial_\phi \nu) + (\partial_\phi \bar{\nu})\right]^2$$

$$A_{\mathrm{D}}[g] = \Gamma[\nu] + \bar{\Gamma}[\bar{\nu}] + K[\nu,\bar{\nu}] + I_{\mathrm{Lio}}[\omega,\hat{g}]$$

$$A_{\mathrm{F}}[g] \equiv \Gamma[\nu] + \bar{\Gamma}[\bar{\nu}] = A_{\mathrm{D}}[g] - (K[\nu,\bar{\nu}] + I_{\mathrm{Lio}}[\omega,\hat{g}])$$

$$C^{\mathrm{F}}_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta A_{\mathrm{F}}[g]}{\delta g^{ab}}$$

$$C^{\mathrm{F}}_{--} = -\frac{c}{24\pi}\frac{1}{F_t'(\phi)^2}\{F_t(\phi),\phi\}, C^{\mathrm{F}}_{++} = -\frac{c}{24\pi}\frac{1}{\bar{F}_t'(\phi)^2}\{\bar{F}_t(\phi),\phi\}, C^{\mathrm{F}}_{-+} = 0$$

$$\{F(\phi),\phi\} = \left(\frac{F''(\phi)}{F'(\phi)}\right)' - \frac{1}{2}\left(\frac{F''(\phi)}{F'(\phi)}\right)^2$$

$$T^{\mathrm{F}}_{--}(x^-,x^+) = T_{--}(F_t(\phi)) - \frac{c}{24\pi}\frac{1}{F_t'(\phi)^2}\{F_t(\phi),\phi\}$$

$$T^{\mathrm{F}}_{++}(x^-,x^+) = T_{++}(\bar{F}_t(\phi)) - \frac{c}{24\pi}\frac{1}{\bar{F}_t'(\phi)^2}\{\bar{F}_t(\phi),\phi\}$$

$$T^{\mathrm{F}}_{-+}(x^-,x^+) = 0$$

Scheme	Diff $\ltimes$ Weyl Ward identities
Diff. invariant	$(\partial_t - \nu\,\partial_\phi - 2\partial_\phi \nu)\,(F_t'^2 T^{\mathrm{D}}_{--}) - \frac{c}{96\pi}\, e^{2\omega}\,(\partial_t - \bar{\nu}\,\partial_\phi)\,R = 0$ $(\partial_t - \bar{\nu}\,\partial_\phi - 2\partial_\phi \bar{\nu})(\bar{F}_t'^2 T^{\mathrm{D}}_{++}) - \frac{c}{96\pi}\, e^{2\omega}\,(\partial_t - \nu\,\partial_\phi)\,R = 0$ $F_t'\bar{F}_t'\, T^{\mathrm{D}}_{-+} + \frac{c}{96\pi}\, e^{2\omega} R = 0$
Chirally split	$(\partial_t - \nu\,\partial_\phi - 2\partial_\phi \nu)\,(F_t'^2 T^{\mathrm{F}}_{--}) + \frac{c}{24\pi}\,\partial_\phi^3 \nu = 0$ $(\partial_t - \bar{\nu}\,\partial_\phi - 2\partial_\phi \bar{\nu})\,(\bar{F}_t'^2 T^{\mathrm{F}}_{++}) + \frac{c}{24\pi}\,\partial_\phi^3 \bar{\nu} = 0$ $F_t'\bar{F}_t'\, T^{\mathrm{F}}_{-+} = 0$

$$T_{--}^{\mathrm{F}}(x^-,x^+) = T_{--}(x^-) + \frac{c}{24\pi}\{f_t(x^-),x^-\}$$
$$T_{++}^{\mathrm{F}}(x^-,x^+) = T_{++}(x^+) + \frac{c}{24\pi}\{\bar{f}_t(x^+),x^+\}$$
$$T_{-+}^{\mathrm{F}}(x^-,x^+) = 0$$

$$[T(\phi_1),T(\phi_2)] = -i\big(T(\phi_1)+T(\phi_2)\big)\delta'_{2\pi}(\phi_1-\phi_2) + \frac{ic}{24\pi}\delta'''_{2\pi}(\phi_1-\phi_2),$$

$$\delta_{2\pi}(\phi) \equiv \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\phi}$$

$$T(\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} L_n e^{in\phi}, [L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{n,-m}$$

$$T_{--}(\phi) = T(\phi)\otimes 1, T_{++}(\phi) = 1\otimes \bar{T}(\phi),$$

$$V_{F_t}T(\phi)V_{F_t}^{\dagger} = F_t'(\phi)^2 T(F_t(\phi)) - \frac{c}{24\pi}\{F_t(\phi),\phi\}$$

$$\left(V_{F_t}\otimes \bar{V}_{\bar{F}_t}\right)T_{--}(\phi)\left(V_{F_t}^{\dagger}\otimes \bar{V}_{\bar{F}_t}^{\dagger}\right) = F_t'(\phi)^2 T_{--}^{\mathrm{F}}(x^-,x^+)$$

$$\left(V_{F_t}\otimes \bar{V}_{\bar{F}_t}\right)T_{--}(\phi)\left(V_{F_t}^{\dagger}\otimes \bar{V}_{\bar{F}_t}^{\dagger}\right) \neq F_t'(\phi)^2 T_{--}^{\mathrm{D}}(x^-,x^+)$$

$$I[\Phi,g] = \int d^2x\sqrt{-g}\mathcal{L} = -\frac{1}{2}\int d^2x\sqrt{-g}g^{ab}\partial_a\Phi\partial_b\Phi$$

$$T_{\pm\pm}^{\mathrm{cl}} = (\partial_{\pm}\Phi)^2, T_{-+}^{\mathrm{cl}} = 0$$

$$\Pi(\phi,t) = -\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\dot{\Phi}(\phi,t)} = -F_t'(\phi)\partial_-\Phi + \bar{F}_t'(\phi)\partial_+\Phi$$

$$[\Phi_{\mathrm{H}}(\phi_1,t),\Pi_{\mathrm{H}}(\phi_2,t)] = i\delta_{2\pi}(\phi_1-\phi_2)$$

$$\Phi_{\mathrm{H}}(\phi,t) = \Phi_-(x^-) + \Phi_+(x^+) = \Phi_-(F_t(\phi)) + \Phi_+(\bar{F}_t(\phi)),$$

$$\Pi_{\mathrm{H}}(\phi,t) = -F_t'(\phi)\Pi_-(F_t(\phi)) + \bar{F}_t'(\phi)\Pi_+(\bar{F}_t(\phi)).$$

$$\Pi_-(\phi) = -\frac{1}{\sqrt{2}}(J(\phi)\otimes\mathbf{1}), \Pi_+(\phi) = \frac{1}{\sqrt{2}}(\mathbf{1}\otimes\bar{J}(\phi)),$$

$$[J(\phi_1),J(\phi_2)] = -i\delta'_{2\pi}(\phi_1-\phi_2)$$

$$T_{--}^{\mathrm{cl}}(\phi,t) = \Pi_-^{\mathrm{cl}}(F_t(\phi))^2, T_{++}^{\mathrm{cl}}(\phi,t) = \Pi_+^{\mathrm{cl}}(\bar{F}_t(\phi))^2.$$

$$\frac{1}{4}[J(\phi_1),J(\phi_2)]_+ = -\frac{1}{4\pi}\frac{1}{4\sin^2\left(\frac{\phi_1-\phi_2}{2}\right)} + T(\phi_1) + \frac{1}{2\pi}\frac{1}{24} + \mathcal{O}(\phi_1-\phi_2), \phi_2\to\phi_1$$

$$F_t'(\phi)^2 T_{--}^{\mathrm{F}}(\phi,t) \equiv \lim_{\epsilon\to 0}\left[\frac{1}{2}F_t'(\phi+\epsilon)F_t'(\phi)[\Pi_-(F_t(\phi+\epsilon)),\Pi_-(F_t(\phi))]_+ + \frac{1}{4\pi\epsilon^2}\right]$$

$$H_{\mathrm{cl}}(t) \equiv \frac{d}{ds} I[\Phi_s, g; M_s]\Big|_{s=t}$$

$$H_{\mathrm{cl}}(t) = -\int_0^{2\pi} d\phi \sqrt{\gamma} \zeta^a n^b T_{ab}^{\mathrm{cl}}$$

$$\zeta^a = \delta_t^a, n_a = \frac{\partial_a t}{\sqrt{-g^{ab}\partial_a t \partial_b t}}$$

$$H_{\mathrm{cl}}(t) =- \int_0^{2\pi} d\phi \dot{F}_t(\phi) F_t'(\phi) T_{--}^{\mathrm{cl}}(x^-, x^+) + \int_0^{2\pi} d\phi \dot{\bar{F}}_t(\phi) \bar{F}_t'(\phi) T_{++}^{\mathrm{cl}}(x^-, x^+)$$
$$+ \int_0^{2\pi} d\phi \big(\dot{F}_t(\phi) \bar{F}_t'(\phi) - \dot{\bar{F}}_t(\phi) F_t'(\phi)\big) T_{-+}^{\mathrm{cl}}(x^-, x^+)$$

$$\partial_t |\Psi(t)\rangle = -i H_{\mathrm{S}}(t) |\Psi(t)\rangle$$

$$U(t) \equiv \overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds H_{\mathrm{S}}(s)\right) = \vec{\mathcal{T}}\exp\left(-i\int_0^t ds H_{\mathrm{H}}(s)\right)$$

$$H_{\mathrm{H}}(t) \equiv U(t)^\dagger H_{\mathrm{S}}(t) U(t)$$

$$H_{\mathrm{H}}(t) =- \int_0^{2\pi} d\phi v(\phi, t) F_t'(\phi)^2 T_{--}(F_t(\phi)) + \int_0^{2\pi} d\phi \bar{v}(\phi, t) \bar{F}_t'(\phi)^2 T_{++}(\bar{F}_t(\phi))$$
$$+ \frac{c}{24\pi} \int_0^{2\pi} d\phi v(\phi, t) \{F_t(\phi), \phi\} - \frac{c}{24\pi} \int_0^{2\pi} d\phi \bar{v}(\phi, t) \{\bar{F}_t(\phi), \phi\}$$

$$H_{\mathrm{S}}(t) = -\int_0^{2\pi} d\phi v(\phi, t) T_{--}(\phi) + \int_0^{2\pi} d\phi \bar{v}(\phi, t) T_{++}(\phi)$$

$$U(t) = \overleftarrow{\mathcal{T}}\exp\left(i\int_0^t ds \int_0^{2\pi} d\phi\, v(\phi, t)\, T(\phi)\right) \otimes \overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds \int_0^{2\pi} d\phi\, \bar{v}(\phi, t)\, \overline{T}(\phi)\right)$$

$$v(\phi, t) = -(\dot{f}_t \circ f_t^{-1})(\phi), \bar{v}(\phi, t) = -\left(\dot{\bar{f}}_t \circ \bar{f}_t^{-1}\right)(\phi).$$

$$U(t) = V_{f_t} \otimes \bar{V}_{\bar{f}_t}$$

$$V_{f_t} V_{h_t} = e^{iB(f_t, h_t)} V_{f_t \circ h_t}, \bar{V}_{f_t} \bar{V}_{h_t} = e^{-iB(f_t, h_t)} \bar{V}_{f_t \circ h_t}.$$

$$B(f_t, h_t) = b(f_t, h_t) + \frac{c}{48\pi} \int_0^{2\pi} d\phi \frac{h_t''(\phi)}{h_t'(\phi)} \log\, f_t'(h_t(\phi))$$

$$a(f_t) = \frac{c}{48\pi} \int_0^t ds \int_0^{2\pi} d\phi \frac{\dot{F}_s(\phi)}{F_s'(\phi)} \left(\frac{F_s''(\phi)}{F_s'(\phi)}\right)'$$

$$|\Psi(t)\rangle = e^{iB(f_t, h^{-1}) - iB(\bar{f}_t, \bar{h}^{-1})} \big(V_{f_t \circ h^{-1}} \otimes \bar{V}_{\bar{f}_t \circ \bar{h}^{-1}}\big) |0\rangle,$$

$$H_{\mathrm{S}}(t) = \int_0^{2\pi} d\phi \frac{T_{--}(\phi)}{p'(\phi)} + \int_0^{2\pi} d\phi \frac{T_{++}(\phi)}{\bar{p}'(\phi)}$$

$$H_{\mathrm{S}}(t) = (V_P \otimes \bar{V}_{\bar{P}}) H_0 (V_P \otimes \bar{V}_{\bar{P}})^\dagger + \frac{c}{24\pi} \int_0^{2\pi} d\phi (\{P(\phi), \phi\} + \{\bar{P}(\phi), \phi\})$$

$$H_0 \equiv \int_0^{2\pi} d\phi (T_{--}(\phi) + T_{++}(\phi)) = L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0$$

$$U(t) = e^{-\frac{ict}{24\pi}\int_0^{2\pi} d\phi(\{P(\phi),\phi\}+\{\bar{P}(\phi),\phi\})} (V_P \otimes \bar{V}_{\bar{P}}) e^{-iH_0 t} (V_P \otimes \bar{V}_{\bar{P}})^\dagger$$

$$|\Psi(t)\rangle = e^{-\frac{ict}{24\pi}\int_0^{2\pi} d\phi(\{P(\phi),\phi\}+\{\bar{P}(\phi),\phi\}-1)} |\Psi(0)\rangle$$

$$T^{\mathrm{F}}_{\pm\pm}(x^-, x^+) = T_{\pm\pm}(\phi \pm t) = e^{itH_0} T_{\pm\pm}(\phi) e^{-itH_0}$$

$$T^{\mathrm{H}}_{\pm\pm}(\phi, t) = U(t)^\dagger T^{\mathrm{S}}_{\pm\pm}(\phi) U(t), T^{\mathrm{H}}_{-+}(\phi, t) = 0$$

$$T^{\mathrm{S}}_{--}(\phi) = T(\phi) \otimes 1, T^{\mathrm{S}}_{++}(\phi) = 1 \otimes \bar{T}(\phi), T^{\mathrm{S}}_{-+}(\phi) = 0$$

$$T^{\mathrm{H}}_{--}(\phi, t) = F_t'(\phi)^2 T^{\mathrm{F}}_{--}(x^-, x^+), T^{\mathrm{H}}_{++}(\phi, t) = \bar{F}_t'(\phi)^2 T^{\mathrm{F}}_{++}(x^-, x^+)$$

$$\partial_t T^{\mathrm{H}}_{--} = \left(\nu \partial_\phi + 2\partial_\phi \nu\right) T^{\mathrm{H}}_{--} - \frac{c}{24\pi} \partial^3_\phi \nu, \partial_t T^{\mathrm{H}}_{++} = \left(\bar{\nu} \partial_\phi + 2\partial_\phi \bar{\nu}\right) T^{\mathrm{H}}_{++} - \frac{c}{24\pi} \partial^3_\phi \bar{\nu}$$

$$V_{F_t} J(\phi) V_{F_t}^\dagger = F_t'(\phi) J(F_t(\phi))$$

$$\Pi_S(\phi) = -\Pi_-(\phi) + \Pi_+(\phi)$$

$$T^S_{\pm\pm}(\phi) \equiv \lim_{\epsilon \to 0} \left(\frac{1}{2} [\Pi_\pm(\phi + \epsilon), \Pi_\pm(\phi)]_+ + \frac{1}{4\pi\epsilon^2} \right)$$

$$\langle A(\phi_1, t_1) \cdots A(\phi_n, t_n) \rangle \equiv \langle 0 | \mathcal{T}\{A_{\mathrm{H}}(\phi_1, t_1) \cdots A_{\mathrm{H}}(\phi_n, t_n)\} | 0 \rangle,$$

$$\langle A(\phi, t) \rangle = \langle \Psi(t) | A_{\mathrm{S}}(\phi, t) | \Psi(t) \rangle$$

$$\langle A(\phi_1, t_1) \cdots A(\phi_n, t_n) \rangle = \langle 0 | A_{\mathrm{H}}(\phi_1, t_1) \cdots A_{\mathrm{H}}(\phi_n, t_n) | 0 \rangle.$$

$$\langle T_{--}(\phi, t) \rangle = -\frac{c}{48\pi} + \frac{c}{24\pi} \{f_t(x^-), x^-\}, \langle T_{++}(\phi, t) \rangle = -\frac{c}{48\pi} + \frac{c}{24\pi} \{\bar{f}_t(x^+), x^+\},$$

$$\langle 0 | L_n | 0 \rangle = -\frac{c}{24} \delta_{n,0}$$

$$\langle H(t) \rangle = \frac{c}{24\pi} \int_0^{2\pi} d\phi \nu(\phi, t) \left(\frac{1}{2} F_t'(\phi)^2 + \{F_t(\phi), \phi\} \right) - (F_t \leftrightarrow \bar{F}_t)$$

$$F_t(\phi) = p(\phi) - t, \bar{F}_t(\phi) = \bar{p}(\phi) + t$$

$$\langle H(t) \rangle = -\frac{c}{12} + \frac{c}{24\pi} \int_0^{2\pi} d\phi (\{P(\phi), \phi\} + \{\bar{P}(\phi), \phi\})$$

$$\langle H(t) \rangle = -\frac{c}{12} + \frac{c}{48\pi} \int_0^{2\pi} d\phi \left(\frac{\nu'(\phi)^2}{\nu(\phi)} - \frac{\bar{\nu}'(\phi)^2}{\bar{\nu}(\phi)} \right)$$

$$(\psi\mathcal{O})(x) = e^{-\Delta\chi(D(x)))}\mathcal{O}(D(x))$$

$$\mathcal{O}_{\mathrm{H}}(\phi,t) = e^{-\Delta\omega(\phi,t)}F_t'(\phi)^{\Delta/2}\bar{F}_t'(\phi)^{\Delta/2}\mathcal{O}_{\Delta/2}(F_t(\phi)) \otimes \overline{\mathcal{O}}_{\Delta/2}(\bar{F}_t(\phi)),$$

$$\langle\mathcal{O}(\phi,t)\mathcal{O}(\theta,s)\rangle = \langle 0|\mathcal{O}_{\mathrm{H}}(\phi,t)\mathcal{O}_{\mathrm{H}}(\theta,s)|0\rangle$$

$$\langle 0|\mathcal{O}_{\Delta/2}(\phi_1)\mathcal{O}_{\Delta/2}(\phi_2)|0\rangle = \lim_{\varepsilon\to 0^+}\frac{1}{(2\pi)^2}\frac{b_{\mathcal{O}}}{\left[2i\sin\left(\frac{\phi_1-\phi_2+i\varepsilon}{2}\right)\right]^{\Delta}}$$

$$\langle\mathcal{O}(\phi,t)\mathcal{O}(\theta,s)\rangle = \frac{b_{\mathcal{O}}}{(2\pi)^4}\left[\frac{e^{-2(\omega(\phi,t)+\omega(\theta,s))}F_t'(\phi)\bar{F}_t'(\phi)F_s'(\theta)\bar{F}_s'(\theta)}{-16\sin^2\left(\frac{F_t(\phi)-F_s(\theta)}{2}\right)\sin^2\left(\frac{\bar{F}_t(\phi)-\bar{F}_s(\theta)}{2}\right)}\right]^{\Delta/2}$$

$$S(t) \equiv -\mathrm{Tr}(\rho(t)\log\,\rho(t)) = -\lim_{n\to 1^+}\partial_n\mathrm{Tr}\rho(t)^n$$

$$(g_{\mathrm{e}})_{ab}dx^a dx^b \equiv e^{2\sigma_{\mathrm{e}}(w,\bar{w})}dwd\bar{w}$$

$$\sigma_{\mathrm{e}}(x^-,x^+) = \sigma(x^-,x^+)$$

$$\mathcal{R}_{(S^1\times\mathbb{R},g_{\mathrm{e}})}(w_1,\bar{w}_1;w_2,\bar{w}_2) \equiv \frac{Z[\mathcal{P}^*g_{\mathrm{e}};S^1\times\mathbb{R}]}{Z[g_{\mathrm{e}};S^1\times\mathbb{R}]^n}$$

$$Z[g_{\mathrm{e}};M] \equiv \int\ [d\Phi_{\mathrm{e}}]e^{-I_{\mathrm{E}}[\Phi_{\mathrm{e}},g_{\mathrm{e}};M]}$$

$$\mathrm{Tr}\rho(t)^n = \mathcal{R}_{(S^1\times\mathbb{R},g)}\left(F_t(\phi_1),\bar{F}_t(\phi_1);F_t(\phi_2),\bar{F}_t(\phi_2)\right)$$

$$\mathcal{C}^w(w,\bar{w}) = C(w) \equiv \tan\frac{w}{2}, \mathcal{C}^{\bar{w}}(w,\bar{w}) = \bar{C}(\bar{w}) \equiv \tan\frac{\bar{w}}{2}$$

$$\mathcal{R}_{(S^1\times\mathbb{R},g_{\mathrm{e}})}(w_1,\bar{w}_1;w_2,\bar{w}_2) = \mathcal{R}_{\left(\mathbb{R}^2,(\mathcal{C}^{-1})^*g_{\mathrm{e}}\right)}\left(\mathcal{C}(w_1,\bar{w}_1);\mathcal{C}(w_2,\bar{w}_2)\right)$$

$$(g_{\mathrm{e}})_{ab}dx^a dx^b = e^{2\chi_{\mathrm{e}}}dwd\bar{w}$$

$$\frac{Z_{\mathrm{D}}[g_{\mathrm{e}};\mathbb{R}^2]}{Z_{\mathrm{D}}[\eta_{\mathrm{e}};\mathbb{R}^2]} = e^{I_{\mathrm{Lio}}[\chi_{\mathrm{e}},\eta_{\mathrm{e}};\mathbb{R}^2]}$$

$$\mathcal{P}^w(w,\bar{w}) = P(w) \equiv \left(\frac{w-w_1}{w-w_2}\right)^n, \mathcal{P}^{\bar{w}}(w,\bar{w}) = \bar{P}(\bar{w}) \equiv \left(\frac{\bar{w}-\bar{w}_1}{\bar{w}-\bar{w}_2}\right)^n$$

$$(\mathcal{P}^*g_{\mathrm{e}})_{ab}(x)dx^a dx^b = e^{2\tilde{\chi}_{\mathrm{e}}(w,\bar{w})}dwd\bar{w}$$

$$\tilde{\chi}_{\mathrm{e}}(w,\bar{w}) = \kappa_{\mathrm{e}}(w,\bar{w}) + \Sigma_{\mathrm{e}}(w,\bar{w})$$

$$\kappa_{\mathrm{e}}(w,\bar{w}) \equiv \frac{1}{2}\log\left(P'(w)\bar{P}'(\bar{w})\right), \Sigma_{\mathrm{e}}(w,\bar{w}) \equiv \chi_{\mathrm{e}}(P(w),\bar{P}(\bar{w}))$$

$$\tilde{\chi}_{\mathrm{e}}(w,\bar{w}) = \frac{\gamma_i}{2}\log|w-w_i|^2 + \mathcal{O}(1), w\to w_i$$

$$Z_{\mathrm{D}}^{\mathrm{reg}}[\mathcal{P}^*g_{\mathrm{e}};\mathbb{R}^2] \equiv Z_{\mathrm{D}}[\mathcal{P}^*g_{\mathrm{e}};\mathbb{R}^2\setminus\mathcal{B}]$$

$$\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(\mathbb{R}^2,g_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2) \equiv \frac{Z^{\text{reg}}_{\text{D}}[\mathcal{P}^* g_e;\mathbb{R}^2]}{Z^{\text{reg}}_{\text{D}}[g_e;\mathbb{R}^2]^n},$$

$$\frac{Z^{\text{reg}}_{\text{D}}[\mathcal{P}^* g_e;\mathbb{R}^2]}{Z^{\text{reg}}_{\text{D}}[\eta_e;\mathbb{R}^2]} = e^{I^{\text{reg}}_{\text{Lio}}[\tilde{\chi}_e,\eta_e;\mathbb{R}^2]}$$

$$\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(\mathbb{R}^2,g_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2) = e^{-\Delta_n(\chi_e(w_1,\bar{w}_1)+\chi_e(w_2,\bar{w}_2))}\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(\mathbb{R}^2,\eta_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2)$$

$$\Delta_n \equiv \frac{c}{12}\frac{\gamma_i(\gamma_i+2)}{\gamma_i+1} = \frac{c}{12}\left(n-\frac{1}{n}\right).$$

$$\chi_e(w,\bar{w}) = \sigma_e(C^{-1}(w),\bar{C}^{-1}(\bar{w})) + \frac{1}{2}\log\,(C^{-1})'(w) + \frac{1}{2}\log\,(\bar{C}^{-1})'(\bar{w})$$

$$\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(S^1\times\mathbb{R},g_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2)$$
$$= \mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(\mathbb{R}^2,\eta_e)}\big(C(w_1),\bar{C}(\bar{w}_1);C(w_2),\bar{C}(\bar{w}_2)\big)\prod_{i=1}^{2}\left[e^{-2\sigma_e(w_i,\bar{w}_i)}C'(w_i)\bar{C}'(\bar{w}_i)\right]^{\Delta_n/2}$$

$$\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(\mathbb{R}^2,\eta_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2) = b(n)\left[\frac{\tilde{\delta}_1\tilde{\delta}_2}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)}\right]^{\Delta_n}$$

$$\mathcal{R}^{\tilde{\delta}_1,\tilde{\delta}_2}_{(S^1\times\mathbb{R},g_e)}(w_1,\bar{w}_1;w_2,\bar{w}_2) = b(n)\left[\frac{\prod_{i=1}^{2}\tilde{\delta}_i^2 e^{-2\sigma_e(w_i,\bar{w}_i)}C'(w_i)\bar{C}'(w_i)}{\big(C(w_1)-C(w_2)\big)^2\big(\bar{C}(\bar{w}_1)-\bar{C}(\bar{w}_2)\big)^2}\right]^{\Delta_n/2}.$$

$$\text{Tr}\rho(t)^n|_{\text{D}} = b(n)\left[\frac{\tilde{\delta}_1^2\tilde{\delta}_2^2 e^{-2(\omega(\phi_1,t)+\omega(\phi_2,t))}F_t'(\phi_1)\bar{F}_t'(\phi_1)F_t'(\phi_2)\bar{F}_t'(\phi_2)}{16\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)\sin^2\left(\frac{\bar{F}_t(\phi_1)-\bar{F}_t(\phi_2)}{2}\right)}\right]^{\Delta_n/2},$$

$$\text{Tr}\rho(t)^n|_{\text{D}} = b(n)\left[\frac{\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2^2}{16\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)\sin^2\left(\frac{\bar{F}_t(\phi_1)-\bar{F}_t(\phi_2)}{2}\right)}\right]^{\Delta_n/2},$$

$$S_{\text{D}}(t) = \frac{c}{6}\big(\omega(\phi_1,t)+\omega(\phi_2,t)\big) + \frac{c}{12}\left[\log\left(\frac{4\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2F_t'(\phi_1)F_t'(\phi_2)}\right) + (F_t\leftrightarrow\bar{F}_t)\right] - b'(1),$$

$$S_{\text{D}}(t)|_{\text{flat}} = \frac{c}{12}\log\left[\frac{4}{\varepsilon_1\varepsilon_2}\sin^2\left(\frac{F_t(\phi_1)-F_t(\phi_2)}{2}\right)\right] + (F_t\leftrightarrow\bar{F}_t) - b'(1)$$

$$\text{Tr}\rho(t)^n|_{\text{D}} = \langle 0|\mathcal{T}^{\text{H}}_n(\phi_1,t)\tilde{\mathcal{T}}^{\text{H}}_n(\phi_2,t)|0\rangle,$$

$$Z^{\text{reg}}_{\text{F}}[g_e;S^1\times\mathbb{R}] \equiv Z^{\text{reg}}_{\text{D}}[g_e;S^1\times\mathbb{R}]e^{-I^{\text{reg}}_{\text{Lio}}[\omega_e,\hat{g}_e;S^1\times\mathbb{R}]}$$

$$\frac{Z_{\mathrm{D}}^{\mathrm{reg}}[g_{\mathrm{e}}; S^1 \times \mathbb{R}]}{Z_{\mathrm{D}}^{\mathrm{reg}}[\hat{g}_{\mathrm{e}}; S^1 \times \mathbb{R}]} = e^{I_{\mathrm{Lio}}^{\mathrm{reg}}[\omega_{\mathrm{e}}, \hat{g}_{\mathrm{e}}; S^1 \times \mathbb{R}]}$$

$$Z_{\mathrm{F}}^{\mathrm{reg}}[g_{\mathrm{e}}; S^1 \times \mathbb{R}] = Z_{\mathrm{D}}^{\mathrm{reg}}[\hat{g}_{\mathrm{e}}; S^1 \times \mathbb{R}]$$

$$S(t) = \frac{c}{12}\log\left(\frac{4\sin^2\left(\frac{F_t(\phi_1) - F_t(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2 F_t'(\phi_1)F_t'(\phi_2)}\right) + (F_t \leftrightarrow \bar{F}_t) - b'(1),$$

$$S(t) = \frac{c}{12}\log\left(\frac{4\sin^2\left(\frac{p(\phi_1) - p(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2 p'(\phi_1)p'(\phi_2)}\right) + (p \leftrightarrow \bar{p}) - b'(1).$$

$$\mathcal{T}_n^{\mathrm{S}}(\phi) = \mathcal{O}_{\Delta_n/2}(\phi) \otimes \mathcal{O}_{\Delta_n/2}(\phi)$$

$$\mathrm{Tr}\rho(t)^n = \langle \Psi(t) | \mathcal{T}_n^{\mathrm{S}}(\phi) \tilde{\mathcal{T}}_n^{\mathrm{S}}(\phi) | \Psi(t) \rangle$$

$$S_{\mathrm{D}}(t) - S(0) = \frac{c}{6}\big(\omega(\phi_1, t) + \omega(\phi_2, t)\big)$$

$$S(t) = \frac{c}{6}\big(\rho(\phi_1, t) + \rho(\phi_2, t)\big) + \frac{c}{12}\log\left[\frac{4}{\varepsilon_1\varepsilon_2}\sin^2\left(\frac{h(\phi_1 - t) - h(\phi_2 - t)}{2}\right)\right] + (h \leftrightarrow \bar{h}),$$

$$G_{\mu\nu}(X)dX^\mu dX^\nu = \frac{\ell^2}{z^2}(dz^2 + dx^- dx^+)$$

$$\lim_{z\to 0}\frac{z^2}{\ell^2}G_{ab}(X)dX^a dX^b = \eta_{ab}dx^a dx^b = dx^- dx^+$$

$$\lim_{z\to 0}\frac{z^2}{\ell^2}\tilde{G}_{ab}(X)dX^a dX^b = g_{ab}(x)dx^a dx^b = e^{2\sigma}dx^- dx^+$$

$$\tilde{G}_{\mu\nu}(X) = \frac{\partial D^\rho}{\partial x^\mu}\frac{\partial D^\sigma}{\partial x^\nu}G_{\rho\sigma}(D(X)).$$

$$C_\pm(x^\pm) = \tan\frac{x^\pm}{2}$$

$$D^z(z, x^-, x^+) = ze^{-2\sigma(x^-, x^+)}C_-'(x^-)C_+'(x^+) + \mathcal{O}(z^2)$$
$$D^\pm(z, x^-, x^+) = C_\pm(x^\pm) + \mathcal{O}(z^2)$$

$$P^z(z, x^-, x^+) = \frac{ze^{-\sigma}(\partial_- C_- \partial_+ C_+)^{3/2}}{\partial_- C_- \partial_+ C_+ + \frac{1}{4}z^2 e^{-2\sigma}(\mathcal{D}_- \partial_- C_-)(\mathcal{D}_+ \partial_+ C_+)}$$

$$P^\pm(z, x^-, x^+) = C_\pm(x^\pm) + \frac{\frac{1}{2}z^2 e^{-2\sigma}(\partial_\pm C_\pm)^2(\mathcal{D}_\mp \partial_\mp C_\mp)}{\partial_- C_- \partial_+ C_+ + \frac{1}{4}z^2 e^{-2\sigma}(\mathcal{D}_- \partial_- C_-)(\mathcal{D}_+ \partial_+ C_+)}$$

$$\tilde{G}_{\mu\nu}(X)dX^\mu dX^\nu = \ell^2\left[\frac{1}{z^2}(dz^2 + e^{2\sigma}dx^- dx^+) + g_{(2)ab}dx^a dx^b + z^2 g_{(4)ab}dx^a dx^b\right]$$

$$g_{(2)\pm\pm} = \partial_\pm^2\sigma - (\partial_\pm\sigma)^2 - \frac{1}{2}\{C_\pm(x^\pm), x^\pm\}, g_{(2)-+} = \partial_-\partial_+\sigma$$

$$g_{(4)\pm\pm} = e^{-2\sigma} g_{(2)-+} g_{(2)\pm\pm}, g_{(4)-+} = \frac{1}{2} e^{-2\sigma}\left(g_{(2)--} g_{(2)++} + g_{(2)-+}^2\right)$$

$$I_{\mathrm{EH}}[G] = \frac{1}{2\kappa}\int_{z>\epsilon} d^3X\sqrt{-G}\left(\mathcal{R} + \frac{2}{\ell^2}\right) + \frac{1}{\kappa}\int_{z=\epsilon} d^2x\sqrt{-\gamma}\left(K - \frac{1}{\ell}\right)$$

$$T_{ab}^{\mathrm{BY}} \equiv \lim_{\epsilon\to 0}\frac{-2}{\sqrt{-\gamma}}\frac{\delta I_{\mathrm{EH}}[\tilde{G}]}{\delta\gamma^{ab}} = -\frac{1}{\kappa}\left(K_{ab} - K\gamma_{ab} + \frac{1}{\ell}\gamma_{ab}\right)$$

$$T_{ab}^{\mathrm{BY}} = \frac{\ell}{\kappa}\left(g_{(2)ab} - g_{ab}g^{cd}g_{(2)cd}\right)$$

$$T_{\pm\pm}^{\mathrm{BY}} = C_{\pm\pm}^{\mathrm{D}} - \frac{c}{24\pi}\{C_\pm(x^\pm), x^\pm\}, T_{-+}^{\mathrm{BY}} = -C_{-+}^{\mathrm{D}} - \frac{c}{12\pi} g_{-+}\nabla^2\sigma,$$

$$g_{(2)\pm\pm} = \frac{12\pi}{c} C_{\pm\pm}^{\mathrm{D}} - \frac{1}{2}\{C_\pm(x^\pm), x^\pm\}, g_{(2)-+} = \frac{12\pi}{c} C_{-+}^{\mathrm{D}} + g_{-+}\nabla^2\sigma$$

$$T_{\pm\pm}^{\mathrm{BY}}(x^-, x^+) = \langle 0|T_{\pm\pm}(x^\pm)|0\rangle + C_{\pm\pm}^{\mathrm{D}}(x^-, x^+),$$

$$T_{\pm\pm}^{\mathrm{BY}}(x^-, x^+) = \langle 0|T_{\pm\pm}^{\mathrm{D}}(x^-, x^+)|0\rangle.$$

$$\langle\mathcal{O}(\phi_1, t_1)\mathcal{O}(\phi_2, t_2)\rangle = e^{-\Delta\tilde{L}_{\mathrm{reg}}(\phi_1,t_1;\phi_2,t_2)}, \Delta\to\infty$$

$$L(X_1, X_2) = \log\left(\frac{1+\sqrt{1-\xi_{12}^2}}{\xi_{12}}\right), \xi_{12} = \frac{2z_1z_2}{z_1^2 + z_2^2 + (x_1^- - x_2^-)(x_1^+ - x_2^+)}$$

$$\tilde{L}(X_1; X_2) = L\big(P(X_1); P(X_2)\big)$$

$$\tilde{\xi}_{12} \equiv \frac{2P^z(X_1)P^z(X_2)}{P^z(X_1)^2 + P^z(X_2)^2 + \big(P^-(X_1) - P^-(X_2)\big)\big(P^+(X_1) - P^+(X_2)\big)}$$

$$\tilde{L}_{\mathrm{reg}}(\phi_1, t_1; \phi_2, t_2) \equiv \tilde{L}\left(\varepsilon_1, F_{t_1}(\phi_1), \bar{F}_{t_1}(\phi_1); \varepsilon_2, F_{t_2}(\phi_2), \bar{F}_{t_2}(\phi_2)\right).$$

$$\tilde{L}_{\mathrm{reg}} = \frac{1}{2}\log\left[\frac{16}{\varepsilon_1^2\varepsilon_2^2} e^{2\sigma(\phi_1,t_1)+2\sigma(\phi_2,t_2)}\sin^2\left(\frac{x_1^- - x_2^-}{2}\right)\sin^2\left(\frac{x_1^+ - x_2^+}{2}\right)\right] + \cdots,$$

$$e^{-\Delta\tilde{L}_{\mathrm{reg}}} = \left[\frac{\varepsilon_1^2\varepsilon_2^2 e^{-2(\omega(\phi_1,t_1)+\omega(\phi_2,t_2))} F_{t_1}'(\phi_1)\bar{F}_{t_1}'(\phi_1)F_{t_2}'(\phi_2)\bar{F}_{t_2}'(\phi_2)}{16\sin^2\left(\frac{F_{t_1}(\phi_1) - F_{t_2}(\phi_2)}{2}\right)\sin^2\left(\frac{\bar{F}_{t_1}(\phi_1) - \bar{F}_{t_2}(\phi_2)}{2}\right)}\right]^{\Delta/2},$$

$$S_{\mathrm{RT}}(t) = \frac{\mathrm{Area}(\mathcal{S})}{4G_N}$$

$$S_{\mathrm{RT}}(t) = \frac{c}{6}\big(\omega(\phi_1, t) + \omega(\phi_2, t)\big) + \frac{c}{12}\left[\log\left(\frac{4\sin^2\left(\frac{F_t(\phi_1) - F_t(\phi_2)}{2}\right)}{\varepsilon_1\varepsilon_2 F_t'(\phi_1)F_t'(\phi_2)}\right) + (F_t \leftrightarrow \bar{F}_t)\right].$$

$$I[\Phi, g; M] = \int_M d^d x \sqrt{-g} \mathcal{L}$$

$$\delta I[\Phi, g; M] = \int_M d^d x \sqrt{-g} \left(-\frac{1}{2} T_{ab}^{\mathrm{cl}} \delta g^{ab} + E_\alpha \delta \Phi^\alpha\right) + \int_{\partial M} d^{d-1} \hat{x} \sqrt{\gamma} n_a \Pi_\alpha^a \delta \Phi^\alpha,$$

$$E_\alpha = \frac{\partial \mathcal{L}}{\partial \Phi^\alpha} - \nabla_a \left(\frac{\partial \mathcal{L}}{\partial(\nabla_a \Phi^\alpha)}\right), \Pi_\alpha^a = -\frac{\partial \mathcal{L}}{\partial(\nabla_a \Phi^\alpha)}.$$

$$\delta_B I[\Phi, g; M] = -\int_{\partial M} d^{d-1} \hat{x} \sqrt{\gamma} \mathcal{L} n_a \delta B^a$$

$$I[D^* \Phi, D^* g; D^{-1}(M)] = I_M[\Phi, g; M]$$

$$(D^* g)_{ab}(x) = \frac{\partial D^c}{\partial x^a} \frac{\partial D^d}{\partial x^b} g_{cd}(D(x))$$

$$\delta_D \Phi^\alpha = \pounds_\xi \Phi^\alpha, \delta_D g^{ab} = \pounds_\xi g^{ab} = -\nabla^a \xi^b - \nabla^b \xi^a, \delta_D B^a(\hat{x}) = -\xi^a(B(\hat{x})),$$

$$\begin{aligned} 0 = \delta_D I[\Phi, g; M] = & \int_M d^d x \sqrt{-g} \left(-\xi^a \nabla^b T_{ab}^{\mathrm{cl}} + E_\alpha \pounds_\xi \Phi^\alpha\right) \\ & - \int_{\partial M} d^{d-1} \hat{x} \sqrt{\gamma} \left(\xi^a n^b T_{ab}^{\mathrm{cl}} - n_a \xi^a \mathcal{L} - n_a \Pi_\alpha^a \pounds_\xi \Phi^\alpha\right) \end{aligned}$$

$$\nabla^b T_{ab}^{\mathrm{cl}} = E_\alpha \pounds_a \Phi^\alpha$$

$$\left(n_a \Pi_\alpha^a \pounds_\xi \Phi^\alpha + n_a \xi^a \mathcal{L}\right)\Big|_{\partial M} = \xi^a n^b T_{ab}^{\mathrm{cl}}\Big|_{\partial M},$$

$$I[\widehat{\Phi}, \hat{g}; M] = I_M[\Phi, g; M]$$

$$\hat{g}_{ab}(x) = e^{2\chi(x)} g_{ab}(x), \widehat{\Phi}^\alpha(x) = e^{-\Delta_\Phi \chi(x)} \Phi^\alpha(x)$$

$$\delta_\chi g^{ab} = -2\delta\chi g^{ab}, \delta_\chi \Phi^\alpha = -\Delta_\Phi \delta\chi \Phi^\alpha$$

$$0 = \delta_\chi I_M = \int_M d^d x \sqrt{-g} \delta\chi \left(g^{ab} T_{ab}^{\mathrm{cl}} - \Delta_\Phi E_\alpha \Phi^\alpha\right) - \int_{\partial M} d^{d-1} \hat{x} \sqrt{\gamma} \delta\chi \Delta_\Phi n_a \Pi_\alpha^a \Phi^\alpha$$

$$g^{ab} T_{ab}^{\mathrm{cl}} = \Delta_\Phi E_\alpha \Phi^\alpha$$

$$n_a \Pi_\alpha^a \Phi^\alpha|_{\partial M} = 0$$

$$\Phi_s(\phi, s) = \Psi(\phi), \Phi_s(\phi, 0) = \Psi_0(\phi)$$

$$H_{\mathrm{cl}}(t) \equiv \frac{d}{ds} I[\Phi_s, g; M_s]\Big|_{s=t}$$

$$\frac{d}{ds} B_s^a(\phi) = \delta_t^a, \frac{d}{ds} \Phi_s(\phi, t) = \dot{\Phi}_s(\phi, t), \frac{d}{ds} g^{ab}(\phi, t) = 0,$$

$$H_{\mathrm{cl}}(t) = \int_{m_t} d^{d-1} \phi \sqrt{\gamma} (n_a \Pi_\alpha^a \dot{\Phi}_t^\alpha - n_t \mathcal{L}) - \int_{m_0} d^{d-1} \phi \sqrt{\gamma} n_a \Pi_\alpha^a \dot{\Phi}_t^\alpha$$

$$0=\frac{d}{ds}\Psi(\phi)=\frac{d}{ds}\Phi_s(\phi,s)=\dot{\Phi}_s(\phi,s)+\partial_s\Phi_s(\phi,s)$$

$$\left.\frac{d}{ds}\Phi_s(\phi,t)\right|_{s=t}=\dot{\Phi}_t(\phi,t)=-\partial_t\Phi_t(\phi,t)=\xi^a\partial_a\Phi_t(\phi,t)=\pounds_\xi\Phi_t(\phi,t)$$

$$H_{\mathrm{cl}}(t)=\int_{m_t}d^{d-1}\phi\sqrt{\gamma}\left(n_a\Pi^a_\alpha\pounds_\xi\Phi+n_a\xi^a\mathcal{L}\right)$$

$$H_{\mathrm{cl}}(t)=\int_{m_t}d^{d-1}\phi\sqrt{\gamma}\xi^a n^b T^{\mathrm{cl}}_{ab}$$

$$\nabla^b T^{\mathrm{cl}}_{ab}(x)=E_\alpha(\Phi(x))\pounds_a\Phi^\alpha(x),g^{ab}(x)T^{\mathrm{cl}}_{ab}(x)=\Delta_\Phi E_\alpha(\Phi(x))\Phi^\alpha(x)$$

$$\nabla^b T^{\mathrm{reg}}_{ab}(x)=\frac{1}{2}[E_\alpha(\Phi(x)),\pounds_a\Phi^\alpha(x+\epsilon)]_+$$
$$g^{ab}(x)T^{\mathrm{reg}}_{ab}(x)=\frac{\Delta_\Phi}{2}[E_\alpha(\Phi(x)),\Phi^\alpha(x+\epsilon)]_+$$

$$[A(x_1),B(x_2)]_+\equiv A(x_1)B(x_2)+A(x_2)B(x_1)$$

$$\mathcal{T}\{\Phi(x)\Phi(x')\}=G_{\mathrm{F}}(x,x')+\Phi^2(x)+\cdots\ x'\to x$$

$$\mathcal{T}\{\Phi(\phi_1,t_1)\Phi(\phi_2,t_2)\}\equiv\Phi(\phi_1,t_1)\Phi(\phi_2,t_2)\Theta(t_1-t_2)+\Phi(\phi_2,t_2)\Phi(\phi_1,t_1)\Theta(t_2-t_1)$$

$$E_\alpha G_{\mathrm{F}}(x,x')=\frac{1}{\sqrt{-g}}\delta^{(d)}(x-x')$$

$$[\Phi(x),\Phi(x')]_+=2\mathrm{Re}G_{\mathrm{F}}(x,x')+\Phi^2(x)+\cdots,x'\to x$$

$$\partial_s\mathcal{G}(x,x';s)=-E_\alpha\mathcal{G}(x,x';s),\mathcal{G}(x,x';0)=\frac{1}{\sqrt{-g}}\delta^{(d)}(x-x')$$

$$G_{\mathrm{F}}(x,x')=\lim_{\lambda\to0^+}\int_\lambda^\infty ds\mathcal{G}(x,x';s)$$

$$E_\alpha G_{\mathrm{F}}(x,x')=\lim_{\lambda\to0^+}\mathcal{G}(x,x';\lambda)$$

$$\nabla^b T^{\mathrm{reg}}_{ab}(x)=\lim_{\lambda\to0^+}\mathrm{Re}\partial_a\mathcal{G}(x,x';\lambda),g^{ab}(x)T^{\mathrm{reg}}_{ab}(x)=\Delta_\Phi\lim_{\lambda\to0^+}\mathrm{Re}\mathcal{G}(x,x';\lambda),$$

$$T^{\mathrm{ren}}_{ab}(x)\equiv\lim_{\epsilon\to0}\left(T^{\mathrm{reg}}_{ab}(x)+T^{\mathrm{ct}}_{ab}(x)\right),$$

$$\nabla^b T^{\mathrm{ren}}_{ab}(x)=\frac{1}{2}\lim_{\lambda\to0^+}\mathrm{Re}\partial_a\mathcal{G}(x,x;\lambda)+\nabla^b T^{\mathrm{ct}}_{ab}(x),$$
$$g^{ab}(x)T^{\mathrm{ren}}_{ab}(x)=\Delta_\Phi\lim_{\lambda\to0^+}\mathrm{Re}\mathcal{G}(x,x;\lambda)+g^{ab}(x)T^{\mathrm{ct}}_{ab}(x),$$

$$\frac{d}{dx}B(x,x)=2\lim_{x'\to x}\frac{\partial}{\partial x'}B(x,x')$$

$$\mathcal{G}(x,x';s)=\frac{1}{(4\pi is)^{d/2}}e^{-\frac{\sigma(x,x')}{2is}}\Omega(x,x';s),$$

$$\Omega(x,x';s)=\sum_{n=0}^{\infty}a_n(x,x')(is)^n$$

$$\mathcal{G}(x,x;\epsilon)=\mathcal{D}(x;\epsilon)+\frac{1}{(4\pi)^{d/2}}a_{d/2}(x,x)+\mathcal{O}(\epsilon),\epsilon\to 0^+$$

$$\nabla^b T_{ab}^{\mathrm{ct}}(x)=-\frac{1}{2}\partial_a\mathcal{D}(x;\epsilon),g^{ab}(x)T_{ab}^{\mathrm{ct}}(x)=-\Delta_\Phi\mathcal{D}(x;\epsilon).$$

$$\nabla^b T_{ab}^{\mathrm{ren}}(x)=\frac{1}{2(4\pi)^{d/2}}\partial_a a_{d/2}(x,x),g^{ab}(x)T_{ab}^{\mathrm{ren}}(x)=\frac{\Delta_\Phi}{(4\pi)^{d/2}}a_{d/2}(x,x).$$

$$T_{ab}^{\mathrm{D}}\equiv T_{ab}^{\mathrm{ren}}-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}a_{d/2}(x,x)g_{ab}.$$

$$\nabla^b T_{ab}^{\mathrm{D}}=0,g^{ab}T_{ab}^{\mathrm{D}}=\frac{1}{(4\pi)^{d/2}}\left(\Delta_\Phi-\frac{d}{2}\right)a_{d/2}(x,x).$$

$$I[\Phi,g]=-\frac{1}{2}\int d^dx\sqrt{-g}\left(g^{ab}\partial_a\Phi\partial_b\Phi+\xi\Phi^2R\right)$$

$$E(\Phi)=(\nabla^2-\xi R)\Phi,$$

$$\begin{aligned}T_{ab}^{\mathrm{cl}}=&(1-2\xi)\partial_a\Phi\partial_b\Phi-\frac{1}{2}(1-4\xi)g_{ab}g^{cd}\partial_c\Phi\partial_d\Phi\\&+\xi\Phi\left(R_{ab}-\frac{1}{2}Rg_{ab}-2(\nabla_a\nabla_b-g_{ab}\nabla^2)\right)\Phi\end{aligned}$$

$$\xi=\frac{d-2}{4(d-1)},\Delta_\Phi=\frac{d-2}{2}.$$

$$\nabla^b T_{ab}^{\mathrm{D}}=0,g^{ab}T_{ab}^{\mathrm{D}}=-\frac{1}{(4\pi)^{d/2}}a_{d/2}(x,x)$$

$$a_0(x,x)=1,a_1(x,x)=\left(\frac{1}{6}-\xi\right)R$$

$$a_2(x,x)=\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2+\frac{1}{180}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}\right)+\left(\frac{1}{30}-\frac{1}{6}\xi\right)\nabla^2R$$

$$a_1(x,x)=-\frac{d-4}{12(d-1)}R$$

$$a_2(x,x)=\frac{(d-4)^2}{288(d-1)^2}R^2+\frac{1}{180}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}\right)-\frac{d-6}{120(d-1)}\nabla^2R$$

$$a_1(x,x)=\frac{1}{6}R,a_2(x,x)=\frac{1}{72}R^2+\frac{1}{180}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}\right)+\frac{1}{30}\nabla^2R,$$

$$a_1(x,x)=0, a_2(x,x)=\frac{1}{180}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}+\nabla^2 R\right).$$

$$\mathcal{G}(x,x';s)=\begin{cases}\frac{1}{4\pi s}+\frac{1}{24\pi}R+\mathcal{O}(s), & d=2\\ \frac{1}{16\pi^2 s^2}+\frac{1}{180}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}+\nabla^2 R\right)+\mathcal{O}(s), & d=4\end{cases}$$

$$g^{ab}T^{\mathrm{D}}_{ab}=\begin{cases}-\frac{1}{24\pi}R, & d=2,\\ -\frac{1}{2880\pi^2}\left(R_{abcd}R^{abcd}-R_{ab}R^{ab}+\nabla^2 R\right), & d=4.\end{cases}$$

$$C^{\mathrm{D}}_{ab}=-\frac{2}{\sqrt{-g}}\frac{\delta A_{\mathrm{D}}[g]}{\delta g^{ab}},$$

$$A_{\mathrm{D}}[\eta]=A_{\mathrm{D}}[e^{-2\sigma}g]=I_{\mathrm{Lio}}[-\sigma,g]+A_{\mathrm{D}}[g],$$

$$A_{\mathrm{D}}[g]=-I_{\mathrm{Lio}}[-\sigma,g]+A_{\mathrm{D}}[\eta]$$

$$C^{\mathrm{D}}_{ab}=\left(-C^{\mathrm{Lio}}_{ab}+\frac{\delta\sigma}{\delta g^{ab}}E_{\mathrm{Lio}}\right)\Big|_{\chi=-\sigma},$$

$$C^{\mathrm{Lio}}_{ab}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta I_{\mathrm{Lio}}[\chi,g]}{\delta g^{ab}}, E_{\mathrm{Lio}}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta I_{\mathrm{Lio}}[\chi,g]}{\delta\chi}$$

$$C^{\mathrm{Lio}}_{ab}=-\frac{c}{12\pi}\left[\nabla_a\chi\nabla_b\chi-\nabla_a\nabla_b\chi+g_{ab}\left(\nabla^2\chi-\frac{1}{2}\nabla^c\chi\nabla_c\chi\right)\right],$$

$$E_{\mathrm{Lio}}=\frac{c}{12\pi}(-2\nabla^2\chi+R)$$

$$E_{\mathrm{Lio}}\,|_{\chi=-\sigma}=-\frac{c}{6\pi}\nabla^2(\chi+\sigma)\Big|_{\chi=-\sigma}=0$$

$$C^{\mathrm{D}}_{ab}=\frac{c}{12\pi}\left[\nabla_a\sigma\nabla_b\sigma+\nabla_a\nabla_b\sigma-g_{ab}\left(\nabla^2\sigma+\frac{1}{2}\nabla^c\sigma\nabla_c\sigma\right)\right].$$

$$\begin{aligned}\nabla^b C^{\mathrm{D}}_{-b}&=2e^{-2\sigma}[\partial_+C^{\mathrm{D}}_{--}+(\partial_--2\partial_-\sigma)C^{\mathrm{D}}_{-+}]\\ \nabla^b C^{\mathrm{D}}_{+b}&=2e^{-2\sigma}[\partial_-C^{\mathrm{D}}_{++}+(\partial_+-2\partial_+\sigma)C^{\mathrm{D}}_{-+}]\end{aligned}$$

$$\partial_+C^{\mathrm{D}}_{--}+(\partial_--2\partial_-\varphi)C^{\mathrm{D}}_{-+}=0, \partial_-C^{\mathrm{D}}_{++}+(\partial_+-2\partial_+\sigma)C^{\mathrm{D}}_{-+}=0, C^{\mathrm{D}}_{-+}=-\frac{c}{96\pi}e^{2\sigma}R.$$

$$\partial_+C^{\mathrm{D}}_{--}-\frac{c}{96\pi}e^{2\sigma}\partial_-R=0, \partial_-C^{\mathrm{D}}_{++}-\frac{c}{96\pi}e^{2\sigma}\partial_+R=0, C^{\mathrm{D}}_{-+}=-\frac{c}{96\pi}e^{2\sigma}R$$

$$\partial_-=-\frac{1}{F_t'(\phi)}\frac{\partial_t-\bar{\nu}\partial_\phi}{\bar{\nu}-\nu}, \partial_+=\frac{1}{\bar{F}_t'(\phi)}\frac{\partial_t-\nu\partial_\phi}{\bar{\nu}-\nu}$$

$$\begin{aligned}F_t'(\phi)^2\bar{F}_t'(\phi)(\bar{\nu}-\nu)\partial_+C^{\mathrm{D}}_{--}&=\left(\partial_t-\nu\partial_\phi-2\partial_\phi\nu\right)\left(F_t'^2C^{\mathrm{D}}_{--}\right),\\ \bar{F}_t'(\phi)^2F_t'(\phi)(\bar{\nu}-\nu)\partial_-C^{\mathrm{D}}_{++}&=\left(\partial_t-\bar{\nu}\partial_\phi-2\partial_\phi\bar{\nu}\right)\left(\bar{F}_t'^2C^{\mathrm{D}}_{++}\right).\end{aligned}$$

$$\left(\partial_t - \nu\partial_\phi - h\partial_\phi\nu\right)F_t'(\phi)^h = 0, \left(\partial_t - \bar{\nu}\partial_\phi - h\partial_\phi\bar{\nu}\right)\bar{F}_t'(\phi)^h = 0$$

$$\begin{aligned}
\left(\partial_t - \nu\partial_\phi - 2\partial_\phi\nu\right)\left(F_t'^2 C_{--}^{\mathrm{D}}\right) - \frac{c}{96\pi}e^{2\omega}\left(\partial_t - \bar{\nu}\partial_\phi\right)R &= 0\\
\left(\partial_t - \bar{\nu}\partial_\phi - 2\partial_\phi\bar{\nu}\right)\left(\bar{F}_t'^2 C_{++}^{\mathrm{D}}\right) - \frac{c}{96\pi}e^{2\omega}\left(\partial_t - \nu\partial_\phi\right)R &= 0\\
F_t'\bar{F}_t' C_{-+}^{\mathrm{D}} + \frac{c}{96\pi}e^{2\omega}R &= 0
\end{aligned}$$

$$I_{\mathrm{Lio}}[\varphi,\eta] = \frac{c}{24\pi}\int d^2x\sqrt{-\eta}\eta^{ab}\partial_a\varphi\partial_b\varphi$$

$$I_{\mathrm{Lio}}[\varphi,\eta] = \frac{c}{12\pi}\int dx^- dx^+ \partial_-\varphi\partial_+\varphi$$

$$\begin{aligned}
&I_{\mathrm{Lio}}[\varphi,\eta]\\
&= -\frac{c}{48\pi}\int dx^- dx^+ \left(\partial_-\log F_t'\partial_+\log F_t' + \partial_-\log \bar{F}_t'\partial_+\log \bar{F}_t' + 2\partial_-\log \bar{F}_t'\partial_+\log F_t'\right)
\end{aligned}$$

$$\begin{aligned}
F_t'\bar{F}_t'(\bar{\nu} - \nu)\partial_-\log F_t'\partial_+\log F_t' &= -\left(\partial_\phi\nu\right)\left(\frac{\partial_\phi\nu}{\bar{\nu} - \nu} - \partial_\phi\log F_t'\right),\\
F_t'\bar{F}_t'(\bar{\nu} - \nu)\partial_-\log \bar{F}_t'\partial_+\log \bar{F}_t' &= -\left(\partial_\phi\bar{\nu}\right)\left(\frac{\partial_\phi\bar{\nu}}{\bar{\nu} - \nu} + \partial_\phi\log \bar{F}_t'\right),\\
F_t'\bar{F}_t'(\bar{\nu} - \nu)\partial_-\log \bar{F}_t'\partial_+\log F_t' &= -\frac{\left(\partial_\phi\nu\right)\left(\partial_\phi\bar{\nu}\right)}{\bar{\nu} - \nu}
\end{aligned}$$

$$\begin{aligned}
F_t'\bar{F}_t'(\bar{\nu} - \nu)&\left(\partial_-\log F_t'\partial_+\log F_t' + \partial_-\log \bar{F}_t'\partial_+\log \bar{F}_t' + 2\partial_-\log \bar{F}_t'\partial_+\log F_t'\right)\\
&= \left(\partial_\phi\nu\right)\partial_\phi\log F_t' - \left(\partial_\phi\bar{\nu}\right)\partial_\phi\log \bar{F}_t' - \frac{\left(\partial_\phi\nu + \partial_\phi\bar{\nu}\right)^2}{\bar{\nu} - \nu}
\end{aligned}$$

$$I_{\mathrm{Lio}}[\varphi,\eta] = -\frac{c}{48\pi}\int d\phi dt\left[\left(\partial_\phi\nu\right)\partial_\phi\log F_t' - \left(\partial_\phi\bar{\nu}\right)\partial_\phi\log \bar{F}_t' - \frac{\left(\partial_\phi\nu + \partial_\phi\bar{\nu}\right)^2}{\bar{\nu} - \nu}\right].$$

$$C_{ab}^{\mathrm{F}} = -\frac{2}{\sqrt{-g}}\frac{\delta A_{\mathrm{F}}[g]}{\delta g^{ab}},$$

$$\begin{aligned}
-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{--}} &= -\frac{1}{F_t^2}\left(\frac{\delta}{\delta\nu} + \frac{1}{\bar{\nu} - \nu}\frac{\delta}{\delta\omega}\right)\\
-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{++}} &= \frac{1}{\bar{F}_t^2}\left(\frac{\delta}{\delta\bar{\nu}} - \frac{1}{\bar{\nu} - \nu}\frac{\delta}{\delta\omega}\right)\\
-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{-+}} &= \frac{1}{F_t'\bar{F}_t'}\frac{1}{\bar{\nu} - \nu}\frac{\delta}{\delta\omega}
\end{aligned}$$

$$C_{--}^{\mathrm{F}} = -\frac{1}{F_t'^2}\frac{\delta\Gamma[\nu]}{\delta\nu}, C_{++}^{\mathrm{F}} = \frac{1}{\bar{F}_t'^2}\frac{\delta\bar{\Gamma}[\bar{\nu}]}{\delta\bar{\nu}}, C_{-+}^{\mathrm{F}} = 0$$

$$\delta\Gamma = \frac{c}{48\pi}\int d\phi dt\left(\delta\nu\partial_\phi^2\log F_t' + \partial_\phi^2\nu\delta\log F_t'\right)$$

$$\left(\partial_t - \nu\partial_\phi\right)\log F_t' = \partial_\phi\nu$$

$$\partial_\phi^2 \nu = (\partial_t - \nu\partial_\phi)\partial_\phi \log F_t' - \partial_\phi \nu \partial_\phi \log F_t'$$

$$\frac{c}{48\pi}\int d\phi dt \partial_\phi^2 \nu \delta \log F_t' = \frac{c}{48\pi}\int d\phi dt \left((\partial_t - \nu\partial_\phi)\partial_\phi \log F_t' \delta \log F_t' - \partial_\phi \nu \partial_\phi \log F_t' \delta \log F_t'\right)$$

$$\begin{aligned}\frac{c}{48\pi}\int d\phi dt \partial_\phi^2 \nu \delta \log F_t' &= \frac{c}{48\pi}\int d\phi dt (\partial_t \partial_\phi \log F_t' \delta \log F_t' + \nu \partial_\phi \log F_t' \partial_\phi \delta \log F_t') \\ &= -\frac{c}{48\pi}\int d\phi dt \partial_\phi \log F_t' (\partial_t - \nu\partial_\phi)\delta \log F_t'\end{aligned}$$

$$(\partial_t - \nu\partial_\phi)\delta \log F_t' = \delta\nu \partial_\phi \log F_t' + \partial_\phi \delta\nu$$

$$\begin{aligned}\frac{c}{48\pi}\int d\phi dt \partial_\phi^2 \nu \delta \log F_t' &= -\frac{c}{48\pi}\int d\phi dt \partial_\phi \log F_t' (\delta\nu \partial_\phi \log F_t' + \partial_\phi \delta\nu) \\ &= \frac{c}{48\pi}\int d\phi dt \delta\nu \left(-(\partial_\phi \log F_t')^2 + \partial_\phi^2 \log F_t'\right)\end{aligned}$$

$$\delta\Gamma = \frac{c}{48\pi}\int d\phi dt \delta\nu \left(2\partial_\phi^2 \log F_t' - (\partial_\phi \log F_t')^2\right) = \frac{c}{24\pi}\int d\phi dt \delta\nu \{F_t(\phi), \phi\}$$

$$C_{--}^{\mathrm{F}} = -\frac{c}{24\pi}\frac{1}{F_t'(\phi)^2}\{F_t(\phi), \phi\}, C_{++}^{\mathrm{F}} = -\frac{c}{24\pi}\frac{1}{\bar{F}_t'(\phi)^2}\{\bar{F}_t(\phi), \phi\}, C_{-+}^{\mathrm{F}} = 0$$

$$\nabla^a C_{\pm a}^{\mathrm{F}} = 2e^{-2\sigma}\partial_\mp C_{\pm\pm}^{\mathrm{F}}$$

$$\begin{aligned}\nabla^a C_{-a}^{\mathrm{F}} &= \frac{2e^{-2\omega}}{\bar{\nu} - \nu}\frac{1}{F_t'}(\partial_t - \nu\partial_\phi - 2\partial_\phi \nu)(F_t'^2 C_{--}^{\mathrm{F}}) \\ \nabla^a C_{+a}^{\mathrm{F}} &= -\frac{2e^{-2\omega}}{\bar{\nu} - \nu}\frac{1}{\bar{F}_t'}(\partial_t - \bar{\nu}\partial_\phi - 2\partial_\phi \bar{\nu})(\bar{F}_t'^2 C_{++}^{\mathrm{F}})\end{aligned}$$

$$(\partial_t - \nu\partial_\phi - 2\partial_\phi \nu)\{F_t(\phi), \phi\} = \partial_\phi^3 \nu$$

$$\nabla^a C_{-a}^{\mathrm{F}} = -\frac{c}{24\pi}\frac{2e^{-2\omega}}{\bar{\nu} - \nu}\frac{1}{F_t'}\partial_\phi^3 \nu, \nabla^a C_{+a}^{\mathrm{F}} = \frac{c}{24\pi}\frac{2e^{-2\omega}}{\bar{\nu} - \nu}\frac{1}{\bar{F}_t'}\partial_\phi^3 \bar{\nu}$$

$$(\partial_t - \nu\partial_\phi - 2\partial_\phi \nu)(F_t'^2 C_{--}^{\mathrm{F}}) = -\frac{c}{24\pi}\partial_\phi^3 \nu, (\partial_t - \bar{\nu}\partial_\phi - 2\partial_\phi \bar{\nu})(\bar{F}_t'^2 C_{++}^{\mathrm{F}}) = -\frac{c}{24\pi}\partial_\phi^3 \bar{\nu}$$

$$I[\Phi, g] = \int d^2x\sqrt{-g}\mathcal{L} = -\frac{1}{2}\int d^2x\sqrt{-g}g^{ab}\partial_a \Phi \partial_b \Phi$$

$$I[\Phi, g] = -\int d\phi dt \frac{1}{\bar{\nu} - \nu}[(\partial_t - \nu\partial_\phi)\Phi][(\partial_t - \bar{\nu}\partial_\phi)\Phi]$$

$$T_{\pm\pm}^{\mathrm{cl}} = (\partial_\pm \Phi)^2, T_{-+}^{\mathrm{cl}} = 0$$

$$\Pi(\phi, t) \equiv \left.\frac{I[\Phi_s, g; M_s]}{\delta\Psi_s(\phi)}\right|_{s=t} = -\sqrt{\gamma}n_a \left.\frac{\partial\mathcal{L}}{\partial(\partial_a \Phi)}\right|_{m_t},$$

$$\Pi(\phi,t)=-\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\dot{\Phi}(\phi,t)}=\frac{(\partial_t-\nu\partial_\phi)\Phi}{\bar{\nu}-\nu}+\frac{(\partial_t-\bar{\nu}\partial_\phi)\Phi}{\bar{\nu}-\nu}=-F_t'(\phi)\partial_-\Phi+\bar{F}_t'(\phi)\partial_+\Phi$$

$$[\Phi_{\mathrm{H}}(\phi_1,t),\Pi_{\mathrm{H}}(\phi_2,t)]=i\delta_{2\pi}(\phi_1-\phi_2)$$

$$\Phi_{\mathrm{H}}(\phi,t)=\Phi_-(x^-)+\Phi_+(x^+)=\Phi_-(F_t(\phi))+\Phi_+(\bar{F}_t(\phi))$$

$$\Pi_{\mathrm{H}}(\phi,t)=-F_t'(\phi)\Pi_-(x^-)+\bar{F}_t'(\phi)\Pi_+(x^+)$$
$$\Pi_-(x^-)\equiv\partial_-\Phi_-(x^-)=\Phi_-'(F_t(\phi)),\Pi_+(x^+)\equiv\partial_+\Phi_+(x^+)=\Phi_+'(\bar{F}_t(\phi))$$

$$[\Phi_\pm(\phi_1),\Pi_\pm(\phi_2)]=\pm\frac{i}{2}\delta_{2\pi}(\phi_1-\phi_2)\mp\frac{1}{2}\frac{i}{2\pi},[\Phi_\pm(\phi_1),\Pi_\mp(\phi_2)]=\mp\frac{1}{4}\frac{i}{2\pi}$$

$$\begin{aligned}&[\Phi_{\mathrm{H}}(\phi_1,t),\Pi_{\mathrm{H}}(\phi_2,t)]\\&=-F_t'(\phi_2)\big[\Phi_-\big(F_t(\phi_1)\big),\Pi_-\big(F_t(\phi_2)\big)\big]+\bar{F}_t'(\phi_2)\big[\Phi_+\big(\bar{F}_t(\phi_1)\big),\Pi_+\big(\bar{F}_t(\phi_2)\big)\big]\\&\quad-F_t'(\phi_2)\big[\Phi_+\big(F_t(\phi_1)\big),\Pi_-\big(F_t(\phi_2)\big)\big]+\bar{F}_t'(\phi_2)\big[\Phi_-\big(\bar{F}_t(\phi_1)\big),\Pi_+\big(\bar{F}_t(\phi_2)\big)\big]\end{aligned}$$

$$[\Phi_{\mathrm{H}}(\phi_1,t),\Pi_{\mathrm{H}}(\phi_2,t)]=\frac{i}{2}F_t'(\phi_2)\delta_{2\pi}\big(F_t(\phi_1)-F_t(\phi_2)\big)+\frac{i}{2}\bar{F}_t'(\phi_2)\delta_{2\pi}\big(\bar{F}_t(\phi_1)-\bar{F}_t(\phi_2)\big)$$

$$f'(\phi_2)\delta_{2\pi}\big(f(\phi_1)-f(\phi_2)\big)=\delta_{2\pi}(\phi_1-\phi_2)$$

$$[\Pi_\pm(\phi_1),\Pi_\pm(\phi_2)]=\pm\frac{i}{2}\delta_{2\pi}'(\phi_1-\phi_2),[\Pi_-(\phi_1),\Pi_+(\phi_2)]=0$$

$$\Pi_-(\phi)=-\frac{1}{\sqrt{2}}(J(\phi)\otimes\mathbf{1}),\Pi_+(\phi)=\frac{1}{\sqrt{2}}(\mathbf{1}\otimes\bar{J}(\phi)),$$

$$[J(\phi_1),J(\phi_2)]=-i\delta_{2\pi}'(\phi_1-\phi_2).$$

$$\Phi_\pm(x^\pm)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}(Q\pm Px^\pm)+\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\pi n}}\left[a_n^\pm e^{\mp inx^\pm}+\left(a_n^\pm\right)^\dagger e^{\pm inx^\pm}\right]$$

$$\Pi_\pm(x^\pm)=\pm\frac{1}{2}\frac{1}{\sqrt{2\pi}}P+\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\sqrt{\frac{n}{2\pi}}\left[\mp a_n^\pm e^{\mp inx^\pm}\pm\left(a_n^\pm\right)^\dagger e^{\pm inx^\pm}\right].$$

$$a_n^-\equiv a_n\otimes 1,a_n^+\equiv 1\otimes a_n$$

$$\left[a_n,a_m^\dagger\right]=\delta_{nm},[Q,P]=i$$

$$J(\phi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}J_ne^{in\phi}$$

$$J_0=\sqrt{\pi}P,J_n=\sqrt{2\pi}\times\begin{cases}-i\sqrt{n}a_n, & n\geq 1\\ i\sqrt{-n}a_{-n}^\dagger, & n\leq -1\end{cases}.$$

$$[J_n,J_m]=2\pi n\delta_{n,-m}$$

$$\mathcal{N}\{J_nJ_m\}\equiv J_nJ_m\Theta(-n)+J_mJ_n\Theta(n),$$

$$L_n \equiv \frac{1}{2\pi}\frac{1}{2}\sum_{m=-\infty}^{\infty} \mathcal{N}\{J_{n-m}J_m\} - \frac{1}{24}\delta_{n,0}$$

$$T(\phi) = \frac{1}{(2\pi)^2}\frac{1}{2}\sum_{n,m=-\infty}^{\infty} \mathcal{N}\{J_nJ_m\}e^{i(n+m)\phi} - \frac{1}{2\pi}\frac{1}{24}$$

$$[J_n, J_m]_+ = 2\pi|n|\delta_{n,-m} + 2\mathcal{N}\{J_nJ_m\},$$

$$\frac{1}{4}[J(\phi_1), J(\phi_2)]_+ = \frac{1}{4\pi}\frac{1}{2}\sum_{n=-\infty}^{\infty} |n|e^{in(\phi_1-\phi_2)} + \frac{1}{(2\pi)^2}\frac{1}{2}\sum_{n,m=-\infty}^{\infty} \mathcal{N}\{J_nJ_m\}e^{in\phi_1+im\phi_2}$$

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} |n|e^{in\phi} = -\frac{1}{4\sin^2\left(\frac{\phi}{2}\right)} \equiv \partial_{\phi_1}\mathcal{P}\left(\frac{1}{2\tan\left(\frac{\phi}{2}\right)}\right)$$

$$\frac{1}{4}[J(\phi_1), J(\phi_2)]_+ = -\frac{1}{4\pi}\frac{1}{4\sin^2\left(\frac{\phi_1-\phi_2}{2}\right)} + T(\phi_1) + \frac{1}{2\pi}\frac{1}{24} + \mathcal{O}(\phi_1-\phi_2), \phi_2 \to \phi_1$$

$$T(\phi) = \lim_{\epsilon\to 0}\left(\frac{1}{4}[J(\phi+\epsilon), J(\phi)]_+ + \frac{1}{4\pi\epsilon^2}\right)$$

$$\lim_{\epsilon\to 0}\left(-\frac{1}{4\pi}\frac{1}{4\sin^2\left(\frac{\epsilon}{2}\right)} + \frac{1}{4\pi\epsilon^2}\right) = -\frac{1}{2\pi}\frac{1}{24}$$

$$T_{--}^{\mathrm{cl}}(\phi, t) = \Pi_-(F_t(\phi))^2, T_{++}^{\mathrm{cl}}(\phi, t) = \Pi_+(\bar{F}_t(\phi))^2.$$

$$F_t'(\phi)^2 T_{--}^{\mathrm{F}}(\phi, t) \equiv \lim_{\epsilon\to 0}\left[\frac{1}{2}F_t'(\phi+\epsilon)F_t'(\phi)[\Pi_-(F_t(\phi+\epsilon)), \Pi_-(F_t(\phi))]_+ + \frac{1}{4\pi\epsilon^2}\right]$$

$$F_t'(\phi)^2 T_{--}^{\mathrm{F}}(\phi, t) = \lim_{\epsilon\to 0}\left[\frac{1}{4}F_t'(\phi+\epsilon)F_t'(\phi)[J(F_t(\phi+\epsilon)), J(F_t(\phi))]_+ \otimes \mathbf{1} + \frac{1}{4\pi\epsilon^2}\right]$$

$$\lim_{\epsilon\to 0}\left[-\frac{1}{4\pi}\frac{F_t'(\phi+\epsilon)F_t'(\phi)}{4\sin^2\left(\frac{F_t(\phi+\epsilon)-F_t(\phi)}{2}\right)} + \frac{1}{4\pi\epsilon^2}\right] = -\frac{1}{2\pi}\frac{1}{24}F_t'(\phi)^2 - \frac{1}{24\pi}\{F_t(\phi), \phi\}$$

$$F_t'(\phi)^2 T_{--}^{\mathrm{F}}(\phi, t) = F_t'(\phi)^2 T(F_t(\phi)) \otimes \mathbf{1} - \frac{1}{24\pi}\{F_t(\phi), \phi\}$$

$$(\psi\mathcal{O})_{a_1\ldots a_n}(x) = e^{-\Delta_{\mathrm{w}}\chi(D(x))}\frac{\partial D^{b_1}}{\partial x^{a_1}}\cdots\frac{\partial D^{b_n}}{\partial x^{a_n}}\mathcal{O}_{b_1\ldots b_n}(D(x))$$

$$\mathcal{O}_{-\ldots-+\ldots+}(x^-, x^+) = e^{-\Delta_{\mathrm{w}}\sigma(x^-,x^+)}\mathcal{O}^{\eta}_{-\ldots-+\ldots+}(x^-, x^+)$$

$$\psi_{\mathcal{C}}\eta = \mathcal{C}^*(e^{2\omega_{\mathcal{C}}}\eta) = \eta$$

$$\mathcal{C}(x^-, x^+) = \left(C(x^-), \bar{C}(x^+)\right) e^{-2\omega_{\mathcal{C}}\left(C(x^-),\bar{C}(x^+)\right)} = C'(x^-)\bar{C}'(x^+)$$

$$\frac{\partial \mathcal{C}^a}{\partial x^b} = \partial_\pm C_\pm(x^\pm)\delta^a_\pm \delta^\pm_b$$

$$(\psi_\mathcal{C}\mathcal{O})^\eta_{-\cdots-+\cdots+}(x^-,x^+) = C'_-(x^-)^h C'_+(x^+)^{\bar{h}} \mathcal{O}^\eta_{-\cdots-+\cdots+}\big(C_-(x^-),C_+(x^+)\big)$$

$$h = \frac{\Delta_\mathrm{w}}{2} + n_-, \bar{h} = \frac{\Delta_\mathrm{w}}{2} + n_+$$

$$\mathcal{O}^\eta_{-\cdots-+\cdots+}(x^-,x^+) = \mathcal{O}_h(x^-) \otimes \mathcal{O}_{\bar{h}}(x^+)$$

$$V_F\mathcal{O}_h(\phi)V_F^\dagger = F'(\phi)^h \mathcal{O}_h(F(\phi)), \overline{\mathcal{O}}_h(\phi) \equiv \mathcal{O}_h(-\phi)$$

$$\big(\psi_\mathcal{C}\mathcal{O}^\eta_{-\cdots-+\cdots+}\big)(x^-,x^+) = (V_C \otimes \bar{V}_{\bar{C}})\mathcal{O}^\eta_{-\cdots-+\cdots+}(x^-,x^+)(V_C \otimes \bar{V}_{\bar{C}})^\dagger$$

$$\mathcal{O}_{-\cdots-+\cdots+}(x^-,x^+) = e^{-\Delta_\mathrm{w}\omega(x^-,x^+)} F'_t(\phi)^{\frac{\Delta_\mathrm{w}}{2}} \bar{F}'_t(\phi)^{\frac{\Delta_\mathrm{w}}{2}} \mathcal{O}_h(x^-) \otimes \mathcal{O}_{\bar{h}}(x^+)$$

$$\mathcal{O}^\mathrm{H}_{-\cdots-+\cdots+}(\phi,t) \equiv F'_t(\phi)^{n_-} \bar{F}'_t(\phi)^{n_+} \mathcal{O}_{-\cdots-+\cdots+}(x^-,x^+),$$

$$\mathcal{O}^\mathrm{H}_{-\cdots-+\cdots+}(\phi,t) = e^{-\Delta_\mathrm{w}\omega(\phi,t)} F'_t(\phi)^h \bar{F}'_t(\phi)^{\bar{h}} \mathcal{O}_h(F_t(\phi)) \otimes \mathcal{O}_{\bar{h}}(\bar{F}_t(\phi))$$

$$\mathcal{O}^\mathrm{S}_{-\cdots-+\cdots+}(\phi,t) = e^{-\Delta_\mathrm{w}\omega(\phi,t)} \mathcal{O}_h(\phi) \otimes \mathcal{O}_{\bar{h}}(\phi)$$

$$\eta_{ab}(x)dx^a dx^b = dwd\bar{w} = dr^2 + r^2 d\theta^2$$

$$\mathrm{Vol}(\mathbb{D},\eta) = \int_\mathbb{D} d^2x\sqrt{\eta} = \int_0^{2\pi} d\theta \int_0^1 drr = \pi$$

$$\mathcal{P}^w(w,\bar{w}) = P(w) = w^n, \mathcal{P}^{\bar{w}}(w,\bar{w}) = \bar{P}(\bar{w}) = \bar{w}^n$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = \int_\mathbb{D} d^2x\sqrt{\mathcal{P}^*\eta}$$

$$(\mathcal{P}^*\eta)_{ab}dx^a dx^b = P'(w)\bar{P}'(\bar{w})dwd\bar{w} = n^2|w|^{n-1}dwd\bar{w} = n^2 r^{2(n-1)}(dr^2 + r^2 d\theta^2)$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = \int_0^{2\pi} d\theta \int_0^1 drrn^2 r^{2(n-1)} = 2\pi n^2 \int_0^1 drr^{2n-1} = n\pi$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = n\mathrm{Vol}(\mathbb{D},\eta)$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = \int_\mathbb{D} d^2x\sqrt{\mathcal{P}^*\eta} = \int_\mathbb{D} d^2x\sqrt{\eta}\det\{\partial_a\mathcal{P}^b\}$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = \int_{\mathcal{P}^{-1}(\mathbb{D})} d^2x\sqrt{\eta} = \int_{\mathcal{P}^{-1}(\mathbb{D})} d\theta drr$$

$$\mathrm{Vol}(\mathbb{D},\mathcal{P}^*\eta) = \int_0^{2\pi n} d\theta \int_0^1 drr = n\pi$$

$$g_{ab}dx^a dx^b = e^{2\sigma(w,\bar{w})}\eta_{ab}dx^a dx^b, \eta_{ab}dx^a dx^b = dwd\bar{w}$$

$$\tilde{g}_{ab}dx^a dx^b = e^{2\tilde{\sigma}(w,\bar{w})}\eta_{ab}dx^a dx^b, \tilde{\eta}_{ab}dx^a dx^b = e^{2\kappa(w,\bar{w})}dwd\bar{w}$$

$$\tilde{\sigma}(w,\bar{w}) = \kappa(w,\bar{w}) + \Sigma(w,\bar{w})$$

$$\kappa(w,\bar{w}) \equiv \frac{1}{2}\log\left(P'(w)\bar{P}'(\bar{w})\right), \Sigma(w,\bar{w}) \equiv \sigma(P(w),\bar{P}(\bar{w}))$$

$$\tilde{\sigma}(w,\bar{w}) = \frac{\gamma}{2}\log|w|^2 + \sigma(0) + \log(\gamma+1) + \mathcal{O}(w^{\gamma+1}) + \mathcal{O}(\bar{w}^{\gamma+1}), w, \bar{w} \to 0$$

$$\tilde{g}_{ab}dx^a dx^b = (\gamma+1)^2|w|^{2\gamma}e^{2\sigma(0)}dwd\bar{w} + \cdots, w, \bar{w} \to 0$$

$$\tilde{g}_{ab}dx^a dx^b = (\gamma+1)^2 r^{2\gamma}e^{2\sigma(0)}(dr^2 + r^2 d\theta^2) + \cdots, r \to 0$$

$$\tilde{r} = e^{\sigma(0)}r^{\gamma+1}, \tilde{\theta} = (\gamma+1)\theta$$

$$\tilde{g}_{ab}dx^a dx^b = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \cdots$$

$$\frac{Z_{\mathrm{D}}^{\mathrm{reg}}[\tilde{g};\mathbb{R}^2]}{Z_{\mathrm{D}}^{\mathrm{reg}}[\eta;\mathbb{R}^2]} = e^{I_{\mathrm{Lio}}^{\mathrm{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2]}, \frac{Z_{\mathrm{D}}^{\mathrm{reg}}[\tilde{\eta};\mathbb{R}^2]}{Z_{\mathrm{D}}^{\mathrm{reg}}[\eta;\mathbb{R}^2]} = e^{I_{\mathrm{Lio}}^{\mathrm{reg}}[\kappa,\eta;\mathbb{R}^2]}, \frac{Z_{\mathrm{D}}^{\mathrm{reg}}[g;\mathbb{R}^2]}{Z_{\mathrm{D}}^{\mathrm{reg}}[\eta;\mathbb{R}^2]} = e^{I_{\mathrm{Lio}}^{\mathrm{reg}}[\sigma,\eta;\mathbb{R}^2]}$$

$$\frac{Z_{\mathrm{D}}^{\mathrm{reg}}[\tilde{g};\mathbb{R}^2]}{Z_{\mathrm{D}}^{\mathrm{reg}}[g;\mathbb{R}^2]^n} = e^{I_{\mathrm{Lio}}^{\mathrm{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] - I_{\mathrm{Lio}}^{\mathrm{reg}}[\kappa,\eta;\mathbb{R}^2] - nI_{\mathrm{Lio}}^{\mathrm{reg}}[\sigma,\eta;\mathbb{R}^2]}\frac{Z_{\mathrm{D}}^{\mathrm{reg}}[\tilde{\eta};\mathbb{R}^2]}{Z_{\mathrm{D}}^{\mathrm{reg}}[\eta;\mathbb{R}^2]^n}$$

$$I_{\mathrm{Lio}}^{\mathrm{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] = \frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}\left(\eta^{ab}\partial_a\tilde{\sigma}\partial_b\tilde{\sigma} + \tilde{\sigma}R_\eta\right)$$

$$B_{\delta_i}(w_i;g) \equiv \left\{(w,\bar{w}) \in \mathbb{R}^2 \mid L_g(w_i,\bar{w}_i;w,\bar{w}) < \delta_i\right\}$$

$$r = \varepsilon \equiv e^{-\frac{\sigma(0)}{\gamma+1}}\tilde{\delta}^{\frac{1}{\gamma+1}}$$

$$I_{\mathrm{Lio}}^{\mathrm{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] = -\frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}\tilde{\sigma}\nabla^2\tilde{\sigma} + \frac{c}{24\pi}\int_{\partial B_{\tilde{\delta}}(0;\tilde{g})} dx\sqrt{h}n^a\tilde{\sigma}\partial_a\tilde{\sigma}$$

$$\sqrt{h} = \varepsilon + \mathcal{O}(\varepsilon^2), n^a = -\delta_r^a + \mathcal{O}(\varepsilon^2), \varepsilon \to 0$$

$$\begin{aligned}\int_{\partial B_{\tilde{\delta}}(0;\tilde{g})} dx\sqrt{h}n^a\tilde{\sigma}\partial_a\tilde{\sigma} &= -\left(\gamma\log\varepsilon + \log(\gamma+1) + \sigma(0) + \mathcal{O}(\varepsilon^{\gamma+1})\right) \\ &\quad \times \int_0^{2\pi} d\theta\varepsilon\partial_r\left(\gamma\log r + \log(\gamma+1) + \sigma(0) + \mathcal{O}(r^{\gamma+1})\right)\Big|_{r=\varepsilon} \\ &= -2\pi\left(\gamma^2\log\varepsilon + \gamma\log(\gamma+1) + \gamma\sigma(0) + \mathcal{O}(\varepsilon^{\gamma+1})\right)\end{aligned}$$

$$\begin{aligned}&I_{\mathrm{Lio}}^{\mathrm{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] \\ &= -\frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}\tilde{\sigma}\nabla^2\tilde{\sigma} - \frac{c}{12}\left[\frac{\gamma^2}{\gamma+1}\log\tilde{\delta} + \frac{\gamma}{\gamma+1}\sigma(0) + \gamma\log(\gamma+1) + \mathcal{O}(\tilde{\delta})\right]\end{aligned}$$

$$\begin{aligned}&I_{\mathrm{Lio}}^{\mathrm{reg}}[\kappa,\eta;\mathbb{R}^2] \\ &= -\frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{\eta})} d^2x\sqrt{\eta}\kappa\nabla^2\kappa - \frac{c}{12}\left[\frac{\gamma^2}{\gamma+1}\log\tilde{\delta} + \gamma\log(\gamma+1) + \mathcal{O}(\tilde{\delta})\right].\end{aligned}$$

$$I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2] = \frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}\left(\eta^{ab}\partial_a\Sigma\partial_b\Sigma - 2\Sigma\nabla^2\kappa\right)$$
$$= -\frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}(\Sigma\nabla^2\Sigma + 2\Sigma\nabla^2\kappa)$$

$$\sqrt{\tilde{\eta}}\tilde{\eta}^{ab} = \sqrt{\eta}\eta^{ab}, \sqrt{\tilde{\eta}}R_{\tilde{\eta}} = \sqrt{\eta}\left(R_\eta - 2\nabla^2\kappa\right) = -\sqrt{\eta}2\nabla^2\kappa, \nabla_a$$

$$I_{\text{Lio}}^{\text{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] - I_{\text{Lio}}^{\text{reg}}[\kappa,\eta;\mathbb{R}^2] - I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2]$$
$$= -\frac{c}{12}\frac{\gamma}{\gamma+1}\sigma(0) + \frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}(\Sigma\nabla^2\kappa - \kappa\nabla^2\Sigma)$$

$$\tilde{\sigma}\nabla^2\tilde{\sigma} - (\Sigma\nabla^2\Sigma + 2\Sigma\nabla^2\kappa) - \kappa\nabla^2\kappa = -(\Sigma\nabla^2\kappa - \kappa\nabla^2\Sigma).$$

$$\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\eta}(\Sigma\nabla^2\kappa - \kappa\nabla^2\Sigma)$$
$$= \int_{\partial B_{\tilde{\delta}}(0;\tilde{g})} dx\sqrt{h}n^a(\Sigma\partial_a\kappa - \kappa\partial_a\Sigma)$$
$$= -\int_0^{2\pi} d\theta\varepsilon(\Sigma\partial_r(\gamma\log r + \log(\gamma+1) + \cdots) - (\gamma\log r + \log(\gamma+1) + \cdots)\partial_r\Sigma)\Big|_{r=\varepsilon}$$
$$= -2\pi\gamma\sigma(0) + \mathcal{O}(\tilde{\delta}) + \mathcal{O}\left(\tilde{\delta}\frac{1}{\gamma+1}\log\tilde{\delta}\right)$$

$$\lim_{\tilde{\delta}\to 0}\left(I_{\text{Lio}}^{\text{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] - I_{\text{Lio}}^{\text{reg}}[\kappa,\eta;\mathbb{R}^2] - I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2]\right) = -\frac{c}{12}\frac{\gamma(\gamma+2)}{\gamma+1}\sigma(0)$$

$$I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2] = \frac{c}{24\pi}\int_{\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})} d^2x\sqrt{\tilde{\eta}}\left(\tilde{\eta}^{ab}\partial_a\Sigma\partial_b\Sigma + \Sigma R_{\tilde{\eta}}\right)$$

$$I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2] = \frac{c}{24\pi}\int_{\mathcal{P}^{-1}\left(\mathbb{R}^2\backslash B_{\tilde{\delta}}(0;\tilde{g})\right)} d^2y\sqrt{\eta}\left(\eta^{ab}\partial_a\sigma\partial_b\sigma + \sigma R_\eta\right)$$

$$I_{\text{Lio}}^{\text{reg}}[\Sigma,\tilde{\eta};\mathbb{R}^2] = nI_{\text{Lio}}^{\text{reg}}[\sigma,\eta;\mathbb{R}^2]$$

$$\lim_{\tilde{\delta}\to 0}\left(I_{\text{Lio}}^{\text{reg}}[\tilde{\sigma},\eta;\mathbb{R}^2] - I_{\text{Lio}}^{\text{reg}}[\kappa,\eta;\mathbb{R}^2] - nI_{\text{Lio}}[\sigma,\eta;\mathbb{R}^2]\right) = -\frac{c}{12}\sum_{i=1}^{k}\frac{\gamma_i(\gamma_i+2)}{\gamma_i+1}\sigma(w_i,\bar{w}_i)$$

$$\mathcal{R}_{(\mathbb{R}^2,\eta)}^{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k}(w_1,\bar{w}_1;\dots;w_k,\bar{w}_k) = \frac{Z_{\text{D}}^{\text{reg}}[\mathcal{P}^*\eta;\mathbb{R}^2]}{Z_{\text{D}}^{\text{reg}}[\eta;\mathbb{R}^2]^n},$$

$$\mathcal{R}_{(\mathbb{R}^2,e^2\chi\eta)}^{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k}(w_1,\bar{w}_1;\dots;w_k,\bar{w}_k) = e^{-\frac{c}{12}\sum_{i=1}^{k}\frac{\gamma_i(\gamma_i+2)}{\gamma_i+1}\chi(w_i,\bar{w}_i)}\mathcal{R}_{(\mathbb{R}^2,\eta)}^{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k}(w_1,\bar{w}_1;\dots;w_k,\bar{w}_k)$$

$$\mathcal{R}_{(\mathbb{R}^2,D^*\eta)}^{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k}(w_1,\bar{w}_1;\dots;w_k,\bar{w}_k) = \mathcal{R}_{(D(\mathbb{R}^2),\eta)}^{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k}\left(D(w_1,\bar{w}_1);\dots;D(w_k,\bar{w}_k)\right)$$

$$\mathcal{R}_{\left(\mathbb{R}^2,D^*(e^{2\chi}\eta)\right)}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}(w_1,\bar{w}_1;w_2,\bar{w}_2)$$
$$= e^{-\Delta_n\left((\chi\circ D)(w_1,\bar{w}_1)+(\chi\circ D)(w_2,\bar{w}_2)\right)}\mathcal{R}_{(D(\mathbb{R}^2),\eta)}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}\left(D(w_1,\bar{w}_1);D(w_2,\bar{w}_2)\right)$$

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

$$\mathcal{C}(w,\bar{w}) = (C(w),\bar{C}(\bar{w})), \chi(C(w),\bar{C}(\bar{w})) = -\frac{1}{2}\log\ C'(w) - \frac{1}{2}\log\ C'(\bar{w})$$

$$(\mathcal{C}^* e^{2\chi}\eta)_{ab}(x) = e^{2\chi(\mathcal{C}(x))}\frac{\partial \mathcal{C}^c}{\partial x^a}\frac{\partial \mathcal{C}^d}{\partial x^b}\eta_{cd}(\mathcal{C}(x)) = \eta_{ab}(x)$$

$$\mathcal{R}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}_{(\mathbb{R}^2,\eta)}(w_1,\bar{w}_1;w_2,\bar{w}_2)$$
$$= [C'(w_1)\bar{C}'(\bar{w}_1)C'(w_2)\bar{C}'(\bar{w}_2)]^{\Delta_n/2}\mathcal{R}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}_{(\mathcal{C}(\mathbb{R}^2),\eta)}\left(C(w_1),\bar{C}(\bar{w}_1);C(w_2),\bar{C}(\bar{w}_2)\right)$$

$$C(w) = \frac{aw+b}{cw+d}, \bar{C}(\bar{w}) = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

$$\mathcal{R}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}_{(\mathbb{R}^2,\eta)}(w_1,\bar{w}_1;w_2,\bar{w}_2) = \frac{B_n(\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{[(w_1-w_2)(\bar{w}_1-\bar{w}_2)]^{\Delta_n}}$$

$$\mathcal{R}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}_{(\mathbb{R}^2,\eta)}(w_1,\bar{w}_1;w_2,\bar{w}_2) \equiv \mathcal{R}[\eta;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]$$

$$\mathcal{R}[\lambda^2\eta;\lambda\tilde{\varepsilon}_1,\lambda\tilde{\varepsilon}_2] = \mathcal{R}[\eta;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]$$

$$\mathcal{R}[\eta;\lambda^{-1}\tilde{\varepsilon}_1,\lambda^{-1}\tilde{\varepsilon}_2] = \lambda^{-\Delta_n}\mathcal{R}[\eta;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]$$

$$\mathcal{R}^{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}_{(\mathbb{R}^2,\eta)}(w_1,\bar{w}_1;w_2,\bar{w}_2) = b(n)\left[\frac{\tilde{\varepsilon}_1\tilde{\varepsilon}_2}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)}\right]^{\Delta_n}$$

Modelo Holográfico AdS/CFT en gravedad cuántica.

	a_Σ	d_2
Bosonic	$\frac{N^3}{24}(1-y_+^2)$	$-\frac{N^3}{6}(q_1+q_2)$
Fermionic	$\frac{N^3}{32} - \frac{1}{96}\Sigma_a\left(\frac{1+2k_a}{k_a^2}N_a^3 + \Sigma_{b=a+1}^n N_a k_b\left(\frac{N_a^2}{k_a^2} + 3\frac{N_b^2}{k_b^2}\right)\right)$	$-\frac{1}{24}\left(N^3 - \Sigma_a\frac{N_a^3}{k_a^2}\right)$

$$S_{\mathrm{OS}}^{(\mathrm{ren})}\Big|_{\log} = \frac{N^3\left(4q_1q_2 - 2(q_1+q_2)y_+(1-y_+) + 5y_+(1-y_+^2)\right)}{1920y_+}$$

$$T^\mu{}_\mu = \frac{1}{4\pi^2}(-a_{4d}E_4 + c|W|^2).$$

$$T^\mu{}_\mu\Big|_{\Sigma_4} \supset \frac{1}{(4\pi)^2}(-a_\Sigma\bar{E}_4 + d_2\mathcal{J}_2 + \cdots)$$

$$\langle T^{ab}\rangle = -h_T\frac{(d-\mathfrak{d}-1)\delta^{ab}}{|x_\perp|^d}, \langle T^{ij}\rangle = h_T\frac{(\mathfrak{d}+1)\delta^{ij}-d\frac{x_\perp^i x_\perp^j}{|x_\perp|^2}}{|x_\perp|^d},$$

$$h_T = -\frac{\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}}(d-1)}d_2$$

$$h_T = -\frac{1}{5\pi^3}d_2$$

$$\int_{-\infty}^{\infty} d\lambda\langle\Psi|T_{\mu\nu}|\Psi\rangle v^\mu v^\nu \geq 0$$

$$S_{\mathrm{EE}}[\Sigma]|_{\log} = -4\left[a_\Sigma + \frac{1}{4}\frac{(d-4)(d-5)}{d-1}d_2\right]\log\left(\frac{R}{\epsilon}\right)$$

$$4a_\Sigma + \frac{2}{5}d_2 = -R\partial_R(S_{\mathrm{EE}}[\Sigma]-S_{\mathrm{EE}}[\emptyset])|_{R\to 0}$$

$$S = -\frac{1}{16\pi G_N^{(7)}}\int d^7x\sqrt{|g|}\left(\mathcal{R}-\frac{1}{2}\left|\partial_\mu\Phi_I\right|^2-\hat{g}^2V(\Phi)-\frac{1}{4}\sum_{I=1}^{2} e^{\vec{a}_I\vec{\Phi}}F_I^2\right)$$

$$V = -4e^{-\frac{1}{2}(\vec{a}_1+\vec{a}_2)\vec{\Phi}}-2\left(e^{\frac{1}{2}(\vec{a}_1+2\vec{a}_2)\vec{\Phi}}+e^{\frac{1}{2}(2\vec{a}_1+\vec{a}_2)\vec{\Phi}}\right)+\frac{1}{2}e^{2(\vec{a}_1+\vec{a}_2)\vec{\Phi}}$$

$$ds_7^2 = (yP(y))^{\frac{1}{5}}ds_{\mathrm{AdS}_5}^2+\frac{y(yP(y))^{\frac{1}{5}}}{4Q(y)}dy^2+\frac{yQ(y)}{(yP(y))^{\frac{4}{5}}}dz^2$$

$$P(y) = H_1(y)H_2(y)$$
$$Q(y) = -y^3+\mu y^2+\frac{\hat{g}^2}{4}P(y)$$

$$A_I = \left(\sqrt{1-\frac{\mu}{q_I}}\frac{q_I}{H_I(y)}+a_I\right)dz$$

$$q_I = y_+\left(\frac{3\hat{n}+1}{\hat{n}\hat{g}^2}-y_+\pm\frac{2}{\hat{g}}\sqrt{\frac{(1+3\hat{n})^2}{4\hat{g}^2\hat{n}^2}-y_+}\right)$$

$$\frac{q_1+q_2}{2y_+} = \left(\frac{3\hat{n}+1}{\hat{n}\hat{g}^2}-y_+\right)\geq\left(\frac{3\hat{n}+1}{\hat{n}\hat{g}^2}-y_{+,\max}\right)=\frac{(\hat{n}-1)(3\hat{n}+1)}{4\hat{g}^2\hat{n}^2}\geq 0.$$

$$ds_{11}^2 = \tilde{\Delta}^{1/3} ds_7^2 + \hat{g}^{-2} \tilde{\Delta}^{-2/3} \left[X_0^{-1} d\mu_0^2 + \sum_{I=1}^{2} X_I^{-1} \left(d\mu_I^2 + \mu_I^2 (d\phi_I + \hat{g} A_I)^2 \right) \right]$$

$$\star_{11} F_4 = 2\hat{g} \sum_{i=0}^{2} \left(X_i^2 \mu_i^2 - \tilde{\Delta} X_i \right) \Upsilon_7 + \hat{g} \tilde{\Delta} X_0 \Upsilon_7 + \frac{1}{2\hat{g}} \sum_{i=0}^{2} \star_7 \, \mathrm{dln} \, X_i \wedge d\left(\mu_i^2 \right)$$

$$+ \frac{1}{2\hat{g}^2} \sum_{I=1}^{2} X_I^{-2} d(\mu_I)^2 \wedge (d\phi_I + \hat{g} A_I) \wedge \star_7 F_I$$

$$X_1 = \frac{(y H_2(y))^{\frac{2}{5}}}{H_1(y)^{\frac{3}{5}}}, X_2 = \frac{(y H_1(y))^{\frac{2}{5}}}{H_2(y)^{\frac{3}{5}}}, X_0 = (X_1 X_2)^{-2}, \tilde{\Delta} = \sum_{i=1}^{2} X_i \mu_i^2$$

$$\mu_0 = \sin \psi \cos \zeta, \mu_1 = \sin \zeta, \mu_2 = \cos \psi \cos \zeta$$

$$ds_{11}^2 = \hat{f}_{\mathrm{AdS}}^2 ds_{\mathrm{AdS}_5}^2 + \hat{f}_y^2 dy^2 + \hat{f}_z^2 dz^2 + \hat{f}_{\phi_i}^2 d\phi_i^2 + \hat{f}_{z\phi_i}^2 dz d\phi_i + \hat{f}_\psi^2 d\psi^2 + \hat{f}_\zeta^2 d\zeta^2 + \hat{f}_{\psi\zeta} d\psi d\zeta$$

$$\sin x \equiv s_x, \cos x \equiv c_x$$

$$\frac{\star_{11} F_4}{\kappa^2} = -2\left(\hat{H}\left(X_0 + 2(X_1 + X_2) \right) - 2X_0^2 + 2(X_0^2 - X_1^2) s_\zeta^2 + 2(X_0^2 - X_2^2) c_\psi^2 c_\zeta^2 \right) \Upsilon_7$$

$$+ \frac{c_\zeta^2 c_\psi s_\psi}{2X_0 X_2} (X_2 \star_7 dX_0 - X_0 \star_7 dX_2) \wedge d\psi + \frac{c_\zeta s_\zeta}{2X_1} \star_7 dX_1 \wedge d\zeta$$

$$- \frac{c_\zeta s_\zeta}{2X_0 X_2} \left(X_2 s_\psi^2 \star_7 dX_0 + X_0 c_\psi^2 \star_7 dX_2 \right) \wedge d\zeta + \frac{c_\zeta s_\zeta}{4X_1^2} d\zeta \wedge (d\phi_1 + 2A_1) \wedge \star_7 dA_1$$

$$- \frac{c_\zeta c_\psi}{4X_2^2} \left(c_\zeta s_\psi d\psi + s_\zeta c_\psi d\zeta \right) \wedge (d\phi_2 + 2A_2) \wedge \star_7 dA_2$$

$$\hat{H} = \frac{X_2 \left(H_2 - q_2 c_\psi^2 \right) c_\zeta^2}{y^2} + X_1 s_\zeta^2$$

$$ds_{11}^2 = \kappa_{11}^{\frac{2}{3}} \left(\frac{\dot{V}\sigma}{2V''} \right)^{\frac{1}{3}} \left(4 ds_{\mathrm{AdS}_5}^2 + \frac{2V''\dot{V}}{\sigma} d\Omega_2^2 + \frac{2(2\dot{V} - \ddot{V})}{\dot{V}\sigma} \left(d\beta + \frac{2\dot{V}\dot{V}'}{2\dot{V} - \ddot{V}} d\chi \right)^2 \right.$$

$$\left. + \frac{2V''}{\dot{V}} \left(dr^2 + \frac{2\dot{V}}{2\dot{V} - \ddot{V}} r^2 d\chi^2 + d\eta^2 \right) \right)$$

$$\equiv f_{\mathrm{AdS}}^2 ds_{\mathrm{AdS}_5}^2 + f_{\mathbb{S}^2} d\Omega_2^2 + f_\beta^2 d\beta^2 + f_\chi^2 d\chi^2 + f_{\beta\chi}^2 d\beta d\chi + f_3^2 (dr^2 + d\eta^2)$$

$$C_3 = \frac{2\kappa_{11}}{\sigma} \left((\dot{V}\dot{V}' - \sigma\eta) d\beta - 2\dot{V}^2 V'' d\chi \right) \wedge \Upsilon_{\mathbb{S}^2}$$

$$V' \equiv \partial_\eta V, \dot{V} \equiv r\partial_r(V), \sigma \equiv V''(2\dot{V} - \ddot{V}) + (\dot{V}')^2.$$

$$\ddot{V}(r,\eta) + r^2 V''(r,\eta) = 0$$

$$\varpi(\eta) = \lim_{r \to 0^+} \dot{V}(r,\eta)$$

$$V(r,\eta) = -\frac{1}{2}\int d\eta' G(r,\eta,\eta')\varpi(\eta')$$

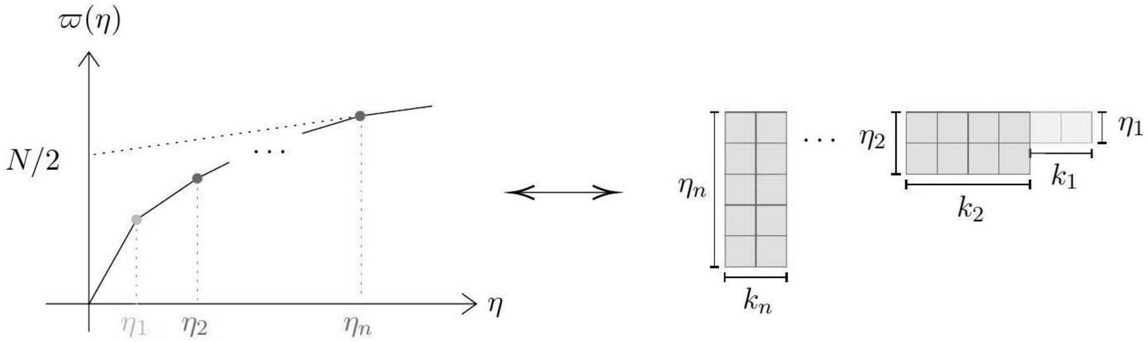
$$G(r,\eta,\eta') = \frac{1}{\sqrt{r^2 + (\eta-\eta')^2}} - \frac{1}{\sqrt{r^2 + (\eta+\eta')^2}}$$

$$\varpi(\eta) = \begin{cases} \left(1 + \frac{1}{\sqrt{1-4q_1}}\right)\eta, & \eta \in \left[0, \frac{N}{2}\sqrt{1-4q_1}\right] \\ \eta + N/2, & \eta \in \left[\frac{N}{2}\sqrt{1-4q_1}, \infty\right) \end{cases}$$

$$\begin{aligned} \varpi_a(\eta) &= \left(1 + \sum_{b=a+1}^{n} k_b\right)\eta + \sum_{b=1}^{a} \eta_b k_b \\ &\equiv p_{a+1}\eta + \delta_{a+1} \end{aligned}$$

$$m_j = \sum_{a=1}^{n} (p_a - p_{a+1})\eta_a^j = \sum_{a=1}^{n} k_a \eta_a^j$$

$$m_1 = \frac{N}{2} \text{ and } m_3 = \sum_a \frac{N_a^3}{8k_a^2}$$



$$ds_{11}^2 = g_{\mathrm{AdS}_7\times\mathbb{S}^4} + h_{11}.$$

$$ds_7^2 = \left(1 + \frac{\bar{\varsigma}}{5}\right) g_{\mathrm{AdS}_7} + \bar{h}_7$$

$$\bar{\varsigma} = \frac{3}{4}\int_{\mathbb{S}^4} \sqrt{g_{\mathbb{S}^4}} h^{ab} g_{ab}^{(0)}$$

$$ds_7^2 = \frac{L^2}{u^2}(du^2 + g)$$

$$g = g_{(0)} + g_{(2)}u^2 + g_{(4)}u^4 + g_{(6)}u^6 + h_{(6)}u^6 \log u^2 + \cdots$$

$$\begin{aligned} \langle T_{ij}\rangle dx^i dx^j &= \frac{3L^5}{8\pi G_N^{(7)}}\left(g_{(6)} - A_{(6)} + \frac{S}{24}\right) \\ &= \frac{N^3}{4\pi^3}\left(g_{(6)} - A_{(6)} + \frac{S}{24}\right) \end{aligned}$$

$$\frac{1}{G_N^{(7)}}=\frac{\mathrm{vol}(\mathbb{S}^4)}{G_N^{(11)}},G_N^{(11)}=2^4\pi^7\ell_P^9,L^3=\pi N\ell_P^3,\mathrm{vol}(\mathbb{S}^4)=\frac{L^4\pi^2}{6}$$

$$\begin{aligned}
F_a^{MNP}F_{bMNP}&\sim c_\theta^2 g_{ab}+\cdots,\\
F_{\varphi_1}^{MNP}F_{\varphi_1MNP}&\sim s_\theta^2+\cdots,\\
F_\theta^{MNP}F_{\theta MNP}&\sim 1+\cdots,\\
F_z^{MNP}F_{zMNP}&\sim q_1^2(13-5c_{2\theta})u^8+\cdots,\\
F_z^{MNP}F_{\varphi_1MNP}&\sim q_1 s_\theta^2 u^4+\cdots,\\
F_y^{MNP}F_{yMNP}&\sim q_1^2 s_{2\theta}^2 u^{12}+\cdots,
\end{aligned}$$

$$\begin{aligned}
ds_{\mathrm{FG}}^2=&\frac{L^2}{u^2}\left(du^2+\hat{\alpha}_{\mathrm{AdS}}ds_{\mathrm{AdS}_5}^2+\hat{\alpha}_z dz^2\right)+L^2 s_\theta^2\hat{\alpha}_{z\varphi_1}dzd\varphi_1+L^2c_\aleph^2c_\theta^2\hat{\alpha}_{z\varphi_2}dzd\varphi_2\\
&+\frac{L^2}{4}\left(\hat{\alpha}_\theta d\theta^2+s_\theta^2\hat{\alpha}_{\varphi_1}d\varphi_1^2+c_\theta^2\left(\hat{\alpha}_\aleph d\aleph^2+c_\aleph^2\hat{\alpha}_{\varphi_2}d\varphi_2^2\right)+\hat{\alpha}_{\theta\aleph}d\theta d\aleph\right)
\end{aligned}$$

$$ds^2=\left(g_{\mu\nu}^{(0)}+h_{\mu\nu}\right)dx^\mu dx^\nu$$

$$g_{\mu\nu}^{(0)}dx^\mu dx^\nu=\frac{L^2du^2}{u^2}+\frac{L^2}{u^2}\left(\left(1+\frac{u^2}{2}+\frac{u^4}{16}\right)ds_{\mathrm{AdS}_5}^2+\left(1-\frac{u^2}{2}+\frac{u^4}{16}\right)dz^2\right)+\frac{L^2}{4}d\Omega_4^2$$

$$\bar{h}_7=-\frac{2L^2(q_1+q_2)}{15}u^4\left(ds_{\mathrm{AdS}_5}^2-5dz^2\right)$$

$$\varsigma=\frac{10q_2c_{2\aleph}c_\theta^2+5(q_2-2q_1)c_{2\aleph}+2q_1-3q_2}{8}u^4+\cdots$$

$$\langle T_{ij}\rangle dx^idx^j=\frac{N^3}{192\pi^3}\left[1-\frac{32}{5}(q_1+q_2)\right]\left(ds_{\mathrm{AdS}_5}^2-5dz^2\right)$$

$$\left\langle T_{ij}^{(\mathrm{vac})}\right\rangle dx^idx^j=\frac{N^3}{192\pi^3}\left(ds_{\mathrm{AdS}_5}^2-5dz^2\right)$$

$$\Delta\langle T_{ij}\rangle dx^idx^j=-\frac{N^3(q_1+q_2)}{30\pi^3}\left(ds_{\mathrm{AdS}_5}^2-5dz^2\right)$$

$$h_T=\frac{N^3(q_1+q_2)}{30\pi^3}$$

$$d_2=-\frac{1}{6}N^3(q_1+q_2)$$

$$\begin{aligned}
\frac{F_4}{2\kappa_{11}}=&\left[c_\omega^2s_\omega^3\frac{5m_3-2m_1^3}{\varrho^3}d\varrho\wedge dz+3c_\omega s_\omega^2m_1d\omega\wedge dz+s_\omega^3\frac{4(m_3-m_1^3)}{\varrho^3}d\varrho\wedge d\varphi\right.\\
&\left.+c_\omega s_\omega^2\frac{6(m_1^3-m_3)}{\varrho^2}d\omega\wedge d\varphi\right]\wedge\mathrm{vol}(\mathbb{S}^2)+\cdots
\end{aligned}$$

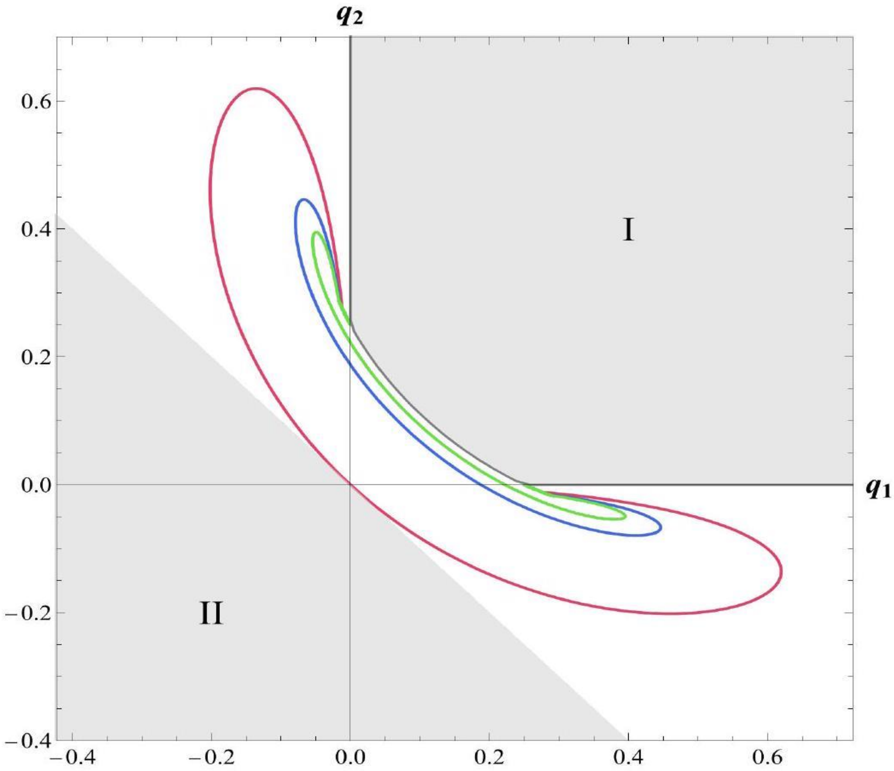


Figura 24. Torsión del espacio – tiempo cuántico. Curvatura por gravedad cuántica.

$$\begin{aligned} &F_{uMNP}F_u^{MNP} \sim s_{2\theta}^2(m_1^3 - m_3)^2u^6 + \cdots \\ &F_{zMNP}F_z^{MNP} \sim (13 - 5c_{2\theta})(m_1^3 - m_3)^2u^8 + \cdots \\ &F_{\varphi MNP}F_z^{MNP} \sim (m_1^3 - m_3)s_\theta^2u^4 + \cdots \\ &F_{aMNP}F_b^{MNP} \sim g_{\mathbb{S}^4} + \cdots \end{aligned}$$

$$h_{11} = \frac{L^2}{u^2}\left(\alpha_{\mathrm{AdS}} - 1 - \frac{u^2}{2} - \frac{u^4}{16}\right)ds_{\mathrm{AdS}_5}^2 + \frac{L^2}{u^2}\left(\alpha_z - 1 + \frac{u^2}{2} - \frac{u^4}{16}\right)dz^2$$
$$+\frac{L^2}{4}(\alpha_\theta - 1)d\theta^2 + \frac{L^2s_\theta^2}{4}(\alpha_\varphi - 1)d\varphi^2 + \frac{L^2c_\theta^2}{4}(\alpha_{\mathbb{S}^2} - 1)d\Omega_2^2 + L^2s_\theta^2\alpha_{z\varphi}dzd\varphi$$

$$\bar{h}_7 = L^2\frac{m_3 - m_1^3}{30m_1^3}u^4ds_{\mathrm{AdS}_5}^2 + L^2\frac{m_1^3 - m_3}{6m_1^3}u^4dz^2 + \cdots$$

$$\varsigma = (1 - 5c_{2\theta})\frac{m_1^3 - m_3}{16m_1^3}u^4 - 11(1 - 5c_{2\theta})\frac{m_1^3 - m_3}{216m_1^3}u^6 + \cdots$$

$$g_7 = \left(1 + \frac{\bar{\varsigma}}{5}\right)g^{(0)} + \bar{h}_7$$

$$ds_7^2 = \frac{L^2}{u^2}\left[du^2 + \left(1 + \frac{u^2}{2} + \frac{u^4}{16} + \frac{(m_3 - m_1^3)u^6}{30m_1^3}\right)ds_{\mathrm{AdS}_5}^2\right.$$
$$\left. + \left(1 - \frac{u^2}{2} + \frac{u^4}{16} + \frac{(m_1^3 - m_3)u^6}{6m_1^3}\right)dz^2\right]$$

$$\langle T_{ij}\rangle dx^i dx^j = -\frac{N^3(3m_1^3 - 8m_3)}{960\pi^3 m_1^3}\left(ds_{\mathrm{AdS}_5}^2 - 5dz^2\right)$$

$$\Delta\langle T_{ij}\rangle dx^i dx^j = -\frac{N^3(m_1^3 - m_3)}{120\pi^3 m_1^3}\left(ds_{\mathrm{AdS}_5}^2 - 5dz^2\right)$$

$$h_T = \frac{m_1^3 - m_3}{15\pi^3}$$

$$\begin{aligned} d_2 &= -\frac{m_1^3 - m_3}{3} \\ &= -\frac{1}{24}\left(N^3 - \sum_a \frac{N_a^3}{k_a^2}\right), \end{aligned}$$

$$S_{\mathrm{EE}} = \frac{\mathcal{A}_{\min}}{4G_N}$$

$$ds_{\mathrm{AdS}_5}^2 = \frac{1}{w^2}\left(dw^2 - dt^2 + dr_{\parallel}^2 + r_{\parallel}^2 d\Omega_2^2\right)$$

$$\mathcal{A}_{\min}[\Sigma] = 8\pi^4 L^9 R \int_{\epsilon_w}^{\infty} dw \frac{\sqrt{R^2 - w^2}}{w^3}\mathcal{I} = 4\pi^4\left(\frac{R^2}{\epsilon_w^2} - \log\frac{2R}{\epsilon_w} + \cdots\right)\mathcal{I}$$

$$\mathcal{I} \equiv \int d\psi d\zeta \int_{y_+}^{\Lambda_y(\epsilon_u,\psi,\zeta)} dy \hat{f}_{\mathrm{AdS}}^3 f_y \sqrt{\left(4\hat{f}_\psi^2\hat{f}_\zeta^2 - \hat{f}_{\psi\zeta}^4\right)\left(\hat{f}_{\phi_1}^2\hat{f}_{z\phi_2}^4 + \hat{f}_{\phi_2}^2\hat{f}_{z\phi_1}^4 - 4\hat{f}_{\phi_1}^2\hat{f}_{\phi_2}^2\hat{f}_z^2\right)}$$

$$\mathcal{I} = \frac{1}{8}\int d\psi d\zeta c_\psi c_\zeta^2 s_\zeta \int_{y_+}^{\Lambda_y(\epsilon_u,\psi,\zeta)} dy y$$

$$\Lambda_y(\epsilon_u,\psi,\zeta) = \frac{1}{\epsilon_u^2} + \frac{1}{2} + \frac{3 - 10q_1 - 9q_2 - 2q_2 c_{2\psi} c_\zeta^2 + (2q_1 - q_2)c_{2\zeta}}{48}\epsilon_u^2 + \cdots$$

$$\mathcal{I} = \frac{1}{24\epsilon_u^4} + \frac{1}{24\epsilon_u^2} + \frac{1}{960}\left(15 - 16(q_1 + q_2) - 40y_+^2\right) + \cdots$$

$$\mathcal{A}_{\min}[\Sigma] - \mathcal{A}_{\min}[\emptyset] = -\frac{\pi^4 L^9}{30}\left(2q_1 + 2q_2 + 5(y_+^2 - 1)\right)\left(\frac{R^2}{\epsilon_w^2} - \log\frac{2R}{\epsilon_w} + \cdots\right)$$

$$-R\partial_R\left(S_{\mathrm{EE}}[\Sigma] - S_{\mathrm{EE}}[\emptyset]\right)\big|_{R\to 0} = -\frac{N^3}{30}\left(2(q_1 + q_2) + 5(y_+^2 - 1)\right)$$

$$a_\Sigma = \frac{N^3}{24}\left(1 - y_+^2\right)$$

$$y_+ \le \frac{3\hat{n} + 1}{4\hat{n}} \le 1$$

$$ds_{11}^2 = f_{\mathrm{AdS}}^2 ds_{\mathrm{AdS}_5}^2 + f_{\mathbb{S}^2} d\Omega_2^2 + f_z^2 dz^2 + f_\varphi^2 d\varphi^2 + f_{z\varphi}^2 dz d\varphi + f_\varrho^2 d\varrho^2 + f_\omega^2 d\omega^2$$

$$\mathcal{A}_{\min}[\Sigma] = 32\pi^4 R \int dw \frac{\sqrt{R^2 - w^2}}{w^3} \mathcal{I}[\Sigma]$$

$$\mathcal{I}[\Sigma] \equiv \int_0^{\pi/2} d\omega \int_0^{\Lambda_\varrho(\epsilon_u,\omega)} f_{\mathrm{AdS}}^3 f_{\mathbb{S}^2}^2 f_\omega f_\varrho \sqrt{4 f_z^2 f_\varphi^2 - f_{z\varphi}^4}$$

$$\Lambda_\varrho(\epsilon_u,\omega) = \frac{2m_1}{\epsilon_u^2} + \frac{2m_1^3 s_\omega^2 - (1+5c_{2\omega})m_3}{48m_1^2}\epsilon_u^2 + s_\omega^2 \frac{m_3 - m_1^3}{36m_1^2}\epsilon_u^4 + \cdots$$

$$\mathcal{A}_{\min}[\Sigma] = 16\pi^4 \left(\frac{R^2}{\epsilon_w^2} - \log\frac{2R}{\epsilon_w} + O(\epsilon_w^0)\right)\mathcal{I}[\Sigma]$$

$$\mathcal{I}[\Sigma] = 64\kappa_{11}^3 \int_0^{\pi/2} d\omega \int_0^{\Lambda_\varrho(\epsilon_u,\omega)} d\varrho \varrho^2 c_\omega \dot{V} V''$$

$$\mathcal{I}[\Sigma] = -32\kappa_{11}^3 \int_0^{\Lambda_\eta} d\eta \int_0^{\Lambda_r} dr \partial_r \dot{V}^2$$

$$\Lambda_r = \Lambda_\varrho(\epsilon_u,\omega)c_\omega, \Lambda_\eta = \Lambda_\varrho(\epsilon_u,\omega)s_\omega$$

$$\frac{\mathcal{I}[\Sigma]}{32\kappa_{11}^3} = \underbrace{\int_0^{\eta_n} d\eta \varpi(\eta)^2}_{I_1} + \underbrace{\int_{\eta_n}^{\Lambda_\varrho(\epsilon_u,\pi/2)} d\eta(\eta + m_1)^2}_{I_2} - \underbrace{\int_{\omega=0}^{\omega=\pi/2} \dot{V}^2\Big|_{\Lambda_r} d\left(\Lambda_\rho(\epsilon_u,\omega)\right)}_{I_3}$$

$$I_2 = \frac{8m_1^2}{3\epsilon_u^6} + \frac{4m_1^3}{\epsilon_u^4} + \frac{13m_1^3 + 2m_3}{6\epsilon_u^2} + \frac{8m_3 + m_1^3 - 18m_1^2\eta_n - 18m_1\eta_n^2 - 6\eta_n^3}{18} + \cdots$$

$$I_3 = \frac{8m_1^3}{3\epsilon_u^6} + \frac{8m_1^3}{3\epsilon_u^4} + \frac{5m_1^3 + 2m_3}{6\epsilon_u^2} + \frac{m_1^3 + 14m_3}{45} + \cdots$$

$$I_2 - I_3 = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} + \frac{4m_3 + m_1^3 - 10\eta_n(\eta_n^2 + 3\eta_n m_1 + 3m_1^2)}{30} + \cdots$$

$$I_1 = \frac{1}{3}\sum_{a=0}^{n-1} \left(p_{a+1}^2(\eta_{a+1}^3 - \eta_a^3) + 3\delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) + 3\delta_{a+1}^2(\eta_{a+1} - \eta_a)\right)$$

$$\frac{\mathcal{I}[\Sigma]}{32\kappa_{11}^3} = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} + \frac{4m_3 + m_1^3}{30} + \frac{1}{3}\sum_{a=0}^{n} (p_{a+1}^2\eta_{a+1}^3 - \eta_a^3)$$
$$+ \sum_{a=0}^{n} \delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) + \sum_{a=0}^{n} \delta_{a+1}^2(\eta_{a+1} - \eta_a)$$

$$\frac{\mathcal{I}[\emptyset]}{32\kappa_{11}^3} = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} - \frac{5m_1^3}{6} + \cdots$$

$$\frac{\mathcal{I}[\Sigma]-\mathcal{I}[\emptyset]}{32\kappa_{11}^3}=\frac{2m_3+13m_1^3}{15}+\frac{1}{3}\sum_{a=0}^{n} p_{a+1}^2(\eta_{a+1}^3-\eta_a^3)+\sum_{a=0}^{n}\delta_{a+1}p_{a+1}(\eta_{a+1}^2-\eta_a^2)$$
$$+\sum_{a=0}^{n}\delta_{a+1}^2(\eta_{a+1}-\eta_a)\Bigg)$$

$$\mathcal{A}_{\min}[\Sigma]-\mathcal{A}_{\min}[\emptyset]=2^9\pi^4\kappa_{11}^3\left(\frac{R^2}{\epsilon_w^2}-\log\frac{2R}{\epsilon_w}+O(1)\right)(\mathcal{I}[\Sigma]-\mathcal{I}[\emptyset])$$

$$R\partial_R(S_{\mathrm{EE}}[\Sigma]-S_{\mathrm{EE}}[\emptyset])=-(\mathcal{I}[\Sigma]-\mathcal{I}[\emptyset]),$$

$$G_N^{(11)}=2^{13}\pi^4\kappa_{11}^3\ \kappa_{11}=L^3/8N$$

$$d_2=-\frac{1}{3}(m_1^3-m_3)$$

$$a_\Sigma=\frac{(\sum_{a=1}^{n}\ k_a\eta_a)^3}{4}+\frac{1}{12}\sum_{a=0}^{n}\left(p_{a+1}^2(\eta_{a+1}^3-\eta_a^3)+3\delta_{a+1}p_{a+1}(\eta_{a+1}^2-\eta_a^2)+3\delta_{a+1}^2(\eta_{a+1}-\eta_a)\right)$$

$$a_\Sigma=\frac{N^3}{32}-\frac{1}{96}\sum_{a=1}^{n}\left(\frac{1+2k_a}{k_a^2}N_a^3+\sum_{b=a+1}^{n}N_ak_b\left(\frac{N_a^2}{k_a^2}+3\frac{N_b^2}{k_b^2}\right)\right)$$

$$a_\Sigma|_{n=1}=\frac{N^3}{48}\left(1+2q_1-\sqrt{1-4q_1}\right)$$

$$c_{4\ \mathrm{d}}=\frac{2^5\pi^3\kappa_{11}^3}{(2\pi\ell_P)^9}\int_{\mathcal{M}_6}\left(\frac{\dot{V}\sigma}{2V''}\right)^{\frac{3}{2}}$$

$$ds_{11}^2=\left(\frac{\kappa_{11}^2\dot{V}\sigma}{2V''}\right)^{\frac{1}{3}}(ds_{\mathrm{AdS}_5}^2+ds_{\mathcal{M}_6}^2)$$

$$S=\frac{1}{16\pi G_N^{(11)}}\int_{\mathcal{M}}d^{11}x\sqrt{-g_{11}}\left(\mathcal{R}-\frac{1}{48}F_{MNPQ}F^{MNPQ}\right)+\frac{1}{8\pi G_N^{(11)}}\int_{\partial\mathcal{M}}K\Upsilon_{\partial\mathcal{M}}+S_{\mathrm{CS}}$$

$$\sqrt{-g_{11}}\left(\mathcal{R}-\frac{1}{48}F_{MNPQ}F^{MNPQ}\right)d^{11}x=-\frac{1}{3}F_4\wedge\star F_4$$

$$d\star F_4=0$$

$$S_{\mathrm{OS}}=\frac{1}{16\pi G_N^{(11)}}\int_{\partial\mathcal{M}}\left(2K\Upsilon_{\partial\mathcal{M}}-\frac{1}{3}F_4\wedge C_6\right)=:S_{\mathrm{OS,GHY}}+S_{\mathrm{OS,bulk}}$$

$$C_6=L^6\Bigg\{\frac{1}{2}q_2c_\zeta^2c_\psi^2d\phi_2+\frac{1}{2}q_1s_\zeta^2d\phi_1+\Bigg[y(y^2+q_2)-\frac{c_\zeta^2}{2y}(q_2c_{2\psi}(y(y-a_2-1)+q_1)$$
$$+2q_1y(a_1-y+1)+q_2y(y-a_2-1)-q_2q_1)]dz\}\wedge\Upsilon_{\mathrm{AdS}_5}$$

$$C_6(y_+) = L^6\left\{\frac{1}{2}q_2 c_\zeta^2 c_\psi^2 d\phi_2 + \frac{1}{2}q_1 s_\zeta^2 d\phi_1 + y_+ H_2(y_+)dz\right\}\wedge \Upsilon_{\mathrm{AdS}_5}$$

$$\Lambda_5 = -zL^6 y_+ H_2(y_+)\Upsilon_{\mathrm{AdS}_5}$$

$$\Lambda_5 \to \Lambda_5 + zL^6 c_\zeta^2 c_\psi^2 a_2 q_2 + zL^6 s_\zeta^2 a_1 q_1$$

$$\tilde{C}_6 = L^6\left[\left(\frac{1}{u^6}+\frac{3}{2u^4}-\frac{1}{16u^2}\left(2q_1-3(5+q_2)+10q_2c_{2\aleph}c_\theta^2+5(q_2-2q_1)c_{2\theta}\right)\right)dz\right]\wedge\Upsilon_{\mathrm{AdS}_5}+\cdots$$

$$F_4 = \frac{L^3}{8}\{\left[\left[3c_\theta^2 s_\theta d\varphi_1\wedge d\theta + \frac{c_\theta^3}{2}\left(5s_\theta^2(2q_1-q_2c_{2\aleph}-q_2)du\wedge d\varphi_1 + 16q_1 du\wedge dz\right)u^3\right]\wedge\Upsilon_{\mathbb{S}^2}\right.$$
$$\left.+\frac{s_\theta|c_\aleph|}{2}du\wedge d\varphi_1\wedge\left(5q_2c_\theta^2 s_{2\aleph}d\theta\wedge d\varphi_2 + 8q_2 dz\wedge\left(\frac{2s_\aleph}{c_\aleph}d\theta - s_{2\theta}d\aleph\right)\right)u^3\right\}+\cdots$$

$$\Upsilon_{\partial\mathcal{M}} = \frac{L^{10}}{16}\left(\frac{1}{\epsilon_u^6}+\frac{1}{\epsilon_u^4}+\frac{5}{16\epsilon_u^2}+\frac{5\left(5c_{2\theta}(q_2-2q_1)+2q_1+q_2\left(10c_\theta^2c_{2\aleph}-3\right)\right)}{432}\right)\Upsilon_{\mathrm{AdS}_5}\wedge dz\wedge\Upsilon_{\mathbb{S}^4}+\cdots$$

$$K = -\frac{6}{L}+\frac{2\epsilon_u^2}{L}-\frac{3\epsilon_u^4}{4L}+\frac{(25c_{2\theta}(q_2-2q_1)+10q_1+50q_2c_\theta^2c_{2\aleph}-15q_2+9)\epsilon_u^6}{72L}+\cdots$$

$$S_{\mathrm{OS,GHY}} = -\mathrm{vol}(\mathrm{AdS}_5)\frac{\pi^2L^9}{8G_N^{(11)}}\left(\frac{2}{\epsilon_u^6}+\frac{4}{3\epsilon_u^4}+\frac{5}{24\epsilon_u^2}\right)+\cdots$$

$$S_{\mathrm{OS,\,bulk}} = -\mathrm{vol}(\mathrm{AdS}_5)\frac{\pi^2L^9}{16G_N^{(11)}}\left(\frac{2}{3\epsilon_u^6}+\frac{1}{\epsilon_u^4}+\frac{5}{8\epsilon_u^2}-\frac{2q_1(q_2+y_+(2+3y_+))}{15y_+}\right.$$
$$\left.-\frac{32q_2+48q_2y_++80y_+^3-25}{120}\right)+\cdots$$

$$S_{\mathrm{OS}}-S_{\mathrm{OS}}^{(\mathrm{vac})} = \frac{\mathrm{vol}(\mathrm{AdS}_5)\pi^2L^9}{120y_+G_N^{(11)}}\left(q_1q_2+(q_1+q_2)y_+(2+3y_+)+5y_+(y_+^3-1)\right)$$
$$= -\frac{\mathrm{vol}(\mathrm{AdS}_5)\pi^2L^9}{24G_N^{(11)}}\left(1-y_+^2-\frac{1}{5}\left(\frac{2q_1^2}{q_1+y_+^2}+\frac{2q_2^2}{q_2+y_+^2}\right)\right)$$
$$= -\frac{\mathrm{vol}(\mathrm{AdS}_5)\pi^2L^9}{24G_N^{(11)}}\left(1-y_+^2-\frac{1}{5}(2a_1q_1+2a_2q_2)\right)$$

$$S_{\mathrm{OS}}^{(\mathrm{ren})}\Big|_{\log} = -\frac{N^3}{1920y_+}\left(q_1q_2+(q_1+q_2)y_+(2+3y_+)+5y_+(y_+^3-1)\right)$$
$$= \frac{N^3\left(4q_1q_2-2(q_1+q_2)y_+(1-y_+)+5y_+(1-y_+^2)\right)}{1920y_+}$$

$$\varphi_I = \tilde{\varphi}_I + 2a_In_Iz$$

$$\frac{3L^9}{8}\left(n_2q_2a_2c_\zeta^2c_\psi^2+n_1a_1q_1s_\zeta^2\right)\Upsilon_{\mathbb{S}^4}\wedge\Upsilon_{\mathrm{AdS}_5}\wedge dz+\cdots\subset F_4\wedge C_6$$

$$S_{\text{OS}} \longmapsto S_{\text{OS}} - \frac{\text{vol}(\text{AdS}_5)\pi^2 L^9}{24G_N^{(11)}}\frac{1}{5}(2n_1 a_1 q_1 + 2n_2 a_2 q_2)$$

$$\mathcal{Z}_{\mathbb{S}^1_\beta \times \mathbb{S}^3} = e^{-\beta E_C}\mathcal{J}$$

$$E_C[\emptyset] \equiv \frac{\mathfrak{c}}{24}, \text{ where } \mathfrak{c} = N(N^2-1)(\mathfrak{b}+\mathfrak{b}^{-1})^2 + N - 1$$

$$\begin{aligned} E_C[\Sigma]_{\vartheta,\vec{\varpi}} - E_C[\emptyset] &= \frac{1}{2}(\mathfrak{b}+\mathfrak{b}^{-1})^2\left[(\hat{\varrho}_{\mathfrak{l}},\hat{\varrho}_{\mathfrak{l}}) - \left(\hat{\varrho}_{\mathfrak{g}},\hat{\varrho}_{\mathfrak{g}}\right)\right] + \frac{1}{2}(\vec{\varpi},\vec{\varpi}), \\ &= -\frac{1}{6}\left(N^3 - \sum_{a=1}^{n} N_a^3 - 3(\vec{\varpi},\vec{\varpi})\right). \end{aligned}$$

$$(\hat{\varrho}_{\mathfrak{l}},\hat{\varrho}_{\mathfrak{l}}) = \frac{1}{12}\sum_{a=1}^{n}(N_a^3 - N_a), \left(\hat{\varrho}_{\mathfrak{g}},\hat{\varrho}_{\mathfrak{g}}\right) = \frac{1}{12}(N^3 - N)$$

$$E_C[\Sigma]_{\vartheta,\vec{0}} - E_C[\emptyset] = 4d_2|_{k_a\to 1},$$

$$a_\Sigma = \frac{9k_{rrr} - 3k_r}{32},$$

$$r_\Sigma = \frac{2}{3}\left(2r_{6d} - M_\varphi\right)$$

$$k_{rrr} = \frac{2}{27}(\mathfrak{n}_v - \mathfrak{n}_h) + \frac{8}{9}\mathfrak{n}_v, k_r = \frac{2}{3}(\mathfrak{n}_v - \mathfrak{n}_h),$$

$$\hat{f}_{\text{AdS}}^2 = \kappa^{2/3}\left[\frac{c_\zeta^2(q_1+y^2)\left(q_2 - q_2 c_{2\psi} + 2y^2\right)}{2y} + y(q_2+y^2)s_\zeta^2\right]^{1/3}$$

$$\hat{f}_y^2 = \kappa^{2/3}\frac{\hat{f}_{\text{AdS}}^2 y}{4(q_1+y^2)(q_2+y^2) - 4y^3},$$

$$\begin{aligned} \hat{f}_z^2 = \kappa^{2/3}&\left[\frac{c_\zeta^2\left(c_{2\psi}\left((a_2+1)^2 q_2 y + a_2^2 y^3 - q_2(q_1+y^2)\right) + (a_2+1)^2 q_2 y + (a_2^2-2)y^3\right)}{2y\hat{f}_{\text{AdS}}^4}\right. \\ &\left. + \frac{s_\zeta^2\left(y\left((a_1^2-1)y + (q_2+y^2)\right) + (a_1+1)^2 q_1\right)}{\hat{f}_{\text{AdS}}^4} + \frac{c_\zeta^2(q_1+y^2)(q_2+2y^2)}{2\hat{f}_{\text{AdS}}^4 y}\right] \end{aligned}$$

$$\hat{f}_{\phi_1}^2 = \kappa^{2/3}\frac{(q_1+y^2)s_\zeta^2}{4\hat{f}_{\text{AdS}}^4},$$

$$\hat{f}_{\phi_2}^2 = \kappa^{2/3}\frac{c_\psi^2 c_\zeta^2(q_2+y^2)}{4\hat{f}_{\text{AdS}}^4},$$

$$\hat{f}_{z\phi_1}^2 = \kappa^{2/3}\frac{s_\zeta^2(a_1 q_1 + a_1 y^2 + q_1)}{\hat{f}_{\text{AdS}}^4},$$

$$\hat{f}^2_{z\phi_2} = \kappa^{2/3}\frac{c^2_\psi c^2_\zeta(a_2 q_2 + a_2 y^2 + q_2)}{\hat{f}^4_{\mathrm{AdS}}}$$
$$\hat{f}^2_\psi = \kappa^{2/3}\frac{c^2_\zeta\left(q_2 - q_2 c_{2\psi} + 2y^2\right)}{8\hat{f}^4_{\mathrm{AdS}}}$$
$$\hat{f}^2_\zeta = \kappa^{2/3}\frac{q_1 c_{2\zeta} + 2q_2 c^2_\psi s^2_\zeta + q_1 + 2y^2}{8\hat{f}^4_{\mathrm{AdS}}}$$
$$\hat{f}^2_{\psi\zeta} = \kappa^{2/3}\frac{q_2 c_\psi c_\zeta s_\psi s_\zeta}{2\hat{f}^4_{\mathrm{AdS}}}$$

$$\hat{f}^2_y dy^2 + \hat{f}^2_\psi d\psi^2 + \hat{f}^2_\zeta d\zeta^2 + \hat{f}^2_{\psi\zeta} d\psi d\zeta = \frac{L^2}{u^2}du^2 + \frac{L^2}{4}\left(c^2_\theta \hat{\alpha}_\aleph d\aleph^2 + \hat{\alpha}_\theta d\theta^2 + \hat{\alpha}_{\theta\aleph} d\theta d\aleph\right)$$

$$y = \frac{1}{u^2} + \frac{1}{2} + \frac{(2q_1 - q_2)c_{2\theta} - 2q_2 c_{2\aleph} c^2_\theta - 10q_1 - 9q_2 + 3}{48}u^2 + \cdots$$
$$\psi = \aleph + \frac{q_2 s_{2\aleph}}{24}u^4 + \cdots$$
$$\zeta = \theta - \frac{s_{2\theta}\left(q_1 - q_2 c^2_\aleph\right)}{24}u^4 + \cdots$$

$$ds^2_{\mathrm{FG}} = \frac{L^2}{u^2}\left(du^2 + \hat{\alpha}_{\mathrm{AdS}} ds^2_{\mathrm{AdS}_5} + \hat{\alpha}_z dz^2\right) + L^2 s^2_\theta \hat{\alpha}_{z\varphi_1} dz d\varphi_1 + L^2 c^2_\aleph c^2_\theta \hat{\alpha}_{z\varphi_2} dz d\varphi_2$$
$$+ \frac{L^2}{4}\left(\hat{\alpha}_\theta d\theta^2 + s^2_\theta \hat{\alpha}_{\varphi_1} d\varphi^2_1 + c^2_\theta\left(\hat{\alpha}_\aleph d\aleph^2 + c^2_\aleph \hat{\alpha}_{\varphi_2} d\varphi^2_2\right) + \hat{\alpha}_{\theta\aleph} d\theta d\aleph\right)$$

$$\phi_I = \varphi_I - 2a_I z$$

$$\hat{\alpha}_{\mathrm{AdS}} = 1 + \frac{u^2}{2} + \frac{3 - 2q_1 + 3q_2 - 10q_2 c_{2\aleph} c^2_\theta + 5(2q_1 - q_2)c_{2\theta}}{48}u^4 + \cdots$$
$$\hat{\alpha}_z = 1 - \frac{u^2}{2} + \frac{3 - 2q_1 + 3q_2 - 10q_2 c_{2\aleph} c^2_\theta + 5(2q_1 - q_2)c_{2\theta}}{48}u^4 + \cdots$$
$$\hat{\alpha}_{\varphi_1} = 1 + \frac{10q_2 c_{2\aleph} c^2_\theta + 5(q_2 - 2q_1)c_{2\theta} + 14q_1 - 11q_2}{24}u^4 + \cdots$$

$$\hat{\alpha}_{\varphi_2} = 1 + \frac{10q_2 c_{2\aleph} c^2_\theta + 5(q_2 - 2q_1)c_{2\theta} - 6q_1 + 9q_2}{24}u^4 + \cdots,$$
$$\hat{\alpha}_{z\varphi_1} = q_1 u^4 - q_1 u^6 + \cdots,$$
$$\hat{\alpha}_{z\varphi_2} = q_2 u^4 - q_2 u^6 + \cdots,$$
$$\hat{\alpha}_\theta = 1 + \frac{5q_2 c_{2\aleph} + 2q_1 - 3q_2}{12}u^4 + \cdots,$$
$$\hat{\alpha}_\aleph = 1 + \frac{5(q_2 - 2q_1)c_{2\theta} - 10q_2 c_{2\aleph} s^2_\theta - 6q_1 - q_2}{24}u^4 + \cdots,$$
$$\hat{\alpha}_{\theta\aleph} = \frac{5q_2 s_{2\theta} s_{2\aleph}}{12}u^4 + \cdots$$

$$a_I = -\frac{q_I}{q_I + y^2_+}$$

$$\varpi_a(\eta) = p_{1+a}\eta + \delta_{1+a}$$

$$-\frac{1}{2}\int d\eta' G(r,\eta,\eta')\varpi_a(\eta') = \frac{p_{1+a}}{2}\Bigg(\sqrt{r^2+(\eta+\eta')^2}-\sqrt{r^2+(\eta-\eta')^2}$$
$$-\eta\tanh^{-1}\left(\frac{\eta+\eta'}{\sqrt{r^2+(\eta+\eta')^2}}\right)+\eta\tanh^{-1}\left(\frac{\eta-\eta'}{\sqrt{r^2+(\eta-\eta')^2}}\right)\Bigg)$$
$$+\frac{\delta_{1+a}}{2}\left(\tanh^{-1}\left(\frac{\eta+\eta'}{\sqrt{r^2+(\eta+\eta')^2}}\right)+\tanh^{-1}\left(\frac{\eta-\eta'}{\sqrt{r^2+(\eta-\eta')^2}}\right)\right)$$

$$f_3^2(dr^2+d\eta^2)\to f_\varrho^2 d\varrho^2+f_\omega^2 d\omega^2$$

$$\begin{aligned}
\dot{V} &= \varrho s_\omega + m_1 s_\omega - \frac{m_3 c_\omega^2 s_\omega}{2\varrho^2} + \frac{m_5(7c_{2\omega}-1)c_\omega^2 s_\omega}{16\varrho^4}+\cdots\\
\ddot{V} &= -m_1 c_\omega^2 s_\omega + \frac{m_3(5c_{2\omega}+1)c_\omega^2 s_\omega}{4\varrho^2} - \frac{m_5(28c_{2\omega}+63c_{4\omega}+29)c_\omega^2 s_\omega}{64\varrho^4}+\cdots\\
\dot{V}' &= 1+\frac{m_1 c_\omega^2}{\varrho}+\frac{m_3(3-5c_{2\omega})c_\omega^2}{4\varrho^3}+\frac{3m_5(21c_{4\omega}-28c_{2\omega}+15)c_\omega^2}{64\varrho^5}+\cdots\\
V'' &= \frac{m_1 s_\omega}{\varrho^2}-\frac{m_3(5c_{2\omega}+1)s_\omega}{4\varrho^4}+\frac{m_5(28c_{2\omega}+63c_{4\omega}+29)s_\omega}{64\varrho^6}+\cdots
\end{aligned}$$

$$\sigma = 1+\frac{2m_1}{\varrho}-\frac{m_1^2(c_{2\omega}-3)}{2\varrho^2}+\frac{m_3(1-3c_{2\omega})}{2\varrho^3}+\frac{m_3 m_1(1-12c_{2\omega}+3c_{4\omega})}{8\varrho^4}+\cdots$$

$$\begin{aligned}
\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}}f_{\mathrm{Ads}}^2 &= 4\varrho+4m_1+\frac{5m_3c_{2\omega}+4m_1^3s_\omega^2+m_3}{3m_1\varrho}+\frac{4(m_3-m_1^3)s_\omega^2}{3\varrho^2}+\cdots\\
\frac{(2m_1)^{1/3}}{s_\omega^2\kappa_{11}^{2/3}}f_{\mathbb{S}^2}^2 &= 2m_1-\frac{(1+5c_{2\omega})m_3+4s_\omega^2m_1^3}{3\varrho^2}+\frac{8m_1(m_1^3-m_3)s_\omega^2}{3\varrho^3}+\cdots\\
\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}}f_\varrho^2 &= \frac{2m_1}{\varrho^2}-\frac{(1+5c_{2\omega})m_3-2s_\omega^2m_1^3}{3\varrho^4}+\frac{4m_1(m_3-m_1^3)s_\omega^2}{3\varrho^5}+\cdots\\
\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}}f_\beta^2 &= 4\varrho+m_1(c_{2\omega}-3)+\frac{(1+5c_{2\omega})m_3+4m_1^3s_\omega^2}{3m_1\varrho}+\cdots\\
\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}}f_\chi^2 &= 4\varrho+4m_1c_{2\omega}+\frac{5m_3c_{2\omega}+4m_1^3s_\omega^2+m_3}{3m_1\varrho}+\cdots\\
\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}}f_{\beta\chi}^2 &= 8\varrho-8m_1s_\omega^2+\frac{2(5m_3c_{2\omega}+4m_1^3s_\omega^2+m_3)}{3m_1\varrho}+\cdots
\end{aligned}$$

$$f_\varrho^2d\varrho^2+f_\omega^2d\omega^2=\frac{L^2}{u^2}du^2+\frac{L^2}{4}\alpha_\theta d\theta^2$$

$$\begin{aligned}
\rho &= \frac{2m_1}{u^2}+\frac{2m_1^3c_\theta^2+m_3(5c_{2\theta}-1)}{48m_1^2}u^2+\frac{(m_3-m_1^3)c_\theta^2}{36m_1^2}u^4+\cdots\\
\omega &= \theta+\frac{\pi}{2}-\frac{(m_1^3+5m_3)s_{2\theta}}{96m_1^3}u^4+\frac{(m_1^3-m_3)s_{2\theta}}{216m_1^3}u^6+\cdots
\end{aligned}$$

$$\chi=(1+\mathcal{C}_z)z+a_\varphi\varphi,\beta=-\mathcal{C}_zz+b_\varphi\varphi$$

$$\frac{f_\varphi^2}{L^2} = \frac{\left(a_\varphi + b_\varphi\right)^2}{u^2} - \frac{1}{8}\left(\left(2a_\varphi + b_\varphi\right)^2 c_{2\theta} + b_\varphi\left(4a_\varphi + 3b_\varphi\right)\right) + \cdots$$

$$\frac{f_{z\varphi}^2}{L^2} = \frac{2\left(a_\varphi + b_\varphi\right)}{u^2} + \frac{1}{4}\left(2a_\varphi \mathcal{C}_z + b_\varphi(\mathcal{C}_z - 2) - \left(2a_\varphi + b_\varphi\right)(\mathcal{C}_z + 2)c_{2\theta}\right) + \cdots$$

$$\frac{f_z^2}{L^2} = \frac{1}{u^2} + \frac{1}{8}\left(\mathcal{C}_z(\mathcal{C}_z + 4) - (\mathcal{C}_z + 2)^2 c_{2\theta}\right) + \cdots$$

$$ds_{\mathrm{FG}}^2 = \frac{L^2}{u^2}\left(du^2 + \alpha_{\mathrm{AdS}} ds_{\mathrm{AdS}_5}^2 + \alpha_z dz^2\right) + L^2 s_\theta^2 \alpha_{z\varphi} dz d\varphi + \frac{L^2}{4}\left(s_\theta^2 \alpha_\varphi d\varphi^2 + c_\theta^2 \alpha_{\mathbb{S}^2} d\Omega_2^2 + \alpha_\theta d\theta^2\right)$$

$$\alpha_{\mathrm{AdS}} = 1 + \frac{u^2}{2} + \frac{1}{96}\left(10c_\theta^2 + \frac{m_3(1 - 5c_{2\theta})}{m_1^3}\right)u^4 + \frac{(m_3 - m_1^3)c_\theta^2}{18m_1^3}u^6 \ldots$$

$$\alpha_z = 1 - \frac{u^2}{2} + \frac{1}{96}\left(10c_\theta^2 + \frac{m_3(1 - 5c_{2\theta})}{m_1^3}\right)u^4 + \frac{(m_3 - m_1^3)(5c_{2\theta} - 13)}{72m_1^3}u^6 + \cdots$$

$$\alpha_\varphi = 1 + \frac{(m_3 - m_1^3)(5c_{2\theta} - 7)}{48m_1^3}u^4 + \frac{(m_1^3 - m_3)(10c_{2\theta} - 17)}{108m_1^3}u^6 + \cdots$$

$$\alpha_{\mathbb{S}^2} = 1 + \frac{(m_3 - m_1^3)(5c_{2\theta} + 3)}{48m_1^3}u^4 + \frac{(m_1^3 - m_3)(5c_{2\theta} + 4)}{54m_1^3}u^6 + \cdots$$

$$\alpha_{z\varphi} = \frac{m_1^3 - m_3}{4m_1^3}u^4 - \frac{m_1^3 - m_3}{4m_1^3}u^6 + \cdots$$

$$\alpha_\theta = 1 + \frac{m_1^3 - m_3}{24m_1^3}u^4 + \frac{(m_3 - m_1^3)(5c_{2\theta} + 9)}{216m_1^3}u^6 + \cdots$$

$$ds_{11}^2 = L^2\left(dx^2 + \cosh^2(x) ds_{\mathrm{AdS}_5}^2 + \sinh^2(x) dz^2\right) + \frac{L^2}{4} d\Omega_4^2$$

$$F_4 = -\frac{3L^3}{8}\Upsilon_{\mathbb{S}^4}$$

$$\star_{11} F_4 = 6L^6 \cosh^5(x) \sinh(x) dx \wedge dz \wedge \Upsilon_{\mathrm{AdS}_5}$$

$$x = -\ln(u/2)$$

$$ds_{11}^2 = \frac{L^2}{u^2}\left(du^2 + \left(1 + \frac{u^2}{2} + \frac{u^4}{16}\right)ds_{\mathrm{AdS}_5}^2 + \left(1 - \frac{u^2}{2} + \frac{u^4}{16}\right)dz^2\right) + \frac{L^2}{4}d\Omega_4^2$$

$$\star_{11} F_4 = 6L^6\left(\frac{1}{u^7} + \frac{1}{u^5} + \frac{5}{16u^3} - \frac{5u}{256} - \frac{u^3}{256} - \frac{u^5}{4096}\right) du \wedge dz \wedge \Upsilon_{\mathrm{AdS}_5} + \cdots$$

$$S_{\mathrm{OS,bulk}}^{(\mathrm{vac})} = -\frac{L^9\pi^2}{8G_N^{(11)}}\mathrm{vol}(\mathrm{AdS}_5)\left(\frac{1}{3\epsilon_u^6} + \frac{1}{2\epsilon_u^4} + \frac{5}{16\epsilon_u^2} - \frac{11}{48}\right) + \cdots$$

$$S_{\mathrm{OS}}^{(\mathrm{vac})} = -\frac{\pi^2 L^9}{8G_N^{(11)}}\mathrm{vol}(\mathrm{AdS}_5)\left(\frac{1}{3\epsilon_u^6} + \frac{5}{3\epsilon_u^5} + \frac{1}{2\epsilon_u^4} + \frac{1}{\epsilon_u^3} + \frac{5}{16\epsilon_u^2} + \frac{5}{48\epsilon_u} - \frac{11}{48}\right) + \cdots$$

$$C_6|_{u=\epsilon_u} = 3L^6\left(\frac{1}{3\epsilon_u^6}+\frac{1}{2\epsilon_u^4}+\frac{5}{16\epsilon_u^2}-\frac{11}{48}+\frac{5\epsilon_u^2}{256}+\frac{\epsilon_u^4}{512}+\frac{\epsilon_u^6}{12288}\right)dz\wedge \Upsilon_{\mathrm{AdS}_5}$$

$$ds^2_{\mathrm{AdS}_5} = dx^2+\sinh^2(x)d\Omega_4^2$$

$$\mathrm{vol}(\mathrm{AdS}_5) = \frac{8\pi^2}{3}\int_0^{\Lambda_x} dx\sinh^4(x) = \frac{2\pi^2}{3\epsilon_x^4}-\frac{4\pi^2}{3\epsilon_x^2}-\pi^2\log\frac{\epsilon_x}{2}+\cdots$$

$$S_{\mathrm{CT},1} = -\frac{1}{4}\int d\Omega_4\sqrt{|g_{\epsilon_x}|} = -\frac{2\pi^2}{3\epsilon_x^4}+\frac{2\pi^2}{3\epsilon_x^2}-\frac{\pi^2}{4}+\cdots$$
$$S_{\mathrm{CT},2} = \frac{1}{48}\int d\Omega_4\sqrt{|g_{\epsilon_x}|}\mathcal{R}_{\epsilon_x} = \frac{2\pi^2}{3\epsilon_x^2}-\frac{\pi^2}{3}+\cdots$$

$$\sqrt{|g_{\epsilon_x}|} = (1-\epsilon_x)^4\sqrt{|g_{\mathbb{S}^4}|}/16\epsilon_x^2, \mathcal{R}_{\epsilon_x} = 12\operatorname{csch}^2(\epsilon_x)$$

$$\mathrm{vol}(\mathrm{AdS}_5) = -\pi^2\log\frac{\epsilon_x}{2}+\cdots$$

$$c_\Sigma = \frac{9k_{rrr}-5k_r}{32}$$

$$\frac{ds^2}{2\pi}=\frac{|h|}{\sqrt{2hh''-(h')^2}}ds^2(\mathrm{AdS}_3)+\sqrt{2hh''-(h')^2}\left[\frac{1}{4|h|}dr^2+\frac{2}{|h''|}ds^2(\mathbb{CP}^3)\right]$$
$$e^{-\Phi} = \frac{(|h''|)^{\frac{3}{2}}}{2\sqrt{\pi}(2hh''-(h')^2)^{\frac{1}{4}}}, H_3 = dB_2, B_2 = 4\pi\left(-(r-l)+\frac{h'}{h''}\right)J$$

$$ds^2(\mathbb{CP}^3)= d\xi^2+\frac{1}{4}\cos^2\xi ds^2(\mathrm{S}_1^2)+\frac{1}{4}\sin^2\xi ds^2(\mathrm{S}_2^2)+\frac{1}{4}\sin^2\xi\cos^2\xi(d\psi+\eta_1+\eta_2)^2, d\eta_i = -\mathrm{vol}(\mathrm{S}_i^2)$$
$$J = \frac{1}{4}\sin^2\xi\mathrm{vol}(\mathrm{S}_2^2)-\frac{1}{4}\cos^2\xi\mathrm{vol}(\mathrm{S}_1^2)-\frac{1}{2}\sin\xi\cos\xi d\xi\wedge(d\psi+\eta_1+\eta_2)$$

$$F_0= -\frac{1}{2\pi}h''', F_2 = B_2F_0+2(h''-(r-l)h''')J$$
$$F_4= \pi d\left(h'+\frac{hh'h''}{2hh''-(h')^2}\right)\wedge\mathrm{vol}(\mathrm{AdS}_3)+B_2\wedge F_2-\frac{1}{2}B_2\wedge B_2F_0$$
$$-4\pi\big(2h'+(r-l)(-2h''+(r-l)h''')\big)J\wedge J$$

$$h''' = -2\pi F_0.$$

$$\hat{f} = e^{-B_2}\wedge f$$

$$\hat{f}_2 = 2(h''-(r-l)h''')J,$$
$$\hat{f}_4 = -4\pi\left(2h'+(r-l)\big((r-l)h'''-2h''\big)\right)J\wedge J,$$
$$\hat{f}_6 = \frac{16\pi^2}{3}\left(6h-(r-l)\big(6h'+(r-l)((r-l)h'''-3h'')\big)\right)J\wedge J\wedge J.$$

$$d\hat{f}_{2n} = -\frac{1}{2\pi}(4\pi)^n(r-l)^n h''''\frac{1}{n!}dr\wedge J^n$$

$$h = c_1 + c_2 r^3, c_{1,2} \neq 0$$
$$h = c_1 + c_2 r + \frac{c_2^2}{4c_1} r^2 + c_3 r^3, c_{1,2,3} \neq 0$$
$$h = (c_1 + c_2 r) r^2, c_{1,2} \neq 0$$

$$dF_0 = \Delta F_0 \delta(r - r_0)$$

$$(h, (h')^2, h'')$$

$$r_0 = l$$

$$ds_{10}^2 = a(\zeta, \vec{\theta})\left(dx_{1,d}^2 + b(\zeta) d\zeta^2\right) + g_{ij}(\zeta, \vec{\theta}) d\theta^i d\theta^j, \Phi = \Phi(\zeta, \vec{\theta})$$

$$c_{hol} = \frac{3d^d}{G_N} \frac{b(\zeta)^{d/2} (\hat{H})^{\frac{2d+1}{2}}}{(\hat{H}')^d}$$

$$\hat{H} = \left(\int d\vec{\theta} e^{-2\Phi} \sqrt{\det[g_{ij}] a(\zeta, \vec{\theta})^d}\right)^2$$

$$c_{hol} = \frac{1}{2} \int dr (2hh'' - (h')^2)$$

$$h(r) = Q_2 - Q_4 r + \frac{1}{2} Q_6 r^2,$$

$$\sinh \mu = \frac{Q_6 r - Q_4}{\sqrt{2Q_2 Q_6 - Q_4^2}}$$

$$ds^2 = \frac{4\pi \sqrt{2Q_2 Q_6 - Q_4^2}}{Q_6} \left(\frac{1}{4} ds^2(\mathrm{AdS}_4) + ds^2(\mathbb{CP}^3)\right)$$

$$ds^2(\mathrm{AdS}_4) = d\mu^2 + \cosh^2 \mu ds^2(\mathrm{AdS}_3), e^{-\Phi} = \frac{Q_6^{\frac{3}{2}}}{2\sqrt{\pi}(2Q_2 Q_6 - Q_4^2)^{\frac{1}{4}}}$$

$$L = \left(\frac{32\pi^2}{Q_6^2}\left(Q_2 Q_6 - \frac{1}{2} Q_4^2\right)\right)^{1/4}$$

$$\frac{1}{2\pi} \int_{\mathbb{CP}^1} \hat{f}_2 = Q_6, \frac{1}{(2\pi)^3} \int_{\mathbb{CP}^2} \hat{f}_4 = Q_4, \frac{1}{(2\pi)^5} \int_{\mathbb{CP}^3} \hat{f}_6 = Q_2$$

$$B_2 = -4\pi \frac{Q_4}{Q_6} J, b = -\frac{Q_4}{Q_6},$$

$$Q_2 \to Q_2 - Q_4 + \frac{1}{2} Q_6$$
$$Q_4 \to Q_4 - Q_6$$

$$Q_2 = N + \frac{k}{12}$$
$$Q_4 = M - \frac{k}{2}$$
$$Q_6 = k$$

$$N \to N + k - M, M \to M - k,$$

$$c_{hol} = \frac{1}{2}(2Q_2Q_6 - Q_4^2) = Nk - \frac{1}{2}M(M-k) - \frac{1}{24}k^2,$$

$$c_{hol}^{(3d)} = \frac{(2Q_2Q_6 - Q_4^2)^{3/2}}{Q_6} = \frac{L^6Q_6^2}{64\pi^3} = \frac{1}{k}\left(2Nk - M(M-k) - \frac{1}{12}k^2\right)^{3/2}$$

$$Q_2^M = Q_2 + bQ_4 + \frac{1}{2}b^2Q_6 = N - \frac{1}{24}k - \frac{1}{2}\frac{M^2}{k} + \frac{1}{2}M$$

$$c_{\text{hol}}^{(3d)} = 2\sqrt{2k}(Q_2^M)^{3/2} = 2^{3/2}k^2\hat{\lambda}^{3/2}$$

$$h_l(r) = Q_2^l - Q_4^l(r-l) + \frac{1}{2}Q_6^l(r-l)^2 - \frac{1}{6}Q_8^l(r-l)^3$$

$$B_2 = 4\pi\left(-(r-l) + \frac{h_l'}{h_l''}\right)J$$

$$b = \frac{1}{4\pi^2}\int_{\mathbb{CP}^1} B_2$$

$$2\pi F_0 = Q_8^l, \frac{1}{2\pi}\int_{\mathbb{CP}^1}\hat{f}_2 = Q_6^l$$
$$\frac{1}{(2\pi)^3}\int_{\mathbb{CP}^2}\hat{f}_4 = Q_4^l, \frac{1}{(2\pi)^5}\int_{\mathbb{CP}^3}\hat{f}_6 = Q_2^l$$

$$h > 0, h'' > 0$$

$$Q_2^l \geq 0, Q_6^l \geq 0, Q_8^l \leq 0, 2Q_2^lQ_6^l \geq \left(Q_4^l\right)^2,$$
$$Q_2^l \geq 0, Q_6^l > Q_8^l \geq 0, 2\left(Q_6^l - Q_8^l\right)\left(Q_2^l + \frac{1}{2}Q_6^l - Q_4^l - \frac{1}{6}Q_8^l\right) \geq \left(Q_4^l - Q_6^l + \frac{Q_8^l}{2}\right)^2,$$

$$h_{l-1}'(l) = \pm h_l'(l)$$

$$Q_2^l = Q_2^{l-1} - Q_4^{l-1} + \frac{1}{2}Q_6^{l-1} - \frac{1}{6}Q_8^{l-1},$$
$$Q_4^l = Q_4^{l-1} - Q_6^{l-1} + \frac{1}{2}Q_8^{l-1},$$
$$Q_6^l = Q_6^{l-1} - Q_8^{l-1},$$

$$B_2^l = 4\pi \frac{-Q_4^l + \frac{1}{2}Q_8^l(r-l)^2}{Q_6^l - Q_8^l(r-l)} J$$

$$Q_2^l = Q_2^{\tilde{l}} - \Delta l Q_4^{\tilde{l}} + \frac{1}{2}(\Delta l)^2 Q_6^{\tilde{l}} - \frac{1}{6}\sum_{i=1}^{\Delta l} (1 - 3i + 3i^2) Q_8^{l-i}$$
$$Q_4^l = Q_4^{\tilde{l}} - \Delta l Q_6^{\tilde{l}} + \frac{1}{2}\sum_{i=1}^{\Delta l} (2i-1) Q_8^{l-i}$$
$$Q_6^l = Q_6^{\tilde{l}} - \sum_{i=1}^{\Delta l} Q_8^{l-i}$$

$$Q_4^l = -\left(Q_4^{l-1} - Q_6^{l-1} + \frac{1}{2}Q_8^{l-1}\right)$$

$$b^l(l) - b^{l-1}(l) = \frac{2Q_4^{l-1} - Q_6^{l-1}}{Q_6^{l-1} - Q_8^{l-1}}$$

$$2Q_4^l = -(z-1)Q_6^l$$
$$2Q_4^{l-1} = (z+1)Q_6^{l-1} - zQ_8^{l-1}$$

$$Q_2^l = Q_2^{\tilde{l}} - \Delta l Q_4^{\tilde{l}} + \frac{1}{2}(\Delta l)^2 Q_6^{\tilde{l}} - \frac{1}{6}\sum_{i=1}^{\Delta l} (1 - 3i + 3i^2) Q_8^{l-i}$$
$$Q_4^l = \frac{1}{2}(1-z)\left(Q_6^{\tilde{l}} - \sum_{i=1}^{\Delta l} Q_8^{l-i}\right)$$
$$Q_6^l = Q_6^{\tilde{l}} - \sum_{i=1}^{\Delta l} Q_8^{l-i}$$

$$2Q_4^{\tilde{l}} - (2\Delta l + z - 1)Q_6^{\tilde{l}} + \sum_{i=2}^{\Delta l} (2(i-1) + z)Q_8^{l-i} + zQ_8^{l-1} = 0$$

$$Q_4^0 = Q_6^0 = 0$$
$$2Q_2^0 Q_6^0 = (Q_4^0)^2$$
$$Q_2^0 = Q_4^0 = 0$$

$$r = -\infty: Q_8^l = 0, \text{ for } l < l_0$$

$$r = +\infty, Q_8^l = 0, l \geq P$$

$$h = Q_2^{P+1} - Q_4^{P+1}(r-P-1) + \frac{1}{2}Q_6^{P+1}(r-P-1)^2 - \frac{1}{3!}Q_8^P(r-P-1)^3$$

$$Q_4^{P+1} = Q_6^{P+1} = 0$$
$$2Q_2^{P+1}Q_6^{P+1} = (Q_4^{P+1})^2$$
$$Q_2^{P+1} = Q_4^{P+1} = 0$$

$$2Q_2^{P+1}+\frac{1}{3}(1+3P+6P^2)Q_6^0=\sum_{i=1}^{P}(2P-i)iQ_8^{P-i}$$

$$Q_4^0=(P+1)Q_6^0-\frac{1}{2}Q_8^P-\frac{1}{2}\sum_{i=1}^{P}(2i+1)Q_8^{P-i}$$
$$6Q_2^0=3(P+1)^2Q_6^0-(2+3P)\sum_{i=0}^{P}Q_8^i+\sum_{i=1}^{P}i(i-(2P+1))Q_8^{P-i}$$
$$Q_6^0>\sum_{i=0}^{P}Q_8^i=Q_8^P+\sum_{i=1}^{P}Q_8^{i-1}$$

$$(\mathrm{AdS}_4,\mathrm{O2})\rightarrow(\mathrm{AdS}_4,\mathrm{O2},\mathrm{Monopole})$$

$$Q_6^0=\sum_{i=0}^{P}Q_8^i\geq 0, Q_2^0\geq PQ_4^0+\frac{1-3P^2}{6}\sum_{i=1}^{P}Q_8^i+3\sum_{i=1}^{P}i(i-1)Q_8^{i-1}$$
$$Q_4^0=-Q_8^P+\frac{1}{2}(1+2P)\sum_{i=1}^{P}Q_8^i-\sum_{i=1}^{P}iQ_8^{P-1}, z=0$$

$$(\mathrm{D8/O8},\mathrm{AdS}_4,\mathrm{O2})\rightarrow(\mathrm{D8/O8},\mathrm{AdS}_4,\mathrm{O2})$$

$$Q_2^P=Q_2^{P-1}-Q_4^{P-1}+\frac{1}{2}Q_6^{P-1}-\frac{1}{3!}Q_8^{P-1}, Q_8^P=-Q_8^{P-1}$$
$$Q_4^P=-Q_4^{P-1}+Q_6^{P-1}-\frac{1}{2}Q_8^{P-1}, Q_6^P=Q_6^{P-1}-Q_8^{P-1}$$

$$Q_8^{P-1}\leq 0, Q_6^{P-1}\geq 0, Q_2^{P-1}\geq 0, 2Q_2^{P-1}Q_6^{P-1}\geq (Q_4^{P-1})^2$$
$$Q_2^{P-1}-Q_4^{P-1}+\frac{1}{2}Q_6^{P-1}-\frac{1}{3!}Q_8^{P-1}\geq 0$$

$$Q_2^{P+1}=Q_2^{P-1}, Q_4^{P+1}=-Q_4^{P-1}, Q_6^{P+1}=Q_6^{P-1}$$

$$Q_2^{P+n}=Q_2^{P-n}, Q_4^{P+n}=-Q_4^{P-n}, Q_6^{P+n}=Q_6^{P-n}, Q_8^{P+n}=-Q_8^{P-n-1}$$

$$2Q_4^0-(2(P-1)+z-1)Q_6^0+\sum_{i=2}^{P-1}(2(i-1)+z)Q_8^{P-i}+zQ_8^{P-1}=0$$

$$h=Q_2^0-(-Q_4^0)(r-2P)+\frac{1}{2}Q_6^0(r-2P)^2-\frac{1}{3!}(-Q_8^0)(r-2P)^3$$

$$\mathrm{D8/O8}\rightarrow\mathrm{D8/O8},\ \mathrm{Monopole}\ \rightarrow\ \mathrm{Monopole}, \mathrm{O2}\rightarrow\mathrm{O2}, \mathrm{AdS}_4\rightarrow\mathrm{AdS}_4$$

$$c_{hol}^l=\frac{1}{2}\int_l^{l+1}dr(2h_lh_l''-(h_l')^2)=2Q_2^lQ_6^l-\left(Q_4^l\right)^2-Q_8^l\left(Q_2^l-\frac{1}{3}Q_4^l+\frac{1}{12}Q_6^l-\frac{1}{60}Q_8^l\right)$$

$$c^{P+n}=c^{P-n-1}, n=0,\dots,P-1\ \Rightarrow\sum_{l=0}^{2P-1}c^l=2\sum_{l=0}^{P-1}c^l$$

$$\frac{ds^2}{2\pi} = \frac{h}{\sqrt{2hh''-(h')^2}}ds^2(\mathrm{AdS}_3) + \frac{2h''}{\sqrt{2hh''-(h')^2}}\left[\left(\frac{h'}{h''}-r\right)d\xi - \frac{d\psi}{2\pi\sin\xi\cos\xi}\right]^2$$
$$+\frac{\sqrt{2hh''-(h')^2}}{4h}dr^2 + \frac{2\sqrt{2hh''-(h')^2}}{h''}\left(d\xi^2 + \frac{1}{4}\cos^2\xi ds^2(\mathrm{S}_1^2) + \frac{1}{4}\sin^2\xi ds^2(\mathrm{S}_2^2)\right)$$

$$e^{\Phi} = \frac{2}{\sin\xi\cos\xi h''}$$
$$B_2 = \pi\left(\frac{h'}{h''}-r\right)[-\cos^2\xi \mathrm{vol}(\mathrm{S}_1^2) + \sin^2\xi \mathrm{vol}(\mathrm{S}_2^2)] + (\eta_1+\eta_2)\wedge d\psi$$
$$H_3 = 2\pi\sin\xi\cos\xi\left[\left(\frac{h'}{h''}-r\right)d\xi - \frac{d\psi}{2\pi\sin\xi\cos\xi}\right]\wedge[\mathrm{vol}(\mathrm{S}_1^2)+\mathrm{vol}(\mathrm{S}_2^2)]$$
$$+\frac{\pi h'h'''}{(h'')^2}dr\wedge[\cos^2\xi\mathrm{vol}(\mathrm{S}_1^2) - \sin^2\xi\mathrm{vol}(\mathrm{S}_2^2)]$$

$$F_1 = \sin\xi\cos\xi(h''-rh''')d\xi - \frac{h'''}{2\pi}d\psi$$
$$F_3 = \pi\sin\xi\cos\xi F_{3,\xi\psi}\wedge[\cos^2\xi\mathrm{vol}(\mathrm{S}_1^2) - \sin^2\xi\mathrm{vol}(\mathrm{S}_2^2)]$$
$$F_5 = 2\pi^2\sin\xi\cos\xi F_{5,\xi\psi}^e\wedge\mathrm{vol}(\mathrm{AdS}_3)\wedge dr + \pi^2\sin^3\xi\cos^3\xi F_{5,\xi\psi}^m\wedge\mathrm{vol}(\mathrm{S}_1^2)\wedge\mathrm{vol}(\mathrm{S}_2^2)$$

$$F_{3,\xi\psi} = \frac{h'(h''+rh''')-r(h'')^2}{h''}d\xi - \frac{(h'')^2-h'h'''}{h''}\frac{d\psi}{2\pi\sin\xi\cos\xi}$$
$$F_{5,\xi\psi}^e = \left(rh' - 2h - \frac{hh'(h'-rh'')}{2hh''-(h')^2}\right)'d\xi + \frac{1}{2}\left(3h' + \frac{(h')^3}{2hh''-(h')^2}\right)'\frac{d\psi}{2\pi\sin\xi\cos\xi}$$
$$F_{5,\xi\psi}^m = \frac{-6h(h'')^2+3(h')^2h''+r(h')^2h'''}{(h'')^2}d\xi + \frac{(h')^2h'''}{(h'')^2}\frac{d\psi}{2\pi\sin\xi\cos\xi}$$

$$h''' = -2\pi F_0$$

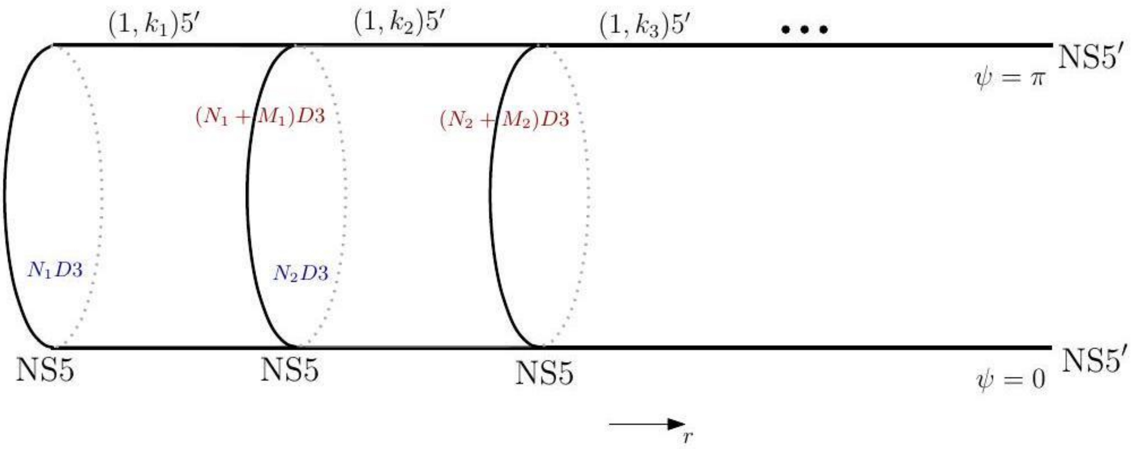
$$h''' = 0 \Rightarrow h = c_2 + c_4 r + \frac{c_6}{2}r^2$$

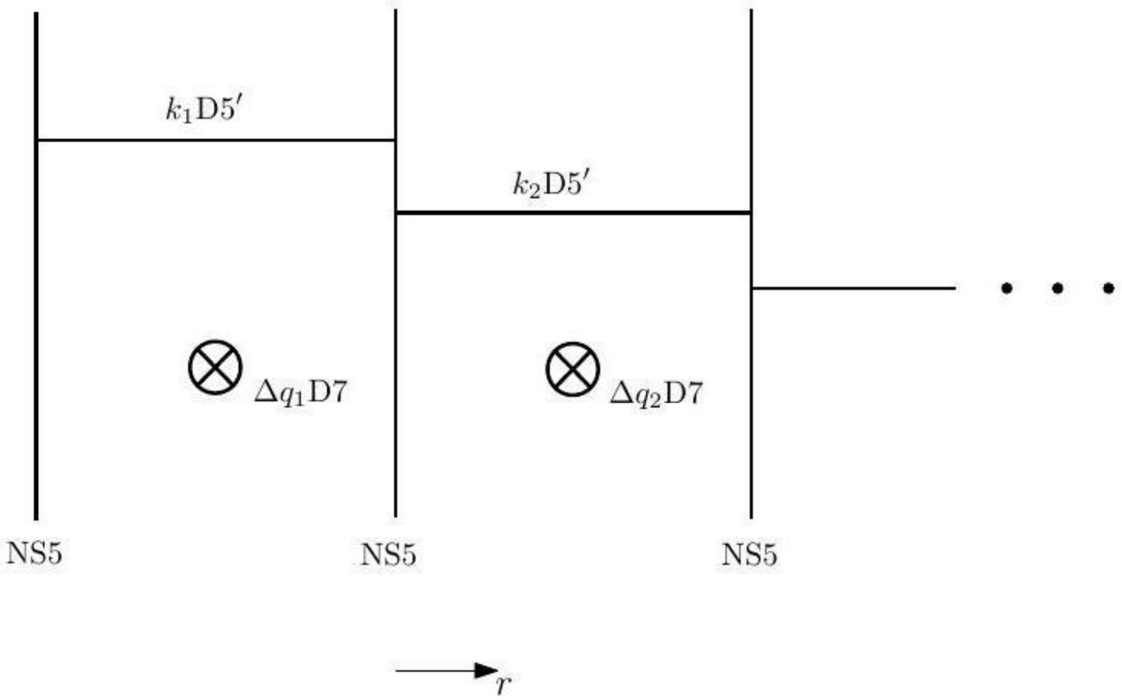
$$\frac{ds^2}{2\pi} = \frac{\sqrt{2c_2c_6 - c_4^2}}{2c_6}\left(ds^2(\mathrm{AdS}_4) + 4d\xi^2 + \cos^2\xi ds^2(\mathrm{S}_1^2) + \sin^2\xi ds^2(\mathrm{S}_2^2)\right)$$
$$+\frac{2c_4^2}{c_6\sqrt{2c_2c_6-c_4^2}}\left(d\xi - \frac{c_6}{2\pi c_4\sin\xi\cos\xi}d\psi\right)^2$$

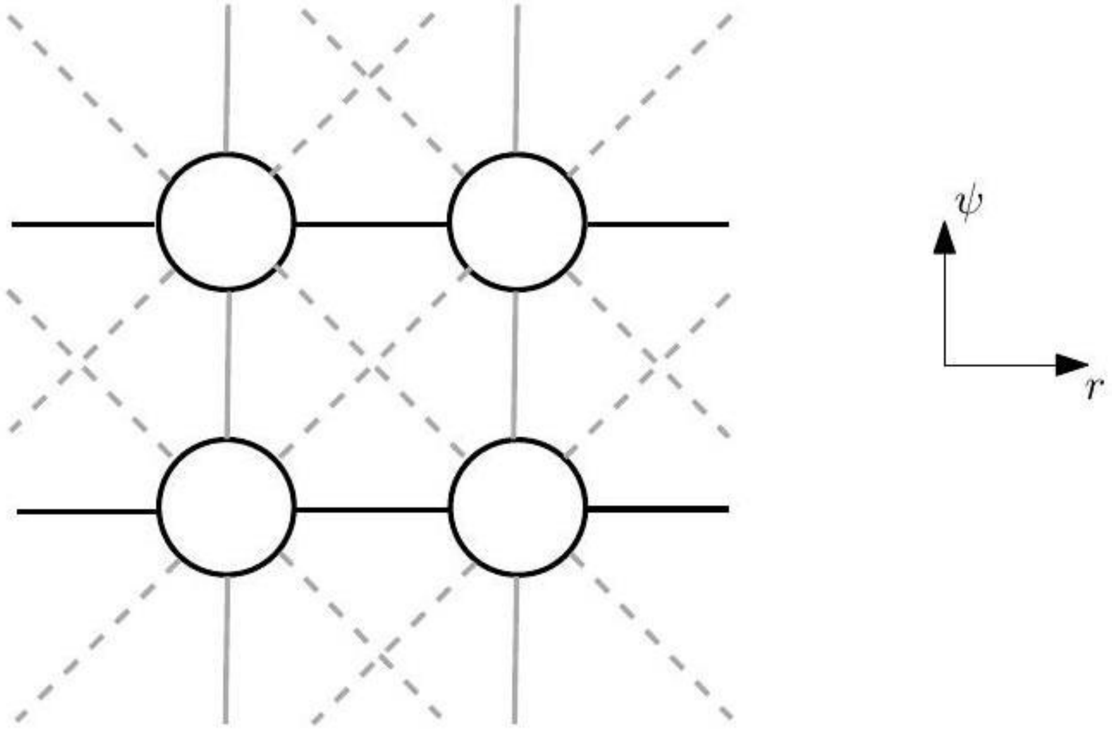
$$e^{\Phi} = \frac{2}{c_6\sin\xi\cos\xi}$$
$$B_2 = \frac{\pi c_4}{c_6}[-\cos^2\xi\mathrm{vol}(\mathrm{S}_1^2) + \sin^2\xi\mathrm{vol}(\mathrm{S}_2^2)] + (\eta_1+\eta_2)\wedge d\psi$$
$$H_3 = \left(\frac{2\pi c_4}{c_6}\sin\xi\cos\xi d\xi - d\psi\right)\wedge\left(\mathrm{vol}(\mathrm{S}_1^2)+\mathrm{vol}(\mathrm{S}_2^2)\right)$$

$$
\begin{aligned}
F_1 &= c_6 \sin\,\xi \cos\,\xi d\xi \\
F_3 &= \frac{c_6}{2}\left(\frac{2\pi c_4}{c_6}\sin\,\xi\cos\,\xi d\xi - d\psi\right)\wedge\left[\cos^2\,\xi \mathrm{vol}(\,\mathrm{S}_1^2) - \sin^2\,\xi \mathrm{vol}(\,\mathrm{S}_2^2)\right] \\
F_5 &= -\frac{3\pi}{2}\sqrt{2c_2c_6 - c_4^2}\mathrm{vol}(\mathrm{AdS}_4)\wedge\left(\frac{2\pi c_4}{c_6}\sin\,\xi\cos\,\xi d\xi - d\psi\right) \\
&\quad + \frac{3\pi^2}{c_6}\sqrt{2c_2c_6 - c_4^2}\sin^3\,\xi\cos^3\,\xi d\xi\wedge\mathrm{vol}(\,\mathrm{S}_1^2)\wedge\mathrm{vol}(\mathrm{S}_2^2)
\end{aligned}
$$

$$
\psi \to \psi + \frac{2\pi c_4}{c_6}\sin\,\xi\cos\,\xi d\xi
$$







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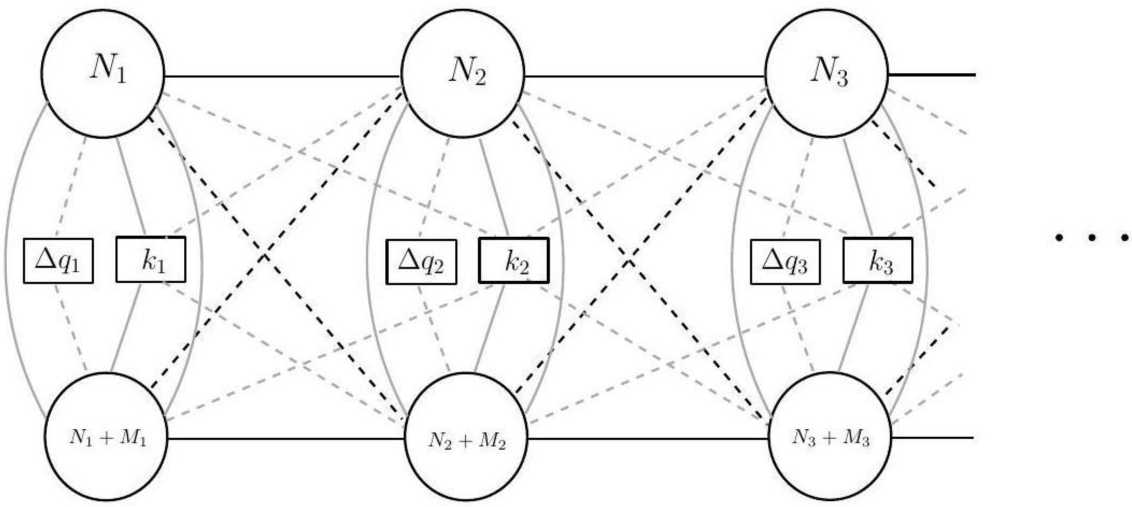
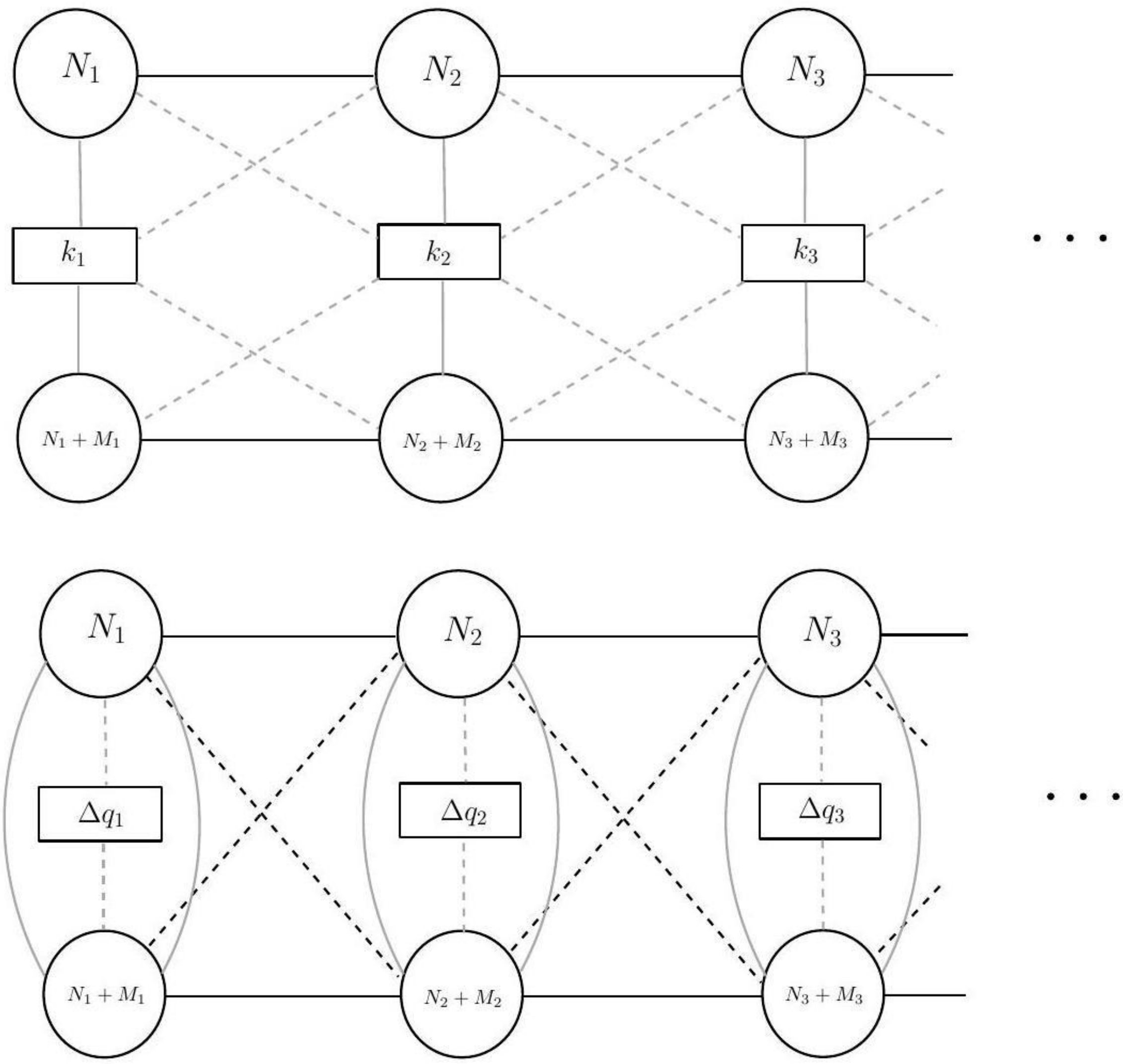


N_1
N_2
N_3
Δq_1
k_1
Δq_2
k_2
Δq_3
k_3
$N_1 + M_1$
$N_2 + M_2$
$N_3 + M_3$



Figuras 25, 26, 27, 28, 29 y 30. Simetría compacta en un plano cuántico – relativista.

$$2M_l - M_{l-1} - M_{l+1} + k_l - \frac{1}{2}k_{l-1} - \frac{1}{2}k_{l+1} - \frac{1}{2}\Delta q_l = 0$$

$$Q_2 = N + \frac{k}{12}$$
$$Q_4 = M - \frac{k}{2} + \frac{q}{12}$$
$$Q_6 = k$$
$$Q_8 = -q$$

$$N_l = N_{l-1} - M_{l-1} + k_{l-1}$$
$$M_l + \frac{q_l}{12} = M_{l-1} + \frac{q_{l-1}}{12} - k_{l-1}$$
$$k_l = k_{l-1} + q_{l-1}$$

$$2k_l = k_{l-1} + k_{l+1} + \Delta q_l$$

$$2M_l = M_{l-1} + M_{l+1} - \Delta k_l,$$

$$2M_l - M_{l-1} + M_{l+1} = 0$$

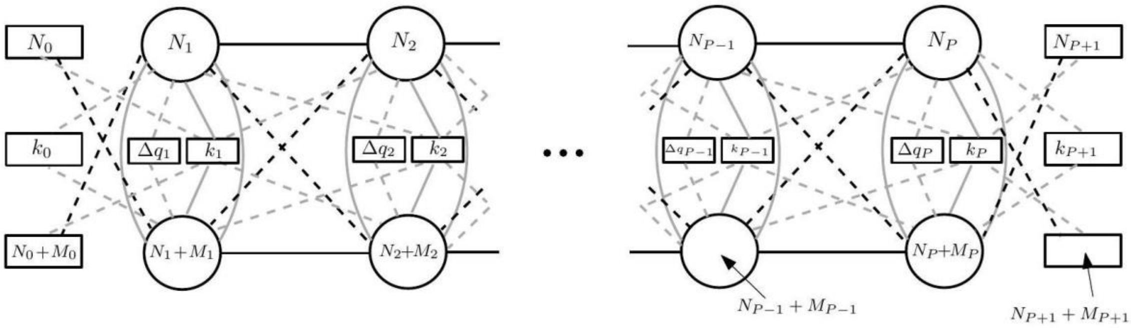


Figura 31. Interacciones de una partícula supermasiva con o sin interferencia gravitónica.

$$N \to N - M + k, M \to M - k, k \to k + q$$

$$c_{hol} = \frac{1}{2}\sum_{l=0}^{P}\left(2N_l k_l - M_l^2 + M_l k_l - \frac{1}{12}k_l^2 + q_l\left(N_l - \frac{1}{2}M_l + \frac{5}{12}k_l - \frac{13}{720}q_l\right)\right).$$

$$c_R = \frac{k(3k+13)}{k+3},$$

$$k = \mathrm{Tr}[\gamma_3 Q_R^2]$$

$$k = \sum_{l=0}^{P}\left(2N_l k_l + M_l k_l - M_l^2\right),$$

$$Q_p^e = \frac{\hat{f}_{p+2}^e}{(2\pi)^p}, Q_p^m = \frac{\hat{f}_{8-p}^m}{(2\pi)^{7-p}}$$

$$Q = \int \sum_{k=0}^{4}(-1)^{k+1}Q_{(2k)}^e Q_{(2k)}^m = \frac{1}{(2\pi)^7}\int \sum_{k=1}^{4}(-1)^{k+1}\hat{f}_{2k+2}^e \hat{f}_{8-2k}^m$$

$$Q = \int \frac{\mathrm{vol}(\mathrm{AdS}_3)}{6\pi} \wedge \frac{\mathrm{vol}(\mathbb{CP}^3)}{\pi^3/6} \wedge dr\left(2hh'' - (h')^2 + \partial_r\left(hh' - \frac{h'''h^3}{2hh'' - (h')^2}\right)\right)$$

$$\hat{f}_4^e = \frac{-h\pi}{2hh'' - (h')^2}\left(\frac{(h')^3 h'''}{2hh'' - (h')^2} - 3(h'')^2\right)\mathrm{vol}(\mathrm{AdS}_3) \wedge dr$$

$$\hat{f}_6^e = \frac{4h\pi^2}{2hh'' - (h')^2}\left(\frac{2h(h')^2 h'''}{2hh'' - (h')^2} - 3h'h''\right)\mathrm{vol}(\mathrm{AdS}_3) \wedge J \wedge dr$$

$$\hat{f}_8^e = \frac{-8h\pi^3}{2hh'' - (h')^2}\left(\frac{4h'h'''h^2}{2hh'' - (h')^2} - 2(hh'' + (h')^2)\right)\mathrm{vol}(\mathrm{AdS}_3) \wedge J \wedge J \wedge dr$$

$$\hat{f}_{10}^e = \frac{32h\pi^4}{2hh'' - (h')^2}\left(\frac{8h^3 h'''}{3(2hh'' - (h')^2)} - 2hh'\right)\mathrm{vol}(\mathrm{AdS}_3) \wedge J \wedge J \wedge J \wedge dr$$

$$\mathfrak{t} = \mathfrak{g} \oplus \mathfrak{h},$$

$$\langle \mu \rangle = J$$

$$\mathfrak{g} \to \mathfrak{su}(2)_X \oplus \mathfrak{f}$$

$$\mathfrak{f} = \bigoplus_i \mathfrak{j}_i,$$

$$\mathbf{adj} \to \bigoplus_\ell V_\ell \otimes R_\ell,$$

$$R_\ell = \bigotimes_i R_\ell^{(i)}$$

$$\hat{A}(T)\mathrm{tr}_\rho e^{iF}\big|_{6\text{-form}}$$

$$\mathrm{Tr}F^2 = \frac{1}{h_{\mathfrak{g}}^\vee}\mathrm{tr}_{\mathbf{adj}}F^2$$

$$\mathrm{tr}_\rho F^2 = A_\rho \mathrm{Tr}F^2$$

$$c_2(F) = \frac{1}{4}\mathrm{Tr}F^2$$

$$\mathrm{tr}_{\boldsymbol{d}} F^2 = A_{\boldsymbol{d}} \mathrm{Tr}F^2 = \frac{d(d^2-1)}{12}\mathrm{Tr}F^2$$

$$\hat{A}(T) = 1 - \frac{1}{24}p_1(T) + \cdots$$

$$I_6^{\mathrm{NG}} = H\left(\frac{1}{12}c_1(r)p_1(T) - \frac{1}{3}c_1(r)^3\right) + H_v c_1(r)c_2(R') + \sum_i H_i c_1(r)c_2(F_i)$$

$$H = \frac{1}{2}\sum_\ell (d_\ell - 1)\mathrm{dim}R_\ell, H_v = \frac{1}{6}\sum_\ell d_\ell(d_\ell - 1)(d_\ell - 2)\mathrm{dim}R_\ell$$

$$H_i = 2\sum_\ell (d_\ell - 1)A_{R_\ell^{(i)}}\frac{\mathrm{dim}R_\ell}{\mathrm{dim}R_\ell^{(i)}}$$

$$I_6 = 24(c^{\mathrm{UV}} - a^{\mathrm{UV}})\left(\frac{1}{12}c_1(r)p_1(T) - \frac{1}{3}c_1(r)^3\right) - 4(2a^{\mathrm{UV}} - c^{\mathrm{UV}})c_1(r)c_2(R)$$
$$+k_G^{\mathrm{UV}}c_1(r)c_2(F) + \sum_a k_a^{\mathrm{UV}}c_1(r)c_2(H_a)$$

$$I_6 = 24(c^{\mathrm{IR}} - a^{\mathrm{IR}})\left(\frac{1}{12}c_1(r)p_1(T) - \frac{1}{3}c_1(r)^3\right) - 4(2a^{\mathrm{IR}} - c^{\mathrm{IR}})c_1(r)c_2(R')$$
$$+\sum_i k_i^{\mathrm{IR}}c_1(r)c_2(F_i) + \sum_a k_a^{\mathrm{IR}}c_1(r)c_2(H_a)$$

$$\begin{aligned}
24(c^{\mathrm{IR}} - a^{\mathrm{IR}}) &= 24(c^{\mathrm{UV}} - a^{\mathrm{UV}}) - H \\
4(2a^{\mathrm{IR}} - c^{\mathrm{IR}}) &= 4(2a^{\mathrm{UV}} - c^{\mathrm{UV}}) - I_X k_G^{\mathrm{UV}} + H_v \\
k_i^{\mathrm{IR}} &= I_i k_G^{\mathrm{UV}} - H_i \\
k_a^{\mathrm{IR}} &= k_a^{\mathrm{UV}}
\end{aligned}$$

$$[1^{m_1}, \cdots, N^{m_N}]$$

$$\sum_i i m_i = N.$$

$$\mathfrak{su}(N) \to \mathfrak{su}(2)_X \oplus \mathfrak{f}.$$

$$\mathfrak{f} = \bigoplus_{i=1}^{N} \mathfrak{su}(m_i)$$

$$I_X = \frac{1}{6}\sum_{i=1}^{N} m_i i(i^2 - 1), I_i = i$$

$$N \longrightarrow \bigoplus_{i=1}^{N} V_i \otimes R_i$$

$$N \otimes \bar{N} = \mathrm{adj} \oplus 1$$

$$V_i \otimes V_j = \bigoplus_{k=1}^{\min(i,j)} V_{i+j+1-2k},$$

$$\boldsymbol{N} \otimes \overline{\boldsymbol{N}} \to \bigoplus_{i,j=1}^{N} \bigoplus_{k=1}^{\min(i,j)} (V_{i+j+1-2k}, R_i \otimes \bar{R}_j)$$

$$H=\frac{1}{2}\sum_{i,j=1}^{N} m_i m_j (ij-\min(i,j))$$
$$H_v=\frac{1}{6}\sum_{i,j=1}^{N} m_i m_j (ij-\min(i,j))(i(i-2)+j(j-2)+2\min(i,j)-1)$$
$$H_i=2\sum_{j=1}^{N} m_j (ij-\min(i,j))$$

$$A_{\text{fund}}=\frac{1}{2}, A_{\overline{\text{fund}}}=\frac{1}{2}, A_{\text{adj}}=h^{\vee}_{\mathfrak{su}(m_i)}$$

$$[1^{m_1},2^{m_2},\cdots,N^{m_N}] \text{ such that } \sum_{i=1}^{N} i m_i = N$$

$$i \text{ even } \implies m_i \text{ even }.$$

$$\mathfrak{so}(N)\to \mathfrak{su}(2)_X \oplus \mathfrak{f}$$

$$\mathfrak{f}=\bigoplus_{i=1}^{N} \mathfrak{j}_i \text{ where } \mathfrak{j}_i=\begin{cases}\mathfrak{usp}(m_i) & \text{if } i \text{ even}\\ \mathfrak{so}(m_i) & \text{if } i \text{ odd}\end{cases}$$

$$N\longrightarrow \bigoplus_{i=1}^{N} V_i\otimes R_i$$

$$I_X=\frac{1}{12}\sum_{i=1}^{N} m_i i(i^2-1), I_i=\begin{cases} i/2 & \text{if } i \text{ even}\\ i & \text{if } i \text{ odd}\end{cases}$$

$$\mathbf{adj}=\mathrm{ASym}(N\otimes N),$$

$$\mathbf{adj}\to \bigoplus_{\substack{i,j=1\\ j>i}}^{N} (V_i\otimes V_j, R_i\otimes R_j)\oplus \bigoplus_{i=1}^{N}\big(\mathrm{ASym}(V_i\otimes V_i), \mathrm{Sym}(R_i\otimes R_i)\big)$$
$$\oplus \bigoplus_{i=1}^{N}\big(\mathrm{Sym}(V_i\otimes V_i), \mathrm{ASym}(R_i\otimes R_i)\big)$$

$$\mathrm{ASym}(V_{2i}\otimes V_{2i})=\bigoplus_{k=1}^{i} V_{4i+1-4k}, \mathrm{ASym}(V_{2i-1}\otimes V_{2i-1})=\bigoplus_{k=1}^{i-1} V_{4i-1-4k}$$
$$\mathrm{Sym}(V_{2i}\otimes V_{2i})=\bigoplus_{k=1}^{i} V_{4i+3-4k}, \mathrm{Sym}(V_{2i-1}\otimes V_{2i-1})=\bigoplus_{k=1}^{i} V_{4i+1-4k}$$

$$\mathrm{ASym}(R_i\otimes R_i)=\boldsymbol{A}^{\mathbf{2}}\oplus \mathbf{1}, \mathrm{Sym}(R_i\otimes R_i)=\mathbf{adj}.$$

$$\mathrm{ASym}(R_i \otimes R_i) = \mathbf{adj}, \mathrm{Sym}(R_i \otimes R_i) = \boldsymbol{S}^{\mathbf{2}} \oplus \mathbf{1}.$$

$$H = \frac{1}{2}\left(\left[\frac{1}{2}\sum_{i,j=1}^{N} m_i m_j (ij - \min(i,j))\right] - \frac{N}{2} + \frac{1}{2}\sum_{\substack{i=1 \\ i \text{ odd}}}^{N} m_i\right)$$

$$H_v = \frac{1}{2}\left[\frac{1}{6}\sum_{i,j=1}^{N} m_i m_j (ij - \min(i,j))(i(i-2) + j(j-2) + 2\min(i,j) - 1)\right]$$
$$+ \frac{1}{12}\sum_{i=1}^{N} (i + 6i^2 - 4i^3) m_i - \frac{1}{12}\sum_{\substack{i=1 \\ i \text{ odd}}}^{N} 3m_i$$

$$H_i = \begin{cases} \left[2\sum_{j=1}^{N} (ij - \min(i,j)) m_j\right] - 4(i-1) & \text{if } i \text{ odd} \\ \frac{1}{2}\left[2\sum_{j=1}^{N} (ij - \min(i,j)) m_j\right] - 2i & \text{if } i \text{ even.} \end{cases}$$

$$\mathfrak{so}(m_i): \quad A_{\text{vector}} = 1, \quad A_{\text{anti-sym}} = h^{\vee}_{\mathfrak{so}(m_i)}, \qquad A_{\mathbf{sym}} = h^{\vee}_{\mathfrak{so}(m_i)} + 4,$$
$$\mathfrak{usp}(m_i): \quad A_{\text{fund}} = \frac{1}{2}, \quad A_{\text{anti-sym}} = h^{\vee}_{\mathfrak{usp}(m_i)} - 2, \quad A_{\mathbf{sym}} = h^{\vee}_{\mathfrak{usp}(m_i)}.$$

$$[1^{m_1}, 2^{m_2}, \cdots, N^{m_N}] \text{ such that } \sum_{i=1}^{N} i m_i = N \text{ and } i \text{ odd} \implies m_i \text{ even}.$$

$$\mathfrak{f} = \bigoplus_{i=1}^{N} \mathfrak{j}_i \text{ where } \mathfrak{j}_i = \begin{cases} \mathfrak{usp}(m_i) & \text{if } i \text{ odd} \\ \mathfrak{so}(m_i) & \text{if } i \text{ even} \end{cases}$$

$$N \to \bigoplus_{i=1}^{N} V_i \otimes R_i.$$

$$I_X = \frac{1}{6}\sum_{i=1}^{N} m_i i(i^2 - 1), I_i = \begin{cases} 2i & \text{if } i \text{ even} \\ i & \text{if } i \text{ odd} \end{cases}$$

$$\mathbf{adj} = \mathrm{Sym}(\boldsymbol{N} \otimes \boldsymbol{N}),$$

$$\mathbf{adj} \to \bigoplus_{\substack{i,j=1 \\ j>i}}^{N} (V_i \otimes V_j, R_i \otimes R_j) \oplus \bigoplus_{i=1}^{N} (\mathrm{ASym}(V_i \otimes V_i), \mathrm{ASym}(R_i \otimes R_i))$$
$$\oplus \bigoplus_{i=1}^{N} (\mathrm{Sym}(V_i \otimes V_i), \mathrm{Sym}(R_i \otimes R_i))$$

$$H = \frac{1}{2}\left(\left[\frac{1}{2}\sum_{i,j=1}^{N} m_i m_j (ij - \min(i,j))\right] + \frac{N}{2} - \frac{1}{2}\sum_{\substack{i=1 \\ i \text{ odd}}}^{N} m_i\right)$$

$$H_v = \frac{1}{2}\left[\frac{1}{6}\sum_{i,j=1}^{N} m_i m_j (ij - \min(i,j))(i(i-2) + j(j-2) + 2\min(i,j) - 1)\right]$$
$$- \frac{1}{12}\sum_{i=1}^{N}(i + 6i^2 - 4i^3) m_i + \frac{1}{12}\sum_{\substack{i=1 \\ i \text{ odd}}}^{N} 3m_i$$

$$H_i = \begin{cases} \frac{1}{2}\left[2\sum_{j=1}^{N}(ij - \min(i,j))m_j\right] + 2(i-1) & \text{if } i \text{ odd} \\ \left[2\sum_{j=1}^{N}(ij - \min(i,j))m_j\right] + 4i & \text{if } i \text{ even.} \end{cases}$$

$$\mathcal{S}_{\mathrm{j}}\langle C_{g,n}\rangle\{O_1, \cdots, O_n\},$$

$$P = [p_1, \cdots, p_N], P' = [p'_1, \cdots, p'_N]$$

$$P' \prec P \Leftrightarrow \sum_{i=j}^{N}(p'_i - p_i) \geq 0 \text{ for } 1 \leq j \leq N$$

$$\Phi_z = \sum_{p \geq q \geq 0} \frac{T_q}{z^{1+q/b}}$$

$$(\mathfrak{j}, o, \mathfrak{g}, b, p, O) \rightarrow \mathcal{D}_p^b(G, O)$$

$$(\mathfrak{j}, o, \mathfrak{g}, b, p, O_{\text{full}}) \rightarrow \mathcal{D}_p^b(G)$$

$$a = \frac{1}{24}(g-1)\left(8d_{\mathrm{j}}h_{\mathrm{j}}^{\vee} + 5r_{\mathrm{j}}\right) + n\delta a(O_{\text{full}}) + m\delta a\left(\tilde{O}_{\text{full}}\right)$$
$$c = \frac{1}{6}(g-1)\left(2d_{\mathrm{j}}h_{\mathrm{j}}^{\vee} + r_{\mathrm{j}}\right) + n\delta c(O_{\text{full}}) + m\delta c\left(\tilde{O}_{\text{full}}\right)$$

j	o	$\mathfrak{g}$	b	p	O
$\mathfrak{su}(n)$	1	$\mathfrak{su}(n)$	n $n-1$	$\mathbb{N}\mathbb{N}$	A-part. of n
$\mathfrak{su}(2n + 1)$	$\mathbb{Z}_2$	$\mathfrak{usp}'(2n)$	$4n + 22n$	$\mathbb{N}_{\text{odd}}$ $\mathbb{N}$	C-part. of $2n$

$\mathfrak{su}(2n)$	$\mathbb{Z}_2$	$\mathfrak{so}(2n+1)$	$4n-22n$	$\mathbb{N}_{\text{odd}}\,\mathbb{N}$	B-part. of $2n+1$
$\mathfrak{so}(2n)$	1	$\mathfrak{so}(2n)$	$2n-2$ n	$\mathbb{N}\mathbb{N}$	D-part. of $2n$
$\mathfrak{so}(2n+2)$	$\mathbb{Z}_2$	$\mathfrak{usp}(2n)$	$2n+22n$	$\mathbb{N}_{\text{odd}}\,\mathbb{N}$	C-part. of $2n$

$\mathfrak{j}$	o	h_t	$\text{Cas}_t(J,o)$	$\mathfrak{g}$	$\text{Cas}(G)$
$\mathfrak{su}(2n+1)\mathfrak{su}(2n)$ $\mathfrak{so}(2n+2)$				$\mathfrak{su}(n)$	$\{2,3,\cdots,n\}$
	$\mathbb{Z}_2$	$4n+2$	$\{3,5,\cdots,2n+1\}$	$\mathfrak{so}(2n+1)$	$\{2,4,\cdots,2n\}$
	$\mathbb{Z}_2$	$4n-2$	$\{3,5,\cdots,2n-1\}$	$\mathfrak{so}(2n)$	$\{2,4,\cdots,2n-2,n\}$
	$\mathbb{Z}_2$	$2n+2$	$\{n+1\}$	$\mathfrak{usp}(2n)$	$\{2,4,\cdots,2n\}$

$$\delta a(O_{\text{full}}) = \frac{1}{48}\left(d_{\mathfrak{j}}(8h_{\mathfrak{j}}^{\vee}-5)+5r_{\mathfrak{j}})\right), \quad \delta c(O_{\text{full}}) = \frac{1}{12}\left(d_{\mathfrak{j}}(2h_{\mathfrak{j}}^{\vee}-1)+r_{\mathfrak{j}})\right),$$
$$\delta a(\tilde{O}_{\text{full}}) = \frac{1}{48}(8d_{\mathfrak{j}}h_{\mathfrak{j}}^{\vee}-5d_{\mathfrak{g}}+5r_{\mathfrak{j}}), \quad \delta c(\tilde{O}_{\text{full}}) = \frac{1}{12}(2d_{\mathfrak{j}}h_{\mathfrak{j}}^{\vee}-d_{\mathfrak{g}}+r_{\mathfrak{j}}).$$

$$\mathfrak{f} = \mathfrak{j}_{k_{\mathfrak{j}}}^{\oplus n} \oplus \mathfrak{g}_{\tilde{k}_{\mathfrak{g}}}^{\oplus m}$$

$$k_{\mathfrak{j}} = 2h_{\mathfrak{j}}^{\vee}, \tilde{k}_{\mathfrak{g}} = 2h_{\mathfrak{g}}^{\vee}$$

$$o = 1, \mathfrak{j} = \mathfrak{g},$$

$$k_{\mathfrak{g}} = 2\left(h_{\mathfrak{g}}^{\vee} - \frac{b}{p}\right)$$

$$12c = \frac{k_{\mathfrak{g}}d_{\mathfrak{g}}}{2h_{\mathfrak{g}}^{\vee}-k_{\mathfrak{g}}} - f$$

$$\mathfrak{g} \oplus \mathfrak{u}(1)^{\oplus f}$$

$$4(2a-c)=\sum_{u}\left(2\Delta(u)-1\right)$$

$$\{\Delta(u)\}=\left\{j-\frac{b}{p}s>1\,\middle|\, j\in \mathrm{Cas}(G), s\geq 1\right\}$$

$$o\neq 1, \mathfrak{j}\neq \mathfrak{g},$$

$$k_{\mathfrak{g}}=2\left(h_{\mathfrak{g}}^{\vee}-\frac{1}{m}\frac{b}{p}\right)$$

$$12c=\frac{k_{\mathfrak{g}}d_{\mathfrak{g}}}{2h_{\mathfrak{g}}^{\vee}-k_{\mathfrak{g}}}-f$$

$$\left\{j-\frac{b}{p}s>1\,\middle|\, j\in \mathrm{Cas}(G), s\geq 1\right\}\sqcup\left\{j_t-\frac{b}{p}\frac{2s-1}{2}>1\,\middle|\, j_t\in \mathrm{Cas}_t(J,o), s\geq 1\right\}.$$

$$\begin{aligned}
\mathrm{d}s_{11}^2 &= \kappa^{2/3}\mathrm{e}^{2\lambda}\left[4\,\mathrm{d}s_{\mathrm{AdS}_5}^2+y^2\mathrm{e}^{-6\lambda}\,\mathrm{d}s^2(S^2)+\mathrm{d}s^2(\mathcal{M}_4)\right]\\
\mathrm{d}s^2(\mathcal{M}_4) &= 4\left(1-y^2\mathrm{e}^{-6\lambda}\right)+\frac{\mathrm{e}^{-6\lambda}}{1-y^2\mathrm{e}^{-6\lambda}}\left(\mathrm{d}y^2+\mathrm{e}^{D}\left(\mathrm{d}x_1^2+\mathrm{d}x_2^2\right)\right)\\
G_4 &= \kappa\mathrm{dvol}(S^2)\wedge\left[\mathcal{D}\chi\wedge\mathrm{d}\left(y^3e^{-6\lambda}\right)+y\left(1-y^2e^{-6\lambda}\right)\mathrm{d}v-\frac{1}{2}\partial_y\mathrm{e}^{D}\,\mathrm{d}x_1\wedge\,\mathrm{d}x_2\right]\\
\mathrm{e}^{-6\lambda} &= -\frac{\partial_y D}{y\left(1-y\partial_y D\right)}, \mathcal{D}\chi=\mathrm{d}\chi+v, v=\frac{1}{2}\left(\partial_{x_2}D\,\mathrm{d}x_1-\partial_{x_1}D\,\mathrm{d}x_2\right)
\end{aligned}$$

$$\left(\partial_{x_1}^2+\partial_{x_2}^2\right)D+\partial_y^2\mathrm{e}^{D}=0$$

$$x_1+ix_2=re^{i\beta}$$

$$r^2e^{D}=\rho^2, y=\rho\partial_\rho V\equiv\dot{V}, \log r=\partial_\eta V\equiv V'$$

$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho V\right)+\partial_\eta^2 V=0$$

$$\begin{aligned}
\mathrm{d}s_{11}^2 &= \kappa^{2/3}\left(\frac{\dot{V}\tilde{\Delta}}{2V''}\right)^{1/3}\left[4\,\mathrm{d}s_{\mathrm{AdS}_5}^2+\frac{2V''\dot{V}}{\tilde{\Delta}}\,\mathrm{d}s^2(S^2)+\mathrm{d}s_4^2\right]\\
\mathrm{d}s_4^2 &= \frac{2(2\dot{V}-\tilde{V})}{\dot{V}\tilde{\Delta}}\mathcal{D}\beta^2+\frac{2V''}{\dot{V}}\left(\mathrm{d}\rho^2+\mathrm{d}\eta^2+\frac{2\dot{V}}{2\dot{V}-\ddot{V}}\rho^2\,\mathrm{d}\chi^2\right)\\
C_3 &= 2\kappa\left[-\frac{2\dot{V}^2V''}{\tilde{\Delta}}\mathrm{d}\chi+\left(\frac{\dot{V}\dot{V}'}{\tilde{\Delta}}-\eta\right)\mathrm{d}\beta\right]\wedge\mathrm{dvol}(S^2)\\
\mathcal{D}\beta &= \mathrm{d}\beta+\frac{2\dot{V}\dot{V}'}{2\dot{V}-\ddot{V}}\,\mathrm{d}\chi, \tilde{\Delta}=(2\dot{V}-\ddot{V})V''+(\dot{V}')^2
\end{aligned}$$

$$\lambda(\eta)\equiv\dot{V}(\eta,\rho=0)$$

$$\lambda_i(\eta)=\hat{\alpha}_i\eta+\hat{\beta}_i, i-1\leq\eta\leq i$$

$$\mathrm{AdS}_5 \times \mathbb{CP}^1 \ltimes \tilde{S}^4$$

$$\mathcal{V}_1(\eta,\rho) = -\eta \log\, \rho$$
$$\mathcal{V}_2(\eta,\rho{:}\,n) = \frac{\eta}{\widetilde{w}(\eta,\rho{:}\,n)} - \frac{\eta}{2}\log\frac{\widetilde{w}(\eta,\rho{:}\,n)+1}{\widetilde{w}(\eta,\rho{:}\,n)-1}$$
$$\mathcal{V}_3(\eta,\rho{:}\,n) = -\frac{n}{2}\log\frac{n\widetilde{w}(\eta,\rho{:}\,n)+\eta}{n\widetilde{w}(\eta,\rho{:}\,n)-\eta}$$

$$\widetilde{w}(\eta,\rho{:}\,n) = \frac{1}{\sqrt{2}n}\sqrt{n^2+\rho^2+\eta^2+\sqrt{4n^2\rho^2+(\rho^2+\eta^2-n^2)^2}}$$

$$\lambda_1(\eta) = -\eta$$
$$\lambda_2(\eta) = \begin{cases}\eta & 0 \le \eta \le n\\ 0 & 0 < n \le \eta\end{cases}$$
$$\lambda_3(\eta) = \begin{cases}0 & 0 \le \eta \le n\\ n & 0 < n \le \eta\end{cases}$$

$$\lambda_{\mathrm{reg}}^{\mathrm{block}}(\eta) = \begin{cases}\eta & 0 \le \eta \le n\\ n & n \le \eta\end{cases}$$

$$V_{\mathrm{reg},[i^{m_i}]} = \sum_{i=1}^{N} m_i \mathcal{V}_{\mathrm{reg}}(\eta,\rho{:}\,i),$$

$$\lambda_{\mathrm{reg}}(\eta) = \eta\sum_{j=i}^{N} m_j + \sum_{j=1}^{i-1} i m_j, i-1 \le \eta \le i,$$

$$V_{\mathrm{irreg}}(\eta,\rho{:}\,p) = -\frac{N}{p}\eta\log\,\rho, \Rightarrow\ \lambda_{\mathrm{irreg}}(\eta) = -\frac{N}{p}\eta$$

$$V_{\mathrm{tot}}(\eta,\rho{:}\,p,[i^{m_i}]) = V_{\mathrm{irreg}}(\eta,\rho{:}\,p) + V_{\mathrm{reg}}(\eta,\rho{:}\,[i^{m_i}])$$

$$\lambda(\eta) = \eta\left(\sum_{j=i}^{N} m_j - \frac{N}{p}\right) + \sum_{j=1}^{i-1} j m_j, i-1 \le \eta \le i.$$

$$N = \sum_{j=1}^{N} m_j \min(j,p)$$

$$\dot{V}_{\mathrm{tot}}(\eta,\rho{:}\,p,[i^{m_i}]) = -\frac{N}{p}\eta + \eta\sum_{i=1}^{N}\frac{m_i}{\widetilde{w}(\eta,\rho{:}\,i)}$$

$$0 = -\frac{N}{p} + \sum_{i=1}^{N}\frac{m_i}{\widetilde{w}(\eta,\rho{:}\,i)}$$

$$\left(m_i^2p^2 - N^2\right)\eta^2 + m_i^2p^2 = \frac{m_i^2 i^2 p^2\left(m_i^2p^2 - N^2\right)}{N^2}$$

$$x_i \to -x_i$$

$$C \rightarrow -C.$$

$$\int_{\hat{S}} G = \frac{1}{2} \int_{S} G$$

$$2 \int_{\hat{S}} \frac{G}{\left(2\pi \ell_p\right)^3} = \int_{\hat{S}} w_4(X) \bmod 2$$

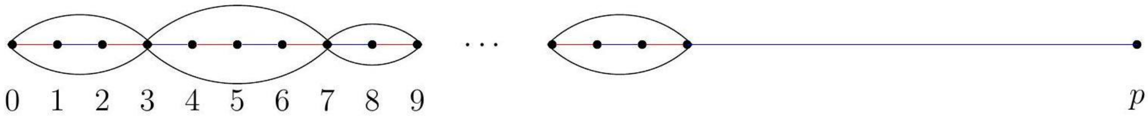
$$\int_{\hat{S}} \frac{1}{2\pi \ell_p^3} G = \frac{1}{2}\left(\int_{S^4_{(1)}} \frac{1}{2\pi \ell_p^3} G + \int_{S^4_{(2)}} \frac{1}{2\pi \ell_p^3} G\right) = \frac{1+1}{2} = 1$$

$$2 \int_{S} \frac{1}{\left(2\pi \ell_p\right)^3} G = 1 \bmod 2$$

$$x \rightarrow -x, \beta \rightarrow \beta + \pi$$

$$\theta \equiv \frac{1}{2\pi} \int_{\mathbb{RP}^2} B \bmod 2, \varphi \equiv \frac{1}{2\pi} \int_{S^1} C_1 \bmod 2$$

theory realization	$O4$	(θ, φ)	Gauge group	Flavor
$\mathbb{R}^5/\mathbb{Z}_2 \times S^1$	$O4^-$	(0,0)	$SO($ even $)$	USp
$\mathbb{R}^5/\mathbb{Z}_2 \times S^1$ and pair of fixed M5s	$O4^+$	(1,0)	$USp($ even $)$	SO
$(\mathbb{R}^5 \times S^1)/\mathbb{Z}_2$	$\widetilde{O4}^-$	(0,1)	$SO($ odd $)$	USp
$(\mathbb{R}^5 \times S^1)/\mathbb{Z}_2$ and one trapped M5	$\widetilde{O4}^+$	(1,1)	$USp'($ even $)$	SO



$$\lambda(\eta) = \tilde{\lambda} - \frac{b}{s_g m p}\eta + \eta \sum_{j=i}^{N} m_j + \sum_{j=1}^{i-1} j m_j, i - 1 \leq \eta \leq i$$

$$I_{12} = -\frac{1}{6} E_4 \wedge E_4 \wedge E_4 - E_4 \wedge X_8$$

$$\int_{S^4} E_4 = N$$

$$\hat{E}_4 \equiv \frac{1}{\left(2\pi \ell_p\right)^3} G_4 - \frac{1}{2} w_4 = E_4 - \frac{1}{2} w_4$$

$$\mathcal{I}_{12} = -\frac{1}{6}\hat{E}_4 \wedge \hat{E}_4 \wedge \hat{E}_4 - \hat{E}_4 \wedge X_8$$

$$\begin{aligned}
48(c-a) &= t_{\mathfrak{g}} \sum_{i=1}^{N} m_i(\lambda(i) + \alpha_i) \\
2a - c &= \frac{t_{\mathfrak{g}}}{4}\left[\sum_{i=1}^{N} \int_{i-1}^{i} (\lambda(\eta) + \alpha_i)^2 \, \mathrm{d}\eta + \int_{N}^{p} (\lambda(\eta) + \alpha_{N+1})^2 \, \mathrm{d}\eta\right] + 2t_{\mathfrak{g}}(c-a) \\
k_i &= 2s_{\mathfrak{g}}(\lambda(i) + \alpha_i)
\end{aligned}$$

$$\mathcal{D}_p^{b=h_t}(G \neq USp', O_{\text{full}}),$$

$$a^{\text{UV}} = \frac{(4p-1)(p-1)}{48p}\dim(\mathfrak{g})$$

$$\deg(p^{n_p} N^{n_N} M^{n_M}) = n_p + n_N + \begin{cases} n_M & \text{if } n_M > 0 \\ 0 & \text{if } n_M < 0 \end{cases}$$

$$\mathcal{D}_p^{b=N}\left(SU(N), \left[M^{N/M}\right]\right)$$

$$a^{\text{UV}} = \frac{1}{48}\frac{(4p-1)(p-1)}{p}(N^2-1), c^{\text{UV}} = \frac{1}{12}(p-1)(N^2-1)$$

$$\mathfrak{f}_k^{\text{UV}} = \mathfrak{su}(N)_{2\left(\frac{p-1}{p}\right)N}$$

$$\mathfrak{su}(N) \to \mathfrak{su}(2)_X \oplus \mathfrak{su}(m_M = N/M).$$

$$I_X = \frac{N(M^2-1)}{6}, I_M = M$$

$$H = \frac{N^2(M-1)}{2M}, H_v = \frac{N^2(2M^2(M-2) + M + 1)}{6M}, H_M = 2N(M-1).$$

$$\begin{aligned}
12c^{\text{IR}} &= 12c^{\text{UV}} - 3I_X k^{\text{UV}} + 3H_v - H \\
&= pN^2 + \frac{M^2N^2}{p} - 2MN^2 - p - \frac{N^2}{p} + \frac{N^2}{M} + 1 \\
48a^{\text{IR}} &= 48a^{\text{UV}} - 12I_X k^{\text{UV}} + 12H_v - 2H \\
&= 4N^2p - 4p + \frac{4M^2N^2}{p} - \frac{3N^2}{p} - \frac{1}{p} - 8MN^2 + \frac{3N^2}{M} + 5
\end{aligned}$$

$$48\left(c^{\text{IR}} - a^{\text{IR}}\right) = \frac{N^2}{M} - \frac{N^2}{p} + \frac{1}{p} - 1.$$

$$\begin{aligned}
\mathfrak{f}_{k_M^{\text{IR}}}^{\text{IR}} &= \mathfrak{su}(N/M)_{k_M^{\text{IR}}}, \\
k_M^{\text{IR}} &= I_M k^{\text{UV}} - H_M = 2N\left(1 - \frac{M}{p}\right).
\end{aligned}$$

$$\lambda(\eta)=\begin{cases}\dfrac{N(p_f-M)}{Mp_f}\eta & 0\le\eta\le M\\ \dfrac{N}{p_f}(p_f-\eta) & M\le\eta\le p_f,\end{cases}$$

$$12\left(2a^{\mathrm{IR}}-c^{\mathrm{IR}}\right)^{\mathrm{inflow}}\Big|_{2\partial}=-3\int_0^p\lambda(\eta)^2\,\mathrm{d}\eta=-\frac{N^2}{p}(p-M)^2.$$

$$12\left(2a^{\mathrm{IR}}-c^{\mathrm{IR}}\right)^{\mathrm{inflow}}\Big|_{X_8}=-\frac{N}{2M}\lambda(M)=-\frac{N^2}{2}\left(\frac{1}{M}-\frac{1}{p}\right).$$

$$12\left(2a^{\mathrm{IR}}-c^{\mathrm{IR}}\right)+12\left(2a^{\mathrm{IR}}-c^{\mathrm{IR}}\right)^{\mathrm{inflow}}=-p+\frac{3}{2}-\frac{1}{2p}.$$

$$48\left(c^{\mathrm{IR}}-a^{\mathrm{IR}}\right)^{\mathrm{inflow}}\Big|_{X_8}=-\frac{N}{M}\lambda(M)=-N^2\left(\frac{1}{M}-\frac{1}{p}\right)$$

$$48(c-a)+48(c-a)^{\mathrm{inflow}}\Big|_{X_8}=\frac{1}{p}-1$$

$$\left(k_M^{\mathrm{IR}}\right)^{\mathrm{inflow}}=-2N\left(1-\frac{M}{p}\right)\Rightarrow k_M^{\mathrm{IR}}+\left(k_M^{\mathrm{IR}}\right)^{\mathrm{inflow}}=0$$

$$\mathcal{D}_p^{b=N-2}\left(SO(N),\left[M^{N/M}\right]\right)$$

$$\mathcal{D}_p^{b=N-2}\left(SO(N),\left[1^N\right]\right),$$

$$\gcd(p,N-2)=1$$

$$\mathcal{D}_p^{b=2N-4}\left(SO(N),\left[M^{N/M}\right]\right),$$

$$\mathcal{D}_p^{b=2N-4}\left(SO(N),\left[1^N\right]\right)$$

$$\begin{aligned}a^{\mathrm{UV}}&=\frac{1}{48}\frac{(4p-1)(p-1)}{p}\frac{N(N-1)}{2}\\ c^{\mathrm{UV}}&=\frac{1}{12}(p-1)\frac{N(N-1)}{2}\end{aligned}$$

$$\mathfrak{f}_{k^{\mathrm{UV}}}^{\mathrm{UV}}=\mathfrak{so}(N)_{2\left(\frac{p-1}{p}\right)(N-2)}$$

$$\mathfrak{so}(N)\to\mathfrak{su}(2)_X\oplus\mathfrak{f}_M\ \text{ where }\ \mathfrak{f}_M=\begin{cases}\mathfrak{usp}(N/M) & \text{if } M \text{ even}\\ \mathfrak{so}(N/M) & \text{if } M \text{ odd}\end{cases}$$

$$I_X=\frac{1}{12}N(M^2-1),I_M=\begin{cases}M/2 & \text{if } M \text{ even}\\ M & \text{if } M \text{ odd}\end{cases}$$

$$\mathrm{adj}\to(3\oplus7\oplus\cdots\oplus2M-1,1\oplus A^2)\oplus(1\oplus5\oplus\cdots\oplus2M-3,S^2).$$

$$\mathrm{adj}\to(3\oplus7\oplus\cdots\oplus2M-3,1\oplus S^2)\oplus(1\oplus5\oplus\cdots\oplus2M-1,A^2),$$

$$H = \frac{1}{2}\left[\frac{N^2(M-1)}{2M}\right] - \frac{N}{4} + \begin{cases} \frac{N}{4M} & \text{if } M \text{ odd} \\ 0 & \text{if } M \text{ even} \end{cases}$$

$$H_v = \frac{1}{2}\left[\frac{N^2(M-1)(2M(M-1)-1)}{6M}\right] + \frac{N(1+6M-4M^2)}{12} - \begin{cases} \frac{N}{4M} & \text{if } M \text{ odd} \\ 0 & \text{if } M \text{ even} \end{cases}$$

$$H_M = \begin{cases} 2N(M-1) - 4(M-1) & \text{if } M \text{ odd} \\ N(M-1) - 2M & \text{if } M \text{ even} \end{cases}$$

$$12c^{\mathrm{IR}} = \frac{N^2 p}{2} - \frac{Np}{2} + \frac{M^2N^2}{2p} - \frac{N^2}{2p} - \frac{M^2N}{p} + \frac{N}{p}$$

$$-MN^2 + \frac{N^2}{2M} + \frac{3MN}{2} - \begin{cases} \frac{N}{M} & \text{if } M \text{ odd} \\ 0 & \text{if } M \text{ even} \end{cases}$$

$$48a^{\mathrm{IR}} = 2N^2p - 2Np + \frac{2M^2N^2}{p} - \frac{3N^2}{2p} - \frac{4M^2N}{p} + \frac{7N}{2p}$$

$$-4MN^2 + \frac{3N^2}{2M} + 6MN - \begin{cases} \frac{7N}{2M} & \text{if } M \text{ odd} \\ 0 & \text{if } M \text{ even} \end{cases}$$

$$48\left(c^{\mathrm{IR}} - a^{\mathrm{IR}}\right) = \frac{N^2}{2M} + \frac{N}{2p} - \frac{N^2}{2p} - \begin{cases} \frac{N}{2M} & \text{if } M \text{ odd} \\ 0 & \text{if } M \text{ even} \end{cases}$$

$$k_M^{\mathrm{IR}} = N + \frac{2M}{p} - \frac{NM}{p},$$

$$k_M^{\mathrm{IR}} = 2N + \frac{4M}{p} - \frac{2NM}{p} - 4.$$

$$\lambda(\eta) = -2 + \eta\left(\sum_{j=i}^{N} m_j - \frac{b}{mps_G}\right) + \sum_{j=1}^{i-1} jm_j,$$

$$12\left(2a^{\mathrm{IR}} - c^{\mathrm{IR}}\right)^{\mathrm{inflow}}\Big|_{2\partial} = -\frac{3}{2}\int_0^p \lambda(\eta)^2 d\eta$$

$$= MN^2 - \frac{M^2N^2}{2p} - \frac{N^2p}{2} + \frac{M^2N}{p} - 3MN + 2Np - 2p$$

$$48\left(c^{\mathrm{IR}} - a^{\mathrm{IR}}\right)^{\mathrm{inflow}} = -\frac{1}{2}\sum_{i=1}^{N} m_i(\lambda(i) + \alpha_i)$$

$$48\left(c^{\mathrm{IR}} - a^{\mathrm{IR}}\right)^{\mathrm{inflow}} = -\frac{1}{2}\left[-\frac{2N}{M} + N^2 - \frac{N^2(M-1)}{M} - \frac{Nb}{mps_G} + \frac{N}{M}\alpha_M\right].$$

$$48\left(c^{\mathrm{IR}} - a^{\mathrm{IR}}\right) = \frac{N^2}{2M} - \frac{N^2}{2p} + \frac{N}{p} + \frac{N}{2M}(\alpha_M - 2),$$

$$48\left(c^{\mathrm{IR}}-a^{\mathrm{IR}}\right)+48\left(c^{\mathrm{IR}}-a^{\mathrm{IR}}\right)^{\text{inflow}}=-\frac{N}{2p}.$$

$$k_i^{\text{inflow}}=-2s_G^{(i)}\lambda(i)$$

$$k_M^{\text{inflow}}=-\begin{cases} N+\dfrac{2M}{p}-\dfrac{MN}{p}+2 & \text{if } M \text{ even}\\ 2N+\dfrac{4M}{p}-\dfrac{2MN}{p}+4 & \text{if } M \text{ odd}\end{cases}$$

$$k_M^{\mathrm{IR}}+k_M^{\text{inflow}}=\begin{cases}2 & \text{if } M \text{ even}\\ 0 & \text{if } M \text{ odd}\end{cases}$$

$$\mathcal{D}_p^b\left(USp(N),\left[M^{N/M}\right]\right) \text{ and } \mathcal{D}_p^b\left(USp'(N),\left[M^{N/M}\right]\right)$$

$$\mathcal{D}_p^{b=N+2}\left(USp(N),\left[M^{N/M}\right]\right).$$

$$a^{\mathrm{UV}}=\frac{1}{48}\frac{(4p-1)(p-1)}{p}\frac{N(N+1)}{2},c^{\mathrm{UV}}=\frac{1}{12}(p-1)\frac{N(N+1)}{2},$$

$$\mathfrak{f}_k^{\mathrm{UV}}=\mathfrak{usp}(N)_{\frac{p-1}{p}(N+2)}$$

$$\mathfrak{usp}(N)\to\mathfrak{su}(2)_X\oplus\mathfrak{f}_M \text{ where } \mathfrak{f}_M=\begin{cases}\mathfrak{usp}(N/M) & \text{if } M \text{ odd}\\ \mathfrak{so}(N/M) & \text{if } M \text{ even},\end{cases}$$

$$I_X=\frac{N(M^2-1)}{6}$$

$$\text{adj}\to(3\oplus 7\oplus\cdots\oplus 2M-1,1\oplus S^2)\oplus(1\oplus 5\oplus\cdots\oplus 2M-3,A^2).$$

$$\text{adj}\to(3\oplus 7\oplus\cdots\oplus 2M-3,1\oplus A^2)\oplus(1\oplus 5\oplus\cdots\oplus 2M-1,S^2),$$

$$H=\frac{1}{2}\left[\frac{N^2(M-1)}{2M}\right]+\frac{N}{4}-\begin{cases}\dfrac{N}{4M} & \text{if } M \text{ odd}\\ 0 & \text{if } M \text{ even}\end{cases}$$

$$H_v=\frac{1}{2}\left[\frac{N^2(M-1)(2M(M-1)-1)}{6M}\right]-\frac{N(1+6M-4M^2)}{12}+\begin{cases}\dfrac{N}{4M} & \text{if } M \text{ odd}\\ 0 & \text{if } M \text{ even}\end{cases}$$

$$H_M=\begin{cases}N(M-1)+2(M-1) & \text{if } M \text{ odd}\\ 2N(M-1)+4M & \text{if } M \text{ even}\end{cases}$$

$$12c^{\mathrm{IR}}=\frac{N^2p}{2}+\frac{Np}{2}+\frac{M^2N^2}{2p}-\frac{N^2}{2p}+\frac{M^2N}{p}-\frac{N}{p}$$
$$-MN^2+\frac{N^2}{2M}-\frac{3MN}{2}+\begin{cases}\dfrac{N}{M} & \text{if } M \text{ odd}\\ 0 & \text{if } M \text{ even}\end{cases}$$
$$48a^{\mathrm{IR}}=2N^2p+2Np+\frac{2M^2N^2}{p}-\frac{3N^2}{2p}+\frac{4M^2N}{p}-\frac{7N}{2p}$$
$$-4MN^2+\frac{3N^2}{2M}-6MN+\begin{cases}\dfrac{7N}{2M} & \text{if } M \text{ odd}\\ 0 & \text{if } M \text{ even}\end{cases}$$

$$48\left(c^{\mathrm{IR}}-a^{\mathrm{IR}}\right)=\frac{N^2}{2M}-\frac{N}{2p}-\frac{N^2}{2p}+\begin{cases}\dfrac{N}{2M} & \text{if } M \text{ odd}\\ 0 & \text{if } M \text{ even}\end{cases}$$

$$k_M^{\mathrm{IR}}=2N-\frac{4M}{p}-\frac{2MN}{p}$$

$$k_M^{\mathrm{IR}}=N-\frac{2M}{p}-\frac{MN}{p}+2=(N+2)\frac{p-M}{p}.$$

$$b=N+2, m=2, s_G=\frac{1}{2}$$

$$\lambda(\eta)=2+\eta\left(\sum_{j=i}^{N}m_j-\frac{N+2}{p}\right)+\sum_{j=1}^{i-1}jm_j,$$

$$\begin{aligned}12\left(2a^{\mathrm{IR}}-c^{\mathrm{IR}}\right)^{\mathrm{inflow}}\Big|_{2\partial}&=-\frac{3}{2}\int_0^p\lambda(\eta)^2d\eta\\&=MN^2-\frac{M^2N^2}{2p}-\frac{N^2p}{2}-\frac{M^2N}{p}+3MN-2Np-2p\end{aligned}$$

$$48\left(c^{\mathrm{IR}}-a^{\mathrm{IR}}\right)^{\mathrm{inflow}}=-\frac{1}{2}\frac{N}{M}(\lambda(M)+\alpha_M)=-\frac{N^2}{2M}+\frac{N}{p}+\frac{N^2}{2p}-\frac{N}{M}\left(1+\frac{\alpha_M}{2}\right).$$

$$k_M^{\mathrm{inflow}}=-2s_G^{(M)}\lambda(i)=\begin{cases}-4-2N+\dfrac{4M}{p}+\dfrac{2MN}{p} & \text{if } M \text{ even}\\ -2-N+\dfrac{2M}{p}+\dfrac{MN}{p} & \text{if } M \text{ odd}\end{cases}$$

$$k_M^{\mathrm{IR}}+k_M^{\mathrm{inflow}}=\begin{cases}-4 & \text{if } M \text{ even}\\ 0 & \text{if } M \text{ odd}\end{cases}$$

$$\mathcal{D}_p^b(SO(2N)), \mathcal{D}_p^b(SO(2N+1)), \mathcal{D}_p^b(USp(2N)), \mathcal{D}_p^b(USp'(2N))$$

$$\mathcal{D}_p^b(G,O),$$

$$\mathcal{D}_p(\mathrm{SU}(N),[1^{m_1},2^{m_2},\cdots,N^{m_N}]),$$

$$\mathcal{D}_p\left(SU(p\ell+N),\left[1^{m_1},2^{m_2},\cdots,N^{m_N},p^{\ell}\right]\right)$$

$$k_p=p\left(\frac{2(p-1)}{p}(p\ell+N)\right)-2((p-1)N+\ell p(p-1))=0$$

$$\lambda(\eta)=\eta\left(\sum_{j=i}^{N+p\ell}m_j-\frac{N+p\ell}{p}\right)+\sum_{j=1}^{i-1}jm_j.$$

$$\chi_T=\chi_E+\beta_E, \beta_T=\beta_E$$

$$Z_{S^1\times S^2} = \frac{1}{(N!)^p} \sum_{\mathfrak{m}_1,\dots,\mathfrak{m}_p\in\mathbb{Z}^N} \int_{\mathcal{C}} \prod_{a=1}^{p} \left[\prod_{i=1}^{N} \frac{dx_{a,i}}{2\pi \mathrm{i} x_{a,i}} (x_{a,i})^{k_a \mathfrak{m}_{a,i}+\mathfrak{t}_a} (\xi_a)^{\mathfrak{m}_{a,i}} \prod_{i\neq j}^{N} \left(1-\frac{x_{a,i}}{x_{a,j}}\right)\right]$$

$$\times \prod_{i,j=1}^{N} \left[\prod_{\text{bi-fund pairs } \Psi_{(a,b)}\&\Psi_{(b,a)}} \left(\frac{\sqrt{\frac{x_{a,i}}{x_{b,j}} y_{\Psi(a,b)}}}{1-\frac{x_{a,i}}{x_{b,j}} y_{\Psi(a,b)}} \right)^{\mathfrak{m}_{a,i}-\mathfrak{m}_{b,j}-\mathfrak{n}_{\Psi(a,b)}+1} \right.$$

$$\left. \times \left(\frac{\sqrt{\frac{x_{b,j}}{x_{a,i}} y_{\Psi(b,a)}}}{1-\frac{x_{b,j}}{x_{a,i}} y_{\Psi(b,a)}} \right)^{-\mathfrak{m}_{a,i}+\mathfrak{m}_{b,j}-\mathfrak{n}_{\Psi(b,a)}+1} \right]$$

$$\times \prod_{i,j=1}^{N} \left[\prod_{\text{adjoint } \Psi_{(a,a)}} \left(\frac{\sqrt{\frac{x_{a,i}}{x_{a,j}} y_{\Psi(a,a)}}}{1-\frac{x_{a,i}}{x_{a,j}} y_{\Psi(a,a)}} \right)^{\mathfrak{m}_{a,i}-\mathfrak{m}_{a,j}-\mathfrak{n}_{\Psi(a,a)}+1} \right]$$

$$\times \prod_{i=1}^{N} \left[\prod_{\text{fund } \Psi_a} \left(\frac{\sqrt{x_{a,i} y_{\Psi_a}}}{1-x_{a,i} y_{\Psi_a}} \right)^{\mathfrak{m}_{a,i}-\mathfrak{n}_{\Psi_a}+1} \prod_{\text{anti-fund } \tilde{\Psi}_a} \left(\frac{\sqrt{\frac{1}{x_{a,i}} y_{\tilde{\Psi}_a}}}{1-\frac{1}{x_{a,i}} y_{\tilde{\Psi}_a}} \right)^{-\mathfrak{m}_{a,i}-\mathfrak{n}_{\Psi_a}+1} \right]$$

$$W = \mathrm{Tr}\left[\sum_{q=1}^{N_f} \tilde{\psi}_q \Psi_3 \psi_q + \Psi_3[\Psi_1, \Psi_2] \right].$$

$$Z^{\mathrm{ADHM}}_{S^1\times S^2}(N, N_f, \Delta, \mathfrak{n}) = \frac{1}{N!} \sum_{\mathfrak{m}\in\mathbb{Z}^N} \oiiint_{\mathcal{C}}^{\boxdot} \prod_{i=1}^{N} \frac{dx_i}{2\pi \mathrm{i} x_i} x_i^{\mathfrak{t}} \xi^{\mathfrak{m}_i} \prod_{i\neq j}^{N} \left(1-\frac{x_i}{x_j}\right)$$

$$\times \prod_{I=1}^{3} \prod_{i,j=1}^{N} \left(\frac{\sqrt{\frac{x_i}{x_j} y_I}}{1-\frac{x_i}{x_j} y_I} \right)^{\mathfrak{m}_i-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}} \left(\frac{\sqrt{\frac{x_j}{x_i} y_I}}{1-\frac{x_j}{x_i} y_I} \right)^{-\mathfrak{m}_i-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}}$$

$$\times \prod_{i=1}^{N} \left(\frac{\sqrt{x_i y_q}}{1-x_i y_q} \right)^{N_f(\mathfrak{m}_i-\mathfrak{n}_q+1)} \left(\frac{\sqrt{\frac{1}{x_i} y_{\tilde{q}}}}{1-\frac{1}{x_i} y_{\tilde{q}}} \right)^{N_f(-\mathfrak{m}_i-\mathfrak{n}_{\tilde{q}}+1)}$$

$$\Delta_q + \Delta_{\tilde{q}} + \Delta_3 = \Delta_1 + \Delta_2 + \Delta_3 = 2, \mathfrak{n}_q + \mathfrak{n}_{\tilde{q}} + \mathfrak{n}_3 = \mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3 = 2$$

$$y_{\Psi_{(a,b)}} \to y_I = e^{\mathrm{i}\pi\Delta_I}, y_{\Psi_a} \to y_q = e^{\mathrm{i}\pi\Delta_q}$$
$$y_{\tilde{\Psi}_a} \to y_{\tilde{q}} = e^{\mathrm{i}\pi\Delta_{\tilde{q}}}, \xi_a \to \xi = e^{\mathrm{i}\pi\left(\Delta_m - N - 1 + 2\left\lfloor\frac{N+1}{2}\right\rfloor\right)}.$$

$$\Delta_1 = \Delta_2 = \frac{1}{2}, \quad \Delta_3 = 1, \quad \Delta_m = 0$$
$$\mathfrak{n}_1 = \mathfrak{n}_2 = \frac{1}{2}, \quad \mathfrak{n}_3 = 1, \quad \mathfrak{t} = 0$$

$$\Delta = (\Delta_I, \Delta_q, \Delta_{\tilde{q}}, \Delta_m), \mathfrak{n} = (\mathfrak{n}_I, \mathfrak{n}_q, \mathfrak{n}_{\tilde{q}}, \mathfrak{t})$$

$$Z^{\mathrm{ADHM}}_{S^1\times S^2}(N,N_f,\Delta,\mathfrak{n}) = \frac{1}{N!}\oiiint_{\mathcal{C}} \prod_{i=1}^{N}\frac{dx_i}{2\pi \mathrm{i} x_i}x_i^{\mathfrak{t}}\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{N}\frac{\left(e^{\mathrm{i}B_i}\right)^M}{e^{\mathrm{i}B_i}-1}$$

$$\times\prod_{I=1}^{3}\prod_{i,j=1}^{N}\left(\frac{\sqrt{\frac{x_i}{x_j}y_I}}{1-\frac{x_i}{x_j}y_I}\right)^{-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}}\left(\frac{\sqrt{\frac{x_j}{x_i}y_I}}{1-\frac{x_j}{x_i}y_I}\right)^{-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}}$$

$$\times\prod_{i=1}^{N}\left(\frac{\sqrt{x_i y_q}}{1-x_i y_q}\right)^{N_f(-\mathfrak{n}_q+1)}\left(\frac{\sqrt{\frac{1}{x_i}y_{\tilde q}}}{1-\frac{1}{x_i}y_{\tilde q}}\right)^{N_f(-\mathfrak{n}_{\tilde q}+1)}$$

$$e^{\mathrm{i}B_i} = \xi\left(\frac{\sqrt{x_i y_q}}{1-x_i y_q}\right)^{N_f}\left(\frac{\sqrt{\frac{1}{x_i}y_{\tilde q}}}{1-\frac{1}{x_i}y_{\tilde q}}\right)^{-N_f}\prod_{I=1}^{3}\prod_{j=1}^{N}\left(\frac{x_i-x_j y_I}{x_j-x_i y_I}\right)$$

$$Z^{\mathrm{ADHM}}_{S^1\times S^2}(N,N_f,\Delta,\mathfrak{n}) = y_q^{\frac{NN_f(1-\mathfrak{n}_q)}{2}}\, y_{\tilde q}\frac{NN_f(1-\mathfrak{n}_{\tilde q})}{2}\prod_{I=1}^{3} y_I^{-\frac{N^2}{2}\mathfrak{n}_I}$$

$$\times\sum_{\{x_i\}\in\mathrm{BAE}}\frac{1}{\det\mathbb{B}}\frac{\prod_{i=1}^{N} x_i^{N+\mathfrak{t}}\prod_{i\neq j}(1-x_i/x_j)}{\prod_{I=1}^{3}\prod_{i,j=1}^{N}(x_j-x_i y_I)^{-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}}(x_i-x_j y_I)^{-\frac{1}{2}\mathfrak{n}_I+\frac{1}{2}}}$$

$$\times\prod_{i=1}^{N}\frac{x_i^{N_f\left(1-\frac{\mathfrak{n}_q+\mathfrak{n}_{\tilde q}}{2}\right)}}{(1-x_i y_q)^{N_f(1-\mathfrak{n}_q)}(x_i-y_{\tilde q})^{N_f(1-\mathfrak{n}_{\tilde q})}}$$

$$2\pi n_i = \pi\left(\Delta_m - N - 1 + 2\left\lfloor\frac{N+1}{2}\right\rfloor\right) + \mathrm{i}N_f\left(\mathrm{Li}_1\left(e^{\mathrm{i}(-u_i+\pi\Delta_{\tilde q})}\right) - \mathrm{Li}_1\left(e^{\mathrm{i}(-u_i-\pi\Delta_q)}\right)\right)$$
$$+\frac{N_f\pi}{2}(2-\Delta_q-\Delta_{\tilde q}) + \mathrm{i}\sum_{I=1}^{3}\sum_{j=1}^{N}\left(\mathrm{Li}_1\left(e^{\mathrm{i}(u_j-u_i+\pi\Delta_I)}\right)-\mathrm{Li}_1\left(e^{\mathrm{i}(u_j-u_i-\pi\Delta_I)}\right)\right)$$
$$+N\pi \quad \text{with} \quad n_i\in\mathbb{Z}.$$

$$\mathcal{V} = \sum_{i=1}^{N}\left(2n_i + N + 1 - 2\left\lfloor\frac{N+1}{2}\right\rfloor - \Delta_m\right)\pi u_i$$
$$+\frac{1}{2}\sum_{I=1}^{3}\sum_{i,j=1}^{N}\left[\mathrm{Li}_2\left(e^{\mathrm{i}(u_j-u_i+\pi\Delta_I)}\right)-\mathrm{Li}_2\left(e^{\mathrm{i}(u_j-u_i-\pi\Delta_I)}\right)\right]$$
$$+N_f\sum_{i=1}^{N}\left[\mathrm{Li}_2\left(e^{\mathrm{i}(-u_i+\Delta_{\tilde q})}\right)-\mathrm{Li}_2\left(e^{\mathrm{i}(-u_i-\Delta_q)}\right)\right]-\frac{N_f\pi}{2}\sum_{i=1}^{N}(2-\Delta_q-\Delta_{\tilde q})u_i$$

$$\mathrm{Li}_n\left(e^{2\pi\mathrm{i}x}\right)+(-1)^n\mathrm{Li}_n\left(e^{-2\pi\mathrm{i}x}\right) = -\frac{(2\pi\mathrm{i})^n}{n!}B_n(x) \quad \text{where} \begin{cases}0\le \mathrm{Re}[x]<1 & \mathrm{Im}[x]\ge 0\\ 0<\mathrm{Re}[x]\le 1 & \mathrm{Im}[x]<0\end{cases}$$

$$0 < \mathrm{Re}\left[u_j - u_i + \pi\Delta_I\right] < 2\pi, -2\pi < \mathrm{Re}\left[u_j - u_i - \pi\Delta_I\right] < 0$$

$$\mathbb{B} = \frac{\partial\left(e^{\mathrm{i}B_1}, \cdots, e^{\mathrm{i}B_N}\right)}{\partial(\log\ x_1, \cdots, \log\ x_N)}, \ \mathbb{B}|_{\mathrm{BAE}} = \frac{\partial(B_1, \cdots, B_N)}{\partial(u_1, \cdots, u_N)}.$$

$$\mathbb{B}_{i,j}\big|_{\mathrm{BAE}} = \delta_{ij}\left[x_i \sum_{I=1}^{3} \sum_{k=1}^{N} \left(\frac{1}{x_i - x_k y_I} + \frac{y_I}{x_k - x_i y_I}\right) + N_f x_i \left(\frac{1}{x_i - y_{\tilde{q}}} - \frac{1}{x_i - y_q^{-1}}\right)\right]$$
$$+ x_j \sum_{I=1}^{3} \left(\frac{-y_I}{x_i - x_j y_I} - \frac{1}{x_j - x_i y_I}\right)$$

$$\log\ Z_{S^1 \times S^2} = \mathcal{B}_1 + \log\left[1 + e^{\mathcal{B}_2 - \mathcal{B}_1} + \cdots\right]$$

$$\log\ Z_{S^1 \times \Sigma_{\mathfrak{g}}}^{\mathrm{ADHM}}(N, N_f, \Delta, \mathfrak{n}) = (1 - \mathfrak{g}) \log\ Z_{S^1 \times S^2}^{\mathrm{ADHM}}\left(N, N_f, \Delta, \frac{\mathfrak{n}}{1 - \mathfrak{g}}\right),$$

$$u_i = \mathrm{i}N^{\frac{1}{2}} t_i + v_i$$

$$u(t(i)) = \mathrm{i}N^{\frac{1}{2}} t(i) + v(t(i)) = u_i$$

$$\rho(t) = \frac{1}{N - 1} \sum_{i=1}^{N} \delta(t - t_i) \ \Leftrightarrow \ di = (N - 1)\rho(t) dt$$

$$\int_{t_{\ll}}^{t_{\gg}} dt \rho(t) = 1$$

$$\mathcal{V} = -\mathrm{i}N^{\frac{3}{2}} \pi \Delta_m \int_{t_{\ll}}^{t_{\gg}} dt t \rho(t) + \mathrm{i}N^{\frac{3}{2}} \frac{4\pi^3}{3} \sum_{I=1}^{3} B_3\left(\frac{\Delta_I}{2}\right) \int_{t_{\ll}}^{t_{\gg}} dt \rho(t)^2$$
$$+ \mathrm{i}N^{\frac{3}{2}} \frac{N_f \pi}{2} \left(2 - \Delta_q - \Delta_{\tilde{q}}\right) \int_{t_{\ll}}^{t_{\gg}} dt \rho(t) |t|$$

$$\rho(t) = \frac{2\pi\mu - N_f\left(2 - \Delta_q - \Delta_{\tilde{q}}\right)|t| + 2\Delta_m t}{2\pi^2 \Delta_1 \Delta_2 \Delta_3}$$
$$t_{\gg} = \frac{2\pi\mu}{N_f\left(2 - \Delta_q - \Delta_{\tilde{q}}\right) - 2\Delta_m}$$
$$t_{\ll} = -\frac{2\pi\mu}{N_f\left(2 - \Delta_q - \Delta_{\tilde{q}}\right) + 2\Delta_m}$$

$$\mu = \sqrt{2N_f \Delta_1 \Delta_2 \frac{\Delta_3}{2 - \Delta_q - \Delta_{\tilde{q}}} \left(\frac{2 - \Delta_q - \Delta_{\tilde{q}}}{2} - \frac{\Delta_m}{N_f}\right)\left(\frac{2 - \Delta_q - \Delta_{\tilde{q}}}{2} + \frac{\Delta_m}{N_f}\right)}.$$

$$n_i = \left\lfloor \frac{N + 1}{2} \right\rfloor - i$$

$$0 < \mathrm{Re}\left[u_j - u_i + \pi\Delta_I\right] < 2\pi, -2\pi < \mathrm{Re}\left[u_j - u_i - \pi\Delta_I\right] < 0$$
$$0 < \mathrm{Re}\left[u_i + \pi\Delta_q\right] < 2\pi, 0 < \mathrm{Re}\left[-u_i + \pi\Delta_{\tilde{q}}\right] < 2\pi$$

$$\mathcal{V} = \mathrm{i}N^{\frac{3}{2}}\frac{2\pi^2\mu}{3}$$

$$\begin{aligned}\log\, Z_{S^1\times S^2}^{\mathrm{ADHM}}(N, N_f, \Delta, \mathfrak{n}) &= -\frac{\pi\mu}{3}N^{\frac{3}{2}}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a} + o\left(N^{\frac{3}{2}}\right)\\ &= -\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}N^{\frac{3}{2}}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a} + o\left(N^{\frac{3}{2}}\right)\end{aligned}$$

$$\begin{aligned}\tilde{\Delta}_a &= \left(\Delta_1, \Delta_2, \frac{2-\Delta_q-\Delta_{\tilde{q}}}{2} - \frac{\Delta_m}{N_f}, \frac{2-\Delta_q-\Delta_{\tilde{q}}}{2} + \frac{\Delta_m}{N_f}\right)\\ \tilde{\mathfrak{n}}_a &= \left(\mathfrak{n}_1, \mathfrak{n}_2, \frac{2-\mathfrak{n}_q-\mathfrak{n}_{\tilde{q}}}{2} + \frac{\mathfrak{t}}{N_f}, \frac{2-\mathfrak{n}_q-\mathfrak{n}_{\tilde{q}}}{2} - \frac{\mathfrak{t}}{N_f}\right)\end{aligned}$$

$$\begin{aligned}&\log Z_{S^1\times S^2}^{\mathrm{ADHM}}(N, N_f, \Delta, \mathfrak{n})\\ &= -\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\sum_{a=1}^{4}\tilde{\mathfrak{n}}_a\left[\frac{1}{\tilde{\Delta}_a}\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{3}{2}} + \left(\mathfrak{c}_a(\tilde{\Delta})N_f + \frac{\mathfrak{d}_a(\tilde{\Delta})}{N_f}\right)\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{1}{2}}\right]\\ &\quad -\frac{1}{2}\log\, \hat{N}_{N_f,\tilde{\Delta}} + \hat{f}_0(N_f, \tilde{\Delta}, \tilde{\mathfrak{n}}) + \hat{f}_{\mathrm{np}}(N, N_f, \tilde{\Delta}, \tilde{\mathfrak{n}})\\ &\quad + \frac{\mathrm{i}N\pi\left(N_f\left(2-\mathfrak{n}_q-\mathfrak{n}_{\tilde{q}}-(\mathfrak{n}_{\tilde{q}}-\mathfrak{n}_q)\Delta_m\right) + \mathfrak{t}(\Delta_{\tilde{q}}-\Delta_q) - 2\right)}{2}\end{aligned}$$

$$\hat{N}_{N_f,\tilde{\Delta}} = N - \frac{2-\Delta_q-\Delta_{\tilde{q}}}{\Delta_3}\frac{N_f}{24} + \frac{N_f}{12}\left(\frac{1}{\tilde{\Delta}_1} + \frac{1}{\tilde{\Delta}_2}\right) + \frac{1}{12N_f}\left(\frac{1}{\tilde{\Delta}_3} + \frac{1}{\tilde{\Delta}_4}\right)$$

$$\begin{aligned}\mathfrak{c}_a(\tilde{\Delta}) = &\Bigg(-\frac{1}{\tilde{\Delta}_1}\frac{(\tilde{\Delta}_2+\tilde{\Delta}_3+\tilde{\Delta}_4)(\tilde{\Delta}_1+\tilde{\Delta}_2)}{8\tilde{\Delta}_1\tilde{\Delta}_2}, -\frac{1}{\tilde{\Delta}_2}\frac{(\tilde{\Delta}_1+\tilde{\Delta}_3+\tilde{\Delta}_4)(\tilde{\Delta}_1+\tilde{\Delta}_2)}{8\tilde{\Delta}_1\tilde{\Delta}_2}\\ &-\frac{\tilde{\Delta}_3+\tilde{\Delta}_4}{8\tilde{\Delta}_1\tilde{\Delta}_2}, -\frac{\tilde{\Delta}_3+\tilde{\Delta}_4}{8\tilde{\Delta}_1\tilde{\Delta}_2}\Bigg)\\ \mathfrak{d}_a(\tilde{\Delta}) = &\Bigg(-\frac{(\tilde{\Delta}_1+\tilde{\Delta}_2)(\tilde{\Delta}_2+\tilde{\Delta}_3+\tilde{\Delta}_4)(\tilde{\Delta}_1+\tilde{\Delta}_3+\tilde{\Delta}_4)}{8\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}\\ &-\frac{(\tilde{\Delta}_1+\tilde{\Delta}_2)(\tilde{\Delta}_2+\tilde{\Delta}_3+\tilde{\Delta}_4)(\tilde{\Delta}_1+\tilde{\Delta}_3+\tilde{\Delta}_4)}{8\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}\\ &-\frac{1}{\tilde{\Delta}_3}\frac{(\tilde{\Delta}_3+\tilde{\Delta}_4)\left((\tilde{\Delta}_1+\tilde{\Delta}_2)(\tilde{\Delta}_2+\tilde{\Delta}_3)(\tilde{\Delta}_3+\tilde{\Delta}_1) + (\tilde{\Delta}_1\tilde{\Delta}_2+\tilde{\Delta}_2\tilde{\Delta}_3+\tilde{\Delta}_3\tilde{\Delta}_1)\tilde{\Delta}_4\right)}{8\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}\\ &-\frac{1}{\tilde{\Delta}_4}\frac{(\tilde{\Delta}_3+\tilde{\Delta}_4)\left((\tilde{\Delta}_1+\tilde{\Delta}_2)(\tilde{\Delta}_2+\tilde{\Delta}_4)(\tilde{\Delta}_4+\tilde{\Delta}_1) + (\tilde{\Delta}_1\tilde{\Delta}_2+\tilde{\Delta}_2\tilde{\Delta}_4+\tilde{\Delta}_4\tilde{\Delta}_1)\tilde{\Delta}_3\right)}{8\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}\Bigg)\end{aligned}$$

$$\tilde{\Delta}_a\left(\mathfrak{c}_a(\tilde{\Delta}) + \mathfrak{d}_a(\tilde{\Delta})\right) = -\frac{\prod_{b\neq a}(\tilde{\Delta}_a+\tilde{\Delta}_b)}{8\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}\sum_{b\neq a}\tilde{\Delta}_b$$

$$
\begin{aligned}
&\operatorname{Re}\log\, Z^{\text{ADHM}}_{S^1\times S^2}(N,N_f,\Delta,\mathfrak{n})\\
&=f^{(\text{lmf})}_{3/2}(N_f,\Delta,\mathfrak{n})N^{\frac{3}{2}}+f^{(\text{lmf})}_{1/2}(N_f,\Delta,\mathfrak{n})N^{\frac{1}{2}}+f^{(\text{lmf})}_{\log}(N_f,\Delta,\mathfrak{n})\log\, N\\
&\quad+f^{(\text{lmf})}_{0}(N_f,\Delta,\mathfrak{n})+\sum_{s=1}^{L} f^{(\text{lmf})}_{-s/2}(N_f,\Delta,\mathfrak{n})N^{-\frac{s}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{n} =&(\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3,\mathfrak{n}_q,\mathfrak{n}_{\tilde{q}},\mathfrak{t})\\
&\in\left\{\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{1}{4},1,\frac{1}{2}\right)\right.\\
&\left.\left(\frac{1}{4},\frac{5}{8},\frac{9}{8},\frac{3}{8},\frac{1}{2},\frac{1}{4}\right),\left(\frac{1}{2},\frac{5}{8},\frac{7}{8},\frac{1}{4},\frac{7}{8},\frac{1}{4}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Re}\log\, Z^{\text{ADHM}}_{S^1\times S^2}(N,N_f,\Delta,\mathfrak{n})\\
&=\sum_{a=1}^{4}\left(f^{(\text{lmf})}_{3/2,a}(N_f,\Delta)N^{\frac{3}{2}}+f^{(\text{lmf})}_{1/2,a}(N_f,\Delta)N^{\frac{1}{2}}\right)\tilde{\mathfrak{n}}_a+f^{(\text{lmf})}_{\log}(N_f,\Delta,\mathfrak{n})\log\, N\\
&\quad+f^{(\text{lmf})}_{0}(N_f,\Delta,\mathfrak{n})+\sum_{s=1}^{L} f^{(\text{lmf})}_{-s/2}(N_f,\Delta,\mathfrak{n})N^{-\frac{s}{2}}
\end{aligned}
$$

$$
f^{(\text{lmf})}_{3/2,a}(N_f,\Delta)\simeq-\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\frac{1}{\tilde{\Delta}_a},
$$

$$
f^{(\text{lmf})}_{\log}(N_f,\Delta,\mathfrak{n})\simeq-\frac{1}{2},
$$

$$
\begin{aligned}
&f^{(\text{lmf})}_{\log}(N_f,\Delta,\mathfrak{n})\log\, N+\sum_{s=1}^{\infty} f^{(\text{lmf})}_{-s}(N_f,\Delta,\mathfrak{n})N^{-s}\\
&\simeq-\frac{1}{2}\log\left(N-\frac{2-\Delta_q-\Delta_{\tilde{q}}}{\Delta_3}\frac{N_f}{24}+\frac{N_f}{12}\left(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2}\right)+\frac{1}{12N_f}\left(\frac{1}{\tilde{\Delta}_3}+\frac{1}{\tilde{\Delta}_4}\right)\right)\\
&=-\frac{1}{2}\log\, \hat{N}_{N_f,\tilde{\Delta}}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Re}\log\, Z^{\text{ADHM}}_{S^1\times S^2}(N,N_f,\Delta,\mathfrak{n})+\frac{1}{2}\log\hat{N}_{N_f,\tilde{\Delta}}\\
&=\sum_{a=1}^{4}\left(\hat{f}^{(\text{lmf})}_{3/2,a}(N_f,\Delta)\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{3}{2}}+\hat{f}^{(\text{lmf})}_{1/2,a}(N_f,\Delta)\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{1}{2}}\right)\tilde{\mathfrak{n}}_a+\hat{f}^{(\text{lmf})}_{0}(N_f,\Delta,\mathfrak{n}),
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}^{(\text{lmf})}_{3/2,a}(N_f,\Delta)&\simeq-\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\frac{1}{\tilde{\Delta}_a}\\
\hat{f}^{(\text{lmf})}_{1/2,a}(N_f,\Delta)&\simeq-\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\left(\mathfrak{c}_a(\tilde{\Delta})N_f+\frac{\mathfrak{d}_a(\tilde{\Delta})}{N_f}\right)
\end{aligned}
$$

$$\mathrm{Re}\log\, Z^{\mathrm{ADHM}}_{S^1\times S^2}(N,N_f)+\frac{\pi\sqrt{2N_f}}{3}\left[\left(\hat{N}_{N_f}\right)^{\frac{3}{2}}-\left(\frac{N_f}{2}+\frac{5}{2N_f}\right)\left(\hat{N}_{N_f}\right)^{\frac{1}{2}}\right]+\frac{1}{2}\log\,\hat{N}_{N_f}$$
$$=\hat{f}_0(N_f)+\hat{f}_{\mathrm{np}}(N,N_f)$$

$$\hat{N}_{N_f}=N+\frac{7N_f}{24}+\frac{1}{3N_f},$$

$$\log\left|\hat{f}_{\mathrm{np}}(N+10,N_f)-\hat{f}_{\mathrm{np}}(N,N_f)\right|\simeq\log\left|\hat{f}_{\mathrm{np}}(N,N_f)\right|+\mathcal{O}(N^0),$$

$$\log\left|\hat{f}_{\mathrm{np}}(N,N_f)\right|=\hat{f}^{(\mathrm{lmf})}_{\mathrm{np},1/2}(N_f)N^{\frac{1}{2}}+\hat{f}^{(\mathrm{lmf})}_{\mathrm{np,log}}(N_f)\log\,N+\sum_{s=0}^{L}\hat{f}^{(\mathrm{lmf})}_{\mathrm{np},-s}(N_f)N^{-s/2},$$

$$\hat{f}^{(\mathrm{lmf})}_{\mathrm{np},1/2}(k)\simeq-2\pi\sqrt{\frac{2}{N_f}}$$

$$\hat{f}_{\mathrm{np}}(N,N_f)=e^{-2\pi\sqrt{2N/N_f}}+\mathcal{O}(\log\,N).$$

$$R_{\mathrm{np},1/2}=\frac{\hat{f}^{(\mathrm{lmf})}_{\mathrm{np},1/2}(N_f)-\left(-2\pi\sqrt{2/N_f}\right)}{-2\pi\sqrt{2/N_f}},$$

	$R_{\mathrm{np},1/2}$
$N_f=1$	7.372×10^{-10}
$N_f=2$	-1.059×10^{-9}
$N_f=3$	-5.557×10^{-11}
$N_f=4$	1.907×10^{-11}

$$W=\mathrm{Tr}\left[\left(A_1B_2-A_2B_1-\sum_{s=1}^{r_1}\tilde{\psi}_1^{(s)}\psi_1^{(s)}\right)^2-\left(A_1B_2-A_2B_1+\sum_{s=1}^{r_2}\tilde{\psi}_2^{(s)}\psi_2^{(s)}\right)^2\right]$$

$$Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r,\Delta,\mathfrak{n})\frac{1}{(N!)^2}\sum_{\mathfrak{m},\tilde{\mathfrak{m}}\in\mathbb{Z}^N}\int_{\mathcal{C}}\prod_{i=1}^{N}\frac{dx_i}{2\pi\mathrm{i}x_i}\frac{d\tilde{x}_i}{2\pi\mathrm{i}\tilde{x}_i}x_i^{k\mathfrak{m}_i}\tilde{x}_i^{-k\tilde{\mathfrak{m}}_i}(-1)^{(N+1)(\mathfrak{m}_i-\tilde{\mathfrak{m}}_i)}\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)$$

$$\times\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_a}}{1-\frac{x_i}{\tilde{x}_j}y_a}\right)^{\mathfrak{m}_i-\tilde{\mathfrak{m}}_j-\mathfrak{n}_a+1}\prod_{b=3,4}\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i}y_b}}{1-\frac{\tilde{x}_j}{x_i}y_b}\right)^{\tilde{\mathfrak{m}}_j-\mathfrak{m}_i-\mathfrak{n}_b+1}$$

$$\times\prod_{i=1}^{N}\left(\frac{\sqrt{x_iy_{q_1}}}{1-x_iy_{q_1}}\right)^{r_1(\mathfrak{m}_i-\mathfrak{n}_{q_1}+1)}\left(\frac{\sqrt{\frac{1}{x_i}y_{\tilde{q}_1}}}{1-\frac{1}{x_i}y_{\tilde{q}_1}}\right)^{r_1(-\mathfrak{m}_i-\mathfrak{n}_{\tilde{q}_1}+1)}$$

$$\times\prod_{i=1}^{N}\left(\frac{\sqrt{\tilde{x}_iy_{q_2}}}{1-\tilde{x}_iy_{q_2}}\right)^{r_2(\tilde{\mathfrak{m}}_i-\mathfrak{n}_{q_2}+1)}\left(\frac{\sqrt{\frac{1}{\tilde{x}_i}y_{\tilde{q}_2}}}{1-\frac{1}{\tilde{x}_i}y_{\tilde{q}_2}}\right)^{r_2(-\tilde{\mathfrak{m}}_i-\mathfrak{n}_{\tilde{q}_2}+1)}$$

$$\begin{aligned}1&=\mathfrak{n}_1+\mathfrak{n}_4=\mathfrak{n}_2+\mathfrak{n}_4, \quad 1=\mathfrak{n}_{q_1}+\mathfrak{n}_{\tilde{q}_1}=\mathfrak{n}_{q_2}+\mathfrak{n}_{\tilde{q}_2}\\ 1&=\Delta_1+\Delta_4=\Delta_2+\Delta_3, \quad 1=\Delta_{q_1}+\Delta_{\tilde{q}_1}=\Delta_{q_2}+\Delta_{\tilde{q}_2}\end{aligned}$$

$$y_{\Psi_{(a,b)}}\to y_a=e^{\mathrm{i}\pi\Delta_a}, y_{\Psi_a}\to y_{q_n}=e^{\mathrm{i}\pi\Delta_{q_n}}, y_{\tilde{\Psi}_a}\to y_{\tilde{q}_n}=e^{\mathrm{i}\pi\Delta_{\tilde{q}_n}}$$

$$\Delta_a=\Delta_{q_n}=\Delta_{\tilde{q}_n}=\frac{1}{2}, \mathfrak{n}_a=\mathfrak{n}_{q_n}=\mathfrak{n}_{\tilde{q}_n}=\frac{1}{2}$$

$$\Delta=(\Delta_a,\Delta_{q_n},\Delta_{\tilde{q}_n}), \mathfrak{n}=(\mathfrak{n}_a,\mathfrak{n}_{q_n},\mathfrak{n}_{\tilde{q}_n})$$

$$Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r,\Delta,\mathfrak{n})=\frac{1}{(N!)^2}\int_{\mathcal{C}}\prod_{i=1}^{N}\frac{dx_i}{2\pi\mathrm{i}x_i}\frac{d\tilde{x}_i}{2\pi\mathrm{i}\tilde{x}_i}\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)$$

$$\times\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_a}}{1-\frac{x_i}{\tilde{x}_j}y_a}\right)^{1-\mathfrak{n}_a}\prod_{b=3,4}\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i}y_b}}{1-\frac{\tilde{x}_j}{x_i}y_b}\right)^{1-\mathfrak{n}_b}$$

$$\times\prod_{i=1}^{N}\left(\frac{\sqrt{x_iy_{q_1}}}{1-x_iy_{q_1}}\right)^{r_1(1-\mathfrak{n}_{q_1})}\left(\frac{\sqrt{\frac{1}{x_i}y_{\tilde{q}_1}}}{1-\frac{1}{x_i}y_{\tilde{q}_1}}\right)^{r_1(1-\mathfrak{n}_{\tilde{q}_1})}$$

$$\times\prod_{i=1}^{N}\left(\frac{\sqrt{\tilde{x}_iy_{q_2}}}{1-\tilde{x}_iy_{q_2}}\right)^{r_2(1-\mathfrak{n}_{q_2})}\left(\frac{\sqrt{\frac{1}{\tilde{x}_i}y_{\tilde{q}_2}}}{1-\frac{1}{\tilde{x}_i}y_{\tilde{q}_2}}\right)^{r_2(1-\mathfrak{n}_{\tilde{q}_2})}$$

$$\times\prod_{i=1}^{N}\frac{\left(e^{\mathrm{i}B_i}\right)^M}{e^{\mathrm{i}B_i}-1}\prod_{j=1}^{N}\frac{\left(e^{\mathrm{i}\tilde{B}_j}\right)^M}{e^{\mathrm{i}\tilde{B}_j}-1}$$

$$e^{\mathrm{i}B_i} = (-1)^{N+1}\sigma_i x_i^k \left(\frac{\sqrt{x_i y_{q_1}}}{1 - x_i y_{q_1}}\right)^{r_1} \left(\frac{\sqrt{\frac{1}{x_i} y_{\tilde{q}_1}}}{1 - \frac{1}{x_i} y_{\tilde{q}_1}}\right)^{-r_1} \prod_{j=1}^{N} \frac{\left(1 - \frac{\tilde{x}_j}{x_i} y_3\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_4\right)}{\left(1 - \frac{\tilde{x}_j}{x_i} y_1^{-1}\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_2^{-1}\right)},$$

$$e^{\mathrm{i}\tilde{B}_j} = (-1)^{N+1}\tilde{\sigma}_j \tilde{x}_j^k \left(\frac{\sqrt{\tilde{x}_j y_{q_2}}}{1 - \tilde{x}_j y_{q_2}}\right)^{-r_2} \left(\frac{\sqrt{\frac{1}{\tilde{x}_j} y_{\tilde{q}_2}}}{1 - \frac{1}{\tilde{x}_j} y_{\tilde{q}_2}}\right)^{r_2} \prod_{i=1}^{N} \frac{\left(1 - \frac{\tilde{x}_j}{x_i} y_3\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_4\right)}{\left(1 - \frac{\tilde{x}_j}{x_i} y_1^{-1}\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_2^{-1}\right)}.$$

$$\sigma_i = \tilde{\sigma}_j = (-1)^N$$

$$\begin{aligned}
&Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r,\Delta,\mathfrak{n}) \\
&= \prod_{a=1}^{2} y_{q_a}^{\frac{1}{2}Nr_a(1-\mathfrak{n}_{qa})} y_{\tilde{q}_a}^{\frac{1}{2}Nr_a(1-\mathfrak{n}_{\tilde{q}a})} \times \prod_{a=1}^{4} y_a^{-\frac{N^2}{2}\mathfrak{n}_a} \\
&\times \sum_{\{x_i,\tilde{x}_j\}\in\mathrm{BAE}} \left[\frac{1}{\det\mathbb{B}} \frac{\prod_{i=1}^{N} x_i^N \tilde{x}_i^N \prod_{i\neq j}^{N} \left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right)}{\prod_{i,j=1}^{N} \prod_{a=1,2} (\tilde{x}_j - x_i y_a)^{1-\mathfrak{n}_a} \prod_{3,4} (x_i - \tilde{x}_j y_a)^{1-\mathfrak{n}_a}}\right. \\
&\left.\times \prod_{i=1}^{N} \frac{x_i^{\frac{1}{2}r_1}}{(1 - x_i y_{q_1})^{r_1(1-\mathfrak{n}_{q_1})} (x_i - y_{\tilde{q}_1})^{r_1(1-\mathfrak{n}_{\tilde{q}_1})}} \frac{\tilde{x}_i^{\frac{1}{2}r_2}}{(1 - \tilde{x}_i y_{q_2})^{r_2(1-\mathfrak{n}_{q2})} (\tilde{x}_i - y_{\tilde{q}_2})^{r_2(1-\mathfrak{n}_{\tilde{q}2})}}\right]
\end{aligned}$$

$$\begin{aligned}
2\pi n_i =& \pi + k u_i + \mathrm{i}\sum_{j=1}^{N}\left[\sum_{a=3,4} \mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i + \pi\Delta_a)}\right) - \sum_{a=1,2} \mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i - \pi\Delta_a)}\right)\right] \\
&+ \mathrm{i}r_1\left[\mathrm{Li}_1\left(e^{\mathrm{i}(-u_i + \pi\Delta_{\tilde{q}_1})}\right) - \mathrm{Li}_1\left(e^{\mathrm{i}(-u_i - \pi\Delta_{q_1})}\right) + \frac{\mathrm{i}\pi}{2}(\Delta_{q_1} + \Delta_{\tilde{q}_1} - 2)\right] (n_i \in \mathbb{Z}) \\
2\pi \tilde{n}_j =& \pi + k\tilde{u}_j + \mathrm{i}\sum_{i=1}^{N}\left[\sum_{a=3,4} \mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i + \pi\Delta_a)}\right) - \sum_{a=1,2} \mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i - \pi\Delta_a)}\right)\right] \\
&- \mathrm{i}r_2\left[\mathrm{Li}_1\left(e^{\mathrm{i}(-\tilde{u}_j + \pi\Delta_{\tilde{q}_2})}\right) - \mathrm{Li}_1\left(e^{\mathrm{i}(-\tilde{u}_j - \pi\Delta_{q_2})}\right) + \frac{\mathrm{i}\pi}{2}(\Delta_{q_2} + \Delta_{\tilde{q}_2} - 2)\right] \ (\tilde{n}_j \in \mathbb{Z})
\end{aligned}$$

$$\mathbb{B} = \frac{\partial\left(e^{\mathrm{i}B_1}, \cdots, e^{\mathrm{i}B_N}, e^{\mathrm{i}\tilde{B}_1}, \cdots, e^{\mathrm{i}\tilde{B}_N}\right)}{\partial(\log x_1, \cdots, \log x_N, \log \tilde{x}_1, \cdots, \log \tilde{x}_N)}$$

$$\mathbb{B}|_{\mathrm{BAE}} = \frac{\partial\left(B_1, \cdots, B_N, \tilde{B}_1, \cdots, \tilde{B}_N\right)}{\partial(u_1, \cdots, u_N, \tilde{u}_1, \cdots, \tilde{u}_N)}$$

$$\log Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r) = -\frac{2\pi(k+r)}{3\sqrt{2k+r}} N^{\frac{3}{2}} + o\left(N^{\frac{3}{2}}\right),$$

$$\begin{aligned}
\log Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r) =& -\frac{2\pi(k+r)}{3\sqrt{2k+r}}\left(\left(\hat{N}_{k,r}\right)^{\frac{3}{2}} - \left(\frac{r}{4} + \frac{3k+2r}{(k+r)^2}\right)\left(\hat{N}_{k,r}\right)^{\frac{1}{2}}\right) \\
&- \frac{1}{2}\log \hat{N}_{k,r} + \hat{f}_0(k,r) + \hat{f}_{\mathrm{np}}(N,k,r) + \frac{\mathrm{i}Nr\pi}{2}
\end{aligned}$$

$$\hat{N}_{k,r} = N + \frac{7r-2k}{48} + \frac{2}{3(k+r)}$$

$$\hat{f}_{\text{np}}(N,k,r) = e^{-\frac{4\pi}{(1+r/k)\sqrt{2k+r}}\sqrt{N}+\mathcal{O}(\log N)} \quad (r \geq 1)$$

$$W = \mathrm{Tr}[\Psi_1^3 + \Psi_2^3 + \Psi_1(A_1B_2 + A_2B_1) + \Psi_2(B_2A_1 + B_1A_2)]$$

$$W = \mathrm{Tr}\left[\sum_{q=1}^{N_f} \tilde{\psi}_q(\Psi_1\Psi_2 + \Psi_2\Psi_1 - \Psi_3^2)\psi_q + \Psi_3[\Psi_1,\Psi_2]\right]$$

$$\Delta_1 + \Delta_2 = \frac{4}{3}, \Delta_3 = \frac{2}{3}, \Delta_q + \Delta_{\tilde{q}} = \frac{2}{3}, \mathfrak{n}_1 + \mathfrak{n}_2 = \frac{4}{3}, \mathfrak{n}_3 = \frac{2}{3}, \mathfrak{n}_q + \mathfrak{n}_{\tilde{q}} = \frac{2}{3}$$

$$\Delta_1 = \Delta_2 = \frac{2}{3}, \Delta_m = 0, \mathfrak{n}_1 = \mathfrak{n}_2 = \frac{2}{3}, \mathfrak{t} = 0$$

$$\begin{aligned}\log Z_{S^1\times S^2}^{V^{5,2}}(N,N_f,\Delta,\mathfrak{n}) &= -\frac{\pi\mu}{3}N^{\frac{3}{2}}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a} + o\left(N^{\frac{3}{2}}\right)\\ &= -\frac{\pi\sqrt{N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}N^{\frac{3}{2}}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a} + o\left(N^{\frac{3}{2}}\right)\end{aligned}$$

$$\begin{aligned}&\mathrm{Re}\log Z_{S^1\times S^2}^{V^{5,2}}(N,N_f,\Delta,\mathfrak{n})\\ &= -\frac{\pi\sqrt{N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a}\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{3}{2}}\\ &\quad -\frac{\pi\sqrt{N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\left(\sum_{I=1}^{2}\left(\mathfrak{a}_I N_f + \frac{\mathfrak{b}_I}{N_f}\right)\mathfrak{n}_I + \frac{\tilde{\Delta}_3-\tilde{\Delta}_4}{3\tilde{\Delta}_3^2\tilde{\Delta}_4^2}\frac{\mathfrak{t}}{N_f^2}\right)\left(\hat{N}_{N_f,\tilde{\Delta}}\right)^{\frac{1}{2}}\\ &\quad -\frac{1}{2}\log \hat{N}_{N_f,\tilde{\Delta}} + \hat{f}_0(N_f,\tilde{\Delta},\tilde{\mathfrak{n}}) + \hat{f}_{\text{np}}(N,N_f,\tilde{\Delta},\tilde{\mathfrak{n}})\end{aligned}$$

$$\begin{aligned}\mathfrak{a}_I(\tilde{\Delta}) &= -\frac{1}{\tilde{\Delta}_I}\frac{2-\left(\frac{2}{3}-\tilde{\Delta}_I\right)}{4\tilde{\Delta}_1\tilde{\Delta}_2}\\ \mathfrak{b}_I(\tilde{\Delta}) &= -\frac{2}{3\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4} - \frac{3}{4\tilde{\Delta}_3\tilde{\Delta}_4} - \frac{\left(\tilde{\Delta}_3-\tilde{\Delta}_4\right)^2}{8\tilde{\Delta}_3^2\tilde{\Delta}_4^2}\end{aligned}$$

$$\begin{aligned}\mathfrak{n} &= (\mathfrak{n}_1,\mathfrak{n}_2,\mathfrak{n}_3,\mathfrak{n}_q,\mathfrak{n}_{\tilde{q}},\mathfrak{t})\\ &\in \left\{\left(\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},0\right),\left(\frac{1}{2},\frac{5}{6},\frac{2}{3},\frac{1}{4},\frac{5}{12},\frac{1}{2}\right),\left(\frac{2}{5},\frac{14}{15},\frac{2}{3},\frac{1}{2},\frac{1}{6},\frac{1}{3}\right)\right.\\ &\quad\left.\left(\frac{1}{4},\frac{13}{12},\frac{2}{3},\frac{3}{8},\frac{7}{24},\frac{1}{4}\right),\left(\frac{1}{3},1,\frac{2}{3},\frac{1}{2},\frac{1}{6},\frac{2}{5}\right)\right\}\end{aligned}$$

$$\mathrm{Re}\log\, Z_{S^1\times S^2}^{V^{5,2}}(N,N_f)+\frac{16\pi\sqrt{N_f}}{27}\left[\left(\hat{N}_{N_f}\right)^{\frac{3}{2}}-\left(\frac{9N_f}{16}+\frac{27}{16N_f}\right)\left(\hat{N}_{N_f}\right)^{\frac{1}{2}}\right]+\frac{1}{2}\log\,\hat{N}_{N_f}$$
$$=\hat{f}_0(N_f)+\hat{f}_{\mathrm{np}}(N,N_f)$$

$$\log\left|\hat{f}_{\mathrm{np}}(N,N_f)\right|=\hat{f}_{\mathrm{np},1/2}^{(\mathrm{lmf})}(N_f)N^{\frac{1}{2}}+\hat{f}_{\mathrm{np,log}}^{(\mathrm{lmf})}(N_f)\log\,N+\sum_{s=0}^{L}\hat{f}_{\mathrm{np},-s}^{(\mathrm{lmf})}(N_f)N^{-s/2}$$

$$\hat{f}_{\mathrm{np},1/2}^{(\mathrm{lmf})}(N_f)\simeq-2\pi\sqrt{\frac{1}{N_f}}$$

$$\hat{f}_{\mathrm{np}}(N,N_f)=e^{-2\pi\sqrt{N/N_f}+\mathcal{O}(\log\,N)}.$$

$$R_{\mathrm{np},1/2}=\frac{\hat{f}_{\mathrm{np},1/2}^{(\mathrm{lmf})}(N_f)-\left(-2\pi\sqrt{1/N_f}\right)}{-2\pi\sqrt{1/N_f}},$$

	$R_{\mathrm{np},1/2}$
$N_f=1$	-9.786×10^{-10}
$N_f=2$	2.603×10^{-12}
$N_f=3$	-1.162×10^{-11}
$N_f=4$	6.036×10^{-9}
$N_f=5$	-6.007×10^{-8}

$$W=\mathrm{Tr}\left[A_1B_1A_2B_2-A_1B_2A_2B_1+\sum_{s=1}^{N_f}\tilde{\psi}_1^{(s)}A_1\psi_1^{(s)}+\sum_{s=1}^{N_f}\tilde{\psi}_2^{(s)}A_2\psi_2^{(s)}\right]$$

$$Z_{S^1\times S^2}^{Q^{1,1,1}}(N,N_f,\Delta,\mathfrak{n})$$
$$=\frac{1}{(N!)^2}\sum_{\mathfrak{m},\tilde{\mathfrak{m}}\in\mathbb{Z}^N}\oiiint_{\mathcal{C}}\prod_{i=1}^{N}\frac{dx_i}{2\pi\mathrm{i}x_i}\frac{d\tilde{x}_i}{2\pi\mathrm{i}\tilde{x}_i}(-1)^{(N+N_f-2\lfloor N_f/2\rfloor)(\mathfrak{m}_i-\tilde{\mathfrak{m}}_i)}\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)$$
$$\times\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_a}}{1-\frac{x_i}{\tilde{x}_j}y_a}\right)^{\mathfrak{m}_i-\tilde{\mathfrak{m}}_j-\mathfrak{n}_a+1}\prod_{a=3,4}\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i}y_a}}{1-\frac{\tilde{x}_j}{x_i}y_a}\right)^{-\mathfrak{m}_i+\tilde{\mathfrak{m}}_j-\mathfrak{n}_a+1}$$
$$\times\prod_{i=1}^{N}\prod_{n=1,2}\left(\frac{\sqrt{\frac{1}{x_i}y_{\tilde{q}_n}}}{1-\frac{1}{x_i}y_{\tilde{q}_n}}\right)^{N_f(-\mathfrak{m}_i-\mathfrak{n}_{\tilde{q}n}+1)}\times\prod_{j=1}^{N}\prod_{n=1,2}\left(\frac{\sqrt{\tilde{x}_j y_{q_n}}}{1-\tilde{x}_j y_{q_n}}\right)^{N_f(\tilde{\mathfrak{m}}_j-\mathfrak{n}_{q_n}+1)}$$

$$2=\sum_{a=1}^{4}\mathfrak{n}_a,\quad 2=\mathfrak{n}_1+\mathfrak{n}_{q_1}+\mathfrak{n}_{\tilde{q}_1}=\mathfrak{n}_2+\mathfrak{n}_{q_2}+\mathfrak{n}_{\tilde{q}_2},$$
$$2=\sum_{a=1}^{4}\Delta_a,\quad 2=\Delta_1+\Delta_{q_1}+\Delta_{\tilde{q}_1}=\Delta_2+\Delta_{q_2}+\Delta_{\tilde{q}_2},$$

$$y_{\Psi_{(a,b)}}\to y_a=e^{\mathrm{i}\pi\Delta_a},y_{\Psi_a}\to y_{q_n}=e^{\mathrm{i}\pi\Delta_{q_n}},y_{\tilde{\Psi}_a}\to y_{\tilde{q}_n}=e^{\mathrm{i}\pi\Delta_{\tilde{q}_n}}$$

$$\Delta_1=\Delta_2\,,\qquad \Delta_3=\Delta_4\,,\qquad \Delta_{q_{1,2}}=\Delta_{\tilde{q}_{1,2}}\,,$$
$$\mathfrak{n}_1=\mathfrak{n}_2\,,\qquad \mathfrak{n}_3=\mathfrak{n}_4\,,\qquad \mathfrak{n}_{q_{1,2}}=\mathfrak{n}_{\tilde{q}_{1,2}}\,.$$

$$\Delta=\left(\Delta_a,\Delta_{q_{1,2}},\Delta_{\tilde{q}_{1,2}}\right),\mathfrak{n}=\left(\mathfrak{n}_a,\mathfrak{n}_{q_{1,2}},\mathfrak{n}_{\tilde{q}_{1,2}}\right)$$

$$Z_{S^1\times S^2}^{Q^{1,1,1}}(N,N_f,\Delta,\mathfrak{n})$$
$$=\prod_{n=1}^{2}y_{q_n}^{\frac{1}{2}Nk(1-\mathfrak{n}_{q_n})}y_{\tilde{q}_n}^{\frac{1}{2}Nk(1-\mathfrak{n}_{\tilde{q}_n})}\times\prod_{a=1}^{4}y_a^{-\frac{N^2}{2}\mathfrak{n}_a}$$
$$\times\sum_{\{x_i,\tilde{x}_j\}\in\mathrm{BAE}}\left[\frac{1}{\det\mathbb{B}}\frac{\prod_{i=1}^{N}x_i^N\tilde{x}_i^N\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)}{\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\tilde{x}_j-x_iy_a\right)^{1-\mathfrak{n}_a}\prod_{a=3,4}\left(x_i-\tilde{x}_jy_a\right)^{1-\mathfrak{n}_a}}\right.$$
$$\left.\times\prod_{i=1}^{N}\prod_{n=1}^{2}\frac{x_i^{N_f(1-\mathfrak{n}_{\tilde{q}n})/2}}{\left(x_i-y_{\tilde{q}_n}\right)^{N_f(1-\mathfrak{n}_{\tilde{q}_n})}}\frac{\tilde{x}_i^{N_f(1-\mathfrak{n}_{q_n})/2}}{\left(1-\tilde{x}_iy_{q_n}\right)^{N_f(1-\mathfrak{n}_{q_n})}}\right]$$

$$2\pi\left(n_i - \frac{N_f}{2} + \left\lfloor\frac{N_f}{2}\right\rfloor\right) = \mathrm{i}\sum_{j=1}^{N}\left[\sum_{a=3,4}\mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i + \pi\Delta_a)}\right) - \sum_{a=1,2}\mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i - \pi\Delta_a)}\right)\right]$$

$$+\mathrm{i}N_f\sum_{n=1}^{2}\mathrm{Li}_1\left(e^{\mathrm{i}(-u_i + \pi\Delta_{\tilde{q}_n})}\right) - \frac{N_f\pi}{2}\sum_{n=1}^{2}\Delta_{\tilde{q}_n} + N_f u_i \ (n_i \in \mathbb{Z})$$

$$2\pi\left(\tilde{n}_j - \frac{N_f}{2} + \left\lfloor\frac{N_f}{2}\right\rfloor\right) = \mathrm{i}\sum_{i=1}^{N}\left[\sum_{a=3,4}\mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i + \pi\Delta_a)}\right) - \sum_{a=1,2}\mathrm{Li}_1\left(e^{\mathrm{i}(\tilde{u}_j - u_i - \pi\Delta_a)}\right)\right]$$

$$+\mathrm{i}N_f\sum_{n=1}^{2}\mathrm{Li}_1\left(e^{\mathrm{i}(-\tilde{u}_j - \pi\Delta_{q_n})}\right) + \frac{N_f\pi}{2}\sum_{n=1}^{2}\left(\Delta_{q_n} - 2\right) + N_f\tilde{u}_j \ (\tilde{n}_j \in \mathbb{Z})$$

$$\log\ Z_{S^1\times S^2}^{Q^{1,1,1}}(N, N_f, \Delta, \mathfrak{n}) = -\frac{4\pi\sqrt{N_f}}{3\sqrt{3}}N^{\frac{3}{2}} + o\left(N^{\frac{3}{2}}\right)$$

$$\mathrm{Re}\log\ Z_{S^1\times S^2}^{Q^{1,1,1}}(N, N_f, \Delta_1, \mathfrak{n}) = -\frac{4\pi\sqrt{N_f}}{3\sqrt{3}}\left(\left(\hat{N}_{N_f}\right)^{\frac{3}{2}} - \left(\frac{N_f}{4} + \frac{3}{4N_f}\right)\left(\hat{N}_{N_f}\right)^{\frac{1}{2}}\right)$$

$$-\frac{1}{2}\log\ \hat{N}_{N_f} + \hat{f}_0(N_f) + \hat{f}_{\mathrm{np}}(N, N_f)$$

$$\hat{N}_{N_f} = N + \frac{N_f}{6}.$$

$$\left(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4, \mathfrak{n}_{q_1}, \mathfrak{n}_{\tilde{q}_1}, \mathfrak{n}_{q_2}, \mathfrak{n}_{\tilde{q}_2}\right)$$
$$\in\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{7}{12}, \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, 1\right), \left(\frac{1}{4}, \frac{2}{3}, \frac{3}{8}, \frac{17}{24}, \frac{2}{3}, \frac{13}{12}, \frac{1}{2}, \frac{5}{6}\right)\right.$$
$$\left.\left(\frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{23}{30}, \frac{2}{5}, \frac{19}{15}, \frac{3}{4}, \frac{17}{20}\right), \left(\frac{1}{3}, \frac{3}{4}, \frac{7}{8}, \frac{1}{24}, \frac{7}{8}, \frac{19}{24}, \frac{1}{3}, \frac{11}{12}\right), \left(\frac{5}{8}, \frac{2}{3}, \frac{1}{4}, \frac{11}{24}, \frac{4}{5}, \frac{23}{40}, \frac{1}{3}, 1\right)\right\}$$

$$\hat{f}_{\mathrm{np}}(N, N_f) = e^{-2\pi\sqrt{\frac{N}{3N_f}} + \mathcal{O}(\log\ N)}$$

$$Z_{S_b^3}^{\mathrm{ADHM}}(N, N_f, \mu)$$
$$= \frac{1}{N!}\int d^N x\prod_{i<j}^{N} 2\sinh\left(\pi b x_{ij}\right) 2\sinh\left(\pi b^{-1} x_{ij}\right)\prod_{i,j=1}^{N} s_b\left(\mu - x_{ij}\right)$$
$$\times\prod_{i,j=1}^{N} s_b\left(\frac{\mathrm{i}(b + b^{-1})}{4} - \frac{\mu}{2} + x_{ij}\right) s_b\left(\frac{\mathrm{i}(b + b^{-1})}{4} - \frac{\mu}{2} - x_{ij}\right)$$
$$\times\left[\prod_{i=1}^{N} s_b\left(\frac{\mathrm{i}(b + b^{-1})}{4} - \frac{\mu}{2} + x_i\right) s_b\left(\frac{\mathrm{i}(b + b^{-1})}{4} - \frac{\mu}{2} - x_i\right)\right]^{N_f}$$

$$Z_{S^3}^{\mathrm{ADHM}}(N, N_f, 0)$$
$$= \left(\frac{2}{\pi^2 N_f}\right)^{-\frac{1}{3}} e^{\frac{1}{2}\left(\mathcal{A}(N_f) + \mathcal{A}(1)N_f^2\right)}\mathrm{Ai}\left[\left(\frac{2}{\pi^2 N_f}\right)^{-\frac{1}{3}}\left(N + \frac{N_f}{8} - \frac{1}{2N_f}\right)\right] + \mathcal{O}\left(e^{-\sqrt{N}}\right)$$

$$Z^{\text{ADHM}}_{S^3_b}\left(N,N_f,\pm\frac{\mathrm{i}(b-b^{-1})}{2}\right)=Z^{\text{ADHM}}_{S^3}(N,N_f,0)$$

$$Z^{\text{ADHM}}_{S^3_b}\left(N,N_f,\pm\frac{\mathrm{i}(b-b^{-1})}{2}\right)$$
$$=\left(\frac{2}{\pi^2N_f}\right)^{-\frac{1}{3}}e^{\frac{1}{2}\left(\mathcal{A}(N_f)+\mathcal{A}(1)N_f^2\right)}\mathrm{Ai}\left[\left(\frac{2}{\pi^2N_f}\right)^{-\frac{1}{3}}\left(N+\frac{N_f}{8}-\frac{1}{2N_f}\right)\right]+\mathcal{O}\left(e^{-\sqrt{N}}\right)$$

$$Z^{\text{ADHM}}_{S^3_b}(N,N_f,\mu)=C_b(N_f,\mu)^{-\frac{1}{3}}e^{\mathcal{A}_b(N_f,\mu)}\mathrm{Ai}\left[C_b(N_f,\mu)^{-\frac{1}{3}}\left(N-B_b(N_f,\mu)\right)\right]+\mathcal{O}\left(e^{-\sqrt{N}}\right)$$

$$C_b(N_f,\mu)=\frac{2}{\pi^2N_f}\frac{1}{\left(\frac{b+b^{-1}}{2}+\mathrm{i}\mu\right)^2\left(\frac{b+b^{-1}}{2}-\mathrm{i}\mu\right)^2}$$
$$B_b(N_f,\mu)=N_f\left(\frac{1}{24}-\frac{\frac{(b+b^{-1})^2(b^2+b^{-2})}{48}+\frac{b^2+4+b^{-2}}{12}\mu^2}{\left(\frac{b+b^{-1}}{2}+\mathrm{i}\mu\right)^2\left(\frac{b+b^{-1}}{2}-\mathrm{i}\mu\right)^2}\right)$$
$$-\frac{1}{N_f}\frac{\frac{(b+b^{-1})^2(b^2-8+b^{-2})}{48}+\frac{b^2-4+b^{-2}}{12}\mu^2}{\left(\frac{b+b^{-1}}{2}+\mathrm{i}\mu\right)^2\left(\frac{b+b^{-1}}{2}-\mathrm{i}\mu\right)^2}$$

$$c_T=-\frac{32}{\pi^2}\frac{\partial^2\log Z^{\text{ADHM}}_{S^3_b}(N,N_f,0)}{\partial b^2}\bigg|_{b=1}=-\frac{32}{\pi^2}\frac{\partial^2\log Z^{\text{ADHM}}_{S^3}(N,N_f,\mu)}{\partial\mu^2}\bigg|_{\mu=0}$$

$$c_T=-\frac{64N_f^{\frac{1}{2}}(2\pi)^{\frac{2}{3}}\left(N+\frac{3N_f}{8}+\frac{3}{4N_f}\right)\mathrm{Ai}'\left[\left(\frac{2}{\pi^2N_f}\right)^{-\frac{1}{3}}\left(N+\frac{N_f}{8}-\frac{1}{2N_f}\right)\right]}{3\pi^2\mathrm{Ai}\left[\left(\frac{2}{\pi^2N_f}\right)^{-\frac{1}{3}}\left(N+\frac{N_f}{8}-\frac{1}{2N_f}\right)\right]}$$
$$+(N-\text{ independent constant })+\mathcal{O}\left(e^{-\sqrt{N}}\right)$$

$$N-B_b(N_f,\mu)\xrightarrow[b\to 0]{\frac{\mu}{b+b^{-1}}\text{ fixed}}N-\frac{N_f}{24}+\frac{N_f}{12}\left(\frac{1}{\frac{1}{2}+\frac{\mathrm{i}\mu}{b+b^{-1}}}+\frac{1}{\frac{1}{2}-\frac{\mathrm{i}\mu}{b+b^{-1}}}\right)$$
$$+\frac{1}{12N_f}\left(\frac{1}{\frac{1}{2}+\frac{\mathrm{i}\mu}{b+b^{-1}}}+\frac{1}{\frac{1}{2}-\frac{\mathrm{i}\mu}{b+b^{-1}}}\right)$$

$$\tilde{\Delta}_1=\tilde{\Delta}_3=\frac{1}{2}-\frac{\mathrm{i}\mu}{b+b^{-1}},\tilde{\Delta}_2=\tilde{\Delta}_4=\frac{1}{2}+\frac{\mathrm{i}\mu}{b+b^{-1}}$$

$$B_b(k,\Delta) = \frac{k}{24} + \frac{1}{12k}\left[\frac{4-\sum_{a=1}^{4}\Delta_a^2}{(b+b^{-1})^2\Delta_1\Delta_2\Delta_3\Delta_4} - \sum_{a=1}^{4}\frac{1}{\Delta_a}\right]$$

$$N - B_b(k,\Delta) \overset{\Delta \text{ fixed}}{\underset{b\to 0}{\to}} \hat{N}_\Delta = N - \frac{k}{24} + \frac{k}{12}\sum_{a=1}^{4}\frac{1}{\Delta_a}$$

$$Z_{S^3}^{N^{0,1,0}}(N,k,r) = C(k,r)^{-\frac{1}{3}} e^{\mathcal{A}(k,r)} \mathrm{Ai}\left[C(k,r)^{-\frac{1}{3}}(N - B(k,r))\right] + \mathcal{O}\left(e^{-\sqrt{N}}\right)$$

$$C(k,r) = \frac{2k+r}{\pi^2(k+r)^2}$$

$$B(k,r) = \frac{2k-3r}{48} + \frac{k}{3(k+r)^2}$$

$$Z_{S_b^3} = C_b^{-\frac{1}{3}} e^{\mathcal{A}_b(k)} \mathrm{Ai}\left[C_b^{-\frac{1}{3}}(N - B_b)\right] + \mathcal{O}\left(e^{-\sqrt{N}}\right),$$

$$C_b(k,\Delta) = \frac{2}{\pi^2 k}\frac{(b+b^{-1})^{-4}}{\prod_{a=1}^{4}\Delta_a}, B_b(k,\Delta) = \frac{k}{24} + \frac{\alpha(\Delta,b)}{k},$$

$$\Delta_1 = \frac{1}{2} - \mathrm{i}\frac{m_1+m_2+m_3}{b+b^{-1}}, \quad \Delta_2 = \frac{1}{2} - \mathrm{i}\frac{m_1-m_2-m_3}{b+b^{-1}},$$
$$\Delta_3 = \frac{1}{2} + \mathrm{i}\frac{m_1+m_2-m_3}{b+b^{-1}}, \quad \Delta_4 = \frac{1}{2} + \mathrm{i}\frac{m_1-m_2+m_3}{b+b^{-1}},$$

$$\alpha(\Delta,b) = -\frac{1}{12}\sum_{a=1}^{4}\Delta_a^{-1} + \frac{1-\frac{1}{4}\sum_a \Delta_a^2}{3(b+b^{-1})^2\prod_{a=1}^{4}\Delta_a}$$

$$\Delta_1 = 1, \Delta_2 + \Delta_3 + \Delta_4 = 1$$

$$F_{S^1\times\Sigma_g} = \frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\sum_{a=1}^{4}\frac{\mathfrak{n}_a}{\Delta_a}\left(\hat{N}_\Delta^{\frac{3}{2}} - \frac{\mathfrak{c}_a}{k}\hat{N}_\Delta^{\frac{1}{2}}\right) + \frac{1-\mathfrak{g}}{2}\log \hat{N}_\Delta - \hat{f}_0(k,\Delta,\mathfrak{n})$$

$$\mathfrak{c}_a = \frac{\prod_{b\neq a}(\Delta_a + \Delta_b)}{8\Delta_1\Delta_2\Delta_3\Delta_4}\sum_{b\neq a}\Delta_b$$

$$\hat{N}_\Delta \equiv N - \frac{k}{24} + \frac{1}{12k}\sum_{a=1}^{4}\Delta_a^{-1}$$

$$\mathfrak{n}_1 = 1 - \mathfrak{g}, \mathfrak{n}_2 + \mathfrak{n}_3 + \mathfrak{n}_4 = 1 - \mathfrak{g}.$$

$$\Delta_2 = \Delta_3 = \Delta_4 = \frac{1}{3}, \mathfrak{n}_2 = \mathfrak{n}_3 = \mathfrak{n}_4 = \frac{(1-\mathfrak{g})}{3}.$$

$$
\begin{aligned}
&\log \mathcal{I}_{\mathrm{ABJM}}(N,k,\omega,\Delta,\mathfrak{n}) \\
&\quad = -\frac{2}{\omega}\left[\frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\hat{N}_{k,\Delta}^{\frac{3}{2}} + \hat{g}_0(k,\Delta)\right] \\
&\qquad + \left[-\frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\sum_{a=1}^{4}\frac{\mathfrak{n}_a}{\Delta_a}\left(\hat{N}_{k,\Delta}^{\frac{3}{2}} - \frac{\mathfrak{c}_a(\Delta)}{k}\hat{N}_{k,\Delta}^{\frac{1}{2}}\right) - \frac{1}{2}\log\,\hat{N}_{k,\Delta} + \hat{f}_0(k,\Delta,\mathfrak{n})\right] \\
&\qquad + \mathcal{O}\left(e^{-\sqrt{N}}\right) + \mathcal{O}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\log\, Z^{\mathrm{ADHM}}_{S^1\times\Sigma_{\mathfrak{g}}}(N,N_f)}{\mathfrak{g}-1} &= \frac{\pi\sqrt{2N_f}}{3}\left[\left(N+\frac{7N_f}{24}+\frac{1}{3N_f}\right)^{\frac{3}{2}} - \left(\frac{N_f}{2}+\frac{5}{2N_f}\right)\left(N+\frac{7N_f}{24}+\frac{1}{3N_f}\right)^{\frac{1}{2}}\right] \\
&\quad + \frac{1}{2}\log\left(N+\frac{7N_f}{24}+\frac{1}{3N_f}\right) - \hat{f}_0(N_f) - \hat{f}_{\mathrm{np}}(N,N_f) \\
\frac{\log\, Z^{N^{0,1,0}}_{S^1\times\Sigma_{\mathfrak{g}}}(N,k)}{\mathfrak{g}-1} &= \frac{4\pi\sqrt{k}}{3\sqrt{3}}\left[\left(N+\frac{5k}{48}+\frac{1}{3k}\right)^{\frac{3}{2}} - \left(\frac{k}{4}+\frac{5}{4k}\right)\left(N+\frac{5k}{48}+\frac{1}{3k}\right)^{\frac{1}{2}}\right] \\
&\quad + \frac{1}{2}\log\left(N+\frac{5k}{48}+\frac{1}{3k}\right) - \hat{f}_0(k) - \hat{f}_{\mathrm{np}}(N,k) \\
\frac{\log\, Z^{V^{5,2}}_{S^1\times\Sigma_{\mathfrak{g}}}(N,N_f)}{\mathfrak{g}-1} &= \frac{16\pi\sqrt{N_f}}{27}\left[\left(N+\frac{N_f}{6}+\frac{1}{4N_f}\right)^{\frac{3}{2}} - \left(\frac{9N_f}{16}+\frac{27}{16N_f}\right)\left(N+\frac{N_f}{6}+\frac{1}{4N_f}\right)^{\frac{1}{2}}\right] \\
&\quad + \frac{1}{2}\log\left(N+\frac{N_f}{6}+\frac{1}{4N_f}\right) - \hat{f}_0(N_f) - \hat{f}_{\mathrm{np}}(N,N_f) \\
\frac{\log\, Z^{Q^{1,1,1}}_{S^1\times\Sigma_{\mathfrak{g}}}(N,N_f)}{\mathfrak{g}-1} &= \frac{4\pi\sqrt{N_f}}{3\sqrt{3}}\left[\left(N+\frac{N_f}{6}\right)^{\frac{3}{2}} - \left(\frac{N_f}{4}+\frac{3}{4N_f}\right)\left(N+\frac{N_f}{6}\right)^{\frac{1}{2}}\right] \\
&\quad + \frac{1}{2}\log\left(N+\frac{N_f}{6}\right) - \hat{f}_0(N_f) - \hat{f}_{\mathrm{np}}(N,N_f)
\end{aligned}
$$

$$
\begin{aligned}
ds^2_{11} &= \frac{L^2}{4}ds^2_4 + L^2 ds^2_6 + L^2\left(d\psi + \sigma + \frac{1}{4}A\right)^2 \\
G &= \frac{3L^3}{8}\mathrm{vol}_4 - \frac{L^3}{4}J\wedge *_4 F
\end{aligned}
$$

$$
\begin{aligned}
ds^2_4 &= U(r)d\tau^2 + U(r)^{-1}dr^2 + r^2 ds^2_{\Sigma_{\mathfrak{g}}} \\
U(r) &= \left(r+\frac{\kappa}{2r}\right)^2 - \frac{Q^2}{4r^2} \\
F &= \frac{Q}{r^2}d\tau\wedge dr \pm \kappa\mathrm{vol}_{\Sigma_{\mathfrak{g}}}
\end{aligned}
$$

$$
I^{\mathrm{reg}}_{2\partial} = (1-\mathfrak{g})\frac{\pi L^2}{2G_N}
$$

$$
I^{\mathrm{reg}}_{2\partial} = (1-\mathfrak{g})\sqrt{\frac{2\pi^6}{27\mathrm{vol}[Y_7]}}N^{\frac{3}{2}} + o\left(N^{\frac{3}{2}}\right)
$$

$$\log Z^{\mathrm{SCFT}}_{S^1\times\Sigma_g} = (g-1)\sqrt{\frac{2\pi^6}{27\mathrm{vol}[Y_7]}}N^{\frac{3}{2}} + o\left(N^{\frac{3}{2}}\right) = -I^{\mathrm{reg}}_{2\partial}$$

$$\mathcal{S} = \mathrm{Re}\log Z^{\mathrm{SCFT}}_{S^1\times\Sigma_g}$$

$\mathcal{F}$	χ
1	1
$\frac{1}{4}(b+b^{-1})^2$	1
$1-g$	$2(1-g)$
$\frac{(\omega+1)^2}{2\omega}$	2
$\frac{1}{8\omega}\left[\left(\frac{1}{n_+}+\frac{1}{n_-}\right)\omega+2\right]^2+\frac{\omega}{8}\left[\frac{1}{n_+}-\frac{1}{n_-}\right]^2$	$\frac{1}{n_+}+\frac{1}{n_-}$

$$-\log Z^{\mathrm{SCFT}}_{M_3} = \pi\mathcal{F}\left(AN^{\frac{3}{2}}+BN^{\frac{1}{2}}\right) - \pi(\mathcal{F}-\chi)CN^{\frac{1}{2}} + o\left(N^{\frac{1}{2}}\right),$$

SCFT	SUSY	A	B/A	C/A
ABJM	$\mathcal{N}=6$	$\frac{\sqrt{2k}}{3}$	$-\frac{k}{16}-\frac{1}{2k}$	$-\frac{3}{2k}$
ADHM	$\mathcal{N}=4$	$\frac{\sqrt{2N_f}}{3}$	$\frac{3N_f}{16}-\frac{3}{4N_f}$	$-\frac{N_f}{4}-\frac{5}{4N_f}$
$N^{0,1,0}$	$\mathcal{N}=3$	$\frac{2(k+r)}{3\sqrt{2k+r}}$	$\frac{3r-2k}{32}-\frac{k}{2(k+r)^2}$	$-\frac{r}{8}-\frac{3k+2r}{2(k+r)^2}$
mABJM	$\mathcal{N}=2$	$\frac{4\sqrt{2k}}{9\sqrt{3}}$	$-\frac{k}{16}-\frac{1}{k}$	$-\frac{9}{4k}$

$$V^{5,2}:\quad A=\frac{16}{27}\sqrt{N_f},\qquad B+C=-\frac{5N_f^2+21}{27\sqrt{N_f}}$$
$$Q^{1,1,1}:\quad A=\frac{4}{3\sqrt{3}}\sqrt{N_f},\qquad B+C=-\frac{1}{\sqrt{3N_f}}$$

Caso 1. $\Delta=\left(\Delta_1,1-\Delta_1,1,\frac{1}{2},\frac{1}{2},0\right)$

$$N_f\in\{1,2,3,4\}\quad \&\quad \Delta_1=\frac{1}{2}$$
$$N_f\in\{1,2,3\}\quad \&\quad \Delta_1\in\left\{\frac{3}{8},\frac{2}{5},\frac{5}{12},\frac{3}{7}\right\}$$

Caso 2. $\Delta=\left(\Delta_1,1-\Delta_1,1,\Delta_q,1-\Delta_q,0\right)$

$$N_f=1\ \left(\Delta_1,\Delta_q\right)\in\left\{\left(\frac{3}{8},\frac{2}{5}\right),\left(\frac{2}{5},\frac{5}{12}\right),\left(\frac{5}{12},\frac{3}{7}\right),\left(\frac{3}{7},\frac{3}{8}\right)\right\}$$

Caso 3. $\Delta=\left(\Delta_1,1-\Delta_1,1,\frac{1}{2},\frac{1}{2},\Delta_m\right)$

$$N_f\in\{1,2,3\}\ \&\ (\Delta_1,\Delta_m)=\left\{\left(\frac{3}{8},\frac{N_f}{8}\right),\left(\frac{3}{8},\frac{N_f}{10}\right)\right\}$$
$$N_f\in\{2,3\}\ \&\ (\Delta_1,\Delta_m)\in\left\{\left(\frac{2}{5},\frac{N_f}{10}\right),\left(\frac{2}{5},\frac{N_f}{12}\right),\left(\frac{5}{12},\frac{N_f}{12}\right)\right.$$
$$\left.\left(\frac{5}{12},\frac{N_f}{14}\right),\left(\frac{3}{7},\frac{N_f}{14}\right),\left(\frac{3}{7},\frac{N_f}{16}\right)\right\}$$

Caso 4. $\Delta=\left(\Delta_1,\Delta_2,2-\Delta_1-\Delta_2,\Delta_q,\Delta_1+\Delta_2-\Delta_q,\Delta_m\right)$

$$N_f=3\qquad \otimes\qquad \left(\Delta_1,\Delta_2,\Delta_q,\Delta_m\right)=\left(\frac{4}{7},\frac{4}{7},\frac{3}{7},-\frac{N_f}{7}\right)$$
$$N_f\in\{2,3\}\qquad \circledcirc\qquad \left(\Delta_1,\Delta_2,\Delta_q,\Delta_m\right)\in\left\{\left(\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{N_f}{5}\right),\left(\frac{3}{10},\frac{5}{10},\frac{1}{5},\frac{N_f}{5}\right)\right.$$
$$\left.\left(\frac{4}{10},\frac{5}{10},\frac{3}{10},\frac{3N_f}{20}\right),\left(\frac{3}{8},\frac{17}{40},\frac{2}{5},\frac{3N_f}{40}\right)\right\}$$
$$N_f\in\{1,2,3\}\qquad \odot\qquad \left(\Delta_1,\Delta_2,\Delta_q,\Delta_m\right)=\left(\frac{1}{\pi},\frac{2}{\pi},\frac{3}{2\pi},N_f\left(1-\frac{3}{\pi}\right)\right)$$
$$N_f\in\{3,4\}\qquad \oplus\qquad \left(\Delta_1,\Delta_2,\Delta_q,\Delta_m\right)=\left(\frac{1}{\pi},\frac{2}{\pi},\frac{e}{2\pi},N_f\left(1-\frac{3}{\pi}\right)\right)$$

$$\hat{f}_{3/2}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right)=\sum_{a=1}^{4}\hat{f}_{3/2,a}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right)\tilde{\mathfrak{n}}_a,$$
$$\hat{f}_{1/2,a}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right)=\sum_{a=1}^{4}\hat{f}_{1/2,a}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right)\tilde{\mathfrak{n}}_a,$$

$$\hat{f}_{3/2}(N_f,\Delta,\mathfrak{n}) = -\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a},$$

$$\hat{f}_{1/2}(N_f,\Delta,\mathfrak{n}) = -\frac{\pi\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\sum_{a=1}^{4}\left(\mathfrak{c}_a(\tilde{\Delta})N_f+\frac{\mathfrak{d}_a(\tilde{\Delta})}{N_f}\right)\tilde{\mathfrak{n}}_a,$$

$$R_X(N_f,\Delta,\mathfrak{n}) = \frac{\hat{f}_X^{(\mathrm{lmf})}(N_f,\Delta,\mathfrak{n})-\hat{f}_X(N_f,\Delta,\mathfrak{n})}{\hat{f}_X(N_f,\Delta,\mathfrak{n})}\ (X\in\{3/2,1/2\}),$$

Caso 1. $\Delta_1 = \frac{1}{2}\varphi^{\dagger}\mathfrak{n} = \left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},0\right)$

	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
$N_f=1$	2.436×10^{-39}	5.319×10^{-37}	-3.0459513105331823845	7.834×10^{-36}
$N_f=2$	-6.565×10^{-29}	-1.915×10^{-26}	-2.8393059176753911173	2.859×10^{-25}
$N_f=3$	4.214×10^{-25}	1.188×10^{-22}	-3.3892805274389775678	2.159×10^{-21}
$N_f=4$	1.620×10^{-22}	4.069×10^{-20}	-4.3655284762174631267	9.220×10^{-19}

Caso 2. $N_f = 1\varphi^{\ddagger}(\Delta_1,\Delta_q) = \left(\frac{3}{8},\frac{2}{5}\right)$

$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
-3.594×10^{-33}	-7.605×10^{-31}	-3.1653887845699785480	1.122×10^{-29}
8.694×10^{-33}	1.792×10^{-30}	-3.1419129712303989770	2.612×10^{-29}
2.285×10^{-33}	4.750×10^{-31}	-3.1592865927006754811	6.990×10^{-30}
1.188×10^{-32}	2.427×10^{-30}	-3.1388618752957474436	3.522×10^{-29}
-6.909×10^{-34}	-1.449×10^{-31}	-3.1623376886353270145	2.113×10^{-30}

Caso 3. $N_f = 2\varphi^{\dagger}(\Delta_1, \Delta_m) = \left(\frac{5}{12}, \frac{N_f}{14}\right)$

$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
8.370×10^{-24}	2.381×10^{-21}	-2.9294387406755353595	3.385×10^{-20}
1.364×10^{-23}	4.159×10^{-21}	-2.6816709273752314462	5.589×10^{-20}
1.359×10^{-23}	3.802×10^{-21}	-3.0545474608660212803	5.577×10^{-20}
4.598×10^{-24}	1.351×10^{-21}	-2.7562653234541241733	1.832×10^{-20}

Caso 4. $N_f = 3\varphi^{\ddagger}(\Delta_1, \Delta_2, \Delta_q, \Delta_m) = \left(\frac{1}{\pi}, \frac{2}{\pi}, \frac{3}{2\pi}, N_f\left(1 - \frac{3}{\pi}\right)\right)$

$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
-2.023×10^{-18}	-4.727×10^{-16}	-4.4813859779853576433	8.868×10^{-15}
-1.879×10^{-18}	-4.836×10^{-16}	-3.5203924078852770102	7.756×10^{-15}
-2.948×10^{-18}	-6.613×10^{-16}	-4.7777616683409756638	1.286×10^{-14}
-1.710×10^{-19}	-4.255×10^{-17}	-3.6883756122275059505	6.527×10^{-16}
-2.484×10^{-18}	-5.685×10^{-16}	-4.6295738231631666536	1.086×10^{-14}

$$k \in \{1,2,3,4\}, \frac{r}{k} \in \left\{\frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3\right\}.$$

$$\hat{f}_{3/2}^{(\mathrm{lmf})}(k,r), \hat{f}_{1/2}^{(\mathrm{lmf})}(k,r)$$

$$\log Z_{S^1\times S^2}^{N^{0,1,0}}(N,k,r) + \frac{1}{2}\log \hat{N}_{k,r} = \hat{f}_{3/2}^{(\mathrm{lmf})}(k,r)\left(\hat{N}_{k,r}\right)^{\frac{3}{2}} + \hat{f}_{1/2}^{(\mathrm{lmf})}(k,r)\left(\hat{N}_{k,r}\right)^{\frac{1}{2}} + \hat{f}_0^{(\mathrm{lmf})}(k,r),$$

$$\hat{f}_{3/2}(k,r) = -\frac{2\pi(k+r)}{3\sqrt{2k+r}}$$
$$\hat{f}_{1/2}(k,r) = \frac{2\pi(k+r)}{3\sqrt{2k+r}}\left(\frac{r}{4} + \frac{3k+2r}{(k+r)^2}\right)$$

$$R_X(k,r) = \frac{\hat{f}_X^{(\mathrm{lmf})}(k,r) - \hat{f}_X(k,r)}{\hat{f}_X(k,r)}\ (X \in \{3/2,1/2\}),$$

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
(1,1/2)	-1.842×10^{-16}	-7.982×10^{-14}	-2.4445122210251198105	7.956×10^{-13}
(2,1)	-1.382×10^{-21}	-9.963×10^{-19}	-1.5990156188560014468	9.539×10^{-18}
(3,3/2)	-1.529×10^{-18}	-1.301×10^{-15}	-1.474778063088432060	1.174×10^{-14}
(4,2)	-1.430×10^{-16}	-1.249×10^{-13}	-1.6979156145862367914	1.187×10^{-12}

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
(1,2/3)	-3.264×10^{-17}	-1.557×10^{-14}	-2.3523428919766891206	1.568×10^{-13}
(2,4/3)	-4.072×10^{-19}	-3.009×10^{-16}	-1.6252242716209757969	2.823×10^{-15}
(3,2)	-1.090×10^{-16}	-8.815×10^{-14}	-1.6284176444001315906	8.459×10^{-13}
(4,8/3)	-4.916×10^{-15}	-3.849×10^{-12}	-2.0166679918730627992	4.102×10^{-11}

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
(1,1)	-3.573×10^{-18}	-1.959×10^{-15}	-2.2479735914758641588	2.024×10^{-14}
(2,2)	-4.454×10^{-16}	-3.269×10^{-13}	-1.7710441322786798897	3.135×10^{-12}
(3,3)	-3.989×10^{-14}	-2.834×10^{-11}	-2.0958128593131947881	3.097×10^{-10}
(4,4)	-8.360×10^{-13}	-5.303×10^{-10}	-2.9167150281215899207	6.910×10^{-9}

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
(1,3/2)	-3.293×10^{-16}	-2.036×10^{-13}	-2.2194141302255562257	1.941×10^{-12}
(2,3)	-2.845×10^{-13}	-1.923×10^{-10}	-2.1883848791741933989	1.976×10^{-9}
(3,9/2)	-1.086×10^{-11}	-6.292×10^{-9}	-3.1734834121743210878	8.184×10^{-8}
(4,6)	-1.205×10^{-10}	-5.835×10^{-8}	-4.9043313527433843627	9.564×10^{-7}

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(1\mathrm{mf})}$	σ_0
(1,2)	-3.601×10^{-14}	-2.338×10^{-11}	-2.2917317046811495268	2.099×10^{-10}
(2,4)	-1.842×10^{-11}	-1.112×10^{-8}	-2.8180835746366396624	1.235×10^{-7}
(3,6)	-4.050×10^{-10}	-1.949×10^{-7}	-4.6848075856487257662	2.909×10^{-6}

(4,8)	-2.996×10^{-9}	-1.167×10^{-6}	-7.6419214327213593603	2.248×10^{-5}

(k,r)	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
(1,3)	-2.018×10^{-11}	-1.291×10^{-8}	-2.6498077170116517266	1.104×10^{-7}
(2,6)	-2.996×10^{-9}	-1.441×10^{-6}	-4.6689712958005925271	1.821×10^{-5}
(3,9)	-3.148×10^{-8}	-1.126×10^{-5}	-8.9730889697708208865	2.015×10^{-4}
(4,12)	-1.397×10^{-7}	-3.939×10^{-5}	-15.329031790765612168	9.246×10^{-4}

(k,r)	$R_{\mathrm{np},1/2}$	(k,r)	$R_{\mathrm{np},1/2}$	(k,r)	$R_{\mathrm{np},1/2}$
(1,1)	3.549×10^{-10}	(1,2)	-3.737×10^{-11}	(1,3)	-5.599×10^{-8}
(2,2)	-2.420×10^{-8}	(2,4)	-3.968×10^{-7}	(2,6)	2.636×10^{-8}
(3,3)	4.451×10^{-5}	(3,6)	-8.227×10^{-6}	(3,9)	-3.638×10^{-5}
(4,4)	-3.138×10^{-4}	(4,8)	8.126×10^{-5}	(4,12)	1.289×10^{-4}

Caso 1. $\Delta=\left(\Delta_1,\frac{4}{3}-\Delta_1,\frac{2}{3},\frac{1}{3},\frac{1}{3},0\right)$

$$\begin{array}{lcl} N_f\in\{1,2,3,4,5\} & \bowtie & \Delta_1=\dfrac{2}{3} \\ N_f\in\{1,2,3\} & \rightthreetimes & \Delta_1\in\left\{\dfrac{1}{2},\dfrac{5}{9}\right\} \\ N_f=1 & \leftthreetimes & \Delta_1=\dfrac{7}{12} \end{array}$$

Caso 2. $\Delta=\left(\Delta_1,\frac{4}{3}-\Delta_1,\frac{2}{3},\frac{1}{3},\frac{1}{3},\Delta_m\right)$

$$N_f \in \{1,2,3\} \,\&\, (\Delta_1,\Delta_m) = \left\{\left(\frac{1}{2},\frac{N_f}{6}\right),\left(\frac{1}{2},\frac{N_f}{9}\right),\left(\frac{5}{9},\frac{N_f}{9}\right)\right.$$
$$\left.\left(\frac{5}{9},\frac{N_f}{12}\right),\left(\frac{7}{12},\frac{N_f}{12}\right),\left(\frac{7}{12},\frac{N_f}{15}\right)\right\}$$
$$N_f \in \{1,2\} \,\&\, (\Delta_1,\Delta_m) \in \left\{\left(\frac{5}{12},\frac{N_f}{12}\right),\left(\frac{7}{15},\frac{N_f}{15}\right),\left(\frac{13}{24},\frac{N_f}{10}\right)\right\}$$

Caso 3. $\Delta = \left(\Delta_1,\frac{4}{3}-\Delta_1,\frac{2}{3},\Delta_q,\frac{2}{3}-\Delta_q,\Delta_m\right)$

$$N_f \in \{1,2\} \,\&\, \left(\Delta_1,\Delta_q,\Delta_m\right) = \left(\frac{2}{3}-\frac{1}{2\pi},\frac{1}{6},N_f\left(\frac{2}{3}-\frac{2}{\pi}\right)\right)$$
$$N_f \in \{3,4\} \,\&\, \left(\Delta_1,\Delta_q,\Delta_m\right) = \left(\frac{2}{3}-\frac{1}{2\pi},\frac{1}{4},N_f\left(\frac{2}{3}-\frac{2}{\pi}\right)\right)$$

$$\hat{f}_{3/2}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right),\hat{f}_{1/2}^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right),$$

$$\hat{f}_{3/2}\left(N_f,\Delta,\mathfrak{n}\right) = -\frac{\pi\sqrt{N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\sum_{a=1}^{4}\frac{\tilde{\mathfrak{n}}_a}{\tilde{\Delta}_a}$$
$$\hat{f}_{1/2}\left(N_f,\Delta,\mathfrak{n}\right) = -\frac{\pi\sqrt{N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\left(\sum_{I=1}^{2}\left(\mathfrak{a}_I N_f+\frac{\mathfrak{b}_I}{N_f}\right)\mathfrak{n}_I+\frac{\tilde{\Delta}_3-\tilde{\Delta}_4}{3\tilde{\Delta}_3^2\tilde{\Delta}_4^2}\frac{\mathfrak{t}}{N_f^2}\right)$$

$$R_X\left(N_f,\Delta,\mathfrak{n}\right) = \frac{\hat{f}_X^{(\mathrm{lmf})}\left(N_f,\Delta,\mathfrak{n}\right)-\hat{f}_X\left(N_f,\Delta,\mathfrak{n}\right)}{\hat{f}_X\left(N_f,\Delta,\mathfrak{n}\right)}\ (X \in \{3/2,1/2\}),$$

Caso 1. $\Delta_1 = \frac{2}{3}\,\phi^{\otimes}\mathfrak{n} = \left(\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},0\right)$

	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
$N_f = 1$	2.446×10^{-28}	7.122×10^{-26}	-2.7620858097124988759	9.445×10^{-25}
$N_f = 2$	-4.779×10^{-22}	-1.594×10^{-19}	-3.2671366455659909955	2.399×10^{-18}
$N_f = 3$	-7.188×10^{-19}	-2.104×10^{-16}	-4.7624014875151824187	4.110×10^{-15}

$N_f = 4$	1.210×10^{-16}	2.994×10^{-14}	-7.0161650179852435476	7.423×10^{-13}
$N_f = 5$	9.005×10^{-15}	1.895×10^{-12}	-9.9830275183613382542	5.851×10^{-11}

Caso 2. $N_f = 2\phi^{\otimes}(\Delta_1, \Delta_q) = \left(\frac{1}{2}, \frac{N_f}{9}\right)$

$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\text{lmf})}$	σ_0
1.078×10^{-16}	3.448×10^{-14}	-3.4528096365944814244	4.489×10^{-13}
1.732×10^{-16}	5.639×10^{-14}	-3.3715743804211707717	7.244×10^{-13}
6.362×10^{-17}	2.046×10^{-14}	-3.3447537752182470037	2.564×10^{-13}
-5.510×10^{-17}	-1.751×10^{-14}	-3.2966941000920532717	2.195×10^{-13}

Caso 3. $N_f = 3\phi^{\otimes}(\Delta_1, \Delta_q, \Delta_m) = \left(\frac{2}{3} - \frac{1}{2\pi}, \frac{1}{4}, N_f\left(\frac{2}{3} - \frac{2}{\pi}\right)\right)$

$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\text{lmf})}$	σ_0
1.584×10^{-15}	4.419×10^{-13}	-5.0895614683300330156	7.852×10^{-12}
1.597×10^{-15}	4.479×10^{-13}	-4.9454216523869612663	7.691×10^{-12}

6.608×10^{-17}	1.862×10^{-14}	-4.8692433262566337160	2.645×10^{-13}
-1.782×10^{-15}	-4.984×10^{-13}	-4.7512952786913154264	8.264×10^{-12}
-3.923×10^{-16}	-1.098×10^{-13}	-4.8145318465752665874	1.898×10^{-12}

$$Z^{Q^{1,1,1}}_{S^1\times S^2}(N, N_f, \Delta, \mathfrak{n}) = \frac{1}{(N!)^2} \sum_{\mathfrak{m},\tilde{\mathfrak{m}}\in\mathbb{Z}^N} \oiiint_{\mathcal{C}}^{\triangle} \prod_{i=1}^{N} \frac{dx_i}{2\pi \mathrm{i} x_i} \frac{d\tilde{x}_i}{2\pi \mathrm{i} \tilde{x}_i} \prod_{i\neq j}^{N} \left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right)$$

$$\times \prod_{i,j=1}^{N} \prod_{a=1,2} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_a}}{1 - \frac{x_i}{\tilde{x}_j} y_a}\right)^{1-\mathfrak{n}_a} \prod_{a=3,4} \left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_a}}{1 - \frac{\tilde{x}_j}{x_i} y_a}\right)^{1-\mathfrak{n}_a}$$

$$\times \prod_{i=1}^{N} \prod_{n=1,2} \left(\frac{\sqrt{\frac{1}{x_i} y_{\tilde{q}_n}}}{1 - \frac{1}{x_i} y_{\tilde{q}_n}}\right)^{N_f(1-\mathfrak{n}_{\tilde{q}_n})} \prod_{j=1}^{N} \prod_{n=1,2} \left(\frac{\sqrt{\tilde{x}_j y_{q_n}}}{1 - \tilde{x}_j y_{q_n}}\right)^{N_f(1-\mathfrak{n}_{q_n})}$$

$$\times \prod_{i=1}^{N} \frac{\left(e^{\mathrm{i}B_i}\right)^M}{e^{\mathrm{i}B_i} - 1} \times \prod_{j=1}^{N} \frac{\left(e^{\mathrm{i}\tilde{B}_j}\right)^M}{e^{\mathrm{i}\tilde{B}_j} - 1}$$

$$e^{\mathrm{i}B_i} = (-1)^{N+N_f-2\lfloor N_f/2 \rfloor} \sigma_i \prod_{n=1,2} \left(\frac{\sqrt{\frac{1}{x_i} y_{\tilde{q}_n}}}{1 - \frac{1}{x_i} y_{\tilde{q}_n}}\right)^{-N_f} \prod_{j=1}^{N} \frac{\left(1 - \frac{\tilde{x}_j}{x_i} y_3\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_4\right)}{\left(1 - \frac{\tilde{x}_j}{x_i} y_1^{-1}\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_2^{-1}\right)}$$

$$e^{\mathrm{i}\tilde{B}_j} = (-1)^{N+N_f-2\lfloor N_f/2 \rfloor} \tilde{\sigma}_j \prod_{n=1,2} \left(\frac{\sqrt{\tilde{x}_j y_{q_n}}}{1 - \tilde{x}_j y_{q_n}}\right)^{-N_f} \prod_{i=1}^{N} \frac{\left(1 - \frac{\tilde{x}_j}{x_i} y_3\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_4\right)}{\left(1 - \frac{\tilde{x}_j}{x_i} y_1^{-1}\right)\left(1 - \frac{\tilde{x}_j}{x_i} y_2^{-1}\right)}$$

$$\sigma_i = \prod_{j=1}^{N} \frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_1}}{-\frac{x_i}{\tilde{x}_j}y_1} \frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_2}}{-\frac{x_i}{\tilde{x}_j}y_2} \frac{1}{\sqrt{\frac{\tilde{x}_j}{x_i}y_3}} \frac{1}{\sqrt{\frac{\tilde{x}_j}{x_i}y_4}} \in \{-1,1\}$$

$$\tilde{\sigma}_j = \prod_{i=1}^{N} \frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_1}}{-\frac{x_i}{\tilde{x}_j}y_1} \frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_2}}{-\frac{x_i}{\tilde{x}_j}y_2} \frac{1}{\sqrt{\frac{\tilde{x}_j}{x_i}y_3}} \frac{1}{\sqrt{\frac{\tilde{x}_j}{x_i}y_4}} \in \{-1,1\}$$

$$\sigma_i = \tilde{\sigma}_j = (-1)^N$$

$$\begin{aligned} \mathcal{V} =& \sum_{i=1}^{N} \left[\frac{N_f}{2}(\tilde{u}_i^2 - u_i^2) - 2\pi\left(\tilde{n}_i - \frac{N_f}{2} + \left\lfloor\frac{N_f}{2}\right\rfloor\right)\tilde{u}_i + 2\pi\left(n_i - \frac{N_f}{2} + \left\lfloor\frac{N_f}{2}\right\rfloor\right)u_i\right] \\ &+ \sum_{i,j=1}^{N} \left[\sum_{a=3,4} \mathrm{Li}_2\left(e^{\mathrm{i}(\tilde{u}_j - u_i + \pi\Delta_a)}\right) - \sum_{a=1,2} \mathrm{Li}_2\left(e^{\mathrm{i}(\tilde{u}_j - u_i - \pi\Delta_a)}\right)\right] \\ &+ N_f \sum_{i=1}^{N}\sum_{n=1}^{2} \left[\mathrm{Li}_2\left(e^{\mathrm{i}(-u_i + \pi\Delta_{\tilde{q}_n})}\right) - \mathrm{Li}_2\left(e^{\mathrm{i}(-\tilde{u}_i - \pi\Delta_{q_n})}\right) + \frac{\pi}{2}u_i\Delta_{\tilde{q}_n} + \frac{\pi}{2}\tilde{u}_i(\Delta_{q_n} - 2)\right]. \end{aligned}$$

$$\begin{aligned} \rho(t) &= \frac{2\mu - k|t|}{\pi^2} \quad (t_< < t < t_>), \\ \delta v(t) &= \pi\Delta_1 - \frac{\pi\mu}{2\mu - k|t|} \quad (t_< < t < t_>), \end{aligned}$$

$$t_\ll = -\frac{\mu}{N_f}, t_\gg = \frac{\mu}{N_f}.$$

$$\mu = \sqrt{\frac{N_f}{3}}\pi.$$

$$n_i = 1 - i - \left\lfloor\frac{N_f}{2}\right\rfloor, \tilde{n}_j = j - N - \left\lfloor\frac{N_f}{2}\right\rfloor$$

$$\begin{aligned} 0 &< \mathrm{Re}[\tilde{u}_j - u_i + \pi\Delta_a] < 2\pi, & -2\pi &< \mathrm{Re}[\tilde{u}_j - u_i - \pi\Delta_a] < 0 \\ 0 &< \mathrm{Re}[\tilde{u}_i + \pi\Delta_{q_n}] < 2\pi, & 0 &< \mathrm{Re}[-u_i + \pi\Delta_{\tilde{q}_n}] < 2\pi. \end{aligned}$$

$$v_i + \tilde{v}_i|_{\text{initial condition}} = \pi(2 - \Delta_1 - \Delta_{q_1} - \Delta_{q_2}) = 0,$$

$$\begin{aligned} &N_f \in \{1,2,3,4,5\} \quad \dagger \quad \Delta_1 = \frac{1}{2} \\ &N_f \in \{1,2,3\} \quad \star \quad \Delta_1 \in \left\{\frac{3}{8}, \frac{5}{12}, \frac{3}{7}\right\} \end{aligned}$$

$$\hat{f}_{3/2}^{(\mathrm{lmf})}(N_f, \Delta, \mathfrak{n}), \hat{f}_{1/2}^{(\mathrm{lmf})}(N_f, \Delta, \mathfrak{n}),$$

$$\hat{f}_{3/2}(N_f,\Delta,\mathfrak{n}) = -\frac{4\pi\sqrt{N_f}}{3\sqrt{3}}$$
$$\hat{f}_{1/2}(N_f,\Delta,\mathfrak{n}) = \frac{4\pi\sqrt{N_f}}{3\sqrt{3}}\left(\frac{N_f}{4}+\frac{3}{4N_f}\right).$$

$$R_X(N_f,\Delta,\mathfrak{n}) = \frac{\hat{f}_X^{(\mathrm{lmf})}(N_f,\Delta,\mathfrak{n}) - \hat{f}_X(N_f,\Delta,\mathfrak{n})}{\hat{f}_X(N_f,\Delta,\mathfrak{n})}\ (X \in \{3/2,1/2\}),$$

	$R_{3/2}$	$R_{1/2}$	$\hat{f}_0^{(\mathrm{lmf})}$	σ_0
$N_f = 1$	1.503×10^{-17}	9.883×10^{-15}	-2.1415723730798296354	6.340×10^{-14}
$N_f = 2$	-3.490×10^{-14}	-2.641×10^{-11}	-2.0385864384989237526	1.780×10^{-10}
$N_f = 3$	-1.753×10^{-12}	-1.168×10^{-9}	-2.2368141361938090934	9.784×10^{-9}
$N_f = 4$	5.722×10^{-12}	3.251×10^{-9}	-2.6005901148883909862	2.981×10^{-8}
$N_f = 5$	2.396×10^{-10}	1.157×10^{-7}	-3.1045097958934355205	1.393×10^{-6}

$$R_{\mu\nu}^{(6)} = 8g_{\mu\nu}^{(6)}$$

$$ds^2_{\mathrm{SE}_7} = (d\psi+\sigma)^2 + ds_6^2$$

$$\sigma = \frac{1}{2}(\cos^2\xi - \sin^2\xi)d\varphi + \frac{1}{2}\cos^2\xi\cos\theta_1 d\phi_1 + \frac{1}{2}\sin^2\xi\cos\theta_2 d\phi_2$$
$$ds_6^2 = d\xi^2 + \cos^2\xi\sin^2\xi\left(d\varphi + \frac{1}{2}\cos\theta_1 d\phi_2 - \frac{1}{2}\cos\theta_2 d\phi_2\right)^2$$
$$+\frac{1}{4}\cos^2\xi(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{4}\sin^2\xi(d\theta_2^2+\sin^2\theta_2 d\phi_2^2)$$

$$0 \le 2\psi \pm \varphi < 4\pi, 0 \le \xi \le \frac{\pi}{2}, 0 \le \theta_{1,2} \le \pi, 0 \le \phi_{1,2} < 2\pi$$

$$X_1 = e^{\mathrm{i}\left(\psi+\frac{\varphi+\phi_1}{2}\right)}\cos\xi\cos\frac{\theta_1}{2}$$
$$X_2 = e^{\mathrm{i}\left(\psi+\frac{\varphi-\phi_1}{2}\right)}\cos\xi\sin\frac{\theta_1}{2}$$
$$X_3 = e^{\mathrm{i}\left(\psi-\frac{\varphi-\phi_2}{2}\right)}\sin\xi\cos\frac{\theta_2}{2}$$
$$X_4 = e^{\mathrm{i}\left(\psi-\frac{\varphi+\phi_2}{2}\right)}\sin\xi\sin\frac{\theta_2}{2}$$

$$(X_3, X_4) \sim e^{\frac{2\pi \mathrm{i}}{N_f}}(X_3, X_4)$$

$$2\psi - \varphi \sim 2\psi - \varphi + \frac{4\pi}{N_f}$$

$$\mathrm{vol}\left[S^7/\mathbb{Z}_{N_f}\right] = \frac{\pi^4}{3N_f}$$

$$\begin{aligned}
\sigma =& -\frac{1}{2}\cos\ \mu \sin\ \theta(\cos\ \phi\sigma_1 - \sin\ \phi\sigma_2) + \frac{1}{4}\cos\theta(1 + \cos^2\ \mu)\sigma_3 - \frac{1}{2}\cos\ \theta d\phi \\
ds_6^2 =& \frac{1}{4}\left(d\theta - \cos\ \mu(\sin\ \phi\sigma_1 + \cos\ \phi\sigma_2)\right)^2 \\
&+ \frac{1}{4}\sin^2\ \theta\left(d\phi - \cos\ \mu\cot\ \theta(\cos\ \phi\sigma_1 - \sin\ \phi\sigma_2) - \frac{1}{2}(1 + \cos^2\ \mu)\sigma_3\right)^2 \\
&+ \frac{1}{2}\left(d\mu^2 + \frac{1}{4}\sin^2\ \mu(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}\sin^2\ \mu\cos^2\ \mu\sigma_3^2\right)
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= \sin\ \beta d\alpha - \cos\ \beta\sin\ \alpha d\gamma \\
\sigma_2 &= \cos\ \beta d\alpha + \sin\ \beta\sin\ \alpha d\gamma \\
\sigma_3 &= d\beta + \cos\ \alpha d\gamma
\end{aligned}$$

$$\begin{aligned}
&0 \le \psi < \pi, \quad 0 \le \mu \le \frac{\pi}{2}, \quad 0 \le \theta < \pi, 0 \le \phi < 2\pi \\
&0 \le \alpha < \pi, \quad 0 \le \beta < 2\pi, \quad 0 \le \gamma < 4\pi
\end{aligned}$$

$$\mathrm{vol}[N^{0,1,0}/\mathbb{Z}_k] = \frac{\pi^4}{8k}$$

$$\begin{aligned}
\sigma =& \frac{3}{8}\cos\ \alpha(d\beta - \cos\ \theta_1 d\phi_1 - \cos\ \theta_2 d\phi_2) \\
ds_6^2 =& \frac{3}{8}d\alpha^2 + \frac{3}{32}\sin^2\ \alpha(d\beta - \cos\ \theta_1 d\phi_1 - \cos\ \theta_2 d\phi_2)^2 \\
&+ \frac{3}{32}(1 + \cos^2\ \alpha)(d\theta_1^2 + \sin^2\ \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\ \theta_2 d\phi_2^2) \\
&- \frac{3}{16}\sin^2\ \alpha\cos\ \beta(d\theta_1 d\theta_2 - \sin\ \theta_1\sin\ \theta_2 d\phi_1 d\phi_2) \\
&+ \frac{3}{16}\sin^2\ \alpha\sin\ \beta(\sin\ \theta_2 d\theta_1 d\phi_2 + \sin\ \theta_1 d\theta_2 d\phi_1)
\end{aligned}$$

$$0 \le \psi < \frac{3}{2}\pi, 0 \le \alpha \le \frac{\pi}{2}, 0 \le \beta < 4\pi, 0 \le \theta_{1,2} \le \pi, 0 \le \phi_{1,2} < 2\pi$$

$$\phi_2 \sim \phi_2 + \frac{2\pi}{N_f}$$

$$\mathrm{vol}\left[V^{5,2}/\mathbb{Z}_{N_f}\right] = \frac{27\pi^4}{128N_f}$$

$$\sigma = \frac{1}{4}\sum_{i=1}^{3} \cos\ \theta_i d\phi_i$$

$$ds_6^2 = \frac{1}{8}\sum_{i=1}^{3} \left(d\theta_i^2 + \sin^2\ \theta_i d\phi_i^2\right)$$

$$0 \leq \psi < \pi, 0 \leq \theta_{1,2,3} \leq \pi, 0 \leq \phi_{1,2,3} < 2\pi$$

$$(\phi_2, \phi_3) \sim (\phi_2, \phi_3) + \left(\frac{2\pi}{N_f}, \frac{2\pi}{N_f}\right)$$

$$\mathrm{vol}\left[Q^{1,1,1}/\mathbb{Z}_{N_f}\right] = \frac{\pi^4}{8N_f}$$

$$v_i|_{\text{initial condition}} = \frac{\pi\left(\Delta_{\tilde{q}} - \Delta_q\right)}{2}$$

$$a(t) = e^{iH(t-t')}a(t')e^{-iH(t-t')};\ t, t' \in \mathbb{R}$$

$$\omega(a) = \mathrm{Tr}(\rho_\omega a);\ \mathrm{Tr}(\rho_\omega) = 1$$

$$\omega_{\xi_{\hat{a}}}(\cdot) = \frac{1}{\omega\left(\pi_{\xi_{\hat{a}}}\right)}\omega\left(\pi_{\xi_{\hat{a}}} \cdot \pi_{\xi_{\hat{a}}}\right)$$

$$P(\xi_{\hat{a}}) = \omega\left(\pi_{\xi_{\hat{a}}}\right)$$

$$\mathcal{A}_{\geq t'} \subsetneq \mathcal{A}_{\geq t} \text{ for } t' \geq t$$

$$\mathcal{C}_{\omega\geq t}(\mathcal{A}_{\geq t}) = \{X \in \mathcal{A}_{\geq t} \mid \omega_t([X, A]) = 0;\ \forall A \in \mathcal{A}_{\geq t}\}$$

$$h_{TFD} = \beta H_R - \beta H_L$$

$$\Delta_{TFD}^{-iu} A(t, \vec{x}) \Delta_{TFD}^{iu} = A(t + \beta u, \vec{x}), \Delta_{TFD}^{-iu} \mathcal{M}_R \Delta_{TFD}^{iu} = \mathcal{M}_R, \forall u \in \mathbb{R}$$

$$\mathcal{M}_R|_{\geq t} \subsetneq \mathcal{M}_R|_{\geq t'}, \text{ for } t > t'$$

$$|\Psi_{\hat{\omega}}\rangle = |TFD\rangle \otimes f(X)$$

$$\mathcal{N}_{R,\geq t} = \mathcal{N}_{R,\geq t-s} = \mathcal{N}_R, \forall s > 0$$

$$\mathcal{Y}_{R,\geq t}^H \subsetneq \mathcal{Y}_{R,\geq t'}^H \text{ for } t > t'$$

$$\mathcal{C}_{TFD,k}(\mathcal{Y}_R^H) = \{W(h)w(h) \mid h \in H\}$$

$$X = \int_{\mathcal{U}} d\vec{x} d\tau \phi(\tau, \vec{x}) f(\tau, \vec{x})$$

$$\mathcal{O}_S \ni Y \mapsto Y(t) \in \mathcal{B}(\mathcal{H}_S)$$

$$Y = F\left[\int_{\mathcal{U}} d\vec{x} d\tau \phi(\tau, \vec{x}) f(\tau, \vec{x})\right] \mapsto F\left[\int_{\mathcal{U}} d\vec{x} d\tau \phi(\tau, \vec{x}) f(\tau + t, \vec{x})\right] = Y(t)$$

$$H_S(t) = H \ \forall t \in \mathbb{R}$$

$$Y(t_2) = e^{iH(t_2-t_1)} Y(t_1) e^{-iH(t_2-t_1)}$$

$$\mathcal{A}_{\geq t} = \overline{\bigvee_{I \subset [t,\infty)} \mathcal{Y}_I}^{\text{weak}}$$

$$\mathcal{A}_{\geq t} \subsetneqq \mathcal{A}_{\geq t'}, \text{ for } t > t'$$

$$\mathcal{A} = \overline{\bigvee_{t \in \mathbb{R}} \mathcal{A}_{\geq t}}^{\|\cdot\|}$$

$$\rho_\omega(t) = e^{iH(t-t')} \rho_\omega(t') e^{-iH(t-t')}$$

$$\tilde{\rho}_\omega(t) = \frac{1}{\mathrm{Tr}(\tilde{\rho}_\omega \pi_a(t))} \pi_a(t) \rho_\omega \pi_a(t)$$

$$\{\pi_\xi \mid \xi \in \mathcal{X}\} \subset \mathcal{A}_{\geq t}$$

$$\pi_\xi \pi_\eta = \delta_{\xi\eta} \pi_\xi; \ \pi_\xi^* = \pi_\xi = \pi_\xi^2; \ \sum_{\xi \in \mathcal{X}} \pi_\xi = 1$$

$$\omega_t := \omega|_{\mathcal{A}_{\geq t}}$$

$$\omega_t(A) = \sum_{\xi \in \mathcal{X}} \omega(\pi_\xi A \pi_\xi) \ \forall A \in \mathcal{A}_{\geq t}$$

$$\mathcal{C}_{\omega_t}(\mathcal{A}_{\geq t}) = \{X \in \mathcal{A}_{\geq t} \mid \omega_t([X,A]) = 0; \ \forall A \in \mathcal{A}_{\geq t}\}$$

$$Z_{\omega_t}(\mathcal{A}_{\geq t}) = \mathcal{C}_{\omega_t}(\mathcal{A}_{\geq t}) \cap \mathcal{C}_{\omega_t}(\mathcal{A}_{\geq t})' = \{Y \in \mathcal{C}_{\omega_t}(\mathcal{A}_{\geq t}) \mid [Y,X] = 0; \ \forall X \in \mathcal{C}_{\omega_t}(\mathcal{A}_{\geq t})\}$$

$$0 < \omega(\pi_i) < 1; \ i = 1,2$$

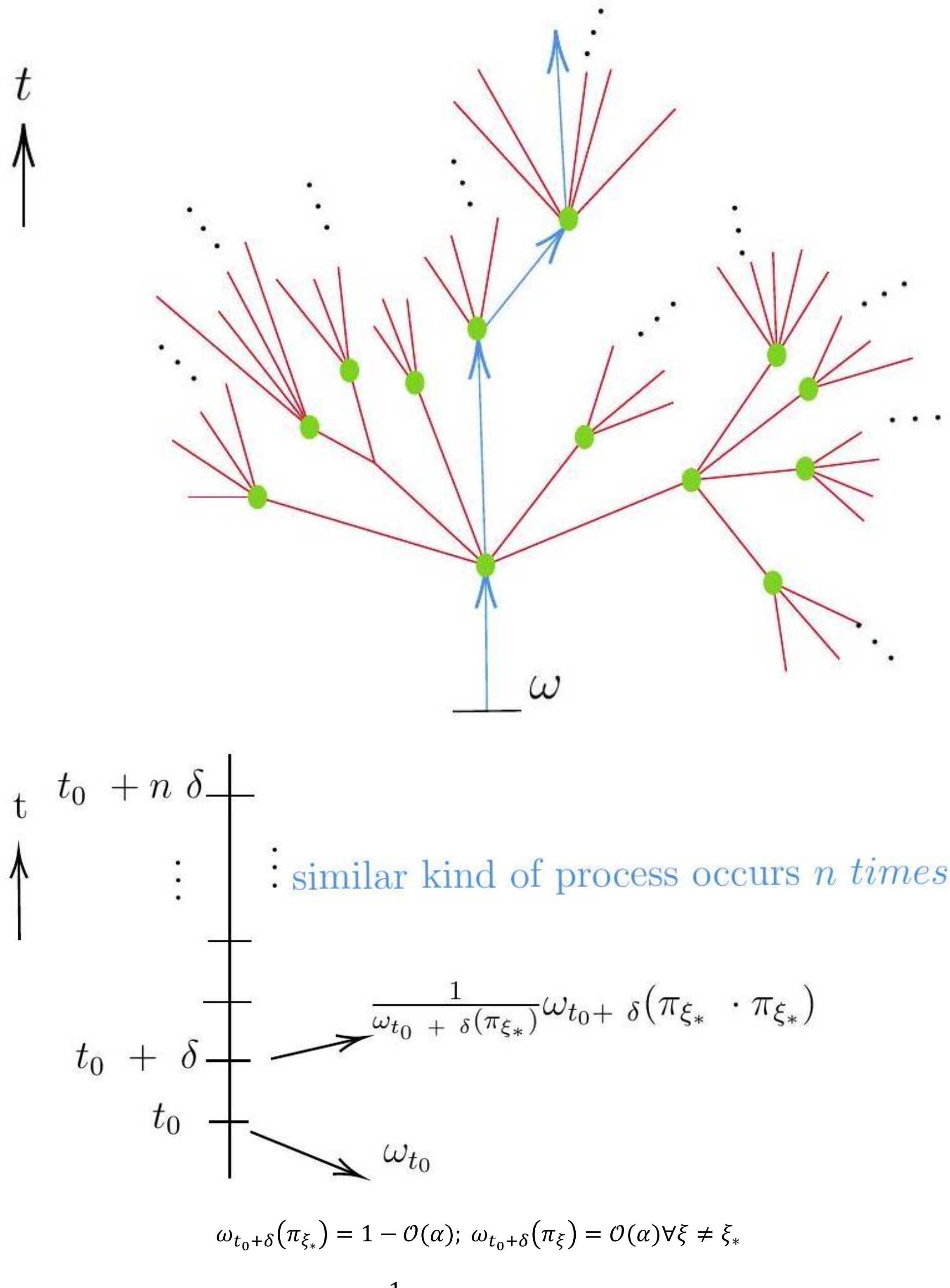
$$\{\pi_\xi \mid \xi \in \mathcal{X}_{\omega_t}\} :: Z_{\omega_t(\mathcal{A}_{\geq t})}$$

$$\begin{aligned}
\omega_t(A) &= \sum_{\xi,\eta \in \mathcal{X}_{\omega_t}} \omega(\pi_\xi A \pi_\eta) \\
&= \sum_{\xi,\eta \in \mathcal{X}_{\omega_t}} \omega(\pi_\eta \pi_\xi A) = \sum_{\xi,\eta \in \mathcal{X}_{\omega_t}} \omega(\delta_{\eta\xi} \pi_\xi A) \\
&= \sum_{\xi \in \mathcal{X}_{\omega_t}} \omega(\pi_\xi A) = \sum_{\xi \in \mathcal{X}_{\omega_t}} \omega(\pi_\xi^2 A) \\
&= \sum_{\xi \in \mathcal{X}_{\omega_t}} \omega(\pi_\xi A \pi_\xi) \qquad \forall A \in \mathcal{A}_{\geq t}
\end{aligned}$$

$$\omega_{t+\epsilon,\xi_*}(\cdot) = \frac{1}{\omega_{t+\epsilon}(\pi_{\xi_*})} \omega_{t+\epsilon}(\pi_{\xi_*} \cdot \pi_{\xi_*})$$

$$P(\xi_*, t) = \omega_t(\pi_{\xi_*})$$

Figura. Comportamiento holográfico de una partícula supermasiva.



$$\omega_{t_0+\delta}(\pi_{\xi_*}) = 1 - \mathcal{O}(\alpha);\ \omega_{t_0+\delta}(\pi_{\xi}) = \mathcal{O}(\alpha) \forall \xi \neq \xi_*$$

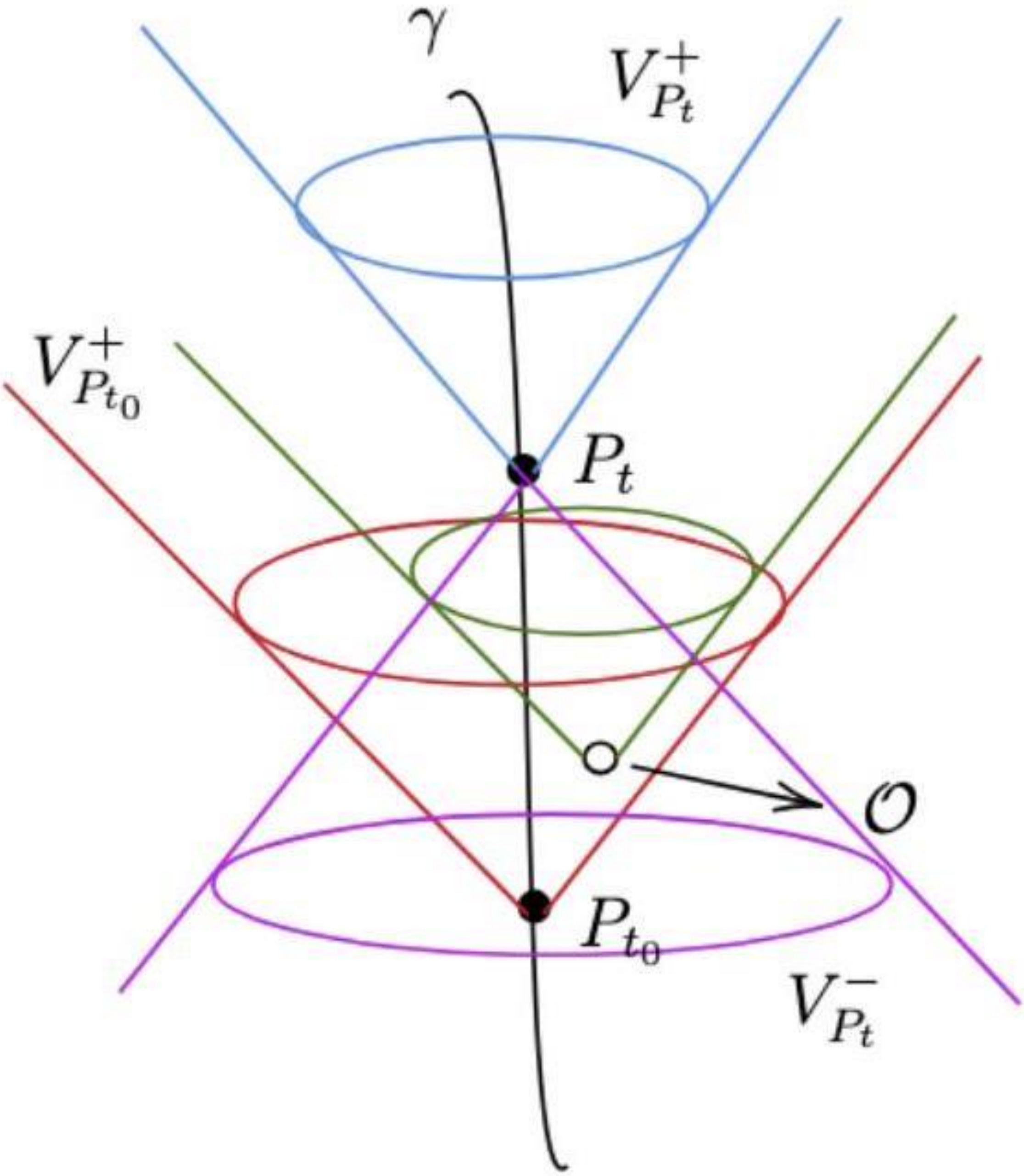
$$\frac{1}{\omega_{t_0+\delta}}\omega_{t_0+\delta}(\pi_{\xi_*} \cdot \pi_{\xi_*})$$

$$\omega_{t_0+\delta} \text{ and } \frac{1}{\omega_{t_0+\delta}(\pi_{\xi_*})}\omega_{t_0+\delta}(\pi_{\xi_*} \cdot \pi_{\xi_*})$$

$$\frac{1}{\omega_{t_0+\delta}(\pi_{\xi_*,n})}\omega_{t_0+\delta}(\pi_{\xi_*,n}\cdot\pi_{\xi_*,n})$$

$$\mathcal{A}=\overline{\bigvee_{P\in\mathbb{M}^4}\mathcal{A}_P}$$

Figura. Deformación del espacio – tiempo cuántico por interacción de la partícula oscura o supermasiva en relación a los niveles adimensionales.



$$\mathcal{A}_{P_t}\subsetneqq\mathcal{A}_{P_{t_0}}, P_t, P_{t_0}$$

$$[\pi_\xi^P,\pi_\eta^{P'}]=0$$

$$A_\mu, \Phi^I, \psi^a_\alpha, \bar{\psi}_{\dot{\alpha}a}$$

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1}, \Phi^I \to U\Phi^I U^{-1}, \psi^a_\alpha \to U\psi^a_\alpha U^{-1}, \bar{\psi}_{\dot{\alpha}a} \to U\bar{\psi}_{\dot{\alpha}a}U^{-1}$$

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}(\partial A)^2 + A^4 + \cdots\right)$$

$$\mathfrak{H}_{\text{connected vacuum diagram}} \sim N^F (g^2)^{E-V}$$

$$\mathfrak{H}_{\text{connected vacuum diagram}} \sim (g^2 N)^{E-V} N^{F-E+V} \sim \lambda^{E-V} N^\chi \sim \lambda^{E-V} N^{2-2k}$$

$$\log Z = \sum_k N^{2-2k} f_k(\lambda)$$

$$Z'[J_i] = \int DA_\mu \int D\hat{\Phi}\exp\left\{i\int d^dx\mathcal{L}[A_\mu, \hat{\Phi}] + iN\int d^dx J_i \mathcal{O}_i\right\} = \int DA_\mu D\Phi\exp\left\{i\int d^dx N\mathrm{Tr}[\dots.]\right\}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle^c = \frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_n}\log Z'[J_i]\bigg|_{J_i=0}\frac{1}{(iN)^n} = \sum_k N^{2-2k-n} h_k(\lambda)$$

$$\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle^c_\beta = \sum_k N^{2-2k-n} p_n(k,\beta)$$

$$G^c_\beta(\tau_1, \dots, \tau_n) = \sum_{m_1,\cdots,m_n=-\infty}^{\infty} G^c_0(\tau_1 - m_1\beta, \dots, \tau_n - m_n\beta)$$

$$\langle \mathcal{O}_1\cdots\mathcal{O}_n\rangle^c_{\Omega,\beta} \sim N^{2-n}$$

$$\langle \mathcal{O}\rangle^c_{\Omega,\beta} \sim N, \langle \mathcal{O}_1\mathcal{O}_2\rangle^c_{\Omega,\beta} \sim \mathcal{O}(1), \langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_{n\geq 3}\rangle^c_{\Omega,\beta} \to 0 \text{ as } N \to \infty$$

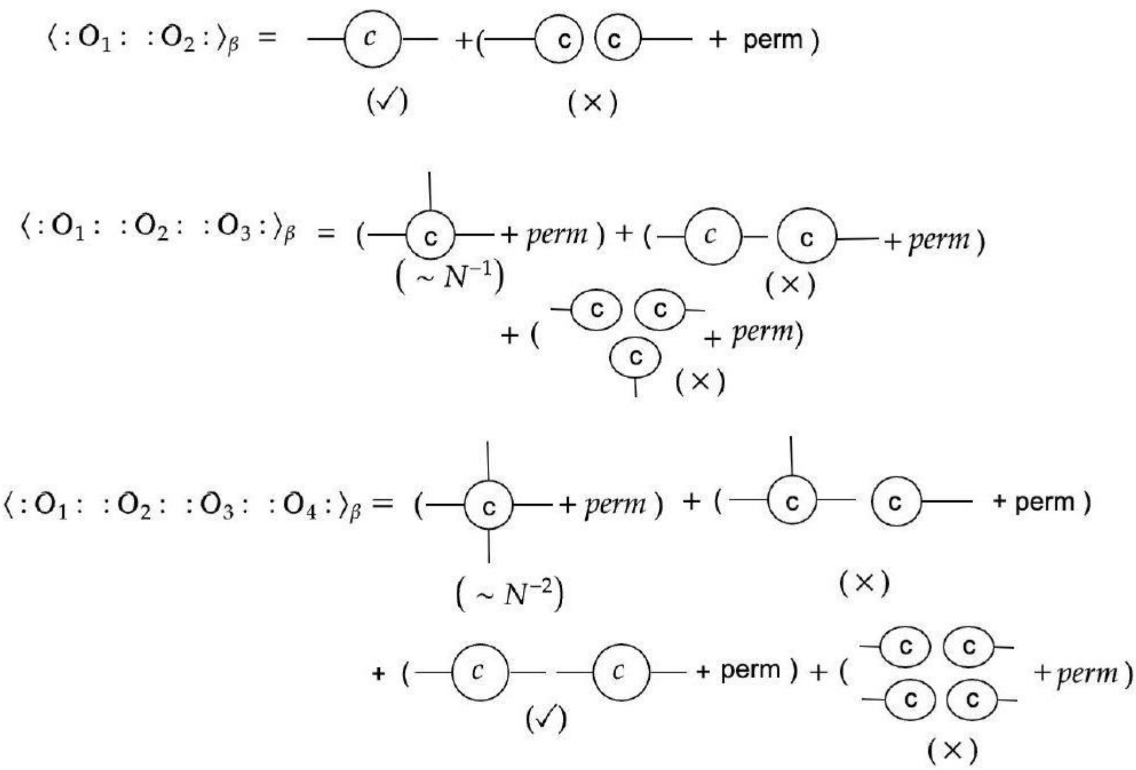
$$:\mathcal{O}:\,:= \mathcal{O} - \langle \mathcal{O}\rangle_\beta$$

$$\langle :\mathcal{O}: \rangle_\beta = 0$$

$$:\mathcal{O}:\,= \mathrm{Tr}\left[O - \langle \widetilde{\mathcal{O}}\rangle_\beta \mathbb{1}\right]$$

$$\langle :\mathcal{O}_1:\cdots:\mathcal{O}_n:\rangle_\beta = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\{i_1\cdots i_n\}\in S_n} \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\rangle_\beta \cdots \langle \mathcal{O}_{i_{n-1}}\mathcal{O}_{i_n}\rangle_\beta & \text{for } n \text{ even} \end{cases}$$

$$\omega_{TFD}(\cdot) = \frac{1}{Z_\beta}\mathrm{Tr}\left(e^{-\beta H}\cdot\right) = \langle TFD|\cdot|TFD\rangle$$



$$|TFD\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_i e^{-\beta E_i/2} |i\rangle_R |i\rangle_L$$

$$\tilde{O} = \frac{1}{N} \mathrm{Tr} O$$

$$0 \leq \omega_{TFD}\big((A+\mathbb{1})(A^*+\mathbb{1})\big) = \omega_{TFD}(AA^*) + 1 + \omega_{TFD}(A) + \omega_{TFD}(A^*)$$

$$0 \leq \omega_{TFD}\big((iA+\mathbb{1})(-iA^*+\mathbb{1})\big) = \omega_{TFD}(AA^*) + 1 + i\omega_{TFD}(A) - i\omega_{TFD}(A^*)$$

$$\omega_{TFD}(A) = \langle TFD|\pi_{TFD}(A)|TFD\rangle$$

$$\langle A \mid B\rangle := \omega_{TFD}(A^*B)$$

$$\mathcal{N} = \{A \in \mathcal{A}_R \mid \omega_{TFD}(A^*A) = 0\}$$

$$\langle A \mid B\rangle = 0 \; \forall B \in \mathcal{A}_R, \forall A \in \mathcal{N}$$

$$0 \leq |\omega_{TFD}(A^*B)|^2 = |\langle A \mid B\rangle|^2 \leq \|A\|^2 \|B\|^2 = 0$$

$$\mathcal{H}_{TFD} = \overline{\mathcal{A}_R/\mathcal{N}} = \bar{D}$$

$$D = \{[A] \in \mathcal{H}_{TFD} \mid A \in \mathcal{A}_R\}$$

$$\pi_{TFD}(A)[B] := [AB] \;\; \text{on } D$$

$$\|AB\|^2 = \omega_{TFD}((AB)^*AB) = \omega_{TFD}(B^*A^*AB) = \langle B \mid A^*AB\rangle = 0$$

$$\pi_{TFD}[A]\pi_{TFD}[B][C] = [ABC] = \pi_{TFD}(AB)[C]$$

$$\langle B|\pi_{TFD}(A)^*|C\rangle = \langle AB\rangle C = \omega_{TFD}(B^*A^*C) = \langle B \mid A^*C\rangle = \langle B \mid \pi_{TFD}(A^*)C\rangle$$

$$\begin{aligned}\langle U\pi_{TFD}(A)TFD \mid U\pi_{TFD}(B)TFD\rangle_{\mathcal{H}'_{TFD}} &= \langle \pi'_{TFD}(A)TFD' \mid \pi'_{TFD}(B)TFD'\rangle \\ &= \omega_{TFD}(A^*B) \\ &= \langle \pi_{TFD}(A)TFD \mid \pi_{TFD}(B)TFD\rangle_{\mathcal{H}_{TFD}}\end{aligned}$$

$$\mathcal{M}_R = \pi_{TFD}(\mathcal{A}_R)''$$

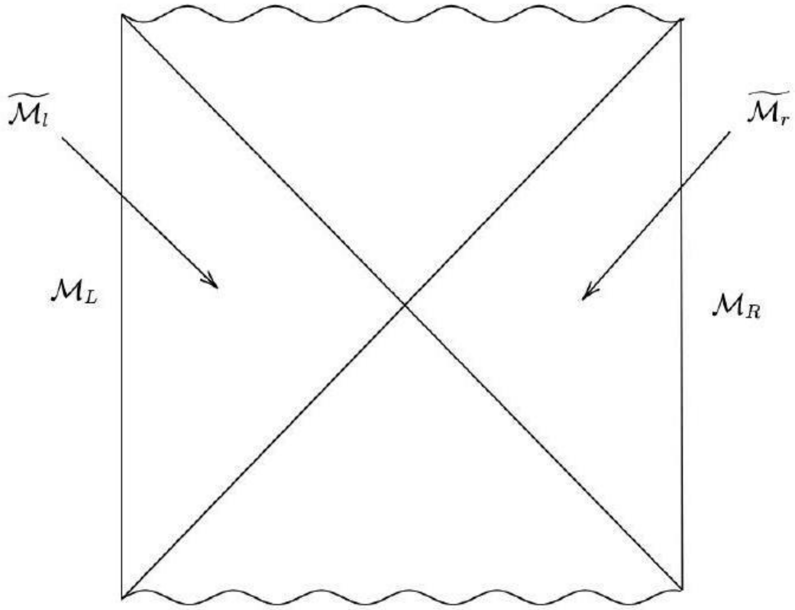
$$G_E(\tau,\omega) = \sum_{m=-\infty}^{\infty} \rho_0(\omega)e^{-\omega|\tau-m\beta|} = \rho_0(\omega)\left(\frac{e^{-\omega\tau}+e^{\omega(\tau-\beta)}}{1-e^{-\omega\beta}}\right)$$

$$G_E(\tau,\omega) = \rho(\omega)\left(\frac{e^{-\omega\tau}+e^{\omega(\tau-\beta)}}{1-e^{-\omega\beta}}\right)$$

$$\rho(\omega) = \theta(\omega)\rho_0(\omega) - \theta(-\omega)\rho_0(-\omega)$$

$$\rho(\omega) = \left(1-e^{-\beta\omega}\right)\sum_{m.n} \delta(\omega - E_{nm})e^{-\beta E_m}\rho_{mn}$$

$$\rho_0(\omega) \propto \theta(\omega)\sum_{l=0}^{\infty} c_l\delta(\omega-\Delta-2l) - \theta(-\omega)\sum_{l=0}^{\infty} c_l\delta(\omega+\Delta+2l)$$



$$\mathcal{M}_R = \widetilde{\mathcal{M}_r}, \mathcal{M}_L = \widetilde{\mathcal{M}_l}$$

$$\mathcal{X}_{\mathcal{U}} = \pi_\Psi\left(\lim_{N\to\infty,\Psi}\mathcal{P}_{\mathcal{U}}\mathcal{B}^N\right) \supseteq \mathcal{P}_{\hat{\mathcal{U}}}\pi_\Psi\left(\lim_{N\to\infty}\mathcal{B}^N\right) = \mathcal{Y}_{\hat{\mathcal{U}}}$$

$$\hat{\mathcal{O}} = \mathcal{O} - \langle\mathcal{O}\rangle_\Omega$$

$$H = \frac{N}{\lambda}\int_{S^3} d^3x\sqrt{g}\mathrm{Tr}\left[F_{0i}F_{oj}g^{ij}+\cdots\right] = N\mathrm{Tr}[\cdots]$$

$$A(t) = e^{i(t-t')H}A(t')e^{-i(t-t')H}$$

$$\mathcal{A}(\mathcal{O}) = \{\Phi(f) \mid \mathrm{supp}(f) \in \mathcal{O}\}$$

$$[\phi(x), \phi(y)] = \Delta_+(x,y) - \Delta_-(x,y) = \Delta(x,y)$$

$$P_x\Delta_{+/-}(x,y) = \delta(x-y), P_x\Delta(x,y) = 0$$

$$\Delta_f(x) = \int d^d y \Delta(x,y) f(y)$$

$$\mathrm{Supp}(\Delta_f) \subset J^+(\mathrm{Supp}(f)) \bigcup J^-(\mathrm{Supp}(f))$$

$$\Delta(f,g) = \int d^d x \int d^d y f(x)\Delta(x,y)g(y), f,g \in C_0^\infty(\mathbb{M}^d)$$

$$0 \to C_0^\infty(\mathbb{M}^d) \xrightarrow{P} C_0^\infty(\mathbb{M}^d) \xrightarrow{\Delta} C_{sc}^\infty(\mathbb{M}^d) \xrightarrow{P} C_{sc}^\infty(\mathbb{M}^d) \to 0$$

$$C_0^\infty(\mathbb{M}^d)/\mathrm{Im}P = C_0^\infty(\mathbb{M}^d)/\ker\Delta = \mathrm{Im}\Delta = \ker P = \{g \in C_{sc}^\infty(\mathbb{M}^4) \mid Pg = 0\} = \mathcal{E}_{sc}(\mathbb{M}^d) = \mathrm{Sol}(\mathbb{M}^d)$$

$$\Delta_{Pf}(x) = \int d^d x \int d^d y \Delta(x,y)(P_y f)(y) = \int d^d x \int d^d y (P_y\Delta(x,y)) f(y) = 0$$

$$f_U = P\chi g$$

$$h - f_U = P(\Delta_+ h - \chi\Delta h) = P((1-\chi)\Delta_+ h + \chi\Delta_- h)$$

$$\mathrm{Supp}((1-\chi)\Delta_+ h) \subset J^-(\Sigma_1) \bigcap J^+(\mathrm{Supp}(h)), \mathrm{Supp}(\chi\Delta_- h) \subset J^+(\Sigma_1) \bigcap J^-(\mathrm{Supp}(h))$$

$$\Phi(h) = \Phi(f_U) + \Phi(\Delta_+ h - \chi\Delta h) = \Phi(f_U)$$

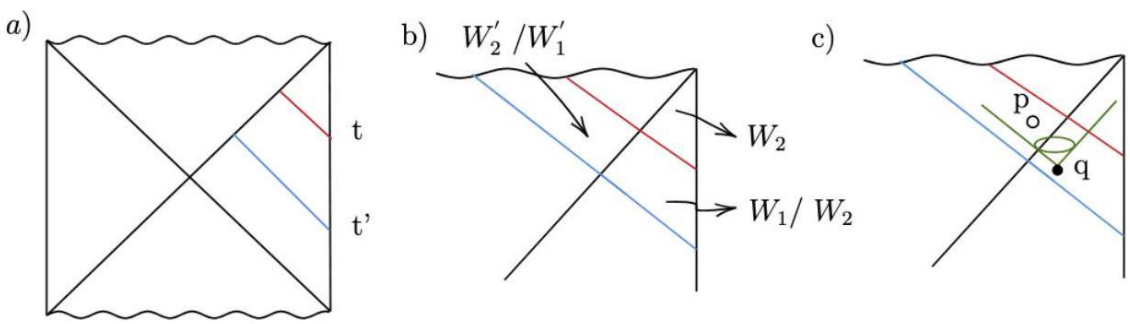
$$\mathcal{A}(\mathbb{M}^d) = \mathcal{A}(U)$$

$$\mathcal{A}(U) = \mathcal{A}(V)$$

$$A(u) = \Delta_{TFD}^{-iu} A \Delta_{TFD}^{iu}, \Delta_{TFD}^{-iu}\mathcal{M}_R\Delta_{TFD}^{iu} = \mathcal{M}_R \ \forall u \in \mathbb{R}$$

$$\mathcal{A} = \overline{\cup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}$$

$$\mathcal{M}_{I\times S^3} = \mathcal{M}|_{I\times S^3} \equiv \mathcal{M}_I$$



$$\mathcal{M}_{(t,\infty)\times S^3} := \mathcal{M}_{\geq t} \subsetneqq \mathcal{M}_{\geq t'} := \mathcal{M}_{(t',\infty)\times S^3}, \forall t > t'$$

$$\widetilde{\mathcal{M}}_{r,\geq t} = \widetilde{\mathcal{M}}(W_2), \widetilde{\mathcal{M}}_{r,\geq t'} = \widetilde{\mathcal{M}}(W_1)$$

$$\widetilde{\mathcal{M}}(W_2) \subsetneqq \widetilde{\mathcal{M}}(W_1)$$

$$\mathcal{M}_{R,\geq t} \subsetneqq \mathcal{M}_{R,\geq t'}, \text{ for } t > t'$$

$$\mathcal{M}'_{I=\left(-\frac{\mathcal{T}}{2},\frac{\mathcal{T}}{2}\right)} \cap \mathcal{M}$$

$$\omega_\Psi([\mathcal{O}(g),\mathcal{O}(f)]) = \int dtdt' g(t) f(t') \langle\Psi|[\mathcal{O}(t),\mathcal{O}(t')]|\Psi\rangle = \int dtdt' g(t)\rho(t,t')f(t')$$

$$\mathcal{O}(f) = \int dt\mathcal{O}(t)f(t), \text{Supp}(f) \in I$$

$$\int dtdt' g(t)\rho(t,t')f(t) = 0, \text{ for all } f \text{ with Supp}(f) \in I$$

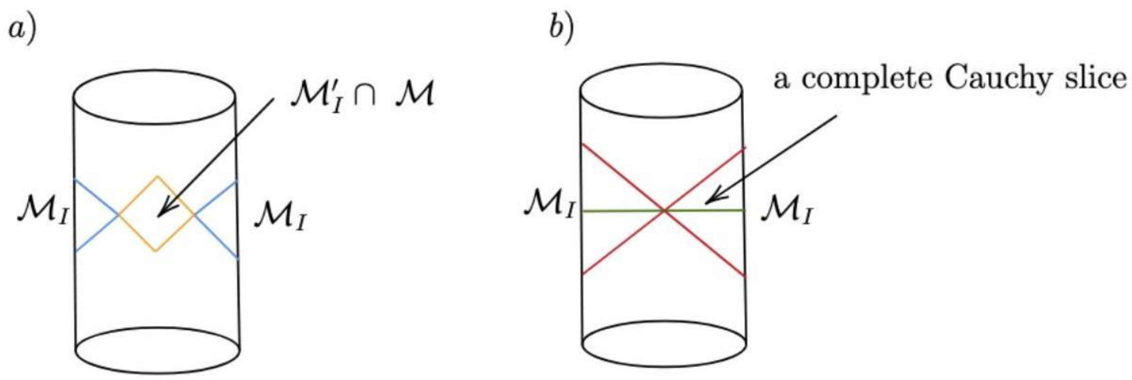
$$\text{Supp}(g * \rho) \in I^c$$

$$(g * \rho)(\omega) = -g(\omega)\rho(\omega)$$

$$\int d\omega g(\omega)\rho(\omega)f(-\omega) = 0$$

$$\rho(\omega) = \rho_0(\omega) = \sum_{n=-\infty}^{\infty} a_n \delta(\omega - 2n - \Delta)$$

$$(g * \rho)(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta(\omega - 2n - \Delta)$$



$$\mathcal{M}_{\geq t} = \mathcal{M}_{\geq t'}, \forall t, t' \in \mathbb{R}$$

$$\mathcal{C}_{TFD}(\mathcal{M}_R) = \{A \in \mathcal{M}_R \mid \omega_{TFD}([A,B]) = 0, \forall B \in \mathcal{M}_R\}$$

$$\mathcal{C}_{\text{fixed}}(\mathcal{M}_R) = \{A \in \mathcal{M}_R \mid \sigma_u^{TFD}(A) = A, \forall u \in \mathbb{R}\}$$

$$F_{AB}(t) = \omega(A\alpha_t(B))$$

$$F_{AB}(t+i\beta)=\omega(\alpha_t(B)A)$$

$$\sigma_u^{\omega}(\mathcal{A})=\Delta_{\omega}^{-iu}\mathcal{A}\Delta_{\omega}^{iu}=\mathcal{A}\ \forall u\in\mathbb{R}$$

$$G_{AB}(u)=\omega(A\sigma_u^{\omega}(B))$$

$$G_{AB}(u+i)=\omega(\sigma_u^{\omega}(B)A)$$

$$-\log\,\Delta_{\omega}=h_{\omega}=\beta H$$

$$\sigma_u^{\omega}(B)=e^{ih_{\omega}u}Be^{-ih_{\omega}u}=e^{iH(\beta u)}Be^{-iH(\beta u)}=\alpha_{t=\beta u}(B)$$

$$h_{TFD}=\beta(H_R-H_L)$$

$$\begin{aligned}\omega_{TFD}(\sigma_u^{TFD}(A)B)&=\omega_{TFD}(A\sigma_{-u}^{TFD}(B))\\&=\omega_{TFD}(\sigma_{-u}^{TFD}(B)A)\\&=\omega_{TFD}\big(A\sigma_{-u+i}^{TFD}(B)\big)\\&=\omega_{TFD}\big(\sigma_{u-i}^{TFD}(A)B\big)\end{aligned}$$

$$\omega_{TFD}(AB)=\omega_{TFD}(\sigma_u^{TFD}(A)B)=\omega_{TFD}\big(B\sigma_{u+i}^{TFD}(A)\big)$$

$$\omega_{TFD}(AB)=\omega_{TFD}\big(B\sigma_{u+i}^{TFD}(A)\big)=\omega_{TFD}(B\sigma_0^{TFD}(A))=\omega_{TFD}(BA)$$

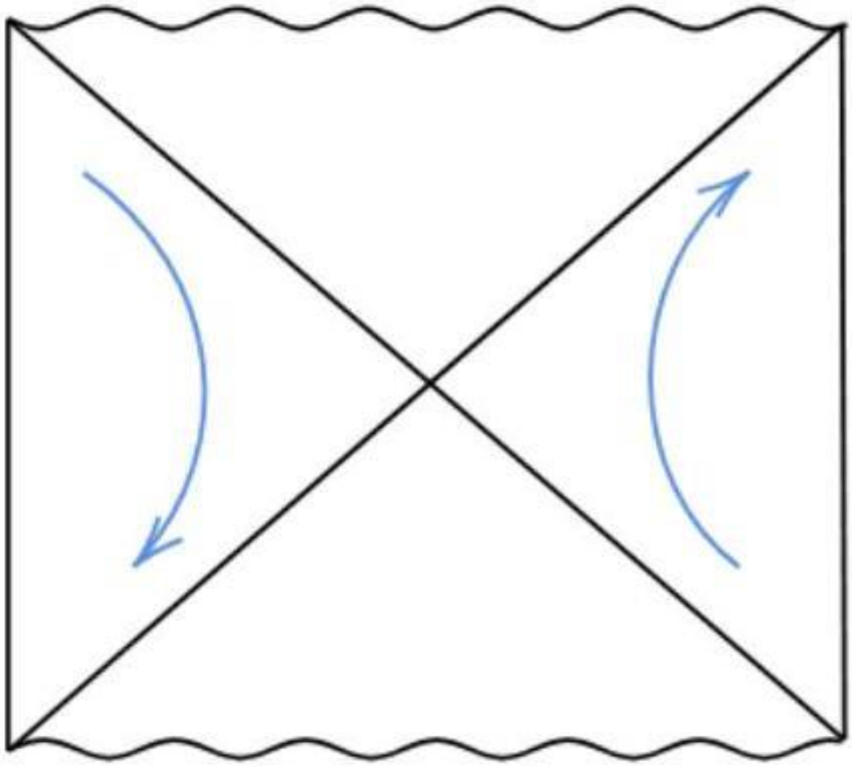
$$\mathcal{C}_{\text{fixed}}\,(\mathcal{M}_R,TFD)\subset\mathcal{C}_{TFD}(\mathcal{M}_R)$$

$$-\log\,\Delta_{TFD}=\frac{2\pi}{\kappa}(H_R-H_L)=h_r-h_l=\hat{h}$$

$$h_r=\int_{\Sigma_r}d\Sigma^{\mu}V^{\nu}T_{\mu\nu},h_l=-\int_{\Sigma_l}d\Sigma^{\mu}V^{\nu}T_{\mu\nu}$$

$$\mathcal{C}_{\text{fixed}}\,(\mathcal{M}_R,TFD)=\mathbb{C}\mathbb{1}$$

Figura. Curvatura paralela.



$$\mathcal{N}_R = \mathcal{M}_R \rtimes \mathcal{A}_{X+h_{TFD}} = \{\mathcal{M}_R, X + h_{TFD}\}''$$

$$\mathcal{N}_R' = \mathcal{N}_L = e^{iPh_{TFD}}\mathcal{M}_L e^{-iPh_{TFD}} \rtimes \mathcal{A}_X = \left\{e^{iPh_{TFD}}\mathcal{M}_L e^{-iPh_{TFD}}, X\right\}''$$

$$\mathcal{N}_R = e^{-iPh_{TFD}/2}\mathcal{M}_R e^{iPh_{TFD}/2} \rtimes \mathcal{A}_{X+\frac{h_{TFD}}{2}}, \mathcal{N}_L = e^{iPh_{TFD}/2}\mathcal{M}_L e^{-iPh_{TFD}/2} \rtimes \mathcal{A}_{X-\frac{h_{TFD}}{2}}$$

$$|TFD, g\rangle = |TFD\rangle \otimes g(X)$$

$$\int dX|g(X)|^2 = 1$$

$$g(X) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-X^2/4\sigma^2}$$

$$\begin{aligned}\hat{\omega}\left(e^{is(h_{TFD}+X)}\left(ae^{iu(h_{TFD}+X)}\right)\right) &= \omega_{TFD}\left(e^{ish_{TFD}}ae^{iuh_{TFD}}\right)\int dX|g(X)|^2 e^{i(u+s)X} \\ &= \omega_{TFD}\left(ae^{i(u+s)h_{TFD}}\right)\int dX|g(X)|^2 e^{i(u+s)X} \\ &= \hat{\omega}\left(\left(ae^{iu(h_{TFD}+X)}\right)e^{is(h_{TFD}+X)}\right)\end{aligned}$$

$$\mathcal{C}_{TFD,g}(\mathcal{N}_R) \supseteq \left\{e^{is(h_{TFD}+X)}, \forall s \in \mathbb{R}\right\}$$

$$\mathcal{N}_{R,I} = \mathcal{M}_{R,I} \rtimes \mathcal{A}_{h_{TFD}+X}$$

$$e^{-is(h_{TFD}+X)}\mathcal{N}_{R,\geq t}e^{is(h_{TFD}+X)} = \mathcal{N}_{R,\geq t-s} = \mathcal{M}_{R,\geq t-s} \rtimes \mathcal{A}_{h_{TFD}+X}$$

$$\mathcal{N}_{R,\geq t} = \mathcal{N}_{R,\geq t-s} = \mathcal{N}_R, \forall s > 0$$

$$G = (\mathrm{Spin}(4) \times SU(4)_R)/\mathbb{Z}_2$$

$$\hat{Q}^a = Q_R^a - Q_L^{*a}$$

$$\hat{Q}^a|TFD\rangle = 0$$

$$G_{\mathcal{M}} = \frac{G_L \times G_R}{G_D} \cong G$$

$$g \to g_L^{-1} g g_R$$

$$G_g = \{g_L \times g_R \in G_L \times G_R \mid g_L^{-1} g g_R = g \text{ or } g_R = g g_L g^{-1}\} \cong G_D$$

$$\hat{\mathcal{H}} = \Gamma_{L^2}(\mathcal{V})$$

$$\hat{\mathcal{H}} = \mathcal{H}_{TFD} \otimes L^2(G_{\mathcal{M}})$$

$$L_{g_L,g_R}(\Psi_L(g)) = W(g_R)\Psi_L(g_L^{-1} g g_R)$$

$$R_{g_L,g_R}(\Psi_R(g)) = W(g_L)\Psi_R(g_L^{-1} g g_R)$$

$$\Psi_R(g) = (\Lambda\Psi_L)(g) = W(g)\Psi_L(g), R_{g_L,g_R} = \Lambda L_{g_L,g_R}\Lambda^{-1}$$

$$\mathcal{Y}_R = \widetilde{\mathcal{M}}_r \rtimes G = \mathcal{M}_R \rtimes G$$

$$|\widetilde{TFD}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_i e^{-\beta E_i/2} g(R_i)|i\rangle_R |i\rangle_L$$

$$\mathcal{Y}_{R,I} = \mathcal{M}_{R,I} \rtimes G$$

$$\mathcal{Y}_{R,\geq t} = \mathcal{M}_{R,\geq t} \rtimes G \subsetneq \mathcal{M}_{R,\geq t'} = \mathcal{Y}_{R,\geq t'} \quad \text{for } t > t'$$

$$|\Psi_{\hat{\omega}}\rangle = |TFD\rangle \otimes f(g) \equiv |TFD, f\rangle$$

$$\int_G d\mu(g)|f(g)|^2 = 1$$

$$w(g_1)f(g) = f(gg_1)$$

$$\big(aW(g_1)w(g_1)\big) \circ \big(bW(g_2)w(g_2)\big) = aW(g_1)bW(g_2)w(g_1)w(g_2) = a\sigma_g(b)W(g_1g_2)w(g_1g_2)$$

$$\begin{aligned}\hat{\omega}\Big(\big(W(g_1)w(g_1)\big)\big(aW(g_2)w(g_2)\big)\Big) &= \omega_{TFD}\big(W(g_1)aW(g_2)\big)\int_G d\mu(g)\bar{f}(g)f(gg_1g_2) \\ &= \omega_{TFD}\big(aW(g_2)W(g_1)\big)\int_G d\mu(g)\bar{f}(g)f(gg_1g_2) \\ &\overset{if[g_1,G]=0}{=} \hat{\omega}\Big(\big(aW(g_2)w(g_2)\big)\big(W(g_1)w(g_1)\big)\Big)\ \forall g_2 \in G\end{aligned}$$

$$\mathcal{C}_{TFD,f}(\mathcal{Y}_R) = \mathbb{C}\mathbb{1}$$

$$\mathfrak{h} = \{J_1, J_2, R_1, R_2, R_3\}$$

$$\mathcal{Y}_R^H = \mathcal{M}_R \rtimes H$$

$$\hat{J}^i = J_R^i - J_L^i, \hat{R}^\alpha = R_R^\alpha - R_L^\alpha$$

$$\hat{\mathcal{H}}^H = \mathcal{H}_{TFD} \otimes L^2(H)$$

$$\int_H d\mu(h)|k(h)|^2 = 1$$

$$[aW(h_1)w(h_1), bW(h_2)w(h_2)] = \left(a\sigma_{h_1}(b) - b\sigma_{h_2}(a)\right)W(h_1h_2)w(h_1h_2), \forall b \in \mathcal{M}_R, \forall h_2 \in H$$

$$\omega_{TFD,k}([aW(h_1)w(h_1), bW(h_2)w(h_2)]) = \omega_{TFD}\left(a\sigma_{h_1}(b) - \sigma_{h_2^{-1}}(b)a\right)\int_H d\mu(h)\bar{k}(h)k(hh_1h_2)$$

$$\mathcal{C}_{TFD,k}(\mathcal{Y}_R^H) = \{W(h)w(h) \mid h \in H\}$$

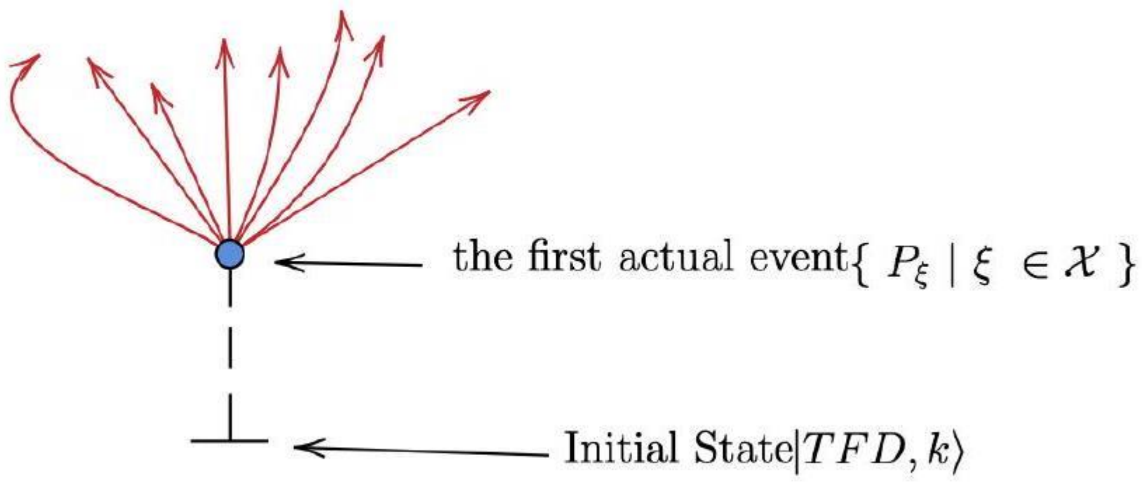
$$Z_{TFD,k}(\mathcal{Y}_R^H) = \mathcal{C}_{TFD,k}(\mathcal{Y}_R^H)$$

$$\tilde{J}^i = \int_{\lambda\in\mathbb{R}} \tilde{\jmath}^i(\lambda)dP(\lambda), \tilde{R}^\alpha = \int_{\lambda\in\mathbb{R}} \tilde{r}^\alpha(\lambda)dP(\lambda)$$

$$P_\xi = \int_{\lambda\in\mathbb{R}} \chi_\xi(\lambda)dP(\lambda)$$

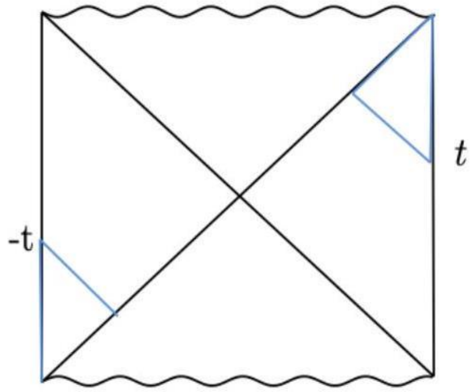
$$\chi_\xi = \begin{cases} 1 & \text{if} \quad \lambda \in I_\xi \\ 0 & \text{if} \quad \lambda \notin I_\xi \end{cases}$$

$$|\Psi(\xi_*)\rangle = \frac{1}{\langle TFD, k|P_{\xi_*}|TFD, k\rangle} P_{\xi_*}|TFD, k\rangle$$



$$\mathcal{C}_{TFD,k}((\mathcal{Y}_R^H \vee \mathcal{Y}_L^H)'') = \{W(h_R)w(h_R), w(h_L) \mid h_R \in H_R, h_L \in H_L\}$$

$$w(h_L)f(h) = f(h_L^{-1}h), f(h) \in L^2(H)$$



$$\Phi(\vec{g},\boldsymbol{\chi}) = \Phi(\vec{g}\cdot{}_{\mathrm{d}}h,\boldsymbol{\chi}), \forall h \in \mathrm{SU}(2)$$

$$\Phi(\vec{g},\boldsymbol{\chi}) = \sum_{\iota}\sum_{\vec{j}}\sum_{\vec{m},\vec{n}} \Phi^{j_1,\ldots,j_4;\iota}_{m_1,\ldots,m_4}(\boldsymbol{\chi})\left[\prod_{i=1}^{4}\sqrt{d(j_i)}D^{j_i}_{m_i n_i}(g_i)\mathcal{I}^{j_1,\ldots,j_4;\iota}_{n_1,\ldots,n_4}\right.$$
$$\equiv \sum_{\vec{\kappa}} \Phi_{\vec{\kappa}}(\boldsymbol{\chi})\psi_{\vec{\kappa}}(\vec{g})$$

$$\left[\hat{\varphi}(\vec{g},\boldsymbol{\chi}),\hat{\varphi}^{\dagger}(\vec{g}',\boldsymbol{\chi}')\right] = \mathbb{1}_G(\vec{g},\vec{g}')\delta(\boldsymbol{\chi}-\boldsymbol{\chi}')$$
$$[\hat{\varphi}(\vec{g},\boldsymbol{\chi}),\hat{\varphi}(\vec{g}',\boldsymbol{\chi}')] = \left[\hat{\varphi}^{\dagger}(\vec{g},\boldsymbol{\chi}),\hat{\varphi}^{\dagger}(\vec{g}',\boldsymbol{\chi}')\right] = 0$$

$$\left[\hat{\varphi}_{\vec{\kappa}}(\boldsymbol{\chi}),\hat{\varphi}^{\dagger}_{\vec{\kappa}'}(\boldsymbol{\chi}')\right] = \delta_{\vec{\kappa},\vec{\kappa}'}\delta(\boldsymbol{\chi}-\boldsymbol{\chi}')$$
$$[\hat{\varphi}_{\vec{\kappa}}(\boldsymbol{\chi}),\hat{\varphi}_{\vec{\kappa}'}(\boldsymbol{\chi}')] = \left[\hat{\varphi}^{\dagger}_{\vec{\kappa}}(\boldsymbol{\chi}),\hat{\varphi}^{\dagger}_{\vec{\kappa}'}(\boldsymbol{\chi}')\right] = 0$$

$$|\vec{\kappa};\chi\rangle \equiv \hat{\varphi}^{\dagger}_{\vec{\kappa}}(\chi)|0\rangle,$$

$$|\sigma\rangle = \mathcal{N}_{\sigma}\exp\left[\int \mathrm{d}\vec{g}\ \mathrm{d}\boldsymbol{\chi}\sigma(\vec{g},\boldsymbol{\chi})\hat{\varphi}^{\dagger}(\vec{g},\boldsymbol{\chi})\right]|0\rangle, |\mathcal{N}_{\sigma}|^2 = \exp\left[-\int \mathrm{d}\vec{g}\ \mathrm{d}\boldsymbol{\chi}|\sigma(\vec{g},\boldsymbol{\chi})|^2\right]$$

$$\sigma(\vec{g},\boldsymbol{\chi}) = \sigma(h\cdot \mathrm{d}\vec{g},\boldsymbol{\chi}), \forall h \in \mathrm{SU}(2)$$

$$\sigma_{\kappa}(\boldsymbol{\chi}) = \sigma_j(\boldsymbol{\chi})(\mathcal{I}^*)^{jjjj,\iota_+}_{m_1m_2m_3m_4}$$

$$\hat{O}^{(n,m)} = \int \left[\prod_{i=1}^{n} \mathrm{d}\vec{g}_i\ \mathrm{d}\boldsymbol{\chi}_i\hat{\varphi}^{\dagger}(\vec{g}_i,\boldsymbol{\chi}_i)\right]\left[\prod_{j=1}^{m} \mathrm{d}\vec{g}'_j\mathrm{d}\boldsymbol{\chi}'_j\hat{\varphi}(\vec{g}'_j,\boldsymbol{\chi}'_j)\right]O^{(n,m)}$$

$$\hat{N} = \int \mathrm{d}\vec{g}\ \mathrm{d}\boldsymbol{\chi}\hat{\varphi}^{\dagger}(\vec{g},\boldsymbol{\chi})\hat{\varphi}(\vec{g},\boldsymbol{\chi})$$
$$\hat{V} = \int \mathrm{d}\vec{g}\ \mathrm{d}\vec{g}'\mathrm{d}\boldsymbol{\chi}\hat{\varphi}^{\dagger}(\vec{g},\boldsymbol{\chi})\mathcal{V}(\vec{g},\vec{g}')\hat{\varphi}(\vec{g}',\boldsymbol{\chi})$$
$$\hat{X}^a = \int \mathrm{d}\vec{g}\ \mathrm{d}\boldsymbol{\chi}\chi^a\hat{\varphi}^{\dagger}(\vec{g},\boldsymbol{\chi})\hat{\varphi}(\vec{g},\boldsymbol{\chi})$$

$$\sigma_{\epsilon,\pi_0}(\vec{g},\chi,\boldsymbol{\psi}) = \eta_{\epsilon}(\chi-\chi_0,\pi_0)\tilde{\sigma}(\vec{g},\chi,\boldsymbol{\psi})$$

$$\eta_{\epsilon}(\chi-\chi_0,\pi_0) = \mathcal{N}_{\varepsilon}\exp\left\{\left(-\frac{(\chi-\chi_0)^2}{2\varepsilon}\right)\right\}\exp\left\{\left(-\mathrm{i}\pi_0(\chi-\chi_0)\right)\right\}$$

$$\langle\hat{O}\rangle_{\sigma_\epsilon,\pi_\psi} = O[\tilde{\sigma}](\chi_0)$$

$$S = K + U + U^*$$

$$S_\chi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$
$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + V(\psi)\right)$$

$$S_\psi^T = \sum_l \frac{\tilde{V}_l}{2}\left(\frac{\psi_{v_l} - \psi_{v_l'}}{L}\right)^2 + \sum_v V_v V(\psi)$$

$$\mathcal{K}_\psi = \exp\left\{\left(i\frac{\tilde{V}_l}{2}\left(\frac{\psi_{v_l} - \psi_{v_l'}}{L}\right)^2\right)\right\}, \mathcal{U}_\psi = \exp\{(\mathrm{i}V_v V(\psi))\}$$

$$K = \int \mathrm{d}\vec{g}\,\mathrm{d}\vec{h}\int \mathrm{d}\chi\,\mathrm{d}\chi'\int \mathrm{d}\psi\,\mathrm{d}\psi'\varphi^*(\vec{g},\chi,\psi)\times$$
$$\times\mathcal{K}(\vec{g},\vec{h};(\chi-\chi')^2;(\psi-\psi')^2)\varphi(\vec{h},\chi',\psi')$$

$$K = \int \mathrm{d}\vec{g}\,\mathrm{d}\vec{h}\int \mathrm{d}\chi\,\mathrm{d}\chi'\int \mathrm{d}\pi_\psi\varphi^*(\vec{g},\chi,\pi_\psi)\mathcal{K}(\vec{g},\vec{h};(\chi-\chi')^2;\pi_\psi)\varphi(\vec{h},\chi',\pi_\psi)$$

$$\mathcal{K}(\vec{g},\vec{h};\chi^2;\pi_\psi) = \sum_{n=0}^{\infty}\frac{\mathcal{K}^{(2n)}(\vec{g},\vec{h};\pi_\psi)}{(2n)!}\chi^{2n}$$

$$U_{\mathrm{S}}[\varphi] = \int \mathrm{d}\chi\,\mathrm{d}\psi\int\left(\prod_{a=1}^{l+1}\mathrm{d}\vec{g}_a\right)\mathcal{U}_{\mathrm{S}}(\vec{g}_1,\dots,\vec{g}_{l+1},\psi)\prod_{a=1}^{l+1}\varphi(\vec{g}_a,\chi,\psi)$$

$$U_{\mathrm{S}} = \int \mathrm{d}\chi\,\mathrm{d}\pi_\psi\int\left(\prod_{a=1}^{l+1}\mathrm{d}\vec{g}_a\right)\mathcal{U}_{\mathrm{S}}(\vec{g}_1,\dots,\vec{g}_{l+1},\pi_\psi)\left(\varphi(\vec{g}_1,\chi) * \dots * \varphi(\vec{g}_{l+1},\chi)\right)(\pi_\psi)$$

$$\psi^n \leftrightarrow \mathrm{i}^n\frac{\partial^n}{\partial\pi_\psi^n}$$

$$U_{\mathrm{T}} = \int \mathrm{d}\chi\,\mathrm{d}\psi\int\left(\prod_{a=1}^{l+1}\mathrm{d}\vec{g}_a\right)\mathcal{U}_{\mathrm{S}}(\vec{g}_1,\dots,\vec{g}_{l+1},\psi)\prod_{a=1}^{\frac{(l+1)}{2}}\varphi^*(\vec{g}_a,\chi,\psi)\prod_{a=1}^{\frac{(l+1)}{2}}\varphi(\vec{g}_a,\chi,\psi)$$

$$U_{\mathrm{T}} = \int \mathrm{d}\chi\,\mathrm{d}\pi_\psi\int\left(\prod_{a=1}^{l+1}\mathrm{d}\vec{g}_a\right)\mathcal{U}_{\mathrm{S}}(\vec{g}_1,\dots,\vec{g}_{l+1},\pi_\psi)\left(\underset{a\,=\,1}{\overset{\frac{(l+1)}{2}}{*}}\varphi^*(\vec{g}_a,\chi)\underset{a=1}{\overset{\frac{(l+1)}{2}}{}}\varphi(\vec{g}_a,\chi)\right)(\pi_\psi)$$

$$0 = \langle\sigma_{\epsilon\epsilon'}|\frac{\delta S[\phi,\phi^\dagger]}{\delta\varphi^\dagger(\vec{g},\chi_0,\pi_{\psi 0})}|\sigma_{\epsilon\epsilon'}\rangle,$$

$$\sigma_{\epsilon\epsilon'}(\vec{g},\chi,\pi_\psi)=\eta_\epsilon(\chi-\chi_0,\pi_0)\tilde{\eta}_{\varepsilon'}(\pi_\psi-\pi_{\psi0})\tilde{\sigma}(\vec{g},\chi,\pi_\psi),$$

$$0=\tilde{\sigma}_j''-2\mathrm{i}\tilde{\pi}_0\tilde{\sigma}_j'-E^2\tilde{\sigma}_j-(\omega_{\mathrm{S},j}\bar{\tilde{\sigma}}_j)\bar{\tilde{\sigma}}_j^{\,l-1},$$
$$0=\tilde{\sigma}_j''-2\mathrm{i}\tilde{\pi}_0\tilde{\sigma}_j'-E^2\tilde{\sigma}_j-(\omega_{\mathrm{T},j}\bar{\tilde{\sigma}}_j)\bar{\tilde{\sigma}}_j^{\frac{l-3}{2}}\tilde{\sigma}_j^{\frac{l+1}{2}},$$

$$\tilde{\pi}_0=\frac{\pi_0}{\epsilon\pi_0^2-1},E_j^2=\epsilon^{-1}\frac{2}{\epsilon\pi_0^2-1}+\frac{B_j}{A_j},$$

$$0=\rho_j''-\left((\theta_j')^2-2\tilde{\pi}_0\theta_j'+E_j^2\right)\rho_j-\mathrm{Re}\left[P_j^{\mathrm{S/T}}\right]$$
$$0=\rho_j\theta_j''+2\rho_j'(\theta_j'-\tilde{\pi}_0)-\mathrm{Im}\left[P_j^{\mathrm{S/T}}\right]$$

$$\begin{cases}P_j^{\mathrm{S}}=\omega_{\mathrm{S},j}(\rho_j\mathrm{e}^{-\mathrm{i}\theta_j})\rho_j^{l-1}\mathrm{e}^{-l\mathrm{i}\theta_j},\\ P_j^{\mathrm{T}}=\omega_{\mathrm{T},j}(\rho_j\mathrm{e}^{-\mathrm{i}\theta_j})\rho_j^{l-1}\mathrm{e}^{\mathrm{i}\theta_j},\end{cases}$$

$$\tilde{Q}_j'=\rho_j\mathrm{Im}\left[P_j^{\mathrm{S/T}}\right],$$
$$\tilde{\mathcal{E}}_j'=2\left(\tilde{\mathcal{E}}_j-\frac{\tilde{Q}_j^2}{\rho_j^2}+\mu_j^2\rho_j^2\right)^{\frac{1}{2}}\mathrm{Re}\left[P_j^{\mathrm{S/T}}\right]+2\frac{\tilde{Q}_j}{\rho_j}\mathrm{Im}\left[P_j^{\mathrm{S/T}}\right].$$

$$\omega_{\mathrm{S/T},j}(\psi)=\sum_n a_{\mathrm{S/T},j}^{(n)}\psi^n\overset{\text{Fourier}}{\rightarrow}\sum_n\tilde{a}_{\mathrm{S/T},j}^{(n)}\partial_{\pi_\psi}^n$$

$$\rho_j=\bar{\rho}_j+\varepsilon\delta\rho_j+\mathcal{O}(\varepsilon^2),\theta_j=\bar{\theta}_j+\varepsilon\delta\theta_j+\mathcal{O}(\varepsilon^2)$$

$$0=\bar{\rho}_j''-\bar{\rho}_j(\bar{\theta}_j')^2+2\tilde{\pi}_0\bar{\rho}_j\bar{\theta}_j'-E_j^2\bar{\rho}_j$$
$$0=\bar{\rho}_j\bar{\theta}_j''+2\bar{\rho}_j'(\bar{\theta}_j'-\tilde{\pi}_0)$$

$$0=\delta\rho_j''-2\bar{\rho}_j\bar{\theta}_j'\delta\theta_j'-\delta\rho_j(\bar{\theta}_j')^2+2\tilde{\pi}_0(\delta\rho_j\bar{\theta}_j'+\bar{\rho}_j\delta\theta_j')-E_j^2\delta\rho_j-\mathrm{Re}[\bar{P}_j]$$
$$0=\bar{\rho}_j\delta\theta_j''+\delta\rho_j\bar{\theta}_j''+2\bar{\rho}_j'\delta\theta_j'+2\delta\rho_j'(\bar{\theta}_j'-\tilde{\pi}_0)-\mathrm{Im}[\bar{P}_j]$$

$$\bar{Q}_j=\bar{\rho}_j^2(\bar{\theta}_j'-\tilde{\pi}_0),\bar{\mathcal{E}}_j=(\bar{\rho}_j')^2+\frac{\bar{Q}_j^2}{\bar{\rho}_j^2}-\mu_j^2\bar{\rho}_j^2$$

$$\mu_j^2\equiv E_j^2-\tilde{\pi}_0^2$$

$$\bar{\rho}_j^2=-\frac{\bar{\mathcal{E}}_j}{2\mu_j^2}+\alpha_j\mathrm{e}^{2\mu_j\chi_0}+\beta_j\mathrm{e}^{-2\mu_j\chi_0}$$

$$\bar{\rho}_j\simeq A_j\mathrm{e}^{\mu_j\chi_0}$$
$$\bar{\theta}_j\simeq\tilde{\pi}_0\chi_0-\frac{\bar{Q}_j}{2\mu_j\bar{\rho}_j^2}+C$$

$$\partial_{\pi_{\psi0}}\left(\bar{\rho}_j\mathrm{e}^{\mathrm{i}\bar{\theta}_j}\right)\simeq\bar{\rho}_j\mathrm{e}^{\mathrm{i}\bar{\theta}_j}\left(\partial_{\pi_{\psi0}}\mu_j\right)\chi_0$$

$$
\begin{aligned}
0 =&\left(\delta\rho_j''' - \mu_j^2(2-X_-^2)\delta\rho_j' - \mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right]'\right)\\
&+3\mu_j X_-\left(\delta\rho_j'' - \frac{1}{3}\mu_j^2(2X_-^2+4X_+-X_-X_+-2)\delta\rho_j - \mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)\\
&+\left(\frac{2\bar{Q}_j}{\bar{\rho}_j^2}\right)^2\left(\delta\rho_j' - \mu_j X_-\delta\rho_j - \frac{\bar{\rho}_j^2}{2\bar{Q}_j}\mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right),\\
0 =&\left((\bar{\rho}_j\delta\theta_j)''' - \mu_j^2(2-X_-^2)(\bar{\rho}_j\delta\theta_j)' - \mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right]'\right)\\
&+3\mu_j X_-\left((\bar{\rho}_j\delta\theta_j)'' - \frac{1}{3}\mu_j^2(6-X_-X_+-2X_-^2)(\bar{\rho}_j\delta\theta_j) - \mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)\\
&+\left(\frac{2\bar{Q}_j}{\bar{\rho}_j^2}\right)^2\left((\bar{\rho}_j\delta\theta_j)' - \mu_j X_-(\bar{\rho}_j\delta\theta_j) + \frac{\bar{\rho}_j^2}{2\bar{Q}_j}\mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right),
\end{aligned}
$$

$$
\begin{aligned}
\bar{P}_j^{\mathrm{S/T}} &= \sum_n \mathfrak{a}_{\mathrm{S/T},j}^{(n)}\chi_0^n\bar{\rho}_j^l \times \begin{cases} e^{-i(l+1)\bar{\theta}_j}, \\ 1, \end{cases}\\
\mathfrak{a}_{\mathrm{S/T},j}^{(n)} &\equiv z\tilde{a}_{\mathrm{S/T},j}^{(n)}\left(\partial_{\pi_{\psi 0}}\mu_j\right)^n
\end{aligned}
$$

$$
X_\pm \equiv \frac{\alpha_j\mathrm{e}^{2\mu_j\chi_0} \pm \beta_j\mathrm{e}^{-2\mu_j\chi_0}}{\bar{\rho}_j^2}
$$

$$
\begin{aligned}
0 &= \left(\delta\rho_j'' - \mu_j^2\delta\rho_j - \mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)' + 3\mu_j\left(\delta\rho_j'' - \mu_j^2\delta\rho_j - \mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)\\
0 &= \left((\bar{\rho}_j\delta\theta_j)'' - \mu_j^2(\bar{\rho}_j\delta\theta_j) - \mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)' + 3\mu_j\left((\bar{\rho}_j\delta\theta_j)'' - \mu_j^2(\bar{\rho}_j\delta\theta_j) - \mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta\rho_j'' - \mu_j^2\delta\rho_j - \mathrm{Re}\left[\bar{P}_j^{\mathrm{S/T}}\right] &\simeq C_j\mathrm{e}^{-3\mu_j\chi_0} \simeq 0\\
(\bar{\rho}_j\delta\theta_j)'' - \mu_j^2(\bar{\rho}_j\delta\theta_j) - \mathrm{Im}\left[\bar{P}_j^{\mathrm{S/T}}\right] &\simeq D_j\mathrm{e}^{-3\mu_j\chi_0} \simeq 0
\end{aligned}
$$

$$
\begin{aligned}
0 &= \delta\rho_{\mathrm{S},j}'' - \mu_j^2\delta\rho_{\mathrm{S},j} - \sum_n \mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n\mathrm{e}^{l\mu_j\chi_0}\cos\left((l+1)\bar{\theta}_j\right)\\
0 &= (\bar{\rho}_j\delta\theta_{\mathrm{S},j})'' - \mu_j^2\bar{\rho}_j\delta\theta_{\mathrm{S},j} + \sum_n \mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n\mathrm{e}^{l\mu_j\chi_0}\sin\left((l+1)\bar{\theta}_j\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta\rho_{\mathrm{S},j} = \sum_n \Bigg(&\frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n}{4\mu_j}\mathrm{e}^{-\mu_j\chi_0}\left(\frac{\Gamma(n+1,h_1\chi_0)}{(h_1\chi_0)^n h_1} + \frac{\Gamma(n+1,h_1^*\chi_0)}{(h_1^*\chi_0)^n h_1^*}\right)\\
&-\frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n}{4\mu_j}\mathrm{e}^{\mu_j\chi_0}\left(\frac{\Gamma(n+1,h_2\chi_0)}{(h_2\chi_0)^n h_2} + \frac{\Gamma(n+1,h_2^*\chi_0)}{(h_2^*\chi_0)^n h_2^*}\right)\Bigg)\\
\bar{\rho}_j\delta\theta_{\mathrm{S},j} = \mathrm{i}\sum_n \Bigg(&\frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n}{4\mu_j}\mathrm{e}^{-\mu_j\chi_0}\left(-\frac{\Gamma(n+1,h_1\chi_0)}{(h_1\chi_0)^n h_1} + \frac{\Gamma(n+1,h_1^*\chi_0)}{(h_1^*\chi_0)^n h_1^*}\right)\\
&-\frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\alpha_j^l\chi_0^n}{4\mu_j}\mathrm{e}^{\mu_j\chi_0}\left(-\frac{\Gamma(n+1,h_2\chi_0)}{(h_2\chi_0)^n h_2} + \frac{\Gamma(n+1,h_2^*\chi_0)}{(h_2^*\chi_0)^n h_2^*}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \delta\rho_j'' - \mu_j^2\delta\rho_j - \mathfrak{a}_{\mathrm{S}}^{(n)}A_j^l\chi_0^n\mathrm{e}^{l\mu_j\chi_0}\cos\left((l+1)\bar{\theta}_j\right)\\
0 &= (\bar{\rho}_j\delta\theta_j)'' - \mu_j^2\bar{\rho}_j\delta\theta_j + \mathfrak{a}_{\mathrm{S}}^{(n)}A_j^l\chi_0^n\mathrm{e}^{l\mu_j\chi_0}\sin\left((l+1)\bar{\theta}_j\right)
\end{aligned}
$$

$$\delta\rho_{S,j} = \sum_n \frac{\mathfrak{a}_S^{(n)}}{2\mu_j}\bar{\rho}_j^l n!\,\mathrm{Re}\left[e^{-i(l+1)\mu_j\chi_0}\sum_{k=0}^{n}\frac{\chi_0^k}{k!}\left(h_1^{k-n-1}-h_2^{k-n-1}\right)\right]$$

$$\bar{\rho}_j\delta\theta_{S,j} = \sum_n \frac{\mathfrak{a}_S^{(n)}}{2\mu_j}\bar{\rho}_j^l n!\,\mathrm{Im}\left[e^{-i(l+1)\mu_j\chi_0}\sum_{k=0}^{n}\frac{\chi_0^k}{k!}\left(-h_1^{k-n-1}+h_2^{k-n-1}\right)\right].$$

$$\frac{(l+1)\mathfrak{a}_{S,j}^{(n)}\alpha_j^l\chi_0^n}{|h_1|^2|h_2|^2}e^{l\mu_j\chi_0}\left(2l\sin\left((l+1)\tilde{\pi}_0\chi_0\right)+\left((l-1)\mu_j^2-(l+1)\tilde{\pi}_0^2\right)\cos\left((l+1)\tilde{\pi}_0\chi\right)\right),$$

$$\tilde{\sigma}_{S,j} \simeq \bar{\rho}_j e^{i\bar{\theta}_j}\left(1+\varepsilon\sum_n \mathfrak{a}_{S,j}^{(n)}e^{-h_2\chi_0}\left(\sum_{k=0}^{n}\frac{n!}{k!}\chi_0^k\frac{1}{h_1^{n-k-1}h_2^{n-k-1}}\sum_{m=0}^{n-k}\left(\frac{h_2}{h_1}\right)^m\right)\right).$$

$$0 = \delta\rho_{T,j}'' - \mu_j^2\delta\rho_{T,j} - \sum_n \mathfrak{a}_{T,j}^{(n)}\chi_0^n\alpha_j^l e^{l\mu_j\chi_0}$$

$$0 = \left(\bar{\rho}_j\delta\theta_{T,j}\right)'' - \mu_j^2\left(\bar{\rho}_j\delta\theta_{T,j}\right)$$

$$\delta\rho_{T,j} = \sum_n \frac{\mathfrak{a}_{T,j}^{(n)}\alpha_j^l\chi_0^n}{2\mu_j}\left(e^{-\mu_j\chi_0}\frac{\Gamma(n+1,h_+\chi_0)}{(h_+\chi_0)^n h_+} - e^{\mu_j\chi_0}\frac{\Gamma(n+1,h_-\chi_0)}{(h_-\chi_0)^n h_-}\right),$$

$$\delta\rho_{T,j} = \sum_n \frac{\mathfrak{a}_S^{(n)}}{2\mu_j}\bar{\rho}_j^l n!\sum_{k=0}^{n}\frac{\chi_0^k}{k!}\left(h_+^{k-n-1}-h_-^{k-n-1}\right)$$

$$\langle\hat{V}\rangle_{\sigma_{\epsilon\epsilon'}} \equiv V = \sum_j V_j\tilde{\sigma}_j^*\tilde{\sigma}_j = \sum_j V_j\rho_j^2$$

$$\left(\frac{\bar{V}'}{\bar{V}}\right)^2 = 4\left(\frac{\sum_j V_j\bar{\rho}_j\,\mathrm{sign}(\bar{\rho}_j')\left(\bar{\mathcal{E}}_j-\frac{\bar{Q}_j^2}{\bar{\rho}_j^2}+\mu_j^2\bar{\rho}_j^2\right)^{\frac{1}{2}}}{\sum_j V_j\bar{\rho}_j^2}\right)^2$$

$$\frac{\bar{V}''}{\bar{V}} = 4\frac{\sum_j V_j\left(\frac{\bar{\mathcal{E}}_j}{2}+\mu_j^2\bar{\rho}_j^2\right)}{\sum_j V_j\bar{\rho}_j^2}$$

$$\delta\left(\left(\frac{V'}{V}\right)^2\right) = 4\frac{\bar{V}'}{\bar{V}^3}\sum_j V_j\left(\bar{V}\left(\bar{\rho}_j\delta\rho_j'+\bar{\rho}_j'\delta\rho_j\right)-\bar{V}'\bar{\rho}_j\delta\rho_j\right)$$

$$\delta\left(\frac{V''}{V}\right) = 2\frac{1}{\bar{V}^2}\sum_j V_j\left(\bar{V}\left(\bar{\rho}_j\delta\rho_j''+\bar{\rho}_j''\delta\rho_j+2\bar{\rho}_j'\delta\rho_j'\right)-\bar{V}''\bar{\rho}_j\delta\rho_j\right)$$

$$G(n,h,\chi) = \frac{\Gamma((n+1),h\chi)}{(h\chi)^n h}, h_\pm = -(l\pm1)\mu_j, h_{1/2} = h_\pm + i(l+1)\tilde{\pi}_0$$

$$\delta\rho_{\mathrm{S},j} = \sum_n \frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\chi_0^n}{4\mu_j}\bar{\rho}_j^l\left(\mathrm{e}^{h_+\chi_0}\left(G(n,h_1)+G(n,h_1^*)\right)-\mathrm{e}^{h_-\chi_0}\left(G(n,h_2)+G(n,h_2^*)\right)\right)$$

$$\delta\rho_{\mathrm{S},j}' = -\mu_j\delta\rho_{\mathrm{S},j} - \sum_n \frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\chi_0^n A^{l-1}}{2}\bar{\rho}_j\left(G(n,h_2)+G(n,h_2^*)\right)$$

$$\delta\rho_{\mathrm{T},j} = \sum_n \frac{\mathfrak{a}_{\mathrm{T},j}^{(n)}\chi_0^n}{2\mu_j}\bar{\rho}_j^l\left(\mathrm{e}^{h_+}G(n,h_+)-\mathrm{e}^{h_-}G(n,h_-)\right)$$

$$\delta\rho_{\mathrm{T},j}' = -\mu_j\delta\rho_j - \sum_n \mathfrak{a}_{\mathrm{T},j}^{(n)}\chi_0^n A^{l-1}\bar{\rho}_j G(n,h_-)$$

$$\begin{aligned}\bar{\Psi} &= \bar{N}\sum_j \partial_{\pi_{\psi 0}}\bar{\theta}_j \\ &= \sum_j\left(\frac{\bar{Q}_j}{\mu_j}\partial_{\pi_{\psi 0}}\mu_j\chi_0 - \frac{1}{2}\partial_{\pi_{\psi 0}}\left(\frac{\bar{Q}_j}{\mu_j}\right)+\bar{N}\partial_{\pi_{\psi 0}}c_j\right) \\ &\equiv \bar{\psi}+\Delta\bar{\Psi},\end{aligned}$$

$$\bar{\psi} \equiv \sum_j\left(\frac{\bar{Q}_j}{\mu_j}\partial_{\pi_{\psi 0}}\mu_j\chi_0 - \frac{1}{2}\partial_{\pi_{\psi 0}}\left(\frac{\bar{Q}_j}{\mu_j}\right)\right), \Delta\bar{\Psi} \equiv \sum_j \bar{N}\partial_{\pi_{\psi 0}}c_j$$

$$\delta\Psi = \sum_j\left(\delta N\left(\partial_{\pi_{\psi 0}}\bar{\theta}_j\right)+\bar{N}\left(\partial_{\pi_{\psi 0}}\delta\theta_j\right)\right) \equiv \delta\psi+\Delta\Psi$$

$$\delta\psi \equiv \left(\delta N\left(\partial_{\pi_{\psi 0}}\bar{\theta}_j\right)\right)'' = 2\left(\frac{\delta\rho_j''}{\bar{\rho}_j} - 2\mu_j\frac{\delta\rho_j'}{\bar{\rho}_j}+\mu_j^2\frac{\delta\rho_j}{\bar{\rho}_j}\right)\bar{\psi}+4\left(\frac{\delta\rho_{j_o}'}{\bar{\rho}_{j_o}}-\mu_{j_o}\frac{\delta\rho_{j_o}}{\bar{\rho}_{j_o}}\right)\bar{\psi}'$$

$$\Delta\Psi \equiv \left(\bar{N}\partial_{\pi_{\psi 0}}\delta\theta_j\right)'' = \bar{N}\left(4\mu_j^2\left(\partial_{\pi_{\psi 0}}\delta\theta_j\right)+4\mu_j\left(\partial_{\pi_{\psi 0}}\delta\theta_j\right)'+\left(\partial_{\pi_{\psi 0}}\delta\theta_j\right)''\right)$$

$$\delta\theta_{\mathrm{S},j} = \mathrm{i}\sum_n \frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\chi_0^n}{4\mu_j}\bar{\rho}_j^{l-1}\left(\mathrm{e}^{h_+\chi_0}\left(-G(n,h_1)+G(n,h_1^*)\right)-\mathrm{e}^{h_-\chi_0}\left(-G(n,h_2)+G(n,h_2^*)\right)\right)$$

$$\delta\theta_{\mathrm{S},j}' = -\mathrm{i}\sum_n \frac{\mathfrak{a}_{\mathrm{S},j}^{(n)}\chi_0^n A_j^{l+1}}{2}\bar{\rho}^{-2}\left(-G(n,h_2)+G(n,h_2^*)\right)$$

$$\omega_{\mathrm{S/T},j}(\psi) \equiv \omega_{\mathrm{S/T},j}(V_\psi) = \omega_{\mathrm{S/T},j}(0)+\left.\frac{\partial\omega_{\mathrm{S/T},j}}{\partial V_\psi}\right|_{V_\psi=0}V_\psi.$$

$$\begin{aligned}\left(\frac{V'}{3V}\right)^2 &= \frac{8\pi G}{3}\left(1+\frac{\Pi_\psi^2}{\Pi_\chi^2}+\frac{V^2}{\Pi_\chi^2}(\Lambda+V_\psi)\right) \\ 0 &= \psi''+\frac{V^2}{\Pi_\chi^2}\frac{\mathrm{d}V_\psi}{\mathrm{d}\psi}\end{aligned}$$

$$\left(\frac{\bar{V}'}{3\bar{V}}\right)^2 = \frac{8\pi G}{3}\left(1+\frac{\Pi_\psi^2}{\Pi_\chi^2}\right), \bar{\psi}'' = 0,$$

$$\left(\frac{\bar{V}'}{\bar{V}}\right)^2 = 4\mu_{j_o}^2 = \frac{\bar{V}''}{\bar{V}}$$

$$\left(\frac{\bar{V}'}{\bar{V}}\right)^2 = 24\pi G\left(1+\frac{\Pi_\psi^2}{\Pi_\chi^2}\right) = \frac{\bar{V}''}{\bar{V}}.$$

$$\mu_{j_o}^2(\pi_{\psi 0}) = 6\pi G\left(1+\pi_{\psi 0}^2/\tilde{\pi}_0^2\right)$$

$$\delta\left(\left(\frac{V'}{3V}\right)^2\right) = \frac{8\pi G}{3}\frac{\bar{V}^2}{\Pi_\chi^2}(\Lambda + V_\psi)$$

$$\delta\psi'' = -\frac{\bar{V}^2}{\Pi_\chi^2}\frac{\mathrm{d}V_\psi}{\mathrm{d}\psi}$$

$$\delta\rho_{\mathrm{T},j} \simeq \frac{\bar{\rho}_j^l}{\mu_j^2(l^2-1)}\sum_n \mathfrak{a}_{\mathrm{T},j}^{(n)}\chi_0^n \equiv \frac{\bar{\rho}_j^l}{\mu_j^2(l^2-1)}(\tilde{\Lambda}+\mathcal{V}),$$

$$\delta\left(\left(\frac{V'}{3V}\right)^2\right) = \frac{8}{9}\frac{1}{l+1}\frac{\bar{V}^{(l-1)/2}}{V_{j_o}^{(l-1)/2}}(\tilde{\Lambda}+\mathcal{V})$$

$$\tilde{\Lambda} = \Lambda 18\pi G\frac{V_{j_o}^2}{\Pi_\chi^2},$$

$$\mathcal{V} = V_\psi 18\pi G\frac{V_{j_o}^2}{\Pi_\chi^2}.$$

$$\delta\psi'' = 2\frac{l-1}{l+1}\bar{\rho}_{j_o}^{l-1}(\tilde{\Lambda}+\mathcal{V})\bar{\psi} \overset{l=5}{=} \frac{4}{3}\frac{\bar{V}^2}{V_{j_o}^2}(\tilde{\Lambda}+\mathcal{V})\bar{\psi}$$

$$m_\psi^2 = -\frac{4}{3}\frac{\tilde{\Lambda}}{V_{j_o}^2}$$

$$-\frac{\bar{V}^2}{\Pi_\chi^2}\frac{\mathrm{d}V_\psi}{\mathrm{d}\bar{\psi}} = \frac{4}{3}\frac{\bar{V}^2}{V_{j_o}^2}\mathcal{V}\bar{\psi} = 24\pi G\frac{\bar{V}^2}{\Pi_\chi^2}V_\psi\bar{\psi},$$

$$V_\psi = V_0\exp\left[-\bar{\psi}^2(12\pi G)\right],$$

$$\mathcal{V} = V_\psi 18\pi\bar{G}\frac{\bar{V}_{j_o}^2}{\bar{\Pi}_\chi^2} + \frac{8\pi\delta G}{3}(1+\mathfrak{p}^2)$$

$$\delta G = -\left(\frac{\mathrm{d}V_\psi}{\mathrm{d}\bar{\psi}} + 12\pi\bar{G}V_\psi\bar{\psi}\right)\left(\frac{8\pi\bar{\Pi}_\chi^2}{3\bar{V}_{j_o}^2}\bar{\psi}(1+\mathfrak{p}^2)\right)^{-1}$$

$$\delta\rho_{\mathrm{S},j} = \bar{\rho}_j^l\sum_n \mathfrak{a}_{\mathrm{S},j}^{(n)}\chi_0^n\mathrm{Re}\left(\frac{e^{-i(l+1)\tilde{\pi}_0\chi_0}}{h_1h_2}\right) \equiv \bar{\rho}_j^l(\tilde{\Lambda}+\mathcal{V})F_j(\tilde{\pi}_0\chi_0)$$

$$F_j = \frac{\cos\left((l+1)\tilde{\pi}_0\chi_0 + \vartheta\right)}{|h_1 h_2|}, h_1 h_2 = |h_1 h_2| e^{\mathrm{i}\vartheta}$$

$$\delta\left(\left(\frac{V'}{3V}\right)^2\right) = \frac{8}{9}\mu_{j_o}\frac{\bar{V}^2}{V_{j_o}^2}\left(F'_{j_o} + 4\mu_{j_o}F_{j_o}\right)(\tilde{\Lambda} + \mathcal{V})$$

$$\mu_{j_o}\left(F'_{j_o} + 4\mu_j F_{j_o}\right)\mathcal{V} = 3\pi G\frac{V_{j_o}^2}{\Pi_\chi^2}V_\psi$$

$$f_{\mathfrak{p}} = M_{\mathrm{Pl}}\frac{|\mathfrak{p}|}{(l+1)\sqrt{6\pi(1+\mathfrak{p}^2)}} < \frac{M_{\mathrm{Pl}}}{\sqrt{6\pi}(l+1)},$$

$$\delta\psi = \sum_j \delta N\left(\partial_{\pi_{\psi 0}}\bar{\theta}_j\right)$$

$$\delta\psi'' = 2\frac{\bar{V}^2}{V_{j_o}^2}\left(-36F_{j_o}\beta_{j_o} + 8\mu_{j_o}F'_{j_o}\right)(\tilde{\Lambda} + \mathcal{V})\bar{\psi}$$

$$\frac{\mathrm{d}V_\psi}{\mathrm{d}\bar{\psi}} = 6\pi G V_\psi \mathcal{F}_{j_o}\bar{\psi}, \mathcal{F}_{j_o} \equiv \frac{-36F_{j_o}\mathfrak{r}_{j_o} + 8\mu_{j_o}F'_{j_o}}{\mu_{j_o}\left(F'_{j_o} + 4\mu_j F_{j_o}\right)}$$

$$\mathcal{F}_{j_o} = 12\left[1 - \frac{2}{3}\cot\left((l+1)\tilde{\pi}_0\chi_0 + \vartheta\right)\right]^{-1}$$

$$\delta G = -\left(\frac{\mathrm{d}V_\psi}{\mathrm{d}\bar{\psi}} + 12\pi\bar{G}\mathcal{F}_{j_o}V_\psi\bar{\psi}\right)\left(\frac{8\pi\bar{\Pi}_\chi^2}{3\bar{V}_{j_o}^2}\bar{\psi}(1+\mathfrak{p}^2)\right)^{-1}$$

$$\begin{aligned}
U &= \int \mathrm{d}\chi\,\mathrm{d}\psi \int \left(\prod_{a=1}^{l+1} \mathrm{d}\vec{g}^a\right)\mathcal{U}(\vec{g}^1, \dots, \vec{g}^{l+1}, \psi)\prod_{a=1}^{l+1} \varphi(\vec{g}^a, \chi, \psi) \\
&= \int \mathrm{d}\chi\,\mathrm{d}\pi_\psi \delta(\pi_\psi) \int \left(\prod_{a=1}^{l+1} \mathrm{d}\vec{g}^a\right) \int \left(\prod_{b=1}^{l+1} \mathrm{d}\pi_b\right)\left(\mathcal{U} * \varphi(\vec{g}^1, \chi) * \dots * \varphi(\vec{g}^{l+1}, \chi)\right)(\pi_\psi) \\
&= \int \mathrm{d}\chi\,\mathrm{d}\pi_\psi \delta(\pi_\psi) \int \left(\prod_{a=1}^{l+1} \mathrm{d}\vec{g}^a\right) \int \left(\prod_{b=1}^{l+1} \mathrm{d}\pi_b\right) \times \\
&\quad \times \mathcal{U}(\vec{g}^1, \dots, \vec{g}^{l+1}, \pi_\psi - \pi_1)\varphi(\vec{g}^1, \chi, \pi_1 - \pi_2) \dots \varphi(\vec{g}^{l+1}, \chi, \pi_{l+1}) \\
&= \int \mathrm{d}\chi \int \left(\prod_{a=1}^{l+1} \mathrm{d}\vec{g}^a\right) \int \left(\prod_{b=1}^{l+1} \mathrm{d}\pi_b\right) \times \\
&\quad \times \mathcal{U}(\vec{g}^1, \dots, \vec{g}^{l+1}, -\pi_1)\varphi(\vec{g}^1, \chi, \pi_1 - \pi_2) \dots \varphi(\vec{g}^{l+1}, \chi, \pi_{l+1})
\end{aligned}$$

$$\frac{\delta\bar{U}}{\delta\varphi^\dagger(\vec{g},\chi,\psi)} = \int \mathrm{d}\chi \int \left(\prod_{a=1}^{l+1} \mathrm{d}\vec{g}^a\right)\int \left(\prod_{b=1}^{l+1} \mathrm{d}\pi_b\right)([\mathrm{I}]+[\mathrm{II}]+\cdots+[\mathrm{l}+1])$$

$$[\mathrm{I}] = \delta\big(\pi-(\pi_1-\pi_2)\big)\delta(\vec{g}-\vec{g}^1)\left(\tilde{\overline{\mathcal{U}}}\varphi^\dagger\right)(\vec{g}^2,\chi,\pi_2-\pi_3)\varphi^\dagger(\vec{g}^3,\chi,\pi_3-\pi_4)\ldots$$
$$\ldots\varphi^\dagger\big(\vec{g}^{l+1},\chi,\pi_{l+1}\big)$$
$$[\mathrm{II}] = (\overline{\mathcal{U}}\varphi^\dagger)(g_1,\chi,\pi_1-\pi_2)\delta\big(\pi-(\pi_2-\pi_3)\big)\delta(\vec{g}-\vec{g}^2)\varphi^\dagger(\vec{g}^3,\chi,\pi_3-\pi_4)\ldots$$
$$\ldots\varphi^\dagger\big(\vec{g}^{l+1},\chi,\pi_{l+1}\big)$$
$$\ldots$$
$$[\mathrm{l}+1] = (\overline{\mathcal{U}}\varphi^\dagger)(g_1,\chi,\pi_1-\pi_2)\varphi^\dagger(g_2,\chi,\pi_2-\pi_3)\ldots$$
$$\ldots\varphi^\dagger\big(\vec{g}^l,\chi,\pi_l-\pi_{l+1}\big)\delta(\pi-\pi_{l+1})\delta\big(\vec{g}-\vec{g}^{l+1}\big)$$

$$\sum_n c_n\psi^n \leftrightarrow \sum_n \tilde{c}_n\partial^n_{\pi_\psi}$$

$$\bar{u}\big(\vec{g}_1,\ldots,\vec{g}^l,\pi_1\big) = \overline{\mathcal{U}}\big(\vec{g},\vec{g}^1,\ldots,\vec{g}^l,-\pi_1\big)+\overline{\mathcal{U}}\big(\vec{g}^1,\vec{g},\vec{g}^2,\ldots,\vec{g}^l,-\pi_1\big)+\cdots+\overline{\mathcal{U}}\big(\vec{g}^1,\ldots,\vec{g}^l,\vec{g},-\pi_1\big),$$

$$\frac{\delta\bar{U}}{\delta\varphi^\dagger(\vec{g},\chi,\psi)} = \int \mathrm{d}\chi \int \left(\prod_{a=1}^{l} \mathrm{d}\vec{g}^a\right)\int \left(\prod_{b=1}^{l} \mathrm{d}\pi_b\right)(\bar{u}\varphi^\dagger)(\vec{g}^1,\chi,-\pi-\pi_1)\times$$
$$\times\varphi^\dagger(\vec{g}^2,\chi,\pi_1-\pi_2)\ldots\varphi^\dagger\big(\vec{g}^l,\chi,\pi_l\big)$$

$$\sigma_{\epsilon\epsilon'}(\vec{g},\chi,\pi) = \tilde{\sigma}(\vec{g},\chi,\pi)\eta_\epsilon\big(\chi-\chi_0,\pi_{\chi 0}\big)\eta_{\epsilon'}\big(\pi-\pi_{\psi 0},\psi_0\big)$$
$$\eta_\epsilon(\chi-\chi_0,\pi_0) = \mathcal{N}_\epsilon \exp\left\{\left(-\frac{(\chi-\chi_0)^2}{2\epsilon}\right)\right\}\exp\left\{\left(-\mathrm{i}\pi_0(\chi-\chi_0)\right)\right\}$$
$$\eta_{\epsilon'}\big(\pi-\pi_{\psi 0},\psi_0\big) = \mathcal{N}_{\epsilon'} \exp\left\{\left(-\frac{\left(\pi-\pi_{\psi 0}\right)^2}{2\epsilon'}\right)\right\}\exp\left\{\left(-\mathrm{i}\psi_0\big(\pi-\pi_{\psi 0}\big)\right)\right\}$$

$$\simeq \int\left(\prod_{a=1}^{l}\mathrm{d}\vec{g}^a\right)\left(\bar{u}\left(\tilde{\sigma}\big(\vec{g}^1,\chi,-\pi-(l-1)\pi_{\psi 0}\big)\eta_{\epsilon'}\big(-\pi-l\pi_{\psi 0},\psi_0\big)\right)\right)\prod_{b=1}^{l-1}\tilde{\sigma}\big(\vec{g}^b,\chi_0,\pi_{\psi 0}\big)$$
$$\simeq \int\left(\prod_{a=1}^{l}\mathrm{d}\vec{g}^a\right)\eta_{\epsilon'}\big(-\pi-l\pi_{\psi 0},\psi_0\big)(\bar{u}\tilde{\sigma})\big(\vec{g}^1,\chi,-\pi-(l-1)\pi_{\psi 0}\big)\prod_{b=1}^{l-1}\tilde{\sigma}\big(\vec{g}^b,\chi_0,\pi_{\psi 0}\big)$$

$$\Gamma(z) = \int_0^\infty \mathrm{d}x x^{z-1}\mathrm{e}^{-x}$$

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$$
$$\gamma(s,x) = \int_0^x t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

$$\Gamma(n,f(x)) = n!\,\mathrm{e}^{-f(x)}\sum_{k=0}^{n-1}\frac{f^k(x)}{k!}$$

$$\frac{\Gamma(s,f(x))}{f^{s-1}(x)\mathrm{e}^{-f(x)}} \overset{f(x)\to\infty}{\rightarrow} 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Gamma(s,f(x)) = -\mathrm{e}^{-f(x)}(f(x))^{s-1}f'(x)$$

Modelo de gravedad cuántica bajo la métrica de Yang – Mills - BRST.

$$\int d^8zu(\Phi)\left(-\frac{D^2}{4\,\square}\right)v(\Phi) = \int d^6zu(\Phi)v(\Phi)$$

$$\mathcal{S}_c = \int d^4x\mathcal{L} = \mathrm{tr}\int d^8z\bar{\Phi}e^{-2gV}\Phi + \frac{1}{2}\mathrm{tr}\int d^6z\mathcal{W}^2$$
$$+\int d^8z(\bar{\Psi}_1e^{-gV}\Psi_1 - \bar{\Psi}_2e^{gV}\Psi_2) + \lambda\mathrm{tr}\int d^6z\Psi_1\Phi\Psi_2 + \text{ h.c.}$$

$$W_{\text{tree}} = \lambda\mathrm{tr}\int d^6z\Psi_1\Phi\Psi_2$$

$$e^{2gV'} \to e^{-g\bar{\Lambda}}e^{2gV}e^{g\Lambda}$$

$$\delta V = iL_{gV/2}\left(\Lambda + \bar{\Lambda} + \coth\left(L_{gV/2}(\Lambda - \bar{\Lambda})\right)\right).$$

$$\delta\Phi = ig\Lambda\Phi, \delta\Psi_1 = \frac{i}{2}g\Lambda\Psi_1, \delta\Psi_2 = -\frac{i}{2}g\Lambda\Psi_2$$

$$\mathcal{S}_{GF} = -\frac{1}{16\xi}\mathrm{tr}\int d^8zD^2V\bar{D}^2V$$

$$\mathcal{S}_{FP} = \mathrm{tr}\int d^8z\left[\bar{c}'c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c + \bar{c}] + \cdots\right]$$

$$\mathcal{S}_0 = \mathcal{S}_c + \mathcal{S}_{GF} + \mathcal{S}_{FP}$$

$$\Phi \to \Phi + \sqrt{\hbar}\phi, \Psi_I \to \Psi_I + \sqrt{\hbar}\psi_I$$

$$\begin{aligned}\mathcal{S}^{(2)} = \mathrm{tr}&\int d^8z(\bar{\phi}\phi - 2g\bar{\Phi}v\phi + 2g\bar{\phi}v\Phi)\\ +\mathrm{tr}&\int d^8z(\bar{\psi}_1\psi_1 - g\bar{\Psi}_1v\psi_1 - g\bar{\psi}_1v\Psi_1)\\ +\mathrm{tr}&\int d^8z(\bar{\psi}_2\psi_2 + g\bar{\Psi}_2v\psi_2 + g\bar{\psi}_2v\Psi_2)\\ -\lambda\mathrm{tr}&\int d^6z(\psi_1\Phi\psi_2 + \Psi_1\phi\psi_2 + \psi_1\phi\Psi_2 + \text{ h.c. })\\ +\mathrm{tr}&\int d^8z\left(-\frac{1}{2}v\,\hat{\square}\,v + cc' + c'c\right)\end{aligned}$$

$$\partial_a\Phi = \partial_a\Psi_I = \partial_a\bar{\Phi} = \partial_a\bar{\Psi}_I = 0$$

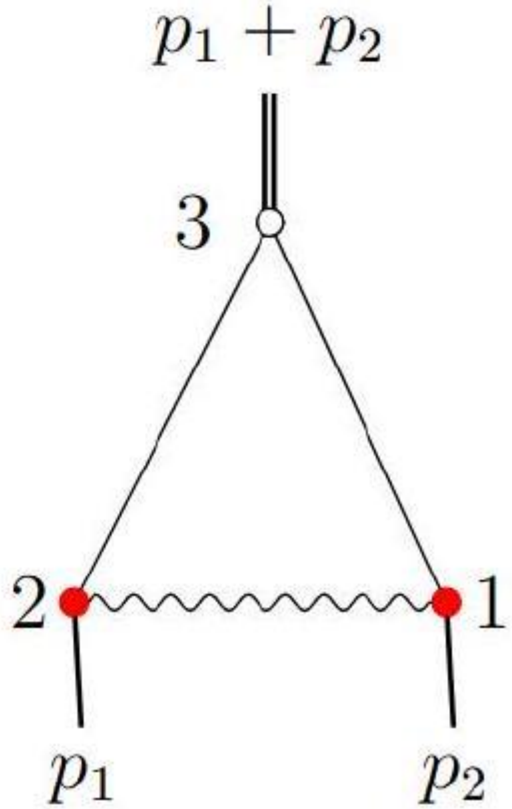
$$\Gamma\left[\Phi, \Psi_i \mid \bar{\Phi}, \bar{\Psi}_i\right] = \int d^8z(\mathbf{K} + \mathbf{A}) + \left(\int d^6z\mathbf{W} + \text{ h.c.}\right)$$

$$\Gamma[\Phi,\Psi_I \mid \bar{\Phi},\bar{\Psi}_I] = \sum_{L=1}^{\infty} \hbar^L \Gamma^{(L)}[\Phi,\Psi_I \mid \bar{\Phi},\bar{\Psi}_I]$$

$$\mathbf{W}[\Phi,\Psi_I] = \sum_{L=1}^{\infty} \hbar^L \mathbf{W}^{(L)}[\Phi,\Psi_I]$$

$$n_{D^2} + 1 = n_{\bar{D}^2},$$

$$\mathbf{W}^{(1)} = \lim_{p_1,p_2\to 0} \frac{\lambda g^2}{4}(2C_F - C_A)\int \prod_{l=1}^{3} d^8 z_l \Phi(z_1)\Psi_1(z_2)\Psi_2(z_3)$$
$$\left\{\frac{1}{\Box_2}\delta_{1,2}\frac{D_1^2\bar{D}_3}{16\,\Box_1}\delta_{1,3}\frac{D_2^2}{4}\delta_{2,3}\frac{1}{\Box_2}\right\}.$$



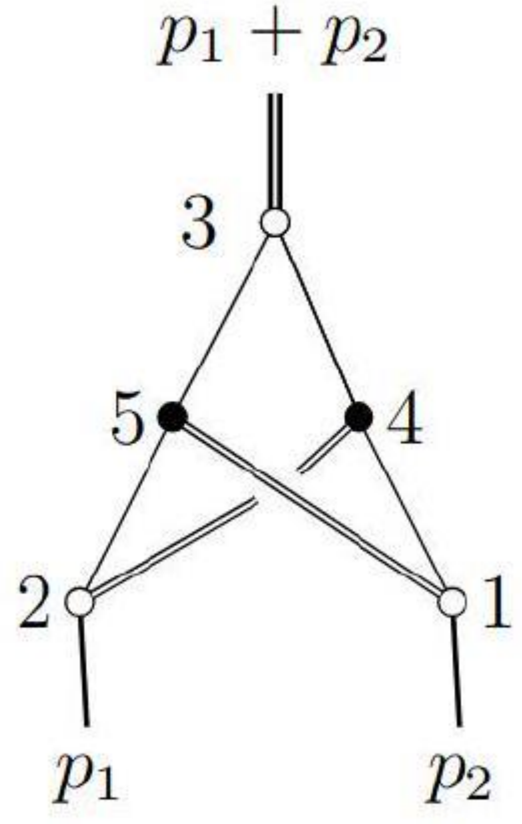
$$\Psi_1(y_1,\theta)\Psi_1(y_2,\theta)\Phi(x,\theta) \simeq [\Psi_1\Phi\Psi_2](x,\theta),$$

$$\mathbf{W}^{(1)} = \frac{\hbar}{(4\pi)^2}\frac{g^2}{2}(C_F - C_A/2)\Upsilon^{(1)} W_{\text{tree}}$$

$$\Upsilon^{(1)} = \lim_{p_1,p_2\to 0}\int d^4q \frac{(p_1+p_2)^2}{q^2(q-p_1)^2(q_1+p_2)^2} = \int_0^1 d\tau \frac{2\log(\tau)}{\tau^2-\tau+1}$$

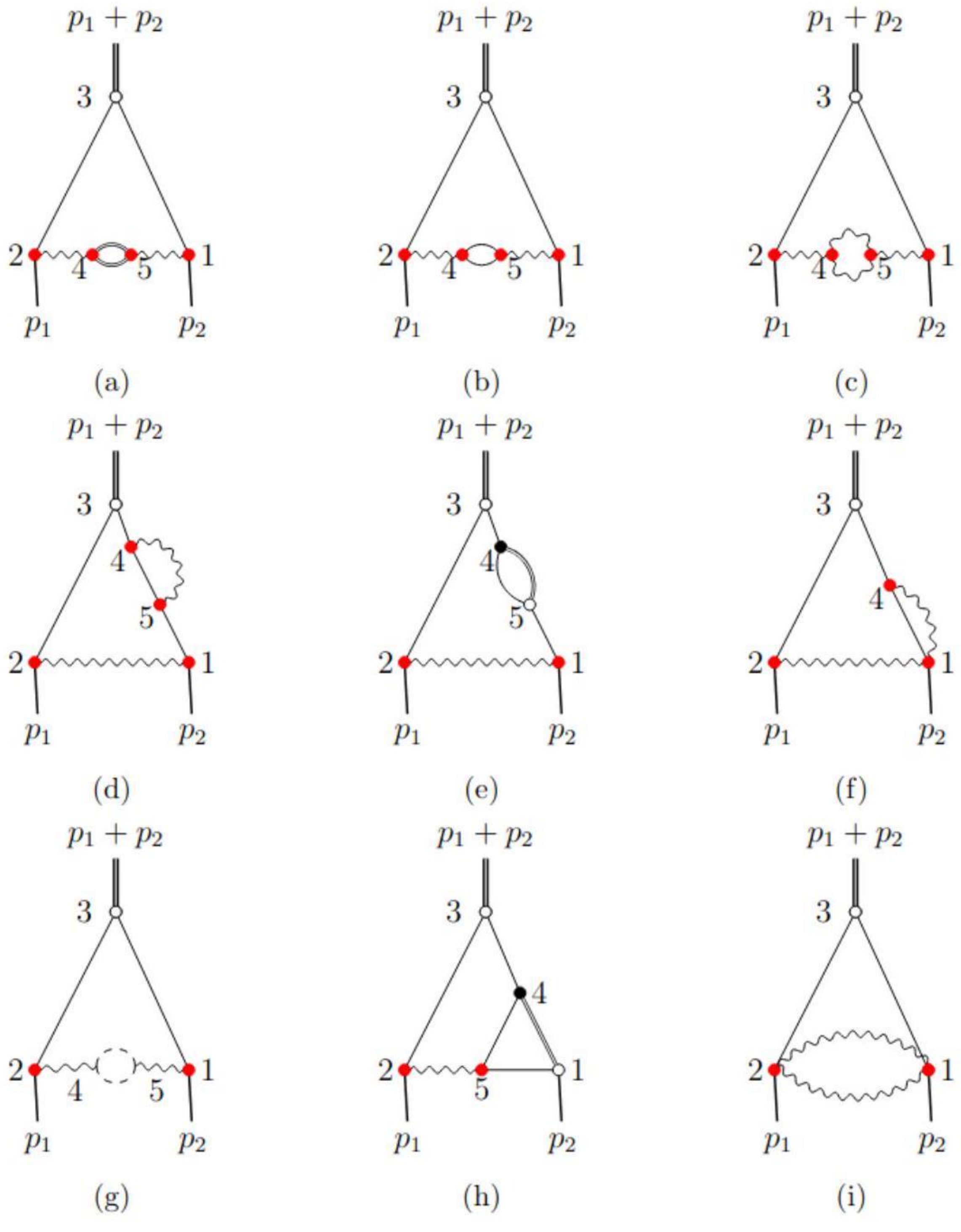
$$\mathbf{W}^{(2)} = \lim_{p_1,p_2\to 0} \frac{g^4}{16}(C_A - C_F)\int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_3)\Phi(z_4)\Psi_2(z_5)\left\{\frac{1}{\Box_1}\delta_{1,3}\frac{D_2^2\bar{D}_3^2}{16\,\Box_2}\delta_{3,2}\right.$$
$$\left.\frac{1}{16\,\Box_2}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\Box_1}\delta_{1,4}\frac{D_1^2\bar{D}_5^2}{16\,\Box_1}\delta_{1,5}\frac{D_2^2}{4\,\Box_2}\delta_{2,5}\right\}$$

$$I^{(2)} = \lim_{p_{1,2}\to 0}\int \frac{d^4 q_1}{(4\pi)^4}\frac{d^4 q_2}{(4\pi)^4}\frac{q_1^2 p_1^2 + q_2^2 p_2^2 - 2p_1 p_2 (q_1 q_2)}{q_1^2 q_2^2 (q_1+q_2)^2 (q_1-p_1)^2 (q_2-p_2)^2 (q_1+q_2-p_1-p_2)^2}$$



$$\mathbf{W}^{(2)} = \frac{\hbar^2}{(4\pi)^4}\frac{3}{8}(C_A - C_F)|\lambda|^4 \zeta(3) \times W_{tree}$$

$$\mathbf{W}_{div}^{(2),A} + \mathbf{W}_{div}^{(2),B} + \mathbf{W}_{div}^{(2),G} = \lim_{p_1,p_2\to 0}\left(2N_f T_F - C_A\right)(2C_F - C_A)\frac{g^4\lambda}{4}\times$$
$$\times\int \prod_{l=1}^{5} d^8 z_l \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{D_1^2\bar{D}_3^2}{16\,\Box_1}\delta_{1,3}\frac{D_2^2}{4\,\Box_2}\delta_{3,2}\frac{1}{\Box_4}\delta_{2,4}\frac{\bar{D}_4^2 D_5^2}{16\,\Box_4}\delta_{4,5}\frac{\bar{D}_5^2 D_4^2}{16\,\Box_5}\delta_{5,4}\frac{1}{\Box_1}\delta_{5,1}\right\}$$



$$\mathbf{W}_{div}^{(2),C} \sim \lim_{p_1,p_2\to 0} \frac{g^4\lambda}{4} \int \prod_{l=1}^{5} d^8 z_l \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3) \left\{ \frac{D_1^2 \bar{D}_3^2}{16\,\Box_1} \delta_{1,3} \right.$$
$$\left. \frac{D_2^2}{4\,\Box_2} \delta_{3,2} \frac{1}{\Box_4} \delta_{2,4} \frac{1}{\Box_4} \delta_{4,5} \frac{1}{\Box_5} \delta_{5,4} \frac{1}{\Box_1} \delta_{5,1} \right\} = 0$$

$$\mathbf{W}_{div}^{(2),I} = 0.$$

$$\mathbf{W}_{div,1}^{(2)} = \mathrm{A} + \mathrm{B} + \mathrm{G} = \frac{1}{4} g^4 \left(2N_f T_F - C_A\right)\left(2C_F - C_A\right) \times J_{1,1}^{(1)} \Upsilon^{(1)} \times W_{\text{tree}}$$

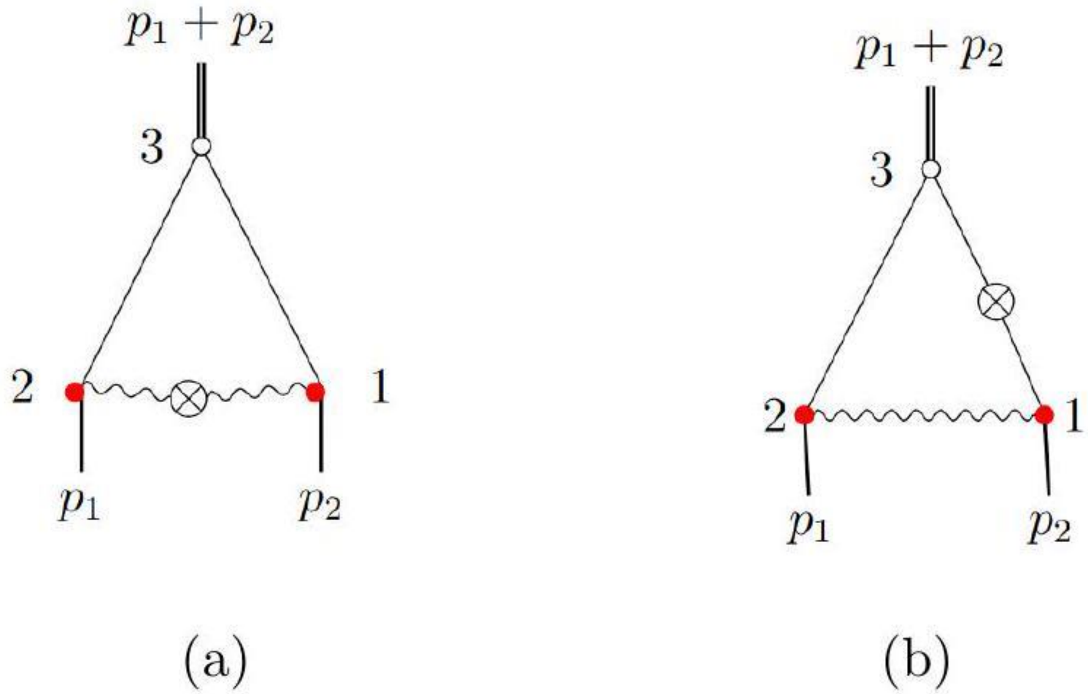
$$J_{1,1}^{(1)}(k) = \left(\frac{1}{\epsilon} + 2 + O(\epsilon^1)\right)(k^2/\mu^2)^{-\epsilon}$$

$$\mathbf{W}_{div,2}^{(2)} = 2\mathrm{D} + 2\mathrm{E} = \frac{1}{8} g^2 (2C_F - C_A)(|\lambda|^2 - g^2) \times J_{1,1}^{(1)} \Upsilon^{(1)} \times W_{\text{tree}} .$$

$$\mathbf{W}_{div}^{(2)} = \left(g^4\{2N_f T_F - C_A\} + g^2(|\lambda^2| - g^2)\right)\frac{1}{4}(2C_F - C_A) J_{1,1}^{(1)} \Upsilon^{(1)} \times W_{\text{tree}}$$

$$\beta(g) = \frac{2g^3}{(4\pi)^2}(2N_f T_F - C_A).$$

$$\sum_i m_i T_F(R_i) = C_A$$

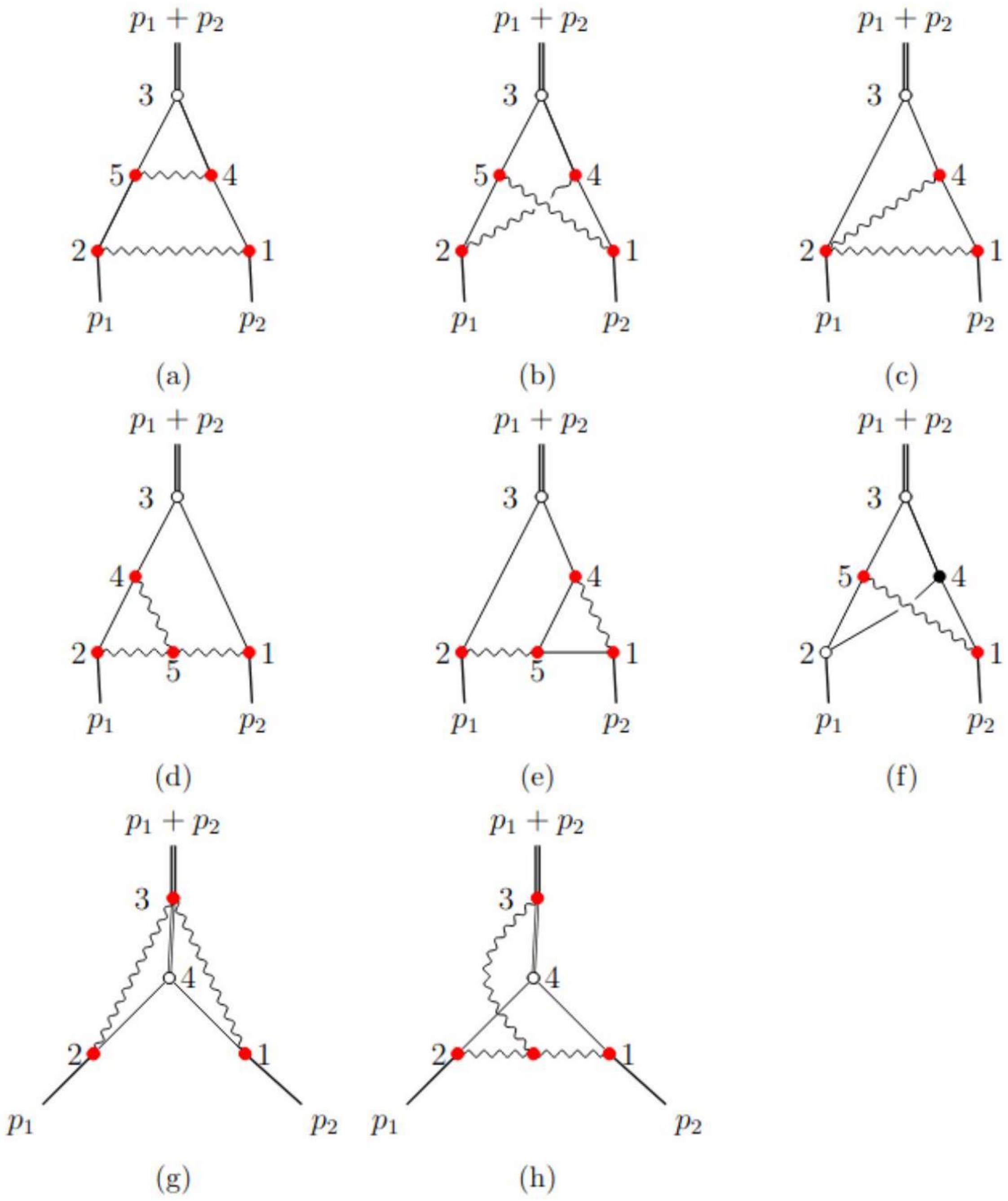


$$\mathbf{W}_{fin}^{(2),A} = \lim_{p_1,p_2 \to 0} \frac{g^4}{16}(C_A - 2C_F)^2 \int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_1}\delta_{2,1} \frac{D_1^2 \bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\frac{D_4^2 \bar{D}_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_5^2}{4\,\Box_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{16\,\Box_5}\delta_{5,2}\frac{1}{\Box_4}\delta_{4,5}\right\}.$$

$$J_a^{(2)} = \lim_{p_1,p_2 \to 0} \int \frac{d^4 q_1}{(4\pi)^2}\frac{d^4 q_2}{(4\pi)^2}\frac{q_1^2 (p_1 + p_2)^2}{q_1^2 (q_1 - p_1)^2 (q_2 - p_2)^2 q_2^2 (q_2 - p_2)^2 (q_1 - p_2)^2} = 6\zeta(3);$$

$$\mathbf{W}_{\text{fin}}^{(2),A} = \frac{3g^4}{8}(C_A - 2C_F)^2 \zeta(3) \times W_{\text{tree}} .$$

$$\mathbf{W}_{fin}^{(2),B} = \lim_{p_1,p_2\to 0} \frac{g^4}{8}(C_A - C_F)\int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\Box_4}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\Box_4}\delta_{1,4}\right.$$
$$\left.\frac{\bar{D}_4^2 D_3^2}{16\,\Box_4}\delta_{4,3}\frac{D_5^2}{4\,\Box_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{16\,\Box_5}\delta_{5,2}\frac{1}{\Box_5}\delta_{1,5}\right\}$$



(a) (b) (c) (d) (e) (f) (g) (h)

$$J_b^{(2)} = \lim_{p_1,p_2\to 0}\int \frac{d^4 q_1}{(4\pi)^2}\frac{d^4 q_2}{(4\pi)^2}\frac{-q_1^2 p_1^2 - q_2^2 p_2^2 + 2(q_1 q_2)(p_1 p_2)}{q_1^2(q_1 - p_1)^2(q_1 + q_2 - p_1)^2 q_2^2(q_2 + p_2)^2(q_1 + q_2 + p_1)^2}$$

$$\mathbf{W}_{fin}^{(2),B} = -\frac{3g^4}{2}(C_A - C_F)\zeta(3)\times W_{tree}$$

$$\mathbf{W}_{fin}^{(2),C} = \lim_{p_1,p_2\to 0}\frac{g^4}{8N^2}\int\prod_{l=1}^{5} d^8 z_l \lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\square_1}\delta_{2,1}\frac{D_1^2\bar{D}_4^2}{16\,\square_4}\delta_{1,4}\right.$$
$$\left.\frac{\bar{D}_4^2 D_3^2}{16\,\square_4}\delta_{4,3}\frac{D_2^2}{4\,\square_3}\delta_{3,2}\frac{1}{\square_4}\delta_{2,4}\right\}$$

$$J_c^{(2)} = \lim_{p_1,p_2\to 0}\int\frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{q_1^2(p_1+p_2)^2}{q_1^2(q_1+p_1)^2(q_2+p_1)^2(q_2-q_1)^2(q_2-p_2)^2}$$

$$\mathbf{W}_{fin}^{(2),C} = \frac{3g^4}{2}(C_A-2C_F)\zeta(3)\times W_{tree}$$

$$\mathbf{W}_{fin}^{(2),D} = \lim_{p_1,p_2\to 0}\frac{g^4}{8}\int\prod_{l=1}^{5} d^8 z_l \lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\square_5}\delta_{5,1}\frac{D_1^2\bar{D}_3^2}{16\,\square_1}\delta_{1,3}\right.$$
$$\left.\frac{D_4^2}{4\,\square_3}\delta_{3,4}\frac{\bar{D}_4^2 D_2^2}{16\,\square_4}\delta_{4,2}\frac{1}{\square_4}\delta_{2,5}\frac{1}{\square_4}\delta_{5,4}\right\}$$

$$J_d^{(2)} = \lim_{p_1,p_2\to 0}\int\frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{(p_1+p_2)^2}{q_1^2(q_1+p_1)^2(q_1-p_2)^2(q_2-p_2)^2 q_2^2(q_1-q_2)^2}$$

$$\mathbf{W}_{\mathrm{fin}}^{(2),D} = \frac{g^4}{8}\Upsilon^{(2)}\times W_{\mathrm{tree}}$$

$$\Upsilon^{(2)} = \int_0^1 d\tau\,\frac{2\log^3(\tau)}{\tau^2-\tau+1}$$

$$\mathbf{W}^{(2),E} = \lim_{p_1,p_2\to 0}\frac{g^4}{8}(C_A-2C_F)\int\prod_{l=1}^{5} d^8 z_l \lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\square_4}\delta_{1,4}\frac{D_1^2\bar{D}_5^2}{16\,\square_5}\delta_{1,5}\right.$$
$$\left.\frac{D_5^2\bar{D}_4^2}{16\,\square_5}\delta_{5,4}\frac{D_4^2\bar{D}_3^2}{16\,\square_4}\delta_{4,3}\frac{D_2^2}{4\,\square_3}\delta_{3,2}\frac{1}{\square_5}\delta_{2,5}\right\}$$

$$J_e^{(2)} = \lim_{p_1,p_2\to 0}\int\frac{d^4q_1}{(4\pi)^2}\frac{d^4q_2}{(4\pi)^2}\frac{-q_1^2p_1^2+(q_1-q_2)^2p_2^2-2(q_1q_2)(p_1p_2)}{q_1^2(q_1-p_1)^2(q_2-q_1+p_1)^2(q_2-p_2)^2q_2^2(q_2+p_1)^2}$$

$$\mathbf{W}_{fin}^{(2),E}(\Phi) = -\frac{3g^4}{4}(C_A-2C_F)\zeta(3)\times W_{tree}$$

$$\mathbf{W}_{fin}^{(2),F}(\Phi) \sim \lim_{p_1,p_2\to 0} g^4\int\prod_{l=1}^{5} d^8 z_l \lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\square_4}\delta_{2,4}\frac{D_1^2\bar{D}_4^2}{16\,\square_4}\delta_{1,4}\right.$$
$$\left.\frac{\bar{D}_4^2 D_3^2}{16\,\square_4}\delta_{4,3}\frac{D_5^2}{4\,\square_5}\delta_{3,5}\frac{\bar{D}_5^2 D_2^2}{16\,\square_5}\delta_{5,2}\frac{1}{\square_5}\delta_{1,5}\right\}$$

$$J_f^{(2)} = 0$$

$$\mathbf{W}_{fin}^{(2),G} \sim \lim_{p_1,p_2\to 0} \frac{g^4}{4}\int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{D_1^2\bar{D}_4^2}{16\,\square_4}\delta_{1,4}\frac{\bar{D}_4^2 D_2^2}{16\,\square_4}\delta_{4,2}\right.$$
$$\left.\frac{1}{\square_3}\delta_{2,3}\frac{1}{\square_1}\delta_{3,1}\frac{D_3^2}{4\,\square_4}\delta_{3,4}\right\} = 0$$

$$\mathbf{W}^{(2),H} \sim \lim_{p_1,p_2\to 0} \frac{g^4}{8}\int \prod_{l=1}^{5} d^8 z_l \lambda \Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\left\{\frac{1}{\square_5}\delta_{3,5}\frac{D_1^2\bar{D}_4^2}{16\,\square_4}\delta_{1,4}\frac{\bar{D}_4^2 D_2^2}{16\,\square_4}\delta_{4,2}\right.$$
$$\left.\frac{1}{\square_3}\delta_{2,5}\frac{1}{\square_1}\delta_{5,3}\frac{D_3^2}{4\,\square_4}\delta_{3,4}\right\} = 0$$

$$\mathbf{W}_{fin}^{(2)} = \left(\frac{1}{8}g^4\Upsilon^{(2)} + \frac{3}{4}\left(\frac{N^2+1}{N^2}|\lambda|^4 - g^4\right)\zeta(3)\right)\times W_{tree}$$

$$\mathbf{W}_{\mathrm{fin},\mathcal{N}=2}^{(2)} = g^4\left(\frac{1}{8}\Upsilon^{(2)} + \frac{3}{4N^2}\zeta(3)\right)\times W_{\mathrm{tree}}$$

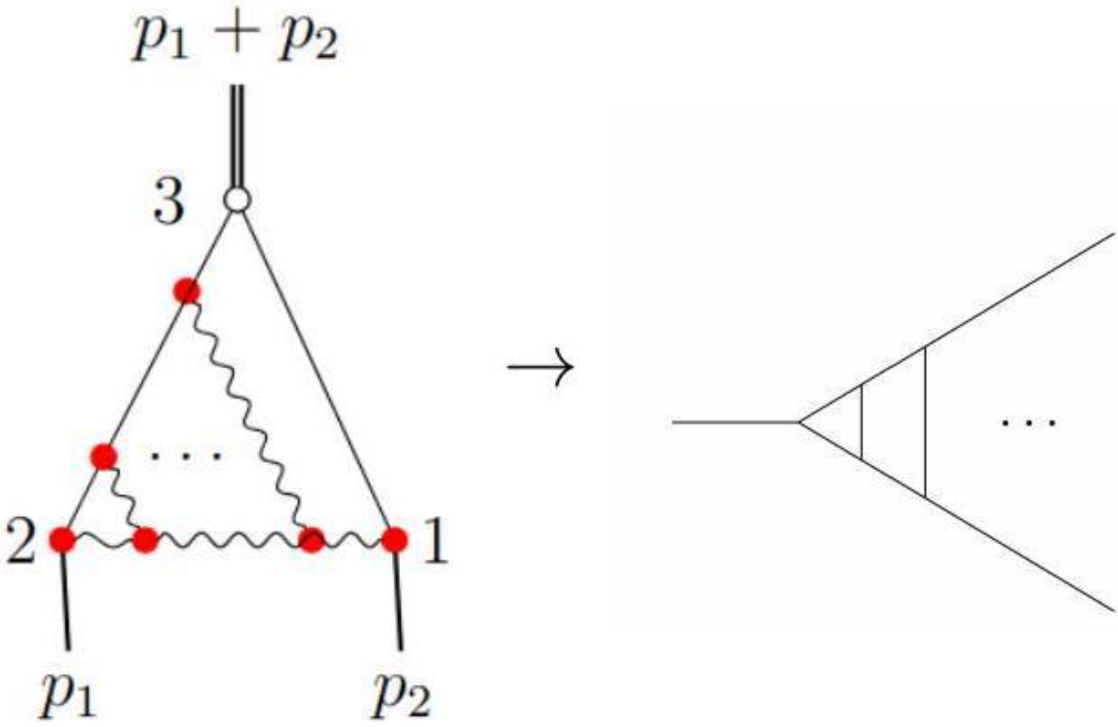
$$\mathbf{W}^{(2)} = \mathbf{W}_{div}^{(2)} + \mathbf{W}_{fin}^{(2)} = \left(\left(g^4\{2N_f T_F - N\} + g^2(|\lambda^2| - g^2)\right)J_{1,1}^{(1)}\Upsilon^{(1)}\right.$$
$$\left. + \frac{1}{2}g^4\Upsilon^{(2)} + \frac{3}{2}\left(\frac{N^2+1}{N^2}|\lambda|^4 - g^4\right)\zeta(3)\right)\times\frac{1}{4}(2C_F - C_A)W_{\mathrm{tree}}$$

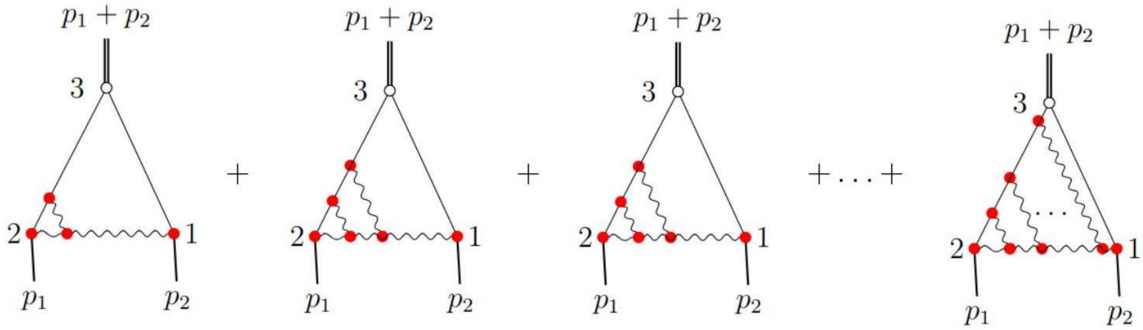
$$\Phi = \left(z_\phi^{1/2}\right)\Phi_R, \bar{\Phi} = \left(z_\phi^{1/2}\right)\bar{\Phi}_R$$
$$\Psi_I = \left(z_\psi^{1/2}\right)\Psi_{I,R}, \bar{\Phi} = \left(z_\psi^{1/2}\right)\bar{\Psi}_{I,R}$$
$$V = z_V^{1/2}V_R, g = z_g g_R, \lambda = z_\lambda^{3/2}\lambda$$

$$z_\phi^{1/2} = z_\psi^{-1}$$

$$\mathbf{W}_R^{(2)} = \left(\frac{1}{2}g_R^4\Upsilon^{(2)} + \frac{3}{2}\left(\frac{N^2+1}{N^2}|\lambda|_R^4 - g_R^4\right)\zeta(3) + 2u\right)\times\frac{1}{4}(2C_F - C_A)W_{\mathrm{tree},R}.$$

$$W_{\mathrm{tree},R} = \mathrm{tr}\int d^6 z \lambda_R \Psi_{1,R}\Phi_R\Psi_{2,R}, u = \left(g_R^4\{2N_f T_F - C_A\} + g_R^2\left(|\lambda_R^2| - g_R^2\right)\right)\Upsilon^{(1)}$$





$$\mathbf{W}'^{(m)} = \lim_{p_1,p_2\to 0}(-1)^{m+1}\frac{g^{2m}}{2^{m+2}}N^{m-2}\int\prod_{l=1}^{2m+1} d^8z_l\lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)$$

$$\times\left\{\frac{1}{\Box_5}\delta_{5,1}\frac{D_1^2\bar{D}_3^2}{16\,\Box_1}\delta_{1,3}\frac{D_4^2}{4\,\Box_3}\delta_{3,4}\frac{\bar{D}_4^2D_6^2}{16\,\Box_4}\delta_{4,6}\cdots\right.$$

$$\cdots\frac{\bar{D}_{2m-2}^2D_{2m}^2}{\Box_{2m-2}}\delta_{2m-2,2m}\frac{\bar{D}_{2m}^2D_2^2}{\Box_{2m}}\delta_{2m,2}\frac{1}{\Box_{2m+1}}\delta_{2m+1,2}\cdots$$

$$\left.\frac{1}{\Box_{2m}}\delta_{2m+1,2m}\cdots\frac{1}{\Box_4}\delta_{5,4}\right\}$$

$$\mathbf{W}'^{(m)} = (-1)^{m-1}\frac{g^{2m}}{2^{m+2}}N^{m-2}\Upsilon^{(m)}\times W_{\text{tree}}$$

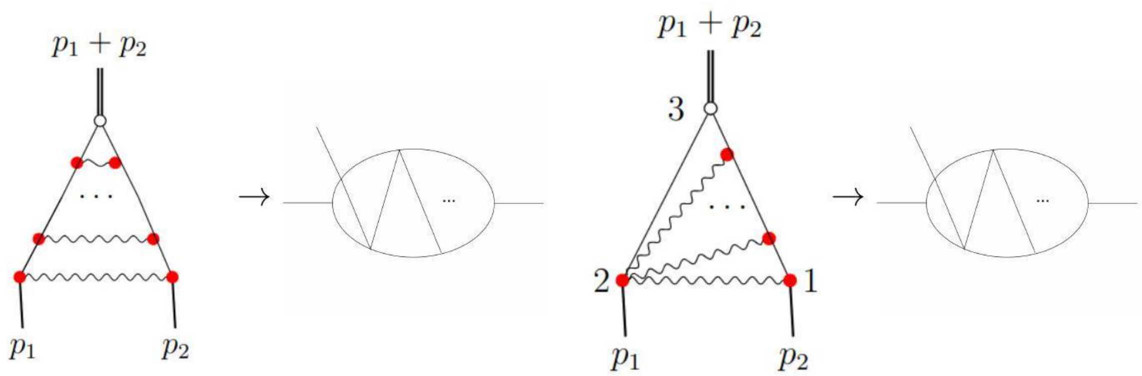
$$\mathbf{W}'^{lead} = \frac{y}{2N^2}\int_0^1 d\tau\frac{\log(\tau)(1-\tau)}{(1+y\log^2(\tau))(1+\tau^3)}\times W_{\text{tree}} = \Upsilon^{\text{tot}}\times W_{\text{tree}}$$

$$\Upsilon^{tot} = \frac{1}{4N^2}\sum_{m=1}^{\infty}((\pi-2\text{Si}(x))\sin(x)-2\text{Ci}(x)\cos(x))U_m(1/2)$$

$$\Upsilon^{tot}|_{\hbar\to\infty} \simeq \frac{1}{2N^2}\log\left(\frac{12e^{\gamma_E}}{\sqrt{\hbar}gN}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}\right) + O(\hbar^{-2})$$

$$Z(L+1) = 4C_L\sum_{p=1}^{\infty}\frac{(-1)^{(p-1)(L+1)}}{p^{2(L+1)-3}} = \begin{cases} 4C_L\zeta(2L-1) \text{ for } L = 2N+1 \\ 4C_L\left(1-2^{2(1-L)}\right)\zeta(2L-1) \text{ for } L = 2N \end{cases}$$

$$\mathbf{W}^{\text{sub}} \sim g^{2L}c_L/N^L \times Z(L+1) \times W_{\text{tree}},$$



$$\sigma^\mu = (\sigma_0, -\vec{\boldsymbol{\sigma}}) \text{ and } \bar{\sigma}^\mu = (\sigma_0, \vec{\boldsymbol{\sigma}})$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}, \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}\ \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma.$$

$$\left(\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu\right)^\alpha_\beta = -2\eta_{\mu\nu}\delta^\alpha_\beta \text{ and } \left(\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu\right)^{\dot{\alpha}}_{\dot{\beta}} = -2\eta_{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}$$

$$\begin{gathered} \text{tr}(\text{ odd number of } \sigma's) = 0 \\ \text{tr}(\sigma^\mu\bar{\sigma}^\nu) = \text{tr}(\bar{\sigma}^\mu\sigma^\nu) = -2\eta^{\mu\nu} \end{gathered}$$

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \bar{D}_{\dot{\alpha}} = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu$$

$$\begin{gathered} \{D_\alpha, D_\beta\} = 0, \{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma)^\mu_{\alpha\dot{\beta}}\partial_\mu \\ D^2\bar{D}_{\dot{\alpha}}D^2 = 0, \bar{D}^2 D_\alpha\bar{D}^2 = 0 \\ D^\alpha\bar{D}^2 D_\alpha = \bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}} \\ D^2\bar{D}^2 + \bar{D}^2 D^2 - 2D^\alpha\bar{D}^2 D_\alpha = 16\,\square \\ D^2\bar{D}^2 D^2 = 16\,\square\, D^2, \bar{D}^2 D^2\bar{D}^2 = 16\,\square\,\bar{D}^2 \\ [D^2, \bar{D}_{\dot{\alpha}}] = -4i\partial_{\alpha\dot{\alpha}}D^\alpha, [\bar{D}^2, D_\alpha] = 4i\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}} \end{gathered}$$

$$\begin{gathered} \int d^2\theta = -\frac{1}{4}D^2, \int d^2\bar{\theta} = \frac{1}{4}\bar{D}^2 \\ d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}d\theta_\alpha d\theta_\beta, \int d\theta_\alpha\theta^\beta = \delta^\beta_\alpha \end{gathered}$$

$$J^{(1)}_{\alpha,\beta}(k) = \int \frac{d^d q}{(q-k)^{2\alpha}q^{2\beta}} = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-d/2)}(k^2/\mu^2)^{d/2-\alpha-\beta},$$

$$J^{(1)}_{1,1}(k) = (k^2/\mu^2)^{-\epsilon}\left(\frac{1}{\epsilon} + 2 + O(\epsilon^1)\right).$$

$$\overset{\Rightarrow}{=} = \langle \phi_a \bar{\phi}_b \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\,\Box} \delta_{ab} \delta^8(z_1 - z_2)$$
$$-\langle \psi_i \bar{\psi}_j \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\,\Box} \delta_{ij} \delta^8(z_1 - z_2)$$
$$\sim m = \langle v_a v_b \rangle = \left(-\frac{D^\alpha \bar{D}^2 D_\alpha}{8\,\Box^2} + \xi \frac{\{D^2, \bar{D}^2\}}{16\,\Box^2} \right) \delta_{ab} \delta^8(z_1 - z_2) \overset{\xi=1}{=} \frac{1}{16\,\Box} \delta_{ab} \delta^8(z_1 - z_2)$$
$$= \langle c'_a c_b \rangle = \langle c'_b c_a \rangle = \frac{1}{16\,\Box} \delta_{ab} \delta^8(z_1 - z_2)$$

$$J^{(2)} = \lim_{p_1, p_2 \to 0} \int \frac{d^4 q_1}{(4\pi)^2} \frac{d^4 q_2}{(4\pi)^2} \frac{q_1^2 (p_1 + p_2)^2}{q_1^2 (q_1 - p_1)^2 (q_2 - p_2)^2 q_2^2 (q_2 - p_2)^2 (q_1 - p_2)^2} = 6\zeta(3).$$

$$J^{(3)} = 20\zeta(5), J^{(4)} = \frac{441}{8} \zeta(7),$$

$$\Upsilon_L(u, v) = \frac{\Gamma(1 - \epsilon)}{u^{2(1-\epsilon)}} \sum_{l=0}^{\infty} \frac{\epsilon^l}{l!} \Upsilon_L^{(l)}(z_1, z_2)$$

$$\Upsilon_L^{(l)}(z_1, z_2) = \sum_{f=0}^{L} \frac{\left(-\ln(z_1 z_2)\right)^f (2L - f)}{f!\,(L - f)!} \sum_{m=0}^{l} \frac{(-1)^m l!}{m!\,(l - m)!} \mathbf{Z}_m(z_1, z_2; 2L + l - f)$$

$$\mathbf{Z}_m(z_1, z_2; k) = \frac{\Gamma(k - m)}{(z_1 - z_2)} \sum_{\{n_i\}=1}^{\infty} \frac{\left(z_1^{n_0} - z_2^{n_0}\right)}{\left(\sum_0^m n_i\right)^{k-m}} \left(\prod_{i=1}^{m} \frac{z_1^{n_i} + z_2^{n_i}}{n_i} \right)$$

$$\Upsilon^{(l)} = \int_0^1 d\tau \frac{2\log^{2l-1}(\tau)}{\tau^2 - \tau + 1}$$
$$= \frac{1}{2^{2l-1} 3^{2l}} \left(\psi^{(2l+1)}\left(\frac{2}{3}\right) - \psi^{(2l+1)}\left(\frac{1}{3}\right) - \psi^{(2l+1)}\left(\frac{1}{6}\right) + \psi^{(2l+1)}\left(\frac{5}{6}\right) \right)$$

$$(\delta A_{\mu\nu\sigma} / \delta A_{\alpha\beta\gamma}) = \frac{1}{3!} \left[\delta_\mu^\alpha \left(\delta_\nu^\beta \delta_\sigma^\gamma - \delta_\sigma^\beta \delta_\nu^\gamma \right) + \delta_\nu^\alpha \left(\delta_\sigma^\beta \delta_\mu^\gamma - \delta_\mu^\beta \delta_\sigma^\gamma \right) + \delta_\sigma^\alpha \left(\delta_\mu^\beta \delta_\nu^\gamma - \delta_\nu^\beta \delta_\mu^\gamma \right) \right]$$

$$\mathcal{L}_{(NG)} = \frac{1}{2} B_2 (\partial \cdot \phi) - \frac{1}{4} B_2^2 + \frac{1}{2} B_3 (\partial \cdot \tilde{\phi}) - \frac{1}{4} B_3^2 + \frac{1}{2} B^2 - B(\partial \cdot A)$$
$$+ \frac{1}{2} B_1^2 + B_1 \left(\frac{1}{3!} \varepsilon^{\mu\nu\sigma\rho} \partial_\mu A_{\nu\sigma\rho} \right) - \frac{1}{4} \left(B_{\mu\nu} \right)^2 + \frac{1}{2} B_{\mu\nu} \left[\partial_\sigma A^{\sigma\mu\nu} + \frac{1}{2} (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) \right]$$
$$- \frac{1}{4} \left(\mathcal{B}_{\mu\nu} \right)^2 + \frac{1}{2} \mathcal{B}_{\mu\nu} \left[\varepsilon^{\mu\nu\sigma\rho} \partial_\sigma A_\rho + \frac{1}{2} (\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu) \right],$$

$$\mathcal{L}_{(0)} = \frac{1}{4} (\partial \cdot \phi)^2 + \frac{1}{4} (\partial \cdot \tilde{\phi})^2 - \frac{1}{2} (\partial \cdot A)^2 - \frac{1}{2} \left(-\frac{1}{3!} \varepsilon^{\mu\nu\sigma\rho} \partial_\mu A_{\nu\sigma\rho} \right)^2$$
$$+ \frac{1}{4} \left[\partial_\sigma A^{\sigma\mu\nu} + \frac{1}{2} (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) \right]^2 + \frac{1}{4} \left[\varepsilon^{\mu\nu\sigma\rho} \partial_\sigma A_\rho + \frac{1}{2} (\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu) \right]^2.$$

$$\mathcal{L}_{(FP)} = \frac{1}{2} \left[(\partial_\mu \bar{C}_{\nu\sigma} + \partial_\nu \bar{C}_{\sigma\mu} + \partial_\sigma \bar{C}_{\mu\nu})(\partial^\mu C^{\nu\sigma}) + (\partial_\mu \bar{C}^{\mu\nu} + \partial^\nu \bar{C}_1) f_\nu \right.$$
$$- (\partial_\mu C^{\mu\nu} + \partial^\nu C_1) \bar{F}_\nu + (\partial \cdot \bar{\beta}) B_4 - (\partial \cdot \beta) B_5 - B_4 B_5 - 2\bar{F}^\mu f_\mu$$
$$\left. - (\partial_\mu \bar{\beta}_\nu - \partial_\mu \bar{\beta}_\nu)(\partial^\mu \beta^\nu) - \partial_\mu \bar{C}_2 \partial^\mu C_2 \right] - \partial_\mu \bar{C} \partial^\mu C$$

$$
\begin{aligned}
&s_b A_{\mu\nu\sigma} = \partial_\mu C_{\nu\sigma} + \partial_\nu C_{\sigma\mu} + \partial_\sigma C_{\mu\nu}, s_b C_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, s_b \bar{C}_{\mu\nu} = B_{\mu\nu} \\
&s_b A_\mu = \partial_\mu C, s_b \bar{C} = B, s_b \bar{\beta}_\mu = \bar{F}_\mu, s_b \beta_\mu = \partial_\mu C_2, s_b \bar{B}_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu \\
&s_b \bar{C}_2 = B_5, s_b C_1 = -B_4, s_b \bar{C}_1 = B_2, s_b \phi_\mu = f_\mu, s_b F_\mu = -\partial_\mu B_4 \\
&s_b \bar{f}_\mu = \partial_\mu B_2, s_b\left[C_2, C, f_\mu, \bar{F}_\mu, \tilde{\phi}_\mu, B, B_1, B_2, B_3, B_4, B_5, B_{\mu\nu}, \mathcal{B}_{\mu\nu}, \overline{\mathcal{B}}_{\mu\nu}\right] = 0,
\end{aligned}
$$

$$
\begin{aligned}
s_b \mathcal{L}_{(B)} = \frac{1}{2}\partial_\mu[&(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})B_{\nu\sigma} + B^{\mu\nu} f_\nu - B_5 \partial^\mu C_2 \\
&+ B_2 f^\mu + B_4 \bar{F}^\mu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)\bar{F}_\nu] - \partial_\mu[B \partial^\mu C]
\end{aligned}
$$

$$
\delta A^{(3)} = - * d * \left[\frac{1}{3!} A_{\mu\nu\sigma}(dx^\mu \wedge dx^\nu \wedge dx^\sigma)\right] = -\frac{1}{2}\left(\partial^\sigma A_{\sigma\mu\nu}\right)(dx^\mu \wedge dx^\nu)
$$

$$
\begin{aligned}
&s_{ab} A_{\mu\nu\sigma} = \partial_\mu \bar{C}_{\nu\sigma} + \partial_\nu \bar{C}_{\sigma\mu} + \partial_\sigma \bar{C}_{\mu\nu}, s_{ab} \bar{C}_{\mu\nu} = \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, s_{ab} C_{\mu\nu} = \bar{B}_{\mu\nu} \\
&s_{ab} A_\mu = \partial_\mu \bar{C}, s_{ab} C = -B, s_{ab} \beta_\mu = F_\mu, s_{ab} \bar{\beta}_\mu = \partial_\mu \bar{C}_2, s_{ab} B_{\mu\nu} = \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu \\
&s_{ab} C_2 = B_4, s_{ab} C_1 = -B_2, s_{ab} \bar{C}_1 = -B_5, s_{ab} \phi_\mu = \bar{f}_\mu, s_{ab} \bar{F}_\mu = -\partial_\mu B_5 \\
&s_{ab} f_\mu = -\partial_\mu B_2, s_{ab}\left[\bar{C}_2, \bar{C}, \bar{f}_\mu, F_\mu, \tilde{\phi}_\mu, B, B_1, B_2, B_3, B_4, B_5, \bar{B}_{\mu\nu}, \overline{\mathcal{B}}_{\mu\nu}, \mathcal{B}_{\mu\nu}\right] = 0,
\end{aligned}
$$

$$
\begin{aligned}
s_{ab} \mathcal{L}_{(\bar{B})} = \frac{1}{2}\partial_\mu\big[&-(\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})\bar{B}_{\nu\sigma} + \bar{B}^{\mu\nu} \bar{f}_\nu + B_4 \partial^\mu \bar{C}_2 \\
&- B_5 F^\mu + B_2 \bar{f}^\mu - \left(\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu\right)F_\nu\big] - \partial_\mu[B \partial^\mu \bar{C}].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{(ng)} &= \frac{1}{2}B^2 - B(\partial \cdot A) + \frac{1}{2}B_2(\partial \cdot \phi) - \frac{1}{4}B_2^2 + \frac{1}{2}B_3(\partial \cdot \tilde{\phi}) - \frac{1}{4}B_3^2 \\
&+ \frac{1}{2}B_1^2 + B_1\left(\frac{1}{3!}\varepsilon^{\mu\nu\sigma\rho}\partial_\mu A_{\nu\sigma\rho}\right) - \frac{1}{4}\left(\bar{B}_{\mu\nu}\right)^2 - \frac{1}{2}\bar{B}_{\mu\nu}\left[\partial_\sigma A^{\sigma\mu\nu} - \frac{1}{2}(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu)\right] \\
&- \frac{1}{4}\left(\overline{\mathcal{B}}_{\mu\nu}\right)^2 - \frac{1}{2}\overline{\mathcal{B}}_{\mu\nu}\left[\varepsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho - \frac{1}{2}\left(\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{(fp)} = \frac{1}{2}\big[&\left(\partial_\mu \bar{C}_{\nu\sigma} + \partial_\nu \bar{C}_{\sigma\mu} + \partial_\sigma \bar{C}_{\mu\nu}\right)(\partial^\mu C^{\nu\sigma}) - \left(\partial_\mu \bar{C}^{\mu\nu} - \partial^\nu \bar{C}_1\right)F_\nu \\
&+ \left(\partial_\mu C^{\mu\nu} - \partial^\nu C_1\right)\bar{f}_\nu + (\partial \cdot \bar{\beta})B_4 - (\partial \cdot \beta)B_5 - B_4 B_5 - 2\bar{f}^\mu F_\mu \\
&- \left(\partial_\mu \bar{\beta}_\nu - \partial_\mu \bar{\beta}_\nu\right)(\partial^\mu \beta^\nu) - \partial_\mu \bar{C}_2 \partial^\mu C_2\big] - \partial_\mu \bar{C} \partial^\mu C
\end{aligned}
$$

$$
\begin{aligned}
&\{s_b, s_{ab}\} A_{\mu\nu\sigma} = \partial_\mu(B_{\nu\sigma} + \bar{B}_{\nu\sigma}) + \partial_\nu\left(B_{\sigma\mu} + \bar{B}_{\sigma\mu}\right) + \partial_\sigma\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) \\
&\{s_b, s_{ab}\} C_{\mu\nu} = \partial_\mu(f_\nu + F_\nu) - \partial_\nu\left(f_\mu + F_\mu\right) \\
&\{s_b, s_{ab}\} \bar{C}_{\mu\nu} = \partial_\mu\left(\bar{f}_\nu + \bar{F}_\nu\right) - \partial_\nu\left(\bar{f}_\mu + \bar{F}_\mu\right)
\end{aligned}
$$

$$
B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, f_\mu + F_\mu = \partial_\mu C_1, \bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1
$$

$$
\begin{aligned}
&s_{(a)b}\left[B_{\mu\nu} + \bar{B}_{\mu\nu} - \left(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu\right)\right] = 0, s_{(a)b}\left[f_\mu + F_\mu - \partial_\mu C_1\right] = 0 \\
&s_{(a)b}\left[\bar{f}_\mu + \bar{F}_\mu - \partial_\mu \bar{C}_1\right] = 0, s_{(a)b}\left[\mathcal{B}_{\mu\nu} + \overline{\mathcal{B}}_{\mu\nu} - \left(\partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu\right)\right] = 0,
\end{aligned}
$$

$$\partial_\mu\left(\partial^\mu\bar{C}^{\nu\lambda}+\partial^\nu\bar{C}^{\lambda\mu}+\partial^\lambda\bar{C}^{\mu\nu}\right)+\frac{1}{2}\left(\partial^\nu\bar{F}^\lambda-\partial^\lambda\bar{F}^\nu\right)=0$$
$$\partial_\mu\left(\partial^\mu\bar{C}^{\nu\lambda}+\partial^\nu\bar{C}^{\lambda\mu}+\partial^\lambda\bar{C}^{\mu\nu}\right)-\frac{1}{2}\left(\partial^\nu\bar{f}^\lambda-\partial^\lambda\bar{f}^\nu\right)=0$$
$$\partial_\mu\left(\partial^\mu C^{\nu\lambda}+\partial^\nu C^{\lambda\mu}+\partial^\lambda C^{\mu\nu}\right)+\frac{1}{2}\left(\partial^\nu f^\lambda-\partial^\lambda f^\nu\right)=0$$
$$\partial_\mu\left(\partial^\mu C^{\nu\lambda}+\partial^\nu C^{\lambda\mu}+\partial^\lambda C^{\mu\nu}\right)-\frac{1}{2}\left(\partial^\nu F^\lambda-\partial^\lambda F^\nu\right)=0$$

$$\partial_\mu\left(\bar{f}_\nu+\bar{F}_\nu\right)-\partial_\nu\left(\bar{f}_\mu+\bar{F}_\mu\right)=0,\partial_\mu\left(f_\nu+F_\nu\right)-\partial_\nu\left(f_\mu+F_\mu\right)=0$$

$$f_\mu+F_\mu=\pm\partial_\mu C_1,\bar{f}_\mu+\bar{F}_\mu=\pm\partial_\mu\bar{C}_1$$

$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\mathcal{B}_{\sigma\rho}+\partial^\mu B=0,\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu B_{\sigma\rho}+\partial^\mu B_1=0$$

$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\overline{\mathcal{B}}_{\sigma\rho}-\partial^\mu B=0,\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\bar{B}_{\sigma\rho}-\partial^\mu B_1=0$$

$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\left[\mathcal{B}_{\sigma\rho}+\overline{\mathcal{B}}_{\sigma\rho}\right]=0,\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\left[B_{\sigma\rho}+\bar{B}_{\sigma\rho}\right]=0$$

$$B_{\mu\nu}+\bar{B}_{\mu\nu}=\pm\left(\partial_\mu\phi_\nu-\partial_\nu\phi_\mu\right),\mathcal{B}_{\mu\nu}+\overline{\mathcal{B}}_{\mu\nu}=\pm\left(\partial_\mu\tilde{\phi}_\nu-\partial_\nu\tilde{\phi}_\mu\right)$$

$$J^\mu_{(b)}=s_b\Phi_i\frac{\partial\mathcal{L}_{(B)}}{\partial\left(\partial_\mu\Phi_i\right)}+B\partial^\mu C+\frac{1}{2}[B_5\partial^\mu C_2+(\partial^\mu\beta^\nu-\partial^\nu\beta^\mu)\bar{F}_\nu]$$
$$-\frac{1}{2}[(\partial^\mu C^{\nu\sigma}+\partial^\nu C^{\sigma\mu}+\partial^\sigma C^{\mu\nu})B_{\nu\sigma}+B^{\mu\nu}f_\nu+B_2f^\mu+B_4\bar{F}^\mu]$$

$$\begin{aligned}J^\mu_{(b)}=\frac{1}{2}\big[&\varepsilon^{\mu\nu\sigma\rho}B_1\left(\partial_\nu C_{\sigma\rho}\right)+(\eta^{\mu\nu}B^{\sigma\rho}+\eta^{\mu\sigma}B^{\rho\nu}+\eta^{\mu\rho}B^{\nu\sigma})\left(\partial_\nu C_{\sigma\rho}\right)+B^{\mu\nu}f_\nu\\&+\varepsilon^{\mu\nu\sigma\rho}(\partial_\nu C)\mathcal{B}_{\sigma\rho}-B_5\partial^\mu C_2-\left(\partial^\mu\bar{\beta}^\nu-\partial^\nu\bar{\beta}^\mu\right)\partial_\nu C_2-\left(\partial^\mu\beta^\nu-\partial^\nu\beta^\mu\right)\bar{F}_\nu\\&+B_2f^\mu+B_4\bar{F}^\mu-(\partial^\mu\bar{C}^{\nu\sigma}+\partial^\nu\bar{C}^{\sigma\mu}+\partial^\sigma\bar{C}^{\mu\nu})(\partial_\nu\beta_\sigma-\partial_\sigma\beta_\nu)\big]-B\partial^\mu C\end{aligned}$$

$$(\partial^\mu B^{\nu\sigma}+\partial^\nu B^{\sigma\mu}+\partial^\sigma B^{\mu\nu})-\varepsilon^{\mu\nu\sigma\rho}\partial_\rho B_1=0,\partial_\mu B^{\mu\nu}+\partial^\nu B_2=0$$

$$\begin{aligned}Q_b=\int d^3x\Big[&\frac{1}{2}\{\varepsilon^{0ijk}B_1\left(\partial_iC_{jk}\right)+\left(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i}\right)B_{ij}+B^{0i}f_i\\&+\varepsilon^{0ijk}(\partial_iC)\mathcal{B}_{jk}-B_5\partial^0C_2-\left(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\right)\partial_iC_2-\left(\partial^0\beta^i-\partial^i\beta^0\right)\bar{F}_i\\&+B_2f^0+B_4\bar{F}^0-\left(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\right)\left(\partial_i\beta_j-\partial_j\beta_i\right)\}-B\partial^0C\Big]\end{aligned}$$

$$s_b\Phi_i=\pm i[\Phi_i,Q_b]_{(\pm)}$$

$$\begin{aligned}s_bQ_b=+i\{Q_b,Q_b\}\equiv-\frac{1}{2}\int d^3x\big[&\left(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i}\right)\left(\partial_i\beta_j-\partial_j\beta_i\right)\\&+\left(\partial^0\bar{F}^i-\partial^i\bar{F}^0\right)\partial_iC_2\big]\neq0\end{aligned}$$

$$\int d^3x\left[\frac{1}{2}\varepsilon^{0ijk}(\partial_iC)\mathcal{B}_{jk}-B\partial^0C\right]$$

$$\frac{1}{2}\int d^3x\left[-\frac{1}{2}\varepsilon^{0ijk}C\big(\partial_i\mathcal{B}_{jk}\big)-B\dot{C}\right]$$

$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\mathcal{B}_{\sigma\rho}+\partial^\mu B=0$$

$$\int d^3x(\dot{B}C-B\dot{C})$$

$$\frac{1}{2}\int d^3x\varepsilon^{0ijk}B_1\big(\partial_iC_{jk}\big)\equiv-\frac{1}{2}\int d^3x\varepsilon^{0ijk}(\partial_iB_1)C_{jk}$$

$$\varepsilon^{\mu\nu\sigma\rho}\partial_\rho B_1=(\partial^\mu B^{\nu\sigma}+\partial^\nu B^{\sigma\mu}+\partial^\sigma B^{\mu\nu})\Longrightarrow\varepsilon^{oijk}\partial_kB_1=\big(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i}\big)$$

$$-\frac{1}{2}\int d^3x\varepsilon^{0ijk}(\partial_iB_1)C_{jk}=-\frac{1}{2}\int d^3x(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i})C_{ij}$$

$$-\frac{1}{2}\int d^3x\big(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i}\big)\big(\partial_i\beta_j-\partial_j\beta_i\big)$$

$$+\frac{1}{2}\int d^3x\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\big(\partial_i\beta_j-\partial_j\beta_i\big)$$
$$-\int d^3x\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\big(\partial_i\beta_j-\partial_j\beta_i\big)$$

$$+\frac{1}{2}\int d^3x\big[\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\big(\partial_i\beta_j-\partial_j\beta_i\big)-\big(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i}\big)C_{ij}\big]$$

$$-2\int d^3x\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\big(\partial_i\beta_j\big)$$

$$+2\int d^3x\partial_i\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\beta_j=\int d^3x\big(\partial^0\bar{F}^j-\partial^j\bar{F}^0\big)\beta_j$$

$$s_b\left[\int d^3x\big(\partial^0\bar{F}^j-\partial^j\bar{F}^0\big)\beta_j\right]=-\int d^3x\big(\partial^0\bar{F}^j-\partial^j\bar{F}^0\big)\partial_iC_2$$

$$-\frac{1}{2}\int d^3x\big(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\big)\partial_iC_2=+\int d^3x\big(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\big)\partial_iC_2$$
$$-\frac{3}{2}\int d^3x\big(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\big)\partial_iC_2$$

$$+\frac{3}{2}\int d^3x\partial_i\big(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\big)C_2=+\frac{3}{2}\int d^3x\dot{B}_5C_2$$

$$Q_B=\int d^3x\Big[\frac{1}{2}\{(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})B_{ij}-\big(\partial^0B^{ij}+\partial^iB^{j0}+\partial^jB^{0i}\big)C_{ij}+B^{0i}f_i$$
$$+3\dot{B}_5C_2-B_5\dot{C}_2-\big(\partial^0\beta^i-\partial^i\beta^0\big)\bar{F}_i+\big(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i}\big)\big(\partial_i\beta_j-\partial_j\beta_i\big)$$
$$+B_4\bar{F}^0+B_2f^0\}+(\dot{B}C-B\dot{C})+\big(\partial^0\bar{F}^i-\partial^i\bar{F}^0\big)\beta_i+\big(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0\big)\partial_iC_2\Big]$$

$$\Pi^{\mu}_{(A1)} = \frac{\partial \mathcal{L}_{(B)}}{\partial(\partial_0 A_\mu)} = \frac{1}{2}\varepsilon^{0\mu\nu\sigma}\mathcal{B}_{\nu\sigma} - \eta^{0\mu}B \Longrightarrow \Pi^{0}_{(A1)} = -B, \Pi^{i}_{(A1)} = \frac{1}{2}\varepsilon^{0ijk}\mathcal{B}_{jk}$$

$$\Pi^{\mu\nu\sigma}_{(A3)} = \frac{\partial \mathcal{L}_{(B)}}{\partial(\partial_0 A_{\mu\nu\sigma})} = \frac{1}{3!}[\varepsilon^{0\mu\nu\sigma}B_1 + (\eta^{0\mu}B^{\nu\sigma} + \eta^{0\nu}B^{\sigma\mu} + \eta^{0\sigma}B^{\mu\nu})] \Longrightarrow$$

$$\Pi^{0ij}_{(A3)} = \frac{1}{3!}B_{ij}, \Pi^{ijk}_{(A3)} = \frac{1}{3!}\varepsilon^{0ijk}B_1$$

$$\frac{1}{3!}\varepsilon^{\mu\nu\sigma\rho}\partial_\rho B_1 = \frac{1}{3!}(\partial^\mu B^{\nu\sigma} + \partial^\nu B^{\sigma\mu} + \partial^\sigma B^{\mu\nu})$$

$$\Longrightarrow \partial_k \Pi^{kij}_{(A3)} = \frac{1}{3!}(\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i})$$

$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu \mathcal{B}_{\sigma\rho} = \partial^\mu B \Longrightarrow \partial_i \Pi^{i}_{(A1)} = \dot{B}$$

$$B\|\text{phys}\rangle = 0, \Longrightarrow (\partial\cdot A)\|\text{phys}\rangle = 0 \Longrightarrow \Pi^{0}_{(A1)}\|\text{phys}\rangle = 0,$$

$$\dot{B}\|\text{phys}\rangle = 0, \Longrightarrow \frac{1}{2}\varepsilon^{0ijk}\partial_i \mathcal{B}_{jk}\|\text{phys}\rangle = 0 \Longrightarrow \partial_i \Pi^{i}_{(A1)}\|\text{phys}\rangle = 0,$$

$$\frac{1}{2}B_{ij}\|\text{phys}\rangle = 0, \Longrightarrow 3\Pi^{0ij}_{(A3)}\|\text{phys}\rangle = 0$$

$$\frac{1}{2}(\partial^0 B^{ij} + (\partial^i B^{j0} + (\partial^j B^{0i})\|\text{phys}\rangle = 0, \Longrightarrow \frac{1}{2}\varepsilon^{0ijk}\partial_k B_1\|\text{phys}\rangle = 0$$

$$\Longrightarrow 3\partial_k \Pi^{kij}_{(A3)}\|\text{phys}\rangle = 0$$

$$J^{\mu}_{(ab)} = s_{ab}\Phi_i \frac{\partial \mathcal{L}_{(\bar{B})}}{\partial(\partial_\mu \Phi_i)} + B\partial^\mu \bar{C} - \frac{1}{2}[B_4\partial^\mu \bar{C}_2 + \bar{B}^{\mu\nu}\bar{f}_\nu + B_2\bar{f}^\mu]$$
$$+\frac{1}{2}[(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\bar{B}_{\nu\sigma} + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)F_\nu + B_5 F^\mu]$$

$$J^{\mu}_{(ab)} = \frac{1}{2}[\varepsilon^{\mu\nu\sigma\rho}B_1(\partial_\nu \bar{C}_{\sigma\rho}) - (\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})\bar{B}_{\nu\sigma} + \bar{B}^{\mu\nu}\bar{f}_\nu$$
$$-\varepsilon^{\mu\nu\sigma\rho}(\partial_\nu \bar{C})\overline{\mathcal{B}}_{\sigma\rho} + B_4\partial^\mu \bar{C}_2 - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)\partial_\nu \bar{C}_2 - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)F_\nu$$
$$-B_5F^\mu + B_2\bar{f}^\mu + (\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})(\partial_\nu \bar{\beta}_\sigma - \partial_\sigma \bar{\beta}_\nu)] - B\partial^\mu \bar{C}$$

$$(\partial^\mu \bar{B}^{\nu\sigma} + \partial^\nu \bar{B}^{\sigma\mu} + \partial^\sigma \bar{B}^{\mu\nu}) + \varepsilon^{\mu\nu\sigma\rho}\partial_\rho B_1 = 0, \partial_\mu \bar{B}^{\mu\nu} + \partial^\nu B_2 = 0$$

$$Q_{ab} = \int d^3x \Big[\frac{1}{2}\{\varepsilon^{0ijk}B_1(\partial_i \bar{C}_{jk}) - (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i})\bar{B}_{ij} + \bar{B}^{0i}\bar{f}_i$$
$$-\varepsilon^{0ijk}(\partial_i \bar{C})\overline{\mathcal{B}}_{jk} + B_4\partial^0 \bar{C}_2 - (\partial^0 \beta^i - \partial^i \beta^0)\partial_i \bar{C}_2 - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)F_i$$
$$-B_5F^0 + B_2\bar{f}^0 + (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})(\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i)\} - B\partial^0 \bar{C}\Big]$$

$$s_{ab}Q_{ab} = +i\{Q_{ab}, Q_{ab}\} \equiv \frac{1}{2}\int d^3x[(\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i})(\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i)$$
$$-(\partial^0 F^i - \partial^i F^0)\partial_i \bar{C}_2] \neq 0$$

$$\int d^3x\Big[-\frac{1}{2}\varepsilon^{0ijk}(\partial_i \bar{C})\overline{\mathcal{B}}_{jk} - B\partial^0 \bar{C}\Big]$$

$$\int d^3x\left[+\frac{1}{2}\varepsilon^{0ijk}\bar{C}(\partial_i\bar{\mathcal{B}}_{jk})-B\partial^0\bar{C}\right]\equiv\int d^3x(\dot{B}\bar{C}-B\dot{\bar{C}})$$

$$\frac{1}{2}\int d^3x\left[\varepsilon^{0ijk}B_1(\partial_i\bar{C}_{jk})=-\frac{1}{2}\int d^3x[\varepsilon^{0ijk}(\partial_iB_1)\bar{C}_{jk}\right.$$

$$-\frac{1}{2}\int d^3x\left[\varepsilon^{0ijk}(\partial_iB_1)\bar{C}_{jk}=+\frac{1}{2}\int d^3x(\partial^0\bar{B}^{ij}+\partial^i\bar{B}^{j0}+\partial^j\bar{B}^{0i})\bar{C}_{ij}\right.$$

$$\frac{1}{2}\int d^3x(\partial^0\bar{B}^{ij}+\partial^i\bar{B}^{j0}+\partial^j\bar{B}^{0i})(\partial_i\bar{\beta}_j-\partial_j\bar{\beta}_i)$$

$$-\frac{1}{2}\int d^3x(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})(\partial_i\bar{\beta}_j-\partial_j\bar{\beta}_i)$$
$$+\int d^3x(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})(\partial_i\bar{\beta}_j-\partial_j\bar{\beta}_i)$$

$$+2\int d^3x(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})(\partial_i\bar{\beta}_j)=-2\int d^3x\partial_i(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})\bar{\beta}_j$$

$$+\int d^3x(\partial^0F^i-\partial^iF^0)\bar{\beta}_i$$

$$-\int d^3x(\partial^0F^i-\partial^iF^0)\partial_i\bar{C}_2$$

$$+\int d^3x(\partial^0\beta^i-\partial^i\beta^0)\partial_i\bar{C}_2-\frac{3}{2}\int d^3x(\partial^0\beta^i-\partial^i\beta^0)\partial_i\bar{C}_2$$

$$+\frac{3}{2}\int d^3x\partial_i(\partial^0\beta^i-\partial^i\beta^0)\bar{C}_2=-\frac{3}{2}\int d^3x(\dot{B}_4\bar{C}_2)$$

$$\begin{aligned}Q_{AB}=\int d^3x\Big[&\frac{1}{2}\{B_4\dot{\bar{C}}_2-3\dot{B}_4\bar{C}_2-(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i})\bar{B}_{ij}-(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0)F_i\\&+(\partial^0\bar{B}^{ij}+\partial^i\bar{B}^{j0}+\partial^j\bar{B}^{0i})\bar{C}_{ij}-(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})(\partial_i\bar{\beta}_j-\partial_j\bar{\beta}_i)-B_2\bar{f}^0\\&+\bar{B}^{0i}\bar{f}_i-B_5F^0\}+(\dot{B}\bar{C}-B\dot{\bar{C}})+(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0)\partial_i\bar{C}_2+(\partial^0F^i-\partial^iF^0)\bar{\beta}_i\Big].\end{aligned}$$

$$\Pi^{\mu\nu\sigma}_{(A3)}=\frac{\partial\mathcal{L}_{(\bar{B})}}{\partial(\partial_0A_{\mu\nu\sigma})}=\frac{1}{3!}[\varepsilon^{0\mu\nu\sigma}B_1-(\eta^{0\mu}\bar{B}^{\nu\sigma}+\eta^{0\nu}\bar{B}^{\sigma\mu}+\eta^{0\sigma}\bar{B}^{\mu\nu})]\Longrightarrow$$
$$\Pi^{0ij}_{(A3)}=-\frac{1}{3!}\bar{B}_{ij},\Pi^{ijk}_{(A3)}=\frac{1}{3!}\varepsilon^{0ijk}B_1$$
$$\Pi^{\mu}_{(A1)}=\frac{\partial\mathcal{L}_{(\bar{B})}}{\partial(\partial_0A_\mu)}=-\frac{1}{2}\varepsilon^{0\mu\nu\sigma}\bar{\mathcal{B}}_{\nu\sigma}-\eta^{0\mu}B\Longrightarrow\Pi^0_{(A1)}=-B,\Pi^i_{(A1)}=-\frac{1}{2}\varepsilon^{0ijk}\bar{\mathcal{B}}_{jk}$$

$$\frac{1}{3!}\varepsilon^{\mu\nu\sigma\rho}\partial_\rho B_1=-\frac{1}{3!}(\partial^\mu\bar{B}^{\nu\sigma}+\partial^\nu\bar{B}^{\sigma\mu}+\partial^\sigma\bar{B}^{\mu\nu})$$
$$\Longrightarrow\partial_k\Pi^{kij}_{(A3)}=-\frac{1}{3!}(\partial^0\bar{B}^{ij}+\partial^i\bar{B}^{j0}+\partial^j\bar{B}^{0i})$$
$$\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\partial_\nu\bar{\mathcal{B}}_{\sigma\rho}=-\partial^\mu B\Longrightarrow\partial_i\Pi^i_{(A1)}=+\dot{B}$$

$$B\|\text{phys}\rangle = 0 \Longrightarrow \Pi^0_{(A1)}\|\text{phys}\rangle = 0, \dot{B}\|\text{phys}\rangle = 0 \Longrightarrow \partial_i \Pi^i_{(A1)}\|\text{phys}\rangle = 0$$

$$\frac{1}{2}\left(\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i}\right)\|\text{phys}\rangle = 0 \implies -3\partial_k \Pi^{kij}_{(A3)}\|\text{phys}\rangle = 0$$

$$-\frac{1}{2}\bar{B}_{ij}\|\text{phys}\rangle = 0 \implies 3\Pi^{0ij}_{(A3)}\|\text{phys}\rangle = 0$$

$$\Pi^{\mu}_{(\phi)} = \frac{\partial \mathcal{L}_{(\bar{B})}}{\partial\left(\partial_0 \phi_\mu\right)} = -\frac{1}{2}\bar{B}^{0\mu} + \frac{1}{2}\eta^{0\mu}B_2 \Longrightarrow \Pi^0_{(\phi)} = +\frac{1}{2}B_2, \Pi^i_{(\phi)} = -\frac{1}{2}\bar{B}^{0i}$$

$$\mathcal{L}^{(D)}_{(0)} = \frac{1}{48}H^{\mu\nu\sigma\rho}H_{\mu\nu\sigma\rho} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

$$\Pi^{\mu\nu\sigma}_{(A3)} = \frac{\partial \mathcal{L}^{(D)}_{(0)}}{\partial\left(\partial_0 A_{\mu\nu\sigma}\right)} \equiv \frac{1}{3!}H^{0\mu\nu\sigma} \implies \Pi^{0ij}_{(A3)} = \frac{1}{3!}H^{00ij} \approx 0$$

$$\Pi^{\mu}_{(A1)} = \frac{\partial \mathcal{L}^{(D)}_{(0)}}{\partial\left(\partial_0 A_\mu\right)} \equiv -F^{0\mu} \implies \Pi^0_{(A1)} = -F^{00} \approx 0$$

$$\partial_\mu H^{\mu\nu\sigma\rho} = 0 \Longrightarrow \partial_0 H^{00jk} + \partial_i H^{i0jk} = 0$$

$$\partial_\mu F^{\mu\nu} = 0 \Longrightarrow \partial_0 F^{00} + \partial_i F^{i0} = 0$$

$$\frac{\partial \Pi^{0jk}_{(A3)}}{\partial t} = \frac{1}{3!}\partial_i H^{0ijk} \approx 0 \implies \frac{\partial \Pi^{0jk}_{(A3)}}{\partial t} \equiv \partial_i \Pi^{ijk}_{(A3)} \approx 0$$

$$\frac{\partial \Pi^0_{(A1)}}{\partial t} = \partial_i F^{i0} \approx 0 \implies \frac{\partial \Pi^0_{(A1)}}{\partial t} \equiv \partial_i \Pi^i_{(A1)} \approx 0$$

$$\mathcal{L}^{(D=4)}_{(0)} = -\frac{1}{2}\left(-\frac{1}{3!}\varepsilon^{\mu\nu\sigma\rho}\partial_\mu A_{\nu\sigma\rho}\right)^2 + \frac{1}{4}\left(\varepsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho\right)^2$$

$$\Pi^{\mu\nu\sigma}_{(A3)} = \frac{\partial \mathcal{L}^{(D=4)}_{(0)}}{\partial\left(\partial_0 A_{\mu\nu\sigma}\right)} \equiv \frac{1}{3!}\varepsilon^{0\mu\nu\sigma}(H_{0123}) \Longrightarrow \Pi^{0ij}_{(A3)} = \frac{1}{3!}\varepsilon^{00ij}(H_{0123}) \approx 0$$

$$\Pi^{\mu}_{(A1)} = \frac{\partial \mathcal{L}^{(D=4)}_{(0)}}{\partial\left(\partial_0 A_\mu\right)} \equiv -F^{0\mu} \implies \Pi^0_{(A1)} = -F^{00} \approx 0$$

$$\Pi^{\mu}_{(A1)} = \frac{\partial \mathcal{L}^{(D=4)}_{(0)}}{\partial\left(\partial_0 A_\mu\right)} = \frac{1}{2}\varepsilon^{\alpha\beta0\mu}\varepsilon_{\alpha\beta\sigma\rho}\partial^\sigma A^\rho \equiv -F^{0\mu}$$

$$\Pi^{ijk}_{(A3)} = \frac{1}{3!}\varepsilon^{0ijk}(H_{0123}), \Pi^i_{(A1)} = -F^{0i} \equiv F^{i0}$$

$$B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad f_\mu + F_\mu = \partial_\mu C_1$$

$$\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = \partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu, \quad \bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1$$

$$\mathcal{L}_{(FP)} - \mathcal{L}_{(fp)} = \left(\partial_\mu \bar{C}^{\mu\nu}\right)(f_\nu + F_\nu) - \left(\partial_\mu C^{\mu\nu}\right)\left(\bar{f}_\nu + \bar{F}_\nu\right) + 2\bar{f}^\mu F_\mu$$

$$+\left(\partial^\mu \bar{C}_1\right)\left(f_\mu - F_\mu\right) + \left(\partial^\mu C_1\right)\left(\bar{f}_\mu - \bar{F}_\mu\right) - 2\bar{F}^\mu f_\mu$$

$$(\partial_\mu \bar{C}^{\mu\nu})(f_\nu + F_\nu - \partial_\nu C_1) + \partial_\mu[\bar{C}^{\mu\nu}(\partial_\nu C_1)]$$
$$-(\partial_\mu C^{\mu\nu})(\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1) - \partial_\mu[C^{\mu\nu}(\partial_\nu \bar{C}_1)],$$

$$(2f^\mu - \partial^\mu C_1)[\bar{f}_\mu + \bar{F}_\mu - \partial_\mu \bar{C}_1] + (2\bar{f}^\mu - \partial^\mu \bar{C}_1)[f_\mu + F_\mu - \partial_\mu C_1]$$

$$\mathcal{L}_{(FP)} - \mathcal{L}_{(fp)} = (\partial_\mu \bar{C}^{\mu\nu} + 2\bar{f}^\nu - \partial^\nu \bar{C}_1)[\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1]$$
$$-(\partial_\mu C^{\mu\nu} - 2f^\nu + \partial^\nu C_1)[\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1]$$

$$\mathcal{L}_{(NG)} - \mathcal{L}_{(ng)} = \frac{1}{4}\left[(\bar{B}_{\mu\nu})^2 - (B_{\mu\nu})^2\right] + \frac{1}{2}(\partial_\sigma A^{\sigma\mu\nu})(\bar{B}_{\mu\nu} + B_{\mu\nu}) + \frac{1}{4}(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu)[B_{\mu\nu} - \bar{B}_{\mu\nu}]$$
$$+\frac{1}{4}\left[(\overline{\mathcal{B}}_{\mu\nu})^2 - (\mathcal{B}_{\mu\nu})^2\right] + \frac{1}{2}(\varepsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho)[\mathcal{B}_{\mu\nu} - \overline{\mathcal{B}}_{\mu\nu}] + \frac{1}{4}(\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu)[\mathcal{B}_{\mu\nu} - \overline{\mathcal{B}}_{\mu\nu}].$$

$$\mathcal{L}_{(NG)} - \mathcal{L}_{(ng)} = \partial_\mu\left[(\partial_\sigma A^{\sigma\mu\nu})\phi_\nu + \varepsilon^{\mu\nu\sigma\rho}(\partial_\sigma A_\rho)\tilde{\phi}_\nu\right]$$
$$+\frac{1}{2}\left[\partial_\sigma A^{\sigma\mu\nu} + \frac{1}{2}(\bar{B}^{\mu\nu} - B^{\mu\nu})\right]\left[B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)\right]$$
$$+\frac{1}{2}\left[\varepsilon^{\mu\nu\sigma\rho}\partial_\sigma A_\rho + \frac{1}{2}\left(\overline{\mathcal{B}}^{\mu\nu} - \mathcal{B}^{\mu\nu}\right)\right]\left[\mathcal{B}_{\mu\nu} + \overline{\mathcal{B}}_{\mu\nu} - (\partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu)\right].$$

$$s_{ab}\mathcal{L}_{(B)} = \frac{1}{2}\partial_\mu\left[B_2 \bar{f}^\mu + (\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})B_{\nu\sigma} + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)f_\nu\right.$$

$$-\bar{B}^{\mu\nu}\bar{F}_\nu - C^{\mu\nu}\partial_\nu B_5 + \bar{C}^{\mu\nu}\partial_\nu B_2 - F^\mu B_5 - B_4 \partial^\mu \bar{C}_2] + \partial_\mu[(\partial_\sigma A^{\sigma\mu\nu})\bar{f}_\nu - B\partial^\mu \bar{C}]$$
$$+\frac{1}{2}[\bar{B}^{\mu\nu}\partial_\mu[\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1] + (f^\mu + F^\mu - \partial^\mu C_1)\partial_\mu B_5 - (\bar{f}^\mu + \bar{F}^\mu - \partial^\mu \bar{C}_1)\partial_\mu B_2]$$
$$-\frac{1}{2}[(\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\partial_\mu[f_\nu + F_\nu - \partial_\nu C_1] + \{B^{\mu\nu} + \bar{B}^{\mu\nu} - (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu)\}\partial_\mu \bar{f}_\nu]$$
$$-\frac{1}{2}(\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})\partial_\mu[B_{\nu\sigma} + \bar{B}_{\nu\sigma} - (\partial_\nu \phi_\sigma - \partial_\sigma \phi_\nu)].$$

$$s_b\mathcal{L}_{(\bar{B})} = \frac{1}{2}\partial_\mu\left[B_2 f^\mu - (\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\bar{B}_{\nu\sigma} + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)\bar{f}_\nu\right.$$
$$-B^{\mu\nu}F_\nu - \bar{C}^{\mu\nu}\partial_\nu B_4 - C^{\mu\nu}\partial_\nu B_2 + \bar{F}^\mu B_4 - B_5 \partial^\mu C_2] - \partial_\mu[(\partial_\sigma A^{\sigma\mu\nu})f_\nu + B\partial^\mu C]$$
$$+\frac{1}{2}[B^{\mu\nu}\partial_\mu[f_\nu + F_\nu - \partial_\nu C_1] - (f^\mu + F^\mu - \partial^\mu C_1)\partial_\mu B_2 - (\bar{f}^\mu + \bar{F}^\mu - \partial^\mu \bar{C}_1)\partial_\mu B_4]$$
$$-\frac{1}{2}[(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)\partial_\mu[\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1] + \{B^{\mu\nu} + \bar{B}^{\mu\nu} - (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu)\}\partial_\mu f_\nu]$$
$$+\frac{1}{2}(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\partial_\mu[B_{\nu\sigma} + \bar{B}_{\nu\sigma} - (\partial_\nu \phi_\sigma - \partial_\sigma \phi_\nu)]$$

$$f_\mu = \frac{1}{2}(\partial^\nu C_{\nu\mu} + \partial_\mu C_1), \qquad \bar{F}_\mu = \frac{1}{2}(\partial^\nu \bar{C}_{\nu\mu} + \partial_\mu \bar{C}_1),$$
$$\bar{f}_\mu = -\frac{1}{2}(\partial^\nu \bar{C}_{\nu\mu} - \partial_\mu \bar{C}_1), \quad F_\mu = -\frac{1}{2}(\partial^\nu C_{\nu\mu} - \partial_\mu C_1).$$

$$f_\mu + F_\mu = \partial_\mu C_1, \bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1$$

$$B_{\mu\nu} = \partial^\sigma A_{\sigma\mu\nu} + \frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu), \mathcal{B}_{\mu\nu} = \varepsilon_{\mu\nu\sigma\rho}\partial^\sigma A^\rho + \frac{1}{2}(\partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu),$$
$$\bar{B}_{\mu\nu} = -\partial^\sigma A_{\sigma\mu\nu} + \frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu), \overline{\mathcal{B}}_{\mu\nu} = -\varepsilon_{\mu\nu\sigma\rho}\partial^\sigma A^\rho + \frac{1}{2}(\partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu),$$

$$C_{\mu\nu} \rightarrow e^{+\Sigma}C_{\mu\nu}, \bar{C}_{\mu\nu} \rightarrow e^{-\Sigma}\bar{C}_{\mu\nu}, \beta_\mu \rightarrow e^{+2\Sigma}\beta_\mu, \bar{\beta}_\mu \rightarrow e^{-2\Sigma}\bar{\beta}_\mu$$
$$f_\mu \rightarrow e^{+\Sigma}f_\mu, \bar{f}_\mu \rightarrow e^{-\Sigma}\bar{f}_\mu, F_\mu \rightarrow e^{+\Sigma}F_\mu, \bar{F}_\mu \rightarrow e^{-\Sigma}\bar{F}_\mu$$

$$C_2 \rightarrow e^{+3\Sigma}C_2, \bar{C}_2 \rightarrow e^{-3\Sigma}\bar{C}_2, C \rightarrow e^{+\Sigma}C, \bar{C} \rightarrow e^{-\Sigma}\bar{C}$$
$$C_1 \rightarrow e^{+\Sigma}C_1, \bar{C}_1 \rightarrow e^{-\Sigma}\bar{C}_1, B_4 \rightarrow e^{+2\Sigma}B_4, B_5 \rightarrow e^{-2\Sigma}B_5$$
$$\Phi \rightarrow e^0\Phi \left(\Phi = A_{\mu\nu\sigma}, B_{\mu\nu}, \mathcal{B}_{\mu\nu}, \bar{B}_{\mu\nu}, \overline{\mathcal{B}}_{\mu\nu}, A_\mu, B, B_1, B_2, B_3, \phi_\mu, \tilde{\phi}_\mu\right),$$

$$s_g C_{\mu\nu} = +C_{\mu\nu}, s_g\bar{C}_{\mu\nu} = -\bar{C}_{\mu\nu}, s_g\beta_\mu = +2\beta_\mu, s_g\bar{\beta}_\mu = -2\bar{\beta}_\mu$$
$$s_g f_\mu = +f_\mu, s_g\bar{f}_\mu = -\bar{f}_\mu, s_g F_\mu = +F_\mu, s_g\bar{F}_\mu = -\bar{F}_\mu$$
$$s_g C_2 = +3C_2, s_g\bar{C}_2 = -3\bar{C}_2, s_g C = +C, s_g\bar{C} = -\bar{C}$$
$$s_g C_1 = +C_1, s_g\bar{C}_1 = -\bar{C}_1, s_g B_4 = +2B_4, s_g B_5 = -2B_5, s_g\Phi = 0,$$

$$J^\mu_{(g)} = \frac{1}{2}[(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\bar{C}_{\nu\sigma} + (\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})C_{\nu\sigma}$$
$$-2\left(\partial^\mu\bar{\beta}^\nu - \partial^\nu\bar{\beta}^\mu\right)\beta_\nu + 2(\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\bar{\beta}_\nu - C^{\mu\nu}\bar{F}_\nu - \bar{C}^{\mu\nu}f_\nu - C_1\bar{F}^\mu$$
$$-\bar{C}_1 f^\mu + 3C_2\partial^\mu\bar{C}_2 + 3\bar{C}_2\partial^\mu C_2 - 2\beta^\mu B_5 - 2\bar{\beta}^\mu B_4\big] + C\partial^\mu\bar{C} + \bar{C}\partial^\mu C$$

$$J^\mu_{(\bar{g})} = \frac{1}{2}[(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\bar{C}_{\nu\sigma} + (\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})C_{\nu\sigma}$$
$$-2\left(\partial^\mu\bar{\beta}^\nu - \partial^\nu\bar{\beta}^\mu\right)\beta_\nu + 2(\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\bar{\beta}_\nu + C^{\mu\nu}\bar{f}_\nu + \bar{C}^{\mu\nu}F_\nu - C_1\bar{f}^\mu$$
$$-\bar{C}_1 F^\mu + 3C_2\partial^\mu\bar{C}_2 + 3\bar{C}_2\partial^\mu C_2 - 2\beta^\mu B_5 - 2\bar{\beta}^\mu B_4\big] + C\partial^\mu\bar{C} + \bar{C}\partial^\mu C$$

$$\Box\, C = 0, \Box\, \bar{C} = 0, \Box\, C_2 = 0, \Box\, \bar{C}_2 = 0, (\partial\cdot\bar{F}) = 0, (\partial\cdot f) = 0$$
$$B_4 = -(\partial\cdot\beta), B_5 = (\partial\cdot\bar{\beta}), f_\mu = \frac{1}{2}\left(\partial^\nu C_{\nu\mu} + \partial_\mu C_1\right), \bar{F}_\mu = \frac{1}{2}\left(\partial^\nu\bar{C}_{\nu\mu} + \partial_\mu\bar{C}_1\right)$$
$$\partial_\mu(\partial^\mu\beta^\nu - \partial^\nu\beta^\mu) - \partial^\nu B_4 = 0, \partial_\mu\left(\partial^\mu\bar{\beta}^\nu - \partial^\nu\bar{\beta}^\mu\right) + \partial^\nu B_5 = 0$$
$$\partial_\mu(\partial^\mu\bar{C}^{\nu\sigma} + \partial^\nu\bar{C}^{\sigma\mu} + \partial^\sigma\bar{C}^{\mu\nu}) + \frac{1}{2}(\partial^\nu\bar{F}^\sigma - \partial^\sigma\bar{F}^\nu) = 0$$
$$\partial_\mu(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}) + \frac{1}{2}(\partial^\nu f^\sigma - \partial^\sigma f^\nu) = 0.$$

$$F_\mu = -\frac{1}{2}\left(\partial^\nu C_{\nu\mu} - \partial_\mu C_1\right), \bar{f}_\mu = -\frac{1}{2}\left(\partial^\nu\bar{C}_{\nu\mu} - \partial_\mu\bar{C}_1\right), (\partial\cdot F) = 0, (\partial\cdot\bar{f}) = 0$$
$$\partial_\mu(\partial^\mu\bar{C}^{\nu\sigma} + \partial^\nu\bar{C}^{\sigma\mu} + \partial^\sigma\bar{C}^{\mu\nu}) - \frac{1}{2}\left(\partial^\nu\bar{f}^\sigma - \partial^\sigma\bar{f}^\nu\right) = 0$$
$$\partial_\mu(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}) - \frac{1}{2}(\partial^\nu F^\sigma - \partial^\sigma F^\nu) = 0$$

$$Q_{(g)}=\int d^3x\Big\{\frac{1}{2}\big[(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i})C_{ij}+(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})\bar{C}_{ij}$$
$$-2(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0)\beta_i+2(\partial^0\beta^i-\partial^i\beta^0)\bar{\beta}_i-\frac{1}{2}C^{0i}\bar{F}_i+\frac{1}{2}C^{i0}\bar{F}_i-\frac{1}{2}\bar{C}^{0i}f_i+\frac{1}{2}\bar{C}^{i0}f_i$$
$$-C_1\bar{F}^0-\bar{C}_1f^0+3C_2\dot{\bar{C}}_2+3\bar{C}_2\dot{C}_2-2\beta^0B_5-2\bar{\beta}^0B_4\big]+C\dot{\bar{C}}+\bar{C}\dot{C}\Big\}$$
$$Q_{(\bar{g})}=\int d^3x\Big\{\frac{1}{2}\big[(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{0i})C_{ij}+(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{0i})\bar{C}_{ij}$$
$$-2(\partial^0\bar{\beta}^i-\partial^i\bar{\beta}^0)\beta_i+2(\partial^0\beta^i-\partial^i\beta^0)\bar{\beta}_i+\frac{1}{2}C^{0i}\bar{f}_i-\frac{1}{2}C^{io}\bar{f}_i+\frac{1}{2}\bar{C}^{0i}F_i-\frac{1}{2}\bar{C}^{i0}F_i$$
$$-C_1\bar{f}^0-\bar{C}_1F^0+3C_2\dot{\bar{C}}_2+3\bar{C}_2\dot{C}_2-2\beta^0B_5-2\bar{\beta}^0B_4\big]+C\dot{\bar{C}}+\bar{C}\dot{C}\Big\}$$

$$\Pi^{\mu}_{(\beta)}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0\beta_\mu)}=-\frac{1}{2}(\partial^0\bar{\beta}^\mu-\partial^\mu\bar{\beta}^0)-\frac{1}{2}(\eta^{0\mu}B_5)\Longrightarrow$$
$$\Pi^{0}_{(\beta)}=-\frac{1}{2}B_5,\Pi^{i}_{(\beta)}=-\frac{1}{2}(\partial^0\bar{\beta}^\mu-\partial^\mu\bar{\beta}^0),\Pi_{(C)}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0C)}=\dot{\bar{C}},$$
$$\Pi^{\mu}_{(\bar{\beta})}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0\bar{\beta}_\mu)}=-\frac{1}{2}(\partial^0\beta^\mu-\partial^\mu\beta^0)+\frac{1}{2}(\eta^{0\mu}B_4)\Longrightarrow$$
$$\Pi^{0}_{(\bar{\beta})}=+\frac{1}{2}B_4,\Pi^{i}_{(\bar{\beta})}=-\frac{1}{2}(\partial^0\bar{\beta}^\mu-\partial^\mu\bar{\beta}^0),\Pi_{(\bar{C})}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0\bar{C})}=-\dot{C},$$
$$\Pi_{(\bar{C}_2)}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0\bar{C}_2)}=-\frac{1}{2}\dot{C}_2,\Pi_{(C_2)}=\frac{\partial\mathcal{L}_{(FP,fp)}}{\partial(\partial_0C_2)}=\frac{1}{2}\dot{\bar{C}}_2,$$

$$\Pi^{\mu\nu}_{(C)}(FP)=\frac{\partial\mathcal{L}_{(FP)}}{\partial(\partial_0C_{\mu\nu})}=-\frac{1}{2}\Big[(\partial^0\bar{C}^{\mu\nu}+\partial^\mu\bar{C}^{\nu0}+\partial^\nu\bar{C}^{0\mu})+\frac{1}{2}(\eta^{0\mu}\bar{F}^\nu-\eta^{0\nu}\bar{F}^\mu)\Big]\Longrightarrow$$
$$\Pi^{ij}_{(C)}(FP)=-\frac{1}{2}(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{i0}),\Pi^{0i}_{(C)}(FP)=\frac{1}{4}\bar{F}_i,\Pi^{io}_{(C)}(FP)=-\frac{1}{4}\bar{F}_i$$
$$\Pi^{\mu\nu}_{(\bar{C})}(FP)=\frac{\partial\mathcal{L}_{(FP)}}{\partial(\partial_0\bar{C}_{\mu\nu})}=\frac{1}{2}\Big[(\partial^0C^{\mu\nu}+\partial^\mu C^{\nu0}+\partial^\nu C^{0\mu})+\frac{1}{2}(\eta^{0\mu}f^\nu-\eta^{0\nu}f^\mu)\Big]\Longrightarrow$$
$$\Pi^{ij}_{(\bar{C})}(FP)=\frac{1}{2}(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{i0}),\Pi^{0i}_{(\bar{C})}(FP)=-\frac{1}{4}f_i,\Pi^{io}_{(\bar{C})}(FP)=\frac{1}{4}f_i$$
$$\Pi_{(\bar{C}_1)}(FP)=\frac{\partial\mathcal{L}_{(FP)}}{\partial(\partial_0\bar{C}_1)}=+\frac{1}{2}f^0,\Pi_{(C_1)}(FP)=\frac{\partial\mathcal{L}_{(FP)}}{\partial(\partial_0C_1)}=-\frac{1}{2}\bar{F}^0,$$

$$\Pi^{\mu\nu}_{(C)}(fp)=\frac{\partial\mathcal{L}_{(fp)}}{\partial(\partial_0C_{\mu\nu})}=-\frac{1}{2}\Big[(\partial^0\bar{C}^{\mu\nu}+\partial^\mu\bar{C}^{\nu0}+\partial^\nu\bar{C}^{0\mu})-\frac{1}{2}(\eta^{0\mu}\bar{f}^\nu-\eta^{0\nu}\bar{f}^\mu)\Big]\Longrightarrow$$
$$\Pi^{ij}_{(C)}(fp)=-\frac{1}{2}(\partial^0\bar{C}^{ij}+\partial^i\bar{C}^{j0}+\partial^j\bar{C}^{i0}),\Pi^{0i}_{(C)}(fp)=-\frac{1}{4}\bar{f}_i,\Pi^{io}_{(C)}(fp)=+\frac{1}{4}\bar{f}_i$$
$$\Pi^{\mu\nu}_{(\bar{C})}(fp)=\frac{\partial\mathcal{L}_{(fp)}}{\partial(\partial_0\bar{C}_{\mu\nu})}=\frac{1}{2}\Big[(\partial^0C^{\mu\nu}+\partial^\mu C^{\nu0}+\partial^\nu C^{0\mu})-\frac{1}{2}(\eta^{0\mu}F^\nu-\eta^{0\nu}F^\mu)\Big]\Longrightarrow$$
$$\Pi^{ij}_{(\bar{C})}(fp)=\frac{1}{2}(\partial^0C^{ij}+\partial^iC^{j0}+\partial^jC^{i0}),\Pi^{0i}_{(\bar{C})}(fp)=+\frac{1}{4}F_i,\Pi^{io}_{(\bar{C})}(fp)=-\frac{1}{4}F_i$$

$$\Pi_{(\bar{C}_1)}(fp)=\frac{\partial\mathcal{L}_{(fp)}}{\partial(\partial_0\bar{C}_1)}=\frac{1}{2}F^0,\Pi_{(C_1)}(fp)=\frac{\partial\mathcal{L}_{(FP)}}{\partial(\partial_0C_1)}=-\frac{1}{2}\bar{f}^0$$

$$Q_{(g)}=\int d^3x\Big\{-\Big[\Pi^{ij}_{(C)}(FP)\Big]C_{ij}+\Big[\Pi^{ij}_{(\bar{C})}(FP)\Big]\bar{C}_{ij}+2\big[\Pi^{i}_{(\beta)}\big]\beta_i-2\Big[\Pi^{i}_{(\bar{\beta})}\Big]\bar{\beta}_i$$
$$+\big[\Pi^{0i}_{(C)}(FP)\big]C_{0i}+\big[\Pi^{i0}_{(C)}(FP)\big]C_{i0}+\Big[\Pi^{0i}_{(\bar{C})}(FP)\Big]\bar{C}_{0i}-\Big[\Pi^{i0}_{(\bar{C})}(FP)\Big]\bar{C}_{i0}$$
$$-\big[\Pi_{(C_1)}(FP)\big]C_1+\big[\Pi_{(\bar{C}_1)}(FP)\big]\bar{C}_1-3\big[\Pi_{(C_2)}\big]C_2+3\big[\Pi_{(\bar{C}_2)}\big]\bar{C}_2$$
$$+2\big[\Pi^{0}_{(\beta)}\big]\beta_0-2\Big[\Pi^{0}_{(\bar{\beta})}\Big]\bar{\beta}_0-\big[\Pi_{(C)}\big]C+\big[\Pi_{(\bar{C})}\big]\bar{C}\Big\}$$
$$Q_{(\bar{g})}=\int d^3x\Big\{-\Big[\Pi^{ij}_{(C)}(fp)\Big]C_{ij}+\Big[\Pi^{ij}_{(\bar{C})}(fp)\Big]\bar{C}_{ij}+2\big[\Pi^{i}_{(\beta)}\big]\beta_i-2\Big[\Pi^{i}_{(\bar{\beta})}\Big]\bar{\beta}_i$$
$$+\big[\Pi^{0i}_{(C)}(fp)\big]C_{0i}+\big[\Pi^{i0}_{(C)}(fp)\big]C_{i0}+\Big[\Pi^{0i}_{(\bar{C})}(fp)\Big]\bar{C}_{0i}-\Big[\Pi^{i0}_{(\bar{C})}(fp)\Big]\bar{C}_{i0}$$
$$-\big[\Pi_{(C_1)}(fp)\big]C_1+\big[\Pi_{(\bar{C}_1)}(fp)\big]\bar{C}_1-3\big[\Pi_{(C_2)}\big]C_2+3\big[\Pi_{(\bar{C}_2)}\big]\bar{C}_2$$
$$+2\big[\Pi^{0}_{(\beta)}\big]\beta_0-2\Big[\Pi^{0}_{(\bar{\beta})}\Big]\bar{\beta}_0-\big[\Pi_{(C)}\big]C+\big[\Pi_{(\bar{C})}\big]\bar{C}\Big\}$$

$$s_gC_{ij}(\vec{x},t)=+i\big[C_{ij}(\vec{x},t),Q_{(g,\bar{g})}\big]=-i\int d^3y\big[C_{ij}(\vec{x},t),\Pi^{kl}_{(C)}(\vec{y},t)C_{kl}(\vec{y},t)\big]$$
$$\equiv -i\int d^3y\{C_{ij}(\vec{x},t),\Pi^{kl}_{(C)}(\vec{y},t)\}C_{kl}(\vec{y},t)+i\int d^3y\Pi^{kl}_{(C)}(\vec{y},t)\{C_{ij}(\vec{x},t),C_{kl}(\vec{y},t)\},$$

$$\{C_{ij}(\vec{x},t),\Pi^{kl}_{(C)}(\vec{y},t)\}=\frac{i}{2}\big(\delta^k_i\delta^l_j-\delta^l_i\delta^k_j\big)\delta^{(3)}(\vec{x}-\vec{y})$$

$$s_g\beta_i(\vec{x},t)=-i\big[\beta_i(\vec{x},t),Q_{(g,\bar{g})}\big]=-i\int d^3y\Big[\beta_i(\vec{x},t),2\Pi^{j}_{(\beta)}(\vec{y},t)\beta_j(\vec{y},t)\Big]$$
$$\equiv -2i\int d^3y\Big[\beta_i(\vec{x},t),\Pi^{j}_{(\beta)}(\vec{y},t)\Big]\beta_j(\vec{y},t)-2i\int d^3y\Pi^{j}_{(\beta)}(\vec{y},t)\big[\beta_i(\vec{x},t),\beta_j(\vec{y},t)\big].$$

$$\Big[\beta_i(\vec{x},t),\Pi^{j}_{(\beta)}(\vec{y},t)\Big]=i\delta^j_i\delta^{(3)}(\vec{x}-\vec{y})$$

$$s_gQ_b=-i\big[Q_b,Q_g\big]=+Q_b\Longrightarrow i\big[Q_g,Q_b\big]=+Q_b,$$
$$s_gQ_{ab}=-i\big[Q_{ab},Q_g\big]=-Q_{ab}\Longrightarrow i\big[Q_g,Q_{ab}\big]=-Q_{ab},$$
$$s_gQ_B=-i\big[Q_B,Q_g\big]=+Q_B\Longrightarrow i\big[Q_g,Q_B\big]=+Q_B,$$
$$s_gQ_{AB}=-i\big[Q_{AB},Q_g\big]=-Q_{AB}\Longrightarrow i\big[Q_g,Q_{AB}\big]=-Q_{AB},$$

$$Q_B^2=0,Q_{AB}^2=0,i\big[Q_g,Q_B\big]=+Q_B,i\big[Q_g,Q_{AB}\big]=-Q_{AB}$$

$$\frac{z_ez'_m-z_mz'_e}{N}+n_en'_m-n_mn'_e\in\mathbb{Z}$$

$$(z_e,z_m,n_e,n_m)=\left(kn_1+\ell n_2,\frac{N}{rk}n_2,n_1+rn_3,-\frac{1}{r}(n_2-rn_4)\right)$$

$$G[k,r,\ell]=U(1)^{(1)}_e\times U(1)^{(1)}_m\times\mathbb{Z}^{(1)}_{N/r\mathrm{gcd}(k,N/rk,\ell)}\times\mathbb{Z}^{(1)}_{gcd(k,N/rk,\ell)}$$

$$\left(\mathbb{Z}^{(1)}_{N/rk}\times\mathbb{Z}^{(1)}_{N/r\mathrm{gcd}(N/kr,\ell)}\right)/\mathbb{Z}^{(1)}_{N/krgcd(N/kr,\ell)}$$

$$G[k,r,0]=U(1)^{(1)}_e\times U(1)^{(1)}_m\times\mathbb{Z}^{(1)}_{N/rk}\times\mathbb{Z}^{(1)}_k$$

$$S_{anomaly}[B_2, C_2] = \frac{N}{2\pi}\int_{M_5} dB_2 \wedge C_2$$

$$S_{anomaly}[\mathrm{B}_2, \mathrm{C}_2] = \frac{2\pi N}{N_e N_m}\int_{M_5} \delta\, \mathrm{B}_2 \cup \mathrm{C}_2$$

$$\mathrm{B}_2 = \frac{N_e}{2\pi} B_2, \mathrm{C}_2 = \frac{N_m}{2\pi} C_2$$

$$S_{\text{anomaly}}\,[\mathrm{B}_2, C_2] = \frac{N}{N_e}\int \delta\, \mathrm{B}_2 \wedge C_2$$

$$S_{\text{anomaly}}\,[B_2, \mathrm{C}_2] = \frac{N}{N_m}\int dB_2 \wedge \mathrm{C}_2$$

$$U(1)^{(1)} \times U(1)^{(1)} \times \mathbb{Z}_{N_e}^{(1)} \times \mathbb{Z}_{N_m}^{(1)} = U(1)^{(1)} \times U(1)^{(1)} \times \mathbb{Z}_{N/rk}^{(1)} \times \mathbb{Z}_k^{(1)},$$

$$1 \to \mathbb{Z}_{N_e} \to U(1) \times \mathbb{Z}_{gcd(N,N_e)} \to U(1) \to 1$$

$$e(\mathrm{B}_2) = \frac{\gcd(N, N_e)}{N_e}\delta\, \mathrm{B}_2$$

$$e(\mathrm{C}_2) = \frac{\gcd(N, N_m)}{N_m}\delta \mathrm{C}_2$$

$$S_{\text{anomaly}} = \frac{2\pi N}{\gcd(N, N_e) N_m}\int_{M_5} e(\mathrm{B}_2) \cup \mathrm{C}_2 = -\frac{2\pi N}{\gcd(N, N_m) N_e}\int_{M_5} \mathrm{B}_2 \cup e(\mathrm{C}_2)$$

$$\begin{aligned}
S_{\text{anomaly}} &= \frac{2\pi N}{\gcd(N, N_e) N_m}\int_{M_5} e(\mathrm{B}_2) \cup \mathrm{C}_2 \\
&= \frac{2\pi N}{\gcd(N, N_e) N_m}\int_{M_5} (1 - yN_m) e(\mathrm{B}_2) \cup \mathrm{C}_2 \\
&= \frac{2\pi N N_e x}{\gcd(N, N_e) N_m}\int_{M_5} e(\mathrm{B}_2) \cup \mathrm{C}_2 \\
&= \frac{2\pi N x}{N_m}\int_{M_5} \delta\, \mathrm{B}_2 \cup \mathrm{C}_2 \\
&= -\frac{2\pi N x}{N_m}\int_{M_5} \mathrm{B}_2 \cup \delta \mathrm{C}_2 \\
&= -\frac{2\pi N x}{\gcd(N, N_m)}\int_{M_5} \mathrm{B}_2 \cup e(\mathrm{C}_2) \\
&= 0 \mathrm{mod} 2\pi
\end{aligned}$$

$$U(1)_e^{(1)} \times U(1)_m^{(1)} \times \mathbb{Z}_{gcd(N,N_e)}^{(1)} \times \mathbb{Z}_{gcd(N,N_m)}^{(1)}.$$

$$S_{sym}^{u(N)}[B_2, C_2, h_2, f_2] = \frac{1}{2\pi}\int_{M_5} [h_2 \wedge dB_2 + f_2 \wedge dC_2 + NC_2 \wedge dB_2]$$

$$\delta B_2 = d\Lambda_1^B, \delta C_2 = d\Lambda_1^C, \delta f_2 = d\lambda_1^f, \delta h_2 = d\lambda_1^h.$$

$$U_B[n_B,\Sigma_2]=e^{in_B\oint_{\Sigma_2}B_2}\ ,\quad U_C[n_C,\Sigma_2]=e^{in_C\oint_{\Sigma_2}C_2}\quad n_B,n_C\in\mathbb{Z}$$
$$U_h[\alpha_h,\Sigma_2]=e^{i\alpha_h\oint_{\Sigma_2}h_2}\ ,\quad U_f[\alpha_f,\Sigma_2]=e^{i\alpha_f\oint_{\Sigma_2}f_2}\quad \alpha_h,\alpha_f\in\mathbb{R}.$$

$$U_B[Nn,\Sigma_2]U_f[-n,\Sigma_2]=1, U_C[Nn,\Sigma_2]U_h[n,\Sigma_2]=1$$

$$\tilde{U}_B[n_B]:=U_B[n_B]U_f\left[-\frac{n_B}{N}\right],\tilde{U}_C[n_C]:=U_C[n_C]U_h\left[\frac{n_C}{N}\right],$$

$$\begin{aligned}\langle U_B[n_B,\Sigma_2]U_h[\alpha_h,\Sigma_2']\rangle &= e^{-2\pi i n_B\alpha_h L(\Sigma_2,\Sigma_2')}\\ \langle U_C[n_C,\Sigma_2]U_f[\alpha_f,\Sigma_2']\rangle &= e^{-2\pi i n_C\alpha_f L(\Sigma_2,\Sigma_2')}\\ \langle U_h[\alpha_h,\Sigma_2]U_f[\alpha_f,\Sigma_2']\rangle &= e^{2\pi i N\alpha_h\alpha_f L(\Sigma_2,\Sigma_2')}\end{aligned}$$

$$\begin{aligned}\langle \tilde{U}_B[n_B,\Sigma_2]\tilde{U}_C[n_C,\Sigma_2']\rangle &= e^{2\pi i\frac{n_B n_C}{N}L(\Sigma_2,\Sigma_2')}\\ \langle U_h[\alpha_h,\Sigma_2]U_f[\alpha_f,\Sigma_2']\rangle &= e^{2\pi i N\alpha_h\alpha_f L(\Sigma_2,\Sigma_2')}\end{aligned}$$

$$S_{sym}^{u(N)}[\tilde{B}_2,\tilde{C}_2,h_2,f_2]=\frac{1}{2\pi}\int_{M_5}\left[\frac{1}{N}h_2\wedge df_2+N\tilde{C}_2\wedge d\tilde{B}_2\right]$$

$$U[n_B,n_C,\alpha_h,\alpha_f;\Sigma_2]=\tilde{U}_B[n_B;\Sigma_2]\tilde{U}_C[n_C;\Sigma_2]U_h[\alpha_h;\Sigma_2]U_f[\alpha_f;\Sigma_2].$$

$$\frac{n_B n_C'-n_C n_B'}{N}+N(\alpha_h\alpha_f'-\alpha_f\alpha_h')\in\mathbb{Z}$$

$$n_B=z_e, n_C=z_m, \alpha_f=\frac{n_e}{N}, \alpha_h=-n_m$$

$$\begin{aligned}n_B &= kn_1+\ell n_2\\ n_C &= \frac{N}{rk}n_2\\ \alpha_f &= \frac{1}{N}(n_1+rn_3)\\ \alpha_h &= \frac{1}{r}(n_2-rn_4).\end{aligned}$$

$$\begin{aligned}\tilde{U}_B[k] &= U_f\left[-\frac{1}{N}\right]\\ \tilde{U}_C\left[\frac{N}{r\mathrm{gcd}(k,\ell)}\right] &= \tilde{U}_B\left[-\frac{\ell k}{\mathrm{gcd}(k,\ell)}\right]U_h\left[-\frac{k}{r\mathrm{gcd}(k,\ell)}\right]\\ &= U_f\left[+\frac{\ell}{N\mathrm{gcd}(k,\ell)}\right]U_h\left[-\frac{k}{r\mathrm{gcd}(k,\ell)}\right]\end{aligned}$$

$$\tilde{U}_B[\ell]\tilde{U}_C\left[\frac{N}{rk}\right]=U_h\left[-\frac{1}{r}\right],$$

$$\left(\mathbb{Z}_k^{(1)}\times\mathbb{Z}_{N/r\mathrm{gcd}(k,\ell)}^{(1)}\right)/\mathbb{Z}_{k/\mathrm{gcd}(k,\ell)}^{(1)}\cong\mathbb{Z}_{N/r\mathrm{gcd}(k,N/rk,\ell)}^{(1)}\times\mathbb{Z}_{\mathrm{gcd}(k,N/rk,\ell)}^{(1)}$$

$$S_{su(N)}[B_2, C_2] = \frac{N}{2\pi}\int_{AdS_5} C_2 \wedge dB_2$$

$$S'_{su(N)}[B_2, C_1, h_2, f_3] = \frac{1}{2\pi}\int_{AdS_5} [h_2 \wedge dB_2 + f_3 \wedge (dC_1 + NB_2)]$$

$$\delta B_2 = d\Lambda_1^B, \delta C_1 = d\Lambda_0^C - N\Lambda_1^B, \delta f_3 = d\lambda_2^f, \delta h_2 = d\lambda_1^h + N\lambda_2^f.$$

$$U_f[\alpha_f, \Sigma_3] = e^{i\alpha_f \oint_{\Sigma_3} f_3}, U_B[n_B, \Sigma_2] = e^{in_B \oint_{\Sigma_2} B_2}$$

$$U_h[\alpha_h, \Sigma_2] = e^{i\alpha_h \oint_{\Sigma_2} h_2}, U_C[n_C, \Sigma_1] = e^{in_C \oint_{\Sigma_1} C_1}$$

$$\begin{aligned} \hat{U}_h[\alpha_h, \Sigma_2, \gamma_3] &= U_h[\alpha_h, \Sigma_2] U_f[-N\alpha_h, \gamma_3] = e^{i\alpha_h \oint_{\Sigma_2} h_2 - iN\alpha_h \int_{\gamma_3} f_3} \\ \hat{U}_C[n_C, \Sigma_1, \gamma_2] &= U_C[n_C, \Sigma_1] U_B[Nn_C, \gamma_2] = e^{in_C \oint_{\Sigma_1} C_1 + iNn_C \int_{\gamma_2} B_2} \end{aligned}$$

$$\left(\hat{U}_h\left[\frac{n}{N}, \Sigma_2\right]\right)^N = \hat{U}_h[n, \Sigma_2] = 1$$

$$\begin{aligned} \left\langle U_f[\alpha_f, \Sigma_3] \hat{U}_C[n_C, \Sigma_1] \right\rangle &= e^{2\pi i n_C \alpha_f L(\Sigma_3, \Sigma_1)} \\ \left\langle \hat{U}_h[\alpha_h, \Sigma_2] U_B[n_B, \Sigma_2'] \right\rangle &= e^{2\pi i n_B \alpha_h L(\Sigma_2, \Sigma_2')} \end{aligned}$$

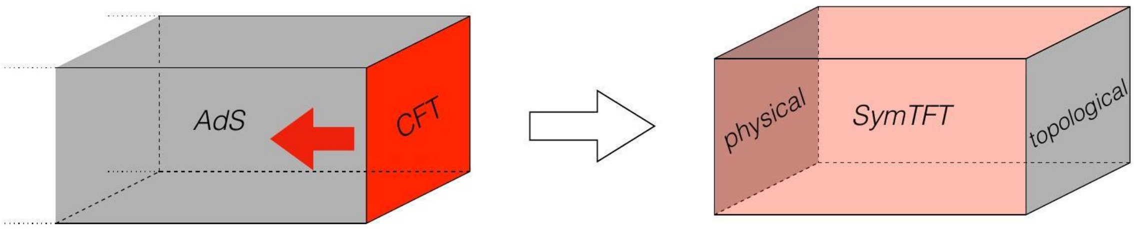
$$n_B \alpha_h' - \alpha_h n_B' + n_C \alpha_f' - \alpha_f n_C' \in \mathbb{Z}$$

$$\begin{aligned} S &= \frac{1}{2\pi}\int_{M_5} [h_2 \wedge dB_2 + f_3 \wedge (dC_1 + NB_2) \\ &+ g_2 \wedge dG_2 + f_2 \wedge dF_2 + G_2 \wedge dF_2 + f_3 \wedge G_2] \end{aligned}$$

$$\delta G_2 = d\Lambda_1^G, \delta F_2 = d\Lambda_1^F, \delta f_2 = d\lambda_1^f, \delta g_2 = d\lambda_1^g + \lambda_2^f$$

$$\delta C_1 = d\Lambda_0^C - N\Lambda_1^B - \Lambda_1^G$$

$$S = \frac{1}{2\pi}\int_{M_5} [(h_2 - Ng_2) \wedge dB_2 + f_2 dF_2 + NF_2 dB_2]$$



$$S_5[B_2, C_2] = \int_{AdS_5} \left[\frac{1}{2g^2}(|dB_2|^2 + |dC_2|^2) + \frac{N}{2\pi} C_2 \wedge dB_2\right],$$

$$S_{IIB}^{IR}[B_2, C_2] = \frac{N}{2\pi}\int_{AdS_5} C_2 \wedge dB_2,$$

$$S_{d+1}[A_1] = \int_{AdS_{d+1}} \frac{1}{2g^2} dA_1 \wedge * \, dA_1 = \int_{AdS_{d+1}} \frac{1}{2g^2} \frac{L^{d-3}}{z^{d-3}} dA_1 \wedge \tilde{*} \, dA_1$$

$$S_{d+1}[A_1, f_{d-1}] = \frac{1}{2\pi} \int_{AdS_{d+1}} \left[f_{d-1} \wedge dA_1 - \frac{g^2}{4\pi} \frac{z^{d-3}}{L^{d-3}} f_{d-1} \wedge \tilde{*} \, f_{d-1} \right]$$

$$f_{d-1} = \frac{2\pi}{g^2} \frac{L^{d-3}}{z^{d-3}} \tilde{*} \, dA_1$$

$$S_{d+1}^{IR}[A_1, f_{d-1}] = \frac{1}{2\pi} \int_{AdS_{d+1}} f_{d-1} \wedge dA_1$$

$$f_{d-1} \to f_{d-1} + d\lambda_{d-2}$$

$$\begin{aligned} S_5[B_2, C_2, h_2, f_2] \ &= \frac{1}{2\pi} \int_{AdS_5} [h_2 \wedge dB_2 + f_2 \wedge dC_2 + NC_2 \wedge dB_2 \\ &\quad - \frac{g^2}{4\pi} \frac{L}{z} (h_2 \wedge \tilde{*} \, h_2 + f_2 \wedge \tilde{*} \, f_2) \Big]. \end{aligned}$$

$$h_2 = \frac{2\pi}{g^2} \frac{z}{L} \tilde{*} \, dB_2, f_2 = \frac{2\pi}{g^2} \frac{z}{L} \tilde{*} \, dC_2$$

$$\begin{aligned} S_{\overline{\text{D6}}}^{\text{vac}} \ &= \alpha \int_{S^5 \times \Sigma_2} \left(dC_6 + \frac{1}{2\pi} dC_4 \wedge (B_2 + F_2^{wv}) + \cdots \right) \\ &= \alpha \int_{\Sigma_2} \left(\frac{1}{g^2} * dC_2 + \frac{N}{2\pi} (B_2 + F_2^{wv}) \right) \end{aligned}$$

$$\begin{aligned} d\left(\frac{1}{g^2} * dB_2 - \frac{N}{2\pi} C_2 \right) &= 0 \\ d\left(\frac{1}{g^2} * dC_2 + \frac{N}{2\pi} B_2 \right) &= 0 \end{aligned}$$

$$U_f[\alpha_f, \Sigma_2] = e^{i\alpha_f \oint_{\Sigma_2} \langle \frac{1}{g^2} * dC_2 + \frac{N}{2\pi} B_2 \rangle}, U_h[\alpha_h, \Sigma_2] = e^{i\alpha_h \oint_{\Sigma_2} \langle \frac{1}{g^2} * dB_2 - \frac{N}{2\pi} C_2 \rangle}$$

$$\begin{aligned} S_{d+1}[A_{p+1}] &= \int_{AdS_{d+1}} \frac{1}{2g^2} dA_{p+1} \wedge * \, dA_{p+1} \\ &= \int_{AdS_{d+1}} \frac{1}{2g^2} \frac{L^{d-2p-3}}{z^{d-2p-3}} dA_{p+1} \wedge \tilde{*} \, dA_{p+1} \end{aligned}$$

$$S_{d+1}[A_{p+1}, f_{d-p-1}] = \frac{1}{2\pi} \int_{AdS_{d+1}} \left[f_{d-p-1} \wedge dA_{p+1} - \frac{g^2}{4\pi} \frac{z^{d-2p-3}}{L^{d-2p-3}} f_{d-p-1} \wedge \tilde{*} \, f_{d-p-1} \right]$$

$$S_{d+1}^{IR}[A_{p+1}, f_{d-p-1}] = \frac{1}{2\pi} \int_{AdS_{d+1}} f_{d-p-1} \wedge dA_{p+1}$$

$$S_{d+1}[A_{d-p-2}] = \int_{AdS_{d+1}} \frac{1}{2\tilde{g}^2} dA_{d-p-2} \wedge * \, dA_{d-p-2}$$

$$= \int_{AdS_{d+1}} \frac{1}{2\tilde{g}^2} \frac{z^{d-2p-3}}{L^{d-2p-3}} dA_{d-p-2} \wedge \tilde{*} \, dA_{d-p-2}$$

$$S_{d+1}[A_{d-p-2}, f_{p+2}] = \int_{AdS_{d+1}} \left[\frac{1}{2\pi} f_{p+2} \wedge dA_{d-p-2} + \frac{1}{2\tilde{g}^2} \frac{L^{d-2p-3}}{z^{d-2p-3}} f_{p+2} \wedge \tilde{*} \, f_{p+2} \right]$$

$$S_{d+1}^{IR}[A_{d-p-2}, f_{p+2}] = \frac{1}{2\pi} \int_{AdS_{d+1}} f_{p+2} \wedge dA_{d-p-2}$$

$$S_5[B_2, C_1, h_2, f_3] = \frac{1}{2\pi} \int_{M_5} \Big[h_2 \wedge dB_2 - f_3 \wedge (dC_1 + NB_2)$$

$$- \frac{g^2}{4\pi} \frac{L}{z} h_2 \wedge \tilde{*} \, h_2 + \frac{\pi}{2g^2} \frac{z}{L} f_3 \wedge \tilde{*} \, f_3 \Big]$$

Simetrías de calibre AdS/CFT/SU(4) en gravedad cuántica.

$$Z(N, b, \boldsymbol{\xi}) = \mathcal{C}^{-1/3} e^{\mathcal{A}} \mathrm{Ai}\left[\mathcal{C}^{-1/3}(N - \mathcal{B})\right] \left(1 + \mathcal{O}\left(e^{-\#\sqrt{N}}\right)\right)$$

$$Z(N, b, \boldsymbol{\xi}) = \frac{1}{(N!)^p} \int \prod_{r=1}^{p} \left[\prod_{i=1}^{N} \left(\frac{d\mu_{r,i}}{2\pi} e^{\frac{\mathrm{i}k_r}{4\pi}\mu_{r,i}^2 - \Delta_{m,r}\mu_{r,i}} \right) \prod_{i>j}^{N} \left(2\sinh \frac{b\mu_{r,ij}}{2} 2\sinh \frac{\mu_{r,ij}}{2b} \right) \right]$$

$$\times \prod_{\Psi} \prod_{\rho_\Psi} s_b \left(\frac{\mathrm{i}Q}{2}(1 - \Delta_\Psi) - \frac{\rho_\Psi(\mu)}{2\pi} \right)$$

	$\alpha(\boldsymbol{\xi})$	$\beta(\boldsymbol{\xi})$	$\gamma(\boldsymbol{\xi})$
ABJM	$\frac{4\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}$	$\frac{k}{24} - \frac{1}{12k}\sum_{a=1}^{4}\frac{1}{\Delta_a}$	$-\frac{4-\sum_{a=1}^{4}\Delta_a^2}{16k\Delta_1\Delta_2\Delta_3\Delta_4}$
ADHM	$\frac{4\sqrt{2N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}$	$\frac{N_f}{24} - \frac{N_f}{12}(\frac{1}{\tilde{\Delta}_1} + \frac{1}{\tilde{\Delta}_2}) - \frac{1}{12N_f}(\frac{1}{\tilde{\Delta}_3} + \frac{1}{\tilde{\Delta}_4})$	$-\frac{N_f}{8\tilde{\Delta}_1\tilde{\Delta}_2} + \frac{\tilde{\Delta}_1^2+\tilde{\Delta}_2^2-2(\tilde{\Delta}_1+\tilde{\Delta}_2)+\tilde{\Delta}_1\tilde{\Delta}_2}{8N_f\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}$
$N^{0,1,0}$	$\frac{2(k+N_f)}{3\sqrt{2k+N_f}}$	$-\frac{7N_f-2k}{48} - \frac{2}{3(k+N_f)}$	$-\frac{N_f}{4} - \frac{3k+2N_f}{(k+N_f)^2}$
$V^{5,2}$	$\frac{16\sqrt{N_f}}{27}$	$-\frac{N_f}{6} - \frac{1}{4N_f}$	$-\frac{9N_f}{16} - \frac{27}{16N_f}$
$Q^{1,1,1}$	$\frac{4\sqrt{N_f}}{3\sqrt{3}}$	$-\frac{N_f}{6}$	$-\frac{N_f}{4} - \frac{3}{4N_f}$

$$Z(N, b, \boldsymbol{\xi}) = \mathcal{C}^{-1/3} e^{\mathcal{A}} \mathrm{Ai}\left[\mathcal{C}^{-1/3}(N - \mathcal{B})\right] \left(1 + \mathcal{O}\left(e^{-\#\sqrt{N}}\right)\right)$$

$$\mathcal{B}(b, \boldsymbol{\xi}) = \beta(\boldsymbol{\xi}) - \frac{4}{3Q^2}\gamma(\boldsymbol{\xi})$$

$$\mathcal{C}(b, \boldsymbol{\xi}) = \left(\frac{8}{3\pi Q^2 \alpha(\boldsymbol{\xi})}\right)^2$$

$$-\log Z(N, b, \boldsymbol{\xi}) = \frac{2}{3}\mathcal{C}^{-1/2}N^{3/2} - \mathcal{C}^{-1/2}\mathcal{B}N^{1/2} + \frac{1}{4}\log N + \mathcal{O}(N^0)$$

$$
\begin{aligned}
\mathrm{Ai}[z] &= \frac{\exp\left[-\frac{2}{3}z^{3/2}\right]}{2\sqrt{\pi}z^{1/4}} \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n u_n z^{-3n/2} \\
u_n &= \frac{(6n-5)(6n-3)(6n-1)}{216(2n-1)n} u_{n-1} \; (n \geq 1, u_0 = 1)
\end{aligned}
$$

$$
C_T = -\frac{32}{\pi^2}\frac{\partial^2 \log Z_{S_b^3}}{\partial b^2}\bigg|_{b=1}
$$

$$
\begin{aligned}
C_T(N,\xi) =& -\frac{32}{9\pi^2}\Bigg\{\frac{\mathrm{Ai}'[\mathcal{C}^{-1/3}(N-\mathcal{B})]}{\mathcal{C}^{7/3}\mathrm{Ai}[\mathcal{C}^{-1/3}(N-\mathcal{B})]}\mathfrak{f}(N,b,\xi) \\
&+ \frac{\mathrm{Ai}[\mathcal{C}^{-1/3}(N-\mathcal{B})]^2(N-\mathcal{B}) - \mathcal{C}^{1/3}\mathrm{Ai}'[\mathcal{C}^{-1/3}(N-\mathcal{B})]^2}{\mathcal{C}^3\mathrm{Ai}[\mathcal{C}^{-1/3}(N-\mathcal{B})]^2}\mathfrak{g}(N,b,\xi)\Bigg\}\aleph'''\big|_{b=1} \\
&- \frac{128}{3\pi^2} - \frac{32}{\pi^2}\partial_b^2\mathcal{A}\Big|_{b=1} + \mathcal{O}\left(e^{-\sqrt{N}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{f}(N,b,\xi) = 4(N-\mathcal{B})(\partial_b\mathcal{C})^2 - 9\mathcal{C}^2\partial_b^2\mathcal{B} + 3\mathcal{C}\left(2\partial_b\mathcal{C}\partial_b\mathcal{B} - (N-\mathcal{B})\partial_b^2\mathcal{C}\right) \\
&\mathfrak{g}(N,b,\xi) = [(N-\mathcal{B})\partial_b\mathcal{C} + 3\mathcal{C}\partial_b\mathcal{B}]^2
\end{aligned}
$$

$$
\partial_b\mathcal{B}|_{b=1} = \partial_b\mathcal{C}|_{b=1} = 0,
$$

$$
\mathfrak{f}(N,1) = \frac{32}{81\pi^4\alpha^4}(6N - 6\beta - \gamma)
$$

$$
\begin{aligned}
C_T(N) =& -\frac{32}{3\pi^{4/3}}\left(\frac{2\alpha^2}{3}\right)^{1/3}(6N - 6\beta - \gamma)\frac{\mathrm{Ai}'\left[\left(\frac{3\pi\alpha}{2}\right)^{2/3}\left(N-\beta+\frac{\gamma}{3}\right)\right]}{\mathrm{Ai}\left[\left(\frac{3\pi\alpha}{2}\right)^{2/3}\left(N-\beta+\frac{\gamma}{3}\right)\right]} \\
&- \frac{128}{3\pi^2} - \frac{32}{\pi^2}\partial_b^2\mathcal{A}\Big|_{b=1} + \mathcal{O}\left(e^{-\sqrt{N}}\right)
\end{aligned}
$$

$$
\frac{\mathrm{Ai}'[a(N-b)]}{\mathrm{Ai}[a(N-b)]} = -\sqrt{a}N^{1/2} + \frac{b\sqrt{a}}{2}N^{-1/2} + \cdots
$$

	r	p	q
ABJM	$\frac{2}{\pi^2 Q^4 \Delta_1\Delta_2\Delta_3\Delta_4}$	$-\frac{1}{12}\sum_{a=1}^4 \frac{1}{\Delta_a} + \frac{4-\sum_{a=1}^4 \Delta_a^2}{12Q^2\Delta_1\Delta_2\Delta_3\Delta_4}$	$\frac{1}{24}$
ADHM	$\frac{2}{\pi^2 Q^4 \tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}$	$-\frac{1}{12}(\frac{1}{\tilde{\Delta}_3} + \frac{1}{\tilde{\Delta}_4}) - \frac{\tilde{\Delta}_1^2+\tilde{\Delta}_2^2-2(\tilde{\Delta}_1+\tilde{\Delta}_2)+\tilde{\Delta}_1\tilde{\Delta}_2}{6Q^2\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}$	$\frac{1}{24} - \frac{1}{12}(\frac{1}{\tilde{\Delta}_1} + \frac{1}{\tilde{\Delta}_2}) + \frac{1}{6Q^2\tilde{\Delta}_1\tilde{\Delta}_2}$
$N^{0,1,0}$	$\frac{12}{\pi^2 Q^4}$	$-\frac{1}{3} + \frac{5}{3Q^2}$	$-\frac{5}{48} + \frac{1}{3Q^2}$
$V^{5,2}$	$\frac{81}{4\pi^2 Q^4}$	$-\frac{1}{4} + \frac{9}{4Q^2}$	$-\frac{1}{6} + \frac{3}{4Q^2}$
$Q^{1,1,1}$	$\frac{12}{\pi^2 Q^4}$	$\frac{1}{Q^2}$	$-\frac{1}{6} + \frac{1}{3Q^2}$

$$
N, K \to \infty, N/K = \lambda
$$

$$
\mathcal{C} = \frac{r}{K}, \mathcal{B} = \frac{p}{K} + qK,
$$

$$-\log\, Z(N,\lambda) = \sum_{g\geq 0} F_g(\lambda) N^{2-2g},$$

$$F_g(\lambda) = \sum_{n\geq 0} \left\{ \frac{2}{3\sqrt{r}} \mathcal{F}_{g,n}(r,p,q) \frac{(-q)^n}{n!} \prod_{\ell=0}^{n-1} \left(\frac{3}{2} - \ell\right) + \mathcal{G}_{g,n}(r,p,q)\lambda^{-3/2} \right\} \lambda^{2g-1/2-n} - A_g \lambda^{2g-2}$$
$$+\delta_{g,1}\left[\frac{1}{4}\log(r) + \left(\frac{1}{4} + A_{\log}\right)\log(\lambda) + \frac{1}{2}\log(4\pi)\right]$$

$$\mathcal{F}_{g,n}(r,p,q) = \begin{cases} 1 & \text{for } g = 0 \\ npq^{-1} & \text{for } g = 1 \\ \displaystyle\sum_{m=0}^{\lfloor \frac{g}{2} \rfloor} F_{g,n}^{(m)}(r,p,q) & \text{for } g > 1 \end{cases}$$

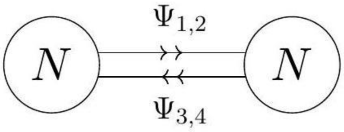
$$F_{g,n}^{(m)}(r,p,q) = \mathcal{P}^{(m)} \frac{n(n-1)\dots(n-g+1-m)}{(g-2m)!} r^m p^{g-2m} q^{-g-m}$$

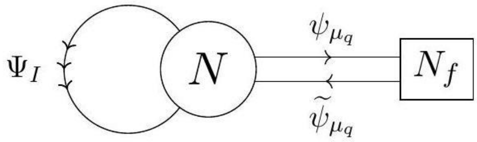
$$\mathcal{G}_{g,n\neq 0}(r,p,q) = \begin{cases} 0 & \text{for } g = 0 \\ -\dfrac{q^n}{4n} & \text{for } g = 1 \\ \displaystyle\sum_{m=1}^{\lfloor \frac{g}{2} \rfloor} G_{g,n}^{(m)}(r,p,q) & \text{for } g > 1 \end{cases} \quad \text{together with } \mathcal{G}_{g,0} = 0$$

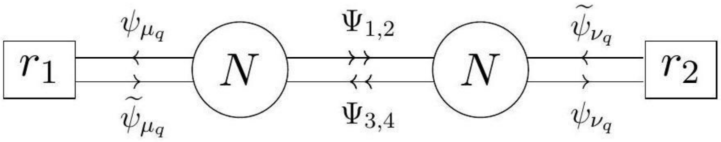
$$G_{g,n}^{(m)}(r,p,q) = \mathcal{Q}^{(m)} \frac{(n-1)\dots(n-g+3-m)}{(g-2m+1)!} r^{m-1} p^{g-2m+1} q^{n-g+2-m}$$

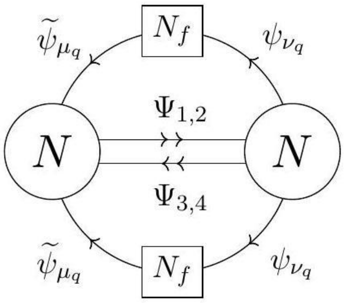
$$\mathcal{A} = \sum_{g\geq 0} A_g K^{2-2g} + A_{\log}\log(K)$$

$$F_0(\lambda) = \lambda^{-2}\left[\frac{2}{3\sqrt{r}}\hat{\lambda}^{3/2} - A_0\right] + \mathcal{O}\left(e^{-\sqrt{\lambda}}\right)$$
$$F_1(\lambda) = -\frac{p}{\sqrt{r}}\hat{\lambda}^{1/2} - A_1 + A_{\log}\log\lambda + \frac{1}{4}\log(16\pi^2 r) + \frac{1}{4}\log\hat{\lambda} + \mathcal{O}\left(e^{-\sqrt{\lambda}}\right)$$
$$F_2(\lambda) = \lambda^2\left[\frac{p^2}{4\sqrt{r}}\hat{\lambda}^{-1/2} - \frac{p}{4}\hat{\lambda}^{-1} + \frac{5\sqrt{r}}{48}\hat{\lambda}^{-3/2} - A_2\right] + \mathcal{O}\left(e^{-\sqrt{\lambda}}\right)$$









$$Z^{\mathrm{ABJM}}(N,b,k,\Delta)=\frac{1}{(N!)^2}\int\left(\prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\frac{d\nu_i}{2\pi}\right)e^{\frac{\mathrm{i}k}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)}$$
$$\times\prod_{i>j}2\sinh\frac{b\mu_{ij}}{2}2\sinh\frac{\mu_{ij}}{2b}2\sinh\frac{b\nu_{ij}}{2}2\sinh\frac{\nu_{ij}}{2b}$$
$$\times\prod_{i,j=1}^{N}\prod_{a=1}^{2}s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta_a)-\frac{\mu_i-\nu_j}{2\pi}\right)\prod_{a=3}^{4}s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta_a)+\frac{\mu_i-\nu_j}{2\pi}\right)$$

$$\sum_{a=1}^{4}\Delta_a=2.$$

$$\Delta_1=\frac{1}{2}-\mathrm{i}\frac{m_1+m_2+m_3}{Q},\Delta_2=\frac{1}{2}-\mathrm{i}\frac{m_1-m_2-m_3}{Q}$$
$$\Delta_3=\frac{1}{2}+\mathrm{i}\frac{m_1+m_2-m_3}{Q},\Delta_4=\frac{1}{2}+\mathrm{i}\frac{m_1-m_2+m_3}{Q}$$

$$\Delta_1\leftrightarrow\Delta_2,\Delta_3\leftrightarrow\Delta_4,(\Delta_1,\Delta_2)\leftrightarrow(\Delta_3,\Delta_4).$$

$$\boldsymbol{\Delta}=\boldsymbol{\Delta}_{\mathrm{Nosaka}}\equiv(\Delta_1,\Delta_2,1-\Delta_2,1-\Delta_1)$$

$$\mathcal{A}^{\mathrm{ABJM}}(1,k,\Delta_{\mathrm{Nosaka}})=\frac{1}{4}\sum_{a=1}^{4}A(2\Delta_a k),$$

$$A(k)\equiv\frac{2\zeta(3)}{\pi^2k}\left(1-\frac{k^3}{16}\right)+\frac{k^2}{\pi^2}\int_0^\infty dx\,\frac{x}{e^{kx}-1}\log\left(1-e^{-2x}\right)$$
$$=-\frac{\zeta(3)}{8\pi^2}k^2+2\zeta'(-1)+\frac{1}{6}\log\frac{4\pi}{k}+\sum_{g\geq2}\left(\frac{2\pi}{k}\right)^{2g-2}\frac{(-1)^g4^{g-1}\left|B_{2g}B_{2g-2}\right|}{g(2g-2)(2g-2)!}$$

$$
\begin{aligned}
A(-k) &= -\frac{2\zeta(3)}{\pi^2 k}\left(1+\frac{k^3}{16}\right)-\frac{k^2}{\pi^2}\int_0^\infty dx\left[x+\frac{x}{e^{kx}-1}\right]\log\left(1-e^{-2x}\right)\\
&= -\frac{2\zeta(3)}{\pi^2 k}\left(1+\frac{k^3}{16}\right)-\frac{k^2}{\pi^2}\left[-\frac{1}{4}\zeta(3)+\int_0^\infty dx\frac{x}{e^{kx}-1}\log\left(1-e^{-2x}\right)\right]\\
&= -A(k)
\end{aligned}
$$

$$
\mathcal{A}^{\mathrm{ABJM}}(\sqrt{3},1,\boldsymbol{\Delta}_{\mathrm{sc}})=\frac{3}{4}A(2)-\frac{1}{4}A(6)=-\frac{\zeta(3)}{3\pi^2}+\frac{1}{6}\log\, 3.
$$

$$
\Delta_{\mathrm{Fermi}}^{(N_f=1)}\equiv\left(\frac{1}{2}+\mathrm{i}\frac{2m}{Q},\frac{1}{2}-\mathrm{i}\frac{2\zeta}{Q},\frac{1}{2}-\mathrm{i}\frac{2m}{Q},\frac{1}{2}+\mathrm{i}\frac{2\zeta}{Q}\right)\Bigg|_{m=\frac{(b^2-3)\mathrm{i}}{4b}}
$$

$$
\begin{aligned}
\mathcal{A}^{\mathrm{ABJM}}\left(b,1,\Delta_{\mathrm{Fermi}}^{(N_f=1)}\right)=\frac{1}{4}\bigg[&A\left(\frac{b^2+1-4\mathrm{i}b\zeta}{2}\right)+A\left(\frac{b^2+1+4\mathrm{i}b\zeta}{2}\right)\\
&+A(b^2-1)-A(2b^2)\bigg]
\end{aligned}
$$

$$
\zeta=\frac{(b^{-2}-3)\mathrm{i}}{4b^{-1}}
$$

$$
\mathcal{A}^{\mathrm{ABJM}}\left(b,1,\Delta_{\mathrm{Fermi}}^{(N_f=1)}\right)\Bigg|_{\zeta=\frac{(b^{-2}-3)\mathrm{i}}{4b^{-1}}}=\frac{1}{4}\left[A(1-b^2)+A(b^2-1)\right]=0
$$

$$
\begin{aligned}
-\log\, Z^{\mathrm{ABJM}}(N,1,k,\boldsymbol{\Delta}) =& \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{\left(\lambda-\frac{1}{24}\right)^{\frac{3}{2}}}{\lambda^2}N^2+\frac{\mathfrak{c}^{\mathrm{ABJM}}(1,\boldsymbol{\Delta})}{\lambda^2}N^2\\
&+\mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right)+o(N^2)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{c}^{\mathrm{ABJM}}(1,\boldsymbol{\Delta}) =& \frac{\zeta(3)}{8\pi^2}\Bigg[4-\sum_{a=1}^{4}\Delta_a^2+\frac{4}{\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}}\sum_{a=1}^{4}\frac{\prod_{b=1}^4\Delta_b}{\Delta_a}\\
&-\left(\frac{4}{\Delta_{13}\Delta_{24}}+\frac{4}{\Delta_{14}\Delta_{23}}\right)\sum_{a=1}^{4}\frac{\prod_{b=1}^4\Delta_b}{\Delta_a}\Bigg]
\end{aligned}
$$

$$
\mathcal{A}^{\mathrm{ABJM}}(1,k,\boldsymbol{\Delta})=-\mathfrak{c}^{\mathrm{ABJM}}(1,\boldsymbol{\Delta})k^2+o(k^2)
$$

$$
\begin{aligned}
&F^{\mathrm{ABJM}}\left(N,b,k,m_1,m_2,\mathrm{i}\frac{b-b^{-1}}{2}\right)\\
&=F^{\mathrm{ABJM}}\left(N,1,k,\frac{b^{-1}(m_1+m_2)+b(m_1-m_2)}{2},\frac{b^{-1}(m_1+m_2)-b(m_1-m_2)}{2},0\right)
\end{aligned}
$$

$$
\begin{aligned}
-\log\, Z^{\mathrm{ABJM}}(N,b,k,\Delta_{\mathrm{sc}}) =& \frac{\pi\sqrt{2k}}{3}\left[\frac{Q^2}{4}\left(N^{3/2}+\frac{16-k^2}{16k}N^{1/2}\right)-\frac{2}{k}N^{1/2}\right]\\
&+\frac{1}{4}\log\, N+\mathcal{O}(N^0)
\end{aligned}
$$

$$
F^{\mathrm{ABJM}}=\frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}N^{3/2}
$$

$$
\Delta_2+\Delta_3+\Delta_4=1
$$

$$\beta(\boldsymbol{\xi}) = \frac{k}{8}\left(1-\frac{2M}{k}\right)^2 - \frac{k}{12} - \frac{1}{12k}\sum_{a=1}^{4}\frac{1}{\Delta_a}$$

$$Z^{\mathrm{ADHM}}(N,b,N_f,\widetilde{\boldsymbol{\Delta}}) = \frac{1}{N!}\int \prod_{i=1}^{N}\left(\frac{d\mu_i}{2\pi}e^{-\Delta_m\mu_i}\right)\prod_{i>j}^{N} 2\sinh\frac{b\mu_{ij}}{2}\, 2\sinh\frac{\mu_{ij}}{2b}$$
$$\times\prod_{a=1}^{3}\prod_{i,j=1}^{N} s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta_a)-\frac{\mu_{ij}}{2\pi}\right)$$
$$\times\prod_{i=1}^{N}\prod_{q=1}^{N_f}\left[s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\Delta_{\mu_q}\right)-\frac{\mu_i}{2\pi}\right)s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\widetilde{\Delta}_{\mu_q}\right)+\frac{\mu_i}{2\pi}\right)\right]$$

$$\widetilde{\Delta} = \left(\Delta_1,\Delta_2,\frac{2-\Delta-\widetilde{\Delta}}{2}-\frac{2}{Q}\frac{\Delta_m}{N_f},\frac{2-\Delta-\widetilde{\Delta}}{2}+\frac{2}{Q}\frac{\Delta_m}{N_f}\right)$$

$$\sum_{I=1}^{3}\Delta_I = \Delta_3+\Delta+\widetilde{\Delta} = 2 \;\rightarrow\; \sum_{a=1}^{4}\widetilde{\Delta}_a = 2$$

$$\widetilde{\Delta}_{\text{Nosaka-like}} \equiv \left(\widetilde{\Delta}_1, 1-\widetilde{\Delta}_1, \widetilde{\Delta}_3, 1-\widetilde{\Delta}_3\right)$$

$$\mathcal{A}^{\mathrm{ADHM}}(1,N_f,\widetilde{\Delta}_{\text{Nosaka-like}}) = \frac{N_f^2}{2}\left(A(2\widetilde{\Delta}_1)+A(2\widetilde{\Delta}_2)\right)$$
$$+\frac{1}{4}\left(A(2\widetilde{\Delta}_3N_f)+A(2\widetilde{\Delta}_3N_f)\right)$$

$$\mathcal{A}^{\mathrm{ADHM}}(\sqrt{3},1,\widetilde{\boldsymbol{\Delta}}_{\mathrm{sc}}) = \frac{3}{4}A(2)-\frac{1}{4}A(6) = -\frac{\zeta(3)}{3\pi^2}+\frac{1}{6}\log 3.$$

$$\widetilde{\Delta}^{(N_f=1)}_{\mathrm{Fermi}} \equiv \left.\left(\frac{1}{2}+\mathrm{i}\frac{2m}{Q},\frac{1}{2}-\mathrm{i}\frac{2m}{Q},\frac{1}{2}-\mathrm{i}\frac{2\zeta}{Q},\frac{1}{2}+\mathrm{i}\frac{2\zeta}{Q}\right)\right|_{m=\frac{(b^2-3)\mathrm{i}}{4b}}$$

$$\mathcal{A}^{\mathrm{ADHM}}\left(b,1,\widetilde{\Delta}^{(N_f=1)}_{\mathrm{Fermi}}\right) = \mathcal{A}^{\mathrm{ABJM}}\left(b,1,\Delta^{(N_f=1)}_{\mathrm{Fermi}}\right)$$
$$= \frac{1}{4}\left[A\left(\frac{b^2+1-4\mathrm{i}b\zeta}{2}\right)+A\left(\frac{b^2+1+4\mathrm{i}b\zeta}{2}\right)+A(b^2-1)-A(2b^2)\right].$$

$$-\log\, Z^{\text{ADHM}}(N,1,N_f,\tilde{\Delta}) = \frac{4\pi\sqrt{2\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\tilde{\Delta}_4}}{3}\frac{\left(\lambda-\frac{1-2(\tilde{\Delta}_1+\tilde{\Delta}_2)+\tilde{\Delta}_1\tilde{\Delta}_2}{24\tilde{\Delta}_1\tilde{\Delta}_2}\right)^{\frac{3}{2}}}{\lambda^2}N^2 + \frac{\mathfrak{c}^{\text{ADHM}}(1,\tilde{\Delta})}{\lambda^2}N^2 + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right) + o(N^2)$$

$$\mathfrak{c}^{\text{ADHM}}(1,\tilde{\Delta}_{\text{Nosaka-like}}) = \frac{\mathcal{A}(2\tilde{\Delta}_1)+\mathcal{A}(2\tilde{\Delta}_2)}{4} - \frac{\zeta(3)}{8\pi^2}\left(\tilde{\Delta}_3^2+\tilde{\Delta}_4^2\right)$$

$$\mathcal{A}^{\text{ADHM}}(1,N_f,\tilde{\mathbf{\Delta}}) = -\mathfrak{c}^{\text{ADHM}}(1,\tilde{\mathbf{\Delta}})N_f^2 + o(N_f^2)$$

$$-\log\, Z^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{sc}}) = \frac{\pi\sqrt{2N_f}}{3}\left[\frac{Q^2}{4}\left(N^{3/2}+\frac{8+7N_f^2}{16N_f}N^{1/2}\right)-\frac{N_f^2+5}{4N_f}N^{1/2}\right] + \mathcal{O}(\log\, N)$$

$$Z^{N^{0,1,0}}(N,b,k,r_1,r_2,\mathbf{\Delta}) = \frac{1}{(N!)^2}\int\left(\prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\frac{d\nu_i}{2\pi}\right)e^{\frac{\mathrm{i}k}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)}$$
$$\times\prod_{i>j}2\sinh\frac{b\mu_{ij}}{2}2\sinh\frac{\mu_{ij}}{2b}2\sinh\frac{b\nu_{ij}}{2}2\sinh\frac{\nu_{ij}}{2b}$$
$$\times\prod_{i,j=1}^{N}\prod_{a=1}^{2}s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta_a)-\frac{\mu_i-\nu_j}{2\pi}\right)\prod_{a=3}^{4}s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta_a)+\frac{\mu_i-\nu_j}{2\pi}\right)$$
$$\times\prod_{q=1}^{r_1}\prod_{i=1}^{N}s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\Delta_{\mu_q}\right)-\frac{\mu_i}{2\pi}\right)s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\tilde{\Delta}_{\mu_q}\right)+\frac{\mu_i}{2\pi}\right)$$
$$\times\prod_{q=1}^{r_2}\prod_{i=1}^{N}s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\Delta_{\nu_q}\right)-\frac{\nu_i}{2\pi}\right)s_b\left(\frac{\mathrm{i}Q}{2}\left(1-\tilde{\Delta}_{\nu_q}\right)+\frac{\nu_i}{2\pi}\right)$$

$$\Delta_a = \Delta_{\mu_q} = \tilde{\Delta}_{\mu_q} = \Delta_{\nu_q} = \tilde{\Delta}_{\nu_q} = \frac{1}{2}$$

$$r_1 = r_2 = \frac{N_f}{2}$$

$$-\log\, Z^{N^{0,1,0}(N,b,k,N_f)=\frac{2\pi(k+N_f)}{3\sqrt{2k+N_f}}}\left[\frac{Q^2}{4}\left(N^{3/2}+\left(\frac{7N_f-2k}{32}+\frac{1}{k+N_f}\right)N^{1/2}\right) - \left(\frac{N_f}{8}+\frac{3k+2N_f}{2(k+N_f)^2}\right)N^{1/2}\right] + \mathcal{O}(\log\, N)$$

$$\Delta_1+\Delta_2 = \frac{4}{3}, \Delta_3 = \frac{2}{3}, \Delta_{\mu_q}+\tilde{\Delta}_{\mu_q} = \frac{2}{3}$$

$$(\Delta_1,\Delta_2,\Delta_3) = \left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right), \left(\Delta_{\mu_q},\tilde{\Delta}_{\mu_q}\right) = \left(\frac{1}{3},\frac{1}{3}\right)$$

$$-\log\, Z^{V^{5,2}}(1,N_f) = \frac{16\sqrt{2}\pi}{27}\frac{\left(\lambda-\frac{1}{48}\right)^{\frac{3}{2}}}{\lambda^2}N^2 + \frac{\mathfrak{c}^{V^{5,2}}(1)}{\lambda^2}N^2 + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right) + o(N^2)$$

$$\mathcal{A}^{V^{5,2}}(1,N_f) = -\mathfrak{c}^{V^{5,2}}(1)N_f^2 + o(N_f^2)$$

$$-\log\, Z^{V^{5,2}}(N,b,N_f) = \frac{16\pi\sqrt{2N_f}}{27}\left[\frac{Q^2}{4}\left(N^{3/2} + \left(\frac{N_f}{4} + \frac{3}{8N_f}\right)N^{1/2}\right) - \frac{9(N_f^2+3)}{32N_f}N^{1/2}\right] + \mathcal{O}(\log\, N)$$

$$Z^{Q^{1,1,1}}(N,b,N_f,\mathbf{\Delta}) = \frac{1}{(N!)^2}\int\left(\prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\frac{d\nu_i}{2\pi}\right)\prod_{i>j} 2\sinh\frac{b\mu_{ij}}{2}2\sinh\frac{\mu_{ij}}{2b}2\sinh\frac{b\nu_{ij}}{2}2\sinh\frac{\nu_{ij}}{2b}$$
$$\times\prod_{i,j=1}^{N}\prod_{a=1}^{2} s_b\left(\frac{iQ}{2}(1-\Delta_a) - \frac{\mu_i-\nu_j}{2\pi}\right)\prod_{a=3}^{4} s_b\left(\frac{iQ}{2}(1-\Delta_a) + \frac{\mu_i-\nu_j}{2\pi}\right)$$
$$\times\prod_{q=1}^{N_f}\prod_{i=1}^{N} s_b\left(\frac{iQ}{2}\left(1-\tilde{\Delta}_{\mu_q}\right) + \frac{\mu_i}{2\pi}\right)s_b\left(\frac{iQ}{2}\left(1-\Delta_{\nu_q}\right) - \frac{\nu_i}{2\pi}\right)$$
$$\times\prod_{q=N_f+1}^{2N_f}\prod_{i=1}^{N} s_b\left(\frac{iQ}{2}\left(1-\tilde{\Delta}_{\mu_q}\right) + \frac{\mu_i}{2\pi}\right)s_b\left(\frac{iQ}{2}\left(1-\Delta_{\nu_q}\right) - \frac{\nu_i}{2\pi}\right)$$

$$\Delta_a = \frac{1}{2}, \tilde{\Delta}_{\mu_q} = \Delta_{\nu_q} = \frac{3}{4},$$

$$-\log\, Z^{Q^{1,1,1}}(1,N_f) = \frac{4\pi}{3\sqrt{3}}\frac{\left(\lambda + \frac{1}{12}\right)^{\frac{3}{2}}}{\lambda^2}N^2 + \frac{\mathfrak{c}^{Q^{1,1,1}}(1)}{\lambda^2}N^2 + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right) + o(N^2)$$

$$\mathcal{A}^{Q^{1,1,1}}(1,N_f) = -\mathfrak{c}^{Q^{1,1,1}}(1)N_f^2 + o(N_f^2).$$

$$-\log\, Z^{Q^{1,1,1}}(N,b,N_f) = \frac{4\pi\sqrt{N_f}}{3\sqrt{3}}\left[\frac{Q^2}{4}\left(N^{3/2} + \frac{N_f}{4}N^{1/2}\right) - \frac{N_f^2+3}{8N_f}N^{1/2}\right] + \mathcal{O}(\log\, N)$$

$$\tilde{\Delta}_{\text{Fermi}} \equiv \left(\frac{1}{2} + i\frac{2m}{Q}, \frac{1}{2} - i\frac{2m}{Q}, \frac{1}{2} - i\frac{2\zeta}{QN_f}, \frac{1}{2} + i\frac{2\zeta}{QN_f}\right)\Bigg|_{m=\frac{(b^2-3)i}{4b}}$$

$$Z^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{Fermi}}) = \frac{1}{N!}\int\prod_{i=1}^{N} d\mu_i \det_{i,j=1}^{N}\langle\mu_i|\hat{\rho}|\mu_j\rangle$$

$$\hat{\rho} = e^{\frac{ib\zeta}{2}\hat{q}_s}\left(\frac{b\hat{q}}{2\pi} + \frac{iQ}{4}\right)^{N_f}\frac{e^{-\frac{\hat{p}}{2b^2}}}{2\cosh\frac{\hat{p}}{2}}\frac{1}{s_b\left(\frac{b\hat{q}}{2\pi} - \frac{iQ}{4}\right)^{N_f}}e^{\frac{ib\zeta}{2}\hat{q}}$$

$$\mathcal{Z}(s) = \text{Tr}[\hat{\rho}^s] = \int\frac{dpdq}{2\pi\hbar}(\rho^s)_W(p,q) = \sum_{n=0}^{\infty}\hbar^{2n}\underbrace{\int\frac{dpdq}{2\pi\hbar}(\rho^s)_W^{(n)}(p,q)}_{=\mathcal{Z}^{(n)}(s)}$$

$$e^{\mathcal{J}^{\text{ADHM}}(\mu;b,N_f,\tilde{\Delta}_{\text{Fermi}})} = 1 + \sum_{N=1}^{\infty} Z^{\text{ADHM}}(N;b,N_f,\tilde{\Delta}_{\text{Fermi}})e^{\mu N}$$

$$\mathcal{J}^{\text{ADHM}}(\mu;b,N_f,\tilde{\Delta}_{\text{Fermi}}) = -\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{ds}{2\pi\mathrm{i}}\Gamma(s)\Gamma(-s)\mathcal{Z}(s)e^{s\mu}\ (c\in(0,1))$$

$$Z^{\text{ADHM}}(N;b,N_f,\tilde{\Delta}_{\text{Fermi}}) = \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}} \frac{d\mu}{2\pi\mathrm{i}}\exp\left[\mathcal{J}^{\text{ADHM}}(\mu;b,N_f,\tilde{\Delta}_{\text{Fermi}}) - \mu N\right]$$

$$\mathcal{Z}^{(0)}(s) = \int \frac{dpdq}{2\pi\hbar}\left(e^{-\mathrm{i}b\zeta q+\left(\frac{1}{2b^2}+\frac{1}{2}\right)p} + \sum_{n=0}^{N_f}\binom{N_f}{n}e^{\left((2n-N_f)\frac{b^2}{2}-\mathrm{i}b\zeta\right)q+\left(\frac{1}{2b^2}-\frac{1}{2}\right)p}\right)^{-s}$$

$$\mathcal{A}^{\text{ADHM}}(b,N_f,\tilde{\Delta}_{\text{Fermi}}) = \frac{1}{4}\Bigg[A\left(\frac{(b^2+1)N_f-4\mathrm{i}b\zeta}{2}\right) + A\left(\frac{(b^2+1)N_f+4\mathrm{i}b\zeta}{2}\right)$$
$$+N_f^2\left(A(b^2-1)-A(2b^2)\right)\Bigg]$$

$$F_{\text{num}}^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{Fermi}}).$$

$$D^{\text{ADHM}}(N,b,N_f,\tilde{\boldsymbol{\Delta}}_{\text{Fermi}}) \equiv F_{\text{num}}^{\text{ADHM}}(N,b,N_f,\tilde{\boldsymbol{\Delta}}_{\text{Fermi}})$$
$$+\log\left[\mathcal{C}^{-1/3}\text{Ai}\left[\mathcal{C}^{-1/3}(N-\mathcal{B})\right]\right]$$

$$D^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{Fermi}}) \overset{!}{=} -\mathcal{A}^{\text{ADHM}}(b,N_f,\tilde{\Delta}_{\text{Fermi}}) + \mathcal{O}\left(e^{-\#\sqrt{N}}\right).$$

$$R^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{Fermi}}) \equiv \frac{D^{\text{ADHM}}(N,b,N_f,\tilde{\Delta}_{\text{Fermi}})}{-\mathcal{A}^{\text{ADHM}}(b,N_f,\tilde{\Delta}_{\text{Fermi}})} - 1 \to 0,$$

$$R^{\text{ADHM}}(20,\sqrt{5},2,\tilde{\Delta}_{\text{Fermi}})\Big|_{\zeta=\frac{i}{10}} = -1.168\times10^{-9}$$
$$R^{\text{ADHM}}(20,\sqrt{7},3,\tilde{\Delta}_{\text{Fermi}})\Big|_{\zeta=\frac{1}{5}} = -1.595\times10^{-6}$$

$$\log\left|D^{\text{ADHM}}(N,b,N_f,\tilde{\boldsymbol{\Delta}}_{\text{Fermi}}) + \mathcal{A}^{\text{ADHM}}(b,N_f,\tilde{\boldsymbol{\Delta}}_{\text{Fermi}})\right|$$
$$= \mathrm{e}_{1/2}^{(\text{lmf})}N^{1/2} + \mathrm{e}_{\log}^{(\text{lmf})}\log N + \sum_{g=0}^{3}\mathrm{e}_g^{(\text{lmf})}N^{-g/2}$$

(b^2, N_f, ζ)	$\mathrm{e}_{1/2}^{(\text{lmf})}$	Standard error	(b^2, N_f, ζ)	$\mathrm{e}_{1/2}^{(\text{lmf})}$	Standard error
(3, 2, 0)	-6.27478965	1.698×10^{-4}	(5, 2, 0)	-3.61888083	2.913×10^{-4}
(3, 5, 0)	-3.95851786	3.338×10^{-4}	(5,5,0)	-2.29316899	4.984×10^{-4}

$(3, 2, \frac{i}{5})$	-5.28818832	8.471×10^{-4}	$(5,2, \frac{i}{5})$	-3.18353969	2.519×10^{-3}
$(3, 5, \frac{i}{5})$	-3.59847102	2.322×10^{-3}	$(5,5, \frac{i}{5})$	-2.15336338	1.505×10^{-3}

$$F(N, b, \boldsymbol{\xi}) = -\log\ Z(N, b, \boldsymbol{\xi}) = \sum_{g=0}^{\infty} N^{2-2\,g} F_g(b, \boldsymbol{\xi}) + \log\ N,$$

$$F_0^{\mathrm{ABJM}}(b, \lambda, \boldsymbol{\Delta}) = \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3} \frac{Q^2}{4} \frac{\left(\lambda - \frac{1}{24}\right)^{3/2}}{\lambda^2} + \frac{\mathrm{c}^{\mathrm{ABJM}}(b, \boldsymbol{\Delta})}{\lambda^2} + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right)$$

$$\mathrm{c}^{\mathrm{ABJM}}(b, \boldsymbol{\Delta}) = \frac{\zeta(3)}{8\pi^2}\left[\frac{Q^2}{4}\left(4 - \sum_{a=1}^{4} \Delta_a^2\right)\right.$$

$$\left. + \left[\frac{4}{\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}} - Q^2\left(\frac{1}{\Delta_{13}\Delta_{24}} + \frac{1}{\Delta_{14}\Delta_{23}}\right)\right] \sum_{a=1}^{4} \frac{\prod_{b=1}^{4} \Delta_b}{\Delta_a}\right]$$

$$\mathcal{A}^{\mathrm{ABJM}}(b, k, \boldsymbol{\Delta}) = -\mathrm{c}^{\mathrm{ABJM}}(b, \boldsymbol{\Delta}) k^2 + o(k^2).$$

$$Z_{S_b^3}^{\mathrm{ABJM}}(N, b, k, \Delta_{\mathrm{Nosaka}}) = \frac{1}{(N!)^2} \int \left(\prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \frac{d\nu_i}{2\pi}\right) e^{\frac{ik}{4\pi}\sum_{i=1}^{N} (\mu_i^2 - \nu_i^2)}$$

$$\times \prod_{i>j} 2\sinh \frac{b\mu_{ij}}{2} 2\sinh \frac{\mu_{ij}}{2b} 2\sinh \frac{b\nu_{ij}}{2} 2\sinh \frac{\nu_{ij}}{2b}$$

$$\times \prod_{i,j=1}^{N} \mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 + m_2}{2}\right) \mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 - m_2}{2}\right)$$

$$= \frac{1}{(N!)^2} \int \left(\prod_{i=1}^{N} d\mu_i d\nu_i\right) \exp\left[-S_{\mathrm{eff}}^{\mathrm{ABJM}}[\mu, \nu; N, b, \lambda, \Delta_{\mathrm{Nosaka}}]\right]$$

$$0 = \frac{\mathrm{i}k}{2\pi}\mu_i + \sum_{j=1}^{N}\left[\frac{b}{2}\coth\frac{b\mu_{ij}}{2} + \frac{1}{2b}\coth\frac{\mu_{ij}}{2b}\right]$$

$$+ \frac{1}{2\pi}\sum_{j=1}^{N}\left[\frac{\mathcal{D}_b'\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 + m_2}{2}\right)}{\mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 + m_2}{2}\right)} + \frac{\mathcal{D}_b'\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 - m_2}{2}\right)}{\mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 - m_2}{2}\right)}\right]$$

$$0 = -\frac{\mathrm{i}k}{2\pi}\nu_j + \sum_{i=1}^{N}\left[\frac{b}{2}\coth\frac{b\nu_{ji}}{2} + \frac{1}{2b}\coth\frac{\nu_{ji}}{2b}\right]$$

$$- \frac{1}{2\pi}\sum_{i=1}^{N}\left[\frac{\mathcal{D}_b'\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 + m_2}{2}\right)}{\mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 + m_2}{2}\right)} + \frac{\mathcal{D}_b'\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 - m_2}{2}\right)}{\mathcal{D}_b\left(\frac{\mu_i - \nu_j}{2\pi} + \frac{m_1 - m_2}{2}\right)}\right]$$

$$S_{\mathrm{cl}}^{\mathrm{ABJM}}(N, b, \lambda, \Delta_{\mathrm{Nosaka}}) \equiv S_{\mathrm{eff}}^{\mathrm{ABJM}}[\mu^\star, \nu^\star; N, b, \lambda, \Delta_{\mathrm{Nosaka}}]$$

$$S_{\mathrm{cl}}^{\mathrm{ABJM}}(N,b,\lambda,\Delta_{\mathrm{Nosaka}}) = F_0^{\mathrm{ABJM}}(b,\lambda,\Delta_{\mathrm{Nosaka}})N^2 + o(N^2).$$

$$\begin{aligned}
&S_{\mathrm{cl}}^{\mathrm{ABJM}}(N,b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}}) \\
&= S_2^{\mathrm{ABJM,(lmf)}}(b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}})N^2 + S_{1-\log}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}})N\log N \\
&+S_1^{\mathrm{ABJM,(lmf)}}(b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}})N + S_{\log}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}})\log N \\
&+\sum_{L=0}^{20} S_{-L}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\boldsymbol{\Delta}_{\mathrm{Nosaka}})N^{-L}
\end{aligned}$$

$$S_{1-\log}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}}) \approx -2, S_{\log}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}}) \approx -\frac{1}{6}$$

$$\begin{aligned}
&S_{\mathrm{cl}}^{\mathrm{ABJM}}(N,b,\lambda,\Delta_{\mathrm{Nosaka}}) + 2N\log N + \frac{1}{6}\log N \\
&= S_2^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}})N^2 + S_1^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}})N \\
&+\sum_{L=0}^{22} S_{-L}^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}})N^{-L}
\end{aligned}$$

$$F_0^{\mathrm{ABJM}}(b,\lambda,\Delta_{\mathrm{Nosaka}}) = \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{Q^2}{4}\frac{\left(\lambda-\frac{1}{24}\right)^{3/2}}{\lambda^2} + \frac{\mathfrak{c}^{\mathrm{ABJM}}(b,\Delta_{\mathrm{Nosaka}})}{\lambda^2} + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right)$$

$$\begin{aligned}
\mathfrak{c}^{\mathrm{ABJM}}(b,\Delta_{\mathrm{Nosaka}}) = \frac{\zeta(3)}{8\pi^2}\Bigg[&-\frac{1}{8}(b+b^{-1})^2(b^2-4+b^{-2}) - \frac{b^2+b^{-2}}{2}(m_1^2+m_2^2) \\
&+\frac{(b-b^{-1})^2}{4}\left(m_2^2+\frac{(b-b^{-1})^2}{4}\right)\frac{2\left(1+\frac{4}{(b+b^{-1})^2}(m_1^2+m_2^2)\right)}{1+\frac{4}{(b+b^{-1})^2}m_2^2}\Bigg]
\end{aligned}$$

$$F_0^{\mathrm{ABJM}}(b,\lambda,\Delta) = \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{Q^2}{4}\lambda^{-1/2} + o\left(\lambda^{-1/2}\right)$$

$$F_0^{\mathrm{ABJM}}(1,\lambda,\boldsymbol{\Delta}) = \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{\left(\lambda-\frac{1}{24}\right)^{3/2}}{\lambda^2} + \frac{\mathfrak{c}^{\mathrm{ABJM}}(1,\boldsymbol{\Delta})}{\lambda^2} + \mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right),$$

$$\frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{Q^2}{4}\frac{\left(\lambda-\frac{1}{24}\right)^{3/2}}{\lambda^2}$$

$$\mathfrak{c}^{\mathrm{ABJM}}(b,\boldsymbol{\Delta})\Big|_{m_3\to \mathrm{i}\frac{b-b^{-1}}{2}} = \mathfrak{c}^{\mathrm{ABJM}}(1,\boldsymbol{\Delta})\Big|_{(m_1,m_2,m_3)\to\left(\frac{b^{-1}m_+-bm_-}{2},\frac{b^{-1}m_++bm_-}{2},0\right)},$$

$$\widetilde{\boldsymbol{\Delta}}_{\mathrm{sc}} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

$$Z_{S_b^3}^{\mathrm{ADHM}}(N,b,N_f)=\frac{1}{N!}\int\prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\prod_{i>j}^{N}2\sinh\frac{b\mu_{ij}}{2}2\sinh\frac{\mu_{ij}}{2b}\prod_{i,j=1}^{N}\mathcal{D}_b\left(\frac{\mu_{ij}}{2\pi}\right)\times\prod_{i=1}^{N}\mathcal{D}_b\left(\frac{\mu_i}{2\pi}\right)^{N_f}$$

$$=\frac{1}{N!}\int\left(\prod_{i=1}^{N}d\mu_i\right)\exp\left[-S_{\mathrm{eff}}[\mu;N,b,\lambda]\right]$$

$$F_0^{\mathrm{ADHM}}(b,\lambda)=\frac{\pi\sqrt{2}}{3}\frac{Q^2}{4}\frac{\left(\lambda-\frac{1}{24}+\frac{b^2+b^{-2}}{3Q^2}\right)^{3/2}}{\lambda^2}+\frac{\mathfrak{c}^{\mathrm{ADHM}}(b)}{\lambda^2}+\mathcal{O}\left(e^{-\#\sqrt{\lambda}}\right),$$

$$\mathcal{A}^{\mathrm{ADHM}}(b,N_f)=-\mathfrak{c}^{\mathrm{ADHM}}(b)N_f^2+o(N_f^2),$$

$$\mathfrak{c}^{\mathrm{ADHM}}(b)=\begin{cases}\frac{\zeta(3)}{16\pi^2}-\frac{1}{2}A(1)=\frac{\zeta(3)}{8\pi^2}-\frac{1}{8}\log 2 & (b=1)\\ \frac{\zeta(3)}{4\pi^2}-\frac{1}{4}(A(2)-A(6))=\frac{\zeta(3)}{3\pi^2}-\frac{1}{6}\log 3 & (b=\sqrt{3})\end{cases}$$

$$0=\sum_{j=1}^{N}\left[\frac{b}{2}\coth\frac{b\lambda_{ij}}{2}+\frac{1}{2b}\coth\frac{\lambda_{ij}}{2b}\right]+\frac{1}{\pi}\sum_{j=1}^{N}\frac{\mathcal{D}_b'\left(\frac{\lambda_{ij}}{2\pi}\right)}{\mathcal{D}_b\left(\frac{\lambda_{ij}}{2\pi}\right)}+\frac{N_f}{2\pi}\frac{\mathcal{D}_b'\left(\frac{\lambda_i}{2\pi}\right)}{\mathcal{D}_b\left(\frac{\lambda_i}{2\pi}\right)},$$

$$S_{\mathrm{cl}}^{\mathrm{ADHM}}(N,b,\lambda)\equiv S_{\mathrm{eff}}^{\mathrm{ADHM}}[\mu^\star;N,b,\lambda]=F_0^{\mathrm{ADHM}}(b,\lambda)N^2+o(N^2)$$

$$S_{\mathrm{cl}}^{\mathrm{ADHM}}(N,b,\lambda)+N\log N+\frac{1}{12}\log N$$
$$=S_2^{\mathrm{ADHM,(lmf)}}(b,\lambda)N^2+S_1^{\mathrm{ADHM,(lmf)}}(b,\lambda)N+\sum_{L=0}^{22}S_{-L}^{\mathrm{ADHM,(lmf)}}(b,\lambda)N^{-L},$$

$$S_{1-\log}^{\mathrm{ADHM,(lmf)}}(b,\lambda)\approx-1,S_{\log}^{\mathrm{ADHM,(lmf)}}(b,\lambda)\approx-\frac{1}{12},$$

$$F_{S^1\times_\omega S^2}(N,\omega,\boldsymbol{\xi})\Big|_{\omega=b^2}=2F_{S_b^3}(N,b,\boldsymbol{\xi})+\mathcal{O}(b^2)$$

$$F=-\log Z$$

$$F_{S^1\times_\omega S^2}(N,\omega,\xi)=\frac{2}{\omega}\left[\frac{\pi\alpha(\boldsymbol{\xi})}{4}(N-\beta(\boldsymbol{\xi}))^{\frac{3}{2}}+\hat{g}_0(\boldsymbol{\xi})+\mathcal{O}\left(e^{-\#\sqrt{N}}\right)\right]$$
$$+\left[\pi\alpha(\boldsymbol{\xi})\left((N-\beta(\boldsymbol{\xi}))^{\frac{3}{2}}+\gamma(\boldsymbol{\xi})(N-\beta(\boldsymbol{\xi}))^{\frac{1}{2}}\right)+\frac{1}{2}\log(N-\beta(\boldsymbol{\xi}))\right.$$
$$\left.-\hat{f}_0(\boldsymbol{\xi})+\mathcal{O}\left(e^{-\#\sqrt{N}}\right)\right]+\mathcal{O}(\omega)$$

$$F_{S_b^3}(N,b,\boldsymbol{\xi})\Big|_{(5.1)+(5.2)}=\frac{1}{b^2}\left[\frac{\pi\alpha}{4}(N-\beta)^{\frac{3}{2}}+\hat{g}_0(\boldsymbol{\xi})\right]+\left[\frac{\pi\alpha}{2}\left(\alpha(N-\beta)^{\frac{3}{2}}+\gamma(N-\beta)^{\frac{1}{2}}\right)\right.$$
$$\left.+\frac{1}{4}\log(N-\beta)-\frac{1}{2}\hat{f}_0(\boldsymbol{\xi})\right]+\mathcal{O}\left(b^2,e^{-\#\sqrt{N}}\right)$$

$$F_{S_b^3}(N,b,\boldsymbol{\xi})\Big|_{(2.2)} = \frac{1}{b^2}\left[\frac{\pi\alpha}{4}(N-\beta)^{\frac{3}{2}}\right] + \log\ b + \left[\frac{\pi\alpha}{2}\left(\alpha(N-\beta)^{\frac{3}{2}} + \gamma(N-\beta)^{\frac{1}{2}}\right)\right.$$
$$\left. + \frac{1}{4}\log(N-\beta) + \frac{1}{2}\log\frac{32}{3\alpha}\right] - \mathcal{A}(b,\boldsymbol{\xi}) + \mathcal{O}\left(b^2, e^{-\#\sqrt{N}}\right)$$

$$\mathcal{A}(b,\boldsymbol{\xi}) \stackrel{!}{=} -\frac{1}{b^2}\hat{g}_0(\boldsymbol{\xi}) + \log\ b + \frac{1}{2}\left[\hat{f}_0(\boldsymbol{\xi}) + \log\frac{32}{3\alpha(\boldsymbol{\xi})}\right] + \mathcal{O}(b^2)$$

$$\mathcal{A}^{(\mathrm{num})}(N,b,\boldsymbol{\xi}) = \log Z_{S_b^3}^{(\mathrm{num})}(N,b,\boldsymbol{\xi}) - \log\left[\mathcal{C}^{-1/3}\mathrm{Ai}\left[\mathcal{C}^{-1/3}(N-\mathcal{B})\right]\right]$$
$$= -\frac{1}{b^2}\hat{g}_0^{\mathrm{num}}(\boldsymbol{\xi}) + \mathcal{O}(\log\ b)$$

$\widetilde{\boldsymbol{\Delta}} = \widetilde{\boldsymbol{\Delta}}_{\mathrm{sc}}$	$\hat{g}_0^{\mathrm{ADHM,(num)}}(N_f, \widetilde{\boldsymbol{\Delta}})$	$\hat{g}_0^{\mathrm{ADHM}}(N_f, \widetilde{\boldsymbol{\Delta}})$
$N_f = 1$	-0.1838410233	-0.183841023338
$N_f = 2$	-0.33723359	-0.337233589618
$N_f = 3$	-0.62879449	-0.628794493649
$N_f = 4$	-1.043299	-1.043299506902
$N_f = 5$	-1.57819	-1.578200494091
$N_f = 6$	-2.2327	-2.232772433361

$\widetilde{\boldsymbol{\Delta}} = \left(\frac{3}{7},\frac{4}{7},\frac{1}{2},\frac{1}{2}\right)$	$\hat{g}_0^{\mathrm{ADHM,(num)}}(N_f, \widetilde{\boldsymbol{\Delta}})$	$\hat{g}_0^{\mathrm{ADHM}}(N_f, \widetilde{\boldsymbol{\Delta}})$
$N_f = 1$	-0.184873819	-0.184873819329
$N_f = 2$	-0.3430694	-0.343069446632
$N_f = 3$	-0.6425169	-0.642516986634

$\widetilde{\boldsymbol{\Delta}} = \left(\frac{3}{8},\frac{5}{8},\frac{2}{5},\frac{3}{5}\right)$	$\hat{g}_0^{\mathrm{ADHM,(num)}}(N_f,\widetilde{\boldsymbol{\Delta}})$	$\hat{g}_0^{\mathrm{ADHM}}(N_f,\widetilde{\boldsymbol{\Delta}})$
$N_f = 1$	-0.18922588	-0.189225885117
$N_f = 2$	-0.3541838	-0.354183848923
$N_f = 3$	-0.66685	-0.666855367656

	$\hat{g}_0^{V^{5,2},\mathrm{(num)}}(N_f)$	$\hat{g}_0^{V^{5,2}}(N_f)$
$N_f = 1$	-0.1394556	-0.139455670622
$N_f = 2$	-0.188884	-0.188884534562
$N_f = 3$	-0.2965	-0.296573183635

$$
\begin{aligned}
&\mathrm{c}^{\mathrm{ABJM}}(b,\boldsymbol{\Delta})\\
&= \frac{1}{b^2}\frac{\zeta(3)}{8\pi^2}\left[\frac{4-\sum_{a=1}^4 \Delta_a^2}{4} - \left(\frac{1}{\Delta_{13}\Delta_{24}}+\frac{1}{\Delta_{14}\Delta_{23}}\right)\sum_{a=1}^4 \frac{\prod_{b=1}^4 \Delta_b}{\Delta_a}\right]\\
&+\frac{\zeta(3)}{8\pi^2}\left[\frac{4-\sum_{a=1}^4 \Delta_a^2}{2} + \left(\frac{4}{\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}} - \frac{1}{2}\left(\frac{1}{\Delta_{13}\Delta_{24}}+\frac{1}{\Delta_{14}\Delta_{23}}\right)\right)\sum_{a=1}^4 \frac{\prod_{b=1}^4 \Delta_b}{\Delta_a}\right] + \mathcal{O}(b^2)
\end{aligned}
$$

$$
\begin{aligned}
\hat{g}_0^{\mathrm{ABJM}}(k,\boldsymbol{\Delta}) &= \hat{g}_{0,2}^{\mathrm{ABJM}}(\boldsymbol{\Delta})\frac{\zeta(3)}{8\pi^2}k^2 + \mathcal{O}(\log\, k)\\
\hat{g}_{0,2}^{\mathrm{ABJM}}(\boldsymbol{\Delta}) &= \frac{4-\sum_{a=1}^4 \Delta_a^2}{4} - \left(\frac{1}{\Delta_{13}\Delta_{24}}+\frac{1}{\Delta_{14}\Delta_{23}}\right)\sum_{a=1}^4 \frac{\prod_{b=1}^4 \Delta_b}{\Delta_a}\\
\hat{f}_0^{\mathrm{ABJM}}(k,\boldsymbol{\Delta}) &= \hat{f}_{0,2}^{\mathrm{ABJM}}(\boldsymbol{\Delta})\frac{\zeta(3)}{8\pi^2}k^2 + \mathcal{O}(\log\, k)\\
\hat{f}_{0,2}^{\mathrm{ABJM}}(\boldsymbol{\Delta}) &= -\left(4-\sum_{a=1}^4 \Delta_a^2\right) + \left(\frac{1}{\Delta_{13}\Delta_{24}}+\frac{1}{\Delta_{14}\Delta_{23}} - \frac{8}{\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}}\right)\sum_{a=1}^4 \frac{\prod_{b=1}^4 \Delta_b}{\Delta_a}
\end{aligned}
$$

$$
\Xi(\mu,b,\boldsymbol{\xi}) = \sum_{N\geq 0} Z(N,b,\boldsymbol{\xi})e^{N\mu} \quad \text{with} \quad Z(0,b,\boldsymbol{\xi}) = 1
$$

$$
\mathcal{J}(\mu,b,\boldsymbol{\xi}) = \log\, \Xi(\mu,b,\boldsymbol{\xi})
$$

$$e^{\mathcal{J}(\mu,b,\xi)} = \sum_{n\in\mathbb{Z}} e^{J(\mu+2\pi \mathrm{i}n,b,\xi)}$$

$$Z(N,b,\boldsymbol{\xi}) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\mu \exp\left[J(\mu,b,\boldsymbol{\xi}) - N\mu\right]$$

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp\left(\mathrm{i}\frac{t^3}{3} + \mathrm{i}tz\right)$$

$$J(\mu,b,\boldsymbol{\xi}) = \frac{\mathcal{C}(b,\boldsymbol{\xi})}{3}\mu^3 + \mathcal{B}(b,\boldsymbol{\xi})\mu + \mathcal{A}(b,\boldsymbol{\xi}) + \mathcal{O}(e^{-\mu})$$

$$\log Z \sim N^{\frac{p}{p-1}} \text{ as } N \to +\infty$$

$$-\log z_i = \sum_{j=1}^{\mathrm{g}_\Sigma} r_{ij}\mu_j + \sum_{k=1}^{b_2(X)-\mathrm{g}_\Sigma} m_{ik}\log \lambda_k$$

$$t_i(\vec{z}) = -\log z_i - \Pi_i(\vec{z})$$

$$t_i(\vec{\mu},\vec{\lambda}) = \sum_{j=1}^{\mathrm{g}_\Sigma} r_{ij}\mu_j + \sum_{k=1}^{b_2(X)-\mathrm{g}_\Sigma} m_{ik}\log \lambda_k + \mathcal{O}(e^{-\mu_j})$$

$$F(\vec{t},g_{\mathrm{top}}) = \frac{1}{6\left(g_{\mathrm{top}}\right)^2}\sum_{i,j,k=1}^{b_2(X)} c_{ijk}t_it_jt_k + \sum_{i=1}^{b_2(X)} d_it_i + \sum_{\mathrm{g}\geq 2} c_{\mathrm{g}}\left(g_{\mathrm{top}}\right)^{2\mathrm{g}-2} + F^{\mathrm{GV}}(\vec{t},g_{\mathrm{top}})$$

$$F^{\mathrm{GV}}(\vec{t},g_{\mathrm{top}}) = \sum_{\mathrm{g}\geq 0}\sum_{\vec{d}\in H_2(X,\mathbb{Z})}\sum_{w\geq 1}\frac{n_{\mathrm{g}}^{\vec{d}}}{w}\left(2\sin\frac{wg_{\mathrm{top}}}{2}\right)^{2\mathrm{g}-2} e^{-w\vec{d}\cdot\vec{t}}$$

$$F(\vec{t},\epsilon_1,\epsilon_2) = \sum_{\mathrm{g},n\geq 0} F_{\mathrm{g},n}(\vec{t})\left(g_{\mathrm{top}}\right)^{2\mathrm{g}-2}\hbar^{2n}$$

$$F^{\mathrm{NS}}(\vec{t},\hbar) = -\lim_{\epsilon_2\to 0}\epsilon_2 F(\vec{t},\epsilon_1,\epsilon_2) = \sum_{n\geq 0} F_n^{\mathrm{NS}}(\vec{t})\hbar^{2n-1}$$

$$F^{\mathrm{NS}}(\vec{t},\hbar) = \frac{1}{6\hbar}\sum_{i,j,k=1}^{b_2(X)} c_{ijk}t_it_jt_k + \hbar\sum_{i=1}^{b_2(X)} d_i^{\mathrm{NS}}t_i + F^{\mathrm{GV,ref}}(\vec{t},\hbar)$$

$$F^{\mathrm{GV,\,ref}}(\vec{t},\hbar) = \sum_{2j_L,2j_R\geq 0}\sum_{\vec{d}}\sum_{w\geq 1}(-1)^{2(j_L+j_R)}\frac{N_{j_L,j_R}^{\vec{d}}}{w^2}\frac{\sin\left(\frac{\hbar w}{2}(2j_L+1)\right)\sin\left(\frac{\hbar w}{2}(2j_R+1)\right)}{2\sin^3\left(\frac{\hbar w}{2}\right)}e^{-w\vec{d}\cdot\vec{t}}$$

$$t_i(\vec{z},\hbar) = -\log z_i - \Pi_i(\vec{z},\hbar)$$

$$J_X(\vec{\mu},\vec{\lambda},\hbar) = \frac{1}{2\pi}\left[\sum_{i=1}^{b_2(X)} t_i(\hbar)\partial_{t_i}F^{\mathrm{NS}}(\vec{t}(\hbar),\hbar) + \hbar^2\partial_\hbar\left(\frac{1}{\hbar}F^{\mathrm{NS}}(\vec{t}(\hbar),\hbar)\right) + \frac{4\pi^2}{\hbar}\sum_{i=1}^{b_2(X)} d_i t_i(\hbar)\right] + \mathcal{A}(\vec{\lambda},\hbar) + F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}\vec{t}(\hbar) + \mathrm{i}\pi\vec{B}, \frac{4\pi^2}{\hbar}\right)$$

$$J_X(\vec{\mu},\vec{\lambda},\hbar) = \frac{1}{12\pi\hbar}\sum_{i,j,k=1}^{b_2(X)} c_{ijk}t_it_jt_k + \sum_{i=1}^{b_2(X)}\left(\frac{2\pi}{\hbar}d_i + \frac{\hbar}{2\pi}d_i^{\mathrm{NS}}\right)t_i + \mathcal{A}(\vec{\lambda},\hbar) + \mathcal{O}\left(e^{-t_i}, e^{-2\pi t_i/\hbar}\right)$$

$$O_j(x,y) + \kappa_j = 0, j = 1,\dots,\mathfrak{g}_\Sigma$$

$$\hat{\rho}_j = \mathrm{O}_j^{-1}, j = 1,\dots,\mathfrak{g}_\Sigma$$

$$\Xi_X(\vec{\mu},\vec{\lambda},\hbar) = \det\left(1 + \sum_{j=1}^{\mathfrak{g}_\Sigma}\kappa_j\hat{\rho}_j\right)$$

$$\Xi_X(\vec{\mu},\vec{\lambda},\hbar) = \sum_{\vec{n}\in\mathbb{Z}^{\mathfrak{g}_\Sigma}} \exp\left[J_X(\vec{\mu} + 2\mathrm{i}\pi\vec{n},\vec{\lambda},\hbar)\right]$$

$$\Xi_X(\vec{\mu},\vec{\lambda},\hbar) = \sum_{N_1\geq 0}\dots\sum_{N_{\mathfrak{g}_\Sigma}\geq 0} Z_X(\vec{N},\vec{\lambda},\hbar)\kappa_1^{N_1}\dots\kappa_{\mathfrak{g}_\Sigma}^{N_{\mathfrak{g}_\Sigma}}$$

$$Z_X(\vec{N},\vec{\lambda},\hbar) = \frac{1}{(2\mathrm{i}\pi)^{\mathfrak{g}_\Sigma}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \dots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{\mathfrak{g}_\Sigma}\exp\left[J_X(\vec{\mu},\vec{\lambda},\hbar) - \sum_{j=1}^{\mathfrak{g}_\Sigma} N_j\mu_j\right]$$

$$s_b(z) \equiv \frac{\Gamma_2\left(\frac{Q}{2} + \mathrm{i}z; b, b^{-1}\right)}{\Gamma_2\left(\frac{Q}{2} - \mathrm{i}z; b, b^{-1}\right)} = \prod_{m,n=0}^{\infty}\frac{mb + nb^{-1} + \frac{Q}{2} - \mathrm{i}z}{mb + nb^{-1} + \frac{Q}{2} + \mathrm{i}z},$$

$$\zeta_r(s,z;\vec{\omega}) \equiv \sum_{n_1,\cdots,n_r=0}^{\infty}\frac{1}{(\vec{n}\cdot\vec{\omega} + z)^s}$$

$$\Gamma_r(z;\vec{\omega}) \equiv \exp\left[\frac{\partial}{\partial s}\zeta_r(s,z;\vec{\omega})\Big|_{s=0}\right] = \prod_{n_1,\cdots,n_r=0}^{\infty}\frac{1}{(\vec{n}\cdot\vec{\omega} + z)}$$

$$s_b(z)s_b(-z) = 1$$

$$\overline{s_b(z)} = s_b(-\bar{z})$$

$$s_b\left(\frac{\mathrm{i}}{2}b^{\pm 1} + z\right)s_b\left(\frac{\mathrm{i}}{2}b^{\pm 1} - z\right) = \frac{1}{2\cosh\left(\pi b^{\pm 1}z\right)}$$

$$\log\, s_b(x) = \sum_{c=1}^{pq-1} \Big[\frac{1}{4}\log\,(pq)B_2(v) + v\log\,\Gamma(v) - \log\,G(v+1)$$
$$+(N(c)-v)\left(\log\,\Gamma(v) - \frac{1}{2}\log\,(2\pi) + \frac{1}{2}\log\,(pq)B_1(v)\right)\Big]$$
$$-(x\to -x)\ \left(v \equiv \frac{Q/2+\mathrm{i}x}{\sqrt{pq}} + \frac{c}{pq}\right)$$

$$N(c) = \{k,l\in\mathbb{Z}_{\geq 0} \mid kp+lq=c\} = 0 \text{ or } 1$$

$$\log\, s_b(x) = \frac{1}{2\pi\mathrm{i}}\int \frac{dt}{t}(\gamma+\log\, t)\frac{1}{(1-e^{bt})(1-e^{t/b})}\left[e^{(Q/2+\mathrm{i}x)t} - e^{(Q/2-\mathrm{i}x)t}\right]$$

$$\mathcal{D}_b(x) \equiv \frac{s_b(x+\mathrm{i}Q/4)}{s_b(x-\mathrm{i}Q/4)} = s_b\left(\frac{\mathrm{i}Q}{4}+x\right)s_b\left(\frac{\mathrm{i}Q}{4}-x\right)$$

$$\mathcal{D}_b(x) = \prod_{\ell=1}^{n}\frac{1}{2\cosh\left(\frac{\pi}{b}x + \frac{\pi\mathrm{i}}{b^2}\left(\frac{n+1}{2}-\ell\right)\right)}\ (b^2=2n-1, n\in\mathbb{N})$$

$$[\hat{q},\hat{p}] = \mathrm{i}\hbar\,,\quad \hat{q}|q\rangle = q|q\rangle\,,\qquad \hat{p}|p\rangle = p|p\rangle\,,\qquad 1 = \int_{-\infty}^{\infty} dq|q\rangle\langle q| = \int_{-\infty}^{\infty} dp|p\rangle\langle p|$$
$$\langle q|q'\rangle = \delta(q-q')\,,\quad \langle p|p'\rangle = \delta(p-p')\,,\quad \langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\mathrm{i}pq/\hbar}\,.$$

$$A_W(p,q) \equiv \int_{-\infty}^{\infty} dq'\left\langle q-\frac{q'}{2}\right|\hat{A}\left|q+\frac{q'}{2}\right\rangle e^{ipq'/\hbar}$$

$$\mathrm{Tr}\hat{A} = \int_{-\infty}^{\infty}\frac{dpdq}{2\pi\hbar}A_W(p,q)$$

$$(\hat{A}\hat{B})_W(p,q) = A_W(p,q)\star B_W(p,q)$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\left(\partial_p^k\partial_q^{n-k}A_W(p,q)\right)\left(\partial_p^{n-k}\partial_q^k B_W(p,q)\right)$$

$$\star = \exp\left[\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)\right]$$

$$Z_N = \frac{1}{N!}\int_{-\infty}^{\infty} d^Nx \det_{i,j=1} f(x_i,x_j)\ (Z_0=1)$$

$$e_N(\lambda_1,\cdots,\lambda_m) = \sum_{1\leq j_1<j_2<\cdots<j_N\leq m}\lambda_{j_1}\cdots\lambda_{j_N}.$$

$$
\begin{aligned}
\Xi(z) &= \sum_{N=0}^{\infty} Z_N z^N \\
&\approx 1 + z\sum_i A_{ii} + \frac{z^2}{2!}\sum_{i,j}\begin{vmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{vmatrix} + \frac{z^3}{3!}\sum_{i,j,k}\begin{vmatrix} A_{ii} & A_{ij} & A_{ik} \\ A_{ji} & A_{jj} & A_{jk} \\ A_{ki} & A_{kj} & A_{kk} \end{vmatrix} + \cdots \\
&= \det(1 + zA) \\
&= \prod_{a=1}^{m}(1 + \lambda_a z) = \sum_{N=0}^{\infty} e_N(\lambda_1, \dots, \lambda_m) z^N
\end{aligned}
$$

$$
Z_N = \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{z^{N+1}} \Xi(z) = r^{-N} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-\mathrm{i}N\theta} \Xi(re^{\mathrm{i}\theta})
$$

$$
-N\log r + \log \Xi(r),
$$

Algorithm 1 Bornemann

Input: A positive integer N, a function $f(x, y)$, a set of points $\{x_1, \dots, x_m\}$
Step 1: Construct the $m \times m$ matrix A, $A_{ab} = \Delta x f(x_a, x_b)$
Step 2: Calculate its eigenvalues $\lambda_1, \dots, \lambda_m$
Step 3: Evaluate $Z_N = e_N(\lambda_1, \dots, \lambda_m)$

Algorithm 2 Numerical calculation of elementary symmetric polynomials

Input: A positive integer N, a set of positive real numbers $\{\lambda_1, \dots, \lambda_m\}$
Step 1: Define $\Xi(z) = \prod_{a=1}^{m}(1 + \lambda_a z)$
Step 2: Determine the 'good' radius r as $r \leftarrow \underset{r}{\operatorname{argmin}}(-N \log r + \log \Xi(r))$
Step 3: Evaluate $I = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-\mathrm{i}N\theta} \Xi(re^{\mathrm{i}\theta})$ with the radius chosen in Step 2
Output: $\log Z_N = \log e_N(\lambda_1, \dots, \lambda_m) = -N \log r + \log I$

$$
f(x, y) = \frac{1}{\sqrt{2\cosh(\pi x)}} \frac{1}{\sqrt{2\cosh(\pi y)}} \frac{1}{2\cosh(\pi(x - y))}.
$$

N	Numerical F_N	Analytic F_N
1	1.386294361	1.386294361
2	3.917318608	3.917318608
10	45.49866122	45.49866122
20	130.2063341	130.2063341

40	371.0289665	371.0289665
100	1474.4973	N /A

$$N = 1 \sim 20, b^2 \in \{3,5,7\}, N_f \in \{1,2,3,4,5\}, \zeta \in \left\{0, \frac{1}{5}, \frac{\mathrm{i}}{5}, \frac{\mathrm{i}}{10}\right\}$$

$\zeta = 0$	$b = \sqrt{3}$	$b = \sqrt{5}$	$b = \sqrt{7}$
$N_f = 1$	2.301×10^{-12}	-1.240×10^{-10}	-3.206×10^{-9}
$N_f = 2$	-2.015×10^{-9}	-2.201×10^{-8}	3.743×10^{-7}
$N_f = 3$	5.868×10^{-9}	-2.268×10^{-7}	1.433×10^{-5}
$N_f = 4$	3.469×10^{-8}	2.935×10^{-7}	2.244×10^{-5}
$N_f = 5$	-1.078×10^{-6}	-1.333×10^{-6}	-1.555×10^{-4}

$\zeta = \frac{1}{5}$	$b = \sqrt{3}$	$b = \sqrt{5}$	$b = \sqrt{7}$
$N_f = 1$	-7.197×10^{-10}	1.988×10^{-8}	-1.113×10^{-8}
$N_f = 2$	-3.665×10^{-10}	-1.090×10^{-8}	-7.673×10^{-8}
$N_f = 3$	-1.949×10^{-9}	-1.886×10^{-7}	-1.595×10^{-6}
$N_f = 4$	-4.454×10^{-9}	-3.156×10^{-6}	-3.100×10^{-5}
$N_f = 5$	-1.008×10^{-8}	-1.625×10^{-5}	-2.761×10^{-4}

$\zeta = \frac{i}{5}$	$b = \sqrt{3}$	$b = \sqrt{5}$	$b = \sqrt{7}$
$N_f = 1$	-1.345×10^{-9}	-8.574×10^{-10}	-1.490×10^{-8}
$N_f = 2$	2.498×10^{-9}	-2.417×10^{-8}	-5.890×10^{-7}
$N_f = 3$	1.620×10^{-8}	-3.621×10^{-7}	-1.729×10^{-6}
$N_f = 4$	-2.979×10^{-8}	-3.326×10^{-6}	-8.288×10^{-5}
$N_f = 5$	1.002×10^{-6}	-8.411×10^{-6}	-1.471×10^{-4}

$\zeta = \frac{i}{10}$	$b = \sqrt{3}$	$b = \sqrt{5}$	$b = \sqrt{7}$
$N_f = 1$	-4.244×10^{-10}	-1.739×10^{-9}	-5.772×10^{-9}
$N_f = 2$	-9.711×10^{-11}	-1.168×10^{-9}	-7.800×10^{-8}
$N_f = 3$	3.599×10^{-9}	-4.187×10^{-8}	-4.511×10^{-6}
$N_f = 4$	-1.380×10^{-5}	-4.786×10^{-4}	3.858×10^{-5}
$N_f = 5$	-2.661×10^{-7}	-8.303×10^{-6}	-1.340×10^{-4}

Caso I. $m_1 = m_2 = m_3 = 0$

i) $\lambda \in \{30,32,34,36,38,40\}, \quad b = \sqrt{7}$
ii) $\lambda \in \{30,35,40\}, \quad b \in \{\sqrt{3}, \sqrt{5}, \sqrt{9}\}$.

Caso II. $m_2 = m_3 = 0$ with $\lambda = 30$

iii) $b = \sqrt{3}, \quad \Delta_{\text{Nosaka}} \in \left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\}$,
iv) $b = \sqrt{5}, \quad \Delta_{\text{Nosaka}} \in \left\{\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\}$,
v) $b \in \{\sqrt{7}, \sqrt{9}\}, \quad \Delta_{\text{Nosaka}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right)$.

Caso III. $m_1 = m_3 = 0$ with $\lambda = 30$

$$\text{vi) } b = \sqrt{3}, \qquad \Delta_{\text{Nosaka}} \in \left\{\left(\frac{5}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8}\right),\left(\frac{3}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5}\right),\left(\frac{7}{12},\frac{5}{12},\frac{5}{12},\frac{7}{12}\right)\right\},$$

$$\text{vii) } b \in \{\sqrt{5},\sqrt{7}\}, \quad \Delta_{\text{Nosaka}} = \left(\frac{5}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8}\right).$$

$b = \sqrt{7}, \mathbf{\Delta}_{\text{Nosaka}} = \mathbf{\Delta}_{\text{sc}}$	$S_{2,\text{ lead}}^{\text{ABJM,(lmf)}}(b, \Delta_{\text{Nosaka}})$	$S_{2,\text{ lead}}^{\text{ABJM}}(b, \mathbf{\Delta}_{\text{Nosaka}})$
	3.38505366724720	$\frac{\pi\sqrt{2}}{3}\frac{Q^2}{4} = 3.38505366716828$
	$\mathfrak{f}^{\text{ABJM,(lmf)}}(b, \mathbf{\Delta}_{\text{Nosaka}})$	$\mathfrak{f}^{\text{ABJM}}(b, \mathbf{\Delta}_{\text{Nosaka}})$
	-0.285715226522448	$1 - \frac{(b-b^{-1})^2}{4} = -\frac{2}{7} = -0.285714285714286$
$b = \sqrt{5}, \mathbf{\Delta}_{\text{Nosaka}} = \left(\frac{5}{8},\frac{2}{5},\frac{3}{5},\frac{3}{8}\right)$	$S_{2,\text{ lead}}^{\text{ABJM,(lmf)}}(b, \Delta_{\text{Nosaka}})$	$S_{2,\text{ lead}}^{\text{ABJM}}(b, \mathbf{\Delta}_{\text{Nosaka}})$
	2.52893330316739	$\frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{Q^2}{4} = 2.52893330317466$
	$\mathfrak{f}^{\text{ABJM,(lmf)}}(b, \mathbf{\Delta}_{\text{Nosaka}})$	$\mathfrak{f}^{\text{ABJM}}(b, \mathbf{\Delta}_{\text{Nosaka}})$
	0.293303408399649	$\frac{1782111}{6076000} = 0.293303324555629$

Caso IV. $m_3 = 0$

$$\text{iix) } b = \sqrt{3}, \lambda = 30, \qquad \Delta_{\text{Nosaka}} \in \left\{\left(\frac{3}{5},\frac{5}{12},\frac{2}{5},\frac{7}{12}\right),\left(\frac{7}{12},\frac{3}{8},\frac{5}{12},\frac{5}{8}\right)\right\},$$

$$\text{ix) } b = \sqrt{5}, \lambda \in \{30,32,34,36,38,40\}, \quad \Delta_{\text{Nosaka}} = \left(\frac{5}{8},\frac{2}{5},\frac{3}{8},\frac{3}{5}\right),$$

$$\text{x) } b = \sqrt{7}, \lambda = 30, \qquad \Delta_{\text{Nosaka}} = \left(\frac{5}{8},\frac{2}{5},\frac{3}{8},\frac{3}{5}\right).$$

$$S_2^{\text{ABJM,(lmf)}}(b,\lambda,\Delta_{\text{Nosaka}}) = S_{2,\text{ lead}}^{\text{ABJM,(lmf)}}(b,\Delta_{\text{Nosaka}})\frac{\left(\lambda - \frac{1}{24}\right)^2}{\lambda^2} + \mathfrak{f}^{\text{ABJM,(lmf)}}(b,\Delta_{\text{Nosaka}})\frac{\zeta(3)}{8\pi^2\lambda^2}$$

$$\mathfrak{f}^{\mathrm{ABJM,(lmf)}}(b,\Delta_{\mathrm{Nosaka}}) \equiv \frac{8\pi^2\lambda^2}{\zeta(3)}\left[S_2^{\mathrm{ABJM,(lmf)}}(b,\lambda,\Delta_{\mathrm{Nosaka}}) - \frac{4\pi\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\frac{Q^2}{4}\frac{\left(\lambda-\frac{1}{24}\right)^{3/2}}{\lambda^2}N^2\right]$$

Data	$\mathfrak{f}^{\mathrm{ABJM,(lmf)}}(b,\mathbf{\Delta}_{\mathrm{Nosaka}})$	$\mathfrak{f}^{\mathrm{ABJM}}(b,\mathbf{\Delta}_{\mathrm{Nosaka}})$
iii) $\mathbf{\Delta}_{\mathrm{Nosaka}} = \left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$	1.16666654026091	$\frac{7}{6} = 1.16666666666667$
iv) $\mathbf{\Delta}_{\mathrm{Nosaka}} = \left(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$	0.577777833211990	$\frac{26}{45} = 0.577777777777778$
vi) $\mathbf{\Delta}_{\mathrm{Nosaka}} = \left(\frac{3}{5},\frac{2}{5},\frac{3}{5},\frac{2}{5}\right)$	0.720000000024472	$\frac{18}{25} = 0.720000000000000$
vii) $\mathbf{\Delta}_{\mathrm{Nosaka}} = \left(\frac{5}{8},\frac{3}{8},\frac{5}{8},\frac{3}{8}\right)$	-0.142856901982585	$-\frac{1}{7} = -0.142857142857143$
iix) $\mathbf{\Delta}_{\mathrm{Nosaka}} = \left(\frac{3}{5},\frac{5}{12},\frac{7}{12},\frac{2}{5}\right)$	0.712043477818766	$\frac{3344219}{4696650} = 0.712043477798005$

i) $\lambda = 30$, $b \in \{1,\sqrt{3}\}$,
ii) $\lambda \in \{30,32,34,36,38,40\}$, $b = \sqrt{5}$,
iii) $\lambda \in \{30,35,40\}$, $b \in \{\sqrt{7},\sqrt{9},\sqrt{11}\}$.

$$\mathfrak{f}^{\mathrm{ADHM}}(b=\sqrt{5}) = -0.247365050537827.$$

(b,λ)	$\mathfrak{c}^{\mathrm{ADHM,(lmf)}}(b,\lambda)$	$\mathfrak{c}^{\mathrm{ADHM}}(b)$
(1,30)	-0.07141916904079654044 7	$\frac{\zeta(3)}{8\pi^2} - \frac{1}{8}\log 2 = -0.0714191690407965 28287$
$(\sqrt{3},30)$	-0.142504105366827254 19	$\frac{\zeta(3)}{3\pi^2} - \frac{1}{6}\log 3 = -0.142504105366827254 19$

$$\frac{\left(\lambda-\frac{1}{24}+\frac{b^2+b^{-2}}{3Q^2}\right)^{3/2}}{\lambda^2},\ \text{and}\ \frac{1}{\lambda^2},$$

$$c^{\mathrm{ADHM,(lmf)}}(b,\lambda)\equiv\lambda^2\left[S_2^{\mathrm{ADHM,(lmf)}}(b,\lambda)-\frac{\pi\sqrt{2}}{3}\frac{Q^2}{4}\frac{\left(\lambda-\frac{1}{24}+\frac{b^2+b^{-2}}{3Q^2}\right)^{3/2}}{\lambda^2}N^2\right],$$

b	$c^{\mathrm{ADHM,(lmf)}}(b,\lambda=30)$	$c^{\mathrm{ADHM,(lmf)}}(b,\lambda=35)$	$c^{\mathrm{ADHM,(lmf)}}(b,\lambda=40)$
$\sqrt{7}$	-0.359318492567472	-0.359318479059387	-0.359318430945854
$\sqrt{9}$	-0.474110276869651	-0.474110155382705	-0.474110482846968
$\sqrt{11}$	-0.590332572015681	-0.590333574547422	-0.590335953236448

$$\sinh\pi b^2 y\approx\frac{1}{2}\mathrm{sign}(y)e^{\pi b^2|y|},$$

$$\log\ s_b\left(\frac{\mathrm{i}Q}{2}(1-\Delta)+by\right)=\mathcal{L}(y,\Delta)b^2+O(b)^0.$$

$$\mathcal{L}(x,\Delta)=\frac{\pi\mathrm{i}}{2}B_2\left(1-\frac{\Delta}{2}-\mathrm{i}x\right)-\frac{\mathrm{i}}{2\pi}\mathrm{Li}_2\left(e^{-\pi\mathrm{i}\Delta+2\pi x}\right),$$

$$Z^{\mathrm{ADHM}}\approx\frac{b^N}{N!}\int\ d^N y\prod_{i=1}^{N}\ e^{-\pi b^2\chi_m y_i}\prod_{i<j}^{N}\ \left[\mathrm{sign}(y_{ij})e^{\pi b^2|y_{ij}|}2\sinh\left(\pi y_{ij}\right)\right]$$
$$\times\prod_{a=1}^{3}\prod_{i,j=1}^{N}\ \exp\left[b^2\mathcal{L}(-y_{ij},\Delta_a)\right]\times\prod_{q=1}^{N_f}\prod_{i=1}^{N}\ \exp\ \left[b^2\left(\mathcal{L}\left(-y_i,\Delta_{\mu_q}\right)+\mathcal{L}\left(y_i,\tilde{\Delta}_{\mu_q}\right)\right)\right]$$

$$\log\ Z^{\mathrm{ADHM}}\approx\rho b^2+O(b)^0,$$

$$\rho=\max_y f(y),$$

$$f(y)=-\sum_{i=1}^{N}\ \pi\chi_m y_i+\sum_{i<j}^{N}\ \pi|y_{ij}|$$
$$+\sum_{a=1}^{3}\ \sum_{i,j=1}^{N}\ \mathcal{L}(-y_{ij},\Delta_a)+\sum_{q=1}^{N_f}\ \sum_{i=1}^{N}\ \left[\mathcal{L}\left(-y_i,\Delta_{\mu_q}\right)+\mathcal{L}\left(y_i,\tilde{\Delta}_{\mu_q}\right)\right].$$

$$\log\ Z^{\mathrm{ADHM}}\approx\rho b^{-2}+O(b)^0.$$

N	ρ	$y_\star$
2	-1.39079	$(-0.132328, 0.132328)$
3	-2.37148	$(-0.233632, 0., 0.233632)$
$\vdots$	$\vdots$	$\vdots$
10	-12.6388	$(-0.713068, -0.487065, \ldots, 0.487065, 0.713068)$

$$\hat{g}_0^{\text{num}} = -\rho - \frac{\pi\alpha}{4}(N-\beta)^{3/2}.$$

N	$\hat{g}_0^{\text{num}}$
2	-0.18383993271077603
3	-0.1838409213600123
4	-0.18384101054942192
5	-0.18384102137056857
6	-0.18384102298546967
7	-0.1838410232669867
8	-0.18384102332245966
9	-0.18384102333451757
10	-0.18384102333738106

$$\hat{g}_0^{\text{ABJM,(num)}}(k,\mathbf{\Delta}) \equiv \frac{1}{2\pi}\mathrm{Im}\mathcal{V}^{\text{ABJM}} - \frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\left(N - \frac{k}{24} + \frac{1}{12k}\sum_{a=1}^{4}\frac{1}{\Delta_a}\right)^{3/2}$$

$$\hat{g}_0^{\text{ABJM,(num)}}(k,\mathbf{\Delta}) = \hat{g}_{0,2}^{\text{ABJM,(num)}}(\mathbf{\Delta})\frac{\zeta(3)}{8\pi^2}k^2 + \sum_{g=0}^{23}\hat{g}_{0,-2g}^{\text{ABJM,(num)}}(\mathbf{\Delta})k^{-2g}$$

	$\hat{g}_{0,2}^{\mathrm{ABJM,(num)}}(\mathbf{\Delta})$	$\hat{g}_{0,2}^{\mathrm{ABJM}}(\mathbf{\Delta})$
$\mathbf{\Delta}=\left(\frac{3}{7},\frac{1}{2},\frac{1}{2},\frac{4}{7}\right)$	-0.244884877027426	$-\frac{18719}{76440}=-0.244884877027734$
$\mathbf{\Delta}=\left(\frac{4}{9},\frac{5}{9},\frac{1}{2},\frac{1}{2}\right)$	-0.248447234644140	$-\frac{52001}{209304}=-0.248447234644345$
$\mathbf{\Delta}=\left(\frac{5}{11},\frac{5}{11},\frac{6}{11},\frac{6}{11}\right)$	-0.243801652892334	$-\frac{59}{242}=-0.243801652892562$
$\mathbf{\Delta}=\left(\frac{5}{12},\frac{7}{12},\frac{5}{12},\frac{7}{12}\right)$	-0.243055555554787	$-\frac{35}{144}=-0.243055555555556$
$\mathbf{\Delta}=\left(\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{4}{5}\right)$	-0.213333333331287	$-\frac{16}{75}=-0.213333333333333$
$\mathbf{\Delta}=\left(\frac{15}{40},\frac{17}{40},\frac{21}{40},\frac{27}{40}\right)$	-0.218610304168689	$-\frac{511723}{2340800}=-0.218610304169515$

$$
\begin{aligned}
\hat{f}_0^{\mathrm{ABJM,(\,from\ TTI\,)}}(k,\mathbf{\Delta}) &= \overbrace{\left[-\sum_{a=1}^{4}\hat{f}_{0,2,a}(\mathbf{\Delta})\Delta_a\right]}^{=\hat{f}_{0,2}^{\mathrm{ABJM,(\,from\ TTI)}}(\mathbf{\Delta})}\frac{\zeta(3)}{8\pi^2}k^2+\mathcal{O}(\log\ k),\\
\hat{f}_{0,2,1}(\mathbf{\Delta}) &= \Delta_1+\frac{\Delta_1\Delta_3}{\Delta_1+\Delta_4}+\frac{\Delta_1\Delta_4}{\Delta_1+\Delta_3}+\frac{\Delta_1\Delta_4(\Delta_2+\Delta_3)}{(\Delta_1+\Delta_4)^2}+\frac{\Delta_1\Delta_3(\Delta_2+\Delta_4)}{(\Delta_1+\Delta_3)^2}\\
&\quad -\frac{2\Delta_3\Delta_4}{(\Delta_1+\Delta_3)(\Delta_1+\Delta_4)}-\frac{\Delta_2^2(\Delta_1-\Delta_2)}{(\Delta_2+\Delta_3)(\Delta_2+\Delta_4)}\\
&\quad +\frac{\Delta_2\Delta_3(\Delta_1+\Delta_4)}{(\Delta_1+\Delta_3)(\Delta_2+\Delta_3)}+\frac{\Delta_2\Delta_4(\Delta_1+\Delta_3)}{(\Delta_1+\Delta_4)(\Delta_2+\Delta_4)}\\
\hat{f}_{0,2,2}(\mathbf{\Delta}) &= \hat{f}_{0,2,1}(\mathbf{\Delta})\big|_{\Delta_1\leftrightarrow\Delta_2}\\
\hat{f}_{0,2,3}(\mathbf{\Delta}) &= \hat{f}_{0,2,1}(\mathbf{\Delta})\big|_{(\Delta_1,\Delta_2)\leftrightarrow(\Delta_3,\Delta_4)}\\
\hat{f}_{0,2,4}(\mathbf{\Delta}) &= \hat{f}_{0,2,1}(\mathbf{\Delta})\big|_{(\Delta_1,\Delta_2)\leftrightarrow(\Delta_4,\Delta_3)}
\end{aligned}
$$

$$
\hat{f}_{0,2}^{\mathrm{ABJM}}(\mathbf{\Delta})=\hat{f}_{0,2}^{\mathrm{ABJM,(\,from\ TTI\,)}}(\mathbf{\Delta})
$$

$$
\mathbf{\Delta}=(\Delta_1,\Delta_2,\Delta_3,\Delta_4)\leftrightarrow(\tilde{\Delta}_1,\tilde{\Delta}_3,\tilde{\Delta}_2,\tilde{\Delta}_4)=\tilde{\mathbf{\Delta}}
$$

$$
\begin{tikzcd}[column sep=small]
 & \overbrace{\mathbb{R}^{1,10+517\,|\,\mathbf{32}}}^{\mathfrak{M}} \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[dr, two heads] & & \mathbb{R}^{1,10+517} \arrow[dl, two heads] \\
 & \mathbb{R}^{1,10} &
\end{tikzcd}
$$

Basic M-algebra

Super tangent space

Exceptional tangent space

Ordinary tangent space

$$
\begin{tikzcd}
\underset{\text{Lie groups}}{\text{super-}} \quad \mathrm{sLieGrp} \arrow[r, "\substack{\text{super-Lie}\\ \text{differentiation}}"] & \mathrm{sLieAlg} \quad \underset{\text{Lie algebras}}{\text{super-}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Hidden M-group} \quad \widehat{\mathcal{M}} \arrow[d, two heads] & \widehat{\mathfrak{M}} \quad \text{Hidden M-algebra} \arrow[d, two heads] \\
\text{Basic M-group} \quad \mathcal{M} & \mathfrak{M} \quad \text{Basic M-algebra}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}^k \arrow[r, hook] \arrow[d, equal] & \widehat{\mathcal{M}} \arrow[r, two heads] \arrow[d] & \widehat{\mathcal{M}}/\mathbb{Z}^k \arrow[d, two heads] \\
\mathbb{Z}^k \arrow[r, hook] & \mathcal{M} \arrow[r, two heads] & \mathcal{M}/\mathbb{Z}^k
\end{tikzcd}
$$

Toroidally compactified super-exceptional spacetime

$$(0 \leq k \leq 528)$$

$$\underbrace{[Q_\alpha, Q_\beta]}_{\substack{\text{super-bracket of}\\ \text{super-charges}}} = -2\Gamma^{a}_{\alpha\beta} \underbrace{P_a}_{\substack{\text{space-time}\\ \text{momenta}}} + 2\Gamma^{a_1 a_2}_{\alpha\beta} \underbrace{Z_{a_1 a_2}}_{\substack{\text{M2-brane}\\ \text{charges}}} - 2\Gamma^{a_1\cdots a_5}_{\alpha\beta} \underbrace{Z_{a_1\cdots a_5}}_{\substack{\text{M5-brane}\\ \text{charges}}} .$$

$$\mathrm{d}e^{a} = +(\bar{\psi}\Gamma^{a}\psi), \mathrm{d}e_{a_1 a_2} = -(\bar{\psi}\Gamma_{a_1 a_2}\psi), \mathrm{d}e_{a_1\cdots a_5} = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)$$

$$
\begin{tikzcd}
\widehat{\mathfrak{M}} = \mathbb{R}^{1,10+517\,|\,\mathbf{32}\oplus\mathbf{32}} \arrow[d, two heads] \arrow[dd, bend left, two heads, "\widehat{\phi}"] \\
\mathfrak{M} = \mathbb{R}^{1,10+517\,|\,\mathbf{32}} \arrow[d, two heads, "\phi"] \\
\mathbb{R}^{1,10\,|\,\mathbf{32}}
\end{tikzcd}
$$

$$\mathrm{d}\phi = 2(1+s)\Gamma_a \psi e^a + \Gamma^{a_1 a_2}\psi e_{a_1 a_2} + 2\frac{6+s}{6!}\Gamma^{a_1 \cdots a_5}\psi e_{a_1 \cdots a_5}$$

$$\hat{P}_3 \propto e_{a_1 a_2} e^{a_1} e^{a_2} + \text{ several more terms}$$

$$\mathrm{d}\hat{P}_3 = \hat{\phi}^* G_4, \text{ where } G_4 := \frac{1}{2}(\bar{\psi}\Gamma_{a_1 a_2}\psi) e^{a_1} e^{a_2}$$

$$\Sigma^{1,5\,|\,2\cdot\mathbf{8}} \xhookrightarrow{\;\;\phi^{\mathrm{M5}}\;\;} X^{1,10\,|\,\mathbf{32}}$$

$$\begin{tikzcd}
\underset{\substack{\text{Super-exceptional} \\ \text{Minkowski spacetime}}}{\mathbb{R}^{1,10+517\,|\,\mathbf{32}\oplus\mathbf{32}}} & \underset{\substack{\text{Hidden} \\ \text{M-group}}}{\widehat{\mathcal{M}}} \arrow[l, mapsto, "\substack{\text{underlying} \\ \text{super-manifold}}"'] \arrow[r, mapsto, "\substack{\text{underlying} \\ \text{super-Lie algebra}}"] & \underset{\substack{\text{Hidden} \\ \text{M-algebra}}}{\widehat{\mathfrak{M}}\,.}
\end{tikzcd}$$

$$\mathbb{R}^{1,10|\mathbf{32}} \simeq \mathbb{R}\langle \underbrace{(Q_\alpha)_{\alpha=1}^{32}}_{\deg=(0,\,\mathrm{odd})}, \underbrace{(P_a)_{a=0}^{10}}_{\deg=(0,\,\mathrm{evn})} \rangle$$

$$[Q_\alpha, Q_\beta] = -2\Gamma^a_{\alpha\beta} P_a.$$

$$\mathrm{CE}\left(\mathbb{R}^{1,10|\mathbf{32}}\right) \simeq \mathbb{R}[\underbrace{(\psi^\alpha)_{\alpha=0}^{32}}_{\deg=(1,\,\mathrm{odd})}, \underbrace{(e^a)_{a=0}^{10}}_{\deg=(1,\,\mathrm{evn})}]$$

$$\begin{aligned} \mathrm{d}\psi &= 0 \\ \mathrm{d}e^a &= (\bar{\psi}\Gamma^a \psi). \end{aligned}$$

$$0 \to \mathbb{R}^{1,10} \hookrightarrow \mathbb{R}^{1,10|\mathbf{32}} \longrightarrow \mathbb{R}^{0|\mathbf{32}} \to 0$$

$$(\bar{\psi}\Gamma^{a_1 a_2}\psi), (\bar{\psi}\Gamma^{a_1 \cdots a_5}\psi) \in \mathrm{CE}\left(\mathbb{R}^{0|\mathbf{32}}\right), a_i \in \{0, \cdots, 10\}$$

$$\mathfrak{M} \simeq \mathbb{R}\langle \underbrace{(Q_\alpha)_{\alpha=1}^{32}}_{\deg=(0,\,\mathrm{odd})}, \underbrace{(P_a)_{a=0}^{10}}_{\deg=(0,\,\mathrm{evn})}, \underbrace{\left(Z_{a_1 a_2} = Z_{[a_1 a_2]}\right)_{a=0}^{10}}_{\deg=(0,\,\mathrm{evn})}, \underbrace{\left(Z_{a_1 \cdots a_5} = Z_{[a_1 \cdots a_5]}\right)_{a=0}^{10}}_{\deg=(0,\,\mathrm{evn})} \rangle$$

$$[Q_\alpha, Q_\beta] = -2\Gamma^a_{\alpha\beta} P_a + 2\Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} - 2\Gamma^{a_1 \cdots a_5}_{\alpha\beta} Z_{a_1 \cdots a_5}$$

$$\mathrm{CE}(\mathfrak{M}) \simeq \mathbb{R}[\underbrace{(\psi^\alpha)_{\alpha=1}^{32}}_{\deg=(1,\,\mathrm{odd})}, \underbrace{(e^a)_{a=0}^{10}}_{\deg=(1,\,\mathrm{evn})}, \underbrace{\left(e_{a_1 a_2} = e_{[a_1 a_2]}\right)_{a_i=0}^{10}}_{\deg=(1,\,\mathrm{evn})}, \underbrace{\left(e_{a_1 \cdots a_5} = e_{[a_1 \cdots a_5]}\right)_{a_i=0}^{10}}_{\deg=(1,\,\mathrm{evn})}],$$

$$\begin{aligned} \mathrm{d}\psi &= 0 \\ \mathrm{d}e^a &= +(\bar{\psi}\Gamma^a \psi) \\ \mathrm{d}e_{a_1 a_2} &= -(\bar{\psi}\Gamma_{a_1 a_2}\psi) \\ \mathrm{d}e_{a_1 \cdots a_5} &= +(\bar{\psi}\Gamma_{a_1 \cdots a_5}\psi) \end{aligned}$$

$$e^{\alpha\beta} := \frac{1}{32}\left(e^a \Gamma_a^{\alpha\beta} + \frac{1}{2} e^{a_1 a_2} \Gamma_{a_1 a_2}^{\alpha\beta} + \frac{1}{5!} e^{a_1 \cdots a_5} \Gamma_{a_1 \cdots a_5}^{\alpha\beta}\right)$$

$$e^{\alpha\beta} = e^{\beta\alpha}$$

$$\begin{aligned} \mathrm{d}\psi^{\alpha} &= 0 \\ \mathrm{d}e^{\alpha\beta} &= \psi^{\alpha}\psi^{\beta} \end{aligned}$$

$$\begin{aligned} g: \mathrm{CE}(\mathfrak{M}) &\longrightarrow \mathrm{CE}(\mathfrak{M}) \\ \psi^{\alpha} &\longmapsto g^{\alpha}_{\alpha'}\psi^{\alpha'} \\ e^{\alpha\beta} &\longmapsto g^{\alpha}_{\alpha'}g^{\beta}_{\beta'}e^{\alpha'\beta'} \end{aligned}$$

$$\begin{aligned} e^{a} &= \Gamma^{a}_{\alpha\beta}e^{\alpha\beta} \\ e^{a_1a_2} &= -\Gamma^{a_1a_2}_{\alpha\beta}e^{\alpha\beta} \\ e^{a_1\cdots a_5} &= \Gamma^{a_1\cdots a_5}_{\alpha\beta}e^{\alpha\beta}. \end{aligned}$$

$$\begin{aligned} \mathrm{d}e^{\alpha\beta} &= \frac{1}{32}\left(\Gamma^{\alpha\beta}_{a}(\bar{\psi}\Gamma^{a}\psi) - \frac{1}{2}\Gamma^{\alpha\beta}_{a_1a_2}(\bar{\psi}\Gamma^{a_1a_2}\psi) + \frac{1}{5!}\Gamma^{\alpha\beta}_{a_1\cdots a_5}(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi)\right) \\ &= \psi^{\alpha}\psi^{\beta} \end{aligned}$$

$$g := \exp\left(\sum_{p=1}^{5}\frac{1}{p!}A_{a_1\cdots a_p}\Gamma^{a_1\cdots a_p}\right) \in \mathrm{SL}(32) \subset \mathrm{GL}(32) \subset \mathrm{End}_{\mathbb{R}}(\mathbf{32})$$

$$\begin{aligned} \psi^{\alpha}\left(\Gamma^{a_1\cdots a_p}_{\alpha'\beta'}g^{\alpha'}_{\alpha}g^{\beta'}_{\beta}\right)\phi^{\beta} &= \left(g^{\alpha'}_{\alpha}\psi^{\alpha}\right)\Gamma^{a_1\cdots a_p}_{\alpha'\beta'}\left(g^{\beta'}_{\beta}\phi^{\beta}\right) \\ &= -\left(\overline{(g\cdot\psi)}\Gamma^{a_1\cdots a_p}(g\cdot\phi)\right) \\ &= -\left(\bar{\psi}(\bar{g}\cdot\Gamma^{a_1\cdots a_p}\cdot g)\phi\right) \\ &= \psi^{\alpha}(\bar{g}\cdot\Gamma^{a_1\cdots a_p}\cdot g)_{\alpha\beta}\phi^{\beta} \end{aligned}$$

$$\mathrm{Spin}(1{,}10) \longleftrightarrow \mathrm{SL}(32)$$

$$\begin{aligned} g &= \exp(r\Gamma_{10}) && \text{for } r \in \mathbb{R} \\ &= \cosh(r)\mathrm{id} + \sinh(r)\Gamma_{10} \end{aligned}$$

$$\begin{aligned} e^{10} = \Gamma^{10}_{\alpha\beta}e^{\alpha\beta} &\mapsto \left(\exp(-r\Gamma_{10})\cdot\Gamma^{10}\cdot\exp(r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} && = \Gamma^{10}_{\alpha\beta}e^{\alpha\beta} = e^{10} \\ e^{a} = \Gamma^{a}_{\alpha\beta}e^{\alpha\beta} &\mapsto \left(\exp(-r\Gamma_{10})\cdot\Gamma^{a}\cdot\exp(r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} && = \left(\Gamma^{a}\cdot\exp(2r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} = \cosh(2r)e^{a} - \sinh(2r)e^{a10} \\ e^{a10} = -\Gamma^{a10}_{\alpha\beta}e^{\alpha\beta} &\mapsto -\left(\exp(-r\Gamma_{10})\cdot\Gamma^{a10}_{\alpha\beta}\cdot\exp(r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} && = -\left(\Gamma^{a10}_{\alpha\beta}\cdot\exp(2r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} = \cosh(2r)e^{a10} - \sinh(2r)e^{a} \\ e^{ab} = -\Gamma^{ab}_{\alpha\beta}e^{\alpha\beta} &\mapsto -\left(\exp(-r\Gamma_{10})\cdot\Gamma^{ab}\cdot\exp(r\Gamma_{10})\right)_{\alpha\beta}e^{\alpha\beta} && -\Gamma^{ab}_{\alpha\beta}e^{\alpha\beta} = e^{ab} \end{aligned}$$

$$\begin{aligned} e^{a_1\cdots a_5} &\mapsto \cosh(2r)e^{a_1\cdots a_5} + \sinh(2r)\frac{1}{5!}\epsilon^{a_1\cdots a_5 10 b_1\cdots b_5}e_{b_1\cdots b_5} \\ e^{a_1\cdots a_4 10} &\mapsto e^{a_1\cdots a_4 10}. \end{aligned}$$

$$\tilde{e}_a := e_{a10},$$

$$\mathrm{CE}(\widehat{\mathfrak{M}}) \equiv \mathbb{R}[\underbrace{(e^{a})^{10}_{a=0}}_{\deg=(1,0)}, \underbrace{\left(e_{a_1a_2} = e_{[a_1a_2]}\right)^{10}_{a_i=0}}_{\deg=(1,0)}, \underbrace{\left(e_{a_1\cdots a_5} = e_{[a_1\cdots a_5]}\right)^{10}_{a_i=0}}_{\deg=(1,0)}, \underbrace{(\psi^{\alpha})^{32}_{\alpha=1}}_{\deg=(1,1)}, \underbrace{(\phi^{\alpha})^{32}_{\alpha=1}}_{\deg=(1,1)}]$$

$$
\begin{aligned}
&\mathrm{d}\psi = 0\\
&\mathrm{d}e^{a} = +(\bar{\psi}\Gamma^{a}\psi)\\
&\mathrm{d}e_{a_1a_2} = -(\bar{\psi}\Gamma_{a_1a_2}\psi)\\
&\mathrm{d}e_{a_1\cdots a_5} = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)\\
&\mathrm{d}\phi = \delta\Gamma_a\psi e^a + \gamma_1\Gamma^{a_1a_2}\psi e_{a_1a_2} + \gamma_2\Gamma^{a_1\cdots a_5}\psi e_{a_1\cdots a_5},
\end{aligned}
$$

$$\delta + 10\cdot\gamma_1 - 6!\cdot\gamma_2 = 0$$

$$-\mathrm{d}^2\phi = \delta\Gamma_a\psi(\bar{\psi}\Gamma^a\psi) - \gamma_1\Gamma_{a_1a_2}\psi(\bar{\psi}\Gamma^{a_1a_2}\psi) + \gamma_2\Gamma_{a_1\cdots a_5}\psi(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi).$$

$$
\begin{aligned}
\delta\frac{1}{11}\Gamma^a\Gamma_a\Xi^{(32)} - \gamma_1\frac{1}{11}\Gamma^{a_1a_2}\Gamma_{a_1a_2}\Xi^{(32)} + \gamma_2\frac{-1}{77}\Gamma^{a_1\cdots a_5}\Gamma_{a_1\cdots a_5}\Xi^{(32)} &= 0\\
\delta\,\Gamma^a\Xi^{(320)}_a - \gamma_1\frac{-2}{9}\Gamma^{a_1a_2}\Gamma_{[a_1}\Xi^{(320)}_{a_2]} + \gamma_2\frac{5}{9}\Gamma^{a_1\cdots a_5}\Gamma_{[a_1\cdots a_4}\Xi^{(320)}_{a_5]} &= 0\\
-\gamma_1\Gamma^{a_1a_2}\Xi^{(1408)}_{a_1a_2} + \gamma_2 2\Gamma^{a_1\cdots a_5}\Gamma_{[a_1a_2a_3}\Xi^{(1408)}_{a_4a_5]} &= 0\\
\gamma_2\Gamma^{a_1\cdots a_5}\Xi^{(4224)}_{a_1\cdots a_5} &= 0
\end{aligned}
$$

$$
\begin{pmatrix}
\delta(s) & = 2(1+s)\\
\gamma_1(s) & = 1\\
\gamma_2(s) & = 2\left(\frac{1}{5!}+\frac{s}{6!}\right)
\end{pmatrix}, s\in\mathbb{R}
$$

$$
\begin{array}{ccc}
\widehat{\mathfrak{M}} & \xrightarrow{\ \phi_{\mathrm{ex}}\ } & \mathbb{R}^{1,10|\mathbf{32}}\\
\mathrm{CE}(\mathbb{R}^{1,10|\mathbf{32}}) & \xleftarrow{\ \phi^*_{\mathrm{ex}}\ } & \mathrm{CE}(\widehat{\mathfrak{M}})\\
e^a & \longleftarrow & e^a\\
\psi^\alpha & \longleftarrow & \psi^\alpha\,.
\end{array}
$$

$$\widehat{\mathfrak{M}} \simeq \mathbb{R}\langle \underbrace{(P_a)_{a=0}^{10}, \left(Z_{a_1a_2} = Z_{[a_1a_2]}\right)_{a_i=0}^{10}, \left(Z_{a_1\cdots a_5} = Z_{[a_1\cdots a_5]}\right)_{a_i=0}^{10}}_{\mathrm{deg}=(0,\,\mathrm{evn}\,)}, \underbrace{(Q_\alpha)_{\alpha=1}^{32}, (O_\alpha)_{\alpha=1}^{32}}_{\mathrm{deg}=(0,\,\mathrm{odd}\,)}\rangle$$

$$
\begin{aligned}
[Q_\alpha, Q_\beta] &= -2\Gamma^a_{\alpha\beta}P_a + 2\Gamma^{a_1a_2}_{\alpha\beta}Z_{a_1a_2} - 2\Gamma^{a_1\cdots a_5}_{\alpha\beta}Z_{a_1\cdots a_5}\\
[P_a, Q_\alpha] &= \delta\Gamma_a{}^{\beta}{}_{\alpha}O_\beta\\
[Z_{a_1a_2}, Q_\alpha] &= \gamma_1\Gamma_{a_1a_2}{}^{\beta}{}_{\alpha}O_\beta\\
[Z_{a_1\cdots a_5}, Q_\alpha] &= \gamma_2\Gamma_{a_1\cdots a_5}{}^{\beta}{}_{\alpha}O_\beta
\end{aligned}
$$

$$
\begin{aligned}
\left[Q_\gamma, [Q_\alpha, Q_\beta]\right] &= \left[Q_\gamma, -2\Gamma^a_{\alpha\beta}P_a + 2\Gamma^{a_1a_2}_{\alpha\beta}Z_{a_1a_2} - 2\Gamma^{a_1\cdots a_5}_{\alpha\beta}Z_{a_1\cdots a_5}\right]\\
&= \underbrace{2\left(\delta\Gamma^a_{\alpha\beta}\Gamma_a{}^{\delta}{}_{\gamma} - \gamma_1\Gamma^{a_1a_2}_{\alpha\beta}\Gamma_{a_1a_2}{}^{\delta}{}_{\gamma} + \gamma_2\Gamma^{a_1\cdots a_5}_{\alpha\beta}\Gamma_{a_1\cdots a_5}{}^{\delta}{}_{\gamma}\right)}_{=:[QQQ]^{\delta}_{\gamma\alpha\beta}}O_\delta
\end{aligned}
$$

$$s = 0 \;\Rightarrow\; \left[Q^\gamma, [Q_\alpha, Q_\beta]\right] = 64\left(\delta^\gamma_\beta O_\alpha + \delta^\gamma_\alpha O_\beta\right)$$

$$
\begin{aligned}
\delta_\alpha^\delta \delta_\gamma^\beta &= \frac{1}{32}\sum_{p=0}^{5} \frac{(-1)^{p(p-1)/2}}{p!} \mathrm{Tr}\left(\delta_\alpha^{\cdot} \delta_{\cdot}^{\beta} \cdot \Gamma_{a_1\cdots a_p}\right)(\Gamma^{a_1\cdots a_p})_\gamma^\delta \\
&= \frac{1}{32}\sum_{p=0}^{5} \frac{(-1)^{p(p-1)/2}}{p!} \left(\delta_\alpha^{\delta'} \delta_{\gamma'}^{\beta} \left(\Gamma_{a_1\cdots a_p}\right)^{\gamma'} \delta'\right)(\Gamma^{a_1\cdots a_p})_\gamma^\delta \\
&= \frac{1}{32}\sum_{p=0}^{5} \frac{(-1)^{p(p-1)/2}}{p!} \left(\Gamma_{a_1\cdots a_p}\right)_\alpha^\beta (\Gamma^{a_1\cdots a_p})_\gamma^\delta
\end{aligned}
$$

$$
\begin{aligned}
\eta_{\delta(\alpha}\eta_{\beta)\gamma} &= \frac{1}{2}(\eta_{\delta\alpha}\eta_{\beta\gamma} + \eta_{\delta\beta}\eta_{\alpha\gamma}) \\
&= \frac{1}{32}\left((\Gamma_a)_{\alpha\beta}(\Gamma^a)_{\gamma\delta} - \frac{1}{2}\left(\Gamma_{a_1a_2}\right)_{\alpha\beta}(\Gamma^{a_1a_2})_{\gamma\delta} + \frac{1}{5!}\left(\Gamma_{a_1\cdots a_5}\right)_{\alpha\beta}(\Gamma^{a_1\cdots a_5})_{\gamma\delta}\right) \\
&= \frac{1}{64}\left(\delta(\Gamma_a)_{\alpha\beta}(\Gamma^a)_{\gamma\delta} - \gamma_1\left(\Gamma_{a_1a_2}\right)_{\alpha\beta}(\Gamma^{a_1a_2})_{\gamma\delta} + \gamma_2\left(\Gamma_{a_1\cdots a_5}\right)_{\alpha\beta}(\Gamma^{a_1\cdots a_5})_{\gamma\delta}\right)
\end{aligned}
$$

$$
\left[Q^\gamma, [Q_\alpha, Q_\beta]\right] = 65\left(\delta_\alpha^\gamma O_\beta + \delta_\beta^\gamma O_\alpha\right) + \left(4s\Gamma_{\alpha\beta}^{a}\Gamma_a^{\gamma\delta} + \frac{4s}{6!}\Gamma_{\alpha\beta}^{a_1\cdots a_5}\Gamma_{a_1\cdots a_5}^{\gamma\delta}\right) O_\delta
$$

$$
g \equiv \Gamma \equiv \left(\Gamma_{10}{}^{\alpha}{}_{\beta}\right) \in \mathrm{GL}(32)
$$

$$
\left(\overline{\Gamma^{10}\psi}\Gamma^{10}\phi\right) \underset{\bowtie}{=} (\bar{\psi}(-\Gamma^{10})\Gamma^{10}\phi) \underset{\boxtimes}{=} -(\bar{\psi}\phi)
$$

$$
\Gamma^\alpha{}_{\alpha'}\eta_{\alpha\beta}\Gamma^\beta{}_{\beta'} \underset{\boxminus}{=} \Gamma_{\beta\alpha'}\Gamma^\beta{}_{\beta'} \underset{\boxdot}{=} \Gamma_{\alpha'\beta}\Gamma^\beta{}_{\beta'} \underset{\otimes}{=} \eta_{\alpha\alpha'}\Gamma^\alpha{}_\beta\Gamma^\beta{}_{\beta'} \underset{\odot}{=} \eta_{\alpha\alpha'}\delta_{\beta'}^\alpha = \eta_{\beta'\alpha'} \underset{\ominus}{=} -\eta_{\alpha'\beta'}
$$

$$
\Gamma^\alpha{}_{\alpha'}e^{\alpha'\beta'}\Gamma^\beta{}_{\beta'} \underset{\circledcirc}{=} -\Gamma^\alpha{}_{\alpha'}e^{\alpha'}{}_{\beta'}\Gamma^{\beta\beta'} \underset{\blacksquare}{=} -\Gamma^\alpha{}_{\alpha'}e^{\alpha'}{}_{\beta'}\Gamma^{\beta'\beta}
$$

$$
\begin{aligned}
\Gamma_a e^a &\mapsto -(\Gamma_{10}\cdot\Gamma_a\cdot\Gamma_{10})e^a &&= +\sum_{a\neq 10}\Gamma_a e^a - \Gamma_{10}e^{10} \\
\Gamma_{a_1a_2}e^{a_1a_2} &\mapsto -\left(\Gamma_{10}\cdot\Gamma_{a_1a_2}\cdot\Gamma_{10}\right)e^{a_1a_2} &&= -\sum_{a_i\neq 10}\Gamma_{a_1a_2}e^{a_1a_2} + 2\sum_{a\neq 10}\Gamma_{a10}e^{a10} \\
\Gamma_{a_1\cdots a_5}e^{a_1\cdots a_5} &\mapsto -\left(\Gamma_{10}\cdot\Gamma_{a_1\cdots a_5}\cdot\Gamma_{10}\right)e^{a_1\cdots a_5} &&= +\sum_{a_i\neq 10}\Gamma_{a_1\cdots a_5}e^{a_1\cdots a_5} + 5\sum_{a\neq 10}\Gamma_{a_1\cdots a_4 10}e^{a_1\cdots a_4 10}
\end{aligned}
$$

$$
\begin{tikzcd}
\psi \arrow[r, mapsto, "\Gamma_{10}"] \arrow[d, mapsto, "\mathrm{d}"] & \Gamma_{10}\psi \arrow[d, mapsto, "\mathrm{d}"] \\
0 \arrow[r, mapsto, "\Gamma_{10}"] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
e^{a} \arrow[r, mapsto, "\Gamma_{10}"] \arrow[dd, mapsto, "\mathrm{d}"'] & \begin{cases} +e^{a} & \text{for } a \neq 10 \\ -e^{a} & \text{otherwise} \end{cases} \arrow[d, mapsto, "\mathrm{d}"] \\
 & \begin{cases} +(\overline{\psi}\,\Gamma^{a}\,\psi) & \text{for } a \neq 10 \\ -(\overline{\psi}\,\Gamma^{a}\,\psi) & \text{otherwise} \end{cases} \arrow[d, equal] \\
(\overline{\psi}\,\Gamma^{a}\,\psi) \arrow[r, mapsto, "\Gamma_{10}"] & (\overline{\Gamma_{10}\psi}\,\Gamma^{a}\,\Gamma_{10}\psi)
\end{tikzcd}
$$

$$
\begin{tikzcd}
e^{a_1 a_2} \arrow[r, mapsto, "\Gamma_{10}"] \arrow[dd, mapsto, "\mathrm{d}"'] & \begin{cases} -e^{a_1 a_2} & \text{for } a_i \neq 10 \\ +e^{a_1 a_2} & \text{otherwise} \end{cases} \arrow[d, mapsto, "\mathrm{d}"] \\
 & \begin{cases} +(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi) & \text{for } a_i \neq 10 \\ -(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi) & \text{otherwise} \end{cases} \arrow[d, equal] \\
-(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi) \arrow[r, mapsto, "\Gamma_{10}"] & -(\overline{\Gamma_{10}\psi}\,\Gamma^{a_1 a_2}\,\Gamma_{10}\psi)
\end{tikzcd}
$$

$$
\begin{tikzcd}
e^{a_1 \cdots a_5} \arrow[r, mapsto, "\Gamma_{10}"] \arrow[dd, mapsto, "\mathrm{d}"'] & \begin{cases} +e^{a_1 \cdots a_5} & \text{for } a_i \neq 10 \\ -e^{a_1 \cdots a_5} & \text{otherwise} \end{cases} \arrow[d, mapsto, "\mathrm{d}"] \\
 & \begin{cases} +(\overline{\psi}\,\Gamma^{a_1 \cdots a_5}\,\psi) & \text{for } a_i \neq 10 \\ -(\overline{\psi}\,\Gamma^{a_1 \cdots a_5}\,\psi) & \text{otherwise} \end{cases} \arrow[d, equal] \\
(\overline{\psi}\,\Gamma^{a_1 \cdots a_5}\,\psi) \arrow[r, mapsto, "\Gamma_{10}"] & (\overline{\Gamma_{10}\psi}\,\Gamma^{a_1 \cdots a_5}\,\Gamma_{10}\psi)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\phi \arrow[r, mapsto, "\Gamma_{10}"] \arrow[dd, mapsto, "\mathrm{d}"'] & -\Gamma_{10}\phi \arrow[d, mapsto, "\mathrm{d}"] \\
 & \begin{array}{l} -\delta\ \Gamma_{10}\Gamma_{a}\psi\, e^{a} \\ -\gamma_1\, \Gamma_{10}\Gamma_{a_1 a_2}\psi\, e^{a_1 a_2} \\ -\gamma_2\, \Gamma_{10}\Gamma_{a_1 \cdots a_5}\psi\, e^{a_1 \cdots a_5} \end{array} \arrow[d, equal] \\
\begin{array}{l} \delta\ \Gamma_{a}\psi\, e^{a} \\ +\gamma_1\, \Gamma_{a_1 a_2}\psi\, e^{a_1 a_2} \\ +\gamma_2\, \Gamma_{a_1 \cdots a_5}\psi\, e^{a_1 \cdots a_5} \end{array} \arrow[r, mapsto, "\Gamma_{10}"] & \begin{array}{l} \delta\ \big(+ \sum_{a \neq 0} \Gamma_{a}\Gamma_{10}\psi\, e^{a} - \Gamma_{10}\Gamma_{10}\psi\, e^{10} \big) \\ +\gamma_1 \big(- \sum_{a_i \neq 0} \Gamma_{a_1 a_2}\Gamma_{10}\psi\, e^{a_1 a_2} + 2\Gamma_{a\,10}\Gamma_{10}\psi\, e^{a\,10} \big) \\ +\gamma_2 \big(+ \sum_{a_i \neq 0} \Gamma_{a_1 \cdots a_5}\Gamma_{10}\psi\, e^{a_1 \cdots a_5} - 5\Gamma_{a_1 \cdots a_4\,10}\Gamma_{10}\psi\, e^{a_1 \cdots a_4\,10} \big) \end{array}
\end{tikzcd}
$$

$$
\begin{aligned}
G_4 &\equiv \frac{1}{2}(\bar{\psi}\Gamma_{a_1 a_2}\psi)e^{a_1}e^{a_2} \\
&\overset{\Gamma_{10}}{\longmapsto} \frac{1}{2}\sum_{a_i \neq 0} (\overline{\Gamma_{10}\psi}\Gamma_{a_1 a_2}\Gamma_{10}\psi)e^{a_1}e^{a_2} - (\overline{\Gamma_{10}\psi}\Gamma_{a_1 10}\Gamma_{10}\psi)e^{a_1}e^{10} \\
&= -\frac{1}{2}\sum_{a_i \neq 0} (\bar{\psi}\Gamma_{a_1 a_2}\psi)e^{a_1}e^{a_2} - (\bar{\psi}\Gamma_{a_1 10}\psi)e^{a_1}e^{10} \\
&= -G_4
\end{aligned}
$$

$$\hat{P}_3 \overset{\Gamma_{10}}{\to} -\hat{P}_3,$$

$$
\begin{aligned}
\mathrm{d}\left(-\frac{1}{2}e_{a_1 a_2}e^{a_1}e^{a_2}\right) &= \phi_{\mathrm{ex}}^{*}G_4 + \cdots \\
\mathrm{d}\left(\frac{1}{2}(\bar{\psi}\Gamma_{a}\phi)e^{a}\right) &= \phi_{\mathrm{ex}}^{*}G_4 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\hat{P}_3 := \;& \alpha_0 e_{a_1 a_2} e^{a_1} e^{a_2} && \\
& + \alpha_1 e^{a_1}{}_{a_2} e^{a_2}{}_{a_3} e^{a_3}{}_{a_1} && + \beta_1 (\bar{\psi}\Gamma_a \phi) e^a \\
& + \alpha_2 e^{a_1 \cdots a_4 b_1} e_{b_1}^{b_2} e_{b_2 a_1 \cdots a_4} && + \beta_2 (\bar{\psi}\Gamma_{a_1 a_2} \phi) e^{a_1 a_2} \\
& + \alpha_3 \epsilon_{a_1 \cdots a_5 b_1 \cdots b_5 c} e^{a_1 \cdots a_5} e^{b_1 \cdots b_5} e^c && + \beta_3 (\bar{\psi}\Gamma_{a_1 \cdots a_5} \phi) e^{a_1 \cdots a_5} \\
& + \alpha_4 \epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1 \cdots c_5} e^{a_1 a_2 a_3 d_1 d_2} e_{d_1 d_2}^{b_1 b_2 b_3} e^{c_1 \cdots c_5} &&
\end{aligned}
$$

$$
\mathrm{d}\hat{P}_3 = \phi_{\mathrm{ex}}^* G_4 \in \mathrm{CE}(\widehat{\mathfrak{M}})
$$

$$
\begin{aligned}
\alpha_0 &= \frac{1}{2}\frac{-1}{5}\frac{6+2s+s^2}{s^2} && \\
\alpha_1 &= \frac{1}{2}\frac{1}{15}\frac{6+2s}{s^2} & \beta_1 &= -1\frac{1}{10\gamma_1}\frac{3-2s}{s^2} \\
\alpha_2 &= \frac{1}{2}\frac{1}{6!}\frac{(6+s)^2}{s^2} & \beta_2 &= -1\frac{1}{20\gamma_1}\frac{3+s}{s^2} \\
\alpha_3 &= \frac{1}{2}\frac{1}{5\cdot 5!\cdot 6!}(6+s)^2 & \beta_3 &= -1\frac{3}{10\cdot 6!\cdot \gamma_1}\frac{6+s}{s^2} \\
\alpha_4 &= \frac{1}{2}\frac{-1(6+s)^2}{9\cdot 5!\cdot 6!\, s^2}. &&
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}\hat{P}_3 = \\
&\alpha_0\big(-(\bar{\psi}\Gamma_{a_1 a_2}\psi) e^{a_1} e^{a_2} - 2 e_{a_1 a_2} (\bar{\psi}\Gamma^{a_1}\psi) e^{a_2}\big) \\
&+\alpha_1\big(-3(\bar{\psi}\Gamma^{a_1}{}_{a_2}\psi) e^{a_2}{}_{a_3} e^{a_3}{}_{a_1}\big) \\
&+\alpha_2\big(2(\bar{\psi}\Gamma^{a_1\cdots a_4 b_1}\psi) e_{b_1}{}^{b_2} e_{b_2 a_1\cdots a_4} + (\bar{\psi}\Gamma_{b_1}{}^{b_2}\psi) e^{a_1\cdots a_4 b_1} e_{b_2 a_1 \cdots a_4}\big) \\
&+\alpha_3\big(2\epsilon_{a_1\cdots a_5 b_1\cdots b_5 c}(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi) e^{b_1\cdots b_5} e^c + \epsilon_{a_1\cdots a_5 b_1 \cdots b_5 c}(\bar{\psi}\Gamma^c\psi) e^{a_1\cdots a_5} e^{b_1\cdots b_5}\big) \\
&+\alpha_4(\underbrace{2\epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1\cdots c_5}(\bar{\psi}\Gamma^{a_1 a_2 a_3 d_1 d_2}\psi) e_{d_1 d_2}{}^{b_1 b_2 b_3} e^{c_1\cdots c_5} + \epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1 \cdots c_5}(\bar{\psi}\Gamma^{c_1\cdots c_5}\psi) e^{a_1 a_2 a_3 d_1 d_2} e_{d_1 d_2}{}^{b_1 b_2 b_3}}) \\
&\overset{\triangle}{=} 3\epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1 \cdots c_5}(\bar{\psi}\Gamma^{c_1\cdots c_5}\psi) e^{a_1 a_2 a_3 d_1 d_1} e_{d_1 d_1}{}^{b_1 b_2 b_3} \\
&+\beta_1(\underbrace{\delta(\bar{\psi}\Gamma_a\Gamma_b\psi) e^a e^b}_{\delta(\bar{\psi}\Gamma_{ab}\psi) e^a e^b} + \underbrace{\gamma_1(\bar{\psi}\Gamma_a\Gamma_{b_1 b_2}\psi) e^a e^{b_1 b_2}}_{2\gamma_1(\bar{\psi}\Gamma^b\psi) e_a e^{ab}} + \underbrace{\gamma_2(\bar{\psi}\Gamma_a\Gamma_{b_1\cdots b_5}\psi) e^a e^{b_1\cdots b_5}}_{\frac{\gamma_2}{5!}\epsilon_{ab_1\cdots b_5 c_1 \cdots c_5}(\bar{\psi}\Gamma_{c_1\cdots b_5}\psi) e^a e^{b_1\cdots b_5}} \\
&+(\bar{\phi}\Gamma_a\psi)(\bar{\psi}\Gamma^a\psi)\big) \\
&+\beta_2(\underbrace{\delta(\bar{\psi}\Gamma_{a_1 a_2}\Gamma_b\psi) e^{a_1 a_2} e^b}_{2\delta(\bar{\psi}\Gamma_a\psi) e^{ab} e_b} + \underbrace{\gamma_1(\bar{\psi}\Gamma_{a_1 a_2}\Gamma_{b_1 b_2}\psi) e^{a_1 a_2} e^{b_1 b_2}}_{4\gamma_1(\bar{\psi}\Gamma_a{}^b\psi) e^a{}_c e^c{}_b} + \underbrace{\gamma_2(\bar{\psi}\Gamma_{a_1 a_2}\Gamma_{b_1\cdots b_5}\psi) e^{a_1 a_2} e^{b_1\cdots b_5}}_{10\gamma_2(\bar{\psi}\Gamma_{ab_1\cdots b_4}\psi) e^{ac} e_c{}^{b_1\cdots b_4}} \\
&-(\bar{\phi}\Gamma_{a_1 a_2}\psi)(\bar{\psi}\Gamma^{a_1 a_2}\psi)\big) \\
&+\beta_3(\underbrace{\delta(\bar{\psi}\Gamma_{a_1\cdots a_5}\Gamma_b\psi) e^{a_1\cdots a_5} e^b}_{\frac{\delta}{5!}\epsilon_{a_1\cdots a_5 b c_1\cdots c_5}(\bar{\psi}\Gamma_{c_1\cdots c_5}\psi) e^{a_1\cdots a_5} e^b} + \underbrace{\gamma_1(\bar{\psi}\Gamma_{a_1\cdots a_5}\Gamma_{b_1 b_2}\psi) e^{a_1\cdots a_5} e^{b_1 b_2}}_{10\gamma_1(\bar{\psi}\Gamma_{a_1\cdots a_4 b}\psi) e^{a_1\cdots a_4 c} e_c{}^b} + \underbrace{\gamma_2(\bar{\psi}\Gamma_{a_1\cdots a_5}\Gamma_{b_1\cdots b_5}\psi) e^{a_1\cdots a_5} e^{b_1\cdots b_5}}_{\gamma_2\epsilon_{a_1\cdots a_5 b_1\cdots b_5 c}(\bar{\psi}\Gamma_c\psi) e^{a_1\cdots a_5} e^{b_1\cdots b_5}} \\
&+(\bar{\phi}\Gamma_{a_1\cdots a_5}\psi)(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi)\big), -\frac{200}{5!}\gamma_2\epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1\cdots c_5}(\bar{\psi}\Gamma^{c_1\cdots c_5}\psi) e^{a_1 a_2 a_3 d_1 d_2} e_{d_1 d_2}{}^{b_1 b_2 b_3} \\
&+600\gamma_2(\bar{\psi}\Gamma_a{}^b\psi) e^{ac_1\cdots c_4} e_{c_1\cdots c_4 b}
\end{aligned}
$$

$$\mathrm{d}\,\widehat{P}_3 \;=\; \tfrac{1}{2}\big(\,\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)\,e^{a_1}\,e^{a_2} \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rcl} -\alpha_0 + \delta\,\beta_1 &=& \tfrac{1}{2} \\ -2\,\alpha_0 \;+\; 2\,\gamma_1\beta_1 \;+\; 2\,\delta\beta_2 &=& 0 \\ -3\,\alpha_1 \;-\; 4\gamma_1\beta_2 &=& 0 \\ 2\,\alpha_2 + 10\,\gamma_2\,\beta_2 + 10\,\gamma_1\beta_3 &=& 0 \\ \alpha_2 \;+\; 600\,\gamma_2\,\beta_3 &=& 0 \\ 2\,\alpha_3 + \tfrac{\gamma_2}{5!}\,\beta_1 + \tfrac{\delta}{5!}\,\beta_3 &=& 0 \\ \alpha_3 \;+\; \gamma_2\,\beta_3 &=& 0 \\ 3\,\alpha_4 \;-\; \tfrac{200}{5!}\gamma_2\,\beta_3 &=& 0 \\ \beta_1 + 10\cdot\beta_2 - 6!\cdot\beta_3 &=& 0\,, \end{array} \right.$$

$$(\bar{\psi}\Gamma_a{}^b\psi)e^a{}_c e^c{}_b = -(\bar{\psi}\Gamma^{a_1}{}_{a_2}\psi)e^{a_2}{}_{a_3}e^{a_3}{}_{a_1}.$$

$$\begin{aligned} &\epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5}\big(\bar{\psi}\Gamma^{a_1a_2a_3d_1d_2}\psi\big)e^{b_1b_2b_3}_{d_1d_2}e^{c_1\cdots c_5} \\ &= \epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5}(\bar{\psi}\Gamma^{c_1\cdots c_5}\psi)e^{a_1a_2a_3d_1d_2}e^{b_1b_2b_3}_{d_1d_2}. \end{aligned}$$

$$\begin{aligned} &\epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5}\big(\bar{\psi}\Gamma^{a_1a_2a_3d_1d_2}\psi\big)e^{b_1b_2b_3}_{d_1d_2}e^{c_1\cdots c_5} \\ &= -\frac{1}{6!}\epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5}\epsilon^{a_1a_2a_3d_1d_2f_1\cdots f_6}\big(\bar{\psi}\Gamma_{f_1\cdots f_6}\psi\big)e^{d_1d_2b_1b_2b_3}e^{c_1\cdots c_5} \\ &= \frac{3!\cdot 8!}{6!}\delta^{d_1d_2f_1\cdots f_6}_{b_1b_2b_3c_1\cdots c_5}\big(\bar{\psi}\Gamma_{f_1\cdots f_6}\psi\big)e_{d_1d_2}{}^{b_1b_2b_3}e^{c_1\cdots c_5} \\ &= \frac{3!\cdot 8!}{6!}\binom{5}{2}\frac{2!\cdot 6!}{8!}\delta^{d_1d_2}_{c_4c_5}\delta^{f_1\cdots f_6}_{b_1b_2b_3c_1c_2c_3}\big(\bar{\psi}\Gamma_{f_1\cdots f_6}\psi\big)e^{b_1b_2b_3}_{d_1d_2}e^{c_1\cdots c_5} \\ &= 120\cdot\big(\bar{\psi}\Gamma_{b_1b_2b_3c_1c_2c_3}\psi\big)e^{b_1b_2b_3}_{d_1d_2}e^{d_1d_2c_3\cdots c_5} \\ &= \frac{120}{5!}\epsilon_{b_1b_2b_3c_1c_2c_3a_1\cdots a_5}(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi)e^{b_1b_2b_3}{}_{d_1d_2}e^{d_1d_2c_3\cdots c_5} \end{aligned}$$

$$\tilde{G}_7 := (\phi_{\mathrm{ex}}^* G_7) - \frac{1}{2}\hat{P}_3(\phi_{\mathrm{ex}}^* G_4), \text{ where } G_7 := \frac{1}{5!}\big(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi\big)e^{a_1}\cdots e^{a_5} \in \mathrm{CE}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)$$

$$\mathrm{d}\tilde{G}_7 = 0$$

$$\mathrm{d}G_7 = \frac{1}{2}G_4G_4$$

$$\begin{aligned} \mathrm{d}\psi^\alpha &= 0 \\ \mathrm{d}e^\alpha{}_\beta &= \psi^\alpha\psi_\beta \\ \mathrm{d}\phi^\alpha &= 64e^\alpha{}_\beta\psi^\beta \end{aligned}$$

$$\begin{aligned} \mathrm{Sp}(32,\mathbb{R})\times\mathrm{CE}\big(\mathbb{R}^{1,10|32}\big) &\longrightarrow \mathrm{CE}\big(\mathbb{R}^{1,10|32}\big) \\ (g,\psi^\alpha) &\longmapsto g^\alpha_{\alpha'}\psi^{\alpha'} \\ \big(g,e^{\alpha\beta}\big) &\longmapsto g^\alpha_{\alpha'}g^\beta_{\beta'}e^{\alpha'\beta'} \\ (g,\phi^\alpha) &\longmapsto g^\alpha_{\alpha'}\phi^{\alpha'} \end{aligned}$$

$$
\begin{aligned}
(\mathrm{d}\phi)_\gamma &= \delta(\Gamma_a\psi)_\gamma e^a + \gamma_1\left(\Gamma_{a_1a_2}\psi\right)_\gamma e^{a_1a_2} + \gamma_2\left(\Gamma_{a_1\cdots a_5}\psi\right)_\gamma e^{a_1\cdots a_5} \\
&= \left(\delta(\Gamma_a)_{\gamma\delta}\Gamma^a_{\alpha\beta} - \gamma_1\left(\Gamma_{a_1a_2}\right)_{\gamma\delta}\Gamma^{a_1a_2}_{\alpha\beta} + \gamma_2\left(\Gamma_{a_1\cdots a_5}\right)_{\gamma\delta}\Gamma^{a_1\cdots a_5}_{\alpha\beta}\right)\psi^\delta e^{\alpha\beta} \\
&= 64\eta_{\delta(\alpha}\eta_{\beta)\gamma}\psi^\delta e^{\alpha\beta} \\
&= +64\psi_\alpha e^\alpha{}_\gamma \\
&= -64\psi^\alpha e_{\alpha\gamma} \\
&= +64 e_{\gamma\alpha}\psi^\alpha
\end{aligned}
$$

$$
\begin{aligned}
g\colon \mathrm{CE}(\widehat{\mathfrak{M}}) &\longrightarrow \mathrm{CE}(\widehat{\mathfrak{M}}) \\
\psi^\alpha &\longmapsto g^\alpha_{\alpha'}\psi^{\alpha'} \\
e^{\alpha\beta} &\longmapsto g^\alpha_{\alpha'}g^\beta_{\beta'}e^{\alpha'\beta'} \\
\phi^\alpha &\longmapsto g^\alpha_{\alpha'}\phi^{\alpha'}
\end{aligned}
$$

$$
g\colon e^\alpha{}_\beta \longmapsto g^\alpha_{\alpha'}e^{\alpha'}{}_{\beta'}\bar{g}^{\beta'}_\beta,
$$

$$
(g^\alpha_{\alpha'}e^{\alpha'\gamma'})g^\gamma_{\gamma'}\eta_{\gamma\beta} = (g^\alpha_{\alpha'}e^{\alpha'\gamma'})\eta_{\gamma'\beta'}\bar{g}^{\beta'}_\beta,
$$

$$
\mathrm{CSp}(2n) := \{g \in \mathrm{GL}(n) \mid \eta(g(-), g(-)) = \lambda(g)\cdot\eta(-,-), \lambda(g) \in \mathbb{R}^\times\}
$$

$$
0 \longrightarrow \mathrm{Sp}(2n) \hookrightarrow \mathrm{CSp}(2n) \xrightarrow{\lambda} \mathbb{R}^\times \longrightarrow 0.
$$

$$
\begin{aligned}
\mathrm{CSp}(32)\times\mathrm{CE}(\widehat{\mathfrak{M}}) &\longrightarrow \mathrm{CE}(\widehat{\mathfrak{M}}) \\
(g,\psi^\alpha) &\longmapsto g^\alpha_{\alpha'}\psi^\alpha \\
(g,e^{\alpha\beta}) &\longmapsto g^\alpha_{\alpha'}g^\beta_{\beta'}e^{\alpha'\beta'} \\
(g,\phi^\alpha) &\longmapsto \lambda(g)\cdot g^\alpha_{\alpha'}\phi^{\alpha'}
\end{aligned}
$$

$$
g^\alpha_{\alpha'}\eta_{\alpha\beta}g^\beta_{\beta'} = \lambda(g)\cdot\eta_{\alpha'\beta'}.
$$

$$
\begin{tikzcd}
\phi^\alpha \arrow[rr, mapsto, "g"] \arrow[dd, mapsto, "\mathrm{d}"] & & \lambda(g)g^\alpha_{\alpha'}\,\phi^{\alpha'} \arrow[d, mapsto, "\mathrm{d}"] \\
& & -2\lambda(g)\,g^\alpha_{\alpha'}\,e^{\alpha'\beta'}\eta_{\beta'\gamma'}\psi^{\gamma'} \arrow[d, phantom, "="] \\
-2\,e^{\alpha\beta}\eta_{\beta\gamma}\psi^\gamma \arrow[rr, "g"] & & -2\,g^\alpha_{\alpha'}e^{\alpha'\beta'}g^\beta_{\beta'}\,\eta_{\beta\gamma}\,g^\gamma_{\gamma'}\psi^{\gamma'}\,.
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Pin}^{+}(1,10) \arrow[r, hook] & \mathrm{CSp}(32) \arrow[r, hook] & \mathrm{Aut}\big(\widehat{\mathfrak{M}}\big) \\
\mathrm{Spin}(1,10) \arrow[r, hook] \arrow[u, hook] & \mathrm{Sp}(32) \arrow[r, hook] \arrow[u, hook] & \mathrm{Aut}\big(\widehat{\mathfrak{M}}\big) \arrow[u, equal]
\end{tikzcd} .
$$

$$
\begin{aligned}
\Omega_3 := \lim_{s\to 0} s^2 \cdot \hat{P}_3 = & -\frac{3}{5} e_{a_1a_2} e^{a_1} e^{a_2} \\
& + \frac{1}{5} e^{a_1}{}_{a_2} e^{a_2}{}_{a_3} e^{a_3}{}_{a_1} \\
& + \frac{18}{6!} e^{a_1\cdots a_4 b_1} e_{b_1}^{b_2} e_{b_2 a_1\cdots a_4} \\
& + \frac{18}{5\cdot 5!\cdot 6!} \epsilon_{a_1\cdots a_5 b_1\cdots b_5 c} e^{a_1\cdots a_5} e^{b_1\cdots b_5} e^{c} \\
& - \frac{2}{5!\cdot 6!} \epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5} e^{a_1a_2a_3d_1d_2} e_{d_1d_2}^{b_1b_2b_3} e^{c_1\cdots c_5} \\
& - \frac{3}{10} (\bar{\psi}\Gamma_a \phi) e^{a} \\
& - \frac{3}{20} (\bar{\psi}\Gamma_{a_1a_2} \phi) e^{a_1a_2} \\
& - \frac{3}{10\cdot 5!} (\bar{\psi}\Gamma_{a_1\cdots a_5} \phi) e^{a_1\cdots a_5}
\end{aligned}
$$

$$
\mathrm{d}\Omega_3 = 0
$$

$$
\mathrm{CE}(\mathrm{II}\mathfrak{A}) \simeq \mathbb{R}_{\mathrm{d}} \left[
\begin{array}{c}
(\psi^{\alpha})_{\alpha=1}^{32} \\
(e^{a})_{a=1}^{9} \\
(\tilde{e}_{a})_{a=1}^{9} \\
\big(e_{a_1a_2} = e_{[a_1a_2]}\big)_{a_i=0}^{9} \\
\big(e_{a_1\cdots a_4} = e_{[a_1\cdots a_4]}\big)_{a_i=0}^{9} \\
\big(e_{a_1\cdots a_5} = e_{[a_1\cdots a_5]}\big)_{a_i=0}^{9}
\end{array}
\right] \Big/ \left(
\begin{array}{ll}
\mathrm{d}\psi & = 0 \\
\mathrm{d}e^{a} & = +(\bar{\psi}\Gamma^{a}\psi) \\
\mathrm{d}\tilde{e}_{a} & = -(\bar{\psi}\Gamma_{a}\Gamma_{10}\psi) \\
\mathrm{d}e_{a_1a_2} & = -(\bar{\psi}\Gamma_{a_1a_2}\psi) \\
\mathrm{d}e_{a_1\cdots a_4} & = +(\bar{\psi}\Gamma_{a_1\cdots a_4}\Gamma_{10}\psi) \\
\mathrm{d}e_{a_1\cdots a_5} & = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)
\end{array}
\right)
$$

$$
\begin{aligned}
(\mathrm{II}\mathfrak{A})_{\mathrm{bos}} & \simeq_{\mathbb{R}} \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^{*} \oplus \wedge^{2}(\mathbb{R}^{1,9})^{*} \oplus \wedge^{4}(\mathbb{R}^{1,9})^{*} \oplus \wedge^{5}(\mathbb{R}^{1,9})^{*} \\
& \simeq_{\mathbb{R}} \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^{*} \oplus \wedge^{2}(\mathbb{R}^{9})^{*} \oplus \wedge^{8}(\mathbb{R}^{9}) \oplus \wedge^{4}(\mathbb{R}^{9})^{*} \oplus \wedge^{6}(\mathbb{R}^{9}) \oplus \wedge^{5}(\mathbb{R}^{1,9})^{*} ,
\end{aligned}
$$

$$
\wedge^{p}\left(\mathbb{R}^{1,d}\right)^{*} \simeq_{\mathbb{R}} \underbrace{\wedge^{p}\left(\mathbb{R}^{d}\right)^{*}}_{\text{spatial}} \oplus \underbrace{\wedge^{1+d-p}\left(\mathbb{R}^{d}\right)}_{\substack{\text{dualized} \\ \text{temporal}}}.
$$

$$\mathrm{CE}(\widehat{\mathrm{II}\mathfrak{A}}) \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^{\alpha})_{\alpha=1}^{32} \\ (e^{a})_{a=1}^{9} \\ (\tilde{e}_{a})_{a=1}^{9} \\ \left(e_{a_1a_2} = e_{[a_1a_2]}\right)_{a_i=0}^{9} \\ \left(e_{a_1\cdots a_4} = e_{[a_1\cdots a_4]}\right)_{a_i=0}^{9} \\ \left(e_{a_1\cdots a_5} = e_{[a_1\cdots a_5]}\right)_{a_i=0}^{9} \\ (\phi^{\alpha})_{\alpha=1}^{32} \end{array}\right] \Bigg/ \left(\begin{array}{c} \mathrm{d}\psi = 0 \\ \mathrm{d}e^{a} = +(\bar{\psi}\Gamma^{a}\psi) \\ \mathrm{d}\tilde{e}_{a} = -(\bar{\psi}\Gamma_{a}\Gamma_{10}\psi) \\ \mathrm{d}e_{a_1a_2} = -(\bar{\psi}\Gamma_{a_1a_2}\psi) \\ \mathrm{d}e_{a_1\cdots a_4} = +(\bar{\psi}\Gamma_{a_1\cdots a_4}\Gamma_{10}\psi) \\ \mathrm{d}e_{a_1\cdots a_5} = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi) \\ \mathrm{d}\phi = \Gamma_{a_1a_2}\psi e^{a_1a_2} + 2\Gamma_{a10}\psi\tilde{e}^{a} \\ +\frac{10}{6!}\Gamma_{a_1\cdots a_5}\psi e^{a_1\cdots a_5} \\ +\frac{50}{6!}\Gamma_{a_1\cdots a_4 10}\psi e^{a_1\cdots a_4} \end{array}\right)$$

$$\begin{aligned} \mathfrak{M}_{\mathrm{bos}} &\simeq_{\mathbb{R}} \mathbb{R}^{1,10} \oplus \wedge^2 (\mathbb{R}^{1,10})^* \oplus \wedge^5 (\mathbb{R}^{1,10})^* \\ &\simeq_{\mathbb{R}} \mathbb{R} \oplus \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^* \oplus \wedge^2 (\mathbb{R}^{1,9})^* \oplus \wedge^4 (\mathbb{R}^{1,9})^* \oplus \wedge^5 (\mathbb{R}^{1,9})^* \\ &\simeq_{\mathbb{R}} \mathbb{R} \oplus (\mathrm{II}\mathfrak{A})_{\mathrm{bos}} \end{aligned}$$

$$\wedge^p \left(\mathbb{R}^{1,d}\right)^* \simeq_{\mathbb{R}} \wedge^{p-1} \left(\mathbb{R}^{1,d-1}\right)^* \oplus \wedge^p \left(\mathbb{R}^{1,d-1}\right)^*$$

$$\begin{tikzcd} \widehat{\mathfrak{M}} \arrow[r, two heads] \arrow[d, two heads] & \widehat{\mathrm{II}\mathfrak{A}} \arrow[r, "(\bar{\psi}\Gamma^{10}\psi)"] \arrow[d, two heads] & b\mathbb{R} \arrow[d, equal] \\ \mathfrak{M} \arrow[r, two heads] & \mathrm{II}\mathfrak{A} \arrow[r, "(\bar{\psi}\Gamma^{10}\psi)"] & b\mathbb{R} \end{tikzcd}$$

$$\begin{array}{rcccl} & \psi & \longleftarrow\!\shortmid & \psi & \\ & e^{a} & \longleftarrow\!\shortmid & e^{a} & \\ \text{wrapped M2-brane charges} & e_{a10} & \longleftarrow\!\shortmid & \tilde{e}_{a} & \text{string charges / doubled spacetime} \\ & e_{a_1a_2} & \longleftarrow\!\shortmid & e_{a_1a_2} & \\ & e_{a_1\cdots a_4 10} & \longleftarrow\!\shortmid & e_{a_1\cdots a_4} & \\ & e_{a_1\cdots a_5} & \longleftarrow\!\shortmid & e_{a_1\cdots a_5}\,. & \end{array}$$

$$\mathrm{CE}(\mathfrak{Brn}) \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^{\alpha})_{\alpha=1}^{32} \\ \left(e_{a_1a_2} = e_{[a_1a_2]}\right)_{a_i=0}^{10} \\ \left(e_{a_1\cdots a_5} = e_{[a_1\cdots a_5]}\right)_{a_i=0}^{10} \end{array}\right] \Bigg/ \left(\begin{array}{ll} \mathrm{d}\psi & = 0 \\ \mathrm{d}e_{a_1a_2} & = -(\bar{\psi}\Gamma_{a_1a_2}\psi) \\ \mathrm{d}e_{a_1\cdots a_5} & = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi) \end{array}\right)$$

$$\mathrm{CE}(\widehat{\mathfrak{Brn}}) \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^{\alpha})_{\alpha=1}^{32} \\ \left(e_{a_1a_2} = e_{[a_1a_2]}\right)_{a_i=0}^{10} \\ \left(e_{a_1\cdots a_5} = e_{[a_1\cdots a_5]}\right)_{a_i=0}^{10} \\ (\phi^{\alpha})_{\alpha=1}^{32} \end{array}\right] \Bigg/ \left(\begin{array}{ll} \mathrm{d}\psi & = 0 \\ \mathrm{d}e_{a_1a_2} & = -(\bar{\psi}\Gamma_{a_1a_2}\psi) \\ \mathrm{d}e_{a_1\cdots a_5} & = +(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi) \\ \mathrm{d}\phi & = \Gamma^{a_1a_2}\psi e_{a_1a_2} + \frac{10}{6!}\Gamma^{a_1\cdots a_5}\psi e_{a_1\cdots a_5} \end{array}\right)$$

$$\mathrm{Der}(\mathrm{CE}(\widehat{\mathfrak{M}})) \simeq \mathrm{CE}(\widehat{\mathfrak{M}})\langle \underbrace{\partial_\psi}_{(-1,\,\mathrm{odd})}, \underbrace{\partial_{e^a}}_{(-1,\,\mathrm{evn})}, \underbrace{\partial_{e_{a_1a_2}}}_{(-1,\,\mathrm{evn})}, \underbrace{\partial_{e_{a_1\cdots a_5}}}_{(-1,\,\mathrm{evn})}, \underbrace{\partial_\phi}_{(-1,\,\mathrm{odd})} \rangle.$$

$$\begin{aligned} \mathrm{d} = {} & (\bar\psi\Gamma^a\psi)\partial_{e^a} - (\bar\psi\Gamma_{a_1a_2}\psi)\partial_{e_{a_1a_2}} + (\bar\psi\Gamma_{a_1\cdots a_5}\psi)\partial_{e_{a_1\cdots a_5}} \\ & + \left(\delta\Gamma_a\psi e^a + \gamma_1\Gamma_{a_1a_2}\psi e^{a_1a_2} + \gamma_2\Gamma_{a_1\cdots a_5}\psi e^{a_1\cdots a_5}\right)\partial_\phi \end{aligned}$$

$$p_*^M : \mathrm{CE}(\widehat{\mathfrak{M}}) \longrightarrow \mathrm{CE}(\widehat{\mathrm{II}\mathfrak{A}})$$

$$p_*^M = \partial_{e^{10}}$$

$$p_*^M\phi_{\mathrm{ex}}^* G_4 \equiv p_*^M\left(\frac{1}{2}(\bar\psi\Gamma_{a_1a_2}\psi)e^{a_1}e^{a_2}\right) = -\underbrace{\sum_{a<10}(\bar\psi\Gamma_{a10}\psi)e^a}_{H_3^A}$$

$$\underbrace{p_*^M\hat P_3}_{\hat P_2} = -2\alpha_0\underbrace{\sum_{a<10} e_{a10}e^a}_{P_2} + \alpha_3\epsilon_{a_1\cdots a_5b_1\cdots b_5 10}e^{a_1\cdots a_5}e^{b_1\cdots b_5} + \beta_1(\bar\psi\Gamma_{10}\phi),$$

$$p_*^M\left(-\frac{1}{2}e_{a_1a_2}e^{a_1}e^{a_2}\right) = \underbrace{\sum_{a<10} e_{10a}e^a}_{P_2} \underset{(19)}{=} e^a\tilde e_a$$

$$[\mathrm{d}, p_*^M] \equiv \mathrm{d}\circ p_*^M + p_*^M \circ \mathrm{d}$$

$$[\mathrm{d}, p_*^M] = -\delta(\Gamma_{10}\psi)\partial_\phi$$

$$\begin{aligned} [\mathrm{d}, p_*^M]\hat P_3 = {} & \beta_1\delta(\bar\psi\Gamma_a\Gamma_{10}\psi)e^a + \beta_2\delta\left(\bar\psi\Gamma_{a_1a_2}\Gamma_{10}\psi\right)e^{a_1a_2} + \beta_3\delta\left(\bar\psi\Gamma_{a_1\cdots a_5}\Gamma_{10}\psi\right)e^{a_1\cdots a_5} \\ = {} & \beta_1\delta\underbrace{\sum_{a<10}(\bar\psi\Gamma_{a10}\psi)e^a}_{H_3^A} - 2\beta_2\delta\underbrace{(-1)\sum_{a<10}(\bar\psi\Gamma^a\psi)e_{a10}}_{H_3^{\tilde A}} + \beta_3\delta\underbrace{\sum_{a_i<10}(\bar\psi\Gamma_{a_1\cdots a_5 10}\psi)e^{a_1\cdots a_5}}_{=:H_3^C} \end{aligned}$$

$$\begin{aligned} \mathrm{d}\hat P_2 & \equiv \mathrm{d}\left(p_*^M\hat P_3\right) \\ & = -p_*^M\,\mathrm{d}\hat P_3 + [\mathrm{d}, p_*^M]\hat P_3 \\ & = -p_*^M\phi_{\mathrm{ex}}^* G_4 + [\mathrm{d}, p_*^M]\hat P_3 \\ & = (1+\beta_1\delta)H_3^A - 2\beta_2\delta H_3^{\tilde A} + 2\beta_3\delta H_3^C \\ & = \begin{cases} H_3^A & \text{for } s = -1 \\ \dfrac{17}{12}H_3^A + \dfrac{1}{12}H_3^{\tilde A} & \text{for } s = -6 \\ \dfrac{2s^2}{5}H_3^A + \dfrac{3s^2}{5}H_3^{\tilde A} - \dfrac{6s^2}{5\cdot 5!}H_3^C & \text{for } s \to 0 \end{cases} \end{aligned}$$

dimensional reduction

$$\mathrm{d}\,\widehat{P}_3 = \phi_{\mathrm{ex}}^{*} G_4 \in \mathrm{CE}(\widehat{\mathfrak{M}})$$

$$\mathrm{d}\,\widehat{P}_2 = \phi_{\mathrm{ex}}^{*} H_3^A \in \mathrm{CE}(\widehat{\mathrm{II}\mathfrak{A}})$$

$$P_6 \in \mathrm{CE}(\widehat{\mathfrak{M}})^{\mathrm{Spin}(1,10)}$$

$$\mathrm{d}P_6 = \underbrace{\frac{1}{5!}(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)e^{a_1}\cdots e^{a_5}}_{G_7} - \frac{1}{2}\underbrace{\frac{1}{2}(\bar{\psi}\Gamma_{a_1a_2}\psi)e^{a_1}e^{a_2}}_{G_4}\underbrace{\left(\frac{1}{2}e^{a_1}e_{a_1a_2}e^{a_2} + \cdots\right)}_{\widehat{P}_3}.$$

$$\begin{aligned} P_6 := &\frac{1}{5!}e_{a_1\cdots a_5}e^{a_1}\cdots e^{a_5} \\ &+r(\bar{\psi}\Gamma_{a_1\cdots a_4}\phi)e^{a_1}\cdots e^{a_4} \\ &+\cdots, \end{aligned}$$

$$\begin{aligned} \mathrm{d}P_6 =&\left(\frac{1}{5!}(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)e^{a_1}\cdots e^{a_5} - \frac{1}{4!}e_b a_1\cdots a_4(\bar{\psi}\Gamma^b\psi)e^{a_1}\cdots e^{a_4}\right) \\ &+r(-\underbrace{(\bar{\psi}\Gamma_{a_1\cdots a_4}\Gamma^{b_1b_2}\psi)e_{b_1b_2}a^{a_1}\cdots e^{a_4}}_{\substack{(\bar{\psi}\Gamma_{a_1\cdots a_4b_1b_2}\psi)e^{b_1b_2}e^{a_1}\cdots e^{a_4} \\ -12\left((\bar{\psi}\Gamma_{a_1a_2}\psi)e^{a_1}e^{a_2}\right)e_{b_1b_2}e^{b_1}e^{b_2}}} - \underbrace{\frac{10}{6!}\Gamma_{a_1\cdots a_4}\Gamma^{b_1\cdots b_5}\psi\Big)e_{b_1\cdots b_1}e^{a_1}\cdots e^{a_4}}_{\substack{(\bar{\psi}\Gamma_{a_1\cdots a_4b_1\cdots b_5}\psi)e^{b_1\cdots b_5}e^{a_1}\cdots e^{a_4} \\ -120(\bar{\psi}\Gamma_{a_3a_4b_3b_4b_5}\psi)e^{c_1c_2b_3b_4b_5}e_{c_1}e_{c_2}e^{a_3}e^{a_4} \\ +120(\bar{\psi}\Gamma_{b_5}\psi)e^{c_1\cdots c_4b_5}e_{c_1}\cdots e_{c_4}}} \\ &+4(\bar{\psi}\Gamma_{ba_1a_2a_3}\phi)(\bar{\psi}\Gamma^b\psi)e^{a_1}e^{a_2}e^{a_3}) + \cdots, \end{aligned}$$

$$-r\frac{1200}{6!} - \frac{1}{4!} = 0 \Leftrightarrow r = -\frac{6!}{1200\cdot 4!} = -\frac{1}{40}.$$

$$\mathrm{d}P_6 = \frac{1}{5!}(\bar{\psi}\Gamma_{a_1\cdots a_5}\psi)e^{a_1}\cdots e^{a_5} - \underbrace{\frac{12\cdot 8}{40}}_{\neq 1}\frac{1}{2}(\underbrace{\frac{1}{2}(\bar{\psi}\Gamma_{a_1a_2}\psi)e^{a_1}e^{a_2}}_{G_4}\underbrace{\frac{1}{2}e^{b_1}e_{b_1b_2}e^{b_2}}_{\widehat{P}_3-\cdots}) + \cdots$$

$$\begin{aligned} \mathrm{d}\phi_{(0)} &= 2\left(\Gamma_a\psi e^a + \frac{1}{2}\Gamma_{a_1a_2}\psi e^{a_1a_2} + \frac{1}{5!}\Gamma_{a_1\cdots a_5}\psi e^{a_1\cdots a_5}\right) \\ \mathrm{d}\phi_{(-6)} &= -10\Gamma_a\psi e^a + \Gamma_{a_1a_2}\psi e^{a_1a_2}. \end{aligned}$$

$$\begin{aligned} 0 &= \Gamma_{ab}\psi(\bar{\psi}\Gamma^b\psi) + \Gamma^b\psi(\bar{\psi}\Gamma_{ab}\psi) \\ 0 &= \Gamma_{a_1\cdots a_4b}\psi(\bar{\psi}\Gamma^b\psi) - \Gamma_{[a_1a_2}\psi(\bar{\psi}\Gamma_{a_3a_4]}\psi) + 6\Gamma^b\psi(\bar{\psi}\Gamma_{a_1\cdots a_4b}\psi) \end{aligned}$$

$$\begin{aligned} 0 &= \delta'\Gamma_{ab}\psi(\bar{\psi}\Gamma^b\psi) - \gamma_1'\Gamma^b\psi(\bar{\psi}\Gamma_{ab}\psi) \\ 0 &= \delta''\Gamma_{a_1\cdots a_4b}\psi(\bar{\psi}\Gamma^b\psi) - \gamma_1''\Gamma_{[a_1a_2}\psi(\bar{\psi}\Gamma_{a_3a_4]}\psi) + \gamma_2''\Gamma^b\psi(\bar{\psi}\Gamma_{a_1\cdots a_4b}\psi). \end{aligned}$$

$$\begin{aligned}\Gamma_{ab}\psi(\bar\psi\Gamma^b\psi) &= \underbrace{\Gamma_{ab}}_{\Gamma_a\Gamma_b-\eta_{ab}}\left(\frac{1}{11}\Gamma^b\Xi^{(32)}+\Xi^{(320)b}\right)\\ &= \frac{10}{11}\Gamma_a\Xi^{(32)}-\Xi^{(320)}_a\\ \Gamma^b\psi(\bar\psi\Gamma_{ab}\psi) &= \Gamma^b\left(\frac{1}{11}\Gamma_{ab}\Xi^{(32)}-\frac{2}{9}\Gamma_{[a}\Xi^{(320)}_{b]}\right)\\ &= -\frac{10}{11}\Gamma_a\Xi^{(32)}-\frac{1}{9}(\Gamma^b\Gamma_a+\Gamma_a\underbrace{\Gamma^b)\Xi^{(320)}_b}_{0}+\frac{1}{9}\Gamma^b\Gamma_b\Xi^{(320)}_a\\ &= -\frac{10}{11}\Gamma_a\Xi^{(32)}+\Xi^{(320)}_a\end{aligned}$$

$$\begin{aligned}\left(\frac{10}{11}\delta'+\frac{10}{11}\gamma_1'\right)\Gamma_a\Xi^{(32)}&=0\\ (-\delta'-\gamma_1')\,\Xi^{(320)}_a&=0\end{aligned}$$

$$\begin{aligned}\Gamma_{a_1\cdots a_4 b}\psi(\bar\psi\Gamma^b\psi) &= \frac{1}{11}\Gamma_{a_1\cdots a_4 b}\Gamma^b\Xi^{(32)}+\Gamma_{a_1\cdots a_4 b}\Xi^{(320)b}\\ &= \frac{7}{11}\Gamma_{a_1\cdots a_4}\Xi^{(32)}-4\Gamma_{[a_1a_2}\Xi^{(320)}_{a_3a_4]},\\ \Gamma_{[a_1a_2}\psi(\bar\psi\Gamma_{a_3a_4]}\psi) &= \Gamma_{[a_1a_2}\left(\frac{1}{11}\Gamma_{a_3a_4]}\Xi^{(32)}-\frac{2}{9}\Gamma_{a_3}\Xi^{(320)}_{a_4]}+\Xi^{(1408)}_{a_3a_4]}\right)\\ &= \frac{1}{11}\Gamma_{a_1\cdots a_4}\Xi^{(32)}-\frac{2}{9}\Gamma_{[a_1a_2a_3}\Xi^{(320)}_{a_4]}+\Gamma_{[a_1a_2}\Xi^{(1408)}_{a_3a_4]},\\ \Gamma^b\psi(\bar\psi\Gamma_{a_1\cdots a_4 b}\psi) &= -\frac{1}{77}\Gamma^b\Gamma_{a_1\cdots a_4 b}\Xi^{(32)}+\frac{5}{9}\Gamma^b\Gamma_{[a_1\cdots a_4}\Xi^{(320)}_{b]}+2\Gamma^b\Gamma_{[a_1a_2a_3}\Xi^{(1408)}_{a_4b]}\\ &\quad -\frac{1}{11}\Gamma_{a_1\cdots a_4}\Xi^{(32)}+\frac{24}{9}\Gamma_{[a_1a_2a_3}\Xi^{(320)}_{a_4]}+6\Gamma_{[a_1a_2}\Xi^{(1408)}_{a_3a_4]},\end{aligned}$$

$$\begin{aligned}\left(\frac{7}{11}\delta''-\frac{1}{11}\gamma_1''-\frac{1}{11}\gamma_2''\right)\Gamma_{a_1\cdots a_4}\Xi^{(32)}&=0\\ \left(-4\delta''+\frac{2}{9}\gamma_1''+\frac{24}{9}\gamma_2''\right)\Gamma_{[a_1a_2a_3}\Xi^{(320)}_{a_4]}&=0\\ (-\gamma_1''+6\gamma_2'')\Gamma_{[a_1a_2}\Xi^{(1408)}_{a_3a_4]}&=0\end{aligned}$$

$$\mathrm{d}\big(\Gamma_{ab}e\Gamma^b\psi+\Gamma^b e\Gamma_{ab}\psi\big)=0.$$

$$(\psi^\alpha_a)_{\substack{\alpha\in\{1,\dots,32\}\\ a\in\{0,1,\dots,10\}}}\ \text{ in deg}=(1,\text{odd})$$

$$\mathrm{d}\psi_a=\frac{1}{16}\big(\Gamma_{ab}e\Gamma^b\psi+\Gamma^b e\Gamma_{ab}\psi\big).$$

$$\begin{aligned}&\mathrm{d}\psi_a\\ &=\Gamma_{ab}e\Gamma^b\psi+\Gamma^b e\Gamma_{ab}\psi\\ &=\frac{1}{16}\Gamma_{ab}\left(\Gamma_c\psi e^c+\frac{1}{2}\Gamma_{c_1c_2}\psi e^{c_1c_2}+\frac{1}{5!}\Gamma_{c_1\cdots c_5}\psi e^{c_1\cdots c_5}\right)\Gamma^b\psi+\frac{1}{16}\Gamma^b\left(\Gamma_c\psi e^c+\frac{1}{2}\Gamma_{c_1c_2}\psi e^{c_1c_2}+\frac{1}{5!}\Gamma_{c_1\cdots c_5}\psi e^{c_1\cdots c_5}\right)\Gamma_{ab}\psi\\ &=\Gamma_{ac}\psi e^c-\Gamma_c\psi e^{ac}+0\end{aligned}$$

$$\begin{aligned}\mathrm{d}(\Gamma^a\psi_a)&=16\Gamma^a(\Gamma_{ac}\psi e^c-\Gamma_c\psi e^{ac})\\ &=16(10\Gamma_c\psi e^c-\Gamma_{ac}\psi e^{ac})\end{aligned}$$

$$(\bar{\psi}\Gamma^{ab}\psi_a)e_b - (\bar{\psi}\Gamma_b\psi_a)e^{ab} \in \mathrm{CE}(\widehat{\mathfrak{M}}).$$

$$\begin{aligned}
&\mathrm{d}\Big((\bar{\psi}_a\Gamma^{ab}\psi)e_b - (\bar{\psi}_a\Gamma_b\psi)e^{ab}\Big)\\
&= \underbrace{\Big((\bar{\psi}_a\Gamma^{ab}\psi)(\bar{\psi}\Gamma_b\psi) + (\bar{\psi}_a\Gamma_b\psi)(\bar{\psi}\Gamma^{ab}\psi)\Big)}_{=0} - \Big((\bar{\psi}\Gamma^{ab}\,\mathrm{d}\psi_a)e_b - (\bar{\psi}\Gamma_b\,\mathrm{d}\psi_a)e^{ab}\Big)
\end{aligned}$$

$$\begin{aligned}
\hat{P}_3 :=\; &\alpha_0 e_{a_1a_2}e^{a_1}e^{a_2}\\
&+\alpha_1 e^{a_1}{}_{a_2}e^{a_2}{}_{a_3}e^{a_3}{}_{a_1}\\
&+\alpha_2 e^{a_1\cdots a_4 b_1}e_{b_1}^{b_2}e_{b_2a_1\cdots a_4}\\
&+\alpha_3 \epsilon_{a_1\cdots a_5 b_1\cdots b_5 c}e^{a_1\cdots a_5}e^{b_1\cdots b_5}e^{c}\\
&+\alpha_4 \epsilon_{a_1a_2a_3b_1b_2b_3c_1\cdots c_5}e^{a_1a_2a_3d_1d_2}e_{d_1d_2}^{b_1b_2b_3}e^{c_1\cdots c_5}\\
&+\beta_1(\bar{\psi}\Gamma_a\phi)e^{a}\\
&+\beta_2\big(\bar{\psi}\Gamma_{a_1a_2}\phi\big)e^{a_1a_2}\\
&+\beta_3\big(\bar{\psi}\Gamma_{a_1\cdots a_5}\phi\big)e^{a_1\cdots a_5}\\
&+\beta_1'\Big((\bar{\psi}\Gamma^{ab}\psi_a)e_b - (\bar{\psi}\Gamma_b\psi)e^{ab}\Big)
\end{aligned}$$

$$\begin{aligned}
\alpha_0 &= -1/20\\
\alpha_1 &= -1/60\\
\alpha_2 &= 0\\
\alpha_3 &= 0\\
\alpha_4 &= 0\\
\beta_1 &= 0\\
\beta_2 &= 0\\
\beta_3 &= 0\\
\beta_1' &= -1/20
\end{aligned}$$

$$\begin{aligned}
&\mathrm{d}\Big((\bar{\psi}\Gamma^{ab}\psi_a)e_b - (\bar{\psi}\Gamma_b\psi_a)e^{ab}\Big)\\
&= -(\bar{\psi}\Gamma^{ab}\,\mathrm{d}\psi_a)e_b + (\bar{\psi}\Gamma_b\,\mathrm{d}\psi_a)e^{ab})\\
&= -\Big(\bar{\psi}\Gamma^{ab}(\Gamma_{ac}\psi e^{c} - \Gamma^{c}\psi e_{ac})\Big)e_b + \Big(\bar{\psi}\Gamma_b(\Gamma_{ac}\psi e^{c} - \Gamma^{c}\psi e_{ac})\Big)e^{ab}\Big) = -9(\bar{\psi}\Gamma_{bc}\psi)e^{b}e^{c} + (\bar{\psi}\Gamma^{a}\psi)e_{ab}e^{b} + (\bar{\psi}\Gamma_a\psi)e_{ab}e^{b} + (\bar{\psi}\Gamma^{bc}\psi)e_{ca}e_b^{a}\\
&= -(\bar{\psi}\Gamma^{ab}\Gamma_{ac}\psi)e^{c}e_b + (\bar{\psi}\Gamma^{ab}\Gamma^{c}\psi)e_{ac}e_b + (\bar{\psi}\Gamma_b\Gamma_{ac}\psi)e^{c}e^{ab} - (\bar{\psi}\Gamma_b\Gamma^{c}\psi)e_{ac}e^{ab}\\
&= -(-9)(\bar{\psi}\Gamma_{bc}\psi)e^{c}e^{b} + (\bar{\psi}\Gamma^{a}\psi)e_{ac}e^{c} - (\bar{\psi}\Gamma_a\psi)e_b e^{ab} - (\bar{\psi}\Gamma^{bc}\psi)e_{ac}e_b^{a}
\end{aligned}$$

$$\begin{aligned}
(\bar{\psi}\Gamma^{ab}\Gamma^{c}\psi)e_{ac}e_b &= \big(\bar{\psi}(\eta^{bc}\Gamma^{a} - \eta^{ac}\Gamma^{b} + \Gamma^{abc})\psi\big)e_{ac}e_b\\
&= \big(\bar{\psi}\eta^{bc}\Gamma^{a}\psi\big)e_{ac}e_b
\end{aligned}$$

$$\mathrm{d}\,\widehat{P}_3 \;=\; \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)\,e^{a_1}\,e^{a_2} \quad\Leftrightarrow\quad \left\{
\begin{aligned}
-\alpha_0 + \delta\,\beta_1 \boxed{-\,9\,\beta_1'} &= \tfrac{1}{2}\\
-2\,\alpha_0 \;+\; 2\,\gamma_1\beta_1 \;+\; 2\,\delta\beta_2 \boxed{+\,2\beta_1'} &= 0\\
-3\,\alpha_1 \;-\; 4\gamma_1\beta_2 \boxed{+\,\beta_1'} &= 0\\
2\,\alpha_2 + 10\,\gamma_2\,\beta_2 + 10\,\gamma_1\beta_3 &= 0\\
\alpha_2 \;+\; 600\,\gamma_2\,\beta_3 &= 0\\
2\,\alpha_3 + \tfrac{\gamma_2}{5!}\,\beta_1 + \tfrac{\delta}{5!}\,\beta_3 &= 0\\
\alpha_3 \;+\; \gamma_2\,\beta_3 &= 0\\
3\,\alpha_4 \;-\; \tfrac{200}{5!}\gamma_2\,\beta_3 &= 0\\
\beta_1 + 10\cdot\beta_2 - 6!\cdot\beta_3 &= 0\,,
\end{aligned}\right.$$

$$\begin{array}{ccc} \mathrm{sSmthMfd} & \xhookrightarrow{C^\infty(-)} & \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}} \\ X \equiv B|V_{\mathrm{odd}} & \longmapsto & \wedge^{\bullet}_{C^\infty(B)}\Gamma_B(V^*) = \Gamma_B(\wedge^{\bullet}_B V^*) \end{array}$$

$$\{f: X^{(1)} \to X^{(2)}\} \simeq \{C^\infty(X^{(1)}) \leftarrow C^\infty(X^{(2)}): f^*\}.$$

$$\begin{tikzcd} \mathrm{SmthMfd} \arrow[r, hook] \arrow[d, hook, "C^\infty(-)"] & \mathrm{sSmthMfd} \arrow[d, hook, "C^\infty(-)"] \\ \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}} \arrow[r, hook] & \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}} \end{tikzcd}$$

$$C^\infty(\mathbb{R}^{0|q}) := \wedge^{\bullet}_{\mathbb{R}}(\mathbb{R}^q)^* \simeq \mathbb{R}[\vartheta^1, \cdots, \vartheta^q], \quad \underset{i}{\forall} \ \deg(\vartheta^i) = \mathrm{odd}$$

$$\vartheta^{i_1 i_2 \cdots i_n} := \vartheta^{i_1}\vartheta^{i_2} \cdots \vartheta^{i_n} = \epsilon^{i_1 i_2 \cdots i_n}\vartheta^1\vartheta^2 \cdots \vartheta^n \in C^\infty(\mathbb{R}^{0|q}).$$

$$\mathrm{sPnt} \hookrightarrow \mathrm{sMfd}$$

$$\mathbb{R}^{p|q} = \mathbb{R}^p \times \mathbb{R}^{0|q}$$

$$C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^{0|q}) \simeq C^\infty(\mathbb{R}^p)[\vartheta^1, \cdots, \vartheta^q].$$

$$T_{\mathrm{odd}}X := X|TX, \qquad C^\infty(T_{\mathrm{odd}}X) = \Omega^{\bullet}_{\mathrm{dR}}(X)$$

$$\begin{array}{ccc} \mathbb{R}^{0|0} & \xrightarrow{x_0} & X \\ \mathbb{R} & \xleftarrow{(x_0)^*} & C^\infty(\mathbb{R}^d) \otimes \mathbb{R}[\theta^1, \cdots, \theta^N] \\ x_0^a & \longleftarrow\!\!| & x^a \\ 0 & \longleftarrow\!\!| & \theta^\alpha \end{array}$$

$$
\begin{array}{ccc}
\mathbb{R}^{0|1} & \xrightarrow{\quad (x_0,\theta_1) \quad} & X \\
\mathbb{R}[\vartheta^1] & \xleftarrow{\quad (x_0,\theta_1)^* \quad} & C^\infty(\mathbb{R}^d)\otimes\mathbb{R}[\theta^1,\cdots,\theta^N] \\
x_0^a & \longmapsfrom & x^a \\
\theta_1^\alpha\vartheta^1 & \longmapsfrom & \theta^\alpha
\end{array}
$$

$$
C^\infty\big(\overset{\rightsquigarrow}{T}_{\mathrm{odd}}\,\mathbb{R}^{d|N}\big) \;\simeq\; C^\infty\big(\mathbb{R}^{d+N}\big)
$$

$$
\begin{tikzcd}
\mathbb{R}^{0\,|\,1} \arrow[r, "{(x_0,\theta_1)}"] \arrow[rr, bend left, "{(x'_0,\theta'_1)\,\equiv\,f_*(x_0,\theta_1)}"] & X \arrow[r, "f"] & X'
\end{tikzcd}
$$

$$
\underbrace{f^\alpha_{\beta_1}(x_0)}_{\text{Only linear contribution is seen on this super-point}}\theta_1^{\beta_1}\cdot\vartheta^1 \quad\longmapsfrom\quad \underbrace{\textstyle\sum_k f^\alpha_{\beta_1\cdots\beta_{2k+1}}(x)}_{\text{Full polynomial effect of map on odd coordinates}}\cdot\,\theta^{\beta_1\cdots\beta_{2k+1}} \quad\longmapsfrom\quad \theta'^{\,\alpha}
$$

$$
\begin{array}{ccc}
\mathbb{R}^{0|q} & \xrightarrow{\quad \left(x_{i_1\cdots i_{2k}},\,\theta_{i_1\cdots i_{2k+1}}\right)_{k\leq q/2} \quad} & X \\
\mathbb{R}[\vartheta^1,\cdots,\vartheta^q] & \longleftarrow & C^\infty(\mathbb{R}^d)\otimes\mathbb{R}[\theta^1,\cdots,\theta^N] \\
\sum_k x^a_{i_1\cdots i_{2k}}\vartheta^{i_1\cdots i_{2k}} & \longmapsfrom & x^a \\
\sum_k \theta^\alpha_{i_1\cdots i_{2k+1}}\vartheta^{i_1\cdots i_{2k+1}} & \longmapsfrom & \theta^\alpha
\end{array}
$$

$$
C^\infty\left(\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}}\,\mathbb{R}^{d|N}\right) \;\simeq\; C^\infty\left(\mathbb{R}^{\left(d\sum_k\binom{q}{2k}+N\sum_k\binom{q}{2k+1}\right)}\right)
$$

$$
\begin{tikzcd}
\mathbb{R}^{0|q} \arrow[r, "{\left(x_{i_1\cdots i_{2k}},\,\theta_{i_1\cdots i_{2k+1}}\right)_{k=0}^{\lfloor q/2\rfloor}}"] \arrow[rr, bend left, "{f_*\left(x_{i_1\cdots i_{2k}},\,\theta_{i_1\cdots i_{2k+1}}\right)_{k=0}^{\lfloor q/2\rfloor}}"] & \mathbb{R}^{d|N} \arrow[r, "f"] & \mathbb{R}^{d|N}
\end{tikzcd}
$$

$$
\begin{array}{rcccc}
\begin{aligned}
&f^a_b\textstyle\sum_{k=0}^{\lfloor q/2\rfloor} x^b_{i_1\cdots i_{2k}}\vartheta^{i_1\cdots i_{2k}} \\
&+f^a_{b_1b_2}\textstyle\sum_{k=0}^{\lfloor q/2\rfloor}\sum_{k'=0}^{k} x^{b_1}_{i_1\cdots i_{2k'}}x^{b_2}_{i_{2k'+1}\cdots i_{2k}}\vartheta^{i_1\cdots i_{2k}} \\
&+f^a_{\beta_1\beta_2}\textstyle\sum_{k=0}^{\lfloor q/2\rfloor}\sum_{k'=0}^{k-1}\theta^{\beta_1}_{i_1\cdots i_{2k'+1}}\theta^{\beta_2}_{i_{2k'+2}\cdots i_{2k}}\vartheta^{i_1\cdots i_{2k}}
\end{aligned}
& \longmapsfrom & f^a_b x^b + f^a_{b_1b_2}x^{b_1}x^{b_2} + f^a_{\beta_1\beta_2}\theta^{\beta_1}\theta^{\beta_2} & \longmapsfrom & x^a \\[2ex]
\begin{aligned}
&f^a_\beta\textstyle\sum_{k=1}^{\lfloor q/2\rfloor}\theta^\beta_{i_1\cdots i_{2k+1}}\vartheta^{i_1\cdots i_{2k+1}} \\
&+f^\alpha_{b\beta}\textstyle\sum_{k=1}^{\lfloor q/2\rfloor}\sum_{k'=1}^{k} x^b_{i_1\cdots i_{2k'}}\theta^\beta_{i_{2k'+1}\cdots i_{2k+1}}\vartheta^{i_1\cdots i_{2k+1}}
\end{aligned}
& \longmapsfrom & f^\alpha_\beta\theta^\beta + f^\alpha_{b\beta}x^b\theta^\beta & \longmapsfrom & \theta^\alpha\,.
\end{array}
$$

$$
\begin{tikzcd}
\mathrm{sSmthMfd} \arrow[rr, "\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}}(-)"] & & \mathrm{SmthMfd} & & \\
X \arrow[d, "f"] & \longmapsto & \overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} X \arrow[d, "\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} f"] & x^a_{i_1 \cdots i_{2k}} \arrow[d, mapsto] & \theta^\alpha_{i_1 \cdots i_{2k+1}} \arrow[d, mapsto] \\
Y & \longmapsto & \overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} Y & f_* x^a_{i_1 \cdots i_{2k}} & f_* \theta^\alpha_{i_1 \cdots i_{2k+1}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\phi^j_i \vartheta^i & \mathrm{sPnt}^{\mathrm{op}} \arrow[rr, "\overset{\rightsquigarrow}{T}{}^{(-)}_{\mathrm{odd}} X"] & & \mathrm{SmthMfd} & & \\
 & C^\infty(\mathbb{R}^{0|q}) & \longmapsto & C^\infty(\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} X) \arrow[d, "\phi^*"] & x^a_{i_1 \cdots i_{2k}} \arrow[d, mapsto] & \theta^\alpha_{i_1 \cdots i_{2k+1}} \arrow[d, mapsto] \\
\vartheta^j \arrow[uu, mapsto] & C^\infty(\mathbb{R}^{0|q'}) \arrow[u, "\phi"] & \longmapsto & C^\infty(\overset{\rightsquigarrow}{T}{}^{(q')}_{\mathrm{odd}} X) & x^a_{j_1 \cdots j_{2k}} \phi^{j_1}_{i_1} \cdots \phi^{j_{2k}}_{i_{2k}} & \theta^\alpha_{j_1 \cdots j_{2k}} \phi^{j_1}_{i_1} \cdots \phi^{j_{2k}}_{i_{2k}}
\end{tikzcd}
$$

$$
G \times G \xrightarrow{\mathrm{prd}} G \;, \qquad * \xrightarrow{\mathrm{e}} G \;, \qquad G \xrightarrow{\mathrm{inv}} G
$$

Associativity

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{prd} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mathrm{prd}"] & G \times G \arrow[d, "\mathrm{prd}"] \\
G \times G \arrow[r, "\mathrm{prd}"] & G
\end{tikzcd}
$$

Unitality

$$
\begin{tikzcd}
G \arrow[r, "\sim"] \arrow[d, "\wr"] & G \times * \arrow[r, "\mathrm{id} \times \mathrm{e}"] & G \times G \arrow[dd, "\mathrm{prd}"] \\
* \times G \arrow[d, "\mathrm{e} \times \mathrm{id}"] & & \\
G \times G \arrow[rr, "\mathrm{prd}"] & & G \arrow[from=1-1, equal]
\end{tikzcd}
$$

Invertibility

$$
\begin{tikzcd}
G \arrow[r, shift left, "(\mathrm{id},\mathrm{inv})"] \arrow[r, shift right, "(\mathrm{inv},\mathrm{id})"'] \arrow[d, "\exists!"] & G \times G \arrow[d, "\mathrm{prd}"] \\
* \arrow[r, "\mathrm{e}"] & G
\end{tikzcd}
$$

$$
\mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1
$$

$$
\begin{tikzcd}
\mathbb{R} \times \mathbb{R} \arrow[r, "+"] & \mathbb{R} \\
\dot{x} + x & x \,. \arrow[l, mapsto]
\end{tikzcd}
$$

$$
C^\infty(\mathbb{Z}) \simeq \mathbb{R}^{\mathbb{Z}} \simeq \{f \equiv (f(n) \in \mathbb{R})_{n \in \mathbb{Z}}\}
$$

$$
(f \cdot g)(n) := f(n) \cdot g(n), (f + g)(n) := f(n) + g(n).
$$

$$
x(n) := n.
$$

$$
\begin{array}{ccc}
\mathbb{Z}\times\mathbb{Z} & \xrightarrow{\;+\;} & \mathbb{Z} \\
\dot{x}+x & \longleftarrow\!\shortmid & x\,,
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{Z} & \hookrightarrow & \mathbb{R} \\
x & \longleftarrow\!\shortmid & x\,,
\end{array}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{po})"] & \mathbb{R} \arrow[d, two heads] \\
1 \arrow[r, hook] & S^1
\end{tikzcd}
\qquad
\begin{tikzcd}
C^\infty(\mathbb{Z}) & C^\infty(\mathbb{R}) \arrow[l, two heads] \\
\mathbb{R} \arrow[u, hook] \arrow[ur, phantom, "(\mathrm{pb})"] & C^\infty(\mathbb{R})_{\mathbf{prdc}} \arrow[l, two heads] \arrow[u, hook] = C^\infty(S^1)
\end{tikzcd}
$$

$$
\begin{array}{ccc}
\mathbb{R}^{1,10\,|\,\mathbf{32}}\times\mathbb{R}^{1,10\,|\,\mathbf{32}} & \xrightarrow{\mathrm{prd}} & \mathbb{R}^{1,10\,|\,\mathbf{32}} \\
x'^a + x^a - (\bar{\theta}'\,\Gamma^a\,\theta) & \xleftarrow{\mathrm{prd}^*} & x^a \\
\theta'+\theta & \xleftarrow{\mathrm{prd}^*} & \theta\,,
\end{array}
\qquad
\begin{array}{ccc}
* & \xrightarrow{\mathrm{e}} & \mathbb{R}^{1,10\,|\,\mathbf{32}} \\
0 & \xleftarrow{\mathrm{e}^*} & x^a \\
0 & \xleftarrow{\mathrm{e}^*} & \theta
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{R}^{1,10\,|\,\mathbf{32}} & \xrightarrow{\mathrm{inv}} & \mathbb{R}^{1,10\,|\,\mathbf{32}} \\
-x^a & \xleftarrow{\mathrm{inv}^*} & x^a \\
-\theta^\alpha & \xleftarrow{\mathrm{inv}^*} & \theta^\alpha
\end{array}
$$

$$
\begin{aligned}
e^a &:= \mathrm{d}x^a + (\bar{\theta}\Gamma^a\,\mathrm{d}\theta) \\
\psi &:= \mathrm{d}\theta
\end{aligned}
$$

$$
\begin{tikzcd}
C^\infty(\mathbb{R}^{1,10\,|\,\mathbf{32}})\,\widehat{\otimes}\,\Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^{1,10\,|\,\mathbf{32}}) & \Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^{1,10\,|\,\mathbf{32}})\,\widehat{\otimes}\,\Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^{1,10\,|\,\mathbf{32}}) \arrow[l, two heads] & \Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^{1,10\,|\,\mathbf{32}}) \arrow[l, "\mathrm{prd}^*"'] \arrow[ll, bend right, "\mathrm{act}^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{R}^{1,10\,|\,\mathbf{32}}\times T_{\mathrm{odd}}\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, hook] \arrow[rr, bend left, "\mathrm{act}"] & T_{\mathrm{odd}}\mathbb{R}^{1,10\,|\,\mathbf{32}}\times T_{\mathrm{odd}}\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, "\mathrm{prd}_*"] & T_{\mathrm{odd}}\mathbb{R}^{1,10\,|\,\mathbf{32}}.
\end{tikzcd}
$$

$$
\begin{aligned}
\mathrm{act}^*e^a &= \mathrm{act}^*\Big(\,\mathrm{d}x^a + (\bar{\theta}\Gamma^a\,\mathrm{d}\theta)\Big) \\
&= \mathrm{d}\,\mathrm{act}^*x^a + \big(\overline{\mathrm{act}^*\theta}\,\Gamma^a\mathrm{d}\,\mathrm{act}^*\theta\big) \\
&= \mathrm{d}\Big(x'^a + x^a - (\bar{\theta}'\Gamma^a\theta)\Big) + \Big((\bar{\theta}'+\bar{\theta})\Gamma^a\,\mathrm{d}(\theta'+\theta)\Big) \\
&= \mathrm{d}x^a - (\bar{\theta}'\Gamma^a\,\mathrm{d}\theta) + (\bar{\theta}'\Gamma^a\,\mathrm{d}\theta) + (\bar{\theta}\Gamma^a\,\mathrm{d}\theta) \\
&= \mathrm{d}x^a + (\bar{\theta}\Gamma^a\,\mathrm{d}\theta) \\
&= e^a
\end{aligned}
\qquad
\begin{aligned}
\mathrm{act}^*\psi &= \mathrm{act}^*\,\mathrm{d}\theta \\
&= \mathrm{d}\,\mathrm{act}^*\theta \\
&= \mathrm{d}(\theta'+\theta) \\
&= \mathrm{d}\theta \\
&= \psi
\end{aligned}
$$

Associativity

$$\begin{tikzcd}
x''^a + x'^a + (\overline{\theta}''\,\Gamma^a\,\theta') + x^a + (\overline{(\theta''+\theta')}\,\Gamma^a\,\theta) \arrow[d, equal] & & x'^a + x^a + (\overline{\theta}'\,\Gamma^a\,\theta) \arrow[ll, mapsto] \\
x''^a + x'^a + x_a + (\overline{\theta}'\,\Gamma^a\,\theta) + (\overline{\theta}''\,\Gamma^a\,(\theta'+\theta)) & & \\
x''^a + x^a + (\overline{\theta}''\,\Gamma^a\,\theta) \arrow[u, mapsto] & & x^a \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
\theta''+\theta'+\theta & \theta'+\theta \arrow[l, mapsto] \\
\theta''+\theta \arrow[u, mapsto] & \theta \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

Unitality

$$\begin{tikzcd}
x^a \arrow[drr, equal] & x'^a \arrow[l, mapsto] & x'^a + x^a + (\overline{\theta}'\,\Gamma^a\,\theta) \arrow[l, mapsto] \\
x'^a \arrow[u, mapsto] & & \\
x'^a + x^a + (\overline{\theta}'\,\Gamma^a\,\theta) \arrow[u, mapsto] & & x^a \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
\theta \arrow[drr, equal] & \theta' \arrow[l, mapsto] & \theta'+\theta \arrow[l, mapsto] \\
\theta \arrow[u, mapsto] & & \\
\theta'+\theta \arrow[u, mapsto] & & \theta \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}$$

Invertibility

$$\begin{tikzcd}
& x^a - x^a - (\overline{\theta}\,\Gamma^a\,\theta) & x'^a + x^a + (\overline{\theta}\,\Gamma^a\,\theta) \arrow[l, mapsto] \\
0 \arrow[ur, equal] & & \\
0 \arrow[u, mapsto] & & x^a \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \theta - \theta & \theta'+\theta \arrow[l, mapsto] \\
0 \arrow[ur, equal] & & \\
0 \arrow[u, mapsto] & & \theta \arrow[ll, mapsto] \arrow[uu, mapsto]
\end{tikzcd}$$

$$\mathfrak{g}_{(q)} := C^\infty\left(\mathbb{R}^{0|q}, \mathfrak{g}\right)_{\mathrm{evn}} := \left(C^\infty\left(\mathbb{R}^{0|q}\right) \otimes_{\mathbb{R}} \mathfrak{g}\right)_{\mathrm{evn}} \simeq \mathbb{R}\left\langle \vartheta^{i_1\cdots i_n} \otimes T \mid n \in \mathbb{N}, \begin{matrix} T \in \mathfrak{g}_{\mathrm{evn}} & \text{for } n \text{ even} \\ T \in \mathfrak{g}_{\mathrm{odd}} & \text{for } n \text{ odd} \end{matrix} \right\rangle$$

$$\left[\vartheta^{i_1\cdots i_n} T, \vartheta^{i'_1\cdots i'_n} T'\right] := \vartheta^{i_1\cdots i_n i'_1\cdots i'_n}[T, T']$$

$$\begin{tikzcd}
\mathfrak{g}_{(q)} & & \mathfrak{g}_{(r)} \arrow[ll, "f^*"'] \\
f^*\left(\vartheta^{i_1\cdots i_n}\right) \otimes T & & \vartheta^{i_1\cdots i_n} \otimes T \arrow[ll, mapsto]
\end{tikzcd}$$

$$\begin{aligned} \mathfrak{g}\colon \ \mathrm{sPnt}^{\mathrm{op}} &\longrightarrow \mathrm{LieAlg}_{\mathbb{R}} \\ \mathbb{R}^{0|q} &\longmapsto \mathfrak{g}_{(q)} \end{aligned}$$

$$\mathfrak{g} \in \mathrm{sLieAlg}^{\mathrm{nil}}_{\mathbb{R}} \ \Rightarrow \ \mathfrak{g}\colon \mathrm{sPnt}^{\mathrm{op}} \to \mathrm{LieAlg}^{\mathrm{nil}}_{\mathbb{R}}.$$

$$\mathrm{prd}(T_1, T_2) = T_1 + T_2 + \frac{1}{2}[T_1, T_2] + \frac{1}{12}\left([T_1, [T_1, T_2]] + [T_2, [T_2, T_1]]\right) + \frac{1}{2}\Big[T_2[T_1, [T_2, T_1]]\Big] + \cdots$$

$$\int \ : \ \mathrm{LieAlg}^{\mathrm{nil}}_{\mathbb{R}} \xrightarrow{\ \sim\ } \mathrm{LieGrp}^{\mathrm{unip}}$$

$$\mathbb{R}^{1,10|32}_{(2)} \simeq \mathbb{R}\left\langle (P_a)_{a=0}^{10}, (\vartheta^{12} P_a)_{a=0}^{10}, (\vartheta^1 Q_\alpha)_{\alpha=1}^{32}, (\vartheta^2 Q_\alpha)_{\alpha=1}^{32} \right\rangle$$

$$\left[\vartheta^i Q_\alpha, \vartheta^j Q_\alpha\right] = -2\Gamma^a_{\alpha\beta}\vartheta^{ij} P_a$$

$$\mathbb{R}^{1,10|\mathbf{32}}_{(2)} \simeq \left\{ \begin{array}{cc|rl} x^a & P_a & x^a \in \mathbb{R} & a \in \{0,1,\cdots,10\} \\ + x^a_{i_1 i_2} & \vartheta^{i_1 i_2} P_a & x^a_{i_1 i_2} = -x^a_{i_2 i_1} \in \mathbb{R} \;, & \alpha \in \{1,2,\cdots,32\} \\ + \;\theta^\alpha_i & \vartheta^i Q_\alpha & \theta^\alpha_i \in \mathbb{R} & i_1, i_2 \in \{1,2\} \end{array} \right\}$$

$$\begin{array}{ccc} \mathbb{R}^{1,10|\mathbf{32}}_{(2)} \times \mathbb{R}^{1,10|\mathbf{32}}_{(2)} & \xrightarrow{\quad \mathrm{prd}_{(2)} \quad} & \mathbb{R}^{1,10|\mathbf{32}}_{(2)} \\ \left(\begin{array}{cc} \dot{x}^a & P_a \\ + \dot{x}^a_{i_1 i_2} & \vartheta^{i_1 i_2} P_a \\ + \;\dot{\theta}^\alpha_i & \vartheta^i Q_\alpha \end{array} , \begin{array}{cc} x^b & P_b \\ + x^b_{j_1 j_2} & \vartheta^{j_1 j_2} P_b \\ + \;\theta^\beta_j & \vartheta^j Q_\beta \end{array} \right) & \longmapsto & \left(\begin{array}{lr} (\dot{x}^a + x^a) & P_a \\ + \left(\dot{x}^a_{ij} + x^a_{ij} - \dot{\theta}^\alpha_i \theta^\beta_j \, \Gamma^a_{\alpha\beta}\right) & \vartheta^{ij} P_a \\ + \left(\dot{\theta}^\alpha_i + \theta^\alpha_i\right) & \vartheta^i Q_\alpha \end{array} \right) \end{array}$$

$$\mathrm{prd}\left(\dot{\theta}^\alpha_i \vartheta^i Q_\alpha, \theta^\beta_j \vartheta^j Q_\beta\right) = \dot{\theta}^\alpha_i \vartheta^i Q_\alpha + \theta^\beta_j \vartheta^j Q_\beta + \dot{\theta}^\alpha_i \theta^\beta_j \frac{1}{2} \underbrace{\left[\vartheta^i Q_\alpha, \vartheta^j Q_\beta\right]}_{-2\Gamma^a_{\alpha\beta}\vartheta^{ij} P_a} + \underbrace{\cdots}_{0}.$$

$$\mathbb{R}^{1,10|\mathbf{32}}_{(q)} \simeq \mathbb{R}\left\langle \left(\vartheta^{i_1 \cdots i_{2k}} P_a\right)_{\substack{a \in \{0,\cdots,10\}, \\ 0 \leq k \leq q/2 \\ i_j \in \{1,\cdots,q\}}} \;,\; \left(\vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha\right)_{\substack{a \in \{0,\cdots,10\}, \\ o \leq k \leq (q-1)/2 \\ i_j \in \{1,\cdots,q\}}} \right\rangle$$

$$\left[\vartheta^{i_1 \cdots i_{2k'+1}} Q_\alpha, \vartheta^{j_1 \cdots j_{2k+1}} Q_\beta\right] = \Gamma^a_{\alpha\beta} \vartheta^{i_1 \cdots i_{2k'+1} j_1 \cdots j_{2k+1}} P_a,$$

$$\mathbb{R}^{1,10|\mathbf{32}}_{(q)} \simeq \left\{ \begin{array}{lr|rcll} \sum_k x^a_{i_1 \cdots i_{2k}} & \vartheta^{i_1 \cdots i_{2k}} P_a & x^a_{i_1 \cdots i_{2k}} & = & x^a_{[i_1 \cdots i_{2k}]} \in \mathbb{R} & a \in \{0,1,\cdots,10\} \\ + \sum_k \theta^\alpha_{i_1 \cdots i_{2k+1}} & \vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha & \theta^\alpha_{i_1 \cdots i_{2k+1}} & = & \theta^\alpha_{[i_1 \cdots i_{2k+1}]} \in \mathbb{R} \;, & \alpha \in \{1,2,\cdots,32\} \\ & & & & & i_j \in \{1,2,\cdots,q\} \end{array} \right\}$$

$$\begin{array}{ccc} \mathbb{R}^{1,10|\mathbf{32}}_{(q)} \times \mathbb{R}^{1,10|\mathbf{32}}_{(q)} & & \left(\begin{array}{lr} \sum_k \dot{x}^a_{i_1 \cdots i_{2k}} & \vartheta^{i_1 \cdots i_{2k}} P_a \\ + \sum_k \dot{\theta}^\alpha_{i_1 \cdots i_{2k+1}} & \vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha \end{array} , \begin{array}{lr} \sum_k x^a_{i_1 \cdots i_{2k}} & \vartheta^{i_1 \cdots i_{2k}} P_a \\ + \sum_k \theta^\alpha_{i_1 \cdots i_{2k+1}} & \vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha \end{array} \right) \\ \Big\downarrow {\scriptstyle \mathrm{prd}_{(q)}} & & \Big\downarrow \\ \mathbb{R}^{1,10|\mathbf{32}}_{(q)} & & \left(\begin{array}{lr} \sum_k \left(\dot{x}^a_{i_1 \cdots i_{2k}} + x^a_{i_1 \cdots i_{2k}} - \sum_{k=0}^{k-1} \dot{\theta}^\alpha_{i_1 \cdots i_{2k+1}} \theta^\beta_{i_{2k+2} \cdots i_{2k}} \Gamma^a_{\alpha\beta}\right) & \vartheta^{i_1 \cdots i_{2k}} P_a \\ + \sum_k \left(\dot{\theta}^\alpha_{i_1 \cdots i_{2k+1}} + \theta^\alpha_{i_1 \cdots i_{2k+1}}\right) & \vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha \end{array} \right) \end{array}$$

$$\begin{array}{ccc} C^\infty\left(\mathbb{R}^{1,10|\mathbf{32}}_{(q)} \times \mathbb{R}^{1,10|\mathbf{32}}_{(q)}\right) & \dot{x}^a_{i_1 \cdots a_{2k}} + x^a_{i_1 \cdots a_{2k}} - \sum_{k=0}^{k-1} \dot{\theta}^\alpha_{i_1 \cdots i_{2k+1}} \theta^\beta_{i_{2k+2} \cdots i_{2k}} \Gamma^a_{\alpha\beta} & \dot{\theta}^\alpha_{i_1 \cdots i_{2k+1}} + \theta^\alpha_{i_1 \cdots i_{2k+1}} \\ \Big\uparrow {\scriptstyle \mathrm{prd}^*_{(q)}} & \Big\uparrow & \Big\uparrow \\ C^\infty\left(\mathbb{R}^{1,10|\mathbf{32}}_{(q)}\right) & x^a_{i_1 \cdots i_{2k}} & \theta^\alpha_{i_1 \cdots i_{2k+1}} \end{array}$$

$$
\begin{array}{ccc}
\mathrm{sPnt}^{\mathrm{op}} & & C^\infty(\mathbb{R}^{0|q}) \xrightarrow{\quad \vartheta^i \,\mapsto\, \phi^i_j \vartheta^j \quad} C^\infty(\mathbb{R}^{0|r}) \\
\big\downarrow {\scriptstyle \mathbb{R}^{1,10|\mathbf{32}}} & & \big\downarrow \\
\mathrm{LieAlg}^{\mathrm{nil}}_{\mathbb{R}} & & \mathbb{R}^{1,10|\mathbf{32}}_{(q)} \xrightarrow[\vartheta^{i_1 \cdots i_{2k+1}} Q_\alpha \,\mapsto\, \left(\phi^{i_1}_{j_1} \cdots \phi^{i_{2k+1}}_{j_{2k+1}}\right) \vartheta^{j_1 \cdots j_{2k+1}} Q_\alpha]{\vartheta^{i_1 \cdots i_{2k}} P_a \,\mapsto\, \left(\phi^{i_1}_{j_1} \cdots \phi^{i_{2k}}_{j_{2k}}\right) \vartheta^{j_1 \cdots j_{2k}} P_a} \mathbb{R}^{1,10|\mathbf{32}}_{(r)} \\
\big\downarrow {\scriptstyle \int} & & \big\downarrow \\
\mathrm{LieGrp}^{\mathrm{unip}} & & \mathbb{R}^{1,10|\mathbf{32}}_{(q)} \xrightarrow{\quad x^a_{i_1 \cdots i_{2k}} \,\mapsto\, \left(\phi^{i_1}_{j_1} \cdots \phi^{i_{2k}}_{i_{jk}}\right) x^a_{j_1 \cdots j_{2k}} \quad} \mathbb{R}^{1,10|\mathbf{32}}_{(q)}
\end{array}
$$

$$
\begin{tikzcd}
 & \mathrm{LieGrp} \arrow[d, two heads] \\
\mathrm{sPnt}^{\mathrm{op}} \arrow[ur, dashed] \arrow[r] & \mathrm{LieAlg}_{\mathbb{R}} \\
\mathbb{R}^{0|q} \arrow[r, mapsto] & \mathbb{R}^{1,10|\mathbf{32}}_{(q)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Odd tangents of super-Lie group structure} \arrow[r, phantom, "\text{naturally isomorphic to}"] & \text{integration of system of Lie algebras of probes by any super-point} \\
C^\infty\left(\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} \mathbb{R}^{1,10|\mathbf{32}}\right) \arrow[r, "\sim"] \arrow[d, "\left(\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} \mathrm{prd}\right)^*"] & C^\infty\left(\mathbb{R}^{1,10|\mathbf{32}}_{(q)}\right) \arrow[d, "\mathrm{prd}^*_{(q)}"] \\
C^\infty\left(\overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} \mathbb{R}^{1,10|\mathbf{32}} \times \overset{\rightsquigarrow}{T}{}^{(q)}_{\mathrm{odd}} \mathbb{R}^{1,10|\mathbf{32}}\right)_{\mathrm{evn}} \arrow[r, "\sim"] & C^\infty\left(\mathbb{R}^{1,10|\mathbf{32}}_{(q)} \times \mathbb{R}^{1,10|\mathbf{32}}_{(q)}\right).
\end{tikzcd}
$$

$$
\begin{array}{ccccc}
\mathbb{R}^{0|2} & \xrightarrow{\left((x', x'_{12}, \theta'_1, \theta'_2),\, (x, x_{12}, \theta_1, \theta_2)\right)} & \mathbb{R}^{1,10|\mathbf{32}} \times \mathbb{R}^{1,10|\mathbf{32}} & \xrightarrow{\mathrm{prd}} & \mathbb{R}^{1,10|\mathbf{32}} \\
(x'^a + x'^a_{ij} \vartheta^{ij}) + (x^a + x^a_{ij} \vartheta^{ij}) - \Gamma^a_{\alpha\beta} (\theta'^{\alpha}_i \vartheta^i)(\theta^\beta_j \vartheta^j) & \longleftarrow & x'^a + x^a - \Gamma^a_{\alpha\beta}\, \theta'^\alpha \theta^\beta & \longleftarrow & x^a \\
= (x'^a + x^a) + \left(x'^a_{ij} + x^a_{ij} - \theta'^\alpha \theta^\beta \Gamma^a_{\alpha\beta}\right) \vartheta^{ij} & & & & \\
(\theta'^\alpha_i + \theta^\alpha_i) \vartheta^i & \longleftarrow & \theta'^\alpha + \theta^\alpha & \longleftarrow & \theta^\alpha
\end{array}
$$

$$
\widehat{\mathcal{M}}_{(q)} \;\simeq\; \left\{ \begin{array}{ll|ll} \sum_k x^a_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} P_a & x^a_{i_1\cdots i_{2k}} & \in \mathbb{R} \\ + \sum_k b^{a_1 a_2}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} Z_{a_1 a_2} & b^{a_1 a_2}_{i_1\cdots i_{2k}} & \in \mathbb{R} \\ + \sum_k b^{a_1\cdots a_5}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} Z_{a_1\cdots a_5} & b^{a_1\cdots a_5}_{i_1\cdots i_{2k}} & \in \mathbb{R} \\ + \sum_k \theta^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} Q_\alpha & \theta^\alpha_{i_1\cdots i_{2k+1}} & \in \mathbb{R} \\ + \sum_k \xi^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} O_\alpha & \xi^\alpha_{i_1\cdots i_{2k+1}} & \in \mathbb{R} \end{array} \;,\; \begin{array}{l} a_j \in \{0, \cdots, 10\} \\ \alpha \in \{0, \cdots, 32\} \\ i_j \in \{1, \cdots, q\} \\ 0 \leq k \leq q/2 \end{array} \right\}
$$

$$
\widehat{\mathcal{M}}_{(q)} \times \widehat{\mathcal{M}}_{(q)} \xrightarrow{\;\;\mathrm{prd}_{(q)}\;\;} \widehat{\mathcal{M}}_{(q)}
$$

$$
\left(\begin{array}{lll} \sum_k \dot{x}^a_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & P_a \\ + \sum_k \dot{b}^{a_1 a_2}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & Z_{a_1 a_2} \\ + \sum_k \dot{b}^{a_1\cdots a_5}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & Z_{a_1\cdots a_5} \\ + \sum_k \dot{\theta}^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} & Q_\alpha \\ + \sum_k \dot{\xi}^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} & O_\alpha \end{array} \;,\; \begin{array}{lll} \sum_k x^a_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & P_a \\ + \sum_k b^{a_1 a_2}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & Z_{a_1 a_2} \\ + \sum_k b^{a_1\cdots a_5}_{i_1\cdots i_{2k}} & \vartheta^{i_1\cdots i_{2k}} & Z_{a_1\cdots a_5} \\ + \sum_k \theta^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} & Q_\alpha \\ + \sum_k \xi^\alpha_{i_1\cdots i_{2k+1}} & \vartheta^{i_1\cdots i_{2k+1}} & O_\alpha \end{array} \right)
$$

$$
\Big\downarrow
$$

$$
\left(\begin{array}{ll} & \sum_k \Big(\dot{x}^a_{i_1\cdots i_{2k}} + \dot{x}^a_{i_{i_1}\cdots i_{2k}} - \sum_{\dot{k}=0}^{k-1} \dot{\theta}^\alpha_{i_1\cdots i_{2\dot{k}+1}} \theta^\beta_{i_{2\dot{k}+2}\cdots i_{2k}} \Gamma^a_{\alpha\beta} \Big) \;\; \vartheta^{i_1\cdots i_{2k}} P_a \\ + & \sum_k \Big(\dot{b}^{a_1 a_2}_{i_1\cdots i_{2k}} + b^{a_1 a_2}_{i_1\cdots i_{2k}} + \sum_{\dot{k}=0}^{k-1} \dot{\theta}^\alpha_{i_1\cdots i_{2\dot{k}+1}} \theta^\beta_{i_{2\dot{k}+2}\cdots i_{2k}} \Gamma^{a_1 a_2}_{\alpha\beta} \Big) \;\; \vartheta^{i_1\cdots i_{2k}} Z_{a_1 a_2} \\ + & \sum_k \Big(\dot{b}^{a_1\cdots a_5}_{i_1\cdots i_{2k}} + b^{a_1\cdots a_5}_{i_1\cdots i_{2k}} - \sum_{\dot{k}=0}^{k-1} \dot{\theta}^\alpha_{i_1\cdots i_{2\dot{k}+1}} \theta^\beta_{i_{2\dot{k}+2}\cdots i_{2k}} \Gamma^{a_1\cdots a_5}_{\alpha\beta} \Big) \;\; \vartheta^{i_1\cdots i_{2k}} Z_{a_1\cdots a_5} \\ + & \sum_k \Big(\dot{\theta}^\alpha_{i_1\cdots i_{2k+1}} + \dot{\theta}^\alpha_{i_1\cdots i_{2k+1}} \Big) \vartheta^{i_1\cdots i_{2k+1}} Q_\alpha \\ + \sum_k & \left(\dot{\xi}^\alpha_{i_1\cdots i_{2k+1}} + \xi^\alpha_{i_1\cdots i_{2k+1}} \begin{array}{l} + \sum_{\dot{k}=0}^{k} \dot{x}^a_{i_1\cdots i_{2\dot{k}}} \theta^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\delta}{2} \Gamma_a{}^\alpha{}_\beta \\ + \sum_{\dot{k}=0}^{k} \dot{b}^{a_1 a_2}_{i_1\cdots i_{2\dot{k}}} \theta^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\gamma_1}{2} \Gamma_{a_1 a_2}{}^\alpha{}_\beta \\ + \sum_{\dot{k}=0}^{k} \dot{b}^{a_1\cdots a_5}_{i_1\cdots i_{2\dot{k}}} \theta^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\gamma_2}{2} \Gamma_{a_1\cdots a_5}{}^\alpha{}_\beta \\ - \sum_{\dot{k}=0}^{k} x^a_{i_1\cdots i_{2\dot{k}}} \dot{\theta}^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\delta}{2} \Gamma_a{}^\alpha{}_\beta \\ - \sum_{\dot{k}=0}^{k} b^{a_1 a_2}_{i_1\cdots i_{2\dot{k}}} \dot{\theta}^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\gamma_1}{2} \Gamma_{a_1 a_2}{}^\alpha{}_\beta \\ - \sum_{\dot{k}=0}^{k} b^{a_1\cdots a_5}_{i_1\cdots i_{2\dot{k}}} \dot{\theta}^\beta_{i_{2\dot{k}+1}\cdots i_{2k+1}} \frac{\gamma_2}{2} \Gamma_{a_1\cdots a_5}{}^\alpha{}_\beta \end{array} + \cdots \right) \vartheta^{i_1\cdots i_{2k+1}} O_\alpha \\ + & \frac{1}{12} \sum_k \sum_{\ddot{k}=0}^{k-1} \sum_{\dot{k}=\ddot{k}}^{k-1} \left(\begin{array}{l} \dot{\theta}^\alpha_{i_1\cdots i_{2\ddot{k}+1}} \dot{\theta}^{\alpha'}_{i_{2\ddot{k}+2}\cdots i_{2\dot{k}+2}} \theta^\beta_{i_{2\dot{k}+3}\cdots i_{2k+1}} \\ + \theta^\alpha_{i_1\cdots i_{2\ddot{k}+1}} \theta^{\alpha'}_{i_{2\ddot{k}+2}\cdots i_{2\dot{k}+2}} \dot{\theta}^\beta_{i_{2\dot{k}+3}\cdots i_{2k+1}} \end{array} \right) [QQQ]^\delta_{\alpha\alpha'\beta} \, \vartheta^{i_1\cdots i_{2k+1}} O_\delta \end{array} \right)
$$

$$
\widehat{\mathcal{M}} \;:=\; \mathbb{R}^{528\,|\,64} \;\simeq\; \left(\begin{array}{rl|rcl} x^a & P_a & & & x^a \in \mathbb{R} \\ + b_{a_1 a_2} & Z^{a_1 a_2} & b_{a_1 a_2} & = & b_{[a_1 a_2]} \in \mathbb{R} \\ + b_{a_1\cdots a_5} & Z^{a_1\cdots a_5} & b_{a_1\cdots a_5} & = & b_{[a_1\cdots a_5]} \in \mathbb{R} \\ +\theta^\alpha & Q_\alpha & & & \theta^\alpha \in \mathbb{R} \\ +\xi^\alpha & O_\alpha & & & \xi^\alpha \in \mathbb{R} \end{array} \;,\; \begin{array}{l} a_i \in \{0, 1, \cdots, 10\} \\ \alpha \in \{1, 2, \cdots, 32\} \end{array} \right)
$$

$$
\begin{array}{ccc}
\widehat{\mathcal{M}} \times \widehat{\mathcal{M}} & \xrightarrow{\quad\quad\quad\quad\text{prd}\quad\quad\quad\quad} & \widehat{\mathcal{M}} \\
x'^a + x^a - \left(\bar{\theta}' \Gamma^a \theta\right) & \longleftarrow & x^a \\
b'_{a_1a_2} + b_{a_1a_2} + \left(\bar{\theta}' \Gamma_{a_1a_2} \theta\right) & \longleftarrow & b_{a_1a_2} \\
b'_{a_1\cdots a_5} + b_{a_1\cdots a_5} - \left(\bar{\theta}' \Gamma_{a_1\cdots a_5} \theta\right) & \longleftarrow & b_{a_1\cdots a_5} \\
\theta' + \theta & \longleftarrow & \theta \\
\left.\begin{array}{c}
\xi' + \xi \\
+\frac{\delta}{2} x'^a \Gamma_a \theta + \frac{\gamma_1}{2} b'^{a_1a_2} \Gamma_{a_1a_2} \theta + \frac{\gamma_2}{2} b'^{a_1\cdots a_5} \Gamma_{a_1\cdots a_5} \theta \\
-\frac{\delta}{2} x^a \Gamma_a \theta' - \frac{\gamma_1}{2} b^{a_1a_2} \Gamma_{a_1a_2} \theta' - \frac{\gamma_2}{2} b^{a_1\cdots a_5} \Gamma_{a_1\cdots a_5} \theta' \\
+\frac{1}{12} [QQQ](\theta', \theta', \theta) + \frac{1}{12} [QQQ](\theta, \theta, \theta')
\end{array}\right\} & \longleftarrow & \xi
\end{array}
$$

$$
\mathrm{prd}\left(x'^i T_i, x^i T_i\right) = x'^i + x^i + \frac{1}{2} f^i_{jk} x'^j x^k + \frac{1}{12} f^i_{jk} f^k_{k_1k_2} x'^j x'^{k_1} x^{k_2} + \frac{1}{12} f^i_{jk} f^k_{k_1k_2} x^j x^{k_1} x'^{k_2}
$$

$$
e^i = \mathrm{d}x^i - \frac{1}{2} f^i_{jk} x^j \, \mathrm{d}x^k + \frac{1}{6} f^i_{jk'} f^{k'}_{kl} \, x^j x^k \, \mathrm{d}x^l
$$

$$
\begin{aligned}
e^i &= \mathrm{d}x^i \left(\frac{1 - \exp\left(-\mathrm{ad}X\right)}{\mathrm{ad}X} \left(\partial_{x^k}\right) \right) \mathrm{d}x^k \\
&:= \mathrm{d}x^i \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-\mathrm{ad}X)^n \left(\partial_{x^k}\right) \right) \mathrm{d}x^k
\end{aligned}
$$

$$
\begin{aligned}
e^a &= \mathrm{d}x^a + \left(\bar{\theta} \Gamma^a \, \mathrm{d}\theta\right) \\
e_{a_1a_2} &= \mathrm{d}b_{a_1a_2} - \left(\bar{\theta} \Gamma_{a_1a_2} \, \mathrm{d}\theta\right) \\
e_{a_1\cdots a_5} &= \mathrm{d}b_{a_1\cdots a_5} + \left(\bar{\theta} \Gamma_{a_1\cdots a_5} \, \mathrm{d}\theta\right) \\
\psi &= \mathrm{d}\theta \\
\phi &= \mathrm{d}\xi - \frac{1}{2} \delta \left(x^a \Gamma_a \, \mathrm{d}\theta - \Gamma_a \theta (\, \mathrm{d}x^a)\right) \\
&\quad - \frac{1}{2} \gamma_1 \left(x^{a_1a_2} \Gamma_{a_1a_2} \, \mathrm{d}\theta - \Gamma_{a_1a_2} \theta (\, \mathrm{d}x^{a_1a_2})\right) \\
&\quad - \frac{1}{2} \gamma_2 \left(x^{a_1\cdots a_5} \Gamma_{a_1\cdots a_5} \, \mathrm{d}\theta - \Gamma_{a_1\cdots a_5} \theta (\, \mathrm{d}x^{a_1\cdots a_5})\right) \\
&\quad + \frac{2}{6} \left(\delta \Gamma^a_{\alpha\beta} \Gamma_{a\gamma} - \gamma_1 \Gamma^{a_1a_2}_{\alpha\beta} \Gamma_{a_1a_2\gamma} + \gamma_2 \Gamma^{a_1\cdots a_5}_{\alpha\beta} \Gamma_{a_1\cdots a_5\gamma}\right) \theta^\gamma \theta^\alpha \mathrm{d}\theta^\beta
\end{aligned}
$$

$$
\begin{array}{ccc}
\mathbb{Z}^{528} & \hookrightarrow & \widehat{\mathcal{M}} \\
x^a & \longleftarrow & x^a \\
b_{a_1a_2} & \longleftarrow & b_{a_1a_2} \\
b_{a_1\cdots a_5} & \longleftarrow & b_{a_1\cdots a_5} \\
0 & \longleftarrow & \theta \\
0 & \longleftarrow & \xi\,.
\end{array}
$$

$$
\begin{tikzcd}
\mathbb{Z}^{528} \arrow[r, hook] \arrow[d, equal] & \mathbb{R}^{528} \arrow[r, hook] \arrow[d, equal] & \widehat{\mathcal{M}} \arrow[d, two heads] \\
\mathbb{Z}^{528} \arrow[r, hook] & \mathbb{R}^{528} \arrow[r, hook] & \mathcal{M}
\end{tikzcd}
$$

$$(\eta_{ab})_{a,b=0}^{d} = \left(\eta^{ab}\right)_{a,b=0}^{d} := (\mathrm{diag}(-1,+1,+1,\cdots,+1))_{a,b=0}^{d}.$$

$$V^a := V_b \eta^{ab}, V_a = V^b \eta_{ab}$$

$$V_{[a_1 \cdots a_p]} := \frac{1}{p!} \sum_{\sigma \in \mathrm{Sym}(n)} (-1)^{|\sigma|} V_{a_{\sigma(1)} \cdots a_{\sigma(p)}}.$$

$$\epsilon_{012\cdots} := +1 \text{ hence } \epsilon^{012\cdots} := -1.$$

$$\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} := \delta^{[a_1}_{[b_1} \cdots \delta^{a_p]}_{b_p]} = \delta^{a_1}_{[b_1} \cdots \delta^{a_p}_{b_p]} = \delta^{[a_1}_{b_1} \cdots \delta^{a_p]}_{b_p}$$

$$V_{a_1 \cdots a_p} \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} = V_{[b_1 \cdots b_p]} \text{ and } \epsilon^{c_1 \cdots c_p a_1 \cdots a_q} \epsilon_{c_1 \cdots c_p b_1 \cdots b_q} = -p! \cdot q! \, \delta^{a_1 \cdots a_q}_{b_1 \cdots b_q}.$$

$$\mathrm{deg}_1 = (n_1, \sigma_1), \mathrm{deg}_2 = (n_2, \sigma_2) \in \mathbb{Z} \times \mathbb{Z}_2 \ \Rightarrow \ \mathrm{sgn}(\mathrm{deg}_1, \mathrm{deg}_2) := (-1)^{n_1 \cdot n_2 + \sigma_1 \cdot \sigma_2}.$$

$$v_1 \cdot v_2 = (-1)^{\mathrm{sgn}(\mathrm{deg}_1, \mathrm{deg}_2)} v_2 \cdot v_1,$$

$$\mathbb{R}[(v_i)_{i \in I}] := \mathrm{Sym}(\mathbb{R}\langle (v_i)_{i \in I} \rangle).$$

$$(\mathrm{d}v_i)_{i \in I}$$

$$e_i \longmapsto \mathrm{d}e_i \,,\ \mathrm{d}e_i \longmapsto 0$$

$$\mathbb{R}_{\mathrm{d}}[(v_i)_{i \in I}] := (\mathrm{Sym}(\mathbb{R}\langle (v_i)_{i \in I}, (\mathrm{d}v_i)_{i \in I} \rangle), \mathrm{d}).$$

$$\Gamma_a : \mathbf{32} \longrightarrow \mathbf{32}$$

$$(\overline{(-)}(-)) : 32 \otimes 32 \longrightarrow \mathbb{R}$$

$$\psi^\alpha \eta_{\alpha\beta} \phi^\beta := (\bar{\psi}\phi)$$

$$\eta_{\alpha\beta} = -\eta_{\beta\alpha}$$

$$\psi_\alpha := \psi^{\alpha'} \eta_{\alpha'\alpha}, \psi^\alpha = \psi_{\alpha'} \eta^{\alpha'\alpha}, \psi_\alpha \phi^\alpha = -\psi^\beta \eta_{\beta\alpha} \eta^{\alpha\gamma} \phi_\gamma = -\psi^\alpha \phi_\alpha$$

$$\Gamma_{a_1 \cdots a_k} := \frac{1}{k!} \sum_{\sigma \in \mathrm{Sym}(k)} \mathrm{sgn}(\sigma) \Gamma_{a_{\sigma(1)}} \cdot \Gamma_{a_{\sigma(2)}} \cdots \Gamma_{a_{\sigma(n)}} :$$

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = +2\eta_{ab} \mathrm{id}_{\mathbf{32}}$$

$$\Gamma^{a_j\cdots a_1}\Gamma_{b_1\cdots b_k} = \sum_{l=0}^{\min(j,k)} \pm\, l!\binom{j}{l}\binom{k}{l}\delta^{[a_1\cdots a_l}_{[b_1\cdots b_l}\Gamma^{a_j\cdots a_{l+1}]}_{b_{l+1}\cdots b_k]}$$

$$\Gamma_{a_1\cdots a_{11}} = \epsilon_{a_1\cdots a_{11}}\mathrm{id}_{\mathbf{32}}$$

$$\mathrm{Tr}(\Gamma_{a_1\cdots a_p}) = \begin{cases} 32 & | \quad p=0 \\ 0 & | \quad p>0 \end{cases}$$

$$\Gamma^{a_1\cdots a_p} = \frac{(-1)^{(p+1)(p-2)/2}}{(11-p)!}\epsilon^{a_1\cdots a_p b_1\cdots a_{11-p}}\Gamma_{b_1\cdots b_{11-p}}$$

$$\Gamma^{a_1\cdots a_{11}} = \epsilon^{a_1\cdots a_{11}}\mathrm{Id}_{\mathbf{32}}, \quad \Gamma^{a_1\cdots a_6} = +\frac{1}{5!}\epsilon^{a_1\cdots a_6 b_1\cdots b_5}\Gamma_{b_1\cdots b_5},$$
$$\Gamma^{a_1\cdots a_{10}} = \epsilon^{a_1\cdots a_{10}b}\Gamma_b, \quad \Gamma^{a_1\cdots a_5} = -\frac{1}{6!}\epsilon^{a_1\cdots a_5 b_1\cdots b_6}\Gamma_{b_1\cdots b_6}.$$

$$\overline{\Gamma_a} = -\Gamma_a \text{ in that } \underset{\phi,\psi\in\mathbf{32}}{\forall}\left(\overline{(\Gamma_a\phi)}\psi\right) = -\left(\bar{\phi}(\Gamma_a\psi)\right),$$

$$\overline{\Gamma_{a_1\cdots a_p}} = (-1)^{p+p(p-1)/2}\Gamma_{a_1\cdots a_p}$$

$$\mathrm{End}_{\mathbb{R}}(\mathbf{32}) = \left\langle 1, \Gamma_{a_1}, \Gamma_{a_1a_2}, \Gamma_{a_1a_2a_3}, \Gamma_{a_1\cdots a_4}, \Gamma_{a_1\cdots a_5}\right\rangle_{a_i=0,1,\cdots},$$

$$\mathrm{Hom}_{\mathbb{R}}\left((\mathbf{32}\otimes\mathbf{32})_{\mathrm{sym}}, \mathbb{R}\right) \simeq \left\langle \left(\overline{(-)}\Gamma_a(-)\right), \left(\overline{(-)}\Gamma_{a_1a_2}(-)\right), \left(\overline{(-)}\Gamma_{a_1\cdots a_5}(-)\right)\right\rangle_{a_i=0,1,\cdots}$$

$$\Gamma^{a}_{\alpha\beta} = \Gamma^{a}_{\beta\alpha}, \Gamma^{a_1a_2}_{\alpha\beta} = \Gamma^{a_1a_2}_{\beta\alpha}, \Gamma^{a_1\cdots a_5}_{\alpha\beta} = \Gamma^{a_1\cdots a_5}_{\beta\alpha},$$

$$\mathrm{Hom}_{\mathbb{R}}\left((\mathbf{32}\otimes\mathbf{32})_{\mathrm{skew}}, \mathbb{R}\right) \simeq \left\langle \left(\overline{(-)}(-)\right), \left(\overline{(-)}\Gamma_{a_1a_2a_3}(-)\right), \left(\overline{(-)}\Gamma_{a_1\cdots a_4}(-)\right)\right\rangle_{a_i=0,1,\cdots}$$

$$\eta_{\alpha\beta} = -\eta_{\beta\alpha}, \Gamma^{a_1a_2a_3}_{\alpha\beta} = -\Gamma^{a_1a_2a_3}_{\beta\alpha}, \Gamma^{a_1\cdots a_5}_{\alpha\beta} = -\Gamma^{a_1\cdots a_5}_{\beta\alpha}$$

$$\phi = \frac{1}{32}\sum_{p=0}^{5}\frac{(-1)^{p(p-1)/2}}{p!}\mathrm{Tr}\left(\phi\circ\Gamma_{a_1\cdots a_p}\right)\Gamma^{a_1\cdots a_p};$$

$$(\bar{\phi}_1\psi)(\bar{\psi}\phi_2) = \frac{1}{32}\left((\bar{\psi}\Gamma^a\psi)(\bar{\phi}_1\Gamma_a\phi_2) - \frac{1}{2}(\bar{\psi}\Gamma^{a_1a_2}\psi)(\bar{\phi}_1\Gamma_{a_1a_2}\phi_2) + \frac{1}{5!}(\bar{\psi}\Gamma^{a_1\cdots a_5}\psi)(\bar{\phi}_1\Gamma_{a_1\cdots a_5}\phi_2)\right).$$

$$(32\otimes 32)_{\mathrm{sym}} \cong 11\oplus 55\oplus 462$$
$$(32\otimes 32\otimes 32)_{\mathrm{sym}} \cong 32\oplus 320\oplus 1408\oplus 4424$$
$$(32\otimes 32\otimes 32\otimes 32)_{\mathrm{sym}} \cong 1\oplus 165\oplus 330\oplus 462\oplus 65\oplus 429\oplus 1144\oplus 17160\oplus 32604.$$

$$\left\langle \Xi^{\alpha}_{a_1\cdots a_p} = \Xi^{\alpha}_{[a_1\cdots a_p]}\right\rangle_{a_i\in\{0,\cdots,10\},\alpha\in\{1,\cdots 32\}} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(1,10))$$
$$\text{with } \Gamma^{a_1}\Xi_{a_1a_2\cdots a_p} = 0$$

$$\begin{aligned}
\psi(\overline{\psi}\,\Gamma_a\,\psi) &= \tfrac{1}{11}\,\Gamma_a\,\Xi^{(32)} + \Xi^{(320)}_a\,, \\
\psi(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi) &= \tfrac{1}{11}\,\Gamma_{a_1a_2}\,\Xi^{(32)} - \tfrac{2}{9}\,\Gamma_{[a_1}\,\Xi^{(320)}_{a_2]} + \Xi^{(1408)}_{a_1a_2}\,, \\
\psi(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi) &= -\tfrac{1}{77}\Gamma_{a_1\cdots a_5}\Xi^{(32)} + \tfrac{5}{9}\Gamma_{[a_1\cdots a_4}\Xi^{(320)}_{a_5]} + 2\,\Gamma_{[a_1a_2a_3}\,\Xi^{(1408)}_{a_4a_5]} + \Xi^{(4224)}_{a_1\cdots a_5}
\end{aligned}$$

$$[-,-]\colon \mathfrak{g} \vee \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\begin{tikzcd}
\wedge^1 \mathfrak{g}^* \arrow[r, "-[\text{-},\text{-}]^*"] \arrow[d, hook] & \wedge^2 \mathfrak{g}^* \arrow[d, hook] \\
\wedge^\bullet \mathfrak{g}^* \arrow[r, "\mathrm{d}"] & \wedge^\bullet \mathfrak{g}^*
\end{tikzcd}$$

$$\mathrm{CE}\big(\mathfrak{g},\,[\text{-},\text{-}]\big) \;:=\; \big(\wedge^\bullet\,\mathfrak{g}^*,\,\mathrm{d}\big)$$

$$\mathrm{sLieAlg}_{\mathbb{R}} \xhookrightarrow{\;\mathrm{CE}\;} \mathrm{sDGCAlg}^{\mathrm{op}}_{\mathbb{R}}$$

$$\mathrm{CE}(\mathfrak{g},[-,-]) \simeq (\mathbb{R}[t^1,\cdots,t^1],\mathrm{d})$$

	Super Lie algebra	Super dgc-algebra
Generators	$\big(\underbrace{T_i}_{\deg=(0,\sigma_i)}\big)_{i=1}^{n}$	$\big(\underbrace{t^i}_{\deg=(1,\sigma_i)}\big)_{i=1}^{n}$
Relations	$[T_i,T_j] = f_{ij}^k\,T_k$	$\mathrm{d}\,t^k = -\frac{1}{2}f_{ij}^k\,t^i t^j$

$$\left[\vartheta^{i_1\cdots i_n}T,\vartheta^{i'_1\cdots i'_{n'}}T'\right]_{\mathrm{sgn}} := \vartheta^{i'_1\cdots i'_{n'}i_1\cdots i_n}[T,T'] = (-1)^{nn'}\vartheta^{i_1\cdots i_n i'_1\cdots i'_{n'}}[T,T']$$

$$\begin{array}{ccc}
(\mathfrak{g}_{(p)},\ [-,-]) & \xrightarrow[\sim]{\vartheta^{i_1\cdots i_n}T\ \mapsto\ \vartheta^{i_n\cdots i_1}T} & (\mathfrak{g}_{(p)},\ [-,-]_{\mathrm{sgn}}) \\
(\vartheta^{i_1\cdots i_n}T,\ \vartheta^{i_1\cdots i'_{n'}}T') & \longmapsto & (\vartheta^{i_n\cdots i_1}T,\ \vartheta^{i'_{n'}\cdots i_1}T') \\
\big\downarrow & & \big\downarrow \\
\vartheta^{i_1\cdots i_n\, i'_1\cdots i'_{n'}}[T,T'] & \longmapsto & \vartheta^{i'_{n'}\cdots i'_1\, i_n\cdots i_1}[T,T']\,.
\end{array}$$

$$\mathrm{d}\star J_{q+1} = 2\kappa G_{q+2}\wedge\star\, \mathcal{J}_{2q+2}$$
$$\mathrm{d}\star \mathcal{J}_{2q+2} = 0$$

$$\mathrm{d}\star J_3 = -G_4\wedge\star\, \mathcal{J}_6$$
$$\mathrm{d}\star \mathcal{J}_6 = 0$$

$$\delta A^I_{p_I+1} = \mathrm{d}\Lambda^I_{p_I}$$

$$\delta A^I_{p_I+1} = \mathrm{d}\Lambda^I_{p_I} + \sum_J \Lambda^J_{p_J}\wedge \mathfrak{F}_{p_I+1-p_J}(A) + \perp_{\text{non-linear}}$$

$$\mathcal{G}^I_{p_I+2} = \mathrm{d}A^I_{p_I+1} + \mathfrak{R}^I_{p_I+2}(A)$$

$$\mathcal{H}^{(n+1)} = \left(\prod_j U(1)^{(p_j)}\right)\times_\kappa U(1)^{(n)}$$

$$\delta B_2 = \mathrm{d}\Lambda_1 + \frac{1}{2\pi}\sum_{I,J}\kappa_{IJ}\Lambda^I_0\,\mathrm{d}A^J_1$$
$$\delta A^I_1 = \mathrm{d}\Lambda^I_0$$

$$U(1)^{(0)}_I\times U(1)^{(0)}_J\times U(1)^{(0)}_C$$

$$j^I_{p_I+1}\int\ \mathcal{D}\phi\exp\left(-iS(\phi,A)\right) A^I_{p_I+1}\phi\ S(\phi,A)\ \delta S\sim\int\ \delta A^I_{p_I+1}\wedge\star\, j^I_{p_I+1}$$

$$H^{(2q+2)} = U(1)^{(q)}\times_\kappa U(1)^{(2q+1)}$$

$$\delta A_{q+1} = \mathrm{d}\Lambda_q$$
$$\delta B_{2q+2} = \mathrm{d}\Lambda_{2q+1} - \kappa\mathrm{d}\Lambda_q\wedge A_{q+1}$$

$$\mathcal{G}_{2q+3} = \mathrm{d}B_{2q+2} + (-1)^{q+1}\kappa A_{q+1}\wedge G_{q+2}$$

$$\mathrm{d}\mathcal{G}_{2q+3} = (-1)^{q+1}\kappa G_{q+2}\wedge G_{q+2},\ B_{2q+2}\to B_{2q+2} - \kappa/2A_{q+1}\wedge A_{q+1}$$

$$\delta S\sim\int\ \delta A_{q+1}\wedge\star\, J_{q+1} + \left(\delta B_{2q+2} + \kappa A_{q+1}\wedge\delta A_{q+1}\right)\wedge\star\,\mathcal{J}_{2q+2}$$

$$\begin{aligned} \mathrm{d}\star J_{q+1} &= 2\kappa G_{q+2}\wedge\star \mathcal{J}_{2q+2} \\ \mathrm{d}\star \mathcal{J}_{2q+2} &= 0 \end{aligned}$$

$$U(1)^{(q)} \times {}_{\kappa}U(1)^{(2q+1)}$$

$$\delta\phi_q = -c_\phi \Lambda_q$$

$$F_{q+1} = \mathrm{d}\phi_q + c_\phi A_{q+1}$$

$$\mathrm{d}F_{q+1} = c_\phi G_{q+2}$$

$$\delta\Phi_{2q+1} = -c_\Phi \Lambda_{2q+1} - \kappa \frac{c_\Phi}{c_\phi}\mathrm{d}\phi_q \wedge \Lambda_q$$

$$\mathcal{F}_{2q+2} = \mathrm{d}\Phi_{2q+1} + c_\Phi B_{2q+2} + \kappa\frac{c_\Phi}{c_\phi} A_{q+1}\wedge \mathrm{d}\phi_q$$

$$\mathrm{d}\mathcal{F}_{2q+2} = c_\Phi \mathcal{G}_{2q+3} + \kappa\frac{c_\Phi}{c_\phi} G_{q+2}\wedge F_{q+1}$$

$$S = \int_M \star\mathcal{L}\left(F_{q+1}, \mathcal{F}_{2q+2}\right)$$

$$S = -\int_M g_{(q)} F_{q+1}\wedge\star F_{q+1} + g_{(2q+1)}\mathcal{F}_{2q+2}\wedge\star\mathcal{F}_{2q+2}$$

$$\begin{aligned} \mathrm{d}\left(\frac{\delta\star\mathcal{L}}{\delta F_{q+1}}\right) &= \kappa\frac{c_\Phi}{c_\phi} G_{q+2}\wedge\frac{\delta\star\mathcal{L}}{\delta\mathcal{F}_{2q+2}} \\ \mathrm{d}\left(\frac{\delta\star\mathcal{L}}{\delta\mathcal{F}_{2q+2}}\right) &= 0 \end{aligned}$$

$$\begin{aligned} \star J_{q+1} &= c_\phi\frac{\delta\star\mathcal{L}}{\delta F_{q+1}} + \kappa\frac{c_\Phi}{c_\phi}F_{q+1}\wedge\frac{\delta\star\mathcal{L}}{\delta\mathcal{F}_{2q+2}} \\ \star\mathcal{J}_{2q+2} &= c_\Phi\frac{\delta\star\mathcal{L}}{\delta\mathcal{F}_{2q+2}} \end{aligned}$$

$$\nabla_\mu T^{\mu\nu} = \frac{1}{(q+1)!} G_{q+2}^{\nu\mu_1\dots\mu_{q+1}} J_{q+1,\mu_1\dots\mu_{q+1}} + \frac{1}{(2q+2)!}\mathcal{G}_{2q+3}^{\nu\mu_1\dots\mu_{2q+2}}\mathcal{J}_{2q+2,\mu_1\dots\mu_{2q+2}}$$

$$S = \int_M \star\hat{\mathcal{L}}\left(F_{q+1}\right) + Q_{(2q+1)}\int_M \mathcal{F}_{2q+2}$$

$$\star J_{q+1} = c_\phi\frac{\delta\star\hat{\mathcal{L}}}{\delta F_{q+1}} + Q_{(2q+1)}\frac{\kappa}{c_\phi}F_{q+1},$$

$$\mathrm{d}\star J_{q+1} = 2\kappa Q_{(2q+1)} G_{q+2}$$

$$\begin{aligned} S_{\text{bulk}} &= -Q_{(2q+1)}\int \mathcal{G}_{2q+3} \\ &= \int_{\text{bulk}} \kappa Q_{(2q+1)} A_{q+1}\wedge \mathrm{d}A_{q+1} - Q_{(2q+1)}\mathrm{d}B_{2q+2} \end{aligned}$$

$$\star F_{q+1} = \pm F_{q+1} + \cdots$$

$$\star F_{q+1} = \frac{c_\phi}{2\kappa Q_{(2q+1)}} J_{q+1}$$

$$J_{q+1} = c_\phi f F_{q+1} + Q_{(2q+1)} \frac{\kappa}{c_\phi} \star F_{q+1} \overset{\text{self-duality}}{\Longrightarrow} \star F_{q+1} = \frac{c_\phi^2 f}{\kappa Q_{(2q+1)}} F_{q+1}$$

$$F_{q+1} = \frac{c_\phi}{2\kappa Q_{(2q+1)}} \star J_{q+1} \Bigg|_{\star F_{q+1} \to \frac{c_\phi}{2\kappa Q_{(2q+1)}} J_{q+1} \text{ recursively}}$$

$$T^{ab} K^I_{ab} = \frac{1}{(q+1)!} G^{Ia\ldots}_{q+2} J_{q+1,a\ldots} + \frac{1}{(2q+2)!} \mathcal{G}^{Ia\ldots}_{2q+3} \mathcal{J}_{2q+2,a\ldots},$$

$$\delta V_2 = -\Lambda_2$$

$$S = -Q_{(5)} \int_M \star \sqrt{1 + \mathcal{Y}(F_3)} + Q_{(5)} \int_M \mathcal{F}_6$$

$$\mathrm{d} \star \left(\frac{1}{\sqrt{1 + \mathcal{Y}(F_3)}} \frac{\delta \mathcal{Y}(F_3)}{\delta F_3} \right) = G_4,$$

$$\star F_3 = \frac{1}{\sqrt{1 + \mathcal{Y}(F_3)}} \frac{\delta \mathcal{Y}(F_3)}{\delta F_3}$$

$$\mathcal{Y}(F_3) = \frac{1}{12} F_{abc} F^{abc} + \frac{1}{288} \left(F_{abc} F^{abc} \right)^2 - \frac{1}{96} \left(F_{abc} F^{hbc} F_{hde} F^{ade} \right)$$

$$\star F_{abc} = \frac{1}{\sqrt{1 + \mathcal{Y}(F_3)}} \left(F_{abc} + \frac{1}{12} F_{deh} F^{deh} F_{abc} - \frac{1}{4} F_{de[a} F_{bc]h} F^{hde} \right)$$

$$£_\beta A_{q+1} = £_\beta B_{2q+2} = £_\beta g_{\mu\nu} = 0$$

$$\mathcal{Z}\left[A_{q+1}, B_{2q+1}, g_{\mu\nu}\right] = \int \mathcal{D}\phi \, \mathrm{e}^{-S_E(\phi; A_{q+1}, B_{2q+1}, g_{\mu\nu})}$$

$$£_\beta \phi = 0$$

$$£_\beta \Lambda_q = £_\beta \Lambda_{2q+1} = 0$$

$$\xi = -u \wedge i_u \xi + \xi_\Sigma,$$

$$* \xi = \star (u \wedge \xi), \int_\Sigma X = -\int_M u \wedge X = * f \int_\Sigma * f = \int_M \star f$$

$$\delta \varphi_{q-1} = -i_\beta \Lambda_q.$$

$$\frac{\mu_q}{T} = -\mathrm{d}\varphi_{q-1} + i_\beta A_{q+1},$$

$$\mathrm{d}\left(\frac{\mu_q}{T}\right) = -i_\beta G_{q+2},$$

$$S_E = \int_\Sigma * \hat{P}(T, \mu_q) + Q_{(2q+1)} \int_\Sigma \psi_{2q+1},$$

$$\frac{1}{T}\psi_{2q+1} = i_\beta B_{2q+2} + 2\kappa\, \mathrm{d}\varphi_{q-1} \wedge A_{q+1,\Sigma} - \kappa i_\beta A_{q+1} \wedge A_{q+1,\Sigma} - \kappa \mathrm{d}(Tu) \wedge \varphi_{q-1} \wedge\ \mathrm{d}\varphi_{q-1},$$

$$\begin{aligned}\frac{1}{T}\delta\psi_{2q+1} &= i_\beta \mathrm{d}\Lambda_{2q+1} + 2\kappa\, \mathrm{d}\varphi_{q-1} \wedge\ \mathrm{d}\Lambda_q + \kappa \mathrm{d}\left(Tu \wedge i_\beta \Lambda_q\right) \wedge \mathrm{d}\varphi_{q-1} + \kappa \mathrm{d}\left(Tu \wedge \varphi_{q-1}\right) \wedge \mathrm{d}i_\beta \Lambda_q \\ &= \mathrm{d}\left(-i_\beta \Lambda_{2q+1} + \kappa \varphi_{q-1} \wedge \left(\mathrm{d}\Lambda_q\right)_\Sigma + \kappa \Lambda_{q,\Sigma} \wedge\ \mathrm{d}\varphi_{q-1}\right)\end{aligned}$$

$$S_E = \int_\Sigma * P(T, \mu_q, \mu_{2q+1})$$

$$\begin{aligned}\delta P &= s\delta T + n_q \cdot \delta\mu_q + n_{2q+1} \cdot \delta\mu_{2q+1} - \frac{1}{2} r^{\mu\nu} \delta g_{\mu\nu} \\ \epsilon &= Ts + \mu_q \cdot n_q + \mu_{2q+1} \cdot n_{2q+1} - P\end{aligned}$$

$$r^{\mu\nu} = \frac{1}{(q-1)!}\left(n_q\right)^{\mu}_{\ \rho\ldots}\left(\mu_q\right)^{\nu\rho\ldots} + \frac{1}{(2q)!}\left(n_{2q+1}\right)^{\mu}_{\ \rho\ldots}\left(\mu_{2q+1}\right)^{\nu\rho\ldots}$$

$$\begin{aligned}J_{q+1} &= u \wedge n_q - (-)^d 2\kappa * \left(\mu_q \wedge * n_{2q+1}\right) \\ \mathcal{J}_{2q+2} &= u \wedge n_{2q+1}\end{aligned}$$

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - r^{\mu\nu} + 2v^{(\mu} u^{\nu)}$$

$$*v = -\kappa\, \mu_q \wedge \mu_q \wedge * n_{2q+1}$$

$$\mathrm{d} * n_q = 2\kappa\left(G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge * n_{2q+1}$$

$$\iota_\beta F_{q+1} = \frac{\mu_q}{T}$$

$$\mathrm{d} * \tilde{n}_p = (-)^d\ \mathrm{d}(Tu) \wedge \frac{\mu_q}{T} + (-)^d G_{q+2,\Sigma},$$

$$S_E = \int_\Sigma * \hat{P}(T, \mu_q, \tilde{n}_p) + Q_{(2q+2)} \int_\Sigma \psi_{2q+1}$$

$$S_E = \int_\Sigma * P(T, \mu_q, \mu_{2q+1}, \tilde{n}_p)$$

$$\begin{aligned}\delta P &= s\delta T + n_q \cdot \delta\mu_q + n_{2q+1} \cdot \delta\mu_{2q+1} - * \tilde{\mu}_p \cdot \delta * \tilde{n}_p - \frac{1}{2} r^{\mu\nu} \delta g_{\mu\nu} \\ \epsilon &= Ts + \mu_q \cdot n_q + \mu_{2q+1} \cdot n_{2q+1} - P\end{aligned}$$

$$r^{\mu\nu} = \frac{1}{(q-1)!}(n_q)^{\mu}_{\rho\ldots}(\mu_q)^{\nu\rho\ldots} + \frac{1}{(2q)!}(n_{2q+1})^{\mu}_{\rho\ldots}(\mu_{2q+1})^{\nu\rho\ldots} - \frac{1}{q!}(*\, n_p)^{\mu}_{\rho\ldots}(*\, \mu_p)^{\nu\rho\ldots}$$

$$J_{q+1} = u \wedge n_q - (-)^d * \tilde{\mu}_p - (-)^d 2\kappa * (\mu_q \wedge * \, n_{2q+1})$$
$$\mathcal{J}_{2q+2} = u \wedge n_{2q+1}$$

$$*\, v = -\mu_q \wedge \tilde{\mu}_p - \kappa \mu_q \wedge \mu_q \wedge * \, n_{2q+1}$$

$$\mathrm{d} * n_q = \mathrm{d}(Tu) \wedge \frac{\tilde{\mu}_p}{T} + 2\kappa \left(G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge * \, n_{2q+1}$$
$$\mathrm{d}\frac{\tilde{\mu}_p}{T} = 0$$

$$\delta\varphi_{2q} = -i_\beta \Lambda_{2q+1} + \kappa \varphi_{q-1} \wedge (\,\mathrm{d}\Lambda_q)_\Sigma + \kappa \Lambda_{q,\Sigma} \wedge \,\mathrm{d}\varphi_{q-1}$$

$$\frac{\mu_{2q+1}}{T} = i_\beta B_{2q+2} - \mathrm{d}\varphi_{2q} + 2\kappa\, \mathrm{d}\varphi_{q-1} \wedge A_{q+1,\Sigma} - \kappa i_\beta A_{q+1} \wedge A_{q+1,\Sigma} - \kappa \mathrm{d}(Tu) \wedge \varphi_{q-1} \wedge \,\mathrm{d}\varphi_{q-1}$$

$$\mathrm{d}\left(\frac{\mu_{2q+1}}{T}\right) = -i_\beta G_{2q+3} - \kappa \left(2G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge \frac{\mu_q}{T}$$

$$\mathrm{d} * n_{2q+1} = 0$$

$$*\, P = \frac{\chi}{2}\mu_q \wedge * \, \mu_q - \frac{f}{2} F_{q+1} \wedge * \, F_{q+1} + \frac{\chi'}{2}\mu_{2q+1} \wedge * \, \mu_{2q+1}.$$

$$\varphi_{2q} = i_\beta \Phi_{2q+1} - \kappa \varphi_{q-1} \wedge (\,\mathrm{d}\phi_q)_\Sigma$$

$$i_\beta \mathcal{F}_{2q+2} = \frac{\mu_{2q+1}}{T} + \kappa \frac{\mu_q}{T} \wedge F_{q+1,\Sigma}$$

$$\mathrm{d} * \tilde{n}_{p-q-1} = -\mathcal{G}_{2q+3,\Sigma} - \mathrm{d}(Tu) \wedge \frac{\mu_{2q+1}}{T} - (-)^d \kappa \left(G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge * \, \tilde{n}_p$$

$$S_E = \int_{\Sigma_{d-1}} \frac{\chi}{2}\mu_q \wedge * \, \mu_q - \frac{f}{2} F_{q+1} \wedge * \, F_{q+1} + \frac{\chi'}{2}\mu_{2q+1} \wedge * \, \mu_{2q+1} - \frac{f'}{2}\mathcal{F}_{2q+2} \wedge * \, \mathcal{F}_{2q+2}$$

$$S_E = \int_\Sigma * \, P\left(\mu_q, \mu_{2q+1}, \tilde{n}_p, \tilde{n}_{p-q-1}\right)$$

$$\delta P = s\delta T + n_q \cdot \delta\mu_q + n_{2q+1} \cdot \delta\mu_{2q+1} - * \, \tilde{\mu}_p \cdot \delta * \tilde{n}_p - * \, \tilde{\mu}_{p-q-1} \cdot \delta * \tilde{n}_{p-q-1} - \frac{1}{2} r^{\mu\nu}\delta g_{\mu\nu}$$
$$\epsilon = Ts + \mu_q \cdot n_q + \mu_{2q+1} \cdot n_{2q+1} - P$$

$$r^{\mu\nu} = \frac{1}{(q-1)!}(n_q)^{\mu}_{\rho\ldots}(\mu_q)^{\nu\rho\ldots} + \frac{1}{(2q)!}(n_{2q+1})^{\mu}_{\rho\ldots}(\mu_{2q+1})^{\nu\rho\ldots} - \frac{1}{q!}(*\, n_p)^{\mu}_{\rho\ldots}(*\, \mu_p)^{\nu\rho\ldots} - \frac{1}{(2q+1)!}(*\, \tilde{n}_{p-q-1})^{\mu}_{\rho\ldots}(*\, \tilde{\mu}_{p-q-1})^{\nu\rho\ldots}$$

$$J_{q+1} = u \wedge n_q - (-1)^d * \tilde{\mu}_p - (-1)^d 2\kappa * \left(\mu_q \wedge * n_{2q+1}\right) + \kappa * \left(* \tilde{n}_p \wedge \tilde{\mu}_{p-q-1}\right)$$
$$\mathcal{J}_{2q+2} = u \wedge n_{2q+1} + * \tilde{\mu}_{p-q-1}$$

$$* v = -\mu_q \wedge \tilde{\mu}_p - \kappa\mu_q \wedge \mu_q \wedge * n_{2q+1} + \mu_{2q+1} \wedge \tilde{\mu}_{p-q-1} + \kappa(-1)^d \mu_q \wedge * \tilde{n}_p \wedge \tilde{\mu}_{p-q-1}$$

$$\mathrm{d} * n_q = \mathrm{d}(Tu) \wedge \frac{\tilde{\mu}_p}{T} + 2\kappa \left(G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge * n_{2q+1} - \kappa(-1)^d\, \mathrm{d}(Tu) \wedge * \tilde{n}_p \wedge \frac{\tilde{\mu}_{p-q-1}}{T}$$
$$\mathrm{d}\frac{\tilde{\mu}_p}{T} = -\kappa \left(G_{q+2,\Sigma} + \mathrm{d}(Tu) \wedge \frac{\mu_q}{T}\right) \wedge \frac{\tilde{\mu}_{p-q-1}}{T}$$
$$\mathrm{d} * n_{2q+1} = \mathrm{d}(Tu) \wedge \frac{\tilde{\mu}_{p-q-1}}{T}$$
$$\mathrm{d}\frac{\tilde{\mu}_{p-q-1}}{T} = 0$$

$$\tilde{n}_p = \frac{1}{2\kappa Q_{(2q+1)}} n_q \quad \Longrightarrow \quad \tilde{n}_p = \frac{1}{2\kappa Q_{(2q+1)}} \frac{\delta S_E}{\delta \mu_q}\bigg|_{T,\tilde{n}_p,\mu_{2q+1}}$$
$$\tilde{\mu}_p = 0 \quad \Longrightarrow \quad 0 = \frac{\delta S_E}{\delta \tilde{n}_p}\bigg|_{T,\mu_q,\mu_{2q+1}}$$

$$* F_{q+1} = \frac{1}{2\kappa Q_{(2q+1)}} * \left(\frac{\delta * \hat{P}(T,\mu_q)}{\delta \mu_q}\right), \iota_u F_{q+1} = \mu_q$$

$$S_E = \int_\Sigma * \hat{P}(T,\mu_2) + Q_{(5)} \int_\Sigma \mu_5$$

$$\frac{\mu_2}{T} = -\mathrm{d}\varphi_1 + i_\beta A_3$$
$$\frac{\mu_5}{T} = -\mathrm{d}\varphi_4 + i_\beta B_6 - \mathrm{d}\varphi_1 \wedge A_{3,\Sigma} + \frac{1}{2} i_\beta A_3 \wedge A_{3,\Sigma} + \frac{1}{2}\, \mathrm{d}(Tu) \wedge \varphi_1 \wedge\, \mathrm{d}\varphi_1$$

$$* F_3 = \frac{-1}{Q_{(5)}} * \left(\frac{\delta * \hat{P}(T,\mu_2)}{\delta \mu_2}\right), \iota_u F_3 = \mu_2$$

$$J_3 = u \wedge n_2 + Q_{(5)} * \mu_2$$
$$\mathcal{J}_6 = Q_{(5)} u \wedge * 1$$
$$T^{\mu\nu} = (\epsilon + \hat{P}) u^\mu u^\nu + \hat{P} g^{\mu\nu} - n_2^{\mu}{}_{\rho} \mu_2^{\nu\rho} + Q_{(5)} * (\mu_2 \wedge \mu_2)^{(\mu} u^{\nu)}$$

$$\delta\hat{P} = s\delta T + \frac{1}{2} n_2^{\mu\nu} \delta\mu_{2\mu\nu} - \frac{1}{2} n_2^{\mu}{}_{\rho} \mu_2^{\nu\rho} \delta g_{\mu\nu}$$
$$\epsilon = Ts + \frac{1}{2} \mu_{2\mu\nu} n_2^{\mu\nu} - \hat{P}$$

$$\hat{p}(T,\Phi_{(2)}) = -\frac{1}{3} Q_{(5)} \frac{1 + 3\left(\tanh^2 \alpha - \Phi_{(2)}^2\right)\cosh^2 \alpha}{\cosh^2 \alpha \sqrt{\tanh^2 \alpha - \Phi_{(2)}^2}}$$

$$\frac{Q_{(5)} \cosh \alpha}{\sqrt{\tanh^2 \alpha - \Phi_{(2)}^2}} = \frac{3\Omega_4}{16\pi G_N} \left(\frac{3}{4\pi T}\right)^3$$

$$\tilde{J}_{p+1} = \star F_{q+1}$$
$$\tilde{\mathcal{J}}_{p-q} = \star\left(\mathcal{F}_{2q+2} - \frac{\tilde{\kappa}}{c_\phi} F_{q+1} \wedge F_{q+1}\right)$$

$$\mathrm{d}\star\tilde{J}_{p+1} = (-)^p c_\phi G_{q+2}$$
$$\mathrm{d}\star\tilde{\mathcal{J}}_{p-q} = -c_\Phi \mathcal{G}_{2q+3} - (-)^p 2\tilde{\kappa} G_{q+2} \wedge\star \tilde{J}_{p+1}$$

$$\left(\left(U(1)^{(q)} \times_\kappa U(1)^{(2q+1)}\right) \times \tilde{U}(1)^{(p-q-1)}\right) \times_{\tilde{\kappa}} \tilde{U}(1)^{(p)}.$$

$$S \sim \int \left(\tilde{A}_{p+1} + \tilde{\kappa} A_{q+1} \wedge \tilde{B}_{p-q}\right) \wedge\star \tilde{J}_{p+1} + \tilde{B}_{p-q} \wedge\star \tilde{\mathcal{J}}_{p-q}$$
$$\sim \int (-)^p \left(\tilde{A}_{p+1} + \tilde{\kappa} A_{q+1} \wedge \tilde{B}_{p-q}\right) \wedge F_{q+1}$$
$$-\tilde{B}_{p-q} \wedge \left(\mathcal{F}_{2q+2} - \frac{\tilde{\kappa}}{c_\phi} F_{q+1} \wedge F_{q+1}\right)$$

$$\delta A_{q+1} = \mathrm{d}\Lambda_q$$
$$\delta B_{2q+2} = \mathrm{d}\Lambda_{2q+1} - \kappa \mathrm{d}\Lambda_q \wedge A_{q+1}$$
$$\delta \tilde{A}_{p+1} = \mathrm{d}\tilde{\Lambda}_p + (-)^p \tilde{\kappa}\, \mathrm{d}\tilde{\Lambda}_{p-q-1} \wedge A_{q+1} - \tilde{\kappa} \mathrm{d}\Lambda_q \wedge \tilde{B}_{p-q}$$
$$\delta \tilde{B}_{p-q} = \mathrm{d}\tilde{\Lambda}_{p-q-1}$$

$$\delta_\Lambda S = -\int (-)^q c_\phi G_{q+2} \wedge \tilde{\Lambda}_p - c_\Phi \mathcal{G}_{2q+3} \wedge \tilde{\Lambda}_{p-q-1} + 2(-)^p c_\phi \tilde{\kappa} \tilde{\Lambda}_{p-q-1} \wedge A_{q+1} \wedge G_{q+2}$$

$$\tilde{G}_{p+2} = \mathrm{d}\tilde{A}_{p+1} - \tilde{\kappa} \tilde{B}_{p-q} \wedge F_{q+2} - \tilde{\kappa} A_{q+1} \wedge \tilde{G}_{p-q+1}$$
$$\tilde{\mathcal{G}}_{p-q+1} = \mathrm{d}\tilde{B}_{p-q}$$

$$S_{\mathrm{bulk}} = \int_{\mathrm{bulk}} c_\phi G_{q+2} \wedge \tilde{A}_{p+1} + c_\Phi \left(\mathcal{G}_{2q+3} + \frac{1}{2} \kappa A_{q+1} \wedge G_{q+2}\right) \wedge \tilde{B}_{p-q}$$

$$\delta S_{\mathrm{tot}} = \int \delta A_{q+1} \wedge\star J_{q+1} + \left(\delta B_{2q+2} + \kappa A_{q+1} \wedge \delta A_{q+1}\right) \wedge\star \mathcal{J}_{2q+2}$$
$$+\left(\delta \tilde{A}_{p+1} + \tilde{\kappa} A_{q+1} \wedge \delta \tilde{B}_{p-q} - \tilde{\kappa} \delta A_{q+1} \wedge \tilde{B}_{p-q}\right) \wedge\star \tilde{J}_{p+1} + \delta \tilde{B}_{p-q} \wedge\star \tilde{\mathcal{J}}_{p-q}$$
$$+\int_{\mathrm{bulk}} \delta A_{q+1} \wedge c_\phi \tilde{G}_{p+2} - \left(\delta B_{2q+2} + \kappa A_{q+1} \wedge \delta A_{q+1}\right) \wedge c_\Phi \tilde{\mathcal{G}}_{p-q+1}$$
$$+\left(\delta \tilde{A}_{p+1} + \tilde{\kappa} A_{q+1} \wedge \delta \tilde{B}_{p-q} - \tilde{\kappa} \delta A_{q+1} \wedge \tilde{B}_{p-q}\right) \wedge c_\phi G_{q+2}$$
$$+\delta \tilde{B}_{p-q} \wedge (-)^p c_\Phi \mathcal{G}_{2q+3}$$

$$\star J_{q+1} \sim c_\phi \tilde{A}_{p+1} + \frac{1}{2} c_\Phi \kappa A_{q+1} \wedge \tilde{B}_{p-q} + \Omega \tilde{\kappa} \tilde{B}_{p-q} \wedge\star \tilde{J}_{p+1}$$
$$\star \mathcal{J}_{2q+2} \sim c_\Phi \tilde{B}_{p-q}$$

$$\mathrm{d}\star J_{q+1} = c_\phi \tilde{G}_{p+2} + 2\kappa G_{q+2} \wedge\star \mathcal{J}_{2q+2} + 2\tilde{\kappa} \tilde{\mathcal{G}}_{p-q+1} \wedge\star \tilde{J}_{p+1}$$
$$\mathrm{d}\star \mathcal{J}_{2q+2} = c_\Phi \tilde{\mathcal{G}}_{p-q+1}$$
$$\mathrm{d}\star \tilde{J}_{p+1} = (-)^p c_\phi G_{q+2}$$
$$\mathrm{d}\star \tilde{\mathcal{J}}_{p-q} = -c_\Phi \mathcal{G}_{2q+3} - (-)^p 2\tilde{\kappa} G_{q+2} \wedge\star \tilde{J}_{p+1}$$

$$S=\hat{S}[F_{q+1}]+Q_{(2q+1)}\int B_{2q+2}+\frac{\kappa}{c_\phi}A_{q+1}\wedge F_{q+1}+\int \tilde{A}_{q+1}\wedge F_{q+1}$$

$$S_{\text{bulk}}=\int_{\text{bulk}} CA_{q+1}\wedge G_{q+2}+\left(c_\phi-\kappa_\times Q_{(2q+1)}\right)\tilde{A}_{q+1}\wedge G_{q+2}$$

$$\delta A_{q+1}=\mathrm{d}\Lambda_q$$
$$\delta\tilde{A}_{q+1}=\mathrm{d}\tilde{\Lambda}_q$$
$$\delta B_{2q+2}=\mathrm{d}\Lambda_{2q+1}-\left(\kappa-\frac{C}{Q_{(2q+1)}}\right)\mathrm{d}\Lambda_q\wedge A_{q+1}-\kappa_\times\mathrm{d}\tilde{\Lambda}_q\wedge A_{q+1}$$

$$\begin{aligned}\delta S_{\text{tot}}=\int &\delta A_{q+1}\wedge\star J_{q+1}+\delta\tilde{A}_{q+1}\wedge\star\tilde{J}_{q+1}\\&+\left(\delta B_{2q+2}+\left(\kappa-\frac{C}{Q_{(2q+1)}}\right)A_{q+1}\wedge\delta A_{q+1}+\kappa_\times\tilde{A}_{q+1}\wedge\delta A_{q+1}\right)\wedge\star\mathcal{J}_{2q+2}\\&+\int_{\text{bulk}}\delta A_{q+1}\wedge\left(2CF_{q+2}+C_\times\tilde{F}_{q+2}\right)+\delta\tilde{A}_{q+1}\wedge C_\times G_{q+2}\end{aligned}$$

$$\mathrm{d}\star J_{q+1}=c_\phi\tilde{G}_{q+2}+2\kappa Q_{(2q+1)}G_{q+2}$$
$$\mathrm{d}\star\tilde{J}_{q+1}=c_\phi G_{q+2}$$

$$\tilde{J}_{q+1}=\frac{c_\phi}{2\kappa Q_{(2q+1)}}J_{q+1}$$

$$\mathrm{d}\star j_2=0$$
$$\mathrm{d}\star J_2=H_3\wedge\star J_4-\tilde{F}_5\wedge\star j_6$$
$$\mathrm{d}\star J_4=F_3\wedge\star j_6+H_3\wedge\star\mathcal{J}_6$$
$$\mathrm{d}\star j_6=0$$
$$\mathrm{d}\star\mathcal{J}_6=H_3\wedge\star\mathcal{J}_8$$
$$\mathrm{d}\star\mathcal{J}_8=0$$

$$X_{[a_1\ldots a_n]}=\tfrac{1}{n!}\left(X_{a_1\ldots a_n}+\text{signed permutations }\right)$$

$$\textstyle\int_\Sigma T\,\mathrm{d}X=-\int_M Tu\wedge\ \mathrm{d}X=\int_M\ \mathrm{d}(Tu\wedge X)-\int_M\ \mathrm{d}(Tu)\wedge X$$

$$\mathrm{d}(Tu)\wedge X=-Tu\wedge i_\beta\mathrm{d}(Tu)\wedge X=-Tu\wedge \pounds_\beta(Tu)\wedge X=0$$

$$A\cdot B=\tfrac{1}{p!}A^{\mu\cdots}B_{\mu\ldots}$$

$$\tilde{\Lambda}_{2q+1}=\Lambda_{2q+1}+\kappa\Lambda_q\wedge A_{q+1},\ \tilde{\varphi}_{2q}=\varphi_{2q}-\kappa\varphi_{q-1}\wedge A_{q+1,\Sigma}$$

$$\delta\tilde{\varphi}_{2q}=-i_\beta\tilde{\Lambda}_{2q+1}-\kappa\Lambda_{q,\Sigma}\wedge\tfrac{\mu_q}{T}$$

$$\mathcal{P}=c_\phi G_{q+2}\wedge\tilde{G}_{p+2}-c_\Phi\mathcal{G}_{2q+3}\wedge\tilde{\mathcal{G}}_{p-q+1}$$

$$S = \int \; d\tau \left[P^a \partial_\tau X_a + p_\alpha \partial_\tau \theta^\alpha + w_\alpha \partial_\tau \lambda^\alpha - \frac{1}{2}P^2\right]$$

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^a\theta)_\alpha P_a$$

$$Q_0\Psi \;= 0 \to D_{(\alpha}A_{\beta\delta\epsilon)} = (\gamma^a)_{(\alpha\beta}A_{a\delta\epsilon)}$$
$$\delta\Psi \;= Q_0\Lambda \to \delta A_{\alpha\beta\delta} = D_{(\alpha}\Lambda_{\beta\delta)}$$

$$\Psi = (\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)C_{abc} + (\lambda\gamma^{ab}\theta)(\lambda\gamma_b\theta)(\lambda\gamma^c\theta)h_{ac} + (\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)(\theta\gamma_{bc}\psi_a) - (\lambda\gamma^a\theta)(\lambda\gamma^{bc}\theta)(\lambda\gamma_b\theta)(\theta\gamma_c\psi_a) + O(\theta^5)$$

$$\partial^d\partial_{[d}C_{abc]} = 0, \Box\, h_{bc} - 2\partial^a\partial_{(b}h_{c)a} + \partial_b\partial_c(\eta^{ad}h_{ad}) = 0, (\gamma^{abc})_{\alpha\beta}\partial_b\psi_c^\beta = 0$$

$$\delta C_{abc} = \partial_a B_{bc}, \delta h_{ab} = \partial_{(a}t_{b)}, \delta\psi_a^\alpha = \partial_a\kappa^\beta$$

$$S = \int \; d\tau \left[P^a \partial_\tau X_a + p_\alpha \partial_\tau \theta^\alpha + w_\alpha \partial_\tau \lambda^\alpha + \bar{w}^\alpha\partial_\tau\bar{\lambda}_\alpha + s^\alpha\partial_\tau r_\alpha - \frac{1}{2}P^2\right]$$

$$Q = Q_0 + s$$

$$b = \frac{1}{2\eta}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^c d)P_c + \frac{2}{\eta^2}L^{(1)}_{ab,cd}\big[(\lambda\gamma^a d)(\lambda\gamma^{bcd}d) + 2(\lambda\gamma^{abcef}\lambda)N^d{}_e P_f + \frac{2}{3}(\delta^b_e\delta^d_f - \eta^{bd}\eta_{ef})(\lambda\gamma^{aecgh}\lambda)N_{gh}P^f\big] - \frac{4}{3\eta^3}L^{(2)}_{ab,cd,ef}\big[(\lambda\gamma^{abcgh}\lambda)(\lambda\gamma^{def}d)N_{gh} - 12\left[(\lambda\gamma^{abceg}\lambda)\eta^{fh} - \frac{2}{3}\eta^{f[a}(\lambda\gamma^{bce]gh}\lambda)\right](\lambda\gamma^d d)N_{gh}\big] + \frac{8}{3\eta^4}L^{(3)}_{ab,cd,ef,gh}(\lambda\gamma^{abcij}\lambda)\left[(\lambda\gamma^{defgk}\lambda)\eta^{hl} - \frac{8}{3}\eta^{h[d}(\lambda\gamma^{efgk]l}\lambda)\right]\{N_{ij}, N_{kl}\}$$

$$\eta = (\bar{\lambda}\gamma^{ab}\bar{\lambda})(\lambda\gamma_{ab}\lambda), N_{ab} = \frac{1}{2}(\lambda\gamma^{ab}w)$$

$$L^{(n)}_{a_0b_0,a_1b_1,\ldots,a_1b_1} = \left(\bar{\lambda}\gamma_{[[a_0b_0}\bar{\lambda}\right)\left(\bar{\lambda}\gamma_{a_1b_1}r\right)\ldots\left(\bar{\lambda}\gamma_{a_nb_n]}\right)r\Big)$$

$$b = P^a\bar{\Sigma}_a - \frac{4}{\eta}(\bar{\lambda}\gamma^{ab}r)(\lambda\gamma_a^c\lambda)\bar{\Sigma}_c\bar{\Sigma}_b - \frac{2}{\eta}(\bar{\lambda}r)(\lambda\gamma^{ab}\lambda)\bar{\Sigma}_a\bar{\Sigma}_b$$

$$\bar{\Sigma}^i = \frac{1}{2\eta}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^i d) + \frac{4}{\eta^2}L^{(1)}_{ab,cd}(\lambda\gamma^{abcei}\lambda)N_e^d + \frac{4}{3\eta^2}L^{(1)}_{ab,c}{}^i(\lambda\gamma^{abcde}\lambda)N_{de} - \frac{4}{3\eta^2}L^{(1)}_{ad,c}{}^d(\lambda\gamma^{aicde}\lambda)N_{de}$$

$$\{Q, \bar{\Sigma}^a\} = \frac{P^a}{2} + \frac{1}{\eta}\left[(\bar{\lambda}\gamma^{cb}\bar{\lambda})(\lambda\gamma_{ba}\lambda) - (\bar{\lambda}\gamma^{ab}\bar{\lambda})(\lambda\gamma_{bc}\lambda)\right]P^c - \frac{2}{\eta}(\bar{\lambda}\gamma^{ba}r)(\lambda\gamma_b^c\lambda)\bar{\Sigma}_c - \frac{4}{\eta}(\bar{\lambda}\gamma^{bc}r)(\lambda\gamma_b{}^a\lambda)\bar{\Sigma}_c + \frac{2}{\eta}(\bar{\lambda}r)(\lambda\gamma^{ab}\lambda)\bar{\Sigma}_b - \frac{2}{\eta^2}(\bar{\lambda}\gamma^{cd}r)(\lambda\gamma_{cd}\lambda)(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\lambda\gamma_{be}\lambda)\bar{\Sigma}^e$$

$$[Q, \mathbf{C}_\alpha] \;= -\frac{1}{3}d_\alpha - (\gamma^a\lambda)_\alpha\mathbf{C}_a$$
$$\{Q, \mathbf{C}_a\} \;= \frac{1}{3}P_a + (\lambda\gamma^{ab}\lambda)\mathbf{\Phi}_b$$
$$[Q, \mathbf{\Phi}^a] \;= (\lambda\gamma^a\mathbf{\Phi})$$
$$[Q, \mathbf{\Phi}^\alpha] \;= \frac{1}{4}(\lambda\gamma^{ab})^\alpha\mathbf{\Omega}_{ab}$$

$$\mathbf{C}_\alpha = \tfrac{1}{3} K_\alpha{}^\beta w_\beta$$
$$\mathbf{C}^a = \tfrac{1}{\eta}(\lambda\gamma^{abc})^\alpha(\bar\lambda\gamma_{bc}\bar\lambda)\left[\tfrac{1}{3}d_\alpha + [Q,\mathbf{C}_\alpha]\right]$$
$$\mathbf{\Phi}^a = \tfrac{2}{\eta}(\bar\lambda\gamma^{ab}\bar\lambda)\left[\tfrac{1}{3}P_b - \{Q,\mathbf{C}_b\}\right]$$
$$\mathbf{\Phi}^\alpha = -\tfrac{2}{\eta}(\gamma^{abc}\lambda)^\alpha(\bar\lambda\gamma_{bc}r)\mathbf{\Phi}_a$$

$$K_\alpha{}^\beta = -\tfrac{1}{6\eta}(\lambda\gamma^{ab})^\beta(\bar\lambda\gamma^{cd}\bar\lambda)(\lambda\gamma_{abcd})_\alpha - \tfrac{4}{3\eta}(\lambda\gamma^{ab})^\beta(\lambda\gamma_b^d)_\alpha(\bar\lambda\gamma_{ad}\bar\lambda) - \tfrac{2}{3\eta}(\lambda\gamma^{cd})^\beta\lambda_\alpha(\bar\lambda\gamma_{cd}\bar\lambda)$$
$$+\tfrac{1}{3\eta}\lambda^\beta(\lambda\gamma^{cd})_\alpha(\bar\lambda\gamma_{cd}\bar\lambda)$$

$$\xi_a^\alpha\mathbf{C}_\alpha = 0, (\bar\lambda\gamma^{ab}\bar\lambda)\mathbf{C}_a = 0, (\bar\lambda\gamma^a)_\alpha\mathbf{\Phi}_a = 0, R_\alpha{}^\beta\mathbf{\Phi}^\alpha = 0$$

$$\xi_a^\beta = \tfrac{1}{2}(\gamma_{abc})^{\beta\delta}\lambda_\delta(\bar\lambda\gamma^{bc}\bar\lambda)$$
$$R_\alpha{}^\beta = \left[-\tfrac{1}{2}(\lambda\gamma^b)_\alpha(\lambda\gamma^c)^\beta - \tfrac{1}{4}(\lambda\gamma^{bk}\lambda)(\gamma^c\gamma^k)_\alpha{}^\beta + \tfrac{1}{2}(\lambda\gamma^{bk})_\alpha(\lambda\gamma^{ck})^\beta - \tfrac{1}{2}(\lambda\gamma^{bc})_\alpha\lambda^\beta\right](\bar\lambda\gamma_{bc}\bar\lambda)$$

$$\lambda^\alpha K_\alpha{}^\beta = \lambda^\beta, (\lambda\gamma^{ab})^\alpha K_\alpha{}^\beta = (\lambda\gamma^{ab})^\beta, (\lambda\gamma^a)_\beta K_\alpha{}^\beta = 0, K_\alpha{}^\beta K_\beta{}^\delta = K_\alpha{}^\delta$$

$$K_\alpha^\beta = \delta_\alpha^\beta + \tfrac{1}{\eta}(\lambda\gamma^{abc})^\beta(\bar\lambda\gamma_{bc}\bar\lambda)(\lambda\gamma_a)_\alpha$$

$$\mathbf{C}_\alpha = \tfrac{w_\alpha}{3} + \tfrac{1}{3\eta}(\lambda\gamma^{abc}w)(\bar\lambda\gamma_{bc}\bar\lambda)(\lambda\gamma_a)_\alpha$$
$$\mathbf{C}_a = \tfrac{1}{3\eta}(\bar\lambda\gamma^{bc}\bar\lambda)(\lambda\gamma_{abc}d) - \tfrac{2}{3\eta}(\bar\lambda\gamma^{bc}r)(\lambda\gamma_{abc}w) + \tfrac{2}{3\eta^2}\phi(\bar\lambda\gamma^{bc}\bar\lambda)(\lambda\gamma_{abc}w)$$
$$+\tfrac{4}{3\eta^2}(\lambda\gamma_{ac}\lambda)(\bar\lambda\gamma^{bc}\bar\lambda)(\bar\lambda\gamma^{de}r)(\lambda\gamma_{bde}w)$$
$$\mathbf{\Phi}^a = \tfrac{2}{3}\Big[\tfrac{1}{\eta}(\bar\lambda\gamma^{ab}\bar\lambda)P_b - \tfrac{2}{\eta^2}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\lambda\gamma_{bcd}d) + \left\{s,\tfrac{2}{\eta^2}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)\right\}(\lambda\gamma_{bcd}w)$$
$$-\tfrac{8}{\eta^3}(\lambda\gamma^a\xi_b)(\bar\lambda\gamma^{cb}r)(\bar\lambda\gamma^{de}r)(\lambda\gamma_{cde}w)\Big]$$
$$\mathbf{\Phi}^\alpha = \tfrac{8}{3}\xi_a^\alpha\Big[\tfrac{1}{\eta^2}(\bar\lambda\gamma^{ab}r)P_b - \tfrac{4}{\eta^4}(\bar\lambda\gamma^{ab}r)(\lambda\gamma_{cb}\lambda)(\bar\lambda\gamma^{cd}\bar\lambda)(\bar\lambda\gamma^{ef}r)(\lambda\gamma_{def}d)$$
$$-\left(\tfrac{8}{\eta^4}(\bar\lambda\gamma^{ab}r)(\lambda\gamma_{cb}\lambda)(\bar\lambda\gamma^{cd}r)(\bar\lambda\gamma^{ef}r) - \tfrac{16}{\eta^5}(\bar\lambda\gamma^{ab}r)\phi(\lambda\gamma_{cb}\lambda)(\bar\lambda\gamma^{cd}\bar\lambda)(\bar\lambda\gamma^{ef}r)\right)(\lambda\gamma_{def}w)\Big]$$

$$\mathbf{C}_\alpha = -\tfrac{1}{9\eta}N^{ab}(\bar\lambda\gamma^{cd}\bar\lambda)(\lambda\gamma_{abcd})_\alpha - \tfrac{8}{9\eta}N^{ab}(\lambda\gamma_b^d)_\alpha(\bar\lambda\gamma_{ad}\bar\lambda) - \tfrac{4}{9\eta}N^{cd}\lambda_\alpha(\bar\lambda\gamma_{cd}\bar\lambda)$$
$$+\tfrac{2}{9\eta}J(\lambda\gamma^{cd})_\alpha(\bar\lambda\gamma_{cd}\bar\lambda)$$
$$\mathbf{C}_a = \tfrac{1}{3\eta}(\lambda\gamma_{abc}d)(\bar\lambda\gamma^{bc}\bar\lambda) + \tfrac{8}{3\eta^2}(\bar\lambda\gamma^{bc}\bar\lambda)(\bar\lambda\gamma^{de}r)(\lambda\gamma_{bcdfa}\lambda)N_e^f + \tfrac{8}{9\eta^2}(\bar\lambda\gamma^{bc}\bar\lambda)(\bar\lambda r)(\lambda\gamma_{abcde}\lambda)N^{de}$$
$$+\tfrac{4}{9\eta^2}(\bar\lambda\gamma_{bc}\bar\lambda)(\bar\lambda\gamma_{da}r)(\lambda\gamma^{bcdef}\lambda)N_{ef}$$
$$\mathbf{\Phi}^a = \tfrac{2}{3}\Big[\tfrac{1}{\eta}(\bar\lambda\gamma^{ab}\bar\lambda)P_b - \tfrac{2}{\eta^2}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\lambda\gamma_{bcd}d) - \tfrac{16}{\eta^3}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\bar\lambda\gamma^{ef}r)(\lambda\gamma_{bcdeg}\lambda)N_f^g$$
$$-\tfrac{8}{\eta^3}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\bar\lambda r)(\lambda\gamma_{bcdef}\lambda)N^{ef}\Big]$$
$$\mathbf{\Phi}^\alpha = \tfrac{8}{3}\xi_a^\alpha\Big[\tfrac{1}{\eta^2}(\bar\lambda\gamma^{ab}r)P_b - \tfrac{2}{\eta^3}(\bar\lambda\gamma^{ab}r)(\bar\lambda\gamma^{cd}r)(\lambda\gamma_{bcd}d) - \tfrac{8}{\eta^4}(\bar\lambda\gamma^{ab}r)(\bar\lambda\gamma^{cd}r)(\bar\lambda r)(\lambda\gamma_{bcdef}\lambda)N^{ef}$$
$$-\tfrac{16}{\eta^4}(\bar\lambda\gamma^{ab}r)(\bar\lambda\gamma^{cd}r)(\bar\lambda\gamma^{ef}r)(\lambda\gamma_{bcdeg}\lambda)N_f^g\Big]$$

$$\mathbf{C}_\alpha\Psi = C_\alpha + (\lambda\gamma^a)_\alpha\rho_a$$

$$\mathbf{C}_a\Psi = C_a + (\lambda\gamma_{ac}\lambda)s^c - Q\rho_a$$

$$\mathbf{\Phi}^a\Psi = \Phi^a + (\lambda\gamma^a\kappa) + Qs^a$$

$$\mathbf{\Phi}^\alpha\Psi = \Phi^\alpha + \left(\lambda\gamma^{ab}\right)^\alpha f_{ab} + \lambda^\alpha f + Q\kappa^\beta$$

$$\Phi^\alpha = \lambda^\beta h_\beta{}^\alpha, f_{ab} = \frac{2}{3\eta}\left(\bar{\lambda}\gamma_{ab}\bar{\lambda}\right)\lambda_\delta\tau^\delta + \frac{4}{3\eta}\left(\bar{\lambda}\gamma_{k[a}\bar{\lambda}\right)\left(\lambda\gamma^k{}_{b]}\right)_\alpha\tau^\alpha + \frac{1}{6\eta}\left(\bar{\lambda}\gamma^{cd}\bar{\lambda}\right)(\lambda\gamma_{cdab})_\alpha\tau^\alpha, f =$$

$$-\frac{1}{3\eta}\left(\bar{\lambda}\gamma_{ab}\bar{\lambda}\right)\left(\lambda\gamma^{ab}\right)_\delta\tau^\delta, \tau^\alpha = \Phi^\alpha + Q\kappa^\alpha$$

$$R_\alpha^\beta = \left[\frac{1}{12}\left(\lambda\gamma^{abcd}\right)_\alpha(\lambda\gamma_{ab})^\beta + \frac{2}{3}\left(\lambda\gamma^{kd}\right)_\alpha(\lambda\gamma_k^c)^\beta + \frac{1}{3}\lambda_\alpha\left(\lambda\gamma^{cd}\right)^\beta - \frac{1}{6}\left(\lambda\gamma^{cd}\right)_\alpha\lambda^\beta\right]\left(\bar{\lambda}\gamma_{cd}\bar{\lambda}\right)$$

$$b = \frac{3}{2}P^a\mathbf{C}_a + \frac{3}{2}(\lambda\gamma^a d)\mathbf{\Phi}_a - \frac{3}{2}(\lambda\gamma^a w)(\lambda\gamma_a\mathbf{\Phi})$$

$$b = \frac{3}{2}P^a\mathbf{C}_a + \frac{3}{2}(\lambda\gamma^a d)\mathbf{\Phi}_a - \frac{1}{2}N^{ab}(\lambda\gamma_{ab}\mathbf{\Phi})$$

$$\{Q,b\} = \frac{1}{2}P^2 + \frac{3}{2}\left(\lambda\gamma^{ab}\lambda\right)P_a\mathbf{\Phi}_b + \frac{3}{2}\left(\lambda\gamma^{ab}\lambda\right)P_b\mathbf{\Phi}_a - \frac{3}{2}(\lambda\gamma^a d)(\lambda\gamma_a\mathbf{\Phi}) + \frac{3}{2}(\lambda\gamma^a d)(\lambda\gamma_a\mathbf{\Phi}) = \frac{1}{2}P^2$$

$$\begin{aligned} b &= -\frac{1}{\eta}\left(\bar{\lambda}\gamma^{ab}\bar{\lambda}\right)P_a(\lambda\gamma_b d) + \frac{1}{2\eta}\left(\bar{\lambda}\gamma^{bc}\bar{\lambda}\right)P^a(\lambda\gamma_{abc}d) + O(r) \\ &= \frac{1}{2\eta}\left(\bar{\lambda}\gamma_{bc}\bar{\lambda}\right)\left(\lambda\gamma^{bc}\gamma^a d\right)P_a + O(r) \end{aligned}$$

$$\begin{aligned} b =& \frac{1}{2\eta}\left(\bar{\lambda}\gamma_{bc}\bar{\lambda}\right)\left(\lambda\gamma^{bc}\gamma^a d\right)P_a + \frac{4}{\eta^2}L^{(1)}_{bc,de}\left(\lambda\gamma^{bcdfa}\lambda\right)P_aN_f^e + \frac{4}{3\eta^2}L^{(1)}_{bf,cf}\left(\lambda\gamma^{abcde}\lambda\right)P_aN_{de} \\ &+ \frac{4}{3\eta^2}L^{(1)}_{bc,da}\left(\lambda\gamma^{bcdef}\lambda\right)P^aN_{ef} + \frac{2}{\eta^2}L^{(1)}_{ab,cd}(\lambda\gamma^a d)\left(\lambda\gamma^{bcd}d\right) - \frac{16}{\eta^3}L^{(2)}_{ab,cd,ef}\left(\lambda\gamma^{bcdeg}\lambda\right)N_g^f(\lambda\gamma^a d) \\ &- \frac{8}{\eta^3}L^{(2)}_{ag,cd,bg}\left(\lambda\gamma^{bcdef}\lambda\right)N_{ef}(\lambda\gamma^a d) - \frac{4}{3\eta^3}L^{(2)}_{ij,ab,cd}\left(\lambda\gamma^{aijkl}\lambda\right)\left(\lambda\gamma^{bcd}d\right)N_{kl} \\ &- \frac{16}{3\eta^4}L^{(3)}_{ig,ab,cd,jg}\left(\lambda\gamma^{bcdef}\lambda\right)\left(\lambda\gamma^{aijkl}\lambda\right)N_{kl}N_{ef} - \frac{32}{3\eta^4}L^{(3)}_{ij,ab,cd,ef}\left(\lambda\gamma^{aijkl}\lambda\right)\left(\lambda\gamma^{bcdeg}\lambda\right)N_{kl}N_g^f \end{aligned}$$

$$U^{(2)} = \{b,\Psi\}$$

$$\begin{aligned} U^{(2)} =& \frac{3}{2}P^aC_a + \frac{3}{2}(\lambda\gamma^a d)\Phi_a - \frac{1}{2}N^{ab}(\lambda\gamma_{ab}\Phi) + Q\left[-\frac{3}{2}P^a\rho_a - \frac{3}{2}(\lambda\gamma^a d)s_a - \frac{3}{2}(\lambda\gamma^a w)(\lambda\gamma_a\kappa)\right] \\ &+ \frac{3}{2}\mathbf{C}^a\partial_a\Psi + \frac{3}{2}\mathbf{\Phi}^a(\lambda\gamma_a D)\Psi + \frac{9}{2}(\lambda\gamma^a C)(\lambda\gamma_a\mathbf{\Phi}) \end{aligned}$$

$$\begin{aligned} \frac{3}{2}\mathbf{C}^a\partial_a\Psi + \frac{3}{2}\mathbf{\Phi}^a(\lambda\gamma_a D)\Psi + \frac{9}{2}(\lambda\gamma^a C)(\lambda\gamma_a\mathbf{\Phi}) =& \frac{3}{2}P^aC_a + \frac{3}{2}(\lambda\gamma^a d)\Phi_a - \frac{3}{2}(\lambda\gamma^a w)(\lambda\gamma_a\Phi) \\ &+ Q\left[-\frac{9}{2}\mathbf{C}^aC_a - \frac{9}{2}\mathbf{\Phi}^a(\lambda\gamma_a C) + \frac{9}{2}(\lambda\gamma^a\mathbf{C})\Phi_a\right] \end{aligned}$$

$$\begin{aligned} U^{(2)} =& 3P^aC_a + 3(\lambda\gamma^a d)\Phi_a - 3(\lambda\gamma^a w)(\lambda\gamma_a\Phi) + Q\left[-\frac{3}{2}P^a\rho_a - \frac{3}{2}(\lambda\gamma^a d)s_a - \frac{3}{2}(\lambda\gamma^a w)(\lambda\gamma_a\kappa)\right. \\ &\left. - \frac{9}{2}\mathbf{C}^aC_a - \frac{9}{2}\mathbf{\Phi}^a(\lambda\gamma_a C) + \frac{9}{2}(\lambda\gamma^a\mathbf{C})\Phi_a\right] \end{aligned}$$

$$\begin{aligned} b\Psi =& \frac{3}{2}\partial^aC_a + \frac{3}{2}(\lambda\gamma^a D)\Phi_a + \frac{3}{2}(\lambda\gamma^a D)(\lambda\gamma_a\kappa) - \frac{3}{2}(\lambda\gamma^a\partial_\lambda)(\lambda\gamma_a\Phi) - \frac{3}{2}(\lambda\gamma^a\partial_\lambda)(\lambda\gamma_a Q\kappa) \\ &+ Q\left[-\frac{3}{2}\partial^a\rho_a - \frac{3}{2}(\lambda\gamma^a D)s_a\right] \end{aligned}$$

$$b\Psi = Q\left[-\frac{3}{2}(\lambda\gamma_a h^a) - \frac{3}{2}\partial^a\rho_a - \frac{3}{2}(\lambda\gamma^a D)s_a - \frac{3}{2}(\lambda\gamma^a\partial_\lambda)(\lambda\gamma_a\kappa)\right]$$

$$Q\Psi + \frac{\kappa}{2}(\lambda\gamma_{ab}\lambda)\mathbf{\Phi}^a\Psi\mathbf{\Phi}^b\Psi + \frac{\kappa}{2}\Psi\{Q,\mathbf{T}\}\Psi - \kappa^2(\lambda\gamma_{ab}\lambda)\mathbf{T}\Psi\mathbf{\Phi}^a\Psi\mathbf{\Phi}^b\Psi = 0$$

$$\mathbf{T} = \frac{32}{9\eta^3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)N_{ab}$$

$$\Psi = \sum_{\mathcal{P}} \Psi_{\mathcal{P}} e^{ik_{\mathcal{P}}\cdot X}$$

$$\begin{aligned} Q\Psi_{p_1} &= 0 \\ Q\Psi_{p_1p_2} &= -\kappa(\lambda\gamma_{ab}\lambda)\mathbf{\Phi}^a\Psi_{p_1}\mathbf{\Phi}^b\Psi_{p_2} - \frac{\kappa}{2}\Psi_{p_1}\{Q,\mathbf{T}\}\Psi_{p_2} - \frac{\kappa}{2}\Psi_{p_2}\{Q,\mathbf{T}\}\Psi_{p_1} \\ Q\Psi_{p_1p_2p_3} &= -\sum_{\mathcal{P}=\mathcal{QUR}} \kappa\left[(\lambda\gamma_{ab}\lambda)\mathbf{\Phi}^a\Psi_{\mathcal{Q}}\mathbf{\Phi}^b\Psi_{\mathcal{R}} + \frac{1}{2}\Psi_{\mathcal{Q}}\{Q,\mathbf{T}\}\Psi_{\mathcal{R}}\right] \\ &\quad + \sum_{\mathcal{P}=\mathcal{QURUS}} \kappa^2(\lambda\gamma_{ab}\lambda)\mathbf{T}\Psi_{\mathcal{Q}}\mathbf{\Phi}^a\Psi_{\mathcal{R}}\mathbf{\Phi}^b\Psi_{\mathcal{S}} \\ &\vdots \end{aligned}$$

$$Q\tilde{\Psi}_{p_1p_2} = -\kappa(\lambda\gamma_{ab}\lambda)\Phi^a_{p_1}\Phi^b_{p_2}$$

$$\begin{aligned}\tilde{\Psi}_{p_1p_2} =& \Psi_{p_1p_2} + \kappa(\lambda\gamma_{ab}\lambda)s^a_{p_1}\Phi^b_{p_2} - \kappa(\lambda\gamma_{ab}\lambda)\Phi^a_{p_1}s^b_{p_2} \\ &+\kappa(\lambda\gamma_{ab}\lambda)s^a_{p_1}Qs^b_{p_2} - \frac{\kappa}{2}\Psi_{p_1}\mathbf{T}\Psi_{p_2} - \frac{\kappa}{2}\Psi_{p_2}\mathbf{T}\Psi_{p_1}\end{aligned}$$

$$\tilde{\Psi}_{p_1p_2} = -\frac{2\kappa}{k^2_{p_1p_2}} b\left[(\lambda\gamma_{ab}\lambda)\Phi^a_{p_1}\Phi^b_{p_2}\right]$$

$$b\left[(\lambda\gamma_{ab}\lambda)\Phi^a_{p_1}\Phi^b_{p_2}\right] = \tilde{C}_{p_1p_2} + Q\Lambda_{p_1p_2}$$

$$\Lambda_{p_1p_2} = -\frac{2}{k^2_{p_1p_2}} b(\tilde{C}_{p_1p_2})$$

$$\begin{aligned}\tilde{C}_{p_1p_2} =& \frac{1}{2}\Big[(\lambda\gamma^{bc}\lambda)h_{p_1,ab}k^a_{p_2}\Phi_{p_2,c} + \Omega_{p_1,ab}k^a_{p_2}C^b_{p_2} - (\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta C^{ab}_{p_2} \\ &+(\lambda\gamma^{bc}\lambda)h_{p_2,ab}k^a_{p_1}\Phi_{p_1,c} + \Omega_{p_2,ab}k^a_{p_1}C^b_{p_1} - (\lambda\gamma^b)_\delta\lambda^\alpha T_{p_2,\alpha a}{}^\delta C^{ab}_{p_1}\Big]\end{aligned}$$

$$T_{\alpha a}{}^\delta = \frac{1}{36}\left[(\gamma^{bcd})_\alpha{}^\delta H_{abcd} + \frac{1}{8}(\gamma_a{}^{bcde})_\alpha{}^\delta H_{bcde}\right]$$

$$\begin{aligned}Q\tilde{C}_{p_1p_2} =& \frac{1}{2}\Big[(\lambda\gamma^{bc}\lambda)(k_{p_1}\cdot k_{p_2})\Phi_{p_1,b}\Phi_{p_2,c} - (\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}\delta^{[a}k^{[a}_{p_2}C^{c]}_{p_2} \\ &+Q\left[-(\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta C^{ab}_{p_2}\right] + (1\leftrightarrow 2)\Big]\end{aligned}$$

$$\begin{aligned}Q\tilde{C}_{p_1p_2} =& \frac{1}{2}\Big[(\lambda\gamma^{bc}\lambda)(k_{p_1}\cdot k_{p_2})\Phi_{p_1,b}\Phi_{p_2,c} + (\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta QC^{ab}_{p_2} \\ &+(\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta\left[(\lambda\gamma^{[bc}\lambda)h^{a]}{}_{c,p_2} - (\lambda\gamma^{ab})_\beta\Phi^\beta_{p_2}\right] \\ &+Q\left[-(\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta C^{ab}_{p_2}\right] + (1\leftrightarrow 2)\Big] \\ =& \frac{1}{2}\left[(\lambda\gamma^{bc}\lambda)(k_{p_1}\cdot k_{p_2})\Phi_{p_1,b}\Phi_{p_2,c} - (\lambda\gamma_b)_\delta\lambda^\alpha T_{p_1,\alpha a}{}^\delta(\lambda\gamma^{ab})_\beta\Phi^\beta_{p_2} + (1\leftrightarrow 2)\right]\end{aligned}$$

$$(\lambda\gamma_b)_\delta T_{\alpha a}{}^\delta\lambda^\alpha = \frac{1}{12}\left[(\lambda\gamma^{de}\lambda)H_{abde} + \frac{1}{24}(\lambda\gamma_{ab}{}^{cdef}\lambda)H_{cdef}\right]$$

$$(\lambda\gamma_{ab})_\alpha(\lambda\gamma^{abcdef}\lambda) = -24(\lambda\gamma^{[ab})_\alpha(\lambda\gamma^{cd]}\lambda)$$

$$Q\tilde{C}_{p_1p_2} = (\lambda\gamma^{bc}\lambda)(k_{p_1}\cdot k_{p_2})\Phi_{p_1,b}\Phi_{p_2,c}$$

$$\mathcal{D}T^A = E^B R_B{}^A, \mathcal{D}R_A{}^B = 0$$

$$\mathcal{D}\mathcal{F}_{A_1\dots A_m}{}^{B_1\dots B_n} = d\mathcal{F}_{A_1\dots A_m}{}^{B_1\dots B_n} - \Omega_{A_1}{}^C\mathcal{F}_{CA_2\dots A_m}{}^{B_1\dots B_n} + \dots + \mathcal{F}_{A_1\dots A_m}{}^{C\dots B_n}\Omega_C{}^{B_1} + \cdots$$

$$[\nabla_A, \nabla_B\} = -T_{AB}^C\nabla_C - 2\Omega_{[AB\}}^C\nabla_C$$
$$R_{AB,C}{}^D = 2\nabla_{[A}\Omega_{B\}C}{}^D + T_{AB}{}^F\Omega_{FC}{}^D + \Omega_{[AB\}}{}^F\Omega_{FC}{}^D$$

$$\nabla_A = D_A - h_A^B D_B$$

$$D_A = \hat{E}_A{}^M\partial_M, h_A{}^B = \hat{E}_A{}^M E_M^{(1)B} = -E_A^{(1)M}\hat{E}_M{}^B, \left(\hat{E}_A{}^M, \hat{E}_M{}^B\right)\left(E_A^{(1)M}, E_M^{(1)A}\right) T_{\alpha\beta}{}^\delta = T_{a\alpha}{}^c =$$

$$T_{ab}{}^c = G_{\alpha\beta\delta\epsilon} = G_{a\alpha\beta\delta} = G_{abc\alpha}T_{\alpha\beta}{}^a = (\gamma^a)_{\alpha\beta}, G_{\alpha\beta ab} = (\gamma_{ab})_{\alpha\beta}$$

$$2D_{(\alpha}h_{\beta)}^a - 2h_{(\alpha}^\delta(\gamma^a)_{\beta)\delta} + h_b^a\left(\gamma^b\right)_{\alpha\beta} = 0$$
$$2D_{(\alpha}h_{\beta)}^\delta - 2\Omega_{(\alpha\beta)}^\delta + (\gamma^a)_{\alpha\beta}h_a^\delta = 0$$
$$\partial_a h_\alpha^\beta - D_\alpha h_a^\beta - T_{a\alpha}^\beta - \Omega_{a\alpha}^\beta = 0$$
$$\partial_a h_\alpha^b - D_\alpha h_a^b - h_a^\beta\left(\gamma^b\right)_{\beta\alpha} + \Omega_{\alpha a}^b = 0$$
$$\partial_a h_b^\alpha - \partial_b h_a^\alpha - T_{ab}^\alpha = 0$$
$$\partial_a h_b^c - \partial_b h_a^c - 2\Omega_{ab}^c = 0$$

$$H_{ABCD} = \hat{E}_{[D}{}^Q\hat{E}_C{}^P\hat{E}_B{}^N\hat{E}_{A\}}{}^M G_{MNPQ}$$

$$H_{ABCD} = 4D_{[A}C_{BCD\}} + 6\hat{T}_{[AB}{}^E C_{ECD\}}, C_{ABC} = \hat{E}_{[C}{}^P\hat{E}_B{}^N\hat{E}_{A\}}{}^M F_{MNP}$$

$$4D_{(\alpha}C_{\beta\delta\epsilon)} + 6(\gamma^a)_{(\alpha\beta}C_{a\delta\epsilon)} = 0$$
$$\partial_a C_{\alpha\beta\delta} - 3D_{(\alpha}C_{a\beta\delta)} + 3\left(\gamma^b\right)_{(\alpha\beta}C_{ba\delta)} = 3(\gamma_{ab})_{(\alpha\beta}h_{\delta)}^b$$
$$2\partial_{[a}C_{b]\alpha\beta} + 2D_{(\alpha}C_{\beta)ab} + (\gamma^c)_{\alpha\beta}C_{cab} = 2\left(\gamma_{[b}^c\right)_{\alpha\beta}h_{a]c} + 2(\gamma_{ab})_{(\alpha\delta}h_{\beta)}^\delta$$
$$3\partial_{[a}C_{bc]\alpha} - D_\alpha C_{abc} = 3\left(\gamma_{[ab}\right)_{\alpha\beta}h_{c]}^\beta$$

$$3QC_\epsilon + D_\epsilon\Psi = -3(\lambda\gamma^a)_\epsilon C_a$$

$$[Q, \mathbf{C}_\epsilon] = -\tfrac{1}{3}d_\epsilon - (\lambda\gamma^a)_\epsilon\mathbf{C}_a$$

$$M_\alpha{}^\beta = \delta_\alpha^\beta - \tfrac{1}{4\alpha}\left(\bar{\lambda}\gamma_c\right)^\beta(\lambda\gamma^c)_\alpha - \tfrac{1}{2\eta\alpha}\left(\bar{\lambda}\gamma_a\right)^\beta\left(\lambda\gamma^{ab}\lambda\right)\left(\bar{\lambda}\gamma_{cb}\bar{\lambda}\right)(\lambda\gamma^c)_\alpha + \tfrac{1}{8\alpha}\left(\bar{\lambda}\gamma_{cd}\right)^\beta\left(\lambda\gamma^{cd}\right)_\alpha$$
$$+\tfrac{1}{8\eta\alpha}\left(\bar{\lambda}\gamma_{ab}\right)^\beta\left(\bar{\lambda}\gamma_{cd}\bar{\lambda}\right)\left(\lambda\gamma^{ab}\lambda\right)\left(\lambda\gamma^{cd}\right)_\alpha - \tfrac{1}{2\eta\alpha}\left(\bar{\lambda}\gamma_{ac}\right)^\beta\left(\bar{\lambda}\gamma_{bd}\bar{\lambda}\right)\left(\lambda\gamma^{ab}\lambda\right)\left(\lambda\gamma^{cd}\right)_\alpha$$

$$M_\alpha^\beta = \delta_\alpha^\beta - \tfrac{1}{4\alpha}\left(\bar{\lambda}\gamma_c\right)^\beta(\lambda\gamma^c)_\alpha - \tfrac{1}{2\eta\alpha}\left(\bar{\lambda}\gamma_a\right)^\beta\left(\lambda\gamma^{ab}\lambda\right)\left(\bar{\lambda}\gamma_{cb}\bar{\lambda}\right)(\lambda\gamma^c)_\alpha$$
$$+\tfrac{1}{8\eta\alpha}\left(\bar{\lambda}\gamma_{ab}\right)^\beta\left(\bar{\lambda}\gamma_{cd}\bar{\lambda}\right)\left(\lambda\gamma^{abcde}\lambda\right)(\lambda\gamma_e)_\alpha$$

$$\left(\gamma^{[ab}\right)_{(\delta\epsilon}\left(\gamma^{cd]}\right)_{\mu)\alpha} = -\tfrac{1}{6}(\gamma_k)_{(\delta\epsilon}\left(\gamma^{abcdk}\right)_{\mu)\alpha} - \tfrac{1}{6}\left(\gamma^{abcdk}\right)_{(\delta\epsilon}(\gamma_k)_{\mu)\alpha}$$

$$\left(\gamma^{ab}\right)_\alpha{}^\beta(\gamma_{ab})_\delta{}^\epsilon = 2(\gamma^a)_\alpha{}^\beta(\gamma_a)_\delta{}^\epsilon + 4(\gamma^a)_\alpha{}^\epsilon(\gamma_a)_\delta{}^\beta + 4(\gamma^a)_{\alpha\delta}(\gamma_a)^{\epsilon\beta} - 4\delta_\alpha^\epsilon\delta_\delta^\beta + 4C_{\alpha\delta}C^{\epsilon\beta}$$

$$\frac{1}{8\eta\alpha}(\bar{\lambda}\gamma_{ab}w)(\lambda\gamma^{abcde}\lambda)(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha = \frac{1}{8\eta\alpha}(\lambda\gamma_{ab}w)(\bar{\lambda}\gamma^{ab}\gamma^{cde}\lambda)(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha + \frac{1}{4\alpha}(\bar{\lambda}\gamma_a w)(\lambda\gamma^a)_\alpha$$
$$-\frac{1}{\eta}(\lambda\gamma_a w)(\bar{\lambda}\gamma^{ac}\bar{\lambda})(\lambda\gamma_c)_\alpha - \frac{1}{\eta}(w\gamma^{cde}\lambda)(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha$$
$$-\frac{1}{2\eta\alpha}(\bar{\lambda}\gamma_a w)(\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{cb}\bar{\lambda})(\lambda\gamma^c)_\alpha = \frac{1}{\eta\alpha}(\lambda\gamma_a\bar{\lambda})(\lambda\gamma^{ab}w)(\bar{\lambda}\gamma_{cb}\bar{\lambda})(\lambda\gamma^c)_\alpha + \frac{1}{\eta}(\lambda\gamma_a w)(\bar{\lambda}\gamma^{ac}\bar{\lambda})(\lambda\gamma_c)_\alpha$$

$$M_\alpha{}^\beta = \delta_\alpha^\beta + \frac{1}{\eta}(\lambda\gamma^{cde})^\beta(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha + \frac{1}{\eta\alpha}(\lambda\gamma_a\bar{\lambda})(\lambda\gamma^{ab})^\beta(\bar{\lambda}\gamma_{cb}\bar{\lambda})(\lambda\gamma^c)_\alpha$$
$$+\frac{1}{8\eta\alpha}(\lambda\gamma_{ab})^\beta(\bar{\lambda}\gamma^{ab}\gamma^{cde}\lambda)(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha$$

$$M_{1,\alpha}{}^\beta = \frac{1}{\eta\alpha}(\lambda\gamma_a\bar{\lambda})(\lambda\gamma^{ab})^\beta(\bar{\lambda}\gamma_{cb}\bar{\lambda})(\lambda\gamma^c)_\alpha, M_{2,\alpha}{}^\beta = \frac{1}{8\eta\alpha}(\lambda\gamma_{ab})^\beta(\bar{\lambda}\gamma^{ab}\gamma^{cde}\lambda)(\bar{\lambda}\gamma_{cd}\bar{\lambda})(\lambda\gamma_e)_\alpha$$

$$(\lambda\gamma^a)_\beta M_{1,\alpha}{}^\beta = 0, (\lambda\gamma^a)_\beta M_{2,\alpha}{}^\beta = 0$$
$$M_{1,\alpha}{}^\alpha = 0, M_{2,\alpha}{}^\alpha = 0$$

$$(\gamma_a)_{(\epsilon\alpha}(\gamma^{abc})_{\delta)\rho} = -(\gamma^{[b})_{(\epsilon\alpha}(\gamma^{c]})_{\delta)\rho} + (\gamma^{[b}{}_k)_{(\epsilon\alpha}(\gamma^{c]k})_{\delta)\rho} + (\gamma^{bc})_{(\epsilon\alpha}C_{\delta)\rho}$$

$$(\lambda\gamma^a)_\alpha(\lambda\gamma^{abc})_\rho = -(\lambda\gamma^{[b})_\alpha(\lambda\gamma^{c]})_\rho - \frac{1}{2}(\lambda\gamma^{[b}{}_k\lambda)(\gamma^{c]k})_{\alpha\rho} + (\lambda\gamma^{[b}{}_k)_\alpha(\lambda\gamma^{c]k})_\rho$$
$$-\frac{1}{2}(\lambda\gamma^{bc}\lambda)C_{\alpha\rho} - (\lambda\gamma^{bc})_\alpha\lambda_\rho$$

$$(\lambda\gamma_a)_\alpha(\lambda\gamma^{abc})^\beta = -\delta_\alpha^\beta(\lambda\gamma^{bc}\lambda) + (\lambda\gamma_a)^\beta(\lambda\gamma^a\gamma^{bc})_\alpha - (\lambda\gamma^{[b}{}_k)_\alpha(\lambda\gamma^{c]k})^\beta$$
$$-(\lambda\gamma_k\gamma^{[c})_\alpha(\lambda\gamma^{b]k})^\beta + (\lambda\gamma^{bc})_\alpha\lambda^\beta$$

$$K_\alpha{}^\beta = \frac{1}{\eta}\left[(\lambda\gamma_a)^\beta(\lambda\gamma^a\gamma^{bc})_\alpha - \lambda_\alpha(\lambda\gamma^{bc})^\beta + (\lambda\gamma^{bc})_\alpha\lambda^\beta - 2(\lambda\gamma^{[b}{}_k)_\alpha(\lambda\gamma^{c]k})^\beta\right](\bar{\lambda}\gamma_{bc}\bar{\lambda})$$

$$(\lambda\gamma^k)^\beta(\lambda\gamma_k)_\epsilon = -\frac{1}{6}(\lambda\gamma^{ab})^\beta(\lambda\gamma_{ab})_\epsilon - \frac{2}{3}\lambda_\epsilon\lambda^\beta$$

$$K_\alpha{}^\beta = -\frac{1}{6\eta}(\lambda\gamma^{ab})^\beta(\bar{\lambda}\gamma^{cd}\bar{\lambda})(\lambda\gamma_{abcd})_\alpha - \frac{4}{3\eta}(\lambda\gamma^{ck})_\alpha(\lambda\gamma_k{}^d)^\beta(\bar{\lambda}\gamma_{cd}\bar{\lambda}) - \frac{2}{3\eta}(\lambda\gamma^{cd})^\beta\lambda_\alpha(\bar{\lambda}\gamma_{cd}\bar{\lambda})$$
$$+\frac{1}{3\eta}\lambda^\beta(\lambda\gamma^{cd})_\alpha(\bar{\lambda}\gamma_{cd}\bar{\lambda})$$

$$ds^2_{\text{bound.}} = -dt^2 + d\varphi^2 + dz^2$$

$$\Phi_{\text{M}} = \oiiint A_\varphi d\varphi$$

$$\mathcal{S} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R - \sum_{i=1}^3 \frac{(\partial\Phi_i)^2}{2} + \frac{2}{L^2}\cosh(\Phi_i) - \frac{1}{4}\sum_{i=1}^4 X_i^{-2}\bar{F}_i^2\right)$$

$$\bar{F}_i = d\bar{A}_i, X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\Phi}}, \vec{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$$

$$\vec{a}_1 = (1,1,1), \vec{a}_2 = (1,-1,-1), \vec{a}_3 = (-1,1,-1), \vec{a}_4 = (-1,-1,1)$$

$$ds^2_{11} = \tilde{\Delta}^{2/3}ds_4^2 + 4L^2\tilde{\Delta}^{-1/3}\sum_{i=1}^4 X_i^{-1}\left(d\mu_i^2 + \mu_i^2\left(d\varphi_i + \frac{1}{2L}\bar{A}_i\right)^2\right)$$
$$F = -\frac{1}{L}\epsilon_4\sum_{i=1}^4 (X_i^2\mu_i^2 - \tilde{\Delta}X_i) + LX_i^{-1\star_4}dX_i \wedge d\mu_i^2 -$$
$$-4L^2\sum_i X_i^{-2}\mu_i d\mu_i \wedge \left(d\varphi_i + \frac{1}{2L}\bar{A}_i\right)\wedge^{\star_4}\bar{F}_i$$

$$\Phi_a = \sqrt{\frac{2}{3}}\phi, \bar{F}_1 = \sqrt{2}F^1, \bar{F}_2 = \bar{F}_3 = \bar{F}_4 = \sqrt{\frac{2}{3}}F^2$$

$$\mathcal{S} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}(\partial\phi)^2 + \frac{3}{L^2}\cosh\left(\sqrt{\frac{2}{3}}\phi\right) - \frac{1}{4}e^{3\sqrt{\frac{2}{3}}\phi}(F^1)^2 - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\phi}(F^2)^2\right)$$

$$\partial_\mu\left(e^{3\sqrt{\frac{2}{3}}\phi}\sqrt{-g}F^{1\mu\nu}\right) = 0, \partial_\mu\left(e^{-\sqrt{\frac{2}{3}}\phi}\sqrt{-g}F^{2\mu\nu}\right) = 0,$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = e^{3\sqrt{\frac{2}{3}}\phi}T^1_{\mu\nu} + e^{-\sqrt{\frac{2}{3}}\phi}T^2_{\mu\nu} + T^\phi_{\mu\nu}$$

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + \frac{\sqrt{6}}{L^2}\sinh\left(\sqrt{\frac{2}{3}}\phi\right) = \frac{1}{2\sqrt{6}}\left(3e^{3\sqrt{\frac{2}{3}}\phi}(F^1)^2 - e^{-\sqrt{\frac{2}{3}}\phi}(F^2)^2\right),$$

$$T^\Lambda_{\mu\nu} = F^\Lambda_{\mu\rho}F^{\Lambda\rho}_\nu - \frac{1}{4}g_{\mu\nu}F^\Lambda_{\rho\sigma}F^{\Lambda\rho\sigma}$$

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}(\partial\phi)^2 + \frac{3}{L^2}\cosh\left(\sqrt{\frac{2}{3}}\phi\right)\right)$$

$$\begin{aligned}\delta\Psi^A_\mu =& \partial_\mu\epsilon^A + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon^A - \frac{1}{2L}\left(\frac{1}{\sqrt{2}}A^1_\mu + \sqrt{\frac{3}{2}}A^2_\mu\right)i(\sigma^2)^A{}_B\epsilon^B + \\ &+ \frac{1}{8}\left(\frac{1}{\sqrt{2}}F^1_{\nu\rho}e^{\sqrt{\frac{3}{2}}\phi} + \sqrt{\frac{3}{2}}F^2_{\nu\rho}e^{-\frac{\phi}{\sqrt{6}}}\right)\gamma^{\nu\rho}\gamma_\mu\varepsilon^{AB}\epsilon_B \\ &+ \frac{1}{2}\mathcal{W}\gamma_\mu\delta^{AB}\epsilon_B\end{aligned}$$

$$\begin{aligned}\delta\lambda^A =& -\gamma^\mu\partial_\mu\phi\epsilon^A + \frac{1}{2\sqrt{2}}\left(-\sqrt{\frac{3}{2}}F^1_{\nu\rho}e^{\sqrt{\frac{3}{2}}\phi} + \frac{1}{\sqrt{2}}F^2_{\nu\rho}e^{-\frac{\phi}{\sqrt{6}}}\right)\gamma^{\nu\rho}\varepsilon^{AB}\epsilon_B \\ &- \frac{1}{2L}\sqrt{\frac{3}{2}}\left(e^{-\sqrt{\frac{3}{2}}\phi} - e^{+\frac{\phi}{\sqrt{6}}}\right)\delta^{AB}\epsilon_B\end{aligned}$$

$$\mathcal{W} = \frac{e^{-\sqrt{\frac{3}{2}}\phi} + 3e^{\frac{\phi}{\sqrt{6}}}}{4L}$$

$$\epsilon^A = \mathrm{Re}\epsilon^A + i\mathrm{Im}\epsilon^A$$

$$\chi_{\mathrm{R}} = \mathrm{Re}\epsilon^1 + i\mathrm{Re}\epsilon^2, \chi_{\mathrm{I}} = \mathrm{Im}\epsilon^1 + i\mathrm{Im}\epsilon^2$$

$$\epsilon^A_{(\mathrm{M})} = \epsilon_A + \epsilon^A = 2\mathrm{Re}\epsilon^A$$

$$\gamma^5\epsilon^A_{(\mathrm{M})} = \epsilon_A - \epsilon^A = -2i\mathrm{Im}\epsilon^A$$

$$\mathrm{Im}\epsilon^A = i\gamma^5\mathrm{Re}\epsilon^A$$

$$\chi_{\mathrm{I}} \equiv \mathrm{Im}\epsilon^1 + i\mathrm{Im}\epsilon^2 = i\gamma^5(\mathrm{Re}\epsilon^1 + i\mathrm{Re}\epsilon^2) = i\gamma^5\chi_{\mathrm{R}}$$

$$\chi_{\mathrm{R}} \to e^{i\Theta}\chi_{\mathrm{R}} \Rightarrow \chi_{\mathrm{I}} \to e^{i\Theta}\chi_{\mathrm{I}},$$

$$\epsilon^A = S(\Theta)^A{}_B \mathring{\epsilon}^B, S(\Theta)^A{}_B = \begin{pmatrix} \cos(\Theta) & -\sin(\Theta) \\ \sin(\Theta) & \cos(\Theta) \end{pmatrix}$$

$$\epsilon^1_{(\mathrm{M})} = 2\mathrm{Re}\chi_{\mathrm{R}}, \epsilon^2_{(\mathrm{M})} = 2\mathrm{Im}\chi_{\mathrm{R}}$$

$$\epsilon^1 = \frac{(\mathbb{1}-\gamma^5)}{2}\epsilon^1_{(\mathrm{M})} = (\mathbb{1}-\gamma^5)\mathrm{Re}\chi_{\mathrm{R}}$$
$$\epsilon^2 = \frac{(\mathbb{1}-\gamma^5)}{2}\epsilon^2_{(\mathrm{M})} = (\mathbb{1}-\gamma^5)\mathrm{Im}\chi_{\mathrm{R}}$$

$$0 = \partial_\mu\chi_{\mathrm{R}} + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\chi_{\mathrm{R}} + \frac{i}{2L}\left(\frac{1}{\sqrt{2}}A_\mu^1 + \sqrt{\frac{3}{2}}A_\mu^2\right)\chi_{\mathrm{R}}$$
$$-\frac{i}{8}\left(\frac{1}{\sqrt{2}}F_{\nu\rho}^1 e^{\sqrt{\frac{3}{2}}\phi} + \sqrt{\frac{3}{2}}F_{\nu\rho}^2 e^{-\frac{\phi}{\sqrt{6}}}\right)\gamma^{\nu\rho}\gamma_\mu\chi_{\mathrm{R}} + \frac{1}{2}\mathcal{W}\gamma_\mu\chi_{\mathrm{R}}$$
$$0 = -\gamma^\mu\partial_\mu\phi\chi_{\mathrm{R}} - \frac{i}{2\sqrt{2}}\left(-\sqrt{\frac{3}{2}}F_{\nu\rho}^1 e^{\sqrt{\frac{3}{2}}\phi} + \frac{1}{\sqrt{2}}F_{\nu\rho}^2 e^{-\frac{\phi}{\sqrt{6}}}\right)\gamma^{\nu\rho}\chi_{\mathrm{R}}$$
$$-\frac{1}{2L}\sqrt{\frac{3}{2}}\left(e^{-\sqrt{\frac{3}{2}}\phi} - e^{+\frac{\phi}{\sqrt{6}}}\right)\chi_{\mathrm{R}}$$

$$t \to i\varphi, \varphi \to it, Q_\Lambda \to iQ_\Lambda$$

$$e^0 = \sqrt{\Upsilon(x)}dt, e^1 = \sqrt{\frac{\Upsilon(x)}{f(x)}}\eta dx, e^2 = \sqrt{\Upsilon(x)f(x)}d\varphi, e^3 = \sqrt{\Upsilon(x)}dz$$
$$\phi = \sqrt{\frac{3}{2}}\ln(x), A^1 = Q_1(x^{-2} - x_0^{-2})d\varphi, A^2 = Q_2(x^2 - x_0^2)d\varphi$$

$$\Upsilon(x) = \frac{4L^2x}{(x^2-1)^2\eta^2}, f(x) = 1 + \frac{\eta^2(x^2-1)^3(3Q_1^2 - x^2Q_2^2)}{6L^2x^2}$$

$$x = 1 \pm \left(\frac{L^2}{\eta\rho} - \frac{L^6}{8\eta^3\rho^3}\right) + \frac{L^8}{8\eta^4\rho^4} + O(\rho^{-5})$$

$$\Upsilon(x) = \frac{\rho^2}{L^2} + O(\rho^{-2})$$
$$g_{\varphi\varphi} = \Upsilon(x)f(x) = \frac{\rho^2}{L^2} - \frac{\mu}{\rho} + O(\rho^{-2})$$
$$\mu = \mp\frac{4L^2}{3\eta}(3Q_1^2 - Q_2^2)$$

$$\phi = L^2\frac{\phi_0}{\rho} + L^4\frac{\phi_1}{\rho^2} + O(\rho^{-3})$$

$$\phi_0 = \pm\frac{\sqrt{6}}{2\eta}, \phi_1 = -\frac{\sqrt{6}}{4\eta^2}$$

$$f(x_0) = 0$$

$$\Delta^{-1} = \left|\frac{1}{4\pi\eta}\frac{df}{dx}\right|_{x=x_0} = \left|\frac{\eta(x_0^2-1)^2}{4\pi L^2x_0^3}(Q_1^2(1+2x_0^2) - Q_2^2x_0^4)\right|.$$

$$\Phi_{\mathrm{M}}^1 = \int F^1 = \oint A^1 = Q_1\Delta(1 - x_0^{-2}) \equiv 2\pi L\psi_1$$
$$\Phi_{\mathrm{M}}^2 = \int F^2 = \oint A^2 = Q_2\Delta(1 - x_0^2) \equiv 2\pi L\psi_2$$

$$\langle\mathcal{O}\rangle = \phi_0 = \pm\frac{\sqrt{6}}{2}\frac{\pi x_0|\psi_1^2(1+2x_0^2)-\psi_2^2|}{\Delta}$$

$$\langle T_{tt}\rangle = -\frac{\mu}{2\kappa L^2}, \langle T_{zz}\rangle = \frac{\mu}{2\kappa L^2}, \langle T_{\varphi\varphi}\rangle = -\frac{\mu}{\kappa L^2}$$

$$\langle J_1^\nu\rangle = \frac{\delta\mathcal{S}}{\delta A_\nu^1} = -\frac{1}{\kappa}N_\mu e^{3\sqrt{\frac{2}{3}}\phi}F^{1\mu\nu}\sqrt{|h|} = \frac{2Q_1}{\eta\kappa}\delta_\varphi^\nu$$

$$\langle J_2^\nu\rangle = \frac{\delta\mathcal{S}}{\delta A_\nu^2} = -\frac{1}{\kappa}N_\mu e^{-\sqrt{\frac{2}{3}}\phi}F^{2\mu\nu}\sqrt{|h|} = -\frac{2Q_2}{\eta\kappa}\delta_\varphi^\nu$$

$$q_1 \equiv \frac{\Delta^2}{4\pi^2 L}\frac{Q_1}{\eta}, q_2 \equiv \frac{\Delta^2}{4\pi^2 L}\frac{Q_2}{\eta}$$

$$Q_1 = \frac{2\pi L\psi_1}{\Delta(1-x_0^{-2})}, Q_2 = \frac{2\pi L\psi_2}{\Delta(1-x_0^2)}, \eta = \frac{x_0^{-1}\Delta}{\pi|\psi_1^2(1+2x_0^2)-\psi_2^2|}$$

$$f(x_0) = 1 + \frac{2}{3}\frac{3(1-x_0^2)^2\psi_1^2-(1-x_0^2)(3\psi_1^2-\psi_2^2)}{x_0^2\left(2(1-x_0^2)\psi_1^2-(3\psi_1^2-\psi_2^2)\right)^2}.$$

$$P(x_0) = 4\psi_1^4x_0^6 + 2\psi_1^2(2\psi_1^2 - 2\psi_2^2 + 1)x_0^4 + \left(\psi_1^4 + \psi_2^4 - 2\psi_1^2\psi_2^2 - 2\psi_1^2 - \frac{2}{3}\psi_2^2\right)x_0^2 + \frac{2}{3}\psi_2^2 = 0$$

$$36\psi_1^6 - 36\psi_2^6 + 108\psi_1^4\psi_2^2 - 108\psi_1^2\psi_2^4 - 141\psi_1^4 - 258\psi_1^2\psi_2^2 + 75\psi_2^4 + 132\psi_1^2 - 52\psi_2^2 + 12 = 0.$$

$$\eta = \frac{3}{2\pi}\Delta\frac{|x_0^4q_2^2-2x_0^2q_1^2-q_1^2|}{x_0(x_0^2-1)(x_0^2q_2^2-3q_1^2)}$$

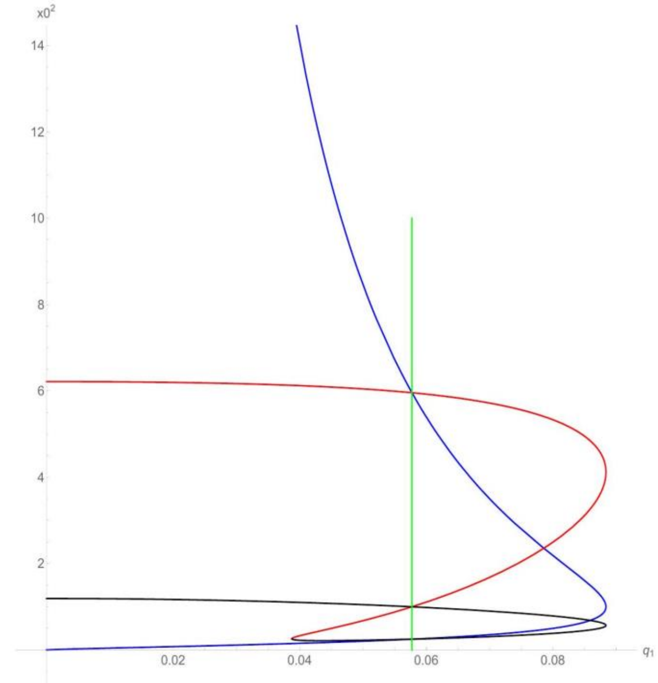
$$f(x_0) = 1 + \frac{(x_0^4q_2^2-2x_0^2q_1^2-q_1^2)^4}{x_0^6(x_0^2-1)(3q_1^2-q_2^2x_0^2)^3}\frac{3^3}{2} = 0$$

$$q_2^2 = \frac{2}{27}\frac{(x_0^2-1)}{x_0^4}\lambda^3 + q_1^2\frac{(1+2x_0^2)}{x_0^4}$$

$$8\lambda^9(\lambda^2-1)(\lambda^2+\lambda+1) + 4\times 3^4q_1^2\lambda^6 - 2\times 3^7q_1^4\lambda^3 + 3^9q_1^6 = 0$$

$$q_1^2 = \frac{2}{3^3}\left(\lambda^3 - \lambda^4 e^{\frac{2\pi i}{3}k}\right)$$

Figuras. Simetría de gauge y escalares de una partícula supermasiva.



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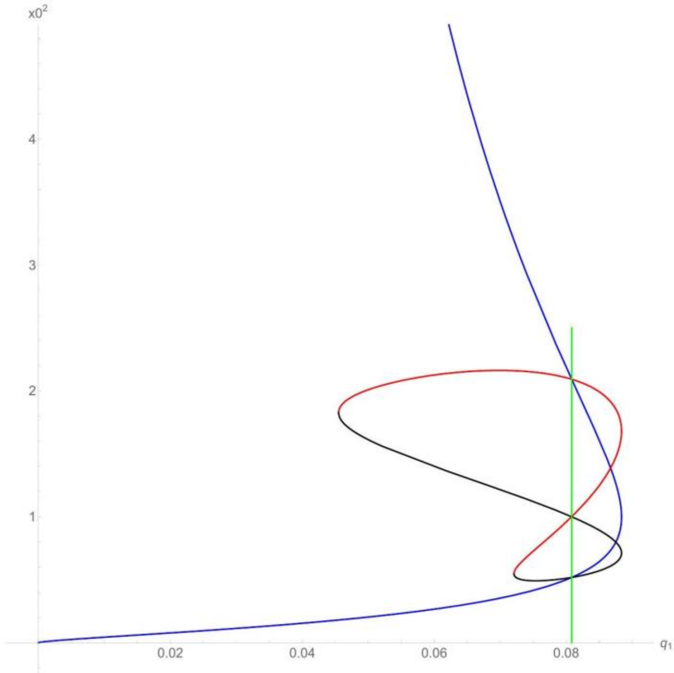
12
10
8
6
4
2
0.02
0.04
0.06
0.08
q_1



x0²
4
3
2
1
0.02
0.04
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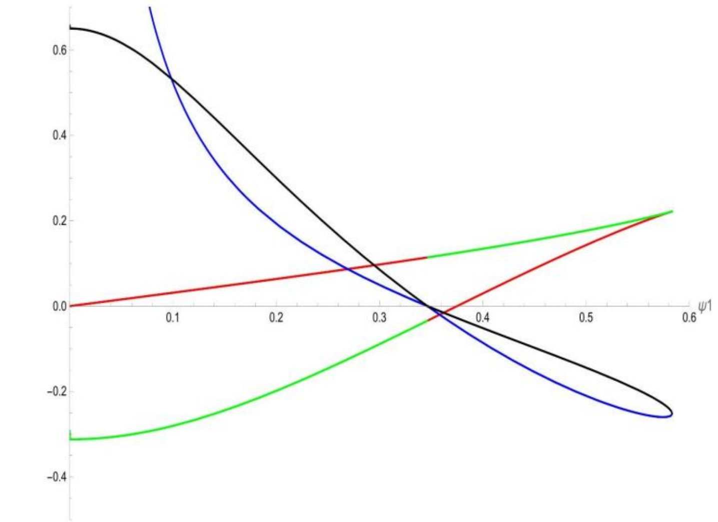
0.08
q_1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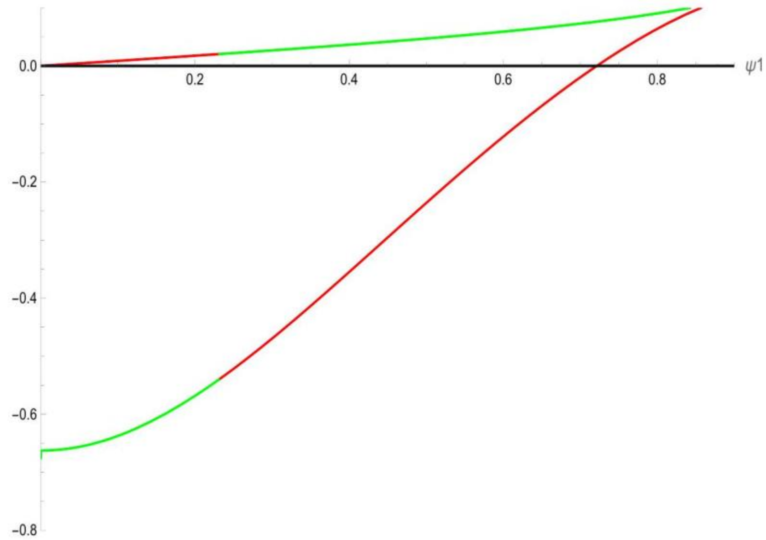
0.4
0.2
0.0
−0.2
−0.4
0.1
0.2
0.3
0.4
0.5
0.6
ψ1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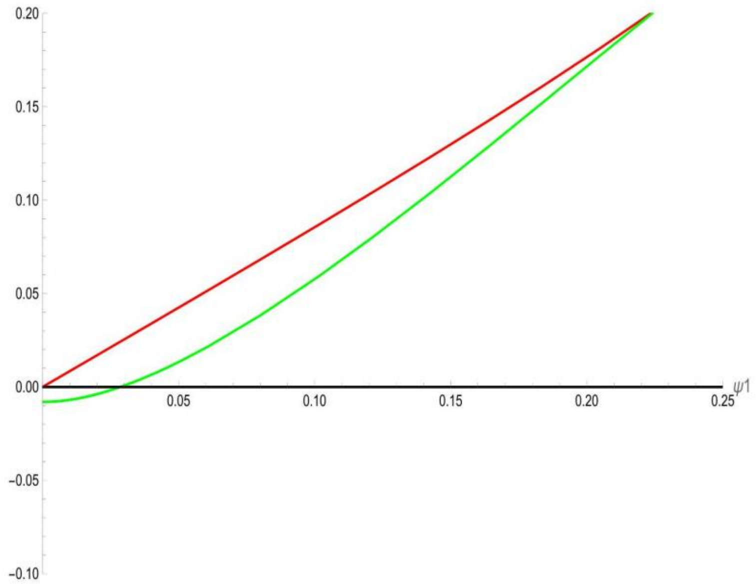
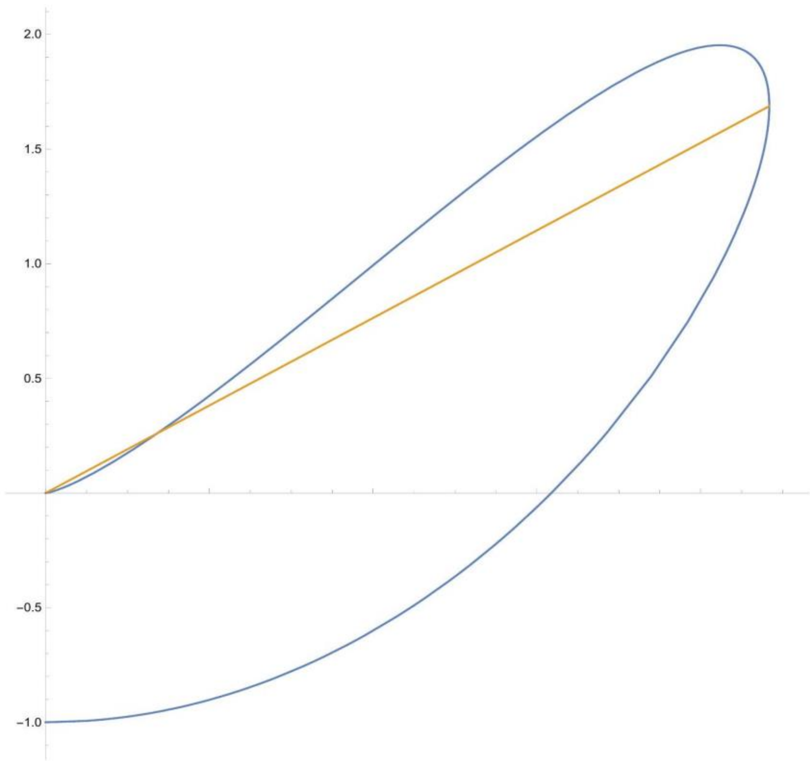
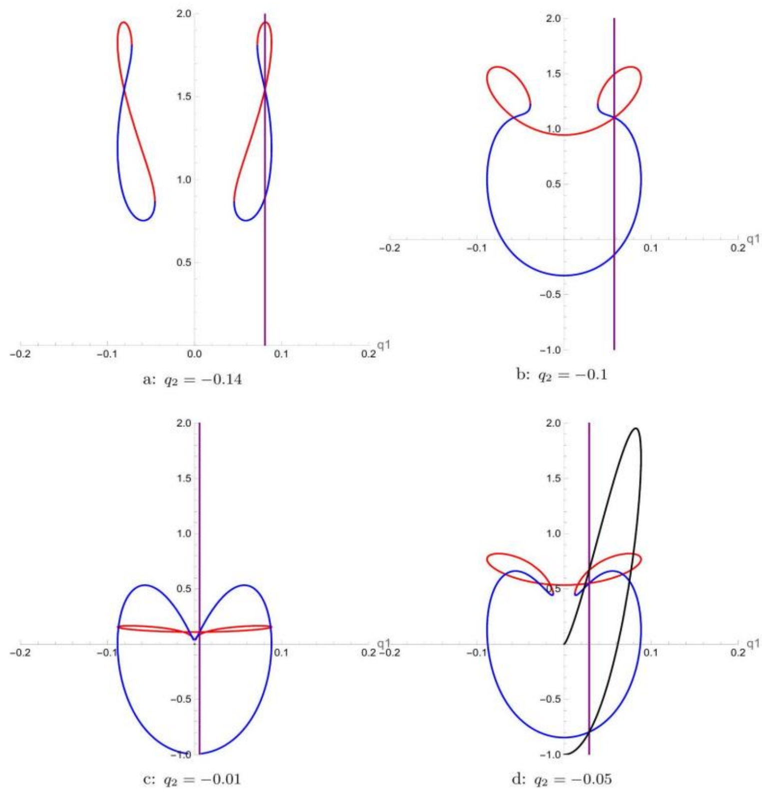
−0.2
−0.4
−0.6
−0.8
0.2
0.4
0.6
0.8
ψ1


0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
0.05
0.10
0.15
0.20
0.25
ψ1





$$x^2 = 1 - \frac{\alpha}{r}$$

$$x_0^2 = 1 - \frac{\alpha}{r_0}$$

$$e^0 = \sqrt{\Upsilon(r)}dt, e^1 = \sqrt{\frac{\Upsilon(r)}{f(r)}}\frac{\alpha\eta}{2r^{3/2}(r-\alpha)^{1/2}}dr, e^2 = \sqrt{\Upsilon(r)f(r)}\,d\varphi, e^3 = \sqrt{\Upsilon(r)}dz$$

$$\phi = \sqrt{\frac{3}{8}}\ln\left(1-\frac{\alpha}{r}\right), A^1 = \frac{Q_1\alpha\left(\frac{1}{r}-\frac{1}{r_0}\right)}{\left(1-\frac{\alpha}{r}\right)\left(1-\frac{\alpha}{r_0}\right)}d\varphi, A^2 = -Q_2\alpha\left(\frac{1}{r}-\frac{1}{r_0}\right)d\varphi$$

$$\Upsilon(r) = \frac{4L^2\sqrt{1-\frac{\alpha}{r}}r^2}{\alpha^2\eta^2}, f(r) = 1-\frac{\eta^2\alpha^3\left(3Q_1^2-Q_2^2+Q_2^2\frac{\alpha}{r}\right)}{6L^2r^3\left(1-\frac{\alpha}{r}\right)}$$

$$\phi = 0, A^1 = \frac{1}{\sqrt{3}}A^2 = 2L^2\tilde{Q}_1\left(\frac{1}{r}-\frac{1}{r_0}\right)d\varphi$$

$$e^0 = \frac{r}{L}dt, e^1 = \frac{dr}{\sqrt{f_0(r)}}, e^2 = \sqrt{f_0(r)}d\varphi, e^3 = \frac{r}{L}dz$$

$$f_0(r) = \frac{r^2}{L^2}f(r) = \frac{r^2}{L^2}-\frac{\mu}{r}-\frac{8L^4\tilde{Q}_1^2}{r^2},$$

$$P(x_0) = 2\psi_1^2(1-x_0^2)^2(1+2\psi_1^2x_0^2),$$

$$f(y_0) = 1+\frac{2}{3\psi_1^2}\frac{3y_0^2-y_0}{(2y_0-1)^2}$$

$$f(x_0) = 1-\frac{3^3}{2}\frac{\left(q_2^2-3q_1^2-2(1-x_0^2)(q_2^2-q_1^2)+(1-x_0^2)^2q_2^2\right)^4}{x_0^6(1-x_0^2)\left(3q_1^2-q_2^2+q_2^2(1-x_0^2)\right)^3},$$

$$\eta = \frac{3\Delta}{2\pi}\frac{\left|(q_2^2-3q_1^2)-2(1-x_0^2)(q_2^2-q_1^2)+(1-x_0^2)^2q_2^2\right|}{x_0(1-x_0^2)\left(3q_1^2-q_2^2+q_2^2(1-x_0^2)\right)}$$

$$f(x_0) = 1-\frac{q_1^2}{2x_0^6}(1+3x_0^2)^4 = 0$$

$$\eta = \frac{\Delta}{2\pi}\frac{1+3x_0^2}{x_0(1-x_0^2)}$$

$$\Delta\varphi = \pi\sqrt{\frac{L^3}{Q}}$$

$$Q = 2\sqrt{2}L^2\tilde{Q}_1$$

$$\Delta = \pi\sqrt{\frac{\eta L}{2\sqrt{2}Q_1}}$$

$$A^1 = \frac{2\pi L\psi_1}{\Delta}, A^2 = \frac{2\pi L\psi_2}{\Delta}$$

$$ds^2 = \frac{L^2}{z^2}(dz^2+dy^a dy_a)$$

$$-(X^{-1})^2 + X^aX_a + (X^3)^2 = -L^2.$$

$$X^{-1} = \frac{1}{2z}(z^2+L^2+y^ay_a), X^a = \frac{Ly^a}{z}, X^3 = \frac{1}{2z}(z^2-L^2+y^ay_a),$$

$$\frac{\partial}{\partial y^1} = L^{-1}(X^{-1} - X^3)\frac{\partial}{\partial X^1} + L^{-1}X^1\left(\frac{\partial}{\partial X^{-1}} + \frac{\partial}{\partial X^3}\right),$$

$$\bar{\varphi}_1 \sim \bar{\varphi}_1 + 2\pi\frac{\psi_1}{\sqrt{2}}, \bar{\varphi}_{i\neq 1} \sim \bar{\varphi}_{i\neq 1} + 2\pi\frac{\psi_2}{\sqrt{6}}$$

$$Q_1 = -\frac{1}{\sqrt{3}}Q_2 \implies \mu = 0$$

$$f(x) = 1 - \frac{(x^2-1)^4}{x^2}\frac{\eta^2 Q_2^2}{6L^2},$$

$$\chi_{\mathrm{R}_{(1)}} = e^{i\omega\varphi}\begin{pmatrix}\alpha_-(x)\\0\\0\\-i\alpha_+(x)\end{pmatrix}, \quad \chi_{\mathrm{R}_{(2)}} = e^{i\omega\varphi}\begin{pmatrix}0\\\alpha_+(x)\\-i\alpha_-(x)\\0\end{pmatrix},$$
$$\chi_{\mathrm{I}_{(1)}} = e^{i\omega\varphi}\begin{pmatrix}0\\\alpha_-(x)\\-i\alpha_+(x)\\0\end{pmatrix}, \quad \chi_{\mathrm{I}_{(2)}} = e^{i\omega\varphi}\begin{pmatrix}-\alpha_+(x)\\0\\0\\i\alpha_-(x)\end{pmatrix},$$

$$\alpha_\pm(x) = \frac{x^{1/4}}{(x^2-1)^{1/2}}\sqrt{1 \pm f(x)^{1/2}}, \omega = -\frac{\pi}{\Delta}$$

$$\epsilon^1_{(k)} = \mathrm{Re}\chi_{\mathrm{R}_{(k)}} + i\mathrm{Re}\chi_{\mathrm{I}_{(k)}} = (\mathbb{1} - \gamma^5)\mathrm{Re}\chi_{\mathrm{R}_{(k)}}$$
$$\epsilon^2_{(k)} = \mathrm{Im}\chi_{\mathrm{R}_{(k)}} + i\mathrm{Im}\chi_{\mathrm{I}_{(k)}} = (\mathbb{1} - \gamma^5)\mathrm{Im}\chi_{\mathrm{R}_{(k)}}$$

$$\epsilon^1_{(1)} = \begin{pmatrix}\cos(\omega\varphi)\alpha_-(x)\\i\cos(\omega\varphi)\alpha_-(x)\\i\sin(\omega\varphi)\alpha_+(x)\\\sin(\omega\varphi)\alpha_+(x)\end{pmatrix}, \quad \epsilon^2_{(1)} = \begin{pmatrix}\sin(\omega\varphi)\alpha_-(x)\\i\sin(\omega\varphi)\alpha_-(x)\\-i\cos(\omega\varphi)\alpha_+(x)\\-\cos(\omega\varphi)\alpha_+(x)\end{pmatrix}$$
$$\epsilon^1_{(2)} = \begin{pmatrix}-i\cos(\omega\varphi)\alpha_+(x)\\\cos(\omega\varphi)\alpha_+(x)\\\sin(\omega\varphi)\alpha_-(x)\\-i\sin(\omega\varphi)\alpha_-(x)\end{pmatrix}, \quad \epsilon^2_{(2)} = \begin{pmatrix}-i\sin(\omega\varphi)\alpha_+(x)\\\sin(\omega\varphi)\alpha_+(x)\\-\cos(\omega\varphi)\alpha_-(x)\\i\cos(\omega\varphi)\alpha_-(x)\end{pmatrix}$$

$$\gamma^5\epsilon^A_{(k)} = -\epsilon^A_{(k)}$$

$$\psi_1 = \frac{\psi_2 x_0^{-2}}{\sqrt{3}}$$

$$\psi_1 = \sqrt{2} - \sqrt{3}\psi_2, \quad x_0^2 = \frac{\psi_2}{\sqrt{6}-3\psi_2}, \quad 0 < \psi_2 < \frac{\sqrt{6}}{3}$$
$$\psi_1 = -\sqrt{2} - \sqrt{3}\psi_2, \quad x_0^2 = -\frac{\psi_2}{\sqrt{6}+3\psi_2}, \quad -\frac{\sqrt{6}}{3} < \psi_2 < 0$$

$$\delta\psi^A_\mu = \partial_\mu\epsilon^A + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon^A - \frac{1}{2\sqrt{2}L}(A^1_\mu + \sqrt{3}A^2_\mu)\epsilon^{AB}\epsilon^B + \frac{W}{2}\gamma_\mu\epsilon_A$$
$$\delta\psi^{I_\alpha}_\mu = \partial_\mu\epsilon^{I_\alpha} + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon^{I_\alpha} - \frac{1}{2\sqrt{2}L}\left(A^1_\mu - \frac{1}{\sqrt{3}}A^2_\mu\right)\epsilon^{I_\alpha J_\alpha}\epsilon^{J_\alpha} + \frac{W}{2}\gamma_\mu\epsilon_{I_\alpha}, (\alpha \neq 1)$$

$$\epsilon^A(\varphi, r) = e^{i\omega_1\varphi}\mathring{\epsilon}^A(r), \epsilon^{I_\alpha}(\varphi, r) = e^{i\omega_2\varphi}\mathring{\epsilon}^{I_\alpha}(r)\ (\alpha \neq 1)$$

$$\omega_1 = \frac{\pi}{\sqrt{2}\Delta}\left(\psi_1 + \sqrt{3}\psi_2\right), \omega_2 = \frac{\pi}{\sqrt{2}\Delta}\left(\psi_1 - \frac{1}{\sqrt{3}}\psi_2\right)$$

$$\psi_1 = \frac{3m+n}{2\sqrt{2}}, \psi_2 = \frac{1}{2}\sqrt{\frac{3}{2}}(n-m)$$

$$q_1 = -\frac{q_2}{\sqrt{3}}$$

$$q_2 = \pm\sqrt{6}\frac{x_S^3}{\left(3x_S^2+1\right)^2}$$

$$\frac{S_E}{V} = I_{\text{bulk}} + I_{\text{GH}} + I_{\text{BK}} + I_{\text{ct}} + I_{\phi},$$

$$V = \beta\Delta\Delta_z, \Delta_z = \int \ dz$$

$$I_{\text{bulk}} = \lim_{\epsilon\to 1^-}\frac{1}{\kappa}\int_{x_0}^{\epsilon} dx\sqrt{g_E}\left(-\frac{R}{2} + \frac{1}{2}(\partial\phi)^2 - \frac{3}{L^2}\cosh\left(\sqrt{\frac{2}{3}}\phi\right) - \frac{1}{4}e^{3\sqrt{\frac{2}{3}}\phi}(F^1)^2 - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\phi}(F^2)^2\right)$$

$$I_{\text{GH}} = -\frac{1}{\kappa}\lim_{\epsilon\to 1^-} K\sqrt{h}$$
$$I_{\text{BK}} = \frac{2}{\kappa L}\lim_{\epsilon\to 1^-}\sqrt{h}$$
$$I_{\text{ct}} = \frac{1}{2\kappa L}\lim_{\epsilon\to 1^-}\sqrt{h}\phi^2$$
$$I_{\phi} = -\frac{L^2}{\kappa}\frac{1}{3\sqrt{6}}\phi_0^3$$

$$I_{\text{bulk}} = \frac{1}{\kappa}\lim_{\rho\to\infty}\left(\frac{\rho^3}{L^4} + \frac{3\rho}{8\eta^2} - \frac{\mu}{L^2} + \frac{L^2}{2\eta^3}\right)$$
$$I_{\text{GH}} = -\frac{1}{\kappa}\lim_{\rho\to\infty}\left(\frac{3\rho^3}{L^4} + \frac{9\rho}{8\eta^2} - \frac{3\mu}{2L^2} + \frac{3L^2}{2\eta^3}\right)$$
$$I_{\text{BK}} = \frac{1}{\kappa}\lim_{\rho\to\infty}\left(\frac{2\rho^3}{L^4} - \frac{\mu}{L^2}\right)$$
$$I_{\text{ct}} = \frac{1}{\kappa}\lim_{\rho\to\infty}\left(\frac{3\rho}{4\eta^2} + \frac{3L^2}{4\eta^3}\right)$$
$$I_{\phi} = \frac{L^2}{\kappa}\frac{1}{4\eta^3}$$

$$\frac{S_E}{V} = -\frac{\mu}{2L^2\kappa}$$

$$\frac{S_E}{V} = \frac{G_\phi}{\Delta\Delta_z} = \frac{M}{\Delta\Delta_z} = -\frac{\mu}{2\kappa L^2} = \pm\frac{2}{3\eta\kappa}\left(3Q_1^2 - Q_2^2\right) =$$
$$= \pm\frac{8\pi^3 L^2}{3\Delta^3\kappa}\frac{x_0\left|2x_0^2\psi_1^2 + \psi_1^2 - \psi_2^2\right|\left(3\psi_1^2 x_0^4 - \psi_2^2\right)}{\left(x_0^2-1\right)^2}$$

$$G_0 = -\frac{32}{27}\frac{\pi^3 L^2}{\Delta^3\kappa}\Delta\Delta_z$$

$$\frac{G_\phi}{|G_0|} = \pm\frac{9}{4}\frac{x_0\left|2x_0^2\psi_1^2 + \psi_1^2 - \psi_2^2\right|\left(3\psi_1^2 x_0^4 - \psi_2^2\right)}{\left(x_0^2-1\right)^2}$$

$$\frac{F_\phi}{V} = \frac{S_{\text{E}}}{V} - \langle J_\Lambda^\nu \rangle A_\nu^\Lambda \Big|_{x\to 1} =$$
$$= -\frac{\mu}{2L^2\kappa} - \left\langle J_1^\phi \right\rangle Q_1(1 - x_0^{-2}) - \left\langle J_2^\phi \right\rangle Q_2(1 - x_0^2) =$$
$$= \pm \frac{2}{3\eta\kappa}(3Q_1^2 - Q_2^2) \pm \frac{2Q_1^2}{\eta\kappa}(1 - x_0^{-2}) \mp \frac{2Q_2^2}{\eta\kappa}(1 - x_0^2),$$

$$q_1 = \frac{\Delta^2}{4\pi^2 L}\frac{Q}{\sqrt{8}L^2}$$

$$\frac{F_{\text{EM}}}{\Delta\Delta_z} = -\frac{m}{2\kappa L^2} + \frac{2Q^2}{\kappa L^2 r_0} = \frac{2\pi^3 L^2}{\Delta^3 \kappa} X^2(5 - 4X)$$

$$\frac{F_\phi}{|G_0|} = \frac{27}{\sqrt{2}}|q_1|.$$

$$\frac{1}{2\sqrt{2}L}\lim_{r\to\infty}\left(A_\varphi^1 + \sqrt{3}A_\varphi^2\right)\Delta = \pi n, n \in \mathbb{Z}, n \text{ odd}$$

$$q_1^2 = 2^{-7}n^2X^2, X \equiv \frac{r_0\Delta}{\pi L^2}$$

$$X = 1, \frac{1}{3}$$

$$q_1^2 = 2^{-7} \text{ or } q_1^2 = \frac{2^{-7}}{9}$$

$$\frac{F_{\text{EM}}}{F_\phi} = \frac{(5-4X)X^2}{\sqrt{(4-3X)X^3}}$$

$$\frac{F_{\text{EM}}}{F_\phi}\bigg|_{X=\frac{1}{3}} = \frac{(5-4X)X^2}{\sqrt{(4-3X)X^3}}\bigg|_{X=\frac{1}{3}} = \frac{11}{9} > 1$$

$$\gamma^0 = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma^1 = -\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \gamma^2 = i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\epsilon_{(\text{M})}^A = \epsilon^A + \epsilon_A$$

$$\gamma^5\epsilon^A = -\epsilon^A, \gamma^5\epsilon_A = \epsilon_A$$

$$C(\bar{\epsilon}_A)^T = \epsilon_A \;\Leftrightarrow\; C\left((\epsilon^A)^\dagger\gamma^0\right)^T = \epsilon_A \;\Leftrightarrow\; (\epsilon^A)^* = \epsilon_A$$

$$\delta\Psi_\mu^A = D_\mu\epsilon^A + \frac{1}{4}T_{\nu\rho}^+\gamma^{\nu\rho}\gamma_\mu\varepsilon^{AB}\epsilon_B + \mathbb{S}^{AB}\gamma_\mu\epsilon_B$$
$$\delta\lambda^{iA} = -\partial_\mu z^i\gamma^\mu\epsilon^A + \frac{1}{2}g^{i\bar{J}}\bar{f}_{\bar{J}}^\Lambda\mathcal{I}_{\Lambda\Sigma}F_{\mu\nu}^{\Sigma-}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B$$

$$D_\mu\epsilon^A = \partial_\mu\epsilon^A + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon^A + \frac{i}{2}(\sigma^2)_B^A\mathbb{A}_\mu^M\theta_M\epsilon^B + \frac{i}{2}\mathcal{Q}_\mu\epsilon^A$$

$$\mathcal{Q}_\mu = \frac{i}{2}\left(\partial_{\bar{\imath}}\mathcal{K}\partial_\mu \bar{z}^{\bar{\imath}} - \partial_i\mathcal{K}\partial_\mu z^i\right)$$

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm i^\star F_{\mu\nu}\right), \mathbb{F}^M_{\mu\nu} = \left(F^\Lambda_{\mu\nu}, G_{\Lambda\mu\nu}\right), \gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}$$
$$T_{\mu\nu} = L^\Lambda \mathfrak{I}_{\Lambda\Sigma} F^\Sigma_{\mu\nu} = \frac{1}{2i} L^\Lambda (\mathfrak{N} - \overline{\mathfrak{N}})_{\Lambda\Sigma} F^\Sigma_{\mu\nu} = -\frac{i}{2}\left(M_\Sigma F^\Sigma_{\mu\nu} - L^\Lambda G_{\Lambda\mu\nu}\right) = \frac{i}{2}\mathcal{V}^M \mathcal{C}_{MN}\mathbb{F}^N_{\mu\nu}$$
$$T^+_{\mu\nu} = \bar{L}^\Lambda \mathfrak{I}_{\Lambda\Sigma} F^{\Sigma+}_{\mu\nu} = -\frac{i}{2}\overline{\mathcal{V}}^M \mathcal{C}_{MN}\mathbb{F}^{N+}_{\mu\nu}, \mathcal{V}^M = e^{\frac{\mathcal{K}}{2}}\Omega^M = \left(L^\Lambda, M_\Lambda\right)$$
$$\mathbb{S}^{AB} = \frac{i}{2}(\sigma^2)^A{}_C \varepsilon^{BC}\mathcal{W}, \mathcal{W} = \mathcal{V}^M\theta_M, \mathcal{U}^M_i = \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)\mathcal{V}^M = \left(f^\Lambda_i, h_{i\Lambda}\right)$$
$$W^{iAB} = i(\sigma^2)_C{}^B \varepsilon^{CA}\theta_M g^{i\bar{\jmath}}\overline{\mathcal{U}}^M_{\bar{\jmath}}, g_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}\mathcal{K}$$

$$\overline{\mathfrak{N}}_{\Lambda\Sigma}F^{\Sigma-} = G^-_\Lambda,\ L^\Lambda \mathfrak{N}_{\Lambda\Sigma} = M_\Sigma$$

$$\mathfrak{N}_{\Lambda\Sigma} = \partial_{\bar{\Lambda}}\partial_{\bar{\Sigma}}\overline{\mathcal{F}} + 2i\frac{\mathrm{Im}[\partial_\Lambda\partial_\Gamma\mathcal{F}]\mathrm{Im}[\partial_\Sigma\partial_\Delta\mathcal{F}]L^\Gamma L^\Delta}{\mathrm{Im}[\partial_\Delta\partial_\Gamma\mathcal{F}]L^\Delta L^\Gamma}$$

$$\partial_\Lambda = \frac{\partial}{\partial\mathcal{X}^\Lambda}, \partial_{\bar{\Lambda}} = \frac{\partial}{\partial\overline{\mathcal{X}}^\Lambda}$$

$$\mathcal{F}\left(\mathcal{X}^\Lambda\right) = -\frac{i}{4}(\mathcal{X}^0)^{\frac{1}{2}}(\mathcal{X}^1)^{\frac{3}{2}}$$

$$\theta_M = ((\theta_2/3)^{-3}(4L)^{-4}, \theta_2, 0,0)$$

$$x^2_{0,1} = -W + Z\cos(\theta), x^2_{0,2} = -W + Z\cos\left(\theta + \frac{2\pi}{3}\right), x^2_{0,3} = -W + Z\cos\left(\theta + \frac{4\pi}{3}\right)$$
$$W = \frac{\psi_1^{-2}}{6}(2\psi_1^2 - 2\psi_2^2 + 1), Z = \left(W^2 + \psi_1^{-2} - \frac{\psi_1^{-4}\psi_2^2}{9} + \frac{1}{12}\psi_1^{-4}\right)^{1/2}$$
$$\cos(3\theta) = \frac{W^3}{Z^3} - \frac{\psi_1^{-2}}{Z^3} - \frac{1}{2^3 3^2}\frac{(4\psi_2^2-3)(2\psi_2^2-1)}{\psi_1^6 Z^3} + \frac{1}{2^2 3^2}\frac{16\psi_2^2-21}{\psi_1^4 Z^3}.$$

$$ds^2_{S^7\times[0,\Delta]} = F_1\sum_{I=1}^4 \left(d\mu_I^2 + \mu_I^2(d\varphi_I - \chi_I d\varphi)^2\right) + F_2 d\varphi^2$$

$$(S^7 \times [0,\Delta])/\sim$$

$$(p, \varphi = 0) \in S^7 \times [0,\Delta] \sim (\mathfrak{M}\cdot p, \varphi = \Delta) \in S^7 \times [0,\Delta]$$

$$\hat{\mathbb{L}}(\mathbf{X}) \equiv \begin{pmatrix} (1 - \mathbf{X}^T\mathbf{X})^{\frac{1}{2}} & -\mathbf{X}^T \\ \mathbf{X} & (1 - \mathbf{X}\mathbf{X}^T)^{\frac{1}{2}} \end{pmatrix} \in \frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}$$

$$\mathbf{X}^T = \left(\mu_1\cos(\varphi_1), \mu_1\sin(\varphi_1), \mu_2\cos(\varphi_2), \mu_2\sin(\varphi_2), \mu_3\cos(\varphi_3), \mu_3\sin(\varphi_3), \mu_4\cos(\varphi_4)\right)$$

$$\mathbf{X}_0 \equiv \mathbf{X}|_{\varphi_i=0}$$

$$\mathcal{T}(\varphi_I) \equiv e^{-(\mathbf{e}_{23}\varphi_1 + \mathbf{e}_{45}\varphi_2 + \mathbf{e}_{67}\varphi_3 - \mathbf{e}_{18}\varphi_4)}, \mathbb{L}_0(\mu_I) \equiv \hat{\mathbb{L}}\left(\mathbf{X}_0(\mu_I)\right)$$

$$\left(\mathbf{e}_{ij}\right)^{k\ell} = \delta^k_i\delta^\ell_j - \delta^k_j\delta^\ell_j, i, j, \cdots = 1, \dots, 8$$

$$\mathbb{L}(\mu_I, \varphi_I) \equiv \mathcal{T}(\varphi_I)\cdot\mathbb{L}_0(\mu_I) = \hat{\mathbb{L}}\left(\mathbf{X}(\mu_I, \varphi_I)\right)\cdot\mathbf{h},$$

$$\mathbb{L}'(\mu_I, \varphi_I, \varphi) \equiv \mathcal{T}(-\chi_I \varphi) \cdot \mathbb{L}(\mu_I, \varphi_I) = \mathcal{T}(\varphi_I - \chi_I \varphi) \cdot \mathbb{L}_0(\mu_I).$$

$$\mathbb{L}'(\mu_I, \varphi_I, \varphi + \Delta) = \mathfrak{M} \cdot \mathbb{L}'(\mu_I, \varphi_I, \varphi)$$

$$\mathfrak{M} \equiv \mathcal{T}(-\chi_I \Delta)$$

$$ds^2_{\text{bound.}} = -dt^2 + d\varphi_1^2 + d\varphi_2^2$$

$$ds^2 = f_1(r) d\varphi_1^2 + \frac{dr^2}{f_1(r)} + \frac{r^2}{L^2}(-dt^2 + d\varphi_2^2)$$
$$ds^2 = f_2(r) d\varphi_2^2 + \frac{dr^2}{f_2(r)} + \frac{r^2}{L^2}(-dt^2 + d\varphi_1^2)$$

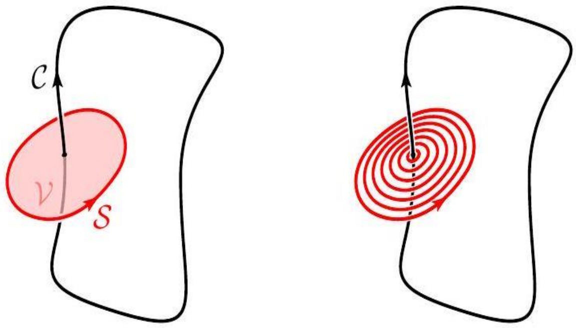
$$f_{1,2}(r) = \frac{r^2}{L^2} - \frac{\mu_{1,2}}{r}$$

$$A^a = A^a{}_\mu dx^\mu \text{ and } B_a = \frac{1}{2} B_{a\mu\nu} dx^\mu \wedge dx^\nu$$

$$L = \frac{1}{g^2} B_a \wedge F^a - \frac{1}{2g^2} B_a \wedge * B^a \text{ where } F^a = dA^a + \frac{1}{2} f^a_{bc} A^b \wedge A^c$$

$$W_\rho(\mathcal{C}) = \mathrm{tr}_\rho \mathrm{Pexp}\left(\oint_{\mathcal{C}} A \right)$$

$$U_\alpha(\mathcal{S})$$

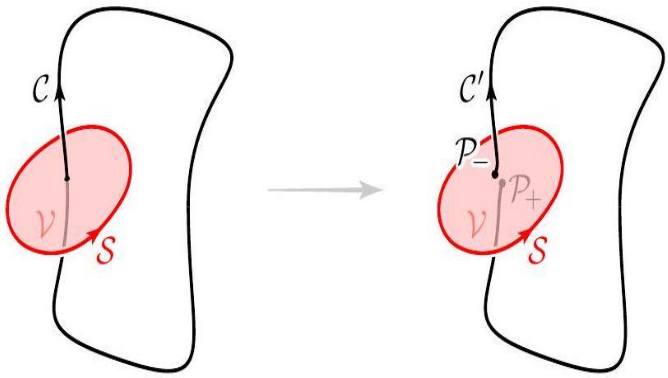


$$W_\rho(\mathcal{C}) \mapsto \rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})} W_\rho(\mathcal{C}),$$

$$\mathrm{link}(\mathcal{C}, \mathcal{S}) = \mathrm{int}(\mathcal{C}, \mathcal{V}) \text{ where } \mathcal{S} = \partial \mathcal{V}$$

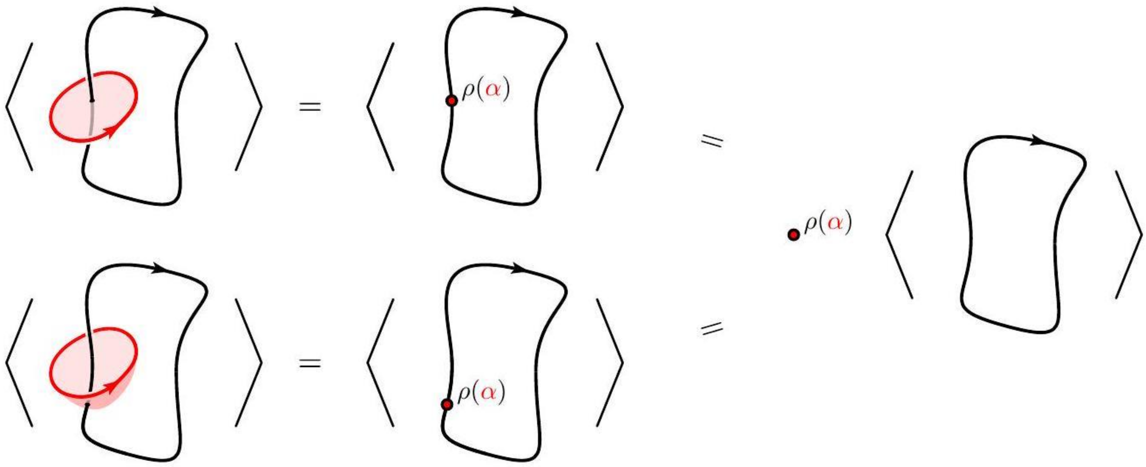
$$A^a \mapsto (\Omega^{-1} A \Omega)^a + (\Omega^{-1} d\Omega)^a \text{ and } B_a \mapsto (\Omega^{-1} B \Omega)_a$$

$$\lim_{\mathcal{P}_\pm \to \mathcal{P}} \Omega(\mathcal{P}_+) \Omega^{-1}(\mathcal{P}_-) = \alpha \text{ where } \alpha \in Z(G)$$



$$W_\rho(\mathcal{C}) \mapsto \lim_{\mathcal{C}'\to\mathcal{C}} \mathrm{tr}_\rho\left[\mathrm{Pexp}\left(\int_{\mathcal{C}'} \Omega^{-1}A\Omega + \Omega^{-1}d\Omega\right)\right]$$
$$= \lim_{\mathcal{C}'\to\mathcal{C}} \mathrm{tr}_\rho\left[\Omega^{-1}(\mathcal{P}_-)\mathrm{Pexp}\left(\int_{\mathcal{C}'} A\right)\Omega(\mathcal{P}_+)\right]$$

$$W_\rho(\mathcal{C}) \mapsto \lim_{\mathcal{C}'\to\mathcal{C}} \mathrm{tr}_\rho\left[\alpha\mathrm{Pexp}\left(\int_{\mathcal{C}'} A\right)\right] = \rho(\alpha)W_\rho(\mathcal{C}),$$



$$\langle U_\alpha(\mathcal{S})W_\rho(\mathcal{C})\rangle = \rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})}\langle W_\rho(\mathcal{C})\rangle,$$

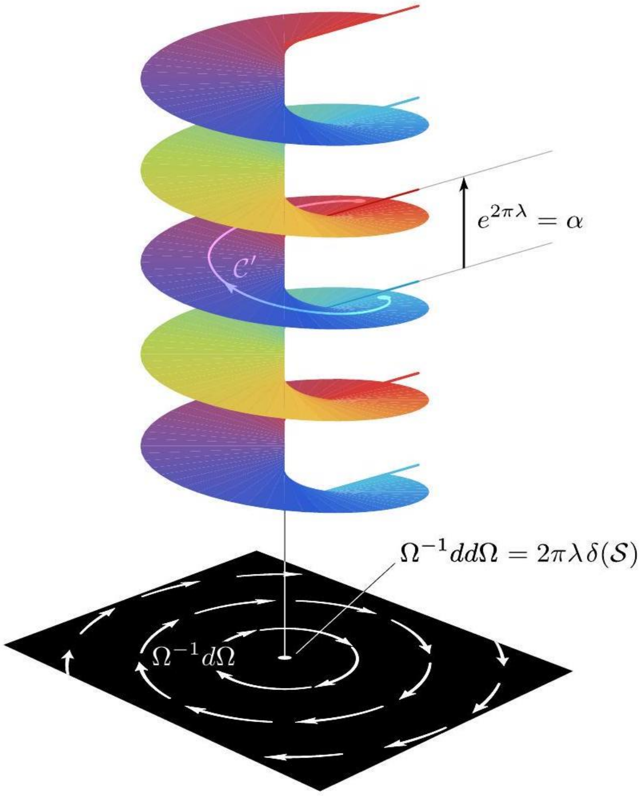
$$\langle U_\alpha(\mathcal{S})W_\rho(\mathcal{C})\rangle = \int \mathcal{D}A\mathcal{D}B e^{-S} U_\alpha(\mathcal{S})W_\rho(\mathcal{C})$$

$$U_\alpha(\mathcal{S}) = \exp\left(\frac{2\pi}{g^2}\int_{\mathcal{S}} \lambda^a B_a\right) \text{ where } e^{2\pi\lambda} = \alpha \in Z(G)$$

$$F^a \mapsto (\Omega^{-1}F\Omega)^a + (\Omega^{-1}dd\Omega)^a$$

$$(\Omega^{-1}dd\Omega)^a = 2\pi\lambda^a\delta(\mathcal{S}) \text{ for any } \lambda^a \text{ such that } e^{2\pi\lambda} = \alpha \in Z(G)$$

$$\int_{\mathcal{M}} \varphi \wedge \delta(\mathcal{S}) = \int_{\mathcal{S}} \varphi$$



$$U_\alpha(\mathcal{S}) = \exp\left(\frac{1}{g^2}\int_{\mathcal{M}} B_a \wedge 2\pi\lambda^a \delta(\mathcal{S})\right) = \exp\left(\frac{1}{g^2}\int_{\mathcal{M}} B_a \wedge (\Omega^{-1} dd\Omega)^a\right)$$

$$e^{-S} U_\alpha(\mathcal{S}) = \exp\left(-\frac{1}{g^2}\int_{\mathcal{M}} B_a \wedge (F - \Omega^{-1} dd\Omega)^a - \frac{1}{2} B_a \wedge * B^a\right)$$

$$e^{-S} U_\alpha(\mathcal{S}) \mapsto e^{-S}$$

$$\langle U_\alpha(\mathcal{S}) W_\rho(\mathcal{C})\rangle \mapsto \int \mathcal{D}A\mathcal{D}B e^{-S} \rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})} W_\rho(\mathcal{C})$$

$$\langle U_\alpha(\mathcal{S})\mathcal{O}\rangle = \langle \mathcal{O}_\Omega\rangle$$

$$\alpha_1 = e^{2\pi\Omega_2^{-1}\lambda_1\Omega_2} \text{ and } \alpha_2 = e^{2\pi\lambda_2}.$$

$$\Omega = \Omega_1\Omega_2 \implies \Omega^{-1} dd\Omega = 2\pi(\Omega_2^{-1}\lambda_1\Omega_2 + \lambda_2)^a \delta(\mathcal{S})$$

$$\Omega = \exp\left(\frac{k}{N} c\phi\right) \text{ where } e^{2\pi c} = \mathbb{1},$$

$$c^i{}_j = i\mathrm{diag}(1,1,\cdots,-N+1),$$

$$\begin{aligned} A^a &= (\Omega^{-1} d\Omega)^a = \frac{k}{N} c^a d\phi = \frac{k}{N} c^a \frac{xdy - ydx}{x^2+y^2} \\ F^a &= (\Omega^{-1} dd\Omega)^a = \frac{k}{N} c^a dd\phi = \frac{k}{N} 2\pi c^a \delta(x)\delta(y) dx \wedge dy \end{aligned}$$

$$\mathrm{tr}_{\mathrm{fund}} \,\mathrm{Pexp}\left(\frac{k}{N} c \int_0^{2\pi} d\phi\right) = e^{2\pi i k/N} N$$

$$(\Omega(\phi = 2\pi)\Omega^{-1}(\phi = 0))^i{}_j = e^{2\pi i k/N} \delta^i{}_j$$

$$\Omega = \exp\left(\frac{k}{N}c\phi\right) \text{ where } \tan\phi = \frac{y-Y(z)}{x-X(z)},$$

$$\delta(x-X(z))\delta(y-Y(z))d(x-X(z))\wedge d(y-Y(z))$$

$$\mathcal{M} = \mathcal{M}_3 \times S^1, t \sim t+\beta$$

$$A^a = n\frac{2\pi c^a}{\beta}dt \text{ where } n \in \mathbb{Z}$$

$$\Omega = \exp\left(\frac{2\pi kc}{N}\frac{t}{\beta}\right)$$

$$A^a = \left(n+\frac{k}{N}\right)\frac{2\pi c^a}{\beta}dt$$

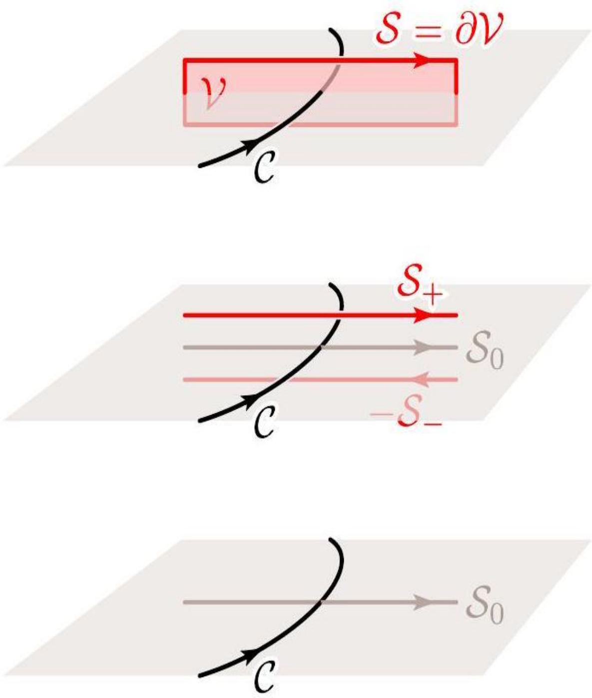
$$x = \frac{a\sinh\tau}{\cosh\tau - \cos\sigma}\cos\phi, y = \frac{a\sinh\tau}{\cosh\tau-\cos\sigma}\sin\phi, z = \frac{a\sin\sigma}{\cosh\tau-\cos\sigma}$$

$$\Omega = \exp\left(\frac{k}{N}c\sigma\right)$$

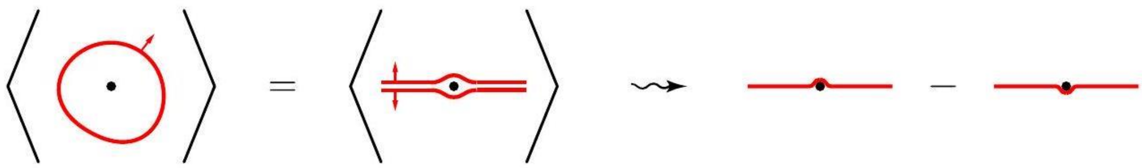
$$\frac{1}{g^2}\left[\frac{1}{2}B_{a\mu\nu}\left(\partial_\rho A_\sigma^a + \frac{1}{2}f_{bc}^a A_\rho^b A_\sigma^c\right)\epsilon^{\mu\nu\rho\sigma} - \frac{1}{4}B_{a\mu\nu}B^{a\mu\nu}\right],$$

$$E_a^i = \frac{1}{2}B_{ajk}\epsilon^{ijk}$$

$$\left[A^a{}_i(x), A^b{}_j(x')\right] = 0$$
$$\left[E^i{}_a(x), A^b{}_j(x')\right] = g^2\delta^i{}_j\delta^b{}_a\delta^{(3)}(x-x'),$$
$$\left[E^i{}_a(x), E^j{}_b(x')\right] = 0,$$



$$\text{link}(\mathcal{C},\mathcal{S}) = \text{int}(\mathcal{C},\mathcal{V}) = \text{int}_3(\mathcal{C},\mathcal{S}_0),$$



$$U_\alpha(\mathcal{S}_0)W_\rho(\mathcal{C})U_\alpha^{-1}(\mathcal{S}_0) = \rho(\alpha)^{\mathrm{int}_3(\mathcal{C},\mathcal{S}_0)}W_\rho(\mathcal{C}),$$

$$U_\alpha(\mathcal{S}_0) = \exp\left(\tfrac{\pi}{g^2}\int_{\mathcal{S}_0} dx^j \wedge dx^k \epsilon_{ijk} E^i{}_a \lambda^a\right) \text{ where } e^{2\pi\lambda} = \alpha$$

$$\frac{\pi}{g^2}\int_{\mathcal{S}_0} dx^i \wedge dx'^j \lambda^b(x')\epsilon_{ijl}[E^l{}_b(x'), A^a{}_k(x)]$$
$$= 2\pi \int d\sigma_1 d\sigma_2 \frac{\partial X^i}{\partial \sigma_1}\frac{\partial X^j}{\partial \sigma_2}\lambda^a(X)\epsilon_{ijk}\delta^{(3)}(x - X)$$

$$U_\alpha(\mathcal{S}_0)A^a{}_i U_\alpha^{-1}(\mathcal{S}_0) = \left(A^a + 2\pi\lambda^a\delta_3(\mathcal{S}_0)\right)_i,$$
$$U_\alpha(\mathcal{S}_0)E^i{}_a U_\alpha{}^{-1}(\mathcal{S}_0) = E^i{}_a.$$

$$U_\alpha(\mathcal{S}_0)W_\rho(\mathcal{C})U_\alpha^{-1}(\mathcal{S}_0) = \rho(\alpha)^{\mathrm{int}_3(\mathcal{C},\mathcal{S}_0)}W_\rho(\mathcal{C}),$$

$$\omega^{AB} = \omega^{AB}{}_\mu dx^\mu \text{ and } e^A = e^A{}_\mu dx^\mu$$

$$L = \tfrac{1}{4g^2}\epsilon_{ABCD}e^A \wedge e^B \wedge R^{CD} - \tfrac{\Lambda}{24g^2}\epsilon_{ABCD}e^A \wedge e^B \wedge e^C \wedge e^D$$

$$R_B^A = d\omega_B^A + \omega_C^A \wedge \omega_B^C,$$

$$D(e^A \wedge e^B) = 0 \text{ and } e^{[B} \wedge R^{CD]} = \tfrac{\Lambda}{3}e^B \wedge e^C \wedge e^D$$

$$g_{\mu\nu} = \delta_{AB}e_\mu^A e_\nu^B$$

$$B_{AB} = \tfrac{1}{2}\epsilon_{ABCD}e^C \wedge e^D$$

$$L = \tfrac{1}{g^2}B_a \wedge R^a - \tfrac{\Lambda}{6g^2}B_a \wedge\star B^a$$

$$B_{AB} = \tfrac{1}{2}\epsilon_{ABCD}e^C \wedge e^D \text{ and } \star B^{AB} = e^A \wedge e^B.$$

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \text{ or } \mathrm{SO}(4) \cong \tfrac{\mathrm{Spin}(4)}{\mathbb{Z}_2} \text{ or } \tfrac{\mathrm{SO}(4)}{\mathbb{Z}_2} \cong \tfrac{\mathrm{Spin}(4)}{\mathbb{Z}_2\times\mathbb{Z}_2},$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \text{ or } \mathbb{Z}_2 \text{ or } \mathbb{1},$$

$$\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C}) \text{ or } \mathrm{SO}^+(3,1) \cong \tfrac{\mathrm{Spin}(3,1)}{\mathbb{Z}_2}$$

$$\mathbb{Z}_2 \text{ or } \mathbb{1},$$

$$W_\rho(\mathcal{C}) = \mathrm{tr}_\rho \mathrm{Pexp}\left(\oint_{\mathcal{C}}\omega\right)$$

$$U_\alpha(\mathcal{S})$$

$$W_\rho(\mathcal{C}) \mapsto \rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})} W_\rho(\mathcal{C})$$

$$\omega^A{}_B \mapsto (\Omega^{-1})^A{}_C \omega^C{}_D \Omega^D{}_B + (\Omega^{-1})^A{}_C d\Omega^C{}_B \ \text{ and } \ e^A \mapsto (\Omega^{-1})^A{}_B e^B$$

$$B^A_B{}_B \mapsto (\Omega^{-1})^A{}_C B^C{}_D \Omega^D{}_B,$$

$$\lim_{\mathcal{P}_\pm \to \mathcal{P}} \Omega(\mathcal{P}_+)\Omega^{-1}(\mathcal{P}_-) = \alpha \ \text{ where } \ \alpha \in Z(G) \ \text{ and } \ \mathcal{P} \subset \mathcal{V}$$

$$\begin{aligned} W_\rho(\mathcal{C}) \mapsto & \lim_{\mathcal{C}' \to \mathcal{C}} \mathrm{tr}_\rho \left[\mathrm{Pexp}\left(\int_{\mathcal{C}'} \Omega^{-1}\omega\Omega + \Omega^{-1} d\Omega \right) \right] \\ &= \lim_{\mathcal{C}' \to \mathcal{C}} \mathrm{tr}_\rho \left[\Omega^{-1}(\mathcal{P}_-) \mathrm{Pexp}\left(\int_{\mathcal{C}'} \omega \right) \Omega(\mathcal{P}_+) \right] \\ &= \lim_{\mathcal{C}' \to \mathcal{C}} \mathrm{tr}_\rho \left[\alpha \mathrm{Pexp}\left(\int_{\mathcal{C}'} \omega \right) \right] = \rho(\alpha) W_\rho(\mathcal{C}) \end{aligned}$$

$$(\Omega^{-1})^A{}_C dd\Omega^C{}_B = 2\pi\lambda^A{}_B \delta(\mathcal{S}) \ \text{ for any } \lambda^A{}_B \text{ such that } \ e^{2\pi\lambda} = \alpha \in Z(G),$$

$$U_\alpha(\mathcal{S}) = \exp\left(\tfrac{\pi}{g^2} \textstyle\int_{\mathcal{S}} \lambda^{AB} B_{AB} \right) \ \text{ where } \ e^{2\pi\lambda} = \alpha \in Z(G)$$

$$U_\alpha(\mathcal{S}) = \exp\left(\tfrac{1}{4G_\mathrm{N}} \textstyle\int_{\mathcal{S}} \tfrac{1}{2} \star \lambda_{AB} (e^A \wedge e^B) \right),$$

$$\begin{aligned} \widetilde{U}(\mathcal{S}) &= \exp\left(-\frac{1}{4G_\mathrm{N}} \int_{\mathcal{S}} \tilde{\lambda}^{\dot\beta}_{\dot\alpha} e^{\dot\alpha\alpha} \wedge e_{\alpha\dot\beta} \right) \ \text{ where } \ \left(e^{2\pi\tilde\lambda} \right)^{\dot\alpha}_{\dot\beta} = -\delta^{\dot\alpha}_{\dot\beta} \\ U(\mathcal{S}) &= \exp\left(-\frac{1}{4G_\mathrm{N}} \int_{\mathcal{S}} \lambda_\alpha{}^\beta e_{\beta\dot\alpha} \wedge e^{\dot\alpha\alpha} \right) \ \text{ where } \ \left(e^{2\pi\lambda} \right)_\alpha{}^\beta = -\delta_\alpha{}^\beta \end{aligned}$$

$$\langle U_\alpha(\mathcal{S}) W_\rho(\mathcal{C}) \rangle = \rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})} \langle W_\rho(\mathcal{C}) \rangle$$

$$\langle U_\alpha(\mathcal{S}) W_\rho(\mathcal{C}) \rangle = \int \mathcal{D}\omega \mathcal{D}e e^{-S} U_\alpha(\mathcal{S}) W_\rho(\mathcal{C})$$

$$e^{-S} U_\alpha(\mathcal{S}) = \exp\left(-\tfrac{1}{4g^2} \textstyle\int_{\mathcal{M}} \epsilon_{ABCD} e^A \wedge e^B \wedge (R - \Omega^{-1} dd\Omega)^{CD} + \cdots \right)$$

$$R^A_B \mapsto (\Omega^{-1})^A_C R^C_D \Omega^D_B + (\Omega^{-1})^A_C dd\Omega^C_B$$

$$e^{-S} U_\alpha(\mathcal{S}) \mapsto e^{-S}$$

$$\langle U_\alpha(\mathcal{S})W_\rho(\mathcal{C})\rangle \mapsto \int \mathcal{D}\omega\mathcal{D}ee^{-S}\rho(\alpha)^{\mathrm{link}(\mathcal{C},\mathcal{S})}W_\rho(\mathcal{C})$$

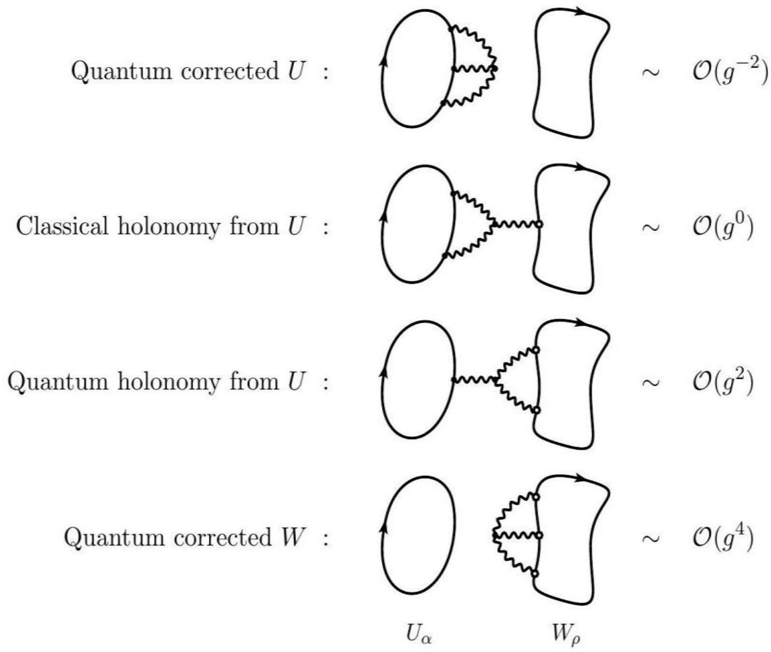
$$-\frac{1}{g^2}\star R_{AB}\wedge e^B = \frac{1}{3!}|e|T^\mu{}_A\epsilon_{\mu\nu\rho\sigma}dx^\nu\wedge dx^\rho\wedge dx^\sigma$$

$$T^\mu_\kappa = -\frac{1}{8G_\mathrm{N}}\frac{1}{|e|}\star\lambda_{\kappa\nu}\delta(\mathcal{S})_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$
$$= -\frac{1}{4G_\mathrm{N}}\star\lambda_{\kappa\nu}\int d\sigma_1 d\sigma_2\delta^{(4)}(x-X)\left(\frac{\partial X^\mu}{\partial\sigma_1}\frac{\partial X^\nu}{\partial\sigma_2}-\frac{\partial X^\nu}{\partial\sigma_1}\frac{\partial X^\mu}{\partial\sigma_2}\right)$$

$$-\frac{1}{g^2}R_{AB}\wedge e^B = \frac{1}{3!}|e|T^{\star\mu}{}_A\epsilon_{\mu\nu\rho\sigma}dx^\nu\wedge dx^\rho\wedge dx^\sigma$$

$$T^{\star\mu}_\kappa = -\frac{1}{8G_\mathrm{N}}\frac{1}{|e|}\lambda_{\kappa\nu}\delta(\mathcal{S})_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

$$\langle U_\alpha(\mathcal{S})W_\rho(\mathcal{C})\rangle = \int \mathcal{D}\omega\mathcal{D}ee^{-S}U_\alpha(\mathcal{S})W_\rho(\mathcal{C})$$



$$\Omega = \exp\left(\frac{k}{2}c\phi\right) \quad \text{where} \quad e^{2\pi c} = \mathbb{1}$$

$$c^A_B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega^A_B = (\Omega^{-1}d\Omega)^A{}_B = \frac{k}{2}c^A{}_B d\phi = \frac{k}{2}c^A{}_B\frac{xdy-ydx}{x^2+y^2}$$
$$R^A{}_B = (\Omega^{-1}dd\Omega)^A{}_B = \frac{k}{2}c^A{}_B dd\phi = \frac{k}{2}2\pi c^A{}_B\delta(x)\delta(y)dx\wedge dy,$$

$$\mathrm{tr}_{\mathrm{vec}}\mathrm{Pexp}\left(\frac{k}{2}c\int_0^{2\pi}d\phi\right) = 4(-1)^k$$

$$c_\alpha{}^\beta = 0 \ \text{ and } \ \tilde{c}^{\dot\alpha}{}_{\dot\beta} = -i(\sigma_1)^{\dot\alpha}{}_{\dot\beta}$$

$$\mathrm{tr}_{\mathrm{sp}}\mathrm{Pexp}\left(\frac{k}{2}c\int_0^{2\pi} d\phi\right)= 2$$

$$\mathrm{tr}_{\widetilde{\mathrm{sp}}}\mathrm{Pexp}\left(\frac{k}{2}\tilde{c}\int_0^{2\pi} d\phi\right)= 2(-1)^k$$

$$ds^2 = \frac{1}{f(r)^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\ \theta d\phi^2 + f(r)^2 dt^2$$

$$f(r) = \sqrt{1 - \frac{2G_{\mathrm{N}}M}{r} + \frac{r^2}{l^2}},$$

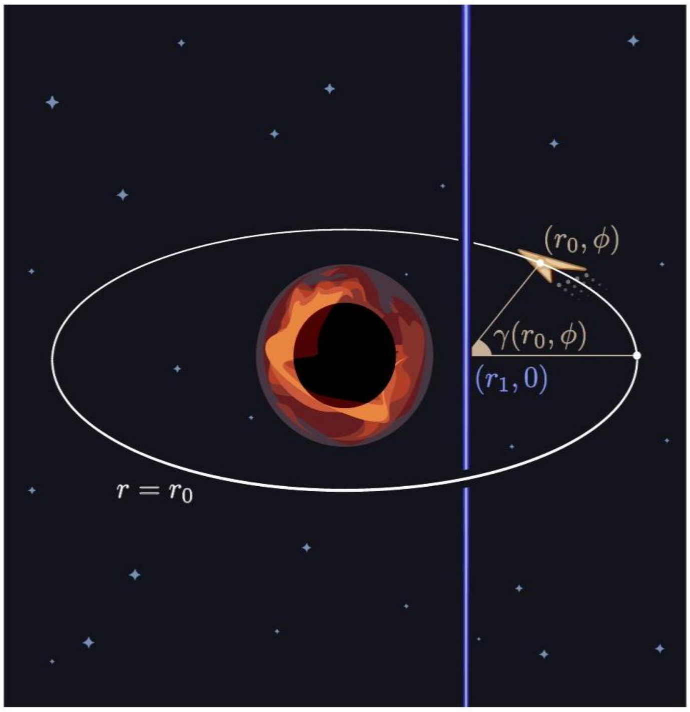
$$W_{\mathrm{vec}}(\mathcal{C}) = \mathrm{tr}_{\mathrm{vec}}\mathrm{Pexp}\left(\int_0^{2\pi}\ \omega_\phi(r_0,\pi/2,\phi,0)d\phi\right)$$

$$\omega^A_{B\phi}(r,\pi/2,\phi,t) = \begin{pmatrix} 0 & 0 & f(r) & 0 \\ 0 & 0 & 0 & 0 \\ -f(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W(\mathcal{C}) = 2 + 2\cos\left(2\pi f(r_0)\right)$$

$$\omega^A{}_{B\phi}(r,\pi/2,\phi,t) = \frac{(c_+ + c_-)^A{}_B}{2}f(r),$$

$$(c_+)^A{}_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } (c_-)^A{}_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$



$$\Omega = \exp\left(\frac{c}{2}\gamma(r,\phi)\right)\ \text{ where }\ e^{2\pi c} = \mathbb{1},$$

$$\tan\ \gamma(r,\phi) = \frac{r\sin\ \phi}{r\cos\ \phi - r_1}.$$

$$\omega^A_{B\phi}(r,\pi/2,\phi,t) = \frac{(c_+ + c_-)^A{}_B}{2} f(r) \mapsto \frac{(c_+ + c_-)^A{}_B}{2} f(r) + \frac{(c_+)^A{}_B}{2}\frac{\partial\gamma}{\partial\phi}(r,\phi),$$

$$\begin{aligned} W_{\text{vec}}(\mathcal{C}) \mapsto & \text{trexp}\left(\frac{c_+ + c_-}{2} 2\pi f(r_0) + \frac{c_+}{2} 2\pi n\right) \\ &= 2\cos\ n\pi + 2\cos\left(2\pi f(r_0) + n\pi\right) \\ &= \left(2 + 2\cos\left(2\pi f(r_0)\right)\right)\cdot(-1)^n \\ &= W_{\text{vec}}(\mathcal{C})\cdot(-1)^n \end{aligned}$$

$$n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\gamma}{\partial\phi}(r,\phi)d\phi = \begin{cases} 1 & \text{if} \quad r_1 < r_0 \\ 0 & \text{if} \quad r_1 > r_0 \end{cases}$$

$$c^A_B = \begin{pmatrix} 0 & n_3 & -n_2 & n_1 \\ -n_3 & 0 & n_1 & n_2 \\ n_2 & -n_1 & 0 & n_3 \\ -n_1 & -n_2 & -n_3 & 0 \end{pmatrix} = n_1(t^+_1)^A{}_B + n_2(t^+_2)^A{}_B + n_3(t^+_3)^A{}_B.$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\frac{f(r)}{2} - \begin{pmatrix} (n_2 n_1 \mathbf{s}^2 - n_3\mathbf{sc})(t^+_1)^A{}_B \\ +\left(\frac{1}{2} - (1 - n_2{}^2)\mathbf{s}^2\right)(t^+_2)^A{}_B \\ +(n_2 n_3 \mathbf{s}^2 + n_1\mathbf{sc})(t^+_3)^A{}_B \end{pmatrix} f(r) + \frac{c^A{}_B}{2}\frac{\partial\gamma}{\partial\phi}(r,\phi)$$

$$\frac{1}{g^2}\left[\frac{1}{2}B_{a\mu\nu}\left(\partial_\rho\omega^a_\sigma + \frac{1}{2}f^a_{bc}\omega^b_\rho\omega^c_\sigma\right)\epsilon^{\mu\nu\rho\sigma} - \frac{\Lambda}{6}B_{a\mu\nu}\star B^a_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\right]$$

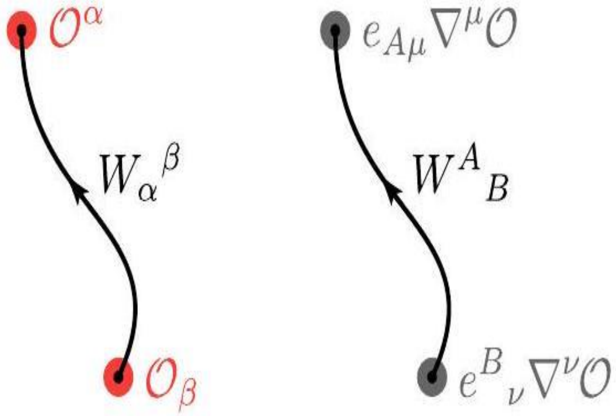
$$E^i_a = \frac{1}{2}B_{ajk}\epsilon^{ijk}$$

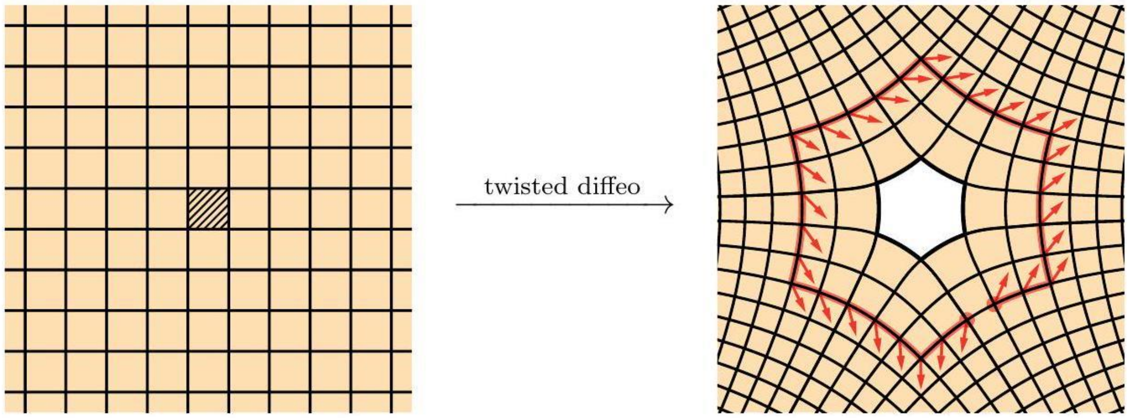
$$\begin{aligned} &\left[\omega^a{}_i(x), \omega^b{}_j(x')\right] = 0 \\ &\left[E^i{}_a(x), \omega^b{}_j(x')\right] = g^2\delta^i{}_j\delta^b{}_a\delta^{(3)}(x - x') \\ &\left[E^i{}_a(x), E^j{}_b(x')\right] = 0 \end{aligned}$$

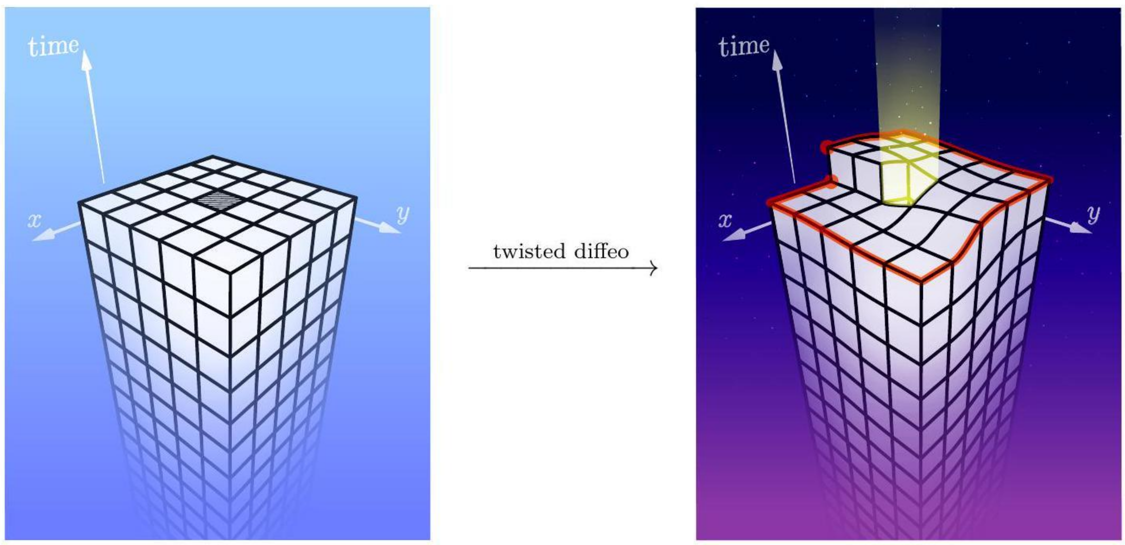
$$U_\alpha(\mathcal{S}_0)W_\rho(\mathcal{C})U^{-1}_\alpha(\mathcal{S}_0) = \rho(\alpha)^{\text{int}_3(\mathcal{C},\mathcal{S}_0)}W_\rho(\mathcal{C}),$$

$$U_\alpha(\mathcal{S}_0) = \exp\left(\frac{\pi}{g^2}\int_{\mathcal{S}_0} dx^j\wedge dx^k\epsilon_{ijk}E^i_a\lambda^a\right) \quad \text{where} \quad e^{2\pi\lambda} = \alpha$$

$$U_\alpha(\mathcal{S}_0)W_\rho(\mathcal{C})U^{-1}_\alpha(\mathcal{S}_0) = \text{tr}_\rho\text{Pexp}\left(\oint_{\mathcal{C}}\omega + 2\pi\lambda\delta_3(\mathcal{S}_0)\right) = \rho(\alpha)^{\text{int}_3(\mathcal{C},\mathcal{S}_0)}W_\rho(\mathcal{C}),$$







$$\epsilon^{1234} = +1, dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} = \epsilon^{\mu\nu\rho\sigma} d^4x, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = +4!$$

$$\delta_b^a = \frac{1}{2}(t^a)_{AB}(t_b)^{AB} \text{ and } (t_a)^{AB}(t^a)_{CD} = 2\delta_C^{[A}\delta^{B]}D.$$

$$X^{AB} = (t_a)^{AB}X^a, X^a = \frac{1}{2}(t^a)_{AB}X^{AB}$$
$$Y_{AB} = Y_a(t^a)_{AB}, Y_a = \frac{1}{2}Y_{AB}(t_a)^{AB}$$

$$Y_a X^a = \frac{1}{2} Y_{AB} X^{AB}$$

$$(t_a)^{AB} f^a_{bc} X^b_1 X^c_2 = X^{AC}_1 \delta_{CD} X^{DB}_2 - X^{AC}_2 \delta_{CD} X^{DB}_1$$

$$\delta_{ab} = \frac{1}{2} \delta_{AC} \delta_{BD} (t_a)^{AB} (t_b)^{CD}, \delta^{ab} = \frac{1}{2} (t^a)_{AB} \left(t^b\right)_{CD} \delta^{AC} \delta^{BD}, \delta^{ac} \delta_{cb} = \delta^a_b$$

$$\star_{ab} = \frac{1}{4} \epsilon_{ABCD} (t_a)^{AB} (t_b)^{CD}, \star^{ab} = \frac{1}{4} (t^a)_{AB} \left(t^b\right)_{CD} \epsilon^{ABCD}, \star^{ac} \star_{cb} = \delta^a{}_b.$$

$$\left(\star_{ab} X^b\right) (t^a)_{AB} = \frac{1}{2} \epsilon_{ABCD} X^{CD} = \star X_{AB}$$
$$(t_a)^{AB} \left(\star^{ab} Y_b\right) = \frac{1}{2} \epsilon^{ABCD} Y_{CD} = \star Y^{AB}$$

$$\star^{ad} \delta_{dc} \star^{ce} \delta_{eb} = \delta^a{}_b, f^a{}_{bc} \star^{be} \star^{cf} = f^a{}_{bc} \delta^{be} \delta^{cf}$$

$$\delta^a{}_b = (\tilde{t}^a)_{\dot{\alpha}\dot{\beta}} (\tilde{t}_b)^{\dot{\alpha}\dot{\beta}} + (t^a)_{\alpha\beta} (t_b)^{\alpha\beta}$$
$$(\tilde{t}_a)^{\dot{\alpha}\dot{\beta}} (\tilde{t}^a)_{\dot{\gamma}\dot{\delta}} = \delta^{(\dot{\alpha}}{}_{\dot{\gamma}} \delta^{\dot{\beta})} \dot{\delta}, (t_a)^{\alpha\beta} (t^a)_{\gamma\delta} = \delta_\gamma{}^{(\alpha} \delta_\delta{}^{\beta)}$$

$$\tilde{X}^{\dot{\alpha}\dot{\beta}} = (\tilde{t}_a)^{\dot{\alpha}\dot{\beta}} X^a, \quad X^a = (\tilde{t}^a)_{\dot{\alpha}\dot{\beta}} \tilde{X}^{\dot{\alpha}\dot{\beta}}, \quad \tilde{Y}_{\dot{\alpha}\dot{\beta}} = Y_a (\tilde{t}^a)_{\dot{\alpha}\dot{\beta}}, \quad Y_a = \tilde{Y}_{\dot{\alpha}\dot{\beta}} (\tilde{t}_a)^{\dot{\alpha}\dot{\beta}},$$
$$X^{\alpha\beta} = (t_a)^{\alpha\beta} X^a, \quad X^a = (t^a)_{\alpha\beta} X^{\alpha\beta}, \quad Y_{\alpha\beta} = Y_a (t^a)_{\alpha\beta}, \quad Y_a = Y_{\alpha\beta} (t_a)^{\alpha\beta}.$$

$$X^a = (\tilde{t}^a)_{\dot{\alpha}\dot{\beta}} \tilde{X}^{\dot{\alpha}\dot{\beta}} + (t^a)_{\alpha\beta} X^{\alpha\beta}, Y_a X^a = \tilde{Y}_{\dot{\alpha}\dot{\beta}} \tilde{X}^{\dot{\alpha}\dot{\beta}} + Y_{\alpha\beta} X^{\alpha\beta}$$

$$(t_a)^{\dot{\alpha}\dot{\beta}} f^a{}_{bc} X^b_1 X^c_2 = \left(\tilde{X}_1\right)^{\dot{\alpha}\dot{\gamma}} \tilde{\epsilon}_{\dot{\gamma}\dot{\delta}} \left(\tilde{X}_2\right)^{\dot{\delta}\dot{\beta}} - \left(\tilde{X}_2\right)^{\dot{\alpha}\dot{\gamma}} \tilde{\epsilon}_{\dot{\gamma}\dot{\delta}} \left(\tilde{X}_1\right)^{\dot{\delta}\dot{\beta}}$$
$$(t_a)^{\alpha\beta} f^a{}_{bc} X^b_1 X^c_2 = (X_1)^{\alpha\beta} \epsilon_{\beta\delta} (X_2)^{\delta\gamma} - (X_2)^{\alpha\beta} \epsilon_{\beta\delta} (X_1)^{\delta\gamma}$$

$$\star_{ab} \left(\tilde{t}^b\right)_{\dot{\alpha}\dot{\beta}} = +\tilde{\epsilon}_{\dot{\alpha}\dot{\gamma}} \tilde{\epsilon}_{\dot{\beta}\dot{\delta}} (\tilde{t}_a)^{\dot{\gamma}\dot{\delta}}, \star_{ab} \left(t^b\right)_{\alpha\beta} = -\epsilon_{\alpha\gamma} \epsilon_{\beta\delta} (t_a)^{\gamma\delta}.$$

$$(\tilde{t}^a)_{\dot{\alpha}\dot{\beta}} \left(\tilde{t}^b\right)_{\dot{\gamma}\dot{\delta}} \tilde{\epsilon}^{\dot{\alpha}\dot{\gamma}} \tilde{\epsilon}^{\dot{\beta}\dot{\delta}} = \frac{1}{2} \left(\delta^{ab} + \star^{ab}\right), (t^a)_{\alpha\beta} \left(t^b\right)_{\gamma\delta} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} = \frac{1}{2} \left(\delta^{ab} - \star^{ab}\right).$$

$$(\tilde{t}^{AB})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left(\tilde{\epsilon} \tilde{\sigma}^{[A} \sigma^{B]}\right)_{\dot{\alpha}\dot{\beta}}, (t^{AB})_{\alpha\beta} = \frac{1}{2} \left(\sigma^{[A} \tilde{\sigma}^{B]} \epsilon\right)_{\alpha\beta}$$

$$v_A (\sigma^A)_{\alpha\dot{\alpha}} = \begin{pmatrix} iv_4 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & iv_4 - v_3 \end{pmatrix}, v_A (\tilde{\sigma}^A)^{\dot{\alpha}\alpha} = \begin{pmatrix} iv_4 - v_3 & -v_1 + iv_2 \\ -v_1 - iv_2 & iv_4 + v_3 \end{pmatrix},$$

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \tilde{\psi}^{\dot{\alpha}} = \tilde{\epsilon}^{\dot{\alpha}\dot{\beta}} \tilde{\psi}_{\dot{\beta}}, \tilde{\psi}_{\dot{\alpha}} = \tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} \tilde{\psi}^{\dot{\beta}}$$

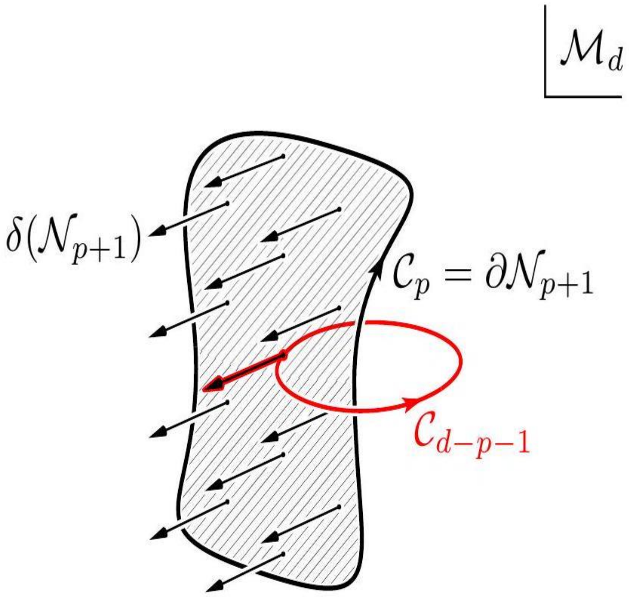
$$\omega^a = (\tilde{t}^a)_{\dot{\alpha}\dot{\beta}} \tilde{\omega}^{\dot{\alpha}\dot{\beta}} + (t^a)_{\alpha\beta} \omega^{\alpha\beta}$$

$$\tilde{R}^{\dot{\alpha}}{}_{\dot{\beta}} = d\tilde{\omega}^{\dot{\alpha}}{}_{\dot{\beta}} - \tilde{\omega}^{\dot{\alpha}}{}_{\dot{\gamma}} \wedge \tilde{\omega}^{\dot{\gamma}}{}_{\dot{\beta}}, R_\alpha{}^\beta = d\omega_\alpha{}^\beta + \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta.$$

$$e^{\dot{\alpha}\alpha} = -\frac{1}{2} (\tilde{\sigma}_A)^{\dot{\alpha}\alpha} e^A$$

$$\tilde{B}^{\dot{\alpha}\dot{\beta}} = \epsilon_{\alpha\beta} e^{\dot{\alpha}\alpha} \wedge e^{\dot{\beta}\beta}, B^{\alpha\beta} = -\tilde{\epsilon}_{\dot{\alpha}\dot{\beta}} e^{\dot{\alpha}\alpha} \wedge e^{\dot{\beta}\beta}.$$

$$-\frac{1}{g^2} \left(e^{\dot{\alpha}\alpha} \wedge e_{\alpha\dot{\beta}} \wedge \tilde{R}^{\dot{\beta}} \dot{\alpha} + e_{\beta\dot{\alpha}} \wedge e^{\dot{\alpha}\alpha} \wedge R_\alpha{}^\beta\right) + \frac{\Lambda}{3g^2} e_{\alpha\dot{\alpha}} \wedge e^{\dot{\alpha}\beta} \wedge e_{\beta\dot{\beta}} \wedge e^{\dot{\beta}\alpha}.$$



$$\int_{\mathcal{N}_p} \alpha^{(p)} = \int_{\mathcal{M}_d} \alpha^{(p)} \wedge \delta(\mathcal{N}_p)$$

$$\frac{1}{(d-p)!}\left[\int d^p\sigma\,\delta^{(d)}(x - X(\sigma))\,\frac{\partial X^{\lambda_1}}{\partial\sigma^1}\cdots\frac{\partial X^{\lambda_p}}{\partial\sigma^p}\right]\epsilon_{\lambda_1\cdots\lambda_p\mu_1\cdots\mu_{d-p}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{d-p}}$$

$$\mathrm{int}(\mathcal{N}_p, \mathcal{N}_{d-p}) = \int_{\mathcal{M}_d} \delta(\mathcal{N}_p) \wedge \delta(\mathcal{N}_{d-p})$$

$$\mathrm{int}(\mathcal{N}_p, \mathcal{N}_{d-p}) = (-1)^{p(d-p)}\mathrm{int}(\mathcal{N}_{d-p}, \mathcal{N}_p)$$

$$\mathrm{link}_*(\partial\mathcal{N}_{p+1}, \mathcal{C}_{d-p-1}) = \mathrm{int}(\mathcal{N}_{p+1}, \mathcal{C}_{d-p-1}) = \int_{\mathcal{C}_{d-p-1}} \delta(\mathcal{N}_{p+1})$$

$$\mathrm{link}(\partial\mathcal{N}_{d-p}, \mathcal{C}_p) = \mathrm{link}_*(\partial\mathcal{N}_{d-p}, \mathcal{C}_p)$$
$$\mathrm{link}(\mathcal{C}_{d-p-1}, \mathcal{C}_p) = (-1)^{dp+1}\mathrm{link}(\mathcal{C}_p, \mathcal{C}_{d-p-1})$$

$$\mathrm{link}(\mathcal{C}_{d-p-1}, \partial\mathcal{N}_{p+1}) = (-1)^{d-p}\mathrm{int}(\mathcal{C}_{d-p-1}, \mathcal{N}_{p+1})$$

$$d\delta(\mathcal{N}_p) = (-1)^p\delta(\partial\mathcal{N}_p)$$
$$\mathrm{link}(\partial\mathcal{N}_{p+1}, \partial\mathcal{N}_{d-p}) = (-1)^{dp+1}\mathrm{link}(\partial\mathcal{N}_{d-p}, \partial\mathcal{N}_{p+1})$$

$$\mathcal{C}_p = \mathcal{K}_p \times \{t_0\}$$

$$(-1)^{d-p}\mathcal{N}_{d-p} = \mathcal{Y}_{d-p-1} \times \mathcal{I}$$

$$\partial\mathcal{N}_{d-p} = (-\mathcal{Y}_{d-p-1} \times \{t_+\}) \cup (\mathcal{Y}_{d-p-1} \times \{t_-\}) \cup \Gamma,$$

$$(-1)^{d-p}\mathrm{link}(\partial\mathcal{N}_{d-p}, \mathcal{C}_p) = \mathrm{int}(\mathcal{Y}_{d-p-1} \times \mathcal{I}, \mathcal{C}_p) = (-1)^p\mathrm{int}_{d-1}(\mathcal{Y}_{d-p-1}, \mathcal{K}_p)$$

$$\mathrm{int}_{d-1}(\mathcal{Y}_{d-p-1}, \mathcal{K}_p) = \int_{\mathcal{X}_{d-1}} \delta_{d-1}(\mathcal{Y}_{d-p-1}) \wedge \delta_{d-1}(\mathcal{K}_p)$$

$$\epsilon_{i_1\cdots i_{d-p-1}dj_1\cdots j_p} = (-1)^p\epsilon_{i_1\cdots i_{d-p-1}j_1\cdots j_pd} = (-1)^p\epsilon_{i_1\cdots i_{d-p-1}j_1\cdots j_p}$$

$$S = \int_{\mathcal{M}_d} B^{(d-p-1)} \wedge F^{(p+1)} + f\left(B^{(d-p-1)}\right)$$

$$W_q\left(\partial\mathcal{N}_{p+1}\right) = \exp\left(q\int_{\mathcal{N}_{p+1}} F^{(p+1)}\right)$$

$$W_q\left(\partial\mathcal{N}_{p+1}\right) \mapsto \exp\left(q\varepsilon \mathrm{int}\left(\mathcal{C}_{d-p-1}, \mathcal{N}_{p+1}\right)\right) W_q\left(\partial\mathcal{N}_{p+1}\right)$$

$$-S \mapsto -S + \varepsilon\int_{\mathcal{M}_d} B^{(d-p-1)} \wedge \delta\left(\mathcal{C}_{d-p-1}\right)$$

$$U_\varepsilon\left(\mathcal{C}_{d-p-1}\right) = \exp\left(\varepsilon\int_{\mathcal{C}_{d-p-1}} B^{(d-p-1)}\right)$$

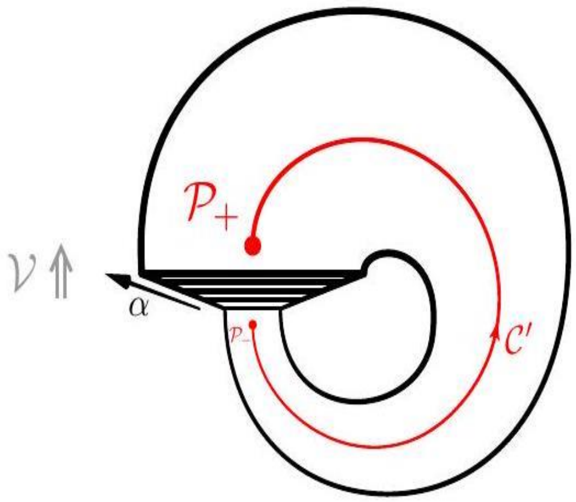
$$\left\langle U_\varepsilon\left(\mathcal{C}_{d-p-1}\right) W_q\left(\partial\mathcal{N}_{p+1}\right)\right\rangle = \exp\left(q\varepsilon \mathrm{int}\left(\mathcal{C}_{d-p-1}, \mathcal{N}_{p+1}\right)\right)\left\langle W_q\left(\partial\mathcal{N}_{p+1}\right)\right\rangle.$$

$$\left\langle U_\varepsilon\left(\mathcal{C}_{d-p-1}\right) W_q\left(\mathcal{C}_p\right)\right\rangle = \exp\left((-1)^{d-p} q\varepsilon \mathrm{link}\left(\mathcal{C}_{d-p-1}, \mathcal{C}_p\right)\right)\left\langle W_q\left(\mathcal{C}_p\right)\right\rangle,$$

$$\mathcal{C}_p \leftrightarrow \mathcal{C}, \mathcal{C}_{d-p-1} \leftrightarrow \mathcal{S}, \mathcal{N}_{d-p} \leftrightarrow \mathcal{V}.$$

$$e^{-\varepsilon Q(\mathcal{Y}_{d-p-1})} W_q\left(\mathcal{K}_p\right) e^{\varepsilon Q(\mathcal{Y}_{d-p-1})} = \exp\left((-1)^p q\varepsilon \mathrm{int}_{d-1}\left(\mathcal{Y}_{d-p-1}, \mathcal{K}_p\right)\right) W_q\left(\mathcal{C}_p\right)$$

$$Q\left(\mathcal{Y}_{d-p-1}\right) = \int_{\mathcal{Y}_{d-p-1}} B^{(d-p-1)}$$



$$\left[A_{i_1\cdots i_p}, B_{k_1\cdots k_{d-p-1}}\right]\epsilon^{k_1\cdots k_{d-p-1}j_1\cdots j_p} = (d-p-1)!\,p!\,(-1)^p \delta^{j_1}{}_{[i_1}\cdots\delta^{j_p}{}_{i_p]}$$
$$\Rightarrow \left[A_{i_1\cdots i_p}, Q\left(\mathcal{Y}_{d-p-1}\right)\right] = (-1)^p\left(\delta_{d-1}\left(\mathcal{Y}_{d-p-1}\right)\right)_{i_1\cdots i_p}$$

$$\mathcal{K}_p \leftrightarrow \mathcal{C}, \mathcal{Y}_{d-p-1} \leftrightarrow -\mathcal{S}_0.$$

$$\lim_{\mathcal{P}_\pm\to\mathcal{P}} \Omega(\mathcal{P}_+)\Omega^{-1}(\mathcal{P}_-) = \alpha \in Z(G)$$
$$\Leftrightarrow\ \exists\lambda^a \text{ s.t. } (\Omega^{-1}dd\Omega)^a = \lambda^a\delta(\partial\mathcal{V}_{d-1}) \text{ and } e^{2\pi\lambda} = \alpha^{(-1)^d},$$

$$\lim_{\mathcal{C}_1'\to\mathcal{C}_1} \mathrm{Pexp}\left(\oint_{\mathcal{C}_1'} \Omega^{-1}d\Omega\right) = \lim_{\mathcal{C}_1'\to\mathcal{C}_1} \Omega^{-1}(\mathcal{P}_-)\Omega(\mathcal{P}_+) = \alpha$$

$$\mathrm{Pexp}\left(\oiiint_{\partial\mathcal{N}_2}\Omega^{-1}d\Omega\right)=\mathrm{Pexp}\left(\int_{\mathcal{N}_2}\Omega^{-1}dd\Omega\right)=\exp\left(\int_{\mathcal{M}_d}\Omega^{-1}dd\Omega\wedge\delta(\mathcal{N}_2)\right)$$

$$\lim_{\mathcal{P}_\pm\to\mathcal{P}}\chi(\mathcal{P}_+)-\chi(\mathcal{P}_-)=(-1)^d 2\pi\lambda \iff dd\chi=2\pi\lambda\delta(\partial\mathcal{V}_{d-1})$$

$$d\chi+(-1)^d 2\pi\lambda\delta(\mathcal{V}_{d-1})=df$$

CONCLUSIONES.

De los resultados obtenidos en el apartado anterior, se concluye que: 1) el espacio – tiempo cuántico es susceptible de deformación, no necesariamente extrema, lo que aparece como un curvatura hipergeométrica; 2) la gravedad cuántica, en escenarios entrópicos, puede formar membranas que no necesariamente se despliegan en dimensiones más altas, sino que, se tratan de desdoblamientos dentro de un mismo espacio – tiempo cuántico, por lo que, no se forman dimensiones en sí mismas, sino distorsiones de realidad espacial y temporal; 3) en un escenario de gravedad cuántica, la formación de agujeros negros cuánticos es posible, especialmente cuando la partícula supermasiva colapsa o colisiona con otra, en tanto que, es imposible la formación de agujeros cuánticos de gusano, pues la gravedad no es extrema aunque entrópica; 4) la partícula supermasiva o llamada también partícula oscura, a propósito de su interacción, produce gravedad cuántica. Cuando la partícula supermasiva, por sí misma, tuerce el espacio – tiempo cuántico, esto se denomina gravedad cuántica endógena, en tanto que, cuando la gravedad cuántica se produce por la interacción de la partícula supermasiva y el gravitón, entonces se vuelve exógena por permeabilización del espacio – tiempo cuántico por un campo gravitónico; 5) la simetría de calibre, se torna esencial para efectos de conciliar la relatividad general y especial y la mecánica cuántica, aunque no en circunstancias extremas o primitivas; 6) la formación de materia oscura, es inevitable en gravedad cuántica, fundamentalmente por interacción de la partícula supermasiva y la formación de agujeros negros cuánticos que devoran materia y energía a escala microscópica, volviéndola en materia pura y oscura, extremadamente densa, en la que, la intervención de la gravedad es unitaria.

ACLARACIONES FINALES:

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo † será reemplazado por este símbolo†o por este símbolo †, equivaliendo lo mismo.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
†	†
	†

2. En todos los casos, este símbolo ‡, será reemplazado por este símbolo ‡ o por este símbolo ‡.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
‡	‡
	‡

3. En todos los casos, se añadirá y por ende, se calculará la magnitud§que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo • podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

REFERENCIAS BIBLIOGRÁFICAS.

Christian Sämann y Martin Wolf, Supersymmetric Yang-Mills Theory as Higher Chern-Simons Theory, arXiv:1702.04160v2 [hep-th] 26 Jul 2017.

Johannes Moerland, Spinorial Superspaces and Super Yang-Mills Theories, arXiv:2411.06165v2 [hep-th] 30 Apr 2025.

Murat Koloğlu, Quantum Vacua of 2d Maximally Supersymmetric Yang-Mills Theory, arXiv:1609.08232v1 [hep-th] 27 Sep 2016.

Kazuya Yonekura, Supersymmetric gauge theory, (2,0) theory and twisted 5d Super-Yang-Mills, arXiv:1310.7943v2 [hep-th] 23 Jan 2014.

G. Bossard, P.S. Howe, U. Lindström, K.S. Stelle y L. Wulff, Integral invariants in maximally supersymmetric Yang-Mills theories, arXiv:1012.3142v1 [hep-th] 14 Dec 2010.

S. James Gates Jr, Yangrui Hu y S.-N. Hazel Mak, Weyl Covariance, and Proposals for Superconformal Prepotentials in 10D Superspaces, arXiv:2007.05097v4 [hep-th] 14 Jan 2021.

Ilya Bakhmatova1, Aybike Catal-Ozerb, Nihat Sadik Degerc, Kirill Gubarevd, Edvard T. Musaevd, Generalized 11D supergravity equations from tri-vector deformations, arXiv:2209.01423v3 [hep-th] 22 Sep 2022.

Andrés Anabalón, Antonio Gallerati, Simon Ross y Mario Trigiante, Supersymmetric solitons in gauged N = 8 supergravity, arXiv:2210.06319v2 [hep-th] 7 Feb 2023.

Nabamita Banerjee, Tabasum Rahnuma y Ranveer Kumar Singh, Soft and Collinear Limits in N = 8 Supergravity using Double Copy Formalism, arXiv:2212.11480v2 [hep-th] 22 Apr 2023.

Max Guillen, Taming the 11D pure spinor b-ghost, arXiv:2212.13653v1 [hep-th] 28 Dec 2022.

Falk Hassler y Yuho Sakatani, All maximal gauged supergravities with uplift, arXiv:2212.14886v2 [hep-th] 20 Sep 2024.

Nikolay Bobev, Junho Hong y Valentin Reys, Large N Partition Functions of 3d Holographic SCFTs, arXiv:2304.01734v3 [hep-th] 3 Nov 2023.

Maor Ben-Shahar y Max Guillen, Superspace Expansion of the 11D Linearized Superfields in The Pure Spinor Formalism, and The Covariant Vertex Operator, arXiv:2305.19898v1 [hep-th] 31 May 2023.

Pietro Capuozzo, John Estes, Brandon Robinson y Benjamin Suzzoni, Holographic Weyl Anomalies for 4d Defects in 6d SCFTs, arXiv:2310.17447v3 [hep-th] 20 Feb 2024.

Christopher Couzens, Monica Jinwoo Kang, Craig Lawrie y Yein Lee, Holographic duals of Higgsed $\mathcal{D}_p^b(BCD)$, arXiv:2312.12503v1 [hep-th] 19 Dec 2023.

Jay Armas, Giorgos Batzios y Akash Jain, Higher-group global symmetry and the bosonic M5 brane, arXiv:2402.19458v2 [hep-th] 20 Mar 2024.

Clifford Cheung, Maria Derda, Joon-Hwi Kim, Vinicius Nevoa, Ira Rothstein y Nabha Shah, Generalized Symmetry in Dynamical Gravity, arXiv:2403.01837v1 [hep-th] 4 Mar 2024.

Jiaming Shi y Zehua Xiao, Cosmological Inflation and Dark Sector from 11D Supergravity, arXiv:2404.11933v3 [hep-th] 8 Jul 2024.

Yolanda Lozanoa, Niall T. Macphersona, Nicolò Petric y Anayeli Ramírez, Holographic 1/2-BPS surface defects in ABJM, arXiv:2404.17469v2 [hep-th] 15 May 2024.

Igal Arav, Jerome P. Gauntlett, Yusheng Jiao, Matthew M. Roberts y Christopher Rosen, Superconformal Monodromy Defects in ABJM and mABJM Theory, arXiv:2408.11088v1 [hep-th] 20 Aug 2024.

Sat Paul, Jitendra Kumar, Sunil Kumar Maurya, Sourav Choudhary y Sweeti Kiroriwal, A generalized solution for anisotropic compact star model in F(Q) gravity, arXiv:2409.16334v1 [gr-qc] 24 Sep 2024.

Bao-Jun Cai y Bao-An Li, New Insights into Supradense Matter from Dissecting Scaled Stellar Structure Equations, arXiv:2409.18854v2 [astro-ph.HE] 11 Dec 2024.

Juliano C. S. Neves y Fernando G. Gardim, Stars and quark stars in bumblebee gravity, arXiv:2409.20360v3 [gr-qc] 12 Feb 2025.

Elly Bayona, Hernando Quevedo y Miguel Alcubierre, Spherically symmetric collapse: Initial configurations, arXiv:2410.17537v1 [gr-qc] 23 Oct 2024.

Grigorios Giotopoulos, Hisham Sati y Urs Schreiber, The Hidden M-Group, arXiv:2411.11963v1 [hep-th] 18 Nov 2024.

Naoki Onishi, General Relativistic Hartree-Fock Calculations for Neutron Star, arXiv:2412.02990v1 [gr-qc] 4 Dec 2024.

Hiroki Sakamoto y Masahiko Taniguchi, Neutron star in Logarithmic model of Cartan F(R) gravity, arXiv:2412.03219v3 [gr-qc] 11 Mar 2025.

Charul Rathod y M. Mishra, Cooling of Neutron Stars through Emission of Neutrinos and Photons: Effects of Modified Gravity and Magnetic Field using TOV Equations, arXiv: 2412.04520v1 [gr-qc] 5 Dec 2024.

Heng-Yu Chen, Nick Dorey, Sanefumi Moriyama, Rishi Mouland y Canberk Şanlı, Giant Gravitons and Volume Minimisation, arXiv:2412.05357v1 [hep-th] 6 Dec 2024.

Ivano Basile, Luca Buoninfante, Francesco Di Filippo, Benjamin Knorr, Alessia Platania y Anna Tokareva, Lectures in Quantum Gravity, arXiv:2412.08690v2 [hep-th] 21 Feb 2025.

Boyang Sun y James M. Lattimer, Correlations between the Neutron Star Mass-Radius Relation and the Equation of State of Dense Matter, arXiv:2412.14645v2 [astro-ph.SR] 14 Apr 2025.

Nikolay Bobev, Pieter-Jan De Smet, Junho Hong, Valentin Reys y Xuao Zhanga, An Airy Tale at Large N, arXiv:2502.04606v2 [hep-th] 23 Jul 2025.

Xuao Zhang, New G2-invariant supersymmetric AdS4 vacua in 11d supergravity and holography, arXiv:2503.21116v1 [hep-th] 27 Mar 2025.

George Tringas and Timm Wrase, Scale separation from O-planes, arXiv:2504.15436v1 [hep-th] 21 Apr 2025.

M. Zeeshan Gul, M. Sharif, Adeeba Arooj y Baiju Dayanandan, Impact of Non-metricity and Matter Source on the Geometry of Anisotropic Spheres, arXiv:2504.21348v1 [gr-qc] 30 Apr 2025.

M. Sharif e Iqra Ibrar, Charged Anisotropic Pulsar SAX J1748.9-2021 in Non-Riemannian Geometry, arXiv:2505.00758v1 [gr-qc] 1 May 2025.

Juan M. Z. Pretel, Takol Tangphati, Izzet Sakall y Ayan Banerjee, White dwarfs in regularized 4D Einstein-Gauss-Bonnet gravity, arXiv:2505.08153v1 [gr-qc] 13 May 2025.

Gianguido Dall'Agata y Fabio Zwirner, Supersymmetry-breaking compactifications
on Riemann-flat manifolds, arXiv:2507.02339v2 [hep-th] 4 Aug 2025.

R. P. Malik, Nilpotency Property, Physicality Criteria, Constraints and Standard BRST Algebra: A 4D Field-Theoretic System, arXiv:2507.17590v2 [hep-th] 3 Aug 2025.

Oren Bergman, Eduardo Garcia-Valdecasas, Francesco Mignosa y Diego Rodriguez-Gomez, The SymTFT of u(N) Yang-Mills Theory and Holography, arXiv:2508.00992v1 [hep-th] 1 Aug 2025.

Bik Soon Sia, Principle of Diminishing Potentialities in Large N Algebra, arXiv:2508.11688v1 [hep-th] 11 Aug 2025.

I.L. Buchbinder, R.M. Iakhibbaev, A.I. Mukhaeva, D.I. Kazakov1 y D.M. Tolkachev, Chiral effective potential in 4D, N = 1 supersymmetric gauge theories, arXiv:2508.14002v1 [hep-th] 19 Aug 2025.

Tom Ladstättera y Luca Marchetti, Interacting Scalar Field Cosmology from Full Quantum Gravity, arXiv:2508.16194v1 [gr-qc] 22 Aug 2025.

Johanna Erdmenger, Jani Kastikainen y Tim Schuhmann, Driven inhomogeneous CFT as a theory in curved space-time, arXiv:2508.18350v1 [hep-th] 25 Aug 2025.

Daniel Baumann, Kurt Hinterbichler, Callum R. T. Jones, Austin Joyce, Hayden Lee, Jiajie Mei y Nathan Meurrens, Constraints on Long-Range Forces in De Sitter Space, arXiv:2508.18353v1 [hep-th] 25 Aug 2025.

C.M. Hull, De Sitter Space in Supergravity and M Theory, arXiv:hep-th/0109213v3 23 Oct 2001.

Renata Kallosh y Lev Kofman, Superconformal Symmetry, Supergravity and Cosmology, arXiv:hep-th/0006179v3 6 Jul 2004.

David Geissbühler, Double Field Theory and N = 4 Gauged Supergravity, arXiv:1109.4280v2 [hep-th] 5 Dec 2011.

Tobias Hurth, Peter van Nieuwenhuizen y Andrew Waldron, On a possible new R^2 theory of supergravity, arXiv:hep-th/9702052v1 6 Feb 1997.

Hitoshi NISHINO, Supergravity Theories in D ≥ 12 Coupled to Super p-Branes, arXiv:hep-th/9807199v2 26 Oct 1998.

Gabriel Larios y Oscar Varela, Minimal D = 4 N = 2 supergravity from D = 11: An M-theory free lunch, arXiv:1907.11027v2 [hep-th] 25 Oct 2019.

A. Candiello y K. Lechner, Duality in Supergravity Theories, arXiv:hep-th/9309143v1 29 Sep 1993.

Jan Louis y Severin Lüst, Classification of maximally supersymmetric backgrounds in supergravity theories, arXiv:1607.08249v2 [hep-th] 10 Feb 2017.

Yifan Wang y Xi Yin, Supervertices and Non-renormalization Conditions in Maximal Supergravity Theories, arXiv:1505.05861v2 [hep-th] 29 May 2015.

KEVIN COSTELLO y SI LI, TWISTED SUPERGRAVITY AND ITS QUANTIZATION, arXiv:1606.00365v1 [hep-th] 1 Jun 2016.

Samuel Monnier y Gregory W. Moore, Remarks on the Green-Schwarz terms of six-dimensional supergravity theories, arXiv:1808.01334v2 [hep-th] 25 Nov 2018.

Per Berglund, Tristan Hübsch y Djordje Minic, On de Sitter Spacetime and String Theory, arXiv:2212.06086v3 [hep-th] 8 Feb 2023.

Fien Apers, Joseph P. Conlon, Edmund J. Copeland, Martin Mosnya y Filippo Revelloc, String Theory and the First Half of the Universe, arXiv:2401.04064v2 [hep-th] 15 Feb 2024.

Albuja Bustamante, M. I. (2024). Demostración del Espectro Hamiltoniano para un Campo de Yang-Mills no Abeliano que Poseen una Brecha de Masa Finita con Respecto al Estado de Vacío. *Ciencia Latina Revista Científica Multidisciplinar*, *8*(1).

Manuel Ignacio Albuja Bustamante , Teoría de Campos: Reforzamiento Teórico – Matemático al Modelo Estándar de Partículas, bajo la estructura ecuacional de Yang – Mills , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 2 (2024).

Manuel Ignacio Albuja Bustamante, La brecha de masa y la curvatura de los campos cuánticos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 4 (2024).

Manuel Ignacio Albuja Bustamante, Formalización Matemática y en Física de Partículas, en Relación a la Brecha de Masa y la Curvatura Geométrica de los Campos Cuánticos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 5 (2024).

Manuel Ignacio Albuja Bustamante, Campos Cuánticos Relativistas: Aproximaciones Teórico – Matemáticas Relativas a los Espacios Cuánticos Geométricamente Deformados o Perforados por Partículas y Antipartículas Supermasivas y Masivas E Hiperpartículas y Suprapartículas , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 1 (2025).

Manuel Ignacio Albuja Bustamante, Teoría Cuántica de Campos Relativistas. Formalización Teórica , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 3 (2025)

Manuel Ignacio Albuja Bustamante, Supergravedad cuántica relativista. Teorización inicial , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Supergravedad cuántica relativista. Dimensiones en R^η. , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Gravedad cuántica relativista , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Agujeros Negros Cuántico-Relativistas Masivos o Supermasivos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025).

Manuel Ignacio Albuja Bustamante, Cromodinámica Cuántica Relativista , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025).

Albuja Bustamante, M. I. (2025). Las partículas supermasivas: naturaleza fenomenológica de la partícula oscura. *Ciencia Latina Revista Científica Multidisciplinar*, *9*(3).