Ciencia Latina Revista Científica Multidisciplinar, Ciudad de México, México.
ISSN 2707-2207 / ISSN 2707-2215 (en línea), septiembre-octubre 2025,
Volumen 9, Número 5.

https://doi.org/10.37811/cl_rcm.v9i5

AGUJEROS CUÁNTICOS: APROXIMACIONES TEÓRICAS RESPECTO DE LOS AGUJEROS NEGROS CUÁNTICOS, AGUJEROS BLANCOS CUÁNTICOS Y AGUJEROS CUÁNTICOS DE GUSANO EN RELACIÓN A CAMPOS CUÁNTICOS RELATIVISTAS O CURVOS Y EN CONDICIONES DE GRAVEDAD Y SUPERGRAVEDAD CUÁNTICAS. VOLUMEN II

QUANTUM HOLES: THEORETICAL APPROACHES TO QUANTUM BLACK HOLES, WHITE QUANTUM HOLES, AND QUANTUM WORMHOLES IN RELATION TO RELATIVISTIC OR CURVED QUANTUM FIELDS AND UNDER CONDITIONS OF QUANTUM GRAVITY AND SUPERGRAVITY. VOLUME II

Manuel Ignacio Albuja Bustamante
Investigador independiente

DOI: https://doi.org/10.37811/cl_rcm.v9i5.19807

AGUJEROS CUÁNTICOS: APROXIMACIONES TEÓRICAS RESPECTO DE LOS AGUJEROS NEGROS CUÁNTICOS, AGUJEROS BLANCOS CUÁNTICOS Y AGUJEROS CUÁNTICOS DE GUSANO EN RELACIÓN A CAMPOS CUÁNTICOS RELATIVISTAS O CURVOS Y EN CONDICIONES DE GRAVEDAD Y SUPERGRAVEDAD CUÁNTICAS. VOLUMEN II

Manuel Ignacio Albuja Bustamante[1]
ignaciomanuelalbujabustamante@gmail.com
https://orcid.org/0009-0005-0115-767X
Investigador Independiente

RESUMEN

Los agujeros negros cuánticos, son sin duda, un campo de estudio fascinante en la teoría cuántica de campos relativistas. Éstos, según la TCCR (Teoría Cuántica de Campos Relativistas), se forman por partículas supermasivas o en su defecto, por superpartículas, de tal suerte que, en escenarios de supergravedad o gravedad cuánticas, según sea el caso, su formación es inminente en la primera y potencial en la segunda. En este trabajo, abordaré más a detalle, la formación de agujeros negros cuánticos supermasivos, especialmente en escenarios de extrema gravitación e intentaremos descifrar la singularidad de los mismos, incluyendo la formación de agujeros cuánticos de gusano, que en sí, son disímiles a los agujeros negros cuánticos. Estos agujeros negros, de carácter subatómico, interactúan con campos cuánticos específicos, con la particularidad, de que devoran energía a escala microscópica, lo que explicaría, en forma exponencial, la formación de materia oscura.

Palabras Clave: materia oscura, agujero negro cuántico, relatividad cuántica, agujeros cuánticos de gusano, partícula oscura, partícula estrella.

[1] Autor principal
Correspondencia: ignaciomanuelalbujabustamante@gmail.com

QUANTUM HOLES: THEORETICAL APPROACHES TO QUANTUM BLACK HOLES, WHITE QUANTUM HOLES, AND QUANTUM WORMHOLES IN RELATION TO RELATIVISTIC OR CURVED QUANTUM FIELDS AND UNDER CONDITIONS OF QUANTUM GRAVITY AND SUPERGRAVITY. VOLUME II

ABSTRACT

Quantum black holes are undoubtedly a fascinating field of study in the quantum theory of relativistic fields. These, according to the TCCR (Quantum Theory of Relativistic Fields), are formed by supermassive particles or, failing that, by superparticles, in such a way that, in scenarios of supergravity or quantum gravity, as the case may be, their formation is imminent in the former and potential in the latter. In this work, I will address in more detail the formation of supermassive quantum black holes, especially in extreme gravitation scenarios and we will try to decipher their singularity, including the formation of quantum wormholes, which in themselves, are dissimilar to quantum black holes. These black holes, of a subatomic nature, interact with specific quantum fields, with the particularity that they devour energy on a microscopic scale, which would explain, exponentially, the formation of dark matter.

Keywords: dark matter, quantum black hole, quantum relativity, wormholes, dark particle, star particle.

Artículo recibido 11 agosto 2025
Aceptado para publicación: 13 setiembre 2025

INTRODUCCIÓN

Los agujeros negros cuánticos, son aquellos fenómenos propios de la dualidad holográfica. Se forman principalmente por la deformación del espacio – tiempo cuántico, sea por una partícula supermasiva o en su defecto, por una superpartícula, aunque es inevitable en el segundo caso, sea por colapso o colisión. Estos agujeros negros cuánticos, a diferencia de los que ocurren a escala cosmológica, devoran energía para convertirla en materia oscura. En este trabajo se consolidará el modelo matemático de un agujero negro cuántico y de un agujero cuántico de gusano, éste último el mismo que se explica a través del puente de Einstein – Rosen, como ha quedado referido en trabajos anteriores, lo que además, explicaría la superposición y el entrelazamiento cuánticos. Se ha dicho que estos fenómenos microscópicos, ocurren por colapso o colisión de una partícula supermasiva o de una superpartícula, según sea el caso, sin embargo, los agujeros negros cuánticos también pueden colisionar, formando así, ondas cuánticas o denominadas también "ondas cuánticas gravitacionales", como se ha teorizado en trabajos anteriores. En este artículo, abordaré los fenómenos antes referidos, con especial énfasis, en la descripción matemática de los mismos, en un espacio de Hilbert – Einstein, tanto en supergravedad como en gravedad cuánticas, según corresponda, incluyendo la morfología propiamente de un agujero negro cuántico y un agujero cuántico de gusano.

RESULTADOS Y DISCUSIÓN.

Retomo el desarrollo matemático desplegado en el volumen I de este manuscrito:

Modelo Matemático para describir agujeros negros cuánticos masivos y supermasivos en condiciones de gravedad y supergravedad cuánticas relativistas. Este modelo es aplicable también para describir tanto agujeros blancos cuánticos de distinta clase como agujeros cuánticos de gusano.

$$H(\Delta, \omega) = -\frac{N^2}{2}\frac{\Delta^3}{\omega^2},$$

$$a = \frac{\pi + \omega_I}{1 - 5\omega_I}$$

$$F(\omega_I,\tau)=\frac{\pi(\pi-5\omega_I)(\omega_I+\pi)}{18\omega_I^2}\left(-\sqrt{3}\sqrt{\frac{-3\omega_I^2-2\pi\omega_I+\pi^2}{(\pi-5\omega_I)^2}}+\frac{2\tau\omega_I(\omega_I+\pi)(\pi-2\omega_I)^2}{\pi(\pi-5\omega_I)(\pi-3\omega_I)}\right.$$

$$\left.+\frac{4(\pi-2\omega_I)^2}{\sqrt{3}\pi(\pi-5\omega_I)\sqrt{\frac{\pi-3\omega_I}{\omega_I+\pi}}}\right)$$

$$\tau=-\frac{\sqrt{\frac{4\pi}{\omega_I+\pi}-3}}{\sqrt{3}\omega_I}$$

$$\omega_{I,C}=\frac{(\sqrt{33}-9)\pi}{5\sqrt{33}-21}.$$

$$F(\omega_I,\tau,\alpha,\lambda_1,\lambda_2)=F(\omega_I,\tau)+\frac{2\alpha}{27(\pi-3\omega_I)^{3/2}\omega_I^2(\pi^2-3\omega_I^2)}\left(f_{\lambda_3}+f_{\lambda_4}\right)$$

$$f_{\lambda_3}=6\lambda_1\big(-18\tau\sqrt{\pi-3\omega_I}\,\omega_I^7+18\omega_I^6\big(2\sqrt{3}\sqrt{\omega_I+\pi}-\pi\tau\sqrt{\pi-3\omega_I}\big)$$
$$+3\pi\omega_I^5\big(2\pi\tau\sqrt{\pi-3\omega_I}-31\sqrt{3}\sqrt{\omega_I+\pi}\big)+3\pi^2\omega_I^4\big(20\pi\tau\sqrt{\pi-3\omega_I}+23\sqrt{3}\sqrt{\omega_I+\pi}\big)$$
$$+2\pi^3\omega_I^3\big(9\pi\tau\sqrt{\pi-3\omega_I}-25\sqrt{3}\sqrt{\omega_I+\pi}\big)+30\pi^4\omega_I^2\big(\sqrt{3}\sqrt{\omega_I+\pi}-\pi\tau\sqrt{\pi-3\omega_I}\big)$$
$$+3\pi^5\omega_I\big(2\pi\tau\sqrt{\pi-3\omega_I}-3\sqrt{3}\sqrt{\omega_I+\pi}\big)+\sqrt{3}\pi^6\sqrt{\omega_I+\pi}\big)$$
$$f_{\lambda_4}=\lambda_2(\omega_I+\pi)(\pi^2-3\omega_I^2)\big(24\tau\sqrt{\pi-3\omega_I}\,\omega_I^4+\pi\omega_I^2\big(37\sqrt{3}\sqrt{\omega_I+\pi}-18\pi\tau\sqrt{\pi-3\omega_I}\big)$$
$$+2\pi^2\omega_I\big(3\pi\tau\sqrt{\pi-3\omega_I}-5\sqrt{3}\sqrt{\omega_I+\pi}\big)-48\sqrt{3}\sqrt{\omega_I+\pi}\,\omega_I^3+\sqrt{3}\pi^3\sqrt{\omega_I+\pi}\big)$$

$$\mathcal{S}=\frac{\pi^2[r_+^4+a^4+a^2(q+2r_+^2)]}{2Gr_+(1-a^2)^2}(1+4\lambda_2\alpha)+\lambda_1\alpha\Delta\mathcal{S}$$

$$\Delta\mathcal{S}=F_S\{a^4(a^2+q)^6(a^2+2q)+a^4(a^2+q)^4(2a^6-5a^2q+3a^4q-3q^2)r_+^2$$
$$+a^2(a^2+q)^3[-7a^8+a^{10}-38a^2q^2-4q^3-2a^4q(37+8q)-2a^6(14+13q)]r_+^4$$
$$-a^2(a^2+q)^2[14a^{10}+13q^3+a^2q^2(104+5q)+3a^8(32+7q)+16a^6(7+13q)+a^4q(211+108q)]r_+^6$$
$$-a^2(a^2+q)[108a^{10}+7a^{12}+q^3(23+2q)+5a^2q^2(31+4q)+4a^8(77+38q)+6a^4q(55+53q)$$
$$+a^6(210+593q+34q^2)]r_+^8-a^2[40a^{12}+2(11-4q)q^3+2a^{10}(140+9q)+a^2q^2(155+48q)$$
$$+a^8(476+451q)+a^4q(347-2(-260+q)q)+a^6(224+q(920+153q))]r_+^{10}-a^2[84a^{10}$$
$$+a^8(322-15q)+(14-23q)q^2+14a^6(27+20q)+a^4(140+(409-55q)q)$$
$$+a^2q(110+(65-17q)q)]r_+^{12}+[-56a^{10}+3q^3+6a^4(-2+3q)(4+7q)+4a^8(-21+25q)$$
$$+a^6(-112+73q)+a^2q(-13+q(28+3q))]r_+^{14}+[70a^8+3q^2+44a^4(1+4q)+a^6(196+135q)$$
$$+a^2(-7+45q+48q^2)]r_+^{16}+[168a^6+3q(3+q)+a^2(42+75q)+a^4(232+78q)]r_+^{18}$$
$$+[140a^4+9(1+q)+a^2(105+17q)]r_+^{20}+2(9+28a^2)r_+^{22}+9r_+^{24}\}$$

$$F_{\mathcal{S}}=-\frac{2\pi^2}{Gr_+^3(1-a^2)^2(a^2+r_+^2)^2\mathcal{D}}$$

$$
\begin{aligned}
\mathcal{D} =& a^2(a^2+q)^3(a^2+2q)+a^2(a^2+q)^2(5a^2+a^4+6q)r_+^2 \\
&+(a^2+q)[3a^6+8a^2q-q^2+a^4(10+q)]r_+^4-[-10a^4-2a^6+2a^8+a^2(-8+q)q+q^2]r_+^6 \\
&-[2a^4+8a^6+(-1+q)q+a^2(-5+4q)]r_+^8-(-1+3a^2+12a^4+2q)r_+^{10} \\
&-(1+8a^2)r_+^{12}-2r_+^{14}
\end{aligned}
$$

$$
Q=\frac{\sqrt{3}\pi q}{4G(1-a^2)^2}(1+4\lambda_2\alpha)+\lambda_1\alpha\Delta Q
$$

$$
\begin{aligned}
\Delta Q=& F_Q\{a^4q^8+2a^2(a^2-2r_+^2)(1+r_+^2)^3(a^2+r_+^2)^7(a^2+r_+^2-2r_+^4)+a^2q^7(9a^4+2a^2r_+^2+r_+^4) \\
&+q^4(a^2+r_+^2)^2[3r_+^{10}+5a^8(21+14r_+^2+r_+^4)-a^4r_+^4(87+193r_+^2+3r_+^4) \\
&+a^2r_+^6(-22-3r_+^2+10r_+^4)-a^6r_+^2(40+213r_+^2+24r_+^4)]+q^6[-3r_+^8+a^6r_+^2(19-22r_+^2) \\
&-a^2r_+^6(9+2r_+^2)+a^8(35+4r_+^2)-6a^4(r_+^4+2r_+^6)]+q^2(a^2+r_+^2)^4[a^8(1+r_+^2)(49+31r_+^2+2r_+^4) \\
&+3r_+^8(1+3r_+^2+8r_+^4)-a^4r_+^4(70+273r_+^2+151r_+^4+2r_+^6)-a^6r_+^2(29+125r_+^2+84r_+^4+6r_+^6) \\
&+a^2r_+^6(11-27r_+^2+54r_+^4+38r_+^6)]+q^5(a^2+r_+^2)[-107a^6r_+^4+a^2r_+^6(-32+3r_+^2) \\
&-21a^4r_+^4(2+3r_+^2)+a^8(77+26r_+^2)-3(r_+^8+r_+^{10})] \\
&+q(1+r_+^2)(a^2+r_+^2)^6[a^6(1+r_+^2)(15+4r_+^2)+3a^2r_+^4(-3-9r_+^2+2r_+^4) \\
&-a^4r_+^2(21+49r_+^2+22r_+^4)+3(r_+^6+3r_+^8+8r_+^{10})]+q^3(a^2+r_+^2)^3[a^8(91+100r_+^2+21r_+^4) \\
&+a^2r_+^6(13-15r_+^2+44r_+^4)-a^4r_+^4(99+323r_+^2+68r_+^4)-a^6r_+^2(53+221r_+^2+72r_+^4) \\
&+3(r_+^{10}+r_+^{12})]\}.
\end{aligned}
$$

$$
F_Q=-\frac{\pi}{\sqrt{3}G(1-a^2)^2r_+^4(a^2+r_+^2)^3\mathcal{D}}.
$$

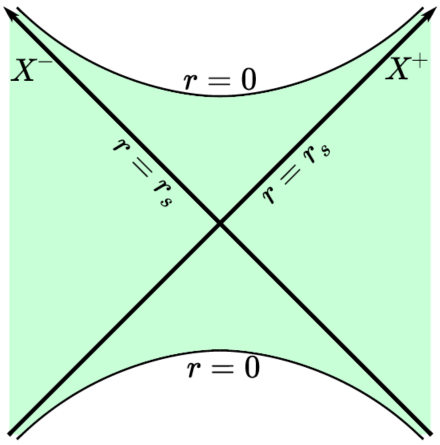
$$
J=\frac{a\pi(1+4\lambda_2\alpha)}{4(1-a^2)^3Gr_+^2}[(a^2+q)^2+(a^4+q+a^2(2+q))r_+^2+(1+2a^2)r_+^4+r_+^6]+\lambda_1\alpha\Delta J,
$$

$$
\begin{aligned}
\Delta J=& F_J\{-2a^2q^8+q^6[-49a^6+a^4(33+7a^2)r_+^2+51a^2(1+a^2)r_+^4+(9+7a^2)r_+^6+3r_+^8] \\
&+q^7[4r_+^4+a^4(-15+r_+^2)+a^2r_+^2(5+r_+^2)]-(1+r_+^2)^2(a^2+r_+^2)^7[a^4(1-3r_+^2) \\
&-2a^2r_+^2(4+19r_+^2+7r_+^4)+r_+^4(7+13r_+^2+18r_+^4)]+q^5[-2r_+^8(-5+r_+^2)+a^4r_+^4(233+281r_+^2-4r_+^4) \\
&+a^2r_+^6(93+47r_+^2-2r_+^4)+a^8(-91+21r_+^2+8r_+^4)+a^6r_+^2(91+317r_+^2+38r_+^4)] \\
&+q^4[r_+^{10}(-1+r_+^2)+a^2r_+^8(71+103r_+^2-66r_+^4)+2a^4r_+^6(181+361r_+^2-14r_+^4) \\
&+a^{10}(-105+35r_+^2+34r_+^4)+5a^8r_+^2(27+169r_+^2+60r_+^4)+2a^6r_+^4(265+715r_+^2+152r_+^4)] \\
&+q(1+r_+^2)(a^2+r_+^2)^5[a^6(-9+16r_+^2+9r_+^4)-2r_+^6(3+17r_+^2+26r_+^4) \\
&+2a^4r_+^2(29+140r_+^2+90r_+^4+7r_+^6)-a^2r_+^4(19-22r_+^2+57r_+^4+34r_+^6)] \\
&+q^2(a^2+r_+^2)^3[8a^4r_+^4(13+82r_+^2+58r_+^4-2r_+^6)+a^8(-35+21r_+^2+53r_+^4+9r_+^6) \\
&-r_+^8(1+25r_+^2+57r_+^4+21r_+^6)-4a^2r_+^6(7+4r_+^2+46r_+^4+37r_+^6)+4a^6r_+^2(40+215r_+^2+209r_+^4+46r_+^6)] \\
&+q^3(a^2+r_+^2)^2[-2r_+^8(4+3r_+^2+23r_+^4)+a^4r_+^4(209+791r_+^2+231r_+^4-39r_+^6) \\
&+5a^2r_+^6(3+17r_+^2-31r_+^4-5r_+^6)+a^8(-77+35r_+^2+59r_+^4+3r_+^6) \\
&+a^6r_+^2(269+1119r_+^2+687r_+^4+53r_+^6)]\}
\end{aligned}
$$

$$
F_J=\frac{a\pi}{2G(1-a^2)^3(a^2+r_+^2)^2r_+^4\mathcal{D}}.
$$

$$
ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega_2^2
$$

$$
f(r)\equiv 1-\frac{2GM}{r}.
$$

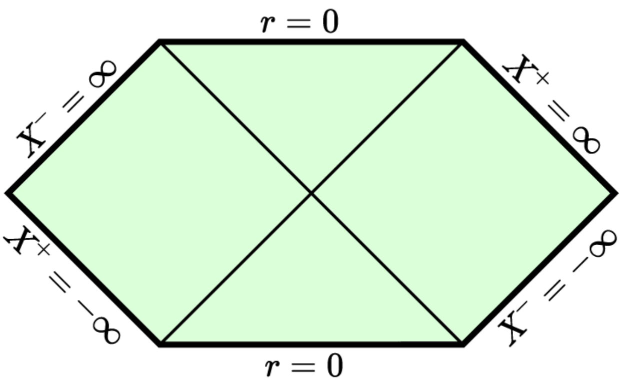


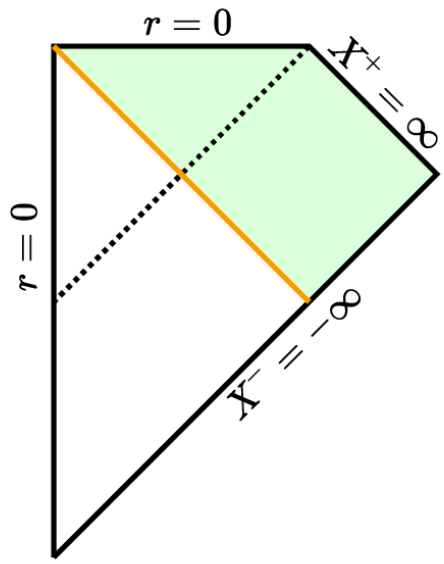
$$X^{\pm} \equiv \pm r_s e^{\frac{r_* \pm t}{2r_s}},$$

$$r_* \equiv r + r_s \log\left(\frac{r - r_s}{r_s}\right)$$

$$ds^2 = -\frac{2r_s}{r} e^{-r/r_s}(dX^+ dX^- + dX^- dX^+) + r^2 d\Omega_2^2.$$

$$X^+ X^- = (r_s - r) r_s e^{r/r_s}.$$





$$\mathcal{L} = \frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu},$$

$$g_{\mu\nu} = g_{\mu\nu}^{cl} + \sqrt{16\pi G}h_{\mu\nu},$$

$$\frac{1}{16\pi G}\int d^4\sqrt{-g}R = \frac{1}{16\pi G}\int d^4x\sqrt{-g^{cl}}R^{cl} + \int d^4x\sqrt{-g^{cl}}(\partial h\partial h + Rh^2 + \sqrt{16\pi G}h\partial h\partial h + 16\pi G h^2\partial h\partial h + \cdots)$$

$$\ell_p \equiv \sqrt{\frac{8\pi G\hbar}{c^3}} \approx 8\times 10^{-35}\ \text{nm}$$

$$H = \frac{1}{2}\int d^3x\left(\dot{\phi}^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi\right)$$

$$K_x = \int d^3x(xT_{00} + tT_{0x})$$

$$(t\pm x)' = (t\pm x)e^{\pm\lambda}$$

$$K_x^R = \int_0^\infty dx\int dydz(xT_{00} + tT_{0x})$$

$$K_x^L = -\int_{-\infty}^0 dx\int dydz(xT_{00} + tT_{0x})$$

$$|\Omega\rangle \propto \sum_i e^{-\pi\omega_i}|i^*\rangle_L|i\rangle_R$$

$$\rho_R \propto e^{-2\pi K_x^R}$$

$$T_{Unruh} = \frac{a}{2\pi} = \frac{\hbar a}{2\pi k_B c},$$

$$X^{\pm\prime} = X^{\pm} e^{\pm\frac{b}{2r_s}}$$

$$H = \frac{K}{2r_s}.$$

$$T_{\text{Hawking}} = \frac{1}{4\pi r_s} = \frac{1}{8\pi GM} = \frac{\hbar c^3}{8\pi k_B GM},$$

$$\frac{dS}{dE} = \frac{1}{T}$$

$$S_{BH} = \frac{4\pi r_s^2}{4G} = \frac{\text{Area}}{4G}$$

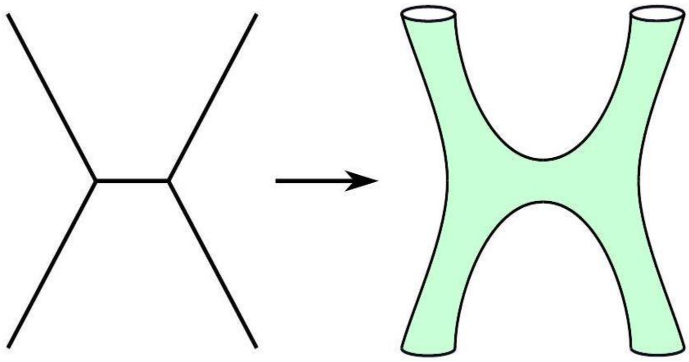
$$\frac{dM}{dt} \sim -(4\pi r_s^2) T_{\text{Hawking}}^4 \sim -\frac{1}{G^2 M^2}$$

$$M(t) = \left(M_0^3 - \frac{Ct}{G^2}\right)^{1/3},$$

$$t_{\text{evap}} \sim G^2 M_0^3$$

$$S_{cg} \sim \frac{1}{GM} \times G^2 M^3 \sim GM^2 \sim \frac{(GM)^2}{G} \sim \frac{\text{Area}}{G},$$

$$S = -\frac{1}{2\pi \ell_s^2} \int d^2x \sqrt{-\gamma} \left(1 + \frac{\Phi_0 \ell_s^2}{2} R\right)$$



$$g \equiv e^{\Phi_0}$$

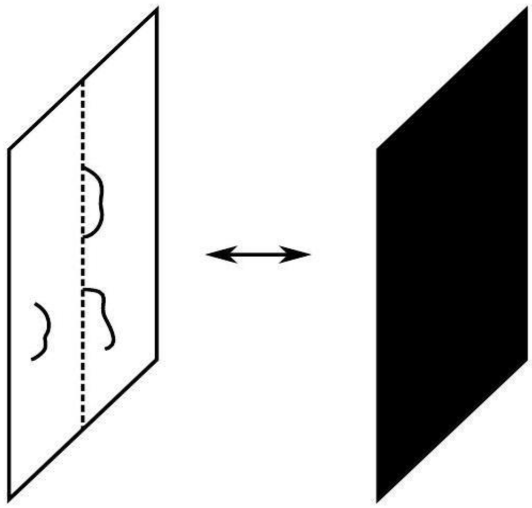
$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{24} H_{\mu\nu\sigma} H^{\mu\nu\sigma}\right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{48} \tilde{F}_{\mu\nu\rho\sigma} \tilde{F}^{\mu\nu\rho\sigma}\right] - \frac{1}{4\kappa_{10}^2} \int B \wedge \tilde{F} \wedge \tilde{F} + \cdots$$

$$2\kappa_{10}^2 = (2\pi)^7 \ell_s^8$$

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-g}\left(R - \frac{1}{48}T_{\alpha\beta\rho\sigma}T^{\alpha\beta\rho\sigma}\right) - \frac{1}{12\kappa_{11}^2}\int M \wedge T \wedge T + \cdots$$

$$r_B = \frac{\ell_s^2}{r_A},$$

$$S = q_1 \int A$$



$$ds^2 = \frac{1}{\sqrt{Z_p(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{Z_p(r)}\left(dr^2 + r^2 d\Omega_{8-p}^2\right)$$
$$e^{2\Phi} = g^2 Z_p(r)^{\frac{3-p}{2}}$$
$$A_{p+1} = g^{-1}\left(\frac{1}{Z_p(r)} - 1\right)dx^0 \wedge \ldots \wedge dx^p$$

$$Z_p(r) \equiv 1 + \frac{(4\pi)^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right) gN\ell_s^{7-p}}{r^{7-p}}.$$

$$p_5 = \frac{2\pi m_5}{L}$$

$$S = \frac{\text{Area}}{4G} = \frac{2\pi\ \text{Area}}{g^2\kappa_{10}^2} = 2\pi\sqrt{N_1 N_5 m_5}.$$

$$\rho \approx e^{2\pi\sqrt{N_1 N_5 m_5}}$$

$$ds^2 = \frac{1}{\sqrt{1+\left(\frac{\ell_{ads}}{r}\right)^4}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{1+\left(\frac{\ell_{ads}}{r}\right)^4}\left(dr^2 + r^2 d\Omega_5^2\right)$$
$$e^{2\Phi} = g^2$$
$$C_4 = -g^{-1}\frac{\ell_{ads}^4}{\ell_{ads}^4 + r^4}dx_0 \wedge \ldots \wedge dx^4$$

$$\ell_{ads}^4 \equiv 4\pi g N \ell_s^4$$

$$ds^2 \approx \left(\frac{r}{\ell_{ads}}\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{\ell_{ads}}{r}\right)^2 dr^2 + \ell_{ads}^2 d\Omega_5^2$$

$$g_{YM}^2 = 4\pi g.$$

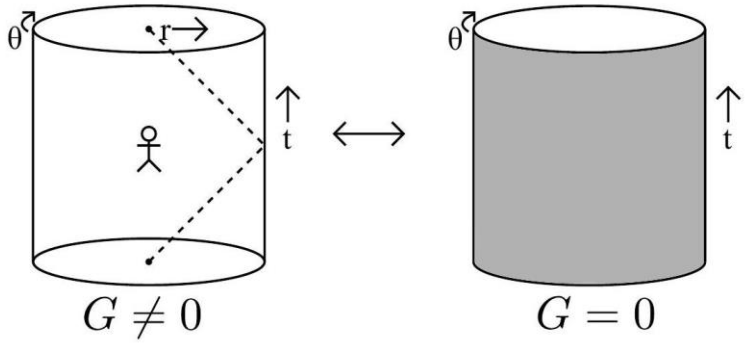
$$\mathcal{L} = -\frac{1}{2g_{YM}^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \sum_i \mathrm{Tr}\left(D_\mu X_i D^\mu X_i\right) + \frac{g_{YM}^2}{2}\sum_{i,j}\mathrm{Tr}\left(\left[X_i, X_j\right]^2\right) + \cdots$$

$$\ell_{10}^8 \equiv g^2\kappa_{10}^2 = \frac{g^2}{2}(2\pi)^7\ell_s^8$$

$$\frac{\ell_{ads}}{\ell_{10}} \sim N^{1/4}.$$

$$\frac{\ell_{ads}}{\ell_s} \sim (g_{YM}^2 N)^{1/4}$$

$$N \gg 1$$
$$g_{YM}^2 N \gg 1.$$



$$ds^2 = -\left(1+\frac{r^2}{\ell_{ads}^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{\ell_{ads}^2}} + r^2 d\Omega_{d-2}^2$$

$$T_1^2 + T_2^2 - \vec{X}^2 = \ell_{ads}^2$$

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$x^{\mu\prime} = \lambda x^\mu$$

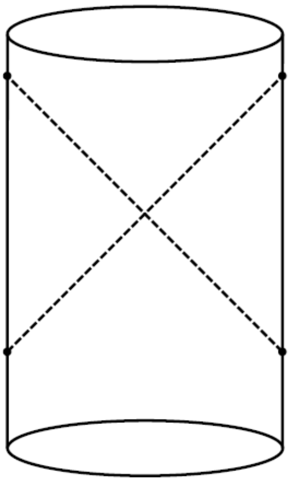
$$x^{\mu\prime} = \frac{x^{\mu} + b^{\mu} x^{\nu} x_{\nu}}{1 + 2b_{\alpha} x^{\alpha} + b_{\beta} b^{\beta} x_{\gamma} x^{\gamma}}:$$

$$e^{iD\alpha} O(x) e^{-iD\alpha} = e^{\alpha\Delta} O(e^{\alpha} x)$$
$$e^{iK_{\mu} a^{\mu}} O(0) e^{-iK_{\mu} a^{\mu}} = O(0),$$

$$\int \mathcal{D}\phi_i e^{iS(\phi_i,\Lambda)} O_{i_1}^{bulk}(t_1,\Omega_1) \ldots O_{i_n}^{bulk}(t_n,\Omega_n) = \langle O_{i_1}(t_1,\Omega_1) \ldots O_{i_n}(t_n,\Omega_n) \rangle_{CFT}$$

$$O_i^{\text{bulk}}(t,\Omega) \equiv \lim_{r\to\infty} r^{\Delta_i} \phi_i(r,t,\Omega)$$

$$g_{\mu\nu} \leftrightarrow T_{\mu\nu}$$



$$= \langle \mathcal{O}(X_1) \mathcal{O}(X_2) \mathcal{O}(X_3) \mathcal{O}(X_4) \rangle_{\text{CFT}}$$

$$A_{\mu} \leftrightarrow J_{\mu}$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2$$

$$f(r) = \frac{r^2}{\ell_{ads}^2} + 1 - \frac{16\pi GM}{(d-2)\Omega_{d-2} r^{d-3}}.$$

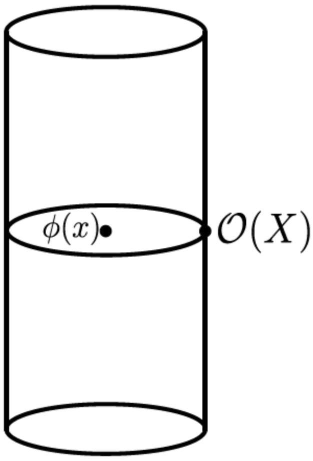
$$T = \frac{(d-3) + (d-1) r_s^2 / \ell_{ads}^2}{4\pi r_s}.$$

$$S \propto E^{\frac{d-2}{d-1}}$$

$$|TFD\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |i^*\rangle_L |i\rangle_R$$

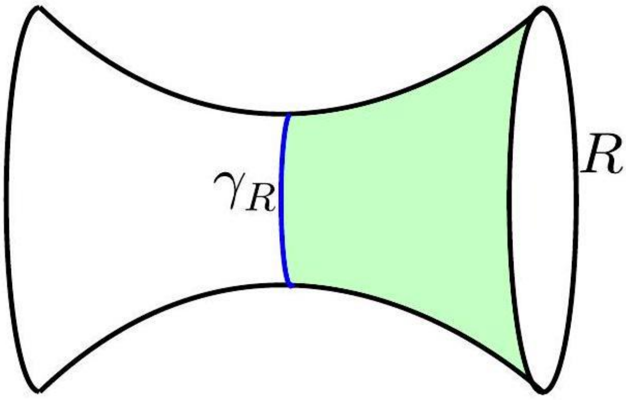
$$C(t) \equiv \frac{1}{Z} \text{Tr}\left(e^{-\beta H} O(t) O(0)\right)$$

$$\lim_{T\to\infty} \int_0^T dt |C(t)|^2 \approx e^{-2S}$$



$$[\phi(x), O(X)] = 0$$

$$S(\rho_R) = \min_{\gamma} \frac{\text{Area}(\gamma)}{4G},$$



$$S(\rho_R) = \min_{\gamma} \left(\text{ext}_{\gamma} \left(\frac{\text{Area}(\gamma) + \cdots}{4G} + S_{bulk}(\rho_H) \right) \right)$$

$$\rho \approx e^{\sqrt{\frac{\pi c_L L E}{3}}}$$

$$M \geq |\mathcal{Z}_1|, \ldots, M \geq |\mathcal{Z}_{[\mathcal{N}/2]}|$$

$$\delta_\epsilon F = 0$$

$$\sum_B \mathbf{E}_B \frac{\delta \delta_\epsilon B}{\delta F} = 0$$

$$M = |\mathcal{Z}(\phi_\infty, q, p)|$$

$$S = S(q, p),$$

$$M = |\mathcal{Z}_{\text{fake}}(\phi_\infty, p, q)|,$$

$$S = \int d^4x\sqrt{|g|}\{R + \mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu} \star F^{\Sigma}{}_{\mu\nu}\}$$

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left[\frac{r_0^4}{\sinh^4(r_0\rho)}d\rho^2 + \frac{r_0^2}{\sinh^2(r_0\rho)}d\Omega_{(2)}^2\right]$$

$$r_0 = 2ST$$

$$S_{\text{eff}}[U,\phi^i] = \int d\rho\left[\dot{U}^2 + \frac{1}{2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2U}V_{\text{bh}}(\phi)\right]$$

$$V_{\text{bh}}(\phi) = -\frac{1}{2}\mathcal{Q}^M\mathcal{M}_{MN}\mathcal{Q}^N, (\mathcal{M}_{MN}) \equiv \begin{pmatrix} I_{\Lambda\Sigma} + R_{\Lambda\Gamma}I^{\Gamma\Omega}R_{\Omega\Sigma} & -R_{\Lambda\Gamma}I^{\Gamma\Sigma} \\ -I^{\Lambda\Omega}R_{\Omega\Sigma} & I^{\Lambda\Sigma} \end{pmatrix}$$

$$\dot{U}^2 + \frac{1}{2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2U}V_{\text{bh}}(\phi) = r_0^2$$

$$\partial_i V_{\text{bh}}|_{\phi=\phi_{\text{h}}} = 0,$$

$$S/\pi = -V_{\text{bh}}(\phi_{\text{h}},\mathcal{Q})$$

$$M^2_{\text{doubleextremal}} = -V_{\text{bh}}(\phi_{\text{h}},\mathcal{Q}) = S/\pi$$

$$M^2 + \frac{1}{2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\text{bh}}(\phi_\infty,\mathcal{Q}) = r_0^2 \geq 0$$

$$\Sigma^i = \Sigma^i(\phi_\infty,\mathcal{Q},M)$$

$$\mathcal{Z}(\mathcal{Q},Z,Z^*) \equiv \mathcal{V}_M\mathcal{Q}^M$$

$$-V_{\text{bh}}(Z,Z^*,\mathcal{Q}) = |\mathcal{Z}|^2 + 4\mathcal{G}^{ij^*}\partial_i|\mathcal{Z}|\partial_{j^*}|\mathcal{Z}|$$

$$(M = |\mathcal{Z}(\phi_\infty,\mathcal{Q})| \equiv |\mathcal{Z}_\infty|)$$

$$S/\pi = |\mathcal{Z}_{\text{h}}|^2\ \partial_i\ |\ \mathcal{Z}||_{\text{h}} = 0$$

$$\partial_i\partial_{j^*}V_{\text{bh}}\Big|_{Z_{\text{h susy}}} = 2\mathcal{G}_{ij^*}V_{\text{bh}}\Big|_{Z_{\text{h susy}}}, \partial_i\partial_{j^*}|\mathcal{Z}|\Big|_{Z_{\text{h susy}}} = \frac{1}{2}\mathcal{G}_{ij^*}|\mathcal{Z}|\Big|_{Z_{\text{h susy}}},$$

$$S[U,Z^i] = \int d\rho[(\dot{U} \pm e^U|\mathcal{Z}|)^2 + \mathcal{G}_{ij^*}(\dot{Z}^i \pm 2e^U\partial^i|\mathcal{Z}|)(\dot{Z}^{*j^*} \pm 2e^U\partial^{j^*}|\mathcal{Z}|)]$$

$$\dot{\mathcal{U}} = e^U|\mathcal{Z}|, \dot{Z}^i = 2e^U\partial^i|\mathcal{Z}|.$$

$$\frac{\partial S}{\partial M} = \frac{1}{T},$$

$$S_{BH} = \frac{k_B}{G\hbar}\frac{1}{4}\text{Area}_H$$

$$ds^2 = -dt^2\left(1-\frac{2m}{\rho}+\frac{q}{\rho^2}\right)+d\rho^2\left(1-\frac{2m}{\rho}+\frac{q}{\rho^2}\right)^{-1}+\rho^2 d\Omega^2$$

$$d\Omega^2 = (d\theta^2 + \sin^2\ \theta d\phi^2)$$

$$\Delta \equiv -2m\rho + q^2 + \rho^2$$

$$\rho_\pm = m \pm \sqrt{m^2 - q^2}$$

$$m \geq |q|$$

$$m = |q|\ ;\ \rho = r + m\ ;\ r^2 = \vec{x}\cdot\vec{x}$$

$$\begin{aligned} ds^2 &= -dt^2\left(1+\frac{q}{r}\right)^{-2}+\left(1+\frac{q}{r}\right)^2(dr^2+r^2d\Omega^2) \\ &= -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}\cdot d\vec{x} \end{aligned}$$

$$H(\vec{x}) = \left(1+\frac{q}{\sqrt{\vec{x}\cdot\vec{x}}}\right)$$

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}d\vec{x}^2$$

$$U(r) = \sum_{i=1}^{p} \alpha_i \log\ H_i(\vec{x})$$

$$M_{\text{Black Hole}} \sim M_{\text{Planck}}$$

$$(\mathbb{P}_{BPS}Q_{SUSY})\ |\ \text{BPS state}\ >= 0$$

$$\{\bar{Q}_{A\alpha}, \bar{Q}_{B\beta}\} = \mathrm{i}(\mathbf{C}\gamma^\mu)_{\alpha\beta}P_\mu\delta_{AB} - \mathbf{C}_{\alpha\beta}\mathbf{Z}_{AB}$$
$$(A, B = 1, \dots, 2p)$$

$$\bar{Q}_A \equiv Q_A^\dagger\gamma_0 = Q_A^T\mathbf{C}$$

$$\mathbf{Z}_{AB} = \begin{pmatrix} \epsilon Z_1 & 0 & \dots & 0 \\ 0 & \epsilon Z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \epsilon Z_p \end{pmatrix}$$

$$A = (a, I)\ ;\ a, b, \dots = 1,2\ ;\ I, J, \dots = 1, \dots, p$$

$$\{\bar{Q}_{aI|\alpha}, \bar{Q}_{bJ|\beta}\} = \mathrm{i}(C\gamma^\mu)_{\alpha\beta}P_\mu\delta_{ab}\delta_{IJ} - C_{\alpha\beta}\epsilon_{ab}\times\mathbb{Z}_{IJ}$$

$$\bar{Q}_{aI} \equiv Q_{aI}^\dagger\gamma_0 = Q_{aI}^T C$$

$$M \geq |Z_I|\ \forall Z_I, I = 1, \dots, p$$

$$\bar{S}_{aI|\alpha}^\pm = \frac{1}{2}(\bar{Q}_{aI}\gamma_0 \pm \mathrm{i}\epsilon_{ab}\bar{Q}_{bI})_\alpha$$

$$\bar{S}^{\pm}_{aI} = \bar{Q}_{bI}\mathbb{P}^{\pm}_{ba}$$
$$\mathbb{P}^{\pm}_{ba} = \frac{1}{2}(\mathbf{1}\delta_{ba} \pm \mathrm{i}\epsilon_{ba}\gamma_0)$$

$$\{\bar{S}^{\pm}_{aI}, \bar{S}^{\pm}_{bJ}\} = \pm\epsilon_{ac}C\mathbb{P}^{\pm}_{cb}(M \mp Z_I)\delta_{IJ}$$

$$(M \pm Z_I) \mid \text{BPS state}, i\rangle = 0$$

$$\bar{S}^{\pm}_{aI} \mid \text{BPS state}, i\rangle = 0$$

$$\begin{aligned} \xi^{\mu}\gamma_{\mu}\epsilon_{aI} &= \mathrm{i}\varepsilon_{ab}\epsilon^{bI} && ; I = 1, \dots, n_{\max} \\ \epsilon_{aI} &= 0 && ; I > n_{\max} \end{aligned}$$

$$\text{Area}_H = \int_{\rho=\rho_+} \sqrt{g_{\theta\theta}g_{\phi\phi}}\,d\theta d\phi = 4\pi\rho_+^2 = 4\pi\left(m + \sqrt{m^2 - |q|^2}\right)^2$$

$$\frac{\text{Area}_H}{4\pi} = |q|^2$$

$$S_{BH} = \mathcal{S}(q, p)$$

$$A_{[p+1]} \equiv A_{M_1 \dots M_{p+1}} dX^{M_1} \wedge \dots \wedge dX^{M}_{p+1}$$

$$I_D = \int d^Dx\sqrt{-g}\left[R - \frac{1}{2}\nabla_M\phi\nabla^M\phi - \frac{1}{2n!}e^{a\phi}F^2_{[p+2]}\right]$$

$$F_{[p+2]} \equiv dA_{[p+1]}$$

$$n = p + 2 \;\text{ and }\; a$$

$$\Delta = a^2 + 2\frac{d\tilde{d}}{D-2}$$

$$ds^2 = \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} dx^{\mu} \otimes dx^{\nu}\eta_{\mu\nu} - \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{4d}{\Delta(D-2)}} dy^m \otimes dy^n\delta_{mn}$$
$$F = \lambda(-)^{p+1}\epsilon_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} \wedge \frac{y^m dy^m}{r}\left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-2} \frac{1}{r^{\tilde{d}+1}}$$
$$e^{\phi(r)} = \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-\frac{2a}{\Delta}}$$

$$\lambda = 2\frac{\tilde{d}k}{\sqrt{\Delta}}$$

$$ds^2 = \left(1 + \frac{k}{r^{d}}\right)^{-\frac{4d}{\Delta(D-2)}} dx^{\mu} \otimes dx^{\nu}\eta_{\mu\nu} - \left(1 + \frac{k}{r^{d}}\right)^{\frac{4\tilde{d}}{\Delta(D-2)}} dy^m \otimes dy^n\delta_{mn}$$
$$\tilde{F}_{[D-n]} = \lambda\epsilon_{\mu_1 \dots \mu_{\tilde{d}} p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{\tilde{d}}} \wedge \frac{y^p}{r^{d+2}}$$
$$e^{\phi(r)} = \left(1 + \frac{k}{r^{d}}\right)^{\frac{2a}{\Delta}}$$

$$\lambda = -2\frac{\tilde{d}k}{\sqrt{\Delta}}$$

$$D = 10\ d = 2\ \tilde{d} = 6\ a = 1\ \Delta = 4\ \lambda = \pm 6k$$

$$ds^2 = \exp\left[2U(r)\right]dx^\mu \otimes dx^\nu - \exp\left[-\frac{2}{3}U(r)\right]dy^m \otimes dy^m$$

$$\exp\left[2U(r)\right] = \left(1 + \frac{k}{r^6}\right)^{-3/4}$$

$$F = 6k\epsilon_{\mu\nu}dx^\mu \wedge dx^\nu \wedge \frac{y^m dy^m}{r}\left(1 + \frac{k}{r^6}\right)^{-2}\frac{1}{r^7}$$

$$\exp\left[\phi(r)\right] = \left(1 + \frac{k}{r^6}\right)^{-1/2}$$

$$\delta\psi_\mu = \nabla_\mu \epsilon + \frac{1}{96}\exp\left[\frac{1}{2}\phi\right](\Gamma_{\lambda\rho\sigma\mu} + 9\Gamma_{\lambda\rho}g_{\sigma\mu})F^{\lambda\rho\sigma}\epsilon$$

$$\delta\chi = \mathrm{i}\frac{\sqrt{2}}{4}\partial^\mu\phi\Gamma_\mu\epsilon - \mathrm{i}\frac{\sqrt{2}}{24}\exp\left[-\frac{1}{2}\phi\right]\Gamma_{\mu\nu\rho}\epsilon F^{\mu\nu\rho}$$

$$\epsilon = \left(1 + \frac{k}{r^6}\right)^{-3/16}\epsilon_0 \otimes \eta_0$$

$$\gamma_3\epsilon_0 = \epsilon_0\ \Sigma_{10}\eta_0 = \eta_0$$

$$ds_{11}^2 = \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-\frac{\tilde{d}}{9}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{d}{9}} dX^I dX^J \delta_{IJ}$$

$$\mathcal{I}_{2-\text{brane}} = ISO(1,2) \otimes SO(8)$$

$$ds^2 = \rho^2(-dt^2 + d\vec{z}\cdot d\vec{z}) + \rho^{-2}d\rho^2 + d\Omega_7^2$$

$$dy^I dy^J \delta_{IJ} = dr^2 + r^2 d\Omega_7^2$$

$$M_2^{hor} = AdS_4 \times S^7$$

$$M_p^{hor} = AdS_{p+2} \times S^{D-p-2}$$

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}\left[dr^2 + r^2\lambda^{-2}ds_{G/H}^2\right]$$
$$A_{\mu_1\dots\mu_d} = \epsilon_{\mu_1\dots\mu_d}e^{C(r)}$$
$$\phi = \phi(r)$$

$$M_p^{hor} = AdS_{p+2} \times \left(\frac{G}{H}\right)_{D-p-2}$$

$$SO(2, p+1) \times G$$

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN}$$
$$\nabla_{M_1}\left(e^{a\phi}F^{M_1\ldots M_n}\right) = 0$$
$$\Box\ \phi = \frac{a}{2n!}F^2$$

$$S_{MN} = \frac{1}{2(n-1)!}e^{a\phi}\left[F_{M\ldots}F_N\cdots - \frac{n-1}{n(D-2)}F^2 g_{MN}\right]$$

$$E^{\underline{\mu}} = e^A dx^\mu;\ \ E^{\underline{\bullet}} = e^B dr;\ \ \ E^{\underline{m}} = e^B r\lambda^{-1}\mathbf{E}^{\underline{m}};$$
$$g_{\mu\nu} = e^{2A}\eta_{\mu\nu};\ \ \ g_{\bullet\bullet} = e^{2B},\ \ \ g_{mn} = e^{2B}r^2\lambda^{-2}\mathbf{g}_{mn}$$

$$dE^{\underline{M}} + \omega^{\underline{M}} \wedge E^{\underline{N}} = 0$$

$$\omega^{\underline{\mu\nu}} = 0,\ \ \omega^{\underline{\mu\bullet}} = e^{-B}A'E^{\underline{\mu}},\ \ \omega^{\underline{\mu n}} = 0,\ \ \omega^{\underline{mn}} = \boldsymbol{\omega}^{\underline{mn}},\ \ \omega^{\underline{m\bullet}} = \exp[-B]\,(B' + r^{-1})E^{\underline{m}}$$

$$R\underline{\underline{MN}} = d\omega\underline{\underline{MN}} + \omega\underline{\underline{M}} \wedge \omega\underline{\underline{SN}}$$

$$R_{\mu\nu} = -\frac{1}{2}\eta_{\mu\nu}e^{2(A-B)}[A'' + d(A')^2 + \tilde{d}A'B' + (\tilde{d}+1)r^{-1}A']$$
$$R_{\bullet\bullet} = -\frac{1}{2}[d(A'' + (A')^2 - A'B') + (\tilde{d}+1)(B'' + r^{-1}B')]$$
$$R_{mn} = -\frac{1}{2}\mathbf{g}_{mn}\frac{r^2}{\lambda^2}[dA'(B' + r^{-1}) + r^{-1}B' + B'' + \tilde{d}(B' + r^{-1})^2] + \mathbf{R}_{mn}$$

$$A'' + d(A')^2 + \tilde{d}A'B' + (\tilde{d}+1)A'r^{-1} = \frac{\tilde{d}}{2(D-2)}S^2$$
$$d[A'' + (A')^2 - A'B'] + (\tilde{d}+1)[B'' + r^{-1}B'] = \frac{\tilde{d}}{2(D-2)}S^2 - \frac{(\phi')^2}{2}$$
$$\mathbf{g}_{mn}\left[dA'(B' + r^{-1}) + r^{-1}B' + B'' + \tilde{d}(B' + r^{-1})^2\right] - 2\mathbf{R}_{mn} =$$
$$-\frac{d}{2(D-2)}\mathbf{g}_{mn}S^2$$

$$C'' + (\tilde{d}+1)r^{-1}C' + \left(\tilde{d}B' - dA' + C' + a\phi'\right)C' = 0$$
$$\phi'' + (\tilde{d}+1)r^{-1}\phi' + \left[dA' + \tilde{d}B'\right]\phi' = -\frac{a}{2}S^2$$

$$S \equiv e^{\frac{1}{2}a\phi + C - dA}C'$$

$$I_{11} = \int\ d^{11}x\sqrt{-g}\left(R - \frac{1}{48}F_{[4]}^2\right) + \frac{1}{6}\int\ F_{[4]} \wedge F_{[4]} \wedge A_{[3]}$$

$$\delta\psi_M = \widetilde{D}_M\epsilon$$

$$\widetilde{D}_M = \partial_M + \frac{1}{4}\omega_M^{AB}\Gamma_{AB} - \frac{1}{288}[\Gamma_M^{PQRS} + 8\Gamma^{PQR}\delta_M^S]F_{PQRS}$$

$$\delta\psi_{M|\psi=0} = \widetilde{D}_M\epsilon = 0$$

$$\Gamma_A = [\gamma_\mu \otimes \mathbb{1}_8, \gamma_3 \otimes \mathbb{1}_8, \gamma_5 \otimes \Gamma_m]$$

$$\epsilon = \varepsilon \otimes \eta(r, y)$$

$$\widetilde{D}_\mu = \partial_\mu + \frac{1}{2}e^{-B-2A}\gamma_\mu\gamma_3[e^{3A}A' - \frac{i}{3}e^C C'\gamma_3\gamma_5] \otimes \mathbb{1}_8$$
$$\widetilde{D}_\bullet = \partial_r + \frac{i}{6}e^{-3A}C'e^C\gamma_3\gamma_5 \otimes \mathbb{1}_8$$
$$\widetilde{D}_m = \mathcal{D}_m^{G/H} + \frac{r}{2\lambda}[(B' + r^{-1})i\gamma_3\gamma_5 + \frac{1}{6}e^{C-3A}C'] \otimes \Gamma_m$$

$$(1_4 - i\gamma_3\gamma_5)\varepsilon = 0;\ 3e^{3A}A' = e^C C'$$

$$\partial_r\eta + \frac{1}{6}C'\eta = 0$$

$$\eta(r, y) = e^{-C(r)/6}\eta_\circ(y)$$

$$B = -\frac{1}{6}C + \mathfrak{G}$$
$$\left[\mathcal{D}_m^{G/H} + \frac{1}{2\lambda}\Gamma_m\right]\eta_\circ = 0$$

$$e \equiv \frac{1}{2\lambda}$$

$$\mathcal{M}_{11} = AdS_4 \times \left(\frac{G}{H}\right)_7$$

$$F_{\mu_1\mu_2\mu_3\mu_4} = e\epsilon_{\mu_1\mu_2\mu_3\mu_4}$$

$$A'' + 7r^{-1}A' = \frac{1}{3}S^2$$
$$(A')^2 = \frac{1}{6}S^2$$
$$\mathbf{R}_{mn} = \frac{3}{\lambda^2}\mathbf{g}_{mn}$$

$$\nabla^2 A - 3(A')^2 \equiv A'' + \frac{7}{r}A' - 3(A')^2 = 0$$

$$\nabla^2 e^{-3A} = 0$$

$$e^{-3A(r)} = H(r) = 1 + \frac{k}{r^6}$$

$$A(r) = -\frac{\tilde{d}}{18}\ln\left(1 + \frac{k}{r^{\tilde{d}}}\right) = -\frac{1}{3}\ln\left(1 + \frac{k}{r^6}\right)$$
$$B(r) = \frac{d}{18}\ln\left(1 + \frac{k}{r^{\tilde{d}}}\right) = \frac{1}{6}\ln\left(1 + \frac{k}{r^6}\right)$$
$$C(r) = 3A(r)$$

$$ds^2 = e^{2U(r)}dt^2 - e^{-2U(r)}d\vec{x}^2\ ;\ (r^2 = \vec{x}^2)$$

$$F^\Lambda = \frac{p^\Lambda}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \frac{\ell^\Lambda(r)}{r^3}e^{2U}dt \wedge \vec{x}\cdot d\vec{x}$$

$$\phi^I = \phi^I(r)\ ;\ (I = 1, \dots m)$$

$$f_{SUSY} = \frac{N}{2n_{\max}}\ ;\ (f_{SUSY}) = \frac{1}{2}\ \text{or}\ \frac{1}{4}\ \text{or}\ \frac{1}{8}$$

$$\gamma^0\xi_A = \mathrm{i}\mathbb{C}_{AB}\xi^B;\ A,B = 1,\dots,n_{\max};\ 2 \le n_{\max} \le N$$
$$\xi_A = 0\ ;\ A = n_{\max} + 1, \dots, N$$
$$\mathbb{C}_{AB} = -\mathbb{C}_{AB} = n_{\max}$$

$$d = 1; \tilde{d} = 1$$
$$p = 0; \Delta = a^2 + 1$$

$$ds^2 = \left(1 + \frac{k}{r}\right)^{-\frac{2}{a^2+1}} dt^2 - \left(1 + \frac{k}{r}\right)^{\frac{2}{a^2+1}} d\vec{x}^2$$

$$U(r) = -\frac{1}{a^2 + 1}\log\ H(r)$$

$$H(r) \equiv \left(1 + \frac{k}{r}\right)$$
$$\Delta_3 H(r) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} H(r) = 0$$

$$\phi(r) = \begin{cases} -\frac{2a}{a^2+1}\ \log\ H(r) \\ \frac{2a}{a^2+1}\ \log\ H(r) \end{cases}$$

$$F = -2\frac{k}{\sqrt{a^2+1}}\frac{1}{r^3}\left(1 + \frac{k}{r}\right)^{-2} dt \wedge \vec{x}\cdot\vec{x}$$
$$F = -2\frac{k}{\sqrt{a^2+1}}\frac{1}{r^3}\epsilon_{abc}\, x^a\, dx^b \wedge dx^c$$

$$\begin{aligned}\ell(r) &= r^3 \frac{2\sqrt{a^2+1}}{a^2-1} \frac{d}{dr} [H(r)]^{\frac{1-a^2}{1+a^2}} \\ p &= -\frac{4}{\sqrt{a^2+1}} k\end{aligned}$$

$$a = \begin{cases} \sqrt{3} \implies \Delta = 4 \;;\; U = -\frac{1}{4}\log H(r) \;;\; \phi = \mp\frac{\sqrt{3}}{2}\log H \;;\; \begin{cases}\ell = 2r^3 \frac{d}{dr} H^{-\frac{1}{2}} \\ p = -k\end{cases} \\ 1 \implies \Delta = 2 \;;\; U = -\frac{1}{2}\log H(r) \;;\; \phi = \mp \log H \;;\; \begin{cases}\ell = -2r^3 \frac{d}{dr} H^{-1} \\ p = -2\sqrt{2}\,k\end{cases} \\ 0 \implies \Delta = 1 \;;\; U = -\log H(r) \;;\; \phi = 0 \;;\; \begin{cases}\ell = -2r^3 \frac{d}{dr} H \\ p = -4\,k\end{cases}\end{cases}$$

$$\Delta = 4 \times 2 \times f_{SUSY}; \begin{cases} f_{SUSY} = \frac{1}{2} \leftrightarrow a = \sqrt{3} \\ f_{SUSY} = \frac{1}{4} \leftrightarrow a = 1 \\ f_{SUSY} = \frac{1}{8} \leftrightarrow a = 0 \end{cases}$$

$$ds^2_{RN} = \left(1 + \frac{k}{r}\right)^{-2} dt^2 - \left(1 + \frac{k}{r}\right)^2 d\vec{x}^2$$

$$AdS_2 \times S^2$$

$$ds^2_{BR} = \frac{1}{m^2_{BR}} r^2 dt^2 - m^2_{BR} \frac{dr^2}{r^2} - m^2_{BR}(\sin^2(\theta) d\phi^2 + d\theta^2)$$

$$m_{BR} = |k|$$

$$\mathbb{P}^{\pm}_{AB} = \frac{1}{2}(\mathbb{1}\delta^B_A \pm \mathrm{i}\mathbb{C}_{AB}\gamma_0)$$

$$\mathcal{L} = \int \sqrt{-g} d^4x \left(2R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma|\mu\nu} + \frac{1}{6} g_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J + \right.$$
$$\left. + \frac{1}{2} \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma}\right)$$

$$\mathcal{N}_{\Lambda\Sigma}(\phi)$$

$$A^\Lambda \equiv A^\Lambda_\mu dx^\mu \; (\Lambda = 1, \ldots, \bar{n})$$

$$F^\Lambda \equiv dA^\Lambda \equiv \mathcal{F}^\Lambda_{\mu\nu} dx^\mu \wedge dx^\nu$$
$$\mathcal{F}^\Lambda_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu)$$
$${}^\star F^\Lambda \equiv \tilde{\mathcal{F}}^\Lambda_{\mu\nu} dx^\mu \wedge dx^\nu$$
$$\tilde{\mathcal{F}}^\Lambda_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\Lambda|\rho\sigma}$$

$$\mathrm{d}^4x \equiv -\frac{1}{4!}\varepsilon_{\mu_1\ldots\mu_4} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_4}$$

$$F^\Lambda \wedge F^\Sigma = \varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^\Sigma_{\rho\sigma} d^4x \; ; \; F^\Lambda \wedge^\star F^\Sigma = -2\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\Sigma|\mu\nu} d^4x$$

$$\mathcal{S} = \frac{1}{2}\int \left\{\left[\gamma_{\Lambda\Sigma}(\phi)F^\Lambda \wedge\star F^\Sigma + \theta_{\Lambda\Sigma}(\phi)F^\Lambda \wedge F^\Sigma\right] + g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J \, \mathrm{d}^4x\right\}$$

$$\left(j\mathcal{F}^\Lambda\right)_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda|\rho\sigma}$$

$$(G, K) \equiv G^T K \equiv \sum_{\Lambda=1}^{\bar{n}} G^\Lambda_{\mu\nu} K^{\Lambda|\mu\nu}$$

$$\mathcal{L}^{(tot)} = \mathcal{F}^T(-\gamma \otimes \mathbb{1} + \theta \otimes j)\mathcal{F} + \frac{1}{2} g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J$$

$$\mathcal{F}^\pm = \frac{1}{2}(\mathcal{F} \pm \mathrm{i}j\mathcal{F})$$
$$j\mathcal{F}^\pm = \mp\mathrm{i}\mathcal{F}^\pm$$

$$\mathcal{N} = \theta - \mathrm{i}\gamma$$
$$\overline{\mathcal{N}} = \theta + \mathrm{i}\gamma,$$

$$\mathcal{L}_{vec} = \mathrm{i}\left[\mathcal{F}^{-T}\overline{\mathcal{N}}\mathcal{F}^- - \mathcal{F}^{+T}\mathcal{N}\mathcal{F}^+\right]$$

$$\tilde{\mathcal{G}}^\Lambda_{\mu\nu} \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^\Lambda_{\mu\nu}} \leftrightarrow \mathcal{G}^{\mp\Lambda}_{\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{\mp\Lambda}_{\mu\nu}}$$

$$j\mathcal{G} \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^T} = -(\gamma \otimes \mathbb{1} - \theta \otimes j)\mathcal{F}$$

$$\partial^\mu\tilde{\mathcal{F}}^\Lambda_{\mu\nu} = 0$$
$$\partial^\mu\tilde{\mathcal{G}}^\Lambda_{\mu\nu} = 0$$

$$\partial^\mu \mathrm{Im}\mathcal{F}^{\pm\Lambda}_{\mu\nu} = 0$$
$$\partial^\mu \mathrm{Im}\mathcal{G}^{\pm\Lambda}_{\mu\nu} = 0$$

$$\mathbf{V} \equiv \begin{pmatrix} j\mathcal{F} \\ j\mathcal{G} \end{pmatrix}$$

$$\begin{pmatrix} j\mathcal{F} \\ j\mathcal{G} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} j\mathcal{F} \\ j\mathcal{G} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2\bar{n}, \mathbb{R})$$

$$\partial\mathbf{V} = 0 \iff \partial\mathbf{V}' = 0$$

$$\mathcal{F} = (\mathcal{F}^+ + \mathcal{F}^-) \; ; \; \mathcal{G} = (\mathcal{G}^+ + \mathcal{G}^-)$$

$$\mathcal{G}^{+} = \mathcal{N}\mathcal{F}^{+}\ \mathcal{G}^{-} = \overline{\mathcal{N}}\mathcal{F}^{-}$$

$$\begin{pmatrix}\mathcal{F}^{+}\\ \mathcal{G}^{+}\end{pmatrix}' = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathcal{F}^{+}\\ \mathcal{N}\mathcal{F}^{+}\end{pmatrix};\ \begin{pmatrix}\mathcal{F}^{-}\\ \mathcal{G}^{-}\end{pmatrix}' = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathcal{F}^{-}\\ \overline{\mathcal{N}}\mathcal{F}^{-}\end{pmatrix}$$

$$\Lambda = \begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

$$\Lambda \equiv \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \mathrm{Sp}(2\bar{n},\mathbb{R}) \subset GL(2\bar{n},\mathbb{R})$$

$$\Lambda \in Sp(2\bar{n},\mathbb{R}) \longrightarrow \Lambda^{T}\begin{pmatrix}\mathbf{0} & \mathbb{1}\\ -\mathbb{1} & \mathbf{0}\end{pmatrix}\Lambda = \begin{pmatrix}\mathbf{0} & \mathbb{1}\\ -\mathbb{1} & \mathbf{0}\end{pmatrix}$$

$$A^{T}C - C^{T}A = B^{T}D - D^{T}B = 0\ A^{T}D - C^{T}B = 1$$

$$\iota_{\delta}: \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}) \longrightarrow GL(2\bar{n},\mathbb{R})$$

$$\forall\ \xi\ \in \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}): \phi^{I} \overset{\xi}{\to} \phi^{I'}$$
$$\exists\ \iota_{\delta}(\xi)\ = \begin{pmatrix}A_{\xi} & B_{\xi}\\ C_{\xi} & D_{\xi}\end{pmatrix} \in GL(2\bar{n},\mathbb{R})$$

$$\xi: \begin{cases}\phi \longrightarrow \xi(\phi)\\ \mathbf{V} \longrightarrow \iota_{\delta}(\xi)\mathbf{V}\\ \mathcal{N}(\phi) \longrightarrow \mathcal{N}'(\xi(\phi))\end{cases}$$

$$\mathcal{L}_{\mathrm{vec}}' = \mathrm{i}\left[\mathcal{F}^{-T}(A+B\overline{\mathcal{N}})^{T}\overline{\mathcal{N}}'(A+B\overline{\mathcal{N}})\mathcal{F}^{-} - \mathcal{F}^{+T}(A+B\mathcal{N})^{T}\mathcal{N}'(A+B\mathcal{N})\mathcal{F}^{+}\right]$$

$$\mathcal{N}' \equiv \mathcal{N}'(\xi(\phi)) = (C + D\mathcal{N}(\phi))(A + B\mathcal{N}(\phi))^{-1}$$

$$\overline{\mathcal{N}}' \equiv \overline{\mathcal{N}}'(\xi(\phi)) = (C + D\overline{\mathcal{N}}(\phi))(A + B\overline{\mathcal{N}}(\phi))^{-1}$$

$$\iota_{\delta}: \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}) \longrightarrow \mathrm{Sp}(2\bar{n},\mathbb{R})$$

$$\mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}) \supset \mathrm{Tor}(\mathcal{M}_{\mathrm{scalar}}) \equiv \ker\iota_{\delta}$$

$$\dim\mathrm{Tor}(\mathcal{M}_{\mathrm{scalar}}) = \infty$$

$$\begin{aligned}\mathrm{Im}\mathcal{F}^{-\Lambda}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} \rightarrow & \mathrm{Im}\tilde{\mathcal{F}}^{-\Lambda}\tilde{\mathcal{G}}_{\Sigma}^{-}\\ & = \mathrm{Im}\big(\mathcal{F}^{-\Lambda}\mathcal{G}_{\Lambda}^{-} + 2\mathcal{F}^{-\Lambda}(C^{T}B)_{\Lambda}{}^{\Sigma}\mathcal{G}_{\Sigma}^{-}\\ & + \mathcal{F}^{-\Lambda}(C^{T}A)_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} + \mathcal{G}_{\Lambda}^{-}(D^{T}B)^{\Lambda\Sigma}\mathcal{G}_{\Sigma}^{-}\big)\end{aligned}$$

$$\xi^{\star}: T\mathcal{M}_{\mathrm{scalar}} \rightarrow T\mathcal{M}_{\mathrm{scalar}}$$

$$\forall X, Y \in T\mathcal{M}_{\mathrm{scalar}}$$
$$g(X,Y) = g(\xi^{\star}X, \xi^{\star}Y)$$

$$\mathcal{N}(\xi(\phi)) = \big(C_{\xi} + D_{\xi}\mathcal{N}(\phi)\big)\big(A_{\xi} + B_{\xi}\mathcal{N}(\phi)\big)^{-1}$$

$$\iota_{\delta}: \mathrm{Diff}\left(\frac{\mathcal{G}}{\mathcal{H}}\right) \longrightarrow \mathrm{Sp}(2\bar{n},\mathbb{R})$$

$$\frac{\mathcal{G}_1\mathbf{2}_1}{\mathcal{H}(\mathbf{u}^1\cdot}\mathcal{J}\left(\frac{\mathcal{G}}{\mathcal{H}}\right)=\mathcal{G}$$

$$\iota_\delta:\mathcal{G}\longrightarrow \mathrm{Sp}(2\bar{n},\mathbb{R})$$

$$\Lambda^T\mathbb{C}\Lambda=\mathbb{C}$$

$$Sp(2\bar{n},\mathbb{R})\sim Usp(\bar{n},\bar{n})\equiv Sp(2\bar{n},\mathbb{C})\cap U(\bar{n},\bar{n})$$

$$\mathcal{S}^T\mathbb{C}\mathcal{S}=\mathbb{C}\,;\;\mathcal{S}^\dagger\mathbb{H}\mathcal{S}=\mathbb{H}$$

$$\mathbb{H}\equiv\begin{pmatrix}\mathbb{1} & \mathbf{0}\\ \mathbf{0} & -\mathbb{1}\end{pmatrix}$$

$$\mathcal{S}=\begin{pmatrix}T & V^\star\\ V & T^\star\end{pmatrix}$$

$$T^\dagger T-V^\dagger V=\mathbb{1}\,;\;T^\dagger V^\star-V^\dagger T^\star=\mathbf{0}$$

$$\mathcal{C}\equiv\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{1} & \mathrm{i}\mathbb{1}\\ \mathbb{1} & -\mathrm{i}\mathbb{1}\end{pmatrix}$$

$$\mathcal{S}=\mathcal{C}\Lambda\mathcal{C}^{-1}$$

$$T=\frac{1}{2}(A+D)-\frac{\mathrm{i}}{2}(B-C)\,;\;V=\frac{1}{2}(A-D)-\frac{\mathrm{i}}{2}(B+C)$$

$$\mathcal{G}\overset{\iota_\delta}{\to}Usp(\bar{n},\bar{n})\,;\;\mathcal{G}\supset\mathcal{H}\overset{\iota_\delta}{\to}U(\bar{n})\subset Usp(\bar{n},\bar{n})$$

$$L(\phi)\longrightarrow\mathcal{O}(\phi)=\begin{pmatrix}U_0(\phi) & U_1^\star(\phi)\\ U_1(\phi) & U_0^\star(\phi)\end{pmatrix}\in Usp(\bar{n},\bar{n})$$

$$\mathcal{O}(\phi')=\mathcal{O}(\phi)\begin{pmatrix}W & \mathbf{0}\\ \mathbf{0} & W^\star\end{pmatrix}$$

$$\mathcal{S}_\xi=\begin{pmatrix}T_\xi & V_\xi^\star\\ V_\xi & T_\xi^\star\end{pmatrix}$$

$$\mathcal{S}_\xi\mathcal{O}(\phi)=\mathcal{O}(\xi(\phi))\begin{pmatrix}W(\xi,\phi) & \mathbf{0}\\ \mathbf{0} & W^\star(\xi,\phi)\end{pmatrix}$$

$$U_0^\dagger(\xi(\phi))+U_1^\dagger(\xi(\phi))=W\left[U_0^\dagger(\phi)(A^T+\mathrm{i}B^T)+U_1^\dagger(\phi)(A^T-\mathrm{i}B^T)\right]$$
$$U_0^\dagger(\xi(\phi))-U_1^\dagger(\xi(\phi))=W\left[U_0^\dagger(\phi)(D^T-\mathrm{i}C^T)-U_1^\dagger(\phi)(D^T+\mathrm{i}C^T)\right]$$

$$\mathcal{N}\equiv\mathrm{i}\left[U_0^\dagger+U_1^\dagger\right]^{-1}\left[U_0^\dagger-U_1^\dagger\right]$$

$$\mathcal{N}^T=\mathcal{N}$$

$$\mathcal{ST}[m,n]\equiv\frac{SU(1,1)}{U(1)}\otimes\frac{SO(m,n)}{SO(m)\otimes SO(n)}$$

$$S = \mathcal{A} - \mathrm{i}\exp[D]$$
$$\partial^{\mu}\mathcal{A} \equiv \varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}B_{\rho\sigma}$$

$$\frac{SU(1,1)}{U(1)} \iiint \frac{SO(m,n)}{SO(m) \otimes SO(n)}$$

$$\mathbf{2\bar{n}} \xrightarrow{\mathcal{G}} \oplus_{i=1}^{\ell} \mathbf{D}_i$$

$$\forall \mathcal{S} \in \mathrm{Sp}(2\bar{n}, \mathbb{R}): \iota'_{\mathcal{S}} \equiv \mathcal{S} \circ \iota_{\mathcal{S}} \circ \mathcal{S}^{-1}$$

$$L \in SO(m,n) \Leftrightarrow L^T \eta L = \eta$$

$$\#\text{vector fields} = m \oplus n$$

$$\mathbf{2m+2n} \overset{SO(m,n)}{\rightarrow} \mathbf{m+n} \oplus \mathbf{m+n}$$

$$\forall L \in SO(m,n) \overset{\iota_{\delta}}{\hookrightarrow} \begin{pmatrix} L & \mathbf{0} \\ \mathbf{0} & (L^T)^{-1} \end{pmatrix} \in Sp(2m+2n, \mathbb{R})$$

$$\mathbf{2m+2n} \overset{SL(2,\mathbb{R})}{\rightarrow} \oplus_{i=1}^{m+n} \mathbf{2}$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \overset{\iota_{\delta}}{\hookrightarrow} \begin{pmatrix} a\,\mathbb{1} & b\,\eta \\ c\,\eta & d\,\mathbb{1} \end{pmatrix} \in Sp(2m+2n, \mathbb{R})$$

$$\forall L \in SO(m,n) \overset{\iota_{\delta}}{\hookrightarrow} \begin{pmatrix} \frac{1}{2}(L+\eta L\eta) & \frac{1}{2}(L-\eta L\eta) \\ \frac{1}{2}(L-\eta L\eta) & \frac{1}{2}(L+\eta L\eta) \end{pmatrix} \in Usp(m+n, m+n)$$
$$\forall \begin{pmatrix} t & v^{\star} \\ v & t^{\star} \end{pmatrix} \in SU(1,1) \overset{\iota_{\delta}}{\hookrightarrow} \begin{pmatrix} \mathrm{Re}t\mathbb{1} + \mathrm{i}\mathrm{Im}t\eta & \mathrm{Re}v\mathbb{1} - \mathrm{i}\mathrm{Im}v\eta \\ \mathrm{Re}v\mathbb{1} + \mathrm{i}\mathrm{Im}v\eta & \mathrm{Re}t\mathbb{1} - \mathrm{i}\mathrm{Im}t\eta \end{pmatrix} \in Usp(m+n, m+n)$$

$$M(S) \equiv \frac{1}{n(S)} \begin{pmatrix} \mathbb{1} & \frac{\mathrm{i}-S}{\mathrm{i}+S} \\ \frac{\mathrm{i}+\bar{S}}{\mathrm{i}-\bar{S}} & \mathbb{1} \end{pmatrix} : \; n(S) \equiv \sqrt{\frac{4\mathrm{Im}S}{1+|S|^2+2\mathrm{Im}S}}$$

$$L(X) \equiv \begin{pmatrix} (\mathbb{1}+XX^T)^{1/2} & X \\ X^T & (\mathbb{1}+X^TX)^{1/2} \end{pmatrix}$$

$$\iota_{\delta}(M(S)) \circ \iota_{\delta}(L(X)) = \begin{pmatrix} U_0(S,X) & U_1^{\star}(S,X) \\ U_1(S,X) & U_0^{\star}(S,X) \end{pmatrix} \in Usp(n+m, n+m)$$

$$\mathcal{N} = \mathrm{i}\mathrm{Im}S\eta L(X)L^T(X)\eta + \mathrm{Re}S\eta$$

$$\mathcal{N} = \mathrm{iIm}SL(X)'L^{T'}(X) + \mathrm{Re}S\eta$$

$$\mathcal{M}_{\text{scalar}} = \mathcal{SM} \otimes \mathcal{HM}$$
$$\dim_{\mathbf{C}}\mathcal{SM} = n_v$$
$$\dim_{\mathbf{R}}\mathcal{HM} = 4n_h$$

$$\Lambda, \Sigma, \Gamma, \ldots = 0,1, \ldots, n$$
$$I, J, K = 1, \ldots, 2n$$

$$c_1(\mathcal{L}) = \frac{i}{2\pi}\bar{\partial}(h^{-1}\partial h) = \frac{i}{2\pi}\bar{\partial}\partial\log\, h$$

$$c_1(\mathcal{L}) = \frac{i}{2\pi}\bar{\partial}\partial\log\, \|\xi(z)\|^2$$

$$\|\xi(z)\|^2 = h(z,\bar{z})\bar{\xi}(\bar{z})\xi(z)$$

$$c_1(\mathcal{L}) = [K]$$

$$K = \frac{i}{2\pi}g_{ij^\star}dz^i \wedge d\bar{z}^{j^\star} = \frac{i}{2\pi}\bar{\partial}\partial\log\, \|W(z)\|^2$$

$$g_{ij^\star} = \partial_i\partial_{j^\star}\mathcal{K}(z,\bar{z})$$

$$h(z,\bar{z}) = \exp\,(\mathcal{K}(z,\bar{z}))$$

$$\log\, \|W(z)\|^2 = \mathcal{K}(z,\bar{z}) + \log\, |W(z)|^2 = G(z,\bar{z})$$

$$\theta \equiv h^{-1}\partial h = \frac{1}{h}\partial_i h dz^i\,;\ \bar{\theta} \equiv h^{-1}\bar{\partial}h = \frac{1}{h}\partial_{i^\star}hd\bar{z}^{i^\star}$$

$$[\bar{\partial}\theta] = c_1(\mathcal{L}) = [K]$$

$$\theta = \partial\mathcal{K} = \partial_i\mathcal{K}dz^i; \bar{\theta} = \bar{\partial}\mathcal{K} = \partial_{i^\star}\mathcal{K}d\bar{z}^{i^\star}$$

$$\theta = \bar{\theta} = 0$$

$$\begin{aligned} \#\,\text{vector fields} \equiv \bar{n} &= n \\ \#\,\text{vector multiplets} \equiv n &= \dim_{\mathbf{C}}\mathcal{M} \\ \text{rank}\,\mathcal{SV} \equiv 2\bar{n} &= 2n \end{aligned}$$

$$\begin{aligned} \#\,\text{vector fields} \equiv \bar{n} &= n+1 \\ \#\,\text{vector multiplets} \equiv n &= \dim_{\mathbf{C}}\mathcal{M} \\ \text{rank}\,\mathcal{SV} \equiv 2\bar{n} &= 2n+2 \end{aligned}$$

$$\mathcal{K}_\beta = \mathcal{K}_\alpha + f_{\alpha\beta} + \bar{f}_{\alpha\beta}$$

$$U(1) :: \mathcal{Q} \bowtie \mathrm{Im}\theta = -\frac{\mathrm{i}}{2}(\theta - \bar{\theta})$$

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\partial_i \mathcal{K} dz^i - \partial_{i^*} \mathcal{K} d\bar{z}^{i^*}\right)$$

$$\nabla\Phi = (d + ip\mathcal{Q})\Phi$$

$$\nabla_i \Phi = \left(\partial_i + \frac{1}{2} p \partial_i \mathcal{K}\right)\Phi \; ; \; \nabla_{i^*}\Phi = \left(\partial_{i^*} - \frac{1}{2} p \partial_{i^*} \mathcal{K}\right)\Phi$$

$$\widetilde{\Phi} = e^{-p\mathcal{K}/2}\Phi$$

$$\nabla_i \widetilde{\Phi} = (\partial_i + p\partial_i \mathcal{K})\widetilde{\Phi}; \; \nabla_{i^*}\widetilde{\Phi} = \partial_{i^*}\widetilde{\Phi}$$

$$\partial_{m^*} W_{ijk} = 0 \; \partial_m W_{i^* j^* k^*} = 0$$
$$\nabla_{[m} W_{i]jk} = 0 \; \nabla_{[m} W_{i^*]j^* k^*} = 0$$
$$\mathcal{R}_{i^* j \ell^* k} = g_{\ell^* j} g_{k i^*} + g_{\ell^* k} g_{j i^*} - e^{2\mathcal{K}} W_{i^* \ell^* s^*} W_{tkj} g^{s^* t}$$

$$C_{ijk} = W_{ijk} e^{\mathcal{K}} \; ; \; C_{i^* j^* k^*} = W_{i^* j^* k^*} e^{\mathcal{K}}$$

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \; \Lambda, \Sigma = 0,1, \dots, n$$

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j$$

$$e^{f_{ij} + f_{jk} + f_{ki}} = 1$$
$$M_{ij} M_{jk} M_{ki} = 1$$

$$i\langle \Omega \mid \bar{\Omega} \rangle \equiv -i\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \bar{\Omega}$$

$$K = \frac{i}{2\pi} \partial \bar{\partial} \log\left(\mathrm{i}\langle \Omega \mid \bar{\Omega} \rangle\right)$$

$$\mathcal{K} = -\log\left(\mathrm{i}\langle \Omega \mid \bar{\Omega} \rangle\right) = -\log\left[\mathrm{i}\left(\bar{X}^\Lambda F_\Lambda - \bar{F}_\Sigma X^\Sigma\right)\right]$$

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathcal{K}/2}\Omega = e^{\mathcal{K}/2} \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}$$

$$1 = \mathrm{i}\langle V \mid \bar{V} \rangle = \mathrm{i}\left(\bar{L}^\Lambda M_\Lambda - \bar{M}_\Sigma L^\Sigma\right)$$

$$\nabla_{i^*} V = \left(\partial_{i^*} - \frac{1}{2} \partial_{i^*} \mathcal{K}\right) V = 0$$

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathcal{K}\right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix}$$

$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^*} \bar{U}_{\ell^*}$$

$$\overline{\mathcal{T}\mathcal{M}}^3 \otimes \mathcal{L}^2$$

$$\begin{aligned}
\nabla_i V &= U_i \\
\nabla_i U_j &= \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star} \\
\nabla_{i^\star} U_j &= g_{i^\star} V \\
\nabla_{i^\star} V &= 0
\end{aligned}$$

$$\bar{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \bar{L}^\Sigma \;;\; h_{\Sigma|i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma$$

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \bar{L}^\Lambda \end{pmatrix} \;;\; h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \bar{M}_\Lambda \end{pmatrix}$$

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ (f^{-1})_\Sigma^I$$

$$\begin{aligned}
\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} L^\Lambda \bar{L}^\Sigma &= -\frac{1}{2} \\
\langle V, U_i \rangle &= \langle V, U_{i^\star} \rangle = 0 \\
U^{\Lambda\Sigma} \equiv f_i^\Lambda f_{j^\star}^\Sigma g^{ij^\star} &= -\frac{1}{2} (\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma \\
g_{ij^\star} &= -\mathrm{i} \langle U_i \mid \bar{U}_{j^\star} \rangle = -2 f_i^\Lambda \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f_{j^\star}^\Sigma; \\
C_{ijk} &= \langle \nabla_i U_j \mid U_k \rangle = f_i^\Lambda \partial_j \overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma = (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f_i^\Lambda \partial_j f_k^\Sigma
\end{aligned}$$

n	G/H	$Sp(2n+2)$	symp rep of G
1	$\frac{SU(1,1)}{U(1)}$	$Sp(4)$	$\underline{\mathbf{4}}$
n	$\frac{SU(1,n)}{SU(n)\times U(1)}$	$Sp(2n+2)$	$\underline{\mathbf{n+1}} \oplus \underline{\mathbf{n+1}}$
$n+1$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\times SO(n)}$	$Sp(2n+4)$	$\underline{\mathbf{2}} \otimes (\underline{\mathbf{n+2}} \oplus \underline{\mathbf{n+2}})$
6	$\frac{Sp(6,\mathbb{R})}{SU(3)\times U(1)}$	$Sp(14)$	$\underline{\mathbf{14}}$
9	$\frac{SU(3,3)}{S(U(3)\times U(3))}$	$Sp(20)$	$\underline{\mathbf{20}}$
15	$\frac{SO^\star(12)}{SU(6)\times U(1)}$	$Sp(32)$	$\underline{\mathbf{32}}$
27	$\frac{E_{7(-6)}}{E_6\times SO(2)}$	$Sp(56)$	$\underline{\mathbf{56}}$

$$\mathcal{K} \to \mathcal{K} + f(z) + \bar{f}(\bar{z})$$

$$X^\Lambda \to X^\Lambda e^{-f}$$

$$e_i^I(z) = \partial_i \left(\frac{X^I}{X^0}\right) \; ; \; a = 1, \dots, n$$

$$0 = \langle V \mid U_i \rangle = X^\Lambda \partial_i F_\Lambda - \partial_i X^\Lambda F_\Lambda$$

$$X^\Sigma \partial_\Sigma F_\Lambda(x) = F_\Lambda(X)$$

$$F_\Lambda(X) = \frac{\partial}{\partial X^\Lambda} F(X)$$

$$t^I \equiv \frac{X^I}{X^0}$$

$$\mathcal{F}(t) \equiv (X^0)^{-2} F(X)$$

$$\mathcal{K}(t,\bar{t}) = -\log \mathrm{i}\left[2(\mathcal{F} - \overline{\mathcal{F}}) - \left(\partial_I \mathcal{F} + \partial_{I^\star} \overline{\mathcal{F}}\right)\left(t^I - \bar{t}^{I^\star}\right)\right]$$

$$W_{IJK} = \partial_I \partial_J \partial_K \mathcal{F}(t)$$

$$e_i^I(z) = \partial_i \left(\frac{X^I}{X^0}\right) = A + B\overline{\mathcal{N}} \frac{\partial^2 F}{\partial X^I \partial X^J}$$

$$\Phi^\Lambda(X) \equiv \frac{1}{\sqrt{2}}\left(L_0^\Lambda + \mathrm{i} L_1^\Lambda\right);\ (\Lambda = 0,1, a\ a = 2, \dots, n+1)$$

$$\eta_{\Lambda\Sigma} = \mathrm{diag}(+,+,-,\dots,-)$$

$$L^\Lambda{}_0 L^\Sigma{}_0 \eta_{\Lambda\Sigma} = 1\,;\ L^\Lambda{}_0 L^\Sigma{}_1 \eta_{\Lambda\Sigma} = 0\,;\ L^\Lambda{}_1 L^\Sigma{}_1 \eta_{\Lambda\Sigma} = 1$$

$$\bar{\Phi}^\Lambda \Phi^\Sigma \eta_{\Lambda\Sigma} = 1\,;\ \Phi^\Sigma \eta_{\Lambda\Sigma} = 0$$

$$\Phi^\Lambda = \frac{X^\Lambda}{\sqrt{\bar{X}^\Lambda X^\Sigma \eta_{\Lambda\Sigma}}}$$

$$X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} = 0$$

$$X^\Lambda(y) \equiv \begin{pmatrix} 1/2(1+y^2) \\ \mathrm{i}/2(1-y^2) \\ y^a \end{pmatrix}$$

$$\Omega(y,S) = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = \begin{pmatrix} X^\Lambda(y) \\ S\eta_{\Lambda\Sigma} X^\Sigma(y) \end{pmatrix}$$

$$\mathcal{K} = \mathcal{K}_1(S) + \mathcal{K}_2(y) = -\log\left[\mathrm{i}(\bar{S} - S)\right] - \log X^T \eta X$$

$$g_{ij^\star} = \begin{pmatrix} g_{S\bar{S}} & \mathbf{0} \\ \mathbf{0} & g_{a\bar{b}} \end{pmatrix} \begin{cases} g_{S\bar{S}} = \partial_S \partial_{\bar{S}} \mathcal{K}_1 = \dfrac{-1}{(\bar{S} - S)^2} \\ g_{a\bar{b}}(y) = \partial_a \partial_{\bar{b}} \mathcal{K}_2 \end{cases}$$

$$C_{Sab} = -\exp\left[\mathcal{K}\right]\delta_{ab},$$

$$\mathcal{N}_{\Lambda\Sigma} = (S - \bar{S})\frac{X_\Lambda \bar{X}_\Sigma + \bar{X}_\Lambda X_\Sigma}{\bar{X}^T \eta X} + \bar{S}\eta_{\Lambda\Sigma}$$

$$\Phi^\Lambda \bar{\Phi}^\Sigma + \Phi^\Sigma \bar{\Phi}^\Lambda = \frac{1}{2} L_\Gamma^\Lambda L_\Delta^\Sigma \left(\delta^{\Gamma\Delta} + \eta^{\Gamma\Delta}\right)$$

$$X^1 = \frac{1}{2}(1+y^2) = -\frac{1}{2}\left(1 - \eta_{ij} t^i t^j\right)$$

$$X^2 = i\frac{1}{2}(1-y^2) = t^2$$

$$X^a = y^a = t^{2+a}\quad a = 1, \dots, n-1$$

$$X^{a=n} = y^n = \frac{1}{2}\left(1 + \eta_{ij} t^i t^j\right)$$

$$\eta_{ij} = \mathrm{diag}(+,-,\dots,-) i,j = 2,\dots,n+1$$

$$\mathcal{C}\begin{pmatrix} X^{\Lambda} \\ S\eta_{\Lambda\Sigma}X^{\Lambda} \end{pmatrix} = \exp\,[\varphi(t)]\begin{pmatrix} 1 \\ S \\ t^i \\ 2\mathcal{F} - t^i\frac{\partial}{\partial t^i}\mathcal{F} - S\frac{\partial}{S}\mathcal{F} \\ S\frac{\partial}{S}\mathcal{F} \\ \frac{\partial}{\partial t^i}\mathcal{F} \end{pmatrix}$$

$$\mathcal{F}(S,t) = \frac{1}{2}S\eta_{ij}t^i t^j = \frac{1}{2}d_{IJK}t^I t^J t^K$$
$$t^1 = S$$
$$d_{IJK} = \begin{cases} d_{1jk} = \eta_{ij} \\ 0 \ \ \text{otherwise} \end{cases}$$

$$W_{IJK} = d_{IJK} = \frac{\partial^3\mathcal{F}(S,t^i)}{\partial t^I \partial t^J \partial t^K}$$

$$F(X) = \frac{1}{3!}\frac{d_{IJK}X^I X^J X^K}{X_0}$$

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathcal{L}_{\text{ungauged}}^{\text{SUGRA|dark particle}} = \sqrt{-g}\Big[R[g] + g_{ij^\star}(z,\bar{z})\partial^\mu z^i \partial_\mu \bar{z}^{j^\star} + \mathrm{i}\big(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{-\Lambda}\mathcal{F}^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{+\Lambda}\mathcal{F}^{+\Sigma|\mu\nu}\big)\Big]$$

$$\gamma_5\psi_{A\mu} = \psi_{A\mu}$$

$$\gamma_5\psi_\mu^A = -\psi_\mu^A$$

$$\gamma_5\lambda^{iA} = \lambda^{iA}$$

$$\lambda_A^{i^\star} = -\gamma_5\lambda_A^{i^\star}$$

$$\delta\Psi_{A\mu} = \mathcal{D}_\mu\epsilon_A + \epsilon_{AB}T_{\mu\nu}^{-}\gamma^\nu\epsilon^B$$
$$\delta\lambda^{iA} = \mathrm{i}\nabla_\mu z^i\gamma^\mu\epsilon^A + G_{\mu\nu}^{-i}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB}$$

$$\gamma_5\epsilon_A = \epsilon_A\,;\ \gamma_5\epsilon^A = -\epsilon^A$$

$$
\begin{aligned}
T^{-}_{\mu\nu} &= 2\mathrm{i}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}L^{\Sigma}F^{\Lambda-}_{\mu\nu}\\
T^{+}_{\mu\nu} &= 2\mathrm{i}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\bar{L}^{\Sigma}F^{\Lambda+}_{\mu\nu}\\
G^{i-}_{\mu\nu} &= -g^{ij^\star}\bar{f}^{\Gamma}_{j^\star}(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}F^{\Lambda-}_{\mu\nu}\\
G^{i^\star+}_{\mu\nu} &= -g^{i^\star j}f^{\Gamma}_{j}(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}F^{\Lambda+}_{\mu\nu}
\end{aligned}
$$

$$
E^{-}= \mathrm{i}\frac{e^{2U(r)}}{r^3}dt\wedge\vec{x}\cdot d\vec{x}+\frac{1}{2}\frac{x^a}{r^3}dx^b\wedge dx^c\epsilon_{abc}
$$

$$
2\pi=\int_{S^2_\infty}E^{-}
$$

$$
E^{-}_{\mu\nu}\gamma^{\mu\nu}=2\mathrm{i}\frac{e^{2U(r)}}{r^3}\gamma_a x^a\gamma_0\frac{1}{2}[\mathbf{1}+\gamma_5]
$$

$$
t^{\Lambda}(r)=2\pi\big(p^{\Lambda}+\mathrm{i}\ell^{\Lambda}(r)\big)
$$

$$
F^{-|\Lambda}=\frac{t^{\Lambda}}{4\pi}E^{-}
$$

$$
\begin{aligned}
F^{\Lambda} &= 2\mathrm{Re}F^{-|\Lambda}=\frac{p^{\Lambda}}{2r^3}\epsilon_{abc}x^a dx^b\wedge dx^c-\frac{\ell^{\Lambda}(r)}{r^3}e^{2\mathcal{U}}dt\wedge\vec{x}\cdot d\vec{x}\\
\tilde{F}^{\Lambda} &= -2\mathrm{Im}F^{-|\Lambda}=-\frac{\ell^{\Lambda}(r)}{2r^3}\epsilon_{abc}x^a dx^b\wedge dx^c-\frac{p^{\Lambda}}{r^3}e^{2\mathcal{U}}dt\wedge\vec{x}\cdot d\vec{x}
\end{aligned}
$$

$$
\begin{aligned}
q_{\Lambda} &\equiv \frac{1}{4\pi}\int_{S^2_\infty}G_{\Lambda|\mu\nu}dx^{\mu}\wedge dx^{\nu}\\
p^{\Sigma} &\equiv \frac{1}{4\pi}\int_{S^2_\infty}F^{\Sigma}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}
\end{aligned}
$$

$$
4\pi\ell^{\Lambda}(r)=-\int_{S^2_r}\tilde{F}^{\Lambda}=2\mathrm{Im}t^{\Lambda}
$$

$$
\begin{aligned}
0 &= \mathcal{D}_{\mu}\xi_A+\epsilon_{AB}T^{-}_{\mu\nu}\gamma^{\nu}\xi^{B}\\
0 &= \mathrm{i}\nabla_{\mu}z^i\gamma^{\mu}\xi^{A}+G^{-i}_{\mu\nu}\gamma^{\mu\nu}\xi_B\epsilon^{AB}
\end{aligned}
$$

$$
\begin{aligned}
&\xi_A(r)= e^{f(r)}\chi_A\ \chi_A=\mathfrak{G}\\
&\gamma_0\chi_A=\pm\mathrm{i}\epsilon_{AB}\chi^{B}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dz^i}{dr} &=\mp\left(\frac{e^{U(r)}}{4\pi r^2}\right)g^{ij^\star}\bar{f}^{\Lambda}_{j^\star}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}t^{\Sigma}=\\
&\mp\left(\frac{e^{U(r)}}{4\pi r^2}\right)g^{ij^\star}\nabla_{j^\star}\bar{Z}(z,\bar{z},p,q)\\
\frac{dU}{dr} &=\mp\left(\frac{e^{U(r)}}{r^2}\right)\left(M_{\Sigma}p^{\Sigma}-L^{\Lambda}q_{\Lambda}\right)=\mp\left(\frac{e^{U(r)}}{r^2}\right)Z(z,\bar{z},p,q)
\end{aligned}
$$

$$
V=\left(L^{\Lambda}(z,\bar{z}),M_{\Sigma}(z,\bar{z})\right)
$$

$$
Z(z,\bar{z},p,q)\equiv\left(M_{\Sigma}p^{\Sigma}-L^{\Lambda}q_{\Lambda}\right)
$$

$$Z^i(z,\bar{z},p,q) \equiv g^{ij^\star}\nabla_{j^\star}\bar{Z}(z,\bar{z},p,q)$$

$$0 = \bar{h}_{j^\star|t^\star}t^{\star\Sigma} - \bar{f}^\Lambda_{j^\star}\mathcal{N}_{\Lambda\Sigma}t^{\star\Sigma}$$
$$0 = M_\Sigma t^{\star\Sigma} - L^\Lambda\mathcal{N}_{\Lambda\Sigma}t^{\star\Sigma}$$

$$q_\Lambda = \frac{1}{2\pi}\mathrm{Re}(\mathcal{N}(z(r),\bar{z}(r))t(r))_\Lambda$$

$$\partial_a\left(\sqrt{-g}\tilde{G}^{a0|\Lambda}(r)\right) = 0 \Rightarrow \partial_r\mathrm{Re}(\mathcal{N}(z(r),\bar{z}(r))t(r))^\Lambda = 0$$

$$0 = -g^{ij^\star}\bar{f}^\Gamma_{j^\star}(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}t^\Lambda(0)$$
$$\frac{dU}{dr} \cong \mp\left(\frac{e^{U(r)}}{r^2}\right)Z\left(z_{fix},\bar{z}_{fix},p,q\right)$$

$$0 = -\frac{1}{2}\frac{t^\Lambda}{4\pi} - \frac{Z_{fix}\bar{L}^\Lambda_{fix}}{8\pi}$$

$$p^\Lambda = \mathrm{i}\left(Z_{fix}\bar{L}^\Lambda_{fix} - \bar{Z}_{fix}L^\Lambda_{fix}\right)$$
$$q_\Sigma = \mathrm{i}\left(Z_{fix}\bar{M}^{fix}_\Sigma - \bar{Z}_{fix}M^{fix}_\Sigma\right)$$
$$Z_{fix} = M^{fix}_\Sigma p^\Sigma - L^\Lambda_{fix}q_\Lambda$$

$$L^\Lambda_{fix} = L^\Lambda(p,q) \longrightarrow Z_{fix} = Z(p,q) = \mathfrak{G}$$

$$\pm\frac{dU}{dr} = \frac{Z(p,q)}{r^2}e^{U(r)}$$

$$\exp\left[U(r)\right] \overset{r\to 0}{\to} \Re + \frac{Z(p,q)}{r}$$

$$m^2_{BR} = |Z(p,q)|^2$$

$$\mathrm{Area}_H = \int_{r=0}\sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi = 4\pi m^2_{BR}$$

$$\frac{\mathrm{Area}_H}{4\pi} = |Z(p,q)|^2$$

$$\mathrm{Area}_H(z,\bar{z}) = 4\pi|Z(z,\bar{z},p,q)|^2$$
$$\frac{\delta\mathrm{Area}_H}{\delta z} = 0 \longrightarrow z = z_{fix}$$

$$\mathbb{L}_{Sp}(\phi) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\mathbb{L}_{Usp} = \frac{1}{\sqrt{2}}\begin{pmatrix} f+\mathrm{i}h & \bar{f}+\mathrm{i}\bar{h} \\ f-\mathrm{i}h & \bar{f}-\mathrm{i}\bar{h} \end{pmatrix} = \mathcal{C}\mathbb{L}_{Sp}\mathcal{C}^{-1}$$

$$f=\frac{1}{\sqrt{2}}(A-\mathrm{i}B)$$
$$h=\frac{1}{\sqrt{2}}(C-\mathrm{i}D)$$

$$\left\langle\begin{matrix}\mathrm{i}\left(f^{\dagger}h-h^{\dagger}f\right) &=\mathbb{1}\\ \left(f^{t}h-h^{t}f\right) &=0\end{matrix}\right\rangle$$

$$H=H_{Aut}\otimes H_{\mathrm{matter}}$$
$$H_{Aut}=SU(N)\otimes U(1)$$

$$f=\left(f_{AB}^{\Lambda},f_{I}^{\Lambda}\right)$$
$$h=\left(h_{\Lambda AB},h_{\Lambda I}\right)$$

$$\left(f_{AB}^{\Lambda}\right)^{*}=f^{\Lambda AB}$$

$$\mathcal{N}=hf^{-1},\mathcal{N}=\mathcal{N}^{t}$$

$$(f^{t})^{-1}=\mathrm{i}(\mathcal{N}-\overline{\mathcal{N}})\bar{f}$$

$$f_{AB\Lambda}\equiv(f^{-1})_{AB\Lambda}=\mathrm{i}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}\bar{f}_{AB}^{\Sigma}$$
$$f_{I\Lambda}\equiv(f^{-1})_{I\Lambda}=\mathrm{i}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}\bar{f}_{I}^{\Sigma}$$

$$\gamma_{5}\chi_{ABC}=\chi_{ABC}$$
$$\gamma_{5}\chi^{ABC}=-\chi^{ABC}$$

$$\delta\psi_{A\mu}=\nabla_{\mu}\epsilon_{A}-\frac{1}{4}T_{AB|\rho\sigma}^{(-)}\gamma^{\rho\sigma}\gamma_{\mu}\epsilon^{B}=0$$
$$\delta\chi_{ABC}=4\mathrm{i}P_{ABCD|i}\partial_{\mu}\Phi^{i}\gamma^{\mu}\epsilon^{D}-3T_{[AB|\rho\sigma}^{(-)}\gamma^{\rho\sigma}\epsilon_{C]}=0$$

$$\delta\lambda_{A}^{I}=aP_{AB,i}^{I}\partial_{a}\phi^{i}\gamma^{a}\epsilon^{B}+bT_{ab}^{-I}\gamma^{ab}\epsilon_{A}$$

$$T_{AB}^{-}=\mathrm{i}\left(\bar{f}^{-1}\right)_{AB\Lambda}F^{-\Lambda}=f_{AB}^{\Lambda}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma}=h_{\Lambda AB}F^{-\Lambda}-f_{AB}^{\Lambda}\mathcal{G}_{\Lambda}^{-}$$
$$T_{I}^{-}=\mathrm{i}\left(\bar{f}^{-1}\right)_{I\Lambda}F^{-\Lambda}=f_{I}^{\Lambda}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}F^{-\Sigma}=h_{\Lambda I}F^{-\Lambda}-f_{I}^{\Lambda}\mathcal{G}_{\Lambda}^{-}$$
$$\bar{T}^{+AB}=(T_{AB}^{-})^{*}$$
$$\bar{T}^{+I}=(T_{I}^{-})^{*}$$

$$T_{AB}=T_{AB}^{+}+T_{AB}^{-}$$

$$T_{I}=T_{I}^{+}+T_{I}^{-}$$

$$T_{AB}^{+}=h_{\Lambda AB}F^{+\Lambda}-f_{AB}^{\Lambda}\mathcal{G}_{\Lambda}^{+}=0$$
$$T_{I}^{+}=h_{\Lambda I}F^{+\Lambda}-f_{I}^{\Lambda}\mathcal{G}_{\Lambda}^{+}=0$$

$$Z_{AB}=\int_{S^2}T_{AB}=\int_{S^2}(T_{AB}^{+}+T_{AB}^{-})=\int_{S^2}T_{AB}^{-}=h_{\Lambda AB}(r)p^{\Lambda}-f_{AB}^{\Lambda}(r)q_{\Lambda}$$
$$Z_{I}=\int_{S^2}T_{I}=\int_{S^2}(T_{I}^{+}+T_{I}^{-})=\int_{S^2}T_{I}^{-}=h_{\Lambda I}(r)p^{\Lambda}-f_{I}^{\Lambda}(r)q_{\Lambda}\ (N\leq 4)$$

$$q_\Lambda = \int_{S^2} \mathcal{G}_\Lambda, p^\Lambda = \int_{S^2} F^\Lambda$$

$$d\Gamma + \Gamma \wedge \Gamma = 0$$

$$\Gamma \equiv \mathbb{L}_{Usp}^{-1} d\mathbb{L}_{Usp} = \begin{pmatrix} \mathrm{i}(f^\dagger dh - h^\dagger df) & \mathrm{i}(f^\dagger d\bar{h} - h^\dagger d\bar{f}) \\ -\mathrm{i}(f^t dh - h^t df) & -\mathrm{i}(f^t d\bar{h} - h^t d\bar{f}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \overline{\mathcal{P}} \\ \mathcal{P} & \bar{\Omega} \end{pmatrix}$$

$$\nabla V = dV - V\omega, \omega = \begin{pmatrix} \omega_{CD}^{AB} & 0 \\ 0 & \omega_J^I \end{pmatrix}$$

$$\nabla(\omega)(f + \mathrm{i}h) = (\bar{f} + \mathrm{i}\bar{h})\mathcal{P}$$
$$\nabla(\omega)(f - \mathrm{i}h) = (\bar{f} - \mathrm{i}\bar{h})\mathcal{P}$$

$$\nabla(\omega) f_{AB}^\Lambda = \bar{f}_I^\Lambda P_{AB}^I + \frac{1}{2}\bar{f}^{\Lambda CD} P_{ABCD}$$
$$\nabla(\omega) f_I^\Lambda = \frac{1}{2}\bar{f}^{\Lambda AB} P_{ABI} + \bar{f}^{\Lambda J} P_{JI}$$

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{ICD} & P_{IJ} \end{pmatrix}$$

$$H_{\mathrm{Aut}} \times H_{\mathrm{matter}}$$

$$\left(f_{AB}^\Lambda, f_I^\Lambda\right)^\star = \left(\bar{f}^{\Lambda AB}, \bar{f}^{\Lambda I}\right)$$

$$\nabla(\omega) Z_{AB} = \bar{Z}_I P_{AB}^I + \frac{1}{2}\bar{Z}^{CD} P_{ABCD}$$
$$\nabla(\omega) Z_I = \frac{1}{2}\bar{Z}^{AB} P_{ABI} + \bar{Z}_J P_I^J$$

$$G/H = SU(3,n)/SU(3) \times SU(n) \times U(1) \left[\frac{36}{6}\right]$$

$$P_{ABCD} = P_{IJ} = 0$$

$$N = 4, G/H = SU(1,1)/U(1) \otimes O(6,n)/O(6) \times O(n)[\overline{37}]$$

$$P_{ABCD} = \epsilon_{ABCD} P, P_{IJ} = \bar{P}\delta_{IJ}$$

$$\frac{SU(\overline{1},\overline{1})}{U(1)} O(6,n)/O(6) \times O(n)$$

$$\frac{1}{2} Z_{AB}\bar{Z}^{AB} + Z_I \bar{Z}^I = -\frac{1}{2} Q^t \mathcal{M}(\mathcal{N}) Q$$

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}$$
$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$$

$$
\begin{aligned}
ff^{\dagger} &= -\mathrm{i}(\mathcal{N}-\overline{\mathcal{N}})^{-1} \\
hh^{\dagger} &= -\mathrm{i}\left(\overline{\mathcal{N}}^{-1}-\mathcal{N}^{-1}\right)^{-1} \equiv -\mathrm{i}\mathcal{N}(\mathcal{N}-\overline{\mathcal{N}})^{-1}\overline{\mathcal{N}} \\
hf^{\dagger} &= \mathcal{N}ff^{\dagger} \\
fh^{\dagger} &= ff^{\dagger}\overline{\mathcal{N}}
\end{aligned}
$$

$$|Z|^2+|Z_i|^2 \equiv |Z|^2+Z_i g^{i\bar{j}}\bar{Z}_{\bar{j}} = -\frac{1}{2}Q^t\mathcal{M}Q$$

$$V(\phi,Q) = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q$$

$$S_{eff} \equiv \int \mathcal{L}_{eff}(\tau)d\tau\,;\ \tau = -\frac{1}{r}$$

$$\mathcal{L}_{eff}(\tau) = \left(\frac{dU}{d\tau}\right)^2 + g_{IJ}\frac{d\phi^I}{d\tau}\frac{d\phi^J}{d\tau} + e^{2U}V(\phi,Q)$$

$$\left(\frac{dU}{d\tau}\right)^2 g_{IJ}\frac{d\phi^I}{d\tau}\frac{d\phi^J}{d\tau} - e^{2U}V(\phi,Q) \cong 0$$

$$\phi^I = \mathfrak{R} = \phi^I_\infty$$

$$\left(\frac{dU}{d\tau}\right)^2 = e^{2U}V(\phi,Q)$$

$$e^{2U(\tau)} \overset{\tau\to\infty}{\to} \frac{1}{m_{BR}^2}\frac{1}{\tau^2} = \frac{4\pi}{\mathrm{Area}_H}\frac{1}{\tau^2}$$

$$V(\phi_H,Q) = \frac{\mathrm{Area}\ _H}{4\pi}$$

$$\lim_{\tau\to\infty} g_{IJ}\frac{d\phi^I}{d\tau}\frac{d\phi^J}{d\tau}e^{2U}\tau^4 < \infty$$

$$g_{IJ}\frac{d\phi^I}{d\tau}\frac{d\phi^J}{d\tau}\frac{4\pi}{\mathrm{Area}\ _H}\tau^2 \overset{\tau\to\infty}{\to} X^2 = \mathfrak{E}\mathfrak{M}$$

$$\tau\frac{d\phi^I}{d\tau} \overset{\tau\to\infty}{\to} \mathcal{E} \to\ \phi^I \sim \mathcal{M}\times\log\tau$$

$$\left.\frac{\partial V}{\partial\phi^I}\right|_H = 0$$

$$\frac{D^2}{d\tau^2}\phi^I = \frac{1}{2}e^{2U}\frac{\partial V}{\partial\phi^I}g^{IJ}$$

$$\frac{d^2}{d\tau^2}\phi^I \cong \frac{1}{2}\frac{\partial V}{\partial\phi^I}g^{IJ}\frac{4\pi}{\mathrm{Area}_H}\frac{1}{\tau^2}$$

$$\phi^I \sim \frac{2\pi}{\mathrm{Area}_H}\frac{\partial V}{\partial\phi^I}g^{IJ}\log\tau + \phi^I_H$$

$$
\begin{aligned}
0 &= \frac{d}{dz^i}\left(Z\bar{Z} + g^{ij^\star} Z_i \bar{Z}_{j^\star}\right) \\
&= \nabla_i Z\bar{Z} + \nabla_i Z_j \bar{Z}_{j^\star} g^{jk^\star} + Z_j \nabla_i \bar{Z}_{k^\star} g^{jk^\star} \\
&= \nabla_i Z\bar{Z} + \mathrm{i} C_{ijm} \bar{Z}_{\ell^\star} \bar{Z}_{k^\star} g^{\ell^\star m} g^{jk^\star} + Z_j Z \delta_j^i (4.3.15)
\end{aligned}
$$

$$Z_i = \nabla_i Z$$

$$Z_i = \nabla_i Z = 0 \longrightarrow \partial_i |Z|^2 = 0$$

$$V^{SUSY}(\phi, \vec{Q}) = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I$$

$$
\begin{aligned}
0 &= dV^{SUSY}(\phi, \vec{Q}) \\
&= \frac{1}{2} \nabla Z_{AB} \bar{Z}^{AB} + \frac{1}{2} Z_{AB} \nabla \bar{Z}^{AB} + \nabla Z_I \bar{Z}^I + Z_I \nabla \bar{Z}^I
\end{aligned}
$$

$$
\begin{aligned}
0 =& \left[\frac{1}{2}\left(\bar{Z}^I P_{I|AB} + \frac{1}{2} \bar{Z}^{CD} P_{ABCD}\right) \bar{Z}^{AB} + \text{ c.c}\right] \\
&+ \left[\frac{1}{2}\left(\bar{Z}^{AB} P_{AB|I} + \bar{Z}^J P_{J|I}\right) \bar{Z}^J + \text{ c.c}\right]
\end{aligned}
$$

$$
\begin{aligned}
Z_I^{fix} &= 0 \\
Z_{[AB}^{fix} Z_{CD]}^{fix} &= 0
\end{aligned}
$$

$$Z_1^{fix} \equiv Z_{12}^{fix}, Z_2^{fix} \equiv Z_{34}^{fix}, \dots, Z_{N/2}^{fix} \equiv Z_{N-1N}^{fix}$$

$$\frac{\text{Area}_H}{4\pi} = \sum_{i=1}^{p} \left|Z_i^{fix}\right|^2$$

$$\frac{\text{Area}_H}{4\pi} = \left|Z_1^{fix}\right|^2 > 0$$

$$\frac{\text{Area}_H}{4\pi} = S(p, q)$$

$$
\begin{aligned}
V(\phi, \vec{Q}) &\equiv \frac{1}{2} \bar{Z}^{AB}(\phi) Z_{AB}(\phi) \\
&= \frac{1}{2} \vec{Q}^T \left[\mathbb{L}_{Sp}^{-1}(\phi)\right]^T \mathbb{L}_{Sp}^{-1}(\phi) \vec{Q}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{L}_{Sp}(\mathcal{A}\phi) = A \mathbb{L}_{Sp}(\phi) W_A(\phi) \\
&W_A(\phi) \in H \subset U(\bar{n}) \subset Sp(2\bar{n}, \mathbb{R}) \implies [W_A(\phi)]^T W_A(\phi) = \mathbb{1}
\end{aligned}
$$

$$U(\bar{n}) = SO(2\bar{n}) \cap Sp(2\bar{n}, \mathbb{R})$$

$$
\begin{aligned}
V(\mathcal{A}\phi, A\vec{Q}) &= \vec{Q}^T A^T \left[\mathbb{L}_{Sp}^{-1}(\mathcal{A}\phi)\right]^T \mathbb{L}_{Sp}^{-1}(\mathcal{A}\phi) A \vec{Q} \\
&= \vec{Q}^T A^T [A^{-1}]^T \left[\mathbb{L}_{Sp}^{-1}(\phi)\right]^T [W_A^{-1}(\phi)]^T W_A^{-1}(\phi) \mathbb{L}_{Sp}^{-1}(\phi) A^{-1} A \vec{Q} \\
&= V(\phi, \vec{Q})
\end{aligned}
$$

$$\phi^{fix} = \phi(p,q)$$

$$\frac{\text{Area}_H}{4\pi} \equiv S(p,q) = \sqrt{Q^\Lambda Q^\Sigma Q^\Delta Q^\Gamma d_{\Lambda\Sigma\Delta\Gamma}}$$

$$S^2(p,q) = I = \sum_i \alpha_i I_i(Z(p,q,\phi))$$

$$\frac{\partial}{\partial \phi^I} I = 0$$

$$I_1 \equiv (Z_{AB}\bar{Z}^{BA})^2$$
$$I_2 \equiv Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA}$$
$$I_3 \equiv \frac{1}{2}\left(\frac{1}{2!\,4!}\epsilon^{ABCDEFGH}Z_{AB}Z_{CD}Z_{EF}Z_{GH} + \text{ c.c }\right)$$

$$I_1 = 4\left(\sum_{i=1}^{4} |Z_i|^2\right)^2$$
$$I_2 = 2\sum_{i<j} |Z_i|^2|Z_j|^2$$
$$I_3 = \frac{1}{2}\left(\prod_{i=1}^{4} Z_i + \prod_{i=1}^{4} \bar{Z}_i\right)$$

$$P_1 = P_{1234}\,;\; P_2 = P_{1256}\,;\; P_3 = P_{1278}$$

$$\nabla Z_1 = P_1\bar{Z}_2 + P_2\bar{Z}_3 + P_3\bar{Z}_4$$
$$\nabla Z_2 = P_1\bar{Z}_1 + P_3\bar{Z}_3 + P_2\bar{Z}_4$$
$$\nabla Z_3 = P_2\bar{Z}_1 + P_3\bar{Z}_2 + P_1\bar{Z}_4$$
$$\nabla Z_4 = P_3\bar{Z}_1 + P_2\bar{Z}_2 + P_1\bar{Z}_3$$

$$d\left(\sum_{i=1}^{3} \alpha_i I_i\right) = 0$$

$$\begin{aligned}\left(\frac{\text{Area}_H}{4\pi}\right)^2 &= \frac{1}{4}I_1 - I_2 + 8I_3 \\ &= \sum_{i=1}^{4} (|Z_i|^2)^2 - 2\sum_{i<j} |Z_i|^2|Z_j|^2 + 4\left(\prod_{i=1}^{4} Z_i + \prod_{i=1}^{4} \bar{Z}_i\right) \\ &= (\rho_1+\rho_2+\rho_3+\rho_4)(\rho_1+\rho_2-\rho_3-\rho_4) \\ &\quad \times (\rho_1-\rho_2+\rho_3-\rho_4)(\rho_1-\rho_2-\rho_3+\rho_4) \\ &\quad +8\rho_1\rho_2\rho_3\rho_4(\cos\,\theta - 1)\end{aligned}$$

$$Z_i = \rho_i \exp\,[\mathrm{i}\theta]\,;\; \rho_i \in \mathbb{R}$$

$$\dim_{\mathbf{R}} \frac{SU(8)}{(SU(2))^4} = 51$$

$$\rho_1 > \rho_2 > \rho_3 > \rho_4 \geq 0 \quad \text{Area}_{\,H} > 0 \quad \frac{1}{8}\text{BPS}$$
$$\rho_1 = \rho_2 > \rho_3 = \rho_4 \geq 0 \quad \text{Area}_{\,H} = 0 \quad \frac{1}{4}\text{BPS}$$
$$\rho_1 = \rho_2 = \rho_3 = \rho_4 \geq 0 \quad \text{Area}_{\,H} = 0 \quad \frac{1}{2}\text{BPS}$$

$$\rho_1^\infty \neq \rho_2^\infty \neq \rho_3^\infty \neq \rho_4^\infty \,;\, \theta^\infty$$

$$\rho_1^H \neq 0\,;\, \rho_2^H = \rho_3^H = \rho_4^H = 0\,;\, \theta^H = 0$$

$$\begin{aligned}(\rho_1^H)^2 =&(\rho_1^\infty + \rho_2^\infty + \rho_3^\infty + \rho_4^\infty)(\rho_1^\infty + \rho_2^\infty - \rho_3^\infty - \rho_4^\infty)\\ &\times(\rho_1^\infty - \rho_2^\infty + \rho_3^\infty - \rho_4^\infty)(\rho_1^\infty - \rho_2^\infty - \rho_3^\infty + \rho_4^\infty)\\ &+8\rho_1^\infty\rho_2^\infty\rho_3^\infty\rho_4^\infty(\cos\,\theta^\infty - 1)\end{aligned}$$

$$\mu^I \equiv \phi^I(\infty)$$

$$\mathcal{M}_{\text{scalar}}^{(N=8)} = \frac{E_{7(7)}}{SU(8)},$$

$$\frac{U_D}{H_D} = \frac{E_{r+1(r+1)}}{H_{r+1}}$$

$D=9$	$E_{2(2)} \equiv SL(2,\mathbb{R}) \otimes O(1,1)$	$H = O(2)$	$\dim_{\mathbf{R}}(U/H) = 3$
$D=8$	$E_{3(3)} \equiv SL(3,\mathbb{R}) \otimes Sl(2,\mathbb{R})$	$H = O(2)\otimes O(3)$	$\dim_{\mathbf{R}}(U/H) = 7$
$D=7$	$E_{4(4)} \equiv SL(5,\mathbb{R})$	$H = O(5)$	$\dim_{\mathbf{R}}(U/H) = 14$
$D=6$	$E_{5(5)} \equiv O(5,5)$	$H = O(5)\otimes O(5)$	$\dim_{\mathbf{R}}(U/H) = 25$
$D=5$	$E_{6(6)}$	$H = Usp(8)$	$\dim_{\mathbf{R}}(U/H) = 42$
$D=4$	$E_{7(7)}$	$H = SU(8)$	$\dim_{\mathbf{R}}(U/H) = 70$
$D=3$	$E_{8(8)}$	$H = O(16)$	$\dim_{\mathbf{R}}(U/H) = 128$

$$\mathcal{M} = \exp\,(\,\text{Solv}\,)$$

$$\mathcal{D}^{(n)}\mathbb{G}_s = 0$$
$$\mathcal{D}\mathbb{G}_s = [\mathbb{G}_s,\mathbb{G}_s] \quad ; \quad \mathcal{D}^{(k+1)}\mathbb{G}_s = [\mathcal{D}^{(k)}\mathbb{G}_s, \mathcal{D}^{(k)}\mathbb{G}_s]$$

$$\begin{aligned}\mathcal{M} \sim \mathcal{G}_s &= \exp(\mathbb{G}_s)\\ g_{|e\in\mathcal{M}} &= h\end{aligned}$$

$$\mathbb{G} = \mathbb{H} + \mathbb{G}_s \qquad \dim\,\mathbb{G}_s = \dim\,\mathcal{M}$$

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K}$$

$$\mathbb{G}_s = \mathcal{H}_K \oplus \{ \sum_{\alpha\in\Phi^+} E_\alpha \cap \mathbb{G}\}$$

$$\begin{aligned} SL(2,\mathbb{R})/SO(2) &= \exp(Solv) \\ Solv &= \{\sigma_3, \sigma_+\} \\ [\sigma_3, \sigma_+] &= 2\sigma_+ \end{aligned}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{ST}[2,2] &= \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2)}{SO(2)\times SO(2)} \\ &\sim \frac{SU(1,1)}{U(1)} \otimes \frac{SU(1,1)}{U(1)} \otimes \frac{SU(1,1)}{U(1)} \\ &\sim \left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3 \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{STU} &= \left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3 = \exp(Solv_{STU}) \\ Solv_{STU} &= F_1 \oplus F_2 \oplus F_3 \end{aligned}$$

$$F_i = \{h_i, g_i\} \quad ; \quad [h_i, g_i] = 2g_i$$

$$[F_i, F_j] = 0$$

$$\begin{aligned} \mathcal{M}_{T^6/Z_3} &= \frac{SU(3,3)}{SU(3)\times U(3)} = \exp(Solv) \\ Solv &= \mathrm{Solv}_{STU} \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \end{aligned}$$

$$\begin{aligned} &SU(3,3) = [SU(3)_1 \oplus SU(3)_2 \oplus U(1)] \oplus \mathrm{Solv} \\ &Solv = F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \\ &F_i = \{\mathrm{h}_i,\ \mathrm{g}_i\}\ i = 1,2,3 \\ &\mathbf{X} = \mathbf{X}^+ \oplus \mathbf{X}^-, \mathbf{Y} = \mathbf{Y}^+ \oplus \mathbf{Y}^-, \mathbf{Z} = \mathbf{Z}^+ \oplus \mathbf{Z}^- \\ &[\mathrm{h}_i,\ \mathrm{g}_i] = 2\ \mathrm{g}_i\ i = 1,2,3 \\ &[F_i, F_j] = 0\ i \neq j \\ &[\mathrm{h}_3, \mathbf{Y}^\pm] = \pm\mathbf{Y}^\pm, [\mathrm{h}_3, \mathbf{X}^\pm] = \pm\mathbf{X}^\pm \\ &[\mathrm{h}_2, \mathbf{Z}^\pm] = \pm\mathbf{Z}^\pm, [\mathrm{h}_2, \mathbf{X}^\pm] = \mathbf{X}^\pm \\ &[\mathrm{h}_1, \mathbf{Z}^\pm] = \mathbf{Z}^\pm, [\mathrm{h}_1, \mathbf{Y}^\pm] = \mathbf{Y}^\pm \\ &[\mathrm{g}_1, \mathbf{X}] = [\mathrm{g}_1, \mathbf{Y}] = [\mathrm{g}_1, \mathbf{Z}] = 0 \\ &[\mathrm{g}_2, \mathbf{X}] = [\mathrm{g}_2, \mathbf{Y}] = [\mathrm{g}_2, \mathbf{Z}^+] = 0, [\mathrm{g}_2, \mathbf{Z}^-] = \mathbf{Z}^+ \\ &[\mathrm{g}_3, \mathbf{Y}^+] = [\mathrm{g}_3, \mathbf{X}^+] = [\mathrm{g}_3, \mathbf{Z}] = 0 \end{aligned}$$

$$[\mathrm{g}_3, \mathbf{Y}^-] = \mathbf{Y}^+; [\mathrm{g}_3, \mathbf{X}^-] = \mathbf{X}^+$$
$$[F_1, \mathbf{X}] = [F_2, \mathbf{Y}] = [F_3, \mathbf{Z}] = 0$$
$$[\mathbf{X}^-, \mathbf{Z}^-] = \mathbf{Y}^-$$

$$\mathrm{h}_1 = H_{\alpha_1}\ \mathrm{g}_1 = \mathrm{i}E_{\alpha_1}$$
$$\mathrm{h}_2 = H_{\alpha_3}\ \mathrm{g}_2 = \mathrm{i}E_{\alpha_3}$$
$$\mathrm{h}_3 = H_{\alpha_5}\ \mathrm{g}_3 = \mathrm{i}E_{\alpha_5}$$
$$\mathbf{X}^+ = \begin{pmatrix} \mathbf{X}_1^+ = \mathrm{i}\left(E_{-\alpha_4} + E_{\alpha_3+\alpha_4+\alpha_5}\right) \\ \mathbf{X}_2^+ = E_{\alpha_3+\alpha_4+\alpha_5} - E_{-\alpha_4} \end{pmatrix}$$
$$\mathbf{X}^- = \begin{pmatrix} \mathbf{X}_1^- = \mathrm{i}\left(E_{\alpha_3+\alpha_4} + E_{-(\alpha_4+\alpha_5)}\right) \\ \mathbf{X}_2^- = E_{\alpha_3+\alpha_4} - E_{-(\alpha_4+\alpha_5)} \end{pmatrix}$$
$$\mathbf{Y}^+ = \begin{pmatrix} \mathbf{Y}_1^+ = \mathrm{i}\left(E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + E_{-(\alpha_2+\alpha_3+\alpha_4)}\right) \\ \mathbf{Y}_2^+ = E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} - E_{-(\alpha_2+\alpha_3+\alpha_4)} \end{pmatrix}$$
$$\mathbf{Y}^- = \begin{pmatrix} \mathbf{Y}_1^- = \mathrm{i}\left(E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} + E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}\right) \\ \mathbf{Y}_2^- = E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)} \end{pmatrix}$$
$$\mathbf{Z}^+ = \begin{pmatrix} \mathbf{Z}_1^+ = \mathrm{i}\left(E_{\alpha_1+\alpha_2+\alpha_3} + E_{-\alpha_2}\right) \\ \mathbf{Z}_2^+ = E_{\alpha_1+\alpha_2+\alpha_3} - E_{-\alpha_2} \end{pmatrix}$$
$$\mathbf{Z}^- = \begin{pmatrix} \mathbf{Z}_1^- = \mathrm{i}\left(E_{\alpha_1+\alpha_2} + E_{-(\alpha_2+\alpha_3)}\right) \\ \mathbf{Z}_2^- = E_{\alpha_1+\alpha_2} - E_{-(\alpha_2+\alpha_3)} \end{pmatrix}$$

$$\tilde{g}_1 = \frac{1}{2}\left(\lambda + \frac{1}{2}\left(H_{c_1} - H_{d_1} + H_{c_1+c_2} - H_{d_1+d_2}\right)\right)$$
$$\tilde{g}_2 = \frac{1}{2}\left(\lambda + \frac{1}{2}\left(H_{c_1} - H_{d_1} - 2\left(H_{c_1+c_2} - H_{d_1+d_2}\right)\right)\right)$$
$$\tilde{g}_3 = \frac{1}{2}\left(\lambda + \frac{1}{2}\left(-2\left(H_{c_1} - H_{d_1}\right) + \left(H_{c_1+c_2} - H_{d_1+d_2}\right)\right)\right)$$

$$\mathbf{20} \overset{SL(2,\mathbb{R})^3}{\longrightarrow} (\mathbf{2},\mathbf{2},\mathbf{2}) \oplus 2 \times [(\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})]$$

$$\{C(n)\} = \left\{\frac{H_{c_1}}{2}, \frac{H_{c_1+c_2}}{2}, \frac{H_{d_1}}{2}, \frac{H_{d_1+d_2}}{2}, \lambda\right\}$$
$$C(n) \cdot \left|v_x^{\Lambda'}\right\rangle = v_{(n)}^{\Lambda'}\left|v_y^{\Lambda'}\right\rangle$$
$$C(n) \cdot \left|v_y^{\Lambda'}\right\rangle = -v_{(n)}^{\Lambda'}\left|v_x^{\Lambda'}\right\rangle$$

$$\vec{v}^{\Lambda'} = v^{\Lambda'}\left(\frac{H_{c_1}}{2}, \frac{H_{c_1+c_2}}{2}, \frac{H_{d_1}}{2}, \frac{H_{d_1+d_2}}{2}, \lambda\right)$$

$$v^0 = \left\{0,0,0,0,\frac{3}{2}\right\}$$

$$v^1 = \left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}$$

$$v^2 = \left\{0,\frac{1}{2},0,-\frac{1}{2},\frac{1}{2}\right\}$$

$$v^3 = \left\{\frac{1}{2},0,-\frac{1}{2},0,\frac{1}{2}\right\}$$

$$v^4 = \left\{\frac{1}{2},0,0,-\frac{1}{2},\frac{1}{2}\right\}$$

$$v^5 = \left\{0,\frac{1}{2},-\frac{1}{2},0,\frac{1}{2}\right\}$$

$$v^6 = \left\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$$

$$v^7 = \left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},0,-\frac{1}{2}\right\}$$

$$v^8 = \left\{0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$$

$$v^9 = \left\{\frac{1}{2},\frac{1}{2},0,\frac{1}{2},-\frac{1}{2}\right\}$$

$$H_{\text{matter}} = SU(3)_1 \oplus SU(3)_2$$

$$\left|v_{x,y}^{\Lambda}\right\rangle [SL(2,\mathbb{R})]^3 Sp(8)$$

$$|v_x^1\rangle = \left\{0,0,0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\}$$

$$|v_x^2\rangle = \left\{0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$$

$$|v_x^3\rangle = \left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0\right\}$$

$$|v_x^4\rangle = \left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0\right\}$$

$$\left|v_y^1\right\rangle = \left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0\right\}$$

$$\left|v_y^2\right\rangle = \left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0\right\}$$

$$\left|v_y^3\right\rangle = \left\{0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\}$$

$$\left|v_y^4\right\rangle = \left\{0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}$$

$$h_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \; g_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} ; \; g_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} ; \; g_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Solv}(\mathcal{U}/\mathcal{H}) = \mathrm{Solv}(S/\mathcal{H}_S) \oplus \mathrm{Solv}(T/\mathcal{H}_T) \oplus \mathcal{W}$$

$$\mathcal{M}_{\text{scalar}} = E_{r+1(r+1)}/\mathcal{H}_{r+1}$$

$$\mathrm{Solv}\left(E_{r+1(r+1)}/\mathcal{H}_{r+1}\right) = O(1,1) \oplus \mathrm{Solv}\left(\frac{SO(r,r)}{SO(r) \times SO(r)}\right) \oplus \mathcal{W}_{n_{r+1}}$$

$$\mathcal{W}_{n_{r+1}} \equiv \mathrm{spin}[r,r]$$

$$\mathrm{Solv}\left(\frac{E_{7(7)}}{SU(8)}\right) = \mathrm{Solv}\left(\frac{SL(2,R)}{O(2)}\right) \oplus \mathrm{Solv}\left(\frac{SO(6,6)}{SO(6) \times SO(6)}\right) \oplus \mathcal{W}_{32}$$

$$S \otimes T = O(1,1) \otimes GL(r)$$

$$\mathrm{Solv}\left(E_{r+1(r+1)}/\mathcal{H}_{r+1}\right) = O(1,1) \oplus \mathrm{Solv}\left(\frac{GL(r)}{SO(r)}\right) \oplus \widetilde{\mathcal{W}}_{n_{r+1}}$$

$$\mathrm{Solv}_{r+1} \equiv \mathrm{Solv}\left(E_{r+1(r+1)}/\mathcal{H}_{r+1}\right) = \mathrm{Solv}(SL(2,R)/O(2)) \oplus \mathrm{Solv}\left(\frac{GL(r)}{SO(r)}\right) \oplus \overline{\mathcal{W}}_{n_{r+1}}$$

$$\mathbb{L} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ \hline f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{array}\right)$$

$$(h,f) = \left(h_{\Lambda\Sigma|AB}, f^{\Lambda\Sigma}_{AB}\right)$$

$$\mathbb{L}^{-1}d\mathbb{L} = \left(\begin{array}{c|c} \delta^{[A}{}_{[C}\Omega^{B]}{}_{D]} & \overline{P}^{ABCD} \\ \hline P_{ABCD} & \delta_{[A}{}^{[C}\overline{\Omega}_{B]}{}^{D]} \end{array}\right)$$

$$P_{ABCD} = \frac{1}{24}\epsilon_{ABCDEFGH}\bar{P}^{EFGH}$$

$$\mathcal{L} = \int \sqrt{-g}d^4x\left(2R + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}F^{\Lambda\Sigma}_{\mu\nu}F^{\Gamma\Delta|\mu\nu} + \frac{1}{6}P_{ABCD,i}\bar{P}^{ABCD}_{j}\partial_\mu\Phi^i\partial^\mu\Phi^j \right.$$
$$\left. + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}F^{\Lambda\Sigma}_{\mu\nu}F^{\Gamma\Delta}_{\rho\sigma}\right)$$

$$R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb}$$

$$\mathcal{N} = hf^{-1} \rightarrow \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} = h_{\Lambda\Sigma|AB}f^{-1}_{\Gamma\Delta}{}^{AB}_{\Gamma\Delta}$$

$$F^{\pm\Lambda\Sigma} = \frac{1}{2}(F \pm \mathrm{i} \star F)^{\Lambda\Sigma}$$

$$G^-_{\Lambda\Sigma} = \overline{\mathcal{N}}_{\Lambda\Sigma|\Gamma\Delta}F^{-\Gamma\Delta}$$
$$G^+_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}F^{+\Gamma\Delta}$$

$$G^{\pm}_{\Lambda\Sigma} = \frac{i}{2}\frac{\delta\mathcal{L}}{\delta F^{\pm\Lambda\Sigma}}$$

$$U\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F' \\ G' \end{pmatrix} \; ; \; U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
$$A^tC - C^tA = 0$$
$$B^tD - D^tB = 0$$
$$A^tD - C^tB = \mathbf{1}$$

$$\mathbb{L}_{Usp} = \mathcal{C}\mathbb{L}_{Sp}\mathcal{C}^{-1} \qquad \mathcal{C} = \begin{pmatrix} \mathbb{1} & \mathrm{i}\mathbb{1} \\ \mathbb{1} & -\mathrm{i}\mathbb{1} \end{pmatrix}$$

$$\delta\chi_{ABC} = 4\mathrm{i}P_{ABCD|i}\partial_\mu\Phi^i\gamma^\mu\epsilon^D - 3T^{(-)}_{[AB|\rho\sigma}\gamma^{\rho\sigma}\epsilon_{C]} = 0$$
$$\delta\psi_{A\mu} = \nabla_\mu\epsilon_A - \frac{1}{4}T^{(-)}_{AB|\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon^B = 0$$

$$\nabla_\mu\epsilon_A = \partial_\mu\epsilon_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\epsilon_A - \Omega^B_A\epsilon_B$$

$$T_{AB}^{(-)} = \left(h_{\Lambda\Sigma AB}(\Phi)F^{-\Lambda\Sigma} - f_{AB}^{\Lambda\Sigma}(\Phi)G_{\Lambda\Sigma}^{-}\right)$$

$$T_{AB}^{+} = 0 \rightarrow T_{AB}^{-} = T_{AB}\ \bar{T}_{AB}^{-} = 0 \rightarrow \bar{T}_{AB}^{+} = \bar{T}_{AB}$$

$$Z_{AB} = \int_{S^2} T_{AB} = h_{\Lambda\Sigma|AB}p^{\Lambda\Sigma} - f_{AB}^{\Lambda\Sigma}q_{\Lambda\Sigma}$$

$$p^{\Lambda\Sigma} = \int_{S^2} F^{\Lambda\Sigma}$$

$$q_{\Lambda\Sigma} = \int_{S^2} \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} \star F^{\Gamma\Delta}$$

$$\begin{array}{rcll} \chi^{\mu}\,\gamma_{\mu}\,\epsilon_A & = & \mathrm{i}\mathbb{C}_{AB}\,\epsilon^B & ; \quad A, B = 1, \ldots, n_{max} \\ \epsilon_A & = & 0 & ; \quad A > n_{max} \end{array}$$

$$g \in G_{stab}(\vec{Q}) \subset E_{7(7)} \Leftrightarrow g\vec{Q} = \vec{Q}$$

SUSY	Central Charge	Stabilizer $\equiv G_{stab}$	Normalizer $\equiv G_{norm}$
1/2	$Z_1 = Z_2 = Z_3 = Z_4$	$E_{6(6)}$	$O(1,1)$
1/4	$Z_1 = Z_2 \neq Z_3 = Z_4$	$SO(5,5)$	$SL(2,\mathbb{R}) \times O(1,1)$
1/8	$Z_1 \neq Z_2 \neq Z_3 \neq Z_4$	$SO(4,4)$	$SL(2,\mathbb{R})^3$

$$[G_{\text{norm}}, G_{\text{stab}}] = 0$$

$$\gamma^0\,\epsilon_A = \mathrm{i}\mathbb{C}_{AB}\,\epsilon^B \quad ; \quad A, B = 1, \ldots, 8$$

$$\mathbf{70} \overset{Usp(8)}{\rightarrow} \mathbf{42} \oplus \mathbf{1} \oplus \mathbf{27}$$

$$\begin{array}{rcl} \text{Solv}_7 & = & \text{Solv}_6 \oplus O(1,1) \oplus \mathbb{D}_6 \\ 70 & = & 42 + 1 + 27 \end{array}$$

$$\begin{array}{rcl} Solv_7 & = & Solv_6 \oplus O(1,1) \oplus \mathbb{D}_6 \\ 70 & = & 42 + 1 + 27 \end{array}$$

$$\text{Solv}_7 \equiv \text{Solv}\left(\frac{E_{7(7)}}{\text{SU}(8)}\right)$$

$$\text{Solv}_6 \equiv \text{Solv}\left(\frac{E_{6(6)}}{Usp(8)}\right)$$

$$\text{dimSolv}_7 = 70;\ \text{rankSolv}_7 = 7$$

$$\text{dimSolv}_6 = 42;\ \text{rankSolv}_6 = 6$$

$$\mathbf{56} \overset{Usp(8)}{\rightarrow} (\mathbf{1},\mathbf{27}) \oplus (\mathbf{1},\mathbf{27}) \oplus (\mathbf{2},\mathbf{1})$$

$$\mathbf{56} \overset{Usp(8)}{\rightarrow} \mathbf{48} \oplus \mathbf{8}$$

$$\begin{aligned}\gamma^0 \epsilon_a &= \mathrm{i}\mathbb{C}_{ab}\,\epsilon^b \quad ; \quad a,b = 1,\ldots,4 \\ \epsilon_X &= 0; \quad X = 5,\ldots,8\end{aligned}$$

$$\begin{aligned}\mathrm{Solv}_{\ 7} &= \mathrm{Solv}_{\ S} \oplus \mathrm{Solv}_{\ T} \oplus \mathcal{W}_{32} \\ 70 &= 2 + 36 + 32\end{aligned}$$

$$\mathrm{Solv}_S \equiv \mathrm{Solv}\left(\frac{SL(2,R)}{U(1)}\right)$$

$$\mathrm{Solv}_T \equiv \mathrm{Solv}\left(\frac{SO(6{,}6)}{SO(6)\times SO(6)}\right)$$

$$\begin{aligned}&\dim \mathrm{Solv}_S = 2;\ \mathrm{rank}\ \mathrm{Solv}_{\ S} = 1 \\ &\dim \mathrm{Solv}_{\ T} = 36;\ \mathrm{rank}\ \mathrm{Solv}_{\ T} = 6\end{aligned}$$

$$\mathrm{Solv}_T = \mathrm{Solv}_{T5} \oplus \mathrm{Solv}_{T1}$$

$$\mathrm{Solv}_{T5} \equiv \mathrm{Solv}\left(\frac{SO(5{,}6)}{SO(5)\times SO(6)}\right)$$

$$\mathrm{Solv}_{T1} \equiv \mathrm{Solv}\left(\frac{SO(1{,}6)}{SO(6)}\right)$$

$$\mathbf{70} \overset{Usp(4)\times SU(4)\times U(1)}{\rightarrow} (\mathbf{1},\mathbf{1},\mathbf{1}+\overline{\mathbf{1}}) \oplus (\mathbf{5},\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4},\mathbf{1})$$

$$\begin{aligned}\mathrm{Solv}_{\ 7} &= \mathrm{Solv}_{\ S} \oplus \mathrm{Solv}_{\ T5} \oplus \mathrm{Solv}_{\ T1} \oplus \mathcal{W}_{32} \\ 70 &= 2 + 30 + 6 + 32\end{aligned}$$

$$\begin{aligned}&\mathbf{56} \overset{Usp(4)\times SU(4)}{\rightarrow} (\mathbf{4},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{4},) \oplus (\mathbf{5},\mathbf{4}) \oplus (\mathbf{4},\mathbf{6}) \\ &\mathbf{8}^{Usp(4)\times SU(4)}(\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4},)\end{aligned}$$

$$G_{\mathrm{norm}}/H_{\mathrm{norm}} = \frac{Sl(2,\mathbb{R})}{U(1)} \times O(1{,}1)$$

$$\equiv \mathrm{Solv}\big(E_{6(4)}/SU(2)\times SU(6)\big)$$

$$\begin{aligned}\exp\,[Solv_3] &= \mathscr{k} \\ \exp\,[Solv_4] &= \mathfrak{q}\end{aligned}$$

$$\mathrm{Solv}_{STU} \equiv \mathrm{Solv}\left(\frac{SL(2,\mathbb{R})^3}{U(1)^3}\right) \subset \mathrm{Solv}_3$$

$$\vec{Q} \equiv \begin{pmatrix} p^{\vec{\Lambda}} \\ q_{\vec{\Sigma}} \end{pmatrix}$$

$$\begin{pmatrix} t^{\vec{\Lambda}_1} = p^{\vec{\Lambda}_1} + \mathrm{i}q_{\vec{\Lambda}_1} \\ \bar{t}_{\vec{\Lambda}_1} = p^{\vec{\Lambda}_1} - \mathrm{i}q_{\vec{\Lambda}_1} \end{pmatrix}$$

$$\vec{Q} \to \vec{Q}^N \equiv \begin{pmatrix} t^0_{(1,1,1)} \\ t^1_{(1,1,15)} \\ t^2_{(1,1,15)} \\ t^3_{(1,1,15)} \\ 0 \\ \cdots \\ 0 \\ \overline{t^0_{(1,1,1)}} \\ \overline{t^1_{(1,1,15)}} \\ \overline{t^2_{(1,1,15)}} \\ \overline{t^3_{(1,1,15)}} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

$$U(1) \times SU(2) \times SU(6) \subset SU(8) \subset E_{7(7)}$$

$$\mathbf{56}_{\text{ real}} = (\mathbf{1},\mathbf{1},\mathbf{1})_{\text{comp.}} \oplus (\mathbf{1},\mathbf{2},\mathbf{6})_{\text{comp.}} \oplus (\mathbf{1},\mathbf{1},\mathbf{15})_{\text{comp}}$$

$$\vec{Z} \equiv (Z^{AB}, Z_{CD})$$

$$\vec{Z} \to \vec{Z}^N \equiv \begin{pmatrix} z^0_{(1,1,1)} \\ z^1_{(1,1,15)} \\ z^2_{(1,1,15)} \\ z^3_{(1,1,15)} \\ 0 \\ :: \\ 0 \\ \overline{z^0_{(1,1,1)}} \\ \overline{z^1_{(1,1,15)}} \\ \overline{z^2_{(1,1,15)}} \\ \overline{z^3_{(1,1,15)}} \\ 0 \\ :: \\ 0 \end{pmatrix}$$

$$\mathbf{56} \to (\mathbf{8}_{\mathrm{v}},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{8}_{\mathrm{s}},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{8}_{\mathrm{s}'},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})$$

$$\mathbf{5} = 56 - \dim \frac{SU(8)}{SU(2)^4}$$

$$\begin{aligned} V(\phi) &\equiv \frac{1}{2}\bar{Z}^{AB}(\phi) Z_{AB}(\phi) \\ &= \frac{1}{2}\vec{Q}^T [\mathbb{L}^{-1}(\phi)]^T \mathbb{L}^{-1}(\phi)\vec{Q} \end{aligned}$$

$$q_f \leftrightarrow \frac{\partial}{\partial q_f} V(\phi) = 0$$

$$\begin{gathered}
\mathrm{Solv}_{\ 3} = F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z} \\
F_i = \{\mathrm{h}_i,\ \mathrm{g}_i\}\ i = 1,2,3 \\
\mathbf{X} = \mathbf{X}^+ \oplus \mathbf{X}^- = \mathbf{X}_{NS} \oplus \mathbf{X}_{RR} \\
\mathbf{Y} = \mathbf{Y}^+ \oplus \mathbf{Y}^- = \mathbf{Y}_{NS} \oplus \mathbf{Y}_{RR} \\
\mathbf{Z} = \mathbf{Z}^+ \oplus \mathbf{Z}^- = \mathbf{Z}_{NS} \oplus \mathbf{Z}_{RR}
\end{gathered}$$

$$\begin{aligned}
\mathrm{Solv}(SU(3,3)_1) &= F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X}_{NS} \oplus \mathbf{Y}_{NS} \oplus \mathbf{Z}_{NS} \\
\mathrm{Solv}(SL(2,\mathbb{R})^3) &= F_1 \oplus F_2 \oplus F_3 \\
\mathcal{W}_{12} &= \mathbf{X}_{RR} \oplus \mathbf{Y}_{RR} \oplus \mathbf{Z}_{RR} \\
\dim(F_i) &= 2; \dim\left(\mathbf{X}_{NS/RR}\right) = \dim(\mathbf{X}^\pm) = 4 \\
\dim\left(\mathbf{Y}_{NS/RR}\right) &= \dim(\mathbf{Y}^\pm) = \dim\left(\mathbf{Z}_{NS/RR}\right) = \dim(\mathbf{Z}^\pm) = 4 \\
[\ \mathrm{h}_i,\ \mathrm{g}_i] &= \mathrm{g}_i\ i = 1,2,3 \\
\left[F_i, F_j\right] &= 0\ i \neq j \\
[\ \mathrm{h}_3, \mathbf{Y}^\pm] &= \pm\frac{1}{2}\mathbf{Y}^\pm \\
[\mathrm{h}_3, \mathbf{X}^\pm] &= \pm\frac{1}{2}\mathbf{X}^\pm \\
[\mathrm{h}_2, \mathbf{Z}^\pm] &= \pm\frac{1}{2}\mathbf{Z}^\pm \\
[\mathrm{g}_3, \mathbf{Y}^+] &= [\mathrm{g}_2, \mathbf{Z}^+] = [\mathrm{g}_3, \mathbf{X}^+] = 0 \\
[\ \mathrm{g}_3, \mathbf{Y}^-] &= \mathbf{Y}^+; [\mathrm{g}_2, \mathbf{Z}^-] = \mathbf{Z}^+; [\mathrm{g}_3, \mathbf{X}^-] = \mathbf{X}^+ \\
[F_1, \mathbf{X}] &= [F_2, \mathbf{Y}] = [F_3, \mathbf{Z}] = 0 \\
[\mathbf{X}^-, \mathbf{Z}^-] &= \mathbf{Y}^-
\end{aligned}$$

$$\begin{aligned}
\mathrm{Solv}_4 =& F_0 \oplus F_1' \oplus F_2' \oplus F_2' \\
& \oplus \mathbf{X}_{NS}' \oplus \mathbf{Y}_{NS}' \oplus \mathbf{Z}_{NS}' \oplus \mathcal{W}_{20} \\
\mathrm{Solv}(SL(2,\mathbb{R})) \oplus \mathrm{Solv}(SU(3,3)_2) =& [F_0] \oplus [F_1' \oplus F_2' \oplus F_2' \\
& \oplus \mathbf{X}_{NS}' \oplus] \\
F_0 =& \{\mathrm{h}_0,\ \mathrm{g}_0\}\ [\mathrm{h}_0,\ \mathrm{g}_0] = \mathrm{g}_0 \\
F_i' =& \{\mathrm{h}_i', \mathrm{g}_i'\}\ i = 1,2,3 \\
[F_0, \mathrm{Solv}(SU(3,3)_2)] =& 0;\ [\mathrm{h}_0, \mathcal{W}_{20}] = \frac{1}{2}\mathcal{W}_{20} \\
[\ \mathrm{g}_0, \mathcal{W}_{20}] =& [\mathrm{g}_0, \mathrm{Solv}(SU(3,3)_2)] = 0 \\
[\mathrm{Solv}(SL(2,\mathbb{R})) \oplus \mathrm{Solv}(SU(3,3)_2), \mathcal{W}_{20}] =& \mathcal{W}_{20}
\end{aligned}$$

$$\mathrm{Solv}_{STU} \equiv \mathrm{Solv}(SL(2,\mathbb{R})^3)$$

$$\begin{aligned}
&\mathrm{Solv}(O(4,4)) \subset \mathrm{Solv}_4 \\
&\mathrm{Solv}(O(4,4)) = F_0 \oplus F_1' \oplus F_2' \oplus F_3' \oplus \mathcal{W}_8
\end{aligned}$$

$$F_0 \oplus F_1' \oplus F_2' \oplus F_3'$$

$$\mathcal{SK}_{15} \equiv \exp\left[\ \mathrm{Solv}_{\ 3}\right] = \frac{SO^\star(12)}{U(1) \times SU(6)}$$

$$G^L = O(4,4) \text{ of } \vec{Q}^N$$

$$\vec{Q}'^N = G^N \cdot \vec{Q}^N \; \vec{Q}^N = G^L \cdot \vec{Q}^N$$
$$\vec{Q}'^N = G^L \cdot \vec{Q}'^N \Rightarrow [G^N, G^L] = 0$$

$$\mathrm{Solv}_{\;STU} \equiv \mathrm{Solv}(SL(2,\mathbb{R})^3)$$

$$E_{7(7)} \supset E_{6(6)} \times O(1,1) \frac{E_{7(7)}}{SU(8)}$$

$$\mathbf{70} \overset{Usp(8)}{=} \mathbf{42} \oplus \mathbf{27} \oplus \mathbf{1}$$
$$\mathbf{28} \overset{Usp(8)}{=} \mathbf{27} \oplus \mathbf{1}$$

$$P_{ABCD} = \overset{\circ}{P}_{ABCD} + \frac{3}{2} C_{[AB} \overset{\circ}{P}_{CD]} + \frac{1}{16} C_{[AB} C_{CD]} P$$
$$T_{AB} = \overset{\circ}{T}_{AB} + \frac{1}{8} C_{AB} T$$

$$4P_{,a}\gamma^a\gamma^0 - 6T_{ab}\gamma^{ab} = 0$$

$$-4\left(\overset{\circ}{P}_{ABCD,a} + \frac{3}{2}\overset{\circ}{P}_{[CD,a}C_{AB]}\right) C^{DL}\gamma^a\gamma^0 - 3\overset{\circ}{T}_{[AB}\delta^L_{C]}\gamma^{ab} = 0$$

$$\overset{\circ}{P}_{AB} = \overset{\circ}{T}_{AB} = 0,$$

$$\overset{\circ}{P}_{ABCD} = 0$$

$$\overset{\circ}{P}_{ABCD|i}\partial_\mu \Phi^i \gamma^\mu \epsilon^D = 0$$
$$\overset{\circ}{P}_{AB|i}\partial_\mu \Phi^i \gamma^\mu \epsilon^B = 0$$
$$\overset{\circ}{T}_{AB} = 0$$

$$\left(\hat{P}\partial_\mu \Phi \gamma^\mu - \frac{3}{2}\mathrm{i}T^{(-)}_{\rho\sigma}\gamma^{\rho\sigma}\gamma^0\right)\epsilon_A = 0$$

$$F^{-\Lambda\Sigma} = \frac{1}{4\pi} t^{\Lambda\Sigma}(r) E^{(-)}$$
$$t^{\Lambda\Sigma}(r) = 2\pi(g + \mathrm{i}\ell(r))^{\Lambda\Sigma}$$

$$T^-_{ab} = \mathrm{i}t^{\Lambda\Sigma}(r)E^-_{ab}C^{AB}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}_{AB}$$

$$\gamma_{ab}E^{\mp}_{ab} = 2\mathrm{i}\frac{e^{2U}}{r^3}x^i\gamma^0\gamma^i\left(\frac{\pm 1 + \gamma_5}{2}\right)$$

$$\frac{d\Phi}{dr} = -\frac{\sqrt{3}}{4}\ell(r)^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}f^{\Gamma\Delta}{}_{AB}\frac{e^U}{r^2}$$

$$\hat{P} = 4\sqrt{3}$$

$$P^{(\text{singlet})}_{ABCD} = \frac{1}{16}PC_{[AB}C_{CD]} = \frac{\sqrt{3}}{4}C_{[AB}C_{CD]}d\Phi.$$

$$\omega^{0i} = \frac{dU}{dr}\frac{x^i}{r}e^{U(r)}V^0$$

$$\omega^{ij} = 2\frac{dU}{dr}\frac{x_k}{r}\eta^{k[i}V^{j]}e^U$$

$$V^0 = e^U dt, V^i = e^{-U}dx^i$$

$$\epsilon_A = e^{f(r)}\zeta_A$$

$$\left\{\frac{df}{dr}\frac{x^i}{r}e^{f+U}\delta_A^B V^i + \Omega_{A,\alpha}^B \partial_i \Phi^\alpha e^f V^i \right.$$
$$\left. -\frac{1}{4}\left(2\frac{dU}{dr}\frac{x^i}{r}e^U e^f\left(\gamma^0\gamma^i V^0 + \gamma^{ij}V_j\right)\right)\delta_A^B + \delta_A^B T_{ab}^- \gamma^{ab}\gamma^c\gamma^0 V_c\right\}\zeta_B = 0$$

$$\frac{dU}{dr} = -\frac{1}{8}\ell(r)^{\Lambda\Sigma}\frac{e^U}{r^2}C^{AB}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}{}_{AB}$$

$$\Phi = 2\sqrt{3}U$$

$$\mathbb{P} = \begin{pmatrix} 0 & \bar{P}_{ABCD} \\ P_{ABCD} & 0 \end{pmatrix}$$

$$\mathbb{K} = \frac{\sqrt{3}}{4}\left(\begin{array}{c|c} 0 & C^{[AB}C^{HL]} \\ \hline C_{[CD}C_{RS]} & 0 \end{array}\right)$$

$$\mathbb{P}_1 = \frac{1}{8}C^{AB}C_{RS}$$

$$\mathbb{P}_{27} = (\delta_{RS}^{AB} - \frac{1}{8}C^{AB}C_{RS})$$

$$\begin{aligned} \exp(\Phi\mathbb{K}) &= \cosh\left(\frac{1}{2\sqrt{3}}\Phi\right)\mathbb{P}_{27} + \frac{3}{2}\sinh\left(\frac{1}{2\sqrt{3}}\Phi\right)\mathbb{P}_{27}\mathbb{K}\mathbb{P}_{27} \\ &+ \cosh\left(\frac{\sqrt{3}}{2}\Phi\right)\mathbb{P}_1 + \frac{1}{2}\sinh\left(\frac{\sqrt{3}}{2}\Phi\right)\mathbb{P}_1\mathbb{K}\mathbb{P}_1 \end{aligned}$$

$$\mathbb{P}_1 \exp[\Phi \mathbb{K}]\mathbb{P}_1 = \cosh(\frac{\sqrt{3}}{2}\Phi)\mathbb{P}_1 + \frac{1}{2}\sinh(\frac{\sqrt{3}}{2}\Phi)\mathbb{P}_1\mathbb{K}\mathbb{P}_1$$

$$\mathbb{L}_{singlet} = \frac{1}{8}\left(\begin{array}{c|c} \cosh(\frac{\sqrt{3}}{2}\Phi)C^{AB}C_{CD} & \sinh(\frac{\sqrt{3}}{2}\Phi)C^{AB}C^{FG} \\ \hline \sinh(\frac{\sqrt{3}}{2}\Phi)C_{CD}C_{LM} & \cosh(\frac{\sqrt{3}}{2}\Phi)C_{CM}C^{FG} \end{array}\right)$$

$$f = \frac{1}{8\sqrt{2}}e^{\frac{\sqrt{3}}{2}\Phi}C^{AB}C_{CD}$$
$$h = -\mathrm{i}\frac{1}{8\sqrt{2}}e^{-\frac{\sqrt{3}}{2}\Phi}C_{AB}C_{CD}$$

$$\mathcal{N}_{ABCD} = -\mathrm{i}\frac{1}{8}e^{-\sqrt{3}\Phi}C_{AB}C_{CD}$$

$$\bar{U}(r), \bar{\ell}(r) = C_{\Lambda\Sigma}\ell^{\Lambda\Sigma}(r)$$

$$\frac{dU}{dr} = \frac{1}{8\sqrt{2}}\frac{\ell(r)}{r^2}\exp(-2U)$$

$$\mathcal{L} = 2R - e^{-\sqrt{3}\Phi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi$$

$$U = -\frac{1}{4}\log H(r)$$
$$\Phi = -\frac{\sqrt{3}}{2}\log H(r)$$
$$\ell = 2r^3\frac{d}{dr}(H(r))^{-\frac{1}{2}} = k \times (H(r))^{-\frac{3}{2}}$$

$$U'' + \frac{2}{r}U' - (U)^2 = \frac{1}{4}(\Phi')^2$$

$$\frac{d}{dr}\left(e^{-\sqrt{3}\Phi}\ell(r)\right) = 0$$

$$\Phi'' + \frac{2}{r}\Phi' = -e^{-\sqrt{3}\Phi+2U}\ell(r)^2\frac{1}{r^4}$$

$$\ell(r) = e^{\sqrt{3}\Phi(r)}$$

$$U = -\frac{1}{4}\log H(x)$$
$$\Phi = -\frac{\sqrt{3}}{2}\log H(x)$$
$$\ell = H(x)^{-\frac{3}{2}}$$

$$H(x) \equiv 1 + \sum_i \frac{k_i}{\vec{x} - \vec{x}_i^0}$$

$$H(x) = 1 + \frac{k}{r}$$

$$E_{(7,7)} \supset O(5,5) \times SL(2,\mathbb{R}) \times O(1,1) = G_{\text{stab}} \times G_{\text{norm}}$$

$$\frac{SL(2,\mathbb{R})}{U(1)} \times O(1,1)$$

$$SU(8) \to Usp(4) \times SU(4) \times U(1)$$

$$SU(4) \times Usp(4), P_{ABCD}$$

$$\mathbf{70} \xrightarrow{Usp(4)\times SU(4)} (\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4}) \oplus (\mathbf{5},\mathbf{6}) \oplus (\mathbf{1},\mathbf{6}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{4}}) \oplus (\overline{\mathbf{1}},\overline{\mathbf{1}})$$

$$\mathbf{28} \xrightarrow{Usp(4)\times SU(4)} (\mathbf{1},\mathbf{6}) \oplus (\mathbf{4},\mathbf{4}) \oplus (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$$

$$\begin{aligned} \delta\chi_{XYZ} &= 0 \\ \delta\chi_{aXY} &= 0 \\ \delta\mathring{\chi}_{abX} &= C^{ab}\delta\chi_{abX} = 0 \\ \delta\chi_{abc} &= C_{[ab}\delta\chi_{c]} = 0 \end{aligned}$$

$$P_{XYZa,\alpha}\partial_\mu\Phi^\alpha = 0$$

$$P_{Xabc} \equiv P_{X[a}C_{bc]} = 0$$

$$\begin{aligned} P_{XY,i}\partial_\mu\Phi^i\gamma^\mu\gamma^0\epsilon_a &= T_{XY\mu\nu}\gamma^{\mu\nu}\epsilon_a \\ \mathring{P}_{XYab,i}\partial_\mu\Phi^i &= 0 \end{aligned}$$

$$P_{XYab} = \mathring{P}_{XYab} + \frac{1}{4}C_{ab}P_{XY}$$

$$\begin{aligned} P_{abcd} &= C_{[ab}C_{cd]}P \\ T_{ab} &= \mathring{T}_{ab} + \frac{1}{4}C_{ab}T \end{aligned}$$

$$\begin{aligned} \mathring{T}_{ab} &= 0 \\ P_{,i}\partial_\mu\Phi^i\gamma^\mu\gamma^0 - \frac{3}{16}T_{\mu\nu}\gamma^{\mu\nu}\epsilon_a &= 0 \end{aligned}$$

$$\begin{aligned} P_{,i}\frac{d\Phi^i}{dr} &= \mathrm{i}\frac{3}{8}(p + \mathrm{i}\ell(r))^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}{}_{AB}C^{AB}\frac{e^U}{r^2} \\ P_{XY,i}\frac{d\Phi^i}{dr} &= 2\mathrm{i}(p + \mathrm{i}\ell(r))^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}{}_{XY}\frac{e^U}{r^2} \end{aligned}$$

$$P_1\frac{d\Phi^1}{dr} = -2\ell(r)^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}_{XY}\frac{e^U}{r^2}$$

$$\Omega^a_X = 0;\ T_{Xa} = 0$$

$$\frac{dU}{dr} = -\frac{1}{4}\ell(r)^{\Lambda\Sigma}\frac{e^U}{r^2}C^{ab}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}_{ab}$$

$$P_2\frac{d\Phi_2}{dr} = -\frac{3}{8}\ell(r)^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f^{\Gamma\Delta}_{XY}C^{XY}\frac{e^U}{r^2}$$

$$\varpi_{AB} = -\varpi_{BA} = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right) \quad ; \quad \Omega_{AB} = -\Omega_{BA} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & C \end{array}\right)$$

$$C^T = -C\ ;\ C^2 = -\mathbb{1}$$

$$A \in Usp(4) \times Usp(4) \subset SU(8)\ \leftrightarrow\ A^\dagger\varpi A = \varpi \text{ and } A^\dagger\Omega A = \Omega$$

$$\tau^{\pm}_{AB} \equiv \frac{1}{2}(\varpi_{AB} \pm \Omega_{AB}) = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & \pm C \end{array}\right)$$

$$\pi_{AB} = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & 0 \end{array}\right) \quad ; \quad \Pi_{AB} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1} \end{array}\right)$$

$$\pi_{AB} = -\varpi_{AC}\varpi_{CB}\ ;\ \Pi_{AB} = -\Omega_{AC}\Omega_{CB}$$

$$\mathbf{28} \overset{Usp(4)\times Usp(4)}{\Longrightarrow} (\mathbf{1},\mathbf{0}) \oplus (\mathbf{0},\mathbf{1}) \oplus (\mathbf{5},\mathbf{0}) \oplus (\mathbf{0},\mathbf{5}) \oplus (\mathbf{4},\mathbf{4})$$

$$\begin{aligned}
\mathbb{P}^{(1,0)}_{AB\ RS} &= \frac{1}{4}\varpi_{AB}\varpi_{RS} \\
\mathbb{P}^{(0,1)}_{AB\ RS} &= \frac{1}{4}\Omega_{AB}\Omega_{RS} \\
\mathbb{P}^{(5,0)}_{AB\ RS} &= \frac{1}{2}(\pi_{AR}\pi_{BS} - \pi_{AS}\pi_{BR}) - \frac{1}{4}\varpi_{AB}\varpi_{RS} \\
\mathbb{P}^{(0,5)}_{AB\ RS} &= \frac{1}{2}(\Pi_{AR}\Pi_{BS} - \Pi_{AS}\Pi_{BR}) - \frac{1}{4}\Omega_{AB}\Omega_{RS} \\
\mathbb{P}^{(4,4)}_{AB\ RS} &= \frac{1}{8}(\pi_{AR}\Pi_{BS} + \Pi_{AR}\pi_{BS} - \pi_{AS}\Pi_{BR} - \Pi_{AS}\pi_{BR})
\end{aligned}$$

$$
\begin{aligned}
\ell^{AB}_{RS} &\equiv \frac{1}{2}(\pi_{AR}\pi_{BS} - \pi_{AS}\pi_{BR}) \\
L^{AB}_{RS} &\equiv \frac{1}{2}(\Pi_{AR}\Pi_{BS} - \Pi_{AS}\Pi_{BR}) \\
U^{ABCD} &\equiv \varpi^{[AB}\varpi^{CD]} = \frac{1}{3}[\varpi^{AB}\varpi^{CD} + \varpi^{AC}\varpi^{DB} + \varpi^{AD}\varpi^{BC}] \\
W^{ABCD} &\equiv \Omega^{[AB}\Omega^{CD]} = \frac{1}{3}[\Omega^{AB}\Omega^{CD} + \Omega^{AC}\Omega^{DB} + \Omega^{AD}\Omega^{BC}] \\
Z^{ABCD} &\equiv \varpi^{[AB}\Omega^{CD]} = \frac{1}{6}[\varpi^{AB}\Omega^{CD} + \varpi^{AC}\Omega^{DB}\varpi^{AD}\Omega^{BC} \\
&\quad + \Omega^{AB}\varpi^{CD} + \Omega^{AC}\varpi^{DB} + \Omega^{AD}\varpi^{BC}]
\end{aligned}
$$

$$
\begin{aligned}
Z_{ABRS}\, Z_{RSUV} &= \frac{4}{9}\left(\mathbb{P}^{(1,0)}_{AB\ UV} + \mathbb{P}^{(0,1)}_{AB\ UV} + \mathbb{P}^{(4,4)}_{AB\ UV}\right) \\
U_{ABRS}\, U_{RSUV} &= \frac{4}{9}\,\ell^{AB}_{UV} \\
W_{ABRS}\, W_{RSUV} &= \frac{4}{9}\,L^{AB}_{UV}
\end{aligned}
$$

$$
SL(2,\mathbb{R}) \text{ algebra} \longrightarrow \begin{cases} [L_+\,,\, L_-] = 2\,L_0 \\ [L_0\,,\, L_\pm] = \pm\, L_\pm \end{cases}
$$

$$
O(1,1) \text{ algebra} \longrightarrow \mathcal{C}
$$

$$
\begin{aligned}
L_0 &= \left(\begin{array}{c|c} 0 & \frac{3}{4}\left(U^{ABFG} + W^{ABFG}\right) \\ \hline \frac{3}{4}\left(U_{LMCD} + W_{LMCD}\right) & 0 \end{array}\right) \\
L_\pm &= \left(\begin{array}{c|c} \pm\frac{\mathrm{i}}{2}\left(\ell^{AB}_{CD} - L^{AB}_{CD}\right) & \mathrm{i}\,\frac{3}{4}\left(U^{ABFG} - W^{ABFG}\right) \\ \hline -\mathrm{i}\,\frac{3}{4}\left(U_{LMCD} - W_{LMCD}\right) & \mp\frac{\mathrm{i}}{2}\left(\ell^{LM}_{FG} - L^{LM}_{FG}\right) \end{array}\right) \\
\mathcal{C} &= \left(\begin{array}{c|c} 0 & \frac{3}{4}Z^{ABFG} \\ \hline \frac{3}{4}Z_{LMCD} & 0 \end{array}\right)
\end{aligned}
$$

$$
\mathbb{L}(\Phi_1, \Phi_2) \equiv \exp\left[\Phi_1\mathcal{C} + \Phi_2 L_0\right]
$$

$$\mathbb{P}^{(1,0)}_{AB\ RS}\,\frac{3}{4}\,Z^{RSUV}\,\mathbb{P}^{(1,0)}_{UV\ PQ} = 0$$

$$\mathbb{P}^{(0,1)}_{AB\ RS}\,\frac{3}{4}\,Z^{RSUV}\,\mathbb{P}^{(0,1)}_{UV\ PQ} = 0$$

$$\mathbb{P}^{(1,0)}_{AB\ RS}\,\frac{3}{4}\,Z^{RSUV}\,\mathbb{P}^{(0,1)}_{UV\ PQ} = \frac{1}{8}\varpi_{AB}\,\Omega_{PQ}$$

$$\mathbb{P}^{(0,1)}_{AB\ RS}\,\frac{3}{4}\,Z^{RSUV}\,\mathbb{P}^{(1,0)}_{UV\ PQ} = \frac{1}{8}\Omega_{AB}\,\varpi_{PQ}$$

$$\mathbb{P}^{(1,0)}_{AB\ RS}\,\frac{3}{4}\,\left(U^{RSUV}+W^{RSUV}\right)\,\mathbb{P}^{(1,0)}_{UV\ PQ} = \frac{1}{2}\mathbb{P}^{(1,0)}_{AB\ PQ}$$

$$\mathbb{P}^{(1,0)}_{AB\ RS}\,\frac{3}{4}\,\left(U^{RSUV}+W^{RSUV}\right)\,\mathbb{P}^{(0,1)}_{UV\ PQ} = 0$$

$$\mathbb{P}^{(0,1)}_{AB\ RS}\,\frac{3}{4}\,\left(U^{RSUV}+W^{RSUV}\right)\,\mathbb{P}^{(0,1)}_{UV\ PQ} = \frac{1}{2}\mathbb{P}^{(0,1)}_{AB\ PQ}$$

$$\mathbb{P}^{(0,1)}_{AB\ RS}\,\frac{3}{4}\,Z^{RSUV}\,\mathbb{P}^{(1,0)}_{UV|PQ} = 0$$

$$\Phi^{\pm} = \frac{\Phi_2 \pm \Phi_1}{2}$$

$$\exp\left[\Phi_1\mathcal{C}+\Phi_2 L_0\right] =$$
$$\left(\begin{array}{c|c} \cosh\Phi^{+}\,\frac{1}{2}\,\tau^{+}_{AB}\,\tau^{+}_{CD}+\cosh\Phi^{-}\,\frac{1}{2}\,\tau^{-}_{AB}\,\tau^{-}_{CD} & \sinh\Phi^{+}\,\frac{1}{2}\,\tau^{+}_{AB}\,\tau^{+}_{CD}+\sinh\Phi^{-}\,\frac{1}{2}\,\tau^{-}_{AB}\,\tau^{-}_{CD} \\ \hline \sinh\Phi^{+}\,\frac{1}{2}\,\tau^{+}_{AB}\,\tau^{+}_{CD}+\sinh\Phi^{-}\,\frac{1}{2}\,\tau^{-}_{AB}\,\tau^{-}_{CD} & \cosh\Phi^{+}\,\frac{1}{2}\,\tau^{+}_{AB}\,\tau^{+}_{CD}+\cosh\Phi^{-}\,\frac{1}{2}\,\tau^{-}_{AB}\,\tau^{-}_{CD} \end{array}\right)$$

$$f_{ABCD} = \frac{1}{\sqrt{2}}\left(\exp\left[\Phi^{+}\right]\frac{1}{2}\tau^{+}_{AB}\tau^{+}_{CD}+\exp\left[\Phi^{-}\right]\frac{1}{2}\tau^{-}_{AB}\tau^{-}_{CD}\right)$$

$$h_{ABCD} = -\frac{\mathrm{i}}{\sqrt{2}}\left(\exp\left[-\Phi^{+}\right]\frac{1}{2}\tau^{+}_{AB}\tau^{+}_{CD}+\exp\left[-\Phi^{-}\right]\frac{1}{2}\tau^{-}_{AB}\tau^{-}_{CD}\right)$$

$$\mathcal{N}_{ABCD} = -\frac{\mathrm{i}}{4}\left(\exp\left[-2\Phi^{+}\right]\frac{1}{2}\tau^{+}_{AB}\tau^{+}_{CD}+\exp\left[-2\Phi^{-}\right]\frac{1}{2}\tau^{-}_{AB}\tau^{-}_{CD}\right)$$

$$\mathbb{L}^{-1}(\Phi_1,\Phi_2)\,d\,\mathbb{L}(\Phi_1,\Phi_2) =$$
$$\frac{3}{4}\left(\begin{array}{c|c} 0 & d\Phi_1\,Z^{ABFG}+d\Phi_2\,\left(U^{ABFG}+W^{ABFG}\right) \\ \hline d\Phi_1\,Z^{LMCD}+d\Phi_2\,\left(U^{LMCD}+W^{LMCD}\right) & 0 \end{array}\right)$$

$$P^{ABCD} = d\Phi_1\frac{3}{4}Z^{ABCD}+d\Phi_2\frac{3}{4}\left(U^{ABCD}+W^{ABCD}\right)$$

$$P^{ABCD}_{\mu}P^{\mu}_{ABCD} = \frac{3}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1+3\partial_\mu\Phi_2\partial^\mu\Phi_2$$

$$A_\mu^{AB} = \tau_{AB}^{+} \frac{1}{2\sqrt{2}} \mathcal{A}_\mu^1 + \tau_{AB}^{-} \frac{1}{2\sqrt{2}} \mathcal{A}_\mu^2$$

$$\mathcal{L}_{\text{red}}^{1/4} = \sqrt{-g}\Big[2R[g] + \frac{1}{4}\partial_\mu \Phi_1 \partial^\mu \Phi_1 + \frac{1}{2}\partial_\mu \Phi_2 \partial^\mu \Phi_2$$
$$-\exp\,[-\Phi_1 - \Phi_2]\big(F_{\mu\nu}^1\big)^2 - \exp\,[\Phi_1 - \Phi_2]\big(F_{\mu\nu}^2\big)^2$$

$$\Phi_1 = \sqrt{2}h_1\,;\ \Phi_2 = h_2$$

$$\mathcal{L}_{\text{red}}^{1/4} = \sqrt{-g}\Big[2R[g] + \frac{1}{2}\partial_\mu h_1 \partial^\mu h_1 + \frac{1}{2}\partial_\mu h_2 \partial^\mu h_2$$
$$-\exp\,[-\sqrt{2}h_1 - h_2]\big(F_{\mu\nu}^1\big)^2 - \exp\,[\sqrt{2}h_1 - h_2]\big(F_{\mu\nu}^2\big)^2$$

$$U'' + \frac{2}{r}U' - (U')^2 = \frac{1}{4}(h_1')^2 + \frac{1}{4}(h_2')^2$$

$$\sqrt{2}h_1' - h_2' = -\frac{\ell_2'}{\ell_2}$$
$$-\sqrt{2}h_1' - h_2' = -\frac{\ell_1'}{\ell_1},$$

$$h_1'' + \frac{2}{r}h_1' = \frac{1}{\sqrt{2}r^4}\Big(e^{\sqrt{2}h_1 - h_2 + 2U}\ell_2^2 - e^{-\sqrt{2}h_1 - h_2 + 2U}\ell_1^2\Big)$$
$$h_2'' + \frac{2}{r}h_2' = \frac{1}{2r^4}\Big(e^{\sqrt{2}h_1 - h_2 + 2U}\ell_2^2 + e^{-\sqrt{2}h_1 - h_2 + 2U}\ell_1^2\Big)$$

$$\ell_1(r) = \ell_1 e^{\sqrt{2}h_1 + h_2}$$
$$\ell_2(r) = \ell_2 e^{-\sqrt{2}h_1 + h_2}$$

$$\ell_1 \equiv \ell_1(\infty)$$
$$\ell_2 \equiv \ell_2(\infty)$$
$$h_1 = -\frac{1}{\sqrt{2}}\log\,\frac{H_1}{H_2}$$
$$h_2 = -\frac{1}{2}\log\,H_1 H_2$$
$$U = -\frac{1}{4}\ln\,H_1 H_2$$

$$H_1(r) = 1 + \sum_i \frac{k_i^1}{\vec{x} - \vec{x}_i^{(1)}}$$
$$H_2(r) = 1 + \sum_i \frac{l_i^2}{\vec{x} - \vec{x}_i^{(2)}}$$

$$ds^2 = (H_1H_2)^{-1/2}dt^2 - (H_1H_2)^{1/2}(dr^2 + r^2 d\Omega_2)$$
$$h_1 = -\frac{1}{\sqrt{2}}\log\frac{H_1}{H_2}$$
$$h_2 = -\frac{1}{2}\log[H_1H_2]$$
$$F^{1,2} = -dt \wedge \overrightarrow{dx} \cdot \frac{\vec{\partial}}{\partial x}\left(H_{1,2}\right)^{-1}$$

$$\frac{16}{3}P_2\frac{d\Phi_2}{dr} \pm P_1\frac{d\Phi_1}{dr} = -2\ell(r)^{\Lambda\Sigma}\tau^{(\pm)AB}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f_{AB}^{\Gamma\Delta}\frac{e^U}{r^2}$$

$$P_2 d\Phi_2 = \frac{3}{8}C^{ab}C^{cd}P_{abcd} = \frac{3}{8}U^{ABCD}P_{ABCD} = \frac{3}{4}d\Phi_2$$
$$P_1 d\Phi_1 = C^{XY}C^{ab}P_{XYab} = Z^{ABCD}P_{ABCD} = 2d\Phi_1$$

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}f_{AB}^{\Gamma\Delta} = -\frac{1}{2\sqrt{2}}\left(e^{-\Phi_+}\tau^{(+)\Lambda\Sigma}\tau^{(+)AB} + e^{-\Phi_-}\tau^{(-)\Lambda\Sigma}\tau^{(-)AB}\right)$$

$$4\frac{d\Phi_2}{dr} \pm 2\frac{d\Phi_1}{dr} = 4\sqrt{2}\tau_{\Lambda\Sigma}^{(\pm)}\ell^{\Lambda\Sigma}\frac{e^{U-\Phi_\pm}}{r^2}$$

$$\tau_{\Lambda\Sigma}^{(+)}q^{\Lambda\Sigma} = \frac{1}{\sqrt{2}}\ell_1(r)$$
$$\tau_{\Lambda\Sigma}^{(-)}q^{\Lambda\Sigma} = \frac{1}{\sqrt{2}}\ell_2(r)$$

$$\ell_1 = -r^2\frac{H_1'}{H_1^2}\sqrt{H_1H_2}$$
$$\ell_2 = -r^2\frac{H_2'}{H_2^2}\sqrt{H_1H_2}$$

$$\mathcal{M} = E_{7(7)}/SU(8)$$

$$\mathcal{M}_{T^6/\mathbb{Z}_3} = [SU(3,3)/SU(3)\times U(3)] \times \mathcal{M}_{\mathrm{Quat}}$$

$$M_{ADM} = \left|Z\left(a_i, b_i, p^\Lambda, q_\Lambda\right)\right|.$$

$$|Z|_{\min}\left(p^\Lambda, q_\Lambda\right) = \left|Z\left(a_i^{fix}, b_i^{fix}, p^\Lambda, q_\Lambda\right)\right|$$
$$0 = \frac{\partial}{\partial a_i}|Z|_{a=b=fixed} = \frac{\partial}{\partial a_i}|Z|_{a=b=fixed}$$

$$\mathcal{M}_{T_6/Z_3} = SU(3,3)/SU(3)\times U(3)$$

$$H_{\mathrm{matter}} = [SO(2)]^2 = \{\tilde{g}_1 - \tilde{g}_2, \tilde{g}_1 - \tilde{g}_3\}$$

$$\mathbb{K} = \mathrm{Solv}_{STU} + \mathrm{Solv}_{STU}^{\dagger}[SL(2,\mathbb{R})]^3$$

$$\underbrace{\{\mathbb{K}^1,\mathbb{K}^2,\mathbb{K}^3,\mathbb{K}^4,\mathbb{K}^5,\mathbb{K}^6\}}_{\{\mathbb{K}^{\widehat{\alpha}}\}} \quad \leftrightarrow \quad \underbrace{\{|v_x^1\rangle,|v_y^2\rangle,|v_y^3\rangle,|v_y^1\rangle,|v_x^2\rangle,|v_x^3\rangle\}}_{\{|v^{\widehat{\alpha}}\rangle\}}$$

$$\{\phi^\alpha\} \equiv \{-2a_1,-2a_2,-2a_3,\log(-b_1),\log(-b_2),\log(-b_3),\}$$

$$\begin{aligned}\mathbb{L}(a_i,b_i) &= \exp(T_\alpha\phi^\alpha) = \\ &(1-2a_1g_1)\cdot(1-2a_2g_2)\cdot(1-2a_3g_3)\cdot\exp\left(\sum_i \log(-b_i)h_i\right) \\ \mathbb{P}^{\widehat{\alpha}} &= \frac{1}{2\sqrt{2}}\mathrm{Tr}\left(\mathbb{K}^{\widehat{\alpha}}\mathbb{L}^{-1}d\mathbb{L}\right) = \{-\frac{da_1}{2b_1},-\frac{da_2}{2b_2},-\frac{da_3}{2b_3},\frac{db_1}{2b_1},\frac{db_2}{2b_2},\frac{db_3}{2b_3}\}\end{aligned}$$

$$\begin{aligned}\left(\mathbb{P}^{\alpha}_{\widehat{\alpha}}\langle v^{\widehat{\alpha}}|\mathbb{L}^t\mathbb{C}\mathbf{M}\right) &= \sqrt{2}\begin{pmatrix}\mathrm{Re}(g^{ij^\star}(\overline{h}_{j^\star|\Lambda})),-\mathrm{Re}(g^{ij^\star}(\overline{f}^{\Sigma}_{j^\star})) \\ \mathrm{Im}(g^{ij^\star}(\overline{h}_{j^\star|\Lambda})),-\mathrm{Im}(g^{ij^\star}(\overline{f}^{\Sigma}_{j^\star}))\end{pmatrix} \\ \begin{pmatrix}\langle v_y^0|\mathbb{L}^t\mathbb{C}\mathbf{M} \\ \langle v_x^0|\mathbb{L}^t\mathbb{C}\mathbf{M}\end{pmatrix} &= \sqrt{2}\begin{pmatrix}\mathrm{Re}(M_\Lambda),-\mathrm{Re}(L^\Sigma) \\ \mathrm{Im}(M_\Lambda),-\mathrm{Im}(L^\Sigma)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}|\vec{Q}\rangle_{Sp(8)_D} &= \mathbf{M}\cdot|\vec{Q}\rangle_{sc} \\ \mathbf{M} &= \begin{pmatrix}0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&-1&0&0\\0&0&-1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&-1&0&0&0&0\\0&1&0&0&0&0&0&0\end{pmatrix} \in Sp(8,\mathbb{R})\end{aligned}$$

$$\begin{aligned}\frac{d\phi^\alpha}{dr} &= \left(\mp\frac{e^U}{r^2}\right)\frac{1}{2\sqrt{2\pi}}\mathbb{P}^{\alpha}_{\widehat{\alpha}}\langle v^{\widehat{\alpha}}|\mathbb{L}^t\mathbb{C}\mathbf{M}|\vec{Q}\rangle_{sc} \\ \frac{dU}{dr} &= \left(\mp\frac{e^U}{r^2}\right)\frac{1}{2\sqrt{2\pi}}\langle v_y^0|\mathbb{L}^t\mathbb{C}\mathbf{M}|\vec{Q}\rangle_{sc} \\ 0 &= \langle v_x^0|\mathbb{L}^t\mathbb{C}\mathbf{M}|t\rangle_{sc}\end{aligned}$$

$$(a_1+\mathrm{i}b_1)_{fix}=\frac{p^\Lambda q_\Lambda-2p^1q_1-\mathrm{i}\sqrt{f(p,q)}}{2p^2p^3-2p^0q_1}$$
$$(a_2+\mathrm{i}b_2)_{fix}=\frac{p^\Lambda q_\Lambda-2p^2q_2-\mathrm{i}\sqrt{f(p,q)}}{2p^1p^3-2p^0q_2}$$
$$(a_3+\mathrm{i}b_3)_{fix}=\frac{p^\Lambda q_\Lambda-2p^3q_3-\mathrm{i}\sqrt{f(p,q)}}{2p^1p^2-2p^0q_3}$$

$$S^2(p,q)=f(p,q)=-(p^0q_0-p^1q_1+p^2q_2+p^3q_3)^2+4(p^2p^3-p^0q_1)(p^1q_0+q_2q_3)$$

$$\{z^i\}=\{S,T,U\},\Omega(z)=\begin{pmatrix}X^\Lambda(z)\\F_\Sigma(z)\end{pmatrix}$$
$$X^\Lambda(z)=\begin{pmatrix}1\\S\\T\\U\end{pmatrix}$$
$$F_\Sigma(z)=\partial_\Sigma F(X)$$
$$\mathcal{K}(z,\bar z)=-\log(8|\mathrm{Im}S\,\mathrm{Im}T\,\mathrm{Im}U|)$$
$$g_{ij^\star}(z,\bar z)=\partial_i\partial_{j^\star}\mathcal{K}(z,\bar z)=\mathrm{diag}\{-(\bar S-S)^{-2},-(\bar T-T)^{-2},-(\bar U-U)^{-2}\}$$
$$\mathcal{N}_{\Lambda\Sigma}=\bar F_{\Lambda\Sigma}+2\mathrm{i}\frac{\mathrm{Im}F_{\Lambda\Omega}\mathrm{Im}F_{\Sigma\Pi}L^\Omega L^\Pi}{L^\Omega L^\Pi\mathrm{Im}F_{\Omega\Pi}}$$
$$F_{\Lambda\Sigma}(z)=\partial_\Lambda\partial_\Sigma F(X)$$
$$F(X)=\frac{X^1X^2X^3}{X^0}$$

$$V(z,\bar z)=\begin{pmatrix}L^\Lambda(z,\bar z)\\M_\Sigma(z,\bar z)\end{pmatrix}=e^{\mathcal{K}(z,\bar z)/2}\Omega(z,\bar z)$$
$$U_i(z,\bar z)=\begin{pmatrix}f_i^\Lambda(z,\bar z)\\h_{i|\Sigma}(z,\bar z)\end{pmatrix}=\nabla_iV(z,\bar z)=\left(\partial_i+\frac{\partial_i\mathcal{K}}{2}\right)V(z,\bar z)$$
$$\bar U_{i^\star}(z,\bar z)=\begin{pmatrix}\bar f_{i^\star}^\Lambda(z,\bar z)\\\bar h_{i^\star|\Sigma}(z,\bar z)\end{pmatrix}=\nabla_{i^\star}\bar V(z,\bar z)=\left(\partial_{i^\star}+\frac{\partial_{i^\star}\mathcal{K}}{2}\right)\bar V(z,\bar z)$$
$$M_\Sigma(z,\bar z)=\mathcal{N}_{\Sigma\Lambda}(z,\bar z)L^\Lambda(z,\bar z)$$
$$h_{i|\Sigma}(z,\bar z)=\overline{\mathcal{N}}_{\Sigma\Lambda}(z,\bar z)f_i^\Lambda(z,\bar z)$$

$$\mathrm{Re}\mathcal{N}=\begin{pmatrix}2a1a2a3&-(a2a3)&-(a1a3)&-(a1a2)\\-(a2a3)&0&a3&a2\\-(a1a3)&a3&0&a1\\-(a1a2)&a2&a1&0\end{pmatrix}$$
$$\mathrm{Im}\mathcal{N}=\begin{pmatrix}\frac{a1^2b2b3}{b1}+\frac{b1(a3^2b2^2+(a2^2+b2^2)b3^2)}{b2b3}&-\frac{a1b2b3}{b1}&-\frac{a2b1b3}{b2}&-\frac{a3b1b2}{b3}\\-\frac{a1b2b3}{b1}&\frac{b2b3}{b1}&0&0\\-\frac{a2b1b3}{b2}&0&\frac{b1b3}{b2}&0\\-\frac{a3b1b2}{b3}&0&0&\frac{b1b2}{b3}\end{pmatrix}$$

$$\frac{dS}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\sqrt{\left|\frac{\mathrm{Im}(S)}{2\mathrm{Im}(T)\mathrm{Im}(U)}\right|}\left[q_0 + \bar{U}q_3 - \bar{U}p^2S + q_1S + \bar{T}(-(\bar{U}p1) + q_2 + \bar{U}p^0S - p^3S)\right]$$

$$\frac{dT}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\sqrt{\left|\frac{\mathrm{Im}(T)}{2\mathrm{Im}(S)\mathrm{Im}(U)}\right|}[q_0 + \bar{U}q_3 - \bar{U}p^1T + q_2T$$
$$+\bar{S}(-(\bar{U}p^2) + q_1 + \bar{U}p^0T - p^3T)]$$

$$\frac{dU}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\sqrt{\left|\frac{\mathrm{Im}(U)}{2\mathrm{Im}(S)\mathrm{Im}(T)}\right|}[q_0 + \bar{t}q_2 - \bar{T}p^1U + q_3U$$
$$+\bar{S}(-(\bar{T}p^3) + q_1 + \bar{T}p^0U - p^2U)]$$

$$\frac{d\mathcal{U}}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\left(\frac{1}{2\sqrt{2}(|\mathrm{Im}(S)\mathrm{Im}(T)\mathrm{Im}(U)|)^{1/2}}\right)[q_0 + S(TUp^0 - Up^2 - Tp^3 + q_1)$$
$$+T(-(Up^1) + q_2) + Uq_3]$$

$$Z(z,\bar{z},p,q) = -\left(\frac{1}{2\sqrt{2}(|\mathrm{Im}(S)\mathrm{Im}(T)\mathrm{Im}(U)|)^{1/2}}\right)[q_0 + S(TUp^0 - Up^2 - Tp^3 + q_1) +$$
$$T(-(Up^1) + q_2) + Uq_3]$$

$$\partial_\mu(\sqrt{-g}\tilde{G}^{\mu\nu}) = 0$$
$$\partial_\mu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 0$$

$$-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}g_{ij^\star}\partial_\nu\overline{z}^{j^\star}\right) + \partial_i(g_{kj^\star})\partial_\mu z^k\partial_\nu\overline{z}^{j^\star}g^{\mu\nu} +$$
$$(\partial_i\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}F^{\Sigma|\cdot\cdot} + (\partial_i\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}\widetilde{F}^{\Sigma|\cdot\cdot} = 0$$

$$\frac{e^{2\mathcal{U}}}{4b_i^2}\left(a_i'' + 2\frac{a_i'}{r} - 2\frac{a_i'b_i'}{b_i}\right) = -\frac{1}{2}\left((\partial_{a_i}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}F^{\Sigma|\cdot\cdot} + (\partial_{a_i}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}\widetilde{F}^{\Sigma|\cdot\cdot}\right)$$
$$\frac{e^{2\mathcal{U}}}{4b_i^2}\left(b_i'' + 2\frac{b_i'}{r} + \frac{(a_i'^2 - b_i'^2)}{b_i}\right) = -\frac{1}{2}\left((\partial_{b_i}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}F^{\Sigma|\cdot\cdot} + (\partial_{b_i}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^{\Lambda}_{\cdot\cdot}\widetilde{F}^{\Sigma|\cdot\cdot}\right)$$

$$R_{MN} = -g_{ij^\star}\partial_M z^i \partial_N \bar{z}^{j^\star} + S_{MN}$$
$$S_{MN} = -2\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\left(F^{\Lambda}_{M.}F^{\Sigma|\cdot}_{N} - \frac{1}{4}g_{MN}F^{\Lambda}_{..}F^{\Sigma|..}\right)$$
$$-2\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\left(F^{\Lambda}_{M.}\widetilde{F}^{\Sigma|\cdot}_{N} - \frac{1}{4}g_{MN}F^{\Lambda}_{..}\widetilde{F}^{\Sigma|..}\right)$$

$$\mathcal{U}'' + \frac{2}{r}\mathcal{U}' = -2e^{-2\mathcal{U}}S_{\underline{00}}$$
$$(\mathcal{U}')^2 + \sum_i \frac{1}{4b_i^2}\left((b_i')^2 + (a_i')^2\right) = -2e^{-2\mathcal{U}}S_{\underline{00}}$$

$$S_{\underline{00}} = -\frac{2e^{4\mathcal{U}}}{(8\pi)^2 r^4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\left(p^\Lambda p^\Sigma + \ell(r)^\Lambda \ell(r)^\Sigma\right)$$

$$\ell^\Lambda(r) = \mathrm{Im}\mathcal{N}^{-1|\Lambda\Sigma}\left(q_\Sigma - \mathrm{Re}\mathcal{N}_{\Sigma\Omega}p^\Omega\right)$$

$$\frac{da}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\frac{1}{\sqrt{-2b}}\left(bq - 2abp + (a^2 b + b^3)p^0\right)$$
$$\frac{db}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\frac{1}{\sqrt{-2b}}\left(3aq - (3a^2 + b^2)p + (a^3 + ab^2)p^0 + q_0\right)$$
$$\frac{d\mathcal{U}}{dr} = \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\left(\frac{1}{2\sqrt{2}(-b)^{3/2}}\right)\left(3aq - (3a^2 - 3b^2)p + (a^3 - 3ab^2)p^0 + q_0\right)$$
$$0 = 3bq - 6abp + (3a^2 b - b^3)p^0$$

$$a_{fix} = \frac{pq + p^0 q_0}{2p^2 - 2p^0 q}$$
$$b_{fix} = -\frac{\sqrt{f(p,q,p^0,q_0)}}{2(p^2 - p^0 q)}$$
where $f(p,q,p^0,q_0) = 3p^2q^2 + 4p^3q_0 - 6pp^0qq_0 - p^0(4q^3 + p^0q_0^2)$

$$Z_{fix}(p,q,p^0,q_0) = Z\left(a_{fix}, b_{fix}, p, q, p^0, q_0\right)$$

$$Z_{fix}(p,q,p^0,q_0) = |Z_{fix}|e^{\theta}$$
$$|Z_{fix}(p,q,p^0,q_0)| = f(p,q,p^0,q_0)^{1/4}$$
$$\sin\theta = \frac{p^0 f(p,q,p^0,q_0)^{1/2}}{2(p^2 - qp^0)^{3/2}}$$
$$\cos\theta = \frac{-2p^3 + 3pp^0q + p^{0^2}q_0}{2(p^2 - p^0q)^{3/2}}$$

$$S^2(p,q,p^0,q_0) = \left|Z_{fix}(p,q,p^0,q_0)\right|^4 = f(p,q,p^0,q_0)$$

$$a_{fix}= \frac{q}{2p}$$

$$b_{fix}= -\frac{\sqrt{3q^2+4q_0p}}{2p}$$

$$I_4 = (3q^2p^2+4q_0p^3)$$

$$\frac{db}{dr}= \pm\frac{e^{\mathcal{U}}}{r^2\sqrt{2b}}\left(q_0+\frac{3q^2}{4p}-b^2p\right)$$

$$\frac{d\mathcal{U}}{dr} = \pm\frac{e^{\mathcal{U}}}{r^2(2b)^{3/2}}\left(q_0+\frac{3q^2}{4p}+3b^2p\right)$$

$$b(r)= -\sqrt{\frac{(A_1+k_1/r)}{(A_2+k_2/r)}}$$

$$e^{\mathcal{U}}= \left(\left(A_2+\frac{k_2}{r}\right)^3(A_1+k_1/r)\right)^{-1/4}$$

$$k_1= \pm\frac{\sqrt{2}(3q^2+4q_0p)}{4p}$$

$$k_2 = \pm\sqrt{2}p$$

$$b(r)\to -\left(\frac{k_1}{k_2}\right)^{1/2} = b_{fix}$$

$$e^{\mathcal{U}(r)}\to r(k_1k_2^3)^{-1/4} = rf^{-1/4}$$

$$a= a_{fix} = \frac{q}{2p}$$

$$b= -\sqrt{\frac{(1+k_1/r)}{(1+k_2/r)}}$$

$$e^{\mathcal{U}} = [(1+k_1/r)(1+k_2/r)^3]^{-1/4}$$

$$\mathrm{Re}\mathcal{N} = \begin{pmatrix} 2a^3 & -a^2 & -a^2 & -a^2 \\ -a^2 & 0 & a & a \\ -a^2 & a & 0 & a \\ -a^2 & a & a & 0 \end{pmatrix}, \mathrm{Im}\mathcal{N} = \begin{pmatrix} 3a^2b+b^3 & -(ab) & -(ab) & -(ab) \\ -(ab) & b & 0 & 0 \\ -(ab) & 0 & b & 0 \\ -(ab) & 0 & 0 & b \end{pmatrix}$$

$$\ell^{\Lambda}(r) = \begin{pmatrix} \frac{-3a^2p+3aq+q_0}{b^3} \\ \frac{-3a^3p+b^2q+3a^2q+a(-2b^2p+q_0)}{b^3} \\ \frac{-3a^3p+b^2q+3a^2q+a(-2b^2p+q_0)}{b^3} \\ \frac{-3a^3p+a^4p^0+b^2q+3a^2q+a(-2b^2p+q_0)}{b^3} \end{pmatrix}$$

$$\left(a'' - 2\frac{a'b'}{b} + 2\frac{a'}{r}\right) = 0$$

$$\left(b'' + 2\frac{b'}{r} + \frac{(a'^2 - b'^2)}{b}\right) = -\frac{b^2 e^{2\mathcal{U}}\left(p^2 - \frac{(-3a^2p + 3aq + q_0)^2}{b^6}\right)}{r^4}$$

$$\frac{(k_2 - k_1)e^{4\mathcal{U}}\left(k_1 + k_2 + \frac{2k_1k_2}{r}\right)}{2br^4}$$

$$\mathcal{U}'' + \frac{2}{r}\mathcal{U}' = (\mathcal{U}')^2 + \frac{3}{4b^2}((b')^2 + (a')^2)$$

$$\mathcal{U}'' + \frac{2}{r}\mathcal{U}' = -2e^{-2\mathcal{U}}\underline{S_0}$$

$$\frac{3(k_2 - k_1)^2}{16r^4(H_1)^2(H_2)^2}$$

$$S_{BH} = \frac{1}{4\pi}(3p^2q^2 + 4q_0p^3)^{1/2}$$

$$\frac{da_1}{dr} = \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[-b_1q_1 + b_2q_2 + b_3q_3 + (-(a_2a_3b_1) + a_1a_3b_2 + a_1a_2b_3 + b_1b_2b_3)p^0 + +(-(a_3b_2) - a_2b_3)p^1 + (a_3b_1 - a_1b_3)p^2 + (a_2b_1 - a_1b_2)p^3]$$

$$\frac{db_1}{dr} = \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[a_1q_1 + a_2q_2 + a_3q_3 + (a_1a_2a_3 + a_3b_1b_2 + a_2b_1b_3 - a_1b_2b_3)p^0 + +(-(a_2a_3) + b_2b_3)p^1 - (a_1a_3 + b_1b_3)p^2 - (a_1a_2 + b_1b_2)p^3 + q_0]$$

$$\frac{da_2}{dr} = (1,2,3) \rightarrow (2,1,3)$$

$$\frac{db_2}{dr} = (1,2,3) \rightarrow (2,1,3)$$

$$\frac{da_3}{dr} = (1,2,3) \rightarrow (3,2,1)$$

$$\frac{db_3}{dr} = (1,2,3) \rightarrow (3,2,1)$$

$$\frac{d\mathcal{U}}{dr} = \pm\frac{e^{\mathcal{U}(r)}}{r^2}\frac{1}{2\sqrt{2}(-b_1b_2b_3)^{1/2}}[a_1q_1 + a_2q_2 + a_3q_3 + (a_1a_2a_3 - a_3b_1b_2 - a_2b_1b_3 - a_1b_2b_3)p^0 + -(a_2a_3 - b_2b_3)p^1 - (a_1a_3 - b_1b_3)p^2 - (a_1a_2 - b_1b_2)p^3 + q_0]$$

$$0b_1q_1 + b_2q_2 + b_3q_3 + (a_2a_3b_1 + a_1a_3b_2 + a_1a_2b_3 - b_1b_2b_3)p^0 - (a_3b_2 + a_2b_3)p^1 - (a_3b_1 + a_1b_3)p^2 - (a_2b_1 - a_1b_2)p^3$$

$$\left(a_1'' - 2\frac{a_1' b_1'}{b_1} + 2\frac{a_1'}{r}\right) = \frac{-2b_1 e^{2\mathcal{U}}}{r^4}\left[a_1 b_2 b_3\left(p^{1^2} - \ell(r)_1^2\right)\right.$$
$$+b_2(-(b_3 p^1 p^2) + b_3 \ell(r)_1 \ell(r)_2) +$$
$$+b_1(-2a_2 a_3 p^1 \ell(r)_1 + a_3 p^3 \ell(r)_1 + a_2 p^4 \ell(r)_1 + a_3 p^1 \ell(r)_3 +$$
$$\left.-p^4 \ell(r)_3 + a_2 p^1 \ell(r)_4 - p^3 \ell(r)_4)\right]$$

$$\left(b_1'' + 2\frac{b_1'}{r} + \frac{(a_1'^2 - b_1'^2)}{b_1}\right) = -\frac{e^{2\mathcal{U}}}{b_2 b_3 r^4}\left[-\left(a_1^2 b_2^2 b_3^2 p^{1^2}\right) + b_1^2 b_2^2 b_3^2 p^{1^2}\right.$$
$$+2a_1 b_2^2 b_3^2 p^1 p^2 +$$
$$-b_2^2 b_3^2 p^2\ {}^2 b_1^2 b_3^2 p^3\ {}^2 b_1^2 b_2^2 p^{4^2}$$
$$+a_1^2 b_2^2 b_3^2 \ell(r)_1^2 +$$
$$-b_1^2 b_2^2 b_3^2 \ell(r)_1^2 + a_3^2 b_1^2 b_2^2 (p^{12} - \ell(r)_1^2)$$
$$+a_2^2 b_1^2 b_3^2$$
$$\left(p^{1^2} - \ell(r)_1^2\right) - 2a_1 b_2^2 b_3^2 \ell(r)_1 \ell(r)_2 + b_2^2 b_3^2 \ell(r)_2^2 +$$
$$-b_1^2 b_3^2 \ell(r)_3^2 + 2a_2 b_1^2 b_3^2 (-(p^1 p^3) + \ell(r)_1 \ell(r)_3) +$$
$$\left.-b_1^2 b_2^2 \ell(r)_4^2 + 2a_3 b_1^2 b_2^2 (-(p^1 p^4) + \ell(r)_1 \ell(r)_4)\right]$$

$$\left(a_2'' - 2\frac{a_2' b_2'}{b_2} + 2\frac{a_2'}{r}\right) = (1,2,3) \to (2,1,3)$$
$$\left(b_2'' + 2\frac{b_2'}{r} + \frac{(a_2'^2 - b_2'^2)}{b_2}\right) = (1,2,3) \to (2,1,3)$$
$$\left(a_3'' - 2\frac{a_3' b_3'}{b_3} + 2\frac{a_3'}{r}\right) = (1,2,3) \to (3,2,1)$$
$$\left(b_3'' + 2\frac{b_3'}{r} + \frac{(a_3'^2 - b_3'^2)}{b_3}\right) = (1,2,3) \to (3,2,1)$$

$$\mathcal{U}'' + \frac{2}{r}\mathcal{U}' = -2e^{-2\mathcal{U}} S_{00}$$
$$(\mathcal{U}')^2 + \sum_i \frac{1}{4b_i^2}((b_i')^2 + (a_i')^2) = 2e^{-2\mathcal{U}} S_{00}$$

$$S_{00} = \frac{e^{4\mathcal{U}}}{4b_1 b_2 b_3 r^4}\left(a_1^2 b_2^2 b_3^2 p_1^2 + b_1^2 b_2^2 b_3^2 p_1^2 - 2a_1 b_2^2 b_3^2 p_1 p_2 + b_2^2 b_3^2 p_2^2 + b_1^2 b_3^2 p_3^2\right.$$
$$+b_1^2 b_2^2 p_4^2 + a_1^2 b_2^2 b_3^2 \ell(r)_1^2 + b_1^2 b_2^2 b_3^2 \ell(r)_1^2 + a_3^2 b_1^2 b_2^2 (p_1^2 + \ell(r)_1^2)$$
$$+a_2^2 b_1^2 b_3^2 (p_1^{2^2} + \ell(r)_1^2\Big) - 2a_1 b_2^2 b_3^2 \ell(r)_1 \ell(r)_2 + b_2^2 b_3^2 \ell(r)_2^2 + b_1^2 b_3^2 \ell(r)_3^2$$
$$\left.-2a_2 b_1^2 b_3^2 (p_1 p_3 + \ell(r)_1 \ell(r)_3) + b_1^2 b_2^2 \ell(r)_4^2 - 2a_3 b_1^2 b_2^2 (p_1 p_4 + \ell(r)_1 \ell(r)_4)\right)$$

$$\ell_\Lambda(r) = \frac{1}{b_1 b_2 b_3}\begin{pmatrix} [q_1 + a_1(a_2 a_3 p^1 - a_3 p^3 - a_2 p^4 + q_2) \\ +a_2(-(a_3 p^2) + q_3) + a_3 q_4] \\ [a_1{}^2(a_2 a_3 p^1 - a_3 p^3 - a_2 p^4 + q_2) + b_1{}^2(a_2 a_3 p^1 - a_3 p^3 - a_2 p^4 + q_2) \\ +a_1(q_1 + a_2(-(a_3 p^2) + q_3) + a_3 q_4)] \\ \begin{bmatrix} a_1\left(a_2{}^2(a_3 p^1 - p^4) + b_2{}^2(a_3 p^1 - p^4) + a_2(-(a_3 p^3) + q_2)\right) \\ +a_2{}^2(-(a_3 p^2) + q_3) + b_2{}^2(-(a_3 p^2) + q_3) + a_2(q_1 + a_3 q_4) \end{bmatrix} \\ \underline{[a_3 q_1 + a_1(-(a_3{}^2 p^3) - b_3{}^2 p^3 + a_2(a_3{}^2 p^1 + b_3{}^2 p^1 - a_3 p^4) + a_3 q_2)} \\ -a_2(a_3{}^2 p^2 + b_3{}^2 p^2 - a_3 q_3) + a_3{}^2 q_4 + b_3{}^2 q_4] \end{pmatrix}$$

$$
\begin{aligned}
\frac{da_1}{dr} &= \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[-b_1q_1 + b_2q_2 + b_3q_3 + \Big(-(a_2\,a_3\,b_1) \\
&\quad +a_1\,a_3\,b_2 + a_1\,a_2\,b_3 + b_1\,b_2\,b_3\Big)\,p^0 + \\
&\quad +(-(a_3\,b_2) - a_2\,b_3)\,p^1 + (a_3\,b_1 - a_1\,b_3)\,p^2 \\
&\quad +(a_2\,b_1 - a_1\,b_2)\,p^3] \\
\frac{db_1}{dr} &= \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[a_1q_1 + a_2q_2 + a_3q_3 \\
&\quad +(a_1\,a_2\,a_3 + a_3\,b_1\,b_2 + a_2\,b_1\,b_3 - a_1\,b_2\,b_3)\,p^0 + \\
&\quad +(-(a_2\,a_3) + b_2\,b_3)\,p^1 - (a_1\,a_3 + b_1\,b_3)\,p^2 \\
&\quad -(a_1\,a_2 + b_1\,b_2)\,p^3 + q_0] \\
\frac{da_2}{dr} &= (1,2,3) \to (2,1,3) \\
\frac{db_2}{dr} &= (1,2,3) \to (2,1,3) \\
\frac{da_3}{dr} &= (1,2,3) \to (3,2,1) \\
\frac{db_3}{dr} &= (1,2,3) \to (3,2,1) \\
\frac{d\mathcal{U}}{dr} &= \pm\frac{e^{\mathcal{U}(r)}}{r^2}\frac{1}{2\sqrt{2}(-b_1b_2b_3)^{1/2}}[a_1q_1 + a_2q_2 + a_3q_3 \\
&\quad +(a_1\,a_2\,a_3 - a_3\,b_1\,b_2 - a_2\,b_1\,b_3 - a_1\,b_2\,b_3)\,p^0 + \\
&\quad -(a_2\,a_3 - b_2\,b_3)\,p^1 - (a_1\,a_3 - b_1\,b_3)\,p^2 - (a_1\,a_2 - b_1\,b_2)\,p^3 + q_0] \\
0 &= b_1q_1 + b_2q_2 + b_3q_3 + (a_2\,a_3\,b_1 + a_1\,a_3\,b_2 + a_1\,a_2\,b_3 - b_1\,b_2\,b_3)\,p^0 \\
&\quad -(a_3\,b_2 + a_2\,b_3)\,p^1 \\
&\quad -(a_3\,b_1 + a_1\,b_3)\,p^2 - (a_2\,b_1 - a_1\,b_2)\,p^3
\end{aligned}
$$

$$
\begin{aligned}
\left(a_1'' - 2\frac{a_1' b_1'}{b_1} + 2\frac{a_1'}{r}\right) &= \frac{-2\, b_1\, e^{2\mathcal{U}}}{r^4}\,[a_1\, b_2\, b_3\, ({p^1}^2 - {\ell(r)_1}^2) \\
&\quad + b_2\, (-(b_3\, p^1\, p^2) + b_3\, \ell(r)_1\, \ell(r)_2) + \\
&\quad + b_1\, (-2\, a_2\, a_3\, p^1\, \ell(r)_1 + a_3\, p^3\, \ell(r)_1 + a_2\, p^4\, \ell(r)_1 + a_3\, p^1\, \ell(r)_3 + \\
&\quad - p^4\, \ell(r)_3 + a_2\, p^1\, \ell(r)_4 - p^3\, \ell(r)_4)] \\
\left(b_1'' + 2\frac{b_1'}{r} + \frac{(a_1'^2 - b_1'^2)}{b_1}\right) &= -\frac{e^{2\mathcal{U}}}{b_2\, b_3\, r^4}\,[-({a_1}^2\, {b_2}^2\, {b_3}^2\, {p^1}^2) + {b_1}^2\, {b_2}^2\, {b_3}^2\, {p^1}^2 \\
&\quad + 2\, a_1\, {b_2}^2\, {b_3}^2\, p^1\, p^2 + \\
&\quad - {b_2}^2\, {b_3}^2\, {p^2}^2 + {b_1}^2\, {b_3}^2\, {p^3}^2 + {b_1}^2\, {b_2}^2\, {p^4}^2 \\
&\quad + {a_1}^2\, {b_2}^2\, {b_3}^2\, {\ell(r)_1}^2 + \\
&\quad - {b_1}^2\, {b_2}^2\, {b_3}^2\, {\ell(r)_1}^2 + {a_3}^2\, {b_1}^2\, {b_2}^2\, ({p^1}^2 - {\ell(r)_1}^2) \\
&\quad + {a_2}^2\, {b_1}^2\, {b_3}^2 \\
&\quad ({p^1}^2 - {\ell(r)_1}^2) - 2\, a_1\, {b_2}^2\, {b_3}^2\, \ell(r)_1\, \ell(r)_2 + {b_2}^2\, {b_3}^2\, {\ell(r)_2}^2 + \\
&\quad - {b_1}^2\, {b_3}^2\, {\ell(r)_3}^2 + 2\, a_2\, {b_1}^2\, {b_3}^2\, (-(p^1\, p^3) + \ell(r)_1\, \ell(r)_3) + \\
&\quad - {b_1}^2\, {b_2}^2\, {\ell(r)_4}^2 + 2\, a_3\, {b_1}^2\, {b_2}^2\, (-(p^1\, p^4) + \ell(r)_1\, \ell(r)_4)] \\
\left(a_2'' - 2\frac{a_2' b_2'}{b_2} + 2\frac{a_2'}{r}\right) &= (1,2,3) \to (2,1,3) \\
\left(b_2'' + 2\frac{b_2'}{r} + \frac{(a_2'^2 - b_2'^2)}{b_2}\right) &= (1,2,3) \to (2,1,3) \\
\left(a_3'' - 2\frac{a_3' b_3'}{b_3} + 2\frac{a_3'}{r}\right) &= (1,2,3) \to (3,2,1) \\
\left(b_3'' + 2\frac{b_3'}{r} + \frac{(a_3'^2 - b_3'^2)}{b_3}\right) &= (1,2,3) \to (3,2,1)
\end{aligned}
$$

$$
\ell_\Lambda(r) = \frac{1}{b_1\, b_2\, b_3}
\left(\begin{array}{l}
\big[q_1 + a_1\ (a_2\, a_3\, p^1 - a_3\, p^3 - a_2\, p^4 + q_2) \\
\quad + a_2\ (-(a_3\, p^2) + q_3) + a_3\, q_4\big] \\
\hline
\big[{a_1}^2\ (a_2\, a_3\, p^1 - a_3\, p^3 - a_2\, p^4 + q_2) + {b_1}^2\ (a_2\, a_3\, p^1 - a_3\, p^3 - a_2\, p^4 + q_2) \\
\quad + a_1\ (q_1 + a_2\ (-(a_3\, p^2) + q_3) + a_3\, q_4)\big] \\
\hline
\big[a_1\ \left({a_2}^2\ (a_3\, p^1 - p^4) + {b_2}^2\ (a_3\, p^1 - p^4) + a_2\ (-(a_3\, p^3) + q_2)\right) \\
\quad + {a_2}^2\ (-(a_3\, p^2) + q_3) + {b_2}^2\ (-(a_3\, p^2) + q_3) + a_2\ (q_1 + a_3\, q_4)\big] \\
\hline
\big[a_3\, q_1 + a_1\ \left(-({a_3}^2\, p^3) - {b_3}^2\, p^3 + a_2\ \left({a_3}^2\, p^1 + {b_3}^2\, p^1 - a_3\, p^4\right) + a_3\, q_2\right) \\
\quad - a_2\ \left({a_3}^2\, p^2 + {b_3}^2\, p^2 - a_3\, q_3\right) + {a_3}^2\, q_4 + {b_3}^2\, q_4\big]
\end{array}\right)
$$

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{2}|e_{\mathrm{D}}|R,$$

$$\mathcal{L}_{\mathrm{scal}} = \frac{1}{2}|e_{\mathrm{D}}|\mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial_\nu\phi^u g^{\mu\nu} - |e_{\mathrm{D}}|V(\phi),$$

$$\mathcal{L}_{\mathrm{vect}} = \frac{1}{4}|e_{\mathrm{D}}|g^{\mu\rho}g^{\nu\sigma}F^{\Lambda}_{\mu\nu}\mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Sigma}_{\rho\sigma} + \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}\mathcal{R}_{\Lambda\Sigma}(\phi)F^{\Sigma}_{\rho\sigma},$$

$$\phi'^s = g\star\phi = \phi'^s(\phi^u)$$

$$A^{\Lambda}_{\mu} \to A^{\Lambda}_{\mu} + \partial_\mu\zeta^{\Lambda}$$

$$\mathcal{L}_{\text{black particle}} = \mathcal{L}_{\text{white particle}} + \mathcal{L}_{\mathrm{scal}} + \mathcal{L}_{\mathrm{vect}}$$

$$ds^2 = \left(1 - \frac{2r_{\mathrm{M}}}{r} + \frac{r_{\mathrm{Q}}^2}{r^2}\right)dt^2 - \left(1 - \frac{2r_{\mathrm{M}}}{r} + \frac{r_{\mathrm{Q}}^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

$$r_{\mathrm{M}} = \frac{\mathrm{G_N}}{c^2}M;\ r_{\mathrm{Q}}^2 = \frac{\mathrm{G_N}}{4\pi c^4}Q^2$$

$$r_{\pm} = r_{\mathrm{M}} \pm \sqrt{r_{\mathrm{M}}^2 - r_{\mathrm{Q}}^2}\ \text{ if } r_{\mathrm{M}} > r_{\mathrm{Q}}$$

$$\delta M = \frac{\kappa}{8\pi\mathrm{G_N}}\delta A_{\mathrm{H}} + \frac{1}{c^2}\Omega_{\mathrm{H}}\delta\mathcal{J}_{\mathrm{H}} + \Phi\delta Q$$

$$T = \frac{\kappa\hbar}{2\pi k_{\mathrm{B}}c}$$

$$\mathcal{S} = \frac{k_{\mathrm{B}}}{4\ell_{\mathrm{P}}^2}A_{\mathrm{H}}$$

$$M^2 \geq \frac{p^2 + q^2}{2}.$$

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = 2\delta^{AB}P_\mu\Gamma^\mu_{\alpha\beta} + Z^{AB}\delta_{\alpha\beta},$$

$$Z^{AB} = -Z^{BA}, A, B = 1, \ldots, \mathcal{N}.$$

$$M \geq |z_\ell|, \ell = 1, \ldots, \frac{\mathcal{N}}{2},$$

$$\frac{1}{e_{\mathrm{D}}}\mathcal{L}_{(4)} = -\frac{R}{2} + \frac{1}{2}\mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial^\mu\phi^u + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8e_{\mathrm{D}}}\mathcal{R}_{\Lambda\Sigma}(\phi)\varepsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}$$

$$F^{\Lambda}_{\mu\nu} = \partial_\mu A^{\Lambda}_\nu - \partial_\nu A^{\Lambda}_\mu, e_{\mathrm{D}} = \sqrt{|\mathrm{Det}(g_{\mu\nu})|}$$

$$p \in \mathcal{M} \longrightarrow g_p H \subset G.$$

$$\mathcal{M} \sim G/H,$$

$$\dim(\mathcal{M}) = \dim(G) - \dim(H)$$

$$p \in \mathcal{M} \;\longrightarrow L(\phi^s) \in g_p H \subset G.$$

$$p' = g \star p$$

$$\phi' = g \star \phi = \phi'^s(\phi^u)$$

$$L(g \star \phi) g L(\phi)$$

$$g L(\phi) = L(g \star \phi) h(\phi, g)$$

$$\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{K}$$

$$[\mathfrak{H}, \mathfrak{H}] \subseteq \mathfrak{H}$$

$$[\mathfrak{H}, \mathfrak{K}] \subseteq \mathfrak{K}$$

$$[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K} \oplus \mathfrak{H},$$

$$[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{H};$$

$$H \in \mathfrak{H} \Rightarrow H^\dagger = -H, K \in \mathfrak{K} \Rightarrow K^\dagger = K$$

$$\left[K_{\underline{s}}, K_{\underline{u}}\right] = \mathcal{C}_{\underline{su}}{}^{q} H_q.$$

$$L(\phi^s) = \exp\left(\phi^s K_s\right)$$

$$G_{\text{solv}} = \exp\left(\mathfrak{S}\right)$$

$$\mathbf{D}\mathfrak{g} \equiv [\mathfrak{g}, \mathfrak{g}], \mathbf{D}^n\mathfrak{g} \equiv [\mathbf{D}^{n-1}\mathfrak{g}, \mathbf{D}^{n-1}\mathfrak{g}].$$

$$\left\{L_s(\phi_p)\right\} = G_{\text{solv}}, p \in \mathcal{M}$$

$$\mathcal{M} \sim G_{\text{solv}},$$

$$\forall g \in G_{\text{semi}} \Rightarrow g = sh \text{ with } s \in G_{\text{solv}}, h \in H$$

$$\Omega = L^{-1} dL$$

$$\Omega(\phi) = \sigma^A(\phi) T_A = L(\phi)^{-1} dL(\phi) = V^{\underline{s}}(\phi) K_{\underline{s}} + w^r(\phi) H_r = \wp(\phi) + w(\phi),$$

$$\Omega(\phi) = \Omega_s(\phi) d\phi^s, V^{\underline{u}}(\phi) = V_{s}{}^{\underline{\underline{u}}}(\phi) d\phi^s, \wp(\phi) = V^{\underline{s}}(\phi) K_{\underline{s}}, w(\phi) = w^q(\phi) H_q.$$

$$d\Omega = dL^{-1} \wedge dL = dL^{-1} L L^{-1} \wedge dL = -L^{-1} dL \wedge L^{-1} dL = -\Omega \wedge \Omega$$

$$d\Omega + \Omega \wedge \Omega = 0$$

$$L(g \star \phi) = g L(\phi) h^{-1}$$

$$\Omega(g \star \phi) = h L(\phi)^{-1} g^{-1} d(g L(\phi) h^{-1}) = h L(\phi)^{-1} (dL(\phi)) h^{-1} + h dh^{-1}$$

$$\Omega(g\star\phi) = \wp(g\star\phi) + w(g\star\phi) = V^{\underline{s}}(g\star\phi)K_{\underline{s}} + w^{r}(g\star\phi)H_{r}$$
$$= h\big(V^{\underline{s}}(\phi)K_{\underline{s}}\big)h^{-1} + h(w^{u}(\phi)H_{u})h^{-1} + hdh^{-1} =$$
$$= h\wp(\phi)h^{-1} + hw(\phi)h^{-1} + hdh^{-1}$$

$$\wp(g\star\phi) = h\wp(\phi)h^{-1}$$
$$w(g\star\phi) = hw(\phi)h^{-1} + hdh^{-1}$$

$$\mathcal{D}\wp \equiv d\wp + w\wedge\wp + \wp\wedge w = 0$$
$$R(w) \equiv dw + w\wedge w = \wp\wedge\wp$$

$$R(w) = \frac{1}{2}R_{su}d\phi^{s}\wedge d\phi^{u} \Rightarrow R_{su} = -[\wp_{s},\wp_{u}] \in \mathfrak{H}.$$

$$\kappa_{\underline{su}} \equiv k\mathrm{Tr}\big(K_{\underline{s}}K_{\underline{u}}\big)$$

$$\mathcal{G}_{su}(\phi) = V_{s}{}^{\underline{s}}(\phi)V_{u}{}^{\underline{u}}(\phi)\kappa_{\underline{su}} \;\Leftrightarrow\; ds^{2}(\phi) = \mathcal{G}_{su}(\phi)d\phi^{s}d\phi^{u} = k\mathrm{Tr}(\wp(\phi)^{2}),$$

$$\wp = \wp_{s}d\phi^{s}$$

$$\mathcal{L}_{\mathrm{scal}} = \frac{e_{\mathrm{D}}}{2}\mathcal{G}_{su}(\phi)\partial_{\mu}\phi^{s}\partial^{\mu}\phi^{u} = \frac{e_{\mathrm{D}}}{2}k\mathrm{Tr}[\wp_{s}(\phi)\wp_{u}(\phi)]\partial_{\mu}\phi^{s}\partial^{\mu}\phi^{u},$$

$$[t_{\alpha},t_{\beta}] = f_{\alpha\beta}^{\gamma}t_{\gamma},$$

$$g \approx \mathbb{1} + \epsilon^{\alpha}t_{\alpha},$$

$$\phi^{s} \to \phi^{s} + \epsilon^{\alpha}k_{\alpha}^{s}(\phi)$$

$$[k_{\alpha},k_{\beta}] = -f_{\alpha\beta}^{\gamma}k_{\gamma}.$$

$$\mathcal{G}_{\Lambda\mu\nu} \equiv -e_{\mathrm{D}}\varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}^{\Lambda}} = \mathcal{R}_{\Lambda\Sigma}F_{\mu\nu}^{\Sigma} - \mathcal{I}_{\Lambda\Sigma}{}^{*}F_{\mu\nu}^{\Sigma},$$

$${}^{*}F_{\mu\nu}^{\Lambda} \equiv \frac{e_{\mathrm{D}}}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}$$

$$\tilde{\mathcal{D}}_{\mu}(\partial^{\mu}\phi^{s}) = \frac{1}{4}\mathcal{G}^{su}\big(F_{\mu\nu}^{\Lambda}\partial_{u}\mathcal{I}_{\Lambda\Sigma}F^{\Sigma\mu\nu} + F_{\mu\nu}^{\Lambda}\partial_{u}\mathcal{R}_{\Lambda\Sigma}{}^{*}F^{\Sigma\mu\nu}\big)$$

$$\nabla_{\mu}\big({}^{*}F^{\Lambda\mu\nu}\big) = 0, \nabla_{\mu}\big({}^{*}\mathcal{G}^{\Lambda\mu\nu}\big) = 0$$

$$\tilde{\mathcal{D}}_{\mu}(\partial_{\nu}\phi^{s}) \equiv \nabla_{\mu}(\partial_{\nu}\phi^{s}) + \tilde{\Gamma}_{vu}^{s}\partial_{\mu}\phi^{v}\partial_{\nu}\phi^{u}$$

$${}^{**}F^{\Lambda} = (\mathcal{I}^{-1})^{\Lambda\Sigma}\big(\mathcal{R}_{\Sigma\Pi}F^{\Pi} - \mathcal{G}_{\Sigma}\big)$$
$${}^{**}\mathcal{G}_{\Lambda} = (\mathcal{R}\mathcal{I}^{-1}\mathcal{R} + \mathcal{I})_{\Lambda\Sigma}F^{\Sigma} - (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma}\mathcal{G}_{\Sigma}$$

$$\mathbb{F}^{M} \equiv \begin{pmatrix} F_{\mu\nu}^{\Lambda} \\ \mathcal{G}_{\Lambda\mu\nu} \end{pmatrix}$$

$${}^{*}\mathbb{F} = -\mathbb{C}\mathcal{M}(\phi^{s})\mathbb{F},$$

$$\mathbb{C} \equiv \mathbb{C}^{MN} \equiv \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{pmatrix} \equiv \begin{pmatrix} \mathbb{0}_{n_v} & \mathbb{1}_{n_v} \\ -\mathbb{1}_{n_v} & \mathbb{0}_{n_v} \end{pmatrix}$$

$$\mathcal{M}(\phi) \equiv \mathcal{M}(\phi)_{MN} \equiv \begin{pmatrix} (\mathcal{R}\mathcal{I}^{-1}\mathcal{R} + \mathcal{I})_{\Lambda\Sigma} & -(\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Gamma} \\ -(\mathcal{I}^{-1}\mathcal{R})^{\Xi}{}_{\Sigma} & (\mathcal{I}^{-1})^{\Xi\Gamma} \end{pmatrix}$$

$$\nabla_\mu({}^*\mathbb{F}^{\mu\nu}) = 0 \;\Leftrightarrow\; \nabla_\mu(\mathbb{C}\mathcal{M}(\phi)\mathbb{F}^{\mu\nu}) = 0 \;\Leftrightarrow\; d\mathbb{F} = 0$$

$$\tilde{\mathcal{D}}_\mu(\partial^\mu\phi^s) = \frac{1}{8}\mathcal{G}^{su}\left(\mathbb{F}_{\mu\nu}\right)^T\partial_u\mathcal{M}(\phi)\mathbb{F}^{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(\mathrm{S})}_{\mu\nu} + T^{(\mathrm{V})}_{\mu\nu}$$

$$T^{(\mathrm{S})}_{\mu\nu} = \mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial_\nu\phi^u - \frac{1}{2}g_{\mu\nu}\mathcal{G}_{su}(\phi)\partial_\rho\phi^s\partial^\rho\phi^u$$
$$T^{(\mathrm{V})}_{\mu\nu} = \left(F_{\mu\rho}\right)^T\mathcal{I}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}\left(F_{\rho\sigma}\right)^T\mathcal{I}F^{\rho\sigma}$$

$$T^{(\mathrm{V})}_{\mu\nu} = \frac{1}{2}\left(\mathbb{F}_{\mu\rho}\right)^T\mathcal{M}(\phi)\mathbb{F}^\rho_\nu$$

$$R = \mathcal{G}_{su}(\phi)\partial_\rho\phi^s\partial^\rho\phi^u$$

$$R_{\mu\nu} = \mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial_\nu\phi^u + \frac{1}{2}\left(\mathbb{F}_{\mu\rho}\right)^T\mathcal{M}(\phi)\mathbb{F}_\nu{}^\rho$$

$$\tilde{\mathcal{D}}_\mu(\partial^\mu\phi^s) = \frac{1}{8}\mathcal{G}^{su}\left(\mathbb{F}_{\mu\nu}\right)^T\partial_u\mathcal{M}(\phi)\mathbb{F}^{\mu\nu}$$
$$R_{\mu\nu} = \mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial_\nu\phi^u + \frac{1}{2}\left(\mathbb{F}_{\mu\rho}\right)^T\mathcal{M}(\phi)\mathbb{F}_\nu{}^\rho$$
$$d\mathbb{F} = 0 \Longrightarrow \nabla_\mu(\mathbb{C}\mathcal{M}(\phi)\mathbb{F}^{\mu\nu}) = 0$$

$$\mathcal{M} \in \mathrm{Sp}(2n_\mathrm{v}, \mathbb{R}) \colon \mathcal{M}^T\mathbb{C}\mathcal{M} = \mathcal{M}\mathbb{C}\mathcal{M}^T = \mathbb{C}$$

$$\mathcal{M}(\phi)_{MP}\mathbb{C}^{PL}\mathcal{M}(\phi)_{LN} = \mathbb{C}_{MN} \;\Longrightarrow\; \mathcal{M}(\phi)^{-1} = -\mathbb{C}\mathcal{M}(\phi)\mathbb{C}$$

$$\mathcal{R}_\mathrm{v}[g] \equiv \mathcal{R}_\mathrm{v}[g]^M{}_N$$

$$\mathcal{M}(g \star \phi) = \mathcal{R}_\mathrm{v}[g]^{-T}\mathcal{M}(\phi)\mathcal{R}_\mathrm{v}[g]^{-1}.$$

$$g \in G \leftrightarrow \mathcal{R}_\mathrm{v}[g] \in \mathrm{Sp}(2n_\mathrm{v}, \mathbb{R}) \;\Longrightarrow\; G \overset{\mathcal{R}_\mathrm{v}}{\longleftrightarrow} \mathrm{Sp}(2n_\mathrm{v}, \mathbb{R}),$$

$$\mathcal{R}_\mathrm{v}[g_1 \cdot g_2] = \mathcal{R}_\mathrm{v}[g_1]\mathcal{R}_\mathrm{v}[g_2],$$
$$\mathcal{R}_\mathrm{v}[g]\mathbb{C}\mathcal{R}_\mathrm{v}[g]^T = \mathcal{R}_\mathrm{v}[g]^T\mathbb{C}\mathcal{R}_\mathrm{v}[g] = \mathbb{C}.$$

$$\mathcal{R}_{\mathrm{v}^*} = \mathcal{R}_\mathrm{v}^{-T}$$

$$\mathcal{R}_{\mathrm{v}^*}[g] = (\mathcal{R}_{\mathrm{v}^*}[g]_M{}^N) = \mathcal{R}_\mathrm{v}[g]^{-T} = -\mathbb{C}\mathcal{R}_\mathrm{v}[g]\mathbb{C}$$
$$\Longrightarrow \mathcal{R}_{\mathrm{v}^*}[g]_M{}^N = \mathbb{C}_{MP}\mathcal{R}_\mathrm{v}[g]^P{}_Q\mathbb{C}^{NQ},$$

$$\mathcal{M}(\phi) \overset{E_S}{\to} \mathcal{M}'(\phi) = (E_S)^T \mathcal{M}(\phi) E_S$$

$$L(\phi) \overset{\mathcal{R}_v}{\to} \mathcal{R}_v[L(\phi)] \in \mathrm{Sp}(2n_v, \mathbb{R}).$$

$$\mathcal{R}_v[H] \nsubseteq \mathrm{SO}(2n_v)$$

$$\mathcal{A} \equiv \mathcal{A}^N{}_{\underline{M}} \in \mathrm{Sp}(2n_v, \mathbb{R})/\mathrm{U}(n)$$

$$\underline{\mathcal{R}}_v[H] \equiv \mathcal{A}^{-1}\mathcal{R}_v[H]\mathcal{A} \subset \mathrm{SO}(2n_v)$$

$$\mathbb{L}(\phi) \equiv \mathbb{L}(\phi)^M{}_{\underline{N}}$$

$$\mathbb{L}(\phi) \equiv \mathcal{R}_v[L(\phi)]\mathcal{A} \implies \mathbb{L}(\phi)^M{}_{\underline{N}} \equiv \mathcal{R}_v[L(\phi)]^M{}_N \mathcal{A}^N{}_{\underline{N}},$$

$$\mathbb{L}(\phi)_M\underline{N} \equiv \mathbb{C}_{MP}\mathbb{C}\underline{NQ}\mathbb{L}(\phi)^P\underline{Q}.$$

$$\forall g \in G: \mathcal{R}_v[g]\mathbb{L}(\phi) = \mathbb{L}(g \star \phi)\underline{\mathcal{R}}_v[h]$$

$$\mathcal{M}(\phi)_{MN} = \mathbb{C}_{MP}\mathbb{L}(\phi)^P{}_{\underline{L}}\mathbb{L}(\phi)^R{}_{\underline{L}}\mathbb{C}_{RN} \implies \mathcal{M}(\phi) = \mathbb{C}\mathbb{L}(\phi)\mathbb{L}(\phi)^T\mathbb{C},$$

$$\forall g \in G: \mathcal{M}(g \star \phi) = \mathbb{C}L(g \star \phi)L(g \star \phi)^T\mathbb{C} = \mathcal{R}_v[g]^{-T}\mathcal{M}(\phi)\mathcal{R}_v[g]^{-1}$$

$$g \in G: \begin{cases} \phi^s & \overset{g}{\to} \quad g \star \phi^s \\ \mathbb{F}^M_{\mu\nu} & \overset{g}{\to} \quad \mathbb{F}'^M_{\mu\nu} = \mathcal{R}_v[g]^M{}_N \mathbb{F}^N_{\mu\nu} \end{cases}$$

$$g \in G: \begin{cases} \phi^s & \to g \star \phi^s \\ \begin{pmatrix} F^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix} & \to \mathcal{R}_v[g] \cdot \begin{pmatrix} F^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix} = \begin{pmatrix} A_g{}^{\Lambda\Sigma} & B_g{}^{\Lambda\Sigma} \\ C_{g\Lambda\Sigma} & D_g{}^{\Sigma} \end{pmatrix} \begin{pmatrix} F^\Sigma \\ \mathcal{G}_\Sigma \end{pmatrix} \end{cases} \text{(curvature)}$$

$$\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \overset{\mathrm{U}(1)}{\to} \begin{pmatrix} F'_{\mu\nu} \\ {}^*F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix}$$

$$\begin{pmatrix} F^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix} \overset{\mathcal{R}_v[g]}{\to} \begin{pmatrix} F'^\Lambda \\ \mathcal{G}'_\Lambda \end{pmatrix} = \begin{pmatrix} A_g{}^\Lambda{}_\Sigma F^\Sigma & +B_g{}^{\Lambda\Sigma}\mathcal{G}_\Sigma \\ C_{g\Lambda\Sigma}F^\Sigma & +D_g{}^\Sigma\mathcal{G}_\Sigma \end{pmatrix}$$

$$\mathcal{L}_{\text{scal}} = \frac{e_D}{2}\mathcal{G}_{su}(\phi)\partial_\mu\phi^s\partial^\mu\phi^u = \frac{e_D}{8}k\mathrm{Tr}\left(\mathcal{M}^{-1}\partial_\mu\mathcal{M}\mathcal{M}^{-1}\partial^\mu\mathcal{M}\right)$$

$$\mathfrak{N}_{\Lambda\Sigma} \equiv \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma}$$

$$\mathfrak{N}(g \star \phi) = \left(C_g + D_g\mathfrak{N}(\phi)\right)\left(A_g + B_g\mathfrak{N}(\phi)\right)^{-1},$$

$$e_\Lambda \equiv \frac{1}{4\pi}\int_{\mathbb{S}^2} \mathcal{G}_\Lambda = \frac{1}{8\pi}\int_{\mathbb{S}^2} \mathcal{G}_{\Lambda\mu\nu} dx^\mu \wedge dx^\nu$$
$$m^\Lambda \equiv \frac{1}{4\pi}\int_{\mathbb{S}^2} F^\Lambda = \frac{1}{8\pi}\int_{\mathbb{S}^2} F^\Lambda_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$\Gamma^M = \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} = \frac{1}{4\pi}\int_{\mathbb{S}^2} \mathbb{F}^M$$

$$(\Gamma_2{}^M)^T \mathbb{C} \Gamma_1{}^M = m_2{}^\Lambda e_{1\Lambda} - m_1{}^\Lambda e_{2\Lambda} = \frac{1}{2\pi}\hbar c n;\ (n \in \mathbb{Z})$$

$$g \in G_{\mathrm{el}}: \mathcal{R}_{\mathrm{v}}[g] = \begin{pmatrix} A^{\Lambda}{}_{\Sigma} & \mathbb{O} \\ C_{\Lambda\Sigma} & D_\Lambda{}^\Sigma \end{pmatrix}$$

$$g \in G_{\mathrm{el}}: \quad \begin{aligned} F^\Lambda &\to F'^\Lambda = A^\Lambda{}_\Sigma F^\Sigma \\ \mathcal{G}_\Lambda &\to \mathcal{G}'_\Lambda = C_{\Lambda\Sigma} F^\Sigma + D_\Lambda{}^\Sigma \mathcal{G}_\Sigma \end{aligned}$$

$$\mathcal{I}_{\Lambda\Sigma} \to D_\Lambda{}^\Pi D_\Sigma{}^\Delta \mathcal{I}_{\Pi\Delta};\ \mathcal{R}_{\Lambda\Sigma} \to D_\Lambda{}^\Pi D_\Sigma{}^\Delta \mathcal{R}_{\Pi\Delta} + C_{\Lambda\Pi} D_\Sigma{}^\Pi$$

$$\delta\mathcal{L}_{\mathrm{BOS}} = \frac{1}{8} C_{\Lambda\Pi} D_\Sigma{}^\Pi \varepsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

$$\mathcal{N} = 2: \mathcal{M}_{\mathrm{scal}} = \mathcal{M}_{\mathrm{SK}} \times \mathcal{M}_{\mathrm{QK}}$$

$$g \in \text{ isom. of } \mathcal{M}_{\mathrm{QK}} \implies \mathcal{R}_{\mathrm{v}}[g] = \mathbb{1}$$

$$G: \begin{cases} \delta\mathbb{L} = \Lambda^\alpha t_\alpha \mathbb{L} \\ \delta\mathbb{F}^M_{\mu\nu} = -\Lambda^\alpha (t_\alpha)_N^M \mathbb{F}^N_{\mu\nu} \end{cases}$$

$$[t_\alpha, t_\beta] = f_{\alpha\beta}^\gamma t_\gamma,$$

$$(t_\alpha)_M{}^N \mathbb{C}_{NP} = (t_\alpha)_P{}^N \mathbb{C}_{NM} \quad M, N, \ldots = 1, \ldots, 2n_{\mathrm{v}}$$

$$(t_\alpha)_{MN} = (t_\alpha)_{NM}$$

$$H = H_{\mathrm{R}} \times H_{\mathrm{matt}},$$

$$G \rightsquigarrow \mathbb{L}(\phi) \rightsquigarrow H.$$

$$(\partial\mathbf{b}) \cdot \mathbb{L}(\phi) \cdot \mathbf{f} = \mathbf{d}(\phi, \partial\mathbf{b}) \cdot \mathbf{f}$$

$$g \in G: \begin{cases} \phi^s & \overset{g}{\rightarrow} & g \star \phi^s \\ \mathbb{F}^M_{\mu\nu} & \overset{g}{\rightarrow} & \mathbb{F}'^M_{\mu\nu} = \mathcal{R}_{\text{v}}[g]^M{}_N \mathbb{F}^N_{\mu\nu} \\ \mathbf{f} & \overset{g}{\rightarrow} & \mathbf{f}' = h(\phi, g) \star \mathbf{f} \end{cases}$$

$$\delta_\epsilon \mathbf{b} = \bar{\epsilon}\mathbf{f}, \delta_\epsilon \mathbf{f} = \epsilon \partial \mathbf{b}$$

$$\delta_\epsilon \mathbf{f} = \mathbf{d}(\phi, \partial \mathbf{b})\epsilon$$

$$ds^2 = f(r)^2 dt^2 - f(r)^{-2} dr^2 - h(r)^2 (d\theta^2 + \sin^2(\theta) d\varphi^2)$$

$$\text{white particles} = 0, \phi^s = \phi^s(r)$$

$$\Gamma^M \equiv (m^\Lambda, e_\Lambda)$$

$$\mathbb{F}^M = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ \mathcal{G}_{\Lambda\mu\nu} \end{pmatrix} \frac{dx^\mu \wedge dx^\nu}{2} = \frac{1}{h^2} \mathbb{C}\mathcal{M}(\phi)\Gamma^M dt \wedge dr + \Gamma^M \sin(\theta) d\theta \wedge d\varphi$$

$$\begin{aligned} (\mathbb{F}_{\mu\nu})^T \partial_s \mathcal{M} \mathbb{F}^{\mu\nu} &= 2(\mathbb{F}_{tr})^T \partial_s \mathcal{M} \mathbb{F}_{tr} g^{tt} g^{rr} + 2(\mathbb{F}_{\theta\varphi})^T \partial_s \mathcal{M} \mathbb{F}_{\theta\varphi} g^{\theta\theta} g^{\varphi\varphi} \\ &= -\frac{2}{h^4} \Gamma^T \mathcal{M} \mathbb{C}^T \partial_s \mathcal{M} \mathbb{C} \mathcal{M} \Gamma + \frac{2}{h^4} \Gamma^T \partial_s \mathcal{M} \Gamma = \\ &= \frac{4}{h^4} \Gamma^T \partial_s \mathcal{M} \Gamma = -\frac{8}{h^4} \partial_s V_{\text{BH}} \end{aligned}$$

$$V_{\text{BH}}(\phi, \Gamma) \equiv -\frac{1}{2} \Gamma^T \mathcal{M}(\phi) \Gamma > 0$$

$$(f^2 h^2 \phi'^s)' + \tilde{\Gamma}^s_{uv} \phi'^u \phi'^v = \frac{1}{h^2} \mathcal{G}^{su} \partial_u V_{\text{BH}},$$

$$f'(r) \equiv \frac{d}{dr}, \text{ while } \partial_{s,u,v,\dots} \text{ indicates } \frac{\partial}{\partial \phi^{s,u,v,\dots}}$$

$$\frac{d\tau}{dr} = \frac{1}{f^2 h^2}$$

$$\dot{f}(\tau) \equiv \frac{df(\tau)}{d\tau}$$

$$\ddot{\phi}^s + \tilde{\Gamma}^s{}_{uv} \dot{\phi}^u \dot{\phi}^v = f^2 \mathcal{G}^{su} \partial_u V_{\text{BH}}$$

$$\begin{aligned} R_{rr} &= \mathcal{G}_{su} \phi'^s \phi'^u - \frac{1}{f^2 h^4} V_{\text{BH}}, \\ R_{tt} &= \frac{f^2}{h^4} V_{\text{BH}}, R_{\theta\theta} = \frac{1}{h^2} V_{\text{BH}}, R_{\varphi\varphi} = \frac{\sin^2(\theta)}{h^2} V_{\text{BH}}, \end{aligned}$$

$$R^t_t = \frac{1}{f^2} R_{tt} = \frac{1}{h^4} V_{\text{BH}} = \frac{1}{h^2} R_{\theta\theta} = -R^\theta_\theta$$

$$\begin{aligned} R^t_t = -R^\theta_\theta \Rightarrow \frac{(f f' h^2)'}{h^2} &= \frac{1}{h^2}(1 - (f^2 h h')') \\ \Rightarrow (f^2 h^2)'' &= 2 \end{aligned}$$

$$f^2h^2 = (r-r_0)^2 - c_{\mathrm{ex}}^2 = (r-r_+)(r-r_-)$$
$$r_\pm \equiv r_0 \pm c_{\mathrm{ex}}$$

$$\frac{d\tau}{dr} = \frac{1}{f^2h^2} = \frac{1}{(r-r_0)^2 - c_{\mathrm{ex}}^2} \Longrightarrow r - r_0 = -c_{\mathrm{ex}}\coth(c_{\mathrm{ex}}\tau)$$

$$\tau = \frac{1}{2c_{\mathrm{ex}}}\log\left(\frac{r-r_+}{r-r_-}\right)$$

$$\frac{d\tau}{dr} = \frac{1}{(r-r_0)^2 - c_{\mathrm{ex}}^2} = \frac{\sinh^2(c_{\mathrm{ex}}\tau)}{c_{\mathrm{ex}}^2}$$

$$f(r)^2 = e^{2U(r)}$$
$$h(r)^2 = e^{-2U(r)}(r-r_+)(r-r_-) = e^{-2U(r)}\frac{c_{\mathrm{ex}}^2}{\sinh^2(c_{\mathrm{ex}}\tau)}.$$

$$ds^2 = e^{2U}dt^2 - e^{-2U}(dr^2 + (r-r_+)(r-r_-)d\Omega^2)$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)d\varphi^2$$

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left(\frac{c_{\mathrm{ex}}^4}{\sinh^4(c_{\mathrm{ex}}\tau)}d\tau^2 + \frac{c_{\mathrm{ex}}^2}{\sinh^2(c_{\mathrm{ex}}\tau)}d\Omega^2\right)$$

$$ff'h^2 = \frac{\dot{f}}{f} = \dot{U},$$

$$\ddot{U} = e^{2U}V_{\mathrm{BH}}.$$

$$R_{tt} = \frac{1}{h^4}\ddot{U}, R_{\tau\tau} = 2c_{\mathrm{ex}}^2 - 2\dot{U}^2 + \ddot{U}, R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2(\theta)} = \frac{1}{f^2h^2}\ddot{U}.$$

$$\dot{U}^2 + \frac{1}{2}\mathcal{G}_{su}\dot{\phi}^s\dot{\phi}^u - e^{2U}V_{\mathrm{BH}} = c_{\mathrm{ex}}^2,$$

$$\ddot{U} = e^{2U}V_{\mathrm{BH}},$$
$$\ddot{\phi}^s + \tilde{\Gamma}^s{}_{uv}\dot{\phi}^u\dot{\phi}^v = e^{2U}\mathcal{G}^{su}\partial_u V_{\mathrm{BH}},$$
$$\dot{U}^2 + \frac{1}{2}\mathcal{G}_{su}\dot{\phi}^s\dot{\phi}^u - e^{2U}V_{\mathrm{BH}} = c_{\mathrm{ex}}^2.$$

$$\mathcal{S}_{\mathrm{eff}} = \int \mathcal{L}_{\mathrm{eff}}d\tau = \int \left(\dot{U}^2 + \frac{1}{2}\mathcal{G}_{su}(\phi)\dot{\phi}^s\dot{\phi}^u + e^{2U}V_{\mathrm{BH}}(\phi,\Gamma^M)\right)d\tau$$

$$\frac{d\mathcal{H}}{d\tau} = 0 \Rightarrow \mathcal{H} = \mathfrak{G}$$

$$\mathcal{H} = \dot{U}^2 + \frac{1}{2}\mathcal{G}_{su}(\phi)\dot{\phi}^s\dot{\phi}^u - e^{2U}V_{\mathrm{BH}}(\phi,\Gamma) = c_{\mathrm{ex}}^2$$

$$M_{\mathrm{ADM}} = \frac{c^2}{8\pi \mathrm{G_N}}\int_{\mathbb{S}^2_\infty} e_{\mathrm{D}}\varepsilon_{\theta\varphi\mu\nu}\nabla^\mu\xi^\nu d\theta d\varphi$$

$$M_{\mathrm{ADM}}=\frac{c^2}{\mathrm{G_N}}\lim_{\tau\to 0^-}\dot{U}$$

$$U(0)=0;\ \dot{U}(0)=\frac{\mathrm{G_N}}{c^2}M_{\mathrm{ADM}};\ \phi^s(0)=\phi_0^s;\ \dot{\phi}^s(0)=\dot{\phi}_0^s$$

$$\tau\to 0\colon \frac{\mathrm{G_N^2}}{c^4}M_{\mathrm{ADM}}^2+\frac{1}{2}\mathcal{G}_{su}(\phi_0)\dot{\phi}_0^s\dot{\phi}_0^u-\frac{8\pi\mathrm{G_N}}{c^4}V_{\mathrm{BH}}(\phi_0,\Gamma)=c_{\mathrm{ex}}^2.$$

$$A_{\mathrm{H}}=\lim_{\tau\to-\infty}\int_{S_2}\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\,d\theta d\varphi=\lim_{\tau\to-\infty}4\pi e^{-2U}\frac{c_{\mathrm{ex}}^2}{\sinh^2(c_{\mathrm{ex}}\tau)}$$

$$r\to r_{\mathrm{H}}=r_+\colon e^{-2U}\sim\frac{A_{\mathrm{H}}}{4\pi}\frac{\sinh^2(c_{\mathrm{ex}}\tau)}{c_{\mathrm{ex}}^2}=\frac{r_{\mathrm{H}}^2}{(r-r_+)(r-r_-)}$$

$$ds^2=\frac{(r-r_+)(r-r_-)}{r_{\mathrm{H}}^2}dt^2-\frac{r_{\mathrm{H}}^2}{(r-r_+)(r-r_-)}dr^2-r_{\mathrm{H}}^2d\Omega^2$$

$$\kappa^2=-\frac{c^4}{2}\nabla^\mu\xi^\nu\nabla_\mu\xi_\nu$$

$$\kappa=\frac{c^2c_{\mathrm{ex}}}{r_{\mathrm{H}}^2}$$

$$T=\frac{\hbar c}{2\pi k_{\mathrm{B}}}\frac{c_{\mathrm{ex}}}{r_{\mathrm{H}}^2},$$

$$\mathcal{S}=\frac{k_{\mathrm{B}}c^3A_{\mathrm{H}}}{4\mathrm{G_N}\hbar}$$

$$c_{\mathrm{ex}}=2\frac{\mathrm{G_N}}{c^4}\mathcal{S}T$$

$$d\rho^2=e^{-2U}dr^2$$

$$\lim_{\rho\to\rho_{\mathrm{H}}}\phi^s(\rho)=\phi_*^s;\ \ |\phi_*^s|<\infty$$

$$c_{\mathrm{ex}}\to 0\colon \tau=-\frac{1}{r}$$

$$(r-r_0)\rightsquigarrow r$$

$$\tau\to-\infty\colon e^{-2U}\sim\lim_{c_{\mathrm{ex}}\to 0}r_{\mathrm{H}}^2\frac{\sinh^2(c_{\mathrm{ex}}\tau)}{c_{\mathrm{ex}}^2}=r_{\mathrm{H}}^2\tau^2$$

$$\tau\to-\infty\colon e^{-U}\sim-\tau r_{\mathrm{H}},\dot{U}\sim-\frac{1}{\tau},\ddot{U}\sim\frac{1}{\tau^2}.$$

$$\tau \to -\infty: d\rho = e^{-U} dr = \lim_{c_{\text{ex}} \to 0} e^{-U} c_{\text{ex}}^2 \frac{d\tau}{\sinh^2 (c_{\text{ex}}\tau)} = \\ = e^{-U} \frac{d\tau}{\tau^2} \sim -r_{\text{H}} \frac{d\tau}{\tau}$$

$$c_{\text{ex}} \to 0, \tau \to -\infty: \rho = -r_{\text{H}} \log(-\tau)$$

$$\lim_{\rho \to -\infty} \phi^s(\rho) = \phi_*^s, |\phi_*^s| < \infty$$

$$\lim_{\rho \to -\infty} \frac{d^\ell}{d\rho^\ell} \phi(\rho) = 0$$

$$\lim_{\tau \to -\infty} \tau \dot{\phi}^s = \lim_{\tau \to -\infty} \tau^2 \ddot{\phi}^s = 0$$

$$\tau \to -\infty: \tau^2 \ddot{\phi}^s + \tilde{\Gamma}_{uv}^s (\tau \dot{\phi}^u)(\tau \dot{\phi}^v) = \frac{1}{r_{\text{H}}^2} \mathcal{G}^{su} \partial_u V_{\text{BH}}$$

$$\lim_{\phi \to \phi_*} \partial_u V_{\text{BH}} = \partial_s V_{\text{BH}}(\phi_*^s, e, m) = 0$$

$$\phi_*^s = \phi_*^s(e, m)$$

$$V_{\text{BH}}^{(\text{ex})} = V_{\text{BH}}(\phi_*, e, m) = V_{\text{BH}}^{(\text{ex})}(e, m).$$

$$\tau \to -\infty: \quad \frac{1}{\tau^2} = \ddot{U} = e^{2U} V_{\text{BH}}^{(\text{ex})} = \frac{1}{r_{\text{H}}^2 \tau^2} V_{\text{BH}}^{(\text{ex})} \\ \Rightarrow V_{\text{BH}}^{(\text{ex})} = r_{\text{H}}^2.$$

$$A_{\text{H}} = 4\pi V_{\text{BH}}^{(\text{ex})}(e, m) = A_{\text{H}}(e, m),$$

$$ds^2 = \left(\frac{r}{r_{\text{H}}}\right)^2 dt^2 - \left(\frac{r}{r_{\text{H}}}\right)^{-2} dr^2 - r_{\text{H}}^2 d\Omega^2$$

$$\phi^s(\tau = 0) = \phi_*^s$$

$$\partial_s V_{\text{BH}}(\phi_*^s, e, m) = 0$$

$$\tau \to -\infty: d\rho = e^{-U} dr \sim -r_{\text{H}} \frac{\sinh(c_{\text{ex}}\tau)}{c_{\text{ex}}} \frac{dr}{d\tau} d\tau \sim \\ \sim -\frac{c_{\text{ex}}}{\sinh(c_{\text{ex}}\tau)} d\tau \sim 2c_{\text{ex}} e^{c_{\text{ex}}\tau} d\tau$$

$$c_{\text{ex}} \neq 0, \tau \to -\infty: \rho(\tau) = 2e^{c_{\text{ex}}\tau}$$

$$\lim_{\rho \to 0} \phi^s(\rho) = \phi_*^s, |\phi_*^s| < \infty \Rightarrow \lim_{\rho \to -\infty} \frac{d^\ell}{d\rho^\ell} \phi(\rho) = 0.$$

$$g \in G: \begin{cases} U(\tau) \\ \phi^s(\tau) \\ \Gamma^M \\ M_{\text{ADM}} \end{cases} \xrightarrow{g} \begin{cases} U'(\tau) = U(\tau) \\ \phi'^s(\tau) = g \star \phi^s(\tau) \\ \Gamma'^M = \mathcal{R}_{\text{v}}[g]\Gamma \\ M_{\text{ADM}} \end{cases}$$

$$\Xi = \{U(\tau), \phi^s(\tau)\}$$

$$\Xi' = \{U'(\tau) = U(\tau), \phi'^s(\tau)\}$$

$$\phi_0' = g \star \phi_0$$

$$V_{\mathrm{BH}}(\phi, \Gamma) \equiv -\frac{1}{2}\Gamma^T \mathcal{M}(\phi)\Gamma,$$

$$V_{\mathrm{BH}}(\phi, \Gamma) \overset{g}{\to} V'_{\mathrm{BH}}(g \star \phi, \mathcal{R}_{\mathrm{v}}[g]\Gamma) = -\frac{1}{2}\Gamma^T \mathcal{R}_{\mathrm{v}}^T \mathcal{R}_{\mathrm{v}}^{-T} \mathcal{M}(\phi) \mathcal{R}_{\mathrm{v}}^{-1} \mathcal{R}_{\mathrm{v}} \Gamma = \\ = V_{\mathrm{BH}}(\phi, \Gamma)$$

$$\phi_*^s(\Gamma) = \phi_*^s(e, m)$$

$$\partial_s V_{\mathrm{BH}}(\phi_*(\Gamma), \Gamma) = 0,$$

$$\partial_s V_{\mathrm{BH}}(\phi_*, \Gamma) = 0 \Leftrightarrow \partial_s V_{\mathrm{BH}}(g \star \phi_*, \mathcal{R}_{\mathrm{v}}[g]\Gamma) = 0,$$

$$g \star \phi_*$$

$$V(\phi', \mathcal{R}_{\mathrm{v}}[g]\Gamma)$$

$$\phi_*(\mathcal{R}_{\mathrm{v}}[g]\Gamma)$$

$$g \star \phi_*^s(\Gamma) = \phi_*^s(\mathcal{R}_{\mathrm{v}}[g]\Gamma)$$

$$V_{\mathrm{BH}}^{(\mathrm{ex})}(\Gamma) = V_{\mathrm{BH}}(\phi_*(\Gamma), \Gamma) = V_{\mathrm{BH}}(g \star \phi_*(\Gamma), \mathcal{R}_{\mathrm{v}}[g]\Gamma) = V_{\mathrm{BH}}(\phi_*(\mathcal{R}_{\mathrm{v}}[g]\Gamma), \mathcal{R}_{\mathrm{v}}[g]\Gamma) \\ = V_{\mathrm{BH}}{}^{(\mathrm{ex})}(\mathcal{R}_{\mathrm{v}}[g]\Gamma)$$

$$I_4(\Gamma) = I_4(e, m)$$

$$\mathcal{R}_{\mathrm{v}}[T_{\mathcal{A}}] \equiv (T_{\mathcal{A}})_M{}^N$$

$$I_4(\Gamma) = -\frac{n_{\mathrm{v}}(2n_{\mathrm{v}} + 1)}{6\dim(G)} (T_{\mathcal{A}})_{MN} (T^{\mathcal{A}})_{PQ} \Gamma^M \Gamma^N \Gamma^P \Gamma^Q$$

$$\eta_{\mathcal{AB}} \equiv (T_{\mathcal{A}})_M{}^N (T_{\mathcal{B}})_N{}^M$$

$$V_{\mathrm{BH}}^{(\mathrm{ex})} = \sqrt{|I_4|},$$

$$A_{\mathrm{H}}^{(\mathrm{ex})} = 4\pi \left(\frac{8\pi \mathrm{G_N}}{c^4} \sqrt{|I_4|} \right)$$

$$\mathcal{S}^{(\mathrm{ex})} = \frac{k_{\mathrm{B}}}{\ell_{\mathrm{P}}^2} \pi \left(\frac{8\pi \mathrm{G_N}}{c^4} \sqrt{|I_4|} \right)$$

$$\frac{1}{e_{\mathrm{D}}} \mathcal{L}_{\dagger} = -\frac{R}{2} + \frac{1}{2} \mathcal{G}_{su}(\phi) \partial_\mu \phi^s \partial^\mu \phi^u + \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) F_{\mu\nu}^{\Lambda} F^{\Sigma\mu\nu} + \frac{1}{8e_{\mathrm{D}}} \mathcal{R}_{\Lambda\Sigma}(\phi) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{\Lambda} F_{\rho\sigma}^{\Sigma},$$

$$\mathcal{M}_{\mathrm{scal}}^{\boxminus} = \frac{G_{(4)}}{H_{(4)}},$$

$$ds^2 = e^{2U}\left(dt + \omega_\varphi d\varphi\right)^2 - e^{-2U} g_{ij}^{\boxtimes} dx^i dx^j;$$

$$A_{\boxdot}^{\Lambda} = A_0^{\Lambda}(dt+\omega) + A_{\boxplus}^{\Lambda}, A_{\ddagger}^{\Lambda} \equiv A_i^{\Lambda} dx^i$$
$$\mathbb{F}^M = \begin{pmatrix} F_{\oplus}^{\Lambda} \\ \mathcal{G}_{\Lambda} \end{pmatrix} = d\mathcal{Z}^M \wedge (dt+\omega) + e^{-2U}\mathbb{C}^{MN}\mathcal{M}_{(4)NP} {}^{\star_3} d\mathcal{Z}^P$$
$$da = -e^{4U\dagger_3} d\omega - \mathcal{Z}^T \mathbb{C} d\mathcal{Z}$$

$$F_{\ominus}^{\Lambda} = dA_{\otimes}^{\Lambda}, \mathcal{G}_{\Lambda} = -\frac{1}{2} * \left(\frac{\partial \mathcal{L}_{\odot}}{\partial F_{\circledcirc}^{\Lambda}}\right)$$

$$\frac{1}{e_{\mathrm{D}}^{\dagger}}\mathcal{L}_{(3)} = -\frac{R^{\triangle}}{2} + \frac{1}{2}\hat{\mathcal{G}}_{ab}(z)\partial_i z^a \partial^i z^b$$
$$= -\frac{R^{\triangle}}{2} + \Big(\partial_i U \partial^i U + \frac{1}{2}\mathcal{G}_{su}\partial_i \phi^s \partial^i \phi^u + \frac{1}{2} e^{-2U}\partial_i \mathcal{Z}^T \mathcal{M}_{\star} \partial^i \mathcal{Z}$$
$$+ \frac{1}{4} e^{-4U}(\partial_i a + \mathcal{Z}^T \mathbb{C} \partial_i \mathcal{Z})(\partial^i a + \mathcal{Z}^T \mathbb{C} \partial^i \mathcal{Z})\Big)$$

$$e_{\mathrm{D}}^{\circ} \equiv \sqrt{\mathrm{Det}(g_{ij}^{\Lambda})}$$

$$\mathcal{M}_{\mathrm{scal}}^{\vee} = \frac{G_{\cap}}{H_{\cup}^{*}},$$

$$\mathcal{M}_{\mathrm{scal}}^{\sqcup} \subset \mathcal{M}_{\mathrm{scal}}^{\sqcap}$$

$$L(\Phi^I) = \exp(-aT.)\exp\left(\sqrt{2}\mathcal{Z}^M T_M\right)\exp(\phi^r T_r)\exp(2UH_0)$$

$$\hat{\zeta}(A) = -\eta A^{\dagger}\eta$$

$$\Phi_0 = \lim_{r\to\infty}\Phi^I(r,\theta)$$

$$\mathcal{M}_{\blacksquare} \equiv \mathcal{M}_{\blacksquare}(\Phi^I) \equiv L\eta L^{\dagger} = \mathcal{M}_{\blacksquare}^{\dagger}$$

$$\hat{J}_i \equiv \frac{1}{2}\partial_i \Phi^I \mathcal{M}_{\blacksquare}^{-1}\partial_I \mathcal{M}_{\blacksquare}$$

$$\mathcal{Q} = \frac{1}{4\pi}\int_{S_2}^{*_3} \hat{J} = \frac{1}{4\pi}\int \sqrt{e_{\mathrm{D}}^{\blacksquare}}\hat{J}^r d\theta d\varphi$$

$$\mathcal{Q}_{\psi} = -\frac{3}{4\pi}\int_{S_2}^{\infty} \psi_{[i}\hat{J}_{j]} dx^i \wedge dx^j = \frac{3}{8\pi}\int_{S_2}^{\infty} g_{\varphi\varphi}^{\#}\hat{J}_{\theta} d\theta d\varphi$$

$$M_{\mathrm{ADM}} = k\mathrm{Tr}(H_0^{\dagger}\mathcal{Q}), N_{\mathrm{NUT}} = -k\mathrm{Tr}(T.^{\dagger}\mathcal{Q}), \Sigma_s = k\mathrm{Tr}(T_s^{\dagger}\mathcal{Q})$$
$$\Gamma^M = \sqrt{2}k\mathbb{C}^{MN}\mathrm{Tr}(T_N^{\dagger}\mathcal{Q}), \mathcal{J}_{\psi} = k\mathrm{Tr}(T.^{\dagger}\mathcal{Q}_{\psi})$$

$$\forall g \in G_{\perp}: \mathcal{M}_{\uplus}(\Phi^{I}) \overset{g}{\rightarrow} \mathcal{M}_{(3)}(\Phi'^{I}) = g\mathcal{M}_{\uplus}(\Phi^{I})g^{\dagger}$$

$$\forall g \in G_3: \mathcal{Q} \overset{g}{\rightarrow} \mathcal{Q}' = (g^{-1})^{\dagger}\mathcal{Q}g^{\dagger}, \mathcal{Q}_{\psi} \overset{g}{\rightarrow} \mathcal{Q}'_{\psi} = (g^{-1})^{\dagger}\mathcal{Q}_{\psi}g^{\dagger}$$

$$\mathbb{J}_M \equiv \frac{1}{2}\left(T_M + T_M^{\dagger}\right)$$

$$H_3^{\mathrm{c}} = \mathrm{U}(1)_{\mathrm{E}} \times H_4$$

$$\mathbb{K}_M \equiv \frac{1}{2}\left(T_M - T_M^{\dagger}\right)$$

$$\mathcal{Q}_{\psi} = \alpha h^{-1}\mathcal{Q}h;\ h \in \mathrm{U}(1)_{\mathrm{E}},$$

$$\mathcal{Q}^3 = \bar{q}^2\mathcal{Q}, \bar{q}^2 = \frac{k}{2}\mathrm{Tr}(\mathcal{Q}^2) = m^2$$

$$\mathcal{Q}_{\psi}{}^3 = \alpha^2\bar{q}^2\mathcal{Q}_{\psi}, \alpha^2 = \frac{\mathrm{Tr}\left(\mathcal{Q}_{\psi}{}^2\right)}{\mathrm{Tr}(\mathcal{Q}^2)}$$

$$\mathcal{Q}_{\psi}{}^2\mathcal{Q} = \alpha^2\bar{q}^2\mathcal{Q};\ \mathcal{Q}^2\mathcal{Q}_{\psi} = \bar{q}^2\mathcal{Q}_{\psi}$$

$$c_{\mathrm{ex}}^2 = m^2 - \alpha^2 = \frac{k}{2}\mathrm{Tr}(\mathcal{Q}^2) - \frac{\mathrm{Tr}\left(\mathcal{Q}_{\psi}^2\right)}{\mathrm{Tr}(\mathcal{Q}^2)}$$

$$T = \frac{c_{\mathrm{ex}}}{2\pi\alpha|\omega_{\mathrm{H}}|} = \frac{c_{\mathrm{ex}}}{2\mathcal{S}},$$

$$\mathcal{S} = \frac{k_{\mathrm{B}}c^3}{\mathrm{G_N}\hbar}\frac{A_{\mathrm{H}}}{4} = \frac{A_{\mathrm{H}}}{4} = \pi\alpha|\omega_{\mathrm{H}}|,$$

$$\omega_{\mathrm{H}} = \lim_{r \to r_+}\omega_{\varphi};\ r_+ = m + c_{\mathrm{ex}}$$

$$m^2 \geq \alpha^2 \implies \frac{k}{2}\mathrm{Tr}(\mathcal{Q}^2) \geq \frac{\mathrm{Tr}\left(\mathcal{Q}_{\psi}^2\right)}{\mathrm{Tr}(\mathcal{Q}^2)}$$

$$\forall g \in G_4: \begin{cases} \phi_0^s \overset{g}{\rightarrow} \phi_0'^s \\ \Gamma^M \overset{g}{\rightarrow} \Gamma'^M \end{cases}$$

$$\forall g \in G_4: \mathcal{J}_{\psi}(\phi_0^s, \Gamma) \overset{g}{\rightarrow} \mathcal{J}_{\psi}(\phi_0'^s, \Gamma')$$

$$\begin{aligned} \mathcal{J}_{\psi}(\phi_0'^s, \Gamma') &= k\mathrm{Tr}\left(T_.^{\dagger}\mathcal{Q}'_{\psi}\right) = k\mathrm{Tr}\left(T_.^{\dagger}(g^{-1})^{\dagger}\mathcal{Q}_{\psi}g^{\dagger}\right) \\ &= k\mathrm{Tr}\left(T_.^{\dagger}\mathcal{Q}_{\psi}\right) = \mathcal{J}_{\psi}(\phi_0^s, \Gamma) \end{aligned}$$

$$\mathbb{J}_M = \left(\mathbb{J}_{\Lambda}, \mathbb{J}^{\Lambda}\right) \in \mathfrak{H}_3^*$$

$$\Gamma^M = \left(p^{\Lambda}, q_{\Lambda}\right)$$

$$\mathcal{N} = 8\text{: p} = \text{rank}\left(\frac{\text{SO}^*(16)}{\text{U}(8)}\right) = 4,$$

$$\mathcal{N} = 4\text{: p} = \text{rank}\left(\frac{\text{SO}(6,2) \times \text{SO}(2,6+n)}{\text{SO}(2)^2 \times \text{SO}(6) \times \text{SO}(6+n)}\right) = 4$$

$$\text{p} = \text{rank}\big(H_{(3)}/H^{\text{c}}_{(3)}\big) = 4$$

$$\mathbb{J}_M = \frac{1}{2}\big(T_M + (T_M)^\dagger\big) = \frac{1}{2}\Big(E_{\gamma_M} + \big(E_{\gamma_M}\big)^\dagger\Big)$$

$$\gamma_{\ell_1} \cdot \gamma_{\ell_2} \propto \delta_{\ell_1\ell_2}\text{: } J_\ell = \frac{1}{2}\Big(E_{\gamma_\ell} + \big(E_{\gamma_\ell}\big)^\dagger\Big)$$

$$\mathcal{M}^4_{\text{scal}} = \frac{G_4}{H_4} = \frac{\text{SL}(2,\mathbb{R})^3}{\text{SO}(2)^3}$$

$$\mathcal{M}^3_{\text{scal}} = \frac{G_3}{H_3} = \frac{\text{SO}(4,4)}{\text{SO}(2,2) \times \text{SO}(2,2)}$$

$$\Gamma_M = \mathbb{C}_{MN}\Gamma^N = \big(q_\Lambda, -p^\Lambda\big) \leftrightarrow \{\gamma_M\}$$

$$\vec{\gamma}_M = \left(\gamma_M[H_0], \frac{\gamma_M\big[H_{\alpha_1}\big]}{2}, \frac{\gamma_M\big[H_{\alpha_2}\big]}{2}, \frac{\gamma_M\big[H_{\alpha_3}\big]}{2}\right)$$

$$\{\vec{\gamma}_a\} = \left\{\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\right\}$$
$$\{\vec{\gamma}_{a+4}\} = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right\}$$

$$\{\gamma_\ell\} = \{\gamma_1, \gamma_6, \gamma_7, \gamma_8\}$$
$$\Gamma_M = (0, p_1, p_2, p_3, q_0, 0,0,0) \equiv \{q_0, p^i\}, (i = 1,2,3)$$

$$\{\gamma_{\ell'}\} = \{\gamma_2, \gamma_3, \gamma_4, \gamma_5\}$$
$$\Gamma_M = (p_0, 0,0,0,0, q_1, q_2, q_3) \equiv \{p^0, q_i\}, (i = 1,2,3)$$

$$\Gamma'_M \longrightarrow \{\gamma_1, \gamma_6, \gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_7, \gamma_8\}$$

$$\Gamma''_M \leftrightarrow \{\gamma_5, \gamma_2, \gamma_3, \gamma_4, \gamma_1, \gamma_6, \gamma_7, \gamma_8\}$$

$$\{J_\ell\}_{\ell=1,6,7,8};\ \{J_{\ell'}\}_{\ell'=2,3,4,5}$$

$$\mathcal{H} \in \exp\big(\mathbb{J}^{(N)}\big)\text{: } \mathcal{H} = \begin{cases} \exp\left(\sum_\ell \log(\beta_\ell) J_\ell\right) & \{q_0, p^i\} \\ \exp\left(\sum_{\ell'} \log(\beta_{\ell'}) J_{\ell'}\right) & \{p^0, q_i\} \end{cases}$$

$$\begin{aligned}\mathcal{Q} &\overset{\mathcal{H}}{\to} \mathcal{Q}' = (\mathcal{H}^{-1})^{\dagger}\mathcal{Q}\mathcal{H}^{\dagger},\\ \mathcal{Q}_{\psi} &\overset{\mathcal{H}}{\to} \mathcal{Q}'_{\psi} = (\mathcal{H}^{-1})^{\dagger}\mathcal{Q}_{\psi}\mathcal{H}^{\dagger}.\end{aligned}$$

$$\beta_{\ell} \to m^{\sigma_{\ell}}\beta_{\ell}, \alpha \to m\Omega, (\ell = 1,6,7,8),$$

$$\beta_{\ell'} \to m^{\sigma_{\ell'}}\beta_{\ell'}, \alpha \to m\Omega, (\ell' = 2,3,4,5),$$

$$\begin{aligned}I_4 > 0: \quad \mathcal{Q}_{\psi}^{(0)} = 0 \to \mathcal{J}_{\psi}^{(\mathrm{ex})} = 0\\ I_4 < 0: \quad \mathcal{Q}_{\psi}^{(0)} \neq 0 \to \mathcal{J}_{\psi}^{(\mathrm{ex})} \neq 0\end{aligned}$$

$$\mathcal{J}_{\psi}^{(\mathrm{ex})} = \frac{\Omega}{4}(1-\varepsilon)\sqrt{|I_4|}$$

$$I_4 = \varepsilon|I_4|(\varepsilon = \pm 1)$$

$$\Omega = \mathcal{J}_{\psi}^{\mathrm{Kerr}}/m^2$$

$$\Gamma_M^{(\mathrm{ex})} = \left(p^{(\mathrm{ex})\Lambda}, q_{\Lambda}^{(\mathrm{ex})}\right)$$

$$\begin{aligned}\mathcal{S}^{(\mathrm{ex})} &= \pi \lim_{m\to 0} \alpha|\omega_{\mathrm{H}}| = \pi \lim_{m\to 0} m\Omega|\omega_{\mathrm{H}}| = \pi\sqrt{|I_4| - 4\left(\mathcal{J}_{\psi}^{(\mathrm{ex})}\right)^2}\\ &= \pi\sqrt{|I_4|\left(1 - \frac{\Omega^2}{2}(1-\varepsilon)\right)}\end{aligned}$$

$$\mathbb{M}_4^{(1,3)} \times \mathcal{M}$$

$$\dim(G_{\mathrm{g}}) \leq n_{\mathrm{v}}.$$

$$\boldsymbol{\Omega}_{\mathbf{g}} = \boldsymbol{\Omega}_{\mathbf{g}\mu}dx^{\mu};\ \boldsymbol{\Omega}_{\mathbf{g}\mu} \equiv gA_{\mu}^{\hat{\Lambda}}X_{\hat{\Lambda}}$$

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = f_{\hat{\Lambda}\hat{\Sigma}}^{\hat{\Gamma}}X_{\hat{\Gamma}},$$

$$(X_{\hat{\Lambda}})_{\hat{N}}^{\hat{M}} = \begin{pmatrix} X_{\hat{\Lambda}}{}^{\hat{\Lambda}}{}_{\hat{\Sigma}} & \mathbb{O}\\ X_{\hat{\Lambda}\hat{\Gamma}\hat{\Sigma}} & X_{\hat{\Lambda}\hat{\Gamma}}{}^{\hat{\Delta}}\end{pmatrix}$$

$$\left(\mathbf{n}_{\mathrm{v}} = \mathrm{coadj}(G_{\mathrm{g}})\right)$$

$$f_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} = -X_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}},$$

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = -X_{\hat{\Lambda}\hat{\Sigma}}^{\hat{\Gamma}}X_{\hat{\Gamma}}$$

$$X_{(\hat{\Gamma}\hat{\Sigma})}^{\hat{\Lambda}} = 0$$

$$\delta_{\hat{\Lambda}}X_{\hat{\Sigma}} \equiv [X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] + X_{\hat{\Lambda}\hat{\Sigma}}^{\hat{\Gamma}}X_{\hat{\Gamma}} = 0$$

$$\boldsymbol{\Omega}_{\mathbf{g}} \longrightarrow \boldsymbol{\Omega}'_{\mathbf{g}} = \mathbf{g}\boldsymbol{\Omega}_{\mathbf{g}}\mathbf{g}^{-1} + d\mathbf{g}\mathbf{g}^{-1} = gA'^{\hat{\Lambda}}X_{\hat{\Lambda}}$$

$$\mathbf{g}(x) \equiv \mathbb{1} + g\zeta^{\hat{\Lambda}}(x)X_{\hat{\Lambda}}$$

$$\delta A_{\mu}^{\hat{\Lambda}} = D_{\mu}\zeta^{\hat{\Lambda}} \equiv \partial_{\mu}\zeta^{\hat{\Lambda}} + gA_{\mu}^{\hat{\Sigma}}X_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}}\zeta^{\hat{\Gamma}}$$

$$\mathcal{F} = F^{\hat{\Lambda}}X_{\hat{\Lambda}} = \frac{1}{2}F_{\mu\nu}^{\hat{\Lambda}}dx^{\mu} \wedge dx^{\nu}X_{\hat{\Lambda}} \equiv \frac{1}{g}\left(d\Omega_{\mathbf{g}} - \Omega_{\mathbf{g}} \wedge \Omega_{\mathbf{g}}\right)$$

$$F_{\mu\nu}^{\hat{\Lambda}} = \partial_{\mu}A_{\nu}^{\hat{\Lambda}} - \partial_{\nu}A_{\mu}^{\hat{\Lambda}} + gX_{\hat{\Gamma}\hat{\Sigma}}^{\hat{\Lambda}}A_{\mu}^{\hat{\Gamma}}A_{\nu}^{\hat{\Sigma}}$$

$$\mathbf{g}(x) \in G_{\mathbf{g}}$$

$$\mathcal{F} \to \mathcal{F}' = \mathbf{g}\mathcal{F}\mathbf{g}^{-1}$$

$$D\mathcal{F} \equiv d\mathcal{F} - \Omega_{\mathbf{g}} \wedge \mathcal{F} + \mathcal{F} \wedge \Omega_{\mathbf{g}} = 0 \;\Rightarrow\; DF^{\hat{\Lambda}} \equiv dF^{\hat{\Lambda}} + gX_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}}A^{\hat{\Sigma}} \wedge F^{\hat{\Lambda}} = 0$$

$$\partial_{\mu}A_{\nu}^{\hat{\Lambda}} - \partial_{\nu}A_{\mu}^{\hat{\Lambda}} \longrightarrow \partial_{\mu}A_{\nu}^{\hat{\Lambda}} - \partial_{\nu}A_{\mu}^{\hat{\Lambda}} + gX_{\hat{\Gamma}\hat{\Sigma}}^{\hat{\Lambda}}A_{\mu}^{\hat{\Gamma}}A_{\nu}^{\hat{\Sigma}}$$

$$\partial_{\mu} \;\longrightarrow\; D_{\mu} = \partial_{\mu} - gA_{\mu}^{\hat{\Lambda}}X_{\hat{\Lambda}}$$

$$D^{2} = -g\mathcal{F} = -gF^{\hat{\Lambda}}X_{\hat{\Lambda}} \;\Rightarrow\; \left[D_{\mu}, D_{\nu}\right] = -gF_{\mu\nu}^{\hat{\Lambda}}X_{\hat{\Lambda}}$$

$$\partial_{\mu}\phi^{s} \longrightarrow D_{\mu}\phi^{s} = \partial_{\mu}\phi^{s} - gA_{\mu}^{\hat{\Lambda}}k_{\hat{\Lambda}}^{s}(\phi),$$

$$\Omega_{\mu} = L^{-1}\partial_{\mu}L \;\longrightarrow\; \hat{\Omega}_{\mu} \equiv L^{-1}D_{\mu}L = L^{-1}\left(\partial_{\mu} - gA_{\mu}^{\hat{\Lambda}}X_{\hat{\Lambda}}\right)L = \hat{\wp}_{\mu} + \hat{w}_{\mu}$$

$$\hat{\wp}_{\mu} = \wp_{\mu} - gA_{\mu}^{\hat{\Lambda}}\wp_{\hat{\Lambda}};\; \hat{w}_{\mu} = w_{\mu} - gA_{\mu}^{\hat{\Lambda}}w_{\hat{\Lambda}}$$

$$\wp_{\hat{\Lambda}} \equiv L^{-1}X_{\hat{\Lambda}}L\big|_{\mathfrak{K}};\; w_{\hat{\Lambda}} \equiv L^{-1}X_{\hat{\Lambda}}L\big|_{\mathfrak{H}}$$

$$\hat{\Omega}_{\mu}(g \star \phi) = h\hat{\Omega}_{\mu}(\phi)h^{-1} + hdh^{-1} \Rightarrow \begin{cases}\hat{\wp}(g \star \phi) = h\hat{\wp}(\phi)h^{-1} \\ \hat{w}(g \star \phi) = h\hat{w}(\phi)h^{-1} + hdh^{-1}\end{cases}$$

$$d\hat{\Omega} + \hat{\Omega} \wedge \hat{\Omega} = -gL^{-1}\mathcal{F}L$$

$$\begin{aligned}\mathcal{D}\hat{\wp} &\equiv d\hat{\wp} + \hat{w} \wedge \hat{\wp} + \hat{\wp} \wedge \hat{w} = -gF^{\hat{\Lambda}}\wp_{\hat{\Lambda}} \\ \hat{R}(\hat{w}) &\equiv d\hat{w} + \hat{w} \wedge \hat{w} = -\hat{\wp} \wedge \hat{\wp} - gF^{\hat{\Lambda}}w_{\hat{\Lambda}}\end{aligned}$$

$$\hat{R}(\hat{w}) = \frac{1}{2}R_{su}\mathcal{D}\phi^{s} \wedge \mathcal{D}\phi^{u} - gF^{\hat{\Lambda}}w_{\hat{\Lambda}}.$$

$$D_{\mu}\xi = \nabla_{\mu}\xi + \hat{w}_{\mu} \star \xi$$

$$\wp_{\mu} \to \hat{\wp}_{\mu};\; w_{\mu} \to \hat{w}_{\mu}$$

$$\delta\mathcal{L}_{\mathrm{BOS}} = \frac{g}{8}\zeta^{\hat{\Lambda}}(x)X_{\hat{\Lambda}\hat{\Gamma}\hat{\Sigma}}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{\hat{\Gamma}}F_{\rho\sigma}^{\hat{\Sigma}}$$

$$\mathcal{L}_{\mathrm{top}} = \frac{1}{3}g\varepsilon^{\mu\nu\rho\sigma}X_{\hat{\Lambda}\hat{\Gamma}\hat{\Sigma}}A_{\mu}^{\hat{\Lambda}}A_{\nu}^{\hat{\Sigma}}\left(\partial_{\rho}A_{\sigma}^{\hat{\Gamma}} + \frac{3}{8}gX_{\hat{\Delta}\hat{\Pi}}^{\hat{\Gamma}}A_{\rho}^{\hat{\Delta}}A_{\sigma}^{\hat{\Pi}}\right)$$

$$X_{(\hat{\Lambda}\hat{\Gamma}\hat{\Sigma})} = 0$$

$$\mathcal{L}_{\text{RS}} = ie_{\text{D}}\bar{\psi}^A_\mu \gamma^{\mu\nu\rho} D_\nu \psi_{A\rho} + \text{ h.c.}$$

$$\delta\psi_\mu = D_\mu \epsilon + \cdots,$$

$$\delta\mathcal{L}_{\text{RS}} = \cdots + 2ie_{\text{D}}\bar{\psi}^A_\mu \gamma^{\mu\nu\rho} D_\nu D_\rho \epsilon_A + \text{ h.c. } = -ige_{\text{D}}\bar{\psi}^A_\mu \gamma^{\mu\nu\rho} F^{\hat{\Lambda}}_{\nu\rho} \left(w_{\hat{\Lambda}}\epsilon\right)_A + \text{ h.c.}$$

$$\delta\lambda^I = i\hat{\wp}^{IA}_\mu \gamma^\mu \epsilon_A + \cdots,$$

$$\delta\mathcal{L}'_{\text{kin}} = \cdots + ige_{\text{D}}\bar{\lambda}_I \gamma^{\mu\nu} F^{\hat{\Lambda}}_{\mu\nu} \wp^{IA}_{\hat{\Lambda}} \epsilon_A + \text{ h.c.}$$

$$\mathcal{L}_{\text{gaug}} = \mathcal{L}^{\dagger}_{\text{gaug}} + \Delta\mathcal{L}^{\star}_{\text{gaug}} + \Delta\mathcal{L}^{\otimes}_{\text{gaug}}$$

$$e_{\text{D}}^{-1}\Delta\mathcal{L}^{\triangle}_{\text{gaug}} = g\left(2\bar{\psi}^A_\mu \gamma^{\mu\nu} \psi^B_\nu \mathbb{S}_{AB} + i\bar{\lambda}^I \gamma^\mu \psi_{\mu A} \mathbb{N}^A_I + \bar{\lambda}^I \lambda^J \mathbb{M}_{IJ}\right) + \text{ h.c.}$$

$$e_{\text{D}}^{-1}\Delta\mathcal{L}^{\boxtimes}_{\text{gaug}} = -g^2 V(\phi)$$

$$\delta_\epsilon \psi_{\mu A} = D_\mu \epsilon_A + ig\mathbb{S}_{AB}\gamma_\mu \epsilon^B + \cdots$$
$$\delta_\epsilon \lambda_I = g\mathbb{N}^A_I \epsilon_A + \cdots$$

$$\delta_B{}^A V(\phi) = g^2(\mathbb{N}_I{}^A \mathbb{N}^I{}_B - 12\mathbb{S}^{AC}\mathbb{S}_{BC})$$

$$\mathbb{N}^I{}_A \equiv (\mathbb{N}_I{}^A)^* \text{ and } \mathbb{S}^{AB} \equiv (\mathbb{S}_{AB})^*$$

$$X_{\hat{\Lambda}} = \theta^\sigma_{\hat{\Lambda}} t_\sigma;\ \theta^\sigma_{\hat{\Lambda}} \in \mathbf{n}_{\text{v}} \times \text{Adj}(G_{\text{el}})$$

$$X_M = \left(X_\Lambda, X^\Lambda\right),$$

$$\begin{pmatrix} X_{\hat{\Lambda}} \\ 0 \end{pmatrix} = E \begin{pmatrix} X_\Lambda \\ X^\Lambda \end{pmatrix}$$

$$A^{\hat{\Lambda}} X_{\hat{\Lambda}} = A^{\hat{\Lambda}} E_{\hat{\Lambda}}{}^\Lambda X_\Lambda + A^{\hat{\Lambda}} E_{\hat{\Lambda}\Lambda} X^\Lambda = A^\Lambda_\mu X_\Lambda + A_{\Lambda\mu} X^\Lambda = \mathbb{A}^M_\mu X_M,$$

$$\mathbb{A}^M_\mu \equiv \left(A^\Lambda_\mu, A_{\Lambda\mu}\right)$$

$$A^\Lambda_\mu = E_{\hat{\Lambda}}{}^\Lambda A^{\hat{\Lambda}}_\mu, A_{\Lambda\mu} = E_{\hat{\Lambda}\Lambda} A^{\hat{\Lambda}}_\mu$$

$$X_M = \Theta_M{}^\alpha t_\alpha, \alpha = 1, \dots, \dim(G).$$

$$\theta_\Lambda{}^\sigma \longrightarrow \Theta_M{}^\alpha \equiv \left(\theta^{\Lambda\alpha}, \theta_\Lambda{}^\alpha\right);\ \Theta_M{}^\alpha \in \mathcal{R}_{\text{v}^*} \times \text{Adj}(G),$$

$$\theta^\alpha_{\hat{\Lambda}} = E^M_{\hat{\Lambda}} \Theta^\alpha_M, 0 = E^{\hat{\Lambda}M} \Theta^\alpha_M$$

$$\dim\left(G_{\mathbf{g}}\right) = \text{rank}(\theta) = \text{rank}(\Theta)$$

$$\Theta_\Lambda{}^\alpha \Theta^{\Lambda\beta} - \Theta_\Lambda{}^\beta \Theta^{\Lambda\alpha} = 0 \implies \mathbb{C}^{MN} \Theta_M{}^\alpha \Theta_N{}^\beta = 0$$

$$\dim\left(G_{\mathbf{g}}\right) = \text{rank}(\Theta) \leq n_{\text{v}},$$

$$X_{MN}{}^{P} \equiv \Theta_M{}^{\alpha} t_{\alpha N}{}^{P} = E^{-1}{}_M{}^{\hat{M}} E^{-1}{}_N^{\hat{N}} X_{\hat{M}\hat{N}}^{\hat{P}} E_{\hat{P}}{}^{P}$$

$$X_{MNP} \equiv X_{MN}{}^{Q} \mathbb{C}_{QP} = X_{MPN}$$

$$X_{(MNP)} = 0 \quad \Longrightarrow \quad \begin{cases} 2\, X_{(\Lambda\Sigma)}{}^{\Gamma} = X^{\Gamma}{}_{\Lambda\Sigma}\,, \\ 2\, X^{(\Lambda\Sigma)}{}_{\Gamma} = X_{\Gamma}{}^{\Lambda\Sigma}\,, \\ X_{(\Lambda\Sigma\Gamma)} = 0\,. \end{cases}$$

$$[X_M, X_N] = -X_{MN}^{P} X_P \implies \Theta_M^{\alpha}\Theta_N^{\beta} f_{\alpha\beta}^{\gamma} + \Theta_M^{\alpha} t_{\alpha N}^{P} \Theta_P^{\gamma} = 0$$

$$\delta_M \Theta_N{}^{\alpha} = 0$$

$$X_{(MNP)} = 0,$$
$$\mathbb{C}^{MN}\Theta_M^{\alpha}\Theta_N^{\beta} = 0,$$
$$[X_M, X_N] = -X_{MN}^{P} X_P.$$

$$\mathcal{R}_{\mathrm{v}^*} \times \mathrm{Adj}(G) \xrightarrow{G} \mathcal{R}_{\Theta} + \cdots$$

$$\mathbb{P}_{\Theta} \cdot \Theta = \Theta$$

$$X_{(MN)}{}^{P} X_P = 0$$

$$\langle \phi^s(x) \rangle \equiv \phi_0^s \equiv \phi_0,$$

$$\left.\frac{\partial V}{\partial \phi^s}\right|_{\phi_0} = 0.$$

$$\Lambda = V(\phi_0)$$

$$R_{\mu\nu\rho\sigma} = -\frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}),$$

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}$$

$$\forall \mathbf{g} \in G: V(\mathbf{g} \star \phi, \mathbf{g} \star \Theta) = V(\phi, \Theta).$$

$$\left.\frac{\partial}{\partial \phi^s} V(\phi, \Theta)\right|_{\phi_0} = 0$$

$$\forall \mathbf{g} \in G: \left.\frac{\partial}{\partial \phi^s} V(\phi, \mathbf{g} \star \Theta)\right|_{\mathbf{g}\star\phi_0} = 0$$

$$\Theta' = L(\phi_0)^{-1} \star \Theta$$

$$\mathbb{N}(\mathcal{O}, \Theta), \mathbb{S}(\mathcal{O}, \Theta), \mathbb{M}(\mathcal{O}, \Theta)$$

$$\delta_\epsilon \mathrm{f}(x) = \langle 0|[\bar{\epsilon}Q, \hat{\mathrm{f}}(x)]|0\rangle = 0$$

$$\begin{aligned} \delta\psi_{\mu A} &= D_\mu \epsilon_A + ig\mathbb{S}_{AB}\gamma_\mu \epsilon^B = 0 \\ \delta\lambda_I &= g\mathbb{N}_I{}^A \epsilon_A = 0 \end{aligned}$$

$$0 = \nabla_{[\mu}\delta\psi_{\nu]A}$$

$$\frac{1}{e_{\mathrm{D}}}\mathcal{L}_{\text{dark particle}} = -\frac{R}{2} + g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8e_{\mathrm{D}}}\mathcal{R}_{\Lambda\Sigma}(z,\bar{z})\varepsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma} - V(z,\bar{z})$$

$$F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$$

$$\Omega^M = \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix}, \Lambda = 0, \dots, n_{\mathrm{v}}$$

$$\begin{aligned} \mathcal{K}(z,\bar{z}) &= -\log\,[i\bar{\Omega}^T \mathbb{C}\Omega] = -\log\left[i\left(\overline{\mathcal{X}}^\Lambda \mathcal{F}_\Lambda - \mathcal{X}^\Lambda \overline{\mathcal{F}}_\Lambda\right)\right] \\ g_{i\bar{j}} &= \partial_i \partial_{\bar{j}}\mathcal{K} \end{aligned}$$

$$\mathrm{Sp}(2(n_{\mathrm{v}}+1), \mathbb{R})$$

$$\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) - f(z) - \bar{f}(\bar{z})$$

$$\mathcal{F}_\Lambda = \frac{\partial \mathcal{F}}{\partial \mathcal{X}^\Lambda}$$

$$\Phi(z,\bar{z}) \to e^{ip\mathrm{Im}[f]}\Phi(z,\bar{z})$$

$$\begin{aligned} \mathcal{D}_i^{[\mathrm{U}(1)]}\Phi &\equiv \left(\partial_i + \frac{p}{2}\partial_i \mathcal{K}\right)\Phi \\ \mathcal{D}_{\bar{\imath}}^{[\mathrm{U}(1)]}\Phi &\equiv \left(\partial_{\bar{\imath}} - \frac{p}{2}\partial_{\bar{\imath}}\mathcal{K}\right)\Phi \end{aligned}$$

$$\mathcal{V}^M = e^{\frac{\mathcal{K}}{2}}\Omega^M = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix}$$

$$\mathcal{D}_{\bar{\imath}}\mathcal{V}^M = \left(\partial_{\bar{\imath}} - \frac{1}{2}\partial_{\bar{\imath}}\mathcal{K}\right)\mathcal{V}^M = 0$$

$$\mathcal{D}_i\mathcal{V}^M = \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)\mathcal{V}^M = \begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix} \equiv \mathcal{U}_i^M$$

$$\mathcal{V}^M \to e^{i\mathrm{Im}[f]}\mathcal{V}^M$$

$$\mathcal{V}^T\mathbb{C}\overline{\mathcal{V}} = i$$

$$\begin{aligned} \mathcal{D}_i\mathcal{U}_j &= i\mathcal{C}_{ijk}g^{k\bar{k}}\overline{\mathcal{U}}_{\bar{k}} \\ \mathcal{D}_i\overline{\mathcal{U}}_{\bar{j}} &= g_{i\bar{j}}\overline{\mathcal{V}} \\ \mathcal{V}^T\mathbb{C}\mathcal{U}_i &= 0 \\ \mathcal{U}_i^T\mathbb{C}\overline{\mathcal{U}}_{\bar{j}} &= -ig_{i\bar{j}} \end{aligned}$$

$$g^{i\bar{j}}\mathcal{U}_i^M\overline{\mathcal{U}}_{\bar{j}}^N = -\frac{1}{2}\mathcal{M}^{MN} - \frac{i}{2}\mathbb{C}^{MN} - \mathcal{V}^M\overline{\mathcal{V}}^N$$

$$\mathcal{M}^{MN} = -\mathbb{C}^{MP}\mathcal{M}_{PQ}\mathbb{C}^{QN}$$

$$\mathcal{M}_{\text{scal}} = \mathcal{M}_{\text{SK}} \times \mathcal{M}_{\text{QK}},$$

$$H = H_{\text{R}} \times H_{\text{matt}},$$

$$H = H^{\text{SK}} \times H^{\text{QK}}$$

$$H^{\text{SK}} = \text{U}(1) \times H_{\text{matt}}^{\text{SK}}, H^{\text{QK}} = \text{SU}(2) \times H_{\text{matt}}^{\text{QK}}$$

$$V = \left(g^{i\bar{j}}\mathcal{U}_i^M\overline{\mathcal{U}}_{\bar{j}}^N - 3\mathcal{V}^M\overline{\mathcal{V}}^N\right)\theta_M\theta_N = -\frac{1}{2}\theta_M\mathcal{M}^{MN}\theta_N - 4\mathcal{V}^M\overline{\mathcal{V}}^N\theta_M\theta_N$$

$$\mathcal{W} = \mathcal{V}^M\theta_M$$

$$V = g^{i\bar{J}}\mathcal{D}_i\mathcal{W}\mathcal{D}_{\bar{j}}\overline{\mathcal{W}} - 3|\mathcal{W}|^2$$

$$V = 4g^{i\bar{j}}\partial_i\mathcal{W}\partial_{\bar{j}}\mathcal{W} - 3\mathcal{N}^2.$$

$$\mathbb{F}_{\mu\nu}^M = \begin{pmatrix} F_{\mu\nu}^\Lambda \\ \mathcal{G}_{\Lambda\mu\nu} \end{pmatrix}$$

$$d\mathbb{F}^M = 0,\ {}^*\mathbb{F}^M = -\mathbb{C}^{MP}\mathcal{M}_{PN}(z,\bar{z})\mathbb{F}^N$$

$$\nabla_\mu\left(\partial^\mu z^i\right) + \tilde{\Gamma}_{jk}^i\partial_\mu z^j\partial^\mu z^k - \frac{1}{8}g^{i\bar{j}}\mathbb{F}_{\mu\nu}^M\partial_{\bar{j}}\mathcal{M}_{MN}(z,\bar{z})\mathbb{F}^{N\mu\nu} + g^{i\bar{j}}\partial_{\bar{j}}V = 0$$

$$R_{\mu\nu} = 2\partial_{(\mu}z^i\partial_{\nu)}\bar{z}^{\bar{j}}g_{i\bar{j}} + \frac{1}{2}\mathbb{F}_{\mu\rho}^M\mathcal{M}_{MN}(z,\bar{z})\mathbb{F}_{\nu}^{N}{}_{\nu}^{\rho} - g_{\mu\nu}V$$

$$z^i \to z'^i\left(z^j\right): \begin{cases} \mathcal{V}^M(z',\bar{z}') = e^{i\text{Im}(f)}(S^{-1})_N^M\mathcal{V}^N(z,\bar{z}) \\ \theta_M \to \theta'_M = S_M^N\theta_N \\ \mathbb{F}^M \to \mathbb{F}^M = (S^{-1})_N^M\mathbb{F}^N \end{cases}$$

$$S \in \text{Sp}(2(n_{\text{v}}+1),\mathbb{R})$$

$$ds^2 = e^{2U(r)}dt^2 - e^{-2U(r)}\left(dr^2 + e^{2\Psi(r)}d\Sigma_\kappa^2\right)$$

$$d\Sigma_\kappa^2 = d\vartheta^2 + f_\kappa^2(\vartheta)d\varphi^2$$

$$\Sigma_\kappa = \{\mathbb{S}^2, \mathbb{H}^2\}$$

$$f_\kappa(\vartheta) = \frac{1}{\sqrt{\kappa}}\sin\left(\sqrt{\kappa}\vartheta\right) = \begin{cases} \sin(\vartheta), & \kappa = 1 \\ \sinh(\vartheta), & \kappa = -1 \end{cases}$$

$$\mathbb{F}^M = \begin{pmatrix} F^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix} = e^{2(U-\Psi)}\mathbb{C}^{MP}\mathcal{M}_{PN}\Gamma^N dt \wedge dr + \Gamma^M f_\kappa(\vartheta)d\vartheta \wedge d\varphi = d\mathbb{A}^M$$

$$e_\Lambda \equiv \frac{1}{\mathrm{vol}(\Sigma_\kappa)}\int_{\Sigma_\kappa} \mathcal{G}_\Lambda$$
$$m^\Lambda \equiv \frac{1}{\mathrm{vol}(\Sigma_\kappa)}\int_{\Sigma_\kappa} F^\Lambda$$

$$\mathrm{vol}(\Sigma_\kappa) = \int f_\kappa(\vartheta) d\vartheta \wedge d\varphi$$

$$\Gamma^M = \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} = \frac{1}{\mathrm{vol}(\Sigma_\kappa)}\int_{\Sigma_\kappa} \mathbb{F}^M$$

$$\mathcal{S}_{\mathrm{eff}} = \int dr \mathcal{L}_{\mathrm{eff}} = \int dr\left[e^{2\Psi}\left(U'^2 - \Psi'^2 + g_{i\bar{\jmath}} z'^i \bar{z}'^{\bar{\jmath}}\right) - V_{\mathrm{eff}}\right]$$

$$V_{\mathrm{eff}} = -e^{2(U-\Psi)} V_{\mathrm{BH}} - e^{-2(U-\Psi)} V + \kappa$$

$$V_{\mathrm{BH}} = -\frac{1}{2}\Gamma^T \mathcal{M}\Gamma,$$

$$\delta\psi_{\mu A} = D_\mu \epsilon_A + i T^-_{\mu\nu}\gamma^\nu \varepsilon_{AB}\epsilon^B + i\mathbb{S}_{AB}\gamma_\mu \epsilon^B$$
$$\delta\lambda^{iA} = i\partial_\mu z^i \gamma^\mu \epsilon^A - \frac{1}{2} g^{i\bar{\jmath}} \bar{f}^\Lambda_{\bar{\jmath}} \mathcal{I}_{\Lambda\Sigma} F^{-\Sigma}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B$$

$$\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}$$

$$D_\mu \epsilon_A = \partial_\mu \epsilon_A + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon_A + \frac{i}{2}(\sigma^2)_A^B \mathbb{A}_\mu^M \theta_M \epsilon_B + \frac{i}{2}\mathcal{Q}_\mu \epsilon_A$$

$$\mathcal{Q}_\mu = \frac{i}{2}\left(\partial_{\bar{\imath}}\mathcal{K}\partial_\mu \bar{z}^{\bar{\imath}} - \partial_i \mathcal{K}\partial_\mu z^i\right)$$

$$F^\pm_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm {}^*F_{\mu\nu}\right), \mathcal{G}^\pm_{\mu\nu} = \frac{1}{2}\left(\mathcal{G}_{\mu\nu} \pm {}^*\mathcal{G}_{\mu\nu}\right)$$
$$T_{\mu\nu} = L^\Lambda \mathcal{I}_{\Lambda\Sigma} F^\Sigma_{\mu\nu} = \frac{1}{2i} L^\Lambda (\mathfrak{N} - \overline{\mathfrak{N}})_{\Lambda\Sigma} F^\Sigma_{\mu\nu} = -\frac{i}{2}\left(M_\Sigma F^\Sigma_{\mu\nu} - L^\Lambda \mathcal{G}_{\Lambda\mu\nu}\right) = \frac{i}{2}\mathcal{V}^M \mathbb{C}_{MN}\mathbb{F}^N_{\mu\nu}$$
$$T^-_{\mu\nu} = L^\Lambda \mathcal{I}_{\Lambda\Sigma} F^{-\Sigma}_{\mu\nu} = \frac{i}{2}\mathcal{V}^M \mathbb{C}_{MN}\mathbb{F}^{-N}_{\mu\nu}$$
$$T_{i\mu\nu} = \mathcal{D}_i T_{\mu\nu} = f_i^\Lambda \mathcal{I}_{\Lambda\Sigma} F^\Sigma_{\mu\nu} = -\frac{i}{2}\left(h_{i\Sigma} F^\Sigma_{\mu\nu} - f_i^\Lambda \mathcal{G}_{\Lambda\mu\nu}\right) = \frac{i}{2}\mathcal{U}_i^M \mathbb{C}_{MN}\mathbb{F}^N_{\mu\nu}$$
$$\mathbb{S}_{AB} = \frac{i}{2}(\sigma^2)_A{}^C \varepsilon_{BC}\theta_M \mathcal{V}^M = \frac{i}{2}(\sigma^2)_A{}^C \varepsilon_{BC}\mathcal{W}$$
$$W^{iAB} = i(\sigma^2)_C{}^B \varepsilon^{CA}\theta_M g^{i\bar{\jmath}}\overline{\mathcal{U}}^M_{\bar{\jmath}}$$

$$\overline{\mathfrak{N}}_{\Lambda\Sigma} F^{-\Sigma} = \mathcal{G}^-_\Lambda, L^\Lambda \mathfrak{N}_{\Lambda\Sigma} = M_\Sigma$$

$$\mathfrak{N} = \mathcal{R} + i\mathcal{I}$$

$$\mathfrak{N}_{\Lambda\Sigma} = \partial_{\bar{\Lambda}}\partial_{\bar{\Sigma}}\overline{\mathcal{F}} + 2i\frac{\mathrm{Im}[\partial_\Lambda \partial_\Gamma \mathcal{F}]\mathrm{Im}[\partial_\Sigma \partial_\Delta \mathcal{F}] L^\Gamma L^\Delta}{\mathrm{Im}[\partial_\Delta \partial_\Gamma \mathcal{F}] L^\Delta L^\Gamma},$$

$$\partial_\Lambda = \frac{\partial}{\partial \mathcal{X}^\Lambda}, \partial_{\bar{\Lambda}} = \frac{\partial}{\partial \overline{\mathcal{X}}^\Lambda}$$

$$\mathcal{Z} = \frac{1}{\mathrm{vol}(\Sigma_\kappa)} \int_{\Sigma_\kappa} T = \mathcal{V}^M \mathbb{C}_{MN} \Gamma^N = L^\Lambda e_\Lambda - M_\Lambda q^\Lambda$$
$$\mathcal{Z}_i = \frac{1}{\mathrm{vol}(\Sigma_\kappa)} \int_{\Sigma_\kappa} T_i = \mathcal{D}_i \mathcal{Z} = f_i^\Lambda e_\Lambda - h_{\Lambda i} q^\Lambda$$

$$V_{\mathrm{BH}} = |\mathcal{D}\mathcal{Z}| - |\mathcal{Z}|^2.$$

$$U' = e^{U-2\Psi} \mathrm{Re}\left[e^{-i\alpha}\mathcal{Z}\right] + e^{-U} \mathrm{Im}\left[e^{-i\alpha}\mathcal{W}\right]$$
$$\Psi' = 2e^{-U} \mathrm{Im}\left[e^{-i\alpha}\mathcal{W}\right]$$

$$z'^i = e^{-U} e^{i\alpha} g^{i\bar{j}} \mathcal{D}_{\bar{j}} \left(e^{2U-2\Psi} \overline{\mathcal{Z}} - i\overline{\mathcal{W}}\right),$$

$$\gamma^0 \epsilon_A = i e^{i\alpha} \varepsilon_{AB} \epsilon^B$$
$$\gamma^1 \epsilon_A = e^{i\alpha} \delta_{AB} \epsilon^B$$

$$\epsilon_A = \chi_A e^{\frac{1}{2}(U - i\int dr\mathcal{B})}$$
$$\epsilon^A = i e^{-i\alpha} \varepsilon^{AB} \gamma^0 \epsilon_B$$

$$\partial_r \chi_A = 0$$
$$\mathcal{B} = \mathcal{Q}_r + 2e^{-U} \mathrm{Re}\left[e^{-i\alpha}\mathcal{W}\right]$$

$$\partial_r \alpha = -\mathcal{B}.$$

$$\mathrm{Im}\left[e^{-i\alpha}\mathcal{Z}\right] = -e^{2\Psi-2U} \mathrm{Re}\left[e^{-i\alpha}\mathcal{W}\right],$$

$$\mathbb{A}_t^M \theta_M = 2e^U \mathrm{Re}\left[e^{-i\alpha}\mathcal{W}\right]$$
$$\mathbb{A}_r^M = 0$$
$$\mathbb{A}_\vartheta^M = 0$$
$$\mathbb{A}_\varphi^M = -\frac{\Gamma^M}{\kappa} \cos\left(\sqrt{\kappa}\vartheta\right)$$

$$\Gamma^M \theta_M = \kappa.$$

$$\mathcal{M}_{\mathrm{scal}}^3 = \frac{G_3}{H_3^*}$$

$$\mathcal{M}_{\mathrm{scal}}^{(3)} \supset \mathcal{U} \equiv e^J.$$

$$L(\Phi^I) = \exp\left(-aT_\cdot\right) \exp\left(\sqrt{2}\mathcal{Z}^M T_M\right) \exp\left(\phi^s T_s\right) \exp\left(2UH_0\right)$$

$$[H_0, T_M] = \frac{1}{2} T_M; \quad [H_0, T_s] = [T_\cdot, T_s] = 0; \quad [H_0, T_\cdot] = T_\cdot;$$
$$[T_M, T_N] = \mathbb{C}_{MN} T_\cdot; \quad [T_s, T_M] = (T_s)^N{}_M T_N; \quad [T_s, T_u] = -(T_{su})^{s'} T_{s'}$$

$$\mathcal{N} = 2 \Rightarrow n_v = \frac{n_s}{2} + 1 \Rightarrow \dim\left(\mathcal{M}_{\text{scal}}^{(3)}\right) = 4n_v$$

$$\zeta(\mathfrak{H}_3^*) = \mathfrak{H}_3^*$$

$$\zeta(X) = -\eta X^\dagger \eta,$$

$$(\eta = \eta^\dagger, \eta^2 = \mathbb{1})$$

$$\mathfrak{g}_3 = \mathfrak{H}_3^* \oplus \mathfrak{K}_3^*,$$

$$\zeta: \zeta(\mathfrak{H}_3^*) = \mathfrak{H}_3^*, \zeta(\mathfrak{K}_3^*) = -\mathfrak{K}_3^*,$$

$$[\mathfrak{H}_3^*, \mathfrak{H}_3^*] \subset \mathfrak{H}_3^*, [\mathfrak{H}_3^*, \mathfrak{K}_3^*] \subset \mathfrak{K}_3^*, [\mathfrak{K}_3^*, \mathfrak{K}_3^*] \subset \mathfrak{H}_3^*.$$

$$H_{(3)}^* = \mathrm{SL}(2, \mathbb{R}) \times G'_{(4)},$$

$$\mathfrak{g}_3 = \mathfrak{H}_3 \oplus \mathfrak{K}_3,$$

$$\tau(X) = -X^\dagger.$$

$$\mathcal{M}_{(3)}(\Phi^I) = L(\Phi^I) \eta L(\Phi^I)^\dagger.$$

$$\mathcal{V}^\mathcal{A} = \mathcal{V}_I^\mathcal{A} d\phi^I$$

$$L^{-1} dL = \mathcal{V}^\mathcal{A} T_\mathcal{A} = \wp + w; \; \mathcal{A} = 1, \ldots, \dim\left(\mathcal{M}_{\text{scal}}^{(3)}\right),$$

$$\wp = \mathcal{V}^\mathcal{A} \mathbb{K}_\mathcal{A} \mathfrak{K}_3^*$$

$$\mathbb{K}_\mathcal{A} = \frac{1}{2}(T_\mathcal{A} + \eta T_\mathcal{A}^\dagger \eta),$$

$$g_{\mathcal{A}\mathcal{B}} = k \mathrm{Tr}[\mathbb{K}_\mathcal{A} \mathbb{K}_\mathcal{B}]$$

$$k = \frac{1}{2\mathrm{Tr}(H_0^2)}$$

$$ds^2 = k\mathrm{Tr}(\wp^2) = g_{\mathcal{A}\mathcal{B}} \wp^\mathcal{A} \wp^\mathcal{B}$$
$$= 2dU^2 + \mathcal{G}_{su} d\phi^s d\phi^u + \frac{1}{2} e^{-4U} \omega^2 + e^{-2U} d\mathcal{Z}^T \mathcal{M}_{(4)}(\phi^s) d\mathcal{Z}$$

$$\omega = da + \mathcal{Z}^T \mathbb{C} d\mathcal{Z}.$$

$$\mathcal{M}_{\text{scal}} = \mathcal{M}_{\text{SK}} \times \mathcal{M}_{\text{QK}}.$$

$$\mathcal{M}_{\text{scal}}^{(4)} = \frac{G_{(4)}}{H_{(4)}} = \left(\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)}\right)^3$$

$$ds^2_{(4)\mathrm{STU}} = \mathcal{G}_{su} d\phi^s d\phi^u = 2g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}} = -2\sum_{a=1}^{3} \frac{dz^a d\bar{z}^{\bar{a}}}{(z^a - \bar{z}^{\bar{a}})^2} = \sum_{\jmath=1}^{3} e_i{}^{\jmath} \bar{e}_{\bar{\imath}}^{\bar{\jmath}} dz^i d\bar{z}^{\bar{\imath}}.$$

$$\phi^s = \{\epsilon_i, \varphi_i\} \implies z_i = \epsilon_i - ie^{\varphi_i}.$$

$$e^{-\mathcal{K}} = 8e^{\varphi_1+\varphi_2+\varphi_3},$$

$$\mathcal{F}(z) = z_1 z_2 z_3.$$

$$\Omega^M(z) = \{1, z_1, z_2, z_3, -z_1 z_2 z_3, z_2 z_3, z_1 z_3, z_1 z_2\},$$

$$\mathcal{V}^M(z,\bar{z}) = e^{\frac{\mathcal{K}}{2}} \Omega^M(z).$$

$$\mathcal{D}_i \mathcal{V} := \partial_i \mathcal{V} + \frac{\partial_i \mathcal{K}}{2} \mathcal{V},$$

$$\begin{aligned}
\mathcal{Z} &= \mathcal{V}^T \mathbb{C}\Gamma \\
&= e^{\frac{\mathcal{K}}{2}}(-q_0 - q_1 z_1 - q_2 z_2 + p^3 z_1 z_2 - q_3 z_3 + p^2 z_1 z_3 + p^1 z_2 z_3 - p^0 z_1 z_2 z_3) \\
\mathcal{Z}_1 &= e_1{}^i \mathcal{D}_i \mathcal{V}^T \mathbb{C}\Gamma \\
&= -ie^{\frac{\mathcal{K}}{2}}(q_0 + q_2 z_2 + q_3 z_3 - p^1 z_2 z_3 + q_1 \bar{z}_1 - p^3 z_2 \bar{z}_1 - p^2 z_3 \bar{z}_1 + p^0 z_2 z_3 \bar{z}_1) \\
\mathcal{Z}_2 &= e_2{}^i \mathcal{D}_i \mathcal{V}^T \mathbb{C}\Gamma \\
&= -ie^{\frac{\mathcal{K}}{2}}(q_0 + q_1 z_1 + q_3 z_3 - p^2 z_1 z_3 + q_2 \bar{z}_2 - p^3 z_1 \bar{z}_2 - p^1 z_3 \bar{z}_2 + p^0 z_1 z_3 \bar{z}_2) \\
\mathcal{Z}_3 &= e_3{}^i \mathcal{D}_i \mathcal{V}^T \mathbb{C}\Gamma \\
&= -ie^{\frac{\mathcal{K}}{2}}(q_0 + q_1 z_1 + q_2 z_2 - p^3 z_1 z_2 + q_3 \bar{z}_3 - p^2 z_1 \bar{z}_3 - p^1 z_2 \bar{z}_3 + p^0 z_1 z_2 \bar{z}_3)
\end{aligned}$$

$$\begin{aligned}
I_4(p,q) =& -(p^0)^2 q_0^2 - 2q_0(-2p^1 p^2 p^3 + p^0 q_3 p^3 + p^0 p^1 q_1 + p^0 p^2 q_2) - (p^1)^2 q_1^2 \\
& -(p^2 q_2 - p^3 q_3)^2 + 2q_1\big(p^1 p^3 q_3 + q_2(p^1 p^2 - 2p^0 q_3)\big).
\end{aligned}$$

$$\mathcal{M}^{(3)}_{\mathrm{scal}} = \frac{G_{(3)}}{H^*_{(3)}} = \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(2,2) \times \mathrm{SO}(2,2)}.$$

$$[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha};\ [E_\alpha, E_{-\alpha}] = H_\alpha,$$

$$E_{-\alpha} = E_\alpha^\dagger = E_\alpha^T.$$

$$T_s = \{E_{\alpha_i}, H_{\alpha_i}/2\}$$

$$L_{(4)}(\phi^s) = \exp\,(\phi^s T_s) = \prod_{i=1}^{3} e^{\epsilon_i E_{\alpha_i}} e^{\varphi_i \frac{H_{\alpha_i}}{2}}$$

$$\mathfrak{h}_{\mathrm{B}} = \mathrm{Span}(T_{\mathcal{A}}), T_{\mathcal{A}} = \{H_0, T_{\cdot}, T_s, T_M\}$$

$$H_0 = \frac{H_{\beta_0}}{2};\ T_{\cdot} = E_{\beta_0};\ T_M = E_{\gamma_M}$$

$$[T_s, T_M] = -(T_s)_M^N T_N$$

$$\mathcal{M}_{(4)MN} = -\sum_{P=1}^{8} \left(L_{(4)}\right)_{M}{}^{P}\left(L_{(4)}\right)_{N}{}^{P}$$

$$\phi^{s}T_{s} = \sum_{i=1}^{3} \epsilon_{i}E_{\alpha_{i}} + \varphi_{i}\frac{H_{\alpha_{i}}}{2} = \begin{pmatrix} A & B \\ \mathbb{0} & -A^{T} \end{pmatrix},$$

$$A=\begin{pmatrix} \frac{\varphi_1}{2}+\frac{\varphi_2}{2}+\frac{\varphi_3}{2} & -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ 0 & -\frac{\varphi_1}{2}+\frac{\varphi_2}{2}+\frac{\varphi_3}{2} & 0 & 0 \\ 0 & 0 & \frac{\varphi_1}{2}-\frac{\varphi_2}{2}+\frac{\varphi_3}{2} & 0 \\ 0 & 0 & 0 & \frac{\varphi_1}{2}+\frac{\varphi_2}{2}-\frac{\varphi_3}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & -\epsilon_2 \\ 0 & -\epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & -\epsilon_1 & 0 \end{pmatrix}$$

$$\frac{1}{e_{\mathrm{D}}}\mathcal{L}_{(4)} = -\frac{R}{2} + \frac{1}{2}g^{\mu\nu}\langle J_{\mu}, J_{\nu}\rangle + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8e_{\mathrm{D}}}\mathcal{R}_{\Lambda\Sigma}(\phi)\varepsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}$$

$$J_{\mu} = \frac{1}{2}\mathcal{M}^{-1}\partial_{\mu}\mathcal{M}$$

$$\mathcal{M}^{(4)}_{\text{scal}} = \frac{G_{(4)}}{H_{(4)}},$$

$$g_{\mu\nu} = \begin{pmatrix} \Upsilon & \Upsilon\omega_{j} \\ \Upsilon\omega_{i} & -\Upsilon^{-1}g^{(3)}_{ij} + \Upsilon\omega_{i}\omega_{j} \end{pmatrix},$$

$$A^{\Lambda}_{\mu} = \left(A^{\Lambda}_{0}, A^{\Lambda}_{0}\omega_{i} + A^{\Lambda}_{i}\right) = \left(\mathcal{Z}^{\Lambda}, \mathcal{Z}^{\Lambda}\omega_{i} + A^{\Lambda}_{i}\right)$$

$$\begin{aligned} \frac{1}{e^{(3)}_{\mathrm{D}}}\tilde{\mathcal{L}} =& +\frac{R^{(3)}}{2} - \frac{1}{2}g^{(3)ij}\langle \hat{J}_{i}, \hat{J}_{j}\rangle - \frac{1}{2\Upsilon}\mathcal{I}_{\Lambda\Sigma}(\phi)\partial_{i}\mathcal{Z}^{\Lambda}\partial^{i}\mathcal{Z}^{\Sigma} - \frac{1}{4\Upsilon^{2}}\partial_{i}\Upsilon\partial^{i}\Upsilon \\ & + \frac{\Upsilon^{2}}{8}\omega_{ij}\omega^{ij} + \frac{\Upsilon}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)\left(F^{\Lambda}_{ij} + \omega_{ij}\mathcal{Z}^{\Lambda}\right)\left(F^{\Sigma ij} + \omega^{ij}\mathcal{Z}^{\Sigma}\right) \\ & + \frac{1}{2e^{(3)}_{\mathrm{D}}}\mathcal{R}_{\Lambda\Sigma}(\phi)\varepsilon^{ijk}\left(F^{\Lambda}_{ij} + \omega_{ij}\mathcal{Z}^{\Lambda}\right)\partial_{k}\mathcal{Z}^{\Sigma} \end{aligned}$$

$$\omega_{ij} = \partial_{i}\omega_{j} - \partial_{j}\omega_{i}, F^{\Lambda}_{ij} = \partial_{i}A^{\Lambda}_{j} - \partial_{j}A^{\Lambda}_{i}$$

$$\left\{g^{(3)}_{ij}, \Upsilon, \omega_{i}, A^{\Lambda}_{0}, A^{\Lambda}_{i}, \phi^{s}\right\}$$

$$\nabla_{i}\left(\Upsilon\mathcal{I}(\phi)\left(F^{ij} + \omega^{ij}\mathcal{Z}\right) + \frac{1}{e^{(3)}_{\mathrm{D}}}\mathcal{R}(\phi)\varepsilon^{ijk}\partial_{k}\mathcal{Z}\right) = 0$$

$$\nabla_{i}\left(\frac{\Upsilon^{2}}{2}\omega^{ij} + \Upsilon\mathcal{I}(\phi)\mathcal{Z}^{T}\left(F^{ij} + \omega^{ij}\mathcal{Z}\right) + \frac{1}{e^{(3)}_{\mathrm{D}}}\mathcal{R}(\phi)\varepsilon^{ijk}\mathcal{Z}^{T}\partial_{k}\mathcal{Z}\right) = 0$$

$$\tilde{\mathcal{L}}' = \tilde{\mathcal{L}} + \frac{1}{2}\varepsilon^{ijk}\mathcal{Z}_\Lambda \partial_i F^\Lambda_{jk} + \frac{1}{4}\varepsilon^{ijk}\left(\mathcal{Z}_\Lambda \mathcal{Z}^\Lambda - a\right)\partial_i \omega_{jk}$$

$$\omega^{ij} = \frac{1}{e_{\mathrm{D}}^{(3)}}\varepsilon^{ijk}\frac{1}{\Upsilon^2}\varpi_k$$

$$F^{\Lambda ij} + \omega^{ij}\mathcal{Z}^\Lambda = \frac{1}{e_{\mathrm{D}}^{(3)}\Upsilon}\varepsilon^{ijk}\mathcal{J}^{-1\Lambda\Sigma}(\phi)\left(\mathcal{R}_{\Sigma\Pi}(\phi)\partial_k \mathcal{Z}^\Pi - \partial_k \mathcal{Z}_\Sigma\right)$$

$$\varpi_i = -\partial_i a - \left(\mathcal{Z}^\Lambda \partial_i \mathcal{Z}_\Lambda - \mathcal{Z}_\Lambda \partial_i \mathcal{Z}^\Lambda\right)$$

$$\frac{1}{e_{\mathrm{D}}^{(3)}}\mathcal{L}_{(3)} = \frac{R^{(3)}}{2} - \frac{1}{2}g^{(3)ij}\langle \hat{J}_i, \hat{J}_j\rangle - \frac{1}{4\Upsilon^2}\left(\partial_i \Upsilon \partial^i \Upsilon + \varpi_i \varpi_j\right) - \frac{1}{2\Upsilon}\partial_i \mathcal{Z}^M \mathcal{M}_{(4)MN}\partial^i \mathcal{Z}_N$$
$$\equiv \frac{R^{(3)}}{2} - \frac{1}{2}\hat{G}_{ab}(z)\partial_i z^a \partial^i z^b$$

$$\varpi_i = -\partial_i a - \mathcal{Z}^M \mathbb{C}_{MN}\partial_i \mathcal{Z}^N$$

$$\mathcal{M}^{(3)}_{\mathrm{scal}} = \frac{G_{(3)}}{H_{(3)}}, (\Upsilon > 0)$$

$$\mathcal{M}^{(3)}_{\mathrm{scal}} = \frac{G_{(3)}}{H^*_{(3)}}, (\Upsilon < 0)$$

$$\mathcal{M}^{(3)}_{\mathrm{scal}} = G_{(3)}/H^*_{(3)}$$

$$\mathcal{M}_{(3)} \equiv \mathcal{M}_{(3)}(\Phi) \equiv L\eta L^\dagger = \mathcal{M}^\dagger_{(3)}$$

$$\frac{1}{e_{\mathrm{D}}^{(3)}}\mathcal{L}_{(3)} = \frac{R^{(3)}}{2} - \frac{1}{2}\langle \hat{J}_i, \hat{J}^j\rangle \equiv$$
$$\equiv \frac{R^{(3)}}{2} - \frac{\hat{\kappa}}{8}g^{(3)ij}\mathrm{Tr}\left(\mathcal{M}^{-1}_{(3)}\partial_i \mathcal{M}_{(3)}\mathcal{M}^{-1}_{(3)}\partial_j \mathcal{M}_{(3)}\right)$$

$$\hat{J}_i = \frac{1}{2}\mathcal{M}^{-1}_{(3)}\partial_i \mathcal{M}_{(3)}$$

$$R^{(3)}_{ij} = \langle \hat{J}_i, \hat{J}_j\rangle,$$
$$\nabla^i \hat{J}_i = 0,$$

$$V_{\mathrm{BH}} \to \frac{8\pi \mathrm{G_N}}{c^4}V_{\mathrm{BH}}$$

$$A_{\mathrm{H}} = 4\pi \frac{8\pi \mathrm{G_N}}{c^4}V_{\mathrm{BH}}{}^{(\mathrm{ex})}(e, m)$$

$$\bar{q}^2 = \frac{k}{2}\mathrm{Tr}(Q^2) = m^2 + N^2_{\mathrm{NUT}} - \frac{p^2 + q^2}{2}$$

$$\omega_{(p)} = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p}dx^{\mu_1} \wedge \ldots dx^{\mu_p}$$

$$S = \frac{k_B}{l_P^2}\frac{A}{4}$$

$$e^{-1}\mathcal{L} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}dr^2 + r^2 d\Omega^2$$

$$F = P\sin\,\theta d\theta \wedge d\phi + Qdt \wedge \frac{dr}{r^2}$$

$$\frac{1}{4\pi}\int_{S^2} F = P, \frac{1}{4\pi}\int_{S^2} \star F = Q$$

$$e^{2U(r)} = 1 - \frac{2M}{r} + \frac{P^2 + Q^2}{r^2},$$

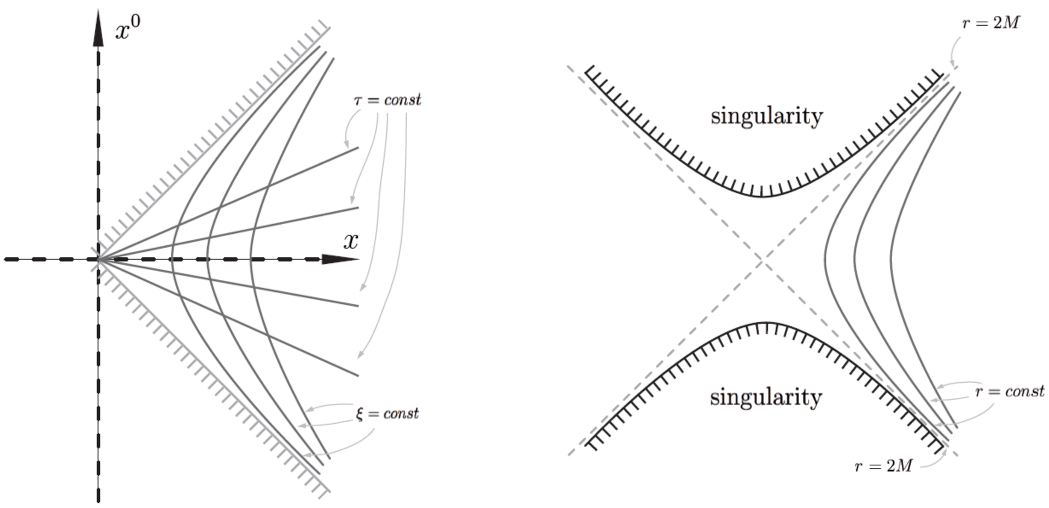
$$R_{\mu\nu}R^{\mu\nu} = 4\frac{(Q^2 + P^2)^2}{r^8} \overset{r\to 0}{\to} \infty$$

$$e^{2U} = 0 \;\Leftrightarrow\; r_\pm = M \pm \sqrt{M^2 - (P^2 + Q^2)}.$$

$$c = r_+ - r_- = \sqrt{M^2 - (P^2 + Q^2)}$$

$$\sqrt{g_{tt}}dt = \sqrt{g_{rr}}dr$$

$$t_{12} = \int_{r_1}^{r_2} \sqrt{\frac{g_{rr}}{g_{tt}}}d\tilde{r} = \int_{r_1}^{r_2} e^{-2U(\tilde{r})}d\tilde{r}$$



$$e^{2U} = \frac{(r - r_+)(r - r_-)}{r^2} \overset{r\to r_+}{\to} \frac{r_+ - r_-}{r_+^2}\rho,$$

$$ds^2 \to -\frac{r_+ - r_-}{r_+^2}\rho dt^2 + \frac{r_+^2}{r_+ - r_-}\frac{d\rho^2}{\rho} + r_+^2 d\Omega^2,$$

$$\rho = e^{2\alpha\xi}, t = \frac{1}{4\alpha^2}\tau, \alpha = \frac{\sqrt{r_+ - r_-}}{2r_+}.$$

$$ds_{NH}^2 = e^{2\alpha\xi}(-d\tau^2 + d\xi^2) + r_+^2 d\Omega^2.$$

$$x(x^0) = \frac{1}{\alpha}\sqrt{1 + \alpha^2 (x^0)^2}$$

$$x^0(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$$

$$\alpha^2 = \left[-\frac{1}{2}\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu\right]_{r=r_+}$$

$$T = \frac{\alpha}{2\pi}$$

$$Z = \mathrm{Tr} e^{-\beta H}$$

$$\tilde{\xi} = e^{\alpha\xi}/\alpha$$

$$ds^2 = d\tilde{\xi}^2 + \alpha^2 \tilde{\xi}^2 d\tilde{\tau}^2$$

$$\beta = \frac{1}{T} = \frac{2\pi}{\alpha} = \frac{4\pi r_+^2}{r_+ - r_-}$$

$$\frac{dS_{BH}}{dM} = \frac{1}{T}.$$

$$S_{BH} = \pi r_+^2 = \pi\left[M + \sqrt{M^2 - (P^2 + Q^2)}\right]^2$$

$$S_{BH} = \frac{A}{4}$$

$$S = \log\ \Omega$$

$$S_{BH} = \log\ \Omega(M, Q, P).$$

$$dM = TdS + \psi dQ + \chi dP + \Omega dJ - g_{ij}\Sigma^i d\phi^j,$$

$$(M_1 + M_2)^2 \geq M_1^2 + M_2^2$$

$$T = \frac{\alpha}{2\pi} = \frac{c}{2S} \Rightarrow c = 2ST.$$

$$\text{Extremality} \Leftrightarrow c = 2ST = 0 \Rightarrow T = 0.$$

$$e^{2U} = \frac{(r_+ - r_-)^2}{r^2} \to \frac{\rho^2}{r_+^2}$$

$$z = -\frac{M^2}{\rho}$$

$$M = \sqrt{P^2 + Q^2}$$

$$ds_{NH}^2 = M^2\left(\frac{-dt^2 + dz^2}{z^2}\right) + M^2 d\Omega^2$$

$$S = \pi\sqrt{P^2 + Q^2}$$

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}_3^2$$

$$\triangle_3 H = 0$$

$$H = 1 + \sum_i \frac{m_i}{|\vec{x} - \vec{x}_i|}, m_i = \sqrt{p_i^2 + q_i^2}$$

$$\mathrm{AdS}_{p+2} \times S^{D-p-2}$$

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{c^4}{\sinh^4(cz)}dz^2 + \frac{c^2}{\sinh^2(cz)}d\Omega^2\right]$$

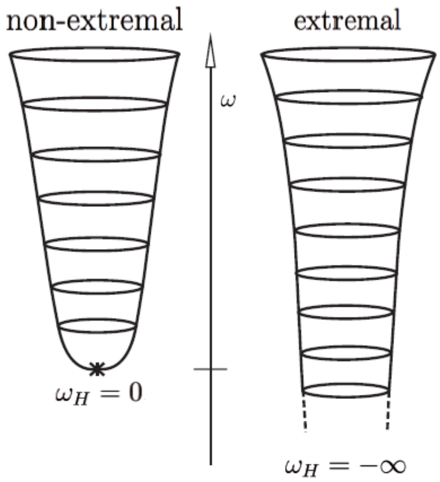
$$ds_{c=0}^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{dz^2}{z^4} + \frac{1}{z^2}d\Omega^2\right]$$

$$ds_{c=0}^2 = -e^{2U}dt^2 + e^{-2U}d\vec{x}^2$$

$$e^{-2U}\frac{c^2}{\sinh^2(cz)} \overset{z\to-\infty}{\rightarrow} \frac{A}{4\pi} = r_H^2,$$

$$e^{-2U}\frac{c^4}{\sinh^4(cz)}dz^2 \longrightarrow \frac{A}{4\pi}4c^2e^{2cz}dz^2 \equiv r_H^2 d\omega^2.$$

$$L = \int_{\omega_H}^{\omega_0} r_H d\omega = r_H\omega_0 < \infty$$



$$\frac{e^{-2U}}{z^2} \longrightarrow \frac{A}{4\pi} = r_H^2.$$

$$e^{-2U}\frac{dz^2}{z^4} \longrightarrow \frac{A}{4\pi}\frac{dz^2}{z^2} = r_H^2 d\omega^2,$$

$$\omega = -\log(-z).$$

$$L = \int_{\omega_H}^{\omega_0} r_H d\omega = +\infty$$

$$e^{-1}\mathcal{L} = R - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\phi)\frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}$$

$$\frac{1}{4\pi}\int_{S^2} F^\Lambda = p^\Lambda, \frac{1}{4\pi}\int_{S^2} G_\Lambda = q_\Lambda$$

$$S_{EM} = \int \left[\mathcal{I}_{\Lambda\Sigma}F^\Lambda \wedge\star F^\Sigma - \mathcal{R}_{\Lambda\Sigma}F^\Lambda \wedge F^\Sigma\right]$$

$$\begin{cases} dF^\Lambda = 0 \\ dG_\Lambda = d\left(\mathcal{R}_{\Lambda\Sigma}F^\Sigma - \mathcal{I}_{\Lambda\Sigma}\star F^\Sigma\right) = 0 \end{cases}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} \rightarrow \begin{pmatrix} F' \\ G' \end{pmatrix} = S\begin{pmatrix} F \\ G \end{pmatrix}.$$

$$G_\Lambda \equiv -\frac{\delta\mathcal{L}}{\delta F^\Lambda}$$

$$A^\Lambda = \chi^\Lambda(r)dt - p^\Lambda\cos\,\theta d\phi$$

$$A_\Lambda = \psi_\Lambda(r)dt - q_\Lambda\cos\,\theta d\phi$$

$$\chi^{\Lambda\prime} = e^{2U}\mathcal{I}^{-1\Lambda\Sigma}\left(q_\Sigma - \mathcal{R}_{\Sigma\Gamma}p^\Gamma\right)$$

$$\phi^i = \phi^i(r), U = U(r)$$

$$L_{1d} = (U')^2 + \frac{1}{2}g_{ij}\phi^{i\prime}\phi^{j\prime} + e^{2U}V_{BH} - c^2,$$

$$V_{BH} = -\frac{1}{2}Q^T\mathcal{M}Q$$

$$\mathcal{M} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}$$

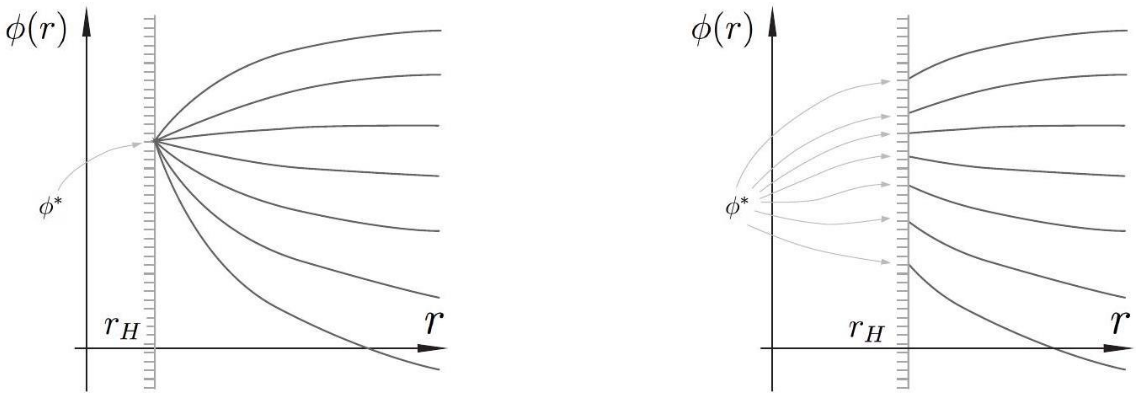
$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$$

$$(U')^2 + \frac{1}{2}g_{ij}\phi^{i\prime}\phi^{j\prime} = e^{2U}V_{BH} + c^2$$

$$U'' = e^{2U} V_{BH}$$
$$\phi^{i''} + \Gamma_{jk}{}^{i} \phi^{j'} \phi^{k'} = e^{2U} g^{ij} \partial_j V_{BH}$$

$$\phi^{i'} \overset{z \to -\infty}{\to} 0.$$

$$\partial_i V_{BH}(\phi^{i*}, q, p) = 0$$



$$\phi^{i*} = \phi^{i*}(q, p).$$

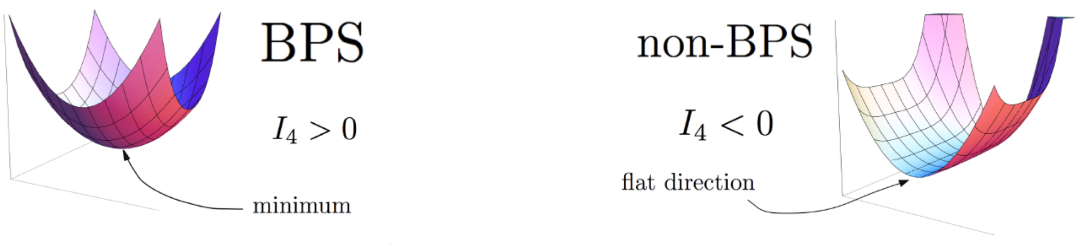
$$V_{BH}^* = V_{BH}(\phi^{i*}(q, p), q, p).$$

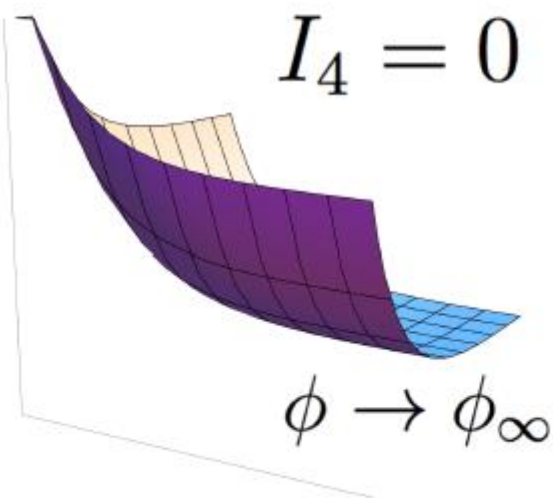
$$U \to -\log\left(\sqrt{V_{BH}^*} z\right).$$

$$r_H = \sqrt{V_{BH}^*}$$

$$S_{BH} = \frac{A}{4} = \pi V_{BH}^*(q, p).$$

$$r_{\pm} = M \pm \sqrt{M^2 - V_{BH}(\phi_\infty, p, q) + \frac{1}{2} g_{ij}(\phi_\infty) \Sigma^i \Sigma^j}.$$





$$\begin{pmatrix} g_{\mu\nu} \\ \psi_\mu^A \\ A_\mu^0 \end{pmatrix}, \quad \begin{pmatrix} A_\mu^i \\ \lambda_A^i \\ z^i \end{pmatrix}, \quad \begin{pmatrix} \zeta^\alpha \\ q^u \end{pmatrix},$$

gravity, n_V vector multiplets, n_H hyper multiplets.

$$\mathcal{M}_{\text{scalar}} = \mathcal{M}_{SK} \otimes \mathcal{M}_{QK}$$

$$K = -\log\, i\left(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\right) = -\log\, i\langle \Omega, \bar{\Omega} \rangle$$

$$\langle A, B \rangle = A^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B$$

$$\begin{pmatrix} X \\ F \end{pmatrix}_\alpha = S_{\alpha\beta} e^{h_{\alpha\beta}(z)} \begin{pmatrix} X \\ F \end{pmatrix}_\beta,$$

$$S_{\alpha\beta} \in \mathrm{Sp}(2n_V + 2, \mathbb{R})$$

$$K_\alpha \to K_\beta + h_{\alpha\beta} + \bar{h}_{\alpha\beta}.$$

$$t^i = \frac{X^i}{X^0}$$

$$F(\lambda X) = \lambda^2 F(X)$$

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma} X^\Sigma$$

$$R_{\Lambda\Sigma} = \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}, I_{\Lambda\Sigma} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$$

$$F = -iX^0 X^1$$

$$\Omega = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix}, K = -\log\, 2(z + \bar{z}).$$

$$F=\frac{X^1X^2X^3}{X^0}$$

$$\Omega=\begin{pmatrix}1\\ s\\ t\\ u\\ -stu\\ tu\\ su\\ st\end{pmatrix}, K=-\log\left[-i(s-\bar{s})(t-\bar{t})(u-\bar{u})\right],$$

$$X^\Lambda=\int_{A_\Lambda}\Omega, F_\Lambda=\int_{B^\Lambda}\Omega$$

$$K=-\log\, i\int_{CY}\Omega\wedge\bar{\Omega}$$

$$F_5=F^\Sigma\wedge\alpha_\Sigma-G_\Sigma\wedge\beta^\Sigma$$

$$\frac{X^i}{X^0}=\int_{C_i}J_c,\frac{F_i}{F_0}=\int_{D^i}J_c\wedge J_c$$

$$K=-\log\left[\frac{4}{3}\int_{CY}J\wedge J\wedge J\right]$$

$$V_{BH}=|Z|^2+4g^{i\bar{j}}\partial_i|Z|\bar{\partial}_{\bar{j}}|Z|,$$

$$Z=e^{-K/2}\left(X^\Lambda q_\Lambda-p^\Lambda F_\Lambda\right)=e^{-K/2}\langle\Omega,Q\rangle$$

$$\mathcal{L}=(U')^2+g_{i\bar{j}}z^{i\prime}\bar{z}^{\bar{j}\prime}+e^{2U}\left(|Z|^2+4g^{i\bar{j}}\partial_i|Z|\bar{\partial}_{\bar{j}}|Z|\right),$$

$$H=0\ \Leftrightarrow\ (U')^2+g_{i\bar{j}}z^{i\prime}\bar{z}^{\bar{j}\prime}=e^{2U}\left(|Z|^2+4g^{i\bar{j}}\partial_i|Z|\bar{\partial}_{\bar{j}}|Z|\right)$$

$$\begin{cases}U'=-e^U|Z|\\ z^{i\prime}=-2e^Ug^{i\bar{j}}\bar{\partial}_{\bar{j}}|Z|\end{cases}$$

$$M_{ADM}=|Z|_\infty$$

$$z^{i\prime}=-e^{U-i\alpha}g^{i\bar{j}}\bar{D}_{\bar{j}}\bar{Z}$$

$$\alpha'+Q=0$$

$$Q=\mathrm{Im}z^{i\prime}\partial_iK$$

$$\partial_i|Z|_*=0\ \Leftrightarrow\ z^{i\prime}=0.$$

$$\partial_iV_{BH}=|Z|\partial_i|Z|+\partial_i\partial_j|Z|\bar{\partial}^j|Z|+\partial_j|Z|\partial_i\bar{\partial}^j|Z|=0$$

$$z^{i*}=z^{i*}(p,q)$$

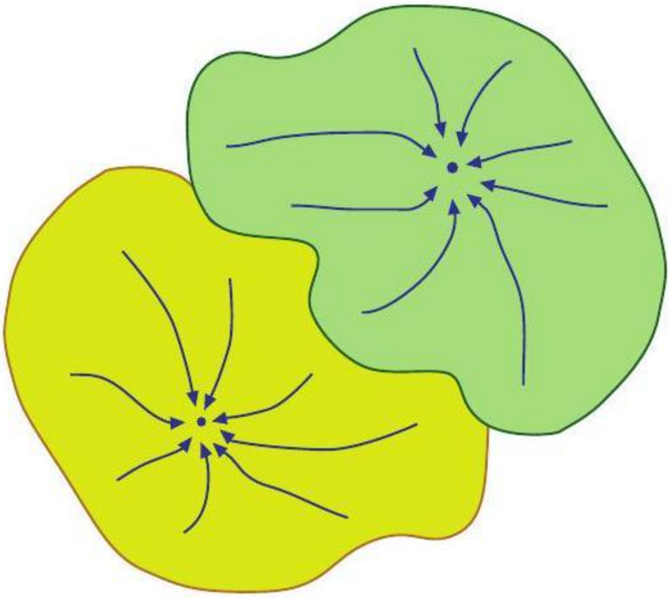
$$(e^{-U})' = |Z|_* \Rightarrow e^{-U} \to |Z|_* z,$$

$$ds^2 = -\frac{r^2}{|Z|_*^2} dt^2 + \frac{|Z|_*^2}{r^2} \left[dr^2 + r^2 d\Omega_{S^2}^2\right]$$

$$S_{BH} = \frac{A}{4} = \pi |Z|_*^2 = \pi V_{BH}^*$$

$$|Z|_* = |Z|\big(p, q, z_*^i(p, q)\big)$$

$$\partial_i \bar{\partial}_{\bar{j}} |Z| = g_{i\bar{j}} |Z| > 0.$$



$$\mathbb{R}^2 \text{ is } f = e^{-2x} - e^{-x-y^2} y^2$$

$$\mu' = |Z| > 0$$

$$\mu \frac{d}{d\mu} z^i = g^{i\bar{j}} \bar{\partial}_{\bar{j}} \log |Z|.$$

$$T_{\mu\nu} \zeta^\mu \zeta^\nu \geq 0, \forall \zeta \mid \zeta^2 = 0$$

$$A' \sim -R_{rr} g^{rr} + R_{tt} g^{tt}$$

$$(\zeta^t)^2 = -g^{tt} \text{ and } (\zeta^r)^2 = g^{rr}$$

$$A' \sim -R_{rr} g^{rr} + R_{tt} g^{tt} = -T_{\mu\nu} \zeta^\mu \zeta^\nu \leq 0.$$

$$\mathcal{L} = (\phi')^2 + V(\phi), H = (\phi')^2 - V(\phi) = 0$$

$$S = \int dt \left(\phi' \pm \sqrt{V}\right)^2 \mp 2 \int dt \phi' \sqrt{V}$$

$$\phi' \pm \sqrt{V} = 0.$$

$$\mathcal{L} = |\vec{\phi}'|^2 + V(\vec{\phi}), H = |\vec{\phi}'|^2 - V(\vec{\phi}) = 0$$

$$S = \int dt|\vec{\phi}' \pm \vec{n}\sqrt{V}|^2 \mp 2\int dt\vec{n}\cdot\vec{\phi}'\sqrt{V}$$

$$\vec{n} = \frac{\nabla_\phi \mathcal{W}}{\sqrt{V}}.$$

$$V(\phi) = |\nabla_\phi \mathcal{W}|^2.$$

$$\vec{\phi} = \{U, z^i\}$$

$$e^{2U}V_{BH} = \partial_U(e^U W)^2 + 4\partial_i(e^U W)g^{i\bar{j}}\bar{\partial}_{\bar{j}}(e^U W),$$

$$V_{BH} = W^2 + 4\partial_i W g^{i\bar{j}}\bar{\partial}_{\bar{j}}W.$$

$$\begin{cases} U' = -e^U W \\ z^{i\prime} = -2e^U g^{i\bar{j}}\bar{\partial}_{\bar{j}}W \end{cases}$$

$$\partial_i W_* = 0$$

$$S_{BH} = \frac{A}{4} = \pi W_*^2 = \pi V_{BH}^*.$$

$$\mathcal{H}\left(\phi^i, \frac{\partial\mathcal{W}}{\partial\phi^i}\right) = 0$$

$$\mathcal{H} = V(\phi, Q) - \left|\frac{\partial\mathcal{W}}{\partial\phi}\right|^2$$

$$\pi^i = \frac{\partial w}{\partial\phi^i} \text{ and } \pi^i = G_{ij}\phi^{j\prime} = \frac{\delta\mathcal{L}}{\delta\phi^{i\prime}}$$

$$\mathcal{W}(\phi) = \mathcal{W}_0 + \int_{\tau_0}^{\tau} \mathcal{L}(\phi, \phi')d\tau$$

$$F = -iX^0X^1$$

$$Z = \frac{q_0 + ip^1 + (q_1 + ip^0)z}{\sqrt{2(z+\bar{z})}}$$

$$V_{BH} = \frac{(p^1)^2 - iq_1(z-\bar{z})p^1 + q_0^2 + ip^0q_0(z-\bar{z}) + ((p^0)^2 + (q_1)^2)z\bar{z}}{z+\bar{z}}.$$

$$z^{\pm} = \frac{\pm(p^0p^1 + q_0q_1) + i(p^0q_0 - p^1q_1)}{(p^0)^2 + (q_1)^2}$$

$$\mathfrak{E}\{\mathfrak{H}(V_{BH})\} = \pm\frac{1}{p^0p^1 + q_0q_1}\{((p^0)^2 + (q_1)^2)^2, ((p^0)^2 + (q_1)^2)^2\}$$

$$\mathcal{W} = \frac{|-q_0 + ip^1 + (q_1 - ip^0)z|}{\sqrt{2(z+\bar{z})}},$$

$$Z = \frac{zq_1 + p^0 z^3}{\sqrt{-i(z-\bar{z})^3}}$$

$$z^* = -i\sqrt{-\frac{q_1}{3p^0}}$$

$$W = \frac{|zq_1 + p^0 z^2 \bar{z}|}{\sqrt{-i(z-\bar{z})^3}},$$

$$z^* = -i\sqrt{\frac{q_1}{3p^0}}.$$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = S\begin{pmatrix} p \\ q \end{pmatrix},$$

$$CY_6 = \frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2} \simeq (T^2)^3$$

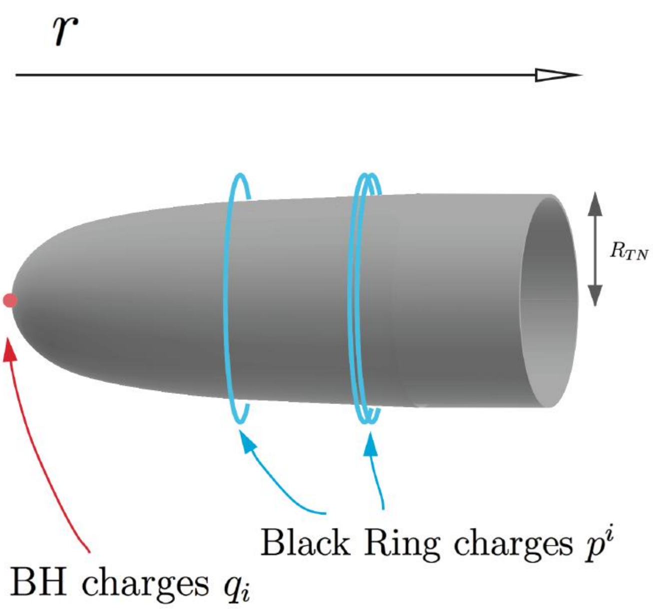
$$ds_{11}^2 = -Z^{-2}(dt+\omega)^2 + Zds_4^2(x) + \sum_I \frac{Z}{Z_I} ds_{T^2}^2.$$

$$Z = (Z_1 Z_2 Z_3)^{1/3}.$$

$$C_3 = \sum_I \left(\frac{-dt+\omega}{Z_I} + a_I\right) \wedge dT_I$$

$$\begin{aligned} da_I &= \star_4 da_I \\ d\star_4 dZ_I &= \frac{|\epsilon_{IJK}|}{2} da_J \wedge \star_4 da_k \\ d\omega + \star_4 d\omega &= Z_I da_I \end{aligned}$$

$$ds_4^2 = \frac{1}{V}(d\psi + \vec{A})^2 + V d\vec{x}_3^2$$



$$\star\, d\vec{A} = \pm dV$$

$$d\vec{x}_3^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$V = h^0 + \frac{p^0}{r}$$

$$R_{TN} = \frac{1}{\sqrt{h^0}}.$$

$$\begin{aligned} a_I &= C^I\left(d\psi - \vec{A}^0\right) + \vec{A}^I \\ \omega &= \mu\left(d\psi - \vec{A}^0\right) + \vec{\omega} \end{aligned}$$

$$z^I = C^I - i\frac{X^I}{\Delta^2},$$

$$\Delta^4 = \frac{Z^{2/3}V}{VZ^3 - V^2\mu^2}.$$

$$z^i \to \frac{a_i z^i + b_i}{c_i z^i + d_i},$$

$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SU(1,1)_i$$

$$M_I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M_I = \begin{pmatrix} 1 & \lambda_I \\ 0 & 1 \end{pmatrix}$$

$$M_I = \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix}$$

$$A^\Lambda \sim \left\{ \overrightarrow{A^0}, \overrightarrow{A^I} \right\}$$

$$\begin{aligned} \star_3 \, d\vec{A}^\Lambda &= dH^\Lambda, \\ \star_3 \, d\vec{A}_\Lambda &= dH_\Lambda, \end{aligned}$$

$$H = h + \sum_i \frac{Q_i}{|\vec{x} - \vec{x}_i|},$$

$$\star \, d\vec{\omega} = \langle dH, H \rangle$$

$$\sum_j \frac{\langle Q_i, Q_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\mathrm{Im}\left(e^{-i\alpha} Z(q_i) \right)_\infty$$

$$\begin{aligned} d\vec{A}^I &= C^I dV - V dC^I \\ d \star_3 dZ_I &= \frac{|\epsilon_{IJK}|}{2} V d \star_3 d(C^J C^K) \\ \star_3 \, d\omega &= d(\mu V) - V Z_I dC^I \end{aligned}$$

$$e^{-4U} = 4VZ_1Z_2Z_3 - b^2.$$

$$d \star d(V\mu) = d(VZ_3) \wedge \star \, dC^3$$

$$ds_4^2 = (V_1V_2)^{-1}(d\psi - \vec{A})^2 + V_1V_2 d\vec{x}_3^2$$

$$V_1 = C^3, V_2 = Z_3$$

$$\star \, d\vec{A} = V_2 dV_1 - V_1 dV_2$$

$$\mathcal{L}_D = R \star 1 - \frac{1}{2}\sum_{i=1}^{2} \star \, \mathrm{d}\varphi_i \wedge \, \mathrm{d}\varphi_i - \frac{1}{2}\sum_{I=1}^{2} X_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} X_1^{-2} X_2^{-2} \star H_{(3)} \wedge H_{(3)}$$

$$\begin{aligned} &X_1 = \mathrm{e}^{-\varphi_1/\sqrt{2(D-2)} - \varphi_2/\sqrt{2}}, X_2 = \mathrm{e}^{-\varphi_1/\sqrt{2(D-2)} + \varphi_2/\sqrt{2}} \\ &F_{(2)}^I = \mathrm{d}A_{(1)}^I, H_{(3)} = \mathrm{d}B_{(2)} - \frac{1}{2} A_{(1)}^1 \wedge \, \mathrm{d}A_{(1)}^2 - \frac{1}{2} A_{(1)}^2 \wedge \, \mathrm{d}A_{(1)}^1. \end{aligned}$$

$$\begin{aligned} \mathcal{L}_D = & R \star 1 - \frac{1}{2}\sum_{i=1}^{2} \star \, \mathrm{d}\varphi_i \wedge \, \mathrm{d}\varphi_i - \frac{1}{2}\sum_{I=1}^{2} X_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} X_1^2 X_2^2 \star F_{(D-3)} \wedge F_{(D-3)} \\ & + (-1)^{D-1} F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(D-4)} \end{aligned}$$

$$F_{(D-3)} = \mathrm{d}A_{(D-4)} = X_1^{-2} X_2^{-2} \star H_{(3)}$$

$$\mathcal{L}_{D+1} = R \star 1 - \frac{1}{2} \star \, \mathrm{d}\phi_1 \wedge \, \mathrm{d}\phi_1 - \frac{1}{2} \mathrm{e}^{2\sqrt{2/(D-1)}\phi_1} \star H_{(3)} \wedge H_{(3)}$$

$$\mathrm{d}s_{D+1}^2 = \mathrm{e}^{-\sqrt{2(D-2)/(D-1)}\phi_2} \left(\mathrm{d}z + A_{(1)}^2 \right)^2 + \mathrm{e}^{\sqrt{2/(D-1)(D-2)}\phi_2} \, \mathrm{d}s_D^2$$

$$\hat{H}_{(3)} = \mathrm{e}^{-3\phi_2/\sqrt{2(D-1)(D-2)}} H_{(3)} + \mathrm{e}^{(D-4)\phi_2/\sqrt{2(D-1)(D-2)}} F^1_{(2)} \wedge \left(\mathrm{d}z + A^2_{(1)} \right)$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \frac{1}{\sqrt{D-1}} \begin{pmatrix} \sqrt{D-2} & -1 \\ 1 & \sqrt{D-2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\begin{pmatrix} t \\ z \end{pmatrix} \to \begin{pmatrix} \cosh\ \delta_1 & \sinh\ \delta_1 \\ \sinh\ \delta_1 & \cosh\ \delta_1 \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

$$\mathrm{d}s^2 = \sum_{\mu=1}^{n} \left(\frac{X_\mu}{U_\mu} \mathcal{A}_\mu^2 + \frac{U_\mu}{X_\mu} \mathrm{d}x_\mu^2 \right)$$

$$U_\mu = \prod_{\nu=1}^{n}{}' (x_\nu^2 - x_\mu^2), \quad X_\mu = -\prod_{k=1}^{n-1} (a_k^2 - x_\mu^2) + 2m_\mu x_\mu,$$

$$\tilde{\gamma}_i = \prod_{\nu=1}^{n} (a_i^2 - x_\nu^2), \quad \mathcal{A}_\mu = \mathrm{d}t - \sum_{i=1}^{n-1} \frac{\tilde{\gamma}_i}{a_i^2 - x_\mu^2} \mathrm{d}\tilde{\phi}_i, \quad \tilde{\phi}_i = \frac{\phi_i}{a_i \prod'^{\,n-1}_{k=1} (a_i^2 - a_k^2)}$$

$$\mathrm{d}s^2 = \sum_{\mu=1}^{n} \left(\frac{X_\mu}{U_\mu} \mathcal{A}_\mu^2 + \frac{U_\mu}{X_\mu} \mathrm{d}x_\mu^2 \right) - \frac{\prod_{k=1}^{n} a_k^2}{\prod_{\mu=1}^{n} x_\mu^2} \left(\mathrm{d}t - \sum_{i=1}^{n} \frac{\tilde{\gamma}_i}{a_i^2} \mathrm{d}\tilde{\phi}_i \right)^2$$

$$U_\mu = \prod_{\nu=1}^{n}{}' (x_\nu^2 - x_\mu^2), \quad X_\mu = \frac{1}{x_\mu^2} \prod_{k=1}^{n} (a_k^2 - x_\mu^2) + 2m_\mu,$$

$$\tilde{\gamma}_i = a_i^2 \prod_{\nu=1}^{n} (a_i^2 - x_\nu^2), \quad \mathcal{A}_\mu = \mathrm{d}t - \sum_{i=1}^{n} \frac{\tilde{\gamma}_i}{a_i^2 - x_\mu^2} \mathrm{d}\tilde{\phi}_i, \quad \tilde{\phi}_i = \frac{\phi_i}{a_i \prod'^{\,n}_{k=1} (a_i^2 - a_k^2)}$$

$$B^{(\mu)}_{(1)} = \frac{x_\mu}{U_\mu} \mathcal{A}_\mu$$

$$G^{(\mu)}_{(2)} = \mathrm{d}B^{(\mu)}_{(1)}$$

$$B^{(\mu)}_{(1)} = \frac{1}{U_\mu} \mathcal{A}_\mu$$

$$G^{(\mu)}_{(2)} = \mathrm{d}B^{(\mu)}_{(1)}$$

$$\sum_{\mu=1}^{n} B^{(\mu)}_{(1)} = 0$$

$$B^{(\mu)} = \frac{x_\mu}{U_\mu}$$

$$B^{(\mu)} = \frac{1}{U_\mu}$$

$$G_{(1)}^{(\mu)} = \mathrm{d}B^{(\mu)}$$

$$\sum_{\mu=1}^{n} B^{(\mu)} = 0$$

$$B_{(2)}^{(\mu)} = \frac{x_\mu}{U_\mu}\mathrm{d}t \wedge \mathcal{A}_\mu$$

$$B_{(2)}^{(\mu)} = \frac{1}{U_\mu}\mathrm{d}t \wedge \mathcal{A}_\mu$$

$$H_{(3)}^{(\mu)} = \mathrm{d}B_{(2)}^{(\mu)}$$

$$\sum_{\mu=1}^{n} B_{(2)}^{(\mu)} = 0$$

$$F_{(D-3)}^{(\mu)} = \mathrm{d}A_{(D-4)}^{(\mu)} \text{ with } H_{(3)}^{(\mu)} =\star\, F_{(D-3)}^{(\mu)}$$

$$\mathcal{A}_\mu = \sum_{k=0}^{n-1} A_\mu^{(k)}\mathrm{d}\psi_k, A_\mu^{(k)} = \sum_{\substack{\nu_1<\nu_2<\cdots<\nu_k \\ \nu_i\neq\mu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2$$

$$A_{(D-4)}^{(\mu)} = \frac{\prod_{\rho=1}^{n} x_\rho}{U_\mu x_\mu}\left(\sum_{\nu\neq\mu} \frac{x_\nu^2 - x_\mu^2}{x_\nu}\mathrm{d}x_\nu \wedge \mathcal{A}_{\mu\nu}\right)^{n-2}$$

$$A_{(D-4)}^{(\mu)} = \frac{1}{U_\mu}\sum_{k=1}^{n} A_\mu^{(k-1)}\mathrm{d}\psi_k \wedge \left(\sum_{\nu\neq\mu} \frac{x_\nu^2 - x_\mu^2}{x_\nu}\mathrm{d}x_\nu \wedge \mathcal{A}_{\mu\nu}\right)^{n-2}$$

$$\mathcal{A}_{\mu\nu} = \sum_{k=1}^{n-1} A_{\mu\nu}^{(k-1)}\mathrm{d}\psi_k, A_{\mu\nu}^{(k)} = \sum_{\substack{\nu_1<\nu_2<\cdots<\nu_k \\ \nu_i\neq\mu,\nu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2$$

$$\sum_{\mu=1}^{n} A_{(D-4)}^{(\mu)} = \mathrm{d}\left(\frac{1}{2}\sum_{k=0}^{n} A^{(k)}\mathrm{d}\psi_k\right) \wedge \mathrm{d}\left(\sum_{\mu} x_\mu \mathrm{d}x_\mu \wedge \mathcal{A}_\mu\right)^{n-2} \wedge\ \mathrm{d}\psi_n$$

$$X = X_1 = X_2 = \mathrm{e}^{-\varphi_1/\sqrt{2(D-2)}}$$

$$A_{(1)} = A_{(1)}^1 = A_{(1)}^2$$

$$\mathrm{d}s^2 = H^{2/(D-2)}\left\{\sum_{\mu=1}^{n}\left[\frac{X_\mu}{U_\mu}\left(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2m_\nu s^2 x_\nu}{HU_\nu}\mathcal{A}_\nu\right)^2 + \frac{U_\mu}{X_\mu}\mathrm{d}x_\mu^2\right]\right\}$$

$$X = H^{-1/(D-2)}, A_{(1)} = \sum_{\mu=1}^{n}\frac{2m_\mu scx_\mu}{HU_\mu}\mathcal{A}_\mu, B_{(2)} = \sum_{\mu=1}^{n}\frac{2m_\mu s^2 x_\mu}{HU_\mu}\mathrm{d}t \wedge \sum_{i=1}^{n-1}\frac{\tilde{\gamma}_i}{z_{i\mu}^2}\,\mathrm{d}\tilde{\phi}_i$$

$$U_\mu = \prod_{\nu=1}^{n}(x_\nu^2 - x_\mu^2), X_\mu = -\prod_{k=1}^{n-1}(a_k^2 - x_\mu^2) + 2m_\mu x_\mu, H = 1 + \sum_{\mu=1}^{n}\frac{2m_\mu s^2 x_\mu}{U_\mu}$$

$$\tilde{\gamma}_i = \prod_{\nu=1}^{n}(a_i^2 - x_\nu^2), z_{i\mu} = a_i^2 - x_\mu^2, \mathcal{A}_\mu = \mathrm{d}t - \sum_{i=1}^{n-1}\frac{\tilde{\gamma}_i}{a_i^2 - x_\mu^2}\,\mathrm{d}\tilde{\phi}_i$$

$$A_{(D-4)} = \sum_{\mu=1}^{n}\frac{2\mathrm{i}m_\mu s^2 \prod_{\rho=1}^{n} x_\rho}{(n-2)!\,U_\mu x_\mu}\left(\sum_{\nu\neq\mu}\frac{x_\nu^2 - x_\mu^2}{x_\nu}\mathrm{d}x_\nu \wedge \mathcal{A}_{\mu\nu}\right)^{n-2}$$

$$\mathrm{d}s^2 = H^{2/(D-2)}\left\{\sum_{\mu=1}^{n}\left[\frac{X_\mu}{U_\mu}\left(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2m_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)^2 + \frac{U_\mu}{X_\mu}\mathrm{d}x_\mu^2\right] - \frac{\prod_{k=1}^{n} a_k^2}{\prod_{\mu=1}^{n} x_\mu^2}\left(\mathrm{d}t - \sum_{i=1}^{n}\frac{\tilde{\gamma}_i}{a_i^2}\,\mathrm{d}\tilde{\phi}_i - \sum_{\nu=1}^{n}\frac{2m_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)^2\right\}$$

$$X = H^{-1/(D-2)}, A_{(1)} = \sum_{\mu=1}^{n}\frac{2m_\mu sc}{HU_\mu}\mathcal{A}_\mu, B_{(2)} = \sum_{\mu=1}^{n}\frac{2m_\mu s^2}{HU_\mu}\mathrm{d}t \wedge \sum_{i=1}^{n}\frac{\tilde{\gamma}_i}{z_{i\mu}^2}\,\mathrm{d}\tilde{\phi}_i$$

$$U_\mu = \prod_{\nu=1}^{n}(x_\nu^2 - x_\mu^2), X_\mu = \frac{1}{x_\mu^2}\prod_{k=1}^{n}(a_k^2 - x_\mu^2) + 2m_\mu, H = 1 + \sum_{\mu=1}^{n}\frac{2m_\mu s^2}{U_\mu}$$

$$\tilde{\gamma}_i = a_i^2\prod_{\nu=1}^{n}(a_i^2 - x_\nu^2), z_{i\mu}^2 = a_i^2 - x_\mu^2, \mathcal{A}_\mu = \mathrm{d}t - \sum_{i=1}^{n}\frac{\tilde{\gamma}_i}{a_i^2 - x_\mu^2}\,\mathrm{d}\tilde{\phi}_i$$

$$\mathrm{d}s^2 = H^{2/(D-2)}\left\{\sum_{\mu=1}^{n}\left[\frac{X_\mu}{U_\mu}\left(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2m_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)^2 + \frac{U_\mu}{X_\mu}\mathrm{d}x_\mu^2\right] - \frac{\prod_{i=1}^{n} a_i^2}{\prod_{\mu=1}^{n} x_\mu^2}\left(\sum_{k=0}^{n} A^{(k)}\mathrm{d}\psi_k - \sum_{\nu=1}^{n}\frac{2m_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)^2\right\}$$

$$A^{(k)} = \sum_{\nu_1<\nu_2<\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2$$

$$A_{(D-4)} = \sum_{\mu=1}^{n}\frac{2m_\mu s^2 \prod_{i=1}^{n} a_i}{(n-2)!\,U_\mu}\sum_{k=1}^{n} A_\mu^{(k-1)}\mathrm{d}\psi_k \wedge \left(\sum_{\nu\neq\mu}\frac{x_\nu^2 - x_\mu^2}{x_\nu}\mathrm{d}x_\nu \wedge \mathcal{A}_{\mu\nu}\right)^{n-2}$$

$$\nabla_{(a}K_{bc)} = 0$$

$$\nabla_{(a}Q_{bc)} = q_{(a}g_{bc)}$$

$$q_a = \frac{1}{D+2}\left(\partial_a Q_b^b + 2\nabla_b Q_a^b\right)$$

$$K_{ab} = Q_{ab} - q g_{ab}$$

$$K_{ab} = k^1{}_{(a} k^2{}_{b)}$$

$$\nabla_{(a}K_{a_1\ldots a_k)} = 0$$

$$\nabla_{(a}Q_{a_1\ldots a_k)} = q_{(aa_1\ldots a_{k-2}}g_{a_{k-1}a_k)}$$

$$\mathrm{d}\tilde{s}^2 = \mathrm{e}^{2\Omega}\,\mathrm{d}s^2$$

$$\mathrm{d}\tilde{s}^2 = \mathrm{e}^{2\Omega}\,\mathrm{d}s^2$$

$$\nabla^{(a_1}Q^{a_2\ldots a_{k+1})} = q^{(a_1\ldots a_{k-1}}g^{a_k a_{k+1})}$$

$$\tilde{\nabla}^{(a_1}Q^{a_2\ldots a_{k+1})} = \tilde{q}^{(a_1\ldots a_{k-1}}g^{a_k a_{k+1})}, \tilde{q}^{a_1\ldots a_{k-1}} = q^{a_1\ldots a_{k-1}} + kQ^{a_1\ldots a_{k-1}b}\partial_b\Omega$$

$$q^a = Q^{ab}\partial_b q$$

$$\mathrm{d}\tilde{s}^2 = \mathrm{e}^{-q}\,\mathrm{d}s^2$$

$$A^{a_1\ldots a_m} = A^{(a_1\ldots a_m)} \text{ and } B^{a_1\ldots a_n} = B^{(a_1\ldots a_n)}$$

$$\begin{aligned}[A,B]_{\mathrm{SN}}^{a_1\ldots a_{m+n-1}} &= mA^{b(a_1\ldots a_{m-1}}\nabla_b B^{a_m\ldots a_{m+n-1})} - nB^{b(a_1\ldots a_{m-1}}\nabla_b A^{a_m\ldots a_{m+n-1})} \\ &= mA^{b(a_1\ldots a_{m-1}}\partial_b B^{a_m\ldots a_{m+n-1})} - nB^{b(a_1\ldots a_{m-1}}\partial_b A^{a_m\ldots a_{m+n-1})}\end{aligned}$$

$$\mathcal{H}\left(x^a, \frac{\partial W}{\partial x^a}\right) = -\frac{1}{2}\mu^2,$$

$$\tilde{\mathcal{H}} = \mathcal{H}/\Lambda$$

$$\mathcal{H} = \frac{1}{2}g^{ab}\frac{\partial W}{\partial x^a}\frac{\partial W}{\partial x^b}$$

$$\mathrm{d}s^2 = \sum_{\mu=1}^{n}\left(e^{\mu}e^{\mu} + e^{\hat{\mu}}e^{\hat{\mu}}\right) - \varepsilon e^{\hat{0}}e^{\hat{0}}$$

$$e^{\mu} = H^{1/(D-2)}\sqrt{\frac{U_\mu}{X_\mu}}\mathrm{d}x_\mu, e^{\hat{\mu}} = H^{1/(D-2)}\sqrt{\frac{X_\mu}{U_\mu}}\left(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)$$

$$e^{\hat{0}} = H^{1/(D-2)}\frac{c}{P}\left(\sum_{k=0}^{n}A^{(k)}\mathrm{d}\psi_k - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)$$

$$N_\mu = m_\mu x_\mu^{1-\varepsilon}$$

$$N_\mu = m_\mu x_\mu$$

$$c=\prod_{i=1}^{n} a_i, P=\prod_{\rho=1}^{n} x_\rho P^2=A^{(n)}$$

$$\mathrm{d}\tilde{s}^2=H^{-2/(D-2)}\mathrm{d}s^2$$

$$\mathrm{d}\tilde{s}^2=\sum_{\mu=1}^{n}\left(\tilde{e}^{\mu}\tilde{e}^{\mu}+\tilde{e}^{\hat{\mu}}\tilde{e}^{\hat{\mu}}\right)-\varepsilon e^{\hat{0}}\tilde{e}^{\hat{0}}$$

$$\tilde{e}^{\mu}=\sqrt{\frac{U_\mu}{X_\mu}}\mathrm{d}x_\mu, \tilde{e}^{\hat{\mu}}=\sqrt{\frac{X_\mu}{U_\mu}}\left(\mathcal{A}_\mu-\sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)$$
$$\tilde{e}^{\hat{0}}=\frac{c}{P}\left(\sum_{k=0}^{n} A^{(k)}\mathrm{d}\psi_k-\sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\right)$$

$$\tilde{e}^{A}=H^{-1/(D-2)}e^{A}$$

$$\mathcal{L}_D=R\star 1+\cdots$$

$$\mathrm{d}\tilde{s}^2=H^{-2/(D-2)}\mathrm{d}s^2$$

$$\mathrm{d}\tilde{e}^{A}+\tilde{\omega}^{A}{}_{B}\wedge\tilde{e}^{B}=0$$

$$\omega_{AB}=-\omega_{BA}$$

$$\tilde{\omega}_{\mu\nu}=(1-\delta_{\mu\nu})\left(-\sqrt{\frac{X_\nu}{U_\nu}}\frac{x_\nu}{x_\mu^2-x_\nu^2}\tilde{e}^{\mu}-\sqrt{\frac{X_\mu}{U_\mu}}\frac{x_\mu}{x_\mu^2-x_\nu^2}\tilde{e}^{\nu}\right)$$
$$\tilde{\omega}_{\mu\hat{\nu}}=\delta_{\mu\nu}\left[-H\partial_\mu\left(\frac{1}{H}\sqrt{\frac{X_\mu}{U_\mu}}\right)\tilde{e}^{\hat{\mu}}+\frac{1}{2}\sum_{\rho\neq\mu}\sqrt{\frac{X_\mu}{U_\mu}}\partial_\mu\left[\log\left(HU_\rho\right)\right]\tilde{e}^{\hat{\rho}}-\varepsilon\frac{c}{P}\left(\frac{1}{x_\mu}+\frac{1}{2}\partial_\mu\log H\right)\tilde{e}^{\hat{0}}\right]$$
$$+(1-\delta_{\mu\nu})\left(-\sqrt{\frac{X_\mu}{U_\mu}}\frac{x_\mu}{x_\mu^2-x_\nu^2}\tilde{e}^{\hat{\nu}}+\frac{1}{2}\sqrt{\frac{X_\nu}{U_\nu}}\partial_\mu\left[\log\left(HU_\nu\right)\right]\tilde{e}^{\hat{\mu}}\right)$$
$$\tilde{\omega}_{\hat{\mu}\hat{\nu}}=(1-\delta_{\mu\nu})\left(-\frac{1}{2}\sqrt{\frac{X_\nu}{U_\nu}}\partial_\mu\left[\log\left(HU_\nu\right)\right]\tilde{e}^{\mu}+\frac{1}{2}\sqrt{\frac{X_\mu}{U_\mu}}\partial_\nu\left[\log\left(HU_\mu\right)\right]\tilde{e}^{\nu}\right)$$
$$\tilde{\omega}_{\mu\hat{0}}=-\frac{c}{P}\left(\frac{1}{x_\mu}+\frac{1}{2}\partial_\mu\log H\right)\tilde{e}^{\hat{\mu}}+\sqrt{\frac{X_\mu}{U_\mu}}\frac{1}{x_\mu}\tilde{e}^{\hat{0}}, \tilde{\omega}_{\hat{\mu}\hat{0}}=\frac{c}{P}\left(\frac{1}{x_\mu}+\frac{1}{2}\partial_\mu\log H\right)\tilde{e}^{\mu},$$

$$\tilde{\omega}^{A}{}_{B}=\tilde{\omega}^{A}{}_{BC}\tilde{e}^{C}$$

$$\widetilde{K}^{(j)}=\sum_{\mu=1}^{n}A_\mu^{(j)}\left(\tilde{e}^{\mu}\tilde{e}^{\mu}+\tilde{e}^{\hat{\mu}}\tilde{e}^{\hat{\mu}}\right)-\varepsilon A^{(j)}\tilde{e}^{\hat{0}}\tilde{e}^{\hat{0}}$$

$$Q^{(j)}=H^{2/(D-2)}\left(\sum_{\mu=1}^{n}A_\mu^{(j)}\left(e^{\mu}e^{\mu}+e^{\hat{\mu}}e^{\hat{\mu}}\right)-\varepsilon A^{(j)}e^{\hat{0}}e^{\hat{0}}\right)$$

$$q^{(j)}{}_{\mu} = \frac{2}{D-2} H^{(4-D)/(D-2)} \partial_{\mu} \left(A_{\mu}^{(j)} H \right)$$

$$q^{(1)} = \frac{2m_1 s^2 x_1 x_2^2}{x_2^2 - x_1^2} + \frac{2m_2 s^2 x_2 x_1^2}{x_1^2 - x_2^2}$$

$$K = Q^{(1)} - q^{(1)} \mathrm{d}s^2 = x_2(x_2 + 2m_2 s^2)\left(e^1 e^1 + e^{\hat{1}} e^{\hat{1}}\right) + x_1(x_1 + 2m_1 s^2)\left(e^2 e^2 + e^{\hat{2}} e^{\hat{2}}\right)$$

$$\mathrm{d}\tilde{s}^2 = H^{-2/(D-2)} \mathrm{d}$$

$$\left(\frac{\partial}{\partial \tilde{s}}\right)^2 = \sum_{\mu=1}^{n} \left[\frac{X_\mu}{U_\mu} \left(\frac{\partial}{\partial x_\mu}\right)^2 + \frac{1}{X_\mu U_\mu} \left(\sum_{k=0}^{n-1+\varepsilon} \left(-x_\mu^2\right)^{n-1-k} \frac{\partial}{\partial \psi_k} + 2N_\mu s^2 \frac{\partial}{\partial \psi_0} \right)^2 \right] - \frac{\varepsilon}{c^2 P^2} \left(\frac{\partial}{\partial \psi_n}\right)^2$$

$$(\partial/\partial s)^2 = H^{-2/(D-2)} (\partial/\partial \tilde{s})^2$$

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} \tilde{g}^{ab} \partial_a S \partial_b S = 0$$

$$S = \frac{1}{2} \mu^2 \lambda + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{n-1+\epsilon} \Psi_k \psi_k$$

$$\sum_{\mu=1}^{n} \frac{F_\mu(x_\mu)}{U_\mu} = -\mu^2 + \frac{\varepsilon \Psi_n^2}{c^2 P^2}$$

$$F_\mu = X_\mu \left(S_\mu'\right)^2 + \frac{1}{X_\mu} \left(\sum_{k=0}^{n-1+\varepsilon} \left(-x_\mu^2\right)^{n-1-k} \Psi_k + 2N_\mu s^2 \Psi_0 \right)^2,$$

$$F_\mu = \sum_{k=0}^{n-1+\varepsilon} c_k \left(-x_\mu^2\right)^{n-1-k}$$

$$c_n = -\Psi_n^2 / c^2$$

$$\left(S_\mu'\right)^2 = -\frac{1}{X_\mu^2} \left(\sum_{k=0}^{n-1+\varepsilon} \left(-x_\mu^2\right)^{n-1-k} \Psi_k + 2N_\mu s^2 \Psi_0 \right)^2 + \frac{1}{X_\mu} \sum_{k=0}^{n-1+\varepsilon} c_k \left(-x_\mu^2\right)^{n-1-k},$$

$$g^{ab} = H^{-2/(D-2)} \tilde{g}^{ab}$$

$$\sum_{\mu=1}^{n} \frac{F_\mu}{U_\mu} = -H^{2/(D-2)} \mu^2 + \frac{\varepsilon \Psi_n^2}{c^2 P^2}$$

$$F_\mu = -\mu^2\left(-x_\mu^2 + 2m_\mu s^2 x_\mu\right) + c_1$$

$$\left(S_\mu'\right)^2 = -\frac{1}{X_\mu^2}\left[\left(-x_\mu^2 + 2m_\mu s^2 x_\mu\right)\Psi_0 + \Psi_1\right]^2 + \frac{1}{X_\mu}\left[-\mu^2\left(-x_\mu^2 + 2m_\mu s^2 x_\mu\right) + c_1\right]$$

$$\Box\, \Phi = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\Phi\right) = \mu^2\Phi$$

$$g = \det(g_{ab})$$

$$\det(g_{ab}) = H^{4/(D-2)}U^2(cP)^{2\varepsilon}, U = \prod_{\mu<\nu}\left(x_\mu^2 - x_\nu^2\right)$$

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu)\prod_{k=0}^{n-1+\varepsilon} \mathrm{e}^{\mathrm{i}\Psi_k\psi_k}$$

$$\sum_{\mu=1}^{n}\frac{G_\mu(x_\mu)}{U_\mu} = H^{2/(D-2)}\mu^2$$

$$G_\mu = \frac{\left(X_\mu R_\mu'\right)'}{R_\mu} + \frac{\varepsilon X_\mu R_\mu'}{x_\mu R_\mu} - \frac{1}{X_\mu}\left(\sum_{k=0}^{n-1+\varepsilon}\left(-x_\mu^2\right)^{n-1-k}\Psi_k + 2N_\mu s^2\Psi_0\right)^2 + \frac{\varepsilon\Psi_n^2}{c^2x_\mu^2}$$

$$G_\mu = \sum_{k=1}^{n-1} b_k\left(-x_\mu^2\right)^{n-1-k}$$

$$b_n = \Psi_n^2/c^2$$

$$\left(X_\mu R_\mu'\right)' + \frac{\varepsilon X_\mu R_\mu'}{x_\mu} - \frac{R_\mu}{X_\mu}\left(\sum_{k=0}^{n-1+\varepsilon}\left(-x_\mu^2\right)^{n-1-k}\Psi_k + 2N_\mu s^2\Psi_0\right)^2 - \sum_{k=1}^{n-1+\varepsilon} b_k\left(-x_\mu^2\right)^{n-1-k}R_\mu = 0.$$

$$G_\mu = \mu^2\left(-x_\mu^2 + 2m_\mu s^2 x_\mu\right) + b_1$$

$$\left(X_\mu R_\mu'\right)' - \frac{R_\mu}{X_\mu}\left[\left(-x_\mu^2 + 2m_\mu s^2x_\mu\right)\Psi_0 + \Psi_1\right]^2 - \left[\mu^2\left(-x_\mu^2 + 2m_\mu s^2 x_\mu\right) + b_1\right]R_\mu = 0$$

$$\mathrm{d}s^2 = -e^0e^0 + \sum_{A=1}^{D-1} e^Ae^A$$

$$e^0 = H^{-1/2}\sqrt{\frac{R}{r^2+y^2}}\left(\mathrm{d}t' + y_1y_2\,\mathrm{d}\psi\right), e^1 = H^{1/2}\sqrt{\frac{r^2+y^2}{R}}\,\mathrm{d}r$$

$$e^2 = H^{-1/2}\sqrt{\frac{Y}{r^2+y^2}}\left(\mathrm{d}t' - r_1r_2\,\mathrm{d}\psi\right), e^3 = H^{1/2}\sqrt{\frac{r^2+y^2}{Y}}\,\mathrm{d}y$$

$$R = a^2 + r^2 - 2mr + g^2r_1r_2(r_1r_2 + a^2), Y = a^2 - y^2 + 2\ell y + g^2y_1y_2(y_1y_2 - a^2),$$
$$r_I = r + 2ms_I^2, y_I = y + 2\ell s_I^2, H = \frac{r_1r_2 + y_1y_2}{r^2+y^2}.$$

$$\tilde{e}^A = H^{-1/2} e^A$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0 + \tilde{e}^1\tilde{e}^1) - r^2(\tilde{e}^2\tilde{e}^2 + \tilde{e}^3\tilde{e}^3)$$

$$Q = H[y^2(-e^0e^0 + e^1e^1) - r^2(e^2e^2 + e^3e^3)]$$

$$q = \frac{r_1r_2y^2 - y_1y_2r^2}{r^2 + y^2}$$

$$K = y_1y_2(-e^0e^0 + e^1e^1) - r_1r_2(e^2e^2 + e^3e^3)$$

$$e^0 = \frac{H^{-1/2}}{1 - ry}\sqrt{\frac{R}{r^2 + y^2}}(\,\mathrm{d}t' + y_1y_2\,\mathrm{d}\psi), e^1 = \frac{H^{1/2}}{1 - ry}\sqrt{\frac{r^2 + y^2}{R}}\,\mathrm{d}r$$

$$e^2 = \frac{H^{-1/2}}{1 - ry}\sqrt{\frac{Y}{r^2 + y^2}}(\,\mathrm{d}t' - r_1r_2\,\mathrm{d}\psi), e^3 = \frac{H^{1/2}}{1 - ry}\sqrt{\frac{r^2 + y^2}{Y}}\,\mathrm{d}y$$

$$R = \gamma - 2mr + \epsilon r^2 - 2\ell r^3 - \gamma r^4, Y = \gamma + 2\ell y - \epsilon y^2 + 2my^3 - \gamma y^4$$

$$r_I = r + 2ms_I^2, y_I = y + 2\ell s_I^2, H = \frac{r_1r_2 + y_1y_2}{r^2 + y^2}$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0 + \tilde{e}^1\tilde{e}^1) - r^2(\tilde{e}^2\tilde{e}^2 + \tilde{e}^3\tilde{e}^3)$$

$$Q = H(1 - ry)^{-2}[y^2(-e^0e^0 + e^1e^2) - r^2(e^2e^2 + e^3e^3)]$$

$$e^0 = H^{-2/3}\sqrt{\frac{R}{r^2 + y^2}}\mathcal{A}, e^1 = H^{1/3}\sqrt{\frac{r^2 + y^2}{R}}\,\mathrm{d}r$$

$$e^2 = H^{1/3}\sqrt{\frac{Y}{r^2 + y^2}}\left(\,\mathrm{d}t' - r^2\,\mathrm{d}\psi_1 - \frac{q}{H(r^2 + y^2)}\mathcal{A}\right), e^3 = H^{1/3}\sqrt{\frac{r^2 + y^2}{Y}}\,\mathrm{d}y$$

$$e^4 = H^{1/3}\frac{ab}{ry}\left(\,\mathrm{d}t' + (y^2 - r^2)\mathrm{d}\psi_1 - r^2y^2\,\mathrm{d}\psi_2 - \frac{q}{H(r^2 + y^2)}\mathcal{A}\right)$$

$$R = \frac{(1 + g^2r^2)(r^2 + a^2)(r^2 + b^2)}{r^2} + qg^2(2r^2 + a^2 + b^2) + q^2g^2 - 2m$$

$$Y = -\frac{(1 - g^2y^2)(a^2 - y^2)(b^2 - y^2)}{y^2}$$

$$H = 1 + \frac{q}{r^2 + y^2}, q = 2ms^2, s = \sinh\,\delta, \mathcal{A} = \mathrm{d}t' + y^2\,\mathrm{d}\psi_1$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0 + \tilde{e}^1\tilde{e}^1) - r^2(\tilde{e}^2\tilde{e}^2 + \tilde{e}^3\tilde{e}^3) + (y^2 - r^2)\tilde{e}^4\tilde{e}^4$$

$$Q = H^{2/3}[y^2(-e^0e^0 + e^1e^1) - r^2(e^2e^2 + e^3e^3) + (y^2 - r^2)e^4e^4]$$

$$e^0 = \sqrt{\frac{R}{r^2+y^2}}\mathcal{A}, e^1 = \sqrt{\frac{r^2+y^2}{R}}\,\mathrm{d}r, e^2 = \sqrt{\frac{Y}{r^2+y^2}}(\,\mathrm{d}t' - r^2\,\mathrm{d}\psi_1)$$

$$e^3 = \sqrt{\frac{r^2+y^2}{Y}}\,\mathrm{d}y, e^4 = \frac{ab}{ry}\left(\,\mathrm{d}t' + (y^2-r^2)\mathrm{d}\psi_1 - r^2y^2\,\mathrm{d}\psi_2 + \frac{qy^2}{ab(r^2+y^2)}\mathcal{A}\right)$$

$$R = \frac{(1+g^2r^2)(r^2+a^2)(r^2+b^2)+2abq+q^2}{r^2} - 2m, Y = -\frac{(1-g^2y^2)(a^2-y^2)(b^2-y^2)}{y^2}$$

$$q = 2ms^2, s = \sinh\,\delta, \mathcal{A} = \mathrm{d}t' + y^2\,\mathrm{d}\psi_1$$

$$K = y^2(-e^0e^0+e^1e^1) - r^2(e^2e^2+e^3e^3) + (y^2-r^2)e^4e^4$$

$$e^0 = H_1^{-2/3}H_3^{1/6}\sqrt{\frac{R}{r^2+y^2}}\frac{r}{\sqrt{r^2+\gamma_r}}(\,\mathrm{d}t' + y^2\,\mathrm{d}\psi_1), e^1 = H_1^{1/3}H_3^{1/6}\sqrt{\frac{r^2+y^2}{R}}\,\mathrm{d}r$$

$$e^2 = H_1^{-2/3}H_3^{1/6}\sqrt{\frac{Y}{r^2+y^2}}\frac{y}{\sqrt{y^2+\gamma_y}}[\,\mathrm{d}t' - (r^2+2ms_1^2)\mathrm{d}\psi_1], e^3 = H_1^{1/3}H_3^{1/6}\sqrt{\frac{r^2+y^2}{Y}}\,\mathrm{d}y$$

$$e^4 = \frac{H_1^{1/3}H_3^{-1/3}ab}{\sqrt{(r^2+\gamma_r)(y^2+\gamma_y)}}\left\{\left[r^2+\gamma_r+\left(1+\frac{2ms_3c_3}{ab}\right)(y^2+\gamma_y)\right]\frac{(\mathrm{d}t'+y^2\,\mathrm{d}\psi_1)}{H_1(r^2+y^2)}\right.$$
$$\left. -(r^2+\gamma_r)[\mathrm{d}\psi_1 + (y^2+\gamma_y)\mathrm{d}\psi_2]\right\}$$

$$R = \frac{(r^2+\tilde{a}^2)(r^2+\tilde{b}^2)+g^2(r^2+\tilde{a}^2+2ms_1^2)(r^2+\tilde{b}^2+2ms_1^2)(r^2+\gamma_r)}{r^2} - 2m$$

$$Y = -\frac{[1-g^2(y^2+\gamma_y)](\tilde{a}^2-y^2)(\tilde{b}^2-y^2)}{y^2}$$

$$\tilde{a} = ac_3 + bs_3, \tilde{b} = bc_3 + as_3, \gamma_r = -\gamma_y + 2ms_3^2, \gamma_y = -[(a^2+b^2)s_3^2 + 2abs_3c_3]$$

$$H_I = 1 + \frac{2ms_I^2}{r^2+y^2}, s_I = \sinh\,\delta_I, c_I = \cosh\,\delta_I$$

$$x \to r, y \to \mathrm{i}y, t \to t', \sigma \to -\psi_1, \chi \to -ab\psi_2$$

$$\tilde{e}^A = H_1^{-1/3}H_3^{-1/6}e^A$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0 + \tilde{e}^1\tilde{e}^1) - r^2(\tilde{e}^2\tilde{e}^2 + \tilde{e}^3\tilde{e}^3) + \left(\frac{y^2+\gamma_y}{H_3} - r^2\right)\tilde{e}^4\tilde{e}^4$$

$$Q = H_1^{2/3}H_3^{1/3}\left[y^2(-e^0e^0+e^1e^1) - r^2(e^2e^2+e^3e^3) + \left(\frac{y^2+\gamma_y}{H_3} - r^2\right)e^4e^4\right]$$

$$
\left(\frac{\partial}{\partial \tilde{s}}\right)^2 = -\frac{(1+\gamma_r/r^2)}{(r^2+y^2)R}\left[(r^2+2ms_1^2)\frac{\partial}{\partial t'}+\frac{\partial}{\partial \psi_1}+\left(1+\frac{2ms_3c_3}{ab}\right)\frac{1}{r^2+\gamma_r}\frac{\partial}{\partial \psi_2}\right]^2
+\frac{R}{r^2+y^2}\left(\frac{\partial}{\partial r}\right)^2+\frac{(1+\gamma_y/y^2)}{(r^2+y^2)Y}\left(y^2\frac{\partial}{\partial t'}-\frac{\partial}{\partial \psi_1}+\frac{1}{y^2+\gamma_y}\frac{\partial}{\partial \psi_2}\right)^2
+\frac{Y}{r^2+y^2}\left(\frac{\partial}{\partial y}\right)^2+\frac{H_3}{a^2b^2(r^2+\gamma_r)(y^2+\gamma_y)}\left(\frac{\partial}{\partial \psi_2}\right)^2
$$

$$
e^0 = H^{-3/4}\sqrt{\frac{R}{(r^2+y^2)(r^2+z^2)}}\mathcal{A}, e^1 = H^{1/4}\sqrt{\frac{(r^2+y^2)(r^2+z^2)}{R}}\,\mathrm{d}r
$$

$$
e^2 = H^{1/4}\sqrt{\frac{Y}{(r^2+y^2)(y^2-z^2)}}\left[\mathrm{d}t'+(z^2-r^2)\mathrm{d}\psi_1-r^2z^2\,\mathrm{d}\psi_2-\frac{qr}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right]
$$

$$
e^3 = H^{1/4}\sqrt{\frac{(r^2+y^2)(y^2-z^2)}{Y}}\,\mathrm{d}y
$$

$$
e^4 = H^{1/4}\sqrt{\frac{Z}{(r^2+z^2)(z^2-y^2)}}\left[\mathrm{d}t'+(y^2-r^2)\mathrm{d}\psi_1-r^2y^2\,\mathrm{d}\psi_2-\frac{qr}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right]
$$

$$
e^5 = H^{1/4}\sqrt{\frac{(r^2+z^2)(z^2-y^2)}{Z}}\,\mathrm{d}z
$$

$$
R = (r^2+a^2)(r^2+b^2)+g^2[r(r^2+a^2)+q][r(r^2+b^2)+q]-2mr
$$
$$
Y = -(1-g^2y^2)(a^2-y^2)(b^2-y^2), Z = -(1-g^2z^2)(a^2-z^2)(b^2-z^2)
$$
$$
H = 1+\frac{qr}{(r^2+y^2)(r^2+z^2)}, q = 2ms^2, s = \sinh\,\delta
$$
$$
\mathcal{A} = \mathrm{d}t'+(y^2+z^2)\mathrm{d}\psi_1+y^2z^2\,\mathrm{d}\psi_2
$$

$$
\widetilde{K}^{(1)} = (y^2+z^2)(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1)+(z^2-r^2)(\tilde{e}^2\tilde{e}^2+\tilde{e}^3\tilde{e}^3)+(y^2-r^2)(\tilde{e}^4\tilde{e}^4+\tilde{e}^5\tilde{e}^5)
$$
$$
\widetilde{K}^{(2)} = y^2z^2(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1)-r^2z^2(\tilde{e}^2\tilde{e}^2+\tilde{e}^3\tilde{e}^3)-r^2y^2(\tilde{e}^4\tilde{e}^4+\tilde{e}^5\tilde{e}^5)
$$

$$
Q^{(1)} = H^{1/2}[(y^2+z^2)(-e^0e^0+e^1e^1)+(z^2-r^2)(e^2e^2+e^3e^3)+(y^2-r^2)(e^4e^4+e^5e^5)]
$$
$$
Q^{(2)} = H^{1/2}[y^2z^2(-e^0e^0+e^1e^1)-r^2z^2(e^2e^2+e^3e^3)-r^2y^2(e^4e^4+e^5e^5)]
$$

$$e^0 = H^{-4/5}\sqrt{\frac{R}{(r^2+y^2)(r^2+z^2)}}\mathcal{A}, e^1 = H^{1/5}\sqrt{\frac{(r^2+y^2)(r^2+z^2)}{R}}\,\mathrm{d}r$$

$$e^2 = H^{1/5}\sqrt{\frac{Y}{(r^2+y^2)(y^2-z^2)}}\left[\mathrm{d}t' + (z^2-r^2)\mathrm{d}\psi_1 - r^2z^2\,\mathrm{d}\psi_2 - \frac{q}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right]$$

$$e^3 = H^{1/5}\sqrt{\frac{(r^2+y^2)(y^2-z^2)}{Y}}\,\mathrm{d}y$$

$$e^4 = H^{1/5}\sqrt{\frac{Z}{(r^2+z^2)(z^2-y^2)}}\left[\mathrm{d}t' + (y^2-r^2)\mathrm{d}\psi_1 - r^2y^2\,\mathrm{d}\psi_2 - \frac{q}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right]$$

$$e^5 = H^{1/5}\sqrt{\frac{(r^2+z^2)(z^2-y^2)}{Z}}\,\mathrm{d}z$$

$$e^6 = H^{1/5}\frac{a_1a_2a_3}{ryz}\Big[\,\mathrm{d}t' + (y^2+z^2-r^2)\mathrm{d}\psi_1 + (y^2z^2-r^2y^2-r^2z^2)\mathrm{d}\psi_2 - r^2y^2z^2\,\mathrm{d}\psi_3 - \frac{q}{H(r^2+y^2)(r^2+z^2)}\left(1+\frac{gy^2z^2}{a_1a_2a_3}\right)\mathcal{A}\Big]$$

$$R = \frac{1+g^2r^2}{r^2}\prod_{i=1}^{3}\left(r^2+a_i^2\right) + qg^2(2r^2+a_1^2+a_2^2+a_3^2) - \frac{2qga_1a_2a_3}{r^2} + \frac{q^2g^2}{r^2} - 2m$$

$$Y = \frac{1-g^2y^2}{y^2}\prod_{i=1}^{3}\left(a_i^2-y^2\right), Z = \frac{1-g^2z^2}{z^2}\prod_{i=1}^{3}\left(a_i^2-z^2\right), H = 1 + \frac{q}{(r^2+y^2)(r^2+z^2)}$$

$$q = 2ms^2, s = \sinh\,\delta, \mathcal{A} = \mathrm{d}t' + (y^2+z^2)\mathrm{d}\psi_1 + y^2z^2\,\mathrm{d}\psi_2$$

$$\begin{aligned}\widetilde{K}^{(1)} =&(y^2+z^2)(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1) + (z^2-r^2)(\tilde{e}^2\tilde{e}^2+\tilde{e}^3\tilde{e}^3) + (y^2-r^2)(\tilde{e}^4\tilde{e}^4+\tilde{e}^5\tilde{e}^5)\\ &+(y^2+z^2-r^2)\tilde{e}^6\tilde{e}^6\\ \widetilde{K}^{(2)} =&y^2z^2(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1) - r^2z^2(\tilde{e}^2\tilde{e}^2+\tilde{e}^3\tilde{e}^3) - r^2y^2(\tilde{e}^4\tilde{e}^4+\tilde{e}^5\tilde{e}^5)\\ &+(y^2z^2-r^2y^2-r^2z^2)\tilde{e}^6\tilde{e}^6\end{aligned}$$

$$\begin{aligned}Q^{(1)} =&H^{2/5}[(y^2+z^2)(-e^0e^0+e^1e^1) + (z^2-r^2)(e^2e^2+e^3e^3) + (y^2-r^2)(e^4e^4+e^5e^5)\\ &+(y^2+z^2-r^2)e^6e^6]\\ Q^{(2)} =&H^{2/5}[y^2z^2(-e^0e^0+e^1e^1) - r^2z^2(e^2e^2+e^3e^3) - r^2y^2(e^4e^4+e^5e^5)\\ &+(y^2z^2-r^2y^2-r^2z^2)e^6e^6]\end{aligned}$$

$$\tilde{e}^{\hat{1}a}\partial_a = \frac{1}{\sqrt{X_1U_1}}\left(\sum_{k=0}^{3}\,(-x_1^2)^{2-k}\frac{\partial}{\partial\psi_k} + 2ms^2\frac{\partial}{\partial\psi_0} + \frac{2ms^2g}{a_1a_2a_3x_1^2}\frac{\partial}{\partial\psi_3}\right)$$

$$K_a{}^b\mathcal{E}_b = \lambda\mathcal{E}_a$$

$$e^0 = (H_1H_2)^{-1/4}\left(1+\frac{2mr(c_1-c_2)^2(a^2-y^2)}{H_1H_2(r^2+y^2)^2}\right)^{-1/2}\sqrt{\frac{R}{r^2+y^2}}\left(\mathrm{d}t-\frac{a^2-y^2}{a}\,\mathrm{d}\phi\right),$$

$$e^1 = (H_1H_2)^{1/4}\sqrt{\frac{r^2+y^2}{R}}\,\mathrm{d}r,$$

$$e^2 = (H_1H_2)^{1/4}\left(1+\frac{2mr(c_1-c_2)^2(a^2-y^2)}{H_1H_2(r^2+y^2)^2}\right)^{1/2}\sqrt{\frac{Y}{r^2+y^2}}\left[\mathrm{d}t-\frac{r^2+a^2}{a}\,\mathrm{d}\phi -\left(1-\frac{1+2mr(c_1c_2-1)/(r^2+y^2)}{H_1H_2[1+2mr(c_1-c_2)^2(a^2-y^2)/H_1H_2(r^2+y^2)^2]}\right)\left(\mathrm{d}t-\frac{a^2-y^2}{a}\,\mathrm{d}\phi\right)\right],$$

$$e^3 = (H_1H_2)^{1/4}\sqrt{\frac{r^2+y^2}{Y}}\,\mathrm{d}y,$$

$$R = r^2+a^2-2mr, Y = a^2-y^2, H_I = 1+\frac{2ms_I^2r}{r^2+y^2}, s_I = \sinh\,\delta_I, c_I = \cosh\,\delta_I$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1)-\left(r^2+\frac{2mr(c_1-c_2)^2(a^2-y^2)}{H_1H_2(r^2+y^2)}\right)\tilde{e}^2\tilde{e}^2-r^2\tilde{e}^3\tilde{e}^3$$

$$Q = (H_1H_2)^{1/2}\left[y^2(-e^0e^0+e^1e^1)-\left(r^2+\frac{2mr(c_1-c_2)^2(a^2-y^2)}{H_1H_2(r^2+y^2)}\right)e^2e^2-r^2e^3e^3\right]$$

$$e^0 = \sqrt{\frac{WR}{W^2+(a^2-y^2)V}}\left(\mathrm{d}t-\frac{a^2-y^2}{a}\,\mathrm{d}\phi\right), e^1 = \sqrt{\frac{W}{R}}\,\mathrm{d}r$$

$$e^2 = \sqrt{\frac{W^2+(a^2-y^2)V}{W(r^2+y^2)}}\sqrt{\frac{Y}{r^2+y^2}}\left[\mathrm{d}t-\frac{r^2+a^2}{a}\,\mathrm{d}\phi -\left(1-\frac{(r^2+y^2)[r^2+y^2+2mr(c_{1234}-s_{1234}-1)+4m^2s_{1234}^2]}{W^2+(a^2-y^2)V}\right)\left(\mathrm{d}t-\frac{a^2-y^2}{a}\,\mathrm{d}\phi\right)\right]$$

$$e^3 = \sqrt{\frac{W}{Y}}\,\mathrm{d}y$$

$$R = r^2+a^2-2mr, Y = a^2-y^2$$
$$V = 2mr[(c_{13}-c_{24})^2-(s_{13}-s_{24})^2]+4m^2[(s_{13}-s_{24})^2-(s_{13}c_{24}-s_{24}c_{13})^2]$$
$$W^2 = (r_1r_3+y^2)(r_2r_4+y^2)-4m^2(s_{13}c_{24}-s_{24}c_{13})^2y^2, r_I = r+2ms_I^2$$
$$s_{I\ldots J} = s_I\ldots s_J, c_{I\ldots J} = c_I\ldots c_J, s_I = \sinh\,\delta_I, c_I = \cosh\,\delta_I$$

$$\tilde{e}^A = \sqrt{(r^2+y^2)/W}e^A$$

$$\widetilde{K} = y^2(-\tilde{e}^0\tilde{e}^0+\tilde{e}^1\tilde{e}^1)-\left(r^2+\frac{(r^2+y^2)(a^2-y^2)V}{W^2}\right)\tilde{e}^2\tilde{e}^2-r^2\tilde{e}^3\tilde{e}^3$$

$$Q = \frac{W}{r^2+y^2}\left[y^2(-e^0e^0+e^1e^1)-\left(r^2+\frac{(r^2+y^2)(a^2-y^2)V}{W^2}\right)e^2e^2-r^2e^3e^3\right]$$

$$[\cosh(\delta_1-\delta_3)-\cosh(\delta_2-\delta_4)][\cosh(\delta_1+\delta_3)-\cosh(\delta_2+\delta_4)]=0$$

$$\{[1+\cosh(\delta_1+\delta_3)][1+\cosh(\delta_2-\delta_4)]-[1+\cosh(\delta_2+\delta_4)][1+\cosh(\delta_1-\delta_3)]\}$$
$$\times\{[1-\cosh(\delta_1-\delta_3)][1-\cosh(\delta_2+\delta_4)]-[1-\cosh(\delta_2-\delta_4)][1-\cosh(\delta_1+\delta_3)]\}=0$$

$$y^2=a^2\cos^2\theta+b^2\sin^2\theta, s_1=\mathrm{sh}_{e1}, s_2=\mathrm{sh}_{e2}, s_3=\mathrm{sh}_{e}, H_I=1+2ms_I^2/(r^2+y^2)$$

$$\bar{\Delta}=H_1H_2H_3(r^2+y^2)^3$$

$$\mathrm{d}\tilde{s}^2=(H_1H_2H_3)^{-1/3}\,\mathrm{d}s^2$$

$$\tilde{K}^{ab}=\tilde{Q}^{ab}-y^2\tilde{g}^{ab}$$

$$\tilde{Q}^{ab}\partial_a\partial_b=-y^2\left(\frac{\partial}{\partial t}\right)^2+\frac{a^2-b^2}{a^2-y^2}\left(\frac{\partial}{\partial\phi}\right)^2+\frac{b^2-a^2}{b^2-y^2}\left(\frac{\partial}{\partial\psi}\right)^2-\frac{(a^2-y^2)(b^2-y^2)}{y^2}\left(\frac{\partial}{\partial y}\right)^2.$$

$$\mathrm{d}s^2=-f^2(\mathrm{d}t+\omega)^2+f^{-1}\,\mathrm{d}\bar{s}_4^2$$

$$J=\bar{e}^1\wedge\bar{e}^2+\bar{e}^3\wedge\bar{e}^4$$

$$\mathrm{d}\bar{s}_4^2=\Delta_r\left(\sin^2\theta\frac{\mathrm{d}\phi}{\Xi_a}+\cos^2\theta\frac{\mathrm{d}\psi}{\Xi_b}\right)^2+\frac{r^2}{\Delta_r}\mathrm{d}r^2+r^2\sin^2\theta\cos^2\theta\Delta_\theta\left(\frac{\mathrm{d}\phi}{\Xi_a}-\frac{\mathrm{d}\psi}{\Xi_b}\right)^2+\frac{r^2}{\Delta_\theta}\mathrm{d}\theta^2$$

$$\Delta_r=g^2r^4+(1+ag+bg)^2r^2, \Delta_\theta=1-a^2g^2\cos^2\theta-b^2g^2\sin^2\theta, \Xi_a=1-a^2g^2, \Xi_b=1-b^2g^2$$

$$J=\frac{1}{2}\,\mathrm{d}\left[r^2\left(\sin^2\theta\frac{\mathrm{d}\phi}{\Xi_a}+\cos^2\theta\frac{\mathrm{d}\psi}{\Xi_b}\right)\right]$$

$$e^0=(H_1H_2H_3)^{-1/3}(\mathrm{d}t+\omega_\phi\mathrm{d}\phi+\omega_\psi\mathrm{d}\psi)$$
$$e^1=(H_1H_2H_3)^{1/6}\frac{r}{\sqrt{R}}\,\mathrm{d}r, e^2=(H_1H_2H_3)^{1/6}\sqrt{R}\left(\frac{(a^2-y^2)\mathrm{d}\phi}{\Xi_a(a^2-b^2)}+\frac{(b^2-y^2)\mathrm{d}\psi}{\Xi_b(b^2-a^2)}\right)$$
$$e^3=(H_1H_2H_3)^{1/6}\frac{r}{\sqrt{Y}}\,\mathrm{d}y, e^4=(H_1H_2H_3)^{1/6}ry\sqrt{Y}\left(\frac{\mathrm{d}\phi}{\Xi_a(a^2-b^2)}+\frac{\mathrm{d}\psi}{\Xi_b(b^2-a^2)}\right)$$

$$\omega_\phi=-\frac{g(a^2-y^2)}{\Xi_a(a^2-b^2)r^2}\left[(r^2+y^2)^2+\left(a^2+2ab+\frac{2(a+b)}{g}\right)(r^2+y^2)+\frac{\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1}{2g^4}-\frac{a^2b^2}{2}+\frac{b^2-a^2}{2g^2}\right]$$
$$\omega_\psi=-\frac{g(b^2-y^2)}{\Xi_b(b^2-a^2)r^2}\left[(r^2+y^2)^2+\left(b^2+2ab+\frac{2(a+b)}{g}\right)(r^2+y^2)+\frac{\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1}{2g^4}-\frac{a^2b^2}{2}+\frac{a^2-b^2}{2g^2}\right]$$

$$R=g^2r^4+(1+ag+bg)^2r^2, Y=-\frac{(1-g^2y^2)(a^2-y^2)(b^2-y^2)}{y^2}, H_I=1+\frac{g^2y^2+\nu_I}{g^2r^2}$$

$$\nu_I=\sqrt{\Xi_a\Xi_b}(1+g^2\mu_I)-1$$

$$y^2=a^2\cos^2\theta+b^2\sin^2\theta$$

$$\nu_1+\nu_2+\nu_3=2(ag+bg+abg^2).$$

$$\tilde{e}^A=(H_1H_2H_3)^{-1/6}e^A$$

$$\tilde{K}=\frac{(1-g^2y^2)[1-(a^2-y^2)(b^2-y^2)/r^4]}{H_1H_2H_3g^2}\tilde{e}^0\tilde{e}^0 +\frac{1}{(H_1H_2H_3)^{1/2}gr}\left[r\sqrt{R}(\tilde{e}^0\tilde{e}^2+\tilde{e}^2\tilde{e}^0)+y\sqrt{Y}(\tilde{e}^0\tilde{e}^4+\tilde{e}^4\tilde{e}^0)\right]+r^2(\tilde{e}^3\tilde{e}^3+\tilde{e}^4\tilde{e}^4)$$

$$Q=\frac{(1-g^2y^2)[1-(a^2-y^2)(b^2-y^2)/r^4]}{H_1H_2H_3g^2}e^0e^0 +\frac{1}{(H_1H_2H_3)^{1/2}gr}\left[r\sqrt{R}(e^0e^2+e^2e^0)+y\sqrt{Y}(e^0e^4+e^4e^0)\right]+r^2(e^3e^3+e^4e^4)$$

$$\mathrm{d}s^2=(H_1H_2)^{1/(D-2)}\mathrm{d}\tilde{s}^2$$

$$\begin{aligned}\mathrm{d}\tilde{s}^2=&\sum_\mu\frac{U_\mu}{X_\mu}\mathrm{d}x_\mu^2-\left(1-\sum_\mu\frac{2N_\mu}{U_\mu}\right)\frac{\mathrm{d}t^2}{H_1H_2}-c_1c_2\sum_i\sum_\mu\frac{2N_\mu\tilde{\gamma}_i}{z_{i\mu}^2U_\mu}\frac{2\,\mathrm{d}t\,\mathrm{d}\tilde{\phi}_i}{H_1H_2}\\&+\sum_i\Bigg[\frac{H_1H_2B_i}{\tilde{\gamma}_i}+\sum_\mu\left(\frac{2N_\mu}{U_\mu z_{i\mu}^4}-\frac{4N_\mu^2s_1^2s_2^2}{U_\mu^2z_{i\mu}^4}\right)\\&+\sum_{\mu<\nu}\left(\frac{4N_\mu N_\nu(s_1^2+s_2^2)x_{\mu\nu}^4}{U_\mu U_\nu z_{i\mu}^4z_{i\nu}^4}-\frac{8N_\mu N_\nu s_1^2s_2^2}{U_\mu U_\nu z_{i\mu}^2z_{i\nu}^2}\right)+\sum_{\mu\neq\nu}\frac{8N_\mu^2N_\nu s_1^2s_2^2x_{\mu\nu}^4}{U_\mu^2U_\nu z_{i\mu}^4z_{i\nu}^4}\\&-\sum_{\mu<\nu<\rho}\frac{16N_\mu N_\nu N_\rho s_1^2s_2^2}{U_\mu U_\nu U_\rho z_{i\mu}^2z_{i\nu}^2z_{i\rho}^2}\left(\frac{x_{\nu\mu}^2x_{\rho\mu}^2}{z_{i\mu}^2}+\frac{x_{\rho\nu}^2x_{\mu\nu}^2}{z_{i\nu}^2}+\frac{x_{\mu\rho}^2x_{\nu\rho}^2}{z_{i\rho}^2}\right)\Bigg]\tilde{\gamma}_i^2\frac{\mathrm{d}\tilde{\phi}_i^2}{H_1H_2}\\&+\sum_{i<j}\Bigg[\sum_\mu\left(\frac{2N_\mu}{z_{i\mu}^2z_{j\mu}^2U_\mu}-\frac{4N_\mu^2s_1^2s_2^2}{z_{i\mu}^2z_{j\mu}^2U_\mu^2}\right)+\sum_{\mu<\nu}\frac{4N_\mu N_\nu(s_1^2+s_2^2)x_{\mu\nu}^4}{z_{i\mu}^2z_{i\nu}^2z_{j\mu}^2z_{j\nu}^2U_\mu U_\nu}\\&-\sum_{\mu<\nu}\frac{4N_\mu N_\nu s_1^2s_2^2}{U_\mu U_\nu}\left(\frac{1}{z_{i\mu}^2z_{j\nu}^2}+\frac{1}{z_{i\nu}^2z_{j\mu}^2}\right)+\sum_{\mu\neq\nu}\frac{8N_\mu^2N_\nu s_1^2s_2^2x_{\mu\nu}^4}{z_{i\mu}^2z_{i\nu}^2z_{j\mu}^2z_{j\nu}^2U_\mu^2U_\nu}\\&+\sum_{\mu<\nu<\rho}\frac{8N_\mu N_\nu N_\rho s_1^2s_2^2}{z_{i\mu}^2z_{i\nu}^2z_{i\rho}^2z_{j\mu}^2z_{j\nu}^2z_{j\rho}^2U_\mu U_\nu U_\rho}\Big[(z_{i\nu}^2z_{j\rho}^2+z_{i\rho}^2z_{j\nu}^2)x_{\nu\mu}^2x_{\rho\mu}^2\\&+(z_{i\rho}^2z_{j\mu}^2+z_{i\mu}^2z_{j\rho}^2)x_{\rho\nu}^2x_{\mu\nu}^2+(z_{i\mu}^2z_{j\nu}^2+z_{i\nu}^2z_{j\mu}^2)x_{\mu\rho}^2x_{\nu\rho}^2\Big]\Bigg]\tilde{\gamma}_i\tilde{\gamma}_j\frac{2\,\mathrm{d}\tilde{\phi}_i\,\mathrm{d}\tilde{\phi}_j}{H_1H_2}\end{aligned}$$

$$A_{(1)}^1=\sum_\mu\frac{2N_\mu s_1}{H_1U_\mu}\left(c_1\,\mathrm{d}t-c_2\sum_i\frac{\tilde{\gamma}_i}{z_{i\mu}^2}\,\mathrm{d}\tilde{\phi}_i\right),A_{(1)}^2=\sum_\mu\frac{2N_\mu s_2}{H_2U_\mu}\left(c_2\,\mathrm{d}t-c_1\sum_i\frac{\tilde{\gamma}_i}{z_{i\mu}^2}\,\mathrm{d}\tilde{\phi}_i\right)$$

$$X_I=\frac{(H_1H_2)^{(D-3)/2(D-2)}}{H_I},B_{(2)}=\sum_\mu\frac{N_\mu s_1s_2}{U_\mu}\left(\frac{1}{H_1}+\frac{1}{H_2}\right)\mathrm{d}t\wedge\sum_i\frac{\tilde{\gamma}_i}{z_{i\mu}^2}\,\mathrm{d}\tilde{\phi}_i$$

$$U_\mu = \prod_{\nu=1}^{n} (x_\nu^2 - x_\mu^2), X_\mu = -\frac{1}{(-x_\mu^2)^\varepsilon} \prod_{k=1}^{n-1+\varepsilon} (a_k^2 - x_\mu^2) + 2N_\mu, x_{\mu\nu}^2 = x_\mu^2 - x_\nu^2$$

$$H_I = 1 + \sum_{\mu=1}^{n} \frac{2N_\mu s_I^2}{U_\mu}, N_\mu = m_\mu x_\mu^{1-\varepsilon}, s_I = \sinh\ \delta_I, c_I = \cosh\ \delta_I$$

$$\tilde{\gamma}_i = a_i^{2\varepsilon} \prod_{\mu=1}^{n} (a_i^2 - x_\mu^2), z_{i\mu}^2 = a_i^2 - x_\mu^2, B_i = \prod_{k=1}^{n-1+\varepsilon} (a_i^2 - a_k^2)$$

$$t = t' + \frac{2\ell^2 \cosh\ \alpha}{a'} \phi', \phi = \frac{\sqrt{a'^2 + \ell^2}}{a'} \phi', \cos\ \theta = \frac{y' - \ell}{\sqrt{a'^2 + \ell^2}}, a = \sqrt{a'^2 + \ell^2}.$$

$$\mathrm{d}s^2 = (H_1 H_2)^{1/2}\, \mathrm{d}\tilde{s}^2$$

$$x_1 = y, x_2 = \mathrm{i}r, m_1 = -\ell, m_2 = -\mathrm{i}M, \delta_1 = \alpha, \delta_2 = 0$$

$$\mathcal{M} \xrightarrow{\pi} \mathcal{M}_4,$$

$$\mathcal{M}_6(p) = \pi^{-1}(p)$$

$$\mathcal{M}_6(p) \cong \mathcal{M}_6(q), \forall p, q \in \mathcal{M}_4$$

$$\delta_\epsilon \phi_b \sim \bar{\epsilon}(p)\big(\phi_b + \bar{\phi}_f \phi_f\big)\phi_f, \delta_\epsilon \phi_f \sim \partial\epsilon(p) + \big(\phi_b + \bar{\phi}_f \phi_f\big)\epsilon(p)\phi_f$$

$$\phi_b \to \phi_b, \phi_f \to -\phi_f.$$

$$\mathbf{R}(\nabla) = \mathbf{T}.$$

$$\mathbb{Z}_2 \to \mathrm{Spin}(1,3) \xrightarrow{\rho} \mathrm{SO}(1,3).$$

$$\mathrm{F}(T\mathcal{M}) = \{(p, \mathrm{e}_1, \dots, \mathrm{e}_4): p \in \mathcal{M}, (\mathrm{e}_1, \dots, \mathrm{e}_4) \,\|T_p\mathcal{M}\|\}$$

$$\pi_\mathrm{F}: \mathrm{F}(T\mathcal{M}) \to \mathcal{M} \text{ by } \pi(p, \mathrm{e}_1, \cdots, \mathrm{e}_4) = p$$

$$\mathbf{e}'_a = A_a^b \mathbf{e}_b, a = 1, \cdots, 4$$

$$\mathrm{F_{SO}}(T\mathcal{M}) = \{(p, \mathrm{e}_1, \dots, \mathrm{e}_4): p \in \mathcal{M}, (\mathrm{e}_1, \dots, \mathrm{e}_4)\langle T_p\mathcal{M} | \mathbf{g}(\mathbf{e}_a, \mathbf{e}_b)\rangle = \eta_{ab}, \det_\partial(\mathbf{e}) > 0\}$$

$$\pi_\mathrm{SO}: \mathrm{F_{SO}}(T\mathcal{M}) \to \mathcal{M} \text{ by } \pi(p, \mathrm{e}_1, \cdots, \mathrm{e}_4) = p$$

$$\mathbf{e}'_a = O_a^b \mathbf{e}_b, a = 1, \cdots, 4$$

$$\pi_\mathrm{SO} \circ \Omega_\mathrm{P} = \pi_\mathrm{P}$$

$$\Omega_\mathrm{P}(p, O) = \Omega_\mathrm{P}(p)\rho(O), \forall p \in \mathrm{P}, \forall O \in \mathrm{Spin}(1,3)$$

$$\mathrm{S} = (\mathrm{P} \times V)/\mathrm{Spin}(1,3).$$

$$\pi_\mathrm{S}: (\mathrm{P} \times V)/\mathrm{Spin}(1,3) \to \mathrm{P}/\mathrm{Spin}(1,3)$$

$$\mathbf{g} = \eta(\mathbf{e}, \mathbf{e}),$$

$$S[\mathbf{e}, \omega] = \frac{1}{2}\int \ R_{a_1 a_2}(\omega) \wedge \mathbf{e}_{a3} \wedge \mathbf{e}_{a4} \epsilon^{a_1 a_2 a_3 a_4}$$

$$\bar{\psi} \not{\bar{\phi}} \psi, \text{ where } \mathcal{D}\psi \sim \partial\psi + \omega\psi$$

$$\omega = \omega_{\text{LC}} + \text{K},$$

$$f: (\mathcal{M}, \mathbf{g}) \to (\mathcal{M}, \tilde{\mathbf{g}} = (f^{-1})^* \mathbf{g})$$

$$\mathbb{TM} = T\mathcal{M} \oplus T^*\mathcal{M} \text{ over } \mathcal{M}$$

$$\mathbb{X} = X + \xi$$

$$\xi \in T^*\mathcal{M}$$

$$\text{G}(\mathbb{X}, \mathbb{Y}) = \frac{1}{2}(\xi(Y) + \eta(X)), \forall \mathbb{X} = X + \xi, \mathbb{Y} = Y + \eta \in \mathbb{TM}$$

$$\text{J}: T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M} \oplus T^*\mathcal{M}$$

$$\Lambda^2 T^*\mathcal{M} \oplus T\mathcal{M} \oplus T^*\mathcal{M}$$

$$\mathcal{M} = \mathcal{M}_4 \times \mathcal{M}_6,\ \text{g}(x, y) = \text{g}_4(x) + \text{g}_6(y)$$

$$\mathbf{g}(y) = \eta + \mathbf{g}_{CY}(y)$$

$$\mathbf{g} = g_{tt}(r) dt \otimes dt - g_{rr}(r) dr \otimes dr - r^2\, \text{h}_{S^2}$$

$$\text{h}_{S^2} = d\theta \otimes d\theta + \sin^2\ \theta d\phi \otimes d\phi$$

$$g_{tt}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, g_{rr}(r) = g_{tt}^{-1}(r)$$

$$g_{tt}(r) = \frac{(r - r_+)(r - r_-)}{r^2}, r_\pm = m \pm \sqrt{m^2 - q^2}$$

$$S = \frac{A}{4} = \pi r_+^2, T = \frac{1}{2\pi}\left(\frac{r_0}{r_+}\right)^2,$$

$$\mathbf{g} = e^{2U} dt \otimes dt - e^{-2U}\left[\frac{r_0^4}{\sinh^4\ r_0\tau} d\tau \otimes d\tau + \frac{r_0^2}{\sinh^2\ r_0\tau}\ \text{h}_{S^2}\right]$$

$$\mathbf{g} = e^{2U} dt \otimes dt - e^{-2U}\left[\frac{1}{\tau^4} d\tau \otimes d\tau + \frac{1}{\tau^2}\mathbf{h}_{S^2}\right] = e^{2U} dt \otimes dt - e^{-2U}[\delta_{ij} dx^i \otimes dx^j]$$

$$\lim_{\tau\to-\infty} e^{-2U} = \frac{A}{4\pi}\lim_{\tau\to-\infty}\tau^2,\ \lim_{\tau\to-\infty}\tau\frac{d\phi^i}{d\tau} = 0, i = 1, \cdots, n_v$$

$$\lim_{\tau\to-\infty}\left[\frac{d^2\phi^i}{d\tau^2}+\frac{4\pi}{A}g^{ij}(\phi_{\mathrm{h}})\partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})\frac{1}{\tau^2}\right]=0$$

$$\lim_{\tau\to-\infty}\phi^i=\lim_{\tau\to-\infty}\left[-\frac{4\pi}{A}g^{ij}(\phi_{\mathrm{h}})\partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})\log(-\tau)+c_1^i\tau+c_2^i\right]$$

$$\lim_{\tau\to-\infty}\frac{d\phi^i}{d\tau}=\lim_{\tau\to-\infty}\left[-\frac{4\pi}{A}g^{ij}(\phi_{\mathrm{h}})\partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})\frac{1}{\tau}+c_1^i\right]$$

$$c_1^i=0, c_2^i=\phi_{\mathrm{h}}^i, \mathcal{G}^{ij}(\phi_{\mathrm{h}})\partial_j V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})=0$$

$$\partial_i V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})=0$$

$$S=\pi V_{\mathrm{bh}}(\phi_{\mathrm{h}},\mathcal{Q})$$

$$\phi_{\mathrm{h}}=\phi_{\mathrm{h}}(\mathcal{Q}), S=\pi V_{\mathrm{bh}}(\phi_{\mathrm{h}}(\mathcal{Q}),\mathcal{Q})$$

$$\delta_\epsilon\phi_b=0, \delta_\epsilon\phi_f=0$$

$$T\mathcal{M}\otimes\mathbb{C}\to\mathcal{M}$$

$$(T\mathcal{M}\otimes\mathbb{C})_p=T\mathcal{M}_p\otimes\mathbb{C}$$

$$\mathcal{J}(v\otimes c)=\mathcal{J}v\otimes c, v\in T\mathcal{M}, c\in\mathbb{C}.$$

$$T_p\mathcal{M}\otimes\mathbb{C}$$

$$T_{(1,0)}\mathcal{M}=\{v\in T\mathcal{M}\otimes\mathbb{C}\mid \mathcal{J}v=iv,\forall v\in T\mathcal{M}\otimes\mathbb{C}\},$$
$$T_{(0,1)}\mathcal{M}=\{v\in T\mathcal{M}\otimes\mathbb{C}\mid \mathcal{J}v=-iv,\forall v\in T\mathcal{M}\otimes\mathbb{C}\}.$$

$$\pi_{(1,0)}: T\mathcal{M}\otimes\mathbb{C}\to T_{(1,0)}\mathcal{M}$$
$$v\to\frac{1}{2}(v\otimes 1-\mathcal{J}v\otimes i)$$
$$\pi_{(0,1)}: T\mathcal{M}\otimes\mathbb{C}\to T_{(0,1)}\mathcal{M}$$
$$v\to\frac{1}{2}(v\otimes 1+\mathcal{J}v\otimes i)$$

$$\pi_{(1,0)}\circ\mathcal{J}=-i\pi_{(0,1)}$$

$$T\mathcal{M}\cong T_{(1,0)}\mathcal{M}\cong\overline{T_{(0,1)}\mathcal{M}}$$
$$(\pi_{(1,0)},\pi_{(0,1)}): T\mathcal{M}\otimes\mathbb{C}\cong T_{(1,0)}\mathcal{M}\oplus T_{(0,1)}\mathcal{M}.$$

$$T^*\mathcal{M}\otimes\mathbb{C}\to\mathcal{M}$$

$$T^*\mathcal{M}\cong T^{(1,0)}\mathcal{M}\cong T^{(0,1)}\mathcal{M}$$
$$(\pi^{(1,0)},\pi^{(0,1)}): T^*\mathcal{M}\otimes\mathbb{C}\cong T^{(1,0)}\mathcal{M}\oplus T^{(0,1)}\mathcal{M}.$$

$$T^{(1,0)}\mathcal{M}=\{\alpha\in T^*\mathcal{M}\otimes\mathbb{C}\mid \alpha(\mathcal{J}v)=i\alpha(v),\forall v\in T\mathcal{M}\otimes\mathbb{C}\}$$
$$T^{(0,1)}\mathcal{M}=\{v\alpha\in T^*\mathcal{M}\otimes\mathbb{C}\mid \alpha(\mathcal{J}v)=-i\alpha(v),\forall v\in T\mathcal{M}\otimes\mathbb{C}\}$$

$$\pi^{(1,0)}: T^*\mathcal{M} \otimes \mathbb{C} \to T^{(1,0)}\mathcal{M}$$
$$\alpha \to \frac{1}{2}(\alpha \otimes 1 - \alpha \otimes i \circ \mathcal{J})$$

$$\pi^{(0,1)}: T^*\mathcal{M} \otimes \mathbb{C} \to T^{(0,1)}\mathcal{M}$$
$$\alpha \to \frac{1}{2}(\alpha \otimes 1 + \alpha \otimes i \circ \mathcal{J})$$

$$\Omega^k(\mathcal{M}, \mathbb{C}) \equiv \Gamma\left(\Lambda^k(T^*\mathcal{M} \otimes \mathbb{C})\right)$$

$$\Lambda^k(T^*\mathcal{M} \otimes \mathbb{C}) = \Lambda^k\left(T^{(0,1)}\mathcal{M} \oplus T^{(1,0)}\mathcal{M}\right) = \bigoplus_{l+m=k} \Lambda^l\left(T^{(0,1)}\mathcal{M}\right) \wedge \Lambda^m\left(T^{(1,0)}\mathcal{M}\right)$$

$$\bigoplus_{l+m=k} \Lambda^l\left(T^{(0,1)}\mathcal{M}\right) \wedge \Lambda^m\left(T^{(1,0)}\mathcal{M}\right)$$

$$\Omega^{(l,m)}(\mathcal{M}, \mathbb{C}) = \Gamma\left(\Lambda^l\left(T^{(0,1)}\mathcal{M}\right) \wedge \Lambda^m\left(T^{(1,0)}\mathcal{M}\right)\right)$$

$$\Omega^k(\mathcal{M}, \mathbb{C}) = \bigoplus_{l+m=k} \Omega^{(l,m)}(\mathcal{M}, \mathbb{C})$$

$$\pi^{(l,m)} : \Lambda^k(T^*\mathcal{M} \otimes \mathbb{C}) \to \Lambda^l\left(T^{(0,1)}\mathcal{M}\right) \wedge \Lambda^m\left(T^{(1,0)}\mathcal{M}\right)$$

$$\partial \equiv \pi^{(l+1,m)} \circ d: \Omega^{(l,m)}(\mathcal{M}, \mathbb{C}) \to \Omega^{(l+1,m)}(\mathcal{M}, \mathbb{C})$$
$$\bar{\partial} \equiv \pi^{(l,m+1)} \circ d: \Omega^{(l,m)}(\mathcal{M}, \mathbb{C}) \to \Omega^{(l,m+1)}(\mathcal{M}, \mathbb{C})$$

$$\mathcal{G}(\mathcal{J}\,\cdot, \mathcal{J}\,\cdot) = \omega(\mathcal{J}\,\cdot, \mathcal{J}^2\,\cdot) = \omega(\cdot, \mathcal{J}\,\cdot) = \mathcal{G}(\cdot,\cdot)$$

$$N(u,v) \equiv [\mathcal{J}u, \mathcal{J}v] - \mathcal{J}[u, \mathcal{J}v] - \mathcal{J}[\mathcal{J}u, v] - [u,v] = 0 \forall u, v \in T\mathcal{M}$$

$$T_p^*\mathcal{M} = \mathbb{R} - \mathrm{Span}\left[dx^i, dy^i\right]_p$$
$$T_p^*\mathcal{M} \otimes \mathbb{C} = \mathbb{C} - \mathrm{Span}\left[dx^i, dy^i\right]_p$$

$$dz^i = dx^i + i dy^i \text{ and } d\bar{z}^{\bar{\imath}} = dx^i - i dy^i$$

$$T_p^*\mathcal{M} \otimes \mathbb{C} = \mathbb{C} - \mathrm{Span}\left[dz^i\right]_p \oplus \mathbb{C} - \mathrm{Span}[d\bar{z}^{\bar{\imath}}]_p = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$$

$$dz^i \circ \mathcal{J} = i dz^i$$

$$d\bar{z}^{\bar{\imath}} \circ \mathcal{J} = -i d\bar{z}^{\bar{\imath}}$$

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right), \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right),$$

$$\partial f = \frac{\partial f}{\partial z^i} dz^i, \bar{\partial} f = \frac{\partial f}{\partial \bar{z}^{\bar{\imath}}} d\bar{z}^{\bar{\imath}}.$$

$$\mathcal{J}_p^*\omega_p(u,v) = \omega_p\left(\mathcal{J}_p u, \mathcal{J}_p v\right) = \mathcal{G}_p\left(v, \mathcal{J}_p u\right) = \omega_p(u,v), \forall p \in \mathcal{M} \text{ and } u, v \in T_p\mathcal{M}$$

$$T\mathcal{M} \otimes \mathbb{C}$$

$$\omega \in \Omega^2(\mathcal{M},\mathbb{C}) = \Omega^{(2,0)}(\mathcal{M},\mathbb{C}) \oplus \Omega^{(1,1)}(\mathcal{M},\mathbb{C}) \oplus \Omega^{(0,2)}(\mathcal{M},\mathbb{C}).$$

$$\omega \in \Omega^{(1,1)}(\mathcal{M},\mathbb{C}),$$

$$[\omega] \in H^{(1,1)}_{\text{Dolbeault}}(\mathcal{M})$$

$$dz^i \wedge dz^j \text{ or } d\bar{z}^{\bar{\imath}} \wedge d\bar{z}^{\bar{\jmath}}$$

$$\omega = i h_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$$

$$\mathcal{G} = h_{i\bar{\jmath}}\left(dz^i \otimes d\bar{z}^{\bar{\jmath}} + dz^{\bar{\jmath}} \otimes d\bar{z}^i\right).$$

$$\mathcal{G} = \partial_i \partial_j \mathcal{K}\left(dz^i \otimes d\bar{z}^{\bar{\jmath}} + dz^{\bar{\jmath}} \otimes d\bar{z}^i\right)$$

$$\Gamma_{jk}{}^{i} = \mathcal{G}^{i\bar{\imath}} \partial_j \mathcal{G}_{\bar{\imath}k}, \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = \mathcal{G}^{\bar{\imath}i} \partial_{\bar{\jmath}} \mathcal{G}_{\bar{k}i}$$

$$R_{i\bar{\imath}} = \partial_i \partial_{\bar{\imath}}\left(\frac{1}{2}\log\ \det\mathcal{G}\right)$$

$$\mathcal{K}' = \mathcal{K} + f + \bar{f}$$

$$\left\{\mathcal{L}^{(q,\bar{q})}, q \in \mathbb{R}, \bar{q} \in \mathbb{R}\right\}$$

$$W_{(\alpha)} = e^{-(qf+\bar{q}\bar{f})/2} W_{(\beta)}, \mathcal{K}_{(\alpha)} = \mathcal{K}_{(\alpha)} + f + \bar{f}$$

$$\mathcal{Q} \equiv (2i)^{-1}\left(dz^i \partial_i \mathcal{K} - d\bar{z}^{\bar{\imath}} \partial_{\bar{\imath}} \mathcal{K}\right),$$

$$\mathfrak{D}_i \equiv \nabla_i + iq\mathcal{Q}_i, \mathfrak{D}_{\bar{\imath}} \equiv \nabla_{\bar{\imath}} - i\bar{q}\mathcal{Q}_{\bar{\imath}},$$

$$\mathcal{Q}_{(\alpha)} = \mathcal{Q}_{(\beta)} - \frac{i}{2}\partial f$$

$$\mathcal{SV} = \mathcal{SM} \otimes \mathcal{L} \xrightarrow{\pi} \mathcal{M}$$

$$\omega = i\partial\bar{\partial}\log\left(i\Omega_M \bar{\Omega}^M\right)$$

$$\Omega_M \bar{\Omega}^M = \langle \Omega \mid \bar{\Omega} \rangle$$

$$\Omega^M = \left(\mathcal{X}^\Lambda, \mathcal{F}_\Lambda\right)^T$$

$$\Omega = \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \Omega \mid \bar{\Omega} \rangle & \equiv \overline{\mathcal{X}}^\Lambda \mathcal{F}_\Lambda - \mathcal{X}^\Lambda \overline{\mathcal{F}}_\Lambda = -ie^{-\mathcal{K}} \\ \partial_{\bar{\imath}} \Omega & = 0 \\ \langle \partial_i \Omega \mid \Omega \rangle & = 0 \end{cases}$$

$$\mathcal{V} = e^{\frac{\kappa}{2}} \Omega^3$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \mathcal{V} \mid \overline{\mathcal{V}} \rangle & \equiv \overline{\mathcal{L}}^\Lambda \mathcal{M}_\Lambda - \mathcal{L}^\Lambda \overline{\mathcal{M}}_\Lambda = -i \\ \mathfrak{D}_{\bar{\imath}} \mathcal{V} & = \left(\partial_{\bar{\imath}} + \frac{1}{2} \partial_{\bar{\imath}} \mathcal{K} \right) \mathcal{V} = 0 \\ \langle \mathfrak{D}_i \mathcal{V} \mid \mathcal{V} \rangle & = 0 \end{cases}$$

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Sigma i} \end{pmatrix}, \overline{\mathcal{U}}_{\bar{\imath}} = \overline{\mathcal{U}_i}$$

$$\mathfrak{D}_{\bar{\imath}} \mathcal{U}_i = \mathcal{G}_{i\bar{\imath}} \mathcal{V} \langle \mathcal{U}_i \mid \overline{\mathcal{U}}_{\bar{\imath}} \rangle = i \mathcal{G}_{i\bar{\imath}}$$
$$\langle \mathcal{U}_i \mid \overline{\mathcal{V}} \rangle = 0, \langle \mathcal{U}_i \mid \mathcal{V} \rangle = 0$$

$$\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = -\langle \mathcal{U}_j \mid \mathcal{U}_i \rangle$$

$$\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = \langle \mathcal{U}_j \mid \mathcal{U}_i \rangle = 0.$$

$$\mathcal{A} = i \langle \mathcal{A} \mid \overline{\mathcal{V}} \rangle \mathcal{V} - i \langle \mathcal{A} \mid \mathcal{V} \rangle \overline{\mathcal{V}} + i \langle \mathcal{A} \mid \mathcal{U}_i \rangle \mathcal{G}^{i\bar{\imath}} \overline{\mathcal{U}_i} - i \langle \mathcal{A} \mid \bar{\mathcal{U}}_{\bar{\imath}} \rangle \mathcal{G}^{i\bar{\imath}} \mathcal{U}_i.$$

$$\mathcal{C}_{ijk} \equiv \langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{U}_k \rangle \rightarrow \mathfrak{D}_i \mathcal{U}_j = i \mathcal{C}_{ijk} \mathcal{G}^{k\bar{l}} \overline{\mathcal{U}}_{\bar{l}}$$

$$\mathfrak{D}_{\bar{\imath}} \mathcal{C}_{jkl} = 0, \mathfrak{D}_{[i} \mathcal{C}_{j]kl} = 0$$

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma, h_{\Lambda i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_i$$

$$\begin{aligned} \mathcal{L}^\Lambda \Im \mathrm{m} \mathcal{N}_{\Lambda\Sigma} \overline{\mathcal{L}}^\Sigma &= -\frac{1}{2} \\ \mathcal{L}^\Lambda \Im \mathrm{m} \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma &= \mathcal{L}^\Lambda \Im \mathrm{m} \mathcal{N}_{\Lambda\Sigma} \bar{f}_{\bar{\imath}} = 0 \\ f_i^\Lambda \Im \mathrm{m} \mathcal{N}_{\Lambda\Sigma} \bar{f}_{\bar{\imath}} &= -\frac{1}{2} \mathcal{G}_{i\bar{\imath}} \end{aligned}$$

$$\begin{aligned} (\partial_i \mathcal{N}_{\Lambda\Sigma}) \mathcal{L}^\Sigma &= -2i \Im\,\mathrm{m}(\mathcal{N})_{\Lambda\Sigma} f_i^\Sigma \\ \partial_i \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma{}_j &= -2 \mathcal{C}_{ijk} \mathcal{G}^{k\bar{k}} \Im\,\mathrm{m} \mathcal{N}_{\Lambda\Sigma} \bar{f}_{\bar{k}} \\ \mathcal{C}_{ijk} &= f^\Lambda{}_i f^\Sigma{}_j \partial_k \overline{\mathcal{N}}_{\Lambda\Sigma} \\ \mathcal{L}^\Sigma \partial_{\bar{\imath}} \mathcal{N}_{\Lambda\Sigma} &= 0 \\ \partial_{\bar{\imath}} \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma &= 2i \mathcal{G}_{i\bar{\imath}} \Im\,\mathrm{m} \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma \end{aligned}$$

$$U^{\Lambda\Sigma} \equiv f^\Lambda{}_i \mathcal{G}^{i\bar{\imath}} \bar{f}^{\bar{\nu}}{}_{\bar{\imath}} = -\frac{1}{2} \Im m(\mathcal{N})^{-1|\Lambda\Sigma} - \overline{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma$$

$$\begin{aligned} \mathcal{T}_\Lambda &\equiv 2i \mathcal{L}_\Lambda = 2i \mathcal{L}^\Sigma \Im \mathrm{m} \mathcal{N}_{\Sigma\Lambda} \\ \mathcal{T}^i{}_\Lambda &\equiv -\bar{f}_\Lambda{}^i = -\mathcal{G}^{i\bar{\jmath}} \overline{f^\Sigma}{}_{\bar{\jmath}} \Im\,\mathrm{m} \mathcal{N}_{\Sigma\Lambda} \end{aligned}$$

$$\begin{aligned} \partial_i \mathcal{N}_{\Lambda\Sigma} &= 4 \mathcal{T}_{i(\Lambda} \mathcal{T}_{\Sigma)} \\ \partial_{\bar{\imath}} \mathcal{N}_{\Lambda\Sigma} &= 4 \bar{\mathcal{C}}_{\bar{\imath}\bar{\jmath}\bar{k}} \mathcal{T}^{\bar{\imath}}_{(\Lambda} \mathcal{T}^{\bar{\jmath}}_{\Sigma)} \end{aligned}$$

$$e^{-\mathcal{K}} = -2 \Im\,\mathrm{m} \mathcal{N}_{\Lambda\Sigma} \mathcal{X}^\Lambda \overline{\mathcal{X}}^\Sigma$$

$$\partial_i \mathcal{X}^\Lambda [2 \mathcal{F}_\Lambda - \partial_\Lambda (\mathcal{X}^\Sigma \mathcal{F}_\Sigma)] = 0$$

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})$$

$$\mathcal{N}_{\Lambda\Sigma} = \overline{\mathcal{F}}_{\Lambda\Sigma} + 2i\frac{\Im\,\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\chi^{\Lambda'}\Im\mathrm{m}\mathcal{F}_{\Sigma\Sigma'}\chi^{\Sigma'}}{\chi^{\Omega}\Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\chi^{\Omega'}}$$

$$\mathcal{C}_{ijk} = e^{\kappa}\partial_i\mathcal{X}^{\Lambda}\partial_j\mathcal{X}^{\Sigma}\partial_k\mathcal{X}^{\Omega}\mathcal{F}_{\Lambda\Sigma\Omega}$$

$$s_p(p) = p, \left(ds_p\right)_p = -\mathrm{I}$$

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$$

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h},\mathfrak{t}] \subset \mathfrak{t}, [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{h},$$

$$f \equiv \left(f_{IJ}^{\Lambda}, f_i^{\Lambda}\right), h \equiv \left(h_{\Lambda IJ}, h_{\Lambda i}\right)$$

$$G \times_H \mathbb{R}^{2n} \to G/H$$

$$\bar{n} = n + \frac{\mathcal{N}(\mathcal{N}-1)}{2}$$

$$U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} f+ih & \bar{f}+i\bar{h} \\ f-ih & \bar{f}-i\bar{h} \end{pmatrix}.$$

$$U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{\dagger} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^{T} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} f^{\dagger}-ih^{\dagger} & -(f^{\dagger}+ih^{\dagger}) \\ -(f-ih) & f+ih \end{pmatrix}$$

$$i\left(f^{\dagger}h - h^{\dagger}f\right) = 1, f^{T}h - h^{T}f = 0$$

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda IJ} \end{pmatrix}, \mathcal{V}_i = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda i} \end{pmatrix}$$

$$\left\langle \mathcal{V}_{IJ} \mid \overline{\mathcal{V}}^{KL} \right\rangle = -2i\delta^{KL}{}_{IJ}$$
$$\left\langle \mathcal{V}_{i} \mid \overline{\mathcal{V}}^{j} \right\rangle = -i\delta_i^j$$

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \Omega & \bar{P} \\ P & \bar{\Omega} \end{pmatrix}$$

$$\Omega = \begin{pmatrix} \Omega^{KL}{}_{IJ} & \Omega^{j}{}_{IJ} \\ \Omega^{KL}{}_{i} & \Omega^{j}{}_{i} \end{pmatrix} = \begin{pmatrix} i\left\langle d\mathcal{V}_{IJ} \mid \overline{\mathcal{V}}^{KL}\right\rangle & i\left\langle d\mathcal{V}_{IJ} \mid \overline{\mathcal{V}}^{j}\right\rangle \\ i\left\langle d\mathcal{V}_{i} \mid \overline{\mathcal{V}}^{KL}\right\rangle & i\left\langle d\mathcal{V}_{i} \mid \overline{\mathcal{V}}^{j}\right\rangle \end{pmatrix},$$

$$P = \begin{pmatrix} P_{KLIJ} & P_{jIJ} \\ P_{KLi} & P_{ij} \end{pmatrix} = \begin{pmatrix} -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_{IJ} \mid \mathcal{V}_{j}\rangle \\ -i\langle d\mathcal{V}_{i} \mid \mathcal{V}_{KL}\rangle & -i\langle d\mathcal{V}_{i} \mid \mathcal{V}_{j}\rangle \end{pmatrix}.$$

$$\mathcal{N} = hf^{-1} = \mathcal{N}^{T},$$

$$\mathfrak{D}h_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}\mathfrak{D}f^\Lambda, h_\Lambda = \mathcal{N}_{\Lambda\Sigma}f^\Sigma,$$

$$-\frac{1}{2}(\Im\,\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma} = \frac{1}{2}f^{\Lambda}{}_{IJ}\bar{f}^{\Sigma IJ} + f^{\Lambda}{}_{i}\bar{f}^{\Sigma i},$$

$$\frac{1}{2}|\mathcal{V}_{IJ}\rangle\left\langle\overline{\mathcal{V}}^{IJ}\right| - \frac{1}{2}\left|\overline{\mathcal{V}}^{IJ}\right\rangle\langle\mathcal{V}_{IJ}| + |\mathcal{V}_i\rangle\left\langle\overline{\mathcal{V}}^{i}\right| - \left|\overline{\mathcal{V}}^{i}\right\rangle\langle\mathcal{V}_i| = i.$$

$$\mathfrak{D}\mathcal{V} = d\mathcal{V} - \mathcal{V}\Omega,$$

$$\Omega^{KL}{}_{i} = \Omega^{j}{}_{IJ} = 0,$$

$$\begin{aligned} P_{IJKL} &= -2f_{IJ}^{\Lambda}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathfrak{D}f_{KL}^{\Sigma} \\ P_{iIJ} &= -2f_{i}^{\Lambda}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathfrak{D}f_{IJ}^{\Sigma} \\ P_{ij} &= -2f_{i}^{\Lambda}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathfrak{D}f_{j}^{\Sigma} \end{aligned}$$

$$\begin{aligned} \mathfrak{D}f_{IJ}^{\Lambda} &= \bar{f}^{\Lambda i}P_{iIJ} + \frac{1}{2}\bar{f}^{\Lambda KL}P_{IJKL} \\ \mathfrak{D}f_{i}^{\Lambda} &= \bar{f}^{\Lambda j}P_{ij} + \frac{1}{2}\bar{f}^{\Lambda IJ}P_{iIJ} \end{aligned}$$

$$\langle\mathfrak{D}\mathcal{V} \mid \overline{\mathcal{V}}\rangle = 0, \langle\mathfrak{D}\mathcal{V} \mid \mathcal{V}\rangle = \langle d\mathcal{V} \mid \mathcal{V}\rangle = iP.$$

$$\bar{P}^{IJKLA}P_{MNOPA} = 4!\,\delta^{IJKL}{}_{MNOP}, \bar{P}^{iIJA}P_{jKLA} = 2\delta^{i}{}_{j}\delta^{IJ}{}_{KL}.$$

$$\begin{aligned} \left\langle\mathfrak{D}_A\mathcal{V}_{IJ} \mid \mathfrak{D}_B\overline{\mathcal{V}}^{KL}\right\rangle &= \frac{i}{2}P_{IJMNA}\bar{P}^{KLMN}{}_{B} + iP_{iIJA}\bar{P}^{iKL}{}_{B} \\ \left\langle\mathfrak{D}_A\mathcal{V}_{IJ} \mid \mathfrak{D}_B\overline{\mathcal{V}}^{i}\right\rangle &= \frac{i}{2}P_{IJKLA}\bar{P}^{iKL}{}_{B} + iP_{jIJA}\bar{P}_{B}^{ij} \\ \left\langle\mathfrak{D}_A\mathcal{V}_{i} \mid \mathfrak{D}_B\overline{\mathcal{V}}^{j}\right\rangle &= \frac{i}{2}P_{iIJA}\bar{P}^{iIJ}{}_{B} + iP_{ikA}\bar{P}_{B}^{jk} \end{aligned}$$

$$\langle\mathfrak{D}_A\mathcal{V}_{IJ} \mid \mathfrak{D}_B\mathcal{V}_{KL}\rangle = \langle\mathfrak{D}_A\mathcal{V}_{IJ} \mid \mathfrak{D}_B\mathcal{V}_{i}\rangle = \langle\mathfrak{D}_A\mathcal{V}_{i} \mid \mathfrak{D}_B\mathcal{V}_{j}\rangle = 0$$

$$d\mathcal{N} = 4i\Im\,\mathrm{m}\mathcal{N}\mathfrak{D}ff^{\dagger}\Im\mathrm{m}\mathcal{N}$$

$$d\mathcal{N}_{\Lambda_\Sigma} = i\Im\,\mathrm{m}\mathcal{N}_{\Gamma(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Omega}\left[P_{IJKL}\bar{f}^{\Gamma IJ}\bar{f}^{\Omega KL} + 4P_{iIJ}\bar{f}^{\Gamma i}\bar{f}^{\Omega IJ} + 4P_{ij}\bar{f}^{\Gamma i}\bar{f}^{\Omega j}\right]$$

$$\begin{aligned} \bar{P}^{IJKLA}\frac{\partial}{\partial\phi^A}\mathcal{N}_{\Lambda\Sigma} &= 4!\,i\Im\,\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta}\bar{f}^{\Omega[IJ|}\bar{f}^{\Delta|KL]} \\ \bar{P}^{iIJA}\frac{\partial}{\partial\phi^A}\mathcal{N}_{\Lambda\Sigma} &= 8i\Im\,\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta}\bar{f}^{\Omega i}\bar{f}^{\Delta IJ} \\ \bar{P}^{IJKLA}\frac{\partial}{\partial\phi^A}\overline{\mathcal{N}}_{\Lambda\Sigma} &= -4i\Im\,\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta}\bar{P}^{IJKLA}\bar{P}^{ij}{}_{A}f^{\Omega}{}_{i}f^{\Delta}{}_{j} \\ \bar{P}^{iIJA}\frac{\partial}{\partial\phi^A}\overline{\mathcal{N}}_{\Lambda\Sigma} &= -4i\Im\,\mathrm{m}\mathcal{N}_{\Omega(\Lambda}\Im\mathrm{m}\mathcal{N}_{\Sigma)\Delta}\bar{P}^{iIJA}\bar{P}^{jk}{}_{A}f^{\Omega}{}_{i}f^{\Delta}{}_{j} \end{aligned}$$

$$\begin{aligned} R_{IJ}^{KL} &= d\Omega_{IJ}^{KL} + \frac{1}{2}\Omega_{MN}^{KL} \wedge \Omega_{IJ}^{MN} \\ &= -\frac{1}{2}\bar{P}^{KLMN} \wedge P_{MNIJ} - \bar{P}^{iKL} \wedge P_{iIJ} \\ &= -i\left\langle \mathfrak{D}\mathcal{V}_{IJ} \mid \mathfrak{D}\overline{\mathcal{V}}^{KL} \right\rangle \\ R_i^j &= d\Omega_i^j + \Omega_k^j \wedge \Omega_i^k = -\frac{1}{2}\bar{P}^{jIJ} \wedge P_{iIJ} - \bar{P}^{ik} \wedge P_{ik} \\ &= -i\left\langle \mathfrak{D}\mathcal{V}_i \mid \mathfrak{D}\overline{\mathcal{V}}^j \right\rangle \end{aligned}$$

$$\frac{1}{2}P_{IJKL} \wedge \bar{P}^{iKL} + P_{jIJ} \wedge \bar{P}^{ij} = -i\langle \mathfrak{D}\mathcal{V}_{IJ} \mid \mathfrak{D}\bar{\mathcal{V}}^i \rangle = 0$$

$$I = \int d^4x\sqrt{|g|}\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 2\Im\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\mu\nu}\}$$

$$\mathcal{E}_a{}^{\mu} \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}, \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta\phi^i}, \mathcal{E}_\Lambda{}^{\mu} \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu},$$

$$\mathcal{B}^{\Lambda\mu} \equiv \nabla_\nu {}^\star F^{\Lambda\nu\mu}.$$

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= G_{\mu\nu} + \mathcal{G}_{ij}\left[\partial_\mu\phi^i\partial_\nu\phi^j - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi^i\partial^\rho\phi^j\right] \\ &\quad +8\Im\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}{}_{\mu}{}^{\rho}F^{\Sigma-}{}_{\nu\rho} \\ \mathcal{E}_i &= \nabla_\mu(\mathcal{G}_{ij}\partial^\mu\phi^i) - \frac{1}{2}\partial_i\mathcal{G}_{jk}\partial_\rho\phi^j\partial^\rho\phi^k + \partial_i[\tilde{F}_\Lambda{}^{\mu\nu\star}F^\Lambda{}_{\mu\nu}], \\ \mathcal{E}_\Lambda{}^{\mu} &= \nabla_\nu {}^\star\tilde{F}_\Lambda{}^{\nu\mu} \end{aligned}$$

$$\tilde{F}_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^\star F^\Lambda{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^\star F^\Sigma{}_{\mu\nu}$$

$$\varepsilon_\mu^M \equiv \begin{pmatrix} \mathcal{B}^\Lambda{}_\mu \\ \varepsilon_{\Lambda\mu} \end{pmatrix}$$

$$\varepsilon_\mu^M = 0$$

$$\mathcal{E}_\mu^M = 0 \Rightarrow \mathfrak{A}^M{}_N\mathcal{E}_\mu^N = 0, \mathfrak{A} \in \mathrm{GL}(2n_v + 2, \mathbb{R}).$$

$$\mathfrak{A} = \begin{pmatrix} D & C \\ B & A \end{pmatrix},$$

$$F^M \equiv \begin{pmatrix} F^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix}, F'^M = \mathfrak{A}^M{}_N F^N.$$

$$\tilde{F}'_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S'}{\delta^\star F'^\Lambda{}_{\mu\nu}}$$

$$i\colon \mathrm{Diff}(\mathcal{M}_{\text{scalar}}) \to GL(2n_v + 2, \mathbb{R}),$$

$$\xi \in \mathrm{Diff}(\mathcal{M}_{\text{scalar}})$$

$$\mathfrak{i}(\xi) \in \mathrm{GL}(2n_v + 2, \mathbb{R})$$

$$\{\phi, F^M, \mathcal{N}_{\Lambda\Sigma}\} \overset{\xi}{\rightarrow} \{\phi', (\mathfrak{i}(\xi))^M{}_N F^N, \mathcal{N}'_{\Lambda\Sigma}(\phi')\}$$

$$\mathcal{N}'(\phi') = (A\mathcal{N}(\phi) + B)(C\mathcal{N}(\phi) + D)^{-1}$$

$$A^T C = C^T A, D^T B = B^T D, A^T D - C^T B = 1$$

$$i: \mathrm{Diff}(\mathcal{M}_{\text{scalar}}) \rightarrow \mathrm{Sp}(2n_v + 2, \mathbb{R})$$

$$\Im\mathrm{m}\mathcal{N}' = (C\mathcal{N}^* + D)^{-1T} \Im\,\mathrm{m}\mathcal{N}(C\mathcal{N} + D)^{-1}, F'^{\Lambda+} = (C\mathcal{N}^* + D)_{\Lambda\Sigma} F^{\Sigma+}$$

$$\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda+}{}_{\mu}{}^{\rho} F^{\Lambda-}{}_{\nu\rho}$$

$$I_{\text{einstein}} = \int d^4x\sqrt{|g|}\{+2\Im\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}{}_{\mu\nu} F^{\Sigma\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\mu\nu}\}$$

$$I_{\text{Scalars}} = \int d^4x\sqrt{|g|}\{\mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\}$$

$$\mathfrak{i}: \mathrm{Isometries}(\mathcal{M}_{\text{scalar}}, \mathcal{G}_{ij}) \rightarrow \mathrm{Sp}(2n_v + 2, \mathbb{R})$$

$$S = \int d^4x\sqrt{|g|}\big(R + \mathcal{G}_{i\bar{j}}(z,\bar{z})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} + h_{uv}(q)\partial_\mu q^u \partial^\mu q^v$$
$$+2\Im\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z}) F^{\Lambda}{}_{\mu\nu} F^{\Sigma\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z}) F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\mu\nu}\big)$$

$$\mathfrak{D}_\mu\partial^\mu q^u = \nabla_\mu\partial^\mu q^u + \Gamma_{vw}{}^u \partial^\mu q^v \partial_\mu q^w = 0$$

$$\mathcal{SV} = \mathcal{SM} \otimes \mathcal{L}$$

$$\mathrm{Sp}(2n_v + 2, \mathbb{R}) \otimes U(1)$$

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma, h_{\Lambda i} = \mathcal{N}^*{}_{\Lambda\Sigma} f^\Sigma{}_i$$

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + i\frac{\chi\zeta(3)}{2(2\pi)^3} + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}}\right)$$

$$Li_3(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^3}$$

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + i\frac{\chi\zeta(3)}{2(2\pi)^3} + \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right)$$

$$\mathcal{F}_0 = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k$$

$$\mathcal{F}_0 = -t^3$$

$$\mathcal{G}_{t\bar{t}} = \frac{-3}{(t-\bar{t})^2}, \qquad \mathcal{V}^T = (1, t, t^3, -3t^2)$$

$$\mathcal{M} = \frac{SU(1,1)}{U(1)}$$

$$\mathrm{Re}\mathcal{N}_{IJ} = \begin{pmatrix} -2\mathfrak{R}^3 & 3\mathfrak{R}^2 \\ 3\mathfrak{R}^2 & -6\mathfrak{R} \end{pmatrix},$$

$$\mathrm{Im}\mathcal{N}_{IJ} = \begin{pmatrix} -(\mathfrak{I}^3 + 3\mathfrak{R}^2\mathfrak{I}) & 3\mathfrak{R}\mathfrak{I} \\ 3\mathfrak{R}\mathfrak{I} & -3\mathfrak{I} \end{pmatrix},$$

$$\mathfrak{R} = \mathrm{Re}(t) \text{ and } \mathfrak{I} = \mathrm{Im}(t)$$

$$S = \int d^4x\sqrt{|g|}\left(R - \frac{3\partial_\mu t \partial^\mu \bar{t}}{(t-\bar{t})^2} + 2\mathfrak{I}\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^{\Lambda}{}_{\mu\nu}F^{\Sigma\mu\nu} - 2\mathfrak{R}\mathrm{e}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\mu\nu}\right)$$

$$I_{\text{Scalars}} = \int d^4x\sqrt{|g|}\{\mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\}$$

$$\mathcal{M}_{\text{scalar}} = \frac{G}{H}$$

$$H = H_{\mathrm{A}} \times H_{\mathrm{M}},$$

$$G \times_H \mathbb{R}^{2n} \rightarrow \frac{G}{H},$$

$$\{e^a{}_\mu, \psi_{I\mu}, A^{IJ}{}_\mu, \chi_{IJK}, \chi^{IJKLM}, P_{IJKL\mu}\}, I, J, \cdots = 1, \cdots, N,$$

$$\{A_{i\mu}, \lambda_{iI}, \lambda_i^{IJK}, P_{iIJ\mu}\}$$

$\psi_{I\mu}, \chi_{IJK}, \chi^{IJKLM}, \lambda_{iI}, \lambda_i{}^{IJK}$ $(\bar{n} \equiv n + \frac{\mathcal{N}(\mathcal{N}-1)}{2})$ symplectic sections $(\Lambda = 1, \dots \bar{n})\mathcal{V}_{IJ}$

$$\mathcal{N} = 4{:}{:}\ P^{*iIJ} = \frac{1}{2}\varepsilon^{IJKL}P_{iKL}$$

$$\mathcal{N} = 6{:}{:}\ P^{*IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4}P_{K_1\cdots K_4}$$

$$\mathcal{N} = 8{:}{:}\ P^{*I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4}P_{J_1\cdots J_4}$$

$$A^{\Lambda}{}_\mu \equiv \frac{1}{2}f^{\Lambda}{}_{IJ}A^{IJ}{}_\mu + f^{\Lambda}{}_i A^i{}_\mu.$$

$$S = \int d^4x\sqrt{|g|}\Big[R + 2\mathfrak{I}\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\mathfrak{R}\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu} \star F^{\Sigma}{}_{\mu\nu}$$
$$+ \frac{2}{4!}\alpha_1 P^{*IJKL}{}_\mu P_{IJKL}{}^\mu + \alpha_2 P^{*iIJ}{}_\mu P_{iIJ}{}^\mu\Big]$$

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T$$

$\mathcal{N}$	G/H	R
$\mathcal{N}=3$	$\frac{SU(3,n_v)}{SU(3)\times SU(n_v)}$	$(\mathbf{3}+\mathbf{n_v})_\mathbf{c}$
$\mathcal{N}=4$	$\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(6,n_v)}{SO(6)\times SO(n_v)}$	$(\mathbf{2},\mathbf{6}+\mathbf{n_v})$
$\mathcal{N}=5$	$\frac{SU(1,5)}{U(5)}$	**20**
$\mathcal{N}=6$	$\frac{SO^*(12)}{U(6)}$	**32**
$\mathcal{N}=8$	$\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$	**56**

$$I=\int d^4x\sqrt{|g|}\{R+\mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j+2\Im\,\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\mu\nu}-2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\mu\nu}\}$$

$$ds^2=e^{2U}dt^2-e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}$$

$$\begin{aligned}\nabla_{\underline{m}}(\mathcal{G}_{AB}\partial^{\underline{m}}\tilde{\phi}^B)-\frac{1}{2}\partial_A\mathcal{G}_{BC}\partial_{\underline{m}}\tilde{\phi}^B\partial^{\underline{m}}\tilde{\phi}^C&=0\\ R_{\underline{mn}}+\mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial_{\underline{n}}\tilde{\phi}^B&=0\\ \partial_{[\underline{m}}\psi^\Lambda\partial_{\underline{n}}\chi_\Lambda&=0\end{aligned}$$

$$\mathcal{G}_{AB}\equiv\begin{pmatrix}2&&\\&\mathcal{G}_{ij}&\\&&4e^{-2U}\mathcal{M}_{MN}\end{pmatrix}$$

$$(\mathcal{M}_{MN})\equiv\begin{pmatrix}(\Im+\Re\Im^{-1}\Re)_{\Lambda\Sigma}&-(\Re\Im^{-1})_{\Lambda}{}^{\Sigma}\\-(\Im^{-1}\Re)^{\Lambda}{}_{\Sigma}&(\Im^{-1})^{\Lambda\Sigma}\end{pmatrix},\Re_{\Lambda\Sigma}\equiv\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma},\Im_{\Lambda\Sigma}\equiv\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma},$$

$$\tilde{\phi}^A=(U,\phi^i,\psi^\Lambda,\chi_\Lambda)$$

$$I=\int d^3x\sqrt{|\gamma|}\{R+\mathcal{G}_{AB}\partial_{\underline{m}}\tilde{\phi}^A\partial^{\underline{m}}\tilde{\phi}^B\}$$

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{d\tau^2}{W_\kappa^4} + \frac{d\Omega_\kappa^2}{W_\kappa^2},$$

$$\begin{aligned} d\Omega_{(1)}^2 &\equiv d\theta^2 + \sin^2\,\theta d\phi^2 \\ d\Omega_{(-1)}^2 &\equiv d\theta^2 + \sinh^2\,\theta d\phi^2 \\ d\Omega_{(0)}^2 &\equiv d\theta^2 + d\phi^2 \end{aligned}$$

$$\begin{aligned} W_1 &= \frac{\sinh\,r_0\tau}{r_0} \\ W_{-1} &= \frac{\cosh\,r_0\tau}{r_0} \\ W_0^\pm &= ae^{\mp r_0\tau} \end{aligned}$$

$$\begin{aligned} \frac{d}{d\tau}\left(\mathcal{G}_{AB}\frac{d\tilde{\phi}^B}{d\tau}\right) - \frac{1}{2}\partial_A\mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} &= 0 \\ \mathcal{G}_{BC}\frac{d\tilde{\phi}^B}{d\tau}\frac{d\tilde{\phi}^C}{d\tau} &= 2r_0^2 \end{aligned}$$

$$\begin{aligned} U'' + e^{2U}V_{\text{bh}} &= 0 \\ (U')^2 + \frac{1}{2}\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime} + e^{2U}V_{\text{bh}} &= r_0^2 \\ (\mathcal{G}_{ij}\phi^{j\prime})' - \frac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\prime}\phi^{k\prime} + e^{2U}\partial_iV_{\text{bh}} &= 0 \end{aligned}$$

$$-V_{\text{bh}}(\phi,\mathcal{Q}) \equiv -\frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN}, (\mathcal{Q}^M) \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$$

$$I_{\text{eff}}[U,\phi^i] = \int d\tau\left\{(U')^2 + \frac{1}{2}\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime} - e^{2U}V_{\text{bh}}\right\}$$

$$\begin{aligned} ds^2 &= e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} \\ \gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} &= \left(\frac{r_0}{\sinh\,r_0\tau}\right)^2\left[\left(\frac{r_0}{\sinh\,r_0\tau}\right)^2 d\tau^2 + d\Omega_{(2)}^2\right] \end{aligned}$$

$$A(\tau_0) = 4\pi f^2(\tau_0)e^{-2U(\tau_0)}$$

$$f(\tau) \equiv \frac{r_0}{\sinh\,r_0\tau}$$

$$A_\pm = \lim_{\tau_0\to\mp\infty} A(\tau_0)$$

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Phi\right) = 0$$

$$e^{-2U}\partial_t^2\Phi - e^{2U}f^{-4}\partial_\tau^2\Phi - e^{2U}f^{-2}\Delta_{S^2}\Phi = 0$$

$$\Delta_{S^2}\Phi = \frac{1}{\sin\,\theta}\partial_\theta(\sin\,\theta\partial_\theta\Phi) + \frac{1}{\sin^2\,\theta}\partial_\phi^2\Phi$$

$$\Phi = e^{-i\omega t}R(\tau)Y_m^l(\theta,\phi)$$

$$\Delta_{S^2}Y_m^l(\theta,\phi) = -l(l+1)Y_m^l(\theta,\phi)$$

$$\omega^2 e^{-4U}f^2R(\tau) + f^{-2}\partial_\tau^2 R(\tau) = l(l+1)R(\tau)$$

$$\mathcal{K}_4\Phi = l(l+1)\Phi$$

$$\mathcal{K}_4 \equiv -e^{-4U}f^2\partial_t^2 + f^{-2}\partial_\tau^2$$

$$L_m = a_{mt}\partial_t + a_{m\tau}\partial_\tau, m = 0, \pm 1$$

$$a_{mt}(t,\tau), a_{m\tau}(t,\tau)$$

$$[L_m, L_n] = (m-n)L_{m+n}, m = 0, \pm 1$$

$$\mathcal{H}^2 \equiv L_0^2 - \frac{1}{2}(L_1L_{-1} + L_{-1}L_1) = \mathcal{K}_4$$

$$\begin{aligned} L_1 &= l(t)[-m(\tau)\partial_t + n(\tau)\partial_\tau] \\ L_0 &= -\frac{c}{r_0}\partial_t \\ L_{-1} &= -l^{-1}(t)[m(\tau)\partial_t + n(\tau)\partial_\tau] \end{aligned}$$

$$\begin{aligned} m^2\partial_t \log\, l + n\partial_\tau m &= \frac{c}{r_0} \\ \frac{c}{r_0}\partial_t \log\, l &= 1 \end{aligned}$$

$$\begin{aligned} m &= h\partial_\tau n \\ m^2 &= e^{-4U}f^2 + (c/r_0)^2 \\ n^2 &= f^{-2} \end{aligned}$$

$$l(t) = c_0 e^{r_0 t/c}, n^2(\tau) = f^{-2}, m(\tau) = h\cosh\,(r_0\tau)$$

$$c^2 = (e^{-2U}f^2)^2$$

$$(e^{-2U}f^2)^2 \overset{\tau\to\mp\infty}{\sim} \left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau}) = c^2 + \mathcal{O}(e^{\pm r_0\tau})$$

$$\begin{aligned} L_1^\pm &= -\frac{e^{r_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh\,(r_0\tau)\partial_t + \sinh\,(r_0\tau)\partial_\tau\right) \\ L_0^\pm &= -\frac{S_\pm}{r_0\pi}\partial_t \\ L_{-1}^\pm &= -\frac{e^{-r_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh\,(r_0\tau)\partial_t - \sinh\,(r_0\tau)\partial_\tau\right) \end{aligned}$$

$$S_\pm = \frac{A_\pm}{4}$$

$$\mathcal{H}^{\pm 2} \equiv \left(L_0^\pm\right)^2 - \frac{1}{2}\left(L_1^\pm L_{-1}^\pm + L_{-1}^\pm L_1^\pm\right)$$

$$\mathcal{K}_4\Phi = \{-e^{-4U}f^2\partial_t^2 + f^{-2}\partial_\tau^2\}\Phi \overset{\tau\to\mp\infty}{\rightarrow} f^{-2}\{-(S_\pm/\pi)^2\partial_t^2 + \partial_\tau^2\}\Phi = \mathcal{H}^{\pm 2}\Phi$$

$$L_m^\pm = -\frac{e^{mr_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh(mr_0\tau)\partial_t + \sinh(mr_0\tau)\partial_\tau\right)$$

$$\begin{aligned} S_\pm &= S_\mathrm{R} \pm S_\mathrm{L} \\ \frac{1}{T_\pm} &= \frac{1}{2}\left(\frac{1}{T_\mathrm{R}} \pm \frac{1}{T_\mathrm{L}}\right) \end{aligned}$$

$$S_+ = \frac{\pi^2}{12}(c_\mathrm{R}T_R + c_\mathrm{L}T_\mathrm{L})$$

$$2S_\pm T_\pm = r_0$$

$$4S_{\mathrm{L,R}}T_{\mathrm{L,R}} = r_0$$

$$S_\mathrm{L} \to 0, T_\mathrm{R} \to 0, T_\pm \to 0, S_\pm \to S_\mathrm{R}$$

$$c = \frac{12}{\pi^2}\frac{S_\mathrm{R}}{T_\mathrm{L}}$$

$$(\mathcal{V}^M{}_{IJ}) = \begin{pmatrix} f^{ij}{}_{IJ} \\ h_{ijIJ} \end{pmatrix},$$

$$\left\langle \mathcal{V}_{IJ} \mid \overline{\mathcal{V}}^{KL}\right\rangle = \frac{1}{2}\bar{f}^{ijKL}h_{ijIJ} - \frac{1}{2}\bar{h}_{ij}{}^{KL}f^{ij}{}_{IJ} = -2i\delta^{KL}{}_{IJ}, \left\langle \mathcal{V}_{IJ} \mid \mathcal{V}_{KL}\right\rangle = 0,$$

$$\langle \mathcal{A} \mid \mathcal{B}\rangle \equiv \mathcal{A}_M\mathcal{B}^M \equiv \mathcal{A}^N\mathcal{B}^M\Omega_{MN},$$

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{1}_{28\times 28} \\ -\mathbb{1}_{28\times 28} & 0 \end{pmatrix}$$

$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}_{IJ}^M\frac{M^{IJ}}{|M|^2}, |M|^2 = M_{IJ}M^{IJ}$$

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U}d\vec{x}^2$$

$$\begin{aligned} e^{-2U} &= |M|^{-2} = \langle \mathcal{R} \mid \mathcal{I}\rangle = \frac{1}{2}\mathcal{I}^{ij}\mathcal{R}_{ij} - \frac{1}{2}\mathcal{I}_{ij}\mathcal{R}^{ij} \\ (d\omega)_{mn} &= 2\epsilon_{mnp}\langle \mathcal{I} \mid \partial_p\mathcal{I}\rangle \end{aligned}$$

$$\mathcal{F} = -\frac{1}{2}d(\mathcal{R}\hat{V}) - \frac{1}{2}\star(\hat{V} \wedge d\mathcal{I}), \hat{V} = \sqrt{2}e^{2U}(dt + \omega)$$

$$P_{IJKL}\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\mathcal{J}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}, \text{ and } P_{IJKL}\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}$$

$$\mathcal{J}^I{}_J \equiv \frac{2M^{IK}M_{JK}}{|M|^2}, \mathcal{J}^I{}_J = \delta^I{}_J - \tilde{\mathcal{J}}^I{}_J$$

$$\mathcal{Q} \equiv \begin{pmatrix} p^{ij} \\ q_{ij} \end{pmatrix}$$

$$J_4(Q) = p^{ij}q_{jk}p^{kl}q_{li} - \frac{1}{4}\left(p^{ij}q_{ij}\right)^2 + \frac{1}{96}\varepsilon_{ijklmnpq}p^{ij}p^{kl}p^{mn}p^{pq} + \frac{1}{96}\varepsilon^{ijklmnpq}q_{ij}q_{kl}q_{mn}q_{pq}.$$

$$S = \pi\sqrt{|J_4(Q)|}$$

$$\mathcal{R}^M(\mathcal{I}) \sim (\mathcal{I},\mathcal{I},\mathcal{I})^M$$

$$(\mathcal{I},\mathcal{I},\mathcal{I})^{ij} = \frac{1}{2}\mathcal{I}^{ik}\mathcal{I}_{kl}\mathcal{I}^{lj} + \frac{1}{8}\mathcal{I}^{ij}\mathcal{I}_{kl}\mathcal{I}^{kl} - \frac{1}{96}\varepsilon^{ijklmnpq}\mathcal{I}_{kl}\mathcal{I}_{mn}\mathcal{I}_{pq}$$

$$(\mathcal{I},\mathcal{I},\mathcal{I})_{ij} = -\frac{1}{2}\mathcal{I}_{ik}\mathcal{I}^{kl}\mathcal{I}_{lj} - \frac{1}{8}\mathcal{I}_{ij}\mathcal{I}_{kl}\mathcal{I}^{kl} + \frac{1}{96}\varepsilon_{ijklmnpq}\mathcal{I}^{kl}\mathcal{I}^{mn}\mathcal{I}^{pq}$$

$$\mathcal{R}^M(\mathcal{I}) = \beta\frac{(\mathcal{I},\mathcal{I},\mathcal{I})^M}{\sqrt{J_4(\mathcal{I})}}$$

$$\langle\tilde{Q} \mid Q\rangle = 2J_4(Q),$$

$$\langle(Q,Q,Q) \mid Q\rangle = J_4(Q)$$

$$\tilde{\tilde{Q}} = -Q,$$

$$e^{-2U} = \langle\mathcal{R} \mid \mathcal{I}\rangle$$

$$J_4(\tilde{Q}) = J_4(Q)$$

$$e^{-2U} = \beta\sqrt{J_4(H)}$$

$$(d\omega)_{mn} = \varepsilon_{mnp}\left(\mathcal{I}_{ij}\partial_p\mathcal{I}^{ij} - \mathcal{I}^{ij}\partial_p\mathcal{I}_{ij}\right)$$

$$2\mathcal{R}_M(\mathcal{I}) = \frac{\partial e^{-2U}}{\partial\mathcal{I}^M}$$

$$\begin{aligned} J_4'(Q_1,Q_2,Q_3,Q_4) \equiv & \frac{1}{6}\mathrm{Tr}_{SL(8,\mathbb{R})}\{p_1\cdot q_2\cdot p_3\cdot q_4 + p_1\cdot q_3\cdot p_4\cdot q_2 + p_1\cdot q_4\cdot p_2\cdot q_3 + (p\leftrightarrow q)\} \\ & -\frac{1}{12}\{[Q_1 \mid Q_2][Q_3 \mid Q_4] + [Q_1 \mid Q_3][Q_2 \mid Q_4] + [Q_1 \mid Q_4][Q_2 \mid Q_3]\} \\ & +\frac{1}{4}\left[\mathrm{Pf}_{SL(8,\mathbb{R})}\|p_1p_2p_3p_4 \mid +(p\leftrightarrow q)\right] \end{aligned}$$

$$[Q_1 \mid Q_2] \equiv -\frac{1}{2}\mathrm{Tr}_{SL(8,\mathbb{R})}[p_1\cdot q_2 + (p\leftrightarrow q)]$$

$$\mathrm{Pf}\|p_1p_2p_3p_4\| \equiv \frac{1}{4!}\varepsilon_{ijklmnop}p_1^{ij}p_2^{kl}p_3^{mn}p_4^{op}$$

$$\mathrm{Pf}\|q_1q_2q_3q_4\| \equiv \frac{1}{4!}\varepsilon^{ijklmnop}q_{1ij}q_{2kl}q_{3mn}q_{4op}.$$

$$\mathbb{K}_{MNPQ\ 1}{}^{M}Q_2{}^{N}Q_3{}^{P}Q_4{}^{Q} \equiv J_4'(Q_1,Q_2,Q_3,Q_4)$$

$$\mathbb{K}_{MNPQ} = \mathbb{K}_{(MNPQ)}$$

$$J_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q},\mathcal{Q}) = J_4(\mathcal{Q}) = \mathbb{K}_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

$$(\mathcal{Q},\mathcal{Q},\mathcal{Q})^M = \mathbb{K}^M{}_{NPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

$$\mathcal{R}_M = \beta\frac{\mathbb{K}_{MNPQ}H^NH^PH^Q}{\sqrt{J_4(H)}}$$

$$e^{-2U} = \beta\sqrt{\mathbb{K}_{MNPQ}H^MH^NH^PH^Q}$$

$$H^M = A^M + \frac{\mathcal{Q}^M/\sqrt{2}}{r}, r \equiv |\vec{x}|$$

$$|M_\infty|^{-2} = \langle\mathcal{R}_\infty \mid \mathcal{I}_\infty\rangle = e^{-2U_\infty} = 1,$$

$$\mathbb{K}_{MNPQ}A^MA^NA^PA^Q = \beta^{-2}$$

$$0 = \langle A \mid \mathcal{Q}\rangle = \Im\mathrm{m}\left(\mathcal{Z}_{\infty IJ}M_\infty^{IJ}\right)$$

$$\mathcal{Z}_{IJ} \equiv \langle\mathcal{V}_{IJ} \mid \mathcal{Q}\rangle$$

$$\mathcal{Z} \equiv \frac{1}{\sqrt{2}}\mathcal{Z}_{IJ}\frac{M^{IJ}}{|M|^2}$$

$$N = \Im\mathrm{m}\mathcal{Z}_\infty = 0$$

$$M = |\mathcal{Z}_\infty| = \Re\mathrm{e}\mathcal{Z}_\infty = \frac{1}{\sqrt{2}}\langle\mathcal{R}_\infty \mid \mathcal{Q}\rangle = \frac{1}{\sqrt{2}}\beta^2\mathbb{K}_{MNPQ}A^MA^NA^P\mathcal{Q}^Q$$

$$e^{-2U} = \sqrt{1 + \frac{4M}{r} + \frac{3\beta^2\mathbb{K}_{MNPQ}A^MA^N\mathcal{Q}^P\mathcal{Q}^Q}{r^2} + \frac{\sqrt{2}\beta^2\mathbb{K}_{MNPQ}A^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q}{r^3} + \frac{\beta^2J_4(\mathcal{Q})/4}{r^4}}.$$

$$H^M = A^M + \sum_a \frac{\mathcal{Q}_a^M/\sqrt{2}}{|\vec{x} - \vec{x_a}|}$$

$$\langle A \mid \mathcal{Q}_a\rangle + \sum_b \frac{\langle\mathcal{Q}_b \mid \mathcal{Q}_a\rangle/\sqrt{2}}{|\vec{x}_a - \vec{x}_b|} = 0$$

$$M_a \equiv 2\sqrt{2}\mathbb{K}_{MNPQ}A^MA^NA^P\mathcal{Q}_a^Q$$

$$|\vec{x} - \vec{x}_a|^{-n}|\vec{x} - \vec{x}_b|^{-m}$$

$$I_{+2} = \mathbb{K}_{MNPQ}\mathcal{Q}_a^M\mathcal{Q}_a{}^N\mathcal{Q}_a{}^P\mathcal{Q}_a{}^Q = J_4'(\mathcal{Q}_a,\mathcal{Q}_a,\mathcal{Q}_a,\mathcal{Q}_a) = J_4(\mathcal{Q}_a)$$
$$I_{+1} = \mathbb{K}_{MNPQ}\mathcal{Q}_a{}^M\mathcal{Q}_a{}^N\mathcal{Q}_a{}^P\mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a,\mathcal{Q}_a,\mathcal{Q}_a,\mathcal{Q}_b)$$
$$I_0 = \mathbb{K}_{MNPQ}\mathcal{Q}_a{}^M\mathcal{Q}_a{}^N\mathcal{Q}_b{}^P\mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a,\mathcal{Q}_a,\mathcal{Q}_b,\mathcal{Q}_b)$$
$$I_{-1} = \mathbb{K}_{MNPQ}\mathcal{Q}_a{}^M\mathcal{Q}_b{}^N\mathcal{Q}_b{}^P\mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_a,\mathcal{Q}_b,\mathcal{Q}_b,\mathcal{Q}_b)$$
$$I_{-2} = \mathbb{K}_{MNPQ}\mathcal{Q}_b{}^M\mathcal{Q}_b{}^N\mathcal{Q}_b{}^P\mathcal{Q}_b{}^Q = J_4'(\mathcal{Q}_b,\mathcal{Q}_b,\mathcal{Q}_b,\mathcal{Q}_b), = J_4(\mathcal{Q}_b)$$

$$\exists!\, \Omega_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times\ {}_a\mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \Omega_{MN} = -\langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

$$\exists!\, \mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s$$

$$\mathbf{q}(\mathcal{Q}) \equiv \varsigma \mathbb{K}_{MNPO} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^O$$

$$\langle t(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3), \mathcal{Q}_4 \rangle = \mathrm{q}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)$$

$$\langle t(\mathcal{Q}_1, \mathcal{Q}_1, \mathcal{Q}_2), t(\mathcal{Q}_2, \mathcal{Q}_2, \mathcal{Q}_2) \rangle = -2 \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \mathbf{q}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_2, \mathcal{Q}_2)$$

$$\mathcal{R}^M(\mathcal{I}) = \beta(\mathcal{I}, \mathcal{I}, \mathcal{I})^M$$

$$\mathbb{K}_{MNPQ} = \alpha \mathbb{S}_{M(N} \mathbb{S}_{PQ)}$$

$$\mathbf{Q}_1^i \overline{\mathbf{Q}}_2^{\bar{J}} \eta_{i\bar{J}} = \mathbb{S}_{MN} \mathcal{Q}_1^M \mathcal{Q}_2^N + i \mathbb{C}_{MN} \mathcal{Q}_1^M \mathcal{Q}_2^N$$

$$q(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}, \mathcal{Q}) = \gamma (\mathbb{S}_{MN} \mathcal{Q}^M \mathcal{Q}^N)^2$$

$$e^{-2U} \sim |\mathbb{S}_{MN} H^M H^N|$$

$$G = SL(2, \mathbb{R}) \times SO(6, n_v) \to \mathcal{N} = 4$$

$$G = SL(2, \mathbb{R}) \times SO(2, n_v) \to \mathcal{N} = 2$$

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U} \delta_{mn} dx^m dx^n$$

$$\mathcal{R} + i\mathcal{I} = \mathcal{V}/X$$

$$\begin{aligned} e^{-2U} &= \langle \mathcal{R} \mid \mathcal{I} \rangle \\ (d\omega)_{xy} &= 2\epsilon_{xyz} \langle \mathcal{I} \mid \partial^z \mathcal{I} \rangle \\ \mathcal{I}^M &= a^M - \frac{\mathcal{Q}^M}{\sqrt{2}} \tau \end{aligned}$$

$$F = -\frac{1}{\sqrt{2}} \{ d[e^{2U} \mathcal{R}(dt + \omega)] - {}^\star[e^{-2U} d\mathcal{I} \wedge (dt + \omega)] \}$$

$$z^i = (\mathcal{V}/X)^i / (\mathcal{V}/X)^0$$

$$\mathcal{I}^M = a^M - \frac{\mathcal{Q}^M}{\sqrt{2}} \tau$$

$$(\mathcal{Q}^M) \equiv \left(p^\Lambda, q_\Lambda \right)^T$$

$$(\mathcal{Q}^M) = \left(p^\Lambda, q_\Lambda \right)^{\mathrm{T}}$$

$$(H^M) = \left(H^\Lambda, H_\Lambda \right)^{\mathrm{T}}$$

$$X = \frac{1}{\sqrt{2}} e^{U + i\alpha},$$

$$\mathcal{F}_\Lambda \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{X}^\Lambda} \text{ and } \mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}, \text{ we have: } \mathcal{F}_\Lambda = \mathcal{F}_{\Lambda\Sigma}\mathcal{X}^\Sigma$$

$$(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} = e^{\mathcal{K}/2} \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix},$$

$$\frac{\mathcal{M}_\Lambda}{X} = \mathcal{F}_{\Lambda\Sigma} \frac{\mathcal{L}^\Sigma}{X}.$$

$$\mathcal{R}^M = \Re\mathrm{e}\mathcal{V}^M/X, \mathcal{I}^M = \Im\mathrm{m}\mathcal{V}^M/X$$

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

$$\mathcal{R}_M = \Omega_{MN}\mathcal{R}^N, \mathcal{R}^M = \mathcal{R}_N\Omega^{NM}$$

$$\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})\mathcal{I}^N$$

$$\mathcal{M}(\mathcal{A}) \equiv \begin{pmatrix} \Im\mathrm{m}\mathcal{A}_{\Lambda\Sigma} + \Re\mathrm{e}\mathcal{A}_{\Lambda\Omega}\Im\mathrm{m}\mathcal{A}^{-1|\Omega\Gamma}\Re\mathrm{e}\mathcal{A}_{\Gamma\Sigma} & -\Re\mathrm{e}\mathcal{A}_{\Lambda\Omega}\Im\mathrm{m}\mathcal{A}^{-1|\Omega\Sigma} \\ -\Im\mathrm{m}\mathcal{A}^{-1|\Lambda\Omega}\Re\mathrm{e}\mathcal{A}_{\Omega\Sigma} & \Im\mathrm{m}\mathcal{A}^{-1|\Lambda\Sigma} \end{pmatrix},$$

$$d\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})d\mathcal{I}^N.$$

$$\frac{\partial \mathcal{I}^M}{\partial \mathcal{R}_N} = \frac{\partial \mathcal{I}^N}{\partial \mathcal{R}_M} = -\frac{\partial \mathcal{R}^M}{\partial \mathcal{I}_N} = -\frac{\partial \mathcal{R}^N}{\partial \mathcal{I}_M} = -\mathcal{M}^{MN}(\mathcal{F})$$

$$\mathcal{I}^M(X, Z, X^*, Z^*) = H^M.$$

$$z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} \text{ and } e^{-2U} = \frac{1}{2|X|^2} = \mathcal{R}_M\mathcal{I}^M$$

$$\dot{\alpha} = 2|X|^2\dot{H}^M H_M - \mathcal{Q}_\star, \text{ where } \mathcal{Q}_\star = \frac{1}{2i}\dot{z}^i\partial_i\mathcal{K} + \text{ c.c.}$$

$$\mathcal{Q}_\star = \frac{1}{2i}\dot{z}^i\partial_i\mathcal{K} + \text{ c.c.}$$

$$I_{\text{FGK}}[U, z^i] = \int d\tau \left\{(\dot{U})^2 + \mathcal{G}_{ij^*}\dot{z}^i\dot{z}^{*j^*} - \frac{1}{2}e^{2U}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N + r_0^2\right\}$$

$$\mathrm{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U} = \frac{1}{2|X|^2}$$

$$\partial_M \mathrm{W} \equiv \frac{\partial \mathrm{W}}{\partial H^M} = 2\tilde{H}_M$$
$$\partial^M \mathrm{W} \equiv \frac{\partial \mathrm{W}}{\partial \tilde{H}_M} = 2H^M$$
$$\partial_M\partial_N \mathrm{W} = -2\mathcal{M}_{MN}(\mathcal{F})$$
$$\mathrm{W}\partial_M\partial_N \log \mathrm{W} = 2\mathcal{M}_{MN}(\mathcal{N}) + 4\mathrm{W}^{-1}H_M H_N$$

$$-\mathcal{M}_{MN}(\mathcal{N}) = \mathcal{M}_{MN}(\mathcal{F}) + 4\mathcal{V}_{(M}\mathcal{V}^*_{N)}$$

$$\mathcal{G}_{ij^*} = -i\mathcal{D}_i\mathcal{V}_M\mathcal{D}_{j^*}\mathcal{V}^{*M}$$

$$-I_{\text{eff}}[H] = \int d\tau \left\{\frac{1}{2}\partial_M\partial_N \log \mathrm{W}\left(\dot{H}^M\dot{H}^N + \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) - \Lambda - r_0^2\right\}$$

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathrm{W}}\right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathrm{W}}\right)^2$$

$$\mathcal{H} \equiv -\frac{1}{2}\partial_M\partial_N \log \mathrm{W}\left(\dot{H}^M\dot{H}^N - \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\frac{\dot{H}^M H_M}{\mathrm{W}}\right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathrm{W}}\right)^2 - r_0^2 = 0$$

$$\frac{1}{2}\partial_P\partial_M\partial_N \log \mathrm{W}\left(\dot{H}^M\dot{H}^N - \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \partial_P\partial_M \log \mathrm{W}\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0$$

$$\frac{1}{2}\partial_M \log \mathrm{W}(\ddot{H}^M - r_0^2 H^M) + \left(\frac{\dot{H}^M H_M}{\mathrm{W}}\right)^2 = 0$$

$$\dot{H}^M H_M = 0$$

$$H^M = A^M - \frac{1}{\sqrt{2}}B^M\tau$$

$$\begin{aligned} \partial_P[V_{\text{bh}}(H,\mathcal{Q}) - V_{\text{bh}}(H,B)] &= 0 \\ V_{\text{bh}}(H,\mathcal{Q}) - V_{\text{bh}}(H,B) &= 0 \\ A^M B_M &= 0 \end{aligned}$$

$$\tilde{\mathcal{Z}}(\phi,B) \equiv \langle \mathcal{V} \mid B\rangle,$$

$$-V_{\text{bh}}(\phi,\mathcal{Q}) = |\tilde{\mathcal{Z}}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\tilde{\mathcal{Z}}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*$$

$$\begin{aligned} \mathrm{W}(A) &= 1 \\ z_\infty^i &= \frac{\tilde{H}^i(A) + iA^i}{\tilde{H}^0(A) + iA^0} \end{aligned}$$

$$X = \frac{1}{\sqrt{2}}e^{U+i\alpha}$$

$$H^M = \sqrt{2}e^{-U}\Im\mathrm{m}\left(e^{-i\alpha}\mathcal{V}^M\right)$$

$$A^M = \sqrt{2}\Im\mathrm{m}\left(e^{-i\alpha_\infty}\mathcal{V}_\infty^M\right)$$

$$A_M B^M = \langle H \mid B\rangle = \Im\mathrm{m}\langle \mathcal{V}/X \mid B\rangle = \Im\mathrm{m}(\tilde{\mathcal{Z}}/X) = e^{-U}\Im\mathrm{m}\left(e^{-i\alpha}\tilde{\mathcal{Z}}\right) = 0$$

$$e^{i\alpha} = \pm\tilde{\mathcal{Z}}/|\tilde{\mathcal{Z}}|,$$

$$A^M = \pm\sqrt{2}\Im\mathrm{m}\left(\frac{\tilde{\mathcal{Z}}_\infty^*}{|\tilde{\mathcal{Z}}_\infty|}\mathcal{V}_\infty^M\right)$$

$$S/\pi = \frac{1}{2}\mathrm{W}(B) = -V_{\text{bh}}(B,\mathcal{Q}) = |\tilde{\mathcal{Z}}(B,B)|^2$$

$$\partial_i \left|\tilde{\mathcal{Z}}(\phi_h, B)\right| = 0$$

$$\partial_M V_{\text{bh}}(B, \mathcal{Q}) = 0$$

$$\tilde{\mathcal{Q}}_M = \frac{1}{2}\frac{\partial\, \mathrm{W}(\mathcal{Q})}{\partial \mathcal{Q}^M}$$

$$\tilde{\tilde{\mathcal{Q}}}_{\mathcal{M}} = -\mathcal{Q}_{\mathcal{M}}$$

$$W(\tilde{\mathcal{Q}}) = W(\mathcal{Q})$$

$$B^M = \tilde{\mathcal{Q}}^M$$

$$M = \dot{U}(0) = \frac{1}{\sqrt{2}}\langle \mathcal{R}(A) \mid B\rangle = \pm|\tilde{\mathcal{Z}}(A, B)|.$$

$$M^2 + \left[\mathcal{G}^{ij^*}\mathcal{D}_i\tilde{\mathcal{Z}}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*\right]_\infty + V_{\text{bh}\infty} = 0$$

$$\Sigma_i = \mathcal{D}_i\hat{\mathcal{Z}}\big|_\infty$$

$$\begin{aligned}\mathcal{D}X^{-1} &= i\langle \mathcal{V} \mid \mathcal{V}^*\rangle \mathcal{D}X^{-1} = i\langle \mathcal{D}(\mathcal{V}/X) \mid \mathcal{V}^*\rangle = i\langle d(\mathcal{V}/X) \mid \mathcal{V}^*\rangle \\ &= i\langle d(\mathcal{V}/X) - d(\mathcal{V}/X)^* \mid \mathcal{V}^*\rangle = -2\langle dH \mid \mathcal{V}^*\rangle \\ &= +\sqrt{2}\tilde{Z}^*(\phi, B)d\tau\end{aligned}$$

$$\langle \mathcal{D}\mathcal{V} \mid \mathcal{V}^*\rangle\langle \mathcal{D}\mathcal{V}^* \mid \mathcal{V}^*\rangle = \langle \mathcal{V}^* \mid \mathcal{V}^*\rangle$$

$$\frac{de^{-U}}{d\tau} = |\tilde{\mathcal{Z}}(\phi, B)|.$$

$$\begin{aligned}dz^i &= i\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k\mathcal{V}\rangle dz^k = iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k(\mathcal{V}/X)\rangle dz^k \\ &= iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \partial_k(\mathcal{V}/X)\rangle dz^k = iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X)\rangle \\ &= iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X) - d(\mathcal{V}/X)^*\rangle = -2X\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid dH\rangle \\ &= +\sqrt{2}X\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid B\rangle d\tau = \sqrt{2}X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*(\phi, B)d\tau\end{aligned}$$

$$\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^* = \mathcal{D}_{j^*}\frac{|\tilde{\mathcal{Z}}|^2}{\tilde{\mathcal{Z}}} = \frac{2|\tilde{\mathcal{Z}}|\partial_{j^*}|\tilde{\mathcal{Z}}|}{\tilde{\mathcal{Z}}} = 2e^{-i\alpha}\partial_{j^*}|\tilde{\mathcal{Z}}|$$

$$\frac{dz^i}{d\tau} = 2e^U\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}|$$

$$\begin{aligned}\ddot{U} + e^{2U}V_{\text{bh}}(\phi, B) &= 0 \\ \ddot{z}^i + \Gamma^i_{jk}\dot{z}^j\dot{z}^k + e^{2U}\partial^i V_{\text{bh}}(\phi, B) &= 0\end{aligned}$$

$$V_{\text{bh}}(\phi, B) = V_{\text{bh}}(\phi, \mathcal{Q})$$

$$H^M(\tau) = A^M \cosh\, r_0\tau + \frac{B^M}{r_0}\sinh\, r_0\tau$$

$$\frac{1}{2}\partial_P\partial_M\partial_N\log\ \mathrm{W}[B^M B^N - r_0^2 A^M A^N] - \partial_P(V_{\mathrm{bh}}(\phi,\mathcal{Q})/\mathrm{W})\ = 0$$
$$-\frac{1}{2}\partial_M\partial_N\log\ \mathrm{W}[B^M B^N - r_0^2 A^M A^N] - V_{\mathrm{bh}}(\phi,\mathcal{Q})/\mathrm{W}\ = 0$$
$$A^M B_M\ = 0$$

$$\tilde{\mathcal{Z}}(\phi,B) \equiv \langle \mathcal{V} \mid B\rangle, \tilde{\mathcal{Z}}(\phi,B_\pm) \equiv \langle \mathcal{V} \mid B_\pm\rangle,$$

$$B_\pm^M \equiv \lim_{\tau\to\mp\infty}\frac{r_0 H^M(\tau)}{\sinh\ r_0\tau} = B^M \mp r_0 A^M$$

$$\frac{A_{\mathrm{h}\pm}}{4\pi} = \mathrm{W}(B_\pm).$$

$$\frac{A_{\mathrm{h}\pm}}{4\pi}\ = -V_{\mathrm{bh}}(B_\pm) \pm 2r_0\mathcal{M}_{MN}[\mathcal{F}(B_\pm)]A^M B_\pm^N = \mathrm{W}(B_\pm)$$
$$\partial_P V_{\mathrm{bh}}(B_\pm)\ = \pm 2r_0\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M B_\pm^N = -2r_0^2\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M A^N$$

$$H^M\partial_P\mathcal{M}_{MN}(\mathcal{F}) = 0$$

$$\rho \equiv \frac{\sinh\ r_0\tau}{r_0\cosh\ r_0\tau}\ f(\rho) \equiv \frac{1}{\sqrt{1-r_0^2\rho^2}} = \cosh\ r_0\tau$$

$$H^M = f(\rho)(A^M + B^M\rho) \equiv f(\rho)\hat{H}^M$$

$$\frac{de^{-\hat{U}}}{d\rho}\ = \sqrt{2}|\tilde{\mathcal{Z}}(\phi,B)|$$
$$\frac{dz^i}{d\rho}\ = -2\sqrt{2}e^{\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}(\phi,B)|$$

$$\frac{d^2\hat{U}}{d\rho^2} + e^{2\hat{U}}V_{\mathrm{bh}}(\phi,\sqrt{2}B)\ = 0$$
$$\frac{d^2z^i}{d\rho^2} + \Gamma_{kl}{}^i\frac{dz^k}{d\rho}\frac{dz^l}{d\rho} + e^{2\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\mathrm{bh}}(\phi,\sqrt{2}B)\ = 0$$

$$\left(\frac{d\hat{U}}{d\rho}\right)^2 + \mathcal{G}_{ij^*}\frac{dz^i}{d\rho}\frac{dz^{*j^*}}{d\rho} + e^{2\hat{U}}V_{\mathrm{bh}}(\phi,\sqrt{2}B) = 0$$

$$d/drho = f^2 d/d\tau$$

$$\ddot{U} - \frac{2\sqrt{2}\rho}{f}e^U|\mathcal{Z}(\phi,\sqrt{2}B)| + \frac{r_0^2}{f^2} + \frac{e^{2U}}{f^2}V_{\mathrm{bh}}(\phi,\sqrt{2}B),$$

$$e^{2U}V_{\mathrm{bh}}(\phi,\mathcal{Q}) = \frac{e^{2U}}{f^2}V_{\mathrm{bh}}(\phi,\sqrt{2}B) - \frac{2\sqrt{2}r_0^2\rho}{f}e^U|\mathcal{Z}(\phi,\sqrt{2}B)| + \frac{r_0^2}{f^2}$$

$$\partial_i\left\{e^{2U}V_{\mathrm{bh}}(\phi,\mathcal{Q}) - \frac{e^{2U}}{f^2}V_{\mathrm{bh}}(\phi,\sqrt{2}B) + \frac{4\sqrt{2}r_0^2\rho}{f}e^U|\mathcal{Z}(\phi,\sqrt{2}B)|\right\} = 0$$

$$V_{\text{bh}}(\phi_\infty, \mathcal{Q}) - V_{\text{bh}}(\phi_\infty, \sqrt{2}B) = r_0^2$$

$$z_\infty^i = z_{\text{h}}^i$$

$$B^M \propto A^M$$

$$\partial_K V_{\text{bh}}(\phi_\infty, \mathcal{Q}) = 0$$

$$-V_{\text{bh}}(\phi_\infty, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B, B)|^2$$

$$H^M(\tau) = A^M \left[\cosh r_0\tau - \mathrm{W}^{1/2}(B)\frac{\sinh r_0\tau}{r_0}\right]$$

$$B_\pm^M = -[\mathrm{W}^{1/2}(B) \pm r_0]A^M$$

$$\mathrm{W}(B_\pm) = [\mathrm{W}^{1/2}(B) \pm r_0]^2.$$

$$\mathrm{W}(B) = r_0^2 - V_{\text{bh}}(\phi_\infty, \mathcal{Q})$$

$$M = \mathrm{W}^{1/2}(B)$$

$$H^M(\tau) = A^M \left[\cosh r_0\tau - M\frac{\sinh r_0\tau}{r_0}\right]$$

$$S_\pm = \pi[M \pm r_0]^2$$

$$r_0 = \sqrt{M^2 - |\tilde{\mathcal{Z}}(B, B)|^2}$$

$$\mathcal{F} = -\frac{1}{3!}\kappa_{ijk}^0 z^i z^j z^k + \frac{ic}{2} + \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right),$$

$$^2c = \frac{\chi\zeta(3)}{(2\pi)^3}$$

$$\mathcal{F}_{\text{P}} = -\frac{1}{3!}\kappa_{ijk}^0 z^i z^j z^k + \frac{ic}{2},$$

$$\mathcal{F}_{\text{NP}} = \frac{i}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i z^i}\right)$$

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa_{ijk}^0 \frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic(\mathcal{X}^0)^2}{2} + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}}\right)$$

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0}$$

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa_{ijk}^0 \frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic}{2}(\mathcal{X}^0)^2$$

$$z^i = \frac{\chi^i}{\chi^0}$$

$$\mathcal{E}_P = \frac{1}{2}\partial_P\partial_M\partial_N\log\ \mathrm{W}\left[\dot{H}^M\dot{H}^N - \frac{1}{2}Q^M Q^N\right] + \partial_P\partial_M\log\ \mathrm{W}\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0$$

$$\mathcal{H} \equiv -\frac{1}{2}\partial_M\partial_N\log\ \mathrm{W}\left(\dot{H}^M\dot{H}^N - \frac{1}{2}Q^M Q^N\right) + \left(\frac{\dot{H}^M H_M}{\mathrm{W}}\right)^2 - \left(\frac{Q^M H_M}{\mathrm{W}}\right)^2 - r_0^2 = 0$$

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathrm{W}}\right)^2 + \left(\frac{Q^M H_M}{\mathrm{W}}\right)^2$$

$$\mathrm{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U}$$

$$H^0 = H_0 = H_i = 0, p^0 = p_0 = q_i = 0$$

$$\mathrm{W}(H) = \alpha\left|\kappa^0_{ijk}H^iH^jH^k\right|^{2/3}$$

$$\alpha = \frac{(3!\,c)^{1/3}}{2}$$

$$V_{\mathrm{bh}} = \frac{\mathrm{W}(H)}{4}\partial_{ij}\log\ \mathrm{W}(H)Q^iQ^j$$

$$z^i = i(3!\,c)^{1/3}\frac{H^i}{\left(\kappa^0_{ijk}H^iH^jH^k\right)^{1/3}}$$

$$\kappa^0_{ijk}\Im\,\mathrm{m}z^i\Im\,\mathrm{m}z^j\Im\,\mathrm{m}z^k > \frac{3c}{2}$$

$$e^{-\mathcal{K}} = 6c$$

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, r_0 = 0$$

$$H^M = H^M(a,b)$$

$$\{H^P = 0, Q^P = 0\} \Rightarrow \mathcal{E}_P = 0$$

$$c > 0 \Rightarrow h^{11} > h^{21}.$$

$$S_{susy} = \pi\alpha\left|\kappa^0_{ijk}p^ip^jp^k\right|^{2/3}$$

$$\mathrm{W}(H) = \alpha|H^1H^2H^3|^{2/3},$$

$$z^i = ic^{1/3}\frac{H^i}{(H^1H^2H^3)^{1/3}}$$

$$H^i = a^i \cosh(r_0\tau) + \frac{b^i}{r_0}\sinh(r_0\tau), b^i = s_b^i\sqrt{r_0^2(a^i)^2 + \frac{(p^i)^2}{2}}$$

$$a^i = -s_b^i \frac{\Im\,\mathrm{m}z_\infty^i}{\sqrt{3c}}$$

$$M = \frac{r_0}{3}\sum_i \sqrt{1 + \frac{3c(p^i)^2}{2r_0^2\left(\Im\,\mathrm{m}z_\infty^i\right)^2}}$$

$$S_\pm = r_0^2\pi\prod_i\left(\sqrt{1 + \frac{3c(p^i)^2}{2r_0^2\left(\Im\,\mathrm{m}z_\infty^i\right)^2}} \pm 1\right)^{2/3}$$

$$S_+S_- = \frac{\pi^2\alpha^2}{4}\prod_i (p^i)^{4/3}$$

$$H^0 = H_0 = H_i = 0, p^0 = q_0 = q_i = 0$$

$$\begin{pmatrix} iH^i \\ \mathcal{R}_i \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^i \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^i} \end{pmatrix}, \begin{pmatrix} \mathcal{R}^0 \\ 0 \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X}\begin{pmatrix} \mathcal{X}^0 \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^0} \end{pmatrix},$$

$$e^{-2U} = \mathcal{R}_i(H)H^i, z^i = i\frac{H^i}{\mathcal{R}^0(H)}$$

$$\frac{\partial F(H)}{\partial \mathcal{R}^0} = 0$$

$$F(H) = \frac{i}{3!}\kappa_{ijk}^0\frac{H^iH^jH^k}{\mathcal{R}^0} + \frac{ic(\mathcal{R}^0)^2}{2} + \frac{i(\mathcal{R}^0)^2}{(2\pi)^3}\sum_{\{d_i\}} n_{\{d_i\}} Li_3\left(e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}}\right)$$

$$\mathcal{R}_i = -i\frac{\partial F(H)}{\partial H^i},$$

$$-\frac{1}{3!}\kappa_{ijk}^0\frac{H^iH^jH^k}{(\mathcal{R}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[Li_3\left(e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}}\right) + Li_2\left(e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}}\right)\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right]\right] = 0.$$

$$\lim_{|w|\to 0} Li_s(w) = w, \forall s \in \mathbb{N}$$

$$w = e^{-2\pi d_i \S m z^i}, \forall\{d_i\} \in (\mathbb{Z}^+)^{h^{1,1}}$$

$$-\frac{1}{3!}\kappa_{ijk}^0\frac{H^iH^jH^k}{(\mathcal{R}^0)^3} + c + \frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}\left[e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}} + e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right]\right] = 0$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\mathcal{R}^0)^3}+\frac{1}{4\pi^3}\sum_{\{d_i\}} n_{\{d_i\}}e^{-2\pi d_i\frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi d_i H^i}{\mathcal{R}^0}\right]=0$$

$$-\frac{1}{3!}\kappa^0_{ijk}\frac{H^iH^jH^k}{(\mathcal{R}^0)^3}+\frac{\hat{n}}{4\pi^3}e^{-2\pi\hat{d}_i\frac{H^i}{\mathcal{R}^0}}\left[\frac{\pi\hat{d}_i H^i}{\mathcal{R}^0}\right]=0, \Im\,\mathrm{m}z^i \gg 1$$

$$\mathcal{R}^0=\frac{\pi\hat{d}_l H^l}{W_a\left(s_a\sqrt{\frac{3\hat{n}\left(\hat{d}_n H^n\right)^3}{2\kappa^0_{ijk}H^iH^jH^k}}\right)}$$

$$\mathcal{R}_i=\frac{1}{2}\kappa^0_{ijk}\frac{H^jH^k}{\pi\hat{d}_l H^l}W_a\left(s_a\sqrt{\frac{3\hat{n}\left(\hat{d}_m H^m\right)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)$$

$$e^{-2U}=\mathrm{W}(H)=\frac{\kappa^0_{ijk}H^iH^jH^k}{2\pi\hat{d}_m H^m}W_a\left(s_a\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)$$

$$z^i=i\frac{H^i}{\pi\hat{d}_m H^m}W_a\left(s_a\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr}H^pH^qH^r}}\right)$$

$$\mathrm{sign}[W_a(x)]=\mathrm{sign}[x], a=0,-1, x\in D^a_{\mathbb{R}}$$

$$s_0\equiv\mathrm{sign}\left[\kappa^0_{ijk}\frac{H^iH^jH^k}{\hat{d}_m H^m}\right]$$

$$s_{-1}\equiv -1$$

$$H^i=a^i-\frac{p^i}{\sqrt{2}}\tau, r_0=0$$

$$S=\frac{1}{2}\kappa^0_{ijk}\frac{p^ip^jp^k}{\hat{d}_m p^m}W_a(s_a\beta)$$

$$\beta=\sqrt{\frac{3\hat{n}\left(\hat{d}_l p^l\right)^3}{2\kappa^0_{pqr}p^pp^qp^r}}$$

$$M=\dot{U}(0)=\frac{1}{2\sqrt{2}}\left[\frac{3\kappa^0_{ijk}p^ia^ja^k}{\kappa^0_{pqr}a^pa^qa^r}\left[1-\frac{1}{1+W_a(s_a\alpha)}\right]\right.$$
$$\left.-\frac{d_lp^l}{d_na^n}\left[1-\frac{3}{2\left(1+W_a(s_a\alpha)\right)}\right]\right]$$

$$\alpha=\sqrt{\frac{3\hat{n}(d_la^l)^3}{2\kappa^0_{pqr}a^pa^qa^r}}$$

$$\sim \left|d_i \Im m z^i\right| e^{-2\pi d_i \Im m z^i}$$

$$W_a(x) \gg 1,$$

$$\mathrm{Arg}[W_a] \in \left|-\frac{1}{e}, 0\right\rangle$$

$$\begin{aligned} e^{-2U} &= \frac{\kappa^0_{ijk} H^i H^j H^k}{2\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right) \\ z^i &= i\frac{H^i}{\pi \hat{d}_m H^m} W_0\left(\sqrt{\frac{3\hat{n}\left(\hat{d}_l H^l\right)^3}{2\kappa^0_{pqr} H^p H^q H^r}}\right) \end{aligned}$$

$$\frac{\kappa^0_{ijk} H^i H^j H^k}{2\pi \hat{d}_n H^n} > 0 \forall \tau \in (-\infty, 0]$$

$$\frac{\kappa^0_{ijk} a^i a^j a^k}{2\pi \hat{d}_m a^m} W_0(\alpha) = 1$$

$$e^{-2U} \overset{\tau\to-\infty}{\to} \frac{\kappa^0_{ijk} p^i p^j p^k}{8\pi \hat{d}_m p^m} W_0(\beta)\tau^2.$$

$\mathrm{Arg}[W_0]|_{\tau=0} = \mathrm{Arg}[W_{-1}]|_{\tau=0} = -1/e$ and $\mathrm{Arg}[W_0]|_{\tau\to-\infty} = \mathrm{Arg}[W_{-1}]|_{\tau\to-\infty} = \beta, \beta \in$

$(-1/e, 0)$

$$Li_w(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^w}, z, w \in \mathbb{C}$$

$$Li_{w-1}(z) = z\frac{\partial Li_w(z)}{\partial z}$$

$$Li_1(z) = -\log(1-z)$$

$$Li_0(z) = \frac{z}{1-z}, Li_{-n}(z) = \left(z\frac{\partial}{\partial z}\right)^n \frac{z}{1-z}.$$

$$Li_w(z) = \int_0^z \frac{Li_{w-1}(s)}{s} ds$$

$$z = W(z)e^{W(z)}, \forall z \in \mathbb{C}.$$

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}, \forall z \notin \{0, -1/e\}; \left.\frac{dW(z)}{dz}\right|_{z=0} = 1,$$

$$\lim_{x\to-1/e} \frac{dW_0(x)}{dx} = \infty, \lim_{x\to-1/e} \frac{dW_{-1}(x)}{dx} = -\infty.$$

$$\langle \mathcal{A} \mid \mathcal{B} \rangle \equiv \mathcal{B}^\Lambda \mathcal{A}_\Lambda - \mathcal{B}_\Lambda \mathcal{A}^\Lambda$$

$$e^{-4U}f^2 = f^{-2}(e^{-2U}f^2)^2 \sim f^{-2}\left[\left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau})\right] \sim f^{-2}\left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau})$$

$$e^{2\pi i d_i z^i} \ll \pi\left|d_i \Im\,\mathrm{m} z^i\right| e^{2\pi i d_i z^i} \text{ for } \Im\mathrm{m} z^i \gg 1$$

$$\dot{x}(\tau) \overset{\tau\to 0}{\rightarrow} 0$$

$$\left|W'_{0,-1}(x)\right| \overset{x\to -1/e}{\rightarrow} \infty$$

$$\Omega_H \beta = 2\pi i.$$

$$\Omega = \frac{1}{\beta}\frac{\partial S_{\text{wald}}}{\partial J},$$

$$\Omega = -\Omega_H = -2\pi i/\beta$$

$$I = S_{wald} + 2\pi i J$$

$$\int d^4x\sqrt{-\det g}R$$

$$\varepsilon^{0123} = 1, \varepsilon_{0123} = -1$$

$$T^{ab} = -\frac{i}{2}\varepsilon^{abcd}T_{cd}, \bar{T}^{ab} = \frac{i}{2}\varepsilon^{abcd}\bar{T}_{cd}$$

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$$

$$F^{-I}_{ab} = \frac{1}{2}\left(F^I_{ab} - \frac{i}{2}\varepsilon_{abcd}F^{Icd}\right)$$

$$\hat{A} = T_{ab}T^{ab}$$

$$F\left(\{\lambda X^I\}, \lambda^2\hat{A}\right) = \lambda^2 F\left(\{X^I\}, \hat{A}\right)$$

$$F_I = \frac{\partial F}{\partial X^I}, F_{\hat{A}} = \frac{\partial F}{\partial \hat{A}}$$

$$S = \int d^4x\sqrt{-\det g}\mathcal{L}$$

$$8\pi\mathcal{L} = \Big[-iF_I\bar{X}^I\left(\frac{1}{6}R - D\right) + \frac{1}{2}i\hat{F}^{-ab}F_{\hat{A}I}\left(F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T_{ab}\right) + \frac{1}{2}iF_{\hat{A}}\hat{C} + \frac{i}{4}F_{\hat{A}\hat{A}}\hat{F}^{-ab}\hat{F}^{-}_{ab}$$
$$+\text{c.c.}\,\Big] + \chi\left(\frac{1}{6}R + \frac{1}{2}D\right) + \cdots$$

$$\hat{C} = 64\mathcal{R}(M)^{-}{}_{ef}{}^{gh}\mathcal{R}(M)^{-ef}{}_{gh} + 16f_a{}^{c}T^{ab}\bar{T}_{cb} + \cdots,$$
$$\hat{F}^{-ab} = -16\mathcal{R}(M)_{cd}{}^{ab}T^{cd},$$
$$f_\mu{}^{a} = \frac{1}{2}R_\mu{}^{a} - \frac{1}{4}\left(D + \frac{1}{3}R\right)e_\mu{}^{a} + \cdots,$$
$$\mathcal{R}(M)_{ab}{}^{cd} = R_{ab}{}^{cd} - 2R_{[a}^{c}\delta_{b]}^{d]} + \frac{R}{3}\delta_{[a}^{c}\delta_{b]}^{d} + D\delta_{[a}^{c}\delta_{b]}^{d} + \cdots,$$
$$\mathcal{R}(M)^{-ef}{}_{gh} = \frac{1}{2}\left(\mathcal{R}(M)^{ef}{}_{gh} - \frac{i}{2}\varepsilon_{ghab}\mathcal{R}(M)^{efab}\right).$$

$$ds^2 = -e^{2g}(dt + \sigma_\alpha dx^\alpha)^2 + e^{-2g}ds_{\text{base}}^2$$

$$e_\mu^0 dx^\mu = e^g(dt + \sigma_\alpha dx^\alpha), e_\mu^p dx^\mu = e^{-g}\hat{e}_\alpha^p dx^\alpha \quad \text{for } p = 1,2,3$$

$$\mathcal{T}_\alpha \equiv \frac{1}{4}e^{-g}\hat{e}_\alpha^p \bar{h}T_{p0}$$

$$\hat{\nabla}_p \equiv \hat{E}_p^\alpha \nabla_\alpha, T_p \equiv \hat{E}_p^\alpha \mathcal{T}_\alpha$$

$$R(\sigma)^m = \varepsilon^{mnp}\hat{E}_n{}^{\alpha}\hat{E}_p{}^{\beta}\partial_\alpha\sigma_\beta.$$

$$\bar{h}T_{p0} = 4e^g T_p$$

$$\bar{h}T_{pq} = -i\varepsilon_{pq0r}\bar{h}T^{0r} = 4ie^g\varepsilon_{pq}^r T_r$$

$$h\bar{T}_{p0} = 4e^g\bar{T}_p, h\bar{T}_{pq} = -4ie^g\varepsilon_{pq}^r\bar{T}_r$$

$$T_m = \hat{\nabla}_m g - \frac{1}{2}iK_m, K_m \equiv e^{2g}R(\sigma)_m = e^{2g}\varepsilon_m^{np}\hat{E}_n^\alpha\hat{E}_p^\beta\partial_\alpha\sigma_\beta$$

$$\bar{T}_m = \hat{\nabla}_m g + \frac{1}{2}iK_m$$

$$\mathcal{R}(M)_{pq0r} = \frac{1}{2}i\varepsilon_{pq}^s e^{2g}\left[\hat{\nabla}_r T_s + 2T_r T_s - \delta_{rs}T_m T_m\right] + \text{ c.c.}$$
$$\mathcal{R}(M)_{0rpq} = \frac{1}{2}i\varepsilon_{pq}{}^{s}e^{2g}\left[\hat{\nabla}_s T_r + 2T_r T_s - \delta_{rs}T_m T_m\right] + \text{ c.c.}$$
$$\mathcal{R}(M)_{0p0q} = -\frac{1}{2}e^{2g}\left[\hat{\nabla}_q T_p + 2T_q T_p - \delta_{pq}T_m T_m\right] + \text{ c.c.}$$
$$\mathcal{R}(M)_{pqrs} = \frac{1}{2}e^{2g}\varepsilon_{rs}{}^{v}\varepsilon_{pq}{}^{u}\left[\hat{\nabla}_v T_u + 2T_v T_u - \delta_{uv}T_m T_m\right] + \text{ c.c.}$$

$$F_{0p}^{-I} = -e^g\left[\hat{\nabla}_p\left(\bar{h}X^I\right) + \left(\hat{\nabla}_p g\right)h\bar{X}^I - \frac{1}{2}ie^{2g}R(\sigma)_p\left(\bar{h}X^I + h\bar{X}^I\right)\right]$$

$$-ie^{-g}\left(\bar{h}X^I - h\bar{X}^I\right) = H^I, -ie^{-g}\left(\bar{h}F_I - h\bar{F}_I\right) = H_I,$$

$$i(\bar{X}^I F_I - X^I\bar{F}_I) + \frac{1}{2}\chi = -128ie^{3g}\hat{\nabla}^p\left[e^{-g}\hat{\nabla}_p g(F_{\hat{A}} - \bar{F}_{\hat{A}})\right] - 32ie^{6g}\left(R(\sigma)_p\right)^2(F_{\hat{A}} - \bar{F}_{\hat{A}})$$
$$-64e^{4g}R(\sigma)_p\hat{\nabla}^p(F_{\hat{A}} + \bar{F}_{\hat{A}})$$

$$(\bar{h}X^I - h\bar{X}^I)\hat{\nabla}_p(\bar{h}F_I - h\bar{F}_I) - (\bar{h}F_I - h\bar{F}_I)\hat{\nabla}_p(\bar{h}X^I - h\bar{X}^I) - \frac{1}{2}\chi e^{2g}R(\sigma)_p$$
$$=128e^{2g}\hat{\nabla}^q\left[2\hat{\nabla}_{[p}g\hat{\nabla}_{q]}(F_{\hat{A}} + \bar{F}_{\hat{A}}) + i\hat{\nabla}_{[p}\left(e^{2g}R(\sigma)_{q]}(F_{\hat{A}} - \bar{F}_{\hat{A}})\right)\right].$$

$$ds^2_{\text{base}} = \frac{r^2 - b^2\cos^2\theta}{r^2 - b^2}dr^2 + (r^2 - b^2\cos^2\theta)d\theta^2 + (r^2 - b^2)\sin^2\theta d\phi^2$$

$$\hat{e}^1_\alpha dx^\alpha = dr\left(\frac{r^2 - b^2\cos^2\theta}{r^2 - b^2}\right)^{1/2}, \hat{e}^2_\alpha dx^\alpha = (r^2 - b^2\cos^2\theta)^{1/2}d\theta$$
$$\hat{e}^3_\alpha dx^\alpha = (r^2 - b^2)^{1/2}\sin\theta d\phi$$

$$\sigma_\alpha dx^\alpha = \omega_\phi d\phi$$

$$E_0^\mu\frac{\partial}{\partial x^\mu} = e^{-g}\frac{\partial}{\partial t}, E_1{}^\mu\frac{\partial}{\partial x^\mu} = e^g\left(\frac{r^2 - b^2\cos^2\theta}{r^2 - b^2}\right)^{-1/2}\frac{\partial}{\partial r}$$
$$E_2{}^\mu\frac{\partial}{\partial x^\mu} = e^g(r^2 - b^2\cos^2\theta)^{-1/2}\frac{\partial}{\partial\theta}, E_3{}^\mu\frac{\partial}{\partial x^\mu} = \frac{e^g}{(r^2 - b^2)^{1/2}\sin\theta}\left(\frac{\partial}{\partial\phi} - \omega_\phi\frac{\partial}{\partial t}\right).$$

$$\hat{E}_1^\alpha\frac{\partial}{\partial x^\alpha} = \left(\frac{r^2 - b^2\cos^2\theta}{r^2 - b^2}\right)^{-1/2}\frac{\partial}{\partial r}, \hat{E}_2^\alpha\frac{\partial}{\partial x^\alpha} = (r^2 - b^2\cos^2\theta)^{-1/2}\frac{\partial}{\partial\theta}$$
$$\hat{E}_3^\alpha\frac{\partial}{\partial x^\alpha} = \frac{1}{(r^2 - b^2)^{1/2}\sin\theta}\frac{\partial}{\partial\phi}$$

$$\partial_\theta\omega_\phi = 0 \text{ at } r = b$$

$$\frac{\partial}{\partial t} - \omega_\phi^{-1}\frac{\partial}{\partial\phi}$$

$$\Omega_H = -\omega_H^{-1},\ \omega_H \equiv \omega_\phi\big|_{r=b}$$

$$-ie^{-g}(\bar{h}X^I - h\bar{X}^I) = \hat{h}^I + \frac{\gamma_N^I}{r_N} + \frac{\gamma_S^I}{r_S}$$
$$-ie^{-g}(\bar{h}F_I - h\bar{F}_I) = \hat{h}_I + \frac{\gamma_{NI}}{r_N} + \frac{\gamma_{SI}}{r_S}$$

$$r_N = r - b\cos\theta, r_S = r + b\cos\theta$$

$$\gamma_N^I + \gamma_S^I = P^I, \gamma_{NI} + \gamma_{SI} = Q_I$$

$$-ie^{-g}\bar{h}X^I = \frac{\gamma_N^I}{r_N} + \mathcal{O}(1), -ie^{-g}\bar{h}F_I = \frac{\gamma_{NI}}{r_N} + \mathcal{O}(1), e^{-2g}\bar{h}^2\hat{A} = -\frac{64}{r_N^2},\ \text{as } r_N \to 0$$
$$ie^{-g}h\bar{X}^I = \mathcal{O}(1), ie^{-g}h\bar{F}_I = \mathcal{O}(1), e^{-2g}h^2\bar{\hat{A}} = \mathcal{O}\left(\frac{1}{r_N}\right),\ \text{as } r_N \to 0$$
$$ie^{-g}h\bar{X}^I = \frac{\gamma_S^I}{r_S} + \mathcal{O}(1), ie^{-g}h\bar{F}_I = \frac{\gamma_{SI}}{r_S} + \mathcal{O}(1), e^{-2g}h^2\bar{\hat{A}} = -\frac{64}{r_S^2},\ \text{as } r_S \to 0$$
$$-ie^{-g}\bar{h}X^I = \mathcal{O}(1), -ie^{-g}\bar{h}F_I = \mathcal{O}(1), e^{-2g}\bar{h}^2\hat{A} = \mathcal{O}\left(\frac{1}{r_S}\right),\ \text{as } r_S \to 0$$

$$\bar{h}e^{-g} = \frac{1}{r_N} \text{ as } r_N \to 0$$

$$he^{-g} = \frac{1}{r_S} \text{ as } r_S \to 0$$

$$\langle \hat{h}, \gamma_N + \gamma_S \rangle = 0$$

$$\langle a, b \rangle \equiv a^I b_I - a_I b^I$$

$$i(\bar{X}^I F_I - X^I \bar{F}_I) + \frac{1}{2}\chi \simeq 0$$

$$(\bar{h}X^I - h\bar{X}^I)\hat{\nabla}_p(\bar{h}F_I - h\bar{F}_I) - (\bar{h}F_I - h\bar{F}_I)\hat{\nabla}_p(\bar{h}X^I - h\bar{X}^I) - \frac{1}{2}\chi e^{2g} R(\sigma)_p \simeq 0$$

$$S_{\text{extremal}} = -i\pi Q_I \gamma_N^I - 2\pi i F(\{i\gamma_N^I\}, -64) + i\pi Q_I \gamma_S^I + 2\pi i \bar{F}(\{-i\gamma_S^I\}, -64)$$

$$F_I(\{i\gamma_N^I\}, -64) = i\gamma_{NI}, \bar{F}_I(\{-i\gamma_S^I\}, -64) = -i\gamma_{SI}$$

$$\begin{aligned} 2F(\{i\gamma_N^I\}, -64) &= i\gamma_N^I F_I(\{i\gamma_N^I\}, -64) - 128 F_{\hat{A}}(\{i\gamma_N^I\}, -64) \\ &= -\gamma_N^I \gamma_{NI} - 128 F_{\hat{A}}(\{i\gamma_N^I\}, -64) \\ 2\bar{F}(\{-i\gamma_S^I\}, -64) &= -i\gamma_S^I \bar{F}_I(\{-i\gamma_S^I\}, -64) - 128 \bar{F}_{\hat{A}}(\{-i\gamma_S^I\}, -64) \\ &= -\gamma_S^I \gamma_{SI} - 128 \bar{F}_{\hat{A}}(\{-i\gamma_S^I\}, -64) \end{aligned}$$

$$\begin{aligned} S_{\text{extremal}} &= -i\pi(\gamma_{NI} + \gamma_{SI})\gamma_N^I + \pi i \left(\gamma_N^I \gamma_{NI} + 128 F_{\hat{A}}(\{i\gamma_N^I\}, -64)\right) \\ &\quad + i\pi(\gamma_{NI} + \gamma_{SI})\gamma_S^I - \pi i \left(\gamma_S^I \gamma_{SI} + 128 \bar{F}_{\hat{A}}(\{-i\gamma_S^I\}, -64)\right) \\ &= -i\pi\langle \gamma_N, \gamma_S \rangle + 128\pi i [F_{\hat{A}}(\{i\gamma_N^I\}, -64) - \bar{F}_{\hat{A}}(\{-i\gamma_S^I\}, -64)]. \end{aligned}$$

$$Y^I \equiv \bar{h}e^{-g} X^I, \Upsilon = \bar{h}^2 e^{-2g} \hat{A}$$

$$F(Y, \Upsilon) = \bar{h}^2 e^{-2g} F(X, \hat{A})$$

$$\begin{aligned} F_I' &= \frac{\partial F(Y, \Upsilon)}{\partial Y^I} = \bar{h}e^{-g} F_I, F_\Upsilon' = \frac{\partial F(Y, \Upsilon)}{\partial \Upsilon} = F_{\hat{A}} \\ F_{\Upsilon I}' &= \frac{\partial^2 F(Y, \Upsilon)}{\partial \Upsilon \partial Y^I} = \bar{h}^{-1} e^g F_{\hat{A}I}, F_{\Upsilon\Upsilon}' = \frac{\partial^2 F(Y, \Upsilon)}{\partial \Upsilon^2} = \bar{h}^{-2} e^{2g} F_{\hat{A}\hat{A}}. \end{aligned}$$

$$\hat{\nabla}_1 g = \hat{E}_1^r \partial_r g = 0, \hat{\nabla}_3 g = \hat{E}_3^\phi \partial_\phi g = 0$$

$$K_1 = e^{2g} \hat{E}_2^\theta \hat{E}_3^\phi (\partial_\theta \sigma_\phi - \partial_\phi \sigma_\theta) = 0, K_3 = e^{2g} \hat{E}_1^r \hat{E}_2^\theta (\partial_r \sigma_\theta - \partial_\theta \sigma_r) = 0$$

$$\begin{aligned} &T_1 = \bar{T}_1 = 0, T_2 = \hat{\nabla}_2 g - \frac{i}{2} K_2, K_2 = -e^{2g} \hat{E}_1^r \hat{E}_3^\phi \partial_r \omega_\phi = -\frac{e^{2g}}{b\sin^2 \theta} \partial_r \omega_\phi \\ &T_3 = \bar{T}_3 = 0, \bar{T}_2 = \hat{\nabla}_2 g + \frac{i}{2} K_2, \text{ for } r = b \end{aligned}$$

$$-\left\langle \hat{h}+\frac{\gamma_N}{r_N}+\frac{\gamma_S}{r_S}, d\left(\hat{h}+\frac{\gamma_N}{r_N}+\frac{\gamma_S}{r_S}\right)\right\rangle + \star\, d\sigma = 0$$

$$\sigma = \omega_\phi d\phi$$

$$\omega_\phi = \langle \gamma_N, \gamma_S \rangle \frac{b\sin^2\theta}{r^2 - b^2\cos^2\theta} + \langle \hat{h}, \gamma_N \rangle \left(\frac{r\cos\theta - b}{r - b\cos\theta} - \frac{r\cos\theta + b}{r + b\cos\theta}\right)$$

$$\omega_H \equiv \omega_\phi\big|_{r=b} = b^{-1}\langle \gamma_N, \gamma_S \rangle - 2\langle \hat{h}, \gamma_N \rangle$$

$$\partial_r \omega_\phi\big|_{r=b} = -2b^{-2}\csc^2\theta \langle \gamma_N, \gamma_S \rangle + 2b^{-1}(1+\cos^2\theta)\csc^2\theta \langle \hat{h}, \gamma_N \rangle.$$

$$\partial_r \omega_\phi\big|_{r=b} \simeq -2b^{-1}\theta^{-2}\omega_H \text{ for } \theta \simeq 0$$
$$\simeq -2b^{-1}(\pi-\theta)^{-2}\omega_H \text{ for } \theta \simeq \pi.$$

$$e^{-2g} = \frac{C_N}{r_N} = \frac{C_N}{r - b\cos\theta}$$

$$\partial_\theta g = \frac{\sin\theta}{2(1-\cos\theta)},$$

$$\hat{\nabla}_2 g = \hat{E}_2^\theta \partial_\theta g = \frac{1}{2b(1-\cos\theta)} \simeq \frac{1}{b\theta^2}.$$

$$K_2 = -\frac{e^{2g}}{b\sin^2\theta}\partial_r\omega_\phi \simeq b^{-1}C_N^{-1}\theta^{-2}\omega_H \text{ for } r = b, \theta \simeq 0$$

$$T_2 = \hat{\nabla}_2 g - \frac{i}{2}K_2 \simeq b^{-1}\theta^{-2}\left(1 - \frac{i\omega_H}{2C_N}\right), \bar{T}_2 \simeq b^{-1}\theta^{-2}\left(1 + \frac{i\omega_H}{2C_N}\right)$$

$$\Upsilon = \bar{h}^2 e^{-2g} T_{ab} T^{ab} = -64 T_2^2 = -64 b^{-2}\theta^{-4}\left(1 - \frac{i\omega_H}{2C_N}\right)^2,$$
$$\bar{\Upsilon} = -64 b^{-2}\theta^{-4}\left(1 + \frac{i\omega_H}{2C_N}\right)^2.$$

$$\bar{\Upsilon} = h^2 e^{-2g}\overline{\hat{A}}$$

$$r_N = b(1-\cos\theta) = b\theta^2/2$$

$$\left(1 + \frac{i\omega_H}{2C_N}\right)^2 = 0.$$

$$C_N = -i\omega_H/2.$$

$$\Upsilon \simeq -256 b^{-2}\theta^{-4} \simeq -64 r_N^{-2},$$

$$C_S = -i\omega_H/2.$$

$$\left(1 + \frac{i\omega_H}{2C_S}\right)^2 = 0,$$

$$e^{-2g} = -\frac{i\omega_H}{2(r + b\cos\,\theta)},\ \partial_\theta g|_{r=b} = -\frac{\sin\,\theta}{2(1+\cos\,\theta)},\ \text{as } r_S \to 0.$$

$$\beta = \frac{2\pi i}{\Omega_H} = -2\pi i\omega_H > 0,$$

$$D' = D - \frac{1}{6}R$$

$$8\pi\mathcal{L} = iF_I\bar{X}^I D' + \frac{1}{2}i\hat{F}^{-ab}F_{\hat{A}I}\left(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}\right) + \frac{1}{2}iF_{\hat{A}}\hat{C} + \frac{i}{4}F_{\hat{A}\hat{A}}\hat{F}^{-ab}\hat{F}_{ab}^{-} + \text{c.c.}$$
$$+\chi\left(\frac{1}{4}R + \frac{1}{2}D'\right) + \cdots$$
$$f_\mu{}^a = \frac{1}{2}R_\mu{}^a - \frac{1}{4}\left(D' + \frac{1}{2}R\right)e_\mu{}^a + \cdots$$
$$\mathcal{R}(M)_{ab}{}^{cd} = R_{ab}{}^{cd} - 2R_{[a}^{c}\delta_{b]}^{d]} + \frac{R}{2}\delta_{[a}^{c}\delta_{b]}^{d} + D'\delta_{[a}^{c}\delta_{b]}^{d} + \cdots$$

$$\int d^4x\sqrt{-\det g}R$$

$$\varepsilon_{ab} = E_a^t E_b^r \varepsilon_{tr} - (a \leftrightarrow b), \varepsilon_{tr} = \frac{\sqrt{-\det g}}{\sqrt{\det g_H}}$$

$$\varepsilon_{13} = \varepsilon^{13} = i, \varepsilon_{31} = \varepsilon^{31} = -i$$

$$S_{wald} = \frac{\pi}{2}\varepsilon_{ab}\varepsilon_{cd}\int_H \frac{\partial\mathcal{L}}{\partial R_{abcd}}d\theta d\phi\sqrt{\det g_H}$$

$$\frac{\partial R_{abcd}}{\partial R_{pqrs}} = \delta_a^p\delta_b^q\delta_c^r\delta_d^s - \delta_a^q\delta_b^p\delta_c^r\delta_d^s - \delta_a^p\delta_b^q\delta_c^s\delta_d^r + \delta_a^q\delta_b^p\delta_c^s\delta_d^r$$

$$-(16\pi G_N)^{-1}\int\sqrt{-\det g}Rd^4x$$

$$S_{wald} = \frac{i}{4}\int_H \frac{\partial(8\pi\mathcal{L})}{\partial R_{3131}}d\theta d\phi\omega_H b\sin\,\theta$$

$$S_{wald} = \frac{i}{4}\omega_H b\int_H \mathcal{I}\sin\,\theta d\theta d\phi$$

$$\mathcal{I} \equiv 8\pi\frac{\partial\mathcal{L}}{\partial R_{3131}}$$
$$= \frac{1}{4}\chi\frac{\partial R}{\partial R_{3131}} + \left[\frac{1}{2}iF_{\hat{A}}\frac{\partial\hat{C}}{\partial R_{3131}} + \text{ c.c.}\right] + \left[\frac{1}{2}i\frac{\partial\hat{F}^{-ab}}{\partial R_{3131}}F_{\hat{A}I}\left(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}\right) + \text{ c.c.}\right]$$
$$+\left[\frac{i}{2}F_{\hat{A}\hat{A}}\frac{\partial\hat{F}^{-ab}}{\partial R_{3131}}\hat{F}_{ab}^{-} + \text{c.c.}\right]$$
$$= \mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$$

$$\mathcal{J}_0 = \frac{1}{4}\chi\frac{\partial R}{\partial R_{3131}},$$
$$\mathcal{J}_1 = \left[64iF'_\Upsilon\frac{\partial\mathcal{R}(M)^{-cdab}}{\partial R_{3131}}\mathcal{R}(M)^-_{cdab} + 8iF'_\Upsilon\frac{\partial f_a^c}{\partial R_{3131}}T^{ab}\bar{T}_{cb} + \text{ c.c.}\right],$$
$$\mathcal{J}_2 = \left[-8i\frac{\partial\mathcal{R}(M)^{cdab}}{\partial R_{3131}}\bar{h}T_{cd}e^{-g}F'_{\Upsilon I}\left(F^{-I}_{ab} - \frac{1}{4}e^g\bar{Y}^I\bar{h}T_{ab}\right) + \text{ c.c.}\right],$$
$$\mathcal{J}_3 = \left[128ie^{-2g}F'_{\Upsilon\Upsilon}\frac{\partial\mathcal{R}(M)^{cdab}}{\partial R_{3131}}\bar{h}T_{cd}\mathcal{R}(M)^{c'd'}{}_{ab}\bar{h}T_{c'd'} + \text{ c.c.}\right].$$

$$\mathcal{J}_0 = \frac{1}{2}\chi = -1$$

$$\frac{\partial f_{ac}}{\partial R_{3131}} = \frac{1}{2}[\delta_a^3\delta_c^3 + \delta_a^1\delta_c^1] - \frac{1}{4}\eta_{ac}$$
$$\frac{\partial\mathcal{R}(M)_{abcd}}{\partial R_{3131}} = \delta_a^3\delta_b^1\delta_c^3\delta_d^1 - \delta_a^1\delta_b^3\delta_c^3\delta_d^1 - \delta_a^3\delta_b^1\delta_c^1\delta_d^3 + \delta_a^1\delta_b^3\delta_c^1\delta_d^3$$
$$-\left[\left\{\frac{1}{2}(\delta_a^3\delta_c^3 + \delta_a^1\delta_c^1)\eta_{bd} - \frac{1}{4}\eta_{ac}\eta_{bd}\right\}\right.$$
$$\left.-(a\leftrightarrow b) - (c\leftrightarrow d) + (a\leftrightarrow b, c\leftrightarrow d)\right].$$

$$\frac{\partial\mathcal{R}(M)^-_{abcd}}{\partial R_{3131}}\mathcal{R}(M)^{-abcd} = 2\mathcal{R}(M)^-_{3131} - 2\mathcal{R}(M)^-_{0202}$$
$$= \mathcal{R}(M)_{3131} - \mathcal{R}(M)_{0202} - i(\mathcal{R}(M)_{0231} + \mathcal{R}(M)_{3102})$$
$$= e^{2g}\left(2\hat{\nabla}_2T_2 + 2T_2^2\right)$$

$$\frac{\partial f_{ab}}{\partial R_{3131}}T^{ac}\bar{T}_c^b = 16e^{2g}\bar{T}_2T_2$$

$$\mathcal{J}_1 = 64iF'_\Upsilon e^{2g}\left(2\hat{\nabla}_2T_2 + 2T_2^2 + 2\bar{T}_2T_2\right) + \text{ c.c.}$$

$$\mathcal{J}_1 = 128iF'_\Upsilon\hat{E}_2{}^\theta\partial_\theta(e^{2g}T_2) + \text{ c.c..}$$

$$\mathcal{J}_1 = 128i\hat{E}_2^\theta\partial_\theta(e^{2g}T_2F'_\Upsilon) - 128ie^{2g}T_2F'_{\Upsilon I}\hat{\nabla}_2Y^I - 128ie^{2g}T_2F'_{\Upsilon\Upsilon}\hat{\nabla}_2\Upsilon + \text{ c.c.}$$
$$= 128i\hat{E}_2^\theta\partial_\theta(e^{2g}T_2F'_\Upsilon) - 128ie^{2g}T_2F'_{\Upsilon I}\hat{\nabla}_2Y^I - 256ie^{2g}\Upsilon F'_{\Upsilon\Upsilon}\hat{\nabla}_2T_2 + \text{ c.c.},$$

$$\Upsilon = \bar{h}^2e^{-2g}T_{ab}T^{ab} = -64T_2^2.$$

$$\mathcal{J}_2 = 128iF'_{\Upsilon I}e^gT_2\left(\hat{\nabla}_2(e^gY^I) - \frac{1}{2}iK_2e^gY^I\right) + \text{ c.c.}$$
$$= 128iF'_{\Upsilon I}e^{2g}T_2\hat{\nabla}_2Y^I - 256i\Upsilon F'_{\Upsilon\Upsilon}e^{2g}T_2^2 + \text{ c.c.}$$

$$F'_{\Upsilon I}Y^I + 2F'_{\Upsilon\Upsilon}\Upsilon = 0$$

$$\mathcal{J}_3 = 256ie^{2g}\Upsilon F'_{\Upsilon\Upsilon}\left(\hat{\nabla}_2T_2 + T_2^2\right) + \text{ c.c.},$$

$$\mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = -1 + \left[128i\hat{E}_2^\theta\partial_\theta(e^{2g}T_2F'_\Upsilon) + \text{ c.c.}\right]$$
$$= -1 + \left[128i\frac{1}{b\sin\,\theta}\partial_\theta(e^{2g}T_2F_{\hat{A}}) - 128i\frac{1}{b\sin\,\theta}\partial_\theta(e^{2g}\bar{T}_2\bar{F}_{\hat{A}})\right],$$

$$F'_{\Upsilon} = F_{\hat{A}}, \bar{F}'_{\Upsilon} = \bar{F}_{\hat{A}}$$

$$S_{\text{wald}} = -i\pi\omega_H b - 32\pi\omega_H \left[\left[F_{\hat{A}} \left(\frac{1}{b\sin\,\theta} \partial_\theta e^{2g} + i \frac{e^{4g}}{b\sin^2\,\theta} \partial_r \omega_\phi \right) \right]_0^\pi - \left[\bar{F}_{\hat{A}} \left(\frac{1}{b\sin\,\theta} \partial_\theta e^{2g} - i \frac{e^{4g}}{b\sin^2\,\theta} \partial_r \omega_\phi \right) \right]_0^\pi \right]_{r=b}$$

$$S_{wald} = -i\pi\big(\langle\gamma_N, \gamma_S\rangle - 2b\langle\hat{h}, \gamma_N\rangle\big) + [128\pi i F_{\hat{A}}|_N - 128\pi i \bar{F}_{\hat{A}}|_S]$$

$$\omega_\phi \simeq -\frac{2b\sin^2\,\theta}{r}\langle\hat{h}, \gamma_N\rangle.$$

$$J = \frac{r\omega_\phi}{2G_N \sin^2\,\theta} = -b\langle\hat{h}, \gamma_N\rangle$$

$$S_{wald} + 2\pi i J = -i\pi\langle\gamma_N, \gamma_S\rangle + [128\pi i F_{\hat{A}}|_N - 128\pi i \bar{F}_{\hat{A}}|_S]$$

$$\begin{aligned} \mathcal{J} &\equiv 8\pi \frac{\partial \mathcal{L}}{\partial R_{3131}} \\ &= \frac{1}{6}(\chi - e^{-K}) \frac{\partial R}{\partial R_{3131}} \\ &+ \left[\frac{1}{2} i \frac{\partial \hat{F}^{-ef}}{\partial R_{3131}} F_{\hat{A}I} \left(F_{ef}^{-I} - \frac{1}{4} \bar{X}^I T_{ef} \right) + \frac{1}{2} i F_{\hat{A}} \frac{\partial \hat{C}}{\partial R_{3131}} + \text{ c.c.} \right] \\ &+ \left[\frac{i}{2} F_{\hat{A}\hat{A}} \frac{\partial \hat{F}^{-ef}}{\partial R_{3131}} \hat{F}_{ef}^{-} + \text{c.c.} \right] \end{aligned}$$

$$e^{-K} = -i(X^I \bar{F}_I - \bar{X}^I F_I)$$

$$\frac{\partial f_{ac}}{\partial R_{3131}} = \frac{1}{2}[\delta_a^3 \delta_c^3 + \delta_a^1 \delta_c^1] - \frac{1}{6}\eta_{ac}$$

$$\begin{aligned} \frac{\partial \mathcal{R}(M)_{abcd}}{\partial R_{3131}} &= \delta_a^3 \delta_b^1 \delta_c^3 \delta_d^1 - \delta_a^1 \delta_b^3 \delta_c^3 \delta_d^1 - \delta_a^3 \delta_b^1 \delta_c^1 \delta_d^3 + \delta_a^1 \delta_b^3 \delta_c^1 \delta_d^3 \\ &- \left[\left\{ \frac{1}{2} (\delta_a^3 \delta_c^3 + \delta_a^1 \delta_c^1) \eta_{bd} - \frac{1}{6} \eta_{ac} \eta_{bd} \right\} \right. \\ &\left. - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d) \right] \end{aligned}$$

$$\mathcal{J}_1' = \frac{i}{2} F_{\hat{A}} \frac{\partial \hat{C}}{\partial R_{3131}} + \text{ c.c..}$$

$$\begin{aligned} \frac{\partial \mathcal{R}(M)_{abcd}^-}{\partial R_{3131}} \mathcal{R}(M)^{-abcd} &= 4\mathcal{R}(M)_{3131}^- - \widetilde{D} \\ &= e^{2g}\big(2\hat{\nabla}_2 T_2 + 2T_2^2\big) - \widetilde{D} \end{aligned}$$

$$\mathcal{R}(M)_{abcd}^- \eta^{bd} + (a \leftrightarrow c) = \frac{3}{2} \widetilde{D} \eta_{ac}$$

$$\widetilde{D} = \frac{2}{3}e^{2g}\left(\hat{\nabla}^p T_p - T_p^2\right)$$
$$= \frac{2}{3}e^{2g}\left(\hat{\nabla}^p \bar{T}_p - \bar{T}_p^2\right)$$
$$= \frac{2}{3}e^{2g}\left[\hat{\nabla}_p^2 g - \left(\hat{\nabla}_p g\right)^2 + \frac{1}{4}e^{4g}\left(R(\sigma)_p\right)^2\right]$$

$$\mathcal{I}_1' = -64iF_\Upsilon'\widetilde{D} + 128iF_\Upsilon' e^{2g}\left(\hat{\nabla}_2 T_2 + T_2^2 + T_2\bar{T}_2\right) + \text{ c.c.}$$
$$= -64iF_\Upsilon'\widetilde{D} + \text{ c.c. } + \mathcal{I}_1$$

$$\mathcal{I}_0' = \frac{1}{6}(\chi - e^{-K})\frac{\partial R}{\partial R_{3131}} = -\frac{1}{3}(2 + e^{-K}) = -1 - \left(\frac{e^{-K} - 1}{3}\right),$$

$$\mathcal{I}_0'' = -\left(\frac{e^{-K} - 1}{3}\right)$$

$$e^{-K} - 1 = -128ie^{3g}\hat{\nabla}^p\left[e^{-g}\hat{\nabla}_p g(F_{\hat{A}} - \bar{F}_{\hat{A}})\right] - 32ie^{6g}\left(R(\sigma)_p\right)^2(F_{\hat{A}} - \bar{F}_{\hat{A}})$$
$$-64e^{4g}R(\sigma)_p\hat{\nabla}^p(F_{\hat{A}} + \bar{F}_{\hat{A}})$$

$$e^{-K} - 1 = -192iF_\Upsilon'\widetilde{D} - 128ie^{2g}T_2F_{\Upsilon I}'\hat{\nabla}_2 Y^I - 128ie^{2g}T_2F_{\Upsilon\Upsilon}'\hat{\nabla}_2\Upsilon + \text{ c.c.},$$

$$\mathcal{I}_0'' = 64iF_\Upsilon'\widetilde{D} + \frac{128}{3}ie^{2g}T_2F_{\Upsilon I}'\hat{\nabla}_2 Y^I + \frac{128}{3}ie^{2g}T_2F_{\Upsilon\Upsilon}'\hat{\nabla}_2\Upsilon + \text{ c.c.}$$
$$= 64iF_\Upsilon'\widetilde{D} + \text{ c.c. } + \frac{1}{3}(\mathcal{I}_2 + \mathcal{I}_3)$$

$$\mathcal{I}_2' = \frac{i}{2}\frac{\partial \hat{F}^{-ef}}{\partial R_{3131}}F_{\hat{A}I}\left(F_{ef}^{-I} - \frac{1}{4}\bar{X}^I T_{ef}\right) + \text{ c.c.}$$
$$= \frac{256i}{3}F_{\Upsilon I}'e^{2g}T_2\left(T_2Y^I + \hat{\nabla}_2 Y^I\right) + \text{ c.c.}$$
$$= \frac{2}{3}\mathcal{I}_2$$

$$\mathcal{I}_3' = \frac{i}{2}F_{\hat{A}\hat{A}}\frac{\partial \hat{F}^{-ef}}{\partial R_{3131}}\hat{F}_{ef}^{-} + \text{c.c.}$$
$$= -\frac{2 \times (128)^2 i}{3}e^{2g}F_{\Upsilon\Upsilon}'T_2^2\left(\hat{\nabla}_2 T_2 + T_2^2\right) + \text{ c.c.}$$
$$= \frac{2}{3}\mathcal{I}_3$$

$$\sum_{i=0}^{3}\mathcal{I}_i' = \sum_{i=0}^{3}\mathcal{I}_i$$

$$x^1 = \sqrt{r^2 - b^2}\sin\,\theta\cos\,\phi, x^2 = \sqrt{r^2 - b^2}\sin\,\theta\sin\,\phi, x^3 = r\cos\,\theta$$

$$r_N = \sqrt{(x^1)^2 + (x^2)^2 + (x^3 - b)^2} = r - b\cos\,\theta$$
$$r_S = \sqrt{(x^1)^2 + (x^2)^2 + (x^3 + b)^2} = r + b\cos\,\theta$$

$$ds^2 = e^{2g}(d\tau + \omega_E d\phi)^2 + e^{-2g}\left[\frac{r^2 - b^2\cos^2\ \theta}{r^2 - b^2}dr^2 + (r^2 - b^2\cos^2\ \theta)d\theta^2 + (r^2 - b^2)\sin^2\ \theta d\phi^2\right]$$

$$\tilde{\phi} = \phi + \omega_0^{-1}\tau$$

$$\begin{aligned} ds^2 &= e^{2g}\left(d\tau\left(1 - \frac{\omega_E}{\omega_0}\right) + \omega_E d\tilde{\phi}\right)^2 + e^{-2g}\left[\frac{r^2 - b^2\cos^2\ \theta}{r^2 - b^2}dr^2 + (r^2 - b^2\cos^2\ \theta)d\theta^2\right] \\ &+e^{-2g}(r^2 - b^2)\sin^2\ \theta\left(d\tilde{\phi} - \omega_0^{-1}d\tau\right)^2 \\ &= d\tilde{\phi}^2[e^{2g}\omega_E^2 + e^{-2g}(r^2 - b^2)\sin^2\ \theta] + d\theta^2 e^{-2g}(r^2 - b^2\cos^2\ \theta) + e^{-2g}\frac{r^2 - b^2\cos^2\ \theta}{r^2 - b^2}dr^2 \\ &+d\tau^2\left[e^{2g}\left(1 - \frac{\omega_E}{\omega_0}\right)^2 + e^{-2g}(r^2 - b^2)\sin^2\ \theta\omega_0^{-2}\right] \\ &+2d\tau d\tilde{\phi}\left[e^{2g}\omega_E\left(1 - \frac{\omega_E}{\omega_0}\right) - e^{-2g}(r^2 - b^2)\sin^2\ \theta\omega_0^{-1}\right] \end{aligned}$$

$$\rho = \sqrt{r - b}$$

$$\omega_E = \omega_0(1 + \rho f_1(\theta) + \rho^2 f_2(\theta) + \cdots), g = g_0(\theta) + g_1(\theta)\rho + g_2(\theta)\rho^2 + \cdots$$

$$\begin{aligned} ds^2 &= e^{2g_0}\omega_0^2\left(1 + 2g_1\rho + 2f_1\rho + \mathcal{O}(\rho^2)\right)d\tilde{\phi}^2 + e^{-2g_0}b^2\sin^2\ \theta\left(1 - 2g_1\rho + \mathcal{O}(\rho^2)\right)d\theta^2 \\ &+2e^{-2g_0}b\left(1 - 2g_1\rho + \mathcal{O}(\rho^2)\right)\sin^2\ \theta d\rho^2 \\ &+\rho^2 d\tau^2(e^{2g_0}f_1(\theta)^2 + 2be^{-2g_0}\omega_0^{-2}\sin^2\ \theta + \mathcal{O}(\rho)) \\ &-2d\tau d\tilde{\phi}e^{2g_0}\omega_0 f_1\rho(1 + \mathcal{O}(\rho)) \end{aligned}$$

$$e^{2g_0}f_1(\theta)$$

$$\sin^2\ \theta(d\rho^2 + c^2\rho^2\tau^2)$$

$$\rho = \sqrt{x^2 + y^2} \text{ and } \tau = c^{-1}\tan^{-1}(y/x)$$

$$-2e^{2g_0}\omega_0 f_1 c^{-1}d\tilde{\phi}\frac{xdy - ydx}{\sqrt{x^2 + y^2}}$$

$$d\tilde{\phi}^2 e^{2g_0}\omega_0^2\left(1 + 2g_1\sqrt{x^2 + y^2} + \cdots\right)$$

$$A_n \rightarrow A_n + d\Lambda_{n-1}$$

$$A_n = \frac{1}{p!}A_{\alpha_1 \dots \alpha_n}dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_n}$$

$$\int A_n = \int A_{0123\dots(n-1)}d^n x$$

$$d \star F_{D-(n+1)} = \star J_{D-n}$$

$$\star F_{\alpha_1 \dots \alpha_{d-p}} = \frac{1}{p!}\epsilon_{\alpha_1 \dots \alpha_{d-p}}{}^{\beta_1 \dots \beta_p}F_{\beta_1 \dots \beta_p}$$

$$\int_{\mathcal{M}} A_n \wedge \star J_{D-n} \propto \int_{\mathcal{M}} \sqrt{-g} A_{\alpha_1 \ldots \alpha_n} J^{\alpha_1 \ldots \alpha_n}$$

$$Q_{D-n} = \int_{\partial B} \star F_{D-(n+1)}$$

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}xe \left[\left(R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{2^3 \cdot 4!} \left(\bar{\psi}_\alpha \Gamma^{\alpha\beta\gamma\delta\sigma\lambda} \psi_\lambda + 12 \bar{\psi}^\beta \Gamma^{\gamma\delta} \psi^\sigma \right) (F + \hat{F})_{\beta\gamma\delta\sigma} - \bar{\psi}_\alpha \Gamma^{\alpha\beta\gamma} D_\beta \left(\frac{1}{2} (\omega + \hat{\omega}) \right) \psi_\gamma \right] - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4$$

$$|F_m|^2 = \frac{1}{m!} g^{\alpha_1 \beta_1} \ldots g^{\alpha_m \beta_m} F_{\alpha_1 \ldots \alpha_m} F_{\beta_1 \ldots \beta_m}$$

$$\hat{\omega}_{\alpha ab} = \omega_{\alpha ab} + \frac{1}{8} \bar{\psi}^\beta \Gamma_{\beta\alpha ab\gamma} \psi^\gamma$$
$$\hat{F}_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} - 3 \bar{\psi}_{[\alpha} \Gamma_{\beta\gamma} \psi_{\alpha]}$$

$$S_{\text{dark particle}} = \frac{1}{2\kappa_{11}^2} \int d^{11}x (-g)^{1/2} \left(R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4$$

$$d(\star F_4 + (\mathfrak{G}) A_3 \wedge F_4) = \star J_3.$$

$$\delta e_\alpha^a = \frac{1}{2} \bar{\eta} \Gamma^a \psi_\alpha,$$
$$\delta A_{\alpha\beta\gamma} = -\frac{3}{2} \bar{\eta} \Gamma_{[\alpha\beta} \psi_{\gamma]},$$
$$\delta \psi_\alpha = D_\alpha(\hat{\omega}) \eta + \frac{\sqrt{2}}{(4!)^2} \left(\Gamma_\alpha^{abcd} - 8 e_\alpha^a \Gamma^{bcd} \right) \eta \hat{F}_{abcd} \equiv \hat{D}_\alpha \eta,$$

$$A_3 \to A_3 + d\Lambda_2$$

$$\{Q_A^I, \bar{Q}^{JB}\}_+ = -2 P_\mu \Gamma_A^{\mu B} \delta^{IJ} - 2i Z^{IJ} \delta_A^B$$

$$\{Q_A^I, Q^{\dagger JB}\}_+ = 2T \delta^{IJ} \delta_A^B + 2i Z^{IJ} \Gamma_A^{0B}$$

$$T \geq |q|.$$

$$ds^2 = -\left(1 - \frac{2GM}{R} + \frac{GQ^2}{R^2}\right) dt^2 + \frac{1}{1 - \frac{2GM}{R} + \frac{GQ^2}{R^2}} dR^2 + R^2 d\Omega_2^2$$

$$\sqrt{(\text{ mass })(\text{length})}$$

$$F = Q^2 / r^2$$

$$GM^2 = Q^2$$

$$r_0 = GM = \sqrt{G}Q$$

$$ds^2 = -(1 - r_0/R)^2 dt^2 + (1 - r_0/R)^{-2} dR^2 + R^2 d\Omega_2^2$$

$$z = r_0(1 - r_0/R)^{-1}$$

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + r_0^2 d\Omega_2^2 + O(z^{-4})$$

$$ds^2 = -f^{-2}dt^2 + f^2 \sum_{i=1}^{3} dx^i dx^i$$

$$f = 1 + r_0/r$$

$$A_t = f^{-1}$$

$$\partial_x^2 f := \sum_{i=1}^{3} \partial_i \partial_i f = -4\pi\delta^{(3)}(x)$$

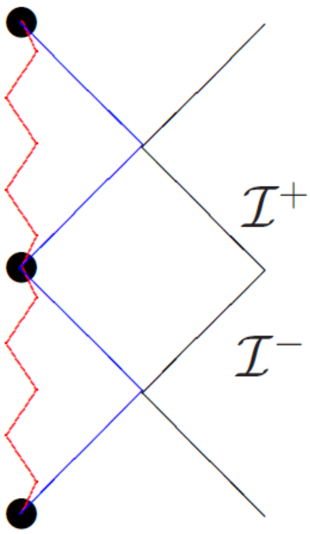
$$dx^2 := \sum_i dx^i dx^i$$

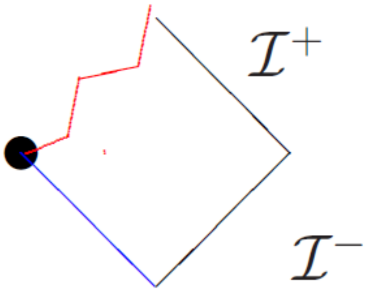
$$\rho = -\frac{1}{4\pi}\nabla^2 f$$

$$\rho_0 + \sum_{k=1}^{N} r_k \delta(x - x_k)$$

$$ds^2 = -f^{-2}dt^2 + f \sum_{i=1}^{4} dx^i dx^i = -f^{-2}dt^2 + f dx^2,$$

$$\partial_x^2 f = -2\Omega_3 \left(\rho_0 + \sum_{k=1}^{N} r_k^2 \delta(x - x_k) \right) \Omega_3$$

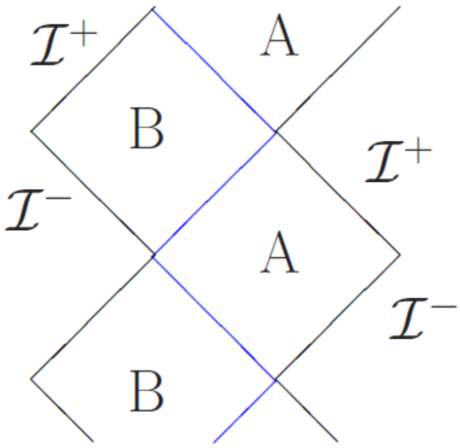




$$dx_\parallel^2 = -\left(dx_\parallel^0\right)^2 + \left(dx_\parallel^1\right)^2 + \left(dx_\parallel^2\right)^2$$

$$A_3 = H_2^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2}$$
$$ds^2 = H_2^{-2/3} dx_\parallel^2 + H_2^{1/3} dx_\perp^2$$

$$dA = F = -\frac{1}{4!}\partial_{x_\perp^i} H_5 \epsilon^i_{jklm} dx^j \wedge dx^k \wedge dx^l \wedge dx^m$$
$$ds^2 = H_5^{-1/3} dx_\parallel^2 + H_5^{2/3} dx_\perp^2$$



$$ds^2 = -dt^2 + dx_\perp^2 + dz^2 + (H_{AS} - 1)(dt - dz)^2$$

$$\partial_\perp^2 H_{AS} = -7\Omega_9\rho$$

$$ds^2 = dx_\parallel^2 + H_{KK}dx_\perp^2 + H_{KK}^{-1}\left(d\theta + a_i dx_\perp^i\right)^2$$

$$\partial_\perp^2 H_{KK} = -4\pi\rho$$

$$\partial_{x_\perp^i} H_{KK} = \epsilon_{ijk}\partial_{x_\perp^j} a_k$$

$$\rho = \frac{L}{4\pi}\delta^{(3)}(x_\perp)$$

$$A = H_x^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2} + H_y^{-1} dt \wedge dy_{\parallel,1} \wedge dy_{\parallel,2}$$
$$ds^2 = -H_x^{-2/3}H_y^{-2/3}dt^2 + H_x^{-2/3}H_y^{1/3}dx_\parallel^2 + H_x^{1/3}H_y^{-2/3}dy_\parallel^2$$
$$+H_x^{1/3}H_y^{1/3}dx_\perp^2$$

$$\partial_{x_\perp}^2 H_x = -7\Omega_8\rho_x$$

$$\partial_{x_\perp}^2 H_y = -7\Omega_8\rho_y$$

$$\rho_x = \rho_y = r_0^4\delta^{(6)}(x_\perp)$$

$$x_\perp = 0 \text{ like } |x_\perp|^{-4}\left(z_\parallel^1, z_\parallel^2\right)$$

$$A = H_x^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2} + H_y^{-1} dt \wedge dy_{\parallel,1} \wedge dy_{\parallel,2} + H_z^{-1} dt \wedge dz_{\parallel,1} \wedge dz_{\parallel,2}$$
$$ds^2 = -H_x^{-2/3}H_y^{-2/3}H_z^{-2/3}dt^2 + H_x^{-2/3}H_y^{1/3}H_z^{1/3}dx_\parallel^2 + H_x^{1/3}H_y^{-2/3}H_z^{1/3}dy_\parallel^2$$
$$+H_x^{1/3}H_y^{1/3}H_z^{-2/3}dz_\parallel^2 + H_x^{1/3}H_y^{1/3}H_z^{1/3}dx_\perp^2$$

$$\partial_{x_\perp}^2 H_{x,y,z}(x_\perp) = -2\Omega_3\rho_{x,y,z}(x_\perp)$$

$$A = \Omega_3 r_x r_y r_z L_{1x}L_{2x}L_{1y}L_{2y}L_{1z}L_{2z}$$

$$16\pi G_{11} = 2\kappa_{11}^2 = (2\pi)^8 l_p^9$$

$$Q_x / L_{1y}L_{2y}L_{1z}L_{2z}$$

$$A/4G_{11} = 2\pi\sqrt{Q_x Q_y Q_z}$$

$$E^2 = p^2 + (k/L)^2$$

$$\theta_{,\mu}\lambda^\mu = \lambda^\mu\lambda_\mu$$

$$g^{ab} = x^a{}_{,\mu}x^b{}_{,\nu}g^{\mu\nu}$$

$$A_1^a = -x_{,\mu}^a\theta_{,\nu}g^{\mu\nu}$$

$$Le^{4\phi/3} = \lambda^\mu g_{\mu\nu}\lambda^\nu$$

$$x_{,\nu}^a g^{\nu\mu}\lambda_\mu = \lambda^\mu\theta_{,\mu} = \lambda^\mu\lambda_\mu\theta_{,\mu} - \lambda_\mu$$

$$c_a = -g_{ab}A_1^b$$

$$\theta_{,\mu}\theta_{,}^{\mu} = \lambda^\mu\lambda_\mu + A_{1a}A_1^a$$

$$A_1^a \to A_1^a + \Lambda_{,b}g^{ab}; \text{ i.e., } A_{1a} \to A_{1a} + \Lambda_{,a}$$

$$ds_{11}^2 = g_{ab}dx^a dx^b + e^{4\phi/3}[d\theta + A_{1a}dx^a]^2$$

$$\hat{A}_3^{abc} = A_3^{\mu\nu\rho}x_{,\mu}^a x_{,\nu}^b x_{,\rho}^c$$

$$A_2^{ab} = A_3^{\mu\nu\rho}x_{,\mu}^a x_{,\nu}^b \lambda_\rho$$

$$A_1 \to A_1 + d\Lambda_0$$

$$A_3 = \frac{1}{3!}\tilde{A}_{3abc}dx^a \wedge dx^b \wedge dx^c + \frac{1}{2!}A_{2ab}dx^a \wedge dx^b \wedge d\theta$$

$$A_3 \to A_2 \wedge d\Lambda_0$$

$$A_3 \to A_3 + d\Lambda_2 \text{ and } A_2 \to A_2 + d\Lambda_1$$

$$\begin{aligned} S_{9+1,\text{ white particle}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\Big[\sqrt{-g}\left(e^{2\phi/3}R - \frac{1}{2}e^{2\phi}|F_2|^2\right) \\ -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(e^{-2\phi/3}|F^2|_3 + e^{2\phi/3}|\tilde{F}_4|^2\right) \\ -\frac{1}{4\kappa_{10}^2}\int A_2 \wedge F_4 \wedge F_4\Big] \end{aligned}$$

$$F_n = dA_{n-1}$$

$$\tilde{F}_4 = dA_3 - A_1 \wedge F_3$$

$$\kappa_{10}^2 = \kappa_{11}^2/L$$

$$g_E = e^{\phi/6}g$$

$$\begin{aligned} S_{\text{dark particle}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_E}\left(R_E - \frac{1}{2}\partial_a\phi\partial^a\phi\right) \\ -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g_E}\left(e^{3\phi/2}|F_2|^2 + e^{-\phi}|F^2|_3 + e^{\phi/2}|\tilde{F}_4|^2\right) \\ -\frac{1}{4\kappa_{10}^2}\int A_2 \wedge F_4 \wedge F_4 \end{aligned}$$

$$\begin{aligned} S_{\text{white particle}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_S}e^{-2\phi}\left(R_S + 4\partial_a\phi\partial^a\phi - \frac{1}{2}|F_3|^2\right) \\ -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g_S}\left(|F_2|^2 + |\tilde{F}_4^2|\right) \\ -\frac{1}{4\kappa_{10}^2}\int A_2 \wedge F_4 \wedge F_4 \end{aligned}$$

$$c = \hbar = 1$$

$$2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8$$

$$dA_{8-n} = \star F_{n+1}$$

$$ds^2_{\text{supergravity}} = H_p^{-1/2} dx_\parallel^2 + H_p^{1/2} dx_\perp^2$$
$$A_{p+1} = H_p^{-1} dx_\parallel^0 \wedge \ldots \wedge dx_\parallel^p$$
$$e^{2\phi} = H_p^{(3-p)/2}$$

$$\partial_\perp^2 H_p = -(7-p)\Omega_{8-p} r_0^{7-p} \delta^{(9-p)}(x_\perp)$$

$$ds^2_{\text{supergravity}} = H_p^{-1/2}\left(-f dt^2 + \sum_{i=1}^{p} \left(dx_\parallel^i\right)^2\right) + H_p^{1/2}\left(\frac{dr^2}{f} + r^2 d\Omega_{8-p}^2\right)$$
$$A_{p+1} = \left[1 + \coth\beta\left(H_p^{-1} - 1\right)\right] dx_\parallel^0 \wedge \ldots \wedge dx_\parallel^p$$
$$e^{2\phi} = H_p^{(3-p)/2}$$

$$H_p = 1 + \frac{\sinh^2 \beta r_+^{7-p}}{r^{7-p}}, f = 1 - \frac{r_+^{7-p}}{r^{7-p}}$$

$$Q = \frac{(7-p)\Omega_{8-p}}{2\kappa_{10}^2} r_+^{7-p} \sinh\beta \cosh\beta$$

$$M/V_p = \frac{(8-p)\Omega_{8-p} r_+^{7-p}}{2\kappa_{10}^2}\left(1 + \frac{7-p}{8-p}\sinh^2\beta\right)$$
$$T = \frac{7-p}{4\pi r_+ \cosh\beta}$$
$$S/V_p = \frac{4\pi\Omega_{8-p}}{2\kappa^2}\cosh\beta r_+^{(8-p)}$$

$$\beta \to \infty, r_+ \to 0$$

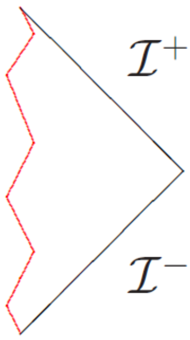
$$M/V_p$$

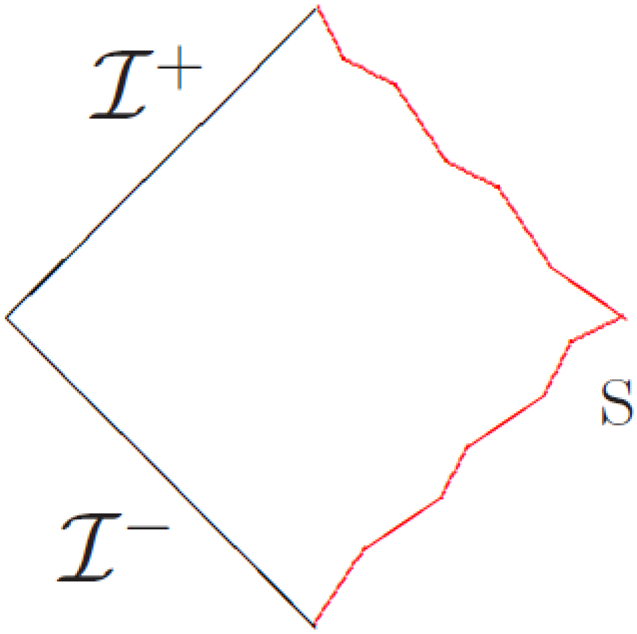
$$M/V_p \to Q \text{ and } S \to 0$$

$$ds^2_{\text{supergravity}} = H_F^{-1} dx_\parallel^2 + dx_\perp^2$$
$$A_2 = H_F^{-1} dx_\parallel^0 \wedge dx_\parallel^1$$
$$e^{2\phi} = H_F^{-1}$$

$$ds^2_{\text{supergravity}} = dx_\parallel^2 + H_5 dx_\perp^2$$

$$F_3 = -\frac{1}{3!}\partial_{x_\perp^i} H_5 \epsilon_{ijkl} dx_\perp^j \wedge dx_\perp^k \wedge dx_\perp^l$$
$$e^{2\phi} = H_5$$





$$\sqrt{-g_S}|F_5|^2$$

$$\sqrt{-g}|F_2|^2$$

$$\mathrm{AdS}_5 \times S^5$$

$$\tau = A_0 + ie^{-\phi}$$

$$\mathcal{M}_{ij} = \frac{1}{\mathrm{Im}\tau}\begin{bmatrix} |\tau|^2 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{bmatrix}$$

$$A_2^i = \begin{bmatrix} B_2 \\ A_2 \end{bmatrix}$$

$$S_{\text{white particle}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_E}\left(R_E - \frac{1}{12}F_{abc}^i\mathcal{M}_{ij}F^{jabc} + \frac{1}{4}\left(\partial^a\mathcal{M}_{ij}\partial_a\mathcal{M}^{-1ij}\right)\right) - \frac{1}{8\kappa_{10}^2}\int d^{10}x\sqrt{-g_S}\left|\tilde{F}_5\right|^2 - \frac{1}{4\kappa_{10}^2}\int A_4 \wedge H_3 \wedge F_3$$

$$F_3^i = \frac{1}{3!}F_{abc}^i dx^a \wedge dx^b \wedge dx^c = dA_2^i$$

$$\tilde{F}_5 = F_5 + \frac{1}{2}\epsilon_{ij}A_2^i \wedge F_3^j$$

$$\tau \to \frac{a\tau + b}{c\tau + d}, F_3^i \to \Lambda_j^i F_3^j$$
$$\tilde{F}_5 \to \tilde{F}_5, g_E \to g_E$$

$$\Lambda_j^i = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\det\Lambda = 1$$

$$\Lambda_j^i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\tilde{g}_{zz} = 1/g_{zz}, \tilde{g}_{z\alpha} = B_{z\alpha}/g_{zz},$$
$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \left(g_{z\alpha}g_{z\beta} - B_{z\alpha}B_{z\alpha}\right)/g_{zz}, \tilde{B}_{z\alpha} = g_{z\alpha}/g_{zz}$$
$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} - \left(g_{z\alpha}B_{\beta z} - g_{z\beta}B_{\alpha z}\right)/g_{zz}, \tilde{\phi} = \phi + \log\, g_{zz}$$

$$\frac{4\pi^2 l_s^2}{L}$$

$$\tilde{F}_{n,\alpha_1\ldots\alpha_n} = (\mathfrak{S}\mathfrak{G})F_{n+1,z\alpha_1\ldots\alpha_n},$$
$$\tilde{F}_{n,z\alpha_1\ldots\alpha_{n-1}} = (\mathfrak{S}\mathfrak{G})F_{n-1,\alpha_1\ldots\alpha_{n-1}}.$$

$$\widehat{\delta_0\phi}(x) = \hat{\phi}(x) - \phi_0(x)$$

$$\widehat{\delta_n\phi} = \phi(x) - \bar{\phi}_n$$

$$\alpha' \propto l_s^2$$

$$16\pi G = 2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8$$

$$GT \sim ng_s$$

$$S = 2\pi\sqrt{Q_z n/6}$$

$$S = 2\pi\sqrt{Q_y Q_z}$$

$$S = A/4G_{11}$$

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left[\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \partial\psi_\mu\right]$$

$$T = \frac{1}{2\pi\alpha'}$$

$$E^2 = \vec{P}^2 + \left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'}N_L$$
$$E^2 = \vec{P}^2 + \left(\frac{n}{R} + \frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'}N_R$$

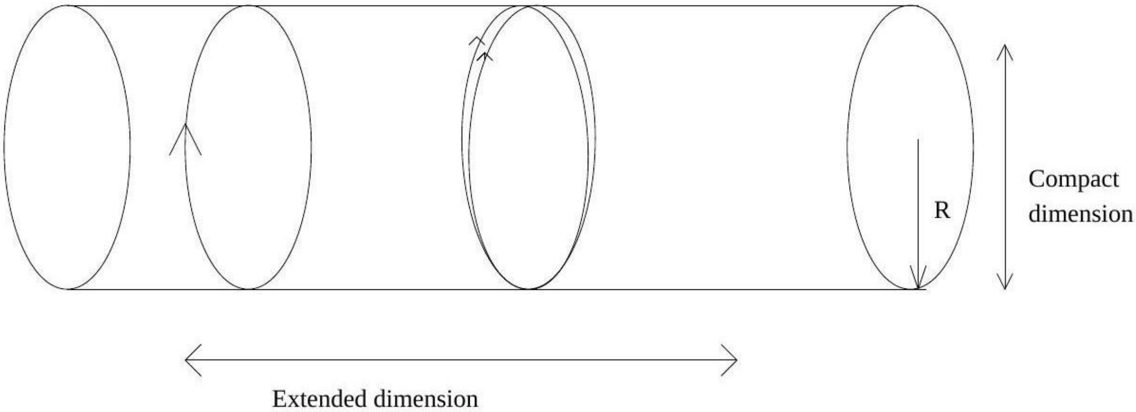
$$P_{9R}^2 - P_{9L}^2 = 4nm = 4(N_L - N_R),$$

$$P_{9L,R} = \frac{n}{R} \mp \frac{mR}{\alpha'}$$

$$\{Q_\alpha^L, Q_\beta^L\} = \Gamma_{\alpha\beta}^\mu P_{\mu L}, \{Q_\alpha^R, Q_\beta^R\} = \Gamma_{\alpha\beta}^\mu P_{\mu R}$$

$$P^0 \geq |\vec{P}_L|, P^0 \geq |\vec{P}_R|$$

$$R \to R' = \frac{\alpha'}{R}, g \to g' = \frac{g\sqrt{\alpha'}}{R}.$$



$$S = \frac{1}{16\pi G_N^{10}} \int d^{10}x\sqrt{-G}\left[e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{3}H^2\right) - \alpha' G^2 - \frac{\alpha'}{12}F'^2 - \frac{\alpha'}{288}\epsilon^{\mu_1 \dots \mu_{10}} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} B_{\mu_9\mu_{10}}\right]$$

$$S \sim \int \sqrt{g}R$$

$$g_E = e^{-\phi/2}G$$

$$F = dA = * F$$

$$g \to g' = \frac{1}{g}, R_i \to R_i = \frac{R_i}{\sqrt{g}}$$

$$g = e^{\phi_\infty}$$

$$G_N^{10} = 8\pi^6 g^2 \alpha'^4$$

$$\mu_p \int_{V_{p+1}} A_{p+1}$$

$$d\tilde{A}_{7-p} = * dA_{p+1}$$

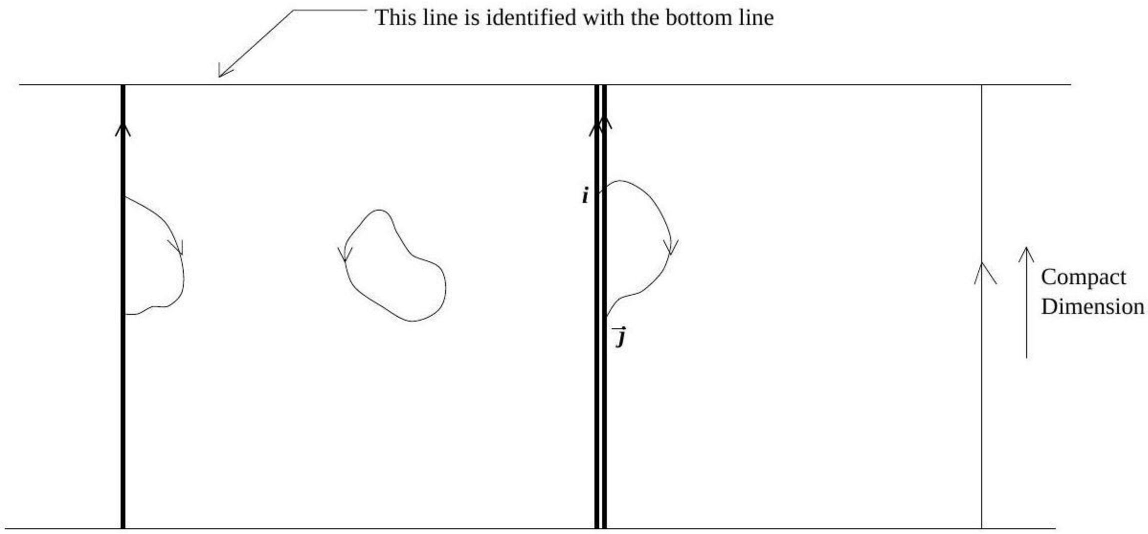
$$\partial_\sigma X^\mu = 0 \text{ for } \mu = 0, \dots, p,$$
$$X^\mu = 0 \text{ for } \mu = p+1, \dots, 9.$$

$$P^2 = \frac{4}{\alpha'} N_{\text{open}} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$E = \frac{R_9}{\alpha' g} + \sum_i \epsilon_i = E_0 + \frac{N_L + N_R}{R_9}$$

$$N_R = \sum_{n>0} nN_n, N_L = \sum_{n<0} nN_n$$

$$T_D = \frac{1}{2\pi g \alpha'}$$



$$S = \frac{1}{4g} \int d^{10}x \mathrm{Tr}\left[F_{\mu\nu} F^{\mu\nu}\right] + \text{ white particles}$$

$$A_\mu(x_9 + 2\pi R_9) = U A_\mu(x_9) U^{-1}$$

$$V = \sum_{IJ} \mathrm{Tr}\left[A_I, A_J\right]^2$$

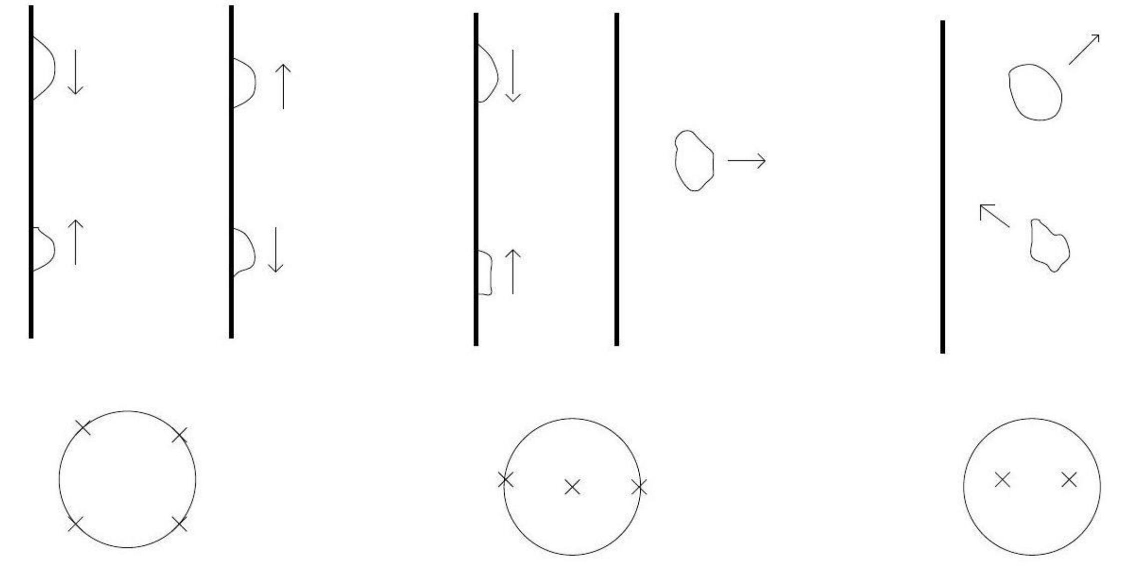
$$A_I = \begin{pmatrix} f_I^1(u,v) & & \\ & \cdot & \\ & & \cdot \\ & & & f_I^Q(u,v) \end{pmatrix},$$

$$4\partial_u \partial_v (\delta A_I)_{mn} - \left(f_J^n - f_J^m\right)^2 (\delta A_I)_{mn} = 0$$

$$f_I^n(x_9 + 2\pi R_9) = f_I^{n+1}(x_9)$$

$$E = \frac{R_9 Q}{g\alpha'} + \frac{N_L' + N_R'}{QR_9}$$

$$P = (N_L' - N_R')/QR_9 = N/R_9$$



Scattering of open string excitations on a D-brane.

An excited D-brane decays by emitting a closed string.

Scattering of closed strings off a D-brane.

$$S_R(z) = \pm\Gamma^0 \cdots \Gamma^p S_L(\bar{z})|_{z=\bar{z}}.$$

$$\epsilon_R = \Gamma^0 \cdots \Gamma^p \epsilon_L$$

$$r_s^{7-p} \sim g$$

$$r_s^{7-p} = Qg$$

$$M = c_p Q_p + c_{p-4} Q_{p-4}$$

$$M \sim \sqrt{c_p' Q_p^2 + c_{p-2}' Q_{p-2}^2}$$

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2$$
$$\Delta = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)$$

$$M = \frac{1}{2G_N^4}(r_+ + r_-), Q = \frac{1}{G_N^4}\sqrt{r_+ r_-}$$

$$ds^2 = h[f_s^{-1}(-dt^2 + dx_9^2) + dx_1^2 + \cdots + dx_8^2]$$

$$\delta\lambda = \left[\partial_\mu \phi \gamma^\mu \Gamma_{11} + \frac{1}{6} H_{\mu\nu\rho} \gamma^{\mu\nu\rho}\right]\eta,$$
$$\delta\psi_\mu = \left[\partial_\mu + \frac{1}{4}(\omega_\mu^{ab} + H_\mu^{ab}\Gamma_{11})\Gamma_{ab}\right]\eta,$$

$$\eta = \epsilon_R + \epsilon_L$$

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}, \gamma^\mu = e_a^\mu \Gamma^a \text{ and } \gamma^{\mu_1 \cdots \mu_n}$$

$$ds^2 = f_f^{-1}(-dt^2 + dx_9^2) + dx_1^2 + \cdots + dx_8^2$$
$$B_{09} = \frac{1}{2}\left(f_f^{-1} - 1\right)$$
$$e^{-2(\phi - \phi_\infty)} = f_f$$

$$\epsilon_{R,L} = f_f^{-1/4}\epsilon_{R,L}^0, \Gamma^0\Gamma^9\epsilon_R^0 = \epsilon_R^0, \Gamma^0\Gamma^9\epsilon_L^0 = -\epsilon_L^0$$

$$f_f = 1 + \frac{Q_f}{r^6},$$

$$c_f^{10} = \frac{8G_N^{10}}{\alpha' 6\omega_7},$$

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$f_f = \sum_i c_f^{10}/(\vec{r} - \vec{r}_i)^6$$

$$ds^2 = -dt^2 + f_{s5}(dx_1^2 + \cdots + dx_4^2) + dx_5^2 + \cdots + dx_9^2$$
$$e^{-2(\phi - \phi_\infty)} = f_{s5}^{-1}$$
$$H_{ijk} = (dB)_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial_l f_{s5}, i, j, k, l = 1,2,3,4$$

$$f_{s5} = 1 + \frac{c_{s5}}{(x_1^2 + \cdots + x_4^2)}$$

$$c_5^s = \alpha' \begin{bmatrix} {}_4[1] \\ -1 \end{bmatrix}$$

$$\epsilon_L^0 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4\epsilon_L^0, \epsilon_R^0 = -\Gamma^1\Gamma^2\Gamma^3\Gamma^4\epsilon_R^0$$

$$ds^2 = f_p^{-1/2}\left(-dt^2 + dx_1^2 + \cdots + dx_p^2\right) + f_p^{1/2}\left(dx_{p+1}^2 + \cdots + dx_9^2\right)$$
$$e^{-2\phi} = f_p^{\frac{p-3}{2}}$$
$$A_{0\cdots p} = -\frac{1}{2}\left(f_p^{-1} - 1\right)$$

$$f_p = 1 + nc_p^{10}/r^{7-p}$$

$$ds^2 = f_f^{-1}du\left[dv + \tilde{k}(r)du + 2F'^i(u)dy^i\right] + dy^i dy^i$$
$$B_{uv} = -\frac{1}{4}\left(f_f^{-1} - 1\right)$$
$$B_{ui} = f_f^{-1}F'^i(u)$$
$$e^{-2\phi} = f_f$$

$$y^i = x^i - F^i(u), v = \tilde{v} + \int^u \left[F'^i(u_0)\right]^2 \left[F'^i(u_0)\right]^2 du_0$$

$$ds^2 = f_f^{-1}(\vec{r},u)du\left[d\tilde{v} - 2\left(f_f(\vec{r},u) - 1\right)F^i(u)dx^i + k(\vec{r},u)du\right] + dx^i dx^i$$

$$k(\vec{r},u) = \tilde{k}(\vec{r},u) + \left(f_f - 1\right)(F'(u))^2$$

$$B_{uv} = -\frac{1}{4}\left(f_f^{-1}(\vec{r},u) - 1\right)$$

$$B_{ui} = \left(f_f^{-1}(\vec{r},u) - 1\right)F^i(u)$$

$$f_f(\vec{r},u) = f_f(\vec{r} - \vec{F}(u))$$

$$k(\vec{r},u) = k(\vec{r} - \vec{F}(u))$$

$$k(r) = P(u)2\pi\alpha' c_f^{10}/r^6$$

$$\epsilon_R^0 = \Gamma^0\Gamma^9\epsilon_R^0, \epsilon_L^0 = 0$$

$$g_E = e^{-\phi/2}G_{\text{supergravity}}$$

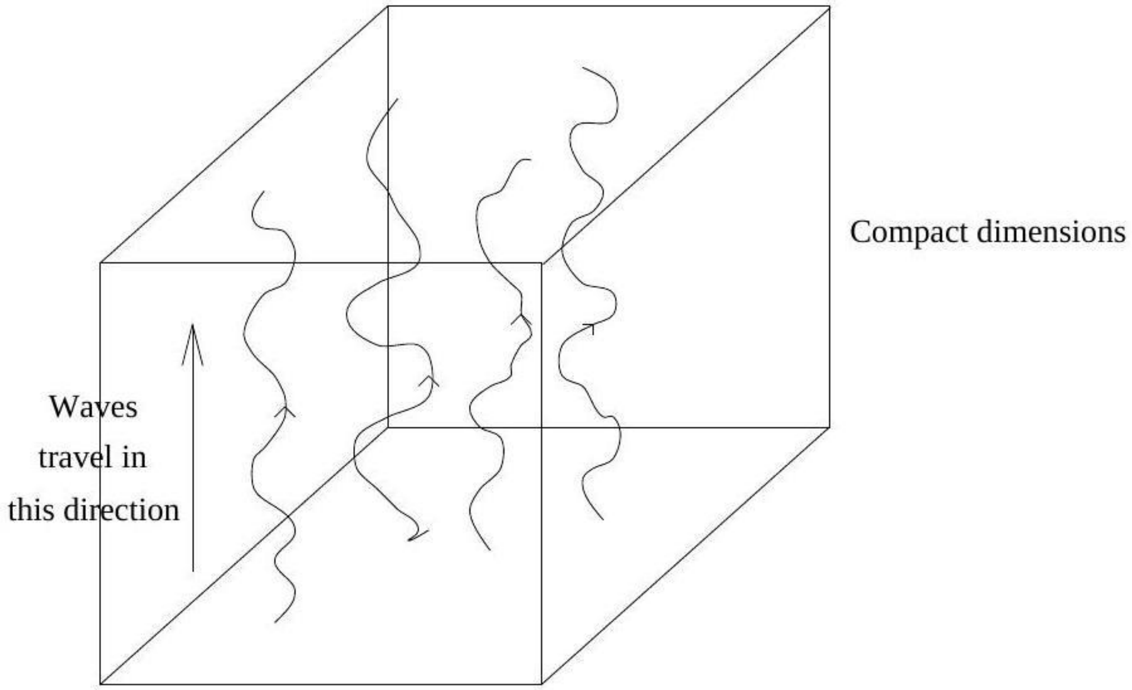
$$g_{E\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$P_i = \frac{m}{2\pi\alpha'}F'^i(u)$$

$$(\Theta_{\alpha\beta}) = \begin{pmatrix} \frac{m}{2\pi\alpha'} + P(u) & -P(u) \\ -P(u) & -\frac{m}{2\pi\alpha'} + P(u) \end{pmatrix}.$$

$$M^{12} \sim (f'^1 f^2 - f'^2 f^1)(u)$$

$$\frac{P(u)m}{2\pi\alpha} = \frac{m^2}{(2\pi\alpha')^2}F'^i(u)^2.$$



$$K = \frac{c_P N}{r^{7-p}}$$

$$c_P = \frac{\alpha'}{R_9^2} c_f$$

$$-dt^2 + dx_9^2 \to -dt^2 + dx_9^2 + k(dt - dx_9)^2$$

$$\epsilon_R^0 = \Gamma^0 \Gamma^9 \epsilon_R^0, \epsilon_L^0 = \Gamma^0 \Gamma^9 \epsilon_L^0$$

$$f = 1 + \sum_i c_p / (\vec{r} - \vec{r}_i)^{7-p}$$

$$f = 1 + \sum_{\vec{n} \in \text{ Lattice}} \frac{c}{(\vec{r} - 2\pi R_i \vec{n})^{7-p}}$$

$$f = 1 + c_P^{(d)} / r^{d-3}$$

$$f_f^{(d)} = 1 + \frac{c_f^{(d)} m}{\rho^{d-3}}, \text{ where } c_f^{(d)} = \frac{16\pi G_N^d R_9}{\alpha'(d-3)\omega_{d-2}}$$

$$e^{-2\phi_d} = e^{-2\phi_{10}} \sqrt{G_{99}} = \sqrt{f_f^{(d)}(1 + k^d)}$$

$$ds_E^2 = -\frac{1}{\left[f_f^{(d)}(1 + k^d)\right]^{\frac{d-3}{d-2}}} dt^2 + \left[f_f^{(d)}(1 + k^d)\right]^{\frac{1}{d-2}} d\vec{x}^2$$

$$g_E = e^{-\phi/2} G$$

$$\tilde{g}_E = e^{-\frac{(\phi-\phi_\infty)}{2}} G = g^{1/2} g_E$$

$$M_E = g^{1/4} M$$

$$G_N^d = G_N^{10}/V_{10-d}$$

$$M_f = \frac{R_9}{\alpha'}.$$

$$g'^{1/4} M' = M'_E = M_E = g^{1/4} \frac{R_9}{\alpha'}$$

$$M^{1D} = \frac{R_9}{g\alpha'}$$

$$M = \frac{R_9}{g\alpha'} = \frac{R_9 R'_8}{g'\alpha'^{3/2}}$$

$$M^{pD} = \frac{R_{10-p} \cdots R_9}{g\alpha'^{(p+1)/2}}$$

$$M^{s5} = \frac{R_5 \cdots R_9}{g^2 \alpha'^3}$$

$$\tilde{g}_{E00} \sim \frac{16\pi G_N^d M}{(d-2)\omega_{d-2}} \frac{1}{r^{d-3}} = \frac{d-3}{d-2} \frac{c^{(d)}}{r^{d-3}}$$

$$S_n, \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$G_N^{10} = 8\pi^6 g^2 \alpha'^4$$

$$c_1^{(5)} = \frac{4G_N^5 R_9}{\pi\alpha' g}, c_5^{(5)} = g\alpha', c_P^{(5)} = \frac{4G_N^5}{\pi R_9}$$

$$c_2^{(4)} = \frac{4G_N^4 R_4 R_9}{g\alpha'^{3/2}}, \quad c_5^{(4)} = \frac{\alpha'}{2R_4}$$
$$c_6^{(4)} = \frac{g\alpha'^{1/2}}{2}, \quad c_P^{(4)} = \frac{4G_N^4}{R_9}$$

$$ds_{str}^2 = f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} (-dt^2 + dx_9^2 + k(dt - dx_9)^2)$$
$$+ f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dx_1^2 + \cdots + dx_4^2) + f_1^{\frac{1}{2}} f_5^{-\frac{1}{2}} (dx_5^2 + \cdots + dx_8^2)$$
$$e^{-2(\phi_{10}-\phi_\infty)} = f_5 f_1^{-1}$$
$$B'_{09} = \frac{1}{2}(f_1^{-1} - 1)$$
$$H'_{ijk} = (dB')_{ijk} = \frac{1}{2} \epsilon_{ijkl} \partial_l f_5, i, j, k, l = 1,2,3,4$$

$$f_1 = 1 + \frac{c_1^{(5)} Q_1}{x^2}, f_5 = 1 + \frac{c_5^{(5)} Q_5}{x^2}, k = \frac{c_P^{(5)} N}{x^2}$$

$$H_3^{\prime RR} = dB_2^{\prime RR}$$

$$F_2 = *_5 H_3^{\prime RR}. Q_5$$

$$\epsilon_L \left(\Gamma^{11} \epsilon_{R,L} = \epsilon_{R,L}\right)$$

$$\epsilon_R = \Gamma^0 \Gamma^9 \epsilon_L, \epsilon_R = \Gamma^0 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_L$$

$$\Gamma^0 \Gamma^9 \epsilon_R = \epsilon_R, \Gamma^0 \Gamma^9 \epsilon_L = \epsilon_L$$

$$\epsilon_L = \epsilon_R = \epsilon_{SO(1,1)}^{+} \epsilon_{SO(4)}^{+} \epsilon_{SO(4)}^{+}$$

$$g_E^5 = e^{-4\phi_5/3} G_{\text{string}}^5$$

$$ds_E^2 = -\frac{1}{(f_1 f_5 (1+k))^{\frac{2}{3}}} dt^2 + (f_1 f_5 (1+k))^{\frac{1}{3}} (dx_1^2 + \cdots + dx_4^2)$$

$$S_e = \frac{A_H}{4G_N^5} = 2\pi \sqrt{N Q_1 Q_5}$$

$$c_P N = c_1 Q_1 = c_5 Q_5 = r_e^2$$

$$e^{-2\phi_{10}} = f_p^{\frac{p-3}{2}}$$

$$ds^2 = -\lambda dt^2 + \lambda^{-1} dr^2 + r^2 d\Omega_3^2$$
$$\lambda = \left(1 - \frac{r_+^2}{r^2}\right)\left(1 - \frac{r_-^2}{r^2}\right)$$

$$M = \frac{3\pi}{8G_N^5}(r_+^2 + r_-^2), Q = \frac{3\pi}{4G_N^5} r_+ r_-$$

$$e^{-2(\phi - \phi_\infty)} = \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)\left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1}$$
$$ds_{str}^2 = \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1/2}\left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{-1/2} [-dt^2 + dx_9^2$$
$$+ \frac{r_0^2}{r^2} (\cosh \sigma dt + \sinh \sigma dx_9)^2 + \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)(dx_5^2 + \cdots + dx_8^2)]$$
$$+ \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{1/2}\left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{1/2}\left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right]$$

$$Q_1 = \frac{V}{4\pi^2 g}\int e^{\phi_6} * H' = \frac{V r_0^2}{2g}\sinh\ 2\alpha$$
$$Q_5 = \frac{1}{4\pi^2 g}\int H' = \frac{r_0^2}{2g}\sinh\ 2\gamma$$
$$N = \frac{R^2 V r_0^2}{2g^2}\sinh\ 2\sigma$$

$$ds_5^2 = -\lambda^{-2/3}\left(1-\frac{r_0^2}{r^2}\right)dt^2 + \lambda^{1/3}\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2\right]$$

$$\lambda = \left(1+\frac{r_0^2\sinh^2\ \alpha}{r^2}\right)\left(1+\frac{r_0^2\sinh^2\ \gamma}{r^2}\right)\left(1+\frac{r_0^2\sinh^2\ \sigma}{r^2}\right)$$

$$E = \frac{RVr_0^2}{2g^2}(\cosh\ 2\alpha + \cosh\ 2\gamma + \cosh\ 2\sigma).$$

$$S = \frac{A_{10}}{4G_N^{10}} = \frac{A_5}{4G_N^5} = \frac{2\pi RVr_0^3}{g^2}\cosh\ \alpha\cosh\ \gamma\cosh\ \sigma.$$

$$T = \frac{1}{2\pi r_0\cosh\ \alpha\cosh\ \gamma\cosh\ \sigma}$$

$$P_1 = \frac{RVr_0^2}{2g^2}\left[\cosh\ 2\sigma - \frac{1}{2}(\cosh\ 2\alpha + \cosh\ 2\gamma)\right],$$
$$P_2 = \frac{RVr_0^2}{2g^2}(\cosh\ 2\alpha - \cosh\ 2\gamma).$$

$$E_{ext} = \frac{R|Q_1|}{g} + \frac{RV|Q_5|}{g} + \frac{|N|}{R},$$
$$S_{ext} = 2\pi\sqrt{|Q_1 Q_5 N|}$$
$$T_{ext} = 0$$
$$P_{1ext} = \frac{|N|}{R} - \frac{R|Q_1|}{2g} - \frac{RV|Q_5|}{2g},$$
$$P_{2ext} = \frac{R|Q_1|}{g} - \frac{RV|Q_5|}{g}.$$

$$M = \frac{R}{g}, P_1 = -\frac{R}{2g}, P_2 = \frac{R}{g}$$

$$M = \frac{RV}{g}, P_1 = -\frac{RV}{2g}, P_2 = -\frac{RV}{g}$$

$$M = \frac{1}{R}, P_1 = \frac{1}{R}, P_2 = 0$$

$$N_1 = \frac{Vr_0^2}{4g}e^{2\alpha}$$
$$N_{\bar{1}} = \frac{Vr_0^2}{4g}e^{-2\alpha}$$
$$N_5 = \frac{r_0^2}{4g}e^{2\gamma}$$
$$N_{\bar{5}} = \frac{r_0^2}{4g}e^{-2\gamma}$$
$$N_R = \frac{r_0^2R^2V}{4g^2}e^{2\sigma}$$
$$N_L = \frac{r_0^2R^2V}{4g^2}e^{-2\sigma}$$

$$E = \frac{R}{g}(N_1 + N_{\bar{1}}) + \frac{RV}{g}(N_5 + N_{\bar{5}}) + \frac{1}{R}(N_R + N_L)$$

$$V = \left(\frac{N_1N_{\bar{1}}}{N_5N_{\bar{5}}}\right)^{1/2}$$
$$R = \left(\frac{g^2N_RN_L}{N_1N_{\bar{1}}}\right)^{1/4}$$

$$\frac{R}{g}(N_1 + N_{\bar{1}}) = \frac{RV}{g}(N_5 + N_{\bar{5}}) = \frac{1}{R}(N_R + N_L).$$

$$S = 2\pi\left(\sqrt{N_1} + \sqrt{N_{\bar{1}}}\right)\left(\sqrt{N_5} + \sqrt{N_{\bar{5}}}\right)\left(\sqrt{N_L} + \sqrt{N_R}\right)$$

$$S = 2\pi|T_{ABC}V^AV^BV^C|^{1/2}$$

$$S = 2\pi\sum_{i,j,k}\left|T_{ABC}V_i^AV_j^BV_k^C\right|^{1/2}$$

$$J_1 = \frac{1}{2}(F_R + F_L), J_2 = \frac{1}{2}(F_R - F_L)$$

$$[1]\begin{bmatrix}1-1\\5\end{bmatrix}, [1]\begin{bmatrix}1-6\\6\\6\\4\end{bmatrix}$$

$$S_{ext} = 2\pi\sqrt{NQ_1Q_5 - J_1J_2}$$

$$e^{-2\phi} = f_2^{-1/2}f_6^{3/2}$$

$$e^{-2\phi} = f_2^{-1/2}f_6^{3/2}f_{s5}^{-1}$$

$$T^6 = T^4 \times S_1' \times S_1$$

$$\Gamma^4\epsilon_R = \epsilon_L = \epsilon_{SO(1,1)}^{+}\epsilon_{SO(4)}^{+}\epsilon_{SO(4)}^{+}$$

$$ds_{str}^2 = f_2^{-\frac{1}{2}} f_6^{-\frac{1}{2}}(-dt^2 + dx_9^2 + k(dt - dx_9)^2) + f_{s5} f_2^{-\frac{1}{2}} f_6^{-\frac{1}{2}} dx_4^2$$
$$+ f_2^{\frac{1}{2}} f_6^{-\frac{1}{2}}(dx_5^2 + \cdots + dx_8^2) + f_{s5} f_2^{\frac{1}{2}} f_6^{\frac{1}{2}}(dx_1^2 + \cdots + dx_3^2)$$
$$e^{-2(\phi_{10} - \phi_\infty)} = f_{s5}^{-1} f_2^{-\frac{1}{2}} f_6^{\frac{3}{2}},$$
$$H_{ij4} = \frac{1}{2}\epsilon_{ijk}\partial_k f_{s5} \; i,j,k = 1,2,3$$
$$C_{049} = \frac{1}{2}(f_2^{-1} - 1),$$
$$(dA)_{ij} = \frac{1}{2}\epsilon_{ijk}\partial_k f_6 \; i,j,k = 1,2,3$$

$$f_2 = 1 + \frac{c_2^{(4)} Q_2}{r}, f_5 = 1 + \frac{c_{s5}^{(4)} Q_5}{r}, f_6 = 1 + \frac{c_6^{(4)} Q_6}{r}, k = \frac{c_P^{(4)} N}{r}$$

$$S = \frac{A_4}{4G_N^4} = 2\pi\sqrt{Q_2 Q_5 Q_6 N}$$

$$ds^2 = -\chi^{-1/2}(r)\left(1 - \frac{r_0}{r}\right)dt^2 + \chi^{1/2}(r)\left[\left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$
$$\chi(r) = \left(1 + \frac{r_0 \sinh^2 \alpha_2}{r}\right)\left(1 + \frac{r_0 \sinh^2 \alpha_5}{r}\right)\left(1 + \frac{r_0 \sinh^2 \alpha_6}{r}\right)\left(1 + \frac{r_0 \sinh^2 \alpha_p}{r}\right)$$

$$Q_2 = \frac{r_0 V}{g}\sinh 2\alpha_2$$
$$Q_5 = r_0 R_4 \sinh 2\alpha_5$$
$$Q_6 = \frac{r_0}{g}\sinh 2\alpha_6$$
$$N = \frac{r_0 V R_9^2 R_4}{g^2}\sinh 2\alpha_p$$

$$G_N^4 = g^2/(8VR_4R_9)$$

$$M = \frac{r_0 V R_4 R_9}{g^2}(\cosh 2\alpha_2 + \cosh 2\alpha_5 + \cosh 2\alpha_6 + \cosh 2\alpha_p)$$

$$S = \frac{A_4}{4G_N^4} = \frac{8\pi r_0^2 V R_4 R_9}{g^2}\cosh\alpha_2 \cosh\alpha_5 \cosh\alpha_6 \cosh\alpha_p$$

$$P_1 = \frac{r_0 V R_4 R_9}{g^2}(\cosh 2\alpha_2 + \cosh 2\alpha_6 - \cosh 2\alpha_5 - \cosh 2\alpha_p)$$
$$P_2 = \frac{r_0 V R_4 R_9}{g^2}(\cosh 2\alpha_2 - \cosh 2\alpha_6)$$
$$P_3 = \frac{r_0 V R_4 R_9}{g^2}(\cosh 2\alpha_5 - \cosh 2\alpha_p)$$

$$M = P_1 = P_2 = \frac{R_4 R_9}{g}, P_3 = 0$$

$$M = P_1 = -P_2 = \frac{VR_4R_9}{g}, P_3 = 0$$

$$M = -P_1 = P_3 = \frac{VR_9}{g^2}, P_2 = 0$$

$$M = -P_1 = -P_3 = \frac{1}{R_9}, P_2 = 0$$

$$N_R = \frac{r_0 V R_9^2 R_4}{2g^2} e^{2\alpha_p}, \qquad N_L = \frac{r_0 V R_9^2 R_4}{2g^2} e^{-2\alpha_p}$$
$$N_2 = \frac{r_0 V}{2g} e^{2\alpha_2}, \qquad N_{\bar{2}} = \frac{r_0 V}{2g} e^{-2\alpha_2}$$
$$N_5 = \frac{r_0 R_4}{2} e^{2\alpha_5}, \qquad N_{\bar{5}} = \frac{r_0 R_4}{2} e^{-2\alpha_5}$$
$$N_6 = \frac{r_0}{2g} e^{2\alpha_6}, \qquad N_{\bar{6}} = \frac{r_0}{2g} e^{-2\alpha_6}$$

$$M = \frac{1}{R_1}(N_R + N_L) + \frac{R_9R_4}{g}(N_2 + N_{\bar{2}}) + \frac{VR_9}{g^2}(N_5 + N_{\bar{5}}) + \frac{VR_9R_4}{g}(N_6 + N_{\bar{6}}),$$

$$V = \sqrt{\frac{N_2 N_{\bar{2}}}{N_6 N_{\bar{6}}}}, R_4 = \sqrt{\frac{N_5 N_{\bar{5}}}{g^2 N_6 N_{\bar{6}}}}, R_9^2 R_4 = \sqrt{\frac{g^2 N_R N_L}{N_2 N_{\bar{2}}}}.$$

$$S = 2\pi\left(\sqrt{N_R} + \sqrt{N_L}\right)\left(\sqrt{N_2} + \sqrt{N_{\bar{2}}}\right)\left(\sqrt{N_5} + \sqrt{N_{\bar{5}}}\right)\left(\sqrt{N_6} + \sqrt{N_{\bar{6}}}\right).$$

$$S = 2\pi \sum_{i,j,k,l} \sqrt{T_{ABCD} V_i^A V_j^B V_k^C V_l^D},$$

$$S = 2\pi\left(\sqrt{N_R N_2 N_5 N_6} + \sqrt{N_L N_2 N_5 N_6 - J^2}\right)$$

$$D_{IJ} = \frac{1}{2}\epsilon_{IJKL} D_{KL}$$

$$V \sim \sum_a D_{12}^{a2} + D_{13}^{a2} + D_{14}^{a2}$$

$$D^a \sim \phi^\dagger T^a \phi'$$

$$V = \mathrm{Tr} F_{IJ} F^{IJ} = \sum_{IJ} \mathrm{Tr}[A_I, A_J]^2 = \sum_a D_{12}^{a2} + D_{13}^{a2} + D_{14}^{a2}$$

$$D_{IJ}^a T^a = [A_I, A_J] + \frac{1}{2}\epsilon_{IJKL}[A_K, A_L]$$

$$D_{IJ}^{a} = f_{bc}^{a}\left(A_I^b A_J^c + \frac{1}{2}\epsilon_{IJKL}A_K^b A_L^c\right) + \chi^\dagger T^a \Gamma_{IJ}\chi$$
$$V = \sum_{aIJ} D_{IJ}^{a2}$$

$$S = \frac{1}{g}\int \mathrm{Tr}\left(F_{\alpha\beta}F^{\alpha\beta}\right) + \mathrm{Tr}\left(F'_{\alpha\beta}F'^{\alpha\beta}\right) + \mathrm{Tr}[(\partial_\alpha A_I + [A_\alpha, A_I])^2]$$
$$+\mathrm{Tr}[(\partial_\alpha A'_I + [A'_\alpha, A'_I])^2] + |(\partial_\alpha + A_\alpha^a T^a + A'^a_\alpha T^a)\chi|^2 + \sum_{aIJ} D_{IJ}^{a2}$$

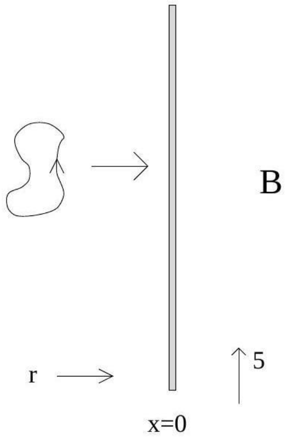
$$\mathrm{SO}(4)_E \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$$

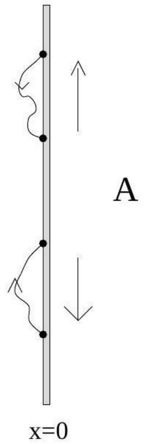
$$S_e = \sqrt{\pi(2N_B + N_F)EL/6} = 2\pi\sqrt{Q_1 Q_5 N},$$

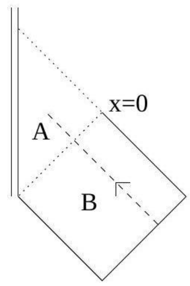
$$E = \frac{l^2}{2}$$

$$(T^4)^{Q_1Q_5}/S(Q_1Q_5)$$

$$S = 2\pi\sqrt{N'} = 2\pi\sqrt{NQ_1Q_5}$$







Penrose Diagram

$$\sqrt{N + \delta N_R} - \sqrt{N}$$

$$\delta N_R/N_L$$

$$\sqrt{\delta N_R/N_L}$$

$$\left.\frac{\Delta S}{S_e}\right|_{\text{oscill}} = \sqrt{\frac{\delta N_R}{N}}.$$

$$\left.\frac{\Delta S}{S}\right|_{\text{gravity}} = \sqrt{\frac{\delta N_{\bar{1}}}{Q_1}}$$

$$\left.\frac{\Delta S}{S}\right|_{\text{total}} = \sqrt{\frac{\delta N_R}{N}} + \sqrt{\frac{\delta N_{\bar{1}}}{Q_1}} + \sqrt{\frac{\delta N_{\bar{5}}}{Q_5}}$$

$$\left.\frac{\delta S_e}{S}\right|_{\text{supergravity}} = \frac{3}{\sqrt{2}}\sqrt{\frac{\delta M}{M_e}}$$

$$d\Gamma \sim \frac{d^4k}{k_0}\frac{1}{p_0^R p_0^L VR}\delta\left(k_0 - (p_0^R + p_0^L)\right)\sum_{i,f}\left|\left\langle \Psi_f \middle| H_{int} \middle| \Psi_i \right\rangle\right|^2$$

$$\mathcal{A} \sim g k_0^2.$$

$$\rho_R(n) = \frac{1}{N_i}\sum_i \langle \Psi_i | a_n^{R\dagger} a_n^R | \Psi_i \rangle$$

$$E_R = \delta N_R / R_9 = \delta N_R' / R_9 Q_1 Q_5$$

$$Z = \sum_{N'} q^{N'} d(N') = \sum_{N'} q^{N'} e^{2\pi\sqrt{N'}}$$

$$\delta N_R Q_1 Q_5 = \delta N_R' = q\frac{\partial}{\partial q}\log Z$$

$$\log q = -\pi\sqrt{1/Q_1 Q_5 \delta N_R}$$

$$\rho_R(k_0) = \frac{q^n}{1-q^n} = \frac{e^{\frac{-k_0}{2T_R}}}{1-e^{\frac{-k_0}{2T_R}}}.$$

$$T_R = \frac{1}{\pi}\frac{1}{R}\sqrt{\frac{\delta N_R}{Q_1 Q_5}}.$$

$$T_R = \frac{r_0^2 R_9 V}{2 g S_e}$$

$$T_L = \frac{1}{\pi}\frac{1}{R}\sqrt{\frac{N}{Q_1 Q_5}}.$$

$$\rho_L \sim \frac{2T_L}{k_0} = \frac{2}{\pi k_0 R}\sqrt{\frac{N}{Q_1 Q_5}}.$$

$$d\Gamma \sim \frac{d^4k}{k_0}\frac{1}{p_0^R p_0^L RV}|\mathcal{A}|^2 Q_1 Q_5 R \rho_R(k_0)\rho_L(k_0)$$

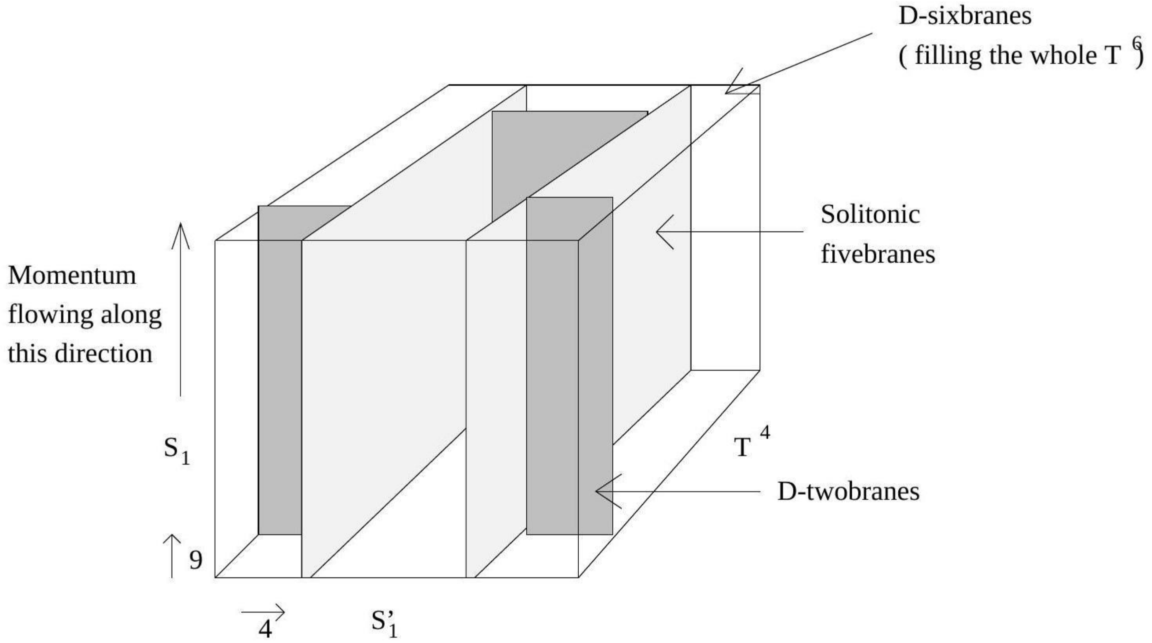
$$\sum_{n} \delta(k_0 - 2n/RQ_1Q_5) \sim RQ_1Q_5$$

$$d\Gamma \sim \frac{g^2}{RV}\sqrt{Q_1Q_5N}\frac{e^{\frac{-k_0}{2T_R}}}{1-e^{\frac{-k_0}{2T_R}}}d^4k \sim (\text{ Area })\frac{e^{\frac{-k_0}{2T_R}}}{1-e^{\frac{-k_0}{2T_R}}}d^4k$$

$$T_H = 2T_R$$

$$\delta M_{\min} \sim \frac{G_N^5}{r_e^4}$$

$$\delta M_{\min} \sim \frac{1}{Q_1Q_5R}$$



$$S = 2\pi\sqrt{\frac{(2N_B + N_F)ER_9}{12}}$$

$$S_{\text{stat}} = 2\pi\sqrt{Q_2Q_5Q_6N}$$

$$\delta M \sim \frac{G_N^4}{r_e^3} \sim \frac{1}{RQ_2Q_5Q_6}$$

$$S = 2\pi\sqrt{N_2N_5N_6}(\sqrt{N_R} + \sqrt{N_L})$$

$$S = 2\pi\sqrt{\frac{c}{6}}\left(\sqrt{N_R} + \sqrt{\tilde{N}_L}\right)$$

$$\tilde{n}_L = n_L - 6J^2/c$$

$$S = 2\pi\sqrt{J^2 + NQ_2Q_5Q_6}$$

$$G_{\mu\nu} = 2\partial_{[\mu}A_{\mu]}, H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}, F'_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]} + 8A_{[\mu}H_{\nu\rho\sigma]}$$

$$\mathcal{L} = \int\ d^2\Theta 2\mathcal{E}\left[-\frac{1}{8}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)N(\mathcal{R},\overline{\mathcal{R}}) + \mathcal{F}(\mathcal{R})\right] + \text{ h.c.}$$

$$N = \frac{12}{M^2}\mathcal{R}\overline{\mathcal{R}} - \frac{\xi}{2}(\mathcal{R}\overline{\mathcal{R}})^2, \mathcal{F} = \alpha + 3\beta\mathcal{R},$$

$$\begin{aligned} e^{-1}\mathcal{L} =& -\frac{1}{12}\left[3(\beta+\bar{\beta}) - \frac{24}{M^2}|X|^2 + 11\xi|X|^4 - \frac{2}{9}\left(\frac{6}{M^2} - \xi|X|^2\right)b_m b^m\right]\left(R + \frac{2}{3}b_m b^m\right) \\ &+ \left(\frac{6}{M^2} - \xi|X|^2\right)\left(\frac{1}{72}R^2 - 2\partial_m X\partial^m\bar{X} + \frac{1}{18}(\nabla_m b^m)^2 - \frac{1}{162}(b_m b^m)^2\right) \\ &+ \frac{i}{2}(\beta-\bar{\beta})\nabla_m b^m - \frac{i}{3}\left(\frac{12}{M^2} - \xi|X|^2\right)b^m(\bar{X}\partial_m X - X\partial_m\bar{X}) - U(X,\bar{X}), \end{aligned}$$

$$U = -6(\alpha\bar{X} + \bar{\alpha}X) - 6(\beta+\bar{\beta})|X|^2 - \frac{48}{M^2}|X|^4 + 18\xi|X|^6$$

$$\frac{M^4\xi}{144} \equiv \zeta \text{ and } |X| \equiv \frac{M}{2\sqrt{6}}\sigma$$

$$e^{-1}\mathcal{L} = \frac{1}{2}f(R,\sigma) - \frac{1}{2}(1-\zeta\sigma^2)(\partial\sigma)^2 - U,$$

$$f(R,\sigma) = \left(1 + \frac{1}{6}\sigma^2 - \frac{11}{24}\zeta\sigma^4\right)R + \frac{1}{6M^2}(1-\zeta\sigma^2)R^2$$
$$U = \frac{1}{2}M^2\sigma^2\left(1 - \frac{1}{6}\sigma^2 + \frac{3}{8}\zeta\sigma^4\right)$$

$$e^{-1}\mathcal{L} = \frac{1}{2}[f_\chi(R-\chi) + f] - \frac{1}{2}(1-\zeta\sigma^2)(\partial\sigma)^2 - U,$$

$$f_\chi \equiv \frac{\partial f}{\partial\chi}$$

$$f \equiv f(\chi,\sigma)$$

$$g_{mn} \to f_\chi^{-1}g_{mn}, e \to f_\chi^{-2}e$$
$$ef_\chi R \to eR - \frac{3}{2}ef_\chi^{-2}\left(\partial f_\chi\right)^2$$

$$f_\chi = A + B\chi \text{ with}$$
$$A \equiv 1 + \frac{1}{6}\sigma^2 - \frac{11}{24}\zeta\sigma^4 \text{ and } B \equiv \frac{1}{3M^2}(1-\zeta\sigma^2),$$

$$f_\chi = \exp\left[\sqrt{\frac{2}{3}}\varphi\right]$$

$$\chi = \frac{1}{B}\left(e^{\sqrt{\frac{2}{3}}\varphi} - A\right) \text{ and } f = \frac{1}{2B}\left(e^{2\sqrt{\frac{2}{3}}\varphi} - A^2\right)$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}(\partial\sigma)^2 - V$$

$$V = \frac{1}{4B}\left(1 - Ae^{-\sqrt{\frac{2}{3}}\varphi}\right)^2 + e^{-2\sqrt{\frac{2}{3}}\varphi}U$$

$$= \frac{3M^2}{4(1-\zeta\sigma^2)}\left[1 - e^{-\sqrt{\frac{2}{3}}\varphi} - \frac{1}{6}\sigma^2\left(1 - \frac{11}{4}\zeta\sigma^2\right)e^{-\sqrt{\frac{2}{3}}\varphi}\right]^2 + \frac{M^2}{2}e^{-2\sqrt{\frac{2}{3}}\varphi}\sigma^2\left(1 - \frac{1}{6}\sigma^2 + \frac{3}{8}\zeta\sigma^4\right)$$

$$\sigma^2 > 1/\zeta$$

$$\sigma^2 = 1/\zeta$$

$$\sigma^2 < 1/\zeta)$$

$$\mathcal{L} = \int d^2\Theta 2\mathcal{E}\left\{-\frac{1}{8}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)N(\mathbf{S},\bar{\mathbf{S}}) + \mathcal{F}(\mathbf{S}) + 6\mathbf{T}(\mathbf{S} - \mathcal{R})\right\} + \text{ h.c.}$$

$$\int d^2\Theta 2\mathcal{E}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)(\mathbf{T} + \bar{\mathbf{T}}) + \text{ h.c. } = -16\int d^2\Theta 2\mathcal{E}\mathcal{R}\mathbf{T} + \text{ h.c.}$$

$$\mathcal{L} = \int d^2\Theta 2\mathcal{E}\left\{\frac{3}{8}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)\left[\mathbf{T} + \bar{\mathbf{T}} - \frac{1}{3}N(\mathbf{S},\bar{\mathbf{S}})\right] + \mathcal{F}(\mathbf{S}) + 6\mathbf{T}\mathbf{S}\right\} + \text{ h.c.}$$

$$\mathcal{L} = \int d^2\Theta 2\mathcal{E}\left[\frac{3}{8}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)e^{-K/3} + W\right] + \text{ h.c.}$$

$$K = -3\log(\mathbf{T} + \bar{\mathbf{T}} - \tilde{N}), \tilde{N} \equiv \frac{1}{3}N = \mathbf{S}\bar{\mathbf{S}} - \frac{3}{2}\zeta(\mathbf{S}\bar{\mathbf{S}})^2$$
$$W = 3M\mathbf{S}\left(\mathbf{T} - \frac{1}{2}\right)$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - K_{i\bar{j}}\partial_m\Phi^i\partial^m\bar{\Phi}^j - e^K\left(K^{i\bar{j}}D_iWD_{\bar{j}}\bar{W} - 3|W|^2\right),$$

$$K_{i\bar{j}} = \begin{pmatrix} K_{T\bar{T}} & K_{T\bar{S}} \\ K_{S\bar{T}} & K_{S\bar{S}} \end{pmatrix} = \frac{3}{P^2}\begin{pmatrix} 1 & -\tilde{N}_{\bar{S}} \\ -\tilde{N}_S & \tilde{N}_S\tilde{N}_{\bar{S}} + P\tilde{N}_{S\bar{S}} \end{pmatrix}$$

$$K^{i\bar{j}} = \begin{pmatrix} K^{T\bar{T}} & K^{T\bar{S}} \\ K^{S\bar{T}} & K^{S\bar{S}} \end{pmatrix} = \frac{P}{3}\begin{pmatrix} P + \tilde{N}^{S\bar{S}}\tilde{N}_S\tilde{N}_{\bar{S}} & \tilde{N}^{S\bar{S}}\tilde{N}_S \\ \tilde{N}^{S\bar{S}}\tilde{N}_{\bar{S}} & \tilde{N}^{S\bar{S}} \end{pmatrix}.$$

$$e^{-1}\mathcal{L}_{\text{kin}} = -\frac{3}{P^2}\left[\partial T\partial\bar{T} - \tilde{N}_S\partial S\partial\bar{T} - \tilde{N}_{\bar{S}}\partial T\partial\bar{S} + \left(\tilde{N}_S\tilde{N}_{\bar{S}} + P\tilde{N}_{S\bar{S}}\right)\partial S\partial\bar{S}\right]$$

$$|S| = \sigma/\sqrt{6}$$

$$P = \exp\left[\sqrt{\frac{2}{3}}\varphi\right]$$

$$e^{-1}\mathcal{L}_{\text{kin}} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}(\partial\sigma)^2$$

$$V = \frac{1}{4B}(1-Ax)^2 + x^2U, \text{ where } \begin{cases} A = 1 + \frac{1}{6}\sigma^2 - \frac{11}{24}\zeta\sigma^4, \\ B = \frac{1}{3M^2}(1-\zeta\sigma^2), \\ U = \frac{M^2}{2}\sigma^2\left(1 - \frac{1}{6}\sigma^2 + \frac{3}{8}\zeta\sigma^4\right). \end{cases}$$

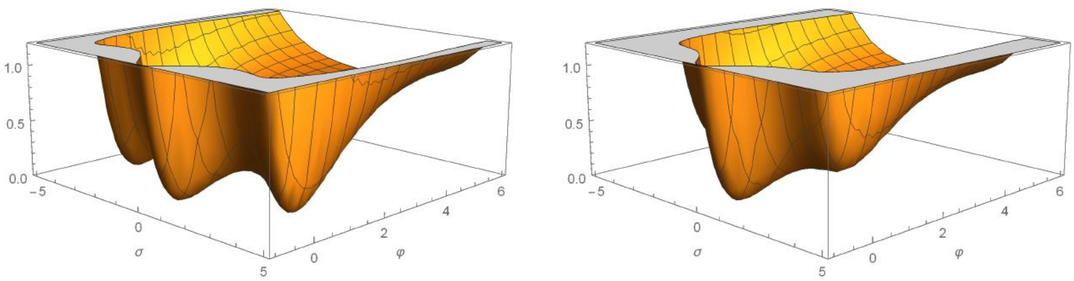
$$\partial_x V = \frac{A}{2B}(Ax-1) + 2xU = 0$$

$$\partial_\sigma V = \frac{2xA'B + (1-Ax)B'}{4B^2}(Ax-1) + x^2U' = 0$$

$$Ax = 1, U = U' = 0$$

$$\sigma^2 = \frac{2}{27\zeta}(2 \pm \sqrt{4 - 162\zeta})$$

$$\sigma^2 = \frac{2}{9\zeta}(1 \pm \sqrt{1 - 54\zeta})$$



$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{2}G_{AB}\partial\phi^A\partial\phi^B - V,$$

$$G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & (1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi} \end{pmatrix}.$$

$$\Box\, \phi^C + \Gamma^C_{AB}\partial\phi^A\partial\phi^B = G^{AC}\partial_A V,$$

$$\Box \equiv \nabla_m\nabla^m$$

$$\Gamma^\sigma_{\sigma\varphi} = -\frac{1}{\sqrt{6}}, \Gamma^\varphi_{\sigma\sigma} = \frac{1}{\sqrt{6}}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}, \Gamma^\sigma_{\sigma\sigma} = -\frac{\zeta\sigma}{1-\zeta\sigma^2}.$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{1}{\sqrt{6}}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}\dot{\sigma}^2 + \partial_\varphi V = 0$$

$$\ddot{\sigma} + 3H\dot{\sigma} - \frac{\zeta\sigma\dot{\sigma}^2}{1-\zeta\sigma^2} - \sqrt{\frac{2}{3}}\dot{\varphi}\dot{\sigma} + \frac{e^{\sqrt{\frac{2}{3}}\varphi}}{1-\zeta\sigma^2}\partial_\sigma V = 0$$

$$3H^2 = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}\dot{\sigma}^2 + V$$

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}\dot{\sigma}^2$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{\dot{\tilde{H}}}{\tilde{H}^2}$$

$$\Sigma^A \equiv \frac{\dot{\phi}^A}{|\dot{\phi}|}, \Omega^A \equiv \frac{\omega^A}{|\omega|},$$

$$|a| \equiv \sqrt{G_{AB}a^A a^B}$$

$$\omega^A \equiv \dot{\Sigma}^A + \Gamma^A_{BC}\Sigma^B\dot{\phi}^C \begin{cases} \omega^\varphi = \dot{\Sigma}^\varphi + \frac{1}{\sqrt{6}}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}\Sigma^\sigma\dot{\sigma} \\ \omega^\sigma = \dot{\Sigma}^\sigma - \frac{1}{\sqrt{6}}(\Sigma^\varphi\dot{\sigma} + \Sigma^\sigma\dot{\varphi}) - \frac{\zeta\sigma}{1-\zeta\sigma^2}\Sigma^\sigma\dot{\sigma} \end{cases}$$

$$\mathcal{M}^A_B \equiv G^{AC}\nabla_B\partial_C V - R^A_{CDB}\dot{\phi}^C\dot{\phi}^D$$

$$R^\varphi_{\sigma\sigma\varphi} = \frac{1}{6}(1-\zeta\sigma^2)e^{-\sqrt{\frac{2}{3}}\varphi}, R^\sigma_{\varphi\varphi\sigma} = \frac{1}{6}.$$

$$\eta_{\Sigma\Sigma} \equiv \frac{\mathcal{M}^A_B\Sigma_A\Sigma^B}{V}, \eta_{\Omega\Omega} \equiv \frac{\mathcal{M}^A_B\Omega_A\Omega^B}{V},$$

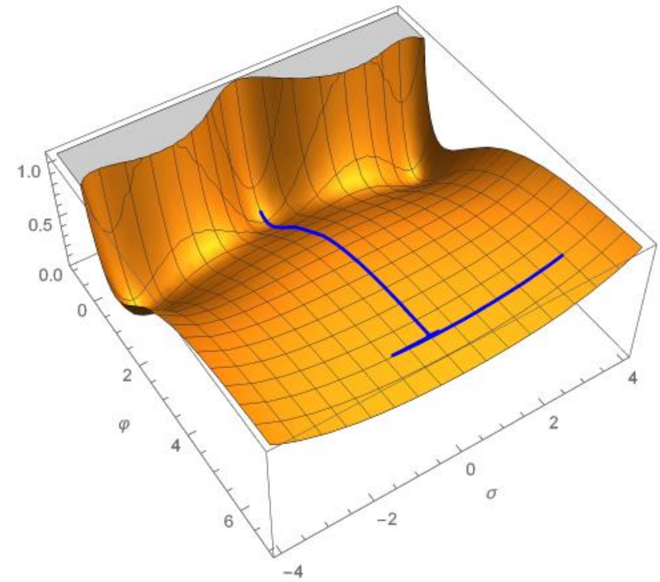
$$T_{SS}(t_1,t_2) \equiv \exp\left[\int_{t_1}^{t_2} dt'\beta(t')H(t')\right]$$

$$T_{RS}(t_1,t_2) \equiv 2\int_{t_1}^{t_2} dt'|\omega(t')|T_{SS}(t_1,t_2)$$

$$\beta(t) \equiv -2\epsilon + \eta_{\Sigma\Sigma} - \eta_{\Omega\Omega} - \frac{4|\omega|^2}{3H^2}$$

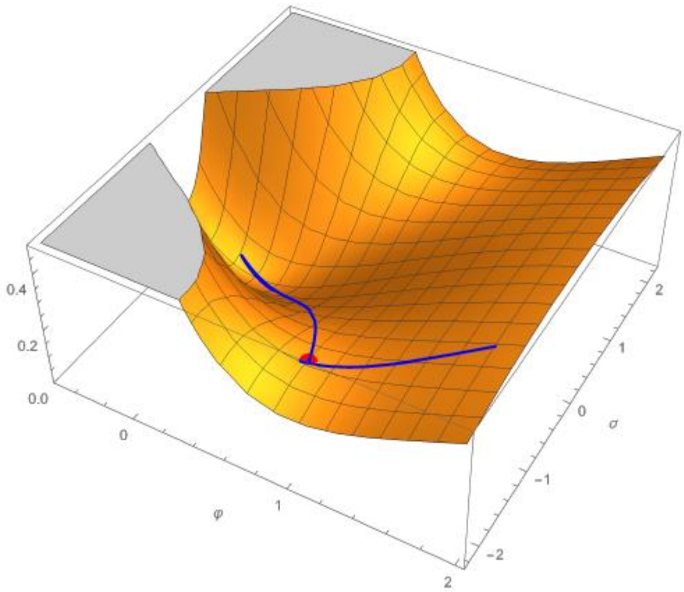
$$n_s = 1 - 6\epsilon + 2\eta_{\Sigma\Sigma} \text{ and } r = \frac{16\epsilon}{1+T_{RS}^2}$$

$$n_s = 0.9649 \pm 0.0042(1\sigma\text{CL}) \text{ and } r < 0.064(2\sigma\text{CL}).$$



$$\varphi(0) = 5, \dot{\sigma}(0) = 79.784527415607, \sigma(0) = \dot{\varphi}(0) = 0.$$

$$N = \frac{12}{M^2}|\mathcal{R}|^2 - \frac{72}{M^4}\zeta|\mathcal{R}|^4 - \frac{768}{M^6}\gamma|\mathcal{R}|^6,$$
$$\mathcal{F} = -3\mathcal{R} + \frac{3\sqrt{6}}{M}\delta\mathcal{R}^2,$$



$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\varphi)^2 - \frac{3M^2}{2}Be^{-\sqrt{\frac{2}{3}}\varphi}(\partial\sigma)^2 - \frac{1}{4B}\left(1 - Ae^{-\sqrt{\frac{2}{3}}\varphi}\right)^2 - e^{-2\sqrt{\frac{2}{3}}\varphi}U$$

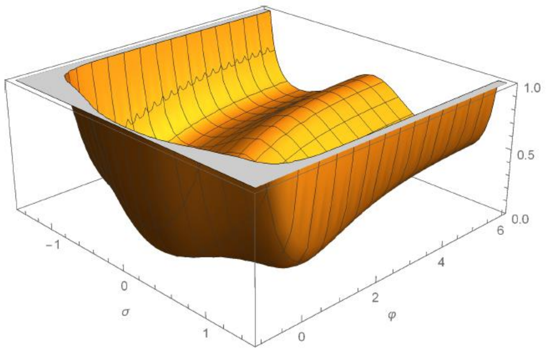
$$
\begin{aligned}
A &= 1 - \delta\sigma + \frac{1}{6}\sigma^2 - \frac{11}{24}\zeta\sigma^4 - \frac{29}{54}\gamma\sigma^6 \\
B &= \frac{1}{3M^2}(1 - \zeta\sigma^2 - \gamma\sigma^4) \\
U &= \frac{M^2}{2}\sigma^2\left(1 + \frac{1}{2}\delta\sigma - \frac{1}{6}\sigma^2 + \frac{3}{8}\zeta\sigma^4 + \frac{25}{54}\gamma\sigma^6\right)
\end{aligned}
$$

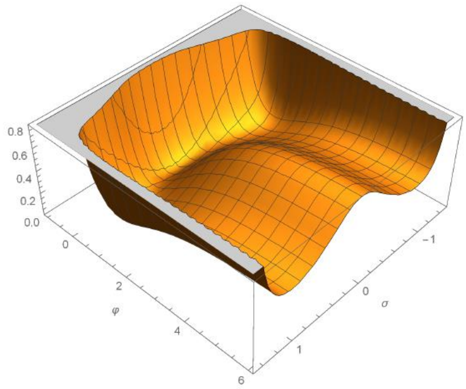
$$0 = \ddot{\varphi} + 3H\dot{\varphi} + \frac{1}{\sqrt{6}}(1 - \zeta\sigma^2 - \gamma\sigma^4)e^{-\sqrt{\frac{2}{3}}\varphi}\dot{\sigma}^2 + \partial_\varphi V$$

$$0 = \ddot{\sigma} + 3H\dot{\sigma} - \frac{\zeta\sigma + 2\gamma\sigma^3}{1 - \zeta\sigma^2 - \gamma\sigma^4}\dot{\sigma}^2 - \sqrt{\frac{2}{3}}\dot{\varphi}\dot{\sigma} + \frac{e^{\sqrt{\frac{2}{3}}\varphi}}{1 - \zeta\sigma^2 - \gamma\sigma^4}\partial_\sigma V$$

$$0 = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(1 - \zeta\sigma^2 - \gamma\sigma^4)e^{-\sqrt{\frac{2}{3}}\varphi}\dot{\sigma}^2 + \dot{H}$$

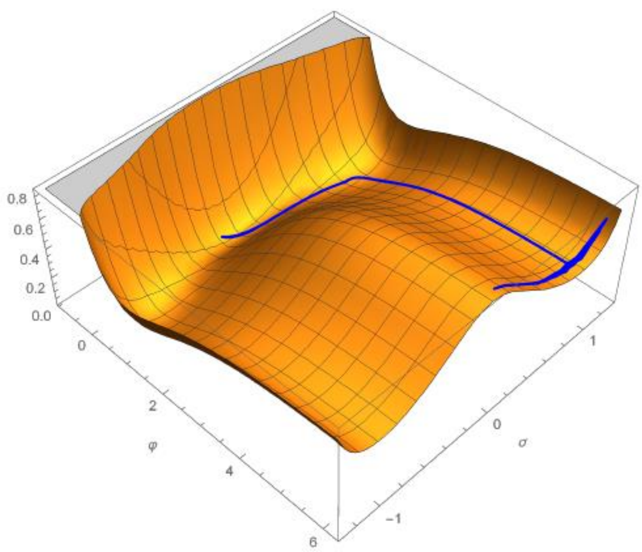
$$0 = V - 3H^2 - \dot{H}$$





$$n_s \approx 0.9545 \text{ and } r_{\max} \approx 0.006$$

$$\partial_\varphi V = \partial_\sigma V = \mathbf{H} = 0$$



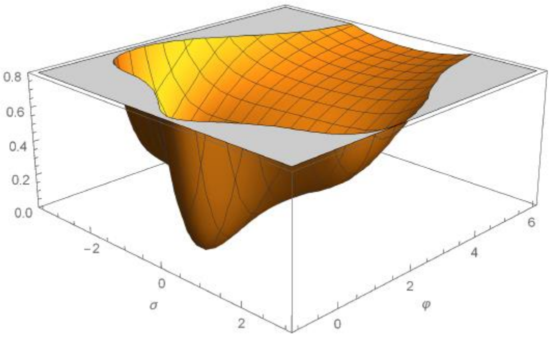
$$M_{\text{PBH}} \simeq \frac{M_{\text{Pl}}^2}{H(t_*)} \exp\left[2(N_{\text{end}} - N_*) + \int_{t_*}^{t_{\text{exit}}} \epsilon(t) H(t) dt\right],$$

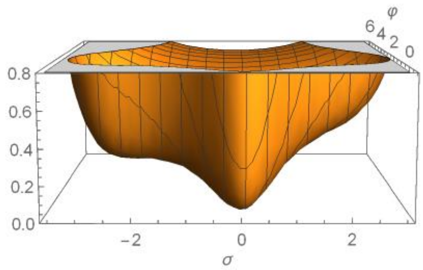
$$\tilde{M}_{\text{PBH}} \simeq 10^{20} \left(\frac{7 \times 10^{12}}{k\text{Mpc}}\right)^2 \text{ g}, \beta_f(k) \simeq \frac{\sigma(k)}{\sqrt{2\pi}\delta_c} e^{-\frac{\delta_c^2}{2\sigma^2(k)}}$$

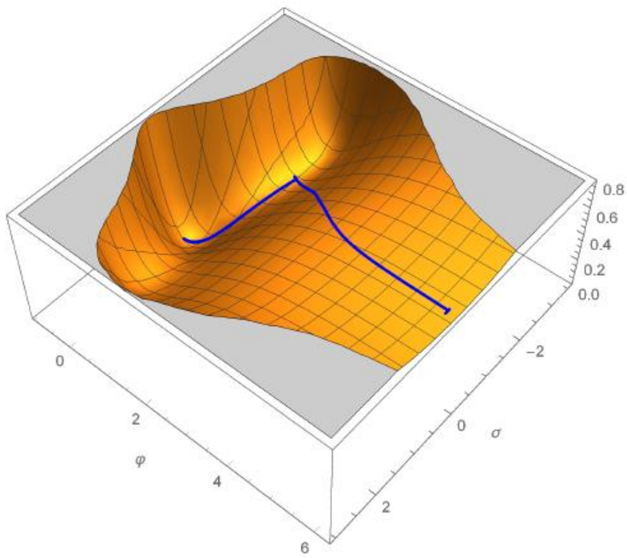
$$\sigma^2(k) = \frac{16}{81} \int \frac{dq}{q} \left(\frac{q}{k}\right)^4 e^{-q^2/k^2} P_\zeta(q)$$

$$\frac{\Omega_{\text{PBH}}(k)}{\Omega_{\text{DM}}} \equiv f(k) \simeq \frac{1.2 \times 10^{24} \beta_f(k)}{\sqrt{\tilde{M}_{\text{PBH}}(k) \text{g}^{-1}}}$$

$$f_{\text{tot}} = \int d\left(\log \tilde{M}_{\text{PBH}}\right) f\left(\tilde{M}_{\text{PBH}}\right)$$







$$\mathcal{L} = \int d^2\Theta 2\mathcal{E}\left[\frac{3}{8}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right) e^{-K\left(\mathbf{\Phi}^i, \overline{\mathbf{\Phi}}^i\right)/3} + W\left(\mathbf{\Phi}^i\right)\right] + \text{ h.c.}$$

$$\mathcal{D}^2 \equiv \mathcal{D}^\alpha \mathcal{D}_\alpha$$

$$\overline{\mathcal{D}}^2 \equiv \overline{\mathcal{D}}_{\dot{\alpha}}\overline{\mathcal{D}}^{\dot{\alpha}}$$

$$\mathbf{\Phi} = \Phi + \sqrt{2}\Theta\chi + \Theta^2 F$$

$$2\mathcal{E} = e[1 + i\Theta\sigma^m\bar{\psi}_m + \Theta^2(6\bar{X} - \bar{\psi}_m\bar{\sigma}^{mn}\bar{\psi}_n)]$$

$$\mathcal{R} = X + \Theta\left(-\frac{1}{6}\sigma^m\bar{\sigma}^n\psi_{mn} - i\sigma^m\bar{\psi}_m X - \frac{i}{6}\psi_m b^m\right) +$$
$$+\Theta^2\left(-\frac{1}{12}R - \frac{i}{6}\bar{\psi}^m\bar{\sigma}^n\psi_{mn} - 4X\bar{X} - \frac{1}{18}b_m b^m + \frac{i}{6}\nabla_m b^m +\right.$$
$$\left.+\frac{1}{2}\bar{\psi}_m\bar{\psi}^m X + \frac{1}{12}\psi_m\sigma^m\bar{\psi}_n b^n - \frac{1}{48}\varepsilon^{abcd}(\bar{\psi}_a\bar{\sigma}_b\psi_{cd} + \psi_a\sigma_b\bar{\psi}_{cd})\right)$$

$$e \equiv \det(e_m^a)$$

$$\psi_{mn} \equiv \tilde{D}_m\psi_n - \tilde{D}_n\psi_m$$

$$\tilde{D}_m\psi_n \equiv \partial_m\psi_n + \psi_n\omega_m$$

$$e^{-1}\mathcal{L} = \frac{1}{2}R - K_{i\bar{j}}\partial_m\Phi^i\partial^m\bar{\Phi}^j - e^K\left(K^{i\bar{j}}D_iWD_{\bar{j}}\bar{W} - 3|W|^2\right)$$

$$K_{i\bar{j}} \equiv \frac{\partial^2 K}{\partial\Phi^i\partial\bar{\Phi}^j}, K^{i\bar{j}} \equiv K_{i\bar{j}}^{-1}, D_iW \equiv \frac{\partial W}{\partial\Phi^i} + W\frac{\partial K}{\partial\Phi^i}$$

$$\beta_{\text{iso}} = \frac{T_{\text{SS}}^2}{1 + T_{\text{SS}}^2 + T_{\text{RS}}^2} = \mathcal{O}(e^{-1200})$$

$$x_i = \chi_i, y_i = \mathrm{e}^{-\varphi_i}$$

$$z_i = x_i + \mathrm{i}y_i$$

$$\mathcal{L}_4 = R\star 1 - \frac{1}{2}\sum_{i=1}^{3}(\star\ \mathrm{d}\varphi_i\wedge\ \mathrm{d}\varphi_i + \mathrm{e}^{2\varphi_i}\star\ \mathrm{d}\chi_i\wedge\ \mathrm{d}\chi_i) - \frac{1}{2}\mathrm{e}^{-\varphi_1}\left(\mathrm{e}^{\varphi_2+\varphi_3}\star\mathcal{F}^1\wedge\mathcal{F}^1 + \mathrm{e}^{\varphi_2-\varphi_3}\star\tilde{\mathcal{F}}_2\wedge\tilde{\mathcal{F}}_2\right.$$
$$\left.+\mathrm{e}^{-\varphi_2+\varphi_3}\star\tilde{\mathcal{F}}_3\wedge\tilde{\mathcal{F}}_3 + \mathrm{e}^{-\varphi_2-\varphi_3}\star\mathcal{F}^4\wedge\mathcal{F}^4\right) + \chi_1\left(F^1\wedge F^4 + \tilde{F}_2\wedge\tilde{F}_3\right)$$

$$\begin{aligned}
&\mathcal{F}^1 = F^1 + \chi_3\tilde{F}_2 + \chi_2\tilde{F}_3 - \chi_2\chi_3F^4, \quad && \mathcal{F}^4 = F^4 \\
&\tilde{\mathcal{F}}_2 = \tilde{F}_2 - \chi_2F^4, && \tilde{\mathcal{F}}_3 = \tilde{F}_3 - \chi_3F^4
\end{aligned}$$

$$\chi_1\left(\mathcal{F}^1\wedge\mathcal{F}^4 + \tilde{\mathcal{F}}_2\wedge\tilde{\mathcal{F}}_3\right)$$

$$\mathcal{L}_4 = \mathrm{d}^4x\sqrt{-g}\left[R - \frac{1}{2}f_{AB}(\Phi)\partial_\mu\Phi^A\partial^\mu\Phi^B - \frac{1}{4}k_{IJ}(\Phi)\mathbf{F}_{\mu\nu}^I\mathbf{F}^{J\mu\nu} + \frac{1}{4}h_{IJ}(\Phi)\epsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\mu\nu}^I\mathbf{F}_{\rho\sigma}^J\right],$$

$$\Phi^A = (\varphi_1, \varphi_2, \varphi_3, \chi_1, \chi_2, \chi_3)$$

$$\mathbf{A}^I = \left(A^1, \tilde{A}_2, \tilde{A}_3, A^4\right)$$

$$f_{AB} = \mathrm{diag}(1,1,1,\mathrm{e}^{2\varphi_1},\mathrm{e}^{2\varphi_2},\mathrm{e}^{2\varphi_3}), h_{IJ} = -\frac{\chi_1}{2}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\mathcal{M}_i = \frac{1}{y_i}\begin{pmatrix} 1 & x_i \\ x_i & x_i^2 + y_i^2 \end{pmatrix} = \begin{pmatrix} e^{\varphi_i} & \chi_i e^{\varphi_i} \\ \chi_i e^{\varphi_i} & e^{-\varphi_i} + \chi_i^2 e^{\varphi_i} \end{pmatrix}.$$

$$\mathrm{SL}(2,\mathbb{R})_1 \times \mathrm{SL}(2,\mathbb{R})_2 \times \mathrm{SL}(2,\mathbb{R})_3$$

$$\mathcal{M}_i \to \omega_i^T \mathcal{M}_i \omega_i,$$

$$\omega_i = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, ad - bc = 1$$

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\sum_{i=1}^{3} (\star\ d\varphi_i \wedge\ d\varphi_i + e^{2\varphi_i} \star\ d\chi_i \wedge\ d\chi_i) = \frac{1}{4}\sum_{i=1}^{3} \mathrm{Tr}\big(\star\ d\mathcal{M}_i^{-1} \wedge\ d\mathcal{M}_i\big),$$

$$-\tilde{A}_1 \wedge\ dF^1 = -\tilde{F}_1 \wedge F^1 + d\big(\tilde{A}_1 \wedge F^1\big)$$

$$\tilde{F}_1 - \chi_1 F^4 = -e^{-\varphi_1+\varphi_2+\varphi_3} \star \mathcal{F}^1.$$

$$\tilde{F}_4 - \chi_1 F^1 = e^{-\varphi_1}\big(-e^{-\varphi_2-\varphi_3} \star \mathcal{F}^4 + \chi_2\chi_3 e^{\varphi_2+\varphi_3} \star \mathcal{F}^1 + \chi_2 e^{\varphi_2-\varphi_3} \star \tilde{\mathcal{F}}_2 + \chi_3 e^{-\varphi_2+\varphi_3} \star \tilde{\mathcal{F}}_3\big).$$

$$A^2 \wedge\ d\tilde{F}_2 = F^2 \wedge \tilde{F}_2 - d\big(A^2 \wedge \tilde{F}_2\big),$$

$$F^2 + \chi_1 \tilde{F}_3 = e^{-\varphi_1+\varphi_2}\big(e^{-\varphi_3} \star \tilde{\mathcal{F}}_2 + \chi_3 e^{\varphi_3} \star \mathcal{F}^1\big)$$
$$F^3 + \chi_1 \tilde{F}_2 = e^{-\varphi_1+\varphi_3}\big(e^{-\varphi_2} \star \tilde{\mathcal{F}}_3 + \chi_2 e^{\varphi_2} \star \mathcal{F}^1\big)$$

$$\mathcal{L}_4 = R \star 1 - \frac{1}{2}\sum_{i=1}^{3} (\star\ d\varphi_i \wedge\ d\varphi_i + e^{2\varphi_i} \star\ d\chi_i \wedge\ d\chi_i) - \frac{1}{2} e^{-\varphi_1-\varphi_2-\varphi_3} \star F^4 \wedge F^4$$
$$-\frac{1}{2}\sum_{i=1}^{3} e^{2\varphi_i-\varphi_1-\varphi_2-\varphi_3} \star \big(\tilde{F}_i - \chi_i F^4\big) \wedge \big(\tilde{F}_i - \chi_i F^4\big) + \chi_1\chi_2\chi_3 F^4 \wedge F^4$$
$$-\big(\chi_1\chi_2\tilde{F}_3 + \chi_2\chi_3\tilde{F}_1 + \chi_3\chi_1\tilde{F}_2\big) \wedge F^4 + \chi_1\tilde{F}_2 \wedge \tilde{F}_3 + \chi_2\tilde{F}_3 \wedge \tilde{F}_1 + \chi_3\tilde{F}_1 \wedge \tilde{F}_2$$

$$e^{\varphi_1-\varphi_2-\varphi_3} \star \big(\tilde{F}_1 - \chi_1 F^4\big) = F^1 + \chi_3\tilde{F}_2 + \chi_2\tilde{F}_3 - \chi_2\chi_3 F^4$$
$$e^{\varphi_2-\varphi_3-\varphi_1} \star \big(\tilde{F}_2 - \chi_2 F^4\big) = F^2 + \chi_1\tilde{F}_3 + \chi_3\tilde{F}_1 - \chi_3\chi_1 F^4$$
$$e^{\varphi_3-\varphi_1-\varphi_2} \star \big(\tilde{F}_3 - \chi_3 F^4\big) = F^3 + \chi_2\tilde{F}_1 + \chi_1\tilde{F}_2 - \chi_1\chi_2 F^4$$

$$\tilde{F}_4 = -e^{-\varphi_1-\varphi_2-\varphi_3} \star F^4 + \sum_{i=1}^{3} e^{2\varphi_i-\varphi_1-\varphi_2-\varphi_3}\chi_i \star \big(\tilde{F}_i - \chi_i F^4\big) + 2\chi_1\chi_2\chi_3 F^4$$
$$-\big(\chi_2\chi_3\tilde{F}_1 + \chi_3\chi_1\tilde{F}_2 + \chi_1\chi_2\tilde{F}_3\big)$$

$$A^0 \equiv -\tilde{A}_4, \tilde{A}_0 \equiv A^4$$

$$\tilde{F}_\Lambda = d\tilde{A}_\Lambda$$

$$\mathcal{L}_4 = R \star 1 - 2g_{i\bar{j}} \star\ dX^i \wedge\ d\bar{X}^{\bar{j}} + \frac{1}{2}\tilde{F}_\Lambda \wedge \tilde{G}^\Lambda$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

$$F(X) = -\frac{X^1 X^2 X^3}{X^0}$$

$$A/\sqrt{3} \equiv A^1 = A^2 = A^3, \varphi/\sqrt{3} \equiv \varphi_1 = \varphi_2 = \varphi_3 \text{ and } \chi/\sqrt{3} \equiv \chi_1 = \chi_2 = \chi_3$$

$$\mathcal{L}_4 = R \star 1 - \frac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \mathrm{e}^{2\varphi/\sqrt{3}} \star \mathrm{d}\chi \wedge \mathrm{d}\chi - \frac{1}{2} \mathrm{e}^{-\varphi/\sqrt{3}} \star \left(\tilde{F} - \chi F^4\right) \wedge \left(\tilde{F} - \chi F^4\right)$$
$$- \frac{1}{2} \mathrm{e}^{-\sqrt{3}\varphi} \star F^4 \wedge F^4 + \frac{\chi}{\sqrt{3}} \left(\tilde{F} \wedge \tilde{F} - \chi \tilde{F} \wedge F^4 + \frac{\chi^2}{3} F^4 \wedge F^4 \right)$$

$$\mathcal{L}_4 = R \star 1 - \frac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \mathrm{e}^{-\sqrt{3}\varphi} \star F \wedge F.$$

$$\mathcal{L}_4 = R \star 1 - \frac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \mathrm{e}^{2\varphi} \star \mathrm{d}\chi \wedge \mathrm{d}\chi - \mathrm{e}^{-\varphi} \left(\star F^1 \wedge F^1 + \star \tilde{F}_2 \wedge \tilde{F}_2 \right) + \chi \left(F^1 \wedge F^1 + \tilde{F}_2 \wedge \tilde{F}_2 \right)$$

$$\mathcal{L}_4 = R \star 1 - \frac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \mathrm{e}^{2\varphi} \star \mathrm{d}\chi \wedge \mathrm{d}\chi - \mathrm{e}^{-\varphi} \star F \wedge F + \chi F \wedge F,$$

$$\mathcal{L}_4 = R \star 1 - 2 \star F \wedge F.$$

$$\mathcal{L}_5 = R \star 1 - \frac{1}{2} \sum_{i=1}^{3} h_i^{-2} \left(\star \mathrm{d}h_i \wedge \mathrm{d}h_i + \star \tilde{F}_i \wedge \tilde{F}_i \right) + \tilde{F}_1 \wedge \tilde{F}_2 \wedge \tilde{A}_3$$

$$h_1 = \mathrm{e}^{-\varphi_1'/\sqrt{6} - \varphi_2'/\sqrt{2}}, \qquad h_2 = \mathrm{e}^{-\varphi_1'/\sqrt{6} + \varphi_2'/\sqrt{2}}, \qquad h_3 = \mathrm{e}^{2\varphi_1'/\sqrt{6}}$$

$$h_1 = \mathrm{e}^{2\phi_2/\sqrt{6}}, h_2 = \mathrm{e}^{\phi/\sqrt{2} - \phi_2/\sqrt{6}}, h_3 = \mathrm{e}^{-\phi/\sqrt{2} - \phi_2/\sqrt{6}}.$$

$$\frac{1}{2} \sum_{i=1}^{3} h_i^{-2} \star \mathrm{d}h_i \wedge \mathrm{d}h_i = \frac{1}{2} \sum_{i=1}^{2} \star \mathrm{d}\varphi_i' \wedge \mathrm{d}\varphi_i' = \frac{1}{2} (\star \mathrm{d}\phi \wedge \mathrm{d}\phi + \star \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_2).$$

$$\tilde{F}_3 = \mathrm{d}\tilde{A}_3 = -h_1^{-2} h_2^{-2} \star \mathcal{H}$$

$$\mathrm{d}\mathcal{H} = -\tilde{F}_1 \wedge \tilde{F}_2$$

$$\mathcal{L}_5 = R \star 1 - \frac{1}{2} \sum_{i=1}^{3} h_i^{-2} \star \mathrm{d}h_i \wedge \mathrm{d}h_i - \frac{1}{2} \sum_{i=1}^{2} h_i^{-2} \star \tilde{F}_i \wedge \tilde{F}_i - \frac{1}{2} h_1^{-2} h_2^{-2} \star \mathcal{H} \wedge \mathcal{H}.$$

$$\mathrm{d}s_5^2 = f^{-1} \mathrm{d}s^2 + f^2 (\mathrm{d}z_5 - A^4)^2, \tilde{A}_{(5\,\mathrm{d})i} = \tilde{A}_i + \chi_i (\mathrm{d}z_5 - A^4)$$

$$f h_i = \mathrm{e}^{-\varphi_i}$$

$$\widetilde{A_2} = \tilde{A}_3 \text{ and } h_2 = h_3$$

$$h_1 = \mathrm{e}^{2\varphi/\sqrt{6}}$$

$$\mathcal{L}_5 = R \star 1 - \frac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \mathrm{e}^{-4\varphi/\sqrt{6}} \star \tilde{F}_1 \wedge \tilde{F}_1 - \mathrm{e}^{2\varphi/\sqrt{6}} \star \tilde{F}_2 \wedge \tilde{F}_2 + \tilde{F}_2 \wedge \tilde{F}_2 \wedge \tilde{A}_1$$

$$\mathcal{L}_5 = R \star 1 - \frac{3}{2} \star \tilde{F} \wedge \tilde{F} + \tilde{F} \wedge \tilde{F} \wedge \tilde{A}$$

$$\mathcal{L}_6 = R \star 1 - \frac{1}{2} \star \, \mathrm{d}\phi \wedge \, \mathrm{d}\phi - \frac{1}{2} \mathrm{e}^{-\sqrt{2}\phi} \star H \wedge H$$

$$\mathrm{d}s^2_{(6d)} = \mathrm{e}^{\phi_2/\sqrt{6}} \, \mathrm{d}s^2 + \mathrm{e}^{-3\phi_2/\sqrt{6}} \left(\mathrm{d}z_6 + \tilde{A}_1 \right)^2, B_{(6d)} = B + \tilde{A}_2 \wedge \left(\mathrm{d}z_6 + \tilde{A}_1 \right),$$

$$H_{(6d)} = \mathcal{H} + \tilde{F}_2 \wedge \left(\mathrm{d}z_6 + \tilde{A}_1 \right), \mathcal{H} = \mathrm{d}B - \tilde{A}_2 \wedge \tilde{F}_1, \tilde{F}_i = \mathrm{d}\tilde{A}_i$$

$$H = \star H$$

$$\mathcal{L}_6 = R \star 1 - \frac{1}{2} \star H \wedge H$$

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2_6 + \mathrm{e}^{\phi/\sqrt{2}} (\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}X_3^2 + \mathrm{d}X_4^2), \Phi = \frac{\phi}{\sqrt{2}}, C \equiv B.$$

$$\mathcal{L}_{11} = R \star 1 - \frac{1}{2} \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{6} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}$$

$$\mathrm{d}s^2_{11} = \mathrm{d}s^2_5 + h_1(\mathrm{d}X_1^2 + \mathrm{d}X_2^2) + h_2(\mathrm{d}X_3^2 + \mathrm{d}X_4^2) + h_3\left(\mathrm{d}X_5^2 + \mathrm{d}X_6^2 \right)$$
$$\mathcal{A} = \tilde{A}_1 \wedge \, \mathrm{d}X_1 \wedge \, \mathrm{d}X_2 + \tilde{A}_2 \wedge \, \mathrm{d}X_3 \wedge \, \mathrm{d}X_4 + \tilde{A}_3 \wedge \, \mathrm{d}X_5 \wedge \, \mathrm{d}X_6$$

$$\mathrm{SO}(4,4)/\mathrm{SL}(2,\mathbb{R})^4$$

$$G/K = \mathrm{SO}(4,4)/\mathrm{SL}(2,\mathbb{R})^4$$

$$\varphi^a = \{U, \sigma, x_i, y_i, \zeta^I, \tilde{\zeta}_I\}$$

$$\mathcal{L}_3 = R \star_3 1 - \frac{1}{2} G_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b \star_3 1$$

$$\mathrm{d}s^2_{G/K} = \sum_i \frac{\mathrm{d}x_i^2 + \mathrm{d}y_i^2}{y_i^2} + 4 \, \mathrm{d}U^2 + \frac{\mathrm{e}^{-4U}}{4} \left(\mathrm{d}\sigma + \sum_I \left(\tilde{\zeta}_I \, \mathrm{d}\zeta^I - \zeta^I \, \mathrm{d}\tilde{\zeta}_I \right) \right)^2$$
$$- \mathrm{e}^{-2U} \sum_{I,J} \left(\frac{Y Y_I Y_J}{X_{IJ}} \, \mathrm{d}\zeta^I \, \mathrm{d}\zeta^J + \frac{Y}{X_{IJ} Y_I Y_J} \, \mathrm{d}\tilde{\zeta}_I \, \mathrm{d}\tilde{\zeta}_J + \frac{X X_{IJ} Y_I}{Y_J} 2 \, \mathrm{d}\zeta^I \, \mathrm{d}\tilde{\zeta}_J \right)$$

$$X_{12} = \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}}{x_1 x_2}, \quad X_{13} = \frac{\sqrt{(x_1^2 + y_1^2)(x_3^2 + y_3^2)}}{x_1 x_3}$$
$$X_{23} = \frac{\sqrt{(x_2^2 + y_2^2)(x_3^2 + y_3^2)}}{x_2 x_3}, \quad X_{11} = X_{22} = X_{33} = X_{44} = 1$$

$$Y_i = \frac{\sqrt{x_i^2 + y_i^2}}{[(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)]^{1/4}}, \qquad Y_4 = -[(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)]^{1/4}$$
$$X = \frac{x_1 x_2 x_3}{y_1 y_2 y_3}, \qquad Y = \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)}}{y_1 y_2 y_3}$$

$$Y_1Y_2Y_3Y_4 = -1, X^2X_{12}X_{13}X_{14} = Y^2, \frac{1}{X^2} = \left(\frac{Y^2}{X^2X_{12}^2} - 1\right)\left(\frac{Y^2}{X^2X_{13}^2} - 1\right)\left(\frac{Y^2}{X^2X_{14}^2} - 1\right)$$

$$\mathrm{d}\left[\mathrm{e}^{-4U} \star_3 \left(\mathrm{d}\sigma + \sum_I \left(\tilde{\zeta}_I\, \mathrm{d}\zeta^I - \zeta^I\, \mathrm{d}\tilde{\zeta}_I\right)\right)\right] = 0$$

$$\mathrm{d}\omega_3 = -\frac{\mathrm{e}^{-4U}}{2} \star_3 \left(\mathrm{d}\sigma + \sum_I \left(\tilde{\zeta}_I\, \mathrm{d}\zeta^I - \zeta^I\, \mathrm{d}\tilde{\zeta}_I\right)\right)$$

$$\mathrm{d}A^I_{(3\mathrm{d})} = -\zeta^I\, \mathrm{d}\omega_3 + \mathrm{e}^{-2U} \star_3 \sum_J \left(\frac{Y}{X_{IJ}Y_IY_J}\, \mathrm{d}\tilde{\zeta}_J + \frac{XX_{IJ}Y_J}{Y_I}\, \mathrm{d}\zeta^J\right)$$

$$\mathrm{d}\tilde{A}_{I(3\mathrm{d})} = -\tilde{\zeta}_I\, \mathrm{d}\omega_3 - \mathrm{e}^{-2U} \star_3 \sum_J \left(\frac{YY_IY_J}{X_{IJ}}\, \mathrm{d}\zeta^J + \frac{XX_{IJ}Y_I}{Y_J}\, \mathrm{d}\tilde{\zeta}_J\right)$$

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\,\mathrm{d}t + \omega_3)^2 + \mathrm{e}^{-2U}\, \mathrm{d}s_3^2$$

$$A^I = \zeta^I(\,\mathrm{d}t + \omega_3) + A^I_{(3\mathrm{d})}, \tilde{A}_I = \tilde{\zeta}_I(\,\mathrm{d}t + \omega_3) + \tilde{A}_{I(3\mathrm{d})}$$

$$\zeta^0 \equiv -\tilde{\zeta}_4, \tilde{\zeta}_0 \equiv \zeta^4$$

$$\zeta^\Lambda = (\zeta^0, \zeta^1, \zeta^2, \zeta^3), \tilde{\zeta}_\Lambda = (\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3)$$

$$\mathrm{d}s^2_{G/K} = \sum_i \frac{\mathrm{d}x_i^2 + \mathrm{d}y_i^2}{y_i^2} + 4\,\mathrm{d}U^2 + \frac{\mathrm{e}^{-4U}}{4}\left(\mathrm{d}\sigma - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda\right)^2 + \mathrm{e}^{-2U}\Big[(\mathrm{Im}\mathcal{N})^{\Lambda\Sigma}\mathrm{d}\tilde{\zeta}_\Lambda \mathrm{d}\tilde{\zeta}_\Sigma$$
$$+((\mathrm{Im}\mathcal{N})^{-1})_{\Lambda\Sigma}\left(\mathrm{d}\zeta^\Lambda - (\mathrm{Re}\mathcal{N})^{\Lambda\Gamma}\mathrm{d}\tilde{\zeta}_\Gamma\right)\left(\mathrm{d}\zeta^\Sigma - (\mathrm{Re}\mathcal{N})^{\Sigma\Delta}\mathrm{d}\tilde{\zeta}_\Delta\right)\Big]$$

$$\mathcal{N} = \begin{pmatrix} -2x_1x_2x_3 - \mathrm{i}y_1y_2y_3\left(1 + \sum_{i=1}^{3}\frac{x_i^2}{y_i^2}\right) & x_2x_3 + \mathrm{i}\frac{x_1y_2y_3}{y_1} & x_1x_3 + \mathrm{i}\frac{x_2y_1y_3}{y_2} & x_1x_2 + \mathrm{i}\frac{x_3y_1y_2}{y_3} \\ x_2x_3 + \mathrm{i}\frac{x_1y_2y_3}{y_1} & -\mathrm{i}\frac{y_2y_3}{y_1} & -x_3 & -x_2 \\ x_1x_3 + \mathrm{i}\frac{x_2y_1y_3}{y_2} & -x_3 & -\mathrm{i}\frac{y_1y_3}{y_2} & -x_1 \\ x_1x_2 + \mathrm{i}\frac{x_3y_1y_2}{y_3} & -x_2 & -x_1 & -\mathrm{i}\frac{y_1y_2}{y_3} \end{pmatrix}.$$

$$\mathcal{N} = -\mathrm{i}\mathbb{I} + O(1/r)$$

$$\mathrm{d}\omega_3 = -\frac{1}{2}\mathrm{e}^{-4U} \star_3 \left(\,\mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda\right)$$

$$\mathrm{d}A^\Lambda_{(3\mathrm{d})} = -\zeta^\Lambda \mathrm{d}\omega_3 - \mathrm{e}^{-2U} \star_3 \left[(\mathrm{Im}\mathcal{N})^{\Lambda\Sigma}\mathrm{d}\tilde{\zeta}_\Sigma + (\mathrm{Re}\mathcal{N})^{\Lambda\Gamma}((\mathrm{Im}\mathcal{N})^{-1})_{\Gamma\Sigma}\left(\mathrm{d}\zeta^\Sigma - (\mathrm{Re}\mathcal{N})^{\Sigma\Delta}\mathrm{d}\tilde{\zeta}_\Delta\right)\right]$$
$$\mathrm{d}\tilde{A}_{\Sigma(3\mathrm{d})} = -\tilde{\zeta}_\Sigma \mathrm{d}\omega_3 + \mathrm{e}^{-2U} \star_3 ((\mathrm{Im}\mathcal{N})^{-1})_{\Sigma\Lambda}\left(\mathrm{d}\zeta^\Lambda - (\mathrm{Re}\mathcal{N})^{\Lambda\Sigma}\mathrm{d}\tilde{\zeta}_\Sigma\right)$$

$$(\zeta^1, \tilde{\zeta}_1) = (\sigma_2, -\psi_2), (\zeta^2, \tilde{\zeta}_2) = (\psi_1, \sigma_1), (\zeta^3, \tilde{\zeta}_3) = (\psi_3, \sigma_3), (\zeta^4, \tilde{\zeta}_4) = (\sigma_4, -\psi_4)$$

$$\sigma = -2\chi_4 - \zeta^1\tilde{\zeta}_1 + \zeta^2\tilde{\zeta}_2 + \zeta^3\tilde{\zeta}_3 - \zeta^4\tilde{\zeta}_4$$

$$H_0 = E_{33} + E_{44} - E_{77} - E_{88}, \quad H_1 = E_{33} - E_{44} - E_{77} + E_{88}$$
$$H_2 = E_{11} + E_{22} - E_{55} - E_{66}, \quad H_3 = E_{11} - E_{22} - E_{55} + E_{66}$$

$$E_0 = E_{47} - E_{38}, E_1 = E_{87} - E_{34}, E_2 = E_{25} - E_{16}, E_3 = E_{65} - E_{12},$$
$$E^{Q_1} = E_{45} - E_{18}, E^{Q_2} = E_{32} - E_{67}, E^{Q_3} = E_{36} - E_{27}, E^{Q_4} = E_{41} - E_{58},$$
$$E^{P^1} = E_{57} - E_{31}, E^{P^2} = E_{46} - E_{28}, E^{P^3} = E_{42} - E_{68}, E^{P^4} = E_{17} - E_{35},$$

$$F_0 = E_{74} - E_{83}, \quad F_1 = E_{78} - E_{43}, \quad F_2 = E_{52} - E_{61}, \quad F_3 = E_{56} - E_{21},$$
$$F^{Q_1} = E_{54} - E_{81}, \quad F^{Q_2} = E_{23} - E_{76}, \quad F^{Q_3} = E_{63} - E_{72}, \quad F^{Q_4} = E_{14} - E_{85},$$
$$F^{P^1} = E_{75} - E_{13}, \quad F^{P^2} = E_{64} - E_{82}, \quad F^{P^3} = E_{24} - E_{86}, \quad F^{P^4} = E_{71} - E_{53},$$

$$\left(E^{Q_I}, E^{P^I}, F^{Q_I}, F^{P^I}\right)\left(E_{q_\Lambda}, E_{p^\Lambda}, F_{q_\Lambda}, F_{p^\Lambda}\right)$$

$$\left(E_{q_i}, E_{p^i}\right) = \left(E^{P^i}, -E^{Q_i}\right), \qquad \left(E_{q_0}, E_{p^0}\right) = \left(E^{Q_4}, E^{P^4}\right)$$
$$\left(F_{q_i}, F_{p^i}\right) = \left(F^{P^i}, -F^{Q_i}\right), \qquad \left(F_{q_0}, F_{p^0}\right) = \left(F^{Q_4}, F^{P^4}\right)$$

$$H_\Lambda^\sharp = H_\Lambda, E_\Lambda^\sharp = F_\Lambda, F_\Lambda^\sharp = E_\Lambda,$$

$$(E^{Q_I})^\sharp = -F^{Q_I}, \left(E^{P^I}\right)^\sharp = -F^{P^I}, (F^{Q_I})^\sharp = -E^{Q_I}, \left(F^{P^I}\right)^\sharp = -E^{P^I}.$$

$$k_\Lambda = E_\Lambda - F_\Lambda, k^{Q_I} = E^{Q_I} + F^{Q_I}, k^{P^I} = E^{P^I} + F^{P^I}$$

$$p_\Lambda = E_\Lambda + F_\Lambda, \quad p^{Q_I} = E^{Q_I} - F^{Q_I}, \quad p^{P^I} = E^{P^I} - F^{P^I}$$

$$A^\sharp = \eta A^T \eta^{-1},$$

$$\eta = \mathrm{diag}(-1, -1, 1, 1, -1, -1, 1, 1)$$

$$\mathfrak{sl}(2, \mathbb{R})^4 = \mathfrak{so}(2,2)^2$$

$$\mathcal{V} = \exp(-UH_0)\exp\left(\frac{1}{2}\sum_i \varphi_i H_i\right)\exp\left(-\sum_i \chi_i E_i\right)\exp\left[-\sum_I \left(\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I}\right)\right]\exp\left(-\frac{1}{2}\sigma E_0\right)$$
$$= \exp(-UH_0)\exp\left[-\frac{1}{2}\sum_i (\log y_i) H_i\right]\exp\left(-\sum_i x_i E_i\right)\exp\left[\sum_\Lambda \left(-\tilde{\zeta}_\Lambda E_{q_\Lambda} + \zeta^\Lambda E_{p^\Lambda}\right)\right]\exp\left(-\frac{1}{2}\sigma E_0\right)$$

$$\mathrm{d}s^2_{G/K} = \mathrm{Tr}(P_* P_*), P_* = \frac{1}{2}\left(\theta + \theta^\sharp\right)$$

$$\mathcal{M} = \mathcal{V}^\sharp \mathcal{V}$$

$$-\frac{1}{2} G_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b \star_3 1 = -\frac{1}{8}\mathrm{Tr}[\star_3 (\mathcal{M}^{-1}\,\mathrm{d}\mathcal{M}) \wedge (\mathcal{M}^{-1}\,\mathrm{d}\mathcal{M})].$$

$$\mathcal{V} \to k\mathcal{V}g$$

$$\mathcal{M} \to g^\sharp \mathcal{M} g,$$

$$e^{-4U} = \mathcal{M}_{33}\mathcal{M}_{44} - \mathcal{M}_{34}^2$$

$$x_1 = \frac{\mathcal{M}_{34}}{\mathcal{M}_{33}}, y_1^{-1} = e^{2U}\mathcal{M}_{33}$$

$$\frac{1}{y_2y_3} = \mathcal{M}_{11} + e^{4U}(\mathcal{M}_{33}\mathcal{M}_{41}^2 + \mathcal{M}_{44}\mathcal{M}_{31}^2 - 2\mathcal{M}_{31}\mathcal{M}_{34}\mathcal{M}_{41})$$
$$\frac{x_2}{y_2y_3} = \mathcal{M}_{16} + e^{4U}(\mathcal{M}_{34}\mathcal{M}_{41}\mathcal{M}_{63} + \mathcal{M}_{31}\mathcal{M}_{34}\mathcal{M}_{64} - \mathcal{M}_{31}\mathcal{M}_{44}\mathcal{M}_{63} - \mathcal{M}_{33}\mathcal{M}_{41}\mathcal{M}_{64})$$
$$\frac{x_3}{y_2y_3} = \mathcal{M}_{12} + e^{4U}(\mathcal{M}_{31}\mathcal{M}_{32}\mathcal{M}_{44} + \mathcal{M}_{33}\mathcal{M}_{41}\mathcal{M}_{42} - \mathcal{M}_{31}\mathcal{M}_{34}\mathcal{M}_{42} - \mathcal{M}_{32}\mathcal{M}_{34}\mathcal{M}_{41})$$
$$\frac{x_3^2 + y_3^2}{y_2y_3} = \mathcal{M}_{22} + \frac{\mathcal{M}_{32}^2}{\mathcal{M}_{33}} + e^{4U}\frac{(\mathcal{M}_{32}\mathcal{M}_{34} - \mathcal{M}_{33}\mathcal{M}_{42})^2}{\mathcal{M}_{33}}$$

$$\begin{aligned}
e^{-4U}\zeta^1 &= \mathcal{M}_{35}\mathcal{M}_{34} - \mathcal{M}_{45}\mathcal{M}_{33}, & e^{-4U}\tilde{\zeta}_1 &= \mathcal{M}_{31}\mathcal{M}_{44} - \mathcal{M}_{41}\mathcal{M}_{34}\\
e^{-4U}\zeta^2 &= \mathcal{M}_{42}\mathcal{M}_{34} - \mathcal{M}_{32}\mathcal{M}_{44}, & e^{-4U}\tilde{\zeta}_2 &= \mathcal{M}_{64}\mathcal{M}_{33} - \mathcal{M}_{63}\mathcal{M}_{34}\\
e^{-4U}\zeta^3 &= \mathcal{M}_{63}\mathcal{M}_{44} - \mathcal{M}_{64}\mathcal{M}_{34}, & e^{-4U}\tilde{\zeta}_3 &= \mathcal{M}_{32}\mathcal{M}_{34} - \mathcal{M}_{42}\mathcal{M}_{33}\\
e^{-4U}\zeta^4 &= \mathcal{M}_{31}\mathcal{M}_{34} - \mathcal{M}_{41}\mathcal{M}_{33}, & e^{-4U}\tilde{\zeta}_4 &= \mathcal{M}_{35}\mathcal{M}_{44} - \mathcal{M}_{45}\mathcal{M}_{34}
\end{aligned}$$

$$\begin{aligned}
\sigma &= \frac{2\mathcal{M}_{38}}{\mathcal{M}_{33}} + \frac{e^{4U}}{\mathcal{M}_{33}}(\mathcal{M}_{33}\mathcal{M}_{35}\mathcal{M}_{41} + \mathcal{M}_{31}\mathcal{M}_{33}\mathcal{M}_{45} + 2\mathcal{M}_{32}\mathcal{M}_{34}\mathcal{M}_{63} - \mathcal{M}_{33}\mathcal{M}_{42}\mathcal{M}_{63}\\
&\quad -\mathcal{M}_{32}\mathcal{M}_{33}\mathcal{M}_{64} - 2\mathcal{M}_{31}\mathcal{M}_{34}\mathcal{M}_{35})\\
&= \frac{2\mathcal{M}_{38}}{\mathcal{M}_{33}} - \zeta^4\tilde{\zeta}_4 - \zeta^1\tilde{\zeta}_1 + \zeta^2\tilde{\zeta}_2 + \zeta^3\tilde{\zeta}_3 + 2x_1\tilde{\zeta}_2\tilde{\zeta}_3 - 2x_1\zeta^4\zeta^1
\end{aligned}$$

$$d(\mathcal{M}^{-1} \star_3 \, d\mathcal{M}) = 0$$

$$d\mathcal{N} = \mathcal{M}^{-1} \star_3 \, d\mathcal{M}$$

$$\mathcal{N} \to g^{-1}\mathcal{N}g$$

$$\begin{aligned}
d\mathcal{N} = \mathcal{M}^{-1} \star_3 \, d\mathcal{M} &= d\omega_3 F_0 + \sum_I \left(dA^I_{(3\,d)}F^{P^I} - d\tilde{A}_{I(3\,d)}F^{Q_I}\right) + \cdots\\
&= d\omega_3 F_0 + \sum_\Lambda \left(d\tilde{A}_{\Lambda(3\,d)}F_{p^\Lambda} + dA^\Lambda_{(3\,d)}F_{q_\Lambda}\right) + \cdots,
\end{aligned}$$

$$\omega_3 = \mathcal{N}_{74}$$

$$\begin{aligned}
A^1_{(3\,d)} &= \mathcal{N}_{75}, & A^2_{(3\,d)} &= \mathcal{N}_{64}, & A^3_{(3\,d)} &= \mathcal{N}_{24}, & A^4_{(3\,d)} &= \mathcal{N}_{71}\\
\tilde{A}_{1(3\,d)} &= \mathcal{N}_{81}, & \tilde{A}_{2(3\,d)} &= \mathcal{N}_{76}, & \tilde{A}_{3(3\,d)} &= \mathcal{N}_{72}, & \tilde{A}_{4(3\,d)} &= \mathcal{N}_{85}
\end{aligned}$$

$$ds_3^2 = dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta \, d\phi^2) + O(r^{-2})dr^2 + O(r^0)d\theta^2 + O(r^0)d\phi^2$$

$$\begin{aligned}
e^{2U} &= 1 - \frac{2M}{r} + O(r^{-2}), & \zeta^I &= \frac{Q_I}{r} + O(r^{-2}), \varphi_i = \frac{\Sigma_i}{r} + O(r^{-2})\\
\omega_3 &= \left(2N\cos\theta + 2J\frac{\sin^2\theta}{r} + O(r^{-2})\right)d\phi, & \tilde{\zeta}_I &= \frac{P^I}{r} + O(r^{-2}), \chi_i = \frac{\Xi_i}{r} + O(r^{-2})
\end{aligned}$$

$$\star_3 1 \sim r^2\sin\theta \, dr \wedge d\theta \wedge d\phi, \star 1 \sim r^2\sin\theta \, dt \wedge dr \wedge d\theta \wedge d\phi$$

$$\star_3 \; \mathrm{d}\omega_3 = -\frac{2N}{r^2}\,\mathrm{d}r - \frac{1}{2}\,\mathrm{d}\left(\frac{4J\cos\,\theta + c}{r^2}\right) + O(r^{-3}),$$

$$\sigma = -\frac{4N}{r} + \frac{4J\cos\,\theta + c}{r^2} + O(r^{-3})$$

$$M = -\lim_{r\to\infty}(rU), Q_I = \lim_{r\to\infty}(r\zeta^I), \Sigma_i = \lim_{r\to\infty}(r\varphi_i), J = \lim_{r\to\infty}\left(\frac{r(\omega_{3\phi} - 2N\cos\theta)}{2\sin^2\theta}\right),$$
$$N = -\frac{1}{4}\lim_{r\to\infty}(r\sigma), P^I = \lim_{r\to\infty}(r\tilde{\zeta}_I), \Xi_i = \lim_{r\to\infty}(r\chi_i).$$

$$(Q_I, P^I) = (-\tilde{P}_I, \tilde{Q}^I)$$

$$(Q_0, P^0) = (-\tilde{Q}^4, -\tilde{P}_4), (\tilde{Q}^0, \tilde{P}_0) = (Q_4, P^4)$$

$$\mathcal{M} = \mathbb{I} + \frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}^{(2)}}{r^2} + O(r^{-3})$$

$$\mathcal{Q} = 2MH_0 + 2Np_0 - \sum_{I=1}^{4}(Q_I p^{Q_I} + P^I p^{P_I}) + \sum_{i=1}^{3}(\Sigma_i H_i - \Xi_i p_i)$$
$$= 2MH_0 + 2Np_0 + \sum_{\Lambda=0}^{3}(-Q_\Lambda p_{p^\Lambda} + P^\Lambda p_{q_\Lambda}) + \sum_{i=1}^{3}(\Sigma_i H_i - \Xi_i p_i)$$

$$\mathcal{M} \to g^\sharp \mathcal{M} g$$

$$g^\sharp g = \mathbb{I}$$

$$\frac{1}{4}\mathrm{Tr}(\mathcal{Q}^2) = 4(M^2 + N^2) - \sum_{I=1}^{4}[(Q_I)^2 + (P^I)^2] + \sum_{i=1}^{3}(\Sigma_i^2 + \Xi_i^2)$$

$$\mathcal{Q}^{(2)} = (-2J\cos\,\theta + a_0)p_0 + \cdots,$$

$$\mathcal{Q}_{\partial_\phi} \equiv -\frac{3}{8\pi}\int_{S^2_\infty}(\partial_\phi)_\mu \mathcal{M}^{-1}\partial_\nu\mathcal{M}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

$$\mathcal{Q}_{\partial_\phi} = -\frac{3}{4}\int_0^\pi \mathrm{d}\theta \sin^2\,\theta\partial_\theta\mathcal{Q}^{(2)} = -2Jp_0 + \cdots$$

$$\frac{1}{16}\mathrm{Tr}\left(\mathcal{Q}^2_{\partial_\phi}\right) = J^2 + \cdots$$

$$\mathrm{d}s^2 = -\frac{r^2 - 2mr - n^2}{r^2 + n^2}(\,\mathrm{d}t + 2n\cos\,\theta\,\mathrm{d}\phi)^2 + \frac{r^2 + n^2}{r^2 - 2mr - n^2}\,\mathrm{d}r^2 + (r^2 + n^2)(\mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2),$$

$$\mathrm{e}^{-2U} = \frac{r^2 + n^2}{r^2 - 2mr - n^2} \qquad \omega_3 = 2n\cos\,\theta\,\mathrm{d}\phi$$
$$\mathrm{d}s_3^2 = \mathrm{d}r^2 + (r^2 - 2mr - n^2)(\mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2)$$

$$\sigma = -\frac{4n(r-m)}{r^2+n^2}$$

$$\overline{\mathcal{M}} = (r^2 - 2mr - n^2)\mathcal{M}$$

$$\mathrm{d}s^2 = -\frac{R}{r^2+u^2}\left(\mathrm{d}\bar{t} - \frac{\bar{a}^2-u^2}{\bar{a}}\,\mathrm{d}\bar{\phi}\right)^2 + \frac{U}{r^2+u^2}\left(\mathrm{d}\bar{t} - \frac{r^2+\bar{a}^2}{\bar{a}}\,\mathrm{d}\bar{\phi}\right)^2 + (r^2+u^2)\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right),$$

$$R = r^2 + \bar{a}^2 - 2mr, U = \bar{a}^2 - u^2 + 2nu$$

$$\frac{\phi}{a} = \frac{\bar{\phi}}{\bar{a}}, t = \bar{t} + \frac{2n^2}{\bar{a}}\bar{\phi}, u = n + a\cos\,\theta$$

$$\bar{a}^2 = a^2 - n^2,\ \mathrm{d}t + \omega_3 = \mathrm{d}\bar{t} + \bar{\omega}_3$$

$$\begin{aligned} \mathrm{d}s_3^2 &= \frac{RU}{\bar{a}^2}\,\mathrm{d}\bar{\phi}^2 + (R-U)\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right), \mathrm{e}^{-2U} = \frac{r^2+u^2}{R-U} \\ \bar{\omega}_3 &= \frac{(r^2+\bar{a}^2)U - (\bar{a}^2-u^2)R}{\bar{a}(R-U)}\mathrm{d}\bar{\phi} = \frac{2(mrU+nuR)}{\bar{a}(R-U)}\mathrm{d}\bar{\phi}. \end{aligned}$$

$$\sigma = \frac{4(mu-nr)}{r^2+u^2}$$

$$\overline{\mathcal{M}} \equiv (R-U)\mathcal{M}$$

$$\overline{\mathcal{M}} = \begin{pmatrix} R-U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R-U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2+u^2 & 0 & 0 & 0 & 0 & 2(mu-\ldots \\ 0 & 0 & 0 & r^2+u^2 & 0 & 0 & -2(mu-nr) & 0 \\ 0 & 0 & 0 & 0 & R-U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R-U & 0 & 0 \\ 0 & 0 & 0 & -2(mu-nr) & 0 & 0 & (r-2m)^2+(u-2n)^2 & 0 \\ 0 & 0 & 2(mu-nr) & 0 & 0 & 0 & 0 & (r-2m)^2+\ldots \end{pmatrix}$$

$$\partial_u \mathcal{N}_\phi = -\frac{R}{a}\mathcal{M}^{-1}\partial_r\mathcal{M}, \partial_r\mathcal{N}_\phi = \frac{U}{a}\mathcal{M}^{-1}\partial_u\mathcal{M}$$

$$\mathcal{N}_\phi = \omega_{3\phi}(F_0 + E_0) - \frac{4(m^2U + n^2R)}{a(R-U)}E_0 + \frac{2(muR - nrU)}{a(R-U)}H_0$$

$$g = \exp\left(-\sum_I \gamma_I k^{P^I}\right)\exp\left(-\sum_I \delta_I k^{Q_I}\right)$$

$$k=\begin{pmatrix} c_{\gamma1}c_{\gamma4} & 0 & s_{\gamma1}c_{\gamma4} & 0 & s_{\gamma1}s_{\gamma4} & 0 & -c_{\gamma1}s_{\gamma4} & 0 \\ 0 & c_{\gamma2}c_{\gamma3} & 0 & -c_{\gamma2}s_{\gamma3} & 0 & s_{\gamma2}s_{\gamma3} & 0 & s_{\gamma2}c_{\gamma3} \\ s_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}s_{\gamma4} & 0 \\ 0 & -c_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}s_{\gamma3} \\ s_{\gamma1}s_{\gamma4} & 0 & c_{\gamma1}s_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 & -s_{\gamma1}c_{\gamma4} & 0 \\ 0 & s_{\gamma2}s_{\gamma3} & 0 & -s_{\gamma2}c_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} & 0 & c_{\gamma2}s_{\gamma3} \\ -c_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 \\ 0 & s_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} \end{pmatrix}$$

$$\times\begin{pmatrix} c_{\delta1}c_{\delta4} & 0 & 0 & -c_{\delta1}s_{\delta4} & s_{\delta1}s_{\delta4} & 0 & 0 \\ 0 & c_{\delta2}c_{\delta3} & -s_{\delta2}c_{\delta3} & 0 & 0 & s_{\delta2}s_{\delta3} & c_{\delta2}s_{\delta3} \\ 0 & -s_{\delta2}c_{\delta3} & c_{\delta2}c_{\delta3} & 0 & 0 & -c_{\delta2}s_{\delta3} & -s_{\delta2}s_{\delta3} \\ -c_{\delta1}s_{\delta4} & 0 & 0 & c_{\delta1}c_{\delta4} & -s_{\delta1}c_{\delta4} & 0 & 0 \\ s_{\delta1}s_{\delta4} & 0 & 0 & -s_{\delta1}c_{\delta4} & c_{\delta1}c_{\delta4} & 0 & 0 \\ 0 & s_{\delta2}s_{\delta3} & -c_{\delta2}s_{\delta3} & 0 & 0 & c_{\delta2}c_{\delta3} & s_{\delta2}c_{\delta3} \\ 0 & c_{\delta2}s_{\delta3} & -s_{\delta2}s_{\delta3} & 0 & 0 & s_{\delta2}c_{\delta3} & c_{\delta2}c_{\delta3} \\ s_{\delta1}c_{\delta4} & 0 & 0 & -s_{\delta1}s_{\delta4} & c_{\delta1}s_{\delta4} & 0 & 0 \\ c_{\delta1}c_{\delta4} & 0 & 0 & 0 & & & 0 \end{pmatrix}$$

$$s_{\delta I}=\sinh\ \delta_I, c_{\delta I}=\cosh\ \delta_I, s_{\delta I\ldots J}=s_{\delta I}\ldots s_{\delta J}, c_{\delta I\ldots J}=c_{\delta I}\ldots c_{\delta J}$$

$$k^\sharp k=\mathbb{I}$$

$$k=\mathrm{e}^{\beta k_0}$$

$$M=m\cos\left(2\beta\right)$$

$$N=m\sin\left(2\beta\right)$$

$$\mathcal{M}=k^\sharp\mathcal{M}_{\mathrm{KTN}}k,$$

$$M=m\mu_1+n\mu_2, N=m\nu_1+n\nu_2$$

$$\mu_1=1+\sum_I\left(\frac{s_{\delta I}^2+s_{\gamma I}^2}{2}-s_{\delta I}^2s_{\gamma I}^2\right)+\frac{1}{2}\sum_{I,J}s_{\delta I}^2s_{\gamma J}^2, \mu_2=\sum_I s_{\delta I}c_{\delta I}\left(\frac{s_{\gamma I}}{c_{\gamma I}}c_{\gamma1234}-\frac{c_{\gamma I}}{s_{\gamma I}}s_{\gamma1234}\right)$$

$$\nu_1=\sum_I s_{\gamma I}c_{\gamma I}\left(\frac{c_{\delta I}}{s_{\delta I}}s_{\delta1234}-\frac{s_{\delta I}}{c_{\delta I}}c_{\delta1234}\right), \nu_2=\iota-D$$

$$\iota=c_{\delta1234}c_{\gamma1234}+s_{\delta1234}s_{\gamma1234}+\sum_{I<J}c_{\delta1234}\frac{s_{\delta IJ}}{c_{\delta IJ}}\frac{c_{\gamma IJ}}{s_{\gamma IJ}}s_{\gamma1234}$$

$$D=c_{\delta1234}s_{\gamma1234}+s_{\delta1234}c_{\gamma1234}+\sum_{I<J}c_{\delta1234}\frac{s_{\delta IJ}}{c_{\delta IJ}}\frac{s_{\gamma IJ}}{c_{\gamma IJ}}c_{\gamma1234}$$

$$n_0\equiv-m\frac{\nu_1}{\nu_2}$$

$$Q_I=2\frac{\partial M}{\partial\delta_I}, P^I=-2\frac{\partial N}{\partial\delta_I}$$

$$Q_I=m\rho_I^1+n\rho_I^2, P^I=m\pi_1^I+n\pi_2^I$$

$$\rho_I^1 = 2\frac{\partial \mu_1}{\partial \delta_I}, \rho_I^2 = 2\frac{\partial \mu_2}{\partial \delta_I}, \pi_1^I = -2\frac{\partial \nu_1}{\partial \delta_I}, \pi_2^I = -2\frac{\partial \nu_2}{\partial \delta_I}$$

$$\rho_I^1 = 2s_{\delta I}c_{\delta I}\left(1 - s_{\gamma I}^2 + \sum_{J\neq I} s_{\gamma J}^2\right), \rho_I^2 = 2(1+2s_{\delta I}^2)\left(\frac{s_{\gamma I}}{c_{\gamma I}}c_{\gamma 1234} - \frac{c_{\gamma I}}{s_{\gamma I}}s_{\gamma 1234}\right)$$

$$\pi_1^I = 2\left[s_{\gamma I}c_{\gamma I}(c_{\delta 1234} - s_{\delta 1234}) + \sum_{J\neq I} s_{\gamma J}c_{\gamma J}\left(c_{\delta 1234}\frac{s_{\delta IJ}}{c_{\delta IJ}} - s_{\delta 1234}\frac{c_{\delta IJ}}{s_{\delta IJ}}\right)\right]$$

$$\pi_2^I = -2\left\{(c_{\gamma 1234} - s_{\gamma 1234})\left(c_{\delta 1234}\frac{s_{\delta I}}{c_{\delta I}} - s_{\delta 1234}\frac{c_{\delta I}}{s_{\delta I}}\right)\right.$$
$$\left. + \sum_{J\neq I}\left[c_{\gamma 1234}\frac{s_{\gamma IJ}}{c_{\gamma IJ}}\left(\frac{c_{\delta J}}{s_{\delta J}}s_{\delta 1234} - \frac{s_{\delta J}}{c_{\delta J}}c_{\delta 1234}\right) + s_{\gamma 1234}\frac{c_{\gamma IJ}}{s_{\gamma IJ}}\left(\frac{s_{\delta J}}{c_{\delta J}}c_{\delta 1234} - \frac{c_{\delta J}}{s_{\delta J}}s_{\delta 1234}\right)\right]\right\}.$$

$$J = (\nu_2 m - \nu_1 n)a$$

$$\mathrm{d}s^2 = -\frac{r^2 - 2mr - n_0^2}{W_0}\,\mathrm{d}t^2 + W_0\left(\frac{\mathrm{d}r^2}{r^2 - 2mr - n_0^2} + \mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2\right).$$

$$W_0^2(r) = \overline{\mathcal{M}}_{33}\overline{\mathcal{M}}_{44} - \overline{\mathcal{M}}_{34}^2.$$

$$\mathrm{d}\tilde{A}_{I(3\,\mathrm{d})} = \tilde{P}_I(r)\sin\,\theta\mathrm{d}\theta\wedge\mathrm{d}\phi$$

$$\tilde{A}_{I(3\,\mathrm{d})} = \tilde{P}_I\cos\,\theta\,\mathrm{d}\phi$$

$$\mathrm{d}s^2 = -\frac{R-U}{W}\left(\,\mathrm{d}t + \omega_{3\phi}\,\mathrm{d}\phi\right)^2 + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U} + \frac{RU}{a^2(R-U)}\mathrm{d}\phi^2\right),$$

$$W^2(r,u) = \overline{\mathcal{M}}_{33}\overline{\mathcal{M}}_{44} - \overline{\mathcal{M}}_{34}^2.$$

$$\mathrm{d}s^2 = -\frac{R_0(r)}{W_0(r)}(\mathrm{d}t + 2N\cos\,\theta\,\mathrm{d}\phi)^2 + W_0(r)\left(\frac{dr^2}{R_0(r)} + \mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2\right),$$

$$R_0(r) = r^2 - 2mr - n^2, W_0^2(r) = R_0^2(r) + 2R_0(r)(2Mr + V) + (L(r) + 2Nn)^2.$$

$$L(r) = \lambda_1 r + \lambda_0$$

$$\lambda_1 = 2(m\nu_2 - n\nu_1), \lambda_0 = 4(m^2+n^2)D, V = 2(-\mu_2 m + \mu_1 n)n + 2(m^2+n^2)C,$$

$$C = 1 + \sum_I (s_{\delta I}^2 c_{\gamma I}^2 + s_{\gamma I}^2 c_{\delta I}^2) + \sum_{I<J}(s_{\delta IJ}^2 + s_{\gamma IJ}^2) + \sum_{I\neq J} s_{\delta I}^2 s_{\gamma J}^2 + \sum_I \sum_{J<K}(s_{\delta I}^2 s_{\gamma JK}^2 + s_{\gamma I}^2 s_{\delta JK}^2)$$
$$+2\sum_{I<J}\left(s_{\delta 1234}c_{\delta 1234}\frac{s_{\gamma IJ}}{c_{\delta IJ}}\frac{c_{\gamma IJ}}{s_{\delta IJ}} + s_{\delta 1234}^2\frac{s_{\gamma IJ}^2}{s_{\delta IJ}^2} + s_{\delta IJ}s_{\gamma IJ}c_{\delta IJ}c_{\gamma IJ} + s_{\delta IJ}^2 s_{\gamma IJ}^2\right) - \nu_1^2 - \nu_2^2.$$

$$\theta \in [0,\pi], \phi \sim \phi + 2\pi$$

$$A^I = \zeta^I(r)(\mathrm{d}t + 2N\cos\,\theta\,\mathrm{d}\phi) + P^I\cos\,\theta\,\mathrm{d}\phi, \tilde{A}_I = \tilde{\zeta}_I(r)(\mathrm{d}t + 2N\cos\,\theta\,\mathrm{d}\phi) - Q_I\cos\,\theta\,\mathrm{d}\phi$$

$$\zeta^I = \frac{1}{2W_0^2}\frac{\partial W_0^2}{\partial \delta_I} = \frac{1}{W_0^2(r)}\left[R(r)\left(Q_I r + \frac{\partial V}{\partial \delta_I}\right) + (L(r) + 2Nn)\left(\frac{\partial L(r)}{\partial \delta_I} - P^I n\right)\right].$$

$$\tilde{\zeta}_I = \frac{R(r)\left(P^I r + \tilde{V}_I\right) + (L(r) + 2Nn)\left(\tilde{L}_I(r) + Q_I n\right)}{W_0^2(r)},$$

$$\tilde{L}_I(r) = (m\rho_I^2 - n\rho_I^1)r - 4(m^2 + n^2)\tilde{D}_I, \tilde{V}_I = (n\pi_1^I - m\pi_2^I)n + 2(m^2 + n^2)\tilde{C}_I$$

$$\tilde{D}_I = \frac{s_{\gamma I}}{c_{\gamma I}} c_{\gamma 1234} s_{\delta I}^2 - \frac{c_{\gamma I}}{s_{\gamma I}} s_{\gamma 1234} c_{\delta I}^2$$

$$\tilde{C}_I = (s_{\delta 1234} - c_{\delta 1234})\tilde{C}_{II} + 2 s_{\gamma I} c_{\gamma I} s_{\delta 1234}\left(2 + \sum_K s_{\gamma K}^2\right) + \sum_{J \neq I}\left(c_{\delta 1234}\frac{s_{\delta IJ}}{c_{\delta IJ}} - s_{\delta 1234}\frac{c_{\delta IJ}}{s_{\delta IJ}}\right)\tilde{C}_{IJ}$$
$$+2\sum_{J \neq I} s_{\gamma J} c_{\gamma J}\left(\frac{s_{\delta IJ}}{c_{\delta IJ}} c_{\delta 1234}\left(s_{\gamma I}^2 + s_{\gamma J}^2\right) - \frac{c_{\delta IJ}}{s_{\delta IJ}} s_{\delta 1234}\sum_{K \neq I,J} s_{\gamma K}^2\right)$$

$$\tilde{C}_{IJ} = 2\left(1 + 2s_{\delta I}^2\right)s_{\gamma 1234}\left[\left(2 + \sum_{K \neq J}\frac{1}{s_{\gamma K}^2}\right)s_{\gamma 1234}\frac{c_{\gamma J}}{s_{\gamma J}} - \left(1 + 2s_{\gamma J}^2\right)\frac{c_{\gamma 1234}}{s_{\gamma J} c_{\gamma J}}\right]$$
$$+2 s_{\delta I}^2 s_{\gamma J} c_{\gamma J}\left(1 + \sum_K s_{\gamma K}^2\right)$$

$$\mathrm{e}^{\varphi_i} = \frac{r^2 + u^2 + g_i}{W}, \chi_i = \frac{f_i}{r^2 + u^2 + g_i}$$

$$f_i = 2(mr + nu)\xi_{i1} + 2(mu - nr)\xi_{i2} + 4(m^2 + n^2)\xi_{i3}$$
$$g_i = 2(mr + nu)\eta_{i1} + 2(mu - nr)\eta_{i2} + 4(m^2 + n^2)\eta_{i3}$$

$$\xi_{11} = \left[(s_{\delta 123} c_{\delta 4} - c_{\delta 123} s_{\delta 4}) s_{\gamma_1} c_{\gamma_1} + (1 \leftrightarrow 4)\right] - ((1,4) \leftrightarrow (2,3)),$$
$$\xi_{12} = \left[\frac{1}{2}\left(c_{\delta 23} s_{\gamma 14} + c_{\gamma 14} s_{\delta 23}\right)\left(c_{\delta 14} c_{\gamma 23} + s_{\gamma 23} s_{\delta 14}\right) + s_{\delta 1} s_{\gamma 4} c_{\delta 4} c_{\gamma 1}\left(s_{\delta 2} s_{\gamma 2} c_{\delta 3} c_{\gamma 3} + s_{\delta 3} s_{\gamma 3} c_{\delta 2} c_{\gamma 2}\right)\right.$$
$$\left.+(1 \leftrightarrow 4)\right] - ((1,4) \leftrightarrow (2,3)),$$
$$\xi_{13} = \left[\left(s_{\delta 134} c_{\delta 2} c_{\gamma 2}^2 + c_{\delta 134} s_{\delta 2} s_{\gamma 2}^2\right) s_{\gamma_3} c_{\gamma_3} + (2 \leftrightarrow 3)\right] - ((1,4) \leftrightarrow (2,3)),$$

$$\eta_{11} = s_{\delta 2}^2 + s_{\delta 3}^2 + s_{\gamma 1}^2 + s_{\gamma 4}^2 + \left(s_{\delta 2}^2 + s_{\delta 3}^2\right)\left(s_{\gamma 1}^2 + s_{\gamma 4}^2\right) + \left(s_{\delta 2}^2 - s_{\delta 3}^2\right)\left(s_{\gamma 3}^2 - s_{\gamma 2}^2\right),$$
$$\eta_{12} = 2 s_{\delta 2} c_{\delta 2}\left(c_{\gamma 2} s_{\gamma 134} - s_{\gamma 2} c_{\gamma 134}\right) + (2 \leftrightarrow 3),$$
$$\eta_{13} = 2 s_{\delta 23} c_{\delta 23}\left(s_{\gamma 23} c_{\gamma 23} + s_{\gamma 14} c_{\gamma 14}\right) + s_{\delta 23}^2\left(1 + \sum_I s_{\gamma I}^2\right) + \left(s_{\delta 2}^2 + s_{\delta 3}^2 + 2 s_{\delta 23}^2\right)\left(s_{\gamma 14}^2 + s_{\gamma 23}^2\right)$$
$$+ s_{\delta 2}^2 s_{\gamma 2}^2 + s_{\delta 3}^2 s_{\gamma 3}^2 + s_{\gamma 14}^2.$$

$$\mathrm{d}s^2 = -\frac{R - U}{W}(\mathrm{d}t + \omega_3)^2 + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U} + \frac{RU}{a^2(R - U)}\mathrm{d}\phi^2\right),$$

$$R(r) = r^2 - 2mr + a^2 - n^2, U(u) = a^2 - (u - n)^2.$$

$$W^2 = (R - U)^2 + (2Nu + L)^2 + 2(R - U)(2Mr + V)$$
$$\omega_3 = \frac{2N(u - n)R + U(L + 2Nn)}{a(R - U)}\mathrm{d}\phi$$

$$L(r)=2(-n\nu_1+m\nu_2)r+4(m^2+n^2)D, V(u)=2(n\mu_1-m\mu_2)u+2(m^2+n^2)C,$$

$$A^I=-W\frac{\partial}{\partial\delta_I}\left(\frac{\mathrm{dt}+\omega_3}{W}\right)$$

$$A^I=\zeta^I(\mathrm{dt}+\omega_3)+A^I_{(3\text{ d})}$$

$$\zeta^I=\frac{1}{2W^2}\frac{\partial}{\partial\delta_I}(W^2)=\frac{1}{W^2}\left[(R-U)\left(Q_Ir+\frac{\partial V}{\partial\delta_I}\right)+(L+2Nu)\left(\frac{\partial L}{\partial\delta_I}-P^Iu\right)\right],$$

$$A^I_{(3\mathrm{d})}=-\frac{\partial}{\partial\delta_I}\omega_3=\left[P^I(u-n)+\frac{U}{R-U}\left(P^Iu-\frac{\partial L}{\partial\delta_I}\right)\right]\frac{\mathrm{d}\phi}{a}.$$

$$\tilde{A}_I=\tilde{\zeta}_I(\mathrm{dt}+\omega_3)+\tilde{A}_{I(3\text{ d})}$$

$$\tilde{A}_{I(3\mathrm{d})}=-\left(Q_I(u-n)+\frac{U\left(Q_Iu+\tilde{L}_I\right)}{R-U}\right)\frac{\mathrm{d}\phi}{a}$$

$$\tilde{\zeta}_I=\frac{1}{W^2}\left((R-U)\left(P^Ir+\tilde{V}_I\right)+(L+2Nu)\left(\tilde{L}_I+Q_Iu\right)\right)$$

$$\tilde{L}_I(r)=(m\rho_I^2-n\rho_I^1)r-4(m^2+n^2)\tilde{D}_I, \tilde{V}_I(u)=(n\pi_1^I-m\pi_2^I)u+2(m^2+n^2)\tilde{C}_I.$$

$$\mathrm{e}^{\varphi_i}=\frac{r^2+u^2+g_i}{W}, \chi_i=\frac{f_i}{r^2+u^2+g_i},$$

$$f_i=2(mr+nu)\xi_{i1}+2(mu-nr)\xi_{i2}+4(m^2+n^2)\xi_{i3}$$
$$g_i=2(mr+nu)\eta_{i1}+2(mu-nr)\eta_{i2}+4(m^2+n^2)\eta_{i3}$$

$$\bar{Q}_I=\frac{1}{4G}Q_I, \bar{P}^I=\frac{1}{4G}P^I$$

$$M=\frac{m}{G}\left(\mu_1-\frac{\nu_1\mu_2}{\nu_2}\right), \qquad J=\frac{ma}{G}\frac{(\nu_1^2+\nu_2^2)}{\nu_2}$$
$$\bar{Q}_I=\frac{m}{2G}\left(\frac{\partial\mu_1}{\partial\delta_I}-\frac{\nu_1}{\nu_2}\frac{\partial\mu_2}{\partial\delta_I}\right), \qquad \bar{P}^I=\frac{m}{2G}\left(\frac{\nu_1}{\nu_2}\frac{\partial\nu_2}{\partial\delta_I}-\frac{\partial\nu_1}{\partial\delta_I}\right)$$

$$\xi^\mu\partial_\mu=\partial_t+\Omega_+\partial_\phi$$

$$\Omega_+=\frac{a}{L(r_+)}$$

$$S_+=\frac{\pi}{G}L(r_+), T_+=\frac{R'(r_+)}{4\pi L(r_+)}=\frac{r_+-m}{2\pi L(r_+)}$$

$$\Phi_+^I=\xi_+^\mu A_\mu^I$$

$$\Psi_I^+=\xi_+^\mu\tilde{A}_{I\mu}$$

$$\Phi_+^I=\Omega_+A^I_{(3\text{ d})\phi}(r_+)=\frac{1}{L}\left(\frac{\partial L}{\partial\delta_I}-n_0P^I\right)\bigg|_{r=r_+}, \Psi_I^+=\Omega_+\tilde{A}^I_{(3\text{ d})\phi}(r_+)=\frac{\tilde{L}_I+n_0Q_I}{L}\bigg|_{r=r_+}$$

$$\delta M = T_+\delta S_+ + \Omega_+\delta J + \Phi_+^I\delta\bar{Q}_I + \Psi_I^+\delta\bar{P}^I$$

$$M = 2T_+S_+ + 2\Omega_+J + \Phi_+^I\bar{Q}_I + \Psi_I^+\bar{P}^I$$

$$\sum_I (\rho_I^2\pi_1^I - \rho_I^1\pi_2^I) = 8(\mu_1\nu_2 - \mu_2\nu_1 - \iota - D)$$

$$\sum_I \left(Q_I\frac{\partial D}{\partial\delta_I} - P^I\widetilde{D}_I\right) = 4D(\mu_1+1)m + (4D\mu_2 + 2\nu_1)n_0$$

$$S_+ = 2\pi\sqrt{|\diamond|}$$

$$S_+ = 2\pi\sqrt{|\Delta|}$$

$$\Delta(Q_I, P^I) = \frac{1}{16}\left(4(Q_1Q_2Q_3Q_4 + P^1P^2P^3P^4) + 2\sum_{J<K} Q_JQ_KP^JP^K - \sum_J (Q_J)^2(P^J)^2\right)$$

$$Q_I = Q, P^I = P$$

$$\Delta = \frac{1}{4}(Q^2+P^2)^2$$

$$\Delta = \frac{1}{4}Q_1Q_2Q_3Q_4$$

$$\Delta = -\frac{1}{16}(Q_1)^2(P^1)^2$$

$$\big((Q_1,P^1) = (Q_4,P^4) \text{ and } (Q_2,P^2) = (Q_3,P^3)\big)$$

$$\Delta = \frac{1}{4}(Q_1Q_2 + P^1P^2)^2$$

$$\Delta = \frac{1}{32}\epsilon^{ab}\epsilon^{cd}\epsilon^{a'b'}\epsilon^{c'd'}\epsilon^{a''c''}\epsilon^{b''d''}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}$$

$$(\gamma_{000},\gamma_{111}) = (P^4,-Q_4), \quad (\gamma_{100},\gamma_{011}) = (Q_1,-P^1)$$
$$(\gamma_{010},\gamma_{101}) = (Q_2,-P^2), \quad (\gamma_{001},\gamma_{110}) = (Q_3,-P^3)$$

$$\Delta = \frac{1}{32}\epsilon^{a'b'}\epsilon^{c'd'}\epsilon^{a''b''}\epsilon^{c''d''}\epsilon^{ac}\epsilon^{bd}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}$$
$$= \frac{1}{32}\epsilon^{a''b''}\epsilon^{c''d''}\epsilon^{ab}\epsilon^{cd}\epsilon^{a'c'}\epsilon^{b'd'}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}$$

$$\Delta = \frac{m^4(\nu_1^2+\nu_2^2)^2(4\iota D - \nu_1^2)}{\nu_2^4}$$

$$\xi_-^\mu\partial_\mu = \partial_t + \Omega_-\partial_\phi$$

$$S_-T_- \leq 0$$

$$S_- = \frac{\pi}{G} L(r_-)$$

$$\delta M = T_- \delta S_- + \Omega_- \delta J + \Phi_-^I \delta \bar{Q}_I + \Psi_I^- \delta \bar{P}^I$$
$$M = 2T_- S_- + 2\Omega_- J + \Phi_-^I \bar{Q}_I + \Psi_I^- \bar{P}^I$$

$$\frac{S_-}{S_+} = \frac{T_+}{-T_-} = \frac{\Omega_+}{\Omega_-} = \frac{L(r_-)}{L(r_+)},$$

$$S_+ = \frac{\pi^2}{3} c_J \frac{-2T_-}{\Omega_- - \Omega_+} = \frac{\pi^2}{3} c_{Q_I} \frac{-2T_-}{\Phi_-^I - \Phi_+^I} = \frac{\pi^2}{3} c_{P^I} \frac{-2T_-}{\Psi_I^- - \Psi_I^+},$$

$$c_J = 6\frac{\partial \Delta_J}{\partial J}, c_{Q_I} = 6\frac{\partial \Delta_J}{\partial \bar{Q}_I}, c_{P^I} = 6\frac{\partial \Delta_J}{\partial \bar{P}^I}$$

$$\Delta_J = \Delta + J^2$$

$$8\pi^2 J = \Omega_+ S_+ \left(\frac{1}{T_+} + \frac{1}{T_-}\right)$$

$$\frac{A_+ A_-}{64\pi^2 G^2} = J^2 + \Delta(Q_I, P^I) = \Delta_J$$

$$S_L \equiv \frac{1}{2}(S_+ + S_-), S_R \equiv \frac{1}{2}(S_+ - S_-)$$

$$S_L^2 - S_R^2 = 4\pi^2 (J^2 + \Delta)$$

$$A_+ A_- = (8\pi J)^2 + (4\pi Q^2)^2$$

$$\mathcal{A}^2 \geq (8\pi J)^2 + (4\pi Q^2)^2$$

$$A_+ A_- = (8\pi J)^2 + (8\pi)^2 \Delta, \mathcal{A}^2 \geq (8\pi J)^2 + (8\pi)^2 \Delta$$

$$A_+ A_- = (8\pi J)^2 + (8\pi)^2 \diamond, \mathcal{A}^2 \geq (8\pi J)^2 + (8\pi)^2 \diamond$$

$$S_+ = 2\pi \left(\sqrt{\Delta + F} + \sqrt{-J^2 + F}\right)$$

$$F(M, Q_I, P^I) = \frac{m^4 (\nu_1^2 + \nu_2^2)^3}{\nu_2^4}$$

$$\Omega_+ / T_+ = -\Omega_- / T_-$$

$$\partial S_L / \partial J = 0$$

$$\partial (S_R^2) / \partial J = -8\pi^2 J$$

$$S_R = 2\pi \sqrt{-J^2 + F}$$

$$S_L = 2\pi \sqrt{\Delta + F}$$

$$S_+ = S_L + S_R$$

$$\mathcal{M}_i(r,u) = \frac{1}{W}\begin{pmatrix} r^2+u^2+g_i & f_i \\ f_i & (W^2+f_i^2)/(r^2+u^2+g_i) \end{pmatrix}$$

$$\mathcal{M}_i = \mathbb{I} + O(r^{-1})$$

$$M_2 = \frac{1}{16}\gamma_{aa'a''}\left[(\mathcal{M}_1^{-1})^{ab}(\mathcal{M}_2^{-1})^{a'b'}(\mathcal{M}_3^{-1})^{a''b''} - (\mathcal{M}_1^{-1})^{ab}\epsilon^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}(\mathcal{M}_2^{-1})^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}\epsilon^{a'b'}(\mathcal{M}_3^{-1})^{a''b''}\right]\gamma_{bb'b''}$$

$$M_2^\infty = \frac{1}{16}\gamma_{aa'a''}\left(\delta^{ab}\delta^{a'b'}\delta^{a''b''} - \delta^{ab}\epsilon^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}\delta^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}\epsilon^{a'b'}\delta^{a''b''}\right)\gamma_{bb'b''} = \frac{1}{16}\sum_{I,J}\left(Q_I Q_J + P^I P^J\right)$$

$$M_2^\infty = |Z(P,Q,z_\infty)|^2$$

$$Z(P,Q,z,\bar{z}) = \frac{1}{\sqrt{2}}\mathrm{e}^{K(z,\bar{z})/2}\left(X^\Lambda(z)Q_\Lambda - F_\Lambda(z)P^\Lambda\right)$$

$$K = -\log(-8y_1y_2y_3)$$

$$F_\Lambda = \partial_\Lambda F$$

$$M^2 \geq M_2^\infty$$

$$I_2(r,u) = -\frac{1}{4}(\widetilde{P}^\Lambda, \widetilde{Q}_\Lambda)\begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}\begin{pmatrix} \widetilde{P}^\Lambda \\ \widetilde{Q}_\Lambda \end{pmatrix},$$

$$J_2(r,u) = \frac{1}{4}(\widetilde{P}^\Lambda, \widetilde{Q}_\Lambda)\begin{pmatrix} \mathrm{Im}\,F + \mathrm{Re}\,F(\mathrm{Im}\,F)^{-1}\mathrm{Re}\,F & -\mathrm{Re}\,F(\mathrm{Im}\,F)^{-1} \\ -(\mathrm{Im}\,F)^{-1}\mathrm{Re}\,F & (\mathrm{Im}\,F)^{-1} \end{pmatrix}\begin{pmatrix} \widetilde{P}^\Lambda \\ \widetilde{Q}_\Lambda \end{pmatrix},$$

$$F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$$

$$F = -X^1X^2X^3/X^0$$

$$I_2^\infty \equiv I_2(\infty,u) = \frac{1}{4}\sum_I\left[(Q_I)^2 + (P^I)^2\right]$$

$$J_2^\infty \equiv J_2(\infty,u) = \frac{1}{4}\sum_I\left[(Q_I)^2 + (P^I)^2\right] - \frac{1}{8}\sum_{I,J}\left(P_IP_J + Q_IQ_J\right)$$

$$|Z|^2 + |Z_i|^2 = I_2^\infty, -|Z|^2 + |Z_i|^2 = J_2^\infty$$

$$J_2^\infty = I_2^\infty - 2M_2^\infty$$

$$S_2^\infty = \frac{1}{4}G_{AB}\partial_r\Phi^A\partial_r\Phi^B\Big|_{r=\infty}$$

$$S_2^\infty = \frac{1}{4}\sum_i \left(\Sigma_i^2 + \Xi_i^2\right).$$

$$M^2 + N^2 + S_2^\infty = I_2^\infty + 4S_+^2T_+^2.$$

$$M^2 + N^2 + S_2^\infty = I_2^\infty + 4S_+^2\left(T_+^2 + \frac{\Omega_+^2}{4\pi^2}\right)$$

$$4G^2S_+^2\left(T_+^2 + \frac{\Omega_+^2}{4\pi^2}\right) = m^2 + n^2 = 4G^2S_-^2\left(T_-^2 + \frac{\Omega_-^2}{4\pi^2}\right)$$

$$\begin{aligned} M &= m\cosh(2\delta)\cosh(2\gamma) + n\sinh(2\delta)\sinh(2\gamma),\\ N &= n\cosh(2\delta)\cosh(2\gamma) - m\sinh(2\delta)\sinh(2\gamma),\\ Q \equiv Q_I &= m\sinh(2\delta)\cosh(2\gamma) + n\sinh(2\gamma)\cosh(2\delta),\\ P \equiv P^I &= m\sinh(2\gamma)\cosh(2\delta) - n\sinh(2\delta)\cosh(2\gamma),\\ J &= aM, \end{aligned}$$

$$\Delta = \frac{1}{4}(Q^2 + P^2)^2.$$

$$F = M^2(M^2 - Q^2 - P^2).$$

$$M = \frac{1}{2}m\left(c_{\delta 1}^2 c_{\gamma 1}^2 - 1\right), \qquad Q_1 = \frac{2ms_{\delta 1}\left(c_{\delta 1}^2 + s_{\delta 1}^2 s_{\gamma 1}^2\right)}{c_{\delta 1}},$$

$$N = 0, \qquad P^1 = \frac{2ms_{\gamma 1}c_{\gamma 1}}{c_{\delta 1}}$$

$$\Delta = -\frac{1}{16}(Q_1)^2(P^1)^2, F = m^4\frac{c_{\gamma_1}^2}{c_{\delta_1}^4}\left(c_{\delta_1}^2 + s_{\delta_1}^2 s_{\gamma_1}^2\right)^3$$

$$H(\psi) = 2\cos\psi\cos(\psi/3) + 6\sin\psi\sin(\psi/3) - 2$$

$$\sin^2\psi(M, Q_1, P^1) = \frac{54M^2[(Q_1)^2 - (P^1)^2]^2}{[8M^2 + (Q_1)^2 + (P^1)^2]^3}$$

$$0 \le \psi \le \pi/2$$

$$4M \ge \left[(Q_1)^{2/3} + (P^1)^{2/3}\right]^{3/2}$$

$$F = \left[M^2 - \frac{1}{4}(Q_1)^2\right]\left[M^2 - \frac{1}{4}(P^1)^2\right] + \frac{1}{3}\left\{M^2 + \frac{1}{8}[(Q_1)^2 + (P^1)^2]\right\}^2 H\left(\psi(M, Q_1, P^1)\right).$$

$$F = \frac{1}{64}\left[32M^4 - 40M^2(Q_1)^2 - (Q_1)^4 + 4M(4M^2 + 2(Q_1)^2)^{3/2}\right].$$

$$S_+ = 2\pi\left(\sqrt{F - \frac{1}{16}(Q_1)^2(P^1)^2} + \sqrt{F - J^2}\right)$$

$$M=\frac{m}{4}\sum_{I}\cosh(2\delta_I),\qquad N=n(c_{\delta 1234}-s_{\delta 1234}),$$
$$Q_I=m\sinh(2\delta_I),\qquad P^I=2n(c_{\delta 1}s_{\delta 234}-s_{\delta 1}c_{\delta 234}).$$

$$J=ma(c_{\delta 1234}-s_{\delta 1234})$$

$$\Delta=\frac{1}{4}Q_1Q_2Q_3Q_4$$
$$F=\frac{1}{8}\left(m^4-4\Delta+\prod_{I}\sqrt{m^2+Q_I^2}+m^2\sum_{I<J}\sqrt{m^2+Q_I^2}\sqrt{m^2+Q_J^2}\right)$$

$$W^2(r,u)=(r^2-2mr+u^2)(r^2+2(2M-m)r+u^2)+4v_2^2m^2r^2$$

$$ds^2=-\frac{r^2-2mr+u^2}{W(r,u)}(dt+\omega_3)^2+W(r,u)\left(\frac{dr^2}{R(r)}+\frac{du^2}{a^2-u^2}+\frac{R(r)(a^2-u^2)}{a^2(r^2-2mr+u^2)}d\phi^2\right)$$

$$R(r)=r^2-2mr+a^2$$

$$\omega_3=\frac{2v_2m(a^2-u^2)r}{a(r^2-2mr+u^2)}d\phi$$

$$(\delta_1,\gamma_1)=(\delta_4,\gamma_4)\text{ and }(\delta_2,\gamma_2)=(\delta_3,\gamma_3)$$

$$Q_2=Q_1,P^2=P^1$$

$$\Delta r_1=m[\cosh(2\delta_1)\cosh(2\gamma_2)-1]+n\sinh(2\delta_1)\sinh(2\gamma_1),$$
$$\Delta r_2=m[\cosh(2\delta_2)\cosh(2\gamma_1)-1]+n\sinh(2\delta_2)\sinh(2\gamma_2),$$
$$\Delta u_1=n[\cosh(2\delta_1)\cosh(2\gamma_2)-1]-m\sinh(2\delta_1)\sinh(2\gamma_1),$$
$$\Delta u_2=n[\cosh(2\delta_2)\cosh(2\gamma_1)-1]-m\sinh(2\delta_2)\sinh(2\gamma_2),$$

$$r_1=r+\Delta r_1,r_2=r+\Delta r_2,u_1=u+\Delta u_1,u_2=u+\Delta u_2$$

$$W=r_1r_2+u_1u_2$$

$$ds^2=-\frac{R}{W}\left(dt-\frac{a^2-u_1u_2+(\Delta u_1+n)(\Delta u_2+n)}{a}d\phi\right)^2+\frac{W}{R}dr^2$$
$$+\frac{U}{W}\left(dt-\frac{r_1r_2+a^2+(\Delta u_1+n)(\Delta u_2+n)}{a}d\phi\right)^2+\frac{W}{U}du^2.$$

$$A^1=\frac{Q_1r_2}{W}\left(dt-\frac{a^2-u_1u_2+(\Delta u_1+n)(\Delta u_2+n)}{a}d\phi\right)$$
$$-\frac{P^1u_2}{W}\left(dt-\frac{r_1r_2+a^2+(\Delta u_1+n)(\Delta u_2+n)}{a}d\phi\right)+\frac{(\Delta u_2+n)}{2a}\frac{\partial(\Delta u_1)}{\partial\delta_1}d\phi$$

$$\tilde{A}_1 = \frac{P^1 r_1}{W}\left(\mathrm{d}t - \frac{a^2 - u_1 u_2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a} \mathrm{d}\phi \right) + \frac{Q_1 u_1}{W}\left(\mathrm{d}t - \frac{r_1 r_2 + a^2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a} \mathrm{d}\phi \right) + \frac{(\Delta u_1 + n)}{2a}\frac{\partial(\Delta r_1)}{\partial \delta_1} \mathrm{d}\phi$$

$$(\delta_1, \gamma_1) = (\delta_4, \gamma_4) \text{ and } (\delta_2, \gamma_2) = (\delta_3, \gamma_3)$$

$$\mathrm{e}^{\varphi_1} = \frac{r_2^2 + u_2^2}{W}, \chi_1 = \frac{r_2 u_1 - r_1 u_2}{r_2^2 + u_2^2}$$

$$\mathrm{d}s^2 = -\frac{R}{W}(\mathrm{d}\tau + u_1 u_2 \, \mathrm{d}\psi)^2 + \frac{U}{W}(\mathrm{d}\tau - r_1 r_2 \, \mathrm{d}\psi)^2 + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right),$$

$$A^1 = \frac{Q_1 r_2}{W}(\mathrm{d}\tau + u_1 u_2 \, \mathrm{d}\psi) - \frac{P^1 u_2}{W}(\mathrm{d}\tau - r_1 r_2 \, \mathrm{d}\psi)$$
$$\tilde{A}_1 = \frac{P^1 r_1}{W}(\mathrm{d}\tau + u_1 u_2 \, \mathrm{d}\psi) + \frac{Q_1 u_1}{W}(\mathrm{d}\tau - r_1 r_2 \, \mathrm{d}\psi)$$

$$\nu_1 = -\mu_2 = -\frac{1}{2}[\sinh(2\delta_1)\sinh(2\gamma_1) + \sinh(2\delta_2)\sinh(2\gamma_2)],$$
$$\nu_2 = \mu_1 = \frac{1}{2}[\cosh(2\delta_1)\cosh(2\gamma_2) + \cosh(2\delta_2)\cosh(2\gamma_1)].$$

$$\begin{aligned} M &= m + \frac{1}{2}(\Delta r_1 + \Delta r_2), & N &= n + \frac{1}{2}(\Delta u_1 + \Delta u_2) \\ Q_1 &= \frac{\partial M}{\partial \delta_1} = \frac{1}{2}\frac{\partial(\Delta r_1)}{\partial \delta_1}, & P^1 &= -\frac{\partial N}{\partial \delta_1} = -\frac{1}{2}\frac{\partial(\Delta u_1)}{\partial \delta_1} \\ Q_2 &= \frac{\partial M}{\partial \delta_2} = \frac{1}{2}\frac{\partial(\Delta r_2)}{\partial \delta_2}, & P^2 &= -\frac{\partial N}{\partial \delta_2} = -\frac{1}{2}\frac{\partial(\Delta u_2)}{\partial \delta_2} \\ J &= Ma \end{aligned}$$

$$\Delta = \left(\frac{1}{2}I_2^\infty - M_2^\infty\right)^2 = \frac{1}{4}(Q_1 Q_2 + P^1 P^2)^2$$
$$F = \left(M^2 - \frac{1}{2}I_2^\infty\right)^2 - \Delta = (M^2 - M_2^\infty)(M^2 + M_2^\infty - I_2^\infty)$$

$$I_2^\infty = \frac{1}{2}[(Q_1)^2 + (P^1)^2 + (Q_2)^2 + (P^2)^2], M_2^\infty = \frac{1}{4}[(Q_1 + P^1)^2 + (Q_2 + P^2)^2].$$

$$(\delta_1, \gamma_1) = (\delta_4, \gamma_4) \text{ and } \delta_2 = \delta_3 = \gamma_2 = \gamma_3 = 0$$

$$m \sim \epsilon^2, \delta_I \sim \epsilon^0, \mathrm{e}^{\gamma_I} \sim \epsilon^{-1}.$$

$$M^2 = M_2^\infty$$

$$S_+ = 2\pi\sqrt{\Delta},$$

$$S_2^\infty = I_2^\infty - M_2^\infty$$

$$\mathrm{d}s^2 = -r^2 W_0^{-1}(r)\mathrm{d}t^2 + W_0(r) r^{-2}(\mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2),$$

$$a=\sqrt{m^2+n_0^2}$$

$$r=r_+=r_-=m$$

$$M^2=I_2^\infty-S_2^\infty+a^2.$$

$$J/a=m(v_1^2+v_2^2)/v_2$$

$$F=J^2.$$

$$S_+=2\pi\sqrt{\Delta+J^2}.$$

$$t\to r_0\lambda^{-1}t, r\to r_++\lambda r_0 r, \phi\to\phi+\Omega_+^{\text{ext}}\lambda^{-1}r_0 t$$

$$A^I\to A^I-\Phi_{+,\text{ext}}^I\lambda^{-1}r_0\,\mathrm{d}t, \tilde{A}_I\to\tilde{A}_I-\Psi_{I,\text{ext}}^+\lambda^{-1}r_0\,\mathrm{d}t$$

$$\lambda\to 0, \Omega_+^{\text{ext}}, \Phi_{+,\text{ext}}^I, \Psi_{I,\text{ext}}^+$$

$$r_0^2=L(r_+)$$

$$\mathrm{d}s^2=W_+\left(-r^2\,\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}+\frac{\mathrm{d}u^2}{U}+\Gamma^2(\,\mathrm{d}\phi+kr\,\mathrm{d}t)^2\right)$$

$$W_+(u)=W(r_+,u)$$

$$\Gamma^2(u)=\frac{L(r_+)^2U(u)}{a^2W_+^2(u)}, k=2(mv_2-n_0v_1)\Omega_+=\frac{2\pi J}{S_+}$$

$$A^I=f^I(\,\mathrm{d}\phi+kr\,\mathrm{d}t)+\frac{e^I}{k}\,\mathrm{d}\phi, \tilde{A}_I=\tilde{f}_I(\,\mathrm{d}\phi+kr\,\mathrm{d}t)+\frac{\tilde{e}_I}{k}\,\mathrm{d}\phi$$

$$f^I(u)=-\frac{L(r_+)}{a}\left(\zeta^I(r_+,u)+\frac{v_1\pi_1^I+v_2\pi_2^I}{2(v_1^2+v_2^2)}\right),\quad e^I=2(mv_2-n_0v_1)\Phi_+^I-n_0\pi_1^I+m\pi_2^I$$

$$\tilde{f}_I(u)=-\frac{L(r_+)}{a}\left(\tilde{\zeta}_I(r_+,u)-\frac{v_1\rho_I^1+v_2\rho_I^2}{2(v_1^2+v_2^2)}\right),\quad \tilde{e}_I=2(mv_2-n_0v_1)\Psi_I^++n_0\rho_I^1-m\rho_I^2$$

$$S_+=\frac{1}{3}\pi^2c_JT_J$$

$$c_J=12J, T_J=\frac{1}{2\pi k}$$

$$S_+=\frac{1}{3}\pi^2c_{Q_1}T_{Q_1}$$

$$c_{Q_1}=24\frac{\partial\Delta}{\partial Q_1}, T_{Q_1}=\frac{1}{2\pi e^1}$$

$$c_{Q_1}=6Q_2Q_3Q_4+3P^1(P^2Q_2+P^3Q_3+P^4Q_4-P^1Q_1)$$

$$m\sim\epsilon^2m, n\sim\epsilon n, a\sim\epsilon a, \mathrm{e}^{\gamma_1}\sim\epsilon^{-1}\mathrm{e}^{\gamma_1}$$

$$r_+ = -r_- + O(\epsilon^2)$$

$$J^2 + \Delta \leq 0$$

$$F = -\Delta$$

$$S_+ = 2\pi\sqrt{-\Delta - J^2}$$

$$M^2 = I_2^\infty - S_\infty^2$$

$$\tilde{P}_1 = 1, \tilde{P}_2 = \tilde{P}_3, \tilde{Q}_2 = \tilde{Q}_3 \text{ and } P^4 = 0$$

$$L = O(\epsilon^{-1}), V = O(\epsilon^{-2})$$

$$W^2 = r^4 + 4Mr^3 + (M^2 b_1 + b_2 J\cos\,\theta)r^2 + b_3 M^3 r - 4J^2\cos^2\,\theta - 4\Delta, \omega_3 = \frac{2J}{r}\sin^2\,\theta\,\mathrm{d}\phi$$

$$\mathrm{d}s^2 = -\frac{r^2}{W(r,\theta)}\left(\mathrm{d}t + \frac{2J}{r}\sin^2\,\theta\,\mathrm{d}\phi\right)^2 + \frac{W(r,\theta)}{r^2}[\,\mathrm{d}r^2 + r^2(\,\mathrm{d}\theta^2 + \sin^2\,\theta\,\mathrm{d}\phi^2)]$$

$$t \to \lambda^{-1} r_0\sqrt{-\Delta - J^2}t, r \to \lambda r_0 r$$

$$A^I \to A^I - \mathrm{d}\left(\Phi_+^I \lambda^{-1} r_0\sqrt{-\Delta - J^2}t\right), \tilde{A}_I \to \tilde{A}_I - \mathrm{d}\left(\Psi_I^+ \lambda^{-1} r_0\sqrt{-\Delta - J^2}t\right)$$

$$\mathrm{d}s^2 = W_+\left(-r^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \Gamma^2(\,\mathrm{d}\phi - kr\,\mathrm{d}t)^2\right)$$

$$W_+ = 2\sqrt{-\Delta - J^2\cos^2\,\theta}, k = \frac{J}{\sqrt{-\Delta + J^2}}, \Gamma^2 = \sin^2\,\theta\frac{-\Delta - J^2}{-\Delta - J^2\cos^2\,\theta}$$

$$C = -v_1^2 + O(\epsilon^{-2})$$

$$\Phi_+^I = \partial_{\delta_I}\log\,v_1 + O(\epsilon)$$

$$W^2 = 4v_1^2 n^2(n^2 - a^2\cos^2\,\theta) + O(\lambda), \omega_3 = -\frac{2anv_1}{\lambda r_0 r}\sin^2\,\theta\,\mathrm{d}\phi + O(\lambda^0)$$

$$\xi^I = \partial_{\delta_I}\log\,v_1$$

$$A^I = P^I\cos\,\theta\,\mathrm{d}\phi - \partial_{\delta_I}\log\,v_1\omega_3$$

$$A^I = f^I(\,\mathrm{d}\phi - kr\,\mathrm{d}t) - \frac{e^I}{k}\,\mathrm{d}\phi, \quad f^I = \frac{P^I(-\Delta - J^2)(\hat{\pi}^I + J\cos\,\theta)}{J(-\Delta - J^2\cos^2\,\theta)},$$

$$\tilde{A}^I = \tilde{f}_I(\,\mathrm{d}\phi - kr\,\mathrm{d}t) - \frac{\tilde{e}_I}{k}\,\mathrm{d}\phi, \quad \tilde{f}_I = \frac{P_I\hat{\pi}^I}{\sqrt{-\Delta - J^2}}$$

$$J(-\Delta - J^2\cos^2\,\theta) \qquad \tilde{e}_I = \frac{Q_I\hat{\rho}_I}{\sqrt{-\Delta - J^2}}$$

$$\hat{\pi}^I = -2\frac{1}{P^I}\frac{\partial\Delta}{\partial Q_I}, \hat{\rho}_I = -2\frac{1}{Q_I}\frac{\partial\Delta}{\partial P^I}$$

$$S_+ = \frac{1}{3}\pi^2 c_J T_J$$

$$c_J = 12J, T_J = \frac{1}{2\pi k}$$

$$S_+ = \frac{1}{3}\pi^2 c_{Q_I} T_{Q_I} = \frac{1}{3}\pi^2 c_{P^I} T_{P^I}$$

$$c_{Q_I} = -6\frac{\partial\Delta}{\partial\bar{Q}_I}, \qquad T_{Q_I} = \frac{1}{2\pi e^I}$$
$$c_{P^I} = -6\frac{\partial\Delta}{\partial\bar{P}^I}, \qquad T_{P^I} = \frac{1}{2\pi\tilde{e}_I}$$

$$M = \frac{me^{2\gamma}\cosh^2\ \delta}{8}, \bar{Q} = \frac{me^{2\gamma}\sinh^3\ \delta}{8\cosh\ \delta}, \bar{P} = \frac{me^{2\gamma}}{8\cosh\ \delta}$$

$$M^{2/3} = \bar{Q}^{2/3} + \bar{P}^{2/3}$$

$$W_Q = r^2 + 4\bar{Q}^{2/3}\sqrt{\bar{Q}^{2/3} + \bar{P}^{2/3}}r + 8\bar{Q}^{\frac{1}{3}}\bar{P}^{-\frac{1}{3}}(\bar{Q}\bar{P} - J\cos\theta),$$
$$W_P = r^2 + 4\bar{P}^{2/3}\sqrt{\bar{Q}^{2/3} + \bar{P}^{2/3}}r + 8\bar{P}^{1/3}\bar{Q}^{-1/3}(\bar{Q}\bar{P} + J\cos\ \theta).$$

$$\hat{\pi}^1 = \hat{\rho}_1 = \sqrt{-\Delta} = \bar{Q}\bar{P}$$

$$x_i = 0, y_2 = y_3 = \frac{1}{y_1} = \frac{\bar{P}^{2/3}(\bar{Q}\bar{P} + J\cos\ \theta)}{\bar{Q}^{2/3}(\bar{Q}\bar{P} - J\cos\ \theta)}$$

$$\psi \sim \psi + 2\pi$$

$$\mathrm{d}s_5^2 = f^2(\theta)\left[R_\psi \mathrm{d}\psi - \frac{\bar{P}r}{\bar{Q}\bar{P} - J\cos\ \theta}\left(\mathrm{d}t + \frac{J}{r}\sin^2\ \theta\ \mathrm{d}\phi\right) + \bar{P}\cos\ \theta\ \mathrm{d}\phi\right]^2$$
$$+\frac{G(\theta)}{2f(\theta)}\left[-\frac{r^2}{G^2(\theta)}\left(\mathrm{d}t + \frac{J}{r}\sin^2\ \theta\ \mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \sin^2\ \theta\ \mathrm{d}\phi^2\right]$$

$$f(\theta) = \left(\frac{\bar{Q}}{\bar{P}}\right)^{1/3}\left(\frac{\bar{Q}\bar{P} - J\cos\ \theta}{\bar{Q}\bar{P} + J\cos\ \theta}\right)^{1/2}, G(\theta) = \sqrt{\bar{Q}^2\bar{P}^2 - J^2\cos^2\ \theta}$$

$$\mathrm{d}s^2 = \Gamma(\theta)\left(-r^2\ \mathrm{d}\bar{t}^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \sum_{A,B=1}^{2}\gamma_{AB}(\theta)(\mathrm{d}\phi^A - k^A r\ \mathrm{d}\bar{t})(\mathrm{d}\phi^B - k^B r\ \mathrm{d}\bar{t})\right)$$

$$\bar{t} = t/(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}, \phi^1 = \phi, \phi^2 = R_\psi\psi$$

$$\Gamma(\theta) = \frac{\bar{P}^{1/3}(\bar{Q}\bar{P} + J\cos\ \theta)}{2\bar{Q}^{1/3}}, k^1 = \frac{J}{(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}}, k^2 = \frac{\bar{Q}\bar{P}^2}{(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}},$$

$$\gamma_{AB} = \frac{1}{(\bar{Q}\bar{P} + J\cos\ \theta)^2}\begin{pmatrix}\bar{Q}^2\bar{P}^2 - J^2\cos^2\ \theta + (\bar{Q}\bar{P}\cos\ \theta - J)^2 & 2\bar{Q}(\bar{Q}\bar{P}\cos\ \theta - J) \\ 2\bar{Q}(\bar{Q}\bar{P}\cos\ \theta - J) & 2\bar{Q}(\bar{Q}\bar{P} - J\cos\ \theta)/\bar{P}\end{pmatrix}.$$

$$\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)^2$$

$$\xi_{-1}=\partial_t,\xi_0=r\partial_r-t\partial_t$$

$$\xi_1=\left(\frac{1}{2r^2}+\frac{\bar{t}^2}{2}\right)\partial_{\bar{t}}-\bar{t}r\partial_r+\frac{k^1}{r}\partial_\phi+\frac{k^2}{R_\psi r}\partial_\psi$$

$$[\xi_0,\xi_1]=-\xi_1,[\xi_0,\xi_{-1}]=\xi_{-1}$$

$$[\xi_{-1},\xi_1]=-\xi_0$$

$$S_+=\frac{1}{3}\pi^2c_JT_J=\frac{1}{3}\pi^2c_QT_Q$$

$$\begin{aligned}c_J&=12J,\\c_Q&=-6\frac{\partial\Delta}{\partial\bar{Q}},\\T_J&=\frac{1}{2\pi k^1}\\T_Q&=\frac{1}{2\pi k^2}\end{aligned}$$

$$\mathrm{d}s^2=-\frac{R-U}{W}\,\mathrm{d}t^2-\frac{(L_uR+L_rU)}{aW}2\,\mathrm{d}t\,\mathrm{d}\phi+\frac{(W_r{}^2U-W_u{}^2R)}{a^2W}\,\mathrm{d}\phi^2+W\left(\frac{\mathrm{d}r^2}{R}+\frac{\mathrm{d}u^2}{U}\right),$$

$$W^2=(R-U)\left(\frac{W_r^2}{R}-\frac{W_u^2}{U}\right)+\frac{(L_uR+L_rU)^2}{RU}$$

$$\sqrt{-g}=W$$

$$\begin{aligned}L_r(r)&=L+2Nn, & W_r^2(r)&=R^2+4MrR+(L+2Nn)^2\\L_u(u)&=2N(u-n), & W_u^2(u)&=U^2-2UV+4N^2(u-n)^2\end{aligned}$$

$$\mathrm{d}\tilde{s}^2=\frac{r^2+u^2}{W}\,\mathrm{d}s^2,$$

$$(\partial/\partial\tilde{s})^2$$

$$(r^2+u^2)\left(\frac{\partial}{\partial\tilde{s}}\right)^2=R\partial_r^2+U\partial_u^2+\left(\frac{W_u^2}{U}-\frac{W_r^2}{R}\right)\partial_t^2-a\left(\frac{L_r}{R}+\frac{L_u}{U}\right)2\partial_t\partial_\phi+a^2\left(\frac{1}{U}-\frac{1}{R}\right)\partial_\phi^2$$

$$K_{\mu\nu}=K_{(\mu\nu)}$$

$$\nabla_{(\mu}K_{\nu\rho)}=0$$

$$Q_{\mu\nu}=Q_{(\mu\nu)}$$

$$\nabla_{(\mu}Q_{\nu\rho)}=q_{(\mu}g_{\mu\nu)}$$

$$q_\mu=\frac{1}{6}(\partial_\mu Q^\nu{}_\nu+2\nabla_\nu Q^\nu{}_\mu)$$

$$\widetilde{K}^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{r^2+u^2}\left[\left(\frac{u^2W_r^2}{R}+\frac{r^2W_u^2}{U}\right)\partial_t^2 - a\left(\frac{u^2L_r}{R}+\frac{r^2L_u}{U}\right)2\partial_t\partial_\phi + a^2\left(\frac{r^2}{U}-\frac{u^2}{R}\right)\partial_\phi^2 \right.$$
$$\left. -u^2R\partial_r^2 + r^2U\partial_u^2\right]$$

$$Q^{\mu\nu} = \widetilde{K}^{\mu\nu}$$

$$L_r = W_r$$

$$L_u = W_u$$

$$W = W_r + W_u$$

$$ds^2 = -\frac{R}{W_r+W_u}\left(\mathrm{d}t+\frac{W_u}{a}\,\mathrm{d}\phi\right)^2 + \frac{U}{W_r+W_u}\left(\mathrm{d}t-\frac{W_r}{a}\,\mathrm{d}\phi\right)^2 + (W_r+W_u)\left(\frac{\mathrm{d}r^2}{R}+\frac{\mathrm{d}u^2}{U}\right)$$

$$(\delta_1,\gamma_1) = (\delta_4,\gamma_4) \text{ and } (\delta_2,\gamma_2) = (\delta_3,\gamma_3)$$

$$\frac{\partial S}{\partial\lambda} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu S\partial_\nu S = 0$$

$$\partial_\mu S = p_\mu = \mathrm{d}x_\mu/\mathrm{d}\lambda, p_\lambda$$

$$S = \frac{1}{2}\mu^2\lambda - Et + L\phi + S_r(r) + S_u(u)$$

$$p^\mu p_\mu = -\mu^2$$

$$(r^2+u^2)\tilde{g}^{\mu\nu}$$

$$\left(\frac{W_u^2}{U}-\frac{W_r^2}{R}\right)E^2 + 2a\left(\frac{L_r}{R}+\frac{L_u}{U}\right)EL + a^2\left(\frac{1}{U}-\frac{1}{R}\right)L^2 + R\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + U\left(\frac{\mathrm{d}S_u}{\mathrm{d}u}\right)^2 + \mu^2(r^2+u^2) = 0$$

$$\frac{\mathrm{d}S_r}{\mathrm{d}r} = \frac{1}{R}\sqrt{W_r^2E^2 - 2aL_rEL + a^2L^2 - (C+\mu^2r^2)R}$$
$$\frac{\mathrm{d}S_u}{\mathrm{d}u} = \frac{1}{U}\sqrt{-W_u^2E^2 - 2aL_uEL - a^2L^2 + (C-\mu^2u^2)U}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \tilde{g}^{rr}p_r = \frac{R}{r^2+u^2}\frac{\mathrm{d}S_r}{\mathrm{d}r}, \frac{\mathrm{d}u}{\mathrm{d}\lambda} = \tilde{g}^{uu}p_u = \frac{U}{r^2+u^2}\frac{\mathrm{d}S_u}{\mathrm{d}u}.$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \tilde{g}^{tt}p_t + \tilde{g}^{t\phi}p_\phi = \frac{E}{r^2+u^2}\left(\frac{W_r^2}{R}-\frac{W_u^2}{U}\right) - \frac{aL}{r^2+u^2}\left(\frac{L_r}{R}+\frac{L_u}{U}\right)$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \tilde{g}^{t\phi}p_t + \tilde{g}^{\phi\phi}p_\phi = \frac{aE}{r^2+u^2}\left(\frac{L_r}{R}+\frac{L_u}{U}\right) + \frac{a^2L}{r^2+u^2}\left(\frac{1}{U}-\frac{1}{R}\right)$$

$$\square\,\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = \mu^2\Phi$$

$$\Phi = \Phi_r(r)\Phi_u(u)\mathrm{e}^{\mathrm{i}(k\phi-\omega t)}$$

$$\mu^2 W = \frac{\omega^2 W_r^2 - 2a\omega k L_r + a^2k^2}{R} - \frac{\omega^2 W_u^2 + 2a\omega k L_u + a^2k^2}{U} + \frac{1}{\Phi_r}\frac{\mathrm{d}}{\mathrm{d}r}\left(R\frac{\mathrm{d}\Phi_r}{\mathrm{d}r}\right) + \frac{1}{\Phi_u}\frac{\mathrm{d}}{\mathrm{d}u}\left(U\frac{\mathrm{d}\Phi_u}{\mathrm{d}u}\right).$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(R\frac{\mathrm{d}\Phi_r}{\mathrm{d}r}\right) + \left(\frac{\omega^2 W_r^2 - 2a\omega k L_r + a^2k^2}{R} + C\right)\Phi_r = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(U\frac{\mathrm{d}\Phi_u}{\mathrm{d}u}\right) - \left(\frac{\omega^2 W_u^2 + 2a\omega k L_u + a^2k^2}{U} + C\right)\Phi_u = 0$$

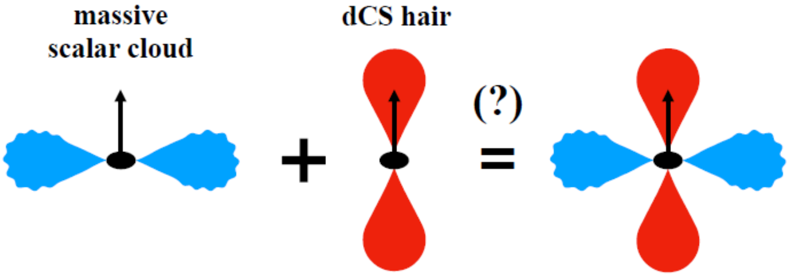
$$x_i = -\chi_i$$

$$\mathcal{F}^1 = \hat{F}_2, \tilde{\mathcal{F}}_2 = \hat{F}_1, \tilde{\mathcal{F}}_3 = \hat{\mathcal{F}}^1, \mathcal{F}^4 = \hat{\mathcal{F}}^2$$

$$(\zeta^{\Lambda}_{\text{ours}}, \tilde{\zeta}^{\text{ours}}_{\Lambda}) = (-\tilde{\zeta}^{\text{theirs}}_{\Lambda}, \zeta^{\Lambda}_{\text{theirs}})$$

$$T_- = (r_- - m)/\left(2\pi L(r_-)\right) \text{and } S_- = \pi L(r_-)/G.$$

$$T_- = (r_- - m)/(2\pi|L(r_-)|) \text{and } S_- = \pi|L(r_-)|/G,$$



$$S = \int d^4x \frac{\sqrt{-g}}{16\pi}\left[R + \frac{\alpha_{\text{CS}}}{4} f(\Theta)^* RR - \frac{1}{2}(\nabla\Theta)^2 - V(\Theta)\right],$$

$$[\alpha_{\text{CS}}] = [L]^2, V(\Theta)$$

$$^*RR = {}^*R^{abcd}R_{bacd} = -\frac{1}{2}\epsilon^{cd}_{ef}R^{abef}R_{abcd}$$

$$^*R^{abcd} = \frac{1}{2}\epsilon^{cd}{}_{ef}R^{abef}$$

$$f(\Theta) \sim \Theta$$

$$f(\Theta) = \Theta, V(\Theta) = \frac{\mu^2}{2}\Theta^2$$

$$\mu \simeq 10^{-10}(M_\odot/M)\text{eV}$$

$$(\Box - \mu^2)\Theta + \frac{\alpha_{\text{CS}}}{4}{}^*RR = 0$$

$$G_{ab} - \frac{1}{2}T^{\Theta}_{ab} + \alpha_{\text{CS}}\mathcal{C}_{ab} = 0$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

$$T_{ab}^{\Theta} = \nabla_a \Theta \nabla_b \Theta - \frac{1}{2} g_{ab}((\nabla\Theta)^2 + \mu^2 \Theta^2).$$

$$\mathcal{C}^{ab} \equiv \mathcal{E}_c \epsilon^{cde(a} \nabla_e R_d^{b)} + \mathcal{F}_{cd}^{*} R^{d(ab)c}$$

$$\mathcal{E}_a = \nabla_a \Theta, \text{ and } \mathcal{F}_{ab} = \nabla_a \nabla_b \Theta$$

$$G_{ab} = \frac{1}{2} T_{ab}^{\text{eff}}$$

$$T_{ab}^{\text{eff}} = T_{ab}^{\Theta} - 2\alpha_{\text{CS}} \mathcal{C}_{ab}$$

$$\Theta \to (\alpha_{\text{CS}}/M^2)\Theta$$

$$\square\, \Theta - \mu^2 \Theta + \frac{\hat{\alpha} M^2}{4} * RR = 0$$
$$G_{ab} = 0$$

$$(\mathcal{M}, g_{ab})(\Sigma_t, \gamma_{ij})\gamma_{ij}$$

$$\gamma_{ab} = g_{ab} + n_a n_b$$

$$\gamma^a{}_b n^b = 0$$

$$n^a n_a = -1$$

$$\gamma^a{}_b = \delta^a{}_b + n^a n_b.$$

$$\begin{aligned} \mathrm{d}s^2 &= g_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b \\ &= -(\alpha^2 - \beta^k \beta_k)\mathrm{d}t^2 + 2\gamma_{ij}\beta^i\, \mathrm{d}t\, \mathrm{d}x^j + \gamma_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \end{aligned}$$

$$K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d = -\frac{1}{2} \mathcal{L}_n \gamma_{ab},$$

$$\begin{aligned} \mathrm{d}s^2 =& -\left(1 - \frac{2Mr_{\text{BL}}}{\Sigma}\right)\mathrm{d}t^2 - \frac{4aMr_{\text{BL}}\sin^2\theta}{\Sigma}\, \mathrm{d}t\, \mathrm{d}\phi \\ &+ \frac{\Sigma}{\Delta}\mathrm{d}r_{\text{BL}}^2 + \Sigma \mathrm{d}\theta^2 + \frac{\mathcal{A}}{\Sigma}\sin^2\theta\, \mathrm{d}\phi^2 \end{aligned}$$

$$\begin{aligned} \Delta &= (r_{\text{BL}} - r_{\text{BL},+})(r_{\text{BL}} - r_{\text{BL},-}) \\ \Sigma &= r_{\text{BL}}^2 + a^2\cos^2\theta \\ \mathcal{A} &= (r_{\text{BL}}^2 + a^2)^2 - \Delta a^2 \sin^2\theta \\ r_{\text{BL},\pm} &= M \pm \sqrt{M^2 - a^2} \end{aligned}$$

$$r_{\mathrm{BL}} = r\left(1 + \frac{r_{\mathrm{BL},+}}{4r}\right)^2$$

$$r_+ = r_{\mathrm{BL},+}/4$$

$$\gamma_{ij} = \mathrm{Diag}\left[\frac{\left(4r + r_{\mathrm{BL},+}\right)^2 \Sigma}{16r^3\left(r_{\mathrm{BL}} - r_{\mathrm{BL},-}\right)}, \Sigma, \frac{\mathcal{A}}{\Sigma}\sin^2\theta\right]$$

$$\alpha = \pm\sqrt{\frac{\Delta\Sigma}{\mathcal{A}}}, \beta^i = \left(0,0,-\frac{2aMr_{\mathrm{BL}}}{\mathcal{A}}\right)$$

$$K_{r\phi} = \alpha\frac{aMr'_{\mathrm{BL}}\sin^2\theta}{\Delta\Sigma^2}\left[2r_{\mathrm{BL}}^2\left(r_{\mathrm{BL}}^2 + a^2\right) + \Sigma\left(r_{\mathrm{BL}}^2 - a^2\right)\right]$$
$$K_{\theta\phi} = -2\alpha\frac{a^3Mr_{\mathrm{BL}}\sin^3\theta}{\Sigma^2}$$

$$r'_{\mathrm{BL}} = \partial_r r_{\mathrm{BL}}$$

$Y^a = (r,\theta,\phi)$ to cartesian coordinates $X^i = (x,y,z)$

$$x = r\cos\phi\sin\theta, y = r\sin\phi\sin\theta, z = r\cos\theta$$

$$\mathcal{O}((\alpha_{\mathrm{CS}}/M^2)^2)$$

$$f'(\Theta) \neq 0$$

$$f'(\Theta_0) = 0$$

$$\gamma_{ij}\,\mathrm{d}X^i\,\mathrm{d}X^j = \psi_0^4\left[\eta_{ij}\,\mathrm{d}X^i\,\mathrm{d}X^j + G(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)^2 + a^2H(x\,\mathrm{d}y - y\,\mathrm{d}x)^2\right]$$

$$\psi_0^4 = \frac{\Sigma}{r^2}, G = \frac{r_{\mathrm{BL}}}{r^2\left(r_{\mathrm{BL}} - r_{\mathrm{BL},-}\right)}, H = \frac{\Sigma + 2Mr_{\mathrm{BL}}}{r^2\Sigma^2}.$$

$$K_{ij} = J_i{}^aJ_j{}^bK_{ab}, \beta_i = J_i{}^a\beta_a.$$

$$J_i{}^a = \partial Y^a/\partial X^i$$

$$K_\Theta = -\mathcal{L}_n\Theta.$$

$$\mathrm{d_t}\Theta = -\alpha K_\Theta$$
$$\mathrm{d_t}K_\Theta = -\alpha D^iD_i\Theta - D^i\alpha D_i\Theta + \alpha\left(KK_\Theta + \mu^2\Theta - \frac{\hat{\alpha}M^2}{4}\,{}^*RR\right).$$

$$\mathrm{d_t} = \left(\partial_t - \mathcal{L}_\beta\right)$$

$${}^*RR = {}^*W_{abcd}W^{bacd} = -16E^{ij}B_{ij},$$

$${}^*W_{abcd} = \frac{1}{2}\epsilon_{cd}{}^{ef}W_{abef}$$

$$E_{ij} = \gamma^{a}{}_{i}\gamma^{b}{}_{j}n^{c}n^{d}W_{acbd}$$

$$B_{ij} = \gamma^{a}{}_{i}\gamma^{b}{}_{j}n^{c}n^{d*}W_{acbd}$$

$$\begin{aligned} E_{ij} &= R_{ij}^{\mathrm{tf}} + \frac{1}{3}A_{ij}K - A_{i}{}^{k}A_{jk} + \frac{1}{3}\gamma_{ij}A_{kl}A^{kl}, \\ B_{ij} &= -\epsilon_{(i|}{}^{kl}D_{l}A_{|j)k}, \end{aligned}$$

$$\gamma_{ij}, K = \gamma^{ij}K_{ij}$$

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K$$

$$\begin{aligned} W &= \gamma^{-\frac{1}{6}}, \tilde{\gamma}_{ij} = W^{2}\gamma_{ij} \\ K &= \gamma^{ij}K_{ij}, \tilde{A}_{ij} = W^{2}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \\ \tilde{\Gamma}^{i} &= \tilde{\gamma}^{kl}\tilde{\Gamma}_{kl}^{i} \end{aligned}$$

$$\begin{aligned} \mathrm{d_t}\Theta &= -\alpha K_{\Theta} \\ \mathrm{d_t}K_{\Theta} &= -W^{2}\tilde{D}^{i}\alpha\tilde{D}_{i}\Theta - \alpha\big(W^{2}\tilde{D}^{i}\tilde{D}_{i}\Theta - W\tilde{D}^{i}\Theta\tilde{D}_{i}W \\ &\quad -KK_{\Theta} - \mu^{2}\Theta + \frac{\hat{\alpha}M^{2}}{4}\,{}^{*}RR\big) \end{aligned}$$

$${}^{*}RR = -16\tilde{\gamma}^{ia}\tilde{\gamma}^{jb}\tilde{E}_{ab}\tilde{B}_{ij}$$

$$\Theta(t=0) = 0 = K_{\Theta}(t=0)$$

$$\Theta(t=0) = A\exp\left\{-\frac{(r-r_{0})^{2}}{\sigma^{2}}\right\}\Sigma_{lm}(\theta,\phi)$$

$$K_{\Theta}(t=0) = -\frac{1}{\alpha}(\partial_{t} - \beta^{a}\partial_{a})\Theta\Big|_{t=0} = \frac{\beta^{\phi}}{\alpha}\partial_{\phi}\Theta\Big|_{t=0} = 0$$

$$\begin{aligned} \Theta &= \hat{\alpha}\frac{a}{M}\left(\frac{5M^{2}}{8r_{\mathrm{BL}}^{2}} + \frac{5M^{3}}{4r_{\mathrm{BL}}^{3}} + \frac{9M^{4}}{4r_{\mathrm{BL}}^{4}}\right)\cos\theta \\ &-\hat{\alpha}\frac{a^{3}}{M^{3}}\left[\left(\frac{M^{2}}{16r_{\mathrm{BL}}^{2}} + \frac{M^{3}}{8r_{\mathrm{BL}}^{3}} + \frac{3M^{4}}{20r_{\mathrm{BL}}^{4}} + \frac{M^{5}}{10r_{\mathrm{BL}}^{5}}\right)\cos\theta\right. \\ &\left.+\left(\frac{3M^{4}}{4r_{\mathrm{BL}}^{4}} + \frac{3M^{5}}{r_{\mathrm{BL}}^{5}} + \frac{25M^{6}}{3r_{\mathrm{BL}}^{6}}\right)\cos^{3}\theta\right] + \mathcal{O}\left(\left(\frac{a}{M}\right)^{5}\right) \end{aligned}$$

$$K_{\Theta}(t=0) = \frac{\beta^{\phi}}{\alpha}\partial_{\phi}\Theta\Big|_{t=0} = 0$$

$$\Theta_{lm} = \exp(-i\omega t)\exp(im\phi)\mathcal{S}_{lm}(\theta)\mathcal{R}_{lm}(r_{\mathrm{BL}}),$$

$$\omega = \bar{\omega} + \iota\bar{\nu}$$

$$\mathcal{R}_{lm}(r_{\mathrm{BL}}) = \Delta_+^{-i\eta}\Delta_-^{i\eta+\chi-1}e^{qr_{\mathrm{BL}}}\sum_{n=0}^{\infty} a_n\left(\frac{\Delta_+}{\Delta_-}\right)^n,$$

$$\Delta_+ = r_{\mathrm{BL}} - r_{\mathrm{BL},+}, \Delta_- = r_{\mathrm{BL}} - r_{\mathrm{BL},-}$$
$$\eta = \frac{2r_{\mathrm{BL},+}(\omega - \omega_{\mathrm{c}})}{r_{\mathrm{BL},+} - r_{\mathrm{BL},-}}, \chi = \frac{\mu^2 - 2\omega^2}{q}$$
$$q = \pm\sqrt{\mu^2 - \omega^2}$$

$$\omega_{\mathrm{c}} = m\Omega_{\mathrm{H}} = m\frac{a}{2Mr_{\mathrm{BL},+}}$$

$$\Theta_{lm}(t=0) = \exp{(im\phi)}\mathcal{S}_{lm}(\theta)\mathcal{R}_{lm}(r_{\mathrm{BL}})$$

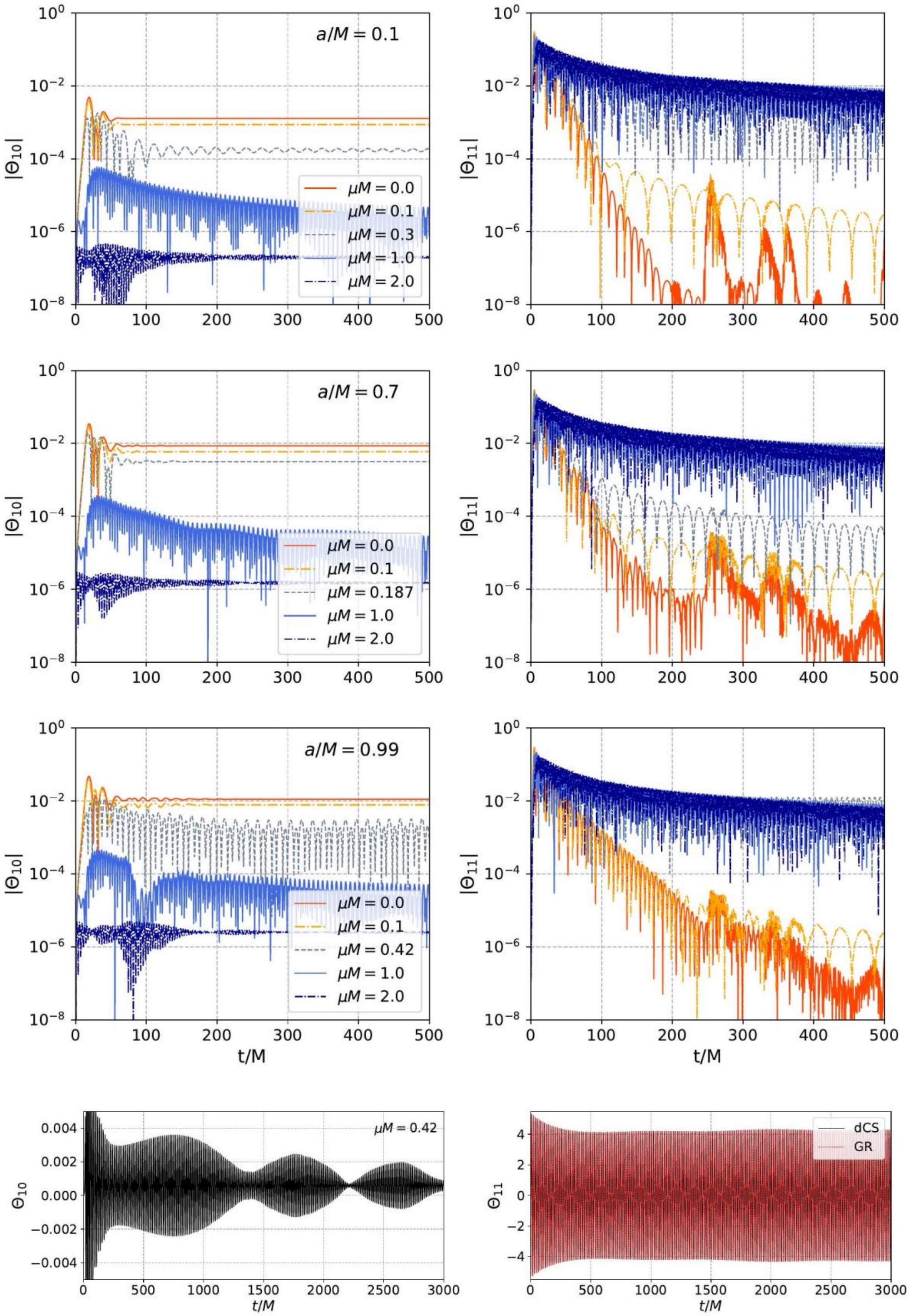
$$K_{\Theta,lm}(t=0) = i\omega\Theta_{lm}(t=0)$$

$$\Theta_{lm}(t, r_{\mathrm{ex}}) = \int \mathrm{d}\Omega\Theta(t, r_{\mathrm{ex}}, \theta, \phi)Y_{lm}^*(\theta, \phi)$$

$$\mathrm{d}x/2^{\mathrm{RL}-1} = \mathrm{d}x/2^6 = 1/64M$$

$$\Delta\Theta_{10}/\Theta_{10,h}$$

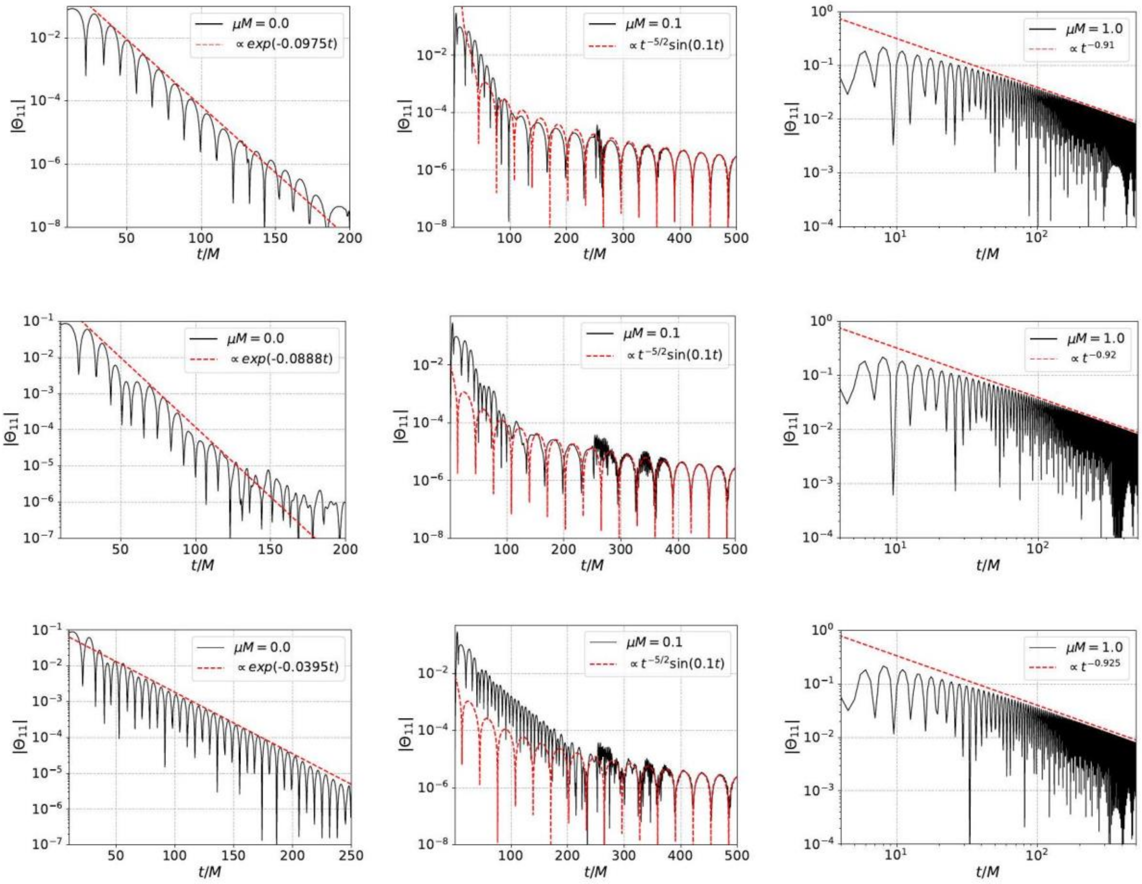
$$\Delta\Theta_{11}/\Theta_{11,h}$$



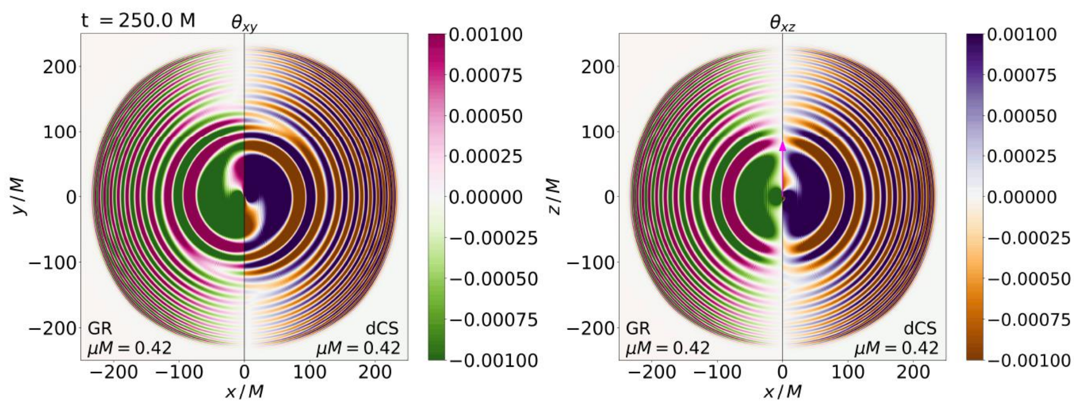
$$\Theta_{11} \propto t^{-(l+3/2)} \sin(\mu t) = t^{-5/2} \sin(0.1t)$$

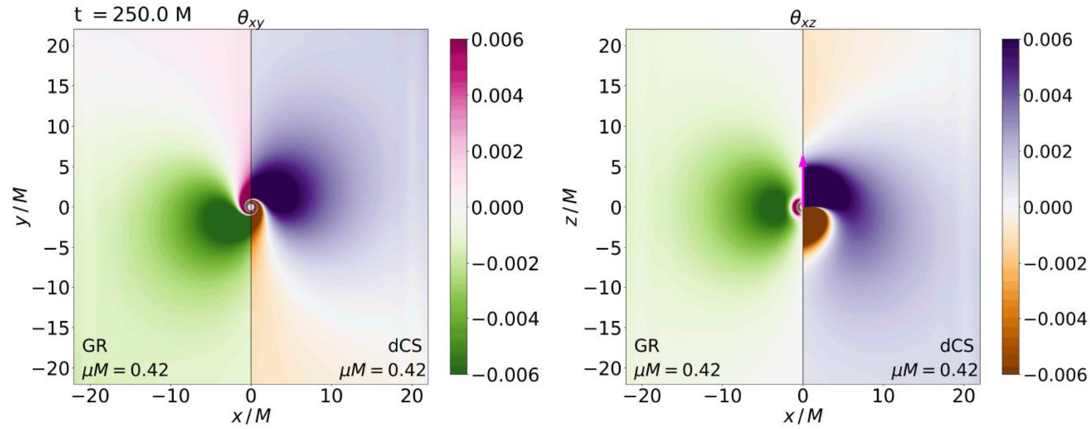
$$1 < t/M < 1/(\mu M)^3$$

$$\propto t^{-p} \sin(\mu t)$$

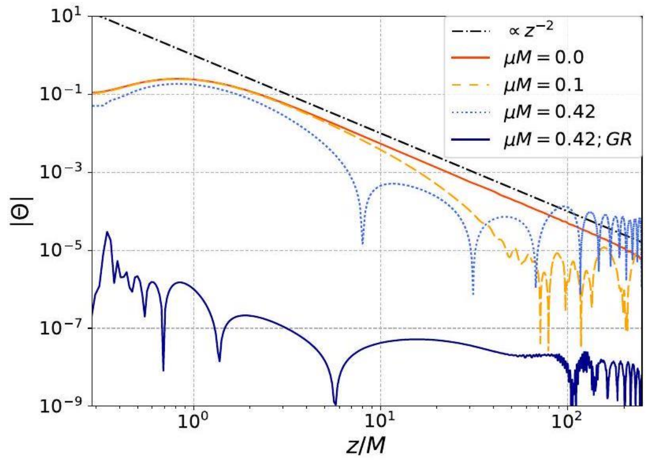


$$\omega_{lmn} \simeq \mu\left(1 - \frac{(\mu M)^2}{2(l+n+1)^2}\right),$$





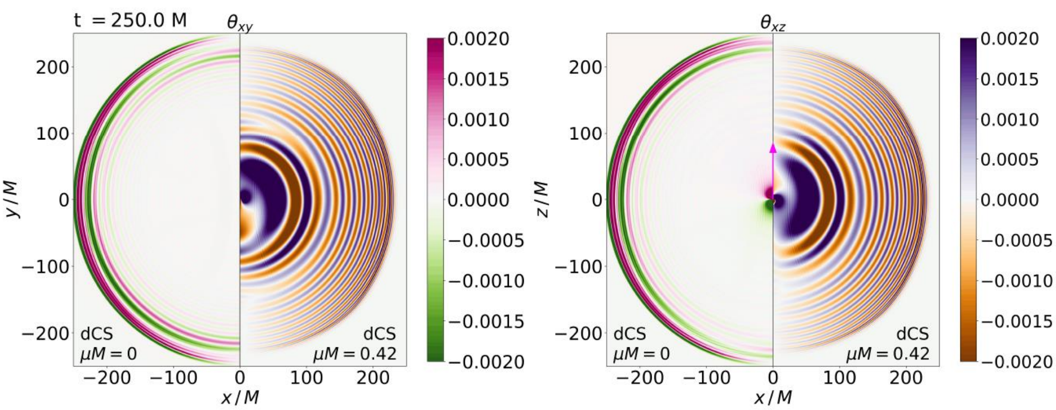
$$r_{\mathrm{QBS,max}} \sim \frac{l(l+1)}{(\mu M)^2} M$$

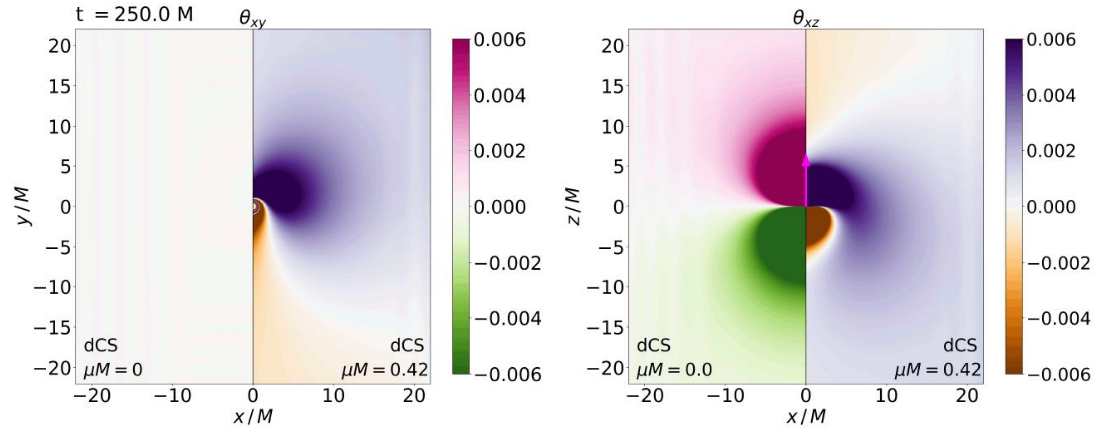


$$\tilde{\Theta}_{lm}(\omega_i) = \frac{1}{N} \sum_{j=0}^{N-1} \Theta_{lm}(t_j) e^{-i\omega_i t_j}$$

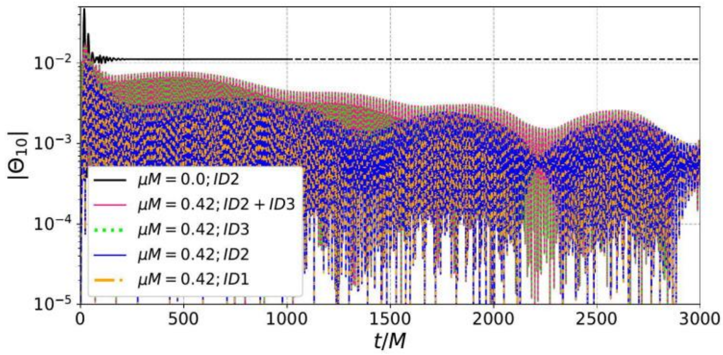
$$P_{lm}(\omega) = \left|\tilde{\Theta}_{lm}(\omega)\right|^2$$

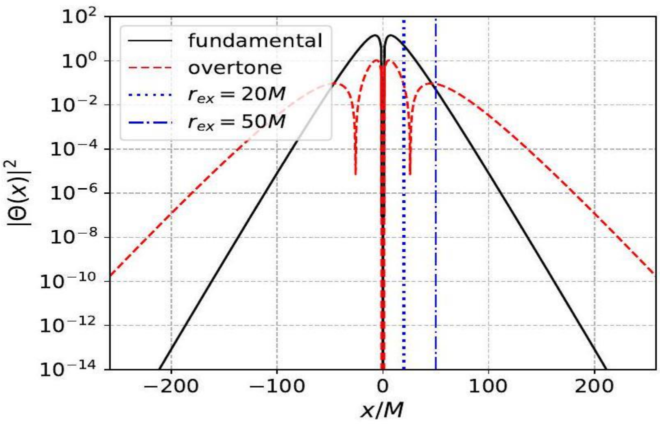
$$\Theta_{lm} \sim \exp\left(-i\bar{\omega}t + \bar{\nu}t\right)$$

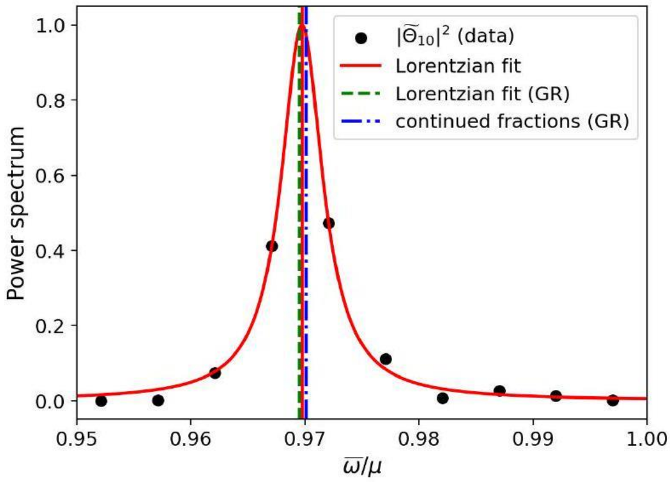


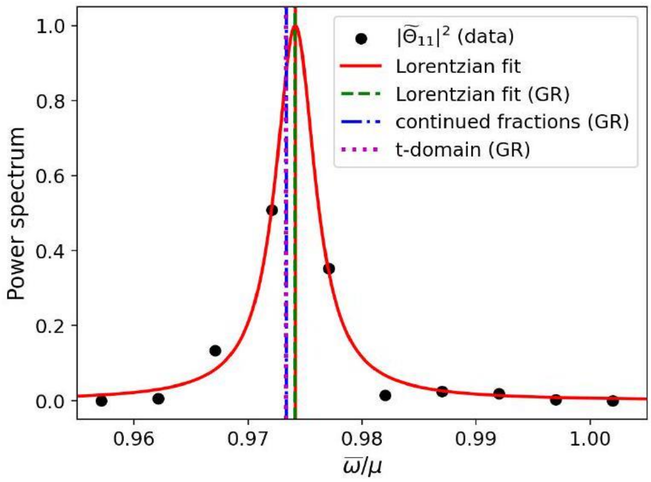


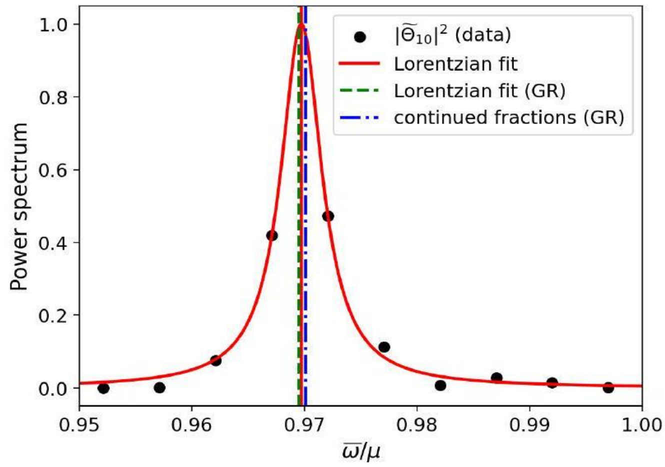
$$\Lambda(\omega; \bar{I}, \bar{\omega}, \bar{\nu}) = \bar{I} \frac{\bar{\nu}^2}{((\omega - \bar{\omega})^2 + \bar{\nu}^2)}$$

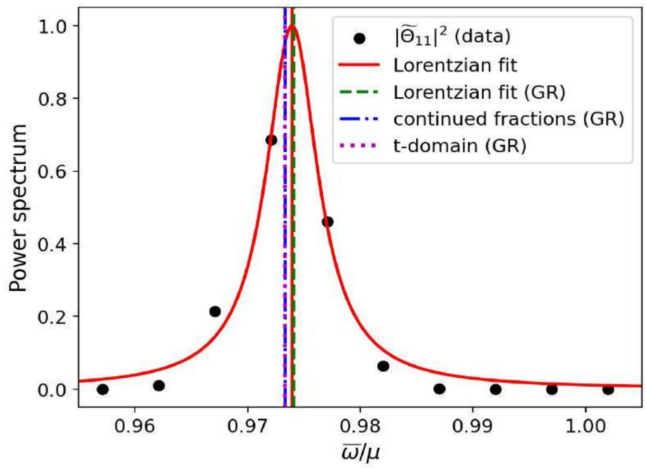


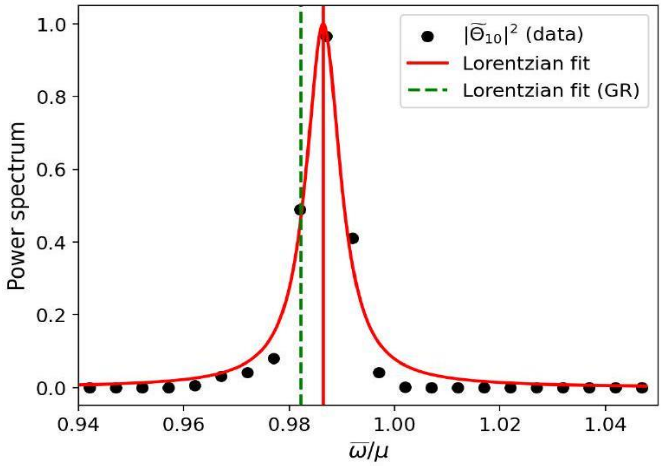


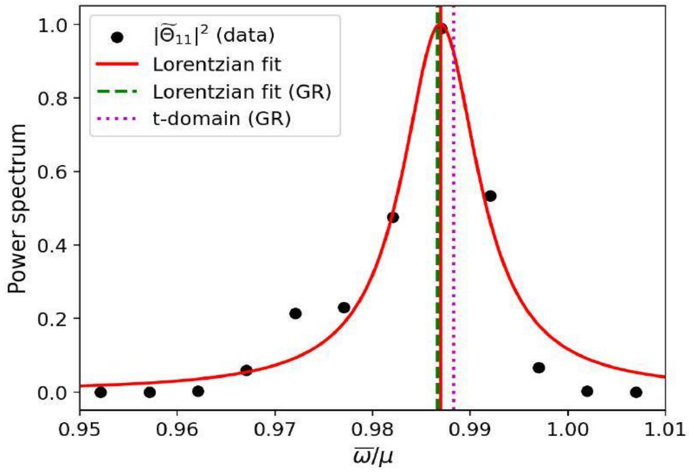

$|\widetilde{\Theta}_{10}|^2$ (data)
Lorentzian fit
Lorentzian fit (GR)
continued fractions (GR)
Power spectrum
$\overline{\omega}/\mu$

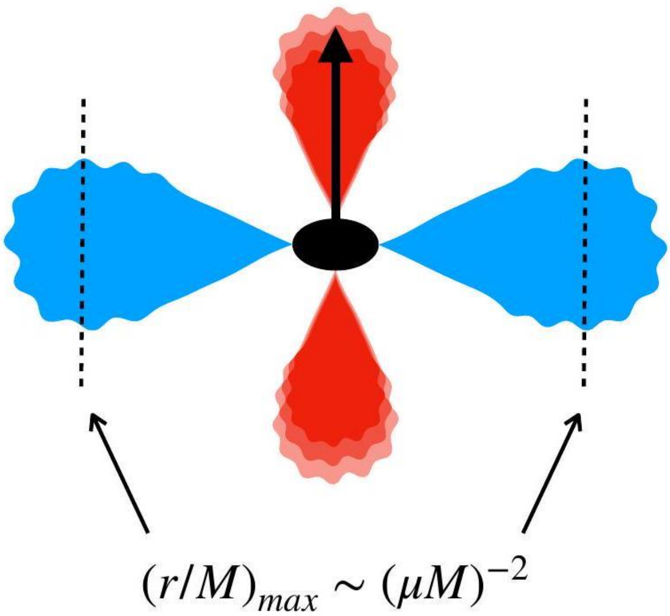

$|\widetilde{\Theta}_{11}|^2$ (data)
Lorentzian fit
Lorentzian fit (GR)
continued fractions (GR)
t-domain (GR)
Power spectrum
$\overline{\omega}/\mu$


$|\widetilde{\Theta}_{10}|^2$ (data)
Lorentzian fit
Lorentzian fit (GR)
continued fractions (GR)
Power spectrum
$\overline{\omega}/\mu$


$|\widetilde{\Theta}_{11}|^2$ (data)
Lorentzian fit
Lorentzian fit (GR)
continued fractions (GR)
t-domain (GR)
Power spectrum
$\overline{\omega}/\mu$







$$(r/M)_{\max} \sim (\mu M)^{-2}$$

$$r \simeq r_{\max}$$

$$r/M \sim (\mu M)^{-2}$$

$$^{*}RR = \sum_{l,m} \mathcal{P}_{lm}(r_{\rm BL}) Y_{lm}(\theta,\phi)$$

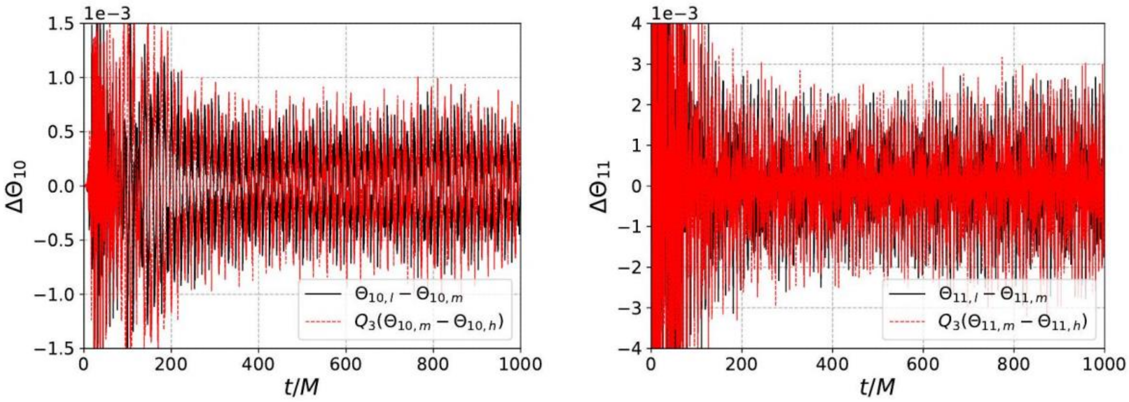
$$\mathcal{P}_{lm} := \int Y^{*}_{lm}(\theta,\phi)(^{*}RR) d(\cos\theta) d\phi$$

$$\int Y^{*}_{lm}(\theta,\phi) Y_{kn}(\theta,\phi) d(\cos\theta) d\phi = \delta_{lk}\delta_{mn}$$

$$^{*}RR = 96M^2 \frac{3r_{\rm BL}^5 a\cos\theta - 10 r_{\rm BL}^3 a^3 \cos^3\theta + 3 r_{\rm BL} a^5 \cos^5\theta}{\Sigma^6},$$

$$^{*}RR = \sum_{j=0}^{\infty} p_j(r_{\rm BL}) Y_{(2j+1)0}(\theta,\phi)$$

$$p_j := 2\pi \int_{-1}^{+1} Y^{*}_{(2j+1),0}(^{*}RR) d(\cos\theta)$$



$$\sigma_{\tilde{P}} = \frac{2\sqrt{\tilde{P}}}{\sqrt{N}} \sigma_{\Theta}.$$

$$\Box\,\Theta - V'(\Theta) + \frac{\alpha_{\rm CS}}{4} f'(\Theta)\, ^{*}RR = 0$$

$$G_{ab} - \frac{1}{2} T^{\Theta}_{ab} + \alpha_{\rm CS} \mathcal{C}_{ab} = 0$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

$$T^{\Theta}_{ab} = \nabla_a\Theta\nabla_b\Theta - g_{ab}\left(\frac{1}{2}(\nabla\Theta)^2 + V(\Theta)\right).$$

$$\mathcal{C}^{ab} \equiv \mathcal{E}_c \epsilon^{cde(a} \nabla_e R_d^{b)} + \mathcal{F}_{cd} * R^{d(ab)c}$$

$$\mathcal{E}_a \equiv f'(\Theta)\nabla_a\Theta,$$
$$\mathcal{F}_{ab} \equiv f'(\Theta)\nabla_a\nabla_b\Theta + f''(\Theta)\nabla_a\Theta\nabla_b\Theta.$$

$$R_{ab} = 0 = \nabla^e R_{ab}$$

$$\mathcal{C}^{ab}_{\mathrm{vac;GR}} = \mathcal{F}_{cd}\ {}^{*}W^{d(ab)c}$$

$$\Theta \rightarrow (\alpha_{\mathrm{CS}}/M^2)\Theta$$

$$\begin{aligned}
\mathrm{d_t}\Theta &= -\alpha K_\Theta \\
\mathrm{d}_t K_\Theta &= -\alpha D^i D_i \Theta - D^i \alpha D_i \Theta \\
&+\alpha\left(K K_\Theta + V'(\Theta) - \frac{\hat{\alpha} M^2}{4} f'(\Theta)^{*}RR\right)
\end{aligned}$$

$$\mathrm{d_t} = \left(\partial_t - \mathcal{L}_\beta\right)$$

$${}^{*}RR = {}^{*}W_{abcd}W^{bacd} = -16E^{ij}B_{ij}$$

$$T^{\mathrm{eff}}_{ab} := T^{\Theta}_{ab} - 2\hat{\alpha}M^2\mathcal{C}^{\mathrm{vac;GR}}_{ab}$$

$$\rho^{\mathrm{eff}} := n^a n^b T^{\mathrm{eff}}_{ab}$$

$$j^{\mathrm{eff}}_i := -\gamma^a_i n^b T^{\mathrm{eff}}_{ab}$$

$$S^{\mathrm{eff}}_{ij} := \gamma^a_i \gamma^b_j T^{\mathrm{eff}}_{ab}$$

$$\begin{aligned}
\rho^{\mathrm{eff}} &= \frac{1}{2}\left(K_\Theta^2 + 2V(\Theta) + D_i\Theta D^i\Theta\right) + 2\hat{\alpha}M^2\left(B^{ij}\mathcal{F}_{ij}\right), \\
j^{\mathrm{eff}}_i &= K_\Theta D_i\Theta + 2\hat{\alpha}M^2\left(B_{ij}\mathcal{F}^j - \epsilon_{ijk}E^{jl}\mathcal{F}_l{}^k\right), \\
S^{\mathrm{eff}}_{ij} &= D_i\Theta D_j\Theta + \frac{1}{2}\gamma_{ij}\left(K_\Theta^2 - D_k\Theta D^k\Theta - 2V(\Theta)\right) \\
&+2\hat{\alpha}M^2\left(2\epsilon_{(ikl)}E_{|j)}{}^l\mathcal{F}^k - 2B_{(i}{}^k\mathcal{F}_{j)k} + \gamma_{ij}B^{kl}\mathcal{F}_{kl}\right. \\
&\left.+B_{ij}(\mathcal{F}_{nn} + \mathrm{tr}\mathcal{F})\right).
\end{aligned}$$

$$\mathcal{F}_{nn} = \mathcal{F}_{ab}n^a n^b, \mathcal{F}_i = -\gamma^a{}_i n^b \mathcal{F}_{ab}$$

$$\mathcal{F}_{ij} = \gamma^a{}_i \gamma^b{}_j \mathcal{F}_{ab}$$

$$\mathrm{tr}\mathcal{F} = \gamma^{ij}\mathcal{F}_{ij}$$

$$\begin{aligned}
\mathcal{F}_{nn} &= \left(D_k D^k\Theta - K K_\Theta - V'(\Theta)\right)f'(\Theta) \\
&+\frac{\hat{\alpha}M^2}{4}\ {}^{*}RRf'(\Theta)^2 + K_\Theta^2 f''(\Theta) \\
\mathcal{F}_i &= \left(D_i K_\Theta - K_{ij}D^j\Theta\right)f'(\Theta) + K_\Theta D_i f''(\Theta), \\
\mathcal{F}_{ij} &= \left(D_i D_j\Theta - K_{ij}K_\Theta\right)f'(\Theta) + D_i\Theta D_j\Theta f''(\Theta)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d_t}\Theta &= -\alpha K_\Theta \\
\mathrm{d_t}K_\Theta &= -W^2\tilde{D}^i\alpha\tilde{D}_i\Theta - \alpha\left(W^2\tilde{D}^i\tilde{D}_i\Theta - W\tilde{D}^i\Theta\tilde{D}_i W\right. \\
&\left.-K K_\Theta - V'(\Theta) + \frac{\hat{\alpha}M^2}{4}\ {}^{*}RRf'(\Theta)\right),
\end{aligned}$$

$${}^{*}RR = -16\tilde{\gamma}^{ia}\tilde{\gamma}^{jb}\tilde{E}_{ab}\tilde{B}_{ij}$$

$$\tilde{E}_{ij} = W^2 E_{ij}$$
$$= W^2 R_{ij}^{\mathrm{tf}} + \frac{1}{3}\tilde{A}_{ij}K - \tilde{A}_i{}^k \tilde{A}_{jk} + \frac{1}{3}\tilde{\gamma}_{ij}\tilde{A}_{kl}\tilde{A}^{kl},$$
$$\tilde{B}_{ij} = W^2 B_{ij}$$
$$= -W\tilde{\epsilon}_{(i|}{}^{kl}\tilde{D}_l\tilde{A}_{|j)k} - \tilde{\epsilon}_{(i|}{}^{kl}\tilde{A}_{|j)l}\tilde{D}_k W.$$

$$\rho^{\mathrm{eff}} = \frac{1}{2}\left(K_\Theta^2 + 2V(\Theta) + W^2(\tilde{D}_i\Theta)(\tilde{D}^i\Theta)\right)$$
$$+2\hat{\alpha}M^2(\tilde{B}^{ij}\tilde{\mathcal{F}}_{ij}),$$

$$j_i^{\mathrm{eff}} = K_\Theta \tilde{D}_i\Theta + 2\hat{\alpha}M^2\left(\tilde{B}_{ij}\tilde{\mathcal{F}}^j - \frac{1}{W}\tilde{\epsilon}_{ijk}\tilde{E}^{jl}\tilde{\mathcal{F}}_l{}^k\right)$$

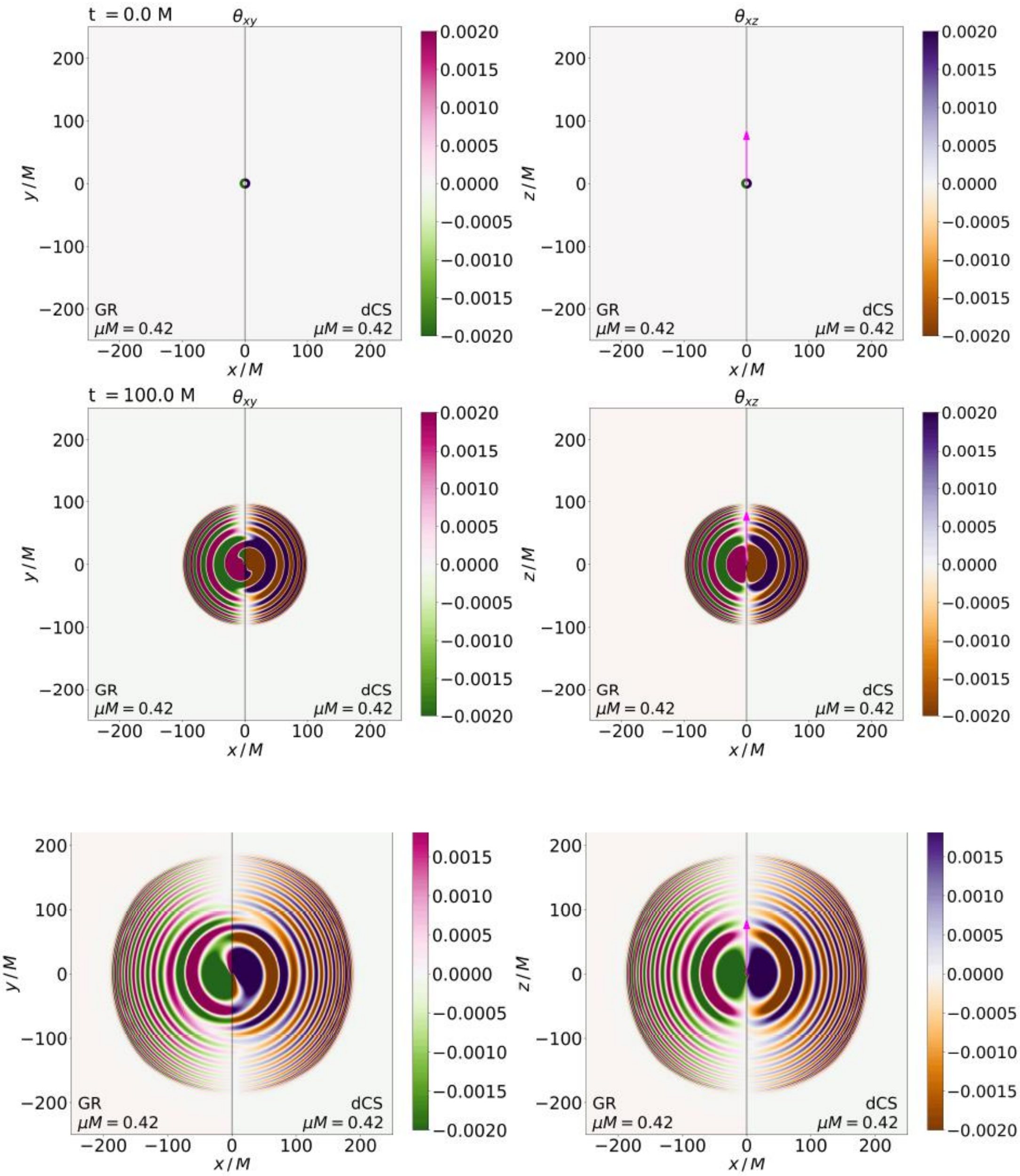
$$S_{ij}^{\mathrm{eff}} = \tilde{D}_i\Theta\tilde{D}_j\Theta + \frac{1}{2W^2}\tilde{\gamma}_{ij}(K_\Theta^2 - W^2\tilde{D}_k\Theta\tilde{D}^k\Theta - 2V($$
$$+\frac{2\hat{\alpha}M^2}{W^2}(2W\tilde{\epsilon}_{(i|kl}\tilde{E}_{|j)}{}^l\tilde{\mathcal{F}}^k - 2\tilde{B}_{(i}{}^k\tilde{\mathcal{F}}_{j)k}$$
$$+\tilde{\gamma}_{ij}\tilde{B}^{kl}\tilde{\mathcal{F}}_{kl} + \tilde{B}_{ij}(\mathcal{F}_{nn} + \mathrm{tr}\mathcal{F})\big)$$

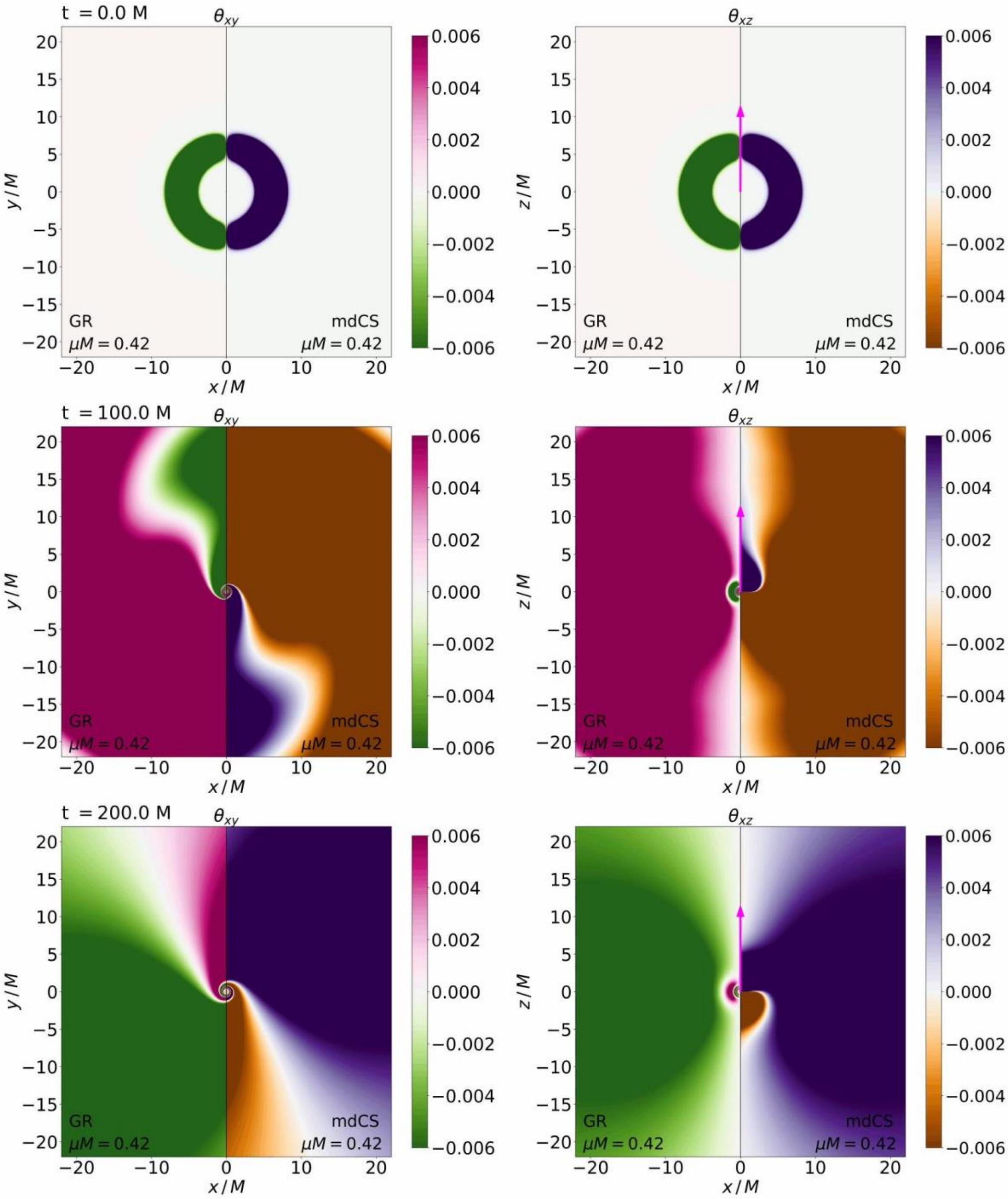
$$\mathcal{F}_{nn} = f'(\Theta)\big(-KK_\Theta + W^2\tilde{D}_k\tilde{D}^k\Theta - W\tilde{D}_k\Theta\tilde{D}^k W$$
$$-V'(\Theta)) + \frac{\hat{\alpha}M^2}{4}\,{}^*RRf'(\Theta)^2 + K_\Theta^2 f''(\Theta)$$
$$\mathcal{F}_i = (\tilde{D}_i K_\Theta)f'(\Theta) - \left(\tilde{A}_{ij} + \frac{1}{3}K\tilde{\gamma}_{ij}\right)(\tilde{D}^j\Theta)f'(\Theta)$$
$$+K_\Theta(\tilde{D}_i\Theta)f''(\Theta)$$
$$\tilde{\mathcal{F}}_{ij} = W^2\mathcal{F}_{ij}$$
$$= -\left(\tilde{A}_{ij} + \frac{1}{3}K\tilde{\gamma}_{ij}\right)K_\Theta f'(\Theta) + W^2(D_iD_j\Theta)f'(\Theta)$$
$$+W^2(\tilde{D}_i\Theta)(\tilde{D}_j\Theta)f''(\Theta),$$

$$\mathcal{F}^i = W^2\tilde{\mathcal{F}}^i$$

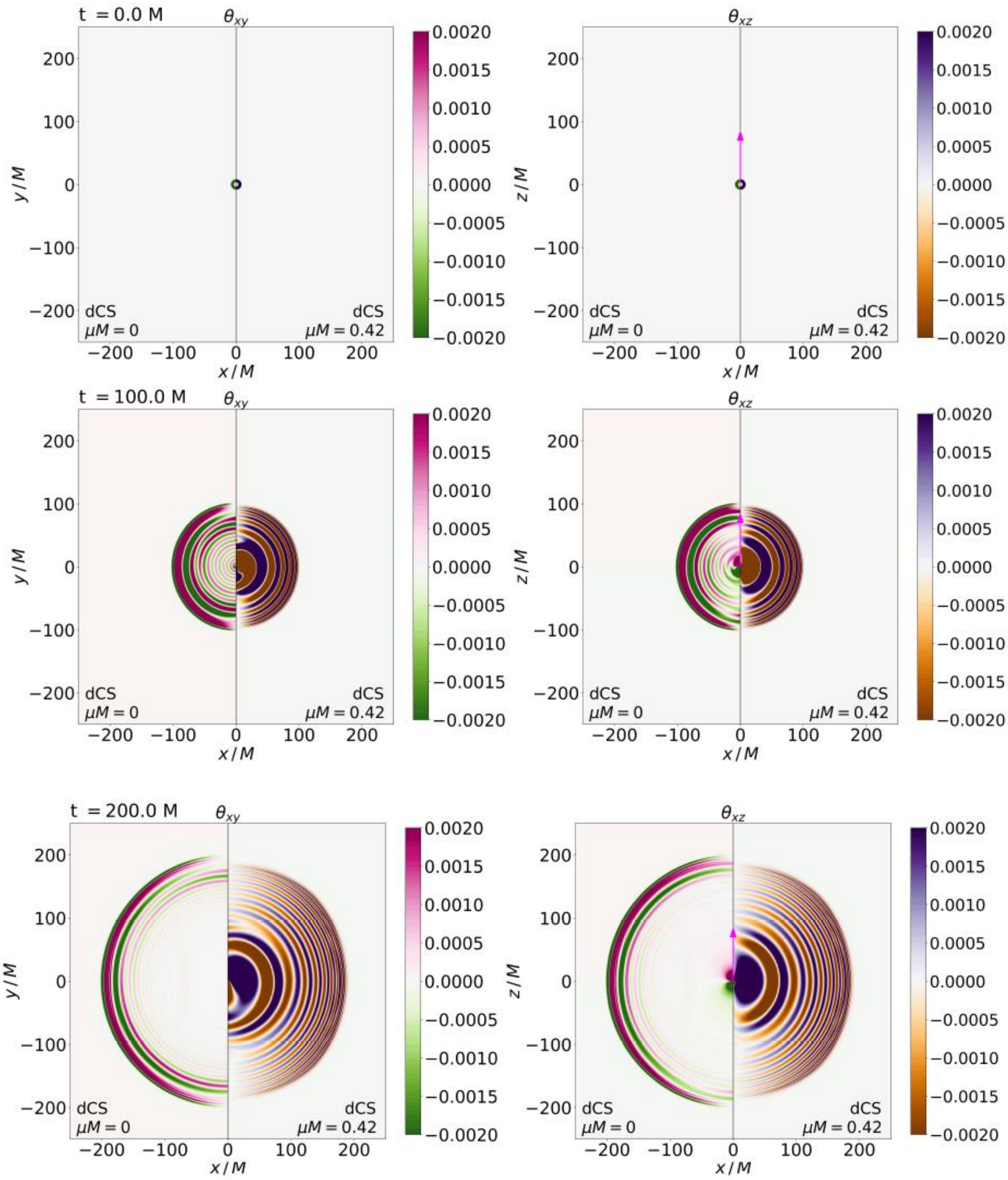
$$\mathrm{tr}\mathcal{F} = \gamma^{ij}\mathcal{F}_{ij} = \tilde{\gamma}^{ij}\tilde{\mathcal{F}}_{ij}$$

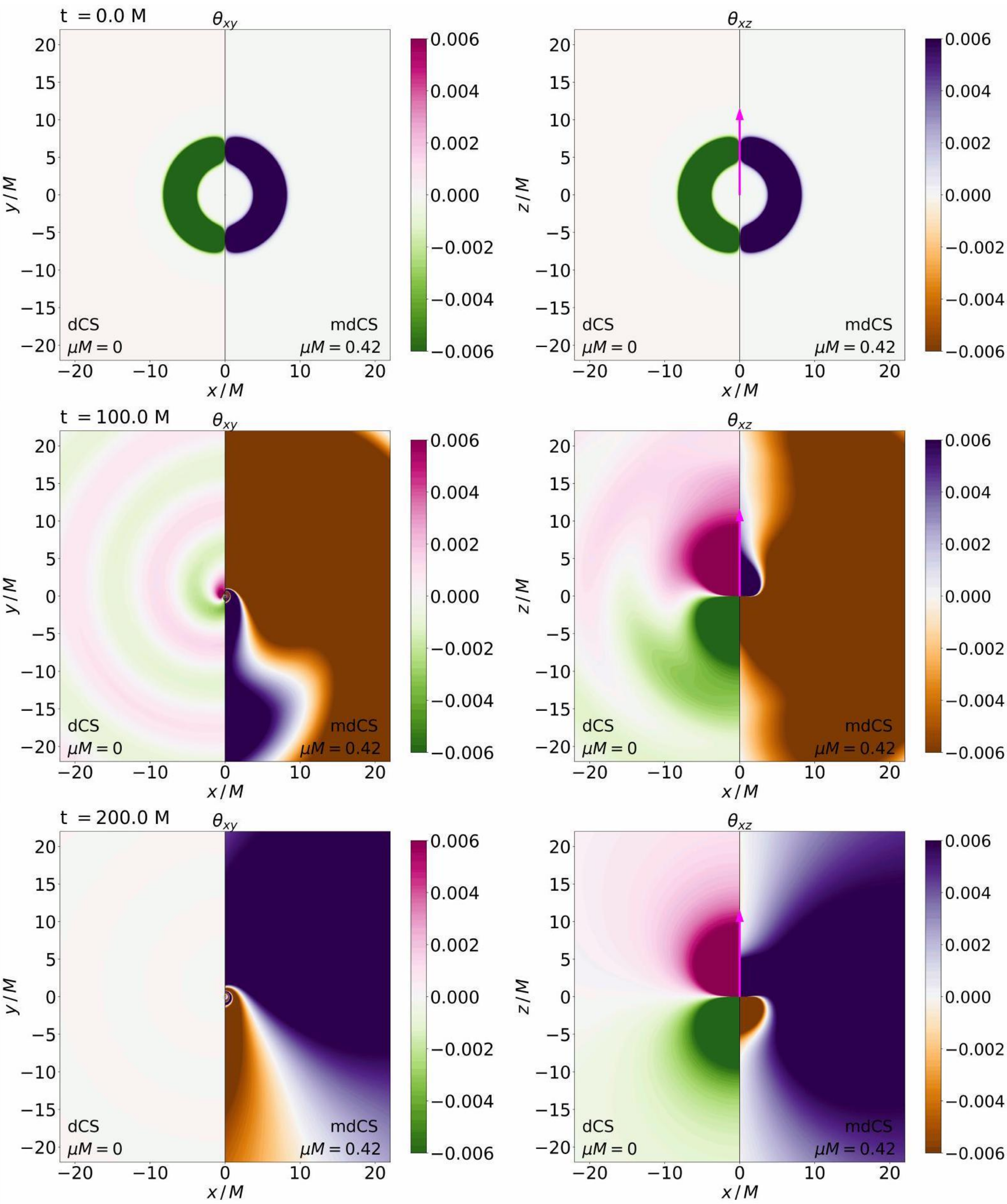
$$D_iD_j\Theta = \frac{\tilde{D}_k\Theta}{W}\left(\tilde{\gamma}_i^k(\tilde{D}_jW) + \tilde{\gamma}_j^k(\tilde{D}_iW) - \tilde{\gamma}_{ij}(\tilde{D}^kW)\right)$$
$$+\tilde{D}_i\tilde{D}_j\Theta$$

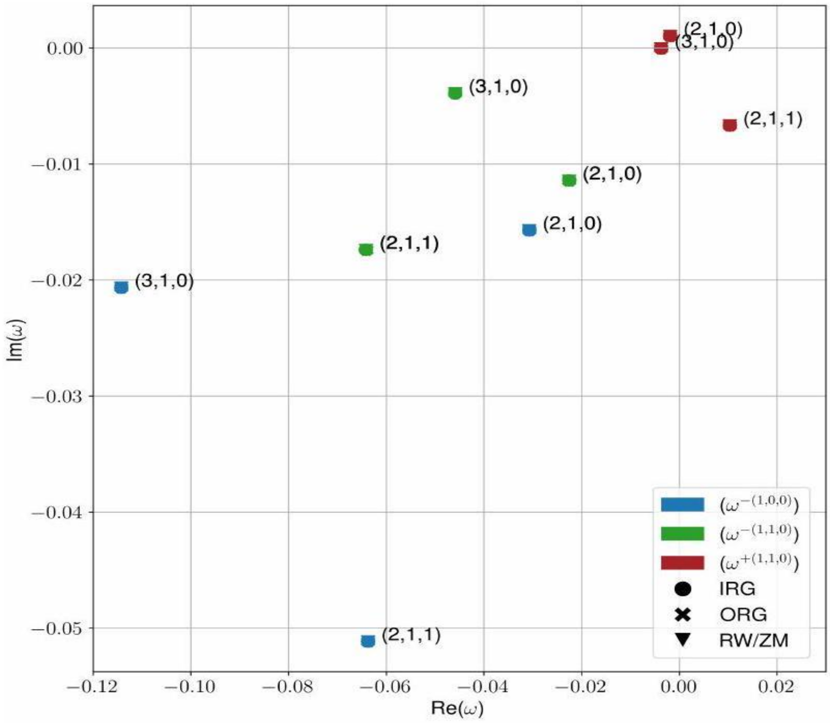

t = 0.0 M
θ_{xy}
θ_{xz}
t = 100.0 M
θ_{xy}
θ_{xz}
y / M
z / M
x / M
GR
dCS
$\mu M = 0.42$
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.0015
−0.0020



t = 0.0 M
θ_{xy}
θ_{xz}
GR
mdCS
$\mu M = 0.42$
t = 100.0 M
t = 200.0 M
y / M
z / M
x / M


t = 0.0 M
t = 100.0 M
t = 200.0 M
θ_xy
θ_xz
y / M
z / M
x / M
dCS
μM = 0
dCS
μM = 0.42
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.0015
−0.0020
−200 −100 0 100 200


t = 0.0 M
t = 100.0 M
t = 200.0 M
θ_{xy}
θ_{xz}
dCS
$\mu M = 0$
mdCS
$\mu M = 0.42$
$y\,/\,M$
$z\,/\,M$
$x\,/\,M$
0.006
0.004
0.002
0.000
−0.002
−0.004
−0.006



$$\square\,\vartheta = -\frac{\alpha}{4} R^*_{\nu\mu\rho\sigma} R^{\mu\nu\rho\sigma},$$

$$R_{\mu\nu} = -\frac{\alpha}{\kappa_g} C_{\mu\nu} + \frac{1}{2\kappa_g} \bar{T}^{\vartheta}_{\mu\nu},$$

$$\kappa_g = (16\pi)^{-1}, \alpha$$

$$\square = \nabla_\mu \nabla^\mu$$

$$^*R^{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon^{\rho\sigma\alpha\beta} R^{\mu\nu}_{\alpha\beta}$$

$$C^{\mu\nu} \equiv (\nabla_\sigma \vartheta) \epsilon^{\sigma\delta\alpha(\mu} \nabla_\alpha R^{\nu)\delta} + (\nabla_\sigma \nabla_\delta \vartheta)^* R^{\delta(\mu\nu)\sigma}$$

$$\bar{T}^{\vartheta}_{\mu\nu} \equiv \left(\nabla_\mu \vartheta\right)(\nabla_\nu \vartheta)$$

$$\vartheta = \zeta \vartheta^{(1,0)} + \zeta \epsilon \vartheta^{(1,1)}$$

$$\Psi_i = \Psi_i^{(0,0)} + \zeta \Psi_i^{(1,0)} + \epsilon \Psi_i^{(0,1)} + \zeta \epsilon \Psi_i^{(1,1)}$$

$$\zeta \equiv \frac{\alpha^2}{\kappa_g M^4}$$

$$\zeta^{1/2} \vartheta \to \vartheta$$

$$\square^{(0,0)}\, \vartheta^{(1,1)} = -\frac{M^2}{16\pi^{\frac{1}{2}}} [R^* R]^{(0,1)} - \square^{(0,1)}\, \vartheta^{(1,0)}$$

$$H_0^{(0,0)} \Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}^{(1,1)}$$

$$H_4^{(0,0)} \Psi_4^{(1,1)} = \mathcal{T}_{\text{geo}}^{(1,1)} + \mathcal{T}^{(1,1)}$$

$$h_{\mu\nu}l^{\nu} = 0, h = 0$$
$$h_{\mu\nu}n^{\nu} = 0, h = 0$$

$$h_{\mu\nu}^{(0,1)} = \mathcal{O}_{\mu\nu}\bar{\Psi}_{\mathrm{H}} + \overline{\mathcal{O}}_{\mu\nu}\Psi_{\mathrm{H}}$$

$$\Psi_0^{(0,1)} = \sum_{\ell,m} {}_2R_{\ell m}^{(0,1)}(r)_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}$$
$$\rho^{-4}\Psi_4^{(0,1)} = \sum_{\ell,m} {}_{-2}R_{\ell m}^{(0,1)}(r)_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}$$

$${}_s\mathcal{Y}_{\ell m}(\theta,\phi) = {}_sS_{\ell m}(\theta)e^{im\phi}$$

$$\bar{\Psi}_{\mathrm{H}} = \sum_{\ell,m} {}_2\hat{R}_{\ell m}(r)_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}$$
$$\bar{\Psi}_{\mathrm{H}} = \sum_{\ell,m} {}_{-2}\hat{R}_{\ell m}(r)_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}$$

$${}_2\hat{R}_{\ell m}(r) = -\frac{2}{\mathfrak{C}}\Delta^2(r)\left(D_{\ell m}^{\dagger}\right)^4\left[\Delta^2(r)\, {}_2R_{\ell m}^{(0,1)}(r)\right]$$
$${}_{-2}\hat{R}_{\ell m}(r) = \frac{32}{\mathfrak{C}}(D_{\ell m})^4\, {}_{-2}R_{\ell m}^{(0,1)}(r)$$

$$D_{\ell m} = \partial_r + i\frac{am - (r^2 + a^2)\omega_{\ell m}}{\Delta(r)},$$
$$D_{\ell m}^{\dagger} = \partial_r - i\frac{am - (r^2 + a^2)\omega_{\ell m}}{\Delta(r)},$$

$$\Delta(r) = r^2 - 2Mr + a^2$$

$$\begin{aligned}\mathfrak{C} =\, & 144M^2\omega_{\ell m}^2 + \left(8 + 6\,{}_sB_{\ell m} + {}_sB_{\ell m}^2\right)^2 - 8\left[-8 + {}_sB_{\ell m}^2(4 + {}_sB_{\ell m})\right]m\gamma_{\ell m} \\ & +4\left[8 - 2\,{}_sB_{\ell m} - {}_sB_{\ell m}^2 + {}_sB_{\ell m}^3 + 2(-2 + {}_sB_{\ell m})(4 + 3\,{}_sB_{\ell m})m^2\right]\gamma_{\ell m}^2 \\ & -8m\left(8 - 12\,{}_sB_{\ell m} + 3{}_sB_{\ell m}^2 + 4(-2 + {}_sB_{\ell m})m^2\right)\gamma_{\ell m}^3 \\ & +2\left(42 - 22{}_sB_{\ell m} + 3{}_sB_{\ell m}^2 + 8(-11 + 3\,{}_sB_{\ell m})m^2 + 8m^4\right)\gamma_{\ell m}^4 \\ & -8m[3{}_sB_{\ell m} + 4(-4 + m^2)]\gamma_{\ell m}^5 + 4(-7 + {}_sB_{\ell m} + 6m^2)\gamma_{\ell m}^6 - 8m\gamma_{\ell m}^7 + \gamma_{\ell m}^8\end{aligned}$$

$$\gamma_{\ell m} = \chi M\omega_{\ell m},\ {}_sB_{\ell m} = {}_sA_{\ell m} + s, s$$

$${}_sA_{\ell m} = \ell(\ell + 1) - s(s + 1) - \frac{2\chi mM\omega s^2}{\ell(\ell + 1)} + \mathcal{O}(\chi^2)$$

$$\mathcal{D}_{\ell m}^{\dagger} \equiv -\frac{2}{\mathfrak{C}}\Delta^2(r)\left(D_{\ell m}^{\dagger}\right)^4\Delta^2(r), \mathcal{D}_{\ell m} \equiv \frac{32}{\mathfrak{C}}(D_{\ell m})^4$$

$${}_s\Psi^{(0,1)} = {}_sR_{\ell m}^{(0,1)}(r)_s\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \eta_{\ell m}\, {}_sR_{\ell -m}^{(0,1)}(r)_s\mathcal{Y}_{\ell -m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t}$$
$${}_s\Psi^{(1,1)} = {}_sR_{\ell m}^{(1,1)}(r)_s\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \eta_{\ell m}\, {}_sR_{\ell -m}^{(1,1)}(r)_s\mathcal{Y}_{\ell -m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t}$$

$$\omega_{\ell -m} = -\bar{\omega}_{\ell m}$$

$$ {}_sR^{(0,1)}_{\ell -m}(r) = (-1)^m \, {}_s\bar{R}^{(0,1)}_{\ell m}(r) $$

$$ {}_sR^{(1,1)}_{\ell m}(r) $$

$$ \begin{aligned} \vartheta^{(1,1)} &= \frac{\Theta^{(1,1)}_{\ell m}(r)}{r}\, {}_0\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \frac{\Theta^{(1,1)}_{\ell -m}(r)}{r}\, {}_0\mathcal{Y}_{\ell -m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t} \\ &\equiv \frac{\Theta^{(1,1)}_{\ell m}(r)}{r}\, {}_0\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \frac{\bar{\Theta}^{(1,1)}_{\ell m}(r)}{r}\, {}_0\overline{\mathcal{Y}}_{\ell m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t} \end{aligned} $$

$$ \bar{\omega}_{\ell m} = \omega_{\ell -m} \text{ and } {}_0\overline{\mathcal{Y}}_{\ell m}(\theta,\phi) = (-1)^m \, {}_0\mathcal{Y}_{\ell -m}(\theta,\phi) $$

$$ \Theta^{(1,1)}_{\ell m}(r) = (-1)^m \bar{\Theta}^{(1,1)}_{\ell -m}(r) $$

$$ \left\{\vartheta^{(1,1)}, \Psi_0^{(1,1)}, \Psi_4^{(1,1)}\right\} $$

$$ {}_0\mathcal{Y}_{\ell m}(\theta,\phi),\ {}_2\mathcal{Y}_{\ell m}(\theta,\phi), \text{ and } {}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi) $$

$$ \begin{aligned} &\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 - 4\chi m M^2\omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell m}\right]\Theta^{(1,1)}_{\ell m}(r) \\ &= V^R_{\ell m}(r) + V^{\square}_{\ell m}(r) + \bar{\eta}_{\ell m}\left(V^{\dagger R}_{\ell -m}(r) + V^{\dagger\square}_{\ell -m}(r)\right), \\ &\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 + 4\chi m M^2\omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell -m}\right]\Theta^{(1,1)}_{\ell -m}(r) \\ &= \eta_{\ell m}\left(V^R_{\ell -m}(r) + V^{\square}_{\ell -m}(r)\right) + V^{\dagger R}_{\ell m}(r) + V^{\dagger\square}_{\ell m}(r), \end{aligned} $$

$$ r_s = 2M \otimes_0 A_{\ell m} $$

$$ \begin{aligned} V^R_{\ell m}(r) &= i\left(g_1^{\ell m}(r)_2\hat{R}_{\ell m}(r) + g_2^{\ell m}(r)_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{00} + \chi\left(g_3^{\ell m}(r)_2\hat{R}_{\ell m}(r) + g_4^{\ell m}(r)_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s} \\ V^{\square}_{\ell m}(r) &= \chi\left(h_1^{\ell m}(r)_2\hat{R}_{\ell m}(r) + h_2^{\ell m}(r)_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s} \end{aligned} $$

$$ V^{\dagger R}_{\ell -m} \boxtimes V^{\dagger\square}_{\ell -m}\left\{\Lambda^{\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 s}\right\} \to \left\{\Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 s}\right\} $$

$$ g_i^{\ell m}(r) = g_i^{\ell m}(r,\omega,M) \text{ and } h_i^{\ell m}(r) = h_i^{\ell m}(r,\omega,M) $$

$$ \begin{aligned} g_1^{\ell m}(r) &\to i\frac{M^2 r^3}{16\pi^{\frac{1}{2}}} g_1^{\ell m}(r), & g_2^{\ell m}(r) &\to i\frac{M^2 r^3}{16\pi^{\frac{1}{2}}} g_2^{\ell m}(r), \\ g_3^{\ell m}(r) &\to -\frac{M^2 r^3}{16\pi^{\frac{1}{2}}} g_3^{\ell m}(r), & g_4^{\ell m}(r) &\to -\frac{M^2 r^3}{16\pi^{\frac{1}{2}}} g_4^{\ell m}(r), \end{aligned} $$

$$ h_1^{\ell m}(r) \to -r^3 h_1^{\ell m}(r), \qquad h_2^{\ell m}(r) \to -r^3 h_2^{\ell m}(r) $$

$$ \bar{g}_i^{\ell -m}(r) = g_i^{\ell m}(r), \bar{h}_i^{\ell -m}(r) = h_i^{\ell m}(r) $$

$$g_i^{\ell m}(r) \circledast h_i^{\ell m}(r) \oplus g_1^{\ell m}(r) \boxplus g_2^{\ell m}(r)$$

$$\mathcal{O}(\chi^1) \triangle g_i^{\ell m}(r) \triangle h_i^{\ell m}(r) \left\{\Lambda_{s_1 s_2}^{\ell_1 \ell_2 m}, \Lambda_{s_1 s_2 c}^{\ell_1 \ell_2 m}, \Lambda_{s_1 s_2 s}^{\ell_1 \ell_2 m}\right\} \triangle \left\{\Lambda_{s_1 s_2}^{\dagger \ell_1 \ell_2 m}, \Lambda_{s_1 s_2 c}^{\dagger \ell_1 \ell_2 m}, \Lambda_{s_1 s_2 s}^{\dagger \ell_1 \ell_2 m}\right\}$$

$$\Lambda_{10s}^{\ell \ell m} = m \Lambda_{10s}^{\ell \ell 1}, \Lambda_{10s}^{\dagger \ell m} = (-1)^{m+1} m \Lambda_{10s}^{\ell \ell 1}\, {}_s R_{\ell m}^{(0,1)}(r)\, {}_s \hat{R}_{\ell m}(r)$$

$$V_{\ell m}^{R}(r) = i\left(g_1^{\ell m}(r)_2 \hat{R}_{\ell m}(r) + g_2^{\ell m}(r)_2 \hat{R}'_{\ell m}(r)\right) + \chi m\left(g_3^{\ell m}(r)_2 \hat{R}_{\ell m}(r) + g_4^{\ell m}(r)_2 \hat{R}'_{\ell m}(r)\right)\Lambda_{10s}^{\ell \ell 1}$$

$$V_{\ell m}^{\square}(r) = \chi m\left(h_1^{\ell m}(r)_2 \hat{R}_{\ell m}(r) + h_2^{\ell m}(r)_2 \hat{R}'_{\ell m}(r)\right)\Lambda_{10s}^{\ell \ell 1},$$
$$V_{\ell -m}^{\dagger R}(r) = -V_{\ell m}^{R}(r), V_{\ell -m}^{\dagger \square}(r) = -V_{\ell m}^{\square}(r).$$

$$\left[r(r-r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0 A_{\ell m}\right] \Theta_{\ell m}^{(1,1)}(r) = (1 - \bar{\eta}_{\ell m})\left(V_{\ell m}^{R}(r) + V_{\ell m}^{\square}(r)\right),$$
$$\left[r(r-r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 + 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0 A_{\ell -m}\right] \Theta_{\ell -m}^{(1,1)}(r) = (\eta_{\ell m} - 1)\left(V_{\ell -m}^{R}(r) + V_{\ell -m}^{\square}(r)\right).$$

$$\left[r(r-r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0 A_{\ell m}\right] \bar{\Theta}_{\ell -m}^{(1,1)}(r) = (-1)^m (1 - \bar{\eta}_{\ell m})\left(V_{\ell m}^{R}(r) + V_{\ell m}^{\square}(r)\right),$$

$$\left[r(r-r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0 A_{\ell m}\right] \Theta_{\ell m}^{(1,1)}(r) = (1 - \bar{\eta}_{\ell m})\left(U_{\ell m}^{R}(r) + U_{\ell m}^{\square}(r)\right),$$
$$\left[r(r-r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 + 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0 A_{\ell -m}\right] \Theta_{\ell -m}^{(1,1)}(r) = (\eta_{\ell m} - 1)\left(U_{\ell -m}^{R}(r) + U_{\ell -m}^{\square}(r)\right),$$

$$U_{\ell m}^{R}(r) = i\left(\mathfrak{g}_1^{\ell m}(r)_{-2} \hat{R}_{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r)_{-2} \hat{R}'_{\ell m}(r)\right) + \chi m\left(\mathfrak{g}_3^{\ell m}(r)_{-2} \hat{R}_{\ell m}(r) + \mathfrak{g}_4^{\ell m}(r)_{-2} \hat{R}'_{\ell m}(r)\right)\Lambda_{10s}^{\ell \ell 1}$$
$$U_{\ell m}^{\square}(r) = \chi m\left(\mathfrak{h}_1^{\ell m}(r)_{-2} \hat{R}_{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r)_{-2} \hat{R}'_{\ell m}(r)\right)\Lambda_{10s}^{\ell \ell 1}$$
$$U_{\ell -m}^{\dagger R}(r) = -U_{\ell m}^{R}(r), U_{\ell -m}^{\dagger \square}(r) = -U_{\ell m}^{\square}(r)$$

$$\bar{\mathfrak{g}}_i^{\ell -m}(r) = \mathfrak{a}_i^{\ell m}(r), \bar{\mathfrak{h}}_i^{\ell -m}(r) = \mathfrak{h}_i^{\ell m}(r).$$

$$H_0^{(0,0)} \Psi_0^{(1,1)} = 2r^2 \left(\mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}_{T_\vartheta}^{(1,1)} + \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}\right.$$
$$\left. + \mathcal{S}_{T_\Psi}^{(1,1)} + \tilde{\mathcal{S}}_{T_\Psi}^{(1,1)}\right)$$

$$H_0^{(0,0)} = H_0^{(0,0,0)} + \chi H_0^{(0,1,0)}$$
$$H_0^{(0,0,0)} = -r(r-r_s)\partial_r^2 - 6(r-M)\partial_r - \frac{C(r)}{r-r_s}$$
$$-\partial_\theta^2 - \cot\theta \partial_\theta + (4 + m^2 + 4m\cos\theta)\csc^2\theta - 6$$
$$H_0^{(0,1,0)} = -4M\left[\frac{m(i(r-M) - M\omega r)}{r(r-r_s)} - \omega\cos\theta\right]$$
$$C(r) = 4i\omega r(r-3M) + \omega^2 r^3.$$

$$\hat{\mathcal{P}} H_0^{(0,0)} = H_0^{(0,0)}$$

$$\hat{\mathcal{P}} f = \hat{\mathcal{C}} \hat{P} f = \hat{\mathcal{C}} f(\pi - \theta, \phi + \pi) = \bar{f}(\pi - \theta, \phi + \pi)$$

$$\mathcal{S}_{\text{geo}}^{(1,1)}, \left\{\mathcal{S}_{T_\vartheta}^{(1,1)}, \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}\right\}, \text{and } \left\{\mathcal{S}_{T_\Psi}^{(1,1)}, \tilde{\mathcal{S}}_{T_\Psi}^{(1,1)}\right\}$$

$$\mathcal{S}_{\text{geo}}^{(1,1)}=-e^{-i\omega_{\ell m}t}H_0^{\ell m(1,0)}\left[{}_2R_{\ell m}^{(0,1)}(r){}_2Y_{\ell m}(\theta,\phi)\right]-\eta_{\ell m}\times(m\to-m),$$

$$H_0^{\ell m(1,0)}=\frac{i\chi mM^4}{448r^9(r-r_s)}(C_1(r)+4i\omega_{\ell m}r^2C_2(r))-\frac{i\chi M^4}{16r^9}\cos\theta\left(C_3(r)-\frac{i\omega_{\ell m}r^2C_4(r)}{2}\right)$$
$$+\frac{i\chi M^4}{32r^8}\left[(r-r_s)C_4(r)\cos\theta\partial_r-\frac{C_5(r)}{2r}\sin\theta\partial_\theta\right]$$

$$\partial_\theta({}_2Y_{\ell m}(\theta,\phi))=\frac{1}{2}\left(\sqrt{(\ell+2)(\ell-1)}{}_1Y_{\ell m}(\theta,\phi)-\sqrt{(\ell+3)(\ell-2)}{}_3Y_{\ell m}(\theta,\phi)\right),$$

$$\mathcal{S}_{\text{geo}}^{(1,1)}=e^{-i\omega_{\ell m}t}\mathcal{O}_{\text{geo}}^{\ell m}\,{}_2R_{\ell m}^{(0,1)}+\eta_{\ell m}\times(m\to-m),$$

$$\mathcal{O}_{\text{geo}}^{\ell m}=-\frac{i\chi mM^4}{448r^9(r-r_s)}\,{}_2Y_{\ell m}(\theta,\phi)(C_1(r)+4i\omega_{\ell m}r^2C_2(r))$$
$$+\frac{i\chi M^4}{16r^9}\cos\theta{}_2Y_{\ell m}(\theta,\phi)\left[C_3(r)-C_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}+\frac{r(r-r_s)}{2}\partial_r\right)\right]$$
$$+\frac{i\chi M^4}{128r^9}C_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\sin\theta{}_1Y_{\ell m}(\theta,\phi)-\sqrt{(\ell+3)(\ell-2)}\sin\theta{}_3Y_{\ell m}(\theta,\phi)\right).$$

$$ {}_sY_{\ell m}(\pi-\theta,\phi+\pi)=(-1)^\ell\,{}_{-s}Y_{\ell m}(\theta,\phi),$$
$$ {}_s\bar{Y}_{\ell m}(\theta,\phi)=(-1)^{m+s}\,{}_{-s}Y_{\ell-m}(\theta,\phi),$$

$$\hat{\mathcal{P}}[\pm 2Y_{\ell m}(\theta,\phi)]=(-1)^{\ell+m}\pm 2Y_{\ell-m}(\theta,\phi)$$
$$\hat{\mathcal{P}}[\sin\theta_{\pm1}Y_{\ell m}(\theta,\phi)]=(-1)^{\ell+m+1}\sin\theta_{\pm1}Y_{\ell-m}(\theta,\phi)$$
$$\hat{\mathcal{P}}[\cos\theta_{\pm2}Y_{\ell m}(\theta,\phi)]=(-1)^{\ell+m+1}\cos\theta_{\pm2}Y_{\ell-m}(\theta,\phi)$$
$$\hat{\mathcal{P}}[\sin\theta_{\pm3}Y_{\ell m}(\theta,\phi)]=(-1)^{\ell+m+1}\sin\theta_{\pm3}Y_{\ell-m}(\theta,\phi)$$

$$\hat{\mathcal{P}}\left[\mathcal{O}_{\text{geo}}^{\ell m}\,{}_2R_{\ell m}^{(0,1)}(r)\right]=(-1)^\ell\mathcal{O}_{\text{geo}}^{\ell-m}\,{}_2R_{\ell-m}^{(0,1)}(r)$$

$$\mathcal{S}_{T_\vartheta}^{(1,1)}=e^{-i\omega_{\ell m}t}\left[\mathcal{A}_1^{\ell m}(r){}_2Y_{\ell m}(\theta,\phi)+i\chi\mathcal{A}_2^{\ell m}(r)\sin\theta{}_1Y_{\ell m}(\theta,\phi)+i\chi\mathcal{A}_3^{\ell m}(r)\cos\theta{}_2Y_{\ell m}(\theta,\phi)\right]$$
$$\tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}=e^{+i\bar{\omega}_{\ell m}t}\left[-\overline{\mathcal{A}}_1^{\ell m}(r){}_{-2}\bar{Y}_{\ell m}(\theta,\phi)+i\chi\overline{\mathcal{A}}_2^{\ell m}(r)\sin\theta{}_{-1}\bar{Y}_{\ell m}(\theta,\phi)-i\chi\overline{\mathcal{A}}_3^{\ell m}(r)\cos\theta{}_{-2}\bar{Y}_{\ell m}(\theta,\phi)\right]$$

$$\mathcal{A}_i^{\ell m}(r)=i\mathcal{O}_i^{\ell m}\Theta_{\ell m}(r)+\alpha_{\ell m}\mathcal{Q}_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)$$

$$\overline{\mathcal{O}}_i^{\ell-m}=\mathcal{O}_i^{\ell m},\overline{\mathcal{Q}}_i^{\ell-m}=\mathcal{Q}_i^{\ell m}$$

$$\mathcal{Q}_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)\partial_r^2\Theta_{\ell m}^{(1,1)}(r)\partial_r\Theta_{\ell m}^{(1,1)}(r)\Theta_{\ell m}(r)\mathcal{S}_{T_\vartheta}^{(1,1)}\ \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}\vartheta^{(1,1)}$$

$$\overline{\mathcal{A}}_i^{\ell-m}(r)=(-1)^{m+1}\left[i\mathcal{O}_i^{\ell m}\Theta_{\ell m}^{(1,1)}(r)+\alpha_{\ell m}\mathcal{Q}_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)\right]$$

$$\Theta_{\ell m}^{(1,1)}(r)=\alpha_{\ell m}\mathcal{H}_\vartheta^{-1}[V_{\ell m}^R(r)+V_{\ell m}^{\square}(r)]$$
$$\equiv i\alpha_{\ell m}\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m}\,{}_2\hat{R}_{\ell m}(r)$$

$$\overline{\mathcal{V}}^{\ell-m}=\mathcal{V}^{\ell m}$$

$$\mathcal{A}_i^{\ell m}(r) \ = \alpha_{\ell m}\hat{A}_i^{\ell m}\ {}_2\hat{R}_{\ell m}(r)$$
$$\overline{\mathcal{A}}_i^{\ell -m}(r) \ = (-1)^{m+1}\alpha_{\ell m}\hat{A}_i^{\ell m}\ {}_2\hat{R}_{\ell m}(r)$$

$$\hat{A}_i^{\ell m} \equiv \mathcal{Q}_i^{\ell m} - \mathcal{O}_i^{\ell m}\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m}$$

$$\hat{\mathcal{P}}\hat{A}_i^{\ell m} = \overline{\hat{A}}_i^{\ell m} = \hat{A}_i^{\ell -m}$$

$$\mathcal{S}_{T_\vartheta}^{(1,1)} = \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}(m \to -m)$$
$$= (1-\bar{\eta}_{\ell m})e^{-i\omega_{\ell m}t}\mathcal{O}_{T_\vartheta}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r)$$

$$\mathcal{O}_{T_\vartheta}^{\ell m} = \left[\ {}_2Y_{\ell m}(\theta,\phi)\hat{A}_1^{\ell m} + i\chi\sin\ \theta_1 Y_{\ell m}(\theta,\phi)\hat{A}_2^{\ell m}\right.$$
$$\left.+ i\chi\cos\ \theta_2 Y_{\ell m}(\theta,\phi)\hat{A}_3^{\ell m}\right]\mathcal{D}_{\ell m}^\dagger$$

$$\mathcal{S}_{T_\vartheta}^{(1,1)} = \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}(m \to -m)$$

$$\bar{\Theta}_{\ell -m}^{(1,1)}(r)\Theta_{\ell m}^{(1,1)}(r)\left\{\Theta_{\ell m}^{(1,1)}(r), \mathcal{S}_{T_\vartheta}^{(1,1)}\right\}\left\{\Theta_{\ell -m}^{(1,1)}(r), \tilde{\mathcal{S}}_{T_\vartheta}^{(1,1)}\right\}\mathcal{S}_{T_\vartheta}^{(1,1)}\Theta_{\ell m}^{(1,1)}(r)$$

$$\hat{\mathcal{P}}\left[\mathcal{O}_{T_\vartheta}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r)\right] = (-1)^\ell \mathcal{O}_{T_\vartheta}^{\ell -m}\ {}_2R_{\ell -m}^{(0,1)}(r)$$

$$\mathcal{S}_{T_\Psi}^{(1,1)} \ = i\chi e^{-i\omega_{\ell m}t}\left[\mathcal{B}_1^{\ell m}(r)\sin\ \theta_1 Y_{\ell m}(\theta,\phi) + \mathcal{B}_2^{\ell m}(r)\cos\ \theta_2 Y_{\ell m}(\theta,\phi) + \mathcal{B}_3^{\ell m}(r)\sin\ \theta_3 Y_{\ell m}(\theta,\phi)\right] + \eta_{\ell m} \times (m \to -m),$$
$$\tilde{\mathcal{S}}_{T_\Psi}^{(1,1)} \ = i\chi e^{+i\bar{\omega}_{\ell m}t}\left[\tilde{\mathcal{B}}_1^{\ell m}(r)\sin\ \theta_{-1}\bar{Y}_{\ell m}(\theta,\phi) + \tilde{\mathcal{B}}_2^{\ell m}(r)\cos\ \theta_{-2}\bar{Y}_{\ell m}(\theta,\phi)\right] + \bar{\eta}_{\ell m} \times (m \to -m),$$

$$\overline{\mathcal{B}}_i^{\ell -m}(r) = \mathcal{B}_i^{\ell m}(r), \overline{\tilde{\mathcal{B}}}_i^{\ell -m}(r) = \tilde{\mathcal{B}}_i^{\ell m}(r)$$

$$\mathcal{B}_i^{\ell m}(r) \equiv \hat{B}_i^{\ell m}\ {}_2\hat{R}_{\ell m}(r), \tilde{\mathcal{B}}_i^{\ell m} \equiv \hat{\tilde{B}}_i^{\ell m}\ {}_2\overline{\hat{R}}_{\ell m}(r),$$

$$\hat{\mathcal{P}}\hat{B}_i^{\ell m} = \hat{B}_i^{\ell -m}, \hat{\mathcal{P}}\hat{\tilde{B}}_i^{\ell m} = \hat{\tilde{B}}_i^{\ell -m}$$

$$\mathcal{S}_{T_\Psi}^{(1,1)} = e^{-i\omega_{\ell m}t}\mathcal{O}_{T_\Psi}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r) + \eta_{\ell m} \times (m \to -m),$$
$$\tilde{\mathcal{S}}_{T_\Psi}^{(1,1)} = \bar{\eta}_{\ell m}e^{-i\omega_{\ell m}t}\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r) + (m \to -m),$$

$$\mathcal{O}_{T_\Psi}^{\ell m} = i\chi\left[\sin\ \theta_1 Y_{\ell m}(\theta,\phi)\hat{B}_1^{\ell m} + \cos\ \theta_2 Y_{\ell m}(\theta,\phi)\hat{B}_2^{\ell m}\right.$$
$$\left.+ \sin\ \theta_3 Y_{\ell m}(\theta,\phi)\hat{B}_3^{\ell m}\right]\mathcal{D}_{\ell m}^\dagger,$$
$$\tilde{\mathcal{O}}_{T_\Psi}^{\ell m} = i\chi\left[-\sin\ \theta_1 Y_{\ell m}(\theta,\phi)\hat{\tilde{B}}_1^{\ell -m} + \cos\ \theta_2 Y_{\ell m}(\theta,\phi)\hat{\tilde{B}}_2^{\ell -m}\right]$$
$$\mathcal{D}_{\ell m}^\dagger,$$

$$\left\{{}_s\overline{\hat{R}}_{\ell -m}(r),\ {}_s\bar{Y}_{\ell -m}(\theta,\phi)\right\}\left\{{}_s\hat{R}_{\ell m}(r),\ {}_sY_{\ell m}(\theta,\phi)\right\}$$

$$\omega_{\ell m} = -\bar{\omega}_{\ell -m}$$

$$\hat{\mathcal{P}}\left[\mathcal{O}_{T_\Psi}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r)\right] = (-1)^\ell \mathcal{O}_{T_\Psi}^{\ell -m}\ {}_2R_{\ell -m}^{(0,1)}(r)$$
$$\hat{\mathcal{P}}\left[\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\ {}_2R_{\ell m}^{(0,1)}(r)\right] = (-1)^\ell \tilde{\mathcal{O}}_{T_\Psi}^{\ell -m}\ {}_2R_{\ell -m}^{(0,1)}(r)$$

$$H_0^{\ell m(0,0)}\left[{}_2R_{\ell m}^{(1,1)}(r){}_2Y_{\ell m}(\theta,\phi)\right]=2r^2\left[\left(\mathcal{O}_{\rm geo}^{\ell m}+\mathcal{O}_{T_\vartheta}^{\ell m}+\mathcal{O}_{T_\Psi}^{\ell m}\right)-\bar{\eta}_{\ell m}\left(\mathcal{O}_{T_\vartheta}^{\ell m}-\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\right)\right]{}_2R_{\ell m}^{(0,1)}(r),$$
$$\eta_{\ell m}H_0^{\ell -m(0,0)}\left[{}_2R_{\ell -m}^{(1,1)}(r){}_2Y_{\ell -m}(\theta,\phi)\right]=2r^2\left[\eta_{\ell m}\left(\mathcal{O}_{\rm geo}^{\ell -m}+\mathcal{O}_{T_\vartheta}^{\ell -m}+\mathcal{O}_{T_\Psi}^{\ell -m}\right)-\left(\mathcal{O}_{T_\vartheta}^{\ell -m}-\tilde{\mathcal{O}}_{T_\Psi}^{\ell -m}\right)\right]{}_2R_{\ell -m}^{(0,1)}(r).$$

$$\bar{\eta}_{\ell m}H_0^{\ell m(0,0)}\left[{}_2R_{\ell m}^{(1,1)}(r){}_2Y_{\ell m}(\theta,\phi)\right]=2r^2\left[\bar{\eta}_{\ell m}\left(\mathcal{O}_{\rm geo}^{\ell m}+\mathcal{O}_{T_\vartheta}^{\ell m}+\mathcal{O}_{T_\Psi}^{\ell m}\right)-\left(\mathcal{O}_{T_\vartheta}^{\ell m}-\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\right)\right]{}_2R_{\ell m}^{(0,1)}(r),$$

$${}_2\bar{R}_{\ell -m}^{(1,1)}(r)=(-1)^m\,{}_2R_{\ell m}^{(1,1)}(r)$$

$$\eta_{\ell m}=\pm 1$$

$$\left[r(r-r_s)\partial_r^2+6(r-M)\partial_r+\frac{4i\omega_{\ell m}r(r-3M)+\omega_{\ell m}^2r^3}{r-r_s}+\frac{4i\chi mM((r-M)+iM\omega_{\ell m}r)}{r(r-r_s)}-{}_2A_{\ell m}\right]{}_2R_{\ell m}^{(1,1)}(r)$$
$$=-2r^2\left[\left(\mathcal{O}_{\rm geo}^{\ell m}+\mathcal{O}_{T_\vartheta}^{\ell m}+\mathcal{O}_{T_\Psi}^{\ell m}\right)\mp\left(\mathcal{O}_{T_\vartheta}^{\ell m}-\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\right)\right]{}_2R_{\ell m}^{(0,1)}(r)$$

$$\mathcal{O}_{\rm geo}^{\ell m}=-\frac{i\chi mM^4}{448r^9(r-r_s)}\left(C_1(r)+4i\omega_{\ell m}r^2C_2(r)\right)+\frac{i\chi mM^4}{16r^9}\left[C_3(r)-C_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}+\frac{r(r-r_s)}{2}\partial_r\right)\right]\Lambda_{22c}^{\ell\ell 1}$$
$$+\frac{i\chi mM^4}{128r^9}C_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\Lambda_{12s}^{\ell\ell 1}-\sqrt{(\ell+3)(\ell-2)}\Lambda_{32s}^{\ell\ell 1}\right)$$
$$\mathcal{O}_{T_\vartheta}^{\ell m}=\left[\hat{A}_1^{\ell m}+i\chi m\Lambda_{12s}^{\ell\ell 1}\hat{A}_2^{\ell m}+i\chi m\Lambda_{22c}^{\ell\ell 1}\hat{A}_3^{\ell m}\right]\mathcal{D}_{\ell m}^{\dagger}$$
$$\mathcal{O}_{T_\Psi}^{\ell m}=i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{B}_1^{\ell m}+\Lambda_{22c}^{\ell\ell 1}\hat{B}_2^{\ell m}+\Lambda_{32s}^{\ell\ell 1}\hat{B}_3^{\ell m}\right]\mathcal{D}_{\ell m}^{\dagger}$$
$$\tilde{\mathcal{O}}_{T_\Psi}^{\ell m}=-i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{\hat{B}}_1^{\ell -m}-\Lambda_{22c}^{\ell\ell 1}\hat{\hat{B}}_2^{\ell -m}\right]\mathcal{D}_{\ell m}^{\dagger}$$

$$\Lambda_{12s}^{\ell\ell m}=m\Lambda_{12s}^{\ell\ell 1},\quad \Lambda_{-12s}^{\dagger\ell m}=(-1)^{m+1}m\Lambda_{12s}^{\ell\ell 1}$$
$$\Lambda_{22c}^{\ell\ell m}=m\Lambda_{22c}^{\ell\ell 1},\quad \Lambda_{-22c}^{\dagger\ell\ell m}=(-1)^{m}m\Lambda_{22c}^{\ell\ell 1}$$
$$\Lambda_{32s}^{\ell\ell m}=m\Lambda_{32s}^{\ell\ell 1},\quad \Lambda_{-32s}^{\dagger\ell\ell m}=(-1)^{m+1}m\Lambda_{32s}^{\ell\ell 1}$$

$$\left\{\hat{A}_i^{\ell m},\hat{B}_i^{\ell m},\hat{\hat{B}}_i^{\ell m}\right\}$$

$$\omega_{\ell m}^{(1,0)}=\omega_{\ell m}^{(1,0,0)}+\chi m\omega_{\ell 1}^{(1,1,0)}+\mathcal{O}(\chi^2)$$

$$\left[r(r-r_s)\partial_r^2-2(r-M)\partial_r-\frac{4i\omega_{\ell m}r(r-3M)-\omega_{\ell m}^2r^3}{r-r_s}-\frac{4i\chi mM((r-M)-iM\omega_{\ell m}r)}{r(r-r_s)}-{}_{-2}A_{\ell m}\right]-{}_2R_{\ell m}^{(1,1)}(r)$$
$$=-2r^6\left[\left(\mathcal{Q}_{\rm geo}^{\ell m}+\mathcal{Q}_{T_\vartheta}^{\ell m}+\mathcal{Q}_{T_\Psi}^{\ell m}\right)\mp\left(\mathcal{Q}_{T_\vartheta}^{\ell m}-\tilde{\mathcal{Z}}_{T_\Psi}^{\ell m}\right)\right]-{}_2R_{\ell m}^{(0,1)}(r),$$

$$\mathcal{Q}_{\rm geo}^{\ell m}=\frac{i\chi mM^4}{448r^{13}(r-r_s)}\left(D_1(r)-4i\omega_{\ell m}r^2D_2(r)\right)+\frac{i\chi mM^4}{16r^{13}}\left[D_3(r)-D_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}-\frac{r(r-r_s)}{2}\partial_r\right)\right]\Lambda_{22c}^{\ell\ell 1}$$
$$-\frac{i\chi mM^4}{128r^{13}}D_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\Lambda_{12s}^{\ell\ell 1}-\sqrt{(\ell+3)(\ell-2)}\Lambda_{32s}^{\ell\ell 1}\right)$$
$$\mathcal{Q}_{T_\vartheta}^{\ell m}=\left[\hat{\mathcal{A}}_1^{\ell m}+i\chi m\Lambda_{12s}^{\ell\ell}\hat{\mathcal{A}}_2^{\ell m}-i\chi m\Lambda_{22c}^{\ell\ell 1}\hat{\mathcal{A}}_3^{\ell m}\right]\mathcal{D}_{\ell m}$$
$$\mathcal{Q}_{T_\Psi}^{\ell m}=i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{\mathcal{B}}_1^{\ell m}-\Lambda_{22c}^{\ell\ell 1}\hat{\mathcal{B}}_2^{\ell m}+\Lambda_{32s}^{\ell\ell 1}\hat{\mathcal{B}}_3^{\ell m}\right]\mathcal{D}_{\ell m}$$
$$\tilde{\mathcal{Q}}_{T_\Psi}^{\ell m}=-i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{\hat{\mathcal{B}}}_1^{\ell -m}+\Lambda_{22c}^{\ell\ell 1}\hat{\hat{\mathcal{B}}}_2^{\ell -m}\right]\mathcal{D}_{\ell m}$$

$$\Lambda_{-1-2s}^{\ell\ell m}=m\Lambda_{12s}^{\ell\ell 1},\qquad \Lambda_{1-2s}^{\dagger\ell\ell m}=(-1)^{m+1}m\Lambda_{12s}^{\ell\ell 1}$$
$$\Lambda_{-2-2c}^{\ell\ell m}=-m\Lambda_{22c}^{\ell\ell 1},\quad \Lambda_{2-2c}^{\dagger\ell\ell m}=(-1)^{m+1}m\Lambda_{22c}^{\ell\ell 1}$$
$$\Lambda_{-3-2s}^{\ell\ell m}=m\Lambda_{32s}^{\ell\ell 1},\qquad \Lambda_{3-2s}^{\dagger\ell\ell m}=(-1)^{m+1}m\Lambda_{32s}^{\ell\ell 1}$$

$$
{}_s\mathcal{H}_{\ell m}^{(0,0)}\ {}_sR_{\ell m}^{(0,1)} = 0
$$
$$
{}_s\mathcal{H}_{\ell m}^{(0,0)}\ {}_sR_{\ell m}^{(1,1)} = {}_s\mathcal{V}_{\ell m}^{(1,0)}\ {}_sR_{\ell m}^{(0,1)}
$$

$$
\begin{aligned}
\omega_{\ell m} &= \omega_{\ell m}^{(0,0)} + \zeta\omega_{\ell m}^{(1,0)} + \mathcal{O}(\zeta^2) \\
&= \left(\omega_{\ell m}^{(0,0,0)} + \chi m \omega_{\ell 1}^{(0,1,0)}\right) + \zeta\left(\omega_{\ell m}^{(1,0,0)} + \chi m \omega_{\ell 1}^{(1,1,0)}\right) \\
&\quad + \mathcal{O}(\zeta^2, \chi^2)
\end{aligned}
$$

$$
\omega_{\ell m}^{(0,1,0)} = m\omega_{\ell 1}^{(0,1,0)}
$$

$$
\omega_{\ell m}^{(1,1,0)} = m\omega_{\ell 1}^{(1,1,0)}
$$

$$
{}_s\mathcal{H}_{\ell m}^{(0,0)}\ {}_sR_{\ell m}^{(1,1)} + \omega_{\ell m}^{(1,0)}\partial_\omega\left({}_s\mathcal{H}_{\ell m}^{(0,0)}\right)\ {}_sR_{\ell m}^{(0,1)} = {}_s\mathcal{V}_{\ell m}^{(1,0)}\ {}_sR_{\ell m}^{(0,1)}
$$

$$
\left\langle \varphi_1(r) \mid {}_s\mathcal{H}_{\ell m}^{(0,0)}\varphi_2(r)\right\rangle = \left\langle {}_s\mathcal{H}_{\ell m}^{(0,0)}\varphi_1(r) \mid \varphi_2(r)\right\rangle,
$$

$$
\langle \varphi_1(r) \mid \varphi_2(r)\rangle = \int_{\mathcal{C}} \Delta^s(r)\varphi_1(r)\varphi_2(r)dr
$$

$$
r_{\mathcal{C}}(\xi) = 2M + \frac{4M^2\xi}{20M^2+\xi^2} + i\left(\frac{\xi^2}{2M} - 2M\right)
$$
$$
\xi \in [-\xi_{\max}, \xi_{\max}]
$$

$$
\left\langle {}_sR_{\ell m}^{(0,1)} \mid {}_s\mathcal{H}_{\ell m}^{(0,0)}\ {}_sR_{\ell m}^{(1,1)}\right\rangle = \left\langle {}_s\mathcal{H}_{\ell m}^{(0,0)}\ {}_sR_{\ell m}^{(0,1)} \mid {}_sR_{\ell m}^{(1,1)}\right\rangle = 0.
$$

$$
\omega_{\ell m}^{(1,0)} = \left\langle {}_s\mathcal{V}_{\ell m}^{(1,0)}\right\rangle / \left\langle \partial_\omega\left({}_s\mathcal{H}_{\ell m}^{(0,0)}\right)\right\rangle
$$

$$
\langle \hat{O}\rangle = \left\langle {}_sR_{\ell m}^{(0,1)} \mid \hat{O}_s R_{\ell m}^{(0,1)}\right\rangle
$$

$$
\mathcal{H}_0^{\ell m(0,0,0)}\ {}_2R_{\ell m}^{(0,0,1)} = 0
$$
$$
\mathcal{H}_0^{\ell m(0,0,0)}\ {}_2R_{\ell m}^{(0,1,1)} = -\mathcal{H}_0^{\ell m(0,1,0)}\ {}_2R_{\ell m}^{(0,0,1)}
$$

$$
\begin{aligned}
\mathcal{H}_0^{\ell m(0,0,0)} &= r(r-r_s)\partial_r^2 + 6(r-M)\partial_r + \frac{4i\omega_{\ell m}r(r-3M) + \omega_{\ell m}^2 r^3}{r-r_s} - \ell(\ell+1) + 6, \\
\mathcal{H}_0^{\ell m(0,1,0)} &= \frac{4imM((r-M) + iM\omega_{\ell m}r)}{r(r-r_s)} + \frac{8mM\omega_{\ell m}}{\ell(\ell+1)},
\end{aligned}
$$

$$
\omega_{0,\ell 1}^{(0,1,0)} = \frac{\left\langle 4iM\left((r-M) + iM\omega_{\ell m}^{(0,0,0)}r\right)/(r-r_s) + 8M\omega_{\ell m}^{(0,0,0)}/(\ell(\ell+1))\right\rangle}{\left\langle\left[4ir(r-3M) + 2\omega_{\ell m}^{(0,0,0)}r^3\right]/(r-r_s)\right\rangle},
$$

$$\delta(\mathrm{Re}[\omega_1],\mathrm{Re}[\omega_2]) = |\mathrm{Re}[\omega_2]/\mathrm{Re}[\omega_1]-1|,$$
$$\delta(\mathrm{Im}[\omega_1],\mathrm{Im}[\omega_2]) = |\mathrm{Im}[\omega_2]/\mathrm{Im}[\omega_1]-1|.$$

$$\delta(\mathrm{Re}[\omega_{\mathrm{Kerr}}],\mathrm{Re}[\omega_{\mathrm{slow}}])$$

$$\delta(\mathrm{Im}[\omega_{\mathrm{Kerr}}],\mathrm{Im}[\omega_{\mathrm{slow}}])$$

$$\mathrm{Re}\left[\omega_{22}^{(0,0)}\right]\mathcal{O}(\zeta^1,\chi^1,\epsilon^1)$$

$$\omega_{0,\ell m}^{\pm(1,0)} = \frac{\langle -2r^2[(\mathcal{O}_{\mathrm{geo}}^{\ell m}+\mathcal{O}_{T\vartheta}^{\ell m}+\mathcal{O}_{T\Psi}^{\ell m})\mp(\mathcal{O}_{T\vartheta}^{\ell m}-\tilde{\mathcal{O}}_{T\Psi}^{\ell m})]\rangle}{\langle\partial_\omega\mathcal{H}_0^{\ell m}\rangle},$$
$$\omega_{4,\ell m}^{\pm(1,0)} = \frac{\langle -2r^6[(\mathcal{Q}_{\mathrm{geo}}^{\ell m}+\mathcal{Q}_{T\vartheta}^{\ell m}+\mathcal{Q}_{T\Psi}^{\ell m})\mp(\mathcal{Q}_{T\vartheta}^{\ell m}-\tilde{\mathcal{Q}}_{T\Psi}^{\ell m})]\rangle}{\langle\partial_\omega\mathcal{H}_4^{\ell m}\rangle},$$

$$\partial_\omega\mathcal{H}_0^{\ell m} = \frac{4ir(r-3M)+2\omega_{\ell m}r^3}{r-r_s}+\chi m\left[\frac{8M}{\ell(\ell+1)}-\frac{4M^2}{r-r_s}\right]$$
$$\partial_\omega\mathcal{H}_4^{\ell m} = \frac{-4ir(r-3M)+2\omega_{\ell m}r^3}{r-r_s}+\chi m\left[\frac{8M}{\ell(\ell+1)}-\frac{4M^2}{r-r_s}\right]$$

$$\omega_{i,\ell m}^{+(1,0)} = \chi m\omega_{i,\ell 1}^{+(1,1,0)}+\mathcal{O}(\chi^2)$$
$$\omega_{i,\ell m}^{-(1,0)} = \omega_{i,\ell m}^{-(1,0,0)}+\chi m\omega_{i,\ell 1}^{-(1,1,0)}+\mathcal{O}(\chi^2)$$

$$\omega_{0,\ell m}^{-(1,0,0)} = \frac{\langle -2r^2(\hat{A}_1^{\ell m}+\hat{A}_1^{\ell m})\mathcal{D}_{\ell m}^\dagger\rangle}{\left\langle\left[4ir(r-3M)+2\omega_{\ell m}^{(0,0,0)}r^3\right]/(r-r_s)\right\rangle_{(93a)}},$$
$$\omega_{4,\ell m}^{-(1,0,0)} = \frac{\langle -2r^6(\hat{\mathcal{A}}_1^{\ell m}+\hat{\mathcal{A}}_1^{\ell m})\mathcal{D}_{\ell m}\rangle}{\left\langle\left[-4ir(r-3M)+2\omega_{\ell m}^{(0,0,0)}r^3\right]/(r-r_s)\right\rangle},$$

$$\omega_{0,\ell m}^{+(1,0,0)} = \omega_{4,\ell m}^{+(1,0,0)} = 0$$

$$\left[r(r-r_s)\partial_r^2+r_s\partial_r+\frac{\omega_{\ell m}^2r^3}{r-r_s}-\frac{r_s}{r}-{}_0A_{\ell m}\right]\Theta_{\ell m}^{(1,1)}(r) = -\frac{2i}{\mathcal{C}_2}(1-\eta_{\ell m})\left(g_1^{\ell m}(r)+g_2^{\ell m}(r)\partial_r\right){}_{-2}R_{\ell m}^{(0,1)}(r),$$

$$g_1^{\ell m}(r) = -3\sqrt{\Lambda_\ell}M^3\frac{(2\omega_{\ell m}^2r^2-8i\omega_{\ell m}r-\ell^2-\ell-4)r^2+2M(9i\omega_{\ell m}r+\ell^2+\ell+10)r-24M^2}{4\sqrt{\pi}r^4(r-r_s)^2},$$
$$g_2^{\ell m}(r) = -3\sqrt{\Lambda_\ell}M^3\frac{i\omega_{\ell m}r^2+r-3M}{2\sqrt{\pi}r^3(r-r_s)},\Lambda_\ell = (\ell+2)(\ell+1)\ell(\ell-1),$$

$$(D_{m\omega})^4\ {}_{-2}R_{\ell m}^{(0,1)}(r) = \mathcal{C}_2\ {}_2\hat{R}_{\ell m}(r)$$
$$\Delta^2\left(D_{m\omega}^\dagger\right)^4\left[\Delta^2\ {}_2R_{\ell m}^{(0,1)}(r)\right] = \mathcal{C}_{-2}\ {}_{-2}\hat{R}_{\ell m}(r)$$

$$\mathcal{C}_2\mathcal{C}_{-2} = \mathfrak{C}$$

$$ {}_2\hat{R}_{\ell m}(r) = -\frac{2}{\mathcal{C}_2} - {}_2R_{\ell m}^{(0,1)}(r) $$
$$ {}_{-2}\hat{R}_{\ell m}(r) = \frac{32}{\mathcal{C}_{-2}}\, {}_2R_{\ell m}^{(0,1)}(r) $$

$$ {}_{\pm 2}R_{\ell m}^{(0,1)} = f_{\pm 2}(r)\left[V_Z(r) + \left(\frac{2}{r^2}(r-3M) \mp 2i\omega_{\ell m}\right)\Lambda_\pm\right]\Psi_{\ell m}^{\mathrm{Rw}(0,1)}(r), $$
$$ V_Z(r) = \left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right), f_2(r) = r^3\Delta^{-2}(r), f_{-2}(r) = r^3, \Lambda_\pm = \frac{d}{dr_*} \mp i\omega_{\ell m} $$

$$ dr_*/dr = (r^2 + a^2)/\Delta(r) $$

$$ \left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega_{\ell m}^2 r^3}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell m}\right]\Theta_{\ell m}^{(1,1)}(r) $$
$$ = -\frac{3i\sqrt{\Lambda_\ell}M^3}{\sqrt{\pi}r^3}\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r) $$

$$ \left[\partial_{r_*}^2 + \frac{r-r_s}{r^3}\left(\frac{\omega_{\ell m}^2 r^3}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell m}\right)\right]\Theta_{\ell m}^{(1,1)}(r) = -\frac{4i}{\mathcal{C}_2}\frac{r-r_s}{r^3}\left(g_1^{\ell m}(r) + g_2^{\ell m}(r)\partial_r\right){}_{-2}R_{\ell m}^{(0,1)}(r) $$

$$ \Theta_{\ell m}^{(1,1)}(\xi) = \frac{\Theta_{\ell m}^R(\xi)\int_{-\infty}^{\xi}\Theta_{\ell m}^L(\xi')\mathcal{S}_\vartheta^{\ell m}(\xi')\partial_{\xi'}r_* d\xi' + \Theta_{\ell m}^L(\xi)\int_{\xi}^{\infty}\Theta_{\ell m}^R(\xi')\mathcal{S}_\vartheta^{\ell m}(\xi')\partial_{\xi'}r_* d\xi'}{\left(\Theta_{\ell m}^L(\xi)\partial_\xi\Theta_{\ell m}^R(\xi) - \Theta_{\ell m}^R(\xi)\partial_\xi\Theta_{\ell m}^L(\xi)\right)\partial_{r_*}\xi} $$

$$ r = r_{\mathcal{C}}(\xi) $$

$$ r = r_{\mathcal{C}}(\xi') $$

$$ \xi = \pm\infty \text{ at } \xi = \pm\xi_{\max} $$

$$ \Theta_{\ell m}^R(\xi\to\infty) \propto e^{i\omega_{\ell m}r_*(\xi)} $$

$$ \Theta_{\ell m}^L(\xi\to-\infty) \propto e^{-i\omega_{\ell m}r_*(\xi)} $$

$$ \xi = \xi_{\max} $$

$$ \xi = -\xi_{\max} $$

$$ \mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m}\, {}_2\mathcal{D}_{\ell m}^\dagger\, {}_2R_{\ell m}^{(0,1)} $$

$$ \hat{A}_1^{\ell m}\mathcal{D}_{\ell m}^\dagger\, {}_2R_{\ell m}^{(0,1)} $$

$$ \mathrm{Im}\left[\omega_{\ell m}^{(0,0)}\right]\omega_{\ell m}^{-(1,0,0)} $$

$$ \omega_{0,\ell m}^{+(1,0)} = \frac{\left\langle -2r^2\left(\mathcal{O}_{\mathrm{geo}}^{\ell m} + \mathcal{O}_{T_\Psi}^{\ell m} + \tilde{\mathcal{O}}_{T_\Psi}^{\ell m}\right)\right\rangle}{\left\langle\partial_\omega\mathcal{H}_0^{\ell m}\right\rangle}, $$
$$ \omega_{4,\ell m}^{+(1,0)} = \frac{\left\langle -2r^6\left(\mathcal{Q}_{\mathrm{geo}}^{\ell m} + \mathcal{Q}_{T_\Psi}^{\ell m} + \tilde{\mathcal{Q}}_{T_\Psi}^{\ell m}\right)\right\rangle}{\left\langle\partial_\omega\mathcal{H}_4^{\ell m}\right\rangle}, $$

$$\{\mathcal{O}_{\text{geo}}^{\ell m}, \mathcal{O}_{T\Psi}^{\ell m}, \tilde{\mathcal{O}}_{T\Psi}^{\ell m}, \mathcal{Q}_{\text{geo}}^{\ell m}, \mathcal{Q}_{T\Psi}^{\ell m}, \tilde{\mathcal{Q}}_{T\Psi}^{\ell m}\}$$

$$_{\pm 2}R_{\ell m}^{(0,1)}(r)\left\{\hat{B}_i^{\ell m}, \hat{\tilde{B}}_i^{\ell m}, \hat{\mathcal{B}}_i^{\ell m}, \hat{\tilde{\mathcal{B}}}_i^{\ell m}\right\}$$

$$\langle \omega_{0,\ell m}^{+(1,0)} \omega_{4,\ell m}^{+(1,0)} \omega_{0,\ell 1}^{+(1,1,0)} \omega_{4,\ell 1}^{+(1,1,0)} \rangle$$

$$\delta \mathrm{Im}(\omega_{\mathrm{ZM}})$$

$$\delta \mathrm{Re}(\omega_{\mathrm{ZM}})$$

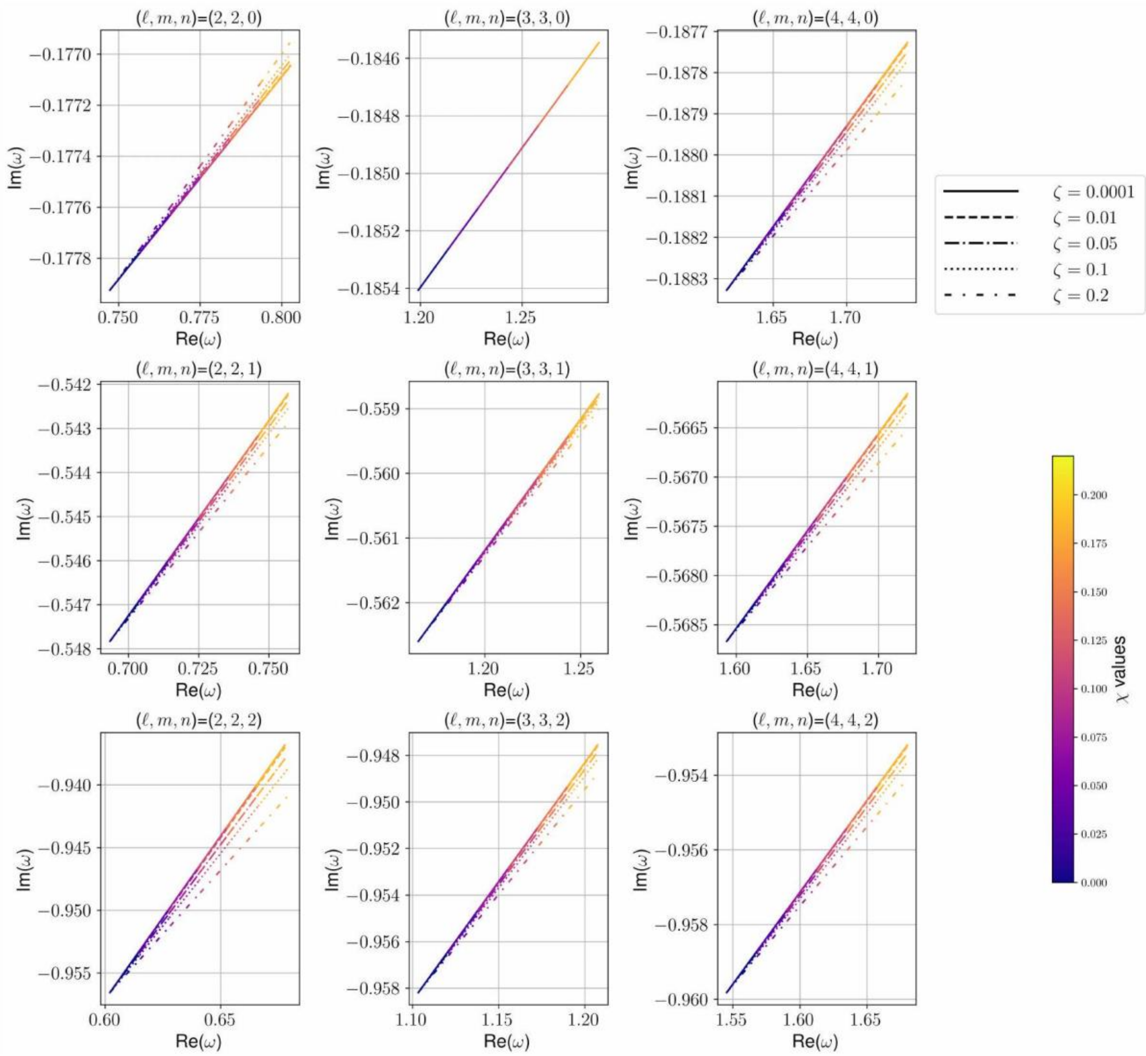
$$\omega_{\ell m}^{+} = \omega_{\ell m}^{(0,0)} + \zeta \omega_{\ell m}^{+(1,0)}$$

$$\omega_{\ell m}^{+(1,0,0)} :: \omega_{\ell 1}^{+(1,1,0)}$$

$$\xi = \xi_{\min} \text{ to } \xi = \xi_{\max}$$

$$S_{\vartheta}^{\ell m}(r) = -\frac{4}{\mathcal{C}_2}\frac{r - r_s}{r^3}\left(V_{\ell m}^{R}(r) + V_{\ell m}^{\square}(r)\right),$$

$$V_{\ell m}^{R}(r) V_{\ell m}^{\square}(r) \blacksquare\, _{-2}R_{\ell m}^{(0,1)}(r)\ _{\pm 2}R_{\ell m}^{(0,1)}(r) \mathcal{O}(\chi^1) r_{\mathcal{C}}(\xi)$$



$$\omega_{\ell m}^{+} = \omega_{\ell m}^{(0,0)} + \zeta \omega_{\ell m}^{+(1,0)}$$

$$\omega_{0,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^2 \left(\mathcal{O}_{\text{geo}}^{\ell m} + 2\mathcal{O}_{T_\vartheta}^{\ell m} + \mathcal{O}_{T\Psi}^{\ell m} - \tilde{\mathcal{O}}_{T\Psi}^{\ell m} \right) \right\rangle}{\left\langle \partial_\omega \mathcal{H}_0^{\ell m} \right\rangle},$$
$$\omega_{4,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^6 \left(\mathcal{Q}_{\text{geo}}^{\ell m} + 2\mathcal{Q}_{T_\vartheta}^{\ell m} + \mathcal{Q}_{T\Psi}^{\ell m} - \tilde{\mathcal{Q}}_{T\Psi}^{\ell m} \right) \right\rangle}{\left\langle \partial_\omega \mathcal{H}_4^{\ell m} \right\rangle},$$

$\{\mathcal{O}_{\text{geo}}^{\ell m}, \mathcal{O}_{T\Psi}^{\ell m}, \tilde{\mathcal{O}}_{T\Psi}^{\ell m}, \mathcal{Q}_{\text{geo}}^{\ell m}, \mathcal{Q}_{T\Psi}^{\ell m}, \tilde{\mathcal{Q}}_{T\Psi}^{\ell m}\}$■${}_{\pm 2}R_{\ell m}^{(0,1)}(r)$ ${}_{\pm 2}R_{\ell m}^{(0,1)}(r)\mathcal{O}_{T_\vartheta}^{\ell m}\mathcal{Q}_{T_\vartheta}^{\ell m}\vartheta^{(1,1)}\omega_{0,\ell m}^{-(1,0,0)}\omega_{4,\ell m}^{-(1,0,0)}\omega_{i,\ell 1}^{-(1,1,0)}\partial_\chi \omega_{i,\ell 1}^{-(1,0)}\omega_{i,\ell 1}^{-(1,0)}$

$$\omega_{i,\ell 1}^{-(1,1,0)} \approx \frac{\omega_{i,\ell 1}^{-(1,0)}(\chi_2) - \omega_{i,\ell 1}^{-(1,0)}(\chi_1)}{\chi_2 - \chi_1}$$

$$\delta\chi = \chi_2 - \chi_1$$

$$\omega_{i,\ell 1}^{-(1,1,0)} \ \Theta_{\ell m}^{(1,1)}(r) \ {}_{\pm 2}R_{\ell m}^{(0,1)}(r)\mathcal{O}(\chi^1) \ {}_{\pm 2}R_{\ell m}^{(0,1,1)}(r)$$

$$\omega_{0,\ell m}^{\pm(1,0)}\omega_{4,\ell m}^{\pm(1,0)}\mathcal{O}(\chi^1)\mathcal{O}(\chi^2)\omega_{0,\ell m}^{-(1,0)}\omega_{4,\ell m}^{-(1,0)}$$

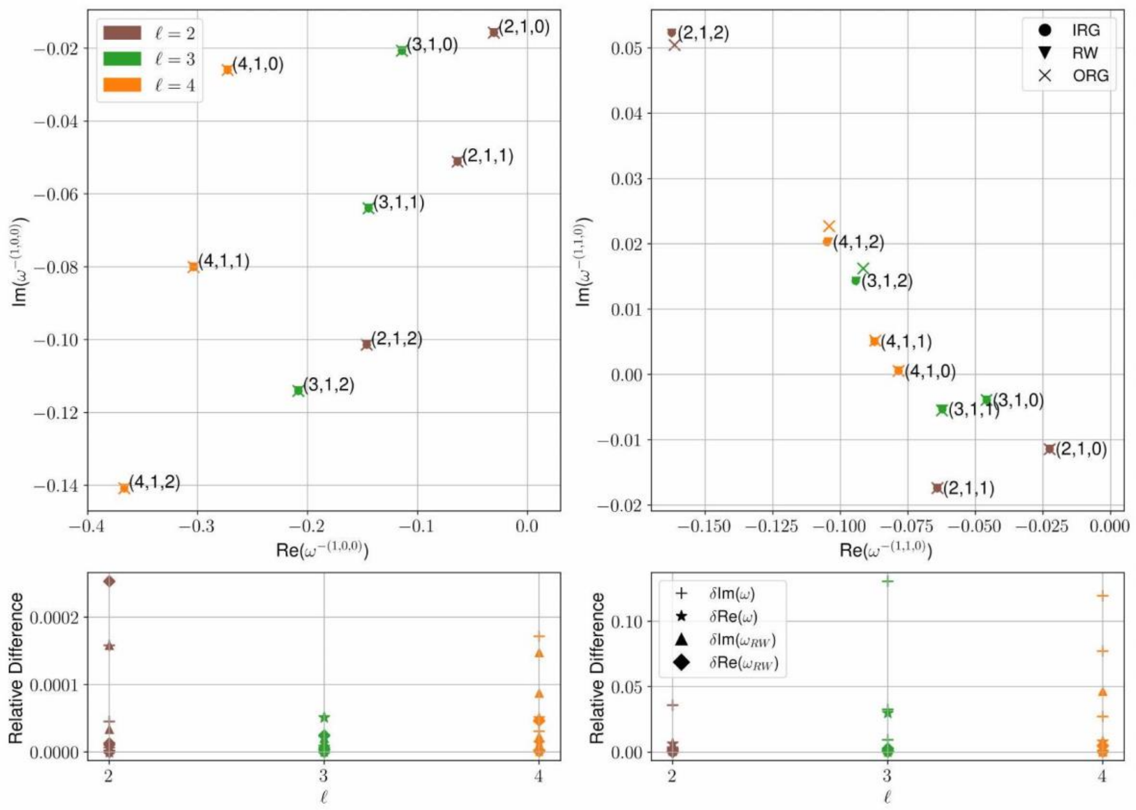
$$\omega_{i,\ell m}^{\pm(1,1,0)}\omega_{i,\ell m}^{\pm(1,0)}\omega_{i,\ell m}^{\pm(1,1,0)}\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}$$

$$\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}$$

$$\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}h_{\mu\nu}^{(0,1)}$$

$$\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}\omega_{0,\ell m}^{+(1,1,0)}\omega_{4,\ell m}^{+(1,1,0)}$$

$$\omega_{0,\ell m}^{-(1,1,0)}\omega_{4,\ell m}^{-(1,1,0)}$$



$$\omega_{\ell 1}^{-(1,0,0)} \omega_{\ell 1}^{-(1,1,0)} \delta\mathrm{Im}(\omega) \delta\mathrm{Re}(\omega)$$

$$\delta\mathrm{Im}(\omega_{\mathrm{RW}}) \delta\mathrm{Re}(\omega_{\mathrm{RW}})$$

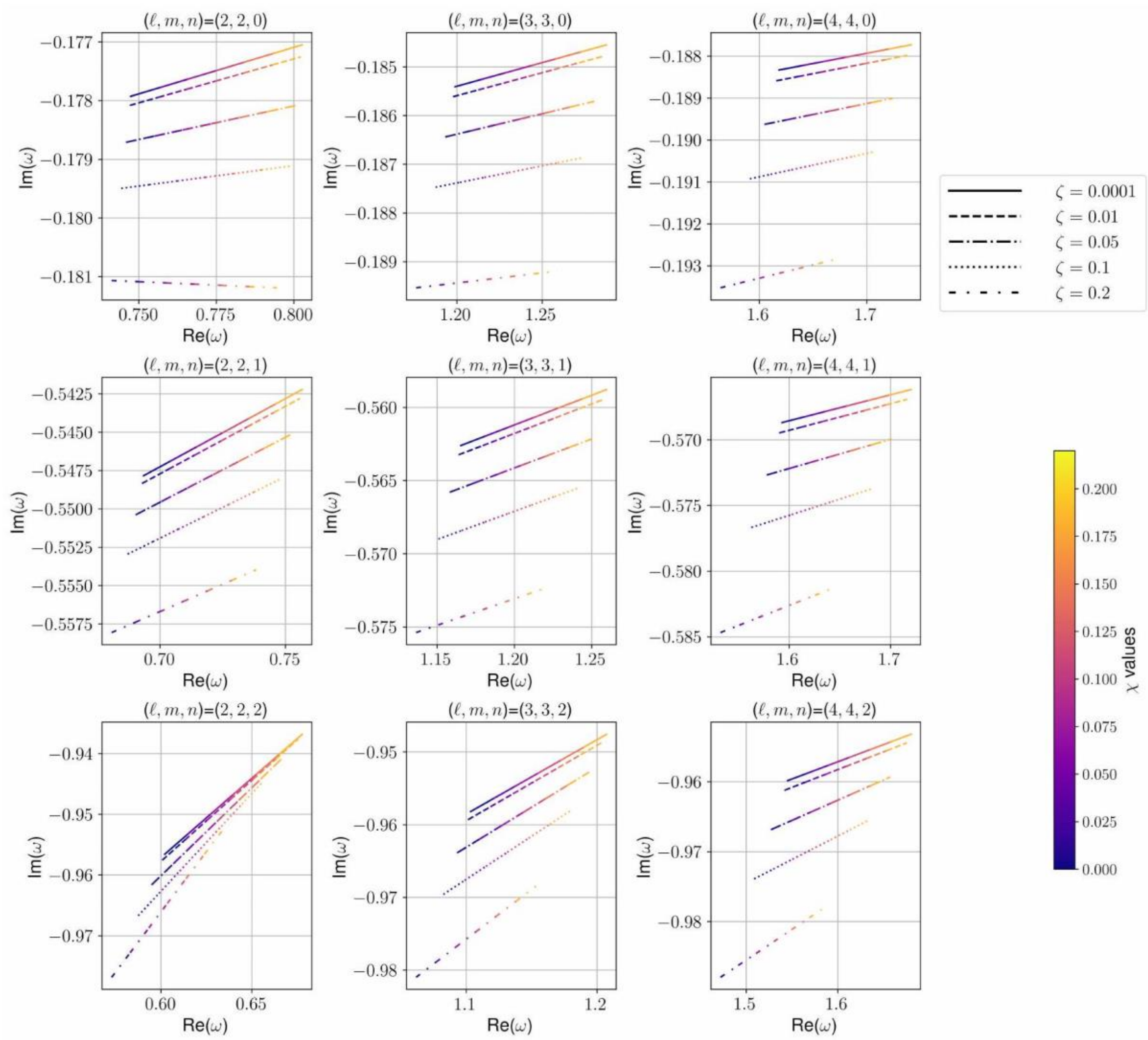
$$\omega_{\ell 1}^{-(1,0,0)} \omega_{\ell 1}^{-(1,1,0)} \quad \pm_2 R_{\ell m}^{(0,1)}(r) \omega_{0,\ell m}^{-(1,1,0)} \omega_{4,\ell m}^{-(1,1,0)}$$

$$\left|\mathrm{Im}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right| \ll \left|\mathrm{Re}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right|$$

$$\left|\mathrm{Im}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right| \mathrm{Im}\left[\omega_{\ell m}^{-(1,1,0)}\right] \mathrm{Re}\left[\omega_{\ell m}^{-(1,1,0)}\right] \left|\mathrm{Im}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right| \left|\mathrm{Re}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right|$$

$$\omega_{\ell 1}^{-(1,1,0)} \omega_{\ell 1}^{-(1,1,0)} \omega_{\ell 1}^{(0,1,0)}$$

$$\omega_{\ell 1}^{-(1,1,0)} \omega_{\ell 1}^{+(1,1,0)} \omega_{\ell 1}^{-(1,0,0)} \omega_{\ell 1}^{-(1,1,0)} \left|\mathrm{Re}\left[\omega_{\ell m}^{+(1,1,0)}\right]\right| \left|\mathrm{Re}\left[\omega_{\ell m}^{-(1,0,0)}\right]\right|, \left|\mathrm{Re}\left[\omega_{\ell m}^{-(1,1,0)}\right]\right|$$



$$\omega_{\ell m}^{-} = \omega_{\ell m}^{(0,0)} + \zeta \omega_{\ell m}^{-(1,0)}$$

$$\omega_{\ell m}^{-} = \omega_{\ell m}^{(0,0)} + \zeta \omega_{\ell m}^{-(1,0)}$$

$$\left(\omega_{0,\ell m}^{\pm(1,0)}\right)\left(\omega_{4,\ell m}^{\pm(1,0)}\right)\omega_{0,\ell m}^{\pm(1,0)}\omega_{4,\ell m}^{\pm(1,0)}$$

$$\left[f(r)^2\partial_r^2 + \frac{2M}{r^2}f(r)\partial_r + \omega^2 - V_{\text{eff}}^{S}(r,\chi,\zeta)\right]\Theta_{\ell m}(r)$$
$$= \zeta^{\frac{1}{2}}M^2\kappa^{\frac{1}{2}}f(r)[g(r) + \chi M m(h(r) + j(r)\partial_r)]\Psi_{\ell m}^{\text{RW}}(r)$$
$$\left[f(r)^2\partial_r^2 + \frac{2M}{r^2}f(r)\partial_r + \omega^2 - V_{\text{eff}}^{A}(r,\chi,\zeta)\right]\Psi_{\ell m}^{\text{RW}}(r)$$
$$= \zeta^{\frac{1}{2}}M^2\kappa^{\frac{1}{2}}f(r)[v(r) + \chi M m(n(r) + p(r)\partial_r)]\Theta_{\ell m}(r)$$
$$\left[f(r)^2\partial_r^2 + \frac{2M}{r^2}f(r)\partial_r + \omega^2 - V_{\text{eff}}^{P}(r,\chi,\zeta)\right]\Psi_{\ell m}^{\text{ZM}}(r) = 0$$

$$V_{\text{eff}}^{S}(r,\chi,\zeta) = V_{\text{eff}}^{S(0,0)}(r,\chi) + \zeta V_{\text{eff}}^{S(1,0)}(r,\chi)$$
$$V_{\text{eff}}^{A}(r,\chi,\zeta) = V_{\text{eff}}^{A(0,0)}(r,\chi) + \zeta V_{\text{eff}}^{A(1,0)}(r,\chi)$$
$$V_{\text{eff}}^{P}(r,\chi,\zeta) = V_{\text{eff}}^{P(0,0)}(r,\chi) + \zeta V_{\text{eff}}^{P(1,0)}(r,\chi)$$

$$\mathcal{H}_S^{\ell m}\Theta_{\ell m}^{(1,1)}(r) \ = M^2\kappa^{\frac{1}{2}}f(r)[g(r)+\chi Mm(h(r)+j(r)\partial_r)]\Psi_{\ell m}^{\mathrm{Rw}(0,1)}(r),$$
$$\mathcal{H}_A^{\ell m}\Psi_{\ell m}^{\mathrm{Rw}(1,1)}(r) \ = M^2\kappa^{\frac{1}{2}}f(r)[v(r)+\chi Mm(n(r)+p(r)\partial_r)]\Theta_{\ell m}^{(1,1)}(r)+V_{\mathrm{eff}}^{A(1,0)}(r,\chi)\Psi_{\ell m}^{\mathrm{Rw}(0,1)}(r),$$
$$\mathcal{H}_P^{\ell m}\Psi_{\ell m}^{\mathrm{ZM}(1,1)}(r) \ = V_{\mathrm{eff}}^{P(1,0)}(r,\chi)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r),$$

$$\mathcal{H}_i^{\ell m}=f(r)^2\partial_r^2+\frac{2M}{r^2}f(r)\partial_r+\omega^2-V_{\mathrm{eff}}^{i(0,0)}(r,\chi)$$

$$\mathcal{H}_S^{\ell m}\Theta_{\ell m}^{(1,1)}(r) \ = M^2\kappa^{\frac{1}{2}}f(r)[g(r)+\chi Mm(h(r)+j(r)\partial_r)]\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r),$$
$$\mathcal{H}_A^{\ell m}\Psi_{\ell m}^{\mathrm{RW}(1,1)}(r) \ = M^2\kappa^{\frac{1}{2}}f(r)[v(r)+\chi Mm(n(r)+p(r)\partial_r)]\Theta_{\ell m}^{(1,1)}(r)+\chi m\kappa[A_2(r)-A_1(r)\partial_r]\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r),$$
$$\mathcal{H}_P^{\ell m}\Psi_{\ell m}^{\mathrm{ZM}(1,1)}(r) \ = \chi m\kappa[P_2(r)-P_1(r)\partial_r]\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r),$$

$$\begin{aligned}\omega_{\vartheta,\ell m} &=\omega_{\vartheta,\ell m}^{(0,0)}+\zeta\omega_{\vartheta,\ell m}^{(1,0)}+\mathcal{O}(\zeta^2)\\ &=\left(\omega_{\vartheta,\ell m}^{(0,0,0)}+\chi m\omega_{\vartheta,\ell 1}^{(0,1,0)}\right)+\zeta\left(\omega_{\vartheta,\ell m}^{(1,0,0)}+\chi m\omega_{\vartheta,\ell 1}^{(1,1,0)}\right)\\ &\quad+\mathcal{O}(\zeta^2,\chi^2)\end{aligned}$$

$$V_{\mathrm{eff}}^{S(1,0)}(r,\chi)\omega_{\vartheta,\ell m}^{(1,0)}\Theta_{\ell m}^{(1,1)}(r)\Theta_{\ell m}^{H(1,1)}(r)$$

$$\omega_{\vartheta,\ell m}^{(0,0)}\ \Psi_{\ell m}^{\mathrm{Rw}(0,1)}(r)\mathcal{O}\left(\zeta^{3/2}\right)\ V_{\mathrm{eff}}^{S(1,0)}(r,\chi)\Theta_{\ell m}^{H(1,1)}(r)\Psi_{\ell m}^{\mathrm{Rw}(1,1)}(r)\omega_{\vartheta,\ell m}^{(0,0)}\omega_{\vartheta,\ell_m}^{(1,0)}\mathcal{O}(\zeta^1,\epsilon^1)$$

$$[R^*R]^{(1,1)}\mathcal{O}\left(\zeta^{3/2}\right)$$

$$\omega_{h,\ell m}^{-(1,0)} \ = \frac{\left\langle V_{\mathrm{eff}}^{A(1,0)}(r,\chi)+M^2\kappa^{\frac{1}{2}}f(r)[v(r)+\chi Mm(n(r)+p(r)\partial_r)]\mathcal{O}^S\right\rangle_h}{\left\langle\partial_\omega\mathcal{H}_A^{\ell m}\right\rangle_h},$$
$$\omega_{h,\ell m}^{+(1,0)} \ = \frac{\left\langle V_{\mathrm{eff}}^{P(1,0)}(r,\chi)\right\rangle_h}{\left\langle\partial_\omega\mathcal{H}_P^{\ell m}\right\rangle_h},$$

$${}_sR_{\ell m}^{(0,1)}\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r)\omega_{h,\ell m}^{-(1,0)}\omega_{h,\ell m}^{+(1,0)}$$

$$\langle\varphi_1(r)\mid\varphi_2(r)\rangle_h=\int_{\mathcal{C}}f(r)\varphi_1(r)\varphi_2(r)dr$$

$$\partial_\omega\mathcal{H}_A^{\ell m} \ = 2\omega_{\ell m}+\chi m\left[\frac{24M^2(3r^2-13Mr+14M^2)}{\ell(\ell+1)\omega^2r^7}-\frac{4M^2}{r^3}\right]$$
$$\partial_\omega\mathcal{H}_P^{\ell m} \ = 2\omega_{\ell m}-\partial_\omega V_{\mathrm{eff}}^{P(0,0)}(r,\chi)$$

$$\partial_\omega V_{\mathrm{eff}}^{P(0,0)}(r,\chi)$$

$$V_{\mathrm{eff}}^{P(0,0)}(r,\chi)$$

$$\Theta_{\ell m}^{(1,1)}(r)\Psi_{\ell m}^{\mathrm{Rw}(0,1)}(r)$$

$$\mathcal{O}^S=M^2\kappa^{\frac{1}{2}}\left(\mathcal{H}_S^{\ell m}\right)^{-1}\{f(r)[g(r)+\chi Mm(h(r)+j(r)\partial_r)]\}$$

$$\left(\mathcal{H}_S^{\ell m}\right)^{-1}\Theta_{\ell m}^{(1,1)}(r)\mathcal{S}_\vartheta^{\ell m}(r)\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r)$$

$$\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r)\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r)$$

$$\Theta_{\ell m}^{(1,1)}(r)\Psi_{\ell m}^{\mathrm{RW}(0,1)}(r)\Psi_{\ell m}^{\mathrm{ZM}(0,1)}(r)$$

$$\delta\left(\mathrm{Re}\left[\omega_{0,\ell m}^{\pm(1,i,0)}\right],\mathrm{Re}\left[\omega_{h,\ell m}^{\pm(1,i,0)}\right]\right)\text{ and }\delta\left(\mathrm{Im}\left[\omega_{0,\ell m}^{\pm(1,i,0)}\right],\mathrm{Im}\left[\omega_{h,\ell m}^{\pm(1,i,0)}\right]\right), i\in\{0,1\}$$

$$\omega_{0,\ell m}^{\pm(1,i,0)}\,\omega_{h,\ell m}^{\pm(1,i,0)}$$

$$\Delta\left(\mathrm{Re}\left[\omega_{\ell \mathrm{m}}^{\pm(1,\mathrm{i},0)}\right]\right):=\delta\left(\mathrm{Re}\left[\omega_{0,\ell m}^{\pm(1,i,0)}\right],\mathrm{Re}\left[\omega_{h,\ell m}^{\pm(1,i,0)}\right]\right)$$
$$\Delta\left(\mathrm{Im}\left[\omega_{\ell \mathrm{m}}^{\pm(1,\mathrm{i},0)}\right]\right):=\delta\left(\mathrm{Im}\left[\omega_{0,\ell m}^{\pm(1,i,0)}\right],\mathrm{Im}\left[\omega_{h,\ell m}^{\pm(1,i,0)}\right]\right)$$

ℓ	Overtones	Δ	$\Delta\left(\mathrm{Im}\left[\omega_{\ell \mathrm{m}}^{-(1,0,0)}\right]\right)$	$\Delta\left(\mathrm{Re}\left[\omega_{\ell 1}^{-(1,1,0)}\right]\right)$	$\Delta\left(\mathrm{Im}\left[\omega_{\ell 1}^{-(1,1,0)}\right]\right)$
	$n=0$	10^{-4}	10^{-5}	0.014	0.004
$\ell=2$	$n=1$	10^{-5}	10^{-4}	0.013	0.045
	$n=2$	0.001	0.003	0.002	0.194
	$n=0$	10^{-6}	10^{-4}	0.021	0.220
$\ell=3$	$n=1$	10^{-4}	0.002	0.078	0.128
	$n=2$	0.003	0.001	0.273	0.310
	$n=0$	10^{-4}	0.002	0.021	4.645
$\ell=4$	$n=1$	10^{-4}	0.009	0.027	0.649
	$n=2$	0.005	0.015	0.500	0.044

$$\left\{\Lambda_{s_1s_2}^{\ell_1\ell_2m},\Lambda_{s_1s_2c}^{\ell_1\ell_2m},\Lambda_{s_1s_2s}^{\ell_1\ell_2m}\right\}\left\{\Lambda_{s_1s_2}^{\dagger\ell_1\ell_2m},\Lambda_{s_1s_2c}^{\dagger\ell_1\ell_2m},\Lambda_{s_1s_2s}^{\dagger\ell_1\ell_2m}\right\}$$

ℓ	Overtones	$\Delta\left(\mathrm{Re}\left[\omega_{\ell m}^{-(1,0,0)}\right]\right)$	$\Delta\left(\mathrm{Im}\left[\omega_{\ell m}^{-(1,0,0)}\right]\right)$	$\Delta\left(\mathrm{Re}\left[\omega_{\ell 1}^{-(1,1,0)}\right]\right)$	$\Delta\left(\mathrm{Im}\left[\omega_{\ell 1}^{-(1,1,0)}\right]\right)$
	$n=0$	10^{-4}	10^{-5}	0.014	0.004
$\ell=2$	$n=1$	10^{-5}	10^{-4}	0.013	0.045
	$n=2$	0.001	0.003	0.002	0.194
	$n=0$	10^{-6}	10^{-4}	0.021	0.220
$\ell=3$	$n=1$	10^{-4}	0.002	0.078	0.128
	$n=2$	0.003	0.001	0.273	0.310
	$n=0$	10^{-4}	0.002	0.021	4.645
$\ell=4$	$n=1$	10^{-4}	0.009	0.027	0.649
	$n=2$	0.005	0.015	0.500	0.044

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = m\Lambda^{\ell\ell 1}_{s_1 s_1 c}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1 \pm 1 s} = m\Lambda^{\ell\ell 1}_{s_1 s_1 \pm 1 s}$$

$$\Lambda^{\ell_1 \ell_2 m}_{s_1 s_2 c} \equiv \int_{S^2} dS \cos\,\theta_{s_1} Y_{\ell_1 m s_2} \bar{Y}_{\ell_2 m}$$
$$\Lambda^{\ell_1 \ell_2 m}_{s_1 s_2 s} \equiv \int_{S^2} dS \sin\,\theta_{s_1} Y_{\ell_1 m s_2} \bar{Y}_{\ell_2 m}$$

$${}_0Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\,\theta,\ {}_{\pm 1}Y_{10}(\theta,\phi) = \pm\sqrt{\frac{3}{8\pi}}\sin\,\theta$$

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = \sqrt{\frac{4\pi}{3}}\int_{S^2} dS_{s_1}\bar{Y}_{\ell m s_1} Y_{\ell m 0} Y_{10}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1 \pm 1 s} = \pm\sqrt{\frac{8\pi}{3}}\int_{S^2} dS_{s_1 \pm 1}\bar{Y}_{\ell m s_1} Y_{\ell m \pm 1} Y_{10}$$

$${}_sY_{\ell m}(\theta,\phi)e^{-is\gamma} = \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-sm}(\phi,\theta,\gamma)$$

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = \frac{2\ell+1}{8\pi^2}\int_0^{2\pi} d\gamma \int_{S^2} dS \bar{D}^{\ell}_{-s_1 m} D^{\ell}_{-s_1 m} D^{1}_{00}$$

$$\Lambda^{\ell\ell m}_{s_1 s_1 \pm 1 s} = \pm\frac{2\ell+1}{4\sqrt{2}\pi^2}\int_0^{2\pi} d\gamma \int_{S^2} dS \bar{D}^{\ell}_{-s_1 \mp 1 m} D^{\ell}_{-s_1 m} D^{1}_{\mp 10}$$

$$\int_0^{2\pi} d\gamma \int_{S^2} dS \bar{D}^{\ell_3}_{s_3 m_3} D^{\ell_2}_{s_2 m_2} D^{\ell_1}_{s_1 m_1} = \frac{8\pi^2}{2\ell_3+1}\delta_{s_1+s_2,s_3}\delta_{m_1+m_2,m_3} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} C^{\ell_3 s_3}_{\ell_1 s_1 \ell_2 s_2}$$

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = C^{\ell m}_{10\ell m} C^{\ell(-s_1)}_{10\ell(-s_1)}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1 \pm 1 s} = \pm\sqrt{2} C^{\ell m}_{10\ell m} C^{\ell(-s_1 \mp 1)}_{1(\mp 1)\ell(-s_1)}$$

$$C^{\ell m}_{10\ell m} = m C^{\ell 1}_{10\ell 1}$$

$$\Lambda^{\ell\ell m}_{-s_1 -s_1 c} = -\Lambda^{\ell\ell m}_{s_1 s_1 c}$$
$$\Lambda^{\ell\ell m}_{-s_1 -s_1 \mp 1 s} = \Lambda^{\ell\ell m}_{s_1 s_1 \pm 1 s}$$

$$\Lambda^{\ell_1 \ell_2 m}_{-s_1 -s_2 c} = (-1)^{s_1+s_2} \int_{S^2} dS \cos\,\theta_{s_1} \bar{Y}_{\ell_1 -m s_2} Y_{\ell_2 -m}$$

$$\Lambda^{\ell_1 \ell_2 m}_{-s_1 -s_2 s} = (-1)^{s_1+s_2} \int_{S^2} dS \sin\,\theta_{s_1} \bar{Y}_{\ell_1 -m s_2} Y_{\ell_2 -m}$$

$$\Lambda^{\ell_1 \ell_2 m}_{-s_1 -s_2 c} = (-1)^{s_1+s_2} \Lambda^{\ell_1 \ell_2 -m}_{s_1 s_2 c}$$
$$\Lambda^{\ell_1 \ell_2 m}_{-s_1 -s_2 s} = (-1)^{s_1+s_2} \Lambda^{\ell_1 \ell_2 -m}_{s_1 s_2 s}$$

$$\Lambda^{\dagger \ell\ell m}_{-s_1 s_2 c} = (-1)^{m+s_1} \Lambda^{\ell\ell m}_{s_1 s_2 c}$$
$$\Lambda^{\dagger \ell m}_{-s_1 s_2 s} = (-1)^{m+s_1} \Lambda^{\ell\ell m}_{s_1 s_2 s}$$

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 c} \equiv \int_{S^2} dS \cos\,\theta_{s_1} \bar{Y}_{\ell_1 -m s_2} \bar{Y}_{\ell_2 m}$$
$$\Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 s} \equiv \int_{S^2} dS \sin\,\theta_{s_1} \bar{Y}_{\ell_1 -m s_2} \bar{Y}_{\ell_2 m}$$

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{-s_1 s_2 c} = (-1)^{m+s_1} \int_{S^2} dS \cos\,\theta_{s_1} Y_{\ell_1 m s_2} \bar{Y}_{\ell_2 m}$$

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{-s_1 s_2 s} = (-1)^{m+s_1} \int_{S^2} dS \sin\,\theta_{s_1} Y_{\ell_1 m s_2} \bar{Y}_{\ell_2 m}$$

$$\left\{\hat{A}^{\ell m}_i, \hat{B}^{\ell m}_i, \hat{\hat{B}}^{\ell m}_i\right\} \left\{\hat{\mathcal{A}}^{\ell m}_i, \hat{\mathcal{B}}^{\ell m}_i, \hat{\hat{\mathcal{B}}}^{\ell m}_i\right\}$$

$$
\begin{aligned}
\hat{A}_1^{\ell m} =& i\left(k_1^{\ell m}(r) + k_2^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m} \\
&+ \frac{1}{1-\bar{\eta}_{\ell m}}\left(k_3^{\ell m}(r) + k_4^{\ell m}(r)\partial_r\right), \\
\hat{A}_2^{\ell m} =& \left(k_5^{\ell m}(r) + k_6^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m} \\
&- \frac{i}{1-\bar{\eta}_{\ell m}}\left(k_7^{\ell m}(r) + k_8^{\ell m}(r)\partial_r\right), \\
\hat{A}_3^{\ell m} =& \left(k_9^{\ell m}(r) + k_{10}^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m} \\
&- \frac{i}{1-\bar{\eta}_{\ell m}}\left(k_{11}^{\ell m}(r) + k_{12}^{\ell m}(r)\partial_r\right), \\
\hat{B}_1^{\ell m} &= -i\left(q_1^{\ell m}(r) + q_2^{\ell m}(r)\partial_r\right), \\
\hat{B}_2^{\ell m} &= -i\left(q_3^{\ell m}(r) + q_4^{\ell m}(r)\partial_r\right), \\
\hat{B}_3^{\ell m} &= -i\left(q_5^{\ell m}(r) + q_6^{\ell m}(r)\partial_r\right), \\
\hat{\hat{B}}_1^{\ell m} &= -i\left(\tilde{q}_1^{\ell m}(r) + \tilde{q}_2^{\ell m}(r)\partial_r\right), \\
\hat{\hat{B}}_2^{\ell m} &= -i\tilde{q}_3^{\ell m}(r),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}^{\ell m} =& g_1^{\ell m}(r) + g_2^{\ell m}(r)\partial_r - i\chi m\Lambda_{10s}^{\ell\ell 1}\left[g_3^{\ell m}(r) + h_1^{\ell m}(r)\right. \\
&\left. + \left(g_4^{\ell m}(r) + h_2^{\ell m}(r)\right)\partial_r\right]
\end{aligned}
$$

$$1/(1-\bar{\eta}_{\ell m})$$

$$k_i^{\ell m}(r) = k_i^{\ell m}(r,\omega,M), q_i^{\ell m}(r) = q_i^{\ell m}(r,\omega,M)$$

$$\tilde{q}_i^{\ell m}(r) = \tilde{q}_i^{\ell m}(r,\omega,M)$$

$$g_i^{\ell m}(r) h_i^{\ell m}(r) \left\{\hat{\mathcal{A}}_i^{\ell m}, \hat{\mathcal{B}}_i^{\ell m}, \hat{\hat{\mathcal{B}}}_i^{\ell m}\right\} \Psi_4^{(1,1)}$$

$$
\begin{aligned}
\hat{\mathcal{A}}_1^{\ell m} =& i\left(\mathfrak{h}_1^{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} \\
&+ \frac{1}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{h}_3^{\ell m}(r) + \mathfrak{k}_4^{\ell m}(r)\partial_r\right), \\
\hat{\mathcal{A}}_2^{\ell m} =& \left(\mathfrak{h}_5^{\ell m}(r) + \mathfrak{k}_6^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} \\
&- \frac{i}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{h}_7^{\ell m}(r) + \mathfrak{k}_8^{\ell m}(r)\partial_r\right), \\
\hat{\mathcal{A}}_3^{\ell m} =& \left(\mathfrak{h}_9^{\ell m}(r) + \mathfrak{h}_{10}^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} \\
&- \frac{i}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{h}_{11}^{\ell m}(r) + \mathfrak{h}_{12}^{\ell m}(r)\partial_r\right), \\
\hat{\mathcal{B}}_1^{\ell m} &= -i\left(\mathfrak{q}_1^{\ell m}(r) + \mathfrak{q}_2^{\ell m}(r)\partial_r\right), \\
\hat{\mathcal{B}}_2^{\ell m} &= -i\left(\mathfrak{q}_3^{\ell m}(r) + \mathfrak{q}_4^{\ell m}(r)\partial_r\right), \\
\hat{\mathcal{B}}_3^{\ell m} &= -i\left(\mathfrak{q}_5^{\ell m}(r) + \mathfrak{q}_6^{\ell m}(r)\partial_r\right), \\
\hat{\hat{\mathcal{B}}}_1^{\ell m} &= -i\left(\tilde{\mathfrak{q}}_1^{\ell m}(r) + \tilde{\mathfrak{q}}_2^{\ell m}(r)\partial_r\right), \\
\hat{\hat{\mathcal{B}}}_2^{\ell m} &= -i\tilde{\mathfrak{q}}_3^{\ell m}(r),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}^{\ell m} =& \mathfrak{g}_1^{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r)\partial_r - i\chi m\Lambda_{10s}^{\ell\ell 1}\left[\mathfrak{g}_3^{\ell m}(r) + \mathfrak{h}_1^{\ell m}(r)\right. \\
&\left. + \left(\mathfrak{g}_4^{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r)\right)\partial_r\right]
\end{aligned}
$$

$$\mathfrak{h}_i^{\ell m}(r) = \hat{\mathfrak{h}}_i^{\ell m}(r,\omega,M), \mathfrak{q}_i^{\ell m}(r) = \mathfrak{q}_i^{\ell m}(r,\omega,M)$$

$$\tilde{\mathfrak{q}}_i^{\ell m}(r) = \tilde{\mathfrak{q}}_i^{\ell m}(r,\omega,M)$$

$$\begin{aligned}
&\left[f(r)^2\partial_r^2 + \left(\frac{2M}{r^2}f(r) + \zeta\chi m\kappa T_1(r)\right)\partial_r + \omega^2 - V_{\mathrm{eff}}^{\mathrm{Smod}}(r,\chi,\zeta)\right]\Theta_{\ell m}(r) \\
&= \zeta^{\frac{1}{2}}M^2\kappa^{\frac{1}{2}}f(r)[g(r) + \chi Mm(h(r) + j(r)\partial_r)]\Psi_{\ell m}^{\mathrm{RWmod}}(r) \\
&\left[f(r)^2\partial_r^2 + \left(\frac{2M}{r^2}f(r) + \zeta\chi m\kappa A_1(r)\right)\partial_r + \omega^2 - V_{\mathrm{eff}}^{\mathrm{Amod}}(r,\chi,\zeta)\right]\Psi_{\ell m}^{\mathrm{RWmod}}(r) \\
&= \zeta^{\frac{1}{2}}M^2\kappa^{\frac{1}{2}}f(r)[v(r) + \chi Mm(n(r) + p(r)\partial_r)]\Theta_{\ell m}(r), \\
&\left[f(r)^2\partial_r^2 + \left(\frac{2M}{r^2}f(r) + \zeta\chi m\kappa P_1(r)\right)\partial_r + \omega^2 - V_{\mathrm{eff}}^{\mathrm{Pmod}}(r,\chi,\zeta)\right]\Psi_{\ell m}^{\mathrm{ZMmod}}(r) = 0,
\end{aligned}$$

$$\begin{aligned}
\Psi_{\ell m}^{\mathrm{RWmod}} &= \Psi_{\ell m}^{\mathrm{RW}}\left(1 - \delta\Psi_{\ell m}^{\mathrm{RW}}\right) \\
\Psi_{\ell m}^{\mathrm{ZMmod}} &= \Psi_{\ell m}^{\mathrm{ZM}}\left(1 - \delta\Psi_{\ell m}^{\mathrm{ZM}}\right)
\end{aligned}$$

$$\begin{aligned}
\delta\Psi_{\ell m}^{\mathrm{RW}} =& -\frac{1}{14\ell(\ell+1)Mr^9\omega}[\pi m(63(83\ell^2 + 83\ell - 640)M^3 r + 32M^2r^2(-5\ell^2 - 5\ell + 63r^2\omega^2 - 35) \\
&+70Mr^3(-3\ell^2 - 3\ell + 12r^2\omega^2 - 4) + 140r^4(-3\ell^2 - 3\ell + 2r^2\omega^2 + 12) + 60480M^4)] \\
\delta\Psi_{\ell m}^{\mathrm{ZM}} =& \frac{1}{14\ell(\ell+1)Mr^9\omega\left((\ell^2 + \ell - 2)r + 6M\right)^2}[\pi m(-144(2730\ell^2 + 2730\ell - 11603)M^5 r \\
&+70(\ell^2 + \ell - 2)Mr^5\left(3\ell^2(4r^2\omega^2 + 3) + 3\ell(4r^2\omega^2 + 3) + 4(5r^2\omega^2 - 42)\right) \\
&+280(\ell^2 + \ell - 2)^2 r^6(r^2\omega^2 - 3) + 6M^4r^2(-5439\ell^4 - 10878\ell^3 + 55307\ell^2 + 60746\ell + 11340r^2\omega \\
&+3M^3r^3(6479\ell^4 + 12958\ell^3 + \ell^2(7812r^2\omega^2 - 17747) + \ell(7812r^2\omega^2 - 24226) - 6504r^2\omega^2 + 37 \\
&+12M^2r^4(3\ell^4(56r^2\omega^2 + 15) + 6\ell^3(56r^2\omega^2 + 15) + \ell^2(296r^2\omega^2 + 625) + 4\ell(32r^2\omega^2 + 145) \\
&-4(57r^2\omega^2 + 1280)\big) - 1469664M^6\big)]
\end{aligned}$$

$$R_{CS} = \frac{1}{2}R^{\mu}_{\ \nu\rho\sigma}\tilde{R}_{\mu}^{\nu\ \rho\sigma}_{\ \ \mu}$$

$$\tilde{R}_{\alpha\beta\gamma\delta} = \frac{1}{2}R_{\alpha\beta}^{\ \ \ \rho\sigma}\varepsilon_{\rho\sigma\gamma\delta},$$

$$\varepsilon_{\rho\sigma\kappa\lambda} = \sqrt{-g(x)}\hat{\epsilon}_{\rho\sigma\kappa\lambda}$$

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - AbR_{CS}\right] = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b)\right] - \int d^4x Ab\hat{R}_{CS}$$

$$A = \sqrt{\frac{2}{3}\frac{\alpha'}{48\kappa}}$$

$$\mathcal{O}\left(\frac{M_p}{M_s^2}\right)$$

$$\mathcal{M}\kappa \gg 1$$

$$\begin{aligned}
G_{\mu\nu} &= \kappa^2 T_{\mu\nu}^b + 4\kappa^2 AC_{\mu\nu}, \\
\Box\, b &= AR_{CS},
\end{aligned}$$

$$T_{\mu\nu}^{b} = \nabla_\mu b \nabla_\nu b - \frac{1}{2} g_{\mu\nu} (\nabla b)^2$$

$$C_{\mu\nu} = -\frac{1}{2} \nabla^\alpha \left[(\nabla^\beta b) \tilde{R}_{\alpha\mu\beta\nu} + (\nabla^\beta b) \tilde{R}_{\alpha\nu\beta\mu} \right]$$

$$\nabla_\mu C^{\mu\nu} = -\frac{1}{4} (\nabla^\nu b) R_{CS}$$

$$\nabla^\mu G_{\mu\nu} = 0, \text{ where } G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$\nabla^\mu T_{\mu\nu} = \Box\, b \nabla^\nu b$$

$$0 = \nabla^\mu G_{\mu\nu} = \kappa^2 \nabla^\mu T_{\mu\nu}^{b} + 4\kappa^2 A \nabla^\mu C_{\mu\nu},$$

$$0 = \kappa^2 \Box\, b \nabla^\nu b + 4\kappa^2 A \nabla^\mu C_{\mu\nu} \overset{\aleph}{\rightarrow} A R_{CS} \nabla^\nu b = -4A \nabla^\mu C_{\mu\nu} \Longrightarrow \nabla^\mu C_{\mu\nu} = -\frac{1}{4} (\nabla^\nu b) R_{CS}.$$

$$\nabla^\mu T_{\mu\nu}^{b} = -4A \nabla^\mu C_{\mu\nu} = A \frac{1}{4} (\nabla^\nu b) R_{CS}.$$

$$ds^2 = -G(r) dt^2 + F(r) dr^2 - 2r^2 a \sin^2\,\theta W(r) dt d\phi + r^2 d\Omega^2.$$

$$A R_{CS} = -aA \cos\,\theta W' \frac{4F^2G^2 + rGF'(rG' - 2G) + F\left(-4G^2 + r^2 (G')^2 + 2rG(G' - rG'')\right)}{F^2 G^2 r^2 \sqrt{FG}},$$

$$\Box\, b = \frac{1}{F}\left[\partial_r^2 + \left(\frac{2}{r} - \frac{F'}{F} + \frac{G'}{2G}\right)\right] b + \frac{1}{r^2} \frac{1}{\sin\,\theta} \partial_\theta [\sin\,\theta \partial_\theta b],$$

$$b = aAu(r) P_1(\cos\,\theta),$$

$$G(r) = \frac{1}{F(r)} = 1 - \frac{2M}{r}, M \equiv G\mathcal{M}$$

$$G = 8\pi\kappa^{-2}$$

$$W(r) = \frac{2M}{r^3} + w(r)$$

$$G_{t\phi} = \kappa^2 T_{t\phi} + 4A\kappa^2 C_{t\phi}$$

$$C_{t\phi} = -\frac{3AM(r - 2M)(ru' - u)}{r^5} a \sin^2\,\theta + \mathcal{O}(a^2)$$

$$\left(\frac{u}{r}\right)' = -\frac{1}{24 A^2 \kappa^2 M} (r^4 w')'$$

$$u(r) = -\frac{r^5 w'}{24A^2\kappa^2 M}$$

$$r^{11}(r-2M)w''' + 2r^{10}(6r-11M)w'' + (28r^{10} - 50Mr^9 - 576A^2\kappa^2M^2r^4)w' + 3456A^2\kappa^2M^3 = 0$$

$$w(r) = \sum_{n=4}^{\infty} \frac{d_n M^{n-2}}{r^n}$$

$$3456A^2\kappa^2M^3 - 162M^7d_9 + 256M^7d_8 + 8M^2d_4r^5 + \sum_{n=-4}^{-1} \frac{M^{n+7}[-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(}{r^n}$$

$$+\sum_{n=1}^{\infty} \frac{M^{n+7}[-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8}] + 576A^2\kappa^2M^{n+3}(n+3)d_{n+3}}{r^n} = 0$$

$d_4 = 0$

$256d_8 - 162d_9 = -3456\frac{A^2\kappa^2}{M^4}$

$-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8} = 0$, for $n = -1, -2, -3, -4$

$-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8} + 576\frac{A^2\kappa^2}{M^4}(n+3)d_{n+3} = 0$ for $n \geq 1$

$$d_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)}d_{n-1} + \frac{576A^2\kappa^2}{n(n-3)M^4}d_{n-6}, \text{ for } n \geq 10$$

$$d_4 = d_5 = 0$$
$$-28d_7 + 48d_6 = 0$$
$$-80d_8 + 126d_7 = 0$$
$$256d_8 - 162d_9 = -3456\frac{A^2\kappa^2}{M^4}$$

$$-2u(r) + 2(r-M)u'(r) + (r^2 - 2Mr)u''(r) = \frac{144M^2}{r^5}$$

$$u(r) = -\frac{5}{4Mr^2} - \frac{5}{2r^3} - \frac{9M}{2r^4}$$

$$u(r) = \frac{1}{24A^2\kappa^2M}\left[4d_4M^2 + \frac{5d_5M^3}{r} + \frac{6d_6M^4}{r^2} + \frac{7d_7M^5}{r^3} + \frac{8d_8M^6}{r^4}\right] + \mathcal{O}(1/r^5)$$

$$d_4 = d_5 = 0, d_6 = -5\gamma^2, d_7 = -\frac{60\gamma^2}{7}, d_8 = -\frac{27\gamma^2}{2}, d_9 = 0, \text{ where } \gamma^2 = \frac{A^2\kappa^2}{M^4}$$

$$d_{10} = d_{11} = 0, d_{12}, d_{13}, d_{14} \sim -\gamma^4, \ldots . d_{21} \sim -\gamma^4 - \gamma^6 \ldots .$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2 - 2r^2a\sin^2\theta W(r)dtd\phi$$

$$W(r) = \frac{2M}{r^3} - \frac{A^2\kappa^2(189M^2 + 120Mr + 70r^2)}{14r^8} + \mathcal{O}(A^{2n}),$$

$$b = aA\cos\,\theta\left(-\frac{5}{4Mr^2} - \frac{5}{2r^3} - \frac{9M}{2r^4}\right) + \mathcal{O}(A^m),\ \text{for}\ m = 2n+1, n \in \mathbb{Z}^+$$

$$4\kappa^2 AC_{t\phi} = a\frac{r-2M}{2}\sum_{n=4}^{\infty}\frac{n(n-3)d_n M^{n-2}}{r^{n+1}}\sin^2(\theta)$$

$$J = \int \star\, d\xi = aM$$

$$T_{\mu\nu}^{eff} = \kappa^2 T_{\mu\nu} + 4\kappa^2 AC_{\mu\nu}$$

$$T_{t\phi}^{eff} = 4\kappa^2 AC_{t\phi} = a\frac{r-2M}{2}\sum_{n=4}^{\infty}\frac{n(n-3)d_n M^{n-2}}{r^{n+1}}\sin^2(\theta).$$

$$l_{\pm}^{\mu} = \left(1,0,0,-\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}\right).$$

$$T_{\mu\nu}^{eff} l_{\pm}^{\mu} l_{\pm}^{\nu} = \pm\frac{a(r-2M)^{3/2}}{2}\sin(\theta)\sum_{n=4}^{\infty}\frac{n(n-3)d_n M^{n-2}}{r^{n+5/2}} + \mathcal{O}(a^2).$$

$$T_{\mu\nu}^{ef} l_{+}^{\mu} l_{+}^{\nu} \leq 0$$

$$T_{\mu\nu}^{ef} l_{-}^{\mu} l_{-}^{\nu} \geq 0$$

$$\Lambda_+ = T_{\mu\nu}^{eff} l_{+}^{\mu} l_{+}^{\nu}/a$$

$$n^{\mu} = \frac{dx^{\mu}}{d\lambda} \Longrightarrow n_{\mu} = \epsilon\partial_{\mu}\lambda \Longrightarrow \partial_{\mu} = \epsilon n_{\mu}\partial_{\lambda}$$

$$A(x) = A^+(x^+)\Theta(\lambda) + A^-(x^-)\Theta(-\lambda)$$

$$[A] = A(\mathcal{V}^+)|_{\Sigma} - A(\mathcal{V}^-)|_{\Sigma}$$

$$[n^{\mu}] = [e_i^{\mu}] = 0$$

$$e_i^{\mu} = \frac{\partial x^{\mu}}{\partial y^i}$$

$$\mathcal{V}^{\pm} \to \Sigma.\,[n^{\mu}] = 0$$

$$g_{\mu\nu} = g_{\mu\nu}^{+}\Theta(\lambda) + g_{\mu\nu}^{-}\Theta(-\lambda)$$

$$\partial_{\rho} g_{\mu\nu} = \partial_{\rho}^{+} g_{\mu\nu}^{+}\Theta(\lambda) + \partial_{\rho}^{-} g_{\mu\nu}^{-}\Theta(-\lambda) + \epsilon n_{\rho}\delta(\lambda)[g_{\mu\nu}].$$

$$[g_{\mu\nu}] = 0$$

$$g_{\mu\nu} = h_{\mu\nu} + \epsilon n_{\mu} n_{\nu}$$

$$[g_{\mu\nu}] = [h_{\mu\nu}]$$

$$h_{ij} = h_{\mu\nu} e_i^{\mu} e_j^{\nu}$$

$$[h_{ij}] = 0$$

$$\Gamma^{\alpha}_{\beta\gamma} = \Theta(\lambda)\Gamma^{+\alpha}_{\beta\gamma} + \Theta(-\lambda)\Gamma^{-\alpha}_{\beta\gamma},$$

$$\partial_{\delta}\Gamma^{\alpha}{}_{\beta\gamma} = \Theta(\lambda)\partial_{\delta}\Gamma^{\alpha}_{+\beta\gamma} + \Theta(-\lambda)\partial_{\delta}\Gamma^{\alpha}_{-\beta\gamma} + \epsilon\delta(\lambda)[\Gamma^{\alpha}{}_{\beta\gamma}]n_{\delta}.$$

$$R^{\alpha}_{\beta\gamma\delta} = \Theta(\lambda)R^{\alpha}_{+\beta\gamma\delta} + \Theta(-\lambda)R^{\alpha}_{-\beta\gamma\delta} + \delta(\lambda)\Re^{\alpha}_{\beta\gamma\delta},$$

$$\Re^{\alpha}{}_{\beta\gamma\delta} = \epsilon\left([\Gamma^{\alpha}{}_{\beta\delta}]n_{\gamma} - [\Gamma^{\alpha}{}_{\beta\gamma}]n_{\delta}\right)$$

$$\Re^{\alpha}{}_{\beta\gamma\delta} = \left(n_{\beta}n_{\gamma}[K^{\alpha}_{\delta}] - n_{\beta}n_{\delta}[K^{\alpha}_{\gamma}] - [K_{\beta\delta}]n^{\alpha}n_{\gamma} + [K_{\beta\gamma}]n^{\alpha}n\delta\right).$$

$$G_{\mu\nu} = G^{+}_{\mu\nu}\Theta(\lambda) + G^{-}_{\mu\nu}\Theta(-\lambda) + \delta(\lambda)\mathfrak{G}_{\mu\nu},$$

$$\mathfrak{G}_{\alpha\beta} = -\epsilon\left([K_{\alpha\beta}] - [K]g_{\alpha\beta}\right) - [K]n_{\alpha}n_{\beta}$$

$$b = b^{+}\Theta(\lambda) + b^{-}\Theta(-\lambda)$$

$$T_{\mu\nu} = \nabla_{\mu}b\nabla_{\nu}b - \frac{1}{2}g_{\mu\nu}(\nabla b)^2$$

$$\partial_{\mu}b = \partial^{+}_{\mu}b^{+}\Theta(\lambda) + \partial^{-}_{\mu}b^{-}\Theta(\lambda) + \epsilon\delta(\lambda)[b] \Longrightarrow [b] = 0$$

$$T_{\mu\nu} = T^{+}_{\mu\nu}\Theta(\lambda) + T^{-}_{\mu\nu}\Theta(-\lambda)$$

$$|g| = -\frac{1}{4}\hat{\varepsilon}^{\mu\nu\rho\sigma}\hat{\varepsilon}^{\alpha\beta\gamma\delta}g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}$$

$$|g| = |g^{+}|\Theta(\lambda) + |g^{-}|\Theta(-\lambda),$$

$$\varepsilon_{\mu\nu\rho\sigma} = \varepsilon^{+}_{\mu\nu\rho\sigma}\Theta(\lambda) + \varepsilon^{-}_{\mu\nu\rho\sigma}\Theta(-\lambda)$$

$$C_{\mu\nu} = \nabla_{\alpha}D^{\alpha}{}_{\mu\nu},$$

$$D^{\alpha}_{\mu\nu} = \left(-\frac{\partial^{\xi}b}{2}\left(\tilde{R}^{\alpha}_{\mu\xi\nu} + \tilde{R}^{\alpha}_{\nu\xi\mu}\right)\right)$$

$$\tilde{R}^{\alpha}{}_{\mu\xi\nu} = \Theta(\lambda)\tilde{R}^{\alpha}_{+\mu\xi\nu} + \Theta(-\lambda)\tilde{R}^{\alpha}_{-\mu\xi\nu} + \delta(\lambda)\Re^{\alpha}{}_{\mu\gamma\delta}\varepsilon^{\gamma\delta}{}_{\xi\nu}.$$

$$D^{\alpha}{}_{\mu\nu} = D^{\alpha}_{+\mu\nu}\Theta(\lambda) + D^{\alpha}_{-\mu\nu}\Theta(-\lambda) + \delta(\lambda)\left(-\frac{\partial^{\xi}b}{2}\Re^{\alpha}{}_{\mu\gamma\delta}\varepsilon^{\gamma\delta}{}_{\xi\nu} + (\mu \leftrightarrow \nu)\right)$$

$$\left(-\frac{\partial^{\xi}b}{2}\Re^{\alpha}{}_{\mu\gamma\delta}\varepsilon^{\gamma\delta}{}_{\xi\nu} + (\mu \leftrightarrow \nu)\right) = 0.$$

$$C_{\mu\nu}=C^{+}_{\mu\nu}\Theta(\lambda)+C^{-}_{\mu\nu}\Theta(-\lambda)+\epsilon\delta(\lambda)n_{\alpha}\left[D^{\alpha}_{\mu\nu}\right]=C^{+}_{\mu\nu}\Theta(\lambda)+C^{-}_{\mu\nu}\Theta(-\lambda)+\delta(\lambda)\mathfrak{C}_{\mu\nu},$$

$$\begin{aligned}[\mathcal{D}^{\alpha\mu\nu}] &= \mathcal{D}^{\alpha\mu\nu}_{+}\big|_{\Sigma}-\mathcal{D}^{\alpha\mu\nu}_{-}|_{\Sigma} \\ &= \left(\frac{-\partial^{+}_{\beta}b^{+}}{2}\tilde{R}^{\alpha\mu\beta\nu}_{+}+\frac{-\partial^{+}_{\beta}b^{+}}{2}\tilde{R}^{\alpha\nu\beta\mu}_{+}-\frac{-\partial^{-}_{\beta}b^{-}}{2}\tilde{R}^{\alpha\mu\beta\nu}_{-}-\frac{-\partial^{-}_{\beta}b^{-}}{2}\tilde{R}^{\alpha\nu\beta\mu}_{-}\right)\Bigg|_{\Sigma} \\ &= -\frac{1}{2}\left[\partial_{\beta}b\tilde{R}^{\alpha\mu\beta\nu}\right]-\frac{1}{2}\left[\partial_{\beta}b\tilde{R}^{\alpha\nu\beta\mu}\right]\end{aligned}$$

$$\mathfrak{C}_{\mu\nu}=-\frac{1}{2}\left[\partial^{\beta}b\tilde{R}^{\alpha}{}_{\mu\beta\nu}n_{\alpha}\right]-\frac{1}{2}\left[\partial^{\beta}b\tilde{R}^{\alpha}{}_{\nu\beta\mu}n_{\alpha}\right].$$

$$\mathfrak{C}_{\mu\nu}n^{\mu}=-\frac{1}{2}\left[\partial^{\beta}b\tilde{R}^{\alpha}{}_{\nu\beta\mu}n^{\mu}n_{\alpha}\right],\mathfrak{C}_{\mu\nu}n^{\nu}=-\frac{1}{2}\left[\partial^{\beta}b\tilde{R}^{\alpha}{}_{\mu\beta\nu}n^{\nu}n_{\alpha}\right].$$

$$\partial_{\beta}b=f(x)n_{\beta}$$

$$\mathfrak{G}_{\mu\nu}-4A\kappa^{2}\mathfrak{C}_{\mu\nu}=\kappa^{2}S_{\mu\nu},$$

$$\begin{aligned}ds^{2}_{\pm} &= -f_{\pm}dt^{2}_{\pm}+\frac{dr^{2}_{\pm}}{f_{\pm}}-2r^{2}_{\pm}\sin^{2}\,\theta_{\pm}a_{\pm}W_{\pm}dt_{\pm}d\phi_{\pm}+r^{2}_{\pm}d\Omega^{2}_{\pm} \\ &= -f_{\pm}dt^{2}_{\pm}+\frac{dr^{2}_{\pm}}{f_{\pm}}+r^{2}_{\pm}d\theta^{2}_{\pm}+r^{2}_{\pm}\sin^{2}\,\theta_{\pm}(d\phi_{\pm}-a_{\pm}W_{\pm}dt_{\pm})^{2}\end{aligned}$$

$$d\psi=d\phi-aW(R)dt$$

$$ds^{2}_{\pm}=-f_{\pm}dt^{2}_{\pm}+\frac{dr^{2}_{\pm}}{f_{\pm}}+r^{2}_{\pm}d\theta^{2}_{\pm}+r^{2}_{\pm}\sin^{2}\,\theta_{\pm}(d\psi_{\pm}-a_{\pm}\Delta W_{\pm}dt_{\pm})^{2}$$

$$\Delta W=W(r)-W(R)$$

$$t_{\pm}=T_{\pm}(\tau),r_{\pm}=R_{\pm}(\tau)$$

$$\begin{aligned}ds^{2}_{-}|_{\Sigma} &= -\left[F(R_{-})\dot{T}^{2}_{-}-\frac{\dot{R}^{2}_{-}}{F(R_{-})}\right]d\tau^{2}+R^{2}_{-}(\tau)d\Omega^{2} \\ ds^{2}_{+}|_{\Sigma} &= -\left[F(R_{+})\dot{T}^{2}_{+}-\frac{\dot{R}^{2}_{+}}{F(R_{+})}\right]d\tau^{2}+R^{2}_{+}(\tau)d\Omega^{2}\end{aligned}$$

$$F_{\pm}=1-2M/R_{\pm}(\tau)$$

$$ds^{2}_{\Sigma}=-d\tau^{2}+R^{2}(\tau)d\theta^{2}+R^{2}(\tau)\sin^{2}\,\theta d\psi^{2}$$

$$\theta=\theta_{+}=\pi-\theta_{-}$$

$$R_{+}=R_{-}=R$$

$$F_{\pm}\dot{T}_{\pm}=\beta_{\pm}=\sqrt{\dot{R}^{2}_{\pm}+F_{\pm}}$$

$$F_{+}=F_{-}\Longrightarrow T_{+}=T_{-}\Longrightarrow\beta_{+}=\beta_{-}$$

$$e_{\tau}^{\mu} = \left(\frac{\beta}{F}, \dot{R}, 0,0\right)$$

$$n_{\mu}^{\pm} = \pm\left(-\dot{R}, \frac{\beta}{F}, 0,0\right)$$

$$n_{\mu}e_{\tau}^{\mu} = 0$$

$$n_{\mu}n^{\mu} = 1$$

$$K_{ij}^{\pm} = -n_{\alpha}^{\pm}\left(\frac{\partial^2 x^{\alpha}}{\partial y^i \partial y^j} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial y^i}\frac{\partial x^{\gamma}}{\partial y^j}\right)$$

$$K_{ij}^{\pm} = \pm\begin{pmatrix} -\frac{\dot{\beta}}{\dot{R}} & 0 & -\frac{a_{\pm}R^2\sin^2\theta W'}{2} \\ 0 & R\beta & 0 \\ -\frac{a_{\pm}R^2\sin^2\theta W'}{2} & 0 & R\sin^2\theta\beta \end{pmatrix}$$

$$-\frac{1}{2}\partial^{\beta} b\tilde{R}^{\alpha}{}_{\mu\beta\nu}n_{\alpha} + (\mu \leftrightarrow \nu)$$

$$b_{\pm} = a_{\pm}Au_{\pm}\cos\theta_{\pm}$$

$$\theta_{+} = \pi - \theta_{-}$$

$$a_{+} = -a_{-}$$

$$[b] \Longrightarrow a_{+} = -a_{-}$$

$$\mathfrak{C}_{\mu\nu} = 0$$

$$S_{ij} = -\frac{\epsilon}{\kappa^2}\left([K_{ij}] - [K]h_{ij}\right),$$

$$S_{ij} = \frac{-\epsilon}{\kappa^2}\begin{pmatrix} 4\frac{\beta}{R} & 0 & 0 \\ 0 & -R^2\left(\frac{2\beta}{R} + \frac{2\dot{\beta}}{\dot{R}}\right) & 0 \\ 0 & 0 & -R^2\sin^2\theta\left(\frac{2\beta}{R} + \frac{2\dot{\beta}}{\dot{R}}\right) \end{pmatrix}$$

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\mu}\Gamma_{\sigma\nu}^{\rho} + \Gamma_{\xi\mu}^{\rho}\Gamma_{\sigma\nu}^{\xi} - \cdots$$

$$\tilde{R}_{\beta\gamma\delta}^{\alpha} = \frac{1}{2}R_{\beta\rho\sigma}^{\alpha}\varepsilon_{\gamma\delta}^{\rho\sigma},$$

$$\varepsilon_{\rho\sigma\kappa\lambda} = \sqrt{-g}$$

$$\varepsilon^{0123} = \frac{-1}{\sqrt{-g}}$$

$$\delta\left(A\int\ d^4x\sqrt{-g}b(x)R^{\mu\nu\rho\sigma}(x)\tilde{R}_{\mu\nu\rho\sigma}(x)\right)\equiv 4A\int\ d^4x\sqrt{-g}C^{\mu\nu}(x)\delta g_{\mu\nu}(x)=-4A\int\ d^4x\sqrt{-g}C_{\mu\nu}(x)\delta g^{\mu\nu}(x)$$

$$C_{\mu\nu}=-\frac{1}{2}\nabla^{\alpha}\left[(\nabla^{\beta}b)\tilde{R}_{\alpha\mu\beta\nu}+(\nabla^{\beta}b)\tilde{R}_{\alpha\nu\beta\mu}\right]$$

$$\varepsilon^{\beta\alpha\rho\sigma}\nabla_{\alpha}R_{\rho\sigma}{}^{\mu\nu}=0$$

$$C_{\mu\nu}=-\frac{1}{4}\nabla_{\alpha}\left[(\nabla^{\beta}b)\left(R^{\alpha}_{\mu\gamma\delta}\varepsilon^{\gamma\delta}_{\beta\nu}+R^{\alpha}_{\nu\gamma\delta}\varepsilon^{\gamma\delta}_{\beta\mu}\right)\right]$$
$$\Longrightarrow g^{\mu\nu}C_{\mu\nu}=\frac{1}{2}\nabla_{\alpha}\left[(\nabla_{\beta}b)(R^{\alpha}_{\mu\gamma\delta})\varepsilon^{\mu\gamma\delta\beta}\right]=0$$

$$R^{\alpha}_{\kappa\lambda\xi}\varepsilon^{\kappa\lambda\xi\sigma}=0$$

$$\begin{aligned}
\nabla_{\mu}C^{\mu\nu}&=-\frac{1}{2}\nabla_{\mu}\nabla_{\rho}\left[(\nabla_{\sigma}b)\left(\tilde{R}^{\rho\mu\sigma\nu}+\tilde{R}^{\rho\nu\sigma\mu}\right)\right]\\
&=-\frac{1}{2}\nabla_{\mu}\nabla_{\rho}\left[(\nabla_{\sigma}b)\tilde{R}^{\rho\mu\sigma\nu}\right]-\frac{1}{2}\nabla_{\mu}\nabla_{\rho}\left[(\nabla_{\sigma}b)\tilde{R}^{\rho\nu\sigma\mu}\right]\\
&=-\frac{1}{2}\left(\nabla_{[\mu}\nabla_{\rho]}\right)\left[(\nabla_{\sigma}b)\tilde{R}^{\rho\mu\sigma\nu}\right]-\frac{1}{2}\left(\nabla_{\mu}\nabla_{\rho}-\nabla_{\rho}\nabla_{\mu}\right)\left[(\nabla_{\sigma}b)\tilde{R}^{\rho\nu\sigma\mu}\right]-\frac{1}{2}\nabla_{\rho}\nabla_{\mu}\left[(\nabla_{\sigma}b)\tilde{R}^{\rho\nu\sigma\mu}\right]\\
&=-\frac{1}{2}\left(\nabla_{\mu}\nabla_{\rho}-\nabla_{\rho}\nabla_{\mu}\right)\left[(\nabla_{\sigma}b)\left(\tilde{R}^{\rho\nu\sigma\mu}+\frac{1}{2}\tilde{R}^{\rho\mu\sigma\nu}\right)\right]\\
&=-\frac{\nabla_{\sigma}b}{2}\left[-R_{\xi\mu}\left(\tilde{R}^{\xi\nu\sigma\mu}+\frac{\tilde{R}^{\xi\mu\sigma\nu}}{2}\right)+R^{\nu}{}_{\xi\mu\rho}\left(\tilde{R}^{\rho\xi\sigma\mu}+\frac{\tilde{R}^{\rho\mu\sigma\xi}}{2}\right)+R_{\xi\rho}\left(\tilde{R}^{\rho\nu\sigma\xi}+\frac{\tilde{R}^{\rho\xi\sigma\nu}}{2}\right)\right]\\
&=-\frac{\partial_{\sigma}b}{2}\left[R^{\nu}{}_{\xi\mu\rho}\left(\tilde{R}^{\rho\xi\sigma\mu}+\frac{\tilde{R}^{\rho\mu\sigma\xi}}{2}\right)\right]=-\frac{\partial_{\sigma}b}{4}\left[\tilde{R}^{\rho\xi\sigma\mu}\left(R^{\nu}{}_{\xi\mu\rho}-R^{\nu}{}_{\rho\mu\xi}\right)+R^{\nu}{}_{\xi\mu\rho}\tilde{R}^{\rho\mu\sigma\xi}\right]\\
&=-\frac{\partial_{\sigma}b}{4}\left[\tilde{R}^{\rho\xi\sigma\mu}R^{\nu}{}_{\mu\xi\rho}+R^{\nu}{}_{\xi\mu\rho}\tilde{R}^{\rho\mu\sigma\xi}\right]=-\frac{\partial_{\sigma}b}{2}\tilde{R}^{\rho\xi\sigma\mu}R^{\nu}{}_{\mu\xi\rho}=-\frac{\partial_{\sigma}b}{2}\tilde{R}^{\rho}{}_{\xi}{}^{\sigma\mu}R^{\xi}{}_{\rho}{}^{\nu}{}_{\mu}=-\frac{1}{4}(\partial^{\nu}b)R_{CS}
\end{aligned}$$

$$\tilde{R}^{\rho}_{\xi}{}_{\xi}{}^{\xi\mu}R_{\rho}{}^{\nu}{}_{\mu}=\frac{1}{4}g^{\sigma\nu}\tilde{R}^{\rho}_{\xi}{}_{\xi}{}^{\lambda\mu}R^{\xi}_{\rho\lambda\mu}=\frac{1}{2}g^{\sigma\nu}R_{CS}$$

$$-2u(r)+2(r-M)u'(r)+(r^2-2Mr)u''(r)=0$$

$$u_1=c_1\left(\frac{r-M}{M}\right)$$

$$u=u_1(r)z(r)$$

$$\frac{1}{M}\left[2(M^2-4Mr+2r^2)z'+r(2M^2-3Mr+r^2)z''\right]=0$$

$$(\ln\, z') = \int \frac{8Mr - 2M^2 - 4r^2}{(r-2M)(r-M)r}dr = -\ln\left[r(r-2M)(r-M)^2\right] + c_2$$

$$\Rightarrow z' = \frac{c_2}{r(r-2M)(r-M)^2}$$

$$z = \frac{c_2}{(r-M)M^2} + \frac{c_2}{2M^3}\ln\left[1 - \frac{2M}{r}\right] + c_3$$

$$u_h = c_1 u_1(r) + c_2 u_2(r)$$

$$u_1(r) = \left(\frac{r-M}{M}\right), u_2(r) = 1 + \frac{1}{2M}(r-M)\ln\left[1 - \frac{2M}{r}\right].$$

$$u(r) = C_1(r)u_1(r) + C_2(r)u_2(r)$$

$$C_1'(r)u_1(r) + C_2'(r)u_2(r) = 0$$

$$C_1'(r)u_1'(r) + C_2'(r)u_2'(r) = \frac{144M^2}{r^5(r^2 - 2Mr)}$$

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{bmatrix}\begin{bmatrix} 0 \\ \dfrac{144M^2}{r^5(r^2-2Mr)} \end{bmatrix}$$

$$C_1(r) = \frac{81M}{2r^4} - \frac{5}{r^3} - \frac{15}{4Mr^2} - \frac{15}{4M^2r} + \ln\left(1 - \frac{2M}{r}\right)\frac{6(4r-3M)}{r^4} - \frac{15}{8M^3}\ln\left(1 - \frac{2M}{r}\right) + c_1$$

$$C_2(r) = \frac{36M}{r^4} - \frac{48}{r^3} + c_2$$

$$u(r) = -c_1 + c_2 - \frac{15}{4M^3} - \frac{9M}{2r^4} - \frac{5}{2r^3} - \frac{5}{4Mr^2} + c_1\frac{r}{M} + \ln\left(1 - \frac{2M}{r}\right)\left[\frac{15}{8M^3} - \frac{c_2}{2}\right]$$

$$+ \ln\left(1 - \frac{2M}{r}\right)\left[\frac{c_2 r}{2M} - \frac{15r}{8M^4}\right]$$

$$u(r) = -\frac{9M}{2r^4} - \frac{5}{2r^3} - \frac{5}{4Mr^2}$$

$$g_{t\phi} = r^2\left(-\frac{2M}{r^3} - w(r)\right)a\sin^2(\theta) \equiv \left(-\frac{2M}{r} - \widetilde{w}(r)\right)a\sin^2(\theta), \widetilde{w}(r) = \sum_{n=4}^{\infty}\frac{d_n M^{n-2}}{r^{n-2}},$$

$$d_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)}d_{n-1} + \frac{576\gamma^2}{n(n-3)}d_{n-6}, \text{ for } n \geq 10$$

$$d_4 = d_5 = 0, d_6 = -5\gamma^2, d_7 = -\frac{60\gamma^2}{7}, d_8 = -\frac{27\gamma^2}{2}, d_9 = 0$$

$$\widetilde{w}(r) = \sum_{n=4}^{\infty}\frac{d_n}{r^{n-2}}$$

$$\tilde{w}(r) = -\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}}$$

$$c_n = a_n c_{n-1} + b_n c_{n-6}, \text{ for } n \geq 10$$

$$a_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)} \quad b_n = \frac{576\gamma^2}{n(n-3)},$$

$$c_4 = c_5 = 0\,, c_6 = 5\gamma^2\,, c_7 = \frac{60\gamma^2}{7}\,, c_8 = \frac{27\gamma^2}{2}\,, c_9 = 0$$

$$\tilde{\Sigma}_N = \sum_{n=4}^{N} \frac{c_n}{r^{n-2}}$$

$$c_n/r^{n-2}$$

$$\tilde{\Sigma}_{N+1} \geq \tilde{\Sigma}_N, \forall N \in \mathbb{N}$$

$$\tilde{\Sigma} = \sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \mathcal{N}$$

$$\sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}}$$

$$\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}}$$

$$\forall r \geq 2$$

$$\sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \mathcal{K}$$

$$r > 2 \rightarrow 1/r^{n-2} < 1/2^{n-2} \rightarrow c_n/r^{n-2} \leq c_n/2^{n-2} \forall n \geq 4$$

$$\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \Rightarrow \sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \mathcal{K}$$

$$\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}}$$

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}}$$

1) $a_n \rightarrow 2$, as $n \rightarrow +\infty$
2) $b_n \rightarrow 0$, as $n \rightarrow +\infty$

$$|a_n| \leq k_1 \text{ and } |b_n| \leq k_2 \forall n \in \mathbb{N}$$

$$|c_{N-6}|, |c_{N-5}|, \dots, |c_{N-1}| \leq N'$$

$$|c_N| \leq |a_N||c_{N-1}| + |b_N||c_{N-6}| \Rightarrow |c_N| \leq (k_1 + k_2)N' \Rightarrow |c_N| \leq \widetilde{\mathcal{M}},$$

$$\widetilde{\mathcal{M}} = (k_1 + k_2)N'$$

$$|c_n| \leq \mathcal{D}, \forall n \geq 10$$

$$\forall n \geq \square$$

$$0 \leq c_n \leq \mathcal{D}, \forall n \geq 4$$

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \mathcal{D} \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^{n-2}$$

$$\Sigma \leq \mathcal{D} \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } -1 < x < 1$$

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n} = \frac{1}{2}$$

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \frac{\mathcal{D}}{2}$$

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

$$T_{\mu\nu} t^\mu t^\nu \geq 0, \forall t \colon t^\mu t_\mu < 0$$

$$T_{\mu\nu} l^\mu l^\nu \geq 0, \forall l \colon l^\mu l_\mu = 0$$

$$\frac{d}{d\tau} \theta_l = -R_{\mu\nu} l^\mu l^\nu + \cdots$$

$$T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} T g_{\mu\nu} t^\mu t^\nu, \forall t \colon t^\mu t_\mu < 0$$

$$\frac{d}{d\tau} \theta_t = -R_{\mu\nu} t^\mu t^\nu + \cdots$$

$$P^\alpha = -T^\alpha{}_\beta t^\beta$$

$$P_\alpha t^\alpha \leq 0 \text{ and } P_\alpha P^\alpha \leq 0$$

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \text{ and } T_{\mu\nu} T^\mu_\alpha t^\nu t^\alpha \leq 0 \forall t \colon t_\mu t^\mu < 0$$

$$T_{\alpha\beta}t^{\alpha}t^{\beta} \geq 0$$

$$[\partial_{\gamma}g_{\alpha\beta}] = n_{\gamma}q_{\alpha\beta} \Longrightarrow q_{\alpha\beta} = \epsilon[\partial_{\gamma}g_{\alpha\beta}]n^{\gamma}$$

$$[\Gamma^{\alpha}_{\beta\gamma}] = \frac{1}{2}(q^{\alpha}_{\beta}n_{\gamma} + q^{\alpha}_{\gamma}n_{\beta} - q_{\beta\gamma}n^{\alpha})$$

$$\mathfrak{R}^{\alpha}_{\beta\gamma\delta} = \frac{\epsilon}{2}(q^{\alpha}{}_{\delta}n_{\beta}n_{\gamma} - q^{\alpha}{}_{\gamma}n_{\beta}n_{\delta} - q_{\beta\delta}n^{\alpha}n_{\gamma} + q_{\beta\gamma}n^{\alpha}n_{\delta})$$

$$G_{\alpha\beta} = G^{+}_{\alpha\beta}\Theta(\lambda) + G^{-}_{\alpha\beta}\Theta(-\lambda) + \delta(\lambda)\mathfrak{G}_{\alpha\beta}$$

$$\mathfrak{G}_{\alpha\beta} = \frac{\epsilon}{2}\left(q_{\mu\alpha}n^{\mu}n_{\beta} + q_{\mu\beta}n^{\mu}n_{\alpha} - qn_{\alpha}n_{\beta} - \epsilon q_{\alpha\beta} - (q_{\mu\nu}n^{\mu}n^{\nu} - \epsilon q)g_{\alpha\beta}\right)$$

$$\mathfrak{G}^{\alpha\beta} = \mathfrak{G}^{ij}e^{\alpha}_{i}e^{\beta}_{j}$$

$$\begin{aligned}
\mathfrak{G}_{ij} &= \mathfrak{G}_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j} \\
&= \frac{\epsilon}{2}\left(-\epsilon q_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j} + \epsilon q_{\mu\nu}(g^{\mu\nu} - \epsilon n^{\mu}n^{\nu})h_{ij}\right) \\
&= \frac{1}{2}\left(-q_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j} + q_{\mu\nu}h^{\mu\nu}h_{ij}\right)
\end{aligned}$$

$$\begin{aligned}
[K_{ij}] &= \left[(\nabla_{\alpha}n_{\beta})e^{\alpha}_{i}e^{\beta}_{j}\right] \\
&\overset{\otimes}{=} [(\nabla_{\alpha}n_{\beta})]e^{\alpha}_{i}e^{\beta}_{j} \\
&= -\left(\left[\Gamma^{\gamma}_{\alpha\beta}\right]n_{\gamma}\right)e^{\alpha}_{i}e^{\beta}_{j} \\
&\overset{\boxtimes}{=} -\left(\frac{1}{2}\left(q^{\gamma}_{\beta}n_{\alpha} + q^{\gamma}_{\alpha}n_{\beta} - q_{\alpha\beta}n^{\gamma}\right)\right)n_{\gamma}e^{\alpha}_{i}e^{\beta}_{j} \\
&= \frac{\epsilon}{2}q_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j}
\end{aligned}$$

$$[K_{ij}] = \frac{\epsilon}{2}q_{\alpha\beta}e^{\alpha}_{i}e^{\beta}_{j}, [K] = \frac{\epsilon}{2}h^{mn}\left(q_{\alpha\beta}e^{\alpha}_{m}e^{\beta}_{n}\right)$$

$$K_{ij} = K_{\mu\nu}e^{\mu}_{i}e^{\mu}_{j}$$

$$q_{\alpha\beta} = \frac{2}{\epsilon}[K_{\alpha\beta}]$$

$$\mathfrak{R}^{\alpha}{}_{\beta\gamma\delta} = \left(n_{\beta}n_{\gamma}[K^{\alpha}_{\delta}] - n_{\beta}n_{\delta}[K^{\alpha}_{\gamma}] - [K_{\beta\delta}]n^{\alpha}n_{\gamma} + [K_{\beta\gamma}]n^{\alpha}n\delta\right),$$

$$I_{CS} = \int d^{4}x\sqrt{-g}bR\tilde{R}$$

$$\delta I_{CS} \sim 2\int d^{4}x\sqrt{-g}bR^{\alpha}_{\beta\kappa\lambda}\varepsilon^{\kappa\lambda\rho\sigma}\left(\nabla_{\rho}\delta\Gamma^{\beta}_{\alpha\sigma}\right)$$

$$\int d\Sigma_{\rho}bR^{\alpha}_{\beta\kappa\lambda}\varepsilon^{\kappa\lambda\rho\sigma}\left(\delta\Gamma^{\beta}_{\alpha\sigma}\right)$$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$$

$$\delta\Gamma^{\beta}_{\alpha\sigma} = -\frac{1}{2}\left[g_{\mu\sigma}\nabla_{\alpha}\left(\delta g^{\mu\beta}\right) + g_{\mu\alpha}\nabla_{\sigma}\left(\delta g^{\mu\beta}\right) - g_{\alpha\nu}g_{\sigma\mu}\nabla^{\beta}(\delta g^{\mu\nu})\right]$$

$$\begin{aligned}\int\ d\Sigma_{\rho} b R^{\alpha}_{\beta\kappa\lambda}\varepsilon^{\kappa\lambda\rho\sigma}\left(\delta\Gamma^{\beta}_{\alpha\sigma}\right) &= -\int\ d\Sigma_{\rho} b\tilde{R}^{\alpha\ \rho\sigma}_{\beta} g_{\mu\sigma}\nabla_{\alpha}\left(\delta g^{\mu\beta}\right) - \int\ d\Sigma_{\rho} b\tilde{R}^{\alpha\ \rho\sigma}_{\beta} g_{\mu\alpha}\nabla_{\sigma}\left(\delta g^{\mu\beta}\right) + \int\ d\Sigma_{\rho} b\tilde{R}^{\alpha\ \rho\sigma}_{\beta} g_{\alpha} \\ &= -\int\ d\Sigma_{\rho} b\tilde{R}^{\alpha}_{\beta\ \mu}\nabla_{\alpha}\left(\delta g^{\mu\beta}\right) - \int\ d\Sigma_{\rho} b\tilde{R}_{\mu\beta}{}^{\rho\alpha}\nabla_{\alpha}\left(\delta g^{\mu\beta}\right) + \int\ d\Sigma_{\rho} b\tilde{R}_{\mu}{}^{\alpha\rho}{}_{\beta}\nabla_{\alpha}\left(\delta g^{\mu\beta}\right. \\ &= \int\ d\Sigma_{\rho} b\left(\tilde{R}_{\mu}{}^{\alpha\rho}{}_{\nu} - \tilde{R}_{\nu}{}^{\rho}{}_{\mu}\right)\nabla_{\alpha}(\delta g^{\mu\nu}) \\ &= \int\ d^3x\sqrt{h}n^{\rho} b\left(\tilde{R}_{\mu\beta\rho\nu} + \tilde{R}_{\nu\beta\rho\mu}\right)g^{\alpha\beta}\nabla_{\alpha}(\delta g^{\mu\nu})\end{aligned}$$

$$\Sigma, \nabla_{\alpha}\delta g^{\mu\nu} = \partial_{\alpha}\delta g^{\mu\nu}$$

$$g^{\alpha\beta} = h^{\alpha\beta} + \epsilon n^{\alpha}n^{\beta}$$

$$\int\ d\Sigma_{\rho} b R^{\alpha}_{\beta\kappa\lambda}\epsilon^{\kappa\lambda\rho\sigma}\left(\delta\Gamma^{\beta}_{\alpha\sigma}\right) = \int\ d^3x\sqrt{h}n^{\rho} b\left(\tilde{R}_{\mu\beta\rho\nu} + \tilde{R}_{\nu\beta\rho\mu}\right)\epsilon n^{\beta}n^{\alpha}\partial_{\alpha}(\delta g^{\mu\nu})$$

$$ds^2 = 2dvdr - r^2X(r,v,x^i)dv^2 + 2r\omega_i(r,v,x^i)dvdx^i + h_{ij}(r,v,x^i)dx^idx^j$$

$$X^{eq} = X^{eq}(rv,x^i), \omega^{eq}_i = \omega^{eq}_i(rv,x^i), h^{eq}_{ij} = h^{eq}_{ij}(rv,x^i).$$

$$\xi = v\partial_v - r\partial_r,$$

$$\mathcal{L}_{\xi} g^{eq}_{\mu\nu}\big|_{r=0} = 0$$

$$A_{\mu} \to A_{\mu} + D_{\mu}\Lambda$$

$$\mathcal{L}_{\xi} A^{eq}_{\mu} + D_{\mu}\Lambda = 0,$$

$$\left(A^{eq}_{\mu}\xi^{\mu} + \Lambda\right)\big|_{r=0} = \left(vA^{eq}_{v} + \Lambda\right)\big|_{r=0} = 0$$

$$\begin{aligned}X = X^{eq}(rv,x^i) + \epsilon\delta X(r,v,x^i), \omega_i = \omega^{eq}_i(rv,x^i) + \epsilon\delta\omega_i(r,v,x^i) \\ h_{ij} = h^{eq}_{ij}(rv,x^i) + \epsilon\delta h_{ij}(r,v,x^i), A_{\mu} = A^{eq}_{\mu}(rv,x^i) + \epsilon\delta A_{\mu}(r,v,x^i)\end{aligned}$$

$$r \to \tilde{r} = \lambda r, v \to \tilde{v} = v/\lambda,$$

$$\mathcal{P} \sim (\partial_r)^{n_1}(\partial_v)^{n_2}\mathcal{Q}$$

$$\mathcal{P}(r,v,x^i) = \mathcal{P}^{eq}(rv,x^i) + \epsilon\delta\mathcal{P}(r,v,x^i)$$

$$\mathcal{P}|_{r=0}\mathcal{P} \sim (\partial_r)^{n_1}(\partial_v)^{n_2}\mathcal{Q}$$

$$\mathcal{P}|_{r=0} \sim \mathcal{O}(\epsilon)$$

$$\mathcal{R} \sim (\partial_r)^{n_1}(\partial_v)^{n_2}\mathcal{Q}_1 \times (\partial_r)^{n_3}(\partial_v)^{n_4}\mathcal{Q}_2$$

$$\mathcal{R}|_{r=0} \sim \mathcal{O}(\epsilon^2)$$

$$\left(A_\mu \xi^\mu + \Lambda\right)\big|_{r=0} = (vA_v + \Lambda)|_{r=0} = \mathcal{O}(\epsilon)$$

$$E_{vv}|_{r=0} = \partial_v \left[\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i \mathcal{J}^i\right] + \mathcal{O}(\epsilon^2)$$

$$S_{\text{tot}} = \int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}(s_{\text{wald}} + s_{\text{cor}})$$

$$s_{\text{wald}} = 1 + s_{\text{wald}}^{HD}$$

$$\partial_v \vartheta = E_{vv}^{HD} + \partial_v \left[\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\left(s_{wald}^{HD} + s_{cor}\right)\right)\right] - T_{vv} + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial S_{tot}}{\partial v} = \int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}\vartheta.$$

$$R_{vv} + E_{vv}^{HD} = T_{vv}$$

$$\mathcal{J}^v|_{eq} = s_{wald}$$

$$v = e^{\zeta(\vec{y})}\tau + \mathcal{O}(\rho), r = e^{-\zeta(\vec{y})}\rho + \mathcal{O}(\rho^2), x^a = y^a + \mathcal{O}(\rho)$$

$$ds^2 = 2d\tau d\rho - \rho^2 \tilde{X}\left(\rho,\tau,y^i\right)d\tau^2 + 2\rho\widetilde{\omega}_i\left(\rho,\tau,y^i\right)d\tau dy^i + \tilde{h}_{ij}\left(\rho,\tau,y^i\right)dy^i dy^j$$

$$\begin{aligned}
\tilde{X}\left(\rho,\tau,y^i\right) =& X + \omega_i\xi_j h^{ij} + \xi_i\xi_j h^{ij} - v\left(\xi_i h^{ij}\partial_v\omega_j + 2\omega_i\xi_j K^{ij} + 2\xi_i\xi_j K^{ij}\right) \\
&+ v^2\left(\xi_i\xi_l h^{kl} h^{ij}\partial_v K_{jk}\right) + \mathcal{O}(\rho) \\
\widetilde{\omega}_i\left(\rho,\tau,y^i\right) =& \omega_i - 2\xi_i + 2v\xi_k h^{jk}K_{ij} + \mathcal{O}(\rho) \\
\tilde{h}_{ij}\left(\rho,\tau,y^i\right) =& h_{ij}\left(v,r,x^i\right) + \mathcal{O}(\rho)
\end{aligned}$$

$$\xi_i = \partial_i\zeta(x^a)$$

$$E_{\mu\nu} \sim \mathcal{O}(l^N/L^{N+2})$$

$$E_{vv}|_{r=0} = \partial_v \left[\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i \mathcal{J}^i\right] + \left(K_{ij} + X_{ij}\right)\left(K^{ij} + X^{ij}\right) + \nabla_i \mathcal{Y}^i + \mathcal{O}(l^N),$$

$$\partial_v \left[\frac{1}{\sqrt{h}} \partial_v \left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i \mathcal{J}^i\right] = -\left(K_{ij} + X_{ij}\right)\left(K^{ij} + X^{ij}\right) - \nabla_i \mathcal{Y}^i + \mathcal{O}(l^N)$$

$$K_{ij} = (1/2)\partial_v h_{ij}$$

$$\mathcal{Y}^i \sim \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}
\frac{\partial S_{tot}}{\partial v} =& \int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}\vartheta = -\int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}\int_v^\infty dv'\partial_{v'}\vartheta \\
=& \int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}\int_v^\infty dv'\left[\partial_{v'}\left(\nabla_i \mathcal{J}^i\right) + \left(K_{ij} + X_{ij}\right)\left(K^{ij} + X^{ij}\right) + \nabla_i \mathcal{Y}^i\right]
\end{aligned}$$

$$\vartheta = (1/\sqrt{h})\partial_v\left(\sqrt{h}\mathcal{J}^v\right)$$

$$\frac{\partial S_{tot}}{\partial v} = \int_{\mathcal{H}_v} d^{d-2}x\sqrt{h}\int_v^{\infty} dv'(K_{ij}+X_{ij})(K^{ij}+X^{ij}) + \mathcal{O}(\epsilon^3)$$

$$\star\, \mathcal{O}(\epsilon^2 l^N)$$

$$L = L(g_{\mu\nu}, \Gamma^{\alpha}_{\beta\mu}, R^{\alpha}_{\beta\mu\nu}, A_\mu, F_{\mu\nu})$$

$$\delta(\sqrt{-g}L) = \sqrt{-g}E^{\mu\nu}\delta g_{\mu\nu} + \sqrt{-g}G^{\mu}\delta A_\mu + \sqrt{-g}D_\mu\Theta^\mu[g_{\mu\nu}, \Gamma^{\alpha}_{\beta\mu}, A_\mu, \delta g_{\mu\nu}, \delta\Gamma^{\alpha}_{\beta\mu}, \delta A_\mu].$$

$$x^\mu \to x^\mu + \zeta^\mu$$

$$A_\mu \to A_\mu + D_\mu\Lambda$$

$$\delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} = D_\mu\zeta_\nu + D_\nu\zeta_\mu$$
$$\delta A_\mu = \mathcal{L}_\zeta A_\mu + D_\mu\Lambda = \zeta^\alpha F_{\alpha\mu} + D_\mu(A_\alpha\xi^\alpha + \Lambda)$$

$$\delta[\sqrt{-g}L] = \sqrt{-g}D_\mu(\zeta^\mu L) + \sqrt{-g}D_\mu\Xi^\mu$$

$$D_\mu[\zeta^\mu L + \Xi^\mu - 2\zeta_\nu E^{\mu\nu} - G^\mu(A_\nu\zeta^\nu + \Lambda) - \Theta^\mu] = -2\zeta_\nu D_\mu E^{\mu\nu} - (A_\nu\zeta^\nu + \Lambda)D_\mu G^\mu + G^\mu\zeta^\nu F_{\nu\mu}$$

$$-2D_\mu E^{\mu\nu} + G_\mu F^{\nu\mu} = 0, \text{ and } D_\mu G^\mu = 0$$

$$\int_{\text{full space-time}} [\zeta_\nu(-2D_\mu E^{\mu\nu} + G_\mu F^{\nu\mu}) - (A_\nu\zeta^\nu + \Lambda)D_\mu G^\mu]$$

$$\Theta^\mu - \Xi^\mu - \zeta^\mu L = -2E^{\mu\nu}\zeta_\nu - G^\mu(A^\nu\zeta_\nu + \Lambda) + D_\nu Q^{\mu\nu}$$

$$\xi = v\partial_v - r\partial_r$$

$$\Theta^\mu\xi_\mu - \Xi^\mu\xi_\mu - \xi^\mu\xi_\mu L = -2E^{\mu\nu}\xi_\mu\xi_\nu - G^\mu\xi_\mu(A^\nu\xi_\nu + \Lambda) + \xi_\mu D_\rho Q^{\mu\rho}$$

$$2vE_{vv} + G_v(vA_v + \Lambda) = (-\Theta^r + \Xi^r + D_\rho Q^{r\rho})\big|_{r=0}$$

$$G_v(vA_v + \Lambda) \sim \mathcal{O}(\epsilon^2)$$

$$S_{IWT} = -2\pi\int_\Sigma \mathbf{Q}_\xi\Big|_{\xi\to 0, D_\alpha\xi_\beta\to\kappa\epsilon_{\alpha\beta}}$$

$$S_{IWT} = -4\pi\int_\Sigma d^{d-2}x\sqrt{h}\tilde{\epsilon}^{\alpha}_{\beta}\frac{\partial L}{\partial R^{\alpha}_{\beta rv}}$$

$$S_{IWT} = -2\int_\Sigma d^{d-2}x\sqrt{h}\left(\frac{\partial L}{\partial R^{v}_{vrv}} - \frac{\partial L}{\partial R^{r}_{rrv}}\right) = \int_\Sigma d^{d-2}x\sqrt{h}s_{IWT}$$

$$\epsilon^{\alpha_1\ldots\alpha_k} = -\frac{1}{\sqrt{h}}\varepsilon^{\alpha_1\ldots\alpha_k}$$

$$\epsilon^{vrx} = -\frac{1}{\sqrt{h}}, \epsilon^{vrxyz} = -\frac{1}{\sqrt{h}}$$

$$I=\int d^3x\sqrt{-g}\mathcal{L},\ \text{with}\ \mathcal{L}=\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu}+\frac{2}{3}\Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\rho\nu}\right)$$

$$E^{\mu\nu}=-\left(\epsilon^{\nu\rho\sigma}D_\rho R_\sigma{}^{\mu}+\epsilon^{\mu\rho\sigma}D_\rho R_\sigma{}^{\nu}\right)$$

$$ds^2=2dvdr-r^2X(r,v,x)dv^2+2r\omega(r,v,x)dvdx+h(r,v,x)dx^2$$

$$\begin{aligned}2E_{vv}=2E^{rr}&=-4\epsilon^{r\rho\sigma}D_\rho R^r_\sigma=4\epsilon^{r\sigma\rho}\left(\partial_\rho R_{\sigma v}-\Gamma^{\alpha}_{\rho\sigma}R_{\alpha v}-\Gamma^{\alpha}_{\rho v}R_{\sigma\alpha}\right)\\&=4\epsilon^{rvx}(\partial_x R_{vv}-\partial_v R_{xv})+\epsilon^{rvx}\left(-\frac{\omega}{h}\partial_v^2h+\frac{\omega}{h^2}(\partial_vh)^2+\frac{1}{h}(\partial_vh)(\partial_v\omega)\right).\end{aligned}$$

$$E_{vv}=\epsilon^{rvx}\left(\partial_v^2\omega+\frac{1}{h}(\partial_v\omega)(\partial_vh)-\frac{1}{h}\partial_v^2\partial_xh+\frac{1}{h}\partial_v\left(\frac{1}{h}(\partial_vh)(\partial_xh)\right)\right).$$

$$A\partial^2{}_vB=\partial_v(A\partial_vB)+\mathcal{O}(\epsilon^2)$$

$$\begin{aligned}E_{vv}&=\partial_v\left(\epsilon^{rvx}\partial_v\omega-\frac{1}{h}\epsilon^{rvx}\partial_v\partial_xh+\frac{1}{h^2}\epsilon^{rvx}(\partial_vh)(\partial_xh)\right)+\mathcal{O}(\epsilon^2)\\&=\partial_v\left(\epsilon^{rvx}\partial_v\omega-\epsilon^{rvx}\partial_x\left(\frac{1}{h}\partial_vh\right)\right)+\mathcal{O}(\epsilon^2)\\&=\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\epsilon^{rvx}\omega\right)-\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\epsilon^{rvx}\frac{1}{h}\partial_vh\right)\right)+\mathcal{O}(\epsilon^2)\end{aligned}$$

$$\mathcal{J}^v=\epsilon^{rvx}\omega,\mathcal{J}^i=-\epsilon^{rvx}\left(\frac{1}{h}\partial_vh\right)$$

$$\begin{aligned}E_{vv}=&\partial_v\left(\epsilon^{rvx}\partial_v\omega-\epsilon^{rvx}\partial_x\left(\frac{1}{h}\partial_vh\right)\right)-\frac{3}{4}\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_vh)^2\right)+\frac{3}{2}\epsilon^{rvx}(\partial_v\omega)\left(\frac{1}{h}\partial_vh\right)\\=&\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\epsilon^{rvx}\omega\right)-\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\epsilon^{rvx}\frac{1}{h}\partial_vh\right)\right)\\&-\frac{3}{4}\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\epsilon^{rvx}\frac{1}{h^2}(\partial_vh)^2\right)+\frac{3}{2}\epsilon^{rvx}(\partial_v\omega)\left(\frac{1}{h}\partial_vh\right)\end{aligned}$$

$$\mathcal{L}=R+l\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\frac{1}{2}R^{\sigma}{}_{\rho\mu\nu}-\frac{1}{3}\Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\rho\nu}\right)$$

$$E^{\mu\nu}=\frac{1}{2}Rg^{\mu\nu}-R^{\mu\nu}-l\left(\epsilon^{\nu\rho\sigma}D_\rho R^{\mu}_{\sigma}+\epsilon^{\mu\rho\sigma}D_\rho R^{\nu}_{\sigma}\right)$$

$$E_{vv}=-R_{vv}+l\epsilon^{rvx}\left(\partial_v^2\omega+\frac{1}{h}(\partial_v\omega)(\partial_vh)-\frac{1}{h}\partial_v^2\partial_xh+\frac{1}{h}\partial_v\left(\frac{1}{h}(\partial_vh)(\partial_xh)\right)\right).$$

$$
\begin{aligned}
E_{vv} =& -R_{vv} + l\epsilon^{rvx}\left(\partial_v^2\omega + \frac{1}{h}(\partial_v\omega)(\partial_v h) - \frac{1}{h}\partial_v^2\partial_x h + \frac{1}{h}\partial_v\left(\frac{1}{h}(\partial_v h)(\partial_x h)\right)\right)\\
=& \partial_v\left(\frac{1}{2}h^{ij}\partial_v h_{ij} + l\epsilon^{rvx}\partial_v\omega - l\epsilon^{rvx}\partial_x\left(\frac{1}{h}\partial_v h\right)\right) + \frac{1}{4}\left(\frac{1}{h}\partial_v h\right)^2\\
&-\frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right) + \frac{3}{2}l\epsilon^{rvx}(\partial_v\omega)\left(\frac{1}{h}\partial_v h\right)\\
=& \partial_v\left(\frac{1}{2}h^{ij}\partial_v h_{ij} + l\epsilon^{rvx}\partial_v\omega - l\epsilon^{rvx}\partial_x\left(\frac{1}{h}\partial_v h\right)\right) - \frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right)\\
&+\left(\frac{1}{2h}\partial_v h + \frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right)^2 - l^2\left(\frac{9}{4}\epsilon^{rvx}\partial_v\omega\right)^2\\
=& \partial_v\left(\frac{1}{2}h^{ij}\partial_v h_{ij} + l\epsilon^{rvx}\partial_v\omega - l\epsilon^{rvx}\partial_x\left(\frac{1}{h}\partial_v h\right)\right) - \frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right)\\
&+\left(\frac{1}{2h}\partial_v h + \frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right)^2 + \mathcal{O}(l^2)
\end{aligned}
$$

$$
\mathcal{J}^v = 1 + l\epsilon^{rvx}\omega, \mathcal{J}^i = -l\epsilon^{rvx}\left(\frac{1}{h}\partial_v h\right)
$$

$$
\left(K_{ij} + X_{ij}\right) \sim \left(\frac{1}{2h}\partial_v h + \frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right), \nabla_i\mathcal{Y}^i \sim -\frac{3}{4}l\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\epsilon^{rvx}\frac{1}{h^2}(\partial_v h)^2\right)
$$

$$
I = \int d^5x\sqrt{-g}\mathcal{L}, \text{ with } \mathcal{L} = 2\epsilon^{\mu\nu\lambda\rho\sigma}F_{\mu\nu}\Gamma^\alpha{}_{\lambda\beta}\left(\frac{1}{2}R^\beta{}_{\alpha\rho\sigma} - \frac{1}{3}\Gamma^\beta{}_{\rho\tau}\Gamma^\tau{}_{\alpha\sigma}\right)
$$

$$
\tilde{\mathcal{L}} = \epsilon^{\mu\nu\rho\sigma\delta}A_\mu R^\alpha{}_{\beta\nu\rho}R^\beta{}_{\alpha\sigma\delta},
$$

$$
\begin{aligned}
\mathcal{L} =& \tilde{\mathcal{L}} + D_\mu B^\mu\\
B^\mu =& 4\epsilon^{\mu\nu\lambda\rho\sigma}A_\nu\Gamma^\alpha{}_{\lambda\beta}\left(\frac{1}{2}R^\beta{}_{\alpha\rho\sigma} - \frac{1}{3}\Gamma^\beta{}_{\rho\tau}\Gamma^\tau{}_{\alpha\sigma}\right).
\end{aligned}
$$

$$
E^{\mu\nu} = \epsilon^{\mu\beta\rho\sigma\delta}D_\alpha\left(R^{\nu\alpha}_{\beta\rho}F_{\sigma\delta}\right) + \epsilon^{\nu\beta\rho\sigma\delta}D_\alpha\left(R^{\mu\alpha}_{\beta\rho}F_{\sigma\delta}\right).
$$

$$
\begin{aligned}
E_{vv} = E^{rr} =& 2\epsilon^{r\beta\rho\sigma\delta}D_\alpha\left(R^{r\alpha}{}_{\beta\rho}F_{\sigma\delta}\right) = 2\epsilon^{r\beta\rho\sigma\delta}\left(F_{\sigma\delta}\left(D_\rho R_{v\beta} - D_\beta R_{v\rho}\right) + R_v^\alpha{}_{\beta\rho}D_\alpha F_{\sigma\delta}\right)\\
=& 2\epsilon^{r\beta\rho\sigma\delta}\left(2F_{\sigma\delta}\left(\partial_\rho R_{v\beta} - \Gamma^\lambda{}_{\rho v}R_{\beta\lambda}\right) + R_v^\alpha{}_{\beta\rho}\left(\partial_\alpha F_{\sigma\delta} - 2\Gamma^\lambda{}_{\alpha\sigma}F_{\lambda\delta}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{r\beta\rho\sigma\delta}R_v{}^\alpha{}_{\beta\rho}\Gamma^\lambda{}_{\alpha\sigma}F_{\lambda\delta} =& 2\epsilon^{rvijk}R_{vnvi}h^{mn}\Gamma^l{}_{mj}F_{lk} + \mathcal{O}(\epsilon^2)\\
=& \epsilon^{rvijk}h^{mn}\hat{\Gamma}^l{}_{mj}F_{kl}\partial^2{}_v h_{ni} + \mathcal{O}(\epsilon^2)\\
\epsilon^{r\beta\rho\sigma\delta}R_v{}^\alpha{}_{\beta\rho}\partial_\alpha F_{\sigma\delta} =& 2\epsilon^{rvijk}\left(R_{vnvi}h^{mn}\partial_m F_{jk} + R_{vrjk}\partial_v F_{vi}\right) + \mathcal{O}(\epsilon^2)\\
=& \epsilon^{rvijk}\left[-h^{mn}\left(\partial_m F_{jk}\right)\left(\partial^2{}_v h_{ni}\right) + 2\left(\partial_j\omega_k\right)\left(\partial_v F_{vi}\right)\right] + \mathcal{O}(\epsilon^2)\\
\epsilon^{r\beta\rho\sigma\delta}F_{\sigma\delta}\Gamma^\lambda{}_{\rho v}R_{\beta\lambda} =& \epsilon^{rvijk}F_{jk}\Gamma^v{}_{iv}R_{vv} + \mathcal{O}(\epsilon^2)\\
=& \frac{1}{4}\epsilon^{rvijk}F_{jk}\omega_i h^{mn}\partial^2{}_v h_{mn} + \mathcal{O}(\epsilon^2)\\
\epsilon^{r\beta\rho\sigma\delta}F_{\sigma\delta}\partial_\rho R_{v\beta} =& \epsilon^{rvijk}F_{jk}(\partial_i R_{vv} - \partial_v R_{vi}) + \mathcal{O}(\epsilon^2)\\
=& \frac{1}{4}\epsilon^{rvijk}F_{jk}(2\partial^2{}_v\omega_i + \omega_i h^{mn}\partial^2{}_v h_{mn})\\
&-\frac{1}{2}\epsilon^{rvijk}F_{jk}h^{nm}\left(\partial_m\partial^2{}_v h_{ni} - \hat{\Gamma}^l{}_{mn}\partial^2{}_v h_{il} - \hat{\Gamma}^l{}_{mi}\partial^2{}_v h_{nl}\right) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$E_{vv} = 2\epsilon^{rvijk}\left[2(\partial_j\omega_k)(\partial_v F_{vi}) - h^{mn}\partial_m(F_{jk}\partial_v^2 h_{ni}) - 2h^{mn}\hat{\Gamma}^l_{mj}F_{kl}\partial_v^2 h_{ni}\right]$$
$$+2\epsilon^{rvijk}F_{jk}\left[\partial_v^2\omega_i + h^{nm}\left(\hat{\Gamma}^l_{mn}\partial_v^2 h_{il} + \hat{\Gamma}^l_{mi}\partial_v^2 h_{nl}\right)\right] + \mathcal{O}(\epsilon^2)$$
$$= 2\partial_v\left[2\epsilon^{rvijk}(\partial_j\omega_k)F_{vi} - \epsilon^{rvijk}h^{mn}\partial_m(F_{jk}\partial_v h_{ni}) - 2\epsilon^{rvijk}h^{mn}\hat{\Gamma}^l_{mj}F_{kl}\partial_v h_{ni}\right]$$
$$+2\partial_v\left[\epsilon^{rvijk}F_{jk}\partial_v\omega_i + \epsilon^{rvijk}F_{jk}h^{nm}\left(\hat{\Gamma}^l_{mn}\partial_v h_{il} + \hat{\Gamma}^l_{mi}\partial_v h_{nl}\right)\right] + \mathcal{O}(\epsilon^2)$$

$$\epsilon^{rvijk}F_{jk}\partial_v\omega_i = \epsilon^{rvijk}\partial_v(F_{jk}\omega_i) - \epsilon^{rvijk}\omega_i\partial_v F_{jk}$$
$$= \epsilon^{rvijk}\partial_v(F_{jk}\omega_i) + \epsilon^{rvijk}\omega_i(\partial_j F_{kv} + \partial_k F_{vj})$$
$$= \epsilon^{rvijk}\left[\partial_v(F_{jk}\omega_i) + 2\partial_i(\omega_j F_{vk}) - 2F_{vi}\partial_j\omega_k\right].$$

$$\epsilon^{rvijk}\hat{\Gamma}^l_{mj}F_{kl}\partial_v h_{ni} = \left(\epsilon^{rvajk}\delta^i_l + \epsilon^{rviak}\delta^j_l + \epsilon^{rvija}\delta^k_l\right)\hat{\Gamma}^l_{mj}F_{ka}\partial_v h_{ni}$$
$$= \epsilon^{rvijk}\left(\hat{\Gamma}^l_{mj}F_{ki}\partial_v h_{nl} + \hat{\Gamma}^l_{ml}F_{kj}\partial_v h_{ni} + \hat{\Gamma}^l_{mj}F_{lk}\partial_v h_{ni}\right),$$
$$2\epsilon^{rvijk}\hat{\Gamma}^l_{mj}F_{kl}\partial_v h_{ni} = \epsilon^{rvijk}F_{jk}\left(\hat{\Gamma}^l_{mi}\partial_v h_{nl} - \hat{\Gamma}^l_{ml}\partial_v h_{ni}\right).$$

$$E_{vv} = 2\partial_v\left[\epsilon^{rvijk}\partial_v(F_{jk}\omega_i) + 2\epsilon^{rvijk}\partial_i(\omega_j F_{vk}) - \epsilon^{rvijk}h^{mn}\partial_m(F_{jk}\partial_v h_{ni})\right]$$
$$+2\partial_v\left[\epsilon^{rvijk}F_{jk}\left(h^{lm}\hat{\Gamma}^n_{ml} + h^{nm}\hat{\Gamma}^l_{ml}\right)(\partial_v h_{in})\right] + \mathcal{O}(\epsilon^2)$$

$$h^{lm}\hat{\Gamma}^n_{ml} + h^{nm}\hat{\Gamma}^l_{ml} = \frac{1}{2}h^{lm}h^{no}(\partial_m h_{ol} + \partial_l h_{om} - \partial_o h_{ml}) + \frac{1}{2}h^{nm}h^{lo}\partial_m h_{ol}$$
$$= h^{lm}h^{no}\partial_m h_{ol} = -\partial_m h^{nm}$$

$$E_{vv} = 2\partial_v\left[\epsilon^{rvijk}\partial_v(F_{jk}\omega_i) + 2\epsilon^{rvijk}\partial_i(\omega_j F_{vk}) - \epsilon^{rvijk}\partial_m(h^{nm}F_{jk}\partial_v h_{ni})\right] + \mathcal{O}(\epsilon^2)$$
$$= 2\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\epsilon^{rvijk}F_{jk}\omega_i\right) + \nabla_i\left(2\epsilon^{rvijk}\omega_j F_{vk} - \epsilon^{rvmjk}h^{ni}F_{jk}\partial_v h_{nm}\right)\right] + \mathcal{O}(\epsilon^2)$$

$$\mathcal{J}^v = 2\epsilon^{rvijk}F_{jk}\omega_i, \mathcal{J}^i = 4\epsilon^{rvijk}\omega_j F_{vk} - 2\epsilon^{rvmjk}h^{ni}F_{jk}\partial_v h_{nm}$$

$$v = e^{\zeta(\vec{y})}\tau + \mathcal{O}(\rho), r = e^{-\zeta(\vec{y})}\rho + \mathcal{O}(\rho^2), x^a = y^a + \mathcal{O}(\rho)$$

$$E_{vv}|_{r=0} = \partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i\mathcal{J}^i\right]$$
$$-\frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right) + \left(\frac{1}{2h}\partial_v h + \frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right)^2 + \mathcal{O}(l^2),$$

$$\mathcal{J}^v = 1 + l\epsilon^{rvx}\omega, \mathcal{J}^i = -l\epsilon^{rvx}\left(\frac{1}{h}\partial_v h\right)$$

$$E_{vv} = \left(\frac{\partial\tau}{\partial v}\right)^2 E_{\tau\tau}, \text{ implying } E_{vv} = e^{-2\zeta}E_{\tau\tau}.$$

$$ds^2 = 2d\tau d\rho - \rho^2\tilde{X}(\rho,\tau,y)d\tau^2 + 2\rho\tilde{\omega}(\rho,\tau,y)d\tau dy + \tilde{h}(\rho,\tau,y)dy^2$$

$$\omega = \tilde{\omega} + 2(\partial_y\zeta) - \tau(\partial_y\zeta)\frac{1}{\tilde{h}}\partial_\tau\tilde{h} + \mathcal{O}(\rho), h = \tilde{h} + \mathcal{O}(\rho)$$

$$\partial_v = e^{-\zeta}\partial_\tau + \mathcal{O}(\rho), \partial_x = \partial_y - \tau(\partial_y\zeta)\partial_\tau + \mathcal{O}(\rho)$$

$$B = e^{-\phi(y)}\tilde{B}$$

$$\partial_x B = \partial_y\left(e^{-\phi}\tilde{B}\right) - \tau\frac{\partial\zeta}{\partial y}\partial_\tau\left(e^{-\phi}\tilde{B}\right) = e^{-\phi}\left(\partial_y\tilde{B} - \partial_\tau\left(\tau\frac{\partial\zeta}{\partial y}\tilde{B}\right) - \frac{\partial(\phi-\zeta)}{\partial y}\tilde{B}\right).$$

$$\partial_x\left(\frac{1}{h}\partial_v h\right) = e^{-\zeta}\left[\partial_y\left(\frac{1}{\tilde{h}}\partial_\tau\tilde{h}\right) - \partial_\tau\left(\tau\frac{\partial\zeta}{\partial y}\frac{1}{\tilde{h}}\partial_\tau\tilde{h}\right)\right]$$

$$\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right) = e^{-2\zeta}\left[\partial_y\left(\frac{1}{\tilde{h}^2}\left(\partial_\tau\tilde{h}\right)^2\right) - \partial_\tau\left(\tau\frac{\partial\zeta}{\partial y}\frac{1}{\tilde{h}^2}\left(\partial_\tau\tilde{h}\right)^2\right) - \frac{\partial\zeta}{\partial y^i}\frac{1}{\tilde{h}^2}\left(\partial_\tau\tilde{h}\right)^2\right].$$

$$\mathcal{J}^v = 1 + l\epsilon^{rvx}\omega = 1 + l\tilde{\epsilon}^{\rho\tau y}\left(\tilde{\omega} + 2\left(\partial_y\zeta\right) - \tau\left(\partial_y\zeta\right)\frac{1}{\tilde{h}}\partial_\tau\tilde{h}\right)$$

$$\mathcal{J}^i = -l\epsilon^{rvx}\left(\frac{1}{h}\partial_v h\right) = -e^{-\zeta}l\tilde{\epsilon}^{\rho\tau y}\left(\frac{1}{\tilde{h}}\partial_\tau\tilde{h}\right)$$

$$\begin{aligned}
\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i\mathcal{J}^i\right] &= \partial_v\left(\frac{1}{2}\frac{\partial_v h}{h} + l\epsilon^{rvx}\partial_v\omega - l\epsilon^{rvx}\partial_x\left(\frac{\partial_v h}{h}\right)\right)\\
&= e^{-2\zeta}\partial_\tau\left[\frac{1}{2}\frac{\partial_\tau\tilde{h}}{\tilde{h}} + l\tilde{\epsilon}^{\rho\tau y}\left\{\partial_\tau\left(\tilde{\omega} + 2\frac{\partial\zeta}{\partial y} - \tau\frac{\partial\zeta}{\partial y}\frac{\partial_\tau\tilde{h}}{\tilde{h}}\right) - \partial_y\left(\frac{\partial_\tau\tilde{h}}{\tilde{h}}\right) + \partial_\tau\left(\tau\frac{\partial\zeta}{\partial y}\frac{\partial_\tau\tilde{h}}{\tilde{h}}\right)\right\}\right]\\
&= e^{-2\zeta}\partial_\tau\left(\frac{1}{2}\frac{\partial_\tau\tilde{h}}{\tilde{h}} + l\tilde{\epsilon}^{\rho\tau y}\partial_\tau\tilde{\omega} - l\tilde{\epsilon}^{\rho\tau y}\partial_y\left(\frac{1}{\tilde{h}}\partial_\tau\tilde{h}\right)\right)
\end{aligned}$$

$$\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{J}^v\right) + \nabla_i\mathcal{J}^i\right] = e^{-2\zeta}\partial_\tau\left[\frac{1}{\sqrt{\tilde{h}}}\partial_\tau\left(\sqrt{\tilde{h}}\tilde{\mathcal{J}}^\tau\right) + \tilde{\nabla}_i\tilde{\mathcal{J}}^i\right]$$

$$\tilde{\mathcal{J}}^\tau = 1 + l\tilde{\epsilon}^{\rho\tau y}\left(\tilde{\omega} + 2\frac{\partial\zeta}{\partial y}\right), \tilde{\mathcal{J}}^i = -l\tilde{\epsilon}^{\rho\tau y}\left(\frac{\partial_v\tilde{h}}{\tilde{h}}\right)$$

$$\tilde{\mathcal{J}}^\tau_{ex} = 2l\tilde{\epsilon}^{\rho\tau y}\frac{\partial\zeta}{\partial y}$$

$$\partial_\tau\left(\sqrt{h}\tilde{\mathcal{J}}^\tau_{ex}\right) = 2l\tilde{\varepsilon}^{\rho\tau y}\partial_\tau\partial_y\zeta = 0$$

$$\int dy\sqrt{h}\tilde{\mathcal{J}}^\tau_{ex} = \int dy 2l\tilde{\varepsilon}^{\rho\tau y}\frac{\partial\zeta}{\partial y} = 0$$

$$\begin{aligned}
&-\frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right) + \left(\frac{1}{2h}\partial_v h + \frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right)^2 + \mathcal{O}(l^2)\\
&= \frac{1}{4}\left(\frac{1}{h}\partial_v h\right)^2 - \frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right) + \frac{3}{2}l\epsilon^{rvx}(\partial_v\omega)\left(\frac{1}{h}\partial_v h\right) + \mathcal{O}(l^2)
\end{aligned}$$

$$\frac{1}{4}\left(\frac{1}{h}\partial_v h\right)^2 = e^{-2\zeta}\frac{1}{4}\left(\frac{1}{\tilde{h}^2}\left(\tilde{\partial}_\tau\tilde{h}\right)^2\right)$$

$$
\begin{aligned}
&\epsilon^{rvx}(\partial_v\omega)\left(\frac{1}{h}\partial_v h\right)-\frac{1}{2}\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right)\\
&=e^{-2\zeta}\tilde{\epsilon}^{\rho\tau y}\left[\tilde{\partial}_\tau\left(\tilde{\omega}+2\frac{\partial\zeta}{\partial y}-\tau\frac{\partial\zeta}{\partial y}\frac{1}{\tilde{h}}\tilde{\partial}_\tau\tilde{h}\right)\left(\frac{1}{\tilde{h}}\tilde{\partial}_\tau\tilde{h}\right)\right]\\
&\quad-\frac{1}{2}e^{-2\zeta}\tilde{\epsilon}^{\rho\tau y}\left[\tilde{\partial}_y\left(\frac{1}{\tilde{h}^2}\left(\tilde{\partial}_\tau\tilde{h}\right)^2\right)-\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y}\frac{1}{\tilde{h}^2}\left(\tilde{\partial}_\tau\tilde{h}\right)^2\right)-\frac{\partial\zeta}{\partial y^i}\frac{1}{\tilde{h}^2}\left(\tilde{\partial}_\tau\tilde{h}\right)^2\right]\\
&=e^{-2\zeta}\tilde{\epsilon}^{\rho\tau y}\left[\left(\tilde{\partial}_\tau\tilde{\omega}\right)\left(\frac{1}{\tilde{h}}\tilde{\partial}_\tau\tilde{h}\right)-\frac{1}{2}\tilde{\partial}_y\left(\frac{1}{\tilde{h}^2}\left(\tilde{\partial}_\tau\tilde{h}\right)^2\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}l\epsilon^{rvx}\partial_x\left(\frac{1}{h^2}(\partial_v h)^2\right)+\left(\frac{1}{2h}\partial_v h+\frac{3}{2}l\epsilon^{rvx}\partial_v\omega\right)^2\\
&=e^{-2\zeta}\left[-\frac{3}{4}l\tilde{\epsilon}^{\rho\tau y}\partial_y\left(\frac{1}{\tilde{h}^2}\left(\partial_\tau\tilde{h}\right)^2\right)+\left(\frac{1}{2\tilde{h}}\partial_\tau\tilde{h}+\frac{3}{2}l\tilde{\epsilon}^{\rho\tau y}\partial_\tau\tilde{\omega}\right)^2\right]+\mathcal{O}(l^2)
\end{aligned}
$$

$$
\partial_i C=\tilde{\partial}_i\left(e^{-\phi}\tilde{C}\right)-\tau\frac{\partial\zeta}{\partial y^i}\tilde{\partial}_\tau\left(e^{-\phi}\tilde{C}\right)=e^{-\phi}\left[\tilde{\partial}_i\tilde{C}-\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^i}\tilde{C}\right)-\frac{\partial(\phi-\zeta)}{\partial y^i}\tilde{C}\right]
$$

$$
\begin{aligned}
&\partial_i\left(\omega_j F_{vk}\right)=e^{-\zeta}\tilde{\partial}_i\left(\omega_j\tilde{F}_{\tau k}\right)-e^{-\zeta}\tilde{\partial}_\tau\left(\frac{\partial\zeta}{\partial y^i}\tau\omega_j\tilde{F}_{\tau k}\right)\\
&=e^{-\zeta}\tilde{\partial}_i\left(\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right)-e^{-\zeta}\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^i}\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right)+\mathcal{O}(\epsilon^2)\\
&\partial_m\left(h^{nm}F_{jk}\partial_v h_{ni}\right)=e^{-\zeta}\tilde{\partial}_m\left(\tilde{h}^{nm}F_{jk}\partial_\tau\tilde{h}_{ni}\right)-e^{-\zeta}\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^m}\tilde{h}^{nm}F_{jk}\partial_\tau\tilde{h}_{ni}\right)\\
&=e^{-\zeta}\tilde{\partial}_m\left(\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right)-e^{-\zeta}\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^m}\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right)+\mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_v\left(F_{jk}\omega_i\right)=e^{-\zeta}\tilde{\partial}_\tau\left[\left(\tau\frac{\partial\zeta}{\partial y^k}\tilde{F}_{\tau j}-\tau\frac{\partial\zeta}{\partial y^j}\tilde{F}_{\tau k}+\tilde{F}_{jk}\right)\left(\tilde{\omega}_i+2\frac{\partial\zeta}{\partial y^i}-\tau\tilde{h}^{mn}\frac{\partial\zeta}{\partial y^n}\partial_\tau\tilde{h}_{im}\right)\right]\\
&=e^{-\zeta}\tilde{\partial}_\tau\left[\left(\tau\frac{\partial\zeta}{\partial y^k}\tilde{F}_{\tau j}-\tau\frac{\partial\zeta}{\partial y^j}\tilde{F}_{\tau k}+\tilde{F}_{jk}\right)\left(\tilde{\omega}_i+2\frac{\partial\zeta}{\partial y^i}\right)-\tau\tilde{h}^{mn}\tilde{F}_{jk}\frac{\partial\zeta}{\partial y^n}\partial_\tau\tilde{h}_{im}\right]+\mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon^{rvijk}\partial_v\left(F_{jk}\omega_i\right)+2\epsilon^{rvijk}\partial_i\left(\omega_j F_{vk}\right)-\epsilon^{rvijk}\partial_m\left(h^{nm}F_{jk}\partial_v h_{ni}\right)\\
&=e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_\tau\left[\left(\tau\frac{\partial\zeta}{\partial y^i}\tilde{F}_{\tau k}-\tau\frac{\partial\zeta}{\partial y^k}\tilde{F}_{\tau i}+\tilde{F}_{ki}\right)\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\right]\\
&-e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_\tau\left[\tau\frac{\partial\zeta}{\partial y^m}\tilde{h}^{mn}\tilde{F}_{jk}\partial_\tau\tilde{h}_{in}\right]\\
&\quad+2e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_i\left(\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right)-2e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^i}\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right)\\
&\quad-e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_m\left(\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right)+e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_\tau\left(\tau\frac{\partial\zeta}{\partial y^m}\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right)+\mathcal{O}(\epsilon^2)\\
&=e^{-\zeta}\tilde{\epsilon}^{\rho\tau ijk}\left(\tilde{\partial}_\tau\left[\tilde{F}_{ki}\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\right]+2\tilde{\partial}_i\left[\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right]-\tilde{\partial}_m\left[\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right]\right)+\mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
E_{vv}&=2\partial_v\left[\epsilon^{rvijk}\partial_v\left(F_{jk}\omega_i\right)+2\epsilon^{rvijk}\partial_i\left(\omega_j F_{vk}\right)-\epsilon^{rvijk}\partial_m\left(h^{nm}F_{jk}\partial_v h_{ni}\right)\right]+\mathcal{O}(\epsilon^2)\\
&=2e^{-2\zeta}\tilde{\partial}_\tau\left(\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_\tau\left[\tilde{F}_{jk}\left(\tilde{\omega}_i+2\frac{\partial\zeta}{\partial y^i}\right)\right]+2\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_i\left[\left(\tilde{\omega}_j+2\frac{\partial\zeta}{\partial y^j}\right)\tilde{F}_{\tau k}\right]\right)\\
&\quad-2e^{-2\zeta}\tilde{\partial}_\tau\left(\tilde{\epsilon}^{\rho\tau ijk}\tilde{\partial}_m\left[\tilde{h}^{nm}\tilde{F}_{jk}\partial_\tau\tilde{h}_{ni}\right]\right)+\mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$E_{vv} = e^{-2\zeta} E_{\tau\tau}$$

$$E_{\tau\tau}|_{\rho=0} = \partial_\tau \left[\frac{1}{\sqrt{\tilde{h}}} \partial_\tau \left(\sqrt{\tilde{h}} \tilde{\jmath}^\tau \right) + \tilde{\nabla}_i \tilde{\jmath}^i \right] + \mathcal{O}(\epsilon^2)$$

$$\tilde{\jmath}^\tau = 2\tilde{\epsilon}^{\rho\tau ijk} \tilde{F}_{jk} \left(\tilde{\omega}_i + 2\frac{\partial\zeta}{\partial y^i} \right), \tilde{\jmath}^i = 4\tilde{\epsilon}^{\rho\tau ijk} \left(\tilde{\omega}_j + 2\frac{\partial\zeta}{\partial y^j} \right) \tilde{F}_{\tau k} - 2\tilde{\epsilon}^{\rho\tau mjk} \tilde{h}^{ni} \tilde{F}_{jk} \partial_\tau \tilde{h}_{nm}$$

$$\tilde{\jmath}^\tau_{ex} = 4\tilde{\epsilon}^{\rho\tau ijk} \tilde{F}_{jk} \frac{\partial\zeta}{\partial y^i}, \tilde{\jmath}^i_{ex} = 8\tilde{\epsilon}^{\rho\tau ijk} \frac{\partial\zeta}{\partial y^j} \tilde{F}_{\tau k}$$

$$\partial_\tau\left(\sqrt{h}\tilde{\jmath}^\tau_{ex}\right) = 4\tilde{\varepsilon}^{\rho\tau ijk} \partial_\tau \tilde{F}_{jk} \frac{\partial\zeta}{\partial y^i} = 4\tilde{\varepsilon}^{\rho\tau ijk} \partial_{[\tau} \tilde{F}_{jk]} \frac{\partial\zeta}{\partial y^i} = 0$$
$$\partial_i\left(\sqrt{h}\tilde{\jmath}^i_{ex}\right) = 8\tilde{\varepsilon}^{\rho\tau ijk} \partial_{[i} \tilde{F}_{\tau k]} \frac{\partial\zeta}{\partial y^j} + 8\tilde{\varepsilon}^{\rho\tau ijk} \tilde{F}_{\tau k} \frac{\partial\zeta}{\partial y^{[i} y^{j]}} = 0$$

$$\frac{1}{\sqrt{\tilde{h}}} \partial_\tau \left(\sqrt{\tilde{h}} \tilde{\jmath}^\tau_{ex} \right) + \tilde{\nabla}_i \tilde{\jmath}^i_{ex} = 0$$

$$\int d^3y \sqrt{\tilde{h}} \tilde{\jmath}^\tau_{ex} = \int d^3y 4\tilde{\varepsilon}^{\rho\tau ijk} \tilde{F}_{jk} \partial_i \zeta = \int d^3y \left[4\partial_i\left(\tilde{\varepsilon}^{\rho\tau ijk} \tilde{F}_{jk}\zeta\right) + 4\tilde{\varepsilon}^{\rho\tau ijk} \partial_i \tilde{F}_{jk}\zeta \right]$$
$$= \int d^3y \sqrt{\tilde{h}} \nabla_i \left[4\tilde{\epsilon}^{\rho\tau ijk} \tilde{F}_{jk} \zeta \right] = 0$$

$$2vE_{vv} + G_v(vA_v + \Lambda) = \left(-\Theta^r + \Xi^r + D_\rho Q^{r\rho}\right)\Big|_{r=0}.$$

$$G_v(vA_v + \Lambda) \sim \mathcal{O}(\epsilon^2)$$

$$\Theta^r|_{r=0} = \Theta_{(1)} + v\Theta_{(2)},\ Q^{rv}|_{r=0} = Q_{(0)} + vQ_{(1)},\ Q^{ri}\big|_{r=0} = J^i_{(1)} + v\partial_v \tilde{J}^i_{(1)}.$$

$$\Theta^r|_{r=0} = (1 + v\partial_v)\mathcal{A}_{(1)} + v\partial_v^2 \mathcal{B}_{(0)}.$$

$$\mathcal{B}_{(0)} \sim X_{(-k+m)} \partial_v^{k-m} Y_{(0)} \sim \mathcal{O}(\epsilon).$$

$$Q^{r\mu} = \tilde{Q}^{r\mu} + vW_v^{r\mu}.$$

$$2E_{vv}|_{r=0} = -\partial_v \left(\frac{1}{\sqrt{h}} \partial_v \left[\sqrt{h} \left(\tilde{Q}^{rv} + \mathcal{B}_{(0)} \right) \right] + \nabla_i \left[\tilde{Q}^{ri} - J^i_{(1)} \right] \right) + \mathcal{O}(\epsilon^2).$$

$$W_v^{ri} = \partial_v J^i_{(1)} + \mathcal{O}(\epsilon^2)$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \mathcal{B}_{(0)}\right), \text{ and } \mathcal{J}^i = -\frac{1}{2}\left(\tilde{Q}^{ri} - J^i_{(1)}\right)$$

$$\delta L = \frac{\partial L}{\partial A_\mu} \delta A_\mu + \frac{\partial L}{\partial F_{\mu\nu}} \delta F_{\mu\nu} + \frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}} \delta \Gamma^\lambda{}_{\mu\nu} + \frac{\partial L}{\partial R^\alpha{}_{\beta\mu\nu}} \delta R^\alpha{}_{\beta\mu\nu} + \frac{\partial L}{\partial g^{\mu\nu}} \delta g^{\mu\nu}.$$

$$E^{\mu\nu}=\frac{1}{2}Lg^{\mu\nu}-g^{\mu\alpha}g^{\nu\beta}\frac{\partial L}{\partial g^{\alpha\beta}}-D_\alpha S^{\alpha\mu\nu},G^\mu=\frac{\partial L}{\partial A_\mu}+2D_\nu\frac{\partial L}{\partial F_{\mu\nu}}$$

$$\Theta^\mu=2\frac{\partial L}{\partial F_{\mu\nu}}\delta A_\nu+2\frac{\partial L}{\partial R^\alpha{}_{\beta\mu\nu}}\delta\Gamma^\alpha{}_{\beta\nu}+S^{\mu\alpha\beta}\delta g_{\alpha\beta}$$

$$S^{\mu\alpha\beta}=\frac{1}{2}D_\nu\left[g^{\alpha\lambda}\left(\frac{\partial L}{\partial R^\lambda{}_{\beta\mu\nu}}+\frac{\partial L}{\partial R^\lambda{}_{\mu\beta\nu}}\right)+g^{\beta\lambda}\left(\frac{\partial L}{\partial R^\lambda{}_{\alpha\mu\nu}}+\frac{\partial L}{\partial R^\lambda{}_{\mu\alpha\nu}}\right)\right.$$
$$\left.-g^{\mu\lambda}\left(\frac{\partial L}{\partial R^\lambda{}_{\beta\alpha\nu}}+\frac{\partial L}{\partial R^\lambda{}_{\alpha\beta\nu}}\right)\right]+\frac{1}{2}\left(g^{\alpha\lambda}\frac{\partial L}{\partial\Gamma^\lambda{}_{\beta\mu}}+g^{\beta\lambda}\frac{\partial L}{\partial\Gamma^\lambda{}_{\alpha\mu}}-g^{\mu\lambda}\frac{\partial L}{\partial\Gamma^\lambda{}_{\beta\alpha}}\right).$$

$$\delta A_\mu=\mathcal{L}_\xi A_\mu+D_\mu\Lambda$$
$$\delta\Gamma^\lambda{}_{\mu\nu}=\mathcal{L}_\xi\Gamma^\lambda{}_{\mu\nu}+\partial_\mu\partial_\nu\xi^\lambda.$$

$$\delta L=\frac{\partial L}{\partial A_\mu}\mathcal{L}_\xi A_\mu+\frac{\partial L}{\partial F_{\mu\nu}}\mathcal{L}_\xi F_{\mu\nu}+\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\mathcal{L}_\xi\Gamma^\lambda{}_{\mu\nu}+\frac{\partial L}{\partial R^\alpha{}_{\beta\mu\nu}}\mathcal{L}_\xi R^\alpha{}_{\beta\mu\nu}+\frac{\partial L}{\partial g^{\mu\nu}}\mathcal{L}_\xi g^{\mu\nu}$$
$$+\frac{\partial L}{\partial A_\mu}D_\mu\Lambda+\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda=\mathcal{L}_\xi L+\frac{\partial L}{\partial A_\mu}D_\mu\Lambda+\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda$$

$$\delta_\xi(\sqrt{-g}L)=\sqrt{-g}D_\mu(\xi^\mu L)+\sqrt{-g}D_\mu\Xi^\mu-\sqrt{-g}\Lambda D_\mu\frac{\partial L}{\partial A_\mu}+\xi^\lambda\partial_\mu\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial\Gamma^\lambda_{\mu\nu}}\right)$$

$$\Xi^\mu=\frac{\partial L}{\partial A_\mu}\Lambda+\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda-\frac{1}{\sqrt{-g}}\xi^\lambda\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\right)$$

$$D_\mu\frac{\partial L}{\partial A_\mu}=0,\partial_\mu\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial\Gamma^\lambda{}_{\mu\nu}}\right)=0$$

$$Q^{\mu\nu}=2\frac{\partial L}{\partial F_{\mu\nu}}(A_\lambda\xi^\lambda+\Lambda)+2\frac{\partial L}{\partial R^\alpha{}_{\beta\mu\nu}}D_\beta\xi^\alpha+\xi_\alpha(S^{\mu\nu\alpha}-S^{\nu\mu\alpha})+\xi^\lambda D_\beta\left[\frac{\partial L}{\partial R^\lambda{}_{\mu\nu\beta}}-\frac{\partial L}{\partial R^\lambda{}_{\nu\mu\beta}}\right],$$

$$L=\epsilon^{\mu\nu\rho\sigma\delta}A_\mu R^\alpha_{\beta\nu\rho}R^\beta_{\alpha\sigma\delta}$$

$$L=L\left(g^{\mu\nu},\Gamma^\alpha{}_{\beta\mu},R^\alpha{}_{\beta\mu\nu},A_\mu,F_{\mu\nu}\right)$$

$$L_n=\tilde{\mathcal{L}}^{\nu_1\cdots\nu_n}\prod_{i=1}^{n}A_{\nu_i}$$

$$D_\mu\frac{\partial L_n}{\partial A_\mu}=0\Longrightarrow nD_\mu\tilde{\mathcal{L}}^{\mu\nu_1\cdots\nu_{n-1}}=0\ \text{ and }\ n(n-1)\tilde{\mathcal{L}}^{\mu\nu\nu_1\cdots\nu_{n-2}}=0$$

$$L=\tilde{\mathcal{L}}+\tilde{\mathcal{L}}^\mu A_\mu$$

$$\tilde{\mathcal{L}}^\mu=\begin{cases}D_\nu\mathcal{B}^{\nu\mu\rho_1\sigma_1\cdots\rho_n\sigma_n}\prod_{i=1}^{n}F_{\rho_i\sigma_i}\ \text{ if }\ 2n+1<D\\ a_g\epsilon^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n}\prod_{i=1}^{n}F_{\rho_i\sigma_i}\ \text{ if }\ 2n+1=D\end{cases}$$

$$L = \mathcal{L} + D_\mu \left[A_\nu \sum_{n=0}^{N-1} \mathcal{B}^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n} \left(\prod_{i=1}^{n} F_{\rho_i\sigma_i} \right) \right] + a_g \epsilon^{\mu\rho_1\sigma_1\cdots\rho_N\sigma_N} A_\mu \prod_{i=1}^{N} F_{\rho_i\sigma_i}$$

$$L = \mathcal{L} + D_\mu (U^{\mu\nu} A_\nu)$$

$$\mathcal{L} = 2\epsilon^{\mu\nu\lambda\rho\sigma} F_{\mu\nu} \Gamma^\alpha{}_{\lambda\beta} \left(\frac{1}{2} R^\beta{}_{\alpha\rho\sigma} - \frac{1}{3} \Gamma^\beta{}_{\rho\tau} \Gamma^\tau{}_{\alpha\sigma} \right)$$

$$U^{\mu\nu} = -4\epsilon^{\mu\nu\lambda\rho\sigma} \Gamma^\alpha_{\lambda\beta} \left(\frac{1}{2} R^\beta_{\alpha\rho\sigma} - \frac{1}{3} \Gamma^\beta_{\rho\tau} \Gamma^\tau_{\alpha\sigma} \right)$$

$$L = \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu F_{\nu\lambda} F_{\rho\sigma}$$

$$L = \underbrace{\mathcal{L}}_{\text{Gauge invariant}} + \underbrace{D_\mu (U^{\mu\nu} A_\nu)}_{\text{total derivative}} + \underbrace{a_g \epsilon^{\mu\rho_1\sigma_1\cdots\rho_N\sigma_N} A_\mu \prod_{i=1}^{N} F_{\rho_i\sigma_i}}_{\text{pure gauge}},$$

$$2vE_{vv} + G_v(vA_v + \Lambda) = \left(-\Theta^r + \Xi^r + D_\rho Q^{r\rho} \right)\big|_{r=0}$$

$$\mathcal{L}^\mu = U^{\mu\nu} A_\nu$$

$$Q_t^{r\rho}\big|_{r=0} = U^{r\lambda} A_\lambda \xi^\rho = v U^{r\lambda} A_\lambda \delta^\rho{}_v$$

$$\Rightarrow \frac{1}{\sqrt{h}} \partial_\rho \left(\sqrt{h} Q_t^{r\rho} \right)\Big|_{r=0} = \frac{1}{\sqrt{h}} (1 + v\partial_v) \left(\sqrt{h} U^{r\rho} A_\rho \right)$$

$$= (1 + v\partial_v)\left(U^{r\rho} A_\rho \right) + \frac{1}{2} v A_v U^{rv} h^{mn} \partial_v h_{mn} + \mathcal{O}(\epsilon^2)$$

$$\Xi_t^r\big|_{r=0} = U^{r\rho} D_\rho \Lambda$$

$$\Theta_t^r\big|_{r=0} = A_v \mathcal{L}_\xi U^{rv} + A_i \mathcal{L}_\xi U^{ri} + U^{rv} \mathcal{L}_\xi A_v + U^{ri} \mathcal{L}_\xi A_i + \frac{1}{2} U^{rv} v A_v h^{mn} \partial_v h_{mn} + \mathcal{O}(\epsilon^2)$$

$$= (1 + v\partial_v)\left(U^{rv} A_v + U^{ri} A_i \right) + U^{r\rho} \partial_\rho \Lambda + \frac{1}{2} U^{rv} v A_v h^{mn} \partial_v h_{mn}$$

$$-\Theta_t^r\big|_{r=0} + \Xi_t^r\big|_{r=0} + \frac{1}{\sqrt{h}} \partial_\rho \left(\sqrt{h} Q_t^{r\rho} \right)\Big|_{r=0} = 0$$

$$L_g = a_g \epsilon^{\mu\rho_1\sigma_1\cdots\rho_N\sigma_N} A_\mu \prod_{i=1}^{N} F_{\rho_i\sigma_i}$$

$$\left[D_\rho Q_g^{r\rho} - \Theta_g^r + \Xi_g^r \right]\Big|_{r=0} - G_g^r \left(A_\rho \xi^\rho + \Lambda \right) = -v \left(\frac{\partial L_g}{\partial A_r} A_v + 2 \frac{\partial L_g}{\partial F_{ri}} F_{vi} \right)$$

$$\left[D_\rho Q_g^{r\rho} - \Theta_g^r + \Xi_g^r \right]\Big|_{r=0} - G_g^r \left(A_\rho \xi^\rho + \Lambda \right) = 0$$

$$\delta \mathcal{L} = \mathcal{L}_\xi \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Gamma^\lambda{}_{\mu\nu}} \partial_\mu \partial_\nu \xi^\lambda$$

$$\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda = D_\mu\Xi^\mu_{\mathcal{L}} + \frac{\xi^\lambda}{\sqrt{-g}}\partial_\mu\partial_\nu\left(\sqrt{-g}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}}\right)$$

$$D_\mu\Xi^\mu_{\mathcal{L}} = \frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda$$

$$D_\mu\Xi^\mu_{\mathcal{L}}\big|_{r=0} = 0$$

$$\Xi^\mu_{\mathcal{L}} = D_\nu q^{\mu\nu}$$

$$q^{r\rho}\ |= \tilde{q}^{r\rho} + v w^{r\rho}_v$$

$$\Xi^r_{\mathcal{L}} = \frac{1}{\sqrt{h}}\partial_\rho\left(\sqrt{h}\tilde{q}^{r\rho}\right) + (1 + v\partial_v)w^{rv}_v + v\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}j^i_{(1)}\right)\right) + \mathcal{O}(\epsilon^2).$$

$$G^\mu_{\mathcal{L}} = \frac{\partial\mathcal{L}}{\partial A_\mu} + 2D_\nu\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}$$

$$\Theta^\mu_{\mathcal{L}} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\delta A_\nu + 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta\mu\nu}}\delta\Gamma^{\alpha}{}_{\beta\nu} + S^{\mu\alpha\beta}_{\mathcal{L}}\delta g_{\alpha\beta}$$

$$Q^{\mu\nu}_{\mathcal{L}} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\left(A_\lambda\xi^\lambda + \Lambda\right) + 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_\beta\xi^\alpha + \xi_\alpha\left(S^{\mu\nu\alpha}_{\mathcal{L}} - S^{\nu\mu\alpha}_{\mathcal{L}}\right) + \xi^\lambda D_\beta\left[\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\mu\nu\beta}} - \frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\nu\mu\beta}}\right]$$

$$\begin{aligned} S^{\mu\alpha\beta}_{\mathcal{L}} = \frac{1}{2}D_\nu\Bigg[& g^{\alpha\lambda}\left(\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\beta\mu\nu}} + \frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\mu\beta\nu}}\right) + g^{\beta\lambda}\left(\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\alpha\mu\nu}} + \frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\mu\alpha\nu}}\right) \\ & - g^{\mu\lambda}\left(\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\beta\alpha\nu}} + \frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\alpha\beta\nu}}\right)\Bigg] + \frac{1}{2}\left(g^{\alpha\lambda}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\beta\mu}} + g^{\beta\lambda}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\alpha\mu}} - g^{\mu\lambda}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\beta\alpha}}\right). \end{aligned}$$

$$\Theta^\mu_{\mathcal{L}} = \Theta^\mu_{\mathcal{L}}\big|_{\text{gauge}} + \Theta^\mu_{\mathcal{L}}\big|_{\text{gravity}}$$

$$\Theta^\mu_{\mathcal{L}}\big|_{\text{gauge}} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\delta A_\nu,\ \Theta^\mu_{\mathcal{L}}\big|_{\text{gravity}} = 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta\mu\nu}}\delta\Gamma^{\alpha}{}_{\beta\nu} + S^{\mu\alpha\beta}_{\mathcal{L}}\delta g_{\alpha\beta}$$

$$Q^{\mu\nu}_{\mathcal{L}} = Q^{\mu\nu}_{\mathcal{L}}\big|_{\text{gauge}} + Q^{\mu\nu}_{\mathcal{L}}\big|_{\text{gravity}}$$

$$Q^{\mu\nu}_{\mathcal{L}}\big|_{\text{gauge}} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\left(A_\lambda\xi^\lambda + \Lambda\right)$$

$$Q^{\mu\nu}_{\mathcal{L}}\big|_{\text{gravity}} = 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_\beta\xi^\alpha + \xi_\alpha\left(S^{\mu\nu\alpha}_{\mathcal{L}} - S^{\nu\mu\alpha}_{\mathcal{L}}\right) + \xi^\lambda D_\beta\left[\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\mu\nu\beta}} - \frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{\nu\mu\beta}}\right].$$

$$\left[D_\rho Q^{r\rho}_{\mathcal{L}} - \Theta^r_{\mathcal{L}}\right]\big|_{\text{gauge}} - G^r_{\mathcal{L}}\left(A_\rho\xi^\rho + \Lambda\right) = \mathcal{O}(\epsilon^2)$$

$$\begin{aligned} Q^{r\rho}_{\mathcal{L}}\big|_{\text{gravity}} &= 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta r\rho}}D_\beta\xi^\alpha + v\left(S^{r\rho r}_{\mathcal{L}} - S^{\rho rr}_{\mathcal{L}}\right) + vD_\beta\left[\frac{\partial\mathcal{L}}{\partial R^{v}{}_{r\rho\beta}} - \frac{\partial\mathcal{L}}{\partial R^{v}{}_{\rho r\beta}}\right] \\ &= 2\frac{\partial\mathcal{L}}{\partial R^{v}{}_{vr\rho}} - 2\frac{\partial\mathcal{L}}{\partial R^{r}{}_{rr\rho}} + v\frac{\partial\mathcal{L}}{\partial R^{m}{}_{rr\rho}}\omega^m - v\frac{\partial\mathcal{L}}{\partial R^{v}{}_{mr\rho}}\omega_m + v\frac{\partial\mathcal{L}}{\partial R^{n}{}_{mr\rho}}h^{nl}\partial_v h_{ml} \\ &\quad + v\left(g^{\rho\lambda}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{rr}} - \frac{\partial\mathcal{L}}{\partial\Gamma^{v}{}_{r\rho}}\right) + 2vD_\beta\left[g^{\rho\lambda}\frac{\partial\mathcal{L}}{\partial R^{\lambda}{}_{rr\beta}} - \frac{\partial\mathcal{L}}{\partial R^{v}{}_{\rho r\beta}}\right] \end{aligned}$$

$$Q_{\mathcal{L}}^{r\rho}\Big|_{\text{gravity}} = \tilde{Q}^{r\rho} + vW_v^{r\rho},$$

$$W_v{}^{ri} = \partial_v J^i{}_{(1)} + \mathcal{O}(\epsilon^2)$$

$$\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}vW_v^{rv}\right) = (1 + v\partial_v)W_v^{rv}$$
$$\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}vW_v^{ri}\right) = v\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}J_{(1)}^i\right)\right) + \mathcal{O}(\epsilon^2)$$

$$D_\rho Q_{\mathcal{L}}^{r\rho}\Big|_{\text{gravity}} = \frac{1}{\sqrt{h}}\partial_\rho\left(\sqrt{h}Q_{\mathcal{L}}^{r\rho}\right)\Big|_{\text{gravity}} = \frac{1}{\sqrt{h}}\partial_\rho\left(\sqrt{h}\tilde{Q}^{r\rho}\right) + (1 + v\partial_v)W_v^{rv}$$
$$+v\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}J_{(1)}^i\right)\right) + \mathcal{O}(\epsilon^2)$$

$$E_{\mathcal{L}}^{r\alpha\beta\nu} = \frac{1}{2}\left(g^{\alpha\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{\beta r v}} + g^{\alpha\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{v r\beta}} + g^{\beta\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{\alpha r v}} + g^{\beta\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{v r\alpha}} - g^{\nu\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{\beta r\alpha}} - g^{\nu\lambda}\frac{\partial\mathcal{L}}{\partial R^\lambda{}_{\alpha r\beta}}\right)$$

$$\Theta_{\mathcal{L}}^r\big|_{\text{gravity}} = -(1 + v\partial_v)\left(E_{\mathcal{L}}^{rmnr}\partial_r h_{mn}\right) + v\partial_v^2\left(J_{(1)}^{mn}\partial_r h_{mn}\right) + \mathcal{O}(\epsilon^2)$$
$$= (1 + v\partial_v)\mathcal{A}_{(1)} + v\partial_v^2\mathcal{B}_{(0)} + \mathcal{O}(\epsilon^2)$$

$$E_{\mathcal{L}}^{rmnr} = \frac{1}{2}\left(h^{ml}\frac{\partial\mathcal{L}}{\partial R^l{}_{rrn}} + h^{nl}\frac{\partial\mathcal{L}}{\partial R^l{}_{rrm}} - \frac{\partial\mathcal{L}}{\partial R^v{}_{rrm}} - \frac{\partial\mathcal{L}}{\partial R^v{}_{mrn}}\right).$$

$$2vE_{vv} = \frac{1}{\sqrt{h}}\partial_\rho\left(\sqrt{h}Q_{\mathcal{L}}^{r\rho}\right) - \Theta_{\mathcal{L}}^r + \Xi_{\mathcal{L}}^r - G_{\mathcal{L}}^r\left(A_\rho\xi^\rho + \Lambda\right)$$
$$= v\partial_v\left(W_v{}^{rv} + w_v{}^{rv} - \mathcal{A}_{(1)} + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left(J^i{}_{(1)} + j^i{}_{(1)}\right)\right) - \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{B}_{(0)}\right)\right)$$
$$+\frac{1}{\sqrt{h}}\partial_\rho\left(\sqrt{h}\left(\tilde{Q}^{r\rho} + \tilde{q}^{r\rho}\right)\right) + W_v{}^{rv} + w_v{}^{rv} - \mathcal{A}_{(1)} + \mathcal{O}(\epsilon^2)$$

$$\mathcal{A}_{(1)} = W_v{}^{rv} + w_v{}^{rv} + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left(\tilde{Q}^{rv} + \tilde{q}^{rv}\right)\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left(\tilde{Q}^{ri} + \tilde{q}^{ri}\right)\right) + \mathcal{O}(\epsilon^2)$$
$$2E_{vv} = \partial_v\left(W_v{}^{rv} + w_v{}^{rv} - \mathcal{A}_{(1)} + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left(J^i{}_{(1)} + j^i{}_{(1)}\right)\right) - \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\mathcal{B}_{(0)}\right)\right) + \mathcal{O}(\epsilon^2)$$

$$2E_{vv} = -\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left(\tilde{Q}^{rv} + \tilde{q}^{rv} + \mathcal{B}_{(0)}\right)\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left(\tilde{Q}^{ri} + \tilde{q}^{ri} - J^i{}_{(1)} - j^i{}_{(1)}\right)\right)\right) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \tilde{q}^{rv} + \mathcal{B}_{(0)}\right), \mathcal{J}^i = -\frac{1}{2}\left(\tilde{Q}^{ri} + \tilde{q}^{ri} - J_{(1)}^i - j_{(1)}^i\right)$$

$$\Theta^r = (1 + v\partial_v)\mathcal{A}_{(1)} + v\partial_v^2\mathcal{B}_{(0)} + \mathcal{O}(\epsilon^2)$$
$$Q^{rv} = \tilde{Q}^{rv} + vW_v^{rv}, Q^{ri} = \tilde{Q}^{ri} + vW_v^{ri}, W_v^{ri} = \partial_v J_{(1)}^i + \mathcal{O}(\epsilon^2)$$
$$\Xi^r = \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\tilde{q}^{rv}\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\tilde{q}^{ri}\right) + (1 + v\partial_v)w_v^{rv} + v\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}j_{(1)}^i\right)\right) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \tilde{q}^{rv} + \mathcal{B}_{(0)}\right), \mathcal{J}^i = -\frac{1}{2}\left(\tilde{Q}^{ri} + \tilde{q}^{ri} - J^i_{(1)} - j^i_{(1)}\right)$$

$$L = \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda$$

$$G^\mu = 2\epsilon^{\mu\nu\lambda} F_{\nu\lambda}, \Theta^\mu = 2\epsilon^{\mu\nu\lambda} A_\lambda \delta A_\nu, \Xi^\mu = \epsilon^{\mu\nu\lambda} F_{\nu\lambda} \Lambda, Q^{\mu\nu} = 2\epsilon^{\mu\nu\lambda} A_\lambda [A_\alpha \xi^\alpha + \Lambda]$$

$$\Longrightarrow D_\rho Q^{r\rho} - \Theta^r + \Xi^r = \mathcal{O}(\epsilon^2)$$

$$\mathcal{L} = \epsilon^{\lambda\mu\nu} \Gamma^\rho{}_{\lambda\sigma} \left(\partial_\mu \Gamma^\sigma{}_{\rho\nu} + \frac{2}{3} \Gamma^\sigma{}_{\mu\tau} \Gamma^\tau{}_{\rho\nu}\right)$$

$$\Theta^\mu = \epsilon^{\alpha\rho\sigma} R^{\mu\beta}{}_{\rho\sigma} \delta g_{\alpha\beta} + \epsilon^{\lambda\mu\nu} \Gamma^\beta{}_{\lambda\alpha} \delta \Gamma^\alpha{}_{\beta\nu}, \Xi^\mu = \epsilon^{\mu\nu\lambda}\left(\partial_\nu \Gamma^\sigma{}_{\rho\lambda}\right)\left(\partial_\sigma \xi^\rho\right),$$
$$Q^{\mu\nu} = \epsilon^{\nu\rho\sigma} R^{\mu\lambda}{}_{\rho\sigma} \xi_\lambda - \epsilon^{\mu\rho\sigma} R^\nu{}_{\lambda\rho\sigma} \xi^\lambda + \epsilon^{\lambda\rho\sigma} R^{\mu\nu}{}_{\rho\sigma} \xi_\lambda + \epsilon^{\mu\nu\lambda} \Gamma^\rho{}_{\lambda\sigma} D_\rho \xi^\sigma.$$

$$\Xi^\mu = \epsilon^{\mu\nu\lambda}\left(\partial_\nu \Gamma^\sigma_{\rho\lambda}\right)\left(\partial_\sigma \xi^\rho\right) + D_\nu L^{\mu\nu} \to \Xi^\mu + D_\nu L^{\mu\nu}$$
$$Q^{\mu\nu} = \epsilon^{\nu\rho\sigma} R^{\mu\lambda}_{\rho\sigma} \xi_\lambda - \epsilon^{\mu\rho\sigma} R^\nu_{\lambda\rho\sigma} \xi^\lambda + \epsilon^{\lambda\rho\sigma} R^{\mu\nu}_{\rho\sigma} \xi_\lambda + \epsilon^{\mu\nu\lambda} \Gamma^\rho_{\lambda\sigma} D_\rho \xi^\sigma - L^{\mu\nu} \to Q^{\mu\nu} - L^{\mu\nu}$$
where $L^{\mu\nu} = \frac{1}{2} \xi^\lambda \left(\epsilon^{\nu\rho\sigma} \partial_\rho \Gamma^\mu_{\sigma\lambda} - \epsilon^{\mu\rho\sigma} \partial_\rho \Gamma^\nu_{\sigma\lambda}\right).$

$$\Theta^r|_{r=0} = (1 + v\partial_v)\left[\epsilon^{rvx} \frac{\partial_v h}{2h}\left(\omega + \frac{\partial_x h}{2h}\right)\right] + \mathcal{O}(\epsilon^2), \Longrightarrow \mathcal{B}_{(0)} = 0$$
$$\Xi^r|_{r=0} = -\frac{1}{\sqrt{h}} \partial_v\left(\sqrt{h} \epsilon^{rvx} \omega\right), \Longrightarrow \tilde{q}^{rv} = -\epsilon^{rvx} \omega, \tilde{q}^{rx} = 0, j^x_{(1)} = 0$$
$$\tilde{Q}^{rv} = -\epsilon^{rvx} \omega, \tilde{Q}^{rx} = 0, W^{rx}_v = \partial_v \left[-\frac{2\epsilon^{rvx} \partial_v h}{h}\right] + \mathcal{O}(\epsilon^2) \Longrightarrow J^x_{(1)} = -\frac{2}{h} \epsilon^{rvx} \partial_v h$$
$$W^{rv}_v = \epsilon^{rvx}\left(2\partial_v \omega + \frac{\omega}{2h} \partial_v h + \frac{1}{4h^2}(\partial_v h)(\partial_x h)\right)$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \tilde{q}^{rv} + \mathcal{B}_{(0)}\right) = \epsilon^{rvx} \omega$$
$$\mathcal{J}^x = -\frac{1}{2}\left(\tilde{Q}^{rx} + \tilde{q}^{rx} - J^x_{(1)} - j^x_{(1)}\right) = -\epsilon^{rvx} \frac{\partial_v h}{h}$$

$$s_{IWT} = -2\left(\frac{\partial L}{\partial R^v_{vrv}} - \frac{\partial L}{\partial R^r_{rrv}}\right) = \epsilon^{rvx} \omega = \mathcal{J}^v$$

$$L = \epsilon^{\mu\nu\rho\sigma\delta} A_\mu R^\alpha_{\beta\nu\rho} R^\beta_{\alpha\sigma\delta}$$

$$L = \mathcal{L} + D_\mu\left(U^{\mu\nu} A_\nu\right)$$

$$\mathcal{L} = 2\epsilon^{\mu\nu\lambda\rho\sigma} F_{\mu\nu} \Gamma^\alpha{}_{\lambda\beta}\left(\frac{1}{2} R^\beta{}_{\alpha\rho\sigma} - \frac{1}{3} \Gamma^\beta{}_{\rho\tau} \Gamma^\tau{}_{\alpha\sigma}\right)$$

$$U^{\mu\nu} = -4\epsilon^{\mu\nu\lambda\rho\sigma} \Gamma^\alpha_{\lambda\beta}\left(\frac{1}{2} R^\beta_{\alpha\rho\sigma} - \frac{1}{3} \Gamma^\beta_{\rho\tau} \Gamma^\tau_{\alpha\sigma}\right)$$

$$\Theta^\mu = -U^{\mu\nu}\delta A_\nu + 2\epsilon^{\mu\nu\tau\rho\sigma}F_{\rho\sigma}\Gamma^\beta_{\tau\alpha}\delta\Gamma^\alpha_{\beta\nu} + 2\epsilon^{\rho\alpha\beta\nu\lambda}R^{\mu\sigma}_{\alpha\beta}F_{\nu\lambda}\delta g_{\rho\sigma}$$
$$\Xi^\mu = 2\epsilon^{\mu\nu\lambda\rho\sigma}F_{\rho\sigma}(\partial_\nu\Gamma^\alpha_{\beta\lambda})(\partial_\alpha\xi^\beta)$$
$$Q^{\mu\nu} = -U^{\mu\nu}(A_\eta\xi^\eta + \Lambda) + 2\epsilon^{\mu\nu\lambda\rho\sigma}F_{\rho\sigma}\Gamma^\alpha_{\lambda\beta}D_\alpha\xi^\beta + 2\xi_\lambda F_{\rho\sigma}\left(\epsilon^{\nu\alpha\beta\rho\sigma}R^{\mu\lambda}_{\alpha\beta} - \epsilon^{\mu\alpha\beta\rho\sigma}R^{\nu\lambda}_{\alpha\beta} + \epsilon^{\lambda\alpha\beta\rho\sigma}R^{\mu\nu}_{\alpha\beta}\right)$$

$$\Theta^\mu \to \Theta^\mu + D_\nu M_1^{\mu\nu}[\delta g_{\alpha\beta}],$$
$$\Xi^\mu \to \Xi^\mu + D_\nu M_2^{\mu\nu}[\xi],$$
$$Q^{\mu\nu} \to Q^{\mu\nu} + M_1^{\mu\nu}[\mathcal{L}_\xi g_{\alpha\beta}] - M_2^{\mu\nu}[\xi],$$
$$\text{where } M_1^{\mu\nu}[\delta g_{\alpha\beta}] = 4\epsilon^{\mu\nu\tau\rho\sigma}A_\rho\Gamma^\beta_{\tau\alpha}\delta\Gamma^\alpha_{\beta\sigma},$$
$$M_2^{\mu\nu}[\xi] = \xi^\lambda F_{\rho\sigma}\left[\epsilon^{\nu\alpha\beta\rho\sigma}\partial_\alpha\Gamma^\mu_{\beta\lambda} - \epsilon^{\mu\alpha\beta\rho\sigma}\partial_\alpha\Gamma^\nu_{\beta\lambda}\right].$$

$$\Theta^r|_{r=0} = -U^{rv}\partial_v(vA_v + \Lambda) + (1 + v\partial_v)\left[\epsilon^{rvijk}F_{jk}(\omega^l\partial_v h_{il} + 2\Gamma^m_{in}\Gamma^n_{mv})\right] + \mathcal{O}(\epsilon^2)$$
$$\Longrightarrow \mathcal{B}_{(0)} = 0$$
$$\Xi^r|_{r=0} = \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[-2\epsilon^{rvijk}F_{jk}\omega_i\right]\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[-4\epsilon^{rvijk}F_{vj}\omega_k\right]\right)$$
$$\Longrightarrow \tilde{q}^{rv} = -2\epsilon^{rvijk}F_{jk}\omega_i, \tilde{q}^{ri} = -4\epsilon^{rvijk}F_{vj}\omega_k, j^i_{(1)} = 0$$
$$Q^{rv} = \tilde{Q}^{rv} + vW_v^{rv} - U^{rv}(vA_v + \Lambda)$$
$$\tilde{Q}^{rv} = -2\epsilon^{rvijk}F_{jk}\omega_i, \tilde{Q}^{ri} = -4\epsilon^{rvijk}F_{vk}\omega_j$$
$$W_v^{ri} = -4\partial_v\left(h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml}\right) + \mathcal{O}(\epsilon^2) \Longrightarrow J^i_{(1)} = -4h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml}$$
$$W_v^{rv} = 8\epsilon^{rvijk}\left(F_{jk}R_{vrvi} + F_{vi}R_{vrjk}\right) + 2\epsilon^{rvijk}F_{jk}\Gamma^\alpha_{i\beta}\Gamma^\beta_{\alpha v}$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \tilde{q}^{rv} + \mathcal{B}_{(0)}\right) = 2\epsilon^{rvijk}F_{jk}\omega_i$$
$$\mathcal{J}^i = -\frac{1}{2}\left(\tilde{Q}^{ri} + \tilde{q}^{ri} - J^i_{(1)} - j^i_{(1)}\right) = 4\epsilon^{rvijk}F_{vk}\omega_j - 2h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml}$$

$$s_{IWT} = -2\left(\frac{\partial L}{\partial R^v_{vrv}} - \frac{\partial L}{\partial R^r_{rrv}}\right) = 2\epsilon^{rvijk}F_{jk}\omega_i = \mathcal{J}^v$$

$$L = L\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, D_{\alpha_1}R_{\mu\nu\rho\sigma}, D_{(\alpha_1}D_{\alpha_2)}R_{\mu\nu\rho\sigma}, \ldots, \phi, D_{\alpha_1}\phi, D_{(\alpha_1}D_{\alpha_2)}\phi, \ldots, F_{\mu\nu}, D_{\alpha_1}F_{\mu\nu}, \ldots\right).$$

$$t^{(k)}_{(a+1)}\Big|_{r=0} = \tilde{T}_{(-k)}\partial_v^{k+a+1}T_{(0)}\Big|_{r=0} + \mathcal{O}(\epsilon^2).$$

$$t^{(k)}_{(a+1)} = \partial_v^{a+1}\left[\sum_{m=0}^{k-1}(-1)^m[\,^{m+a}C_m]\tilde{T}_{(-k+m)}\partial_v^{(k-m)}T_{(0)}\right] + (-1)^k[\,^{k+a}C_a]\tilde{T}_{(0)}\partial_v^{a+1}T_{(0)} + \mathcal{O}(\epsilon^2)$$

$$\delta S_{\alpha_1\alpha_2\ldots\alpha_k}[\delta g_{\alpha\beta} \to \mathcal{L}_\xi g_{\alpha\beta}] = \mathcal{L}_\xi S_{\alpha_1\alpha_2\ldots\alpha_k}.$$

$$\Theta^\mu = 2E_R^{\mu\nu\alpha\beta}D_\beta(\mathcal{L}_\xi g_{\nu\alpha}) + \sum_k \mathcal{T}^{\mu\alpha_1\alpha_2\ldots\alpha_k}\mathcal{L}_\xi S_{\alpha_1\ldots\alpha_k}.$$

$$E_R^{\mu\nu\alpha\beta} = \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}} - D_{\rho_1}\frac{\partial L}{\partial D_{\rho_1}R_{\mu\nu\alpha\beta}} + \cdots + (-1)^m D_{(\rho_1}\ldots D_{\rho_m)}\frac{\partial L}{\partial D_{(\rho_1}\ldots D_{\rho_m)}R_{\mu\nu\alpha\beta}},$$

$$\Theta^r|_{r=0} = (1 + v\partial_v)\mathcal{A}_{(1)} + v\partial_v^2\mathcal{B}_{(0)}.$$

$$\mathcal{B}_{(0)} \sim X_{(-k+m)}\partial_v^{k-m}Y_{(0)} \sim \mathcal{O}(\epsilon)$$

$$Q^{\mu\nu} = W^{\mu\nu\rho}\xi_\rho - 2E_R^{\mu\nu\alpha\beta}D_{[\alpha}\xi_{\beta]}$$

$$Q^{r\rho} = \tilde{Q}^{r\rho} + vW_v^{r\rho}$$

$$D_\mu Q^{r\mu} = \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\tilde{Q}^{rv}\right) + \nabla_i\tilde{Q}^{ri} + v\partial_v\left(\nabla_i J_{(1)}^i\right) + (1 + v\partial_v)W_v^{rv} + \mathcal{O}(\epsilon^2),$$

$$W_v^{ri} = \partial_v J_{(1)}^i + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned} 2vE_{vv} &= \left(-\Theta^r + D_\mu Q^{r\mu}\right)\big|_{r=0} \\ &= -\mathcal{A}_{(1)} + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\tilde{Q}^{rv}\right) + \nabla_i\tilde{Q}^{ri} + W_v^{rv} \\ &\quad + v\partial_v\left[-\mathcal{A}_{(1)} + W_v^{rv} + \nabla_i J_{(1)}^i - \partial_v\mathcal{B}_{(0)}\right] + \mathcal{O}(\epsilon^2) \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{(1)} &= \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\tilde{Q}^{rv}\right) + \nabla_i\tilde{Q}^{ri} + W_v^{rv} \\ 2E_{vv} &= \partial_v\left[-\mathcal{A}_{(1)} + W_v^{rv} + \nabla_i J_{(1)}^i - \partial_v\mathcal{B}_{(0)}\right] \end{aligned}$$

$$2E_{vv}|_{r=0} = -\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left[\sqrt{h}\left(\tilde{Q}^{rv} + \mathcal{B}_{(0)}\right)\right] + \nabla_i\left[\tilde{Q}^{ri} - J_{(1)}^i\right]\right) + \mathcal{O}(\epsilon^2).$$

$$\partial_v^2\mathcal{B}_{(0)} = \partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left[\sqrt{h}\mathcal{B}_{(0)}\right]\right) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{J}^v = -\frac{1}{2}\left(\tilde{Q}^{rv} + \mathcal{B}_{(0)}\right), \text{ and } \mathcal{J}^i = -\frac{1}{2}\left(\tilde{Q}^{ri} - J_{(1)}^i\right)$$

$$2vE_{vv}|_{r=0} = \left(-\Theta^r + D_\mu Q^{r\mu}\right)\big|_{r=0} + \mathcal{O}(\epsilon^2) = \mathcal{U}_{(1)} + v\mathcal{V}_{(2)} + \mathcal{O}(\epsilon^2)$$

$$\mathcal{U}_{(1)} = \mathcal{O}(\epsilon^2),\ 2E_{vv}|_{r=0} = \mathcal{V}_{(2)} + \mathcal{O}(\epsilon^2)$$

$$\psi(\lambda) := \bar{\psi} + \lambda\delta\psi$$

$$\partial_v\left(\frac{1}{\sqrt{\mu}}\frac{\partial}{\partial\lambda}\left(*\,Q_K^{vr}\sqrt{\mu}\right) - v(*\,\theta^r)\left[g; \frac{\partial}{\partial\lambda}g\right]\right)\bigg|_{\lambda=0} + D_A\frac{\partial}{\partial\lambda}\left(*\,Q_K^{Ar}\right)\bigg|_{\lambda=0} = 2v\frac{\partial}{\partial\lambda}E_{vv}\bigg|_{\lambda=0}$$

$$\begin{aligned}
\Theta^r =&(1+v\partial_v)\left(8R_{vijr}K^{ij}-8\bar{K}^{ij}\partial_v K_{ij}\right)+v\partial_v^2\left(8\bar{K}^{ij}K_{ij}\right)+\mathcal{O}(\epsilon^2)\\
Q^{rv} =&8R_{rvrv}-8v\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}R_{rvrv}\right)-4v\nabla_i\left(h^{ij}\partial_v\omega_j+K_k^i\omega^k\right)\\
&+8vK^{ij}R_{vijr}-8v\bar{K}^{ij}\partial_v K_{ij}+\mathcal{O}(\epsilon^2)\\
Q^{ri} =&4h^{ij}\partial_v\omega_j+4K_k^i\omega^k+4v\partial_v\left(K_k^i\omega^k\right)\\
&+v\partial_v\left(-4h^{ij}\partial_v\omega_j+8\nabla_m K^{im}\right)+\mathcal{O}(\epsilon^2)\\
D_\mu Q^{r\mu} =&\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}8R_{rvrv}\right)+\nabla_i\left(4h^{ij}\partial_v\omega_j+4K_k^i\omega^k\right)\\
&+v\partial_v\left[\nabla_i\left(4K_k^i\omega^k-4h^{ij}\partial_v\omega_j+8\nabla_m K^{im}\right)\right]\\
&+(1+v\partial_v)\left[-8\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}R_{rvrv}\right)-4\nabla_i\left(h^{ij}\partial_v\omega_j+K_k^i\omega^k\right)\right.\\
&\left.+8K^{ij}R_{vijr}-8\bar{K}^{ij}\partial_v K_{ij}\right]+\mathcal{O}(\epsilon^2)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}^{rv} =&8R_{rvrv}\,,\tilde{Q}^{ri}=4h^{ij}\partial_v\omega_j+4K_k^i\omega^k,\\
J_{(1)}^i =&4K_k^i\omega^k-4h^{ij}\partial_v\omega_j+8\nabla_m K^{im},\\
W_v^{rv} =&-8\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}R_{rvrv}\right)-4\nabla_i\left(h^{ij}\partial_v\omega_j+K_k^i\omega^k\right)\\
&+8K^{ij}R_{vijr}-8\bar{K}^{ij}\partial_v K_{ij},\\
\mathcal{A}_{(1)} =&8R_{vijr}K^{ij}-8\bar{K}^{ij}\partial_v K_{ij}\,,\mathcal{B}_{(0)}=8\bar{K}^{ij}K_{ij}
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_{(1)} =&-8R_{vijr}K^{ij}+8\bar{K}^{ij}\partial_v K_{ij}+\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}8R_{rvrv}\right)+\nabla_i\left(4h^{ij}\partial_v\omega_j+4K_k^i\omega^k\right)\\
&-8\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}R_{rvrv}\right)-4\nabla_i\left(h^{ij}\partial_v\omega_j+K_k^i\omega^k\right)+8K^{ij}R_{vijr}-8\bar{K}^{ij}\partial_v K_{ij}\\
=&0\\
\mathcal{V}_{(2)} =&\partial_v\left[\frac{1}{\sqrt{h}}\partial_v\left[\sqrt{h}\left(-8R_{rvrv}-8\bar{K}^{ij}K_{ij}\right)\right]+\nabla_i\left(-8h^{ij}\partial_v\omega_j+8\nabla_m K^{im}\right)\right]=2E_{vv}
\end{aligned}$$

$$2vE_{vv}|_{r=0}=\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}Q^{rv}\right)-\Theta^r+\Xi^r\right)\Big|_{r=0}=\mathcal{U}_{(1)}+v\mathcal{V}_{(2)}$$

$$\mathcal{L}=\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu}+\frac{2}{3}\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\rho\nu}\right)$$

$$\begin{aligned}
\Theta^r|_{r=0} &= (1+v\partial_v)\left[\epsilon^{rvx}\frac{\partial_v h}{2h}\left(\omega+\frac{\partial_x h}{2h}\right)\right]+\mathcal{O}(\epsilon^2),\\
\Xi^r|_{r=0} &= -\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\epsilon^{rvx}\omega\right),\\
Q^{rv} &= -\epsilon^{rvx}\omega+v\epsilon^{rvx}\left(2\partial_v\omega+\frac{\omega}{2h}\partial_v h+\frac{1}{4h^2}(\partial_v h)(\partial_x h)\right),\\
Q^{rx} &= v\partial_v\left[-\frac{2}{h}\epsilon^{rvx}\partial_v h\right]+\mathcal{O}(\epsilon^2).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}Q^{rv}\right)=&(1+v\partial_v)\left(\epsilon^{rvx}\left(2\partial_v\omega+\frac{\omega}{2h}\partial_v h+\frac{1}{4h^2}(\partial_v h)(\partial_x h)\right)\right)\\
&+\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}[-\epsilon^{rvx}\omega]\right)+v\partial_v\left(\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\left[-\frac{2}{h}\epsilon^{rvx}\partial_v h\right]\right)\right)+\mathcal{O}(\epsilon^2)
\end{aligned}$$

$$\mathcal{A}_{(1)} = \epsilon^{rvx}\frac{\partial_v h}{2h}\left(\omega + \frac{\partial_x h}{2h}\right), \mathcal{B}_{(0)} = 0$$
$$\tilde{q}^{rv} = -\epsilon^{rvx}\omega, \tilde{q}^{rx} = 0, j_{(1)}^{x} = 0, w_v^{rv} = 0$$
$$\tilde{Q}^{rv} = -\epsilon^{rvx}\omega, \tilde{Q}^{rx} = 0, W_v^{rx} = \partial_v\left[-\frac{2}{h}\epsilon^{rvx}\partial_v h\right] + \mathcal{O}(\epsilon^2) \Longrightarrow J_{(1)}^{x} = -\frac{2}{h}\epsilon^{rvx}\partial_v \hbar$$
$$W_v^{rv} = \epsilon^{rvx}\left(2\partial_v\omega + \frac{\omega}{2h}\partial_v h + \frac{1}{4h^2}(\partial_v h)(\partial_x h)\right)$$

$$\begin{aligned}\mathcal{U}_{(1)} =& \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}[-\epsilon^{rvx}\omega]\right) + \epsilon^{rvx}\left(2\partial_v\omega + \frac{\omega}{2h}\partial_v h + \frac{1}{4h^2}(\partial_v h)(\partial_x h)\right)\\ & -\epsilon^{rvx}\frac{\partial_v h}{2h}\left(\omega + \frac{\partial_x h}{2h}\right) - \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\epsilon^{rvx}\omega\right)\\ =& 0\end{aligned}$$
$$\begin{aligned}\mathcal{V}_{(2)} =& \partial_v\left(\epsilon^{rvx}\left(2\partial_v\omega + \frac{\omega}{2h}\partial_v h + \frac{1}{4h^2}(\partial_v h)(\partial_x h)\right) + \frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\left[-\frac{2}{h}\epsilon^{rvx}\partial_v h\right]\right)\right)\\ & -\partial_v\left(\epsilon^{rvx}\frac{\partial_v h}{2h}\left(\omega + \frac{\partial_x h}{2h}\right)\right)\\ =& \partial_v\left(\frac{1}{\sqrt{h}}\partial_x\left(\sqrt{h}\left[-\frac{2}{h}\epsilon^{rvx}\partial_v h\right]\right) + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}[2\epsilon^{rvx}\omega]\right)\right)\\ =& 2E_{vv}\end{aligned}$$

$$\mathcal{L} = 2\epsilon^{\mu\nu\lambda\rho\sigma}F_{\mu\nu}\Gamma^{\alpha}{}_{\lambda\beta}\left(\frac{1}{2}R^{\beta}{}_{\alpha\rho\sigma} - \frac{1}{3}\Gamma^{\beta}{}_{\rho\tau}\Gamma^{\tau}{}_{\alpha\sigma}\right)$$

$$\begin{aligned}\Theta^r|_{r=0} &= -U^{rv}\partial_v(vA_v + \Lambda) + (1 + v\partial_v)\left[\epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma_{in}^{m}\Gamma_{mv}^{n}\right)\right] + \mathcal{O}(\epsilon^2)\\ \Xi^r|_{r=0} &= \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[-2\epsilon^{rvijk}F_{jk}\omega_i\right]\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[-4\epsilon^{rvijk}F_{vj}\omega_k\right]\right)\\ Q^{rv} &= -2\epsilon^{rvijk}F_{jk}\omega_i + 2v\left(\epsilon^{rvijk}\left(4F_{jk}R_{vrvi} + 4F_{vi}R_{vrjk} + F_{jk}\Gamma_{i\beta}^{\alpha}\Gamma_{\alpha v}^{\beta}\right)\right) - U^{rv}(vA_v + \Lambda)\\ Q^{ri} &= -4\epsilon^{rvijk}F_{vk}\omega_j - 4v\partial_v\left(h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml}\right) + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}Q^{rv}\right) =& -U^{rv}\partial_v(vA_v + \Lambda) + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[2\epsilon^{rvijk}F_{jk}\omega_i\right]\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[4\epsilon^{rvijk}F_{vj}\omega_k\right]\right)\\ & + (1 + v\partial_v)\left(\epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma_{in}^{m}\Gamma_{mv}^{n}\right)\right)\\ & + v\partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[8\epsilon^{rvijk}\omega_j F_{vk} - 4\epsilon^{rvmjk}F_{jk}h^{in}\partial_v h_{nm}\right]\right)\right)\\ & + v\partial_v\left(\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[4\epsilon^{rvijk}F_{jk}\omega_i\right]\right)\right) + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$\mathcal{A}_{(1)} = \epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma^{m}{}_{in}\Gamma^{n}{}_{mv}\right), \mathcal{B}_{(0)} = 0,$$
$$\tilde{q}^{rv} = -2\epsilon^{rvijk}F_{jk}\omega_i, \tilde{q}^{ri} = -4\epsilon^{rvijk}F_{vj}\omega_k, j_{(1)}^{i} = 0, w_v{}^{rv} = 0,$$
$$\tilde{Q}^{rv} = -2\epsilon^{rvijk}F_{jk}\omega_i, \tilde{Q}^{ri} = -4\epsilon^{rvijk}F_{vk}\omega_j,$$
$$W_v^{ri} = -4\partial_v\left(h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml}\right) + \mathcal{O}(\epsilon^2) \Longrightarrow J_{(1)}^{i} = -4h^{im}\epsilon^{rvljk}F_{jk}\partial_v h_{ml},$$
$$W_v^{rv} = 8\epsilon^{rvijk}\left(F_{jk}R_{vrvi} + F_{vi}R_{vrjk}\right) + 2\epsilon^{rvijk}F_{jk}\Gamma_{i\beta}^{\alpha}\Gamma_{\alpha v}^{\beta}.$$

$$
\begin{aligned}
\mathcal{U}_{(1)} =& \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[2\epsilon^{rvijk}F_{jk}\omega_i\right]\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[4\epsilon^{rvijk}F_{vj}\omega_k\right]\right) \\
&+\epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma^m{}_{in}\Gamma^n{}_{mv}\right) - \epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma^m{}_{in}\Gamma^n{}_{mv}\right) \\
&+\frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[-2\epsilon^{rvijk}F_{jk}\omega_i\right]\right) + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[-4\epsilon^{rvijk}F_{vj}\omega_k\right]\right) \\
=&0 \\
\mathcal{V}_{(2)} =& \partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[8\epsilon^{rvijk}\omega_j F_{vk} - 4\epsilon^{rvmjk}F_{jk}h^{in}\partial_v h_{nm}\right]\right) + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[4\epsilon^{rvijk}F_{jk}\omega_i\right]\right)\right) \\
&+\partial_v\left(\epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma^m{}_{in}\Gamma^n{}_{mv}\right)\right) - \partial_v\left(\epsilon^{rvijk}F_{jk}\left(\omega^l\partial_v h_{il} + 2\Gamma^m{}_{in}\Gamma^n{}_{mv}\right)\right) \\
=& \partial_v\left(\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\left[8\epsilon^{rvijk}\omega_j F_{vk} - 4\epsilon^{rvmjk}F_{jk}h^{in}\partial_v h_{nm}\right]\right) + \frac{1}{\sqrt{h}}\partial_v\left(\sqrt{h}\left[4\epsilon^{rvijk}F_{jk}\omega_i\right]\right)\right) \\
=&2E_{vv}\cdot
\end{aligned}
$$

$$
\delta_\xi L = \mathcal{L}_\xi L + \frac{\partial L}{\partial A_\mu}D_\mu\Lambda + \frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda
$$

$$
\frac{\partial L}{\partial A_\mu}D_\mu\Lambda = D_\mu\left(\frac{\partial L}{\partial A_\mu}\Lambda\right) - \Lambda D_\mu\frac{\partial L}{\partial A_\mu}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\mu\partial_\nu\xi^\lambda =& \partial_\mu\left(\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda\right) - \left(\partial_\nu\xi^\lambda\right)\left(\partial_\mu\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right) \\
=& D_\mu\left(\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda\right) - \Gamma^\tau{}_{\mu\tau}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda - \left(\partial_\nu\xi^\lambda\right)\left(\partial_\mu\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right) \\
=& D_\mu\left(\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda\right) - \frac{1}{\sqrt{-g}}\left(\partial_\nu\xi^\lambda\right)\partial_\mu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right) \\
=& D_\mu\left(\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda\right) - \partial_\mu\left[\frac{\xi^\lambda}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right)\right] \\
&+\xi^\lambda\partial_\mu\left[\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right)\right] \\
=& D_\mu\left(\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\partial_\nu\xi^\lambda - \frac{1}{\sqrt{-g}}\xi^\lambda\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right)\right) + \frac{\xi^\lambda}{\sqrt{-g}}\partial_\mu\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}}\right)
\end{aligned}
$$

$$
\delta_\xi(\sqrt{-g}L) = \sqrt{-g}D_\mu(\xi^\mu L) + \sqrt{-g}D_\mu\Xi^\mu - \sqrt{-g}\Lambda D_\mu\frac{\partial L}{\partial A_\mu} + \xi^\lambda\partial_\mu\partial_\nu\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^\lambda_{\mu\nu}}\right)
$$

$$
J^\mu = 2E^{\mu\nu}\xi_\nu + G^\mu(A_\nu\xi^\nu + \Lambda) + \Theta^\mu - \xi^\mu L - \Xi^\mu
$$

$$
J^\mu = D_\nu Q^{\mu\nu}
$$

$$
\begin{aligned}
\Theta^\mu =& \left(g^{\alpha\lambda}\frac{\partial L}{\partial R^\lambda{}_{\beta\mu\nu}} + g^{\alpha\lambda}\frac{\partial L}{\partial R^\lambda{}_{\nu\mu\beta}} - g^{\nu\lambda}\frac{\partial L}{\partial R^\lambda{}_{\beta\mu\alpha}}\right)D_\nu\left(D_\alpha\xi_\beta + D_\beta\xi_\alpha\right) \\
&+2\frac{\partial L}{\partial F_{\mu\nu}}\left[\xi^\lambda F_{\lambda\nu} + D_\nu\left(A_\lambda\xi^\lambda + \Lambda\right)\right] + 2S^{\mu\alpha\beta}D_\alpha\xi_\beta
\end{aligned}
$$

$$\Theta^{\mu} = D_{\nu}\left(2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_{\beta}\xi^{\alpha}\right) - 2\left(D_{\beta}\xi^{\alpha}\right)D_{\nu}\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}} + 2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\nu\mu}}R^{\alpha}{}_{\beta\nu\eta}\xi^{\eta}$$
$$+2\frac{\partial L}{\partial F_{\mu\nu}}\left[\xi^{\lambda}F_{\lambda\nu} + D_{\nu}\left(A_{\lambda}\xi^{\lambda} + \Lambda\right)\right] + 2S^{\mu\alpha\beta}D_{\alpha}\xi_{\beta}$$

$$G^{\mu}\Lambda - \frac{\partial L}{\partial A_{\mu}}\Lambda + 2\frac{\partial L}{\partial F_{\mu\nu}}D_{\nu}\Lambda = 2D_{\nu}\left(\frac{\partial L}{\partial F_{\mu\nu}}\Lambda\right)$$

$$J^{\mu}_{\text{gauge}} = D_{\nu}\left(2\frac{\partial L}{\partial F_{\mu\nu}}\left(A_{\lambda}\xi^{\lambda} + \Lambda\right)\right) + \xi^{\lambda}\left(\frac{\partial L}{\partial A_{\mu}}A_{\lambda} + 2\frac{\partial L}{\partial F_{\mu\nu}}F_{\lambda\nu}\right).$$

$$J^{\mu}_{\text{metric}} = 2E^{\mu\alpha}\xi_{\alpha} - \xi^{\mu}L + 2S^{\mu\nu\alpha}D_{\nu}\xi_{\alpha} - \frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\partial_{\nu}\xi^{\lambda} + \frac{1}{\sqrt{-g}}\xi^{\lambda}\partial_{\nu}\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\right)$$
$$+\left(g^{\alpha\lambda}\frac{\partial L}{\partial R^{\lambda}{}_{\beta\mu\nu}} + g^{\alpha\lambda}\frac{\partial L}{\partial R^{\lambda}{}_{\nu\mu\beta}} - g^{\nu\lambda}\frac{\partial L}{\partial R^{\lambda}{}_{\beta\mu\alpha}}\right)D_{\nu}\left(D_{\alpha}\xi_{\beta} + D_{\beta}\xi_{\alpha}\right)$$
$$= 2S^{\mu\nu\alpha}D_{\nu}\xi_{\alpha} - 2\xi_{\alpha}D_{\nu}S^{\nu\mu\alpha} + D_{\nu}\left(2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_{\beta}\xi^{\alpha}\right) - 2\left(D_{\beta}\xi^{\alpha}\right)D_{\nu}\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}$$
$$+2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\nu\mu}}R^{\alpha}{}_{\beta\nu\eta}\xi^{\eta} - \frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\partial_{\nu}\xi^{\lambda} + \frac{1}{\sqrt{-g}}\xi^{\lambda}\partial_{\nu}\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\right) - 2g^{\mu\nu}\frac{\partial L}{\partial g^{\nu\lambda}}\xi^{\lambda}$$
$$= D_{\nu}\left(2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_{\beta}\xi^{\alpha} + \xi_{\alpha}\left(S^{\mu\nu\alpha} - S^{\nu\mu\alpha}\right)\right) + \left(S^{\mu\nu\alpha} + S^{\nu\mu\alpha}\right)D_{\nu}\xi_{\alpha} - \xi_{\alpha}D_{\nu}\left(S^{\mu\nu\alpha} + S^{\nu\mu\alpha}\right)$$
$$-2\left(D_{\beta}\xi^{\alpha}\right)D_{\nu}\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}} + 2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\nu\mu}}R^{\alpha}{}_{\beta\nu\eta}\xi^{\eta} - \frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\partial_{\nu}\xi^{\lambda}$$
$$+\frac{1}{\sqrt{-g}}\xi^{\lambda}\partial_{\nu}\left(\sqrt{-g}\frac{\partial L}{\partial \Gamma^{\lambda}{}_{\mu\nu}}\right) - 2g^{\mu\nu}\frac{\partial L}{\partial g^{\nu\lambda}}\xi^{\lambda}$$

$$S^{\mu\nu\alpha} + S^{\nu\mu\alpha} = g^{\alpha\lambda}\left(\frac{\partial L}{\partial \Gamma^{\lambda}_{\nu\mu}} + D_{\beta}\left[\frac{\partial L}{\partial R^{\lambda}_{\nu\mu\beta}} + \frac{\partial L}{\partial R^{\lambda}_{\mu\nu\beta}}\right]\right)$$

$$J^{\mu}_{\text{metric}} = D_{\nu}\left(2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_{\beta}\xi^{\alpha} + \xi_{\alpha}\left(S^{\mu\nu\alpha} - S^{\nu\mu\alpha}\right) + \xi^{\lambda}D_{\beta}\left[\frac{\partial L}{\partial R^{\lambda}{}_{\mu\nu\beta}} - \frac{\partial L}{\partial R^{\lambda}{}_{\nu\mu\beta}}\right]\right)$$
$$+\xi^{\lambda}\left(2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\nu\mu}}R^{\alpha}{}_{\beta\nu\lambda} - R^{\mu}{}_{\eta\alpha\beta}\frac{\partial L}{\partial R^{\lambda}{}_{\eta\alpha\beta}} - R_{\alpha\beta\lambda}{}^{\eta}\frac{\partial L}{\partial R^{\eta}{}_{\mu\alpha\beta}}\right)$$
$$+\xi^{\lambda}\left(2\frac{\partial L}{\partial \Gamma^{\eta}{}_{\mu\nu}}\Gamma^{\eta}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\nu\eta}\frac{\partial L}{\partial \Gamma^{\lambda}{}_{\eta\nu}} - 2g^{\mu\nu}\frac{\partial L}{\partial g^{\nu\lambda}}\right)$$

$$J^{\mu} = D_{\nu}Q^{\mu\nu} + \xi^{\lambda}B^{\mu}_{\lambda}$$

$$Q^{\mu\nu} = 2\frac{\partial L}{\partial F_{\mu\nu}}\left(A_{\lambda}\xi^{\lambda} + \Lambda\right) + 2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\mu\nu}}D_{\beta}\xi^{\alpha} + \xi_{\alpha}\left(S^{\mu\nu\alpha} - S^{\nu\mu\alpha}\right) + \xi^{\lambda}D_{\beta}\left[\frac{\partial L}{\partial R^{\lambda}{}_{\mu\nu\beta}} - \frac{\partial L}{\partial R^{\lambda}{}_{\nu\mu\beta}}\right].$$

$$B^{\mu}_{\lambda} = 2\frac{\partial L}{\partial R^{\alpha}{}_{\beta\nu\mu}}R^{\alpha}{}_{\beta\nu\lambda} - R^{\mu}_{\eta\alpha\beta}\frac{\partial L}{\partial R^{\lambda}{}_{\eta\alpha\beta}} - R_{\alpha\beta\lambda}{}^{\eta}\frac{\partial L}{\partial R^{\eta}{}_{\mu\alpha\beta}} + 2\frac{\partial L}{\partial \Gamma^{\eta}{}_{\mu\nu}}\Gamma^{\eta}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\nu\eta}\frac{\partial L}{\partial \Gamma^{\lambda}{}_{\eta\nu}}$$
$$+\frac{\partial L}{\partial A_{\mu}}A_{\lambda} + 2\frac{\partial L}{\partial F_{\mu\nu}}F_{\lambda\nu} - 2g^{\mu\nu}\frac{\partial L}{\partial g^{\nu\lambda}}$$

$$\frac{\partial L_n}{\partial A_\mu} = \sum_{i=1}^{n} \mathcal{L}^{\nu_1\cdots\nu_{i-1}\mu\nu_{i+1}\cdots\nu_n} \prod_{j=1, j\neq i}^{n} A_{\nu_j} = n\mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}} \prod_{i=1}^{n-1} A_{\nu_i}$$

$$\begin{aligned} D_\mu \frac{\partial L_n}{\partial A_\mu} &= n\left(\prod_{i=1}^{n-1} A_{\nu_i}\right)\left(D_\mu \mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}}\right) + n\mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}} \sum_{j=1}^{n-1}\left(\left(D_\mu A_{\nu_j}\right)\left(\prod_{i=1, i\neq j}^{n-1} A_{\nu_i}\right)\right) \\ &= n\left(\prod_{i=1}^{n-1} A_{\nu_i}\right)\left(D_\mu \mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}}\right) + n\left(D_\mu A_\nu\right)\left(\sum_{j=1}^{n-1} \mathcal{L}^{\mu\nu_1\cdots\nu_{j-1}\nu\nu_j\cdots\nu_{n-2}}\left(\prod_{i=1}^{n-2} A_{\nu_i}\right)\right) \\ &= n\left(\prod_{i=1}^{n-1} A_{\nu_i}\right)\left(D_\mu \mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}}\right) + n(n-1)\mathcal{L}^{\mu\nu\nu_1\cdots\nu_{n-2}}\left(D_\mu A_\nu\right)\left(\prod_{i=1}^{n-2} A_{\nu_i}\right) \end{aligned}$$

$$D_\mu \frac{\partial L_n}{\partial A_\mu} = 0 \Longrightarrow nD_\mu \mathcal{L}^{\mu\nu_1\cdots\nu_{n-1}} = 0 \text{ and } n(n-1)\mathcal{L}^{\mu\nu\nu_1\cdots\nu_{n-2}} = 0$$

$$\mathcal{L}^\mu = \sum_n \mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n} \prod_{i=1}^{n} F_{\rho_i\sigma_i}$$

$$\begin{aligned} D_\mu \mathcal{L}^\mu &= \sum_n \left(D_\mu \mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n}\right)\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right) + \sum_n \sum_{j=1}^{n} \mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n}\left(D_\mu F_{\rho_j\sigma_j}\right)\left(\prod_{i=1, i\neq j}^{n} F_{\rho_i\sigma_i}\right) \\ &= \sum_n \left(D_\mu \mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n}\right)\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right) + \sum_n n\mathcal{C}^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_{n-1}\sigma_{n-1}}\left(D_\mu F_{\nu\lambda}\right)\left(\prod_{i=1}^{n-1} F_{\rho_i\sigma_i}\right) \end{aligned}$$

$$D_\mu \mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n} = 0 \text{ and } \mathcal{C}^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_{n-1}\sigma_{n-1}} \text{ is antisymmetric in } (\mu,\nu,\lambda)$$

$$\mathcal{C}^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n} = \begin{cases} D_\nu \mathcal{B}^{\nu\mu\rho_1\sigma_1\cdots\rho_n\sigma_n} & \text{(i) } 2n+1 < D \\ a_g \epsilon^{\mu\rho_1\sigma_1\cdots\rho_n\sigma_n} & \text{(ii) } 2n+1 = D \end{cases}$$

$$\begin{aligned} \mathcal{L}^\mu A_\mu &= A_\nu \sum_{n=0}^{N} \left(D_\mu \mathcal{B}^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n}\right)\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right) = A_\nu D_\mu\left[\sum_{n=0}^{N} \mathcal{B}^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n}\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right)\right] \\ &= D_\mu\left[A_\nu \sum_{n=0}^{N} \mathcal{B}^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n}\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right)\right] - \frac{1}{2}\left(F_{\mu\nu}\right)\sum_{n=0}^{N} \mathcal{B}^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n}\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right) \end{aligned}$$

$$\delta\left(\sqrt{-g}D_\mu \mathcal{L}^\mu\right) = \partial_\mu \delta\left(\sqrt{-g}\mathcal{L}^\mu\right) = \sqrt{-g}D_\mu\left(\delta\mathcal{L}^\mu + \frac{1}{2}\mathcal{L}^\mu g^{\alpha\beta}\delta g_{\alpha\beta}\right)$$

$$\delta\mathcal{L}^\mu + \frac{1}{2}\mathcal{L}^\mu g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\delta\mathcal{L}^\mu\right) = \mathcal{E}^{\mu\rho\sigma}\delta g_{\rho\sigma} + \mathcal{G}^{\mu\nu}\delta A_\nu + D_\nu \theta^{\mu\nu}$$

$$\mathcal{G}^{\mu\nu}=\frac{\partial\mathcal{L}^{\mu}}{\partial A_{\nu}}+2D_{\lambda}\frac{\partial\mathcal{L}^{\mu}}{\partial F_{\nu\lambda}},\mathcal{E}^{\mu\rho\sigma}=\frac{1}{2}\mathcal{L}^{\mu}g^{\rho\sigma}-g^{\rho\alpha}g^{\sigma\beta}\frac{\partial\mathcal{L}^{\mu}}{\partial g^{\alpha\beta}}-D_{\alpha}\mathcal{S}^{\mu\alpha\rho\sigma}$$

$$\theta^{\mu\nu}=2\frac{\partial\mathcal{L}^{\mu}}{\partial F_{\nu\lambda}}\delta A_{\lambda}+2\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\beta\nu\lambda}}\delta\Gamma^{\alpha}{}_{\beta\lambda}+\mathcal{S}^{\mu\nu\rho\sigma}\delta g_{\rho\sigma}$$

$\therefore$

$$\mathcal{S}^{\mu\nu\rho\sigma}=\frac{1}{2}D_{\beta}\left[g^{\rho\alpha}\left(\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\sigma\nu\beta}}+\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\nu\sigma\beta}}\right)+g^{\sigma\alpha}\left(\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\rho\nu\beta}}+\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\nu\rho\beta}}\right)\right.$$
$$\left.-g^{\nu\alpha}\left(\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\sigma\rho\beta}}+\frac{\partial\mathcal{L}^{\mu}}{\partial R^{\alpha}{}_{\rho\sigma\beta}}\right)\right]+\frac{1}{2}\left(g^{\rho\alpha}\frac{\partial\mathcal{L}^{\mu}}{\partial\Gamma^{\alpha}{}_{\sigma\nu}}+g^{\sigma\alpha}\frac{\partial\mathcal{L}^{\mu}}{\partial\Gamma^{\alpha}{}_{\rho\nu}}-g^{\nu\alpha}\frac{\partial\mathcal{L}^{\mu}}{\partial\Gamma^{\alpha}{}_{\sigma\rho}}\right)$$

$$\Theta_t^{\mu}=\delta\mathcal{L}^{\mu}+\frac{1}{2}\mathcal{L}^{\mu}g^{\alpha\beta}\delta g_{\alpha\beta}=\mathcal{E}^{\mu\rho\sigma}\delta g_{\rho\sigma}+\mathcal{G}^{\mu\nu}\delta A_{\nu}+D_{\nu}\theta^{\mu\nu}$$

$$\mathcal{L}_{\xi}\mathcal{L}^{\mu}=\xi^{\nu}D_{\nu}\mathcal{L}^{\mu}-\mathcal{L}^{\nu}D_{\nu}\xi^{\mu}=D_{\nu}(\mathcal{L}^{\mu}\xi^{\nu}-\mathcal{L}^{\nu}\xi^{\mu})-\mathcal{L}^{\mu}D_{\nu}\xi^{\nu}+\xi^{\mu}D_{\nu}\mathcal{L}^{\nu}$$

$$\delta\left(\sqrt{-g}D_{\mu}\mathcal{L}^{\mu}\right)=\sqrt{-g}D_{\mu}(\xi^{\mu}D_{\nu}\mathcal{L}^{\nu})+\sqrt{-g}D_{\mu}\left(\left(\delta-\mathcal{L}_{\xi}\right)\mathcal{L}^{\mu}\right)$$

$$\Xi_t^{\mu}=\delta\mathcal{L}^{\mu}-\mathcal{L}_{\xi}\mathcal{L}^{\mu}=\frac{\partial\mathcal{L}^{\mu}}{\partial A_{\nu}}D_{\nu}\Lambda+\frac{\partial\mathcal{L}^{\mu}}{\partial\Gamma^{\tau}{}_{\rho\sigma}}\partial^{2}{}_{\rho\sigma}\xi^{\tau}$$

$$D_{\nu}Q_t^{\mu\nu}=2E^{\mu\nu}\xi_{\nu}+G^{\mu}(A_{\nu}\xi^{\nu}+\Lambda)+\Theta^{\mu}-\xi^{\mu}L-\Xi^{\mu}$$
$$=\delta\mathcal{L}^{\mu}+\mathcal{L}^{\mu}D_{\nu}\xi^{\nu}-\left(\delta-\mathcal{L}_{\xi}\right)\mathcal{L}^{\mu}-\xi^{\mu}D_{\nu}\mathcal{L}^{\nu}=D_{\nu}(\mathcal{L}^{\mu}\xi^{\nu}-\mathcal{L}^{\nu}\xi^{\mu})$$

$$Q_t^{\mu\nu}=\mathcal{L}^{\mu}\xi^{\nu}-\mathcal{L}^{\nu}\xi^{\mu}$$

$$\Xi_{\mathcal{L}}^{\mu}=\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}}\partial_{\nu}\xi^{\lambda}-\frac{1}{\sqrt{-g}}\xi^{\lambda}\partial_{\nu}\left(\sqrt{-g}\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}}\right).$$

$$\xi:\Xi(\xi_1+\xi_2)=\Xi(\xi_1)+\Xi(\xi_2)$$

$$q^{\mu\nu}=\mathcal{K}^{\mu\nu}{}_{\lambda}\xi^{\lambda}+\mathcal{L}^{\mu\nu\alpha}{}_{\beta}\partial_{\alpha}\xi^{\beta}$$

$$q^{\mu\nu}=\tilde{q}^{\mu\nu}+vw_v^{\mu\nu}$$

$$q_n{}^{\mu\nu}=Q^{\mu\nu\lambda_1\cdots\lambda_n}\prod_{i=1}^{n}A_{\lambda_i}$$

$$D_{\nu}q_n{}^{\mu\nu}=\left(D_{\nu}Q^{\mu\nu\lambda_1\cdots\lambda_n}\right)\left(\prod_{i=1}^{n}A_{\lambda_i}\right)+\sum_{j=1}^{n}Q^{\mu\nu\lambda_1\cdots\lambda_n}\left(D_{\nu}A_{\lambda_j}\right)\left(\prod_{i=1,i\neq j}^{n}A_{\lambda_i}\right)$$
$$=\left(D_{\nu}Q^{\mu\nu\lambda_1\cdots\lambda_n}\right)\left(\prod_{i=1}^{n}A_{\lambda_i}\right)+nQ^{\mu\nu\lambda\lambda_1\cdots\lambda_{n-1}}(D_{\nu}A_{\lambda})\left(\prod_{i=1}^{n-1}A_{\lambda_i}\right)$$

$$Q^{\mu\nu\lambda\lambda_1\cdots\lambda_n}=-Q^{\mu\lambda\nu\lambda_1\lambda_2\cdots\lambda_n}=-Q^{\mu\lambda_1\nu\lambda\lambda_2\cdots\lambda_n}=Q^{\mu\lambda_1\lambda\nu\lambda_2\cdots\lambda_n}=Q^{\mu\lambda\lambda_1\nu\lambda_2\cdots\lambda_n}$$
$$Q^{\mu\nu\lambda\lambda_1\cdots\lambda_n}=Q^{\mu\nu\lambda_1\lambda\cdots\lambda_n}=-Q^{\mu\lambda\lambda_1\nu\cdots\lambda_n}$$

$$Q^{\mu\nu\lambda} = \sum_n \mathcal{M}^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n} \prod_{i=1}^{n} F_{\rho_i\sigma_i}$$

$$D_\nu Q^{\mu\nu\lambda} = \sum_n \left(D_\nu \mathcal{M}^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n}\right)\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right)$$
$$+\sum_n n\mathcal{M}^{\mu\nu\lambda\rho\sigma\rho_1\sigma_1\cdots\rho_{n-1}\sigma_{n-1}}\left(D_\nu F_{\rho\sigma}\right)\left(\prod_{i=1}^{n-1} F_{\rho_i\sigma_i}\right)$$

$$\mathcal{M}^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n} = \begin{cases} D_\tau \mathcal{N}^{\tau\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n} & \text{(i) } 2n+3<D \\ \mathcal{S}\epsilon^{\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n} & \text{(ii) } 2n+3=D \end{cases}$$

$$A_\lambda Q^{\mu\nu\lambda} = \sum_n A_\lambda\left(D_\tau \mathcal{N}^{\tau\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n}\right)\left(\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right)$$
$$= D_\tau\left[\sum_n A_\lambda \mathcal{N}^{\tau\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n}\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right] - \frac{1}{2}\sum_n \mathcal{N}^{\tau\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n} F_{\tau\lambda}\prod_{i=1}^{n} F_{\rho_i\sigma_i}$$

$$q^{\mu\nu} = Q^{\mu\nu} + D_\tau\left(\sum_{n=0}^{N-2} A_\lambda \mathcal{N}^{\tau\mu\nu\lambda\rho_1\sigma_1\cdots\rho_n\sigma_n}\prod_{i=1}^{n} F_{\rho_i\sigma_i}\right)$$

$$\left[D_\rho Q_g^{r\rho} - \Theta_g^r + \Xi_g^r\right]\Big|_{r=0} - G_g^r\left(A_\rho\xi^\rho + \Lambda\right)$$
$$= D_\rho\left[2\frac{\partial L_g}{\partial F_{r\rho}}[vA_v + \Lambda]\right] - 2\frac{\partial L_g}{\partial F_{r\rho}}\left[vF_{v\rho} + D_\rho(vA_v + \Lambda)\right] + \frac{\partial L_g}{\partial A_r}\Lambda$$
$$-\left[\frac{\partial L_g}{\partial A_r} + 2D_\rho\frac{\partial L_g}{\partial F_{r\rho}}\right][vA_v + \Lambda]$$
$$= -v\left(\frac{\partial L_g}{\partial A_r}A_v + 2\frac{\partial L_g}{\partial F_{ri}}F_{vi}\right)$$

$$\left[D_\rho Q_g^{r\rho} - \Theta_g^r + \Xi_g^r\right]\Big|_{r=0} - G_g^r\left(A_\rho\xi^\rho + \Lambda\right)$$
$$= -v\left(a_g\epsilon^{r\rho_1\sigma_1\cdots\rho_N\sigma_N}A_v\prod_{i=1}^{N} F_{\rho_i\sigma_i} + 2Na_g\epsilon^{\mu ri\rho_1\sigma_1\cdots\rho_{N-1}\sigma_{N-1}}A_\mu F_{vi}\prod_{i=1}^{N-1} F_{\rho_i\sigma_i}\right)$$
$$= -v\left(2Na_g\epsilon^{rvij_1k_1\cdots j_{N-1}k_{N-1}}A_v F_{vi}\prod_{a=1}^{N-1} F_{j_ak_a}\right.$$
$$\left.+2Na_g\epsilon^{vrij_1k_1\cdots j_{N-1}k_{N-1}}A_v F_{vi}\prod_{a=1}^{N-1} F_{j_ak_a}\right) = 0$$

$$\left[D_\rho Q_{\mathcal{L}}^{r\rho} - \Theta_{\mathcal{L}}^r\right]\Big|_{\text{gauge}} - G_{\mathcal{L}}^r\left(A_\rho\xi^\rho + \Lambda\right)$$
$$= D_\rho\left[2\frac{\partial\mathcal{L}}{\partial F_{r\rho}}[vA_v + \Lambda]\right] - 2\frac{\partial\mathcal{L}}{\partial F_{r\rho}}\left[vF_{v\rho} + D_\rho(vA_\lambda + \Lambda)\right] - \left[\frac{\partial\mathcal{L}}{\partial A_r} + 2D_\rho\frac{\partial\mathcal{L}}{\partial F_{r\rho}}\right][vA_v + \Lambda]$$
$$= -\left(\frac{\partial\mathcal{L}}{\partial A_r}(vA_v + \Lambda) + 2\frac{\partial\mathcal{L}}{\partial F_{ri}}vF_{vi}\right) = \mathcal{O}(\epsilon^2)$$

$$\Theta^r_{\mathcal{L}}|_{\text{gravity}} = 2\frac{\partial\mathcal{L}}{\partial R^{\alpha}{}_{\beta r v}}\delta\Gamma^{\alpha}{}_{\beta v} + S^{r\alpha\beta}_{\mathcal{L}}\delta g_{\alpha\beta} = E^{r\alpha\beta\nu}_{\mathcal{L}} D_\nu \delta g_{\alpha\beta} + S^{r\alpha\beta}_{\mathcal{L}}\delta g_{\alpha\beta}$$
$$= D_\nu\left(E^{r\alpha\beta\nu}_{\mathcal{L}}\delta g_{\alpha\beta}\right) + \left(S^{r\alpha\beta}_{\mathcal{L}} - D_\nu E^{r\alpha\beta\nu}_{\mathcal{L}}\right)\delta g_{\alpha\beta}$$

$$U^{r\alpha\beta}\delta g_{\alpha\beta} = vU^{rmn}\partial_v h_{mn} = \mathcal{O}(\epsilon^2)$$

$$P^{\mu\nu} = E^{\mu\alpha\beta\nu}_{\mathcal{L}}\delta g_{\alpha\beta}$$

$$\Theta^r_{\mathcal{L}}|_{\text{gravity}} = D_\rho P^{r\rho} + \mathcal{O}(\epsilon^2)$$

$$D_\rho P^{r\rho} = \partial_\rho P^{r\rho} + \Gamma^r_{\lambda\rho}P^{\lambda\rho} + \Gamma^\rho_{\lambda\rho}P^{r\lambda} = \partial_r P^{rr} + \partial_v P^{rv} + \mathcal{O}(\epsilon^2)$$

$$E^{rmnr}_{\mathcal{L}} = \partial_v J_{(1)}{}^{mn} + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}
\partial_v P^{rv} &= (1+v\partial_v)(E^{rmnv}_{\mathcal{L}}\partial_v h_{mn})\\
\partial_r P^{rr} &= E^{rmnr}_{\mathcal{L}}(v\partial^2{}_{vr} - \partial_r)h_{mn}\\
&= -(1+v\partial_v)(E^{rmnr}_{\mathcal{L}}\partial_r h_{mn}) + v\partial_v(E^{rmnr}_{\mathcal{L}}\partial_r h_{mn}) + vE^{rmnr}_{\mathcal{L}}\partial^2{}_{vr}h_{mn}\\
&= -(1+v\partial_v)(E^{rmnr}_{\mathcal{L}}\partial_r h_{mn}) + v\partial_v\left((\partial_v J_{(1)}{}^{mn})(\partial_r h_{mn}) + J_{(1)}{}^{mn}\partial^2{}_{vr}h_{mn}\right) + \mathcal{O}(\epsilon^2)\\
&= -(1+v\partial_v)(E^{rmnr}_{\mathcal{L}}\partial_r h_{mn}) + v\partial_v^2\left(J_{(1)}{}^{mn}\partial_r h_{mn}\right) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

$$\Theta^r_{\mathcal{L}}|_{\text{gravity}} = (1+v\partial_v)\mathcal{A}_{(1)} + v\partial^2{}_v\mathcal{B}_{(0)} + \mathcal{O}(\epsilon^2),$$

$$g^{rr} = r^2X(r,v,x^i) + r^2\omega^i(r,v,x^i)\omega_i(r,v,x^i) = r^2X + r^2\omega^2$$
$$g^{ri} = -r\omega^i \; g^{rv} = 1 \; g^{ij} = h^{ij} \; g^{vv} = g^{vi} = 0$$

$$K_{ij} = \frac{1}{2}\partial_v h_{ij} \; \bar{K}_{ij} = \frac{1}{2}\partial_r h_{ij} \; K^{ij} = -\frac{1}{2}\partial_v h^{ij} \; \bar{K}^{ij} = -\frac{1}{2}\partial_r h^{ij}$$
$$K = \frac{1}{2}h^{ij}\partial_v h_{ij} = \frac{1}{\sqrt{h}}\partial_v\sqrt{h} \; \bar{K} = \frac{1}{2}h^{ij}\partial_r h_{ij} = \frac{1}{\sqrt{h}}\partial_r\sqrt{h}$$

$$\Gamma^v_{vv} = \frac{1}{2}(2rX + r^2\partial_r X) \; \Gamma^v_{vr} = 0 \; \Gamma^v_{vi} = -\frac{1}{2}(\omega_i + r\partial_r\omega_i)$$
$$\Gamma^r_{vv} = \frac{1}{2}\left(2r^3X^2 + 2r^3X\omega^2 + r^4X\partial_r X + r^4\omega^2\partial_r X - r^2\partial_v X - 2r^2\omega^i\partial_v\omega_i - r^3\omega^i\partial_i X\right)$$
$$\Gamma^r_{vr} = -\frac{1}{2}\left(2rX + r^2\partial_r X + r\omega^2 + r^2\omega^i\partial_r\omega_i\right)$$
$$\Gamma^r_{vi} = -\frac{1}{2}\left(r^3X\partial_r\omega_i + r^3\omega^2\partial_r\omega_i + r^2X\omega_i + r^2\omega^2\omega_i + r^2\partial_i X + r^2\omega^j\partial_i\omega_j - r^2\omega^j\partial_j\omega_i\right) - r\omega^j K_{ij}$$
$$\Gamma^i_{vv} = \frac{1}{2}\left(-2r^2\omega^i X - r^3\omega^i\partial_r X + 2h^{ij}r\partial_v\omega_j + r^2h^{ij}\partial_j X\right)$$
$$\Gamma^i_{vj} = \frac{1}{2}r\omega^i\left(\omega_j + r\partial_r\omega_j\right) + \frac{1}{2}h^{ik}\left(r\partial_j\omega_k - r\partial_k\omega_j\right) + h^{ik}K_{jk}$$
$$\Gamma^i_{vr} = \frac{1}{2}h^{ij}\left(r\partial_r\omega_j + \omega_j\right) \; \Gamma^v_{rr} = 0 \; \Gamma^v_{ri} = 0 \; \Gamma^v_{ij} = -\bar{K}_{ij} \; \Gamma^r_{rr} = 0$$
$$\Gamma^i_{rr} = 0 \; \Gamma^r_{ri} = -r\omega^j\bar{K}_{ij} + \frac{1}{2}(\omega_i + r\partial_r\omega_i)$$
$$\Gamma^r_{ij} = \frac{1}{2}\left(r\partial_i\omega_j + r\partial_j\omega_i\right) - r\omega_k\hat{\Gamma}^k_{ij} - (r^2X + r^2\omega^2)\bar{K}_{ij} - K_{ij}$$
$$\Gamma^i_{rj} = h^{ik}\bar{K}_{jk} \; \Gamma^i_{jk} = r\omega^i\bar{K}_{jk} + \hat{\Gamma}^i_{jk}$$

$$R_{rvrv} = X + \frac{1}{4}\omega^2$$
$$R_{vjvr} = \frac{1}{2}\left(\partial_v \omega_j + \omega^k K_{jk}\right)$$
$$R_{vlvm} = -\partial_v K_{lm} + K_{ln} K_m^n$$
$$R_{rv} = -X - \frac{1}{2}\omega^2 - \partial_r K - \bar{K}_{ij} K^{ij} + \frac{1}{2}\nabla^i \omega_i$$
$$R_{jv} = -\frac{1}{2}\partial_v \omega_j + \nabla_n K_j^n - \nabla_j K - \frac{1}{2}\omega_j K$$
$$R_{vv} = -\partial_v K - K_{mn} K^{mn}$$
$$R = \hat{R} - 2X - \frac{3}{2}\omega^2 - 4\partial_r K + 2\left(\nabla^i \omega_i\right) - 2\bar{K}_{ij} K^{ij} - 2\bar{K} K$$

$$\text{In}\mathcal{P} \sim (\partial_r)^{n_1} (\partial_v)^{n_2} \mathcal{Q}, n_1$$

$$T_{vv} \sim \mathcal{O}(\epsilon^2)$$

$$T_{vv} \sim \mathcal{O}(\epsilon)$$

$$E_{\mu\nu} \sim \mathcal{O}(l^N)$$

$$\tilde{\epsilon}^r{}_r = -1 \text{ and } \tilde{\epsilon}^v{}_v = 1$$

$$\mathcal{J}^v \sim \omega \wedge dA$$

$$v = \tau = 0: \tilde{\mathcal{J}}^\tau = \mathcal{J}^v + \nabla_i B^i$$

$$B_2 = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$H_3 = dB_2, \text{ i.e. } H_{\lambda\mu\nu} \equiv \partial_{[\lambda} B_{\mu\nu]}$$

$$B_2 \to B_2 + d \wedge_1$$

$$V = V\left(B_{\mu\nu} B^{\mu\nu} \pm b_{\mu\nu} b^{\mu\nu}\right)$$

$$\delta_{KR}^{nonmin} = \int ed^4x \left[\frac{R}{2\kappa} - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} - V\left(B_{\mu\nu} B^{\mu\nu} \pm b_{\mu\nu} b^{\mu\nu}\right) + \frac{1}{2\kappa}\left(\xi_2 B^{\lambda\nu} B_\nu^\mu R_{\lambda\mu} + \xi_3 B^{\mu\nu} B_{\mu\nu} R\right) + \mathcal{L}^M\right],$$

$$[\xi] = L^2$$

$$B_{\mu\nu} B^{\mu\nu} \pm b_{\mu\nu} b^{\mu\nu}$$

$$\partial_p b_{\mu\nu} = 0$$

$$b^2 = \eta^{\mu\nu} \eta^{\alpha\beta} b_{\mu\alpha} b_{\nu\beta}$$

$$b^2 = b_{\mu\nu} b^{\mu\nu}$$

$$b_2 = -\tilde{E}(x^1) dx^0 \wedge dx^1.$$

$$b_2 = d\tilde{A}_1$$

$$\tilde{A}_1 = \tilde{A}_0(x^1)dx^0$$

$$\tilde{E} = -\partial_1 \widetilde{A_0}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T^{\xi_2}_{\mu\nu} + \kappa T^M_{\mu\nu},$$

$$T^{\xi_2}_{\mu\nu} = \frac{\xi_2}{\kappa}\left[\frac{1}{2}g_{\mu\nu}B^{\alpha\gamma}B^{\beta}_{\gamma}R_{\alpha\beta} - B^{\alpha}_{\mu}B^{\beta}_{\nu}R_{\alpha\beta} - B^{\alpha\beta}B_{\mu\beta}R_{\nu\alpha} - B^{\alpha\beta}B_{\nu\beta}R_{\mu\alpha} + \frac{1}{2}D_\alpha D_\mu\left(B_{\nu\beta}B^{\alpha\beta}\right) + \frac{1}{2}D_\alpha D_\nu\left(B_{\mu\beta}B^{\alpha\beta}\right) - \frac{1}{2}D^2\left(B^{\alpha}_{\mu}B_{\alpha\nu}\right) - \frac{1}{2}g_{\mu\nu}D_\alpha D_\beta\left(B^{\alpha\gamma}B^{\beta}_{\gamma}\right)\right]$$

$$ds^2 = -e^{2\phi(r)}dt^2 + \left(1 - \frac{\Omega(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\ \theta d\phi^2,$$

$$\Omega(r_0) = r_0$$

$$\Omega'(r) < 1$$

$$\Omega(r)/r < 1, r > r_0$$

$$b^2 = g^{\mu\alpha}g^{\nu\beta}b_{\mu\nu}b_{\alpha\beta}$$

$$\tilde{E}(r) = \frac{|b|e^{\phi(r)}}{\sqrt{2\left(1 - \frac{\Omega(r)}{r}\right)}},$$

$$t^\mu = \left(\frac{\partial}{\partial t}\right)^\mu$$

$$\psi^\mu = \left(\frac{\partial}{\partial \phi}\right)^\mu$$

$$\begin{aligned}
G_{tt} &= \frac{\lambda}{4}\left[3R_{tt} - \left(1 - \frac{\Omega(r)}{r}\right)R_{rr}\right] + \kappa T^M_{tt} \\
G_{rr} &= \frac{\lambda}{4}\left[3R_{rr} - \left(1 - \frac{\Omega(r)}{r}\right)^{-1} R_{tt}\right] + \kappa T^M_{rr} \\
G_{\theta\theta} &= \frac{\lambda r^2}{4}\left[R_{tt} - \left(1 - \frac{\Omega(r)}{r}\right)R_{rr}\right] + \kappa T^M_{\theta\theta} \\
G_{\phi\phi} &= \sin^2\ \theta G_{\theta\theta}
\end{aligned}$$

$$\lambda = |b|^2\xi_2$$

$$\phi(r) = A = ג$$

$$R_{rr} = \frac{r\Omega'(r) - \Omega(r)}{r\left(1 - \frac{\Omega(r)}{r}\right)}, R_{\theta\theta} = \frac{r\Omega'(r) + \Omega(r)}{2r},$$

$$\left(T^{\mu}_{\nu}\right)^M = (\rho(r) + P_r(r))u^\mu u_\nu + P_r(r)g^{\nu}_{\mu} + (P_t(r) - P_r(r))\chi^\mu\chi_\nu,$$

$$u^\mu u_\nu = -\frac{1}{2}\chi^\mu \chi_\nu = -1$$

$$(T_t^t)^M = -\rho(r), (T_r^r)^M = P_r(r), \left(T_\theta^\theta\right)^M = \left(T_\phi^\phi\right)^M = P_t(r).$$

$$\begin{aligned}
\rho(r) &= \frac{1}{8\pi}\left[\frac{\Omega'(r)}{r^2} + \frac{\lambda}{4r^3}(r\Omega'(r) - \Omega(r))\right], \\
P_r(r) &= -\frac{1}{8\pi}\left[\frac{\Omega(r)}{r^3} + \frac{3\lambda}{4r^3}(r\Omega'(r) - \Omega(r))\right], \\
P_t(r) &= -\frac{1}{8\pi}\left(1 - \frac{\lambda}{2}\right)\frac{r\Omega'(r) - \Omega(r)}{2r^3}.
\end{aligned}$$

$$\rho(r) = \alpha\left[\left(\frac{r}{r_s}\right)^2 + \beta\right]^\eta,$$

$$\Omega(r) = \frac{16\pi\alpha}{\lambda + 6}\beta^\eta H(r) r^3 + C_1 r^{\frac{\lambda}{\lambda+4}},$$

$$H(r) = {}_2F_1\left[-\eta, \frac{\lambda+6}{\lambda+4}, \frac{2(\lambda+5)}{\lambda+4}, -\frac{r^2}{r_s^2\beta}\right]$$

$$C_1 = \frac{r_0^{\frac{4}{\lambda+4}}}{\lambda+6}[\lambda + 6 - 16\pi\alpha\beta^\eta H(r_0) r_0^2].$$

$$\Omega(r) = \frac{1}{\lambda+6}\left[16\pi\alpha\beta^\eta H(r) r^3 + r^{\frac{\lambda}{\lambda+4}} r_0^{\frac{4}{\lambda+4}}\{\lambda + 6 - 16\pi\alpha\beta^\eta H(r_0) r_0^2\}\right].$$

$$\Omega'(r_0) = \frac{1}{\lambda+4}\left[\lambda + 32\pi\alpha r_0^2\left(\frac{r_0^2}{r_s^2} + \beta\right)^\eta\right] < 1.$$

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)^\gamma\left(1 + \frac{r}{r_s}\right)^{3-\gamma}},$$

$$\Omega(r) = C_2((\lambda+4)r)^{\frac{\lambda}{\lambda+4}} - \frac{32\pi\rho_s G(r) r^3\left(\frac{r}{r_s}\right)^{-\gamma}}{\gamma(\lambda+4) - 2(\lambda+6)},$$

$$G(r) = {}_2F_1\left[3 - \gamma, \frac{2(\lambda+6)}{\lambda+4} - \gamma, \frac{3\lambda+16}{\lambda+4} - \gamma, -\frac{r}{a}\right]$$

$$C_2 = \frac{r_0(r_0(\lambda+4))^{-\frac{\lambda}{\lambda+4}}}{\gamma(\lambda+4) - 2(\lambda+6)}\left[\gamma(\lambda+4) - 2(\lambda+6) + 32\pi r_0^2\rho_s G(r_0)\left(\frac{r_0}{r_s}\right)^{-\gamma}\right].$$

$$\Omega(r) = \frac{r_0\left(\frac{r}{r_0}\right)^{\frac{\lambda}{\lambda+4}}\left[\gamma(\lambda+4) - 2(\lambda+6) + 32\pi\rho_s G(r_0) r_0^2\left(\frac{r_0}{r_s}\right)^{-\gamma}\right] - 32\pi\rho_s G(r) r^3\left(\frac{r}{r_s}\right)^{-\gamma}}{\gamma(\lambda+4) - 2(\lambda+6)}.$$

$$\Omega'(r_0) = \frac{\lambda r_0^3 + r_s^3\left[\lambda + 32\pi r_0^2 \rho_s \left(\frac{r_s + r_0}{r_s}\right)^{\gamma}\left(\frac{r_0}{r_s}\right)^{-\gamma}\right] + 3\lambda r_0 r_s (r_0 + r_s)}{(\lambda + 4)(r_s + r_0)^3} < 1.$$

$$\rho(r) + P_r(r) \geq 0, \rho(r) + P_t(r) \geq 0$$

$$\rho(r) \geq 0, \rho(r) + P_r(r) \geq 0, \rho(r) + P_t(r) \geq 0$$

$$\rho(r) - |P_r(r)| \geq 0, \rho(r) - |P_t(r)| \geq 0$$

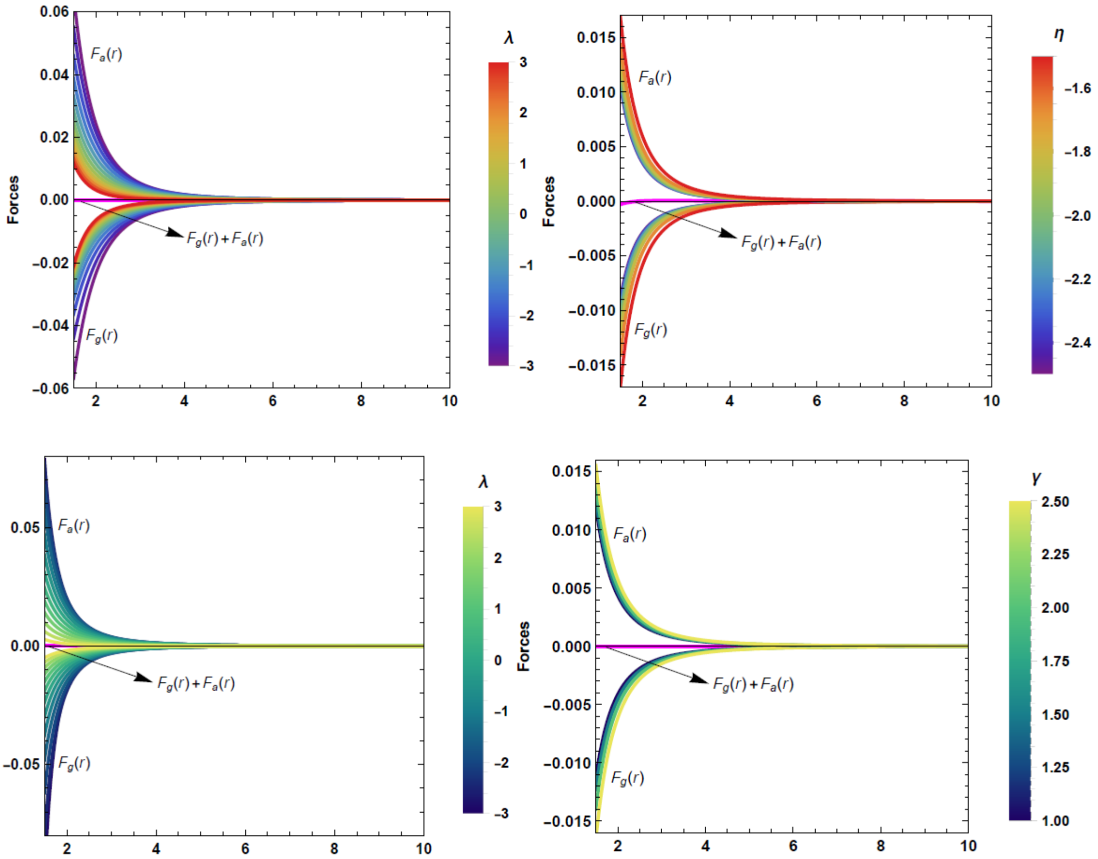
$$\rho(r) + P_r(r) \geq 0, \rho(r) + P_t(r) \geq 0, \rho(r) + P_r(r) + 2P_t(r) \geq 0$$

$$\rho(r) + P_r(r) = \begin{cases} \dfrac{(\lambda - 2)\left(\frac{r_0^2}{r_s^2\beta} + 1\right)^{-\eta}\left(\frac{r^2}{r_s^2\beta} + 1\right)^{-\eta}\left[\mathcal{T}_1 - 16\pi\alpha H(r_0) r_0^{\frac{4}{\lambda+4}+2} r^{\frac{\lambda}{\lambda+4}}\left(\frac{r_0^2}{r_s^2} + \beta\right)^{\eta}\left(\frac{r^2}{r_s^2\beta} + 1\right)^{\eta}\right],}{4\pi(\lambda + 4)(\lambda + 6) r^3} (\mathfrak{K}) \\ \dfrac{(\lambda - 2)\left[32\pi\rho_s r_0^2 G(r_0)(r_s + r)^3 (r_0(\lambda + 4))^{\frac{4}{\lambda+4}}((\lambda + 4)r)^{\frac{\lambda}{\lambda+4}}\left(\frac{r_0}{r_s}\right)^{-\gamma} + (\lambda + 4)\left\{r_0\mathcal{T}_2 + 8\pi\rho_s\mathcal{T}_3 r^3\left(\frac{r}{r_s}\right)^{-\gamma}\right\}\right]}{4\pi r^3(\lambda + 4)^2[\gamma(\lambda + 4) - 2(\lambda + 6)](r_s + r)^3}, (\mathfrak{NFW}). \end{cases}$$

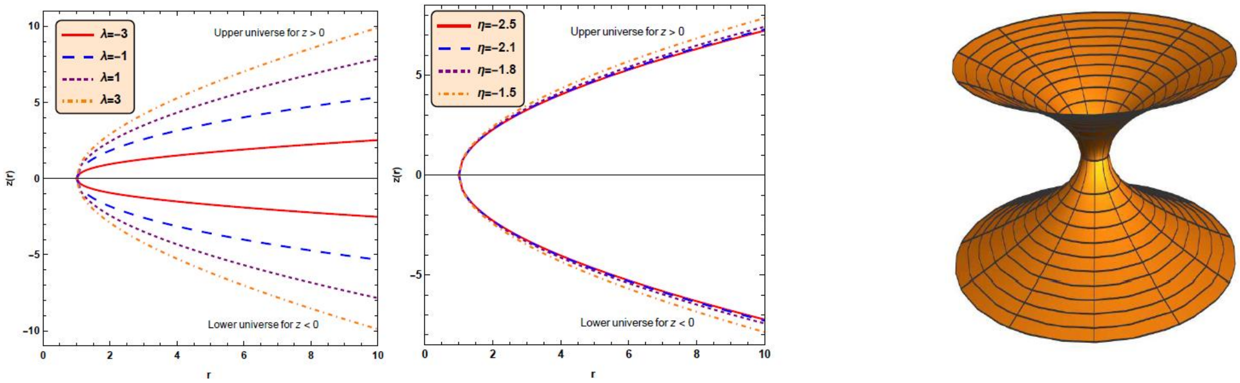
$$\mathcal{T}_1 = \left(\frac{r_0^2}{r_s^2\beta} + 1\right)^{\eta}\left[(\lambda + 6)\left(\frac{r^2}{r_s^2\beta} + 1\right)^{\eta}\left\{r_0^{\frac{4}{\lambda+4}} r^{\frac{\lambda}{\lambda+4}} - 8\pi\alpha r^3\left(\frac{r^2}{r_0^2} + \beta\right)^{\eta}\right\} + 16\pi\alpha H(r) r^3\left(\frac{r^2}{r_s^2} + \beta\right)^{\eta}\right],$$

$$\mathcal{T}_2 = (r_s + r)^3[\gamma(\lambda + 4) - 2(\lambda + 6)]\left(\frac{r}{r_0}\right)^{\frac{\lambda}{\lambda+4}},$$

$$\mathcal{T}_3 = r_s^{3-\gamma}[2(\lambda + 6) - \gamma(\lambda + 4)](r_s + r)^{\gamma} - 4G(r)(r_s + r)^3.$$



$$(\rho(r)+P_r(r))_{r=r_0}=\begin{cases}\dfrac{(\lambda-2)\left[1-8\pi\alpha r_0^2\left(\dfrac{r_0^2}{r_s^2}+\beta\right)^{\eta}\right]}{4\pi r_0^2(\lambda+4)}, & (\mathfrak{K})\\[2ex] \dfrac{(\lambda-2)\left[3r_s^2r_0+3r_sr_0^2+r_0^3+r_s^3\left\{1-8\pi\rho_s r_0^2\left(\dfrac{r_0}{r_s+r_0}\right)^{-\gamma}\right\}\right]}{4\pi r_0^2(\lambda+4)(r_s+r_0)^3}, & (\mathfrak{NFW})\,.\end{cases}$$



$$(\rho(r)+P_r(r))_{r=r_0}=0$$

$$\rho(r)+P_r(r)$$

$$\rho(r)+P_t(r),\rho(r)-P_r(r)$$

$$\rho(r)+P_r(r)+2P_t(r)$$

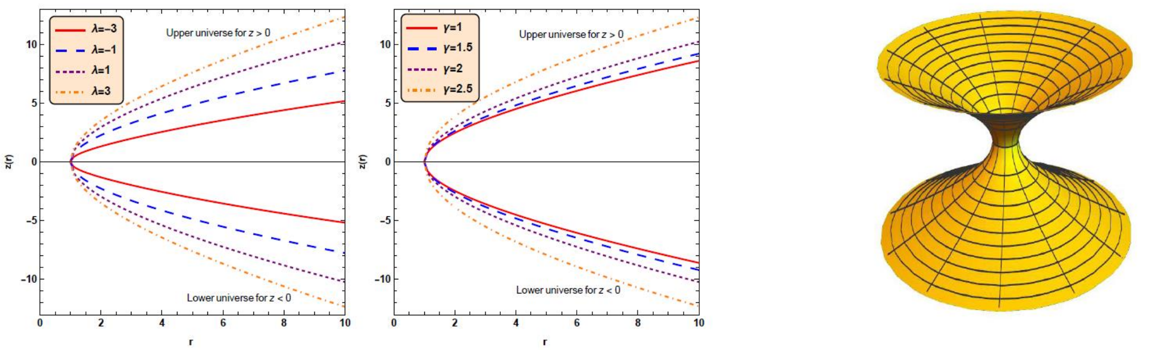
$$\rho(r)-P_t(r)$$

$$\frac{\omega'}{2}[\rho(r)+P_r(r)]+\frac{dP_r(r)}{dr}+\frac{2}{r}[P_r(r)-P_t(r)]=0.$$

$$\omega=2\phi=2A$$

$$F_g(r)=-\frac{\omega'}{2}[\rho(r)+P_r(r)],F_h(r)=-dP_r(r)/dr,F_a(r)=\frac{2}{r}[P_t(r)-P_r(r)].$$

$$F_g(r)+F_h(r)+F_a(r)=0.$$



$$ds^2=-\left(1-\frac{\Omega(r)}{r}\right)^{-1}dr^2-r^2d\phi^2$$

$$ds^2 = -\left[1 + \left(\frac{dz(r)}{dr}\right)^2\right] dr^2 - r^2 d\phi^2.$$

$$\frac{dz(r)}{dr} = \pm\left(\frac{r}{\Omega(r)} - 1\right)^{-\frac{1}{2}}.$$

$$z(r) = \pm\int_{r_0^+}^{\infty} \left(\frac{r}{\Omega(r)} - 1\right)^{-\frac{1}{2}}$$

$$Y_{TF} = 8\pi\Delta(r) - \frac{4\pi}{r^3}\int_0^r \bar{r}^3 \rho'(\bar{r}) d\bar{r}$$

$$\Delta(r) = P_r(r) - P_t(r)$$

$$m_T = (m_T)_R \left(\frac{r}{R}\right)^3 + r^3 \int_0^R \frac{e^{(\nu+\xi)/2}}{\bar{r}} Y_{TF} d\bar{r}.$$

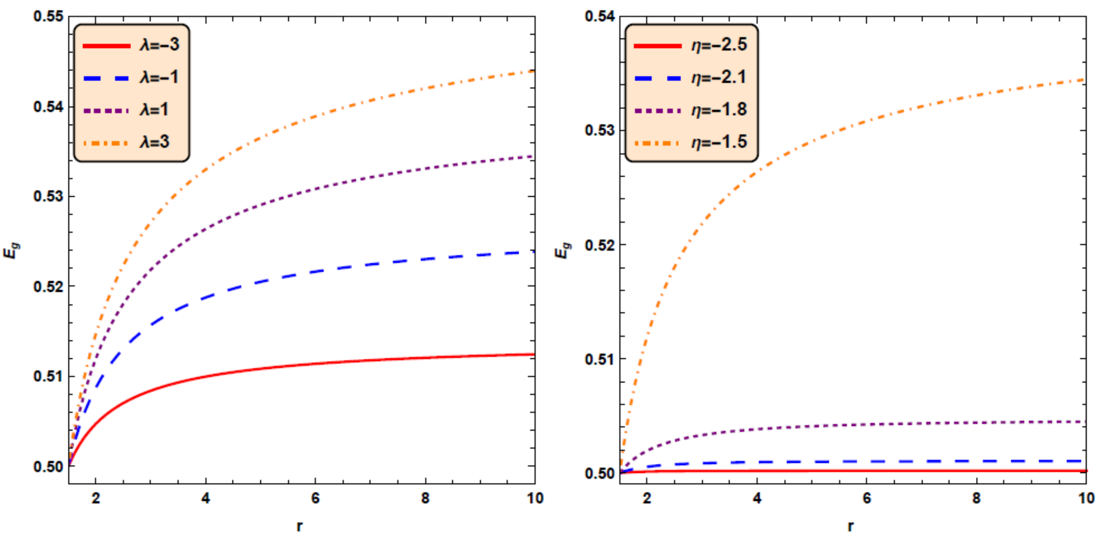
$$\Delta(r) = \frac{1}{2r^3}\int_0^r \bar{r}^3 \rho'(\bar{r}) d\bar{r}$$

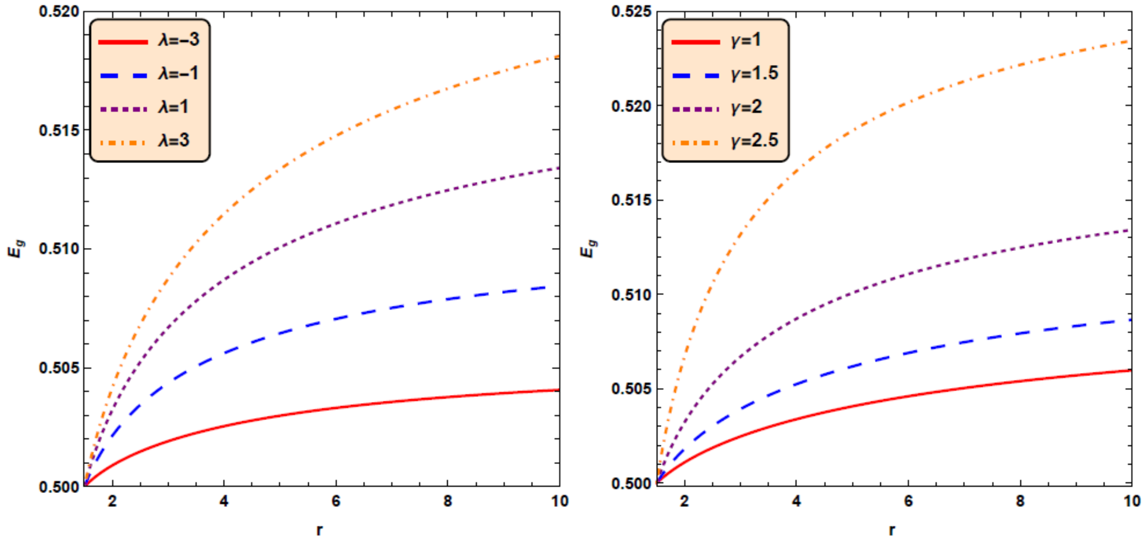
$$Y_{TF} = 8\pi\Delta(r) - \frac{4\pi}{r^3}\int_{r_0}^r \bar{r}^3 \rho'(\bar{r}) d\bar{r}$$

$$M_A = 4\pi\int_{r_0}^r \bar{r}^2 \rho(\bar{r}) d\bar{r}$$

$$M_A = \left[\frac{4}{3}\pi\alpha r^3 \left(\frac{r^2}{r_s^2\beta} + 1\right)^{-\eta} \left(\frac{r^2}{r_s^2} + \beta\right)^{\eta} {}_2F_1\left(\frac{3}{2}, -\eta; \frac{5}{2}; -\frac{r^2}{r_s^2\beta}\right)\right]_{r_0}^{r}.$$

$$M_A = \left[\frac{4\pi\rho_s}{3-\gamma} r^3 \left(\frac{r}{r_s}\right)^{-\gamma} {}_2F_1\left(3-\gamma, 3-\gamma; 4-\gamma; -\frac{r}{r_s}\right)\right]_{r_0}^{r}.$$



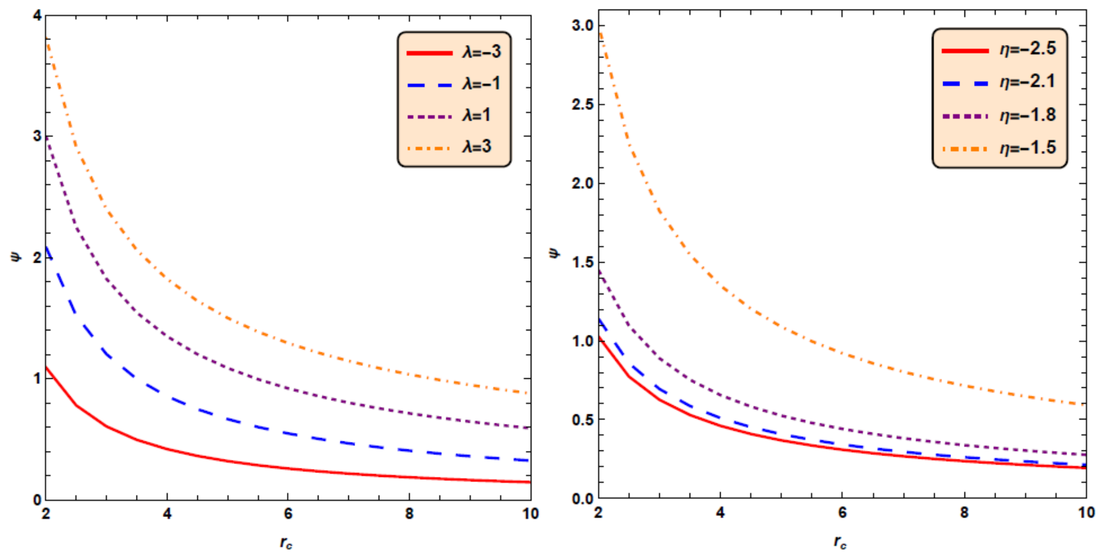


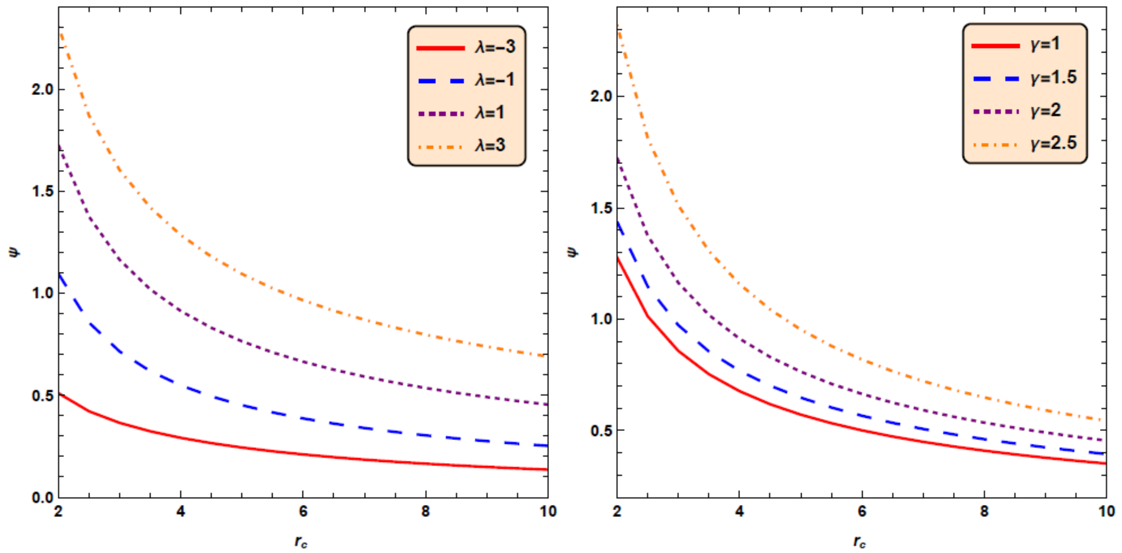
$$E_g = Mc^2 - E_M,$$

$$Mc^2 = \frac{1}{2}\int_{r_0}^{r} (T_t^t)^M r^2 dr + \frac{r_0}{2}$$

$$E_M = \frac{1}{2}\int_{r_0}^{r} r^2 (T_t^t)^M \sqrt{g_{rr}} dr, \text{ with } g_{rr} = \left(1 - \frac{\Omega(r)}{r}\right)^{-1}.$$

$$E_g = \frac{1}{2}\int_{r_0}^{r} [1 - \sqrt{g_{rr}}](T_t^t)^M r^2 dr + \frac{r_0}{2}.$$





$$\psi = -\pi + 2\int_{r_c}^{\infty} \frac{e^{\phi(r)}}{r^2\sqrt{\left(1-\frac{\Omega(r)}{r}\right)\left(\frac{1}{\beta^2}-\frac{e^{2\phi(r)}}{r^2}\right)}}dr$$

$$\beta = r_c e^{-\phi(r_c)}$$

$$f(\mathcal{Q},\mathcal{T}) \equiv \mathcal{Q} + \beta\mathcal{T}$$

$$\mathcal{S} = \int \frac{1}{2\kappa^2} f(\mathcal{Q},\mathcal{T})\sqrt{-g}d^4x + \int \mathcal{L}_m\sqrt{-g}d^4x$$

$$\kappa^2 = 8\pi G/c^4$$

$$\mathcal{Q}_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}$$

$$\mathcal{Q} = -\mathcal{Q}_{\alpha\mu\nu}P^{\alpha\mu\nu} \equiv -g^{\mu\nu}\left(L^{\beta}_{\alpha\mu}L^{\alpha}_{\nu\beta} - L^{\beta}_{\alpha\beta}L^{\alpha}_{\mu\nu}\right)$$

$$P^{\alpha}{}_{\mu\nu} = \frac{1}{4}\left[2\mathcal{Q}_{(\mu}{}^{\alpha}{}_{\nu)} - \mathcal{Q}^{\alpha}{}_{\mu\nu} + \mathcal{Q}^{\alpha}g_{\mu\nu} - \tilde{\mathcal{Q}}^{\alpha}g_{\mu\nu} - \delta^{\alpha}_{(\mu}\mathcal{Q}_{\nu)}\right]$$

$$L^{\beta}{}_{\mu\nu} = \frac{1}{2}\mathcal{Q}^{\beta}{}_{\mu\nu} - \mathcal{Q}_{(\mu}{}^{\beta}{}_{\nu)}$$

$$\mathcal{Q}_\alpha = \mathcal{Q}_\alpha{}^{\mu}{}_{\mu}, \tilde{\mathcal{Q}}_\alpha = \mathcal{Q}^{\mu}{}_{\alpha\mu}$$

$$\frac{-2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}f_{\mathcal{Q}}P^{\alpha}{}_{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}f + f_{\mathcal{T}}\left(\mathcal{T}_{\mu\nu} + \Theta_{\mu\nu}\right) - f_{\mathcal{Q}}\left(P_{\mu\alpha\beta}\mathcal{Q}_{\nu}{}^{\alpha\beta} - 2\mathcal{Q}^{\alpha\beta}{}_{\mu}P_{\alpha\beta\nu}\right) = \kappa^2\mathcal{T}_{\mu\nu}$$

$$f_{\mathcal{Q}} = \frac{\partial f}{\partial \mathcal{Q}} \text{ and } f_{\mathcal{T}} = \frac{\partial f}{\partial \mathcal{T}}$$

$$\Theta_{\mu\nu} = g^{\alpha\beta}\frac{\delta\mathcal{T}_{\alpha\beta}}{\delta g^{\mu\nu}}, \mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}$$

$$ds^2 = e^{2\xi(r)}dt^2 - \left(1 - \frac{h(r)}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\Phi^2$$

$$h(r_0) = r_0$$

$$\frac{h - rh'}{2h^2} > 0$$

$$\frac{h(r)}{r} \to 0 \text{ as } r \to \infty$$

$$T_\mu^\nu = (\sigma c^2 + p_\theta)U_\mu U^\nu - p_\theta \delta_\mu^\nu + (p_r - p_\theta)V_\mu V^\nu$$

$$U_\mu U^\nu = -V_\mu V^\nu = 1$$

$$T = \sigma c^2 - p_r - 2p_\theta$$

$$\mathcal{L}_m = -P$$

$$P \equiv \frac{p_r + 2p_\theta}{3}$$

$$\Theta_{\mu\nu} = -g_{\mu\nu}P - 2T_{\mu\nu}$$

$$\mathcal{Q} = -\frac{h}{r^2}\left[\frac{rh' - h}{r(r-h)} + 2\xi'\right]$$

$$\frac{2(r-h)}{(2r-h)f_\mathcal{Q}}\left[\sigma c^2 - \frac{(r-h)}{\kappa^2 r^3}\left(\frac{hrf_{\mathcal{Q}\mathcal{Q}}\mathcal{Q}'}{r-h} + hf_\mathcal{Q}\left(\frac{r\xi'+1}{r-h} - \frac{2r-h}{2(r-h)^2}\right) + \frac{fr^3}{2(r-h)}\right) + \frac{f_\mathcal{T}(P+\sigma)}{\kappa^2}\right] = \frac{h'}{\kappa^2 r^2}$$

$$\frac{2h}{fr^3}\left[p_r + \frac{(r-h)}{2\kappa^2 r^3}\left(f_\mathcal{Q}\left(\frac{h\left(\frac{rh'-h}{r-h} + 2r\xi' + 2\right)}{r-h} - 4rh'\right) + \frac{2hrf_{\mathcal{Q}\mathcal{Q}}\mathcal{Q}'}{r-h}\right) + \frac{fr^3(r-h)\xi'}{\kappa^2 hr^2} - \frac{f_\mathcal{T}(P-p_r)}{\kappa^2}\right]$$
$$= \frac{1}{\kappa^2}\left[2\left(1 - \frac{h}{r}\right)\frac{\xi'}{r} - \frac{h}{r^3}\right]$$

$$\frac{1}{f_\mathcal{Q}\left(\frac{r}{r-h} + r\xi'\right)}\left[p_\theta + \frac{(r-h)}{4\kappa^2 r^2}\left(f_\mathcal{Q}\left(\frac{4(2h-r)\xi'}{r-h} - 4r(\xi')^2 - 4r\xi''\right) + \frac{2fr^2}{r-b} - 4rf_{\mathcal{Q}\mathcal{Q}}\mathcal{Q}'\xi'\right) - \frac{f_\mathcal{T}(P-p_\theta)}{\kappa^2}\right.$$
$$\left. + \frac{(r-h)}{\kappa^2 r}\left(\xi'' + \xi'^2 - \frac{(rh'-h)\xi'}{2r(r-h)} + \frac{\xi'}{r}\right)f_\mathcal{Q}\left(\frac{r}{r-h} + r\xi'\right)\right] = \frac{1}{\kappa^2}\left(1 - \frac{h}{r}\right)\left[\xi'' + \xi'^2 - \frac{(rh'-h)\xi'}{2r(r-h)} - \frac{rh'-h}{2r^2(r-h)}\right.$$

$$\tilde{\sigma}c^2 = \frac{2(r-h)}{(2r-h)f_\mathcal{Q}}\left[\sigma c^2 - \frac{\left(1 - \frac{h}{r}\right)\left\{\frac{hrf_{\mathcal{Q}\mathcal{Q}}\mathcal{Q}'}{r-h} + \frac{hf_\mathcal{Q}}{r-h} - \frac{h(2r-h)f_\mathcal{Q}}{2(r-h)^2} + \frac{fr^3}{2(r-h)}\right\}}{\kappa^2 r^2} + \frac{f_\mathcal{T}(P+\sigma c^2)}{\kappa^2}\right]$$

$$\tilde{p}_r = \frac{2h}{fr^3}\left[p_r - \frac{f_\mathcal{T}(P-p_r)}{\kappa^2} + \frac{\left(1 - \frac{h}{r}\right)\left\{\frac{hf_\mathcal{Q}\left(\frac{rh'-h}{r-b} + 2\right)}{r-h} + \frac{2hrf_{\mathcal{Q}\mathcal{Q}}\mathcal{Q}'}{r-h}\right\}}{2\kappa^2 r^2}\right]$$

$$\tilde{p_\theta} = \frac{(r-h)}{rf_{\mathcal{Q}}}\left[p_\theta - \frac{f_{\mathcal{T}}(P-p_\theta)}{\kappa^2} + \frac{fr\left(1-\frac{h}{r}\right)}{2\kappa^2(r-h)}\right]$$

$$\tilde{\sigma}c^2 + \tilde{p_j} \geq 0\forall j$$

$$\tilde{\sigma}c^2 - \left|\tilde{p_j}\right| \geq 0\forall j$$

$$\tilde{\sigma}c^2 + \tilde{p_r} \geq 0, \tilde{\sigma}c^2 + \tilde{p_\theta} \geq 0$$

$$\tilde{\sigma}c^2 + \tilde{p_r} + \tilde{p_\theta} \geq 0$$

$$f(\mathcal{Q},\mathcal{T}) = \mathcal{Q} + \beta\mathcal{T}$$

$$\begin{aligned}
\sigma &= \frac{(3+8\beta)h'}{3(1+2\beta)(1+4\beta)\kappa^2c^2r^2}\\
p_r &= -\frac{3(1+4\beta)h - 4\beta rh'}{3(1+2\beta)(1+4\beta)\kappa^2r^3}\\
p_\theta &= \frac{3(1+4\beta)h - (3+4\beta)rh'}{6(1+2\beta)(1+4\beta)\kappa^2r^3}
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}c^2 &= \sigma c^2 + \beta(3\sigma c^2 - p_r/3 - 2p_\theta/3),\\
\tilde{p_r} &= p_r - \beta(\sigma c^2 - 7p_r/3 - 2p_\theta/3),\\
\tilde{p_\theta} &= p_\theta - \beta(\sigma c^2 - p_r/3 - 8p_\theta/3).
\end{aligned}$$

$$\begin{aligned}
&\left[i\frac{\partial}{\partial\tau} + \frac{1}{2}\nabla^2 - aV\right]\psi = 0\\
&\nabla^2 V = 4\pi(|\psi|^2 - 1)
\end{aligned}$$

$$\sigma_{\mathrm{sol}}(r) = \frac{\sigma_c}{[1+\alpha(r/r_c)^2]^8}$$

$$\sigma_c = 2.4\times10^{12}\left(\frac{m_b}{10^{-22}\mathrm{eV}}\right)^{-2}\left(\frac{r_c}{\mathrm{pc}}\right)^{-4}\frac{M_\odot}{\mathrm{pc}^3}$$

$$\sigma_{\mathrm{NFW}}(r) = \frac{\sigma_s}{(r/r_s)(1+r/r_s)^2}$$

$$\begin{aligned}
\sigma_s &= \frac{\zeta(z)\sigma_{m0}}{3}\frac{c^3}{\log(1+c) - \frac{c}{1+c}}\\
\zeta(z) &= \frac{18\pi^2 + 82(\Omega_m(z)-1) - 39(\Omega_m(z)-1)^2}{\Omega_m(z)}
\end{aligned}$$

$$\sigma_{c0} = 3H_0^2/8G$$

$$\sigma_{m0} = \Omega_{m0}\sigma_{c0}$$

$$h(r) = r_0 + A\left[\arctan\left(\sqrt{\alpha}r/r_c\right) - \arctan\left(\sqrt{\alpha}r_0/r_c\right)\right] + B\frac{r-r_0}{(r_c^2+\alpha r^2)^7}\mathcal{F}(r)$$

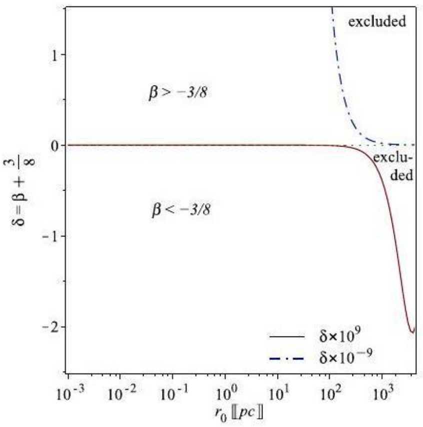
$$A = \frac{99\sigma_s c^2\kappa^2(1+2\beta)(1+4\beta)r_c^3}{2048\alpha\sqrt{\alpha}(3+8\beta)}$$

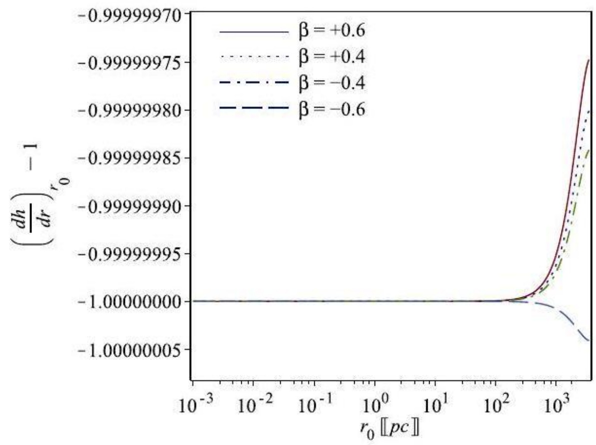
$$B = -\frac{99\sigma_s c^2\kappa^2(1+2\beta)(1+4\beta)r_c^4}{2048\alpha(3+8\beta)(r_c^2+\alpha r_0^2)^7}$$

$$\frac{3\kappa^2\sigma_c c^2 r_0^2 r_c^{16}(1+2\beta)(1+4\beta)}{(3+8\beta)(r_c^2+\alpha r_0^2)^8} < 1$$

$$\beta < -\frac{3}{8} + \frac{4(r_c^2+\alpha r_0^2)^8 - \sqrt{16(r_c^2+\alpha r_0^2)^{16} + 9\kappa^4\sigma_c^2 c^4 r_c^{32} r_0^4}}{24\kappa^2\sigma_c c^2 r_c^{16} r_0^2},$$

$$-3/8 < \beta < -\frac{3}{8} + \frac{4(r_c^2+\alpha r_0^2)^8 + \sqrt{16(r_c^2+\alpha r_0^2)^{16} + 9\kappa^4\sigma_c^2 c^4 r_c^{32} r_0^4}}{24\kappa^2\sigma_c c^2 r_c 16 r_0^2}$$





$$(dh/dr)_{r_0} < 1$$

$$h(r)/r < 1 \text{ at } r > r_0$$

$$h(r)/r \to 0 \text{ as } r \to \infty$$

$$dh/dr < 1 \text{ at } r > r_0$$

$$dh/dr \to 0 \text{ as } r \to \infty$$

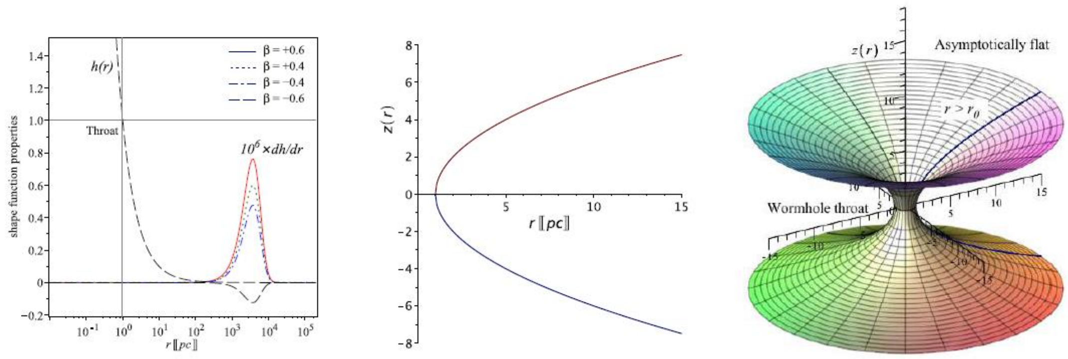
$$ds^2 = \frac{dr^2}{1-h(r)/r} + r^2 d\Phi^2.$$

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\Phi^2$$

$$z(r) = \pm\int_{r_0}^{r} \frac{d\zeta}{\sqrt{\zeta/h(\zeta) - 1}}$$

$$\lim_{r\to r_0} dz/dr \to \infty,$$

$$h(r) = r_0 + \tilde{A}\frac{r - r_0}{r + r_s} + \tilde{B}\ln\left[\frac{r + r_s}{r_0 + r_s}\right],$$



$$\tilde{A} = -\frac{3\sigma_s c^2\kappa^2(1 + 2\beta)(1 + 4\beta)r_s^4}{(3 + 8\beta)(r_0 + r_s)}$$

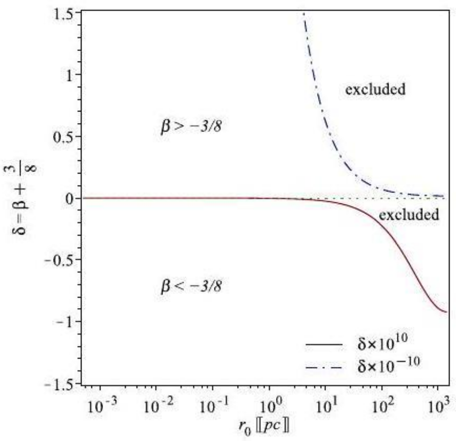
$$\tilde{B} = -\tilde{A}\left(1 + \frac{r_0}{r_s}\right) = \frac{3\sigma_s c^2\kappa^2(1 + 2\beta)(1 + 4\beta)r_s^3}{(3 + 8\beta)}$$

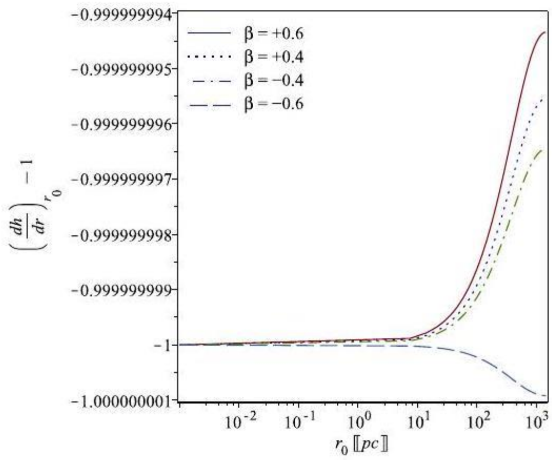
$$\frac{3\kappa^2\sigma_s c^2 r_0 r_s^3(1 + 2\beta)(1 + 4\beta)}{(3 + 8\beta)(r_s + r_0)^2} < 1.$$

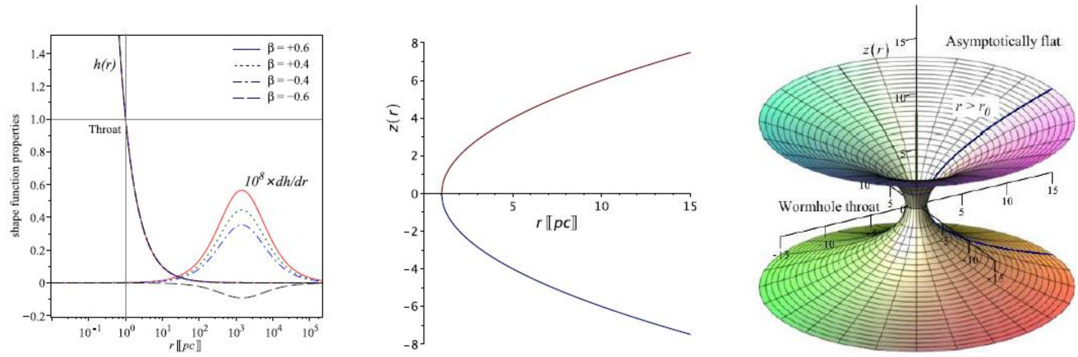
$$\beta < -\frac{3}{8} + \frac{4(r_s + r_0)^2 - \sqrt{16(r_s + r_0)^4 + 9\kappa^4\sigma_s^2 c^4 r_s^6 r_0^2}}{24\kappa^2\sigma_s c^2 r_s^3 r_0},$$

$$-3/8 < \beta < -\frac{3}{8} + \frac{4(r_s + r_0)^2 + \sqrt{16(r_s + r_0)^4 + 9\kappa^4\sigma_s^2 c^4 r_s^6 r_0^2}}{24\kappa^2\sigma_s c^2 r_s^3 r_0}$$

$$\lim_{r \to r_0} dz/dr \to \infty,$$





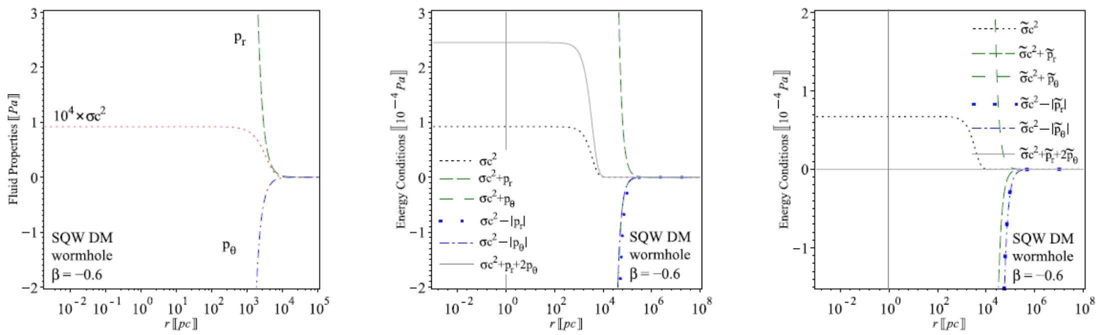


$$\Re_{\alpha\beta}u^{\alpha}u^{\beta} \geq 0 \text{ and } \Re_{\alpha\beta}\ell^{\alpha}\ell^{\beta} \geq 0$$

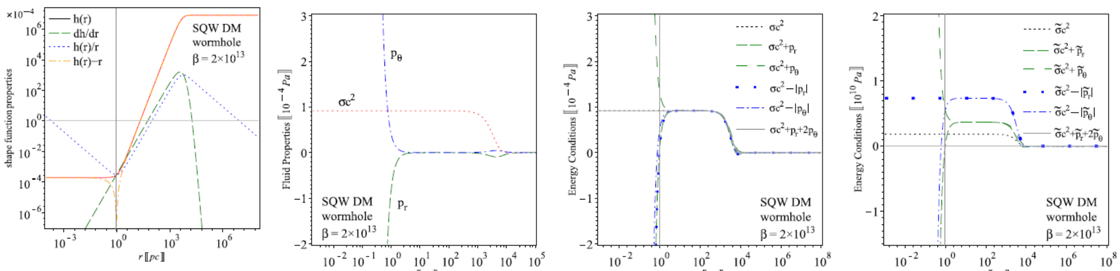
$$\tilde{\mathfrak{T}}^{\alpha}{}_{\beta} = \mathrm{diag}(-\tilde{\rho}c^2, \tilde{p}_r, \tilde{p}_t, \tilde{p}_t)$$

$$\Re_{\alpha\beta} = \kappa\left(\tilde{\mathfrak{T}}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{\mathfrak{T}}\right)$$

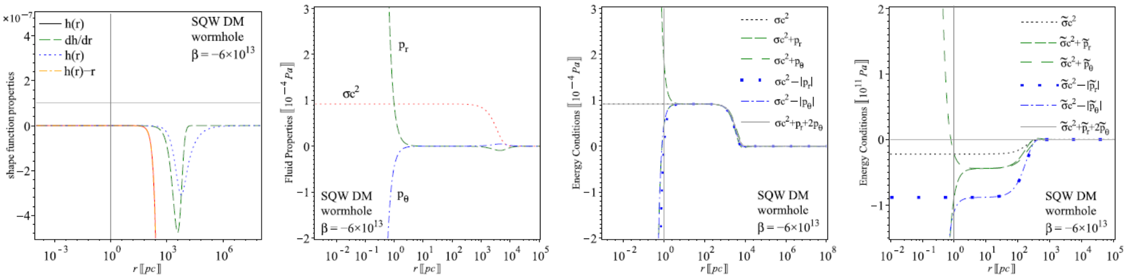
$$\begin{aligned}
\tilde{\sigma}c^2 + \tilde{p}_r &= (1+2\beta)(\sigma c^2 + p_r),\\
\tilde{\sigma}c^2 + \tilde{p}_\theta &= (1+2\beta)(\sigma c^2 + p_\theta),\\
\tilde{\sigma}c^2 - \tilde{p}_r &= \sigma c^2 - p_r + 4\beta(\sigma c^2 - p_r/3 - 2p_\theta/3),\\
\tilde{\sigma}c^2 - \tilde{p}_\theta &= \sigma c^2 - p_\theta + 2\beta(2\sigma c^2 - p_r/3 - 5p_\theta/3),\\
\tilde{\sigma}c^2 + \tilde{p}_r + 2\tilde{p}_\theta &= \sigma c^2 + p_r + 2p_\theta + \frac{8}{3}\beta(p_r + 2p_\theta).
\end{aligned}$$



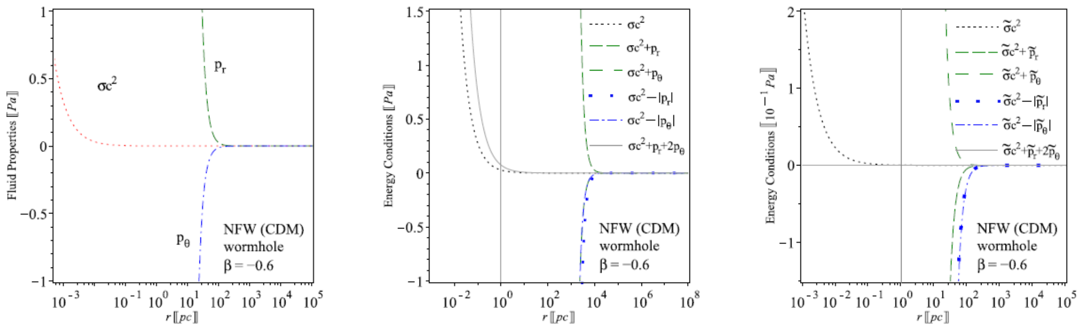
(a) The WH fluid (b) Matter energy conditions (c) Effective energy conditions



(a) The shape function (b) The WH fluid (c) Matter energy conditions (d) Effective energy conditions



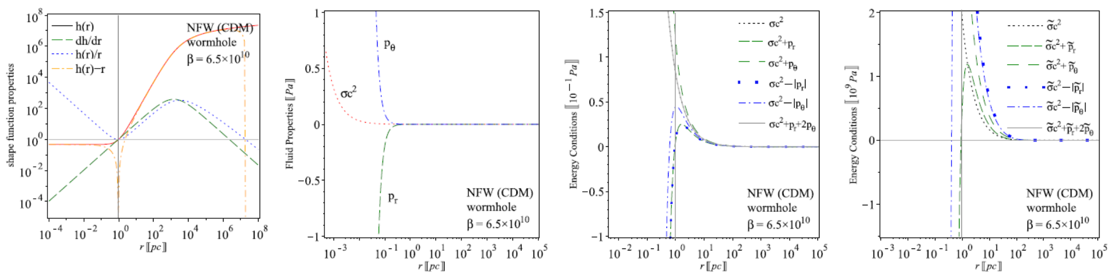
(a) The shape function (b) The WH fluid (c) Matter energy conditions (d) Effective energy conditions



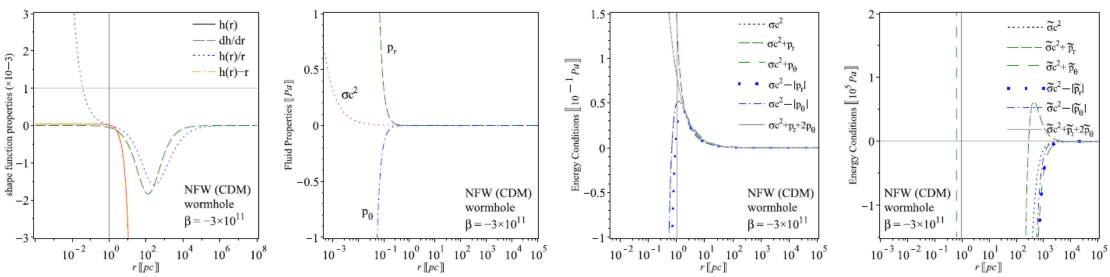
(a) The WH fluid (b) Matter energy conditions (c) Effective energy conditions

$$p_r = -\frac{\left[r_0 + \tilde{A}\dfrac{(r-r_0)}{(r+r_s)} + \tilde{B}\ln\left(\dfrac{r_s+r}{r_s+r_0}\right)\right]}{(1+2\beta)\kappa^2 r^3} + \frac{4\beta\left[(\tilde{A}+\tilde{B})r_s + \tilde{A}r_0 + \tilde{B}r\right]}{3(r_s+r)^2(1+2\beta)(1+4\beta)\kappa^2 r^2},$$

$$p_\theta = \frac{\left[r_0 + \tilde{A}\dfrac{(r-r_0)}{(r+r_s)} + \tilde{B}\ln\left(\dfrac{r_s+r}{r_s+r_0}\right)\right]}{2(1+2\beta)\kappa^2 r^3} + \frac{(3+4\beta)\left[(\tilde{A}+\tilde{B})r_s + \tilde{A}r_0 + \tilde{B}r\right]}{6(r_s+r)^2(1+2\beta)(1+4\beta)\kappa^2 r^2}.$$



(a) The shape function (b) The WH fluid (c) Matter energy conditions (d) Effective energy conditions



(a) The shape function (b) The WH fluid (c) Matter energy conditions (d) Effective energy conditions

$$p_r' = -\frac{3(1+2\beta)(\sigma c^2 + p_r)}{3+7\beta}\xi' + \beta\frac{3\sigma' c^2 - 2p_\theta'}{3+7\beta} + \frac{6(1+2\beta)(p_\theta - p_r)}{(3+7\beta)r}.$$

$$F_h = -p_r', F_g = -\frac{3(1+2\beta)(\sigma c^2 + p_r)}{3+7\beta}\xi', F_c = \beta\frac{3\sigma' c^2 - 2p_\theta'}{3+7\beta}, F_a = \frac{6(1+2\beta)(p_\theta - p_r)}{(3+7\beta)r}.$$

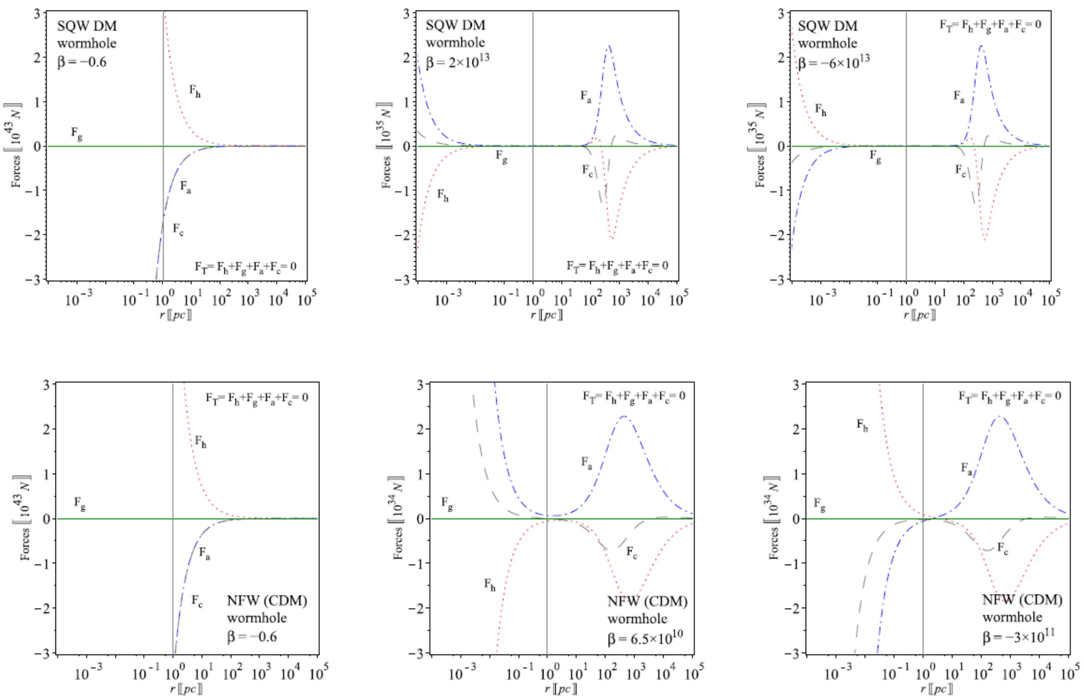
$$p_r' = -(\sigma c^2 + p_r)\xi' + \frac{2}{r}(p_\theta - p_r)$$

$$F_h + F_c + F_a = 0$$

$$\frac{1+2\beta}{3+7\beta} = \frac{2}{7} + \frac{1}{7(3+7\beta)} \to \frac{2}{7}|\beta|$$

$$p_r < p_\theta (p_r > p_\theta)$$

$$\begin{aligned} F_h &= -\frac{3h}{(1+2\beta)\kappa^2 r^4} + \frac{(3+20\beta)h'}{3(1+2\beta)(1+4\beta)\kappa^2 r^3} - \frac{4\beta h''}{3(1+2\beta)(1+4\beta)\kappa^2 r^2} \\ F_c &= \frac{3\beta h}{(1+2\beta)(3+7\beta)\kappa^2 r^4} - \frac{\beta(27+68\beta)h'}{3(1+2\beta)(1+4\beta)(3+7\beta)\kappa^2 r^3} + \frac{4\beta h''}{3(1+2\beta)(1+4\beta)\kappa^2 r^2} \\ F_a &= \frac{9h}{(3+7\beta)\kappa^2 r^4} - \frac{3h'}{(3+7\beta)\kappa^2 r^3} \end{aligned}$$



$$2\mathcal{L} = -\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{h}{r}} + r^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\Phi}^2\right]$$

$$\mathfrak{p}_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = \dot{t} = E, \mathfrak{p}_\Phi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = r^2\sin^2\theta\dot{\Phi} = L$$

$$\dot{t} = E, \dot{\Phi} = \frac{L}{r^2\sin^2\theta}$$

$$\mathfrak{p}_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{1 - \frac{h}{r}}, \mathfrak{p}_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \dot{\theta}$$

$$\frac{\partial S}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},$$

$$S = -Et + L\Phi + S_r(r) + S_\theta(\theta)$$

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = Q - \frac{L^2}{\sin^2\ \theta}, \left(1 - \frac{h}{r}\right)\left(\frac{dS_r}{dr}\right)^2 = E^2 - \frac{Q}{r^2}$$

$$\mathfrak{p}_r = \frac{\partial S}{\partial r} = \frac{dS_r}{dr}$$

$$\mathfrak{p}_\theta = \frac{\partial S}{\partial \theta} = \frac{dS_\theta}{d\theta}$$

$$\frac{1}{\left(1 - \frac{h}{r}\right)^{1/2}} \frac{dr}{d\lambda} = \pm\sqrt{R(r)}, r^2 \frac{d\theta}{d\lambda} = \pm\sqrt{T(\theta)}$$

$$R(r) = E^2 - \frac{Q}{r^2}, T(\theta) = Q - \frac{L^2}{\sin^2\ \theta}$$

$$\zeta = \frac{L}{E}, \eta = \frac{Q}{E^2}$$

$$R(r) = 1 - \frac{\eta}{r^2}, T(\theta) = \eta - \frac{\zeta^2}{\sin^2\ \theta}$$

$$\left(\frac{dr}{d\tilde{\lambda}}\right)^2 + V_{eff} = 0, V_{eff} = -\left(1 - \frac{h}{r}\right) R(r) = -\left(1 - \frac{h}{r}\right)\left(1 - \frac{\eta}{r^2}\right),$$

$$V_{eff} = 0, \frac{dV_{eff}}{dr} = 0, \frac{d^2 V_{eff}}{dr^2} \leq 0$$

$$\kappa^2 \sigma c^2 = \frac{(r-h)}{2r^3}\left[f_Q \left\{\frac{(2r-h)(rh'-h)}{(r-h)^2} + \frac{h(2r\xi'+2)}{r-h}\right\} + \frac{2hrf_{QQ}Q'}{r-h} + \frac{fr^3}{r-h} - \frac{2r^3 f_{\mathcal{T}}(P+\sigma c^2)}{(r-h)}\right]$$
$$\kappa^2 p_r = -\frac{(r-h)}{2r^3}\left[f_Q \left\{\frac{h}{r-h}\left(\frac{rh'-h}{r-b} + 2r\xi' + 2\right) - 4r\xi'\right\} + \frac{2hrf_{QQ}Q'}{r-h} + \frac{fr^3}{r-h} - \frac{2r^3 f_{\mathcal{T}}(P-p_r)}{(r-h)}\right]$$
$$\kappa^2 p_\theta = -\frac{(r-h)}{4r^2}\left[f_Q \left\{\frac{(rh'-h)\left(\frac{2r}{r-h} + 2r\xi'\right)}{r(r-h)} + \frac{4(2h-r)\xi'}{r-h} - 4r\xi'^2 - 4r\xi''\right\} - 4rf_{QQ}Q'\xi' + \frac{2fr^2}{r-h} - \frac{4r^2 f_{\mathcal{T}}(P-p_\theta)}{(r-h)}\right]$$

$$G_{\mu\nu} = \kappa^2\left(\mathcal{T}_{\mu\nu} + \mathcal{T}_{\mu\nu}^{\mathrm{NMG}}\right) = \kappa^2 \tilde{\mathcal{T}}_{\mu\nu}$$

$$\tilde{\mathcal{T}}^{\mu}{}_{\nu} = \mathrm{diag}(\tilde{\sigma}c^2, \tilde{p}_r, \tilde{p}_\theta, \tilde{p}_\theta)$$

$$\tilde{\sigma}c^2=\frac{h'}{\kappa^2r^2}$$
$$\tilde{p}_r=\frac{1}{\kappa^2}\left[2\left(1-\frac{h}{r}\right)\frac{\xi'}{r}-\frac{h}{r^3}\right]$$
$$\tilde{p}_\theta=\frac{1}{\kappa^2}\left(1-\frac{h}{r}\right)\left[\xi''+\xi'^2-\frac{(rh'-h)\xi'}{2r(r-h)}-\frac{rh'-h}{2r^2(r-h)}+\frac{\xi'}{r}\right]$$

$$\tilde{\sigma}c^2=\frac{2(r-h)}{(2r-h)f_Q}\left[\sigma c^2-\frac{1}{\kappa^2r^2}\left(1-\frac{h}{r}\right)\left(\frac{hrf_{QQ}Q'}{r-b}+hf_Q\left(\frac{r\xi'+1}{r-h}-\frac{2r-h}{2(r-h)^2}\right)+\frac{fr^3}{2(r-h)}\right)+\frac{f_T(P+\sigma c^2)}{\kappa^2}\right]$$
$$\tilde{p}_r=\frac{2h}{fr^3}\left[p_r+\frac{1}{2\kappa^2r^2}\left(1-\frac{h}{r}\right)\left(f_Q\left(\frac{h\left(\frac{rh'-h}{r-h}+2r\xi'+2\right)}{r-h}-4r\xi'\right)+\frac{2hrf_{QQ}Qh'}{r-g}\right)+\frac{fr^3(r-h)\xi'}{\kappa^2hr^2}-\frac{f_T(P-p_r)}{\kappa^2}\right]$$
$$\tilde{p}_\theta=\frac{1}{f_Q\left(\frac{r}{r-h}+r\xi'\right)}\left[p_\theta+\frac{1}{4\kappa^2r}\left(1-\frac{h}{r}\right)\left(f_Q\left(\frac{4(2h-r)\xi'}{r-h}-4r(\xi')^2-4r\xi''\right)+\frac{2fr^2}{r-h}-4rf_{QQ}Q'\xi'\right)\right.$$
$$\left.+\frac{1}{\kappa^2}\left(1-\frac{h}{r}\right)\left(\xi''+\xi'^2-\frac{(rh'-h)\xi'}{2r(r-h)}+\frac{\xi'}{r}\right)f_Q\left(\frac{r}{r-h}+r\xi'\right)-\frac{f_T(P-p_\theta)}{\kappa^2}\right]$$

$$\tilde{\sigma}c^2+\tilde{p}_r=\frac{h(h-2r)f_Q^2[h(3h-2r)-hrh']-fr^4(h-r)^2[2hf_{QQ}Q'+r^2\{f-2f_T(P+\sigma c^2)+2\kappa^2p_r\}]}{\kappa^2fr^6(h-2r)(h-r)f_Q}$$
$$+\frac{2(r-h)(\sigma c^2+p_r)}{(2r-h)f_Q}+\frac{hr(h-r)f_Q[2(h-2r)\{Qf_{QQ}Q'+r^2(f_T(p_r-P)+\kappa^2p_r)\}-hfr^2]}{\kappa^2fr^6(h-2r)(h-r)f_Q}'$$
$$\tilde{\sigma}c^2+\tilde{p}_\theta=$$
$$\tilde{\sigma}c^2-\tilde{p}_r=-\frac{2(r-h)(\sigma+p_\theta)}{(2r-h)f_Q}-\frac{2h^2f_Q+r(r-h)[2rf_T(+2\sigma r-hp_\theta+hP+2rp_\theta)-h(4f_{QQ}Q'+fr+2\kappa^2rp_\theta)]}{2\kappa^2r^3(h-2r)f_Q},$$
$$+\frac{2(r-h)(\sigma-p_r)}{(2r-h)f_Q}-\frac{hr(h-r)f_Q[2(h-2r)\{hf_{QQ}Q'+r^2(f_T(p_r-P)+\kappa^2p_r)\}+hfr^2]}{\kappa^2fr^6(h-2r)(h-r)f_Q},$$
$$\tilde{\sigma}c^2-\tilde{p}_\theta=\frac{2(r-h)(\sigma-p_\theta)}{(2r-h)f_Q}-\frac{2h^2f_Q+r(r-h)[2\{rf_T((h-2r)p_\theta-hP+4Pr+2\sigma r)-2bf_{QQ}Q'+\kappa^2hrp_\theta\}+fr(h-4r)]}{2\kappa^2r^3(h-2r)f_Q}$$
$$\tilde{\sigma}c^2+\tilde{p}_r+2\tilde{p}_\theta=+\frac{2(r-h)(p_r+2p_\theta+\sigma)}{(2r-h)f_Q}+\frac{hr}{\kappa^2r^6}\left[\frac{2(hf_{QQ}Q'+r^2\{f_T(p_r-P)+\kappa^2p_r\})}{f}-\frac{hr^2}{h-2r}\right]$$
$$+\frac{h^2f_Q(3h-rh'-2r)}{\kappa^2r^6f(h-r)}+\frac{r^4(h-r)[2rf_T(hP-(h-2r)p_\theta-Pr+\sigma r)-2hf_{QQ}Q'+fr(r-h)-2\kappa^2r(hp_\theta+rp_r)]}{\kappa^2r^6(h-2r)f_Q}$$

- $\tilde{\sigma}\geq 0\Rightarrow\sigma\geq 0$

$$\frac{(2r-h)f_Q}{r-h}$$
$$>\mathfrak{G}\;\frac{2(r-h)\left[\frac{\left(1-\frac{h}{r}\right)\left(\frac{hrf_{QQ}Q'}{r-h}+\frac{hf_Q}{r-h}-\frac{h(2r-h)f_Q}{2(r-h)^2}+\frac{fr^3}{2(r-h)}\right)}{\kappa^2r^2}-\frac{f_T(P+\sigma)}{\kappa^2}\right]}{(2r-h)f_Q}\leq\mathfrak{G}$$

- $\tilde{\sigma}+\tilde{p}_r\geq 0\Rightarrow\sigma+p_r\geq 0$

$$\frac{(2r-h)f_Q}{r-h}>\mathfrak{G}\;\frac{hr(h-r)f_Q\left[2(h-2r)\left(hf_{QQ}Q'+r^2(f_T(p_r-P)+\kappa^2p_r)\right)-bfr^2\right]}{\kappa^2fr^6(b-2r)(b-r)f_Q}$$
$$+\frac{h(h-2r)f_Q^2(h(3h-2r)-hrh')-fr^4(h-r)^2[2hf_{QQ}Q'+r^2(-2f_T(P+\sigma)+f+2\kappa^2p_r)]}{\kappa^2fr^6(h-2r)(h-r)f_Q}\geq\mathfrak{G}$$

- $\tilde{\sigma} + \tilde{p_\theta} \geq 0 \Rightarrow \sigma + p_\theta \geq 0$

$$\frac{(2r-h)f_Q}{r-h} > \mathfrak{G}\, \frac{2h^2 f_Q + r(r-h)\left[2rf_Q(-hp_\theta + hP + 2rp_\theta + 2\sigma r) - h\left(4f_{QQ}Q' + fr + 2\kappa^2 rp_\theta\right)\right]}{2\kappa^2 r^3 (h-2r) f_Q} \leq \mathfrak{G}$$

- $\tilde{\sigma} - \tilde{p_r} \geq 0 \Rightarrow \sigma - p_r \geq 0$

$$\frac{(2r-h)f_Q}{r-h} > \mathfrak{G}\, \frac{hr(h-r)f_Q\left[2(h-2r)\left\{hf_{QQ}Q' + r^2\left(f_{\mathcal{T}}(p_r - P) + \kappa^2 p_r\right)\right\} + bfr^2\right]}{\kappa^2 f r^6 (b-2r)(b-r) f_Q} + \frac{h(h-2r)f_Q^2[h(3h-2r) - hrh'] + fr^4(h-r)^2\left[2hf_{QQ}Q' + r^2(-2f_{\mathcal{T}}(P+\sigma) + f - 2\kappa^2 p_r)\right]}{\kappa^2 f r^6 (b-2r)(h-r) f_Q} \leq \mathfrak{G}$$

- $\tilde{\sigma} - \tilde{p_\theta} \geq 0 \Rightarrow \sigma - p_\theta \geq 0$

$$\frac{(2r-h)f_Q}{r-h} > \mathfrak{G}\, \frac{2h^2 f_Q + r(r-h)\left[2\left(rf_{\mathcal{T}}((h-2r)p_\theta - hP + 4Pr + 2\sigma r) - 2bf_{QQ}Q' + \kappa^2 hrp_\theta\right) + fr(h-4r)\right]}{2\kappa^2 r^3 (h-2r) f_Q} \leq \mathfrak{G}$$

- $\tilde{\sigma} + \tilde{p_r} + 2\tilde{p_\theta} \geq 0 \Rightarrow \sigma + p_r + 2p_\theta \geq 0$

$$\frac{(2r-h)f_Q}{r-h} > \mathfrak{G}\, \frac{hr\left[\dfrac{2\left(hf_{QQ}Q' + r^2\left(f_{\mathcal{T}}(p_r - P) + \kappa^2 p_r\right)\right)}{f} - \dfrac{hr^2}{h-2r}\right]}{\kappa^2 r^6} + \frac{\dfrac{h^2 f_Q(3h - rh' - 2r)}{f(h-r)} + \dfrac{r^4(h-r)\left[2rf_{\mathcal{T}}(hP - (h-2r)p_\theta - Pr + \sigma r) - 2hf_{QQ}Q' + fr(r-h) - 2\kappa^2 r(hp_\theta + rp_r)\right]}{(h-2r)f_Q}}{\kappa^2 r^6} \geq \mathfrak{G}$$

$\mathcal{F}(r) = 680680 r_c^8 r_0{}^{11} r^7 \alpha^9 + 680680 r_c^8 r_0{}^{7} r^{11} \alpha^9 + 3465 r^{13} \alpha^{13} r_0{}^{13} + 3465 r_c^{26} - 48580 r_c^{24} \alpha r_0{}^{2} -$

$48580 r_c^{24} \alpha r^2 - 92323 r_c^{22} \alpha^2 r^4 - 92323 r_c^{22} \alpha^2 r_0{}^{4} - 101376 r_c^{20} \alpha^3 r_0{}^{6} - 101376 r_c^{20} \alpha^3 r^6 -$

$65373 r_c^{18} \alpha^4 r_0{}^{8} - 23100 r_c^{16} \alpha^5 r_0{}^{10} - 3465 r_c^{14} \alpha^6 r_0{}^{12} - 65373 r_c^{18} \alpha^4 r^8 - 23100 r_c^{16} \alpha^5 r^{10} -$

$3465 r_c^{14} \alpha^6 r^{12} + 65373 r_c^4 r_0{}^{9} r^{13} \alpha^{11} + 23100 r_c^2 r_0{}^{13} r^{11} \alpha^{12} - 1155 r_c^2 r_0{}^{12} r^{12} \alpha^{12} +$

$23100 r_c^2 r_0{}^{11} r^{13} \alpha^{12} + 65373 r_c^4 r_0{}^{13} r^9 \alpha^{11} - 7392 r_c^4 r_0{}^{12} r^{10} \alpha^{11} + 154308 r_c^4 r_0{}^{11} r^{11} \alpha^{11} -$

$7392 r_c^4 r_0{}^{10} r^{12} \alpha^{11} - 127820 r_c^8 r_0{}^{10} r^8 \alpha^9 + 92323 r_c^8 r_0{}^{5} r^{13} \alpha^9 - 28952 r_c^8 r_0{}^{6} r^{12} \alpha^9 +$

$1245013 r_c^8 r_0{}^{9} r^9 \alpha^9 + 101376 r_c^6 r_0^7 r^{13} \alpha^{10} + 101376 r_c^6 r_0^{13} r^7 \alpha^1 0 - 19899 r_c^6 r_0^1 2 r^8 \alpha^{10} +$

$437712 r_c^6 r_0^{11} r^9 \alpha^{10} - 47388 r_c^6 r_0{}^{10} r^{10} \alpha^{10} + 437712 r_c^6 r_0{}^{9} r^{11} \alpha^{10} - 19899 r_c^6 r_0{}^{8} r^{12} \alpha^{10} -$

$72835 r_c^{24} \alpha r r_0 - 165088 r_c^{22} \alpha^2 r_0 r^3 - 165088 r_c^{22} \alpha^2 r_0{}^{3} r - 505148 r_c^{22} \alpha^2 r_0^2 r^2 -$

$868912 r_c^{20} \alpha^3 r_0{}^{4} r^2 + 151268 r_c^{20} \alpha^3 r_0{}^{3} r^3 - 222651 r_c^{20} \alpha^3 r_0{}^{5} r - 222651 r_c^{20} \alpha^3 r_0 r^5 -$

$868912 r_c^{20} \alpha^3 r_0{}^{2} r^4 - 1134763 r_c^{18} \alpha^4 r_0{}^{4} r^4 + 804020 r_c^{18} \alpha^4 r_0{}^{5} r^3 - 896280 r_c{}^{18} \alpha^4 r_0{}^{2} r^6 -$

$186648 r_c^{18} \alpha^4 r_0 r^7 + 804020 r_c^{18} \alpha^4 r_0{}^{3} r^5 - 896280 r_c^1 8 \alpha^4 r_0^6 r^2 - 186648 r_c^{18} \alpha^4 r_0{}^{7} r -$

$95865 r_c^{16} \alpha^5 r_0 r^9 - 982072 r_c^{16} \alpha^5 r_0{}^6 r^4 + 1146824 r_c^{16} \alpha^5 r_0{}^3 r^7 - 982072 r_c^{16} \alpha^5 r_0{}^4 r^6 -$

$553476 r_c^{16} \alpha^5 r_0{}^8 r^2 - 95865 r_c^{16} \alpha^5 r_0{}^9 r + 2249233 r_c^{16} \alpha^5 r^5 r_0{}^5 - 553476 r_c^{16} \alpha^5 r_0{}^2 r^8 +$

$1146824 r_c^{16} \alpha^5 r_0{}^7 r^3 - 27720 r_c^{14} \alpha^6 r_0{}^{11} r - 542073 r_c^{14} \alpha^6 r_0{}^4 r^8 + 2689232 r_c^{14} \alpha^6 r_0^5 r^7 -$

$27720 r_c^{14} \alpha^6 r_0 r^{11} + 2689232 r_c^{14} \alpha^6 r_0^7 r^5 + 830760 r_c^{14} \alpha^6 r_0{}^9 r^3 - 189420 r_c^{14} \alpha^6 r_0^2 r^{10} -$

$542073 r_c^{14} \alpha^6 r_0^8 r^4 + 830760 r_c^{14} \alpha^6 r_0{}^3 r^9 - 858928 r_c^{14} \alpha^6 r_0{}^6 r^6 - 189420 r_c^{14} \alpha^6 r_0{}^{10} r^2 +$

$1766023 r_c^{12} r_0{}^5 r^9 \alpha^7 + 1942472 r_c^{10} r_0{}^7 r^9 \alpha^8 - 28952 r_c^8 r_0^{12} r^6 \alpha^9. +92323 r_c^8 r_0{}^{13} r^5 \alpha^9 -$

$522032 r_c^{12} r_0{}^6 r^8 \alpha^7 - 27720 r_c^{12} r_0{}^2 r^{12} \alpha^7 - 172760 r_c^{12} r_0{}^{10} r^4 \alpha^7 + 622076 r_c^{10} r_0{}^{11} r^5 \alpha^8 -$

$24185 r_c^{10} r_0^4 r^{12} \alpha^8 - 186424 r_c^{10} r_0{}^6 r^{10} \alpha^8 - 345583 r_c^{10} r_0{}^8 r^8 \alpha^8 + 622076 r_c^{10} r_0{}^5 r^{11} \alpha^8 +$

$1942472 r_c^{10} r_0^9 r^7 \alpha^8 + 48580 r_c^{10} r_0^3 r^{13} \alpha^8 + 48580 r_c^{10} r_0^{13} r^3 \alpha^8 - 186424 r_c^{10} r_0{}^{10} r^6 \alpha^8 -$

$24185 r_c^{10} r_0{}^{12} r^4 \alpha^8 - 3465 r_c^{12} r_0 r^{13} \alpha^7 + 312340 r_c^{12} r_0{}^3 r^{11} \alpha^7 + 3026128 r_c^{12} r_0^7 r^7 \alpha^7 -$

$27720 r_c^{12} r_0^{12} r^2 \alpha^7 - 3465 r_c^{12} r_0{}^{13} r \alpha^7 + 1766023 r_c^{12} r_0{}^9 r^5 \alpha^7 + 312340 r_c^{12} r_0{}^{11} r^3 \alpha^7 -$

$522032 r_c^{12} r_0{}^8 r^6 \alpha^7 - 172760 r_c^{12} r_0{}^4 r^{10} \alpha^7 - 127820 r_c^8 r_0^8 r^{10} \alpha^9.$

$$p_r = -\frac{A\left[\arctan\left(\frac{\sqrt{\alpha} r}{r_c}\right) - \arctan\left(\frac{\sqrt{\alpha} r_0}{r_c}\right)\right]}{\kappa^2 r^3 (1+2\beta)} + \frac{1}{3(r_c^2 + \alpha r^2)^8 \kappa^2 (1+2\beta)(8\beta+1) r^3} [136\beta r^2 \mathcal{F}(r) \alpha r_0 + 8\beta r^2 \mathcal{F}'(r) r_c^2$$
$$+8\beta r^4 \mathcal{F}'(r)\alpha - 16\beta r \mathcal{F}(r) r_c^2 - 128\beta r^3 \mathcal{F}(r)\alpha + 3\mathcal{F}(r) r_0 \alpha r^2 + 24\beta \mathcal{F}(r) r_0 r_c^2 + 56\beta r^3 A \alpha^{\frac{3}{2}} r_c^{13} + 168\beta r^5 A \alpha^{\frac{5}{2}} r_c^{11}$$
$$+280\beta r^7 A \alpha^{\frac{7}{2}} r_c^9 + 280\beta r^9 A \alpha^{\frac{9}{2}} r_c^7 + 168\beta r^{11} A \alpha^{\frac{11}{2}} r_c^5 + 56\beta r^{13} A \alpha^{\frac{13}{2}} r_c^3 + 8\beta r^{15} A \alpha^{\frac{15}{2}} r_c - 192\beta r_0 r_c^{14} \alpha r^2 - 672\beta r_0 r_c^{12} \alpha^2$$
$$-1344\beta r_0 r_c^{10} \alpha^3 r^6 - 1680\beta r_0 r_c^8 \alpha^4 r^8 - 1344\beta r_0 r_c^6 \alpha^5 r^{10} - 672\beta r_0 r_c^4 \alpha^6 r^{12} - 192\beta r_0 r_c^2 \alpha^7 r^{14} + 8\beta r A \sqrt{\alpha} r_c^{15}$$
$$-8\beta r \mathcal{F}'(r) r_0 r_c^2 - 8\beta r^3 \mathcal{F}'(r) r_0 \alpha - 24 r_0 r_c^{14} \alpha r^2 - 84 r_0 r_c^{12} \alpha^2 r^4 - 168 r_0 r_c^{10} \alpha^3 r^6 - 210 r_0 r_c^8 \alpha^4 r^8 - 168 r_0 r_c^6 \alpha^5 r^{10}$$
$$-84 r_0 r_c{}^4 \alpha^6 r^{12} - 24 r_0 r_c{}^2 \alpha^7 r^{14} - 24\beta r_0 \alpha^8 r^{16} - 3\mathcal{F}(r) r r_c{}^2 - 3\mathcal{F}(r) r^3 \alpha + 3\mathcal{F}(r) r_0 r_c{}^2 - 3 r_0 r_c{}^{16} - 3 r_0 \alpha^8 r^{16} - 24\beta r_0$$

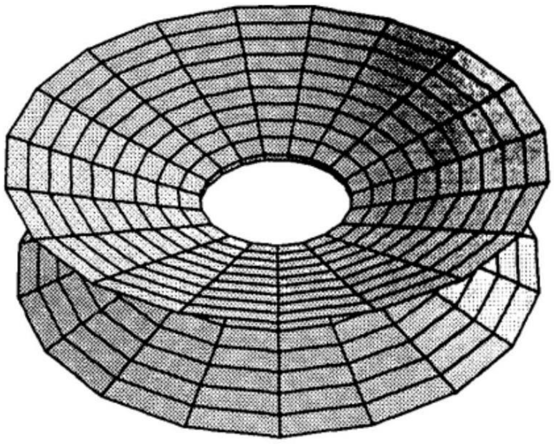
$$p_\theta = \frac{A\left[\arctan\left(\frac{\sqrt{\alpha} r}{r_c}\right) - \arctan\left(\frac{\sqrt{\alpha} r_0}{r_c}\right)\right]}{2\kappa^2 r^3 (1+2\beta)} + \frac{1}{6(r_c^2 + \alpha r^2)^8 \kappa^2 r^3 (1 + 10\beta + 16\beta^2)} [-(8\beta+3) r_c A r^{15}$$
$$-21(8\beta+3) r_c^5 A r^{11} \alpha^{\frac{11}{2}} - 7(8\beta+3) r_c^3 A r^{13} \alpha^{\frac{13}{2}} - 35(8\beta+3) r_c^7 A r^9 \alpha^{9/2} - 7(8\beta+3) r_c^{13} A r^3 \alpha^{3/2} - 21(8\beta +$$
$$-35(8\beta+3) r_c{}^9 A r^7 \alpha^{7/2} + 3 r_0 (8\beta+1)(r_c{}^2 + \alpha r^2)^8 - (8\beta+3)(r - r_0) r (r_c{}^2 + \alpha r^2) \mathcal{F}'(r) - (8\beta+3)$$
$$+((128\beta + 42)\alpha r^3 - r_0(45 + 136\beta)\alpha r^2 + 16\beta r r_c{}^2 - 3 r_0 r_c{}^2 (8\beta+1))\mathcal{F}(r)]$$

$$F_h = \frac{3}{(1+2\beta)\kappa^2 r^4}\left[r_0 + A\left[\arctan\left(\sqrt{\alpha}r/r_c\right) - \arctan\left(\sqrt{\alpha}r_0/r_c\right)\right] + B\frac{r-r_0}{(r_c^2+\alpha r^2)^7}\mathcal{F}(r)\right]$$
$$+\frac{(3+20\beta)}{3(1+2\beta)(1+4\beta)\kappa^2 r^3}\left[\frac{A\sqrt{\alpha}}{r_c\left(1+\frac{r^2\alpha}{r_c^2}\right)} + \frac{\mathcal{F}'(r)(r-r_0)}{(r_c^2+r^2\alpha)^7} + \frac{\mathcal{F}(r)}{(r_c^2+r^2\alpha)^7} - 14\frac{\mathcal{F}(r)(r-r_0)r\alpha}{(r_c^2+r^2\alpha)^8}\right]$$
$$-\frac{4\beta}{3(1+2\beta)(1+4\beta)\kappa^2 r^2}\left[-\frac{2A\alpha^{3/2}r}{r_c^3\left(1+\frac{r^2\alpha}{r_c^2}\right)^2} + \frac{\mathcal{F}''(r)(r-r_0)}{(r_c^2+r^2\alpha)^7} + \frac{2\mathcal{F}'(r)}{(r_c^2+r^2\alpha)^7} - \frac{28\mathcal{F}'(r)(r-r_0)r\alpha}{(r_c^2+r^2\alpha)^8}\right.$$
$$\left.-\frac{28\mathcal{F}(r)r\alpha}{(r_c^2+r^2\alpha)^8} + \frac{224\mathcal{F}(r)(r-r_0)r^2\alpha^2}{(r_c^2+r^2\alpha)^9} - \frac{14\mathcal{F}(r)(r-r_0)\alpha}{(r_c^2+r^2\alpha)^8}\right]$$

$$F_c = \frac{3\beta}{(1+2\beta)(3+7\beta)\kappa^2 r^4}\left[r_0 + A\left[\arctan\left(\sqrt{\alpha}r/r_c\right) - \arctan\left(\sqrt{\alpha}r_0/r_c\right)\right] + B\frac{r-r_0}{(r_c^2+\alpha r^2)^7}\mathcal{F}(r)\right]$$
$$-\frac{\beta(27+68\beta)}{3(1+2\beta)(1+4\beta)(3+7\beta)\kappa^2 r^3}\left[\frac{A\sqrt{\alpha}}{r_c\left(1+\frac{r^2\alpha}{r_c^2}\right)} + \frac{\mathcal{F}'(r)(r-r_0)}{(r_c^2+r^2\alpha)^7} + \frac{\mathcal{F}(r)}{(r_c^2+r^2\alpha)^7} - 14\frac{\mathcal{F}(r)(r-r_0)r\alpha}{(r_c^2+r^2\alpha)^8}\right]$$
$$+\frac{4\beta}{3(1+2\beta)(1+4\beta)\kappa^2 r^2}\left[-\frac{2A\alpha^{3/2}r}{r_c^3\left(1+\frac{r^2\alpha}{r_c^2}\right)^2} + \frac{\mathcal{F}''(r)(r-r_0)}{(r_c^2+r^2\alpha)^7} + \frac{2\mathcal{F}'(r)}{(r_c^2+r^2\alpha)^7} - \frac{28\mathcal{F}'(r)(r-r_0)r\alpha}{(r_c^2+r^2\alpha)^8}\right.$$
$$\left.-\frac{28\mathcal{F}(r)r\alpha}{(r_c^2+r^2\alpha)^8} + \frac{224\mathcal{F}(r)(r-r_0)r^2\alpha^2}{(r_c^2+r^2\alpha)^9} - \frac{14\mathcal{F}(r)(r-r_0)\alpha}{(r_c^2+r^2\alpha)^8}\right]$$

$$F_a = \frac{9}{(3+7\beta)\kappa^2 r^4}\left[r_0 + A\left[\arctan\left(\sqrt{\alpha}r/r_c\right) - \arctan\left(\sqrt{\alpha}r_0/r_c\right)\right] + B\frac{r-r_0}{(r_c^2+\alpha r^2)^7}\mathcal{F}(r)\right]$$
$$-\frac{3}{(3+7\beta)\kappa^2 r^3}\left[\frac{A\sqrt{\alpha}}{r_c\left(1+\frac{r^2\alpha}{r_c^2}\right)} + \frac{\mathcal{F}'(r)(r-r_0)}{(r_c^2+r^2\alpha)^7} + \frac{\mathcal{F}(r)}{(r_c^2+r^2\alpha)^7} - 14\frac{\mathcal{F}(r)(r-r_0)r\alpha}{(r_c^2+r^2\alpha)^8}\right]$$

$$F_h = \frac{3}{(1+2\beta)\kappa^2 r^4}\left[r_0 + \frac{\tilde{A}(r-r_0)}{r+r_s} + \tilde{B}\ln\left(\frac{r+r_s}{r_s+r_0}\right)\right] + \frac{(3+20\beta)\left[\tilde{B}r + (\tilde{A}+\tilde{B})r_s + \tilde{A}r_0\right]}{3(r+r_s)^2(1+2\beta)(1+4\beta)\kappa^2 r^3}$$
$$+\frac{4\beta\left[\tilde{B}r + (2\tilde{A}+\tilde{B})r_s + 2\tilde{A}r_0\right]}{3(r+r_s)^3(1+2\beta)(1+4\beta)(3+7\eta)\kappa^2 r^2}$$
$$F_c = \frac{3\beta}{(1+2\beta)(3+7\beta)\kappa^2 r^4}\left[r_0 + \tilde{A}\frac{r-r_0}{r_s+r} + \tilde{B}\ln\left(\frac{r_s+r}{r_s+r_0}\right)\right] - \frac{\beta(27+68\beta)}{3(1+2\beta)(1+4\beta)(3+7\beta)\kappa^2 r^3}\times$$
$$\left[\frac{(\tilde{A}+\tilde{B})r_s + \tilde{A}r_0 + \tilde{B}r}{(r+r_s)^2}\right] - \frac{4\beta}{3(1+2\beta)(1+4\beta)\kappa^2 r^2}\left[\frac{(2\tilde{A}+\tilde{B})r_s + 2\tilde{A}r_0 + \tilde{B}r}{(r+r_s)^3}\right]$$
$$F_a = \frac{9}{(3+7\beta)\kappa^2 r^4}\left[r_0 + \frac{\tilde{A}(r-r_0)}{r+r_s} + \tilde{B}\ln\left(\frac{r+r_s}{r_s+r_0}\right)\right] - \frac{(3\tilde{A}+3\tilde{B})r_s + 3\tilde{A}r_0 + 3\tilde{B}r}{(r+r_s)^2(3+7\beta)\kappa^2 r^3}$$



$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$

$$dr^k/dr = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$

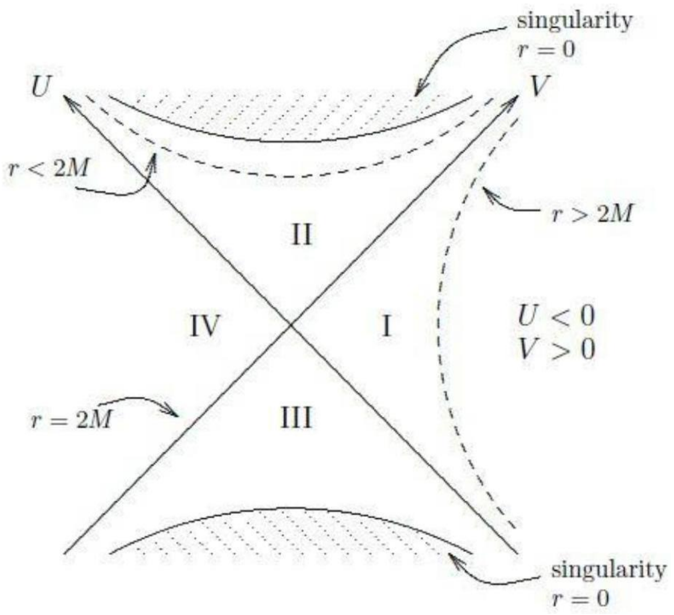
$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2dudr + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$

$$U = -e^{-u/4M}$$

$$V = e^{v/4M}$$

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2(d\theta^2 + \sin^2\ \theta d\varphi^2)$$

$$UV = -\frac{(r - 2M)}{2M}e^{r/2M}$$

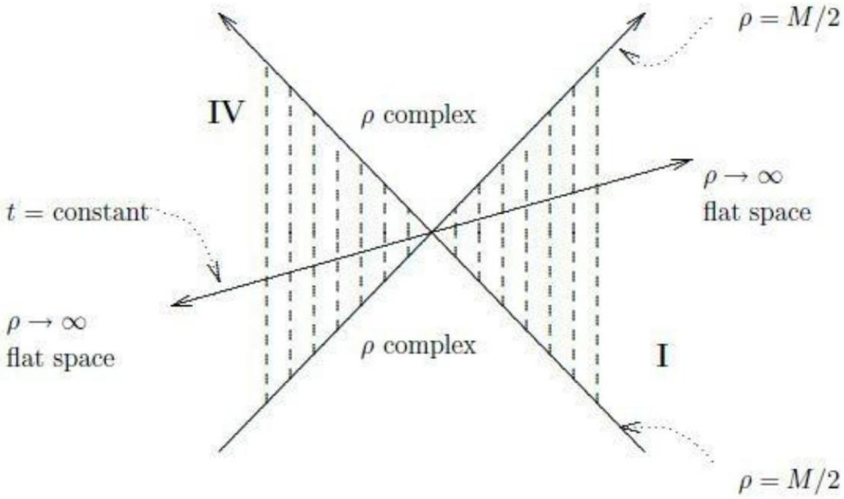


$$\frac{U}{V} = e^{-t/2M}$$

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2$$

$$ds^2 = -\left(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}}\right)^2 dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\ \theta d\varphi^2)\right)$$

$$\rho \to M^2/(4\rho)$$



$$R_{\mu v} - \frac{1}{2}Rg_{\mu v} = 8\pi\left(\nabla_\mu\Phi\nabla_v\Phi - \frac{1}{2}g_{\mu v}\nabla^\sigma\Phi\nabla_\sigma\Phi\right)$$

$$\begin{aligned} ds^2 &= -dt^2 + (d\rho - f(\rho)dt)^2 + r^2(\rho)(d\theta^2 + \sin^2\ \theta d\varphi^2) \\ &= -[1 - f^2(\rho)]dT^2 + \frac{1}{1 - f^2(\rho)}d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\ \theta d\varphi^2) \end{aligned}$$

$$dT = dt + \frac{f(\rho)}{1 - f^2(\rho)} d\rho$$

$$-\infty < t < \infty;\ -\infty < \rho < \infty;\ 0 < \theta < \pi;\ -\pi < \varphi < \pi.$$

$$f(\rho) = -\sqrt{1 - e^{-\left(\frac{2m}{n}\right)\tau(\rho)}}$$

$$r(\rho) = \sqrt{(\rho - m)^2 + a^2 e^{\left(\frac{m}{n}\right)\tau(\rho)}}$$

$$\tau(\rho) = \frac{n}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho - m}{a}\right)\right]$$

$$a = \sqrt{n^2 - m^2}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{\left(1 - \frac{b(r)}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

$$G_{tt} = \frac{b'(r)}{r^2}$$

$$G_{rr} = -\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\phi'(r)}{r}$$

$$G_{\theta\theta} = \left(1 - \frac{b(r)}{r}\right)\left[\phi''(r) - \frac{rb'(r) - b(r)}{2r(r - b(r))}\phi'(r) + \phi'(r)^2 + \frac{\phi'(r)}{r} - \frac{rb'(r) - b(r)}{2r^2(r - b(r))}\right] = G_{\varphi\varphi}$$

$$T_{tt} = \rho(r), T_{rr} = -\tau(r), T_{\theta\theta} = T_{\varphi\varphi} = p(r)$$

$$\rho(r) = \frac{1}{8\pi}\left[\frac{b'(r)}{r^2}\right]$$

$$\tau(r) = \frac{1}{8\pi}\left[\frac{b(r)}{r^3} - 2\left(1 - \frac{b(r)}{r}\right)\frac{\phi'(r)}{r}\right]$$

$$p(r) = \frac{1}{8\pi}\left(1 - \frac{b(r)}{r}\right)\left[\phi''(r) - \frac{rb'(r) - b(r)}{2r(r - b(r))}\phi'(r) + \phi'(r)^2 + \frac{\phi'(r)}{r} - \frac{rb'(r) - b(r)}{2r^2(r - b(r))}\right]$$

$$ds^2 = N^2 dt^2 - \frac{dr^2}{\left(1 - \frac{b}{r}\right)} - r^2 K^2 [d\theta^2 + \sin^2 \theta (d\varphi - \omega dt)^2]$$

$$\xi = \frac{(\tau - \rho)}{|\rho|} = \frac{b/r - b' - r(1 - b/r)\phi'}{|b'|}.$$

$$\frac{d^2 r}{dz^2} = \frac{b - rb'}{b^2} > 0$$

$$\xi = \frac{2b^2}{r|b'|}\frac{d^2r}{dz^2} - \frac{r(1-b/r)\phi'}{|b'|}$$

$$(1-b/r)\phi' \to 0$$

$$\xi(r_0) = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0$$

$$T_\mu{}^\nu = \mathrm{diag}(\rho, -p_1, -p_2, -p_3).$$

$$T_{\mu\nu}k^\mu k^\nu \geq 0$$

$$\rho + p_i \geq 0, \forall i.$$

$$T_{\mu\nu}U^\mu U^\nu \geq 0$$

$$\rho \geq 0 \ \text{ and } \ \rho + p_i \geq 0, \forall i.$$

$$\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right)U^\mu U^\nu \geq 0$$

$$\rho + p_i \geq 0 \ \text{ and } \ \rho + \sum_i p_i \geq 0$$

$$T_{\mu\nu}U^\mu U^\nu \geq 0$$

$$\rho \geq 0 \ \text{ and } \ p_i \in [-\rho, +\rho].$$

$$M_{200} = \frac{4}{3}\pi 200\rho_{\mathrm{crit}} r_{200}^3,$$

$$\rho(r) = \rho_s\left(\frac{r}{r_s}\right)^{-\gamma}\left(1+\frac{r}{r_s}\right)^{\gamma-\beta},$$

$$\rho_{\mathrm{NFW}}(r) = \rho_s\left(\frac{r}{r_s}\right)^{-1}\left(1+\frac{r}{r_s}\right)^{-2}.$$

$$\rho_{\mathrm{PI}}(r) = \frac{\rho_s}{1+\left(\frac{r}{r_s}\right)^2}.$$

$$\rho_{\mathrm{URC}}(r) = \frac{\rho_s r_s^3}{(r+r_s)(r^2+r_s^2)}$$

$$\rho_{\mathrm{PI}}(r) = \frac{\rho_s}{1+r^2/r_c^2}$$

$$\mathcal{Z} = \sum_{N_i,\varepsilon} e^{-\zeta\left(E_{N_i,\varepsilon} - \sum_i N_i\mu_i\right)}$$

$$\zeta = \frac{1}{k_B T}, N_i$$

$$E_{N_i,\varepsilon} = f(N_i, m_i, V, \varepsilon)$$

$$p_0 = -\Xi = \frac{1}{\zeta V}\ln\,\mathcal{Z}$$

$$\bar{E} = -\frac{\partial}{\partial\zeta}\ln\,\mathcal{Z} + \sum_i \bar{N}_i\mu_i$$

$$\bar{N}_i = \frac{1}{\zeta}\left(\frac{\partial}{\partial\mu_i}\ln\,\mathcal{Z}\right)_{T,V,m_j} = -V\left(\frac{\partial\Xi}{\partial\mu_i}\right)_{T,m_j}$$

$$E_0 = \Xi + \sum_i n_i\mu_i$$

$$n_i = \frac{\bar{N}_i}{V} = -\left(\frac{\partial\Xi}{\partial\mu_i}\right)_{T,m_j}$$

$$E^{Bag}_{N_i,\varepsilon} = E_{N_i,\varepsilon} + BV$$

$$\mathcal{Z}^{Bag} = \mathcal{Z}e^{-\zeta BV}$$

$$n_i^{Bag} = \frac{\bar{N}_i}{V} = -\left(\frac{\partial}{\partial\mu_i}(\Xi + B)\right)_{T,m_j,E_{N_i,\varepsilon},B}$$

$$p^{Bag} = -(\Xi + B);\ E^{Bag} = (\Xi + B) + \sum_i n_i\mu_i$$

$$p^{Bag} = p_0 - B;\ E^{Bag} = (E_0 + B)$$

$$p^{Bag} = \frac{1}{3}(\rho - 4B)$$

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}$$

$$\rho_G = \frac{Me^{-\frac{r^2}{4\Theta}}}{8\pi^{3/2}\Theta^{3/2}}$$

$$\rho_L = \frac{\sqrt{\Theta}M}{\pi^2(\Theta + r^2)^2}$$

$$\nabla_\gamma V_\nu^\mu = \partial_\gamma V_\nu^\mu + V_\nu^\delta\Gamma_{\delta\gamma}^\mu - V_\delta^\mu\Gamma_{\nu\gamma}^\delta$$

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$$

$$\nabla_\gamma g_{\mu\nu} = 0$$

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\delta}\left(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\mu\delta} - \partial_\delta g_{\mu\nu}\right)$$

$$R^{\lambda}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\lambda}_{\mu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\tau}_{\mu\sigma}\Gamma^{\lambda}_{\tau\nu} - \Gamma^{\tau}_{\mu\nu}\Gamma^{\lambda}_{\tau\sigma}.$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = -R^{\lambda}_{\mu\nu\lambda}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_m$$

$$ds^2 = dt^2 - a^2(t)d\bar{x}^2 = a^2(\eta)[d\eta^2 - d\bar{x}^2]$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}$$

$$K^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}\left(\mathcal{T}_{\nu\delta\mu} + \mathcal{T}_{\mu\delta\nu} - \mathcal{T}_{\delta\mu\nu}\right),$$

$$L^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}\left(Q_{\delta\mu\nu} - Q_{\mu\nu\delta} - Q_{\nu\delta\mu}\right)$$

$$\hat{R}^{\lambda}_{\mu\nu\sigma} = R^{\lambda}_{\mu\nu\sigma} + \nabla_\nu \hat{\mathcal{D}}^{\lambda}_{\mu\sigma} - \nabla_\sigma \hat{\mathcal{D}}^{\lambda}_{\mu\nu} + \hat{\mathcal{D}}^{\lambda}_{\tau\nu}\hat{\mathcal{D}}^{\tau}_{\mu\sigma} - \hat{\mathcal{D}}^{\lambda}_{\tau\sigma}\hat{\mathcal{D}}^{\tau}_{\mu\nu},$$

$$\hat{\mathcal{D}}^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu} = K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}.$$

$$R^{\lambda}_{\mu\nu\sigma} = K^{\lambda}_{\tau\sigma}K^{\tau}_{\mu\nu} - K^{\lambda}_{\tau\nu}K^{\tau}_{\mu\sigma} + \nabla_\sigma K^{\lambda}_{\mu\nu} - \nabla_\nu K^{\lambda}_{\mu\sigma}.$$

$$R^{\lambda}_{\mu\nu\sigma} = L^{\lambda}_{\tau\sigma}L^{\tau}_{\mu\nu} - L^{\lambda}_{\tau\nu}L^{\tau}_{\mu\sigma} + \nabla_\sigma L^{\lambda}_{\mu\nu} - \nabla_\nu L^{\lambda}_{\mu\sigma}$$

$$R = K^{\lambda}{}_{\nu\mu}K^{\nu\mu}{}_{\lambda} - K^{\lambda}{}_{\nu\lambda}K^{\nu\mu}{}_{\mu} - 2\nabla_\lambda K^{\lambda\mu}{}_{\mu} = \mathcal{T} + 2\nabla_\lambda \mathcal{T}_{\mu}{}^{\mu\lambda},$$

$$R = L^{\lambda\mu\nu}L_{\nu\lambda\mu} - L^{\lambda}_{\lambda\nu}L^{\nu\mu}_{\mu} + \nabla_\mu L^{\lambda\mu}_{\lambda} - \nabla_\lambda L^{\lambda\mu}{}_{\mu} = Q + \nabla_\mu Q^{\lambda\mu}_{\lambda} - \nabla_\lambda Q^{\lambda\mu}{}_{\mu}$$

$$S_{EH} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R$$

$$S_{\mathrm{TEGR}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\mathcal{T}.$$

$$S_{STEGR} = \frac{1}{16\pi}\int d^4x\sqrt{-g}Q$$

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = \eta_{ab}\vartheta^a \otimes \vartheta^b$$

$$\tilde{\Gamma}^{a}_{b} = \omega^{a}_{b} + K^{a}_{b} + L^{a}_{b},$$

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu} = -L^{\alpha}_{\lambda\mu}g_{\alpha\nu} - L^{\alpha}_{\lambda\nu}g_{\alpha\mu}$$

$$Q = -g^{\mu\nu}\left(L^{\alpha}_{\beta\nu}L^{\beta}_{\mu\alpha} - L^{\beta}_{\alpha\beta}L^{\alpha}_{\mu\nu}\right)$$

$$P^{\lambda}_{\mu\nu} = -\frac{1}{2}L^{\lambda}_{\mu\nu} + \frac{1}{4}\left(Q^{\lambda} - \tilde{Q}^{\lambda}\right)g_{\mu\nu} - \frac{1}{4}\delta^{\lambda}_{(\mu}Q_{\nu)}$$

$$Q = -g^{\mu\nu}\left(L^{\alpha}_{\beta\nu}L^{\beta}_{\mu\alpha} - L^{\beta}_{\alpha\beta}L^{\alpha}_{\mu\nu}\right) = -P^{\lambda\mu\nu}Q_{\lambda\mu\nu}.$$

$$S = \frac{1}{16\pi}\int \sqrt{-g}f(Q)d^4x + \int \mathcal{L}_m\sqrt{-g}d^4x$$

$$-\frac{2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}f_Q P^{\alpha}_{\mu\nu}\right) - f_Q\left(P_{\mu\alpha\beta}Q_{\nu}^{\alpha\beta} - 2Q^{\alpha\beta}_{\mu}P_{\alpha\beta\nu}\right) - \frac{1}{2}g_{\mu\nu}f = 8\pi T_{\mu\nu},$$

$$f_Q = \frac{df}{dQ}$$

$$\nabla_\mu\nabla_\gamma\left(\sqrt{-g}f_Q P^{\gamma}_{\mu\nu}\right) = 0.$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \chi^\alpha}\partial_\mu\partial_\nu\chi^\alpha$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Lambda^{\lambda}_{\alpha}\partial_\mu(\Lambda^{-1})^{\alpha}_{\nu},$$

$$L^{\lambda}_{\mu\nu}\Big|_{G=0} = \left(\frac{1}{2}Q^{\lambda}_{\mu\nu} - Q^{\lambda}_{(\mu\nu)}\right)\Big|_{G=0} = 0,$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}\Big|_{G=0} = \Gamma^{\lambda}_{\mu\nu}\Big|_{G=0} + L^{\lambda}_{\mu\nu}\Big|_{G=0} = \Gamma^{\lambda}_{\mu\nu}\Big|_{G=0}.$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu}\Big|_{G=0} = \Gamma^{\lambda}_{\mu\nu}\Big|_{G=0}$$

$$S = \frac{1}{16\pi}\int f(Q,T)\sqrt{-g}d^4x + \int \mathcal{L}_m\sqrt{-g}d^4x$$

$$\frac{-2}{\sqrt{-g}}\nabla\alpha\left(\sqrt{-g}f_Q P^{\alpha}_{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}f + f_T\left(T_{\mu\nu} + \Xi_{\mu\nu}\right) - f_Q\left(P_{\mu\alpha\beta}Q_{\nu}^{\alpha\beta} - 2Q^{\alpha\beta}_{\mu}P_{\alpha\beta\nu}\right) = 8\pi T_{\mu\nu},$$

$$f_Q = \frac{\partial f}{\partial Q}$$

$$f_T = \frac{\partial f}{\partial T}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}$$

$$\Xi_{\mu\nu} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$$

$$\Xi_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu}\mathcal{L}_m - 2T_{\mu}{}^{\nu}$$

$$S = \frac{1}{16\pi}\int f(Q)\sqrt{-g}d^4x + \int \left(\mathcal{L}_m + \mathcal{L}_{\text{quantum hole}}\right)\sqrt{-g}d^4x$$

$$\mathcal{L}_{\text{quantum hole}} = -\frac{\lambda}{4}(\Phi^2 - \eta^2)^2 - \frac{1}{2}\sum_a g^{ij}\partial_i\Phi^a\partial_j\Phi^a$$

$$\Phi^a = \frac{\eta}{r} F(r) x^a$$

$$\sum_a x^a x^a = r^2$$

$$\mathcal{L}_{\text{quantum hole}} = -\left(1 - \frac{b(r)}{r}\right)\frac{\eta^2 (F')^2}{2} - \frac{\eta^2 F^2}{r^2} - \frac{\lambda \eta^4}{4}(F^2 - 1)^2.$$

$$\left(1 - \frac{b(r)}{r}\right)F'' + F'\left[\left(1 - \frac{b(r)}{r}\right)\frac{2}{r} + \frac{1}{2}\left(\frac{b - b'r}{r^2}\right)\right] - F\left[\frac{2}{r^2} + \lambda\eta^2(F^2 - 1)\right] = 0.$$

$$\bar{T}_{ij} = \partial_i \Phi^a \partial_j \Phi^a - \frac{1}{2} g_{ij} g^{\mu v} \partial_\mu \Phi^a \partial_v \Phi^a - \frac{g_{ij}\lambda}{4}(\Phi^2 - \eta^2)^2$$

$$\bar{T}_t^t = -\eta^2\left[\frac{F^2}{r^2} + \left(1 - \frac{b(r)}{r}\right)\frac{(F')^2}{2} + \frac{\lambda\eta^2}{4}(F^2 - 1)^2\right],$$
$$\bar{T}_r^r = -\eta^2\left[\frac{F^2}{r^2} + \left(1 - \frac{b(r)}{r}\right)\frac{(F')^2}{2} + \frac{\lambda\eta^2}{4}(F^2 - 1)^2\right],$$

$$\bar{T}_\theta^\theta = \bar{T}_\varphi^\varphi = -\eta^2\left[\left(1 - \frac{b(r)}{r}\right)\frac{(F')^2}{2} + \frac{\lambda\eta^2}{4}(F^2 - 1)^2\right].$$

$$\bar{T}_t^t = \bar{T}_r^r = -\frac{\eta^2}{r^2}, \bar{T}_\theta^\theta = \bar{T}_\varphi^\varphi = 0$$

$$T_{\mu v} = \mathbb{T}_{\mu v} + \bar{T}_{\mu v}$$

$$\mathbb{T}_\mu^v = (\rho + p_t) u_\mu u^v - p_t \delta_\mu^v + (p_r - p_t) v_\mu v^v$$

$$Q_{rtt} = 2e^{2\phi}\phi', Q_{rrr} = -\frac{(rb' - b)}{(r - b)^2}, Q_{\theta r\theta} = Q_{\theta\theta r} = -\frac{rb}{r - b}, Q_{\varphi r\varphi} = Q_{\varphi\varphi r} = -\frac{rb\sin^2\,\theta}{r - b}$$

$$L_{tr}^t = L_{rt}^t = -\phi', L_{\theta\theta}^r = -b, L_{tt}^r = -\frac{(r - b)}{r} e^{2\phi}\phi', L_{rr}^r = -\frac{(rb' - b}{2r(r - b)}, L_{\varphi\varphi}^r = -b\sin^2\,\theta$$

$$Q = -\frac{b}{r^2}\left[2\phi' + \frac{rb' - b}{r(r - b)}\right].$$

$$8\pi\rho = \frac{(r - b)}{2r^3}\left[f_Q\left(\frac{(2r - b)(rb' - b)}{(r - b)^2} + \frac{b(2r\phi' + 2)}{r - b}\right) + \frac{fr^3}{r - b} + \frac{2brf_{QQ}Q'}{r - b}\right] - \frac{8\pi\eta^2}{r^2},$$
$$8\pi p_r = -\frac{(r - b)}{2r^3}\left[-f_Q\left(\frac{b}{r - b}\left(\frac{rb' - b}{r - b} + 2 + 2r\phi'\right) - 4r\phi'\right) - \frac{fr^3}{r - b} - \frac{2brf_{QQ}Q'}{r - b}\right] + \frac{8\pi\eta^2}{r^2},$$

$$8\pi p_t = \frac{(r - b)}{4r^2}\left[f_Q\left(\frac{(rb' - b)\left(\frac{2r}{r - b} + 2r\phi'\right)}{r(r - b)} + \frac{4(2b - r)\phi'}{r - b} - 4r(\phi')^2 - 4r\phi''\right) + \frac{2fr^2}{r - b} - 4rf_{QQ}Q'\phi'\right]$$

$$f(Q) = \alpha Q,$$

$$\rho = \frac{\alpha b' - 8\pi\eta^2}{8\pi r^2}$$

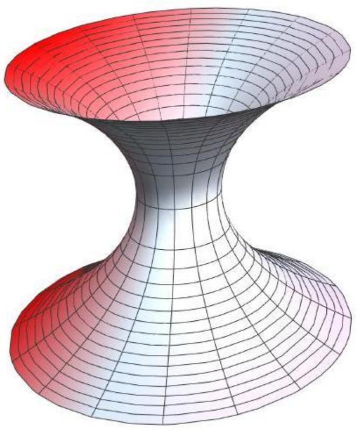
$$p_r = \frac{2\alpha r(b-r)\phi' + \alpha b + 8\pi\eta^2 r}{8\pi r^3}$$

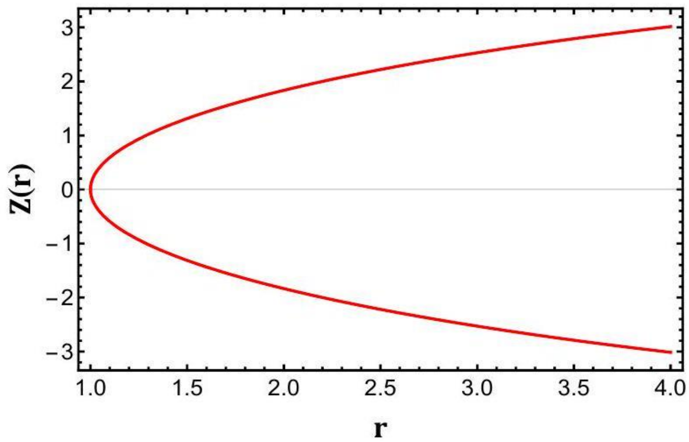
$$p_t = \frac{\alpha(r\phi'+1)(rb' + 2r(b-r)\phi' - b) + 2\alpha r^2(b-r)\phi''}{16\pi r^3}.$$

$$b(r) = \frac{1}{\alpha}\left(8\pi\rho_s r_s^3\left(\tan^{-1}\left(\frac{r_0}{r_s}\right) - \tan^{-1}\left(\frac{r}{r_s}\right)\right) + r_0\big(\alpha - 8\pi(\eta^2 + \rho_s r_s^2)\big) + 8\pi r(\eta^2 + \rho_s r_s^2)\right).$$

$$\frac{dz}{dr} = \pm\frac{1}{\sqrt{\frac{r}{b(r)} - 1}}.$$

$$Z(r) = \pm\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{r}{b(r)} - 1}}$$





$$p_r = \frac{1}{8\pi r^3}\left(8\pi\rho_s r_s^3\left(\tan^{-1}\left(\frac{r_0}{r_s}\right) - \tan^{-1}\left(\frac{r}{r_s}\right)\right) + r_0\big(\alpha - 8\pi(\eta^2 + \rho_s r_s^2)\big) + 8\pi r(2\eta^2 + \rho_s r_s^2)\right)$$

$$p_t = \frac{1}{16r^3}\left(-\frac{r_0\alpha}{\pi} + 8\left(\tan^{-1}\left(\frac{r}{r_s}\right) - \tan^{-1}\left(\frac{r_0}{r_s}\right)\right)\rho_s r_s^3 + 8r_0(\eta^2 + \rho_s r_s^2) - \frac{8\rho_s r r_s^4}{r^2 + r_s^2}\right).$$

$$(\rho + p_r)_{\mathrm{at} r=r_0} = \frac{\frac{\alpha}{8\pi} + \eta^2}{r_0^2} + \frac{\rho_s r_s^2}{r_0^2 + r_s^2}$$

$$(\rho + p_t)_{\mathrm{at}\ r=r_0} = \frac{3\rho_s r_s^2}{2(r_0^2 + r_s^2)} - \frac{\alpha - 8\pi\eta^2}{16\pi r_0^2}.$$

$$p_r = \frac{1}{8\pi r^4}\left(r_0(r-2)\left(\alpha - 8\pi(\eta^2+\rho_s r_s^2)\right) + 8\pi\rho_s r_s^3(r-2)\left(\tan^{-1}\left(\frac{r_0}{r_s}\right) - \tan^{-1}\left(\frac{r}{r_s}\right)\right) + 2r \times (\alpha + 8\pi\eta^2(r-1) + 4\pi\rho_s(r-2)r_s^2)\right)$$

$$p_t = \frac{1}{16r^5}\left(\frac{r_0}{\pi}((r-3)r-2)(8\pi(\eta^2+\rho_s r_s^2)-\alpha) + 8((r-3)r-2)\left(\tan^{-1}\left(\frac{r}{r_s}\right) - \tan^{-1}\left(\frac{r_0}{r_s}\right)\right) \times \rho_s r_s^3 + \frac{8\rho_s r r_s^2}{r^2+r_s^2}(2(r+1)r^2 + (2-(r-3)r)r_s^2) - \frac{2r}{\pi}(r+1)(\alpha - 8\pi\eta^2)\right)$$

$$(\rho + p_r)_{\mathrm{at} r=r_0} = \frac{\frac{\alpha}{8\pi}+\eta^2}{r_0^2} + \frac{\rho_s r_s^2}{r_0^2+r_s^2}$$

$$(\rho + p_t)_{\mathrm{at} r=r_0} = \frac{1}{16r_0^3}\left(\frac{8r_0^2(3r_0-1)\rho_s r_s^2}{r_0^2+r_s^2} + \frac{(r_0-1)(8\pi\eta^2-\alpha)}{\pi}\right)$$

$$\phi(r) = \log\left(1+\frac{r_0}{r}\right)$$

$$p_r = \frac{1}{8\pi r^3(r_0+r)}\left(-r_0^2\alpha + 8\pi r_0^2\eta^2 + 3r_0\alpha r - 8\pi r_0\eta^2 r + 8\pi\rho_s r_s^2(r_0-r)^2 + 8\pi\rho_s r_s^3\left(\tan^{-1}\left(\frac{r}{r_s}\right) \times (r_0-r) - \tan^{-1}\left(\frac{r_0}{r_s}\right)\right) + 16\pi\eta^2 r^2\right)$$

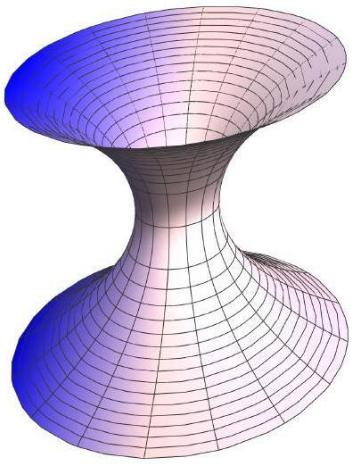
$$p_t = \frac{1}{16\pi r^3(r_0+r)(r^2+r_s^2)}\left(-8\pi\rho_s r_s^2(2r_0^2(r^2+r_s^2) - 3r_0 r(r^2+r_s^2) + r^2 r_s^2) + r_0(\alpha - 8\pi\eta^2) \times (2r_0-3r)(r^2+r_s^2) + \left(\tan^{-1}\left(\frac{r_0}{r_s}\right) - \tan^{-1}\left(\frac{r}{r_s}\right)\right)8\pi\rho_s r_s^3(2r_0-r)(r^2+r_s^2)\right)$$

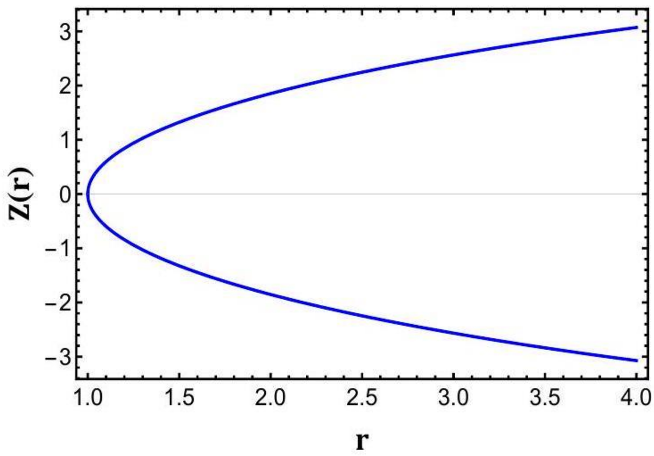
$$(\rho + p_r)_{\mathrm{at} r=r_0} = \frac{\frac{\alpha}{8\pi}+\eta^2}{r_0^2} + \frac{\rho_s r_s^2}{r_0^2+r_s^2}$$

$$(\rho + p_t)_{\mathrm{at}\, r=r_0} = \frac{5\rho_s r_s^2}{4(r_0^2+r_s^2)} - \frac{\alpha - 8\pi\eta^2}{32\pi r_0^2}.$$

$$b(r) = -\frac{8\pi}{\alpha}\left(\frac{\rho_s r_s^4}{r_0+r_s} + \rho_s r_s^3 \log(r_0+r_s) + \eta^2(r_0+r_s)\right) + r_0 + \frac{8\pi}{\alpha}\left(\frac{\rho_s r_s^4}{r+r_s} + \rho_s r_s^3 \log(r+r_s) + \eta^2(r+r_s)\right)$$

$$p_r = \frac{1}{8r^3}\left(\frac{r_0\alpha}{\pi} - 8\eta^2(r_0-2r) + \frac{8\rho_s r_s^4(r_0-r)}{(r_0+r_s)(r+r_s)} + 8\rho_s r_s^3(\log(r+r_s) - \log(r_0+r_s))\right)$$





$$p_t = \frac{1}{16r^3}\left(\frac{8\rho_s r_s^3\left(r_0(r^2 - rr_s - r_s^2) + rr_s(2r + r_s)\right)}{(r_0 + r_s)(r + r_s)^2} + r_0\left(8\eta^2 - \frac{\alpha}{\pi}\right) + 8(\log(r_0 + r_s) - \log(r + r_s))\rho_s r_s^3\right)$$

$$(\rho + p_r)_{\text{at } r=r_0} = \frac{1}{8r_0^2}\left(8\left(\frac{r_0\rho_s r_s^3}{(r_0 + r_s)^2} + \eta^2\right) + \frac{\alpha}{\pi}\right)$$

$$(\rho + p_t)_{\text{atr}=r_0} = \frac{1}{16r_0^2}\left(8\left(\frac{3r_0\rho_s r_s^3}{(r_0 + r_s)^2} + \eta^2\right) - \frac{\alpha}{\pi}\right).$$

$$p_r = \frac{1}{8r^4}\left(\frac{r_0}{\pi}(r - 2)(\alpha - 8\pi\eta^2) + \frac{2r}{\pi}(\alpha + 8\pi\eta^2(r - 1)) + \frac{8\rho_s(r - 2)r_s^4(r_0 - r)}{(r_0 + r_s)(r + r_s)} + 8\rho_s(r - 2)r_s^3 \times \left(\log(r + r_s) - \log(r_0 + r_s)\right)\right)$$

$$p_t = \frac{1}{16\pi r^5(r_0 + r_s)(r + r_s)^2}\left(-r_0^2((r - 3)r - 2)(\alpha - 8\pi\eta^2)(r + r_s)^2 + r_0(8\pi\rho_s r_s^3((r - 1)r^3 + (2) - (r - 3)r)r_s^2 + (2 - (r - 3)r)rr_s) - (\alpha - 8\pi\eta^2)(r + r_s)^2(((r - 3)r - 2)r_s + 2r(r + 1))) + 8\pi\rho_s((r - 3)r - 2)r_s^3(r_0 + r_s)(r + r_s)^2\left(\log(r_0 + r_s) - \log(r + r_s)\right) + 8\pi\rho_s rr_s^4(r(r(2r + r_s - 4) - 3r_s - 2) - 2r_s) - 2r(r + 1)r_s(\alpha - 8\pi\eta^2)(r + r_s)^2\right)$$

$$(\rho + p_r)_{\text{at } r=r_0} = \frac{1}{8r_0^2}\left(8\left(\frac{r_0\rho_s r_s^3}{(r_0 + r_s)^2} + \eta^2\right) + \frac{\alpha}{\pi}\right)$$

$$(\rho + p_t)_{\text{atr}=r_0} = \frac{1}{16r_0^3}\left(\frac{(r_0 - 1)(8\pi\eta^2 - \alpha)}{\pi} + \frac{8r_0(3r_0 - 1)\rho_s r_s^3}{(r_0 + r_s)^2}\right).$$

$$\phi(r) = \log\left(1 + \frac{r_0}{r}\right)$$

$$p_r = -\frac{1}{8\pi r^3(r_0 + r)(r_0 + r_s)(r + r_s)}\left((r_0 + r_s)(r + r_s)\left(r_0\alpha(r_0 - 3r) - 8\pi\eta^2(r_0^2 - r_0 r + 2r^2)\right) + 8\pi\rho_s \times r_s^4(r_0 - r)^2 - 8\pi\rho_s r_s^3\left(\log(r_0 + r_s) - \log(r + r_s)\right)(r_0 - r)(r_0 + r_s)(r + r_s)\right)$$

$$p_t = \frac{1}{16\pi r^3(r_0+r)(r_0+r_s)(r+r_s)^2}\Big(8\pi\rho_s r_s^3\left(2r_0^2 r_s(r+r_s)+r_0 r(r^2-3rr_s-3r_s^2)+r^2 r_s(2r+r_s)\right)$$
$$-8\pi\rho_s r_s^3(2r_0-r)(r_0+r_s)\big(\log(r_0+r_s)-\log(r+r_s)\big)(r+r_s)^2+r_0(\alpha-8\pi\eta^2)(2r_0-3r)$$
$$\times(r_0+r_s)(r+r_s)^2\Big)$$

$$(\rho+p_r)_{\text{at } r=r_0} = \frac{1}{8r_0^2}\left(8\left(\frac{r_0\rho_s r_s^3}{(r_0+r_s)^2}+\eta^2\right)+\frac{\alpha}{\pi}\right)$$

$$(\rho+p_t)_{\text{at } r=r_0} = \frac{1}{32r_0^2}\left(8\left(\frac{5r_0\rho_s r_s^3}{(r_0+r_s)^2}+\eta^2\right)-\frac{\alpha}{\pi}\right).$$

$$b(r) = r-\frac{r^5}{\frac{r_0^4(r_0-\delta)e^{\frac{2}{r}-\frac{2}{r_0}}}{\delta}+r^4}+\delta, 0<\delta<r_0.$$

$$f(Q) = Q+mQ^n.$$

$$\rho = \frac{-1}{16\pi r^6(r-b)^2}\Bigg(\Big(\frac{b(-rb'+2r(b-r)\phi'+b)}{r^3(r-b)}\Big)^{n-2}2m(n-1)nb(r^2b(r(b''-4\phi'+2r\phi''))-b'(4$$
$$\times r\phi'+5)\big)+r^3b'(b'+2r\phi')+rb^2(-r(b''-8\phi'+4r\phi'')+b'(2r\phi'+3)+4)+b^3(2r^2\phi''-4r\phi'$$
$$-3))+r^3(r-b)(b(rb'+2r(b-r)\phi'+b)-2r^2b')\left(1+mn\left(\frac{b(-rb'+2r(b-r)\phi'+b)}{r^3(r-b)}\right)^{n-1}\right)$$
$$-r^3(r-b)\left(b(-rb'+2r(b-r)\phi'+b)+\left(\frac{b\left(\frac{b-rb'}{r^2-rb}-2\phi'\right)}{r^2}\right)^n mr^3(r-b)\right)+16\pi\eta^2r^4(r-b)^2\Bigg)$$

$$p_r = \frac{1}{16\pi}\Bigg(\frac{-mn}{b(-rb'+2r(b-r)\phi'+b)^2}\big((r^2b(2r((n-1)(b''+2r\phi'')-4(n-2)\phi'+14r\phi'^2)+2b'$$
$$\times(2(5-2n)r\phi'-5n+6)+b'^2)+2r^3(b'+2r\phi')((n-1)b'-2r\phi')+2rb^2(r(-(n-1)(b''$$
$$+4r\phi'')+2(4n-9)\phi'-16r\phi'^2)+b'(2(n-3)r\phi'+3n-5)+4n-5)+b^3(4r(\phi'(-2n+3$$
$$\times r\phi'+5)+(n-1)r\phi'')-6n+9))\left(\frac{b}{r^2}\left(\frac{b-rb'}{r^2-rb}-2\phi'\right)\right)^n\Bigg)+\left(\frac{b}{r^2}\left(\frac{b-rb'}{r^2-rb}-2\phi'\right)\right)^n m$$
$$+\frac{2}{r^3}(2r(b-r)\phi'+b+8\pi\eta^2 r)\Bigg)$$

$$p_t = \frac{1}{16\pi r^4 (r-b)^2}\Bigg((1-\frac{b}{r})\Bigg(2m(n-1)n\phi'(\frac{b(-rb'+2r(b-r)\phi'+b)}{r^3(r-b)})^{n-2}(r^2 b(r(b''-4\phi'$$
$$+2r\phi'')-b'(4r\phi'+5))+r^3b'(b'+2r\phi')+rb^2(-r(b''-8\phi'+4r\phi'')+b'(2r\phi'+3)+4)$$
$$+b^3(2r^2\phi''-4r\phi'-3))-r^2\big((b-rb')(r(r-b)\phi'+r)+2r^2(r-b)^2\phi'^2+2(r-b)(r-2b)$$
$$\times\phi' r+2r^2(r-b)^2\phi''\big)\left(mn\left(\frac{b(-rb'+2r(b-r)\phi'+b)}{r^3(r-b)}\right)^{n-1}+1\right)+r^2b(-rb'+2r(b-r)$$
$$\times\phi'+b)+mr^5(r-b)\left(\frac{b}{r^2}\left(\frac{b-rb'}{r^2-rb}-2\phi'\right)\right)^n\Bigg)\Bigg)$$

$$IV = \oint [\rho + p_r] dV$$

$$dV = r^2 dr d\Omega$$

$$\oint dV = 2\int_{r_0}^{\infty} dV = 8\pi \int_{r_0}^{\infty} r^2 dr$$

$$IV = 8\pi \int_{r_0}^{\infty} (\rho + p_r) r^2 dr$$

$$IV = 8\pi \int_{r_0}^{r_1} (\rho + p_r) r^2 dr$$

$$\frac{2(r-b)}{(2r-b)f_Q}\left[\rho - \frac{(r-b)}{8\pi r^3}\left(\frac{brf_{QQ}Q'}{r-b} + bf_Q\left(\frac{r\phi'+1}{r-b} - \frac{2r-b}{2(r-b)^2}\right) + \frac{fr^3}{2(r-b)}\right) + \frac{f_T(P+\rho)}{8\pi}\right] = \frac{b'}{8\pi r^2}$$

$$\frac{2b}{fr^3}\left[p_r + \frac{(r-b)}{16\pi r^3}\left(f_Q\left(\frac{b\left(\frac{rb'-b}{r-b}+2r\phi'+2\right)}{r-b} - 4r\phi'\right) + \frac{2brf_{QQ}Q'}{r-b}\right) + \frac{fr^3(r-b)\phi'}{8\pi br^2} - \frac{f_T(P-p_r)}{8\pi}\right]$$
$$= \frac{1}{8\pi}\left[2\left(1-\frac{b}{r}\right)\frac{\phi'}{r} - \frac{b}{r^3}\right]$$

$$\frac{1}{f_Q\left(\frac{r}{r-b}+r\phi'\right)}\left[p_t + \frac{(r-b)}{32\pi r^2}\left(f_Q\left(\frac{4(2b-r)\phi'}{r-b} - 4r(\phi')^2 - 4r\phi''\right) + \frac{2fr^2}{r-b} - 4rf_{QQ}Q'\phi'\right)\right.$$
$$\left. + \frac{(r-b)}{8\pi r}\left(\phi'' + \phi'^2 - \frac{(rb'-b)\phi'}{2r(r-b)} + \frac{\phi'}{r}\right)f_Q\left(\frac{r}{r-b}+r\phi'\right) - \frac{f_T(P-p_t)}{8\pi}\right]$$
$$= \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left[\phi'' + \phi'^2 - \frac{(rb'-b)\phi'}{2r(r-b)} - \frac{rb'-b}{2r^2(r-b)} + \frac{\phi'}{r}\right]$$

$$\frac{b'}{8\pi r^2} = \tilde{\rho}$$
$$\frac{1}{8\pi}\left[2\left(1-\frac{b}{r}\right)\frac{\phi'}{r} - \frac{b}{r^3}\right] = \tilde{p_r}$$

$$\frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left[\phi'' + \phi'^2 - \frac{(rb'-b)\phi'}{2r(r-b)} - \frac{rb'-b}{2r^2(r-b)} + \frac{\phi'}{r}\right] = \tilde{p}_t$$

$$\tilde{\rho}=\frac{2(r-b)}{(2r-b)f_Q}\left[\rho-\frac{1}{8\pi r^2}\left(1-\frac{b}{r}\right)\left(\frac{brf_{QQ}Q'}{r-b}+bf_Q\left(\frac{r\phi'+1}{r-b}-\frac{2r-b}{2(r-b)^2}\right)+\frac{fr^3}{2(r-b)}\right)+\frac{f_T(P+\rho)}{8\pi}\right]$$

$$\tilde{p_r}=\frac{2b}{fr^3}\left[p_r+\frac{1}{16\pi r^2}\left(1-\frac{b}{r}\right)\left(f_Q\left(\frac{b\left(\frac{rb'-b}{r-b}+2r\phi'+2\right)}{r-b}-4r\phi'\right)+\frac{2brf_{QQ}Q'}{r-b}\right)+\frac{fr^3(r-b)\phi'}{8\pi br^2}-\frac{f_T(P-p_r)}{8\pi}\right]$$

$$\tilde{p}_t=\frac{1}{f_Q\left(\frac{r}{r-b}+r\phi'\right)}\left[p_t+\frac{1}{32\pi r}\left(1-\frac{b}{r}\right)\left(f_Q\left(\frac{4(2b-r)\phi'}{r-b}-4r(\phi')^2-4r\phi''\right)+\frac{2fr^2}{r-b}-4rf_{QQ}Q'\phi'\right)+\frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left(\phi''+\phi'^2-\frac{(rb'-b)\phi'}{2r(r-b)}+\frac{\phi'}{r}\right)f_Q\left(\frac{r}{r-b}+r\phi'\right)-\frac{f_T(P-p_t)}{8\pi}\right]$$

$$8\pi\rho=\frac{(r-b)}{2r^3}\left[f_Q\left(\frac{(2r-b)(rb'-b)}{(r-b)^2}+\frac{b(2r\phi'+2)}{r-b}\right)+\frac{2brf_{QQ}Q'}{r-b}+\frac{fr^3}{r-b}-\frac{2r^3f_T(P+\rho)}{(r-b)}\right]$$

$$8\pi p_r=-\frac{(r-b)}{2r^3}\left[f_Q\left(\frac{b}{r-b}\left(\frac{rb'-b}{r-b}+2r\phi'+2\right)-4r\phi'\right)+\frac{2brf_{QQ}Q'}{r-b}+\frac{fr^3}{r-b}-\frac{2r^3f_T(P-p_r)}{(r-b)}\right]$$

$$8\pi p_t=-\frac{(r-b)}{4r^2}\left[f_Q\left(\frac{(rb'-b)\left(\frac{2r}{r-b}+2r\phi'\right)}{r(r-b)}+\frac{4(2b-r)\phi'}{r-b}-4r(\phi')^2-4r\phi''\right)-4rf_{QQ}Q'\phi'+\frac{2fr^2}{r-b}-\frac{4r^2f_T(P-p_t)}{(r-b)}\right]$$

$$\rho=-\frac{f_Qf_T\left(-rb'(2r(r-b)\phi'+b+2r)+3b^2+4r(b-r)(\phi'(r(b-r)\phi'+3b-2r)+r(b-r)\phi'')\right)}{48\pi r^3(r-b)(f_T+8\pi)}-\frac{24\pi f_Q\left(r(b-2r)b'+b(2r(b-r)\phi'+b)\right)}{48\pi r^3(r-b)(f_T+8\pi)}-\frac{r(b-r)\left(f_T\left(2f_{QQ}Q'(2r(b-r)\phi'+b)+3fr^2\right)+24\pi\left(2bf_{QQ}Q'+fr^2\right)\right)}{48\pi r^3(r-b)(f_T+8\pi)},$$

$$p_r=\frac{f_Qf_T\left(rb'(2r(r-b)\phi'+b+2r)-3b^2-4r(b-r)(\phi'(r(b-r)\phi'+3b-2r)+r(b-r)\phi'')\right)}{48\pi r^3(b-r)(f_T+8\pi)}+\frac{24\pi f_Q\left(brb'-(3b-2r)(2r(b-r)\phi'+b)\right)}{48\pi r^3(b-r)(f_T+8\pi)}-\frac{r(b-r)\left(f_T\left(2f_{QQ}Q'(2r(b-r)\phi'+b)+3fr^2\right)+24\pi\left(2bf_{QQ}Q'+fr^2\right)\right)}{48\pi r^3(b-r)(f_T+8\pi)}$$

$$p_t = -\frac{f_Q f_T\left(-rb'(2r(r-b)\phi' + b + 2r) + 3b^2 + 4r(b-r)(\phi'(r(b-r)\phi' + 3b - 2r) + r(b-r)\phi'')\right)}{48\pi r^3(b-r)(f_T + 8\pi)}$$
$$- \frac{24\pi r f_Q\left(r(b'((b-r)\phi' - 1) + 2(b-r)^2((\phi')^2 + \phi'') + (2r - 5b)\phi') + b(3b\phi' + 1)\right)}{48\pi r^3(b-r)(f_T + 8\pi)}$$
$$- \frac{r(b-r)\left(f_T\left(2bf_{QQ}Q' + 3fr^2\right) + 4r(b-r)f_{QQ}(f_T + 12\pi)Q'\phi' + 24\pi fr^2\right)}{48\pi r^3(b-r)(f_T + 8\pi)}$$

$$f_Q[f_T(r(b+2r)b' - 3b^2) + 24\pi(brb' - b(3b - 2r))] - r(b-r)\left[f_T\left(2bf_{QQ}Q' + 3fr^2\right) + 24\pi\left(2bf_{QQ}Q' + fr^2\right)\right] = \omega\left[f_Q\left(f_T(3b^2 - r(b+2r)b') + 24\pi(r(b-2r)b' + b^2)\right) + r(b-r)\left(f_T\left(2bf_{QQ}Q' + 3fr^2\right) + 24\pi\left(2bf_{QQ}Q' + fr^2\right)\right)\right].$$

$$f(Q,T) = \alpha Q + \beta T$$

$$\rho = \frac{\alpha(12\pi - \beta)b'}{3(4\pi - \beta)(\beta + 8\pi)r^2},$$
$$p_r = -\frac{\alpha(2\beta rb' - 3\beta b + 12\pi b)}{3(4\pi - \beta)(\beta + 8\pi)r^3},$$

$$p_t = -\frac{\alpha((\beta + 12\pi)rb' + 3b(\beta - 4\pi))}{6(4\pi - \beta)(\beta + 8\pi)r^3}.$$

$$b(r) = c_1 r^{\lambda_2},$$

$$\lambda_2 = -\frac{3(\beta - 4\pi)}{\beta(\omega - 2) - 12\pi\omega}$$

Parameters	
ω	β
$(-\infty, -1)$	$\left(-\infty, \frac{12\pi\omega}{\omega - 2}\right) \cup (12\pi, \infty)$
$(-1,2)$	$\left(\frac{12\pi\omega}{\omega - 2}, 12\pi\right)$
2	$(-\infty, 12\pi)$

$(2,\infty)$	$(-\infty, 12\pi) \cup \left(\frac{12\pi\omega}{\omega-2}, \infty\right)$

$$\rho = \frac{\alpha(\beta - 12\pi)r^{\lambda_2-3}}{\Lambda}$$
$$p_r = \frac{\alpha(\beta - 12\pi)\omega r^{\lambda_2-3}}{\Lambda}$$

$$p_t = \frac{\alpha(12\pi(\omega+1) - \beta(\omega-3))r^{\lambda_2-3}}{2\Lambda},$$

$$\Lambda = (\beta + 8\pi)(12\pi\omega - \beta(\omega-2))$$

$$p_t = np_r.$$

$$f_Q[(n-1)f_T(3b^2 - r(b+2r)b') + 24\pi(rb'(r-bn) + b(3bn - 2nr - r))] + r(b-r)\left[2bf_{QQ}Q'((n-1)f_T + 24\pi n) + 3f(n-1)r^2(f_T + 8\pi)\right] = 0.$$

$$f(Q,T) = Q + \lambda_1 Q^2 + \eta_1 T$$

$$b(r) = c_2 r^{\eta_2},$$

$$\eta_2 = \frac{3(4\pi - \beta)(2n+1)}{\beta - 4\beta n + 12\pi},$$

Parameters	
n	β
$(-\infty, -2)$	$\left(\frac{12\pi}{4n-1}, \frac{12\pi n}{n+2}\right)$
-2	$\left(\frac{-4\pi}{3}, \infty\right)$
$\left(-2, \frac{-1}{2}\right]$	$\left(-\infty, \frac{12\pi n}{n+2}\right)\right) \cup \left(\frac{12\pi}{4n-1}, \infty\right)$
$\left(\frac{-1}{2}, \frac{1}{4}\right)$	$\left(-\infty, \frac{12\pi}{4n-1}\right) \cup \left(\frac{12\pi n}{n+2}, \infty\right)$
$\frac{1}{4}$	$\left(\frac{4\pi}{3}, \infty\right)$
$\left(\frac{1}{4}, 1\right)$	$\left(\frac{12\pi n}{n+2}, \frac{12\pi}{4n-1}\right)$
$(1, \infty)$	$\left(\frac{12\pi}{4n-1}, \frac{12\pi n}{n+2}\right)$

$$\rho = \frac{\alpha(12\pi - \beta)(2n+1)r^k}{\Lambda_1},$$
$$p_r = -\frac{3\alpha(\beta + 4\pi)cr^k}{\Lambda_1},$$

$$p_t = -\frac{3\alpha(\beta + 4\pi)cnr^k}{\Lambda_1},$$

$$\Lambda_1 = (\beta + 8\pi)(\beta - 4\beta n + 12\pi)$$

$$k = -\frac{6(\beta + 4\pi)(n-1)}{\beta(4n-1) - 12\pi}$$

$$\frac{\varpi'}{2}(\rho + p_r) + \frac{dp_r}{dr} + \frac{2}{r}(p_r - p_t) = 0,$$

$$F_h = -\frac{dp_r}{dr}, F_g = -\frac{\varpi'}{2}(\rho + p_r), F_a = \frac{2}{r}(p_t - p_r).$$

$$F_h = -\frac{(\beta - 12\pi)\omega\left(\frac{3(\beta - 4\pi)}{12\pi\omega - \beta(\omega - 2)} - 3\right) r^{\frac{3(\beta-4\pi)}{12\pi\omega-\beta(\omega-2)}-4}}{(\beta + 8\pi)(12\pi\omega - \beta(\omega - 2))}$$

$$F_a = \frac{3\alpha(\beta(-\omega) + \beta + 4\pi(3\omega + 1)) r^{\frac{3(\beta-4\pi)}{12\pi\omega-\beta(\omega-2)}-4}}{(\beta + 8\pi)(12\pi\omega - \beta(\omega - 2))}.$$

$$F_h = -\frac{18\alpha(\beta + 4\pi)^2(n-1) r^{-\frac{6(\beta+4\pi)(n-1)}{\beta(4n-1)-12\pi}-1}}{(\beta + 8\pi)(\beta - 4\beta n + 12\pi)(\beta(4n-1) - 12\pi)}$$

$$F_a = -\frac{6\alpha(\beta + 4\pi)(n-1) r^{-\frac{6(\beta+4\pi)(n-1)}{\beta(4n-1)-12\pi}-1}}{(\beta + 8\pi)(\beta - 4\beta n + 12\pi)}.$$

$$\tilde{\rho} = \frac{2(r-b)}{(2r-b)f_Q}\left(\rho - \frac{(r-b)}{8\pi r^2}\left(\frac{brf_{QQ}Q'}{r-b} + \frac{bf_Q}{r-b} - \frac{b(2r-b)f_Q}{2(r-b)^2} + \frac{fr^3}{2(r-b)}\right) + \frac{f_T(P+\rho)}{8\pi}\right),$$

$$\tilde{p}_r = \frac{2b}{fr^3}\left(p_r - \frac{f_T(P - p_r)}{8\pi} + \frac{(r-b)}{16\pi r^3}\left(\frac{bf_Q\left(\frac{rb' - b}{r-b} + 2\right)}{r-b} + \frac{2brf_{QQ}Q'}{r-b}\right)\right),$$

$$\tilde{p}_t = \frac{(r-b)}{rf_Q}\left(p_t - \frac{f_T(P - p_t)}{8\pi} + \frac{fr\left(1 - \frac{b}{r}\right)}{16\pi(r-b)}\right)$$

$$B = \frac{\alpha(-\beta r\omega b' + 2\beta r b' + 12\pi r\omega b' - 3\beta b + 12\pi b)}{12(4\pi - \beta)(\beta + 8\pi)r^3\omega},$$

$$b(r) = r_0\gamma\left(1 - \frac{r_0}{r}\right) + r_0,$$

$$B = \frac{r_0\alpha\big(12\pi(r_0\gamma(\omega - 1) + (\gamma + 1)r) - \beta(r_0\gamma(\omega - 5) + 3(\gamma + 1)r)\big)}{12(4\pi - \beta)(\beta + 8\pi)r^4\omega},$$

$$\rho = \frac{r_0^2\alpha(12\pi-\beta)\gamma}{3(4\pi-\beta)(\beta+8\pi)r^4},$$

$$p_r = \frac{r_0\alpha(-5r_0\beta\gamma-12\pi(-r_0\gamma+\gamma r+r)+3\beta(\gamma+1)r)}{3(4\pi-\beta)(\beta+8\pi)r^4},$$

$$p_t = \frac{r_0\alpha(2r_0\beta\gamma+12\pi(-2r_0\gamma+\gamma r+r)-3\beta(\gamma+1)r)}{6(4\pi-\beta)(\beta+8\pi)r^4}$$

$$b(r) = r_0\left(\frac{r}{r_0}\right)^{\gamma_1},$$

$$b(r) = r_0\sqrt{\frac{r}{r_0}}.$$

$$B = \frac{\alpha(12\pi(\omega+2)-\beta(\omega+4))}{24(4\pi-\beta)(\beta+8\pi)r^2\omega\sqrt{\frac{r}{r_0}}},$$

$$\rho = \frac{12\pi-\beta}{6(4\pi-\beta)(\beta+8\pi)r^2\sqrt{\frac{r}{r_0}}}$$

$$p_r = -\frac{2(6\pi-\beta)}{3(4\pi-\beta)(\beta+8\pi)r^2\sqrt{\frac{r}{r_0}}}$$

$$p_t = \frac{12\pi-7\beta}{12(4\pi-\beta)(\beta+8\pi)r^2\sqrt{\frac{r}{r_0}}}$$

$$F_h = \frac{r_0(4r_0\gamma-3(\gamma+1)r)}{8\pi r^5}$$

$$F_a = \frac{r_0(3(\gamma+1)r-4r_0\gamma)}{8\pi r^5}.$$

$$F_h = -\frac{5}{16\pi r^3\sqrt{\frac{r}{r_0}}}$$

$$F_a = \frac{5}{16\pi r^3\sqrt{\frac{r}{r_0}}},$$

$$\frac{\alpha(12\pi-\beta)b'(r)}{3(4\pi-\beta)(\beta+8\pi)r^2} = \frac{Me^{-\frac{r^2}{4\Theta}}}{8\pi^{3/2}\Theta^{3/2}}$$

$$b(r) = c_1 + \mathcal{K}_1\left(2\sqrt{\pi}\Theta^{3/2}\mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - 2\Theta r e^{-\frac{r^2}{4\Theta}}\right)$$

$$\mathcal{K}_1 = \frac{3(-\beta^2 - 4\pi\beta + 32\pi^2)M}{8\pi^{3/2}\alpha(12\pi - \beta)\Theta^{3/2}}$$

$$\mathrm{erf}(\Theta) = \frac{2}{\sqrt{\pi}}\int_0^{\Theta} e^{-t^2}dt$$

$$c_1 = r_0 - \mathcal{K}_1\left(2\sqrt{\pi}\Theta^{3/2}\mathrm{erf}\left(\frac{r_0}{2\sqrt{\Theta}}\right) - 2\Theta r_0 e^{-\frac{r_0^2}{4\Theta}}\right)$$

$$b(r) = r_0 + 2\Theta\mathcal{K}_1\left[e^{-\frac{r^2+r_0^2}{4\Theta}}\left(e^{\frac{r^2}{4\Theta}}\left(\sqrt{\pi}\sqrt{\Theta}e^{\frac{r_0^2}{4\Theta}}\left(\mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - \mathrm{erf}\left(\frac{r_0}{2\sqrt{\Theta}}\right)\right) + r_0\right) - re^{\frac{r_0^2}{4\Theta}}\right)\right].$$

$$p_r = \frac{e^{-\frac{r^2+r_0^2}{4\Theta}}}{\mathcal{K}_2(\beta + 8\pi)}\left[(\beta + 8\pi)Me^{\frac{r_0^2}{4\Theta}}\left(3\sqrt{\pi}(4\pi - \beta)\Theta^{3/2}e^{\frac{r^2}{4\Theta}}\mathcal{K}_3 - \beta r^3 - 3(\beta - 4\pi)\Theta r\right) + \Theta r_0 e^{\frac{r^2}{4\Theta}} \times\left(-3(4\pi - \beta)(\beta + 8\pi)M - 4\pi^{3/2}\alpha(12\pi - \beta)\sqrt{\Theta}e^{\frac{r_0^2}{4\Theta}}\right)\right]$$

$$p_t = \frac{1}{4\mathcal{K}_2}\left[-6\sqrt{\pi}(4\pi - \beta)\Theta^{3/2}M\mathcal{K}_3 + Me^{-\frac{r^2}{4\Theta}}(6(\beta - 4\pi)\Theta r - (\beta + 12\pi)r^3) + 6(4\pi - \beta) \times \Theta M r_0 e^{-\frac{r_0^2}{4\Theta}} + \frac{8\pi^{3/2}\alpha(12\pi - \beta)\Theta^{3/2}r_0}{\beta + 8\pi}\right],$$

$$\mathcal{K}_2 = 4\pi^{3/2}(12\pi - \beta)\Theta^{3/2}r^3$$

$$\mathcal{K}_3 = \left(\mathrm{erf}\left(\frac{r_0}{2\sqrt{\Theta}}\right) - \mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right)\right)$$

$$\rho + p_r = \frac{e^{-\frac{r^2+r_0^2}{4\Theta}}}{2\mathcal{K}_2(\beta + 8\pi)}\left[3(4\pi - \beta)(\beta + 8\pi)Me^{\frac{r_0^2}{4\Theta}}\left(2\sqrt{\pi}\Theta^{3/2}e^{\frac{r^2}{4\Theta}}\mathcal{K}_3 + r^3 + 2\Theta r\right) - 2\Theta r_0 e^{\frac{r^2}{4\Theta}} \times\left(3(4\pi - \beta)(\beta + 8\pi)M + 4\pi^{3/2}\alpha(12\pi - \beta)\sqrt{\Theta}e^{\frac{r_0^2}{4\Theta}}\right)\right]$$

$$\rho + p_t = \frac{1}{16\pi^{3/2}\Theta^{3/2}r^3}\left[\frac{3(4\pi - \beta)Me^{-\frac{r^2}{4\Theta}}}{12\pi - \beta}\left(-2\sqrt{\pi}\Theta^{3/2}e^{\frac{r^2}{4\Theta}\mathcal{K}3 + r^3 - 2\Theta r}\right) + 2\Theta r_0 \times\left(\frac{4\pi^{3/2}\alpha\sqrt{\Theta}}{\beta + 8\pi} + \frac{3(4\pi - \beta)Me^{-\frac{r_0^2}{4\Theta}}}{12\pi - \beta}\right)\right]$$

$$\rho + p_r|_{r=r_0} = \frac{3(4\pi - \beta)Me^{-\frac{r_0^2}{4\Theta}}}{8\pi^{3/2}(12\pi - \beta)\Theta^{3/2}} - \frac{\alpha}{(\beta + 8\pi)r_0^2}$$

$$\rho + p_t|_{r=r_0} = \frac{3(4\pi - \beta)Me^{-\frac{r_0^2}{4\Theta}}}{16\pi^{3/2}(12\pi - \beta)\Theta^{3/2}} + \frac{\alpha}{2(\beta + 8\pi)r_0^2}$$

$$\rho + p_r + 2p_t = -\frac{\beta Me^{-\frac{r^2}{4\Theta}}}{\pi^{3/2}(24\pi - 2\beta)\Theta^{3/2}}$$

$$\frac{\alpha(12\pi - \beta)b'(r)}{3(4\pi - \beta)(\beta + 8\pi)r^2} = \frac{\sqrt{\Theta}M}{\pi^2(\Theta + r^2)^2}.$$

$$b(r) = \frac{\mathcal{M}_1}{(\Theta + r^2)}\left((\Theta + r^2)\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \sqrt{\Theta}r\right) + c_2,$$

$$\mathcal{M}_1 = \frac{3(4\pi - \beta)(\beta + 8\pi)M}{2\pi^2\alpha(12\pi - \beta)}$$

$$c_2 = r_0 - \frac{\mathcal{M}_1}{(\Theta + r_0^2)}\left((\Theta + r_0^2)\tan^{-1}\left(\frac{r_0}{\sqrt{\Theta}}\right) - \sqrt{\Theta}r_0\right).$$

$$b(r) = r_0 + \frac{\mathcal{M}_1}{(\Theta + r^2)(\Theta + r_0^2)}\left((\Theta + r^2)(\Theta + r_0^2)\mathcal{M}_2 + \sqrt{\Theta}(r - r_0)(rr_0 - \Theta)\right),$$

$$\mathcal{M}_2 = \left(\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \tan^{-1}\left(\frac{r_0}{\sqrt{\Theta}}\right)\right)$$

$$p_r = \frac{1}{2r^3}\left[\frac{M}{\pi^2(12\pi - \beta)(\Theta + r^2)^2}\left(\sqrt{\Theta}r\left(12\pi(\Theta + r^2) - \beta(3\Theta + 7r^2)\right) - 3(4\pi - \beta)(\Theta + r^2)^2\mathcal{M}_2\right)\right.$$
$$\left. + r_0\left(\frac{3(\beta - 4\pi)\sqrt{\Theta}M}{\pi^2(12\pi - \beta)(\Theta + r_0^2)} - \frac{2\alpha}{\beta + 8\pi}\right)\right]$$

$$p_t = -\frac{1}{\mathcal{M}_3}\left[(\Theta + r^2)\left(3(4\pi - \beta)(\beta + 8\pi)M(\Theta + r_0^2)\left(\sqrt{\Theta}r - (\Theta + r^2)\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right)\right) - 3\right.\right.$$
$$\times (4\pi - \beta)(\beta + 8\pi)M(\Theta + r^2)\left(\sqrt{\Theta}r_0 - (\Theta + r_0^2)\tan^{-1}\left(\frac{r_0}{\sqrt{\Theta}}\right)\right) - 2\pi^2\alpha(12\pi - \beta)$$
$$\left.\left.\times r_0(\Theta + r^2)(\Theta + r_0^2)\right) + 2(\beta + 8\pi)(\beta + 12\pi)\sqrt{\Theta}Mr^3(\Theta + r_0^2)\right],$$

$$\mathcal{M}_3 = 4\pi^2(12\pi - \beta)(\beta + 8\pi)r^3(\Theta + r^2)^2(\Theta + r_0^2)$$

$$\rho + p_r|_{r=r_0} = \frac{3(4\pi - \beta)\sqrt{\Theta}M}{\pi^2(12\pi - \beta)(\Theta + r_0^2)^2} - \frac{\alpha}{(\beta + 8\pi)r_0^2}$$

$$\rho + p_t|_{r=r_0} = \frac{1}{2}\left(\frac{3(4\pi - \beta)\sqrt{\Theta}M}{\pi^2(12\pi - \beta)(\Theta + r_0^2)^2} + \frac{\alpha}{(\beta + 8\pi)r_0^2}\right).$$

$$f(Q,T) = Q + \lambda_1 Q^2 + \eta_1 T,$$

$$\rho = \frac{1}{\mathcal{N}}(rb'(b\lambda_1 rb'(-11b\eta_1 + 72\pi b + 8\eta_1 r - 96\pi r) + 2b^2$$
$$\lambda_1(15b(\eta_1 - 8\pi) + 16(12\pi - \eta_1)r) + 4(12\pi - \eta_1)r^3(b - r)^2)$$
$$+b^3\lambda_1(3b(88\pi - 9\eta_1) + 32(\eta_1 - 12\pi)r))$$

$$p_r = \frac{1}{\mathcal{N}}(rb'(b\lambda_1 rb'(-13b\eta_1 + 24\pi b + 16\eta_1 r) + 2b^2\lambda_1(3b$$
$$(3\eta_1 + 8\pi) - 8(\eta_1 + 12\pi)r) - 8\eta_1 r^3(b - r)^2) + b^3\lambda_1(3b$$
$$(\eta_1 - 56\pi) + 8(36\pi - \eta_1)r) - 12b(4\pi - \eta_1)r^3(b - r)^2)$$

$$p_t = \frac{1}{\mathcal{N}}(rb'(b\lambda_1 rb'(4(\eta_1 + 12\pi)r - b(\eta_1 + 24\pi)) + 2b^2$$
$$\lambda_1(9b\eta_1 + 24\pi b - 20\eta_1 r - 48\pi r) - 2(\eta_1 + 12\pi)r^3$$
$$(b - r)^2) + b^3\lambda_1(-3b(11\eta_1 + 8\pi) + 52\eta_1 r + 48\pi r)$$
$$+6b(4\pi - \eta_1)r^3(b - r)^2)$$

$$\mathcal{N} = 12(4\pi - \eta_1)(\eta_1 + 8\pi)r^6(b - r)^2$$

$$\frac{1}{\mathcal{N}}(rb'(b\lambda_1 rb'(-11b\eta_1 + 72\pi b + 8\eta_1 r - 96\pi r) + 2b^2$$
$$\lambda_1(15b(\eta_1 - 8\pi) + 16(12\pi - \eta_1)r) + 4(12\pi - \eta_1)r^3(b - r)^2)$$
$$+b^3\lambda_1(3b(88\pi - 9\eta_1) + 32(\eta_1 - 12\pi)r)) = \frac{Me^{-\frac{r^2}{4\Theta}}}{8\pi^{3/2}\Theta^{3/2}}$$

$$\frac{1}{\mathcal{N}}(rb'(b\lambda_1 rb'(-11b\eta_1 + 72\pi b + 8\eta_1 r - 96\pi r) + 2b^2$$
$$\lambda_1(15b(\eta_1 - 8\pi) + 16(12\pi - \eta_1)r) + 4(12\pi - \eta_1)r^3(b - r)^2)$$
$$+b^3\lambda_1(3b(88\pi - 9\eta_1) + 32(\eta_1 - 12\pi)r)) = \frac{\sqrt{\Theta}M}{\pi^2(\Theta + r^2)^2}$$

$$ds^2 = A(x)dt^2 - B(x)dr^2 - C(x)(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$e^{2\phi(r)} = \left(\frac{r}{b_0}\right)^m, b_0$$

$$A(x) = \left(\frac{r}{b_0}\right)^m;\ B(x) = \left(1 - \frac{b(r)}{r}\right)^{-1};\ C(x) = r^2.$$

$$\alpha(\bar{x}) = \alpha_e + I(\bar{x}),$$

$$\alpha_e = -2\ln\left(\frac{2a}{3} - 1\right) - 0.8056$$

$$I(\bar{x}) = 2\int_{\bar{x}}^{\infty} \frac{\sqrt{B(x)}}{\sqrt{C(x)}\sqrt{\frac{C(x)A(\bar{x})}{C(\bar{x})A(x)} - 1}}dx$$

$$I(\bar{x}) = \int_{\bar{x}}^{a} R(x)dx$$

$$R(x)=\frac{2}{\sqrt{x^2\left[1-\frac{1}{x}\left\{\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2x^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}x\right)-2\mathcal{M}\sqrt{\Theta}x\right)}{(\Theta+(2\mathcal{M})^2x^2)}-\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2x_0^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}x_0\right)-2\mathcal{M}\sqrt{\Theta}x_0\right)}{(\Theta+(2\mathcal{M})^2x_0^2)}+x_0\right\}\right]}}$$

$$\frac{1}{\sqrt{\left(\frac{x^{2-m}}{\bar{x}^{2-m}}-1\right)}}$$

$$\mathcal{N}_1=\frac{3(4\pi-\beta)(\beta+8\pi)M}{4\mathcal{M}\pi^2\alpha(12\pi-\beta)}$$

$$I(\bar{x})=\int_1^{\frac{a}{\bar{x}}}\frac{2}{\sqrt{H(y)}}dy$$

$$H(y)=\left[1-\frac{1}{\bar{x}y}\left\{\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x}y)^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}y\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}y\right)}{(\Theta+(2\mathcal{M})^2(\bar{x}y)^2)}-\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x}y_0)^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}y_0\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}y_0\right)}{(\Theta+(2\mathcal{M})^2(\bar{x}y_0)^2)}+\bar{x}y_0\right\}\right](y^{4-m}-y^2).$$

$$g(y)=1-\frac{1}{\bar{x}y}\left\{\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x}y)^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}y\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}y\right)}{(\Theta+(2\mathcal{M})^2(\bar{x}y)^2)}-\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x}y_0)^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}y_0\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}y_0\right)}{(\Theta+(2\mathcal{M})^2(\bar{x}y_0)^2)}+\bar{x}y_0\right\}$$

$$f(y)=(y^{4-m}-y^2)$$

$$H(y)=(2-m)g(1)(y-1)+\left[\frac{1}{2}(5-m)(2-m)g(1)+(2-m)g'(1)\right](y-1)^2$$

$$g(y)=1-\frac{1}{\bar{x}y}\left\{\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x}y)^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}y\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}y\right)}{(\Theta+(2\mathcal{M})^2(\bar{x}y)^2)}-\right.$$

$$\left.\frac{\mathcal{N}_1\left((\Theta+(2\mathcal{M})^2(\bar{x})^2)\tan^{-1}\left(\frac{2\mathcal{M}}{\sqrt{\Theta}}\bar{x}\right)-2\mathcal{M}\sqrt{\Theta}\bar{x}\right)}{(\Theta+(2\mathcal{M})^2(\bar{x})^2)}+\bar{x}\right\}.$$

$$ds^2=-f(r)dt^2+h(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$$

$$\mathcal{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathcal{R}=8\pi G\mathcal{T}_{\mu\nu}$$

$$\mathcal{T}_\nu^\mu=\mathrm{diag}(-\rho,P_r,P_t,P_t)$$

$$h(r)=\left(1-\frac{2GM(r)}{r}\right)^{-1},$$

$$M(r)=\int_0^r 4\pi\tilde{r}^2\rho(\tilde{r})d\tilde{r}$$

$$-P_r'=\frac{G(\rho+P_r)(M+4\pi r^3P_r)}{r(r-2M)}+\frac{2(P_r-P_t)}{r},$$

$$P_r=\omega\rho+\bar{\omega}\frac{\rho^n}{\rho_0^{n-1}}$$
$$P_t=\omega_1\rho+\omega_2\frac{\rho^m}{\rho_0^{m-1}}$$

$$P_r=-\rho,P_t=\omega_1\rho+\omega_2\frac{\rho^m}{\rho_0^{m-1}}$$

$$\rho(r)=\frac{\rho_0}{\left[c_1(\rho_0r^{2(\omega_1+1)})^{m-1}-\frac{\omega_2}{\omega_1+1}\right]^{\frac{1}{m-1}}},$$

$$\rho(r\to 0)\sim\frac{\rho_0}{\left(\frac{-\omega_2}{\omega_1+1}\right)^{\frac{1}{m-1}}},$$

$$\rho(r\to\infty)\sim\frac{1}{c_1^{\frac{1}{m-1}}r^{2(\omega_1+1)}}$$

$$P_t=0,\to\ r=\left(\frac{\rho_0^{-m+1}}{\omega_1c_1}\right)^{\frac{1}{2(\omega_1+1)(m-1)}}.$$

$$P_t = 2\rho - \frac{3\rho^{\frac{3}{2}}}{\sqrt{\rho_0}} > 0, \rightarrow \begin{cases} 0 < \rho < \frac{4}{9}\rho_0, \\ \left(\frac{1}{2c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}} < r < \infty. \end{cases}$$

$$0 \leq r \leq \left(\frac{1}{2c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$\frac{1}{9}\rho_0 < \rho < \frac{4}{9}\rho_0$$

$$\rho > P_t$$

$$\rho = \frac{16}{81}\rho_0$$

$$r = \left(\frac{5}{4}c_1\sqrt{\rho_0}\right)^{\frac{1}{3}}$$

$$P_{t,\max} = \frac{32}{243}\rho_0$$

$$\frac{4}{81}\rho_0 < \rho < \frac{16}{81}\rho_0$$

$$0 \leq v_s = \sqrt{dP/d\rho} \leq 1$$

$$\frac{1}{9}\rho_0 < \rho < \frac{16}{81}\rho_0$$

$$\left(2/c_1\sqrt{\rho_0}\right)^{\frac{1}{3}} \leq r \leq \left(5/4c_1\sqrt{\rho_0}\right)^{\frac{1}{3}}$$

$$X = (\pi\rho_0 M^2)^{\frac{1}{3}} r/M$$

$$f(r) = 1 - \frac{8\pi\sqrt{\rho_0}}{3c_1 r} + \frac{8\pi\sqrt{\rho_0}}{3c_1 r\left(1 + c_1\sqrt{\rho_0}r^3\right)}.$$

$$f(r) \sim 1 - \frac{8\pi\sqrt{\rho_0}}{3c_1 r} + \frac{8\pi}{3c_1^2 r^4} + O(r^{-7})$$

$$M = 4\pi\int_0^\infty r^2\rho(r)dr = \frac{4\pi\sqrt{\rho_0}}{3c_1}$$

$$R \sim \left(\frac{M}{\rho_0}\right)^{\frac{1}{3}} \sim \left(\frac{4\pi}{3c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}.$$

$$r_0 = \frac{\mathcal{A}^{\frac{1}{3}}}{18c_1} + \frac{128\pi^2\rho_0}{9c_1\mathcal{A}^{\frac{1}{3}}} + \frac{8\pi\sqrt{\rho_0}}{9c_1}$$

$$r_{\pm} = -\frac{\mathcal{A}^{\frac{1}{3}}}{36c_1} - \frac{64\pi^2\rho_0}{9c_1\mathcal{A}^{\frac{1}{3}}} + \frac{8\pi\sqrt{\rho_0}}{9c_1}$$

$$\pm\frac{i\sqrt{3}}{2}\left(\frac{\mathcal{A}^{\frac{1}{3}}}{18c_1} - \frac{128\pi^2\rho_0}{9c_1\mathcal{A}^{\frac{1}{3}}}\right)$$

$$\mathcal{A} = 4096\pi^3\rho_0^{\frac{3}{2}} - \frac{2916c_1^2}{\sqrt{c_1}} + 108c_1\sqrt{\frac{729c_1^2}{\rho_0} - 2048\pi^3}.$$

$$c_{1,\,\mathrm{ext}} = 9.333\rho_0$$

$$M_{\mathrm{ext}} = \frac{0.44881}{\sqrt{\rho_0}}, \rightarrow\ r_{\mathrm{ext}} = \left(\frac{2}{c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}} = \frac{0.5984134}{\sqrt{\rho_0}}.$$

$$f(r) \sim 1 - \frac{8\pi\rho_0}{3}r^2 + \mathcal{O}(r^5).$$

$$\lim_{r\to 0}\mathcal{R} = 32\pi\rho_0 = 4\Lambda_{eff},$$

$$\lim_{r\to 0}\mathcal{K} = \lim_{r\to 0}\mathcal{R}_{abcd}\mathcal{R}^{abcd} = \frac{512\pi^2\rho_0^2}{3} = \frac{8}{3}\Lambda_{eff}^2.$$

$$r = \left(\frac{-1}{c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$r_{sec} = \left(\frac{1}{2c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$\mathcal{R} = 0 \rightarrow r_R = \left(\frac{2}{c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$\lambda_1 = -\frac{8\pi\rho_0\left(1 + c_1^2\rho_0 r^6 - 7c_1\sqrt{\rho_0}r^3\right)}{3\left(1 + c_1\sqrt{\rho_0}r^3\right)^3}$$

$$\lambda_4 = \frac{8\pi\rho_0}{3\left(1 + c_1 r^3\sqrt{\rho_0}\right)}$$

$$\lambda_2 = \lambda_3 = -\lambda_5 = -\lambda_6 = \frac{4\pi\rho_0\left(-2 + c_1 r^3\sqrt{\rho_0}\right)}{3\left(1 + c_1 r^3\sqrt{\rho_0}\right)^2}$$

$$r_{\pm}/M, r_{\mathrm{ext}}/M, r_{\mathrm{rep}}/M, r_{ph}/M$$

$$r_{\mathrm{rep}} = \left(\frac{8 + 3\sqrt{6}}{2c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$r_{\text{dom}} = \left(\frac{7+3\sqrt{5}}{2c_1\sqrt{\rho_0}}\right)^{\frac{1}{3}}$$

$$\mathcal{F} = M - \frac{\mathcal{S}}{\tau} = \frac{r_+}{4} + \frac{\sqrt{9c_1^2 r_+^4 + 96\pi}}{12c_1 r_+} - \frac{\pi r_+^2}{\tau},$$

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_+}, -\cot(\theta)\csc(\theta)\right)$$

$$\tau = -\frac{8\pi r_+^3 c_1\sqrt{9c_1^2 r_+^4 + 96\pi}}{32\pi - c_1 r_+^2\sqrt{9c_1^2 r_+^4 + 96\pi} - 3c_1^2 r_+^4}$$

$$n = (n^r, n^\theta) = (\phi^r + /|\phi|, \phi^\theta/|\phi|)$$

$$\frac{r_+}{r_0} = a\cos(s) + r_c, \theta = b\sin(s) + \frac{\pi}{2}$$

$$\Omega(s) = \oint_C \epsilon_{ab} n^a n^b \partial_s n^b ds$$

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2ML^2}$$

$$P_t = 3\rho - \frac{4\rho^{\frac{3}{2}}}{\sqrt{\rho_0}} > 0 \to \begin{cases} 0 < \rho < \frac{9}{16}\rho_0 \\ \left(\frac{1}{3c_1\sqrt{\rho_0}}\right)^{\frac{1}{4}} < r < \infty \end{cases}$$

$$\rho = \rho_0/4$$

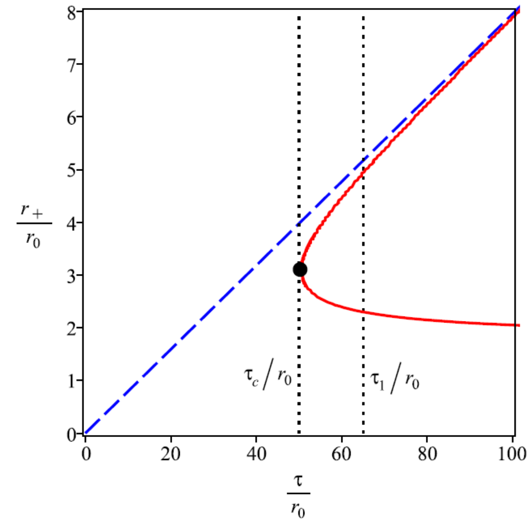
$$r = (1/c_1\sqrt{\rho_0})^{\frac{1}{4}}$$

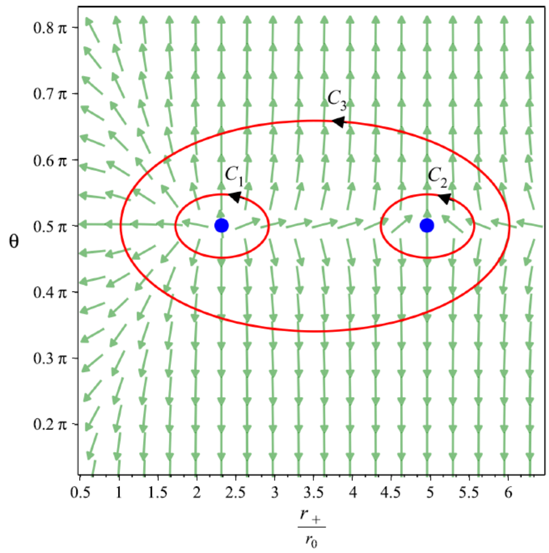
$$P_{t,\max} = \rho_0/4$$

$$\rho > 9\rho_0/16 \text{ or } 0 < r < \left(1/(3c_1\sqrt{\rho_0})\right)^{\frac{1}{4}}$$

$$(1/c_1\sqrt{\rho_0})^{1/4} \le r \le \left(2/(c_1\sqrt{\rho_0})\right)^{1/4}$$

$$X = (\pi\rho_0 M^2)^{\frac{1}{3}} r/M$$





$$f(r) = 1 - \frac{2\pi\rho_0 r^2}{1 + c_1\sqrt{\rho_0}r^4} + \frac{\pi\sqrt{2\rho_0}}{2c_1\mathcal{E}^{\frac{1}{4}}r}\left[\ln\left(\frac{r^2 + \sqrt{2}\mathcal{E}^{\frac{1}{4}}r + \sqrt{\mathcal{E}}}{r^2 - \sqrt{2}\mathcal{E}^{\frac{1}{4}}r + \sqrt{\mathcal{E}}}\right)^{\frac{1}{2}} - \arctan\left(\frac{\sqrt{2}r}{\mathcal{E}^{1/4} - \mathcal{E}^{-1/4}r^2}\right)\right],$$

$$\mathcal{E} = \left(c_1\sqrt{\rho_0}\right)^{-1}$$

$$f(r) \sim 1 - \frac{\pi^2\sqrt{2}\rho_0^{\frac{5}{8}}}{2c_1^{\frac{3}{4}}r} + \mathcal{O}(r^{-3}),$$

$$X = (\pi^2\rho_0 M^2)^{\frac{1}{3}}r/M$$

$$M = 4\pi\int_0^{\infty} r^2\rho(r)dr = \frac{\sqrt{2}\pi^2\rho_0^{\frac{5}{8}}}{4c_1^{\frac{3}{4}}}$$

$$R \sim \left(\frac{M}{\rho_0}\right)^{\frac{1}{3}} = \left(\frac{\sqrt{2}\pi^2}{4c_1^{\frac{3}{4}}\rho_0^{\frac{3}{8}}}\right)^{\frac{1}{3}}.$$

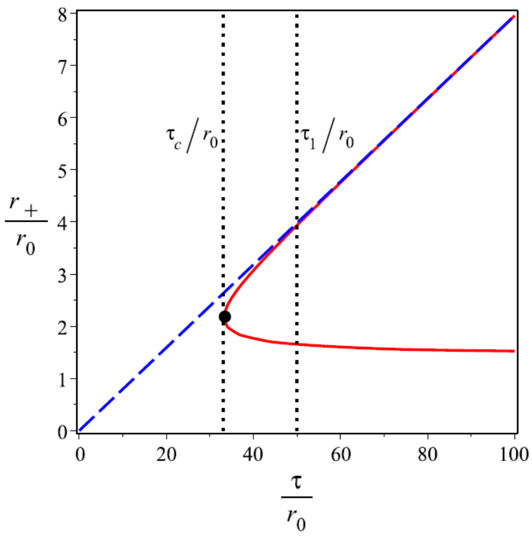
$$f(r) \sim 1 - \frac{8\pi\rho_0}{3}r^2 + \mathcal{O}(r^3).$$

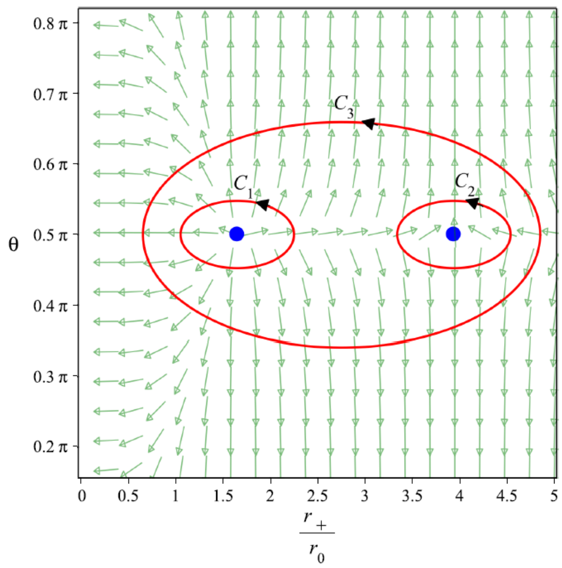
$$\lim_{r\to 0}\mathcal{R} = 32\pi\rho_0 = 4\Lambda_{eff}$$

$$\lim_{r\to 0}\mathcal{K} = \frac{512\pi^2\rho_0^2}{3} = \frac{8}{3}\Lambda_{eff}^2$$

$$\Phi = \frac{R(r)}{r}e^{i\omega t}Y(\theta,\phi),$$

$$\frac{d^2R}{dr_*^2} + \left[\omega^2 - f\left(\frac{l(l+1)}{r^2} + \frac{f'}{r}\right)\right]R = 0$$





$$ds^2 = \eta_{ab} e^a \wedge e^b$$

$$\eta_{ab} = \mathrm{diag}(-1,1,1,1)$$

$$e^a = e^a_b dx^\mu$$

$$de^a + \omega^a_b \wedge e^b = 0$$

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d$$

$$\lambda_1 = \frac{2hff'' + ff'h' - hf'^2}{4f^2},$$

$$\lambda_2 = \lambda_3 = -\frac{h'}{2r}, \lambda_4 = \frac{1-h}{r^2},$$

$$\lambda_5 = \lambda_6 = \frac{hf'}{2fr}.$$

$$\frac{\partial \lambda_i}{\partial r} = 0.$$

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle$$

$$\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - (\langle\hat{A}\rangle)^2}$$

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}$$

$$\Delta x \sim \frac{\blacksquare}{\Delta p}$$

$$\Delta x \sim \frac{\hbar}{\Delta p} + G\Delta p$$

$$[x, p] = i\hbar(1 + \lambda p^2)$$

$$\Delta p_m \sim \sqrt{\hbar/G}$$

$$\Delta x_m \sim \sqrt{\hbar G}$$

$$f(T, \mathcal{T}) = \alpha T + \beta \mathcal{T}$$

$$f(T, \mathcal{T}) = \eta T^2 + \chi \mathcal{T}$$

$$E(a) = -\frac{\hbar c \pi^2 S}{720 a^3}$$

$$\rho_c = \frac{E}{aS} = -\frac{\hbar c \pi^2}{720 a^4}$$

$$p_c = \frac{F_c}{S} = -\frac{1}{S}\frac{dE}{da} = -3\frac{\hbar c \pi^2}{720 a^4}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \lambda(\Delta p)^2]$$

$$[\hat{x}, \hat{p}] = i\hbar(1 + \lambda \hat{p}^2)$$

$$(\Delta x)_{|\psi_x^{ML}\rangle} = \Delta x_0$$

$$\langle \psi_x^{ML} | \hat{x} | \psi_x^{ML} \rangle = x$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar[f(\hat{p}^2)\delta_{ij} + g(\hat{p}^2)\hat{p}_i\hat{p}_j] \; i, j = 1,2, \dots, n$$

$$\langle p \mid \psi_{\boldsymbol{r}}^{ML} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \Omega(p) \exp\left\{-\frac{i}{\hbar}[\boldsymbol{\kappa}(p) \cdot \boldsymbol{r} - \hbar\omega(p)t]\right\}$$

$$p = |\hat{\boldsymbol{p}}|, \omega(p)$$

$$\left[\hat{x} - \langle \hat{x} \rangle + \frac{\langle [\hat{x}, \hat{p}] \rangle}{2(\Delta p)^2}(\hat{p} - \langle \hat{p} \rangle)\right] |\psi\rangle = 0$$

$$f(\hat{p}^2) = \frac{\lambda \hat{p}^2}{\sqrt{1 + 2\lambda \hat{p}^2} - 1}, g(\hat{p}^2) = \lambda$$

$$\kappa_i(p) = \left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)p_i, \omega(p) = \frac{pc}{\hbar}\left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)$$

$$\Omega(p) = \left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)^{\alpha/2}$$

$$\alpha = 1 + \sqrt{1+n/2}$$

$$\int \frac{d^n p}{\sqrt{1+2\lambda p^2}}\left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)^{n+\alpha} |\boldsymbol{p}\rangle\langle\boldsymbol{p}| = \mathbf{1}$$

$$\kappa_i(p) = \left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)p_i, \omega(p) = \frac{pc}{\hbar}\left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)$$

$$\Omega(p) = \left[\Gamma\left(\frac{3}{2}\right)\left(\frac{2\sqrt{2}}{\pi\sqrt{\lambda}}\right)^{1/2}\right]\left(\frac{1}{p}\frac{\lambda p^2}{\sqrt{1+2\lambda p^2}-1}\right)^{1/2} \boldsymbol{J}_{\frac{1}{2}}\left[\frac{\pi\sqrt{\lambda}}{\sqrt{2}}\left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)p\right]$$

$$= \frac{\sqrt{2}}{\pi}\frac{\sqrt{\lambda p^2}}{\sqrt{1+2\lambda p^2}-1}\sin\left(\frac{\sqrt{2}\pi\left(\sqrt{1+2\lambda p^2}-1\right)}{2\sqrt{\lambda p^2}}\right)$$

$$\int \frac{d^n p}{\sqrt{1+2\lambda p^2}}\left(\frac{\sqrt{1+2\lambda p^2}-1}{\lambda p^2}\right)^{n} |\boldsymbol{p}\rangle\langle\boldsymbol{p}| = \mathbf{1}$$

$$f(p^2) = 1 + \lambda p^2, g(p^2) = 0$$

$$\kappa_i(p) = \frac{1}{p\sqrt{\lambda}}\arctan\,(p\sqrt{\lambda})p_i, \omega(p) = \frac{c}{\hbar\sqrt{\lambda}}\arctan\,(p\sqrt{\lambda}), \Omega(p) = 1,$$

$$\int \frac{d^3 p}{1+\lambda p^2}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| = \mathbf{1}$$

$$E_i(a) = -\frac{\hbar c \pi^2}{720}\frac{S}{a^3}\left[1 + \Lambda_i\left(\frac{\hbar^2\lambda}{a^2}\right)\right],$$

$$\Lambda_1 = \pi^2\left(\frac{28+3\sqrt{10}}{14}\right)\ (\mathfrak{KMN}), \Lambda_2 = 4\pi^2\left(\frac{3+\pi^2}{21}\right)\ (\mathfrak{DGS}), \Lambda_3 = \frac{2\pi^2}{3}\ (\blacksquare).$$

$$F(a) = -\frac{dE}{da} = -\frac{3\hbar c\pi^2}{720}\frac{S}{a^4}\left[1 + \frac{5}{3}\Lambda_i\left(\frac{\hbar^2\lambda}{a^2}\right)\right]$$

$$P(a) = \frac{F}{S} = -\frac{3\hbar c\pi^2}{720}\frac{1}{a^4}\left[1 + \frac{5}{3}\Lambda_i\left(\frac{\hbar^2\lambda}{a^2}\right)\right]$$

$$\rho(a) = -\frac{\hbar c\pi^2}{720 a^4}\left[1 + \frac{5}{3}\Lambda_i\left(\frac{\hbar^2\lambda}{a^2}\right)\right].$$

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\Phi^2, \nu(r)$$

$$e^{\lambda(r)} = \left(1 - \frac{b(r)}{r}\right)^{-1}, b(r)$$

$$r = r_0 \Longrightarrow b(r_0) = r_0$$

$$\frac{b(r) - rb'(r)}{b^2(r)} > 0$$

$$\frac{b(r)}{r} \to \infty \text{ as } r \to \infty$$

$$S = \int e\left[\frac{1}{2k^2} f(T, \mathcal{T}) + \mathcal{L}_m\right] d^4x$$

$$e = \det(e_\nu^\mu) = \sqrt{-g}\widehat{g}$$

$$T^\lambda{}_{\mu\nu} = e_a{}^\lambda(\partial_\mu e_\nu^a - \partial_\nu e^a{}_\mu),$$
$$K^{\mu\nu}{}_\lambda = \frac{1}{2}(T_\lambda{}^{\mu\nu} + T^{\nu\mu}{}_\lambda - T^{\mu\nu}{}_\lambda),$$
$$S_\lambda{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\lambda + \delta^\mu{}_\lambda T^{\gamma\nu}{}_\gamma - \delta^\nu{}_\lambda T^{\gamma\mu}{}_\gamma).$$

$$T = T^\lambda{}_{\mu\nu} S_\lambda{}^{\mu\nu}.$$

$$\overset{\text{e-m}}{\mathcal{T}}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}.$$

$$\hat{\mathcal{T}}_\alpha^{\text{e-m}\,\beta} = (\rho + p_t)u_\alpha u^\beta + p_t\delta_\alpha^\beta + (p_r - p_t)v_\alpha v^\beta,$$

$$u_\mu = e^{\frac{\nu}{2}}\delta_\mu^0$$

$$v_\mu = e^{\frac{\lambda}{2}}\delta_\mu^1$$

$$\underset{\mathcal{T}}{\mathcal{T}_\mu}{}^\nu = [-\rho, p_r, p_t, p_t]$$

$$\left[e^{-1}\partial_\epsilon\left(e e_a^\alpha S_\alpha^{\psi\epsilon}\right) + e_a^\alpha T_{\rho\alpha}^\epsilon S_\epsilon^{\rho\psi}\right] f_T + e_a^\alpha S_\alpha^{\psi\epsilon}(f_{TT}\partial_\epsilon T + f_{T\mathcal{T}}\partial_\epsilon \mathcal{T}) + \frac{e_a^\psi f}{4} - \left(\frac{e_a^\alpha \mathcal{T}_\alpha^\psi + p_t e_a^\psi}{2}\right) f_\mathcal{T} = \frac{e_a^\alpha \overset{\text{e}}{\mathcal{T}}_\alpha{}^\psi}{4}$$

$$e_\gamma^\eta = \text{diag}\left(e^{\frac{\nu}{2}}, e^{\frac{\lambda}{2}}, r, r\sin\theta\right),$$

$$e = \det(e^\eta{}_\gamma) = e^{\frac{\nu+\lambda}{2}} r^2 \sin\theta$$

$$f(T, \mathcal{T}) = \alpha T + \beta\mathcal{T}.$$

$$T = -\frac{2e^{-\lambda}}{r}\left(\nu' + \frac{1}{r}\right)$$

$$\rho = -\frac{e^{-\lambda}}{4(\beta-1)(\beta+2)r^2}\left[2\alpha\beta - 4\alpha + \alpha\beta r^2\lambda'\nu' - 2\alpha\beta r^2\nu'' - \alpha\beta r^2(\nu')^2 - \alpha\beta r\lambda' - 2\alpha\beta e^{\lambda} - 3\alpha\beta r\nu' + 4\alpha r\lambda\right.$$

$$p_r = \frac{e^{-\lambda}}{4(\beta-1)(\beta+2)r^2}\left[2\alpha\beta - 4\alpha + \alpha\beta r^2\lambda'\nu' - 2\alpha\beta r^2\nu'' - \alpha\beta r^2(\nu')^2 + 3\alpha\beta r\lambda' - 2\alpha\beta e^{\lambda} + \alpha\beta r\nu' + 4\alpha e^{\lambda}\right.$$

$$p_t = \frac{e^{-\lambda}}{4(\beta-1)(\beta+2)r^2}\left[-2\alpha\beta + \alpha r^2\lambda'\nu' - 2\alpha r^2\nu'' - \alpha r^2(\nu')^2 + \alpha\beta r\lambda' + 2\alpha\beta e^{\lambda} - \alpha\beta r\nu' + 2\alpha r\lambda' - 2\alpha\right.$$

$$\lambda(r) = -\ln\left(1 - \frac{b(r)}{r}\right)$$

$$\rho(r) = \frac{\alpha(\beta-4)rb'(r) + \alpha\beta b(r)}{4(\beta-1)(\beta+2)r^3},$$
$$p_r(r) = \frac{3\alpha\beta rb'(r) + \alpha(4-5\beta)b(r)}{4(\beta-1)(\beta+2)r^3},$$
$$p_t(r) = \frac{\alpha(\beta+2)rb'(r) + \alpha(\beta-2)b(r)}{4(\beta-1)(\beta+2)r^3}.$$

$$b(r) = \frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} - \frac{\pi^2(\beta-1)(\beta+2)}{180\alpha}\left\{\frac{1}{4}\left[\frac{1}{r} - \frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right] + \frac{5}{3}\cdot\frac{\Lambda_i\lambda}{12-2\beta}\left[\frac{1}{r^3} - \frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}$$

$$\gamma = -\frac{\pi^2(\beta-1)(\beta+2)}{180\alpha}, \sigma_i = \frac{5}{3}\cdot\frac{\Lambda_i\lambda}{12-2\beta}$$

$$b(r) = \frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \frac{\gamma}{4}\left[\frac{1}{r} - \frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right] + \gamma\sigma_i\left[\frac{1}{r^3} - \frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]$$

$$b_n(r) = \frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} - \frac{\pi^2(\beta-1)(\beta+2)}{180\alpha}\left\{\sum_{j=0}^{n}\frac{D_i^{(j)}\lambda^j}{4(2j+1)-2j\beta}\left[\frac{1}{r^{2j+1}} - \frac{r_0^{\frac{4(2j+1)-2j\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}$$

$$\rho = -\frac{\pi^2}{720r^4}\left[1 + D_i^{(1)}\left(\frac{\sqrt{\lambda}}{r}\right)^2 + D_i^{(2)}\left(\frac{\sqrt{\lambda}}{r}\right)^4 + \cdots\right]$$

$$D_i^{(0)} = 1, D_i^{(1)} = 5\Lambda_i/3$$

$$\lim_{r\to\infty}\frac{b(r)}{r} \triangleq 0$$

$$\frac{2\beta-4}{\beta-4} > 0$$

$$\alpha \in \mathbb{R} - \{0\}$$
$$\beta \in (-\infty, 2) \cup (4, \infty) - \{6\}$$

$$\rho + p_r = \frac{\alpha}{(\beta+2)r^3}\left(-\left(\frac{2\beta-4}{\beta-4}\right)\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \gamma\left\{\frac{1}{4}\left[-\frac{2}{r} + \left(\frac{2\beta-4}{\beta-4}\right)\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right] + \sigma_i\left[-\frac{4}{r^3} + \left(\frac{2\beta-4}{\beta-4}\right)\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right)$$

$$\rho + p_t = \frac{\alpha}{2(\beta+2)r^3}\left(-\left(\frac{4}{\beta-4}\right)\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \gamma\left\{\frac{1}{\beta-4}\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \sigma_i\left[-\frac{2}{r^3} + \left(\frac{4}{\beta-4}\right)\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right)$$

$$\rho + p_r|_{r=r_0} = -\frac{1}{540(\beta-4)(\beta+2)}\left[1080\alpha(\beta-2)r_0^{-2} + 3\pi^2(\beta-1)(\beta+2)r_0^{-4}\left(1+\frac{5\lambda\Lambda_i}{3r_0^2}\right)\right],$$

$$\rho + p_t|_{r=r_0} = -\frac{1}{1080(\beta-4)(\beta+2)}\left[2160\alpha r_0^{-2} + 3\pi^2(\beta-1)(\beta+2)r_0^{-4}\left(1+\frac{5\lambda\Lambda_i}{3r_0^2}\right)\right],$$

$$\rho(r) + p_r(r) < f(r), f(r) > 0,$$

$$\Delta = \Delta p = p_t - p_r = \frac{\alpha(3b(r) - rb'(r))}{2(\beta+2)r^3}$$

$$\omega_r = \frac{p_r}{\rho}, \omega_t = \frac{p_t}{\rho}$$

$$\Delta = \frac{\alpha}{2(\beta+2)r^3}\left(\left(\frac{4\beta-12}{\beta-4}\right)\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \gamma\left\{\frac{1}{r} + \left(\frac{\beta-3}{\beta-4}\right)\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}} + \sigma_i\left[\frac{6}{r^3} + \left(\frac{4\beta-12}{\beta-4}\right)\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right),$$

$$\omega_r(r_0) = \frac{4320\alpha(\beta-2)r_0^4 + 9\pi^2\beta(\beta+2)r_0^2 + 15\pi^2\beta(\beta+2)\lambda\Lambda_i}{\pi^2(\beta-4)(\beta+2)(3r_0^2 + 5\lambda\Lambda_i)}$$

$$\omega_t(r_0) = \frac{4320\alpha r_0^4 + 3\pi^2(\beta+2)^2 r_0^2 + 5\pi^2(\beta+2)^2\lambda\Lambda_i}{\pi^2(\beta-4)(\beta+2)(3r_0^2 + 5\lambda\Lambda_i)}$$

$$e_\nu^\mu = \begin{pmatrix} e^{\frac{\nu}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\lambda}{2}}\sin\theta\cos\Phi & r\cos\theta\cos\Phi & -r\sin\theta\sin\Phi \\ 0 & e^{\frac{\lambda}{2}}\sin\theta\sin\Phi & r\cos\theta\sin\Phi & r\sin\theta\cos\Phi \\ 0 & e^{\frac{\lambda}{2}}\cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$T(r) = -\frac{2e^{-\lambda}\left(e^{\frac{\lambda}{2}} - 1\right)\left(e^{\frac{\lambda}{2}} - r\nu' - 1\right)}{r^2}.$$

$$\rho = -\frac{e^{-\frac{\lambda}{2}}\left(e^{-\frac{\lambda}{2}} - 1\right)(f_{TT}T' + f_{T\mathcal{T}}\mathcal{T}')}{r} - \frac{f_T}{2}\left(-\frac{e^{-\lambda}(1 - r\lambda')}{r^2} - \frac{1}{r^2} + \frac{T}{2}\right) + \frac{f}{4} + \frac{f_{\mathcal{T}}}{2}(\rho + p_t),$$

$$p_r = \left(\frac{e^{-\lambda}(r\nu' + 1)}{r^2} - \frac{1}{r^2} + \frac{T}{2}\right)\frac{f_T}{2} - \frac{f}{4} - \frac{f_{\mathcal{T}}}{2}(p_t - p_r),$$

$$p_t = \frac{e^{-\lambda}}{2}\left(-\frac{e^{\frac{\lambda}{2}}}{r} + \frac{\nu'}{2} + \frac{1}{r}\right)(f_{TT}T' + f_{T\mathcal{T}}\mathcal{T}') + \left[e^{-\lambda}\left(\left(\frac{\nu'}{4} + \frac{1}{2r}\right)(\nu' - \lambda') + \frac{\nu''}{2}\right) + \frac{T}{2}\right]\frac{f_T}{2} - \frac{f}{4}.$$

$$f(T,\mathcal{T}) = \eta T^n + \chi\mathcal{T} + \phi,$$

$$\eta = n = 1, \chi = \phi = 0$$

$$f(T,\mathcal{T}) = \eta T^2 + \chi\mathcal{T}.$$

$$\rho = -\mathcal{L}_1\left[(3\chi - 4)r^2\lambda'(r) + 2\left(-3\chi + (6\chi - 8\chi r + 10r - 8)e^{\frac{\lambda(r)}{2}} + (-3\chi + 2\chi r - 3r + 4)e^{\lambda(r)} + 9\chi r - 11r\right.\right.$$

$$p_r = -\mathcal{L}_1\left[\chi r^2\lambda'(r) - 2\chi\left(e^{\frac{\lambda(r)}{2}} - 1\right)^2 + 2r\left(5\chi + (2 - 4\chi)e^{\frac{\lambda(r)}{2}} + e^{\lambda(r)} - 3\right)\right]$$

$$p_t = \mathcal{L}_1\left[(5\chi - 6)r^2\lambda'(r) + 2\left(\chi + 2(r - \chi)e^{\frac{\lambda(r)}{2}} + (\chi - 2\chi r + r)e^{\lambda(r)} + (\chi - 3)r\right)\right]$$

$$\mathcal{L}_1 = \frac{\eta e^{-2\lambda(r)}\left(e^{\frac{\lambda(r)}{2}} - 1\right)^2}{(\chi - 2)(\chi - 1)r^5}$$

$$\Box\,\xi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\xi\right) = 0$$

$$\xi(t,r,\theta,\phi) = \frac{1}{r}\sum_{l,m}\psi_l(t,r)Y_{lm}(\theta,\phi),$$

$$\psi_l(t,r) = \hat{\psi}_l(r)e^{-i\omega t}$$

$$\left(\frac{d^2}{dx^2} + \omega^2 - V_l(r)\right)\hat{\psi}_l(x) = 0$$

$$dx = \pm\frac{dr}{e^{\frac{\nu(r)}{2}}\sqrt{1 - \frac{b(r)}{r}}}$$

$$V_l(r) = e^{\nu}\left[\frac{l(l+1)}{r^2} - \frac{b'r - b}{2r^3} + \frac{\nu'}{2r}\left(1 - \frac{b}{r}\right)\right]$$

$$\left(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right)_{,\nu} = 0$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu(t,r,\theta,\phi) = \int d\omega\sum_{l,m}\left[a_{lm}(r)e^{-i\omega t}\begin{pmatrix}0\\0\\\frac{1}{\sin\theta}\partial_\phi Y_{lm}(\theta,\phi)\\-\sin\theta\,\partial_\theta Y_{lm}\end{pmatrix}_{\text{odd}} + e^{-i\omega t}\begin{pmatrix}f_{lm}(r)Y_{lm}\\u_{lm}(r)Y_{lm}\\k_{lm}(r)\partial_\theta Y_{lm}\\k_{lm}(r)\partial_\phi Y_{lm}\end{pmatrix}_{\text{even}}\right].$$

$$\frac{d^2\psi_{lm}^{(1)}}{dx^2} + \left(\omega^2 - e^{\nu}\frac{l(l+1)}{r^2}\right)\psi_{lm}^{(1)} = 0$$

$$V_l^{(1)}(r) = e^{\nu}\frac{l(l+1)}{r^2}, \psi_{lm}^{(1)}(r) = \begin{cases} \frac{r^2}{l(l+1)} e^{-\frac{\nu(r)}{2}}\sqrt{1-\frac{b(r)}{r}}\left(-i\omega u_{lm} - \frac{df_{lm}}{dr}\right) & \text{even} \\ a_{lm} & \text{odd} \end{cases}$$

$$ds^2 = -e^{2\nu_1}dt^2 + e^{2\psi}(d\phi - q_1 dt - q_2 dr - q_3 d\theta)^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2$$

$$e^{2\nu_1} = e^{2\nu}, e^{-2\mu_1} = 1 - \frac{b}{r} = \frac{\Xi}{r^2}, \Xi = r^2 - br, e^{\nu_2} = r, e^{\psi} = r\sin\,\theta, q_1 = q_2 = q_3 = 0$$

$$\frac{e^{\nu}}{\sqrt{\Xi}}\frac{1}{r^3\sin^3\,\theta}\frac{\partial Q}{\partial\theta} = -i\omega q_{1,2} - \omega^2 q_2$$
$$\frac{e^{\nu}\sqrt{\Xi}}{r^3\sin^3\,\theta}\frac{\partial Q}{\partial r} = +i\omega q_{1,3} + \omega^2 q_3$$

$$Q(r,\theta) = Q(r)C_{l+2}^{-3/2}(\theta)$$

$$\left[\frac{d}{d\theta}\sin^{2m}\,\theta\frac{d}{d\theta} + n(n+2m)\sin^{2m}\,\theta\right]C_n^m(\theta) = 0$$

$$\left(\frac{d^2}{dx^2} + \omega^2 - V^{(-)}\right)Z = 0$$

$$V^{(-)}(r) = e^{\nu}\left[\frac{l(l+1)}{r^2} + \frac{b'r - 5b}{2r^3} - \frac{\nu'}{2r}\left(1 - \frac{b}{r}\right)\right]$$

$$V(r) = e^{\nu}\frac{l(l+1)}{r^2} + e^{\nu}(1-s)\left[\frac{\nu'}{2r}\left(1 - \frac{b}{r}\right) + \frac{(1+2s)b - b'r}{2r^3}\right]$$

$$[\gamma^a e_a^{\mu}(\partial_{\mu} + \Gamma_{\mu}) + m]\Upsilon = 0$$

$$\Gamma_{\mu} = \frac{1}{8}[\gamma^a, \gamma^b]e_a^{\nu}e_{b\nu;\mu}$$

$$\partial_x^2\Upsilon_l(x) + \omega^2\Upsilon_l(x) = V_{d\pm}(r)\Upsilon_l(x)$$

$$V_{d\pm}(r) = \frac{k}{r}\left(\frac{ke^{\nu}}{r} \mp \frac{e^{\nu}\sqrt{1-\frac{b}{r}}}{r} \pm e^{\frac{\nu}{2}}\sqrt{1-\frac{b}{r}}\frac{de^{\frac{\nu}{2}}}{dr}\right)$$

$$\Upsilon_+ = A\left(W + \frac{d}{dx}\right)\Upsilon_-, W = \sqrt{e^{\nu/2}\sqrt{1-\frac{b}{r}}}$$

$$\frac{d^2\Psi}{dx^2} + Q(\omega, x)\Psi = 0$$

$$Q(\omega, x) = \omega^2 - V(x)$$

$$\omega^2 = \left[V_0 + (-2V_0'')^{1/2}\widetilde{\Lambda}\right] - i\left(n+\frac{1}{2}\right)(-2V_0'')^{1/2}(1+\widetilde{\Omega})$$

$$\widetilde{\Lambda}(n) = \frac{1}{(-2V_0'')^{1/2}}\left\{\frac{1}{8}\left[\frac{V_0^{(4)}}{V_0''}\right]\left(\frac{1}{4}+\alpha^2\right) - \frac{1}{288}\left[\frac{V_0'''}{V_0''}\right]^2(7+60\alpha^2)\right\}$$

$$\widetilde{\Omega}(n) = \frac{1}{(-2V_0'')}\left\{\frac{5}{6912}\left[\frac{V_0'''}{V_0''}\right]^4(77+188\alpha^2) - \frac{1}{384}\left[\frac{V_0'''^2 V_0^{(4)}}{V_0''^3}\right](51+100\alpha^2) + \frac{1}{2304}\left[\frac{V_0^{(4)}}{V_0''}\right]^2(67+68\alpha^2)\right.$$
$$\left. + \frac{1}{288}\left[\frac{V_0''' V_0^{(5)}}{V_0''^2}\right](19+28\alpha^2) - \frac{1}{288}\left[\frac{V_0^{(6)}}{V_0''}\right](5+4\alpha^2)\right\}$$

$$\alpha = n+\frac{1}{2}, n = \begin{cases} 0,1,2,\ldots, & \mathrm{Re}(\omega) > 0 \\ -1,-2,-3,\ldots, & \mathrm{Re}(\omega) < 0 \end{cases}$$

$$\Psi(x) \sim e^{\pm i\omega x}, x \to \pm\infty$$

$$M_{\text{active}} = \int_{r_0}^{R} 4\pi\rho r^2 dr$$

$$M_{\text{active}} = \frac{\pi^3}{180}\left[\left(\frac{1}{R}-\frac{1}{r_0}\right) + \frac{5}{9}\Lambda_i\lambda\left(\frac{1}{R^3}-\frac{1}{r_0^3}\right)\right].$$

$$E_g = Mc^2 - E_M,$$

$$Mc^2 = \frac{1}{2}\int_{r_0}^{r}\rho r^2 dr + \frac{r_0}{2}, E_M = \frac{1}{2}\int_{r_0}^{r}\sqrt{g_{rr}}\rho r^2 dr$$

$$E_g = \frac{1}{2}\int_{r_0}^{r}(1-\sqrt{g_{rr}})\rho r^2 dr + \frac{r_0}{2},$$

$$g_{rr} = \left(1-\frac{b(r)}{r}\right)^{-1}$$

$$I_V = \oint[\rho(r)+p_r(r)]dV = 2\int_{r_0}^{\infty}[\rho(r)+p_r(r)]dV$$
$$\Rightarrow I_V = 8\pi\int_{r_0}^{\infty}[\rho(r)+p_r(r)]r^2 dr$$

$$I_V = 8\pi\int_{r_0}^{r_1}[\rho(r)+p_r(r)]r^2 dr$$

$$I_V(r_1;r_0) = \frac{8\pi\alpha}{\beta+2}\left[\frac{\gamma}{2}\left(\frac{1}{r_1}-\frac{1}{r_0}\right) + \frac{4\sigma_i\gamma}{3}\left(\frac{1}{r_1^3}-\frac{1}{r_0^3}\right) + \mathcal{M}\left(\frac{1}{r_1^{\frac{\beta}{\beta-4}}} - \frac{1}{r_0^{\frac{\beta}{\beta-4}}}\right)\right].$$

$$\gamma = -\frac{\pi^2(\beta-1)(\beta+2)}{180\alpha}, \sigma_i = \frac{5}{3}\frac{\lambda\Lambda_i}{(12-2\beta)}, \mathcal{M} = \frac{2\beta-4}{\beta}\left(r_0^{\frac{2\beta-4}{\beta-4}} - \frac{\gamma}{4}r_0^{\frac{4}{\beta-4}} - \sigma_i\gamma r_0^{\frac{12-2\beta}{\beta-4}}\right)$$

$$-\frac{dp_r}{dr}-\frac{\nu'}{2}(\rho+p_r)+\frac{2}{r}\Delta-\frac{2\beta}{\beta+2}\left(\frac{1}{4}\frac{d\rho}{dr}-\frac{1}{4}\frac{dp_r}{dr}-\frac{dp_t}{dr}\right)=0$$

$$\mathcal{F}_h+\mathcal{F}_g+\mathcal{F}_a+\mathcal{F}_e=0$$

$$\mathcal{F}_h=-\frac{dp_r}{dr},\mathcal{F}_g=-\frac{\nu'}{2}(\rho+p_r),\mathcal{F}_a=\frac{2}{r}(p_t-p_r),\mathcal{F}_e=-\frac{2\beta}{\beta+2}\left[\frac{1}{4}\frac{d\rho}{dr}-\frac{1}{4}\frac{dp_r}{dr}-\frac{dp_t}{dr}\right]$$

$$\mathcal{F}_h=-\frac{\alpha}{(\beta-1)(\beta+2)r^4}\left(\frac{8(\beta-1)(\beta-2)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}+\gamma\left\{\frac{2\beta-1}{r}-\frac{2(\beta-1)(\beta-2)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right.\right.$$
$$\left.\left.+\sigma_i\left[\frac{21\beta-6}{r^3}-\frac{8(\beta-1)(\beta-2)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right)$$

$$\mathcal{F}_a=\frac{\alpha}{(\beta+2)r^4}\left(\left(\frac{4\beta-12}{\beta-4}\right)\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}+\gamma\left\{\frac{1}{r}+\left(\frac{\beta-3}{\beta-4}\right)\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}+\sigma_i\left[\frac{6}{r^3}+\left(\frac{4\beta-12}{\beta-4}\right)\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right)$$

$$\mathcal{F}_e=\frac{\alpha\beta}{(\beta-1)(\beta+2)r^4}\left(\frac{4(\beta-1)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{2\beta-4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}+\gamma\left\{\frac{1}{r}-\frac{(\beta-1)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{4}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}+\sigma_i\left[\frac{15}{r^3}-\frac{4(\beta-1)(\beta-3)}{(\beta-4)^2}\frac{r_0^{\frac{12-2\beta}{\beta-4}}}{r^{\frac{\beta}{\beta-4}}}\right]\right\}\right)$$

$$ds^2=-e^{\nu(r)}dt^2+e^{\lambda(r)}dr^2+r^2d\theta^2+r^2\sin^2\,\theta d\phi^2$$

$$\{t\in\mathbb{R},r\in[r_0,\infty),\theta\in(0,\pi),\phi\in[0,2\pi)\}$$

$$e^{\lambda(r)}=(1-b(r)/r)^{-1}$$

$$\frac{b(r)}{r}<1\forall r>r_0;b(r_0)=r_0$$
$$\left.\frac{b-b'r}{b^2}\right|_{r_0}>0\rightarrow b'(r_0)<1,$$
$$\lim_{r\to\infty}\frac{b(r)}{r}=0$$

$$\mathcal{S}=\frac{1}{16\pi}\int f(R,\mathcal{L}_m,T)\sqrt{-g}d^4x+\int \mathcal{L}_m\sqrt{-g}d^4x$$

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\lambda}\left(\frac{\partial g_{\gamma\lambda}}{\partial x^{\beta}}+\frac{\partial g_{\lambda\beta}}{\partial x^{\gamma}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\lambda}}\right).$$

$$R_{\mu\nu}=\partial_\lambda\Gamma^{\lambda}_{\mu\nu}-\partial_\nu\Gamma^{\lambda}_{\lambda\mu}+\Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\sigma\lambda}-\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda},$$

$$R=g^{\mu\nu}R_{\mu\nu},$$

$$f_R R_{\mu\nu}-\frac{1}{2}[f-(f_{\mathcal{L}_m}+2f_T)\mathcal{L}_m]g_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R$$
$$=\left[8\pi+\frac{1}{2}(f_{\mathcal{L}_m}+2f_T)\right]T_{\mu\nu}+f_T\tau_{\mu\nu}$$

$$f_R = \frac{\partial f(R, \mathcal{L}_m, T)}{\partial R}, f_{\mathcal{L}_m} = \frac{\partial f(R, \mathcal{L}_m, T)}{\partial \mathcal{L}_m}, f_T = \frac{\partial f(R, \mathcal{L}_m, T)}{\partial T}$$

$$\tau_{\mu\nu} = 2g^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}$$

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}.$$

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p_r, p_t, p_t),$$

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t \delta_{\mu\nu} + (p_r - p_t)v_\mu v_\nu,$$

$$\left(1 - \frac{b}{r}\right)\left[\left\{\Phi'' + \Phi'^2 - \frac{(rb' - b)}{2r(r - b)}\Phi' + \frac{2\Phi'}{r}\right\} f_R - \left\{\Phi' - \frac{(rb' - b)}{2r(r - b)} + \frac{2}{r}\right\} f_R' - f_R''\right] + \frac{1}{2}\left(f - f_{\mathcal{L}_m}\mathcal{L}_m\right) = \frac{1}{2}\rho\left(16\pi + f_{\mathcal{L}_m} + 2f_T\right) + f_T \mathcal{L}_m$$

$$\left(1 - \frac{b}{r}\right)\left[\left\{-(\Phi'' + \Phi'^2) + \frac{(rb' - b)}{2r(r - b)}\left(\Phi' + \frac{2}{r}\right)\right\} f_R + \left\{\Phi' - \frac{(rb' - b)}{2r(r - b)} + \frac{2}{r}\right\} f_R'\right] - \frac{1}{2}\left(f - f_{\mathcal{L}_m}\mathcal{L}_m\right) = \frac{1}{2}p_r\left(16\pi + f_{\mathcal{L}_m} + 2f_T\right) - f_T \mathcal{L}_m$$

$$\left(1 - \frac{b}{r}\right)\left[\left\{-\frac{\Phi'}{r} + \frac{(rb' + b)}{2r^2(r - b)}\right\} f_R + \left\{\Phi' + \frac{2}{r} - \frac{(rb' - b)}{2r(r - b)}\right\} f_R' + f_R''\right] - \frac{1}{2}\left(f - \mathcal{L}_m f_{\mathcal{L}_m}\right) = \frac{1}{2}p_t\left(16\pi + f_{\mathcal{L}_m} + 2f_T\right) - f_T \mathcal{L}_m$$

$$f(R, \mathcal{L}_m, T) = R + \alpha\mathcal{L}_m + \beta T,$$

$$R = \frac{2b'}{r^2} - 2\left\{\Phi'' + \Phi'^2 + \frac{\Phi'}{r}\right\}\left(1 - \frac{b}{r}\right) + \frac{\Phi'}{r^2}(rb' + b - 2r)$$

$$\frac{b'}{r^2} = \rho\left(\lambda - \frac{\beta}{2}\right) - \frac{\beta}{2}(p_r + 2p_t),$$

$$-\frac{b}{r^3} = p_r\left(\lambda + \frac{\beta}{2}\right) + \frac{\beta}{2}(2p_t + \rho)$$

$$\frac{1}{2r^2}\left(\frac{b}{r} - b'\right) = p_t(\lambda + \beta) + \frac{\beta}{2}(p_r + \rho)$$

$$\lambda = 8\pi + \frac{\alpha}{2} + \beta$$

$$\rho = \frac{b'}{\lambda r^2}$$
$$p_r = -\frac{b}{\lambda r^3}$$
$$p_t = \frac{b - b'r}{2\lambda r^3}$$

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}$$

$$\rho(r) = \frac{Me^{-\frac{r^2}{4\Theta}}}{(4\pi\Theta)^{3/2}}$$

$$\sqrt{\Theta} \sim \ell_p$$

$$\rho(r) = \frac{\sqrt{\Theta}M}{\pi^2(\Theta + r^2)^2}$$

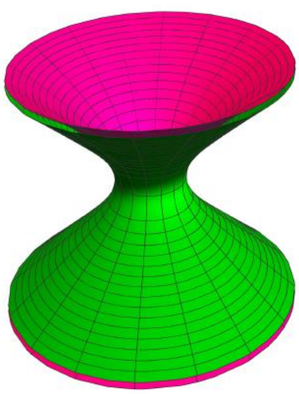
$$Z(r) = \pm\int_{r_0}^{\infty}\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}} \mathrm{d}r$$

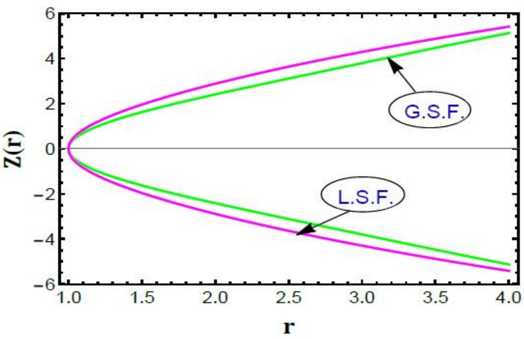
$$\frac{b'(r)}{\lambda r^2} = \frac{Me^{-\frac{r^2}{4\Theta}}}{(4\pi\Theta)^{3/2}}$$

$$b(r) = r_0 + \frac{\lambda M}{4\pi}\left(\mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - \frac{re^{-\frac{r^2}{4\Theta}}}{\sqrt{\pi\Theta}} + \mathcal{K}_G\right)$$

$$\mathcal{K}_G = \frac{r_0 e^{-\frac{r_0^2}{4\Theta}}}{\sqrt{\pi\Theta}} - \mathrm{erf}\left(\frac{r_0}{2\sqrt{\Theta}}\right)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$





$$b'(r) = \frac{\lambda M r^2 e^{-\frac{r^2}{4\Theta}}}{(4\pi\Theta)^{3/2}}.$$

$$\frac{\lambda M}{(4\pi\Theta)^{3/2}} r_0^2 e^{-\frac{r_0^2}{4\Theta}} < 1,$$

$$\beta + \frac{\alpha}{2} < \frac{(4\pi\Theta)^{3/2}}{M r_0^2} e^{\frac{r_0^2}{4\Theta}} - 8\pi.$$

$$\frac{b'(r)}{\lambda r^2} = \frac{\sqrt{\Theta} M}{\pi^2 (\Theta + r^2)^2}.$$

$$b(r) = r_0 + \frac{\lambda M}{2\pi^2}\left(\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \frac{\sqrt{\Theta} r}{\Theta + r^2} + \mathcal{K}_L\right),$$

$$\mathcal{K}_L = \frac{\sqrt{\Theta} r_0}{\Theta + r_0^2} - \tan^{-1}\left(\frac{r_0}{\sqrt{\Theta}}\right)$$

$$b'(r) = \frac{\lambda \sqrt{\Theta} M r^2}{\pi^2 (\Theta + r^2)^2}.$$

$$\beta + \frac{\alpha}{2} < \frac{\pi^2 (\Theta + r_0^2)^2}{\sqrt{\Theta} M r_0^2} - 8\pi,$$

$$\rho + p_r \geq 0, \rho + p_t \geq 0$$

$$\rho \geq 0, \rho + p_r \geq 0, \rho + p_t \geq 0$$

$$\rho \geq 0, \rho - |p_r| \geq 0, \rho - |p_t| \geq 0$$

$$\rho + p_r + 2p_t \geq 0, \rho + p_r \geq 0, \rho + p_t \geq 0$$

$$\frac{b'(r_0)}{\lambda r_0^2} - \frac{1}{\lambda r_0^2} \geq 0$$
$$\frac{b'(r_0)}{2\lambda r_0^2} + \frac{1}{2\lambda r_0^2} \geq 0$$

$$\beta + \alpha/2 < -8\pi.$$

$$\rho \geq -\frac{1}{\lambda r_0^2}$$
$$\leftrightarrow \ \lambda < -\frac{1}{\rho r_0^2}.$$

$$\rho + p_r + 2p_t = 0$$

$$\rho - |p_r|\big|_{r=r_0} = \frac{b'(r_0)}{\lambda r_0^2} - \left|\frac{1}{\lambda r_0^2}\right|$$
$$\rho - |p_t|\big|_{r=r_0} = \frac{b'(r_0)}{\lambda r_0^2} - \left|\frac{1}{2\lambda r_0^2} - \frac{b'(r_0)}{2\lambda r_0^2}\right|.$$

$$\frac{b'(r_0)}{\lambda r_0^2} + \frac{1}{\lambda r_0^2} \geq 0$$

$$\frac{1}{2\lambda r_0^2} - \frac{b'(r_0)}{2\lambda r_0^2} < 0$$

$$\rho - |p_t|\big|_{r=r_0} = \frac{b'(r_0)}{2\lambda r_0^2} + \frac{1}{2\lambda r_0^2}$$

$$p_r = \frac{M}{4\pi r^3}\left(\frac{re^{-\frac{r^2}{4\Theta}}}{\sqrt{\pi\Theta}} - \text{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - \widetilde{\mathcal{K}}_G\right),$$

$$p_t = \frac{M}{8\pi r^3}\left(\text{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - \frac{r(r^2 + 2\Theta)}{2\sqrt{\pi\Theta^3}}e^{-\frac{r^2}{4\Theta}} + \widetilde{\mathcal{K}}_G\right),$$

$$\widetilde{\mathcal{K}}_G = \mathcal{K}_G + \frac{4\pi r_0}{\lambda M}$$

$$\rho + p_r|_{r=r_0} = \frac{Me^{-\frac{r_0^2}{4\Theta}}}{(4\pi\Theta)^{3/2}} - \frac{1}{\lambda r_0^2} \overset{!}{\geq} 0$$
$$\rho + p_t|_{r=r_0} = \frac{Me^{-\frac{r_0^2}{4\Theta}}}{2(4\pi\Theta)^{3/2}} + \frac{1}{2\lambda r_0^2} \overset{!}{\geq} 0$$

$$\beta + \frac{\alpha}{2} < -\frac{(4\pi\Theta)^{3/2}}{Mr_0^2}e^{\frac{r_0^2}{4\Theta}} - 8\pi$$

$$p_r = \frac{M}{2\pi^2 r^3}\left(\frac{\sqrt{\Theta}r}{\Theta + r^2} - \tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \widetilde{\mathcal{K}}_L\right)$$
$$p_t = \frac{M}{4\pi^2 r^3}\left(\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \frac{\sqrt{\Theta}r(\Theta + 3r^2)}{(\Theta + r^2)^2} + \widetilde{\mathcal{K}}_L\right)$$

$$\widetilde{\mathcal{K}}_L = \mathcal{K}_L + \frac{2\pi^2 r_0}{\lambda M}$$

$$\rho + p_r|_{r=r_0} = \frac{\sqrt{\Theta}M}{\pi^2(\Theta + r_0^2)^2} - \frac{1}{\lambda r_0^2} \overset{!}{\geq} 0$$
$$\rho + p_t|_{r=r_0} = \frac{\sqrt{\Theta}M}{2\pi^2(\Theta + r_0^2)^2} + \frac{1}{2\lambda r_0^2} \overset{!}{\geq} 0$$

$$\beta + \frac{\alpha}{2} < -\frac{\pi^2(\Theta + r_0^2)^2}{\sqrt{\Theta}Mr_0^2} - 8\pi$$

$$\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$$

$$\dot{\gamma} = \partial\gamma/\partial s$$

$$\mathcal{L} = -e^{-2\Phi}\frac{\dot{t}^2}{2} + \frac{1}{1-b/r}\frac{\dot{r}^2}{2} + r^2\frac{\dot{\theta}^2}{2} + r^2\sin^2\,\theta\frac{\dot{\phi}^2}{2} = 0$$

$$E := \partial\mathcal{L}/\partial\dot{t} = e^{-2\Phi}\dot{t}$$

$$L := \partial\mathcal{L}/\partial\dot{\phi} = r^2\dot{\phi}$$

$$E^2 = \frac{e^{2\Phi}}{1-b/r}\dot{r}^2 + V_{eff}(r)$$

$$V_{eff}(r) := e^{2\Phi}L^2/r^2$$

$$\mathrm{d}V_{\mathrm{eff}}/\mathrm{d}r \neq 0$$

$$l(r) = \pm\int_{r_0}^{r}\left(1-\frac{b(r')}{r'}\right)^{-\frac{1}{2}}\mathrm{d}r'$$

$$\boldsymbol{\alpha}(r_{tp}) = -\pi + 2\int_{r_{tp}}^{\infty}\frac{e^{\Phi}\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{\sqrt{\frac{r^2}{u^2}-e^{2\Phi}}}\,\mathrm{d}r$$

$$u = r_{tp}e^{-\Phi},$$

$$\frac{v'}{2}(\rho+p_r) + \frac{dp_r}{dr} + \frac{2}{r}(p_r - p_t) = 0$$

$$F_h = -\frac{dp_r}{dr}, F_g = -\frac{v'}{2}(\rho+p_r), F_a = \frac{2}{r}(p_t - p_r)$$

$$F_h + F_a = 0$$

$$\nabla^{\mu}T_{\mu\nu} = -\frac{1}{\lambda}\left[\left(\frac{\alpha}{2}+\beta\right)\nabla_{\nu}\rho + \frac{1}{2}\left(\beta\nabla_{\nu}T - \frac{\alpha}{2}\nabla_{\nu}\rho\right)\right].$$

$$F_m = \frac{\beta}{2\lambda}(\rho' + p_r' + 2p_t')$$

$$F_h = \left(1+\frac{6\Theta}{r^2}\right)\frac{Me^{-\frac{r^2}{4\Theta}}}{r(4\pi\Theta)^{3/2}} - \frac{3M}{4\pi r^4}\left(\widetilde{\mathcal{K}}_G + \mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right)\right)$$

$$F_a = \frac{M}{4\pi r^4}\left(3\mathrm{erf}\left(\frac{r}{2\sqrt{\Theta}}\right) - \frac{r(r^2+6\Theta)}{2\sqrt{\pi\Theta^3}}e^{-\frac{r^2}{4\Theta}} + 3\widetilde{\mathcal{K}}_G\right)$$

$$F_h = \left(3 + \frac{2r^2}{\Theta + r^2}\right)\frac{M\sqrt{\Theta}}{2\pi^2 r^3(\Theta + r^2)}$$
$$-\frac{3M}{2\pi^2 r^4}\left(\widetilde{\mathcal{K}}_L + \tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right)\right)$$
$$F_a = \frac{M}{2\pi^2 r^4}\left(3\tan^{-1}\left(\frac{r}{\sqrt{\Theta}}\right) - \frac{\sqrt{\Theta}r(3\Theta + 5r^2)}{(\Theta + r^2)^2} + 3\widetilde{\mathcal{K}}_L\right)$$

$$\beta + \frac{\alpha}{2} < \frac{(4\pi\Theta)^{3/2}}{Mr_0^2} e^{\frac{r_0^2}{4\Theta}} - 8\pi$$

$$\beta + \frac{\alpha}{2} < \frac{\pi^2(\Theta + r_0^2)^2}{\sqrt{\Theta}Mr_0^2} - 8\pi$$

$$\beta + \alpha/2 \geq -8\pi,$$

$$\beta + \alpha/2 + 8\pi < -\frac{1}{\rho r_0^2}$$

$$S_{NMEYM} = \int d^4x\sqrt{-g}\left\{\frac{R}{8\pi Gc^{-4}} + F_{ik}^{(a)}F^{ik(a)} + \frac{1}{2}\mathcal{R}^{ikmn}F_{ik}^{(a)}F_{mn}^{(a)}\right\}$$

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2}R\left(g^{im}g^{kn} - g^{in}g^{km}\right)$$
$$+\frac{q_2}{2}\left(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}\right) + q_3R^{ikmn}$$

$$ds^2 = F(r)dt^2 - dr^2F(r)^{-1} - r^2d\Omega^2$$

$$\frac{1 - F(r)}{r^2} - \frac{F'(r)}{r} = \frac{8\pi Gc^{-4}\nu^2}{r^4}$$
$$\times\left[\frac{1}{2} - q_1\frac{F'(r)}{r} + (13q_1 + 4q_2 + q_3)\frac{F(r)}{r^2} - \frac{q_1 + q_2 + q_3}{r^2}\right]$$
$$\frac{1 - F(r)}{r^2} - \frac{F'(r)}{r} = \frac{8\pi Gc^{-4}\nu^2}{r^4}$$
$$\times\left[\frac{1}{2} - q_1\frac{F'(r)}{r} - (7q_1 + 4q_2 + q_3)\frac{F(r)}{r^2} - \frac{q_1 + q_2 + q_3}{r^2}\right]$$
$$\frac{F'(r)}{r} + \frac{F''(r)}{2} = \frac{8\pi Gc^{-4}\nu^2}{r^4}$$
$$\times\left[\frac{1}{2} - q_1\frac{F''(r)}{2} - (7q_1 + 4q_2 + q_3)\left(\frac{F'(r)}{r} - \frac{2F(r)}{r^2}\right) + 2\frac{(q_1 + q_2 + q_3)}{r^2}\right]$$

$$q_1 \equiv -q, q_2 = 4q, q_3 = -6q$$

$$\frac{d}{dr}\left[r(F(r) - 1)\left(1 + \frac{2GQ^2q}{r^4}\right)\right] = -\frac{GQ^2}{r^2}$$

$$F(r) = 1 + \frac{r^4}{r^4 + 2qGQ^2}\left(-\frac{2GM}{r} + \frac{GQ^2}{r^2}\right),$$

$$E^2 f(\epsilon)^2 - p^2 g^2(\epsilon) = m^2$$

$$ds^2 = -\frac{F(r)}{f^2(\epsilon)} dt^2 + \frac{dr^2}{F(r) g^2(\epsilon)} + \frac{r^2}{g^2(\epsilon)} d\Omega^2$$

$$\tilde{t} = \frac{t}{f(\epsilon)}, \tilde{r} = \frac{r}{g(\epsilon)}, d\tilde{t} = \frac{dt}{f(\epsilon)}, d\tilde{r} = \frac{dr}{g(\epsilon)}, \tilde{G} = \frac{G}{g(\epsilon)}$$

$$\tilde{G} = \frac{G}{g(\epsilon)}$$

$$f(\epsilon) = g(\epsilon) = \frac{1}{1 + \lambda\epsilon}$$

$$F(\tilde{r}) = 1 + \frac{\tilde{r}^4}{\tilde{r}^4 + 2q\tilde{G}Q^2}\left(-\frac{2\tilde{G}M}{\tilde{r}} + \frac{\tilde{G}Q^2}{\tilde{r}^2}\right),$$

$$F(r,\epsilon) = 1 + \frac{r^4}{r^4 + 2qQ^2 G g^3(\epsilon)}\left(-\frac{2GM}{r} + \frac{GQ^2 g(\epsilon)}{r^2}\right),$$

$$\tilde{Q}^2 = Q^2 g(\epsilon), \tilde{q} = q g(\epsilon)^2$$

$$T_H = \frac{1}{2\pi} \lim_{r \to r_H} \sqrt{\frac{-g^{tt} g^{rr}}{4} \left(\frac{\partial g_{tt}}{\partial r}\right)^2}$$

$$T_H = \frac{1}{4\pi} \sqrt{\left(\frac{\partial F(r,\epsilon)}{\partial r}\right)^2}\Bigg|_{r=r_H}$$

$$M = \frac{r_H^4 + 2Q_m^2 q g(\epsilon)^3 + r_H^2 Q_m^2 g(\epsilon)}{2r_H^3}$$

$$T_H = \frac{-6qQ^2 + (r_H + \lambda)^2[-Q^2 + r_H(r_H + \lambda)]}{4\pi r_H[2qQ^2 + r_H(r_H + \lambda)^3]}$$

$$M = \frac{r_H}{2}\left\{1 + \frac{Q^2[2q + (\lambda + r_H)^2]}{r_H(\lambda + r_H)^3}\right\},$$

$$S = S_0 - \frac{2\pi q Q^2}{r_H^2} + \lambda\pi Q \log\left(\frac{r_H - Q}{r_H + Q}\right),$$

$$C = \frac{dM}{dT} = \frac{dM/dr_H}{dT_H/dr_H}.$$

$$C = -2\pi r_H^2$$

$$T_{\text{quantum hole}} = (f(\epsilon)/g(\epsilon)) T_{\text{gravity}}$$

$$\tilde{G}(\epsilon) = G/g(\epsilon)$$

$$\frac{2\tilde{G}(\epsilon)M}{\tilde{r}(\epsilon)} = \frac{2GM}{r},$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right) = 0$$

$$\frac{d^2\varphi}{dr_*^2} + \left[\omega^2 + V_{eff}\left(r(r_*)\right)\right]\varphi = 0$$

$$dr_*/dr = F(r)^{-1}$$

$$V_{eff}(r) = g(\epsilon)^2\frac{\ell(\ell+1)F(r)}{r^2} + g(\epsilon)^2\frac{F(r)F'(r)}{r}.$$

$$\Gamma_\ell^\omega = \text{sech}^2\left(\int_{r_H}^{\infty}\frac{V_{eff}}{2\omega}dr_*\right),$$

$$\frac{d^2N}{dtd\omega} = \frac{1}{2\pi}\frac{\Gamma_0^\omega}{[\exp(\omega/T) - 1]},$$

$$T_{\mu\nu} = (\rho + p_\perp)t_\mu t_\nu + p_\perp g_{\mu\nu} - (p_\perp - p_r)x_\mu x_\nu,$$

$$t_\mu t^\mu = 1, x_\mu x^\mu = -1$$

$$x^\mu t_\mu = \rho$$

$$\rho^{eff} = -p_r^{eff} = \frac{2Q^2g(\epsilon)}{r^4}\left[\frac{1}{2} + \frac{qg^2(\epsilon)F'(r)}{r} - \frac{3qg^2(\epsilon)F(r)}{r^2} + \frac{3qg^2(\epsilon)}{r^2}\right];$$

$$p_\theta^{eff} = p_\phi^{eff} = \frac{2Q^2g(\epsilon)}{r^4}\left[\frac{1}{2} + \frac{1}{2}qg^2(\epsilon)F''(r) - 3qg^2(\epsilon)\left(\frac{F'(r)}{r} - \frac{2F(r)}{r^2}\right) - \frac{6qg^2(\epsilon)}{r^2}\right],$$

$$T_{\mu\nu}^{eff}l^\mu l^\nu \geq 0 \rightarrow \rho^{eff} + p_i^{eff} \geq 0$$

$$\rho^{eff} + p_r^{eff}$$

$$\rho^{eff} + p_\theta^{eff}$$

$$\rho^{eff} + p_i^{eff} \geq 0$$

$$T_{\mu\nu}^{eff}t^\mu t^\nu \geq 0 \rightarrow \rho^{eff} + p_i^{eff} \geq 0 \text{ and } \rho^{eff} \geq 0$$

$$\rho^{eff} \geq 0 \text{ and } \rho^{eff} - \left|p_i^{eff}\right| \geq 0$$

$$\rho^{eff} + p_i^{eff} \geq 0 \text{ and } \rho^{eff} + p_r^{eff} + p_\theta^{eff} + p_\phi^{eff} \geq 0.$$

$$\rho^{eff} - \left|p_i^{eff}\right| \geq 0$$

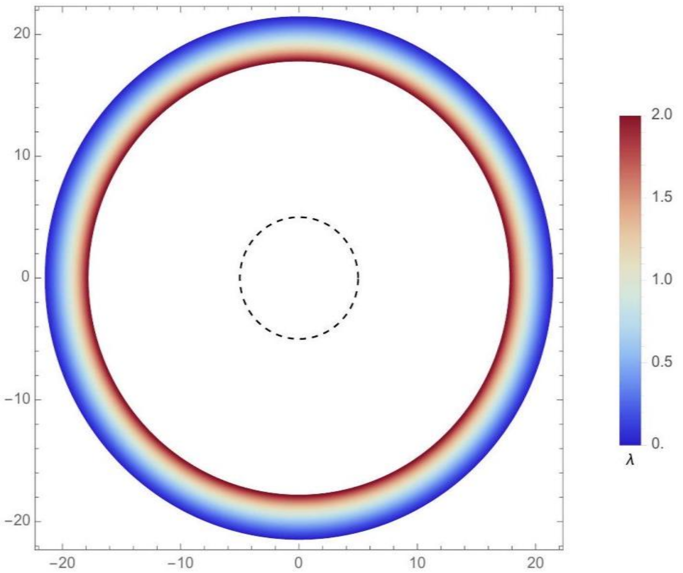
$$R = -\frac{g(\epsilon)^2(-2+2F(r)+4rF'(r)+r^2F''(r))}{r^2}$$
$$= -\frac{8g(\epsilon)^3qQ^2[6g(\epsilon)^4qQ^4-20g(\epsilon)^3MqQ^2r-5g(\epsilon)Q^2r^4+6Mr^5]}{[2g(\epsilon)^3qQ^2+r_H^4]^3}$$

$$R = -\frac{6(\lambda+r_H)^2}{qr_H^2}.$$

$$K = g(\epsilon)^4\left[\frac{4(F(r)-1)^2}{r^4}+\frac{4F'(r)^2}{r^2}+F''(r)^2\right].$$

$$\rho^{eff}+p_r^{eff}+p_\theta^{eff}+p_\phi^{eff}$$

$$K \approx \frac{20}{9q^2}+r\left[\frac{1}{9q^2}-\frac{8}{q^2r_H}-\frac{8r_H}{q^2Q^2}-\frac{8\lambda}{q^2Q^2}-\frac{24}{qr_H(r_H+\lambda)^2}\right].$$



$$R_{sh} \approx R_o \sin\,\alpha_{sh},$$

$$\sin\,\alpha_{sh} = \frac{\gamma(r_{ph})}{\gamma(R_o)},$$

$$\gamma(r) = \sqrt{-\frac{g^{tt}}{g^{\phi\phi}}}.$$

$$\frac{d\gamma^2(r)}{dr} = 0 \text{ at } r = r_{ph}.$$

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} d^nx\sqrt{-g}(R-2\Lambda+L(F))$$

$$L(F)=\begin{cases}4\beta^2\left[1-\sqrt{1+\frac{F^2}{2\beta^2}}\right],\\ 4\beta^2\left[e^{-\frac{F^2}{4\beta^2}}-1\right],\\ -8\beta^2\ln\left[1+\frac{F^2}{8\beta^2}\right],\end{cases}$$

$$F^2=\gamma_{ab}F_{\mu\nu}^{(a)}F^{(b)\mu\nu},\gamma_{ab}\equiv-\frac{\Gamma_{ab}}{|\det\Gamma_{ab}|^{1/N}}$$

$$F_{\mu\nu}^{(a)}=\partial_\mu A_\nu^{(a)}-\partial_\nu A_\mu^{(a)}+\frac{1}{e}C_{bc}^{a}A_\mu^{(b)}A_\nu^{(c)},$$

$$L(F)=\zeta_1\beta^2\mathcal{L}(Y)$$

$$\mathcal{L}(Y)=\begin{cases}1-\sqrt{1+Y},\\ e^{-Y}-1,\\ \ln(1+Y),\end{cases}$$

$$\zeta_1=+4,+4,-8$$

$$Y=\frac{F^2}{2\beta^2},\frac{F^2}{4\beta^2},\frac{F^2}{8\beta^2}$$

$$R_{\mu\nu}=\frac{2\Lambda}{n-2}g_{\mu\nu}+\zeta_2\gamma_{ab}\partial_Y\mathcal{L}(Y)F_\mu^{(a)\lambda}F_{\nu\lambda}^{(b)}+\frac{\zeta_3\beta^2}{n-2}[2Y\partial_Y\mathcal{L}(Y)-\mathcal{L}(Y)]g_{\mu\nu}$$
$$\nabla_\nu\left(\partial_Y\mathcal{L}(Y)F^{(a)\mu\nu}\right)=\frac{1}{e}\partial_Y\mathcal{L}(Y)C_{bc}^{a}A_\nu^{(b)}F^{(c)\nu\mu}$$

$$\zeta_2=-4,-2,+2$$

$$\zeta_3=+4,+4,-8$$

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega_k^2$$

$$d\Omega_k^2=d\theta^2+k^{-1}\sin^2(\sqrt{k}\theta)\left(d\phi_1^2+\sum_{i=2}^{n-3}\Pi_{j=1}^{i-1}\sin^2\phi_j d\phi_i^2\right)$$

$$\theta\in\left[0,\frac{\pi}{2}\right]$$

$$x_1=\frac{r}{\sqrt{k}}\sin(\sqrt{k}\theta)\Pi_{j=1}^{n-3}\sin\phi_j,$$
$$x_i=\frac{r}{\sqrt{k}}\sin(\sqrt{k}\theta)\cos\phi_{n-i-1}\Pi_{j=1}^{n-i-2}\sin(\phi_j),i=2,\ldots,n-2$$
$$x_{n-1}=r\cos(\sqrt{k}\theta),$$

$$A^{(a)} = \frac{e}{r^2}(x_a dx_{n-1} - x_{n-1}dx_a) \text{ for } a = 1, \dots, n-2$$
$$A^{(b)} = \frac{e}{r^2}(x_i dx_j - x_j dx_i) \text{ for } i = 1, \dots, n-3, j = 2, \dots, n-2, \text{ and } i < j$$

$$\gamma_{ab} = \epsilon_a \delta_{ab},$$

$$\epsilon_a = 1 \text{ , for } a = 1, \dots, \frac{(n-1)(n-2)}{2},$$

$$\epsilon_a = \begin{cases} -1 & 1 \le a \le n-2 \\ 1 & n-1 \le a \le \frac{(n-1)(n-2)}{2}. \end{cases}$$

$$f(r) = k - \frac{m}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)} + \begin{cases} \frac{4\beta^2 r^2}{(n-1)(n-2)}\left[1 - \frac{n-1}{r^{n-1}}\int r^{n-2}\sqrt{1+\frac{\eta}{2}}dr\right], \\ -\frac{4\beta^2 r^2}{(n-1)(n-2)}\left[1 - \frac{n-1}{r^{n-1}}\int r^{n-2}\exp\left(-\frac{\eta}{4}\right)dr\right], \\ -\frac{8\beta^2}{(n-2)r^{n-3}}\int r^{n-2}\ln\left[1+\frac{\eta}{8}\right]dr, \end{cases}$$

$$\eta = \frac{(n-2)(n-3)e^2}{\beta^2 r^4}$$

$$f(r) = k - \frac{m}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)} + \begin{cases} \frac{4\beta^2 r^2}{(n-1)(n-2)}\left[1 - {}_2F_1\left(\left[\frac{-1}{2}, \frac{1-n}{4}\right], \left[\frac{5-n}{4}\right], -\frac{\eta}{2}\right)\right], \\ -\frac{4\beta^2 r^2}{(n-1)(n-2)}\left[1 - {}_2F_1\left(\left[\frac{1-n}{4}\right], \left[\frac{5-n}{4}\right], -\frac{\eta}{4}\right)\right], \\ -\frac{8\beta^2 r^2}{(n-1)(n-2)}\ln\left[1+\frac{\eta}{8}\right] - \frac{4(n-3)e^2}{(n-1)(n-5)r^2}\,{}_2F_1\left(\left[1, \frac{5-n}{4}\right], \left[\frac{9-n}{4}\right], -\frac{\eta}{8}\right), \end{cases}$$

$$f(r) = k - \frac{m}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)} + \begin{cases} -\frac{(n-3)e^2}{(n-5)r^2} + \mathcal{O}\left(\frac{1}{\beta^2}\right), & n \neq 5 \\ -\frac{2e^2 \ln(r/r_0)}{r^2} + \mathcal{O}\left(\frac{1}{\beta^2}\right), & n = 5 \end{cases}$$

$$f(r) = 1 - \frac{m}{r^2} - \frac{\Lambda r^2}{6} + \frac{\beta^2 r^2}{3}\left[1 - \sqrt{1+\frac{\eta}{2}}\right] - \frac{e^2}{r^2}\ln\left[\mathrm{r}^2\left(1+\sqrt{1+\frac{\eta}{2}}\right)\right] - \frac{4\beta^2 \mathrm{C}_5}{3\mathrm{r}^2},$$

$$C_5 = -\frac{3e^2}{8\beta^2}(1 + 2\ln(2))$$

$$f(r) = 1 - \frac{m - A_5}{r^2} - \frac{2\sqrt{3}}{3}\beta e + \mathcal{O}(r), A_5 = \frac{1}{2}e^2\left(1 + \ln\left(\frac{4\beta^2}{3e^2}\right)\right)$$

$$f(r) = 1 - \frac{2\sqrt{3}}{3}\beta e$$

$$\left(\frac{4}{l^2} + \frac{4\beta^2}{3}\right) r_{ex}^3 + \left(2 + \frac{4\beta}{3}\sqrt{\beta^2 r_{ex}^2 + 3e^2}\right) r_{ex}$$

$$f(r) = 1 - \frac{m - A_6}{r^3} - \frac{\sqrt{6}}{3}\beta e + \mathcal{O}(r), A_6 = -\frac{12}{5}\sqrt[4]{\frac{6}{\pi^2\beta^2}} e^{5/2}\Gamma\left(\frac{3}{4}\right)^2$$

$$f(r) = 1 - \frac{m - A_5}{r^2} + \mathcal{O}(r), A_5 = \frac{1}{2}e^2\left(1 - \gamma + \ln\left(\frac{2\beta^2}{3e^2}\right)\right)$$

$$f(r) = 1 - \frac{m - A_5}{r^2} + \mathcal{O}(r), A_5 = \frac{1}{2}e^2\left(1 + \ln\left(\frac{4\beta^2}{3e^2}\right)\right)$$

$$f(r) = k - \frac{m}{r^{n-3}} + \frac{4\beta^2 C_n}{(n-2)r^{n-3}} - \frac{2\sqrt{2(n-2)(n-3)}}{(n-2)(n-3)}\beta e + \mathcal{O}(r),$$

$$f(r) = k - \frac{m}{r^{n-3}} + \frac{4\beta^2 C_n}{(n-2)r^{n-3}} + \mathcal{O}(r),$$

$$f(r) = k - \frac{m}{r^{n-3}} - \frac{8\beta^2 C_n}{(n-2)r^{n-3}} + \mathcal{O}(r),$$

$$T_+ = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi} = \frac{k(n-3)}{4\pi r_+} - \frac{\Lambda}{2\pi(n-2)} r_+$$

$$+\begin{cases} +\dfrac{\beta^2 r_+}{\pi(n-2)}\left(1 - \sqrt{1 + \dfrac{(n-2)(n-3)e^2}{2\beta^2 r_+^4}}\right), \\ -\dfrac{\beta^2 r_+}{\pi(n-2)}\left[1 - \exp\left(-\dfrac{(n-2)(n-3)e^2}{4\beta^2 r_+^4}\right)\right], \\ -\dfrac{2\beta^2 r_+}{\pi(n-2)}\ln\left(1 + \dfrac{(n-2)(n-3)e^2}{8\beta^2 r_+^4}\right), \end{cases}$$

$$S = \frac{A}{4} = \frac{r_+^{n-2}}{4}\omega_{n-2},$$

$$ds^2 = \lambda_{ab} dx^a dx^b = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-2}^2,$$

$$V_0(r) = \begin{cases} k - \dfrac{2\Lambda}{(n-1)(n-2)} r^2, & n \neq 5 \\ k - \dfrac{\Lambda}{6} r^2 - \dfrac{2e^2 \ln(r/r_0)}{r^2} & n = 5. \end{cases}$$

$$M = \frac{1}{8\pi}\int_\Sigma d^{n-2}\sqrt{\sigma}\{(K_{ab} - K\lambda_{ab}) - (K_{ab}^0 - K^0\lambda_{ab}^0)\} n^a \xi^b,$$

$$M = \frac{(n-2)\omega_{n-2}}{16\pi} m$$

$$Q = \frac{1}{4\pi} \int d^{n-2}x \sqrt{\mathrm{Tr}\left(F_{\mu\nu}^{(a)} F^{(a)\mu\nu}\right)} = \frac{\sqrt{(n-2)(n-3)}\omega_{n-2}}{4\pi} e.$$

$$dM = TdS + \Phi dQ$$

$$T = \left(\frac{\partial M}{\partial S}\right)_Q$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \begin{cases} -\frac{2\pi Q(n-2)(n-3)r_+^{n-5}}{(n-5)} {}_2F_1\left(\left[\frac{1}{2}, \frac{5-n}{4}\right], \left[\frac{9-n}{4}\right], -8\xi_+\right), \\ -\frac{2\pi Q(n-2)(n-3)r_+^{n-5}}{(n-5)} {}_2F_1\left(\left[\frac{5-n}{4}\right], \left[\frac{9-n}{4}\right], -4\xi_+\right), \\ -\frac{2\pi Q(n-2)(n-3)r_+^{n-5}}{(n-1)(1+2\xi_+)} - \frac{8\pi Q(n-2)(n-3)r_+^{n-5}}{(n-1)(n-5)} {}_2F_1\left(\left[1, \frac{5-n}{4}\right], \left[\frac{9-n}{4}\right], -2\xi_+\right) + \\ \frac{16\pi Q\xi_+(n-2)(n-3)r_+^{n-5}}{(n-1)(n-9)} {}_2F_1\left(\left[2, \frac{9-n}{4}\right], \left[\frac{13-n}{4}\right], -2\xi_+\right). \end{cases}$$

$$\xi_+ = \frac{(n-2)(n-3)\pi^2 Q^2}{\beta^2 r_+^4}$$

$$C_Q = T_+ \left(\frac{\partial S}{\partial T_+}\right)_Q = T_+ \left(\frac{\partial^2 M}{\partial S^2}\right)_Q^{-1}$$

$$H = \begin{bmatrix} \left(\frac{\partial^2 M}{\partial S^2}\right)_Q & \left(\frac{\partial^2 M}{\partial S \partial Q}\right) \\ \left(\frac{\partial^2 M}{\partial Q \partial S}\right)^2 & \left(\frac{\partial^2 M}{\partial Q^2}\right)_S \end{bmatrix}$$

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q > 0$$

$$\left(\frac{\partial^2 M}{\partial Q^2}\right)_S > 0$$

$$dM = TdS + \Phi dQ + VdP + Bd\beta$$

$$P = -\frac{\Lambda}{8\pi}.$$

$$v = \frac{4r_+}{n-2}$$

$$V=\frac{\omega_{n-2}}{n-1}r_+^{n-1},$$

$$B=\frac{\partial M}{\partial\beta}$$

$$=\begin{cases}\frac{e^2(n-2)(n-3)r^{n-5}}{8\pi\beta(n-5)}\ {}_2F_1\Bigl(\Bigl[\frac{1}{2},\frac{5-n}{4}\Bigr],\Bigl[\frac{9-n}{4}\Bigr],-\frac{\eta_+}{2}\Bigr)-\frac{\beta r^{n-1}}{2\pi(n-1)}\Bigl(\ {}_2F_1\Bigl(\Bigl[-\frac{1}{2},\frac{1-n}{4}\Bigr],\Bigl[\frac{5-n}{4}\Bigr],-\frac{\eta_+}{2}\Bigr) \\ \frac{e^2(n-2)(n-3)r^{n-5}}{8\pi\beta(n-5)}\ {}_2F_1\Bigl(\Bigl[\frac{5-n}{4}\Bigr],\Bigl[\frac{9-n}{4}\Bigr],-\frac{\eta_+}{4}\Bigr)+\frac{\beta r^{n-1}}{2\pi(n-1)}\Bigl(\ {}_2F_1\Bigl(\Bigl[\frac{1-n}{4}\Bigr],\Bigl[\frac{5-n}{4}\Bigr],-\frac{\eta_+}{4}\Bigr)-1\Bigr), \\ \frac{(n-2)(n-3)e^2r^{n-5}}{8\pi\beta(n-1)}\Bigl(1+\frac{\eta_+}{8}\Bigr)^{-1}-\frac{\beta r^{n-1}}{\pi(n-1)}\ln\Bigl(1+\frac{\eta_+}{8}\Bigr)-\frac{\eta_+(n-2)(n-3)e^2r^{n-5}}{16\pi(n-1)(n-9)\beta}\ {}_2F_1\Bigl(\Bigl[2,\frac{9-n}{4}\Bigr]\end{cases}$$

$$M=\frac{1}{n-3}[\Phi Q-\beta B-2PV+(n-2)TS],\ \text{for } n\neq 4z+1.$$

$$\frac{\partial P}{\partial v}=0,\frac{\partial^2P}{\partial v^2}=0$$

$$P=\frac{T}{v}-k\frac{n-3}{\pi(n-2)v^2}-\frac{\beta^2}{4\pi}\left(1-\sqrt{1+\frac{128(n-3)e^2}{(n-2)^3\beta^2v^4}}\right).$$

$$x^3+px+q=0$$

$$x=\left[v_c^4+\frac{128(n-3)e^2}{(n-2)^3\beta^2}\right]^{-\frac{1}{2}},p=-\frac{3(n-2)^3\beta^2}{256(n-3)e^2},q=\frac{(n-2)^5\beta^2}{8192(n-3)e^4}$$

$$|x|\leq\frac{(n-2)\sqrt{2(n-2)(n-3)}\beta}{16(n-3)e}$$

$$\beta\geq\beta_0=\frac{\sqrt{(n-2)(n-3)}}{4e}$$

$$x_{k'}=2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{3}{\pi}}\right)-\frac{2\pi k'}{3}\right),k'=0,1,2$$

$$\beta_0=\frac{\sqrt{(n-2)(n-3)}}{4e}\leq\beta\leq\beta_2$$

$$\beta_2=\frac{\sqrt{2(n-2)(n-3)}}{4e}$$

$$\beta\geq\beta_1=\frac{\sqrt{(n-2)(n-3)(6+4\sqrt{3})}}{12e}$$

$$v_c = \left[\frac{1}{x^2} - \frac{128(n-3)e^2}{(n-2)^3\beta^2}\right]^{\frac{1}{4}}, x = \begin{cases} x_1, & \beta \geq \beta_0 \\ x_0, & \beta \in (\beta_0, \beta_2) \end{cases}$$

$$v_c = \frac{4\sqrt{6}e}{n-2} - \frac{7\sqrt{6}(n-3)}{216e\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

$$T_c = \frac{\sqrt{6}(n-3)}{18\pi e} + \frac{\sqrt{6}(n-2)(n-3)^2}{5184e^3\pi\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

$$P_c = \frac{(n-2)(n-3)}{192\pi e^2} + \frac{7(n-2)^2(n-3)^2}{165888\pi e^4\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

$$\rho_c = \frac{P_c V_c}{T_c} = \frac{3}{8} - \frac{(n-2)(n-3)}{768e^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right).$$

$$\rho_c = \frac{P_c v_c}{T_c}$$

$$P = \frac{T}{v} - k\frac{(n-3)}{\pi(n-2)v^2} + \frac{\beta^2}{4\pi}\left[1 - \exp\left(-\frac{64(n-3)e^2}{(n-2)^3\beta^2 v^4}\right)\right],$$

$$P = \frac{T}{v} - k\frac{(n-3)}{\pi(n-2)v^2} + \frac{\beta^2}{2\pi}\ln\left(1 + \frac{32(n-3)e^2}{(n-2)^3\beta^2 v^4}\right),$$

$$\beta_0 = \frac{\sqrt{3}}{2e} \approx \frac{0.86}{e}, \beta_1 = \frac{\sqrt{6(3+2\sqrt{3})}}{6e} \approx \frac{1.03}{e}$$

$$\beta_2 = \frac{\sqrt{6}}{2e} \approx \frac{1.22}{e}$$

$$T_{c,YM} = \sqrt{6}/9\pi e, v_{c,YM} = 4\sqrt{6}e/3, P_c = 1/32\pi e^2$$

$$S(T,V) = \frac{1}{4}\omega_{n-2}^{\frac{1}{n-1}}(V(n-1))^{\frac{n-2}{n-1}}$$

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = 0 \Rightarrow \alpha = 0.$$

$$p = \frac{P}{P_c}, \nu = \frac{v}{v_c}, \tau = \frac{T}{T_c},$$

$$p = \frac{\tau}{\nu\rho_c} + h(\nu),$$

$$\rho_c = \frac{P_c v_c}{T_c}$$

$$h(v) = -k\frac{n-3}{\pi P_c(n-2)v^2 v_c^2} + \frac{1}{P_c} \times \begin{cases} -\frac{\beta^2}{4\pi}\left(1-\sqrt{1+\frac{128(n-3)e^2}{(n-2)^3\beta^2 v^4 v_c^4}}\right), \\ \frac{\beta^2}{4\pi}\left[1-\exp\left(-\frac{64(n-3)e^2}{(n-2)^3\beta^2 v^4 v_c^4}\right)\right], \\ \frac{\beta^2}{2\pi}\ln\left(1+\frac{32(n-3)e^2}{(n-2)^3\beta^2 v^4 v_c^4}\right). \end{cases}$$

$$v = (1+\omega)^{1/z}$$

$$p = 1 + At - Btw - Cw^3 + \mathcal{O}(tw^2, w^4),$$

$$B = \frac{1}{z\rho_c}, C = \frac{1}{z^3}\left(\frac{1}{\rho_c} - \frac{h^{(3)}\big|_{v=1}}{6}\right)$$

$$dP = -P_c(Bt + 3Cw^2)dw$$

$$p = 1 + At - Bt\omega_l - C\omega_l^3 = 1 + At - Bt\omega_s - C\omega_s^3$$

$$0 = \int_{\omega_l}^{\omega_s} \omega dP$$

$$w_s = -\omega_l = \sqrt{-\frac{Bt}{C}}.$$

$$\eta = V_c(\omega_l - \omega_s) = 2V_c\omega_l \propto \sqrt{-t}$$

$$\beta' = \frac{1}{2}.$$

$$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_T \propto -\frac{V_c}{BP_c}\frac{1}{t}$$

$$\hat{F} \equiv \frac{1}{4}F^2$$

$$\mathcal{L}(\hat{F}) \equiv -L(\hat{F})/4$$

$$\mathcal{H} \equiv 2\mathcal{L}_{\hat{F}}\hat{F} - \mathcal{L}$$

$$\mathcal{L}_{\hat{F}} \equiv \frac{\partial \mathcal{L}}{\partial \hat{F}}$$

$$P \equiv \mathcal{L}_{\hat{F}}^2 \hat{F}$$

$$\mathcal{H}_P \equiv \frac{\partial \mathcal{H}}{\partial P}$$

$$\mathcal{H}_P = \begin{cases} \sqrt{1+\frac{2\hat{F}}{\beta^2}}, \\ e^{\frac{\hat{F}}{\beta^2}}, \\ 1+\frac{\hat{F}}{2\beta^2}. \end{cases}$$

$$\hat{F} = \frac{e^2}{2r^4}$$

$$x = \sqrt{-2Q^2P}$$

$$\mathcal{H}_{xx} = \frac{\partial^2 \mathcal{H}}{\partial x^2}$$

$$\mathcal{H}_{xx} = -\frac{1}{Q^2} \times \begin{cases} \left(1+\frac{2\hat{F}}{\beta^2}\right)^{\frac{3}{2}}, \\ e^{\frac{\hat{F}}{\beta^2}}\left(1-\frac{2\hat{F}}{\beta^2}\right)^{-1}, \\ \left(1+\frac{\hat{F}}{2\beta^2}\right)^{2}\left(1-\frac{\hat{F}}{2\beta^2}\right)^{-1}. \end{cases}$$

$$\beta^2 < e^2/r^4 \text{ and } \beta^2 < e^2/4r^4$$

$$C_{23}^1 = C_{31}^2 = C_{12}^3 = -1, \gamma_{ab} = \text{diag}(1,1,1)$$

$$A_\mu^{(i)} = A_\theta^{(i)} d\theta + A_\phi^{(i)} d\phi, i = 1,2,3$$

$$\begin{bmatrix} A_\mu^{(1)} \\ A_\mu^{(2)} \\ A_\mu^{(3)} \end{bmatrix} = e \begin{bmatrix} -\cos\phi & \sin\theta\cos\theta\sin\phi \\ -\sin\phi & -\sin\theta\cos\theta\cos\phi \\ 0 & \sin^2\theta \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \end{bmatrix}$$

$$C_{23}^1 = C_{31}^2 = -C_{12}^3 = 1, \gamma_{ab} = \text{diag}(-1,-1,1),$$

$$\begin{bmatrix} A_\mu^{(1)} \\ A_\mu^{(2)} \\ A_\mu^{(3)} \end{bmatrix} = e \begin{bmatrix} -\cos\phi & \sinh\theta\cosh\theta\sin\phi \\ -\sin\phi & -\sinh\theta\cosh\theta\cos\phi \\ 0 & \sinh^2\theta \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \end{bmatrix}$$

$$C_{24}^1 = C_{35}^1 = C_{41}^2 = C_{36}^2 = C_{51}^3 = C_{62}^3 = 1,$$
$$C_{56}^4 = -C_{21}^4 = C_{64}^5 = -C_{31}^5 = C_{45}^6 = -C_{32}^6 = 1,$$
$$\gamma_{ab} = \text{diag}(1,1,1,1,1,1),$$

$$A_\mu^{(i)} = A_\theta^{(i)} d\theta + A_\phi^{(i)} d\phi + A_\psi^{(i)} d\psi, i = 1,2,3,4,5,6$$

$$\begin{bmatrix} A_\mu^{(1)} \\ A_\mu^{(2)} \\ A_\mu^{(3)} \\ A_\mu^{(4)} \\ A_\mu^{(5)} \\ A_\mu^{(6)} \end{bmatrix} = e \begin{bmatrix} -\sin\phi\cos\psi & -\sin\theta\cos\theta\cos\phi\cos\psi & \sin\theta\cos\theta\sin\phi\sin\psi \\ -\sin\phi\sin\psi & -\sin\theta\cos\theta\cos\phi\sin\psi & -\sin\theta\cos\theta\sin\phi\cos\psi \\ -\cos\phi & \sin\theta\cos\theta\sin\phi & 0 \\ 0 & 0 & -\sin^2\theta\sin^2\phi \\ 0 & \sin^2\theta\cos\psi & -\sin^2\theta\sin\phi\cos\phi\sin\psi \\ 0 & \sin^2\theta\sin\psi & \sin^2\theta\sin\phi\cos\phi\cos\psi \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \\ d\psi \end{bmatrix}$$

$$C_{24}^1 = C_{35}^1 = C_{41}^2 = C_{36}^2 = C_{51}^3 = C_{62}^3 = 1$$
$$C_{56}^4 = C_{21}^4 = C_{64}^5 = C_{31}^5 = C_{45}^6 = C_{32}^6 = 1$$
$$\gamma_{ab} = \mathrm{diag}(-1,-1,-1,1,1,1),$$

$$\begin{bmatrix} A_\mu^{(1)} \\ A_\mu^{(2)} \\ A_\mu^{(3)} \\ A_\mu^{(4)} \\ A_\mu^{(5)} \\ A_\mu^{(6)} \end{bmatrix} = e \begin{bmatrix} -\sin\phi\cos\psi & -\sinh\theta\cosh\theta\cos\phi\cos\psi & \sinh\theta\cosh\theta\sin\phi\sin\psi \\ -\sin\phi\sin\psi & -\sinh\theta\cosh\theta\cos\phi\sin\psi & -\sinh\theta\cosh\theta\sin\phi\cos\psi \\ -\cos\phi & \sinh\theta\cosh\theta\sin\phi & 0 \\ 0 & 0 & \sinh^2\theta\sin^2\phi \\ 0 & -\sinh^2\theta\cos\psi & \sinh^2\theta\sin\phi\cos\phi\sin\psi \\ 0 & -\sinh^2\theta\sin\psi & -\sinh^2\theta\sin\phi\cos\phi\cos\psi \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \\ d\psi \end{bmatrix}$$

$$f(r) = k - \frac{m}{r^2} - \frac{\Lambda r^2}{6}$$
$$+\begin{cases} \frac{\beta^2 r^2}{3}\left[1-\sqrt{1+\frac{\eta}{2}}\right] - \frac{e^2}{r^2}\left(\ln\left[\frac{r^2}{2}\left(1+\sqrt{1+\frac{\eta}{2}}\right)\right] - \frac{1}{2}\right), \\ -\frac{\beta^2 r^2}{3}\left[1-\exp\left(-\frac{\eta}{4}\right)\right] - \frac{e^2}{2r^2}\left[E_i\left(1,\frac{\eta}{4}\right) - 1 + \ln\left(\frac{3e^2}{2\beta^2}\right) + \gamma\right], \\ \frac{e^2}{2r^2}\left[1-4\ln(r)\right] - \frac{2\beta^2 r^2}{3}\left(1+\frac{\eta}{8}\right)\ln\left(1+\frac{\eta}{8}\right), \end{cases}$$

$$E_i(a,z) = z^{a-1}\Gamma(1-a,z)$$

$$f(r) = k - \frac{m}{r^6} - \frac{\Lambda r^2}{28}$$
$$+\begin{cases} \frac{\beta^2 r^2}{14}\left[1-\sqrt{1+\frac{21\eta}{6}}\right] - \frac{3e^2}{4r^2}\sqrt{1+\frac{21\eta}{6}} + \frac{21\eta e^2}{8r^2}\left(\ln\left[\frac{r^2}{2}\left(1+\sqrt{1+\frac{21\eta}{6}}\right)\right] + \frac{1}{4}\right), \\ -\frac{\beta^2 r^2}{14}\left[1-\left(1-\frac{7\eta}{8}\right)\exp\left(-\frac{7\eta}{8}\right)\right] + \frac{21\eta e^2}{16r^2}\left[E_i\left(1,\frac{7\eta}{4}\right) - \frac{3}{2} + \ln\left(\frac{21e^2}{2\beta^2}\right) + \gamma\right], \\ -\frac{21\eta e^2}{64r^2}(1-8\ln(r)) - \frac{3e^2}{4r^2} - \frac{\beta^2 r^2}{7}\left[1-\left(\frac{49\eta^2}{64}\right)\right]\ln\left(1+\frac{7\eta}{8}\right). \end{cases}$$

$$\log Z \approx -\frac{4N^2}{\pi^2}\beta^3$$

$$S(q) = \log Z(\beta) + q\beta \to \pi\left[\frac{q^3}{27N^2}\right]^{\frac{1}{2}}$$

$$\log Z \approx -\frac{4N^2\beta^3}{\pi^2+\gamma^2}$$

$$S(q,j) = \pi\sqrt{\frac{q^3}{27N^2} - j^2}$$

$$\rho(\theta) = \frac{1}{\pi^2}(\pi - |\theta|) \text{ for } -\pi < \theta < \pi, \rho(\theta) = \rho(\theta + 2\pi).$$

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^{3} R_I\Delta_I - \sum_{i=1}^{2} J_i\omega_i}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 0$$

$$[R_I, Q] = +\frac{1}{2}Q, [J_i, Q] = -\frac{1}{2}Q$$

$$r\{Q, Q^\dagger\} \sim rE - (R_1 + R_2 + R_3 + J_1 + J_2) \geq 0$$

$$e^{-\Delta_1} = e^{-\Delta_2} = e^{-\Delta_3} \equiv x^2$$

$$e^{-\omega_1} = e^{-\omega_2} \equiv x^3$$

$$Z(x) = \mathrm{Tr}\left[(-1)^F x^{6\left(\frac{R_1+R_2+R_3}{3} + \frac{J_1+J_2}{2}\right)}\right] \equiv \mathrm{Tr}\left[(-1)^F x^{6(R+J_+)}\right]$$

$$R \equiv \frac{R_1 + R_2 + R_3}{3}, J_+ \equiv \frac{J_1 + J_2}{2}. q \equiv 6(R + J_+)$$

$$e^{-\Delta_I} \equiv x^2 \text{ and } e^{-\omega_1} = x^3 y, e^{-\omega_2} = x^3 y^{-1}$$

$$Z(x,y) = \mathrm{Tr}\left[(-1)^F x^{6(R+J_+)} y^{2J_-}\right]$$

$$J_- \equiv \frac{J_1 - J_2}{2}$$

$$Z(\Delta_I, \omega_i) = \frac{1}{N!}\prod_{a=1}^{N}\int_0^{2\pi}\frac{d\alpha_a}{2\pi} \cdot \prod_{a<b}\left(2\sin\frac{\alpha_{ab}}{2}\right)^2 \cdot \exp\left[\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{a_n(\Delta_I, \omega_i)}{n} e^{in\alpha_{ab}}\right]$$

$$a_n \equiv 1 - \frac{\prod_{I=1}^{3}\left(1 - e^{-n\Delta_I}\right)}{(1 - e^{-n\omega_1})(1 - e^{-n\omega_2})}$$

$$Z(\Delta_I, \omega_i) = \frac{1}{N!}\prod_{a=1}^{N}\int_0^{2\pi}\frac{d\alpha_a}{2\pi}\exp\left[N\sum_{n=1}^{\infty}\frac{a_n}{n}\right]\exp\left[-\sum_{a<b} V(\alpha_a - \alpha_b)\right]$$

$$V(\theta) \equiv -\log\left[4\sin^2\frac{\theta}{2}\right] - \sum_{n=1}^{\infty}\frac{a_n}{n}\left(e^{in\theta} + e^{-in\theta}\right)$$

$$Z(x) = \sum_{q=0}^{\infty}\Omega_q x^q$$

$$\Omega_q = \oint \frac{dx}{2\pi i x} x^{-q} Z(x)$$

$$\Omega_q \sim e^{S(q)+\cdots} + e^{\overline{S(q)}+\cdots} \sim e^{\mathrm{Re}[S(q)]+\cdots} \cos\left[\mathrm{Im}[S(q)] + \cdots\right]$$

$$Z(x) \leftarrow \sum_q e^{\mathrm{Re}(S(q))} e^{\pm i\mathrm{Im}(S(q)) - \mu q}$$

$$\pm \frac{d}{dq} \mathrm{Im}(S(q)) = \mathrm{Im}(\mu)$$

$$\frac{dS(q)}{dq} = \mu \text{ or } \frac{d\overline{S(q)}}{dq} = \mu$$

$$\sum_{b(\neq a)} V'(\alpha_a - \alpha_b) = 0$$

$$\rho(\alpha_a) \equiv \frac{1}{N} \frac{\Delta a}{\Delta \alpha_a}$$

$$\Delta\alpha_a \equiv \alpha_{a+\Delta a} - \alpha_a$$

$$\int_{-\theta_0}^{\theta_0} d\theta \rho(\theta) V'(\alpha - \theta) = 0$$

$$-\sum_a V'(\alpha - \alpha_a) = \int_{-\theta_0}^{\theta_0} \cot\left(\frac{\alpha - \theta}{2}\right) \rho(\theta) d\theta - 2 \sum_{n=1}^{\infty} a_n \rho_n \sin(n\alpha) = 0$$

$$\rho_n \equiv \int_{-\theta_0}^{\theta_0} d\theta \rho(\theta) \cos(n\theta), n = 1,2,\cdots$$

$$\int_{-\theta_0}^{\theta_0} d\theta \rho(\theta) \sin(n\theta) = 0$$

$$\rho(\theta) \overset{\text{real}}{\rightarrow} \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \rho_n \cos(n\theta)$$

$$s \equiv \frac{a}{N} - \frac{1}{2}$$

$$-\frac{1}{2} < s < \frac{1}{2}, \alpha_a$$

$$s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\rho(\alpha_a) = \frac{1}{N} \frac{da}{d\alpha_a}$$

$$s(\alpha) = \int \rho(\alpha) d\alpha + \aleph$$

$$\int_{-\theta_0}^{\theta_0} d\theta \rho(\theta) = 1$$

$$\rho(\theta) = \frac{1}{\pi}\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}} \sum_{n=1}^{\infty} Q_n \cos\left[\left(n - \frac{1}{2}\right)\theta\right]$$

$$Q_n \equiv 2\sum_{l=0}^{\infty} a_{n+l}\rho_{n+l} P_l(\cos\ \theta_0)$$

$$\sum_{l=0}^{\infty} P_l(x) z^l = (1 - 2xz + z^2)^{-\frac{1}{2}}$$

$$Q_1 = Q_0 + 2$$

$$R\vec{\rho} = \vec{\rho}\,, \vec{A}\cdot\vec{\rho} = 1$$

$$R_{ml} = a_l \sum_{k=1}^{l}\left[B^{m+k-\frac{1}{2}}(s^2) + B^{\left|m-k+\frac{1}{2}\right|}(s^2)\right] P_{l-k}(1 - 2s^2)$$

$$A_m = a_m[P_{m-1}(1 - 2s^2) - P_m(1 - 2s^2)]$$

$$s^2 \equiv \sin^2\frac{\theta_0}{2}$$

$$B^{n-\frac{1}{2}}(s^2) = \frac{1}{\pi}\int_{-\theta_0}^{\theta_0} d\theta \sqrt{s^2 - \sin^2\frac{\theta}{2}} \cos\left[\left(n - \frac{1}{2}\right)\theta\right]$$

$$\sum_{n=0}^{\infty} B^{n+\frac{1}{2}}(x) z^n = \frac{\sqrt{(1-z)^2 + 4zx} + z - 1}{2z}$$

$$\det(R - \mathbf{1}) = 0$$

$$\vec{\rho} = M^{-1}\vec{e}_1$$

$$M = \begin{pmatrix} M_{p\times p} & \mathbf{0}_{p\times\infty} \\ L_{\infty\times p} & \mathbf{1}_{\infty\times\infty} \end{pmatrix}$$

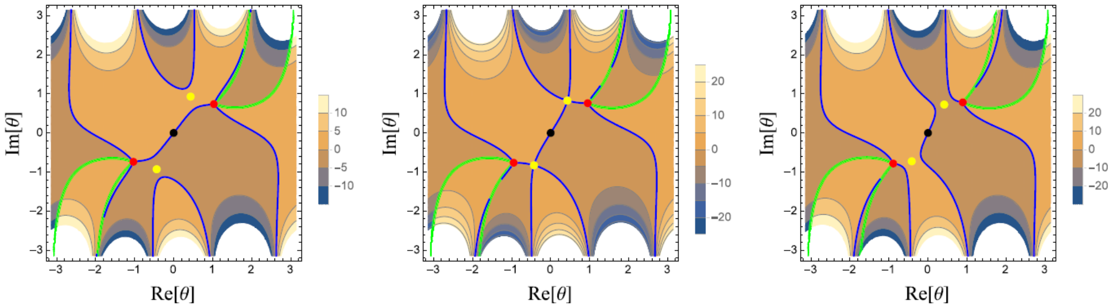
$$\vec{\rho} = \begin{pmatrix} M_{p\times p}^{-1} & \mathbf{0}_{p\times\infty} \\ -L_{\infty\times p} M_{p\times p}^{-1} & \mathbf{1}_{\infty\times\infty} \end{pmatrix} \vec{e}_1$$

$$\log\ Z = \frac{N^2}{2}\int_{-\theta_0}^{\theta_0} d\theta_1 d\theta_2 \log\left(4\sin^2\frac{\theta_1 - \theta_2}{2}\right)\rho(\theta_1)\rho(\theta_2) + N^2 \sum_{n=1}^{p} \frac{a_n}{n}|\rho_n|^2$$

$$\mu = \int_{-\theta_0}^{\theta_0} d\theta \rho(\theta) \log \left[4\sin^2 \frac{\alpha - \theta}{2}\right] + \sum_{n=1}^{p} \frac{2a_n}{n} \rho_n \cos(n\alpha)$$

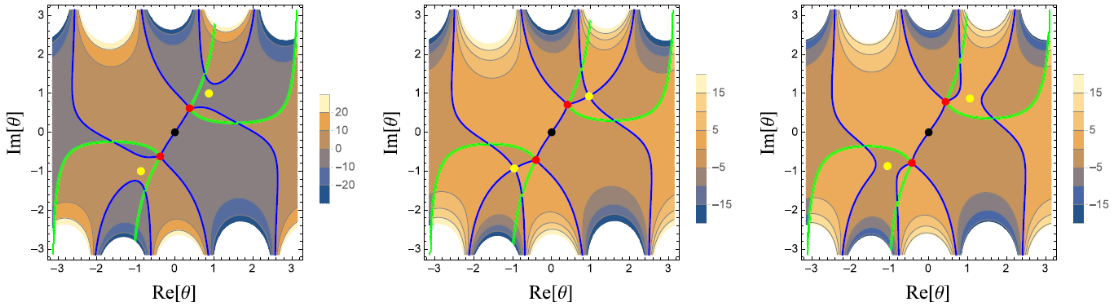
$$\frac{N^2}{2} \int_{-\theta_0}^{\theta_0} d\alpha \rho(\alpha)$$

$$\log Z = \frac{N^2 \mu}{2}$$



$$s(\theta) \approx s(\theta_*) + s'(\theta_*)(\theta - \theta_*) + \frac{1}{2} s''(\theta_*)(\theta - \theta_*)^2 + \cdots,$$

$$\mathrm{Im}[s(\theta_*(x))] = 0.$$



$$s(\theta) = \frac{1}{\pi} \arcsin \left[\frac{\sin \frac{\theta}{2}}{t^{\frac{1}{2}}}\right] + \frac{1}{\pi t} \sin \frac{\theta}{2} \sqrt{t - \sin^2 \frac{\theta}{2}}.$$

$$s(\theta) = \frac{1}{\pi} \arcsin \left[\frac{\sin \frac{\theta}{2}}{t^{\frac{1}{2}}}\right] + \frac{1}{\pi} \sin \frac{\theta}{2} \sqrt{t - \sin^2 \frac{\theta}{2}} \left[\frac{1}{t} - a_2(2 - 8t + 14t^2 - 7t^3) + 2a_2(1 - t)^2 \cos \theta\right]$$

$$s(0) = 0, s(\pm\theta_0) = \pm\frac{1}{2}$$

$$\theta_* = \pi, \theta_1, -\theta_1 \text{ , where } \cos \theta_1 \equiv 1 - \frac{Q_1}{Q_2} = -\frac{1}{2a_2 t(1 - t)^2} + \frac{4 - 10t + 12t^2 - 5t^3}{2(1 - t)^2}.$$

$$s(\theta_1) = \frac{1}{\pi}\arcsin\left[\frac{\sqrt{a_2^{-1} - 2t + 6t^2 - 10t^3 + 5t^4}}{2t(1-t)}\right]$$

$$+\frac{1+t}{2\pi t}\sqrt{-1 + 2a_2(2t - 4t^2 + 6t^3 - 3t^4) - a_2^2(4t^2 - 16t^3 + 36t^4 - 44t^5 + 36t^6 - 20t^7 + 5t^8)}$$

$$S_{\text{eff}} = \frac{N^2}{2}\int d\theta_1 d\theta_2 V(\theta_1 - \theta_2)\rho(\theta_1)\rho(\theta_2)$$

$$\rho(\theta) = \rho(\theta + 2\pi)$$

$$\int_0^{2\pi} d\theta\rho(\theta) = 1$$

$$\rho_n, \rho(\theta) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n\neq 0}\rho_n e^{in\theta}$$

$$\rho_{-n} = \overline{\rho_n}$$

$$S_{\text{eff}} = N^2\sum_{n=1}^{\infty}\frac{1-a_n}{n}|\rho_n|^2$$

$$\frac{1}{N}\text{Tr}_{\text{fund}}\left[P\exp\left(i\oint d\tau A_\tau\right)\right]$$

$$\frac{1}{N}\sum_{a=1}^{N} e^{i\alpha_a} = \rho_1$$

$$Z \sim \int\prod_{n=2}^{\infty} d\rho_n d\rho_{-n}\exp\left[N^2\sum_{n=2}^{\infty}\frac{a_n(x)-1}{n}|\rho_n|^2\right]\int_{f_-(\rho_n)}^{f_+(\rho_n)} d\rho_1\exp\left[N^2(a_1(x)-1)\rho_1^2\right]$$

$$f_-(\rho_n) \leq \rho_1 \leq f_+(\rho_n)$$

$$f_+(\rho_n = 0) = \frac{1}{2}, f_-(\rho_n = 0) = -\frac{1}{2}$$

$$\int_{f_-(\rho_n)}^{f_+(\rho_n)} d\rho_1\exp\left[N^2(a_1(x)-1)\rho_1^2\right]$$

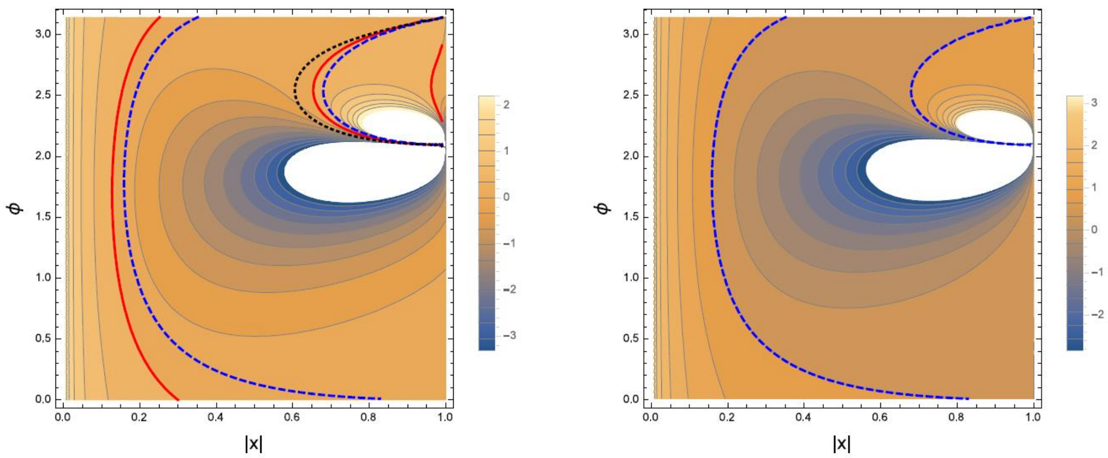
$$\sim \exp\left[N^2(a_1(x)-1)\left(\max|f_\pm|\right)^2\right]$$

$$t \equiv \sin^2\frac{\theta_0}{2}$$

$$R = a_1(2t - t^2), A = 2a_1 t$$

$$t \equiv s^2 = \sin^2\frac{\theta_0}{2}$$

$$t^2 - 2t + \frac{1}{a_1} = 0 \to t\left(= \sin^2 \frac{\theta_0}{2}\right) = t_\pm \equiv 1 \mp \sqrt{1 - \frac{1}{a_1}}.$$



(a) $\frac{1}{N^2}\mathrm{Re}(\log Z_+)$ for the $p = 1$ model

(b) $\frac{1}{N^2}\mathrm{Re}(\log Z_{\mathrm{BH}})$

$$\rho(\theta) = \frac{\cos\frac{\theta}{2}}{\pi t}\sqrt{t - \sin^2\frac{\theta}{2}}.$$

$$s(\theta) = \int d\theta \rho(\theta)$$

$$\log Z_\pm = \frac{N^2 \mu_\pm}{2} = \frac{N^2}{2}\left[-1 + \log t_\pm + \frac{1}{t_\pm}\right]$$

$$a_1 = 1 - \frac{(1 - x^2)^3}{(1 - x^3)^2}$$

$$\log Z_{\mathrm{BH}} = \frac{N^2}{2}\frac{\Delta^3}{\omega^2}, x = e^{-\frac{\omega}{3}+\frac{2\pi i}{3}} = -e^{-\frac{\Delta}{2}}.$$

$$x = |x| e^{i\phi}$$

$$\log Z_+(x) - q\log x$$

$$\rho_1\left(\frac{2\pi}{3} < \phi < \pi\right)$$

$$|x| = 1, \phi = \pi \text{ at } q = 0$$

$$|x| = 1, \phi = \tfrac{2\pi}{3} \text{ at } q = \infty$$

$$(t - 1)^2 \approx -2\beta^3, \log Z \approx -\frac{N^2}{2}\beta^3$$

$$R = \begin{pmatrix} a_1(2t - t^2) & 4a_2 t(1 - t)^2 \\ 2a_1 t(1 - t)^2 & a_2 t(4 - 14t + 20t^2 - 9t^3) \end{pmatrix}, \vec{A} = \left(2a_1 t, 2a_2(2t - 3t^2)\right)$$

$$1-2(a_1+2a_2)t+(a_1+14a_2)t^2-20a_2t^3+3a_2(3+2a_1)t^4-6a_1a_2t^5+a_1a_2t^6=0$$

$$\rho_1=\frac{1-4a_2t+14a_2t^2-20a_2t^3+9a_2t^4}{2a_1t(1-4a_2t^3+3a_2t^4)},\rho_2=\frac{(1-t)^2}{1-4a_2t^3+3a_2t^4}$$

$$Q_1=2a_1\rho_1+2a_2\rho_2(1-2t^2),Q_2=2a_2\rho_2$$

$$\log\ Z=\frac{N^2}{2}\left[(tQ_1+(t-t^2)Q_2)\log\ t-\left(tQ_1+\left(t+\frac{t^2}{2}\right)Q_2\right)+2a_1\rho_1+a_2\rho_2\right]$$

$$-(a_1+2a_2)+(a_1+14a_2)t-30a_2t^2+6(3+2a_1)t^3-15a_1a_2t^4+3a_1a_2t^5=0.$$

$$t\approx .2727+.1198i:\ x\approx .8503\exp[.3793\pi i]\equiv x_1$$
$$t\approx .0016+.2655i:\ x\approx .8003\exp[.7256\pi i]\equiv x_2.$$

$$0.6<|x|<1\text{ and }\frac{2\pi}{3}<\phi<\pi\ )$$

$$\mathrm{Re}(\log\ Z_a)=|x_a|(\phi)$$

$$\min(|x_1|(\phi),|x_2|(\phi))$$

$$S_a(q;x)=\log\ Z_a(x)-q\log\ x\text{ at }q\equiv 3(2R+J_1+J_2)>0$$

$$|x|=1,\phi=\pi$$

$$(t-1)^2\approx -2\beta^3\ ,\log\ Z_1\approx -\frac{N^2}{2}\beta^3.$$

$$|x|=1,\phi=\frac{2\pi}{3}$$

$$|x|=1,\phi=\frac{\pi}{2}$$

$$R=\begin{pmatrix} a_1(2t-t^2) & 4a_2t(1-t)^2 & 3a_3t(1-t)^2(2-5t) \\ 2a_1t(1-t)^2 & a_2t(4-14t+20t^2-9t^3) & 6a_3t(1-t)^2(1-4t+6t^2) \\ a_1t(1-t)^2(2-5t) & 4a_2t(1-t)^2(1-4t+6t^2) & a_3t(6-51t+200t^2-366t^3+312t^4-100t^5) \end{pmatrix}$$
$$\vec{A}=\left(2a_1t,2a_2(2t-3t^2),2a_3t(3-12t+10t^2)\right)$$

$$\begin{gathered}0=1-2(a_1+2a_2+3a_3)t+(a_1+14a_2+51a_3)t^2-20(a_2+10a_3)t^3\\ +(9a_2+6a_1a_2+366a_3+64a_1a_3+50a_2a_3)t^4-(6a_1a_2+312a_3+224a_1a_3+250a_2a_3)t^5\\ +(a_1a_2+100a_3+288a_1a_3+535a_2a_3)t^6-a_3(152a_1+640a_2)t^7+a_3(25a_1+470a_2)t^8\\ -20a_2a_3(10+a_1)t^9+a_2a_3(36+30a_1)t^{10}-12a_1a_2a_3t^{11}+a_1a_2a_3t^{12}.\end{gathered}$$

$$\begin{aligned}\log\ Z=&\frac{N^2}{2}\Big[\{tQ_1+(t-t^2)(Q_2+(1-2t)Q_3)\}\log\ t-t\left(Q_1+\left(1+\frac{t}{2}\right)Q_2+\left(1+\frac{3}{2}t-\frac{5}{3}t^2\right)Q_3\right)\\ &+2a_1\rho_1+a_2\rho_2+\frac{2}{3}a_3\rho_3\Big]\end{aligned}$$

$$(t-1)^2\approx -\beta^3,\log\ Z\approx -\frac{9N^2}{20}\beta^3$$

$$(t-1)^4 \propto \beta^3$$

$$|x| \to 1^{-} p^2 + p$$

$$\det(R-1) = e^{\frac{2\pi i}{3}} e^{-\beta}$$

$$t = \sin^2 \frac{\theta_0}{2}$$

$$\det(R-\mathbf{1}) \sim t^{(2l+m)^2}(1-t)^{l^2+l} \times \ [(2l+1)(2l+m)]\,.$$

$$(2l+m)^2 t^{(2l+m)^2} = 0$$

$$t \sim \beta^2$$
$$t^3 \sim \beta^2$$
$$\therefore$$
$$2(2l+m) - t^{2(2l+m)-3} \sim \beta^2$$
$$2(2l+m) - t^{2(2l+m)-1} \sim \beta^2.$$

$$a_n(x) = 1 - \frac{(1-x^{2n})^3}{(1-x^{3n})^2} = 1 - \frac{\left(1-e^{\frac{4\pi i n}{3}}e^{-2n\beta}\right)^3}{(1-e^{-3n\beta})^2} \sim \begin{cases} \dfrac{\left(1-e^{\frac{4\pi i n}{3}}\right)^3}{9n^2\beta^2} & \text{if } n \notin 3\mathbb{Z}. \\ 1 & \text{if } n \in 3\mathbb{Z} \end{cases}$$

$$V'(\theta) = -\cot\frac{\theta}{2} + 2\sum_{n=1}^{p} a_n(x)\sin(n\theta)$$

$$0 = \sum_{b(\neq a)} V'(\alpha_{ab}) = \sum_{b(\neq a)} \left[-\cos\frac{\alpha_{ab}}{2} + 2\sum_{n=1}^{p} a_n(x)\sin(n\alpha_{ab})\right] \sim \sum_{b(\neq a)} \left[-\frac{2}{\alpha_{ab}} + \frac{\#_p \alpha_{ab}}{\beta^2}\right]$$

$$|x| = 1, \phi = \frac{2\pi}{3} \phi > \frac{2\pi}{3}$$

$$|x| \to 1, \phi \to \frac{2\pi}{3} \phi < \frac{2\pi}{3}$$

$$\det(R-\mathbf{1}) \sim (t-1)^{p^2+p}$$

$$\left.\begin{array}{l}(t-1)^2 \sim \beta^3 \\ (t-1)^4 \sim \beta^3 \\ \vdots \\ (t-1)^{2\lceil \frac{p}{2} \rceil} \sim \beta^3\end{array}\right\}$$

$$\left.\begin{array}{l}(t-1)^{2\lceil \frac{p}{2} \rceil+2} \sim \beta^1 \\ (t-1)^{2\lceil \frac{p}{2} \rceil+4} \sim \beta^1 \\ \vdots \\ (t-1)^{2p} \sim \beta^1\end{array}\right\}$$

$$(t-1)^2 \sim \beta^3, \phi < \pi, \phi > \pi, \phi \to 2\pi - \phi$$

$$(t-1)^{2p} \sim \beta^1 (t-1)^2 \sim \beta^3 p = 3, g_2 (t-1)^2 \sim \beta^3$$

$$P_l(1-2t) = (-1)^l \sum_{n=0}^{l} \frac{(l+n)!}{(n!)^2 (l-n)!} u^n, B^{l+\frac{1}{2}}(t) = \delta_{l,0} + (-1)^l \sum_{n=0}^{l} \frac{(l+n)!}{(n+1)!\, n!\, (l-n)!} u^{n+1}.$$

$$r_{ml}(u) \equiv \sum_{k=1}^{l} \left[B^{m+k-\frac{1}{2}} + B^{\left|m-k+\frac{1}{2}\right|} \right] P_{l-k}$$

$$= \delta_{m,l} - (-1)^{l+m} m l^2 \left[u^2 + \frac{m^2+l^2-2}{3} u^3 + \frac{11 - 8(m^2+l^2) + m^4 + l^4 + 3m^2 l^2}{24} u^4 + \mathcal{O}(u^5) \right]$$

$$0 = R_{ml}\rho_l - \rho_m = [r_{ml}(u) - a_m^{-1}\delta_{ml}](a_l \rho_l)$$

$$= m \left[\frac{1-a_m^{-1}}{m^3} \delta_{m,l} - (-1)^{l+m} \left(u^2 + \frac{m^2+l^2-2}{3} u^3 + \frac{11 - 8(m^2+l^2) + m^4 + l^4 + 3m^2 l^2}{24} u^4 + \mathcal{O}(u^5) \right) \right] l^2 a_l$$

$$a_m^{-1} - 1 \approx 2m^3 \beta^3$$

$$a_m^{-1} - 1 \approx \frac{8m\beta}{9}$$

$$(-1)^l l^2 a_l \rho_l \equiv (v_{\text{odd}}, v_{\text{even}})$$

$$0 \approx 2\beta^3 (v_{\text{odd}})_m + u^2 n_m (n \cdot v) + u^3 (M_1 \cdot v)_m + u^4 (M_2 \cdot v)_m + \mathcal{O}(u^5)$$
$$0 \approx \frac{8\beta}{9m^2} (v_{\text{even}})_m + u^2 n_m (n \cdot v) + u^3 (M_1 \cdot v)_m + u^4 (M_2 \cdot v)_m + \mathcal{O}(u^5)$$

$$n_m = 1, (M_1)_{ml} = \frac{m^2+l^2-2}{3}, (M_2)_{ml} = \frac{11 - 8(m^2+l^2) + m^4 + l^4 + 3m^2 l^2}{24}.$$

$$V(\alpha) = -\log\left[4\sin^2\frac{\alpha}{2}\right] - 2\sum_{n=1}^{p}\frac{a_n(x)}{n}\cos(n\alpha)$$

$$a_n(-e^{-\beta}) = \begin{cases} 1 - \dfrac{2\sinh^3(n\beta)}{\sinh^2\frac{3n\beta}{2}} & \text{for even } n \\ 1 - \dfrac{2\sinh^3(n\beta)}{\cosh^2\frac{3n\beta}{2}} & \text{for odd } n \end{cases}$$

$$\beta = \pm\frac{2\pi i}{3n}$$

$$\beta = \pm\frac{\pi i}{3\left\lceil\frac{p}{2}\right\rceil}$$

$$f(p,\beta) = \sum_{a,b} f_{a,b}\left(\frac{1}{p}\right)^a (p\beta)^b$$

$$0 \approx 2\beta^3(v_{\text{odd}})_m + u^2 n_m(n\cdot v_{\text{odd}}), 0 \approx \frac{8\beta}{9m^2}(v_{\text{even}})_m + u^2 n_m(n\cdot v_{\text{odd}})$$

$$v_{\text{odd}} = v_0\beta^0 + \cdots, v_{\text{even}} = w_0\beta^2 + \cdots$$

$$u \approx u_0\beta^{\frac{3}{2}}, w_0$$

$$2v_0 + u_0^2 n(n\cdot v_0) = 0$$

$$u_0^2 = -\frac{2}{\left\lceil\frac{p}{2}\right\rceil}$$

$$v_m = (-1)^m m^2 a_m\rho_m \approx (-1)^m m^2\rho_m$$

$$2\sum_{l=1}^{\left\lceil\frac{p}{2}\right\rceil}\rho_{2l-1} \approx 1$$

$$\rho_{2n-1} = \frac{\frac{1}{2(2n-1)^2}}{\sum_{l=1}^{\left\lceil\frac{p}{2}\right\rceil}\frac{1}{(2l-1)^2}}$$

$$\rho_{2n-1} = \frac{\frac{1}{2(2n-1)^2}}{\sum_{l=1}^{\infty}\frac{1}{(2l-1)^2}} = \frac{4}{\pi^2}\frac{1}{(2n-1)^2}$$

$$t = \sin^2\frac{\theta_0}{2} \to 1$$

$$\rho(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \rho_n \cos(n\theta) = \frac{1}{2\pi}\left[1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}\right] = \frac{1}{\pi^2}(\pi - |\theta|)$$

$$\rho(\theta)_{\text{Bethe}} \overset{\beta\to 0}{\to} \begin{cases} \frac{1}{\pi} & \text{for } |\theta| < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < |\theta| < \pi \end{cases}$$

$$Z = Z_\infty \text{ at } p \to \infty$$

$$\log Z = -\frac{N^2}{2}\int_{-\theta_0}^{\theta_0} d\theta_1 d\theta_2 V(\theta_1 - \theta_2)\rho(\theta_1)\rho(\theta_2)$$

$$= N^2 \sum_{n=1}^{\infty} \frac{1}{n}\int_{-\theta_0}^{\theta_0} d\theta_1 d\theta_2 [a_n - 1] e^{in(\theta_1 - \theta_2)}\rho(\theta_1)\rho(\theta_2) = N^2 \sum_{n=1}^{\infty} \frac{a_n - 1}{n}(\rho_n)^2$$

$$\rho_{\text{odd}} \sim \mathcal{O}(\beta^0) \text{ and } \rho_{\text{even}} \sim \mathcal{O}(\beta^2)$$

$$a_n(\beta) - 1 \approx \begin{cases} -2n^3\beta^3 & \text{for odd } n \\ -\frac{8n\beta}{9} & \text{for even } n \end{cases}$$

$$\log Z \approx -2N^2\beta^3 \sum_{n=1}^{\infty} (2n-1)^2 \cdot \left(\frac{4}{\pi^2}\frac{1}{(2n-1)^2}\right)^2 = -\frac{4N^2\beta^3}{\pi^2}$$

$$\log Z \sim \frac{N^2}{2}\frac{(2\beta)^3}{(-\pi i + 3\beta)^2}$$

$$\log Z \approx -\frac{4N^2\beta^3}{\pi^2}$$

$$q = 6(R + J_+) \ll N^2$$

$$-\frac{4N^2}{\pi^2}\beta^3 + \beta q \overset{\text{extremize}}{\to} S(q) = \pi\sqrt{\frac{q^3}{27N^2}} = \pi\sqrt{\frac{8(R + J_+)^3}{N^2}},$$

$$S = \pi\sqrt{\frac{8(R_1 + J_+)(R_2 + J_+)(R_3 + J_+)}{N^2}}$$

$$\frac{J_+}{N^2} \sim \left(\frac{R}{N^2}\right)^2 \ll \frac{R}{N^2}$$

$$S \approx \pi\sqrt{\frac{8R_1R_2R_3}{N^2}}.$$

$$S = 2\pi\sqrt{pQ_1Q_2}(R_1R_2R_3)^{\frac{3}{2}}q^{\frac{3}{2}}$$

$$u \approx u_0 \beta^{\frac{3}{4}}$$

$$v_{\text{odd}} \sim \mathcal{O}(\beta^0) \text{ and } v_{\text{even}} \sim \mathcal{O}(\beta^2)$$

$$v_{\text{odd}} = v_0 + v_1 \beta^{\frac{3}{4}} + v_2 \beta^{\frac{3}{2}} + \cdots, u = \beta^{\frac{3}{4}} \left(u_0 + u_1 \beta^{\frac{3}{4}} + u_2 \beta^{\frac{3}{2}} + \cdots \right)$$

$$2\beta^3 v_0 + u_0^2 \beta^{\frac{3}{2}} n \left(n \cdot \left(v_0 + \beta^{\frac{3}{4}} v_1 + \beta^{\frac{3}{2}} v_2 \right) \right) + 2u_0 u_1 \beta^{\frac{9}{4}} n \left(n \cdot \left(v_0 + \beta^{\frac{3}{4}} v_1 \right) \right) + (2u_0 u_2 + u_1^2) \beta^3 n (n \cdot v_0)$$

$$+ u_0^3 \beta^{\frac{9}{4}} M_1 \cdot \left(v_0 + \beta^{\frac{3}{4}} v_1 \right) + 3u_0^2 u_1 \beta^3 M_1 \cdot v_0 + u_0^4 M_2 \cdot v_0$$

$$n \cdot v_0 = \mathcal{O}\left(\beta^{\frac{9}{4}} \right)$$

$$n(n \cdot v_1) + u_0 M_1 \cdot v_0 = 0$$

$$v_1 = v_{1\parallel} + v_{1\perp}$$

$$v_{1\parallel} = c_1 n$$

$$0 = D c_1 n + \frac{u_0}{3} n \sum_{l \in \text{odd}} l^2 \cdot (v_0)_l \rightarrow c_1 = -\frac{u_0}{3D} \sum_{l \in \text{odd}} l^2 (v_0)_l$$

$$D \equiv \left\lceil \frac{p}{2} \right\rceil$$

$$\begin{aligned} 0 =& 2(v_0)_m + u_0^2 n_m (n \cdot v_2) + 2u_0 u_1 n_m (n \cdot v_1) + u_0^3 (M_1 \cdot v_1)_m + 3u_0^2 u_1 M_1 \cdot v_0 + u_0^4 (M_2 \cdot v_0)_m \\ =& \left[2(v_0)_m + \frac{D u_0^3 c_1}{3} m^2 + \frac{u_0^4}{8} m^2 \sum_{l \in \text{odd}} l^2 (v_0)_l \right] \\ &+ n_m \left[u_0^2 (n \cdot v_2) - u_0 u_1 (n \cdot v_1) + \frac{u_0^3}{3} \sum_{l \in \text{odd}} \left(l^2 (v_{1\perp})_l + c_1 (l^2 - 2) + \frac{u_0}{8} (l^4 - 8l^2)(v_0)_l \right) \right]. \end{aligned}$$

$$0 = 2(v_0)_m + \frac{u_0^4}{72} \sum_{l \in \text{odd}} q_m l^2 (v_0)_l$$

$$q_m \equiv m^2 - \frac{1}{D} \sum_{l \in \text{odd}} l^2 = m^2 - \frac{4D^2 - 1}{3}$$

$$\mathcal{M}_{ml} (v_0)_l \equiv \sum_{l \in \text{odd}} \left(m^2 - \frac{4D^2 - 1}{3} \right) l^2 (v_0)_l = -\frac{144}{u_0^4} (v_0)_m$$

$$(v_0)_m \propto q_m = m^2 - \frac{4D^2 - 1}{3},$$

$$-\frac{144}{u_0^4} = \sum_{l \in \text{odd}} l^2 \left(l^2 - \frac{4D^2 - 1}{3} \right) = \frac{16}{45} D (D^2 - 1)(4D^2 - 1).$$

$$(v_{\text{odd}})_m \approx (v_0)_m = (-1)^m m^2 a_m \rho_m \approx -m^2 \rho_m$$

$$\rho_m \approx -\frac{v_m}{m^2} \propto 1 - \frac{4D^2-1}{3m^2}.$$

$$\rho_{2m-1} = \frac{1 - \frac{4\left[\frac{p}{2}\right]^2 - 1}{3(2m-1)^2}}{2\sum_{l=1}^{\left[\frac{p}{2}\right]}\left(1 - \frac{4\left[\frac{p}{2}\right]^2 - 1}{3(2l-1)^2}\right)}$$

$$\rho_{2m-1} = \frac{1 - \frac{4D^2-1}{3(2m-1)^2}}{2D - \frac{\pi^2(4D^2-1)}{12}} \overset{D\to\infty}{\to} \frac{4}{\pi^2(2m-1)^2}$$

$$\sum_{l\in \text{odd}} l^{2k}(v_0)_l = 0 \text{ for } k = 0,1,\cdots,n-2$$

$$\mathcal{M}\cdot v_0 = u_0^{-2n} v_0$$

$$(\mathcal{M})_{ml} \propto \left(m^{2(n-1)} + a_{n-2}m^{2(n-2)} + \cdots a_1 m^2 + a_0\right) l^{2(n-1)}$$

$$\sum_{m\in \text{odd}} m^{2k}\left(m^{2(n-1)} + a_{n-2}m^{2(n-2)} + \cdots a_1 m^2 + a_0\right) = 0 \text{ for } k = 0,1,\cdots,n-2.$$

$$(v_0)_m \propto m^{2(n-1)} + a_{n-2}m^{2(n-2)} + \cdots a_1 m^2 + a_0.$$

$$a_0 \propto D^{2(n-1)}, a_1 \propto D^{2(n-2)}, \cdots, a_{n-2} \propto D^2$$

$$\rho_{2m-1} \overset{D\to\infty}{\to} \frac{4}{\pi^2(2m-1)^2}$$

$$\log Z \approx -\frac{4N^2\beta^3}{\pi^2}$$

$$Z(\beta,\Omega_i) = \text{Tr}\left[e^{-\beta(H-\sum_i \Omega_i J_i)}\right],$$

$$Z(\beta,\gamma) = \text{Tr}\left[(-1)^F\left(-e^{-3\beta}\right)^{(2R+J_1+J_2)} e^{-\gamma(J_1-J_2)}\right]$$

$$x = -e^{-\beta}, y = e^{-\gamma}$$

$$\log Z = \frac{N^2}{2}\frac{\Delta^3}{\omega_1\omega_2} = -\frac{N^2}{2}\frac{8\beta^3}{(\pi+3i\beta)^2+\gamma^2}$$

$$\omega_1 = -\pi i + 3\beta + \gamma, \omega_2 = -\pi i + 3\beta - \gamma$$

$$\Delta = 2\beta$$

$$\log Z \to -\frac{4N^2\beta^3}{\pi^2+\gamma^2}$$

$$q \equiv 3(2R+J_1+J_2), j \equiv J_1-J_2$$

$$S(\beta,\gamma;q,j) = \log Z + \beta q + \gamma j \to q = \frac{12N^2\beta^2}{\pi^2+\gamma^2}, j = -\frac{8N^2\beta^3\gamma}{(\pi^2+\gamma^2)^2}$$

$$\beta = \frac{\pi q^2}{6N^2\sqrt{\frac{q^3}{27N^2}-j^2}}, \gamma = -\frac{\pi j}{\sqrt{\frac{q^3}{27N^2}-j^2}},$$

$$S(q,j) = \pi\sqrt{\frac{q^3}{27N^2}-j^2}$$

$$j^2 \to \frac{q^3}{27N^2}$$

$$\frac{q^4}{N^4} \ll \frac{q^3}{27N^2}-j^2 \ll \frac{q^3}{27N^2}$$

$$\left(\partial_{z_1}\right)^n \hat{O} \otimes e^{n(|\gamma|-3\beta)}$$

$$n > \frac{\mu}{|\gamma|} \gg 1$$

$$F(q,j) \equiv \left(\frac{S}{\pi N^2}\right)^2 = \frac{1}{27}\left(\frac{q}{N^2}\right)^3 - \left(\frac{j}{N^2}\right)^2 \equiv \frac{\hat{q}^3}{27} - \hat{j}^2.$$

$$\hat{q} = \frac{q}{N^2}, \hat{j} = \frac{j}{N^2}$$

$$R_I + R_J \geq 0, R_I + J_i \geq 0, J_1 + J_2 \geq 0$$

$$\Delta R \geq 0, \Delta J_1 + \Delta J_2 \geq 0, \Delta R + \Delta J_i \geq 0$$

$$q' = q - 3(2\Delta R + \Delta J_1 + \Delta J_2), j' = j - (\Delta J_1 - \Delta J_2).$$

$$\Delta F \equiv F(q',j') - F(q,j) \approx 2\hat{j}(\Delta J_1 - \Delta J_2) - \frac{\hat{q}^2}{3}(2\Delta R + \Delta J_1 + \Delta J_2)$$

$$\hat{j} < \frac{\hat{q}^{\frac{3}{2}}}{3\sqrt{3}}$$

$$\hat{j} > \frac{\hat{q}^2}{6}$$

$$\frac{\hat{q}^2}{6} < \hat{j} < \frac{\hat{q}^{\frac{3}{2}}}{3\sqrt{3}}$$

$$\Delta F < -4\hat{j}[\Delta R + \Delta J_2] \leq 0$$

$$\frac{\hat{q}^2}{6} = \hat{j}(1-\epsilon)$$

$$\Delta F = 2\hat{j}[\epsilon \Delta J_1 - (2-2\epsilon)\Delta R - (2-\epsilon)\Delta J_2]$$

$$\Delta J_1 > \frac{2}{\epsilon}(\Delta R + \Delta J_2)$$

$$\frac{\hat{q}^2}{6} < \hat{j} < \frac{\hat{q}^{\frac{3}{2}}}{3\sqrt{3}}$$

$$\hat{j} < \frac{\hat{q}^2}{6}$$

$$\gamma \equiv i\xi e^{in\xi}$$

$$t = \sin^2\frac{\theta_0}{2} t = 1 + u\beta^{\frac{3}{2}} + \cdots$$

$$R_{ml} - \delta_{ml} = (-1)^{m+l+1} m l^2 \left[-\frac{8\left(-e^{i\xi}\right)^m}{(1-(-e^{i\xi})^m)^2}\delta_{ml} + u_0^2\right]\beta^3 + \mathcal{O}\left(\beta^{\frac{9}{2}}\right)$$

$$\det(R-\mathbf{1}) = \#\left(\prod_{n=1}^{p}\frac{-8(-e^{i\xi})^n}{(1-(-e^{i\xi})^n)^2}\right)\left(1-\frac{u_0^2}{8}\sum_{m=1}^{p}\frac{(1-(-e^{i\xi})^m)^2}{(-e^{i\xi})^m}\right)\beta^{3p}$$

$$u_0^2 = -\frac{8}{\sum_{m=1}^{p}\left(2-(-e^{i\xi})^m-(-e^{-i\xi})^m\right)}$$

$$0 = \sum_{l=1}^{p}(-1)^l l^2\left(u_0^2 - 8\frac{\left(-e^{i\xi}\right)^m}{(1-(-e^{i\xi})^m)^2}\delta_{ml}\right)\rho_l$$

$$\frac{(-1)^l}{l^2}\frac{\left(1-\left(-e^{i\xi}\right)^l\right)^2}{(-e^{i\xi})^l}$$

$$\rho_n = -\frac{(-1)^n}{2}\frac{\frac{2-\left(-e^{i\xi}\right)^n-\left(-e^{-i\xi}\right)^n}{n^2}}{\sum_{m=1}^{p}\frac{2-(-e^{i\xi})^m-(-e^{-i\xi})^m}{m^2}} + \mathcal{O}\left(\beta^{\frac{3}{2}}\right)$$

$$\rho_n = -\frac{(-1)^n}{2}\frac{\frac{2-\left(-e^{i\xi}\right)^n-\left(-e^{-i\xi}\right)^n}{n^2}}{\frac{\pi^2}{3} - \mathrm{Li}_2(-e^{i\xi}) - \mathrm{Li}_2(-\mathrm{e}^{-\mathrm{i}\xi})}$$

$$\mathrm{Li}_2(x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$\mathrm{Li}_2\left(-e^{i\xi}\right) + \mathrm{Li}_2\left(-e^{-i\xi}\right) = -\frac{(2\pi i)^2}{2!} B_2\left(\frac{\xi+\pi}{2\pi}\right) = \frac{\xi^2}{2} - \frac{\pi^2}{6}$$

$$B_2(x) \equiv x^2 - x + \frac{1}{6}$$

$$\rho_n = \frac{(-1)^{n-1}}{\pi^2 - \xi^2} \cdot \frac{2 - \left(-e^{i\xi}\right)^n - \left(-e^{-i\xi}\right)^n}{n^2}$$

$$\rho(\theta) \ = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} \rho_n \cos(n\theta)\right)$$

$$= \frac{1}{2\pi}\left[1 + \frac{1}{\pi^2 - \xi^2} \sum_{\pm} \left(\mathrm{Li}_2\left(e^{i(\xi\pm\theta)}\right) + \mathrm{Li}_2\left(e^{-i(\xi\pm\theta)}\right) - 2\mathrm{Li}_2\left(-e^{\pm i\theta}\right)\right)\right]$$

$$\sum_{\pm} \mathrm{Li}_2\left(-e^{\pm i\theta}\right)$$

$$\mathrm{Li}_2\left(e^{ix}\right) + \mathrm{Li}_2\left(e^{-ix}\right) = 2\pi^2 B_2\left(\frac{x}{2\pi} - n\right)$$

$$2\pi n < x < 2\pi(n+1), n \in \mathbb{Z}$$

$$\mathrm{Li}_2\left(e^{ix}\right) + \mathrm{Li}_2\left(e^{-ix}\right) = \begin{cases} 2\pi^2\left[\left(\frac{x}{2\pi}\right)^2 - \frac{x}{2\pi} + \frac{1}{6}\right] & \text{for } 0 < x < 2\pi \\ 2\pi^2\left[\left(\frac{x}{2\pi}\right)^2 + \frac{x}{2\pi} + \frac{1}{6}\right] & \text{for } -2\pi < x < 0 \end{cases}$$

$$|x| = \frac{x^2}{2\pi} + \frac{\pi}{3} - \frac{\mathrm{Li}_2\left(e^{ix}\right) + \mathrm{Li}_2\left(e^{-ix}\right)}{\pi}$$

$$-2\pi < x < 2\pi$$

$$\rho(\theta) = \frac{1}{\pi^2 - \xi^2}\left(\pi - \frac{|\theta+\xi| + |\theta-\xi|}{2}\right)$$

$$\log\ Z = N^2 \sum_{n=1}^{\infty} \frac{a_n - 1}{n} \rho_n^2$$

$$a_n(\beta,\gamma) - 1 = -\frac{\left(1 - e^{-2n\beta}\right)^3}{\left(1 - (-1)^n e^{-n(3\beta+i\xi)}\right)\left(1 - (-1)^n e^{-n(3\beta-i\xi)}\right)} \to -\frac{8n^3\beta^3}{2 - (-e^{i\xi})^n - (-e^{-i\xi})^n}$$

$$\log\ Z = -\frac{4N^2\beta^3}{\pi^2 - \xi^2}$$

$$\log\ Z = -\frac{4N^2\beta^3}{\pi^2 + \gamma^2}$$

$$\rho(\theta) = \frac{1}{\pi^2 - \xi^2}\left[\pi - \frac{\text{sgn}(\text{Re}(\theta+\xi))(\theta+\xi) + \text{sgn}(\text{Re}(\theta-\xi))(\theta-\xi)}{2}\right]$$

$$-\pi < \text{Re}(\theta) < \pi \text{ and } -\pi < \text{Re}(\xi) < \pi$$

$$\text{Re}(\theta) = \pm\text{Re}(\xi)$$

$$\theta = \pm\xi$$

$$s(\theta) = \int d\theta\rho(\theta)$$

$$s(\theta) = \begin{cases} s_1(\theta) \equiv \dfrac{(\theta+\pi)^2}{2(\pi^2-\xi^2)} - \dfrac{1}{2} & \text{for } -\pi < \text{Re}(\theta) < -\text{Re}(\xi) \\ s_2(\theta) \equiv \dfrac{\theta}{\pi+\xi} & \text{for } -\text{Re}(\xi) < \text{Re}(\theta) < \text{Re}(\xi) \\ s_3(\theta) \equiv -\dfrac{(\theta-\pi)^2}{2(\pi^2-\xi^2)} + \dfrac{1}{2} & \text{for } \text{Re}(\xi) < \text{Re}(\theta) < \pi \end{cases}$$

$$s(\pm\pi) = \pm\frac{1}{2}, s(0) = 0$$

$$\log Z = -\frac{4N^2\beta_1\beta_2\beta_3}{\pi^2+\gamma^2}$$

$$S = \pi\sqrt{\frac{8(R_1+J_+)(R_2+J_+)(R_3+J_+)}{N^2} - (J_1-J_2)^2}$$

$$j \approx \frac{q^{\frac{3}{2}}}{3\sqrt{3}N}$$

$$j = \frac{q^2}{6N^2}$$

$$x_I^2 \equiv e^{-\Delta_I}, x_1x_2x_2e^{\gamma} = e^{-\omega_1}$$

$$x_1x_2x_3e^{-\gamma} = e^{-\omega_2}$$

$$a_n = 1 - \frac{\prod_{I=1}^3 \left(1-e^{-n\Delta_I}\right)}{(1-e^{-n\omega_1})(1-e^{-n\omega_2})} = 1 - \frac{\prod_{I=1}^3 1 - x_I^{2n}}{(1-x_1^nx_2^nx_3^ne^{n\gamma})(1-x_1^nx_2^nx_3^ne^{-n\gamma})}$$

$$x_I = -e^{-\beta_I}$$

$$a_n = 1 + \frac{8n^3(-1)^ne^{n\gamma}}{((-1)^{n+1}+e^{n\gamma})^2}\beta_1\beta_2\beta_3 + O(\beta^5)$$

$$(R-1)\rho = 0, A\cdot\rho = 1$$

$$R_{ml} = a_l \sum_{k=1}^{l} \left(B^{m+k-\frac{1}{2}}(t) + B^{\left|m-k+\frac{1}{2}\right|}(t)\right) P_{l-k}(1-2t), A_m = a_m(P_{m-1}(1-2t) - P_m(1-2t)).$$

$$t = 1 + t_1(\beta_1\beta_2\beta_2)^{\frac{1}{2}} + t_2\beta_1\beta_2\beta_3 + \mathcal{O}\left(\beta^{\frac{7}{2}}\right)$$

$$P_l(1-2t) = (-1)^l \left[1 + l(l+1)\left(t_1(\beta_1\beta_2\beta_3)^{\frac{1}{2}} + t_2\beta_1\beta_2\beta_3\right) + \frac{l(l^2-1)(l+2)}{4} t_1^2\beta_1\beta_2\beta_3\right] + \mathcal{O}\left(\beta^{\frac{7}{2}}\right),$$

$$B^{l+\frac{1}{2}}(t) = \delta_{l,0} + (-1)^l \left[t_1(\beta_1\beta_2\beta_3)^{\frac{1}{2}} + t_2\beta_1\beta_2\beta_3 + \frac{l(l+1)}{2} t_1^2\beta_1\beta_2\beta_3\right] + \mathcal{O}\left(\beta^{\frac{7}{2}}\right)$$

$$t = 1 + t_1(\beta_1\beta_2\beta_3)^{\frac{1}{2}} + t_2\beta_1\beta_2\beta_3 + \mathcal{O}\left(\beta^{\frac{7}{2}}\right)$$

$$t = 1 + t_1\beta^{\frac{3}{2}} + t_2\beta^3 + t_3\beta^{\frac{7}{2}} + t_4\beta^4 + t_5\beta^{\frac{9}{2}}$$

$$a_n = 1 + a_n^{(1)}\beta_1\beta_2\beta_3 + \cdots$$

$$R_{ml} \sim \mathcal{O}\left(\beta^{\frac{9}{2}}\right)$$

$$(\beta_1\beta_2\beta_3)^{\frac{1}{2}}\beta_1\beta_2\beta_3$$

$$R_{ml} - \delta_{ml} = (-1)^{m+l+1} m l^2 \left[\frac{-8(-e^\gamma)^m}{(1-(-e^\gamma)^m)^2}\delta_{ml} + t_1^2\right]\beta_1\beta_2\beta_3 + \mathcal{O}\left(\beta^{\frac{9}{2}}\right)$$

$$\det(R-1) =\# \left(\prod_{n=1}^{p} \frac{-8(-e^\gamma)^n}{(1-(-e^\gamma)^n)^2}\right)\left(1 - \frac{t_1^2}{8}\sum_{m=1}^{p} \frac{(1-(-e^\gamma)^m)^2}{(-e^\gamma)^m}\right)(\beta_1\beta_2\beta_3)^p +$$
$$=\# \left(\prod_{n=1}^{p} \frac{-8(-e^\gamma)^n}{(1-(-e^\gamma)^n)^2}\right)\left(1 + \frac{t_1^2}{8}\sum_{m=1}^{p} (2-(-e^\gamma)^m - (-e^\gamma)^{-m})\right)(\beta_1\beta_2\beta_3)^p$$
$$+ \partial\beta \setminus \partial\mathfrak{G}$$

$$t_1^2 = -\left(\frac{\sum_{m=1}^{p} (2-(-e^\gamma)^m - (-e^\gamma)^{-m}}{8}\right)^{-1}$$

$$\rho_n = -\frac{(-1)^n}{2}\frac{\frac{2-(-e^\gamma)^n-(-e^\gamma)^{-n}}{n^2}}{\sum_{l=1}^{p}\frac{2-(-e^\gamma)^l-(-e^\gamma)^l)}{l^2}} + \mathcal{O}\left(\beta^{\frac{3}{2}}\right), n = 1, \cdots, p$$

$$\rho(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} \rho_n \cos n\theta\right) = \frac{1}{2\pi}\left(1 - \frac{\sum_{n=1}^{\infty}\frac{2-(-e^\gamma)^n-(-e^\gamma)^{-n}}{n^2}(-1)^n\cos n\theta}{\sum_{n=1}^{\infty}\frac{2-(-e^\gamma)^n-(-e^\gamma)^n)}{n^2}}\right)$$

$$\frac{\log Z}{N^2} = -\sum_{n=1}^{\infty} \frac{1-a_n}{n} \rho_n^2 = -\frac{2\beta_1\beta_2\beta_3}{\sum_{n=1}^{\infty} \frac{2-(-e^{\gamma})^n-(-e^{\gamma})^{-n}}{n^2}} = -\frac{4\beta_1\beta_2\beta_3}{\pi^2+\gamma^2}$$

$$S(\beta_I,\gamma;q_I,j) \equiv -\frac{4N^2\beta_1\beta_2\beta_3}{\pi^2+\gamma^2} + \sum_{I=1}^{3} \beta_I q_I + \gamma j$$

$$\beta_I = \frac{\pi}{2N^2}\frac{q_1q_2q_3}{q_I}\frac{1}{\sqrt{\frac{q_1q_2q_3}{4N^2}-j^2}}, \gamma = -\frac{\pi j}{\sqrt{\frac{q_1q_2q_3}{N^2}-j^2}}$$

$$S(q_I,j) = \pi\sqrt{\frac{q_1q_2q_3}{N^2}-j^2}$$

$$Z = \int \prod_{a=1}^{N} dz_a \exp\left[-\sum_{a=1}^{N} \log z_a + \sum_{a\neq b} \log\left(1-\frac{z_b}{z_a}\right) + \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{a,b=1}^{N} z_a^n z_b^{-n}\right]$$
$$= \int \prod_{a=1}^{N} dz_a e^{-N^2\left[\frac{1}{N}\sum_{a=1}^{N} \log z_a - \frac{1}{2N^2}\sum_{a\neq b} \log (z_a-z_b)^2 - \frac{1}{N^2}\sum_{n=1}^{\infty} \frac{a_n}{n}\sum_{a,b=1}^{N} z_a^n z_b^{-n}\right]}$$

$$0 = \log z - \int_\gamma dz' \log (z-z')^2 \rho(z') - \sum_{n=1}^{\infty} \frac{a_n}{n}\int_\gamma dz' (z^n z'^{-n} + z^{-n} z'^{n})\rho(z')$$
$$= \log z - \int_\gamma dz' \log (z-z')^2 \rho(z') - \sum_{n=1}^{\infty} \frac{a_n}{n}(z^n\rho_{-n} + z^{-n}\rho_n)$$

$$\log (z-z')^2 = \log (z_+ - z') + \log (z_- - z'), z, z' \in \gamma$$

$$\rho_n \equiv \int_\gamma dz z^n \rho(z), n \in \mathbb{Z}$$

$$W(z) = \log z - \sum_{n=1}^{\infty} \frac{a_n}{n}(z^n\rho_{-n} + z^{-n}\rho_n)$$

$$\rho_n = \int_\gamma dz z^n \rho(z)$$

$$W(z) = \log z - \sum_{n=1}^{p} \frac{\lambda_n}{n}(z^n + z^{-n})$$

$$\lambda_n = a_n\rho_n$$

$$\rho_n = \int_\gamma dz z^n \rho(z) = \int_\gamma dz \frac{z^n+z^{-n}}{2}\rho(z)$$

$$-\frac{\log Z}{N^2} = \int_\gamma dz\rho(z)\log z - \frac{1}{2}\int_{\gamma^{\times 2}} dzdz'\log(z-z')^2\rho(z)\rho(z') - \sum_{n=1}^{p}\frac{a_n}{n}\int_{\gamma^{\times 2}} dzdz'z^n z'^{-n}\rho(z)\rho(z')$$

$$= \int_{\gamma^{\times 2}} dzdz'\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z'}{z}\right)^n\rho(z)\rho(z') - \sum_{n=1}^{p}\frac{a_n}{n}\int_{\gamma^{\times 2}} dzdz'z^n z'^{-n}\rho(z)\rho(z')$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\rho_n\rho_{-n} - \sum_{n=1}^{p}\frac{a_n}{n}\rho_n\rho_{-n}$$

$$\log(z-z')^2 \text{ in } |z| > |z'|$$

$$Z = \prod_{a=1}^{N}\int_\Gamma dz_a \prod_{i<j}(z_a - z_b)^2 \exp\left(-NW(z_a)\right)$$

$$Z = \prod_{a=1}^{N}\int_\Gamma dz_a e^{-N^2 S_N}$$

$$S_N \equiv \frac{1}{N}\sum_a W(z_a) - \frac{1}{2N^2}\sum_{a\neq b}\log(z_a - z_b)^2$$

$$NW'(z_a) + \sum_{b\neq a}\frac{2}{z_b - z_a} = 0, a = 1, \cdots, N$$

$$\rho(s) = \frac{1}{N}\sum_{a=1}^{N}\delta(s - s_a), s \in [0,1]$$

$$\omega(z) = \frac{1}{N}\sum_a \frac{1}{z - z_a}$$

$$\omega(z_+) - \omega(z_-) = -\frac{2\pi i}{N}\sum_{a=1}^{N}\delta(s - s_a) = -2\pi i\rho(s), f(s) = z \in \gamma$$

$$\frac{1}{N}\omega'(z) + \omega(z)^2 - W'(z)\omega(z) = \frac{1}{N}\sum_a \frac{W'(z) - W'(z_a)}{z - z_a}$$

$$\frac{1}{N}\sum_{a=1}^{N}\delta(s - s_a) \longrightarrow \rho(s)ds = \rho(z)dz, \int_\gamma \rho(z)dz$$

$$S[\rho] = \int_\gamma W(z)\rho(z)dz - \frac{1}{2}\int_{\gamma^{\times 2}} dzdz'\log(z-z')^2\rho(z)\rho(z')$$

$$\omega(z) = \int_\gamma P\frac{\rho(z')dz'}{z - z'}$$

$$\begin{matrix}\omega(z_+) = \omega(z) - \pi i\rho(z) \\ \omega(z_-) = \omega(z) + \pi i\rho(z)\end{matrix} \quad z \in \gamma$$

$$\omega(z_+) - \omega(z_-) = -2\pi i \rho(z), z \in \gamma$$

$$\omega(z)^2 - W'(z)\omega(z) = -\int_\gamma \frac{W'(z) - W'(z')}{z - z'} \rho(z') dz' \equiv -P(z)$$

$$\omega(z) = \frac{1}{2}\left(W'(z) \pm \sqrt{W'(z)^2 - 4P(z)}\right).$$

$$\rho(z) = \frac{1}{2\pi}\sqrt{4P(z) - W'(z)^2}, z \in \gamma$$

$$y(z) \equiv W'(z) - 2\omega(z)$$

$$\rho(z) = \pm\frac{y(z_\pm)}{2\pi i}, z \in \gamma$$

$$y(z)^2 = (W'(z))^2 - 4\int_\gamma \frac{W'(z) - W'(z')}{z - z'} \rho(z') dz'$$

$$G(z) \equiv \int_a^z y(z') dz'$$

$$0 = \mathrm{Re} G(z)$$

$$\mathrm{Re} G(z_+ + i dz) = \mathrm{Re} G(z) - \mathrm{Im}(G'(z_+) dz) = -\mathrm{Im}(y(z_+) dz)$$
$$\mathrm{Re} G(z_- - i dz) = \mathrm{Re} G(z) + \mathrm{Im}(G'(z_-) dz) = \mathrm{Im}(y(z_-) dz)$$

$$\rho(z) = \pm\frac{y(z_\pm)}{2\pi i}$$

$$\mathrm{Re} G(z_+ + i dz) = \mathrm{Re} G(z_- - i dz) = -2\pi \mathrm{Re}(\rho(z) dz) < 0$$

$$W(z) = \log z - \sum_{n=1}^{p} \frac{\lambda_n}{n}(z^n + z^{-n})$$

$$\int_\gamma \frac{W'(z) - W'(z')}{z - z'} \rho(z') dz = \int_\gamma \left[-\frac{1}{zz'} + \sum_{n=1}^{p} \lambda_n \left(-\sum_{i+j=n-2} z^i z'^j - z^{-n-1} z'^{-n-1} \sum_{i+j=n} z^i z'^j\right)\right] \rho(z') dz'$$
$$\equiv -\sum_{i=-p-1}^{p-2} \frac{c_i}{4} z^i$$

$$y(z)^2 = (W'(z))^2 + \sum_{i=-p-1}^{p-2} c_i z^i$$

$$\omega(z) = \frac{1}{z} + \sum_{n=1}^{p-1} \frac{1}{z^{n+1}} \int_\gamma z'^n \rho(z') dz' + O(z^{-p-1})$$

$$y(z) = -\frac{1}{z} - \sum_{n=1}^{p} \lambda_n z^{n-1} + \sum_{n=1}^{p-1} \lambda_n z^{-n-1} - 2\sum_{n=1}^{p-1} \rho_n z^{-n-1} + \mathcal{O}(z^{-p-1})$$

$$\rho_n \equiv \int_{\gamma} z^n \rho(z) dz$$

$$c_i = 4\lambda_{i+2} + 4 \sum_{j=1}^{p-i-2} \rho_j \lambda_{i+j+2}, i = -1, \cdots, p-2$$

$$c_{-i-3} = 4 \sum_{j=1}^{p-i-1} \lambda_{i+1+j} \rho_{-j}, i = -1, \cdots, p-2$$

$$y(z)^2 = \left(\frac{1}{z} - \sum_{i=1}^{p} \lambda_i \left(z^{i-1} - z^{-i-1}\right)\right)^2 + 4\sum_{i=1}^{p} \lambda_i z^{i-2} + 4\sum_{i=1}^{p-1} \sum_{j=1}^{p-i} \lambda_{i+j} \left(z^{i-2}\rho_j + z^{-i-2}\rho_{-j}\right) + 4z^{-2} \sum_{j=1}^{p} \lambda_j \rho_{-j}$$

$$y(z)^2 = \lambda_p^2 \frac{\prod_{i=1}^{2p} (z - a_i^-)(z - a_i^+)}{z^{2p+2}}$$

$$y(z)^2 = \lambda_p^2 \frac{(z-a)(z-a^{-1})(z+1)^2 \prod_{i=1}^{p-1} (z-d_i)^2 \left(z - d_i^{-1}\right)^2}{z^{2p+2}}$$

$$(z+1) \prod_{i=1}^{p-1} (z - d_i)\left(z - d_i^{-1}\right) \equiv z^{p-\frac{1}{2}} \left(z^{p-\frac{1}{2}} + z^{-p+\frac{1}{2}} + \sum_{i=1}^{p-1} \frac{Q_i}{2\lambda_p} \left(z^{i-\frac{1}{2}} + z^{-i+\frac{1}{2}}\right) \right)$$

$$A \equiv \frac{a + a^{-1}}{2}$$

$$z^{-\frac{1}{2}} \sqrt{z^2 + 1 - 2Az} \sum_{i=1}^{p} \frac{Q_i}{2} \left(z^{i-\frac{1}{2}} + z^{-i+\frac{1}{2}}\right)$$
$$= \sqrt{\left(1 + \sum_{i=1}^{p} \lambda_i (z^i + z^{-i})\right)^2 + 4\sum_{i=1}^{p-1} \sum_{j=1}^{p-i} \lambda_{i+j} \rho_j (z^i + z^{-i}) + 4\sum_{j=1}^{p} \lambda_j \rho_j}$$

$$(1 + z^2 - 2Az)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(A) z^l$$

$$\sum_{i=1}^{p} \frac{Q_i}{2}\left(z^{i-\frac{1}{2}} + z^{-i+\frac{1}{2}}\right)$$

$$= \sum_{l=0}^{\infty} z^{l+\frac{1}{2}} P_l(A) \sqrt{\left(1 + \sum_{i=1}^{p} \lambda_i (z^i + z^{-i})\right)^2 + 4 \sum_{i=1}^{p-1} \sum_{j=1}^{p-i} \lambda_{i+j}\rho_j (z^i + z^{-i}) + 4 \sum_{j=1}^{p} \lambda_j \rho_j}$$

$$n = -p + \frac{1}{2}, -p + \frac{3}{2}, \cdots, \frac{1}{2}$$

$$z^{-p} \sqrt{\left(z^p \sum_{i=1}^{p} \lambda_i z^{-i}\right)^2 + 2\lambda_p z^p + \mathcal{O}(z^{p+1})} = \sum_{i=1}^{p} \lambda_i z^{-i} + 1 + \mathcal{O}(z)$$

$$\sum_{i=1}^{p} \sum_{l=0}^{i} \lambda_i P_l(A) z^{l-i+\frac{1}{2}} + z^{\frac{1}{2}} + \mathcal{O}(z) = \sum_{i=1}^{p} \sum_{l=0}^{p-i} \lambda_{l+i} P_l(A) z^{-i+\frac{1}{2}} + z^{\frac{1}{2}} + \mathcal{O}(z)$$

$$Q_i = 2 \sum_{l=0}^{p-i} \lambda_{l+i} P_l(A), i = 1, \cdots p$$

$$z^{-i+\frac{1}{2}}, i = 1, \cdots p$$

$$Q_1 = 2 \sum_{i=1}^{p} \lambda_i P_i(A) + 2$$

$$\rho(\theta) = \frac{1}{\pi} \sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}} \sum_{n=1}^{p} Q_n \cos\left(n - \frac{1}{2}\right)\theta, z \quad \begin{array}{l} = e^{i\theta} \\ = e^{i\theta_0} \end{array}$$

$$\lambda_n = a_n \rho_n$$

$$\rho_n = \int_\gamma dz z^n \rho(z)$$

$$\rho_n = \int_\gamma dz z^n \rho(z)$$

$$(R - 1)\rho = 0, A \cdot \rho = 1$$

$$ds^2 = -f^2\left(dt + \omega_\psi d\psi + \omega_\phi d\phi\right)^2 + f^{-1} h_{mn} dx^m dx^n$$
$$h_{mn} dx^m dx^n = r^2\left[\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right] + \mathcal{M}_{ij} d\phi^i d\phi^j, \text{ where } i,j = 1,2, \phi^i = (\phi^1, \phi^2) = (\psi, \phi)$$
$$\mathcal{M} = r^2\begin{pmatrix} \frac{c_\theta^2}{\Xi_b^2}\left(\Xi_b + c_\theta^2(\rho^2 g^2 + 2(1+bg)(a+b)g)\right) & \frac{s_\theta^2 c_\theta^2}{\Xi_a \Xi_b}(\rho^2 g^2 + 2(a+b)g + (a+b)^2 g^2) \\ \frac{s_\theta^2 c_\theta^2}{\Xi_a \Xi_b}(\rho^2 g^2 + 2(a+b)g + (a+b)^2 g^2) & \frac{s_\theta^2}{\Xi_a^2}\left(\Xi_a + s_\theta^2(\rho^2 g^2 + 2(1+ag)(a+b)g)\right) \end{pmatrix}$$
$$\Delta_r = r^2(g^2 r^2 + (1 + ag + bg)^2), \Delta_\theta = 1 - a^2 g^2 \cos^2\theta - b^2 g^2 \sin^2\theta$$
$$\Xi_a = 1 - a^2 g^2, \Xi_b = 1 - b^2 g^2, \rho^2 = r^2 + a^2\cos^2\theta + b^2 \sin^2\theta$$

$$f^{-1} = 1 + \frac{\sqrt{\Xi_a \Xi_b}(1 + g^2\mu) - \Xi_a \cos^2\theta - \Xi_b \sin^2\theta}{g^2 r^2}$$
$$\omega_\psi = -\frac{g\cos^2\theta}{r^2 \Xi_b}\left[\rho^4 + (2r_m^2 + b^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2 b^2 + g^{-2}(a^2 - b^2)\right)\right]$$
$$\omega_\phi = -\frac{g\sin^2\theta}{r^2 \Xi_a}\left[\rho^4 + (2r_m^2 + a^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2 b^2 + g^{-2}(b^2 - a^2)\right)\right]$$

$$r_m^2 = \frac{a+b}{g} + ab, \mu = \frac{1}{3\sqrt{\Xi_a \Xi_b}}\left[2r_m^2 + 3g^{-2}\left(1 - \sqrt{\Xi_a \Xi_b}\right)\right]$$
$$\beta_2 = 3\Xi_a \Xi_b \mu^2 - \frac{6\sqrt{\Xi_a \Xi_b}\left(1 - \sqrt{\Xi_a \Xi_b}\right)}{g^2}\mu + \frac{3\left(1 - \sqrt{\Xi_a \Xi_b}\right)^2}{g^4}$$

$$(c_\theta, s_\theta) \equiv (\cos\theta, \sin\theta)$$

$$U(1)_R \subset U(1)^3 \subset SO(6)$$

$$A = (f - 1)dt + f\left(\omega_\psi d\psi + \omega_\phi d\phi\right) + U_\psi d\psi + U_\phi d\phi$$

$$U_\psi = \frac{g\cos^2\theta}{\Xi_b}\left[\rho^2 + 2r_m^2 + b^2 - \sqrt{\Xi_a \Xi_b}\mu + g^{-2}\left(1 - \sqrt{\Xi_a \Xi_b}\right)\right]$$
$$U_\phi = \frac{g\sin^2\theta}{\Xi_a}\left[\rho^2 + 2r_m^2 + a^2 - \sqrt{\Xi_a \Xi_b}\mu + g^{-2}\left(1 - \sqrt{\Xi_a \Xi_b}\right)\right]$$

$$t = \tilde{t}, \psi = \tilde{\psi} - g\tilde{t}, \phi = \tilde{\phi} - g\tilde{t}$$

$$E \to \tilde{E} = E + g(J_1 + J_2)$$

$$\phi^1 = \psi, \phi^2 = \phi$$

$$\tilde{E} = g(R_1 + R_2 + R_3 + J_1 + J_2)$$

$$R_1 + R_2 + R_3 \equiv U(1)_R$$

$$J_1 \equiv J_\psi, J_2 \equiv J_\phi$$

$$R = \frac{\pi}{4Gg}\left[\mu + \frac{\mu^2 g^2}{2}\right]$$

$$J_1 = \frac{\pi}{4G}\left[\frac{3g\mu^2}{2} + g^3\mu^3 + g^{-3}\left(\sqrt{\frac{\Xi_a}{\Xi_b}} - 1\right)(1+g^2\mu)^3\right]$$

$$J_2 = \frac{\pi}{4G}\left[\frac{3g\mu^2}{2} + g^3\mu^3 + g^{-3}\left(\sqrt{\frac{\Xi_b}{\Xi_a}} - 1\right)(1+g^2\mu)^3\right]$$

$$S = \frac{\pi^2}{2G}\sqrt{(1+3g^2\mu)\mu^3 - \frac{9g^2\mu^4}{4} - \frac{\left(\sqrt{\Xi_a} - \sqrt{\Xi_b}\right)^2}{g^6\sqrt{\Xi_a\Xi_b}}(1+g^2\mu)^3}$$

$$\mathrm{AdS}_5 \times S^5, N^2 = \frac{\pi}{2g^3G}$$

$$S = 2\pi\sqrt{3R^2 - \frac{N^2}{2}(J_1 + J_2)} = 2\pi\sqrt{\frac{R^3 + \frac{N^2}{2}J_1J_2}{\frac{N^2}{2} + 3R}}$$

$$-g^{-1} < a, b < g^{-1}$$

$$\frac{g^{-2}(1+ag+bg)^2}{1-a^2g^2} = \frac{g^{-1}(\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}g) + g^{-2}(1+\tilde{a}g+\tilde{b}g)^2}{1-\tilde{a}^2g^2}$$
$$\frac{g^{-2}(1+ag+bg)^2}{1-b^2g^2} = \frac{g^{-1}(\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}g) + g^{-2}(1+\tilde{a}g+\tilde{b}g)^2}{1-\tilde{b}^2g^2}$$

$$\tilde{a} = \frac{13a^2 + 8ab - 5b^2 + (12a^3 + 20a^2b + 8ab^2)g + (12a^3b + 13a^2b^2)g^2}{12a + 12b + (13a^2 + 20ab + 13b^2)g + (8a^2b + 8ab^2)g^2 - 5a^2b^2g^3}$$
$$\tilde{b} = \frac{-5a^2 + 8ab + 13b^2 + (8a^2b + 20ab^2 + 12b^3)g + (13a^2b^2 + 12ab^3)g^2}{12a + 12b + (13a^2 + 20ab + 13b^2)g + (8a^2b + 8ab^2)g^2 - 5a^2b^2g^3}.$$

$$\tilde{a} + \tilde{b} + \tilde{a}\tilde{b}g > 0$$

$(\hat{a}, \hat{b})$

$\equiv (ag, bg)$

$$\boxtimes \frac{(\hat{a}+\hat{b}+\hat{a}\hat{b})^2\left(32\hat{a}^3(1+\hat{b}) + \hat{b}(32+61\hat{b}+32\hat{b}^2) + \hat{a}^2(61+118\hat{b}+61\hat{b}^2) + 2\hat{a}(16+59\hat{b}+59\hat{b}^2+16\hat{b}^3)\right)}{\left(\hat{b}(12+13\hat{b}) + \hat{a}^2(13+8\hat{b}-5\hat{b}^2) + 4\hat{a}(3+5\hat{b}+2\hat{b}^2)\right)^2}$$

$$\tilde{a}(a) = \begin{cases} \frac{a(4+5a)}{6+4a-a^2} & a = b \\ \frac{a^2}{6-5a^2} & a = -b \end{cases}$$

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}[dr^2 + r^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2)]$$
$$A = (f-1)dt + f\omega, f^{-1} = 1 + \frac{a_-^2 + 8a_+}{12r^2}, \omega = -\frac{a_+a_-}{2r^2}(\cos^2\theta d\psi - \sin^2\theta d\phi)$$

$$ds^2 = -\alpha^2(x)dt^2 + a^2(x)dx^2 + r^2(x)d\Omega^2$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g^2}F_{\mu\nu}^a F_a^{\mu\nu}$$

$$F_{\mu\nu}^a = \nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a + \epsilon_{abc}A_\mu^b A_\nu^c$$

$$A_\mu \equiv T^a A_\mu^a, F_{\mu\nu} \equiv T^a F_{\mu\nu}^a$$

$$\begin{aligned} A_\mu &\to A'_\mu = UA_\mu U^{-1} - i(\nabla_\mu U)U^{-1} \\ F_{\mu\nu} &\to F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \end{aligned}$$

$$U = e^{-i\Lambda}$$

$$\Lambda = \Lambda^a T^a$$

$$\begin{aligned} A_t &= T^3 u \\ A_x &= T^3 v \\ A_\theta &= T^1 w_1 + T^2 w_2 \\ A_\phi &= (-T^1 w_1 + T^2 w_2 + T^3 \cot\theta)\sin\theta \end{aligned}$$

$$\begin{aligned} F_{tx} &= -T^3\partial_x u \\ F_{t\theta} &= -T^1 u w_1 + T^2 u w_2 \\ F_{t\phi} &= -(T^1 u w_2 + T^2 u w_1)\sin\theta \\ F_{x\theta} &= T^1(\partial_x w_2 - v w_1) + T^2(\partial_x w_1 + v w_2) \\ F_{x\phi} &= -[T^1(\partial_x w_1 + v w_2) - T^2(\partial_x w_2 - v w_1)]\sin\theta \\ F_{\theta\phi} &= -T^3(1 - w_1^2 - w_2^2)\sin\theta \end{aligned}$$

$$\begin{aligned} g^2\mathcal{L}_{\text{YM}} = {} & \frac{(\partial_x u)^2}{2\alpha^2 a^2} - \frac{(1 - w_1^2 - w_2^2)^2}{2r^4} + \frac{u^2(w_1^2 + w_2^2)}{\alpha^2 r^2} \\ & - \frac{(\partial_x w_2 - v w_1)^2 + (\partial_x w_1 + v w_2)^2}{a^2 r^2} \end{aligned}$$

$$v \to v' = v - \partial_x\lambda, w \to w' = we^{-i\lambda}$$

$$\mathcal{L}_{\text{YM}} = -\frac{(\partial_x w)^2}{g^2 a^2 r^2} - \frac{(1 - w^2)^2}{2g^2 r^4}$$

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu})}\mathcal{L}_{\text{YM}}$$

$$\det(g_{\mu\nu}) = \alpha^2 a^2 r^4 \sin^2\theta$$

$$0 = \partial_x\left(\frac{\alpha}{a}\partial_x w\right) + \frac{\alpha a}{r^2}w(1 - w^2)$$

$$T_{\mu\nu} = \frac{1}{g^2}g^{\sigma\lambda}F_{\mu\sigma}^a F_{\nu\lambda}^a + g_{\mu\nu}\mathcal{L}_{\text{YM}}$$

$$T_t^t = -\frac{(1-w^2)^2}{2g^2r^4} - \frac{(\partial_x w)^2}{g^2a^2r^2}$$
$$T_x^x = -\frac{(1-w^2)^2}{2g^2r^4} + \frac{(\partial_x w)^2}{g^2a^2r^2}$$
$$T_\theta^\theta = +\frac{(1-w^2)^2}{2g^2r^4}$$

$$T^\phi{}_\phi = T^\theta{}_\theta$$

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2 d\Omega^2$$

$$N(r) = 1 - \frac{2m(r)}{r}$$

$$\partial_r \sigma = 4\pi \frac{r\sigma}{N}(T_r^r - T_t^t)$$
$$\partial_r m = -4\pi r^2 T_t^t$$

$$x \to r, \alpha \to \sigma\sqrt{N}, a \to \frac{1}{\sqrt{N}}.$$

$$\partial_r^2 w = -\left[4\pi r(T_r^r + T_t^t) + \frac{2m}{r^2}\right]\frac{\partial_r w}{N} - \frac{w(1-w^2)}{Nr^2}$$

$$T_t^t = -\frac{(1-w^2)^2}{2g^2r^4} - \frac{N(\partial_r w)^2}{g^2r^2}$$
$$T_r^r = -\frac{(1-w^2)^2}{2g^2r^4} + \frac{N(\partial_r w)^2}{g^2r^2}$$
$$T_\theta^\theta = +\frac{(1-w^2)^2}{2g^2r^4}.$$

$$\bar{r} \equiv gr, \bar{m} \equiv gm$$

$$N(\bar{r}_h) = 0$$

$$\sigma = \sigma_h + \sigma_1(\bar{r} - \bar{r}_h) + \cdots$$
$$w = w_h + w_1(\bar{r} - \bar{r}_h) + \cdots$$
$$N = N_1(\bar{r} - \bar{r}_h) + \cdots$$

$$\sigma_1 = \sigma_h \frac{8\pi w_1^2}{\bar{r}_h}$$
$$w_1 = \frac{\bar{r}_h w_h (w_h^2 - 1)}{\bar{r}_h^2 - 4\pi(w_h^2 - 1)^2}$$
$$N_1 = \frac{1}{\bar{r}_h}\left[1 - \frac{4\pi(w_h^2 - 1)^2}{\bar{r}_h^2}\right]$$

$$\bar{m} = \frac{\bar{r}_h}{2} + \frac{1}{2}(1 - N_1\bar{r}_h)(\bar{r} - \bar{r}_h) + \cdots$$

$$\bar{M} = \bar{m}(\bar{r} \to \infty),$$

$$M_{n=\infty} = \frac{\bar{r}_h}{2} + \frac{2\pi}{\bar{r}_h}.$$

$$\bar{M}_{n=\infty} = \sqrt{4\pi}.$$

$$\bar{\kappa} = \frac{1}{2}\sigma_h N_1,$$

$$G_{\mu\nu} = 8\pi\left(T_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle\right),$$

$$ds^2 = e^{2\nu(x)}(-dt^2 + dx^2) + r^2(x)d\Omega^2$$

$$G_{tt} = \frac{1}{r^2}(2rr'\nu' - r'^2 - 2rr'' + e^{2\nu})$$
$$G_{xx} = \frac{1}{r^2}(2rr'\nu' + r'^2 - e^{2\nu})$$
$$G_{\theta\theta} = re^{-2\nu}(r'' + r\nu'')$$

$$G_{\phi\phi} = G_{\theta\theta}\sin^2\theta$$

$$ds^2 = g_{ab}dx^a dx^b$$

$$\langle \hat{T}_{ab} \rangle^{(\eta\mathrm{D})} = \frac{R^{(\eta\mathrm{D})}}{48\pi} g_{ab} + \frac{1}{48\pi}\left(\mathcal{A}_{ab} - \frac{1}{2} g_{ab}\mathcal{A}_c^c\right)$$

$$\mathcal{A}_{ab} = \frac{4}{|\xi|}\nabla_a\nabla_b|\xi|, |\xi| = \sqrt{-\xi_a\xi^a}$$

$$\langle \hat{T}_{ab} \rangle = F(r)\langle \hat{T}_{ab} \rangle^{(2\mathrm{D})}$$

$$\nabla_\mu \langle \hat{T}^\mu{}_\nu \rangle = 0$$

$$ds^2 = e^{2\nu}(-dt^2 + dx^2)$$

$$\langle \hat{T}_{tt} \rangle = -\frac{F}{24\pi}(\nu'^2 - 2\nu'')$$
$$\langle \hat{T}_{xx} \rangle = -\frac{F}{24\pi}\nu'^2$$
$$\langle \hat{T}_{\theta\theta} \rangle = -\frac{1}{24\pi} r^2 e^{-2\nu}\nu'^2\left(F + \frac{rF'}{2r'}\right)$$

$$\langle \hat{T}_{\phi\phi} \rangle = \langle \hat{T}_{\theta\theta} \rangle \sin^2\theta$$

$$F = \frac{1}{4\pi r^2}$$

$$F + rF'/2r' = \langle \hat{T}_{\theta\theta} \rangle$$

$$\alpha \to e^\nu, a \to e^\nu$$

$$w'' = -\frac{e^{2\nu}}{r^2} w(1-w^2)$$

$$T_t^t = -\frac{(1-w^2)^2}{2g^2r^4} - \frac{w'^2}{g^2r^2e^{2\nu}}$$
$$T_r^r = -\frac{(1-w^2)^2}{2g^2r^4} + \frac{w'^2}{g^2r^2e^{2\nu}}$$
$$T_\theta^\theta = +\frac{(1-w^2)^2}{2g^2r^4}$$

$$r'' - 2r'\nu' = 4\pi r e^{2\nu}(T_t^t - T_r^r) + \frac{Fr}{3}(\nu'^2 - \nu'').$$

$$r'' + r\nu'' = 8\pi r e^{2\nu} T_\theta^\theta - \frac{1}{3} r\nu'^2 \left(F + \frac{rF'}{2r'}\right).$$

$$\nu''\left(1 - \frac{F}{3}\right) = -4\pi e^{2\nu}(T_t^t - T_r^r - 2T_\theta^\theta) - \frac{2r'\nu'}{r} - \nu'^2\left[\frac{F}{3} + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right)\right]$$
$$r''\left(1 - \frac{F}{3}\right) = 4\pi r e^{2\nu}\left(T_t^t - T_r^r - \frac{2F}{3} T_\theta^\theta\right) + 2r'\nu' + \frac{Fr}{3}\nu'^2\left[1 + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right)\right]$$

$$2rr'\nu' + r'^2 - e^{2\nu} = 8\pi r^2 e^{2\nu} T_r^r - \frac{F}{3} r^2 \nu'^2,$$

$$\nu' = -\frac{3r'}{Fr}\left[1 \pm \sqrt{1 + \frac{F}{3r'^2}(8\pi r^2 e^{2\nu} T_r^r - r'^2 + e^{2\nu})}\right].$$

$$\dot{\nu} = \partial_r \nu \text{ and } \dot{w} = \partial_r w. x$$

$$\nu' = r'\dot{\nu}, \nu'' = r''\dot{\nu} + r'^2\ddot{\nu},$$

$$r''\left(1 - \frac{F}{3}\right) = 4\pi r e^{2\nu}\left(T_t^t - T_r^r - \frac{2F}{3} T_\theta^\theta\right) + 2r'^2\dot{\nu} + \frac{Fr}{3} r'^2 \dot{\nu}^2\left[1 + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right)\right]$$

$$0 = \ddot{\nu}\left(1 - \frac{F}{3}\right) + \dot{\nu}^3 r \frac{F}{3}\left(1 + \frac{2F + rF'/r'}{6}\right) + \dot{\nu}^2\left(2 + \frac{F}{3} + \frac{2F + rF'/r'}{6}\right) + \frac{2\dot{\nu}}{r} + 4\pi \frac{e^{2\nu}}{r'^2}\left[(1 + r\dot{\nu})(T_t^t - T_r^r) - 2\left(1 + \frac{F}{3} r\dot{\nu}\right) T_\theta^\theta\right]$$

$$2rr'^2\dot{\nu} + r'^2 - e^{2\nu} = 8\pi r^2 e^{2\nu} T_r^r - \frac{F}{3} r^2 r'^2 \dot{\nu}^2$$

$$T_t^t = -\frac{(1-w^2)^2}{2g^2r^4} - \frac{r'^2\dot{w}^2}{g^2r^2e^{2\nu}}$$
$$T_r^r = -\frac{(1-w^2)^2}{2g^2r^4} + \frac{r'^2\dot{w}^2}{g^2r^2e^{2\nu}}$$

$$r'^2 = e^{2\nu}\left[1 - 4\pi\frac{(1-w^2)^2}{g^2r^2}\right] \times \left(1 + 2r\dot{\nu} + \frac{F}{3}r^2\dot{\nu}^2 - \frac{8\pi\dot{w}^2}{g^2}\right)^{-1}$$

$$\ddot{w} = -\frac{1}{r'^2}\left[r''\dot{w} + \frac{e^{2\nu}}{r^2}w(1-w^2)\right]$$

$$ds^2 = -e^{2\nu(r)}dt^2 + \frac{e^{2\nu(r)}}{b^2(r)}dr^2 + r^2d\Omega^2$$

$$g_{rr} = \frac{e^{2\nu}}{r'^2}$$

$$\nu = \nu_0 + \frac{\nu_1}{r} + \frac{\nu_2}{r^2} + \cdots, w = w_0 + \frac{w_1}{r} + \frac{w_2}{r^2} + \cdots.$$

$$\nu = -\frac{M}{r} - \frac{M^2}{r^2} + O(r^{-3})$$
$$w = \pm\left(1 - \frac{w_1}{r} + \frac{3w_1^2 - 6Mw_1}{4r^2}\right) + O(r^{-3})$$

$$x = \int \frac{dr}{\sigma(r)N(r)}$$

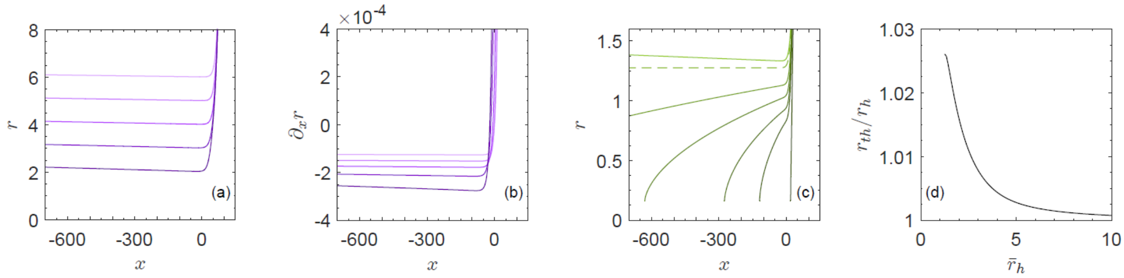
$$\nu = \frac{1}{2}\ln(\sigma^2N)$$

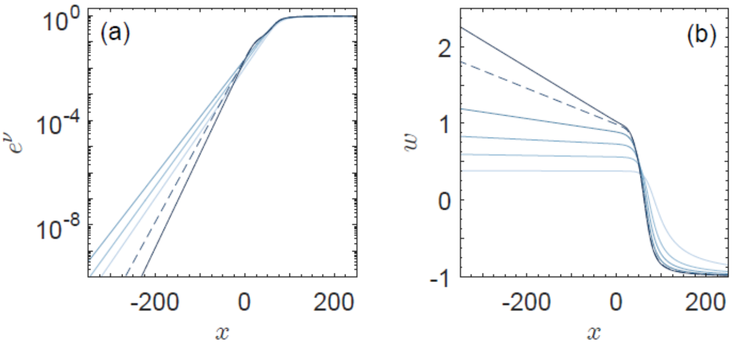
$$\partial_x w = \sigma N \partial_r w$$

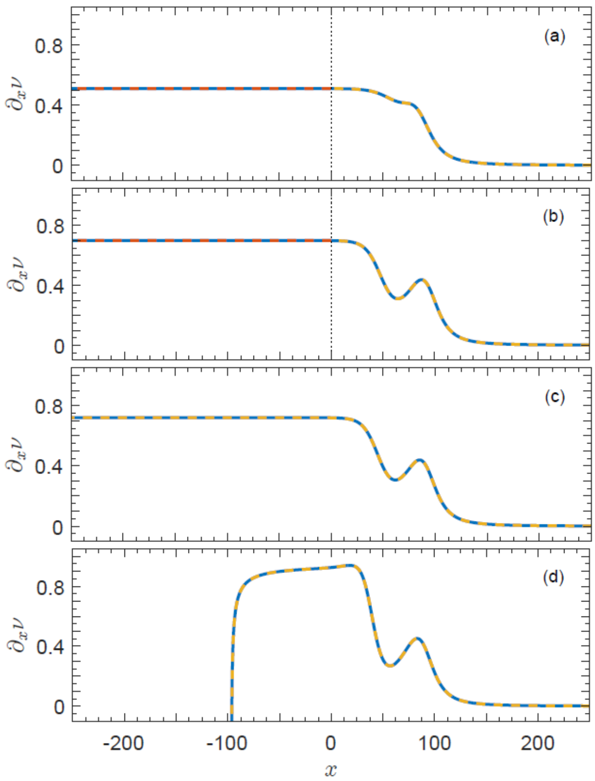
$$\partial_x \nu = \sigma N \partial_r \nu, \partial_x r = \sigma N$$

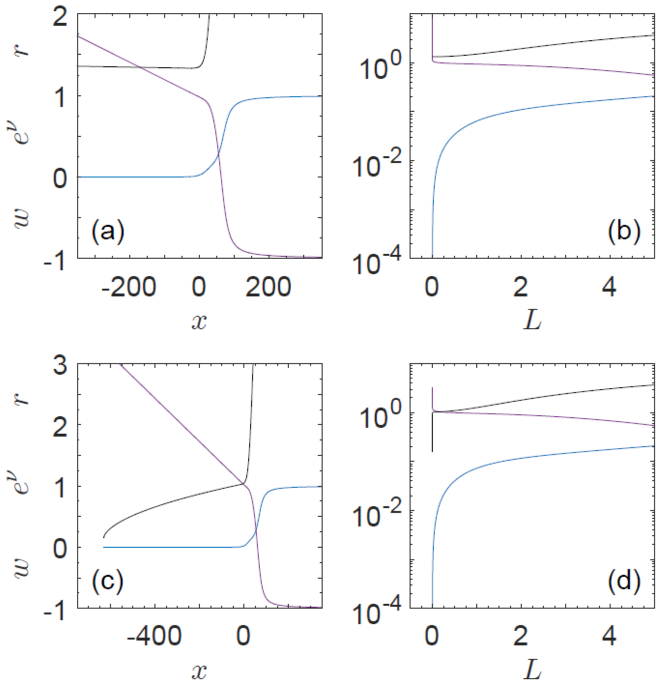
$$\partial_r \nu = 4\pi\frac{r}{N}T_r^r + \frac{m}{r^2N}$$

$$L = \int dx e^{\nu(x)}$$

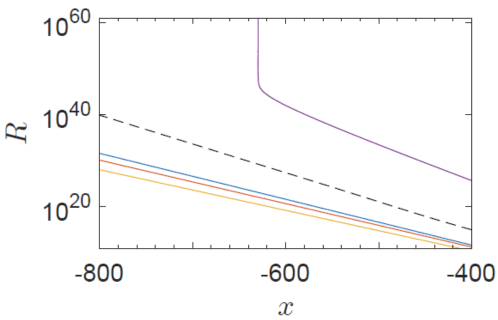



(a)
(b)
e^{ν}
w
x
x

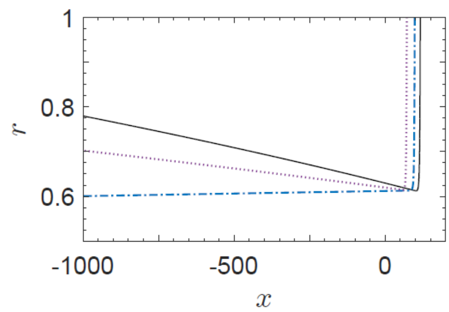

(a)
(b)
(c)
(d)
$\partial_x \nu$
x



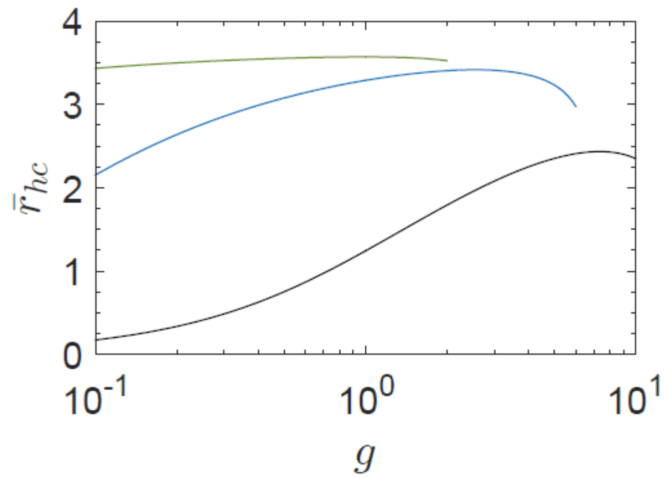
(a)
(b)
(c)
(d)
x
L
w e^{ν} r



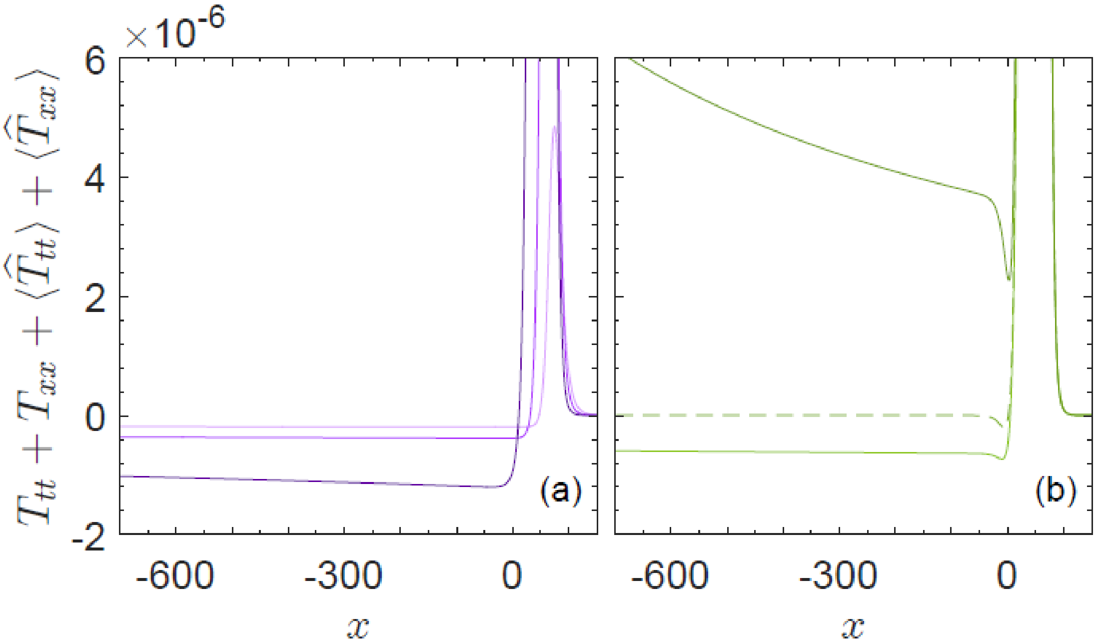
R
x



r
x



$\bar{r}_{hc}$
g



$$T_{tt} + T_{xx} + \langle \hat{T}_{tt} \rangle + \langle \hat{T}_{xx} \rangle < 0$$

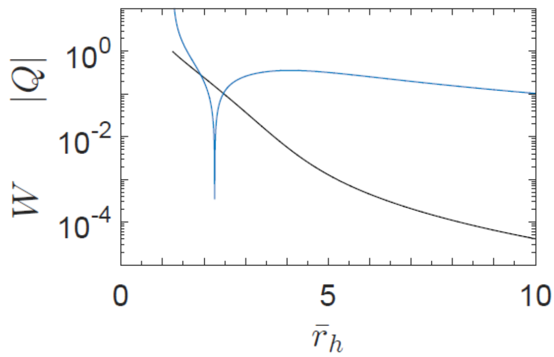
$$\frac{2w'^2}{g^2 r^2} - \frac{F}{12\pi}(\nu'^2 - \nu'') < 0$$

$$\ddot{\nu}\left(1 - \frac{F}{3}\right) \simeq 4\pi r e^{2\nu} \frac{\dot{\nu}}{r'^2}\left[\frac{2w'^2}{g^2 r^2 e^{2\nu}} + \frac{2F}{3}\frac{(1-w^2)^2}{2g^2 r^4}\right]$$
$$-\frac{Fr}{3}\dot{\nu}^3\left[1 + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right)\right],$$

$$\dot{w} = (\partial r/\partial w)^{-1}$$

$$\frac{e^{2\nu}}{r'^2} \simeq \frac{F}{3} r^2 \dot{\nu}^2 \left\{1 + 8\pi\left[\frac{w'^2}{g^2 e^{2\nu}} - \frac{(1-w^2)^2}{2g^2 r^2}\right]\right\}^{-1}$$

$$\frac{d\dot{\nu}}{\dot{\nu}^3} \simeq -dr \frac{Fr/3}{1 - F/3}\left[1 + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right) - W\right]$$



$$W \equiv 4\pi r^2 \left[\frac{2w'^2}{g^2 r^2 e^{2\nu}} + \frac{F}{3}\frac{(1-w^2)^2}{g^2 r^4}\right] \times \left\{1 + 8\pi\left[\frac{w'^2}{g^2 e^{2\nu}} - \frac{(1-w^2)^2}{2g^2 r^2}\right]\right\}^{-1}$$

$$\frac{d\dot{\nu}}{\dot{\nu}^3} \simeq -dr\frac{Fr/3}{1-F/3}\left[1 + \frac{1}{3}\left(F + \frac{rF'}{2r'}\right)\right]$$

$$\nu \simeq \sqrt{\frac{2(12\pi r_{th}^2 - 1)}{r_{th}}}\sqrt{r - r_{th}} + \nu_{th}$$

$$g_{rr} \simeq \frac{F}{3} r^2 \dot{\nu}^2 (1+Q)^{-1}$$
$$Q \equiv 8\pi\left[\frac{w'^2}{g^2 e^{2\nu}} - \frac{(1-w^2)^2}{2g^2 r^2}\right]$$

$$\nu' \simeq 12\pi\sqrt{\frac{2}{3}\frac{w'}{g}},$$

$$\nu \simeq 12\pi\sqrt{\frac{2}{3}\frac{w}{g}} + C$$

$$S = \int \left[\frac{R}{16\pi G} - \frac{1}{2e^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)\right]\sqrt{-g}\,dx^4$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + i\left[A_\mu, A_\nu\right]$$
$$A_\mu = \frac{1}{2}\tau^a A_\mu^a$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa}{e^2}\mathrm{Tr}\left(F_{\mu\alpha}F_\nu^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right),$$
$$\nabla_\mu F^{\mu\nu} + i\left[A_\mu, F^{\mu\nu}\right] = 0.$$

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)\sigma(r)dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$A_\mu dx^\mu = \frac{1-w(r)}{2}\left(\tau_\varphi d\theta - \tau_\theta \sin\theta d\varphi\right)$$

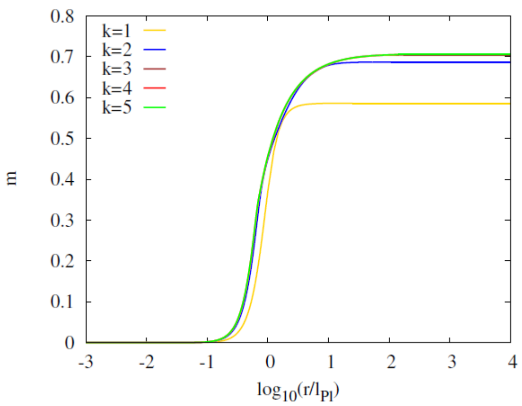
$$m' = \frac{\kappa}{2}\left(1 - \frac{2m}{r}\right)w'^2 + \kappa\frac{(w^2-1)^2}{4r^2}$$
$$\sigma' = \frac{2\kappa\sigma}{r}w'^2$$
$$w'' = \frac{w'}{2r^3\left(1 - \frac{2m}{r}\right)}\left(\kappa(w^2-1)^2 - 4rm\right) + \frac{w(w^2-1)}{r^2\left(1 - \frac{2m}{r}\right)}$$

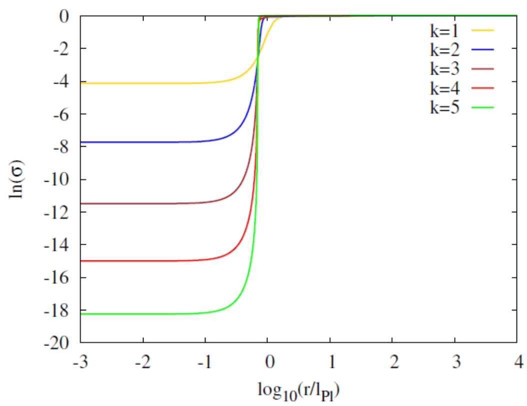
$$
\begin{aligned}
m(r) &= \kappa \bar{W}_2^2 r^3 + \frac{4}{5}\kappa \bar{W}_2^3 r^5 + \mathcal{O}(r^7)\\
\sigma(r) &= \sigma_0 - 2\kappa \bar{W}_2^2 r^2 + \mathcal{O}(r^4)\\
w(r) &= 1 + \bar{W}_2 r^2 + \frac{1}{10}\bar{W}_2^2(3 + 4\kappa \bar{W}_2) r^4 + \mathcal{O}(r^4)
\end{aligned}
$$

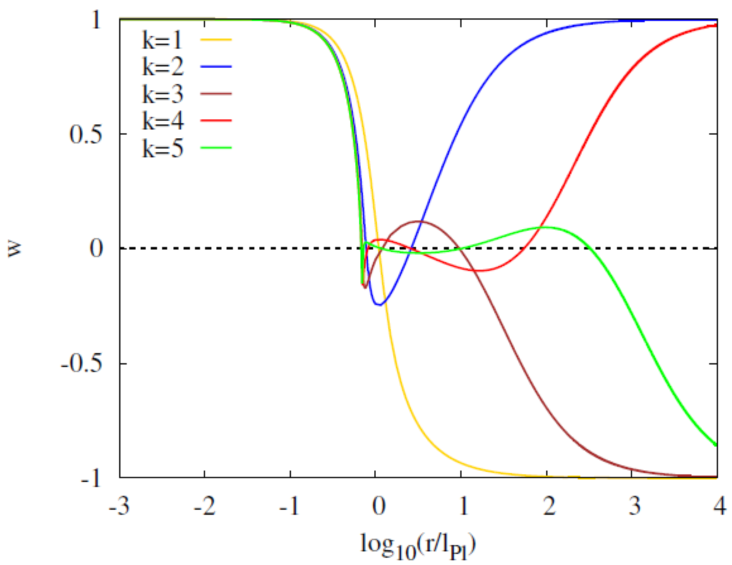
$$
\begin{aligned}
m(r) &= m_h + \frac{\kappa}{4r_h^2}\left(1 - W_h^2\right)^2 (r - r_h) + \mathcal{O}((r - r_h)^2)\\
\sigma(r) &= \sigma_h - \kappa \frac{\widetilde{W}_1^2}{r_h}(r - r_h) + \mathcal{O}((r - r_h)^2)\\
w(r) &= W_h + \widetilde{W}_1 (r - r_h) + \mathcal{O}((r - r_h)^2)
\end{aligned}
$$

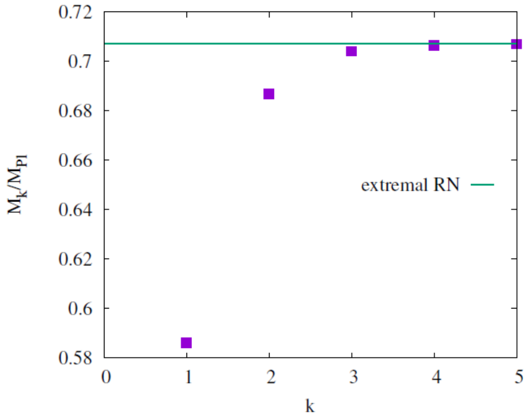
$$
\widetilde{W}_1 = \frac{2r_h W_h\left(1 - W_h^2\right)}{\kappa(1 - W_h^2)^2 - 2r_h^2}
$$

$$
\begin{aligned}
m(r) &= M - \frac{\kappa w_1^2}{2r^3} \mp \frac{\kappa w_1^2(4w_1 \pm 5M)}{4r^4} + \mathcal{O}\left(\frac{1}{r^5}\right),\\
\sigma(r) &= 1 - \frac{\kappa w_1^2}{2r^4} \mp \frac{6\kappa w_1(w_1 \pm 2M)}{5r^5} + \mathcal{O}\left(\frac{1}{r^6}\right),\\
w(r) &= \pm 1 + \frac{w_1}{r} \pm \frac{3w_1(w_1 \pm 2M)}{4r^2} + \mathcal{O}\left(\frac{1}{r^3}\right),
\end{aligned}
$$

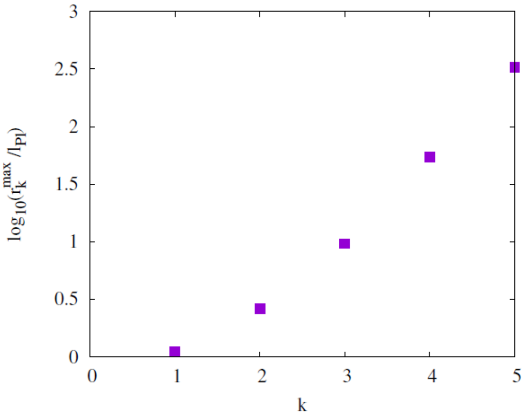




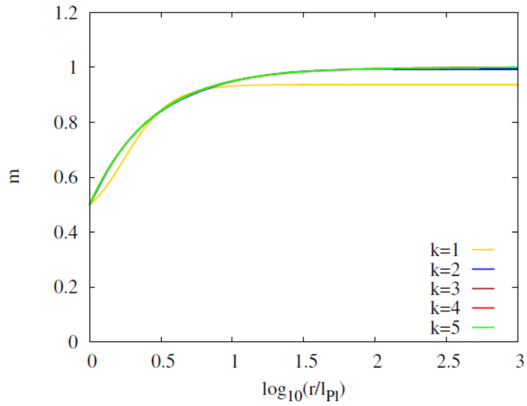




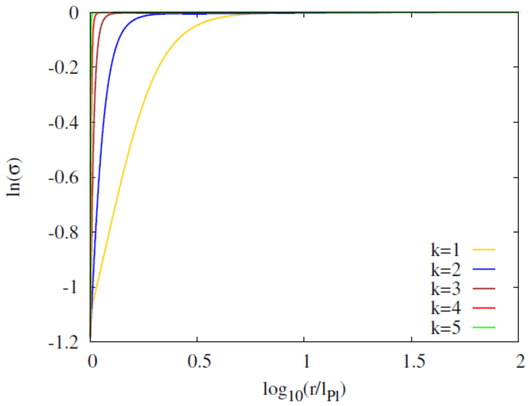
M_k/M_{Pl}
extremal RN
k



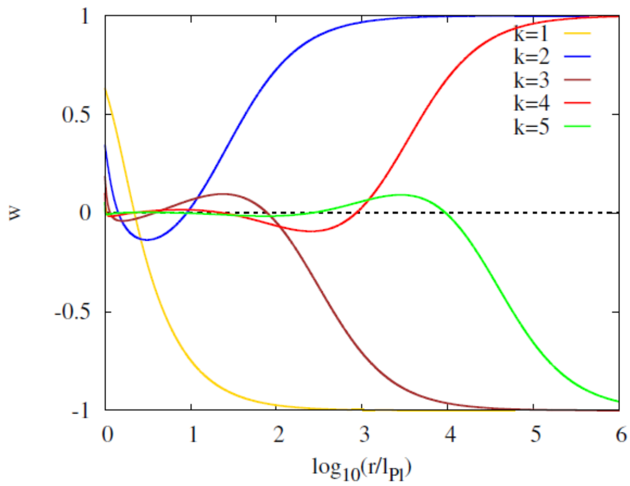
$\log_{10}(r_k^{max}/l_{Pl})$
k



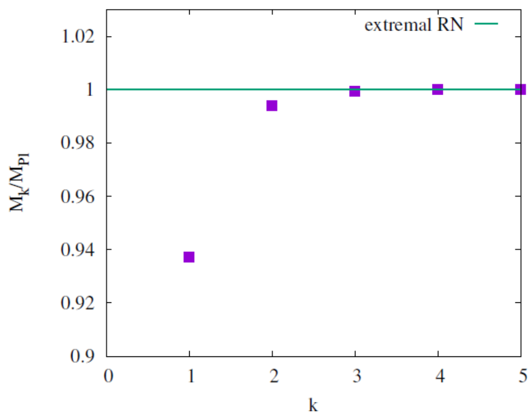
m
k=1
k=2
k=3
k=4
k=5
$\log_{10}(r/l_{Pl})$



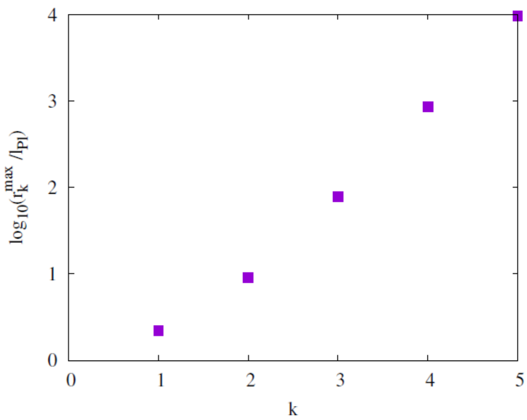
ln(σ)
k=1
k=2
k=3
k=4
k=5
$\log_{10}(r/l_{Pl})$



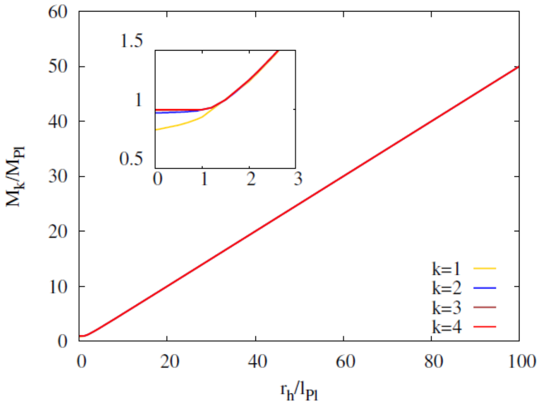
w
k=1
k=2
k=3
k=4
k=5
$\log_{10}(r/l_{Pl})$

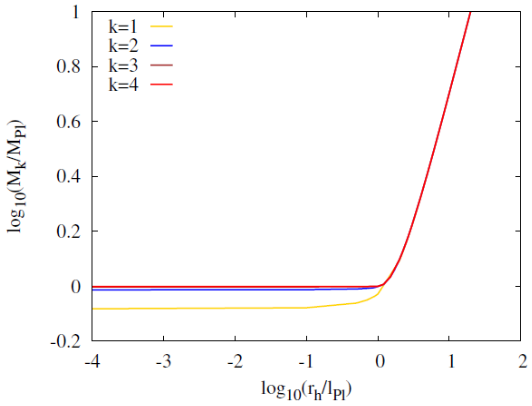


M_k/M_{Pl}
extremal RN
k



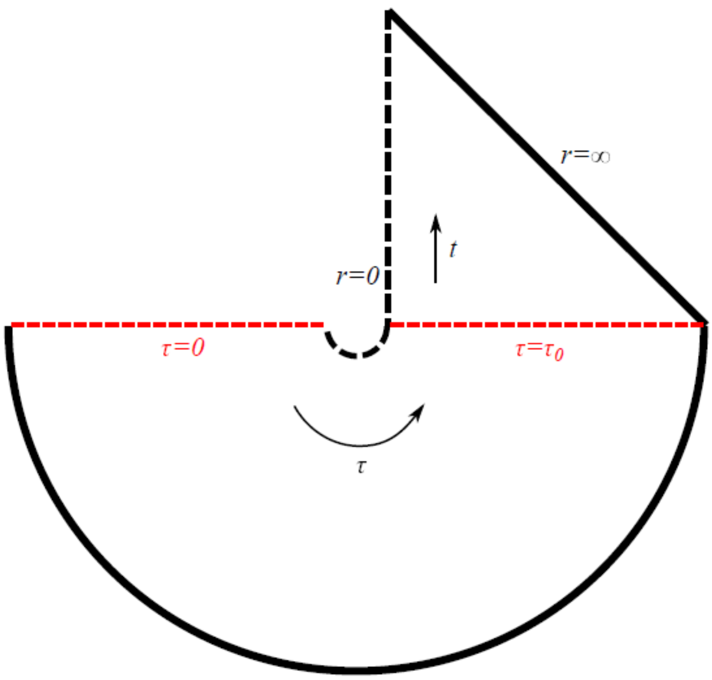
$\log_{10}(r_k^{max}/l_{Pl})$
k





$$m_e(r) = \frac{1}{e} m_1(er),$$

$$m_1(\infty) \simeq M_{\mathrm{Pl}}$$



$$S_{\mathrm{E}} = \int_{\mathcal{M}} \left[-\frac{R}{16\pi} + \frac{1}{2} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \right] \sqrt{+g}\, dx^4 - \frac{1}{8\pi} \int_{\partial\mathcal{M}} (K - K_0) \sqrt{+h}\, d^3x$$

$$\int \left[-\frac{R}{16\pi} \right] \sqrt{+g}\, dx^4 = 4\pi\beta \int_{r_h}^{\infty} dr \sqrt{\sigma} \left(-\frac{2}{\kappa} m' \right) + \frac{\beta}{4} \left((2m - 2rm')\sqrt{\sigma} + (r - 2m) r \frac{\sigma'}{\sqrt{\sigma}} \right) \Bigg|_{r_h}^{\infty}$$

$$= 4\pi\beta \int_{r_h}^{\infty} dr \sqrt{\sigma} \left(-\frac{2}{\kappa} m' \right) + \frac{\beta}{2} (M - m_h)$$

$$\sigma \to 1, \sigma' = O(r^{-5})$$

$$m' = O(r^{-4})$$

$$\int \left[\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)\right]\sqrt{+g}dx^4 = 4\pi\beta\int_{r_h}^{\infty} dr\sqrt{\sigma}\left(\frac{(1-w^2)^2}{2r^2}+\left(1-\frac{2m}{r}\right)w'^2\right)$$

$$= 4\pi\beta\int_{r_h}^{\infty} dr\sqrt{\sigma}\left(\frac{2}{\kappa}m'\right)$$

$$-\frac{1}{8\pi}\int_{\partial\mathcal{M}}(K-K_0)\sqrt{+h}d^3x = \beta\frac{M}{2}$$

$$S_{\mathrm{E}} = \beta M - \frac{\beta}{2}m_h$$

$$\beta = 1/T = 8\pi m_h$$

$$S = 4\pi m_h^2$$

$$TS_{\mathrm{E}} = M - TS$$

$$m_h = M - \omega$$

$$B \equiv S_E(M-\omega) - S_E(M) = \frac{\Delta\mathcal{A}}{4} = 4\pi(M^2-(M-\omega)^2) \simeq 8\pi M\omega = \beta\omega$$

$$\omega_k = M - m_{h,k}$$

$$\Gamma \simeq Ae^{-B}$$

$$\sim V_k^{-1} = r_k^{-3}$$

$$V_k \propto e^{Ck}$$

$$\Gamma_{\mathrm{bh}\to\mathrm{particle}} \simeq e^{-\beta\omega}$$

$$\Gamma_k \simeq e^{-\beta M_k + 4\pi M^2} = e^{-4\pi M_k^2},$$

$$S_1(S) = S_0 - S$$

$$S_2(S) = 0$$

$$S = p_1S_1 + p_2S_2 = p_1(S_0 - S)$$

$$p_1(t) = 1 - p_2(t) \to 0$$

$$\gamma_{ab} \equiv -\frac{\Gamma_{ab}}{|\det\Gamma_{ab}|^{1/N}} \text{ and } \Gamma_{\mathrm{ab}} \equiv \mathrm{C}^{\mathrm{c}}_{\mathrm{ad}}\mathrm{C}^{\mathrm{d}}_{\mathrm{bc}}$$

$$I_{\mathrm{bulk}} = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left\{-2\Lambda + \mathcal{L}_1 + \hat{\mu}_2\mathcal{L}_2 + \hat{\mu}_3\mathcal{L}_3 + \hat{\mu}_4\mathcal{L}_4 - \gamma_{ab}F^{(a)}_{\mu\nu}F^{(b)\mu\nu}\right\}$$

$$\mathcal{L}_2 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

$$\mathcal{L}_3 = R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + \frac{1}{8(2n-1)(n-3)}\left(b_1 R_{abcd}R^{abcd}R + b_2 R_{abcd}R^{abc}{}_e R^{de}\right.$$
$$\left.+ b_3 R_{abcd}R^{ac}R^{bd} + b_4 R_a{}^b R_b{}^c R_c{}^a + b_5 R_a{}^b R_b{}^a R + b_6 R^3\right)$$

$$\mathcal{L}_4 = c_1 R_{abcd}R^{cdef}R^{hg}{}_{ef}R_{hg}{}^{ab} + c_2 R_{abcd}R^{abcd}R_{ef}{}^{ef} + c_3 R R_{ab}R^{ac}R_c{}^b + c_4\left(R_{abcd}R^{abcd}\right)^2$$
$$+c_5 R_{ab}R^{ac}R_{cd}R^{db} + c_6 R R_{abcd}R^{ac}R^{db} + c_7 R_{abcd}R^{ac}R^{be}R^d{}_e + c_8 R_{abcd}R^{acef}R^b{}_e R^d{}_f$$
$$+c_9 R_{abcd}R^{ac}R_{ef}R^{bedf} + c_{10}R^4 + c_{11}R^2 R_{abcd}R^{abcd} + c_{12}R^2 R_{ab}R^{ab}$$
$$+c_{13}R_{abcd}R^{abef}R_{ef}{}^c{}_g R^{dg} + c_{14}R_{abcd}R^{aecf}R_{gehf}R^{gbhd}$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \frac{1}{e}C_{bc}^a A_\mu^{(b)} A_\nu^{(c)}$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left[d\theta^2 + k^{-1}\sin^2(\sqrt{k}\theta)d\Omega_{k,n-2}^2\right]$$

$$x_1 = \frac{r}{\sqrt{k}}\sin(\sqrt{k}\theta)\Pi_{j=1}^{n-2}\sin\phi_j,$$
$$x_l = \frac{r}{\sqrt{k}}\sin(\sqrt{k}\theta)\cos\phi_{n-l}\Pi_{j=1}^{n-l-1}\sin(\phi_j), l = 2, \dots, n-1$$
$$x_n = r\cos(\sqrt{k}\theta),$$

$$A^{(a)} = \frac{e}{r^2}(x_l dx_n - x_n dx_l) \text{ for } a = l = 1, \dots, n-1$$
$$A^{(b)} = \frac{e}{r^2}\left(x_l dx_j - x_j dx_l\right) \text{ for } b = n, \dots, n(n-1)/2, l = 1, \dots, n-2, j = 2, \dots, n-1, \text{ and } l < j.$$

$$\mu_2 \equiv (n-2)(n-3)\hat{\mu}_2$$
$$\mu_3 \equiv \frac{(n-2)(n-5)(3n^2-9n+4)}{8(2n-1)}\hat{\mu}_3$$
$$\mu_4 \equiv n(n-1)(n-3)(n-7)(n-2)^2(n^5 - 15n^4 + 72n^3 - 156n^2 + 150n - 42)\hat{\mu}_4$$

$$\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \Psi + \zeta = 0$$

$$\Psi(r) = [k - f(r)]/r^2$$

$$\zeta = \begin{cases} -\dfrac{2\Lambda}{n(n-1)} - \dfrac{m}{r^n} - \dfrac{(n-2)e^2}{(n-4)r^4}, & n > 4 \\ -\dfrac{\Lambda}{6} - \dfrac{m}{r^4} - \dfrac{2e^2}{r^4}\ln\left(\dfrac{r}{r_0}\right). & n = 4 \end{cases}$$

$$f(r) = k - r^2 \times \begin{cases} -\dfrac{\mu_3}{4\mu_4} + \dfrac{-W + \sqrt{-\left(3A + 2y - \frac{2B}{W}\right)}}{2}, & \mu_4 > 0 \\ -\dfrac{\mu_3}{4\mu_4} + \dfrac{W - \sqrt{-\left(3A + 2y + \frac{2B}{W}\right)}}{2}, & \mu_4 < 0 \end{cases}$$

$$W=\sqrt{A+2y}, A=-\frac{3\mu_3^2}{8\mu_4^2}+\frac{\mu_2}{\mu_4}, y=\begin{cases}-\frac{5}{6}A+U-\frac{P}{3U}, & U\neq 0\\ -\frac{5}{6}A+U-\sqrt[3]{H}, & U=0\end{cases}$$

$$U=\left(-\frac{H}{2}\pm\sqrt{\frac{H^2}{4}+\frac{P^3}{27}}\right)^{\frac{1}{3}}, H=-\frac{A^3}{108}+\frac{AC}{3}-\frac{B^2}{8}, P=-\frac{A^2}{12}-C$$

$$B=\frac{\mu_3^3}{8\mu_4^3}-\frac{\mu_2\mu_3}{2\mu_4^2}+\frac{1}{\mu_4}, C=-\frac{3\mu_3^4}{256\mu_4^4}+\frac{\mu_2\mu_3^2}{16\mu_4^3}-\frac{\mu_3}{4\mu_4^2}+\frac{\zeta}{\mu_4}.$$

$$f(r)=k-\frac{r^2}{2}\left[\mp\sqrt{2\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}+\frac{2\xi}{3\mu_4\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}}}\pm\left(-2\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}-\frac{2\xi}{3\mu_4\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}}\right.\right.$$

$$\left.\left.\pm\frac{2}{\mu_4}\left(2\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}+\frac{2\xi}{3\mu_4\left(\frac{1}{16\mu_4^2}+\sqrt{\Delta}\right)^{\frac{1}{3}}}\right)^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

$$\Delta=\frac{1}{256\mu_4^4}-\frac{\zeta^3}{27\mu_4^3}$$

$$f(r)=k+\zeta r^2+\mu_4\zeta^4r^2+4\mu_4^2\zeta^7r^2+\mathcal{O}\big((\mu_4)^{8/3}\big)$$

$$f(r)=k-\left(\frac{m}{\mu_4}\right)^{\frac{1}{4}}r^{\frac{8-n}{4}}-\frac{(n-2)e^2}{4(n-4)m}\times\left(\frac{m}{\mu_4}\right)^{\frac{1}{4}}r^{\frac{3n-8}{4}}+\mathcal{O}(r)$$

$$R_{abcd}R^{abcd}\propto\sqrt{\frac{m}{\mu_4}}r^{-\frac{n}{2}}$$

$$M=\frac{(n-1)}{16\pi}m,$$

$$m(r_+)=\begin{cases}\frac{\mu_4k^4}{r_+^{8-n}}+\frac{\mu_3k^3}{r_+^{6-n}}+\frac{\mu_2k^2}{r_+^{4-n}}+\frac{k}{r_+^{2-n}}-\frac{2\Lambda}{n(n-1)}r_+^n-\frac{(n-2)e^2}{(n-4)r_+^{4-n}}, & n>4,\\ \frac{\mu_4k^4}{r_+^4}+\frac{\mu_3k^3}{r_+^2}+\mu_2k^2+kr_+^2-\frac{\Lambda}{6}r_+^4-2e^2\ln(r_+), & n=4.\end{cases}$$

$$Q=\frac{1}{4\pi\sqrt{(n-1)(n-2)}}\int d^{n-1}r\sqrt{\mathrm{Tr}\left(F_{\mu\nu}^{(a)}F_{\mu\nu}^{(a)}\right)}=\frac{e}{4\pi}.$$

$$T_+ = \frac{|f'(r_+)|}{4\pi} = \frac{\left|(n-8)\mu_4 k^4 + (n-6)\mu_3 k^3 r_+^2 + (n-4)\mu_2 k^2 r_+^4 + k(n-2) r_+^6 - \frac{2\Lambda r_+^8}{n-1} - (n-2)e^2 r_+^4\right|}{4\pi r_+ \left|(4\mu_4 k^3 + 3k^2 \mu_3 r_+^2 + 2k\mu_2 r_+^4 + r_+^6)\right|}$$

$$S = \frac{r_+^{n-1}}{4} + \frac{(n-1)k\mu_2}{2(n-3)} r_+^{n-3} + \frac{3(n-1)k^2\mu_3}{4(n-5)} r_+^{n-5} + \frac{(n-1)k^3\mu_4}{(n-7)} r_+^{n-7}.$$

$$dM = TdS + UdQ,$$

$$T = \left(\frac{\partial M}{\partial S}\right)_Q$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_S$$

$$U_+ = \begin{cases} -\frac{2\pi Q(n-1)(n-2)}{(n-4)} r_+^{n-4}, & n > 4 \\ -4\pi Q(n-1)\ln(r_+) & n = 4. \end{cases}$$

$$C_e = T\left(\frac{\partial S}{\partial T}\right)_Q = T\left(\frac{\partial^2 M}{\partial S^2}\right)_Q^{-1},$$

$$\det H = \left(\frac{\partial^2 M}{\partial S^2}\right)\left(\frac{\partial^2 M}{\partial Q^2}\right) - \left(\frac{\partial^2 M}{\partial S \partial Q}\right)^2,$$

$$\left(\frac{\partial^2 M}{\partial Q^2}\right) = \begin{cases} -\frac{2\pi(n-1)(n-2)}{(n-4)} r_+^{n-4}, & n > 4, \\ -4\pi(n-1)\ln(r_+) & n = 4, \end{cases}$$

$$P = -\frac{\Lambda}{8\pi}$$

$$V = \frac{r_+^n}{n}$$

$$v = \frac{4r_+}{n-1}$$

$$dM = TdS + UdQ + VdP + \Psi_2 d\hat{\mu}_2 + \Psi_3 d\hat{\mu}_3 + \Psi_4 d\hat{\mu}_4,$$

$$\Psi_2 = \frac{\partial M}{\partial \hat{\mu}_2} = \frac{k^2(n-1)}{16\pi} r_+^{n-4} - \frac{k(n-1)}{2(n-3)} r_+^{n-3} T_+,$$
$$\Psi_3 = \frac{\partial M}{\partial \hat{\mu}_3} = \frac{k^3(n-1)}{16\pi} r_+^{n-6} - \frac{3k^2(n-1)}{4(n-5)} r_+^{n-5} T_+,$$
$$\Psi_4 = \frac{\partial M}{\partial \hat{\mu}_4} = \frac{k^4(n-1)}{16\pi} r_+^{n-8} - \frac{k^3(n-1)}{(n-7)} r_+^{n-7} T_+.$$

$$P = \frac{T}{v} - \frac{k(n-2)}{(n-1)\pi v^2} + \frac{16(n-2)e^2}{(n-1)^3\pi v^4} + \frac{32k\mu_2}{(n-1)^2 v^3}\left(T - \frac{k(n-4)}{2\pi(n-1)v}\right) + \frac{768k^2\mu_3}{(n-1)^4 v^5}\left(T - \frac{k(n-6)}{3\pi(n-1)v}\right) + \frac{16384k^3\mu_4}{(n-1)^6 v^7}\left(T - \frac{k(n-8)}{4\pi(n-1)v}\right)$$

$$\frac{\partial P}{\partial v} = 0, \frac{\partial^2 P}{\partial v^2} = 0$$

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = 0 \Rightarrow \alpha = 0$$

$$p = \frac{P}{P_C}, \nu = \frac{v}{v_C}, \tau = \frac{T}{T_C},$$

$$p = 1 + \frac{t}{\rho_C} - \frac{1}{z\rho_C} tw - Aw^3 + \mathcal{O}(tw^2, w^4),$$

$$A = \frac{1}{z^3}\left(\frac{1}{\rho_c} - \frac{h^{(3)}\big|_{\nu=1}}{6}\right), \rho_C = \frac{P_C v_C}{T_C}$$

$$h(\nu) = \frac{1}{P_C}\Bigg[-\frac{k(n-2)}{(n-1)\pi\nu^2 v_C^2} + \frac{16(n-2)e^2}{(n-1)^3\pi\nu^4 v_C^4} + \frac{32k\mu_2}{(n-1)^2\nu^3 v_C^3}\left(\tau T_C - \frac{k(n-4)}{2\pi(n-1)\nu v_C}\right) + \frac{768k^2\mu_3}{(n-1)^4\nu^5 v_C^5}\left(\tau T_C - \frac{k(n-6)}{3\pi(n-1)\nu v_C}\right) + \frac{16384k^3\mu_4}{(n-1)^6\nu^7 v_C^7}\left(\tau T_C - \frac{k(n-8)}{4\pi(n-1)\nu v_C}\right)\Bigg].$$

$$p = 1 + \frac{t}{\rho_C} - \frac{1}{z\rho_C} tw_l - Aw_l^3 = 1 + \frac{t}{\rho_C} - \frac{1}{z\rho_C} tw_s - Aw_s^3$$

$$0 = \int_{\omega_l}^{\omega_s} \omega dP$$

$$w_s = -\omega_l = \sqrt{-\frac{t}{z\rho_C A}},$$

$$\eta = v_C(\omega_l - \omega_s) = 2v_C\omega_l \propto \sqrt{-t} \Rightarrow \beta = \frac{1}{2}$$

$$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_T \propto \frac{\rho_C}{P_C}\frac{1}{t} \Rightarrow \gamma = 1$$

$$p - 1 = -Cw^3 \Rightarrow \delta = 3$$

$$I_{\text{bulk}} = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left\{-2\Lambda + \hat{\mu}_4\mathcal{L}_4 - \gamma_{ab}F_{\mu\nu}^{(a)}F^{(b)\mu\nu}\right\}$$

$$\mu_4\Psi^4 + \zeta = 0$$

$$f_{PY}(r) = k \mp \frac{r^{\frac{3}{2}}}{\mu_4} \times \begin{cases} \left[\mu_4^3\left(\frac{2\Lambda r^2}{n(n-1)} + \frac{(n-2)e^2}{(n-4)r^2} + \frac{m}{r^{n-2}}\right)\right]^{\frac{1}{4}}, & n > 4 \\ \left[\mu_4^3\left(\frac{\Lambda r^2}{6} + \frac{2e^2 \ln\left(\frac{r}{r_0}\right)}{r^2} + \frac{m}{r^2}\right)\right]^{\frac{1}{4}}, & n = 4 \end{cases}$$

$$f_{PY}(r) = k \mp \left(\frac{2\Lambda}{\mu_4 n(n-1)}\right)^{\frac{1}{4}} r^2$$

$$R_{abcd}R^{abcd} \propto \sqrt{\frac{m}{\mu_4}} r^{-\frac{n}{2}}$$

$$m(r_+) = \begin{cases} -\frac{2\Lambda}{n(n-1)} r_+^n - \frac{(n-2)e^2}{(n-4)r_+^{4-n}} + \frac{\mu_4 k^4}{r_+^{8-n}}, & n > 4 \\ -\frac{\Lambda}{6} r_+^4 - 2e^2 \ln(r_+) + \frac{\mu_4 k^4}{r_+^4}, & n = 4 \end{cases}$$

$$T_+ = \frac{|f'(r_+)|}{4\pi} = \left|\frac{k(n-8)}{16\pi r_+} - \frac{\Lambda r_+^7}{8\pi(n-1)\mu_4 k^3} - \frac{(n-2)e^2 r_+^3}{16\pi \mu_4 k^3}\right|$$

$$S = \frac{(n-1)\mu_4 k^3}{(n-7)} r_+^{n-7}$$

$$U_+ = \begin{cases} -\frac{\Lambda r_+^n}{2n} - \frac{(n-1)(n-2)e^2}{4(n-4)r_+^{4-n}} + \frac{(n-1)\mu_4 k^4}{4r_+^{8-n}}, & n > 4 \\ -\frac{\Lambda r_+^4}{8} - \frac{3}{2}e^2 \ln(r_+) + \frac{3\mu_4 k^4}{4r_+^4} & n = 4 \end{cases}$$

$$C_Q = -\frac{16\pi \mu_4^2 k^6 (n-1)^2 r_+^{n-6}}{14\Lambda r_+^8 + 3(n-1)(n-2)e^2 r_+^4 + (n-1)(n-8)\mu_4 k^4} T_+.$$

$$\begin{aligned} b_1 &= 3(3n-5) \\ b_2 &= -24(n-1) \\ b_3 &= 24(n+1) \\ b_4 &= 48(n-1) \\ b_5 &= -12(3n-1) \\ b_6 &= 3(n+1) \end{aligned}$$

$$
\begin{aligned}
c_1 &= -(n-1)(n^7 - 3n^6 - 29n^5 + 170n^4 - 349n^3 + 348n^2 - 180n + 36) \\
c_2 &= -4(n-3)(2n^6 - 20n^5 + 65n^4 - 81n^3 + 13n^2 + 45n - 18) \\
c_3 &= -64(n-1)(3n^2 - 8n + 3)(n^2 - 3n + 3) \\
c_4 &= -(n^8 - 6n^7 + 12n^6 - 22n^5 + 114n^4 - 345n^3 + 468n^2 - 270n + 54) \\
c_5 &= 16(n-1)(10n^4 - 51n^3 + 93n^2 - 72n + 18) \\
c_6 &= -32(n-1)^2(n-3)^2(3n^2 - 8n + 3) \\
c_7 &= 64(n-2)(n-1)^2(4n^3 - 18n^2 + 27n - 9) \\
c_8 &= -96(n-1)(n-2)(2n^4 - 7n^3 + 4n^2 + 6n - 3) \\
c_9 &= 16(n-1)^3(2n^4 - 26n^3 + 93n^2 - 117n + 36) \\
c_{10} &= n^5 - 31n^4 + 168n^3 - 360n^2 + 330n - 90 \\
c_{11} &= 2(6n^6 - 67n^5 + 311n^4 - 742n^3 + 936n^2 - 576n + 126) \\
c_{12} &= 8(7n^5 - 47n^4 + 121n^3 - 141n^2 + 63n - 9) \\
c_{13} &= 16n(n-1)(n-2)(n-3)(3n^2 - 8n + 3) \\
c_{14} &= 8(n-1)(n^7 - 4n^6 - 15n^5 + 122n^4 - 287n^3 + 297n^2 - 126n + 18)
\end{aligned}
$$

$$
\begin{aligned}
&C^1_{23} = C^2_{31} = C^3_{12} = -1, \gamma_{ab} = \mathrm{diag}(1,1,1) \\
&A^{(1)}_\mu = e(-\cos\phi d\theta + \sin\theta\cos\theta\sin\phi d\phi) \\
&A^{(2)}_\mu = -e(\sin\phi d\theta + \sin\theta\cos\theta\cos\phi d\phi) \\
&A^{(3)}_\mu = e\sin^2\theta d\phi
\end{aligned}
$$

$$
\begin{aligned}
&C^1_{23} = C^2_{31} = -C^3_{12} = 1, \gamma_{ab} = \mathrm{diag}(-1,-1,1) \\
&A^{(1)}_\mu = e(-\cos\phi d\theta + \sinh\theta\cosh\theta\sin\phi d\phi) \\
&A^{(2)}_\mu = -e(\sin\phi d\theta + \sinh\theta\cosh\theta\cos\phi d\phi) \\
&A^{(3)}_\mu = e\sinh^2\theta d\phi
\end{aligned}
$$

$$
\begin{aligned}
&C^1_{24} = C^1_{35} = C^2_{41} = C^2_{36} = C^3_{51} = C^3_{62} = 1 \\
&C^4_{56} = -C^4_{21} = C^5_{64} = -C^5_{31} = C^6_{45} = -C^6_{32} = 1 \\
&\gamma_{ab} = \mathrm{diag}(1,1,1,1,1,1)
\end{aligned}
$$

$$
\begin{aligned}
A^{(1)}_\mu &= -e(\sin\phi\cos\psi d\theta + \sin\theta\cos\theta(\cos\phi\cos\psi d\phi - \sin\phi\sin\psi d\psi)) \\
A^{(2)}_\mu &= -e(\sin\phi\sin\psi d\theta + \sin\theta\cos\theta(\cos\phi\sin\psi d\phi + \sin\phi\cos\psi d\psi)) \\
A^{(3)}_\mu &= -e(\cos\phi d\theta - \sin\theta\cos\theta\sin\phi d\phi) \\
A^{(4)}_\mu &= -e\sin^2\theta\sin^2\phi d\psi \\
A^{(5)}_\mu &= e\sin^2\theta(\cos\psi d\phi - \sin\phi\cos\phi\sin\psi d\psi) \\
A^{(6)}_\mu &= e\sin^2\theta(\sin\psi d\phi + \sin\phi\cos\phi\cos\psi d\psi)
\end{aligned}
$$

$$C_{24}^{1}=C_{35}^{1}=C_{41}^{2}=C_{36}^{2}=C_{51}^{3}=C_{62}^{3}=1$$
$$C_{56}^{4}=C_{21}^{4}=C_{64}^{5}=C_{31}^{5}=C_{45}^{6}=C_{32}^{6}=1$$
$$\gamma_{ab}=\mathrm{diag}(-1,-1,-1,1,1,1)$$
$$A_{\mu}^{(1)}=-e(\sin\phi\cos\psi d\theta+\sinh\theta\cosh\theta(\cos\phi\cos\psi d\phi-\sin\phi\sin\psi d\psi))$$
$$A_{\mu}^{(2)}=-e(\sin\phi\sin\psi d\theta+\sinh\theta\cosh\theta(\cos\phi\sin\psi d\phi+\sin\phi\cos\psi d\psi))$$
$$A_{\mu}^{(3)}=-e(\cos\phi d\theta-\sinh\theta\cosh\theta\sin\phi d\phi)$$
$$A_{\mu}^{(4)}=e\sinh^2\theta\sin^2\phi d\psi$$
$$A_{\mu}^{(5)}=-e\sinh^2\theta(\cos\psi d\phi-\sin\phi\cos\phi\sin\psi d\psi)$$
$$A_{\mu}^{(6)}=-e\sinh^2\theta(\sin\psi d\phi+\sin\phi\cos\phi\cos\psi d\psi)$$

$$ds^2=-f(r)dt^2+f(r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)$$

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{Q_{YM}}{r^{4p-1}}$$

$$Q_{YM}\equiv\frac{2p-1}{4p-3}q_{YM}^{2p}$$

$$g^{rr}=0,$$

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{Q_{YM}}{r^{4p-1}}=0.$$

$$\sin\theta=1,\frac{d\theta}{d\tau}=0.$$

$$2\mathcal{L}=g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau},$$

$$\mathcal{L}=\frac{1}{2}\left[-f(r)\dot{t}^2+\frac{1}{f(r)}\dot{r}^2+r^2\dot{\theta}^2+(r^2\sin^2\theta)\dot{\varphi}^2\right].$$

$$p_t=\frac{\partial\mathcal{L}}{\partial\dot{t}}=-f(r)\dot{t}$$
$$p_r=-\frac{\partial\mathcal{L}}{\partial\dot{r}}=-\frac{1}{f(r)}\dot{r}$$
$$p_\theta=-\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=-r^2\dot{\theta}$$
$$p_\varphi=-\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}=-r^2\sin^2\theta\dot{\varphi}$$

$$E=-g_{00}\frac{dt}{d\tau}=f(r)\frac{dt}{d\tau}$$
$$L=g_{33}\frac{d\varphi}{d\tau}=r^2\frac{d\varphi}{d\tau}$$

$$2\mathcal{L}=-\frac{E^2}{f(r)}+\frac{1}{f(r)}\dot{r}^2+\frac{L^2}{r^2}.$$

$$\frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{f(r)} - \frac{L^2}{r^2}.$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{b^2}r^4 - f(r)r^2 \equiv V(r)$$

$$V(r)|_{r=r_{ph}} = 0,$$
$$\left.\frac{dV(r)}{dr}\right|_{r=r_{ph}} = 0.$$

$$\left.\frac{d}{dr}\frac{f(r)}{r^2}\right|_{r=r_{ph}} = 0.$$

$$b_{ph} = \frac{r_{ph}}{\sqrt{f(r_{ph})}}$$

$$V_{\text{eff}}(r) \equiv f(r)\frac{1}{r^2}.$$

$$V_{\text{eff}}(r_{ph}) = \frac{1}{b_{ph}^2}.$$

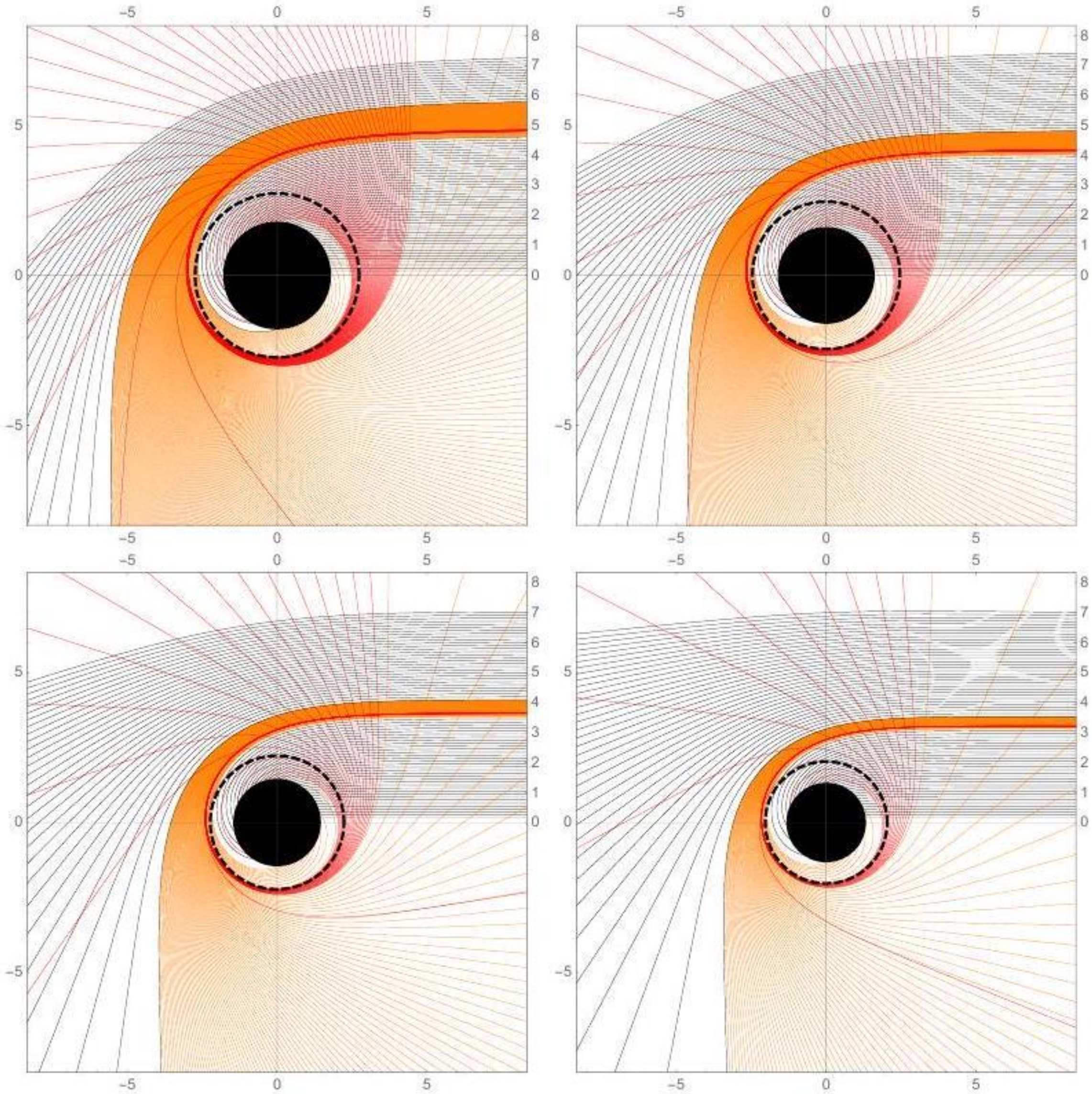
$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{b^2} - f\left(\frac{1}{u}\right)u^2$$

$$n = \frac{\varphi}{2\pi}.$$

$$\varphi = \int_0^{u_h} \frac{1}{\sqrt{\frac{1}{b^2} - f\left(\frac{1}{u}\right)u^2}}du$$

$$\varphi = 2\int_0^{u_{\max}} \frac{1}{\sqrt{\frac{1}{b^2} - f\left(\frac{1}{u}\right)u^2}}du$$

$$n(b) = \frac{2m-1}{4}, m = 1,2,3,\cdots$$



$$r_{\text{quantum hole}} = \frac{4M^4 - 3M^2Q^2(1+Q_{YM}) + M^{\frac{4}{3}}u^{\frac{1}{3}}\left(2M^{\frac{4}{3}} + u^{\frac{1}{3}}\right)}{M^{\frac{5}{3}}(1+Q_{YM})u^{\frac{1}{3}}}$$

$$I_0(r,\nu_0) = g^3 I_e(r,\nu_e)$$

$$g = \frac{\nu_0}{\nu_e} = \sqrt{f(r)}$$

$$I_{obs}(r) = \int I_0(r,\nu_0)d\nu_0 = \int g^4 I_e(r,\nu_e)d\nu_e = f(r)^2 I_{em}(r)$$

$$I_{em}(r) = \int I_e(r,\nu_e)d\nu_e$$

$$I_{obs}(b) = \sum_m f(r)^2 I_{em}(r)\Big|_{r=r_m(b)}$$

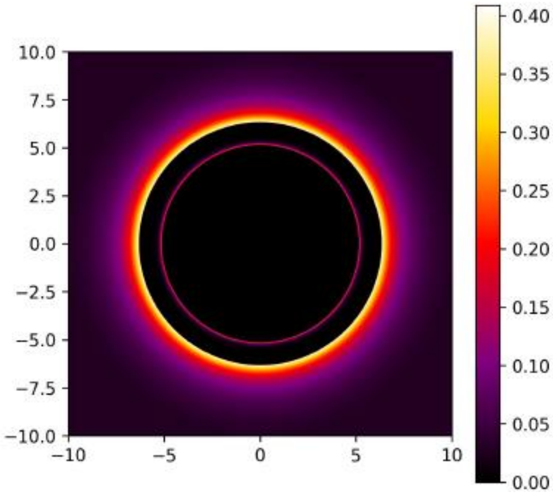
$$\varphi = \frac{2m-1}{2}\pi\frac{dr_m}{db}$$

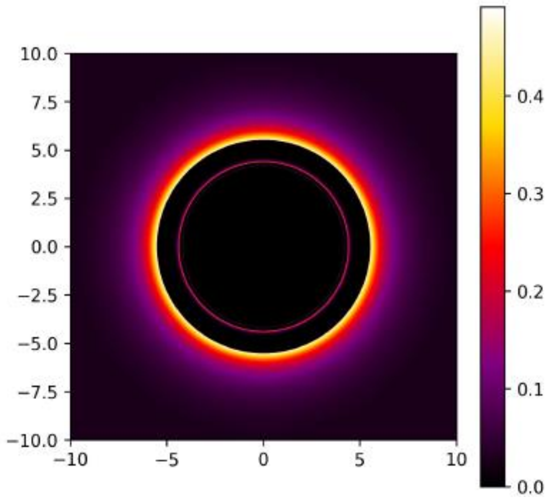
$$r_1(b) = \frac{1}{u\left(\frac{\pi}{2}, b\right)}, b \in (b_1^-, +\infty)$$
$$r_2(b) = \frac{1}{u\left(\frac{3\pi}{2}, b\right)}, b \in (b_2^-, b_2^+)$$
$$r_3(b) = \frac{1}{u\left(\frac{5\pi}{2}, b\right)}, b \in (b_3^-, b_3^+)$$

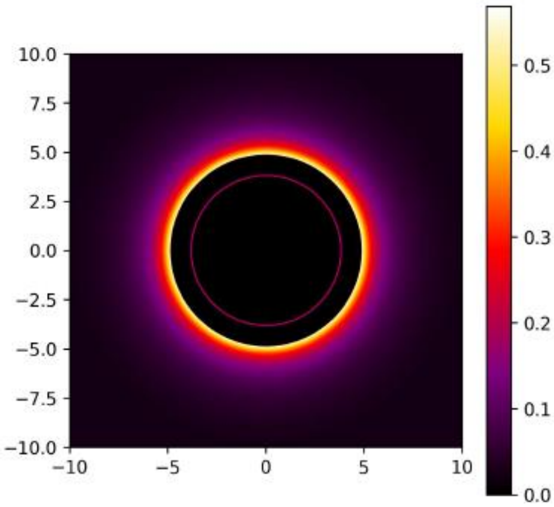
$$I_{em1}(r) = \begin{cases} I_0\left[\frac{1}{r-(r_{isco}-1)}\right]^2, & r > r_{quantum\,hole} \\ 0, & r \leq r_{quantum\,hole} \end{cases}$$

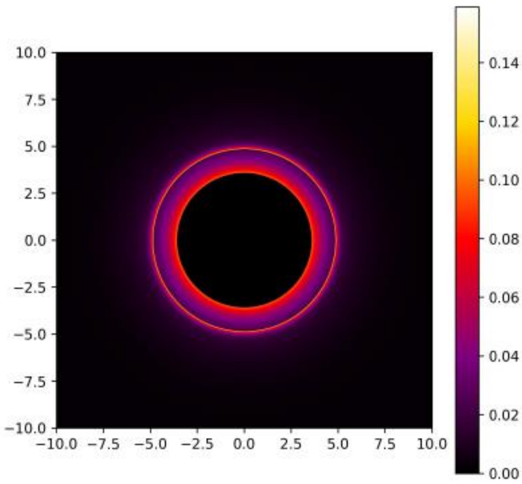
$$I_{em2}(r) = \begin{cases} I_0\left[\frac{1}{r-(r_{ph}-1)}\right]^3, & r > r_{ph} \\ 0, & r \leq r_{ph} \end{cases}$$

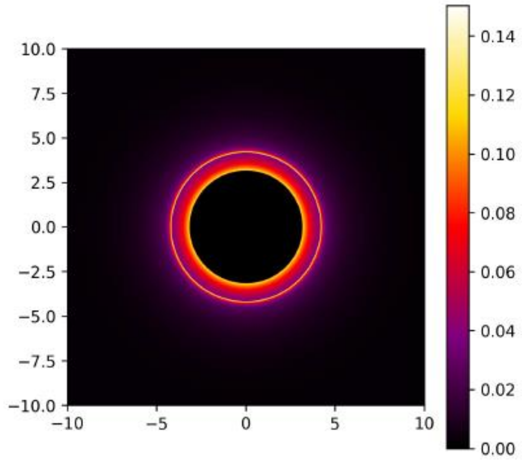
$$I_{em3}(r) = \begin{cases} I_0\frac{\frac{\pi}{2}-\arctan\,\theta[r-(r_{isco}-1)]}{\frac{\pi}{2}-\arctan\,\theta[r_h-(r_{isco}-1)]}, & r > r_h \\ 0, & r \leq r_h \end{cases}$$

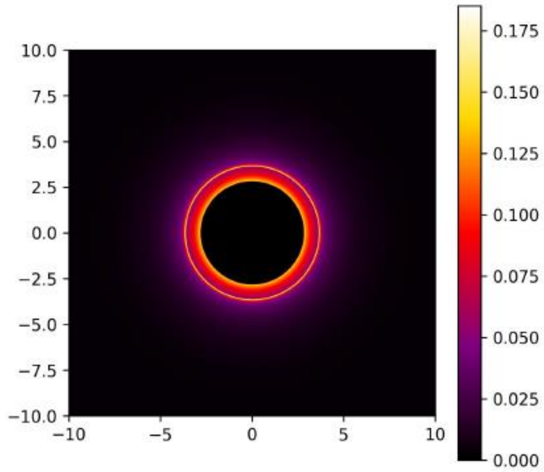



10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10
−5
0
5
10
0.4
0.3
0.2
0.1
0.0


10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10
−5
0
5
10
0.5
0.4
0.3
0.2
0.1
0.0


10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00





$$: A_{\alpha\dot{\beta}}, \lambda_\alpha = \Psi_{4\alpha}, \bar{\lambda}_{\dot{\alpha}} = \bar{\Psi}^4_{\dot{\alpha}}$$
$$: \phi_m = \Phi_{4m}, \bar{\phi}^m = \bar{\Phi}^{4m}, \psi_{m\alpha} = -i\Psi_{m\alpha}, \bar{\psi}^m_{\dot{\alpha}} = i\bar{\Psi}^m_{\dot{\alpha}}$$

$$S_i^\alpha = \left(Q^i_\alpha\right)^\dagger, \bar{S}^{i\dot{\alpha}} = (\bar{Q}_{i\dot{\alpha}})^\dagger$$

$$\left\{Q^i_\alpha, S^\beta_j\right\} = \frac{1}{2} H\delta^i_j\delta^\beta_\alpha + R^i_j\delta^\beta_\alpha + J_\alpha{}^\beta\delta^i_j$$

$$2\left\{Q, Q^\dagger\right\} = H - (R_1 + R_2 + R_3 + J_1 + J_2).$$

$$\bar{\phi}^m, \psi_{m+}, f_{++}, \bar{\lambda}_{\dot{\alpha}}$$

$$g^2_{\mathrm{YM}}, H(g_{\mathrm{YM}}) = \sum_{L=0}^{\infty} g^{2L}_{\mathrm{YM}} H_{(L)}$$

$$\left\{Q(g_{\mathrm{YM}}), Q^\dagger(g_{\mathrm{YM}})\right\} = H(g_{\mathrm{YM}}) - \sum_I R_I - \sum_i J_i = \sum_{L=1}^{\infty} g^{2L}_{\mathrm{YM}} H_{(L)}$$

$$Q\bar{\phi}^m = 0, Q\bar{\lambda}_{\dot{\alpha}} = 0, Q\psi_{m+} = -\frac{i}{2}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p], Qf = -i[\bar{\phi}^m, \psi_{m+}], [Q, D_{+\dot{\alpha}}] = -i[\lambda_{\dot{\alpha}}, \},$$

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\Delta_1 R_1 - \Delta_2 R_2 - \Delta_3 R_3 - \omega_1 J_! - \omega_2 J_2}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 = \omega_1 + \omega_2 (\mathrm{mod} 4\pi i)$$

$$\bar{\phi}^m, \psi_{m+}, f_{++},$$

$$(\bar{\phi}^m, \psi_{m+}, f_{++}) \to (\phi^m, \psi_m, f)$$

$$Z(\Delta_I) = \mathrm{Tr}_{\mathrm{BMN}}\left[(-1)^F e^{-\sum_{I=1}^3 \Delta_I (R_I + J)}\right]$$

$$Z(\Delta_I) = \frac{1}{N!}\int_0^{2\pi} \frac{d^N\alpha \quad \prod_{a\neq b}\left(1 - e^{i\alpha_{ab}}\right)\prod_{a,b=1}^N \prod_{I<J}\left(1 - e^{-\Delta_I - \Delta_J} e^{i\alpha_{ab}}\right)}{(2\pi)^N \prod_{a,b=1}^N \left[\left(1 - e^{-(\Delta_1 + \Delta_2 + \Delta_3)} e^{i\alpha_{ab}}\right)\prod_{I=1}^3 \left(1 - e^{-\Delta_I} e^{i\alpha_{ab}}\right)\right]}$$
$$\times \frac{\left(1 - e^{-(\Delta_1 + \Delta_2 + \Delta_3)}\right)\prod_{I=1}^3 \left(1 - e^{-\Delta_I}\right)}{\prod_{I<J}\left(1 - e^{-\Delta_I - \Delta_J}\right)}$$

$$\left(e^{-\Delta_1}, e^{-\Delta_2}, e^{-\Delta_3}\right) = t^2 (x, y^{-1}, x^{-1} y)$$

$$S(j, N) = N^2 f\left(\frac{j}{N^2}\right)$$

$$\rho(\alpha) = \frac{3}{4\pi^3}(\pi^2 - \alpha^2)$$

$$\log\, Z = -\frac{3N^2}{2\pi^2}\Delta_1\Delta_2\Delta_3$$

$$S_{\mathrm{BMN}}(q_I; \Delta_I) = \log\, Z + \sum_I q_I \Delta_I$$

$$S_{\mathrm{BMN}}(q_I) = 2\pi\sqrt{\frac{2q_1 q_2 q_3}{3N^2}}$$

$$\sim N^2 \sqrt{\epsilon_1 \epsilon_2 \epsilon_3} \propto N^2$$

$$S(q_I) = 2\pi\sqrt{\frac{2q_1 q_2 q_3}{N^2}}$$

$$Z(t) = \sum_j \Omega_j t^j \text{ for large } j = 2(q_1 + q_2 + q_3)$$

$$S_{\mathrm{BMN}}(j, 2) \sim 3\log\, j, S_{\mathrm{BMN}}(j, 3) \sim 2\log\, j, S_{\mathrm{BMN}}(j, 4) \sim 8\log\, j$$

$$S \propto (N^2 - 1)^{\frac{1}{3}} j^{\frac{2}{3}}$$

$$S \propto j^{\frac{n}{n+1}} j^{\frac{0}{0+1}} \log\, j$$

$$u^{i_1 i_2 \cdots i_n} = \mathrm{tr}\left(\bar{\phi}^{(i_1}\bar{\phi}^{i_2} \cdots \bar{\phi}^{i_n)}\right)$$

$$(u_n)^{i_1 \cdots i_n} = \mathrm{tr}\left(\phi^{(i_1} \cdots \phi^{i_n)}\right)$$

$$(v_n)^{i_1 \cdots i_{n-1}}{}_j = \mathrm{tr}\left(\phi^{(i_1} \cdots \phi^{i_{n-1})}\psi_j\right) - \text{'trace'}$$

$$(w_n)^{i_1 \cdots i_{n-1}} = \mathrm{tr}\left(\left(\phi^{(i_1 \cdots i_{n-1})} f + \frac{1}{2}\epsilon^{jk(i_p} \sum_{p=1}^{n-1} \phi^{i_1} \cdots \phi^{i_{p-1}}\psi_j \phi^{i_{p+1}} \cdots \phi^{i_{n-1}}\right)\psi_k\right).$$

$$R_{ij} \equiv \epsilon_{ikm}\epsilon_{jln}(u_2)^{kl}(u_2)^{mn} = Q\left[-i\epsilon_{a_1 a_2 (i}\mathrm{tr}\left(\psi_{j)}\right)\phi^{a_1}\phi^{a_2}\right)\right]$$

$$\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim 0, \mathrm{tr}(XY)\mathrm{tr}(XZ) - \mathrm{tr}(X^2)\mathrm{tr}(YZ) \sim 0,$$

$$\sum_a f_a(\{u_n, v_n, w_n\})R_a(\{u_n, v_n, w_n\})$$

$$u^{ik}R_{jk}(u_2) - \frac{1}{3}\delta^i_j u^{kl}R_{kl}(u_2) = 0$$

$$u^{1i}R_{2i} = u^{11}[u^{23}u^{13} - u^{12}u^{33}] + u^{12}[u^{33}u^{11} - (u^{13})^2] + u^{13}[u^{12}u^{13} - u^{11}u^{23}] = 0.$$

$$G_i(g_i, \lambda_I) \equiv g_i - g_i(\lambda_I) = 0$$

$$\sum_i q_i(g_i, \lambda_I)G_i(g_i, \lambda_I),$$

$$p(g_i, \lambda_I) = \sum_a q_a(g_i, \lambda_I)G_a(g_i, \lambda_I) + r(g_i, \lambda_I),$$

$$Z^{SU(2)}_{\mathrm{grav}} = \frac{1 + 3t^4 - 8t^6 - 6t^{10} + 10t^{12} + 9t^{14} - 9t^{16} + 16t^{18} - 18t^{20} - 3t^{22} + t^{24} - 3t^{26} + 9t^{28} - 2t^{30} + 3t^{32} - 3t^{34}}{(1 - t^4)^3(1 - t^8)^3}.$$

$$f = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & -f_1 - f_2 \end{pmatrix}$$

$$f = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 - f_1 & 0 \\ 0 & 0 & 0 & -f_2 - f_3 \end{pmatrix}$$

$$f = \mathrm{diag}(f_1, f_2, f_3, -f_1 - f_2 - f_3)$$

$$Z - Z_{\mathrm{grav}} = Z_{\mathrm{core}}(\Delta_I) \cdot \prod_{I=1}^{3} \frac{1}{1 - e^{-\Delta_I}e^{-\Delta_1 - \Delta_2 - \Delta_3}} \cdot \prod_{I<J} \left(1 - e^{-\Delta_I - \Delta_J}\right)$$

$$Z_{\mathrm{core}}(\Delta_I) \equiv f(t, x, y)$$

$$e^{-\Delta_1} = t^2 x, e^{-\Delta_2} = t^2 y^{-1}, e^{-\Delta_3} = t^2 x^{-1} y$$

$$f(t,x,y) = \sum_{j=0}^{54} \sum_{\mathbf{R}_j} (-1)^{F(\mathbf{R}_j)} \chi_{\mathbf{R}_j}(x,y) t^j + \mathcal{O}(t^{56})$$

$$\prod_{I<J} \left(1 - e^{-\Delta_1 - \Delta_2}\right)$$

$$[n,0]_j \otimes [2,0]_{10} = [n+2,0]_{j+10} \oplus [n,1]_{j+10} \oplus [n-2,2]_{j+10}$$

$$\begin{aligned} p &= n_\phi - l - 2m_\phi - 3b_\phi + m_\psi \\ q &= n_\psi - l - 2m_\psi - 3b_\psi + m_\phi \end{aligned}$$

$$(R,J) = \left(\frac{1}{3},0\right), \left(\frac{1}{6},\frac{1}{2}\right)$$

$$\begin{aligned} R &= \frac{n_\phi}{3} + \frac{n_\psi}{6} = \frac{p}{3} + \frac{q}{6} + \frac{l}{2} + \frac{m_\phi}{2} + b_\phi + \frac{b_\psi}{2} \\ J &= \frac{n_\psi}{2} + n_f = \frac{q}{2} + n_f + \frac{l}{2} + m_\psi + \frac{3b_\psi}{2} - \frac{m_\phi}{2} \\ j &= 6(R+J) = 2p + 4q + 6n_f + 6l + 6m_\psi + 6b_\phi + 12b_\psi \end{aligned}$$

$$\begin{aligned} 4 &= n_f + l + m_\psi + b_\phi + 2b_\psi, \\ J &= n_f + \frac{l}{2} + m_\psi + \frac{3b_\psi}{2} - \frac{m_\phi}{2}, \\ R &= \frac{n}{3} + 1 + \frac{l}{2} + \frac{m_\phi}{2} + b_\phi + \frac{b_\psi}{2} \geq \frac{n}{3} + 1. \end{aligned}$$

$$l = m_\phi = b_\phi = b_\psi = 0$$

$$J = \frac{j}{6} - R = 5 + \frac{n}{3} - R \leq 4$$

$$n_\psi = l + 2m_\psi + 3b_\psi - m_\phi$$

$$Z - Z_{\text{grav}} = Z_{\text{core}}(\Delta_I) \cdot \prod_{I=1}^{3} \frac{1}{1 - e^{-\Delta_I} e^{-\Delta_1 - \Delta_2 - \Delta_3}} \cdot \prod_{I<J} \left(1 - e^{-\Delta_I - \Delta_J}\right)$$

$$Z_{\text{core}} \stackrel{\text{def}}{=} -\sum_{n=0}^{\infty} t^{24+12n}$$

$$Z - Z_{\text{grav}} = \left[-\chi_{[2,0]}(x,y) t^{28} - \chi_{[3,0]}(x,y) t^{30} + \mathcal{O}(t^{32})\right] \cdot \prod_{I<J} \left(1 - e^{-\Delta_I - \Delta_J}\right)$$

$$O_0 \equiv \epsilon^{abc} (v_2)_a^m (v_2)_b^n \operatorname{tr}\left(\psi_{(c} \psi_m \psi_{n)}\right)$$

$$Q \operatorname{tr}\left(\psi_{(c} \psi_m \psi_{n)}\right) \propto \epsilon_{ab(c} (v_2)_m^a (v_2)_{n)}^b \equiv R(v_2)_{cmn}$$

$$QO_0 \propto \epsilon^{abc} (v_2)_a^m (v_2)^n{}_b R(v_2)_{cmn} = 0$$

$$\epsilon^{abc} (v_2)^m{}_a (v_2)^n{}_b R(v_2)_{cmn}$$

$$R \equiv \frac{R_1 + R_2 + R_3}{3}$$

$$(R, J) = \left(\frac{n}{2}, \frac{8-n}{2}\right)$$

$$u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right)$$
$$v^i_j \equiv \mathrm{tr}\left(\phi^i\psi_j\right) - \frac{1}{3}\delta^i_j \mathrm{tr}(\phi^a\psi_a), v^{ij}{}_k \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \frac{1}{4}\delta^i_k \mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \frac{1}{4}\delta^j_k \mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right)$$
$$w^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$$

$$O^{(2,1)} \equiv 65u^{ij}\left(r^{(2,1)}_{20}\right)_{ij} - 39w^{ij}\left(r^{(1,1)}_{14}\right)_{ij} + 5w^i\left(r^{(1,1)}_{16}\right)_i + 312v^{jk}_i\left(r^{(1,2)}_{16}\right)^i_{jk} + 26v^j_i\left(r^{(1,2)}_{18}\right)^i_j + 6w^i\left(r^{(0,3)}_{16}\right)_i$$

$$R^{(n_f, n_\psi - 1)}_j \equiv iQr^{(n_f, n_\psi)}_j$$

$$O^{(1,2)}_1 \equiv -3v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} - 3u^{(ij}w^{k)}\left(r^{(0,2)}_{12}\right)_{ijk} + \epsilon_{a_1a_2i}u^{a_1j}w^{a_2}\left(r^{(0,2)}_{12}\right)^i_j$$
$$O^{(1,2)}_2 \equiv -9u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 10\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k + 30v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk} + 60u^{(jk}v^{l)}{}_i\left(r^{(0,3)}_{14}\right)^i_{jkl},$$
$$O^{(1,2)}_3 \equiv -3u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 6\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k + 4u^{ijk}\left(r^{(1,2)}_{18}\right)_{ijk} + 14v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk}$$
$$-6w^{ij}\left(r^{(0,2)}_{14}\right)_{ij} - 12\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}\left(r^{(0,2)}_{12}\right)_{ijk} - 4v^j{}_a v^a{}_i\left(r^{(0,2)}_{12}\right)^i_j$$
$$O^{(1,2)}_4 \equiv -3u^{a(i}v^{j)}{}_a\left(r^{(1,1)}_{14}\right)_{ij} + 14\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(1,1)}_{14}\right)^{ij}_k + 8v^{jk}{}_i\left(r^{(1,1)}_{16}\right)^i_{jk} + 42v^{(j}{}_i w^{k)}\left(r^{(0,1)}_{10}\right)^i_{jk}$$
$$+12u^{(ij}w^{k)}\left(r^{(0,2)}_{12}\right)_{ijk} - 24w^{ij}\left(r^{(0,2)}_{14}\right)_{ij} - 36\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}\left(r^{(0,2)}_{12}\right)_{ijk} - 8v^{jk}{}_i\left(r^{(0,3)}_{16}\right)^i_{jk}$$

$$O^{(1,1)}_1 \equiv \epsilon_{a_1a_2i}u^{a_1(j}w^{k)a_2}\left(r^{(0,1)}_{10}\right)^i_{jk}$$
$$O^{(1,1)}_2 \equiv \epsilon_{a_1a_2i}u^{a_1jk}w^{a_2}\left(r^{(0,1)}_{10}\right)^i_{jk}$$
$$O^{(1,1)}_3 \equiv \epsilon_{a_1a_2i}\epsilon_{b_1b_2}u^{a_1b_1}u^{a_2b_2k}\left(r^{(1,1)}_{14}\right)^{ij}_k + 5v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk}$$
$$O^{(0,3)}_1 = -\epsilon_{ia_1a_2}(4u^{a_1b}v^{ja_2}{}_b + 3u^{ja_1b}v^{a_2}{}_b)\left(r^{(0,2)}_{12}\right)^i_j = \frac{1}{2}iQ\left(\left(r^{(0,2)}_{12}\right)^i_j\left(r^{(0,2)}_{12}\right)^j_i\right)$$
$$O^{(0,3)}_2 = -\epsilon_{a_1a_2(i}\left(u^{a_1(k}v^{l)a_2}{}_{j)} + u^{kla_1}v^{a_2}{}_{j)}\right)\left(r^{(0,2)}_{12}\right)^{ij}_{kl} = \frac{1}{2}iQ\left(\left(r^{(0,2)}_{12}\right)^{kl}_{ij}\left(r^{(0,2)}_{12}\right)^{ij}_{kl}\right)$$
$$O^{(0,3)}_3 \equiv -u^{a(i}v^{jk)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk}$$
$$O^{(0,3)}_4 \equiv -\epsilon_{a_1a_2i}u^{a_1b}v^{a_2}{}_b\left(r^{(0,2)}_{14}\right)^i$$
$$O^{(0,3)}_5 \equiv 6v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} + 6u^{a(ij}v^{k)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk} + \epsilon_{a_1a_2i}u^{a_1bj}v^{a_2}{}_b\left(r^{(0,2)}_{12}\right)^i_j$$
$$O^{(0,3)}_6 \equiv 24v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk} + 6u^{a(i}v^{j)}{}_a\left(r^{(0,2)}_{14}\right)_{ij} - \epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(0,2)}_{14}\right)^{ij}_k$$
$$O^{(0,3)}_7 \equiv v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} - 10v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk} + 6u^{a(ij}v^{k)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk} + 10\epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}\left(r^{(0,2)}_{12}\right)^{ij}_{kl},$$
$$O^{(0,3)}_8 \equiv 5v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk} + 9u^{a(ij}v^{k)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk} + 6\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(r^{(0,3)}_{14}\right)^i_{jkl},$$
$$O^{(0,3)}_9 \equiv 6v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} + 12v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk} + 18u^{a(ij}v^{k)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk} - \epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(r^{(0,2)}_{14}\right)^{ij}_k,$$
$$O^{(0,3)}_{10} \equiv 38v^a{}_i v^{jk}{}_a\left(r^{(0,1)}_{10}\right)^i_{jk} + 4v^{(j}{}_a v^{k)a}{}_i\left(r^{(0,1)}_{10}\right)^i_{jk} + 24u^{a(ij}v^{k)}{}_a\left(r^{(0,2)}_{12}\right)_{ijk} + 5u^{(jk}v^{l)}{}_i\left(r^{(0,2)}_{14}\right)^i_{jkl}.$$

$$
\begin{aligned}
O \equiv\ & -6O_6^{(0,3)} \\
= & \ 288v^j{}_a v^{ka}{}_i \epsilon_{c_1c_2(j}\mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^i\psi_{k)}\right) - 72v^a{}_b v^{bk}{}_a \epsilon_{c_1c_2(k}\mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^d\psi_{d)}\right) \\
& +36\epsilon_{a_1a_2i}u^{a_1k}v^{a_2}{}_j\left[2\mathrm{tr}\left(\phi^{(i}\phi^c\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\mathrm{tr}\left(\phi^{(i|}\phi^c\phi^{(j)}\psi_{(c}\psi_{k)}\right)\right. \\
& \left. +9\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^c\psi_{k)}\right)\right] \\
& -9\epsilon_{a_1a_2j}u^{a_1b}v^{a_2}{}_b\left[2\mathrm{tr}\left(\phi^{(j}\phi^c\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\mathrm{tr}\left(\phi^{(j|}\phi^c\phi^{|d)}\psi_{(c}\psi_{d)}\right)\right. \\
& \left. +9\mathrm{tr}\left(\phi^{(j}\phi^d\psi_{(c}\phi^{c)}\psi_{d)}\right) - 6\mathrm{tr}\left(\phi^{(j}\phi^{d)}\psi_{(c}\phi^c\psi_{d)}\right)\right] \\
& -20u^{ai}v^j{}_a\epsilon_{b_1b_2b_3}\left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^{b_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{b_1}\psi_{j)}\phi^{b_2}\phi^{b_3}\right)\right] \\
& -36u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^c\phi^{b_1}\phi^{b_2}\right)\right] \\
& -36u^{ai}v^j{}_a\epsilon_{b_1b_2(}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_j\phi^c\psi_c\phi^{b_1}\phi^{b_2}\right)\right] \\
& -36u^{ai}v^j{}_a\epsilon_{b_1b_2(}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\psi_c\phi^c\right) + \mathrm{tr}\left(\psi_j\right)^{b_1}\phi^c\psi_c\phi^{b_2}\Big) + \mathrm{tr}\left(\psi_j\phi^c\phi^{b_1}\psi_c\phi^{b_2}\right)\right] \\
& -36u^{ai}v^j{}_a\epsilon_{b_1b_2(}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^c\psi_c\right) + \mathrm{tr}\left(\psi_j\phi^{b_1}\phi^c\phi^{b_2}\psi_c\right) + \mathrm{tr}\left(\psi_j\phi^c\phi^{b_1}\phi^{b_2}\psi_c\right)\right] \\
& +12u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[5\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}(\psi_c\phi^c) + 2\mathrm{tr}\left(\psi_{j)}\phi^{(b_1}\phi^{c)}\right)\mathrm{tr}\left(\psi_c\phi^{b_2}\right) - 2\mathrm{tr}\left(\psi_j\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^{(b_1}\phi^{c)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
O_1^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + 2\epsilon_{a_1a_2i}u^{a_1b}u^{a_2(j}v^{k)}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} \\
O_2^{(0,2)} &\equiv -6\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - \epsilon_{a_1a_2i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(r_{12}^{(0,1)}\right)^{ij}_{kl} \\
O_3^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - \epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}\left(r_{12}^{(0,1)}\right)^{ij}_{kl} \\
O_4^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(r_{12}^{(0,2)}\right)^{ij}_{kl} \\
O_5^{(0,2)} &\equiv -4\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - 24\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} - \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}\left(r_{14}^{(0,2)}\right) \\
O_6^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + 12\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(r_{10}^{(0,1)}\right)^i_{jk} + 3\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(r_{14}^{(0,2)}\right)^i_{jkl}.
\end{aligned}
$$

$$
\begin{aligned}
O^{(0,1)} \equiv\ & 36\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}\left(r_{10}^{(0,1)}\right)^i_{jk} + 5\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}r_{12}^{(0,1)} \\
& -6\epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(r_{12}^{(0,1)}\right)^{ij}_{kl}
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{tr}(\phi^1\phi^1\psi_1\phi^1\psi_1\psi_1\phi^1\psi_1) + \mathrm{tr}(\phi^2\phi^2\psi_2\phi^2\psi_2\psi_2\phi^2\psi_2) + \mathrm{tr}(\phi^3\phi^3\psi_3\phi^3\psi_3\psi_3\phi^3\psi_3) \\
& \mathrm{tr}(\phi^1\phi^1\phi^2\psi_2)\mathrm{tr}(\psi_1\psi_2)\mathrm{tr}(\phi^2\psi_1) + \mathrm{tr}(\phi^2\phi^2\phi^3\psi_3)\mathrm{tr}(\psi_2\psi_3)\mathrm{tr}(\phi^3\psi_2) + \mathrm{tr}(\phi^3\phi^3\phi^1\psi_1)\mathrm{tr}(\psi_3\psi_1)\mathrm{tr}(\phi^1\psi_3) \\
& +\mathrm{tr}(\phi^3\phi^3\phi^2\psi_2)\mathrm{tr}(\psi_3\psi_2)\mathrm{tr}(\phi^2\psi_3) + \mathrm{tr}(\phi^1\phi^1\phi^3\psi_3)\mathrm{tr}(\psi_1\psi_3)\mathrm{tr}(\phi^3\psi_1) + \mathrm{tr}(\phi^2\phi^2\phi^1\psi_1)\mathrm{tr}(\psi_2\psi_1)\mathrm{tr}(\phi^1\psi_2) \\
& \mathrm{tr}(\phi^2\phi^2\psi_1\psi_2\phi^3)\mathrm{tr}(f\phi^1\phi^1) + \mathrm{tr}(\phi^3\phi^3\psi_2\psi_3\phi^1)\mathrm{tr}(f\phi^2\phi^2) + \mathrm{tr}(\phi^1\phi^1\psi_3\psi_1\phi^2)\mathrm{tr}(f\phi^3\phi^3) \\
& -\mathrm{tr}(\phi^3\phi^3\psi_1\psi_3\phi^2)\mathrm{tr}(f\phi^1\phi^1) - \mathrm{tr}(\phi^1\phi^1\psi_2\psi_1\phi^3)\mathrm{tr}(f\phi^2\phi^2) - \mathrm{tr}(\phi^2\phi^2\psi_3\psi_2\phi^1)\mathrm{tr}(f\phi^3\phi^3)
\end{aligned}
$$

$$
f = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & -f_1 - f_5 \end{pmatrix}
$$

$$
\begin{aligned}
& u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\right) \\
& v^i{}_j \equiv \mathrm{tr}\left(\phi^i\psi_j\right) - \frac{1}{3}\delta^i\mathrm{tr}(\phi^a\psi_a), v^{ij}{}_k \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \frac{1}{4}\delta^i_k\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \frac{1}{4}\delta^j_k\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right) \\
& w^i \equiv \mathrm{tr}\left(f\phi^i + \frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)
\end{aligned}
$$

$$Q\phi^m = 0, Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p], Qf = -i[\phi^m, \psi_m]$$

$$[n,0]: O^{i_1 i_2 i_3 \cdots i_n} \to O^{i_1 i_2 i_3 \cdots i_n}, \qquad [0,n]: O_{i_1 i_2 i_3 \cdots i_n} \to O_{i_1 i_2 i_3 \cdots i_n}$$

$$[1,1]: O^i_j \to O^i_j - \frac{1}{3}\delta^i_j O^a_a$$

$$[2,1]: O^{ij}_k \to O^{ij}_k - \frac{1}{2}\delta^{(i}_k O^{j)a}_a, \qquad [1,2]: O^i_{jk} \to O^i_{jk} - \frac{1}{2}\delta^i_{(j} O^a_{k)a}$$

$$[3,1]: O^{ijk}_l \to O^{ijk}_l - \frac{3}{5}\delta^{(i}_l O^{jk)a}_a, \qquad [1,3]: O^i_{jkl} \to O^i_{jkl} - \frac{3}{5}\delta^i_{(j} O^a_{kl)a}$$

$$[2,2]: O^{ij}_{kl} \to O^{ij}_{kl} - \frac{4}{5}\delta^{(i}_{(k} O^{j)a}_{l)a} + \frac{1}{10}\delta^{(i}_{(k}\delta^{j)}_{l)} O^{a_1 a_2}_{a_1 a_2}$$

$$t^{10}[1,2](u_2 u_3): \left(R^{(0,0)}_{10}\right)^i_{jk} = \epsilon_{a_1 a_2 (j}\epsilon_{k) b_1 b_2} u^{a_1 b_1} u^{i a_2 b_2}$$

$$t^{12}[0,0](u_2 u_2 u_2): R^{(0,0)}_{12} = \epsilon_{a_1 a_2 a_3}\epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} u^{a_3 b_3}$$

$$t^{12}[2,2](u_2 u_2 u_2, u_3 u_3): \left(R^{(0,0)}_{12}\right)^{ij}_{kl} = \epsilon_{a_1 a_2 (k}\epsilon_{l) b_1 b_2}\left(u^{a_1 b_1} u^{a_2 b_2} u^{ij} + 6 u^{a_1 b_1 (i} u^{j) a_2 b_2}\right)$$

$$t^{12}[0,3](u_2 v_3): \left(R^{(0,1)}_{12}\right)_{ijk} = \epsilon_{(i|a_1 a_2}\epsilon_{|j|b_1 b_2} u^{a_1 b_1} v^{a_2 b_2}{}_{|k)}$$

$$t^{12}[1,1](u_2 v_3, u_3 v_2): \left(R^{(0,1)}_{12}\right)^i_j = \epsilon_{j a_1 a_2}(4 u^{a_1 b} v^{i a_2}{}_b + 3 u^{i a_1 b} v^{a_2}{}_b)$$

$$t^{12}[2,2](u_2 v_3, u_3 v_2): \left(R^{(0,1)}_{12}\right)^{ij}_{kl} = \epsilon_{a_1 a_2 (k}\left(u^{a_1 (i} v^{j) a_2}{}_{l)} + u^{ij a_1} v^{a_2}{}_{l)}\right)$$

$$t^{14}[1,0](u_2 u_2 v_2): \left(R^{(0,1)}_{14}\right)^i = \epsilon_{a_1 a_2 a_3} u^{i a_1} u^{b a_2} v^{a_3}{}_b$$

$$t^{14}[0,2](u_2 u_2 v_2, u_3 v_3): \left(R^{(0,1)}_{14}\right)_{ij} = \epsilon_{a_1 a_2 (i|}\left(\epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} v^{b_3}{}_{|j)} - 2\epsilon_{|j) b_1 b_2} u^{a_1 b_1 c} v^{a_2 b_2}{}_c\right)$$

$$t^{14}[2,1](u_2 u_2 v_2, u_3 v_3): \left(R^{(0,1)}_{14}\right)^{ij}_k = \epsilon_{k a_1 a_2}\left(3 u^{(a_1 b} u^{ij)} v^{a_2}{}_b + 4 u^{a_1 b} u^{a_2 (i} v^{j)}{}_b + 24 u^{a_1 b (i} v^{j) a_2}{}_b\right)$$

$$t^{14}[1,3](u_2 u_2 v_2, u_3 v_3): \left(R^{(0,1)}_{14}\right)^i_{jkl} = \epsilon_{(j|a_1 a_2}\epsilon_{|k|b_1 b_2}\left(u^{a_1 b_1} u^{a_2 b_2} v^i{}_{|l)} + 6 u^{i a_1 b_1} v^{a_2 b_2}{}_{|l)}\right)$$

$$t^{14}[3,2](u_2 u_2 v_2, u_3 v_3): \left(R^{(0,1)}_{14}\right)^{ijk}_{lm} = \epsilon_{a_1 a_2 (l}\left(u^{(a_1 i} u^{jk)} v^{a_2}{}_{m)} + 6 u^{a_1 (ij} v^{k) a_2}{}_{m)}\right)$$

$$t^{14}[1,3](v_2 v_3): \left(R^{(0,2)}_{14}\right)^i_{jkl} = \epsilon_{a_1 a_2 (j} v^{a_1}{}_k v^{i a_2}{}_{l)}$$

$$t^{16}[0,1](u_2 v_2 v_2, v_3 v_3): \left(R^{(0,2)}_{16}\right)_i = \epsilon_{i a_1 a_2}(12 u^{bc} v^{a_1}{}_b v^{a_2}{}_c + 13 u^{a_1 b} v^{a_2}{}_c v^c{}_b + 12 v^{a_1 b}{}_c v^{a_2 c}{}_b)$$

$$t^{16}[1,2](u_2 v_2 v_2, v_3 v_3): \left(R^{(0,2)}_{16}\right)^i_{jk} = \epsilon_{a_1 a_2 (j}\left(3 u^{ib} v^{a_1}{}_{k)} v^{a_2}{}_b - 7 u^{i a_1} v^b{}_{k)} v^{a_2}{}_b + 6 u^{a_1 b} v^i{}_{k)} v^{a_2}{}_b + 24 v^{a_1 b}{}_{k)} v^{i a_2}{}_b\right)$$

$$t^{16}[2,3](u_2 v_2 v_2, v_3 v_3): \left(R^{(0,2)}_{16}\right)^{ij}_{klm} = \epsilon_{a_1 a_2 (k}\left(u^{a_1 (i\ j} v^{j)}{}_l v^{a_2}{}_{m)} + 3 v^{a_1 (i}{}_l v^{j) a_2}{}_{m)}\right)$$

$$t^{18}[0,0](u_3 v_2 v_2): R^{(0,2)}_{18} = \epsilon_{a_1 a_2 a_3} u^{a_1 bc} v^{a_2}{}_b v^{a_3}{}_c$$

$$t^{20}[1,0](v_2 v_2 v_3): \left(R^{(0,3)}_{20}\right)^i = 2 v^a{}_c v^b{}_a v^{ic}{}_b - 3 v^i{}_a v^c{}_b v^{ab}{}_c$$

$$t^{22}[2,0](u_2 v_2 v_2 v_2): \left(R^{(0,3)}_{22}\right)^{ij} = u^{ij} v^a{}_b v^b{}_c v^c{}_a - 3 u^{a(i} v^{j)}{}_b v^b{}_c v^c{}_a + 3 u^{ab} v^{(i}{}_a v^{j)}{}_c v^c{}_b$$

$$t^{24}[0,0](u_2 v_2 v_2 v_3): R^{(0,3)}_{24} = \epsilon_{a_1 a_2 a_3} u^{a_1 b} v^{a_2}{}_b v^{a_3 c}{}_d v^d{}_c$$

$$t^{26}[1,0](v_2 v_2 v_2 v_3): \left(R^{(0,4)}_{26}\right)^i = v^i{}_a v^a{}_b v^d{}_c v^{bc}{}_d$$

$$t^{30}[0,0](v_2 v_2 v_2 v_2 v_2): R^{(0,5)}_{30} = v^a{}_b v^b{}_c v^c{}_d v^d{}_e v^e{}_a$$

$$t^{30}[3,0](v_2 v_2 v_2 v_2 v_2): \left(R^{(0,5)}_{30}\right)^{ijk} = \epsilon^{a_1 a_2 (i} v^j{}_{a_1} v^{k)}{}_{a_2} v^b{}_c v^c{}_d v^d{}_b$$

$t^{14}[0,2](v_2v_3,u_2w_3)$: $\left(R_{14}^{(1,0)}\right)_{ij}=\epsilon_{a_1a_2(i}\left(8v^{a_1b}{}_{j)}v^{a_2}{}_b+5\epsilon_{j)b_1b_2}u^{a_1b_1}w^{a_2b_2}\right)$

$t^{14}[2,1](v_2v_3,u_2w_3,u_3w_2)$: $\left(R_{14}^{(1,0)}\right)_k^{ij}=2v^{(i}{}_av^{j)a}{}_k-5v^a{}_kv^{ij}{}_a+3\epsilon_{ka_1a_2}u^{a_1(i}w^{j)a_2}+3\epsilon_{ka_1a_2}u^{ija_1}w^{a_2}$

$t^{16}[0,1](v_3v_3,u_2v_2v_2,u_2u_2w_2)$:

$\left(R_{16}^{(1,0)}\right)_i=\epsilon_{ia_1a_2}\left(48v^{a_1b_1}{}_{b_2}v^{a_2b_2}{}_{b_1}+9u^{b_1b_2}v^{a_1}{}_{b_1}v^{a_2}{}_{b_2}-13\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}w^{b_3}\right)$

$t^{16}[1,2](v_3v_3,u_2v_2v_2,u_3w_3,u_2u_2w_2)$:

$\left(R_{16}^{(1,0)}\right)^i{}_{jk}=\epsilon_{a_1a_2(j|}\left(24v^{ia_1}{}_bv^{ba_2}{}_{|k)}+2u^{ia_1}v^{a_2}{}_bv^b{}_{|k)}-6u^{a_1b}v^{a_2}{}_bv^i{}_{|k)}\right.$

$\left.+6\epsilon_{|k)b_1b_2}u^{ia_1b_1}w^{a_2b_2}+\epsilon_{|k)b_1b_2}u^{a_1b_1}u^{a_2b_2}w^i\right)$

$t^{16}[3,1](v_3v_3,u_2v_2v_2,u_3w_3,u_2u_2w_2)$:

$\left(R_{16}^{(1,0)}\right)^{ijk}=24v^{(ij}{}_av^{k)a}{}_l+7u^{(ij}v^{k)}{}_av^a{}_l-6u^{a(i}v^j{}_av^{k)}{}_l+18\epsilon_{la_1a_2}u^{a_1(ij}w^{k)a_2}+3\epsilon_{la_1a_2}u^{(ij}u^{k)a_1}w^{a_2}$

$t^{16}[1,2](v_2w_3,v_3w_2)$: $\left(R_{16}^{(1,1)}\right)_{jk}^i=\epsilon_{a_1a_2(j}\left(v^{a_1}{}_{k)}w^{a_2i}+v^{a_1i}{}_{k)}w^{a_2}\right)$

$t^{18}[0,0](v_2v_2v_2,u_2v_2w_2)$: $R_{18}^{(1,1)}=v^{a_1}{}_{a_2}v^{a_2}{}_{a_3}v^{a_3}{}_{a_1}-3\epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_bw^{a_3}$

$t^{18}[1,1](v_2v_2v_2,v_3w_3,u_2v_2w_2)$:

$\left(R_{18}^{(1,1)}\right)_j^i=9v^i{}_{a_1}v^{a_1}{}_{a_2}v^{a_2}{}_j-24\epsilon_{ja_1a_2}v^{ia_1}{}_bw^{ba_2}$

$-13\epsilon_{ja_1a_2}u^{ia_1}v^{a_2}{}_bw^b-16\epsilon_{ja_1a_2}u^{ib}v^{a_1}{}_bw^{a_2}+5\epsilon_{ja_1a_2}u^{a_1b}v^i{}_bw^{a_2}$

$t^{18}[0,3](v_2v_2v_2,v_3w_3,u_2v_2w_2)$:

$\left(R_{18}^{(1,1)}\right)_{ijk}=\epsilon_{a_1a_2(i)}\left(3v^{a_1}{}_{|j|}v^{a_2}{}_bv^b{}_{|k)}-3\epsilon_{b_1b_2|j}v^{a_1b_1}{}_{k)}w^{a_2b_2}-\epsilon_{b_1b_2|j}u^{a_1b_1}v^{a_2}{}_{k)}w^{b_2}\right)$

$t^{18}[2,2](v_2v_2v_2,v_3w_3,u_2v_2w_2)$: $\left(R_{18}^{(1,1)}\right)_{ij}^{kl}=2v^{(i}{}_av^{j)}{}_{(k}v^a{}_{l)}-6\epsilon_{a_1a_2(k}v^{a_1(i}{}_{l)}w^{j)a_2}-\epsilon_{a_1a_2(k}u^{ij}v^{a_1}{}_{l)}w^{a_2}$

$t^{20}[0,2](v_2v_2w_2,u_2w_2w_2,w_3w_3)$: $\left(R_{20}^{(2,0)}\right)_{ij}=2\epsilon_{a_1a_2(i)}v^{a_1}{}_bv^b{}_{|j)}w^{a_2}-3\epsilon_{a_1a_2a_3}v^{a_1}{}_iv^{a_2}{}_jw^{a_3}$

$+\epsilon_{ia_1a_2}\epsilon_{jb_1b_2}u^{a_1b_1}w^{a_2}w^{b_2}+3\epsilon_{ia_1a_2}\epsilon_{jb_1b_2}w^{a_1b_1}w^{a_2b_2}$

$$iQr_j^{(n_f,n_\psi)}=R_j^{(n_f,n_\psi-1)}$$

$\left(r_{10}^{(0,1)}\right)_{jk}^i=-2\epsilon_{a_1a_2(j}\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right)$

$r_{12}^{(0,1)}=\epsilon_{a_1a_2a_3}\left[6\mathrm{tr}(\psi_b\phi^{a_1})\mathrm{tr}\left(\phi^b\phi^{a_2}\phi^{a_3}\right)-\mathrm{tr}(\psi_b\phi^{a_1}\phi^{a_2})\mathrm{tr}\left(\phi^b\phi^{a_3}\right)\right]$

$-3\epsilon_{a_1a_2a_3}\left[\mathrm{tr}\left(\psi_b\phi^b\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^b\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^b\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^b\right)\right]$

$\left(r_{12}^{(0,1)}\right)_{kl}^{ij}=-2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right)+7\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right]$

$$\left(r_{12}^{(0,2)}\right)_{ijk}=\frac{1}{2}\epsilon_{a_1a_2(i}\mathrm{tr}\big(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\big),$$

$$\left(r_{12}^{(0,2)}\right)^{i}_{j}=6\mathrm{tr}\big(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\big)-5\mathrm{tr}\big(\phi^{[i}\psi_a\phi^{a]}\psi_j\big),$$

$$\left(r_{12}^{(0,2)}\right)^{ij}_{kl}=\mathrm{tr}\big(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\big),$$

$$\left(r_{14}^{(0,2)}\right)^{i}=3\mathrm{tr}\big(\phi^i\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\big)+2\mathrm{tr}\big(\phi^i\phi^{a_1}\big)\mathrm{tr}\big(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\big)$$
$$-6\mathrm{tr}\big(\phi^i\psi_{a_1}\big)\mathrm{tr}\big(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\big)-\mathrm{tr}\big(\phi^i\psi_{a_1}\psi_{a_2}\big)\mathrm{tr}(\phi^{a_1}\phi^{a_2}),$$

$$\left(r_{14}^{(0,2)}\right)_{ij}=\frac{5}{9}\epsilon_{a_1a_2a_3}\big[2\mathrm{tr}\big(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\big)+\mathrm{tr}\big(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\big)\big]$$
$$+\epsilon_{a_1a_2(i}\big[\mathrm{tr}\big(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\big)+\mathrm{tr}\big(\psi_j\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\big)+\mathrm{tr}\big(\psi_j\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\big)\big]$$
$$+\epsilon_{a_1a_2(i}\big[\mathrm{tr}\big(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\big)+\mathrm{tr}\big(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\big)+\mathrm{tr}\big(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\big)\big]$$
$$+\epsilon_{a_1a_2(i}\Big[\mathrm{tr}\big(\psi_j\big)^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\Big)+\mathrm{tr}\big(\psi_j\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\big)+\mathrm{tr}\big(\psi_j\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\big)\Big]$$
$$+\epsilon_{a_1a_2(i}\big[\mathrm{tr}\big(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\big)+\mathrm{tr}\big(\psi_j\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\big)+\mathrm{tr}\big(\psi_{j)}\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\big)\big]$$
$$-\frac{1}{3}\epsilon_{a_1a_2(i}\big[5\mathrm{tr}\big(\psi_{j)}\phi^{a_1}\phi^{a_2}\big)\mathrm{tr}\big(\psi_{a_3}\phi^{a_3}\big)+2\mathrm{tr}\big(\psi_j\phi^{(a_1}\phi^{a_3)}\big)\mathrm{tr}\big(\psi_{a_3}\phi^{a_2}\big)-2\mathrm{tr}\big(\psi_j\phi^{a_2}\big)\mathrm{tr}\big(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\big)\big],$$

$$\left(r_{14}^{(0,2)}\right)^{ij}_{k}=12\mathrm{tr}\big(\phi^{(i}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\big)+12\mathrm{tr}\big(\phi^{(i|}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\big)+54\mathrm{tr}\big(\phi^{(i}\phi^j\psi_{(a}\phi^{a)}\psi_{k)}\big)-36\mathrm{tr}\big(\phi^{(i}\phi^{j)}\psi_{(a}\phi^a\psi_{k)}\big),$$

$$\left(r_{14}^{(0,2)}\right)^{i}_{jkl}=2\epsilon_{a_1a_2(j}\big[\mathrm{tr}\big(\phi^i\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\big)+3\mathrm{tr}\big(\phi^i\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\big)-2\mathrm{tr}\big(\phi^i\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\big)\big],$$

$$\left(r_{14}^{(0,3)}\right)^{i}_{jkl}=-\frac{1}{2}\mathrm{tr}\big(\phi^i\psi_{(j}\psi_k\psi_{l)}\big),$$

$$\left(r_{16}^{(0,3)}\right)_{i}=\frac{39}{4}\mathrm{tr}\big(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\big)+2\mathrm{tr}\big(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\big)-\frac{61}{4}\mathrm{tr}\big(\psi_i\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\big)+\frac{97}{4}\mathrm{tr}\big(\psi_i\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\big)$$
$$-\frac{41}{4}\mathrm{tr}\big(\psi_i\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\big)-5\mathrm{tr}\big(\psi_i\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\big)-\frac{25}{2}\mathrm{tr}\big(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\big)+2\mathrm{tr}\big(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\big)$$
$$-\frac{61}{4}\mathrm{tr}\big(\psi_i\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\big)-\frac{11}{4}\mathrm{tr}(\phi^{b_1}\phi^{b_2})\mathrm{tr}\big(\psi_i\psi_{b_1}\psi_{b_2}\big)-\frac{27}{2}\mathrm{tr}\big(\psi_{b_1}\psi_{b_2}\big)\mathrm{tr}(\psi_i\phi^{b_1}\phi^{b_2})$$
$$+\frac{29}{4}\mathrm{tr}\big(\phi^{b_2}\psi_{b_2}\big)\mathrm{tr}\big(\psi_i[\psi_{b_1},\phi^{b_1}]\big),$$

$$\left(r_{16}^{(0,3)}\right)^{i}_{jk}=2\mathrm{tr}\big(\psi_{(j}\psi_{k)}\psi_b\phi^b\phi^i\big)-4\mathrm{tr}\big(\psi_{(j}\psi_{k)}\psi_b\phi^i\phi^b\big)-\mathrm{tr}\big(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\big)-4\mathrm{tr}\big(\psi_{(j}\psi_{k)}\phi^{(b}\psi_b\phi^{i)}\big)$$
$$+7\mathrm{tr}\big(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\big)-11\mathrm{tr}\big(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\big)-4\mathrm{tr}\big(\psi_{(j}\psi_{k)}\phi^b\phi^i\psi_b\big)+2\mathrm{tr}\big(\psi_{(j}\psi_{k)}\phi^i\phi^b\psi_b\big)$$
$$+3\mathrm{tr}\big(\psi_{(j|}\psi_b\big)\mathrm{tr}\big(\psi_{|k)}[\phi^b,\phi^i]\big)+6\mathrm{tr}\big(\psi_{(j}\phi^{[b}\big)\mathrm{tr}\big(\{\psi_{k)},\psi_b\}\phi^{i]}\big).$$

$$\left(r_{14}^{(1,1)}\right)_{ij}=5\epsilon_{a_1a_2(i}\mathrm{tr}\big(f\phi^{a_1}\psi_{j)}\phi^{a_2}\big)+\mathrm{tr}\big(\phi^a\{\psi_a,\psi_{(i}\psi_{j)}\}\big)-4\mathrm{tr}\big(\phi^a\psi_{(i}\psi_a\psi_{|j)}\big)$$

$$\left(r_{14}^{(1,1)}\right)^{ij}_{k}=3\mathrm{tr}\big(f\phi^{(i}\phi^{j)}\psi_k\big)-3\mathrm{tr}\big(f\psi_k\phi^{(i}\phi^{j)}\big)+\epsilon^{a_1a_2(i\mathrm{tr}(\phi^j)}\psi_k\psi_{a_1}\psi_{a_2}\big)-\epsilon^{a_1a_2(i\mathrm{tr}(\phi^j)}\psi_{a_1}\psi_{a_2}\psi_k\big)$$

$$\left(r_{16}^{(1,1)}\right)_{i}=13\epsilon_{a_1a_2a_3}\mathrm{tr}(f\psi_i)\mathrm{tr}(\phi^{a_1}\phi^{a_2}\phi^{a_3})+\frac{10}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(\psi_i\phi^{a_2}\phi^{a_3})+\frac{10}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}(f\phi^{a_1}\phi^{a_2})\mathrm{tr}(\psi_i\phi^{a_3})$$
$$+46\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^b)\mathrm{tr}(\psi_b\phi^{a_1}\phi^{a_2})-7\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(\psi_b\phi^{a_2}\phi^b)-7\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^b\phi^{a_1})\mathrm{tr}(\psi_b\phi^{a_2})$$
$$+6\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^{a_1}\phi^{a_2})\mathrm{tr}(\psi_b\phi^b)-\frac{115}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}(f\psi_i\phi^{a_1}\phi^{a_2}\phi^{a_3})-\frac{95}{3}\epsilon_{a_1a_2a_3}\mathrm{tr}(f\phi^{a_1}\psi_i\phi^{a_2}\phi^{a_3})$$
$$+5\epsilon_{a_1a_2a_3}\mathrm{tr}(f\phi^{a_1}\phi^{a_2}\psi_i\phi^{a_3})+36\epsilon_{ia_1a_2}\mathrm{tr}(f\psi_b\phi^{a_1}\phi^{a_2}\phi^b)-43\epsilon_{ia_1a_2}\mathrm{tr}(f\psi_b\phi^{a_1}\phi^b\phi^{a_2})$$
$$+39\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^{a_1}\psi_b\phi^{a_2}\phi^b)-68\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^{a_1}\phi^{a_2}\psi_b\phi^b)+39\epsilon_{ia_1a_2}\mathrm{tr}(f\phi^{a_1}\phi^b\psi_b\phi^{a_2})$$

$$+13\mathrm{tr}(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}) - 31\mathrm{tr}(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_2}\phi^{b_1}\}) + 14\mathrm{tr}(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2})$$
$$-22\mathrm{tr}(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}) + 14\mathrm{tr}(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}),$$
$$\left(r_{16}^{(1,1)}\right)^i_{jk} = \epsilon_{a_1a_2(j}[-4\mathrm{tr}(f\phi^i)\mathrm{tr}(\psi_{k)}\phi^{a_1}\phi^{a_2}) - \mathrm{tr}(\phi^i\phi^{a_2})\mathrm{tr}(f[\psi_{k)},\phi^{a_1}])]$$
$$+\epsilon_{a_1a_2(j}[3\mathrm{tr}(f\phi^{a_1}\{\psi_{k)},\phi^i\}\phi^{a_2}) + 5\mathrm{tr}(f\{\psi_{k)},\phi^{a_1}\phi^i\phi^{a_2}\}) - 4\mathrm{tr}(f\psi_{k)}\phi^i\phi^{a_1}\phi^{a_2}) - 4\mathrm{tr}(f\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)})]$$
$$+2\mathrm{tr}(\psi_{(j}\psi_{k)}\psi_b[\phi^b,\phi^i]) - 3\mathrm{tr}(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}) + 6\mathrm{tr}(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i) - 9\mathrm{tr}(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b)$$
$$-2\mathrm{tr}(\psi_{(j}\psi_{k)}[\phi^b,\phi^i]\psi_b) + \mathrm{tr}(\psi_{(j|}\psi_b)\mathrm{tr}(\psi_{|k)}[\phi^b,\phi^i]) + \mathrm{tr}(\psi_{(j|}\phi^b)\mathrm{tr}(\{\psi_{|k)},\psi_b\}\phi^i),$$
$$\left(r_{16}^{(1,2)}\right)^i_{jk} = -\frac{1}{2}\mathrm{tr}(f\phi^i\psi_{(j}\psi_{k)}) - \frac{1}{2}\mathrm{tr}(f\psi_{(j}\phi^i\psi_{k)}) - \frac{1}{2}\mathrm{tr}(f\psi_{(j}\psi_{k)}\phi^i) - \frac{1}{4}\epsilon^{ia_1a_2}\mathrm{tr}(\psi_{a_1}\psi_{a_2}\psi_{(j}\psi_{k)}),$$
$$\left(r_{18}^{(1,2)}\right)^i_j = -4\mathrm{tr}(f\phi^i\phi^a)\mathrm{tr}(\psi_j\psi_a) - 5\mathrm{tr}(f\phi^a\phi^i)\mathrm{tr}(\psi_j\psi_a)$$
$$-\frac{53}{2}\mathrm{tr}(f\phi^i\psi_j)\mathrm{tr}(\phi^a\psi_a) + 7\mathrm{tr}(f\phi^i\psi_a)\mathrm{tr}(\phi^a\psi_j) + \frac{15}{2}\mathrm{tr}(f\phi^a\psi_j)\mathrm{tr}(\phi^i\psi_a) + 12\mathrm{tr}(f\phi^a\psi_a)\mathrm{tr}(\phi^i\psi_j)$$
$$+2\mathrm{tr}(f\psi_j\phi^i)\mathrm{tr}(\phi^a\psi_a) - 13\mathrm{tr}(f\psi_a\phi^i)\mathrm{tr}(\phi^a\psi_j) + 4\mathrm{tr}(f\psi_j\phi^a)\mathrm{tr}(\phi^i\psi_a)$$
$$+6\mathrm{tr}(f\psi_j\psi_a)\mathrm{tr}(\phi^i\phi^a) + \frac{13}{2}\mathrm{tr}(f\psi_a\psi_j)\mathrm{tr}(\phi^i\phi^a)$$
$$-4\mathrm{tr}(f\phi^i)\mathrm{tr}(\phi^a\psi_j\psi_a) + 14\mathrm{tr}(f\phi^i)\mathrm{tr}(\phi^a\psi_a\psi_j) - 8\mathrm{tr}(f\phi^a)\mathrm{tr}(\phi^i\psi_j\psi_a) - 8\mathrm{tr}(f\phi^a)\mathrm{tr}(\phi^i\psi_a\psi_j)$$
$$-4\mathrm{tr}(f\psi_j)\mathrm{tr}(\psi_a\phi^i\phi^a) - 9\mathrm{tr}(f\psi_a)\mathrm{tr}(\psi_j\phi^i\phi^a) + 6\mathrm{tr}(f\psi_a)\mathrm{tr}(\psi_j\phi^a\phi^i)$$
$$+3\mathrm{tr}(f\phi^i\phi^a\psi_j\psi_a) - \frac{31}{2}\mathrm{tr}(f\phi^i\phi^a\psi_a\psi_j) + 3\mathrm{tr}(f\phi^a\phi^i\psi_j\psi_a) + \frac{5}{2}\mathrm{tr}(f\phi^a\phi^i\psi_a\psi_j)$$
$$+12\mathrm{tr}(f\phi^i\psi_j\phi^a\psi_a) - \frac{13}{2}\mathrm{tr}(f\phi^i\psi_a\phi^a\psi_j) - 6\mathrm{tr}(f\phi^a\psi_j\phi^i\psi_a) - \frac{13}{2}\mathrm{tr}(f\phi^a\psi_a\phi^i\psi_j)$$
$$+18\mathrm{tr}(f\phi^i\psi_j\psi_a\phi^a)$$
$$-12\mathrm{tr}(f\psi_j\phi^i\phi^a\psi_a) + \frac{17}{2}\mathrm{tr}(f\psi_a\phi^i\phi^a\psi_j) - \frac{43}{2}\mathrm{tr}(f\psi_a\phi^a\phi^i\psi_j)$$
$$+\frac{1}{3}\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^i\psi_j)\mathrm{tr}(\psi_{a_1}\psi_{a_2}\psi_{a_3}) - 2\epsilon^{a_1a_2i}\mathrm{tr}(\phi^b\psi_{a_1})\mathrm{tr}(\psi_b\psi_j\psi_{a_2})$$
$$-10\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^i\psi_j\psi_{a_1}\psi_{a_2}\psi_{a_3}) + 8\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^i\psi_{a_1}\psi_j\psi_{a_2}\psi_{a_3}) - 2\epsilon^{a_1a_2a_3}\mathrm{tr}(\phi^i\psi_{a_1}\psi_{a_2}\psi_j\psi_{a_3}),$$
$$\left(r_{18}^{(1,2)}\right)_{ijk}$$
$$= -\epsilon_{a_1a_2(i}\left[\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(\phi^{a_2}\psi_j\psi_{k)}) - \frac{3}{2}\mathrm{tr}(f\psi_j)\mathrm{tr}(\psi_{k)}\phi^{a_1}\phi^{a_2}\right) + 3\mathrm{tr}(f\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}) - 3\mathrm{tr}(f\psi_j\phi^{a_1}\psi_k\phi^{a_2})$$
$$-\frac{1}{2}\mathrm{tr}(\phi^a\psi_a)\mathrm{tr}(\psi_{(i}\psi_j\psi_{k)}) + \frac{3}{2}\mathrm{tr}(\phi^a\psi_{(i|})\mathrm{tr}(\psi_a\psi_{|j}\psi_{k)}) + \frac{1}{2}\mathrm{tr}(\phi^a\psi_{(i}\psi_{j|})\mathrm{tr}(\psi_a\psi_{|k)})$$
$$+\frac{3}{2}\mathrm{tr}(\phi^a\psi_{(i|}\psi_a\psi_{|j}\psi_{k)}) - \frac{3}{2}\mathrm{tr}(\phi^a\psi_{(i}\psi_{j|}\psi_a\psi_{|k)}),$$
$$\left(r_{20}^{(2,1)}\right)_{ij}$$
$$= -\epsilon_{a_1a_2(i}\left[\mathrm{tr}(ff)\mathrm{tr}(\phi^{a_1}\phi^{a_2}\psi_{j)}) + \frac{1}{2}\mathrm{tr}(f\psi_{j)})\right)\mathrm{tr}(f\phi^{a_1}\phi^{a_2}) + 2\mathrm{tr}(f\phi^{a_1})\mathrm{tr}(f[\phi^{a_2},\psi_{j)}])\Big)\Big]$$
$$+\epsilon_{a_1a_2(i}\left[4\mathrm{tr}(ff\phi^{a_1}\phi^{a_2}\psi_{j)}) - \mathrm{tr}(f\phi^{a_1}\phi^{a_2}f\psi_{j)})\right]$$
$$+2\mathrm{tr}(f\phi^a\psi_{(i})\mathrm{tr}(\psi_j\psi_a) - 4\mathrm{tr}(f\psi_{(i}\phi^a)\mathrm{tr}(\psi_{j)}\psi_a)$$
$$-\frac{1}{2}\mathrm{tr}(f\psi_{(i)})(\phi^a\psi_{j)}\psi_a) - \frac{5}{2}\mathrm{tr}(f\psi_{(i|})(\phi^a\psi_a\psi_{|j)}) + 2\mathrm{tr}(f\psi_a)(\phi^a\psi_{(i}\psi_{j)})$$

$$-4\mathrm{tr}\big(f\psi_{(i|}\psi_a\big)\big(\phi^a\psi_{|j)}\big)$$
$$+2\mathrm{tr}\big(f\phi^a\psi_{(i}[\psi_{j)},\psi_a]\big)+4\mathrm{tr}\big(f\phi^a\psi_a\psi_{(i}\psi_{j)}\big)$$
$$+4\mathrm{tr}\big(f\psi_{(i}\phi^a\psi_{j)}\psi_a\big)-3\mathrm{tr}\big(f\psi_{(i|}\phi^a\psi_a\psi_{|j)}\big)-2\mathrm{tr}\big(f\psi_a\phi^a\psi_{(i}\psi_{j)}\big)$$
$$-\mathrm{tr}\big(f\psi_{(i|}\psi_a\phi^a\psi_{|j)}\big)+4\mathrm{tr}\big(f\psi_a\psi_{(i}\phi^a\psi_{j)}\big)$$
$$+\frac{2}{5}\epsilon^{a_1a_2a_3}\big[2\mathrm{tr}\big(\psi_{a_1}\psi_{a_2}\big)\mathrm{tr}\big(\psi_{a_3}\psi_{(i}\psi_{j)}\big)-3\mathrm{tr}\big(\psi_{(i|}\psi_{a_1}\psi_{|j)}\psi_{a_2}\psi_{a_3}\big)\big].$$

$$s_1^{(2,0)}=u^{ij}\left(R_{20}^{(2,0)}\right)_{ij},\qquad s_2^{(2,0)}=w^{ij}\left(R_{14}^{(1,0)}\right)_{ij},\quad s_3^{(2,0)}=w^{i}\left(R_{16}^{(1,0)}\right)_{i}$$
$$s_1^{(1,2)}=v^{jk}{}_i\left(R_{16}^{(1,1)}\right)^i_{jk},\quad s_2^{(1,2)}=v^{j}{}_i\left(R_{18}^{(1,1)}\right)^i_{j},\quad s_3^{(1,2)}=w^{i}\left(R_{16}^{(0,2)}\right)_{i}$$

$$iQO^{(2,1)}\equiv 65s_1^{(2,0)}-39s_2^{(2,0)}+5s_3^{(2,0)}-312s_1^{(1,2)}-26s_2^{(1,2)}+6s_3^{(1,2)}=0.$$

$$s_1^{(1,1)}=u^{a(i}v^{j)}{}_a\left(R_{14}^{(1,0)}\right)_{ij},s_2^{(1,1)}=\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(R_{14}^{(1,0)}\right)^{ij}_{k},s_3^{(1,1)}=v^{jk}{}_i\left(R_{16}^{(1,0)}\right)^i_{jk},$$
$$s_4^{(1,1)}=u^{ijk}\left(R_{18}^{(1,1)}\right)_{ijk},s_5^{(1,1)}=v^{(j}{}_iw^{k)}\left(R_{10}^{(0,0)}\right)^i_{jk},s_6^{(1,1)}=u^{(ij}w^{k)}\left(R_{12}^{(0,1)}\right)_{ijk},$$
$$s_7^{(1,1)}=\epsilon_{a_1a_2i}u^{a_1j}w^{a_2}\left(R_{12}^{(0,1)}\right)^i_{j},s_8^{(1,1)}=w^{ij}\left(R_{14}^{(0,1)}\right)_{ij},$$
$$s_1^{(0,3)}=\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)}{}_{a_2}\left(R_{12}^{(0,1)}\right)_{ijk},s_2^{(0,3)}=v^{j}{}_av^{a}{}_i\left(R_{12}^{(0,1)}\right)^i_{j},$$
$$s_3^{(0,3)}=u^{(jk}v^{k)}{}_i\left(R_{14}^{(0,2)}\right)^i_{jkl},s_4^{(0,3)}=v^{jk}{}_i\left(R_{16}^{(0,2)}\right)^i_{jk}.$$

$$iQO_1^{(1,2)}\equiv 3s_5^{(1,1)}-3s_6^{(1,1)}+s_7^{(1,1)}=0$$
$$iQO_2^{(1,2)}\equiv 9s_1^{(1,1)}-10s_2^{(1,1)}-30s_5^{(1,1)}-60s_3^{(0,3)}=0$$
$$iQO_3^{(1,2)}\equiv 3s_1^{(1,1)}-6s_2^{(1,1)}+4s_4^{(1,1)}-14s_5^{(1,1)}-6s_8^{(1,1)}-12s_1^{(0,3)}-4s_2^{(0,3)}=0$$
$$iQO_4^{(1,2)}\equiv 3s_1^{(1,1)}-14s_2^{(1,1)}-8s_3^{(1,1)}-42s_5^{(1,1)}+12s_6^{(1,1)}-24s_8^{(1,1)}-36s_1^{(0,3)}+8s_4^{(0,3)}=0$$

$$s_1^{(1,0)}=\epsilon_{a_1a_2i}\epsilon_{b_1b_2j}u^{a_1b_1}u^{a_2b_2k}\left(R_{14}^{(1,0)}\right)^{ij}_{k},s_2^{(1,0)}=\epsilon_{a_1a_2i}u^{a_1(j}w^{k)a_2}\left(R_{10}^{(0,0)}\right)^i_{jk},$$
$$s_3^{(1,0)}=\epsilon_{a_1a_2i}u^{a_1jk}w^{a_2}\left(R_{10}^{(0,0)}\right)^i_{jk},$$
$$s_1^{(0,2)}=v^{a}{}_iv^{jk}{}_a\left(R_{10}^{(0,0)}\right)^i_{jk},s_2^{(0,2)}=v^{(j}{}_av^{k)a}{}_i\left(R_{10}^{(0,0)}\right)^i_{jk},s_3^{(0,2)}=u^{a(i}v^{jk)}{}_a\left(R_{12}^{(0,1)}\right)_{ijk},$$
$$s_4^{(0,2)}=u^{a(ij}v^{k)}{}_a\left(R_{12}^{(0,1)}\right)_{ijk},s_5^{(0,2)}=\epsilon_{a_1a_2i}u^{a_1b}v^{a_2j}{}_b\left(R_{12}^{(0,1)}\right)^i_{j},s_6^{(0,2)}=\epsilon_{a_1a_2i}u^{a_1bj}v^{a_2}{}_b\left(R_{12}^{(0,1)}\right)^i_{j},$$
$$s_7^{(0,2)}=\epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(R_{12}^{(0,1)}\right)^{ij}_{kl},s_8^{(0,2)}=\epsilon_{a_1a_2i}u^{a_1kl}v^{a_2}{}_{j)}\left(R_{12}^{(0,1)}\right)^{ij}_{kl},$$
$$s_9^{(0,2)}=\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(R_{14}^{(0,2)}\right)^i_{jkl},s_{10}^{(0,2)}=\epsilon_{a_1a_2i}u^{a_1b}v^{a_2}{}_b\left(R_{14}^{(0,1)}\right)^i,s_{11}^{(0,2)}=u^{a(i}v^{j)}{}_a\left(R_{14}^{(0,1)}\right)_{ij},$$
$$s_{12}^{(0,2)}=\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}\left(R_{14}^{(0,1)}\right)^{ij}_{k},$$
$$s_{13}^{(0,2)}=u^{(jk}v^{l)}{}_i\left(R_{14}^{(0,1)}\right)^i_{jkl}.$$

$$iQO_1^{(1,1)} \equiv s_2^{(1,0)} = 0$$
$$iQO_2^{(1,1)} \equiv s_3^{(1,0)} = 0$$
$$iQO_3^{(1,1)} \equiv s_1^{(1,0)} + 5s_1^{(0,2)} - 2s_2^{(0,2)} = 0$$
$$iQO_1^{(0,3)} \equiv 4s_5^{(0,2)} + 3s_6^{(0,2)} = \left(R_{12}^{(0,1)}\right)_j^i \left(R_{12}^{(0,1)}\right)_i^j = iQ\left[\frac{1}{2}iQ\left(\left(r_{12}^{(0,2)}\right)_j^i \left(r_{12}^{(0,2)}\right)_i^j\right)\right] = 0$$
$$iQO_2^{(0,3)} \equiv s_7^{(0,2)} + s_8^{(0,2)} = \left(R_{12}^{(0,1)}\right)_{kl}^{ij} \left(R_{12}^{(0,1)}\right)_{ij}^{kl} = iQ\left[\frac{1}{2}iQ\left(\left(r_{12}^{(0,2)}\right)_{kl}^{ij} \left(r_{12}^{(0,2)}\right)_{ij}^{kl}\right)\right] = 0$$
$$iQO_3^{(0,3)} \equiv s_3^{(0,2)} = 0$$
$$iQO_4^{(0,3)} \equiv s_{10}^{(0,2)} = 0$$

$$iQO_5^{(0,3)} \equiv 6s_1^{(0,2)} - 6s_4^{(0,2)} - s_6^{(0,2)} = 0$$
$$iQO_6^{(0,3)} \equiv 24s_2^{(0,2)} - 6s_{11}^{(0,2)} + s_{12}^{(0,2)} = 0$$
$$iQO_7^{(0,3)} \equiv s_1^{(0,2)} - 10s_2^{(0,2)} - 6s_4^{(0,2)} - 10s_8^{(0,2)} = 0$$
$$iQO_8^{(0,3)} \equiv 5s_1^{(0,2)} - 2s_2^{(0,2)} - 9s_4^{(0,2)} + 6s_9^{(0,2)} = 0$$
$$iQO_9^{(0,3)} \equiv 6s_1^{(0,2)} + 12s_2^{(0,2)} - 18s_4^{(0,2)} + s_{12}^{(0,2)} = 0$$
$$iQO_{10}^{(0,3)} \equiv 38s_1^{(0,2)} + 4s_2^{(0,2)} - 24s_4^{(0,2)} - 5s_{13}^{(0,2)} = 0$$

$$s_1^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i, s_2^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b}u^{a_2(j}v^{k)}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i,$$
$$s_3^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b\left(R_{10}^{(0,0)}\right)_{jk}^i, s_4^{(0,1)} = \epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}\left(R_{12}^{(0,0)}\right)_{kl}^{ij},$$
$$s_5^{(0,1)} = \epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}\left(R_{12}^{(0,0)}\right)_{kl}^{ij}, s_6^{(0,1)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(R_{12}^{(0,1)}\right)_{kl}^{ij},$$
$$s_7^{(0,1)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}\left(R_{14}^{(0,1)}\right)_k^{ij}, s_8^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}\left(R_{14}^{(0,1)}\right)_{jkl}^i.$$

$$iQO_1^{(0,2)} \equiv s_1^{(0,1)} - 2s_2^{(0,1)} = 0$$
$$iQO_2^{(0,2)} \equiv 6s_3^{(0,1)} + s_4^{(0,1)} = 0$$
$$iQO_3^{(0,2)} \equiv s_1^{(0,1)} + s_5^{(0,1)} = 0$$
$$iQO_4^{(0,2)} \equiv s_1^{(0,1)} + s_6^{(0,1)} = 0$$
$$iQO_5^{(0,2)} \equiv 4s_1^{(0,1)} + 24s_3^{(0,1)} - s_7^{(0,1)} = 0$$
$$iQO_6^{(0,2)} \equiv s_1^{(0,1)} - 12s_3^{(0,1)} + 3s_8^{(0,1)} = 0$$

$$s_1^{(0,0)} = \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2i}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}\left(R_{10}^{(0,0)}\right)_{jk}^i,$$
$$s_2^{(0,0)} = R_{12}^{(0,0)}R_{12}^{(0,0)},$$
$$s_3^{(0,0)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}\left(R_{12}^{(0,0)}\right)_{kl}^{ij},$$
$$s_4^{(0,0)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1(k}u^{l)a_2b_2}\left(R_{12}^{(0,0)}\right)_{kl}^{ij}.$$

$$iQO^{(0,1)} \equiv 36s_1^{(0,0)} + 5s_2^{(0,0)} - 6s_3^{(0,0)} = 0$$

$$p_1 = \epsilon^{a_1 a_2 a_3} v^{(i}{}_{a_1} v^{j}{}_{a_2} v^{k)}{}_{a_3} \left(R_{12}^{(0,1)}\right)_{ijk}, p_2 = v^{j}{}_{a} v^{a}{}_{b} v^{b}{}_{i} \left(R_{12}^{(0,1)}\right)_{j}^{i}, p_3 = v^{(k}{}_{a} v^{l)}{}_{(i} v^{a}{}_{j)} \left(R_{12}^{(0,1)}\right)_{kl}^{ij},$$

$$p_4 = u^{(jk} v^{l)}{}_{a} v^{a}{}_{i} \left(R_{14}^{(0,2)}\right)_{jkl}^{i}, p_5 = u^{a(j} v^{k}{}_{a} v^{l)}{}_{i} \left(R_{14}^{(0,2)}\right)^{i}{}_{kl}, p_6 = v^{(jk}{}_{a} v^{l)a}{}_{i} \left(R_{14}^{(0,2)}\right)_{jkl}^{i},$$

$$p_7 = v^{(k}{}_{(i} v^{lm)}{}_{j)} \left(R_{16}^{(0,2)}\right)_{klm}^{ij}, p_8 = v^{(j}{}_{a} v^{k)a}{}_{i} \left(R_{16}^{(0,2)}\right)_{jk}^{i}, p_9 = v^{a}{}_{i} v^{jk}{}_{a} \left(R_{16}^{(0,2)}\right)_{jk}^{i},$$

$$p_{10} = v^{ia}{}_{b} v^{b}{}_{a} \left(R_{16}^{(0,2)}\right)_{i}, p_{11} = \epsilon_{i a_1 a_2} u^{a_1 b} v^{a_2}{}_{b} \left(R_{20}^{(0,3)}\right)^{i}.$$

$$\begin{aligned}
&5p_1 - 10p_2 - 30p_4 + 8p_{11} = 0\\
&15p_1 + 6p_2 - 40p_3 - 30p_4 + 90p_5 = 0\\
&105p_1 - 336p_2 + 140p_3 - 1050p_4 + 10080p_6 - 900p_7 = 0\\
&15p_1 - 138p_2 - 160p_3 - 570p_4 + 864p_6 - 48p_8 = 0\\
&375p_1 + 6p_2 + 1760p_3 + 150p_4 + 4320p_6 - 120p_9 = 0\\
&55p_1 - 266p_2 - 160p_3 - 1050p_4 + 2880p_6 + 40p_{10} = 0
\end{aligned}$$

$$D_{+\dot{\alpha}} \equiv \partial_{+\dot{\alpha}} - i[A_{+\dot{\alpha}}]$$

$$ds^2 = -\frac{r^{2z}}{L^{2z}} dt^2 + \frac{L^2}{r^2} dr^2 + r^2 d\vec{x}_{n-1}^2,$$

$$t \to \lambda^z t, \vec{x} \to \lambda \vec{x}, r \to \lambda^{-1} r.$$

$$ds^2 = r^{-\frac{2\theta}{n-1}} \left(-\frac{r^{2z}}{L^{2z}} dt^2 + \frac{L^2}{r^2} dr^2 + r^2 d\vec{x}_{n-1}^2 \right)$$

$$F_{i\mu\nu}^{(a)} = \partial_\mu A_{i\nu}^{(a)} - \partial_\nu A_{i\mu}^{(a)} + \frac{1}{e_i} C_{bc}^{a} A_{i\mu}^{(b)} A_{i\nu}^{(c)}$$

$$S = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left(R - V(\Phi) - \frac{4}{n-1} \partial_\mu \Phi \partial^\mu \Phi - \sum_{i=1}^{3} e^{-4\xi_i \Phi/(n-1)} F_i^2 \right)$$

$$F_i^2 = \gamma_{ab} F_{i\mu\nu}^{(a)} F_i^{(b)\mu\nu}$$

$$\gamma_{ab} \equiv -\frac{\Gamma_{ab}}{|\det \Gamma_{ab}|^{1/N}}, \Gamma_{ab} = C_{ad}^{c} C_{bc}^{d}$$

$$R_{\mu\nu} = \frac{2}{n-1} \left[2\partial_\mu \Phi \partial_\nu \Phi + \frac{V(\Phi)}{2} g_{\mu\nu} \right] + 2 \sum_{i=1}^{3} e^{-4\xi_i \Phi/(n-1)} \left[\gamma_{ab} F_{i\mu}^{(a)\lambda} F_{i\nu\lambda}^{(b)} - \frac{1}{2(n-1)} F_i^2 g_{\mu\nu} \right]$$

$$D_\nu \left(e^{-4\xi_i \Phi/(n-1)} F_i^{(a)\mu\nu} \right) = \frac{1}{e_i} e^{-4\xi_i \Phi/(n-1)} C_{bc}^{a} A_{i\nu}^{(b)} F_i^{(c)\nu\mu}$$

$$\nabla^2 \Phi - \frac{n-1}{8} \frac{dV(\Phi)}{d\Phi} + \sum_{i=1}^{3} \frac{\xi_i}{2} e^{-4\xi_i \Phi/(n-1)} F_i^2 = 0$$

$$x_1 = \frac{r}{\sqrt{k}} \sin(\sqrt{k}\theta) \Pi_{j=1}^{n-2} \sin \phi_j,$$
$$x_l = \frac{r}{\sqrt{k}} \sin(\sqrt{k}\theta) \cos \phi_{n-l} \Pi_{j=1}^{n-l-1} \sin(\phi_j), l = 2, \ldots, n-1$$
$$x_n = r\cos(\sqrt{k}\theta),$$

$$A_i^{(a)} = \frac{e_i}{r^2}(x_l dx_n - x_n dx_l) \text{ for } a = l = 1, \ldots, n-1$$
$$A_i^{(b)} = \frac{e_i}{r^2}(x_l dx_j - x_j dx_l) \text{ for } b = n, \ldots, n(n-1)/2, l = 1, \ldots, n-2, j = 2, \ldots, n-1, \text{ and } l < j$$

$$ds^2 = -\frac{r^{2z}}{L^{2z}} f(r) dt^2 + \frac{L^2}{r^2 f(r)} dr^2 + r^2 \left[d\theta^2 + k^{-1} \sin^2(\sqrt{k}\theta) d\Omega_{k,n-2}^2 \right]$$

$$\Phi(r) = \frac{n-1}{2} \sqrt{z-1} \ln\left(\frac{r}{s}\right)$$

$$f(r) = 1 + \frac{kL^2(n-2)}{z(z+n-3)r^2} - \frac{m}{r^{z+n-1}} + \begin{cases} \dfrac{(n-2)L^2 e_3^2}{(z-n+3)r^{2z+2}} s^{2(z-1)}, & \text{for } z \neq n-3 \\ -\dfrac{(n-2)L^2 e_3^2}{r^{2z+2}} \ln\left(\dfrac{r}{r_0}\right) s^{2(z-1)}, & \text{for } z = n-3 \end{cases}$$

$$\xi_1 = -\frac{2}{\sqrt{z-1}}, \xi_2 = -\frac{1}{\sqrt{z-1}}, \xi_3 = \sqrt{z-1}$$
$$e_1^2 = -\frac{2(z-1)\Lambda}{(n-1)(n-2)(z+1)} s^4, e_2^2 = k\frac{z-1}{z} s^2$$
$$\Lambda = -\frac{(n-1)(z+1)(z+n-1)}{4L^2}$$

$$\Phi(r) = (n-1)\sqrt{z-1} \ln(r/s)/2$$

$$\sqrt{z-1} e^{-4\xi_i \Phi/(n-1)}$$

$$f(r) = 1 + \frac{kL^2}{r^2} - \frac{m}{r^n} + \begin{cases} -\dfrac{(n-2)L^2 e_3^2}{(n-4)r^4}, & \text{for } n \neq 4 \\ -\dfrac{(n-2)L^2 e_3^2}{r^4} \ln\left(\dfrac{r}{r_0}\right), & \text{for } n = 4 \end{cases}$$

$$m(r_+) = r_+^{z+n-1} + \frac{L^2(n-2) r_+^{z+n-3}}{z(z+n-3)} + \begin{cases} \dfrac{(n-2)L^2 e_3^2 r_+^{-z+n-3}}{(z-n+3)} s^{2(z-1)}, & \text{for } z \neq n-3 \\ -(n-2)L^2 e_3^2 r_+^{-z+n-3} \ln(r_+) s^{2(z-1)}, & \text{for } z = n-3 \end{cases}$$

$$ds^2 = -A(\mathcal{R}) dt^2 + \frac{d\mathcal{R}^2}{B(\mathcal{R})} + \mathcal{R}^2 d\Omega_{n-1}^2,$$

$$A(\mathcal{R}) = \frac{r(\mathcal{R})^{2z}}{L^{2z}} f(r(\mathcal{R})) \text{ and } B(\mathcal{R}) = \frac{r(\mathcal{R})^2}{L^2} f(r(\mathcal{R})).$$

$$M_T = \frac{1}{8\pi} \int_\Sigma d^{n-1} x \sqrt{\sigma} \{(K_{ab} - K\gamma_{ab})\} n^a \xi^b,$$

$$f_0(r(\mathcal{R})) = 1 + \frac{L^2(n-2)}{z(z+n-3)[r(\mathcal{R})]^2}.$$

$$M_0 = \frac{1}{8\pi}\int_\Sigma d^{n-1}x\sqrt{\sigma}\{(K^0_{ab} - K^0\gamma^0_{ab})\}n^a\xi^b$$

$$M = M_T - M_0 = \frac{(n-1)\omega_{n-1}}{16\pi L^{z+1}}m$$

$$T_+ = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{1}{2}(\nabla_\mu\chi_\nu)(\nabla_\nu\chi_\mu)}\Bigg|_{r=r_+} = \frac{r^{z+1}f'(r)}{4\pi L^{z+1}}\Bigg|_{r=r_+}$$
$$= \frac{(n-2)}{4\pi z L^{z-1}}r_+^{z-2} + \frac{(z+n-1)}{4\pi L^{z+1}}r_+^z - \frac{(n-2)e_3^2 s^{2(z-1)}}{4\pi L^{z-1}r_+^{z+2}}$$

$$S = \frac{A}{4} = \frac{r_+^{n-1}}{4}\omega_{n-1}$$

$$Q = \frac{1}{4\pi\sqrt{(n-1)(n-2)}}\int d^{n-1}x\sqrt{\mathrm{Tr}\left(F^{(a)}_{\mu\nu}F^{(a)}_{\mu\nu}\right)} = \frac{\omega_{n-1}}{4\pi L^{1-z}}e_3$$

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, U = \left(\frac{\partial M}{\partial Q}\right)_S$$

$$dM = TdS + UdQ$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_S = -\frac{(n-1)(n-2)s^{2z-2}}{2r_+^{z-n+3}L^{2z-2}}e_3 \times \begin{cases}\frac{1}{-z+n-3}, & \text{for } z \neq n-3\\ \ln(r_+), & \text{for } z = n-3\end{cases}$$

$$H = \begin{bmatrix}\left(\frac{\partial^2 M}{\partial S^2}\right)_Q & \left(\frac{\partial^2 M}{\partial S\partial Q}\right)\\ \left(\frac{\partial^2 M}{\partial Q\partial S}\right)^2 & \left(\frac{\partial^2 M}{\partial Q^2}\right)_S\end{bmatrix}$$

$$\det H = -\frac{2(n-2)^2 s^{2z-2}}{(-z+n-3)L^{4z-2}}\left(\frac{(z-2)L^2}{zr_+^4} + \frac{z(z+n-1)}{(n-2)r_+^2} + \frac{16(2n-z-4)\pi^2Q^2s^{2z-2}}{L^{2z-4}r_+^{2z+4}}\right), \text{ for } z \neq n-3$$

$$\det H = -\frac{2(n-2)^2 s^{2z-2}\ln(r_+)}{L^{4z-2}}\left(\frac{(z-2)L^2}{zr_+^4} + \frac{z(z+n-1)}{(n-2)r_+^2} + \frac{16(2n-z-4)\pi^2Q^2s^{2z-2}}{L^{2z-4}r_+^{2z+4}}\right)$$
$$-\frac{64(n-2)^2\pi^2Q^2s^{4z-4}}{L^{6z-6}r_+^{2z+4}} \quad \text{for } z = n-3$$

$$dM = TdS + VdP + \Psi dQ^2,$$

$$\Psi = \left(\frac{\partial M}{\partial Q^2}\right)_{S,P} = -\frac{(n-1)(n-2)\pi s^{2z-2}}{(-z+n-3)L^{3z-3}}r_+^{-z+n-3}$$

$$P = \frac{n(n-1)r_+^{z-1}}{16\pi L^{z+1}}.$$

$$(z+n-3)M = (n-1)TS - 2PV + 2z\Psi Q^2, \text{ for } z \neq n-3$$

$$Q^2 = -\frac{L^{3z-3}T}{4\pi(n-2)s^{2z-2}}\left(\frac{(z-n+3)L^{3z-3}\Psi}{(n-1)(n-2)\pi s^{2z-2}}\right)^{\frac{z+2}{-z+n-3}} + \frac{L^{2z-2}}{16\pi^2 z s^{2z-2}}\left(\frac{(z-n+3)L^{3z-3}\Psi}{(n-1)(n-2)\pi s^{2z-2}}\right)^{\frac{2z}{-z+n-3}} + \frac{(z+n-1)L^{2z-4}}{16\pi^2(n-2)s^{2z-2}}\left(\frac{(z-n+3)L^{3z-3}\Psi}{(n-1)(n-2)\pi s^{2z-2}}\right)^{\frac{2(z+1)}{-z+n-3}}$$

$$\left.\frac{\partial Q^2}{\partial \Psi}\right|_{T_C} = 0\,, \left.\frac{\partial^2 Q^2}{\partial \Psi^2}\right|_{T_C} = 0,$$

$$\Psi_C = -\frac{(n-1)(n-2)\pi s^{2z-2}}{(-z+n-3)L^{3z-3}}\left(-\frac{(z-2)(n-2)L^2}{z(z+1)(z+n-1)}\right)^{\frac{-z+n-3}{2}}$$

$$T_C = -\frac{(z+1)(z+n-1)}{(z-2)(z+2)\pi L^{z+1}}\left(-\frac{(z-2)(n-2)L^2}{z(z+1)(z+n-1)}\right)^{\frac{z}{2}}$$

$$G(Q^2,T) = M - TS = \frac{(z+2)T}{4(-z+n-3)}r_+^{n-1} - \frac{(n-1)(n-2)}{8\pi(z+n-3)(-z+n-3)L^{z-1}}r_+^{z+n-3} - \frac{(n-1)(z+1)}{8\pi(-z+n-3)L^{z+1}}r_+^{z+n-1}, \text{ for } z \neq n-3$$

$$ds^2 = r^{2\alpha}\left(-\frac{f(r)r^{2z}}{L^{2z}}dt^2 + \frac{L^2dr^2}{r^2 f(r)} + r^2 d\Omega_k^2\right),$$

$$V(\phi) = 2\Lambda e^{\lambda\phi},$$

$$\Phi(r) = \frac{n-1}{2}\sqrt{(\alpha+1)(\alpha+z-1)}\ln\left(\frac{r}{b}\right)$$

$$f(r) = 1 + \frac{L^2(n-2)}{(z+2\alpha)(z+n-3+(n-1)\alpha)r^2} - \frac{m}{r^{z+(n-1)(\alpha+1)}} + \begin{cases} \dfrac{(n-2)L^2e_3^2}{(\alpha+1)(z-n+3-(n-5)\alpha)r^{2z+2+4\alpha}}b^{2(\alpha+z-1)}, & \text{for } z \neq n-3+(n-5)\alpha \\ -\dfrac{(n-2)L^2e_3^2}{(\alpha+1)r^{2z+2+4\alpha}}\ln\left(\dfrac{r}{r_0}\right)b^{2(\alpha+z-1)}, & \text{for } z = n-3+(n-5)\alpha \end{cases}$$

$$\xi_1 = -\frac{(n-1)\lambda\sqrt{(\alpha+1)(\alpha+z-1)} + 8(\alpha+1)}{4\sqrt{(\alpha+1)(\alpha+z-1)}}, \xi_2 = -\sqrt{\frac{\alpha+1}{\alpha+z-1}}, \xi_3 = \sqrt{\frac{\alpha+z-1}{\alpha+1}}$$

$$e_1^2 = -\frac{2(z-1)\Lambda}{(n-1)(n-2)(2\alpha+z+1)} b^{4(\alpha+1)}, e_2^2 = \frac{\alpha+z-1}{z+2\alpha} b^{2(\alpha+1)}$$

$$\Lambda = -\frac{(n-1)(2\alpha+z+1)(z+(n-1)(\alpha+1))}{4L^2} b^{-2\alpha}$$

$$\lambda = -\frac{4\alpha}{(n-1)\sqrt{(\alpha+1)(\alpha+z-1)}}.$$

$$\Phi(r) = \frac{1}{2}\sqrt{(n-\theta-1)[(z-1)(n-1)-\theta]}\ln\left(\frac{r}{b}\right),$$

$$f(r) = 1 + \frac{L^2(n-1)(n-2)}{[(n-1)z-2\theta](z+n-3-\theta)r^2} - \frac{m}{r^{z+n-\theta-1}}$$

$$+\begin{cases} \frac{(n-1)^2(n-2)L^2e_3^2}{(\theta-n+1)[(n-z-3)(n-1)-(n-5)\theta]r^{2(z+1)-\frac{4\theta}{n-1}}} b^{2(z-1)-\frac{2\theta}{n-1}}, & \text{for } z \neq n-3-\frac{n-5}{n-1}\theta \\ \frac{(n-1)(n-2)L^2e_3^2}{(\theta-n+1)r^{2(z+1)-\frac{4\theta}{n-1}}} b^{2(z-1)-\frac{2\theta}{n-1}}\ln(r), & \text{for } z = n-3-\frac{n-5}{n-1}\theta \end{cases}$$

$$m(r_+) = r_+^{z+n-\theta-1} + \frac{L^2(n-1)(n-2)}{[(n-1)z-2\theta](z+n-3-\theta)} r_+^{z+n-\theta-3}$$

$$+\begin{cases} \frac{(n-1)^2(n-2)L^2e_3^2}{(\theta-n+1)[(n-z-3)(n-1)-(n-5)\theta]} r_+^{n-z-\theta-3+\frac{4\theta}{n-1}} b^{2(z-1)-\frac{2\theta}{n-1}}, \\ \frac{(n-1)(n-2)L^2e_3^2}{(\theta-n+1)} r_+^{n-z-\theta-3+\frac{4\theta}{n-1}} b^{2(z-1)-\frac{2\theta}{n-1}}\ln(r_+), \end{cases} \text{for } z \neq n-3-\frac{n-5}{n-1}\theta$$

$$M = \frac{(n-1-\theta)}{16\pi L^{z+1}} m$$

$$T_+ = \left.\frac{r^{z+1}f'(r)}{4\pi L^{z+1}}\right|_{r=r_+} = \frac{(z+n-\theta-1)r_+^z}{4\pi L^{z+1}} + \frac{(n-1)(n-2)r_+^{z-2}}{4\pi[z(n-1)-2\theta]L^{z-1}} + \frac{(n-1)(n-2)e_3^2 b^{2(z-1)-\frac{2\theta}{n-1}}}{4\pi(\theta-n+1)L^{z-1}r_+^{z+2-\frac{4\theta}{n-1}}}$$

$$S = \frac{r_+^{n-1-\theta}}{4}$$

$$Q = \frac{\omega_{n-1}}{4\pi L^{1-z}} e_3$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_S = -\frac{(n-1)(n-2)b^{2z-2-\frac{2\theta}{n-1}}}{2r_+^{z-n+3+\frac{n-5}{n-1}\theta} L^{2z-2}} e_3 \times \begin{cases} \frac{n-1}{(n-1)(n-z-3)-\theta(n-5)}, & \text{for } z \neq n-3 \\ \ln(r_+), & \text{for } z = n-3 \end{cases}$$

$$\mathrm{Det}H=-\frac{2(n-1)^2(n-2)^2 b^{2z-2-\frac{2\theta}{n-1}} r_+^{\frac{4\theta}{n-1}}}{[(n-1)(n-z-3)-\theta(n-5)](n-\theta-1)L^{4z-2}}\left(\frac{(n-1)(z-2)L^2}{[(n-1)z-2\theta]r_+^4}+\frac{z(z+n-\theta-1)}{(n-2)r_+^2}\right.$$
$$\left.+\frac{16[(n-1)(2n-z-4)-2\theta(n-3)]\pi^2 Q^2 b^{2z-2-\frac{2\theta}{n-1}}}{(n-\theta-1)L^{2z-4} r_+^{2z+4-\frac{4\theta}{n-1}}}\right),\ \text{for } z\neq n-3-\frac{n-5}{n-1}\theta$$

$$\mathrm{Det}H=-\frac{64(n-1)^2(n-2)^2\pi^2 Q^2 b^{4z-4-\frac{4\theta}{n-1}}}{(n-\theta-1)^2 L^{6z-6} r_+^{2z+4-\frac{8\theta}{n-1}}}-\frac{2(n-1)(n-2)^2 b^{2z-2-\frac{2\theta}{n-1}} r_+^{\frac{4\theta}{n-1}}\ln(r_+)}{(n-\theta-1)L^{4z-2}}\left(\frac{(n-1)(z-2)L^2}{[z(n-1)-2\theta]r_+^4}+\right.$$
$$\left.\frac{z(z+n-\theta-1)}{(n-2)r_+^2}+\frac{16[(n-1)(2n-z-4)-2\theta(n-3)]\pi^2 Q^2 b^{2z-2-\frac{2\theta}{n-1}}}{(n-\theta-1)L^{2z-4} r_+^{2z+4-\frac{4\theta}{n-1}}}\right),\ \text{for } z=n-3-\frac{n-5}{n-1}\theta.$$

$$\mathrm{Det}(H)z>n-3-\frac{n-5}{n-1}\theta$$

$$\mathrm{Det}(H)(n-1)(2n-z-4)-2\theta(n-3)$$

$$\mathrm{Det}(H),\ d^2M/dS^2$$

$$z=n-3-\frac{n-5}{n-1}\theta$$

$$\Psi=\left(\frac{\partial M}{\partial Q^2}\right)_{S,P}=-\frac{(n-1)^2(n-2)\pi b^{2z-2-\frac{2\theta}{n-1}}}{(n^2-n(\theta+z+4)+z+5\theta+3)L^{3z-3}} r_+^{-z+n-3-\frac{\theta(n-5)}{n-1}}$$

$$P=\frac{(n-\theta)(n-\theta-1)r_+^{z-1}}{16\pi L^{z+1}}$$

$$(z+n-\theta-3)M=(n-\theta-1)TS-2PV+\frac{2[(n-1)z-2\theta]}{n-1}\Psi Q^2\ \ \text{for } z\neq n-3-\frac{n-5}{n-1}\theta$$

$$Q^2=-\frac{(n-\theta-1)L^{3z-3}T}{4\pi(n-1)(n-2)b^{2z-2-\frac{2\theta}{n-1}}} r_+^{z+2-\frac{4\theta}{n-1}}+\frac{(n-\theta-1)L^{2z-2}}{16\pi^2((n-1)z-2\theta)b^{2z-2-\frac{2\theta}{n-1}}} r_+^{2z-\frac{4\theta}{n-1}}$$
$$+\frac{(z+n-\theta-1)(n-\theta-1)L^{2z-4}}{16\pi^2(n-1)(n-2)b^{2z-2-\frac{2\theta}{n-1}}} r_+^{2z+2-\frac{4\theta}{n-1}}$$

$$G=M-TS$$

$$\Psi_C=-\frac{(n-1)^2(n-2)\pi b^{2(z-1)-\frac{2\theta}{n-1}}}{[(-z+n-3)(n-1)-\theta(n-5)]L^{3z-3}}\left(-\frac{(n-1)(n-2)(z-2)L^2}{z((z+1)(n-1)-2\theta)(z+n-\theta-1)}\right)^{\frac{-z+n-3}{2}-\frac{(n-5)\theta}{2(n-1)}}$$

$$T_C=-\frac{((n-1)(z+1)-2\theta)(z+n-\theta-1)}{(z-2)((n-1)(z+2)-4\theta)\pi L^{z+1}}\left(-\frac{(n-1)(n-2)(z-2)L^2}{z((n-1)(z+1)-2\theta)(z+n-\theta-1)}\right)^{\frac{z}{2}}$$

$$ds^2=-\frac{r^{2z}}{L^{2z}}f(r)dt^2+\frac{L^2}{r^2 f(r)}dr^2+r^2 d\theta^2+r^2\begin{cases}\sin^2\theta d\phi^2, & \text{for } k=1\\ \sinh^2\theta d\phi^2, & \text{for } k=-1\end{cases}$$

$$x_1 = r\sin\,\theta\cos\,\phi,$$
$$x_2 = r\sin\,\theta\sin\,\phi,$$
$$x_3 = r\cos\,\theta,$$

$$A^{(1)} = \frac{e}{r^2}(x_1dx_3 - x_3dx_1)$$
$$A^{(2)} = \frac{e}{r^2}(x_2dx_3 - x_3dx_2)$$
$$A^{(3)} = \frac{e}{r^2}(x_1dx_2 - x_2dx_1)$$

$$A_\mu^{(1)} = e(-\cos\,\phi d\theta + \sin\,\theta\cos\,\theta\sin\,\phi d\phi)$$
$$A_\mu^{(2)} = -e(\sin\,\phi d\theta + \sin\,\theta\cos\,\theta\cos\,\phi d\phi)$$
$$A_\mu^{(3)} = e\sin^2\,\theta d\phi$$

$$C_{23}^1 = C_{31}^2 = C_{12}^3 = -1$$

$$\gamma_{ab} = \mathrm{diag}(1,1,1)$$

$$A_\mu^{(1)} = e(-\cos\,\phi d\theta + \sinh\,\theta\cosh\,\theta\sin\,\phi d\phi)$$
$$A_\mu^{(2)} = -e(\sin\,\phi d\theta + \sinh\,\theta\cosh\,\theta\cos\,\phi d\phi)$$
$$A_\mu^{(3)} = e\sinh^2\,\theta d\phi$$

$$C_{23}^1 = C_{31}^2 = -C_{12}^3 = 1$$

$$\gamma_{ab} = \mathrm{diag}(-1,-1,1)$$

$$ds^2 = -\frac{r^{2z}}{L^{2z}}f(r)dt^2 + \frac{L^2}{r^2f(r)}dr^2 + r^2d\theta^2 + r^2\begin{cases}\sin^2\,\theta(d\phi^2 + \sin^2\,\phi d\psi^2), & \text{for } k = 1\\ \sinh^2\,\theta(d\phi^2 + \sin^2\,\phi d\psi^2), & \text{for } k = -1\end{cases}$$

$$A_\mu^{(1)} = -e(\sin\,\phi\cos\,\psi d\theta + \sin\,\theta\cos\,\theta(\cos\,\phi\cos\,\psi d\phi - \sin\,\phi\sin\,\psi d\psi))$$
$$A_\mu^{(2)} = -e(\sin\,\phi\sin\,\psi d\theta + \sin\,\theta\cos\,\theta(\cos\,\phi\sin\,\psi d\phi + \sin\,\phi\cos\,\psi d\psi))$$
$$A_\mu^{(3)} = -e(\cos\,\phi d\theta - \sin\,\theta\cos\,\theta\sin\,\phi d\phi)$$
$$A_\mu^{(4)} = -e\sin^2\,\theta\sin^2\,\phi d\psi$$
$$A_\mu^{(5)} = e\sin^2\,\theta(\cos\,\psi d\phi - \sin\,\phi\cos\,\phi\sin\,\psi d\psi)$$
$$A_\mu^{(6)} = e\sin^2\,\theta(\sin\,\psi d\phi + \sin\,\phi\cos\,\phi\cos\,\psi d\psi)$$

$$C_{24}^1 = C_{35}^1 = C_{41}^2 = C_{36}^2 = C_{51}^3 = C_{62}^3 = 1, C_{56}^4 = -C_{21}^4 = C_{64}^5 = -C_{31}^5 = C_{45}^6 = -C_{32}^6 = 1,$$

$$A_\mu^{(1)} = -e(\sin\,\phi\cos\,\psi d\theta + \sinh\,\theta\cosh\,\theta(\cos\,\phi\cos\,\psi d\phi - \sin\,\phi\sin\,\psi d\psi))$$
$$A_\mu^{(2)} = -e(\sin\,\phi\sin\,\psi d\theta + \sinh\,\theta\cosh\,\theta(\cos\,\phi\sin\,\psi d\phi + \sin\,\phi\cos\,\psi d\psi))$$
$$A_\mu^{(3)} = -e(\cos\,\phi d\theta - \sinh\,\theta\cosh\,\theta\sin\,\phi d\phi)$$
$$A_\mu^{(4)} = e\sinh^2\,\theta\sin^2\,\phi d\psi$$
$$A_\mu^{(5)} = -e\sinh^2\,\theta(\cos\,\psi d\phi - \sin\,\phi\cos\,\phi\sin\,\psi d\psi)$$
$$A_\mu^{(6)} = -e\sinh^2\,\theta(\sin\,\psi d\phi + \sin\,\phi\cos\,\phi\cos\,\psi d\psi)$$

$$C^1_{24} = C^1_{35} = C^2_{41} = C^2_{36} = C^3_{51} = C^3_{62} = 1, C^4_{56} = C^4_{21} = C^5_{64} = C^5_{31} = C^6_{45} = C^6_{32} = 1$$

$$\gamma_{ab} = \mathrm{diag}(-1,-1,-1,1,1,1)$$

$$S_G = \frac{1}{2}\int \sqrt{-\det(g_{\mu\nu})}\left[R - \frac{\Lambda(D-2)(D-1)}{3} + \alpha\mathcal{L}_{GB} - \mathcal{F}^q\right] d^D x$$

$$\mathcal{L}_{GB} = R^2 - 4R_{\gamma\zeta}R^{\gamma\zeta} + R_{\gamma\zeta\pi\sigma}R^{\gamma\zeta\pi\sigma}$$

$$\mathcal{F} = \sum_{a=1}^{\frac{(D-1)(D-2)}{2}} \mathrm{Tr}\left(F^{(a)}_{\lambda\sigma}F^{\lambda\sigma(a)}\right)$$

$$F^{(a)}_{\gamma\zeta} = \partial_\gamma A^{(a)}_\zeta - \partial_\zeta A^{(a)}_\gamma + \frac{1}{2\sigma}C^{(a)}_{(b)(c)}A^{(b)}_\gamma A^{(c)}_\zeta,$$

$$d\left({}^*\mathbf{F}^{(a)}\mathcal{F}^{q-1}\right) + \frac{1}{\sigma}C^{(a)}_{(b)(c)}\mathcal{F}^{q-1}\left(\mathbf{A}^{(b)} \wedge \mathbf{F}^{(c)}\right) = 0$$

$$G_{\mu\nu} + \frac{(D-2)(D-1)}{3}\Lambda g_{\mu\nu} + \alpha H_{\mu\nu} = T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$H_{\mu\nu} = 2\left(RR_{\mu\nu} - 2R_{\mu\sigma}R^\sigma_\nu - 2R^{\sigma\delta}R_{\mu\sigma\nu\delta} + R^{\sigma\delta\lambda}_\mu R_{\nu\sigma\delta\lambda}\right) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{\mathcal{GB}}$$

$$T^\mu_\nu = 2q\mathcal{F}^{q-1}\mathrm{Tr}\left(F^{(a)}_{\nu\delta}F^{\mu\delta(a)}\right) - \frac{1}{2}\delta^\mu_\nu\mathcal{F}^q$$

$$\mathbf{A}^{(a)} = \frac{Q}{r^2}\left(x_i dx_j - x_j dx_i\right)$$

$$r^2 = \sum_{p=1}^{D-1} x_p^2$$

$$2 \le 2a \le (D-1)(D-2)$$

$$1 \le j \le i-1 \le D-2$$

$$g_{\gamma\zeta}dx^\gamma dx^\zeta = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(h_{rs}dx^r dx^s)$$

$$f(r) = 1 + \frac{r^2\left[1 \pm \sqrt{1 + 2\alpha\left(\frac{4M}{r^3} - \frac{(2Q^2)^q}{(4q-3)r^{4q}}\right)}\right]}{2\alpha}$$

$$f(r) = 1 + \frac{r^2\left[1 \pm \sqrt{1 + \frac{4\alpha}{r^3}\left(2M - \frac{Q^2}{r}\right)}\right]}{2\alpha}.$$

$$f(r) = 1 + \frac{r^2}{2\alpha}\left[1 \pm \sqrt{1 + \frac{8\alpha M}{r^3}}\right]$$

$$f(r) = 1 - \frac{2M}{r} + \frac{2^{q-1}Q^{2q}}{(4q-3)r^{4q-2}},$$

$$\left[\frac{d}{dr}(\varrho(r))\right]_{r=r_{ph}} = 0$$

$$\varrho(r) = \frac{g_{22}}{g_{00}} = \frac{r^2}{f(r)}$$

$$dt - du = \frac{dr}{f(r)}$$

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\Omega_2^2,$$

$$\begin{aligned} l^a &= \delta_r^a \\ n^a &= \delta_u^a - \frac{f(r)}{2}\delta_r^a \\ m^a &= \frac{1}{\sqrt{2}r}\left(\delta_\theta^a + \frac{i}{\sin\,\theta}\delta_\phi^a\right) \end{aligned}$$

$$g^{ab} = -l^a n^b - n^a l^b + m^a \bar{m}^b + \bar{m}^a m^b,$$

$$\begin{aligned} l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0, \\ l_a m^a = l_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0, \\ -l_a n^a = -l^a n_a = m_a \bar{m}^a = m^a \bar{m}_a = 1. \end{aligned}$$

$$\begin{aligned} u' - u &\to -ia\cos\,\theta, \\ r' - r &\to ia\cos\,\theta. \end{aligned}$$

$$\begin{aligned} f(r) &\to \mathcal{A}(r,\theta,a), \\ r^2 &\to \mathcal{B}(r,\theta,a). \end{aligned}$$

$$\begin{aligned} l'^a &= \delta_r^a \\ n'^a &= \delta_u^a - \frac{\mathcal{A}}{2}\delta_r^a \\ m'^a &= \frac{1}{\sqrt{2\mathcal{B}}}\left(ia\sin\,\theta(\delta_u^a - \delta_r^a) + \delta_\theta^a + i\csc\,\theta\delta_\phi^a\right) \end{aligned}$$

$$\begin{aligned} ds^2 =& -\mathcal{A}du^2 - 2dudr + 2a(\mathcal{A}-1)\sin^2\,\theta dud\phi + \mathcal{B}d\theta^2 + 2a\sin^2\,\theta drd\phi \\ &+(\mathcal{B}\sin^2\,\theta - (\mathcal{A}-2)a^2\sin^4\,\theta)d\phi^2 \end{aligned}$$

$$\begin{aligned} dt - du &= \frac{a^2 + r^2}{a^2 + r^2 f(r)}dr \\ d\phi' - d\phi &= \frac{a}{a^2 + r^2 f(r)}dr \end{aligned}$$

$$\mathcal{A} = \frac{r^2 f(r) + a^2\cos^2\ \theta}{\mathcal{B}}, \mathcal{B} = a^2\cos^2\ \theta + r^2$$

$$ds^2 = \frac{a^2\sin^2\ \theta - \Delta(r)}{\rho^2}dt^2 + \frac{\rho^2}{\Delta(r)}dr^2 + \rho^2 d\theta^2 - \frac{\Delta(r)a^2\sin^4\ \theta - (r^2+a^2)^2\sin^2\ \theta}{\rho^2}d\phi^2 - \frac{2a\sin^2\ \theta(a^2+r^2-\Delta(r))}{\rho^2}dtd\phi$$

$$\Delta(r) = a^2 + r^2 f(r) = a^2 + r^2 + \frac{r^4\left[1 \pm \sqrt{1 + 2\alpha\left(\frac{4M}{r^3} - \frac{(2Q^2)^q}{(4q-3)r^{4q}}\right)}\right]}{2\alpha}$$
$$\rho^2 = r^2 + a^2\cos^2\ \theta$$

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$p_\mu = g_{\mu\nu}\dot{x}^\nu$$

$$E := -\frac{\partial\mathcal{L}}{\partial\dot{t}} = -g_{tt}\dot{t} - g_{\phi t}\dot{\phi} = p_t$$
$$l := \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi} = p_\phi$$

$$\dot{x} = \frac{\partial x}{\partial\tau}$$

$$2\frac{\partial\mathcal{S}_\mathcal{J}}{\partial\tau} + g^{\mu\nu}\frac{\partial\mathcal{S}_\mathcal{J}}{\partial x^\mu}\frac{\partial\mathcal{S}_\mathcal{J}}{\partial x^\nu} = 0$$

$$\mathcal{S}_\mathcal{J} = \frac{1}{2}m_p\tau - Et + l\phi + \mathcal{A}_r(r) + \mathcal{A}_\theta(\theta)$$

$$\rho^4\dot{t}^2 = \left[\frac{E(r^2+a^2)^2 - al(r^2+a^2)}{\Delta} + a\sin^2\ \theta(l\csc^2\ \theta - aE)\right]^2$$
$$\rho^4\dot{r}^2 = \mathcal{R}$$
$$\rho^4\dot{\theta}^2 = \Theta$$
$$\rho^4\dot{\phi}^2 = \left[\frac{aE(r^2+a^2) - la^2}{\Delta} + (l\csc^2\ \theta - aE)\right]^2$$

$$\mathcal{R}(r) = (al - (a^2+r^2)E)^2 - \Delta\mathcal{Z} - \Delta(aE-l)^2$$
$$\Theta(\theta) = \cot^2\ \theta(a^2E^2\sin^2\ \theta - l^2) + \mathcal{Z}$$

$$\mathcal{Z} + (aE-l)^2$$

$$V_{eff} = -\frac{\mathcal{R}(r)}{2r^4}$$

$$K_E = \frac{16r^2\Delta(r)}{(\Delta'(r))^2}, L_E = \frac{a^2+r^2}{a} - \frac{4r\Delta(r)}{a\Delta'(r)},$$

$$L_E = l/E$$

$$K_E = (\mathcal{Z} + (l - aE)^2)/E^2$$

$$\xi = l/E$$

$$\eta = \mathcal{Z}/E^2$$

$$\xi = \frac{r^2 + a^2}{a} - \frac{4r\Delta(r)}{a\Delta'(r)}$$

$$\eta = \frac{r^2(8r\Delta(r)\Delta'(r) - 16\Delta(r)(\Delta(r) - a^2) - r^2(\Delta'(r))^2)}{a^2(\Delta'(r))^2}$$

$$\mu = -\lim_{r\to\infty}\left[\frac{d\phi}{dr} r^2 \sin\,\theta\right]_{\theta\to\theta_0}$$

$$\lambda = \lim_{r\to\infty}\left[\frac{d\theta}{dr} r^2\right]_{\theta\to\theta_0}$$

$$\mu = -\xi \csc\,\theta_0$$

$$\lambda = \pm\sqrt{\eta + a^2\cos^2\,\theta_0 - \xi^2\cot^2\,\theta_0}$$

$$\mu = -\xi$$

$$\lambda = \pm\sqrt{\eta}$$

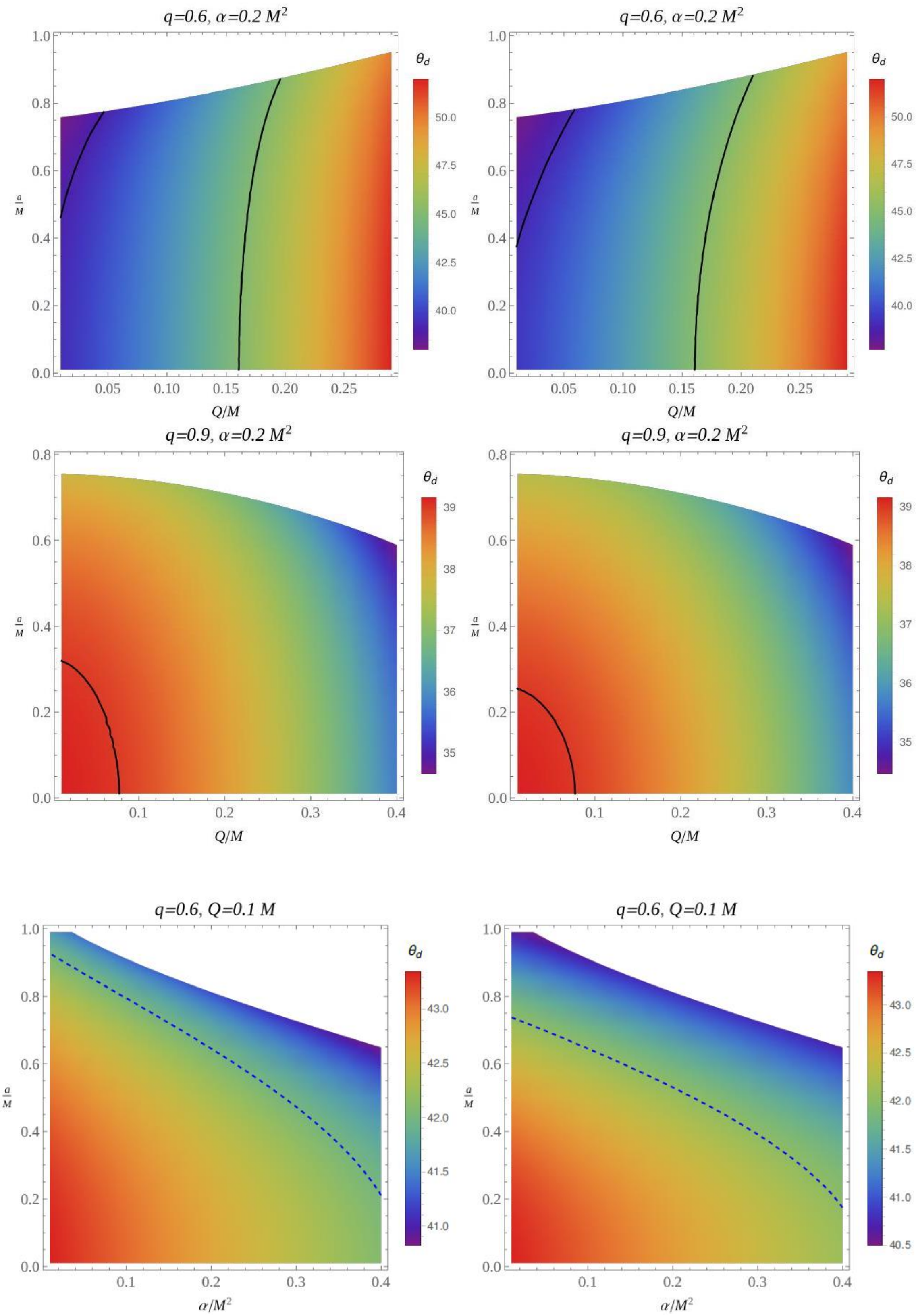
$$R_{sh} = \frac{(\mu_t - \mu_r)^2 + \lambda_t^2}{2|\mu_t - \mu_r|}$$

$$\delta_s = \frac{|\bar{\mu}_l - \mu_l|}{R_{sh}},$$

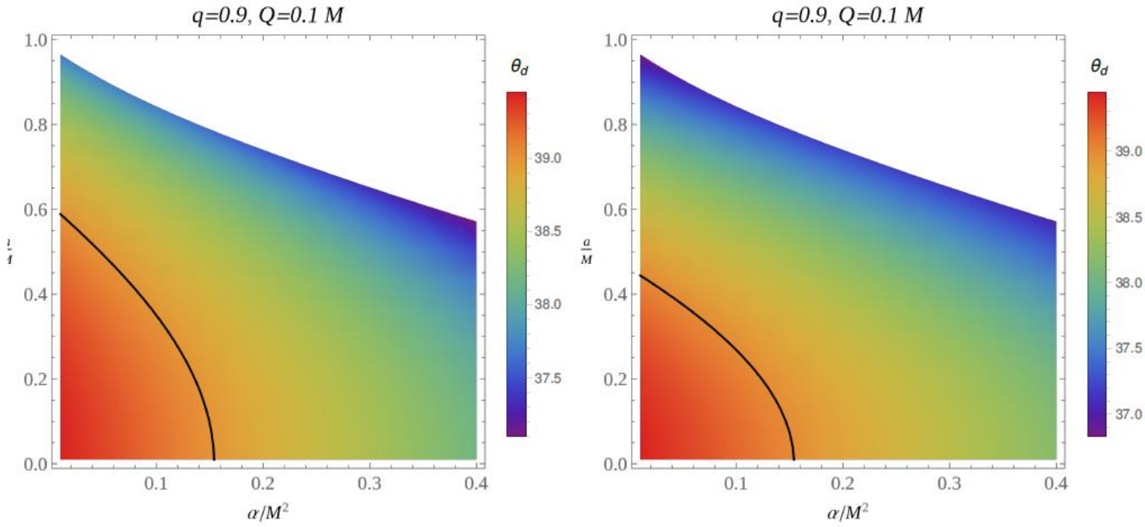
$$A = 2\int_{r_-}^{r_+} \left(\beta(r)\frac{d\alpha(r)}{dr}\right) dr$$

$$D = \frac{\Delta\alpha}{\Delta\beta}$$

$$\theta_d = \frac{2R_a}{d}, R_a^2 = \frac{A}{\pi}$$



q=0.6, α=0.2 M²
q=0.6, α=0.2 M²
q=0.9, α=0.2 M²
q=0.9, α=0.2 M²
q=0.6, Q=0.1 M
q=0.6, Q=0.1 M
θ_d
a/M
Q/M
α/M²

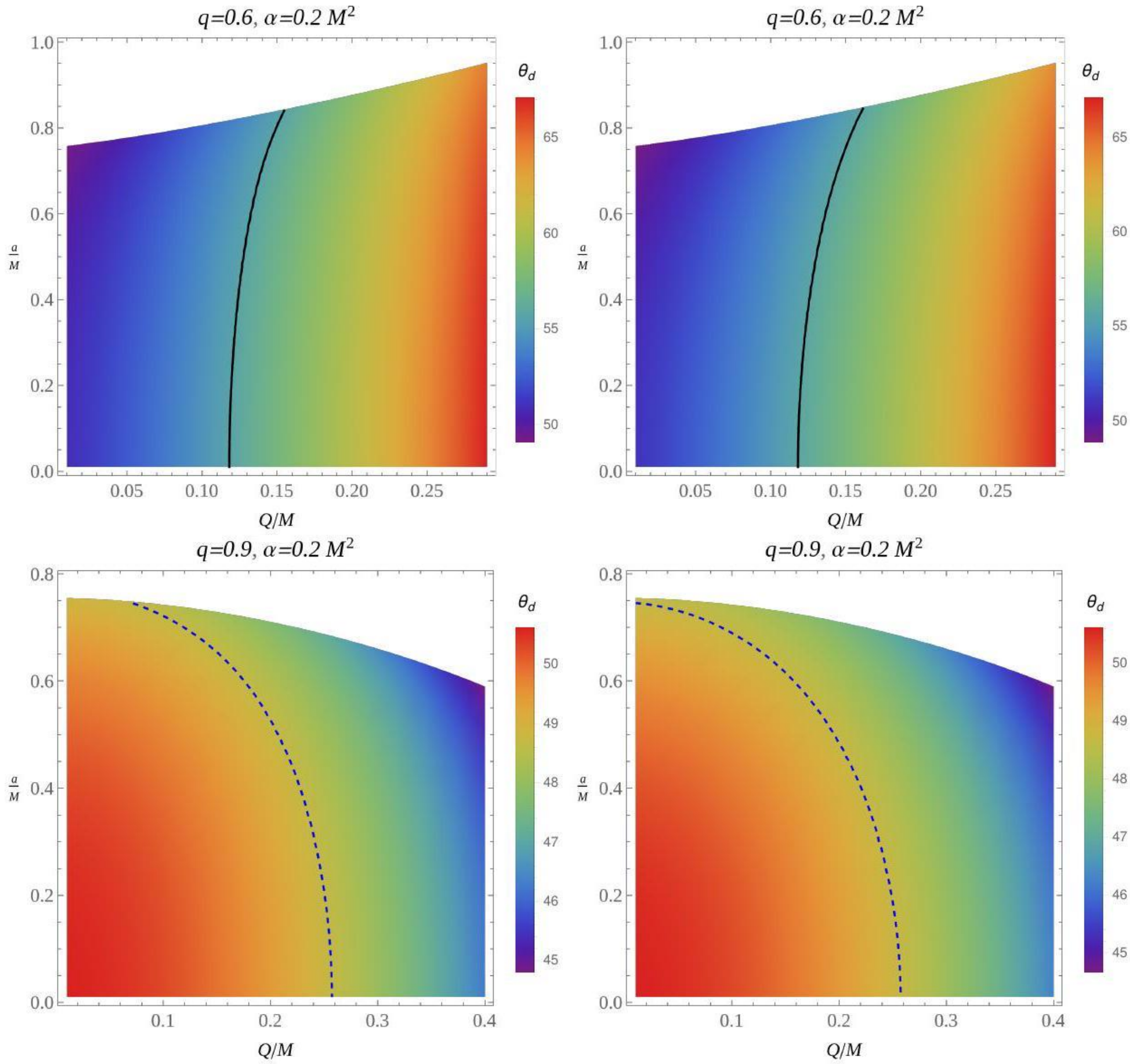


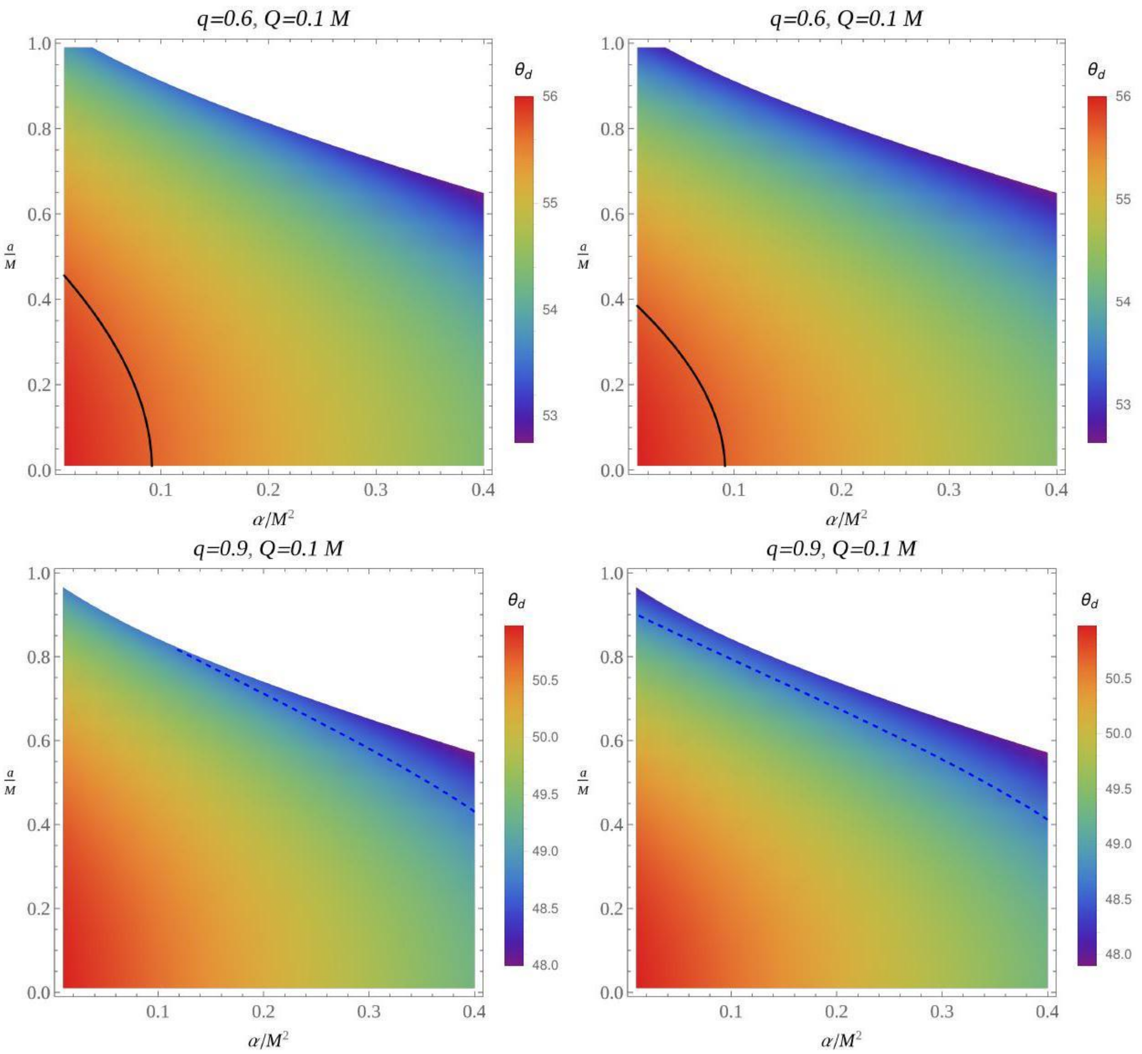
$$\sigma_{lim} \approx \pi R_{sh}^2.$$

$$\mathcal{E}_{\omega t} := \frac{d^2\mathcal{E}(\omega)}{d\omega dt} = \frac{2\pi^2\sigma_{lim}\omega^3}{e^{\omega/T_H}-1} \approx \frac{2\pi^3 R_{sh}^2\omega^3}{e^{\omega/T_H}-1},$$

$$\kappa = \left.\frac{\Delta'(r)}{2(a^2+r^2)}\right|_{r=r_+}$$

$$\kappa = \left.\frac{1}{2}f'(r)\right|_{r=r_+}.$$


q=0.6, α=0.2 M²
q=0.6, α=0.2 M²
q=0.9, α=0.2 M²
q=0.9, α=0.2 M²
θ_d
a/M
Q/M



$$S=\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^{n+1}x\sqrt{-g}\left(R-\frac{4}{n-1}\nabla^{\mu}\Phi\nabla_{\mu}\Phi-V(\Phi)-e^{-4\alpha\Phi/(n-1)}\mathrm{Tr}\left(F_{\mu\nu}^{(a)}F^{(a)\mu\nu}\right)\right)+\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{n}x\sqrt{-h}K$$

$$F_{\mu\nu}^{(a)}=\partial_{\mu}A_{\nu}^{(a)}-\partial_{\nu}A_{\mu}^{(a)}+\frac{1}{2\bar{\kappa}}C_{(b)(c)}^{(a)}A_{\mu}^{(b)}A_{\nu}^{(c)}$$

$$R_{\mu\nu}=\frac{g_{\mu\nu}}{n-1}\left(V(\Phi)-e^{-4\alpha\Phi/(n-1)}\mathrm{Tr}\left(F_{\rho\sigma}^{(a)}F^{(a)\rho\sigma}\right)\right)+\frac{4}{n-1}\partial_{\mu}\Phi\partial_{\nu}\Phi+2e^{-4\alpha\Phi/(n-1)}\mathrm{Tr}\left(F_{\mu\sigma}^{(a)}F_{\nu}^{(a)\sigma}\right)$$

$$\nabla_{\mu}\nabla^{\mu}\Phi=\frac{n-1}{8}\frac{\partial V}{\partial\Phi}-\frac{\alpha}{2}e^{-4\alpha\Phi/(n-1)}\mathrm{Tr}\left(F_{\rho\sigma}^{(a)}F^{(a)\rho\sigma}\right)$$

$$\nabla_{\mu}\left(e^{-4\alpha\Phi/(n-1)}F^{(a)\mu\nu}\right)+\frac{1}{\bar{\kappa}}e^{-4\alpha\Phi/(n-1)}C_{(b)(c)}^{(a)}A_{\mu}^{(b)}F^{(c)\mu\nu}=0$$

$$ds^{2}=-W(r)dt^{2}+\frac{dr^{2}}{W(r)}+r^{2}R^{2}(r)d\Omega_{n-1}^{2}$$

$$\mathbf{A}^{(a)}=\frac{q}{r^{2}}C_{(i)(j)}^{(a)}x^{i}dx^{j},r^{2}=\sum_{j=1}^{n}x_{j}^{2}$$

$$x_1 = r\cos\,\chi_{n-1}\sin\,\chi_{n-2}\ldots\sin\,\chi_1, x_2 = r\sin\,\chi_{n-1}\sin\,\chi_{n-2}\ldots\sin\,\chi_1$$
$$x_3 = r\cos\,\chi_{n-2}\sin\,\chi_{n-3}\ldots\sin\,\chi_1, x_4 = r\sin\,\chi_{n-2}\sin\,\chi_{n-3}\ldots\cos\,\chi_1$$
$$\therefore$$
$$x_n = r\cos\,\chi_1$$

$$d\Omega_{n-1}^2 = d\chi_1^2 + \sum_{j=2}^{n-1}\prod_{i=1}^{j-1}\sin^2\,\chi_i d\chi_j^2$$

$$\mathrm{Tr}\left(F_{\rho\sigma}^{(a)}F^{(a)\rho\sigma}\right) = (n-1)(n-2)\frac{q^2}{r^4R^4}$$

$$V(\Phi) = \Lambda e^{\lambda\Phi} + \Lambda_1 e^{\lambda_1\Phi} + \Lambda_2 e^{\lambda_2\Phi}$$

$$R(r) = e^{2\alpha\Phi/(n-1)}$$

$$W(r) = -mr^{1+(1-n)(1-\gamma)} + \frac{(n-2)(1+\alpha^2)^2}{(1-\alpha^2)(\alpha^2+n-2)}b^{-2\gamma}r^{2\gamma} - \frac{\Lambda(1+\alpha^2)^2}{(n-1)(n-\alpha^2)}b^{2\gamma}r^{2(1-\gamma)} + \frac{(n-2)q^2(1+\alpha^2)^2}{(\alpha^2-1)(n+3\alpha^2-4)}b^{-6\gamma}r^{6\gamma-2}$$

$$\Phi(r) = \frac{\alpha(n-1)}{2(1+\alpha^2)}\ln\left(\frac{b}{r}\right).$$

$$\lambda_0 = \frac{4\alpha}{(n-1)}, \lambda_1 = \frac{4}{\alpha(n-1)}, \lambda_2 = \frac{4(2-\alpha^2)}{\alpha(n-1)}$$
$$\Lambda_1 = \frac{\alpha^2(n-1)(n-2)}{b^2(\alpha^2-1)}, \Lambda_2 = -\frac{\alpha(n-1)(n-2)}{\alpha^2-1}q^2b^{-4}$$

$$W(r) = 1 - \frac{m}{r^{n-2}} + \frac{\text{L}}{n(n-1)}r^2 - \frac{(n-2)q^2}{(n-4)r^2}.$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = (W'')^2 + \frac{(n-1)}{(rR)^2}[((rR)')^2(W')^2 + (2(rR)''W - (rR)'W')^2] + \frac{2(n-1)(n-2)}{(rR)^4}(1-((rR)')^2W)^2.$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \simeq \frac{(n-1)}{(1+\alpha^2)^4}\left((n+1)(2+\alpha^2-n)^2 + 2(n-2) + 4\alpha^2(2-n+2\alpha^2)\right)m^2r^{2(1-n)(1-\gamma)-2}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \simeq \frac{2\text{Ł}^2}{(n-1)^2(n-\alpha^2)^2}(2(1-\alpha^2)^2 + n(n-1) + 2(n-1)(1+\alpha^2)^2)b^{4\gamma}r^{-4\gamma}$$

$$\kappa^2 = -\frac{1}{2}\nabla_a\chi_b\nabla^a\chi^b$$

$$T = \frac{\kappa}{2\pi} = \frac{(1+\alpha^2)}{4\pi}\left(\frac{n-2}{1-\alpha^2}b^{-2\gamma}r_+^{2\gamma-1} - \frac{\Lambda}{n-1}b^{2\gamma}r_+^{1-2\gamma} - \frac{n-2}{1-\alpha^2}q^2b^{-6\gamma}r_+^{6\gamma-3}\right)$$

$$S = \frac{\omega_{n-1}}{4}b^{\gamma(n-1)}r_+^{(n-1)(1-\gamma)}$$

$$dM = TdS$$

$$T = \left(\frac{\partial M}{\partial S}\right)_{\mathrm{L},q},$$

$$M = \frac{(n-1)b^{(n-1)\gamma}\omega_{n-1}}{16\pi(1+\alpha^2)}m,$$

$$M = \frac{(n-1)(1+\alpha^2)b^{(n-1)\gamma}\omega_{n-1}}{16\pi}\left(\frac{(n-2)b^{-2\gamma}}{(1-\alpha^2)(\alpha^2+n-2)}b^{-2\gamma}\left(\frac{4S}{\omega_{n-1}b^{(n-1)\gamma}}\right)^{\frac{n-2-(n-3)\gamma}{(n-1)(1-\gamma)}} - \frac{\Lambda}{(n-1)(n-\alpha^2)}b^{2\gamma}\left(\frac{4S}{\omega_{n-1}b^{(n-1)\gamma}}\right)^{\frac{n-(n+1)\gamma}{(n-1)(1-\gamma)}} + \frac{(n-2)q^2}{(\alpha^2-1)(n+3\alpha^2-4)}b^{-6\gamma}\left(\frac{4S}{\omega_{n-1}b^{(n-1)\gamma}}\right)^{\frac{(n-1)(1-\gamma)+6\gamma-3}{(n-1)(1-\gamma)}}\right)$$

$$C_{\mathrm{E},q} = T\left(\frac{\partial S}{\partial r_+}\right)_{\mathrm{E},q}\left(\frac{\partial T}{\partial r_+}\right)_{\mathrm{E},q}^{-1}$$

$$C_{\mathrm{E},q} = \frac{(n-1)\omega_{n-1}}{4}b^{\gamma(n-1)}r_+^{(n-1)(1-\gamma)}\left(\frac{n-2}{1-\alpha^2}b^{-2\gamma}r_+^{2\gamma-1} - \frac{\Lambda}{n-1}b^{2\gamma}r_+^{1-2\gamma} - \frac{n-2}{1-\alpha^2}q^2b^{-6\gamma}r_+^{3(2\gamma-1)}\right) \times\left(-(n-2)b^{-2\gamma}r_+^{2\gamma-1} - \frac{\Lambda(1-\alpha^2)}{n-1}b^{2\gamma}r_+^{1-2\gamma} - 3(n-2)q^2b^{-6\gamma}r_+^{2(3\gamma-2)}\right)^{-1}$$

$$P = -\frac{\Lambda}{16\pi}\left(\frac{b}{r_+}\right)^{2\gamma}$$

$$V = \left(\frac{\partial H}{\partial P}\right)_S = \left(\frac{\partial M}{\partial P}\right)_S.$$

$$V = \frac{\omega_{n-1}(1+\alpha^2)}{n-\alpha^2}b^{(n-1)\gamma}r_+^{(n-1)(1-\gamma)+1}.$$

$$Q = \frac{1}{4\pi\sqrt{(n-1)(n-2)}}\int_\Sigma d^{n-1}\chi J(\Omega)\sqrt{\mathrm{Tr}\left(F_{\mu\nu}^{(a)}F_{\mu\nu}^{(a)}\right)} = \frac{\omega_{n-1}}{4\pi}q.$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_{S,P}$$

$$dM = TdS + VdP + UdQ$$

$$(n+\alpha^2-2)M = (n-1)TS + 2(\alpha^2-1)VP + (1-\alpha^2)UQ.$$

$$(n-2)M = (n-1)TS - 2VP + UQ.$$

$$P = \frac{(n-1)}{4(1+\alpha^2)}\frac{T}{r_+} - \frac{(n-1)(n-2)}{16\pi(1-\alpha^2)}b^{-2\gamma}r_+^{2(\gamma-1)}\left(1-q^2b^{-4\gamma}r_+^{2(2\gamma-1)}\right)$$

$$[P] = \frac{\hbar c}{l_{Pl}^{n-1}}P, [T] = \frac{\hbar c}{k}T$$

$$P=\frac{T}{v}-\frac{(n-2)(1+\alpha^2)}{4\pi(1-\alpha^2)}\kappa^{2\gamma-1}b^{-2\gamma}v^{2(\gamma-1)}\big(1-q^2b^{-4\gamma}\kappa^{2(2\gamma-1)}v^{2(2\gamma-1)}\big),$$

$$v=\frac{4(1+\alpha^2)}{n-1}r_+$$

$$\kappa=(n-1)/\big(4(1+\alpha^2)\big)$$

$$\left(\frac{\partial P}{\partial v}\right)_T=0,\left(\frac{\partial^2 P}{\partial v^2}\right)_T=0$$

$$v_c=\frac{4(1+\alpha^2)}{n-1}(3(2-\alpha^2)q^2b^{-4\gamma})^{\frac{1}{2(1-2\gamma)}}$$
$$T_c=\frac{(n-1)(n-2)}{12\pi(1-\alpha^2)(1+\alpha^2)}\kappa^{2(\gamma-1)}b^{-2\gamma}v_c^{2\gamma-1}$$

$$P_c=\frac{(n-1)(n-2)(1-\alpha^2)}{16\pi(1+\alpha^2)(2-\alpha^2)}\kappa^{2(\gamma-1)}b^{-2\gamma}v_c^{2(\gamma-1)}$$

$$\rho_c=\frac{P_cv_c}{T_c}=\frac{3(1-\alpha^2)^2}{4(2-\alpha^2)}$$

$$\left.\frac{P_cv_c}{T_c}\right|_{\alpha=0}=\frac{3}{8}$$

$$G(T,P)=\frac{\omega_{n-1}(1+\alpha^2)b^{(n-1)\gamma}}{16\pi}r_+^{(n-1)(1-\gamma)}\left(\frac{n-2}{\alpha^2+n-2}b^{-2\gamma}r_+^{2\gamma-1}-\right.$$
$$\left.\frac{16\pi(1-\alpha^2)P}{(n-1)(n-\alpha^2)}r_+-\frac{3(n-2)}{(n+3\alpha^2-4)}q^2b^{-6\gamma}r_+^{3(2\gamma-1)}\right)$$

$$\kappa_T=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T.$$

$$\kappa_T=\frac{n+\alpha^2}{1+\alpha^2}\left(P-\frac{n-2}{4\pi}\kappa^{2\gamma-1}b^{-2\gamma}v^{2(\gamma-1)}\big(1-3q^2b^{-4\gamma}\kappa^{2(2\gamma-1)}v^{2(2\gamma-1)}\big)\right)^{-1}.$$

$$P_0(v_2-v_1)=\int_{v_1}^{v_2}P\,\mathrm{d}v,$$

$$P_0(v_2-v_1)=T\ln\left(\frac{v_2}{v_1}\right)+\frac{1+\alpha^2}{1-\alpha^2}A\big(v_2^{2\gamma-1}-v_1^{2\gamma-1}\big)-\frac{1+\alpha^2}{3(1-\alpha^2)}B\left(v_2^{3(2\gamma-1)}-v_1^{3(2\gamma-1)}\right),$$

$$A=\frac{(n-2)(1+\alpha^2)}{4\pi(1-\alpha^2)}\kappa^{2\gamma-1}b^{-2\gamma},B=\frac{(n-2)(1+\alpha^2)}{4\pi(1-\alpha^2)}\kappa^{3(2\gamma-1)}b^{-6\gamma}q^2.$$

$$T\left(\frac{1}{v_2}-\frac{1}{v_1}\right)-A\left(v_2^{2(\gamma-1)}-v_1^{2(\gamma-1)}\right)+B\left(v_2^{2(3\gamma-2)}-v_1^{2(3\gamma-2)}\right)=0.$$

$$T=\frac{x}{x-1}\Big(Av_2^{2\gamma-1}\big(1-x^{2(\gamma-1)}\big)-Bv_2^{3(2\gamma-1)}\big(1-x^{2(3\gamma-2)}\big)\Big).$$

$$v_2^{2(2\gamma-1)} = \frac{A}{B}\frac{\left(\frac{2}{1-\alpha^2}(1-x^{2\gamma-1}) - \frac{x}{x-1}\ln\, x\left(1-x^{2(\gamma-1)}\right)\right)}{\left(\frac{2(2-\alpha^2)}{3(1-\alpha^2)}(1-x^{3(2\gamma-1)}) - \frac{x}{x-1}\ln\, x(1-x^{2(3\gamma-2)})\right)}.$$

$$P = \frac{1}{x-1}\left(Av_2^{2(\gamma-1)}(1-x^{2\gamma-1}) - Bv_2^{2(3\gamma-2)}\left(1-x^{3(2\gamma-1)}\right)\right).$$

$$\frac{dP}{dT} = \frac{L}{T(v_2-v_1)},$$

$$L = v_2(1-x)T(x)\frac{dP}{dx}\frac{dx}{dT}$$

$$S(T,V) = \frac{\omega_{n-1}}{4}b^{(n-1)\gamma}\left[\frac{(n-\alpha^2)}{\omega_{n-1}(1+\alpha^2)}b^{-(n-1)\gamma}V\right]^{(n-1)(1-\gamma)/((n-1)(1-\gamma)+1)}$$

$$p = \frac{P}{P_c}, \tau = \frac{T}{T_c}, \nu = \frac{v}{v_c},$$

$$p = \frac{4(2-\alpha^2)}{3(1-\alpha^2)^2}\frac{\tau}{\nu} - \frac{(1+\alpha^2)(2-\alpha^2)}{(1-\alpha^2)}\nu^{2(\gamma-1)}\left(1-\frac{\nu^{2(2\gamma-1)}}{3(2-\alpha^2)}\right)$$

$$p = \frac{1}{\rho_c}\frac{\tau}{\nu} + h(\nu)$$

$$p = 1 + At - Bt\omega - C\omega^3 + \mathcal{O}(t\omega^2, \omega^4)$$

$$A = \frac{1}{\rho_c}, B = \frac{1}{z\rho_c}, C = \frac{1}{z^3}\left(\frac{1}{\rho_c} - \frac{1}{6}h^{(3)}(1)\right),$$

$$z = (n+\alpha^2)/(1+\alpha^2)$$

$$\mathrm{d}P = -P_c(Bt + 3C\omega^2)\mathrm{d}\omega.$$

$$p = 1 + At - Bt\omega_l - C\omega_l^3 = 1 + At - Bt\omega_s - C\omega_s^3$$

$$\omega_s = -\omega_l = \sqrt{-\frac{Bt}{C}}$$

$$\eta = V_c(\omega_l - \omega_s) = 2V_c\omega_l \simeq (-t)^{1/2} \;\Rightarrow\; \bar{\beta} = \frac{1}{2}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \sim \frac{1}{P_c Bt}, \Rightarrow\; \bar{\gamma} = 1$$

$$p - 1 = -C\omega^3, \Rightarrow\; \bar{\delta} = 3$$

CONCLUSIONES.

Los agujeros negros cuánticos se forman en gravedad cuántica o en su defecto en supergravedad cuántica, aunque es inevitable en el segundo caso. La singularidad de un agujero negro cuántico, es en sí, la masa compacta de la partícula colapsada o colisionada, cuyo punto concéntrico es extremadamente denso, capaz de devorar materia y energía, transformándola en materia y energía oscuras, es decir, materia y energía inertes atravesadas por gravedad en condiciones perturbativas o entrópicas. El agujero cuántico de gusano, es distinto al agujero negro cuántico, es decir, no se forma por el colapso o colisión de una partícula oscura o de una partícula blanca, sino por la distorsión del espacio – tiempo cuántico, en condiciones de gravedad exógena, esto es, cuando cualquiera de las partículas antes referidas, interactúan con el gravitón o su supercompañera, el supergravitón o gravitino, formándose así, lo que se conoce también como puente de Einstein – Rosen, lo que explicaría la superposición y entrelazamiento cuánticos desde la perspectiva de la relatividad general. El radio de un agujero negro cuántico, es infinito en proporción a su disco de acreción. El horizonte de eventos de un agujero negro cuántico, inicialmente es débil, más su acaparamiento es mayor en tanto la absorción de materia y energía se vuelve más agresiva. El agujero negro cuántico, sea en gravedad o supergravedad cuánticas, produce materia y energía oscuras, es decir, tejido cuánticamente muerto, sin interacción más que la gravedad en sí misma, sin que sea posible la formación de otro fenómeno subatómico que requiera energía o materia. Un agujero negro cuántico, no puede crecer más allá del límite del campo en el que se ha formado, además que, es capaz de producir radiación bajo parámetros termodinámicos hostiles. Finalmente, es preciso señalar que un agujero blanco cuántico, se forma por el colapso o colisión de una partícula blanca en sentido estricto, más no por una partícula oscura, aunque puede formarse en condiciones de antimateria, lo que causa una supercurvatura y en consecuencia, la formación de un agujero blanco cuántico cuya principal característica es, expulsar materia y energía, sin capacidad de absorción o atracción, por lo que, todo escapa a su horizonte de eventos. La singularidad de un agujero blanco cuántico, es el centro de masa y energía de la partícula colapsada o colisionada, cuya densidad es extremadamente ligera, la misma que expulsa paquetes de energía de manera infinita al igual que repele materia aunque la produce. Por tanto, la gravedad no interactúa en un agujero blanco cuántico, aunque éstos pueden formarse a partir de la distorsión del espacio – tiempo cuántico. El agujero blanco cuántico, es vital para la creación de multidimensiones cuánticas y por ende, de hiperespacios paralelos al espacio – tiempo cuántico de origen, pues se constituye como un punto de salida de un puente de Einstein – Rosen propiamente dicho, arrojando energía y materia, bien sea, por atrapamiento desde el otro extremo del puente, por un agujero negro cuántico, aunque el referido puente, no necesariamente se forma a partir de un agujero negro cuántico, es decir, que su opuesto puede producir un puente ER de manera directa, cuando una partícula blanca o estrella colapsa o colisiona, en cuyo caso, la formación de materia y energía y la repulsión de la misma, se origina desde la singularidad del agujero blanco cuántico, aunque en condiciones extremas y para estos efectos, el agujero blanco cuántico pueda apalancarse de un agujero negro cuántico para efectos de desplegar su fenomenología, es decir, se forma de manera ulterior al agujero blanco cuántico. En todos los casos, el puente ER, es el ducto de comunicación entre ambos agujeros, pero esta configuración no es absoluta, pues un agujero blanco cuántico o su opuesto, pueden formarse a partir de una partícula blanca u oscura, según sea el caso, expulsando o en su defecto, absorbiendo masa y energía, que no se transporta a través del puente ER, sino que interactúa con la singularidad del agujero cuántico de que se trate, sea formando gravedad o reduciéndola a cero.

ACLARACIONES FINALES:

Algunas aclaraciones finales a tener en consideración y aplicar, a propósito de la Teoría Cuántica de Campos Relativistas o Curvos (TCCR) propuesta por este autor:

1. En todos los casos, este símbolo † será reemplazado por este símbolo†o por este símbolo †, equivaliendo lo mismo.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
†	†
	†

2. En todos los casos, este símbolo ‡, será reemplazado por este símbolo‡o por este símbolo ‡.

Símbolo a ser reemplazado.	**Símbolos de reemplazo.**
‡	‡
	‡

3. En todos los casos, se añadirá y por ende, se calculará la magnitud§que equivale a un campo de Yang – Mills y por ende, a la teoría de Yang – Mills en sentido amplio, en relación a la Teoría Cuántica de Campos Relativistas propuesta por este autor.

4. Este símbolo • podrá usarse como exponente u operador, según sea el caso.

Las aclaraciones antes referidas aplican tanto a este trabajo como a todos los trabajos previos y posteriores publicados por este autor, según corresponda.

REFERENCIAS BIBLIOGRÁFICAS.

M. M. Stetsko, Static spherically symmetric Einstein-Yang-Mills-dilaton black hole and its thermodynamics, arXiv:2005.13447v1 [hep-th] 27 May 2020.

M. Zubair, Muhammad Ali Raza, Furkat Sarikulov y Javlon Rayimbaev, 4D Einstein-Gauss-Bonnet Black Hole in Power-Yang-Mills Field: A Shadow Study, arXiv:2305.16888v1 [gr-qc] 26 May 2023.

Fatemeh Naeimipour, Behrouz Mirza y Soudabe Nasirimoghadam, Lifshitz and hyperscaling violated Yang-Mills-dilaton black holes, arXiv:2112.08260v1 [hep-th] 15 Dec 2021.

Jaehyeok Choi, Sunjin Choi, Seok Kim, Jehyun Lee y Siyul Lee, Finite N black hole cohomologies, arXiv:2312.16443v2 [hep-th] 29 Sep 2024.

Zuting Luo, Meirong Tang y Zhaoyi Xu, Testing Einstein Maxwell Power-Yang-Mills Hair via Black Hole Photon Rings, arXiv:2501.06690v1 [gr-qc] 12 Jan 2025.

Fatemeh Naeimipour, Behrouz Mirza y Fatemeh Masoumi Jahromi, Yang-Mills black holes in Quasitopological gravity, arXiv:2106.08558v1 [hep-th] 16 Jun 2021.

Pisin Chena, Xiao Yan Chewe, Misao Sasakia y Dong-han Yeom, Yang-Mills instantons as the end point of black hole evaporation, arXiv:2311.00311v1 [gr-qc] 1 Nov 2023.

Ben Kain, Quantum corrected Einstein-Yang-Mills black holes in semiclassical gravity, arXiv:2502.08278v1 [gr-qc] 12 Feb 2025.

Sunjin Choi, Saebyeok Jeong y Seok Kim, The Yang-Mills duals of small AdS black holes, arXiv:2103.01401v3 [hep-th] 12 Sep 2024.

Fatemeh Masoumi Jahromi, Behrouz Mirza, Fatemeh Naeimipour y Soudabe Nasirimoghadam, Nonlinear Yang-Mills black holes, arXiv:2306.17523v1 [gr-qc] 30 Jun 2023.

Celio R. Muniz y Francisco Bento Lustosa, Einstein-Yang-Mills Regular Black Holes in Rainbow Gravity, arXiv:2504.08799v1 [gr-qc] 8 Apr 2025.

Niklas Loewer, Moreshwar Tayde y P.K. Sahoo, A Study of stable wormhole solution with non-commutative geometry in the framework of linear f (R,Lm, T) gravity, arXiv:2409.04172v6 [gr-qc] 1 Apr 2025.

Mohammed Muzakkir Rizwan, Zinnat Hassan, P.K. Sahoo y Ali Övgün, Influence of GUP corrected Casimir energy on zero tidal force wormholes in modified teleparallel gravity with matter coupling, arXiv:2410.05348v2 [gr-qc] 15 Oct 2024.

Seyed Naseh Sajadi, Supakchai Ponglertsakul y Orlando Luongo, Constructing regular black holes from multi-polytropic equations of state, arXiv:2502.02098v1 [gr-qc] 4 Feb 2025.

MORESHWAR JAGADEORAO TAYDE, Study ofWormholes in Symmetric Teleparallel Theories of Gravity, arXiv:2505.04952v1 [gr-qc] 8 May 2025.

G.G.L. Nashed y Waleed El Hanafy, Stable soliton dark matter wormhole in non-minimally coupled f(Q, T) gravity, arXiv:2505.20331v2 [gr-qc] 13 Aug 2025.

Susmita Sarkar y Nayan Sarkar, Deflection of light by dark matter supporting traversable wormholes in the framework of Kalb-Ramond gravity, arXiv:2507.12970v1 [gr-qc] 17 Jul 2025.

Ishan Deo, Prateksh Dhivakar y Nilay Kundua, Entropy-current for dynamical black holes in Chern-Simons theories of gravity, arXiv:2306.12491v2 [hep-th] 22 Nov 2023.

Nikos Chatzifotis, Panos Dorlis, Nick E. Mavromatos y Eleftherios Papantonopoulos, Scalarization of Chern-Simons Kerr Black Hole Solutions and Wormholes, arXiv:2202.03496v2 [gr-qc] 11 Feb 2022.

Dongjun Li, Pratik Wagle, Yanbei Chen y Nicolás Yunes, Perturbations of spinning black holes in dynamical Chern-Simons gravity: Slow rotation quasinormal modes, arXiv:2503.15606v1 [gr-qc] 19 Mar 2025.

Chloe Richards, Alexandru Dima y Helvi Witek, Black Holes in Massive Dynamical Chern-Simons gravity: scalar hair and quasibound states at decoupling, arXiv:2305.07704v1 [gr-qc] 12 May 2023.

David D. K. Chow y Geoffrey Compère, Black holes in N = 8 supergravity from SO(4; 4) hidden symmetries, arXiv:1404.2602v4 [hep-th] 20 Mar 2020.

Yermek Aldabergenov, Andrea Addazi y Sergei V. Ketov, Primordial black holes from modified supergravity, arXiv:2006.16641v3 [hep-th] 26 Sep 2020.

Juan Martín Maldacena, Black Holes in String Theory, arXiv:hep-th/9607235v2 20 Aug 1996.

Donald Marolf, Black holes and branes in supergravity, arXiv:1107.1022v2 [hep-th] 17 Aug 2011.

Subramanya Hegde, Ashoke Sen, P Shanmugapriya y Amitabh Virmani, Supersymmetric Index for Half BPS Black Holes in N=2 Supergravity with Higher Curvature Corrections, arXiv:2411.08260v1 [hep-th] 13 Nov 2024.

Carlos Shahbazi Alonso, Black Holes in Supergravity with Applications to String Theory, arXiv:1307.3064v1 [hep-th] 11 Jul 2013.

David D. K. Chow, Symmetries of supergravity black holes, arXiv:0811.1264v3 [hep-th] 9 Sep 2010.

Gianguido Dall' Agata, Black holes in supergravity: ow equations and duality, arXiv:1106.2611v2 [hep-th] 15 Jun 2011.

Antonio Gallerati, Constructing black hole solutions in supergravity theories, arXiv:1905.04104v6 [hep-th] 16 Jul 2020.

RICCARDO D'AURIA y PIETRO FRE', BPS BLACK HOLES IN SUPERGRAVITY, arXiv:hep-th/9812160v2 9 Feb 1999.

Tomás Ortín, Black hole solutions in theories of Supergravity, arXiv:2412.12020v2 [hep-th] 8 Feb 2025.

Daniel Harlow, Black holes in quantum gravity, arXiv:2304.10367v1 [gr-qc] 19 Apr 2023.

Debabrata Sahu y Chandrasekhar Bhamidipati, Off-shell phase diagram of BPS black holes in AdS5, arXiv:2502.01519v1 [hep-th] 3 Feb 2025.

Dmitri Gal'tsov y Aleksandr Kulitskii, Petrov types, separability and generalized photon surfaces of supergravity black holes, arXiv:2409.13324v1 [gr-qc] 20 Sep 2024.

Rudeep Gaur y Matt Visser, Black holes, white holes, and near-horizon physics, arXiv:2304.10692v3 [gr-qc] 16 May 2024.

Alberto Castellano, Dieter Lüst, Carmine Montella y Matteo Zatti, Quantum Calabi–Yau Black Holes and Non-Perturbative D0-brane Effects, arXiv:2505.15920v1 [hep-th] 21 May 2025.

Monserrat Aguayo, Leonardo Gajardo, Nicolás Grandi, Javier Moreno, Julio Oliva y Martín Reyes, Holographic explorations of regular black holes in pure Gravity, arXiv:2505.11736v2 [hep-th] 16 Jun 2025.

Faramarz Rahmani y Mehdi Sadeghi, The Phase Transition of 4D Yang-Mills Charged GB AdS Black Hole with Cloud of Strings, arXiv:2402.07856v3 [hep-th] 14 Sep 2024.

Jay Armas y Gianbattista-Piero Nicosia, New asymptotically (Anti)-de Sitter black holes in (super)gravity, arXiv:2406.17489v2 [hep-th] 3 Dec 2024.

Siyul Lee, Quantum Anatomy of Supersymmetric Black Holes in AdS Spacetimes, arXiv:2409.03862v1 [hep-th] 5 Sep 2024.

Ankit Anand, Thermodynamic Extremality in Power-law AdS Black Holes: A Universal Perspective, arXiv:2409.07079v2 [hep-th] 7 Mar 2025.

Sunjin Choi, Diksha Jain, Seok Kim, Vineeth Krishna, Eunwoo Lee, Shiraz Minwalla y Chintan Patel, Dual Dressed Black Holes as the end point of the Charged Superradiant instability in N = 4 Yang Mills, arXiv:2409.18178v2 [hep-th] 24 Mar 2025.

Robert de Mello Koch, Minkyoo Kim, Seok Kim, Jehyun Lee y Siyul Lee, Brane-fused black hole operators, arXiv:2412.08695v1 [hep-th] 11 Dec 2024.

IGNACIO ACEVEDO AND CLAUDIO MUÑOS, KINK DYNAMICS FOR THE YANG-MILLS FIELD IN AN EXTREMAL REISSNER-NORDSTR¨OM BLACK HOLE, arXiv:2501.12790v3 [math.AP] 17 Mar 2025.

Zening Yan, Standardized Constraints on the Shadow Radius and the Instability of Scalar, Electromagnetic, p-Form, and Gravitational Perturbations of High-Dimensional Spherically Symmetric Black Holes in Einstein-power-Yang-Mills-Gauss-Bonnet Gravity, arXiv:2506.04703v1 [gr-qc] 5 Jun 2025.

Suvankar Dutta, Shruti Menon y Aayush Srivastav, Bosonization, BTZ Black Hole Microstates, and Logarithmic Correction to Entropy, arXiv:2508.02663v1 [hep-th] 4 Aug 2025.

Luca V. Iliesiu, Sameer Murthy y Gustavo J. Turiaci, Black hole microstate counting from the gravitational path integral, arXiv:2209.13602v1 [hep-th] 27 Sep 2022.

Minwoo Suh, Spindle black holes and mass-deformed ABJM, arXiv:2211.11782v3 [hep-th] 29 May 2024.

Nikolay Bobev, Vasil Dimitrov y Annelien Vekemans, Wrapped M5-branes and AdS5 Black Holes, arXiv:2212.10360v1 [hep-th] 20 Dec 2022.

Nikolay Bobev, Marina David, Junho Hong y Rishi Mouland, AdS7 Black Holes from Rotating M5-branes, arXiv:2307.06364v2 [hep-th] 22 Sep 2023.

Kiril Hristova y Minwoo Suh, Spindle black holes in AdS4×SE7, arXiv:2307.10378v2 [hep-th] 30 Oct 2023.

Zvi Bern, Enrico Herrmann, Radu Roiban, Michael S. Ruf, Alexander V. Smirnov, Vladimir A. Smirnov y Mao Zeng, Amplitudes, Supersymmetric Black Hole Scattering at $\mathcal{O}(G^5)$, and Loop Integration, arXiv:2406.01554v1 [hep-th] 3 Jun 2024.

Nizar Ezroura y Finn Larsena, Supergravity Spectrum of AdS5 Black Holes, arXiv:2408.11529v2 [hep-th] 25 Nov 2024.

Kabir Bajaj, Vipul Kumar, Shiraz Minwalla, Jyotirmoy Mukherjee, Asikur Rahaman, Grey Galaxies in AdS5, arXiv:2412.06904v1 [hep-th] 9 Dec 2024.

Marina David, Victor Godet, Zhihan Liua y Leopoldo A. Pando Zayasa, Non-topological logarithmic corrections in minimal gauged supergravity, arXiv:2112.09444v2 [hep-th] 28 Nov 2023.

Thomas W. Grimm y Washington Taylor, Structure in 6D and 4D N = 1 supergravity theories from F-theory, arXiv:1204.3092v2 [hep-th] 7 Jun 2012.

C.P. Burgess, F. Muia y F. Quevedo, 4D de Sitter from String Theory via 6D Supergravity, arXiv:2408.03852v1 [hep-th] 7 Aug 2024.

Marcel R. R. Hughes y Masaki Shigemori, Fortuity within Supergravity, arXiv:2505.14888v1 [hep-th] 20 May 2025.

L. Andrianopoli, B. L. Cerchiai, P. A. Grassi y M. Trigiante, The Quantum Theory of Chern-Simons Supergravity, arXiv:1903.04431v3 [hep-th] 30 May 2019.

Dionysios Anninos, Pietro Benetti Genolini y Beatrix Mühlmann, dS2 Supergravity, arXiv:2309.02480v3 [hep-th] 17 Jul 2025.

S. Kalyana Rama, LQ Stars: Modifying the Tolman – Oppenheimer – Volkoff equations a la LQC, arXiv:1909.08199v1 [gr-qc] 18 Sep 2019.

Charis Anastopoulos y Ntina Savvidou, Classi_cation theorem and properties of singular solutions to the Tolman-Oppenheimer-Volko_ equation, arXiv:2010.02279v2 [gr-qc] 22 Jan 2021.

Stephen L. Adler y Brent Doherty, Dynamical gravastar simulated horizon from the Tolman-Oppenheimer-Volkoff equation initial value problem with relativistic matter, arXiv:2309.13380v9 [gr-qc] 21 Nov 2024.

Sudhanva Lalit, Alexandra C. Semposki y Joshua M. Maldonado, Star Log-extended eMulation: a method for efficient computation of the Tolman-Oppenheimer-Volkoff equations, arXiv: 2411.10556v2 [astro-ph.HE] 19 Nov 2024.

Sante Carloni y Daniele Vernieri, The covariant Tolman-Oppenheimer-Volkoff equations II: The anisotropic case, arXiv:1709.03996v1 [gr-qc] 12 Sep 2017.

Albuja Bustamante, M. I. (2024). Demostración del Espectro Hamiltoniano para un Campo de Yang-Mills no Abeliano que Poseen una Brecha de Masa Finita con Respecto al Estado de Vacío. *Ciencia Latina Revista Científica Multidisciplinar*, *8*(1).

Manuel Ignacio Albuja Bustamante , Teoría de Campos: Reforzamiento Teórico – Matemático al Modelo Estándar de Partículas, bajo la estructura ecuacional de Yang – Mills , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 2 (2024).

Manuel Ignacio Albuja Bustamante, La brecha de masa y la curvatura de los campos cuánticos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 4 (2024).

Manuel Ignacio Albuja Bustamante, Formalización Matemática y en Física de Partículas, en Relación a la Brecha de Masa y la Curvatura Geométrica de los Campos Cuánticos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 8 Núm. 5 (2024).

Manuel Ignacio Albuja Bustamante, Campos Cuánticos Relativistas: Aproximaciones Teórico – Matemáticas Relativas a los Espacios Cuánticos Geométricamente Deformados o Perforados por Partículas y Antipartículas Supermasivas y Masivas E Hiperpartículas y Suprapartículas , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 1 (2025).

Manuel Ignacio Albuja Bustamante, Teoría Cuántica de Campos Relativistas. Formalización Teórica , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 3 (2025)

Manuel Ignacio Albuja Bustamante, Supergravedad cuántica relativista. Teorización inicial , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Supergravedad cuántica relativista. Dimensiones en R^η. , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Gravedad cuántica relativista , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025)

Manuel Ignacio Albuja Bustamante, Agujeros Negros Cuántico-Relativistas Masivos o Supermasivos , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025).

Manuel Ignacio Albuja Bustamante, Cromodinámica Cuántica Relativista , Ciencia Latina Revista Científica Multidisciplinar: Vol. 9 Núm. 4 (2025).

Albuja Bustamante, M. I. (2025). Las partículas supermasivas: naturaleza fenomenológica de la partícula oscura. *Ciencia Latina Revista Científica Multidisciplinar*, *9*(3).