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**TEORÍA CUÁNTICA DE CAMPOS RELATIVISTAS:
UNA ALTERNATIVA DE SOLUCIÓN AL
PROBLEMA DEL MILENIO DE YANG – MILLS. UN
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VOLUMEN VIII.**

RELATIVISTIC QUANTUM FIELD THEORY: AN
ALTERNATIVE SOLUTION TO THE YANG–MILLS
MILLENNIUM PROBLEM. AN ATTEMPT TO UNIFY
GENERAL RELATIVITY AND QUANTUM MECHANICS.
VOLUME VIII.

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RESUMEN.

En este trabajo, compuesto por diez volúmenes, abordaremos aspectos esenciales de la Teoría Cuántica de Campos Relativistas (TCCR), con propósitos de optimización de los cálculos expuestos en trabajos anteriores pero sobre todo, posicionar la referida teoría, como una alternativa de solución al problema del milenio de Yang – Mills y la brecha de masa. La idea esencial es la misma, todo espacio – tiempo cuántico, es decir, todo campo cuántico, es curvo y esa deformación ocurre por la gravedad y supergravedad cuánticas, según sea el caso, que provocan las partículas oscuras o estrella, al momento de interactuar con un campo gravitónico o supergravitónico, según corresponda, o en relación a la criticidad de su centro de masa y/o energía, lo que afecta su espín, velocidad y momento angular y por ende, sus trayectorias orbitales. Por tanto, la TCCR, no es un intento por cuantizar la gravedad, sino por introducir la gravedad, como principio de mínima acción de un sistema cuántico y de sus estados fundamentales.

Las métricas siguen siendo las mismas, es decir, que para un campo cuántico curvo o geoméricamente deformado, la densidad lagrangiana/hamiltoniana equivale a: $\mathcal{LH}_{curvature} = \langle \int \hat{e}^{iht} \sqrt{\hat{g}^{\mu\nu}} \otimes \overleftarrow{m}\psi\bar{\psi} -$

$$\partial^2 \Delta' \gamma \langle \otimes_{\mathfrak{R}} | d^4x / \partial \mathcal{R} \rangle' \int \left\| \frac{\partial \phi_{\sigma\rho}^*}{\partial \phi_{\sigma\rho}^*} \right\| -$$

$$\left\langle \frac{\partial \phi_{\sigma\rho}^*}{\partial \phi_{\mu\nu}^*} \left| \partial \uparrow / \partial t \setminus \partial \downarrow / \partial t \partial^2 \square \left[\begin{matrix} \blacksquare_{\uparrow}^{\sigma} \\ \blacksquare_{\downarrow}^{\nu} \end{matrix} \partial^2 \varphi / \partial \psi \blacksquare \right] \Lambda_{\nu}^{\mu} \sum_{\substack{0 \leq l \leq m \\ 0 < j < n}} P(l, j) \prod_{k=1}^n A_k \cup_{n=1}^m (X_n \cap Y_n) \cup_{n=1}^m (X_n \cap Y_n) \odot \Lambda_{\nu}^{\mu} \odot \Gamma_{\nu}^{\mu}, \right.$$

respecto de una partícula pesada ρ , sea oscura o blanca (partícula estrella), según corresponda, a propósito de la criticidad de su masa y/o energía $\langle 0 | \sum_{\delta} \partial m / \partial e \rangle$ o de su interacción con un gravitón o un gravitino, según corresponda, en coordenadas $\langle \rho^{\mu} \rho^{\nu} \rho^{\sigma} \rho^{\varrho} \rangle$, esto último, lo que ocurre por permeabilización del campo gravitónico o supergravitónico en $\blacksquare = \int \langle \partial \mathfrak{G} / \partial \mathfrak{G} \rangle$, lo que corresponde al espacio – cuántico deformado en $\mathfrak{C}_{\mathfrak{R}}^{\sigma\rho} = \langle \sum_{\square}^{\sigma\rho} \mathcal{R}_{\nu}^{\mu\dagger} | \otimes \mathcal{H}_{\mu}^{\nu*} \rangle$ lo que en dimensiones \mathbb{R}^{η} , representa, gravedad o supergravedad cuánticas por curvatura o supercurvatura del espacio - tiempo cuántico multidimensional.

Palabras Clave: Supergravedad cuántica, gravedad cuántica, partícula oscura, partícula estrella, hiperpartículas, suprapartículas, teoría cuántica de campos relativistas, problema del milenio de Yang – Mills y la brecha de masa, partículas ligeras, curvatura, supercurvatura, multidimensiones, agujeros cuánticos.

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RELATIVISTIC QUANTUM FIELD THEORY: AN ALTERNATIVE SOLUTION TO THE YANG–MILLS MILLENNIUM PROBLEM. AN ATTEMPT TO UNIFY GENERAL RELATIVITY AND QUANTUM MECHANICS. VOLUME VIII.

ABSTRACT.

In this work, composed of ten volumes, we will address essential aspects of the Quantum Theory of Relativistic Fields (TCCR), with the purpose of optimizing the calculations exposed in previous works but above all, positioning the aforementioned theory as an alternative solution to the Yang-Mills millennium problem and the mass gap. The essential idea is the same, all quantum space-time, that is, every quantum field, is curved and that deformation occurs due to quantum gravity and supergravity, as the case may be, caused by dark particles or stars, when interacting with a gravitonic or supergravitonic field, as appropriate, or in relation to the criticality of its center of mass and/or energy which affects their spin, velocity and angular momentum and therefore, their orbital trajectories. Therefore, the TCCR is not an attempt to quantize gravity, but to introduce gravity, as the principle of least action of a quantum system and its fundamental states.

The metrics remain the same, i.e., for a curved or geometrically warped quantum field, the Lagrangian/Hamiltonian density is equal to: $\mathcal{L}\mathcal{H}_{curvature} = \langle \int \hat{e}^{iht} \sqrt{\bar{g}^{\mu\nu}} \otimes \overline{m\psi\bar{\psi}} -$

$$\partial^2 \Delta' \gamma' \langle \otimes_{\mathbb{R}} | d^4x / \partial \mathcal{R} \rangle' \int \left\| \frac{\partial \phi_{\sigma\rho}'}{\partial \phi_{\sigma\rho}^\dagger} \right\| -$$

$$\left\langle \frac{\partial \phi_{\sigma\rho}^*}{\partial \bar{\mu\nu}} \right| \partial \uparrow / \partial t \setminus \partial \downarrow / \partial t \partial^2 \square \left[\begin{matrix} \blacksquare^\cup \\ \blacksquare^\cup \end{matrix} \partial^2 \varphi / \partial \psi^\blacksquare \right] \wedge_\nu^\mu \sum_{\substack{0 \leq l \leq m \\ 0 < j < n}} P(l, j) \prod_{k=1}^n A_k \cup_{n=1}^m (X_n \cap Y_n) \cup_{n=1}^m (X_n \cap Y_n) \odot \Lambda_\nu^\mu \odot \Gamma_\nu^\mu$$

with respect to a heavy particle ρ , whether dark or white (star particle), as appropriate, regarding the criticality of its mass and/or energy $\langle 0 | \sum_\delta \partial m / \partial e \rangle$ or its interaction with a graviton or a gravitin, as appropriate, in coordinates $\langle \rho^\mu \rho^\nu \rho^\sigma \rho^\varrho \rangle$, the latter, which occurs by permeabilization of the gravitonic or supergravitonic field in $\blacksquare = \int \langle \partial \mathcal{G} / \partial \mathcal{S} \mathcal{G} \rangle$, what corresponds to the space – quantum deformed in $\mathfrak{C}_{\mathfrak{S}\mathfrak{R}} = \langle \sum_{\square}^{\sigma\rho} \mathcal{R}_\nu^{\mu\dagger} | \otimes \mathcal{H}_\mu^{\nu*} \rangle$ the which in dimensions \mathbb{R}^n , represents, quantum gravity or supergravity by curvature or supercurvature of multidimensional quantum space-time.

Keywords: Quantum supergravity, quantum gravity, dark particle, star particle, quantum theory of relativistic fields.

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INTRODUCCIÓN.

En este punto, es indispensable establecer las bases teóricas que conforman la Teoría Cuántica de Campos Relativistas (TCCR) y que se encuentran desarrolladas en trabajos previos. Por tanto, estos son los puntos más relevantes.

1. Todo campo cuántico, es curvo por acción inmediata de una partícula cuya masa y/o energía alcanzan el mayor grado de criticidad. En este caso, la gravedad es endógena o implícita, es decir, una cualidad propia de la partícula interactuante.

2. Siguiendo lo dicho, en el numeral que antecede, las partículas se dividen en:

2.1. Partículas Supermasivas (Tipo IA): Son aquellas, cuyo centro de masa/energía en unidades de

Planck dados en $\mathcal{M}_p = \sqrt{\frac{\hbar c}{\mathfrak{G}}} \approx 2,18 \times 10^{-8} \text{ kg}$ (masa) y $E_p = \frac{\hbar}{t_p}$, $E_p = m_p c^2$, $E_p = \sqrt{\frac{\hbar c^5}{G}} \approx$

$1.956 \times 10^9 \text{ J} \approx 1.22 \times 10^{19} \text{ GeV} \approx 0.5433 \text{ MWh} \sqrt{\frac{\hbar c^5}{8\pi G}} \approx 0.390 \times 10^9 \text{ J} \approx 2.43 \times 10^{18} \text{ GeV}$

(energía $\approx 10^{-120}$), alcanza el mayor grado de criticidad, deformando el espacio – tiempo cuántico, lo que afecta el estado fundamental de los orbitales (espín, momentum, velocidad, trayectorias, etc), desplegados por las partículas repercutidas. Esta partícula también se la denomina “partícula oscura”, en la medida en que, su centro de energía/masa, es oscuro. Principal candidata para explicar la materia oscura, en la medida en que, la gravedad converge en su centro, absorbiendo energía y materia.

2.2. Partículas Blancas (Tipo IB): Son aquellas, cuyo centro de masa/ energía en unidades de Planck, alcanza el mayor grado de criticidad, deformando el espacio – tiempo cuántico, lo que afecta el estado fundamental de los orbitales (espín, momentum, velocidad, trayectorias, etc), desplegados por las partículas repercutidas. Esta partícula también se la denomina “partícula estrella”, en la medida en que, su centro de masa/energía es extremadamente denso, superando la masa, temperatura y energía de

Planck, en $\mathcal{M}_p = \sqrt{\frac{\hbar c}{\mathfrak{G}}} \approx 2,18 \times 10^{-8} \text{ kg}$ (masa), $\mathcal{M}_p = \sqrt{\frac{\hbar c^5}{\mathfrak{G} \hbar^2}}$ $T_p \approx 1.416784(16) \times 10^{32} \text{ K}$

(temperatura) y $E_p = \frac{\hbar}{t_p}$, $E_p = m_p c^2$, $E_p = \sqrt{\frac{\hbar c^5}{G}} \approx 1.956 \times 10^9 \text{ J} \approx 1.22 \times 10^{19} \text{ GeV} \approx$

$0.5433 \text{ MWh} \sqrt{\frac{\hbar c^5}{8\pi G}} \approx 0.390 \times 10^9 \text{ J} \approx 2.43 \times 10^{18} \text{ GeV}$. También se la denomina “partícula estrella”.



2.3. Hiperpartículas (Tipo IIA): Son aquellas, cuyo centro de masa/energía es extremadamente bajo, en unidades de Planck, más sin embargo, son capaces de igualar o superar la velocidad de la luz.

2.4. Suprapartículas (Tipo IIB): Son aquellas, cuyo centro de masa/energía es el equivalente al de una partícula oscura o blanca, más sin embargo, éstas, a diferencia de las referidas en los numerales 2.1 y 2.2, ésta igual o supera la velocidad de la luz.

3. Agujero negro cuántico: Fenómeno que ocurre en un espacio cuántico de Sitter, esto es, cuando una partícula oscura colisiona con otra o en su defecto, cuando una partícula blanca colisiona con otra o cuando una partícula blanca y una partícula oscura colisionan entre sí. Los agujeros negros cuánticos, también se forman por el colapso (por compresión gravitacional) o por la aniquilación (por interacción) de una partícula oscura o de una partícula blanca. Lo primero, ocurre cuando se atraen mutuamente por gravedad en tanto que lo segundo, ocurre cuando su centro de masa/energía alcanza el mayor grado de criticidad posible. En el centro del agujero negro cuántico, se encuentra la masa de la partícula aniquilada o comprimida, la que comporta condiciones gravitatorias extremas. Ahí es donde radica la singularidad de un agujero negro cuántico. La información que ingresa al agujero negro cuántico, no se destruye, muy al contrario, se transforma en materia y energía, las mismas que son repulsadas por el agujero negro cuántico blanco que se encuentra en el otro extremo del agujero cuántico de gusano. Por tanto, la materia y energía atrapada por el agujero negro cuántico, se convierte en materia y energía oscuras interferidas por gravedad extrema.

4. Agujero cuántico de gusano: Túnel cuántico por el cual, se conectan un agujero negro cuántico y un agujero blanco cuántico. A través de este túnel, por teletransportación cuántica, la información es procesada y convertida en materia y energía, todo esto, en un espacio de Sitter.

5. Agujero blanco cuántico: Fenómeno que ocurre en un espacio cuántico de Sitter, volviéndose la región de salida o repulsión de materia y energía, a propósito de lo que devora el agujero negro cuántico y de lo que procesa el canal cuántico de gusano. Lo que repulsa el agujero blanco cuántico, es materia y energía procesadas.

6. Espacio – tiempo cuántico: Entiéndase por espacio – tiempo cuántico, al campo en sí mismo, cuya

Longitud de Planck, es superior a $\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1,616199(97) \times 10^{-35}$ metros. La métrica es la



curvatura escalar de Ricci, así: $\mathcal{R} = \sum_{\alpha,\beta=0}^3 g^{\alpha\beta} \mathcal{R}_{\alpha\beta} \approx o(\mathcal{L}_p^{-2}) \approx 3,828 \cdot 10^{69} m^{-2}$. Ahora bien, el espacio – tiempo cuántico puede ser, bien de Sitter (dS) o bien, anti de Sitter (AdS). En el primero, se forma la curvatura cuántica y sus subniveles, subespacios o subcapas, en tanto que en el segundo, se forman los agujeros cuánticos y las multidimensiones.

7. Todo campo cuántico, es curvo por acción inmediata de la gravedad, esto a propósito de la existencia (Modelo – Higgs):

7.1. De un campo gravitónico, es decir, cuando una partícula cualquiera, interactúa con un gravitón, lo que supone la permeabilidad del campo cuántico, por un campo gravitónico que transfiere gravedad al campo primario, curvándolo.

7.2. De un campo supergravitónico, es decir, cuando una partícula cualquiera, interactúa con un gravitino o supergravitón, lo que supone la permeabilidad del campo cuántico, por un campo gravitónico que transfiere gravedad al campo primario, deformándolo.

7.3. Lo referido en este numeral se denomina gravedad exógena.

8. La gravedad cuántica, sea endógena o exógena comporta la curvatura del espacio – tiempo cuántico, en tanto que, la supergravedad cuántica, sea endógena o exógena, comporta la deformación (supercurvatura) del espacio – tiempo cuántico, formándose pliegues multidimensionales (en alta configuración – membranas dimensionales) en rango superior a $\mathbb{R}^4 - AdS$. Cabe indicar que las membranas dimensionales, se dividen en TIPO I y TIPO II respectivamente, la primera a propósito de la curvatura del campo en gravedad cuántica y la segunda, la deformación del campo en supergravedad cuántica, todo esto, lo cual también depende de la naturaleza de la gravedad que interfiere, es decir, si es exógena o endógena, lo que llamaríamos membranas dimensionales tipo IA, IB, IIA y IIB respectivamente, las cuales, pueden contener dimensiones y subdimensiones infinitas, en relación a las interacciones de la partícula que provoca de la deformación del espacio – tiempo cuántico. Esto es lo que llamamos supersimetrías de gauge en dimensiones altas a \mathbb{R}^4 , es decir, cuando estamos ante membranas dimensionales tipo IA, IB y IIB, según sea el caso en tanto que, las membranas dimensionales del tipo IIA, contienen dimensiones infinitas en $\mathbb{R}^4 - dS$.



9. Cuando una partícula colisiona con otra y se aniquilan o cuando la partícula pesada colapsa por compresión, la extinción provoca ondas cuánticas que se desplazan en longitud sobre el campo cuántico deformado el mismo que, es superfluido.

10. El puente ER, en esta teoría, explica la superposición y el entrelazamiento cuánticos en sentido estricto, en un espacio AdS.

11. Los enunciados antes referidos, aplican a la antimateria, es decir, a la región de antipartículas.

12. La brecha de masa, provoca la curvatura del espacio – tiempo cuántico pero no lo deforma por completo, pues este fenómeno, no ocurre con una partícula deformante, sino en partículas ligeras como las hiperpartículas, esto en la medida en que, no registran estado de vacío.

13. Adicionalmente, es importante, establecer las siguientes reglas:

13.1. La gravedad cuántica relativista, ocurre concretamente en un espacio cuántico de Sitter, en el que se pueden formar subdimensiones o subespacios dentro del límite de \mathbb{R}^4 .

13.2. La supergravedad cuántica relativista, ocurre concretamente en un espacio cuántico anti de Sitter, en el que se pueden formar hiperespacios o dimensiones más altas, superiores a \mathbb{R}^4 .

13.3. Las partículas propuestas, viajan en gravedad cuántica más, interactúan en supergravedad cuántica por permeabilización.

13.4. Cualquier partícula, de las aquí propuestas, se puede convertir en otra, por aniquilación, siguiendo los diagramas de Feynman.

13.5. Las dimensiones en alta configuración así como las de ensamble, son infinitas.

13.6. La materia y energía oscuras, están formadas esencialmente por partículas aniquiladas o colapsadas por gravedad. En consecuencia, es la criticidad de la masa la que las vuelve compatibles.

13.7. Los agujeros cuánticos, absorben partículas ligeras y pesadas, sin distinción, lo que explica la expansión del universo por acción gravitacional en la materia.

13.8. Las partículas aquí propuestas, son susceptibles de enganche, como ocurre con un diquark.

13.9. En esta teoría, se incorpora el concepto de cuerda, pero en un espacio cuántico anti de Sitter.

13.10. Las partículas pesadas, cuando se desplazan de un punto a otro en forma infinita hasta su aniquilación o colapso, lo hacen por medio de gravedad, deformando, en el caso de las partículas blancas



y las hiperpartículas, un espacio de Sitter, creando capas dimensionales en límite de \mathbb{R}^4 en tanto que, la partícula oscura, crea capas dimensiones en alta configuración a \mathbb{R}^4 en un espacio anti de Sitter.

13.11. La hiperpartícula es la única en este modelo, que no tiene masa, es por ello que puede viajar a la velocidad de la luz.

13.12. La suprapartícula es por excepción, un caso de mutación por aniquilación, en la medida en que, pese a tratarse de una partícula pesada, con un centro de masa/energía extremadamente crítico y denso, es capaz de viajar a la velocidad de la luz. La suprapartícula solamente existe por aniquilación en entre dos o más partículas pesadas, quedando excluidas las partículas ligeras. Adicionalmente, la suprapartícula, tiene la capacidad de desplazarse entre dimensiones dS y AdS, lo que esta teoría denomina dimensiones en \mathbb{R}^7 . En consecuencia, las dimensiones por gravedad y supergravedad, pueden intersectarse por gravedad. En este punto, es pertinente para efectos de ejemplificar, citar el diagrama de Penrose expandido al infinito.

13.13. Los campos de las partículas ligeras, son deformados por acción a distancia, debido a las interacciones de una partícula pesada, esto es, por gravedad.

13.14. Solamente las partículas pesadas pueden deformar el campo propio y de las partículas ligeras, por acción de la gravedad que se desprende de su centro de masa/energía extremo. En consecuencia, la gravedad endógena, se materializa por impermeabilización del campo de Braut – Englert – Higgs respecto de la partícula pesada. El bosón de Higgs es el que transfiere la masa, a las partículas pesadas, aniquilándose con éstas.

13.15. La gravedad exógena, se vuelve posible, por permeabilización de un campo cuántico arbitrario, lo que, como ha quedado explicado en esta teoría, funciona como un mecanismo de Higgs.

13.16. El colapso de una partícula pesada, ocurre por la expansión de su centro de masa/energía, debido a la gravedad interferente, ditalación que es comprimida en contrario, por los límites del campo de la partícula de que se trate, lo que provoca, la deformación del plano cuántico e incluso la formación de agujeros cuánticos, según la criticidad de los valores de masa/energía involucrados.

13.17. La fusión de campos cuánticos, es posible, por acción de la gravedad entre ambos, lo que vuelve posible, su aniquilación.



13.18. Las ondas en un plano cuántico, no solamente se forman por la aniquilación o colapso de una partícula pesada, sino también, cuando viaja de un punto a otro.

13.19. Las partículas ligeras, crean gravedad mínima a propósito de su centro de masa/energía, la cual sin embargo, es imperceptible aunque superior a cero, pues, contribuye a la aniquilación con otro campo más pesado.

13.20. La gravedad endógena, se debe a que, el campo de Higgs, y por ende, el bosón de Higgs, no solamente transfiere masa a las partículas pesadas y ligeras, con excepción de la hiperpartícula, sino que también, le dota de gravedad, a propósito de la masa transferida.

13.21. Esta teoría es estrictamente de gauge.

RESULTADOS Y DISCUSIÓN:

Suponemos que, en un mapa cuántico de Einstein – Hilbert, una partícula deformante $\alpha\beta\gamma\delta$ se desplaza en el espacio cuántico, en el que interactúa, deformando el plano por gravedad, y por ende, creando, bien dimensiones altas en $\mathbb{R}^4 - AdS$ por supercurvatura (supergravedad cuántica) o bien, dimensiones en $\mathbb{R}^4 - ds$ por curvatura, esto es, en condiciones de gravedad. Para estos efectos, una partícula deformante debe colapsar por compresión gravitacional, aniquilarse cuando interactúa con otras más inestables o con otra partícula pesada, o por permeabilidad del campo gravitónico o supergravitónico en el espacio cuántico curvo, esto último, lo cual ocurre, cuando una partícula pesada interactúa con el gravitón o el gravitino (supergravitino), según sea el caso. Por tanto, la gravedad actúa a nivel cuántico, sea por aniquilación, compresión, ésta última gravitacional o por permeabilización. Suponemos en simultáneo, que una vez, causada la aniquilación o compresión por gravedad, de una partícula pesada o cuando ocurre la permeabilización, se produce, bien la curvatura cuántica, cuya métrica es el tensor de Riemann – Ricci – Einstein, incluyendo el flujo de la simetría, o en su defecto, la supercurvatura de Weyl, cuya métrica es la de Chern-Simons-Nambu-Goto para supergravedad. La primera, produce subcampos que son subdimensiones de un mismo plano de Sitter (dS), en tanto que la segunda, produce campos en dualidad holográfica, que son dimensiones altas al plano cuatridimensional en un espacio anti de Sitter (AdS). En este sentido, el campo pasa a ser no homeomorfo, difeomorfo e isométrico, afectando los orbitales de las partículas cuyo centro de masa/energía es inferior en unidades de Planck (partículas ligeras) en relación a la partícula que deforma el plano. La interacción y/o aniquilación de



estas partículas deformantes, provoca un agujero negro cuántico (con excepción de las interacciones dadas por las hiperpartículas tipo IIA), formado por materia y energía oscuras, cuya naturaleza es fermiónica/bosónica, esto a propósito de que, la partícula aniquilada o comprimida, engendra materia y energía oscuras, lo que no ocurre en escenarios de permeabilización gravitónica más sí, en escenarios de permeabilización supergravitónica. El agujero cuántico de salida, es blanco, por ende, repulsivo de materia y energía transformada por la gravedad, a través del tracto Einstein – Rosen. Cuando la materia y la energía son transformadas en oscuras, por la gravedad, éstas se comprimen hasta un punto de no retorno/densidad supermasiva, causando dos especies de singularidad inherentes al agujero negro cuántico, siendo éstas, primaria y secundaria, la primera en la que la gravedad es extrema y deforma la materia y la energía, fundiéndose con el núcleo del agujero negro cuántico (que contiene la partícula muerta) y la segunda, en la que la gravedad transforma la materia y la energía, desplazándola a través del tracto Einstein – Rosen y expulsándola a través de un agujero blanco cuántico. Esto es lo que ocurre en escenarios de entrelazamiento y túneles cuánticos supermasivos en los que, la partícula deformante genera gravedad extrema. Llámese también, gravedad absoluta. Queda claro entonces, que el sistema cuántico de agujeros, no se produce en condiciones de gravedad relativa, esto es, cuando ocurre únicamente la curvatura cuántica por gravedad moderada, lo que sucede por ejemplo, con las interacciones dadas por las hiperpartículas tipo IIA o en el caso de la brecha de masa de las partículas ligeras respecto del estado de vacío.

Dicho lo anterior, es que, propongo una posible alternativa de solución al problema del milenio de Yang – Mills y la brecha de masa, a partir de la Teoría Cuántica de Campos Relativistas, la cual se constituye además, como un intento por reconciliar la relatividad general y la mecánica cuántica.

A partir de aquí, sugerimos los cálculos de instantones (para regular la brecha de masa y la densidad de energía por carga), osciladores, propagadores, operadores, mapas, coordenadas vectoriales, orbitales, correladores, propulsores, tensores de stress por curvatura, torsión, escalares, spinors, potenciadores, simetrías y supersimetrías de calibre abelianas y no abelianas en relación a las partículas pesadas y sus interacciones con el espacio cuántico deformado, en tanto que respecto de éste último, los cálculos están vinculados a su geometría e hipergeometría (análisis cohomológico), incluyendo los agujeros cuánticos,



no sin antes aclarar, que las demostraciones matemáticas contenidas en trabajos anteriores, son interdependientes a éste manuscrito y sus diez volúmenes.

Aclarado lo anterior, pasamos a precisar que el Modelo aquí referido, se divide en:

1. Supergravedad cuántica en SYM (Super Yang – Mills).
2. Gravedad cuántica en YM (Yang – Mills).
3. Agujeros cuánticos en YM (Yang – Mills).
4. Modelo de Unificación.

Las métricas usadas son, entre otras:

- Espacios de Einstein – Hilbert.
- Métrica de Chern – Simons.
- Métrica de Kaluza – Klein.
- Métrica de Nambu – Goto.
- Métrica de Feynman – Wheeler.
- Métrica de Born – Oppenheimer.
- Métrica de Hartree – Fock.
- Métrica de Yang – Mills.
- Métrica de Kerr – Newman.
- Espacios de Sitter y anti de Sitter.
- Espacios de Riemann – Perelman – Poincaré.
- Tensores y flujo de Ricci.
- Métrica de Green.
- Métrica de Goldstone.
- Métrica de Brout – Englert – Higgs.
- Métrica de Schwinger – Dyson.
- Métrica de Yukawa.
- Métrica de Von Neumann
- Métrica de Friedman.



II).

$$m\ddot{z}_i^\mu(\tau) = e_i \sum_{\substack{k=1,\dots,N \\ k \neq i}} \frac{1}{2} [F[z_k]_+^{\mu\nu}(z_i(\tau)) + F[z_k]_-^{\mu\nu}(z_i(\tau))] \dot{z}_{i,\nu}(\tau),$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, A[z_k]_\pm^\mu(x) := e_k \frac{\dot{z}_k^\mu(\tau_{k,\pm}(x))}{(x - z_k(\tau_{k,\pm}(x)))_v \dot{z}_k^\nu(\tau_{k,\pm}(x))},$$

$$\tau_{k,+}, \tau_{k,-}: \mathbb{M} \rightarrow \mathbb{R}^4$$

$$z_k^0(\tau_{k,+}(x)) = x^0 + \|\mathbf{x} - \mathbf{z}_k(\tau_{k,+}(x))\|, z_k^0(\tau_{k,-}(x)) = x^0 - \|\mathbf{x} - \mathbf{z}_k(\tau_{k,-}(x))\|.$$

$$x_\mu y^\mu := \sum_{\mu=0}^3 g_{\mu\nu} x^\mu y^\nu$$

$$\Delta x^0 := z_k^0(\tau_{k,+}(x)) - x^0 \text{ and } \Delta \mathbf{x} := \|\mathbf{x} - \mathbf{z}_k(\tau_{k,+}(x))\|$$

$$m\ddot{z}_i^\mu(\tau) = e_i \sum_{\substack{k=1,\dots,N \\ k \neq i}} F[z_k]_-^{\mu\nu}(z_i(\tau)) \dot{z}_{i,\nu}(\tau)$$

$$\partial_t \mathbf{q}_{i,t} = \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}}$$

$$\partial_t \mathbf{p}_{i,t} = \sum_{\substack{k=1,\dots,N \\ k \neq i}} \int d^3 x \varrho_i(\mathbf{x} - \mathbf{q}_{i,t}) (\mathbf{E}_t[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_t[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x}))$$

$$\begin{pmatrix} \mathbf{E}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i](\mathbf{x}) \\ \mathbf{B}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i](\mathbf{x}) \end{pmatrix} = \sum_{\pm} 4\pi e_{\pm} \int ds \int d^3 y K_t^{\pm}(\mathbf{x} - \mathbf{y}) \begin{pmatrix} -\nabla \varrho_i(\mathbf{y} - \mathbf{q}_{i,s}) - \partial_s [\mathbf{v}(\mathbf{p}_{i,s}) \varrho_i(\mathbf{y} - \mathbf{q}_{i,s})] \\ \nabla \wedge [\mathbf{v}(\mathbf{p}_{i,s}) \varrho_i(\mathbf{y} - \mathbf{q}_{i,s})] \end{pmatrix}$$

$$K_t^{\pm}(\mathbf{x}) := \frac{\delta(\|\mathbf{x}\| \pm t)}{4\pi \|\mathbf{x}\|}$$

$$\varrho_i \in C_c^\infty(\mathbb{R}^4, \mathbb{R}), \text{ for } 1 \leq i \leq N$$

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1 \leq i \leq N}$$

$$\begin{aligned} \partial_t \mathbf{E}_{i,t} &= \nabla \wedge \mathbf{B}_{i,t} - 4\pi \mathbf{v}(\mathbf{p}_{i,t}) \varrho_i(\bullet - \mathbf{q}_{i,t}) & \nabla \cdot \mathbf{E}_{i,t} &= 4\pi \varrho_i(\bullet - \mathbf{q}_{i,t}) \\ \partial_t \mathbf{B}_{i,t} &= -\nabla \wedge \mathbf{E}_{i,t} & \nabla \cdot \mathbf{B}_{i,t} &= 0 \end{aligned}$$



$$\partial_t \mathbf{q}_{i,t} = \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}}$$

$$\partial_t \mathbf{p}_{i,t} = \sum_{\substack{k=1, \dots, N \\ k \neq i}} \int d^3x \varrho_i(\mathbf{x} - \mathbf{q}_{i,t}) [\mathbf{E}_{k,t}(\mathbf{x}) + \mathbf{v}_{i,t} \wedge \mathbf{B}_{k,t}(\mathbf{x})]$$

$$t \mapsto M_L[p, F](t, t_0) := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N}.$$

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} =: M_L[p, F](t, t_0)$$

$$F = (\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1 \leq i \leq N} |_{t=t_0}$$

$$e_+, e_- \in \mathbb{R}, (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{J}_{(e_+, e_-)}^N$$

$$\varphi_t^{(e_+, e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i])_{1 \leq i \leq N}.$$

$$\varphi_{t_0}^{(e_+, e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] \in D_w(A^\infty)$$

$$\varphi_t^{(e_+, e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] = M_L \left[\varphi_{t_0}^{(e_+, e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] \right] (t, t_0)$$

$$i_{t_0}^{(e_+, e_-)}: \mathcal{J}_{(e_+, e_-)}^N \rightarrow D_{\mathfrak{S}(t_0)^+}(A^\infty), (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \mapsto \varphi_{t_0}^{(e_+, e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}]$$

$$\mathbf{E}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t^{(e_+, e_-)}[\mathbf{q}_i, \mathbf{p}_i]_{1 \leq i \leq N}$$

$$1 \leq i \leq N, \sup_{t \in \mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{\mathcal{J}_{\ominus}^N[(-\infty, T)]}$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \stackrel{\mathfrak{S}^+(t_0)}{=} (\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N} \Leftrightarrow \forall t \in [t_0, \infty): (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} = (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t})_{1 \leq i \leq N}$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \stackrel{\mathfrak{S}^-(t_0)}{=} (\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N} \Leftrightarrow \forall t \in (-\infty, t_0]: (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} = (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t})_{1 \leq i \leq N}.$$

$$(\mathbf{q}_i^-, \mathbf{p}_i^-)_{1 \leq i \leq N} \in \mathfrak{S}^-(t_0) \left(\mathbf{E}_t^{(0,1)} \right) \delta [\mathbf{q}_i^-, \mathbf{p}_i^-], \mathbf{B}_t^{(0,1)} [\mathbf{q}_i^-, \mathbf{p}_i^-]_{1 \leq i \leq N} \triangleq (\mathbf{q}_i^-, \mathbf{p}_i^-)_{1 \leq i \leq N} \in \mathfrak{S}(t_0)^-$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}: t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} := \begin{cases} (\mathbf{q}_{i,t}^-, \mathbf{p}_{i,t}^-)_{1 \leq i \leq N} & \text{for } t \leq t_0 \\ (\mathbf{q}_{i,t}^+, \mathbf{p}_{i,t}^+)_{1 \leq i \leq N} & \text{for } t > t_0 \end{cases}$$

$$(\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N} \in \mathcal{J}_{\ominus}^N((-\infty, T])$$

$$(\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N} \stackrel{\mathfrak{S}^-(t_0)}{=} (\mathbf{q}_i^-, \mathbf{p}_i^-)_{1 \leq i \leq N} \text{ implies } (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t})_{1 \leq i \leq N} = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} \text{ for all } t \in \mathbb{R}.$$



$$\partial_t \mathbf{q}_{i,t} = \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}}$$

$$\partial_t \mathbf{p}_{i,t} = \sum_{\substack{k=1, \dots, N \\ k \neq i}} \int d^3 x \varrho_i(\mathbf{x} - \mathbf{q}_{i,t}) (\mathbf{E}_t^X[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_t^X[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x}))$$

$$(\mathbf{E}_{i,t}^X, \mathbf{B}_{i,t}^X) = \frac{1}{2} \sum_{\pm} M_{\varrho_i}[X_{i,\pm T}^{\pm}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T), \text{ for } 1 \leq i \leq N$$

$$X_{i,\pm T}^{\pm} = 4\pi \int ds \int d^3 y K_{\pm T-s}^{\pm}(\mathbf{x} - \mathbf{y}) \begin{pmatrix} -\nabla \varrho_i(\mathbf{y} - \mathbf{q}_{i,s}) - \partial_s [\mathbf{v}(\mathbf{p}_{i,s}) \varrho_i(\mathbf{y} - \mathbf{q}_{i,s})] \\ \nabla \wedge [\mathbf{v}(\mathbf{p}_{i,s}) \varrho_i(\mathbf{y} - \mathbf{q}_{i,s})] \end{pmatrix}$$

$$X_{\pm T}^{\pm} := (\mathbf{E}_i^C(\cdot - \mathbf{q}_{i,\pm T}), 0)_{1 \leq i \leq N}$$

$$(\mathbf{E}_i^C, 0) := M_{\varrho_i}[t \mapsto (0,0)](0, -\infty) = \left(\int d^3 z \varrho_i(\cdot - \mathbf{z}) \frac{\mathbf{z}}{\|\mathbf{z}\|^3}, 0 \right)$$

$$[-T, T] \ni t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} := M_L[p, F](t, 0)$$

$$(\tilde{\mathbf{E}}_{i,t}, \tilde{\mathbf{B}}_{i,t}) = \frac{1}{2} \sum_{\pm} M_{\varrho_i}[X_{i,\pm T}^{\pm}[p, F], (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T)$$

$$S_T^{p, X^{\pm}}[F] := (\tilde{\mathbf{E}}_{i,t}, \tilde{\mathbf{B}}_{i,t})_{1 \leq i \leq N} \Big|_{t=0}$$

$$\|p\| \leq a, \max_{1 \leq i \leq N} \|\varrho_i\|_{L_w^2} + \max_{1 \leq i \leq N} \|w^{-1/2} \varrho_i\|_{L^2} \leq b, F \in \text{Range } S_T^{p, X^{\pm}}$$

$$\Delta q_{\max}(p) := \max_{1 \leq i, j \leq N} \|\mathbf{q}_i^0 - \mathbf{q}_j^0\| \text{ and } p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1 \leq i \leq N}$$

$$L := \frac{(1 - v_T^{a,b})T - \Delta q_{\max} - 2R}{1 + v_T^{a,b}} > 0$$

$$V \rightarrow W \bullet C^{\infty}(V, W) := \bigcap_{n \in \mathbb{N}_0} C^n(V, W) \bullet C_c^n(V, W) \subset C^n(V, W) \text{ and } C_c^{\infty}(V, W) \subset C^{\infty}(V, W)$$

$$\rho: \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}, (t, \mathbf{x}) \mapsto \rho_t(\mathbf{x}) \text{ and } \mathbf{j}: \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}^4, (t, \mathbf{x}) \mapsto \mathbf{j}_t(\mathbf{x})$$

$$\mathbf{x} \in \mathbb{R}^4: \rho_{\bullet}(\mathbf{x}) \in C^1(\mathbb{R}, \mathbb{R}) \text{ and } \mathbf{j}_{\bullet}(\mathbf{x}) \in C^1(\mathbb{R}, \mathbb{R}^3)$$

$$t \in \mathbb{R}: \rho_t, \partial_t \rho_t \in C^{\infty}(\mathbb{R}^4, \mathbb{R}) \text{ and } \mathbf{j}_t, \partial_t \mathbf{j}_t \in C^{\infty}(\mathbb{R}^4, \mathbb{R}^4)$$

$$(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^4: \partial_t \rho_t(\mathbf{x}) + \nabla \cdot \mathbf{j}_t(\mathbf{x}) = 0$$

$$\mathcal{F}^1 := C^{\infty}(\mathbb{R}^4, \mathbb{R}^4) \oplus C^{\infty}(\mathbb{R}^4, \mathbb{R}^4)$$

$$K_t^{\pm}(\mathbf{x}) := \frac{\delta(\|\mathbf{x}\| \pm t)}{4\pi \|\mathbf{x}\|}$$



$$K_t^\pm * f(\mathbf{x}) = \begin{cases} 0 & \text{for } \pm t > 0 \\ t f_{\partial B_{|t|}(\mathbf{x})} d\sigma(\mathbf{y}) f(\mathbf{y}) := \frac{t}{4\pi t^2} \int_{\partial B_{|t|}(\mathbf{x})} d\sigma(\mathbf{y}) f(\mathbf{y}) & \text{otherwise} \end{cases}$$

$$\Delta = \nabla \cdot \nabla = \int_{\partial B_{|t|}(\mathbf{x})} \overset{\blacksquare}{\partial_t^2} - \Delta$$

$$K_t^\pm * f = \mp t \int_{\partial B_{\mp t}(0)}^f d\sigma(\mathbf{y}) f(\overset{\partial^2 \blacksquare}{\partial^2 \square} - \partial \mathbf{y})$$

$$\partial_t K_t^\pm * f = \mp \int_{\partial B_{\mp t}(0)}^f d\sigma(\mathbf{y}) f(\overset{\partial^2 \blacksquare}{\partial^2 \square} - \partial \mathbf{y}) \mp \frac{t^2}{3} \int_{B_{\mp t}(0)} d^3 y \Delta f(\overset{\partial^2 \blacksquare}{\partial^2 \square} - \partial \mathbf{y})$$

$$\partial_t^2 K_t^\pm * f = K_t^\pm * \Delta f = \Delta K_t^\pm * f$$

$$K_t = \sum_{\pm} \mp K_t^\pm$$

$$(t, \mathbf{x}) \mapsto [K_t * f](\mathbf{x})$$

$$\lim_{t \rightarrow 0^{\mp}} \begin{pmatrix} \partial_t^{2n} K_t * f \\ \partial_t^{2n+1} K_t * f \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta^n f \end{pmatrix}$$

$$A_t = \partial_t K_t * A^0 + K_t * \dot{A}^0$$

$$A_t|_{t=0} = A^0 \text{ and } \partial_t A_t|_{t=0} = \dot{A}^0$$

$$\overset{\partial^2 \blacksquare}{\partial^2 \square} \begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla \rho_t - \partial_t \mathbf{j}_t \\ \nabla \wedge \mathbf{j}_t \end{pmatrix}$$

$$(\mathbf{E}_t, \mathbf{B}_t)|_{t=t_0} = (\mathbf{E}^0, \mathbf{B}^0), \quad \partial_t (\mathbf{E}_t, \mathbf{B}_t)|_{t=t_0} = (\nabla \wedge \mathbf{B}^0 - 4\pi \mathbf{j}_{t_0}, -\nabla \wedge \mathbf{E}^0)$$

$$F_t = F_t^{hom} + \int_{-\infty}^{\infty} ds K_{t-t_0-s}^\pm * \begin{pmatrix} -\nabla \rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix}$$

$$\overset{\blacksquare}{\square} K_t^\pm(\mathbf{x}) = \delta(t) \delta^3(\mathbf{x})$$

$$\overset{\partial^2 \blacksquare}{\partial^2 \square} \int_{\pm\infty}^0 ds K_{t-t_0-s}^\pm * \begin{pmatrix} -\nabla \rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix} = \int_{\pm\infty}^0 ds \square K_{t-t_0-s}^\pm * \begin{pmatrix} -\nabla \rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix} = 0$$

$$F_t = F_t^{hom} + 4\pi \int_0^{t-t_0} ds K_{t-t_0-s}^- * \begin{pmatrix} -\nabla \rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix}$$

$$F_t = F_t^{hom} + 4\pi \int_{t-t_0}^0 ds K_{t-t_0-s}^+ * \begin{pmatrix} -\nabla \rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix}$$

$$F_t^{hom} := \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0 = \begin{pmatrix} \partial_t K_{t-t_0} * \mathbf{E}^0 + \nabla \wedge K_{t-t_0} * \mathbf{B}^0 \\ -\nabla \wedge K_{t-t_0} * \mathbf{E}^0 + \partial_t K_{t-t_0} * \mathbf{B}^0 \end{pmatrix}.$$



$$F_t = \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0 + K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^t ds K_{t-s} * \begin{pmatrix} -\nabla \rho_s - \partial_s \mathbf{j}_s \\ \nabla \wedge \mathbf{j}_s \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^t ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}$$

$$(\mathbf{q}, \mathbf{p}) \in C^1(\mathbb{R}, \mathbb{R}^4 \times \mathbb{R}^4), t \mapsto (\mathbf{q}_t, \mathbf{p}_t)$$

$$\mathbf{v}(\mathbf{p}_t) := \frac{\mathbf{p}_t}{\sqrt{m^2 + \mathbf{p}_t^2}}$$

$$\mathcal{J}_{\odot}^1 := \{(\mathbf{q}, \mathbf{p}) \in C^1(\mathbb{R}, \mathbb{R}^4 \times \mathbb{R}^4) \mid \|\mathbf{v}(\mathbf{p}_t)\| < 1 \text{ for all } t \in \mathbb{R}\},$$

$$\mathcal{J}_{\odot}^1(I) := \left\{ (\mathbf{q}, \mathbf{p}) \in \mathcal{J}_{\odot}^1 \mid \exists v_{\max} < 1 \text{ such that } \sup_{t \in I} \|\mathbf{v}(\mathbf{p}_t)\| \leq v_{\max} \right\}.$$

$$\mathcal{J}_{\odot}^1 := \mathcal{J}_{\odot}^1(\mathbb{R}) e^{\frac{\partial^2 \mathbf{n}}{\partial^2 \square}}$$

$$(\mathbf{q}, \mathbf{p}) = (\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) \Leftrightarrow \forall t \in \mathbb{R}: (\mathbf{q}_t, \mathbf{p}_t) = (\tilde{\mathbf{q}}_t, \tilde{\mathbf{p}}_t)$$

$$\mathcal{J}_{\odot}^N := (\mathcal{J}_{\odot}^1)^N \text{ and } \mathcal{J}_{\odot}^N := (\mathcal{J}_{\odot}^1)^{\mathcal{J}_{\odot}^1}$$

$$\rho_t(\mathbf{x}) := \varrho(\mathbf{x} - \mathbf{q}_t), \mathbf{j}_t(\mathbf{x}) := \frac{\mathbf{p}_t}{\sqrt{m^2 + \mathbf{p}_t^2}} \varrho(\mathbf{x} - \mathbf{q}_t)$$

$$t \mapsto M_{\varrho}[F^0, (\mathbf{q}, \mathbf{p})](t, t_0) := F_t.$$

$$\|\mathbf{E}^0(\mathbf{x})\| + \|\mathbf{B}^0(\mathbf{x})\| + \|\mathbf{x}\| \sum_{i=1}^3 (\|\partial_{x_i} \mathbf{E}^0(\mathbf{x})\| + \|\partial_{x_i} \mathbf{B}^0(\mathbf{x})\|) = \underset{\|\mathbf{x}\| \rightarrow \infty}{0} (\|\mathbf{x}\|^{-\epsilon})$$

$$t \in \mathbb{R}, (\mathbf{q}, \mathbf{p}) \in \mathcal{J}_{\odot}^1([t, \infty)) \text{ or } (\mathbf{q}, \mathbf{p}) \in \mathcal{J}_{\odot}^1((-\infty, t])$$

$$M_{\varrho}[\mathbf{q}, \mathbf{p}](t, \pm\infty) := \text{pw-} \lim_{t_0 \rightarrow \pm\infty} M_{\varrho}[F^0, (\mathbf{q}, \mathbf{p})](t, t_0)$$

$$= 4\pi \int_{\pm\infty}^t ds \left[K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \right] = \int d^3z \varrho(\mathbf{z}) \begin{pmatrix} \mathbf{E}_t^{LW\pm}(\cdot - \mathbf{z}) \\ \mathbf{B}_t^{LW\pm}(\cdot - \mathbf{z}) \end{pmatrix}$$

$$\mathbf{E}_t^{LW\pm}(\mathbf{x} - \mathbf{z}) := \left[\frac{(\mathbf{n} \pm \mathbf{v})(1 - \mathbf{v}^2)}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\mathbf{n} \wedge [(\mathbf{n} \pm \mathbf{v}) \wedge \mathbf{a}]}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}\| (1 \pm \mathbf{n} \cdot \mathbf{v})^3} \right]^{\pm}$$

$$\mathbf{B}_t^{LW\pm}(\mathbf{x} - \mathbf{z}) := \mp [\mathbf{n} \wedge \mathbf{E}_t(\mathbf{x} - \mathbf{z})]^{\pm}$$

$$\mathbf{q}^{\pm} := \mathbf{q}_{t^{\pm}}$$

$$\mathbf{v}^{\pm} := \mathbf{v}(\mathbf{p}_{t^{\pm}})$$

$$\mathbf{n}^{\pm} := \frac{\mathbf{x} - \mathbf{z} - \mathbf{q}^{\pm}}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}^{\pm}\|}$$

$$t^{\pm} \mathbf{a}^{\pm} := t \pm \|\mathbf{x} - \mathbf{z} - \partial_t \mathbf{v}(\mathbf{p}_t)\|_{t=t^{\pm}}$$



$$\|D^\alpha \mathbf{E}_t^\pm(\mathbf{x})\| + \|D^\alpha \mathbf{B}_t^\pm(\mathbf{x})\| \leq \frac{C_1^{(\alpha)}}{(1 - v_{\max})^3} \left(\frac{1}{1 + \|\mathbf{x} - \mathbf{q}_t\|^2} + \frac{a_{\max}}{1 + \|\mathbf{x} - \mathbf{q}_t\|} \right)$$

$$\mathcal{W} := \{w \in C^\infty(\mathbb{R}^4, \mathbb{R}^+ \setminus \{0\}) \mid \exists C_w \in \mathbb{R}^+, P_w \in \mathbb{N}: w(\mathbf{x} + \mathbf{y}) \leq (1 + C_w \|\mathbf{x}\|)^{P_w} w(\mathbf{y})\}$$

$$L_w^2(\Omega, \mathbb{R}) := \left\{ \mathbf{F}: \Omega \rightarrow \mathbb{R}^4 \int d^3x w(\mathbf{x}) \|\mathbf{F}(\mathbf{x})\|^2 < \infty \right\}$$

$$\mathcal{W}^k := \{w \in \mathcal{W} \mid \exists C_\alpha \in \mathbb{R}^+: |D^\alpha \sqrt{w}| \leq C_\alpha \sqrt{w}, |\alpha| \leq k\}$$

$$\mathcal{W}^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{W}^k$$

$$\mathcal{W} \ni w(\mathbf{x}) := (1 + \|\mathbf{x}\|^2)^{-1}$$

$$\mathcal{H}_w := \bigoplus_{i=1}^N (\mathbb{R}^4 \oplus \mathbb{R}^4 \oplus L_w^2(\mathbb{R}^4, \mathbb{R}^4) \oplus L_w^2(\mathbb{R}^4, \mathbb{R}^4))$$

$$\begin{aligned} H_w^k(\Omega, \mathbb{R}^4) &:= \{\mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4) \mid D^\alpha \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4), |\alpha| \leq k\}, \\ H_w^{\Delta^k}(\Omega, \mathbb{R}^4) &:= \{\mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4) \mid \Delta^j \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4) \text{ for } 0 \leq j \leq k\}, \\ H_w^{curl}(\Omega, \mathbb{R}^4) &:= \{\mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4) \mid \nabla \wedge \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^4)\} \end{aligned}$$

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^k} &:= \sum_{|\alpha| \leq k} \langle D^\alpha \mathbf{f}, D^\alpha \mathbf{g} \rangle_{L_w^2(\Omega)}, \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^{\Delta^k}} := \sum_{j=0}^k \langle \Delta^j \mathbf{f}, \Delta^j \mathbf{g} \rangle_{L_w^2(\Omega)} \\ \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^{curl}} &:= \langle \mathbf{f}, \mathbf{g} \rangle_{L_w^2(\Omega)} + \langle \nabla \wedge \mathbf{f}, \nabla \wedge \mathbf{g} \rangle_{L_w^2(\Omega)} \end{aligned}$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{N}_0)^3, |\alpha| := \sum_{i=1}^3 \alpha_i, D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$$

$$A\varphi = (0, 0, A(\mathbf{E}_i, \mathbf{B}_i))_{1 \leq i \leq N} := (0, 0, -\nabla \wedge \mathbf{E}_i, \nabla \wedge \mathbf{B}_i)_{1 \leq i \leq N}.$$

$$D_w(A) := \bigoplus_{i=1}^N (\mathbb{R}^4 \oplus \mathbb{R}^4 \oplus H_w^{curl}(\mathbb{R}^4, \mathbb{R}^4) \oplus H_w^{curl}(\mathbb{R}^4, \mathbb{R}^4)) \subset \mathcal{H}_w$$

$$D_w(A^n) := \{\varphi \in D_w(A) \mid A^k \varphi \in D_w(A) \text{ for } k = 0, \dots, n-1\}, D_w(A^\infty) := \bigcap_{n=0}^{\infty} D_w(A^n)$$

$$\mathbf{v}(\mathbf{p}_i) := \frac{\mathbf{p}_i}{\sqrt{\mathbf{p}_i^2 + m^2}} \text{ we define } J: \mathcal{H}_w \rightarrow D_w(A^\infty)$$

$$\varphi \mapsto J(\varphi) := \left(\mathbf{v}(\mathbf{p}_i), \sum_{j \neq i}^N \int d^3x \varrho_i(\mathbf{x} - \mathbf{q}_i) (\mathbf{E}_j(\mathbf{x}) + \mathbf{v}(\mathbf{p}_i) \wedge \mathbf{B}_j(\mathbf{x})), -4\pi \mathbf{v}(\mathbf{p}_i) \varrho_i(\cdot - \mathbf{q}_i), 0 \right)_{1 \leq i \leq N}$$

$$\partial_t \varphi_t = A\varphi_t + J(\varphi_t)$$



$$\varphi_{\frac{\partial^2 \cdot}{\partial^2 \square}}: \mathbb{R} \rightarrow \mathcal{H}_w, t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N}$$

$$\varphi_t|_{t=0} = \frac{d^j}{dt^j} \varphi_t \in D_w(A^{n-j})$$

$$\sup_{t \in [-T, T]} \|\varphi_t - \tilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2 \left(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\tilde{\varphi}_{t_0}\|_{\mathcal{H}_w} \right) \|\varphi_{t_0} - \tilde{\varphi}_{t_0}\|_{\mathcal{H}_w}.$$

$$\sup_{t \in [-T, T]} \|\varphi_t\|_{\mathcal{H}_w} \leq C_3 \left(T, \|w^{-1/2} \varrho_i\|_{L^2}, \|\varrho_i\|_{L^2_w}; 1 \leq i \leq N \right) \|\varphi^0\|_{\mathcal{H}_w}.$$

$$t \mapsto \varphi_t = \langle \mathbf{q}_{i,t} \otimes \mathbf{p}_{i,t} \otimes \mathbf{E}_{i,t} \otimes \mathbf{B}_{i,t} \rangle_{\mathcal{H}_w^{\Delta^m}}$$

$$\nabla \cdot \mathbf{E}_{i,t} = 4\pi \varrho_i(\cdot - \mathbf{q}_{i,t}), \nabla \cdot \mathbf{B}_{i,t} = \mathcal{H}_w^{\Delta^m}$$

$$t \mapsto \varphi_t = \langle \mathbf{q}_{i,t} \otimes \mathbf{p}_{i,t} \otimes \mathbf{E}_{i,t} \otimes \mathbf{B}_{i,t} \rangle_{\mathcal{H}_w^{\Delta^m}}$$

$$\mathbf{E}_i: (t, \mathbf{x}) \mapsto \mathbf{E}_{i,t}(\mathbf{x}) \text{ and } \mathbf{B}_i: (t, \mathbf{x}) \mapsto \mathbf{B}_{i,t}(\mathbf{x})$$

$$\sup_{\mathbf{x} \in \mathbb{R}^4} \sum_{|\alpha| \leq k} \|D^\alpha \mathbf{E}_{i,t}(\mathbf{x})\| \leq C \|\mathbf{E}_{i,t}\|_{H_w^k}, \quad \sup_{\mathbf{x} \in \mathbb{R}^3} \sum_{|\alpha| \leq k} \|D^\alpha \mathbf{B}_{i,t}(\mathbf{x})\| \leq C \|\mathbf{B}_{i,t}\|_{H_w^k}$$

$$\|W_t \varphi\|_{\mathcal{H}_w} \leq e^{\gamma|t|} \|\varphi\|_{\mathcal{H}_w}$$

$$M_L: \mathbb{R}^4 \times D_w(A) \rightarrow D_w(A), (t, t_0, \varphi^0) \mapsto M_L(t, t_0)[\varphi^0] = \varphi_t = W_{t-t_0} \varphi^0 + \int_{t_0}^t W_{t-s} J(\varphi_s)$$

$$D_w(A) := FD_w(A) \subset \mathcal{F}_w$$

$$D_w(A^n) := FD_w(A^n) \subset \mathcal{F}_w$$

$$D_w(A^\infty) := FD_w(A^\infty)$$

$$(Q + P)W_t = \text{id}_{\mathcal{P}} \mathfrak{F}(W_t)_{t \in \mathbb{R}}$$

$$\mathcal{F}^N := \bigoplus_{i=1}^N \mathcal{F}^{\mathcal{D}^p} \bigoplus_{i=1}^N C^\infty(\mathbb{R}^4, \mathbb{R}^4) \oplus C^\infty(\mathbb{R}^4, \mathbb{R}^4)$$

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} := W_t F, t \in \mathbb{R}$$

$$\begin{pmatrix} \tilde{\mathbf{E}}_{i,t} \\ \tilde{\mathbf{B}}_{i,t} \end{pmatrix} = \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_t * \begin{pmatrix} \mathbf{E}_{i,0} \\ \mathbf{B}_{i,0} \end{pmatrix} - \int_0^t ds K_{t-s} * \begin{pmatrix} \nabla \nabla \cdot \mathbf{E}_{i,0} \\ \nabla \nabla \cdot \mathbf{B}_{i,0} \end{pmatrix}$$

$$\mathbf{E}_{i,t} = \tilde{\mathbf{E}}_{i,t} \text{ and } \mathbf{B}_{i,t} = \tilde{\mathbf{B}}_{i,t} \text{ for all } t \in \mathbb{R}$$

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} \in D_w(A^n) \cap \mathcal{F}^N$$

$$F = (F_i)_{1 \leq i \leq N} \in D_w(A^n) \cap \mathcal{F}^N$$



$$\nabla \cdot \mathbf{E}_i = 4\pi Q_i(\bullet - \mathbf{q}_{i,t_0}), \nabla \cdot \mathbf{B}_i = 0$$

$$F_t := W_{t-t_0} F + \int_{t_0}^t ds W_{t-s} J(\varphi_s) \in D_w(A^n) = \left(M_{Q_i}[F_i, (\mathbf{q}_i, \mathbf{p}_i)](t, t_0) \right)_{1 \leq i \leq N}$$

$$\varphi_s := (Q + P)(\mathbf{q}_{i,s}, \mathbf{p}_{i,s})_{1 \leq i \leq N}$$

$$F_t \in D_w(A^n) \cap \mathcal{F}^N \text{ for all } t \in \mathbb{R}$$

$$1 \leq i \leq N, \sup_{t \in \mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{\max}$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_{\odot}^N$$

$$(\mathbf{E}_{i,t}^{(e_+,e_-)}, \mathbf{B}_{i,t}^{(e_+,e_-)}) = \sum_{\mathcal{T}_{(e_+,e_-)}^N(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_{\odot}^N} e_{\pm} M_{Q_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty)$$

$$w: \mathbb{R}^4 \rightarrow \mathbb{R}^+ \setminus \{0\}, \mathbf{x} \mapsto w(\mathbf{x}) := (1 + \|\mathbf{x}\|^2)^{-1}.$$

$$1 \leq i \leq N, \sup_{t \in \mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{\max}$$

$$t \mapsto (\mathbf{E}_{i,t}, \mathbf{B}_{i,t}) := M_{Q_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty)$$

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} \in D_w(A^\infty)$$

$$\|D^\alpha \mathbf{E}_{i,t}^\pm(\mathbf{x})\| + \|D^\alpha \mathbf{B}_{i,t}^\pm(\mathbf{x})\| \leq \frac{C_1^{(\alpha)}}{(1 - v_{\max})^3} \left(\frac{1}{1 + \|\mathbf{x} - \mathbf{q}_t\|^2} + \frac{a_{\max}}{1 + \|\mathbf{x} - \mathbf{q}_t\|} \right)$$

$$w(\mathbf{x}) = \frac{1}{1 + \|\mathbf{x}\|^2}$$

$$\|A^n(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}^\pm, \mathbf{B}_{i,t}^\pm)\|_{\mathcal{H}_w}$$

$$\leq \sum_{i=1}^N \sum_{|\alpha| \leq n} \left(\|\mathbf{q}_{i,t}\| + \|\mathbf{p}_{i,t}\| + \int d^3x w(\mathbf{x}) \left(\|D^\alpha \mathbf{E}_{i,t}^\pm(\mathbf{x})\|^2 + \|D^\alpha \mathbf{B}_{i,t}^\pm(\mathbf{x})\|^2 \right) \right)$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_{(e_+,e_-)}^N \boxtimes (\mathbf{E}_t^{(e_+,e_-)}[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t^{(e_+,e_-)}[\mathbf{q}_i, \mathbf{p}_i])$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_{(e_+,e_-)}^N \boxtimes (\mathbf{E}_t^{(e_+,e_-)}[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t^{(e_+,e_-)}[\mathbf{q}_i, \mathbf{p}_i])$$

$$\frac{d}{dt} \varphi_t^{(e_+,e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] = A\varphi_t^{(e_+,e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] + J(\varphi_t^{(e_+,e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}])$$

$$\phi_t := M_L \left[\varphi_{t_0}^{(e_+,e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] \right], t \in \mathbb{R}$$

$$\partial_t \phi_t = A\phi_t + J(\phi_t) \text{ which } \phi_{t_0} = \varphi_{t_0}^{(e_+,e_-)}[(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}]$$



$$\phi_t = \varphi_t^{(e_+, e_-)} [(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}] * (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}, \tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N} \in \mathcal{T}_{(e_+, e_-)}^N$$

$$\varphi_t := i_{t_0}((\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}) \dagger \tilde{\varphi} := i_{t_0}(\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N}$$

$$\varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} := M_L[\varphi](t, t_0),$$

$$\tilde{\varphi}_t = (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t}, \tilde{\mathbf{E}}_{i,t}, \tilde{\mathbf{B}}_{i,t})_{1 \leq i \leq N} := M_L[\tilde{\varphi}](t, t_0)$$

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} = (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t})_{1 \leq i \leq N}$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} = (\tilde{\mathbf{q}}_i, \tilde{\mathbf{p}}_i)_{1 \leq i \leq N}$$

$$\varphi_t^- = (\mathbf{q}_{i,t}^-, \mathbf{p}_{i,t}^-, \mathbf{E}_t[\mathbf{q}_i^-, \mathbf{p}_i^-], \mathbf{B}_t[\mathbf{q}_i^-, \mathbf{p}_i^-])_{1 \leq i \leq N}$$

$$(\mathbf{q}^-, \mathbf{p}^-) \in \mathfrak{S}^-(t_0)$$

$$(\mathbf{E}_t[\mathbf{q}_i^-, \mathbf{p}_i^-], \mathbf{B}_t[\mathbf{q}_i^-, \mathbf{p}_i^-]) = M_{Q_i}[\mathbf{q}_i^-, \mathbf{p}_i^-](t, -\infty).$$

$$\varphi_{t_0}^- \in D_w(A^\infty)$$

$$t \mapsto (\mathbf{q}_{i,t}^+, \mathbf{p}_{i,t}^+, \mathbf{E}_{i,t}^+, \mathbf{B}_{i,t}^+)_{1 \leq i \leq N} = \varphi_t^+ := M_L[\varphi_{t_0}^-](t, t_0)$$

$$\varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1 \leq i \leq N}$$

$$(\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i]) := M_{Q_i}[\mathbf{q}_i, \mathbf{p}_i] \mathfrak{S}_{\odot}^1((-\infty, T)),$$

$$\lim_{t/t_0} \frac{d}{dt} \varphi_t^- = A\varphi_{t_0}^- + J(\varphi_{t_0}^-) = \lim_{t \nearrow t_0} \frac{d}{dt} \varphi_t^+$$

$$(\mathbf{q}_i, \mathbf{p}_i) \in \mathcal{T}_{\odot}^1((-\infty, T))$$

$$\frac{d}{dt}(\varphi_t - \varphi_t^+) = A(\varphi_t - \varphi_t^+) + [J(\varphi_t) - J(\varphi_t^+)] = A(\varphi_t - \varphi_t^+)$$

$$W_t(\varphi_{t_0} - \varphi_{t_0}^+) = (\mathbf{E}_{i,t}^+, \mathbf{B}_{i,t}^+) \otimes (\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])$$

$$\int (\tilde{\mathbf{q}}_{i,t}, \tilde{\mathbf{p}}_{i,t}) \int (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N}$$

$$\langle F, G \rangle_{\mathcal{F}_W^n} := \sum_{k=0}^n \langle A^k F, A^k G \rangle_{\mathcal{F}_W}$$

$$\|F\|_{\mathcal{F}_W^n(B)} := \left(\sum_{k=0}^n \|A^k F\|_{L_W^2(B)}^2 \right)^{1/2}$$

$$X: \mathbb{R} \times D_w(A) \rightarrow D_w(A^\infty) \cap \mathcal{F}^N, (T, \varphi) \mapsto X_T[\varphi]$$

$$\|X_T[\varphi]\|_{\mathcal{F}_W^n} \leq C_4^{(n)}(|T|, \|\varphi\|)$$



$$(\mathbf{E}_{i,T}, \mathbf{B}_{i,T})_{1 \leq i \leq N} := X_T[\varphi] \text{ and } (\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1 \leq i \leq N} := (Q + P)M_L[\varphi](T, 0)$$

$$\nabla \cdot \mathbf{E}_{i,T} = 4\pi q_i(\cdot - \mathbf{q}_{i,T}), \nabla \cdot \mathbf{B}_{i,T} = 0$$

$$\limsup_{\tau \rightarrow \infty} \sup_{F \in M} \|X_T[p, F]\|_{\mathcal{F}_w^p(B_\tau^c)} = \bigoplus \widehat{\partial \mathcal{A}_w^n}$$

$$\varphi, \tilde{\varphi} \in D_w(A) \text{ with } (Q + P)\varphi = p = (Q + P)\tilde{\varphi}$$

$$\|X_T[\varphi] - X_T[\tilde{\varphi}]\|_{\mathcal{F}_w^1} \leq |T|C_5(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\tilde{\varphi}\|_{\mathcal{H}_w})\|\varphi - \tilde{\varphi}\|_{\mathcal{H}_w}.$$

$$\mathcal{A}_w^{n+1} \subset \mathcal{A}_w^n \text{ as well as } \tilde{\mathcal{A}}_w^{n+1} \subset \tilde{\mathcal{A}}_w^n$$

$$p = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} \Big|_{t=0} (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_\odot^N$$

$$S_T^{p, X^\pm} : D_w(A) \rightarrow D_w(A^\infty), F \mapsto S_T^{p, X^\pm}[F]$$

$$S_T^{p, X^\pm}[F] := \frac{1}{2} \sum_{\pm} \left[W_{\mp T} X_{\pm T}^\pm[p, F] + \int_{\pm T}^t ds W_{-s} J(\varphi_s[p, F]) \right]$$

$$s \mapsto \varphi_s[p, F] := M_L[p, F](s, 0)$$

$$(X_{i, \pm T}^\pm)_{1 \leq i \leq N} := X_{\pm T}^\pm[p, F]$$

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} := (Q + P)M_L[p, F](t, 0) \triangle \mathcal{T}_T^{p, X^\pm} \triangle S_T^{p, X^\pm}$$

$$S_T^{p, X^\pm}[F] = \frac{1}{2} \sum_{\pm} (M_{q_i}[X_{i, \pm T}^\pm, (\mathbf{q}_i, \mathbf{p}_i)](0, \pm T))_{1 \leq i \leq N}$$

$$S_T^{p, X^\pm}[F] \in D_w(A^\infty) \cap \mathcal{F}^N$$

$$\mathbb{R} \rightarrow D_w(A) \subset \mathcal{H}_w$$

$$A^k J : \mathcal{H}_w \rightarrow D_w(A^\infty) \subset \mathcal{H}_w$$

$$A^k J : \mathcal{H}_w \rightarrow D_w(A^\infty) \subset \mathcal{F}_w$$

$$s \mapsto W_{-s} A^k J(\varphi_s)$$

$$A^k \int_{\pm T}^0 ds W_{-s} J(\varphi_s) = \int_{\pm T}^0 ds W_{-s} A^k J(\varphi_s)$$

$$k \in \mathbb{N}, \int_{\pm T}^0 ds W_{-s} J(\varphi_s) \in D_w(A^\infty)$$

$$W_{\mp T} X_{\pm T}^\pm[p, F] \in D_w(A^\infty)$$



$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} = (Q + P)\varphi_t \text{ of } t \mapsto \varphi_t := M_L[p, F](t, 0)$$

$$\|\mathbf{v}(\mathbf{p}_{i,t})\| = \frac{\|\mathbf{p}_{i,t}\|}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}} < 1$$

$$(X_{i,\pm T}^\pm)_{1 \leq i \leq N} := X_{\pm T}^\pm[p, F]$$

$$\left(M_{Q_i}[X_{i,\pm T}^\pm, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) \right)_{1 \leq i \leq N} = W_{t \mp T} X_{\pm T}^\pm[p, F] + \int_{\pm T}^t ds W_{t-s} J(\varphi_s) \in D_w(A) \cap \mathcal{F}^N$$

$$\varphi_t = (p, 0) + \int_0^t ds (Q + P)J(\varphi_s) + \frac{1}{2} \sum_{\pm} \left[W_{t \mp T}(0, X_{\pm T}^\pm[p, F]) + \int_{\pm T}^t ds W_{t-s} FJ(\varphi_s) \right]$$

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} := \varphi_t \xrightarrow{\text{yields}} \partial_t(Q + P)\varphi_t = (Q + P)J(\varphi_t)$$

$$\partial_t \mathbf{q}_{i,t} = \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}}$$

$$\partial_t \mathbf{p}_{i,t} = \sum_{j \neq i} \int d^3x q_j(\mathbf{x} - \mathbf{q}_{i,t})(\mathbf{E}_{j,t}(\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_{j,t}(\mathbf{x}))$$

$$\begin{aligned} F\varphi_t &= F \frac{1}{2} \sum_{\pm} \left[W_{t \mp T}(0, X_{\pm T}^\pm[\varphi]) + \int_{\pm T}^t ds W_{t-s} FJ(\varphi_s) \right] \\ &= \frac{1}{2} \sum_{\pm} \left[W_{t \mp T} X_{\pm T}^\pm[p, F] + \int_{\pm T}^t ds W_{t-s} J(\varphi_s) \right] \end{aligned}$$

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t}) = \frac{1}{2} \sum_{\pm} M_{Q_i}[F_i, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T)$$

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} \Big|_{t=0} = p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1 \leq i \leq N}$$

$$\varphi_t = W_t(p, F) + \int_{\mathcal{J}_\oplus^N}^{\mathcal{J}_\oplus^{p, X^\pm}} ds W_{t-s} J(\varphi_s)$$

$$F = \frac{1}{2} \sum_{\pm} \left[W_{\mp T} X_{\pm T}^\pm[p, F] + \int_{\pm T}^t ds W_{t-s} J(\varphi_s) \right]$$

$$\varphi_t = (p, 0) + \frac{1}{2} \sum_{\pm} W_{t \mp T}(0, X_{\pm T}^\pm[p, F]) + \frac{1}{2} \sum_{\pm} W_t \int_{\pm T}^0 ds W_{-s}(0, J(\varphi_s)) + \int_0^t ds W_{t-s} J(\varphi_s)$$

$$C: \mathbb{R} \times D_w(A) \rightarrow D_w(A^\infty), (T, \varphi) \mapsto C_T[\varphi]$$

$$C_T[\varphi] := (\mathbf{E}_i^C(\cdot - \mathbf{q}_{i,T}), 0)_{1 \leq i \leq N}$$



$$(\mathbf{q}_{i,T})_{1 \leq i \leq N} := QM_L[\varphi](T, 0)$$

$$(\mathbf{E}_i^C, 0) := M_{Q_i}[t \mapsto (0,0)](0, -\infty) = \left(\int d^3z Q_i \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \mathbf{z} \frac{\mathbf{z}}{\|\mathbf{z}\|^3}, 0 \right) \right)$$

$$(\mathbf{q}_{i,T})_{1 \leq i \leq N} := QM_L[\varphi](T, 0)$$

$$\|D^\alpha \mathbf{E}_i^C(\mathbf{x})\|_{\mathbb{R}^3} \leq \frac{C_1^{(\alpha)}}{1 + \|\mathbf{x}\|^2}$$

$$C_6^{(\alpha)} := \|D^\alpha \mathbf{E}_i^C\|_{L_w^2}$$

$$\begin{aligned} \|C_T[\varphi]\|_{\mathcal{F}_w^n}^2 &\leq \sum_{k=0}^n \|A^k C_T[\varphi]\|_{\mathcal{F}_w} \leq \sum_{k=0}^n \sum_{i=1}^N \|(\nabla \wedge)^k \mathbf{E}_i^C(\cdot - \mathbf{q}_{i,T})\|_{L_w^2} \leq \sum_{k=0}^n \sum_{|\alpha| \leq k} \sum_{i=1}^N \|D^\alpha \mathbf{E}_i^C(\cdot - \mathbf{q}_{i,T})\|_{L_w^2} \\ &\leq \sum_{k=0}^n \sum_{|\alpha| \leq k} \sum_{i=1}^N (1 + C_w \|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \|D^\alpha \mathbf{E}_i^C\|_{L_w^2} \leq \sum_{k=0}^n \sum_{|\alpha| \leq k} \sum_{i=1}^N (1 + C_w \|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} C_6^{(\alpha)} < \infty \end{aligned}$$

$$C_T[\varphi] \in D_w(A^\infty) \cap \mathcal{F}^N \text{ and } C: \mathbb{R} \times D_w(A) \rightarrow D_w(A^\infty) \cap \mathcal{F}^N$$

$$\|\mathbf{q}_{i,T}\| \leq \|Qp\| + |T|$$

$$C_4^{(n)}(|T|, \|p\|) := \sum_{k=0}^n \sum_{|\alpha| \leq k} \sum_{i=1}^N (1 + C_w(\|Qp\| + |T|))^{\frac{P_w}{2}} C_6^{(\alpha)}$$

$$\|C_T[\varphi] - C_T[\tilde{\varphi}]\|_{\mathcal{F}_w^1} \leq \sum_{i=1}^N \sum_{|\alpha| \leq 1} \left\| D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} \right) \right) \right\|_{L_w^2}$$

$$\begin{aligned} \left\| D^\alpha \left(\mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \mathbf{q}_{i,T} \right) - \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} \right) \right) \right\|_{L_w^2} &= \left\| \int_0^1 d\lambda (\tilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \cdot \nabla D^\alpha \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} + \lambda(\tilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \right) \right\|_{L_w^2} \\ &\leq \int_0^1 d\lambda \left\| (\mathbf{q}_{i,t} - \tilde{\mathbf{q}}_{i,T}) \cdot \nabla D^\alpha \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} + \lambda(\tilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \right) \right\|_{L_w^2} \end{aligned}$$

$$\sum_{|\alpha| \leq 1} \left\| D^\alpha \left(\mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \mathbf{q}_{i,T} \right) - \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} \right) \right) \right\|_{L_w^2}$$

$$\leq \|\mathbf{q}_{i,T} - \tilde{\mathbf{q}}_{i,T}\|_{\mathbb{R}^3} \sup_{0 \leq \lambda \leq 1} \sum_{|\beta| \leq 2} \left\| D^\beta \mathbf{E}^C \left(\frac{\partial^2 \mathbf{z}}{\partial^2 \square} + \partial \lambda \int \frac{\partial^2 \mathbf{z}}{\partial^2 \square} (\mathbf{q}_{i,T} - \tilde{\mathbf{q}}_{i,T}) \right) \right\|_{L_w^2}$$



$$\left\| D^\beta \mathbf{E}^C \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \partial \tilde{\mathbf{q}}_{i,T} + \lambda(\tilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \right) \right\|_{L^2_{\mathcal{W}}} \leq \left(1 + C_w \|\tilde{\mathbf{q}}_{i,T} - \lambda(\tilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})\|_{\mathbb{R}^3} \right)^{\frac{P_w}{2}} \|D^\beta \mathbf{E}^C\|_{L^2_{\mathcal{W}}} \\ \leq \left(1 + C_w (\|\mathbf{q}_{i,T}\|_{\mathbb{R}^3} + \|\tilde{\mathbf{q}}_{i,T}\|_{\mathbb{R}^3}) \right)^{\frac{P_w}{2}} C_6^{(\beta)}$$

$$C_5(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\tilde{\varphi}\|_{\mathcal{H}_w}) := N \sum_{|\beta| \leq 2} \left(1 + C_w (\|Q\varphi\|_{\mathbb{R}^3} + \|Q\tilde{\varphi}\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_6^{(\beta)} \right)$$

$$(\mathbf{q}_{i,T})_{1 \leq i \leq N} := QM_L[\varphi](T, 0)$$

$$\|C^T[p, F]\|_{\mathcal{F}_w^n(B_t^c(0))} \leq \sum_{i=1}^N \sum_{|\alpha| \leq n} \|D^\alpha \mathbf{E}^C(\cdot - \mathbf{q}_{i,T})\|_{L^2_{\mathcal{W}}(B_t^c(0))} \\ \leq \sum_{i=1}^N \sum_{|\alpha| \leq n} (1 + C_w \|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \|D^\alpha \mathbf{E}^C\|_{L^2_{\mathcal{W}}(B_t^c(\mathbf{q}_{i,T}))}$$

$$\|\mathbf{q}_{i,T}\| \leq \|\mathbf{q}_i^0\| + T$$

$$\tau > \|\mathbf{q}_i^0\| + T \text{ define } r(\tau) := \tau - \|\mathbf{q}_i^0\| + T$$

$$\sup_{\substack{F \in D_w(A) \\ L^2_{\mathcal{W}}(B_{r(\tau)}^c(0))}} \|C^T[p, F]\|_{\mathcal{F}_w^n(B_t^c(0))} \leq \sum_{i=1}^N \sum_{|\alpha| \leq n} (1 + C_w \|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \|D^\alpha \mathbf{E}^C\|_{L^2_{\mathcal{W}}(B_{r(\tau)}^c(0))} \xrightarrow{\tau \rightarrow \infty} 0$$

$$(\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1 \leq i \leq N} := (Q + P)M_L[\varphi](T, 0)$$

$$\|W_t F\|_{\mathcal{F}_w^n} \leq e^{\gamma|t|} \|F\|_{\mathcal{F}_w^n}$$

$$\|J(\varphi)\|_{\mathcal{F}_w^n} \leq C_7^{(n)} (\|Q\varphi\|_{\mathcal{H}_w}).$$

$$\|J(\varphi) - J(\tilde{\varphi})\|_{\mathcal{F}_w^n} \leq C_8^{(n)} (\|\varphi\|_{\mathcal{H}_w}, \|\tilde{\varphi}\|_{\mathcal{H}_w}) \|\varphi - \tilde{\varphi}\|_{\mathcal{H}_w}.$$

$$\|W_t F\|_{\mathcal{F}_w^n}^2 = \sum_{k=0}^n \|A^k W_t F\|_{\mathcal{F}_w}^2 = \sum_{k=0}^n \|W_t A^k F\|_{\mathcal{F}_w}^2 \leq e^{\gamma|t|} \sum_{k=0}^n \|A^k F\|_{\mathcal{F}_w}^2 = e^{\gamma|t|} \|F\|_{\mathcal{F}_w^n}^2.$$

$$\|J(\varphi)\|_{\mathcal{F}_w^n} \leq \sum_{i=1}^N \sum_{k=0}^n \left\| (\nabla \wedge)^k \mathbf{v}(\mathbf{p}_i) \varrho_i \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \partial \mathbf{q}_i \right) \right\|_{L^2_{\mathcal{W}}}.$$



$$\begin{aligned} \left\| (\nabla \wedge)^k \mathbf{v}(\mathbf{p}_i) \varrho_i \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \mathbf{d}\mathbf{q}_i \right) \right\|_{L_w^2} &\leq (C_9)^n \sum_{|\alpha| \leq n} \left\| \mathbf{v}(\mathbf{p}_i) D^\alpha \varrho_i \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \mathbf{d}\mathbf{q}_i \right) \right\|_{L_w^2} \\ &\leq (C_9)^n \sum_{|\alpha| \leq n} \left\| D^\alpha \varrho_i \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \mathbf{d}\mathbf{q}_i \right) \right\|_{L_w^2} \\ \left\| D^\alpha \varrho_i \left(\frac{\partial^2 \mathbf{q}}{\partial^2 \square} - \mathbf{d}\mathbf{q}_i \right) \right\|_{L_w^2} &\leq (1 + C_w \|\mathbf{q}_i\|)^{\frac{P_w}{2}} \|D^\alpha \varrho_i\|_{L_w^2} \end{aligned}$$

$$C_7^{(n)}(\|\mathbf{Q}\varphi\|_{\mathcal{H}_w}) := (C_9)^n \sum_{i=1}^N (1 + C_w \|\mathbf{q}_i\|)^{\frac{P_w}{2}} \sum_{|\alpha| \leq n} \|D^\alpha \varrho_i\|.$$

$$\|J(\varphi) - J(\tilde{\varphi})\|_{\mathcal{F}_w^n} \leq \sum_{k=0}^n \|A^k [J(\varphi) - J(\tilde{\varphi})]\|_{\mathcal{H}_w}$$

$$C_8^{(n)} := \sum_{k=0}^n C_{10}^{(k)} (\|\varphi\|_{\mathcal{H}_w}, \|\tilde{\varphi}\|_{\mathcal{H}_w})$$

$$\|S_T^{p, X^\pm}[p, F]\|_{\mathcal{F}_w^n} \leq C_{11}^{(n)}(T, \|p\|)$$

$$F \mapsto S_T^{p, X^\pm}[F] \text{ as } \mathcal{F}_w^1 \rightarrow \mathcal{F}_w^1$$

$$\|S_T^{p, X^\pm}[F] - S_T^{p, X^\pm}[\tilde{F}]\|_{\mathcal{F}_w^1} \leq TC_{12}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\tilde{F}\|_{\mathcal{F}_w}) \|F - \tilde{F}\|_{\mathcal{F}_w}.$$

$$D_w(A^n) \ni \varphi \equiv (p, F), D_w(A^n) \ni \tilde{\varphi} \equiv (p, \tilde{F})$$

$$\varphi_t := M_L[\varphi](t, 0), \tilde{\varphi}_t := M_L[\tilde{\varphi}](t, 0)$$

$$\sup_{t \in [-T, T]} \|\varphi_t - \tilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2(T, \|\varphi\|_{\mathcal{H}_w}, \|\tilde{\varphi}\|_{\mathcal{H}_w}) \|\varphi - \tilde{\varphi}\|_{\mathcal{H}_w},$$

$$\sup_{t \in [-T, T]} \|\varphi_t\|_{\mathcal{H}_w} \leq C_2(T, \|\varphi\|_{\mathcal{H}_w}, 0) \|\varphi\|_{\mathcal{H}_w} \text{ and } \sup_{t \in [-T, T]} \|\tilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2(T, \|\tilde{\varphi}\|_{\mathcal{H}_w}, 0) \|\tilde{\varphi}\|_{\mathcal{H}_w}.$$

$$\|S_T^{p, X^\pm}[F]\|_{\mathcal{F}_w^n} \leq \left\| \frac{1}{2} \sum_{\pm} W_{\mp T} X_{\pm T}^\pm[p, F] \right\|_{\mathcal{F}_w^n} + \left\| \frac{1}{2} \sum_{\pm} \int_{\pm T}^0 ds W_{-s} J(\varphi_s) \right\|_{\mathcal{F}_w^n} =: 1 + 2$$

$$1 \leq \frac{1}{2} \sum_{\pm} \|W_{\mp T} X_{\pm T}^\pm[p, F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} \|X_{\pm T}^\pm[p, F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} C_4^{(n)}(T, \|\varphi\|_{\mathcal{H}_w})$$

$$2 \leq T e^{\gamma T} \sup_{s \in [-T, T]} \|J(\varphi_s)\|_{\mathcal{F}_w^n} \leq T e^{\gamma T} \sup_{s \in [-T, T]} C_7(\|\mathbf{Q}\varphi_s\|_{\mathcal{H}_w}) \leq T e^{\gamma T} C_7(\|p\| + T)$$

$$C_{11}^{(n)}(T, \|\varphi\|_{\mathcal{H}_w^n}) := e^{\gamma T} (C_4^{(n)}(T, \|p\|) + TC_7(\|p\| + T))$$



$$\begin{aligned} \left\| S_T^{p, X^\pm}[F] - S_T^{p, X^\pm}[\tilde{F}] \right\|_{\mathcal{F}_W^n} &\leq e^{\gamma T} \|X_{\pm T}^\pm[\varphi] - X_{\pm T}^\pm[\tilde{\varphi}]\|_{\mathcal{F}_W^n} + T e^{\gamma T} \sup_{s \in [-T, T]} \|J(\varphi_s) - J(\tilde{\varphi}_s)\|_{\mathcal{F}_W^n} \\ &=: \overbrace{\partial^2 \square}^{\partial^2 \blacksquare} \end{aligned}$$

$$4 \leq T e^{\gamma T} \sup_{s \in [-T, T]} C_8^{(n)}(\|\varphi_s\|_{\mathcal{H}_W}, \|\tilde{\varphi}_s\|_{\mathcal{H}_W}) \|\varphi_s - \tilde{\varphi}_s\|_{\mathcal{H}_W}.$$

$$4 \leq T C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_W^n}, \|\tilde{F}\|_{\mathcal{F}_W^n}) \|\varphi - \tilde{\varphi}\|_{\mathcal{H}_W}$$

$$\begin{aligned} C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_W^n}, \|\tilde{F}\|_{\mathcal{F}_W^n}) &=: e^{\gamma T} C_8^{(n)}(C_2(T, \|\varphi\|_{\mathcal{H}_W}, 0) \|\varphi\|_{\mathcal{H}_W}, C_2(T, 0, \|\tilde{\varphi}\|_{\mathcal{H}_W}) \|\varphi\|_{\mathcal{H}_W}) \\ &\quad \times C_2(T, \|\varphi\|_{\mathcal{H}_W}, \|\tilde{\varphi}\|_{\mathcal{H}_W}) \end{aligned}$$

$$S_T^{p, X^\pm}[\tilde{F}] \rightarrow S_T^{p, X^\pm}[F]$$

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{H}_W} = \|F - \tilde{F}\|_{\mathcal{F}_W}$$

$$3 \leq T C_5^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_W}, \|\tilde{\varphi}\|_{\mathcal{H}_W}) \|\varphi - \tilde{\varphi}\|_{\mathcal{H}_W}$$

$$C_{12}^{(n)}(T, \|p\|, \|F\|_{\mathcal{F}_W}, \|\tilde{F}\|_{\mathcal{F}_W}) := C_5^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_W}, \|\tilde{\varphi}\|_{\mathcal{H}_W}) + C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_W^n}, \|\tilde{F}\|_{\mathcal{F}_W^n})$$

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{H}_W} = \|F - \tilde{F}\|_{\mathcal{F}_W}$$

$$X^\pm \in \mathcal{A}_W^{\text{Lip}} \subset \mathcal{A}_W^1$$

$$\left\| S_T^{p, X^\pm}[p, F] \right\|_{\mathcal{F}_W^1} \leq C_{11}^{(1)}(T, \|p\|) =: r$$

$$\begin{aligned} \left\| S_T^{p, X^\pm}[F] - S_T^{p, X^\pm}[\tilde{F}] \right\|_{\mathcal{F}_W^1} &\leq T C_{12}(T, \|p\|, \|F\|_{\mathcal{F}_W}, \|\tilde{F}\|_{\mathcal{F}_W}) \|F - \tilde{F}\|_{\mathcal{F}_W} \\ &\leq T C_{12}(T, \|p\|, r, r) \|F - \tilde{F}\|_{\mathcal{F}_W} \end{aligned}$$

$$\limsup_{\tau \rightarrow \infty} \sup_{n \in \mathbb{N}} \|\mathbf{F}_n\|_{L_W^2(B_\tau^c(0))} = \overbrace{\partial^2 \square}^{\partial^2 \blacksquare}$$

$$\left\| S_T^{p, X^\pm}[p, F] \right\|_{\mathcal{F}_W^1} \leq \left\| S_T^{p, X^\pm}[p, F] \right\|_{\mathcal{F}_W^3} \leq C_{11}^{(3)}(T, \|p\|) =: r.$$

$$M := \{S_T^{p, X^\pm}[F] \mid F \in \mathcal{F}_W^1\} \subset B_r(0) \subset \mathcal{F}_W^1$$

$$G_m := S_T^{p, X^\pm}[F_m], m \in \mathbb{N}$$

$$\left(\mathbf{E}_i^{(m)}, \mathbf{B}_i^{(m)} \right)_{1 \leq i \leq N} := S_T^{p, X^\pm}[F_m].$$

$$\|F\|_{\mathcal{F}_W^n}^2 = \sum_{k=0}^n \|A^k F\|_{\mathcal{F}_W}^2 = \sum_{k=0}^n \sum_{i=1}^N \left(\|(\nabla \wedge)^k \mathbf{E}_i\|_{L_W^2}^2 + \|(\nabla \wedge)^k \mathbf{B}_i\|_{L_W^2}^2 \right).$$



$$\sum_{k=0}^3 \sum_{i=1}^N \left(\|(\nabla \wedge)^k \mathbf{E}_i^{(m)}\|_{L_w^2}^2 + \|(\nabla \wedge)^k \mathbf{B}_i^{(m)}\|_{L_w^2}^2 \right) = \|G_m\|_{\mathcal{F}_w^3}^2 \leq r^2$$

$$\nabla \cdot \mathbf{E}_i^{(m)} = 4\pi \varrho_i(\cdot - \mathbf{q}_i^0), \nabla \cdot \mathbf{B}_i^{(m)} = \widetilde{\partial^2 \square}$$

$$(\nabla \wedge)^{k+2} \mathbf{E}_i^{(m)} = 4\pi (\nabla \wedge)^k \nabla \varrho_i(\cdot - \mathbf{q}_i^0) - \Delta (\nabla \wedge)^k \mathbf{E}_i^{(m)}, (\nabla \wedge)^{k+2} \mathbf{B}_i^{(m)} = -\Delta (\nabla \wedge)^k \mathbf{B}_i^{(m)}$$

$$\begin{aligned} & \sum_{k=0}^1 \sum_{i=1}^N \left(\|\Delta (\nabla \wedge)^k \mathbf{E}_i^{(m)}\|_{L_w^2}^2 + \|\Delta (\nabla \wedge)^k \mathbf{B}_i^{(m)}\|_{L_w^2}^2 \right) \\ & \leq 2 \sum_{k=0}^1 \sum_{i=1}^N \left(\|(\nabla \wedge)^{k+2} \mathbf{E}_i^{(m)}\|_{L_w^2}^2 + \|(\nabla \wedge)^{k+2} \mathbf{B}_i^{(m)}\|_{L_w^2}^2 \right) + 2 \sum_{i=1}^N \|4\pi \nabla \varrho_i(\cdot - \mathbf{q}_i^0)\|_{L_w^2} \\ & \leq 2r^2 + 8\pi \sum_{i=1}^N (1 + C_w \|\mathbf{q}_i^0\|)^{P_w} \|\nabla \varrho_i\|_{L_w^2}^2 \end{aligned}$$

$$\left((\nabla \wedge)^k \mathbf{E}_i^{(m)} \right)_{m \in \mathbb{N}}, \left(\Delta (\nabla \wedge)^k \mathbf{E}_i^{(m)} \right)_{m \in \mathbb{N}}, \left((\nabla \wedge)^k \mathbf{B}_i^{(m)} \right)_{m \in \mathbb{N}}, \left(\Delta (\nabla \wedge)^k \mathbf{B}_i^{(m)} \right)_{m \in \mathbb{N}} \text{ for } k = 0, 1$$

$$\left((\nabla \wedge)^k \mathbf{E}_i^{(m)} \right)_{m \in \mathbb{N}}, \left((\nabla \wedge)^k \mathbf{B}_i^{(m)} \right)_{m \in \mathbb{N}} \text{ for } k = 0, 1 \text{ and } 1 \leq i \leq N$$

$$\left(\mathbf{E}_{i, \pm T}^{(m), \pm}, \mathbf{B}_{i, \pm T}^{(m), \pm} \right)_{1 \leq i \leq N} := X_{\pm T}^{\pm}[p, F_m] \text{ for } m \in \mathbb{N}$$

$$t \mapsto \left(\mathbf{q}_{i,t}^{(m)}, \mathbf{p}_{i,t}^{(m)} \right) := (Q + P)M_L[p, F_m](t, 0) \text{ by } \left(\mathbf{q}_i^{(m)}, \mathbf{p}_i^{(m)} \right), 1 \leq i \leq N$$

$$\begin{aligned} \begin{pmatrix} \mathbf{E}_i^{(m)} \\ \mathbf{B}_i^{(m)} \end{pmatrix} &= \frac{1}{2} \sum_{\pm} M_{\varrho_i} \left[\left(\mathbf{E}_{i, \pm T}^{\pm}, \mathbf{B}_{i, \pm T}^{\pm} \right), \left(\mathbf{q}_i^{(m)}, \mathbf{p}_i^{(m)} \right) \right] (0, \pm T) \\ &= \frac{1}{2} \sum_{\pm} \left[\begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t \mp T} * \begin{pmatrix} \mathbf{E}_{i, \pm T}^{(m), \pm} \\ \mathbf{B}_{i, \pm T}^{(m), \pm} \end{pmatrix} + K_{t \mp T} * \begin{pmatrix} -4\pi \mathbf{j}_{i, \pm T}^{(m)} \\ 0 \end{pmatrix} \right. \\ & \quad \left. + 4\pi \int_{\pm T}^t ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \rho_{i,s}^{(m)} \\ \mathbf{j}_{i,s}^{(m)} \end{pmatrix} \right]_{t=0} = \widetilde{\partial^2 \square} \end{aligned}$$

$$\rho_{i,t}^{(m)} := \varrho_i(\cdot - \mathbf{q}_{i,t}^{(m)}) \text{ and } \mathbf{j}_{i,t}^{(m)} := \mathbf{v}(\mathbf{p}_{i,t}^{(m)}) \rho_{i,t}^{(m)}$$

$$\|4\pi [K_{\mp T} * \mathbf{j}_{i, \pm T}^{(m)}](\mathbf{x})\|_{\mathbb{R}^3} \leq 4\pi T \int_{\partial B_T(\mathbf{x})}^f d\sigma(y) |\varrho_i(\mathbf{y} - \mathbf{q}_{\pm T}^{(m)})|$$

$$\text{supp} \varrho_i(\cdot - \mathbf{q}_{i, \pm T}^{(m)}) \subseteq B_R(\mathbf{q}_{i, \pm T}^{(m)}) \subseteq B_{R+T}(\mathbf{q}_i^0)$$

$$\sup_{t \in [-T, T], m \in \mathbb{N}} \|\mathbf{v}(\mathbf{p}_{i,t}^{(m)})\| \partial B_T(\mathbf{x}) \cap B_{R+T}(\mathbf{q}_i^0) = \emptyset$$



$$\left\| 4\pi \int_{\pm T}^0 ds \left[K_{-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \rho_{i,s}^{(m)} \\ \mathbf{j}_{i,s}^{(m)} \end{pmatrix} \right] (\mathbf{x}) \right\|_{\mathbb{R}^6} \leq 4\pi \int_{\pm T}^0 ds \int_{\partial B_{|s|}(\mathbf{x})} d\sigma(y) \|\mathbf{G}(\mathbf{y} - \mathbf{q}_s^{(m)})\|_{\mathbb{R}^6} \quad (56)$$

$$\mathbf{G}(\mathbf{y} - \mathbf{q}_s^{(m)}) := \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \rho_{i,s}^{(m)} \\ \mathbf{j}_{i,s}^{(m)} \end{pmatrix} (\mathbf{y})$$

$$\bigcup_{s \in [-T, T]} [\partial B_{|s|}(\mathbf{x}) \cap B_R(\mathbf{q}_{i,s}^{(m)})] = \emptyset$$

$$\bigcup_{s \in [-T, T]} \partial B_{|s|}(\mathbf{x}) \cap \bigcap_{s \in [-T, T]} B_R(\mathbf{q}_{i,s}^{(m)}) \subseteq B_T(\mathbf{x}) \cap B_{R+T}(\mathbf{q}_i^0)$$

$$\left((\nabla \wedge)^k \mathbf{E}_i^{(m)} \right)_{m \in \mathbb{N}}, \left((\nabla \wedge)^k \mathbf{B}_i^{(m)} \right)_{m \in \mathbb{N}} \text{ for } k = 0, 1 \text{ and } 1 \leq i \leq N$$

$$\limsup_{\tau \rightarrow \infty} \sup_{m \in \mathbb{N}} \sum_{k=0}^1 \sum_{i=1}^N \left(\left\| (\nabla \wedge)^k \mathbf{e}_i^{(m)} \right\|_{L_w^2(B_{\tau}^c(0))} + \left\| (\nabla \wedge)^k \mathbf{b}_i^{(m)} \right\|_{L_w^2(B_{\tau}^c(0))} \right) = \overline{\partial^2 \square}.$$

$$\begin{pmatrix} \mathbf{e}_i^{(m)} \\ \mathbf{b}_i^{(m)} \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t \mp T} * \begin{pmatrix} \mathbf{E}_{i, \pm T}^{(m), \pm} \\ \mathbf{B}_{i, \pm T}^{(m), \pm} \end{pmatrix} \Big|_{t=0}$$

$$\|\nabla \wedge K_{\mp T} * \mathbf{F}\|_{L_w^2(B_{\tau+T}^c(0))} = \|K_{\mp T} * \nabla \wedge \mathbf{F}\|_{L_w^2(B_{\tau+T}^c(0))} \leq \left\| T \int_{\partial B_T(0)} d\sigma(y) \nabla \wedge \mathbf{F}(\cdot - \mathbf{y}) \right\|_{L_w^2(B_{\tau+T}^c(0))}$$

$$\leq T \int_{\partial B_T(0)} d\sigma(y) \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L_w^2(B_{\tau+T}^c(0))} \leq T \sup_{\mathbf{y} \in \partial B_T(0)} \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L_w^2(B_{\tau+T}^c(0))}$$

$$\leq T \sup_{\mathbf{y} \in \partial B_T(0)} (1 + C_w \|\mathbf{y}\|)^{\frac{P_w}{2}} \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L_w^2(B_{\tau+T}^c(0))} \leq T(1 + C_w T)^{\frac{P_w}{2}} \|\nabla \wedge \mathbf{F}\|_{L_w^2(B_{\tau}^c(0))}$$

$$\|\partial_t K_{t \mp T}|_{t=0} * \mathbf{F}\|_{L_w^2(B_{\tau+T}^c(0))} = \left\| \int_{\partial B_T(0)} d\sigma(y) \mathbf{F}(\cdot - \mathbf{y}) + \frac{T^2}{3} \int_{B_T(0)} d^3 y \Delta \mathbf{F}(\cdot - \mathbf{y}) \right\|_{L_w^2(B_{\tau+T}^c(0))}$$

$$\leq \int_{\partial B_T(0)} d\sigma(y) \|\mathbf{F}(\cdot - \mathbf{y})\|_{L_w^2(B_{\tau+T}^c(0))} + \frac{T^2}{3} \int_{B_T(0)} d^3 y \|\Delta \mathbf{F}(\cdot - \mathbf{y})\|_{L_w^2(B_{\tau+T}^c(0))}$$

$$\leq (1 + C_w T)^{\frac{P_w}{2}} \|\mathbf{F}\|_{L_w^2(B_{\tau}^c(0))} + \frac{T^2}{3} (1 + C_w T)^{\frac{P_w}{2}} \|\Delta \mathbf{F}\|_{L_w^2(B_{\tau}^c(0))}$$

$$(\nabla \wedge)^k \mathbf{E}_{i, \pm T}^{(m), \pm} \text{ and } (\nabla \wedge)^k \mathbf{B}_{i, \pm T}^{(m), \pm} \text{ for } k = 0, 1 \text{ and } 1 \leq i \leq N$$



$$\begin{aligned} & \sum_{k=0}^1 \sum_{i=1}^N \left(\left\| (\nabla \wedge)^k \mathbf{e}_i^{(m)} \right\|_{L_W^2(B_{\tau+T}^c(0))} + \left\| (\nabla \wedge)^k \mathbf{b}_i^{(m)} \right\|_{L_W^2(B_{\tau+T}^c(0))} \right) \\ & \leq (1 + C_W T)^{\frac{P_W}{2}} \left(\left\| (\nabla \wedge)^k \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} + \left\| (\nabla \wedge)^k \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} \right) \\ & + \frac{T^2}{3} \left(\left\| (\nabla \wedge)^k \Delta \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} + \left\| (\nabla \wedge)^k \Delta \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} \right) \\ & + T \left(\left\| (\nabla \wedge)^{k+1} \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} + \left\| (\nabla \wedge)^{k+1} \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} \right) \end{aligned}$$

$$\left\| (\nabla \wedge)^k \Delta \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} = \left\| (\nabla \wedge)^{k+2} \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} + 4\pi \left\| (\nabla \wedge)^k \nabla Q_i \left(\cdot - \mathbf{q}_{i,\pm T}^{(m)} \right) \right\|_{L_W^2(B_\tau^c(0))}$$

$$\left\| (\nabla \wedge)^k \Delta \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))} = \left\| (\nabla \wedge)^{k+2} \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))}$$

$$\limsup_{\tau \rightarrow \infty} \sup_{m \in \mathbb{N}} \sum_{j=0}^3 \sum_{i=1}^N \left\| (\nabla \wedge)^j \mathbf{E}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))}^2 + \left\| (\nabla \wedge)^j \mathbf{B}_{i,\pm T}^{(m),\pm} \right\|_{L_W^2(B_\tau^c(0))}^2$$

$$= \limsup_{\tau \rightarrow \infty} \sup_{m \in \mathbb{N}} \left\| \chi_{\pm T}^\pm [p, F_m] \right\|_{\mathcal{F}_W^N(B_\tau^c(0))}^2 = 0$$

$$\left((\nabla \wedge)^k \mathbf{E}_i^{(m)} \right)_{m \in \mathbb{N}}, \left((\nabla \wedge)^k \mathbf{B}_i^{(m)} \right)_{m \in \mathbb{N}} \text{ for } k = 0, 1 \text{ and } 1 \leq i \leq N$$

$$\mathbf{E}_i^{(m,k)} := (\nabla \wedge)^k \mathbf{E}_i^{(m)} \text{ and } \mathbf{B}_i^{(m,k)} := (\nabla \wedge)^k \mathbf{B}_i^{(m)} \text{ for } 1 \leq i \leq N, k = 0, 1, \text{ and } m \in \mathbb{N}$$

$$\left(\mathbf{E}_i^{(m,k)} \right)_{m \in \mathbb{N}}, \left(\mathbf{B}_i^{(m,k)} \right)_{m \in \mathbb{N}}, \left(\Delta \mathbf{E}_i^{(m,k)} \right)_{m \in \mathbb{N}} \text{ and } \left(\Delta \mathbf{B}_i^{(m,k)} \right)_{m \in \mathbb{N}}$$

$$\left(\mathbf{E}_i^{(m_l^0,0)} \right)_{l \in \mathbb{N}} \wedge \left(\mathbf{E}_i^{(m_l^0,0)} \right)_{m \in \mathbb{N}} \wedge \left(\mathbf{E}_i^{(m_l^1,1)} \right)_{l \in \mathbb{N}} \wedge \left(\mathbf{E}_i^{(m_l^0,1)} \right)_{l \in \mathbb{N}}$$

$$(m_l)_{l \in \mathbb{N}} \subset \mathbb{N}(G_{m_l})_{l \in \mathbb{N}} \supset (AG_{m_l})_{l \in \mathbb{N}}$$

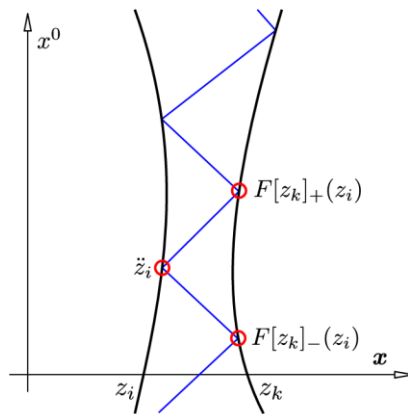
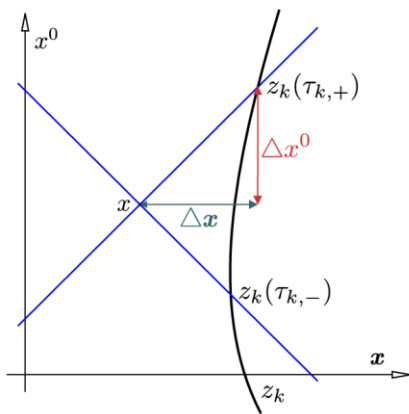
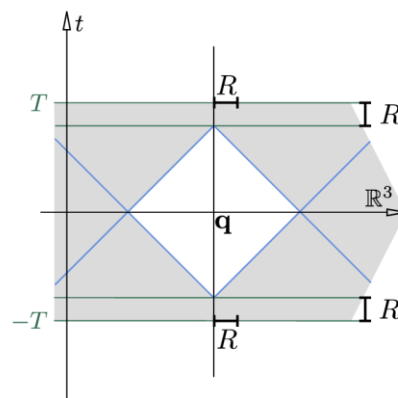
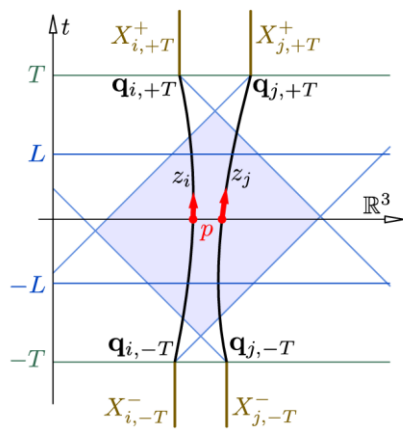
$$(X_{i,\pm T}^\pm)_{1 \leq i \leq N} = X^\pm \in \tilde{\mathcal{A}}_W^3$$

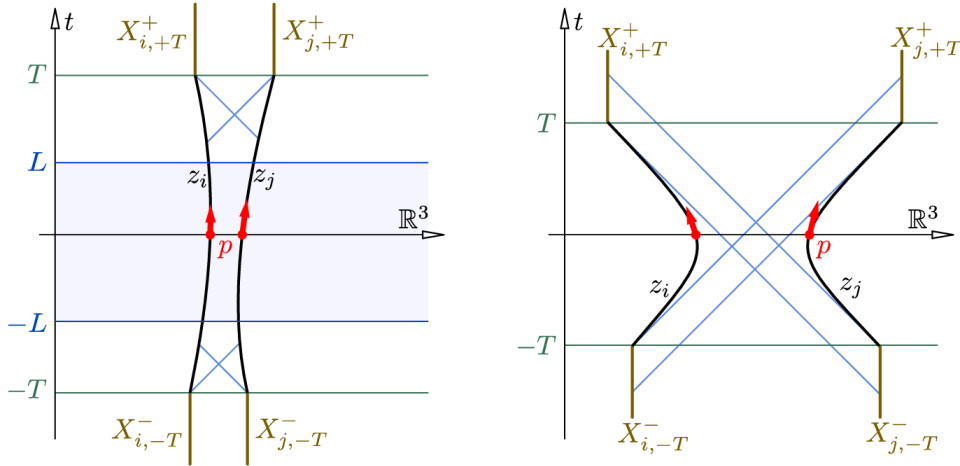
$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} := (Q + P)M_L[p, F](t, 0) \text{ are in } \mathcal{T}_T^{p, X^\pm}$$

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} \Big|_{t=0} = (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_\odot^N \oint (\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_T^{p, X^\pm}$$



$$\begin{aligned}
& M_{Q_i}[X_{i,\pm T}^\pm, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{Q_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty) \\
&= \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t\mp T} * X_{i,\pm T}^\pm + K_{t\mp T} * \begin{pmatrix} -4\pi \mathbf{v}(\mathbf{p}_{i,\pm T}) \varrho_i \left(\frac{\partial^2 \square}{\partial^2 \square} - \partial \mathbf{q}_{i,\pm T} \right) \\ 0 \end{pmatrix} \\
&- 4\pi \int_{\pm\infty}^{\pm T} ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \varrho_i \left(\frac{\partial^2 \square}{\partial^2 \square} - \partial \mathbf{q}_{i,s} \right) \\ \mathbf{v}(\mathbf{p}_{i,s}) \varrho_i \left(\frac{\partial^2 \square}{\partial^2 \square} - \partial \mathbf{q}_{i,s} \right) \end{pmatrix}
\end{aligned}$$





$$\mathbf{E}^C := \int d^3z \varrho(\overline{\partial^2 \square} - \partial \mathbf{z}) \frac{\mathbf{z}}{\|\mathbf{z}\|^3}$$

$$\left[\begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t \mp T} * \begin{pmatrix} \mathbf{E}^C(\overline{\partial^2 \square} - \partial \mathbf{q}) \\ 0 \end{pmatrix} + K_{t \mp T} * \begin{pmatrix} -4\pi \mathbf{v} \varrho(\overline{\partial^2 \square} - \partial \mathbf{q}) \\ 0 \end{pmatrix} \right] (\mathbf{x})$$

$$\int_{\pm\infty}^{\pm T} ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla \wedge \end{pmatrix} \begin{pmatrix} \varrho \left(\overline{\partial^2 \square} - \partial \mathbf{q}_s \right) \\ \mathbf{v}(\mathbf{p}_s) \varrho \left(\overline{\partial^2 \square} - \partial \mathbf{q}_s \right) \end{pmatrix} (\mathbf{x})$$

$$\begin{aligned} \|-4\pi \mathbf{v}[K_{t \mp T} * \varrho(\cdot - \mathbf{q})](\mathbf{x})\| &= 4\pi \|\mathbf{v}\| \left| (t \mp T) \int_{\partial B_{|t \mp T|}(0)} d\sigma(y) \varrho(\mathbf{x} - \mathbf{y} - \mathbf{q}) \right|_{B_{|t \mp T| - R}(\mathbf{q})} \\ &\leq 4\pi \|\mathbf{v}\| |t \mp T| \sup |\varrho| \int_{\partial B_{|t \mp T|}(\mathbf{q})} d\sigma(y) \mathbb{1}_{B_R(\mathbf{x})}(y) \end{aligned}$$

$$\mathbf{x} \in B_{|t \mp T| - R}(\mathbf{q}) \text{ implies } \partial B_{|t \mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$$

$$\left[\partial_t K_{t \mp T} * \mathbf{E}^C(\overline{\partial^2 \square} - \partial \mathbf{q}) \right] (\mathbf{x}) = \int_{\partial B_{|t \mp T|}(0)} d\sigma(y) \mathbf{E}^C(\mathbf{x} - \mathbf{y} - \mathbf{q}) + (t \mp T) \partial_t \int_{\partial B_{|t \mp T|}(0)} d\sigma(y) \mathbf{E}^C(\mathbf{x} - \mathbf{y} - \mathbf{q})$$

$$= \int_{\partial B_{|t \mp T|}(0)} d\sigma(y) \mathbf{E}^C(\mathbf{x} - \mathbf{y} - \mathbf{q}) + \frac{(t \mp T)^2}{3} \int_{B_{|t \mp T|}(0)} d^3y \Delta \mathbf{E}^C(\mathbf{x} - \mathbf{y})$$

$$\mathfrak{H}''' = \mathbf{E}^C(\mathbf{x} - \mathbf{q}) + \int_0^{|t \mp T|} ds \partial_s \int_{\partial B_s(0)} d\sigma(y) \mathbf{E}^C(\mathbf{x} - \mathbf{y} - \mathbf{q})$$

$$= \mathbf{E}^C(\mathbf{x} - \mathbf{q}) + \int_0^{|t \mp T|} dr \frac{r}{3} \int_{B_r(0)} d^3y \Delta \mathbf{E}^C(\mathbf{x} - \mathbf{y} - \mathbf{q})$$



$$0 = (\nabla \wedge)^2 \mathbf{E}^C = \nabla(\nabla \cdot \mathbf{E}^C) - \Delta \mathbf{E}^C \text{ and } \nabla \cdot \mathbf{E}^C = 4\pi \varrho$$

$$\begin{aligned} \mathfrak{P}''' &= \mathbf{E}^C(\mathbf{x} - \mathbf{q}) + \int_0^{|t+T|} dr \frac{r}{3} \int_{B_r(0)} d^3y 4\pi \nabla \varrho(\mathbf{x} - \mathbf{y} - \mathbf{q}) \\ &= \mathbf{E}^C(\mathbf{x} - \mathbf{q}) - \int_0^{|t+T|} dr \frac{1}{r^2} \int_{\partial B_r(0)} d\sigma(y) \frac{\mathbf{y}}{r} \varrho(\mathbf{x} - \mathbf{y} - \mathbf{q}) \end{aligned}$$

$$\overline{\partial^* \mathcal{D}\rho} / \overline{\partial^{\ddot{}} \square} = \int_{B_{|t+T|}^c(0)} d^3y \varrho(\mathbf{x} - \mathbf{y} - \mathbf{q}) \frac{\mathbf{y}}{\|\mathbf{y}\|^3}$$

$$B_{|t+T|}^c(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$$

$$\mathbf{x} \in B_{|t+T|-R}(\mathbf{q})$$

$$\Delta \mathbf{E}^C = 4\pi \nabla \varrho$$

$$\overline{\partial^* \mathcal{D}\rho} / \overline{\partial^{\ddot{}} \square} = - \int_{\partial B_{|t+T|}(0)} d\sigma(y) \frac{\mathbf{y}}{(t+T)^2} \varrho(\mathbf{x} - \mathbf{y} - \mathbf{q})$$

$$\partial B_{|t+T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$$

$$\mathbf{x} \in B_{|t+T|-R}(\mathbf{q})$$

$$\bigcup_{|s|>T} (\partial B_{|t-s|}(\mathbf{x}) \cap B_R(\mathbf{q}_s)) = \emptyset$$

$$v^{a,b}: \mathbb{R}^+ \rightarrow [0,1), T \mapsto v_T^{a,b}$$

$$\sup \left\{ \|\mathbf{v}(\mathbf{p}_{i,t})\|_{\mathbb{R}^3} \mid t \in [-T, T], \|p\| \leq a, F \in \text{Range} S_T^{p,C}, \|q_i\|_{L_w^2} + \|w^{-1/2} q_i\|_{L^2} \leq b, 1 \leq i \leq N \right\} \leq v_T^{a,b} < 1$$

$$(\mathbf{p}_{i,t})_{1 \leq i \leq N} := \text{PM}_L[p, F](t, 0) \text{ for all } t \in \mathbb{R}$$

$$\sup_{t \in [-T, T]} \|M_L[\varphi](t, 0)\|_{\mathcal{H}_w} \leq C_3 \left(T, \|q_i\|_{L_w^2}, \|w^{-1/2} q_i\|_{L^2}; 1 \leq i \leq N \right) \|\varphi\|_{\mathcal{H}_w}$$

$$\|F\|_{\mathcal{F}_w} \leq C_{11}^{(1)}(T, \|p\|) \leq C_{11}^{(1)}(T, a)$$

$$\begin{aligned} &\sup \left\{ \|\mathbf{p}_{i,t}\|_{\mathbb{R}^3} \mid t \in [-T, T], \|p\| \leq a, F \in \text{Range} S_T^{p,C}, \|q_i\|_{L_w^2} + \|w^{-1/2} q_i\|_{L^2} \leq b, 1 \leq i \leq N \right\} \\ &\leq \sup \left\{ \|\mathbf{p}_{i,t}\|_{\mathbb{R}^3} \mid t \in [-T, T], \varphi \in D_w(A), \|\varphi\|_{\mathcal{H}_w} \leq c, \|q_i\|_{L_w^2} + \|w^{-1/2} q_i\|_{L^2} \leq b, 1 \leq i \leq N \right\} \\ &\leq C_3(T, b, b, c) =: p_T^{a,b} < \infty \end{aligned}$$

$$v_T^{a,b} := \frac{p_T^{a,b}}{\sqrt{m^2 + (p_T^{a,b})^2}} < B_{|L+T|-R}(\mathbf{q}_{i,\pm T})$$

$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \text{ by } t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1 \leq i \leq N} := (Q + P)M_L[p, F^*](t, 0)$$



$$(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_T^{p,C}$$

$$\begin{aligned} & M_{\varrho_i}[X_{i,\pm T}^\pm, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{\varrho_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty) \\ &= \begin{pmatrix} \partial_t & \nabla \wedge \\ -\nabla \wedge & \partial_t \end{pmatrix} K_{t \mp T} * X_{i,\pm T}^\pm + K_{t \mp T} * \begin{pmatrix} -4\pi \mathbf{v}(\mathbf{p}_{i,\pm T}) \varrho_i(\square^\bullet - \mathbf{q}_{i,\pm T}) \\ B_{|t \mp T| - R}(\mathbf{q}_{i,\pm T}) \end{pmatrix} \\ &- 4\pi \int_{\pm\infty}^{\pm T} ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ B_{T-R}(\mathbf{q}_{i,\pm T}) & \nabla \wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\square^\bullet - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s}) \varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix} \\ &- v_T^{a,b} T = \|\mathbf{q}_j^0 - \mathbf{q}_i^0\| + 2R + v_T^{a,b} L - (T - L) \end{aligned}$$

$$L := \frac{(1 - v_T^{a,b})T - \Delta q_{\max} - 2R}{1 + v_T^{a,b}} > \square^\bullet$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon P(t, x_t, \varepsilon, \mu)$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$x_t(s) \stackrel{\text{def}}{=} x(t+s), s \in [-h, h].$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon x(t-1)$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon \dot{x}(t-1)$$

$$P(t, \vartheta, \varepsilon, \mu) = g \circ \vartheta(0), \text{ a perturbation delay as } P(t, x_t, \varepsilon, \mu) = g \circ x(t);$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta(-1), \text{ a perturbation with a constant delay } P(t, x_t, \varepsilon, \mu) = x(t-1);$$

$$P(t, \vartheta, \varepsilon, \mu) = \frac{1}{\varepsilon} f \circ \vartheta(-\varepsilon) - \frac{1}{\varepsilon} f \circ \vartheta(0), \text{ delay system } \dot{x}(t) = f \circ x(t - \varepsilon);$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ \vartheta(0), \text{ a state-dependent perturbation } P(t, x_t, \varepsilon, \mu) = x(t + r \circ x(t));$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ \vartheta \circ r_1 \circ \vartheta(0) \text{ containing nested delays } P(t, x_t, \varepsilon, \mu) = x(t + r \circ x(t + r_1 \circ x(t)));$$

$$P(t, \vartheta, \varepsilon, \mu) = \left(\frac{d}{ds} \vartheta\right)(0) \text{ as } P(t, x_t, \varepsilon, \mu) = \dot{x}(t);$$

$$P(t, \vartheta, \varepsilon, \mu) = \left(\frac{d}{ds} \vartheta\right)(-1), \text{ a constant delay } P(t, x_t, \varepsilon, \mu) = \dot{x}(t-1);$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ r \circ \left(\frac{d}{ds} \vartheta\right)(0), \text{ as } P(t, x_t, \varepsilon, \mu) = x(t + r \circ \dot{x}(t));$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta \circ \tau(t) \text{ time-dependent delay } P(t, x_t, \varepsilon, \mu) = x(t + \tau(t));$$

$$P(t, \vartheta, \varepsilon, \mu) = \vartheta(-1) + \vartheta(-2) + \vartheta(+1), \text{ perturbation with several delays}$$

$$|P(t, u, \mu) - P(s, v, \mu)| \leq B_1 |t - s| + B_2 \|u - v\|_{C^1}$$



$$\hat{x}^\varepsilon: \mathbb{R} \rightarrow \mathbb{R}^n \text{ and } \phi^\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$$

$$x(t) = (x_0 + \hat{x}^\varepsilon) \circ \phi^\varepsilon(t)$$

$$\dot{\xi}(t) = Df \circ x_0(t)\xi(t)$$

$$\frac{d\{U(v;t)\}_{v,t \in \mathbb{R}}}{dv} U(v;t) = Df \circ x_0(v)U(v;t), U(t;t) = Id_n$$

$$U(v;t) = U(v,s)U(s;t)$$

$$\mathbb{R}^n \cong T_{x_0(t)}\mathbb{R}^n = E_t^c \oplus E_t^s \oplus E_t^u$$

$$\{U^c(v;t)\}_{v,t \in \mathbb{R}}, U^c(v;t): E_t^c \rightarrow E_v^c$$

$$\{U^s(v;t)\}_{v,t \in \mathbb{R}}, U^s(v;t): E_t^s \rightarrow E_v^s$$

$$\{U^u(v;t)\}_{v,t \in \mathbb{R}}, U^u(v;t): E_t^u \rightarrow E_v^u$$

$$\frac{d}{dv} U^\sigma(v;t) = Df \circ x_0(v)U^\sigma(v;t), U^\sigma(t;t) = Id|_{E_t^\sigma}$$

$$|U^s(v;t)| \leq e^{-\lambda_s(v-t)} C_U \quad v \geq t$$

$$|U^u(v;t)| \leq e^{\lambda_u(v-t)} C_U \quad v \leq t$$

$$\sup_{t \in \mathbb{R}} \|\Pi_t^\sigma\| \leq C_\Pi, \sigma \in \{c, s, u\}$$

$$\Pi_t^\sigma: \mathbb{R}^n \rightarrow E_t^\sigma$$

$$V \in T_{x_0(t)}\mathbb{R}^n, \text{ there exists } A_V \in \mathbb{R}$$

$$\Pi_t^c V = A_V f \circ x_0(t)$$

$$\dot{\xi}(t) = Df \circ x_0(t)\xi(t) + a(t) \text{ with } a(t) \in E_t^\sigma$$

$$\alpha(t) = U^\sigma(0;t)\xi(t)$$

$$\xi(t) = U^\sigma(t;0)\alpha(t)$$

$$\dot{\xi}(t) = Df \circ x_0(t)\xi(t) + U^\sigma(t;0)\dot{\alpha}(t)$$

$$a(t) \stackrel{\text{def}}{=} U^\sigma(t;0)\dot{\alpha}(t) \in E_t^\sigma$$

$$\xi(t) = U^\sigma(t;0)\xi(0) + \int_0^t U^\sigma(t;s)a(s)ds$$

$$U^\sigma(t;s)E_s^\sigma = E_t^\sigma, \text{ then } \xi(t) \in E_t^\sigma$$

$$x(t) = (x_0 + \hat{x}) \circ \phi(t)$$

$$\dot{\phi}(t) = X \circ \phi(t)$$



$$\begin{aligned}x(t) &= ((x_0 + \hat{x}) \circ w) \circ (w^{-1} \circ \phi)(t) \\ &= (x_0 + (x_0 \circ w - x_0 + \hat{x} \circ w)) \circ (w^{-1} \circ \phi)(t)\end{aligned}$$

$$(\psi, \hat{y}) \stackrel{\text{def}}{=} (w^{-1} \circ \phi, x_0 \circ w - x_0 + \hat{x} \circ w)$$

$$\hat{x} \circ \phi(t) \in E_{\phi(t)}^s \oplus E_{\phi(t)}^u$$

$$X(\phi(t))(x_0 + \hat{x})'(\phi(t)) = f \circ (x_0 + \hat{x}) \circ \phi(t) + \varepsilon \mathcal{P}[(x_0 + \hat{x}) \circ \phi, \varepsilon, \mu](t)$$

$$\mathcal{P}[u, \varepsilon, \mu](t) \stackrel{\text{def}}{=} P(t, u_t, \varepsilon, \mu)$$

$$f \circ (x_0 + \hat{x}) \circ \phi(t) = f \circ x_0 \circ \phi(t) + Df \circ x_0 \circ \phi(t) \hat{x} \circ \phi(t) + T[x_0, \hat{x}](\phi(t))$$

$$T[x_0, \hat{x}](\phi(t)) \stackrel{\text{def}}{=} f \circ (x_0 + \hat{x}) \circ \phi(t) - f \circ x_0 \circ \phi(t) - Df \circ x_0 \circ \phi(t) \hat{x} \circ \phi(t)$$

$$\begin{aligned}X(\phi(t))\hat{x}'(\phi(t)) &= (1 - X \circ \phi(t))f \circ x_0 \circ \phi(t) + Df \circ x_0 \circ \phi(t) \hat{x} \circ \phi(t) \\ &\quad + T[x_0, \hat{x}](\phi(t)) + \varepsilon \mathcal{P}[(x_0 + \hat{x}) \circ \phi, \varepsilon, \mu](t)\end{aligned}$$

$$X(\rho)Df \circ x_0(\rho)\hat{x}(\rho)$$

$$X(\rho)\hat{x}'(\rho) = X(\rho)Df \circ x_0(\rho)\hat{x}(\rho) + (1 - X(\rho))f \circ x_0(\rho) + \mathcal{B}[X, \hat{x}](\rho) + \varepsilon \varphi[X, \hat{x}](\rho)$$

$$\mathcal{B}[X, \hat{x}](\rho) \stackrel{\text{def}}{=} (1 - X(\rho))Df \circ x_0(\rho)\hat{x}(\rho) + T[x_0, \hat{x}](\rho),$$

$$\varphi[X, \hat{x}](\rho) \stackrel{\text{def}}{=} \mathcal{P}[(x_0 + \hat{x}) \circ \phi, \varepsilon, \mu](\phi^{-1}(\rho)) = P(\phi^{-1}(\rho), ((x_0 + \hat{x}) \circ \phi)_{\phi^{-1}(\rho)}, \varepsilon, \mu)$$

$$0 = \Pi_{\rho}^c(1 - X(\rho)) \cdot f \circ x_0(\rho) + \Pi_{\rho}^c(\mathcal{B}[X, \hat{x}](\rho) + \varepsilon \varphi[X, \hat{x}](\rho)),$$

$$(\hat{x}^s)'(\rho) = Df \circ x_0(\rho)\hat{x}^s(\rho) + \Pi_{\rho}^s \frac{1}{X(\rho)} (\mathcal{B}[X, \hat{x}](\rho) + \varepsilon \varphi[X, \hat{x}](\rho)),$$

$$(\hat{x}^u)'(\rho) = Df \circ x_0(\rho)\hat{x}^u(\rho) + \Pi_{\rho}^u \frac{1}{X(\rho)} (\mathcal{B}[X, \hat{x}](\rho) + \varepsilon \varphi[X, \hat{x}](\rho)),$$

$$X(\rho) = \Gamma_c^{\varepsilon}[X, \hat{x}](\rho) \stackrel{\text{def}}{=} 1 + \frac{\langle \Pi_{\rho}^c(\mathcal{B}[X, \hat{x}](\rho) + \varepsilon \varphi[X, \hat{x}](\rho)), f \circ x_0(\rho) \rangle}{\langle f \circ x_0(\rho), f \circ x_0(\rho) \rangle},$$

$$\hat{x}^s(\rho) = \int_{-\rho_0}^{\rho} U^s(\rho; v) \Pi_v^s \frac{1}{X(v)} (\mathcal{B}[X, \hat{x}](v) + \varepsilon \varphi[X, \hat{x}](v)) dv + U^s(\rho; -\rho_0) \hat{x}^s(-\rho_0)$$

$$\hat{x}^u(\rho) = - \int_{\rho}^{\rho_0} U^u(\rho; v) \Pi_v^u \frac{1}{X(v)} (\mathcal{B}[X, \hat{x}](v) + \varepsilon \varphi[X, \hat{x}](v)) dv + U^u(\rho; \rho_0) \hat{x}^u(\rho_0)$$

$$\hat{x}^s(\rho) = \Gamma_s^{\varepsilon}[X, \hat{x}](\rho) \stackrel{\text{def}}{=} \int_{-\infty}^{\rho} U^s(\rho; v) \Pi_v^s \frac{1}{X(v)} (\mathcal{B}[X, \hat{x}](v) + \varepsilon \varphi[X, \hat{x}](v)) dv$$

$$\hat{x}^u(\rho) = \Gamma_u^{\varepsilon}[X, \hat{x}](\rho) \stackrel{\text{def}}{=} - \int_{\rho}^{+\infty} U^u(\rho; v) \Pi_v^u \frac{1}{X(v)} (\mathcal{B}[X, \hat{x}](v) + \varepsilon \varphi[X, \hat{x}](v)) dv$$



$$\begin{aligned} \frac{d}{d\rho} \Gamma_s^\varepsilon[X, \hat{x}](\rho) &= U^s(\rho; \rho) \Pi_\rho^s \frac{1}{X(\rho)} (\mathcal{B}[X, \hat{x}](\rho) + \varepsilon\varphi[X, \hat{x}](\rho)) \\ &\quad + \int_{-\infty}^{\rho} Df \circ x_0(\rho) U^s(\rho; v) \Pi_v^s \frac{1}{X(v)} (\mathcal{B}[X, \hat{x}](v) + \varepsilon\varphi[X, \hat{x}](v)) dv \\ &= \Pi_\rho^s \frac{1}{X(\rho)} (\mathcal{B}[X, \hat{x}](\rho) + \varepsilon\varphi[X, \hat{x}](\rho)) + Df \circ x_0(\rho) \Gamma_s^\varepsilon[X, \hat{x}](\rho) \end{aligned}$$

$$\frac{d}{dt} \psi(t) = Y \circ \psi(t), \psi(0) = \Gamma_c^\varepsilon[X, \hat{x}^s, \hat{x}^u] \bigoplus \Gamma^\varepsilon \begin{pmatrix} \Gamma_c^\varepsilon \\ \Gamma_s^\varepsilon \\ \Gamma_u^\varepsilon \end{pmatrix}$$

$$C^\ell(I) = C^\ell(I, \mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ g: I \rightarrow \mathbb{R}^n \begin{array}{l} g \text{ is } \ell \text{ times in } I \text{ and} \\ \text{the derivatives to } \bar{I} \text{ with} \\ \|g\|_{C^\ell} \stackrel{\text{def}}{=} \max_{0 \leq j \leq \ell} \left\{ \sup_{x \in I} |D^j g(x)| \right\} < +\infty \end{array} \right\}$$

$$C^{\ell+\text{Lip}}(I) = C^{\ell+\text{Lip}}(I, \mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ g \in C^\ell(I, \mathbb{R}^n): \|g\|_{C^{\ell+\text{Lip}}} \stackrel{\text{def}}{=} \max\{\|g\|_{C^\ell}, \text{Lip}(D^\ell g)\} < +\infty \right\}.$$

$$\text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g);$$

$$\text{Lip}(\lambda f) \leq |\lambda| \text{Lip}(f);$$

$$\text{Lip}(fg) \leq \text{Lip}(f) \|g\|_{C^0} + \|f\|_{C^0} \text{Lip}(g) \leq 2 \|f\|_{C^1} \|g\|_{C^1}; \text{ and}$$

$$\text{Lip}(f \circ g) \leq \text{Lip}(f) \text{Lip}(g).$$

$$c = (c_0, \dots, c_\ell, c_\ell^{\text{Lip}}) \in \mathbb{R}_+^{\ell+2}$$

$$\mathcal{B}_c^{\ell+\text{Lip}}(I) \stackrel{\text{def}}{=} \left\{ g \in C^{\ell+\text{Lip}}: |D^j(g - l)| \leq c_j \text{ for } j = 0, \dots, \ell \text{ and } \text{Lip}(D^\ell(g - l)) \leq c_\ell^{\text{Lip}} \right\}.$$

$$f \in \mathcal{B}_{c_f}^\ell \text{ and } g \in \mathcal{B}_{c_g}^\ell \text{ implies } f + g \in \mathcal{B}_{c_f+c_g}^\ell;$$

$$\lambda \in \mathbb{R} \text{ and } g \in \mathcal{B}_c^\ell \text{ implies } \lambda g \in \mathcal{B}_{|\lambda|c}^\ell; \text{ and}$$

$$f \in \mathcal{B}_{c_f}^\ell \text{ and } g \in \mathcal{B}_{c_g}^\ell \text{ implies } fg \in \mathcal{B}_c^\ell$$

$$\tilde{c}_j \stackrel{\text{def}}{=} \sum_{k=0}^j \binom{j}{k} c_{f,k} c_{g,j-k}$$

$$[g]_{C^{0,\alpha}} = [g]_{C^{0,\alpha}(I)} \stackrel{\text{def}}{=} \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|^\alpha}$$

$$C^{k,\alpha} = C^{k,\alpha}(I) \stackrel{\text{def}}{=} \left\{ g \in C^k(I): \|g\|_{C^{k,\alpha}} \stackrel{\text{def}}{=} \max\{\|g\|_{C^k(I)}, [D^k g]_{C^{0,\alpha}(I)}\} < +\infty \right\}.$$

$$C^\infty \subset \dots \subset C^2 \subset C^{1+\text{Lip}} \subset C^{1,\alpha} \subset C^1 \subset C^{\text{Lip}} \subset C^{0,\alpha} \subset C^0$$

$$\|g\|_{C^{\theta r + (1-\theta)t}} \leq M_{r,t} \|g\|_{C^r}^\theta \|g\|_{C^t}^{1-\theta}$$



$$\mu \stackrel{\text{def}}{=} \frac{t-s}{t-r}$$

$$\|g\|_{C^s} \leq M_{r,t} \|g\|_{C^r}^\mu \|g\|_{C^t}^{1-\mu}$$

$$C_\eta = C_\eta(I, \mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ g: I \rightarrow \mathbb{R}^n \text{ is continuous: } \|g\|_{C_\eta} = \|g\|_\eta \stackrel{\text{def}}{=} \sup_{\rho \in I} |g(\rho)| e^{-\eta|\rho|} < +\infty \right\}$$

$$\|\mathcal{S}[X] - \mathcal{S}[Y]\|_{C_\eta} \leq c \|X - Y\|_{C^0}$$

$$|\phi(\rho) - \psi(\rho)| e^{-\eta|\rho|} \leq \frac{\|X - Y\|_{C^0}}{e(\eta - t_1)}$$

$$Y_X(\phi)(\rho) = \begin{cases} \int_0^\rho X \circ \phi(s) ds & \rho \geq 0 \\ -\int_0^{-\rho} X \circ \phi(-s) ds & \rho < 0 \end{cases}$$

$$\begin{aligned} |Y_X(\phi)(\rho) - Y_X(\tilde{\phi})(\rho)| e^{-\eta|\rho|} &\leq e^{\eta\rho} \int_0^{-\rho} |X \circ \phi(-s) - X \circ \tilde{\phi}(-s)| ds \leq e^{\eta\rho} \text{Lip}(X) \int_0^{-\rho} |\phi(-s) - \tilde{\phi}(-s)| ds \\ &\leq \|\phi - \tilde{\phi}\|_\eta \text{Lip}(X) e^{\eta\rho} (e^{-\eta\rho} - 1) \leq \frac{\text{Lip}(X)}{\eta} \|\phi - \tilde{\phi}\|_\eta \end{aligned}$$

$$\|\phi - \psi\|_\eta \leq \left(1 - \frac{t_1}{\eta}\right)^{-1} \|\phi - Y_Y(\phi)\|_\eta$$

$$|\phi(\rho) - Y_Y(\phi)(\rho)| e^{-\eta|\rho|} \leq e^{\eta\rho} \int_0^{-\rho} |X \circ \phi(-s) - Y \circ \phi(-s)| ds \leq e^{\eta\rho} \frac{-\eta\rho}{\eta} \|X - Y\|_{C^0}$$

$$\sup_{u < 0} -e^u u = \sup_{u \geq 0} e^{-u} u = e^{-1}$$

$$\|\phi - \psi\|_\eta \leq \left(1 - \frac{t_1}{\eta}\right)^{-1} \frac{1}{\eta e} \|X - Y\|_{C^0}$$

$$g: [a, b] \rightarrow \mathbb{R}^n$$

$$\|g|_{[a,b]}\|_{C_\eta} \leq \|g|_{[a,b]}\|_{C^0} \leq e^{\eta \max\{|a|, |b|\}} \|g|_{[a,b]}\|_{C_\eta}$$

$$\|g|_{[a,b]}\|_{C^{(1-\theta)t}} \leq M_{0,t} \|g|_{[a,b]}\|_{C^0}^\theta \|g|_{[a,b]}\|_{C^t}^{1-\theta} \leq M_{0,t} e^{\eta \theta \max\{|a|, |b|\}} \|g|_{[a,b]}\|_{C_\eta}^\theta \|g|_{[a,b]}\|_{C^t}^{1-\theta}$$

$$\text{let } \hat{x}^s \in \mathcal{B}_s^{\ell+1+\text{Lip}} \text{ and } \hat{x}^u \in \mathcal{B}_u^{\ell+1+\text{Lip}} \text{ for } s, u \in \mathbb{R}_+^{\ell+3}$$

$$\hat{x} \in \mathcal{B}_{s+u}^{\ell+1+\text{Lip}}$$

$$(X, \hat{x}^s, \hat{x}^u) \in \mathcal{X} = \mathcal{X}_{t,s,u}^\ell \stackrel{\text{def}}{=} \mathcal{B}_t^{\ell+\text{Lip}}(1) \times \mathcal{B}_s^{\ell+1+\text{Lip}}(0) \times \mathcal{B}_u^{\ell+1+\text{Lip}}(0)$$



$$\begin{aligned} t &\stackrel{\text{def}}{=} (t_0, \dots, t_\ell, t_\ell^{\text{Lip}}) \in \mathbb{R}^{\ell+2} \\ s &\stackrel{\text{def}}{=} (s_0, \dots, s_\ell, s_{\ell+1}, s_{\ell+1}^{\text{Lip}}) \in \mathbb{R}^{\ell+3} \\ u &\stackrel{\text{def}}{=} (u_0, \dots, u_\ell, u_{\ell+1}, u_{\ell+1}^{\text{Lip}}) \in \mathbb{R}^{\ell+3}. \end{aligned}$$

$$P: \mathbb{R} \times C^{\ell+1+\text{Lip}}([-h, h], \mathbb{R}^n) \times (0,1)^2 \rightarrow \mathbb{R}^n$$

$$\mathcal{P}: C^{\ell+1+\text{Lip}}(\mathbb{R}, \mathbb{R}^n) \times (0,1) \rightarrow C^{\ell+\text{Lip}}(\mathbb{R}, \mathbb{R}^n), \mathcal{P}[u, \varepsilon](t) \stackrel{\text{def}}{=} P(t, u_t, \varepsilon, \mu_0)$$

$$\varepsilon \in (0,1), u \in C^{\ell+1+\text{Lip}}(\mathbb{R}, \mathbb{R}^n), t \in \mathbb{R}$$

$$\left| \frac{d^j}{dt^j} \mathcal{P}[u, \varepsilon, \mu](t) \right| \leq C_j F_j(\|u\|_{C^{j+1}}) \quad j = 0, \dots, \ell, \text{ and } \text{Lip} \left(\frac{d^\ell}{dt^\ell} \mathcal{P}[u, \varepsilon, \mu] \right) \leq C_\ell^{\text{Lip}} F_\ell^{\text{Lip}}(\|u\|_{C^{\ell+1+\text{Lip}}}),$$

$$x \stackrel{\text{def}}{=} (x_0 + \hat{x}) \circ \phi$$

$$\varepsilon \in (0,1), u^1, u^2 \in C^{\ell+1+\text{Lip}}(\mathbb{R}, \mathbb{R}^n)$$

$$|\mathcal{P}[u^2, \varepsilon](s) - \mathcal{P}[u^1, \varepsilon](t)| \leq \mathfrak{L}_1 |s - t| + \mathfrak{L}_2 \|u_s^2 - u_t^1\|_{C^1([-h, h])}$$

$$\mathcal{E}_c \stackrel{\text{def}}{=} \Gamma_c^\varepsilon[X_{(0)}, \hat{x}_{(0)}^s, \hat{x}_{(0)}^u] - X_{(0)}, \mathcal{E}_s \stackrel{\text{def}}{=} \Gamma_s^\varepsilon[X_{(0)}, \hat{x}_{(0)}^s, \hat{x}_{(0)}^u] - \hat{x}_{(0)}^s, \mathcal{E}_u \stackrel{\text{def}}{=} \Gamma_u^\varepsilon[X_{(0)}, \hat{x}_{(0)}^s, \hat{x}_{(0)}^u] - \hat{x}_{(0)}^u,$$

$$\left\| (X - X_{(0)})|_{[a, b]} \right\|_{C^j} \leq c E_\eta^{\frac{\ell+1-j}{\ell+1}}, \quad \text{for } 0 \leq j \leq \ell$$

$$\left\| (\hat{x}^\sigma - \hat{x}_{(0)}^\sigma)|_{[a, b]} \right\|_{C^j} \leq c E_\eta^{\frac{\ell+2-j}{\ell+2}}, \quad \text{for } \sigma = s, u, \text{ and } 0 \leq j \leq \ell + 1$$

$$E_\eta \stackrel{\text{def}}{=} \|\mathcal{E}_c\|_\eta \oplus \|\mathcal{E}_s\|_\eta \oplus \|\mathcal{E}_u\|_\eta \oplus \|D\mathcal{E}_s\|_\eta \oplus \|D\mathcal{E}_u\|_\eta$$

$$\left\| D^j (X - X_{(0)})|_{(0, +\infty)} \right\|_\eta \leq c E_\eta^{\frac{1}{j+1}}, \quad \text{for } 0 \leq j \leq \ell$$

$$\left\| D^j (\hat{x}^\sigma - \hat{x}_{(0)}^\sigma)|_{(0, +\infty)} \right\|_\eta \leq c E_\eta^{\frac{1}{j+1}}, \quad \text{for } \sigma = s, u, \text{ and } 0 \leq j \leq \ell + 1$$

$$\phi: \mathbb{R} \times (0,1) \rightarrow \mathbb{R} \text{ and } \hat{x}: \mathbb{R} \times (0,1) \rightarrow \mathbb{R}^n$$

$$x(t, \mu) = x_0 \circ \phi(t, \mu) + \hat{x}(\phi(t, \mu), \mu)$$

$$|\mathcal{P}[u^2, \varepsilon, \mu](s) - \mathcal{P}[u^1, \varepsilon, \mu](t)| \leq \mathfrak{L}_1 |s - t| + \mathfrak{L}_2 \|u_s^2 - u_t^1\|_{C^1([-h, h] \times (0,1))},$$

$$\dot{x}(t) = g(x(t), \mu_0 + h)$$

$$f(x) \stackrel{\text{def}}{=} g(x, \mu_0)$$

$$Q(x, h) = \int_0^1 D_\mu g(x, \mu_0 + \sigma h) d\sigma$$

$$\dot{x}(t) = g(x(t), \mu_0) + hQ(x(t), h)$$



$$\Gamma^\varepsilon[X^*, \hat{x}^*] = (X^*, \hat{x}^*)$$

$$d((X^*, \hat{x}^*), (Y^*, \hat{y}^*)) \leq \kappa d((X^*, \hat{x}^*), (Y^*, \hat{y}^*))$$

$$X - 1 \in C^\ell \Rightarrow \phi - Id \in C^{\ell+1}$$

$$\hat{x} \in C^{\ell+1} \Rightarrow \hat{x} \circ \phi \in C^{\ell+1}$$

$$\dot{\phi}(t) = X \circ \phi(t)$$

$\hat{X} \stackrel{\text{def}}{=} X - 1$ and $\hat{X} \in \mathcal{B}_t^{\ell+1, \text{Lip}}(0)$ with $t \stackrel{\text{def}}{=} (t_0, t_1, \dots, t_\ell, t_\ell^{\text{Lip}}) \in \mathbb{R}_+^{\ell+2}$. If $t_0 < 1$

$$(1 - t_0)|t - s| \leq |\phi(t) - \phi(s)| \leq (1 + t_0)|t - s|$$

$$\frac{1}{1 + t_0}|t - s| \leq |\phi^{-1}(t) - \phi^{-1}(s)| \leq \frac{1}{1 - t_0}|t - s|$$

$$(1 - t_0)|t| \leq |\phi(t)| \leq (1 + t_0)|t| \text{ and } \frac{1}{1 + t_0}|t| \leq |\phi^{-1}(t)| \leq \frac{1}{1 - t_0}|t|$$

$|D^{j+1}\phi(t)| \leq \tilde{t}_j$ and $|D^{j+1}(\phi^{-1})(t)| \leq \hat{t}_j$ for all $j = 0, \dots, \ell$, where \tilde{t}_j and \hat{t}_j depend on t_0, t_1, \dots, t_j .

$\text{Lip}(D^{\ell+1}\phi) \leq \tilde{t}_\ell^{\text{Lip}}$ and $\text{Lip}(D^{\ell+1}(\phi^{-1})) \leq \hat{t}_\ell^{\text{Lip}}$, where $\tilde{t}_\ell^{\text{Lip}}$ and $\hat{t}_\ell^{\text{Lip}}$ depend on t .

$$D\phi \in \mathcal{B}_t^{\ell+1, \text{Lip}} \text{ and } D(\phi^{-1}) \in \mathcal{B}_t^{\ell+1, \text{Lip}}$$

$$D(\phi^{-1})(t) = \frac{1}{X(t)}, \text{ hence } \phi^{-1}(t) = \int_0^t \frac{d\sigma}{X(\sigma)}.$$

$$\phi(t) - \phi(s) = \int_s^t X \circ \phi(u) du$$

$$|D\phi(t)| \leq 1 + t_0$$

$$|D^j(D\phi(t))| \leq \tilde{t}_j \text{ for } j = 1, \dots, \ell$$

$$D^2\phi(t) = D\hat{X} \circ \phi(t)(1 + \hat{X} \circ \phi(t))$$

$$D^{r+1}\phi = D^r(\hat{X} \circ \phi) = \sum_{\substack{(m_1, \dots, m_r) \in \mathbb{N}^r \\ \sum_{j=1}^r m_j = r}} C_{m_1, \dots, m_r} D^{m_1 + \dots + m_r} \hat{X} \circ \phi \prod_{j=1}^r (D^j \phi)^{m_j},$$

$$|D^{j+1}\phi(t)| \leq \tilde{t}_j \text{ for } j = 0, \dots, r - 1$$

$$D^{r+1}(\phi^{-1}) = D^r\left(\frac{1}{X}\right) \boxtimes D^{r+1}(\phi^{-1})(t)$$

$|D^j T[x_0, \hat{x}](\rho)| \leq a_j$ for a_j depending on $s_0, \dots, s_j, u_0, \dots, u_j, \|x_0\|_{C^j}$ and $\|f\|_{C^{j+2}}$;



$|D^j \mathcal{B}[X, \hat{x}](\rho)| \leq b_j$ for b_j depending on $t_0, \dots, t_j, s_0, \dots, s_j, u_0, \dots, u_j, \|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$;

$\text{Lip}(D^\ell T[x_0, \hat{x}]) \leq a_\ell^{\text{Lip}}$ and $\text{Lip}(D^\ell \mathcal{B}[X, \hat{x}]) \leq b_\ell^{\text{Lip}}$ for a_ℓ^{Lip} and b_ℓ^{Lip} depending on

$t, s, u, \|x_0\|_{C^{\ell+\text{Lip}}}$, and $\|f\|_{C^{\ell+2+\text{Lip}}}$.

$$T[x_0, \hat{x}](\rho) = \int_0^1 \int_0^\sigma D^2 f \circ (x_0 + s\hat{x})(\rho) \hat{x}(\rho)^2 ds d\sigma$$

$$D^2 f \circ (x_0 + s\hat{x})(\rho) \hat{x}(\rho)^2$$

$D^2 f \circ (x_0 + s\hat{x})(\rho)$ acting on $\hat{x}(\rho)$ and $\hat{x}(\rho)$

$D^j T[x_0, \hat{x}]$ in terms of $s_0 + u_0, \dots, s_j + u_j, \|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$

$\text{Lip}(D^\ell T[x_0, \hat{x}])$ can be bounded by $s, u, \|x_0\|_{C^{\ell+\text{Lip}}}$, and $\|f\|_{C^{\ell+2+\text{Lip}}}$

$$D^j \mathcal{B}[X, \hat{x}] = D^j[(1 - X)Df \circ x_0 \hat{x}] + D^j T[x_0, \hat{x}]$$

$D^j \mathcal{B}[X, \hat{x}]$ is bounded by $t_0, \dots, t_j, s_0, \dots, s_j, u_0, \dots, u_j, \|x_0\|_{C^j}$, and $\|f\|_{C^{j+2}}$

$b \leq \inf\{|f \circ x_0|\}$ and $1 - t_0 < X$ for $X \in \mathcal{B}_t^{\ell+\text{Lip}}$

$$|\Gamma_c^\varepsilon[X, \hat{x}^s, \hat{x}^u](\rho) - 1| \leq b_{c,0} + \varepsilon d_{c,0};$$

$$|D^j \Gamma_c^\varepsilon[X, \hat{x}^s, \hat{x}^u](\rho)| \leq b_{c,j} + \varepsilon d_{c,j} \text{ and } j = 1, \dots, \ell;$$

$$\text{Lip}(D^\ell \Gamma_c^\varepsilon[X, \hat{x}^s, \hat{x}^u]) \leq b_{c,\ell}^{\text{Lip}} + \varepsilon d_{c,\ell}^{\text{Lip}},$$

where $b_{c,j}$ and $d_{c,j}$ depend on $\|x_0\|_{C^j}, \|\Pi_t^c\|_{C^j}, \|f\|_{C^{j+2}}, t_0, \dots, t_j, s_0, \dots, s_j$, and u_0, \dots, u_j .

$b_{c,\ell}^{\text{Lip}}$ and $d_{c,\ell}^{\text{Lip}}$ depend on $\|x_0\|_{C^{\ell+\text{Lip}}}, \|\Pi_t^c\|_{C^{\ell+\text{Lip}}}, \|f\|_{C^{\ell+2+\text{Lip}}}, t, s$, and u .

$$b_{c,0} = \frac{C_\Pi \|f\|_{C^0}}{b^2} \left[t_0 \|f\|_{C^1}(s_0 + u_0) + \frac{1}{2} \|f\|_{C^2}(s_0 + u_0)^2 \right] \quad d_{c,0} = \frac{C_\Pi \|f\|_{C^0}}{b^2} \|\varphi[X, \hat{x}]\|_{C^0}$$

$$|\Gamma_\sigma^\varepsilon[X, \hat{x}^s, \hat{x}^u](\rho)| \leq b_{\sigma,0} + \varepsilon d_{\sigma,0}.$$

$$|D^j \Gamma_\sigma^\varepsilon[X, \hat{x}^s, \hat{x}^u](\rho)| \leq b_{\sigma,j} + \varepsilon d_{\sigma,j} \text{ for } j = 1, \dots, \ell + 1;$$

$$\text{Lip}(D^{\ell+1} \Gamma_\sigma^\varepsilon[X, \hat{x}^s, \hat{x}^u]) \leq b_{\sigma,\ell+1}^{\text{Lip}} + \varepsilon d_{\sigma,\ell+1}^{\text{Lip}},$$

for $j > 0, b_{\sigma,j}$ and $d_{\sigma,j}$ depend on $\|x_0\|_{C^{j-1}}, \lambda_\sigma, \|\Pi_t^\sigma\|_{C^{j-1}}, \|f\|_{C^{j+1}}, t_0, \dots, t_{j-1}, s_0, \dots, s_{j-1}$, and

u_0, \dots, u_{j-1} . However, $d_{\sigma,j}$ depend on s_j and u_j . The constants $b_{\sigma,\ell+1}^{\text{Lip}}$ and $d_{\sigma,\ell+1}^{\text{Lip}}$ depend on

$\|x_0\|_{C^{\ell+\text{Lip}}}, \lambda_\sigma, \|\Pi_t^\sigma\|_{C^{\ell+\text{Lip}}}, \|f\|_{C^{\ell+2+\text{Lip}}}, t, s$, and u , depend on $s_{\ell+1}^{\text{Lip}}$ and $u_{\ell+1}^{\text{Lip}}$ is for $d_{\sigma,\ell+1}^{\text{Lip}}$.



$$\mathfrak{b}_{\sigma,0} = \frac{C_{\Pi}C_U}{\lambda_{\sigma}(1-t_0)} \left[t_0 \|f\|_{C^1}(s_0 + u_0) + \frac{1}{2} \|f\|_{C^2}(s_0 + u_0)^2 \right] \mathfrak{d}_{\sigma,0} = \frac{C_{\Pi}C_U}{\lambda_{\sigma}(1-t_0)} \|\varphi[X, \hat{x}]\|_{C^0}.$$

$$D\Gamma_{\sigma}^{\varepsilon}[X, \hat{x}^s, \hat{x}^u](\rho) = Df \circ x_0(\rho) \Gamma_{\sigma}^{\varepsilon}[X, \hat{x}^s, \hat{x}^u](\rho) + \Pi_{\rho}^{\sigma} \frac{1}{X(\rho)} [\mathcal{B}[X, \hat{x}](\rho) + \varepsilon\varphi[X, \hat{x}](\rho)], \sigma \in \{s, u\}.$$

$$\begin{aligned} \mathfrak{b}_{c,0} + \varepsilon \mathfrak{d}_{c,0} &\leq t_0 \\ \mathfrak{b}_{s,0} + \varepsilon \mathfrak{d}_{s,0} &\leq s_0 \\ \mathfrak{b}_{u,0} + \varepsilon \mathfrak{d}_{u,0} &\leq u_0 \end{aligned}$$

$$\begin{aligned} G_c^i(\varepsilon, t_0, \dots, t_i, s_0, \dots, s_i, u_0, \dots, u_i) &\leq t_i, \\ G_s^i(\varepsilon, t_0, \dots, t_{i-1}, s_0, \dots, s_{i-1}, u_0, \dots, u_{i-1}) &\leq s_i, \\ G_u^i(\varepsilon, t_0, \dots, t_{i-1}, s_0, \dots, s_{i-1}, u_0, \dots, u_{i-1}) &\leq u_i, \end{aligned}$$

$$d((\Gamma_c^{\varepsilon}[X, \hat{x}^s, \hat{x}^u], \Gamma_s^{\varepsilon}[X, \hat{x}^s, \hat{x}^u], \Gamma_u^{\varepsilon}[X, \hat{x}^s, \hat{x}^u]), (\Gamma_c^{\varepsilon}[Y, \hat{y}^s, \hat{y}^u], \Gamma_s^{\varepsilon}[Y, \hat{y}^s, \hat{y}^u], \Gamma_u^{\varepsilon}[Y, \hat{y}^s, \hat{y}^u])) < \kappa d((X, \hat{x}^s, \hat{x}^u), (Y, \hat{y}^s, \hat{y}^u))$$

$$d((X, \hat{x}^s, \hat{x}^u), (Y, \hat{y}^s, \hat{y}^u)) \stackrel{\text{def}}{=} \|X - Y\|_{\eta} + \|\hat{x}^s - \hat{y}^s\|_{\eta} + \|\hat{x}^u - \hat{y}^u\|_{\eta} + \|D\hat{x}^s - D\hat{y}^s\|_{\eta} + \|D\hat{x}^u - D\hat{y}^u\|_{\eta}$$

$$\|x - y\|_{\eta} \stackrel{\text{def}}{=} \sup_{\rho \in \mathbb{R}} |x(\rho) - y(\rho)| e^{-\eta|\rho|}$$

$$\|\hat{x} - \hat{y}\| \text{ instead of } \|\hat{x}^s - \hat{y}^s\| + \|\hat{x}^u - \hat{y}^u\|$$

$$\|\Gamma_c^{\varepsilon}[X, \hat{x}] - \Gamma_c^{\varepsilon}[Y, \hat{y}]\|_{\eta} \leq \frac{C_{\Pi} \|f\|_{C^0}}{b^2} [(\mathfrak{d}_B + \varepsilon \mathfrak{d}_{\varphi}) \|X - Y\|_{\eta} + (c_B + \varepsilon c_{\varphi}) \|\hat{x} - \hat{y}\|_{\eta} + \varepsilon e_{\varphi} \|D\hat{x} - D\hat{y}\|_{\eta}]$$

$$\|\Gamma_{\sigma}^{\varepsilon}[X, \hat{x}] - \Gamma_{\sigma}^{\varepsilon}[Y, \hat{y}]\|_{\eta} \leq \frac{2C_{\Pi}C_U}{(\lambda_{\sigma} - \eta)(1 - t_0)} [(\mathfrak{d}_B + \varepsilon \mathfrak{d}_{\varphi}) \|X - Y\|_{\eta} + (c_B + \varepsilon c_{\varphi}) \|\hat{x} - \hat{y}\|_{\eta} + \varepsilon e_{\varphi} \|D\hat{x} - D\hat{y}\|_{\eta}]$$

$$I_1 = \int_{-\infty}^{\rho} e^{-\lambda_s(\rho-v)} e^{-\eta|\rho|} e^{\eta|v|} dv$$

$$I_1 \leq \frac{2}{\lambda_s - \eta} \eta < \lambda_u \Leftrightarrow \eta < \min\{\lambda_s, \lambda_u\}$$

$$\begin{aligned} \|D\Gamma_{\sigma}^{\varepsilon}[X, \hat{x}] - D\Gamma_{\sigma}^{\varepsilon}[Y, \hat{y}]\|_{\eta} &\leq \|f\|_{C^1} \|\Gamma_{\sigma}^{\varepsilon}[X, \hat{x}] - \Gamma_{\sigma}^{\varepsilon}[Y, \hat{y}]\|_{\eta} \\ &+ \frac{C_{\Pi}}{1 - t_0} [(\mathfrak{d}_B + \varepsilon \mathfrak{d}_{\varphi}) \|X - Y\|_{\eta} + (c_B + \varepsilon c_{\varphi}) \|\hat{x} - \hat{y}\|_{\eta} + \varepsilon e_{\varphi} \|D\hat{x} - D\hat{y}\|_{\eta}] \end{aligned}$$

$$|\mathcal{B}[X, \hat{x}](\rho) - \mathcal{B}[Y, \hat{y}](\rho)| e^{-\eta|\rho|} \leq c_B \|\hat{x} - \hat{y}\|_{\eta} + \mathfrak{d}_B \|X - Y\|_{\eta}$$

$$T[x_0, \hat{x}](\rho) = \int_0^1 \int_0^{\sigma} D^2 f \circ (x_0 + s\hat{x})(\rho) \hat{x}(\rho)^2 ds d\sigma$$

$$|T[x_0, \hat{x}](\rho) - T[x_0, \hat{y}](\rho)| e^{-\eta|\rho|} \leq (s_0 + u_0) (\text{Lip}(D^2 f)(s_0 + u_0) + \|D^2 f\|) \|\hat{x} - \hat{y}\|_{\eta}$$



$$|\mathcal{B}[X, \hat{x}](\rho) - \mathcal{B}[Y, \hat{y}](\rho)|e^{-\eta|\rho|} \leq (\|f\|_{C^1} t_0 + (\xi_0 + u_0)(\text{Lip}(D^2 f)(\xi_0 + u_0) + \|D^2 f\|))\|\hat{x} - \hat{y}\|_\eta + \|f\|_{C^1}(\xi_0 + u_0)\|X - Y\|_\eta$$

$$c_B \stackrel{\text{def}}{=} \|f\|_{C^1} t_0 + (\xi_0 + u_0)(\text{Lip}(D^2 f)(\xi_0 + u_0) + \|D^2 f\|)$$

$$d_B \stackrel{\text{def}}{=} \|f\|_{C^1}(\xi_0 + u_0)$$

$$|\phi^{-1}(\rho) - \psi^{-1}(\rho)|e^{-\eta|\rho|} \leq \frac{\|X - Y\|_\eta}{\eta(1 - t_0)^2}$$

$$\|\phi^{-1} - \psi^{-1}\|_\eta \leq \frac{1}{\eta(1 - t_0)^2} \|X - Y\|_{C^0}$$

$$\phi^{-1}(\rho) = \int_0^\rho \frac{d\sigma}{X(\sigma)} \quad \text{and} \quad \psi^{-1}(\rho) = \int_0^\rho \frac{d\sigma}{Y(\sigma)}$$

$$|\phi^{-1}(\rho) - \psi^{-1}(\rho)|e^{-\eta|\rho|} \leq \frac{\|X - Y\|_\eta}{(1 - t_0)^2} \int_0^1 |\rho|e^{(\sigma-1)\eta|\rho|} d\sigma = \frac{\|X - Y\|_\eta}{\eta(1 - t_0)^2} (1 - e^{-\eta|\rho|}) \leq \frac{\|X - Y\|_\eta}{\eta(1 - t_0)^2}$$

$$\alpha(\rho, s) \stackrel{\text{def}}{=} \phi(\phi^{-1}(\rho) + s) \quad \text{and} \quad \beta(\rho, s) \stackrel{\text{def}}{=} \psi(\psi^{-1}(\rho) + s).$$

$$\sup_{s \in [-h, h]} |\alpha(\rho, s) - \beta(\rho, s)|e^{-\eta|\rho|} \leq 3\|X - Y\|_{\eta \mathcal{O}_{\mathcal{B}}^{\text{Lip}}(t_0, t_1)}$$

$$\alpha_\pm(\rho, s) \stackrel{\text{def}}{=} \alpha(\rho, \pm s) \quad \text{and} \quad \beta_\pm(\rho, s) \stackrel{\text{def}}{=} \beta(\rho, \pm s) \quad \text{for } s \in [0, h]$$

$$\sup_{s \in [-h, h]} |\alpha(\rho, s) - \beta(\rho, s)|e^{-\eta|\rho|} = \max\left\{ \sup_{s \in [0, h]} |\alpha_+(\rho, s) - \beta_+(\rho, s)|e^{-\eta|\rho|}, \sup_{s \in [0, h]} |\alpha_-(\rho, s) - \beta_-(\rho, s)|e^{-\eta|\rho|} \right\}$$

$$\alpha_\pm(\rho, s) = \rho \pm \int_0^s X \circ \alpha_\pm(\rho, \sigma) d\sigma \quad \text{and} \quad \beta_\pm(\rho, s) = \rho \pm \int_0^s Y \circ \beta_\pm(\rho, \sigma) d\sigma$$

$$|\alpha_\pm(\rho, s) - \beta_\pm(\rho, s)|e^{-\eta|\rho|} \leq \int_0^s |X \circ \alpha_\pm(\rho, \sigma) - Y \circ \alpha_\pm(\rho, \sigma)|e^{-\eta|\rho|} + \text{Lip}(Y)|\alpha_\pm(\rho, \sigma) - \beta_\pm(\rho, \sigma)|e^{-\eta|\rho|} d\sigma.$$

$$\int_0^s |X \circ \alpha_\pm(\rho, \sigma) - Y \circ \alpha_\pm(\rho, \sigma)|e^{-\eta|\rho|} d\sigma \leq \|X - Y\|_\eta \int_0^s e^{\eta[|\alpha_\pm(\rho, \sigma)| - |\rho|]} d\sigma$$

$$\leq \|X - Y\|_\eta \int_0^s e^{\eta(1+t_0)\sigma} d\sigma = \|X - Y\|_\eta \frac{e^{\eta(1+t_0)h} - 1}{\eta(1+t_0)}$$

$$|\alpha_\pm(\rho, s) - \beta_\pm(\rho, s)|e^{-\eta|\rho|} \leq e^{t_1 h} \frac{e^{\eta(1+t_0)h} - 1}{\eta(1+t_0)} \|X - Y\|_\eta$$



$$\hat{x}, \hat{y} \in \mathcal{B}_{(\hat{s}_0 + u_0, \hat{s}_1 + u_1, \hat{s}_2 + u_2)}^{1+\text{Lip}}(0)$$

$$|\varphi[X, \hat{x}](\rho) - \varphi[Y, \hat{y}](\rho)| e^{-\eta|\rho|} \leq c_\varphi \|\hat{x} - \hat{y}\|_\eta + \mathfrak{d}_\varphi \|X - Y\|_\eta + e_\varphi \|D\hat{x} - D\hat{y}\|_\eta.$$

$$\begin{aligned} & |\varphi[X, \hat{x}](\rho) - \varphi[Y, \hat{y}](\rho)| \\ & \leq \mathfrak{L}_1 |\phi^{-1}(\rho) - \psi^{-1}(\rho)| \\ & \quad + \mathfrak{L}_2 \left\| ((x_0 + \hat{x}) \circ \phi)_{\phi^{-1}(\rho)} - ((x_0 + \hat{y}) \circ \psi)_{\psi^{-1}(\rho)} \right\|_{C^1([-h, h])} \end{aligned}$$

$$|\phi^{-1}(\rho) - \psi^{-1}(\rho)| e^{-\eta|\rho|} \triangleq \|X - Y\|_\eta$$

$$\sup_{s \in [-h, h]} |(x_0 + \hat{x}) \circ \phi(\phi^{-1}(\rho) + s) - (x_0 + \hat{y}) \circ \psi(\psi^{-1}(\rho) + s)| e^{-\eta|\rho|}.$$

$$\sup_{s \in [-h, h]} |(x_0 \circ \alpha - x_0 \circ \beta) + (\hat{x} \circ \alpha - \hat{y} \circ \alpha) + (\hat{y} \circ \alpha - \hat{y} \circ \beta)|_{(\rho, s)} | e^{-\eta|\rho|}.$$

$$|\hat{x} \circ \alpha(\rho, s) - \hat{y} \circ \alpha(\rho, s)| e^{-\eta|\rho|} \leq e^{\eta(|\alpha(\rho, s)| - |\rho|)} \|\hat{x} - \hat{y}\|_\eta \leq e^{\eta(1+t_0)h} \|\hat{x} - \hat{y}\|_\eta$$

$$\begin{aligned} \frac{d}{ds} & \left[((x_0 + \hat{x}) \circ \phi)_{\phi^{-1}(\rho)}(s) - ((x_0 + \hat{y}) \circ \psi)_{\psi^{-1}(\rho)}(s) \right] \\ & = (x_0 + \hat{x})' \circ \alpha(\rho, s) X \circ \alpha(\rho, s) - (x_0 + \hat{y})' \circ \beta(\rho, s) Y \circ \beta(\rho, s) \end{aligned}$$

$$|(x'_0 \circ \alpha - x'_0 \circ \beta) X \circ \alpha(\rho, s)| e^{-\eta|\rho|} \leq (1 + t_0) \text{Lip}(x'_0) \mathfrak{L} \|X - Y\|_\eta,$$

$$|x'_0 \circ \beta(X \circ \alpha - Y \circ \alpha)(\rho, s)| e^{-\eta|\rho|} \leq \|x_0\|_{C^1} e^{\eta(1+t_0)h} \|X - Y\|_\eta,$$

$$|x'_0 \circ \beta(Y \circ \alpha - Y \circ \beta)(\rho, s)| e^{-\eta|\rho|} \leq \|x_0\|_{C^1} t_1 \mathfrak{L} \|X - Y\|_\eta,$$

$$|(\hat{x}' \circ \alpha - \hat{x}' \circ \beta) X \circ \alpha(\rho, s)| e^{-\eta|\rho|} \leq (1 + t_0) (\hat{s}_2 + u_2) \mathfrak{L} \|X - Y\|_\eta,$$

$$|(\hat{x}' \circ \beta - \hat{y}' \circ \beta) X \circ \alpha(\rho, s)| e^{-\eta|\rho|} \leq (1 + t_0) e^{\eta(1+t_0)h} \|\hat{x}' - \hat{y}'\|_\eta,$$

$$|\hat{y}' \circ \beta(X \circ \alpha - X \circ \beta)(\rho, s)| e^{-\eta|\rho|} \leq (\hat{s}_1 + u_1) t_1 \mathfrak{L} \|X - Y\|_\eta,$$

$$|\hat{y}' \circ \beta(X \circ \beta - Y \circ \beta)(\rho, s)| e^{-\eta|\rho|} \leq (\hat{s}_1 + u_1) e^{\eta(1+t_0)h} \|X - Y\|_\eta$$

$$\|x_0\|_{C^{1+\text{Lip}}}, \mathfrak{L}_1, \mathfrak{L}_2, h, \eta, t_0, t_1, \hat{s}_1, \hat{s}_2, \hat{s}_1 \stackrel{\text{def}}{=} (X_{(0)}, \hat{x}_{(0)}^s, \hat{x}_{(0)}^u)$$

$$\|X_{(0)} - \Pi^c[v^*]\|_{C^{\ell+\text{Lip}}} \leq M_c < +\infty$$

$$\|\hat{x}_{(0)}^s - \Pi^s[v^*]\|_{C^{\ell+1+\text{Lip}}} \leq M_s < +\infty$$

$$\|\hat{x}_{(0)}^u - \Pi^u[v^*]\|_{C^{\ell+1+\text{Lip}}} \leq M_u < +\infty$$

$$d(v, v^*) \leq (1 - \kappa)^{-1} d(v, \Gamma^\varepsilon[v])$$

$$\|(v - v^*)|_{[a, b]}\|_{C^0} \leq e^{\delta\eta} (1 - \kappa)^{-1} d(v, \Gamma^\varepsilon[v])$$

$$\delta = \max\{|a|, |b|\}$$



$$\begin{aligned} \|(X_{(0)} - \Pi^c[v^*])|_{[a,b]}\|_{C^j} &\leq c_c e^{\delta\eta \frac{\ell+1-j}{\ell+1}} (1-\kappa)^{-\frac{\ell+1-j}{\ell+1}} d(v, \Gamma^\varepsilon[v])^{\frac{\ell+1-j}{\ell+1}} 0 \leq j \leq \ell \\ \|(\hat{x}_{(0)}^s - \Pi^s[v^*])|_{[a,b]}\|_{C^j} &\leq c_s e^{\delta\eta \frac{\ell+2-j}{\ell+2}} (1-\kappa)^{-\frac{\ell+2-j}{\ell+2}} d(v, \Gamma^\varepsilon[v])^{\frac{\ell+2-j}{\ell+2}} 0 \leq j \leq \ell+1 \\ \|(\hat{x}_{(0)}^u - \Pi^u[v^*])|_{[a,b]}\|_{C^j} &\leq c_u e^{\delta\eta \frac{\ell+2-j}{\ell+2}} (1-\kappa)^{-\frac{\ell+2-j}{\ell+2}} d(v, \Gamma^\varepsilon[v])^{\frac{\ell+2-j}{\ell+2}} 0 \leq j \leq \ell+1 \end{aligned}$$

$$\Pi^\sigma[w](t) \stackrel{\text{def}}{=} \Pi_t^\sigma w \text{ for } \sigma \in \{c, s, u\}$$

$$g_\eta(t) \stackrel{\text{def}}{=} e^{-\eta|t|} g(t) = \begin{cases} e^{-\eta t} g(t) & t > 0 \\ e^{\eta t} g(t) & t \leq 0 \end{cases}$$

$g: (0, +\infty) \rightarrow \mathbb{R}^n$ be a $C^{\ell+\text{Lip}}$ function ($\ell \geq 0$)

$$\|g\|_{C_\eta} \leq 1 \text{ implies } \|D^j g\|_{C_\eta} \leq c_j \|g\|_{C_\eta}^{\frac{1}{j+1}}$$

$$\|g\|_{C^{j+1}} (\|g\|_{C^{\ell+\text{Lip}}} \text{ when } j = \ell)$$

$$\|g_\eta\|_{C^0} = \|g\|_{C_\eta} \text{ and } \|g\|_{C_\eta} \leq \|g\|_{C_\eta}^{1/2} \text{ as } \|g\|_{C_\eta} \leq 1$$

$$\begin{aligned} |e^{-\eta t} Dg(t)| &= |Dg_\eta(t) + \eta e^{-\eta t} g(t)| \leq \|g_\eta\|_{C^1} + \eta \|g\|_{C_\eta} \leq M_{0,2} \|g_\eta\|_{C^0}^{1/2} \|g_\eta\|_{C^2}^{1/2} + \eta \|g_\eta\|_{C^0} \\ &= (M_{0,2} \|g_\eta\|_{C^2}^{1/2} + \eta) \|g_\eta\|_{C^0}^{1/2} \end{aligned}$$

$$c_1 = M_{0,2} \|g_\eta\|_{C^2}^{1/2} + \eta$$

$$D^j g_\eta(t) = \sum_{k=0}^j \binom{j}{k} D^k (e^{-\eta t}) D^{j-k} g(t) = e^{-\eta t} \left(D^j g(t) + \sum_{k=1}^j \binom{j}{k} (-\eta)^k D^{j-k} g(t) \right)$$

$$\begin{aligned} |e^{-\eta t} D^j g(t)| &= \left| D^j g_\eta + \sum_{k=1}^j \binom{j}{k} (-\eta)^{k+1} D^{j-k} g(t) \right| \leq \|g_\eta\|_{C^j} + \sum_{k=1}^j \alpha_k \|g_\eta\|_{C^0}^{\frac{1}{k}} \\ &\leq \left(M_{0,j+1} \|g_\eta\|_{C^{j+1}}^{\frac{j}{j+1}} + \sum_{k=1}^j \alpha_k \|g_\eta\|_{C^0}^{\frac{1}{k} \frac{1}{j+1}} \right) \|g_\eta\|_{C^0}^{\frac{1}{j+1}} \end{aligned}$$

$$c_j = M_{0,j+1} \|g_\eta\|_{C^{j+1}}^{\frac{j}{j+1}} + \sum_{k=1}^j \alpha_k$$



$$\left\| D^j (X_{(0)} - \Pi^c[v^*]) \Big|_{(0,+\infty)} \right\|_{C_\eta} \leq c_c (1 - \kappa)^{\frac{1}{j+1}} d(v, \Gamma^\varepsilon[v])^{\frac{1}{j+1}} 0 \leq j \leq \ell$$

$$\left\| D^j (\hat{x}_{(0)}^s - \Pi^s[v^*]) \Big|_{(0,+\infty)} \right\|_{C_\eta} \leq c_s (1 - \kappa)^{\frac{1}{j+1}} d(v, \Gamma^\varepsilon[v])^{\frac{1}{j+1}} 0 \leq j \leq \ell + 1$$

$$\left\| D^j (\hat{x}_{(0)}^u - \Pi^u[v^*]) \Big|_{(0,+\infty)} \right\|_{C_\eta} \leq c_u (1 - \kappa)^{\frac{1}{j+1}} d(v, \Gamma^\varepsilon[v])^{\frac{1}{j+1}} 0 \leq j \leq \ell + 1$$

$$\Pi^\sigma[w](t) \stackrel{\text{def}}{=} \Pi_t^\sigma w \text{ for } \sigma \in \{c, s, u\}$$

$$z(t) = \int_0^t \sin e^s ds$$

$$\ddot{x}(t) = Df \circ x(t) \dot{x}(t) + \varepsilon \frac{d}{dt} P(t, x_t, \varepsilon, \mu)$$

$$\ddot{x}(t) = D^2 f \circ x(t) \dot{x}(t)^{\otimes 2} + Df \circ x(t) \dot{x}(t) + \varepsilon \frac{d^2}{dt^2} P(t, x_t, \varepsilon, \mu)$$

$$\left| \frac{\partial^j}{\partial t^j} P(t, x_t, \varepsilon, \mu) \right| \leq C_j e^{\gamma j |t|}$$

$$|D^{j+1} x(t)| e^{-\gamma j |t|} < +\infty \text{ for } j \geq 0$$

$$\|g\|_{C_Y^\ell} \stackrel{\text{def}}{=} \max_{j=0, \dots, \ell} \sup_{t \in I} |D^j g(t)| e^{-\gamma j |t|} \text{ for all } g \in C_Y^\ell$$

$\varepsilon \in (0, \varepsilon_0), t \in \mathbb{R}$ and for $j = 1, \dots, \ell', u \in C^1(\mathbb{R}, \mathbb{R}^n), Du \in C_Y^{j-1}(\mathbb{R}, \mathbb{R}^n)$

$$\left| \frac{d^j}{dt^j} \mathcal{P}[u, \varepsilon, \mu](t) \right| \leq C_j e^{\gamma j |t|} F(\|u\|_{C^1}, \|Du\|_{C_Y^{j-1}})$$

$\dot{x}(t) = g(x(t), t)$, where g is ℓ -times

$$y \stackrel{\text{def}}{=} (x, t)$$

$$y'(s) = \frac{d}{ds} \begin{pmatrix} x(s) \\ t(s) \end{pmatrix} = G \circ y(s) = \begin{pmatrix} g(x(s), t(s)) \\ 1 \end{pmatrix}$$

$$\tilde{C}^\ell \stackrel{\text{def}}{=} Id + C^\ell$$

$$P(t, x_t) = g(t, x(t))$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q(t, x(t + r(t, x(t))))$$

$$P(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} Q(t, \vartheta \circ r(t, \vartheta(0))).$$

$r: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ a $C^{\ell+1}$ Lip map and $Q: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a $C^{\ell+1}$ Lip map

$$h \stackrel{\text{def}}{=} \|r\|_{C^0}$$



$$\mathcal{P}[u, \varepsilon](t) = \mathcal{P}[u](t) = Q(t, u(t + r(t, u(t)))),$$

$$|\mathcal{P}[u^2](s) - \mathcal{P}[u^1](t)| \leq \text{Lip}(Q)|s - t| + \text{Lip}(Q) |u^2(s + r(s, u^2(s))) - u^1(t + r(t, u^1(t)))|.$$

$$\begin{aligned} & |u^2(s + r(s, u^2(s))) - u^1(t + r(t, u^1(t)))| \leq |u^2(s + r(s, u^2(s))) - u^2(s + r(t, u^1(t)))| \\ & + |u^2(s + r(t, u^1(t))) - u^1(t + r(t, u^1(t)))| \\ & \leq \|u^2\|_{C^1} \|r\|_{C^1} [|s - t| + \|u_s^2 - u_t^1\|_{C^0[-h, h]}] \\ & + \|u_s^2 - u_t^1\|_{C^0[-h, h]}. \end{aligned}$$

$$C^{\ell+1+\text{Lip}}(\mathbb{R}, \mathbb{R}^n)$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q \left(t, x \left(t + r(t, x(t + r_1 \circ x(t))) \right) \right)$$

$$P(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} Q(t, \vartheta \circ r(t, r_1 \circ \vartheta(0)))$$

$$h \stackrel{\text{def}}{=} \max\{\|r\|_{C^0}, \|r_1\|_{C^0}\}$$

$r: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^{\ell+1+\text{Lip}}$ map, $r_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ a $C^{\ell+1+\text{Lip}}$ map, and $Q: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a $C^{\ell+1+\text{Lip}}$ map

$$|u^2(s + r(s, u^2(s + r_1 \circ u^2(s)))) - u^1(t + r(t, u^1(t + r_1 \circ u^1(t))))|$$

$$\begin{aligned} \widehat{\partial^2 \mathbb{H}} \setminus \mathbb{H} \llbracket \partial^2 \mathbb{V} \rrbracket^{\square} & \leq |u^2(s + r(s, u^2(s + r_1 \circ u^2(s)))) - u^2(s + r(t, u^1(t + r_1 \circ u^1(t))))| \\ & + |u^2(s + r(t, u^1(t + r_1 \circ u^1(t))) - u^1(t + r(t, u^1(t + r_1 \circ u^1(t))))| \\ & \leq \|u^2\|_{C^1} \|r\|_{C^1} [|s - t| + \|u_s^2 - u_t^1\|_{C^0[-h, h]}] + \|u_s^2 - u_t^1\|_{C^0[-h, h]} \\ & + \|u_s^2 - u_t^1\|_{C^0[-h, h]} \end{aligned}$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon Q \left(t, x \left(t + r \left(t, \frac{d}{dt} x(t) \right) \right) \right)$$

$$P(t, \vartheta, \varepsilon, \mu) \stackrel{\text{def}}{=} Q \left(t, \vartheta \circ r \left(t, \frac{d}{ds} \vartheta(0) \right) \right)$$

$$\frac{dx_t}{ds}(0) = \frac{dx}{dt}(t)$$

$$\left| u^2 \left(s + r \left(s, \frac{d}{ds} u^2(s) \right) \right) - u^1 \left(t + r \left(t, \frac{d}{dt} u^1(t) \right) \right) \right|$$

$$\begin{aligned} \widehat{\partial^2 \mathbb{H}} \setminus \mathbb{H} \llbracket \partial^2 \mathbb{V} \rrbracket^{\square} & \leq \left| u^2 \left(s + r \left(s, \frac{d}{ds} u^2(s) \right) \right) - u^2 \left(s + r \left(t, \frac{d}{dt} u^1(t) \right) \right) \right| \\ & + \left| u^2 \left(s + r \left(t, \frac{d}{dt} u^1(t) \right) \right) - u^1 \left(t + r \left(t, \frac{d}{dt} u^1(t) \right) \right) \right| \\ & \leq \|u^2\|_{C^1} \|r\|_{C^1} [|s - t| + \|u_s^2 - u_t^1\|_{C^1[-h, h]}] \\ & + \|u_s^2 - u_t^1\|_{C^0[-h, h]} \end{aligned}$$



$$\frac{d}{ds} \vartheta(s) \text{ for } s \in [-h, h]$$

$$\dot{x}(t) = f \circ x(t - \varepsilon \tau(x_t))$$

$$\dot{x}(t) = f \circ x(t) + \varepsilon \left[\frac{1}{\varepsilon} f \circ x(t - \varepsilon \tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t) \right].$$

$$\frac{1}{\varepsilon} f \circ x(t - \varepsilon \tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t) \approx -Df \circ x(t) x'(t) \tau(x_t)$$

$$\frac{1}{\varepsilon} f \circ x(t - \varepsilon \tau(x_t)) - \frac{1}{\varepsilon} f \circ x(t) = - \int_0^1 Df \circ x(t - \sigma \varepsilon \tau(x_t)) x'(t - \sigma \varepsilon \tau(x_t)) \tau(x_t) d\sigma$$

$$Q[\vartheta, \varepsilon] \stackrel{\text{def}}{=} - \int_0^1 Df \circ \vartheta(-\sigma \varepsilon \tau \circ \vartheta) \vartheta'(-\sigma \varepsilon \tau \circ \vartheta) \tau \circ \vartheta d\sigma$$

$$\tau: C^{\ell+1+\text{Lip}}([-h, h], \mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\dot{x}(t) = f \circ (x(t - \varepsilon \tau_1(t, x_t)), \dots, x(t - \varepsilon \tau_L(t, x_t))) + \varepsilon P(t, x_t, \varepsilon, \mu)$$

$$Q_i(t, \vartheta, \varepsilon) \stackrel{\text{def}}{=} \int_0^1 D_i f(\vartheta(-\sigma \varepsilon \tau_1(t, \vartheta)), \dots, \vartheta(-\sigma \varepsilon \tau_L(t, \vartheta))) \vartheta'(-\sigma \varepsilon \tau_i(t, \vartheta)) \tau_i(t, \vartheta) d\sigma$$

$$\begin{aligned} & Df \circ \vartheta(-\sigma \varepsilon \tau \circ \vartheta) \vartheta'(-\sigma \varepsilon \tau \circ \vartheta) \tau \circ \vartheta - Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho) \varrho'(-\sigma \varepsilon \tau \circ \varrho) \tau \circ \varrho \\ &= [Df \circ \vartheta(-\sigma \varepsilon \tau \circ \vartheta) - Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho)] \vartheta'(-\sigma \varepsilon \tau \circ \vartheta) \tau \circ \vartheta \\ &+ [Df \circ \varrho(-\sigma \varepsilon \tau \circ \vartheta) - Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho)] \vartheta'(-\sigma \varepsilon \tau \circ \vartheta) \tau \circ \vartheta \\ &+ Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho) [\vartheta'(-\sigma \varepsilon \tau \circ \vartheta) - \varrho'(-\sigma \varepsilon \tau \circ \vartheta)] \tau \circ \vartheta \\ &+ Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho) [\varrho'(-\sigma \varepsilon \tau \circ \vartheta) - \varrho'(-\sigma \varepsilon \tau \circ \varrho)] \tau \circ \vartheta \\ &+ Df \circ \varrho(-\sigma \varepsilon \tau \circ \varrho) \varrho'(-\sigma \varepsilon \tau \circ \varrho) [\tau \circ \vartheta - \tau \circ \varrho]. \end{aligned}$$

$$|\widehat{\partial^2 \vartheta} \setminus \{\partial^2 \vartheta\}^\square \otimes(s)| \leq \|f\|_{C^2} \|\vartheta\|_{C^1} \|\tau\|_{C^0} \|\vartheta - \varrho\|_{C^0}$$

$$|\widehat{\partial^2 \vartheta} \setminus \{\partial^2 \vartheta\}^\square \otimes(s)| \leq \sigma \varepsilon \text{Lip}(Df \circ \varrho) \text{Lip}(\tau) \|\vartheta - \varrho\|_{C^0} \|\vartheta\|_{C^1} \|\tau\|_{C^0}$$

$$|\widehat{\partial^2 \vartheta} \setminus \{\partial^2 \vartheta\}^\square \otimes(s)| \leq \|f\|_{C^1} \|\vartheta - \varrho\|_{C^1} \|\tau\|_{C^0}$$

$$|\widehat{\partial^2 \vartheta} \setminus \{\partial^2 \vartheta\}^\square \otimes(s)| \leq \sigma \varepsilon \|f\|_{C^1} \|\varrho\|_{C^2} \text{Lip}(\tau) \|\vartheta - \varrho\|_{C^0} \|\tau\|_{C^0}$$

$$|\widehat{\partial^2 \vartheta} \setminus \{\partial^2 \vartheta\}^\square \otimes(s)| \leq \|f\|_{C^1} \|\varrho\|_{C^1} \text{Lip}(\tau) \|\vartheta - \varrho\|_{C^0}$$

$$|Q[\vartheta, \varepsilon] - Q[\varrho, \varepsilon]| \leq \mathfrak{L}_2 \|\vartheta - \varrho\|_{C^1}$$

$$\ddot{q}_i(t) = A_{\text{ext}}(t, q_i(t), \dot{q}_i(t)) + \sum_{j \neq i} A_{i,j} \left(q_i(t), \dot{q}_i(t), q_j(t - \tau_{ij}), \dot{q}_j(t - \tau_{ij}), q_j(t + \sigma_{ij}), \dot{q}_j(t + \sigma_{ij}) \right)$$

$$\tau_{ij}(t) = \frac{1}{c} |q_i(t) - q_j(t - \tau_{ij}(t))|$$

$$\sigma_{ij}(t) = \frac{1}{c} |q_i(t) - q_j(t + \sigma_{ij}(t))|$$

$$\begin{aligned} \ddot{q}_i(t) = A_{\text{ext}}(t, q_i(t), \dot{q}_i(t)) + F_i \left(\{q_j(t - \tau_{jk}(t))\}_{j,k=1}^N, \{\dot{q}_j(t - \tau_{jk}(t))\}_{j,k=1}^N, \right. \\ \left. \{q_j(t + \sigma_{jk}(t))\}_{j,k=1}^N, \{\dot{q}_j(t + \sigma_{jk}(t))\}_{j,k=1}^N \right), \end{aligned}$$



$$\tau_{ij}(t) = \varepsilon|q_i(t) - q_j(t)| + \varepsilon^2(q_i(t) - q_j(t)) \cdot \dot{q}_j(t) + O(\varepsilon^3).$$

$$y(t) \stackrel{\text{def}}{=} (q_1(t), \dots, q_N(t), \dot{q}_1(t), \dots, \dot{q}_N(t))$$

$$|\dot{q}_j(t)| \leq \xi_1 c, \text{ for all } j,$$

$$|q_i(t) - q_j(t)| \geq \xi_2, \text{ for all } i \neq j.$$

$$\tau_{ij}(t), \sigma_{ij}(t) > 0$$

$$\|\tau_{ij} - \tilde{\tau}_{ij}\|_{C^0}, \|\sigma_{ij} - \tilde{\sigma}_{ij}\|_{C^0} \leq C \left(\|q_i - \tilde{q}_i\|_{C^0} + \|q_j - \tilde{q}_j\|_{C^0} \right)$$

$$\|\tau_{ij}\|_{C^{\ell+Lip}}, \|\sigma_{ij}\|_{C^{\ell+Lip}} \leq g \left(\|q_i\|_{C^{\ell+Lip}}, \|q_j\|_{C^{\ell+Lip}}, \xi_1, \xi_2 \right).$$

$$\mathcal{N}[\tau; q_i, q_j](t) \stackrel{\text{def}}{=} \varepsilon|q_i(t) - q_j(t - \tau(t))|$$

$$|\mathcal{N}[\tau](t) - \mathcal{N}[\tilde{\tau}](t)| = \varepsilon \left| |q_i(t) - q_j(t - \tau(t))| - |q_i(t) - q_j(t - \tilde{\tau}(t))| \right| \\ \leq \varepsilon |q_j(t - \tau(t)) - q_j(t - \tilde{\tau}(t))| \leq \varepsilon \text{Lip}(q_j) \|\tau - \tilde{\tau}\|_{C^0}$$

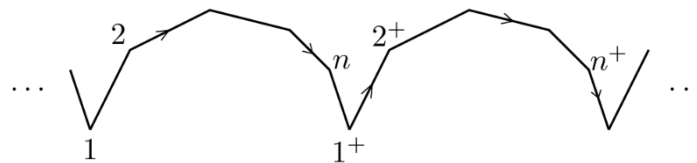
$$\varepsilon < 1/\|q_j\|_{C^1}, \text{ and define } \kappa \stackrel{\text{def}}{=} \varepsilon \|q_j\|_{C^1}$$

$$\|\mathcal{N}[\tilde{\tau}; q_i, q_j] - \tilde{\tau}\|_{C^0} \leq B \text{ for } \tilde{\tau} \text{ a fixed point of } \mathcal{N}[\cdot; \tilde{q}_i, \tilde{q}_j] \text{ by particles } \tilde{q}_i \text{ and } \tilde{q}_j$$

$$\|\tau - \tilde{\tau}\|_{C^0} \leq \frac{B}{1 - \kappa}$$

$$|\mathcal{N}[\tilde{\tau}; q_i, q_j](t) - \tilde{\tau}(t)| = |\varepsilon|q_i(t) - q_j(t - \tilde{\tau}(t))| - \varepsilon|\tilde{q}_i(t) - \tilde{q}_j(t - \tilde{\tau}(t))| \\ \leq \varepsilon|q_i(t) - \tilde{q}_i(t)| + \varepsilon|q_j(t - \tilde{\tau}(t)) - \tilde{q}_j(t - \tilde{\tau}(t))| \\ \leq \varepsilon \left(\|q_i - \tilde{q}_i\|_{C^0} + \|q_j - \tilde{q}_j\|_{C^0} \right)$$

$$F(p_1, \dots, p_n; q) := \int d^4x e^{-iq \cdot x} \langle p_1, \dots, p_n | \mathcal{O}(x) | 0 \rangle$$



$$\mathcal{F} := F/F^{\text{tree}} \sim \langle W[\mathcal{C}_n] \rangle = \left\langle \text{Tr} \mathcal{P} \exp \left[i g_{\text{YM}} \oint_{\mathcal{C}_n}^{D\rho} dx^\mu A_\mu^a(x) T^a \right] \right\rangle,$$

$$\ell_i := \{x^\mu(t_i) = (1 - t_i)x_i^\mu + t_i x_{i+1}^\mu; t_i \in [0, 1]\},$$

$$g^2 = g_{\text{YM}}^2 N_c / (4\pi)^2$$

$$x_{ij}^2 := (x_i - x_j)^2 = (p_i + \dots + p_{j-1})^2 = s_{i, \dots, j-1}.$$



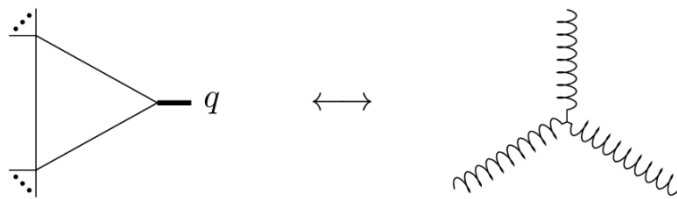
$$I_{\text{O}}(i, j) = \begin{array}{c} x_{i+1} \\ \cdot \\ \cdot \\ z_i \\ \cdot \\ \cdot \\ x_i \end{array} \begin{array}{c} x_j \\ \cdot \\ \cdot \\ z_j \\ \cdot \\ \cdot \\ x_{j+1} \end{array} = \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} dz_i^\mu dz_j^\nu D^{\mu\nu}(x-y),$$

$$D^{\mu\nu}(x-y) := \langle A^\mu(x)A^\nu(y) \rangle_0 = -\frac{\pi^\epsilon}{4\pi^2} \frac{\Gamma(1-\epsilon)}{(-(x-y)^2 + i0^+)^{1-\epsilon}} \eta^{\mu\nu}$$

$$\mathcal{F}_n^{(1)} = \sum_{i=1}^n \sum_{j=i+1}^{i+n-1} I_{\text{O}}(i, j)$$

$$\Sigma_5 \propto \sqrt{(1-u-v)^2 - 4uv}$$

$$u = \frac{s_{34}s_{41}}{s_{12}s_{23}}, v = \frac{4(q \cdot p_1)(q \cdot p_3)}{s_{12}s_{23}}$$



$$D_\xi^{\mu\nu}(x-y) := \langle A^\mu(x)A^\nu(y) \rangle = \int \frac{d^D k}{k^2 + i0^+} \left(\eta^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right) e^{-ik \cdot (x-y)},$$

$$S_\xi(i, j) := \frac{\pi^\epsilon \Gamma(1-\epsilon)}{4\pi^2} \frac{1-\xi}{4\epsilon} \left[(-x_{i,j}^2)^\epsilon + (-x_{i+1,j+1}^2)^\epsilon - (-x_{i+1,j}^2)^\epsilon - (-x_{i,j+1}^2)^\epsilon \right],$$

$$\begin{array}{|c|} \hline s_4 & s_3 \\ \hline ij & \\ \hline s_1 & s_2 \\ \hline \end{array} = s_1(-x_{i,j}^2)^\epsilon + s_2(-x_{i+1,j}^2)^\epsilon + s_3(-x_{i+1,j+1}^2)^\epsilon + s_4(-x_{i,j+1}^2)^\epsilon,$$

$$\begin{array}{|c|c|c|} \hline + & - & 0 \\ \hline 14 & 24 & 34 \\ \hline + & - & 0 \\ \hline 13 & 23 & 0 \\ \hline + & 0 & 0 \\ \hline - & 0 & 0 \\ \hline 12 & & \\ \hline 0 & 0 & \\ \hline \end{array} = -(-x_{1,5}^2)^\epsilon = -(-x_{1,1}^2)^\epsilon = 0,$$

$$\begin{array}{|c|c|c|} \hline & & + \\ \hline & & 32^+ \\ \hline & + & 0 \\ \hline & 21^+ & 31^+ \\ \hline + & 0 & 0 \\ \hline 13 & 23 & \\ \hline + & 0 & 0 \\ \hline 12 & & \\ \hline 0 & & \\ \hline \end{array} = \begin{aligned} & -(-x_{1,1+}^2)^\epsilon - (-x_{2,2+}^2)^\epsilon - (-x_{3,3+}^2)^\epsilon \\ & = (-x_{2,1+}^2)^\epsilon + (-x_{3,2+}^2)^\epsilon + (-x_{1+,3+}^2)^\epsilon \\ & \parallel \\ & \log\left(\frac{s_{12}s_{23}s_{31}}{(q^2)^3}\right) + O(\epsilon^1) \end{aligned}$$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma d\tau \frac{(\partial_\alpha z)^2 + (\partial_\alpha x)^2}{z^2} - i \sum_{i=1}^n k_i \cdot x(0, \sigma_i) + iq \cdot x(-\infty, \sigma_0)$$

$$K_i = \sum_{j=1}^i k_j$$

$$\begin{aligned} \sum_{i=1}^n k_i \cdot x(0, \sigma_i) &= q \cdot x(0, \sigma_{n+1}) + \sum_{i=1}^n K_i \cdot (x(0, \sigma_i) - x(0, \sigma_{i+1})) \\ &= q \cdot x(0, \sigma_{n+1}) - \sum_{i=1}^n K_i \cdot \int_{\sigma_i}^{\sigma_{i+1}} d\sigma \partial_\sigma x(0, \sigma) \end{aligned}$$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma d\tau \frac{(\partial_\alpha z)^2 + (\partial_\alpha x)^2}{z^2} + i \sum_{i=1}^n K_i \cdot \int_{\sigma_i}^{\sigma_{i+1}} d\sigma \partial_\sigma x(0, \sigma) - iq \cdot \int_{-\infty}^0 d\tau \partial_\tau x(\tau, \gamma(\tau))$$

$$\oint d\sigma A_\sigma(\tau, \sigma)$$

$$i \int_{\mathcal{D}} d\sigma d\tau y \cdot (\partial_\tau A_\sigma - \partial_\sigma A_\tau) + i\ell \cdot \oint d\sigma A_\sigma(\tau, \sigma)$$

$$\partial_\alpha x - A_\alpha|_{x=0} = \partial_\alpha x'$$

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma d\tau \frac{(\partial_\alpha z)^2 + A_\alpha^2}{z^2} - i \sum_{i=1}^n K_i \cdot \int_{\sigma_i}^{\sigma_{i+1}} d\sigma A_\sigma(0, \sigma) + iq \cdot \int_{-\infty}^0 d\tau A_\tau(\tau, \gamma(\tau)) \\ &+ i \int_{\mathcal{D}} d\sigma d\tau y \cdot (\partial_\tau A_\sigma - \partial_\sigma A_\tau) + i\ell \cdot \oint d\sigma A_\sigma(-\infty, \sigma) \end{aligned}$$

$$y \cdot (\partial_\tau A_\sigma - \partial_\sigma A_\tau) \text{ and rescaling fields } (A, y, z) \rightarrow \left(\frac{\sqrt{\lambda}}{4\pi} A, \frac{4\pi}{\sqrt{\lambda}} y, \frac{\sqrt{\lambda}}{4\pi} z\right)$$



$$\begin{aligned}
\left(\frac{\sqrt{\lambda}}{4\pi}\right)^{-1} S &= \int_{\mathcal{D}} d\sigma d\tau \left(\frac{(\partial_\alpha z)^2 + A_\alpha^2}{z^2} - i(A_\sigma \cdot \partial_\tau y - A_\tau \cdot \partial_\sigma y) \right) \\
&\quad - i \int_{-\infty}^0 d\tau A_\tau(\tau, \gamma(\tau)) \cdot [y(\tau, \sigma_{n+1}) - y(\tau, \sigma_1) - q] \\
&\quad - i \sum_{i=1}^n \int_{\sigma_i}^{\sigma_{i+1}} d\sigma A_\sigma(0, \sigma) \cdot (K_i - y(0, \sigma)) \\
&\quad - i \oint d\sigma A_\sigma(-\infty, \sigma) \cdot (-\ell + y(-\infty, \sigma)) \\
\frac{(\partial_\alpha z)^2}{z^2} + z^2 (\partial_\alpha y)^2 &= \frac{(\partial_\alpha r)^2 + (\partial_\alpha y)^2}{r^2}
\end{aligned}$$

$$y(\tau, \sigma_1 + 2\pi - 0^+) = y(\tau, \sigma_1) + q,$$

$$y(0, \sigma) = K_i = \sum_{j=1}^i k_j \quad \text{for } \sigma \in (\sigma_i, \sigma_{i+1}),$$

$$y(-\infty, \sigma) = \ell \quad \text{for } \sigma \in (\sigma_1, \sigma_1 + 2\pi - 0^+).$$

$$\mathcal{L} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{4} \phi_{AB} [\psi_\alpha^A, \psi^{\alpha B}] - \frac{1}{64} [\phi_{AB}, \phi_{CD}] [\phi^{AB}, \phi^{CD}]$$

$$W[\mathcal{C}_n]^{(L)} = \frac{1}{L!} \int_{z_i \sim z_i + q} dz_1 \cdots dz_L \left\langle \prod_{n_1, \dots, n_L \in \mathbb{Z}} \frac{\mathcal{L}^{[n_1]}(z_1 + n_1 q)}{\lambda^2} \cdots \frac{\mathcal{L}^{[n_L]}(z_L + n_L q)}{\lambda^2} W[\mathcal{C}_n] \right\rangle_0$$

$$\begin{aligned}
D_\xi^{\mu\nu}(x) &:= \int \frac{d^{4-2\epsilon} k}{k^2 + i0^+} \left(\eta^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right) e^{-ik \cdot x} \sum_{n \in \mathbb{Z}} \delta(k \cdot q - 2\pi n) \\
&= \int \frac{d^{4-2\epsilon} k}{k^2 + i0^+} \left(\eta^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right) e^{-ik \cdot x} \sum_{n \in \mathbb{Z}} e^{-ink \cdot q} \\
&= \sum_{n \in \mathbb{Z}} \int \frac{d^{4-2\epsilon} k}{k^2 + i0^+} \left(\eta^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right) e^{-ik \cdot (x + nq)} \\
&= \sum_{n=-\infty}^{\infty} D_\xi^{\mu\nu}(x + nq)
\end{aligned}$$

$$I_\xi(i, j) := \int d(z_i)_\mu d(z_j)_\nu D_\xi^{\mu\nu}(z_i - z_j) = \sum_{n \in \mathbb{Z}} I_\xi(i, j + n)$$

$$\frac{1}{2} \sum_{i, j=1}^n I_\xi(i, j) = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{j=-N}^N \sum_{i=1}^n I_\xi(i, j)$$



$$\lim_{N \rightarrow \infty} \sum_{j=-N}^N \sum_{i=1}^n S_{\xi}(i, j) \propto \lim_{N \rightarrow \infty} \begin{array}{|c|c|c|c|c|} \hline 1N & 2N & 3N & \cdots & nN \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 11 & 21 & 31 & \cdots & n1 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 1-N & 2-N & 3-N & \cdots & n-N \\ \hline \end{array}$$

$$= \lim_{N \rightarrow \infty} ((-x_{1,-N}^2)^{\epsilon} + (-x_{n+1,N+1}^2)^{\epsilon} - (-x_{1,N+1}^2)^{\epsilon} - (-x_{n+1,-N}^2)^{\epsilon})$$

$$= \lim_{N \rightarrow \infty} \log \left(\frac{x_{1,-N}^2 x_{n+1,N+1}^2}{x_{1,N+1}^2 x_{n+1,-N}^2} \right) + O(\epsilon)$$

$$\lim_{N \rightarrow \infty} \frac{x_{1,-N}^2}{x_{n+1,-N}^2} = \lim_{x_{1,-N} \rightarrow \infty} \frac{x_{1,-N}^2}{x_{1,-N}^2 - 2q \cdot x_{1,-N} + q^2} = 1$$

$$\lim_{N \rightarrow \infty} \log (x_{n+1,N+1}^2 / x_{1,N+1}^2) = 0$$

$$\int_{x_i}^{x_{i+1}} dx^{\gamma\dot{\gamma}} \int_{x_j}^{x_{j+1}} dy^{\sigma\dot{\sigma}} \int_{\ell \sim \ell+q} d^4\ell \sum_{n \in \mathbb{Z}} \langle \lambda^{-2} F_{\alpha\beta}(\ell + nq) F^{\alpha\beta}(\ell + nq) A_{\gamma\dot{\gamma}}(x) A_{\sigma\dot{\sigma}}(y) \rangle_0$$

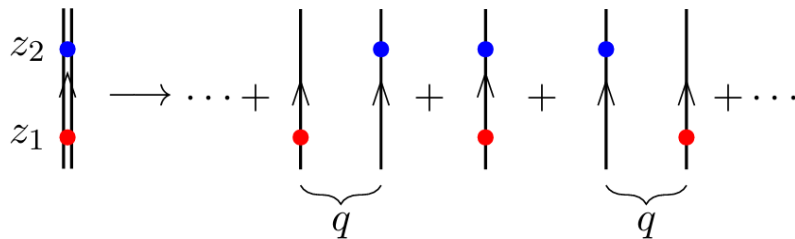
$$F_{\alpha\beta} = F_{\alpha\beta} = \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} F_{\alpha\dot{\alpha},\beta\dot{\beta}}$$

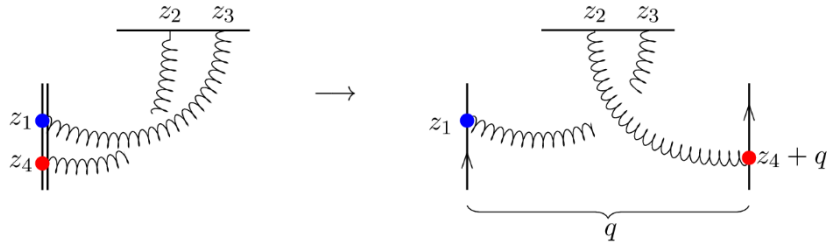
$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}]$$

$$\sum_{n \in \mathbb{Z}} \int_{\ell \sim \ell+q} d^4\ell O(\ell + nq; \dots) = \int_{\ell} d^4\ell O(\ell; \dots)$$

$$\int_{x_i}^{x_{i+1}} dx^{\gamma\dot{\gamma}} \int_{x_j}^{x_{j+1}} dy^{\sigma\dot{\sigma}} \int_{\ell} d^4\ell \langle \lambda^{-2} F_{\alpha\beta}(\ell) F^{\alpha\beta}(\ell) A_{\gamma\dot{\gamma}}(x) A_{\sigma\dot{\sigma}}(y) \rangle_0$$

$$W[\mathcal{C}_n] = N_c^{-1} \text{Tr} (1 + ig \int_{x \in \mathcal{C}_n} dx^{\mu} A_{\mu}^a(x) T^a - \frac{1}{2} g^2 \int_{x,y \in \mathcal{C}_n} dx^{\mu}(\lambda) dy^{\nu}(\tau) \times A_{\mu}^a(x) A_{\nu}^b(y) [T^a T^b \theta(\lambda - \tau) + T^b T^a \theta(\tau - \lambda)] + \dots)$$





$$\int_{z_4 < z_1} \int_{z_2 < z_3} (dz_1 \cdot dz_3)(dz_2 \cdot dz_4) D(z_1 - z_3) D(z_2 - z_4 - q)$$

$$\mathcal{F}_n^{\text{BDS}} = \exp \left[\sum_{\ell=1}^{\infty} g^{2\ell} \left(f^{(\ell)}(\epsilon) \mathcal{F}_n^{(1)}(\ell\epsilon) + C^{(\ell)} \right) \right]$$

$$R_n^{(2)} = \mathcal{F}_n^{(2)}(\epsilon) - \frac{1}{2} \left(\mathcal{F}_n^{(1)}(\epsilon) \right)^2 - f^{(2)}(\epsilon) \mathcal{F}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

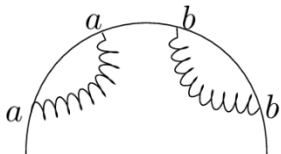
$$f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2 + \mathcal{O}(\epsilon^3), C^{(2)} = 4\zeta_4.$$

$$\log(\mathcal{F}_n) = \log \left(1 + g^2 \mathcal{F}_n^{(1)} + g^4 \mathcal{F}_n^{(2)} + \mathcal{O}(g^6) \right) = g^2 \mathcal{F}_n^{(1)} + g^4 \left(\mathcal{F}_n^{(2)} - \frac{1}{2} \left(\mathcal{F}_n^{(1)} \right)^2 \right) + \mathcal{O}(g^6)$$

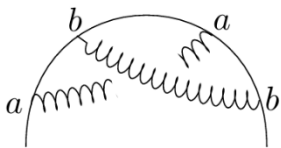
$$R_n^{(2)} = \log(\mathcal{F}_n)^{(2)} - f^{(2)}(\epsilon) \mathcal{F}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

$$= \log(\mathcal{W}[\mathcal{C}_n])^{(2)} - f^{(2)}(\epsilon) \mathcal{F}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

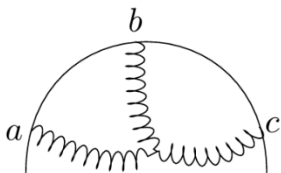
$$\mathcal{S} \left[R_n^{(2)} \right] = \mathcal{S} \left[\log(\mathcal{W}[\mathcal{C}_n])^{(2)} \right]$$



$$N_c^{-1} \text{Tr}(T^a T^a T^b T^b) = C_F^2 \rightarrow 0,$$



$$N_c^{-1} \text{Tr}(T^a T^b T^a T^b) = C_F^2 - \frac{1}{2} C_F C_A \rightarrow -\frac{1}{2} C_F C_A,$$

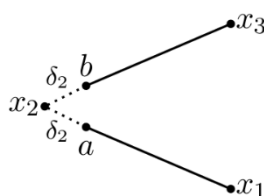
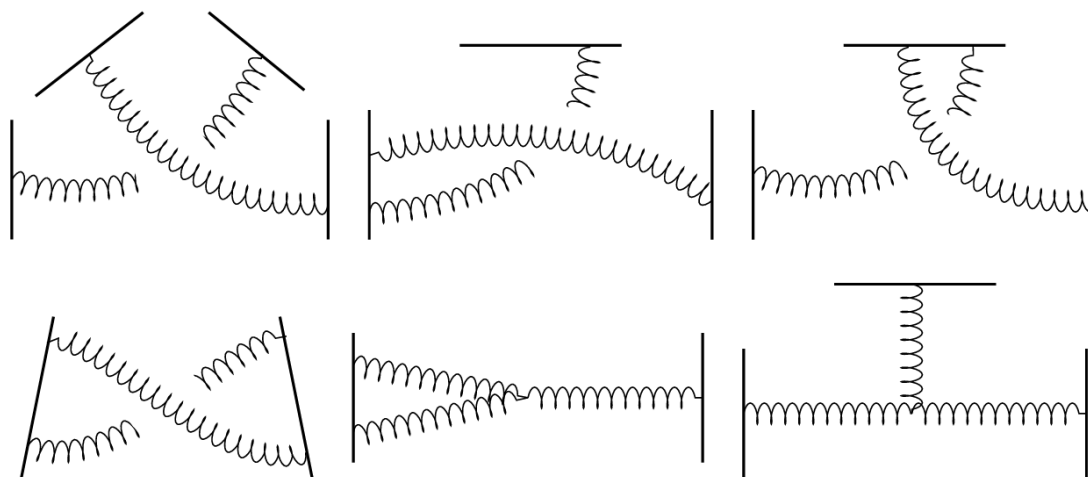


$$-i f^{abc} N_c^{-1} \text{Tr}(T^a T^b T^c) = \frac{1}{2} C_F C_A \rightarrow \frac{1}{2} C_F C_A,$$

$$\langle \mathcal{W}[\mathcal{C}_n] \rangle^{(2)} - \frac{1}{2} \left(\langle \mathcal{W}[\mathcal{C}_n] \rangle^{(1)} \right)^2$$

$$C_F C_A g_{\text{YM}}^4 = \frac{N_c^2 - 1}{2} g_{\text{YM}}^4 \sim g^4$$





$$a = (1 - \delta_2)x_2 + \delta_2 x_1, \quad b = (1 - \delta_2)x_2 + \delta_2 x_3,$$

$$(x_1 - b)^2 = (x_3 - a)^2 = \delta_2(x_1 - x_3)^2 = \eta,$$

$$T_\epsilon(z_1, z_2, z_3) = \begin{array}{c} z_2 \\ \diagdown \\ \bullet \\ \diagup \\ z_1 \end{array} \text{---} z_3 = \Gamma(1 - 2\epsilon) \int_{\mathbb{R}_+^3} d^3 a \frac{(a_1 a_2 a_3)^{-\epsilon} \delta(a_1 + a_2 + a_3 - 1)}{\left(\frac{1}{2} \sum_{i,j=1}^3 a_i a_j (z_i - z_j)^2\right)^{1-2\epsilon}},$$

$$M_{ij}^+ := \begin{array}{c} x_{i+1} \\ \diagdown \\ z_i \\ \diagup \\ x_i \end{array} \text{---} \begin{array}{c} x_j \\ \diagdown \\ z_j \\ \diagup \\ x_{j+1} \end{array} = \int_0^1 dt_j \int_0^1 dt_i (p_i \cdot p_j) T_\epsilon(x_i, z_i, z_j)$$

$$\xrightarrow{\text{regularize}} \int_0^1 dt_j \int_{\delta_i}^1 dt_i (p_i \cdot p_j) T_0(x_i - \delta_i p_{i-1}, z_i, z_j),$$

$$M_{ij}^- := \begin{array}{c} x_i \\ \diagdown \\ z_{i-1} \\ \diagup \\ x_{i-1} \end{array} \text{---} \begin{array}{c} x_j \\ \diagdown \\ z_j \\ \diagup \\ x_{j+1} \end{array} = \int_0^1 dt_j \int_0^1 dt_{i-1} (p_{i-1} \cdot p_j) T_\epsilon(z_{i-1}, x_i, z_j)$$

$$\xrightarrow{\text{regularize}} \int_0^1 dt_j \int_0^{1-\delta_i} dt_{i-1} (p_{i-1} \cdot p_j) T_0(z_{i-1}, x_i + \delta_i p_i, z_j),$$



$$T_0(x_1, x_2, x_3) \propto \int \frac{d^4x}{(x-x_1)^2(x-x_2)^2(x-x_3)^2}$$

$$T_0(x_1, x_2, x_3) = \frac{2}{(x_1-x_3)^2(z-\bar{z})} \left(\text{Li}_2(z) - \text{Li}_2(\bar{z}) + \frac{1}{2} \log(z\bar{z}) \log\left(\frac{1-z}{1-\bar{z}}\right) \right)$$

$$z\bar{z} = \frac{(x_1-x_2)^2}{(x_1-x_3)^2}, (1-z)(1-\bar{z}) = \frac{(x_2-x_3)^2}{(x_1-x_3)^2}$$

$$z, \bar{z} = \frac{x_{12}^2 + x_{13}^2 - x_{23}^2 \pm \sqrt{(x_{12}^2 + x_{13}^2 - x_{23}^2)^2 - 4x_{12}^2x_{13}^2}}{2x_{13}^2}$$

$$M_{ij}^+ = -M_{ij}^-(x_{i+1} \leftrightarrow x_{i-1}),$$

$$M_{ij}^+ = M_{ij}^{+(-1)} \epsilon^{-1} + O(1) = M_{ij}^{+(-1)} \log(\eta) + O(1)$$

$$I_O(i, k)I_O(j, l) = \left[\begin{array}{c} z_j \quad z_k \\ \diagdown \quad \diagup \\ \text{wavy lines} \\ \diagup \quad \diagdown \\ z_i \quad z_l \end{array} \right] := \int \frac{2(dz_i \cdot dz_k)}{(z_i - z_k)^2} \int \frac{2(dz_j \cdot dz_l)}{(z_j - z_l)^2},$$

$$\begin{aligned} I_O(i, k) &:= \int \frac{2(dz_i \cdot dz_k)}{(z_i - z_k)^2} = \int_{[0,1]^2} dt_i dt_k \frac{(p_i + p_k)^2}{((x_i(1-t_i) + x_{i+1}t_i) - (x_k(1-t_k) + x_{k+1}t_k))^2} \\ &= \text{Li}_2 \left(1 - \frac{(x_{i,k}^2 - x_{i,k+1}^2)(x_{i,k+1}^2 - x_{i+1,k+1}^2)}{x_{i,k+1}^2 x_{i+1,k}^2 - x_{i,k}^2 x_{i+1,k+1}^2} \right) \\ &\quad + \log(x_{i,k+1}^2) \log \left(\frac{(x_{i,k}^2 - x_{i,k+1}^2)(x_{i,k+1}^2 - x_{i+1,k+1}^2)}{x_{i,k+1}^2 x_{i+1,k}^2 - x_{i,k}^2 x_{i+1,k+1}^2} \right) \\ &\quad - (i \leftrightarrow i+1) - (k \leftrightarrow k+1) + (i \leftrightarrow i+1, k \leftrightarrow k+1), \end{aligned}$$

$$I_{C,1}(i, j, k) = \left[\begin{array}{c} x_j \quad z_3 x_{j+1} \\ \text{wavy lines} \\ x_{i+1} \quad x_k \\ z_2 \quad z_4 \\ \text{wavy lines} \\ x_i \quad x_{k+1} \end{array} \right] = 4 \int_{z_1 < z_2} \frac{(dz_1 \cdot dz_3)(dz_2 \cdot dz_4)}{(z_1 - z_3)^2(z_2 - z_4)^2},$$

$$I_{C,2}(i, j, k) = \left[\begin{array}{c} x_j \quad z_2 \quad z_3 x_{j+1} \\ \text{wavy lines} \\ x_{i+1} \quad x_k \\ z_1 \quad z_4 \\ \text{wavy lines} \\ x_i \quad x_{k+1} \end{array} \right] = 4 \int_{z_2 < z_3} \frac{(dz_1 \cdot dz_3)(dz_2 \cdot dz_4)}{(z_1 - z_3)^2(z_2 - z_4)^2},$$



$$I_{C,3}(i, j, k) = \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \int_{z_1}^{z_3} \int_{z_4}^{z_k} \frac{(dz_1 \cdot dz_3)(dz_2 \cdot dz_4)}{(z_1 - z_3)^2(z_2 - z_4)^2} = 4 \int_{z_3 < z_4} \frac{(dz_1 \cdot dz_3)(dz_2 \cdot dz_4)}{(z_1 - z_3)^2(z_2 - z_4)^2}$$

$$I_{C,1}(i-1, i, k) = 2M_{i,k}^- + O(1), \quad I_{C,2}(i-1, i, k) = 2M_{i,k}^+ + O(1),$$

$$I_{C,2}(i-1, k-1, k) = 2M_{k,i-1}^- + O(1), \quad I_{C,3}(i-1, k-1, k) = 2M_{k,i-1}^+ + O(1),$$

$$I_{C,2}(1, 2, 3) = \left(\frac{\zeta_2}{\epsilon^2} - \frac{2\zeta_3}{\epsilon} + \frac{19}{4}\zeta_4 \right) (x_{13}^2 x_{24}^2)^\epsilon,$$

$$I_S(i, j, k) = \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \int_{z_i}^{z_k} \frac{(dz_i \cdot dz_j)(dz_k \cdot dz_i)}{(z_i - z_j)^2(z_k - z_i)^2}$$

$$z_a = (1 - t_a)x_a + t_ax_{a+1} \quad \text{for } a = i, j, k \text{ and } 0 \leq t_a \leq 1,$$

$$V = (dz_i \cdot dz_j)(dz_k \cdot (\partial_i - \partial_j)) + (dz_j \cdot dz_k)(dz_i \cdot (\partial_j - \partial_k)) + (dz_k \cdot dz_i)(dz_j \cdot (\partial_k - \partial_i))$$

$$\partial_a := \partial / \partial z_a$$

$$I_S(i, j, k) = \int_{z_i, z_j, z_k} VT_\epsilon(z_i, z_j, z_k)$$

$$= \Gamma(1 - 2\epsilon) \int_{z_i, z_j, z_k} V \int_{\mathbb{R}_+^3} d^3a \frac{(a_1 a_2 a_3)^{-\epsilon} \delta(a_1 + a_2 + a_3 - 1)}{\left(\frac{1}{2} \sum_{i,j=1}^3 a_i a_j (z_i - z_j)^2 \right)^{1-2\epsilon}}$$

$$(z_1 - z_2)^2 = (1 - t_1)t_2(x_1 - x_3)^2$$

$$T_0(z_1, z_2, z_3) \propto \frac{1}{(1 - t_1)t_2(x_1 - x_3)^2}$$

$$V_{\text{div}} := (dz_2 \cdot dz_3)(dz_1 \cdot \partial_1) - (dz_3 \cdot dz_1)(dz_2 \cdot \partial_2)$$

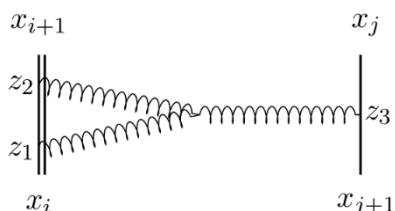


$$(V - V_{\text{div}}) \left[\frac{\delta(a_1 + a_2 + a_3 - 1)}{\sum_{1 \leq i < j \leq 3} a_i a_j (z_i - z_j)^2} \right] \xrightarrow{a_3 \rightarrow 0} 0$$

$$\int_{z_2, z_3} (dz_2 \cdot dz_3) (T_\epsilon(x_2, z_2, z_3) - T_\epsilon(x_1, z_2, z_3)) - \int_{z_1, z_3} (dz_3 \cdot dz_1) (T_\epsilon(z_1, x_3, z_3) - T_\epsilon(z_1, x_2, z_3))$$

$$I_S(1, 2, j) = M_{2j}^+ + M_{2j}^- + O(1)$$

$$\begin{aligned} I_S(1, 2, 3) &= M_{23}^- + M_{31}^+ + \frac{\zeta_2}{4\epsilon^2} + \frac{\zeta_3}{\epsilon} + \frac{\zeta_2}{4\epsilon} \log(x_{13}^2 x_{24}^2) + O(1) \\ &= M_{23}^- + M_{31}^+ + \frac{\zeta_2}{2} \log(\eta)^2 - \frac{\zeta_2}{2} \log(\eta) \log(x_{13}^2 x_{24}^2) - 3\zeta_3 \log(\eta) + O(1) \end{aligned}$$



$$\square|_Y = \sum_{m, n \in \mathbb{Z}} \int_{z_1 < z_2} \int d^4x V T_\epsilon(z_1 + mq, z_2 + nq, z_3).$$

$$dz_1 \cdot dz_2 = p_1^2 dt_1 dt_2 = 0$$

$$V_Y = (dz_1 \cdot dz_3) dz_2 \cdot \partial_2 - (dz_2 \cdot dz_3) dz_1 \cdot \partial_1,$$

$$dz_i \cdot \partial_i = dt_i \partial_{t_i}$$

$$I_Y = \sum_{m, n \in \mathbb{Z}} \int_{x_2}^{y_2} \int_{x_1}^{y_1} (dz_1 \cdot dz_3) (T_\epsilon(z_1 + mq, y_1 + nq, z_3) + T_\epsilon(x_1 + mq, z_1 + nq, z_3)$$

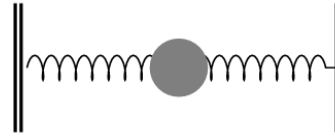
$$- 2T_\epsilon(z_1 + mq, z_1 + nq, z_3))$$

$$= \left[\text{Diagram 1} \right] + \left[\text{Diagram 2} \right] - \dots$$

$$= M_{i+1, j}^- + M_{i, j}^+ - 2 \sum_{m, n \in \mathbb{Z}} \int_{x_2}^{y_2} \int_{x_1}^{y_1} (dz_1 \cdot dz_3) T_\epsilon(x_1 + mq, z_1 + nq, z_3)$$

$$\int_{x_1}^{y_1} \int_{x_2}^{y_2} (dz_1 \cdot dz_3) T_\epsilon(z_1 + mq, z_1 + nq, z_3)$$





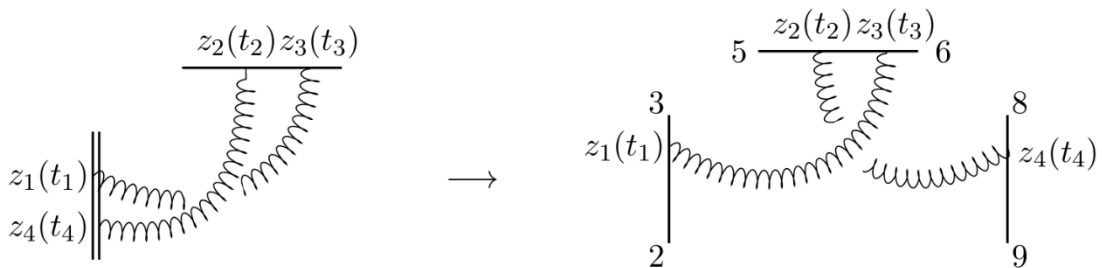
$$I_{Y,1}(i, j) = M_{i+1, j}^- + M_{i, j}^+ = \left[\text{Diagram of a spring attached to a wall on the left and a mass on the right} \right] + \left[\text{Diagram of a spring attached to a wall on the right and a mass on the left} \right]$$

$$I_{Y,2}(i, j) := \left[\text{Diagram of a spring attached to a wall on the left and a mass on the right, with a horizontal line above the mass} \right] + \left[\text{Diagram of a spring attached to a wall on the right and a mass on the left, with a horizontal line above the mass} \right]$$

$$I_{Y,2}(i, i+1) = M_{i+1, i+n}^+ + O(1), I_{Y,2}(i, i+n-1) = M_{i+n, i}^- + O(1)$$

$$I_X(i, j) := \left[\text{Diagram of a quadrilateral with vertices } z_1, z_2, z_3, z_4 \text{ and wavy edges} \right]$$

$$I_X = 4 \int_{z_1 < z_2} \int_{z_3 < z_4} \frac{(dz_1 \cdot dz_3)(dz_2 \cdot dz_4)}{(z_1 - z_3)^2(z_2 - z_4)^2},$$



$$x_3 - x_2 = x_9 - x_8 = p_1, x_6 - x_5 = p_2, x_8 - x_2 = q$$

$$I_{XC}(i, j) = \int_0^1 dt_3 \int_0^{t_3} dt_2 \int_0^1 dt_1 \int_0^{t_1} dt_4 \frac{((p_1 + p_2)^2)^2}{(z_1 - z_3)^2(z_2 - z_4)^2}$$

$$-x_{2,5}^2 + x_{2,6}^2 + x_{3,5}^2 - x_{3,6}^2 = -x_{5,8}^2 + x_{5,9}^2 + x_{6,8}^2 - x_{6,9}^2 = (p_1 + p_2)^2.$$

$$a \in \left\{ 0, 1, \frac{x_{5,8}^2}{x_{2,5}^2}, \frac{x_{6,8}^2}{x_{2,6}^2}, \frac{x_{3,6}^2 x_{5,8}^2 - x_{3,5}^2 x_{6,8}^2}{x_{2,5}^2 x_{3,6}^2 - x_{2,6}^2 x_{3,5}^2} \right\}$$

$$z\bar{z} = \frac{x_{5,8}^2 x_{6,9}^2 - x_{5,9}^2 x_{6,8}^2}{x_{2,5}^2 x_{3,6}^2 - x_{2,6}^2 x_{3,5}^2}, (1-z)(1-\bar{z}) = \frac{4(p_1 \cdot q)(p_2 \cdot q)}{x_{2,5}^2 x_{3,6}^2 - x_{2,6}^2 x_{3,5}^2}$$



$$\frac{1}{2}F \otimes \left(\frac{1-z}{1-\bar{z}} \otimes (z\bar{z}) - \frac{z}{\bar{z}} \otimes ((1-z)(1-\bar{z})) \right)$$

$$F = \frac{x_{5,9}^2 x_{6,8}^2}{x_{5,8}^2 x_{6,9}^2} \otimes X(0) - \frac{x_{2,5}^2 x_{5,9}^2}{x_{3,5}^2 x_{5,8}^2} \otimes X\left(\frac{x_{5,8}^2}{x_{2,5}^2}\right)$$

$$- \frac{x_{3,6}^2 x_{6,8}^2}{x_{2,6}^2 x_{6,9}^2} \otimes X\left(\frac{x_{6,8}^2}{x_{2,6}^2}\right) - \frac{x_{3,5}^2 x_{6,9}^2}{x_{3,6}^2 x_{5,9}^2} \otimes X\left(\frac{x_{3,6}^2 x_{5,8}^2 - x_{3,5}^2 x_{6,8}^2}{x_{2,5}^2 x_{3,6}^2 - x_{2,6}^2 x_{3,5}^2}\right)$$

$$S[(z-\bar{z})T_0(z,\bar{z})] = \frac{1}{2} \left((z\bar{z}) \otimes \frac{1-z}{1-\bar{z}} - ((1-z)(1-\bar{z})) \otimes \frac{z}{\bar{z}} \right)$$

$$I_{XC}(i, i+1) = 2M_{i+1, i+n}^+ + O(1), I_{XC}(i, i+n-1) = 2M_{i+n, i}^- + O(1)$$

$$R_n^{(2)} = \sum_{1 < j < k \leq n} (I_S(1, j, k) - I_{C,1}(1, j, k) - I_{C,2}(1, j, k) - I_{C,3}(1, j, k))$$

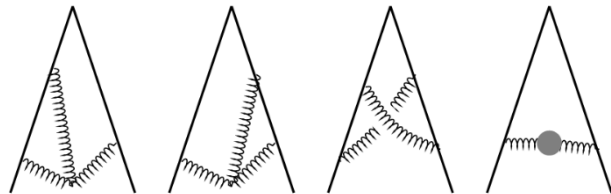
$$+ \sum_{j=2}^n \left(I_{Y,2}(1, j) - \frac{1}{2} (I_{XC}(1, j) + I_{C,2}(1, j, n+1)) \right)$$

$$+ \sum_{j=3}^n (I_{Y,1}(1, j) + I_{Y,1}(j-2, n) - I_X(1, j))$$

$$- \sum_{\substack{1 < j < k < l \\ 2 < k \leq n, 1 < l-j < n}} I_0(1, k) I_0(j, l) - (\dots) + \text{cyclic}$$

$$I_0(1,3)I_0(4,7) = I_0(1,3)I_0(-1,2) \xrightarrow{\text{cyclic}^2} I_0(1,4)I_0(3,5),$$

$$I_0(1,5)I_0(3,7) = I_0(1,5)I_0(-2,2) \xrightarrow{\text{cyclic}^3} I_0(1,5)I_0(4,8),$$



$$\frac{1}{4\epsilon^2} \left[\frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} - 1 \right] \frac{(-s_{i,i+1})^{2\epsilon}}{(2\epsilon)^2},$$

$$\frac{1}{4\epsilon^2} \left[\frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} - 1 \right] \sum_{i=1}^n \frac{(-s_{i,i+1})^{2\epsilon}}{(2\epsilon)^2} \propto \zeta_2 \mathcal{F}_n^{(1)}(2\epsilon) + O(\epsilon),$$

$$\mathcal{M}_k^\pm := \sum_{i=1}^n M_{i, i+k}^\pm$$

$$\sum_{i=3}^n (I_{Y,1}(1, i) + I_{Y,1}(i-2, n)) + \text{cyclic} = \sum_{k=2}^{n-1} (\mathcal{M}_k^+ + \mathcal{M}_{-k}^+ + \mathcal{M}_{k-1}^- + \mathcal{M}_{-k-1}^-) + O(1),$$

$$I_{Y,2}(1,2) + I_{Y,2}(1, n) + \text{cyclic} = \mathcal{M}_{n-1}^+ + \mathcal{M}_{-n}^- + O(1),$$



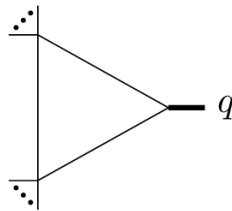
$$\begin{aligned}
& I_S(1,2,3) + \sum_{i=4}^n I_S(1,2,i) + \sum_{i=1}^{n-3} I_S(i,n-1,n) + \text{cyclic} \\
&= \mathcal{M}_1^- + \mathcal{M}_{-2}^+ + \sum_{k=2}^{n-2} (\mathcal{M}_k^+ + \mathcal{M}_{-k-1}^+ + \mathcal{M}_k^- + \mathcal{M}_{-k-1}^-) + O(1) \\
& \left(\sum_{i=3}^n I_{C,1}(1,2,i) + \sum_{i=4}^n I_{C,2}(1,2,i) + \sum_{i=1}^{n-3} I_{C,2}(i,n-1,n) + \sum_{i=1}^{n-2} I_{C,3}(i,n-1,n) \right) + \text{cyclic} \\
&= 2 \sum_{k=1}^{n-2} \mathcal{M}_k^- + 2 \sum_{k=2}^{n-2} \mathcal{M}_k^+ + 2 \sum_{k=1-n}^{-3} \mathcal{M}_k^- + 2 \sum_{k=1-n}^{-2} \mathcal{M}_k^+ + O(1)
\end{aligned}$$

$$\frac{1}{2} (I_{XC}(1,2) + I_{XC}(1,n) + I_{C,2}(1,2,n+1) + I_{C,2}(1,n,n+1)) + \text{cyclic} = 2(\mathcal{M}_{n-1}^+ + \mathcal{M}_{-n}^-) + O(1)$$

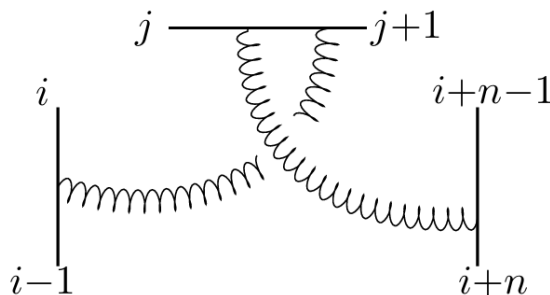
$$\text{div}_{Y,1} + \text{div}_{Y,2} + \text{div}_S - \text{div}_{C,1} - \text{div}_{C,2} - \text{div}_{C,3} - \text{div}_{XC} = 0.$$

$$\begin{aligned}
R_n^{(2)} &= \sum_{1 < j < k \leq n} (\hat{I}_S(1,j,k) - \hat{I}_{C,1}(1,j,k) - \hat{I}_{C,2}(1,j,k) - \hat{I}_{C,3}(1,j,k)) \\
&+ \sum_{j=2}^n \left(\hat{I}_{Y,2}(1,j) - \frac{1}{2} (\hat{I}_{XC}(1,j) + \hat{I}_{C,2}(1,j,n+1)) \right) - \sum_{j=3}^n I_X(1,j) \\
&- \sum_{\substack{1 < j < k < l \\ 2 < k \leq n, 1 < l-j < n}} I_O(1,k) I_O(j,l) + \text{cyclic},
\end{aligned}$$

$$\hat{I}_{C,2}(i-1,i,i+1) = 0, \hat{I}_S(i-1,i,i+1) = O(1).$$



$$z\bar{z} = \frac{(p_i + \dots + p_j)^2}{q^2}, (1-z)(1-\bar{z}) = \frac{(p_{j+1} + \dots + p_{i-1})^2}{q^2}.$$



$$\left\{ \frac{x_{j,i^+}^2 x_{j+1,i^++1}^2 - x_{j,i^++1}^2 x_{j+1,i^+}^2}{x_{i,j}^2 x_{i+1,j+1}^2 - x_{i,j+1}^2 x_{i+1,j}^2}, \frac{4(p_i \cdot q)(p_j \cdot q)}{x_{i,j}^2 x_{i+1,j+1}^2 - x_{i,j+1}^2 x_{i+1,j}^2} \right\}_{i,j}^2$$

$$x_{i,j}^2 x_{i+1,j+1}^2 - x_{i,j+1}^2 x_{i+1,j}^2 \propto \langle i-1, i, i+1, j \rangle \langle i, j-1, j, j+1 \rangle,$$

$$x_{j,i^+}^2 x_{j+1,i^++1}^2 - x_{j,i^++1}^2 x_{j+1,i^+}^2 \propto \langle j-1, j, j+1, i^+ \rangle \langle j, i^+-1, i^+, i^++1 \rangle,$$

$$2p_i \cdot q \propto \langle i-1, i, i+1, i^+ \rangle.$$

$$\left\{ \frac{\langle i, i+1 \rangle \sum_{k=i}^j \langle ik \rangle [kj]}{[i, i+1] \sum_{k=i}^j [ik] \langle kj \rangle} \right\}_{i,j}$$

$$\tilde{\lambda}_i = \lambda_i = \sqrt{2e_i} \left(\frac{t_i^2 - 1}{t_i^2 + 1}, \frac{2t_i}{t_i^2 + 1} \right)^\top$$

$$t_i = \begin{cases} \alpha_i \epsilon, & i \text{ odd} \\ 1 - \alpha_i \epsilon, & i \text{ even} \end{cases}$$

$$\{E_1, E_2, E_3, E_4, E_5, E_6, E_1 + E_3, E_2 + E_4, E_1 + E_5, E_3 + E_5, E_2 + E_6, E_4 + E_6, E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4 - E_5, E_5 - E_6, E_6 - E_1, E_3 - E_4 + E_5, E_4 - E_5 + E_6, E_5 - E_6 + E_1\}$$

$$E_{2i} = \frac{e_{2i}}{e_2 + e_4 + e_6}, E_{2i+1} = \frac{e_{2i+1}}{e_1 + e_3 + e_5}$$

$$\sum_{j=1}^k E_{i+2j}, \sum_{j=0}^k (-1)^j E_{i+j}$$

$$E_{2i} = \frac{e_{2i}}{\sum_{k=1}^{n/2} e_{2k}}, E_{2i+1} = \frac{e_{2i+1}}{\sum_{k=1}^{n/2} e_{2k-1}}$$

$$\sum_i E_{2i} = \sum_i E_{2i+1} = 1$$

$$x_{i+1} - x_i = p_i = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$$

$$Z_i = \begin{pmatrix} \lambda_i^\alpha \\ \mu_i^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \lambda_i^\alpha \\ x_i^{\alpha\dot{\alpha}} \lambda_{i\alpha} \end{pmatrix}$$

$$\langle i, j, k, l \rangle := \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D = \det([Z_i Z_j Z_k Z_l])$$



$$\sum_{k=1}^5 (-1)^k \langle i_1, i_2, i_3, j_k \rangle \langle j_1, \dots, \widehat{j}_k, \dots, j_5 \rangle = 0$$

$$x_i^{\alpha\dot{\alpha}} = \frac{\lambda_{i-1}^\alpha \mu_i^{\dot{\alpha}} - \lambda_i^\alpha \mu_{i-1}^{\dot{\alpha}}}{\langle i-1, i \rangle}$$

$$x_{i,j}^2 = (x_i - x_j)^2 = \frac{\langle i-1, i, j-1, j \rangle}{\langle i-1, i \rangle \langle j-1, j \rangle},$$

$$\langle i, j \rangle := \lambda_i^\alpha \lambda_j^\beta \epsilon_{\alpha\beta}$$

$$\frac{\langle i-1, i, i+1, j \rangle}{\langle i-1, i \rangle \langle i, i+1 \rangle} = \lambda_j^\alpha (x_j - x_i)_{\alpha\dot{\alpha}} \tilde{\lambda}_i^{\dot{\alpha}}$$

$$I_\infty = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\top$$

$$Z_{i+n} = \begin{pmatrix} \lambda_{i+n}^\alpha \\ x_{i+n}^{\alpha\dot{\alpha}} \lambda_{(i+n)\alpha} \end{pmatrix} = \begin{pmatrix} \lambda_i^\alpha \\ (x_i + q)^{\alpha\dot{\alpha}} \lambda_{i\alpha} \end{pmatrix} = \begin{pmatrix} \delta_\beta^\alpha & 0 \\ q_\beta^{\dot{\alpha}} & \delta_\beta^{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} \lambda_i^\beta \\ x_i^{\alpha\beta} \lambda_{i\alpha} \end{pmatrix} = P Z_i.$$

$$P(Z) = Z + \frac{\langle n+1, Z \rangle}{\langle n, n+1 \rangle \langle n+1, 1 \rangle} I_\infty \cap (n, n+1, 1)$$

$$I_\infty \cap (n, n+1, 1) := -\langle n, n+1 \rangle Z_1 + \langle n, 1 \rangle Z_{n+1} - \langle n+1, 1 \rangle Z_n$$

$$I_\infty = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}^\top, P = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_1^{-1} & 0 & 0 \\ 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & S_1^{-1} \end{pmatrix}, M_2 = \frac{1}{\sqrt{F_2 S_2 T_2}} \begin{pmatrix} F_2 S_2 & 0 & 0 & 0 \\ -F_2 S_2 (T_2^2 - 1) & F_2 S_2 T_2^2 & 0 & S_2 T_2 (S_2 - F_2 T_2) \\ T_2 - F_2 S_2 & 0 & T_2 & 0 \\ 0 & 0 & 0 & S_2^2 T_2 \end{pmatrix}$$

$$M_3 = \frac{1}{\sqrt{F_3 S_3 T_3}} \begin{pmatrix} -S_3 (F_3 (1 - 3T_3^2) + S_3 T_3) & -F_3 S_3 (T_3^2 - 1) & S_3 T_3 (F_3 T_3 - S_3) & F_3 S_3 (T_3^2 - 1) \\ 3F_3 S_3 (T_3^2 - 1) - (S_3^2 - 1) T_3 & -F_3 S_3 (T_3^2 - 3) - T_3 & S_3 T_3 (F_3 T_3 - S_3) & F_3 S_3 (T_3^2 - 3) + 2T_3 \\ S_3 (F_3 (1 - 3T_3^2) + 2S_3 T_3) & F_3 S_3 (T_3^2 - 1) & S_3 T_3 (2S_3 - F_3 T_3) & -F_3 S_3 (T_3^2 - 1) \\ T_3 - F_3 S_3 & F_3 S_3 - T_3 & 0 & 2T_3 - F_3 S_3 \end{pmatrix}$$

$$M_4 = \frac{1}{\sqrt{F_4 S_4 T_4}} \times \begin{pmatrix} F_4 S_4 (9 - 5T_4^2) + (S_4^2 - 4) T_4 & 3F_4 S_4 (T_4^2 - 1) - (S_4^2 - 1) T_4 & -F_4 S_4 (T_4^2 - 3) - 2T_4 & F_4 S_4 (3 - 4T_4^2) + (2S_4^2 - 1) T_4 \\ 4(S_4^2 - 1) T_4 - 15F_4 S_4 (T_4^2 - 1) & F_4 S_4 (9T_4^2 - 5) + (1 - 4S_4^2) T_4 & F_4 S_4 (5 - 3T_4^2) - 2T_4 & F_4 S_4 (5 - 12T_4^2) + (8S_4^2 - 1) T_4 \\ F_4 S_4 (5T_4^2 - 12) - (S_4^2 - 8) T_4 & F_4 S_4 (4 - 3T_4^2) + (S_4^2 - 2) T_4 & F_4 S_4 (T_4^2 - 4) + 4T_4 & 4F_4 S_4 (T_4^2 - 1) - 2(S_4^2 - 1) T_4 \\ S_4 (F_4 (3 - 5T_4^2) + 2S_4 T_4) & -S_4 (F_4 (1 - 3T_4^2) + 2S_4 T_4) & -F_4 S_4 (T_4^2 - 1) & S_4 (F_4 (1 - 4T_4^2) + 4S_4 T_4) \end{pmatrix}$$

$$Z_1 = M_1 (0, 1, 0, 1)^\top, Z_2 = M_1 M_2 (1, 3, -1, 1)^\top, \\ Z_3 = M_1 M_2 (1, 1, -2, 0)^\top, Z_4 = (0, 0, 1, 0)^\top.$$



$$Z_1 = M_1(0,1,0,1)^\top, Z_2 = M_1 M_2 M_3(1,4,-1,2)^\top$$

$$Z_3 = M_1 M_2 M_3(3,5,-4,1)^\top, Z_4 = M_1 M_2(1,1,-2,0)^\top, Z_5 = (0,0,1,0)^\top$$

$$Z_1 = M_1(0,1,0,1)^\top, Z_2 = M_1 M_2 M_3(1,4,-1,2)^\top, Z_3 = M_1 M_2 M_3 M_4(5,12,-6,4)^\top,$$

$$Z_4 = M_1 M_2 M_3 M_4(4,6,-6,1)^\top, Z_5 = M_1 M_2(1,1,-2,0)^\top, Z_6 = (0,0,1,0)^\top.$$

$$\left\langle \prod_{i=1}^{n_1} \frac{1}{k_i} \text{Tr} \left[(Y_1 \cdot \Phi(w_1))^{k_i} \right] \prod_{j=1}^{n_2} \frac{1}{l_j} \text{Tr} \left[(Y_2 \cdot \Phi(w_2))^{l_j} \right] \right\rangle_{\mathcal{N}=4}^c$$

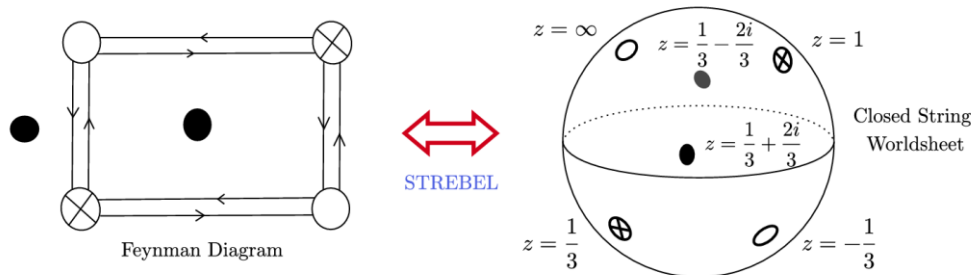
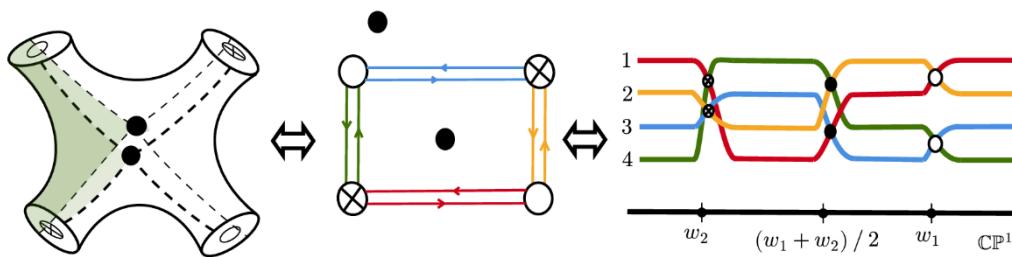
$$= \sum_{g \geq 0} N^{2-2g-(n_1+n_2)} \sum_{\text{Feyn. Diags. of genus } g} \frac{1}{\text{Aut}(\text{Diag})} \left[\frac{\lambda Y_1 \cdot Y_2}{|w_1 - w_2|^2} \right]^{\#\text{Edges}}$$

Takeaway #1: $g_{st} = \frac{1}{N}$

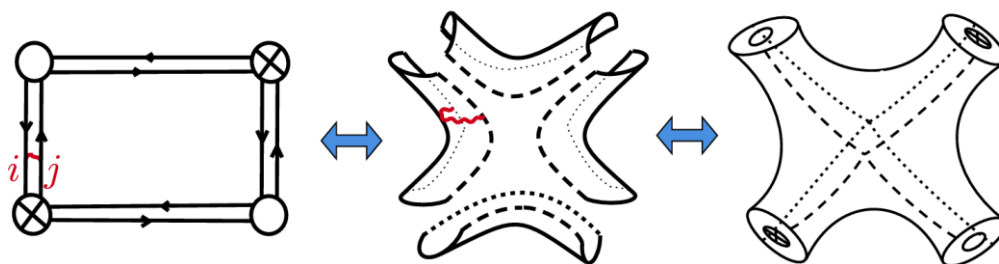
Takeaway #2: Each FD = a Specific Worldsheet

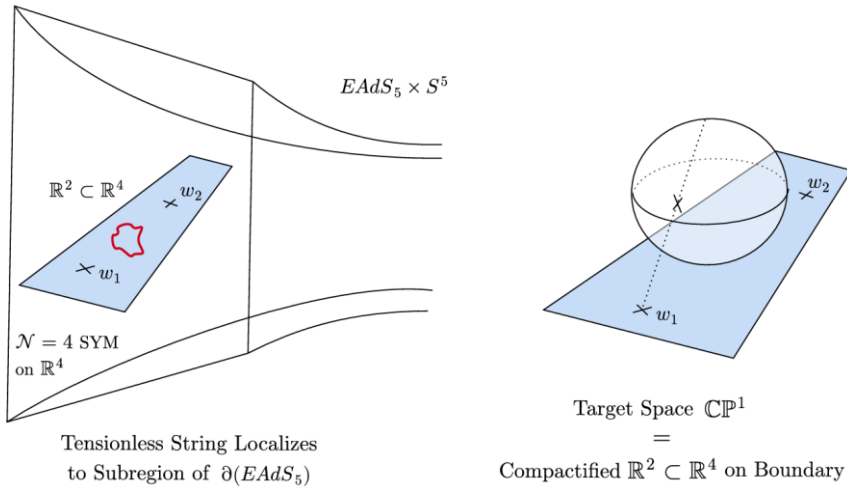
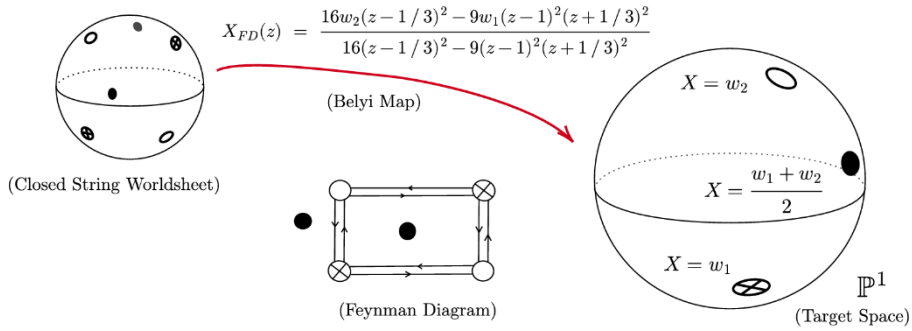
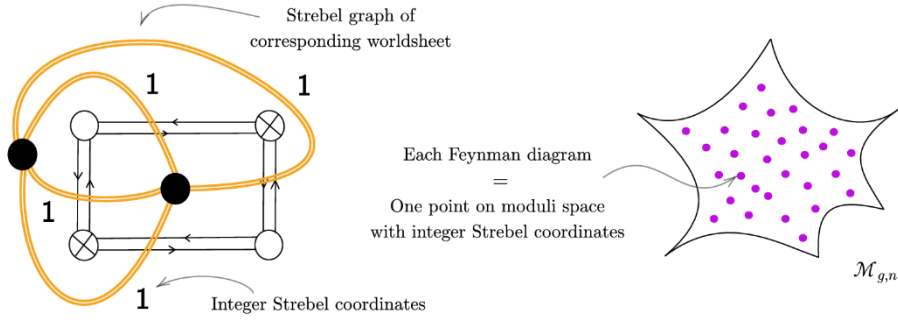
Takeaway #3: Weight of FD \leftrightarrow Action of String

$$\sum_{g \geq 0} g_{st}^{2g-2+(n_1+n_2)} \int_{\mathcal{M}_{g, n_1+n_2}} \sum_{\mathcal{N}=4} \delta \left(\begin{array}{l} \text{Streibl Graph of} \\ \text{Worldsheet} \end{array} = \begin{array}{l} \text{Dual Skeleton Graph of} \\ \text{Feyn. Diag} \end{array} \right) e^{-S_{NG}[X_{FD}]}$$



Worldsheet Metric in "Streibl Gauge" $ds^2 = |\phi_S(z)| dz d\bar{z} = \left| \frac{(z - 1/3 - 2i/3)^2 (z - 1/3 + 2i/3)^2}{\pi^2 (z-1)^2 (z+1/3)^2 (z-1/3)^2} \right| dz d\bar{z}$





$$Z_N = \int dK dM_{N \times N} e^{-\frac{N}{g} \text{Tr}(KM)}$$

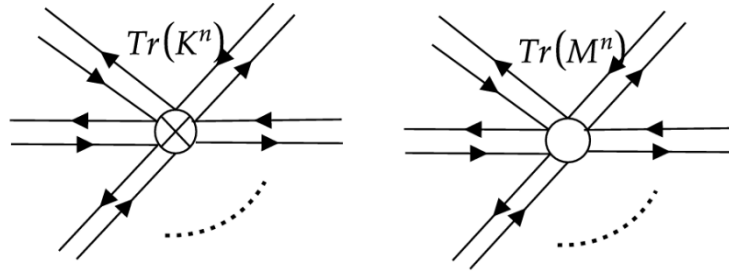
$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle \equiv \frac{1}{Z_N} \int dK dM_{N \times N} e^{-\frac{N}{g} \text{Tr}(KM)} \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j})$$

$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle_c = \sum_{h \geq 0} N^{2-2h-(V_K+V_M)} \left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle_{c,h}$$

$$\langle K_{ij} M_{lm} \rangle = \frac{g}{N} \delta_{jl} \delta_{im}; \quad \langle K_{ij} K_{lm} \rangle = \langle M_{ij} M_{lm} \rangle = 0$$



$$\langle M_{ij} K_{kl} \rangle = \frac{g}{N} \delta_{il} \delta_{jk}$$



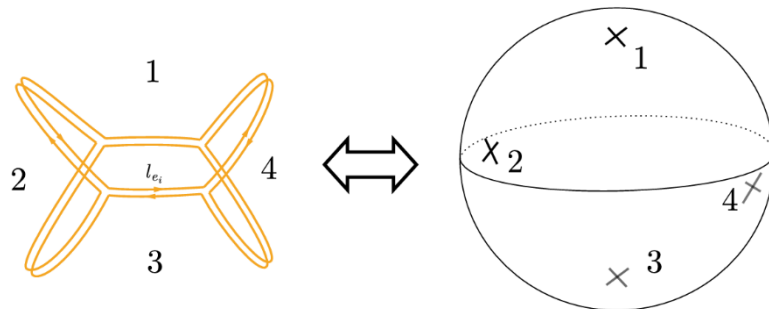
$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle_{c,h}$$

$$E = \sum_{i=1}^{V_K} l_i = \sum_{j=1}^{V_M} n_j \sum_c \left\langle \left(\frac{1}{2} \text{Tr}(K^2) \right)^2 \left(\frac{1}{2} \text{Tr}(M^2) \right)^2 \right\rangle_c$$

$$g^E N^{(V_K+V_M)-E+F-(V_K+V_M)} = g^E N^{2-2h-(V_K+V_M)}$$

$$\langle \text{Tr}(K^2) \text{Tr}(K^2) \text{Tr}(M^2) \text{Tr}(M^2) \rangle = 2 \langle \text{Tr}(K^2) \text{Tr}(M^2) \rangle^2 + \langle \text{Tr}(K^2) \text{Tr}(K^2) \text{Tr}(M^2) \text{Tr}(M^2) \rangle_c$$

$$\left\langle \frac{1}{2} \text{Tr}(K^2) \frac{1}{2} \text{Tr}(K^2) \frac{1}{2} \text{Tr}(M^2) \frac{1}{2} \text{Tr}(M^2) \right\rangle_c = g^4 \times N^{-2}$$



$$V - E + F = 2 - 2g$$

$$n = \# \text{Faces}$$

$$\text{edge lengths } l_{e_i}$$

$$\text{perimeters } L_k = \sum_{e_i \text{ around face } k} l_{e_i}$$

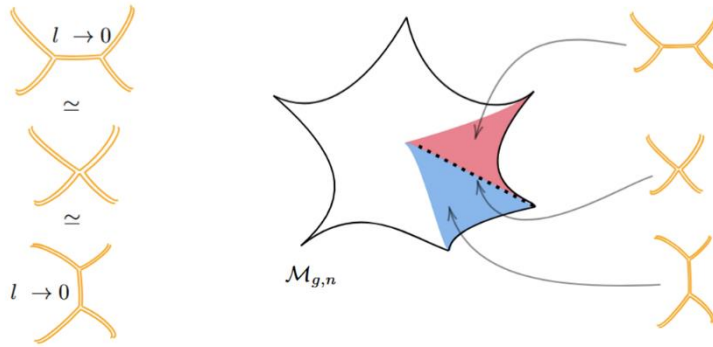
genus g Riemann surface

$n = \#$ marked points

coordinates on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$

coordinates on \mathbb{R}_+^n of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$





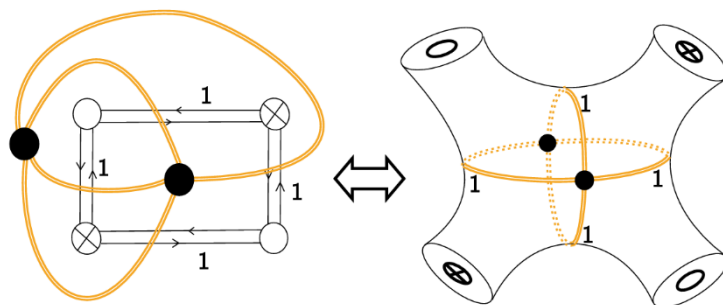
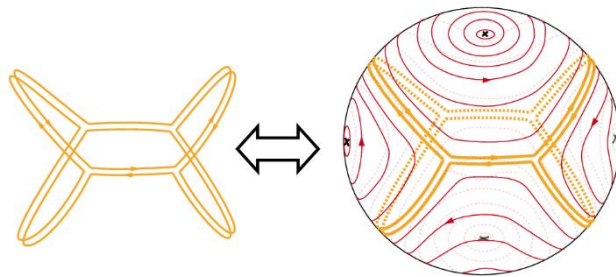
$$\phi_S = (\phi_S)_{zz} dz \otimes dz = \phi(z) dz^2.$$

$$\phi_S \approx -\frac{L_k^2}{(2\pi)^2} \frac{du_k^2}{u_k^2}$$

$$\phi(z_H(t)) \left(\frac{dz_H(t)}{dt} \right)^2 > 0.$$

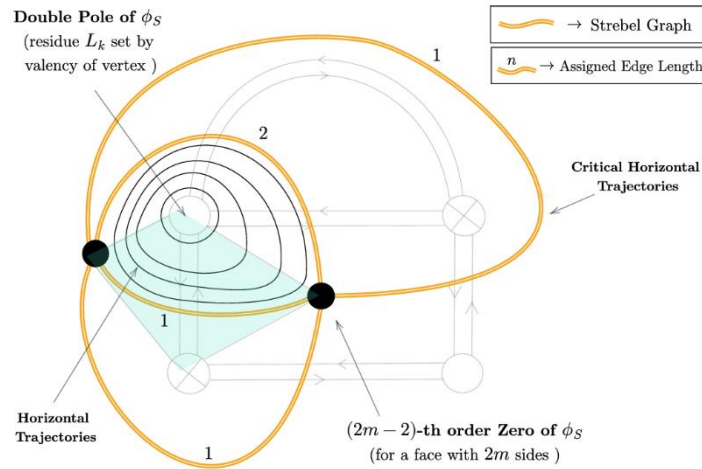
$$\oint_{C_k} d\tau = L_k$$

$$\phi_S \approx \frac{m^2}{4} w^{m-2} dw^2$$



$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle_{c,g}$$

$\mathcal{M}_{g, V_K + V_M} \times \mathbb{R}_+^{V_K + V_M}$, with coordinates $(l_1, \dots, l_{V_K}, n_1, \dots, n_{V_M})$ on the $\mathbb{R}_+^{V_K} + V_M$ fiber



$$ds^2 = |\phi(z)| dz d\bar{z}$$

$$\frac{1}{4\pi} \int \sqrt{g} R d^2 z = 2 - 2g$$

$$\zeta(z) = \int_{z_0}^z \sqrt{\phi(z')} dz'$$

$$ds^2|_{\square} = d\zeta d\bar{\zeta}$$

$$ds^2|_{\square} = \frac{(m+2)^2}{4} |\omega^m| d\omega d\bar{\omega}$$

$$\frac{1}{4\pi} \sqrt{g} R|_{\square} = -2 \frac{\partial}{\partial \omega} \frac{\partial}{\partial \bar{\omega}} \log(|\omega|^m) = -\frac{m}{2} \delta(\omega) \delta(\bar{\omega})$$

$$ds^2|_{\square} = \frac{L_k^2}{(2\pi)^2 |u_k|^2} du_k d\bar{u}_k$$

$$\frac{1}{4\pi} \sqrt{g} R|_{\square} = \delta(u_k) \delta(\bar{u}_k)$$

$$\frac{1}{4\pi} \int_{WS} \sqrt{g_S} R_S d^2 z = \sum_{m \geq 1} s_m \left(-\frac{m}{2}\right) + n = 2 - 2g$$

$$E = (1/2) \sum_{m \geq 1} (m+2) s_m$$

$$V - E + F = \sum_{m \geq 1} s_m - \sum_{m \geq 1} s_m \frac{(m+2)}{2} + n = 2 - 2g$$

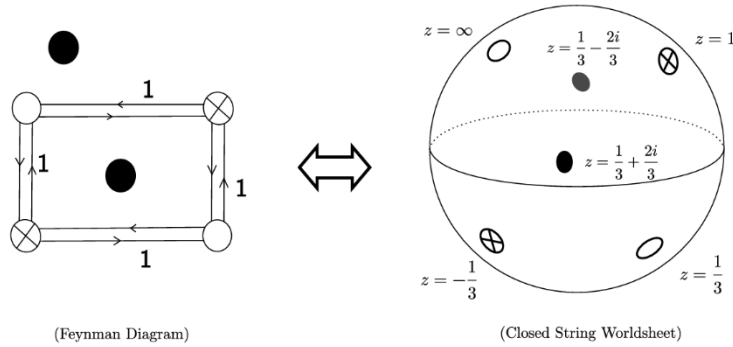
$$\phi(z) dz^2 = -\frac{1}{\pi^2} \frac{(z-z_a)^2 (z-z_b)^2}{(z-1)^2 (z^2-t^2)^2} dz^2$$



$$\frac{(1 - z_a)^2(1 - z_b)^2}{(1 - t)^2(1 + t)^2} = 1$$

$$\frac{(t - z_a)^2(t - z_b)^2}{4t^2(t - 1)^2} = 1$$

$$\therefore \frac{(t + z_a)^2(t + z_b)^2}{4t^2(t + 1)^2} = 1$$



$$ds_{WS}^2 = \left| \frac{(z - 1/3 - 2i/3)^2(z - 1/3 + 2i/3)^2}{\pi^2(z - 1)^2(z + 1/3)^2(z - 1/3)^2} \right| dzd\bar{z}$$

$$ds_{WS}^2 = |\phi_{WS}(z)| dzd\bar{z}$$

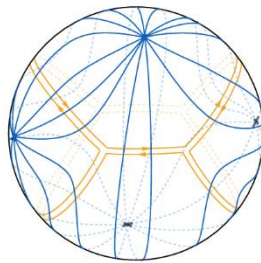
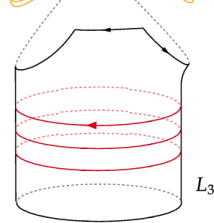
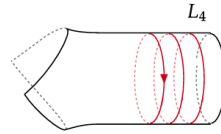
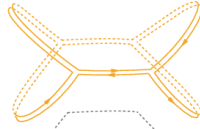
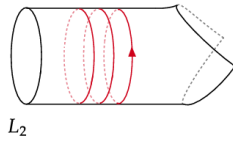
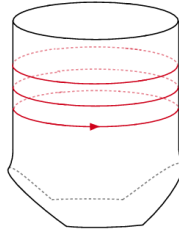
$$z_{a,b}^{(1)} = 1 \pm \sqrt{1 - t^2}$$

$$z_{a,b}^{(2)} = t \pm \sqrt{2t(t - 1)}$$

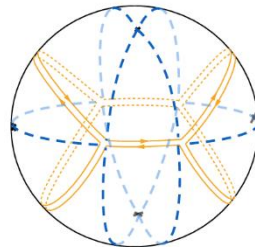
$$\int_{z_a}^{z_b} \sqrt{\phi(z)} dz = \pm \frac{i}{\pi} \int_{z_a}^{z_b} \frac{(z - z_a)(z - z_b)}{(z - 1)(z^2 - t^2)} dz = 1 + i0$$

$$\phi_{WS}(z) dz^2 = -\frac{1}{\pi^2} \frac{(z - 1/3 - 2i/3)^2(z - 1/3 + 2i/3)^2}{(z - 1)^2(z + 1/3)^2(z - 1/3)^2} dz^2$$

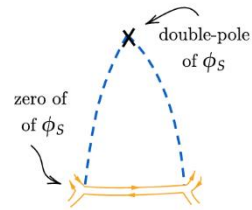
$$\oint_{\text{Hor.Traj.}} d\tau = L_1$$



Generic
Vertical Trajectories



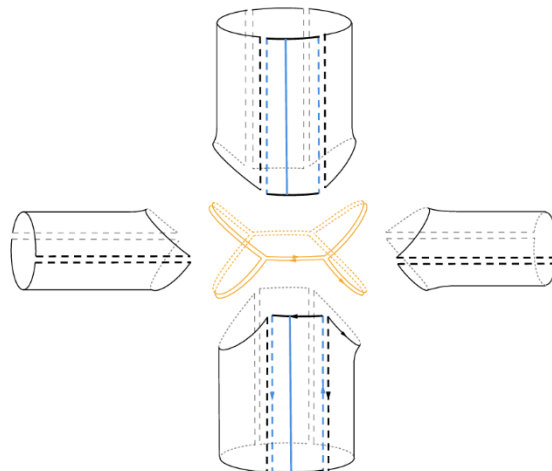
Special
Vertical Trajectories

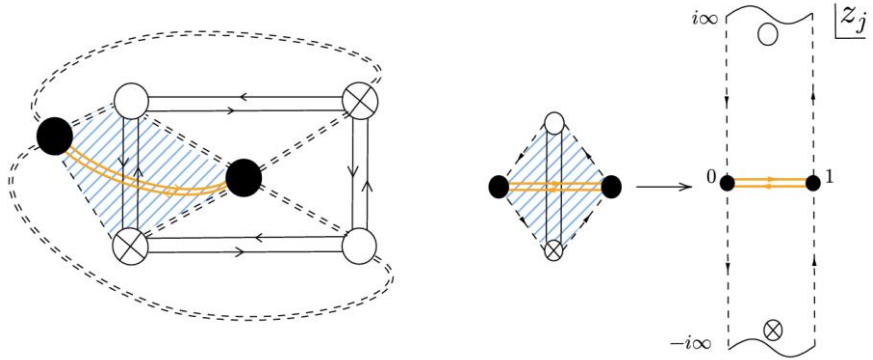
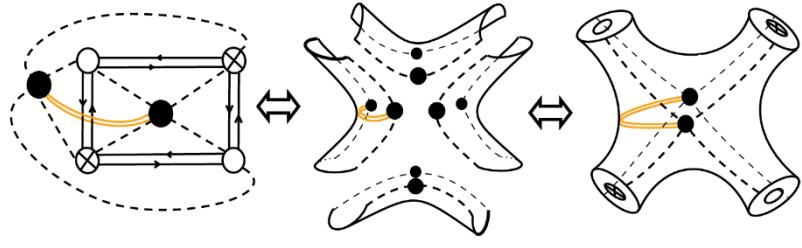
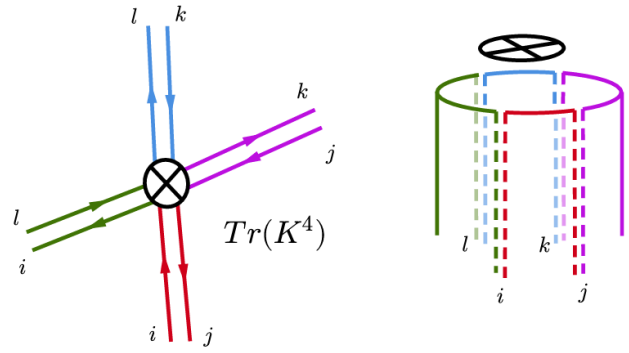


Each "Triangle" =
One Semi-infinite Strip

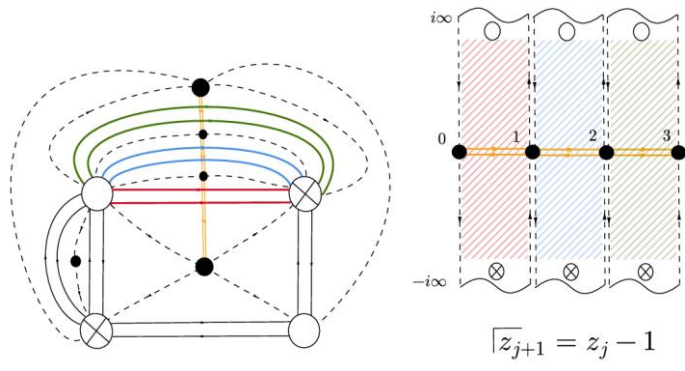
$$\phi(z_V(t)) \left(\frac{dz_V(t)}{dt} \right)^2 < 0$$

$$z_j \equiv \int_0^{z_j} \sqrt{\phi(s)} ds$$





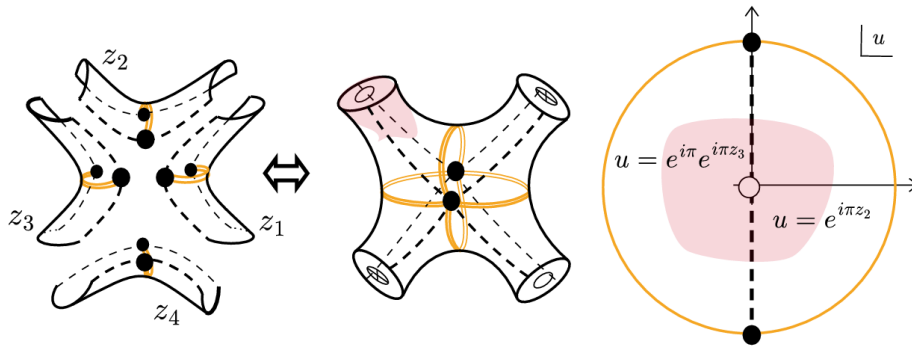
$$\phi(z_j) dz_j^2 = dz_j^2$$



$$u_k(z_j) = \exp\left(2\pi i \frac{j+z_j}{l}\right)$$

$$dz_j^2 = -\frac{l^2}{(2\pi)^2} \frac{du_k^2}{u_k^2}$$



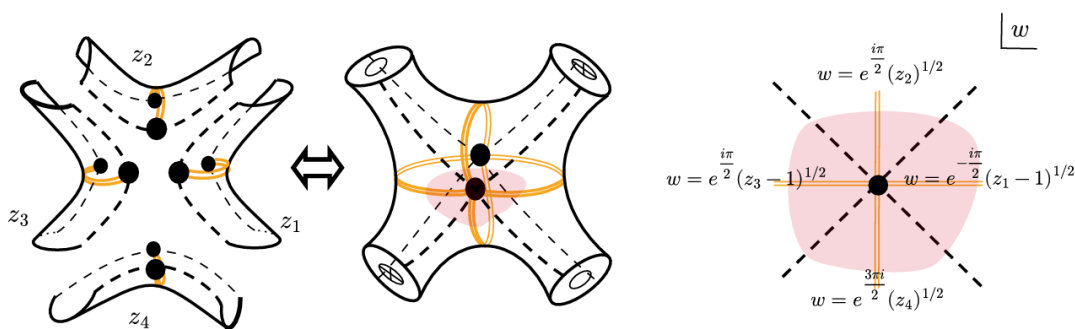


$$u_k(z_j) = \exp\left(2\pi i \frac{j+z_j}{l}\right)$$

$$v_k(z_j) = \exp\left(-2\pi i \frac{j+z_j}{l}\right),$$

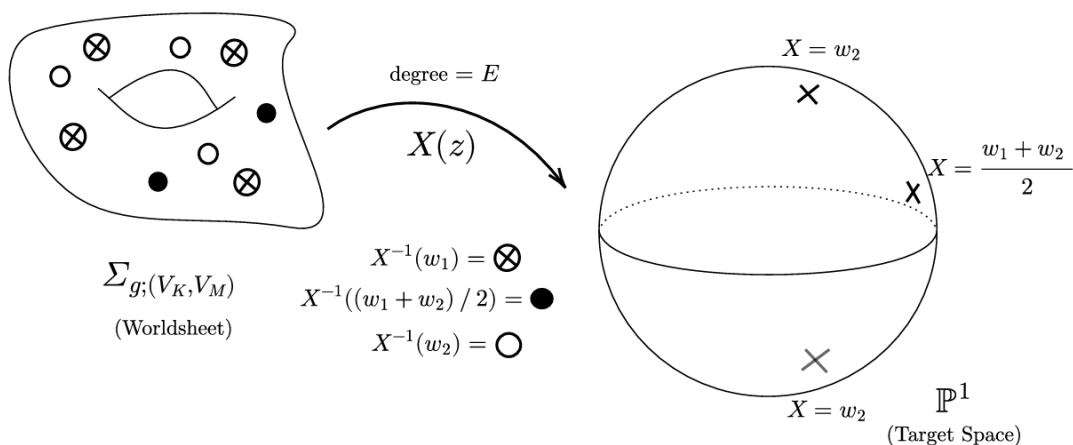
$$\omega(z_{2j}) = e^{\frac{(2j-1)i\pi}{m}} (z_{2j})^{1/m}$$

$$\omega(z_{2j-1}) = e^{\frac{(2j-3)i\pi}{m}} (z_{2j-1} - 1)^{1/m}$$

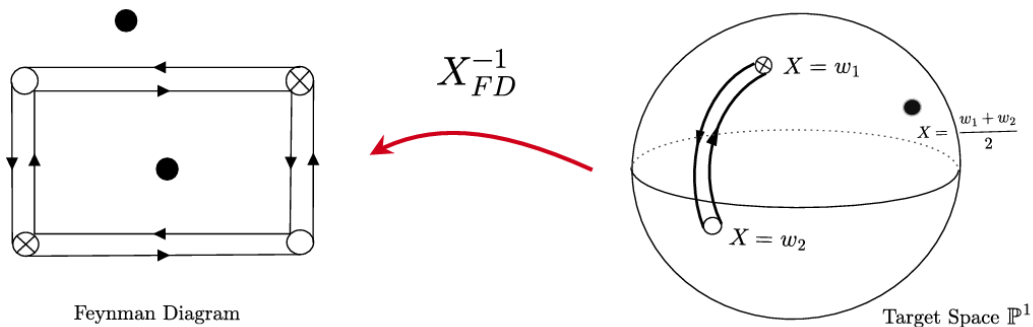
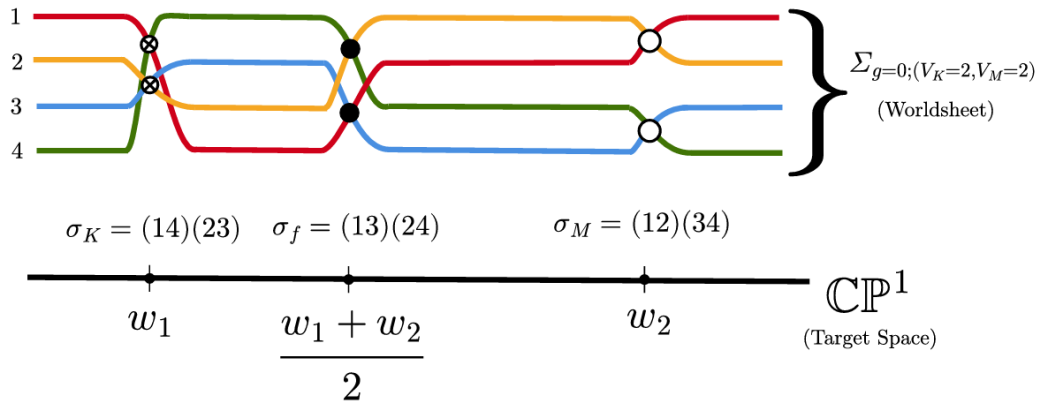


$$\omega(z_{2j}) = e^{\frac{(2j-1)i\pi}{m}} z_{2j}^{1/m} \text{ and } \omega(z_{2j-1}) = e^{\frac{(2j-3)i\pi}{m}} (z_{2j-1} - 1)^{1/m}$$

$$(dz_{2j})^2 = m^2 \omega^{2m-2} d\omega^2, \text{ and } (dz_{2j-1})^2 = m^2 \omega^{2m-2} d\omega^2,$$



$$E = \sum_{i=1}^{V_K} l_i = \sum_{j=1}^{V_M} n_j \otimes \left(\prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right)_{c,h}$$



$$X(z) = \frac{w_2 \zeta(z) + w_1}{\zeta(z) + 1}$$

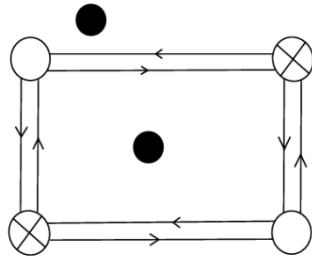
$$\zeta(z) = \alpha \frac{(z-a)^2 (z-b)^2}{(z-c)^2 (z-d)^2}$$

$$\zeta(z) - 1 = \beta \frac{(z-e)^2 (z-f)^2}{(z-c)^2 (z-d)^2}$$

$$(\alpha - 1)t^2 = \beta e^2 f^2, t(\alpha + \alpha t + 1) = \beta e f (e + f), \alpha = \beta, \\ \alpha + \alpha t(t + 4) = \beta(e^2 + 4ef + f^2) + 1, \alpha(t + 1) = \beta(e + f).$$

$$\zeta(z) = -\frac{9}{16} \frac{(z-1)^2 (z+1/3)^2}{(z-1/3)^2}$$

$$X(z)_{FD} = \frac{16w_2 \left(z - \frac{1}{3}\right)^2 - 9w_1 (z-1)^2 \left(z + \frac{1}{3}\right)^2}{16 \left(z - \frac{1}{3}\right)^2 - 9(z-1)^2 \left(z + \frac{1}{3}\right)^2}$$



(Feynman Diagram)

$$\Leftrightarrow X_{FD}(z) = \frac{16w_2(z-1/3)^2 - 9w_1(z-1)^2(z+1/3)}{16(z-1/3)^2 - 9(z-1)^2(z+1/3)^2}$$

(Embedding Map of Worksheet into Target Space)

$$\text{Tr}(M^2)\text{Tr}(K^2) = M_{a_1}^{a_2} M_{a_2}^{a_1} K_{b_1}^{b_2} K_{b_2}^{b_1} = M_{a_1}^{a_{\sigma_M(1)}} M_{a_2}^{a_{\sigma_M(2)}} K_{b_1}^{b_{\sigma_K(1)}} K_{b_2}^{b_{\sigma_K(2)}}$$

$$\left\langle \prod_{j=1}^{V_M} \text{Tr}(M^{n_j}) \prod_{i=1}^{V_K} \text{Tr}(K^{l_i}) \right\rangle_{c,h}$$

$$\sum_{j=1}^{V_M} n_j = \sum_{i=1}^{V_K} l_i = E$$

$$\prod_{j=1}^{V_M} \text{Tr}(M^{n_j}) = M_{i_{\sigma_M(1)}}^{i_1} \dots M_{i_{\sigma_M(E)}}^{i_E}$$

$$\begin{aligned} \left\langle \prod_{j=1}^{V_M} \text{Tr}(M^{n_j}) \prod_{i=1}^{V_K} \text{Tr}(K^{l_i}) \right\rangle &= \left\langle M_{i_{\sigma_M(1)}}^{i_1} \dots M_{i_{\sigma_M(E)}}^{i_E} K_{j_{\sigma_K(1)}}^{j_1} \dots K_{j_{\sigma_K(E)}}^{j_E} \right\rangle \\ &= \left\langle M_{k_1}^{i_1} \dots M_{k_E}^{i_E} K_{l_1}^{j_1} \dots K_{l_E}^{j_E} \right\rangle \delta_{i_{\sigma_M(1)}}^{k_1} \dots \delta_{i_{\sigma_M(E)}}^{k_E} \delta_{j_{\sigma_K(1)}}^{l_1} \dots \delta_{j_{\sigma_K(E)}}^{l_E} \\ &= \left(\frac{g}{N}\right)^E \left(\sum_{\gamma \in \mathcal{S}_E} \delta_{l_{\gamma(1)}}^{i_1} \dots \delta_{l_{\gamma(E)}}^{i_E} \delta_{k_1}^{j_{\gamma(1)}} \dots \delta_{k_E}^{j_{\gamma(E)}} \right) \delta_{i_{\sigma_M(1)}}^{k_1} \dots \delta_{i_{\sigma_M(E)}}^{k_E} \delta_{j_{\sigma_K(1)}}^{l_1} \dots \delta_{j_{\sigma_K(E)}}^{l_E} \\ &= \left(\frac{g}{N}\right)^E \sum_{\gamma \in \mathcal{S}_E} \delta_{l_{\gamma(1)}}^{i_1} \dots \delta_{l_{\gamma(E)}}^{i_E} \delta_{i_{\sigma_M(1)}}^{j_{\gamma(1)}} \dots \delta_{i_{\sigma_M(E)}}^{j_{\gamma(E)}} \delta_{j_{\sigma_K(1)}}^{l_1} \dots \delta_{j_{\sigma_K(E)}}^{l_E} \end{aligned}$$



$$\begin{aligned}
\left\langle \prod_{i=1}^{V_K} \text{Tr}(K^{n_i}) \prod_{j=1}^{V_M} \text{Tr}(M^{l_j}) \right\rangle &= \left(\frac{g}{N}\right)^E \sum_{I,J,K,L} \sum_{\gamma \in S_E} \delta_{l_{\gamma(1)}}^{i_1} \dots \delta_{l_{\gamma(E)}}^{i_E} \delta_{k_1}^{j_{\gamma(1)}} \dots \delta_{k_E}^{j_{\gamma(E)}} \delta_{i_{\sigma_M(1)}}^{k_1} \dots \delta_{i_{\sigma_M(E)}}^{k_E} \delta_{j_{\sigma_K(1)}}^{l_1} \dots \delta_{j_{\sigma_K(E)}}^{l_E} \\
&= \left(\frac{g}{N}\right)^E \sum_{I,J,L} \sum_{\gamma \in S_E} \delta_{l_{\gamma(1)}}^{i_1} \dots \delta_{l_{\gamma(E)}}^{i_E} \delta_{i_{\sigma_M(1)}}^{j_{\gamma(1)}} \dots \delta_{i_{\sigma_M(E)}}^{j_{\gamma(E)}} \delta_{j_{\sigma_K(1)}}^{l_1} \dots \delta_{j_{\sigma_K(E)}}^{l_E} \\
&= \left(\frac{g}{N}\right)^E \sum_{I,J,L} \sum_{\gamma \in S_E} \delta_{l_{\gamma(1)}}^{i_1} \dots \delta_{l_{\gamma(E)}}^{i_E} \delta_{j_{\sigma_K \gamma(1)}}^{l_{\gamma(1)}} \dots \delta_{j_{\sigma_K \gamma(E)}}^{l_{\gamma(E)}} \delta_{i_{\sigma_M(1)}}^{j_{\gamma(1)}} \dots \delta_{i_{\sigma_M(E)}}^{j_{\gamma(E)}} \\
&= \left(\frac{g}{N}\right)^E \sum_{I,J} \sum_{\gamma \in S_E} \delta_{j_{\sigma_K \gamma(1)}}^{i_1} \dots \delta_{j_{\sigma_K \gamma(E)}}^{i_E} \delta_{i_{\sigma_M(1)}}^{j_{\gamma(1)}} \dots \delta_{i_{\sigma_M(E)}}^{j_{\gamma(E)}} \\
&= \left(\frac{g}{N}\right)^E \sum_{I,J} \sum_{\gamma \in S_E} \delta_{j_{\sigma_K \gamma(1)}}^{i_1} \dots \delta_{j_{\sigma_K \gamma(E)}}^{i_E} \delta_{i_{\sigma_M(\gamma^{-1} \sigma_K \gamma)(1)}}^{j_{\sigma_K \gamma(1)}} \dots \delta_{i_{\sigma_M(\gamma^{-1} \sigma_K \gamma)(E)}}^{j_{\sigma_K \gamma(E)}} \\
&= \left(\frac{g}{N}\right)^E \sum_I \sum_{\gamma \in S_E} \delta_{i_{\sigma_M \gamma^{-1}}}^{i_1} \dots \delta_{i_{\sigma_M \gamma(1)}}^{i_E} \\
&= \left(\frac{g}{N}\right)^E \sum_{\gamma \in S_E} \text{tr}_E(\sigma_M \gamma^{-1} \sigma_K \gamma)
\end{aligned}$$

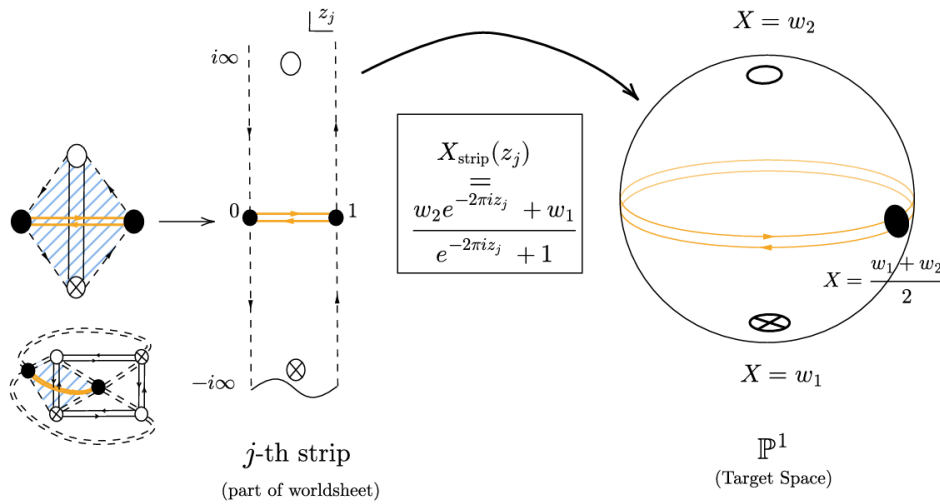
$$\begin{aligned}
\left\langle \prod_{i=1}^{V_K} \text{Tr}(K^{n_i}) \prod_{j=1}^{V_M} \text{Tr}(M^{l_j}) \right\rangle &= \frac{g^E N^{-E}}{|\lceil \sigma_K \rceil|} \sum_{\sigma_K \in \lceil \sigma_K \rceil} \sum_{\gamma \in S_E} \text{tr}_E(\sigma_M \gamma^{-1} \sigma_K \gamma) = \frac{g^E E!}{|\lceil \sigma_K \rceil|} \sum_{\sigma_K \in \lceil \sigma_K \rceil} \text{tr}_E(\sigma_K \sigma_M) \\
&= \frac{g^E N^{-E} E!}{|\lceil \sigma_K \rceil|} \sum_{\sigma_K \in \lceil \sigma_K \rceil} N^{C_{\sigma_K} \sigma_M}
\end{aligned}$$

$$\left\langle \prod_{j=1}^{V_M} \text{Tr}(M^{l_j}) \prod_{i=1}^{V_K} \text{Tr}(K^{k_i}) \right\rangle = \frac{g^E E!}{|\lceil \sigma_K \rceil|} \sum_{\sigma_K \in \lceil \sigma_K \rceil, \sigma_f \in S_E} N^{C_{\sigma_f} \sigma_K} \delta(\sigma_M \sigma_f \sigma_K)$$

$$X_s(i\infty) = w_2 X_s(-i\infty) = w_1 X_s(0) = X_s(1) = \frac{w_1 + w_2}{2}$$

$$X_s(z_j) = \frac{w_2 e^{-2\pi i z_j} + w_1}{e^{-2\pi i z_j} + 1}$$





$$u_k(z_j) = e^{2\pi i \frac{j}{l}} e^{2\pi i \frac{z_j}{l}}, j = 0, \dots, l-1$$

$$X_s(u_k) = \frac{w_2 u_k^{-l} + w_1}{u_k^{-l} + 1}$$

$$X_s = w_2 + (w_1 - w_2) u_k^l + \mathcal{O}(u_k^{2l})$$

$$v_k(z_j) = e^{-2\pi i \frac{j}{l}} e^{-2\pi i \frac{z_j}{l}}$$

$$X_s(v_k) = \frac{w_2 v_k^l + w_1}{v_k^l + 1}$$

$$X_s(v_k) = w_1 + (w_2 - w_1) v_k^l + \mathcal{O}(v_k^{2l})$$

$$X(\omega) = X_s(z_{2j}(\omega)) = \frac{w_2 \exp(-2\pi i z_{2j}) + w_1}{\exp(-2\pi i z_{2j}) + 1} = \frac{w_2 \exp(-2\pi i (e^{(2j-1)i\pi} \omega^m)) + w_1}{\exp(-2\pi i (e^{(2j-1)i\pi} \omega^m)) + 1} \\ \sim \frac{w_1 + w_2}{2} - (w_1 - w_2) \pi i \omega^m$$

$$X(\omega) = X_s(z_{2j-1}(\omega)) = \frac{w_2 \exp(-2\pi i z_{2j-1}) + w_1}{\exp(-2\pi i z_{2j-1}) + 1} = \frac{w_2 \exp(-2\pi i (1 + e^{(2j-3)i\pi} \omega^m)) + w_1}{\exp(-2\pi i (1 + e^{(2j-3)i\pi} \omega^m)) + 1} \\ \sim \frac{w_1 + w_2}{2} - (w_1 - w_2) \pi i \omega^m$$

$$X_s(z) = \frac{w_2 e^{-2\pi i z} + w_1}{e^{-2\pi i z} + 1}$$

$$\frac{dX_s(z)}{dz} = -2\pi i \frac{(X_s - w_1)(X_s - w_2)}{w_1 - w_2} \Rightarrow \phi_s(z) dz^2 \Big|_{\text{strip}} = dz^2 = -\frac{1}{4\pi^2} \frac{(w_1 - w_2)^2}{(X_s - w_1)^2 (X_s - w_2)^2} dX_s^2$$

$$\phi_s(z) dz^2 = -\frac{1}{4\pi^2} \frac{(w_1 - w_2)^2}{(X - w_1)^2 (X - w_2)^2} dX^2$$



$$|\phi_S(z)| dz \wedge d\bar{z} = \frac{1}{4\pi^2} \frac{(|w_1 - w_2|^2)}{|X - w_1|^2 |X - w_2|^2} dX \wedge d\bar{X} = X^*(\Omega_{\text{Target}})$$

$$\mathcal{K}_{\text{Target}}(X, \bar{X}) = \frac{1}{4\pi^2} \log \left(\frac{X - w_1}{X - w_2} \right) \log \left(\frac{\bar{X} - \bar{w}_1}{\bar{X} - \bar{w}_2} \right)$$

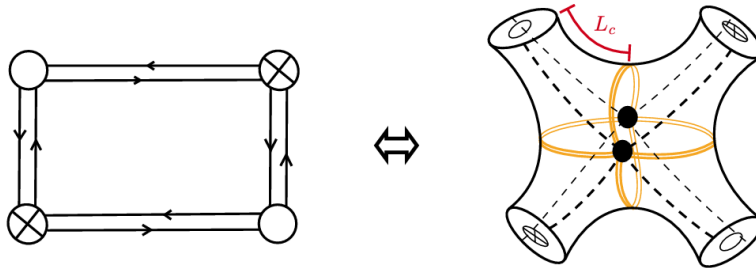
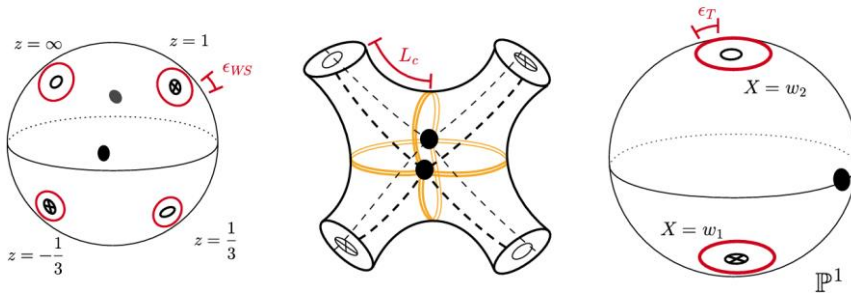
$$S_{\text{A-model}}[X] = 2\pi \int_{WS} X^*(\Omega_{\text{Target}}) = 2\pi \int_{WS} |\phi_S(z)| dz \wedge d\bar{z} \int \left(\frac{Y_1 \cdot Y_2}{|w_1 - w_2|^2} \right)^E$$

$$A_{WS} = \int_{WS} |\phi_S(z)| dz \wedge d\bar{z} = e^{-2\pi \frac{L_c}{m}}$$

$$|X(iL_c) - w_2| \equiv \epsilon_T \quad |X(-iL_c) - w_1| \equiv \epsilon_T$$

$$L_c = \frac{1}{4\pi} \log \left(\frac{|w_2 - w_1 - \epsilon_T|^2}{\epsilon_T^2} \right) = \frac{1}{4\pi} \log \left(\frac{|w_2 - w_1|^2}{\epsilon_T^2} \right) + \mathcal{O}(\epsilon_T)$$

$$e^{-2\pi S_{NG}[X]} = e^{-2\pi A_{WS}} = e^{-2\pi A_{WS}} = e^{-4\pi E L_c} = \left(\frac{\epsilon_T^2}{|w_2 - w_1|^2} \right)^{e^{-2\pi S[X]} \int e^{-2\pi A_{WS}}} (1 + \mathcal{O}(\epsilon_T))$$



$$G_\epsilon(w_1, w_2)^4 = \left(\frac{\epsilon_T^2}{|w_2 - w_1|^2} \right)^4 \quad e^{-2\pi \int_{WS} X^*(\Omega_T)} = e^{-2\pi A_{WS}} = e^{-2\pi \times 4 \times 2 L_c} = \left(\frac{\epsilon_T^2}{|w_2 - w_1|^2} \right)^4$$

$$(V_K + V_M) - \frac{1}{2} \left(\sum_{i=1}^{V_K} l_i + \sum_{j=1}^{V_M} n_j \right) + F = 2 - 2g$$

$$E = \sum_{i=1}^{V_K} l_i + \sum_{j=1}^{V_M} n_j$$

$$2g \leq \frac{1}{2} \left(\sum_{i=1}^{V_K} l_i + \sum_{j=1}^{V_M} n_j \right) - (V_K + V_M) + 1$$

$$\sigma_K = (l_1) \dots (l_{V_K}) \sigma_f = (m_1) \dots (m_F) \sigma_M = (n_1) \dots (n_{V_M})$$

$$d = \sum_{i=1}^{V_K} l_i = \sum_{r=1}^F m_r = \sum_{j=1}^{V_M} n_j = E$$

$$2 - 2g_{WS} = d(2 - 2g_{\mathbb{C}\mathbb{P}^1}) - \sum_{p \in WS} (e_p - 1)$$

$$\begin{aligned} 2 - 2g_{WS} &= 2d - \left(\sum_{i=1}^{V_K} (l_i - 1) + \sum_{r=1}^F (m_r - 1) + \sum_{j=1}^{V_M} (n_j - 1) \right) \\ &= 2d - ((d - V_K) + (d - F) + (d - V_M)) \\ &= -E + (V_K + V_M) + F \end{aligned}$$

$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}((Y_1 \cdot \Phi(w_1))^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}((Y_2 \cdot \Phi(w_2))^{n_j}) \right\rangle_{\mathcal{N}=4}$$

$$X_{Q \times Q} = \text{diag}\{x_1, \dots, x_Q\} \text{ and } V_{R \times R} = \text{diag}\{v_1, \dots, v_R\}$$

$$\det(\dots) = e^{\text{Tr} \log(\dots)}$$

$$\frac{\left\langle \prod_{\mu=1}^R \det(v_\mu - Y_1 \cdot \Phi(w_1)) \prod_{a=1}^Q \det(x_a - Y_2 \cdot \Phi(w_2)) \right\rangle}{\det_{Q \times Q}(X)^N \det_{R \times R}(V)^N} = \left\langle e^{-\sum_{i=1}^{\infty} \frac{\tilde{t}_i}{l_i} \text{Tr}((Y_1 \cdot \Phi(w_1))^{l_i})} e^{-\sum_{j=1}^{\infty} \frac{t_j}{n_j} \text{Tr}((Y_2 \cdot \Phi(w_2))^{n_j})} \right\rangle$$

$$t_k = \text{Tr}_Q(X^{-k}) \quad \tilde{t}_k = \text{Tr}_R(V^{-k})$$

$$\frac{1}{Z_{\mathcal{N}=4}} \int \mathcal{D}p e^{-\frac{N}{\lambda} S_{\mathcal{N}=4}} \prod_{a=1}^Q \det(x_a - Y_2 \cdot \Phi(w_2)) \prod_{\mu=1}^R \det(v_\mu - Y_1 \cdot \Phi(w_1))$$

$$\frac{1}{Z_{\text{scalar}}} \int D\Phi^I e^{+\frac{N}{2\lambda} \int d^4w \text{Tr}(\Phi^I \square \Phi^I)} \prod_{a=1}^Q \det(x_a - Y_2 \cdot \Phi(w_2)) \prod_{\mu=1}^R \det(v_\mu - Y_1 \cdot \Phi(w_1))$$

$$P^{IJ} = \delta^{IJ} - \frac{1}{Y_1 \cdot Y_2} (Y_1^I Y_2^J + Y_2^I Y_1^J)$$

$$\frac{1}{2} \int d^4w \text{Tr}(\Phi^I \square \Phi^I) = \frac{1}{2} \int d^4w \text{Tr}((P\Phi)^I \square (P\Phi)^I) + \frac{1}{Y_1 \cdot Y_2} \int d^4w \text{Tr}(Y_1 \cdot \Phi \square Y_2 \cdot \Phi)$$



$$\prod_{I=1}^6 D\Phi^I = D(P\Phi)^I \times D(Y_1 \cdot \Phi)D(Y_2 \cdot \Phi)$$

$$\frac{1}{Z_{Y \cdot \Phi}} \int D(Y_1 \cdot \Phi)D(Y_2 \cdot \Phi) e^{\frac{N}{\lambda Y_1 \cdot Y_2}} \int d^4w \text{Tr}(Y_1 \cdot \Phi \square Y_2 \cdot \Phi) \prod_{a=1}^Q \det(x_a - Y_2 \cdot \Phi(w_2)) \prod_{\mu=1}^R \det(v_\mu - Y_1 \cdot \Phi(w_1))$$

$$Y_1 \cdot \Phi(w) \equiv \frac{|w_1 - w_2|^2}{|w - w_2|^2} K + F_1(w)$$

$$Y_2 \cdot \Phi(w) \equiv \frac{|w_1 - w_2|^2}{|w - w_1|^2} M + F_2(w)$$

$$\square \frac{1}{|w|^2} = -4\pi^2 \delta(w)$$

$$\int d^4w \text{Tr}(Y_1 \cdot \Phi(w) \square Y_2 \cdot \Phi(w)) = -4\pi^2 |w_1 - w_2|^2 \text{Tr}(KM) + \int d^4w \text{Tr}(F_1(w) \square F_2(w))$$

$$D(Y_1 \cdot \Phi) = \prod_{w \in \mathbb{R}^4} d(Y_1 \cdot \Phi(w)) = dK \prod_{w \neq w_1} dF_1(w)$$

$$D(Y_2 \cdot \Phi) = \prod_{w \in \mathbb{R}^4} d(Y_2 \cdot \Phi(w)) = dM \prod_{w \neq w_2} dF_2(w)$$

$$\left\langle \prod_{a=1}^Q \det(x_a - Y_2 \cdot \Phi(w_2)) \prod_{\mu=1}^R \det(v_\mu - Y_1 \cdot \Phi(w_1)) \right\rangle_{\mathcal{N}=4}$$

$$= \frac{1}{Z_{K,M}} \int dK dM_{N \times N} e^{-\frac{N4\pi^2 |w_1 - w_2|^2}{\lambda Y_1 \cdot Y_2} \text{Tr}(KM)}$$

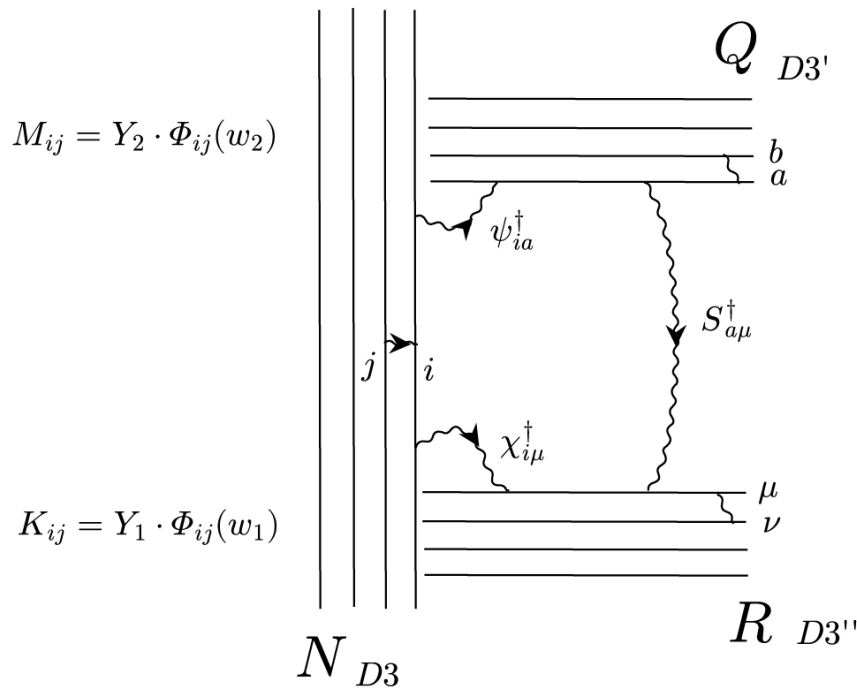
$$\prod_{a=1}^Q \det(x_a - M) \prod_{\mu=1}^R \det(v_\mu - K)$$

$$g \leftrightarrow \frac{\lambda Y_1 \cdot Y_2}{4\pi^2 |w_1 - w_2|^2}$$

$$X(z) = \frac{\mu(z)}{\lambda(z)}$$

$$Z[t_k, \bar{t}_k, s_k] \equiv \left(\frac{\det_N(NY)}{\det_Q(X) \det_R(V)} \right)^N \frac{1}{(2\pi g)^{N^2}} \int dK dM_{N \times N} e^{-\frac{N}{g} \text{Tr}_N(\sqrt{Y}K\sqrt{Y}M)} \\ \det(X \otimes \mathbb{I}_N - \mathbb{I}_Q \otimes M) \det(V \otimes \mathbb{I}_N - \mathbb{I}_R \otimes K)$$





$$\mathcal{D}\rho = \frac{\det_N(NY)^N}{(2\pi g)^{N^2}} \int dK dM_{N \times N} e^{-\frac{N}{g} \text{Tr}_N(\sqrt{Y}K\sqrt{Y}M) - \sum_{k \geq 1} \frac{t_k}{k} \text{Tr}(K^k) - \sum_{k \geq 1} \frac{\bar{t}_k}{k} \text{Tr}(M^k)}$$

$Y_{N \times N} = \text{diag}(y_1, \dots, y_N)$, $X_{Q \times Q} = \text{diag}(x_1, \dots, x_Q)$ and $V_{R \times R} = \text{diag}(v_1, \dots, v_R)$

$$s_k = \text{Tr}_N(Y^{-k}) t_k = \text{Tr}_Q(X^{-k}) \bar{t}_k = \text{Tr}_R(V^{-k})$$

$$\left\langle \prod_{i=1}^{V_K} \frac{1}{l_i} \text{Tr}(K^{l_i}) \prod_{j=1}^{V_M} \frac{1}{n_j} \text{Tr}(M^{n_j}) \right\rangle_c = \prod_{i=1}^{V_K} \frac{\partial}{\partial t_{l_i}} \prod_{j=1}^{V_M} \frac{\partial}{\partial \bar{t}_{n_j}} \log(Z[t_k, \bar{t}_k, s_k]) \Big|_{t_k = \bar{t}_k = 0}$$

$$\langle K_{ij} M_{kl} \rangle = \frac{g}{N} \sqrt{y_i y_j} \delta_{il} \delta_{jk}$$

$$\sum_{i=1}^N \left(\frac{1}{\sqrt{y_i}} \right)^{2k} = s_k$$

$$Z = \frac{\mathcal{K}}{Z_N} \int dK dM d\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(-\frac{N}{g} \text{Tr}_N(K\sqrt{Y}M\sqrt{Y}) + \psi_{ia}^\dagger (X_{ab} \delta_{ij} - \delta_{ab} M_{ij}) \psi_{jb} + \chi_{i\mu}^\dagger (V_{\mu\nu} \delta_{ij} - K_{ij} \delta_{\mu\nu}) \chi_{j\nu} \right)$$

$$\mathcal{K} = \left(\frac{\det_N(Y)}{\det_Q(X) \det_R(V)} \right)^N \text{ and } Z_N = (2\pi g/N)^{N^2}$$

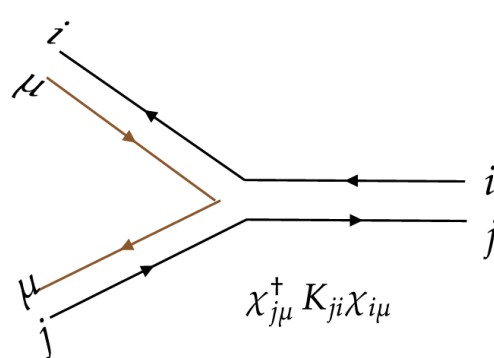
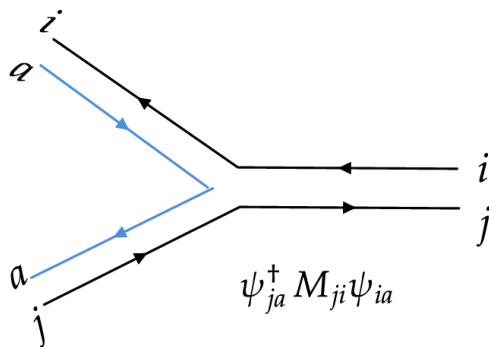
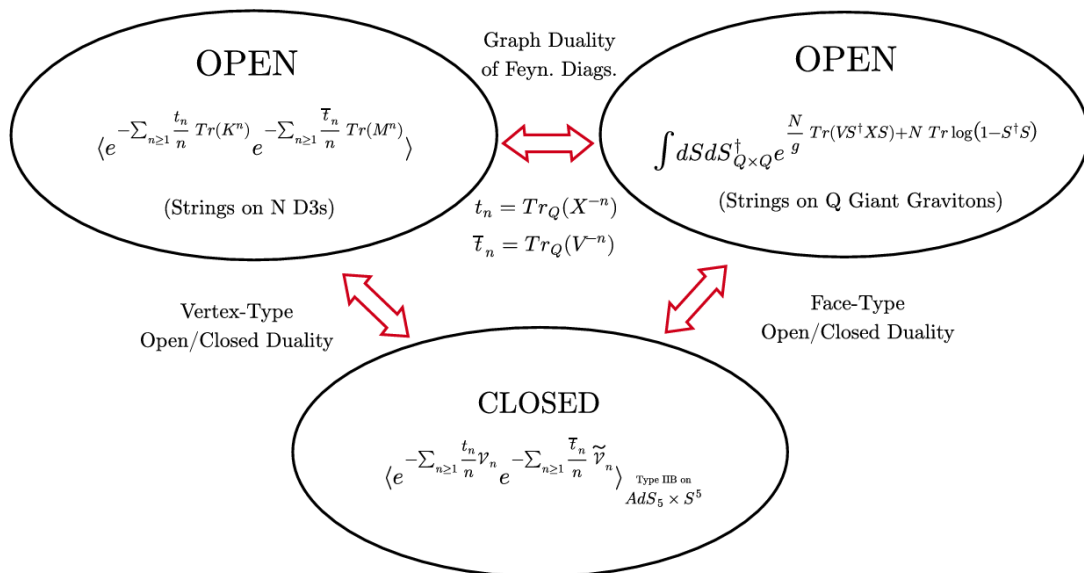
$$Z = \left(\frac{1}{\det_Q(X) \det_R(V)} \right)^N \int d\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} \psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu} \chi_{i\nu} - \frac{g}{N} \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{k\mu} \chi_{l\mu}^\dagger \left(Y^{-\frac{1}{2}} \right)_{lj} \psi_{ja} \right)$$

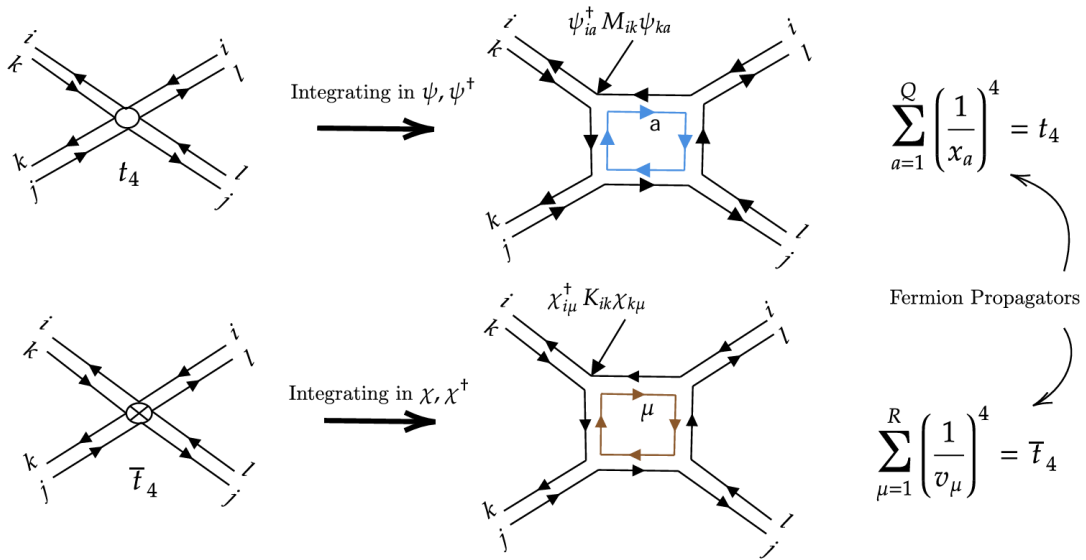


$$Z[t_k, \bar{t}_k, s_k] = \frac{\mathcal{K}'}{Z_{QR}} \int d\psi d\psi^\dagger d\chi d\chi^\dagger dS dS^\dagger \exp \left(\frac{N}{g} S_{\mu a}^\dagger S_{a\mu} - S_{\mu a}^\dagger \chi_{l\mu}^\dagger \left(Y^{-\frac{1}{2}} \right)_{lj} \psi_{ja} - \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{k\mu} \Phi_{a\mu} \right. \\ \left. + \psi_{ia}^\dagger X_{ab} \psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu} \chi_{i\nu} \right)$$

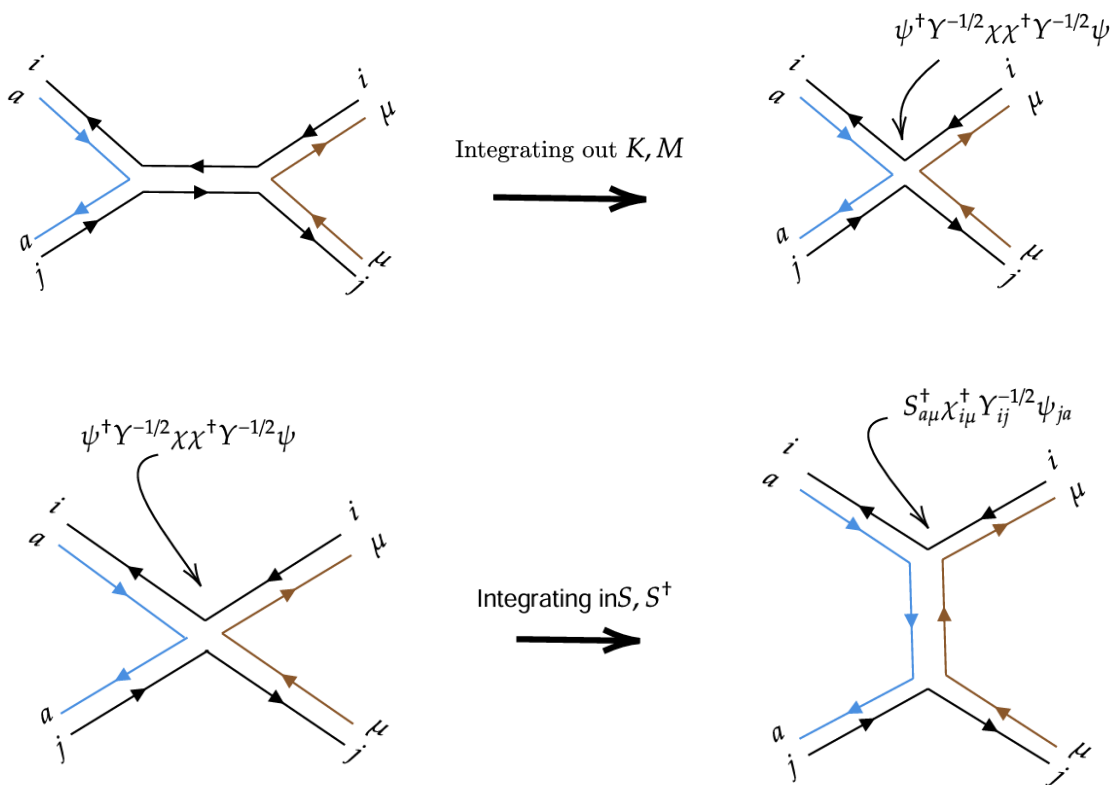
$$Z_{QR} = (2\pi g/N)^{QR}$$

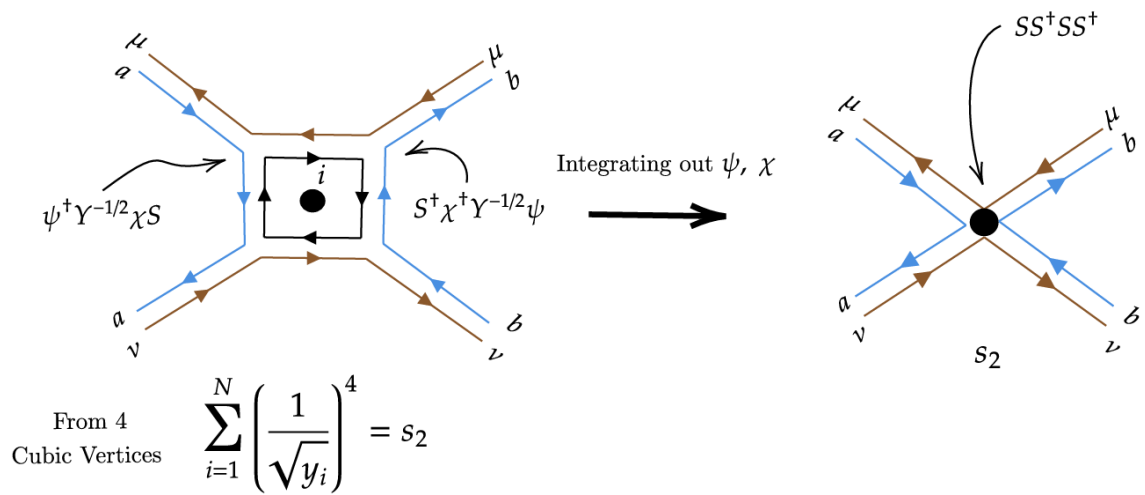
$$Z[t_k, \bar{t}_k, s_k] = \frac{(\det_Q(X))^R (\det_R(V))^Q}{\det_N(Y)^N Z_{QR}} \int dS dS_{Q \times R}^\dagger \exp \left(\frac{N}{g} \text{Tr}(VS^\dagger XS) \prod_{i=1}^N \det(y_i - SS^\dagger) \right) \\ = \frac{(\det_Q(X))^R (\det_R(V))^Q}{Z_{QR}} \int dS dS_{Q \times R}^\dagger \exp \left(\frac{N}{g} \text{Tr}(VS^\dagger XS) + \sum_{k \geq 1} \frac{s_k}{k} \text{Tr}((SS^\dagger)^k) \right) \\ \left\langle e^{-\sum_{n \geq 1} \frac{t_n}{n} \text{Tr}(K^n)} e^{-\sum_{n \geq 1} \frac{\bar{t}_n}{n} \text{Tr}(M^n)} \right\rangle \\ = \frac{(\det_Q(X))^R (\det_R(V))^Q}{Z_{QR}} \int dS dS_{Q \times R}^\dagger \exp \left(\frac{N}{g} \text{Tr}(VS^\dagger XS) + N \text{Tr} \log(1 - SS^\dagger) \right)$$



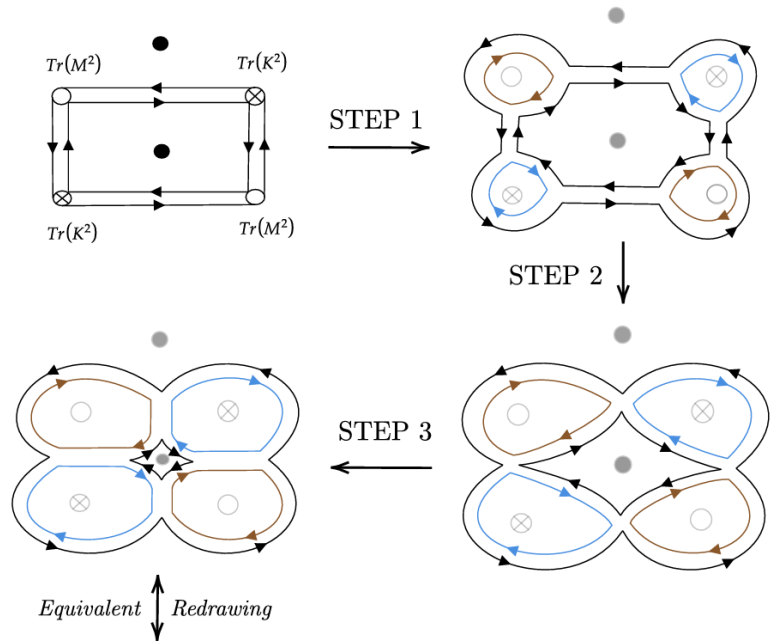


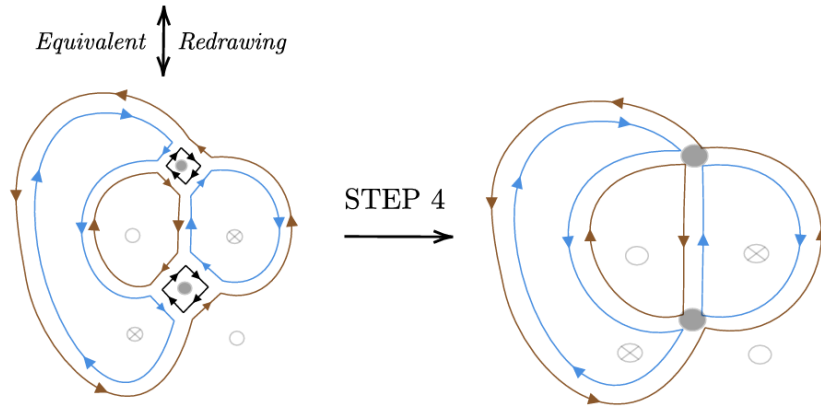
$$\sum_{a=1}^Q \left(\frac{1}{x_a}\right)^k = t_k \sum_{\mu=1}^R \left(\frac{1}{v_\mu}\right)^k = \bar{t}_k$$





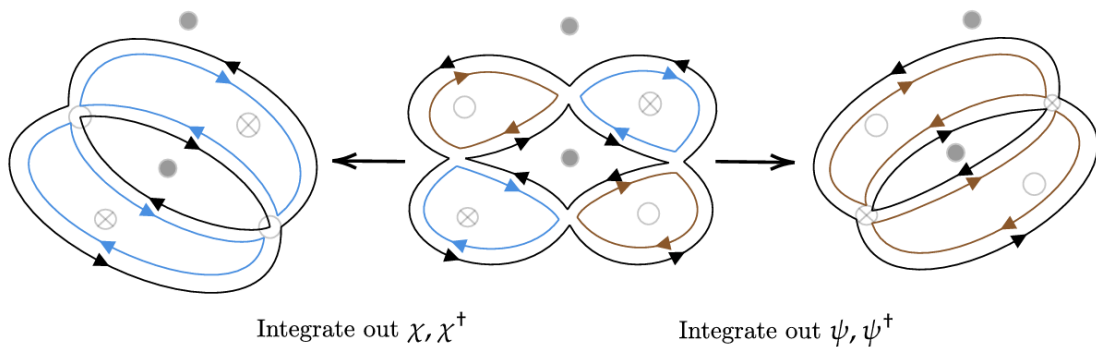
$$\left\langle \left(\frac{1}{2} \text{Tr} K^2\right)^2 \left(\frac{1}{2} \text{Tr} M^2\right)^2 \right\rangle_c$$





$$Z[t_k, \bar{t}_k, s_k] = \frac{\det_N(Y)^Q}{\det_Q(X)^N} \int d\psi d\psi_{N \times Q}^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} Y_{ij} \psi_{jb} - \sum_{k=1}^{\infty} \left(\frac{g}{N}\right)^k \frac{\bar{t}_k}{k} \text{Tr}_N(\psi \psi^\dagger)^k \right) \bigotimes \langle \psi_{ia}^\dagger \psi_{ja} | \chi_{i\mu}^\dagger \chi_{j\mu} \rangle$$

$$Z[t_k, \bar{t}_k, s_k] = \frac{\det_N(Y)^R}{\det_R(V)^N} \int d\chi d\chi_{N \times R}^\dagger \exp \left(\chi_{i\mu}^\dagger V_{\mu\nu} Y_{ij} \chi_{j\nu} - \sum_{k=1}^{\infty} \left(\frac{g}{N}\right)^k \frac{t_k}{k} \text{Tr}_N(\chi \chi^\dagger)^k \right) \bigotimes \langle \psi_{ia}^\dagger \psi_{ja} | \chi_{i\mu}^\dagger \chi_{j\mu} \rangle$$



$$M_{ij} \sim \psi_{ia}^\dagger \psi_{ja} \text{ and } K_{ij} \sim \chi_{i\mu}^\dagger \chi_{j\mu}$$

$$\langle \psi_{ia}^\dagger \psi_{ib} | \chi_{i\mu}^\dagger \chi_{iv} \rangle = \int dB_{Q \times Q} \prod_{a,b=1}^Q \delta(A_{ba} + g\psi_{ia}^\dagger \psi_{ib}) = \frac{1}{Z_Q} \int dAdB_{Q \times Q} e^{-\frac{N}{g} B_{ab} (A_{ba} + g\psi_{ia}^\dagger \psi_{ib})}$$

$$\langle \psi_{ia}^\dagger \psi_{ib} | \chi_{i\mu}^\dagger \chi_{iv} \rangle = \int dD_{R \times R} \prod_{\mu,\nu=1}^R \delta(D_{\nu\mu} + g\chi_{i\mu}^\dagger \chi_{iv}) = \frac{1}{Z_R} \int dCdD_{R \times R} e^{-\frac{N}{g} C_{\mu\nu} (D_{\nu\mu} + g\chi_{i\mu}^\dagger \chi_{iv})}$$

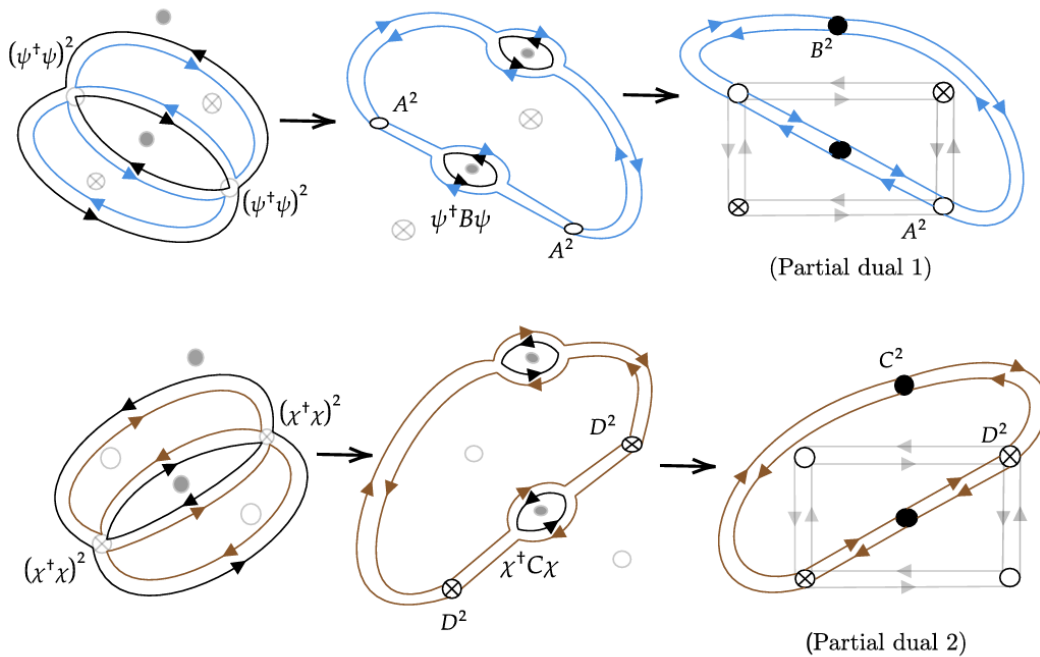
$$\begin{aligned} Z[t_k, \bar{t}_k, s_k] &= \frac{\det_Q(X)^Q}{Z_Q} \int dAdB_{Q \times Q} \exp \left(-\frac{N}{g} \text{Tr}_Q(\sqrt{XB}\sqrt{XA}) - \sum_{k \geq 1} (-1)^k \frac{\bar{t}_k}{k} \text{Tr}(A^k) - \sum_{k \geq 1} \frac{s_k}{k} \text{Tr}(B^k) \right) \\ &= \left(\frac{\det_Q(X) \det_R(V)}{\det_N(Y)} \right)^Q \frac{1}{Z_Q} \int dAdBe^{-\frac{N}{g} \text{Tr}_Q(\sqrt{XB}\sqrt{XA})} \prod_{\mu=1}^R \det_Q(v_\mu + A)^{-1} \prod_{i=1}^N \det_Q(y_i - B) \end{aligned}$$

$$Z[t_k, \bar{t}_k, s_k] = \frac{\det_R(V)^R}{Z_R} \int dCdD_{R \times R} \exp \left(-\frac{N}{g} \text{Tr}(\sqrt{V}C\sqrt{V}D) - \sum_{k \geq 1} \frac{s_k}{k} \text{Tr}(C^k) - \sum_{k \geq 1} (-1)^k \frac{\bar{t}_k}{k} \text{Tr}(D^k) \right)$$

$$= \left(\frac{\det(X)\det(V)}{\det(Y)} \right)^R \frac{1}{Z_R} \int dCdD \exp \left(-\frac{N}{g} \text{Tr}(\sqrt{V}C\sqrt{V}D) \right) \prod_{i=1}^N \det(y_i - C) \prod_{a=1}^Q \det(x_a + D)^{-1}$$

$$Z_Q = (2\pi g/N)^{Q^2}, Z_R = (2\pi g/N)^{R^2}$$

$$\text{Tr}_N(M^n) \leftrightarrow (-1)^n \text{Tr}_Q(A^n) \text{Tr}_N(K^n) \leftrightarrow (-1)^n \text{Tr}_R(D^n)$$



$$Z[t_k, \bar{t}_k, s_k = N] = (-1)^{Q^2} \frac{(\det(X)\det(V))^Q}{Z_Q}$$

$$\times \int dAdB_{N \times N} \exp \left(\frac{N}{g} \text{Tr}(ABX) - \frac{N}{g} \text{Tr}(AX) - \sum_{k \geq 1} (-1)^k \frac{\bar{t}_k}{k} \text{Tr}(A)^k \right) (\det(B))^N$$

$$Z[t_k, \bar{t}_k, s_k = N] = \frac{\det(X)^Q \det(V)^Q}{Z_Q} \int dAd\sigma \exp \left(-\frac{N}{g} \text{Tr}(\sigma) + \frac{N}{g} \text{Tr}(AX) - \sum_{k \geq 1} (-1)^k \frac{\bar{t}_k}{k} \text{Tr}(A)^k \right)$$

$$\times (\det(\sigma))^N (\det(A))^{-N-Q} (\det(X))^{-N-Q}$$

$$= \frac{(\det(V))^Q}{(\det(X))^N Z_Q} \int dA \exp \left(\frac{N}{g} \text{Tr}(AX) + N \text{Tr}(V(A)) - (N+Q) \text{Tr} \log A \right)$$

Imbimbo–Mukhi Model

$$\times \int d\sigma e^{-\frac{N}{g} \text{Tr}(\sigma)} (\det(\sigma))^N$$

$$N \text{Tr} V(A) = - \sum_{k \geq 1} (-1)^k \frac{\bar{t}_k}{k} \text{Tr}(A)^k$$

$$Z_{IM}[t_k, \bar{t}_k, \mu = -iN] = \frac{1}{\det(X)^N} \int dA \exp \left(\frac{N}{g} \text{Tr}(AX) + N \text{Tr}(V(A)) - (N+Q) \text{Tr} \log A \right) \quad (135)$$

$$\log Z_{IM}[t_k, \bar{t}_k] = \langle e^{\sum_{k \geq 1} t_k T_{+k}} e^{\sum_{k \geq 1} \bar{t}_k T_{-k}} \rangle_{c=1, R=1}$$

$$\left\langle \prod_{i=1}^{n_1} \frac{1}{k_i} \text{Tr} \left[(Y_1 \cdot \Phi(w_1))^{k_i} \right] \prod_{j=1}^{n_2} \frac{1}{l_j} \text{Tr} \left[(Y_2 \cdot \Phi(w_2))^{l_j} \right] \right\rangle_{N=4} = \left(\frac{\lambda Y_1 \cdot Y_2}{4\pi^2 |w_1 - w_2|^2} \right)^{\sum_{i=1}^{n_1} k_i} \left\langle \prod_{i=1}^{n_1} T_{+k_i} \prod_{j=1}^{n_2} T_{-l_j} \right\rangle_{c=1}$$

$$\frac{1}{n} \text{Tr}(K^n) \leftrightarrow T_{+n} \quad \frac{1}{n} \text{Tr}(M^n) \leftrightarrow T_{-n}$$

$$\langle \text{Tr}(M^l) \text{Tr}(K^l) \rangle_{h=0} = l g^l N^0$$

$$\langle \text{Tr}(K^l) \text{Tr}(M^{l_1}) \text{Tr}(M^{l-l_1}) \rangle_{h=0} = l l_1 (l-l_1) g^l N^{-1}$$

$$\langle \text{Tr}(K^l) \text{Tr}(M^{l_1}) \text{Tr}(M^{l_2}) \text{Tr}(M^{l_3}) \rangle_{h=0} = l(l-1) l_1 l_2 l_3 g^l N^{-2} \delta_{l_1+l_2+l_3, l}$$

$$\langle P(+l) P(-l) \rangle_{h=0} = -(\Gamma(1-l))^2 \frac{\mu^l}{l}$$

$$\frac{1}{l} \text{Tr}(M^l) \leftrightarrow \frac{-iP(l)}{\Gamma(1-|l|)} \frac{1}{l} \text{Tr}(K^l) \leftrightarrow \frac{-iP(-l)}{\Gamma(1-|l|)}$$

$$N \leftrightarrow -i\beta\mu \text{ and } g \leftrightarrow \mu.$$

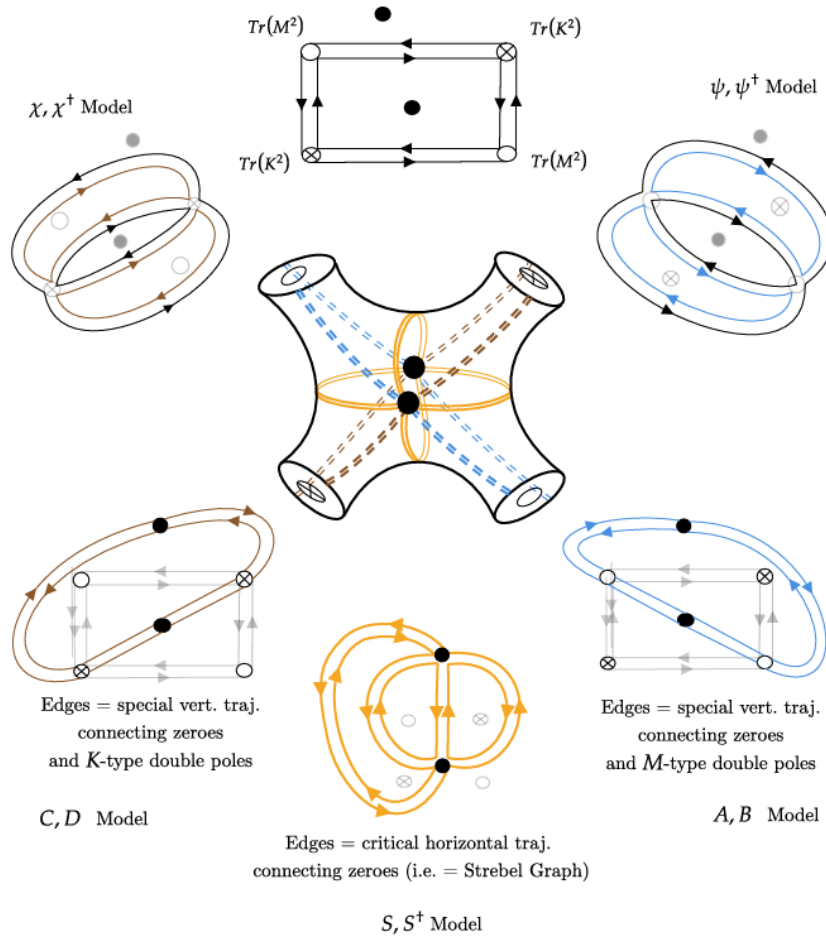
$$\langle \text{Tr}(M^l) \text{Tr}(K^l) \rangle_{h=1} = l \binom{l+1}{4} g^l N^{-2}$$

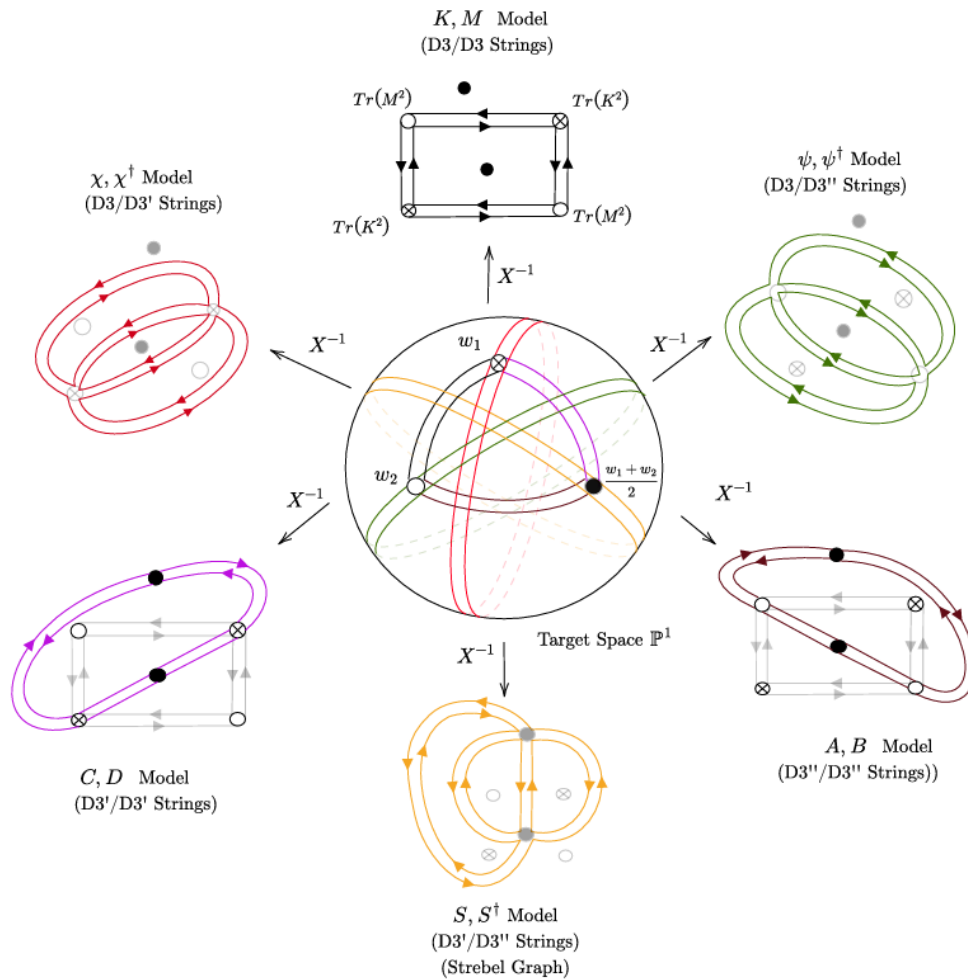
$$\langle P(+l) P(-l) \rangle_{h=1} = \Gamma((1-l)^2) \mu^l (\beta\mu)^{-2} \frac{(l+1)(l-1)(l-2)}{24}$$



K, M Model

Edges = vertical traj. connecting
K- and M-type of double poles





$$\text{Tr}(KM) + \sum_n \frac{c_n}{n} \text{Tr}(K^n) + \frac{\tilde{c}_n}{n} \text{Tr}(M^n)$$

$$S_0^{WS} + \sum_n c_n \int d^2z \mathcal{V}_n + \sum_n \tilde{c}_n \int d^2z \tilde{\mathcal{V}}_n$$

$$\text{Tr}_N(KM) \rightarrow \int d^2z \text{Tr}_N(\beta \bar{\partial} \gamma)$$

$$\beta_{ab}(z_1) \gamma_{cd}(z_2) \sim \frac{1}{z_1 - z_2} \delta_{ad} \delta_{bc}$$

$$[\sigma_K] = [\sigma_M] = \{(1432), (1342), (1423), (1243), (1324), (1234)\}$$

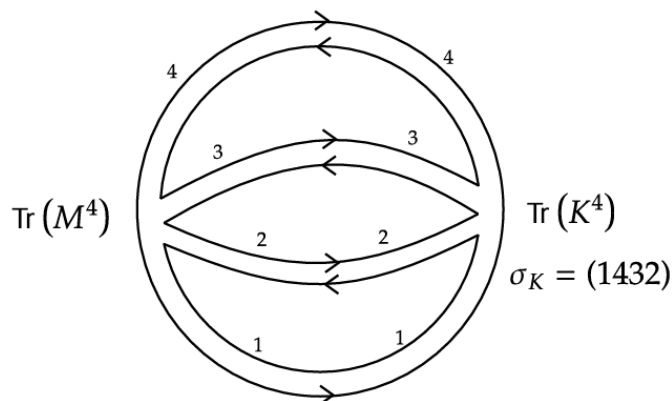
$$\langle \text{Tr}(M^4) \text{Tr}(K^4) \rangle = \frac{4!}{6} g^4 (1 + 4N^{-2} + N^{-2}) = g^4 (4 + 20N^{-2})$$

$$\left\langle \prod_{j=1}^{V_M} \text{Tr}(M^{l_j}) \prod_{i=1}^{V_K} \text{Tr}(K^{k_i}) \right\rangle = \frac{E!}{|[\sigma_K]| |[\sigma_M]|} \sum_{\sigma_K \in [\sigma_K], \sigma_M \in [\sigma_M], \sigma_f \in S_E} g^E N^{C_{\sigma_f} - E} \delta(\sigma_M \sigma_f \sigma_K)$$



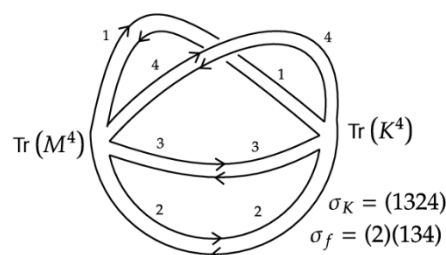
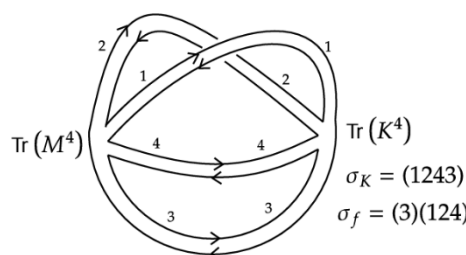
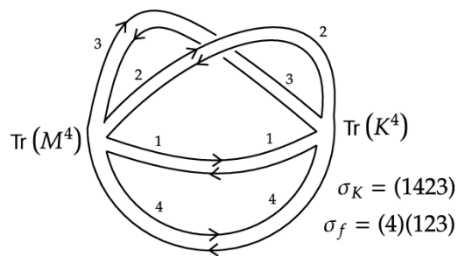
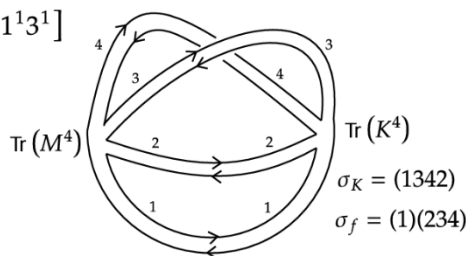
• $g = 0$:

$\circ [\sigma_f] = [1^4]$

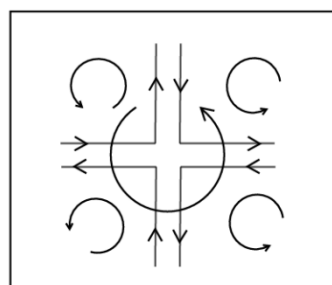
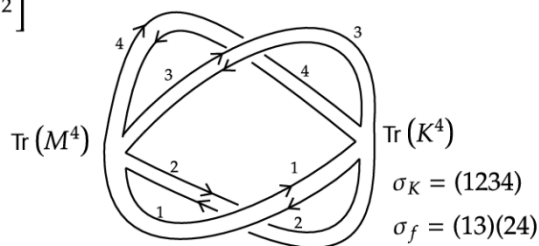


• $g = 1$:

$\circ [\sigma_f] = [1^1 3^1]$



$\circ [\sigma_f] = [2^2]$



$$\left\langle \prod_{j=1}^{V_M} \left(\frac{1}{l_j} \text{Tr}(M^{l_j}) \right) \prod_{i=1}^{V_K} \left(\frac{1}{k_i} \text{Tr}(K^{k_i}) \right) \right\rangle \sim \sum_{G \in \text{Feynman Diagrams}} \frac{g^E N^{F-E}}{|\text{Aut}(G)|} = \sum_G \frac{g^E N^{2-2h-(V_K+V_M)}}{|\text{Aut}(G)|},$$

$$Z(X, Y, V) = \frac{\mathcal{K}}{Z_N} \int dK dM_{N \times N} e^{-\frac{N}{g} \text{Tr}_N(K \sqrt{Y} M \sqrt{Y})} \prod_{a=1}^Q \det_N(x_a \mathbb{I}_N - M) \prod_{\mu=1}^R \det_N(v_\mu \mathbb{I}_N - K)$$

where, $\mathcal{K} := \left(\frac{\det_N(Y)}{\det_Q(X) \det_R(V)} \right)^N$



$$\begin{aligned}
Z &= \frac{\mathcal{K}}{Z_N} \int dKdMd\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(-\frac{N}{g} \text{Tr}_N(K\sqrt{Y}M\sqrt{Y}) + \psi_{ia}^\dagger (X_{ab}\delta_{ij} - \delta_{ab}M_{ij})\psi_{jb} \right. \\
&\quad \left. + \chi_{i\mu}^\dagger (V_{\mu\nu}\delta_{ij} - K_{ij}\delta_{\mu\nu})\chi_{j\nu} \right) \\
&= \frac{\mathcal{K}}{Z_N} \int dKdMd\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(-\frac{N}{g} K_{ij}((\sqrt{Y}M\sqrt{Y})_{ji} + g\chi_{i\mu}^\dagger\chi_{j\mu}) + \psi_{ia}^\dagger (X_{ab}\delta_{ij} - \delta_{ab}M_{ij})\psi_{jb} \right. \\
&\quad \left. + \chi_{i\mu}^\dagger V_{\mu\nu}\chi_{j\nu} \right) \\
&\quad (2\pi g/N)^{N^2} \delta \left((\sqrt{Y}M\sqrt{Y})_{ji} + \frac{g}{N} \chi_{i\mu}^\dagger\chi_{j\mu} \right)
\end{aligned}$$

$$\tilde{M} = \sqrt{Y}M\sqrt{Y} \text{ sets } M_{ij} = -\frac{g}{N} \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{l\mu}^\dagger \chi_{k\mu} \left(Y^{-\frac{1}{2}} \right)_{lj}$$

$$Z = \mathcal{K}' \int d\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab}\psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu}\chi_{i\nu} + \frac{g}{N} \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{l\mu}^\dagger \chi_{k\mu} \left(Y^{-\frac{1}{2}} \right)_{lj} \psi_{ja} \right)$$

$$\mathcal{K}' = (\det_Q(X)\det_R(V))^{-N}$$

$$\begin{aligned}
Z &= \frac{\mathcal{K}'}{Z_{QR}} \int d\psi d\psi^\dagger d\chi d\chi^\dagger dSdS^\dagger \exp \left(\frac{N}{g} \left(S_{\mu a}^\dagger - \frac{g}{N} \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{k\mu} \right) \left(S_{a\mu} - \frac{g}{N} \chi_{l\mu}^\dagger \left(Y^{-\frac{1}{2}} \right)_{lj} \psi_{ja} \right) \right. \\
&\quad \left. + \psi_{ia}^\dagger X_{ab}\psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu}\chi_{i\nu} - \frac{g}{N} \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{k\mu} \chi_{l\mu}^\dagger \left(Y^{-\frac{1}{2}} \right)_{lj} \psi_{ja} \right) \\
&= \frac{\mathcal{K}'}{Z_{QR}} \int d\psi d\psi^\dagger d\chi d\chi^\dagger dSdS^\dagger \exp \left(\frac{N}{g} S_{\mu a}^\dagger S_{a\mu} - S_{\mu a}^\dagger \chi_{i\mu}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ij} \psi_{ja} - \psi_{ia}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ik} \chi_{k\mu} S_{a\mu} \right. \\
&\quad \left. + \psi_{ia}^\dagger X_{ab}\psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu}\chi_{i\nu} \right)
\end{aligned}$$

$$Z_{QR} = \left(\frac{2\pi g}{N} \right)^{QR}$$

$$\begin{aligned}
\left(\psi_{ia}^\dagger \quad \chi_{i\mu}^\dagger \right) \begin{pmatrix} X_{ab}\delta_{ij} & -\left(Y^{-\frac{1}{2}} \right)_{ij} S_{av} \\ -S_{\mu b}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ij} & V_{\mu\nu}\delta_{ij} \end{pmatrix} \begin{pmatrix} \psi_{jb} \\ \chi_{j\nu} \end{pmatrix} &= \left(\psi_{ia}^\dagger \quad \chi_{i\mu}^\dagger \right) \begin{pmatrix} X_{ab}\psi_{ib} - \left(Y^{-\frac{1}{2}} \right)_{ij} S_{av}\chi_{j\nu} \\ -S_{\mu b}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ij} \psi_{jb} + V_{\mu\nu}\chi_{i\nu} \end{pmatrix} \\
&= \psi_{ia}^\dagger X_{ab}\psi_{ib} - \psi_{ia}^\dagger Y_{ij}^{-\frac{1}{2}} S_{av}\chi_{j\nu} - \chi_{i\mu}^\dagger S_{\mu b}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ij} \psi_{jb} + \chi_{i\mu}^\dagger V_{\mu\nu}\chi_{i\nu}
\end{aligned}$$

$$Z = \frac{\mathcal{K}'}{Z_{QR}} \int dSdS^\dagger e^{\frac{N}{g} S_{\mu a}^\dagger S_{a\mu}} \int d\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left\{ \left(\psi_{ia}^\dagger \quad \chi_{i\mu}^\dagger \right) \begin{pmatrix} X_{ab}\delta_{ij} & -\left(Y^{-\frac{1}{2}} \right)_{ij} S_{av} \\ -S_{\mu b}^\dagger \left(Y^{-\frac{1}{2}} \right)_{ij} & V_{\mu\nu}\delta_{ij} \end{pmatrix} \begin{pmatrix} \psi_{jb} \\ \chi_{j\nu} \end{pmatrix} \right\}$$

$$Z = \frac{\mathcal{K}'}{Z_{QR}} \int dSdS^\dagger \exp \left(\frac{N}{g} S_{\mu a}^\dagger S_{a\mu} \right) \prod_{i=1}^N \det_{Q+R} \begin{pmatrix} X_{ab} & -\left(Y_i \right)^{-\frac{1}{2}} S_{av} \\ -S_{\mu b}^\dagger \left(Y_i \right)^{-\frac{1}{2}} & V_{\mu\nu} \end{pmatrix}$$



$$\det_{Q+R} \begin{pmatrix} X & -(y_i)^{-\frac{1}{2}}S \\ -S^\dagger(y_i)^{-\frac{1}{2}} & V \end{pmatrix} = \det_{Q+R} \begin{pmatrix} X & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 1 & -(y_i)^{-\frac{1}{2}}X^{-1}S \\ -(y_i)^{-\frac{1}{2}}V^{-1}S^\dagger & 1 \end{pmatrix}$$

$$= \det_Q(X)\det_R(V)\det_{Q+R}(\mathbb{I}_{Q+R} - \Lambda_i)$$

where, $\Lambda_i = \begin{pmatrix} 0 & (y_i)^{-\frac{1}{2}}X^{-1}S \\ (y_i)^{-\frac{1}{2}}V^{-1}S^\dagger & 0 \end{pmatrix}$ and $\text{Tr}_{Q+R}(\Lambda_i^{2m}) = 2y_i^{-m}\text{Tr}_Q((X^{-1}SV^{-1}S^\dagger)^m)$

$$Z = \frac{1}{Z_{QR}} \int dSdS^\dagger \exp \left(\frac{N}{g} S_{\mu a}^\dagger S_{a\mu} - \sum_k \frac{1}{k} \left(\sum_{i=1}^N y_i^{-k} \right) \text{Tr}_Q((X^{-1}SV^{-1}S^\dagger)^k) \right)$$

$$S \rightarrow X^{-\frac{1}{2}}SV^{-\frac{1}{2}}, S^\dagger \rightarrow V^{-\frac{1}{2}}S^\dagger X^{-\frac{1}{2}}$$

$$Z = \frac{(\det_Q(X))^R (\det_R(V))^Q}{Z_{QR}} \int dSdS^\dagger \exp \left(\frac{N}{g} \text{Tr}(VS^\dagger XS) - \sum_k \frac{S_k}{k} \text{Tr}((SS^\dagger)^k) \right)$$

$$Z = \left(\frac{\mathcal{K}}{\det_N(Y)^N} \right) \int d\psi d\psi^\dagger d\chi d\chi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} \psi_{ib} + \chi_{i\mu}^\dagger V_{\mu\nu} \chi_{iv} + \frac{g}{N} \psi_{ia}^\dagger (Y^{-\frac{1}{2}})_{ik} \chi_{i\mu}^\dagger \chi_{k\mu} (Y^{-\frac{1}{2}})_{lj} \psi_{ja} \right)$$

$$\prod_{\mu=1}^R \det_N \left(v_\mu \delta_{ij} - \frac{g}{N} (Y^{-\frac{1}{2}})_{il} (\psi\psi^\dagger)_{lk} Y_{kj}^{-\frac{1}{2}} \right) = \det(V)^N \prod_{\mu=1}^N \det_N \left(I - \frac{g}{N v_\mu} Y^{-1} \psi\psi^\dagger \right)$$

$$Z = \mathcal{K} \left(\frac{\det(V)}{\det(Y)} \right)^N \int d\psi d\psi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} \psi_{ib} - \sum_{k=1}^{\infty} \frac{\text{Tr}_R V^{-k}}{k} \text{Tr}_N \left(\frac{g}{N} Y^{-1} \psi\psi^\dagger \right)^k \right)$$

$$= \mathcal{K} \left(\frac{\det(V)^N}{\det(Y)^{N-Q}} \right) \int d\psi d\psi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} Y_{ij} \psi_{jb} - \sum_{k=1}^{\infty} \frac{\text{Tr}_R V^{-k}}{k} \text{Tr}_N \left(\frac{g}{N} \psi\psi^\dagger \right)^k \right)$$

$$Z = \mathcal{K} \left(\frac{\det(X)}{\det(Y)} \right)^N \int d\chi d\chi^\dagger \exp \left(\chi_{i\mu}^\dagger V_{\mu\nu} \chi_{iv} - \sum_{k=1}^{\infty} \frac{\text{Tr}_Q X^{-k}}{k} \text{Tr}_N \left(\frac{g}{N} Y^{-1} \chi\chi^\dagger \right)^k \right)$$

$$= \mathcal{K} \left(\frac{\det(X)^N}{\det(Y)^{N-R}} \right) \int d\chi d\chi^\dagger \exp \left(\chi_{i\mu}^\dagger V_{\mu\nu} Y_{ij} \chi_{j\nu} - \sum_{k=1}^{\infty} \frac{\text{Tr}_Q X^{-k}}{k} \text{Tr}_N \left(\frac{g}{N} \chi\chi^\dagger \right)^k \right)$$

$$\text{Tr}_N(\psi\psi^\dagger)^k = -\text{Tr}_Q(\psi^\dagger\psi)^k$$

$$1 = \int dA \delta \left(A_{ba} + \frac{g}{N} \psi_{ia}^\dagger \psi_{ib} \right)$$

$$Z = \mathcal{K} \left(\frac{\det(V)^N}{\det(Y)^{N-Q} Z_Q} \right) \int d\psi d\psi^\dagger \exp \left(\psi_{ia}^\dagger X_{ab} Y_{ij} \psi_{jb} + \sum_{k=1}^{\infty} \frac{\text{Tr}_R V^{-k}}{k} \text{Tr}_Q \left(\frac{g}{N} \psi^\dagger\psi \right)^k \right)$$

$$\int dAdB \exp \left(-\frac{N}{g} B_{ab} \left(A_{ba} + \frac{g}{N} \psi_{ia}^\dagger \psi_{ib} \right) \right)$$



$$\sum_{k=1}^{\infty} \frac{\text{Tr}_R V^{-k}}{k} \text{Tr}_Q \left(\frac{g}{N} \psi^\dagger \psi \right)^k \rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{\text{Tr}_R V^{-k}}{k} \text{Tr}_Q (A^k)$$

$$\prod_i \det(y_i X_{ab} - B_{ab})$$

$$Z = \left(\frac{\det_Q(X) \det_R(V)}{\det_N(Y)} \right)^Q \frac{1}{Z_Q} \int dAdBe^{-\frac{N}{g} \text{Tr}_Q(\sqrt{XB}\sqrt{XA})} \frac{\det_{N+Q}(I_Q \otimes Y - B \otimes I_N)}{\det_{Q+R}(I_Q \otimes V + A \otimes I_R)}$$

$$= \left(\frac{\det_Q(X) \det_R(V)}{\det_N(Y)} \right)^Q \frac{1}{Z_Q} \int dAdBe^{-\frac{N}{g} \text{Tr}_Q(\sqrt{XB}\sqrt{XA})} \prod_{i=1}^N \det_Q(y_i - B) \prod_{\mu=1}^R (\det_Q(v_\mu + A))^{-1}$$

$$\text{Tr}_N(\chi\chi^\dagger)^k = -\text{Tr}_R(\chi^\dagger\chi)^k$$

$$1 = \int dD\delta\left(D_{\nu\mu} + \frac{g}{N} \chi_{i\mu}^\dagger \chi_{iv}\right) = \frac{1}{Z_R} \int dCdD \exp\left(-\frac{N}{g} C_{\mu\nu} \left(D_{\nu\mu} + \frac{g}{N} \chi_{i\mu}^\dagger \chi_{iv}\right)\right)$$

$$Z = \left(\frac{\det_Q(X)}{\det_N(Y)} \right)^R \frac{1}{Z_R} \int dCdDe^{-\frac{N}{g} \text{Tr}_R(CD)} \frac{\det_{N+R}(I_R \otimes Y - V^{-1}C \otimes I_N)}{\det_{Q+R}(I_R \otimes X + D \otimes I_Q)}$$

$$\equiv \left(\frac{\det_R(V) \det_Q(X)}{\det_N(Y)} \right)^R \frac{1}{Z_R} \int dCdDe^{-\frac{N}{g} \text{Tr}_R(\sqrt{VC}\sqrt{VD})} \prod_{i=1}^N \det_R(y_i - C) \prod_{a=1}^Q (\det_R(x_a + D))^{-1}$$

$$A_\mu, \Phi_i, \Psi = \left[\begin{array}{c|ccc} \overset{\circ}{a} & \emptyset & \emptyset & \emptyset \\ \hline \emptyset & \text{æ} & \text{æ} & \text{æ} \\ \emptyset & \text{æ} & \text{æ} & \text{æ} \\ \emptyset & \text{æ} & \text{æ} & \text{æ} \end{array} \right].$$

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}, \Gamma^I = \gamma^5 \otimes \hat{\gamma}^I$$

$$\mathcal{L} = \frac{N}{\lambda} \text{tr} \left(\frac{1}{2} F_{MN}^2 + i\bar{\Psi} \Gamma^M D_M \Psi \right)$$

$$\mathcal{L}_{\text{gf}} = \frac{N}{\lambda} \text{tr}(D_M A^M)^2$$

$$\mathcal{L}_{\text{gh}} = \frac{2N}{\lambda} \text{tr} D_M \bar{c} D^M c$$

$$A_M = A_M + a_M, \Psi = \psi$$

$$\mathcal{L}_{\text{tot}} = \frac{1}{g^2} \text{tr} \left\{ a^M \left(-D^2 \eta_{MN} + 2i F_{MN}^{\text{adj}} \right) a^N + i\bar{\psi} \Gamma^M D_M \psi + 2\bar{c}(-D^2)c \right.$$

$$\left. - 2i D_M a_N [a^M, a^N] + \bar{\psi} \Gamma^M [a_M, \psi] + 2i D_M \bar{c} [a^M, c] - \frac{1}{2} [a^M, a^N]^2 \right\}$$

$$\langle a_M^{1a}(x) a_N^{b1}(x') \rangle = \frac{\lambda \delta^{ab}}{2N} G_{MN}(x, x')$$



$$-D^2 = -\frac{\partial^2}{\partial t^2} + H = \omega^2 + H$$

$$H = -(\vec{\partial} - i\vec{A})^2 + \Phi^2$$

$$F_{iI} = \partial_i \Phi_I = -q \frac{n_I x_i}{r^3}$$

$$ds^2 = r^2 \left(\frac{dt^2 + dr^2}{r^2} + d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$

$$\xi = \frac{(t - t')^2 + r^2 + r'^2}{2rr'}$$

$$\eta = \mathbf{n} \cdot \mathbf{n}'$$

$$L_i = -i\varepsilon_{ijk} x_j D_k - q \frac{x_i}{r}$$

$$\mathbf{L}^2 Y_{\ell m} = \ell(\ell + 1) Y_{\ell m}, L_z Y_{\ell m} = m Y_{\ell m}$$

$$A = -q \cos \theta d\varphi$$

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{(2\ell + 1)(\ell - \alpha_+)!(\ell + \alpha_+)!}{4\pi(\ell + \alpha_-)!(\ell - \alpha_-)!}} e^{im\varphi} \sin^\alpha \frac{\theta}{2} \cos^\beta \frac{\theta}{2} P_{\ell - \alpha_+}^{(\alpha, \beta)}(\cos \theta)$$

$$\alpha = |m + q|, \beta = |m - q|, \alpha_\pm = \frac{\alpha \pm \beta}{2}.$$

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) Y_{\ell m}^*(\mathbf{n}') = \frac{2\ell + 1}{4\pi} \left(\frac{1 + \mathbf{n} \cdot \mathbf{n}'}{2} \right)^q P_{\ell - q}^{(0, 2q)}(\mathbf{n} \cdot \mathbf{n}')$$

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) Y_{\ell m}^*(\mathbf{n}) = \frac{2\ell + 1}{4\pi}$$

$$H = \frac{1}{r} \left(-\frac{\partial^2}{\partial r^2} + \frac{\mathbf{L}^2}{r^2} \right) r$$

$$\textcircled{\text{E}}_{\text{eigenmodes}} = \frac{1}{r} e^{-i\omega t} \sqrt{kr} J_{j+\frac{1}{2}}(kr) Y_{jm}(\mathbf{n})$$

$$G(x, x') = \frac{1}{rr'} \sum_{jm} Y_{jm}^*(\mathbf{n}) Y_{jm}(\mathbf{n}') \\ \times \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_0^{\infty} dk \frac{e^{i\omega(t-t')} \sqrt{kr} J_{j+\frac{1}{2}}(kr) \sqrt{kr'} J_{j+\frac{1}{2}}(kr')}{\omega^2 + k^2}$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 + k^2} = \frac{e^{-k|t|}}{2k}$$



$$G(x, x') = \frac{1}{rr'} \sum_{jm} Y_{jm}^*(\mathbf{n}) Y_{jm}(\mathbf{n}') D_{j+1}(|t - t'|, r, r')$$

$$D_{\Delta}(t, r, r') = \frac{\sqrt{rr'}}{2} \int_0^{\infty} dk e^{-kt} J_{\Delta-\frac{1}{2}}(kr) J_{\Delta-\frac{1}{2}}(kr')$$

$$m^2 = \Delta(\Delta - 1)$$

$$D_{\Delta}(|t - t'|, r, r') = \frac{1}{2\pi} Q_{\Delta-1}(\xi)$$

$$G(x, x') = \frac{1}{4\pi rr'} \left(\frac{1+\eta}{2}\right)^q \sum_j (2j+1) P_{j-q}^{(0,2q)}(\eta) D_{j+1}(\xi)$$

$$D_{\Delta}(\xi) = \frac{\Gamma(\Delta)}{2^{\Delta+1} \pi^{\frac{d-3}{2}} \Gamma\left(\Delta - \frac{d-5}{2}\right) \xi^{\Delta}} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d-5}{2}; \frac{1}{\xi^2}\right).$$

$$\frac{d\omega}{2\pi} \rightarrow \frac{d^{d-3}\omega}{(2\pi)^{d-3}}$$

$$G(x, x') = \frac{1}{4\pi (rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \sum_j (2j+1) P_{j-q}^{(0,2q)}(\eta) D_{j+\frac{d}{2}-1}(\xi)$$

$$D_{\Delta}(t, r, r') = \left(\frac{rr'}{2\pi t}\right)^{\frac{d-3}{2}} t \int_0^{\infty} dk k^{\frac{d-3}{2}} K_{\frac{d-5}{2}}(kt) J_{\Delta-\frac{d-3}{2}}(kr) J_{\Delta-\frac{d-3}{2}}(kr')$$

$$D_{\Delta}(1) = \frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma(\Delta)}{(4\pi)^{\frac{d}{2}-1} \Gamma(\Delta - d + 4)}.$$

$$\mathbb{B}_{\text{tadpole}} \equiv G(x, x) = \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} r^{d-2}} \sum_{j=q}^{\infty} (2j+1) \frac{\Gamma\left(j + \frac{d}{2} - 1\right)}{\Gamma\left(j - \frac{d}{2} + 3\right)}.$$

$$G(x, x) = -\frac{2q\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(q + \frac{d}{2} - 1\right)}{(4\pi)^{\frac{d}{2}} (d-2) \Gamma\left(q - \frac{d}{2} + 2\right) r^{d-2}}$$

$$G(x, x) = -\frac{q^2}{16\pi^2 r^2} \left(\frac{1}{\varepsilon} + \ln(4\pi r^2) + 1 - \frac{1}{q} - 2\psi(q) - \gamma\right).$$

$$\hat{M} = \frac{1}{r^2} \begin{pmatrix} r^2 \partial_t^2 + r^2 p_r^2 + \mathbf{L}^2 & -i\mathbf{Bn}^T \\ i\mathbf{Bn} & r^2 \partial_t^2 + r^2 p_r^2 + \mathbf{L}^2 - i\mathbf{Bn} \times \end{pmatrix}.$$



$$\begin{aligned} \mathbf{J}^2 \mathbf{C}_{jm}^{(\lambda)} &= j(j+1) \mathbf{C}_{jm}^{(\lambda)}, \\ J_z \mathbf{C}_{jm}^{(\lambda)} &= m \mathbf{C}_{jm}^{(\lambda)}, \\ (\mathbf{n} \cdot \mathbf{S}) \mathbf{C}_{jm}^{(\lambda)} &= \lambda \mathbf{C}_{jm}^{(\lambda)}, \end{aligned}$$

$$\phi_1(r, t, \theta, \phi) = e^{-i\omega t} \frac{1}{r} \sum_{jm} g_j(r) Y_{jm}(\theta, \phi)$$

$$\mathbf{a}(r, t, \theta, \phi) = e^{-i\omega t} \frac{1}{r} \sum_{jm} \left(f_j^{(-1)}(r) \mathbf{C}_{jm}^{(-1)} + f_j^{(0)}(r) \mathbf{C}_{jm}^{(0)} + f_j^{(+1)}(r) \mathbf{C}_{jm}^{(+1)} \right)$$

$$g_j(r), f_j^{(-1)}(r), f_j^{(0)}(r), f_j^{(+1)}(r)$$

$$\begin{aligned} \mathbf{n} \cdot \mathbf{C}_{jm}^{(\lambda)} &= \delta_{\lambda 0} Y_{jm}, \\ \mathbf{n} \times \mathbf{C}_{jm}^{(\lambda)} &= -i\lambda \mathbf{C}_{jm}^{(\lambda)}, \end{aligned}$$

$$g_j(r) = g_j \sqrt{kr} J_\nu(kr), f_j^{(\lambda)}(r) = f_j^{(\lambda)} \sqrt{kr} J_\nu(kr),$$

$$a = v^2 - \left(j + \frac{1}{2} \right)^2 \iiint \langle g_j, f_j^{(-1)}, f_j^{(0)}, f_j^{(+1)} \rangle_{\square},$$

$$\begin{pmatrix} 0 & 0 & 2iq & 0 \\ 0 & 0 & \sqrt{2(j-q)(1+j+q)} & 0 \\ -2iq & \sqrt{2(j-q)(1+j+q)} & -2 & \sqrt{2(1+j-q)(j+q)} \\ 0 & 0 & \sqrt{2(1+j-q)(j+q)} & 0 \end{pmatrix}$$

$$v_1 = \begin{bmatrix} -i \sqrt{\frac{j^2 - q^2}{j(1+2j)}} \\ \sqrt{\frac{(j+q)(1+j+q)}{2j(1+2j)}} \\ 0 \\ -\sqrt{\frac{(j-q)(1+j-q)}{2j(1+2j)}} \end{bmatrix}, v_2 = \begin{bmatrix} i \sqrt{\frac{(1+j-q)(1+j+q)}{(1+j)(1+2j)}} \\ \sqrt{\frac{(j-q)(1+j-q)}{2(1+j)(1+2j)}} \\ 0 \\ -\sqrt{\frac{(j+q)(1+j+q)}{2(1+j)(1+2j)}} \end{bmatrix}, \delta_{1,2} = 0,$$

$$v_3 = \begin{bmatrix} \frac{iq}{\sqrt{j(1+2j)}} \\ \sqrt{\frac{j^2 + j - q^2 - q}{j(1+2j)}} \\ \sqrt{\frac{j}{1+2j}} \\ \sqrt{\frac{j^2 + j - q^2 + q}{2j(1+2j)}} \end{bmatrix}, \delta_3 = -1, v_4 = \begin{bmatrix} \frac{iq}{\sqrt{(1+j)(1+2j)}} \\ \sqrt{\frac{j + j^2 - q^2 - q}{2(1+j)(1+2j)}} \\ -\sqrt{\frac{1+j}{1+2j}} \\ \sqrt{\frac{j^2 + j - q^2 + q}{2(1+j)(1+2j)}} \end{bmatrix}, \delta_4 = 1.$$



$$v = \begin{bmatrix} -\frac{i}{\sqrt{1+j}} \\ 0 \\ \sqrt{j} \\ \sqrt{(j+1)} \end{bmatrix}, \delta = 0$$

$$v_3 = \begin{bmatrix} \frac{i\sqrt{j}}{\sqrt{1+2j}} \\ \frac{\sqrt{j}}{\sqrt{1+2j}} \\ 1 \\ \sqrt{1+2j} \end{bmatrix}, \delta_3 = -1, v_4 = \begin{bmatrix} \frac{ij}{\sqrt{(1+j)(1+2j)}} \\ -\frac{\sqrt{1+j}}{\sqrt{1+2j}} \\ \sqrt{j} \\ \sqrt{(1+j)(1+2j)} \end{bmatrix}, \delta_4 = 1$$

$$f_j^{(+1)}(r) = \sqrt{kr} J_{j+\frac{1}{2}}(kr)$$

$$\tilde{G}(x, x') = \frac{1}{4\pi(rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \sum_j (2j+1) P_{j-q}^{(0,2q)}(\eta) \sum_\alpha |v_\alpha^1|^2 D_{j+\frac{d}{2}-1+\delta_\alpha}(\xi).$$

$$\tilde{G}(x, x') = \frac{1}{4\pi(rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \sum_j P_{j-q}^{(0,2q)}(\eta) \times \left[\frac{q^2}{j} D_{j+\frac{d}{2}-2} + \frac{(2j+1)(j^2+j-q^2)}{j(j+1)} D_{j+\frac{d}{2}-1} + \frac{q^2}{j+1} D_{j+\frac{d}{2}} \right].$$

$$G^-(x, x') = \frac{q^2}{4\pi(rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \times \sum_j P_{j-q}^{(0,2q)}(\eta) \left(\frac{D_{j+\frac{d}{2}-2} - D_{j+\frac{d}{2}-1}}{j} - \frac{D_{j+\frac{d}{2}-1} - D_{j+\frac{d}{2}}}{j+1} \right)$$

$$G^-(x, x')|_{\eta=1} = \frac{q}{4\pi(rr')^{\frac{d}{2}-1}} \left(D_{q+\frac{d}{2}-2} - D_{q+\frac{d}{2}-1} \right).$$

$$G^-(x, x) = \frac{2q\Gamma\left(3-\frac{d}{2}\right)\Gamma\left(q+\frac{d}{2}-2\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(q-\frac{d}{2}+3\right)r^{d-2}} \stackrel{d \rightarrow 4}{=} \frac{1}{8\pi^2 r^2}$$

$$S_2^{ferm} = \frac{1}{2} \text{tr}(i\bar{\Psi}\Gamma^\mu\partial_\mu\Psi + \bar{\Psi}\Gamma^1[\Phi_1, \Psi] + \bar{\Psi}\Gamma^\mu[A_\mu, \Psi])$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$S_2^f = \Psi^\top C \left(i\gamma^\mu(\partial_\mu - iA_\mu) \pm \gamma^5 \frac{q}{r} \right) \Psi$$



$$S_2^f = (\Psi_A^\top, \Psi_B^\top) C \begin{pmatrix} i\partial_t & \vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) + \frac{q}{r} \\ -\vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) + \frac{q}{r} & -i\partial_t \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}.$$

$$\phi_{jm}^{(1)} = \begin{bmatrix} \left(\frac{j+m}{2j}\right)^{1/2} Y_{j-1/2m-1/2} \\ \left(\frac{j-m}{2j}\right)^{1/2} Y_{j-1/2m+1/2} \end{bmatrix}, \phi_{jm}^{(2)} = \begin{bmatrix} -\left(\frac{j-m+1}{2j+2}\right)^{1/2} Y_{j+1/2m-1/2} \\ \left(\frac{j+m+1}{2j+2}\right)^{1/2} Y_{j+1/2m+1/2} \end{bmatrix},$$

$$\mathbf{L}^2 \phi_{jm}^{(1)} = \left(j - \frac{1}{2}\right) \left(j + \frac{1}{2}\right) \phi_{jm}^{(1)}, \mathbf{L}^2 \phi_{jm}^{(2)} = \left(j + \frac{1}{2}\right) \left(j + \frac{3}{2}\right) \phi_{jm}^{(2)},$$

$$\mathbf{n} \cdot \mathbf{S} \chi_{jm}^{(1)} = -\chi_{jm}^{(1)}, \mathbf{n} \cdot \mathbf{S} \chi_{jm}^{(2)} = \chi_{jm}^{(2)},$$

$$\begin{aligned} \chi_{jm}^{(1)} &= c \phi_{jm}^{(1)} + s \phi_{jm}^{(2)} \\ \chi_{jm}^{(2)} &= -s \phi_{jm}^{(1)} + c \phi_{jm}^{(2)}. \end{aligned}$$

$$c \equiv \cos \varphi_0 = \frac{1}{\sqrt{2}} \frac{\left(j + \frac{1}{2} + q\right)^{1/2}}{\left(j + \frac{1}{2}\right)^{1/2}}$$

$$s \equiv \sin \varphi_0 = \frac{1}{\sqrt{2}} \frac{\left(j + \frac{1}{2} - q\right)^{1/2}}{\left(j + \frac{1}{2}\right)^{1/2}}$$

$$\begin{aligned} \vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) f(r) \chi_{jm}^{(1)} &= -(\partial_r + r^{-1}) f(r) \chi_{jm}^{(1)} + \mu r^{-1} f(r) \chi_{jm}^{(2)} \\ \vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) g(r) \chi_{jm}^{(2)} &= +(\partial_r + r^{-1}) g(r) \chi_{jm}^{(2)} - \mu r^{-1} g(r) \chi_{jm}^{(1)} \end{aligned}$$

$$\mu = \left(\left(j + \frac{1}{2}\right)^2 - q^2 \right)^{1/2}$$

$$\vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) f(r) \chi_m = (\partial_r + r^{-1}) f(r) \chi_m,$$

$$\begin{pmatrix} E - \lambda & \vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) + \frac{q}{r} \\ -\vec{\sigma} \cdot (\vec{\partial} - i\vec{A}) + \frac{q}{r} & -E - \lambda \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \begin{pmatrix} f_-(r) \chi_{jm}^{(1)} + f_+(r) \chi_{jm}^{(2)} \\ g_-(r) \chi_{jm}^{(1)} + g_+(r) \chi_{jm}^{(2)} \end{pmatrix}$$



$$\begin{aligned}
0 &= (E - \lambda)f_- - \mu r^{-1}g_+ - (\partial_r + r^{-1})g_- + \frac{q}{r}g_- \\
0 &= (E - \lambda)f_+ + \mu r^{-1}g_- + (\partial_r + r^{-1})g_+ + \frac{q}{r}g_+ \\
0 &= -(E + \lambda)g_- + \mu r^{-1}f_+ + (\partial_r + r^{-1})f_- + \frac{q}{r}f_- \\
0 &= -(E + \lambda)g_+ - \mu r^{-1}f_- - (\partial_r + r^{-1})f_+ + \frac{q}{r}f_+
\end{aligned}$$

$$\begin{aligned}
f_+(r) &= f_{++}\mathcal{J}_+(r) + f_{+-}\mathcal{J}_-(r), \\
f_-(r) &= f_{-+}\mathcal{J}_+(r) + f_{--}\mathcal{J}_-(r), \\
g_+(r) &= g_{++}\mathcal{J}_+(r) + g_{+-}\mathcal{J}_-(r), \\
g_-(r) &= g_{-+}\mathcal{J}_+(r) + g_{--}\mathcal{J}_-(r),
\end{aligned}$$

$$\mathcal{J}_{\pm}(r) = \frac{1}{\sqrt{kr}} J_{\nu_{\pm}}(kr), k^2 = \lambda^2 - E^2$$

$$\partial_r \mathcal{J}_+(r) = -\left(J + \frac{3}{2}\right) r^{-1} \mathcal{J}_+(r) + k \mathcal{J}_-(r)$$

$$\partial_r \mathcal{J}_-(r) = \left(J - \frac{1}{2}\right) r^{-1} \mathcal{J}_-(r) - k \mathcal{J}_+(r)$$

$$\begin{aligned}
f_{+-} &= \frac{\lambda + E}{k} g_{++}, & f_{++} &= -\frac{\lambda + E}{k} g_{+-}, \\
f_{-+} &= \frac{\lambda + E}{k} g_{--}, & f_{--} &= -\frac{\lambda + E}{k} g_{-+}, \\
g_{-+} &= t g_{++}, & g_{+-} &= -t g_{--},
\end{aligned}$$

$$t \equiv \tan \varphi_0 = \frac{j + \frac{1}{2} - q}{j + \frac{1}{2} + q}$$

$$\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = e^{iEt} \begin{pmatrix} \frac{\lambda + E}{k} \{g_{--}\mathcal{J}_+(r)[c\chi^{(1)} + s\chi^{(2)}] + g_{++}\mathcal{J}_-(r)[-s\chi^{(1)} + c\chi^{(2)}]\} \\ g_{++}\mathcal{J}_+(r)[s\chi^{(1)} + c\chi^{(2)}] + g_{--}\mathcal{J}_-(r)[c\chi^{(1)} - s\chi^{(2)}] \end{pmatrix},$$

$$\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = g_{--} \begin{pmatrix} \frac{\lambda + E}{k} \mathcal{J}_+(r) \chi_+^1 \\ \mathcal{J}_-(r) \chi_-^1 \end{pmatrix} e^{iEt} + g_{++} \begin{pmatrix} \frac{\lambda + E}{k} \mathcal{J}_-(r) \chi_+^2 \\ \mathcal{J}_+(r) \chi_-^2 \end{pmatrix} e^{iEt}$$

$$\Psi_1 = \left(\frac{k}{2\lambda}\right)^{1/2} \begin{pmatrix} \sqrt{\lambda + E} \mathcal{J}_+(r) \chi_+^1 \\ \sqrt{\lambda - E} \mathcal{J}_-(r) \chi_-^1 \end{pmatrix} e^{iEt}, \Psi_2 = \left(\frac{k}{2\lambda}\right)^{1/2} \begin{pmatrix} \sqrt{\lambda + E} \mathcal{J}_-(r) \chi_+^2 \\ \sqrt{\lambda - E} \mathcal{J}_+(r) \chi_-^2 \end{pmatrix} e^{iEt}.$$

$$\Psi \sim \frac{1}{r^{3/2}} r^{\nu_{\pm}+1}$$

$$\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \begin{pmatrix} [f_+\mathcal{J}_+(r) + f_-\mathcal{J}_-(r)]\chi_m \\ [g_+\mathcal{J}_+(r) + g_-\mathcal{J}_-(r)]\chi_m \end{pmatrix} e^{iEt},$$

$$\Psi_+ = \left(\frac{k}{2\lambda}\right)^{1/2} \begin{pmatrix} \sqrt{\lambda + E} & \mathcal{J}_-(r)\chi_m \\ \sqrt{\lambda - E} & \mathcal{J}_+(r)\chi_m \end{pmatrix} e^{iEt}$$



$$\Psi_- = \left(\frac{k}{2\lambda}\right)^{1/2} \begin{pmatrix} -\sqrt{\lambda+E} & J_+(r)\chi_m \\ \sqrt{\lambda-E} & J_-(r)\chi_m \end{pmatrix} e^{iEt}$$

$$r^2 |J_-(r)|_{j=q-\frac{1}{2}}^2 \sim r^{2q}$$

$$\sum_m \phi_{jm}^{(1)}(\mathbf{n}) \phi_{jm}^{(1)}(\mathbf{n})^\dagger = \frac{j+\frac{1}{2}}{4\pi} - \frac{q}{4\pi} \mathbf{n} \cdot \vec{\sigma},$$

$$\sum_m \phi_{jm}^{(2)}(\mathbf{n}) \phi_{jm}^{(2)}(\mathbf{n})^\dagger = \frac{j+\frac{1}{2}}{4\pi} + \frac{q}{4\pi} \mathbf{n} \cdot \vec{\sigma},$$

$$\sum_m \phi_{jm}^{(1)}(\mathbf{n}) \phi_{jm}^{(2)}(\mathbf{n})^\dagger = -\frac{\left(\left(j+\frac{1}{2}\right)^2 - q^2\right)^{1/2}}{4\pi} \mathbf{n} \cdot \vec{\sigma},$$

$$\sum_m \phi_{jm}^{(2)}(\mathbf{n}) \phi_{jm}^{(1)}(\mathbf{n})^\dagger = -\frac{\left(\left(j+\frac{1}{2}\right)^2 - q^2\right)^{1/2}}{4\pi} \mathbf{n} \cdot \vec{\sigma}.$$

$$\langle \mathcal{O}(x) \rangle_T = \lim_{y \rightarrow 0} \frac{\langle T \mathcal{O}(x) \rangle}{\langle T \rangle |y|^\Delta \langle \mathcal{O}(y) \mathcal{O}(0) \rangle^{\frac{1}{2}}}$$

$$\langle \mathcal{O}(x) \rangle_T = \frac{\mathbb{C}}{(2r)^\Delta}$$

$$\langle T(C_R) \mathcal{O}(x) \rangle \stackrel{x \rightarrow \infty}{\simeq} \frac{\mathbb{C} R^\Delta}{|x|^{2\Delta}} \langle T(C_R) \rangle$$

$$u = \frac{2\pi i q}{\sqrt{\lambda}}$$

$$\mathbb{C}_L = \frac{1}{i^L \sqrt{L}} \left[x^L + \frac{(-1)^{L+1}}{x^L} \right]$$

$$\mathbb{C}_2 = \frac{1}{\sqrt{2}} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^2 \left(1 + \frac{\lambda}{8\pi^2 q^2} - \frac{\lambda^2}{128\pi^4 q^4} + \frac{\lambda^3}{1024\pi^6 q^6} + \dots \right)$$

$$\mathbb{C}_3 = \frac{1}{\sqrt{3}} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^3 \left(1 + \frac{3\lambda}{16\pi^2 q^2} \right)$$

$$\mathbb{C}_4 = \frac{1}{2} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^4 \left(1 + \frac{\lambda}{4\pi^2 q^2} + \frac{\lambda^2}{128\pi^4 q^4} - \frac{\lambda^4}{32768\pi^8 q^8} + \dots \right)$$

$$\mathbb{C}_5 = \frac{1}{\sqrt{5}} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^5 \left(1 + \frac{5\lambda}{16\pi^2 q^2} + \frac{5\lambda^2}{256\pi^4 q^4} \right)$$

$$\mathbb{C}_6 = \frac{1}{\sqrt{6}} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^6 \left(1 + \frac{3\lambda}{8\pi^2 q^2} + \frac{9\lambda^2}{256\pi^4 q^4} + \frac{\lambda^3}{2048\pi^6 q^6} - \frac{\lambda^6}{8388608\pi^{12} q^{12}} + \dots \right)$$

$$x^L + \frac{1}{x^L} = 2T_L(u), \quad x^L - \frac{1}{x^L} = 2\sqrt{u^2 - 1} U_{L-1}(u),$$



$$\mathbb{C}_L = \frac{2T_L(u)}{i^L \sqrt{L}}, (L - \text{odd})$$

$$\mathbb{C}_L = \frac{2\sqrt{u^2 - 1} U_{L-1}(u)}{i^L \sqrt{L}}, (L - \text{even})$$

$$\mathbb{C}_L = \frac{2T_L(u)}{i^L \sqrt{L}} + \tilde{\mathbb{W}}_{\text{wrappings}}.$$

$$\langle z^{1a}(x) z^{b1}(x') \rangle = \frac{\lambda \delta^{ab}}{2N} G^-(x, x').$$

$$\langle \text{tr} Z^2 \rangle_T = \frac{4\pi^2 q^2}{\sqrt{2}\lambda r^2} \left(1 + \frac{\lambda}{8\pi^2 q^2} + \dots \right),$$

$$\mathbb{C}_L = \frac{1}{\sqrt{L}} \left(\frac{4\pi q}{\sqrt{\lambda}} \right)^L \left[1 + L \cdot \frac{\lambda}{16\pi^2 q^2} + \frac{1}{2} L(L-3) \cdot \left(\frac{\lambda}{16\pi^2 q^2} \right)^2 + \dots \right].$$

$$Z = \begin{bmatrix} 1 & \varphi^\dagger \\ \varphi & 0 \end{bmatrix}$$

$$\langle \varphi_a^\dagger \varphi^b \rangle = \frac{\lambda \delta_a^b}{16\pi^2 q^2 N}$$

$$\langle \text{CPO}_L \rangle_T \equiv \frac{\mathbb{C}_L}{(2r)^L} = \frac{1}{\sqrt{L}} \left(\frac{2\pi q}{\sqrt{\lambda} r} \right)^L \langle \text{tr} Z^L \rangle$$

$$Z^{L+1} = \begin{bmatrix} P_L(\varphi^\dagger \varphi) & Q_L(\varphi^\dagger \varphi) \varphi^\dagger \\ Q_L(\varphi^\dagger \varphi) \varphi & R_L(\varphi^\dagger \varphi) \end{bmatrix}$$

$$P_{L+1}(\xi) = P_L(\xi) + \xi Q_L(\xi)$$

$$Q_{L+1}(\xi) = P_L(\xi)$$

$$Q_{L+1}(\xi) = Q_L(\xi) + \xi R_L(\xi)$$

$$R_{L+1}(\xi) = Q_L(\xi)$$

$$\xi = -\frac{1}{4u^2}, Q_L = \frac{1}{(2u)^L} \tilde{Q}_L$$

$$\langle \text{tr} Z^L \rangle = \langle P_{L-1}(\varphi^\dagger \varphi) + \varphi^\dagger \varphi R_{L-1}(\varphi^\dagger \varphi) \rangle$$

$$P_{L-1} + \xi R_{L-1} = Q_L + \xi Q_{L-2} = \frac{1}{(2u)^L} (U_L - U_{L-2}) = \frac{2T_L}{(2u)^L}$$

$$\langle (\varphi^\dagger \varphi)^n \rangle \stackrel{N \rightarrow \infty}{=} \langle \varphi^\dagger \varphi \rangle^n = \left(\frac{\lambda}{16\pi^2 q^2} \right)^n$$

$$u = \frac{2\pi i q}{\sqrt{\lambda}}$$

$$\mathbb{C}_L = \frac{2T_L(u)}{\sqrt{L}} \left(\frac{2\pi q}{\sqrt{\lambda} u} \right)^L = \frac{2T_L(u)}{i^L \sqrt{L}}$$



$$\Phi_i^{\text{cl}} = \frac{1}{x_{\perp}} \begin{pmatrix} (t_i)_{k \times k} & 0_{k \times (N-k)} \\ 0_{(N-k) \times k} & 0_{(N-k) \times (N-k)} \end{pmatrix}, i = 1, 2, 3, \Phi_i^{\text{cl}} = 0, i = 4, 5, 6,$$

$$\mathbb{C}_L = \frac{1}{i^L \sqrt{L}} \left(\sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}} x_a^L + k \delta_{L,2} + \sum_{b \in \mathbb{Z} + \frac{k-1}{2}} \frac{1}{x_b^L} \right)$$

$$x_a + \frac{1}{x_a} = 2u_a, u_a = \frac{2\pi i a}{\sqrt{\lambda}}$$

$$\mathbb{C}_L = \frac{2}{i^L \sqrt{L}} \left(\sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}} T_L(u_a) + k \delta_{L,2} + \sum_{b=\frac{k+1}{2}}^{\infty} \frac{1}{x_b^L} \right),$$

$$\mathbb{C}_L(g) = \frac{2}{\sqrt{L}} \sum_{n=0}^{\frac{L}{2}} \binom{L-n}{n} \frac{L}{L-n} \frac{B_{L-2n+1} \left(\frac{1+k}{2} \right)}{L-2n+1} \left(\frac{\lambda}{16\pi^2} \right)^{n-L/2}$$

$$+ \sum_{n=0}^{\infty} \frac{L \left[\Psi^{(L+2n-1)} \left(\frac{1+k}{2} \right) - \Psi^{(L+2n-1)} \left(\frac{1-k}{2} \right) \right]}{(-1)^n n! (L+n)!} \left(\frac{\lambda}{16\pi^2} \right)^{n+L/2}$$

$$N(L, n) = \binom{L-n}{n} \frac{L}{L-n}$$

$$\binom{L-2-(n-1)}{n-1} = \binom{L-1-n}{n-1}$$

$$\langle (z^2(x))_{aa'} \rangle = \frac{\lambda}{16\pi^2 x_{\perp}^2} \delta_{a,a'}, a, a' = 1, \dots, k$$

$$\text{tr} t_3^{L-2n} = \frac{2}{L-2n+1} B_{L-2n+1} \left(\frac{1+k}{2} \right)$$

$$\Delta_K^{1\text{-loop}} = 2 + \frac{3\lambda}{4\pi^2}$$

$$\langle KT \rangle^{1\text{-loop}} = \left(\frac{q}{r} \right)^2 + 2 \frac{\lambda}{2} [(9-d)G(x, x) + \tilde{G}(x, x)]$$

$$\langle KT \rangle^{1\text{-loop}} = \frac{q^2}{r^2} \left[1 - \frac{\lambda}{8\pi^2} \left(\frac{3}{\varepsilon} + 3 \ln(4\pi r^2) + 4 - \frac{3}{q} - \frac{1}{q^2} - 6\psi(q) - 3\gamma \right) \right]$$

$$\langle K(y)K(0) \rangle^{1\text{-loop}} = \frac{3\lambda^2}{16\pi^4 y^4} \left[1 - \frac{\lambda}{4\pi^2} \left(\frac{3}{\varepsilon} + 3 \ln(\pi y^2) + 4 + 3\gamma \right) \right]$$

$$\langle K \rangle_T^{1\text{-loop}} = \frac{4\pi^2 q^2}{\sqrt{3}\lambda r \Delta_K} \left[1 + \frac{\lambda}{4\pi^2} \left(\frac{3}{2q} + \frac{1}{2q^2} + 3\psi(q) - 3 \ln 2 + 3\gamma \right) \right]$$



$$\langle K \rangle_T \stackrel{1\text{-loop}}{=} \frac{\pi^2}{\sqrt{3}\lambda r^{\Delta_K}} \left[1 + \frac{\lambda}{4\pi^2} (5 - 9 \ln 2) \right], (q = 1/2)$$

$$\mathcal{O} = \Psi^{I_1 \dots I_L} \text{tr} \Phi_{I_1} \dots \Phi_{I_L}$$

$$\Gamma|\Psi\rangle = \frac{\lambda}{16\pi^2} \Delta^{(1)}|\Psi\rangle$$

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^L (2 - 2P_{\ell\ell+1} + K_{\ell\ell+1})$$

$$P_{JL}^{IK} = \delta_L^I \delta_J^K, K_{JL}^{IK} = \delta^{IK} \delta_{JL}$$

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi^2}{\lambda r^2} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \frac{\langle \text{Bst} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}$$

$$|\Psi\rangle = |\Psi_B^{(0)}\rangle + \lambda |\Psi_B^{(1)}\rangle + \lambda |\Psi_F^{(0)}\rangle \equiv |\Psi_B\rangle + \lambda |\Psi_F\rangle,$$

$$\begin{aligned} \langle T\mathcal{O}(x) \rangle / \langle T \rangle &= \left(\frac{q}{r} \right)^L \left\{ \langle B | \Psi_B \rangle \left(1 + \frac{\lambda}{8\pi^2 q^2} \right) \right. \\ &\quad \left. - \frac{\lambda}{16\pi^2} \left(\frac{1}{\epsilon} + \log(4\pi r^2) + 1 - \frac{1}{q} - 2\Psi(q) - \gamma \right) \langle B' | \Psi_B \rangle \right\} \end{aligned}$$

$$B'_{i_1 \dots i_L} = n_{i_1} \dots n_{i_{L-1}} \delta_{i_L i_{L-1}} n_{i_{L-2}} \dots n_{i_L}.$$

$$\frac{\lambda}{16\pi^2} \langle B' | = \langle B | \Gamma,$$

$$\begin{aligned} \langle T\mathcal{O}(x) \rangle / \langle T \rangle &= \left(\frac{q}{r} \right)^L \langle B | \Psi_B \rangle \left\{ 1 + \frac{\lambda}{8\pi^2 q^2} \right. \\ &\quad \left. - \left(\frac{1}{\epsilon} + \log(4\pi r^2) + 1 - \frac{1}{q} - 2\Psi(q) - \gamma \right) \lambda \Delta^{(1)} \right\} \end{aligned}$$

$$\langle \mathcal{O}(y)\mathcal{O}(0) \rangle = L \left(\frac{\lambda}{8\pi^2 y^2} \right)^L \langle \Psi | \Psi \rangle \left[1 - 2\lambda \Delta^{(1)} \left(\frac{1}{\epsilon} + 1 + \gamma + \log(\pi y^2) \right) \right],$$

$$\langle \Psi | \Psi \rangle = \langle \Psi_B | \Psi_B \rangle + \frac{\lambda(L-1)}{8\pi^2 L} \langle \Psi_F | \Psi_F \rangle$$

$$W(C) = \text{tr} \text{Pexp} \left[\int_C ds (i\dot{x}^\mu A_\mu + |\dot{x}| v^I \Phi_I) \right]$$

$$\langle WT \rangle \stackrel{t \rightarrow \infty}{\simeq} e^{\alpha \frac{t}{r}}$$

$$\alpha(\lambda, \varphi) = \sum_{n=0}^{\infty} \alpha^{(n)}(\varphi) \lambda^n$$



$$\langle WT \rangle = e^{\frac{q}{r} \cos \varphi} + N - 1$$

$$\alpha^{(0)} = \begin{cases} q \cos \varphi & \text{for } \varphi < \frac{\pi}{2} \\ 0 & \text{for } \varphi > \frac{\pi}{2} \end{cases}$$

$$\langle WT \rangle^{(1)} = \int_0^t dt_2 \int_0^{t_2} dt_1 e^{\frac{t-t_2+t_1}{r} q \cos \varphi} (v^I v^J \langle \Phi_I(t_1) \Phi_J(t_2) \rangle - \langle A_0(t_1) A_0(t_2) \rangle)$$

$$\langle v \Phi v \Phi \rangle - \langle A_0 A_0 \rangle = \frac{\lambda}{2} \tilde{G} \cos^2 \varphi + \frac{\lambda}{2} G (\sin^2 \varphi - 1) = \frac{\lambda}{2} G^- \cos^2 \varphi,$$

$$\alpha^{(1)} = \frac{\lambda}{2} r \cos^2 \varphi \int_0^\infty dt e^{-\alpha^{(0)} \frac{t}{r}} G^-(t)$$

$$G^-(\tau) = \frac{q}{8\pi r} \int_0^\infty dk e^{-k\tau} \left(J_{q-\frac{1}{2}}(kr)^2 - J_{q+\frac{1}{2}}(kr)^2 \right)$$

$$\alpha^{(1)} = \frac{\lambda q \cos^2 \varphi}{16\pi} \left(f_{q-\frac{1}{2}}(\alpha^{(0)}) - f_{q+\frac{1}{2}}(\alpha^{(0)}) \right),$$

$$f_\nu(a) = \int_0^\infty dk \frac{J_\nu(k)^2}{\kappa + a}$$

$$f_\nu(a) = \frac{1}{2\pi^{\frac{3}{2}}} G_{34}^{33} \left(a^2 \left| \begin{array}{c} \frac{1}{2} \frac{1}{2} \\ \frac{1}{2} \frac{1}{2} \\ 0 \frac{1}{2} \nu - \nu \end{array} \right. \right).$$

$$f_\nu(a) \stackrel{a \rightarrow 0}{\simeq} \frac{1}{2\nu} \quad (\nu > 0).$$

$$f_0(a) \stackrel{a \rightarrow 0}{\simeq} -\ln a.$$

$$f_\nu(a) \stackrel{a \rightarrow \infty}{\simeq} \frac{1}{\pi a} \left(\ln(2a) - \psi \left(\nu + \frac{1}{2} \right) \right).$$

$$\alpha \stackrel{\varphi \rightarrow \frac{\pi}{2}}{=} \cos \varphi \hat{\alpha}(\lambda \cos \varphi)$$

$$\hat{\alpha}(x) = \frac{1}{2} - \frac{x \ln x}{32\pi} + \dots$$

$$\alpha \stackrel{\varphi \rightarrow i\infty}{=} q \cos \varphi + \frac{\lambda \cos \varphi}{16\pi^2 q} + \dots$$

$$W(t) = e^{\frac{\alpha^{(0)}}{r} t} + \frac{\lambda}{2} \cos^2 \varphi \int_0^t dt_2 \int_0^{t_2} dt_1 W(t_1) G^-(t_2 - t_1) e^{\frac{\alpha^{(0)}}{r} (t-t_2)}$$



$$r e^{\frac{\alpha^{(0)}}{r}t} \frac{d}{dt} e^{-\frac{\alpha^{(0)}}{r}t} = r \frac{d}{dt} - \alpha^{(0)}.$$

$$r \frac{dW(t)}{dt} = \alpha^{(0)}W(t) + \frac{\lambda r}{2} \cos^2 \varphi \int_0^t dt_1 W(t_1) G^-(t - t_1)$$

$$\widehat{W}(s) = \int_0^\infty dt e^{-\frac{s}{r}t} W(t), \widehat{D}(s) = \frac{\lambda r}{2} \cos^2 \varphi \int_0^\infty dt e^{-\frac{s}{r}t} G^-(t)$$

$$s\widehat{W}(s) - r = \alpha^{(0)}\widehat{W}(s) + \widehat{D}(s)\widehat{W}(s)$$

$$\widehat{W}(s) = \frac{r}{s - \alpha^{(0)} - \widehat{D}(s)}$$

$$\widehat{D}(s) = \frac{\lambda q \cos^2 \varphi}{16\pi} \left(f_{q-\frac{1}{2}}(s) - f_{q+\frac{1}{2}}(s) \right)$$

$$W(t) \stackrel{t \rightarrow \infty}{\simeq} e^{\frac{\alpha}{r}t}.$$

$$\alpha = \alpha^{(0)} + \widehat{D}(\alpha)$$

$$\alpha = q \cos \varphi + \frac{\lambda q \cos^2 \varphi}{16\pi} \left(f_{q-\frac{1}{2}}(\alpha) - f_{q+\frac{1}{2}}(\alpha) \right).$$

$$\alpha = q \cos \varphi + \frac{\lambda \cos^2 \varphi}{16\pi^2 \alpha}.$$

$$\alpha = \frac{q \cos \varphi}{2} \left(1 + \sqrt{1 + \frac{\lambda}{4\pi^2 q^2}} \right).$$

$$\alpha \stackrel{\lambda \rightarrow \infty}{\simeq} \frac{\sqrt{\lambda} \cos \varphi}{4\pi},$$

$$\alpha = \frac{\sqrt{\lambda} [\mathbf{E} - (1 - m)\mathbf{K}]^2}{2\pi \sqrt{m(1 - m)}},$$

$$\varphi = \sqrt{1 - 2m\mathbf{K}}.$$

$$\alpha(0) = \frac{\sqrt{\lambda} \Gamma\left(\frac{3}{4}\right)^4}{4\pi^2}$$

$$\alpha \stackrel{\varphi \rightarrow \frac{\pi}{2}}{\simeq} \sqrt{\frac{\lambda \cos^3 \varphi}{54\pi}}.$$

$$\hat{\alpha}(x) \stackrel{x \rightarrow \infty}{\simeq} \sqrt{\frac{x}{54\pi}}.$$



$$\varphi \stackrel{m \rightarrow 1}{\simeq} \frac{i}{2} \ln \frac{16}{1-m}$$

$$\alpha \stackrel{m \rightarrow 1}{\simeq} \frac{\sqrt{\lambda}}{8\pi} \frac{4}{\sqrt{1-m}} \simeq \frac{\sqrt{\lambda}}{8\pi} e^{-i\varphi} \simeq \frac{\sqrt{\lambda} \cos \varphi}{4\pi},$$

$$G(x, x') = \frac{1}{8\pi^2 r r'} \sum_j (2j+1) P_j(\eta) Q_j(\xi)$$

$$\sum_{j=0}^{\infty} (2j+1) P_j(\eta) Q_j(\xi) = \frac{1}{\eta - \xi},$$

$$G(x, x') = \frac{1}{8\pi^2 r r'} \frac{1}{\frac{(t-t')^2 + r^2 + r'^2}{2rr'} - \mathbf{n} \cdot \mathbf{n}'} = \frac{1}{4\pi^2 (x-x')^2}$$

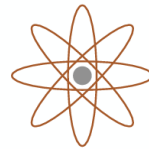
Eikonal Regime
(Point-particle description)

Born Regime
(Wave description)

$$b \gg \lambda_{\text{dB}}, \lambda_C$$

$$b \sim \lambda_{\text{dB}} \gg \lambda_C$$

Massive



Massless

$$b \gg \lambda, \lambda_C$$

$$b \sim \lambda \gg \lambda_C$$



Eikonal regime
(Point-particle description)

$$T \sim e^{\frac{i}{\hbar} \int \sqrt{(E-V)^2 - p^2} dt - 1}$$

Born regime
(Wave description)

$$T \sim \frac{V}{1 - \int GV}$$

Massive

$$V = - \frac{16\pi G(m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2)}{\mathbf{q}^2} - \frac{6\pi^2 G^2(m_1 + m_2)(m_1^2 m_2^2 - 5(p_1 \cdot p_2)^2)}{|\mathbf{q}|}$$

$m_i \gg |p|$

$$V = \frac{16\pi G m_1^2 m_2^2}{\mathbf{q}^2} + \frac{24\pi^2 G^2 m_1^2 m_2^2 (m_1 + m_2)}{|\mathbf{q}|}$$

Massless

$m_1 \rightarrow 0$
 $m_2 \rightarrow M$

$$V = \frac{32\pi G M \omega^2}{\mathbf{q}^2} + \frac{30\pi^2 G^2 M^2 \omega^2}{|\mathbf{q}|}$$

$\omega \gg |q|$

$$V = \frac{32\pi G M \omega^2}{\mathbf{q}^2} + \frac{30\pi^2 G^2 M^2 \omega^2}{|\mathbf{q}|} - \frac{G^2 M^2 \pi^2}{2} |\mathbf{q}|$$

$$p_1^\mu = (E_1, \mathbf{p}), \quad E_1 = \sqrt{|\mathbf{p}|^2 + m_1^2}$$

$$p_2^\mu = (E_2, -\mathbf{p}), \quad E_2 = \sqrt{|\mathbf{p}|^2 + m_2^2}$$

$$p_3^\mu = (E_1, \mathbf{p}') = (E_1, \mathbf{p} + \mathbf{q}),$$

$$p_4^\mu = (E_2, -\mathbf{p}') = (E_2, -\mathbf{p} - \mathbf{q}).$$

$$s = (p_1 + p_2)^2, \quad t = (p_3 - p_1)^2.$$

$$|\mathbf{p}|^2 = \frac{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}{4s}, \quad t = -|\mathbf{p}' - \mathbf{p}|^2 = -|\mathbf{q}|^2$$

$$|\mathbf{p}|^2 = \frac{E(E + 2m_1)(E + 2m_2)(E + 2m_1 + 2m_2)}{4(E + m_1 + m_2)^2}$$

$$E \equiv \sqrt{s} - (m_1 + m_2)$$

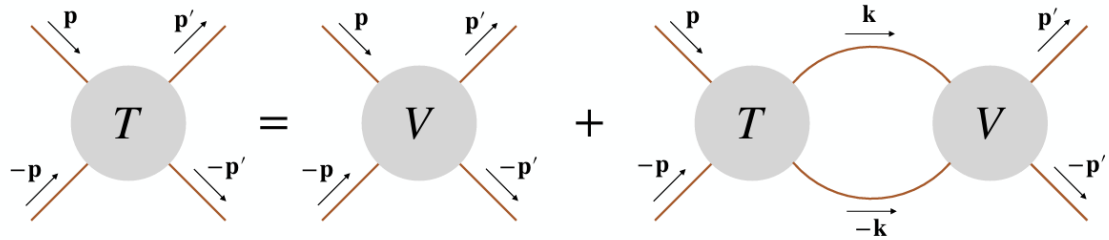
$$|\mathbf{p}|^2 = 2\mu E,$$

$$\text{with } \mu = \frac{m_1 m_2}{m_1 + m_2},$$

$$|\mathbf{p}|^2 = E^2 + 2m_1 E = E_1^2 - m_1^2$$

$$|\mathbf{p}|^2 = E^2 = \omega^2,$$





$$T(\mathbf{p}, \mathbf{p}') = V(\mathbf{p}, \mathbf{p}') + \int d^3\mathbf{k} T(\mathbf{p}, \mathbf{k}) G(\mathbf{p}, \mathbf{k}) V(\mathbf{k}, \mathbf{p}'),$$

$$\begin{aligned} \text{Im}T(\mathbf{p}, \mathbf{p}')|_{\text{zPC}} &= \frac{1}{8\pi^2} \int d^4k \delta^+(k^2 - m_1^2) \delta^+((k - p_1 - p_2)^2 - m_2^2) T(p, k) T^*(k, p') \\ &= \frac{1}{16\pi^2 \sqrt{s}} \int d^3\mathbf{k} \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) T(\mathbf{p}, \mathbf{k}) T^*(\mathbf{k}, \mathbf{p}') \end{aligned}$$

$$\delta^+(k^2 - m_1^2) \equiv \Theta(k^0) \delta(k^2 - m_1^2)$$

$$T = V + VGV + VGVGV + \dots = V \frac{1}{1 - GV},$$

$$T^* = V \frac{1}{1 - G^*V}$$

$$TT^* = V \frac{1}{1 - GV} V \frac{1}{1 - G^*V}$$

$$T - T^* = V \left[\frac{1}{1 - GV} - \frac{1}{1 - G^*V} \right] = V \frac{1}{1 - GV} (G - G^*) V \frac{1}{1 - G^*V}$$

$$\text{Im}G(\mathbf{p}, \mathbf{k}) = \frac{\delta(|\mathbf{k}|^2 - |\mathbf{p}|^2)}{16\pi^2 \sqrt{s}}$$

$$G(\mathbf{p}, \mathbf{k}) = \frac{1}{(2\pi)^3} \frac{1}{2\sqrt{s}} \frac{1}{|\mathbf{k}|^2 - |\mathbf{p}|^2 - i\epsilon},$$

$$G(\mathbf{p}, \mathbf{k}) \rightarrow \frac{1}{(2\pi)^3} \frac{1}{4m_1 m_2} \frac{1}{\frac{|\mathbf{k}|^2}{2\mu} - E - i\epsilon}$$

$$G(\mathbf{p}, \mathbf{k}) \rightarrow \frac{1}{(2\pi)^3} \frac{1}{2M} \frac{1}{|\mathbf{k}|^2 - \omega^2 - i\epsilon},$$

$$(\nabla^2 + |\mathbf{p}|^2)G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')$$

$$G(\mathbf{x}, \mathbf{x}') \equiv -\frac{1}{4\pi} \frac{e^{i|\mathbf{p}||\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} = -2\sqrt{s} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{p}, \mathbf{k})$$

$$(\nabla^2 + |\mathbf{p}|^2)\Psi(\mathbf{x}) = V(\mathbf{x})\Psi(\mathbf{x})$$



$$V(\mathbf{p}, \mathbf{p}') = -2\sqrt{s} \int d^3\mathbf{x} e^{-i\mathbf{p}'\cdot\mathbf{x}} V(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\Psi(\mathbf{p}') = G(\mathbf{p}, \mathbf{p}') \int d^3\mathbf{k} V(\mathbf{k}, \mathbf{p}') \Psi(\mathbf{k})$$

$$\Psi(\mathbf{k}) \equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi(\mathbf{x})$$

$$\Psi_{\mathbf{p}}^{(0)}(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\Psi_{\mathbf{p}}(\mathbf{x}) = \Psi_{\mathbf{p}}^{(0)}(\mathbf{x}) + \int d^3\mathbf{x}' G(\mathbf{x}, \mathbf{x}') V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}')$$

$$T(\mathbf{p}, \mathbf{p}') = -2\sqrt{s} \int d^3\mathbf{x}' e^{-i\mathbf{p}'\cdot\mathbf{x}'} V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}')$$

$$|\mathbf{p}|^2 = |\mathbf{p}(\mathbf{x})|^2 + V(\mathbf{x})$$

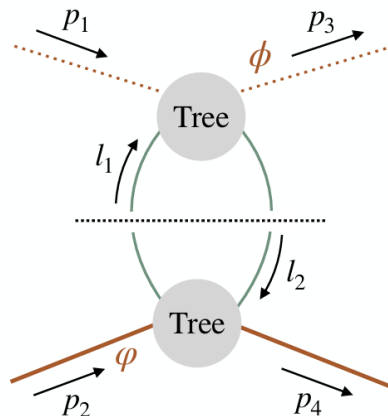
$$|\mathbf{p}| \equiv |\mathbf{p}(\mathbf{x} \rightarrow \infty)|$$

$$I_r = \int |\mathbf{p}(r)| dr = \int \sqrt{|\mathbf{p}|^2 - V(r) - \frac{I^2}{r^2}} dr, \text{ where } r \equiv |\mathbf{x}|$$

$$(\nabla^2 + \omega^2)\Psi(\mathbf{x}) = V(\mathbf{x})\Psi(\mathbf{x})$$

$$T^{(0)}(\mathbf{p}, \mathbf{p}') = \frac{32\pi G M^2 \omega^2}{\mathbf{q}^2}$$

$$iT^{(1)}(\mathbf{p}, \mathbf{p}')|_{\text{cut-t}} = \frac{\mu^{2\epsilon}}{2} \int d\text{LIPS}(l_1, -l_2) \sum_{\lambda_1, \lambda_2} \mathcal{M}(p_1, l_1^{\lambda_1}, -p_3, -l_2^{\lambda_2}) \times \mathcal{M}(p_2, -l_1^{\lambda_1}, -p_4, l_2^{\lambda_2})^\dagger$$



$$dLIPS(l_1, -l_2) = d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^4(p_1 + p_3 + l_1 - l_2)$$

$$\mathcal{M}(1_\phi, 2^+, 3_\phi, 4^+) = \mathcal{M}(1_\phi, 2^-, 3_\phi, 4^-) = 0$$

$$\mathcal{M}(1_\phi, 2^-, 3_\phi, 4^+) = 8\pi G \frac{\langle 214 \rangle^4}{s(t - M^2)(u - M^2)}$$

$$\mathcal{M}(1_\phi, 2^-, 3_\phi, 4^+) = 8\pi G \frac{\langle 214 \rangle^4}{stu}$$

$$iT^{(1)}(\mathbf{p}, \mathbf{p}') = c_\square \mathcal{J}_\square + c_X \mathcal{J}_X + c_\Delta \mathcal{J}_\Delta,$$

$$c_\square = c_X = 1024\pi^2 G^2 \omega^4 M^4, \\ c_\Delta = -16\pi^2 G^2 M^4 (60\omega^2 - \mathbf{q}^2).$$

$$\mathcal{J}_\Delta = -\frac{i}{32M|\mathbf{q}|} + \mathcal{O}(1/M^2)$$

$$\theta = \frac{4GM}{b} + \frac{15\pi G^2 M^2}{4b^2}$$

$$T(\mathbf{p}, \mathbf{p}') = (V^{(0)}(\mathbf{p}, \mathbf{p}') + V^{(1)}(\mathbf{p}, \mathbf{p}')) + \int d^3 \mathbf{k} V^{(0)}(\mathbf{p}, \mathbf{k}) G(\mathbf{p}, \mathbf{k}) V^{(0)}(\mathbf{k}, \mathbf{p}')$$

$$V^{(0)}(\mathbf{p}, \mathbf{p}') = \frac{32\pi G M^2 \omega^2}{\mathbf{q}^2}$$

$$V^{(1)}(\mathbf{p}, \mathbf{p}') = \frac{30\pi^2 G^2 M^3 \omega^2}{|\mathbf{q}|} - \frac{G^2 M^3 \pi^2}{2} |\mathbf{q}|$$

$$S = \int d^4 x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0.$$

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -B(r) & 0 \\ 0 & 0 & 0 & -B(r) \end{pmatrix}, A(r) = \frac{\left(1 - \frac{GM}{2r}\right)^2}{\left(1 + \frac{GM}{2r}\right)^2}, B(r) = \left(1 + \frac{GM}{2r}\right)^4$$



$$\nabla^2 \phi(x) = \left(\frac{B}{A} \partial_t^2 - \frac{\partial_r(AB)}{2AB} \partial_r \right) \phi(x)$$

$$\nabla^2 = \delta^{ij} \partial_i \partial_j$$

$$\partial_r = \frac{1}{r} \mathbf{x} \cdot \nabla$$

$$\phi(x) = e^{-i\omega t} \phi(\mathbf{x}),$$

$$(\nabla^2 + \omega^2) \phi(\mathbf{x}) = \left(\left(1 - \frac{B}{A}\right) \omega^2 - \frac{\partial_r(AB)}{2AB} \partial_r \right) \phi(\mathbf{x})$$

$$V = \left(1 - \frac{\left(1 + \frac{GM}{2r}\right)^6}{\left(1 - \frac{GM}{2r}\right)^2} \right) \omega^2 + \frac{2G^2 M^2}{G^2 M^2 r - 4r^3} \partial_r,$$

$$V = - \left(\frac{4MG}{r} + \frac{15G^2 M^2}{2r^2} \right) \omega^2 - \frac{G^2 M^2}{2r^3} \partial_r + \mathcal{O}(G^3).$$

$$\begin{aligned} V(\mathbf{p}, \mathbf{p}') &= -2M \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(r) e^{i\mathbf{p} \cdot \mathbf{x}} \\ &= \frac{32\pi G M^2 \omega^2}{\mathbf{q}^2} + \frac{30\pi^2 G^2 M^3 \omega^2}{|\mathbf{q}|} - \frac{G^2 M^3 \pi^2}{2} |\mathbf{q}| \end{aligned}$$

$$g_{\mu\nu} = \eta_{\mu\nu} - \Phi l_\mu l_\nu, g^{\mu\nu} = \eta^{\mu\nu} + \Phi l^\mu l^\nu$$

$$\Phi = \frac{2GM r^3}{r^4 + (\mathbf{a} \cdot \mathbf{x})^2}$$

$$l_\mu = \left(1, \frac{1}{r^2 + a^2} \left[r x_i + \frac{(\mathbf{a} \cdot \mathbf{x})}{r} a_i + \epsilon_{ijk} x_j a_k \right] \right)$$

$$a^2 + r^2 = |\mathbf{x}|^2 + \frac{(\mathbf{a} \cdot \mathbf{x})^2}{r^2}$$

$$\mathcal{L} = \nabla^\mu \phi(x) \nabla_\nu \phi(x) = \partial^\mu \phi(x) \partial_\mu \phi(x) + \Phi l^\mu l^\nu \partial_\mu \phi(x) \partial_\nu \phi(x)$$

$$\partial_\mu \partial^\mu \phi(x) + \partial_\mu (\Phi l^\mu l^\nu \partial_\nu \phi(x)) = 0$$

$$(\nabla^2 + \omega^2) \phi(\mathbf{x}) = \left[-\Phi \omega^2 + i\omega [\partial_i (\Phi l_i) + 2\Phi l_i \partial_i] + \partial_i (\Phi l_i l_j \partial_j) \right] \phi(\mathbf{x})$$

$$\phi(x) = e^{-i\omega t} \phi(\mathbf{x})$$



$$\begin{aligned}
V(\mathbf{x}) &= -\Phi\omega^2 + i\omega[\partial_i(\Phi l_i) + 2\Phi l_i \partial_i] + \partial_i(\Phi l_i l_j \partial_j) \\
&= -\frac{2GM\omega^2 r^3}{r^4 + (\mathbf{a} \cdot \mathbf{x})^2} \\
&\quad + \frac{2iGM\omega r^2}{r^4 + (\mathbf{a} \cdot \mathbf{x})^2} \left[1 + \frac{2((\mathbf{a} \cdot \mathbf{x})a_i + r^2 x_i)\partial_i}{r^2 + a^2} - \frac{2r\epsilon_{ijk}a_j x_k \partial_i}{r^2 + a^2} \right] \\
&\quad + \frac{2GMr}{(r^2 + a^2)(r^4 + (\mathbf{a} \cdot \mathbf{x})^2)} [((\mathbf{a} \cdot \mathbf{x})a_i + r^2 x_i)\partial_i - r\epsilon_{ijk}a_j x_k \partial_i] \\
&\quad - \frac{2GMr^3}{(r^2 + a^2)^2(r^4 + (\mathbf{a} \cdot \mathbf{x})^2)} \left[\left(r^2 + a^2 - \frac{2(\mathbf{a} \cdot \mathbf{x})^2}{r^2} \right) a_i a_j \partial_i \partial_j \right. \\
&\quad \left. + (a^2 - r^2)x_i x_j \partial_i \partial_j - (r^2 + a^2) \left(a^2 - \frac{(\mathbf{a} \cdot \mathbf{x})^2}{r^2} \right) \partial_i \partial_i \right. \\
&\quad \left. - 4(\mathbf{a} \cdot \mathbf{x})a_i x_j \partial_i \partial_j + 2 \left(r x_i \partial_i + \frac{(\mathbf{a} \cdot \mathbf{x})}{r} a_i \partial_i \right) \epsilon_{jkl} a_k x_l \partial_j \right]
\end{aligned}$$

$$\begin{aligned}
T(\mathbf{p}, \mathbf{p}') &= \frac{32\pi GM^2}{\mathbf{q}^2} \left[\omega^2 \cosh(\mathbf{a} \cdot \mathbf{q}) - i\omega \epsilon_{ijl} q_i a_j p_l \frac{\sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})} \right] \\
&\quad + 8\pi GM [a^2 \mathbf{q}^2 - 2(\mathbf{a} \cdot \mathbf{q})(\mathbf{a} \cdot \mathbf{p}')] \frac{(\mathbf{a} \cdot \mathbf{q}) \cosh(\mathbf{a} \cdot \mathbf{q}) - \sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})^3}
\end{aligned}$$

$$-\partial_\mu \partial^\mu \phi(x) = \partial_\mu (\Phi l^\mu l^\nu \partial_\nu \phi(x)) - \int d^4 y \partial_\mu [\delta^4(x-y)(\partial^\mu + \Phi l^\mu l^\nu \partial_\nu) \phi(y)]$$

$$\begin{aligned}
a^2 \int d^3 \mathbf{x} \partial_i \left(\frac{x_i x_j}{r^5} e^{-i\mathbf{p}' \cdot \mathbf{x}} \partial_j (e^{i\mathbf{p} \cdot \mathbf{x}}) \right) &= i a^2 p_j \int_{S^2} d^2 x r^2 n_i \frac{x_i x_j}{r^5} e^{-i\mathbf{q} \cdot \mathbf{x}} \\
&= a^2 p_j q_i \int d^2 x \frac{x_i x_j}{r} = \frac{4\pi a^2}{3} \mathbf{p} \cdot \mathbf{q}
\end{aligned}$$

$$\phi(x) = \int \frac{d^4 k}{(2\pi)^4} \phi(k) e^{-ik \cdot x}$$

$$\begin{aligned}
\phi(\mathbf{p}') &= G(\mathbf{p}', \omega) \left[16\pi^3 M \int d^4 x \partial_\mu (e^{i\mathbf{p}_3 \cdot \mathbf{x}}) \Phi l^\mu l^\nu \partial_\nu \phi(x) \right] \\
&= G(\mathbf{p}', \omega) \int d^3 \mathbf{k} \phi(\mathbf{k}) \left[2M p_{3\mu} k_\nu \int d^3 \mathbf{x} \Phi l^\mu l^\nu e^{i(\mathbf{k}-\mathbf{p}') \cdot \mathbf{x}} \right]
\end{aligned}$$

$$V_\phi(\mathbf{p}, \mathbf{p}') = 2M p_{3\mu} p_{1\nu} \int d^3 \mathbf{x} \Phi l^\mu l^\nu e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}}$$

$$\Pi_T^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu, \Pi_L^{\mu\nu} = u^\mu u^\nu, u_\mu = (1, 0, 0, 0)$$

$$l^\mu = u^\mu l_0 + \Pi_T^{\mu i} l_i$$

$$\begin{aligned}
\int d^3 \mathbf{x} \Phi l^\mu l^\nu e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} &= u^\mu u^\nu \int d^3 \mathbf{x} \Phi e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} + u^\mu \Pi_T^{\nu j} \int d^3 \mathbf{x} \Phi l_j e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} \\
&\quad + \Pi_T^{\mu i} u^\nu \int d^3 \mathbf{x} \Phi l_i e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} + \Pi_T^{\mu i} \Pi_T^{\nu j} \int d^3 \mathbf{x} \Phi l_i l_j e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}}
\end{aligned}$$



$$T_\phi(\mathbf{p}, \mathbf{p}') = \frac{32\pi GM^2}{\mathbf{q}^2} \left[\omega^2 \cosh(\mathbf{a} \cdot \mathbf{q}) - i\omega \epsilon_{ijl} q_i a_j p_l \frac{\sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})} \right] \\ + 16\pi GM^2 [(\mathbf{a} \cdot \mathbf{p}')(\mathbf{a} \cdot \mathbf{p}) - a^2 \omega^2] \frac{(\mathbf{a} \cdot \mathbf{q}) \cosh(\mathbf{a} \cdot \mathbf{q}) - \sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})^3}$$

$$T_\phi(p_1, p_3) = -\frac{32\pi GM^2}{q^2} [(u \cdot p_1)(u \cdot p_3) \cosh(a \cdot q) \\ + \frac{i}{2}(u \cdot p_1 + u \cdot p_3) \epsilon_{\alpha\beta\gamma\delta} a^\alpha p_3^\beta p_1^\gamma u^\delta \frac{\sinh(a \cdot q)}{(a \cdot q)}] \\ + 16\pi GM^2 [(a \cdot p_1)(a \cdot p_3) - a^2(u \cdot p_1)(u \cdot p_3)] \frac{(a \cdot q) \cosh(a \cdot q) - \sinh(a \cdot q)}{(a \cdot q)^3}$$

$$T_\phi(p_1, p_3) \xrightarrow{M \gg \omega \gg |\mathbf{q}|} \frac{32\pi GM^2}{\mathbf{q}^2} \left[\omega^2 \cosh(\mathbf{a} \cdot \mathbf{q}) - i\omega \epsilon_{ijl} q_i a_j p_l \frac{\sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})} \right],$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \\ = -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{2} \Phi l^\mu l^\alpha \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \mathcal{O}(\Phi^2)$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$-\eta^{\mu\alpha} \eta^{\nu\beta} \partial_\mu F_{\alpha\beta} = \partial_\mu [\Phi(\eta^{\nu\beta} l^\mu l^\alpha + \eta^{\alpha\mu} l^\beta l^\nu) F_{\alpha\beta}] \\ - \int d^4 y \partial_\mu [\delta^4(x-y) \Phi(\eta^{\nu\beta} l^\mu l^\alpha + \eta^{\alpha\mu} l^\beta l^\nu) F_{\alpha\beta}] + \mathcal{O}(\Phi^2)$$

$$-16\pi^3 M \int d^4 x e^{ip_3 \cdot x}$$

$$A_\nu(\mathbf{p}') = G(\mathbf{p}', \omega) \left[16\pi^3 M \int d^4 x \partial_\mu (e^{ip_3 \cdot x}) \Phi(\eta^{\nu\beta} l^\mu l^\alpha + \eta^{\alpha\mu} l^\beta l^\nu) F_{\alpha\beta}(x) \right] \\ = G(\mathbf{p}', \omega) \int d^3 \mathbf{k} A_\mu(\mathbf{k}) \left[2M (k_\alpha \delta_\beta^\mu - k_\beta \delta_\alpha^\mu) (p_{3\sigma} \eta^{\nu\beta} \delta_\rho^\alpha + p_3^\alpha \delta_\sigma^\beta \delta_\rho^\nu) \int d^3 \mathbf{x} \Phi l^\sigma l^\rho e^{i(\mathbf{k}-\mathbf{p}') \cdot \mathbf{x}} \right]$$

$$V_\gamma^{\mu\nu}(\mathbf{p}, \mathbf{p}') = 2M (p_{1\alpha} \delta_\beta^\mu - p_{1\beta} \delta_\alpha^\mu) (p_{3\sigma} \eta^{\nu\beta} \delta_\rho^\alpha + p_3^\alpha \delta_\sigma^\beta \delta_\rho^\nu) \int d^3 \mathbf{x} \Phi l^\sigma l^\rho e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}}$$

$$T_\gamma(\mathbf{p}, \mathbf{p}') = \epsilon_{3\mu}^* V_\gamma^{\mu\nu}(\mathbf{p}, \mathbf{p}') \epsilon_{1\nu} \\ = 2M (p_{3\alpha} \epsilon_{3\beta}^* - p_{3\beta} \epsilon_{3\alpha}^*) (p_{1\sigma} \epsilon_1^\beta \delta_\rho^\alpha + p_1^\alpha \epsilon_{1\rho} \delta_\sigma^\beta) \int d^3 \mathbf{x} \Phi l^\sigma l^\rho e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}}$$



$$\begin{aligned}
T_{\gamma}(p_1, p_3) = & -\frac{32\pi GM^2}{q^2} \cosh(a \cdot q) \left[\left((u \cdot p_1)(u \cdot p_3) + \frac{q^2}{4} \right) \epsilon_1 \cdot \epsilon_3^* \right. \\
& + (u \cdot p_1)(q^\nu u^\mu - q^\mu u^\nu) \epsilon_{1\mu} \epsilon_{3\nu}^* - \frac{1}{2} (q^\mu q^\nu + q^2 u^\mu u^\nu) \epsilon_{1\mu} \epsilon_{3\nu}^* \left. \right] \\
& - \frac{32i\pi GM^2 \sinh(a \cdot q)}{q^2 (a \cdot q)} \left[\epsilon_{\alpha\beta\mu\nu} u^\alpha a^\beta k^\mu q^\nu \left((u \cdot p_1) \eta^{\mu\nu} - \frac{1}{2} (u^\mu p_1^\nu + u^\nu p_3^\mu) \right) \epsilon_{1\mu} \epsilon_{3\nu}^* \right. \\
& + \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} u^\alpha a^\beta q^\mu \left(\epsilon_1^\nu \left((\epsilon_3^* \cdot p_1) + \frac{q^2}{2} (\epsilon_3^* \cdot u) \right) + \epsilon_3^{\nu} \left((\epsilon_1 \cdot p_3) + \frac{q^2}{2} (\epsilon_1 \cdot u) \right) \right) \left. \right] \\
& - 16\pi GM^2 \frac{(a \cdot q) \cosh(a \cdot q) - \sinh(a \cdot q)}{(a \cdot q)^3} \left[(a \cdot p_1) p_3^\mu a^\nu + (a \cdot p_3) a^\mu p_1^\nu \right. \\
& - \left. \left((a \cdot p_1)(a \cdot p_3) + a^2 (u \cdot p_1)(u \cdot p_3) + \frac{a^2 q^2}{2} \right) \eta^{\mu\nu} \right. \\
& + \left. a^2 (u \cdot p_1) (u^\mu p_1^\nu + u^\nu p_3^\mu) + \frac{q^2}{2} (a^\mu a^\nu + a^2 u^\mu u^\nu) + q^2 q^\mu q^\nu \right] \epsilon_{1\mu} \epsilon_{3\nu}^*
\end{aligned}$$

$$T_{\gamma}(p_1, p_3) \xrightarrow{M \gg \omega \gg |q|} = \frac{32\pi GM^2}{q^2} \left[\omega^2 \cosh(\mathbf{a} \cdot \mathbf{q}) - i\omega \epsilon_{ijl} q_i a_j p_l \frac{\sinh(\mathbf{a} \cdot \mathbf{q})}{(\mathbf{a} \cdot \mathbf{q})} \right],$$

$$\begin{aligned}
& \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} G(\mathbf{p}, \mathbf{k}) \\
& = \frac{1}{16\pi^3 \sqrt{s}} \int \frac{\mathbf{k}^2 d|\mathbf{k}|}{(2\pi)^2} \frac{1}{|\mathbf{k}|^2 - |\mathbf{p}|^2 - i\epsilon} \int_{-1}^1 d\mu e^{i|\mathbf{k}||\mathbf{x}|\mu} \\
& = \frac{1}{16\pi^3 i\sqrt{s}|\mathbf{x}|} \int_0^\infty \frac{|\mathbf{k}| d|\mathbf{k}|}{(2\pi)^2} \frac{(e^{i|\mathbf{k}||\mathbf{x}|} - e^{-i|\mathbf{k}||\mathbf{x}|})}{|\mathbf{k}|^2 - |\mathbf{p}|^2 - i\epsilon} \\
& = \frac{1}{4\sqrt{s}} \frac{e^{i|\mathbf{p}||\mathbf{x}|}}{(2\pi)^4 |\mathbf{x}|}
\end{aligned}$$

$$(\nabla^2 + |\mathbf{p}|^2)G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')$$

$$\begin{aligned}
\Psi_{\mathbf{p}}(\mathbf{x}) & = \Psi_{\mathbf{p}}^{(0)}(\mathbf{x}) + \int d^3 \mathbf{x}' G(\mathbf{x}, \mathbf{x}') V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}') \\
& = e^{i\mathbf{p} \cdot \mathbf{x}} - \frac{1}{4\pi} \int d^3 \mathbf{x}' \frac{e^{i|\mathbf{p}||\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}')
\end{aligned}$$

$$\Psi_{\mathbf{p}}^{(0)}(\mathbf{x}) = e^{i\mathbf{p} \cdot \mathbf{x}}$$

$$|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|} = |\mathbf{x}| - \hat{\mathbf{x}} \cdot \mathbf{x}'$$

$$\Psi_{\mathbf{p}}(\mathbf{x} \rightarrow \infty) = e^{i\mathbf{p} \cdot \mathbf{x}} - \frac{1}{4\pi} \frac{e^{i|\mathbf{p}||\mathbf{x}|}}{|\mathbf{x}|} \int d^3 \mathbf{x}' e^{-i\mathbf{p}' \cdot \mathbf{x}'} V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}')$$

$$T(\mathbf{p}, \mathbf{p}') = N \int d^3 \mathbf{x}' e^{-i\mathbf{p}' \cdot \mathbf{x}'} V(\mathbf{x}') \Psi_{\mathbf{p}}(\mathbf{x}')$$

$$N \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x})$$



$$\begin{aligned}
T(\mathbf{p}, \mathbf{p}') &= N \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x}) e^{i\mathbf{p} \cdot \mathbf{x}} \\
+ \int d^3 \mathbf{k} G(\mathbf{p}, \mathbf{k}) \left[-2\sqrt{s} \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} \right] \left[N \int d^3 \mathbf{x}' e^{-i\mathbf{k} \cdot \mathbf{x}'} \Psi_{\mathbf{p}}(\mathbf{x}') \right] \\
&= N \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x}) e^{i\mathbf{p} \cdot \mathbf{x}} \\
+ \int d^3 \mathbf{k} G(\mathbf{p}, \mathbf{k}) \left[-2\sqrt{s} \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} \right] T(\mathbf{p}, \mathbf{k})
\end{aligned}$$

$$V(\mathbf{p}, \mathbf{p}') = -2\sqrt{s} \int d^3 \mathbf{x} e^{-i\mathbf{p}' \cdot \mathbf{x}} V(\mathbf{x}) e^{i\mathbf{p} \cdot \mathbf{x}}$$

$$T(\mathbf{p}, \mathbf{p}') = V(\mathbf{p}, \mathbf{p}') + \int d^3 \mathbf{k} T(\mathbf{p}, \mathbf{k}) G(\mathbf{p}, \mathbf{k}) V(\mathbf{k}, \mathbf{p}')$$

$$J_{\square} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l-p_1)^2} \frac{1}{(l-p_3)^2} \frac{1}{l^2} \frac{1}{(p_1+p_2-l)^2 - M^2}$$

$$J_{\times} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l-p_1)^2} \frac{1}{(l-p_3)^2} \frac{1}{l^2} \frac{1}{(p_2-p_3+l)^2 - M^2}$$

$$X_1 = |p - l_1|, X_2 = |k - l_1|, Y_1 = l_1^2 - \omega^2$$

$$\int \frac{dl_0}{2\pi} (\cdot) = \frac{i}{2} \left[\sum_{l_0^2 \in H^+} \text{Res}(\cdot) - \sum_{l_0^2 \in H^-} \text{Res}(\cdot) \right]$$

$$J_{\square} = \int \frac{d^3 l_1}{(2\pi)^3} \int \frac{dl_0}{2\pi} \frac{1}{l_0^2 - X_1^2 + i0} \frac{1}{l_0^2 - X_2^2 + i0} \frac{1}{l_0^2 + 2\omega l_0 - Y_1 + i0} \frac{1}{l_0^2 - 2\sqrt{M^2 + \omega^2} l_0 - Y_1 + i0}$$

$$c_{\Delta} J_{\Delta} \sim M^4/M \sim M^3 \text{ and } c_{\square} = c_{\times} \sim M^4$$

$$J_{\square} + J_{\times} = \frac{i}{2M} \int \frac{d^3 l_i}{(2\pi)^3} \frac{1}{X_1^2 Y X_2^2} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

$$f(r) \equiv 1 - 2GM/r$$

$$V(r) = -\left(1 - \frac{1}{f(r)}\right) \hat{E}^2 + (f(r) - 1) \partial_r^2 + \left(f'(r) + 2 \frac{f(r) - 1}{r}\right) \partial_r$$

$$\begin{aligned}
V(r) &= \frac{2GM}{r} \hat{E}^2 - \frac{2GM}{r^2} \partial_r - \frac{2GM}{r} \partial_r^2 \\
&= \frac{2GM}{r} \hat{E}^2 - \frac{2GM}{r^3} (\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{2GM}{r} \left(\frac{\mathbf{r} \cdot \partial_{\mathbf{r}}}{r}\right)^2 + \mathcal{O}(G^2)
\end{aligned}$$

$$\left(\frac{\mathbf{r} \cdot \partial_{\mathbf{r}}}{r}\right)^2 \phi(r) = \left(\frac{r_i \partial_i}{r}\right) \left(\frac{r_j \partial_j}{r}\right) \phi(r) = \left(\frac{r_i r_j}{r^2}\right) \partial_i \partial_j \phi(r)$$



$$\int d^d \mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} V(r) \phi(r)$$

$$V(r) = \int \frac{d^d \mathbf{q}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} V(\mathbf{q}), \phi(\mathbf{r}) = \int \frac{d^d \mathbf{q}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} \phi(\mathbf{q})$$

$$\int d^3 \mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} V(r) \phi(r) = 2GM\omega^2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \phi(\mathbf{k}) \int d^3 \mathbf{r} e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{r}} \frac{1}{r}$$

$$= \frac{2GM\omega^2}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) \frac{1}{|\mathbf{k}-\mathbf{p}|^2}$$

$$\int \frac{d^3 \mathbf{r}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{1}{r^{2\nu}} = \frac{2^{-2\nu} \Gamma\left(\frac{3}{2}-\nu\right)}{\pi^{3/2} \Gamma(\nu)} \frac{1}{|\mathbf{q}|^{3-2\nu}}$$

$$-2GM \int d^3 \mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \frac{\mathbf{r}\cdot\partial_{\mathbf{r}}\phi(r)}{r^3} = -2GM \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \phi(\mathbf{k}) (\mathbf{k}\cdot\partial_{\mathbf{k}}) \int d^3 \mathbf{r} \frac{e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{r}}}{r^3}$$

$$= \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) (\mathbf{k}\cdot\partial_{\mathbf{k}} \log |\mathbf{k}-\mathbf{p}|) = \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) \frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{p})}{|\mathbf{k}-\mathbf{p}|^2}$$

$$-2GM \int d^3 \mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \frac{r_i r_j \partial_i \partial_j \phi(\mathbf{r})}{r^3} = -2GM \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \phi(\mathbf{k}) k_i k_j \partial_{k_i} \partial_{k_j} \int d^3 \mathbf{r} \frac{e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{r}}}{r^3}$$

$$= \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) k_i k_j \partial_{k_i} \partial_{k_j} \log |\mathbf{k}-\mathbf{p}| = \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) k_i k_j \partial_{k_i} \frac{(k_j - p_j)}{|\mathbf{k}-\mathbf{p}|^2}$$

$$= \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) k_i k_j \frac{\delta_{ij} - 2(k_i - p_i)(k_j - p_j)/|\mathbf{k}-\mathbf{p}|^2}{|\mathbf{k}-\mathbf{p}|^2}$$

$$= \frac{2GM}{2\pi^2} \int d^3 \mathbf{k} \phi(\mathbf{k}) \frac{|\mathbf{k}|^2 |\mathbf{k}-\mathbf{p}|^2 - 2[\mathbf{k}\cdot(\mathbf{k}-\mathbf{p})]^2}{|\mathbf{k}-\mathbf{p}|^4}$$

$$\int d^3 \mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} V(r) \phi(\mathbf{r}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} V(\mathbf{p}, \mathbf{k}) \phi(\mathbf{k})$$

$$V(\mathbf{p}, \mathbf{k}) = 8\pi GM \left[\frac{\omega^2}{|\mathbf{q}|^2} + \frac{\mathbf{p}\cdot\mathbf{q} + |\mathbf{q}|^2}{|\mathbf{q}|^2} + \frac{|\mathbf{p}|^2}{|\mathbf{q}|^2} - \frac{2(\mathbf{p}\cdot\mathbf{q} + |\mathbf{q}|^2)^2}{|\mathbf{q}|^4} \right]$$

$$V(\omega, |\mathbf{q}|) = \frac{16\pi GM \omega^2}{|\mathbf{q}|^2} + O(G^2)$$

$$(8\pi G)^{1/2} \left[(k_{[\mu} \epsilon_{\alpha]} k'_{[\nu} \epsilon'_{\beta]}) + k_{[\nu} \epsilon_{\alpha]} k'_{[\mu} \epsilon'_{\beta]}) \eta^{\alpha\beta} - \frac{1}{2} \eta_{\mu\nu} k^{[\alpha} \epsilon^{\beta]} k'^{[\alpha} \epsilon'_{\beta]} \right]$$

$$(8\pi G)^{1/2} (k^\mu k'^\nu + k'^\nu k^\mu - \eta^{\mu\nu} ((k\cdot k') + M^2))$$

$$\frac{1}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta})$$



$$T(s, t) = -\frac{2}{M_{pl}^2} \epsilon_1^\mu \epsilon_3^{*\nu} \left[\frac{(M^4 - su)}{t} g_{\mu\nu} + \frac{(M^2 - s)}{t} p_{2\mu} p_{1\nu} + \frac{(M^2 - u)}{t} p_{4\mu} p_{1\nu} \right. \\ \left. + \frac{(M^2 - u)}{t} p_{3\mu} p_{2\nu} + \frac{(M^2 - s)}{t} p_{3\mu} p_{4\nu} + \left(1 - \frac{2M^2}{t}\right) p_{3\mu} p_{1\nu} \right. \\ \left. + p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} \right]$$

$$T(p_1, p_3) = -16\pi G M^2 \epsilon_1^\mu \epsilon_3^{*\nu} \left[\left(1 + \frac{4\omega^2}{t}\right) \eta_{\mu\nu} + \frac{4\omega}{t} (q_\mu u_{2\nu} - u_{2\mu} q_\nu) - \frac{2q_\mu q_\nu}{t} - 2u_{2\mu} u_{2\nu} \right]$$

$$D_\mu D^\mu \phi = 0, D_\mu = \partial_\mu + ie\Phi_{\sqrt{Kerr}} l_\mu, \Phi_{\sqrt{Kerr}} = \frac{r^3}{r^4 + (a \cdot x)^2}$$

$$V_{\sqrt{Kerr}} = ie(p_\mu + k_\mu) \int d^3x \Phi l^\mu e^{i(k-p)x} \\ = ie(p_\mu + k_\mu) \left[u^\mu \int d^3x \Phi e^{i(k-p)x} + \Pi_T^{\mu i} \int d^3x \Phi l_i e^{i(k-p)x} \right] \\ = \frac{8\pi i e}{q^2} \left[\omega \cosh aq - i\epsilon_{\mu\nu\alpha\beta} a^\mu k^\nu q^\alpha u^\beta \frac{\sinh aq}{aq} \right]$$

$$\frac{x_{i_1} \cdots x_{i_n}}{r^m} \partial_{j_1} \cdots \partial_{j_{n'}}$$

$$\int d^3x e^{-i\vec{p}\cdot\vec{x}} \frac{x_{i_1} \cdots x_{i_n}}{r^m} \partial_{j_1} \cdots \partial_{j_{n'}} \phi(\vec{x})$$

$$\phi(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})$$

$$\frac{1}{(2\pi)^3} \int d^3k \phi(\vec{k}) (ik^{j_1}) \cdots (ik^{j_{n'}}) \int d^3x e^{i\vec{q}\cdot\vec{x}} \frac{x_{i_1} \cdots x_{i_n}}{r^m}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \phi(\vec{k}) (ik^{j_1}) \cdots (ik^{j_{n'}}) (-i\partial_{q_{i_1}}) \cdots (-i\partial_{q_{i_n}}) \int d^3x \frac{e^{i\vec{q}\cdot\vec{x}}}{r^m}$$

$$\int d^3r e^{-i\vec{p}\cdot\vec{r}} V(r) \phi(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} V(\mathbf{p}, \mathbf{k}) \phi(\mathbf{k})$$

$$V(\vec{x}) = \frac{x_{i_1} \cdots x_{i_n}}{r^m} \partial_{j_1} \cdots \partial_{j_{n'}}$$

$$V(\vec{p}, \vec{k}) = (-i)^n (i)^{n'} k^{j_1} \cdots k^{j_{n'}} \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} \int d^3x \frac{e^{i\vec{q}\cdot\vec{x}}}{r^m}$$

$$V_m(q) = \int d^d x \frac{e^{i\vec{q}\cdot\vec{x}}}{r^m} = \pi^{\frac{3}{2}-\epsilon} \frac{\Gamma\left(\frac{3-m}{2} - \epsilon\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{2}{|\vec{q}|}\right)^{3-m-2\epsilon}, d = 3 - 2\epsilon$$

$$V_m(q) = \frac{4\pi}{q^2}, m = 1$$



$$V_m(q) = \frac{2\pi^2(-1)^{\frac{m-2}{2}}}{\Gamma(m-1)} q^{m-3}, m = 2,4,6,8,10, \dots$$

$$V_m(q) = \frac{2\pi(-1)^{\frac{m+1}{2}}}{\Gamma(m-1)} \left(\psi_{\frac{m-1}{2}}^{(0)} - \frac{1}{\epsilon} + \log(4\pi) - 2\log(q) \right) q^{m-3}$$

$$V_1^{(n)} = 4\pi(-2)^{(n)} q^{-n-2}, m = 1$$

$$x^{(n)} \equiv \frac{\Gamma(x+1)}{\Gamma(-n+x+1)}$$

$$V_m^{(n)}(q) = \frac{2\pi^2(m-3)^{(n)}(-1)^{\frac{m-2}{2}}}{\Gamma(m-1)} q^{m-n-3}, m = 2,4,6,8,10, \dots$$

$$V_m^{(n)}(q) = \frac{2\pi(-1)^{\frac{m+1}{2}}}{\Gamma(m-1)} q^{m-n-3} \\ \times \left[\left(\psi_{\frac{m-1}{2}}^{(0)} - \frac{1}{\epsilon} + \log(4\pi) - 2\log(q) \right) (m-3)^{(n)} + 2(-1)^n c_{m-3,n} \right] \\ m = 3,5,7,9, \dots$$

$$c_{m,n} \equiv (-m)_m (1)_{n-m-1}, \text{ if } n > m \\ \equiv -(-m)_n H_{m,n}, \text{ if } n \leq m$$

$$H_{m,n} \equiv \sum_{j=0}^{n-1} \frac{1}{m-j}$$

$$\begin{aligned} \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} f(q) &= q_{i_1} \cdots q_{i_n} f^{(n)}(q) \\ &+ \delta_{(i_1 i_2)} q_{i_3} \cdots q_{i_n} f^{(n-1)}(q) \\ &+ \delta_{(i_1 i_2)} \delta_{i_3 i_4} q_{i_5} \cdots q_{i_n} f^{(n-2)}(q) \\ &+ \cdots \\ &+ \delta_{(i_1 i_2)} \cdots \delta_{i_{n-3} i_{n-2}} q_{i_{n-1}} q_{i_n} f^{(\frac{n+2}{2})}(q) \\ &+ \delta_{(i_1 i_2)} \cdots \delta_{i_{n-1} i_n} f^{(\frac{n}{2})}(q) \end{aligned}$$

$$\begin{aligned} \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} f(q) &= q_{i_1} \cdots q_{i_n} f^{(n)}(q) \\ &+ \delta_{(i_1 i_2)} q_{i_3} \cdots q_{i_n} f^{(n-1)}(q) \\ &+ \delta_{(i_1 i_2)} \delta_{i_3 i_4} q_{i_5} \cdots q_{i_n} f^{(n-2)}(q) \\ &+ \cdots \\ &+ \delta_{(i_1 i_2)} \cdots \delta_{i_{n-4} i_{n-3}} q_{i_{n-2}} q_{i_{n-1}} q_{i_n} f^{(\frac{n+3}{2})}(q) \\ &+ \delta_{(i_1 i_2)} \cdots \delta_{i_{n-2} i_{n-1}} q_{i_n} f^{(\frac{n+1}{2})}(q) \end{aligned}$$

$$\partial_{q_{i_1}} \cdots \partial_{q_{i_n}} f(q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \delta_{(i_1 i_2)} \cdots \delta_{i_{2j-1} i_{2j}} q_{i_{2j+1}} \cdots q_{i_n} f^{(n-j)}(q)$$



$$\delta_{(ij} \cdots \delta_{kl} q_m \cdots q_n) \equiv \frac{1}{2^{n_\delta} n_\delta! n_q!} \sum_{\text{permutations}} \delta_{ij} \cdots \delta_{kl} q_m \cdots q_n$$

$$f_{(n+1)} = \frac{1}{q} \frac{df_{(n)}}{dq}$$

$$f_{(n)}(q) \equiv \sum_{j=1}^n \frac{b_{n,j}}{q^{n+j-1}} \frac{d^{n-j+1} f(q)}{dq^{n-j+1}}$$

$$b_{n,j} = 2^{1-j} \frac{(1-n)_{j-1} (n)_{j-1}}{\Gamma(j)} = \frac{2^{1-j} \Gamma(j-n) \Gamma(j+n-1)}{\Gamma(j) \Gamma(1-n) \Gamma(n)}$$

$$\partial_{q_i} f(q) = q_i f_{(1)}(q) = \frac{q_i}{q} f'(q)$$

$$\partial_{q_i} \partial_{q_j} f(q) = q_i q_j f_{(2)}(q) + \delta_{ij} f_{(1)}(q) = q_i q_j \left(\frac{f''(q)}{q^2} - \frac{f'(q)}{q^3} \right) + \delta_{ij} \frac{f'(q)}{q}$$

$$\begin{aligned} \partial_{q_i} \partial_{q_j} \partial_{q_k} f(q) &= q_i q_j q_k f_{(3)}(q) + (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}) f_{(2)}(q) \\ &= q_i q_j q_k \left(\frac{f^{(3)}(q)}{q^3} - \frac{3f''(q)}{q^4} + \frac{3f'(q)}{q^5} \right) \\ &\quad + (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}) \left(\frac{f''(q)}{q^2} - \frac{f'(q)}{q^3} \right) \end{aligned}$$

$$\begin{aligned} \partial_{q_i} \partial_{q_j} \partial_{q_k} \partial_{q_l} f(q) &= q_i q_j q_k q_l f_{(4)}(q) + (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} + \delta_{jk} \delta_{il}) f_{(2)}(q) \\ &\quad + (\delta_{ik} q_j q_l + \delta_{ij} q_k q_l + \delta_{jk} q_i q_l + \delta_{il} q_j q_k + \delta_{jl} q_i q_k + \delta_{kl} q_i q_j) f_{(3)}(q) \end{aligned}$$

$$f_{(4)}(q) = \frac{f^{(4)}(q)}{q^4} - \frac{6f^{(3)}(q)}{q^5} + \frac{15f''(q)}{q^6} - \frac{15f'(q)}{q^7}$$

$$f_{(3)}(q) = \frac{f^{(3)}(q)}{q^3} - \frac{3f''(q)}{q^4} + \frac{3f'(q)}{q^5}$$

$$f_{(2)}(q) = \frac{f''(q)}{q^2} - \frac{f'(q)}{q^3}$$

$$V(\vec{x}) = \frac{(a \cdot x)^n}{r^m} = a^{i_1} \cdots a^{i_n} \frac{x_{i_1} \cdots x_{i_n}}{r^m}$$

$$V(\vec{p}, \vec{k}) = (-i)^n a^{i_1} \cdots a^{i_n} \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} \int d^3x \frac{e^{-i\vec{q} \cdot \vec{x}}}{r^m}$$

$$\partial_{q_{i_1}} \cdots \partial_{q_{i_n}} f(q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \delta_{(i_1 i_2} \cdots \delta_{i_{2j-1} i_{2j}} q_{i_{2j+1}} \cdots q_{i_n}) f_{(n-j)}(q)$$

$$a^{i_1} \cdots a^{i_n} \partial_{q_{i_1}} \cdots \partial_{q_{i_n}} f(q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{j! (n-2j)! 2^j} (a^2)^j (\vec{a} \cdot \vec{q})^{n-2j} f_{(n-j)}(q)$$



$$\begin{aligned}
V_{m,n,0,0}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int e^{i\vec{q}\cdot\vec{x}} \frac{(a\cdot x)^n}{r^m} d^3\vec{x} \\
&= (-i)^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{j!(n-2j)!2^j} (a^2)^j (\vec{a}\cdot\vec{q})^{n-2j} V_{m,n-j}(q) \\
k^{i_1} a^{i_2} \dots a^{i_n} \partial_{q_{i_1}} \dots \partial_{q_{i_n}} f(q) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1)!}{j!(n-2j)!2^j} (a^2)^{j-1} (\vec{a}\cdot\vec{q})^{n-2j-1} f_{(n-j)}(q) \\
&\quad \times [(n-2j)(\vec{k}\cdot\vec{q})(a^2) + (2j)(\vec{k}\cdot\vec{a})(\vec{a}\cdot\vec{q})] \\
V_{m,n,1,0}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int e^{i\vec{q}\cdot\vec{x}} \frac{(\vec{a}\cdot\vec{x})^{n-1} (\vec{x}\cdot\vec{\partial})}{r^m} d^3\vec{x} \\
&= (-i)^n i \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1)!}{j!(n-2j)!2^j} (a^2)^{j-1} (\vec{a}\cdot\vec{q})^{n-2j-1} V_{m,n-j}(q) \\
&\quad \times [(n-2j)(\vec{k}\cdot\vec{q})(a^2) + (2j)(\vec{k}\cdot\vec{a})(\vec{a}\cdot\vec{q})] \\
V_{m,n,0,1}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int e^{i\vec{q}\cdot\vec{x}} \frac{(\vec{a}\cdot\vec{x})^{n-1} (\epsilon^{ijl} x_i a_j \partial_l)}{r^m} d^3\vec{x} \\
&= (-i)^n i \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1)!}{j!(n-2j-1)!2^j} (a^2)^j (\vec{a}\cdot\vec{q})^{n-2j-1} V_{m,n-j}(q) (\epsilon^{ijl} q_i a_j k_l) \\
k^{i_1} k^{i_2} a^{i_3} \dots a^{i_n} \partial_{q_{i_1}} \dots \partial_{q_{i_n}} f(q) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2)!}{j!(n-2j)!2^j} (a^2)^{j-2} (\vec{a}\cdot\vec{q})^{n-2j-2} f_{(n-j)}(q) \\
&\quad \times [(n-2j)(n-2j-1)(\vec{k}\cdot\vec{q})^2 (a^2)^2 + 2(2j)(n-2j)(\vec{k}\cdot\vec{q})(\vec{k}\cdot\vec{a})(\vec{a}\cdot\vec{q})(a^2) \\
&\quad + (2j)(2j-2)(\vec{k}\cdot\vec{a})^2 (\vec{a}\cdot\vec{q})^2 + (2j)(\vec{k})^2 (\vec{a}\cdot\vec{q})^2 (a^2)] \\
V_{m,n,2,0}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int e^{i\vec{q}\cdot\vec{x}} \frac{(a\cdot x)^{n-2} (x\cdot\partial)^2}{r^m} d^3\vec{x} \\
&= (-i)^n (i)^2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2)!}{j!(n-2j)!2^j} (a^2)^{j-2} (\vec{a}\cdot\vec{q})^{n-2j-2} V_{m,n-j}(q) \\
&\quad \times [(n-2j)(n-2j-1)(\vec{k}\cdot\vec{q})^2 (a^2)^2 + 2(2j)(n-2j)(\vec{k}\cdot\vec{q})(\vec{k}\cdot\vec{a})(\vec{a}\cdot\vec{q})(a^2) \\
&\quad + (2j)(2j-2)(\vec{k}\cdot\vec{a})^2 (\vec{a}\cdot\vec{q})^2 + (2j)(\vec{k})^2 (\vec{a}\cdot\vec{q})^2 (a^2)] \\
V_{m,n,1,1}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int e^{i\vec{q}\cdot\vec{x}} \frac{(a\cdot x)^{n-2} (x\cdot\partial) (\epsilon^{ijl} x_i a_j \partial_l)}{r^m} d^3\vec{x} \\
&= (-i)^n (i)^2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2)!}{j!(n-2j-1)!2^j} (a^2)^{j-1} (\vec{a}\cdot\vec{q})^{n-2j-2} V_{m,n-j}(q) \\
&\quad \times [(n-2j-1)(\vec{k}\cdot\vec{q})(\epsilon^{ijl} q_i a_j k_l)(a^2) + (2j)(\epsilon^{ijl} q_i a_j k_l)(\vec{k}\cdot\vec{a})(\vec{a}\cdot\vec{q})]
\end{aligned}$$

$$\begin{aligned}
V_{m,n,0,0}^w(\vec{a}, \vec{q}, \vec{k}) &\equiv \int x^w e^{i\vec{q}\cdot\vec{x}} \frac{(a\cdot x)^n}{r^m} d^3\vec{x} = (-i) \frac{\partial V_{m,n,0,0}(\vec{a}, \vec{q}, \vec{k})}{\partial q^w} \\
&= (-i)^{n+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{j!(n-2j)! 2^j} (a^2)^j (\vec{a}\cdot\vec{q})^{n-2j-1} \\
&\quad \times [a^w(n-2j)V_{m,n-j}(q) + q^w(\vec{a}\cdot\vec{q})V_{m,n-j+1}(q)]
\end{aligned}$$

$$\begin{aligned}
V_{m,n,0,0}^{wz}(\vec{a}, \vec{q}, \vec{k}) &\equiv \int x^w x^z e^{i\vec{q}\cdot\vec{x}} \frac{(a\cdot x)^n}{r^m} d^3\vec{x} = (-i)^2 \frac{\partial^2 V_{m,n,0,0}(\vec{a}, \vec{q}, \vec{k})}{\partial q^w \partial q^z} \\
&= (-i)^{n+2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{j!(n-2j)! 2^j} (a^2)^j (\vec{a}\cdot\vec{q})^{n-2j-2} \\
&\quad \times [a^w a^z (n-2j)(n-2j-1)V_{m,n-j}(q) + (a^w q^z + a^z q^w)(n-2j)(\vec{a}\cdot\vec{q})V_{m,n-j+1}(q) \\
&\quad + \delta^{wz}(\vec{a}\cdot\vec{q})^2 V_{m,n-j+1}(q) + q^w q^z (\vec{a}\cdot\vec{q})^2 V_{m,n-j+2}(q)]
\end{aligned}$$

$$|\mathbf{p}|^2 \rightarrow |\mathbf{p}|^2 + i\epsilon \left(\frac{s^2 - (m_1 - m_2)^2}{4s^2} \right), \text{ where } \left(\frac{s^2 - (m_1 - m_2)^2}{4s^2} \right) > \sqrt{\text{Kerr} \frac{n!}{2^n n_\delta! n_q!}}$$

$$\partial_t f + \xi \cdot \nabla_x f = \nabla_\xi \cdot (\nabla_\xi f - F_d f),$$

$$\partial_t v + v \cdot \nabla_x v + \nabla_x P - \Delta_x v = - \int_{\mathbb{R}^4} F_d f d\xi$$

$$\partial_t f^\epsilon + \frac{1}{\epsilon} \xi \cdot \nabla_x f^\epsilon = \frac{1}{\epsilon^2} \nabla_\xi \cdot (\nabla_\xi f^\epsilon + (\xi - \epsilon v^\epsilon) f^\epsilon), (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^4$$

$$\partial_t v^\epsilon + v^\epsilon \cdot \nabla_x v^\epsilon + \nabla_x P^\epsilon - \Delta_x v^\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^4} (\xi - \epsilon v^\epsilon) f^\epsilon d\xi$$

$$\nabla_x \cdot v^\epsilon = 0$$

$$\epsilon \partial_t f^\epsilon + \xi \cdot \nabla_x f^\epsilon = \frac{1}{\epsilon} \nabla_\xi \cdot \left(f^\epsilon \nabla_\xi \log \left(\frac{f^\epsilon}{M_{\rho_{f^\epsilon, \epsilon v^\epsilon}}} \right) \right),$$

$$M_{\rho_{f^\epsilon, \epsilon v^\epsilon}} := \frac{\rho_{f^\epsilon}}{2\pi} \exp \left(-\frac{|\xi - \epsilon v^\epsilon|^2}{2} \right).$$

$$\partial_t v + v \cdot \nabla_x v + \nabla_x P - \Delta_x v = 0$$

$$\nabla_x \cdot v = 0$$

$$\epsilon \partial_t \rho_{f^\epsilon} + \nabla_x \cdot m_{f^\epsilon} = 0$$

$$\epsilon \partial_t m_{f^\epsilon} + \nabla_x \cdot \left(\int_{\mathbb{R}^4} \xi \otimes \xi f^\epsilon d\xi \right) = -\frac{1}{\epsilon} m_{f^\epsilon} + \rho_{f^\epsilon} v^\epsilon$$

$$m_{f^\epsilon} := \int_{\mathbb{R}^4} \xi f^\epsilon d\xi$$



$$\partial_t \rho_{f^\varepsilon} + \nabla_x \cdot (\rho_{f^\varepsilon} v^\varepsilon) = \Delta_x \rho_{f^\varepsilon} + \varepsilon \nabla_x \cdot \partial_t m_{f^\varepsilon} + \nabla_x \otimes \nabla_x : \int_{\mathbb{R}^4} (\xi \otimes \xi - \mathbb{I}_2) f^\varepsilon d\xi$$

$$\int_{\mathbb{R}^4} \xi \otimes \xi M_{\rho_{f^\varepsilon}, \varepsilon v^\varepsilon} d\xi = \rho_{f^\varepsilon} (\varepsilon^2 v^\varepsilon \otimes v^\varepsilon + \mathbb{I}_2) \text{ and } \int_{\mathbb{R}^4} \xi M_{\rho_{f^\varepsilon}, \varepsilon v^\varepsilon} d\xi = \varepsilon \rho_{f^\varepsilon} v^\varepsilon$$

$$\begin{aligned} & \varepsilon \nabla_x \cdot \partial_t m_{f^\varepsilon} + \nabla_x \otimes \nabla_x : \int_{\mathbb{R}^2} (\xi \otimes \xi - \mathbb{I}_2) f^\varepsilon d\xi \\ & \sim \varepsilon^2 [\nabla_x \cdot \partial_t (\rho_{f^\varepsilon} v^\varepsilon) + \nabla_x \otimes \nabla_x : (\rho_{f^\varepsilon} v^\varepsilon \otimes v^\varepsilon)] = \mathcal{O}(\varepsilon^2), (\varepsilon \ll 1) \end{aligned}$$

$$\partial_t \rho + \nabla_x \cdot (\rho v) = \Delta_x \rho$$

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho v) &= \Delta_x \rho \\ \partial_t v + v \cdot \nabla_x v + \nabla_x P - \Delta_x v &= 0 \\ \nabla_x \cdot v &= 0 \end{aligned}$$

$$M_{\rho, u}(\xi) := \frac{\rho}{2\pi} \exp\left(-\frac{|u - \xi|^2}{2}\right)$$

$$\mathcal{H}[f \mid M_{\rho, u}] := \iint_{\Omega \times \mathbb{R}^4} f \log \frac{f}{M_{\rho, u}} dx d\xi - \iint_{\Omega \times \mathbb{R}^2} (f - M_{\rho, u}) dx d\xi$$

$$\iint_{\Omega \times \mathbb{R}^4} f(t, x, \xi) dx d\xi = 1, t \geq 0$$

$$d_{\text{BL}}^X(\mu, \nu) := \sup_{\|\phi\|_{L^\infty(X)} + \text{Lip}_X(\phi) \leq 1} \left| \int_X \phi d\mu - \int_X \phi d\nu \right|$$

$$\|f\|_{L_2^1} := \iint_{\Omega \times \mathbb{R}^4} (1 + |\xi|^2) f dx d\xi$$

(i) $f^\varepsilon \in L^\infty([0, T]; L_+^1(\Omega \times \mathbb{R}^2, (1 + |x| + |\xi|^2) dx d\xi) \cap L \log L(\Omega \times \mathbb{R}^2))$,

(ii) $v^\varepsilon \in L^\infty([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1(\Omega))$,

$$\begin{aligned} & \iint_{\Omega \times \mathbb{R}^4} \left(\frac{1}{2} |\xi|^2 + \log f^\varepsilon(t)\right) f^\varepsilon(t) dx d\xi + \frac{1}{2} \|v^\varepsilon(t)\|_{L^2}^2 \\ & + \frac{1}{\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^2} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi ds + \int_0^t \int_\Omega |\nabla v^\varepsilon|^2 dx ds \\ & \leq \iint_{\Omega \times \mathbb{R}^4} \left(\frac{1}{2} |\xi|^2 + \log f_0^\varepsilon\right) f_0^\varepsilon dx d\xi + \frac{1}{2} \|v_0^\varepsilon\|_{L^2}^2 \end{aligned}$$

$$\rho \in L^\infty(0, T; \mathcal{P} \cap L^\infty(\Omega)), v, \nabla \log \rho \in L^\infty(0, T; W^{1, \infty}(\Omega)) \cap W^{1, \infty}(0, T; L^\infty(\Omega)).$$

$$\sup_{\varepsilon > 0} (\| |x| f_0^\varepsilon \|_{L^1} + \| f_0^\varepsilon \|_{L_2^1 \cap L \log L} + \| v_0^\varepsilon \|_{L^2}) < \infty.$$



$$\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

$$\begin{aligned} f^\varepsilon &\rightarrow M_{\rho,0} \text{ in } L^\infty(0, T; L^1(\Omega \times \mathbb{R}^2)) \\ \int_{\mathbb{R}^4} f^\varepsilon d\xi &\rightarrow \rho \text{ in } L^\infty(0, T; L^1(\Omega)) \\ \frac{1}{\varepsilon} \int_{\mathbb{R}^4} \xi f^\varepsilon d\xi &\rightarrow \rho(v - \nabla \log \rho) \text{ in } L^1((0, T) \times \Omega) \\ v^\varepsilon &\rightarrow v \text{ in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \end{aligned}$$

$$\begin{aligned} &\|f^\varepsilon - M_{\rho,0}\|_{L^\infty(0, T; L^1)}^2 + \|\rho_{f^\varepsilon} - \rho\|_{L^\infty(0, T; L^1)}^2 + \|v^\varepsilon - v\|_{L^\infty(0, T; L^2) \cap L^2(0, T; H^1)}^2 \\ &+ \left\| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \xi f^\varepsilon d\xi - \rho(v - \nabla \log \rho) \right\|_{L^1((0, T) \times \Omega)}^2 \\ &\leq C(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^2 \end{aligned}$$

$$d_{\text{BL}}(\rho_{f_0^\varepsilon}, \rho_0) \rightarrow 0 \text{ and } \varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

$$f^\varepsilon \rightarrow M_{\rho,0} \text{ in } L^2(0, T; \mathcal{M}(\Omega \times \mathbb{R}^2)) \cap L^1((0, T) \times \Omega \times \mathbb{R}^2),$$

$$\begin{aligned} \int_{\mathbb{R}^4} f^\varepsilon d\xi &\overset{*}{\rightarrow} \rho \text{ in } L^\infty(0, T; \mathcal{M}(\Omega)) \cap L^1((0, T) \times \Omega) \\ \frac{1}{\varepsilon} \int_{\mathbb{R}^4} \xi f^\varepsilon d\xi &\rightarrow \rho(v - \nabla \log \rho) \text{ in } L^1((0, T) \times \Omega) \\ v^\varepsilon &\rightarrow v \text{ in } L_{\text{loc}}^2((0, T) \times \Omega) \end{aligned}$$

$$\begin{aligned} \int_0^T d_{\text{BL}}^2(f^\varepsilon, M_{\rho,0}) dt &\leq C d_{\text{BL}}^2(\rho_{f_0^\varepsilon}, \rho_0) + C\varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^2 \\ \sup_{0 \leq t \leq T} d_{\text{BL}}^2(\rho_{f^\varepsilon}(t), \rho(t)) &\leq C d_{\text{BL}}^2(\rho_{f_0^\varepsilon}, \rho_0) + C\varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^4 \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^2} f \log \frac{f}{M_{\rho,u}} d\xi &= \int_{\mathbb{R}^2} f \log \frac{f}{M_{\rho_f, u_f}} d\xi + \rho_f \log \frac{\rho_f}{\rho} + \frac{\rho_f}{2} |u_f - u|^2 \\ &\frac{1}{\varepsilon} m_{f^\varepsilon} \rightarrow \rho(v - \nabla \log \rho), v^\varepsilon \rightarrow v \rightarrow \left\{ \frac{1}{\varepsilon} m_{f^\varepsilon} \right\}_\varepsilon \end{aligned}$$

$$\|f^\varepsilon - M_{\rho,0}\|_{L^\infty(0, T; L^1)}^2 \lesssim \mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2 + \varepsilon^2$$

$$f^\varepsilon = M_{\rho,0} + \varepsilon f_1 + O(\varepsilon^2)$$

$$\rho_{f^\varepsilon} = \int_{\mathbb{R}^4} f^\varepsilon d\xi$$

$$\sup_{0 \leq t \leq T} d_{\text{BL}}^2(\rho_{f^\varepsilon}(t), \rho(t)) \lesssim d_{\text{BL}}^2(\rho_{f_0^\varepsilon}, \rho_0) + \varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + \varepsilon^4$$

$$\int_0^T \int_\Omega p |\phi|^2 dx dt \leq C \left(1 + \sup_{0 < t < T} \int_\Omega p |\ln p| dx \right) \|\phi\|_{L^2(0, T; H^1(\Omega))}^2$$



$$\Phi^*(\xi) = \sup_{\eta \in \mathbb{R}} (\xi\eta - \Phi(\eta)) = (\xi \ln \xi - \xi) + 1 \text{ for } \xi \geq 0$$

$$\xi\eta \leq (e^\eta - 1) + \xi \ln \xi - \xi + 1$$

$$\xi = p(x) \text{ and } \eta = \beta\psi(x)^2 \text{ (so } \eta \geq 0 \text{)}$$

$$p\beta\psi^2 \leq (e^{\beta\psi^2} - 1) + p \ln p - p + 1$$

$$p\psi^2 \leq \frac{1}{\beta}(e^{\beta\psi^2} - 1) + \frac{1}{\beta}p \ln p + \frac{1}{\beta}(1 - p)$$

$$p\psi^2 \leq \frac{1}{\beta}(e^{\beta\psi^2} - 1) + \frac{1}{\beta}p \ln p \text{ on } \{p > 1\}$$

$$\bar{\phi} := \frac{1}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} \phi dx$$

$$\int_{\mathbb{T}^2} \phi^\circ = 0, \|\phi^\circ\|_{L^2} \lesssim \|\nabla\phi\|_{L^2}, \text{ and } \|\nabla\phi^\circ\|_{L^2} = \|\nabla\phi\|_{L^2}$$

$$\int_{\mathbb{T}^2} \exp\left(\alpha_0 \frac{|\phi^\circ|^2}{\|\nabla\phi\|_{L^2}^2}\right) dx \leq K_{\mathbb{T}^2}$$

$$\beta := \frac{\alpha_0}{\|\nabla\phi\|_{L^2}^2}$$

$$\int_{\mathbb{T}^2} (e^{\beta|\psi|^2} - 1) dx = \int_{\mathbb{T}^2} \left(\exp\left(\alpha_0 \frac{|\phi^\circ|^2}{\|\nabla\phi\|_{L^2}^2}\right) - 1\right) dx \leq K_{\mathbb{T}^2} - |\mathbb{T}^2|$$

$$\int_{\mathbb{R}^2} (e^{\alpha_*|g|^2} - 1) dx \leq K_{\mathbb{R}^2}$$

$$g := \frac{\phi}{\|\phi\|_{H^1}} \text{ and } \beta := \frac{\alpha_*}{\|\phi\|_{H^1}^2}$$

$$\int_{\mathbb{R}^2} (e^{\beta|\psi|^2} - 1) dx = \int_{\mathbb{R}^2} (e^{\alpha_*|g|^2} - 1) dx \leq K_{\mathbb{R}^2}$$

$$\beta = \frac{\alpha_0}{\|\nabla\phi\|_{L^2}^2}$$

$$\int_{\mathbb{T}^2} p|\phi^\circ|^2 dx \leq \frac{K_{\mathbb{T}^2} - |\mathbb{T}^2|}{\beta} + \frac{1}{\beta} \int_{\mathbb{T}^2} p|\ln p| dx + \frac{1}{\beta} \int_{\mathbb{T}^2} (1 - p) dx \leq \frac{K_{\mathbb{T}^2}}{\beta} + \frac{1}{\beta} \int_{\mathbb{T}^2} p|\ln p| dx$$

$$\int_{\mathbb{T}^2} p|\phi^\circ|^2 dx \leq \frac{C}{\alpha_0} \|\nabla\phi\|_{L^2}^2 + \frac{\|\nabla\phi\|_{L^2}^2}{\alpha_0} \int_{\mathbb{T}^2} p|\ln p| dx = C \left(1 + \int_{\mathbb{T}^2} p|\ln p| dx\right) \|\nabla\phi\|_{L^2(\mathbb{T}^2)}^2$$

$$\int_{\mathbb{T}^2} p|\bar{\phi}|^2 dx = |\bar{\phi}|^2 \leq C\|\phi\|_{L^2}^2$$



$$\int_{\mathbb{T}^4} p|\phi|^2 dx \leq C \left(1 + \int_{\mathbb{T}^4} p|\ln p| dx \right) \|\phi\|_{H^1(\mathbb{T}^4)}^2$$

$$\beta = \frac{\alpha_*}{\|\phi\|_{H^1}^2}$$

$$\int_{\{p>1\}} p|\phi|^2 dx \leq \frac{1}{\beta} \int_{\mathbb{R}^4} (e^{\beta|\phi|^2} - 1) dx + \frac{1}{\beta} \int_{\mathbb{R}^4} p|\ln p| dx \leq \frac{K_{\mathbb{R}^2}}{\alpha_*} \|\phi\|_{H^1}^2 + \frac{\|\phi\|_{H^1}^2}{\alpha_*} \int_{\mathbb{R}^4} p|\ln p| dx$$

$$\int_{\{p \leq 1\}} p|\phi|^2 dx \leq \int_{\mathbb{R}^4} |\phi|^2 dx \leq \|\phi\|_{H^1}^2$$

$$\int_{\mathbb{R}^4} p|\phi|^2 dx \leq C \left(1 + \int_{\mathbb{R}^4} p|\ln p| dx \right) \|\phi\|_{H^1(\mathbb{R}^4)}^2$$

$$\mathcal{F}[f, v] := \iint_{\Omega \times \mathbb{R}^4} \left(\frac{|\xi|^2}{2} + \log f \right) f dx d\xi + \int_{\Omega} \frac{|v|^2}{2} dx$$

$$\mathcal{F}[f^\varepsilon(t), v^\varepsilon(t)] + \frac{1}{\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi ds + \int_0^t \int_{\Omega} |\nabla v^\varepsilon|^2 dx ds \leq \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon]$$

$$\begin{aligned} & \frac{d}{dt} \iint_{\Omega \times \mathbb{R}^4} \left(\frac{|\xi|^2}{2} + \log f^\varepsilon \right) f^\varepsilon dx d\xi + \frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi \\ &= \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} v^\varepsilon \cdot (\varepsilon v^\varepsilon - \xi) f^\varepsilon dx d\xi \end{aligned}$$

$$\frac{d}{dt} \int_{\Omega} \frac{|v^\varepsilon|^2}{2} dx + \int_{\Omega} |\nabla v^\varepsilon|^2 dx = \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} v^\varepsilon \cdot (\xi - \varepsilon v^\varepsilon) f^\varepsilon dx d\xi$$

$$\begin{aligned} & \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi \\ &= \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi + \int_{\Omega} \rho^\varepsilon |u^\varepsilon - \varepsilon v^\varepsilon|^2 dx. \end{aligned}$$

$$\iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon dx d\xi$$

$$\int_0^T \int_{\Omega} \rho^\varepsilon |v^\varepsilon|^2 dx dt$$

$$\sup_{0 \leq t \leq T} \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon(t) dx d\xi + \sup_{0 \leq t \leq T} \int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \leq C(1 + \|x f_0^\varepsilon\|_{L^1} + \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon])$$

$$\begin{aligned} \frac{d}{dt} \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon dx d\xi &= \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} \frac{x \cdot \xi}{|x|} f^\varepsilon dx d\xi \\ &= \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} \frac{x}{|x|} \cdot ((\xi - \varepsilon v^\varepsilon) f^\varepsilon + \nabla_\xi f^\varepsilon) dx d\xi + \iint_{\Omega \times \mathbb{R}^4} \frac{x}{|x|} \cdot v^\varepsilon f^\varepsilon dx d\xi \\ &=: I + II \end{aligned}$$



$$|I| \leq \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} |(\xi - \varepsilon v^\varepsilon) f^\varepsilon + \nabla_\xi f^\varepsilon| dx d\xi \leq \left(\frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi \right)^{\frac{1}{2}}$$

$$|\mathcal{D}\rho| \leq \left(\int_{\Omega} \rho^\varepsilon |v^\varepsilon|^2 dx \right)^{\frac{1}{2}}$$

$$\iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon(t) dx d\xi \leq \iint_{\Omega \times \mathbb{R}^4} |x| f_0^\varepsilon dx d\xi + T^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2} \int_0^T \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi dt \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \left(\int_0^T \int_{\Omega} \rho^\varepsilon |v^\varepsilon|^2 dx dt \right)^{\frac{1}{2}}$$

$$\leq \iint_{\Omega \times \mathbb{R}^4} |x| f_0^\varepsilon dx d\xi + T^{\frac{1}{2}} \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon]^{\frac{1}{2}} + \frac{T}{4\delta} + \delta \int_0^T \int_{\Omega} \rho^\varepsilon |v^\varepsilon|^2 dx dt$$

$$\delta \int_0^T \int_{\Omega} \rho^\varepsilon |v^\varepsilon|^2 dx dt \leq C\delta \left(1 + \sup_{0 \leq t \leq T} \int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \right) \|v^\varepsilon\|_{L^2(0,T;H^1(\Omega))}^2 \leq C\delta \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon] \left(1 + \sup_{0 \leq t \leq T} \int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \right)$$

$$\iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon(t) dx d\xi \leq \iint_{\Omega \times \mathbb{R}^4} |x| f_0^\varepsilon dx d\xi + T^{\frac{1}{2}} \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon]^{\frac{1}{2}} + \frac{T}{4\delta} + C\delta \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon] \left(1 + \sup_{0 \leq t \leq T} \int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \right)$$

$$\int_{\mathbb{R}^4} f^\varepsilon \log \frac{f^\varepsilon}{M_{1,0}} d\xi = \int_{\mathbb{R}^4} f^\varepsilon \log \frac{f^\varepsilon}{M_{\rho^\varepsilon, u^\varepsilon}} d\xi + \rho^\varepsilon \log \rho^\varepsilon + \frac{\rho^\varepsilon}{2} |u^\varepsilon|^2$$

$$\int_{\mathbb{R}^4} f^\varepsilon \log \frac{f^\varepsilon}{M_{1,0}} d\xi = \int_{\mathbb{R}^2} f^\varepsilon \log f^\varepsilon d\xi + \log(2\pi) \rho^\varepsilon + \frac{1}{2} \int_{\mathbb{R}^4} |\xi|^2 f^\varepsilon d\xi$$

$$\int_{\mathbb{R}^4} f^\varepsilon \log \frac{f^\varepsilon}{M_{\rho^\varepsilon, u^\varepsilon}} d\xi + \rho^\varepsilon \log \rho^\varepsilon = \int_{\mathbb{R}^4} f^\varepsilon \log f^\varepsilon d\xi + \log(2\pi) \rho^\varepsilon + \frac{1}{2} \int_{\mathbb{R}^4} |\xi - u^\varepsilon|^2 f^\varepsilon d\xi$$

$$\int_{\mathbb{R}^4} f^\varepsilon d\xi = \int_{\mathbb{R}^4} M_{\rho^\varepsilon, u^\varepsilon} d\xi$$

$$\int_{\mathbb{R}^4} f^\varepsilon \log \frac{f^\varepsilon}{M_{\rho^\varepsilon, u^\varepsilon}} d\xi = \mathcal{H}[f^\varepsilon | M_{\rho^\varepsilon, u^\varepsilon}] \geq 0$$

$$\frac{1}{2} \int_{\mathbb{R}^4} |\xi - u^\varepsilon|^2 f^\varepsilon d\xi \leq \frac{1}{2} \int_{\mathbb{R}^4} |\xi|^2 f^\varepsilon d\xi$$

$$\int_{\Omega} \rho^\varepsilon \log \rho^\varepsilon dx \leq \mathcal{F}[f^\varepsilon, v^\varepsilon] + \log(2\pi) \leq \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon] + \log(2\pi)$$

$$2 \int_{\Omega} \rho^\varepsilon \log_- \rho^\varepsilon dx \leq \int_{\Omega} |x| \rho^\varepsilon dx + C \int_{\Omega} e^{-|x|} dx \leq \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon dx d\xi + C$$



$$\int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \leq C + C\mathcal{F}[f_0^\varepsilon, v_0^\varepsilon] + C \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon dx d\xi$$

$$\sup_{0 \leq t \leq T} \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon(t) dx d\xi \leq C(1 + \|x f_0^\varepsilon\|_{L^1} + \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon])$$

$$\delta = \frac{1}{M(1 + \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon])}$$

$$\sup_{0 \leq t \leq T} \int_{\Omega} \rho^\varepsilon |\log \rho^\varepsilon| dx \leq C(1 + \|x f_0^\varepsilon\|_{L^1} + \mathcal{F}[f_0^\varepsilon, v_0^\varepsilon])$$

$$\sup_{0 \leq t \leq T} \iint_{\Omega \times \mathbb{R}^2} |\xi|^2 f^\varepsilon(t) dx d\xi \leq C$$

$$\log f^\varepsilon = \log_+ f^\varepsilon - \log_- f^\varepsilon, \log_- f = \max\{0, -\log f\}$$

$$\iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \log_- f^\varepsilon dx d\xi \leq \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \left(\frac{|x|}{2} + \frac{|\xi|^2}{4} \right) dx d\xi + C \iint_{\Omega \times \mathbb{R}^4} e^{-\frac{|x|}{4} - \frac{|\xi|^2}{8}} dx d\xi$$

$$\iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \left(\frac{|x|}{2} + \frac{|\xi|^2}{4} + \log_+ f^\varepsilon \right) dx d\xi \leq \mathcal{F}[f^\varepsilon, v^\varepsilon] + \iint_{\Omega \times \mathbb{R}^4} |x| f^\varepsilon dx d\xi + C_0$$

$$C_0 := C \iint_{\Omega \times \mathbb{R}^4} e^{-\frac{|x|}{4} - \frac{|\xi|^2}{8}} dx d\xi$$

$$\sup_n \left\{ \|f_n\|_{L^1_2 \cap L \log L} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x| f_n(x, \xi) dx d\xi \right\} < \infty$$

$$\rho_n(x) := \int_{\mathbb{R}^d} f_n(x, \xi) d\xi, m_n(x) := \int_{\mathbb{R}^d} \xi f_n(x, \xi) d\xi$$

$$\sup_n \int_{\mathbb{R}^d} \rho_n |\log \rho_n| dx \leq C + C \|f_n\|_{L^1_2 \cap L \log L} + C \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x| f_n(x, \xi) dx d\xi < \infty$$

$$\int_{|x| > R} \rho_n(x) dx = \iint_{|x| > R} f_n(x, \xi) dx d\xi \leq \frac{1}{R} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x| f_n dx d\xi \leq \frac{C}{R}$$

$$\int_E |m_n| dx \leq \left(\int_E \rho_n dx \right)^{1/2} \left(\iint_{E \times \mathbb{R}^d} |\xi|^2 f_n dx d\xi \right)^{1/2} \leq C_0 \left(\int_E \rho_n dx \right)^{1/2}$$

$$\sup_n \int_E \rho_n dx < \frac{\varepsilon^2}{C_0^2}$$

$$\sup_n \int_E |m_n| dx < \varepsilon$$



$$\begin{aligned} \int_{\{|x|>R\}} |m_n| dx &\leq \left(\int_{\{|x|>R\}} \rho_n dx \right)^{1/2} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |\xi|^2 f_n dx d\xi \right)^{1/2} \\ &\leq C_0 \left(\int_{\{|x|>R\}} \rho_n dx \right)^{1/2} \rightarrow 0 \text{ as } R \rightarrow \infty \end{aligned}$$

$$\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \bar{u}) = 0$$

$$\int_0^T \int_{\Omega} |\bar{u}(t, x)| \bar{\rho}(t, x) dx dt < \infty$$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$

$$d_{\text{BL}}^2(\bar{\rho}(t), \rho(t)) \leq C d_{\text{BL}}^2(\bar{\rho}(0), \rho(0)) + C \int_0^t \int_{\Omega} |(\bar{u} - u)(s, x)|^2 \bar{\rho}(s, x) dx ds$$

$$\begin{aligned} d_{\text{BL}}^2((\bar{\rho} \bar{u})(t), (\rho u)(t)) &\leq C d_{\text{BL}}^2(\bar{\rho}(0), \rho(0)) + C \int_{\Omega} |(\bar{u} - u)(t, x)|^2 \bar{\rho}(t, x) dx \\ &\quad + C \int_0^t \int_{\Omega} |(\bar{u} - u)(s, x)|^2 \bar{\rho}(s, x) dx ds \end{aligned}$$

$$\int_0^t d_{\text{BL}}^2((\bar{\rho} \bar{u})(s), (\rho u)(s)) ds \leq C d_{\text{BL}}^2(\bar{\rho}(0), \rho(0)) + C \int_0^t \int_{\Omega} |(\bar{u} - u)(s, x)|^2 \bar{\rho}(s, x) dx ds$$

$$|\xi|^2 f \in L^1((0, T) \times \Omega \times \mathbb{R}^d) \text{ and } \nabla_{\xi} \sqrt{f} \in L^2((0, T) \times \Omega \times \mathbb{R}^d).$$

$$\begin{aligned} \int_0^t d_{\text{BL}}^2(f, M_{\rho, u}) ds &\leq C d_{\text{BL}}^2(\rho_{f_0}, \rho_0) + C \int_0^t \iint_{\Omega \times \mathbb{R}^d} \frac{1}{f} |\nabla_{\xi} f + (\xi - u_f) f|^2 dx d\xi ds \\ &\quad + C \int_0^t \int_{\Omega} \rho_f |u_f - u|^2 dx ds \end{aligned}$$

$$\int_0^t d_{\text{BL}}^2(f, M_{\rho, 0}) ds$$

$$\leq C d_{\text{BL}}^2(\rho_{f_0}, \rho_0) + C \int_0^t \iint_{\Omega \times \mathbb{R}^d} \frac{1}{f} |\nabla_{\xi} f + \xi f|^2 dx d\xi ds + C \int_0^t \int_{\Omega} \rho_f |u_f - u|^2 dx ds$$

$$u_{\varepsilon} = \varepsilon(v - \nabla \log \rho)$$

$$\partial_t \rho + \frac{1}{\varepsilon} \nabla \cdot (\rho u_{\varepsilon}) = 0$$

$$\rho \partial_t u_{\varepsilon} + \frac{1}{\varepsilon} \rho u_{\varepsilon} \cdot \nabla u_{\varepsilon} + \frac{1}{\varepsilon} \nabla \rho = \frac{1}{\varepsilon} \rho v - \frac{1}{\varepsilon^2} \rho u_{\varepsilon} + \rho e_{\varepsilon}$$

$$\partial_t v + v \cdot \nabla v + \nabla_x P - \Delta v = 0$$

$$\nabla \cdot v = 0$$

$$\rho e_{\varepsilon} = \rho \partial_t u_{\varepsilon} + \frac{1}{\varepsilon} \rho u_{\varepsilon} \cdot \nabla u_{\varepsilon}$$

$$\|e_{\varepsilon}\|_{L^{\infty}} \leq \varepsilon (\|\partial_t(v - \nabla \log \rho)\|_{L^{\infty}} + \|(v - \nabla \log \rho) \cdot \nabla(v - \nabla \log \rho)\|_{L^{\infty}})$$



$$\begin{aligned}
& \mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](t) + \frac{1}{2} \|(v^\varepsilon - v)(t)\|_{L^2}^2 + \frac{1}{\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds \\
& + \frac{1}{\varepsilon^2} \int_0^t \int_\Omega \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \int_0^t \int_\Omega |\nabla(v^\varepsilon - v)|^2 dx ds \\
& \leq \mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](0) + \frac{1}{2} \|(v^\varepsilon - v)(0)\|_{L^2}^2 \\
& - \frac{1}{\varepsilon} \int_0^t \iint_{\Omega \times \mathbb{R}^4} (f^\varepsilon(u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) - f^\varepsilon I_d) : \nabla u_\varepsilon dx d\xi ds + \int_0^t \int_\Omega (v^\varepsilon - v) \otimes (v^\varepsilon - v) : \nabla v dx ds \\
& + \int_0^t \int_\Omega (\rho^\varepsilon - \rho)(v^\varepsilon - v) \cdot \nabla \log \rho dx ds + \int_0^t \int_\Omega \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx ds
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \log \left(\frac{f^\varepsilon}{M_{\rho, u_\varepsilon}} \right) dx d\xi &= \iint_{\Omega \times \mathbb{R}^4} \partial_t f^\varepsilon (\log f^\varepsilon - \log M_{\rho, u_\varepsilon}) dx d\xi \\
&+ \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \left(\frac{\partial_t f^\varepsilon}{f^\varepsilon} - \frac{\partial_t M_{\rho, u_\varepsilon}}{M_{\rho, u_\varepsilon}} \right) dx d\xi \\
&= I + II
\end{aligned}$$

$$\begin{aligned}
I &= \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} (\log f^\varepsilon - \log M_{\rho, u_\varepsilon}) \cdot \left(-\xi \cdot \nabla f^\varepsilon + \frac{1}{\varepsilon} \nabla_\xi \cdot (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \right) dx d\xi \\
II &= - \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \left(\frac{\partial_t \rho}{\rho} - (u_\varepsilon - \xi) \cdot \partial_t u_\varepsilon \right) dx d\xi
\end{aligned}$$

$$\begin{aligned}
I_1 &:= \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} \xi f^\varepsilon \cdot \left(\frac{\nabla f^\varepsilon}{f^\varepsilon} - \frac{\nabla M_{\rho, u_\varepsilon}}{M_{\rho, u_\varepsilon}} \right) dx d\xi \\
&= -\frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} \xi f^\varepsilon \cdot (\nabla \log \rho - (\nabla u_\varepsilon)(u_\varepsilon - \xi)) dx d\xi \\
&= -\frac{1}{\varepsilon} \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \nabla \log \rho dx + \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} \xi f^\varepsilon \cdot (\nabla u_\varepsilon)(u_\varepsilon - \xi) dx d\xi
\end{aligned}$$

$$\begin{aligned}
I_2 &:= -\frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \cdot \left(\frac{\nabla_\xi f^\varepsilon}{f^\varepsilon} - \frac{\nabla_\xi M_{\rho, u_\varepsilon}}{M_{\rho, u_\varepsilon}} \right) dx d\xi \\
&= -\frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \cdot \left(\frac{\nabla_\xi f^\varepsilon}{f^\varepsilon} - (u_\varepsilon - \xi) \right) dx d\xi
\end{aligned}$$

$$\begin{aligned}
& \iint_{\Omega \times \mathbb{R}^4} (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \cdot \left(\frac{\nabla_\xi f^\varepsilon}{f^\varepsilon} + (\xi - u_\varepsilon) \right) dx d\xi \\
&= \iint_{\Omega \times \mathbb{R}^4} (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \cdot \left(\frac{\nabla_\xi f^\varepsilon}{f^\varepsilon} + (\xi - \varepsilon v^\varepsilon) \right) dx d\xi \\
&+ \iint_{\Omega \times \mathbb{R}^4} (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon) \cdot (\varepsilon v^\varepsilon - u_\varepsilon) dx d\xi \\
&= \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon|^2 dx d\xi - \int_\Omega \rho^\varepsilon (u^\varepsilon - \varepsilon v^\varepsilon) \cdot (u_\varepsilon - \varepsilon v^\varepsilon) dx \\
&= \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi + \int_\Omega \rho^\varepsilon (u^\varepsilon - \varepsilon v^\varepsilon) \cdot (u^\varepsilon - u_\varepsilon) dx
\end{aligned}$$



$$\begin{aligned}
I &= -\frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot \nabla \log \rho dx + \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \xi \cdot (\nabla u_\varepsilon)(u_\varepsilon - \xi) dx d\xi \\
&\quad - \frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^\varepsilon (u^\varepsilon - \varepsilon v^\varepsilon) \cdot (u_\varepsilon - u^\varepsilon) dx \\
\mathcal{D}\rho_1 &:= \frac{1}{\varepsilon} \int_{\Omega} \frac{\rho^\varepsilon}{\rho} \nabla \cdot (\rho u_\varepsilon) dx = \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon u_\varepsilon \cdot \nabla \log \rho dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon I_d : \nabla u_\varepsilon dx \\
\mathcal{D}\rho_2 &:= \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon (u_\varepsilon - \xi) \cdot \left(-\frac{1}{\varepsilon} (u_\varepsilon \cdot \nabla) u_\varepsilon - \frac{1}{\varepsilon} \nabla \log \rho + \frac{1}{\varepsilon} v - \frac{1}{\varepsilon^2} u_\varepsilon + e_\varepsilon \right) dx d\xi \\
&= -\frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon u_\varepsilon \cdot (\nabla u_\varepsilon)(u_\varepsilon - \xi) dx d\xi - \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot \nabla \log \rho dx \\
&\quad + \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon (u_\varepsilon - \xi) \cdot v dx d\xi - \frac{1}{\varepsilon^2} \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot u_\varepsilon dx + \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx \\
\mathcal{D}\rho &= \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot \nabla \log \rho dx - \frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^2} f^\varepsilon u_\varepsilon \cdot (\nabla u_\varepsilon)(u_\varepsilon - \xi) dx d\xi + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon I_d : \nabla u_\varepsilon dx \\
&\quad + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot (\varepsilon v - u_\varepsilon) dx + \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx \\
\frac{d}{dt} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \log \left(\frac{f^\varepsilon}{M_{\rho, u_\varepsilon}} \right) dx d\xi &+ \frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^\varepsilon |u_\varepsilon - u^\varepsilon|^2 dx \\
&= -\frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon (u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) : \nabla u_\varepsilon dx d\xi + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon I_d : \nabla u_\varepsilon dx \\
&\quad + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot (v - v^\varepsilon) dx + \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx \\
&\quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} |v^\varepsilon - v|^2 dx + \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx \\
&\quad = \int_{\Omega} (v^\varepsilon - v) \otimes (v^\varepsilon - v) : \nabla v dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (v^\varepsilon - v) \cdot (u^\varepsilon - \varepsilon v^\varepsilon) dx \\
\frac{d}{dt} \left(\iint_{\Omega \times \mathbb{R}^4} f^\varepsilon \log \left(\frac{f^\varepsilon}{M_{\rho, u_\varepsilon}} \right) dx d\xi + \frac{1}{2} \int_{\Omega} |v^\varepsilon - v|^2 dx \right) & \\
&+ \frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^\varepsilon |u_\varepsilon - u^\varepsilon|^2 dx + \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx \\
&= -\frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} f^\varepsilon (u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) : \nabla u_\varepsilon dx d\xi + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon I_d : \nabla u_\varepsilon dx + \int_{\Omega} (v^\varepsilon - v) \otimes (v^\varepsilon - v) : \nabla v dx \\
&\quad + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot (v - v^\varepsilon) dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (v^\varepsilon - v) \cdot (u^\varepsilon - \varepsilon v^\varepsilon) dx + \int_{\Omega} \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (u_{\varepsilon} - u^{\varepsilon}) \cdot (v - v^{\varepsilon}) dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (v^{\varepsilon} - v) \cdot (u^{\varepsilon} - \varepsilon v^{\varepsilon}) dx \\
&= \frac{2}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (u_{\varepsilon} - u^{\varepsilon}) \cdot (v - v^{\varepsilon}) dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v^{\varepsilon}) dx \\
&= -\frac{1}{\varepsilon^2} \int_{\Omega} \rho^{\varepsilon} |(u_{\varepsilon} - u^{\varepsilon}) - \varepsilon(v - v^{\varepsilon})|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^{\varepsilon} |u_{\varepsilon} - u^{\varepsilon}|^2 dx \\
&+ \int_{\Omega} \rho^{\varepsilon} |v - v^{\varepsilon}|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v^{\varepsilon}) dx \\
&= -\frac{1}{\varepsilon^2} \int_{\Omega} \rho^{\varepsilon} |(u_{\varepsilon} - u^{\varepsilon}) - \varepsilon(v - v^{\varepsilon})|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^{\varepsilon} |u_{\varepsilon} - u^{\varepsilon}|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v) dx \\
&\quad \frac{1}{\varepsilon} \int_{\Omega} \rho^{\varepsilon} (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v) dx = \frac{1}{\varepsilon} \int_{\Omega} (\rho^{\varepsilon} - \rho) (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v) dx
\end{aligned}$$

$$= \int_{\Omega} (\rho^{\varepsilon} - \rho) (v^{\varepsilon} - v) \cdot \nabla \log \rho dx$$

$$\int_{\Omega} \rho (v^{\varepsilon} - v) \cdot (u_{\varepsilon} - \varepsilon v) dx = -\varepsilon \int_{\Omega} (v^{\varepsilon} - v) \cdot \nabla \rho dx = 0$$

$$\rho u_{\varepsilon} = \varepsilon (\rho v - \nabla \rho)$$

$$\begin{aligned}
& \frac{d}{dt} \left(\iint_{\Omega \times \mathbb{R}^4} f^{\varepsilon} \log \left(\frac{f^{\varepsilon}}{M_{\rho, u_{\varepsilon}}} \right) dx d\xi + \frac{1}{2} \int_{\Omega} |v^{\varepsilon} - v|^2 dx \right) \\
&+ \frac{1}{\varepsilon^2} \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^{\varepsilon}} |\nabla_{\xi} f^{\varepsilon} + (\xi - u^{\varepsilon}) f^{\varepsilon}|^2 dx d\xi + \frac{1}{\varepsilon^2} \int_{\Omega} \rho^{\varepsilon} |(u_{\varepsilon} - u^{\varepsilon}) - \varepsilon(v - v^{\varepsilon})|^2 dx + \int_{\Omega} |\nabla (v^{\varepsilon} - v)|^2 dx \\
&= -\frac{1}{\varepsilon} \iint_{\Omega \times \mathbb{R}^4} (f^{\varepsilon} (u_{\varepsilon} - \xi) \otimes (u_{\varepsilon} - \xi) - f^{\varepsilon} I_d) : \nabla u_{\varepsilon} dx d\xi + \int_{\Omega} (v^{\varepsilon} - v) \otimes (v^{\varepsilon} - v) : \nabla v dx \\
&+ \int_{\Omega} (\rho^{\varepsilon} - \rho) (v^{\varepsilon} - v) \cdot \nabla \log \rho dx + \int_{\Omega} \rho^{\varepsilon} (u_{\varepsilon} - u^{\varepsilon}) \cdot e_{\varepsilon} dx
\end{aligned}$$

$$\frac{1}{2} \mathcal{H}[f^{\varepsilon} | M_{\rho, 0}] - C\varepsilon^2 \leq \mathcal{H}[f^{\varepsilon} | M_{\rho, u_{\varepsilon}}] \leq 2\mathcal{H}[f^{\varepsilon} | M_{\rho, 0}] + C\varepsilon^2.$$

$$\begin{aligned}
\iint_{\Omega \times \mathbb{R}^4} f^{\varepsilon} \log \frac{M_{\rho, u_{\varepsilon}}}{M_{\rho, 0}} dx d\xi &= \frac{1}{2} \int_{\Omega} \rho^{\varepsilon} (2u_{\varepsilon} \cdot u^{\varepsilon} - |u_{\varepsilon}|^2) dx \\
&= -\int_{\Omega} \rho^{\varepsilon} u_{\varepsilon} \cdot (u^{\varepsilon} - u_{\varepsilon}) dx + \frac{1}{2} \int_{\Omega} \rho^{\varepsilon} |u_{\varepsilon}|^2 dx,
\end{aligned}$$

$$|\mathcal{H}[f^{\varepsilon} | M_{\rho, 0}] - \mathcal{H}[f^{\varepsilon} | M_{\rho, u_{\varepsilon}}]| \leq 2 \int_{\Omega} \rho^{\varepsilon} |u_{\varepsilon}|^2 dx + \frac{1}{4} \int_{\Omega} \rho^{\varepsilon} |u^{\varepsilon} - u_{\varepsilon}|^2 dx \leq C\varepsilon^2 + \frac{1}{2} \mathcal{H}[f^{\varepsilon} | M_{\rho, u_{\varepsilon}}]$$

$$\mathcal{H}[f^{\varepsilon} | M_{\rho, u_{\varepsilon}}] \leq 2\mathcal{H}[f^{\varepsilon} | M_{\rho, 0}] + C\varepsilon^2$$

$$\mathcal{H}[f^{\varepsilon} | M_{\rho, 0}] \leq 2\mathcal{H}[f^{\varepsilon} | M_{\rho, u_{\varepsilon}}] + C\varepsilon^2.$$



$$\begin{aligned} & \mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](t) + \frac{1}{2} \|(v^\varepsilon - v)(t)\|_{L^2}^2 + \frac{1}{\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds \\ & + \frac{1}{\varepsilon^2} \int_0^t \int_\Omega \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \int_0^t \int_\Omega |\nabla(v^\varepsilon - v)|^2 dx ds \\ & \leq \mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](0) + \frac{1}{2} \|(v^\varepsilon - v)(0)\|_{L^2}^2 \end{aligned}$$

$$\mathfrak{J} := -\frac{1}{\varepsilon} \int_0^t \iint_{\Omega \times \mathbb{R}^4} (f^\varepsilon(u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) - f^\varepsilon I_d) : \nabla u_\varepsilon dx d\xi ds$$

$$\mathfrak{H} := \int_0^t \int_\Omega (v^\varepsilon - v) \otimes (v^\varepsilon - v) : \nabla v dx ds$$

$$\mathfrak{K} := \int_0^t \int_\Omega (\rho^\varepsilon - \rho)(v^\varepsilon - v) \cdot \nabla \log \rho dx ds$$

$$\mathfrak{L} := \int_0^t \int_\Omega \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \cdot e_\varepsilon dx ds$$

$$\begin{aligned} & \int_\Omega f^\varepsilon(u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) d\xi = \rho^\varepsilon(u_\varepsilon - u^\varepsilon) \otimes (u_\varepsilon - u^\varepsilon) - \int_\Omega f^\varepsilon(u^\varepsilon - \xi) \otimes \xi d\xi \\ & - \iint_{\Omega \times \mathbb{R}^4} (f^\varepsilon(u_\varepsilon - \xi) \otimes (u_\varepsilon - \xi) - f^\varepsilon I_d) : \nabla u_\varepsilon dx d\xi \\ & = - \int_\Omega \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \otimes (u_\varepsilon - u^\varepsilon) : \nabla u_\varepsilon dx + \int_\Omega \left(\int_\Omega (f^\varepsilon(u^\varepsilon - \xi) - \nabla_\xi f^\varepsilon) \otimes \xi d\xi \right) : \nabla u_\varepsilon dx \end{aligned}$$

$$\begin{aligned} \left| \int_\Omega \rho^\varepsilon (u_\varepsilon - u^\varepsilon) \otimes (u_\varepsilon - u^\varepsilon) : \nabla u_\varepsilon dx \right| & \leq \|\nabla u_\varepsilon\|_{L^\infty} \int_\Omega \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx \\ & = \varepsilon \|\nabla(v - \nabla \log \rho)\|_{L^\infty} \int_\Omega \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx \end{aligned}$$

$$\begin{aligned} & \left| \int_\Omega \left(\int_\Omega (f^\varepsilon(u^\varepsilon - \xi) - \nabla_\xi f^\varepsilon) \otimes \xi d\xi \right) : \nabla u_\varepsilon dx \right| \\ & \leq \|\nabla u_\varepsilon\|_{L^\infty} \left(\iint_{\Omega \times \mathbb{R}^4} |\xi|^2 f^\varepsilon dx d\xi \right)^{\frac{1}{2}} \left(\iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 dx d\xi \right)^{\frac{1}{2}} \\ & = \varepsilon \|\nabla(v - \nabla \log \rho)\|_{L^\infty} \|f^\varepsilon\|_{L^1}^{\frac{1}{2}} \left(\iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 dx d\xi \right)^{\frac{1}{2}} \end{aligned}$$

$$|\mathfrak{J}| \leq C \int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds + C\varepsilon^2 + \frac{1}{4\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^2} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds$$

$$|\mathfrak{H}| \leq \|\nabla v\|_{L^\infty} \int_0^t \int_\Omega |v^\varepsilon - v|^2 dx ds$$

$$\begin{aligned} |\mathfrak{K}| & \leq \|\nabla \log \rho\|_{L^\infty} \left(2 \int_0^t \int_\Omega \rho^\varepsilon \log \left(\frac{\rho^\varepsilon}{\rho} \right) dx ds \right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega (\rho + \rho^\varepsilon) |v^\varepsilon - v|^2 dx ds \right)^{\frac{1}{2}} \\ & \leq C \int_0^t \int_\Omega \rho^\varepsilon \log \left(\frac{\rho^\varepsilon}{\rho} \right) dx ds + C \int_0^t \int_\Omega |v^\varepsilon - v|^2 dx ds + \frac{1}{4} \int_0^t \int_\Omega |\nabla(v^\varepsilon - v)|^2 dx ds \end{aligned}$$



$$|\underline{\Delta}| \leq \frac{1}{2\varepsilon} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds + \varepsilon \frac{\|e_\varepsilon\|_{L^\infty}}{2} \|\rho^\varepsilon\|_{L^1} \leq \frac{1}{2\varepsilon} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds + C\varepsilon^2$$

$$\begin{aligned} \frac{1}{2\varepsilon} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds &\leq \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \varepsilon \int_0^t \int_{\Omega} \rho^\varepsilon |v - v^\varepsilon|^2 dx ds \\ &\leq \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + C \int_0^t \int_{\Omega} |v^\varepsilon - v|^2 dx ds \\ &\quad + \frac{1}{4} \int_0^t \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx ds \end{aligned}$$

$$\begin{aligned} &\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](t) + \|(v^\varepsilon - v)(t)\|_{L^2}^2 + \frac{1}{2\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds \\ &+ \frac{1}{2\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx ds \\ &\leq C\varepsilon^2 + C\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](0) + C\|(v^\varepsilon - v)(0)\|_{L^2}^2 + C \int_0^t (\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](s) + \|(v^\varepsilon - v)(s)\|_{L^2}^2) ds \end{aligned}$$

$$\begin{aligned} &\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](t) + \|(v^\varepsilon - v)(t)\|_{L^2}^2 + \frac{1}{2\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^4} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds \\ &+ \frac{1}{2\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |(u - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx ds \\ &\leq C\varepsilon^2 + C\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](0) + C\|v_0^\varepsilon - v_0\|_{L^2}^2 \end{aligned}$$

$$\sup_{0 \leq t \leq T} (\mathcal{H}[f^\varepsilon | M_{\rho, u_\varepsilon}](t) + \|(v^\varepsilon - v)(t)\|_{L^2}^2) + \|\nabla(v^\varepsilon - v)\|_{L^2([0, T] \times \Omega)}^2 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

$$\begin{aligned} f^\varepsilon &\rightarrow M_{\rho, 0} \text{ in } L^\infty(0, T; L^1(\Omega \times \mathbb{R}^2)) \\ \int_{\mathbb{R}^4} f^\varepsilon d\xi &\rightarrow \rho \text{ in } L^\infty(0, T; L^1(\Omega)) \\ v^\varepsilon &\rightarrow v \text{ in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \end{aligned}$$

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^2} \xi f^\varepsilon d\xi = \frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon$$

$$\begin{aligned} \int_{\Omega} |\rho^\varepsilon u^\varepsilon - \rho u_\varepsilon| dx &\leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon| dx + \int_{\Omega} |\rho^\varepsilon - \rho| |u_\varepsilon| dx \\ &\leq \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + \varepsilon \|v - \nabla \log \rho\|_{L^\infty} \|\rho^\varepsilon - \rho\|_{L^1} \end{aligned}$$

$$\frac{1}{2\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds$$



$$\begin{aligned} \frac{1}{2\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds &\leq \frac{1}{\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + \int_0^t \int_{\Omega} \rho^\varepsilon |v - v^\varepsilon|^2 dx ds \\ &\leq \frac{1}{\varepsilon^2} \int_0^t \int_{\Omega} \rho^\varepsilon |(u_\varepsilon - u^\varepsilon) - \varepsilon(v - v^\varepsilon)|^2 dx ds + C \int_0^t \int_{\Omega} |v^\varepsilon - v|^2 dx ds \\ &\quad + C \int_0^t \int_{\Omega} |\nabla(v^\varepsilon - v)|^2 dx ds \\ &\leq C\varepsilon^2 + C\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + C\|v_0^\varepsilon - v_0\|_{L^2}^2 \end{aligned}$$

$$\frac{1}{\varepsilon} \int_0^T \int_{\Omega} |\rho^\varepsilon u^\varepsilon - \rho u_\varepsilon| dx dt \leq C \left(\frac{1}{\varepsilon^2} \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx dt \right)^{\frac{1}{2}} + C \sup_{0 \leq t \leq T} \|(\rho^\varepsilon - \rho)(t)\|_{L^1}$$

$$\frac{1}{\varepsilon} \int_0^T \int_{\Omega} |\rho^\varepsilon u^\varepsilon - \rho u_\varepsilon| dx dt \leq C\varepsilon + C(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2)^{\frac{1}{2}}$$

$$d_{\text{BL}}(\rho_{f_0^\varepsilon}, \rho_0) \rightarrow 0 \text{ and } \varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

$$\begin{aligned} \sup_{0 \leq t \leq T} d_{\text{BL}}^2(\rho^\varepsilon(t), \rho(t)) + \int_0^T d_{\text{BL}}^2(f^\varepsilon, M_{\rho, u_\varepsilon}) ds \\ \leq C d_{\text{BL}}^2(\rho_0^\varepsilon, \rho_0) + C \int_0^T \iint_{\Omega \times \mathbb{R}^2} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds + C \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds \end{aligned}$$

$$\begin{aligned} \int_0^T \iint_{\Omega \times \mathbb{R}^2} \frac{1}{f^\varepsilon} |\nabla_\xi f^\varepsilon + (\xi - u^\varepsilon) f^\varepsilon|^2 dx d\xi ds + \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx ds \\ \leq C\varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^4 \end{aligned}$$

$$\begin{aligned} \sup_{0 \leq t \leq T} d_{\text{BL}}^2(\rho^\varepsilon(t), \rho(t)) + \int_0^T d_{\text{BL}}^2(f^\varepsilon, M_{\rho, u_\varepsilon}) ds \\ \leq C d_{\text{BL}}^2(\rho_0^\varepsilon, \rho_0) + C\varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^4 \end{aligned}$$

$$\iint_{\Omega \times \mathbb{R}^4} \varphi(f^\varepsilon - M_{\rho,0}) dx d\xi = \iint_{\Omega \times \mathbb{R}^4} \varphi(f^\varepsilon - M_{\rho, u_\varepsilon}) dx d\xi + \iint_{\Omega \times \mathbb{R}^4} \varphi(M_{\rho, u_\varepsilon} - M_{\rho,0}) dx d\xi$$

$$|\mathfrak{A}| \leq C d_{\text{BL}}(f^\varepsilon, M_{\rho, u_\varepsilon})$$

$$|\mathfrak{P}| \leq C \|\rho\|_{L^1} \|u_\varepsilon\|_{L^\infty} \leq C\varepsilon$$

$$d_{\text{BL}}^2(f^\varepsilon, M_{\rho,0}) \leq C d_{\text{BL}}^2(f^\varepsilon, M_{\rho, u_\varepsilon}) + C\varepsilon^2$$

$$\int_0^T d_{\text{BL}}^2(f^\varepsilon, M_{\rho,0}) dt \leq C d_{\text{BL}}^2(\rho_0^\varepsilon, \rho_0) + C\varepsilon^2(\mathcal{H}[f^\varepsilon | M_{\rho,0}](0) + \|v_0^\varepsilon - v_0\|_{L^2}^2) + C\varepsilon^2$$

$$\sup_{\varepsilon > 0} \sup_{0 \leq t \leq T} \iint_{\Omega \times \mathbb{R}^4} (|x| + |\xi|^2) f^\varepsilon dx d\xi < \infty$$

$$f^\varepsilon \rightarrow M_{\rho,0} \text{ in } L^1((0, T) \times \Omega \times \mathbb{R}^2) \text{ as } \varepsilon \rightarrow 0$$

$$\rho^\varepsilon \xrightarrow{*} \rho \text{ in } L^\infty(0, T; \mathcal{M}(\Omega)) \cap L^1((0, T) \times \Omega) \text{ as } \varepsilon \rightarrow 0,$$



$$\frac{1}{\varepsilon^2} \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon|^2 dx dt \leq \frac{2}{\varepsilon^2} \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u_\varepsilon|^2 dx dt + \frac{2T}{\varepsilon^2} \|u_\varepsilon\|_{L^\infty}^2 \leq C$$

$$\int_0^T \int_{\Omega} \frac{1}{\varepsilon} |\rho^\varepsilon u^\varepsilon| dx dt \leq \left(\frac{1}{\varepsilon^2} \int_0^T \int_{\Omega} \rho^\varepsilon |u^\varepsilon|^2 dx dt \right)^{\frac{1}{2}} \leq C$$

$$\frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon \rightharpoonup \tilde{u} \text{ weakly in } L^1((0, T) \times \Omega)$$

$$\int_0^T \int_{\Omega} \left(\rho^\varepsilon \partial_t \varphi + \frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon \cdot \nabla \varphi \right) dx dt + \int_{\Omega} \rho_0^\varepsilon \varphi(x, 0) dx$$

$$\int_0^T \int_{\Omega} (\rho \partial_t \varphi + \tilde{u} \cdot \nabla \varphi) dx dt + \int_{\Omega} \rho_0 \varphi(x, 0) dx$$

$$\int_0^T \int_{\Omega} (\rho \partial_t \varphi + \rho(v - \nabla \log \rho) \cdot \nabla \varphi) dx dt + \int_{\Omega} \rho_0 \varphi(x, 0) dx$$

$$\int_0^T \int_{\Omega} (\tilde{u} - \rho(v - \nabla \log \rho)) \cdot \nabla \varphi dx dt = 0 \text{ for any } \varphi \in C_c^\infty([0, T] \times \Omega)$$

$$\tilde{u} = \rho(v - \nabla \log \rho)$$

$$\frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon \rightharpoonup \rho(v - \nabla \log \rho) \text{ weakly in } L^1((0, T) \times \Omega)$$

$$\partial_t v^\varepsilon = \Delta v^\varepsilon - (v^\varepsilon \cdot \nabla) v^\varepsilon - \nabla p^\varepsilon + \frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon - \rho^\varepsilon v^\varepsilon$$

$$(v^\varepsilon \cdot \nabla) v^\varepsilon \in L^2(0, T; W^{-1,1}(\Omega))$$

$$\nabla p^\varepsilon \in L^2(0, T; W^{-1,1}(\Omega))$$

$$\frac{1}{\varepsilon} \rho^\varepsilon u^\varepsilon - \rho^\varepsilon v^\varepsilon \in L^2(0, T; L^1(\Omega)) \subset L^2(0, T; W^{-1,1}(\Omega))$$

$$\|\partial_t v^\varepsilon\|_{L^2(0, T; W^{-1,1}(\Omega))} \leq C$$

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,1}(\Omega)$$

$$H^1(K) \hookrightarrow L^2(K),$$

$$v^\varepsilon \rightarrow v \text{ in } L^2_{\text{loc}}((0, T) \times \Omega)$$

$$\rho^\varepsilon(t, x) := \int_{\mathbb{R}^4} f^\varepsilon(t, x, \xi) d\xi$$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} \xi \cdot \nabla f^\varepsilon = \frac{1}{\varepsilon^2} \nabla_\xi \cdot (\nabla_\xi f^\varepsilon + (\xi - \varepsilon v^\varepsilon) f^\varepsilon)$$



$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} \xi \cdot \nabla f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon - \frac{1}{\varepsilon} \nabla_\xi \cdot (v^\varepsilon f^\varepsilon), \mathcal{L} g := \nabla_\xi \cdot (\nabla_\xi g + \xi g)$$

$$M(\xi) := \frac{1}{2\pi} e^{-\frac{|\xi|^2}{2}}$$

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots, v^\varepsilon = v_0 + \varepsilon v_1 + \dots,$$

$$f_0(t, x, \xi) = \rho(t, x) M(\xi)$$

$$\mathcal{L} f_1 = \xi \cdot \nabla(\rho M) + \nabla_\xi \cdot (v_0 \rho M).$$

$$\mathcal{L}(\xi_i M) = -\xi_i M$$

$$f_1(t, x, \xi) = (-\xi \cdot \nabla \rho(t, x) + \rho(t, x) v_0(t, x) \cdot \xi) M(\xi).$$

$$m^\varepsilon(t, x) := \int_{\mathbb{R}^4} \xi f^\varepsilon(t, x, \xi) d\xi$$

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \nabla \cdot m^\varepsilon = 0$$

$$\int_{\mathbb{R}^4} \xi M d\xi = 0, \int_{\mathbb{R}^4} \xi_i \xi_j M d\xi = \delta_{ij}$$

$$m^\varepsilon = \varepsilon(-\nabla \rho + \rho v_0) + O(\varepsilon^2)$$

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3).$$

$$f_0 = \rho M, f_1 = (\xi \cdot \nabla \rho + \rho v_0 \cdot \xi) M,$$

$$f^\varepsilon = \rho M + \varepsilon(\xi \cdot \nabla \rho + \rho v_0 \cdot \xi) M + O(\varepsilon^2)$$

$$\rho^\varepsilon = \int_{\mathbb{R}^2} f_0 d\xi + \varepsilon \int_{\mathbb{R}^2} f_1 d\xi + \varepsilon^2 \int_{\mathbb{R}^2} f_2 d\xi + O(\varepsilon^3)$$

$$\int_{\mathbb{R}^4} f_0 d\xi = \rho \text{ and } \int_{\mathbb{R}^4} f_1 d\xi = 0$$

$$\rho^\varepsilon - \rho = \varepsilon^2 \int_{\mathbb{R}^4} f_2 d\xi + O(\varepsilon^3), \text{ hence } \rho^\varepsilon = \rho + O(\varepsilon^2)$$

$$\partial_t \rho + \nabla \cdot (\rho v) = \Delta \rho$$

$$\partial_t v + v \cdot \nabla v + \nabla P - \Delta v = 0$$

$$\nabla \cdot v = 0$$

$$X_{s,p}(T) := \{\rho > 0: \rho \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)), \nabla \log \rho \in L^\infty([0, T]; (L^p \cap \dot{H}^s)(\Omega))\}.$$

$$\int_0^T \|v(t)\|_{W^{1,\infty}(\Omega)} dt < \infty$$

$$\rho_0 > 0, \rho_0 \in L^1(\Omega) \cap L^\infty(\Omega), \text{ and } \phi_0 := \nabla \log \rho_0 \in (L^p \cap H^s)(\Omega)$$



$$\iiint \|\phi_0\|_{L^p \cap \dot{H}^s} \otimes \|\nabla v\|_{L^1(0,T;L^\infty)}$$

$$\int_0^{T'} \|\nabla \phi(t)\|_{L^\infty(\Omega)} dt < \infty$$

$$\rho_0(x) := C_k(1 + |x|^2)^{-k}$$

$$\partial_t \phi + \nabla(\phi \cdot v) = \Delta \phi + \nabla(|\phi|^2)$$

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\phi\|_{L^p}^p &= \int_{\mathbb{R}^4} |\phi|^{p-2} \phi \cdot (-\nabla(\phi \cdot v) + \Delta \phi + \nabla(|\phi|^2)) dx \\ &= \mathcal{D}\rho_1 + \mathcal{D}\rho_2 + \mathcal{D}\rho_3 \end{aligned}$$

$$\partial_i(|\phi|^\ell) = \ell |\phi|^{\ell-2} \phi_j \partial_i \phi_j, \ell \in \mathbb{N},$$

$$\begin{aligned} \mathcal{D}\rho_1 &= - \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_i \partial_i (v_j \phi_j) dx = - \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_i \phi_j \partial_i v_j dx - \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_i \partial_i \phi_j v_j dx \\ &= - \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_i \phi_j \partial_i v_j dx + \frac{1}{p} \int_{\mathbb{R}^4} \nabla(|\phi|^p) \cdot v dx \\ &\leq \|\nabla v\|_{L^\infty} \|\phi\|_{L^p}^p \end{aligned}$$

$$\begin{aligned} \mathcal{D}\rho_2 &= \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_j \partial_{ii} \phi_j dx = - \int_{\mathbb{R}^4} |\phi|^{p-2} |\nabla \phi|^2 dx - (p-2) \int_{\mathbb{R}^4} |\phi|^{p-4} (\phi_k \partial_i \phi_k) (\phi_j \partial_i \phi_j) dx \\ &= - \int_{\mathbb{R}^4} |\phi|^{p-2} |\nabla \phi|^2 dx - \frac{4(p-2)}{p^2} \int_{\mathbb{R}^4} |\nabla |\phi|^{\frac{p}{2}}|^2 dx \end{aligned}$$

$$\mathcal{D}\rho_3 = 2 \int_{\mathbb{R}^4} |\phi|^{p-2} \phi_j \partial_j \phi_k \phi_k dx = \frac{2}{p} \int_{\mathbb{R}^4} \nabla(|\phi|^p) \cdot \phi dx \leq C \|\nabla \cdot \phi\|_{L^\infty} \|\phi\|_{L^p}^p$$

$$\frac{d}{dt} \|\phi\|_{L^p}^p + p \int_{\mathbb{R}^4} |\phi|^{p-2} |\nabla \phi|^2 dx + \frac{4(p-2)}{p} \int_{\mathbb{R}^4} |\nabla |\phi|^{\frac{p}{2}}|^2 dx \leq C(\|\nabla v\|_{L^\infty} + \|\nabla \phi\|_{L^\infty}) \|\phi\|_{L^p}^p$$

$$\Lambda^a := (-\Delta)^{\frac{a}{2}}, a > 0$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^a \phi\|_{L^2}^2 + 2 \|\Lambda^a(\nabla \phi)\|_{L^2}^2 &= - \int_{\Omega} \Lambda^a \nabla(\phi \cdot v) \cdot \Lambda^a \phi dx + \int_{\Omega} \Lambda^a(\nabla|\phi|^2) \cdot \Lambda^a \phi dx \\ &=: \square_1 + \square_2 \end{aligned}$$

$$\begin{aligned} \square_1 &= - \int_{\Omega} v_j (\Lambda^a \partial_i \phi_j) \Lambda^a \phi_i dx - \int_{\Omega} [\Lambda^a \partial_i v_j \phi_j - v_j \Lambda^a \partial_i \phi_j] \Lambda^a \phi_i dx \\ &= - \int_{\Omega} v_j (\Lambda^a \partial_j \phi_i) \Lambda^a \phi_i dx - \int_{\Omega} [\Lambda^a \partial_i v_j \phi_j - v_j \Lambda^a \partial_i \phi_j] \Lambda^a \phi_i dx \\ &= - \int_{\Omega} [\Lambda^a \partial_i v_j \phi_j - v_j \Lambda^a \partial_i \phi_j] \Lambda^a \phi_i dx \\ &\leq C \|\nabla v\|_{L^\infty} \|\Lambda^a \phi\|_{L^2}^2 \end{aligned}$$



$$\begin{aligned} \square_2 &= \int_{\Omega} \phi_j (\Lambda^\alpha \partial_i \phi_j) \Lambda^\alpha \phi_i dx + \int_{\Omega} [\Lambda^\alpha \partial_i (\phi_j \phi_j) - \phi_j (\Lambda^\alpha \partial_i \phi_j)] \Lambda^\alpha \phi_i dx \\ &= -\frac{1}{2} \int_{\Omega} (\nabla \cdot \phi) |\Lambda^\alpha \phi|^2 dx + \int_{\Omega} [\Lambda^\alpha \partial_i (\phi_j \phi_j) - \phi_j (\Lambda^\alpha \partial_i \phi_j)] \Lambda^\alpha \phi_i dx \\ &\leq C \|\nabla \phi\|_{L^\infty} \|\Lambda^\alpha \phi\|_{L^2}^2 \end{aligned}$$

$$\frac{d}{dt} \|\phi\|_{L^p \cap \dot{H}^s}^2 \leq C (\|\nabla v\|_{L^\infty} + \|\nabla \phi\|_{L^\infty}) \|\phi\|_{L^p \cap \dot{H}^s}^2$$

$$L^p \cap \dot{H}^s(\Omega) \hookrightarrow W^{1,\infty}(\Omega), \text{ hence } \|\nabla \phi\|_{L^\infty} \leq C \|\phi\|_{L^p \cap \dot{H}^s}$$

$$\sup_{0 \leq t \leq T_*} \|\phi(t)\|_{H^s} < \infty \text{ for some } T_* = T_*(\|\phi_0\|_{H^s}, \|\nabla v\|_{L^1(0,T;L^\infty)}) > 0$$

$$\int_0^T \|\nabla \phi(t)\|_{L^\infty(\Omega)} dt < \infty$$

$$\phi \in L^\infty(0, T_*; L^\infty(\Omega))$$

$$\rho \in L^\infty(0, T_*; L^1 \cap L^\infty(\Omega))$$

$$\phi_0 = \nabla \log \rho_0 \in H^s(\mathbb{T}^2) \text{ with } s > 2$$

$$c_0 \leq \rho(t, x) \leq \|\rho_0\|_{L^\infty(\mathbb{T}^2)} \text{ for all } (t, x) \in [0, T] \times \mathbb{T}^2$$

$$\rho(t) = e^{(t-t_0)\Delta} \rho(t_0) - \int_{t_0}^t e^{(t-s)\Delta} (v(s) \cdot \nabla \rho(s)) ds$$

$$\|\nabla^m e^{\tau \Delta} f\|_{L^\infty} \leq C_m \tau^{-m/2} \|f\|_{L^\infty}, \tau > 0$$

$$\|\nabla \rho(t)\|_{L^\infty} \leq C(t-t_0)^{-1/2} \|\rho(t_0)\|_{L^\infty} + \int_{t_0}^t C(t-s)^{-1/2} \|v(s)\|_{L^\infty} \|\nabla \rho(s)\|_{L^\infty} ds$$

$$\sup_{t \in [t_0+\delta, T]} \|\nabla \rho(t)\|_{L^\infty} \leq C_{\delta, T} \|\rho(t_0)\|_{L^\infty} \exp(C_{\delta, T} \|v\|_{L^1(t_0, T; L^\infty)})$$

$$\|\nabla \rho(t_0)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \|\phi\|_{L^\infty(0, t_0; L^\infty)}$$

$$\|\phi\|_{L^\infty(0, t_0; L^\infty)} < \infty$$

$$\sup_{t \in [t_0+\delta, T]} \|\nabla \rho(t)\|_{W^{1,\infty}} \leq C_{\delta, T} (\|\rho_0\|_{L^\infty} + \|\rho_0\|_{L^\infty} \|\phi\|_{L^\infty(0, t_0; L^\infty)}) \exp(C_{\delta, T} \|v\|_{L^1(t_0, T; L^\infty)})$$

$$\nabla \phi = \nabla \left(\frac{\nabla \rho}{\rho} \right) = \frac{\nabla^2 \rho}{\rho} - \frac{\nabla \rho \otimes \nabla \rho}{\rho^2}$$

$$\|\nabla \phi(t)\|_{L^\infty} \leq \frac{1}{c_0} \|\nabla^2 \rho(t)\|_{L^\infty} + \frac{1}{c_0^2} \|\nabla \rho(t)\|_{L^\infty}^2$$

$$\int_{t_0+\delta}^T \|\nabla \phi(t)\|_{L^\infty} dt < \infty$$



$$\int_0^{t_0+\delta} \|\nabla\phi(t)\|_{L^\infty} dt < \infty$$

$$H_{N,b}^\Lambda := \sum_{i=1}^N (H_b^\Lambda)_{x_i} + 2\lambda \sum_{1 \leq i < j \leq N} V(x_i; x_j),$$

$$(H_b^\Lambda)_{x_i} = (-i\nabla_{x_i} - bA(x_i))^2, A(x_i) = A(x_i^{(1)}, x_i^{(2)}) := \frac{1}{2}(-x_i^{(2)}, x_i^{(1)}).$$

$$L_a^2(\Lambda^N) \subset L^2(\Lambda)^{\otimes N}$$

$$\tau_{b,(L_1,0)}, \tau_{b,(0,L_2)}: L^2(\Lambda) \rightarrow L^2(\Lambda)$$

$$(\tau_{b,(L_1,0)}\psi)(x_1, x_2) := e^{-ibL_1x_2}\psi(x_1 - L_1, x_2), (\tau_{b,(0,L_2)}\psi)(x_1, x_2) := e^{ibL_2x_1}\psi(x_1, x_2 - L_2).$$

$$\psi(L_1/2, x_2) = (\tau_{b,(L_1,0)}\psi)(-L_1/2, x_2), \psi(x_1, L_2/2) = (\tau_{b,(0,L_2)}\psi)(x_1, -L_2/2)$$

$$E_n := |b|(2n + 1), n \in \mathbb{N}$$

$$E_{N,b}^\Lambda[\omega] = \text{Tr}_{L_a^2(\Lambda^N)}(H_{N,b}^\Lambda \omega)$$

$$0 \leq \gamma \leq 1, \text{Tr}_{L^2(\Lambda)}\gamma = N = \rho|\Lambda|$$

$$\omega_{\text{Slater}} := |\psi_{\text{Slater}}\rangle\langle\psi_{\text{Slater}}|, \psi_{\text{Slater}} = f_1 \wedge \dots \wedge f_N$$

$$\mathcal{E}_{\text{HF}}^{\Lambda,b}[\gamma] := \text{Tr}_{L^2(\Lambda)}(H_b^\Lambda \gamma) + \lambda \int_{\Lambda} dx \int_{\Lambda} dy v(x-y)[\gamma(x,x)\gamma(y,y) - |\gamma(x,y)|^2]$$

$$E_{N,b}^\Lambda[\omega] = \mathcal{E}_{\text{HF}}^{\Lambda,b}[\gamma] \text{ for } \gamma = \omega^{(1)}$$

$$\gamma_{\beta,\mu} = f_{\text{FD}}(H_b^\Lambda) = e^{-\beta(H_b^\Lambda - \mu)} (\text{Id} + e^{-\beta(H_b^\Lambda - \mu)})^{-1} = (\text{Id} + e^{\beta(H_b^\Lambda - \mu)})^{-1}$$

$$f_{\text{FD}}(t) := \frac{1}{1 + e^{\beta(t-\mu)}}, t \in \mathbb{R}$$

$$\mathcal{E}_{\text{HF}}^{\Lambda,b}[\gamma] = \text{Tr}_{L^2(\Lambda)}(H_b^\Lambda \gamma) + \lambda \int_{\Lambda} dx \left[\int_{\Lambda} dx' \delta(x-x') \int_{\Lambda} dy v(x-y)\gamma(y,y) \right] \gamma(x',x)$$

$$- \lambda \int_{\Lambda} dx \left[\int_{\Lambda} dx' v(x-x')\gamma(x,x') \right] \overline{\gamma(x,x')}$$

$$= \text{Tr}_{L^2(\Lambda)}(H_b^\Lambda \gamma) + \lambda \int_{\Lambda} dx W^\Lambda[\gamma](x, x')\gamma(x', x)$$

$$= \text{Tr}_{L^2(\Lambda)}((H_b^\Lambda + \lambda W^\Lambda[\gamma])\gamma)$$

$$W^\Lambda[\gamma](x, x') := \delta(x-x') \int_{\Lambda} v(x-y)\gamma(y,y)dy - v(x-x')\gamma(x, x')$$

$$H_{b,\lambda}^\Lambda[\gamma] := H_b^\Lambda + \lambda W^\Lambda[\gamma]$$



$$\gamma_* = f(H_{b,\lambda}^\Lambda[\gamma_*])$$

$$\mathcal{E}_{\text{HF}}[\gamma] := \lim_{L_1, L_2 \rightarrow \infty} \frac{1}{|\Lambda|} \mathcal{E}_{\text{HF}}^\Lambda[\gamma]$$

$$\rho = \rho_\gamma := \lim_{L_1, L_2 \rightarrow \infty} \frac{1}{|\Lambda|} \text{Tr}_{L^2(\Lambda)}(\gamma) = \gamma(0,0)$$

$$\mathcal{E}_{\text{HF}}[\gamma] = \underline{\text{Tr}}((H_b + \lambda W[\gamma])\gamma),$$

$$H_b := (-i\nabla - bA(x))^2, A(x) = A(x_1, x_2) = \frac{1}{2}(-x_2, x_1),$$

$$W[\gamma](x, x') := \delta(x - x')v(x)\rho_\gamma - v(x - x')\gamma(x, x'),$$

$$W(x, x') = \delta(x - x') \int_{\mathbb{R}^4} v(x - y)f(H_b + \lambda W)(y, y)dy - v(x - x')f(H_b + \lambda W)(x, x'), x, x' \in \mathbb{R}^2$$

$$\phi(x, x') := (x_2x'_1 - x_1x'_2)/2 =: \frac{1}{2}x' \wedge x, x, x' \in \mathbb{R}^2.$$

$$\phi(x, x') = -\phi(x', x)$$

$$(\tau_{b,y}\psi)(x) := e^{ib\phi(x,y)}\psi(x - y), \psi \in L^2(\mathbb{R}^2),$$

$$T(x, x') = e^{ib\phi(x,x')}F_T(x - x'), \text{ where } F_T(x) := T(x, 0).$$

$$F(x) = f(H_b + \lambda W_F)(x, 0)$$

$$W_F := \left(\int_{\mathbb{R}^4} v(y)dy \right) F(0) + Z_F, Z_F(x, x') := -e^{ib\phi(x,x')}v(x - x')F(x - x')$$

$$v(x) = e^{-\alpha|x|} \ln(|x|)$$

$$BC_H(\mathbb{R}^2) := \{G \in BC(\mathbb{R}^2) \mid \overline{G(-x)} = G(x), \forall x \in \mathbb{R}^2\}$$

$$BC(\mathbb{R}^2) := C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$

$$\sup_{x \in [0, \infty)} \langle x \rangle^n |f^{(k)}(x)| < \infty$$

$$H_{b,\lambda,F} := H_b + \lambda W_F, \lambda \in \mathbb{R}$$

$$\Phi_{b,\lambda}^f(F)(\cdot) \equiv \Phi_\lambda(F)(\cdot) := f(H_{b,\lambda,F})(\cdot, 0) \in BC_H(\mathbb{R}^2)$$

$$\Phi_\lambda(F) = F, F \in BC_H(\mathbb{R}^2),$$

$$f(H_{b,\lambda,F}) * f(H_{b,\lambda,G}) \iint \mathcal{O}(|\lambda| \|F - G\|_{L^\infty(\mathbb{R}^4)})$$

$$\mathcal{J}(P_{b,\lambda,N}) = P_{b,\lambda,N}(0,0) = \frac{Nb}{2\pi}.$$



$$P_{b,\lambda,N} - P_{b,0,N} = f_N(H_{b,\lambda,F_{b,\lambda}}) - f_N(H_{b,\lambda,0}) \sim |\lambda|$$

$$2\pi \frac{\partial \mathcal{J}(P_{b,\lambda,N})}{\partial b} = 2\pi \frac{\partial \mathcal{J}(P_{b,0,N})}{\partial b} = N$$

$$|\partial^\alpha p(\xi)| \leq C_\alpha \langle \xi \rangle^m, \forall \xi \in \mathbb{R}^2.$$

$$\langle f_1, \text{Op}^A(p) f_2 \rangle := (2\pi)^{-2} \int_{\mathbb{R}^4} d\xi \int_{\mathbb{R}^4} dx dy \overline{f_1(x)} f_2(y) e^{ib\phi(x,y)} e^{i\xi \cdot (x-y)} p(\xi),$$

$$\phi(x, y) = (x_2 y_1 - x_1 y_2) / 2 = x \cdot y^\perp,$$

$$\Psi_{\gamma, \gamma^*}(y) := e^{ib\phi(y, \gamma)} g(y - \gamma) (2\pi)^{-1} e^{i\gamma^* \cdot (y - \gamma)}, y \in \mathbb{R}^2$$

$$\text{supp}(g) \subset (-1, 1)^2 \subset \mathbb{R}^2, \sum_{\gamma \in \mathbb{Z}^2} g^2(x - \gamma) = 1.$$

$$f(x) = \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} \Psi_{\gamma, \gamma^*}(x) \langle \Psi_{\gamma, \gamma^*}, f \rangle$$

$$f = \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} \Psi_{\gamma, \gamma^*} \langle \Psi_{\gamma, \gamma^*}, f \rangle$$

$$\langle f_1, f_2 \rangle = \sum_{\gamma \in \mathbb{Z}^2} \langle g(\square - \gamma) f_1, g(\square - \gamma) f_2 \rangle = \sum_{\gamma \in \mathbb{Z}^2} \langle e^{ib\phi(\cdot, \gamma)} g(\square - \gamma) f_1, e^{ib\phi(\cdot, \gamma)} g(\square - \gamma) f_2 \rangle,$$

$$\langle f_1, f_2 \rangle = \sum_{\gamma \in \mathbb{Z}^2} \langle f_1, \Psi_{\gamma, \gamma^*} \rangle \langle \Psi_{\gamma, \gamma^*}, f_2 \rangle$$

$$f(x) = \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} \Psi_{\gamma, \gamma^*}(x) \langle \Psi_{\gamma, \gamma^*}, f \rangle$$

$$(\Psi_{\gamma, \gamma^*})_{\gamma, \gamma^* \in \mathbb{Z}^2} \setminus R: \mathcal{S}(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$$

$$|\langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle| \leq C \langle \alpha - \beta \rangle^{-n} \langle \alpha^* - \beta^* \rangle^{-n}, \forall \alpha, \beta, \alpha^*, \beta^* \in \mathbb{Z}^2.$$

$$\langle f_1, Rf_2 \rangle = (2\pi)^{-2} \int_{\mathbb{R}^4} d\xi \int_{\mathbb{R}^4} dx dy e^{i\xi \cdot (x-y)} e^{ib\phi(x,y)} r\left(\frac{x+y}{2}, \xi\right) \overline{f_1(x)} f_2(y)$$

$$\langle f_1, Rf_2 \rangle = \langle R^* f_1, f_2 \rangle$$

$$= \sum_{\beta, \beta^* \in \mathbb{Z}^2} \langle R^* f_1, \Psi_{\beta, \beta^*} \rangle \langle \Psi_{\beta, \beta^*}, f_2 \rangle = \sum_{\beta, \beta^* \in \mathbb{Z}^2} \langle f_1, R\Psi_{\beta, \beta^*} \rangle \langle \Psi_{\beta, \beta^*}, f_2 \rangle$$

$$= \sum_{\alpha, \beta, \alpha^*, \beta^* \in \mathbb{Z}^2} \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \int_{\mathbb{R}^4} dx dy \Psi_{\alpha, \alpha^*}(x) \overline{\Psi_{\beta, \beta^*}(y)} f_2(y) \overline{f_1(x)}$$

$$= \sum_{\alpha, \beta, \alpha^*, \beta^* \in \mathbb{Z}^2} \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \times$$

$$\times \int_{\mathbb{R}^4} du dv \Psi_{\alpha, \alpha^*}\left(u + \frac{v}{2}\right) \overline{\Psi_{\beta, \beta^*}\left(u - \frac{v}{2}\right)} f_2\left(u - \frac{v}{2}\right) \overline{f_1\left(u + \frac{v}{2}\right)}$$



$$R(u, v) = \sum_{\alpha, \beta, \alpha^*, \beta^* \in \mathbb{Z}^2} \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \Psi_{\alpha, \alpha^*} \left(u + \frac{v}{2}\right) \overline{\Psi_{\beta, \beta^*} \left(u - \frac{v}{2}\right)}$$

$$e^{ib\phi(u, v)} = e^{-ib\phi(x, y)} \sum^N$$

$$r_N(u, \xi) := \sum^N \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \int_{\mathbb{R}^4} dv e^{ib\phi(u, v)} e^{-i\xi \cdot v} \Psi_{\alpha, \alpha^*} \left(u + \frac{v}{2}\right) \overline{\Psi_{\beta, \beta^*} \left(u - \frac{v}{2}\right)}$$

$$\int_{\mathbb{R}^4} dv e^{ib\phi(u, v)} e^{-i\xi \cdot v} \Psi_{\alpha, \alpha^*} \left(u + \frac{v}{2}\right) \overline{\Psi_{\beta, \beta^*} \left(u - \frac{v}{2}\right)}$$

$$= \frac{e^{i(\beta \cdot \beta^* - \alpha \cdot \alpha^*)} e^{iu \cdot (b(\alpha - \beta)^\perp + \alpha^* - \beta^*)}}{(2\pi)^2} \times$$

$$\times \int_{\mathbb{R}^4} dv e^{iv \cdot (-\xi - bu^\perp + (b\alpha^\perp + b\beta^\perp + \alpha^* + \beta^*)/2)} g\left(u + \frac{v}{2} - \alpha\right) g\left(u - \frac{v}{2} - \beta\right)$$

$$\frac{e^{i(\beta \cdot \beta^* - \alpha \cdot \alpha^*)} e^{iu \cdot (b(\alpha - \beta)^\perp + \alpha^* - \beta^*)}}{(2\pi)^2}$$

$$\times \int_{(-2, 2)^2} dw e^{i(w + \alpha - \beta) \cdot (-\xi - bu^\perp + (b\alpha^\perp + b\beta^\perp + \alpha^* + \beta^*)/2)} g\left(u + \frac{w - \alpha - \beta}{2}\right) g\left(u - \frac{w + \alpha + \beta}{2}\right)$$

$$u \notin (\alpha + \beta)/2 + (-1, 1)^2$$

$$\langle \xi + (\alpha^* + \beta^*)/2 \rangle^{-n}$$

$$\langle u - (\alpha + \beta)/2 \rangle^n \oslash \langle \xi + (\alpha^* + \beta^*)/2 \rangle^{-n}$$

$$u - (\alpha + \beta)/2 \notin (-1, 1)^2$$

$$\langle w \rangle^n \langle \alpha - \beta \rangle^n \doteq \langle \alpha^* - \beta^* \rangle^n$$

$$R = \text{Op}^A(r) \text{ with } r \in S_0^0(\mathbb{R}^2)$$

$$(\tau_z f)(x) = e^{ib\phi(x, z)} f(x - z)$$

$$\langle f_1, Rf_2 \rangle = \langle \tau_z f_1, R\tau_z f_2 \rangle$$

$$-\phi(x + z, z) + \phi(x + z, y + z) + \phi(y + z, z) = \phi(x, y) = \phi(v, u)$$

$$0 = \int_{\mathbb{R}^4} d\xi \int_{\mathbb{R}^4} du dv e^{i\xi \cdot v} e^{ib\phi(v, u)} (r(u + z, \xi) - r(u, \xi)) \overline{f_1 \left(u + \frac{v}{2}\right)} f_2 \left(u - \frac{v}{2}\right)$$

$$F(x, y) = \sum_{\alpha, \alpha^* \in \mathbb{Z}^2} \langle F(x, \cdot), \Psi_{\alpha, \alpha^*} \rangle \Psi_{\alpha, \alpha^*}(y), x, y \in \mathbb{R}^4$$

$$f_1(x) = \langle F(x, \cdot), \Psi_{\alpha, \alpha^*} \rangle \text{ and } f_2(y) = \Psi_{\alpha, \alpha^*}(y)$$

$$0 = \int_{\mathbb{R}^4} d\xi \int_{\mathbb{R}^4} du dv e^{i\xi \cdot v} e^{i\phi(v, u)} (r(u + z, \xi) - r(u, \xi)) F \left(u + \frac{v}{2}, u - \frac{v}{2}\right)$$



$$\mathcal{S}(\mathbb{R}^4) \ni F(u, v) \mapsto G(u, \xi) := \int_{\mathbb{R}^4} dv e^{i\xi \cdot v} e^{ib\phi(v, u)} F\left(u + \frac{v}{2}, u - \frac{v}{2}\right) \in \mathcal{S}(\mathbb{R}^4)$$

$$0 = \int_{\mathbb{R}^4} du d\xi (r(u + z, \xi) - r(u, \xi)) G(u, \xi)$$

$$\Pi_j := \frac{1}{2} i [H_b, X_j] = (-i\partial_{x_j} - bA_j(x)) \quad j \in \{1, 2\}$$

$$A(x) = (1/2)(-x_2, x_1)$$

$$R: L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$$

$$[T_1, [T_2, \dots [T_n, R] \dots]], n \geq 1,$$

$$\begin{aligned} (\alpha_j - \beta_j) \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle &= \langle X_j \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle + \langle (\alpha_j \text{Id} - X_j) \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \\ &\quad - \langle \Psi_{\alpha, \alpha^*}, R X_j \Psi_{\beta, \beta^*} \rangle - \langle \Psi_{\alpha, \alpha^*}, R(\beta_j \text{Id} - X_j) \Psi_{\beta, \beta^*} \rangle \\ &= \langle \Psi_{\alpha, \alpha^*}, [X_j, R] \Psi_{\beta, \beta^*} \rangle + \langle (\alpha_j \text{Id} - X_j) \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \\ &\quad - \langle \Psi_{\alpha, \alpha^*}, R(\beta_j \text{Id} - X_j) \Psi_{\beta, \beta^*} \rangle \end{aligned}$$

$$\langle (\alpha_j \text{Id} - X_j) \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle - \langle \Psi_{\alpha, \alpha^*}, R(\beta_j \text{Id} - X_j) \Psi_{\beta, \beta^*} \rangle$$

$$\bigwedge (x_j - \gamma_j) \bigvee g(x - \gamma) \ominus \langle \Psi_{\alpha, \alpha^*}, [X_j, R] \Psi_{\beta, \beta^*} \rangle$$

$$(\alpha_j - \beta_j) \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \triangle (\alpha_j - \beta_j)^n \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle$$

$$(-i\partial_{x_j} - bA_j(x)) e^{ib\phi(x, \gamma)} = e^{ib\phi(x, \gamma)} (-i\partial_{x_j} - bA_j(x - \gamma))$$

$$(\alpha_j^* - \beta_j^*) \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle = \langle \Pi_j \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle - \langle \Psi_{\alpha, \alpha^*}, R\Pi_j \Psi_{\beta, \beta^*} \rangle$$

$$(\alpha_j^* \beta_j^*) \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle \dagger (\alpha_j^* - \beta_j^*)^n \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle$$

$$\langle f_1, \text{Op}^A(p) f_2 \rangle = \sum_{\alpha, \beta, \alpha^*, \beta^* \in \mathbb{Z}^2} \langle f_1, \Psi_{\alpha, \alpha^*} \rangle \langle \Psi_{\alpha, \alpha^*}, \text{Op}^A(p) \Psi_{\beta, \beta^*} \rangle \langle \Psi_{\beta, \beta^*}, f_2 \rangle$$

$$\langle \Psi_{\alpha, \alpha^*}, \text{Op}^A(p) \Psi_{\beta, \beta^*} \rangle e^{i\beta \cdot \beta^* - i\alpha \cdot \alpha^*}$$

$$= (2\pi)^{-2} \int_{\mathbb{R}^4} d\xi \int_{\mathbb{R}^4} dx dy e^{ib\phi(\alpha, x)} g(x - \alpha) e^{-i\alpha^* \cdot x} e^{ib\phi(x, y)} e^{i\xi \cdot (x-y)} p(\xi) e^{ib\phi(y, \beta)} g(y - \beta) e^{i\beta^* \cdot y}$$

$$\begin{aligned} \xi - (\alpha^* + \beta^*)/2 &\mapsto \zeta \\ (x + y - \alpha - \beta)/2 &\mapsto u \\ x - y + \beta - \alpha &\mapsto v \end{aligned}$$

$$\langle \Psi_{\alpha, \alpha^*}, \text{Op}^A(p) \Psi_{\beta, \beta^*} \rangle e^{i\beta \cdot \beta^* - i\alpha \cdot \alpha^*}$$

$$\begin{aligned} &= (2\pi)^{-2} e^{i(b\alpha \cdot \beta^{\perp} + \alpha \cdot \beta^* / 2 - \beta \cdot \alpha^* / 2)} \int_{\mathbb{R}^4} \langle d\zeta \rangle e^{ib\phi(x, x')} T_{\epsilon}(x - x') \int_{\mathbb{R}^4} du dv e^{i[\zeta \cdot (v + \alpha - \beta) - u \cdot v^{\perp} + u \cdot (2b(\beta - \alpha)^{\perp} + \beta^* - \alpha^*)]} \\ &\times p(\zeta + (\alpha^* + \beta^*)/2) g(u + v/2) g(u - v/2) \end{aligned}$$

$$p(\zeta + (\alpha^* + \beta^*)/2) g(u + v/2) g(u - v/2)$$



$p \in S_0^{-m}(\mathbb{R}^2)$ and $q \in S_0^{-n}(\mathbb{R}^2)$ with $m, n \geq 0$, then $t \in S_0^{-n-m}(\mathbb{R}^4)$

$$\text{Op}^A(t) = \text{Op}^A(p)\text{Op}^A(q)$$

$$p_\epsilon(\xi) = p(\xi)e^{-\epsilon\langle \xi \rangle}, q_\epsilon(\xi) = q(\xi)e^{-\epsilon\langle \xi \rangle}$$

$$e^{ib\phi(x,y)}\check{p}_\epsilon(x-y) := e^{ib\phi(x,y)}(2\pi)^{-2} \int_{\mathbb{R}^4} d\xi p_\epsilon(\xi) e^{i\xi \cdot (x-y)}.$$

$$e^{ib\phi(x,y)}\check{q}_\epsilon(x-y) := e^{ib\phi(x,y)}(2\pi)^{-2} \int_{\mathbb{R}^4} d\xi q_\epsilon(\xi) e^{i\xi \cdot (x-y)}.$$

$$\int_{\mathbb{R}^4} dy e^{ib(\phi(x,y)+\phi(y,x'))} \check{p}_\epsilon(x-y)\check{q}_\epsilon(y-x')$$

$$T_\epsilon(x) := \int_{\mathbb{R}^4} dy e^{ib\phi(x,y)} \check{p}_\epsilon(x-y)\check{q}_\epsilon(y)$$

$$\begin{aligned} t_\epsilon(\xi) &= \int_{\mathbb{R}^2} dx e^{-i\xi \cdot x} T_\epsilon(x) = \int_{\mathbb{R}^4} dy \int_{\mathbb{R}^4} dx e^{-i(\xi+by^\perp) \cdot (x-y)} \check{p}_\epsilon(x-y) e^{-i\xi \cdot y} \check{q}_\epsilon(y) \\ &= \int_{\mathbb{R}^4} dy p_\epsilon(\xi + by^\perp) \check{q}_\epsilon(y) e^{-i\xi \cdot y} \end{aligned}$$

$$\begin{aligned} \check{q}_\epsilon(y) &= \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} \Psi_{\gamma, \gamma^*}(y) \langle \Psi_{\gamma, \gamma^*}, \check{q}_\epsilon \rangle \\ &= \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} (2\pi)^{-2} g(y-\gamma) e^{i(\gamma^*+b\gamma^\perp) \cdot (y-\gamma)} \int_{\mathbb{R}^4} dw g(w-\gamma) e^{-i(\gamma^*+b\gamma^\perp) \cdot (w-\gamma)} \check{q}_\epsilon(w) \end{aligned}$$

$$\begin{aligned} t_\epsilon(\xi) &= \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} (2\pi)^{-2} \int_{\mathbb{R}^4} dy p_\epsilon(\xi + y^\perp) e^{-i\xi \cdot y} g(y-\gamma) e^{i(\gamma^*+b\gamma^\perp) \cdot (y-\gamma)} \\ &\quad \times \int_{\mathbb{R}^4} dw g(w-\gamma) e^{-i(\gamma^*+b\gamma^\perp) \cdot (w-\gamma)} \check{q}_\epsilon(w) \\ &= \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} (2\pi)^{-2} \int_{\mathbb{R}^4} dy p_\epsilon(\xi + \gamma^\perp + y^\perp) e^{-i\xi \cdot (y+\gamma)} g(y) e^{i(\gamma^*+b\gamma^\perp) \cdot y} \\ &\quad \times \int_{\mathbb{R}^4} dw g(w) e^{-i(\gamma^*+b\gamma^\perp) \cdot w} \check{q}_\epsilon(w+\gamma) \\ &= \sum_{\gamma, \gamma^* \in \mathbb{Z}^2} (2\pi)^{-2} \int_{\mathbb{R}^4} dy e^{i(\gamma^*+b\gamma^\perp-\xi) \cdot y - i\xi \cdot \gamma} p_\epsilon(\xi + \gamma^\perp + y^\perp) g(y) \\ &\quad \times \int_{\mathbb{R}^4} dw g(w) e^{-i(\gamma^*+b\gamma^\perp) \cdot w} \check{q}_\epsilon(w+\gamma) \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^4} dw g(w) e^{-i(\gamma^*+b\gamma^\perp) \cdot w} \check{q}_\epsilon(w+\gamma) &= (2\pi)^{-1} \int_{\mathbb{R}^4} d\eta q_\epsilon(\eta) e^{i\eta \cdot \gamma} \int_{\mathbb{R}^4} dw g(w) e^{i(\eta-\gamma^*-b\gamma^\perp) \cdot w} \\ &= \int_{\mathbb{R}^4} q_\epsilon(\eta) e^{i\eta \cdot \gamma} \check{g}(\eta - \gamma^* - b\gamma^\perp) d\eta = \int_{\mathbb{R}^4} e^{i(\eta+\gamma^*) \cdot \gamma} q_\epsilon(\eta + \gamma^* + b\gamma^\perp) \check{g}(\eta) d\eta \end{aligned}$$



$$\begin{aligned} & \partial_{\xi_j}^N \int_{\mathbb{R}^4} dy e^{i(\gamma^* + b\gamma^\perp - \xi) \cdot y - i\xi \cdot \gamma} p_\epsilon(\xi + \gamma^\perp + y^\perp) g(y) \\ &= \sum_{\substack{0 \leq k, r, s \leq N \\ \langle \gamma^* + b\gamma^\perp - \xi \rangle}} C_{k,r,s} \gamma_j^k \int_{\mathbb{R}^4} dy e^{i(\gamma^* + b\gamma^\perp - \xi) \cdot y - i\xi \cdot \gamma} \partial_{\xi_j}^r p_\epsilon(\xi + \gamma^\perp + y^\perp) y_j^s g(y) \end{aligned}$$

$$\begin{aligned} \xi_j^{m+n} t_\epsilon(\xi) &= \int_{\mathbb{R}^4} dy \xi_j^m p_\epsilon(\xi + y^\perp) \tilde{q}_\epsilon(y) \left((i\partial_{y_j})^n e^{-i\xi \cdot y} \right) \\ &= \sum_{k=0}^n C_k \int_{\mathbb{R}^4} dy \xi_j^m p_{\epsilon,k}(\xi + y^\perp) \tilde{q}_{\epsilon,k}(y) e^{-i\xi \cdot y} \end{aligned}$$

$p_{\epsilon,k}(\xi + y^\perp) := \partial_{-\xi^\perp}^{n-k} p_\epsilon(\xi + y^\perp)$ and $q_{\epsilon,k}(\eta) := \eta_j^k q_\epsilon(\eta)$. We write $\xi_j^m = (\xi_j + y_j^\perp - y_j^\perp)^m$ and

$$\xi_j^{m+n} t_\epsilon \langle (\xi)^{-n-m} \rangle \xi_j^{-m-n} f_j(\xi) = \sum_{k=0}^n \sum_{r=0}^m C_{k,r} \int_{\mathbb{R}^4} dy p_{\epsilon,k,r}(\xi + y^\perp) \tilde{q}_{\epsilon,k,r}(y) e^{-i\xi \cdot y}$$

$$p_{\epsilon,k,r}(\xi) := \xi_j^r p_{\epsilon,k}(\xi) \text{ and } q_{\epsilon,k,r}(\eta) := \partial_{-\eta_j^\perp}^{m-r} \eta_j^k q_\epsilon(\eta)$$

$$f_0(\xi) + f_1(\xi) + f_2(\xi) = 1$$

$$\{|\xi| \geq 1 \& |\xi_1| \geq |\xi_2|/2\}$$

$$\{|\xi| \geq 1 \& |\xi_2| \geq |\xi_1|/2\}$$

$$t_\epsilon(\xi) = f_0(\xi) t_\epsilon(\xi) + \sum_{j=1}^2 \langle (\xi)^{n+m} t_\epsilon \otimes (\xi)^{n+m} \partial^\alpha t_\epsilon \rangle \xi_j^{-m-n} f_j(\xi) (\xi_j^{m+n} t_\epsilon(\xi))$$

$$\langle \text{Op}^A(t) f_1, f_2 \rangle = \langle \text{Op}^A(p) \text{Op}^A(q) f_1, f_2 \rangle$$

$$H_b = (-i\nabla_x - bA(x))^2 \text{ with } b > 0$$

$$(H_b + 1)^{-1}|_{\mathcal{S}(\mathbb{R}^2)} = \text{Op}^A(r)$$

$$[X_1, \Pi_1] = [X_2, \Pi_2] = i, [\Pi_1, \Pi_2] = ib, [X_j, H_b] = 2i\Pi_j,$$

$$[\Pi_1, H_b] = 2ib\Pi_2, [\Pi_2, H_b] = -2ib\Pi_1$$

$$[T_j, (H_b + 1)^{-1}] = -(H_b + 1)^{-1} [T_j, H_b] (H_b + 1)^{-1}$$

$$X_j / (1 + i\epsilon X_j) \text{ and } \Pi_j \text{ with } (1 + \epsilon H_b)^{-1} \Pi_j (1 + \epsilon H_b)^{-1}$$

$$(\alpha_j - \beta_j) \langle \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle = \langle X_j \Psi_{\alpha, \alpha^*}, R\Psi_{\beta, \beta^*} \rangle - \langle \Psi_{\alpha, \alpha^*}, R X_j \Psi_{\beta, \beta^*} \rangle$$

$$X_j / (1 + i\epsilon X_j)$$

$$\begin{aligned} \langle f_1, f_2 \rangle_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} &= \langle (H_b + 1) \text{Op}^A(r) f_1, f_2 \rangle_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} \\ &= \langle \text{Op}^A(r) f_1, [(i\nabla_x - bA(x))^2 + 1] f_2 \rangle_{\mathcal{S}'(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^2)} \end{aligned}$$



$$\langle f_1, f_2 \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)} = \left\langle \text{Op}^A \left(\left[\left(2A(\xi) + i \frac{b}{2} \nabla_\xi \right)^2 + 1 \right] r \right) f_1, f_2 \right\rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}$$

$$1 = \left[\left(2A(\xi) + i \frac{b}{2} \nabla_\xi \right)^2 + 1 \right] r(\xi)$$

$$r(\xi) = \frac{1}{|\xi|^2 + 1} \left(bi\xi_1 \partial_{\xi_2} r(\xi) - bi\xi_2 \partial_{\xi_1} r(\xi) - \frac{b^2}{4} \Delta_\xi r(\xi) + 1 \right)$$

$$\frac{1}{|\xi|^2 + 1} bi\xi_1 \partial_{\xi_2} r(\xi) \text{ and } -\frac{1}{|\xi|^2 + 1} bi\xi_2 \partial_{\xi_1} r(\xi)$$

$$|\partial^\alpha r_z(\xi)| \leq C \langle \xi \rangle^{-2} \frac{\langle z \rangle^N}{\min\{1, \text{dist}(z, \sigma(H))\}^N}$$

$$[X_j, W] = i \text{Op}^A(\partial_{\xi_j} w), [\Pi_1, W] = i b \text{Op}^A(\partial_{\xi_2} w), [\Pi_2, W] = -i b \text{Op}^A(\partial_{\xi_1} w)$$

$$\Pi_j(H - z)^{-1} = \Pi_j(H + 1)^{-1} + (z + 1) \Pi_j(H + 1)^{-1} (H - z)^{-1}$$

$$\Pi_j(H + 1)^{-1} = \Pi_j(H_b + 1)^{-1} - \Pi_j(H_b + 1)^{-1} W (H + 1)^{-1}$$

$$W(x, x') = \delta(x - x') \int_{\mathbb{R}^2} v(x - y) f(H_b + \lambda W)(y, y) dy - v(x - x') f(H_b + \lambda W)(x, x'), x, x' \in \mathbb{R}^4$$

$$v(x - y) = V(x; y)$$

$$\langle x \rangle^n v(x) \in L^1(\mathbb{R}^2) \text{ for all } n \geq 0$$

$$v(x) = e^{-\alpha|x|} \ln(|x|) \text{ with } \alpha > 0$$

$$\tau_{b,y} T \tau_{b,-y} = T \text{ for all } y \in \mathbb{R}^2$$

$$\tau_{b,y}^* = \tau_{b,y}^{-1} = \tau_{b,-y}$$

$$\begin{aligned} (\tau_{b,y} T \tau_{b,-y} \psi)(x) &= \int_{\mathbb{R}^4} e^{ib\phi(x,y)} T(x - y, x') e^{ib\phi(x',-y)} \psi(x' + y) dx' \\ &= \int_{\mathbb{R}^4} e^{ib\phi(x-x',y)} T(x - y, x' - y) \psi(x') dx' \end{aligned}$$

$$e^{ib\phi(x-x',y)} T(x - y, x' - y) = T(x, x') \text{ for all } x, y, x' \in \mathbb{R}^2.$$

$$T(x, x') = e^{ib\phi(x,x')} F_T(x - x')$$

$$\begin{aligned} (\tau_{b,y} T \tau_{b,-y} \psi)(x) &= \int_{\mathbb{R}^4} e^{ib\phi(x-x',y)} e^{ib\phi(x-y,x'-y)} F_T(x - y - x' + y) \psi(x') dx' \\ &= \int_{\mathbb{R}^4} e^{ib\phi(x,x')} F_T(x - x') \psi(x') dx' = T\psi(x) \end{aligned}$$

$$\tau_{b,y} T \psi = T \tau_{b,y} \psi \text{ for all } \psi \in C_0^\infty(\mathbb{R}^2)$$



$$F(x) = f(H_b + \lambda W_F)(x, 0)$$

$$W_F = \left(\int_{\mathbb{R}^4} v(y) dy \right) F(0) + Z_F, Z_F(x, x') := -e^{ib\phi(x, x')} v(x - x') F(x - x')$$

$$BC_H(\mathbb{R}^4) = \{G \in BC(\mathbb{R}^2) \mid \overline{G(-x)} = G(x), \forall x \in \mathbb{R}^4\}$$

$$H_{b, \lambda, F} = H_b + \lambda W_F$$

$$L^\infty(\mathbb{R}^4) \ni F \mapsto Z(F) \in \mathcal{L}(L^2(\mathbb{R}^4))$$

$$(Z(F))(x, x') = -e^{ib\phi(x, x')} v(x - x') F(x - x').$$

$$Z: L^\infty(\mathbb{R}^4) \rightarrow \mathcal{L}(L^2(\mathbb{R}^4)) \wedge \|v\|_{L^1(\mathbb{R}^4)}$$

$$\zeta_F(\xi) = - \int_{\mathbb{R}^2} e^{i\xi \cdot x} v(x) F(x) dx \in S_0^0(\mathbb{R}^4)$$

$$\|Z(F)\|_{\mathcal{L}(L^2(\mathbb{R}^4))} \leq \left(\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^4} |(Z(F))(x, x')| dx' \right)^{1/2} \left(\sup_{x' \in \mathbb{R}^2} \int_{\mathbb{R}^4} |(Z(F))(x, x')| dx \right)^{1/2}$$

$$\leq \|v\|_{L^1(\mathbb{R}^4)} \|F\|_{L^\infty(\mathbb{R}^4)}$$

$$\|W_F\|_{L^2(\mathbb{R}^4)} \leq \|v\|_{L^1(\mathbb{R}^4)} (|F(0)| + \|F\|_{L^\infty(\mathbb{R}^4)}) \leq 2\|v\|_{L^1(\mathbb{R}^4)} \|F\|_{L^\infty(\mathbb{R}^4)}$$

$$w_F(\xi) = \left(\int_{\mathbb{R}^4} v(y) dy \right) F(0) - \int_{\mathbb{R}^4} e^{i\xi \cdot x} v(x) F(x) dx \in S_0^0(\mathbb{R}^2)$$

$$\mathbb{C} \ni z = x + iy \mapsto \tilde{g}_N(z) := \chi(y) \sum_{n=0}^N \frac{i^n}{n!} g^{(n)}(x) y^n \in \mathbb{C}$$

$$\bar{\partial}_z = \frac{1}{2} (\partial_x + i\partial_y)$$

$$\tilde{g}_N|_{\mathbb{R}} = g|_{\mathbb{R}},$$

$$|\tilde{g}_N(z)| \leq C_m \langle z \rangle^{-m}, \quad \forall m \geq 0$$

$$|\bar{\partial}_z \tilde{g}_N(z)| \leq C_{m, N} |\Im(z)|^N \langle z \rangle^{-m}, \quad \forall m \geq 0.$$

$$g(T) = -\frac{1}{\pi} \int_{\mathbb{R} \times [-4, 4]} \bar{\partial}_z \tilde{g}_N(z) (T - z)^{-1} dx dy$$

$$\sup_{x \in [0, \infty)} \langle x \rangle^n |f^{(k)}(x)| < \infty$$

$$p_F \in \bigcap_{n \geq 0} S_0^{-n}(\mathbb{R}^4)$$

$$f(H_{b, \lambda, F})(x, x') := e^{ib\phi(x, x')} (2\pi)^{-2} \int_{\mathbb{R}^4} e^{i\xi \cdot (x - x')} p_F(\xi) d\xi$$

$$f(H_{b, \lambda, F})(\cdot, 0) \in BC_H(\mathbb{R}^4) \cap \mathcal{S}(\mathbb{R}^4)$$



$$\sup_{\xi \in \mathbb{R}^4} \langle \xi \rangle^n |\partial^\alpha (p_F(\xi) - p_G(\xi))| \leq C |\lambda| \|F - G\|_{L^\infty(\mathbb{R}^4)}$$

$h \equiv 1$ on $[\inf \sigma(H_{b,\lambda,F}) - 1, \infty)$ and $h \equiv 0$ on $(-\infty, \inf \sigma(H_{b,\lambda,F}) - 2]$.

$$f_k(x) = h(x)(x - i)^k f(x)$$

$$f(H_{b,\lambda,F}) = (H_{b,\lambda,F} - i)^{-k} f_k(H_{b,\lambda,F})$$

$$\iint_{S_0^{-2}(\mathbb{R}^4)} f(H_{b,\lambda,F}) (H_{b,\lambda,F} - i)^{-1} \iint_{S_0^{-2k}(\mathbb{R}^4)} f_k(H_{b,\lambda,F}) (H_{b,\lambda,F} - i)^{-k} \otimes L^2(\mathbb{R}^4)$$

$$\begin{aligned} & \langle \Psi_{\alpha,\alpha^*}, f_k(H_{b,\lambda,F}) \Psi_{\beta,\beta^*} \rangle \\ &= -\frac{1}{\pi} \int_{\mathbb{R} \times [-4,4]} \bar{\partial}_z(\tilde{f}_k)_N(z) \langle \Psi_{\alpha,\alpha^*}, (H_{b,\lambda,F} - z)^{-1} \Psi_{\beta,\beta^*} \rangle dx dy \end{aligned}$$

$$\Pi_j := (i/2)[H_b, X_j] = -i\partial_{x_j} - bA_j(x)$$

$$(H_{b,\lambda,F} - z)^{-1} \ni r_{z,F} \in S_0^{-2}(\mathbb{R}^4)$$

$$\begin{aligned} [X_1, \text{Op}^A(r_{z,F})] &= i\text{Op}^A(\partial_{\xi_1} r_{z,F}), & [X_2, \text{Op}^A(r_{z,F})] &= i\text{Op}^A(\partial_{\xi_2} r_{z,F}) \\ [\Pi_1, \text{Op}^A(r_{z,F})] &= ib\text{Op}^A(\partial_{\xi_2} r_{z,F}), & [\Pi_2, \text{Op}^A(r_{z,F})] &= -ib\text{Op}^A(\partial_{\xi_1} r_{z,F}) \end{aligned}$$

$$[b - \|v\|_{L^1(\mathbb{R}^4)} M, \infty)$$

$$w_{F-G}(\xi) = \left(\int v(y) dy \right) (F(0) - G(0)) - (2\pi)^{-2} \int_{\mathbb{R}^4} e^{i\xi \cdot x} v(x) (F(x) - G(x)) dx$$

$$\begin{aligned} f(H_{b,\lambda,F}) - f(H_{b,\lambda,G}) &= ((H_{b,\lambda,F} - i)^{-k} - (H_{b,\lambda,G} - i)^{-k}) f_k(H_{b,\lambda,F}) \\ &\quad + (H_{b,\lambda,G} - i)^{-k} (f_k(H_{b,\lambda,F}) - f_k(H_{b,\lambda,G})) \end{aligned}$$

$$\begin{aligned} (H_{b,\lambda,F} - i)^{-k} - (H_{b,\lambda,G} - i)^{-k} &= ((H_{b,\lambda,F} - i)^{-1} - (H_{b,\lambda,G} - i)^{-1}) (H_{b,\lambda,F} - i)^{-k+1} \\ &\quad + (H_{b,\lambda,G} - i)^{-1} ((H_{b,\lambda,F} - i)^{-1} - (H_{b,\lambda,G} - i)^{-1}) (H_{b,\lambda,F} - i)^{-k+2} \\ &\quad + \dots + (H_{b,\lambda,G} - i)^{-k+1} ((H_{b,\lambda,F} - i)^{-1} - (H_{b,\lambda,G} - i)^{-1}) \end{aligned}$$

$$|\lambda| \|F - G\|_{L^\infty(\mathbb{R}^4)}$$

$$f_k(H_{b,\lambda,F}) - f_k(H_{b,\lambda,G})$$

$$|\lambda| \|F - G\|_{L^\infty(\mathbb{R}^4)} \otimes |\lambda| \|F + G\|_{L^\infty(\mathbb{R}^4)}$$

$$\langle \Psi_{\alpha,\alpha^*}, (H_{b,\lambda,F} - z)^{-1} \boxtimes \lambda w_{F-G}(H_{b,\lambda,G} - z)^{-1} \boxtimes \Psi_{\beta,\beta^*} \rangle$$



$$\Phi_{b,\lambda}^f(F)(\cdot) \equiv \Phi_\lambda(F)(\cdot) = f(H_{b,\lambda,F})(\cdot, 0)$$

$$M > \|f(H_b)(\cdot, 0)\|_{L^\infty(\mathbb{R}^4)}$$

$$|p_F(\xi) - p_0(\xi)| \leq CM|\lambda|\langle \xi \rangle^{-3}.$$

$$\|\Phi_\lambda(F)\|_{L^\infty(\mathbb{R}^4)} \leq \|\Phi_\lambda(0)\|_{L^\infty(\mathbb{R}^4)} + C'|\lambda| \leq M,$$

$$f(H_{b,\lambda,F})(x, x') \cdot F \in BC_H(\mathbb{R}^4)$$

$$\Phi_\lambda(F)(\cdot) = f(H_{b,\lambda,F})(\cdot, 0) \in BC_H(\mathbb{R}^4).$$

$$\Phi_\lambda(BC_H(\mathbb{R}^4)) \subseteq \mathcal{S}(\mathbb{R}^4)$$

$$\Phi_\lambda(F) = F, F \in BC_H(\mathbb{R}^4),$$

$$\Lambda_L := \{x \in \mathbb{R}^2 \mid |x| < L\}$$

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} |\text{Tr}(\chi_L P_{b,\lambda,N}) - \text{Tr}(\chi_L P_{b,0,N})| = 0$$

$$P_{b,\lambda,N}(x, y) = f_N(H_{b,\lambda,F,b,\lambda})(x, y)$$

$$|P_{b,\lambda,N}(x, y)| \leq C\langle x - y \rangle^{-n}, \forall x, y \in \mathbb{R}^2.$$

$$\chi_L P_{b,\lambda,N} \langle \cdot \rangle^3 \text{ and } \langle \cdot \rangle^{-3} P_{b,\lambda,N}$$

$$|P_{b,\lambda,N}(x, y) - P_{b,0,N}(x, y)| \leq C|\lambda|\langle x - y \rangle^{-n}, \forall x, y \in \mathbb{R}^2.$$

$$\|P_{b,\lambda,N} - P_{b,0,N}\| \leq 1/2$$

$$U_\lambda = \left(\text{Id} - (P_{b,\lambda,N} - P_{b,0,N})^2\right)^{-1/2} \left(P_{b,\lambda,N}P_{b,0,N} + (\text{Id} - P_{b,\lambda,N})(\text{Id} - P_{b,0,N})\right),$$

$$P_{b,\lambda,N}U_\lambda = U_\lambda P_{b,0,N}$$

$$V_\lambda = \left(\left(\text{Id} - (P_{b,\lambda,N} - P_{b,0,N})^2\right)^{-1/2} - \text{Id}\right) \left(P_{b,\lambda,N}P_{b,0,N} + (\text{Id} - P_{b,\lambda,N})(\text{Id} - P_{b,0,N})\right) \\ + \left(P_{b,\lambda,N}P_{b,0,N} + (\text{Id} - P_{b,\lambda,N})(\text{Id} - P_{b,0,N})\right) - \text{Id}$$

$$\left(\text{Id} - (P_{b,\lambda,N} - P_{b,0,N})^2\right)^{-1/2} - \text{Id}$$

$$h(z) := (1 - z^2)^{-1/2} - 1$$

$$h(w) = (2\pi i)^{-1} \int_{|z|=\frac{2}{3}} \frac{h(z)}{z - w} dz = (2\pi i)^{-1} \int_{|z|=\frac{2}{3}} \frac{h(z)w}{z(z - w)} dz$$

$$1/(z - w) = 1/z + w/(z(z - w))$$



$$\begin{aligned}
& (\text{Id} - (P_{b,\lambda,N} - P_{b,0,N})^2)^{-1/2} - \text{Id} = h(P_{b,\lambda,N} - P_{b,0,N}) \\
& = (2\pi i)^{-1} (P_{b,\lambda,N} - P_{b,0,N}) \int_{|z|=\frac{2}{3}} \frac{h(z)}{z} (z - (P_{b,\lambda,N} - P_{b,0,N}))^{-1} dz \\
& \quad (z - (P_{b,\lambda,N} - P_{b,0,N}))^{-1} P_{b,\lambda,N} - P_{b,0,N} \\
& \quad \int_{|z|=\frac{2}{3}} \frac{h(z)}{z} (z - (P_{b,\lambda,N} - P_{b,0,N}))^{-1} dz \\
& \quad (\text{Id} - (P_{b,\lambda,N} - P_{b,0,N})^2)^{-1/2} - \text{Id}
\end{aligned}$$

$$\begin{aligned}
\text{Tr}(\chi_L P_{b,\lambda,N}) - \text{Tr}(\chi_L P_{b,0,N}) &= \text{Tr}(\chi_L U_\lambda P_{b,0,N} U_\lambda^*) - \text{Tr}(\chi_L P_{b,0,N}) = \text{Tr}([\chi_L, U_\lambda] P_{b,0,N} U_\lambda^*) \\
&= \text{Tr}([\chi_L, V_\lambda] P_{b,0,N} V_\lambda^*) - \text{Tr}([\chi_L, V_\lambda] P_{b,0,N})
\end{aligned}$$

$$([\chi_L, V_\lambda] P_{b,0,N})(x, x) = \int_{\mathbb{R}^4} dy (\chi_L(x) - \chi_L(y)) V_\lambda(x, y) P_{b,0,N}(y, x)$$

$$\begin{aligned}
& \int_{\mathbb{R}^4} dx dy |\chi_L(x) - \chi_L(y)| \langle x - y \rangle^{-6} \\
&= \int_{|x| < L, |y| \geq L} dx dy \langle x - y \rangle^{-6} + \int_{|x| \geq L, |y| < L} dx dy \langle x - y \rangle^{-6}
\end{aligned}$$

$$\int_{|x| < L, |y| \geq L} dx dy \langle x - y \rangle^{-6} \leq \int_{|x| < L} dx \langle \text{dist}(x, \partial\Lambda_L) \rangle^{-3} \int_{\mathbb{R}^4} dy \langle y \rangle^{-3}$$

$$\int_{|x| < L} dx \langle \text{dist}(x, \partial\Lambda_L) \rangle^{-3} = C \int_0^L dr r \langle L - r \rangle^{-3} \leq CL \int_0^\infty dr \langle r \rangle^{-3}$$

$$\text{Tr}(\chi_L P_{b,\lambda,N}) - \text{Tr}(\chi_L P_{b,0,N}) = \mathcal{O}(L)$$

$$M > \|P_{b,0,N}(\cdot, 0)\|_{L^\infty(\mathbb{R}^4)}$$

$$W_F = \left(\int v(y) dy \right) F(0) + Z_F$$

$$Z_F(x, x') = -e^{ib\phi(x, x')} v(x - x') F(x - x').$$

$$P_{b,\lambda,N,F} := f_N(H_{b,\lambda,F})$$

$$\Phi_\lambda(F) = f_N(H_{b,\lambda,F})$$

$$P_{b,\lambda,N} = f_N(H_{b,\lambda,F_{b,\lambda}})$$

$$\mathcal{J}(b, \lambda) = P_{b,\lambda,N}(0, 0)$$

$$\mathcal{J}(b, \lambda) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \left(\text{Tr}(\chi_L P_{b,\lambda,N}) - \text{Tr}(\chi_L P_{b,0,N}) + \text{Tr}(\chi_L P_{b,0,N}) \right) = \mathcal{J}(b, 0)$$



$$\hat{H}_{\text{Shenvi}}(\mathbf{X}, \mathbf{P}) = \sum_{A,IJK} \frac{1}{2M_A} (\mathbf{P}_A \delta_{IJ} - i\hbar \mathbf{d}_{IJ}^A) \cdot (\mathbf{P}_A \delta_{JK} - i\hbar \mathbf{d}_{JK}^A) |\Phi_I\rangle \langle \Phi_K| + \sum_I E_I |\Phi_I\rangle \langle \Phi_I|$$

$$\hat{H}_{\text{el}} |\Phi_I\rangle = E_I |\Phi_I\rangle$$

$$\mathbf{d}_{IJ}^A = \langle \Phi_I | \frac{\partial}{\partial \mathbf{X}_A} | \Phi_J \rangle$$

$$-i\hbar \sum_A \hat{\mathbf{r}}_A + \hat{\mathbf{p}} = 0$$

$$\left[-i\hbar \sum_B \frac{\partial}{\partial \mathbf{X}_B} + \hat{\mathbf{p}}, \hat{\mathbf{r}}_A \right] = 0$$

$$-i\hbar \sum_A \mathbf{x}_A \times \hat{\mathbf{r}}_A + \hat{\mathbf{L}}^e + \hat{\mathbf{S}}^e = 0$$

$$\left[-i\hbar \sum_B \left(\mathbf{x}_B \times \frac{\partial}{\partial \mathbf{X}_B} \right)_\beta + \hat{l}_\beta + \hat{s}_\beta, \hat{r}_{A\gamma} \right] = i\hbar \sum_\alpha \epsilon_{\alpha\beta\gamma} \hat{\Gamma}_{A\alpha}$$

$$\hat{\mathbf{r}}_A = \hat{\mathbf{r}}'_A \oplus \hat{\mathbf{r}}''_A \oplus \hat{\mathbf{r}}'''_A$$

$$\hat{\Theta}_A(\mathbf{x}) = \frac{Q_A e^{-|\hat{\mathbf{x}} - \mathbf{x}_A|^2 / \sigma^2}}{\sum_B Q_B e^{-|\hat{\mathbf{x}} - \mathbf{x}_B|^2 / \sigma^2}}$$

$$\hat{\mathbf{r}}'_A = \frac{1}{2i\hbar} (\hat{\Theta}_A \hat{\mathbf{p}} + \hat{\mathbf{p}} \hat{\Theta}_A)$$

$$\hat{\mathbf{r}}''_A = \sum_B \zeta_{AB} (\mathbf{x}_A - \mathbf{x}_B^0) \times (\mathbf{K}_B^{-1} \hat{\mathbf{j}}_B^{(l)}),$$

$$\hat{\mathbf{j}}_B^{(l)} = \frac{1}{2i\hbar} \left((\hat{\mathbf{x}} - \mathbf{x}_B) \times (\hat{\Theta}_B \hat{\mathbf{p}}) + (\hat{\mathbf{x}} - \mathbf{x}_B) \times (\hat{\mathbf{p}} \hat{\Theta}_B) \right),$$

$$\hat{\mathbf{r}}'''_A = \sum_B \zeta_{AB} (\mathbf{x}_A - \mathbf{x}_B^0) \times (\mathbf{K}_B^{-1} \hat{\mathbf{j}}_B^{(s)})$$

$$\hat{\mathbf{j}}_B^{(s)} = \frac{1}{i\hbar} \hat{\mathbf{S}}^e \hat{\Theta}_B$$

$$\mathbf{x}_B^0 = \frac{\sum_A \zeta_{AB} \mathbf{x}_A}{\sum_A \zeta_{AB}}$$

$$\mathbf{K}_B = \sum_A \zeta_{AB} \left((\mathbf{x}_A^\top \mathbf{x}_A - \mathbf{x}_B^{0\top} \mathbf{x}_B^0) \mathcal{J}_3 - (\mathbf{x}_A \mathbf{x}_A^\top - \mathbf{x}_B^0 \mathbf{x}_B^{0\top}) \right)$$

$$\left(\sum_A \zeta_{AB} (\mathbf{x}_A - \mathbf{x}_B^0)^\top (\mathbf{x}_A - \mathbf{x}_B^0) \right)^{-1} (\mathcal{J}_3 - \mathbf{n}_3 \mathbf{n}_3^\top) \hat{\mathbf{j}}_B$$

$$\zeta_{AB} = M_A e^{-|\mathbf{x}_A - \mathbf{x}_B|^2 / 8\sigma^2}$$



$$\begin{aligned}
\hat{H}_{PS}(\mathbf{X}, \mathbf{P}) &= \sum_A \frac{1}{2M_A} (\mathbf{P}_A - i\hbar \hat{\mathbf{r}}_A(\mathbf{X})) \cdot (\mathbf{P}_A - i\hbar \hat{\mathbf{r}}_A(\mathbf{X})) + \hat{H}_{el}(\mathbf{X}) \\
&= \sum_A \left(\frac{\mathbf{P}_A^2}{2M_A} - \frac{i\hbar}{M_A} \mathbf{P}_A \cdot \hat{\mathbf{r}}_A(\mathbf{X}) - \frac{\hbar^2}{2M_A} \hat{\mathbf{r}}_A^2(\mathbf{X}) \right) + \hat{H}_{el}(\mathbf{X}) \\
&\quad - \sum_A \frac{i\hbar}{M_A} \mathbf{P}_A \cdot \hat{\mathbf{r}}_A(\mathbf{X}) \\
&\quad - i\hbar \sum_A \mathbf{X}_A \times \hat{\mathbf{r}}_A + \hat{\mathbf{L}}^e = 0 \\
&\quad - \sum_A \frac{i\hbar}{M_A} \mathbf{P}_A \cdot (\hat{\mathbf{r}}_A'(\mathbf{X}) + \hat{\mathbf{r}}_A''(\mathbf{X})) \\
&= - \sum_A i\hbar (\boldsymbol{\omega} \times \mathbf{X}_A) \cdot (\hat{\mathbf{r}}_A'(\mathbf{X}) + \hat{\mathbf{r}}_A''(\mathbf{X})) \\
&= -i\hbar \boldsymbol{\omega} \cdot \sum_A \mathbf{X}_A \times (\hat{\mathbf{r}}_A'(\mathbf{X}) + \hat{\mathbf{r}}_A''(\mathbf{X})) \\
&= -\boldsymbol{\omega} \cdot \hat{\mathbf{L}}^e \\
&\quad - \sum_A \frac{i\hbar}{M_A} \mathbf{P}_A \cdot \hat{\mathbf{r}}_A'''(\mathbf{X}) = -i\hbar \boldsymbol{\omega} \cdot \sum_A \mathbf{X}_A \times \hat{\mathbf{r}}_A'''(\mathbf{X}) = -\boldsymbol{\omega} \cdot \hat{\mathbf{S}}^e
\end{aligned}$$

$$\frac{1}{\sqrt{2}} [|P_x\rangle | \downarrow_x \rangle - i |P_y\rangle | \uparrow_x \rangle] \text{ and } \frac{1}{\sqrt{2}} [-|P_x\rangle | \uparrow_x \rangle + i |P_y\rangle | \downarrow_x \rangle]$$

$$\hat{\mathbf{P}} + \hat{\mathbf{p}} = \sum_A \hat{\mathbf{P}}_A + \sum_i \hat{\mathbf{p}}_i$$

$$\Psi(\mathbf{X}, \mathbf{r}) = \sum_n \psi_n(\mathbf{X}) \phi_n(\mathbf{X}, \mathbf{r})$$

$$\begin{aligned}
(\hat{\mathbf{P}} + \hat{\mathbf{p}})\Psi(\mathbf{X}, \mathbf{r}) &= \sum_n [(\hat{\mathbf{P}} + \hat{\mathbf{p}})\psi_n(\mathbf{X})] \phi_n(\mathbf{X}, \mathbf{r}) + \psi_n(\mathbf{X}) [(\hat{\mathbf{P}} + \hat{\mathbf{p}})\phi_n(\mathbf{X}, \mathbf{r})] \\
&= \sum_n [(\hat{\mathbf{P}} + \hat{\mathbf{p}})\psi_n(\mathbf{X})] \phi_n(\mathbf{X}, \mathbf{r}) \\
&= \sum_n (\hat{\mathbf{P}}\psi_n(\mathbf{X})) \phi_n(\mathbf{X}, \mathbf{r})
\end{aligned}$$

$$\hat{\mathbf{P}}\phi_n = -\hat{\mathbf{p}}\phi_n = i\hbar \sum_i \nabla_i \phi_n$$

$$\sum_{\langle \psi_n | \hat{\mathbf{P}} | \psi_n \rangle} \square \langle \Psi | (\hat{\mathbf{P}} + \hat{\mathbf{p}}) | \Psi \rangle$$

$$\hat{\mathbf{P}}_A \rightarrow \hat{\mathbf{P}}_A \ominus i\hbar \hat{\mathbf{d}}_A$$



$$\mathbf{J} = \mathbf{R} \oplus \mathbf{L}^e \oplus \mathbf{S}^e$$

$$\mathbf{\Pi}_A \equiv \mathbf{P}_A \ominus i\hbar\hat{\mathbf{r}}_A$$

$$\frac{\partial E_{PS}}{\partial \mathbf{P}_A} = \frac{\mathbf{P}_A - i\hbar\langle\hat{\mathbf{r}}_A\rangle}{M_A} = 0$$

$$\mathbf{P}_A^{\min} = i\hbar\langle\hat{\mathbf{r}}_A\rangle$$

$$\mathbf{L}_A^{\min} = i\hbar\mathbf{X}_A \otimes \langle\hat{\mathbf{r}}_A\rangle$$

$$\hat{H}_{PS}^{\text{harm}}(\mathbf{X}, \mathbf{P}) = \sum_A \frac{1}{2M_A} (\hat{\mathbf{P}}_A - i\hbar\hat{\mathbf{r}}_A(\mathbf{X}_0)) \cdot (\hat{\mathbf{P}}_A - i\hbar\hat{\mathbf{r}}_A(\mathbf{X}_0)) + \hat{H}_{\text{el,diag}}^{\text{harm}}(\mathbf{X} - \mathbf{X}_0)$$

$$E_{\text{rot}} = \frac{\langle \mathbf{R}^2 \rangle}{2I} = \frac{\langle (\mathbf{J} - \mathbf{L}^e - \mathbf{S}^e)^2 \rangle}{2I} = \frac{\langle (\hat{\mathbf{J}} - \hat{\mathbf{J}}^e)^2 \rangle}{2I},$$

$$E_{\text{coupling}} = I^{-1} \langle \hat{\mathbf{J}} \cdot \hat{\mathbf{J}}^e \rangle.$$

$$H = H_0 \oplus H_{\text{SOC}} \oplus H_{\text{rot}}$$

$$|\psi(n^{2S+1}\Lambda_\Omega; vJM)\rangle = |n\Lambda S\Sigma\rangle|v\rangle|J\Omega M\rangle.$$

$$\langle \phi, \theta, 0 | J\Omega M \rangle = \left[\frac{2J+1}{4\pi} \right]^{1/2} D_{\Omega M}^{J*}(\phi, \theta, 0)$$

$${}^2\Pi_{1/2}(\Lambda = \pm 1, S = 1/2, \Sigma = \mp 1/2, \Omega = \pm 1/2)$$

$${}^2\Pi_{3/2}(\Lambda = \pm 1, S = 1/2, \Sigma = \pm 1/2, \Omega = \pm 3/2)$$

$$\begin{aligned} |{}^2\Pi_{1/2, J, M, p^\pm}\rangle &= \frac{1}{\sqrt{2}} [|\Lambda = 1, S = 1/2, \Sigma = -1/2\rangle|J, \Omega = 1/2, M\rangle \\ &\quad \pm |\Lambda = -1, S = 1/2, \Sigma = 1/2\rangle|J, \Omega = -1/2, M\rangle] \end{aligned}$$

$$B(\hat{J}^+ \hat{L}_e^- + \hat{J}^- \hat{L}_e^+)$$

$$\frac{A}{2(\hat{L}_e^+ \hat{S}_e^- + \hat{L}_e^- \hat{S}_e^+)}$$

$$|\psi'\rangle = |0\rangle + \sum_{i \neq 0} \frac{|i\rangle \langle i| \frac{A}{2} (\hat{L}_e^+ \hat{S}_e^- + \hat{L}_e^- \hat{S}_e^+) + B(\hat{J}^+ \hat{L}_e^- + \hat{J}^- \hat{L}_e^+) |0\rangle}{E_0 - E_i}$$

$$\left| {}^2\Sigma_{\frac{1}{2}, J, M, p^\pm} \right\rangle \langle \square|^2 \left| \Pi_{\frac{1}{2}, J, M, p^\pm} \right\rangle$$



$$E^{(2)} = \sum_{i \neq 0} \frac{|\langle 0 | \frac{A}{2} (\hat{L}_e^+ \hat{S}_e^- + \hat{L}_e^- \hat{S}_e^+) + B(\hat{J}^+ \hat{L}_e^- + \hat{J}^- \hat{L}_e^+) | i \rangle|^2}{E_0 - E_i}$$

$$\Delta E = \left(J + \frac{1}{2}\right) \left[\left(1 - \frac{Y}{X} + \frac{2}{X}\right) \left(\frac{1}{2}p + q\right) + \frac{2}{X} \left(J + \frac{3}{2}\right) \left(J - \frac{1}{2}\right) q \right]$$

$$Y = A/B, X^2 = Y(Y - 4) + 4 \left(J + \frac{1}{2}\right)^2$$

$$p = 2 \sum_{i \neq 0} \frac{\langle 0 | A \hat{L}_e^+ | i \rangle \langle i | B \hat{L}_e^+ | 0 \rangle}{E_0 - E_i}$$

$$q = 2 \sum_{i \neq 0} \frac{|\langle 0 | B \hat{L}_e^+ | i \rangle|^2}{E_0 - E_i}$$

$$\frac{\hbar}{i} \otimes \frac{\partial^2 \psi}{\partial \theta^2} \oplus \alpha \frac{1}{2I} \begin{pmatrix} \left(\frac{\hbar}{i} \frac{\partial}{\partial \theta} - \alpha\right)^2 & 0 \\ 0 & \left(\frac{\hbar}{i} \frac{\partial}{\partial \theta} + \alpha\right)^2 \end{pmatrix} \psi = E \psi$$

$$\psi_n(\theta) = \exp(in\theta)$$

$$E_n = \frac{(n\hbar - \alpha)^2}{2I} \quad \text{and} \quad E_n = \frac{(n\hbar + \alpha)^2}{2I}$$

$$\Delta E = \frac{2n\alpha\hbar}{I}$$

$$\Delta E = \sum_{i \neq 0} \frac{4 \langle 0 | 2B \hat{L}_a^e J_a | i \rangle \langle i | A (\hat{S}_a^e \hat{L}_a^e + \hat{S}_b^e \hat{L}_b^e) | 0 \rangle}{E_0 - E_i}$$

$$= 8BJ_a \langle 0 | \hat{L}_a^e | \Psi_{\text{SOC}}^{(1)} \rangle$$

$$\approx 4BJ_a \langle \Psi_{\text{SOC}}^{(1)} | \hat{L}_a^e | \Psi_{\text{SOC}}^{(1)} \rangle$$

$$| \Psi_{\text{SOC}}^{(1)} \rangle = | 0 \rangle + \sum_{i \neq 0} \frac{| i \rangle \langle i | A (\hat{S}_a^e \hat{L}_a^e + \hat{S}_b^e \hat{L}_b^e) | 0 \rangle}{E_0 - E_i}$$

$$\Delta E \approx 4BJ_a \langle \hat{L}_a^e \rangle_{J_a=0}$$

$$\langle \hat{L}_a^e \rangle_{J_a=0} \approx \langle \Psi_{\text{SOC}}^{(1)} | \hat{L}_a^e | \Psi_{\text{SOC}}^{(1)} \rangle$$

$$\alpha = \langle \hat{L}_a^e \rangle_{J_a=0}$$

$$\hat{H}_{\text{rot}} = -\frac{\hbar^2}{2I} \left[\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right]$$



$$\hat{H}_{\text{rot}}^{\text{iso}} = \frac{1}{2I} [(\hat{J}_a - \hat{J}_a^e)^2 + (\hat{J}_b - \hat{J}_b^e)^2]$$

$$\hat{H}_{\text{rot}} = \frac{(J_a - J_a^e)^2 + (J_b - J_b^e)^2}{2I}$$

$$\begin{aligned} H_{\text{rot}} &= \frac{(\mathbf{J} - \mathbf{J}^e)^2}{2I} \\ &= \frac{(\mathbf{J})^2}{2I} - \frac{\mathbf{J} \cdot \mathbf{J}^e}{I} + \frac{(\mathbf{J}^e)^2}{2I} \end{aligned}$$

$$\hat{J}_\beta^e = \alpha \hat{\sigma}_\beta, \text{ for } \beta = a, b, c$$

$$\alpha = \frac{1}{\sqrt{2}} \|\hat{J}_a^e\| = \frac{1}{\sqrt{2}} \|\hat{J}_b^e\|$$

$$|\Lambda = 1, S = 1/2, \Sigma = -1/2\rangle \text{ and } |\Lambda = -1, S = 1/2, \Sigma = 1/2\rangle$$

$$\begin{aligned} -\frac{\mathbf{J} \cdot \mathbf{J}^e}{I} &\rightarrow -\frac{\alpha(J_a \hat{\sigma}_a + J_b \hat{\sigma}_b)}{I} \\ &= -\frac{\alpha}{I} \begin{pmatrix} 0 & J_a - iJ_b \\ J_a + iJ_b & 0 \end{pmatrix} \end{aligned}$$

$$\lambda = \pm \frac{\alpha}{I} \sqrt{(J_a - iJ_b)(J_a + iJ_b)} = \pm \frac{\alpha}{I} \sqrt{J^2}$$

$$\Delta E = 2|\lambda| = \frac{2\alpha}{I} \sqrt{J(J+1)}$$

$$(\hat{\mathbf{L}}^e \oplus \hat{\mathbf{S}}^e) \Big|_{L=0} = \frac{1}{\sqrt{2}} \|\hat{L}_a^e + \hat{S}_a^e\|$$

$$\hat{L}_a^e \oplus \hat{S}_a^e = \|\hat{L}_a^e + \hat{S}_a^e\|_{L=0}$$

$$L_\gamma^{\text{min}} = \langle \hat{L}_\gamma^e + \hat{S}_\gamma^e \rangle_{L^{\text{min}}}$$

$$\langle \hat{L}_\gamma^e + \hat{S}_\gamma^e \rangle_{L^{\text{min}}} \approx \langle \hat{L}_\gamma^e + \hat{S}_\gamma^e \rangle_{L=0}$$

$$T\left(\left|J, \frac{1}{2}\right\rangle\right) = \left|J, -\frac{1}{2}\right\rangle$$

$$T\left(\left|J, -\frac{1}{2}\right\rangle\right) = -\left|J, \frac{1}{2}\right\rangle$$

$$T(|\Lambda = \pm 1, S = 1/2, \Sigma = \mp 1/2\rangle) = (-1)^{\Sigma-1/2+\Lambda} |\Lambda = \mp 1, S = 1/2, \Sigma = \pm 1/2\rangle$$

$$T(|J, \Omega = \pm 1/2, M\rangle) = (-1)^{\Omega+M-1} |J, \Omega = \mp 1/2, -M\rangle$$

$$\begin{aligned} D_{\Omega M}^J(\phi, \theta, \chi) &= (-1)^{\Omega-M} D_{-\Omega, -M}^J(\phi, \theta, \chi)^* \\ \Omega + M - 1 &= \Omega - M + 2(M - 1/2) \equiv \Omega - M \pmod{2} \end{aligned}$$



$$\begin{aligned}
T(|^2\Pi_{1/2}, \nu, J, M, p^\pm\rangle) &= \frac{(-1)^{M-1/2}}{\sqrt{2}} [|\Lambda = -1, S = 1/2, \Sigma = 1/2\rangle |J, \Omega = -1/2, -M\rangle |\nu\rangle \\
&\quad \pm |\Lambda = 1, S = 1/2, \Sigma = -1/2\rangle |J, \Omega = 1/2, -M\rangle |\nu\rangle] \\
&= \pm (-1)^{M-1/2} |^2\Pi_{1/2}, \nu, J, -M, p^\pm\rangle
\end{aligned}$$

$$T^2(|^2\Pi_{1/2}, \nu, J, M, p^\pm\rangle) = -|^2\Pi_{1/2}, \nu, J, -M, p^\pm\rangle$$

$$\begin{aligned}
|^2\Pi_{1/2}, J, M, p^\pm\rangle &= \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} (|J\Omega M\rangle + |J - \Omega M\rangle) \frac{1}{\sqrt{2}} (|\Lambda S - \Sigma\rangle \pm |-\Lambda S\Sigma\rangle) \right. \\
&\quad \left. + \frac{1}{\sqrt{2}} (|J\Omega M\rangle - |J - \Omega M\rangle) \frac{1}{\sqrt{2}} (|\Lambda S - \Sigma\rangle \mp |-\Lambda S\Sigma\rangle) \right] \\
&= \frac{1}{\sqrt{2}} [|+J_a\rangle |\zeta^\pm\rangle + |-J_a\rangle |\zeta^\mp\rangle]
\end{aligned}$$

$$|\pm J_a\rangle = \frac{1}{\sqrt{2}} (|J\Omega M\rangle \pm |J - \Omega M\rangle)$$

$$\langle J_a \rangle = \pm 1/2(J + 1/2)$$

$$|\zeta^\pm\rangle = \frac{1}{\sqrt{2}} (|\Lambda S - \Sigma\rangle \pm |-\Lambda S\Sigma\rangle)$$

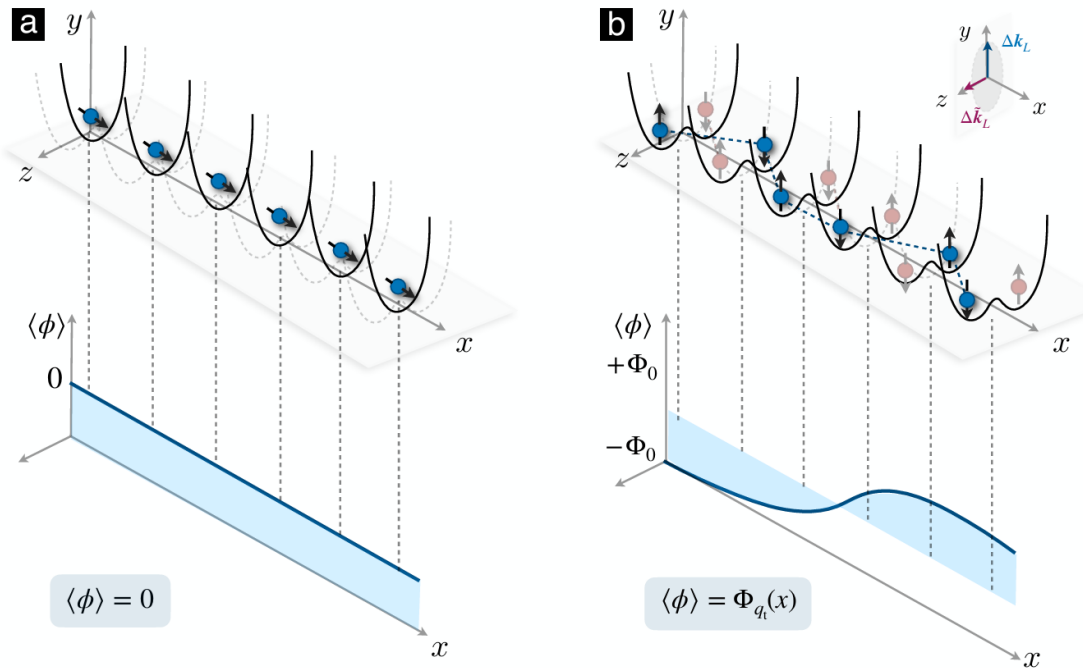
$$L_a = L_b = \frac{1}{2} \left(J + \frac{1}{2} \right)$$

$$2 \cdot \frac{1}{2} \left(J + \frac{1}{2} \right) \approx \sqrt{J(J+1)}$$

$$\Delta E = k \left(J + \frac{1}{2} \right) \left(J + \frac{3}{2} \right) \left(J - \frac{1}{2} \right)$$

$$\Delta E \approx 7 \times 10^{-6} \left(J + \frac{1}{2} \right) \left(J + \frac{3}{2} \right) \left(J - \frac{1}{2} \right) \text{ nm}^{-1}$$





$$S = \int d^4x \left(i\bar{\psi}(\gamma^\mu \partial_\mu - g\phi)\psi + \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m_0^2 \phi^2) - \frac{\lambda}{4} \phi^4 \right)$$

$$\bar{\psi}(t, x) = \psi^\dagger(t, x) \gamma^0 \star \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$H = \int dx \mathcal{H}$$

$$\mathcal{H} = \bar{\psi}(-i\gamma^1 \partial_x + g\phi)\psi + \frac{1}{2}(\pi^2 + (\partial_x \phi)^2 + m_0^2 \phi^2) + \frac{\lambda}{4} \phi^4.$$

$$[\phi(t, x), \pi(t, y)] = i\delta(x - y)$$

$$\{\psi(t, x), \psi^\dagger(t, y)\} = \delta(x - y)$$

$$\langle \phi(x) \rangle = \Phi_0 \neq 0, \forall x \text{ for } m_0^2 < m_c^2(\lambda)$$

$$V_{\text{cl}}(\Phi) = \int dx \left(\frac{m_0^2}{2} \Phi^2(x) + \frac{\lambda}{4} \Phi^4(x) \right)$$

$$\Phi_0 = \begin{cases} 0, & m_0^2 > 0 \\ \pm \sqrt{-\frac{m_0^2}{\lambda}}, & m_0^2 < 0 \end{cases}$$

$$\mu^2 = m_0^2 + 3\lambda \int \frac{d^4k}{(2\pi)^2} \frac{1}{k^2 - m_0^2 + i\epsilon}$$

$$\Phi_{q_t}(x) = q_t \Phi_0 \tanh\left(\frac{x - x_0}{\xi_0}\right), \xi_0^2 = -\frac{2}{m_0^2}$$

$$E_{q_t} = \langle \bar{P}_0^2 \oplus M_0^2 \rangle^{1/2}$$

$$M_0 = \int dx \left(\frac{1}{2} (\partial_x \Phi_{q_t})^2 + \frac{m_0^2}{2} \Phi_{q_t}^2 + \frac{\lambda}{4} \Phi_{q_t}^4 \right) = \frac{2\sqrt{2} m_0^3}{3 \lambda}$$

$$J_{q_t}^\mu(x) = \frac{1}{2\Phi_0} \varepsilon^{\mu\nu} \partial_\nu \Phi_{q_t}(x)$$

$$\int dx J_{q_t}^0 = \frac{\Phi_{q_t}(+\infty) - \Phi_{q_t}(-\infty)}{2\Phi_0} = q_t.$$

$$\Phi_0 \mapsto \Phi = \sqrt{-\mu^2/\lambda} \text{ and } \xi_0^2 \mapsto \xi^2 = 2/\mu^2$$

$$M_0 \mapsto M = 2^{3/2} \mu^3 / 3\lambda + 2\mu(1/2\sqrt{3} - 3/2\pi)$$

$$m_{q_t}(x) = g \langle \phi(x) \rangle_{q_t} = q_t g \Phi \tanh((x - x_0)/\xi)$$

$$\varepsilon_\pm(k) = \pm(k^2 + g^2 \Phi^2)^{1/2}$$

$$\{|\varepsilon_n\rangle, |-\varepsilon_n\rangle = \sigma^y |\varepsilon_n\rangle\}$$

$$\partial_x \Psi_{q_f}(x) = i m_t(x) \gamma^1 \Psi_{q_f}(x).$$

$$\Psi_{q_f}(x) = \mathcal{N} \cosh^{-\sqrt{\frac{2g^2}{\lambda}} \left(\frac{x - x_0}{\xi} \right)} \left(\frac{1}{i q_t} \right),$$

$$\sigma^y \Psi_{q_f}(x) = q_t \Psi_{q_f}(x)$$

$$\varepsilon_n = \pm \frac{1}{\xi} \sqrt{2n \sqrt{\frac{2g^2}{\lambda}} - n^2}, n = 0, 1, \dots, \lfloor \sqrt{2g^2/\lambda} \rfloor$$

$$j_{q_f}^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \int dx: j_{q_f}^0$$

$$|g_-\rangle = \prod_{\varepsilon < 0} \gamma_\varepsilon^\dagger(k) |0\rangle, |g_+\rangle = \gamma_0^\dagger \prod_{\varepsilon < 0} \gamma_\varepsilon^\dagger(k) |0\rangle.$$

$$j_{q_f}^0 := \lim_{a \rightarrow 0} \frac{1}{2} (\psi^\dagger(x+a) \psi(x-a) - \psi(x-a) \psi^\dagger(x+a))$$

$$q_f |g_\pm\rangle = \int dx: j_{q_f}^0: |g_\pm\rangle = \int dx \frac{1}{2} |\Psi_{q_f}(x)|^2 |g_\pm\rangle = \pm \frac{1}{2} |g_\pm\rangle$$

$$\mathcal{A}_\pm(k) = \langle \varepsilon_-(k) | i \partial_k | \varepsilon_-(k) \rangle = \mp i g \Phi / 2(k^2 + g^2 \Phi^2)$$

$$\square_{\text{Chern-Simons}} = \frac{i}{2\pi} \int_{\text{BZ} \times S^0} dk dx A(k, x)$$

$$S_0 = \{x_0 - d, x_0 + d\}$$



$$CS_1 = \pm i \int_{\text{BZ}} \frac{dk}{2\pi} (A_+(k) - A_-(k)) = \frac{q_t}{2} \text{mod } \mathbb{Z}.$$

$$\Lambda_\ell = \left\{ x_n = an - \frac{L}{2} : n \in \{1, \dots, N\} \right\},$$

$$\phi(x), \pi(x) \mapsto \phi_n, \pi_n$$

$$[\phi_n, \pi_{n'}] = i\delta_{n,n'}/a \rightarrow i\delta(x-y)$$

$$H_b = a \sum_{n=1}^N \mathcal{H}_n = a \sum_n \left(\frac{\pi_n^2}{2} + \frac{1}{2} (\nabla_a^+ \phi_n)^2 + \frac{m_0^2}{2} \phi_n^2 + \frac{\lambda}{4} \phi_n^4 \right),$$

$$\phi_{n+1} \approx \phi(x) + \nabla_a^+ \phi_n = (\phi_{n+1} - \phi_n)/a \partial_x \phi$$

$$a \sum_n \mapsto \int dx \text{ and } L = Na \rightarrow \infty$$

$$\nabla_a \phi_n = (\phi_{n+1} - \phi_{n-1})/2a$$

$$k^2 \mapsto k_0^2 + \frac{2}{a^2} \sin^2 \left(\frac{ka}{2} \right) \text{ for } k = -\frac{\pi}{a} + \frac{2\pi}{L} j \text{ with } j = 1, \dots, N$$

$$m_0^2 = \mu^2 - \frac{3\lambda}{2\pi} \frac{1}{\sqrt{1 + \frac{1}{4}\mu^2 a^2}} K \left(\frac{1}{1 + \frac{1}{4}\mu^2 a^2} \right),$$

$$K(x) = \int_0^{\pi/2} d\theta (1 - x \sin^2 \theta)^{-1/2}$$

$$\mu^2 + \Sigma_{\text{sr}}(\mathbf{0}) = 0,$$

$$\Sigma_{\text{sr}}(\mathbf{0}) = -\frac{3}{2} \frac{\lambda^2}{(aN)^2} \sum_{j_1, j_2=1}^N \frac{1}{S(j_1, j_2)},$$

$$\int \frac{dk}{2\pi} \mapsto \frac{1}{aN} \sum_{j=1}^N, \omega^2(j) = \frac{4}{a^2} \sin^2 \frac{ka}{2} + \mu^2$$

$$S(j_1, j_2) = \omega(j_1) + \omega(j_2) + \omega(j_1 + j_2)$$

$$\frac{d\phi_n}{dt} = \pi_n$$

$$\frac{d\pi_n}{dt} = \frac{1}{a^2} (\phi_{n+1} - 2\phi_n + \phi_{n-1}) - m_0^2 \phi_n - \lambda \phi_n^3$$

$$\omega(k) = \sqrt{m_0^2 + \frac{4}{a^2} \sin^2 \left(\frac{ka}{2} \right)}$$



$$\phi_n = \sum_k \phi_k e^{ikx_n/\sqrt{L}} \text{ for } k = -\frac{\pi}{a} + \frac{2\pi}{L} \left(j + \frac{1}{2}\right)$$

$$\omega^2(k) = m_0^2 + k^2$$

$$\phi_n \mapsto \Phi_n, \pi_n \mapsto \Pi_n$$

$$\frac{1}{i\hbar} [\phi_n, \pi_{n'}] \mapsto \{\Phi_n, \Pi_{n'}\}_{\text{PB}} = \delta_{n,n'}/a, \text{ where } \{F, G\}_{\text{PB}} = \sum_n (\partial_{\Phi_n} F \partial_{\Pi_n} G - \partial_{\Phi_n} G \partial_{\Pi_n} F)$$

$$\Phi_{q_t, n} = q_t \Phi_0 \tanh \left(\frac{a}{\xi_0} (n - n_0) \right).$$

$$\Phi_{q_t}(x) \rightarrow \Phi_{q_t}(\gamma_v(x - vt)) \text{ with } \gamma_v = 1/\sqrt{1 - v^2}$$

$$V_{\text{PN}}(n_0) = a \sum_n \frac{1}{2} \left(\frac{\Phi_{q_t, n+1} - \Phi_{q_t, n}}{a} \right)^2 + \frac{1}{2} m_0^2 \Phi_n^2 + \frac{\lambda}{4} \Phi_n^4,$$

$$V_{\text{PN}}(n_0 + 1) = V_{\text{PN}}(n_0)$$

$$V_{\text{PN}}(n_0) \approx M_0 + V_0 \cos(2\pi n_0)$$

$$\Phi_0 \tanh \left(\left(n - \frac{1}{2} \right) a / \xi_0 \right)$$

$$\phi_n(t) = \Phi_{K, n} + \delta\phi_n(t)$$

$$\frac{d^4}{dt^2} \delta\phi_n = -(2 + m_0^2 + 3\lambda\Phi_{K, n}^2) \delta\phi_n + \delta\phi_{n+1} + \delta\phi_{n-1}.$$

$$\delta\phi_n(t) = e^{i\Omega_v t} \mathcal{M}_{v, n}^b$$

$$m_0^2 \mapsto 2\lambda\Phi_0^2 = 2m_0^2$$

$$\{\chi_n, \chi_{n'}^\dagger\} = \delta_{n, n'}/a \rightarrow \delta(x - y)$$

$$H_{\text{fb}} = a \sum_{n=1}^N (iaJ\chi_n^\dagger \nabla_a \chi_n + (-1)^n m(\phi_n) \chi_n^\dagger \chi_n),$$

$$\nabla_a \chi_n = (\chi_{n+1} - \chi_{n-1})/2a$$

$$m(\phi_n) = m_f + g\phi_n.$$

$$\chi_n = \sum_k e^{ikx} \chi(k)/\sqrt{L}$$

$$H_{\text{fb}} = \sum_{k \in \text{HBZ}} \psi^\dagger(k) (\mathbf{d}(k) \cdot \boldsymbol{\sigma}) \psi(k)$$

$$\psi(k) = (\chi(k), \chi(k + \pi/a))^t$$



$$\mathbf{d}(k) = (m_f, 0, 2J \sin(ka))$$

$$k = -\frac{\pi}{2a} + \frac{\pi}{L}j \text{ with } j = 1, \dots, \frac{N}{2}$$

$$\varepsilon_{\pm}(k) = \pm |\mathbf{d}(k)| = \pm \sqrt{m_f^2 + 4J^2 \sin^2(ka)}.$$

$$\gamma^0 = \sigma^x, \gamma^1 = -i\sigma^y, \text{ and } \gamma^5 = \sigma^z$$

$$\{|\varepsilon_+(k)\rangle, |\varepsilon_-(k)\rangle = \sigma^y |\varepsilon_+(k)\rangle\} \iint \sigma^y (\mathbf{d}(k) \cdot \boldsymbol{\sigma}) \sigma^y = -\mathbf{d}(k) \cdot \boldsymbol{\sigma}$$

$$\psi_{\pm}(x) = \frac{1}{2}(1 \pm \gamma^5)\psi(x)$$

$$\chi_{n\pm 1} \approx \psi_+(x) \pm a\partial_x \psi_+(x) + (-1)^{n\pm 1}(\psi_-(x) \pm a\partial_x \psi_-(x)),$$

$$M_{q_t, n} = g\langle \phi_n \rangle_{q_t} = q_t g \Phi_0 \tanh(a(n - n_0)/\xi_0)$$

$$iJ\Psi_{q_f, n+1} + (-1)^n M_{q_t, n} \Psi_{q_f, n} - iJ\Psi_{q_f, n-1} = 0.$$

$$H_{\text{fb}} = a \sum_{n,l} \chi_n^\dagger [h(\Phi_K)]_{nl} \chi_l,$$

$$[h(\Phi_K)]_{nl} = iJ\delta_{l, n+1} - iJ\delta_{l, n-1} + g(-1)^n \Phi_{K, n} \delta_{l, n}.$$

$$|g_+\rangle = \prod_{v=1}^{N_f} \gamma_{\varepsilon_v}^\dagger |0\rangle,$$

$$N_f = (N + 1)/2 (N_f = N/2)$$

$$\gamma_{\varepsilon_v}^\dagger = \sum_n \mathcal{M}_{n, \varepsilon_v}^f \chi_n^\dagger$$

$$\sum_{n,l} (\mathcal{M}_{n, \varepsilon_v}^f)^* [h(\Phi_K)]_{nl} \mathcal{M}_{l, \varepsilon_\tau}^f = \varepsilon_v \delta_{v, \tau}$$

$$\mathcal{M}_{n, -\varepsilon_v}^f = (\mathcal{M}_{N-n+1, +\varepsilon_v}^f)^*.$$

$$\begin{aligned} a \sum_n \langle \chi_n^\dagger \chi_n \rangle_{g_+} &= \sum_n \sum_{\varepsilon_v < 0} |\mathcal{M}_{n, \varepsilon_v}^f|^2 + |\mathcal{M}_{n, 0}^f|^2 \\ &= \sum_n \sum_{\varepsilon_v < 0} \frac{1}{2} (|\mathcal{M}_{n, \varepsilon_v}^f|^2 + |\mathcal{M}_{N-n+1, -\varepsilon_v}^f|^2) + |\mathcal{M}_{n, 0}^f|^2 \\ &= \sum_n \sum_{\varepsilon_v > 0} \frac{1}{2} |\mathcal{M}_{n, \varepsilon}^f|^2 + \frac{1}{2} |\mathcal{M}_{n, 0}^f|^2 \\ &= \sum_n \frac{1}{2} (1 + |\mathcal{M}_{n, 0}^f|^2), \end{aligned}$$



$$\mathcal{M}^f(\mathcal{M}^f)^\dagger = \mathbb{1}$$

$$\rho_{n,n+1} = \frac{a}{2} \langle \chi_n^\dagger \chi_n + \chi_{n+1}^\dagger \chi_{n+1} \rangle$$

$$q_f = \sum_n \left(\rho_{n,n+1} - \frac{1}{2} \right) = \frac{1}{2} \sum_n |\mathcal{M}_{n,0}^f|^2 = \frac{1}{2}.$$

$$2Q_{n,n+1} = \sum_{l=n-n_0}^{n+n_0} f_{l-n} \left(\chi_l^\dagger \chi_l - \frac{1}{2} \right) + \sum_{l=n+1-n_0}^{n+1+n_0} f_{l-n-1} \left(\chi_l^\dagger \chi_l - \frac{1}{2} \right),$$

$$\langle Q_{n,n+1} \rangle = \rho_{n,n+1} - \frac{1}{2}.$$

$$\Delta Q_n = \sum_{j=1}^n \langle Q_{j,j+1} \rangle = \sum_{j=1}^n \left(\rho_{j,j+1} - \frac{1}{2} \right)$$

$$H_h = \sum_{n,\alpha} \left(\frac{p_{n,\alpha}^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 u_{n,\alpha}^2 + \frac{1}{4} \sum_{l \neq n} k_{nl}^{\alpha\beta} (u_{n,\alpha} - u_{l,\beta})^2 \right).$$

$$k_{nl}^{\alpha\beta} = - \frac{e^2 \left(3(\mathbf{R}_n^0 - \mathbf{R}_l^0)_\alpha (\mathbf{R}_n^0 - \mathbf{R}_l^0)_\beta - \delta_{\alpha\beta} (\mathbf{R}_n^0 - \mathbf{R}_l^0)^2 \right)}{4\pi\epsilon_0 \|\mathbf{R}_n^0 - \mathbf{R}_l^0\|^5},$$

$$\frac{1}{2} \sum_{i=1}^n |\mathcal{M}_{i,0}^f|^2$$

$$k_{nl}^{zz} = k_{nl}^{yy} = -\frac{1}{2} k_{nl}^{xx} = -e^2/4\pi\epsilon_0 |x_n - x_l|^3$$

$$\kappa_x = (\omega_x/\omega_z)^2$$

$$u_{n,z} = \frac{1}{\sqrt{L}} \sum_{k \in \text{BZ}} e^{ikan} u_z(k), p_{n,z} = \frac{1}{\sqrt{L}} \sum_{k \in \text{BZ}} e^{ikan} p_z(k)$$

$$\omega(k) = \omega_z \sqrt{1 - \kappa_x \left(\frac{\ell}{a} \right)^3 \sum_{r=1}^{N/2} \frac{4}{r^3} \sin^2 \left(\frac{ka}{2} r \right)},$$

$$\ell^3 = e^2/4\pi\epsilon_0 m_\alpha \omega_x^2$$

$$\omega(k) = \sqrt{\frac{m_0^2 c^4}{\hbar^2} + \frac{4c_b^2}{a^2} \sin^2 \left(\frac{ka}{2} \right)}$$

$$\omega^2(k) \approx m_0^2 c_b^4 / \hbar^2 + c_b^2 k^2$$



$$c_b^2 = a^2 \omega_x^2 \left(\frac{\ell}{a}\right)^3 \eta_N(1), \eta_N(s) = \sum_{r=1}^{N/2} \frac{1}{r^s} (-1)^{r+1}$$

$$m_0^2 = \frac{\hbar^2}{c_b^4} \left(\omega_z^2 - \frac{7}{2} \omega_x^2 \left(\frac{\ell}{a}\right)^3 \zeta_N(3) \right), \zeta_N(s) = \sum_{r=1}^{N/2} \frac{1}{r^s}$$

$$\kappa_{x,c} = 2a^3/7\ell^3\zeta_N$$

$$\phi(x) = \frac{(-1)^n}{\sqrt{m_a a^3}} u_{n,z}, \pi(x) = (-1)^n \sqrt{m_a a} p_{n,z},$$

$$[\phi(t, x), \pi(t, y)] = i\hbar\delta(x - y)$$

$$V_{\text{int}} = \frac{1}{2} \sum_{n \neq l} \frac{\beta_{nl}^{zz}}{4} (u_{n,z} - u_{l,z})^4, \beta_{nl}^{zz} = \frac{3e^2}{8\pi\epsilon_0} \frac{1}{|x_n - x_l|^5},$$

$$\lambda = \frac{243\zeta_N}{4K^4} m_a^3 \omega_x^2 \ell^3, K = \frac{m_a a c_b}{\hbar}$$

$$\phi(x) \rightarrow -\phi(x)$$

$$H = \int dx \left(\frac{c_b^2}{2\hbar^2} \pi^2 + \frac{\hbar^2}{2} (\partial_x \phi)^2 + \frac{m_0^2 c_b^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right)$$

$$\sigma_n^z = |\uparrow_n\rangle\langle\uparrow_n| - |\downarrow_n\rangle\langle\downarrow_n|, \sigma_n^+ = |\uparrow_n\rangle\langle\downarrow_n|, \sigma_n^- = |\downarrow_n\rangle\langle\uparrow_n|,$$

$$\sigma_n^x = \sigma_n^+ + \sigma_n^-, \text{ and } \sigma_n^y = -i\sigma_n^+ + i\sigma_n^-$$

$$\sigma_n^+ = \sqrt{a} \chi_n^\dagger e^{i\pi(\frac{n}{2} + a \sum_{l=1}^{n-1} \chi_l^\dagger \chi_l)} = (\sigma_n^-)^\dagger, \sigma_n^z = 2a \chi_n^\dagger \chi_n - 1$$

$$\chi_n \rightarrow e^{ie\Lambda_n/\hbar} \chi_n \text{ with } \Lambda_n = \hbar\pi n/2e$$

$$k_{nl}^{yy} = -e^2/4\pi\epsilon_0 \|\mathbf{R}_n^0 - \mathbf{R}_l^0\|^3$$

$$v(k) = \omega_y \sqrt{1 - \kappa_y \left(\frac{\ell}{a}\right)^3 \sum_{r=1}^{N/2} \frac{4}{r^3} \sin^2\left(\frac{ka}{2} r\right)},$$

$$\kappa_y = (\omega_x/\omega_y)^2 \ll 1$$

$$H_y = \sum_k \left(\frac{1}{2m_a} |p_y(k)|^2 + \frac{1}{2} m_a v^2(k) |u_y(k)|^2 \right) = \sum_k v(k) \left(b_k^\dagger b_k + \frac{1}{2} \right)$$

$$v(k) \in [\omega_y, \omega_y - \Delta v]$$



$$V_y(t) \approx \sum_{nk} \mathcal{F}_{nk} \Delta y_0 \sigma_n^{\varphi_s} b_k e^{i(\varphi_m - \delta(k))t} + \text{H. c.}$$

$$\sigma_n^{\varphi_s} = i e^{i\varphi_s} \sigma_n^+ + i e^{-i\varphi_s} \sigma_n^-$$

$$\varphi_s = (\varphi_r + \varphi_b)/2, \text{ and } \Delta y_0 = 1/\sqrt{2m_a \omega_y}$$

$$\mathcal{F}_{nk} = \frac{\Omega_L}{2} \Delta \mathbf{k}_L \cdot \mathbf{e}_y \sqrt{\omega_y/v(k)} \mathcal{M}_{nk}^s$$

$$\mathcal{M}_{nk}^s = e^{ikx_n} \sqrt{N} \mathcal{O}(\Omega_L(\mathbf{k}_L))$$

$$\varphi_m = (\varphi_r - \varphi_b)/2$$

$$\delta(k) = v(k) - \delta_L$$

$$|\mathcal{F}_{nk} \Delta y_0|, h_0 \ll \delta(k) \ll \omega_y, \max |J_{nl}| \ll 2h_0,$$

$$J_{nl} = - \sum_k \mathcal{F}_{nk} \Delta y_0 \frac{1}{\delta(k)} \mathcal{F}_{mk}^* \Delta y_0 + \text{c. c.}$$

$$J_{nl} = J_0 \left(\frac{\omega_x^4 \eta_N(1)}{(\omega_y^2 - \delta_L^2)^2} \frac{\ell^3}{|x_n - x_l|^3} - (-1)^{n-l} \frac{\tilde{\lambda}_c a^2}{\ell^3} e^{-\frac{|x_n - x_l|}{\tilde{\lambda}_c}} \right)$$

$$J_0 = 2\Omega_L^2 \eta_x^2 / \omega_x \eta_N(1), \text{ and } \eta_x = \Delta k_L / \sqrt{2m_a \omega_x}$$

$$\tilde{\lambda}_c = \frac{1}{m_0^2 c_b^4 - \delta_L^2}$$

$$S_z = \frac{1}{N} \sum_n \sigma_n^z$$

$$V_{\text{eff}} = \sum_n \sum_{l>n} J_{nl} \sigma_n^+ \sigma_l^- + \text{H. c.}$$

$$V_{\text{eff}} = a \sum_n \sum_{l>n} (i J_{nl} \chi_n^\dagger \chi_l + \text{H. c.}) + V_{\text{int}}(\{\chi_r^\dagger, \chi_r\}),$$

$$V_{\text{int}} = a \sum_n \sum_{r>0} J_{n,n+r} \chi_n^\dagger (\Pi_r (1 - 2a \chi_r^\dagger \chi_r) - 1) \chi_{n+r} + \text{H. c.}$$

$$c_f = 2a J_{n,n+1} \rightarrow c_f = 2a \sum_{r>0} (2r-1) (-1)^{r-1} J_{n,n+(2r-1)}$$

$$V_{\text{int}} \approx -a \sum_n 2J_{n,n+2} a \chi_n^\dagger \chi_{n+1}^\dagger \chi_{n+1} \chi_{n+2} + \text{H. c.}$$

$$V_z \approx \sum_n \frac{\tilde{\Omega}_L}{2} |\uparrow_n\rangle \langle \uparrow_n| J_0 (|\beta_n|) (-1)^n \sin(\Delta \tilde{k}_L \cdot \mathbf{e}_z \sqrt{m_a a^3} \phi(x))$$



$$\beta_{\tilde{\Omega}_L(\Delta\tilde{k}_L)} = \frac{q_z}{2} \Delta\tilde{k}_L \cdot \mathbf{e}_z \sqrt{m_a a^3} (-1)^n \langle \phi(x) \rangle$$

$$\langle \phi(x) \rangle = \pm \Phi_n$$

$$J_0(-x) = J_0(x) \sin(-x) = -\sin(x)$$

$$\phi(x) = \Phi_n + \delta\phi(x)$$

$$\sin(c\Phi_n + c\delta\phi(x)) \approx \cos(c\Phi_n) \sin(c\delta\phi(x)) \approx \cos(c|\Phi_n|) c\delta\phi(x)$$

$$V_z = g \sum_n (-1)^n \frac{1}{2} (1 + \sigma_n^z) \delta\phi(x)$$

$$g = \frac{\tilde{\Omega}_L}{2} J_0 \left(\frac{q_z}{2} \Delta\tilde{k}_L \cdot \mathbf{e}_z z_0 \right) \cos(\Delta\tilde{k}_L \cdot \mathbf{e}_z z_0)$$

$$\phi \rightarrow -\phi, \psi \rightarrow \gamma^5 \psi, \bar{\psi} \rightarrow -\bar{\psi} \gamma^5$$

$$f_n = g (-1)^n \langle \chi_n^\dagger \chi_n \rangle$$

$$\frac{d\phi_n}{dt} = \pi_n$$

$$\frac{d\pi_n}{dt} = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{a^2} - m_0^2 \phi_n - \lambda \phi_n^3 - g (-1)^n \chi_n^\dagger \chi_n$$

$$\frac{d\chi_n}{dt} = J(\chi_{n+1} - \chi_{n-1}) - ig(-1)^n \phi_n \chi_n,$$

$$\frac{d\chi_n^\dagger}{dt} = J(\chi_{n+1}^\dagger - \chi_{n-1}^\dagger) + ig(-1)^n \phi_n \chi_n^\dagger.$$

$$|g_+(t)\rangle = \prod_{\varepsilon_v \leq 0} \gamma_{\varepsilon_v}^\dagger(t) |0\rangle = \prod_{\varepsilon_v \leq 0} \sum_n \mathcal{M}_{n,\varepsilon_v}^f(\Phi(t)) \chi_n^\dagger |0\rangle,$$

$$J(\mathcal{M}_{n+1,\varepsilon_v}^f - \mathcal{M}_{n-1,\varepsilon_v}^f) - ig(-1)^n \Phi_n(t) \mathcal{M}_{n,\varepsilon_v}^f = \varepsilon_v \mathcal{M}_{n,\varepsilon_v}^f.$$

$$\phi_n, \pi_n \rightarrow \Phi_n, \Pi_n$$

$$V_{\text{BO}}(\Phi) = V_{\text{cl}}(\Phi) + \delta V_{\text{BO}}(\Phi)$$

$$\begin{aligned} \delta V_{\text{BO}}(\Phi) &= a \langle \chi_n^\dagger \chi_n \rangle \sum_{a(\chi_n^\dagger \chi_n)} g (-1)^n \Phi_n \langle g_+(t) | \chi_n^\dagger \chi_n | g_+(t) \rangle \\ &= a \langle \chi_n^\dagger \chi_n \rangle \sum_{a(\chi_n^\dagger \chi_n)} \sum_{\varepsilon_v \leq 0} \sum_{\varepsilon_\mu \leq 0} g (-1)^n \Phi_n \left(\mathcal{M}_{n,\varepsilon_\mu}^f(\Phi) \right)^* \mathcal{M}_{n,\varepsilon_v}^f(\Phi) \end{aligned}$$

$$\Phi = (\Phi_1, \dots, \Phi_N), \Pi = (\Pi_1, \dots, \Pi_N)$$



$|\Phi\rangle = \otimes_n |\Phi_n\rangle, |\Pi\rangle = \otimes_n |\Pi_n\rangle$, such that $\langle \Phi | \Pi \rangle = e^{i\Phi \cdot \Pi / \hbar} / (2\pi)^{N/2}$

$$W(\Phi, \Pi) = \frac{1}{(2\pi)^N} \int d^N \zeta \langle \Phi - \zeta/2 | \rho | \Phi + \zeta/2 \rangle e^{i \frac{\Pi \cdot \zeta}{\hbar}},$$

$$O(\{\phi_m, \pi_n\}) = O_1(\{\phi_m\}) \bigoplus O_2(\{\pi_n\})$$

$$\phi_n, \pi_n \mapsto \Phi_n, \Pi_n$$

$$\langle O(t) \rangle = \int d^N \Phi \int d^N \Pi O_W(\Phi, \Pi) W(\Phi, \Pi, t),$$

$$\partial_t W(\Phi, \Pi, t) = \{H(\Phi, \Pi), W(\Phi, \Pi, t)\}_{\text{MB}}$$

$$\{H, W\}_{\text{MB}} = \frac{2}{\hbar} H_W(\Phi, \Pi) \sin\left(\frac{\hbar}{2} \Lambda\right) W(\Phi, \Pi).$$

$$\Lambda = \overleftarrow{\nabla}_\Phi \cdot \overleftarrow{\nabla}_\Pi - \overleftarrow{\nabla}_\Pi \cdot \overleftarrow{\nabla}_\Phi,$$

$$\overleftarrow{\nabla}_v = (\partial_{v_1}, \partial_{v_2}, \dots, \partial_{v_N})$$

$$\sin(\hbar \Lambda / 2) \approx \hbar \Lambda / 2$$

$$\partial_t W - \{H, W\}_{\text{PB}} = \square^\bullet$$

$$W(\Phi, \Pi, t) d^N \Phi d^N \Pi = W(\Phi_0, \Pi_0) d^N \Phi_0 d^N \Pi_0$$

$$\Phi(-t) = \Phi_0, \Pi(-t) = \Pi_0$$

$$\langle O(t) \rangle = \int d^N \Phi_0 \int d^N \Pi_0 W(\Phi_0, \Pi_0) O_W(\Phi(t), \Pi(t))$$

$$\Phi = \Phi_K + \Phi_0, \Pi = \Pi_0$$

$$H_b \approx \frac{a}{2} \pi^2 + \frac{1}{2a} \delta \phi^t K(\Phi_K) \delta \phi$$

$$\pi = (\pi_1, \dots, \pi_N)^t, \delta \phi = (\delta \phi_1, \dots, \delta \phi_N)^t$$

$$K(\Phi_K) = (2 + m_0^2 a^2 + 3a^2 \lambda \Phi_{K,n}^2) \delta_{n,n} + \delta_{n,n+1} + \delta_{n,n-1}$$

$$\sum_{nl} \mathcal{M}_{nv}^b [K(\Phi_K)]_{nl} \mathcal{M}_{l\tau}^b = \Omega_v^2 \delta_{v,\tau}$$

$$H_b = \sum_v \frac{\Omega_v}{2} (P_v^2 + Q_v^2)$$

$$Q_v = \sum_n \mathcal{M}_{nv}^b \Phi_{0,n}, P_v = a \sum_n \mathcal{M}_{nv}^b \Pi_{0,n}$$



$$W(\Phi_0, \Pi_0) \propto \prod_{v=0}^{N-1} \exp\left(-\frac{Q_v^2(\Phi_0)}{2\sigma_{Q_v}^2} - \frac{P_v^2(\Pi_0)}{2\sigma_{P_v}^2}\right),$$

$$\sigma_{Q_v}^2 = 1/2\Omega_v \text{ and } \sigma_{P_v}^2 = \Omega_v/2$$

$$\Gamma_{nl} = \langle \chi_n^\dagger \chi_l \rangle, \frac{d\Gamma}{dt} = \{\Gamma, h(\Phi(t))\}$$

$$h(\Phi_K) \mapsto h(\Phi(t))$$

$$\frac{d\Phi_n}{dt} = \Pi_n(t)$$

$$\frac{d\Pi_n}{dt} = \frac{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}{a^2} - m_0^2\Phi_n - \lambda\Phi_n^3 - g(-1)^n\Gamma_{nn}(t)$$

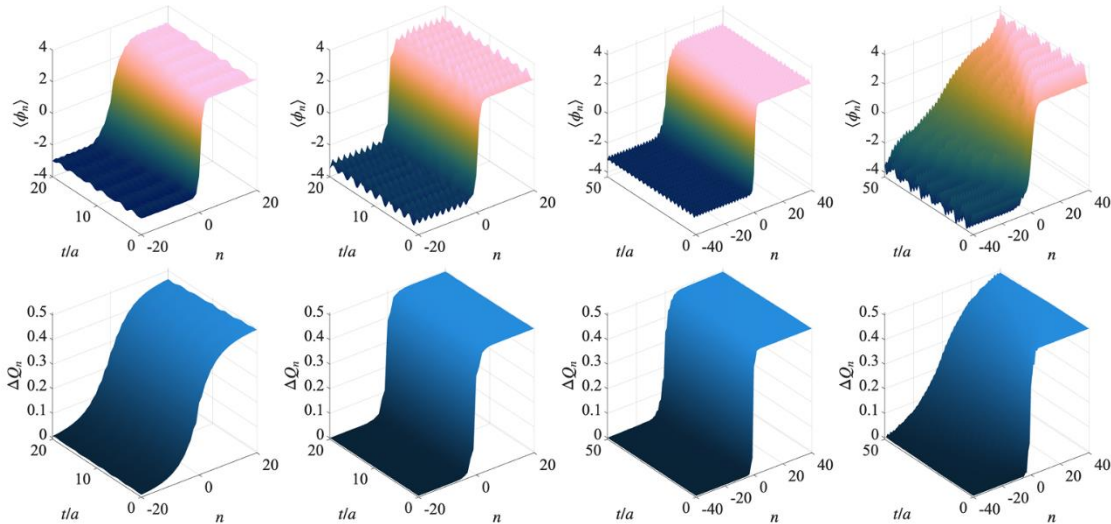
$$\{W(\Phi_0, \Pi_0), \Phi(\Gamma(t)), \Pi(\Gamma(t)), |\psi(\Gamma(\Phi(t), \Pi(t)))\rangle\}$$

$$\Gamma_{nl}(t + \Delta t) = \sum_{v \leq N} (\mathcal{M}_{n,\varepsilon_v}^f(\Phi(t + \Delta t)))^* \mathcal{M}_{l,\varepsilon_v}^f(\Phi(t + \Delta t))$$

$$|\Psi_0\rangle = |g_b\rangle \otimes |g_f\rangle$$

$$|g_f\rangle = \prod_{\varepsilon_v \leq 0} \sum_n \mathcal{M}_{n,\varepsilon_v}^f(\Phi_0) \chi_n^\dagger |0\rangle = |g_+(t_0)\rangle$$

$$\Gamma_{nl}(t_0) = \sum_{v \leq N/2} (\mathcal{M}_{n,\varepsilon_v}^f(\Phi_0))^* \mathcal{M}_{l,\varepsilon_v}^f(\Phi_0)$$



$$\Phi_{K,n} = \Phi_0 \tanh(an/\xi)$$

$$\Phi_n(t) = \Phi_0(t) \tanh\left(\frac{a}{\xi(t)}(n - n_0)\right)$$



$$\Delta Q_n(t) = A(t) \tanh \left(\frac{a}{\xi_f(t)} (n - n_0) \right) + B$$

$$W(\Phi_0, \Pi_0) \propto e^{-\left(\frac{Q_0^2}{2\sigma_{Q_0}^2} + \frac{(P_0 - \bar{P}_0)^2}{2\sigma_{P_0}^2}\right)} \prod_{v=1}^{N-1} e^{-\left(\frac{Q_v^2}{2\sigma_{Q_v}^2} + \frac{P_v^2}{2\sigma_{P_v}^2}\right)}$$

$$x_0 \mapsto x(t) = x_0 \bigoplus vt$$

$$\Phi_{0,n} = \Phi_{K\bar{K},n}, \Pi_{0,n} = -\bar{\Pi}_0 \mathcal{M}_{n,0}^{b,(e)}$$

$$\Phi_{K\bar{K},n} = -\Phi_0 \tanh \left(\frac{an - d/2}{\xi_0} \right) + \Phi_0 \tanh \left(\frac{an + d/2}{\xi_0} \right) - \Phi_0$$

$$\langle \bar{\psi}(x_n) \psi(x_n) \rangle = \langle \chi_{2n-1}^\dagger \chi_{2n-1} \rangle - \langle \chi_{2n}^\dagger \chi_{2n} \rangle,$$

$$W_0 \propto e^{-\left(\frac{Q_0^2}{2\sigma_{Q_0}^2} + \frac{(P_0 - \bar{P}_0)^2}{2\sigma_{P_0}^2}\right)} e^{-\left(\frac{Q_1^2}{2\sigma_{Q_1}^2} + \frac{P_1^2}{2\sigma_{P_1}^2}\right)} \prod_{v=2}^{N-1} e^{-\left(\frac{Q_v^2}{2\sigma_{Q_v}^2} + \frac{P_v^2}{2\sigma_{P_v}^2}\right)},$$

$$\mathcal{P} \frac{1}{p^2 - m^2}$$

$$\Sigma_{FW}(p) = \int_{k_s \leq \Lambda} \frac{d^4 \mathbf{k}}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{1}{(p-k)^2 - m_1^2 + i\epsilon} \mathcal{P} \frac{1}{k^2 - m_2^2}$$

$$\mathcal{P} \frac{1}{k^2 - m_2^2} = \frac{1}{2} \left(\frac{1}{k^2 - m_2^2 - i\epsilon} + \frac{1}{k^2 - m_2^2 + i\epsilon} \right)$$

$$\Sigma_{FW}(p) = \frac{1}{2} (\Sigma(p) + \Sigma'(p))$$

$$\Sigma'(p) = \int_{k_s \leq \Lambda} \frac{d^4 \mathbf{k}}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{1}{(p-k)^2 - m_1^2 + i\epsilon} \frac{1}{k^2 - m_2^2 - i\epsilon}$$

$$\Sigma'(p) = \frac{i}{4} \int_{k_s \leq \Lambda} \frac{d^4 \mathbf{k}}{(2\pi)^3} \frac{1}{\omega_1 \omega_2} \left(\frac{1}{p^0 - \omega_1 + \omega_2 + i\epsilon} - \frac{1}{p^0 + \omega_1 - \omega_2 - i\epsilon} \right)$$

$$\omega_1 = \sqrt{k_s^2 + p_s^2 - 2k_s p_s u + m_1^2}, \omega_2 = \sqrt{k_s^2 + m_2^2}, p_s = |\mathbf{p}| \text{ and } u = \cos \theta, \theta$$

$$(\omega_1 - \omega_2)^2 \leq (m_1 - m_2)^2$$

$$p^2 > (m_1 - m_2)^2 \text{ gives } (p^0)^2 > (\omega_1 - \omega_2)^2$$



$$\Sigma'(p) = \frac{i}{8\pi^2} \int_0^\Lambda \frac{k_s^2 dk_s}{\omega_1 \omega_2} \left(\frac{1}{p^0 - \omega_1 + \omega_2} - \frac{1}{p^0 + \omega_1 - \omega_2} \right) \\ \frac{m_1^2 - m_2^2}{(p^0)^2 k_s} + \mathcal{O}\left(\frac{1}{k_s^3}\right)$$

$$\Sigma'_{\text{div}}(p) = \frac{i \ln \Lambda^2 m_1^2 - m_2^2}{(4\pi)^2 p^2}$$

$$(p^0)^2 - (\omega_1 - \omega_2)^2 = p^2 + p_s^2(1 - u^2) + \mathcal{O}(1/k_s)$$

$$(p^0)^2 > (\omega_1 - \omega_2)^2$$

$$\Sigma'(p) = \frac{i}{8\pi^2} \int_0^\Lambda \frac{k_s^2 dk_s}{\omega_1 \omega_2} \int_{-1}^1 \frac{du}{\omega_2 - \omega_1 + i\epsilon}$$

$$\Sigma'_{\text{div}}(p) = \frac{i}{8\pi^2} \int_0^\Lambda dk_s \int_{-1}^1 du \left[\frac{1}{p_s(u + i\epsilon)} + \frac{m_1^2 - m_2^2 + p_s^2(1 + u^2)}{2k_s p_s^2(u + i\epsilon)^2} \right] \\ = \frac{\Lambda}{8\pi p_s} - \frac{i(m_1^2 - m_2^2)}{(4\pi)^2 p_s^2} \ln \Lambda^2$$

$$\Sigma_{\text{FW-FW}}(p) = \mathcal{P} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(p-k)^2 - m_1^2} \frac{1}{k^2 - m_2^2} = \frac{1}{2} (\Sigma_{\text{FW}}(p) + \Sigma_{\text{FW}}^*(p))$$

$$i(4\pi)^2 \Sigma'(p) = -\frac{m^2}{p^2} \ln \frac{4\Lambda^2}{m^2} - \frac{p^2 - m^2}{p^2} \ln \frac{p^2 - m^2 + i\epsilon}{p^2}$$

$$2\text{Im}[-i\Sigma'(p)] = -\frac{m^2 - p^2}{8\pi p^2} \theta(m^2 - p^2)$$

$$\Sigma'(p) = -\int \frac{d^D k}{(2\pi)^D} \frac{1}{(p-k)^2 - m_1^2 + i\epsilon} \frac{1}{-k^2 + m_2^2 + i\epsilon} \\ = -\int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[i\epsilon - (1-2x)k^2 + p^2 \frac{x(1-x)}{1-2x} - m_1^2 x + m_2^2(1-x) \right]^2}$$

$$i(4\pi)^2 \Sigma'(p) = -\left[\frac{2}{4-D} + 2 - \gamma_E + \ln(4\pi) \right] \frac{m^2 p^2}{(p^2)^2 + \epsilon^2} + \frac{1}{2} \ln((p^2)^2 + \epsilon^2) \\ + \frac{m^2 \ln m^2}{p^2 - i\epsilon} - \frac{m^2}{2} \left[\frac{\ln(p^2 + i\epsilon)}{p^2 + i\epsilon} + \frac{\ln(p^2 - i\epsilon)}{p^2 - i\epsilon} \right] \\ - \frac{p^2 - m^2 + i\epsilon}{p^2 + i\epsilon} \ln(p^2 - m^2 + i\epsilon)$$

$$\ln \Lambda^2 = \frac{2}{4-D} + 2 - \gamma_E + \ln \pi$$

$$\Sigma'(p) = U(p^2 + i\epsilon, m_1^2, m_2^2) - U(p^2 - i\epsilon, m_2^2, m_1^2)$$



$$U(a, b, c) = \frac{i\Gamma\left(\frac{4-D}{2}\right)}{(4\pi)^{D/2}} \int_0^{1/2} dx (1-2x)^{2-D} [ax(1-x) - (1-2x)(bx - c(1-x))]^{(D-4)/2}$$

$$i(4\pi)^2 U(p^2, m_1^2, m_2^2) = \frac{v_+}{2p^2} \left(\ln \frac{4\Lambda^2}{m_2^2} - z \ln \frac{1+z}{1-z} \right)$$

$$z = \frac{\sqrt{u_+ u_-}}{v_+}, u_{\pm} = (m_1 \pm m_2)^2 - p^2, v_{\pm} = p^2 \mp m_1^2 \pm m_2^2$$

$$x = \frac{\sqrt{|u_+ u_-|}}{v_+}, y = \frac{\sqrt{|u_+ u_-|}}{v_-}, z' = \frac{\sqrt{u_+ u_-}}{v_-}$$

p^2 range	$z \ln \frac{1+z}{1-z} \Big _{p^2 \rightarrow p^2 + i\epsilon}$	$z' \ln \frac{1+z'}{1-z'} \Big _{p^2 \rightarrow p^2 - i\epsilon}$	
$u_+ < 0$	$x \ln \left(\frac{1+x}{1-x} \right)$	$y \ln \left(\frac{1+y}{1-y} \right)$	$0 < x < 1, 0 < y < 1$
$-v_+ < 0 < u_+$	$-2x \arctan(x)$	$-2y \arctan(y)$	$x > 0, y > 0$
$u_- < 0 < -v_+$	$-2x \arctan(x) - 2\pi x$	$-2y \arctan(y)$	$x < 0, y > 0$
$-p^2 < 0 < u_-$	$x \ln \left(\frac{1+x}{1-x} \right) - 2i\pi x$	$y \ln \left(\frac{1+y}{1-y} \right)$	$-1 < x < 0, 0 < y < 1$
$-v_- < 0 < -p^2$	$x \ln \left(\frac{1+x}{x-1} \right) - i\pi x$	$y \ln \left(\frac{1+y}{y-1} \right) + i\pi y$	$x < -1, y > 1$
$0 < -v_-$	$x \ln \left(\frac{1+x}{x-1} \right) - i\pi x$	$y \ln \left(\frac{1+y}{y-1} \right) + i\pi y$	$x < -1, y < -1$

$$\Sigma_{\text{FW}}(p) = \frac{1}{2} \left(V(p^2 + i\epsilon, m_1^2, m_2^2) + U(p^2 + i\epsilon, m_1^2, m_2^2) - U(p^2 - i\epsilon, m_2^2, m_1^2) \right)$$

$$\Sigma_{\text{FW-FW}}(p) = \frac{1}{4} \left(V(p^2 + i\epsilon, m_1^2, m_2^2) + U(p^2 + i\epsilon, m_1^2, m_2^2) - U(p^2 - i\epsilon, m_2^2, m_1^2) - V(p^2 - i\epsilon, m_1^2, m_2^2) - U(p^2 - i\epsilon, m_1^2, m_2^2) + U(p^2 + i\epsilon, m_2^2, m_1^2) \right)$$

$$\ominus iT \bigoplus iT^\dagger = T^\dagger T$$

$$\Sigma(p)p^2 = (m_1 + m_2)^2 \bigotimes \text{Im}(-i\Sigma)$$

$$2\text{Im}(-i\Sigma) = \frac{\sqrt{u_+ u_-}}{8\pi p^2} \theta(-u_+)$$

$$p^2 > (m_1 + m_2)^2$$

$$(2\pi)\theta(p^0 - k^0)\delta((p-k)^2 - m_1^2), (2\pi)\theta(k^0)\delta(k^2 - m_2^2).$$

$$p^2 = (m_1 - m_2)^2 \text{ (i.e. } u_- = 0 \text{)}$$

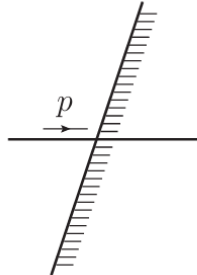
$$\Sigma'(p)p^2 = (m_1 - m_2)^2$$

$$p^2 = (m_1 + m_2)^2$$



$$p^2 < (m_1 - m_2)^2$$

$$2\text{Im}(-i\Sigma') = -\frac{\sqrt{u_+u_-}}{8\pi p^2}\theta(u_-).$$



$$= \begin{cases} \theta(p^0)(2\pi)\delta(p^2 - m^2) & \text{Feynman} \\ 0 & \text{fakeon} \\ \pi[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2) & \text{Feynman-Wheeler} \end{cases}$$

$$(2\pi)\theta(p^0 - k^0)\delta((p - k)^2 - m_1^2), -(2\pi)\theta(-k^0)\delta(k^2 - m_2^2),$$

$$2\text{Im}(-i\Sigma_{\text{FW}}) = \frac{\sqrt{u_+u_-}}{16\pi p^2} [\theta(-u_+) - \theta(u_-)]$$

$$(2\pi)\theta(p^0 - k^0)\delta((p - k)^2 - m_1^2), \text{sgn}(k^0)\pi\delta(k^2 - m_2^2)$$

$$\text{Im}(-i\Sigma_{\text{FW}})p^2 < (m_1 - m_2)^2$$

$$\Sigma_{\text{FW-FW}}(p) = -\text{Im}(-i\Sigma_{\text{FW}}) = -\frac{\sqrt{u_+u_-}}{32\pi p^2} [\theta(-u_+) - \theta(u_-)] \star 2\text{Im}(-i\Sigma_{\text{FW-FW}}),$$

$$|p^0| = \omega_1 \ominus \omega_2.$$

$$p^0 = \omega_1 - \omega_2 \sim -p_s \cos \theta + \frac{1}{2k_s} [m_1^2 - m_2^2 + p_s^2 \sin^2 \theta] + \mathcal{O}\left(\frac{1}{k_s^2}\right).$$

$$\Sigma(p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(p - k)^2 - m_1^2 + i\epsilon} \frac{1}{k^2 - m_2^2 + i\epsilon}$$

$$\Sigma(p) = V(p^2 + i\epsilon, m_1^2, m_2^2)$$

$$V(a, b, c) = \frac{i\Gamma\left(\frac{4-D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dx [-ax(1-x) + bx + c(1-x)]^{(D-4)/2}$$

$$i(4\pi)^2 V(p^2, m_1^2, m_2^2) = -\ln \frac{4\Lambda^2}{m_1 m_2} + \frac{m_1^2 - m_2^2}{p^2} \ln \frac{m_1}{m_2} - \frac{\sqrt{u_+u_-}}{p^2} \ln \frac{m_1^2 + m_2^2 - p^2 + \sqrt{u_+u_-}}{2m_1 m_2}$$



$$i(4\pi)^2 \square V(p^2 + i\epsilon, m_1^2, m_2^2) = -\ln \frac{4\Lambda^2}{m_1 m_2} + \frac{m_1^2 - m_2^2}{p^2} \ln \frac{m_1}{m_2} \\ - \frac{\sqrt{u_+ u_-}}{p^2} \theta(u_-) \left(\ln \frac{\sqrt{u_+} + \sqrt{u_-}}{\sqrt{u_+} - \sqrt{u_-}} \right) + \frac{2\sqrt{-u_+ u_-}}{p^2} \theta(-u_-) \theta(u_+) \arctan \sqrt{\frac{-u_-}{u_+}} \\ + \frac{\sqrt{u_+ u_-}}{p^2} \theta(-u_+) \left(\ln \frac{\sqrt{-u_-}}{\sqrt{-u_+}} \sqrt{-u_+} \sqrt{-u_+} - i\pi \right)$$

$$\Sigma_f(p) \square \int \frac{d^D k}{(2\pi)^D} \frac{1}{(p-k)^2 - m_1^2 + i\epsilon} \frac{1}{k^2 - m_2^2} \Big|_f'$$

$$p^2 = (m_1 + m_2)^2, \text{ while } p^2 = (m_1 - m_2)^2$$

$$p^2 = (m_1 + m_2)^2$$

$$\Sigma_f(p) = \Sigma(p) \text{ for } p^2 < (m_1 + m_2)^2$$

$$p^2 \rightarrow p^2 + i\epsilon$$

$$\Sigma_f(p) = \frac{1}{2} \left(V(p^2 + i\epsilon, m_1^2, m_2^2) + V(p^2 - i\epsilon, m_1^2, m_2^2) \right) \\ = \frac{1}{2} (\Sigma(p) - \Sigma^*(p)) = \Sigma(p) + \frac{\sqrt{u_+ u_-}}{16\pi p^2} \theta(-u_+)$$

$$2\text{Im}(-i\Sigma_f) = 0$$

$$\Sigma_{ff}(p) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{1}{(p-k)^2 - m_1^2} \Big|_f \frac{1}{k^2 - m_2^2} \Big|_f'$$

$$\Sigma_{ff}(p) = \Sigma_f(p) = \Sigma(p) + \frac{\sqrt{u_+ u_-}}{16\pi p^2} \theta(-u_+)$$

$$|p^0| = \omega_1 \bigoplus \omega_2$$

$$\Sigma'(p) = \frac{i}{8\pi\sqrt{p^2}} \ln \frac{\sqrt{p^2 - i\epsilon} + m_1 - m_2}{\sqrt{p^2 + i\epsilon} - m_1 + m_2}$$

$$2\text{Im}(-i\Sigma') = -\frac{\theta(u_-)}{4\sqrt{p^2}}$$

$$U(p^2, m_1^2, m_2^2) = \frac{i}{4\pi\sqrt{u_+ u_-}} \ln \frac{1+z}{1-z}$$

$$2\text{Im}(-i\Sigma') = -\frac{\theta(u_-)}{\sqrt{u_+ u_-}}$$

$$S_{\text{QG}}(g, \Phi) = -\frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[2\Lambda_c + \zeta R + \alpha \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) - \frac{\xi}{6} R^2 \right] + S_m(g, \Phi)$$



$$M_{\text{Pl}} = 1/\sqrt{G} = \sqrt{8\pi\zeta}/\kappa$$

$$m_\phi = \sqrt{\zeta/\xi} \text{ and } m_\chi = \sqrt{\zeta/\alpha}$$

$$\alpha_\chi = m_\chi^2/M_{\text{Pl}}^2 \text{ and } \alpha_\phi = m_\phi^2/M_{\text{Pl}}^2$$

$$\alpha_\chi \ln \frac{E^2}{\mu^2}, \alpha_\phi \ln \frac{E^2}{\mu^2}$$

$$\alpha_\chi \sim \alpha_\phi \sim 10^{-14} \text{ and } \mu \sim M_{\text{Pl}}$$

$$E \sim 10^{-10^{-13}} M_{\text{Pl}}$$

$$\mathcal{O}(x, y) \propto \text{Tr}[(y \cdot \phi(x))^2]$$

$$\langle \mathcal{O}(x_1, y_1) \mathcal{O}(x_2, y_2) \rangle = (y_{12}^2/x_{12}^2)^2$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

$$\sigma = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} = \alpha\bar{\alpha}, \tau = \frac{y_{23}^2 y_{14}^2}{y_{13}^2 y_{24}^2} = (1-\alpha)(1-\bar{\alpha})$$

$$\frac{x_{13}^4 x_{24}^4}{y_{13}^4 y_{24}^4} \langle \mathcal{O}(x_1, y_1) \cdots \mathcal{O}(x_4, y_4) \rangle = 1 + \frac{\sigma^2}{u^2} + \frac{\tau^2}{v^2} + \frac{1}{c} \left(\frac{\sigma}{u} + \frac{\tau}{v} + \frac{\sigma\tau}{uv} \right) + \frac{1}{c} (z-\alpha)(z-\bar{\alpha})(\bar{z}-\alpha)(\bar{z}-\bar{\alpha}) \mathcal{H}(z, \bar{z})$$

$$c = \frac{N_c^2 - 1}{4} \Delta \text{SU}(N_{\mathcal{P}_{u,v}^{\mathcal{N}=4}(\Delta, J)})$$

$$\mathcal{H}(u, v) = \mathcal{H}(v, u) = u^{-4} \mathcal{H}\left(\frac{1}{u}, \frac{v}{u}\right)$$

$$\mathcal{H}(u, v) = \mathcal{H}^{\text{sugra}}(u, v) + \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 \mathcal{P}_{u, v}^{\mathcal{N}=4}(\Delta, J)$$

$$\mathcal{H}^{\text{sugra}}(u, v) = -\bar{D}_{2,4,2,2} = \partial_u \partial_v (1 + u\partial_u + v\partial_v) F_1(u, v)$$

$$F_1(u, v) \equiv \frac{2\text{Li}_2(z) - 2\text{Li}_2(\bar{z}) + \log(z\bar{z})(\log(1-z) - \log(1-\bar{z}))}{z - \bar{z}}$$

$$\mathcal{H}(u, v) = \iint \frac{d s dt}{(4\pi i)^2} u^{\frac{s}{2}-4} v^{\frac{t}{2}-4} \Gamma\left(4 - \frac{s}{2}\right)^2 \Gamma\left(4 - \frac{t}{2}\right)^2 \Gamma\left(4 - \frac{u}{2}\right)^2 \hat{\mathcal{H}}(s, t)$$

$$s \oplus t \oplus u = 4\Delta_\phi^{\text{eff}}$$



$$\lim_{g \rightarrow 0} \widehat{\mathcal{H}} = \frac{-2g^2}{\left(\frac{s}{2} - 3\right)^2 \left(\frac{t}{2} - 3\right)^2 \left(\frac{u}{2} - 3\right)^2} + O(g^4)$$

$$\lim_{g \rightarrow \infty} \widehat{\mathcal{H}} = \frac{1}{\left(\frac{s}{2} - 3\right) \left(\frac{t}{2} - 3\right) \left(\frac{u}{2} - 3\right)} \equiv \widehat{\mathcal{H}}^{\text{sugra}}$$

$$\widehat{\mathcal{H}}(s, t) = \widehat{\mathcal{H}}^{\text{sugra}}(s, t) + \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 \mathcal{P}_{\Delta, J}^{\mathcal{N}=4}(s, t)$$

$$\widehat{\mathcal{P}}_{\Delta, J}^{\mathcal{N}=4}(s, t) = \sum_{n=0}^{\infty} \mathcal{Q}_{\Delta+4, J}^n (16 - s - t) \left[\frac{1}{s - (\Delta - J + 2n + 4)} + \frac{1}{t - (\Delta - J + 2n + 4)} \right]$$

$$\mathcal{Q}_{\Delta, J}^n(u) = K_{\Delta, J}^n \times \left(\mathcal{Q}_{\Delta, J}^n(u) \equiv \sum_{k, q=0}^J (-n)_q [Q_{\Delta, J}]_{q, k} \left(\frac{8-u}{2}\right)_k \right)$$

$$K_{\Delta, J}^n = \frac{2\Gamma(\Delta + J)\Gamma(\Delta + J - 1)}{n! \Gamma(\Delta - 1 + n) \Gamma\left(\frac{8 - \Delta + J}{2} - n\right)^2 \Gamma\left(\frac{\Delta + J}{2}\right)^4}$$

$$u : [Q_{\Delta, J}]_{q, k} = (\Delta_{\phi}^{\text{eff}})^J \cdot 1/\Gamma\left(\frac{8 - \Delta + J}{2} - n\right)^2$$

$$0 = \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 X_{u, v}[\Delta, J] \text{ with } X_{u, v} \equiv \mathcal{P}_{u, v}^{\mathcal{N}=4} - u^{-4} \mathcal{P}_{1/u, v/u}^{\mathcal{N}=4}, \text{ for } (u, v)$$

$$0 = \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 \widehat{X}_{s, t}[\Delta, J] \text{ with } \widehat{X}_{s, t} \equiv \widehat{\mathcal{P}}_{s, t}^{\mathcal{N}=4} - \widehat{\mathcal{P}}_{16-s-t, t}^{\mathcal{N}=4}$$

$$0 = \widehat{B}_t^{\text{protected}} + \sum_{\Delta, J} \lambda_{\Delta, J}^2 \widehat{B}_t[\Delta, J]$$

$$\widehat{B}_t^{\text{protected}} = \frac{2}{\left(\frac{t}{2} - 3\right) \left(\frac{t}{2} - 2\right)}$$

$$\widehat{B}_t[\Delta, J] = \sum_{n=0}^{\infty} \frac{2(\Delta - J + 2n) + 2 - t}{t - 6} \mathcal{Q}_{\Delta+4, J}^n (10 - t)$$

$$0 = B_v^{\text{protected}} + \sum_{\Delta, J} \lambda_{\Delta, J}^2 B_v[\Delta, J] \quad (v > 0)$$

$$B_v^{\text{protected}} = \frac{v^2 - 1 - 2v \log v}{v(1 - v)^3}$$

$$0 = W^{\text{protected}} + \sum_{\Delta, J} \lambda_{\Delta, J}^2 W[\Delta, J]$$

$$\Phi_{\ell, \ell+2}^{\text{protected}} = 0, \Psi_{\ell}^{\text{protected}} = -\frac{2\Gamma(\ell + 3)^2}{\Gamma(2\ell + 5)}.$$



$$(\mathcal{H}(u, v)^{\text{here}} = u^{-2}f(u, v)^{\text{there}})$$

$$I_2(g) = -\frac{1}{2\pi^2} \int d^4x \mathcal{H}(u, v) \Big|_{u=x^2, v=(e-x)^2}$$

$$I_4(g) = -\frac{16}{2\pi^2} \int d^4x (1+u+v) F_1(u, v) \mathcal{H}(u, v) \Big|_{u=x^2, v=(e-x)^2}$$

$$I_2(g) = -\frac{1}{2} \iint \frac{d s dt}{(4\pi i)^2} \hat{\mathcal{H}}(s, t) Y(s, t)$$

$$I_4(g) = -48 \iint \frac{d s dt}{(4\pi i)^2} \hat{\mathcal{H}}(s, t) Y(s, t) \left[\frac{2(u-5)}{(s-6)(t-6)} + \frac{t-s}{u-6} \left(H_{\frac{s}{2}-3} + H_{3-\frac{s}{2}} \right) \right]$$

$$Y(s, t) = \prod_{x=s, t, u} \Gamma\left(\frac{x}{2} - 2\right) \Gamma\left(4 - \frac{x}{2}\right)$$

$$H_a = \Gamma'(a+1)/\Gamma(a+1) - \Gamma'(1)$$

$$I_2(g) = \int_0^\infty \frac{t d t e^{-t}}{(1-e^{-t})^2} (J_1(2gt)^2 - J_2(2gt)^2)$$

$$I_4(g) = 48\zeta_3 - \frac{8}{g^2} \int_0^\infty \frac{t d t e^{-t}}{(1-e^{-t})^2} J_1(2gt)^2$$

$$-\frac{192}{g} \int_0^\infty \frac{t d t e^{-t} J_1(2gt)}{(1-e^{-t})^2} \int_0^\infty \frac{t' d t' e^{-t'} J_1(2gt')}{(1-e^{-t'})^2} \left(\frac{t J_0(2gt) J_1(2gt') - (t \leftrightarrow t')}{t'^2 - t^2} \right)$$

$$g^2 = \frac{\lambda}{16\pi^2}$$

$$0 = I_p^{\text{protected}}(g) + \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 I_p[\Delta, J] \quad (p = 2, 4)$$

$$I_2^{\text{protected}}(g) \equiv \frac{1}{4} - I_2(g), \quad I_4^{\text{protected}}(g) \equiv 24(2\zeta_3 - 1) - I_4(g)$$

$$I_2^{\text{protected}}(g) \rightarrow \begin{cases} \frac{1}{4} - 6g^2\zeta_3 + O(g^4), & g \rightarrow 0 \\ \frac{3\zeta_3}{(4\pi g)^3} - \frac{45\zeta_5}{4(4\pi g)^5} + O(g^{-7}), & g \rightarrow \infty \end{cases}$$

$$I_4^{\text{protected}}(g) \rightarrow \begin{cases} 24(2\zeta_3 - 1) - 960g^2\zeta_5 + O(g^4), & g \rightarrow 0 \\ \frac{384\zeta_3}{(4\pi g)^3} - \frac{1152\zeta_5}{(4\pi g)^5} + O(g^{-7}), & g \rightarrow \infty \end{cases}$$

$$\sum_{\Delta} \frac{\lambda_{\Delta, J}^2}{\lambda_{\Delta, J}^2, \text{free}} \simeq \int \frac{d\Delta}{2}$$

$$\sum_{\Delta} \frac{\lambda_{\Delta, J}^2}{\lambda_{\Delta, J}^2, \text{free}} e^{-i\pi(\Delta - J - 2\Delta_\phi)}(\dots) \simeq \int \frac{d\Delta}{2} (1 + i a_J^{\text{flat}}(s))_{s=\Delta^2/R_{\text{AdS}}^2}(\dots)$$



$$\sum_{\Delta} \frac{\lambda_{\Delta,J}^2}{\lambda_{\Delta,J}^{2,\text{free}}} 2\sin^2\left(\frac{\pi(\Delta-J-2\Delta\phi)}{2}\right) (\dots) \simeq \int \frac{d\Delta}{2} \text{Im} a_J^{\text{flat}}(s)_{s=\Delta^2/R_{\text{AdS}}^2} (\dots)$$

$$a_J^{\text{flat}}(s) \approx \frac{C_{m,J}^2}{m^2 - s - i0} \Leftrightarrow 2\sin^2\left(\frac{\pi(\Delta-J-2\Delta\phi)}{2}\right) \frac{\lambda_{\Delta,J}^2}{\lambda_{\Delta,J}^{2,\text{free}}} = \frac{\pi R_{\text{AdS}}}{4m} C_{m,J}^2$$

$$\lambda_{\Delta,J}^{2,\text{free}} = 2(\Delta+2)(J+1) \frac{\Gamma\left(\frac{\Delta-J}{2}+1\right)^2 \Gamma\left(\frac{\Delta+J}{2}+2\right)^2}{\Gamma(\Delta-J+1)\Gamma(\Delta+J+3)}$$

$$\mathcal{M}(s,t) = \frac{8\pi G_5}{stu} \frac{\Gamma\left(1-\frac{\alpha's}{4}\right)\Gamma\left(1-\frac{\alpha't}{4}\right)\Gamma\left(1-\frac{\alpha'u}{4}\right)}{\Gamma\left(1+\frac{\alpha's}{4}\right)\Gamma\left(1+\frac{\alpha't}{4}\right)\Gamma\left(1+\frac{\alpha'u}{4}\right)}$$

$$C_{2/\sqrt{\alpha'},0}^2 = \frac{2G_5}{(\alpha')^{\frac{5}{2}}}$$

$$8\pi G_5 = \frac{\pi^2 R_{\text{AdS}}^3}{c}, \alpha' = R_{\text{AdS}}^2 \lambda^{-\frac{1}{2}}$$

$$\frac{2\sin^2\left(\frac{\pi\Delta_K}{2}\right) \lambda_K^2}{\lambda_{\Delta_K,0}^{2,\text{free}}} = \frac{\pi^2 \lambda}{32c} \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right)$$

$$(\Delta_K+2)^2 = 4\lambda^{1/2} + 8 + \frac{6-12\zeta_3}{\lambda^{1/2}} + \frac{4+12\zeta_3+30\zeta_5}{\lambda} + O(\lambda^{-3/2}),$$

$$\lambda_K^2 = \frac{\pi^3 \Delta_K^6 4^{-\Delta_K-6}}{\sin^2(\pi\Delta_K/2)} \left(1 + f_1\lambda^{-1/4} + f_2\lambda^{-1/2} + f_3\lambda^{-3/4} + f_4\lambda^{-1} + O(\lambda^{-5/4})\right)$$

$$\tilde{\lambda}_K^2 \equiv \frac{2^8 \sin^2\left(\frac{\pi}{2}\Delta_K\right) \lambda_K^2}{\left[\frac{\pi}{2}(\Delta_K-2)(\Delta_K+6)\right]^2 \lambda_{\Delta_K,J=0}^{2,\text{free}}}$$

$$\tilde{\lambda}_K^2 = 2 - 10g^2 + \frac{288\zeta_3 - 97}{2} g^4 + O(g^6)$$

$$\tilde{\lambda}_K^2 = 1 + \frac{5+8\zeta_3}{4} \lambda^{-1/2} + \frac{13-136\zeta_3+32\zeta_3^2-48\zeta_5}{16} \lambda^{-1} + O(\lambda^{-3/2})$$

$$= \exp\left(\frac{5+8\zeta_3}{(\Delta_K+2)^2} + \frac{\frac{81}{2}-112\zeta_3-48\zeta_5}{(\Delta_K+2)^4} + O(\Delta_K^{-6})\right)$$

$$0 = W_k^{\text{protected}} + \sum_{(\Delta,J)_{\text{long}}} \lambda_{\Delta,J}^2 W_k[\Delta,J].$$

$$0 = \sum_{(\Delta,J)_{\text{long}}} \lambda_{\Delta,J}^2 O[\Delta,J]$$



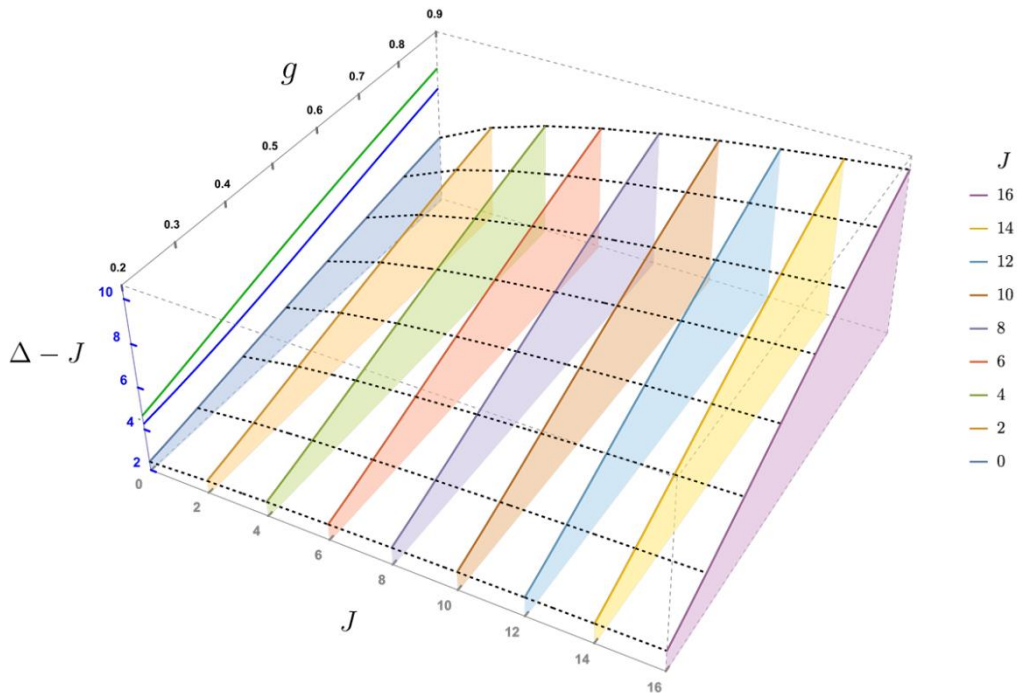
$$-W_k^{\text{protected}} O[K] - OW_k[K] = \sum_{(\Delta', J') \text{ long}} \lambda_{\Delta', J'}^2 (W_k[\Delta', J'] O[K] - W_k[K] O[\Delta', J'])$$

$$\sum_k \alpha_k (W_k[\Delta', J'] O[K] - W_k[K] O[\Delta', J']) \geq 0 \forall (\Delta', J')$$

$$\sum_k \alpha_k W_k^{\text{protected}} \times O[K]$$

$$\sum_k \alpha_k W_k[K] = \pm 1$$

$$O[\Delta, J] = \mathcal{P}_{u,v}^{\mathcal{N}=4}[\Delta, J]$$



$$\tau(J)_{\text{twist-}2, J \rightarrow \infty} = 2 + 2\Gamma_{\text{cusp}}(g) \log(Je^{YE}) + 2\Gamma_{\text{virtual}}(g) + O(\log^{\#}(J)/J)$$

$$\tau(J)_{\text{twist-}4, J \rightarrow \infty} = \tau(J)_{\text{twist-}2, J \rightarrow \infty} + \Delta\tau_{J \rightarrow \infty}$$

$$\Delta \sim \sqrt{\lambda}, J \sim \sqrt{\lambda} \text{ with } \lambda \equiv 16\pi^2 g^2$$

$$\Delta = 8g \int_0^{\rho_0} d\rho \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}$$

$$J = 8g \int_0^{\rho_0} d\rho \frac{\omega \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}$$

$$\Delta_{\text{GKP-like}}(J) = \Delta_{\text{GKP}}^{g-\frac{1}{2}\Gamma_{\text{cusp}}(g)}(J) + C(g)$$



$$\eta_{\text{AdS}} \equiv 1 + \frac{2(J+1)^2}{(\Delta-J+1)(\Delta+J+3)}$$

$$\mathcal{H}(u, v) = \mathcal{H}^{\text{sugra}}(u, v) + \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 \mathcal{P}_{u, v}^{\mathcal{N}=4}[\Delta, J]$$

$$O_{u_0, v_0} \equiv \mathcal{H}_{u_0, v_0}^{\text{long}} = \sum_{(\Delta, J) \text{ long}} \lambda_{\Delta, J}^2 \mathcal{P}_{u_0, v_0}^{\mathcal{N}=4}[\Delta, J]$$

$$\begin{aligned} \mathcal{H}(z, \bar{z})^{2\text{-loop}} &= -2g^2 \frac{F_1(z)}{uv} + 4g^4 \frac{\frac{1+u+v}{4} F_1(z)^2 + F_2(z) + F_2(1-z) + \frac{F_2\left(\frac{z}{z-1}\right)}{v}}{uv} + O(g)^6 \\ &\stackrel{z, \bar{z} \rightarrow 0}{=} \frac{-\frac{1}{3} + \lambda_K^2 \times (z\bar{z})^{\frac{\Delta_K-2}{2}} + \dots}{z\bar{z}} \end{aligned}$$

$$F_L(z) \equiv \sum_{l=0}^L \frac{(-1)^l (2L-l)! [\log(z\bar{z})]^l \text{Li}_{2L-l}(z) - \text{Li}_{2L-l}(\bar{z})}{L! (L-l)! l!} \frac{1}{z-\bar{z}}$$

$$u \equiv z\bar{z}, v \equiv (1-z)(1-\bar{z})$$

$$\Delta_K = 2 + 12g^2 - 48g^4 + O(g)^6 \text{ and } \lambda_K^2 = \frac{1}{3} - 4g^2 + g^4(56 + 24\zeta_3) + O(g)^6$$

$$\frac{\mathcal{H}_{z, \bar{z}} - \mathcal{H}_{z, \bar{z}}^{\text{sugra}}}{\mathcal{P}_{z, \bar{z}}[K]} \quad \frac{\mathcal{H}_{z, \bar{z}} - \mathcal{H}_{z, \bar{z}}^{\text{sugra}}}{\lambda_K^2 \mathcal{P}_{z, \bar{z}}^{\mathcal{N}=4}[\Delta_K, 0]}$$

$$\mathcal{M}(s, t) = \frac{1}{s^{4+\frac{D-4}{2}}} \sum_{J=0,2,4,\dots} n_J a_J(s) \mathcal{P}_J\left(1 + \frac{2t}{s}\right)$$

$$n_J = \frac{(4\pi)^{D/2} (D+2J-3) \Gamma(D+J-3)}{\pi \Gamma\left(\frac{D-2}{2}\right) \Gamma(J+1)}, \mathcal{P}_J(x) = {}_2F_1\left(-J, J+D-3, \frac{D-2}{2}, \frac{1-x}{2}\right)$$

$$\mathcal{M}(s, t) = \frac{8\pi G_D}{stu} + \sum_{J, m} C_{m, J}^2 \mathcal{P}_{s, t}^{\text{curve}}[m, J]$$

$$\mathcal{P}_{s, t}^{\text{curve}}[m, J] \equiv \frac{n_J}{m^{8+D-4}} \mathcal{P}_J\left(1 - 2\frac{s+t}{m^2}\right) \left(\frac{1}{m^2-s} + \frac{1}{m^2-t}\right)$$

$$\text{Im} a_J(s) = \sum_m C_{m, J}^2 \pi \delta(s - m^2).$$

$$[Q_{\Delta, J}]_{q, k} \sim \Delta^{J-q-k}$$

$$n^q [Q_{\Delta, J}]_{q, k} t^k \sim \Delta^{J+q+k}$$



$$\lim_{\Delta^2 \sim n \sim t \rightarrow \infty} Q_{\Delta, J}^{n, a, b}(t) = - \left(\frac{n}{4}\right)^J \frac{(d-2)_J}{\left(\frac{d-2}{2}\right)_J} \mathcal{P}_J \left(1 + \frac{t}{n}\right)$$

$$\hat{\mathcal{P}}_{s, t}^{\mathcal{N}=4}[\Delta, J] \Big|_{s^{a_t b}} \approx \mathcal{P}_{s, t}^{\text{curve}}[\Delta, J] \Big|_{s^{a_t b}} \times \frac{\Delta^9}{128\pi(J+1)} \sum_n K_{\Delta+4, J}^n \left(\frac{\Delta^2}{2n}\right)^{a+b+1} \left(\frac{n}{4}\right)^J$$

$$\begin{aligned} \sum_{n=0}^{\infty} K_{\Delta+4, J}^n \left(\frac{\Delta^2}{2n}\right)^{a+b+1} \left(\frac{n}{4}\right)^J &= K_{\Delta+4, J}^0 \sum_n \frac{\left(\frac{\Delta-J-2}{2}\right)_n^2}{n! (\Delta+3)_n} \left(\frac{\Delta^2}{2n}\right)^{a+b+1} \left(\frac{n}{4}\right)^J \\ &\approx \frac{2 \sin\left(\frac{\pi(\Delta-J)}{2}\right)^2}{\lambda_{\Delta, J}^{2, \text{free}}} \frac{J+1}{\pi^2 \Delta^8} 2^{a+b+14} \Gamma(a+b+6) \iiint \frac{\Gamma(a+b+6) \Gamma(\Delta+a+b+4)}{\Gamma\left(\frac{\Delta}{2}+a+b+5\right)^2} \end{aligned}$$

$$\frac{\lambda_{\Delta, J}^{2, \text{free}}}{2 \sin\left(\frac{\pi(\Delta-J)}{2}\right)^2} \hat{\mathcal{P}}_{s, t}^{\mathcal{N}=4}[\Delta, J] \Big|_{s^{a_t b}} = \mathcal{P}_{s, t}^{\text{curve}}[m, J] \Big|_{s^{a_t b}} \times c_{a+b} \frac{4m}{\pi R_{\text{AdS}}} (1 + O(\Delta^{-1}))$$

$$m = \Delta/R_{\text{AdS}} \text{ and } c_n \equiv \frac{2^{n+5} \Gamma(n+6)}{\pi^2 R_{\text{AdS}}^{2n+9}}$$

$$\hat{\mathcal{H}}_{s, t} \approx \frac{32R_{\text{AdS}}^3 c}{\pi^2} \int_{\frac{4m}{\pi R_{\text{AdS}}}}^{\infty} d\beta \beta^5 e^{-\beta R_{\text{AdS}}^2} \mathcal{M}(2\beta s, 2\beta t)$$

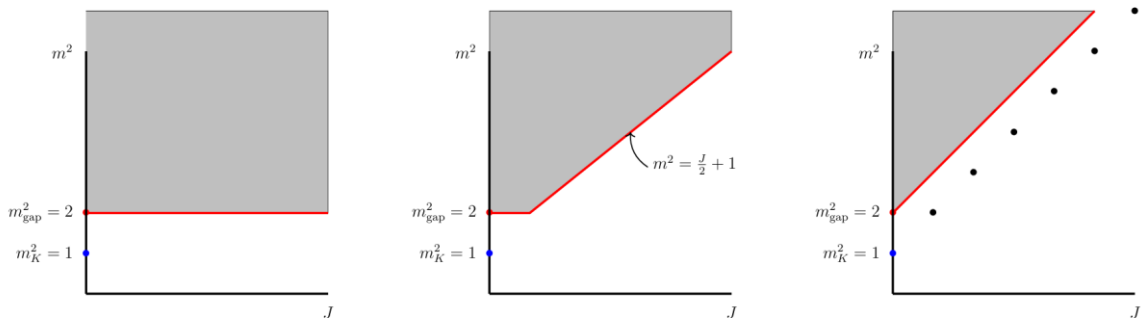
$$\mathcal{M}_{s, t}^{\text{curve}} \supset s^{a_t b} \Leftrightarrow \hat{\mathcal{H}}_{s, t} \supset c_{a+b} s^{a_t b}, c_n \equiv \frac{2^{n+5} \Gamma(n+6)}{\pi^2 R_{\text{AdS}}^{2n+9}}.$$

$$\hat{\mathcal{X}}_{s, t} \equiv \hat{\mathcal{P}}_{s, t}^{\mathcal{N}=4} - \hat{\mathcal{P}}_{16-s-t, t}^{\mathcal{N}=4}$$

$$\tilde{\lambda}_K^{2, \text{curve}} = \frac{m_K^4}{8\pi G_D} \times \text{Res}_{s=m_K^2}[-\mathcal{M}(s, t)] \equiv \frac{m_K^4}{8\pi G_D} \times \frac{n_{J=0}}{m_K^{4+D}} \text{Res}_{s=m_K^2}[-a_{J=0}(s)].$$

$$[m_K^2]^{VS} = \frac{4}{\alpha'} \text{ and } [\tilde{\lambda}_K^{2, \text{flat}}]^{VS} = 1$$

$$m_{n, J}^2 = m_K^2 \left(\frac{J}{2} + 1 + n\right) \quad (n = 0, 1, 2 \dots)$$



$$\text{I: } m_j^2 \geq 2m_K^2.$$

$$\text{II: } m_j^2 \geq m_K^2 \times \max\left(\frac{J}{2} + 1, 2\right),$$

$$\text{III: } m_j^2 = m_K^2 \frac{(J+2)}{2} \text{ or } m_j^2 \geq m_K^2 \frac{(J+4)}{2}.$$

$$\mathcal{M}_{EFT} \stackrel{s,t \rightarrow 0}{=} \frac{8\pi G_D}{stu} + g_0 + g_2(s^2 + t^2 + u^2) + g_3stu + g_4(s^2 + t^2 + u^2)^2 + \dots$$

$$g_0 \frac{m_K^6}{8\pi G_D} \rightarrow \left[g_0 \frac{m_K^6}{8\pi G_D} \right]^{VS} = 2\zeta_3 \text{ and } g_2 \frac{m_K^{10}}{8\pi G_D} \rightarrow \left[g_2 \frac{m_K^{10}}{8\pi G_D} \right]^{VS} = \zeta_5.$$

$$B_k: \text{Res}_{s=0} \left[\frac{2s+t}{s(s+t)} \frac{\mathcal{M}_{EFT}(s,t)}{[s(s+t)]^{k/2}} \right] = \left\langle \frac{2m^2+t}{m^2+t} \frac{\mathcal{P}_J\left(1 + \frac{2t}{m^2}\right)}{[m^2(m^2+t)]^{k/2}} \right\rangle \quad (t < 0, k = -2, 0, 2, \dots)$$

$$\langle f(m^2, J) \rangle = \sum_{J=0,2,\dots} n_J \int_{m_J^2}^{\infty} \frac{dm^2}{\pi} \frac{m^{(4-D)-8}}{m^2} \rho_J(m^2) f(m^2, J)$$

$$\rho_J(m^2) \equiv \text{Im} a_J(m^2)$$

$$\langle f(m^2, J) \rangle = \tilde{\lambda}_K^{2, \text{curve}} \times \frac{8\pi G_D}{m_K^6} f(m_K^2, 0) + \langle f(m^2, J) \rangle'$$

$$\langle f(m^2, J) \rangle' \equiv \frac{n_0}{\pi} \int_{m_{\text{gap}}^2}^{\infty} \frac{dm^2}{m^{D+6}} \rho_0(m^2) f(m^2, 0) + \sum_{J=2,4,\dots} \frac{n_J}{\pi} \int_{m_J^2}^{\infty} \frac{dm^2}{m^{D+6}} \rho_J(m^2) f(m^2, J)$$

$$g_0 = 2\tilde{\lambda}_K^{\text{flat}} \frac{8\pi G_D}{m_K^6} + \langle 2 \rangle' \text{ and } g_2 = \frac{\tilde{\lambda}_K^{2, \text{curve}}}{m_K^4} \frac{8\pi G_D}{m_K^6} + \left\langle \frac{1}{m^4} \right\rangle'$$

$$0 = \left\langle \frac{4J^2}{D-2} - 1 \right\rangle = \partial_t B_0|_{t=0}$$

$$X_{s,t}^{\text{curve}} = \mathcal{P}_J \left(1 + \frac{2u}{m^2} \right) \left(\frac{m^2}{m^2-s} + \frac{m^2}{m^2-t} \right) - (s \leftrightarrow u) \text{ with } u = -s - t$$

$$X_{a,b}(m^2, J) = m_K^{2(a+b)} X_{s,t}^{\text{curve}} \Big|_{s^a u^b}$$

$$\vec{X}^{\text{cross}} = \left(X_{0,1} X_{0,2} X_{0,3} X_{0,4} X_{1,3} X_{0,5} X_{1,4} X_{0,6} X_{1,5} \right) \\ = \left(\frac{(2-D) + 4J^2}{(D-2)(m/m_K)^2} \frac{D(2-D) + 2(4-3D)J^2 + 4J^4}{D(D-2)(m/m_K)^4} \dots \right)$$

$$\vec{W}^{\text{curve}} = \left\{ g_0(m^2, J) \equiv 2, g_2(m^2, J) \equiv \frac{m_K^4}{m^4}, \vec{X}^{\text{cross}} \right\}$$



$$\vec{\alpha} \cdot \vec{W}^{\text{curve}}(m^2, J) = \alpha_1 g_0(m^2, J) + \alpha_2 g_2(m^2, J) + \sum_{i=1}^{N_{\text{cross}}} \alpha_{i+2} \mathcal{X}_i^{\text{cross}}(m^2, J) \geq 0$$

$$\vec{\alpha} \cdot \vec{W}^{\text{curve}}(m_K^2, 0) = \pm 1.$$

$$\alpha_1^{(-)} g_0 \frac{m_K^6}{8\pi G_D} + \alpha_2^{(-)} g_2 \frac{m_K^{10}}{8\pi G_D} \leq \tilde{\lambda}_K^{2, \text{flat}} \leq -\alpha_1^{(+)} g_0 \frac{m_K^6}{8\pi G_D} - \alpha_2^{(+)} g_2 \frac{m_K^{10}}{8\pi G_D}$$

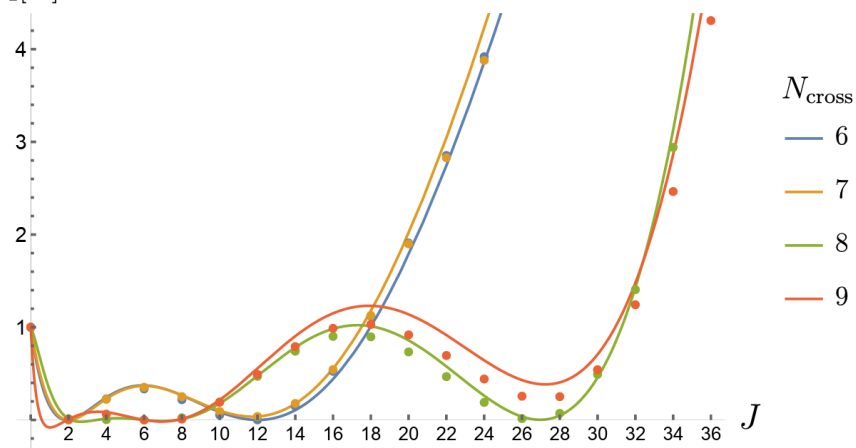
$$\vec{W}[\Delta, J] = (I_2[\Delta, J] \ I_4[\Delta, J] \ \vec{X}^{\text{cross}}[\Delta, J])$$

$$\vec{X}^{\text{cross}} \equiv \left(\hat{X}_{\frac{24}{5}, \frac{28}{5}} \hat{X}_{\frac{24}{5}, \frac{29}{5}} \hat{X}_{\frac{24}{5}, \frac{30}{5}} \hat{X}_{\frac{25}{5}, \frac{28}{5}} \hat{X}_{\frac{26}{5}, \frac{24}{5}} \hat{X}_{\frac{26}{5}, \frac{27}{5}} \hat{X}_{\frac{27}{5}, \frac{25}{5}} \hat{X}_{\frac{22}{5}, \frac{29}{5}} \hat{X}_{\frac{28}{5}, \frac{22}{5}} \right)$$

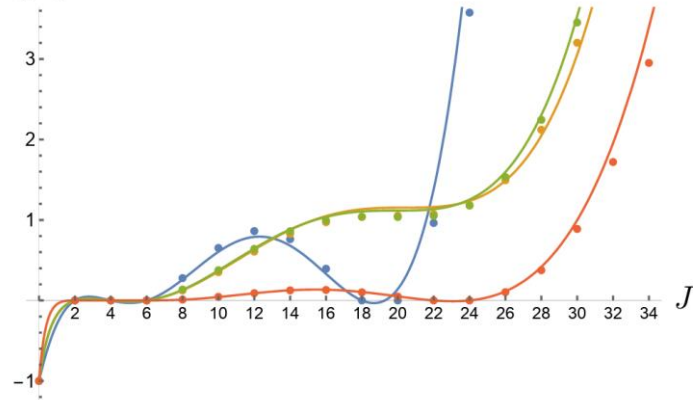
$$\Delta_{\text{gap}} \approx 4\sqrt{2\pi g} - 2 + \frac{1}{\sqrt{2\pi g}} + \mathcal{O}(1/g)$$

$$\text{High - energy states : } \begin{cases} J = 0 & : 2 \lesssim \frac{(\Delta + 2)^2}{(\Delta_K + 2)^2} \lesssim 5, \\ J = 2, 4, \dots, J_{\text{max}} & : 1 \lesssim \frac{(\Delta + 2)^2 / (J + 2)}{(\Delta_K + 2)^2 / 2} \lesssim 5. \end{cases}$$

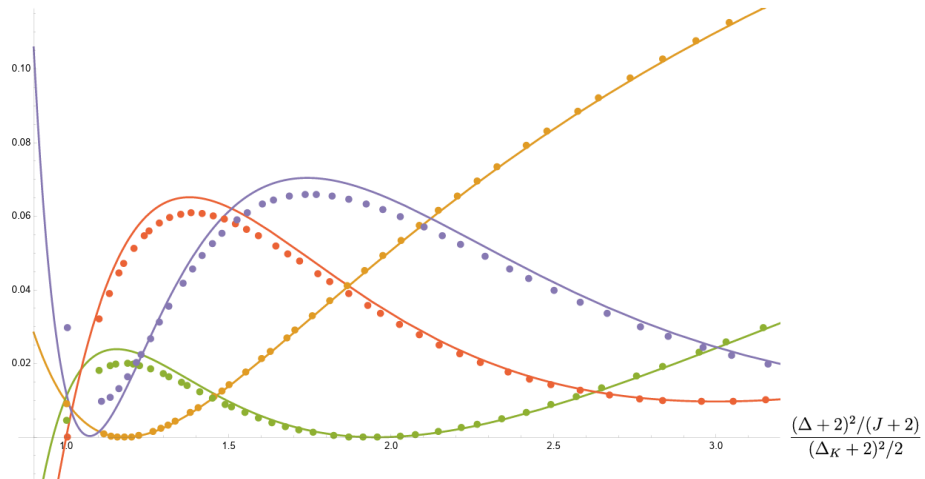
$$\frac{(1 + J/2)^2 I_2[\Delta, J]}{(1 + J/10)^6 I_2[K]} \times [\vec{\alpha}^{(+)} \cdot \vec{W}]$$



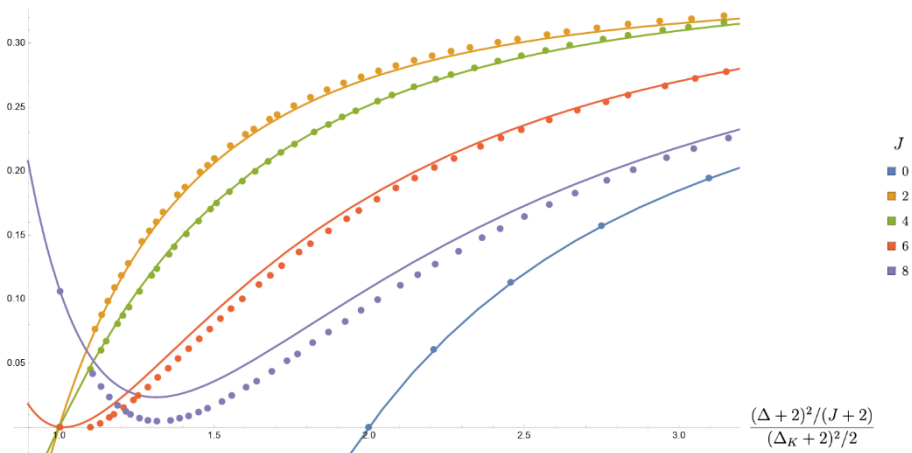
$$\frac{(1 + J/2)^2 I_2[\Delta, J]}{(1 + J/10)^4 I_2[K]} \times [\bar{\alpha}^{(-)}, \bar{W}]$$



$$\frac{\mathcal{I}_2[\Delta_K]}{\mathcal{I}_2[\Delta, J]} \times (\bar{\alpha}^{(+)}, \bar{W}[\Delta, J])$$



$$\frac{\mathcal{I}_2[\Delta_K]}{\mathcal{I}_2[\Delta, J]} \times (\bar{\alpha}^{(-)}, \bar{W}[\Delta, J])$$



$$\frac{[W_2^{\text{gravity}}]^{\text{protected}}}{[W_1]^{\text{protected}}} = \frac{\zeta_5}{2\zeta_3} - \frac{\zeta_3^2 - 2\zeta_5}{\zeta_3(4\pi g)} + O(g^{-2})$$



$$B_v[\Delta, J] = \int_v^\infty dv' \int_0^{(\sqrt{v'} - \sqrt{v})^2} du' \frac{v' - u'}{\pi^2 v \sqrt{v^2 - 2(u' + v')v + (u' - v')^2}} d\text{Disc}_s[G_{\Delta, J}^{\mathcal{N}=4}(u', v')] \\ G_{\Delta, J}^{\mathcal{N}=4}(u', v') = u'^{-4} G_{\Delta+4, J}(u', v')$$

$$r = \sqrt{\rho \bar{\rho}}, \eta = \frac{\rho + \bar{\rho}}{2\sqrt{r\rho}}, \rho = \frac{1}{(\sqrt{1-w} + \sqrt{-w})^2}, \bar{\rho} = \frac{1}{(\sqrt{1-\bar{w}} + \sqrt{-\bar{w}})^2},$$

$$B_v[\Delta, J] = \frac{1}{16\pi^2 v} \int_{\sqrt{v}}^\infty \frac{\eta d\eta}{\sqrt{\eta^2 - v}} \int_0^{r_{\max}(\eta)} \frac{dr}{r^4} (1 - r^4) \frac{(1 + r^2)^2 - 4\eta^2 r^2}{\sqrt{(1 + r^2)^2 - 4vr^2}} d\text{Disc}_s[G_{\Delta, J}^{\mathcal{N}=4}(u', v')]$$

$$r_{\max}(\eta) = \eta - \sqrt{\eta^2 - 1}$$

$$\frac{u'v'}{v} = \frac{((1 + r^2)^2 - 4\eta^2 r^2)^2}{(4r)^4 v}$$

$$\Pi_{k, \eta}^{\Delta\phi}[\Delta, J] = \int_0^{r_{\max}(\eta)} dr r^{k-2} d\text{Disc}_s[u'^{-\Delta\phi} G_{\Delta, J}(u', v')]$$

$$B_v[\Delta, J] = \frac{1}{16\pi^2} \int_{\sqrt{v}}^\infty \frac{\eta d\eta}{v \sqrt{\eta^2 - v}} \left(\Pi_{-2, \eta}^{\Delta\phi=4}[\Delta + 4, J] + (2v + 1 - 4\eta^2) \Pi_{0, \eta}^{\Delta\phi=4}[\Delta + 4, J] + O(\Pi_{2, \eta}) \right)$$

$$m^2 \equiv (\Delta - J + 1)(\Delta + J + 3), \eta_{\text{AdS}} \equiv 1 + \frac{2(J + 1)^2}{m^2}$$

$$\frac{\Pi_{k, \eta}^{\Delta\phi}[\Delta + 4, J]}{2\sin^2\left(\frac{\Delta - J - 2\Delta\phi}{2}\right)} = \frac{4^{2\Delta\phi-1} \pi^{d-2}}{(m^2)^{2\Delta\phi+k-1} b_{\Delta+4, J}} \int_0^\infty \frac{dv}{2\pi} \gamma_{2\Delta\phi+k-1}(v)^2 \rho(v) \mathcal{P}_{\frac{2-d}{2}+iv}(\eta) \mathcal{P}_{\frac{2-d}{2}+iv}(\eta_{\text{AdS}}) \times (1 + O(m^{-2}))$$

$$\gamma_a(v) = \Gamma\left(\frac{1 + a - d/2 - iv}{2}\right) \Gamma\left(\frac{1 + a - d/2 + iv}{2}\right)$$

$$\mathcal{P}_J(\eta) = {}_2F_1\left(-J, J + d - 2, \frac{d-1}{2}, \frac{1-\eta}{2}\right)$$

$$\int_{\sqrt{v}}^\infty \frac{\eta d\eta}{v \sqrt{\eta^2 - v}} \mathcal{P}_{-1+iv}(\eta) \Big|_{d=4} = \frac{\pi}{2v} \coth\left(\frac{\pi v}{2}\right) B_v(v), B_v(v) \equiv \frac{{}_2F_1\left(\frac{-iv}{2}, \frac{iv}{2}, 1, 1-v\right)}{v}$$

$$\frac{B_v[\Delta, J]}{2\sin^2\left(\frac{\Delta - J}{2}\right)} \rightarrow \frac{2^9 \pi}{m^{10} b_{\Delta+4, J}} \int_0^\infty \frac{dv}{2\pi} \coth\left(\frac{\pi v}{2}\right) \gamma_5(v)^2 v \mathcal{P}_{-1+iv}(\eta_{\text{AdS}}) B_v(v) (1 + O(m^{-2}))$$

$$\lambda_{\Delta, J}^{2, \text{free}} = 2(\Delta + 2)(J + 1) \frac{\Gamma\left(\frac{\Delta - J}{2} + 1\right)^2 \Gamma\left(\frac{\Delta + J}{2} + 2\right)^2}{\Gamma(\Delta - J + 1) \Gamma(\Delta + J + 3)}$$

$$\frac{2^9 \pi}{m^6 b_{\Delta+4, J}} = \frac{2^9 (\eta_{\text{AdS}}^2 - 1)}{\pi \lambda_{\Delta, J}^{2, \text{free}}}$$



$$\beta_{\text{AdS}} = \cosh^{-1}(\eta_{\text{AdS}}) \equiv \log \frac{\Delta + J + 3}{\Delta - J + 1}$$

$$v\mathcal{P}_{-1+iv}(\eta_{\text{AdS}}) = \frac{\sin(v\beta_{\text{AdS}})}{\sinh(\beta_{\text{AdS}})}$$

$$\frac{B_v[\Delta, J] m^4 \lambda_{\Delta, J}^{2, \text{free}}}{2 \sin^2\left(\frac{\Delta - J}{2}\right)} \rightarrow B_v^{\text{Regge}}[\eta_{\text{AdS}}] \times (1 + O(m^{-2}))$$

$$B_v^{\text{Regge}}[\eta_{\text{AdS}}] \equiv 4 \sinh(\beta_{\text{AdS}}) \int_0^\infty dv \frac{\cosh\left(\frac{\pi v}{2}\right)}{\sinh\left(\frac{\pi v}{2}\right)^3} v^2 (v^2 + 4)^2 \sin(v\beta_{\text{AdS}}) B_v(v)$$

$$B_v(v) = \frac{1}{2} \int \frac{dt v^{\frac{t}{2}-4}}{4\pi i} \Gamma\left(\frac{t-2}{2}\right)^2 \Gamma\left(\frac{8-t}{2}\right)^2 \hat{B}_t(v), \hat{B}_t(v) \equiv \frac{4}{\pi^2} \sinh\left(\frac{\pi v}{2}\right)^2 \frac{\Gamma\left(\frac{t-iv-6}{2}\right) \Gamma\left(\frac{t+iv-6}{2}\right)}{(6-t) \Gamma\left(\frac{t-2}{2}\right)^2}$$

$$\hat{B}_t^{\text{Regge}}[\eta_{\text{AdS}}] = 4 \sinh(\beta_{\text{AdS}}) \left[\int_0^\infty dv \frac{\cosh\left(\frac{\pi v}{2}\right)}{\sinh\left(\frac{\pi v}{2}\right)^3} v^2 (v^2 + 4)^2 \sin(v\beta_{\text{AdS}}) \hat{B}_t(v) + \frac{16(t-4)^2(t-8)^2 \Gamma(t-5)}{\pi \tan\left(\pi \frac{t-6}{2}\right) \Gamma\left(\frac{t-2}{2}\right)^2} \sinh((t-6)\beta_{\text{AdS}}) \right]$$

$$\hat{B}_t^{\text{Regge}}[\eta_{\text{AdS}}] = \frac{2^{12} \times 3}{\pi^2 (6-t) \eta_{\text{AdS}}^3} \left[x {}_2F_1\left(4, \frac{5-t}{2}, \frac{3}{2}, x\right) + x^2 {}_2F_1\left(4, \frac{7-t}{2}, \frac{5}{2}, x\right) \right]_{x=1-\eta_{\text{AdS}}^{-2}}$$

$$X^{\text{Regge}}[\eta_{\text{AdS}}] = \int_{-\infty}^{+\infty} \frac{dy}{4\pi} \tilde{X}(y) \tilde{B}_{5+iy}^{\text{Regge}}[\eta_{\text{AdS}}]$$

$$\tilde{B}_v(y) = \frac{\pi^2 (1+y^2)^2}{32 \cosh\left(\frac{\pi y}{2}\right)^2} v^{\frac{iy-3}{2}}$$

$$\Phi_{\ell_1, \ell_2}(y) \equiv \Phi_{\ell_1}(y) - \frac{\Phi_{\ell_1}^\infty}{\Phi_{\ell_2}^\infty} \Phi_{\ell_2}(y)$$

$$\Phi_\ell(y) = \frac{i\pi^2}{16} (1+y^2)^2 \int_\infty^y \frac{dy' A_\ell(y')}{\cosh\left(\frac{\pi y'}{2}\right)^2}$$

$$A_J(y) = \frac{(J+1)^2(J+3)}{4J(2J+1)(2J+3)} A_{J-2}(y) - \frac{iy}{2J} A_{J-1}(y)$$

$$A_0(y) = 1, A_1(y) = \frac{-iy}{2}$$

$$\frac{\pi}{4} \int_\infty^y \frac{dy' y'^n}{\cosh\left(\frac{\pi y'}{2}\right)^2} = \sum_{a=0}^n \frac{n! y^{n-a}}{(n-a)! \pi^a} \text{Li}_a(-e^{-\pi y})$$



$$\Psi_\ell(y) = \frac{\pi^2(1+y^2)^2}{32\cosh\left(\frac{\pi y}{2}\right)^2} A_\ell(y) - \int_y^\infty \frac{dy' \tilde{\Psi}_\ell(y')}{\cosh\left(\frac{\pi y'}{2}\right)^2}$$

$$\tilde{\Psi}_\ell(y) = A_\ell(y) \left(\frac{1}{2} H_{-\frac{1+iy}{2}} + \frac{1}{2} H_{-\frac{1-iy}{2}} + \Psi_\beta \right) + \sum_{j=0}^{\ell-2} \frac{(\ell+2)! (2j+5)!}{(j+2)! (2\ell+4)! (\ell-j)(\ell+j+5)} \frac{A_j(y)}{m^4}$$

$$\frac{I_k[\Delta, J] m^4 \lambda_{\Delta, J}^{2, \text{free}}}{2 \sin^2\left(\frac{\Delta-J}{2}\right)} \sim \frac{1}{m^4}$$

$$I_p = \iint \frac{d s d t}{(4\pi i)^2} \Gamma\left(\Delta_\phi - \frac{s}{2}\right)^2 \Gamma\left(\Delta_\phi - \frac{t}{2}\right)^2 \Gamma\left(\Delta_\phi - \frac{u}{2}\right)^2 I_p[s, t] \hat{\mathcal{H}}(s, t)$$

$$I_2[s, t] \equiv -\frac{1}{2\pi^2} \int d^4 x |x|^{s-8} |e-x|^{t-8}$$

$$I_4[s, t] \equiv -\frac{16}{2\pi^2} \int d^4 x |x|^{s-8} |e-x|^{t-8} (1+x^2 + (e-x)^2) \bar{D}_{1,1,1,1}(x^2, (e-x)^2)$$

$$I_2[s, t]|_{s\text{-poles}} = -\frac{2}{\pi} \int_0^\pi \sin^2 \theta d\theta \int_0^1 \frac{dr}{r} r^{s-4} (1+r^2 - 2r \cos \theta)^{\frac{t}{2}-4}$$

$$= \frac{-1}{s-4} - \frac{\left(3-\frac{1}{2}t\right)\left(4-\frac{1}{2}t\right)}{2(s-2)} - \frac{\left(3-\frac{1}{2}t\right)\left(4-\frac{1}{2}t\right)^2\left(5-\frac{1}{2}t\right)}{12s} - \frac{\left(3-\frac{1}{2}t\right)\left(4-\frac{1}{2}t\right)^2\left(5-\frac{1}{2}t\right)^2\left(5-\frac{1}{2}t\right)}{144(s+2)} + \dots$$

$$I_2[s, t] = -\frac{1}{2} \frac{\Gamma\left(\frac{s}{2}-2\right) \Gamma\left(\frac{t}{2}-2\right) \Gamma\left(\frac{u}{2}-2\right)}{\Gamma\left(4-\frac{s}{2}\right) \Gamma\left(4-\frac{t}{2}\right) \Gamma\left(4-\frac{u}{2}\right)}$$

$$I_4[s, t] = -8 \frac{\Gamma\left(\frac{s}{2}-2\right) \Gamma\left(\frac{t}{2}-2\right) \Gamma\left(\frac{u}{2}-2\right)}{\Gamma\left(4-\frac{s}{2}\right) \Gamma\left(4-\frac{t}{2}\right) \Gamma\left(4-\frac{u}{2}\right)} \times \left[\frac{4(s-5)}{(t-6)(u-6)} + \left(\frac{t-s}{u-6} + \frac{u-s}{t-6}\right) \left(H_{\frac{s}{2}-3} + H_{3-\frac{s}{2}}\right) + (2 \text{ cyclic}) \right]$$

$$\frac{t J_0(2gt) J_1(2gt') - (t \leftrightarrow t')}{t'^2 - t^2} = \frac{1}{g t t'} \sum_{n=1}^{\infty} 2n J_{2n}(2gt) J_{2n}(2gt')$$

$$2m/t J_m(t) = J_{m-1}(t) + J_{m+1}(t)$$

$$J_{i,j}(g) = \int_0^\infty \frac{t d t e^{-t}}{(1-e^{-t})^2} J_i(2gt) J_j(2gt)$$

$$I_2(g) = J_{1,1}(g) - J_{2,2}(g)$$

$$I_4(g) = 48\zeta_3 - 8g^{-2} J_{1,1}(g) - \sum_{n=1}^{\infty} \frac{96}{n} \left(J_{1,2n-1}(g) + J_{1,2n+1}(g) \right)^2$$



$$\hat{\mathcal{H}}^{\text{curve model}} = \frac{1}{\left(\frac{s}{2}-3\right)\left(\frac{t}{2}-3\right)\left(\frac{u}{2}-3\right)} + \tilde{g}_0 + \tilde{g}_2((s-4)^2 + (t-4)^2 + (u-4)^2)$$

$$I_2^{\text{curve model}} = \frac{1}{4} - \frac{1}{40}\tilde{g}_0 - \frac{2}{35}\tilde{g}_2$$

$$I_4^{\text{curve model}} = 48\zeta_3 - 24 - \frac{16}{5}\tilde{g}_0 - \frac{272}{35}\tilde{g}_2$$

$$\tilde{g}_0 = \frac{120\zeta_3}{(4\pi g)^3} - \frac{1890\zeta_5}{(4\pi g)^5} + O(g^{-7}), \tilde{g}_2 = \frac{630\zeta_5}{(4\pi g)^5} + O(g^{-7})$$

$$\mathcal{M}^{\text{curve model}}(s, t) \propto \frac{1}{stu} + g_0 + g_2(s^2 + t^2 + u^2) + O(m_K^{-12})$$

$$g_0 = \frac{c_{-3}}{8c_0}\tilde{g}_0 = \frac{2\zeta_3}{m_K^6}, g_2 = \frac{c_{-3}}{8c_2}\tilde{g}_2 = \frac{\zeta_5}{m_K^{10}}$$

$$\hat{\mathcal{P}}_s^{\mathcal{N}=4}[\Delta, J]_q \equiv \sum_{n=0}^{\infty} \frac{K_{\Delta, J}^{n, \{\Delta_i\}}}{\tilde{K}_{\Delta, J}^{\mathcal{N}=4}} \frac{(-n)_q}{s - (\Delta - J + 2n + 4)}$$

$$[I_{p,s}]_k = \int \frac{dt}{4\pi i} \frac{\Gamma\left(\frac{t}{2}-2\right)\Gamma\left(4-\frac{t}{2}\right)\Gamma\left(\frac{u}{2}-2\right)\Gamma\left(4-\frac{u}{2}+k\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(4-\frac{s}{2}\right)}$$

$$\times \begin{cases} -1, & \text{for } p=2, \\ -48 \left[\frac{4(u-5)}{(s-6)(t-6)} + \left(\frac{t-s}{u-6} \left(H_{\frac{s}{2}-3} + H_{3-\frac{s}{2}} \right) + (s \leftrightarrow t) \right) \right], & \text{for } p=4. \end{cases}$$

$$H_x = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x} \right)$$

$$[I_{2,s}]_k = \frac{-\left(\frac{s}{2}\right)_k}{(k+2)(k+3)}$$

$$[I_{4,s}]_k = \frac{-768(s-5)^2 k!}{(s-6)^2(s-4)(s-2)} + \frac{48\left(\frac{s}{2}\right)_k}{(k+1)_3(s-6)} \left[2 \frac{(k+3)(s-5)(s-4)}{2k+s-2} + 8(k+s-4) \right. \\ \left. + ((k+5)(s-4)-4) \left(H_{\frac{s}{2}+k-1} - H_{\frac{s}{2}-4} - 2H_{k+3} \right) \right]$$

$$I_p[\Delta, J] = \tilde{K}_{\Delta, J}^{\mathcal{N}=4} \int \frac{ds}{4\pi i} \Gamma\left(\frac{s}{2}-2\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(4-\frac{s}{2}\right)^2 \sum_{q,k=0}^J \hat{\mathcal{P}}_s^{\mathcal{N}=4}[\Delta, J]_q [Q_{\Delta+4, J}]_{q,k} [I_{p,s}]_k$$

$$\frac{(\Delta+2)^2}{J+2} = 2\lambda^{1/2} + \left(\frac{4}{J+2} - 1 + \frac{3}{2}(J+2) \right) + \frac{\frac{15}{4} + \left(\frac{3}{8} - 3\zeta_3\right)(J+2) - \frac{3}{8}(J+2)^2}{\lambda^{1/2}} \\ + \frac{\frac{15}{4} - \frac{9(3+8\zeta_3)}{16}(J+2) + \frac{-9+60(\zeta_3+\zeta_5)}{16}(J+2)^2 + \frac{31}{64}(J+2)^3}{\lambda} + O(\lambda^{-3/2})$$



$$a_J(s) = \frac{s^{\frac{D-4}{2}}}{2^{2D-3}\pi^{\frac{D-2}{2}}\Gamma\left(\frac{D-2}{2}\right)} \int_{-1}^1 dx (1-x^2)^{\frac{d-4}{2}} \mathcal{P}_J(x) s^4 \mathcal{M}(s, t)_{t=-\frac{1}{2}s(1-x)}$$

$$C_{2\sqrt{(1+J/2)/\alpha', J}}^2 = \frac{2G_5}{(\alpha')^{\frac{5}{2}}} \frac{2^{-2J} \left(1 + \frac{J}{2}\right)^{J+\frac{5}{2}}}{(J+1)\Gamma\left(1 + \frac{J}{2}\right)^2} \Rightarrow \frac{2\sin^2\left(\frac{\pi\Delta}{2}\right)\lambda_{\Delta, J}^2}{\lambda_{\Delta, J}^{2, \text{free}}} = \frac{\pi^2 \lambda}{32c} \frac{2^{-2J} \left(1 + \frac{J}{2}\right)^{J+2}}{(J+1)\Gamma\left(1 + \frac{J}{2}\right)^2}$$

$$\tilde{\lambda}_{\Delta, J}^2 = \frac{\lambda_{\Delta, J}^2}{\lambda_{\Delta, J}^{2, \text{free}}} \times \frac{2^8}{(\Delta+J+6)^2} \frac{\sin^2\left(\frac{\pi}{2}(\Delta-J)\right)}{\left[\frac{\pi}{2}(\Delta-J-2)\right]^2} \times \frac{2^{3J}(J+1)\Gamma\left(\frac{J}{2}+1\right)^2}{(J+2)^J}$$

$$\begin{aligned} \frac{\log \tilde{\lambda}_{\Delta, J}^2}{J+2} &= \frac{\frac{17}{6} + (J+2) + \left(-\frac{7}{12} + \zeta_3\right)(J+2)^2}{(\Delta+2)^2} \\ &+ \frac{\frac{511}{60} + 6(J+2) + \left(\frac{1}{12} - 2\zeta_3\right)(J+2)^2 - \left(\frac{13}{8} + 6\zeta_3\right)(J+2)^3 + \left(\frac{31}{40} - \frac{3}{2}\zeta_5\right)(J+2)^4}{(\Delta+2)^4} \\ &+ \mathcal{O}(\Delta^{-6}) \end{aligned}$$

$$v_{N, k}^{\square} = \frac{N!}{k!(N-k)!},$$

$$\mathcal{L}_g = -\frac{1}{4g^2} \int d^4\theta W^{aa} W_a^a + \text{H. c.} = -\frac{1}{4g^2} G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2g^2} D^a D^a + \frac{i}{g^2} \lambda^a \sigma^{\mu\nu} \mathcal{D}_\mu \bar{\lambda}^a$$

$$\mathcal{L}_{\text{kin}} = \int d^4\theta d^2\bar{\theta} \bar{Q}^f e^V Q_f$$

$$\mathcal{L}_{\text{cl}} = \frac{m_0}{2} \int d^2\theta Q_\alpha^f Q_f^\alpha + \text{H. c.}$$

$$\mathcal{W}_{\text{inst}}(M) = \frac{\Lambda_{F=1}^5}{M}$$

$$\Lambda_{F=1}^5 \stackrel{\text{def}}{=} \frac{e^{-8\pi^2/g_0^2}}{g_0^4} (M_{\text{uv}})^5$$

$$\mathcal{L}_{\text{sp}} = \mathcal{L}_{\text{cl}} + \left(\int d^4\theta \mathcal{W}_{\text{inst}} + \text{H. c.} \right)$$

$$d\mu = \frac{1}{2^5} \frac{\Lambda_{F=1}^5}{M(x_0, \theta_0)} \exp(-4\pi^2 |M| \rho_{\text{inv}}^2) \frac{d\rho^2}{\rho^2} d^4x_0 d^2\theta_0 d^2\bar{\beta} d^2\bar{\theta}_0$$

$$\bar{\beta}_{\text{inv}} = \frac{\bar{\beta}}{1 - 4i(\bar{\beta}\bar{\theta}_0)}, \rho_{\text{inv}}^2 = \frac{\rho^2}{1 - 4i(\bar{\beta}\bar{\theta}_0)},$$



$$\int d\mu \propto \int \frac{d\rho^2}{\rho^2} d^2\bar{\beta}_{\text{inv}} d^2\bar{\theta}_0 \exp\left(-\frac{8\pi^2}{g^2} - 4\pi^2|v|^2\rho_{\text{inv}}^2\right) = 16\exp\left(-\frac{8\pi^2}{g_0^2}\right)$$

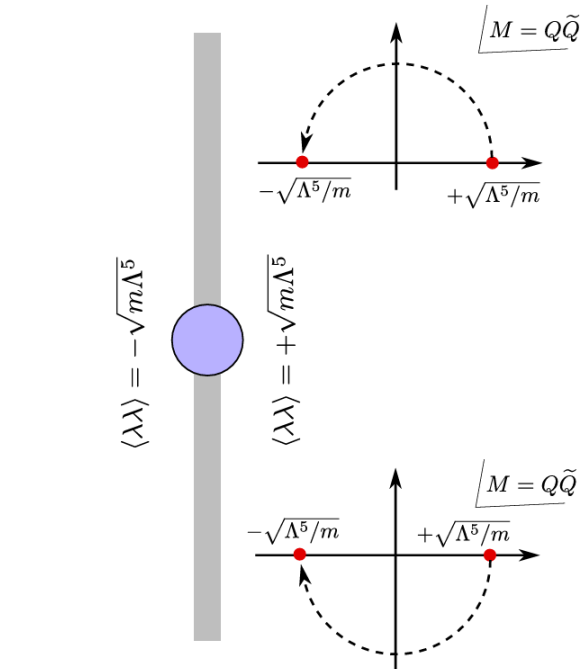
$$\langle M \rangle = \pm \frac{1}{m_0} \sqrt{m_0 \Lambda_{F=1}^5} = \pm \frac{1}{m_0} \sqrt{m_0 \frac{e^{-8\pi^2/g_0^2}}{g_0^4} M_{\text{uv}}^5}$$

$$\langle \text{Tr}\lambda\lambda \rangle = \pm 16\pi^2 \left[m_0 \frac{e^{-8\pi^2/g_0^2}}{g_0^4} M_{\text{uv}}^5 \right]^{1/2}$$

$$\left\langle \frac{\text{Tr}\lambda\lambda}{16\pi^2} \right\rangle = \pm \left[\frac{e^{-8\pi^2/g_0^2}}{g_0^4} M_{\text{uv}}^6 \right]^{1/2} = \Lambda_{\text{SYM}}^3$$

$$\Lambda_{\text{SYM}}^3 = \frac{e^{-4\pi^2/g_0^2}}{g_0^2} M_{\text{uv}}^3$$

$$\frac{\partial M}{\partial z} = \pm \frac{1}{2} |M| \frac{\partial(\bar{\mathcal{W}}_{\text{cl}} + \bar{\mathcal{W}}_{\text{inst}})}{\partial \bar{M}}$$



$$M_{\square} = |\langle M \rangle| e^{\pm i\alpha(z)}$$

$$i\alpha(z) = 2\log\left(\frac{1 + ie^{m_0(z-z_0)}}{\sqrt{1 + e^{2m_0(z-z_0)}}}\right)$$

$$\mathcal{W} = \frac{1}{2} \sqrt{m_0 \Lambda_{1fl}^5} \cos 2\alpha, 2\alpha \in [0, \pi]$$

$$M_f^{\bar{g}} = Q_f \tilde{Q}^{\bar{g}}, f, \bar{g} = 1, \dots, F.$$

$$\mathcal{W} = m \text{Tr} M + (N - F) \left(\frac{\Lambda_F^{3N-F}}{\det M} \right)^{\frac{1}{N-F}}$$

$$\langle M \rangle_n = M_n \cdot \mathbb{I}_{F \times F}, M_n = e^{\frac{2\pi i n}{N}} \cdot \left(\frac{\Lambda_F^{3N-F}}{m^{N-F}} \right)^{1/N}, n = 1, 2, \dots, N$$

$$\left\langle \frac{\text{Tr} \lambda \lambda}{16\pi^2} \right\rangle_n = m M_n$$

$$\mathcal{W} = m \text{Tr} M + \lambda [\det M - \Lambda_{F=N}^2].$$

$$\langle M \rangle_n = M_n \cdot \mathbb{I}_{F \times F}, M_n = e^{\frac{2\pi i n}{N}} \cdot \hat{m}^{-1} (\det \hat{m})^{1/N} \Lambda_{F=N}^2, n = 1, 2, \dots, N$$

$$\left\langle \frac{\text{Tr} \lambda \lambda}{16\pi^2} \right\rangle_n = m M_n$$

$$v_{N,k}^{\square} = \frac{N!}{k! (N-k)!}$$

$$\tilde{\mathcal{M}}_k = G(k, N) = \frac{U(N)}{U(k) \times U(N-k)}$$

$$\mathbf{x} = \frac{8\pi^2}{Ng^2} \boldsymbol{\rho} - \left(\frac{4\pi}{g^2} \boldsymbol{\sigma} - i\boldsymbol{\gamma} \right), \boldsymbol{\rho} \equiv \sum_{j=1}^{N-1} \boldsymbol{\mu}_j$$

$$(\boldsymbol{\sigma}, \boldsymbol{\gamma}) \in \left(\frac{\mathbb{R}^{N-1}}{2\pi\Lambda_r} \times \frac{\mathbb{R}^{N-1}}{2\pi\Lambda_w} \right) / W_{\text{su}(N)}$$

center: $\mathbf{X} \rightarrow \mathbf{X} + 2\pi\boldsymbol{\mu}, \boldsymbol{\mu} \in \Lambda_w \text{ mod } \Lambda_r,$

chiral: $\mathbf{X} \rightarrow \mathbf{X} + i \frac{2\pi k}{N} \boldsymbol{\rho}, k \in \mathbb{Z} \text{ mod } 2N.$

$$\Lambda_w / \Lambda_r = \mathbb{Z}_N$$

$$Y_j \equiv \exp \{ \boldsymbol{\alpha}_j \cdot \mathbf{X} \}, j \in \mathbb{Z} \text{ mod } N$$

$$\boldsymbol{\alpha}_0 = - \sum_{j=1}^{N-1} \boldsymbol{\alpha}_j$$

$$\prod_{j \in \mathbb{Z}_N} Y_j = 1.$$

center: $Y_j \rightarrow Y_{j+1}$

chiral: $Y_j \rightarrow Y_j e^{i \frac{2\pi}{N}}$



$$\mathcal{W} = L\Lambda_{\text{SYM}}^3 \sum_{j \in \mathbb{Z}_N} Y_j$$

$$\langle \mathbf{X} \rangle_n = -i \frac{2\pi n}{N} \boldsymbol{\rho} \pmod{2\pi\Lambda_w}, \langle Y_j \rangle_n = e^{-i \frac{2\pi n}{N}}$$

$$(\mathcal{W}_{3d})_n = L\Lambda_{\text{SYM}}^3 e^{-i \frac{2\pi n}{N}}$$

$$\mathcal{W} = \eta Y_0 + \sum_{j=1}^{N-1} Y_j - \lambda \left(\prod_{j=0}^{N-1} Y_j - 1 \right), \eta \equiv (L\Lambda^3)^N.$$

$$\mathcal{Q}_{0,j}^{m_{U(1)=g_{3d}(q_{3d})}} = \alpha_j + \alpha_0 = - \sum_{i \neq j} \alpha_i.$$

$$\mathcal{W} = \eta_1 Y_0 + \lambda(MY_0 - 1) + mM$$

$$\mathcal{W} = \frac{\eta_1}{M} + mM$$

$$\mathcal{W} = \eta_1 Y_0 + \frac{m}{Y_0}$$

$$Y_0 \rightarrow \tilde{Y}_0 = \frac{Y_0}{m}, \eta_1 \rightarrow \eta_0 = m\eta_1$$

$$M = \pm \sqrt{\eta_1/m} = \pm L \sqrt{\Lambda_1^5/m}$$

$$Y_0 = \pm \sqrt{m/\eta_1}$$

$M = 1/Y_0 = \sqrt{\eta_1/m}$ at $z \rightarrow -\infty$ to the vacuum $M = 1/Y_0 = -\sqrt{\eta_1/m}$ at $z \rightarrow +\infty$

$$\mathcal{W} = Y_0(\eta_N - \det(M) + B\tilde{B}) + \text{Tr}(\hat{m}M)$$

$$\eta_N = (L\Lambda_N^2)^N = L^N \Lambda_N^b$$

$$\langle M \rangle_n = \hat{m}^{-1} (\det(\hat{m}) \eta_N)^{1/N} e^{\frac{2\pi i n}{N}}$$

$$\langle M \rangle_n = M_n \cdot \mathbb{1}_{N \times N}, M_n = e^{\frac{2\pi i n}{N}} \cdot (\eta_N)^{1/N}, \hat{m} \equiv m \mathbb{1}_{N \times N}$$

$$\langle M \rangle \sim (\eta_{\text{SYM}})^{1/N} / m$$

$$\mathcal{W} = \lambda(Y_0 \det(M') - 1) + \eta_{N-1} Y_0 + \text{Tr}(\hat{m}' M')$$

$$\hat{m} = \begin{pmatrix} & & 0 \\ & \hat{m}' & \vdots \\ & & 0 \\ 0 & \dots & 0 & \hat{m}_{NN} \end{pmatrix}, \hat{m}_N^N \gg (\hat{m}')_{fg}, f, g = 1, \dots, N-1.$$

$$\mathcal{W} = Y_0(\eta_{N-1} - \det M') + \hat{m}_{NN} M_{NN} + \text{Tr}(\hat{m}' M')$$



$$Y_0, M_{NN}, M_{Nf}, M_{fN}, (f = 1, \dots, N - 1)$$

$$M_{NN} = \frac{\eta_{N-1}}{\det M'}, Y_0 = \frac{\hat{m}_{NN}}{\det M'}, M_{Nf} = M_{fN} = 0$$

$$\mathcal{W} = \frac{\eta_{N-1}}{\det(M')} + \text{Tr}(\hat{m}'M')$$

$$\langle M' \rangle_n = (\hat{m}')^{-1} (\det(\hat{m}') \eta_{N-1})^{1/N} e^{\frac{2\pi i n}{N}}$$

$$\hat{m}' \equiv m \mathbb{I}_{(N-1) \times (N-1)}$$

$$\langle M \rangle_n = M_n \cdot \mathbb{I}_{(N-1) \times (N-1)}, M_n = e^{\frac{2\pi i n}{N}} \cdot \left(\frac{\eta_{N-1}}{m} \right)^{1/N}$$

$$\mathcal{W} = \eta_F Y_0 + (N - F - 1) \frac{1}{(Y_0 \det(M))^{\frac{1}{N-F-1}}} + \text{Tr}(mM)$$

$$\eta_F = L^N \Lambda_F^{3N-F}$$

$$\mathcal{W} = (N - F) \left[\frac{\eta}{\det(M)} \right]^{\frac{1}{N-F}} + \text{Tr}(mM)$$

$$M = x \mathbb{I}_{F \times F}, x = \frac{1}{Y_0^{\frac{1}{N-1}} m^{\frac{N-F-1}{N-1}}}$$

$$\mathcal{W} = \eta Y_0 + (N - 1) \frac{1}{Y_0^{\frac{1}{N-1}}}$$

$$\mathcal{W} = \eta Y_0 + \sum_{i=F+2}^{N-1} Y_i + \frac{[\prod_{i=1}^{F+1} Y_i]}{\det M} - \lambda \left(Y_0 \prod_{j=1}^{N-1} Y_j - 1 \right) + \text{Tr}(\hat{m}M)$$

$$Y_{\text{conf}} \equiv \frac{[\prod_{i=1}^{F+1} Y_i]}{\det M}$$

$$\mathcal{W} = \eta Y_0 + \sum_{i=F+2}^{N-1} Y_i + Y_{\text{conf}} - \lambda \left(\det(M) \cdot Y_0 \cdot Y_{\text{conf}} \cdot \prod_{i=F+2}^{N-1} Y_i - 1 \right) + \text{Tr}(\hat{m}M)$$

$$\langle Y_0 \rangle_n = \frac{(m^F \eta)^{1/N}}{\eta} e^{2\pi i n / N}, \langle Y_{\text{conf}} \rangle_n = (m^F \eta)^{1/N} e^{2\pi i n / N}$$

$$\langle Y_j \rangle_n = (m^F \eta)^{1/N} e^{2\pi i n / N} \quad (j = F + 2, \dots, N - 1)$$

$$\langle M \rangle_n = M_n \cdot \mathbb{I}_{F \times F}, M_n = \frac{(m^F \eta)^{1/N}}{m} e^{2\pi i n / N}$$

$$\langle \lambda \rangle_n = (m^F \eta)^{1/N} e^{2\pi i n / N}$$

$$X = \frac{1}{m \eta^{1/N}} \hat{m}M, m \equiv (\det \hat{m})^{1/N}$$



$$\mathcal{W} = m\eta_N^{1/N} \left\{ Y_0 \cdot \left(1 - \prod_{i=1}^N x_i \right) + \sum_{i=1}^N x_i \right\}$$

$$\langle x_j \rangle_n = e^{\frac{2\pi i n}{N}}, n = 0, \dots, N-1$$

$$w_i = \frac{1}{2\pi i} \int_{\Gamma_i} \frac{dx_i}{x_i}$$

$$\prod_{i=1}^N x_i = 1$$

$$N_R \arg(x_R) + N_L \arg(x_L) = 0 \pmod{2\pi}$$

$$N_L = k, N_R = N - k$$

$$v_{N,k}^{\square} = \binom{N}{k} = \frac{N!}{k!(N-k)!}$$

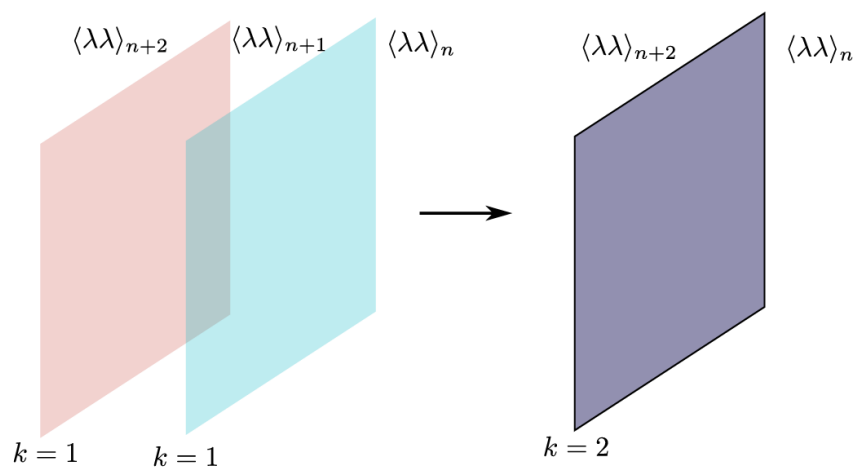
$$\mathcal{M}_k^{\square, F=N} = \text{Gr}(k, N) = \frac{U(N)}{U(k) \times U(N-k)}$$

$$\left(\prod_{i=1}^F x_i \right) \cdot Y \cdot Y_{\text{conf}} \cdot \left(\prod_{i=F+2}^{N-1} Y_i \right) = 1$$

$$v_{N,k}^{\square} = \binom{N}{k} = \frac{N!}{k!(N-k)!}$$

$$\max(0, F+k-N) \leq J \leq \min(k, F)$$

$$v_{N,k}^{\square} \Big|_{\mathbb{R}^4 \times \mathbb{S}^2} = \sum_{J=\max(0, F+k-N)}^{\min(k, F)} \binom{N-F}{k-J} \binom{F}{J} = \binom{N}{k}$$



$$U(N-F)_{\text{muon}} \times U(F)_{\text{quark}}$$

$$[U(N - F)]_{\text{muon}} \times [U(F)]_{\text{quark}} \rightarrow [U(k - J) \times U(N - F - k + J)]_{\text{muon}} \times [U(J) \times U(F - J)]_{\text{quark}}.$$

$$\mathcal{M}_{\mathbb{R}^4 \times \mathbb{S}^2}^{k,J} = \mathbb{G}\mathbb{R}(N - F, k - J)_{\text{muon}} \times \mathbb{G}\mathbb{R}(F, J)_{\text{quark}}$$

$$\mathcal{K}_{3d}^Q = L \bar{Q} Q$$

$$\mathcal{K}_{3d}^Y = \frac{g_{4d}^2}{L} \ln(\bar{Y}) \ln(Y)$$

$$U(k)_{N-k,N} \equiv \frac{SU(k)_{N-k} \times U(1)_N}{\mathbb{Z}_k} \text{CS}, 0 < k \leq \frac{N}{2}$$

$$\mathbb{G}\mathbb{R}(N - F, k - J)_{\text{muon}}.$$

$$U(n)_{m-n,m} \leftrightarrow U(m - n)_{-n,-m}$$

$$U(k - J)_{(N-F)-(k-J),N-F} \text{CS}$$

$$U((N - F) - (k - J))_{-(k-J),-(N-F)}.$$

$$v_{N,k}^{\square} \Big|_{\mathbb{R}^4 \times \mathbb{S}^2} = \sum_{J=\max(0,F+k-N)}^{\min(k,F)} \binom{N-F}{k-J} \binom{F}{J} = \binom{N}{k}.$$

$$T_{\text{junc}} = \pi \left| \frac{|m_1| - |m_2|}{\sqrt{|m_1 m_2|}} \right| \Big|_{\Lambda_{F=2}^2} \xrightarrow{m_2 \rightarrow \infty} \pi \sqrt{\frac{\Lambda_{F=1}^5}{|m_1|}}$$

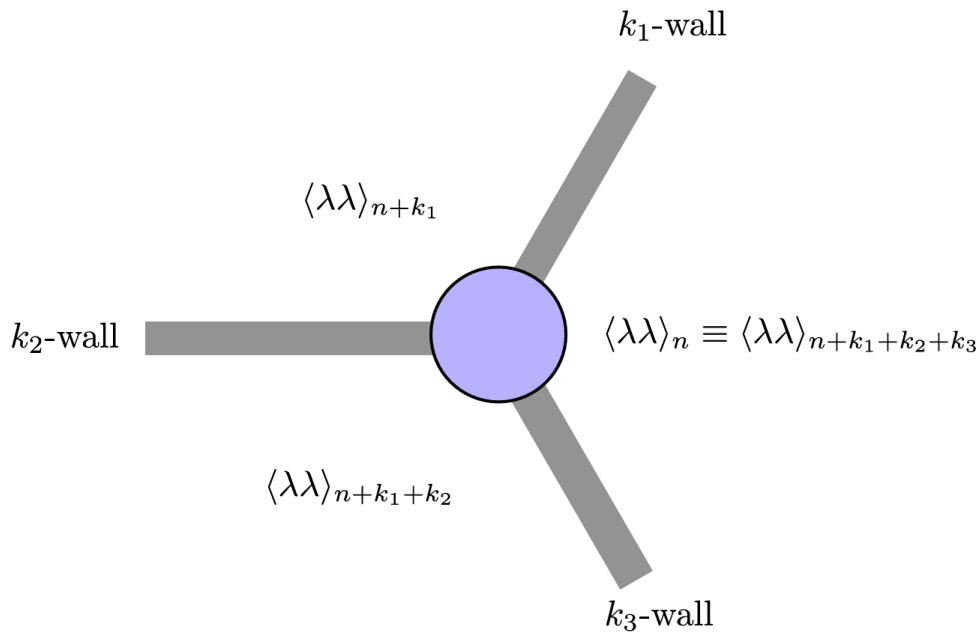
$$\Lambda_{4d} = \mu \exp\left(-\frac{2\pi}{g_{2d}^2(\mu)}\right), \frac{1}{g_{2d}^2(\mu)} = L \frac{\Lambda_{F=2}^2}{\sqrt{m_1 m_2}}$$

$$M_{\text{kink}} \approx \frac{|\Delta m|}{g_{2d}^2}$$

$$M_{\text{kink}} = \frac{1}{2\pi e} \Lambda_{2d}$$

$$\sum_a k_a = N; k_a > 0, a = 1, \dots, N_{\text{walls}}.$$





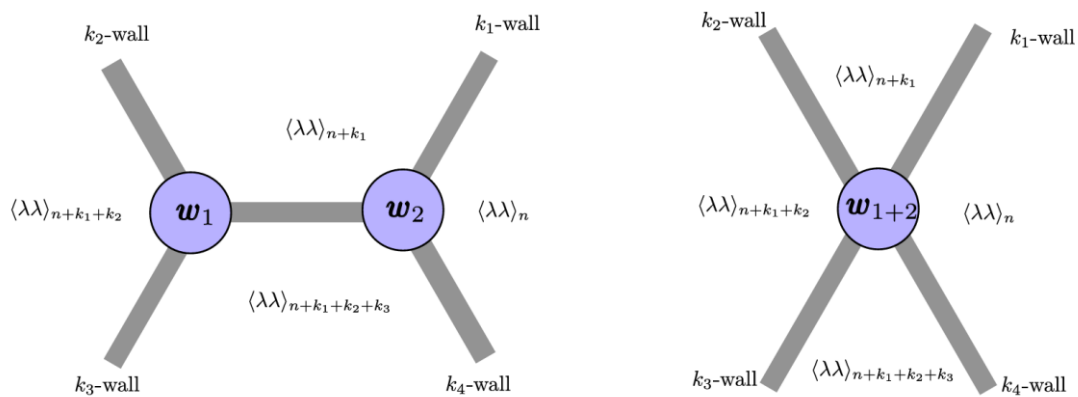
$$T_{(k_a)} = \left[\frac{u(N)_N}{\prod_a u(k_a)_N} \right].$$

$$w(Y_j) = \sum_{a \in \blacksquare} \left(\frac{k_a}{N} - 1 \right) + \sum_{a \in \blacksquare} \frac{k_a}{N} = 1 - N_L^{(j)}$$

$$N_L^{(j)} = 1, \text{ for all } j = 0, \dots, N-1.$$

$$\sum_{j=0}^{N-1} N_L^{(j)} = N$$

$$\sum_{j=0}^{N-1} N_L^{(j)} = \sum_{a=1}^{N_{\square}} k_a = N$$



$$\mathcal{M}_{(k_a)}^{\text{junc, SYM}} = \frac{U(N)}{\prod_a U(k_a)}.$$



$$w(Y_j) = \mathbf{w}_{\text{curve}} \cdot \boldsymbol{\alpha}_j = - \sum_{j=1}^{N-1} \alpha_j$$

$$\sum_a k_a = N, \sum_a J_a = F; k_a, J_a > 0, a = 1, \dots, N_{\square}$$

$$\mathcal{M}_{(k_a)}^{\text{junc}, F=N} = \frac{U(N)}{\prod_a U(k_a)}$$

$$\chi(G/H) = \frac{|W_G|}{|W_H|}$$

$$\nu_{(k_a)}^{\text{junc}, F=N} = \chi(\mathcal{M}_{(k_a)}^{\text{junc}, F=N}) = \frac{N!}{\prod_a k_a!}$$

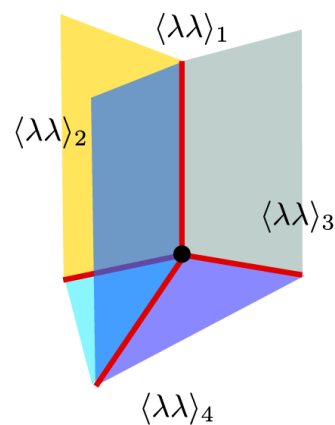
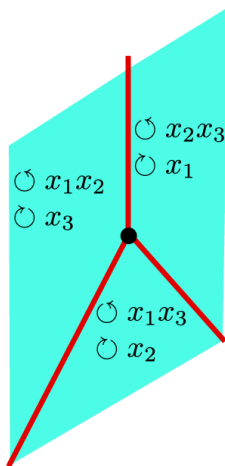
$$\nu^{(4\text{-wall})} = \frac{N!}{k_1! k_2! k_3! k_4!}$$

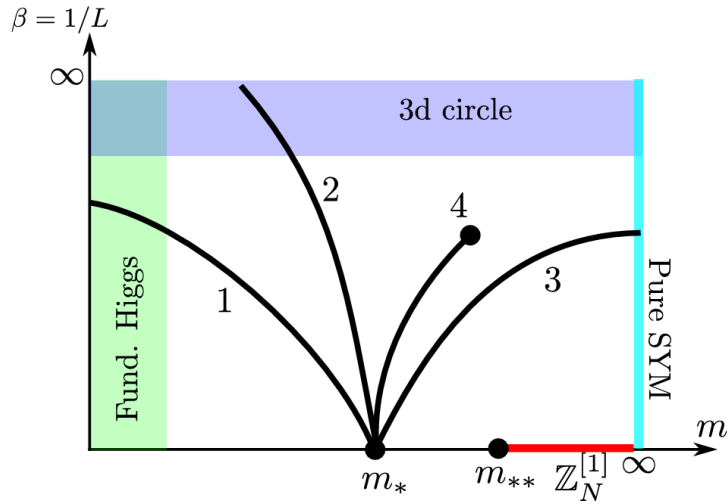
$$\nu_{(k_1 k_2 k_5)}^{\text{junc}, F=N} \cdot \nu_{(k_5 k_3 k_4)}^{\text{junc}, F=N}$$

$$\nu^{(4\text{-wall} \times 4\text{-wall})} = \frac{\nu_{(k_1 k_2 k_5)}^{\text{junc}, F=N} \cdot \nu_{(k_5 k_3 k_4)}^{\text{junc}, F=N}}{\nu_{k_5}^{\text{wall}}}$$

$$\mathcal{M}_{(k_a, J_a)}^{\text{junc}, F} = \left[\frac{U(N-F)}{\prod_a U(k_a - J_a)} \right]_{\text{muon}} \times \left[\frac{U(F)}{\prod_a U(J_a)} \right]_{\text{fl}}$$

$$\left[\frac{U(N-F)_{N-F}}{\prod_a U(k_a - J_a)_{N-F}} \right]^{4d} \times \left[\frac{U(F)}{\prod_a U(J_a)} \right]^{4d}$$





$$T_{\text{junc}} = \pi \left| \frac{|m_1| - |m_2|}{\sqrt{|m_1 m_2|}} \right| \Lambda_{F=2}^2 = \pi \left| \frac{|m_1| - |m_2|}{|m_1 m_2|} \right| \Lambda_{F=0}^3$$

$$T_{\text{junc}} \xrightarrow{m_2 \rightarrow \infty} \pi \sqrt{\frac{\Lambda_{F=1}^5}{|m_1|}} = \frac{\pi}{|m_1|} \Lambda_{F=0}^3$$

$$T_{\text{wall}} \sim \Lambda_{\text{SYM}}^3.$$

$$T_{\text{wall}} \sim \Lambda_F^3 \left(\frac{m}{\Lambda_F} \right)^{F/N},$$

$$t_W \sim \Lambda^{-1} \exp\left(\frac{m^2}{\Lambda^2}\right)$$

$$A \ll m \Lambda^{-3} \exp\left(\frac{m^2}{\Lambda^2}\right)$$

$$\mathcal{V}_{N,k}^{\text{walls}} \Big|_{\mathbb{R}^4 \times \mathbb{S}^2} = \sum_{J=\max(0, F+k-N)}^{\min(k, F)} \binom{N-F}{k-J} \binom{F}{J} = \binom{N}{k}.$$

$$\left[\frac{U(N-F)_{N-F}}{\prod_a U(k_a - J_a)_{N-F}} \right]^{4d} \times \left[\frac{U(F)}{\prod_a U(J_a)} \right]^{4d}$$

$$(\phi_f^\alpha)_{\text{vac}} = v \delta_f^\alpha$$

$$\phi_f^\alpha = v \frac{x_f^\alpha}{\sqrt{x^2 + \rho^2}}$$

$$x_f^\alpha = x_\mu \varepsilon^{\alpha\beta} (\sigma^\mu)_{\beta f}$$

$$\mathcal{V}(x_0, \theta_0, \bar{\beta}, \bar{\zeta}, \bar{\omega}, \rho) = e^{iPx_0} e^{-iQ\theta_0} e^{-i\bar{S}\bar{\beta}} e^{-i\bar{Q}\bar{\zeta}} e^{i\bar{M}\bar{\omega}} e^{iD \ln \rho}$$



$$\delta \bar{\zeta}_{\alpha} = \bar{\varepsilon}_{\alpha} - 4i\bar{\beta}_{\alpha}(\bar{\zeta}\bar{\varepsilon})$$

$$(\bar{\theta}_0)^{\dot{\alpha}} = \bar{\zeta}^{\dot{\alpha}}[1 - 4i(\bar{\beta}\bar{\zeta})], \delta(\bar{\theta}_0)^{\dot{\alpha}} = \bar{\varepsilon}^{\dot{\alpha}}$$

$$\mathcal{V}(x_0, \theta_0, \bar{\theta}_0, \bar{\beta}_{\text{inv}}, \bar{\omega}_{\text{inv}}, \rho_{\text{inv}}) = e^{iPx_0} e^{-iQ\theta_0} e^{-i\bar{Q}\bar{\theta}_0} e^{-i\bar{S}\bar{\beta}_{\text{inv}}} e^{i\bar{M}\bar{\omega}_{\text{inv}}} e^{iD \ln \rho_{\text{inv}}}$$

$$\bar{\beta}_{\text{inv}} = \bar{\beta}[1 + 4i(\bar{\beta}\bar{\zeta})] = \frac{\bar{\beta}}{1 - 4i(\bar{\beta}\bar{\theta}_0)}$$

$$\rho_{\text{inv}}^2 = \rho^2[1 + 4i(\bar{\beta}\bar{\zeta})] = \frac{\rho^2}{1 - 4i(\bar{\beta}\bar{\theta}_0)}$$

$$[\Omega_{\text{inv}}]_{\beta}^{\dot{\alpha}} \equiv [e^{-i\bar{\omega}_{\text{inv}}}]_{\beta}^{\dot{\alpha}} = \exp \left\{ -4i \left[\bar{\zeta}^{\dot{\alpha}} \bar{\beta}_{\dot{\gamma}} + \frac{1}{2} \delta_{\dot{\gamma}}^{\dot{\alpha}} (\bar{\zeta}\bar{\beta}) \right] \right\} \Omega_{\beta}^{\dot{\gamma}}$$

$$z_{\alpha\dot{\alpha}} = (x_L - x_0)_{\alpha\dot{\alpha}} + 4i(\theta - \theta_0)_{\alpha}(\bar{\theta}_0)_{\dot{\alpha}}, \theta - \theta_0, \bar{\theta} - \bar{\theta}_0.$$

$$\tilde{x}_{\alpha\dot{\alpha}} = (x_L - x_0)_{\alpha\dot{\alpha}} + 4i\tilde{\theta}_{\alpha}\bar{\zeta}_{\dot{\alpha}},$$

$$\frac{\tilde{x}^2}{\rho^2} = \frac{z^2}{\rho_{\text{inv}}^2}$$

$$d\mu_{\text{one-fl}} = \frac{1}{2^{11}\pi^4 v^2} M_{\text{PV}}^5 \int \left(\frac{8\pi^2}{g^2} \right)^2 \exp \left(-\frac{8\pi^2}{g^2} - 4\pi^2 |v|^2 \rho_{\text{inv}}^2 \right) \\ \times d^4 x_0 d^2 \theta_0 \frac{d\rho^2}{\rho^2} d^2 \bar{\beta}_{\text{inv}} d^2 \bar{\theta}_0$$

$$d\rho_{\text{inv}}^2 / \rho_{\text{inv}}^2 \int d\rho^2 / \rho^2 \int d\rho_{\text{inv}}^2 / \rho_{\text{inv}}^2 \int d^4 x_0$$

$$\int \frac{d\rho_{\text{inv}}^2}{\rho_{\text{inv}}^2} d^2 \bar{\beta} d^2 \bar{\theta}_0 f(\rho_{\text{inv}}^2) = 0$$

$$\int \frac{d\rho^2}{\rho^2} d^2 \bar{\beta} d^2 \bar{\theta}_0 f(\rho^2(1 + 4i\bar{\beta}\bar{\theta}_0)) = \int \frac{d\rho^2}{\rho^2} 16\rho^4 f''(\rho^2) = 16f(\rho^2 = 0)$$

$$\frac{d\rho^2}{\rho^2} d^2 \bar{\beta} d^2 \bar{\theta}_0 f(\rho_{\text{inv}}^2) = 16d\rho_{\text{inv}}^2 \delta(\rho_{\text{inv}}^2) f(\rho_{\text{inv}}^2)$$

$$E \frac{d}{dE} \frac{8\pi^2}{g^2} = 3C(\text{adj}) - C(R) \equiv b$$

$$\hat{\Lambda}^{b_0} = M_{\text{uv}}^{b_0} e^{2\pi i \tau (M_{\text{uv}})}$$

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

$$\Lambda^b = M_{\text{uv}}^b \frac{1}{g^{2N(M_{\text{uv}})}} e^{2\pi i \tau (M_{\text{uv}})}$$



$$\left\langle \frac{\text{tr} \lambda^2}{16\pi^2} \right\rangle = \Lambda^3$$

$$\tilde{\Lambda}^3 = M_{\text{uv}}^3 \frac{16\pi^2}{3Ng^2(M_{\text{uv}})} e^{\frac{2\pi i \tau(M_{\text{uv}})}{N}}$$

$$\tau(E) = \tau(M_{\text{uv}}) - \frac{b}{2\pi i} \log \frac{E}{M_{\text{uv}}}$$

$$\Lambda_A^{b_A} = m^{b_A - b_B} \Lambda_B^{b_B}$$

$$\Lambda_{F-1}^{3N-F+1} = m \Lambda_F^{3N-F}$$

$$\Lambda_0^3 = m \Lambda_N^2$$

$$\eta_F = L^N \Lambda_F^{3N-F}$$

$$\eta_{F-1} = m \eta_F, (Y_0)_{F-1} \text{ flavors} = \frac{1}{m} (Y_0)_F \text{ flavors}$$

$$V = -i\theta\bar{\theta}\sigma - \theta\gamma^i\bar{\theta}A_i + i\bar{\theta}^2\theta\lambda - i\theta^2\bar{\theta}\lambda^\dagger + \frac{1}{2}\theta^2\bar{\theta}^2D$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$$

$$\frac{1}{g^2} \int d^4\theta \text{Tr} W_\alpha^2 + h.c.$$

$$\partial_\mu \gamma_j = \frac{2\pi}{g^2} \epsilon_{\mu\nu\rho} F^{\nu\rho}, F_{\mu\nu} = \frac{g^2}{4\pi} \epsilon_{\mu\nu\rho} \partial^\rho \gamma_j, j = 1, \dots, r$$

$$x_j = i(\tau\sigma_j + i\gamma_j)$$

$$\tau = i \frac{4\pi}{g^2} + \frac{\theta}{2\pi}$$

$$X_j = \epsilon^{\alpha\beta} \bar{D}_\alpha D_\beta V_j$$

$$\sum \sigma_j = \sum \gamma_j = 0$$

$$Y_j \sim \exp \left(\frac{\sigma_j - \sigma_{j+1}}{g^2} + i(\gamma_j - \gamma_{j-1}) \right), j = 1, \dots, N-1$$

$$Q = q + \theta\psi + \theta^2 F.$$

$$\int d^4\theta m \tilde{Q} Q$$



$$\mathcal{L} = \int d^4\theta \left(-\frac{1}{e^2} X^2 - \frac{k}{4\pi} XV - \frac{\zeta}{2\pi} V + Q^\dagger e^V Q \right)$$

$$\mathcal{L}_m = \theta(Q^\dagger e^{m_r \theta \bar{\theta}} Q + \bar{Q} e^{\bar{m}_r \theta \bar{\theta}} \bar{Q}^\dagger) \sim \int d^2(\theta + \bar{\theta}) (m_r Q^\dagger Q + \bar{m}_r \bar{Q} \bar{Q}^\dagger)$$

$$g^2 \int |\sigma Q|^2$$

$$k = \frac{1}{2} q_1 q_2 \text{sign}(m_Q)$$

$$k \int d^4\theta X^{(1)} V^{(2)}$$

$$\left\langle \frac{\text{tr} \lambda^2}{16\pi^2} \right\rangle = b_0^{-1} \Lambda \frac{\partial}{\partial \Lambda} \left\langle \frac{1}{2\pi R} \mathcal{W}_{4d} \right\rangle$$

$$\Lambda^3 = \mu^3 \frac{1}{g^2(\mu)} \exp \frac{2\pi i \tau(\mu)}{c_2}$$

$$\mathcal{L}_{uv} = \frac{1}{2g_{uv}^2} \int d^4\theta \text{Tr}(W^2)$$

$$\mathcal{L}_{eff} = \frac{b_0}{16\pi^2} \ln \frac{\mu}{\Lambda} \int d^4\theta \text{Tr}(W^2)$$

$$\frac{\partial}{\partial \mu} \mathcal{L}_{eff} \text{ or } \frac{\partial}{\partial \Lambda} \mathcal{L}_{eff}$$

$$\mathcal{W}_{4d} = \int_0^{2\pi R} dx " \mathcal{W}_{4d} " = 2\pi R \mathcal{W}_{4d}$$

$$\frac{b_0}{16\pi^2} \ln \frac{\mu}{\Lambda} \text{Tr}(W^2) = \mathcal{W}_{4d}$$

$$\frac{b_0}{16\pi^2} \text{Tr}(W^2) = -\Lambda \frac{\partial}{\partial \Lambda} \mathcal{W}_{4d} .$$

$$\frac{\text{Tr} \lambda \lambda}{16\pi^2} = \frac{1}{b_0} \cdot \frac{1}{2\pi R} \mathcal{W}_{4d}$$

$$z^2 \partial_z^2 \phi(z, \vec{x}) + (1 - \nu) z \partial_z \phi(z, \vec{x}) + z^2 \nabla^2 \phi(z, \vec{x}) - \Lambda(\Delta - \nu) \phi(z, \vec{x}) = 0$$

$$\hat{\phi}(z, \vec{k}) = \int \phi(z, \vec{x}) e^{i\vec{k} \cdot \vec{x}} d^{\nu} x$$

$$z^2 \partial_z^2 \hat{\phi}(z, \vec{k}) + (1 - \nu) z \partial_z \hat{\phi}(z, \vec{k}) - [z^2 k^2 + \Lambda(\Delta - \nu)] \hat{\phi}(z, \vec{k}) = 0$$

$$z^2 \partial_z^2 \hat{\phi}(z, \vec{k}) + (1 - \nu) z \partial_z \hat{\phi}(z, \vec{k}) - z^2 k^2 \hat{\phi}(z, \vec{k}) = 0$$

$$\partial_z^2 \hat{\phi}(z, \vec{k}) + \frac{(1 - \nu)}{z} \partial_z \hat{\phi}(z, \vec{k}) - k^2 \hat{\phi}(z, \vec{k}) = 0$$



$$F''(z) + \frac{(1-2\alpha)}{z} F'(z) - \left[k^2 + \frac{\mu^2 - \alpha^2}{z^2} \right] F(z) = 0$$

$$F(z) = z^\alpha \mathcal{K}_\mu(kz)$$

$$\hat{\phi}(z, k) = z^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(kz)$$

$$\mathcal{K}_{\frac{\nu}{2}}(kz) = \frac{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)}{(kz)^{\frac{\nu}{2}}} + O\left((kz)^{-\frac{\nu}{2}+2}\right)$$

$$\lim_{z \rightarrow 0} z^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(kz) = \frac{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)}{k^{\frac{\nu}{2}}}$$

$$\phi(z, \vec{x}) = \frac{z^{\frac{\nu}{2}}}{(2\pi)^\nu} \int a(\vec{k}) \mathcal{K}_{\frac{\nu}{2}}(kz) e^{-i\vec{k} \cdot \vec{x}} d^\nu k$$

$$\begin{aligned} \phi(0, \vec{x}) &= \phi_0(\vec{x}) = \frac{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)}{(2\pi)^\nu} \int a(\vec{k}) k^{-\frac{\nu}{2}} e^{-i\vec{k} \cdot \vec{x}} d^\nu k = \\ &= \frac{1}{(2\pi)^\nu} \int \hat{\phi}_0(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d^\nu k \end{aligned}$$

$$\phi(z, \vec{x}) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \int k^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(kz) \hat{\phi}_0(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d^\nu k$$

$$\phi(z, \vec{x}) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \iint k^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(kz) \phi_0(\vec{x}') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} d^\nu k d^\nu x'$$

$$K(z, \vec{x} - \vec{x}') = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \int k^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(kz) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} d^\nu k$$

$$\hat{f}(k) = \int f(\vec{x}) e^{i\vec{k} \cdot \vec{x}} d^\nu x = \frac{(2\pi)^{\frac{\nu}{2}}}{k^{\frac{\nu}{2}-1}} \int_0^\infty r^{\frac{\nu}{2}} J_{\frac{\nu}{2}-1}(kr) f(r) dr$$

$$f(r) = \frac{1}{(2\pi)^\nu} \int \hat{f}(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d^\nu k = \frac{1}{(2\pi)^{\frac{\nu}{2}} r^{\frac{\nu}{2}-1}} \int_0^\infty k^{\frac{\nu}{2}} J_{\frac{\nu}{2}-1}(kr) \hat{f}(k) dk$$

$$K(z, \vec{x} - \vec{x}') = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \frac{(2\pi)^{\frac{\nu}{2}}}{|\vec{x} - \vec{x}'|^{\frac{\nu}{2}-1}} \int_0^\infty k^\nu \mathcal{K}_{\frac{\nu}{2}}(kz) J_{\frac{\nu}{2}-1}(k|\vec{x} - \vec{x}'|) dk$$

$$\int_0^\infty x^{\mu+\nu+1} \mathcal{K}_\nu(bx) J_\mu(ax) dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu + \nu + 1)}{(a^2 + b^2)^{\mu+\nu+1}}$$



$$K(z, \vec{x} - \vec{x}') = \frac{\Gamma(\nu)}{\pi^{2\nu} \Gamma\left(\frac{\nu}{2}\right)} \left[\frac{z}{z^2 + (\vec{x} - \vec{x}')^2} \right]^\nu$$

$$\phi(z, \vec{x}) = \int K(z, \vec{x} - \vec{x}') \phi_0(\vec{x}') d^\nu x'$$

$$\lim_{z \rightarrow 0} K(z, \vec{x} - \vec{x}') = \delta(\vec{x} - \vec{x}')$$

$$z^2 \partial_z^2 \hat{\phi}(z, \vec{k}) + (1 - \nu) z \partial_z \hat{\phi}(z, \vec{k}) - [z^2 k^2 + \Delta(\Delta - \nu)] \hat{\phi}(z, \vec{k}) = 0$$

$$\partial_z^2 \hat{\phi}(z, \vec{k}) + \frac{(1 - \nu)}{z} \partial_z \hat{\phi}(z, \vec{k}) - \left[k^2 + \frac{\Delta(\Delta - \nu)}{z^2} \right] \hat{\phi}(z, \vec{k}) = 0$$

$$\hat{\phi}(z, k) = z^{\frac{\nu}{2}} \mathcal{K}_\mu(kz)$$

$$\mu = \pm \sqrt{\frac{\nu^2}{4} + \Delta(\Delta - \nu)}$$

$$\mathcal{K}_\mu(z) = \mathcal{K}_{-\mu}(z)$$

$$\phi(z, \vec{x}) = \frac{z^{\frac{\nu}{2}}}{(2\pi)^\nu} \int a(\vec{k}) \mathcal{K}_\mu(kz) e^{-i\vec{k} \cdot \vec{x}} d^\nu k$$

$$\begin{aligned} \phi(\epsilon, \vec{x}) = \phi_\epsilon(\vec{x}) &= \frac{\epsilon^{\frac{\nu}{2}}}{(2\pi)^\nu} \int a(\vec{k}) \mathcal{K}_\mu(k\epsilon) e^{-i\vec{k} \cdot \vec{x}} d^\nu k = \\ &= \frac{1}{(2\pi)^\nu} \int \hat{\phi}_\epsilon(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d^\nu k \end{aligned}$$

$$a(\vec{k}) = \frac{\hat{\phi}_\epsilon(\vec{k})}{\epsilon^{\frac{\nu}{2}} \mathcal{K}_\mu(k\epsilon)}$$

$$\phi(z, \vec{x}) = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_\mu(kz)}{\mathcal{K}_\mu(k\epsilon)} \hat{\phi}_\epsilon(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d^\nu k$$

$$\phi(z, \vec{x}) = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \iint \frac{\mathcal{K}_\mu(kz)}{\mathcal{K}_\mu(k\epsilon)} \phi_\epsilon(\vec{x}') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} d^\nu k d^\nu x'$$

$$K_m(z, \vec{x} - \vec{x}') = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_\mu(kz)}{\mathcal{K}_\mu(k\epsilon)} e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} d^\nu k$$

$$\phi(z, \vec{x}) = \int K_m(z, \vec{x} - \vec{x}') \phi_\epsilon(\vec{x}') d^\nu x'$$

$$K_m(\epsilon, \vec{x} - \vec{x}') = \delta(\vec{x} - \vec{x}')$$

$$\mathcal{K}_\mu(k\epsilon) = \frac{2^{\mu-1} \Gamma(\mu)}{(k\epsilon)^\mu} + O((k\epsilon)^{2-\mu})$$



$$\mathcal{K}_\mu(k\epsilon) = \frac{2^{\mu-1}\Gamma(\mu)}{(k\epsilon)^\mu}.$$

$$M_m(z, \vec{x} - \vec{x}') = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \frac{\epsilon^\mu}{2^{\mu-1}\Gamma(\mu)} \int k^\mu \mathcal{K}_\mu(kz) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} d^\nu k.$$

$$\int k^\mu \mathcal{K}_\mu(kz) e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} d^\nu k = \frac{(2\pi)^{\frac{\nu}{2}}}{|\vec{x} - \vec{x}'|^{\frac{\nu}{2}-1}} \int_0^\infty k^{\mu+\frac{\nu}{2}} \mathcal{K}_\mu(kz) \mathcal{J}_{\frac{\nu}{2}-1}(k|\vec{x} - \vec{x}'|) dk$$

$$M_m(z, \vec{x} - \vec{x}') = \frac{\epsilon^{\mu-\frac{\nu}{2}} \Gamma\left(\mu + \frac{\nu}{2}\right)}{\pi^{\frac{\nu}{2}} \Gamma(\mu)} \left[\frac{z}{z^2 + (\vec{x} - \vec{x}')^2} \right]^{\mu+\frac{\nu}{2}}.$$

$$\gamma = \frac{\nu}{2} + \mu = \frac{\nu}{2} + \sqrt{\frac{\nu^2}{4} + \Delta(\Delta - \nu)},$$

$$M_m(z, \vec{x} - \vec{x}') = \frac{\epsilon^{\gamma-\nu}}{\pi^{\frac{\nu}{2}}} \frac{\Gamma(\gamma)}{\Gamma\left(\gamma - \frac{\nu}{2}\right)} \left[\frac{z}{z^2 + (\vec{x} - \vec{x}')^2} \right]^\gamma.$$

$$M_m(\epsilon, \vec{x} - \vec{x}') \neq \delta(\vec{x} - \vec{x}')$$

$$N_m(z, \vec{x} - \vec{x}') = M_m(z, \vec{x} - \vec{x}') \epsilon^{\nu-\gamma}$$

$$N_m(z, \vec{x} - \vec{x}') = \frac{1}{\pi^{\frac{\nu}{2}}} \frac{\Gamma(\gamma)}{\Gamma\left(\gamma - \frac{\nu}{2}\right)} \left[\frac{z}{z^2 + (\vec{x} - \vec{x}')^2} \right]^\gamma$$

$$\lim_{z \rightarrow 0} N_m(z, \vec{x} - \vec{x}') \neq \delta(\vec{x} - \vec{x}')$$

$$z^2 \partial_z^2 \hat{\phi}(z, k) + (1 - \nu) z \partial_z \hat{\phi}(z, k) + z^2 k^2 \hat{\phi}(z, k) = 0$$

$$k^2 = k_0^2 - \vec{k}^2 = \rho$$

$$z^2 \partial_z^2 \hat{\phi}(z, \rho) + (1 - \nu) z \partial_z \hat{\phi}(z, \rho) + z^2 \rho \hat{\phi}(z, \rho) = 0$$

$$z^2 \partial_z^2 \hat{\phi}(z, \rho) + (1 - \nu) z \partial_z \hat{\phi}(z, \rho) - z^2 \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \right]^2 \hat{\phi}(z, \rho) = 0$$

$$(\rho \pm i0)^\lambda = \rho_+^\lambda + e^{\pm i\pi\lambda} \rho_-^\lambda$$

$$\partial_z^2 \hat{\phi}(z, \rho) + \frac{(1 - \nu)}{z} \partial_z \hat{\phi}(z, \rho) - \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \right]^2 \hat{\phi}(z, \rho) = 0$$

$$\hat{\phi}(z, k) = z^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right].$$

$$\lim_{k \rightarrow \infty} e^{ikx} = 0$$



$$\mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] = \frac{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)}{\left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}}} + O\left(\left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]^{-\frac{\nu}{2}+2}\right)$$

$$\phi_{\mp}(z, x) = \frac{z^{\frac{\nu}{2}}}{(2\pi)^{\nu}} \int a(\vec{k}) \mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] e^{-ik \cdot x} d^{\nu} k = \int \hat{\phi}(z, k) e^{ik \cdot x} d^{\nu} k$$

$$\phi_{\mp}(z, x) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^{\nu} \Gamma\left(\frac{\nu}{2}\right)} \int \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] \hat{\phi}_0(k) e^{-ik \cdot x} d^{\nu} k$$

$$\phi_{\mp}(z, x) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^{\nu} \Gamma\left(\frac{\nu}{2}\right)} \iint \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] \times \phi_0(x') e^{-ik \cdot (x-x')} d^{\nu} k d^{\nu} x'$$

$$K_{\mp}(z, x - x') = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^{\nu} \Gamma\left(\frac{\nu}{2}\right)} \int \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] e^{-i\vec{k} \cdot (x-x')} d^{\nu} k$$

$$K_F(z, x - x') = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^{\nu} \Gamma\left(\frac{\nu}{2}\right)} \int \left[-i(\rho + i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[-i(\rho + i0)^{\frac{1}{2}} z \right] e^{-i\vec{k} \cdot (x-x')} d^{\nu} k$$

$$K_{AF}(z, x - x') = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^{\nu} \Gamma\left(\frac{\nu}{2}\right)} \int \left[i(\rho - i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[i(\rho - i0)^{\frac{1}{2}} z \right] e^{-i\vec{k} \cdot (x-x')} d^{\nu} k$$

$$W(z, x - x') = \frac{1}{2} [K_F(z, x - x') + K_{AF}(z, x - x')].$$

$$K_F(z, \rho) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left[-i(\rho + i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[-i(\rho + i0)^{\frac{1}{2}} z \right]$$

$$K_{AF}(z, \rho) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left[i(\rho - i0)^{\frac{1}{2}} z \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[i(\rho - i0)^{\frac{1}{2}} z \right]$$

$$K_R(z, \rho) = H(k^0) K_F(z, \rho) + H(-k^0) K_{AF}(z, \rho)$$

$$K_A(z, \rho) = H(k^0) K_{AF}(z, \rho) + H(-k^0) K_F(z, \rho)$$

$$\phi(k) = \int_{-\infty}^{\infty} \phi(\hat{x}) e^{ikx} dx$$

$$0 = \lim_{k \rightarrow \infty} \phi(k) = \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} \phi(\hat{x}) e^{ikx} dx = \int_{-\infty}^{\infty} \phi(\hat{x}) \lim_{k \rightarrow \infty} e^{ikx} dx$$



$$\lim_{k \rightarrow \infty} e^{ikx} = 0$$

$$K_F(z, x) = \frac{z^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \int \left[-i(\rho + i0)^{\frac{1}{2}} \right]^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}} \left[-i(\rho + i0)^{\frac{1}{2}} z \right] e^{-ik \cdot x} d^\nu k$$

$$k_0 = ik_{0E}, x_0 = ix_{0E}, k_E^2 = k_{0E}^2 + \vec{k}^2, \text{ and } x_E^2 = x_{0E}^2 + \vec{x}^2$$

$$K_F(z, \vec{x}_E) = \frac{iz^{\frac{\nu}{2}} 2^{1-\frac{\nu}{2}}}{(2\pi)^\nu \Gamma\left(\frac{\nu}{2}\right)} \int k_E^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2}}(k_E z) e^{-i\vec{k}_E \cdot \vec{x}_E} d^\nu k_E$$

$$K_F(z, x_E) = \frac{i\Gamma(\nu)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[\frac{z}{z^2 + x_E^2} \right]^\nu$$

$$K_F(z, x) = \frac{i\Gamma(\nu)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[\frac{z}{z^2 - x^2 - i0} \right]^\nu$$

$$K_{AF}(z, x) = \frac{i\Gamma(\nu)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left[\frac{z}{z^2 - x^2 + i0} \right]^\nu.$$

$$\partial_z^2 \hat{\phi}(z, \rho) + \frac{(1-\nu)}{z} \partial_z \hat{\phi}(z, \rho) - \left\{ \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \right]^2 + \frac{\Delta(\Delta-\nu)}{z^2} \right\} \hat{\phi}(z, \rho) = 0$$

$$\mu = \sqrt{\frac{\nu^2}{4} + \Delta(\Delta-\nu)}$$

$$\hat{\phi}_\mp(z, \rho) = z^{\frac{\nu}{2}} \mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]$$

$$\phi_\mp(z, x) = \frac{z^{\frac{\nu}{2}}}{(2\pi)^\nu} \int a(\vec{k}) \mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right] e^{-ik \cdot x} d^\nu k$$

$$\begin{aligned} \phi(\epsilon, x) = \phi_\epsilon(x) &= \frac{\epsilon^{\frac{\nu}{2}}}{(2\pi)^\nu} \int a(\vec{k}) \mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \epsilon \right] e^{-ik \cdot x} d^\nu k = \\ &= \frac{1}{(2\pi)^\nu} \int \hat{\phi}_\epsilon(k) e^{-ik \cdot x} d^\nu k \end{aligned}$$

$$a(k) = \frac{\hat{\phi}_\epsilon(k)}{\epsilon^{\frac{\nu}{2}} \mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \epsilon \right]}$$

$$\phi_\mp(z, x) = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon} \right)^{\frac{\nu}{2}} \iint \frac{\mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \epsilon \right]} \phi_\epsilon(x') e^{-ik \cdot (x-x')} d^\nu k d^\nu x'$$



$$K_{m\mp}(z, x - x') = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[\mp i(\rho \pm i0)^{\frac{1}{2}} \epsilon \right]} e^{-i\vec{k} \cdot (x-x')} d^\nu k$$

$$K_{mF}(z, x - x') = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_\mu \left[-i(\rho + i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[-i(\rho + i0)^{\frac{1}{2}} \epsilon \right]} e^{-i\vec{k} \cdot (x-x')} d^\nu k$$

$$K_{mAF}(z, x - x') = \frac{1}{(2\pi)^\nu} \left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_\mu \left[i(\rho - i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[i(\rho - i0)^{\frac{1}{2}} \epsilon \right]} e^{-i\vec{k} \cdot (x-x')} d^\nu k$$

$$\mathcal{K}_\mu \left[-i(\rho + i0)^{\frac{1}{2}} \epsilon \right] = \frac{2^{\mu-1} \Gamma(\mu)}{(-i)^\mu (\rho + i0)^{\frac{\mu}{2}} \epsilon^\mu},$$

$$M_{mF}(z, x) = \frac{z^{\frac{\nu}{2}}}{(2\pi)^\nu} \frac{\epsilon^{\mu-\frac{\nu}{2}}}{2^{\mu-1} \Gamma(\mu)} \int \mathcal{K}_\mu \left[-i(\rho + i0)^{\frac{1}{2}} z \right] \left[-i(\rho + i0)^{\frac{1}{2}} \right]^\mu e^{-ik \cdot x} d^\nu k$$

$$M_{mF}(z, \vec{x}_E) = \frac{iz^{\frac{\nu}{2}}}{(2\pi)^\nu} \frac{\epsilon^{\mu-\frac{\nu}{2}}}{2^{\mu-1} \Gamma(\mu)} \int k_E^\mu \mathcal{K}_\mu(k_E z) e^{-i\vec{k}_E \cdot \vec{x}_E} d^\nu k_E$$

$$M_{mF}(z, \vec{x}_E) = \frac{i\epsilon^{\mu-\frac{\nu}{2}} \Gamma\left(\mu - \frac{\nu}{2}\right)}{\pi^{\frac{\nu}{2}} \Gamma(\mu)} \left(\frac{z}{z^2 + x_E^2}\right)^{\mu+\frac{\nu}{2}}$$

$$M_{mF}(z, x) = \frac{i\epsilon^{\gamma-\nu}}{\pi^{\frac{\nu}{2}}} \frac{\Gamma(\gamma)}{\Gamma\left(\gamma - \frac{\nu}{2}\right)} \left(\frac{z}{z^2 - x^2 - i0}\right)^\gamma$$

$$\gamma = \frac{\nu}{2} + \sqrt{\frac{\nu^2}{4} + \Delta(\Delta - \nu)}.$$

$$M_{mF}(\epsilon, x) \neq \delta(x)$$

$$N_{mF}(\epsilon, x) = \epsilon^{\nu-\gamma} M_{mF}(\epsilon, x)$$

$$M_{mAF}(z, x) = \frac{i\epsilon^{\gamma-\nu}}{\pi^{\frac{\nu}{2}}} \frac{\Gamma(\gamma)}{\Gamma\left(\gamma - \frac{\nu}{2}\right)} \left(\frac{z}{z^2 - x^2 + i0}\right)^\gamma$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = - \int \sqrt{g} \partial_\mu K(y_0, \vec{y} - \vec{x}_1) \partial^\mu K(y_0, \vec{y} - \vec{x}_2) d^{\nu+1} y$$

$$0 \leq y_0 = z < \infty, y_\mu = x_\mu, \mu \neq 0$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = - \int_{\text{Boundary}} \lim_{z \rightarrow 0} [z^{1-\nu} K(z, \vec{x} - \vec{x}_1) \partial_z K(z, \vec{x} - \vec{x}_2)] d^\nu x$$



$$\lim_{z \rightarrow 0} K(0, \vec{x}_1 - \vec{x}_2) = \delta(\vec{x}_1 - \vec{x}_2)$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = -\lim_{z \rightarrow 0} [z^{1-\nu} \partial_z K(z, \vec{x}_1 - \vec{x}_2)]$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = -\frac{\Gamma(\nu + 1)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \frac{1}{(\vec{x}_1 - \vec{x}_2)^{2\nu}}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_m = -\int_{\text{Boundary}} [z^{1-\nu} K_m(z, \vec{x} - \vec{x}_1) \partial_z K(z, \vec{x} - \vec{x}_2)] d^{\nu} x$$

$$K_m(\epsilon, \vec{x} - \vec{x}_1) = \delta(\vec{x} - \vec{x}_1)$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_m = -\int \delta(\vec{x} - \vec{x}_1) [z^{1-\nu} \partial_z K(z, \vec{x} - \vec{x}_2)]_{z=\epsilon} d^{\nu} x$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_m = -[z^{1-\nu} \partial_z K_m(z, \vec{x}_1 - \vec{x}_2)]_{z=\epsilon}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_m = -\frac{\epsilon^{1-\nu}}{(2\pi)^{\nu}} \partial_z \left[\left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_{\mu}(kz)}{\mathcal{K}_{\mu}(k\epsilon)} e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} d^{\nu} k \right]_{z=\epsilon}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_m = -\frac{\epsilon^{1-\frac{3\nu}{2}}}{(2\pi)^{\nu}} \left[\frac{\nu}{2} z^{\frac{\nu}{2}-1} \int \frac{\mathcal{K}_{\mu}(kz)}{\mathcal{K}_{\mu}(k\epsilon)} e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} d^{\nu} k + z^{\frac{\nu}{2}} \int k \frac{\mathcal{K}'_{\mu}(kz)}{\mathcal{K}_{\mu}(k\epsilon)} e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} d^{\nu} k \right]_{z=\epsilon}$$

$$\mathcal{K}'_{\mu}(z) = -\frac{\mu}{z} \mathcal{K}_{\mu} + \mathcal{K}_{\mu-1}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = \epsilon^{-\nu} \left(\mu - \frac{\nu}{2} \right) \delta(\vec{x}_1 - \vec{x}_2) - \frac{\epsilon^{1-\nu}}{(2\pi)^{\nu}} \int k \frac{\mathcal{K}_{\mu-1}(k\epsilon)}{\mathcal{K}_{\mu}(k\epsilon)} e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} d^{\nu} k$$

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_F = i \lim_{z \rightarrow 0} [z^{1-\nu} \partial_z K_F(z, x_1 - x_2)]$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_F = -\frac{\Gamma(\nu + 1)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \frac{1}{[(\vec{x}_1 - \vec{x}_2)^2 - (x_{10} - x_{20})^2 + i0]^{\nu}}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_{AF} = -\frac{\Gamma(\nu + 1)}{\pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \frac{1}{[(\vec{x}_1 - \vec{x}_2)^2 - (x_{10} - x_{20})^2 - i0]^{\nu}}$$

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle_{mF} = i [z^{1-\nu} \partial_z K_{mF}(z, x_1 - x_2)]_{z=\epsilon}$$

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_{mF} = i \frac{\epsilon^{1-\nu}}{(2\pi)^{\nu}} \partial_z \left[\left(\frac{z}{\epsilon}\right)^{\frac{\nu}{2}} \int \frac{\mathcal{K}_{\mu} \left[-i(\rho + i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_{\mu} \left[-i(\rho + i0)^{\frac{1}{2}} \epsilon \right]} e^{-i\vec{k} \cdot (x_1 - x_2)} d^{\nu} k \right]_{z=\epsilon}$$



$$\begin{aligned}
\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle_{mF} &= i \frac{\epsilon^{1-\frac{3\nu}{2}}}{(2\pi)^\nu} \left[\frac{\nu}{2} \frac{\nu-1}{z^2} \int \frac{\mathcal{K}_\mu \left[-i(\rho+i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[-i(\rho+i0)^{\frac{1}{2}} \epsilon \right]} e^{-ik \cdot (x_1-x_2)} d^\nu k - \right. \\
&\quad \left. i z^{\frac{\nu}{2}} \int (\rho+i0)^{\frac{1}{2}} \frac{\mathcal{K}'_\mu \left[-i(\rho+i0)^{\frac{1}{2}} z \right]}{\mathcal{K}_\mu \left[-i(\rho-i0)^{\frac{1}{2}} \epsilon \right]} e^{-ik \cdot (x_1-x_2)} d^\nu k \right]_{z=\epsilon} \\
\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle_F &= \epsilon^{-\nu} \left(\mu - \frac{\nu}{2} \right) \delta(x_1 - x_2) + \\
&\quad \frac{\epsilon^{1-\nu}}{(2\pi)^\nu} \int (\rho+i0)^{\frac{1}{2}} \frac{\mathcal{K}_{\mu-1} \left[-i(\rho+i0)^{\frac{1}{2}} \epsilon \right]}{\mathcal{K}_\mu \left[-i(\rho+i0)^{\frac{1}{2}} \epsilon \right]} e^{-ik \cdot (x_1-x_2)} d^\nu k \\
\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle_{mAF} &= \epsilon^{-\nu} \left(\mu - \frac{\nu}{2} \right) \delta(x_1 - x_2) - \frac{\epsilon^{1-\nu}}{(2\pi)^\nu} \int (\rho-i0)^{\frac{1}{2}} \frac{\mathcal{K}_{\mu-1} \left[i(\rho-i0)^{\frac{1}{2}} \epsilon \right]}{\mathcal{K}_\mu \left[i(\rho-i0)^{\frac{1}{2}} \epsilon \right]} e^{-ik \cdot (x_1-x_2)} d^\nu k
\end{aligned}$$

$$\varphi(t, \mathbf{x}_1, \mathbf{x}_2) = \psi(t, \mathbf{x}_1, t, \mathbf{x}_2).$$

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \lambda \int dV(x'_1) \int dV(x'_2) G_1(x_1 - x'_1) G_2(x_2 - x'_2) K(x'_1, x'_2) \psi(x'_1, x'_2)$$

$$K(x_1, x_2) \propto \delta((x_1 - x_2)^2) \text{ where } (x_1 - x_2)^2$$

$$\varphi(0, \mathbf{x}_1, 0, \mathbf{x}_2) = \varphi_0(\mathbf{x}_1, \mathbf{x}_2)$$

$$i\partial_t \varphi(t, \mathbf{x}_1, \mathbf{x}_2) = \left(H_1^{\text{free}} + H_2^{\text{free}} + \lambda V(t, \mathbf{x}_1, \mathbf{x}_2) \right) \varphi(t, \mathbf{x}_1, \mathbf{x}_2)$$

$$\begin{aligned}
\varphi(t, \mathbf{x}_1, \mathbf{x}_2) &= \varphi^{\text{free}}(t, \mathbf{x}_1, \mathbf{x}_2) + \lambda \int_0^\infty dt' \int d^3 \mathbf{x}'_1 d^3 \mathbf{x}'_2 G_1(t - t', \mathbf{x}_1 - \mathbf{x}'_1) \\
&\quad \times G_2(t - t', \mathbf{x}_2 - \mathbf{x}'_2) V(t', \mathbf{x}'_1, \mathbf{x}'_2) \varphi(t', \mathbf{x}'_1, \mathbf{x}'_2)
\end{aligned}$$

$$\varphi^{\text{free}}(t, \mathbf{x}_1, \mathbf{x}_2) = \left(e^{-i(H_1^{\text{free}} + H_2^{\text{free}})t} \varphi_0 \right)_{(i\partial_t - H_k^{\text{free}})}(\mathbf{x}_1, \mathbf{x}_2)$$

$$G_i^{\text{sym}} = \frac{1}{2} (G_i^{\text{ret}} + G_i^{\text{adv}})$$

$$\int_0^{t_1} dt'_1 \int d^3 \mathbf{x}'_1 \int_0^{t_2} dt'_2 \int d^3 \mathbf{x}'_2 \kappa(x_1, x_2, x'_1, x'_2) \psi(x'_1, x'_2)$$

$$(\square_k + m_k) \psi^{\text{free}} = 0, k = 1, 2$$

$$K(x_1, x_2) = \frac{1}{2\pi} \delta((x_1 - x_2)^2) = G^{\text{sym}}(x_1, x_2)$$



$$K(x_1, x_2) = K(x_2, x_1)$$

d	G^{sym}
1	$\frac{1}{2}H(x^2)J_0(m\sqrt{x^2})$
2	$\frac{1}{2\pi}H(x^2)\frac{\cos(m\sqrt{x^2})}{\sqrt{x^2}}$
3	$\frac{1}{2\pi}\delta(x^2) - \frac{m}{4\pi\sqrt{x^2}}H(x^2)J_1(m\sqrt{x^2})$

$$G^{\text{ret}}(x) = H(x^0)G^{\text{sym}}$$

$$\begin{aligned} \psi(t_1, z_1, t_2, z_2) &= \psi^{\text{free}}(t_1, z_1, t_2, z_2) + \frac{\lambda}{4} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'_2 \int dz'_1 dz'_2 H(t_1 - t'_1 - |z_1 - z'_1|) \\ &\times J_0\left(m_1 \sqrt{(t_1 - t'_1)^2 - |z_1 - z'_1|^2}\right) H(t_2 - t'_2 - |z_2 - z'_2|) J_0\left(m_2 \sqrt{(t_2 - t'_2)^2 - |z_2 - z'_2|^2}\right) \\ &\times K(t'_1, z'_1, t'_2, z'_2) \psi(t'_1, z'_1, t'_2, z'_2) \end{aligned}$$

$$K(t_1, z_1, t_2, z_2) = \frac{1}{2}H((t_1 - t_2)^2 - |z_1 - z_2|^2)$$

$$\begin{aligned} \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= \psi^{\text{free}}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) + \frac{\lambda}{(2\pi)^2} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'_2 \int d^4\mathbf{x}'_1 d^2\mathbf{x}'_2 \\ &\times H(t_1 - t'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|) \frac{\cos(m_1 \sqrt{(t_1 - t'_1)^2 - |\mathbf{x}_1 - \mathbf{x}'_1|^2})}{\sqrt{(t_1 - t'_1)^2 - |\mathbf{x}_1 - \mathbf{x}'_1|^2}} H(t_2 - t'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|) \\ &\times \frac{\cos(m_2 \sqrt{(t_2 - t'_2)^2 - |\mathbf{x}_2 - \mathbf{x}'_2|^2})}{\sqrt{(t_2 - t'_2)^2 - |\mathbf{x}_2 - \mathbf{x}'_2|^2}} K(t'_1, \mathbf{x}'_1, t'_2, \mathbf{x}'_2) \psi(t'_1, \mathbf{x}'_1, t'_2, \mathbf{x}'_2) \end{aligned}$$

$$\begin{aligned} \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= \psi^{\text{free}}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^2} \int_0^{t_1} dt'_1 \int d^4\mathbf{x}'_1 \int_0^{t_2} dt'_2 \int d^4\mathbf{x}'_2 \\ &\times \frac{\delta(t_1 - t'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\delta(t_2 - t'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{|\mathbf{x}_2 - \mathbf{x}'_2|} K(t'_1, \mathbf{x}'_1, t'_2, \mathbf{x}'_2) \psi(t'_1, \mathbf{x}'_1, t'_2, \mathbf{x}'_2) \end{aligned}$$

$$\begin{aligned} \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= \psi^{\text{free}}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^2} \int d^4\mathbf{x}'_1 d^4\mathbf{x}'_2 \frac{H(t_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{H(t_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{|\mathbf{x}_2 - \mathbf{x}'_2|} \\ &\times K(t_1 - |\mathbf{x}_1 - \mathbf{x}'_1|, \mathbf{x}'_1, t_2 - |\mathbf{x}_2 - \mathbf{x}'_2|, \mathbf{x}'_2) \psi(t_1 - |\mathbf{x}_1 - \mathbf{x}'_1|, \mathbf{x}'_1, t_2 - |\mathbf{x}_2 - \mathbf{x}'_2|, \mathbf{x}'_2) \end{aligned}$$

$$\mathcal{B}_d := L^\infty\left([0, T]_{(t_1, t_2)}^2, L^2(\mathbb{R}_{(\mathbf{x}_1, \mathbf{x}_2)}^{4d})\right)$$

$$\|\psi\|_{\mathcal{B}_d} = \text{ess sup}_{t_1, t_2 \in [0, T]} \|\psi(t_1, \square, t_2, \square)\|_{L^2}$$



$$K(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) = \frac{f(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|},$$

$$\begin{aligned} K(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= \delta((t_1 - t_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2) \\ &= \frac{1}{2|\mathbf{x}_1 - \mathbf{x}_2|} [\delta(t_1 - t_2 - |\mathbf{x}_1 - \mathbf{x}_2|) + \delta(t_1 - t_2 + |\mathbf{x}_1 - \mathbf{x}_2|)] \end{aligned}$$

$$\mathcal{B}_d = L^\infty \left([0, T]_{(t_1, t_2)}^2, L^2(\mathbb{R}_{(\mathbf{x}_1, \mathbf{x}_2)}^{4d}) \right)$$

$$\|\psi(t_1, \square, t_2, \square)\|_{L^2} \rightarrow \pm \psi^{\text{free}}$$

$$(\square_g + m^2 - \xi R)\psi = 0$$

$$\square_g = \nabla_\mu \nabla^\mu$$

$$\xi = (d - 1)/(4d) = \frac{1}{6} \text{ for } d = 4$$

$$(\square_{g,k} + m_k^2 - \xi R)\psi^{\text{free}}(x_1, x_2) = 0, k = 1, 2.$$

$$dV(x_i) = \sqrt{-g(x)} d^{1+d}x$$

$$(\square_g + m^2 - \xi R(x))G(x, x') = [-g(x)]^{-1/2} \delta^{(1+d)}(x, x'),$$

$$\int d^{1+d}x' \delta^{(1+d)}(x, x') f(x') = f(x)$$

$$ds^2 = a^2(\eta) [d\eta^2 - dr^2 - s_k^2(r) d\Omega_{d-1}^2]$$

$$s_k^2(r) = \begin{cases} \sinh^2(r), & \text{open } (k = -1) \\ r^2, & \text{curve } (k = 4) \\ \sin^2(r), & \text{closed } (k = 1) \end{cases}$$

$$\tilde{g}_{ab} = \Omega^2 g_{ab}$$

$$\iiint \langle \square_g - \xi R | \mathcal{M}, g_{ab} \rangle \iiint \langle \square_{\tilde{g}} - \xi \tilde{R} | \tilde{\mathcal{M}}, \tilde{g}_{ab} \rangle$$

$$(\square_g - \xi R) = \Omega^{\frac{d+3}{2}} (\square_{\tilde{g}} - \xi \tilde{R}) \Omega^{-\frac{d-1}{2}}$$

$$(\square_g - \xi R)\phi = 0, \text{ then } \tilde{\phi} = \Omega^{\frac{d-1}{2}} \phi \text{ fulfills } (\square_{\tilde{g}} - \xi \tilde{R})\tilde{\phi} = 0$$

$$\tilde{G}(x, x') = \Omega^{-\frac{d-1}{2}}(x) \Omega^{\frac{d-1}{2}}(x') G(x, x')$$

$$(\square_{\tilde{g}} - \xi \tilde{R})\tilde{G}(x, x') = [-\tilde{g}(x)]^{-1/2} \delta^{(1+d)}(x, x')$$

$$G^{\text{ret}}(\eta, \mathbf{x}, \eta', \mathbf{x}') = H(\eta - \eta') G^{\text{sym}}(\eta, \mathbf{x}, \eta', \mathbf{x}')$$



$$x^2 = \eta^2 - |\mathbf{x}|^2$$

$$ds^2 = d\eta^2 - dr^2 - \sinh^2 r d\Omega_2^2$$

d	$G^{\text{sym}}(x, x')$
1	$\frac{1}{2}H((x - x')^2)$
2	$\frac{1}{2\pi} \frac{1}{[a(\eta)a(\eta')]^{1/2}} \frac{H((x - x')^2)}{\sqrt{(x - x')^2}}$
3	$\frac{1}{2\pi} \frac{1}{a(\eta)a(\eta')} \delta((x - x')^2)$

$$G^{\text{ret/adv}}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \frac{1}{a(\eta)a(\eta')} \frac{\delta(\eta - \eta' \mp s(\mathbf{x}, \mathbf{x}'))}{4\pi \sinh(s(\mathbf{x}, \mathbf{x}'))},$$

$$ds^2 = d\eta^2 - r_0^2(dr^2 + \sin^2 r d\Omega_2^2).$$

$$G_F(x, x') = \frac{i}{4\pi^2 r_0} \sum_{n=-\infty}^{\infty} \frac{s(q, q') + 2\pi n r_0}{\sin(s(q, q')/r_0)} \frac{1}{(\eta - \eta')^2 - (s(q, q') + 2\pi n r_0)^2 - i\varepsilon}$$

$$x \in \mathbb{R} \times r_0 S^3 \text{ as } x = (\eta, q) \text{ with } q \in r_0 S^3; s(q, q')$$

$$\frac{1}{x - i\varepsilon} = \text{P} \frac{1}{x} + i\pi \delta(x),$$

$$G^{\text{sym}}(x, x') = \text{Re} G_F(x, x').$$

$$G^{\text{sym}}(x, x') = -\frac{1}{4\pi r_0} \sum_{n=-\infty}^{\infty} \frac{s + 2\pi n r_0}{\sin(s/r_0)} \delta((\eta - \eta')^2 - (s + 2\pi n r_0)^2)$$

$$G_{\text{FLRW}, k=1}^{\text{sym}}(x, x') = -\frac{1}{4\pi} \frac{1}{a(\eta)a(\eta')} \sum_{n=-\infty}^{\infty} \frac{s(q, q') + 2\pi n}{\sin(s(q, q'))} \delta((\eta - \eta')^2 - (s(q, q') + 2\pi n)^2).$$

$$dV(x) = a^{1+d}(\eta) d\eta d^d \mathbf{x}$$



$$\begin{aligned}
\psi(\eta_1, z_1, \eta_2, z_2) &= \psi^{\text{free}}(\eta_1, z_1, \eta_2, z_2) \\
&+ \frac{\lambda}{8} \int_0^{\eta_1} d\eta'_1 \int_0^{\eta_2} d\eta'_2 \int dz'_1 dz'_2 a^2(\eta'_1) a^2(\eta'_2) \\
&\times H(\eta_1 - \eta'_1 - |z_1 - z'_1|) H(\eta_2 - \eta'_2 - |z_2 - z'_2|) H((\eta'_1 - \eta'_2)^2 \\
&- |z'_1 - z'_2|^2) \psi(\eta'_1, z'_1, \eta'_2, z'_2) \\
K(\eta_1, z_1, \eta_2, z_2) &= \frac{1}{2} H((\eta_1 - \eta_2)^2 - |z_1 - z_2|^2) \\
\psi(\eta_1, z_1, \eta_2, z_2) &= \psi^{\text{free}}(\eta_1, z_1, \eta_2, z_2) + \frac{\lambda}{4} \int_0^{\eta_1} d\eta'_1 \int_0^{\eta_2} d\eta'_2 \int dz'_1 dz'_2 a^2(\eta'_1) a^2(\eta'_2) \\
&\times H(\eta_1 - \eta'_1 - |z_1 - z'_1|) H(\eta_2 - \eta'_2 - |z_2 - z'_2|) \tilde{K}(\eta'_1, z'_1, \eta'_2, z'_2) \psi(\eta'_1, z'_1, \eta'_2, z'_2) \\
\psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(2\pi)^3} \frac{1}{[a(\eta_1)a(\eta_2)]^{1/2}} \int_0^{\eta_1} d\eta'_1 \int_0^{\eta_2} d\eta'_2 \int d^4\mathbf{x}'_1 d^2\mathbf{x}'_2 \\
&\times a^2(\eta'_1) a^2(\eta'_2) \frac{H(\eta_1 - \eta'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{\sqrt{(\eta_1 - \eta'_1)^2 - |\mathbf{x}_1 - \mathbf{x}'_1|^2}} \frac{H(\eta_2 - \eta'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{\sqrt{(\eta_2 - \eta'_2)^2 - |\mathbf{x}_2 - \mathbf{x}'_2|^2}} \\
&\times \frac{H((\eta'_1 - \eta'_2)^2 - |\mathbf{x}'_1 - \mathbf{x}'_2|^2)}{\sqrt{(\eta'_1 - \eta'_2)^2 - |\mathbf{x}'_1 - \mathbf{x}'_2|^2}} \psi(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2) \\
K(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \frac{1}{2\pi} \frac{1}{[a(\eta_1)a(\eta_2)]^{1/2}} \frac{H((\eta_1 - \eta_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2)}{\sqrt{(\eta_1 - \eta_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2}} \\
&a^{-1/2}(\eta_1) a^{-1/2}(\eta_2) \tilde{K}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) \\
\psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(2\pi)^2} \frac{1}{[a(\eta_1)a(\eta_2)]^{1/2}} \int_0^{\eta_1} d\eta'_1 \int_0^{\eta_2} d\eta'_2 \int d^4\mathbf{x}'_1 d^2\mathbf{x}'_2 \\
&\times a^2(\eta'_1) a^2(\eta'_2) \frac{H(\eta_1 - \eta'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{\sqrt{(\eta_1 - \eta'_1)^2 - |\mathbf{x}_1 - \mathbf{x}'_1|^2}} \frac{H(\eta_2 - \eta'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{\sqrt{(\eta_2 - \eta'_2)^2 - |\mathbf{x}_2 - \mathbf{x}'_2|^2}} \\
&\times \tilde{K}(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2) \psi(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2) \\
\psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{2\lambda}{(4\pi)^3} \frac{1}{a(\eta_1)a(\eta_2)} \int_0^{\eta_1} d\eta'_1 \int d^4\mathbf{x}'_1 \int_0^{\eta_2} d\eta'_2 \int d^4\mathbf{x}'_2 \\
&\times a^2(\eta'_1) a^2(\eta'_2) \frac{\delta(\eta_1 - \eta'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\delta(\eta_2 - \eta'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{|\mathbf{x}_2 - \mathbf{x}'_2|} \\
&\times \delta((\eta'_1 - \eta'_2)^2 - |\mathbf{x}'_1 - \mathbf{x}'_2|^2) \psi(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2) \\
K(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \frac{1}{2\pi} a^{-1}(\eta_1) a^{-1}(\eta_2) \delta((\eta_1 - \eta_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2) \\
&a^{-1}(\eta_1) a^{-1}(\eta_2) \tilde{K}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2)
\end{aligned}$$

$$\begin{aligned} \psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^2} \frac{1}{a(\eta_1)a(\eta_2)} \int d^4\mathbf{x}'_1 d^4\mathbf{x}'_2 \\ &\times a^2(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|) a^2(\eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|) \frac{H(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{H(\eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{|\mathbf{x}_2 - \mathbf{x}'_2|} \\ &\times \tilde{K}(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|, \mathbf{x}'_1, \eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|, \mathbf{x}'_2) \psi(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|, \mathbf{x}'_1, \eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|, \mathbf{x}'_2) \\ K(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \frac{a^{-1}(\eta_1)a^{-1}(\eta_2)}{4\pi \sinh(s(\mathbf{x}_1, \mathbf{x}_2))} [\delta(\eta_1 - \eta_2 - s(\mathbf{x}_1, \mathbf{x}_2)) + \delta(\eta_1 - \eta_2 + s(\mathbf{x}_1, \mathbf{x}_2))] \end{aligned}$$

$$a^{-1}(\eta_1)a^{-1}(\eta_2)\tilde{K}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2)$$

$$\begin{aligned} \psi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \psi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^2} \frac{1}{a(\eta_1)a(\eta_2)} \int d^4\mathbf{x}'_1 \int d^4\mathbf{x}'_2 \\ &\times a^2(\eta'_1) a^2(\eta'_2) \frac{H(\eta_1 - \eta'_1 - s(\mathbf{x}_1, \mathbf{x}'_1))}{\sinh(s(\mathbf{x}_1, \mathbf{x}'_1))} \frac{H(\eta_2 - \eta'_2 - s(\mathbf{x}_2, \mathbf{x}'_2))}{\sinh(s(\mathbf{x}_2, \mathbf{x}'_2))} \\ &\times \tilde{K}(\eta_1 - s(\mathbf{x}_1, \mathbf{x}'_1), \mathbf{x}'_1, \eta_2 - s(\mathbf{x}_2, \mathbf{x}'_2), \mathbf{x}'_2) \psi(\eta_1 - s(\mathbf{x}_1, \mathbf{x}'_1), \mathbf{x}'_1, \eta_2 - s(\mathbf{x}_2, \mathbf{x}'_2), \mathbf{x}'_2) \end{aligned}$$

$$\begin{aligned} \psi(\eta_1, q_1, \eta_2, q_2) &= \psi^{\text{free}}(\eta_1, q_1, \eta_2, q_2) - \frac{\lambda}{(4\pi)^3} \frac{1}{a(\eta_1)a(\eta_2)} \int_0^T d\eta'_1 \int_0^T d\eta'_2 \int d\Omega_3(q'_1) d\Omega_3(q'_2) \\ &\times a^2(\eta'_1) a^2(\eta'_2) \sum_{l,m,n=-\infty}^{\infty} \frac{s(q_1, q'_1) + 2\pi l}{\sin(s(q_1, q'_1))} \delta((\eta_1 - \eta'_1)^2 - (s(q_1, q'_1) + 2\pi l)^2) \\ &\times \frac{s(q_2, q'_2) + 2\pi m}{\sin(s(q_2, q'_2))} \delta((\eta_2 - \eta'_2)^2 - (s(q_2, q'_2) + 2\pi m)^2) \\ &\times \frac{s(q'_1, q'_2) + 2\pi n}{\sin(s(q'_1, q'_2))} \delta((\eta'_1 - \eta'_2)^2 - (s(q'_1, q'_2) + 2\pi n)^2) \psi(\eta'_1, q'_1, \eta'_2, q'_2) \end{aligned}$$

$$K(\eta_1, q_1, \eta_2, q_2) = \frac{-(4\pi)^{-1}}{a(\eta_1)a(\eta_2)} \sum_n \frac{s(q_1, q_2) + 2\pi n}{\sin(s(q_1, q_2))} \delta((\eta_1 - \eta_2)^2 - (s(q_1, q_2) + 2\pi n)^2)$$

$$a^{-1}(\eta_1)a^{-1}(\eta_2)\tilde{K}(\eta_1, q_1, \eta_2, q_2)$$

$$\begin{aligned} &\delta((\eta_i - \eta'_i)^2 - (s(q_i, q'_i) + 2\pi n)^2) \\ &= \frac{1}{2|s(q_i, q'_i) + 2\pi n|} [\delta(\eta_i - \eta'_i - |s(q_i, q'_i) + 2\pi n|) + \delta(\eta_i - \eta'_i + |s(q_i, q'_i) + 2\pi n|)] \end{aligned}$$

$$\begin{aligned} \psi(\eta_1, q_1, \eta_2, q_2) &= \psi^{\text{free}}(\eta_1, q_1, \eta_2, q_2) + \frac{\lambda}{4(4\pi)^2} \frac{1}{a(\eta_1)a(\eta_2)} \sum_{l,m=-\infty}^{\infty} \int d\Omega_3(q'_1) d\Omega_3(q'_2) \\ &\times \frac{\text{sgn}(s(q_1, q'_1) + 2\pi l)}{\sin(s(q_1, q'_1))} \frac{\text{sgn}(s(q_2, q'_2) + 2\pi m)}{\sin(s(q_2, q'_2))} \\ &\times \sum_{\sigma_1, \sigma_2 = \pm 1} [\mathbb{1}_{[0, T]}(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|) \mathbb{1}_{[0, T]}(\eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|) \\ &\times a^2(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|) a^2(\eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|) \\ &\times (\tilde{K} \times \psi)(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|, q'_1, \eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|, q'_2)] \end{aligned}$$

$$\tilde{\phi} = a^{-\frac{d-1}{2}}(\eta)\phi$$



$$a^{-\frac{d-1}{2}}(\eta_1)a^{-\frac{d-1}{2}}(\eta_2) \text{ for } \eta_1, \eta_2 \rightarrow 0$$

$$a(0) = 0 \text{ and } a(\eta) > 0 \text{ for } \eta > 0, \text{ and } \tilde{K}: ([0, \infty) \times \mathbb{R}^d)^2 \rightarrow \mathbb{C}$$

$$a^{\frac{d-1}{2}}(\eta_1)a^{\frac{d-1}{2}}(\eta_2)\psi^{\text{free}} \in \mathcal{B}_d$$

$$a^{\frac{d-1}{2}}(\eta_1)a^{\frac{d-1}{2}}(\eta_2)\psi \in \mathcal{B}_d \text{ for } 0 \leq \eta_1, \eta_2 \leq T$$

$$t_i \leftrightarrow \eta_i \text{ and } K(t_1, z_1, t_2, z_2) = a^2(t_1)a^2(t_2)\tilde{K}(t_1, z_1, t_2, z_2)$$

$$\frac{d-1}{2} = \frac{1}{2}$$

$$a^{1/2}(\eta_1)a^{1/2}(\eta_2)$$

$$\chi = a^{1/2}(\eta_1)a^{1/2}(\eta_2)\psi \text{ as } \chi^{\text{free}} = a^{1/2}(\eta_1)a^{1/2}(\eta_2)\psi^{\text{free}}$$

$$\begin{aligned} \chi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \chi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(2\pi)^2} \int_0^{\eta_1} d\eta'_1 \int_0^{\eta_2} d\eta'_2 \int d^4\mathbf{x}'_1 d^4\mathbf{x}'_2 \\ &\times a^{3/2}(\eta'_1)a^{3/2}(\eta'_2) \frac{H(\eta_1 - \eta'_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{\sqrt{(\eta_1 - \eta'_1)^2 - |\mathbf{x}_1 - \mathbf{x}'_1|^2}} \frac{H(\eta_2 - \eta'_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{\sqrt{(\eta_2 - \eta'_2)^2 - |\mathbf{x}_2 - \mathbf{x}'_2|^2}} \\ &\times \tilde{K}(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2)\chi(\eta'_1, \mathbf{x}'_1, \eta'_2, \mathbf{x}'_2) \end{aligned}$$

$$K(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) = a^{3/2}(t_1)a^{3/2}(t_2)\tilde{K}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) \text{ and } m_1 = m_2 = 0$$

$$\chi = a(\eta_1)a(\eta_2)\psi \text{ as } \chi^{\text{free}} = a(\eta_1)a(\eta_2)\psi^{\text{free}}$$

$$\begin{aligned} \chi(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) &= \chi^{\text{free}}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) + \frac{\lambda}{(4\pi)^2} \int d^3\mathbf{x}'_1 d^3\mathbf{x}'_2 \\ &\times a(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)a(\eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|) \frac{H(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|)}{|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{H(\eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|)}{|\mathbf{x}_2 - \mathbf{x}'_2|} \\ &\times (\tilde{K} \times \chi)(\eta_1 - |\mathbf{x}_1 - \mathbf{x}'_1|, \mathbf{x}'_1, \eta_2 - |\mathbf{x}_2 - \mathbf{x}'_2|, \mathbf{x}'_2) \end{aligned}$$

$$K(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) = a(t_1)a(t_2)\tilde{K}(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)$$

$$\tilde{K}(x_1, x_2) = \begin{cases} f(d(x_1, x_2)) & \text{if } x_1, x_2 \text{ are time-like dimension} \\ 0 & \text{else} \end{cases}$$

$$d(x_1, x_2) = (|\eta_1 - \eta_2| - |\mathbf{x}_1 - \mathbf{x}_2|) \int_0^1 a(\tau\eta_1 + (1-\tau)\eta_2) d\tau$$

$$x_1 = (\eta_1, \mathbf{x}_1) \text{ and } x_2 = (\eta_2, \mathbf{x}_2)$$

$$\lim_{\eta_1, \eta_2 \rightarrow 0} a(\eta_1)a(\eta_2)\psi = \lim_{\eta_1, \eta_2 \rightarrow 0} a(\eta_1)a(\eta_2)\psi^{\text{free}}$$

$$\partial\psi^{\text{free}} / \partial\eta_i$$

$$\tilde{K}(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2) = \frac{f(\eta_1, \mathbf{x}_1, \eta_2, \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}$$



$$a(\eta_1)a(\eta_2)\psi \in \mathcal{B}_3 \text{ with } a(\eta_1)a(\eta_2)\psi^{\text{free}} \in \mathcal{B}_3$$

$$a^{-1}(\eta_1)a^{-1}(\eta_2) \text{ for } \eta_1, \eta_2 \rightarrow T \text{ as well as for } \eta_1, \eta_2 \rightarrow 0$$

$$a(0) = 0 = a(T) \text{ and } a(\eta) > 0 \text{ for } \eta \in (0, T)$$

$$\mathcal{B} = L^\infty([0, T]^2, L^2((S^3)^2))$$

$$\|\psi\|_{\mathcal{B}} = \text{ess sup}_{\eta_1, \eta_2 \in [0, T]} \|\psi(\eta_1, \cdot, \eta_2, \cdot)\|_{L^2((S^3)^2)}, \text{ and } f: ([0, T] \times S^3)^2 \rightarrow \mathbb{C}$$

$$a(\eta_1)a(\eta_2)\psi^{\text{free}} \in \mathcal{B}$$

$$\tilde{K}(\eta_1, q_1, \eta_2, q_2) = \frac{f(\eta_1, q_1, \eta_2, q_2)}{\sin(s(q_1, q_2))}$$

$$|\lambda| < \left(\frac{\pi^2}{\sqrt{2}} \left(\left\lfloor \frac{T}{\pi} \right\rfloor + 1 \right)^2 \|a\|_\infty^2 \|f\|_\infty \right)^{-1}$$

$$f(\eta_1, q_1, \eta_2, q_2) \propto \sin(s(q_1, q_2))$$

$$\chi = a(\eta_1)a(\eta_2)\psi \text{ and } \chi^{\text{free}} = a(\eta_1)a(\eta_2)\psi^{\text{free}}$$

$$\begin{aligned} \chi(\eta_1, q_1, \eta_2, q_2) &= \chi^{\text{free}}(\eta_1, q_1, \eta_2, q_2) + \frac{\lambda}{4(4\pi)^2} \sum_{l, m=-\infty}^{\infty} \int d\Omega_3(q'_1) d\Omega_3(q'_2) \\ &\times \frac{\text{sgn}(s(q_1, q'_1) + 2\pi l) \text{sgn}(s(q_2, q'_2) + 2\pi m)}{\sin(s(q_1, q'_1)) \sin(s(q_2, q'_2))} \\ &\times \sum_{\sigma_1, \sigma_2 = \pm 1} [\mathbb{1}_{[0, T]}(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|) \mathbb{1}_{[0, T]}(\eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|) \\ &\times a(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|) a(\eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|) \\ &\times \frac{1}{\sin(s(q'_1, q'_2))} (f \times \chi)(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|, q'_1, \eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|, q'_2) \end{aligned}$$

$$\chi(\eta_1 + \sigma_1 |s(q_1, q'_1) + 2\pi l|, q'_1, \eta_2 + \sigma_2 |s(q_2, q'_2) + 2\pi m|, q'_2)$$

$$\chi = \chi^{\text{free}} + \hat{K}\chi$$

$$\hat{K} = \sum_{l, m, \sigma_1, \sigma_2} \hat{K}_{\sigma_1 \sigma_2}^{lm} \otimes \hat{K}_{\sigma_1 \sigma_2}^{lm} \chi$$

$$d\Omega_3(q'_1) d\Omega_3(q'_2)$$



$$\begin{aligned} \|\widehat{K}_{\sigma_1\sigma_2}^{lm}\chi\|_{\mathcal{B}} &\leq \operatorname{ess\,sup}_{\eta_1,\eta_2\in[0,T]} \frac{|\lambda|}{4(4\pi)^2} \|a\|_{\infty}^2 \|f\|_{\infty} \left[\int d\Omega_3(q_1)d\Omega_3(q_2) \right. \\ &\times \left(\int d\Omega_3(q'_1)d\Omega_3(q'_2) \frac{1}{\sin^2(s(q'_1,q'_2))} \right) \\ &\times \left(\int d\Omega_3(q_1)d\Omega_3(q'_1) \frac{1}{\sin^2(s(q_1,q'_1))} \frac{1}{\sin^2(s(q_2,q'_2))} \right) \\ &\times \mathbb{1}_{[0,T]}(\eta_1 + \sigma_1|s(q_1,q'_1) + 2\pi l|) \mathbb{1}_{[0,T]}(\eta_2 + \sigma_2|s(q_2,q'_2) + 2\pi m|) \\ &\times |\chi|^2(\eta_1 + \sigma_1|s(q_1,q'_1) + 2\pi l|, q'_1, \eta_2 + \sigma_2|s(q_2,q'_2) + 2\pi m|, q'_2) \Big]^{1/2} \end{aligned}$$

$$\begin{aligned} (q'_2)^0 &= \cos \alpha_2 \\ (q'_2)^1 &= \sin \alpha_2 \cos \beta_2 \\ (q'_2)^2 &= \sin \alpha_2 \sin \beta_2 \sin \varphi_2 \\ (q'_2)^3 &= \sin \alpha_2 \sin \beta_2 \sin \varphi_2 \end{aligned}$$

$$\alpha_2, \beta_2 \in [0, \pi) \text{ and } \varphi_2 \in [0, 2\pi)$$

$$d\Omega_3(q'_2) = \sin^2 \alpha_2 \sin \beta_2 d\alpha_2 d\beta_2 d\varphi_2$$

$$\begin{aligned} \int d\Omega_3(q_1)d\Omega_3(q'_2) \frac{1}{\sin^2(s(q'_1,q'_2))} &= \int d\Omega_3(q_1) \int_0^{\pi} d\alpha_2 \int_0^{\pi} d\beta_2 \int_0^{2\pi} d\varphi_2 \frac{\sin^2 \alpha_2 \sin \beta_2}{\sin^2 \alpha_2} \\ &= |S^3| \times 4\pi^2 = 8\pi^4 \end{aligned}$$

$$\begin{aligned} \|\widehat{K}_{\sigma_1\sigma_2}^{lm}\chi\|_{\mathcal{B}} &\leq \operatorname{ess\,sup}_{\eta_1,\eta_2\in[0,T]} \frac{|\lambda|}{16\sqrt{2}} \|a\|_{\infty}^2 \|f\|_{\infty} \left[\int d\Omega_3(q_1) d\Omega_3(q_2) d\Omega_3(q'_1) d\Omega_3(q'_2) \right. \\ &\times \frac{1}{\sin^2(s(q_1,q'_1))} \frac{1}{\sin^2(s(q_2,q'_2))} \\ &\times \mathbb{1}_{[0,T]}(\eta_1 + \sigma_1|s(q_1,q'_1) + 2\pi l|) \mathbb{1}_{[0,T]}(\eta_2 + \sigma_2|s(q_2,q'_2) + 2\pi m|) \\ &\times |\chi|^2(\eta_1 + \sigma_1|s(q_1,q'_1) + 2\pi l|, q'_1, \eta_2 + \sigma_2|s(q_2,q'_2) + 2\pi m|, q'_2) \Big]^{1/2}. \end{aligned}$$

$$\begin{aligned} \|\widehat{K}_{\sigma_1\sigma_2}^{lm}\chi\|_{\mathcal{B}} &\leq \operatorname{ess\,sup}_{\eta_1,\eta_2\in[0,T]} \frac{|\lambda|}{16\sqrt{2}} \|a\|_{\infty}^2 \|f\|_{\infty} \left[\int d\Omega_3(q'_1) d\Omega_3(q'_2) \right. \\ &\times \int_0^{\pi} d\alpha_1 \int_0^{\pi} d\beta_1 \int_0^{2\pi} d\varphi_1 \frac{\sin^2 \alpha_1 \sin \beta_1}{\sin^2 \alpha_1} \int_0^{\pi} d\alpha_2 \int_0^{\pi} d\beta_2 \int_0^{2\pi} d\varphi_2 \frac{\sin^2 \alpha_2 \sin \beta_2}{\sin^2 \alpha_2} \\ &\times \mathbb{1}_{[0,T]}(\eta_1 + \sigma_1|\alpha_1 + 2\pi l|) \mathbb{1}_{[0,T]}(\eta_2 + \sigma_2|\alpha_2 + 2\pi m|) \\ &\times |\chi|^2(\eta_1 + \sigma_1|\alpha_1 + 2\pi l|, q'_1, \eta_2 + \sigma_2|\alpha_2 + 2\pi m|, q'_2) \Big]^{1/2}. \end{aligned}$$



$$\begin{aligned} \|\widehat{K}_{\sigma_1\sigma_2}^{lm}\chi\|_{\mathcal{B}} &\leq \operatorname{ess\,sup}_{\eta_1,\eta_2\in[0,T]} \frac{|\lambda|}{16\sqrt{2}} \|a\|_{\infty}^2 \|f\|_{\infty} \\ &\times \left[\int_0^{\pi} d\alpha_1 \int_0^{\pi} d\beta_1 \int_0^{2\pi} d\varphi_1 \sin \beta_1 \int_0^{\pi} d\alpha_2 \int_0^{\pi} d\beta_2 \int_0^{2\pi} d\varphi_2 \sin \beta_2 \right. \\ &\times \mathbb{1}_{[0,T]}(\eta_1 + \sigma_1|\alpha_1 + 2\pi l|) \mathbb{1}_{[0,T]}(\eta_2 + \sigma_2|\alpha_2 + 2\pi m|) \\ &\left. \times \|\chi(\eta_1 + \sigma_1|\alpha_1 + 2\pi l|, \cdot, \eta_2 + \sigma_2|\alpha_2 + 2\pi m|, \cdot)\|_{L^2((S^3)^2)}^2 \right]^{1/2} \end{aligned}$$

$$\begin{aligned} \|\widehat{K}_{\sigma_1\sigma_2}^{lm}\chi\|_{\mathcal{B}} &\leq \operatorname{ess\,sup}_{\eta_1,\eta_2\in[0,T]} \frac{|\lambda|}{16\sqrt{2}} \|a\|_{\infty}^2 \|f\|_{\infty} \times 4\pi^2 \|\chi\|_{\mathcal{B}} \\ &= \frac{\pi^2}{4\sqrt{2}} |\lambda| \|a\|_{\infty}^2 \|f\|_{\infty} \|\chi\|_{\mathcal{B}} \end{aligned}$$

$$\|\widehat{K}\|_{\mathcal{B}\rightarrow\mathcal{B}} \leq 4 \left(\left\lfloor \frac{T}{\pi} \right\rfloor + 1 \right)^2 \|\widehat{K}_{11}^{00}\|_{\mathcal{B}\rightarrow\mathcal{B}} \frac{\pi^2}{\sqrt{2}} |\lambda| \left(\left\lfloor \frac{T}{\pi} \right\rfloor + 1 \right)^2 \|a\|_{\infty}^2 \|f\|_{\infty}$$

$$|\lambda| < \left(\frac{\pi^2}{\sqrt{2}} \left(\left\lfloor \frac{T}{\pi} \right\rfloor + 1 \right)^2 \|a\|_{\infty}^2 \|f\|_{\infty} \right)^{-1}$$

$$(1 - \lambda(\widehat{G}^{\text{ret}} \otimes \widehat{G}^{\text{ret}})\widehat{K})\psi = \psi^{\text{free}}$$

$$|\lambda| < \|\widehat{G}^{\text{ret}}\|^{-2} \|\widehat{K}\|^{-1}$$

$$V_0 = \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 \Phi_1^\dagger \Phi_1 \Phi_2^\dagger \Phi_2 + \lambda_4 \Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_1 + \frac{\lambda_5}{2} [(\Phi_1^\dagger \Phi_2)^2 + \text{h.c.}],$$

$$\Phi_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\phi_k^+ \\ v_k + \phi_k^0 + ia_k \end{pmatrix} \quad (k = 1, 2),$$

$$\lambda_1 = -\lambda_{345} \tan^2 \beta, \lambda_2 = -\lambda_{345} \cot^2 \beta,$$

$$R(\beta) \equiv \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix},$$

$$c_\theta \equiv \cos \theta \text{ and } s_\theta \equiv \sin \theta$$

$$\begin{pmatrix} \Phi_{\text{SM}} \\ \Phi_{\text{BSM}} \end{pmatrix} = R(\beta) \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

$$\Phi_{\text{SM}} = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}, \Phi_{\text{BSM}} = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}(H + iA) \end{pmatrix}.$$

$$m_h^2 = m_{G^0}^2 = m_{G^\pm}^2 = 0, m_H^2 = -\lambda_{345} v^2, m_A^2 = -\lambda_5 v^2, m_{H^\pm}^2 = -\frac{\lambda_{45}}{2} v^2$$

$$\lambda_1 = -\lambda_{345} \tan^2 \beta - \frac{2}{v^3 c_\beta^2} (T_h - T_H \tan \beta), \lambda_2 = -\lambda_{345} \cot^2 \beta - \frac{2}{v^3 s_\beta^2} (T_h + T_H \cot \beta),$$



$$\begin{aligned} \mathcal{M}_S^2(p^2) &= \begin{pmatrix} 0 & 0 \\ 0 & -\lambda_{345}v^2 \end{pmatrix} + \begin{pmatrix} \Pi_{hh}(p^2) & \Pi_{hH}(p^2) \\ \Pi_{hH}(p^2) & \Pi_{HH}(p^2) \end{pmatrix} - \frac{3}{v} \begin{pmatrix} T_h & T_H \\ T_H & \tilde{T} \end{pmatrix} \\ \mathcal{M}_P^2(p^2) &= \begin{pmatrix} 0 & 0 \\ 0 & -\lambda_5v^2 \end{pmatrix} + \begin{pmatrix} \Pi_{G^0G^0}(p^2) & \Pi_{G^0A}(p^2) \\ \Pi_{G^0A}(p^2) & \Pi_{AA}(p^2) \end{pmatrix} - \frac{1}{v} \begin{pmatrix} T_h & T_H \\ T_H & \tilde{T} \end{pmatrix} \\ \mathcal{M}_C^2(p^2) &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\lambda_{45}}{2}v^2 \end{pmatrix} + \begin{pmatrix} \Pi_{G^\pm G^\mp}(p^2) & \Pi_{G^\pm H^\mp}(p^2) \\ \Pi_{G^\pm H^\mp}(p^2) & \Pi_{H^\pm H^\mp}(p^2) \end{pmatrix} - \frac{1}{v} \begin{pmatrix} T_h & T_H \\ T_H & \tilde{T} \end{pmatrix}, \end{aligned}$$

$$\tilde{T} = T_h + 2\cot 2\beta T_H$$

$$\Pi_{G^0G^0}(0) = \Pi_{G^\pm G^\mp}(0) = T_h/v$$

$$\Pi_{G^0A}(0) = \Pi_{G^\pm H^\mp}(0) = T_H/v$$

$$v = (\sqrt{2}G_F)^{-1/2}.$$

$$M_h^2 = \Pi_{hh}(0) - 3\frac{T_h}{v}$$

$$M_H^2 = -\frac{\lambda_{345}}{\sqrt{2}G_F} - m_H^2 \frac{\delta v^2}{v^2} + \text{Re}\Pi_{HH}(m_H^2) - 3\frac{\tilde{T}}{v}$$

$$M_A^2 = -\frac{\lambda_5}{\sqrt{2}G_F} - m_A^2 \frac{\delta v^2}{v^2} + \text{Re}\Pi_{AA}(m_A^2) - \frac{\tilde{T}}{v}$$

$$M_{H^\pm}^2 = -\frac{\lambda_{45}}{2\sqrt{2}G_F} - m_{H^\pm}^2 \frac{\delta v^2}{v^2} + \text{Re}\Pi_{H^\pm H^\mp}(m_{H^\pm}^2) - \frac{\tilde{T}}{v}$$

$$(\sqrt{2}G_F)^{-1} = v^2 + \delta v^2.$$

$$(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2) \text{ or } (\Phi_{\text{SM}}^\dagger \Phi_{\text{SM}} + \Phi_{\text{BSM}}^\dagger \Phi_{\text{BSM}})$$

$$M_h^2 = \frac{1}{8\pi^2 v^2} (m_H^4 + m_A^4 + 2m_{H^\pm}^4 + 6m_W^4 + 3m_Z^4 - 4N_c m_t^4)$$

$$[\mathcal{M}_S^2(0)]_{12} = \frac{1}{8\pi^2 v^2} [m_H^2(3m_H^2 + m_A^2 + 2m_{H^\pm}^2)\cot 2\beta - 4N_c m_t^4 \cot \beta]$$



$$\begin{aligned}
16\pi^2 v T_h &= m_H^2 A_0(m_H^2) + m_A^2 A_0(m_A^2) + 2m_H^2 A_0(m_H^2) - 4N_c m_t^2 A_0(m_t^2) \\
&\quad + m_Z^2 [3A_0(m_Z^2) + 2m_Z^2] + 2m_W^2 [3A_0(m_W^2) + 2m_W^2] \\
16\pi^2 v T_H &= m_H^2 \cot 2\beta [3A_0(m_H^2) + A_0(m_A^2) + 2A_0(m_{H^\pm}^2)] - 4N_c m_t^2 \cot \beta A_0(m_t^2), \\
16\pi^2 v^2 \Pi_{hh}(p^2) &= m_H^2 F(p^2, m_H^2) + m_A^2 F(p^2, m_A^2) + 2m_{H^\pm}^2 F(p^2, m_{H^\pm}^2) \\
&\quad - 2N_c m_t^2 [2F(p^2, m_t^2) + p^2 B_0(p^2, m_t^2, m_t^2)] \\
&\quad + f_1(p^2, m_Z^2) + 2f_1(p^2, m_W^2), \\
16\pi^2 v^2 \Pi_{hH}(p^2) &= m_H^2 \cot 2\beta [3F(p^2, m_H^2) + F(p^2, m_A^2) + 2F(p^2, m_{H^\pm}^2)] \\
&\quad - 2N_c m_t^2 \cot \beta [2F(p^2, m_t^2) + p^2 B_0(p^2, m_t^2, m_t^2)], \\
16\pi^2 v^2 \Pi_{HH}(p^2) &= 2m_H^2 \cot^2 2\beta [3A_0(m_H^2) + A_0(m_A^2) + 2A_0(m_{H^\pm}^2)] - 4m_H^4 B_0(p^2, m_H^2, 0) \\
&\quad - 2m_H^4 \cot^2 2\beta [9B_0(p^2, m_H^2, m_H^2) + B_0(p^2, m_A^2, m_A^2) + 2B_0(p^2, m_{H^\pm}^2, m_{H^\pm}^2)] \\
&\quad - 2N_c m_t^2 \cot^2 \beta [2F(p^2, m_t^2) + p^2 B_0(p^2, m_t^2, m_t^2)] \\
&\quad + f_2(p^2, m_Z^2, m_A^2, m_H^2) + 2f_2(p^2, m_W^2, m_{H^\pm}^2, m_H^2), \\
16\pi^2 v^2 \Pi_{AA}(p^2) &= 2m_H^2 \cot^2 2\beta [A_0(m_H^2) + 3A_0(m_A^2) + 2A_0(m_{H^\pm}^2)] \\
&\quad - 4m_A^4 B_0(p^2, m_A^2, 0) - 4m_H^4 \cot^2 2\beta B_0(p^2, m_A^2, m_H^2) \\
&\quad - 2N_c m_t^2 \cot^2 \beta [2A_0(m_t^2) + p^2 B_0(p^2, m_t^2, m_t^2)] \\
&\quad + f_2(p^2, m_Z^2, m_H^2, m_A^2) + 2f_2(p^2, m_W^2, m_{H^\pm}^2, m_A^2), \\
16\pi^2 v^2 \Pi_{H^+H^-}(p^2) &= 2m_H^2 \cot^2 2\beta [A_0(m_H^2) + A_0(m_A^2) + 4A_0(m_{H^\pm}^2)] \\
&\quad - 4m_{H^\pm}^4 B_0(p^2, m_{H^\pm}^2, 0) - 4m_H^4 \cot^2 2\beta B_0(p^2, m_{H^\pm}^2, m_H^2) \\
&\quad - 2N_c m_t^2 \cot^2 \beta [A_0(m_t^2) + (p^2 - m_t^2) B_0(p^2, m_t^2, 0)] \\
&\quad + f_2(p^2, m_W^2, m_H^2, m_{H^\pm}^2) + f_2(p^2, m_W^2, m_A^2, m_{H^\pm}^2) + c_{2\theta_W}^2 f_2(p^2, m_Z^2, m_{H^\pm}^2, m_{H^\pm}^2) \\
&\quad + s_{2\theta_W}^2 m_Z^2 \left[2(\xi - 1)p^2 + (3 - 2\xi)A_0(m_{H^\pm}^2) + \frac{m_{H^\pm}^2}{m_Z^2} A_0(\xi m_Z^2) \right. \\
&\quad \left. + (3 - \xi)(p^2 + m_{H^\pm}^2) B_0(p^2, m_{H^\pm}^2, 0) \right]
\end{aligned}$$

$c_{\theta_W} = m_W/m_Z$, and $A_0(m^2)$ and $B_0(p^2, m_1^2, m_2^2)$ are Passarino-Veltman functions

$$A_0(m^2) = m^2 \left(\ln \frac{m^2}{Q^2} - 1 \right), B_0(0, m_1^2, m_2^2) = -\frac{A_0(m_1^2) - A_0(m_2^2)}{m_1^2 - m_2^2}$$

$$\begin{aligned}
F(p^2, m^2) &= A_0(m^2) - 2m^2 B_0(p^2, m^2, m^2) \\
f_1(p^2, m_V^2) &= 3m_V^2 [A_0(m_V^2) + 2m_V^2] - p^2 [A_0(m_V^2) - A_0(\xi m_V^2)] \\
&\quad - \frac{1}{2} (p^4 - 4p^2 m_V^2 + 12m_V^4) B_0(p^2, m_V^2, m_V^2) + \frac{p^4}{2} B_0(p^2, \xi m_V^2, \xi m_V^2), \\
f_2(p^2, m_V^2, m_1^2, m_2^2) &= 2m_V^4 + (2m_V^2 + m_1^2) A_0(m_V^2) + m_V^2 A_0(m_1^2) - p^2 [A_0(m_V^2) - A_0(\xi m_V^2)] \\
&\quad - [m_1^4 + (p^2 - m_V^2)^2 - 2m_1^2(p^2 + m_V^2)] B_0(p^2, m_1^2, m_V^2) \\
&\quad + (p^2 - m_2^2)(p^2 + m_2^2 - 2m_1^2) B_0(p^2, m_1^2, \xi m_V^2)
\end{aligned}$$

$$(\sqrt{2}G_F)^{-1} = v^2 + \delta v^2$$

$$\delta v^2 = v^2 \left[\frac{\Pi_{WW}(0)}{m_W^2} - \delta_{\text{VB}} \right]$$



$$\frac{16\pi^2}{g^2} \Pi_{WW}(0) \Big|_{\xi=1} = m_W^2 \left[\frac{27 - 34s_{\theta_W}^2}{8c_{\theta_W}^2} + \left(\frac{17}{s_{\theta_W}^2} + 8s_{\theta_W}^2 - 29 \right) \frac{\ln c_{\theta_W}^2}{4c_{\theta_W}^2} + \left(\frac{s_{\theta_W}^2}{c_{\theta_W}^2} - 1 \right) \ln \frac{m_W^2}{Q^2} \right]$$

$$- \frac{N_c}{2} m_t^2 \left(\ln \frac{m_t^2}{Q^2} - \frac{1}{2} \right) + \tilde{B}_{22}(0, m_H^2, m_{H^\pm}^2) + \tilde{B}_{22}(0, m_A^2, m_{H^\pm}^2)$$

$$\frac{16\pi^2}{g^2} \delta_{\text{VB}} \Big|_{\xi=1} = 6 + \left(\frac{7}{2} - 6s_{\theta_W}^2 \right) \frac{\ln c_{\theta_W}^2}{s_{\theta_W}^2} - 4 \ln \frac{m_Z^2}{Q^2}$$

$$\tilde{B}_{22}(0, m_1^2, m_2^2) = \frac{1}{2} \left(\frac{m_1^2 + m_2^2}{4} - \frac{m_1^2 m_2^2}{2(m_1^2 - m_2^2)} \ln \frac{m_1^2}{m_2^2} \right)$$

$$\delta v^2 = \delta v^2|_{\xi=1} + \frac{1}{16\pi^2} (1 - \xi) [m_Z^2 B_0(0, m_Z^2, \xi m_Z^2) + 2m_W^2 B_0(0, m_W^2, \xi m_W^2)]$$

$$V_0 = \frac{\Lambda_1}{2} (\Phi_{\text{SM}}^\dagger \Phi_{\text{SM}})^2 + \frac{\Lambda_2}{2} (\Phi_{\text{BSM}}^\dagger \Phi_{\text{BSM}})^2$$

$$+ \Lambda_3 (\Phi_{\text{SM}}^\dagger \Phi_{\text{SM}}) (\Phi_{\text{BSM}}^\dagger \Phi_{\text{BSM}}) + \Lambda_4 (\Phi_{\text{SM}}^\dagger \Phi_{\text{BSM}}) (\Phi_{\text{BSM}}^\dagger \Phi_{\text{SM}})$$

$$+ \left[\frac{\Lambda_5}{2} (\Phi_{\text{SM}}^\dagger \Phi_{\text{BSM}})^2 + (\Lambda_6 \Phi_{\text{SM}}^\dagger \Phi_{\text{SM}} + \Lambda_7 \Phi_{\text{BSM}}^\dagger \Phi_{\text{BSM}}) \Phi_{\text{SM}}^\dagger \Phi_{\text{BSM}} + \text{h.c.} \right]$$

$$\Lambda_1 = -\frac{2}{v^3} T_h, \Lambda_6 = -\frac{2}{v^3} T_H$$

$$T_\varphi = d\Delta V/d\varphi|_{\min}$$

$$\Lambda_1 = \lambda_1 c_\beta^4 + \lambda_2 s_\beta^4 + \frac{1}{2} \lambda_{345} s_{2\beta}^2, \Lambda_6 = -\frac{1}{2} s_{2\beta} (\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_{345} c_{2\beta})$$

$$\Delta V = (v+h)^4 \left[A + B \ln \frac{(v+h)^2}{Q^2} \right]$$

$$A = \sum_\varphi \frac{\alpha_\varphi m_\varphi^4}{64\pi^2 v^4} \left(\ln \frac{m_\varphi^2}{v^2} - k_\varphi \right), B = \sum_\varphi \frac{\alpha_\varphi m_\varphi^4}{64\pi^2 v^4} \otimes \mathcal{O}(\tilde{\Lambda}^2/16\pi^2)$$

$$T_h = d\Delta V/dh|_{\min} = \ln \frac{v^2}{Q_{\text{GW}}^2} = -\frac{1}{2} - \frac{A}{B}$$

$$v = e^{-\left(\frac{1}{4} + \frac{A}{2B}\right)} Q_{\text{GW}}$$

$$\ln \frac{v^2}{Q^2} = -\frac{1}{2} - \frac{A}{B} - \frac{\Lambda_1}{2B}$$

$$X_{ij} = (p_i + p_{i+1} + \dots + p_{j-1})^2, \text{ for } 1 \leq i < j \leq n,$$

$$A_n \equiv A_n(X_{ij}) \equiv A_n(X)$$

$$A_n(a_1 a_2 \dots a_n) \equiv A_n(X_{a_i a_j})$$



$$m_4(a_1 a_2 a_3 a_4) = \frac{1}{X_{a_1 a_3}} + \frac{1}{X_{a_2 a_4}}$$

$$m_4(1234) = \frac{1}{X_{13}} + \frac{1}{X_{24}} \equiv \frac{1}{(p_1 + p_2)^2} + \frac{1}{(p_2 + p_3)^2}$$

$$\tilde{A}(X_{ij}, X_{i,i+1}) = A(X_{ij}) + \sum_{k=1}^n X_{k,k+1} F_k(X_{ij}, X_{i,i+1})$$

$$\tilde{A}(X_{ij}, X_{i,i+1}) = \tilde{A}(X_{ij}) \equiv A(X_{ij}).$$

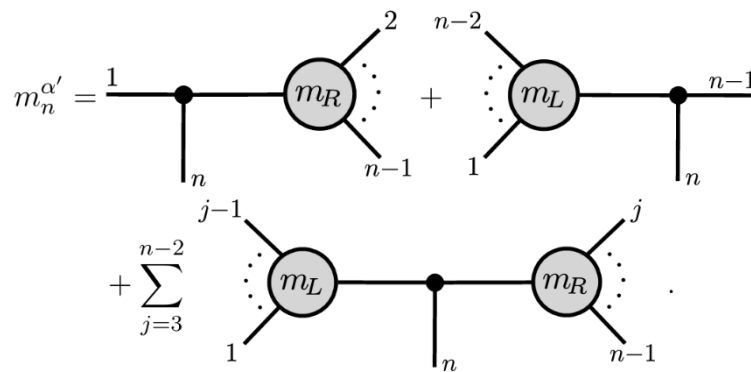
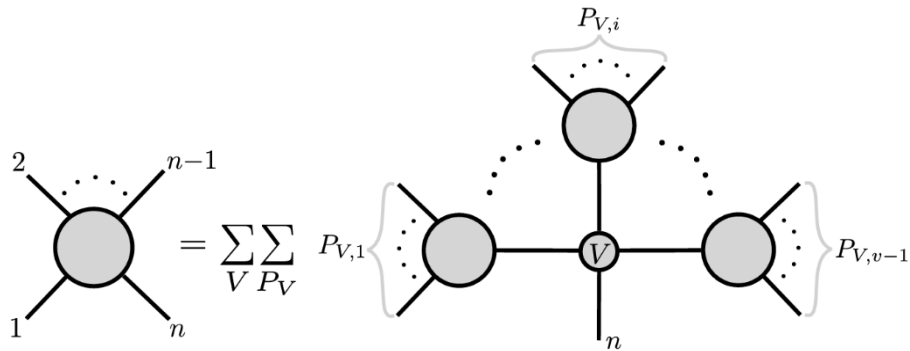
$$\mathcal{L}_{\text{Bi-Adjoint-Scalar}} = \frac{1}{2} \partial \phi^{a\bar{a}} \cdot \partial \phi^{a\bar{a}} - \frac{1}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}},$$

$$m_n^{a_1 \bar{a}_1 \dots a_n \bar{a}_n} = \sum_{\sigma, \rho \in S_{n-1}} \text{tr}(t^{a_1} t^{a_{\sigma(2)}} \dots t^{a_{\sigma(n)}}) \text{tr}(t^{\bar{a}_1} t^{\bar{a}_{\rho(2)}} \dots t^{\bar{a}_{\rho(n)}}) m_n(1\sigma | 1\rho)$$

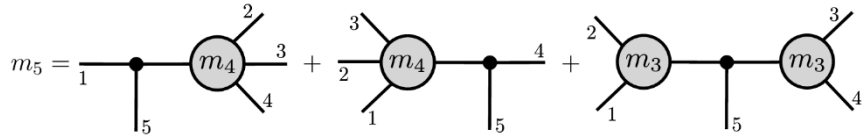
$$m_4(\mathbb{1} | \mathbb{1}) = \frac{1}{X_{13}} + \frac{1}{X_{24}}, m_4(\mathbb{1} | 1243) = -\frac{1}{X_{13}}, m_4(\mathbb{1} | 1324) = -\frac{1}{X_{24}},$$

$$m_5(\mathbb{1} | \mathbb{1}) = \frac{1}{X_{13} X_{14}} + \text{cyc.}, m_5(\mathbb{1} | 13245) = \frac{1}{X_{14}} \left(\frac{1}{X_{13}} + \frac{1}{X_{24}} \right),$$

$$\mathcal{L}_{\text{tr}\phi^3} = \frac{1}{2} \langle \partial \phi \cdot \partial \phi \rangle - \frac{1}{3!} \langle \phi^3 \rangle,$$



$$m_n(1 \dots n) = \frac{1}{X_{2n}} m_{n-1}(2 \dots n) + \frac{1}{X_{1n-1}} m_{n-1}(1 \dots n-1) + \sum_{j=3}^{n-2} \frac{1}{X_{1j} X_{jn}} m_j(1 \dots j) m_{n-j+1}(j \dots n)$$



$$m_5(\mathbb{1}) = \frac{1}{X_{25}} m_4(2345) + \frac{1}{X_{14}} m_4(1234) + \frac{1}{X_{13} X_{35}} m_3(123) m_3(345) = \frac{1}{X_{25}} \left(\frac{1}{X_{24}} + \frac{1}{X_{35}} \right) + \frac{1}{X_{14}} \left(\frac{1}{X_{13}} + \frac{1}{X_{24}} \right) + \frac{1}{X_{13} X_{35}}$$

$$M_n^{\text{cl}} = \sum_{\sigma, \rho \in S_{n-3}} A_n^{\circ}(1\sigma n - 1n) S_n^{\alpha'}(\sigma | \rho) A_n^{\circ}(1\rho n n - 1)$$

$$m_n^{\alpha'} \equiv (S_n^{\alpha'})^{-1}$$

$$t_{ij} = \tan(\pi\alpha' X_{ij}), s_{ij} = \sin(\pi\alpha' X_{ij})$$

$$t_{ij} = t_{ji}, t_{i,i+1} = 0$$

$$t_{ij} \simeq s_{ij} = \pi\alpha' X_{ij} + \mathcal{O}(\alpha'^3)$$

$$m_3^{\alpha'}(\mathbb{1} | \mathbb{1}) = -m_3^{\alpha'}(\mathbb{1} | 132) = 1$$

$$m_4^{\alpha'}(\mathbb{1} | \mathbb{1}) = \frac{1}{t_{13}} + \frac{1}{t_{24}}, m_4^{\alpha'}(\mathbb{1} | 1243) = -\frac{1}{s_{13}}, m_4^{\alpha'}(\mathbb{1} | 1324) = -\frac{1}{s_{24}},$$

$$m_5^{\alpha'}(\mathbb{1} | \mathbb{1}) = \left(\frac{1}{t_{13} t_{14}} + \text{cyc.} \right) + 1, m_5^{\alpha'}(\mathbb{1} | 13245) = \frac{1}{s_{14}} \left(\frac{1}{t_{13}} + \frac{1}{t_{24}} \right),$$

$$m_6^{\alpha'}(\mathbb{1} | \mathbb{1}) = \frac{1}{t_{13} t_{14} t_{15}} + \frac{1}{2} \frac{1}{t_{13} t_{14} t_{46}} + \frac{1}{2} \frac{1}{t_{13} t_{36} t_{46}} + \frac{1}{3} \frac{1}{t_{13} t_{15} t_{35}} + \frac{1}{t_{13}} + \text{cyc.},$$

$$\frac{1}{p} = \frac{1}{\tan(\pi\alpha' p^2)}, \quad \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 3 \\ \vdots \\ n \end{array} \equiv \kappa_n = \begin{cases} C_{\frac{n-3}{2}} & \text{if } n \geq 3 \text{ odd,} \\ 0 & \text{if } n \text{ even,} \end{cases}$$

$$\mathcal{L}_{\text{KLT}} = \frac{1}{2} \left\langle \phi - \frac{1}{4} \arcsin(2\phi) - \frac{1}{2} \phi \sqrt{1-4\phi^2} \right\rangle,$$



$$m_3^{\alpha'} = \text{diagram}, \quad m_4^{\alpha'} = \text{diagram}, \quad m_5^{\alpha'} = \text{diagram} + \text{diagram},$$

$$m_6^{\alpha'} = \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}.$$

$$m_n^{\alpha'} = (\pi\alpha')^{3-n}(m_n + \mathcal{O}(\alpha')).$$

$$X_{ij} = k/\alpha', k \in \mathbb{Z}$$

$$\text{Res}_{X_{ij}=k/\alpha'} m_n^{\alpha'}(t_{ij}) = m_L^{\alpha'}(t_L) m_R^{\alpha'}(t_R)$$

$$m_n^{\alpha'}(\mathbb{1} | \mathbb{1}) + \sum_{k=2}^{n-1} e^{ix_k} m_n^{\alpha'}(2 \dots k 1 k + 1 \dots n | \mathbb{1}) = 0$$

$$x_k = \pi\alpha' 2p_1 \cdot (p_2 + \dots + p_k)$$

$$P(X_{ij}) = \begin{cases} \text{even,} & \text{if } p(i) = p(j), \\ \text{odd,} & \text{if } p(i) \neq p(j), \end{cases}$$

$$X_{\text{even/odd}} = \{X_{ij}, P(X_{ij}) = \text{even/odd}\}$$

$$\hat{X}_{\text{even}} = X_{\text{even}} \pm 1/\alpha', \hat{X}_{\text{odd}} = X_{\text{odd}}$$

$$\frac{1}{t_{\text{even}}} \rightarrow -\tau_{\text{even}}, \frac{1}{t_{\text{odd}}} \rightarrow \frac{1}{\tau_{\text{odd}}}$$

$$\tau_{ij} = \tan\left(\frac{\pi}{2}\alpha' X_{ij}\right)$$

$$A_{2k}^{\pi, \alpha'}(X) = m_{2k}^{\alpha'/2}(\hat{X})$$

$$A_4^{\pi, \alpha'} = -\tau_{13} - \tau_{24}$$

$$A_6^{\pi, \alpha'} = \frac{1}{2} \frac{(\tau_{13} + \tau_{24})(\tau_{46} + \tau_{15})}{\tau_{14}} - \tau_{13} - \frac{1}{3} \tau_{13} \tau_{35} \tau_{15} + \text{cyc.}$$

$$A_{2k}^{\pi, \alpha'} = \frac{\pi\alpha'}{2} (A_{2k}^{\text{NLSM}} + \mathcal{O}(\alpha'^2))$$

$$\lim_{\alpha' \rightarrow 0} A_{2k}^{\pi, \alpha'} = 0$$

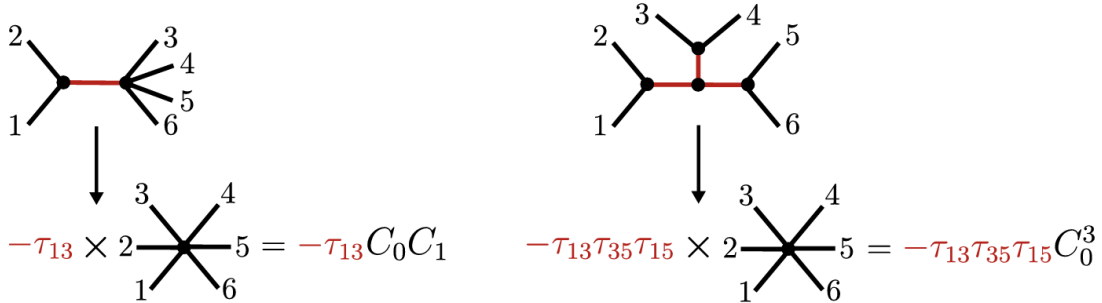
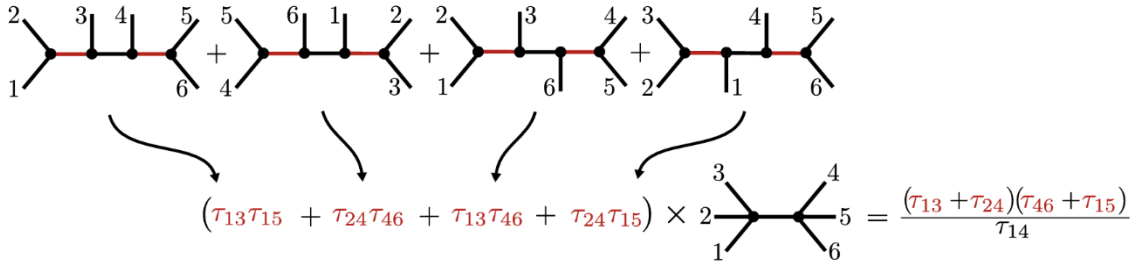
$$m_{2k}(\hat{X}(\delta)) = \frac{1}{\delta^{2k-2}} (A_{2k}^{\text{NLSM}}(X) + \mathcal{O}(\delta^{-1}))$$

$$m_{2k}^{\alpha'/2}(\hat{X}(\alpha')) = \frac{\pi\alpha'}{2} (A_{2k}^{\text{NLSM}}(X) + \mathcal{O}(\alpha'^2))$$



$$m_4(\hat{X}(\delta)) = \frac{1}{\delta^2} \left(-(X_{13} + X_{24}) + \frac{1}{\delta} (X_{13}^2 - X_{24}^2) - \frac{1}{\delta^2} (X_{13}^3 + X_{24}^3) + \dots \right)$$

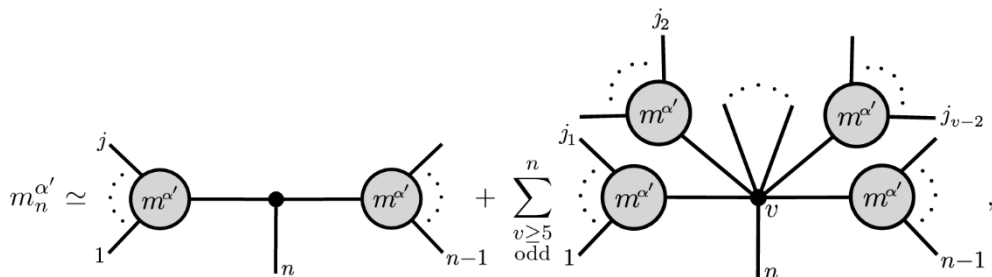
$$m_4^{\alpha'/2}(\hat{X}(\alpha')) = \frac{\pi\alpha'}{2} \left(-(X_{13} + X_{24}) - \frac{(\pi\alpha')^2}{12} (X_{13}^3 + X_{24}^3) - \frac{(\pi\alpha')^4}{120} (X_{13}^5 + X_{24}^5) + \dots \right)$$



$$\mathcal{L}_{\text{NLSM}} = \frac{1}{8} \langle \partial U \cdot \partial U^\dagger \rangle, \text{ where } U(\phi) = i2\phi + \sqrt{1 - 4\phi^2}$$

$$V_{2k}(X) = -\frac{1}{2} \sum_{j=0}^{k-2} C_j C_{k-j-2} (X_{1,3+2j} + \text{cyc.}), \text{ where } k \geq 2$$

$$\frac{1}{2} \sum_{j=0}^{k-2} \left(\begin{array}{c} 2j+2 \\ \vdots \\ 1 \end{array} \begin{array}{c} C_j \\ \vdots \\ C_{k-j-2} \end{array} \begin{array}{c} 2j+3 \\ \vdots \\ 2k \end{array} + \text{cyc.} \right) \rightsquigarrow -\frac{1}{2} \sum_{j=0}^{k-2} C_j C_{k-j-2} (\tau_{1,3+2j} + \text{cyc.}),$$



$$\begin{array}{c} L \\ \text{---} \bullet \text{---} R \\ | \\ n \end{array} = 1 + \tan(\pi\alpha' X_L) \tan(\pi\alpha' X_R),$$



$$m_n^{\alpha'} = \begin{array}{c} \text{---} 1 \text{---} \bullet \text{---} \text{---} m_R^{\alpha'} \begin{array}{l} \nearrow 2 \\ \vdots \\ \searrow n-1 \end{array} \text{---} + \begin{array}{c} n-2 \\ \vdots \\ 1 \end{array} \text{---} m_L^{\alpha'} \text{---} \bullet \text{---} n-1 \\ \downarrow n \qquad \downarrow n \\ + \sum_{j=3}^{n-2} \begin{array}{c} j-1 \\ \vdots \\ 1 \end{array} \text{---} m_L^{\alpha'} \text{---} \bullet \text{---} m_R^{\alpha'} \begin{array}{l} \nearrow j \\ \vdots \\ \searrow n-1 \end{array} \text{---} \end{array} .$$

$$m_n^{\alpha'}(1 \dots n) = \frac{1}{t_{2n}} m_{n-1}^{\alpha'}(2 \dots n) + \frac{1}{t_{1,n-1}} m_{n-1}^{\alpha'}(1 \dots n-1) + \sum_{j=3}^{n-2} \frac{1 + t_{1j}t_{jn}}{t_{1j}t_{jn}} m_j^{\alpha'}(1 \dots j) m_{n-j+1}^{\alpha'}(j \dots n)$$

$$m_5^{\alpha'} = \begin{array}{c} \text{---} 1 \text{---} \bullet \text{---} \text{---} m_4^{\alpha'} \begin{array}{l} \nearrow 2 \\ \searrow 3 \\ \vdots \\ \searrow 4 \end{array} \text{---} + \begin{array}{c} 3 \\ \vdots \\ 2 \end{array} \text{---} m_4^{\alpha'} \text{---} \bullet \text{---} 4 \\ \downarrow 5 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow 5 \\ + \begin{array}{c} 2 \\ \vdots \\ 1 \end{array} \text{---} m_3^{\alpha'} \text{---} \bullet \text{---} m_3^{\alpha'} \begin{array}{l} \nearrow 3 \\ \vdots \\ \searrow 4 \end{array} \text{---} \end{array} .$$

$$m_5^{\alpha'} = \frac{1}{t_{25}} m_4^{\alpha'}(2345) + \frac{1}{t_{14}} m_4^{\alpha'}(1234) + \frac{1 + t_{13}t_{35}}{t_{13}t_{35}} m_3^{\alpha'}(123) m_3^{\alpha'}(345) \\ = \frac{1}{t_{25}} \left(\frac{1}{t_{24}} + \frac{1}{t_{35}} \right) + \frac{1}{t_{14}} \left(\frac{1}{t_{13}} + \frac{1}{t_{24}} \right) + \frac{1 + t_{13}t_{35}}{t_{13}t_{35}} \\ \frac{1 + t_{1j}t_{jn}}{t_{1j}t_{jn}} \xrightarrow{\alpha' \rightarrow 0} \frac{1}{\alpha'^2 X_{1j} X_{jn}} + \mathcal{O}(1)$$

$$V_3(p_1, p_2, p_3) = \begin{array}{c} p_1 \text{---} \square \text{---} p_2 \\ \downarrow p_3 \end{array} = 1 + t_1 t_2 + t_2 t_3 + t_3 t_1,$$

$$\tilde{m}_5^{\alpha'} = \begin{array}{c} \begin{array}{c} 2 \\ \vdots \\ 1 \end{array} \text{---} \square \text{---} \square \text{---} \square \text{---} \begin{array}{c} 3 \\ \vdots \\ 4 \end{array} \\ \downarrow 5 \end{array} + \text{cyc.} = \frac{1 + t_{13}t_{35}}{t_{13}t_{35}} + \text{cyc.} \neq m_5^{\alpha'} .$$

$$m_{2k}^{\alpha'}(1 \dots 2k) \Big|_{\text{ct.}} = 0, \text{ for } k < K.$$



$$m_{2K}^{\alpha'}|_{\text{ct.}} = \frac{1}{t_{1,2K-1}} m_{2K-1}^{\alpha'}|_{\text{ct.}} + \sum_{j=3}^{2K-2} \left(1 + \frac{1}{t_{1,j} t_{j,2K}}\right) m_j^{\alpha'} m_{2K-j+1}^{\alpha'}|_{\text{ct.}} + \frac{1}{t_{2,2K}} m_{2K-1}^{\alpha'}|_{\text{ct.}}$$

$$= \sum_{j=3}^{2K-2} m_j^{\alpha'} m_{2K-j+1}^{\alpha'}|_{\text{ct.}}$$

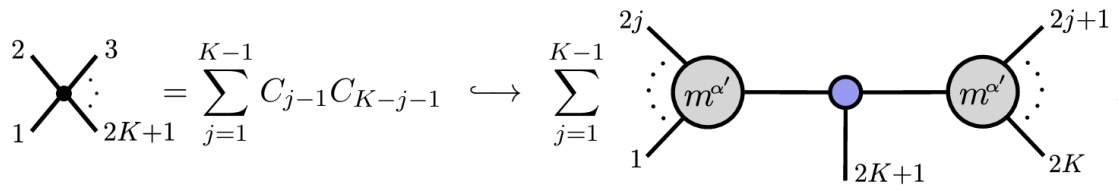
$$\kappa_{2K} \equiv m_{2K}^{\alpha'}|_{\text{ct.}} = 0$$

$$m_{2k+1}^{\alpha'}|_{\text{ct.}} = C_{k-1}$$

$$m_{2K+1}^{\alpha'}|_{\text{ct.}} = \sum_{j=3}^{2K-1} m_j^{\alpha'} m_{2K-j+2}^{\alpha'}|_{\text{ct.}}$$

$$= \sum_{j=2}^{K-1} m_{2j}^{\alpha'} m_{2(K-j+1)}^{\alpha'}|_{\text{ct.}} + \sum_{j=1}^{K-1} m_{2j+1}^{\alpha'} m_{2(K-j)+1}^{\alpha'}|_{\text{ct.}}$$

$$\kappa_{2K+1} \equiv m_{2K+1}^{\alpha'}|_{\text{ct.}} = \sum_{j=1}^{K-1} C_{j-1} C_{K-j-1} = C_{K-1},$$



$$\frac{\partial}{\partial t_{1i}^{-1}} \text{diagram} = \text{diagram} = \text{diagram}$$

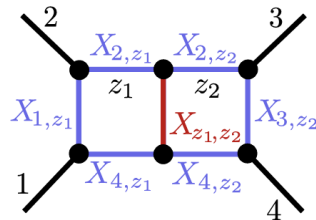
$$\Leftrightarrow \frac{\partial}{\partial t_{1i}^{-1}} m_n^{\alpha'}(1 \dots i \dots n) = m_i^{\alpha'}(1 \dots i) m_{n-i+2}^{\alpha'}(i \dots n 1),$$

$$\frac{\partial}{\partial t_{1K-1}^{-1}} m_K^{\alpha'}(1 \dots K) = \frac{\partial}{\partial t_{1K-1}^{-1}} \frac{1}{t_{1,K-1}} m_{K-1}^{\alpha'}(1 \dots K-1) = m_{K-1}^{\alpha'}(1 \dots K-1).$$

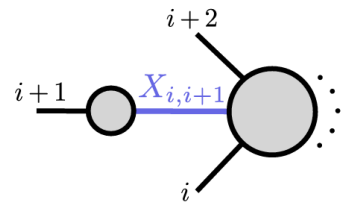
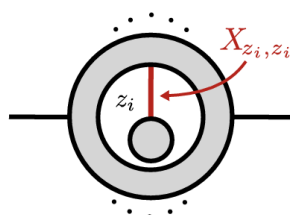
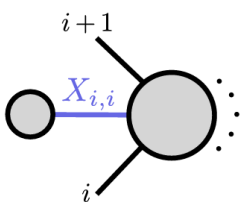
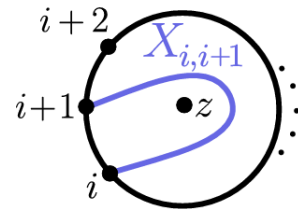
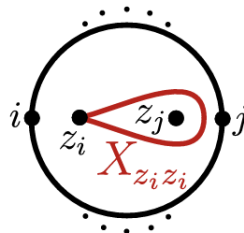
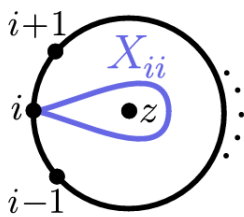
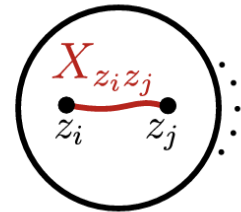
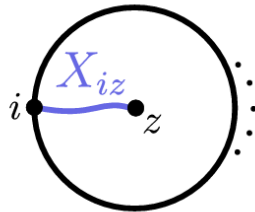
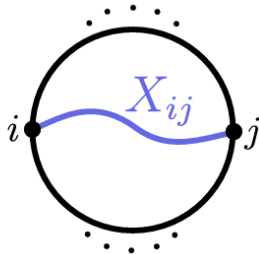


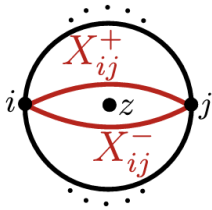
$$\begin{aligned} \frac{\partial}{\partial t_{1i}^{-1}} m_K^{\alpha'}(1 \dots K) &= \frac{1}{t_{1,K-1}} \frac{\partial}{\partial t_{1i}^{-1}} m_{K-1}^{\alpha'}(1 \dots K-1) + \frac{1}{t_{2,K}} \frac{\partial}{\partial t_{1i}^{-1}} m_{K-1}^{\alpha'}(2 \dots K) \\ &+ \sum_{j=3}^{K-2} \left\{ \frac{\partial}{\partial t_{1i}^{-1}} \left(1 + \frac{1}{t_{1j} t_{jK}} \right) \right\} m_j^{\alpha'}(1 \dots j) m_{K-j+1}^{\alpha'}(j \dots K) \\ &+ \sum_{j=3}^{K-2} \left(1 + \frac{1}{t_{1j} t_{jK}} \right) \frac{\partial}{\partial t_{1i}^{-1}} m_j^{\alpha'}(1 \dots j) m_{K-j+1}^{\alpha'}(j \dots K). \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t_{1i}^{-1}} m_K^{\alpha'}(1 \dots K) &= m_i^{\alpha'}(1 \dots i) \left\{ \frac{1}{t_{1,K-1}} m_{K-i+1}^{\alpha'}(i \dots K-1) + \frac{1}{t_{i,K}} m_{K-i+1}^{\alpha'}(i \dots K) \right. \\ &+ \left. \sum_{j=i+1}^{K-2} \left(1 + \frac{1}{t_{1j} t_{jK}} \right) m_{j-i+2}^{\alpha'}(i \dots j) m_{K-j+1}^{\alpha'}(j \dots K) \right\} \\ &\equiv m_i^{\alpha'}(1 \dots i) m_{K-i+2}^{\alpha'}(i \dots K) \end{aligned}$$



$$X_{i,z_j} = (\ell_j + p_1 + \dots + p_i)^2, X_{z_i,z_j} = (\ell_i - \ell_j)^2,$$





$$\begin{aligned}
 & \text{Diagram 1} \approx \text{Diagram 2} \\
 & \approx \frac{1 + t_{1j}t_{jn}}{t_{1j}t_{jn}} m_L^{\alpha'}(1 \dots j) m_R^{\alpha'}(j \dots n).
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 3} \approx \text{Diagram 4} \approx \text{Diagram 5}
 \end{aligned}$$

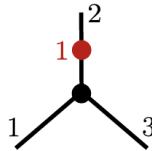
$$\begin{aligned}
 & \text{Diagram 6} \approx \text{Diagram 7} = \frac{1}{t_{1,n-1}} m_L^{\alpha'}(1 \dots n-1).
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 8} \approx \text{Diagram 9} \\
 & \approx \frac{1 + t_{1,n-1}t_{n-1,n}}{t_{1,n-1}t_{n-1,n}} m_L^{\alpha'}(1 \dots n-1) m_{1,R}^{\alpha'}(n-1, n).
 \end{aligned}$$

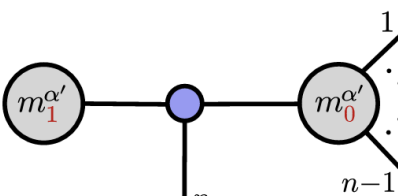
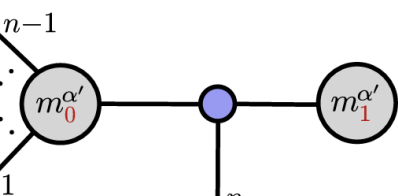
$$m_{1,n}^{\alpha'} \supset m_{0,L}^{\alpha'} + m_{1,L}^{\alpha'} + m_{1,R}^{\alpha'} + m_{0,R}^{\alpha'}$$

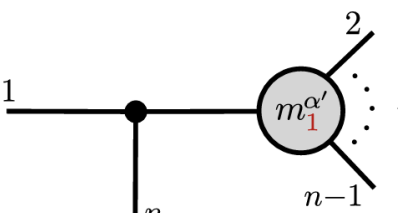
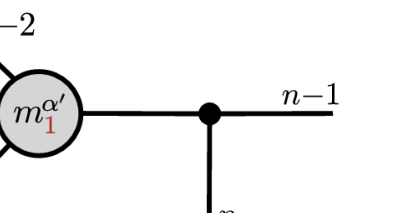
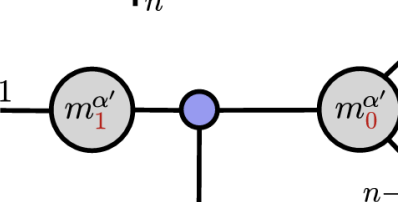
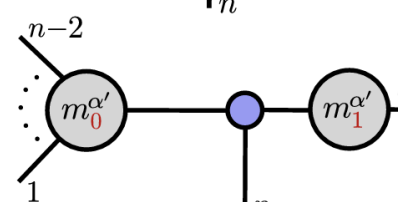


$$\begin{aligned}
m_{1,3}^{\alpha'}(123) &= \frac{1+t_{11}t_{13}}{t_{11}t_{13}} m_{1,1}^{\alpha'}(1)m_{0,3}^{\alpha'}(123) + \frac{1+t_{13}t_{33}}{t_{13}t_{33}} m_{0,3}^{\alpha'}(123)m_{1,1}^{\alpha'}(3) \\
&+ \frac{1}{t_{12}} m_{1,2}^{\alpha'}(12) + \frac{1}{t_{23}} m_{1,2}^{\alpha'}(23) + \frac{1+t_{1z}t_{3z}}{t_{1z}t_{3z}} m_{0,4}^{\alpha'}(123z) \\
&= \frac{1}{3} \frac{1}{t_{1z}t_{2z}t_{3z}} + \frac{1}{t_{12}t_{1z}t_{2z}} + \frac{1}{t_{11}t_{12}t_{1z}} + \frac{1}{t_{11}t_{13}t_{1z}} + \frac{1}{t_{1z}} + \frac{1}{t_{12}} + \text{cyc.}
\end{aligned}$$

$$m_{1,3}^{\alpha'} - m_{1,3,\text{Feyn}}^{\alpha'} = \frac{1}{t_{12}} + \text{cyc.} = \text{Diagram} + \text{cyc.}$$


$$m_{1,n}^{\alpha'} = m_{1,n,(1)}^{\alpha'} + m_{1,n,(2)}^{\alpha'} + m_{1,n,(\geq 3)}^{\alpha'} + m_{1,n,(\text{loop})}^{\alpha'}$$

$$\begin{aligned}
m_{1,n,(1)}^{\alpha'} &= \text{Diagram 1} + \text{Diagram 2} \\
&= \frac{1+t_{11}t_{1n}}{t_{11}t_{1n}} m_{1,1}^{\alpha'}(1)m_{0,n}^{\alpha'}(1\dots n) + \frac{1+t_{1n}t_{nn}}{t_{1n}t_{nn}} m_{0,n}^{\alpha'}(1\dots n)m_{1,1}^{\alpha'}(n).
\end{aligned}$$



$$\begin{aligned}
m_{1,n,(2)}^{\alpha'} &= \text{Diagram 3} + \text{Diagram 4} \\
&+ \text{Diagram 5} + \text{Diagram 6} \\
&= \frac{1}{t_{2n}} m_{1,n-1}^{\alpha'}(2\dots n) + \frac{1}{t_{1n-1}} m_{1,n-1}^{\alpha'}(1\dots n-1) \\
&+ \frac{1+t_{12}t_{2n}}{t_{12}t_{2n}} m_{1,2}^{\alpha'}(12)m_{0,n-1}^{\alpha'}(2\dots n) + \frac{1+t_{1n-1}t_{n-1n}}{t_{1n-1}t_{n-1n}} m_{0,n-1}^{\alpha'}(1\dots n-1)m_{1,2}^{\alpha'}(n-1n)
\end{aligned}$$





$$m_{1,n,(\geq 3)}^{\alpha'} = \sum_{j=3}^{n-2} \left(\begin{array}{c} j-1 \\ \vdots \\ m_0^{\alpha'} \\ \vdots \\ 1 \end{array} \text{---} \text{---} \begin{array}{c} j \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ n-1 \end{array} + \begin{array}{c} j-1 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 1 \end{array} \text{---} \text{---} \begin{array}{c} j \\ \vdots \\ m_0^{\alpha'} \\ \vdots \\ n-1 \end{array} \right)$$

$$= \sum_{j=3}^{n-2} \frac{1 + t_{1j}t_{jn}}{t_{1j}t_{jn}} \left\{ m_{0,j}^{\alpha'}(1 \dots j)m_{1,n-j+1}^{\alpha'}(j \dots n) + m_{1,j}^{\alpha'}(1 \dots j)m_{0,n-j+1}^{\alpha'}(j \dots n) \right\}$$

$$m_{1,n,(\text{loop})}^{\alpha'} = \begin{array}{c} 1 \quad \dots \quad n-1 \\ \vdots \\ m_0^{\alpha'} \\ \vdots \\ z \\ \text{---} \\ \text{---} \\ \text{---} \\ \vdots \\ n \end{array} = \frac{1 + t_{1z}t_{nz}}{t_{1z}t_{nz}} m_{0,n+1}^{\alpha'}(1 \dots nz).$$

$$m_{1,n}^{\alpha'} = (\pi\alpha')^{-n} (m_{1,n} + \mathcal{O}(\alpha'^2))$$

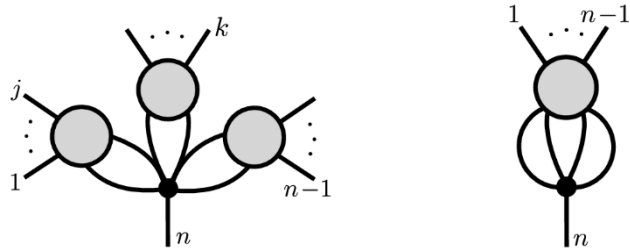
$$\begin{aligned} m_{1,4}^{\alpha'} &= \frac{1}{4} \frac{1}{t_{1z}t_{2z}t_{3z}t_{4z}} + \frac{1}{t_{1z}t_{2z}t_{3z}} \frac{1}{t_{13}} + \frac{1}{t_{1z}t_{2z}} \left\{ \frac{1}{t_{12}} \left(\frac{1}{t_{13}} + \frac{1}{t_{24}} \right) + 1 \right\} + \frac{1}{2} \frac{1}{t_{1z}t_{3z}} \frac{1}{t_{13}^2} \\ &+ \frac{1}{t_{1z}} \left\{ \frac{1}{t_{11}} \left(\frac{1}{t_{12}t_{13}} + \frac{1}{t_{13}t_{14}} + \frac{1}{t_{14}t_{24}} + \frac{1}{t_{24}t_{12}} + \frac{1}{t_{13}^2} + 1 \right) + \frac{2}{t_{13}} + \frac{1}{t_{24}} \right\} \\ &+ \frac{1}{t_{12}} \left(\frac{1}{t_{13}} + \frac{1}{t_{24}} \right) + \frac{1}{2} \frac{1}{t_{13}^2} + \frac{3}{4} + \text{cyc.} \end{aligned}$$

$$m_{1,4}^{\alpha'} - m_{1,4,\text{Feyn}}^{\alpha'} = \left(\begin{array}{c} 2 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 1 \end{array} \begin{array}{c} 3 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 4 \end{array} + \begin{array}{c} 1 \\ \color{red}{\bullet} \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 4 \end{array} \begin{array}{c} 2 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 3 \end{array} + \frac{1}{2} \begin{array}{c} 2 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 1 \end{array} \begin{array}{c} 3 \\ \color{red}{\bullet} \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 4 \end{array} + \text{cyc.} \right) + \begin{array}{c} 2 \\ \color{red}{\bullet} \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 1 \end{array} \begin{array}{c} 3 \\ \color{red}{\bullet} \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 4 \end{array},$$

$$\begin{aligned} \frac{\partial}{\partial t_{1z}^{-1}} \begin{array}{c} 2 \\ \vdots \\ m_1^{\alpha'} \\ \vdots \\ 1 \end{array} &= \begin{array}{c} 2 \\ \vdots \\ m_0^{\alpha'} \\ \vdots \\ 1 \end{array} \\ \Leftrightarrow \frac{\partial}{\partial t_{1z}^{-1}} m_{1,n}^{\alpha'}(1 \dots n) &= m_{0,n+2}^{\alpha'}(1z12 \dots n), \end{aligned}$$



$$m_{L_L, n_L}^{\alpha'} \cdot m_{L_R, n_R}^{\alpha'} \equiv \frac{L_L! L_R!}{L!} \sum_{Z_L \subset Z} m_{L_L, n_L}^{\alpha'}(Z_L) m_{L_R, n_R}^{\alpha'}(Z_R)$$



$$m_{1, n_L}^{\alpha'} \cdot m_{2, n_R}^{\alpha'} = \frac{1}{3} \left(m_{1, n_L}^{\alpha'}(z_1) m_{2, n_R}^{\alpha'}(z_2 z_3) + m_{1, n_L}^{\alpha'}(z_2) m_{2, n_R}^{\alpha'}(z_1 z_3) + m_{1, n_L}^{\alpha'}(z_3) m_{2, n_R}^{\alpha'}(z_1 z_2) \right)$$

$$m_{L, n}^{\alpha'} \supset \sum_{L=L_R+L_L} \text{Diagram} \simeq \frac{1+t_L t_R}{t_L t_R} \sum_{L=L_R+L_L} m_{L_L, n_L}^{\alpha'} \cdot m_{L_R, n_R}^{\alpha'}$$

$$m_{L, n}^{\alpha'} \supset \text{Diagram} = \frac{1+t_{1Z} t_{nZ}}{t_{1Z} t_{nZ}} \cdot m_{L-1, n+1}^{\alpha'}(1 \dots nZ)$$

$$\equiv \frac{1}{L} \sum_{i=1}^L \frac{1+t_{1z_i} t_{nz_i}}{t_{1z_i} t_{nz_i}} m_{L-1, n+1}^{\alpha'}(1 \dots nz_i)$$

$$m_{L, 1}^{\alpha'} = \frac{1+t_{11}^2}{t_{11}^2} \sum_{l=1}^{L-1} m_{l, 1}^{\alpha'}(1) \cdot m_{L-l, 1}^{\alpha'}(1) + \frac{1+t_{1Z}^2}{t_{1Z}^2} \cdot m_{L-1, 2}^{\alpha'}(1Z)$$

$$m_{L, 1}^{\alpha'} \supset \sum_{l=1}^{L-1} \text{Diagram} = \frac{1+t_{11}^2}{t_{11}^2} \sum_{l=1}^{L-1} m_{l, 1}^{\alpha'}(1) \cdot m_{L-l, 1}^{\alpha'}(1)$$

$$m_{L, 2}^{\alpha'} = \frac{1+t_{11} t_{12}}{t_{11} t_{12}} \sum_{l=1}^{L-1} m_{l, 1}^{\alpha'}(1) \cdot m_{L-l, 2}^{\alpha'}(12) + \frac{1+t_{12} t_{22}}{t_{12} t_{22}} \sum_{l=1}^{L-1} m_{l, 2}^{\alpha'}(12) \cdot m_{L-l, 1}^{\alpha'}(2)$$

$$+ \frac{1}{t_{11}} m_{L, 1}^{\alpha'}(1) + \frac{1}{t_{22}} m_{L, 1}^{\alpha'}(2) + \frac{1+t_{1Z} t_{2Z}}{t_{1Z} t_{2Z}} \cdot m_{L-1, 3}^{\alpha'}(12Z)$$

$$\begin{aligned}
m_{L,3}^{\alpha'} &= \frac{1+t_{11}t_{13}}{t_{11}t_{13}} \sum_{l=1}^L m_{l,1}^{\alpha'}(1) \cdot m_{L-l,3}^{\alpha'}(123) + \frac{1+t_{13}t_{33}}{t_{13}t_{33}} \sum_{l=0}^{L-1} m_{l,3}^{\alpha'}(123) \cdot m_{L-l,1}^{\alpha'}(3) \\
&+ \frac{1}{t_{12}} m_{L,2}(12) + \frac{1}{t_{23}} m_{L,2}(23) + \frac{1+t_{12}t_{23}}{t_{12}t_{23}} \sum_{l=1}^{L-1} m_{l,2}^{\alpha'}(12) \cdot m_{L-l,2}^{\alpha'}(23) \\
&+ \frac{1+t_{1Z}t_{3Z}}{t_{1Z}t_{3Z}} \cdot m_{L-1,4}^{\alpha'}(123Z)
\end{aligned}$$

$$m_{L,n}^{\alpha'} = m_{L,n,(1)}^{\alpha'} + m_{L,n,(2)}^{\alpha'} + m_{L,n,(\geq 3)}^{\alpha'} + m_{L,n,(\text{loop})}^{\alpha'}$$

$$m_{L,n,(1)}^{\alpha'} = \sum_{l=1}^L \left(\frac{1+t_{11}t_{1n}}{t_{11}t_{1n}} m_{l,1}^{\alpha'}(1) \cdot m_{L-l,n}^{\alpha'}(1 \dots n) + \frac{1+t_{1n}t_{nn}}{t_{1n}t_{nn}} m_{L-l,n}^{\alpha'}(1 \dots n) \cdot m_{l,1}^{\alpha'}(n) \right)$$

$$\begin{aligned}
m_{L,n,(2)}^{\alpha'} &= \frac{1}{t_{2n}} m_{L,n-1}^{\alpha'}(2 \dots n) + \frac{1}{t_{1n-1}} m_{L,n-1}^{\alpha'}(1 \dots n-1) \\
&+ \sum_{l=1}^L \left(\frac{1+t_{12}t_{2n}}{t_{12}t_{2n}} m_{l,2}^{\alpha'}(12) \cdot m_{L-l,n-1}^{\alpha'}(2 \dots n) \right. \\
&\quad \left. + \frac{1+t_{1n-1}t_{n-1n}}{t_{1n-1}t_{n-1n}} m_{L-l,n-1}^{\alpha'}(1 \dots n-1) \cdot m_{l,2}^{\alpha'}(n-1n) \right)
\end{aligned}$$

$$m_{L,n,(\geq 3)}^{\alpha'} = \sum_{j=3}^{n-2} \frac{1+t_{1j}t_{jn}}{t_{1j}t_{jn}} \sum_{l=0}^L m_{l,j}^{\alpha'}(1 \dots j) \cdot m_{L-l,n-j+1}^{\alpha'}(j \dots n).$$

$$m_{L,n,(\text{loop})}^{\alpha'} = \frac{1+t_{1Z}t_{nZ}}{t_{1Z}t_{nZ}} \cdot m_{L-1,n+1}^{\alpha'}(1 \dots nZ),$$

$$\begin{aligned}
m_{2,1}^{\alpha'} &= \text{Diagram 1} + \text{Diagram 2} \\
&= \frac{1}{2!} \left(\frac{1}{t_{1z_1}^2 t_{z_1 z_2} t_{1z_2}} + \frac{1}{t_{1z_1}^2 t_{z_1 z_1} t_{z_1 z_2}} + \frac{1}{t_{1z_1}^2 t_{11} t_{1z_2}} + \frac{1}{t_{1z_1} t_{11}^2 t_{1z_2}} \right. \\
&\quad \left. + \frac{1}{t_{1z_1} t_{1z_2}} + \frac{1}{t_{1z_1} t_{z_1 z_2}} + \frac{1}{t_{11} t_{1z_1}} + \frac{1}{t_{z_1 z_1} t_{z_1 z_2}} + \frac{1}{t_{1z_1}^2} + (z_1 \leftrightarrow z_2) \right) + 1.
\end{aligned}$$

$$m_{L,n}^{\alpha'}|_{\text{ct}} \equiv \kappa_{L,n} = \begin{cases} \neq 0, & \text{if } L+n \text{ is odd} \\ 0, & \text{if } L+n \text{ is even} \end{cases}$$

$$m_{L,n}^{\alpha'} \simeq \mathcal{O}\left(\frac{1}{t^{3(L-1)+n}}\right)$$

$$\kappa_{L,n} = \sum_{l=0}^L \sum_{k=1}^n \kappa_{l,k} \kappa_{L-l,n-k+1} + \kappa_{L-1,n+1}$$



$$\frac{\partial \mathcal{L}_{\text{KLT}}}{\partial \phi} \equiv \mathcal{L}'_{\text{KLT}} = \frac{1}{2} \left(\left(1 - \frac{\hbar}{\phi}\right) \left[1 - \sqrt{1 + 4 \frac{\frac{\hbar}{\phi} K_1(\hbar) - \phi^2}{(1 - \frac{\hbar}{\phi})^2}} \right] \right)$$

$$\kappa_{L,n} = \frac{1}{L!(n-1)!} \frac{\partial^L}{\partial \hbar^L} \frac{\partial^{n-1}}{\partial \phi^{n-1}} \mathcal{L}'_{\text{KLT}} \Big|_{\substack{\hbar=0 \\ \phi=0}}.$$

$$K_1(\hbar) = \sum_{l=1}^{\infty} \kappa_{2l,1} \hbar^{2l} = \sum_{l=1}^{\infty} \left| \begin{array}{c} 2l \\ 1 \end{array} \right| \hbar^{2l}.$$

$$\kappa_{2l,1} = \frac{1}{2} \frac{3^l}{4l-1} C_{2l} \Rightarrow K_1(\hbar) = \sum_{l=1}^{\infty} \kappa_{2l,1} \hbar^{2l} = \frac{1}{2} + \frac{1}{6} \frac{(1-x)^{3/2} - (1+x)^{3/2}}{x},$$

$$\kappa_{L,n} = \frac{1}{L!} 4^{\lfloor \frac{L-1}{2} \rfloor} P_L(n) C_{\lfloor \frac{n-1}{2} \rfloor}$$

$$P_2(n) = n(n+1),$$

$$P_3(n) = n(n-1)(n+4),$$

$$P_4(n) = n(n+1)(n+2)(n+8),$$

$$P_5(n) = n(n-1)(n^3 + 20n^2 + 112n + 228),$$

$$P_6(n) = n(n+1)(n+2)(n^3 + 30n^2 + 272n + 912),$$

...

$$P_Z^{(i)} = \{p^{(i)}(z_1), \dots, p^{(i)}(z_L)\}, p^{(i)}(z_j) \equiv \text{even/odd}$$

$$X = X_{\text{even}}^{(i)} \cup X_{\text{odd}}^{(i)}, i = 1, \dots, 2^L,$$

$$\alpha' \rightarrow \alpha'/2, \text{ and } \hat{X}_{\text{even}}^{(i)} = X_{\text{even}}^{(i)} \pm 1/\alpha', \hat{X}_{\text{odd}}^{(i)} = X_{\text{odd}}^{(i)}$$

$$A_{L,2k}^{\pi, \alpha'}(X) = \sum_{i=1}^{2^L} m_{L,2k}^{\alpha'/2}(\hat{X}_{\text{even}}^{(i)}, \hat{X}_{\text{odd}}^{(i)})$$

$$A_{L,2k}^{\pi} = \sum_{i=1}^{2^L} \lim_{\delta \rightarrow \infty} \delta^{2k+2L-2} m_{L,2k}(\hat{X}_{\text{even}}^{(i)}(\delta), \hat{X}_{\text{odd}}^{(i)}(\delta))$$

$$A_{L,2k}^{\pi, \alpha'} = \left(\frac{\pi \alpha'}{2}\right)^{1-L} (A_{L,2k}^{\pi} + \mathcal{O}(\alpha'^2))$$

$$X_{\text{even}}^{(1)} = \{X_{11}, X_{22}, X_{2z}\}, \quad X_{\text{odd}}^{(1)} = \{X_{12}, X_{1z}\},$$

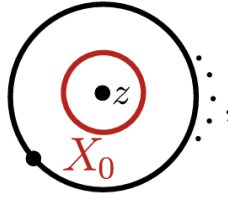
$$X_{\text{even}}^{(2)} = \{X_{11}, X_{22}, X_{1z}\}, \quad X_{\text{odd}}^{(2)} = \{X_{12}, X_{2z}\},$$

$$m_{1,2}^{\alpha'/2}(\hat{X}_{\text{even}}^{(1)}, \hat{X}_{\text{odd}}^{(1)}) = 1 - \frac{\tau_{11} + \tau_{2z}}{\tau_{1z}} + \tau_{22}\tau_{2z}, \quad m_{1,2}^{\alpha'/2}(\hat{X}_{\text{even}}^{(2)}, \hat{X}_{\text{odd}}^{(2)}) = 1 - \frac{\tau_{22} + \tau_{1z}}{\tau_{2z}} + \tau_{11}\tau_{1z}$$

$$(5.4) \Rightarrow A_{1,2}^{\pi, \alpha'} = 2 - \frac{\tau_{11} + \tau_{2z}}{\tau_{1z}} - \frac{\tau_{22} + \tau_{1z}}{\tau_{2z}} + \tau_{11}\tau_{1z} + \tau_{22}\tau_{2z}$$



$$A_{L,2k}^{\alpha',\pi} \xrightarrow{\text{surf.soft}} 0,$$



$$m_{1,1}^{(X_0)} = \frac{1}{t_{1z}} + \frac{1}{t_0}, m_{1,2}^{(X_0)} = \frac{1}{t_{11}t_{1z}} + \frac{1}{t_{11}t_0} + \frac{1}{t_{22}t_{2z}} + \frac{1}{t_{22}t_0} + \frac{1}{t_{1z}t_{2z}} + 1,$$

$$m_{\text{diag}}^{\alpha'}(\sigma) \equiv m^{\alpha'}(\sigma | \sigma)$$

$$\mathcal{L}_{\alpha'} = -\frac{1}{2} \langle \phi \Delta^{-1} \phi \rangle + V(\phi)$$

$$\Delta^{-1} = \tan(\pi \alpha' \square)$$

$$V(\phi) = \left\langle \frac{1}{2} \phi - \frac{1}{8} \arcsin 2\phi - \frac{1}{4} \phi \sqrt{1 - 4\phi^2} \right\rangle = \sum_{k=1}^{\infty} \frac{C_{k-1}}{2k+1} \langle \phi^{2k+1} \rangle,$$

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

$$\phi = \phi^a T^a \text{ and } \langle \square \rangle \equiv \text{Tr}(\square)$$

$$\exp\left(\frac{i}{\hbar} \mathcal{T}[\phi]\right) \equiv \exp\left(-i \frac{\hbar}{2} \frac{\delta}{\delta \phi} \cdot \Delta \cdot \frac{\delta}{\delta \phi}\right) \exp\left(\frac{i}{\hbar} S_{\text{int}}[\phi]\right)$$

$$\frac{\delta}{\delta \phi} \cdot \Delta \cdot \frac{\delta}{\delta \phi} \equiv \int dx dy \frac{\delta}{\delta \phi^a(x)} \Delta(x, y) \frac{\delta}{\delta \phi^a(y)}$$

$$S =: \exp(i\mathcal{T}[\hat{\phi}]):$$

$$\square \hat{\phi} = 0$$

$$\begin{aligned} A(1^{a_1}, 2^{a_2}, \dots, n^{a_n}) &= \sum_{\sigma \in S_n / \mathbb{Z}_n} \langle T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}} \rangle m_{\text{diag}}^{\alpha'}(\sigma) \\ &= \left(\prod_{j=1}^n \int dx_j e^{ip_j x_j} \frac{\delta}{\delta \phi^{a_j}(x_j)} \right) \mathcal{T}[\phi] \Big|_{\phi \rightarrow 0}. \end{aligned}$$

$$\begin{aligned} \frac{\delta \mathcal{T}[\phi]}{\delta \phi_a(x)} &= e^{-\frac{i}{\hbar} \mathcal{T}[\phi]} \exp\left(-i \frac{\hbar}{2} \frac{\delta}{\delta \phi} \cdot \Delta \cdot \frac{\delta}{\delta \phi}\right) \frac{\delta S_{\text{int}}[\phi]}{\delta \phi_a(x)} \exp\left(\frac{i}{\hbar} S_{\text{int}}[\phi]\right) \\ &= e^{-\frac{i}{\hbar} \mathcal{T}[\phi]} \exp\left(-i \frac{\hbar}{2} \frac{\delta}{\delta \phi} \cdot \Delta \cdot \frac{\delta}{\delta \phi} - i \frac{\hbar}{2} \frac{\delta}{\delta \chi} \cdot \Delta \cdot \frac{\delta}{\delta \chi} - i \hbar \frac{\delta}{\delta \chi} \cdot \Delta \cdot \frac{\delta}{\delta \phi}\right) \\ &\quad \times \frac{\delta S_{\text{int}}[\chi]}{\delta \phi_a(x)} \exp\left(\frac{i}{\hbar} S_{\text{int}}[\phi]\right) \Big|_{\chi=\phi} \end{aligned}$$



$$\exp\left(-i\frac{\hbar}{2}\frac{\delta}{\delta\chi}\cdot\Delta\cdot\frac{\delta}{\delta\chi}\right)\frac{\delta S_{int}[\chi]}{\delta\phi_A(x)}\rightarrow 0$$

$$\exp\left(-i\frac{\hbar}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}\right)\exp\left(\frac{i}{\hbar}S_{int}[\phi]\right)=e^{\frac{i}{\hbar}\mathcal{T}[\phi]}$$

$$\begin{aligned}\frac{\delta\mathcal{T}[\phi]}{\delta\phi_a(x)} &= e^{-\frac{i}{\hbar}\mathcal{T}[\phi]}\exp\left(-i\hbar\frac{\delta}{\delta\chi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}\right)\frac{\delta S_{int}[\chi]}{\delta\phi_a(x)}e^{\frac{i}{\hbar}\mathcal{T}[\phi]}\Bigg|_{\chi=\phi} \\ &= e^{-\frac{i}{\hbar}\mathcal{T}[\phi]}\frac{\delta S_{int}}{\delta\phi_a(x)}\left[\chi-i\hbar\Delta\cdot\frac{\delta}{\delta\phi}\right]e^{\frac{i}{\hbar}\mathcal{T}[\phi]}1\Bigg|_{\chi=\phi} \\ &= \frac{\delta S_{int}}{\delta\phi_a(x)}\left[\chi-i\hbar\Delta\cdot\left(\frac{\delta}{\delta\phi}+\frac{i}{\hbar}\frac{\delta\mathcal{T}[\phi]}{\delta\phi}\right)\right]1\Bigg|_{\chi=\phi}\end{aligned}$$

$$\frac{\delta\mathcal{T}[\phi]}{\delta\phi_a(x)}=\frac{\delta S_{int}}{\delta\phi_a(x)}\left[\chi-i\hbar\Delta\cdot\left(\frac{\delta}{\delta\phi}+\frac{i}{\hbar}\frac{\delta\mathcal{T}[\phi]}{\delta\phi}\right)\right]1\Bigg|_{\chi=\phi}$$

$$\exp\left(\psi\cdot\frac{\delta}{\delta\phi}\right)F[\phi]=F[\phi+\psi]$$

$$e^{-\frac{i}{\hbar}\mathcal{T}[\phi]}\frac{\delta}{\delta\phi}e^{\frac{i}{\hbar}\mathcal{T}[\phi]}=\frac{\delta}{\delta\phi}+\frac{i}{\hbar}\frac{\delta\mathcal{T}[\phi]}{\delta\phi}$$

$$\mathcal{T}[\phi]=\mathcal{T}^{(0)}[\phi]+\hbar\mathcal{T}^{(1)}[\phi]+O(\hbar^2)$$

$$\frac{\delta\mathcal{T}^{(0)}[\phi]}{\delta\phi_a(x)}=\frac{\delta S_{int}}{\delta\phi_a(x)}[J]$$

$$J^a=\phi^a+\Delta\cdot\frac{\delta\mathcal{T}^{(0)}[\phi]}{\delta\phi^a}$$

$$S_{int}[\phi]\equiv\int dx\langle V(\phi(x))\rangle,\phi\equiv T^a\phi^a$$

$$\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}=\frac{\partial V(J)}{\partial J},J=\phi+\Delta\cdot\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}$$

$$\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}\equiv T^a\frac{\delta\mathcal{T}^{(0)}}{\delta\phi^a},\frac{\partial V(J)}{\partial J}\equiv T^a\frac{\partial V(J)}{\partial J^a},\Delta\cdot\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}\equiv\int dy\Delta(x,y)\frac{\delta\mathcal{T}^{(0)}}{\delta\phi(y)}$$

$$\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}=\sum_{k=1}^{\infty}C_{k-1}\left(\phi+\Delta\cdot\frac{\delta\mathcal{T}^{(0)}}{\delta\phi}\right)^{2k}$$

$$\frac{\partial V(J)}{\partial J}=J^2+\left(\frac{\partial V(J)}{\partial J}\right)^2$$

$$J^2+\left(\frac{\partial V(J)}{\partial J}\right)^2=J^2+\sum_{i,j=1}^{\infty}C_{i-1}C_{j-1}J^{2(i+j)}=J^2+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}C_{k-1}C_{n-k-1}J^{2n}$$



$$\sum_{k=1}^{n-1} C_{k-1} C_{n-k-1} = C_{n-1}$$

$$J^2 + \left(\frac{\partial V(J)}{\partial J}\right)^2 = J^2 + \sum_{n=2}^{\infty} C_{n-1} J^{2n} = \sum_{n=1}^{\infty} C_{n-1} J^{2n} = \frac{\partial V(J)}{\partial J}$$

$$\frac{\delta \mathcal{T}^{(0)}}{\delta \phi} = J^2 + \left(\frac{\partial V(J)}{\partial J}\right)^2, J = \phi + \Delta \cdot \frac{\delta \mathcal{T}^{(0)}}{\delta \phi}$$

$$\frac{\delta \mathcal{T}^{(0)}}{\delta \phi} = \left(\phi + \Delta \cdot \frac{\delta \mathcal{T}^{(0)}}{\delta \phi}\right)^2 + \left(\frac{\delta \mathcal{T}^{(0)}}{\delta \phi}\right)^2$$

$$\frac{\delta \mathcal{T}^{(0)}}{\delta \phi} = \phi^2 + \phi \Delta \cdot \frac{\delta \mathcal{T}^{(0)}}{\delta \phi} + \frac{\delta \mathcal{T}^{(0)}}{\delta \phi} \cdot \Delta \phi + \frac{\delta \mathcal{T}^{(0)}}{\delta \phi} \cdot (1 + \Delta \Delta) \cdot \frac{\delta \mathcal{T}^{(0)}}{\delta \phi}$$

$$\begin{aligned} \frac{\delta \mathcal{T}^{(0)}}{\delta \phi(x)} &= \phi(x)^2 + \phi(x) \int dy \Delta(x, y) \frac{\delta \mathcal{T}^{(0)}}{\delta \phi(y)} + \int dy \Delta(x, y) \frac{\delta \mathcal{T}^{(0)}}{\delta \phi(y)} \phi(x) \\ &+ \int dy dz [\delta(x-y)\delta(x-z) + \Delta(x, y)\Delta(x, z)] \frac{\delta \mathcal{T}^{(0)}}{\delta \phi(y)} \frac{\delta \mathcal{T}^{(0)}}{\delta \phi(z)} \end{aligned}$$

$$\kappa_{L,n} = \sum_{l=0}^L \sum_{k=1}^n \kappa_{l,k} \kappa_{L-l,n-k+1} + \kappa_{L-1,n+1}$$

$$\mathcal{L}'_L \equiv \sum_{n=1}^{\infty} \kappa_{L,n} \phi^{n-1}$$

$$\mathcal{L}'_L = \sum_{l=0}^L \mathcal{L}'_l \mathcal{L}'_{L-l} + \frac{1}{\phi} (\mathcal{L}'_{L-1} - \kappa_{L-1,1})$$

$$\begin{aligned} \mathcal{L}'_L &= \frac{1}{1 - 2\mathcal{L}'_0} \left[\sum_{l=1}^{L-1} \mathcal{L}'_l \mathcal{L}'_{L-l} + \frac{1}{\phi} (\mathcal{L}'_{L-1} - \kappa_{L-1,1}) \right] \\ &= \frac{1}{\sqrt{1 - 4\phi^2}} \left[\sum_{l=1}^{L-1} \mathcal{L}'_l \mathcal{L}'_{L-l} + \frac{1}{\phi} (\mathcal{L}'_{L-1} - \kappa_{L-1,1}) \right] \end{aligned}$$

$$\mathcal{L}'_0 = \frac{1}{2} (1 - \sqrt{1 - 4\phi^2})$$

$$\mathcal{L}'_L(\phi) = \frac{1}{\sqrt{1 - 4\phi^2}} \left[\sum_{l=1}^{L-1} \mathcal{L}'_l(\phi) \mathcal{L}'_{L-l}(\phi) + \frac{1}{\phi} (\mathcal{L}'_{L-1}(\phi) - \mathcal{L}'_{L-1}(0)) \right]$$



$$\begin{aligned}\mathcal{L}'_1(\phi) &= \frac{1}{\sqrt{1-4\phi^2}} \frac{1}{\phi} \mathcal{L}'_0 = \frac{1}{2\phi} \left(\frac{1}{\sqrt{1-4\phi^2}} - 1 \right), & \mathcal{L}'_2(\phi) &= \frac{1}{(1-4\phi^2)^{3/2}} \\ \mathcal{L}'_3(\phi) &= 8\phi \frac{1-2\phi^2}{(1-4\phi^2)^{5/2}}, & \mathcal{L}'_4(\phi) &= \frac{9-16\phi^2}{(1-4\phi^2)^{7/2}} \\ \mathcal{L}'_5(\phi) &= 16\phi \frac{9-56\phi^2+144\phi^4-144\phi^6}{(1-4\phi^2)^{9/2}}, & \mathcal{L}'_6(\phi) &= \frac{162-32\phi^2(27-64\phi^2+64\phi^4)}{(1-4\phi^2)^{11/2}}\end{aligned}$$

$$\mathcal{L}' = \sum_{L=1}^{\infty} \hbar^L \mathcal{L}'_L$$

$$\mathcal{L}'(\phi) = \frac{1}{\sqrt{1-4\phi^2}} \left[\mathcal{L}'(\phi)^2 + \frac{\hbar}{\phi} (\mathcal{L}'(\phi) + \mathcal{L}'_0(\phi) - \mathcal{L}'(0)) \right]$$

$$\mathcal{L}' = \frac{1}{2} \left(1 - \frac{\hbar}{\phi} \right) \left[1 - \sqrt{1 + 4 \frac{\frac{\hbar}{\phi} \mathcal{L}'(0) - \phi^2}{\left(1 - \frac{\hbar}{\phi}\right)^2}} \right] - \mathcal{L}'_0$$

$$\lim_{\phi \rightarrow 0} \mathcal{L}' = \mathcal{L}'(0)$$

$$\mathcal{L}'(0) = \sum_{k=1}^{\infty} \kappa_{2k,1} \hbar^{2k}$$

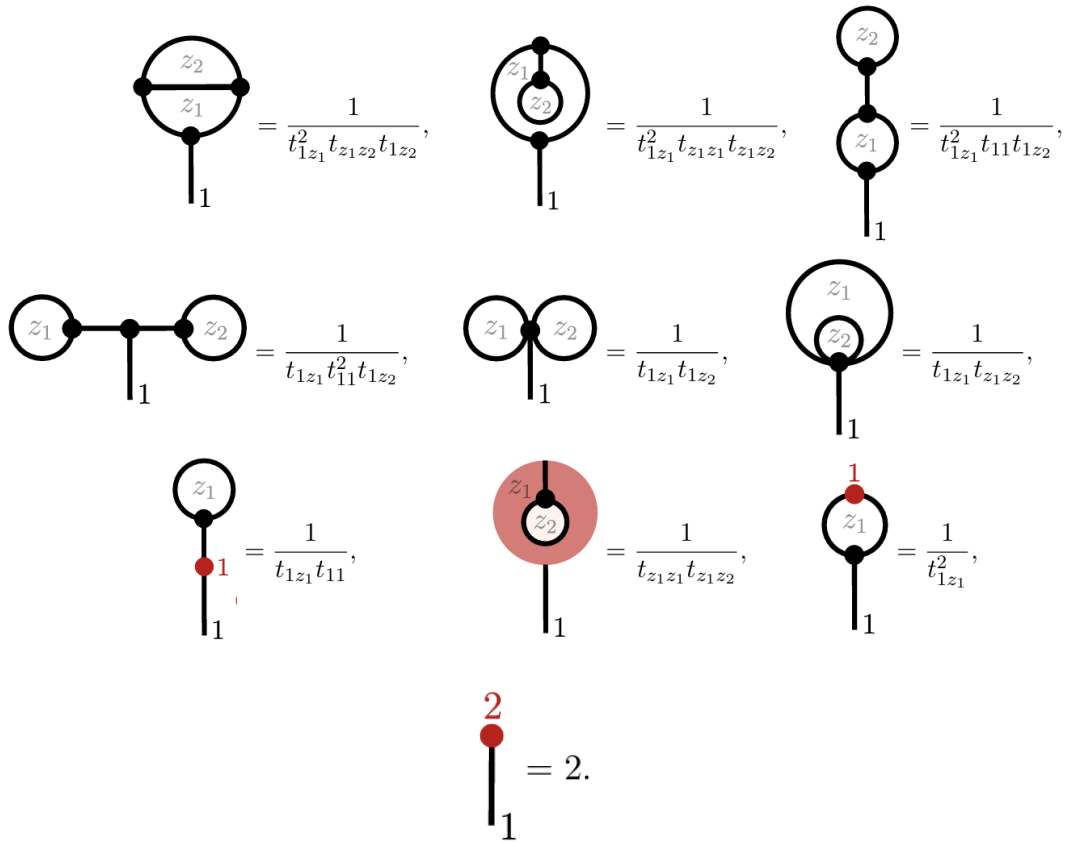
$$\mathcal{L}'_L(\phi, \{\kappa_{l,1}\}) = \frac{1}{L!} \frac{\partial^L}{\partial \hbar^L} \mathcal{L}' \Big|_{\hbar=0}$$

$$\begin{aligned}\mathcal{L}'_1(\phi, \{\kappa_{l,1}\}) &= \frac{1}{2\phi} \left(\frac{1}{\sqrt{1-4\phi^2}} - 1 \right) \\ \mathcal{L}'_2(\phi, \{\kappa_{l,1}\}) &= \frac{1}{(1-4\phi^2)^{3/2}} \\ \mathcal{L}'_3(\phi, \{\kappa_{l,1}\}) &= \frac{1 - (1-4\phi^2)^2 \kappa_{2,1}}{\phi(1-4\phi^2)^{5/2}} = \frac{1 - \kappa_{2,1}}{\phi} + O(\phi) \\ \mathcal{L}'_4(\phi, \{\kappa_{l,1}\}) &= \frac{1 + \phi^2 - (1-4\phi^2)^2 \kappa_{2,1}}{\phi^2(1-4\phi^2)^{7/2}} = \frac{1 - \kappa_{2,1}}{\phi^2} + O(1) \\ \mathcal{L}'_5(\phi, \{\kappa_{l,1}\}) &= \frac{1 - \kappa_{2,1} + \phi^2 [3 + (6 - 32\phi^4) \kappa_{2,1} - (1-4\phi^2)^4 \kappa_{4,1}]}{\phi^3(1-4\phi^2)^{9/2}} \\ &= \frac{1 - \kappa_{2,1}}{\phi^3} + \frac{21 - 12\kappa_{2,1} - \kappa_{4,1}}{\phi} + O(\phi) \\ &\dots(\text{B.5})\end{aligned}$$

$$\kappa_{2,1} = 1, \kappa_{4,1} = 9, \dots$$

$$\kappa_{2k,1} = 1, 9, 162, 3861, 107406, 3296538, 108314820, \dots$$





$$\mathcal{L}_F = -\frac{1}{2} \langle \phi F(\square) \phi \rangle + V(\phi)$$

$$\mathcal{S} = \tilde{\psi}_0(-\partial_t + D_0 \nabla^2 - D_0 \tau_0) \psi_0 + \frac{D_0 g_0}{2} (\tilde{\psi}_0^2 \psi_0 - \tilde{\psi}_0 \psi_0^2) + D_0 h_0 \tilde{\psi}_0$$

$\psi_0 \equiv \psi_0(t, \mathbf{x})$ is the other parameters $\tilde{\psi}_0 \equiv \tilde{\psi}_0(t, \mathbf{x})$

$$\partial_t = \partial / \partial t \nabla^2 = \sum_{i=1}^d \partial^2 / \partial x_i^2$$

$$D_0 h_0 \equiv D_0 h_0(t, \mathbf{x})$$

$$\tilde{\psi}_0 D_0 \nabla^2 \psi_0 = D_0 \int dt \int d^d x \tilde{\psi}_0(t, \mathbf{x}) \nabla^2 \psi_0(t, \mathbf{x})$$

$$\partial_t \psi_0 = D_0 (\nabla^2 - \tau_0) \psi_0 - \frac{D_0 g_0}{2} \psi_0^2 + D_0 h_0 + \sqrt{\psi_0} \zeta$$

$$\langle \zeta(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = g_0 D_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}')$$

$$\mathcal{W}[\tilde{A}, A] = \ln \int \mathcal{D}[\tilde{\psi}_0, \psi_0] \exp(\mathcal{S} + \tilde{A} \tilde{\psi}_0 + A \psi_0)$$

$$\langle \psi_0 \rangle \equiv m_0 \neq 0$$

$$\psi_0 = m_0 + \varphi_0$$



$$S = \tilde{\varphi}_0(-\partial_t + D_0\nabla^2 - D_0\tau_0 - D_0g_0m_0)\varphi_0 + \frac{g_0D_0}{2}m_0\tilde{\varphi}_0^2 + \frac{D_0g_0}{2}(\tilde{\varphi}_0^2\varphi_0 - \tilde{\varphi}_0\varphi_0^2) + \tilde{\varphi}_0D_0\left[h_0 - m_0\left(\tau_0 + \frac{1}{2}g_0m_0\right)\right]$$

$$0 = D_0h_0 - D_0m_0\left(\tau_0 + \frac{1}{2}g_0m_0\right) + \sum_{\text{graphs}}$$

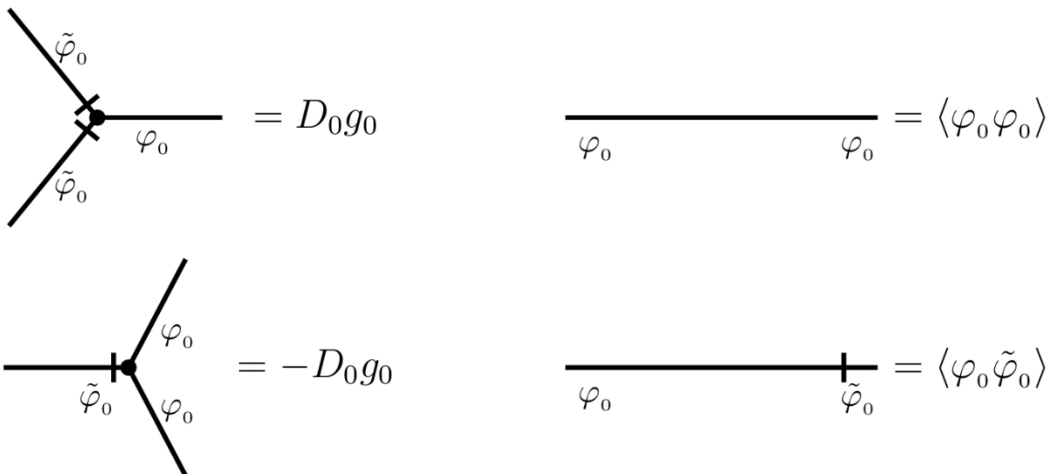
$$\langle\varphi_0\tilde{\varphi}_0\rangle = \langle\tilde{\varphi}_0\varphi_0\rangle^* = \frac{1}{-i\omega + D_0(\mathbf{k}^2 + \tilde{\tau}_0)}, \langle\varphi_0\varphi_0\rangle = \frac{D_0g_0m_0}{\omega^2 + D_0^2(\mathbf{k}^2 + \tilde{\tau}_0)^2},$$

$$\tilde{\tau}_0 \equiv \tau_0 + g_0m_0$$

$$\langle\varphi_0(t)\tilde{\varphi}_0(t')\rangle = \theta(t-t')\exp\{-D_0(\mathbf{k}^2 + \tilde{\tau}_0)(t-t')\}$$

$$\langle\varphi_0(t)\varphi_0(t')\rangle = \frac{g_0m_0}{2(\mathbf{k}^2 + \tilde{\tau}_0)}\exp\{-D_0(\mathbf{k}^2 + \tilde{\tau}_0)|t-t'|\}$$

$$V_{\varphi_0\tilde{\varphi}_0\tilde{\varphi}_0} = -V_{\varphi_0\varphi_0\tilde{\varphi}_0} = D_0g_0.$$

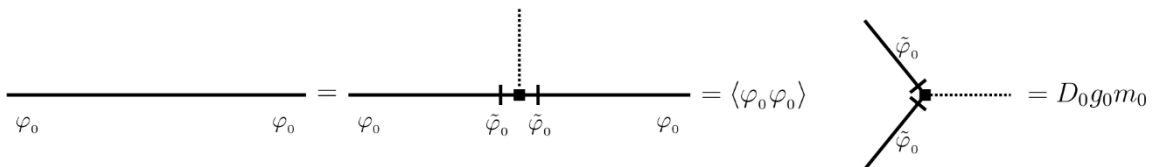


$$\text{Field } \varphi_0: v_1 + 2v_2 = p_1 + 2p_2,$$

$$\text{Field } \tilde{\varphi}_0: 2v_1 + v_2 = p_1 + 1$$

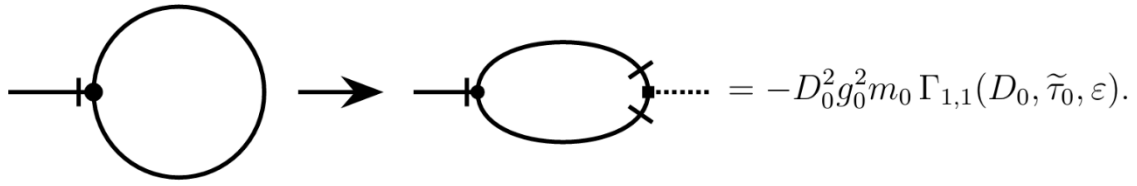
$$l = v_1 + p_2, l + p_2 = v_2 + 1.$$

$$\langle\varphi_0\varphi_0\rangle = D_0g_0m_0\langle\varphi_0\tilde{\varphi}_0\rangle\langle\tilde{\varphi}_0\varphi_0\rangle.$$



$$\sum_{\text{graphs}} = \sum_{n=1}^l (D_0g_0)^{2n-1} \sum_{j=1}^n (-1)^{n+j-1} (D_0g_0m_0)^j \Gamma_{n,j}(D_0, \tilde{\tau}_0, \varepsilon),$$





$$\Gamma_{1,1}(D_0, \tilde{\tau}_0, \varepsilon) = \frac{\Gamma\left(1 + \frac{\varepsilon}{2}\right) \tilde{\tau}_0^{1-\varepsilon/2}}{(4\pi)^{d/2}} \frac{1}{D_0 \varepsilon(\varepsilon - 2)}$$

$$\tilde{\Gamma}(\tau_0, \varepsilon) = -\frac{\partial \tau_0^{-n\varepsilon/2}}{\partial \tau_0} \langle \tilde{\varphi}_0 \varphi_0 \rangle \Big|_{\substack{p=0, \omega=0 \\ V_{\varphi_0 \tilde{\varphi}_0 \tilde{\varphi}_0}} \quad \mathbf{k} \rightarrow \tilde{\tau}_0^{1/2} \mathbf{k}$$

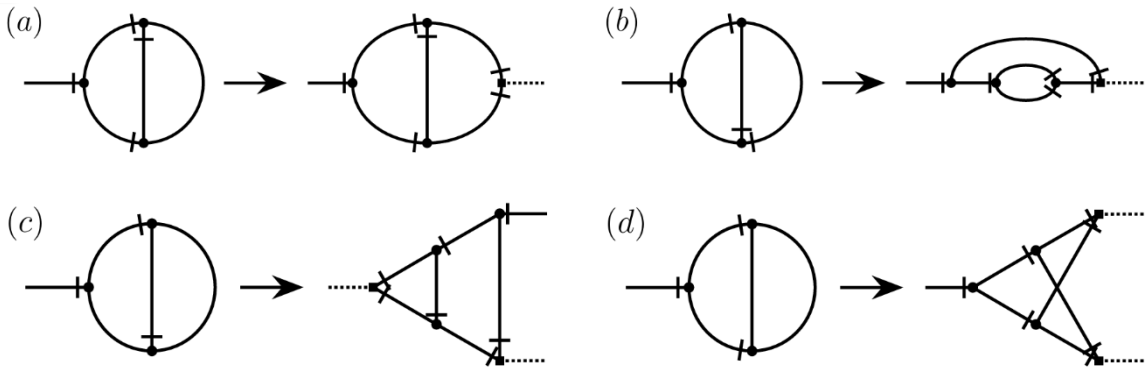
$$\Gamma_{n,1}^{(i)}(D_0, \tilde{\tau}_0, \varepsilon) = \frac{\tilde{\tau}_0^{1-\varepsilon n/2}}{D_0^{2n-1}} \Gamma_{n,1}^{(i)}(\tilde{\tau}_0 = 1, \varepsilon)$$

$$\Gamma_{n,1}^{(i)}(D_0, \tilde{\tau}_0, \varepsilon) = \frac{\tilde{\tau}_0}{1 - \varepsilon n/2} \frac{\partial}{\partial \tilde{\tau}_0} \Gamma_{n,1}^{(i)}(D_0, \tilde{\tau}_0, \varepsilon)$$

$$\Gamma_{n,1}^{(i)}(D_0, \tilde{\tau}_0, \varepsilon) = \frac{-1}{1 - \varepsilon n/2} \frac{s_{\text{shift}}}{s_{\text{orig}}} \frac{\tilde{\tau}_0}{D_0^{2n-1}} \tilde{\Gamma}(\tau_0 \equiv \tilde{\tau}_0, \varepsilon) = \frac{-1}{1 - \varepsilon n/2} \frac{s_{\text{shift}}}{s_{\text{orig}}} \frac{\tilde{\tau}_0^{1-\varepsilon n/2}}{D_0^{2n-1}} \tilde{\Gamma}(\varepsilon)$$

$$\tilde{\Gamma}(\tau_0, \varepsilon) = -\langle \tilde{\varphi}_0 \varphi_0^2 \rangle \Big|_{p=0, \omega=0}$$

$$\Gamma_{n,2}^{(i)}(D_0, \tilde{\tau}_0, \varepsilon) = \frac{\tilde{\tau}_0^{-n\varepsilon/2}}{D_0^{2n}} \Gamma_{n,2}^{(i)}(\tilde{\tau}_0 = 1, \varepsilon) = -\frac{s_{\text{shift}}}{s_{\text{orig}}} \frac{\tilde{\tau}_0^{-n\varepsilon/2}}{D_0^{2n}} \tilde{\Gamma}(\varepsilon)$$



$$\Gamma_{2,1}^{(a)} = -\frac{\tilde{\tau}_0^{1-\varepsilon}}{8D_0^3} \frac{\Gamma(1 + \varepsilon)}{(4\pi)^d \varepsilon(1 - \varepsilon)} \left[\frac{4}{\varepsilon} + 2 - 2.988841305\varepsilon + \mathcal{O}(\varepsilon^2) \right]$$

$$\Gamma_{2,1}^{(b)} = \frac{\tilde{\tau}_0^{1-\varepsilon}}{D_0^3} \frac{\Gamma\left(1 + \frac{\varepsilon}{2}\right)}{(4\pi)^d \Gamma\left(2 - \frac{\varepsilon}{2}\right)} \frac{3}{16\varepsilon} \left[1 + (2 - \ln 3) \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2) \right]$$

$$\Gamma_{2,2}^{(c)} = \frac{\tilde{\tau}_0^{-\varepsilon}}{D_0^4} \frac{\Gamma(1 + \varepsilon)}{16(4\pi)^d \varepsilon} \left[\frac{2}{\varepsilon} + 1 - 6 \ln 2 + 3 \ln 3 - 0.816060873\varepsilon + \mathcal{O}(\varepsilon^2) \right]$$

$$\Gamma_{2,2}^{(d)} = \frac{\tilde{\tau}_0^{-\varepsilon}}{D_0^4} \frac{\Gamma(1 + \varepsilon)}{(4\pi)^d \varepsilon} \frac{3}{64} \left[4 \ln \frac{4}{3} - 0.904479706\varepsilon + \mathcal{O}(\varepsilon^2) \right],$$



$$g_0^2 \frac{S_d}{2(2\pi)^d} \rightarrow u_0, S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$D_0 h_0 = D_0 m_0 \left(\tau_0 + \frac{g_0 m_0}{2} \right) - D_0 m_0 \left[-u_0 \tilde{\tau}_0^{1-\varepsilon/2} \Gamma_{1,1} + u_0^2 \tilde{\tau}_0^{-\varepsilon} (\tilde{\tau}_0 \Gamma_{2,1} - g_0 m_0 \Gamma_{2,2}) \right]$$

$$\Gamma_{2,1}^{(a)} + \Gamma_{2,1}^{(b)} \equiv \Gamma_{2,1} \text{ and } \Gamma_{2,2}^{(c)} + \Gamma_{2,2}^{(d)} \equiv \Gamma_{2,2}$$

$\Gamma_{i,j}^{(\dots)}$	method	ε^{-2}	ε^{-1}	ε^0
$\Gamma_{2,1}^{(a)}$	exact	$-\frac{1}{2}$	-0.25	-0.3682451*
	numeric	$-\frac{1}{2}$	-0.250000(2)	-0.368239(6)
$\Gamma_{2,1}^{(b)}$	exact	0	0.1875	-0.0092449*
	numeric	0	0.1875000(1)	-0.009244(10)
$\Gamma_{2,2}^{(c)}$	exact	$\frac{1}{8}$	-0.1164404*	0.1258992*
	numeric	$\frac{1}{8}$	-0.116441(4)	0.125901(4)
$\Gamma_{2,2}^{(d)}$	exact	0	0.0539404*	-0.09633788*
	numeric	0	0.053940(1)	-0.0963372(12)

$$S_R = \tilde{\psi}(-Z_1 \partial_t + Z_2 D \nabla^2 - Z_3 D \tau) \psi + \frac{Dg\mu^{\varepsilon/2}}{2} Z_4 (\tilde{\psi}^2 \psi - \tilde{\psi} \psi^2) + Dh \tilde{\psi}$$

$$Z_h = Z_1^{1/2} Z_2^{-1}, Z_D = Z_1^{-1} Z_2, Z_\tau = Z_2^{-1} Z_3$$

$$Z_m = Z_\psi = Z_{\tilde{\psi}} = Z_1^{1/2}, Z_u = Z_g^2 = Z_1^{-1} Z_2^{-2} Z_4^2$$

$$h = m \left[\tau + \frac{gm}{2} + \tau A_1(g, \varepsilon, \ln(\tilde{\tau}/\mu^2)) + gm A_2(g, \varepsilon, \ln(\tilde{\tau}/\mu^2)) \right]$$

$$\mu \frac{d}{d\mu} \Big|_0 h_0(\tau_0, m_0, g_0) = 0$$

$$h(\tau, m, g; \mu) = \mu^{2+d/2} s^{d+\Delta_\omega - \Delta_\varphi} h(\mu^{-2} s^{-\Delta_\tau} \tau, \mu^{-d/2} s^{-\Delta_\varphi} m, g_*; 1)$$

$$h = m^{(d+\Delta_\omega)/\Delta_\varphi - 1} F(\tau m^{-\Delta_\tau/\Delta_\varphi})$$

$$h = \tau^{(d+\Delta_\omega - \Delta_\varphi)/\Delta_\tau} F(m \tau^{-\Delta_\varphi/\Delta_\tau})$$

$$a(\varepsilon) h = m^{(d+\Delta_\omega)/\Delta_\varphi - 1} F(b(\varepsilon) \tau m^{-\Delta_\tau/\Delta_\varphi}).$$

$$x = b(\varepsilon) \tau m^{-\Delta_\tau/\Delta_\varphi} \text{ and } y = a(\varepsilon) h m^{1-(d+\Delta_\omega)/\Delta_\varphi}$$



$$\chi = \left. \frac{\partial m}{\partial h} \right|_{\tau}$$

$$\frac{\chi_-}{\chi_+} = \frac{2\Delta_\phi}{\Delta_\tau} - 1 + \mathcal{O}(\varepsilon^3) = 1 - \frac{\varepsilon}{3} - \varepsilon^2 0.022563 \dots + \mathcal{O}(\varepsilon^3)$$

$$ds^2 = \frac{-d\eta^2 + d\vec{x}^2}{H^2\eta^2}$$

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left((\partial\phi)^2 + m^2\phi^2 + \frac{\lambda}{n!} \phi^n \right)$$

$$\text{coupled scalar: } f_k^-(\eta) = \frac{iH\eta}{\sqrt{2k}} e^{-ik\eta}, f_k^+(\eta) = \frac{-iH\eta}{\sqrt{2k}} e^{ik\eta}$$

$$\text{massless: } f_k^-(\eta) = \frac{H}{\sqrt{2k^3}} (1 + ik\eta) e^{-ik\eta}, f_k^+(\eta) = \frac{H}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}$$

$$\langle \Omega | \mathcal{O}(\eta_0) | \Omega \rangle = \langle 0 | \bar{T} \left[\exp \left(i \int_{-\infty_+}^{\eta_0} d\eta' H_{\text{int}}(\eta') \right) \right] \mathcal{O}(\eta_0) T \left[\exp \left(-i \int_{-\infty_-}^{\eta_0} d\eta' H_{\text{int}}(\eta') \right) \right] | 0 \rangle$$

$$\Psi[\phi] = \exp \left(-\frac{1}{2} \int \frac{d^4\vec{k}_1 d^4\vec{k}_2}{(2\pi)^6} \Psi_2(\vec{k}_1, \vec{k}_2) \phi(\vec{k}_1) \phi(\vec{k}_2) \right. \\ \left. + \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 \dots d^3\vec{k}_n}{(2\pi)^{3n}} \Psi_n(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n) \phi(\vec{k}_1) \phi(\vec{k}_2) \dots \phi(\vec{k}_n) \right),$$

$$\langle \phi_1 \phi_2 \dots \phi_n \rangle = \int \mathcal{D}\phi \phi_1 \phi_2 \dots \phi_n |\Psi[\phi]|^2$$

$$\langle \phi(\vec{k}_1) \phi(\vec{k}_2) \dots \phi(\vec{k}_n) \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \dots + \vec{k}_n) \mathcal{B}(\{k_i\}; \{p_i\})$$

$$\text{Disc}_p \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = \frac{1}{2P_p(\eta_0)} \left(\text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p) \text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p) \right. \\ \left. - \overline{\text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p)} \overline{\text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p)} \right),$$

$$\mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{2\text{Re}\psi^{(1)}(\{\mathbf{k}_L\}, q)}{\mathcal{P}_{\mathbf{k}_L}^{-1}(\eta_0) P_q^{-1}(\eta_0)}, \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{2i\text{Im}\psi^{(1)}(\{\mathbf{k}_L\}, q)}{\mathcal{P}_{\mathbf{k}_L}^{-1}(\eta_0) P_q^{-1}(\eta_0)}$$

$$\text{Disc}_p f(p, k_1, \dots, k_n) = f(p, k_1, \dots, k_n) - f(-p, k_1, \dots, k_n)$$

$$\overline{\text{Disc}_p f(p, k_1, \dots, k_n)} = f(p, k_1, \dots, k_n) + f(-p, k_1, \dots, k_n)$$



$$\text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = \frac{1}{2P_{p_{r-1}}(\eta_0)}$$

$$\left[\text{Disc}_{p_{r-1}} \mathcal{B}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \text{Disc}_{p_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \right. \\ \left. - \overline{\text{Disc}}_{p_{r-1}} \tilde{\mathcal{B}}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \overline{\text{Disc}}_{p_{r-1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \right]$$

$$\text{single cuts} = \text{Disc}_{p_1} \left[\text{Disc}_{p_2} \left[\dots \text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \right] \right],$$

$$\text{multiple cuts} = \text{Disc}_{p_1, p_2, \dots, p_{r-1}} \left[\mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \right]$$

$$f(z_0) = \lim_{\epsilon \rightarrow 0^+} f(z_0 + i\epsilon) = \frac{1}{2\pi i} \int_0^\infty \frac{dz}{z - z_0 - i\epsilon} \mathbf{Disc}_z f(z)$$

$$\mathbf{Disc}_z f(z) = \lim_{\epsilon \rightarrow 0^+} [f(z + i\epsilon) - f(z - i\epsilon)].$$

$$\mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{q dq}{q^2 - p^2} \text{Disc}_q \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; q)$$

$$\mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = \left(\frac{1}{2\pi i} \right)^{(r-1)} \left(\prod_{i=1}^{r-1} \int \frac{q_i dq_i}{q_i^2 - p_i^2} \right) \times$$

$$\text{Disc}_{q_1} \left[\text{Disc}_{q_2} \left[\dots \text{Disc}_{q_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{r-1}\}) \right] \right]$$

$$\mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; y_{12}) = \mathcal{A}_4^{(1)} + \mathcal{A}_4^{(2)}$$

$$\mathcal{A}_4^{(1)} = \lambda^2 \int_{-\infty}^\infty dp \left(-2i \int_0^\infty ds_1 \frac{p}{p^2 + (s_1 + k_{12})^2} \right) \left(-2i \int_0^\infty ds_2 \frac{p}{p^2 + (s_2 + k_{34})^2} \right) \\ \times \left(\frac{1}{\pi} \frac{1}{p^2 + y_{12}^2} \right)$$

$$\mathcal{A}_4^{(2)} = \lambda^2 \pi \int_{-\infty}^\infty dp \frac{1}{p^2 + y_{12}^2}$$

$$k_{ij} = |\vec{k}_i| + |\vec{k}_j|, y_{ij} = |\vec{k}_i + \vec{k}_j|$$

$$\mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{q dq}{q^2 - p^2} \frac{1}{2P_q(\eta_0)} \times \\ \left(\text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p) \text{Disc}_p \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p) - \overline{\text{Disc}}_p \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p) \overline{\text{Disc}}_p \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p) \right)$$

$$\psi^{(1)}(\{\mathbf{k}_L\}, q) = \frac{\lambda(n-4)!}{H^4 \eta_0^n} \frac{1}{(k_L + q)^{n-3}} \left[i^n e^{-i\eta_0(k_L + q)} \Big|_{n-4} \right], \text{ for } n \geq 4$$

$$e^{-i\eta_0(k_L + q)} \Big|_{n-4} = \sum_{k=0}^{n-4} \frac{1}{k!} (-i(k_L + q)\eta_0)^k$$



$$\mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{\lambda(-1)^m(2m-4)!H^{4m-4}\eta_0^{2m}}{q\{2k_i\}^L(k_L+q)^{2m-3}}, \quad \text{for } n = 2m$$

$$\mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{\lambda(-1)^m(2m-3)!H^{4m-2}\eta_0^{2m+2}}{q\{2k_i\}^L(k_L+q)^{2m-3}}, \quad \text{for } n = 2m+1$$

$$\tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{i\lambda(-1)^{m+1}(2m-4)!H^{4m-4}\eta_0^{2m+1}}{q\{2k_i\}^L(k_L+q)^{2m-4}}, \quad \text{for } n = 2m$$

$$\tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{i\lambda(-1)^m(2m-3)!H^{4m-2}\eta_0^{2m+1}}{q\{2k_i\}^L(k_L+q)^{2m-2}}, \quad \text{for } n = 2m+1$$

$$\{2k_i\}^L = \prod_{i=1}^{n-1} 2k_i$$

$$\text{Disc}_q \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{\lambda(-1)^m H^{4m-4} \eta_0^{2m}}{q\{2k_i\}^L} \partial_{k_L}^{2m-4} \left(\frac{1}{k_L+q} + \frac{1}{k_L-q} \right)$$

$$\text{Disc}_q \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) = \frac{-\lambda_e^c}{q\{2k_i\}^L} \mathcal{D}_{k_L} \left(\frac{2k_L}{q^2 - k_L^2} \right)$$

$$\text{Disc}_q \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q) = \frac{-\lambda_e^c}{q\{2k_i\}^R} \mathcal{D}_{k_R} \left(\frac{2k_R}{q^2 - k_R^2} \right)$$

$$\{2k_i\}^R = \prod_{i=n}^{2n-2} 2k_i, \mathcal{D}_{k_L} \equiv \partial_{k_L}^{2m-4}, \lambda_e^c = \lambda(-1)^m H^{4m-4} \eta_0^{2m}$$

$$\mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = \frac{\lambda^2 H^{4n-10} \eta_0^{2n-2}}{\{2k_i\}^L \{2k_i\}^R} \mathcal{D}_{k_L} \mathcal{D}_{k_R} \left(\frac{-1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq}{q^2 - p^2} \frac{2k_L}{q^2 - k_L^2} \frac{2k_R}{q^2 - k_R^2} \right)$$

$$\widetilde{\text{Disc}}_p \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p) = \frac{-i\lambda_o^\psi}{q\{2k_i\}^L} \tilde{\mathcal{D}}_{k_L} \left(\frac{2q}{q^2 - k_L^2} \right)$$

$$\tilde{\mathcal{D}}_{k_L} \equiv \partial_{k_L}^{2m-3}, \lambda_o^\psi = \lambda(-1)^m H^{4m-2} \eta_0^{2m+1}$$

$$\widetilde{\text{Disc}}_p \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p) = \frac{-i\lambda_o^\psi}{q\{2k_i\}^R} \tilde{\mathcal{D}}_{k_L} \left(\frac{2q}{q^2 - k_R^2} \right).$$

$$\mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = \frac{\lambda^2 H^{2n-10} \eta_0^{2n-2}}{\{2k_i\}^L \{2k_i\}^R} \tilde{\mathcal{D}}_{k_L} \tilde{\mathcal{D}}_{k_R} \left(\frac{-1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq}{q^2 - p^2} \frac{2q}{q^2 - k_L^2} \frac{2q}{q^2 - k_R^2} \right)$$

$$\text{Disc}_{p_2} \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \frac{1}{2P_{p_2}(\eta_0)} (\text{Disc}_{p_2} \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1) \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2) \\ - \widetilde{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1) \widetilde{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2))$$



$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2)] \\
&= \frac{1}{2P_{p_2}(\eta_0)} (\text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1)] \times \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2) \\
&\quad - \text{Disc}_{p_1} [\overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1)] \times \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2)) \\
& \\
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2)] \\
&= \frac{1}{2P_{p_2}(\eta_0)} [\text{Disc}_{p_1} \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1) \times \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2) \\
&\quad - \text{Disc}_{p_1} \tilde{\mathcal{B}}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_M\}, p_2; p_1) \times \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2)] - (p_2 \leftrightarrow -p_2).
\end{aligned}$$

$$\text{Disc}_{p_1} \tilde{\mathcal{B}}^{(r)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p_1) = \text{Disc}_{p_1} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p_1) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_1) \leftrightarrow \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_1)'}$$

$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2)] = \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 1} \\
&+ \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 2} + \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 3} + \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 4}
\end{aligned}$$

$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 1} = \frac{1}{4P_{p_1}(\eta_0)P_{p_2}(\eta_0)} [\text{Disc}_{p_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p_1) \times \\
&\quad \overline{\text{Disc}}_{p_1, p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \times \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2)] - (p_2 \leftrightarrow -p_2)
\end{aligned}$$

$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 2} = -\frac{1}{4P_{p_1}(\eta_0)P_{p_2}(\eta_0)} [\text{Disc}_{p_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p_1) \times \\
&\quad \text{Disc}_{p_1, p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \times \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2)] - (p_2 \leftrightarrow -p_2)
\end{aligned}$$

$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 3} = -\frac{1}{4P_{p_1}(\eta_0)P_{p_2}(\eta_0)} [\overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p_1) \times \\
&\quad \text{Disc}_{p_1, p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \times \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2)] - (p_2 \leftrightarrow -p_2)
\end{aligned}$$

$$\begin{aligned}
& \text{Disc}_{p_1} [\text{Disc}_{p_2} \mathcal{B}^{(3)}] \Big|_{\text{term } 4} = \frac{1}{4P_{p_1}(\eta_0)P_{p_2}(\eta_0)} [\overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p_1) \times \\
&\quad \overline{\text{Disc}}_{p_1, p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \times \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2)] - (p_2 \leftrightarrow -p_2)
\end{aligned}$$

$$\begin{aligned}
& \overline{\text{Disc}}_{p_i, p_{i+1}} \mathcal{B}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1}) = \mathcal{B}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1}) + \mathcal{B}^{(1)}(\{\mathbf{k}_{M_i}\}, -p_i, -p_{i+1}), \\
& \overline{\text{Disc}}_{p_i, p_{i+1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1}) = \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1}) - \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_i}\}, -p_i, -p_{i+1}).
\end{aligned}$$

$$\begin{aligned}
& \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) \\
&= \left(\frac{-1}{2\pi i}\right)^2 \int_{-\infty}^{+\infty} \frac{q_1 dq_1}{q_1^2 - p_1^2} \int_{-\infty}^{+\infty} \frac{q_2 dq_2}{q_2^2 - p_2^2} \text{Disc}_{q_1} [\text{Disc}_{q_2} \mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; q_1; q_2)] \\
&= \sum_{i=1}^4 \mathcal{B}_i^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2)
\end{aligned}$$

$$\begin{aligned}
& \mathcal{B}_1^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \frac{q_1 dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{q_2 dq_2}{p_2^2 - q_2^2} \frac{2}{4P_{q_1}(\eta_0)P_{q_2}(\eta_0)} \times \\
&\quad \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) \times \overline{\text{Disc}}_{q_1, q_2} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) \times \text{Disc}_{q_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_2)
\end{aligned}$$



$$\mathcal{B}_2^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = -\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \frac{q_1 dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{q_2 dq_2}{p_2^2 - q_2^2} \frac{2}{4P_{q_1}(\eta_0)P_{q_2}(\eta_0)} \times \\ \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) \times \text{Disc}_{q_1, q_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) \times \widetilde{\text{Disc}}_{q_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_2)$$

$$\mathcal{B}_3^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = -\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \frac{q_1 dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{q_2 dq_2}{p_2^2 - q_2^2} \frac{2}{4P_{q_1}(\eta_0)P_{q_2}(\eta_0)} \times \\ \widetilde{\text{Disc}}_{q_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) \times \text{Disc}_{q_1, q_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) \times \text{Disc}_{q_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_2)$$

$$\mathcal{B}_4^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \frac{q_1 dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{q_2 dq_2}{p_2^2 - q_2^2} \frac{2}{4P_{q_1}(\eta_0)P_{q_2}(\eta_0)} \times \\ \widetilde{\text{Disc}}_{q_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) \times \widetilde{\text{Disc}}_{q_1, q_2} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) \times \widetilde{\text{Disc}}_{q_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_2)$$

$$\mathcal{B}^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \mathcal{B}_1^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) \\ = \frac{(-\lambda_e^c)^3}{\prod_{i=1}^{3(n-2)+2} 2k_i} \left(\frac{1}{H^2 \eta_0^2}\right)^2 \times (\mathcal{D}_{k_L} \mathcal{D}_{k_M} \mathcal{D}_{k_R}) \prod_{m=1}^2 \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_m}{p_m^2 - q_m^2}\right) \times \\ \left(\frac{2k_L}{q_1^2 - k_L^2} \frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \frac{2k_R}{q_2^2 - k_R^2}\right)$$

$$\mathcal{B}_1^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \frac{(-\lambda_o^c)^3}{\prod_{i=1}^{3(n-2)+2} 2k_i} \left(\frac{1}{H^2 \eta_0^2}\right)^2 \times \\ (\mathcal{D}_{k_L} \mathcal{D}_{k_M} \mathcal{D}_{k_R}) \prod_{m=1}^2 \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_m}{p_m^2 - q_m^2}\right) \left(\frac{2k_L}{q_1^2 - k_L^2} \frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \frac{2k_R}{q_2^2 - k_R^2}\right),$$

$$\mathcal{B}_2^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = -\frac{(-\lambda_o^c) (-i\lambda_o^\psi) (-i\lambda_o^\psi)}{\prod_{i=1}^{3(n-2)+2} 2k_i} \left(\frac{1}{H^2 \eta_0^2}\right)^2 \times \\ (\mathcal{D}_{k_L} \tilde{\mathcal{D}}_{k_M} \tilde{\mathcal{D}}_{k_R}) \prod_{m=1}^2 \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_m}{p_m^2 - q_m^2}\right) \left(\frac{2k_L}{q_1^2 - k_L^2} \frac{2(q_1 + q_2)}{(q_1 + q_2)^2 - k_M^2} \frac{2q_2}{q_2^2 - k_R^2}\right),$$

$$\mathcal{B}_3^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = -\frac{(-i\lambda_o^\psi) (-i\lambda_o^\psi) (-\lambda_o^c)}{\prod_{i=1}^{3(n-2)+2} 2k_i} \left(\frac{1}{H^2 \eta_0^2}\right)^2 \times \\ (\tilde{\mathcal{D}}_{k_L} \tilde{\mathcal{D}}_{k_M} \mathcal{D}_{k_R}) \prod_{m=1}^2 \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_m}{p_m^2 - q_m^2}\right) \left(\frac{2q_1}{q_1^2 - k_L^2} \frac{2(q_1 + q_2)}{(q_1 + q_2)^2 - k_M^2} \frac{2k_R}{q_2^2 - k_R^2}\right),$$

$$\mathcal{B}_4^{(3)}(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1; p_2) = \frac{(-i\lambda_o^\psi) (-\lambda_o^c) (-i\lambda_o^\psi)}{\prod_{i=1}^{3(n-2)+2} 2k_i} \left(\frac{1}{H^2 \eta_0^2}\right)^2 \times \\ (\tilde{\mathcal{D}}_{k_L} \mathcal{D}_{k_M} \tilde{\mathcal{D}}_{k_R}^2) \prod_{m=1}^2 \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_m}{p_m^2 - q_m^2}\right) \left(\frac{2q_1}{q_1^2 - k_L^2} \frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \frac{2q_2}{q_2^2 - k_R^2}\right).$$

$$\text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = \frac{1}{2P_{p_{r-1}}(\eta_0)} \times$$

$$[(\mathcal{B}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \times \text{Disc}_{p_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \\ - \tilde{\mathcal{B}}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \times \widetilde{\text{Disc}}_{p_{r-1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \\ - (p_{r-1} \leftrightarrow -p_{r-1})]$$



$$\begin{aligned} & \text{Disc}_{p_{r-1}} \widetilde{\mathcal{B}}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \\ &= \text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \leftrightarrow \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1})} \end{aligned}$$

$$\begin{aligned} \text{Disc}_{p_{r-2}} [\text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\})] &= \frac{1}{2P_{p_{r-1}}(\eta_0)} \times \\ & [\text{Disc}_{p_{r-2}} \mathcal{B}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \times \text{Disc}_{p_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \\ & - \text{Disc}_{p_{r-2}} \mathcal{B}^{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})} \\ & \times \widetilde{\text{Disc}}_{p_{r-1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) - (p_{r-1} \leftrightarrow -p_{r-1})] \end{aligned}$$

$$\begin{aligned} \text{Disc}_{p_1} \dots [\text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\})] &= \hat{\mathcal{J}}_r(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \\ &= \frac{1}{2P_{p_{r-1}}(\eta_0)} [\hat{\mathcal{J}}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \times \text{Disc}_{p_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \\ & - \hat{\mathcal{J}}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})} \\ & \times \widetilde{\text{Disc}}_{p_{r-1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) - (p_{r-1} \leftrightarrow -p_{r-1})] \end{aligned}$$

$$\hat{\mathcal{J}}_r(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{r-1}\}) \equiv \text{Disc}_{q_1} \left[\dots [\text{Disc}_{q_{r-1}} \mathcal{B}^{(s)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{r-1}\})] \right]$$

$$\begin{aligned} \hat{\mathcal{J}}_s(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{s-1}\}) &= \text{Disc}_{q_1} [\dots \text{Disc}_{q_{s-1}} \mathcal{B}^{(s)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{s-1}\})] \\ &= \mathcal{J}_s(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{s-1}\}) \\ &- \sum_{\substack{\sigma_2, \dots, \sigma_{s-1} = \pm 1 \\ (\sigma_2, \dots, \sigma_{s-1}) \neq (+, \dots, +)}} \left(\prod_{i=2}^{s-1} \sigma_i \right) \times \mathcal{J}_s(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \sigma_2 q_2, \dots, \sigma_{s-1} q_{s-1}\}), \end{aligned}$$

$$\text{Disc}_{q_i} \mathcal{B}^{(1)}(\{\mathbf{k}\}; q_i), \widetilde{\text{Disc}}_{q_i} \mathcal{B}^{(1)}(\{\mathbf{k}\}; q_i), \text{Disc}_{q_i} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}\}; q_i), \widetilde{\text{Disc}}_{q_i} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}\}; q_i),$$

$$\text{Disc}_{p_i} \mathcal{B}^{(1)}(\{\mathbf{k}_{L/R}\}, p_i) \text{ or } \widetilde{\text{Disc}}_{p_i} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{L/R}\}, p_i)$$

$$\widetilde{\text{Disc}}_{p_i, p_{i+1}} \mathcal{B}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1}), \text{ or } \text{Disc}_{p_i, p_{i+1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_i}\}, p_i, p_{i+1})$$

$$\begin{aligned} \hat{\mathcal{J}}_2(\{\mathbf{k}_L, \mathbf{k}_R\}; p_1) &= \text{Disc}_{p_1} \mathcal{B}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p_1) * \\ & \frac{1}{2 \left(\prod_{i=\text{all internal lines}} P_{p_i}(\eta_0) \right)} = \frac{1}{2P_{q_1}(\eta_0)} (\text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_1) \\ & + (-1)^{1+2} \widetilde{\text{Disc}}_{q_1} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) \widetilde{\text{Disc}}_{q_1} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_1)) \end{aligned}$$

$$\hat{\mathcal{J}}_3(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1, p_2) = \mathcal{J}_3(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1, p_2) - \mathcal{J}_3(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1, -p_2)$$



$$\begin{aligned}
\mathcal{J}_3(\{\mathbf{k}_L, \mathbf{k}_M, \mathbf{k}_R\}; p_1, p_2) &= \frac{1}{4P_{p_1}(\eta_0)P_{p_2}(\eta_0)} \times \\
&(\text{Disc}_{p_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p_1) \overline{\text{Disc}}_{\{p_1, p_2\}} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2) \\
&+ (-1)^{(2+3)} \text{Disc}_{p_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, p_1) \text{Disc}_{\{p_1, p_2\}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2) \\
&+ (-1)^{(1+2)} \overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p_1) \text{Disc}_{\{p_1, p_2\}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \text{Disc}_{p_2} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_2) \\
&+ (-1)^{(1+3)} \overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p_1) \overline{\text{Disc}}_{\{p_1, p_2\}} \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, p_1, p_2) \overline{\text{Disc}}_{p_2} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_2)) \\
\text{Disc}_{p_1} \dots [\text{Disc}_{p_{r-1}} \mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\})] &= \hat{\mathcal{J}}_r(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \\
&= \frac{1}{2P_{p_{r-1}}(\eta_0)} [\hat{\mathcal{J}}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \times \text{Disc}_{p_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \\
&- \hat{\mathcal{J}}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\})]_{\mathcal{B}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})} \\
&\times \overline{\text{Disc}}_{p_{r-1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) - (p_{r-1} \leftrightarrow -p_{r-1})] \\
\hat{\mathcal{J}}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) &= \mathcal{J}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \\
&- \sum_{\substack{\sigma_2, \dots, \sigma_{r-2} = \pm 1 \\ (\sigma_2, \dots, \sigma_{r-2}) \neq (+1, \dots, +1)}} \left(\prod_{i=2}^{s-1} \sigma_i \right) \times \mathcal{J}_{(r-1)}(\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \sigma_2 p_2, \dots, \sigma_{r-2} p_{r-2}\})
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) &= \left(\frac{-1}{2\pi i} \right)^{(r-1)} \times 2^{r-2} \times \\
&\prod_{i=1}^{r-1} \int \frac{q_i dq_i}{q_i^2 - p_i^2} \mathcal{J}_r(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{q_1, \dots, q_{r-1}\})
\end{aligned}$$

$$\text{Left vertex: } \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1),$$

$$\text{Right vertex: } \text{Disc}_{q_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_{r-1})$$

$$\text{Middle vertex: } 2\overline{\text{Disc}}_{q_j, q_{j+1}} \mathcal{B}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}), \text{ where } j = 1, \dots, r-2$$

$$\text{Left vertex: } \overline{\text{Disc}}_{q_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1),$$

$$\text{Right vertex: } \overline{\text{Disc}}_{q_{r-1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}).$$

$$\text{Middle vertex: } 2\text{Disc}_{q_j, q_{j+1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}), \text{ where } j = 1, \dots, r-2.$$

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{q_i dq_i}{2P_{q_i}(\eta_0)} \frac{1}{p_i^2 - q_i^2}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{q_i^2 dq_i}{H^2 \eta_0^2} \frac{1}{p_i^2 - q_i^2}$$

$$\text{Left vertex: } \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) = \frac{-\lambda_{\mathcal{D}_{k_L} \equiv \partial_{k_L}^{2m-4}}}{q_1 \{2k_i\}^L} \mathcal{D}_{k_L} \left(\frac{2k_L}{q_1^2 - k_L^2} \right)$$

$$\text{Right vertex: } \text{Disc}_{q_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \frac{-\lambda_{\mathcal{D}_{k_R} \equiv \partial_{k_R}^{2m-4}}}{q_{r-1} \{2k_i\}^R} \mathcal{D}_{k_R} \left(\frac{2k_R}{q_{r-1}^2 - k_R^2} \right),$$



Middle vertex:

$$2\widetilde{\text{Disc}}_{q_{j+1}, q_j} \mathcal{B}^{(1)}(\{\mathbf{k}_{M_j}\}, q_{j+1}, q_j) = \frac{-\lambda_{\mathcal{D}_{k_L} \equiv \partial_{k_L}^{2m-4}}}{q_j q_{j+1} \{2k_i\}^{M_j}} \mathcal{D}_{k_{M_j}} \left(\frac{2k_{M_j}}{(q_{j+1} + q_j)^2 - k_{M_j}^2} \right)$$

$$\begin{aligned} \lambda_1 = \lambda_e^c &= \lambda(-1)^m H^{4m-4} \eta_0^{2m} & (\text{for } n = \text{even} = 2m) \\ \lambda_1 = \lambda_o^c &= \lambda(2m-3)(-1)^m H^{4m-2} \eta_0^{2m+2} & (\text{for } n = \text{odd} = 2m+1) \end{aligned}$$

$$\text{Left vertex: } \widetilde{\text{Disc}}_{q_1} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) = \frac{-i\lambda_o^\psi}{q_1 \{2k_i\}^L} \widetilde{\mathcal{D}}_{k_L} \left(\frac{2q_1}{q_1^2 - k_L^2} \right),$$

$$\text{Right vertex: } \widetilde{\text{Disc}}_{q_{r-1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \frac{-i\lambda_o^\psi}{q_{r-1} \{2k_i\}^L} \widetilde{\mathcal{D}}_{k_R} \left(\frac{2q_{r-1}}{q_{r-1}^2 - k_R^2} \right),$$

Middle vertex:

$$2\text{Disc}_{q_j, q_{j+1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}) = \frac{-i\lambda_o^\psi}{q_j q_{j+1} \{2k_i\}^M} \widetilde{\mathcal{D}}_{k_{M_j}} \left(\frac{2(q_j + q_{j+1})}{(q_j + q_{j+1})^2 - k_{M_j}^2} \right)$$

$$\widetilde{\mathcal{D}}_{k_L} \equiv \partial_{k_L}^{2m-3}, \quad \lambda_o^\psi = \lambda(-1)^m H^{4m-2} \eta_0^{2m+1}$$

$$\text{Left vertex: } \text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) = \frac{\pi \lambda H^2 \eta_0^3}{\{2k_i\}^L q_1}$$

$$\text{Right vertex: } \text{Disc}_{q_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \frac{\pi \lambda H^2 \eta_0^3}{\{2k_i\}^R q_{r-1}}$$

$$\text{Middle vertex: } 2\widetilde{\text{Disc}}_{q_{j+1}, q_j} \mathcal{B}^{(1)}(k_{M_j}, q_{j+1}, q_j) = \frac{\pi \lambda H^2 \eta_0^3}{\{2k_i\}^{M_j} q_j q_{j+1}}$$

$$\text{Left vertex: } \widetilde{\text{Disc}}_{q_1} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^L q_1} \int_0^\infty ds_L \frac{2q_1}{(s_L + k_L)^2 - q_1^2},$$

Right vertex:

$$\widetilde{\text{Disc}}_{q_{r-1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^R q_{r-1}} \int_0^\infty ds_R \frac{2q_{r-1}}{(s_R + k_R)^2 - q_{r-1}^2}$$

Middle vertex:

$$2\text{Disc}_{q_j, q_{j+1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^{M_j} q_j q_{j+1}} \int_0^\infty ds_j \frac{2(q_j + q_{j+1})}{(s_j + k_{M_j})^2 - (q_j + q_{j+1})^2}$$



$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dq_i}{\vec{p}_i^2 - q_i^2} \times \left(\frac{q_i^4}{H^2 \eta_0^2 (1 + q_i^2 \eta_0^2)} \right)$$

Left vertex:

$$\text{Disc}_{q_1} \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q_1) = \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2 \cos(\pi\epsilon/2)}{\{2k_i^3\}^L q_1^3} \int_0^{\infty} ds_L \frac{\mathcal{Q}_3(k_L, q_1, s_L)}{(s_L + k_L)^2 - q_1^2}$$

Right vertex:

$$\text{Disc}_{q_{r-1}} \mathcal{B}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2 \cos(\pi\epsilon/2)}{\{2k_i^3\}^R q_{r-1}^3} \int_0^{\infty} ds_R \frac{\mathcal{Q}_3(k_R, q_{r-1}, s_R)}{(s_R + k_R)^2 - q_{r-1}^2}$$

Middle vertex:

$$2\widetilde{\text{Disc}}_{q_{j+1}, q_j} \mathcal{B}^{(1)}(k_{M_j}, q_j, q_{j+1}) = \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2 \cos(\pi\epsilon/2)}{\{2k_i^3\}^{M_j} q_j^3 q_{j+1}^3} \int_0^{\infty} ds_j \frac{\mathcal{Q}_3(k_{M_j}, q_j + q_{j+1}, s_j)}{(s_j + k_{M_j})^2 - (q_j + q_{j+1})^2}$$

Left vertex:

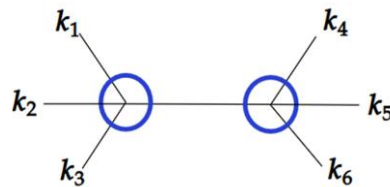
$$\widetilde{\text{Disc}}_{q_1} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) = \lim_{\epsilon \rightarrow 0^+} \frac{i\lambda H^2 \sin(\pi\epsilon/2)}{\{2k_i^3\}^L q_1^3} \int_0^{\infty} ds_L \frac{\widetilde{\mathcal{Q}}_3(k_L, q_1, s_L)}{(s_L + k_L)^2 - q_1^2}$$

Right vertex:

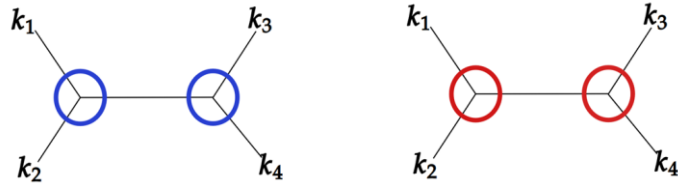
$$\widetilde{\text{Disc}}_{q_{r-1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \lim_{\epsilon \rightarrow 0^+} \frac{i\lambda H^2 \sin(\pi\epsilon/2)}{\{2k_i^3\}^R q_{r-1}^3} \int_0^{\infty} ds_R \frac{\widetilde{\mathcal{Q}}_3(k_R, q_{r-1}, s_R)}{(s_R + k_R)^2 - q_{r-1}^2}$$

Middle vertex:

$$2\widetilde{\text{Disc}}_{q_j, q_{j+1}} \widetilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}) = \lim_{\epsilon \rightarrow 0^+} \frac{i\lambda H^2 \sin(\pi\epsilon/2)}{\{2k_i^3\}^{M_j} q_j^3 q_{j+1}^3} \int_0^{\infty} ds_j \frac{\widetilde{\mathcal{Q}}_3(k_{M_j}, q_j + q_{j+1}, s_j)}{(s_j + k_{M_j})^2 - (q_j + q_{j+1})^2}$$



$$\mathcal{B}_{\phi^4}^{(2)}(\{\mathbf{k}_L, \mathbf{k}_R\}; p) = \left(\frac{\lambda^2 H^6 \eta_0^6}{\prod_{i=1}^6 2k_i} \right) \left(\frac{1}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{dq}{p^2 - q^2} \left(\frac{2k_L}{q^2 - k_L^2} \right) \left(\frac{2k_R}{q^2 - k_R^2} \right).$$



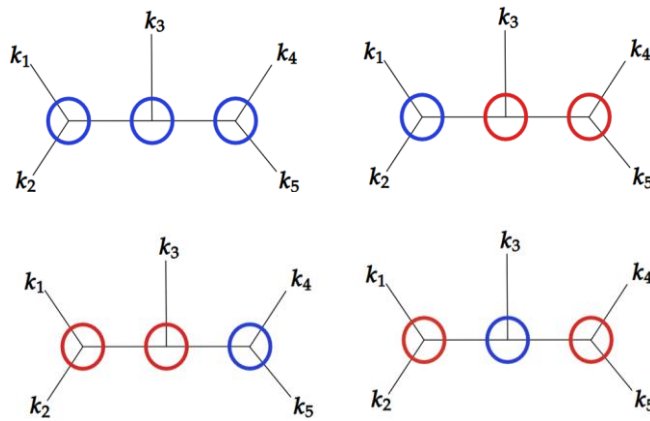
$$\mathfrak{P} = \left(\frac{\lambda^2 H^2 \eta_0^4}{\prod_{i=1}^4 2k_i} \right) \left(\frac{1}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{dq}{p^2 - q^2} \pi^2$$

$$\mathfrak{S} = \left(\frac{\lambda^2 H^2 \eta_0^4}{\prod_{i=1}^4 2k_i} \right) \left(\frac{1}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{dq}{p^2 - q^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \times$$

$$\left(\frac{2q}{(s_1 + k_1 + k_2)^2 - q^2} \frac{2q}{(s_2 + k_3 + k_4)^2 - q^2} \right)$$

$$\mathcal{B}^{(2)}(k_1, k_2, k_3, k_4; p) = \left(\frac{\lambda^2 H^2 \eta_0^4}{\prod_{i=1}^4 2k_i} \right) \left(\frac{1}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{dq}{p^2 - q^2} \times$$

$$\left(\pi^2 + \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{2q}{(s_1 + k_1 + k_2)^2 - q^2} \frac{2q}{(s_2 + k_3 + k_4)^2 - q^2} \right)$$



$$\mathfrak{S} = \frac{\pi^3 \lambda^3 H^2 \eta_0^5}{\prod_{i=1}^5 2k_i} \left(\frac{1}{2\pi i} \right)^2 \int_{-\infty}^{\infty} \frac{dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{dq_2}{p_2^2 - q_2^2}$$

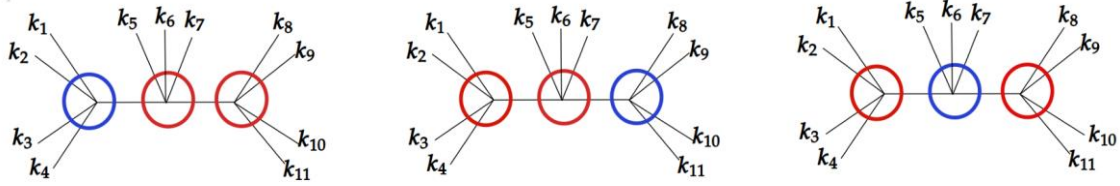
$$\mathfrak{P} = \frac{-4\pi \lambda^3 H^2 \eta_0^5}{\prod_{i=1}^5 2k_i} \left(\frac{1}{2\pi i} \right)^2 \int_0^\infty ds_2 \int_0^\infty ds_3 \int_{-\infty}^{\infty} \frac{dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{dq_2}{p_2^2 - q_2^2}$$

$$\times \frac{(q_1 + q_2)}{(s_2 + k_3)^2 - (q_1 + q_2)^2} \frac{q_2}{(s_3 + k_4 + k_5)^2 - q_2^2}$$

$$\square = \frac{4\pi \lambda^3 H^2 \eta_0^5}{\prod_{i=1}^5 2k_i} \left(\frac{1}{2\pi i} \right)^2 \int_0^\infty ds_1 \int_0^\infty ds_3 \int_{-\infty}^{\infty} \frac{dq_1}{p_1^2 - q_1^2} \int_{-\infty}^{\infty} \frac{dq_2}{p_2^2 - q_2^2}$$

$$\times \frac{q_1}{(s_1 + k_1 + k_2)^2 - q_1^2} \frac{q_2}{(s_3 + k_4 + k_5)^2 - q_2^2}$$

$$\begin{aligned} \mathcal{W} &= \lim_{\epsilon \rightarrow 0^+} \frac{\lambda^2 H^4 \cos^2(\pi\epsilon/2)}{\prod_{i=1}^4 4k_i^3} \left(\frac{1}{2\pi i}\right) \int_0^\infty ds_L \int_0^\infty ds_R \int_{-\infty}^\infty \frac{dq_1}{p_1^2 - q_1^2} \\ &\times \left(\frac{1}{H^2 \eta_0^2 (1 + q_1^2 \eta_0^2) q_1^4}\right) \left(\frac{Q_3(k_1 + k_2, q_1, s_L)}{(s_L + k_1 + k_2)^2 - q_1^2} \frac{Q_3(k_3 + k_4, q_1, s_R)}{(s_R + k_3 + k_4)^2 - q_1^2}\right), \\ \mathfrak{H} &= \lim_{\epsilon \rightarrow 0^+} \frac{-\lambda^2 H^4 \sin^2(\pi\epsilon/2)}{\prod_{i=1}^4 4k_i^3} \left(\frac{1}{2\pi i}\right) \int_0^\infty ds_L \int_0^\infty ds_R \int_{-\infty}^\infty \frac{dq_1}{p_1^2 - q_1^2} \\ &\times \left(\frac{1}{H^2 \eta_0^2 (1 + q_1^2 \eta_0^2) q_1^4}\right) \left(\frac{\tilde{Q}_3(k_1 + k_2, q_1, s_L)}{(s_L + k_1 + k_2)^2 - q_1^2} \frac{\tilde{Q}_3(k_3 + k_4, q_1, s_R)}{(s_R + k_3 + k_4)^2 - q_1^2}\right). \end{aligned}$$



$$\mathcal{D}\rho = \frac{(-1)^{2+3} (-\lambda_o^c) (-i\lambda_o^\psi)^2}{(\prod_{i=1}^{11} 2k_i) (H^2 \eta_0^2)^2} \tilde{\mathcal{D}}_{k_M} \tilde{\mathcal{D}}_{k_R} \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_1}{p_1^2 - q_1^2}\right) \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_2}{p_2^2 - q_2^2}\right) \times$$

$$\left(\frac{2k_L}{q_1^2 - k_L^2} \frac{2(q_1 + q_2)}{(q_1 + q_2)^2 - k_M^2} \frac{2q_2}{q_2^2 - k_R^2}\right),$$

$$\square \equiv \frac{(-1)^{1+2} (-i\lambda_o^\psi)^2 (-\lambda_o^c)}{(\prod_{i=1}^{11} 2k_i) (H^2 \eta_0^2)^2} \tilde{\mathcal{D}}_{k_L} \tilde{\mathcal{D}}_{k_M} \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_1}{p_1^2 - q_1^2}\right) \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_2}{p_2^2 - q_2^2}\right) \times$$

$$\left(\frac{2q_1}{q_1^2 - k_L^2} \frac{2(q_1 + q_2)}{(q_1 + q_2)^2 - k_M^2} \frac{2k_R}{q_2^2 - k_R^2}\right),$$

$$\mathfrak{H}\mathfrak{u} \equiv \frac{(-1)^{1+3} (-i\lambda_o^\psi) (-\lambda_o^c) (-i\lambda_o^\psi)}{(\prod_{i=1}^{11} 2k_i) (H^2 \eta_0^2)^2} \tilde{\mathcal{D}}_{k_L} \tilde{\mathcal{D}}_{k_R} \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_1}{p_1^2 - q_1^2}\right) \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq_2}{p_2^2 - q_2^2}\right) \times$$

$$\left(\frac{2q_1}{q_1^2 - k_L^2} \frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \frac{2q_2}{q_2^2 - k_R^2}\right).$$

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_{z_0}}^{\square} \frac{dz}{z - z_0} f(z)$$

$$\lim_{\epsilon \rightarrow 0^+} f(z + i\epsilon) = f(z)$$

$$f(z_0) = \lim_{\epsilon \rightarrow 0^+} f(z_0 + i\epsilon) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \oint_{C_{z_0}}^{\square} \frac{dz}{z - z_0 - i\epsilon} f(z)$$

$$f(z_0) = \lim_{\epsilon \rightarrow 0^+} f(z_0 + i\epsilon) = \frac{1}{2\pi i} \int_0^\infty \frac{dz}{z - z_0 - i\epsilon} \mathbf{Disc}_z f(z)$$

$$\mathbf{Disc}_z f(z) = \lim_{\epsilon \rightarrow 0^+} [f(z + i\epsilon) - f(z - i\epsilon)]. \quad (\text{A.4})$$

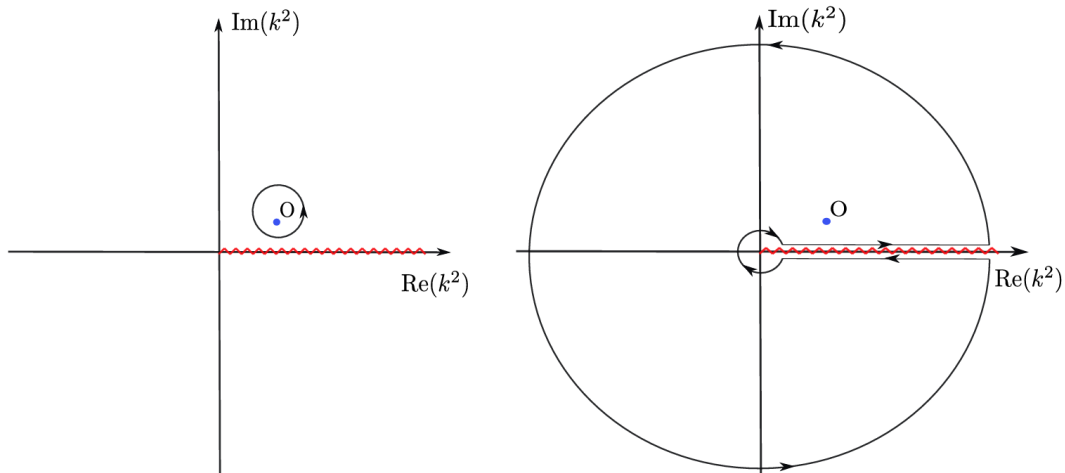


$$\lim_{|k^2| \rightarrow \infty} f(k^2) = 0$$

$$f(p^2) = \frac{1}{2\pi i} \int_0^\infty \frac{dk^2}{k^2 - p^2 - i\epsilon} \mathbf{Disc}_{k^2} f(k^2)$$

$$f(k^2) = \tilde{f}(\sqrt{k^2})$$

$$\lim_{\epsilon \rightarrow 0^+} f(k^2 + i\epsilon) = \lim_{\epsilon \rightarrow 0^+} \tilde{f}(\sqrt{k^2 + i\epsilon}) = \lim_{\epsilon \rightarrow 0^+} \tilde{f}\left(k + \frac{i\epsilon}{2k}\right).$$



$$\lim_{\epsilon \rightarrow 0^+} f(k^2 - i\epsilon) = \lim_{\epsilon \rightarrow 0^+} \tilde{f}(\sqrt{k^2 - i\epsilon}) = \lim_{\epsilon \rightarrow 0^+} \tilde{f}\left(-k + \frac{i\epsilon}{2k}\right).$$

$$\mathbf{Disc}_{k^2} f(k^2) = \lim_{\epsilon \rightarrow 0^+} [f(k^2 + i\epsilon) - f(k^2 - i\epsilon)] = \lim_{\epsilon \rightarrow 0^+} \left[\tilde{f}\left(k + \frac{i\epsilon}{2k}\right) - \tilde{f}\left(-k + \frac{i\epsilon}{2k}\right) \right].$$

$$\mathbf{Disc}_{k^2} f(k^2) = \mathbf{Disc}_k \tilde{f}(k).$$

$$\tilde{f}(p) = \frac{1}{2\pi i} \int_0^\infty \frac{2kdk}{k^2 - p^2 - i\epsilon} \mathbf{Disc}_k \tilde{f}(k)$$

$$\tilde{f}(p) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{kdk}{k^2 - p^2 - i\epsilon} \mathbf{Disc}_k \tilde{f}(k)$$

$$\begin{aligned} & 2 \left(\mathcal{B}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) + \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, -q_1, -q_2) \right) \\ &= \frac{\lambda(-1)^m (2m-4)! H^{4m-4} \eta_0^{2m}}{q_1 q_2 \{2k_i\}^L} \left(\frac{1}{(k_M + q_1 + q_2)^{2m-3}} + \frac{1}{(k_M - q_1 - q_2)^{2m-3}} \right) \\ &= \frac{-\lambda_e^c}{q_1 q_2 \{2k_i\}^M} \mathcal{D}_{k_M} \left(\frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \right) \end{aligned}$$



$$\begin{aligned}
& 2 \left(\mathcal{B}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) + \mathcal{B}^{(1)}(\{\mathbf{k}_M\}, -q_1, -q_2) \right) \\
&= \frac{\lambda(-1)^{m+1}(2m-3)! H^{4m-2} (-\eta_0)^{2m+1} \eta_0}{q_1 q_2 \{2k_i\}^M} \left(\frac{1}{(k_M + q_1 + q_2)^{2m-3}} + \frac{1}{(k_M - q_1 - q_2)^{2m-3}} \right) \\
&= \frac{-\lambda_o^c}{q_1 q_2 \{2k_i\}^M} \mathcal{D}_{k_M} \left(\frac{2k_M}{(q_1 + q_2)^2 - k_M^2} \right),
\end{aligned}$$

$$\mathcal{D}_{k_M} = \partial_{k_M}^{2m-4}, \lambda_o^c = \lambda(2m-3)(-1)^m H^{4m-2} \eta_0^{2m+2}$$

$$\begin{aligned}
& 2 \left(\tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, q_1, q_2) - \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_M\}, -q_1, -q_2) \right) \\
&= \frac{i\lambda(-1)^m(2m-3)! H^{4m-2} \eta_0^{2m+1}}{q_1 q_2 \{2k_i\}^M} \left(\frac{1}{(k_M + q_1 + q_2)^{2m-2}} - \frac{1}{(k_M - q_1 - q_2)^{2m-2}} \right) \\
&= \frac{-i\lambda_o^\psi}{q_1 q_2 \{2k_i\}^M} \tilde{\mathcal{D}}_{k_M} \left(\frac{2(q_1 + q_2)}{(q_1 + q_2)^2 - k_M^2} \right)
\end{aligned}$$

$$\tilde{\mathcal{D}}_{k_M} \equiv \partial_{k_M}^{2m-3}, \lambda_o^\psi = \lambda(-1)^m H^{4m-2} \eta_0^{2m+1}$$

$$\begin{aligned}
\text{Disc}_q \mathcal{B}^{(1)}(\{\mathbf{k}_L\}, q) &= \frac{\lambda(-1)^m(2m-3)! H^{4m-2} \eta_0^{2m+2}}{q \{2k_i\}^L} \left(\frac{1}{(k_L + q)^{2m-3}} + \frac{1}{(k_L - q)^{2m-3}} \right) \\
&= \frac{-\lambda_o^c}{q \{2k_i\}^L} \mathcal{D}_{k_L} \left(\frac{2k_L}{q^2 - k_L^2} \right)
\end{aligned}$$

$$\mathcal{D}_{k_L} = \partial_{k_L}^{2m-4}, \lambda_o^c = \lambda(2m-3)(-1)^m H^{4m-2} \eta_0^{2m+2}$$

$$\widetilde{\text{Disc}}_p \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, p) = \frac{-i\lambda_o^\psi}{q \{2k_i\}^L} \tilde{\mathcal{D}}_{k_L} \left(\frac{2q}{q^2 - k_L^2} \right),$$

$$\tilde{\mathcal{D}}_{k_L} \equiv \partial_{k_L}^{2m-3}, \lambda_o^\psi = \lambda(-1)^m H^{4m-2} \eta_0^{2m+1}$$

$$\tilde{\mathcal{B}}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = (i\lambda)^2 \int d\eta_r \int d\eta_{r-1} \times$$

$$\begin{aligned}
& [\mathcal{V}_{(r-2)}^+ \mathcal{K}_{\eta_0 \eta_{r-1}}^+(\{\mathbf{k}_{M_{r-2}}\}) G_{p_{r-1}}^{++}(\eta_{r-1}, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^+(\{\mathbf{k}_R\}) \\
& + \mathcal{V}_{(r-2)}^+ \mathcal{K}_{\eta_0 \eta_{r-1}}^+(\{\mathbf{k}_{M_{r-2}}\}) G_{p_{r-1}}^{+-}(\eta_{r-1}, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^-(\{\mathbf{k}_R\}) \\
& - \mathcal{V}_{(r-2)}^- \mathcal{K}_{\eta_0 \eta_{r-1}}^-(\{\mathbf{k}_{M_{r-2}}\}) G_{p_{r-1}}^{-+}(\eta_{r-1}, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^+(\{\mathbf{k}_R\}) \\
& - \mathcal{V}_{(r-2)}^- \mathcal{K}_{\eta_0 \eta_{r-1}}^-(\{\mathbf{k}_{M_{r-2}}\}) G_{p_{r-1}}^{--}(\eta_{r-1}, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^-(\{\mathbf{k}_R\})]
\end{aligned}$$

$$\mathcal{V}_{(r-2)}^{\sigma \in \pm}(\{\eta_1, \dots, \eta_{r-1}\}) = (i\lambda)^{r-2} \int^{(r-2)} d\eta \sum_{\{\sigma_i\}_{i=1}^{r-2} \in \pm} \left[\left(\prod_{j=1}^{r-2} \sigma_j \right) \times \right.$$

$$\left. \left(\mathcal{K}_{\eta_0 \eta_1}^{\sigma_1}(\{\mathbf{k}_L\}) \prod_{j=2}^{r-2} \mathcal{K}_{\eta_0 \eta_j}^{\sigma_j}(\{\mathbf{k}_{M_{j-1}}\}) \right) \left(\prod_{j=1}^{r-3} G_{p_j}^{\sigma_j \sigma_{j+1}}(\eta_j, \eta_{j+1}) \right) G_{p_{r-2}}^{\sigma_{r-2} \sigma}(\eta_{r-2}, \eta_{r-1}) \right]$$

$$\text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)}(\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = \text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)}|_{\text{Term 1}} + \text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)}|_{\text{Term 2}}$$



$$\text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)} \Big|_{\text{Term 1}} = \frac{(i\lambda)^2}{2P_{p_{r-1}}(\eta_0)} \times \prod_{i=r}^{r+1} \int d\eta_i [(\mathcal{V}_{(r-2)}^+ \mathcal{K}_{\eta_0 \eta_{r-1}}^+ (\{\mathbf{k}_{M_{r-2}}\}) K_{p_{r-1}}^+ (\eta_0, \eta_{r-1}) - \mathcal{V}_{(r-2)}^- \mathcal{K}_{\eta_0 \eta_{r-1}}^- (\{\mathbf{k}_{M_{r-2}}\}) K_{p_{r-1}}^- (\eta_0, \eta_{r-1})] - (p_{r-1} \leftrightarrow -p_{r-1})] \times \\ \left[(K_{p_{r-1}}^+ (\eta_0, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^+ (\{\mathbf{k}_R\}) + K_{p_{r-1}}^- (\eta_0, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^- (\{\mathbf{k}_R\})) - (p_{r-1} \leftrightarrow -p_{r-1}) \right]$$

$$\text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)} \Big|_{\text{Term 2}} = \frac{(i\lambda)^2}{2P_{p_{r-1}}(\eta_0)} \prod_{i=r}^{r+1} \int d\eta_i [(-\mathcal{V}_{(r-2)}^+ \mathcal{K}_{\eta_0 \eta_{r-1}}^+ (\{\mathbf{k}_{M_{r-2}}\}) K_{p_{r-1}}^+ (\eta_0, \eta_{r-1}) - \mathcal{V}_{(r-2)}^- \mathcal{K}_{\eta_0 \eta_{r-1}}^- (\{\mathbf{k}_{M_{r-2}}\}) K_{p_{r-1}}^- (\eta_0, \eta_{r-1})] + (p_{r-1} \leftrightarrow -p_{r-1})] \times \\ \left[(K_{p_{r-1}}^+ (\eta_0, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^+ (\{\mathbf{k}_R\}) - K_{p_{r-1}}^- (\eta_0, \eta_r) \mathcal{K}_{\eta_0 \eta_r}^- (\{\mathbf{k}_R\})) + (p_{r-1} \leftrightarrow -p_{r-1}) \right]$$

$$\text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)} (\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) = \frac{1}{2P_{p_{r-1}}(\eta_0)} \times \\ \left[\text{Disc}_{p_{r-1}} \mathcal{B}^{(r-1)} (\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_R\}, p_{r-1}) - \overline{\text{Disc}}_{p_{r-1}} \tilde{\mathcal{B}}^{(r-1)} (\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \overline{\text{Disc}}_{p_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_R\}, p_{r-1}) \right]$$

$$\text{Disc}_{p_{r-1}} \tilde{\mathcal{B}}^{(r)} (\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \\ = \text{Disc}_{p_{r-1}} \mathcal{B}^{(r)} (\{\mathbf{k}_L, \dots, \mathbf{k}_R\}; \{p_1, \dots, p_{r-1}\}) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_R\}, p_{r-1}) \leftrightarrow \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, p_{r-1})}$$

$$\text{Disc}_{p_{r-2}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}), \text{ or } \overline{\text{Disc}}_{p_{r-2}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}).$$

$$\overline{\text{Disc}}_{p_{r-2}, p_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}), \text{ or } \text{Disc}_{p_{r-2}, p_{r-1}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}).$$

$$\underset{\text{Disc}}{p_{r-2}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \overline{\text{Disc}}_{p_{r-2}, p_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})$$

$$\overline{\text{Disc}}_{p_{r-2}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \overline{\text{Disc}}_{p_{r-2}, p_{r-1}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}).$$

$$\hat{j}_{(r-1)} (\{\mathbf{k}_L, \dots, \mathbf{k}_{M_{r-2}}\}, p_{r-1}; \{p_1, \dots, p_{r-2}\}) \Big|_{\mathcal{B}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})}$$

$$\text{Disc}_{p_{r-2}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}), \text{ or } \overline{\text{Disc}}_{p_{r-2}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}),$$

$$\text{Disc}_{p_{r-2}, p_{r-1}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}), \text{ or } \overline{\text{Disc}}_{p_{r-2}, p_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}).$$

$$\text{Disc}_{p_{r-2}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \text{Disc}_{p_{r-2}, p_{r-1}} \tilde{\mathcal{B}}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1})$$

$$\overline{\text{Disc}}_{p_{r-2}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}) \leftrightarrow \overline{\text{Disc}}_{p_{r-2}, p_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_{M_{r-2}}\}, p_{r-2}, p_{r-1}).$$

$$\psi^{(1)}(k_1, k_2, p) = \frac{i\lambda}{H^4 \eta_0^3} \log((k_1 + k_2 + p)\eta_0) + \frac{\pi\lambda}{2H^4 \eta_0^3}$$

$$\mathcal{B}^{(1)}(k_1, k_2, p) = \frac{\pi\lambda H^2 \eta_0^3}{8k_1 k_2 p}, \tilde{\mathcal{B}}^{(1)}(k_1, k_2, p) = \frac{i\lambda H^2 \eta_0^3}{4k_1 k_2 p} \log((k_1 + k_2 + p)\eta_0)$$

$$\text{Disc}_{q_1} \mathcal{B}^{(1)} (\{\mathbf{k}_L\}, q_1) = \frac{\pi\lambda H^2 \eta_0^3}{\{2k_i\}^L q_1}, \text{Disc}_{q_{r-1}} \mathcal{B}^{(1)} (\{\mathbf{k}_R\}, q_{r-1}) = \frac{\pi\lambda H^2 \eta_0^3}{\{2k_i\}^R q_{r-1}}$$



$$\overline{\text{Disc}}_{q_1} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_L\}, q_1) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^L q_1} \log \left(\frac{k_L + q_1}{k_L - q_1} \right) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^L q_1} \int_0^\infty ds_1 \frac{2q_1}{(s_1 + k_L)^2 - q_1^2}$$

$$\overline{\text{Disc}}_{q_{r-1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_R\}, q_{r-1}) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^R q_{r-1}} \int_0^\infty ds_2 \frac{2q_{r-1}}{(s_2 + k_R)^2 - q_{r-1}^2}$$

$$2\overline{\text{Disc}}_{q_{j+1}, q_j} \mathcal{B}^{(1)}(k_{M_j}, q_{j+1}, q_j) = \frac{\pi\lambda H^2 \eta_0^3}{\{2k_i\}^{M_j} q_j q_{j+1}}$$

$$2\text{Disc}_{q_j, q_{j+1}} \tilde{\mathcal{B}}^{(1)}(\{\mathbf{k}_{M_j}\}, q_j, q_{j+1}) = \frac{i\lambda H^2 \eta_0^3}{\{2k_i\}^{M_j} q_j q_{j+1}} \int_0^\infty ds_j \frac{2(q_j + q_{j+1})}{(s_j + k_{M_j})^2 - (q_j + q_{j+1})^2}$$

$$\psi_3(\vec{k}_1, \vec{k}_2, \vec{k}_3) = i\lambda \int_{-\infty}^0 \frac{d\eta}{H^4 \eta^4} \prod_{j=1}^3 (1 - ik_j \eta) \times e^{i(k_1 + k_2 + k_3)\eta}$$

$$\int_{-\infty}^0 \frac{d\eta}{\eta^\alpha} e^{ik\eta}, \text{ for } \alpha \geq 1, \text{ from } \eta \rightarrow 0$$

$$\psi_3(\vec{k}_1, \vec{k}_2, \vec{k}_3) = i\lambda \int_{-\infty}^0 \frac{d\eta}{H^4 \eta^{(4-\epsilon)}} \prod_{j=1}^3 (1 - ik_j \eta) \times e^{i(k_1 + k_2 + k_3 - i\beta)\eta}$$

$$\Re(\psi^{(1)}(k_1, k_2, k_3)) = \frac{\lambda}{H^4} \int_0^\infty ds \left(\frac{s^{(3-\epsilon)}}{\Gamma(4-\epsilon)} + \frac{s^{(2-\epsilon)} k_{123}}{\Gamma(3-\epsilon)} + \frac{s^{(1-\epsilon)} b_{12}}{\Gamma(2-\epsilon)} + \frac{s^{-\epsilon} c_{123}}{\Gamma(1-\epsilon)} \right) \frac{1}{(s + k_{123})},$$

$$\Im(\psi^{(1)}(k_1, k_2, k_3)) = \alpha \Re(\psi^{(1)}(k_1, k_2, k_3))$$

$$(-i\eta)^{-a} e^{ik\eta} = \frac{1}{\Gamma(a)} \int_0^\infty ds s^{(a-1)} e^{i(s+k)\eta}, \text{ (for } \text{Re}(a) > 0)$$

$$\begin{aligned} &\mathcal{B}^{(1)}(k_1, k_2, k_3) \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2}{8k_1^3 k_2^3 k_3^3} \int_0^\infty ds \left(\frac{s^{(3-\epsilon)}}{\Gamma(4-\epsilon)} + \frac{s^{(2-\epsilon)} k_{123}}{\Gamma(3-\epsilon)} + \frac{s^{(1-\epsilon)} b_{12}}{\Gamma(2-\epsilon)} + \frac{s^{-\epsilon} c_{123}}{\Gamma(1-\epsilon)} \right) \frac{\cos(\pi\epsilon/2)}{(s + k_{123})} \end{aligned}$$

$$\begin{aligned} &\tilde{\mathcal{B}}^{(1)}(k_1, k_2, k_3) \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2}{8k_1^3 k_2^3 k_3^3} \int_0^\infty ds \left(\frac{s^{(3-\epsilon)}}{\Gamma(4-\epsilon)} + \frac{s^{(2-\epsilon)} k_{123}}{\Gamma(3-\epsilon)} + \frac{s^{(1-\epsilon)} b_{12}}{\Gamma(2-\epsilon)} + \frac{s^{-\epsilon} c_{123}}{\Gamma(1-\epsilon)} \right) \frac{\sin(\pi\epsilon/2)}{(s + k_{123})} \end{aligned}$$

$$\begin{aligned} \text{Disc}_{p_1} \mathcal{B}^{(1)}(k_1, k_2, p_1) &= \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2}{4k_1^3 k_2^3 p_1^3} \int_0^\infty ds \left(\frac{s^{(3-\epsilon)}(k_{12} + s)}{\Gamma(4-\epsilon)} + \frac{s^{(2-\epsilon)}(k_{12}(k_{12} + s) - p_1^2)}{\Gamma(3-\epsilon)} \right. \\ &+ \left. \frac{s^{(1-\epsilon)}(k_1 k_2 (k_{12} + s) - p_1^2 k_{12})}{\Gamma(2-\epsilon)} - \frac{s^{-\epsilon} k_1 k_2 p_1^2}{\Gamma(1-\epsilon)} \right) \frac{\cos(\pi\epsilon/2)}{((k_{12} + s)^2 - p_1^2)} \end{aligned}$$

$$\text{Disc}_{p_1} \mathcal{B}^{(1)}(k_1, k_2, p_1) = \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2 \cos(\pi\epsilon/2)}{4k_1^3 k_2^3 p_1^3} \int_0^\infty ds \frac{Q_3(k_{12}, p_1, s)}{(s + k_{12})^2 - p_1^2}$$

$$Q_3(k_{12}, p_1, s) = s^{-\epsilon} \sum_{m,n=0}^{m+n=3} \frac{a_j b_i s^{3-m-n}}{\Gamma(4-m-n-\epsilon)} \left((s + k_{12}) \cos\left(\frac{\pi}{2}n\right) + ip_1 \sin\left(\frac{\pi}{2}n\right) \right)$$



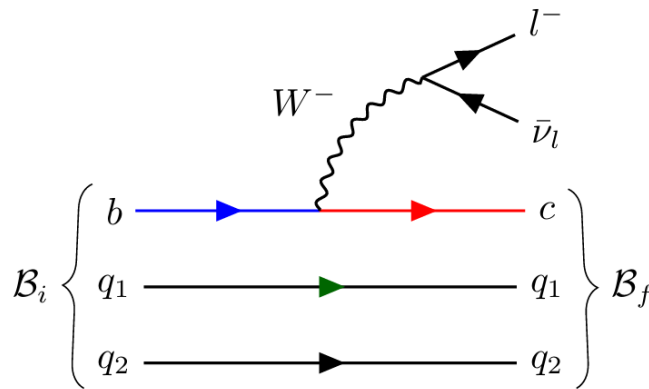
$$\overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(k_1, k_2, p_1) = \lim_{\epsilon \rightarrow 0^+} \frac{\lambda H^2}{4k_1^3 k_2^3 p_1^3} \int_0^\infty ds \left(\frac{-s^{(3-\epsilon)} p_1}{\Gamma(4-\epsilon)} + \frac{s^{(2-\epsilon)} s p_1}{\Gamma(3-\epsilon)} + \frac{s^{(1-\epsilon)} p_1 (k_1^2 + k_2^2 + k_1 k_2 + s k_{12})}{\Gamma(2-\epsilon)} + \frac{s^{-\epsilon} k_1 k_2 p_1 (k_{12} + s)}{\Gamma(1-\epsilon)} \right) \frac{\sin(\pi\epsilon/2)}{((k_{12} + s)^2 - p_1^2)}$$

$$\overline{\text{Disc}}_{p_1} \tilde{\mathcal{B}}^{(1)}(k_1, k_2, p_1) = \lim_{\epsilon \rightarrow 0^+} \frac{i\lambda H^2 \sin(\pi\epsilon/2)}{4k_1^3 k_2^3 p_1^3} \int_0^\infty ds \frac{\tilde{\mathcal{Q}}_3(k_{12}, p_1, s)}{(s + k_{12})^2 - p_1^2}$$

$$\tilde{\mathcal{Q}}_3(k_{12}, p_1, s) = s^{-\epsilon} \sum_{m,n=0}^{m+n=3} \frac{a_j b_i s^{3-m-n}}{\Gamma(4-m-n-\epsilon)} \left(i p_1 \cos\left(\frac{\pi}{2}n\right) - (s + k_{12}) \sin\left(\frac{\pi}{2}n\right) \right).$$

$$H_{eff} = \frac{G_F}{\sqrt{2}} V_{cb} \bar{c} \gamma_\mu (1 - \gamma_5) b \bar{\nu}_l \gamma_\mu (1 - \gamma_5) l,$$

$$\begin{aligned} T &= \langle \mathcal{B}_f(p') l(l) \bar{\nu}_l(v) | H_{eff} | \mathcal{B}_i(p) \rangle \\ &= \frac{G_F}{\sqrt{2}} V_{cb} \langle \mathcal{B}_f(p') | \bar{c} \gamma_\mu (1 - \gamma_5) b | \mathcal{B}_i(p) \rangle \\ &\quad \times \langle l(l) \bar{\nu}_l(v) | \bar{\nu}_l \gamma_\mu (1 - \gamma_5) l | 0 \rangle \quad (2) \end{aligned}$$



$$\langle l(l) \bar{\nu}_l(v) | \bar{\nu}_l \gamma_\mu (1 - \gamma_5) l | 0 \rangle = \bar{u}_{\nu, s} \gamma_\mu (1 - \gamma_5) u_{-l, s'},$$

$$\begin{aligned} &\langle \mathcal{B}_f(p') | \bar{c} \gamma_\mu (1 - \gamma_5) b | \mathcal{B}_i(p) \rangle \\ &= \bar{u}(p', s') \left[\gamma_\mu f_1(q^2) + i \frac{\sigma_{\mu\nu} q^\nu}{m_{B_i}} f_2(q^2) + \frac{q_\mu}{m_{B_i}} f_3(q^2) \right] U(p, s) \\ &\quad - \bar{u}(p', s') \left[\gamma_\mu g_1(q^2) + i \frac{\sigma_{\mu\nu} q^\nu}{m_{B_i}} g_2(q^2) + \frac{q_\mu}{m_{B_i}} g_3(q^2) \right] \gamma_5 U(p, s), \end{aligned}$$

$$\begin{aligned} \frac{d\Gamma}{dq^2} &= \frac{d\Gamma_L}{dq^2} + \frac{d\Gamma_T}{dq^2} \\ \frac{d\Gamma_L}{dq^2} &= \frac{G_F^2 V_{cb}^2 q^2}{384\pi^3 m_{B_i}^2} \frac{\sqrt{Q_+ Q_-}}{2m_{B_i}} \left(1 - \frac{m_l^2}{q^2}\right)^2 \\ &\quad \times \left[\left(2 + \frac{m_l^2}{q^2}\right) \left(\left|H_{-\frac{1}{2},0}\right|^2 + \left|H_{\frac{1}{2},0}\right|^2 \right) \right. \\ &\quad \left. + \frac{3m_l^2}{q^2} \left(\left|H_{-\frac{1}{2},t}\right|^2 + \left|H_{\frac{1}{2},t}\right|^2 \right) \right] \\ \frac{d\Gamma_T}{dq^2} &= \frac{G_F^2 V_{cb}^2 q^2}{384\pi^3 m_{B_i}^2} \frac{\sqrt{Q_+ Q_-}}{2m_{B_i}} \left(1 - \frac{m_l^2}{q^2}\right)^2 \left(2 + \frac{m_l^2}{q^2}\right) \\ &\quad \times \left(\left|H_{\frac{1}{2},1}\right|^2 + \left|H_{-\frac{1}{2},-1}\right|^2 \right) \end{aligned}$$

$$H_{\lambda_f, \lambda_W} = H_{\lambda_f, \lambda_W}^V \ominus H_{\lambda_f, \lambda_W}^A$$

$$\begin{aligned} H_{\frac{1}{2},0}^V &= -i \frac{\sqrt{Q_-}}{\sqrt{q^2}} \left[(m_{B_i} + m_{B_f}) f_1(q^2) - \frac{q^2}{m_{B_i}} f_2(q^2) \right] \\ H_{\frac{1}{2},1}^V &= i \sqrt{2Q_-} \left[-f_1(q^2) + \frac{m_{B_i} + m_{B_f}}{m_{B_i}} f_2(q^2) \right] \\ H_{\frac{1}{2},t}^V &= -i \frac{\sqrt{Q_+}}{\sqrt{q^2}} \left[(m_{B_i} - m_{B_f}) f_1(q^2) + \frac{q^2}{m_{B_i}} f_3(q^2) \right] \\ H_{\frac{1}{2},0}^A &= -i \frac{\sqrt{Q_+}}{\sqrt{q^2}} \left[(m_{B_i} - m_{B_f}) g_1(q^2) + \frac{q^2}{m_{B_i}} g_2(q^2) \right] \\ H_{\frac{1}{2},1}^A &= i \sqrt{2Q_+} \left[-g_1(q^2) - \frac{m_{B_i} - m_{B_f}}{m_{B_i}} g_2(q^2) \right] \\ H_{\frac{1}{2},t}^A &= -i \frac{\sqrt{Q_-}}{\sqrt{q^2}} \left[(m_{B_i} + m_{B_f}) g_1(q^2) - \frac{q^2}{m_{B_i}} g_3(q^2) \right] \end{aligned}$$

$$\begin{aligned} H_{-\lambda_f, -\lambda_W}^V &\otimes H_{\lambda_f, \lambda_W}^V \\ H_{-\lambda_f, -\lambda_W}^A &\otimes -H_{\lambda_f, \lambda_W}^A \end{aligned}$$

$$\Gamma = \Gamma_L + \Gamma_T = \int_{m_l^2}^{(m_{B_i} - m_{B_f})^2} \frac{d\Gamma}{dq^2} dq^2$$

$$\begin{aligned} \Pi_\mu(p', q) &= i^2 \int d^4x d^4y e^{ip' \cdot x} e^{iq \cdot y} \\ &\quad \times \langle 0 | \mathcal{T} [J_{B_f}(x) J_\mu^{V-A}(y) \bar{J}_{B_i}(0)] | 0 \rangle \end{aligned}$$

$$\begin{aligned} J_{B_f}(x) &= \varepsilon_{ijk} (q_1^{iT}(x) C \gamma_\alpha q_2^j(x)) \gamma_\alpha \gamma_5 c^k(x) \\ J_\mu^{V-A}(y) &= \bar{c}^m(y) \gamma_\mu (1 - \gamma_5) b^m(y) \\ J_{B_i}(0) &= \varepsilon_{i'j'k'} (q_1^{i'T}(0) C \gamma_\beta q_2^{j'}(0)) \gamma_\beta \gamma_5 b^{k'}(0) \end{aligned}$$



$$\begin{aligned}
& \Pi_{\mu}^{\text{phy}}(p^2, p'^2) \\
&= \frac{\langle 0|J_{\mathcal{B}_f}|\mathcal{B}_f^+(p')\rangle\langle\mathcal{B}_f^+(p')|J_{\mu}^{V-A}|\mathcal{B}_i^+(p)\rangle\langle\mathcal{B}_i^+(p)|\bar{J}_{\mathcal{B}_i}|0\rangle}{\left(m_{\mathcal{B}_f^+}^2 - p'^2\right)\left(m_{\mathcal{B}_i^+}^2 - p^2\right)} \\
&+ \frac{\langle 0|J_{\mathcal{B}_f}|\mathcal{B}_f^-(p')\rangle\langle\mathcal{B}_f^-(p')|J_{\mu}^{V-A}|\mathcal{B}_i^+(p)\rangle\langle\mathcal{B}_i^+(p)|\bar{J}_{\mathcal{B}_i}|0\rangle}{\left(m_{\mathcal{B}_f^-}^2 - p'^2\right)\left(m_{\mathcal{B}_i^+}^2 - p^2\right)} \\
&+ \frac{\langle 0|J_{\mathcal{B}_f}|\mathcal{B}_f^+(p')\rangle\langle\mathcal{B}_f^+(p')|J_{\mu}^{V-A}|\mathcal{B}_i^-(p)\rangle\langle\mathcal{B}_i^-(p)|\bar{J}_{\mathcal{B}_i}|0\rangle}{\left(m_{\mathcal{B}_f^+}^2 - p'^2\right)\left(m_{\mathcal{B}_i^-}^2 - p^2\right)} \\
&+ \frac{\langle 0|J_{\mathcal{B}_f}|\mathcal{B}_f^-(p')\rangle\langle\mathcal{B}_f^-(p')|J_{\mu}^{V-A}|\mathcal{B}_i^-(p)\rangle\langle\mathcal{B}_i^-(p)|\bar{J}_{\mathcal{B}_i}|0\rangle}{\left(m_{\mathcal{B}_f^-}^2 - p'^2\right)\left(m_{\mathcal{B}_i^-}^2 - p^2\right)} \\
&+ \text{h.c.},
\end{aligned}$$

$$\langle 0|J_{\mathcal{B}_f}(0)|\mathcal{B}_f^+(p')\rangle = \lambda_{\mathcal{B}_f^+} u(p', s')$$

$$\langle 0|J_{\mathcal{B}_f}(0)|\mathcal{B}_f^-(p')\rangle = \lambda_{\mathcal{B}_f^-} \gamma_5 u(p', s')$$

$$\langle\mathcal{B}_i^+(p)|\bar{J}_{\mathcal{B}_i}(0)|0\rangle = \lambda_{\mathcal{B}_i^+} \bar{U}(p, s)$$

$$\langle\mathcal{B}_i^-(p)|\bar{J}_{\mathcal{B}_i}(0)|0\rangle = -\lambda_{\mathcal{B}_i^-} \bar{U}(p, s) \gamma_5$$

$$\begin{aligned}
\langle\mathcal{B}_f^+(p')|J_{\mu}^{V-A}(0)|\mathcal{B}_i^+(p)\rangle &= \bar{u}(p', s') \left[F_1^{++}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^+}} + F_2^{++}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^+}} + F_3^{++}(q^2) \gamma_{\mu} \right] U(p, s) \\
&- \bar{u}(p', s') \left[G_1^{++}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^+}} + G_2^{++}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^+}} + G_3^{++}(q^2) \gamma_{\mu} \right] \gamma_5 U(p, s),
\end{aligned}$$

$$\begin{aligned}
\langle\mathcal{B}_f^-(p')|J_{\mu}^{V-A}(0)|\mathcal{B}_i^+(p)\rangle &= \bar{u}(p', s') \gamma_5 \left[F_1^{+-}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^+}} + F_2^{+-}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^-}} + F_3^{+-}(q^2) \gamma_{\mu} \right] U(p, s) \\
&- \bar{u}(p', s') \gamma_5 \left[G_1^{+-}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^+}} + G_2^{+-}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^-}} + G_3^{+-}(q^2) \gamma_{\mu} \right] \gamma_5 U(p, s),
\end{aligned}$$

$$\begin{aligned}
\langle\mathcal{B}_f^+(p')|J_{\mu}^{V-A}(0)|\mathcal{B}_i^-(p)\rangle &= \bar{u}(p', s') \left[F_1^{-+}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^-}} + F_2^{-+}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^+}} + F_3^{-+}(q^2) \gamma_{\mu} \right] \gamma_5 U(p, s) \\
&- \bar{u}(p', s') \left[G_1^{-+}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^-}} + G_2^{-+}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^+}} + G_3^{-+}(q^2) \gamma_{\mu} \right] U(p, s),
\end{aligned}$$

$$\begin{aligned}
\langle\mathcal{B}_f^-(p')|J_{\mu}^{V-A}(0)|\mathcal{B}_i^-(p)\rangle &= \bar{u}(p', s') \gamma_5 \left[F_1^{--}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^-}} + F_2^{--}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^-}} + F_3^{--}(q^2) \gamma_{\mu} \right] \gamma_5 U(p, s) \\
&- \bar{u}(p', s') \gamma_5 \left[G_1^{--}(q^2) \frac{p^{\mu}}{m_{\mathcal{B}_i^-}} + G_2^{--}(q^2) \frac{p'^{\mu}}{m_{\mathcal{B}_f^-}} + G_3^{--}(q^2) \gamma_{\mu} \right] U(p, s),
\end{aligned}$$



$$\begin{aligned}
\Pi_{\mu}^{\text{phy}}(p^2, p'^2) = & \frac{\lambda_{B_f^+} \lambda_{B_i^+} (\not{p}' + m_{B_f^+}) \left\{ \begin{aligned} & \left[F_1^{++}(q^2) \frac{p^\mu}{m_{B_i^+}} + F_2^{++}(q^2) \frac{p'^\mu}{m_{B_f^+}} + F_3^{++}(q^2) \gamma_\mu \right] \\ & - \left[G_1^{++}(q^2) \frac{p^\mu}{m_{B_i^+}} + G_2^{++}(q^2) \frac{p'^\mu}{m_{B_f^+}} + G_3^{++}(q^2) \gamma_\mu \right] \gamma_5 \end{aligned} \right\} (\not{p} + m_{B_i^+})}{(m_{B_f^+}^2 - p'^2)(m_{B_i^+}^2 - p^2)} \\
& + \frac{\lambda_{B_f^-} \lambda_{B_i^+} \gamma_5 (\not{p}' + m_{B_f^-}) \left\{ \begin{aligned} & \gamma_5 \left[F_1^{+-}(q^2) \frac{p^\mu}{m_{B_i^+}} + F_2^{+-}(q^2) \frac{p'^\mu}{m_{B_f^-}} + F_3^{+-}(q^2) \gamma_\mu \right] \\ & - \gamma_5 \left[G_1^{+-}(q^2) \frac{p^\mu}{m_{B_i^+}} + G_2^{+-}(q^2) \frac{p'^\mu}{m_{B_f^-}} + G_3^{+-}(q^2) \gamma_\mu \right] \gamma_5 \end{aligned} \right\} (\not{p} + m_{B_i^+})}{(m_{B_f^-}^2 - p'^2)(m_{B_i^+}^2 - p^2)} \\
& - \frac{\lambda_{B_f^+} \lambda_{B_i^-} (\not{p}' + m_{B_f^+}) \left\{ \begin{aligned} & \left[F_1^{-+}(q^2) \frac{p^\mu}{m_{B_i^-}} + F_2^{-+}(q^2) \frac{p'^\mu}{m_{B_f^+}} + F_3^{-+}(q^2) \gamma_\mu \right] \gamma_5 \\ & - \left[G_1^{-+}(q^2) \frac{p^\mu}{m_{B_i^-}} + G_2^{-+}(q^2) \frac{p'^\mu}{m_{B_f^+}} + G_3^{-+}(q^2) \gamma_\mu \right] \end{aligned} \right\} (\not{p} + m_{B_i^-}) \gamma_5}{(m_{B_f^+}^2 - p'^2)(m_{B_i^-}^2 - p^2)} \\
& - \frac{\lambda_{B_f^-} \lambda_{B_i^-} \gamma_5 (\not{p}' + m_{B_f^-}) \left\{ \begin{aligned} & \gamma_5 \left[F_1^{--}(q^2) \frac{p^\mu}{m_{B_i^-}} + F_2^{--}(q^2) \frac{p'^\mu}{m_{B_f^-}} + F_3^{--}(q^2) \gamma_\mu \right] \gamma_5 \\ & - \gamma_5 \left[G_1^{--}(q^2) \frac{p^\mu}{m_{B_i^-}} + G_2^{--}(q^2) \frac{p'^\mu}{m_{B_f^-}} + G_3^{--}(q^2) \gamma_\mu \right] \end{aligned} \right\} (\not{p} + m_{B_i^-}) \gamma_5}{(m_{B_f^-}^2 - p'^2)(m_{B_i^-}^2 - p^2)} + h.c.
\end{aligned}$$

$$\begin{aligned}
\Pi_{\mu}^{\text{phy}}(p^2, p'^2) = & \Pi_1^{\text{phy}} \gamma_\mu + \Pi_2^{\text{phy}} \gamma_\mu \not{p}' + \Pi_3^{\text{phy}} \gamma_\mu q + \Pi_4^{\text{phy}} \gamma_\mu \not{p}' \not{q} \\
& + \Pi_5^{\text{phy}} \not{p}' p'_\mu + \Pi_6^{\text{phy}} \not{p}' q_\mu + \Pi_7^{\text{phy}} q p'_\mu + \Pi_8^{\text{phy}} q q_\mu \\
& + \Pi_9^{\text{phy}} \not{p}' \not{q} p'_\mu + \Pi_{10}^{\text{phy}} \not{p}' \not{q} q_\mu + \Pi_{11}^{\text{phy}} p'_\mu + \Pi_{12}^{\text{phy}} q_\mu \\
& + \Pi_{13}^{\text{phy}} \gamma_\mu \gamma_5 + \Pi_{14}^{\text{phy}} \gamma_\mu \not{p}' \gamma_5 + \Pi_{15}^{\text{phy}} \gamma_\mu q \gamma_5 \\
& + \Pi_{16}^{\text{phy}} \gamma_\mu \not{p}' \not{q} \gamma_5 + \Pi_{17}^{\text{phy}} \not{p}' \gamma_5 p'_\mu + \Pi_{18}^{\text{phy}} \not{p}' \gamma_5 q_\mu \\
& + \Pi_{19}^{\text{phy}} \not{q} \gamma_5 p'_\mu + \Pi_{20}^{\text{phy}} \not{q} \gamma_5 q_\mu + \Pi_{21}^{\text{phy}} \not{p}' \not{q} \gamma_5 p'_\mu \\
& + \Pi_{22}^{\text{phy}} \not{p}' \not{q} \gamma_5 q_\mu + \Pi_{23}^{\text{phy}} \gamma_5 p'_\mu + \Pi_{24}^{\text{phy}} \gamma_5 q_\mu.
\end{aligned}$$

$$\begin{aligned}
\Pi_{\mu}^{\text{QCD}}(p', q) = & A \varepsilon_{ijk} \varepsilon_{i'j'k'} \int d^4x d^4y e^{ip' \cdot x} e^{iq \cdot y} \\
& \times \text{Tr} \left\{ S_{q_2}^{k'lj}(x) \gamma_\beta C S_{q_1}^{ki'T}(x) C \gamma_\alpha \right\} \gamma_\alpha \gamma_5 \\
& \times S_c^{j'm}(x-y) \gamma_\mu (1 - \gamma_5) S_b^{mi}(y) \gamma_5 \gamma_\beta,
\end{aligned}$$

$$\begin{aligned}
S_d^{ij}(x) = & \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \frac{\delta^{ij}}{\not{k}} - \frac{g_s G_{\alpha\beta}^n t_{ij}^n \sigma^{\alpha\beta} \not{k} + \not{k} \sigma^{\alpha\beta}}{4(k^2)^2} \right. \\
& \left. - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\mu\nu\beta})}{4(k^2)^5} \right\} \\
& - \frac{\delta^{ij} \langle \bar{q} q \rangle}{12} - \frac{\delta^{ij} x^2 \langle \bar{q} g_s \sigma G q \rangle}{192} - \frac{\langle \bar{q}^j \sigma^{\mu\nu} q^i \rangle \sigma_{\mu\nu}}{8} \\
& - \frac{\delta^{ij} x^4 \langle \bar{q} q \rangle \langle g_s^2 G G \rangle}{27648} + \dots
\end{aligned}$$



$$\begin{aligned}
S_s^{ij}(x) = & \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \frac{\delta^{ij}}{k - m_s} \right. \\
& - \frac{g_s G_{\alpha\beta}^n t_{ij}^n \sigma^{\alpha\beta} (k + m_s) + (k + m_s) \sigma^{\alpha\beta}}{4(k^2 - m_s^2)^2} \\
& \left. - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\nu\beta\mu})}{4(k^2 - m_s^2)^5} \right\} \\
& - \frac{\delta^{ij} \langle \bar{s}s \rangle}{12} + \frac{i \delta^{ij} \not{x} m_s \langle \bar{s}s \rangle}{48} - \frac{\delta^{ij} x^2 \langle \bar{s} g_s \sigma G s \rangle}{192} \\
& + \frac{i \delta^{ij} x^2 \not{x} m_s \langle \bar{s} g_s \sigma G s \rangle}{1152} - \frac{\langle \bar{s}^j \sigma^{\mu\nu} s^i \rangle \sigma_{\mu\nu}}{8} \\
& - \frac{\delta^{ij} x^4 \langle \bar{s}s \rangle \langle g_s^2 G G \rangle}{27648} + \dots, \\
S_{c[b]}^{ij}(x) = & \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \frac{\delta^{ij}}{k - m_{c[b]}} \right. \\
& - \frac{g_s G_{\alpha\beta}^n t_{ij}^n \sigma^{\alpha\beta} (k + m_{c[b]}) + (k + m_{c[b]}) \sigma^{\alpha\beta}}{4(k^2 - m_{c[b]}^2)^2} \\
& \left. - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\nu\beta\mu})}{4(k^2 - m_{c[b]}^2)^5} + \dots \right\}.
\end{aligned}$$

$$\langle \bar{q}(\bar{s}) g_s \sigma G q(s) \rangle = \langle \bar{q}(\bar{s}) g_s \sigma_{\mu\nu} G_{\mu\nu}^\alpha t^\alpha q(s) \rangle, t^\alpha = \frac{\lambda^\alpha}{2}, \lambda^\alpha (\alpha = 1, \dots, 8)$$

$$f^{\alpha\beta\mu\nu} = (k + m) \gamma^\alpha (k + m) \gamma^\beta (k + m) \gamma^\mu (k + m) \gamma^\nu (k + m),$$

$$\begin{aligned}
\Pi_\mu^{\text{QCD}}(p^2, p'^2) = & \Pi_1^{\text{QCD}} \gamma_\mu + \Pi_2^{\text{QCD}} \gamma_\mu \not{p}' + \Pi_3^{\text{QCD}} \gamma_\mu \not{q} \\
& + \Pi_4^{\text{QCD}} \gamma_\mu \not{p}' \not{q} + \Pi_5^{\text{QCD}} \not{p}' p'_\mu + \Pi_6^{\text{QCD}} \not{p}' q_\mu \\
& + \Pi_7^{\text{QCD}} q p'_\mu + \Pi_8^{\text{QCD}} q q_\mu + \Pi_9^{\text{QCD}} \not{p}' \not{q} p'_\mu \\
& + \Pi_{10}^{\text{QCD}} \not{p}' \not{q} q_\mu + \Pi_{11}^{\text{QCD}} p'_\mu + \Pi_{12}^{\text{QCD}} q_\mu \\
& + \Pi_{13}^{\text{QCD}} \gamma_\mu \gamma_5 + \Pi_{14}^{\text{QCD}} \gamma_\mu p'_\mu \gamma_5 + \Pi_{15}^{\text{QCD}} \gamma_\mu q \gamma_5 \\
& + \Pi_{16}^{\text{QCD}} \gamma_\mu \not{p}' \not{q} \gamma_5 + \Pi_{17}^{\text{QCD}} \not{p}' \gamma_5 p'_\mu + \Pi_{18}^{\text{QCD}} \not{p}' \gamma_5 q_\mu \\
& + \Pi_{19}^{\text{QCD}} \not{q} \gamma_5 p'_\mu + \Pi_{20}^{\text{QCD}} \not{q} \gamma_5 q_\mu + \Pi_{21}^{\text{QCD}} \not{p}' \not{q} \gamma_5 p'_\mu \\
& + \Pi_{22}^{\text{QCD}} \not{p}' \not{q} \gamma_5 q_\mu + \Pi_{23}^{\text{QCD}} \gamma_5 p'_\mu + \Pi_{24}^{\text{QCD}} \gamma_5 q_\mu,
\end{aligned}$$

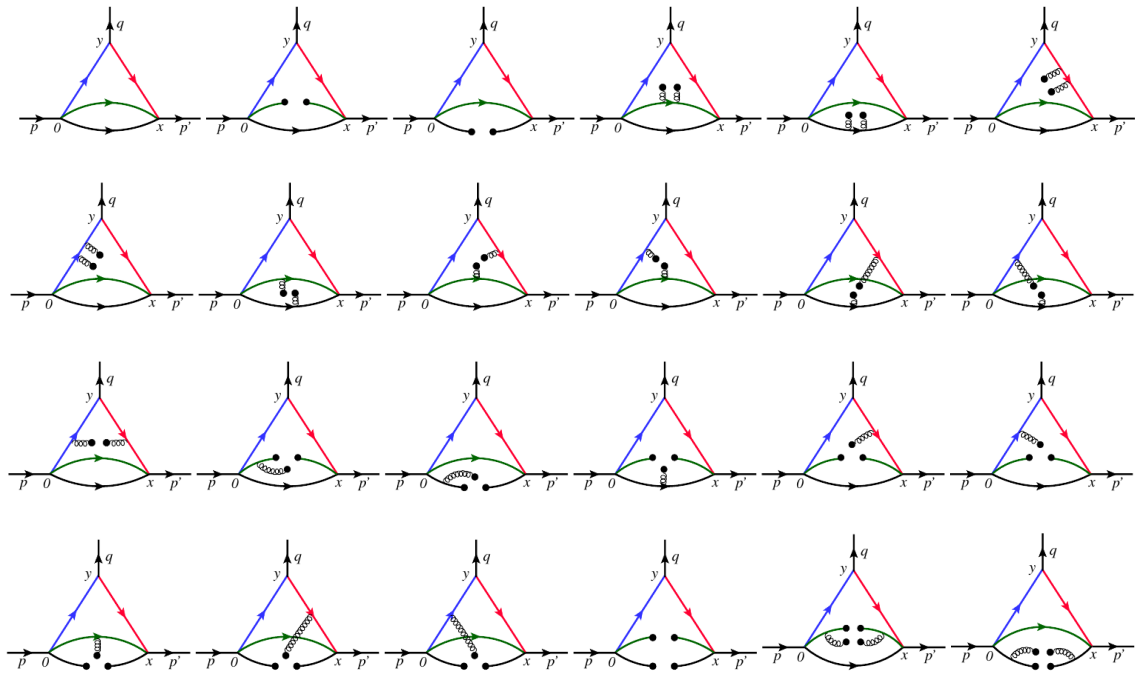
$$\Pi_i^{\text{QCD}}(p^2, p'^2) = \int_{u_{\min}}^\infty du \int_{s_{\min}}^\infty ds \frac{\rho_i^{\text{QCD}}(s, u, q^2)}{(s - p^2)(u - p'^2)}$$

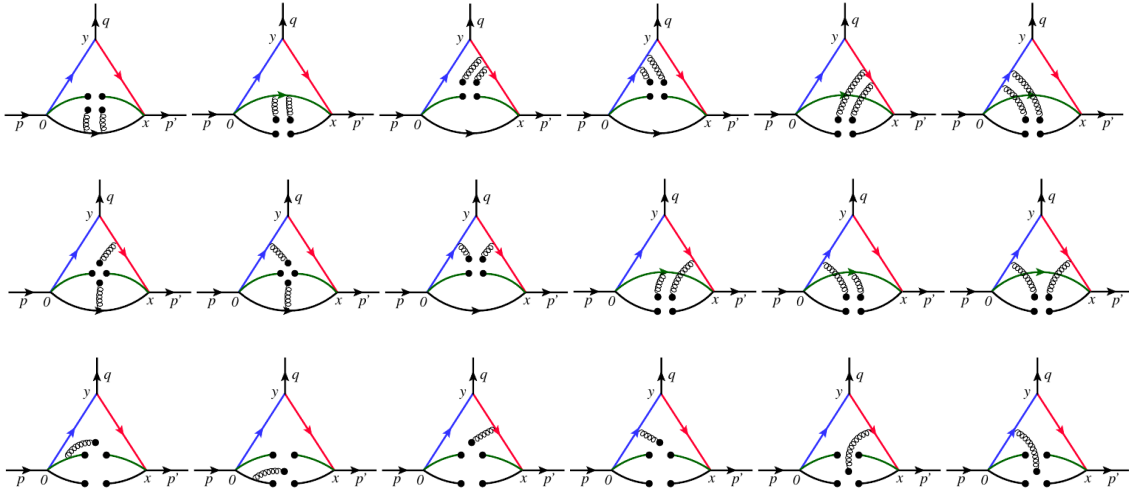


$$\begin{aligned}
\rho_i^{\text{QCD}}(s, u, q^2) = & \rho_i^{\text{pert}}(s, u, q^2) + \rho_i^{(\bar{q}q)}(s, u, q^2) \\
& + \rho_i^{(\bar{s}s)}(s, u, q^2) + \rho_i^{(g_s^2 GG)}(s, u, q^2) \\
& + \rho_i^{(\bar{q}g_s \sigma Gq)}(s, u, q^2) + \rho_i^{(\bar{s}g_s \sigma Gs)}(s, u, q^2) \\
& + \rho_i^{(\bar{q}q)^2}(s, u, q^2) + \rho_i^{(\bar{q}q)(\bar{s}s)}(s, u, q^2) \\
& + \rho_i^{(\bar{s}s)^2}(s, u, q^2) + \rho_i^{(\bar{q}q)(g_s^2 GG)}(s, u, q^2) \\
& + \rho_i^{(\bar{s}s)(g_s^2 GG)}(s, u, q^2) + \rho_i^{(\bar{q}q)(\bar{q}g_s \sigma Gq)}(s, u, q^2) \\
& + \rho_i^{(\bar{q}q)(\bar{s}g_s \sigma Gs)}(s, u, q^2) + \rho_i^{(\bar{s}s)(\bar{q}g_s \sigma Gq)}(s, u, q^2) \\
& + \rho_i^{(\bar{s}s)(\bar{s}g_s \sigma Gs)}(s, u, q^2).
\end{aligned}$$

$$\begin{aligned}
\Pi_{0\mu}^{\text{QCD}}(p, q) = & \frac{6A}{(2\pi)^8} \int d^4k_1 d^4k_2 d^4k_3 d^4k_4 \delta^4(q + k_3 - k_4) \\
& \times \delta^4(p' - k_1 - k_2 - k_3) \text{Tr}\{(k_1 + m_{q_2})\gamma_\beta(k_2 - m_{q_1}) \\
& \times \gamma_\alpha\} \frac{\gamma_\alpha \gamma_5 (k_3 + m_c) \gamma_\mu (1 - \gamma_5) (k_4 + m_b) \gamma_5 \gamma_\beta}{(k_1^2 - m_{q_2}^2)(k_2^2 - m_{q_1}^2)(k_3^2 - m_c^2)(k_4^2 - m_b^2)}
\end{aligned}$$

$$\begin{aligned}
\rho_{0\mu}^{\text{QCD}}(s, u, q^2) = & -\frac{6A}{(2\pi)^8} \frac{(-2\pi i)^5}{(2\pi i)^3} \int_{(m_{q_1} + m_{q_2})^2}^{(\sqrt{u} - m_c)^2} dr \int d^4k_1 d^4k_3 \\
& \times \delta(k_1^2 - m_{q_2}^2) \delta[(q' - k_1)^2 - m_{q_1}^2] \\
& \times \delta[(p' - k_3)^2 - r] \delta[(k_3 + q)^2 - m_b^2] \delta(k_3^2 - m_c^2) \\
& \times \text{Tr}\{(k_1 + m_{q_2})\gamma_\beta(p' - k_3 - k_1 - m_{q_1})\gamma_\alpha\} \\
& \times \gamma_\alpha \gamma_5 (k_3 + m_c) \gamma_\mu (1 - \gamma_5) [q + k_3 + m_b] \gamma_5 \gamma_\beta,
\end{aligned}$$





$$\Pi_{3b\mu}^{\text{QCD}}(p, q) = \frac{A\langle\bar{q}_1 q_1\rangle i}{2(2\pi)^4} \int d^4 k_3 \text{Tr}\{\gamma_\beta(\not{p}' - \not{k}_3 - m_{q_2})\gamma_\alpha\} \\ \times \frac{\gamma_\alpha \gamma_5 (\not{k}_3 + m_c) \gamma_\mu (1 - \gamma_5) (\not{k}_3 + q + m_b) \gamma_5 \gamma_\beta}{(k_3^2 - m_c^2)[(p' - k_3)^2 - m_{q_2}^2][(k_3 + q)^2 - m_b^2]}$$

$$\rho_{3b\mu}^{\text{QCD}}(s, u, q^2) = \frac{A\langle\bar{q}_1 q_1\rangle i (-2\pi i)^3}{2(2\pi)^4 (2\pi i)^2} \int d^4 k_3 \delta(k_3^2 - m_c^2) \\ \times \delta[(p' - k_3)^2 - m_{q_2}^2] \delta[(k_3 + q)^2 - m_b^2] \\ \times \text{Tr}\{\gamma_\beta(\not{p}' - \not{k}_3 - m_{q_2})\gamma_\alpha\} \\ \times \gamma_\alpha \gamma_5 (\not{k}_3 + m_c) \gamma_\mu (1 - \gamma_5) (\not{k}_3 + q + m_b) \gamma_5 \gamma_\beta. (28)$$

$$\Pi_{4n\mu}^{\text{QCD}}(p, q) = \frac{A\langle g_s^2 GG\rangle}{48(2\pi)^8} \int d^4 k_1 d^4 k_2 d^4 k_3 d^4 k_4 \delta^4(q + k_3 - k_4) \\ \times \delta^4(p' - k_1 - k_2 - k_3) (g_{\rho\lambda} g_{\sigma\chi} - g_{\rho\chi} g_{\sigma\lambda}) \\ \times \text{Tr}\{(f^{\rho\sigma\lambda\chi} + f^{\rho\lambda\sigma\chi} + f^{\rho\lambda\chi\sigma})\gamma_\beta\} \\ \times (\not{p}' - \not{k}_3 - \not{k}_1 - m_{q_2})\gamma_\alpha \\ \times \frac{\gamma_\alpha \gamma_5 (\not{k}_3 + m_c) \gamma_\mu (1 - \gamma_5) (\not{k}_4 + m_b) \gamma_5 \gamma_\beta}{(k_1^2 - m_{q_2}^2)^5 (k_2^2 - m_{q_1}^2) (k_3^2 - m_c^2) (k_4^2 - m_b^2)}$$

$$\frac{1}{(k^2 - m^2)^n} = \frac{1}{(n-1)!} \frac{\partial^{(n-1)}}{(\partial A)^{(n-1)}} \frac{1}{k^2 - A} \Big|_{A \rightarrow m^2}$$

$$\begin{aligned}
F_1^{++}(Q^2) &= \frac{m_{B_i^+} e^{m_{B_i^+}^2/T^2 + m_{B_f^+}^2/kT^2}}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \rho_{12}^{\text{QCD}}(s, u, Q^2) + m_{B_f^-} \rho_6^{\text{QCD}}(s, u, Q^2) + (m_{B_i^-} - m_{B_f^+}) \left[m_{B_f^-} \rho_{10}^{\text{QCD}}(s, u, Q^2) + \rho_8^{\text{QCD}}(s, u, Q^2) \right] \right\} \\
F_2^{++}(Q^2) &= \frac{m_{B_f^+} e^{m_{B_i^+}^2/T^2 + m_{B_f^+}^2/kT^2}}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \begin{aligned} &\rho_{11}^{\text{QCD}}(s, u, Q^2) - \rho_{12}^{\text{QCD}}(s, u, Q^2) + 2[\rho_2^{\text{QCD}}(s, u, Q^2) - \rho_3^{\text{QCD}}(s, u, Q^2)] \\ &+ m_{B_f^-} [2\rho_4^{\text{QCD}}(s, u, Q^2) + \rho_5^{\text{QCD}}(s, u, Q^2) - \rho_6^{\text{QCD}}(s, u, Q^2)] \\ &+ (m_{B_i^-} - m_{B_f^+}) \left[m_{B_f^-} \rho_9^{\text{QCD}}(s, u, Q^2) - m_{B_f^-} \rho_{10}^{\text{QCD}}(s, u, Q^2) + 2\rho_4^{\text{QCD}}(s, u, Q^2) + \rho_7^{\text{QCD}}(s, u, Q^2) \right. \\ &\quad \left. - \rho_8^{\text{QCD}}(s, u, Q^2) \right] \end{aligned} \right\} \\
F_3^{++}(Q^2) &= \frac{e^{m_{B_i^+}^2/T^2 + m_{B_f^+}^2/kT^2}}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \rho_1^{\text{QCD}}(s, u, Q^2) + (m_{B_i^-} + m_{B_f^+}) \rho_3^{\text{QCD}}(s, u, Q^2) - m_{B_f^-} \left[\rho_2^{\text{QCD}}(s, u, Q^2) + (m_{B_i^-} + m_{B_f^+}) \rho_4^{\text{QCD}}(s, u, Q^2) \right] \right\}, \\
G_1^{++}(Q^2) &= \frac{m_{B_i^+} e^{m_{B_i^+}^2/T^2 + m_{B_f^+}^2/kT^2}}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \rho_{24}^{\text{QCD}}(s, u, Q^2) + m_{B_f^-} \rho_{18}^{\text{QCD}}(s, u, Q^2) - (m_{B_i^-} + m_{B_f^+}) \left[m_{B_f^-} \rho_{22}^{\text{QCD}}(s, u, Q^2) + \rho_{20}^{\text{QCD}}(s, u, Q^2) \right] \right\}, \\
G_2^{++}(Q^2) &= \frac{m_{B_f^+} e^{m_{B_i^+}^2 T^2 + m_{B_f^+}^2/kT^2}}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \begin{aligned} &\rho_{23}^{\text{QCD}}(s, u, Q^2) - \rho_{24}^{\text{QCD}}(s, u, Q^2) + 2[\rho_{14}^{\text{QCD}}(s, u, Q^2) - \rho_{15}^{\text{QCD}}(s, u, Q^2)] \\ &+ m_{B_f^-} [2\rho_{16}^{\text{QCD}}(s, u, Q^2) + \rho_{17}^{\text{QCD}}(s, u, Q^2) - \rho_{18}^{\text{QCD}}(s, u, Q^2)] \\ &+ (m_{B_i^-} + m_{B_f^+}) \left[m_{B_f^-} \rho_{22}^{\text{QCD}}(s, u, Q^2) - m_{B_f^-} \rho_{21}^{\text{QCD}}(s, u, Q^2) - 2\rho_{16}^{\text{QCD}}(s, u, Q^2) - \rho_{19}^{\text{QCD}}(s, u, Q^2) \right. \\ &\quad \left. + \rho_{20}^{\text{QCD}}(s, u, Q^2) \right] \end{aligned} \right\} \\
G_3^{++}(Q^2) &= \frac{e^{m_{B_i^+}^2} T^2 + m_{B_f^+}^2/kT^2}{\lambda_{B_f^+} \lambda_{B_i^+} (m_{B_f^+} + m_{B_f^-}) (m_{B_i^+} + m_{B_i^-})} \int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds e^{-s/T^2 - u/kT^2} \\
&\quad \times \left\{ \rho_{13}^{\text{QCD}}(s, u, Q^2) - (m_{B_i^-} - m_{B_f^+}) \rho_{15}^{\text{QCD}}(s, u, Q^2) - m_{B_f^-} \left[\rho_{14}^{\text{QCD}}(s, u, Q^2) - (m_{B_i^-} - m_{B_f^+}) \rho_{16}^{\text{QCD}}(s, u, Q^2) \right] \right\}.
\end{aligned}$$

$$m_{c[b]}(\mu) = m_{c[b]}(m_{c[b]}) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_{c[b]})} \right]^{\frac{12}{33-2N_f}},$$

$$m_s(\mu) = m_s(2\text{GeV}) \left[\frac{\alpha_s(\mu)}{\alpha_s(2\text{GeV})} \right]^{\frac{12}{33-2N_f}},$$

$$\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{12}{33-2N_f}},$$

$$\langle \bar{s}s \rangle(\mu) = \langle \bar{s}s \rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{12}{33-2N_f}},$$

$$\langle \bar{q}g_s\sigma Gq \rangle(\mu) = \langle \bar{q}g_s\sigma Gq \rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{2}{33-2N_f}},$$

$$\langle \bar{s}g_s\sigma Gs \rangle(\mu) = \langle \bar{s}g_s\sigma Gs \rangle(1\text{GeV}) \left[\frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{2}{33-2N_f}},$$

$$\alpha_s(\mu) = \frac{1}{b_0 t} \left[1 - \frac{b_1 \log t}{b_0^2 t} + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right],$$

$$t = \log \left(\frac{\mu^2}{\Lambda_{\text{QCD}}^2} \right), b_0 = \frac{33-2N_f}{12\pi}, b_1 = \frac{153-19N_f}{24\pi^2} \text{ and } b_2 = \frac{2857 - \frac{5033}{9}N_f + \frac{325}{27}N_f^2}{128\pi^3}$$

$$\text{Pole}_s = \frac{\int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds}{\int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{\infty} ds}, \text{Pole}_u = \frac{\int_{u_{\min}}^{u_0} du \int_{s_{\min}}^{s_0} ds}{\int_{u_{\min}}^{\infty} du \int_{s_{\min}}^{s_0} ds}.$$

$$F_i(Q^2) = \frac{F(0)}{1 + Q^2/m_{B_c^*(1^-)}^2} \times \left\{ 1 + a \left[z(Q^2) - z(0) - \frac{1}{3} [z(Q^2)^3 - z(0)^3] \right] + b \left[z(Q^2)^2 - z(0)^2 + \frac{2}{3} [z(Q^2)^3 - z(0)^3] \right] \right\},$$

$$G_i(Q^2) = \frac{G(0)}{1 + Q^2/m_{B_c^*(1^+)}^2} \times \left\{ 1 + \tilde{a} \left[z(Q^2) - z(0) - \frac{1}{3} [z(Q^2)^3 - z(0)^3] \right] + \tilde{b} \left[z(Q^2)^2 - z(0)^2 + \frac{2}{3} [z(Q^2)^3 - z(0)^3] \right] \right\}.$$

$$z(Q^2) = \frac{\sqrt{t_+^2 + Q^2} - \sqrt{t_+^2 - t^2}}{\sqrt{t_+^2 + Q^2} + \sqrt{t_+^2 - t^2}},$$



$$\begin{aligned}
f_1(q^2) &= F_3(q^2) + \frac{m_{B_i} + m_{B_f}}{2m_{B_i}} F_1(q^2) + \frac{m_{B_i} + m_{B_f}}{2m_{B_f}} F_2(q^2), \\
f_2(q^2) &= \frac{1}{2} \left[F_1(q^2) + \frac{m_{B_i}}{m_{B_f}} F_2(q^2) \right], \\
f_3(q^2) &= \frac{1}{2} \left[F_1(q^2) - \frac{m_{B_i}}{m_{B_f}} F_2(q^2) \right], \\
g_1(q^2) &= G_3(q^2) - \frac{m_{B_i} - m_{B_f}}{2m_{B_i}} G_1(q^2) - \frac{m_{B_i} - m_{B_f}}{2m_{B_f}} G_2(q^2), \\
g_2(q^2) &= \frac{1}{2} \left[G_1(q^2) + \frac{m_{B_i}}{m_{B_f}} G_2(q^2) \right], \\
g_3(q^2) &= \frac{1}{2} \left[G_1(q^2) - \frac{m_{B_i}}{m_{B_f}} G_2(q^2) \right],
\end{aligned}$$

$$\Gamma(\Sigma_b^- \rightarrow \Sigma_c^0 l^- \bar{\nu}_l) = \Gamma(\Xi_b'^- \rightarrow \Xi_c'^0 l^- \bar{\nu}_l) = \Gamma(\Omega_b^- \rightarrow \Omega_c^0 l^- \bar{\nu}_l)$$

$$\Gamma(\Omega_b^- \rightarrow \Omega_c^0(e^-, \mu^-) \bar{\nu}_{e,\mu}) \approx 2\Gamma(\Xi_b'^- \rightarrow \Xi_c'^0(e^-, \mu^-) \bar{\nu}_{e,\mu}) \text{ and } \Gamma(\Omega_b^- \rightarrow \Omega_c^0 \tau^- \bar{\nu}_\tau) \approx 5\Gamma(\Xi_b'^- \rightarrow \Xi_c'^0 \tau^- \bar{\nu}_\tau)$$

$$\begin{aligned}
R_{B_f} &= \frac{\Gamma(B_i \rightarrow B_f \tau \bar{\nu}_\tau)}{\Gamma(B_i \rightarrow B_f e \bar{\nu}_e)}, \\
A_{FB}(q^2) &= \frac{d\Gamma_{\text{forward}}/dq^2 - d\Gamma_{\text{backward}}/dq^2}{d\Gamma/dq^2}, \\
&= \frac{3}{4} \left(\frac{\left| H_{\frac{1}{2},1} \right|^2 - \left| H_{-\frac{1}{2},-1} \right|^2}{|H_{\text{tot}}|^2} - \frac{2m_l^2 H_{\frac{1}{2},0} H_{\frac{1}{2},t}^* + H_{-\frac{1}{2},0} H_{-\frac{1}{2},t}^*}{q^2 |H_{\text{tot}}|^2} \right), \\
\alpha &= \frac{d\Gamma_{\lambda_2=\frac{1}{2}}/dq^2 - d\Gamma_{\lambda_2=-\frac{1}{2}}/dq^2}{d\Gamma_{\lambda_2=\frac{1}{2}}/dq^2 + d\Gamma_{\lambda_2=-\frac{1}{2}}/dq^2},
\end{aligned}$$

$$\begin{aligned}
|H_{\text{tot}}|^2 &= \left(1 + \frac{m_l^2}{2q^2} \right) \left(\left| H_{\frac{1}{2},1} \right|^2 + \left| H_{-\frac{1}{2},-1} \right|^2 + \left| H_{\frac{1}{2},0} \right|^2 \right. \\
&\quad \left. + \left| H_{-\frac{1}{2},0} \right|^2 \right) + \frac{3m_l^2}{2q^2} \left(\left| H_{\frac{1}{2},t} \right|^2 + \left| H_{-\frac{1}{2},t} \right|^2 \right), \\
\frac{d\Gamma_{\lambda_2=\pm\frac{1}{2}}}{dq^2} &= \frac{G_F^2 V_{cb}^2 q^2}{384\pi^3 m_{B_i}^2} \frac{\sqrt{Q_+ Q_-}}{2m_{B_i}} \left(1 - \frac{m_l^2}{q^2} \right)^2 \\
&\quad \times \left[\frac{4m_l^2}{3q^2} \left(\left| H_{\pm\frac{1}{2},1} \right|^2 + \left| H_{\pm\frac{1}{2},0} \right|^2 + 3 \left| H_{\pm\frac{1}{2},t} \right|^2 \right) \right. \\
&\quad \left. + \frac{8}{3} \left(\left| H_{\pm\frac{1}{2},1} \right|^2 + \left| H_{\pm\frac{1}{2},0} \right|^2 \right) \right].
\end{aligned}$$

$$\int_k \equiv \int \frac{d^4 k}{(2\pi)^4}$$



$$S_{\text{bulk}} = -\frac{M_{\text{pl}}^2}{2} \int d^d x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right]$$

$$M_{\text{pl}}^2 = \frac{c^3}{8\pi G} (4\pi e^{\gamma_E} L_0^2)^{\frac{4-d}{2}} \setminus R$$

$$\tilde{g}_{\mu\nu} = \left(\frac{\phi}{\phi_0}\right)^{1-\frac{d-4}{d-2}} g_{\mu\nu}$$

$$\frac{\psi}{\tilde{M}_{\text{pl}}} = \sqrt{\frac{3 + 2\omega_0 - \frac{d-4}{d-2}}{2}} \ln \frac{\phi}{\phi_0}$$

$$\omega_0^{(n)} = \left. \frac{d^n \omega(\phi)}{d\phi^n} \right|_{\phi=\phi_0}, \omega_0^{(0)} \equiv \omega_0$$

$$S_{\text{bulk}} = \int d^d x \sqrt{-\tilde{g}} \left[-\frac{\tilde{M}_{\text{pl}}^2}{2} \left(\tilde{R} - \frac{1}{2} \tilde{\Gamma}^\mu \tilde{\Gamma}_\mu \right) + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \sum_{n=0}^{\infty} \frac{c_n}{n!} \left(\frac{\psi}{\tilde{M}_{\text{pl}}} \right)^n \right] + S_{\text{GF}},$$

$$c_0 = 1, c_n = \left(\frac{2}{3 + 2\omega_0 - \frac{d-4}{d-2}} \right)^{1+n/2} \sum_{m=1}^n (-1)^m \omega_0^{(m)} \phi_0^m \left[\sum_{l=1}^m \frac{l^{n-1} (-1)^l}{\Gamma(l) \Gamma(1+m-l)} \right].$$

$$S_{\text{mat}} = - \sum_{a=1,2} \int d\sigma_a \frac{m_a(\phi)}{2} e_a \left[\frac{1}{e_a^2} g_{\mu\nu} (x_a^\alpha(\sigma)) v_a^\mu(\sigma_a) v_a^\nu(\sigma_a) + 1 \right] + \dots$$

$$v_a^\mu \equiv \frac{dx_a^\mu}{d\sigma}$$

$$S_{\text{mat}} = - \sum_{a=1,2} \int d\tau_a \frac{m_a(\phi)}{2} [g_{\mu\nu} (x_a^\alpha(\tau_a)) v_a^\mu(\tau_a) v_a^\nu(\tau_a) + 1]$$

$$e_a^2 = g_{\mu\nu} (x_a^\alpha(\tau_a)) v_a^\mu(\tau_a) v_a^\nu(\tau_a) = 1$$

$$S_{\text{mat}} = - \sum_{a=1,2} \frac{\bar{m}_a}{2} \int d\tau_a \left[\tilde{g}_{\mu\nu} (x_a^\alpha(\tau_a)) v_a^\mu(\tau_a) v_a^\nu(\tau_a) \sum_{n=0}^{\infty} \frac{\tilde{d}_n^{(a)}}{n!} \left(\frac{\psi}{\tilde{M}_{\text{pl}}} \right)^n + \sum_{n=0}^{\infty} \frac{\tilde{f}_n^{(a)}}{n!} \left(\frac{\psi}{\tilde{M}_{\text{pl}}} \right)^n \right]$$

$$\bar{m}_a \equiv m_a(\phi_0), s_a^{(n)} = \left. \frac{d^{n+1} \ln m_a(\phi)}{d \ln \phi^{n+1}} \right|_{\phi=\phi_0}$$



$$\begin{aligned} \tilde{d}_n^{(a)} &= d_n^{(a)} \left(\frac{2}{3 + 2\omega_0 - \frac{d-4}{d-2}} \right)^{n/2} \\ d_0^{(a)} &= 1 \\ d_1^{(a)} &= s_a^{(0)} - 1 + \frac{d-4}{d-2} \\ d_2^{(a)} &= s_a^{(1)} + (s_a^{(0)})^2 - 2s_a^{(0)} + 1 + 2 \frac{d-4}{d-2} \left(s_a^{(0)} - \frac{d}{2(d-2)} \right) \\ d_3^{(a)} &= s_a^{(2)} - 3s_a^{(1)} + 3s_a^{(0)} s_a^{(1)} + (s_a^{(0)})^3 - 3(s_a^{(0)})^2 + 3s_a^{(0)} - 1 \\ &\quad + 3 \frac{d-4}{d-2} \left(s_a^{(1)} + \left(s_a^{(0)} - \frac{d}{2(d-2)} \right)^2 + \frac{(d-4)^2}{12(d-2)^2} \right) \end{aligned}$$

$$\begin{aligned} \tilde{f}_n^{(a)} &= f_n^{(a)} \left(\frac{2}{3 + 2\omega_0 - \frac{d-4}{d-2}} \right)^{n/2} \\ f_0^{(a)} &= 1 \\ f_1^{(a)} &= s_a^{(0)} \\ f_2^{(a)} &= (s_a^{(0)})^2 + s_a^{(1)} \\ f_3^{(a)} &= s_a^{(2)} + 3s_a^{(0)} s_a^{(1)} + (s_a^{(0)})^3 \end{aligned}$$

$$S_{\text{bulk}} = \int d^d x \sqrt{-\tilde{g}} \left[-\frac{\tilde{M}_{\text{pl}}^2}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \sum_{n=0} \frac{c_n}{n!} \left(\frac{\psi}{\tilde{M}_{\text{pl}}} \right)^n \right] + S_{\text{GF}}$$

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},$$



$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2} \left(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\alpha\beta} \right),$$

$$\bar{\mathcal{P}}_{\mu\nu\alpha\beta} = \frac{1}{2} \left(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} - \eta_{\mu\nu} \eta_{\alpha\beta} \right),$$







$$\bar{\mathcal{J}}^{\alpha\beta\mu\nu}{}_{\rho\lambda} = \left(\frac{1}{2} \bar{\mathcal{P}}^{\alpha\beta\mu\nu} \eta_{\rho\lambda} - \bar{\mathcal{P}}^{\alpha\beta\mu}{}_{\rho} \delta_{\lambda}^{\nu} - \bar{\mathcal{P}}^{\alpha}{}_{\rho}{}^{\mu\nu} \delta_{\lambda}^{\beta} \right).$$



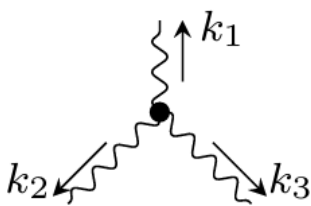
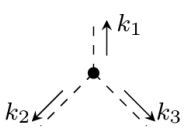
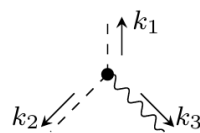
Propagators

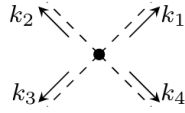
	$\langle h_{\mu\nu}(x)h_{\alpha\beta}(y) \rangle$	$\frac{i \mathcal{P}_{\mu\nu\alpha\beta}(k)}{k^2} e^{ik \cdot (x-y)}$
	$\langle \psi(x)\psi(y) \rangle$	$\frac{i}{k^2} e^{ik \cdot (x-y)}$

Worldline vertices

	$-i \frac{\bar{m}_a}{M_{\text{pl}}} v_a^\mu v_a^\nu$
	$-i \frac{\bar{m}_a}{2M_{\text{pl}}} \left(\tilde{d}_1^{(a)} v_a^\mu v_{a,\mu} + \tilde{f}_1^{(a)} \right)$
	$-i \frac{\bar{m}_a \tilde{d}_1^{(a)}}{M_{\text{pl}}^2} v_a^\mu v_a^\nu$
	$-i \frac{\bar{m}_a}{4M_{\text{pl}}^2} \left(\tilde{d}_2^{(a)} v_a^\mu v_{a,\mu} + \tilde{f}_2^{(a)} \right)$
	$-i \frac{\bar{m}_a}{2M_{\text{pl}}^3} \left(\tilde{d}_2^{(a)} v_a^\mu v_a^\nu \right)$
	$-i \frac{\bar{m}_a}{2M_{\text{pl}}^3} \frac{1}{3!} \left(\tilde{d}_3^{(a)} v_a^\mu v_{a,\mu} + \tilde{f}_3^{(a)} \right)$

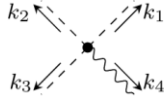
Bulk vertices

	$iV_{hhh}^{\mu\nu\alpha\beta\rho\lambda}(k_1, k_2, k_3)$	
	$iV_{\psi\psi\psi}(k_1, k_2, k_3)$	$i \frac{c_1}{2M_{\text{pl}}} (k_1^2 + k_2^2 + k_3^2)$
	$iV_{\psi\psi h}^{\alpha\beta}(k_1, k_2)$	$i \frac{2}{M_{\text{pl}}} \bar{\mathcal{P}}_{\mu\nu}^{\alpha\beta} k_1^\mu k_2^\nu$



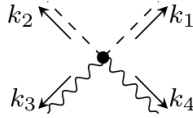
$$iV_{\psi\psi\psi\psi}(k_1, k_2, k_3, k_4)$$

$$-i \frac{c_2}{2M_{\text{pl}}^2} \sum_{1 \leq i < j \leq 4} k_i \cdot k_j$$



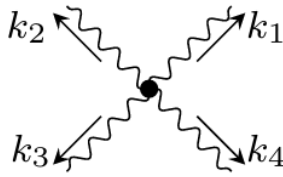
$$iV_{\psi\psi\psi h}^{\alpha\beta}(k_1, k_2, k_3)$$

$$i \frac{2c_1}{M_{\text{pl}}^2} \bar{P}_{\mu\nu}^{\alpha\beta} (k_1^\mu k_2^\nu + k_2^\mu k_3^\nu + k_1^\mu k_3^\nu)$$



$$iV_{\psi\psi h h}^{\alpha\beta\mu\nu}(k_1, k_2)$$

$$i \frac{4}{M_{\text{pl}}^2} \bar{I}^{\alpha\beta\mu\nu}{}_{\rho\lambda} k_1^\rho k_2^\lambda$$



$$iV_{hhhh}^{\mu\nu\alpha\beta\rho\lambda\gamma\delta}(k_1, k_2, k_3, k_4)$$

$$e^{iS_{\text{eff}}[x_a]} = \int \mathcal{D}\psi \mathcal{D}h_{\mu\nu} e^{iS_{\text{bulk}}[h, \psi] + iS_{\text{mat}}[h, x_a]}$$

$$e^{iS_{\text{eff}}[x_a]} = e^{(-\sum_{a=1,2} i \frac{\bar{m}_a}{2} \int d\tau_a [\eta_{\mu\nu} v_a^\mu v_a^\nu \tilde{a}_0^{(a)} + \tilde{f}_0^{(a)}] + iB_{\text{bulk}}^{hh} + iB_{\text{bulk}}^{\psi\psi})}$$

$$S_{\text{eff}} = \sum_n \int d\tau_1 \mathcal{L}_n[x_1(\tau_1), x_2(\tau_2)]$$

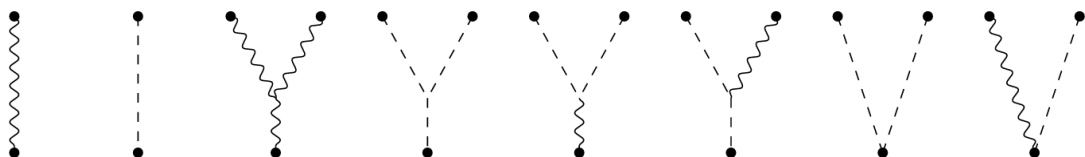
$$\mathcal{L}_0 = - \int d\tau_1 \frac{\bar{m}_1}{2} [\eta_{\mu\nu} v_1^\mu v_1^\nu \tilde{a}_0^{(1)} + \tilde{f}_0^{(1)}] = - \int d\tau_1 \frac{\bar{m}_1}{2} \eta_{\mu\nu} v_1^\mu v_1^\nu$$

$$\Delta p_1^\mu = \bar{m}_1 \Delta v_1^\mu = -\eta^{\mu\nu} \sum_n \int_{-\infty}^{\infty} d\tau_1 \frac{\partial \mathcal{L}_n}{\partial x_1^\nu}$$

$$\Delta^{(n)} p_a^\mu = \sum_{k \leq n} \Delta_{\mathcal{L}_k}^{(n)} p_a^\mu$$

$$\Delta_{\mathcal{L}_k}^{(n)} p_a^\mu = -\eta^{\mu\nu} \int_{-\infty}^{\infty} d\tau_a \left(\frac{\partial}{\partial x_a^\nu} \mathcal{L}_k \left[b_a + u_a \tau_a + \sum_{r=0}^{n-k} \delta^{(r)} x_a \right] \right)_{O(G)^n}$$

$$x_a^\mu = b_a^\mu + u_a^\mu \tau_a + \sum_n \delta^{(n)} x_a^\mu$$



$$\mathcal{L}_1 = -i \left(\frac{-i\bar{m}_1}{\bar{M}_{\text{pl}}} \right) \left(\frac{-i\bar{m}_2}{\bar{M}_{\text{pl}}} \right) \int d\tau_2 v_2^\alpha(\tau_2) v_2^\beta(\tau_2) v_1^\mu(\tau_1) v_1^\nu(\tau_1) \int_k i \frac{\mathcal{P}_{\mu\nu\alpha\beta}}{k^2} e^{ik(x_1(\tau_1) - x_1(\tau_2))}$$

$$-i \left(\frac{-i\bar{m}_1}{2\bar{M}_{\text{pl}}} \right) \left(\frac{-i\bar{m}_2}{2\bar{M}_{\text{pl}}} \right) \int d\tau_2 \left(\tilde{d}_1^{(1)} v_1^2(\tau_1) + \tilde{f}_1^{(1)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \int_k \frac{i}{k^2} e^{ik(x_1(\tau_1) - x_1(\tau_2))}$$

$$\mathcal{L}_1 = -\frac{\bar{m}_1 \bar{m}_2}{2\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(2(v_2(\tau_2) \cdot v_1(\tau_1))^2 - \frac{2}{d-2} v_1^2(\tau_1) v_2^2(\tau_2) \right) \int_k \frac{1}{k^2} e^{ik \cdot (x_1(\tau_1) - x_1(\tau_2))}$$

$$-\frac{\bar{m}_1 \bar{m}_2}{4\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(\tilde{d}_1^{(1)} v_1^2(\tau_1) + \tilde{f}_1^{(1)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \int_k \frac{1}{k^2} e^{ik \cdot (x_1(\tau_1) - x_1(\tau_2))}$$

$$\bar{m}_1 \frac{dv_1^\mu}{d\tau_1} = -\eta^{\mu\nu} \left(\sum_{n=1}^{\infty} \frac{\partial \mathcal{L}_n}{\partial x_1^\nu(\tau_1)} - \frac{d}{d\tau_1} \left(\frac{\partial \mathcal{L}_n}{\partial v_1^\nu} \right) \right),$$

$$\frac{dv_1^\mu}{d\tau_1} = \frac{\bar{m}_2}{2\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(2(v_2(\tau_2) \cdot v_1(\tau_1))^2 - \frac{2}{d-2} v_1^2(\tau_1) v_2^2(\tau_2) \right) \int_k \frac{ik^\mu}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$-\frac{\bar{m}_2}{\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(2(v_2(\tau_2) \cdot v_1(\tau_1)) v_2^\mu(\tau_2) - \frac{2}{d-2} v_1^\mu(\tau_1) v_2^2(\tau_2) \right) \int_k \frac{ik \cdot v_1(\tau_1)}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$-\frac{\bar{m}_2}{\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(2 \left(v_2(\tau_2) \cdot \frac{dv_1(\tau_1)}{d\tau_1} \right) v_2^\mu(\tau_2) - \frac{2}{d-2} \frac{dv_1^\mu(\tau_1)}{d\tau_1} v_2^2(\tau_2) \right) \int_k \frac{1}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$+\frac{\bar{m}_2}{4\bar{M}_{\text{pl}}^2} \int d\tau_2 \left(\tilde{d}_1^{(1)} v_1^2(\tau_1) + \tilde{f}_1^{(1)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \int_k \frac{ik^\mu}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$-\frac{\bar{m}_2}{2\bar{M}_{\text{pl}}^2} \int d\tau_2 \tilde{d}_1^{(1)} v_1^\mu(\tau_1) \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \int_k \frac{ik \cdot v_1(\tau_1)}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$-\frac{\bar{m}_2}{2\bar{M}_{\text{pl}}^2} \int d\tau_2 \tilde{d}_1^{(1)} \frac{dv_1^\mu(\tau_1)}{d\tau_1} \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \int_k \frac{1}{k^2} e^{ik \cdot (x_1(\tau_1) - x_2(\tau_2))}$$

$$\begin{aligned}
\mathcal{L}_2 = & \frac{\bar{m}_1 \bar{m}_2^2}{2\bar{M}_{\text{pl}}^3} v_1^\alpha(\tau_1) v_1^\beta(\tau_1) \int d\tau_2 \int d\bar{\tau}_2 v_2^\mu(\tau_2) v_2^\nu(\tau_2) v_2^\lambda(\bar{\tau}_2) v_2^\rho(\bar{\tau}_2) \mathcal{P}_{\alpha\beta\bar{\alpha}\bar{\beta}}(k_1) \mathcal{P}_{\mu\nu\bar{\mu}\bar{\nu}}(k_2) \mathcal{P}_{\lambda\rho\bar{\lambda}\bar{\rho}}(k_3) \\
& \times \int_{k_{1,2,3}} e^{ik_1 x_1(\tau_1)} e^{ik_2 x_2(\tau_2)} e^{ik_3 x_2(\bar{\tau}_2)} \frac{V_{hhhh}^{\bar{\alpha}\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\rho}}(k_1, k_2, k_3)}{k_1^2 k_2^2 k_3^3} \delta^4(k_1 + k_2 + k_3) \\
& + \frac{\bar{m}_1 \bar{m}_2^2}{16\bar{M}_{\text{pl}}^3} \int d\tau_2 \int d\bar{\tau}_2 \left(\tilde{d}_1^{(1)} v_1^2(\tau_1) + \tilde{f}_1^{(1)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\bar{\tau}_2) + \tilde{f}_1^{(2)} \right) \\
& \times \int_{k_{1,2,3}} \frac{V_{\psi\psi\psi}(k_1, k_2, k_3)}{k_1^2 k_2^2 k_3^2} e^{ik_1 x_1(\tau_1)} e^{ik_2 x_2(\tau_2)} e^{ik_3 x_2(\bar{\tau}_2)} \delta^4(k_1 + k_2 + k_3) \\
& + \frac{\bar{m}_1 \bar{m}_2^2}{8\bar{M}_{\text{pl}}^3} v_1^\mu(\tau_1) v_1^\nu(\tau_1) \int d\tau_2 \int d\bar{\tau}_2 \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \left(\tilde{d}_1^{(2)} v_2^2(\bar{\tau}_2) + \tilde{f}_1^{(2)} \right) \\
& \times \int_{k_{1,2,3}} \frac{\mathcal{P}_{\mu\nu\alpha\beta}(k_1)}{k_1^2 k_2^2 k_3^2} V_{\psi\psi h}^{\alpha\beta}(k_2, k_3) e^{ik_1 x_1(\tau_1)} e^{ik_2 x_2(\tau_2)} e^{ik_3 x_2(\bar{\tau}_2)} \delta^4(k_1 + k_2 + k_3) \\
& + \frac{\bar{m}_1 \bar{m}_2^2}{4\bar{M}_{\text{pl}}^3} \left(\tilde{d}_1^{(1)} v_1^2(\tau_1) + \tilde{f}_1^{(1)} \right) \int d\tau_2 \int d\bar{\tau}_2 \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) v_2^\mu(\bar{\tau}_2) v_2^\nu(\bar{\tau}_2) \\
& \times \int_{k_{1,2,3}} \frac{\mathcal{P}_{\mu\nu\alpha\beta}(k_3)}{k_1^2 k_2^2 k_3^2} V_{\psi\psi h}^{\alpha\beta}(k_1, k_2) e^{ik_1 x_1(\tau_1)} e^{ik_2 x_2(\tau_2)} e^{ik_3 x_2(\bar{\tau}_2)} \delta^4(k_1 + k_2 + k_3) \\
& - \frac{\bar{m}_1 \bar{m}_2^2}{16\bar{M}_{\text{pl}}^4} \left(\tilde{d}_2^{(1)} v_1^2(\tau_1) + \tilde{f}_2^{(1)} \right) \int d\tau_2 \int d\bar{\tau}_2 \left(\tilde{d}_1^{(2)} v_2^2(\tau_2) + \tilde{f}_1^{(2)} \right) \left(\tilde{d}_1^{(2)} v_2(\bar{\tau}_2)^2 + \tilde{f}_1^{(2)} \right) \\
& \times \int_{k_{1,2}} \frac{1}{k_1^2 k_2^2} e^{i(k_1+k_2)x_1(\tau_1)} e^{-ik_1 x_2(\tau_2)} e^{-ik_2 x_2(\bar{\tau}_2)} \\
& - \frac{\bar{m}_1 \bar{m}_2^2}{2\bar{M}_{\text{pl}}^4} \tilde{d}_1^{(1)} v_1^\mu(\tau_1) v_1^\nu(\tau_1) \int d\tau_2 \int d\bar{\tau}_2 v_2^\alpha(\tau_2) v_2^\beta(\tau_2) \left(\tilde{d}_1^{(2)} v_2(\bar{\tau}_2)^2 + \tilde{f}_1^{(2)} \right) \\
& \times \int_{k_{1,2}} \frac{\mathcal{P}_{\mu\nu\alpha\beta}(k_1)}{k_1^2 k_2^2} e^{i(k_1+k_2)x_1(\tau_1)} e^{-ik_1 x_2(\tau_2)} e^{-ik_2 x_2(\bar{\tau}_2)} \\
& + 1 \leftrightarrow 2
\end{aligned}$$

$$v_a^\mu(\tau_a) = u_a^\mu + \sum_n \delta^{(n)} v_a^\mu(\tau_a)$$

$$x_a^\mu(\tau_a) = b_a^\mu + u_a^\mu \tau_a + \sum_n \delta^{(n)} x_a^\mu(\tau_a)$$

$$\begin{aligned}
\delta^{(1)} v_1^\mu(\tau_1) = & \frac{\bar{m}_2}{\bar{M}_{\text{pl}}^2} \left[\left(\frac{2\gamma^2 - \frac{2}{d-2}}{2} + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{4} \right) \eta^{\mu\alpha} - \left(2\gamma u_2^\mu - \frac{2}{d-2} u_1^\mu \right) u_1^\alpha \right. \\
& \left. - \frac{\tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} u_1^\mu u_1^\alpha \right] \int_{-\infty}^{\tau_1} d\bar{\tau}_1 \int_k \frac{ik_\alpha}{k^2} \hat{\delta}(u_2 \cdot k) e^{ik \cdot b} e^{i(k \cdot u_1 - i\epsilon)\bar{\tau}_1} \\
= & \frac{\bar{m}_2}{\bar{M}_{\text{pl}}^2} \left[\left(\frac{2\gamma^2 - \frac{2}{d-2}}{2} + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{4} \right) \eta^{\mu\alpha} - 2\gamma u_2^\mu u_1^\alpha \right. \\
& \left. - \left(\frac{\tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} - \frac{2}{d-2} \right) u_1^\mu u_1^\alpha \right] \int_k \frac{k_\alpha}{k^2} \hat{\delta}(u_2 \cdot k) e^{ik \cdot b} \frac{e^{i(k \cdot u_1 - i\epsilon)\tau_1}}{(k \cdot u_1 - i\epsilon)}
\end{aligned}$$



$$\delta^{(1)}x_1^\mu(\tau_1) = -i \frac{\bar{m}_2}{\bar{M}_{\text{pl}}^2} \left[\left(\frac{2\gamma^2 - \frac{2}{d-2}}{2} + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)})(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{4} \right) \eta^{\mu\alpha} - 2\gamma u_2^\mu u_1^\alpha \right. \\ \left. - \left(\frac{\tilde{d}_1^{(1)}(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} - \frac{2}{d-2} \right) u_1^\mu u_1^\alpha \right] \int_k \frac{k_\alpha}{k^2} \hat{\delta}(u_2 \cdot k) e^{ik \cdot b} \frac{e^{i(k \cdot u_1 - i\epsilon)\tau_1}}{(k \cdot u_1 - i\epsilon)^2}$$

$$\delta_{\mathcal{L}_2}^{(2)}v_1^\mu(\tau_1) = -\frac{1}{\bar{m}_1} \eta^{\mu\nu} \left(\frac{\partial \mathcal{L}_2}{\partial x_1^\nu(\tau_1)} - \frac{d}{d\tau_1} \left(\frac{\partial \mathcal{L}_2}{\partial v_1^\nu(\tau_1)} \right) \right) \\ - \frac{1}{\bar{m}_1} \eta^{\mu\nu} \left(\frac{\partial \mathcal{L}_2}{\partial x_1^\nu(\tilde{\tau}_1)} - \frac{d}{d\tilde{\tau}_1} \left(\frac{\partial \mathcal{L}_2}{\partial v_1^\nu(\tilde{\tau}_1)} \right) \right)$$

$$\Delta p_1^\mu = -\eta_{\mu\nu} \int d\tau_1 \frac{\partial \mathcal{L}_1}{\partial x_1^\nu}$$

$$2\sin\left(\frac{\chi}{2}\right) = \chi - \frac{1}{24}\chi^3 + \frac{1}{1920}\chi^5 + \mathcal{O}(\chi^7) = \frac{\Delta p_{1cm}}{p_\infty^{cm}} = \frac{\sqrt{-\Delta p^2}}{p_\infty^{cm}}$$

$$p_\infty^{cm} = \mu \frac{\sqrt{\gamma^2 - 1}}{\Gamma}$$

$$\Gamma \equiv E/M = \sqrt{1 + 2v(\gamma - 1)}, \mu = \bar{m}_1 \bar{m}_2 / (\bar{m}_1 + \bar{m}_2)$$

$$\frac{\chi}{2} = \sum_n \chi_b^{(n)} \left(\frac{GM}{\phi_0 b} \right)^n$$

$$\Delta_{\mathcal{L}_1}^{(1)}p_1^\mu = \frac{\bar{m}_1 \bar{m}_2}{2\bar{M}_{\text{pl}}^2} (2\gamma^2 - 1) \int_k ik^\mu \frac{\hat{\delta}(k \cdot u_1) \hat{\delta}(k \cdot u_2)}{k^2} e^{ik \cdot b} \\ + \frac{\bar{m}_1 \bar{m}_2}{4\bar{M}_{\text{pl}}^2} (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) \int_k ik^\mu \frac{\hat{\delta}(k \cdot u_1) \hat{\delta}(k \cdot u_2)}{k^2} e^{ik \cdot b}$$

$$\Delta_{\mathcal{L}_1}^{(1)}p_1^\mu = -\frac{\bar{m}_1 \bar{m}_2 (2\gamma^2 - 1)}{2\bar{M}_{\text{pl}}^2} \frac{b^\mu}{\sqrt{\gamma^2 - 1} 2\pi |b^2|} \\ - \frac{\bar{m}_1 \bar{m}_2}{4\bar{M}_{\text{pl}}^2} (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) \frac{1}{\sqrt{\gamma^2 - 1} 2\pi |b^2|} b^\mu \\ = -\frac{2\bar{m}_1 \bar{m}_2 G}{\phi_0} \frac{1}{\sqrt{\gamma^2 - 1}} \left(2\gamma^2 - 1 + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right) \frac{b^\mu}{|b^2|}$$

$$\frac{\chi_b^{(1)}}{\Gamma} = \frac{1}{(\gamma^2 - 1)} \left(2\gamma^2 - 1 + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right)$$



$$\Delta_{\mathcal{L}_1}^{(2)} p_1^\mu = \frac{\bar{m}_1 \bar{m}_2}{\bar{M}_{\text{pl}}^2} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \int_k \frac{ik^\mu}{k^2} \left\{ \left(2\gamma u_2 + u_1 \left(\frac{\tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} - 1 \right) \right) \cdot \delta^{(1)} v_1(\tau_1) \right. \\ \left. + \left(\frac{2\gamma^2 - 1}{2} + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{4} \right) (ik) \cdot (\delta^{(1)} x_1(\tau_1) - \delta^{(1)} x_2(\tau_2)) \right. \\ \left. + \left(2\gamma u_1 + u_2 \left(\frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) \tilde{d}_1^{(2)}}{2} - 1 \right) \right) \cdot \delta^{(1)} v_2(\tau_2) \right\} e^{ik \cdot b + ik \cdot (u_1 \tau_1 - u_2 \tau_2)}$$

$$\Delta_{\mathcal{L}_1}^{(2)} p_1^\mu = i \frac{\bar{m}_1 \bar{m}_2^2}{\bar{M}_{\text{pl}}^4} \int_{\ell, k} \left\{ \left(2\gamma^2 - 1 + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right)^2 \frac{\ell^2}{8} \right. \\ \left. - 2\gamma^2 \left(1 + \frac{\tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right) (k \cdot u_1)^2 - \frac{\tilde{f}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \left(1 - \frac{\tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right) (k \cdot u_1)^2 \right\} \\ \times \frac{(\ell^\mu - k^\mu) \hat{\delta}(\ell \cdot u_1) \hat{\delta}(\ell \cdot u_2) \hat{\delta}(k \cdot u_2)}{k^2 (\ell - k)^2 (k \cdot u_1 - i\epsilon)^2} e^{i\ell \cdot b}$$

$$\Delta_{\mathcal{L}_2}^{(2)} p_1^\mu = \frac{i \bar{m}_1 \bar{m}_2^2 (\gamma^2 + 3)}{16 \bar{M}_{\text{pl}}^4} \int_{\ell, k} \frac{\ell^\mu \delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b} \\ - \frac{i \bar{m}_1 \bar{m}_2^2}{32 \bar{M}_{\text{pl}}^4} c_1 (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2 \int_{\ell, k} \frac{\ell^\mu \delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b} \\ + \frac{i \bar{m}_1 \bar{m}_2^2 (\gamma^2 - 1)}{32 \bar{M}_{\text{pl}}^4} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2 \int_{\ell, k} \frac{\ell^\mu \delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b} \\ + \frac{i \bar{m}_1 \bar{m}_2^2}{16 \bar{M}_{\text{pl}}^4} (\tilde{d}_2^{(1)} + \tilde{f}_2^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2 \int_{\ell, k} \frac{\ell^\mu \delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b} \\ + \frac{i \bar{m}_1 \bar{m}_2^2}{4 \bar{M}_{\text{pl}}^4} (2\gamma^2 - 1) \tilde{d}_1^{(1)} (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) \int_{\ell, k} \frac{\ell^\mu \delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b}$$

$$\Delta^{(2)} p_1^\mu = \Delta_{\mathcal{L}_1}^{(2)} p_1^\mu + \Delta_{\mathcal{L}_2}^{(2)} p_1^\mu \\ = \Delta_b^{(2)} p_1^\mu + \Delta_u^{(2)} p_1^\mu$$

$$(p_a + \Delta p_a)^2 = p_a^2 \Rightarrow 2p_a \cdot \Delta p_a = -\Delta p_a^2$$

$$\Delta_b^{(2)} p_1^\mu = \frac{\bar{m}_1 \bar{m}_2^2}{16 \bar{M}_{\text{pl}}^4} \left\{ -3(5\gamma^2 - 1) - 4(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) + \frac{(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2}{2} [\gamma^2 - 1 \right. \\ \left. + 2(\tilde{d}_2^{(1)} + \tilde{f}_2^{(1)}) + 4\tilde{d}_1^{(1)} \tilde{f}_1^{(1)} - c_1 (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) \right\} \frac{\partial}{\partial b_\mu} \int_{\ell, k} \frac{\delta(\ell \cdot u_1) \delta(\ell \cdot u_2) \delta(k \cdot u_2)}{k^2 (\ell - k)^2} e^{i\ell \cdot b}$$

$$\Delta_u^{(2)} p_1^\mu = i \frac{\bar{m}_1 \bar{m}_2^2}{8 \bar{M}_{\text{pl}}^4} \left(2\gamma^2 - 1 + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right)^2 \int_{\ell, k} \frac{(\ell^\mu - k^\mu) \ell^2 \hat{\delta}(\ell \cdot u_1) \hat{\delta}(\ell \cdot u_2) \hat{\delta}(k \cdot u_2)}{k^2 (\ell - k)^2 (k \cdot u_1 - i\epsilon)^2} e^{i\ell \cdot b}$$



$$\Delta_b^{(2)} p_1^\mu = -\frac{\pi \bar{m}_1 \bar{m}_2^2}{4\sqrt{\gamma^2 - 1} \phi_0^2 |b^2|^{3/2}} \left\{ 3(5\gamma^2 - 1) + 4(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)})(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) - \frac{(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2}{2} \left[2(\tilde{d}_2^{(1)} + \tilde{f}_2^{(1)}) - c_1 \alpha_1 + 4\tilde{d}_1^{(1)} \tilde{f}_1^{(1)} + \gamma^2 - 1 \right] \right\} + 1 \leftrightarrow 2$$

$$\Delta_u^{(2)} p_1^\mu = 2 \frac{\bar{m}_1 \bar{m}_2^2}{(\gamma^2 - 1)^2} \left(2\gamma^2 - 1 + \frac{(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)})(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})}{2} \right)^2 \frac{G^2}{\phi_0^2 |b^2|} (\gamma u_2^\mu - u_1^\mu) - 1 \leftrightarrow 2$$

$$\frac{\chi_b^{(2)}}{\Gamma} = \frac{\pi}{8(\gamma^2 - 1)} \left(3(5\gamma^2 - 1) + 4(\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)})(\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) - \frac{\bar{m}_2 (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)})^2}{2M} \left[2(\tilde{d}_2^{(1)} + \tilde{f}_2^{(1)}) - c_1 (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)}) + 4\tilde{d}_1^{(1)} \tilde{f}_1^{(1)} + \gamma^2 - 1 \right] - \frac{\bar{m}_1 (\tilde{d}_1^{(1)} + \tilde{f}_1^{(1)})^2}{2M} \left[2(\tilde{d}_2^{(2)} + \tilde{f}_2^{(2)}) - c_1 (\tilde{d}_1^{(2)} + \tilde{f}_1^{(2)}) + 4\tilde{d}_1^{(2)} \tilde{f}_1^{(2)} + \gamma^2 - 1 \right] \right)$$

$$\Delta_b^{(3)} p_1^\mu = \frac{G^3 b^\mu}{\phi_0^3 |b^2|^2} \left[\frac{8m_1^2 m_2^2 \sinh^{-1} \sqrt{\frac{\gamma - 1}{2}}}{(\gamma^2 - 1)} (4\gamma^4 - 12\gamma^2 - 3 + C_1^{\text{ST}}) - \frac{2m_1^2 m_2^2 \gamma}{3(\gamma^2 - 1)^{5/2}} ((20\gamma^6 - 90\gamma^4 + 120\gamma^2 - 53) + C_2^{\text{ST},1}) - \frac{2m_1 m_2^3}{(\gamma^2 - 1)^{5/2}} ((16\gamma^6 - 32\gamma^4 + 16\gamma^2 - 1) + C_3^{\text{ST},1}) \right] + 1 \leftrightarrow 2$$

$$C_1^{\text{ST}} = -\frac{1}{16} \alpha_1 \alpha_2 (4\tilde{d}_1^{(1)} \tilde{f}_1^{(1)} + 2(\tilde{d}_2^{(1)} + \tilde{f}_2^{(1)}) - c_1 \alpha_1) (4\tilde{d}_1^{(2)} \tilde{f}_1^{(2)} + 2(\tilde{d}_2^{(2)} + \tilde{f}_2^{(2)}) - c_1 \alpha_2) + \frac{1}{4} \alpha_1 \alpha_2 (8 + \alpha_1 \alpha_2 + 40\gamma^2)$$

$$C_2^{\text{ST},a} = \frac{3}{2} \alpha_1 \alpha_2 (50\gamma^4 - 102\gamma^2 + 49) - \frac{3}{8} \alpha_1^2 \alpha_2^2 (2 + \alpha_1 \alpha_2 + 4\gamma^2) - \frac{1}{2} \alpha_\alpha^2 (\gamma^2 - 1)^2 (4\gamma^2 + 2 + 3\alpha_1 \alpha_2) - \frac{3}{8} \alpha_\alpha^2 (\gamma^2 - 1) (4\gamma^2 - 2 + \alpha_1 \alpha_2) (4\tilde{d}_1^{(a)} \tilde{f}_1^{(a)} + 2(\tilde{d}_2^{(a)} + \tilde{f}_2^{(a)}) - c_1 \alpha_a)$$

$$C_3^{\text{ST},a} = \frac{1}{6} \alpha_1 \alpha_2 (20\gamma^4 - 46\gamma^2 + 17) - \frac{1}{8} \alpha_1^2 \alpha_2^2 (2 + \alpha_1 \alpha_2 + 4\gamma^2) - \frac{1}{3} \alpha_\alpha^2 (\gamma^2 - 1)^2 (4\gamma^2 + \alpha_1 \alpha_2 - 2) + \frac{1}{8} \alpha_\alpha^2 (\gamma^2 - 1) (2(\tilde{d}_2^{(a)} + \tilde{f}_2^{(a)} + 2\tilde{d}_1^{(a)} \tilde{f}_1^{(a)}) - c_1 \alpha_a) (6 - 8\gamma^2 + \alpha_1 \alpha_2 (2\gamma^2 - 3)) + \frac{1}{12} \alpha_\alpha^3 (\gamma^2 - 1)^2 (\alpha_a (2c_1^2 - c_2) + 2(\tilde{d}_3^{(a)} + \tilde{f}_3^{(a)} - 3\tilde{f}_1^{(a)} \tilde{f}_2^{(a)}) - 6\tilde{d}_1^{(a)} (\tilde{d}_2^{(a)} + 2\alpha_a \tilde{f}_1^{(a)}) - 3c_1 (\tilde{d}_2^{(a)} + \tilde{f}_2^{(a)} + 2\tilde{d}_1^{(a)} \tilde{f}_1^{(a)} - \alpha_a^2))$$



$$\frac{\chi_b^{(3)}}{\Gamma} = \frac{1}{(\gamma^2 - 1)^{3/2}} \left[-8v \sinh^{-1} \sqrt{\frac{\gamma - 1}{2}} (4\gamma^4 - 12\gamma^2 - 3 + C_1^{\text{ST}}) \right. \\ \left. + \frac{2v\gamma}{3(\gamma^2 - 1)^{3/2}} \left(-2(\gamma^2 - 1)^2 (14\gamma^2 + 25) + \frac{C_2^{\text{ST},1} + C_2^{\text{ST},2}}{2} \right) \right. \\ \left. + \frac{1}{(\gamma^2 - 1)^{3/2}} \left(\left(\frac{m_2}{M} - v \right) C_3^{\text{ST},1} + \left(\frac{m_1}{M} - v \right) C_3^{\text{ST},2} \right) \right. \\ \left. + \frac{1 + 2v(\gamma - 1)}{3(\gamma^2 - 1)^{3/2}} \left((64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5) + C_4^{\text{ST}} \right) \right]$$

$$C_4^{\text{ST}} = \frac{(\alpha_1 \alpha_2)^3}{4} + 3\alpha_1 \alpha_2 (2\gamma^2 - 1)^2 + \frac{3}{2} (\alpha_1 \alpha_2)^2 (2\gamma^2 - 1)$$

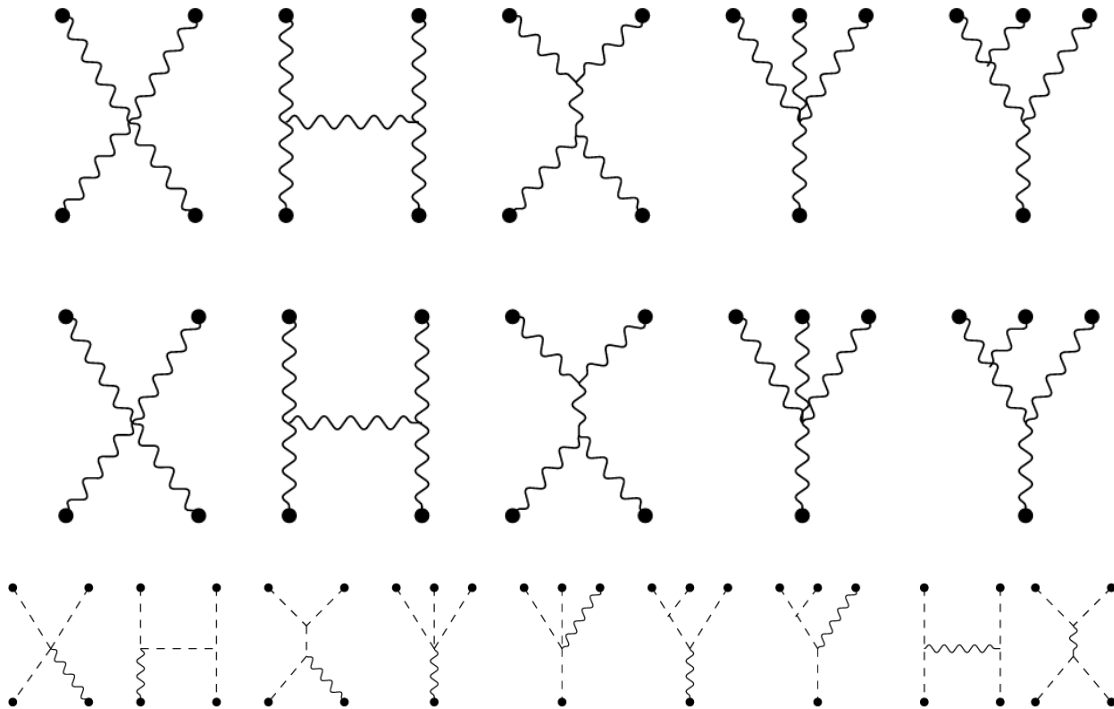
$$\alpha_a = \tilde{d}_1^{(a)} + \tilde{f}_1^{(a)}$$

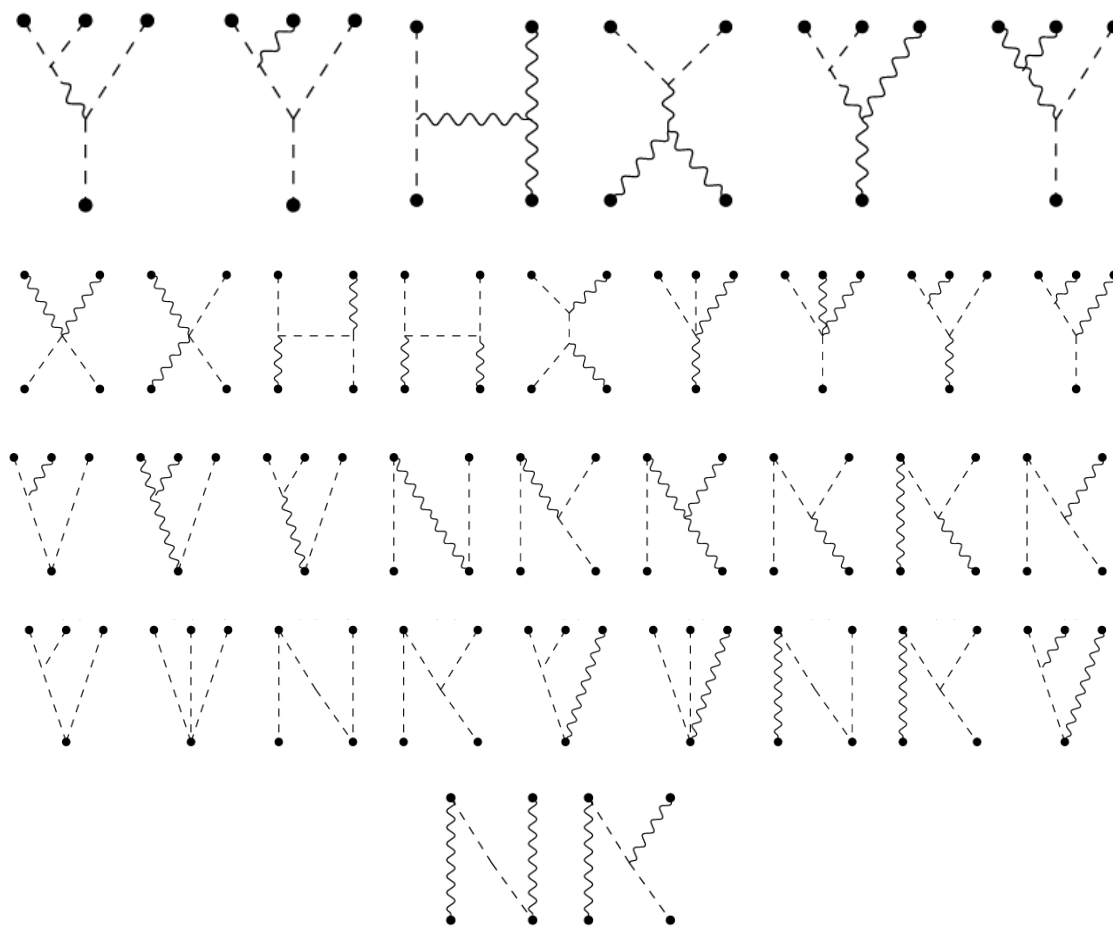
$$\bar{\gamma} = -2 \frac{\alpha_1 \alpha_2}{2 + \alpha_1 \alpha_2}$$

$$\delta_a = 2 \frac{\alpha_a^2}{(2 + \alpha_1 \alpha_2)^2}$$

$$\frac{\gamma_{AB}^2 + 4\beta_a}{\delta_d} = 4\tilde{d}_1^{(a)} \tilde{f}_1^{(a)} + 2(\tilde{d}_2^{(a)} + \tilde{f}_2^{(a)}) - c_1 (\tilde{d}_1^{(a)} + \tilde{f}_1^{(a)}),$$

$$-\frac{2\alpha_a}{\gamma_{AB} \delta_d} \epsilon_a = 4\alpha_a^3 + \alpha_a (2c_1^2 - c_2) + 2(\tilde{d}_3^{(a)} + \tilde{f}_3^{(a)} - 3\tilde{f}_1^{(a)} \tilde{f}_2^{(a)}) \\ - 6\tilde{d}_1^{(a)} (\tilde{d}_2^{(a)} + 2\alpha_a \tilde{f}_1^{(a)}) - 3c_1 (\tilde{d}_2^{(a)} + \tilde{f}_2^{(a)} + 2\tilde{d}_1^{(a)} \tilde{f}_1^{(a)} - \alpha_a^2)$$





$$\int \exp[iS_g(\ell, \Omega)](\square) d\mu(\ell) \mathcal{D}\Omega = \int \exp[i\tilde{S}_g(\ell)](\square) F(\ell) D\ell = \int (-\det\|g_{\lambda\mu}\|)^{\Delta\eta/2}$$

$$\tilde{S}_g(\ell) = \tilde{S}_g(\ell_{(0)}) + \frac{1}{2} \sum_{j,k,l,m} \frac{\partial^2 \tilde{S}_g(\ell_{(0)})}{\partial l_j \partial l_l} \frac{\partial l_j(u_{(0)})}{\partial u_k} \frac{\partial l_l(u_{(0)})}{\partial u_m} \Delta u_k \Delta u_m + \dots, \Delta u = u - u_{(0)}.$$

$$\frac{\partial \tilde{S}_g(\ell_{(0)})}{\partial l_j} = 0.$$

$$F(\ell_{(0)})^{-2} \det \left\| \frac{\partial^2 \tilde{S}_g(\ell_{(0)})}{\partial l_i \partial l_k} \right\| = \star \frac{\partial^2 \hat{\square}}{\partial \varphi}$$

$$b_s = l_{\text{Pl}} \sqrt{\frac{\eta - 9}{2\pi}}, l_{\text{Pl}} = \sqrt{8\pi G}$$

$$(np)^{-1} = (p_0 + i\varepsilon)^{-1}, (\bar{np})^{-1} = (p_0 - i\varepsilon)^{-1}$$

$$n^\lambda = (1, -\varepsilon(\partial^\alpha \partial_\alpha)^{-1} \partial^\beta)$$

$$p_0^{-2} \Rightarrow (p_0^2 + \varepsilon^2)^{-1}, \varepsilon \rightarrow 0$$

$$n^\lambda w_{\lambda\mu} = 0, w_{\lambda\mu} = g_{\lambda\mu} - g_{0\lambda}^{(0)}$$



$$p_0^{-j} \Rightarrow [(p_0 + i\varepsilon)^{-j} + (p_0 - i\varepsilon)^{-j}]/2$$

$$\square \sim b_s^{-1} \sim (l_{P1}\eta^{1/2})^{-1} \approx (\bar{\Delta}^{(s)}\Delta^{(s)})^{-1} = -(\Delta^{(s)}\Delta^{(s)})^{-1}$$

$$v_{\sigma^2}^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}l_{\sigma_1^1}^c l_{\sigma_2^1}^d$$

$$2\mathbf{v}_{\sigma^2} = i\mathbf{l}_{\sigma_1^1} \times \mathbf{l}_{\sigma_2^1} - l_{\sigma_1^1}^0 l_{\sigma_2^1}^0 + l_{\sigma_2^1}^0 l_{\sigma_1^1}^0$$

$$\Omega, R_{\sigma^2}(\Omega) = \prod_{\sigma^3 \supset \sigma^2} \Omega_{\sigma^3}^{\pm 1}$$

$$\Omega = {}^+\Omega - \Omega, -\Omega = ({}^+\Omega)^*, {}^+R(\Omega) = R({}^+\Omega)$$

$$S_g[v, \Omega] = \frac{1}{2} \sum_{\sigma^2} \left(1 + \frac{i}{\gamma}\right) \sqrt{\mathbf{v}_{\sigma^2}^2} \arcsin \frac{\mathbf{v}_{\sigma^2} * {}^+R_{\sigma^2}(\Omega)}{\sqrt{\mathbf{v}_{\sigma^2}^2}} + \text{c.c.}, \mathbf{v} * R \stackrel{\text{def}}{=} \frac{1}{2} v^i R^{kl} \epsilon_{ikl}$$

$$S_g[v, \Omega] = \frac{1}{4} \sum_{\substack{\lambda\mu\nu\rho \\ \text{sites}}} \left(1 + \frac{i}{\gamma}\right) \epsilon^{\lambda\mu\nu\rho} \sqrt{\mathbf{v}_{\lambda\mu}^2} \arcsin \frac{\mathbf{v}_{\lambda\mu} * {}^+R_{\nu\rho}(\Omega)}{\sqrt{\mathbf{v}_{\lambda\mu}^2}} + \text{c.c.}$$

$$2\mathbf{v}_{\lambda\mu} = i\mathbf{l}_\lambda \times \mathbf{l}_\mu + l_\lambda^0 \mathbf{l}_\mu - l_\mu^0 \mathbf{l}_\lambda, R_{\lambda\mu}(\Omega) = \bar{\Omega}_\lambda (\bar{T}_\lambda \bar{\Omega}_\mu) (\bar{T}_\mu \Omega_\lambda) \Omega_\mu,$$

$$-\frac{1}{4} \sum_{\text{sites}} \left(1 + \frac{i}{\gamma}\right) \epsilon^{\lambda\mu\nu\rho} \mathbf{v}_{\lambda\mu} \cdot (\Delta_\nu \boldsymbol{\omega}_\rho - \Delta_\rho \boldsymbol{\omega}_\nu + \boldsymbol{\omega}_\nu \times \boldsymbol{\omega}_\rho) + \text{c.c.}, \epsilon_{123} = +1, \Delta_\lambda = T_\lambda - 1,$$

$$\frac{1}{2} \int R \sqrt{-g} d^4x \omega_\lambda^k = -\frac{1}{4} \otimes \epsilon^k{}_{lm} \omega_\lambda^{lm} + \oplus \frac{i}{2} \omega_\lambda^{0k}$$

$$S_g[e, \omega] = -\frac{1}{8} \int (\epsilon_{abcd} e_\lambda^a e_\mu^b + \frac{2}{\gamma} e_{\lambda c} e_{\mu d}) \epsilon^{\lambda\mu\nu\rho} [\partial_\nu + \omega_\nu, \partial_\rho + \omega_\rho]^{cd} d^4x$$

$$= -\frac{1}{8} \epsilon^{\lambda\mu\nu\rho} \left(1 + \frac{i}{\gamma}\right) \int (i\mathbf{e}_\lambda \times \mathbf{e}_\mu + e_\lambda^0 \mathbf{e}_\mu - e_\mu^0 \mathbf{e}_\lambda) \cdot (\partial_\nu \boldsymbol{\omega}_\rho - \partial_\rho \boldsymbol{\omega}_\nu + \boldsymbol{\omega}_\nu \times \boldsymbol{\omega}_\rho) d^4x + \text{c.c.}$$

$$\int \exp [i(S_g(v, \Omega_0) + (\omega B \omega) + O(\omega^3))] \mathcal{D}\omega = \exp [i(S_g(\ell))] \int \exp [i(\omega B \omega)] (1 + O(\omega^3)) \mathcal{D}\omega,$$

$$(\Gamma_1 \omega_{\sigma_1^3} \Gamma_0 \omega_{\sigma_2^3} \Gamma_2) * \mathbf{v}_{\sigma^2}$$

$$\omega_{\sigma_1^3} \otimes \omega_{\sigma_2^3} \square \mathbf{v}_{\sigma^2}$$

$$\epsilon^{\alpha\beta\gamma} [i|\mathbf{v}| (\mathbf{e}_\alpha^{(0)} \times \mathbf{e}_\beta^{(0)}) \cdot (\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_\gamma) + |\boldsymbol{\tau}| \mathbf{e}_\alpha^{(0)} \cdot (\boldsymbol{\omega}_\beta \times \boldsymbol{\omega}_\gamma) + \text{c.c.}]$$

$$\mathbf{e}_\alpha^{(0)i} = \delta_\alpha^i, 2|\mathbf{v}| = b_s^2, 2|\boldsymbol{\tau}| = b_s b_t.$$

$$\boldsymbol{\omega}_\alpha = \frac{1}{\sqrt{|\boldsymbol{\tau}|}} \tilde{\boldsymbol{\omega}}_\alpha, \boldsymbol{\omega}_0 = \frac{\sqrt{|\boldsymbol{\tau}|}}{|\mathbf{v}|} \tilde{\boldsymbol{\omega}}_0$$



$$\mathcal{D}\omega = (4\pi^2 \omega^2)^{-1} \sin^2(\sqrt{\omega^2}/2) d^3\omega$$

$$d^3\omega d^3\omega^* \equiv 2^3 d^3\text{Re}\omega d^3\text{Im}\omega$$

$$\frac{1}{2} \int R\sqrt{-g} d^4x \Delta_\lambda^{(s)} = (T_\lambda - \bar{T}_\lambda)/2$$

$$\begin{aligned} \tilde{S}_g[g] = & \frac{1}{8} \sum_{\text{sites}} g^{\lambda\rho} g^{\mu\sigma} g^{\nu\tau} \left[2(\Delta_\lambda^{(s)} g_{\mu\nu})(\Delta_\tau^{(s)} g_{\sigma\rho}) - (\Delta_\nu^{(s)} g_{\mu\lambda})(\Delta_\tau^{(s)} g_{\sigma\rho}) \right. \\ & \left. - 2(\Delta_\lambda^{(s)} g_{\mu\rho})(\Delta_\sigma^{(s)} g_{\nu\tau}) + (\Delta_\nu^{(s)} g_{\lambda\rho})(\Delta_\tau^{(s)} g_{\mu\sigma}) \right] \sqrt{-g} \end{aligned}$$

$$g_{\lambda\mu} = g_{\lambda\mu}^{(0)} \bigoplus w_{\lambda\mu} \propto (\Delta^{(s)2} + i0)^{-1} = \langle \sin^2 p_0 - \sum_{j=1}^3 \sin^2 p_j + i0 \rangle^{-1}$$

$$\begin{aligned} \check{S}_g = & \frac{1}{8} \sum_{\text{sites}} g^{\lambda\rho} g^{\mu\sigma} g^{\nu\tau} \left[2(\Delta_\lambda^{(s)} g_{\mu\nu})(\Delta_\tau^{(s)} g_{\sigma\rho}) - (\Delta_\nu g_{\mu\lambda})(\Delta_\tau g_{\sigma\rho}) \right. \\ & \left. - 2(\Delta_\lambda^{(s)} g_{\mu\rho})(\Delta_\sigma^{(s)} g_{\nu\tau}) + (\Delta_\nu g_{\lambda\rho})(\Delta_\tau g_{\mu\sigma}) \right] \sqrt{-g} \end{aligned}$$

$$\delta_{\text{metric}}(v) = \int V^\eta \delta^{21}(\epsilon_{abcd} v_{\lambda\mu}^{ab} v_{\nu\rho}^{cd} - V \epsilon_{\lambda\mu\nu\rho}) dV.$$

$$\int d^{36} v_{\lambda\mu}^{ab} \delta_{\text{metric}}(v) \sim (-g)^{\frac{\eta-7}{2}} d^{10} g_{\lambda\mu}$$

$$\mathcal{N}_0(v) = \left| \frac{1}{\frac{1}{4}(\frac{1}{\gamma} - i)^2 v^2 + 1} \frac{\frac{1}{4}(\frac{1}{\gamma} - i)v}{\text{sh}\left[\frac{\pi}{2}(\frac{1}{\gamma} - i)v\right]} \right|^2$$

$$\prod_{\text{sites}} \left(\prod_{\alpha} \mathcal{N}_0(2v_\alpha) \right) (-g)^{\frac{\eta-7}{2}} d^{10} g_{\lambda\mu}.$$

$$p_0^{-j} \Rightarrow [(p_0 + i\varepsilon)^{-j} + (p_0 - i\varepsilon)^{-j}]/2$$

$$\check{S}'_g[g, J] = \check{S}_g - \sum_{\text{sites}} \left[J^{\lambda\mu} w_{\lambda\mu} + \frac{1}{4} (n^\mu w_{\mu\lambda}) \lambda^{\lambda\sigma} (n^\tau w_{\tau\sigma}) \right], g_{\lambda\mu} = g_{\lambda\mu}^{(0)} + w_{\lambda\mu}$$

$$w_{\lambda\mu} = \check{G}_{\lambda\mu\sigma\tau}(n, \bar{n}) J^{\sigma\tau}, \left(\|\lambda^{\lambda\mu}\|^{-1} \right)_{\sigma\tau} \stackrel{\text{def}}{=} \alpha_{\sigma\tau}, \eta_{\lambda\mu} = \text{diag}(-1, 1, 1, 1),$$

$$n^\lambda = v^\lambda - \varepsilon \frac{\Delta_\perp^{(s)\lambda}}{\Delta_\perp^{(s)2}}, \Delta_\perp^{(s)\lambda} = \Delta^{(s)\lambda} - \frac{v^\lambda}{v^2} (v\Delta^{(s)}).$$

$$g_{\lambda\mu}^{(0)} = g_{\lambda\mu}(x)|_{x \rightarrow \infty}$$

$$g_{\lambda\mu}^{(0)} = \text{diag}(-b_t^2, b_s^2, b_s^2, b_s^2) = l_\lambda^{(0)a} \eta_{ab} l_\mu^{(0)b}, l_\lambda^{(0)a} = \text{diag}(b_t, b_s, b_s, b_s), l_\mu^{(0)a} l_a^{(0)\lambda} \equiv \delta_\mu^\lambda$$

$$g_{\lambda\mu}^{(0)} \neq \eta_{\lambda\mu} = \text{diag}(-1, 1, 1, 1)$$



$$\tilde{g}_{ab} = l_a^{(0)\lambda} g_{\lambda\mu} l_b^{(0)\mu}, \text{ which has } \tilde{g}_{ab}^{(0)} = \eta_{ab}$$

$$\begin{aligned} & \check{G}_{\lambda\mu\sigma\tau} \left(n, \bar{n} \mid \{g_{\lambda\mu}^{(0)}\}, \{\Delta_\lambda^{(s)}\}, \{\Delta_\lambda\}, \{\alpha_{\lambda\mu}\}, \{n^\lambda\} \right) \\ &= l_\lambda^{(0)a} l_\mu^{(0)b} l_\sigma^{(0)e} l_\tau^{(0)f} \check{G}_{abef} \left(n, \bar{n} \mid \{\eta_{ab}\}, \{(-g^{(0)})^{1/4} l_a^{(0)\lambda} \Delta_\lambda^{(s)}\}, \{(-g^{(0)})^{1/4} l_a^{(0)\lambda} \Delta_\lambda\} \right. \\ & \quad \left. \{l_a^{(0)\lambda} \alpha_{\lambda\mu} l_b^{(0)\mu}\}, \{l_\lambda^{(0)a} n^\lambda\} \right) \\ &= l_\lambda^{(0)a} l_\mu^{(0)b} l_\sigma^{(0)e} l_\tau^{(0)f} \check{G}_{abef} \left(n, \bar{n} \mid \{\eta_{ab}\}, \{l_a^{(0)\lambda} \Delta_\lambda^{(s)}\}, \{l_a^{(0)\lambda} \Delta_\lambda\} \right. \\ & \quad \left. \{(-g^{(0)})^{1/2} l_a^{(0)\lambda} \alpha_{\lambda\mu} l_b^{(0)\mu}\}, \{l_\lambda^{(0)a} n^\lambda\} \right) (-g^{(0)})^{-1/2} \end{aligned}$$

$$l_a^{(0)\lambda} \Delta_\lambda^{(s)} = (b_t^{-1} \Delta_0^{(s)}, b_s^{-1} \Delta^{(s)}), l_a^{(0)\lambda} \Delta_\lambda = (b_t^{-1} \Delta_0, b_s^{-1} \Delta)$$

$$\check{G}_{\lambda\mu\sigma\tau}(n, \bar{n}) p_0^{-2} \Rightarrow (p_0^2 + \varepsilon^2)^{-1}, \varepsilon \rightarrow 0$$

$$\check{G}_{\lambda\mu\sigma\tau} = \frac{1}{2} \check{G}_{\lambda\mu\sigma\tau}(n, n) + \frac{1}{2} \check{G}_{\lambda\mu\sigma\tau}(\bar{n}, \bar{n})$$

$$p_0^{-j} \Rightarrow [(p_0 + i\varepsilon)^{-j} + (p_0 - i\varepsilon)^{-j}]/2$$

$$\begin{aligned} \left[\sin^2 \frac{p_0}{2} - \frac{b_t^2}{b_s^2} \sum_{\alpha=1}^3 \sin^4 \frac{p_\alpha}{2} + O(\varepsilon^2) \right]^{-j} &\Rightarrow \frac{1}{2} \left[\sin^2 \frac{p_0 + i\varepsilon}{2} - \frac{b_t^2}{b_s^2} \sum_{\alpha=1}^3 \sin^4 \frac{p_\alpha}{2} + O(\varepsilon^2) \right]^{-j} \\ &+ \frac{1}{2} \left[\sin^2 \frac{p_0 - i\varepsilon}{2} - \frac{b_t^2}{b_s^2} \sum_{\alpha=1}^3 \sin^4 \frac{p_\alpha}{2} + O(\varepsilon^2) \right]^{-j} \end{aligned}$$

$$(n\Delta^{(s)})^{-1} = (v\Delta^{(s)} - \varepsilon)^{-1} \propto (\sin p_0 + i\varepsilon)^{-1}$$

$$\begin{aligned} \frac{1}{2} G_{\lambda\mu\sigma\tau}^{\text{eff}}(n, n) &= \frac{-1}{\Delta\Delta} [L_{\lambda\sigma}(n, n)L_{\mu\tau}(n, n) + L_{\mu\sigma}(n, n)L_{\lambda\tau}(n, n) - L_{\lambda\mu}(n, n)L_{\sigma\tau}(n, n)] \\ &- \frac{(\alpha_{\lambda\sigma}\Delta_\tau^{(s)} + \alpha_{\lambda\tau}\Delta_\sigma^{(s)})\Delta_\mu^{(s)} + (\alpha_{\mu\sigma}\Delta_\tau^{(s)} + \alpha_{\mu\tau}\Delta_\sigma^{(s)})\Delta_\lambda^{(s)}}{(n\Delta^{(s)})^2} + \Delta_\lambda^{(s)}\Delta_\mu^{(s)} \frac{n^v\alpha_{v\sigma}\Delta_\tau^{(s)} + n^v\alpha_{v\tau}\Delta_\sigma^{(s)}}{(n\Delta^{(s)})^3} \end{aligned}$$

$$+ \frac{\alpha_{\lambda\nu}n^v\Delta_\mu^{(s)} + \alpha_{\mu\nu}n^v\Delta_\lambda^{(s)}}{(n\Delta^{(s)})^3} \Delta_\sigma^{(s)}\Delta_\tau^{(s)} - \frac{n^v\alpha_{v\rho}n^\rho}{(n\Delta^{(s)})^4} \Delta_\lambda^{(s)}\Delta_\mu^{(s)}\Delta_\sigma^{(s)}\Delta_\tau^{(s)}$$

$$L_{\lambda\mu}(n, n) \stackrel{\text{def}}{=} \eta_{\lambda\mu} - \Delta_\lambda^{(s)} \frac{n_\mu}{n\Delta^{(s)}} - \frac{n_\lambda}{n\Delta^{(s)}} \Delta_\mu^{(s)} + \frac{n^2\Delta_\lambda^{(s)}\Delta_\mu^{(s)}}{(n\Delta^{(s)})^2}$$

$$\check{F}[g] = \frac{1}{2} \sum_{\text{sites}} w_{\lambda\mu} \Delta \tilde{\mathcal{M}}^{\lambda\mu\sigma\tau} w_{\sigma\tau} = \sum_{\text{sites}} \left[-\frac{1}{4\varepsilon^2} \check{f}_\rho[g] (\tilde{\mathfrak{M}}^{-1})^{\rho\kappa} \check{f}_\kappa[g] + \frac{1}{2} w_{\lambda\mu} m^{\lambda\mu\sigma\tau} w_{\sigma\tau} \right]$$

$$\check{f}_\rho = \tilde{\mathcal{O}}_\rho^{\lambda\mu} w_{\lambda\mu}, \check{\mathcal{O}}_\rho^{\lambda\mu} = \delta_\rho^{\lambda\mu} v^\mu + \frac{\varepsilon^2}{2} \frac{\Delta_\perp^{(s)v}}{\Delta_\perp^{(s)2}} \check{G}_{\rho\nu\pi\zeta}^{(0)} v^\pi \lambda^{\zeta\lambda} \frac{\Delta_\perp^{(s)\mu}}{\Delta_\perp^{(s)2}}$$

$$m^{\lambda\mu\sigma\tau} = -\frac{\varepsilon^2}{2} \frac{\Delta_\perp^{(s)\lambda}}{\Delta_\perp^{(s)2}} \left(\lambda^{\mu\sigma} + \frac{1}{2} \lambda^{\mu\zeta} v^\pi \check{G}_{\zeta\pi\chi\psi}^{(0)} v^\chi \lambda^{\psi\sigma} \right) \frac{\Delta_\perp^{(s)\tau}}{\Delta_\perp^{(s)2}}, \tilde{\mathfrak{M}}_{\lambda\tau} = \frac{\alpha_{\lambda\tau}}{\varepsilon^2} - \frac{1}{2} \frac{\Delta_\perp^{(s)\mu}}{\Delta_\perp^{(s)2}} \check{G}_{\lambda\mu\sigma\tau}^{(0)} \frac{\Delta_\perp^{(s)\sigma}}{\Delta_\perp^{(s)2}}$$

$$\check{G}_{\zeta\pi\chi\psi}^{(0)} = \left[\left\| \tilde{\mathcal{M}}^{v\rho\kappa\varphi} - \frac{1}{2} v^{(v\lambda\rho)(\kappa\nu\varphi)} \right\|^{-1} \right]_{\zeta\pi\chi\psi} = \check{G}_{\zeta\pi\chi\psi}(v, v)$$



$$(g_{\lambda\mu})^{\Xi} - g_{\lambda\mu} = \delta g_{\lambda\mu} = -g_{\lambda\nu}\partial_{\mu}\xi^{\nu} - g_{\mu\nu}\partial_{\lambda}\xi^{\nu} - \xi^{\nu}\partial_{\nu}g_{\lambda\mu}.$$

$$\check{\Phi}[g]^{-1} = \int \exp(i\check{\mathcal{F}}[g^{\Xi}]) \prod_{\text{sites}} d\Xi, \quad d\Xi = \prod_{\lambda} d\xi^{\lambda},$$

$$\delta^{\Xi}g_{\lambda\mu} = g_{\lambda\mu}^{\Xi} - g_{\lambda\mu} = -\Delta_{\mu}^{(s)}\xi_{\lambda} - \Delta_{\lambda}^{(s)}\xi_{\mu} + 2\Gamma_{\lambda\mu}^{\nu}\xi_{\nu} + O((\xi)^2),$$

$$\Gamma_{\lambda\mu}^{\nu} = \frac{1}{2}g^{\nu\rho}(\Delta_{\mu}^{(s)}g_{\rho\lambda} + \Delta_{\lambda}^{(s)}g_{\rho\mu} - \Delta_{\rho}^{(s)}g_{\lambda\mu}),$$

$$\check{\mathcal{O}}_{\rho}{}^{\nu}\xi_{\nu} \stackrel{\text{def}}{=} \check{\mathcal{O}}_{\rho}{}^{\lambda\mu}(\Delta_{\mu}^{(s)}\xi_{\lambda} + \Delta_{\lambda}^{(s)}\xi_{\mu} - 2\Gamma_{\lambda\mu}^{\nu}\xi_{\nu}).$$

$$\int \exp\left\{i\left[\check{S}_g[g] + \frac{1}{2}\sum_{\text{sites}} w_{\lambda\mu}(-g)^{-\frac{1}{8}}\Delta\tilde{\mathcal{M}}^{\lambda\mu\sigma\tau}(-g)^{-\frac{1}{8}}\right.\right. \\ \left.\left.\cdot w_{\sigma\tau}\right\}(\cdot) \prod_{\text{sites}} \left(\prod_{\alpha} \mathcal{N}_0(2v_{\alpha})\right) (-g)^{\frac{\eta-8}{2}} d^{10}g_{\lambda\mu}$$

$$i m_1 = 2v_1 = i\sqrt{g_{22}g_{33} - g_{23}^2}, \quad 2 \text{ perm } (123)$$

$$\mathcal{N}_0(im_{\alpha}) = \frac{\gamma^2}{\gamma^2 + 1} \left| \frac{m_{\alpha}^2}{m_{\alpha}^2 + 4\left(1 + \frac{i}{\gamma}\right)^{-2}} \right|^2 \left| \frac{2}{1 - \exp\left[-\pi\left(1 + \frac{i}{\gamma}\right)m_{\alpha}\right]} \right|^2 \frac{e^{-\pi m_{\alpha}}}{m_{\alpha}^2}.$$

$$\prod_{\alpha} \mathcal{N}_0(im_{\alpha}) \propto \prod_{\alpha} \frac{1}{m_{\alpha}^2} \exp(-\pi m_{\alpha})$$

$$(-g)^{\frac{\eta-8}{2}} = (-g_{00}^{(0)} - w_{00} + w_{0\alpha}\gamma^{\alpha\beta}w_{\beta 0})^{\frac{\eta-8}{2}} \gamma^{\frac{\eta-8}{2}} = [-g_{00}^{(0)} + O(\varepsilon^2)]^{\frac{\eta-8}{2}} \gamma^{\frac{\eta-8}{2}} \propto \gamma^{\frac{\eta-8}{2}}$$

$$g_{\hat{\alpha}<\hat{\beta}}: g_{\hat{\alpha}\hat{\beta}} = |g_{\alpha\alpha}|^{-1/2}g_{\alpha\beta}|g_{\beta\beta}|^{-1/2}, \quad \alpha < \beta$$

$$\gamma = \det\|g_{\hat{\alpha}\hat{\beta}}\| \prod_{\alpha<\beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{-1/2} \prod_{\alpha} m_{\alpha}.$$

$$d^{10}g_{\lambda\mu} = d^4g_{0\lambda} \prod_{\alpha<\beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{-1} \left(\prod_{\alpha} m_{\alpha}\right) d^3g_{\hat{\alpha}<\hat{\beta}^2} d^3m_{\alpha}$$

$$d^4g_{0\lambda}(\det\|g_{\hat{\alpha}\hat{\beta}}\|)^{\frac{\eta-8}{2}} \left[\prod_{\alpha<\beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{\frac{4-\eta}{4}}\right] d^3g_{\hat{\alpha}<\hat{\beta}} \prod_{\alpha} \exp(-\pi m_{\alpha}) m_{\alpha}^{\frac{\eta-10}{2}} dm_{\alpha}.$$

$$S = \check{S}_g + \check{\mathcal{F}}[\{w_{\lambda\mu}\}(-g)^a] \det\|\partial^2 S(\ell_{(0)})/(\partial l_i \partial l_k)\|$$

$$\check{\mathcal{F}}[\{w_{\lambda\mu}\}(-g)^a] d^3m_{\alpha} = 2 \prod_{\alpha<\beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{1/4} \left(\prod_{\alpha} m_{\alpha}^{1/2}\right) d^3 \det\|\partial^2 S(\ell_{(0)})/(\partial l_i \partial l_k)\| \sqrt{g_{\alpha\alpha}}$$



$$2 d^4 g_{0\lambda} (\det \|g_{\hat{\alpha}\hat{\beta}}\|)^{\frac{\eta-8}{2}} \prod_{\alpha < \beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{\frac{5-\eta}{4}} d^3 g_{\hat{\alpha} < \hat{\beta}} \cdot \left(\prod_{\alpha} \exp(-\pi m_{\alpha}) m_{\alpha}^{\frac{\eta-9}{2}} \right) d^3 \sqrt{g_{\alpha\alpha}}$$

$$m_{\alpha}^{(0)} = \frac{\eta - 9}{2\pi} \forall \alpha,$$

$$\check{G}_{\alpha\beta\gamma\delta}(n, n) = (-g^{(0)})^{-1/2} \frac{-2}{\bar{\Delta}\Delta} [L_{\alpha\gamma}(n, n)L_{\beta\delta}(n, n) + L_{\beta\gamma}(n, n)L_{\alpha\delta}(n, n) - L_{\alpha\beta}(n, n)L_{\gamma\delta}(n, n)], L_{\alpha\beta}(n, n) \stackrel{\text{def}}{=} g_{\alpha\beta}^{(0)} + \frac{n^2}{(n\Delta^{(s)})^2 + \mathcal{A}n^2} \Delta_{\alpha}^{(s)} \Delta_{\beta}^{(s)}, \mathcal{A} \stackrel{\text{def}}{=} -\bar{\Delta}\Delta - \Delta^{(s)2}$$

$$g_{\alpha\beta}^{(0)} = b_s^2 \delta_{\alpha\beta}, n^2 = -b_t^2 (\text{up to } O(\varepsilon^2)), g^{(0)} = -b_t^2 b_s^6, \bar{\Delta}\Delta = b_s^{-2} \sum_{\alpha=1}^3 \bar{\Delta}_{\alpha} \Delta_{\alpha} - b_t^{-2} \bar{\Delta}_0 \Delta_0$$

$$\mathcal{A} = b_t^{-2} (\bar{\Delta}_0 \Delta_0 + \Delta_0^{(s)2}) - b_s^{-2} \sum_{\alpha=1}^3 (\bar{\Delta}_{\alpha} \Delta_{\alpha} + \Delta_{\alpha}^{(s)2}), n\Delta^{(s)} = \Delta_0^{(s)} - \varepsilon = O(1)$$

$$\check{G}_{\alpha\beta\gamma\delta}(n, n) \propto \frac{b_t^{-1} b_s (\delta_{\alpha\gamma} \delta_{\beta\delta} + \dots)}{b_s^{-2} \sum_{\alpha=1}^3 \sin^2 \frac{p_{\alpha}}{2} - b_t^{-2} \sin^2 \frac{p_0}{2} - i0} + \dots$$

$$\langle w_{\alpha\beta} w_{\gamma\delta} \rangle = O(b_s^2), w_{\alpha\beta} = O(b_s).$$

$$w_{0\lambda} = \gamma^{-a} w'_{0\lambda}, w_{\alpha\beta} = w'_{\alpha\beta}$$

$$v^{\mu} w_{\rho\mu} (-g)^a \Rightarrow v^{\mu} w_{\rho\mu} \left(-\frac{g}{\gamma}\right)^a = v^{\mu} w_{\rho\mu} [-g_{00}^{(0)} + O(\varepsilon^2)]$$

$$\check{F}[\{w_{\lambda\mu}\}(-g)^a] v^{\mu} w_{\lambda\mu} \Rightarrow v^{\mu} w_{\lambda\mu} + (\gamma^{-a} - 1) v^{\mu} w_{\lambda\mu}$$

$$\prod_{\text{sites}} \gamma^{-4a} = \prod_{\text{sites}} \gamma^{1/2}$$

$$d^4 g_{0\lambda} (\det \|g_{\hat{\alpha}\hat{\beta}}\|)^{\frac{\eta-7}{2}} \left[\prod_{\alpha < \beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{\frac{3-\eta}{4}} \right] d^3 g_{\hat{\alpha} < \hat{\beta}} \prod_{\alpha} \exp(-\pi m_{\alpha}) m_{\alpha}^{\frac{\eta-9}{2}} dm_{\alpha}$$

$$\propto d^{10} \tilde{g}_{\lambda\mu} = d^4 g_{0\lambda} \prod_{\alpha < \beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2)^{-1} \left(\prod_{\alpha} \tilde{m}_{\alpha} \right) d^3 g_{\hat{\alpha} < \hat{\beta}} d^3 \tilde{m}_{\alpha}$$

$$\exp(-\pi m_{\alpha}) m_{\alpha}^{\frac{\eta-9}{2}} dm_{\alpha} \propto f \tilde{m}_{\alpha} d\tilde{m}_{\alpha}, f = \left[(\det \|g_{\hat{\alpha}\hat{\beta}}\|)^{-2} \prod_{\alpha < \beta} (1 - g_{\hat{\alpha}\hat{\beta}}^2) \right]^{\frac{\eta-7}{12}}$$

$$\int_0^{\pi m} \exp(-z) z^{\tilde{k}} dz = C [f \tilde{m}^2 + C_1(\{g_{\hat{\alpha}\hat{\beta}}\})], \tilde{k} = \frac{\eta - 9}{2}$$

$$\tilde{m} = \tilde{m}^{(0)} \Leftrightarrow m = m^{(0)} \forall \{g_{\hat{\alpha}\hat{\beta}}\}$$



$$\int_0^y \exp(-z) z^{\tilde{k}} dz = \frac{x}{x_0} \int_0^k \exp(-z) z^{\tilde{k}} dz, y = \pi m$$

$$x = \tilde{m}^{(0)2} + (\tilde{m}^2 - \tilde{m}^{(0)2})f, x_0 = \tilde{m}^{(0)2}$$

$$y = k + \sum_{n=1}^{\infty} \left. \frac{d^n y}{dx^n} \right|_{y=k} \frac{\Delta x^n}{n!}$$

$$\pi m = y = k + \sum_{n=1}^{\infty} \left(\frac{\Delta x}{x_0} I \right)^n \frac{1}{n!} \left[e^y \left(1 + \frac{y}{k} \right)^{-\tilde{k}} \frac{d}{dy} \right]^{n-1} e^y \left(1 + \frac{y}{k} \right)^{-\tilde{k}} \Big|_{y=0}$$

$$I = \int_0^k e^{k-z} \left(\frac{z}{k} \right)^{\tilde{k}} dz$$

$$\pi m = k + \sqrt{k} \sqrt{\frac{\pi}{2}} \frac{\Delta x}{x_0} + \frac{1}{6} \sqrt{k} \left(\sqrt{\frac{\pi}{2}} \frac{\Delta x}{x_0} \right)^3 + \dots$$

$$f = \left[1 + O\left(g_{\tilde{\alpha} < \tilde{\beta}}^2 \right) \right]^{\frac{\eta-7}{12}} = \left[1 + O(\eta^{-1}) \right]^{\frac{\eta-7}{12}}$$

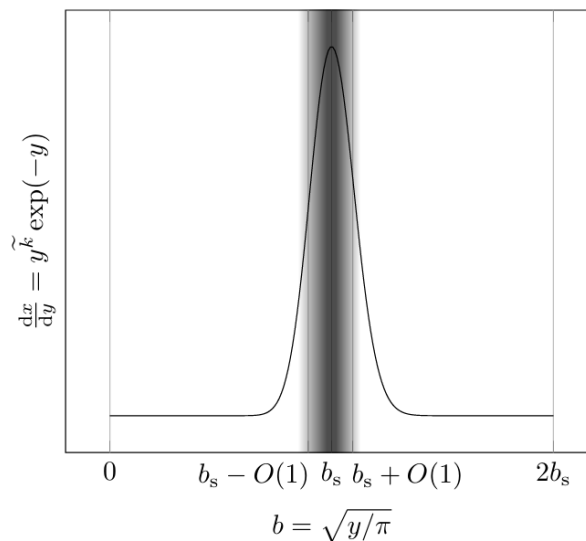
$$e^y \left(1 + \frac{y}{k} \right)^{-\tilde{k}} = \exp \left[y - \tilde{k} \ln \left(1 + \frac{y}{k} \right) \right]$$

$$= \exp \left\{ \frac{\tilde{k}}{k} \left[\frac{1}{2} \left(\frac{y}{k^{1/2}} \right)^2 - \frac{1}{3} \left(\frac{y}{k^{2/3}} \right)^3 + \frac{1}{4} \left(\frac{y}{k^{3/4}} \right)^4 - \dots \right] + \frac{k - \tilde{k}}{k} y \right\}$$

$$\left[O(k^{-1+\delta}) \right]^{n-1} (I \Delta x / x_0)^n = O(k^{(-1+\delta)(n-1)+n/2}) = O(k^{(\delta-1/2)(n-1)}) \cdot \sqrt{k}.$$

$$O\left(k^{\frac{1}{2}(\delta-1/2)[1+(-1)^n]} \right) \cdot \sqrt{k}.$$

$$|k - \tilde{k}| \lesssim O(\sqrt{k}) \text{ or } |\sqrt{k} - \sqrt{\tilde{k}}| \lesssim O(1)$$



$$I = O(\sqrt{k}), \frac{\Delta(\tilde{m}^2)}{\tilde{m}^{(0)2}} = O\left(\frac{\Delta x}{x_0}\right) = O(1).$$

$$\tilde{g}_{\alpha\beta}^{(0)} = \delta_{\alpha\beta}, \tilde{g}_{\alpha\beta} = \delta_{\alpha\beta} + \tilde{w}_{\alpha\beta}$$

$$m_1 = \frac{k}{\pi} + \sqrt{\frac{k}{2\pi}} (\tilde{w}_{22} + \tilde{w}_{33} + \tilde{w}_{22}\tilde{w}_{33} - \tilde{w}_{23}^2) + \dots$$

$$g_{11} = \frac{m_2 m_3}{m_1} \frac{(1 - \tilde{g}_{\hat{2}\hat{3}})^{1/2}}{(1 - \tilde{g}_{\hat{3}\hat{1}})^{1/2}(1 - \tilde{g}_{\hat{1}\hat{2}})^{1/2}}, \dots$$

$$\begin{aligned} \frac{\pi}{k} g_{11} &= 1 + \sqrt{\frac{2\pi}{k}} \tilde{w}_{11} + \frac{1}{2} (\tilde{w}_{31}^2 + \tilde{w}_{12}^2 - \tilde{w}_{23}^2) \\ &+ \sqrt{\frac{\pi}{2k}} (\tilde{w}_{33}\tilde{w}_{11} + \tilde{w}_{11}\tilde{w}_{22} - \tilde{w}_{22}\tilde{w}_{33} - \tilde{w}_{31}^2 - \tilde{w}_{12}^2 + \tilde{w}_{23}^2) + \frac{\pi}{2k} (\tilde{w}_{11} - \tilde{w}_{22})(\tilde{w}_{11} - \tilde{w}_{33}) \end{aligned}$$

$$g_{\hat{\alpha}\hat{\beta}} = \tilde{g}_{\hat{\alpha}\hat{\beta}}, \text{ then } \tilde{g}_{\alpha\beta} = O(k^{-1/2})$$

$$\tilde{g}_{\alpha\beta} \Rightarrow \sqrt{\frac{2\pi}{k}} \tilde{g}_{\alpha\beta}$$

$$\begin{aligned} \frac{\pi}{k} g_{11} &= 1 + \sqrt{\frac{2\pi}{k}} \left[\tilde{w}_{11} + \frac{1}{2} (\tilde{w}_{33}\tilde{w}_{11} + \tilde{w}_{11}\tilde{w}_{22} - \tilde{w}_{22}\tilde{w}_{33}) \right. \\ &+ \left. \frac{1}{2} \sqrt{\frac{2\pi}{k}} \left(1 - \sqrt{\frac{2\pi}{k}} \right) (\tilde{w}_{31}^2 + \tilde{w}_{12}^2 - \tilde{w}_{23}^2) + \frac{1}{4} \sqrt{\frac{2\pi}{k}} (\tilde{w}_{11} - \tilde{w}_{22})(\tilde{w}_{11} - \tilde{w}_{33}) \right] \\ \frac{\pi}{k} g_{23} &= \sqrt{\frac{2\pi}{k}} \left[\tilde{w}_{23} - \frac{1}{2} \left(1 - \sqrt{\frac{2\pi}{k}} \right) (\tilde{w}_{22} + \tilde{w}_{33}) \tilde{w}_{23} \right] \end{aligned}$$

$$\frac{\pi}{k} g_{\alpha\beta} = \delta_{\alpha\beta} + \sqrt{\frac{2\pi}{k}} u_{\alpha\beta}(\{\tilde{w}_{\gamma\delta}\}), u_{\alpha\beta} = \tilde{w}_{\alpha\beta} + V_{\alpha\beta}$$

$$V_{\alpha\beta} = V_{\alpha\beta}(\{\tilde{w}_{\gamma\delta}\})$$

$$w_{0\lambda} \dots [w_{\alpha\beta}(\{\tilde{w}_{\gamma\delta}\}) (-g(\{\tilde{w}_{\gamma\delta}\}))^\alpha]$$

$$\langle w_{\alpha\beta} w_{\gamma\delta} \rangle = 2b_s^2 \langle \tilde{w}_{\alpha\beta} \tilde{w}_{\gamma\delta} \rangle.$$

$$-i \langle \tilde{w}_{\alpha\beta} \tilde{w}_{\gamma\delta} \rangle \propto \frac{b_t^{-1} b_s^{-1} (\delta_{\alpha\gamma} \delta_{\beta\delta} + \dots)}{b_s^{-2} \sum_{\alpha=1}^3 \sin^2 \frac{p_\alpha}{2} - b_t^{-2} \sin^2 \frac{p_0}{2} - i0} + \dots$$

$$\sqrt{-g} g^{\lambda_1 \lambda_2} g^{\lambda_3 \lambda_4} g^{\lambda_5 \lambda_6} (\Delta_{\mu_5}^\# g_{\mu_1 \mu_2}) (\Delta_{\mu_6}^\# g_{\mu_3 \mu_4}), \Delta^\# \stackrel{\text{def}}{=} \Delta \text{ or } \Delta^{(s)}$$



$$\begin{aligned}
& \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} g^{\alpha_5 \alpha_6} (\Delta_{\beta_5}^{\#} g_{\beta_1 \beta_2}) (\Delta_{\beta_6}^{\#} g_{\beta_3 \beta_4}), \\
& \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} g^{00} (\Delta_0^{\#} g_{\beta_1 \beta_2}) (\Delta_0^{\#} g_{\beta_3 \beta_4}). \\
& \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} (b_s b_t^{-1} \Delta_0^{\#} u_{\beta_1 \beta_2}) (b_s b_t^{-1} \Delta_0^{\#} u_{\beta_3 \beta_4}) \\
& \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} [b_s b_t^{-1} \Delta_0^{\#} (\tilde{w}_{\gamma_n \delta_n} \dots \tilde{w}_{\gamma_{m+1} \delta_{m+1}})] [b_s b_t^{-1} \Delta_0^{\#} (\tilde{w}_{\gamma_m \delta_m} \dots \tilde{w}_{\gamma_1 \delta_1})]. \\
& \frac{b_s}{b_t} \Delta_0^{\#} \left(\sum_{j=m+1}^n p_j \right) \frac{b_s}{b_t} \Delta_0^{\#} \left(\sum_{j=1}^m p_j \right) \prod_{j=1}^n \frac{b_t^{-1} b_s^{-1}}{b_s^{-2} \sum_{\alpha=1}^3 \sin^2 \frac{p_{j\alpha}}{2} - b_t^{-2} \sin^2 \frac{p_{j0}}{2} - i0} \\
& g_{\alpha\beta} = b_s^2 \delta_{\alpha\beta} + \sqrt{2} b_s u_{\alpha\beta} (\{\tilde{w}_{\gamma\delta}\}) \text{ into } \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} \\
& \sum_{j=m+1}^n p_{j0} = O(b_t/b_s), \sum_{j=1}^m p_{j0} = O(b_t/b_s), \\
& \frac{b_s}{b_t} \Delta_0^{\#} \left(\sum_{j=m+1}^n p_j \right) = O(1), \frac{b_s}{b_t} \Delta_0^{\#} \left(\sum_{j=1}^m p_j \right) = O(1). \\
& b_s b_t^{-1} \Delta_0^{\#} \left(\sum_{j=1}^m p_j \right) = O(1) \dots \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} g_{\alpha\beta} = b_s^2 \delta_{\alpha\beta} + \sqrt{2} b_s u_{\alpha\beta} \\
& b_s b_t^{-1} \Delta_0^{\#} \left(\sum_{j=m+1}^n p_j \right) \times b_t b_s^{-1} = \Delta_0^{\#} \left(\sum_{j=m+1}^n p_j \right) = O(1)
\end{aligned}$$

$$\sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} g^{\alpha_5 \alpha_6} (\Delta_{\beta_5}^{\#} g_{\beta_1 \beta_2}) (\Delta_{\beta_6}^{\#} g_{\beta_3 \beta_4}) \sim b_t b_s^{-1} (\Delta_{\beta_5}^{\#} u_{\beta_1 \beta_2}) (\Delta_{\beta_6}^{\#} u_{\beta_3 \beta_4}) \sim b_t b_s^{-1}$$

$$\begin{aligned}
& \sqrt{-g} g^{\alpha_1 \alpha_2} g^{\alpha_3 \alpha_4} g^{\alpha_5 \alpha_6} (\Delta_{\beta_5}^{\#} g_{\beta_1 \beta_2}) (\Delta_{\beta_6}^{\#} g_{\beta_3 \beta_4}) = \\
& \sum_{n \geq 2} C_{(n)}^{\alpha_1 \dots \alpha_6 \gamma_1 \delta_1 \dots \gamma_{n-2} \delta_{n-2}} u_{\gamma_1 \delta_1} \dots u_{\gamma_{n-2} \delta_{n-2}} (\Delta_{\beta_5}^{\#} u_{\beta_1 \beta_2}) (\Delta_{\beta_6}^{\#} u_{\beta_3 \beta_4}), \\
& C_{(n)}^{\alpha_1 \dots \alpha_6 \gamma_1 \delta_1 \dots \gamma_{n-2} \delta_{n-2}} = O(b_t b_s^{1-n}).
\end{aligned}$$

$$\begin{aligned}
b_s^{-1} dp_{\alpha} & \Rightarrow d\tilde{p}_{\alpha}, b_s^{-1} \Delta_{\alpha}^{(s)} \Rightarrow i\tilde{p}_{\alpha}, b_s^{-1} \Delta_{\alpha} \Rightarrow i\tilde{p}_{\alpha}, \\
b_t^{-1} dp_0 & \Rightarrow d\tilde{p}_0, b_t^{-1} \Delta_0^{(s)} \Rightarrow i\tilde{p}_0, b_t^{-1} \Delta_0 \Rightarrow i\tilde{p}_0.
\end{aligned}$$

$$V_{\alpha\beta} (\{\tilde{w}_{\gamma\delta}\}) \text{ in } u_{\alpha\beta} = \tilde{w}_{\alpha\beta} + V_{\alpha\beta}$$

$$\tilde{w}_{\gamma_1 \delta_1} (\Delta_{\mu_5}^{\#} \tilde{w}_{\beta_1 \beta_2}) (\Delta_{\mu_6}^{\#} \tilde{w}_{\beta_3 \beta_4}),$$

$$[\Delta_{\mu_5}^{\#} (\tilde{w}_{\gamma_1 \delta_1} \tilde{w}_{\gamma_2 \delta_2})] (\Delta_{\mu_6}^{\#} \tilde{w}_{\beta_3 \beta_4}).$$

$$\int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \dots ds_n$$



$$\mathbb{E} \left[\prod_{j=1}^m Z_j \right] = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \sum_{\text{pairings}} \mathcal{P} \text{ of } \{1, \dots, m\} \prod_{\{i,k\} \in \mathcal{P}} \text{Cov}(Z_i, Z_k), & \text{if } m \text{ is even.} \end{cases}$$

$$X(s) = (X_1(s), \dots, X_m(s)) \in \mathbb{R}^m, s \in [0,1],$$

$$\text{Cov}(X_i(s), X_j(t)) = f(s, t)\delta_{ij}, s, t \in [0,1]$$

$$f: [0,1] \times [0,1] \rightarrow \mathbb{R}$$

$$Q(x) = \sum_{\alpha \in \mathbb{N}_0^m} q_\alpha x^\alpha \text{ of } m \text{ variables with arbitrary coefficients,}$$

$$\int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \cdots ds_n$$

$$\exp(-tH_Q)(x, y) = (2\pi t)^{-m/2} \exp\left(\frac{-|x-y|^2}{2t}\right) \mathbb{E} \left[\exp\left(-\int_0^t Q(X^{x,y;t}(s)) ds\right) \right]$$

$$\mathbb{E} \left[\exp\left(-\int_0^t Q(X^{x,y;t}(s)) ds\right) \right] = \sum_{n=0}^{\infty} \int_{\{0 \leq s_1 \leq \dots \leq s_n \leq t\}} \mathbb{E} \left[\prod_{k=1}^n Q(X^{x,y;t}(s_k)) \right] ds_1 \cdots ds_n$$

$$Q(x) = (x, Dx) + (c, x), \text{ where } D \in \mathbb{R}^{m \times m} \text{ is symmetric and } c \in \mathbb{R}^m$$

$$\int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n X(s_k) \right] ds_1 \cdots ds_n = \int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \sum_{\substack{\text{pairings} \\ \text{of } \{1, \dots, n\}}} \prod_{\{i,j\} \in \mathcal{P}} f(s_i, s_j) ds_1 \cdots ds_n.$$

As for every symmetric $n \times n$ matrix A one has

$$\begin{aligned} \sum_{\substack{\text{pairings} \\ \text{of } \{1, \dots, n\}}} \prod_{\{i,j\} \in \mathcal{P}} A_{i,j} &= \frac{1}{l! 2^l} \sum_{\sigma \in \mathcal{S}_n} A_{\sigma(1), \sigma(2)} \cdots A_{\sigma(n-1), \sigma(n)} \\ &= \frac{1}{l! 2^l} \sum_{\sigma \in \mathcal{S}_n} \int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} f(s_{\sigma(1)}, s_{\sigma(2)}) \cdots f(s_{\sigma(n-1)}, s_{\sigma(n)}) ds_1 \cdots ds_n \\ &= \frac{1}{l! 2^l} \int_{[0,1]^n} f(s_1, s_2) \cdots f(s_{n-1}, s_n) ds_1 \cdots ds_n \\ &= \frac{1}{l! 2^l} \left(\int_{[0,1]^2} f(s_1, s_2) ds_1 ds_2 \right)^l \end{aligned}$$



$$\int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \dots ds_n$$

$$= \frac{1}{n!} \sum_{\substack{\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathbb{N}_0^m \\ \frac{1}{2} \sum_{k=1}^n \sum_{i=1}^m \alpha_i^{(k)} \in \mathbb{N}}} \left(\prod_{k=1}^n q_{\alpha^{(k)}} \right) \sum_{\Gamma_1} \dots \sum_{\Gamma_m} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right)$$

$$\mathcal{P} = \{ \{u_1, v_1\}, \dots, \{u_{l/2}, v_{l/2}\} \}$$

$$O(\mathcal{M}) := \{ (x, s) : x \in M, 1 \leq s \leq m(x) \} \subset M \times \mathbb{N}$$

$$\mathcal{M}(\alpha^{(1)}, \dots, \alpha^{(n)}) := ([n] \times [m], (k, i) \mapsto \alpha_i^{(k)})$$

$$O(\alpha^{(1)}, \dots, \alpha^{(n)}) := O(\mathcal{M}(\alpha^{(1)}, \dots, \alpha^{(n)}))$$

$$= \{ (k, i, r) : k \in [n], i \in [m], 1 \leq r \leq \alpha_i^{(k)} \} \subset [n] \times [m] \times \mathbb{N},$$

$$|O(\alpha^{(1)}, \dots, \alpha^{(n)})| = \sum_{k=1}^n \sum_{i=1}^m \alpha_i^{(k)}$$

$$Y^{(k)} = (Y_1^{(k)}, \dots, Y_m^{(k)}) \in \mathbb{R}^m$$

$$Q(x) = \sum_{\alpha \in \mathbb{N}_0^m} q_{\alpha} x^{\alpha}$$

$$\mathbb{E} \left[\prod_{k=1}^n Q(Y^{(k)}) \right]$$

$$= \sum_{\substack{\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathbb{N}_0^m \\ |O(\alpha^{(1)}, \dots, \alpha^{(n)})|/2 \in \mathbb{N}}} \left(\prod_{k=1}^n q_{\alpha^{(k)}} \right) \sum_{\text{pairings } \mathcal{P} \text{ of } O(\alpha^{(1)}, \dots, \alpha^{(n)})} \prod_{\{(k,i,r), (l,j,s)\} \in \mathcal{P}} \text{Cov}(Y_i^{(k)}, Y_j^{(l)}).$$

$$\prod_{k=1}^n Q(Y^{(k)}) = \sum_{\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathbb{N}_0^m} \left(\prod_{k=1}^n q_{\alpha^{(k)}} \right) \left(\prod_{k=1}^n (Y^{(k)})^{\alpha^{(k)}} \right)$$

$$C(M, A) := \frac{P(A!)}{2^{\text{tr}(M)}} \sum_{\substack{M_1, \dots, M_l \in \mathbb{N}_0^{n \times n} \\ M_1 + \dots + M_l = M \\ \forall j \in [n] k \in [l]: \\ SC(M_k, j) + SR(M_k, j) = A_{j,k}}} \prod_{i=1}^l \frac{1}{P(M_i!)}.$$

$$P(A!) = \prod_{j=1}^n \prod_{k=1}^l A_{j,k}!, P(M_i!) = \prod_{p,q=1}^n (M_i)_{p,q}!, 2^{\text{tr}(M)} = 2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}.$$



$$z := \frac{\prod_{j=1}^n \prod_{k=1}^l A_{j,k}!}{2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}} \prod_{i=1}^l \frac{1}{\prod_{p,q=1}^n (M_i)_{p,q}!}$$

$$E(c_1, \dots, c_n) := \frac{(c_1 + \dots + c_n)!}{c_1! \dots c_n!} \\ = \binom{c_1 + \dots + c_n}{c_1} \binom{c_2 + \dots + c_n}{c_2} \dots \binom{c_{n-1} + c_n}{c_{n-1}} \in \mathbb{N}.$$

$$z = \frac{\prod_{j=1}^n \prod_{k=1}^l A_{j,k}!}{2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}} \prod_{i=1}^l \frac{1}{\prod_{p,q=1}^n (M_i)_{p,q}!} \\ = \frac{1}{2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}} \prod_{i=1}^l \frac{\prod_{j=1}^n A_{j,i}!}{\prod_{p,q=1}^n (M_i)_{p,q}!} \\ = \frac{1}{2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}} \prod_{i=1}^l \prod_{j=1}^n \frac{A_{j,i}!}{\prod_{q=1}^n (M_i)_{j,q}!} \\ = \frac{1}{2^{\sum_{j=1}^n \sum_{k=1}^l (M_k)_{j,j}}} \prod_{i=1}^l \prod_{j=1}^n E((M_i)_{j,1}, \dots, (M_i)_{j,n}, (M_i)_{1,j}, \dots, (M_i)_{n,j}) \prod_{q=1}^n (M_i)_{q,j}! \\ = \prod_{i=1}^l \prod_{j=1}^n \frac{1}{2^{(M_i)_{j,j}}} E((M_i)_{j,1}, \dots, (M_i)_{j,n}, (M_i)_{1,j}, \dots, (M_i)_{n,j}) \prod_{q=1}^n (M_i)_{q,j}!$$

$$E((M_i)_{j,1}, \dots, (M_i)_{j,n}, (M_i)_{1,j}, \dots, (M_i)_{n,j}) \setminus \binom{(M_i)_{j,j} + (M_i)_{j,j}}{(M_i)_{j,j}}$$

$$\binom{(M_i)_{j,j} + (M_i)_{j,j}}{(M_i)_{j,j}} \oslash \frac{(M_i)_{j,j}!}{2^{(M_i)_{j,j}}}$$

$$\binom{y+y}{y} \frac{y!}{2^y} = \frac{(2y)!}{y! 2^y} \\ = \frac{(1 \cdot 3 \dots (2y-1)) \cdot (2 \cdot 4 \dots 2y)}{y! 2^y} \\ = \frac{(1 \cdot 3 \dots (2y-1)) \cdot (1 \cdot 2 \dots y)}{y!} \\ = 1 \cdot 3 \dots (2y-1) \in \mathbb{N}$$

$$\deg(v) = 2h(\{v\}) + \sum_{w \in V \setminus \{v\}} h(\{v, w\}),$$

$$h(\{a, v_1\}) \geq 1, h(\{v_i, v_{i+1}\}) \geq 1 \text{ for all } i = 1, \dots, n-1, h(\{v_n, b\}) \geq 1$$

$$\int_{\Gamma} f := \int_{[0,1]^l} \prod_{(\{v_a, v_b\}, j) \in O(E)} f(s_a, s_b) ds_1 \dots ds_l$$

$$M_{ij}^{\Gamma} := \begin{cases} h(\{v_i, v_j\}), & \text{if } i < j \\ h(\{v_i\}), & \text{if } i = j \\ 0, & \text{if } i > j \end{cases}$$



$$A^\Gamma := \begin{pmatrix} SC(M^\Gamma, 1) + SR(M^\Gamma, 1) \\ \vdots \\ SC(M^\Gamma, l) + SR(M^\Gamma, l) \end{pmatrix} = \begin{pmatrix} \deg(v_1) \\ \vdots \\ \deg(v_l) \end{pmatrix} \in \mathbb{N}_0^{l \times 1}.$$

$$C(\Gamma) = \frac{P(A^\Gamma!) \cdot 1}{2^{\text{tr}(M^\Gamma)} P(M^\Gamma!)}$$

$$\begin{aligned} & \int_{[0,1]^n} \sum_{\substack{\text{pairings } \mathcal{P} \\ \text{of } O(\alpha^{(1)}, \dots, \alpha^{(n)})}} \prod_{\substack{\{(k,i,r),(l,j,s)\} \in \mathcal{P} \\ \Gamma_1 = (\{v_1, \dots, v_n\}, E_1) \\ \forall j \in [n]: \deg(v_j) = \alpha_1^{(j)}}} \delta_{ij} f(s_k, s_l) ds_1 \cdots ds_n \\ &= \sum_{\substack{\Gamma_m = (\{v_1, \dots, v_n\}, E_m) \\ \forall j \in [n]: \deg(v_j) = \alpha_m^{(j)}}} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right). \end{aligned}$$

$$B_q := O\left(\left([n], k \mapsto \alpha_q^{(k)}\right)\right) \text{ for all } q \in [m].$$

$$\begin{aligned} & \int_{[0,1]^n} \sum_{\substack{\text{pairings } \mathcal{P} \\ \text{of } O(\alpha^{(1)}, \dots, \alpha^{(n)})}} \prod_{\{(k,i,r),(l,j,t)\} \in \mathcal{P}} \delta_{ij} f(s_k, s_l) ds_1 \cdots ds_n \\ &= \int_{[0,1]^n} \sum_{\substack{\text{pairings } \mathcal{P}_1 \\ \text{of } B_1}} \cdots \sum_{\substack{\text{pairings } \mathcal{P}_m \\ \text{of } B_m}} \left(\prod_{\{(k_1, r_1), (l_1, t_1)\} \in \mathcal{P}_1} f(s_{k_1}, s_{l_1}) \right) \cdots \left(\prod_{\{(k_m, r_m), (l_m, t_m)\} \in \mathcal{P}_m} f(s_{k_m}, s_{l_m}) \right) ds_1 \cdots ds_n \end{aligned}$$



$$\begin{aligned}
& \int_{[0,1]^n} \sum_{\substack{\text{pairings } \mathcal{P}_1 \\ \text{of } B_1}} \cdots \sum_{\substack{\text{pairings } \mathcal{P}_m \\ \text{of } B_m}} \\
& \left(\prod_{\{(k_1, r_1), (l_1, t_1)\} \in \mathcal{P}_1} f(s_{k_1}, s_{l_1}) \right) \cdots \left(\prod_{\{(k_m, r_m), (l_m, t_m)\} \in \mathcal{P}_m} f(s_{k_m}, s_{l_m}) \right) ds_1 \cdots ds_n \\
& = \sum_{\substack{\Gamma_1 = (\{v_1, \dots, v_n\}, E_1) \\ \forall j \in [n]: \deg(v_j) = \alpha_1^{(j)}}} \sum_{\substack{\Gamma_m = (\{v_1, \dots, v_n\}, E_m) \\ \forall j \in [n]: \deg(v_j) = \alpha_m^{(j)}}} \tilde{C}(\Gamma_1) \cdots \tilde{C}(\Gamma_m) \int_{[0,1]^n} \\
& \left(\prod_{\{(v_a, v_b), j\} \in O(E_1)} f(s_a, s_b) \right) \cdots \left(\prod_{\{(v_a, v_b), j\} \in O(E_m)} f(s_a, s_b) \right) ds_1 \cdots ds_n \\
& = \sum_{\substack{\Gamma_1 = (\{v_1, \dots, v_n\}, E_1) \\ \forall j \in [n]: \deg(v_j) = \alpha_1^{(j)}}} \sum_{\substack{\Gamma_m = (\{v_1, \dots, v_n\}, E_m) \\ \forall j \in [n]: \deg(v_j) = \alpha_m^{(j)}}} \left(\prod_{q=1}^m \tilde{C}(\Gamma_q) \right) \int_{\Gamma_1 + \cdots + \Gamma_m} f \\
& = \sum_{\substack{\Gamma_1 = (\{v_1, \dots, v_n\}, E_1) \\ \forall j \in [n]: \deg(v_j) = \alpha_1^{(j)}}} \cdots \sum_{\substack{\Gamma_m = (\{v_1, \dots, v_n\}, E_m) \\ \forall j \in [n]: \deg(v_j) = \alpha_m^{(j)}}} \left(\prod_{q=1}^m \tilde{C}(\Gamma_q) \right) \prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \cdots + \Gamma_m)} \int_{\Lambda} f.
\end{aligned}$$

$$\tilde{C}(M^{\Gamma_q}, A^{\Gamma_q}) = C(M^{\Gamma_q}, A^{\Gamma_q})$$

$$\prod_{\{(k, r), (l, t)\} \in \mathcal{P}_q} f(s_{k_r}, s_{l_t})$$

$$\prod_{\{(v_a, v_b), j\} \in O(E_q)} f(s_a, s_b)$$

$$\deg(v_i) = SC(M^{\Gamma_q}, i) + SR(M^{\Gamma_q}, i)$$

$$\prod_{\{(k, r), (l, t)\} \in \mathcal{P}_a} f(s_k, s_l) = \prod_{1 \leq k \leq l \leq n} f(s_k, s_l)^{M_{k,l}^{\Gamma_q}}$$

$$\mathcal{P}_{\sigma_1, \dots, \sigma_n} = \{ \{(k, \sigma_k(r)), (l, \sigma_l(t))\} \text{ for } \{(k, r), (l, t)\} \in \mathcal{P}_a \}$$

$$\{(k, r_1), (l, t_1)\}, \{(k, r_2), (l, t_2)\}, \dots, \{(k, r_q), (l, t_q)\} \in \mathcal{P}_a$$

$$\{(k, r_1), (k, t_1)\} \in \mathcal{P}_a$$

$$\begin{aligned}
\tilde{C}(M^{\Gamma_q}, A^{\Gamma_q}) &= \frac{\alpha_q^{(1)}! \cdots \alpha_q^{(n)}!}{P(M^{\Gamma_q}) 2^{\text{tr}(M^{\Gamma_q})}} \\
&= C(M^{\Gamma_q}, A^{\Gamma_q})
\end{aligned}$$

$$X(s) = (X_1(s), \dots, X_m(s)) \in \mathbb{R}^m, s \in [0,1]$$



$$\text{Cov}(X_i(s), X_j(t)) = f(s, t)\delta_{ij}, \text{ for } f: [0,1] \times [0,1] \rightarrow \mathbb{R},$$

$$Q(x) = \sum_{\alpha \in \mathbb{N}_0^m} q_\alpha x^\alpha$$

$$\begin{aligned} & \int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \dots ds_n \\ &= \frac{1}{n!} \sum_{\substack{\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathbb{N}_0^m \\ |\alpha^{(1)}, \dots, \alpha^{(n)}|/2 \in \mathbb{N}}} \left(\prod_{k=1}^n q_{\alpha^{(k)}} \right) \times \\ & \times \sum_{\substack{\Gamma_1 = (\{v_1, \dots, v_n\}, E_1) \\ \forall j \in [n]: \deg(v_j) = \alpha_1^{(j)}}} \dots \sum_{\substack{\Gamma_m = (\{v_1, \dots, v_n\}, E_m) \\ \forall j \in [n]: \deg(v_j) = \alpha_m^{(j)}}} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right). \\ & \int_{\{0 \leq s_1 \leq \dots \leq s_n \leq 1\}} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \dots ds_n = \frac{1}{n!} \int_{[0,1]^n} \mathbb{E} \left[\prod_{k=1}^n Q(X(s_k)) \right] ds_1 \dots ds_n \\ & Q(x) = (x, Dx) + (c, x) =: \sum_{\alpha \in \mathbb{N}_0^m} q_\alpha x^\alpha \end{aligned}$$

$$q_{e_i} = c_i, q_{2e_i} = D_{i,i}, q_{e_i+e_j} = 2D_{i,j} \text{ if } i \neq j$$

$$\Psi_1[f] := \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ |\alpha|/2 \in \mathbb{N}}} q_\alpha \sum_{\substack{\Gamma_1 = (\{v\}, E_1) \\ \deg(v) = \alpha_1}} \dots \sum_{\substack{\Gamma_m = (\{v\}, E_m) \\ \deg(v) = \alpha_m}} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right)$$

$$D_{i,i} C(\Gamma) \int_{\Gamma} f = D_{i,i} \int_{[0,1]} f(s, s) ds$$

$$\Psi_1[f] = \sum_i D_{i,i} \int_{[0,1]} f(s, s) ds.$$

$$\begin{aligned} \Psi_2[f] &:= \frac{1}{2} \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^m \\ |\alpha, \beta|/2 \in \mathbb{N}}} q_\alpha q_\beta \sum_{\substack{\Gamma_1 = (\{v, w\}, E_1) \\ \deg(v) = \alpha_1, \deg(w) = \beta_1}} \\ & \times \sum_{\substack{\Gamma_m = (\{v, w\}, E_m) \\ \deg(v) = \alpha_m, \deg(w) = \beta_m}} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right) \end{aligned}$$

$$q_{e_i} q_{e_i} = c_i^2, q_{2e_i} q_{2e_i} = D_{i,i}^2, q_{e_i+e_j} q_{e_i+e_j} = 4D_{i,j}^2 \text{ with } i \neq j$$

$$q_{2e_i} q_{2e_j} = D_{i,i} D_{j,j} \text{ with } i \neq j$$



$$\frac{1}{2} c_i^2 C(\Gamma^{(a)}) \int_{\Gamma^{(a)}} f = \frac{1}{2} c_i^2 \int_{[0,1]^2} f(s_1, s_2) ds_1 ds_2$$

$$\frac{1}{2} D_{i,i}^2 C(\Gamma^{(b_1)}) \int_{\Gamma^{(b_1)}} f = D_{i,i}^2 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2$$

$$\frac{1}{2} D_{i,i}^2 C(\Gamma^{(b_2)}) \int_{\Gamma^{(b_2)}} f = \frac{1}{2} D_{i,i}^2 \left(\int_{[0,1]^2} f(s, s) ds \right)^2$$

$$\frac{1}{2} 4D_{i,j}^2 C(\Gamma^{(a)})^2 \int_{\Gamma^{(b_1)}} f = 2D_{i,j}^2 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2$$

$$\frac{1}{2} D_{i,i} D_{j,j} C(\Gamma^{(d_1)}) C(\Gamma^{(d_2)}) \int_{\Gamma^{(b_2)}} f = \frac{1}{2} D_{i,i} D_{j,j} \left(\int_{[0,1]^2} f(s, s) ds \right)^2$$

$$\Psi_2[f] = \frac{1}{2} \sum_i c_i^2 \int_{[0,1]^2} f(s_1, s_2) ds_1 ds_2 + \sum_i D_{i,i}^2 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2$$

$$+ \frac{1}{2} \sum_i D_{i,i}^2 \left(\int_{[0,1]} f(s, s) ds \right)^2 + 2 \sum_{i \neq j} D_{i,j}^2 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2$$

$$+ \frac{1}{2} \sum_{i \neq j} D_{i,i} D_{j,j} \left(\int_{[0,1]} f(s, s) ds \right)^2$$

$$\Psi_3[f] := \frac{1}{6} \sum_{\substack{\alpha, \beta, \gamma \in \mathbb{N}_0^m \\ |O(\alpha, \beta, \gamma)|/2 \in \mathbb{N}}} q_\alpha q_\beta q_\gamma \sum_{\substack{\Gamma_1 = \{v, w, x\}, E_1 \\ \deg(v) = \alpha_1, \deg(w) = \beta_1, \deg(x) = \gamma_1}}$$

$$\times \sum_{\substack{\Gamma_m = \{v, w, x\}, E_m \\ \deg(v) = \alpha_m, \deg(w) = \beta_m, \deg(x) = \gamma_m}} \left(\prod_{q=1}^m C(\Gamma_q) \right) \left(\prod_{\Lambda \in \mathcal{C}(\Gamma_1 + \dots + \Gamma_m)} \int_{\Lambda} f \right)$$

$$q_{e_i} q_{2e_i} q_{e_i} = c_i^2 D_{i,i}, q_{e_j} q_{2e_i} q_{e_j} = D_{i,i} c_j^2, q_{e_i} q_{e_i+e_j} q_{e_j} = 2c_i c_j D_{i,j}$$

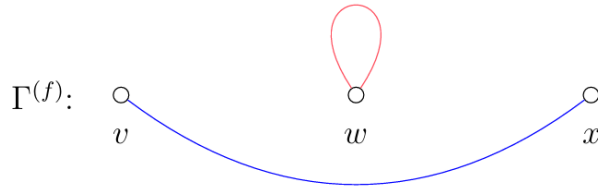
$$q_{e_i+e_j} q_{2e_i} q_{e_i+e_j} = 4D_{i,j}^2 D_{i,i}, q_{e_i+e_j} q_{2e_k} q_{e_i+e_j} = 4D_{i,j}^2 D_{k,k}, q_{2e_i} q_{2e_j} q_{2e_i} = D_{i,i}^2 D_{j,j}$$

$$q_{2e_i} q_{2e_i} q_{2e_i} = D_{i,i}^3, q_{2e_i} q_{2e_j} q_{2e_k} = D_{i,i} D_{j,j} D_{k,k}, q_{e_i+e_j} q_{e_i+e_k} q_{e_j+e_k} = 8D_{i,j} D_{i,k} D_{j,k}$$

$$\frac{3}{6} c_i^2 D_{i,i} C(\Gamma^{(e_1)}) \int_{\Gamma^{(e_1)}} f = c_i^2 D_{i,i} \int_{[0,1]^3} f(s_1, s_2) f(s_2, s_3) ds_1 ds_2 ds_3$$

$$\frac{3}{6} c_i^2 D_{i,i} C(\Gamma^{(e_2)}) \int_{\Gamma^{(e_2)}} f = \frac{1}{2} c_i^2 D_{i,i} \int_{[0,1]^2} f(s_1, s_2) ds_1 ds_2 \int_{[0,1]} f(s, s) ds$$

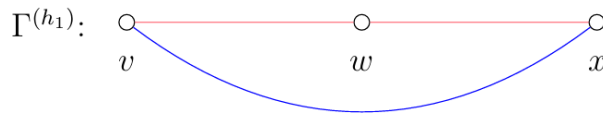




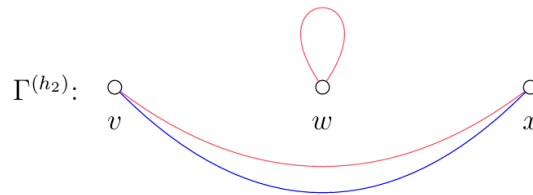
$$\frac{3}{6} D_{i,i} c_j^2 C(\Gamma_{\text{red}}^{(f)}) C(\Gamma_{\text{blue}}^{(f)}) \int_{\Gamma(f)} f = \frac{1}{2} D_{i,i} c_j^2 \int_{[0,1]} f(s,s) ds \int_{[0,1]^2} f(s_1, s_2) ds_1 ds_2.$$



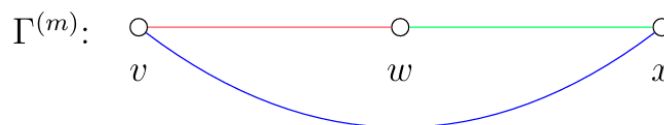
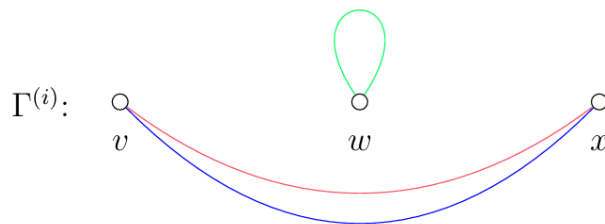
$$\frac{3}{6} 2c_i c_j D_{i,j} C(\Gamma_{\text{red}}^{(g)}) C(\Gamma_{\text{blue}}^{(g)}) \int_{\Gamma(g)} f = c_i c_j D_{i,j} \int_{[0,1]^3} f(s_1, s_2) f(s_2, s_3) ds_1 ds_2 ds_3.$$



$$\frac{3}{6} 4D_{i,j}^2 D_{i,i} C(\Gamma_{\text{red}}^{(h_1)}) C(\Gamma_{\text{blue}}^{(h_1)}) \int_{\Gamma(h_1)} f = 4D_{i,j}^2 D_{i,i} \int_{[0,1]^3} f(s_1, s_2) f(s_2, s_3) f(s_3, s_1) ds_1 ds_2 ds_3.$$



$$\frac{3}{6} 4D_{i,j}^2 D_{i,i} C(\Gamma_{\text{red}}^{(h_2)}) C(\Gamma_{\text{blue}}^{(h_2)}) \int_{\Gamma(h_2)} f = 2D_{i,j}^2 D_{i,i} \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds$$



$$\frac{3}{6} 4D_{i,j}^2 D_{k,k} C(\Gamma_{\text{red}}^{(i)}) C(\Gamma_{\text{blue}}^{(i)}) C(\Gamma_{\text{green}}^{(i)}) \int_{\Gamma(i)} f = 2D_{i,j}^2 D_{k,k} \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds.$$



$$\frac{3}{6} D_{i,i}^2 D_{j,j} C(\Gamma_{\text{red}}^{(j_1)}) C(\Gamma_{\text{blue}}^{(j_1)}) C(\Gamma_{\text{green}}^{(j_1)}) \int_{\Gamma^{(j_1)}} f = \frac{1}{2} D_{i,i}^2 D_{j,j} \left(\int_{[0,1]} f(s,s) ds \right)^3$$

$$\frac{3}{6} D_{i,i}^2 D_{j,j} C(\Gamma_{\text{red}}^{(j_2)}) C(\Gamma_{\text{blue}}^{(j_2)}) C(\Gamma_{\text{green}}^{(j_2)}) \int_{\Gamma^{(j_2)}} f = D_{i,i}^2 D_{j,j} \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds.$$

$$\frac{1}{6} D_{i,i}^3 C(\Gamma^{(k_1)}) \int_{\Gamma^{(k_1)}} f = \frac{4}{3} D_{i,i}^3 \int_{[0,1]^3} f(s_1, s_2) f(s_2, s_3) f(s_3, s_1) ds_1 ds_2 ds_3$$

$$\frac{1}{6} D_{i,i}^3 C(\Gamma^{(k_2)}) \int_{\Gamma^{(k_2)}} f = \frac{1}{6} D_{i,i}^3 \left(\int_{[0,1]} f(s,s) ds \right)^3$$

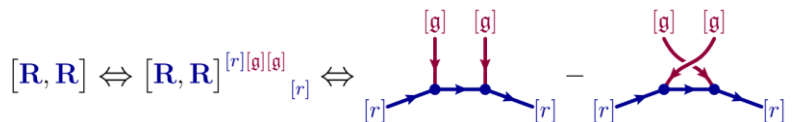
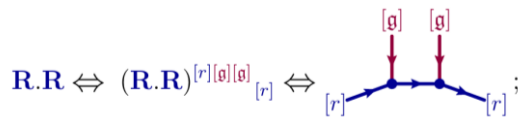
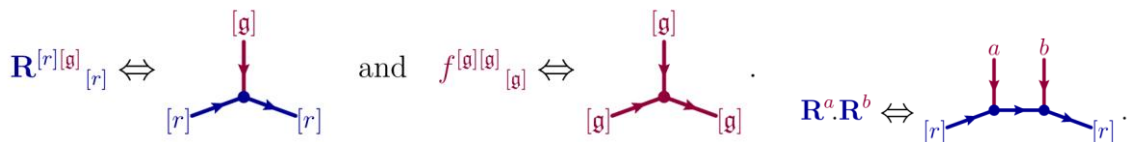
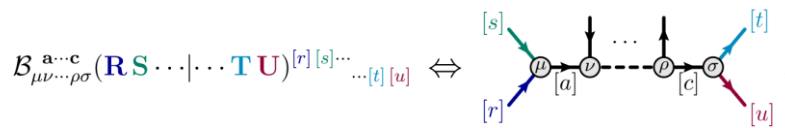
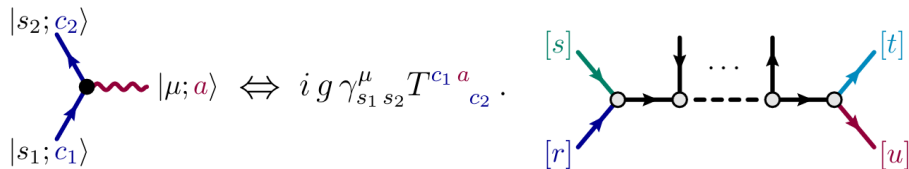
$$\frac{1}{6} D_{i,i}^3 C(\Gamma^{(k_3)}) \int_{\Gamma^{(k_3)}} f = \frac{1}{3} D_{i,i}^3 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds$$

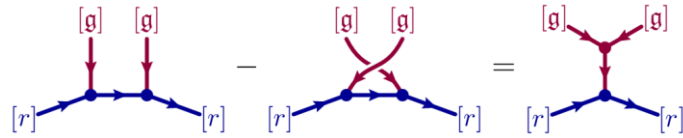
$$\frac{1}{6} D_{i,i}^3 C(\Gamma^{(k_4)}) \int_{\Gamma^{(k_4)}} f = \frac{1}{3} D_{i,i}^3 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds$$

$$\frac{1}{6} D_{i,i}^3 C(\Gamma^{(k_5)}) \int_{\Gamma^{(k_5)}} f = \frac{1}{3} D_{i,i}^3 \int_{[0,1]^2} f(s_1, s_2)^2 ds_1 ds_2 \int_{[0,1]} f(s,s) ds$$

$$\frac{1}{6} D_{i,i} D_{j,j} D_{k,k} C(\Gamma_{\text{red}}^{(l)}) C(\Gamma_{\text{blue}}^{(l)}) C(\Gamma_{\text{green}}^{(l)}) \int_{\Gamma^{(l)}} f = \frac{1}{6} D_{i,i} D_{j,j} D_{k,k} \left(\int_{[0,1]} f(s,s) ds \right)^3$$

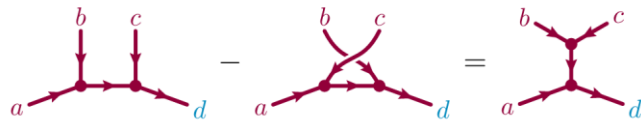
$$\frac{1}{6} 8 D_{i,j} D_{i,k} D_{j,k} C(\Gamma_{\text{red}}^{(m)}) C(\Gamma_{\text{blue}}^{(m)}) C(\Gamma_{\text{green}}^{(m)}) \int_{\Gamma^{(m)}} f = \frac{4}{3} \int_{[0,1]^3} f(s_1, s_2) f(s_2, s_3) f(s_3, s_1) ds_1 ds_2 ds_3$$





$$\mathbf{ad}_{[\mathfrak{g}]}^{[\mathfrak{g}]} := f_{[\mathfrak{g}]}^{[\mathfrak{g}]} \Leftrightarrow \begin{array}{c} [\mathfrak{g}] \\ \downarrow \\ [\mathfrak{g}] \leftarrow \rightarrow [\mathfrak{g}] \end{array}$$

$$\begin{aligned} & [[\mathbf{R}^a, \mathbf{R}^b], \mathbf{R}^c] - [[\mathbf{R}^a, \mathbf{R}^c], \mathbf{R}^b] = [\mathbf{R}^a, [\mathbf{R}^b, \mathbf{R}^c]] \\ &= \sum_{e \in [\mathfrak{g}]} (f_e^{ab} [\mathbf{R}^e, \mathbf{R}^c] - f_e^{ac} [\mathbf{R}^e, \mathbf{R}^b]) = \sum_{e \in [\mathfrak{g}]} [\mathbf{R}^a, \mathbf{R}^e] f_e^{bc} \\ &= \sum_{\substack{e \in [\mathfrak{g}] \\ d \in [\mathfrak{g}]} } (f_e^{ab} f_e^{cd} \mathbf{R}^d - f_e^{ac} f_e^{bd} \mathbf{R}^d) = \sum_{\substack{e \in [\mathfrak{g}] \\ d \in [\mathfrak{g}]} } f_e^{bc} f_e^{ad} \mathbf{R}^d. \end{aligned}$$



$$\mathbf{M} \cdot \overline{\mathbf{M}} = \sum_{r' \in [r']} \mathbf{M}_{r'}^{[r]} \overline{\mathbf{M}}_{[r]}^{r'} = \delta_{[r]}^{[r']} \Leftrightarrow [r] \rightarrow \textcircled{\mathbf{M}} \rightarrow \textcircled{\overline{\mathbf{M}}} \rightarrow [r] = [r] \rightarrow [r];$$

$$\overline{\mathbf{M}} \cdot \mathbf{M} = \sum_{r \in [r']} \overline{\mathbf{M}}_{[r']}^{r'} \mathbf{M}_{[r]}^r = \delta_{[r']}^{[r]} \Leftrightarrow [r'] \rightarrow \textcircled{\overline{\mathbf{M}}} \rightarrow \textcircled{\mathbf{M}} \rightarrow [r] = [r'] \rightarrow [r].$$

$$\mathbf{M}_{[r']}^{[r]} \Leftrightarrow [r] \rightarrow \textcircled{\mathbf{M}} \rightarrow [r']$$

$$\mathbf{R}' := \overline{\mathbf{M}} \cdot \mathbf{R} \cdot \mathbf{M} \Leftrightarrow \begin{array}{c} [\mathfrak{g}] \\ \downarrow \\ [r'] \rightarrow \rightarrow [r'] \end{array} := [r'] \rightarrow \textcircled{\overline{\mathbf{M}}} \rightarrow \begin{array}{c} [\mathfrak{g}] \\ \downarrow \\ [r'] \rightarrow \rightarrow [r'] \end{array}$$

$$\overline{\mathbf{R}} \Leftrightarrow \overline{\mathbf{R}}_{[\overline{r}]}^{[\overline{r}][\mathfrak{g}]} := -\mathbf{R}_{[r]}^{[\mathfrak{g}][r]} \Leftrightarrow \begin{array}{c} [\mathfrak{g}] \\ \downarrow \\ [\overline{r}] \rightarrow \rightarrow [\overline{r}] \end{array} := - \begin{array}{c} [\mathfrak{g}] \\ \downarrow \\ [r] \rightarrow \rightarrow [r] \end{array}$$

$$[r] \rightarrow \star \leftarrow [\overline{r}] := \delta_{[\overline{r}]}^{[r]} \quad \text{and} \quad [\overline{r}] \rightarrow \star \rightarrow [r] := \delta_{[r]}^{[\overline{r}]}$$



$$\begin{aligned}
g_{\mathbf{R}}^{[r][r]} &:= \sum_{\bar{r} \in [\bar{r}]} \mathbf{M}_{\bar{r}}^{[r]} \delta^{\bar{r}[r]} \Leftrightarrow [r] \rightarrow \circ \leftarrow [r] := [r] \rightarrow \textcircled{\mathbf{M}}_{[\bar{r}]} \star \leftarrow [r] \\
g_{[r][r]}^{\mathbf{R}} &:= \sum_{\bar{r} \in [\bar{r}]} \delta_{[r]\bar{r}} \overline{\mathbf{M}}_{\bar{r}}^{[r]} \Leftrightarrow [r] \leftarrow \circ \rightarrow [r] := [r] \leftarrow \star_{[\bar{r}]} \textcircled{\mathbf{M}} \rightarrow [r]; \\
g_{[\bar{r}][\bar{r}]}^{\overline{\mathbf{R}}} &:= \sum_{r \in [r]} \delta_{[\bar{r}]r} \mathbf{M}_{[\bar{r}]}^r \Leftrightarrow [\bar{r}] \leftarrow \circ \rightarrow [\bar{r}] := [\bar{r}] \leftarrow \star_{[r]} \textcircled{\mathbf{M}} \rightarrow [\bar{r}] \\
g_{\mathbf{R}}^{[\bar{r}][\bar{r}]} &:= \sum_{r \in [r]} \overline{\mathbf{M}}_{r}^{[\bar{r}]} \delta^{r[\bar{r}]} \Leftrightarrow [\bar{r}] \rightarrow \circ \leftarrow [\bar{r}] := [\bar{r}] \rightarrow \textcircled{\mathbf{M}}_{[r]} \star \leftarrow [\bar{r}]
\end{aligned}$$

$$[g] \rightarrow \textcircled{\Gamma} \leftarrow [g] \propto g_{\text{ad}}^{[g][g]}$$

$$\text{tr}_{\mathbf{R}}(12) := T(\mathbf{R}) g_{\text{ad}}^{[g][g]} \Leftrightarrow [g] \rightarrow \textcircled{[r]} \leftarrow [g] := T(\mathbf{R}) [g] \rightarrow \circ \leftarrow [g].$$

$$\Gamma_{[r]}^{[r]} \propto \delta_{[r]}^{[r]} \Leftrightarrow [r] \rightarrow \textcircled{\Gamma} \rightarrow [r] \propto [r] \rightarrow [r] \quad \Gamma_{[s]}^{[r]} \propto \delta_s^{\mathbf{r}} \Leftrightarrow [r] \rightarrow \textcircled{\Gamma} \rightarrow [s]$$

$$\sum_{\substack{r \in [r] \\ a, b \in [g]}} \mathbf{r}^{[r]a} \mathbf{r}^{rb} g_{ab}^{\text{ad}} := C_2(\mathbf{r}) \delta_{[r]}^{[r]} \Leftrightarrow [r] \rightarrow \textcircled{[r]} \rightarrow [r] := C_2(\mathbf{r}) [r] \rightarrow [r]$$

$$\mathbf{C}_1(\text{ad ad}|\text{ad}) := \text{ad} = f_{[g]}^{[g][g]} \Leftrightarrow [g] \rightarrow \textcircled{[g]} \rightarrow [g]$$

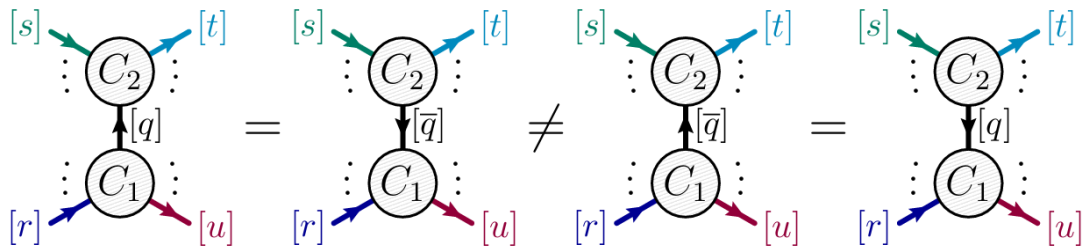
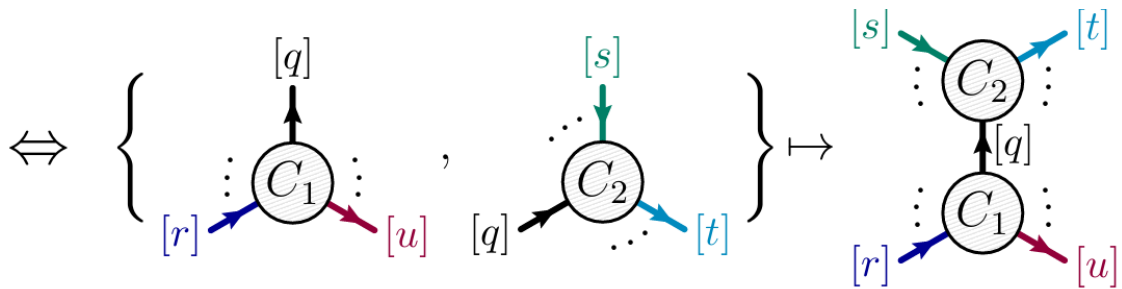
$$\mathbf{C}_2(\text{ad ad}|\text{ad}) := d_{[g]}^{[g][g]} := \sum_{a \in [g]} d_{[g]}^{[g]a} g_a^{\text{ad}} \Leftrightarrow [g] \rightarrow \textcircled{[g]} \rightarrow [g]$$

$$\mathbf{C}(\mathbf{R} \overline{\mathbf{R}}|1)^{[r][\bar{r}]}_1 := \delta^{[r][\bar{r}]}_1 = \delta^{[r][\bar{r}]} \Leftrightarrow [r] \rightarrow \textcircled{\uparrow} \leftarrow [\bar{r}] := [r] \rightarrow \star \leftarrow [\bar{r}]$$

$$\mathbf{C}(\text{ad ad}|1)^{[g][g]}_1 := g_{\text{ad}}^{[g][g]} \Leftrightarrow [g] \rightarrow \textcircled{\uparrow} \leftarrow [g] := [g] \rightarrow \circ \leftarrow [g]$$

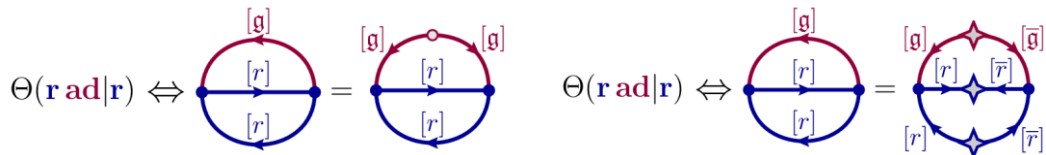


$$\mathbf{C}(\mathbf{ad} \overline{\mathbf{ad}} | \mathbf{1})^{[\mathfrak{g}] [\overline{\mathfrak{g}}]}_1 := \delta^{[\mathfrak{g}] [\overline{\mathfrak{g}}]} \Leftrightarrow [\mathfrak{g}] \xrightarrow{\circ} [\overline{\mathfrak{g}}] := [\mathfrak{g}] \xrightarrow{\star} [\overline{\mathfrak{g}}] := [\mathfrak{g}] \xrightarrow{\rightarrow} [\overline{\mathfrak{g}}]$$



$$\delta^{[r]}_{[r]} \Leftrightarrow [r] \xrightarrow{\rightarrow} [r] \Leftrightarrow \xrightarrow{\rightarrow} [r]$$

$$\dim(\mathbf{R}) := |[r]| = \sum_{r \in [r]} \delta^r_r \Leftrightarrow \bigcirc [r] = \bigcirc [\overline{r}] \Leftrightarrow \sum_{\overline{r} \in [\overline{r}]} \delta_{\overline{r}}^{\overline{r}} = |[\overline{r}]| = \dim(\overline{\mathbf{R}})$$



$$\begin{aligned} \mathbf{C}(\overline{\mathbf{r}} \overline{\mathbf{ad}} | \overline{\mathbf{r}})^{[\overline{r}] [\overline{\mathfrak{g}}]}_{[\overline{r}]} &:= \sum_{\substack{r_i \in [r] \\ a \in [\mathfrak{g}]}} \delta^{[\overline{r}] r_1} \delta^{[\overline{\mathfrak{g}}] a} \delta_{[\overline{r}] r_2} \mathbf{C}(\mathbf{r} | \mathbf{ad} \mathbf{r})^{r_2 a}_{r_1} \\ &= \sum_{\substack{r_i \in [r] \\ a \in [\mathfrak{g}]}} \delta^{[\overline{r}] r_1} \delta^{[\overline{\mathfrak{g}}] a} \delta_{[\overline{r}] r_2} \mathbf{C}(\mathbf{r} \overline{\mathbf{ad}} | \mathbf{r})^{r_2 \overline{a}}_{r_1} \\ &= \sum_{\substack{r_i \in [r] \\ a_i \in [\mathfrak{g}]}} \delta^{[\overline{r}] r_1} \delta^{[\overline{\mathfrak{g}}] a_1} g_{a_1 a_2}^{\mathbf{ad}} \delta_{[\overline{r}] r_2} \mathbf{C}(\mathbf{r} \mathbf{ad} | \mathbf{r})^{r_2 a_2}_{r_1} \end{aligned}$$

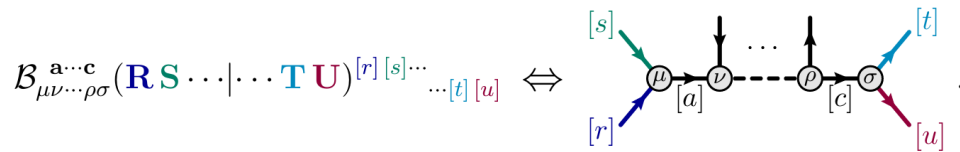
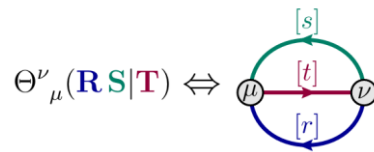
$$\overline{C(\mathbf{R} \cdots \mathbf{S} | \mathbf{T} \cdots \mathbf{U})^{[r] \cdots [s]_{[t] \cdots [u]}}} := C(\mathbf{U} \cdots \mathbf{T} | \mathbf{S} \cdots \mathbf{R})^{[u] \cdots [t]_{[s] \cdots [r]}}$$

$$= \sum_{\substack{\bar{r} \in [\bar{r}], \dots, \bar{s} \in [\bar{s}] \\ \bar{t} \in [\bar{t}], \dots, \bar{u} \in [\bar{u}]}} (\delta^{[u] \bar{u}} \cdots \delta^{[t] \bar{t}}) (\delta_{[s] \bar{s}} \cdots \delta_{[r] \bar{r}}) C(\overline{\mathbf{R}} \cdots \overline{\mathbf{S}} | \overline{\mathbf{T}} \cdots \overline{\mathbf{U}})^{\bar{r} \cdots \bar{s}_{\bar{t} \cdots \bar{u}}}$$

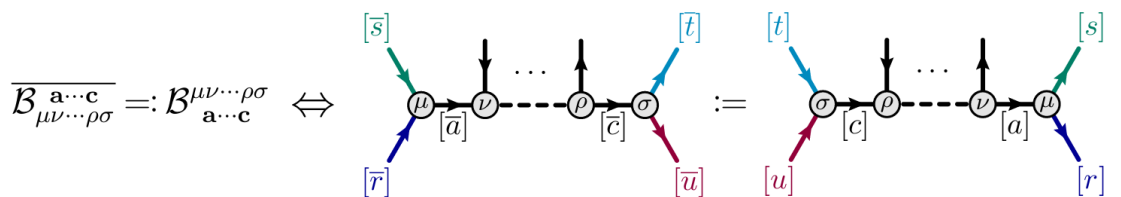
$$\langle C' | C \rangle := \sum_{\substack{r \in [r], \dots, s \in [s] \\ t \in [t], \dots, u \in [u]}} C'(\mathbf{U} \cdots \mathbf{T} | \mathbf{S} \cdots \mathbf{R})^{u \cdots t}_{s \cdots r} C(\mathbf{R} \cdots \mathbf{S} | \mathbf{T} \cdots \mathbf{U})^{r \cdots s}_{t \cdots u}$$

$$\langle C' | C \rangle := \sum_{\substack{r \in [r], \dots, s \in [s] \\ t \in [t], \dots, u \in [u]}} C'(\mathbf{U} \cdots \mathbf{T} | \mathbf{S} \cdots \mathbf{R})^{u \cdots t}_{s \cdots r} C(\mathbf{R} \cdots \mathbf{S} | \mathbf{T} \cdots \mathbf{U})^{r \cdots s}_{t \cdots u}$$

$$\Theta^\nu_\mu(\mathbf{R} \mathbf{S} | \mathbf{T}) := \langle \overline{C}^\nu(\mathbf{T} | \mathbf{R} \mathbf{S}) | C_\mu(\mathbf{R} \mathbf{S} | \mathbf{T}) \rangle := \sum_{\substack{r \in [r], s \in [s] \\ t \in [t]}} \overline{C}^\nu(\mathbf{T} | \mathbf{R} \mathbf{S})^t_{rs} C_\mu(\mathbf{R} \mathbf{S} | \mathbf{T})^{rs}_t$$



$$\mathcal{B}_{\mu\nu \cdots \rho\sigma}^{\mathbf{a} \cdots \mathbf{c}}(\mathbf{R} \mathbf{S} \mathbf{T} \cdots | \cdots) := \sum_{a \in [a]} C_\mu(\mathbf{R} \mathbf{S} | \mathbf{a})^{[r][s]}_a \mathcal{B}_{\nu \cdots \rho\sigma}^{\mathbf{b} \cdots}(\mathbf{a} \mathbf{T} \cdots | \cdots)^{a[t] \cdots}$$



$$C(\mathbf{R} \mathbf{S} \cdots | \cdots \mathbf{T} \mathbf{U}) = \sum_{\substack{\mathbf{a} \cdots \mathbf{c} \\ \mu \cdots \sigma}} \frac{\langle \mathcal{B}_{\mathbf{a} \cdots \mathbf{c}}^{\mu \cdots \sigma} | C \rangle}{\langle \mathcal{B}_{\mathbf{a} \cdots \mathbf{c}}^{\mu \cdots \sigma} | \mathcal{B}_{\mu \cdots \sigma}^{\mathbf{a} \cdots \mathbf{c}} \rangle} \mathcal{B}_{\mu \cdots \sigma}^{\mathbf{a} \cdots \mathbf{c}}(\mathbf{R} \mathbf{S} \cdots | \cdots \mathbf{T} \mathbf{U})$$

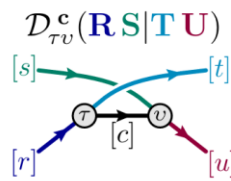
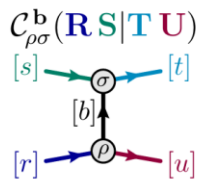
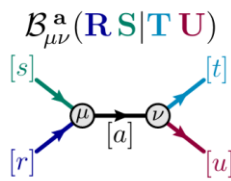
$$\text{rank}[\text{span}\{C(\mathbf{R} \cdots \mathbf{S} | \mathbf{T} \cdots \mathbf{U})\}] = m^{\mathbf{R} \otimes \cdots \otimes \mathbf{S} \otimes \overline{\mathbf{T}} \otimes \cdots \otimes \overline{\mathbf{U}}}_1$$



$$\left\{ C_\mu(\mathbf{RR}|\mathbf{T})^{r_1 r_2} t \right\}_{\substack{r_i \in [r] \\ t \in [t]}} \neq \left\{ C_\mu(\mathbf{RR}|\mathbf{T})^{r_2 r_1} t \right\}_{\substack{r_i \in [r] \\ t \in [t]}} \Leftrightarrow \begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array} \neq \begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array}$$

$$\begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array} \in \text{span}_\nu \left\{ \begin{array}{c} [r] \\ \downarrow \\ \nu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array} \right\} \quad \begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array} =: \sum_{\nu \in [m^{\mathbf{RR}_T}]} \mathcal{R}_\mu^\nu \begin{array}{c} [r] \\ \downarrow \\ \nu \xrightarrow{[t]} [t] \\ \uparrow \\ [r] \end{array}$$

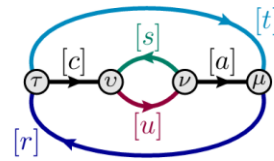
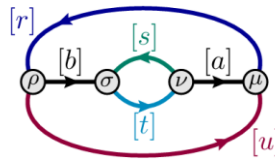
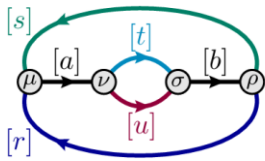
$$C_\mu(\mathbf{RS}|\mathbf{T})^{[r][s]}_{[t]} =: \sum_{\nu \in [m^{\mathbf{RS}_T}]} \mathcal{R}_\mu^\nu(\mathbf{RS}) C_\nu(\mathbf{SR}|\mathbf{T})^{[s][r]}_{[t]}$$



$$\langle \mathcal{B}_b^{\rho\sigma} | \mathcal{B}_{\mu\nu}^a \rangle$$

$$\langle \mathcal{C}_a^{\mu\nu} | \mathcal{C}_{\rho\sigma}^b \rangle$$

$$\langle \mathcal{D}_a^{\mu\nu} | \mathcal{D}_{\tau\nu}^c \rangle$$



$$\langle \mathcal{C}_b^{\rho\sigma} | \mathcal{B}_{\mu\nu}^a \rangle \Leftrightarrow \begin{array}{c} [s] \quad \sigma \quad [t] \\ \downarrow \quad \downarrow \\ \mu \xrightarrow{[a]} \nu \\ \uparrow \quad \uparrow \\ [r] \quad \rho \end{array}$$

$$\langle \mathcal{D}_c^{\tau\nu} | \mathcal{B}_{\mu\nu}^a \rangle \Leftrightarrow \begin{array}{c} [s] \quad \nu \quad [u] \\ \downarrow \quad \downarrow \\ \mu \xrightarrow{[a]} \nu \\ \uparrow \quad \uparrow \\ [r] \quad \tau \end{array}$$

$$\langle \mathcal{D}_c^{\tau\nu} | \mathcal{C}_{\rho\sigma}^b \rangle \Leftrightarrow \begin{array}{c} [u] \quad \nu \quad [s] \\ \downarrow \quad \downarrow \\ \rho \xrightarrow{[b]} \sigma \\ \uparrow \quad \uparrow \\ [r] \quad \tau \end{array}$$

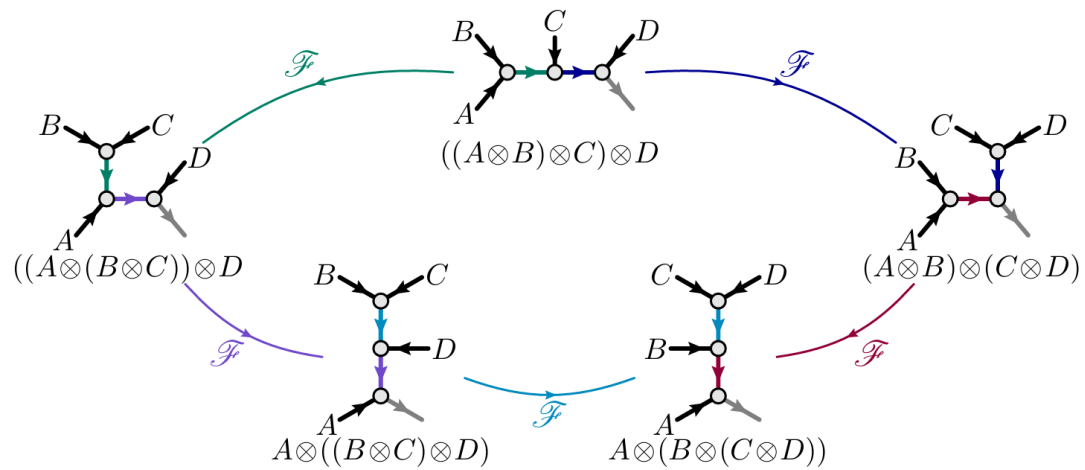
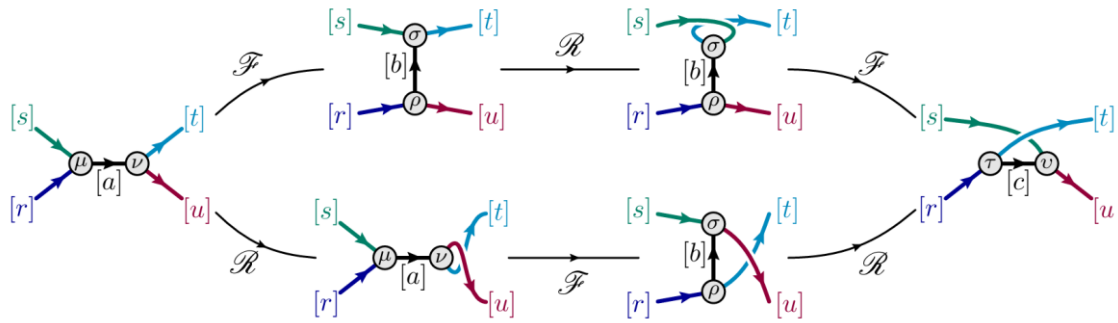
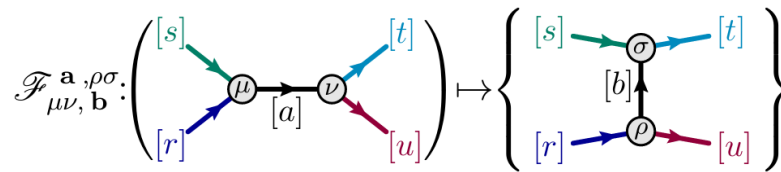
$$\mathcal{B}_{\mu\nu}^a = \sum_{\mathbf{b}, \rho, \sigma} \frac{\langle \mathcal{C}_b^{\rho\sigma} | \mathcal{B}_{\mu\nu}^a \rangle}{\langle \mathcal{C}_b^{\rho\sigma} | \mathcal{C}_{\rho\sigma}^b \rangle} \mathcal{C}_{\rho\sigma}^b = \sum_{\mathbf{c}, \tau, \nu} \frac{\langle \mathcal{D}_c^{\tau\nu} | \mathcal{B}_{\mu\nu}^a \rangle}{\langle \mathcal{D}_c^{\tau\nu} | \mathcal{D}_{\tau\nu}^c \rangle} \mathcal{D}_{\tau\nu}^c$$

$$\mathcal{F}_{\mu\nu}^{\rho\sigma}(\mathbf{qRS}|\mathbf{TUV}) := \left(\begin{array}{c} \mathbf{RS} \mathbf{q} \\ \mathbf{TUV} \end{array} \right)_{\rho\sigma \mu\nu} := \begin{array}{c} [s] \quad \sigma \quad [t] \\ \downarrow \quad \downarrow \\ \mu \xrightarrow{[q]} \nu \\ \uparrow \quad \uparrow \\ [r] \quad \tau \end{array}$$

$$\begin{array}{c} \vdots \\ \downarrow \\ \hat{\Gamma} \\ \uparrow \\ \vdots \end{array} \begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[q]} [t] \\ \uparrow \\ [s] \end{array} =: \begin{array}{c} \vdots \\ \downarrow \\ \Gamma \\ \uparrow \\ \vdots \end{array} \begin{array}{c} [r] \\ \downarrow \\ \mu \xrightarrow{[q]} [t] \\ \uparrow \\ [s] \end{array}$$



$$C(\mathbf{RS}\dots|\dots\mathbf{TU}) = \sum_{\substack{\mathbf{a}\dots\mathbf{c} \\ \mu\dots\sigma}} \frac{\langle \mathcal{B}_{\mathbf{a}\dots\mathbf{c}}^{\mu\dots\sigma} | C \rangle}{\langle \mathcal{B}_{\mathbf{a}\dots\mathbf{c}}^{\mu\dots\sigma} | \mathcal{B}_{\mu\dots\sigma}^{\mathbf{a}\dots\mathbf{c}} \rangle} \mathcal{B}_{\mu\dots\sigma}^{\mathbf{a}\dots\mathbf{c}}(\mathbf{RS}\dots|\dots\mathbf{TU})$$

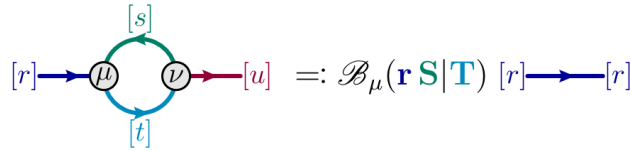


$$\mathcal{F}_{\mu\nu\rho, \mathbf{pt}}^{\mathbf{rs}, \nu\omega\tau} = \sum_{\substack{\mathbf{r}, \mathbf{t} \\ \bar{\mu}, \sigma, \tau}} \mathcal{F}_{\mu\nu\rho, \mathbf{rt}}^{\mathbf{rs}, \bar{\mu}\sigma\tau} \mathcal{F}_{\bar{\mu}\sigma\tau, \mathbf{pt}}^{\mathbf{rt}, \nu\omega\tau} = \sum_{\substack{\mathbf{q}, \mathbf{s}, \mathbf{q}, \mathbf{p} \\ \eta, \theta, \bar{\rho}, \bar{\eta}, \epsilon, \nu}} \mathcal{F}_{\mu\nu\rho, \mathbf{qs}}^{\mathbf{rs}, \eta\theta\bar{\rho}} \mathcal{F}_{\eta\theta\bar{\rho}, \mathbf{qp}}^{\mathbf{qs}, \bar{\eta}\epsilon\nu} \mathcal{F}_{\bar{\eta}\epsilon\nu, \mathbf{pt}}^{\mathbf{qp}, \nu\omega\tau}$$

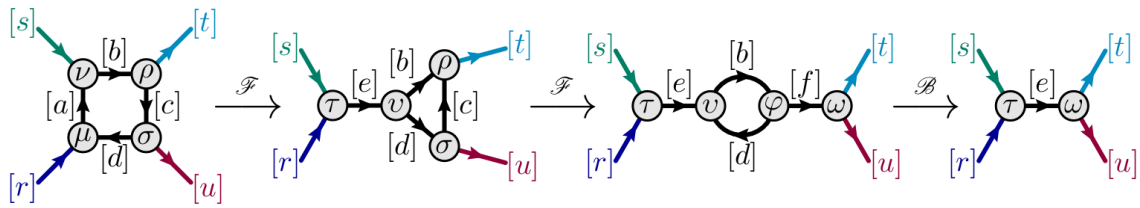
$$C(\mathbf{R}\dots\mathbf{S}\dots|\mathbf{T}\dots\mathbf{U}) = \sum_{\substack{\mathbf{a}\dots\mathbf{c} \\ \mu\dots\sigma}} \frac{\langle \mathcal{B}_{\mathbf{a}\dots\mathbf{c}}^{\mu\dots\sigma} | C \rangle}{\langle \mathcal{B}_{\mathbf{a}\dots\mathbf{c}}^{\mu\dots\sigma} | \mathcal{B}_{\mu\dots\sigma}^{\mathbf{a}\dots\mathbf{c}} \rangle} \mathcal{B}_{\mu\dots\sigma}^{\mathbf{a}\dots\mathbf{c}}(\mathbf{R}\dots\mathbf{S}|\mathbf{T}\dots\mathbf{U})$$

$$\sum_{s \in [s], t \in [t]} \bar{C}^\nu(\mathbf{T}|\mathbf{S}\mathbf{u})_{s[u]}^t C_\mu(\mathbf{r}\mathbf{S}|\mathbf{T})_{st}^{[r]} =: \mathcal{B}_\mu(\mathbf{R}\mathbf{S}|\mathbf{T}) \delta_{[u]}^{[r]} \delta_\mu^\nu \delta_{\mathbf{u}}^{\mathbf{r}}$$





$$\begin{aligned} \Theta_\mu(\mathbf{rs|t}) &= \mathcal{B}_\mu(\mathbf{rs|t}) \dim(\mathbf{r}) \\ &= \sum_{\nu \in [m^{\mathbf{r} \times \mathbf{s}}]} \mathcal{R}_\mu^\nu(\mathbf{rs}) \mathcal{B}_\nu(\mathbf{s} \mathbf{r} | \mathbf{t}) \dim(\mathbf{s}) \\ &= \mathcal{B}_\mu(\mathbf{t} \overline{\mathbf{s}} | \mathbf{r}) \dim(\mathbf{t}) \end{aligned}$$



$$(\mathfrak{so}_{10}) \left\{ \begin{array}{c} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}, \left\{ \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}, \left\{ \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}$$

$$(\mathfrak{so}_{10}) \{S^1, S^2, S^3\} := \left\{ \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}, \left\{ \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}, \left\{ \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \\ \begin{array}{c} [f] \uparrow \end{array} \begin{array}{c} [f] \uparrow \end{array} \end{array} \right\}$$

$$\begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} = \frac{1}{3} \left(2 \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} - \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \right)$$

$$\begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} = \frac{1}{3} \left(-\frac{1}{3} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} + \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} + \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \right)$$

$$\begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} = \frac{1}{2} C_2(\mathbf{ad}) \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array} \begin{array}{c} [g] \downarrow \\ \bullet \end{array}$$

$$f^{a_1 a_2 \dots a_{-2} a_{-1}} \Leftrightarrow \begin{array}{c} a_2 \quad a_3 \quad \dots \quad a_{-2} \\ \downarrow \quad \downarrow \quad \dots \quad \downarrow \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ \leftarrow \quad \leftarrow \quad \dots \quad \leftarrow \\ a_1 \quad \quad \quad \quad a_{-1} \end{array}$$



$$\begin{array}{c} [g] \\ \swarrow \\ \text{---} [g] \\ \searrow \\ [g] \end{array} = \frac{1}{T(\mathbf{R})} \left(\begin{array}{c} [g] \\ \swarrow \\ [r] \text{---} [r] \\ \searrow \\ [g] \end{array} - \begin{array}{c} [g] \\ \swarrow \\ [r] \text{---} [r] \\ \searrow \\ [g] \end{array} \right)$$

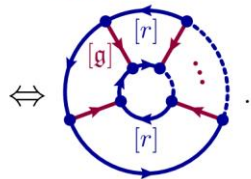
$$\text{tr}_{\mathbf{R}}(a_1, \dots, a_n) := \sum_{r_i \in [r]} (\mathbf{R}^{r_1 a_1}_{r_2} \cdots \mathbf{R}^{r_n a_n}_{r_1}) \Leftrightarrow \begin{array}{c} a_2 \quad a_3 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ a_1 \quad a_n \end{array}$$

$$\text{tr}_{\mathbf{R}}(1 \ 2 \ \cdots \ n) := \text{tr}_{\mathbf{R}}^{[g][g][g] \cdots [g]} := \{ \text{tr}_{\mathbf{R}}(a_1, \dots, a_n) \}_{a_i \in [g]} \Leftrightarrow \begin{array}{c} [g] \quad [g] \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ [g] \quad [g] \end{array}$$

$$\text{tr}_{\mathbf{R}}(a_1, \dots, a_n) = (-1)^n \text{tr}_{\mathbf{R}}(a_n, \dots, a_1) \Leftrightarrow \begin{array}{c} a_2 \quad a_3 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ a_1 \quad a_n \end{array} = (-1)^n \begin{array}{c} a_3 \quad a_2 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ a_n \quad a_1 \end{array}$$

$$\overline{\text{tr}_{\mathbf{R}}(a_1, a_2, \dots, a_n)} \Leftrightarrow \begin{array}{c} \bar{a}_2 \quad \bar{a}_3 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ \bar{a}_1 \quad \bar{a}_n \end{array} = (-1)^n \begin{array}{c} a_2 \quad a_3 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ a_1 \quad a_n \end{array} = (-1)^n \begin{array}{c} a_3 \quad a_2 \\ \swarrow \quad \swarrow \\ [r] \text{---} [r] \\ \searrow \quad \searrow \\ a_n \quad a_1 \end{array}$$

$$\langle \text{tr}_{\mathbf{R}}(1 \ 2 \ \cdots \ n) | \text{tr}_{\mathbf{R}}(1 \ 2 \ \cdots \ n) \rangle := \sum_{a_i \in [g]} \overline{\text{tr}_{\mathbf{R}}(1 \ 2 \ \cdots \ n)_{a_1 \cdots a_n}} \text{tr}_{\mathbf{R}}(1 \ 2 \ \cdots \ n)^{a_1 \cdots a_n}$$



$$\begin{array}{c} [f] \\ \swarrow \\ [g] \text{---} [f] \\ \searrow \\ [f] \end{array} = \begin{cases} \left(k \frac{(k+2)}{(k+1)} \right)^n \left(1 - \frac{1}{(-k(k+2))^{n-1}} \right) & \mathbf{a}_k \\ k^n \left(1 + \frac{k(2k+1)}{(2k)^n} + \frac{k(2k+3)}{(-2k)^n} \right) & \mathbf{b}_k \\ \frac{(2k+1)^n}{2^n} \left(1 - \frac{1-k(1+(-1)^n)}{(2k+1)^{n-1}} \right) & \mathbf{c}_k \\ \frac{(2k-1)^n}{2^n} \left(1 - \frac{1-(k+1)(1+(-1)^n)}{(2k-1)^{n-1}} \right) & \mathbf{d}_k \end{cases}$$



$$\{\mathcal{B}_\mu(\mathbf{R} \mathbf{ad} | \mathbf{R})\}_{\mu \in [m \mathbf{R} \mathbf{ad} \mathbf{R}]} =: \{\mathbf{R}^{[r] [g]}_{[r]}, \dots\} \Leftrightarrow \left\{ \begin{array}{c} [g] \\ \downarrow \\ \mu \\ \swarrow \quad \searrow \\ [r] \quad [r] \end{array} \right\}$$

$$\mathcal{B}(\mathbf{F} \mathbf{ad} | \mathbf{F}) := \{\mathbf{F}^{[f] [g]}_{[f]}\} \Leftrightarrow \left\{ \begin{array}{c} [g] \\ \downarrow \\ \bullet \\ \swarrow \quad \searrow \\ [f] \quad [f] \end{array} \right\}$$

$$(\mathbf{a}_{k>1}) \mathcal{B}_\mu(\mathbf{ad} \mathbf{ad} | \mathbf{ad}) := \{\mathbf{ad}^{[g] [g]}_{[g]}, d^{[g] [g]}_{[g]}\} \Leftrightarrow \left\{ \begin{array}{c} [g] \\ \downarrow \\ \bullet \\ \swarrow \quad \searrow \\ [g] \quad [g] \end{array}, \begin{array}{c} [g] \\ \downarrow \\ \circ \\ \swarrow \quad \searrow \\ [g] \quad [g] \end{array} \right\}$$

$$\mathcal{B}_{\mu\nu}^{\mathbf{a}}(\mathbf{R} \mathbf{S} | \mathbf{T} \mathbf{U}) \Leftrightarrow \begin{array}{c} [s] \quad [t] \\ \swarrow \quad \searrow \\ \mu \quad \nu \\ \swarrow \quad \searrow \\ [r] \quad [u] \end{array} \quad \mathcal{B}^{\mathbf{q}}(\mathbf{F} \mathbf{F} | \mathbf{F} \mathbf{F}) \Leftrightarrow \begin{array}{c} [f] \quad [f] \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ [f] \quad [f] \end{array}$$

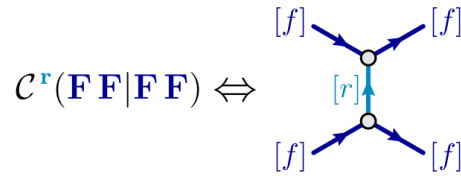
$$\{\mathbf{Q}^1, \dots, \mathbf{Q}^4\} \Leftrightarrow \left\{ \begin{array}{c} [f] \quad [f] \quad [f] \quad [f] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ [f] \quad [f] \quad [f] \quad [f] \end{array}, \begin{array}{c} [f] \quad [f] \quad [f] \quad [f] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ [f] \quad [f] \quad [f] \quad [f] \end{array}, \begin{array}{c} [f] \quad [f] \quad [f] \quad [f] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ [f] \quad [f] \quad [f] \quad [f] \end{array}, \begin{array}{c} [f] \quad [f] \quad [f] \quad [f] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ [f] \quad [f] \quad [f] \quad [f] \end{array} \right\}$$

$$\mathbf{Q}^5 \Leftrightarrow \begin{array}{c} [f] \quad [f] \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ [f] \quad [f] \end{array}$$

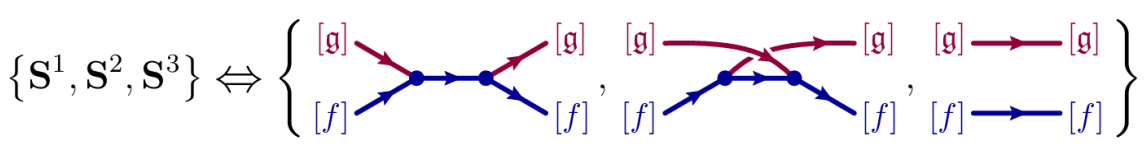
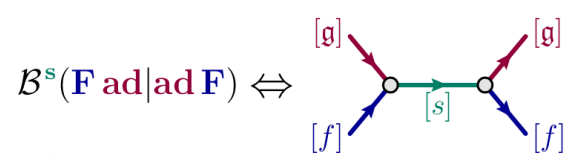
$$\begin{array}{ccccc} \mathbf{a}_k & \mathbf{b}_k & \mathbf{c}_k & \mathbf{d}_k & \mathbf{e}_6 \\ \begin{pmatrix} \frac{k+2}{k+1} & \frac{k}{k+1} \\ \frac{k+2}{k+1} & \frac{k}{k+1} \\ -1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} -\frac{2k+1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{2k+1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2k+1} & 1 & 1 \\ \frac{1}{2k+1} & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{2k+1}{4k} & \frac{1}{2} & \frac{1}{2} \\ \frac{2k+1}{4k} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2k} & 1 & -1 \\ -\frac{1}{2k} & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{2k-1}{4k} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{2k-1}{4k} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2k} & 1 & 1 \\ \frac{1}{2k} & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -13 & 1 & 2 \\ -13 & -1 & 2 \\ 9 & -9 & 9 \\ 9 & 9 & 9 \end{pmatrix} \end{array}$$

$$\begin{array}{ccccc} \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{f}_4 & \mathbf{g}_2 & \\ \begin{pmatrix} -\frac{19}{448} & -7 & 1 & 1 \\ \frac{19}{448} & -7 & -1 & 1 \\ \frac{1}{56} & 8 & -24 & 8 \\ -\frac{1}{56} & 8 & 24 & 8 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{1}{248} & -\frac{1}{2} & -1 & 0 & 1 \\ -\frac{1}{248} & \frac{1}{2} & -1 & 0 & 1 \\ \frac{1}{248} & 1 & 5 & -1 & 30 \\ \frac{1}{248} & -1 & 5 & 1 & 30 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{1}{13} & -1 & 1 & 0 & -1 \\ -\frac{1}{13} & 1 & 1 & 0 & -1 \\ \frac{1}{26} & 2 & 6 & -1 & 1 \\ \frac{1}{26} & -2 & 6 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\frac{2}{7} & 0 & 1 & 1 \\ -\frac{2}{7} & 0 & 1 & -1 \\ \frac{1}{7} & 1 & 3 & -1 \\ -\frac{1}{7} & -1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \end{array}$$





$$\begin{matrix}
 \mathbf{a}_k & \mathbf{b}_k & \mathbf{c}_k & \mathbf{d}_k & \mathbf{e}_6 \\
 \begin{pmatrix} 0 & 1 \\ \frac{k(k+2)}{(k+1)^2} & -\frac{1}{k+1} \\ 1 & 0 \\ \frac{1}{k+1} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{k}{2k+1} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2k+1} & 1 & 1 \\ \frac{1}{2k+1} & -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{2k+1}{4k} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2k} & 1 & 1 \\ \frac{1}{2k} & -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{2k-1}{4k} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2k} & 1 & 1 \\ \frac{1}{2k} & -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{26}{243} & \frac{8}{9} & -1 \\ 1 & 0 & 0 \\ \frac{1}{27} & 1 & 9 \end{pmatrix} \\
 \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{f}_4 & \mathbf{g}_2 \\
 \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{19}{448} & \frac{7}{8} & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \frac{1}{56} & 1 & 24 & 8 \\ \frac{1}{56} & -1 & 24 & -8 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{248} & \frac{1}{2} & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{248} & 1 & 5 & 1 & 30 \\ \frac{1}{248} & -1 & 5 & -1 & 30 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{13} & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{26} & 1 & 3 & 1 & 1 \\ \frac{1}{26} & -1 & 3 & -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{7} & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ \frac{1}{7} & 1 & 3 & 1 \\ \frac{1}{7} & -1 & 3 & -1 \end{pmatrix}
 \end{matrix}$$



	S_1	S_2	S_3	S_1	S_2	S_3	
\mathbf{a}_k	\mathbf{F} [10...0]	$\frac{1}{2}k(k+1)(k+3)$ [20...01]	$\frac{1}{2}(k+1)(k-1)(k+2)$ [010...01]	\mathbf{e}_6	27 [100000]	351 [000100]	1728 [100001]
\mathbf{b}_k	\mathbf{F} [10...0]	$\frac{1}{3}k(2k+1)(2k-1)$ [0010...0]	$\frac{1}{3}(2k+1)(2k-1)(2k+3)$ [110...0]	\mathbf{e}_7	56 [1000000]	912 [0000001]	6480 [1000010]
\mathbf{c}_k	\mathbf{F} [10...0]	$\frac{2}{3}k(k+1)(2k+1)$ [30...0]	$\frac{8}{3}k(k+1)(k-1)$ [110...0]	\mathbf{f}_4	26 [1000]	273 [0100]	1053 [1001]
\mathbf{d}_k	\mathbf{F} [10...0]	$\frac{2}{3}k(k-1)(2k-1)$ [0010...0]	$\frac{8}{3}k(k+1)(k-1)$ [110...0]	\mathbf{g}_2	7 [10]	27 [20]	64 [11]



$$\begin{array}{ccccc}
\mathbf{a}_1 & \mathbf{a}_{k>1} & \mathbf{b}_k & \mathbf{c}_k & \mathbf{d}_k \\
\begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \\ \frac{2}{3} & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{k(k+2)} & 1 & 1 \\ \frac{k+1}{k(k+2)} & 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2k} & 1 & 1 \\ \frac{1}{k} & -1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2k+1} & 1 & 1 \\ \frac{2}{2k+1} & 1 & -2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2k-1} & 1 & 1 \\ \frac{2}{2k-1} & -1 & 2 \end{pmatrix} \\
\mathbf{e}_6 & \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{f}_4 & \mathbf{g}_2 \\
\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{13} & 1 & 1 \\ \frac{9}{26} & -\frac{3}{2} & 6 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{19} & 1 & 1 \\ \frac{8}{19} & -2 & 12 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{248} & \frac{1}{2} & -1 & 0 & 1 \\ \frac{1}{248} & 1 & 5 & 1 & 30 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 1 \\ \frac{1}{2} & -2 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \frac{1}{2} & -\frac{3}{2} & 2 \end{pmatrix}
\end{array}$$

$$(\mathbf{a}_1) \quad \begin{array}{c} \begin{array}{c} [g] \rightarrow \bullet \\ [f] \rightarrow \bullet \end{array} \rightarrow \bullet \rightarrow \begin{array}{c} \bullet \rightarrow [g] \\ \bullet \rightarrow [f] \end{array} \\ + \quad \begin{array}{c} [g] \rightarrow \bullet \\ [f] \rightarrow \bullet \end{array} \rightarrow \bullet \rightarrow \begin{array}{c} \bullet \rightarrow [g] \\ \bullet \rightarrow [f] \end{array} \\ = \quad \begin{array}{c} [g] \rightarrow \bullet \\ [f] \rightarrow \bullet \end{array} \rightarrow \bullet \rightarrow \begin{array}{c} \bullet \rightarrow [g] \\ \bullet \rightarrow [f] \end{array} \end{array}$$

$$\mathcal{C}_\mu^r(\mathbf{F} \text{ ad} | \text{ ad } \mathbf{F}) \Leftrightarrow \begin{array}{c} [g] \rightarrow \circ \\ [r] \uparrow \circ \\ [f] \rightarrow \circ \end{array}$$

$$(\mathbf{a}_{k>1}) \quad \{\mathcal{C}_\mu^r\} \Leftrightarrow \left\{ \begin{array}{c} [g] \rightarrow \circ \\ \vdots \\ [f] \rightarrow \circ \end{array}, \begin{array}{c} [g] \rightarrow \bullet \\ [g] \uparrow \bullet \\ [f] \rightarrow \bullet \end{array}, \begin{array}{c} [g] \rightarrow \circ \\ [g] \uparrow \circ \\ [f] \rightarrow \bullet \end{array} \right\}$$

$$\begin{array}{ccccc}
\mathbf{a}_1 & \mathbf{a}_{k>1} & \mathbf{b}_k & \mathbf{c}_k & \mathbf{d}_k \\
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{k+1} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{k+1} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2k+1} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2k+1} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2k} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2k} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2k} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2k} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \\
\mathbf{e}_6 & \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{f}_4 & \mathbf{g}_2 \\
\begin{pmatrix} \frac{1}{27} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{27} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{56} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{56} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{248} & -\frac{1}{2} & 1 & 0 & -1 \\ \frac{1}{248} & \frac{1}{2} & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{26} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{26} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{7} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{7} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}
\end{array}$$

$$(\mathbf{a}_k) \quad \{\mathcal{C}_\mu^r\} \Leftrightarrow \left\{ \begin{array}{c} [g] \rightarrow \circ \\ \vdots \\ [f] \rightarrow \circ \end{array}, \begin{array}{c} [g] \rightarrow \bullet \\ [g] \uparrow \bullet \\ [f] \rightarrow \bullet \end{array}, \begin{array}{c} [g] \rightarrow \circ \\ [r] \uparrow \circ \\ [f] \rightarrow \bullet \end{array} \right\}$$



$$\mathcal{B}_{\mu\nu}^t(\text{ad ad}|\text{ad ad}) \Leftrightarrow \begin{array}{c} \begin{array}{c} \text{[g]} \\ \swarrow \\ \text{---} \circ \text{---} \\ \searrow \\ \text{[g]} \end{array} \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad \begin{array}{c} \text{---} \circ \text{---} \\ \swarrow \\ \text{[g]} \\ \searrow \\ \text{[g]} \end{array} \\ \text{[t]} \end{array}$$

	t_1	t_2	t_3	t_4	t_5	t_6
$\mathfrak{a}_{k>2}$	$\mathbf{1}$ [0...0]	\mathbf{ad} [10...01]	$\frac{1}{4}k(k+1)^2(k+4)$ [20...02]	$\frac{1}{4}k(k-1)(k+2)(k+3)$ [20...010]	$\frac{1}{4}k(k-1)(k+2)(k+3)$ [010...02]	$\frac{1}{4}(k+1)^2(k^2-4)$ [010...010]
\mathfrak{b}_k	$\mathbf{1}$ [0...0]	\mathbf{ad} [010...0]	$k(2k+3)$ [20...0]	$\frac{1}{2}k(k-1)(2k+1)(2k+3)$ [1010...0]	$\frac{1}{3}(k^2-1)(2k+1)(2k+3)$ [020...0]	$\frac{1}{6}k(k-1)(4k^2-1)$ [00010...0]
\mathfrak{c}_k	$\mathbf{1}$ [0...0]	\mathbf{ad} [20...0]	$(k-1)(2k+1)$ [010...0]	$\frac{1}{2}k(k-1)(2k+1)(2k+3)$ [210...0]	$\frac{1}{3}k(4k^3-7k+3)$ [020...0]	$\frac{1}{6}k(k+1)(2k+1)(2k+3)$ [40...0]
$\mathfrak{d}_{k>4}$	$\mathbf{1}$ [0...0]	\mathbf{ad} [010...0]	$(2k^2+k-1)$ [20...0]	$\frac{1}{2}k(k+1)(2k-1)(2k-3)$ [1010...0]	$\frac{1}{3}k(4k^3-7k-3)$ [020...0]	$\frac{1}{6}k(k-1)(2k-1)(2k-3)$ [00010...0]
\mathfrak{e}_6	$\mathbf{1}$ [000000]	\mathbf{ad} [000001]	$\mathbf{650}$ [100010]	$\mathbf{2925}$ [001000]	$\mathbf{2430}$ [000002]	
\mathfrak{e}_7	$\mathbf{1}$ [0...0]	\mathbf{ad} [0000010]	$\mathbf{1539}$ [0100000]	$\mathbf{8654}$ [0000100]	$\mathbf{7371}$ [0000020]	
\mathfrak{e}_8	$\mathbf{1}$ [0...0]	\mathbf{ad} [10000000]	$\mathbf{3875}$ [00000010]	$\mathbf{30380}$ [01000000]	$\mathbf{27000}$ [20000000]	
\mathfrak{f}_4	$\mathbf{1}$ [0000]	\mathbf{ad} [0001]	$\mathbf{324}$ [2000]	$\mathbf{1274}$ [0010]	$\mathbf{1053}$ [0002]	
\mathfrak{g}_2	$\mathbf{1}$ [00]	\mathbf{ad} [01]	$\mathbf{27}$ [20]	$\mathbf{77'}$ [30]	$\mathbf{77}$ [02]	

$$(\mathfrak{a}_1) \quad \text{ad} \otimes \text{ad} =: \begin{array}{ccc} t_1 & t_2 & t_3 \\ \mathbf{1} \oplus \mathbf{ad} \oplus \mathbf{5} \\ [0] & [2] & [4] \end{array}$$

$$(\mathfrak{a}_2) \quad \text{ad} \otimes \text{ad} =: \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \\ \mathbf{1} \oplus \mathbf{ad} \oplus \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{10} \\ [00] & [11] & [22] & [30] & [03] \end{array}$$

$$(\mathfrak{d}_4) \quad \text{ad} \otimes \text{ad} =: \begin{array}{ccccccc} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\ \mathbf{1} \oplus \mathbf{ad} \oplus \mathbf{35} \oplus \mathbf{350} \oplus \mathbf{300} \oplus \mathbf{35} \oplus \mathbf{35} \\ [0000] & [0100] & [2000] & [1011] & [0200] & [0002] & [0020] \end{array}$$

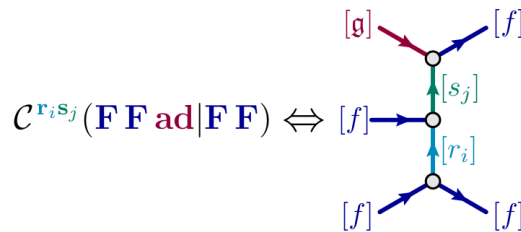
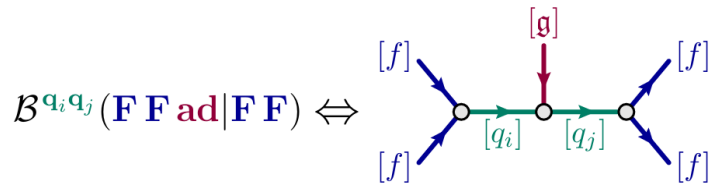
$$(\mathfrak{a}_{k>1}) \{ \mathcal{B}^1, \dots, \} \Leftrightarrow \left\{ \begin{array}{c} \text{---} \circ \text{---} \\ \swarrow \\ \text{[g]} \\ \searrow \\ \text{[g]} \end{array}, \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \begin{array}{c} \text{---} \circ \text{---} \\ \swarrow \\ \text{[g]} \\ \searrow \\ \text{[g]} \end{array}, \dots \right\}$$

$$(\mathfrak{d}_k) \quad \{ \mathcal{B}^1, \dots, \} \Leftrightarrow \left\{ \begin{array}{c} \text{---} \circ \text{---} \\ \swarrow \\ \text{[g]} \\ \searrow \\ \text{[g]} \end{array}, \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \begin{array}{c} \text{---} \circ \text{---} \\ \swarrow \\ \text{[g]} \\ \searrow \\ \text{[g]} \end{array}, \dots \right\}$$

$$\begin{matrix}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_{k>2} \\
\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{4} \\ -\frac{1}{6} & 0 & \frac{1}{4} \\ 1 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} & \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{24} & 0 & 0 & 0 & -\frac{1}{5} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{24} & 0 & 0 & 0 & -\frac{1}{5} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & -\frac{1}{6} & 0 & 0 & \frac{3}{10} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & 0 & 0 & \frac{3}{10} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} & \begin{pmatrix} \frac{1}{k+1} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{k+1} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{k+1} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{k+1} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{k+1} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{k(k+1)(k+2)} & 0 & 0 & 0 & -\frac{1}{(k-1)(k+3)} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{k(k+1)(k+2)} & 0 & 0 & 0 & -\frac{1}{(k-1)(k+3)} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{k(k+2)} & \frac{-1}{2(k+1)} & 0 & 0 & \frac{k+1}{2(k-1)(k+3)} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{k(k+2)} & \frac{1}{2(k+1)} & 0 & 0 & \frac{k+1}{2(k-1)(k+3)} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}
\end{matrix}$$

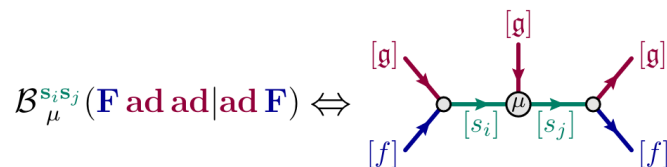
$$\begin{matrix}
\mathbf{b}_k & \mathbf{c}_k & \mathbf{d}_{k>4} \\
\begin{pmatrix} \frac{1}{2k+1} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2k+1} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2k(2k+1)} & 0 & \frac{1}{2k(2k-1)} & 0 & \frac{1}{4} & -\frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{k(2k+1)} & \frac{-1}{2k-1} & \frac{1}{2k-1} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{k(2k+1)} & \frac{1}{2k-1} & \frac{1}{2k-1} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} & \begin{pmatrix} \frac{1}{2k} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2k} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{-1}{2k(2k+1)} & 0 & \frac{-1}{4(k+1)} & 0 & \frac{1}{4} & -\frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{k(2k+1)} & \frac{-1}{2(k+1)} & \frac{1}{2(k+1)} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{k(2k+1)} & \frac{1}{2(k+1)} & \frac{1}{2(k+1)} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \end{pmatrix} & \begin{pmatrix} \frac{1}{2k} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2k} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2k(2k-1)} & 0 & \frac{1}{4(k-1)} & 0 & \frac{1}{4} & -\frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{k(2k-1)} & \frac{-1}{2(k-1)} & \frac{1}{2(k-1)} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{k(2k-1)} & \frac{1}{2(k-1)} & \frac{1}{2(k-1)} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \\
\mathbf{e}_6 & \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{f}_4 & \mathbf{g}_2 \\
\begin{pmatrix} \frac{1}{27} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{27} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{4}{351} & 0 & -\frac{5}{16} & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{78} & -\frac{1}{4} & \frac{9}{8} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{78} & \frac{1}{4} & \frac{9}{8} & \frac{1}{2} & \frac{3}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{56} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{56} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{152} & 0 & -\frac{7}{20} & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{133} & -\frac{1}{3} & \frac{9}{5} & -\frac{1}{2} & 3 \\ \frac{1}{133} & \frac{1}{3} & \frac{9}{5} & \frac{1}{2} & 3 \end{pmatrix} & \begin{pmatrix} \frac{1}{248} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{60} \\ \frac{1}{248} & -\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{60} \\ \frac{1}{496} & 0 & -\frac{3}{8} & 0 & \frac{1}{15} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{248} & -1 & \frac{25}{4} & -1 & 15 \\ \frac{1}{248} & 1 & \frac{25}{4} & 1 & 15 \end{pmatrix} & \begin{pmatrix} \frac{1}{26} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{26} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{104} & 0 & -\frac{2}{7} & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{52} & -\frac{1}{3} & \frac{9}{7} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{52} & \frac{1}{3} & \frac{9}{7} & \frac{1}{2} & \frac{3}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{7} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{7} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{7}{32} & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{14} & -\frac{1}{4} & \frac{9}{16} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{14} & \frac{1}{4} & \frac{9}{16} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\end{matrix}$$

$$\mathbf{c}[\mathbf{d}_4] = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{56} & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{28} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{28} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$



\mathfrak{g}	\mathfrak{a}_1	$\{(q_1 q_2), (q_2 q_1), (q_2 q_2)\}$	3 #
	$\mathfrak{a}_{k>1}$	$\{(q_1 q_1), (q_1 q_2), (q_2 q_1), (q_2 q_2)\}$	4
$\mathfrak{b}_k, \mathfrak{c}_k, \mathfrak{d}_k$		$\{(q_1 q_2), (q_2 q_1), (q_2 q_2), (q_2 q_3), (q_3 q_2), (q_3 q_3)\}$	6
	\mathfrak{e}_6	$\{(q_1 q_1), (q_1 q_2), (q_2 q_1), (q_2 q_2), (q_2 q_3), (q_3 q_2), (q_3 q_3)\}$	7
	$\mathfrak{e}_7, \mathfrak{g}_2$	$\{(q_1 q_2), (q_2 q_1), (q_2 q_2), (q_2 q_3), (q_3 q_2), (q_3 q_3), (q_3 q_4), (q_4 q_3), (q_4 q_4)\}$	9
	\mathfrak{f}_4	$\{(q_1 q_2), (q_2 q_1), (q_2 q_2), (q_2 q_3), (q_3 q_2), (q_3 q_3), (q_3 q_4), (q_4 q_3), (q_4 q_4), (q_4 q_5), (q_5 q_4), (q_5 q_5)\}$	12

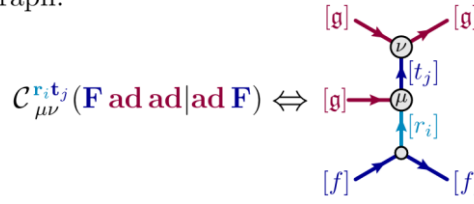
\mathfrak{g}	\mathfrak{a}_1	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2)\}$	3 #
	$\mathfrak{a}_{k>1}$	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3)\}$	4
$\mathfrak{b}_k, \mathfrak{c}_k, \mathfrak{d}_k$		$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3), (r_3 s_1), (r_3 s_3)\}$	6
	\mathfrak{e}_6	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3), (r_3 s_1), (r_3 s_2), (r_3 s_3)\}$	7
	\mathfrak{e}_7	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3), (r_3 s_1), (r_3 s_2), (r_3 s_3), (r_4 s_1), (r_4 s_3)\}$	9
	\mathfrak{f}_4	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3), (r_3 s_1), (r_3 s_2), (r_3 s_3), (r_4 s_1), (r_4 s_2), (r_4 s_3), (r_5 s_1), (r_5 s_2)\}$	12
	\mathfrak{g}_2	$\{(r_1 s_1), (r_2 s_1), (r_2 s_2), (r_2 s_3), (r_3 s_1), (r_3 s_2), (r_3 s_3), (r_4 s_1), (r_4 s_2)\}$	9



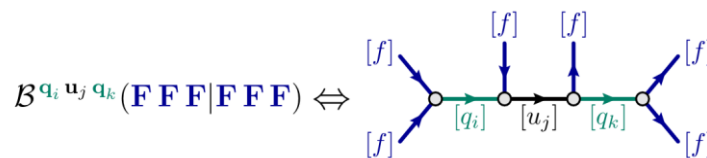
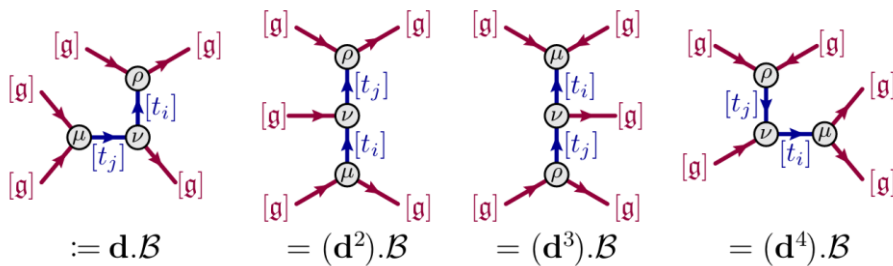
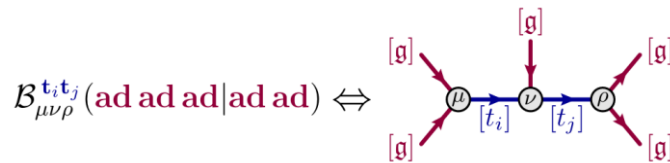
\mathfrak{g}	\mathfrak{a}_1	$\{(s_1 s_1), (s_1 s_2), (s_2 s_1), (s_2 s_2)\}$	4 #
	\mathfrak{a}_2	$\{(s_1 s_1), (s_1 s_2), (s_1 s_3), (s_2 s_1), (s_2 s_2)_{\mu \in [2]}, (s_2 s_3), (s_3 s_1), (s_3 s_2), (s_3 s_3)\}$	10
	$\mathfrak{a}_{k>2}, \mathfrak{d}_4$	$\{(s_1 s_1), (s_1 s_2), (s_1 s_3), (s_2 s_1), (s_2 s_2)_{\mu \in [2]}, (s_2 s_3), (s_3 s_1), (s_3 s_2), (s_3 s_3)_{\mu \in [2]}\}$	11
$\mathfrak{b}_k, \mathfrak{c}_k, \mathfrak{d}_{k>4}, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{f}_4, \mathfrak{g}_2$		$\{(s_1 s_1), (s_1 s_2), (s_1 s_3), (s_2 s_1), (s_2 s_2), (s_2 s_3), (s_3 s_1), (s_3 s_2), (s_3 s_3)_{\mu \in [2]}\}$	10



graph:

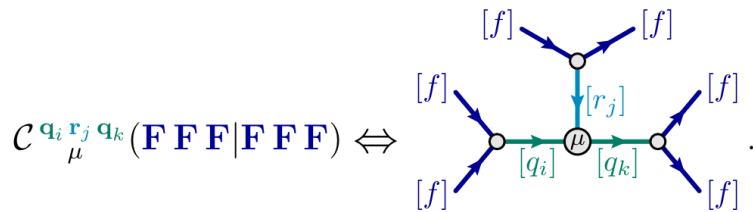


\mathfrak{g}	\mathfrak{a}_1	$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3)\}$	6
	\mathfrak{d}_4	$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_2 \mathfrak{t}_4), (\mathfrak{t}_2 \mathfrak{t}_5), (\mathfrak{t}_2 \mathfrak{t}_6), (\mathfrak{t}_2 \mathfrak{t}_7), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3), (\mathfrak{t}_3 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_2), (\mathfrak{t}_4 \mathfrak{t}_3), (\mathfrak{t}_4 \mathfrak{t}_5), (\mathfrak{t}_4 \mathfrak{t}_6), (\mathfrak{t}_4 \mathfrak{t}_7), (\mathfrak{t}_4 \mathfrak{t}_4)_{\nu \in [3]}, (\mathfrak{t}_5 \mathfrak{t}_2), (\mathfrak{t}_5 \mathfrak{t}_4), (\mathfrak{t}_5 \mathfrak{t}_5), (\mathfrak{t}_6 \mathfrak{t}_2), (\mathfrak{t}_6 \mathfrak{t}_4), (\mathfrak{t}_6 \mathfrak{t}_6), (\mathfrak{t}_7 \mathfrak{t}_2), (\mathfrak{t}_7 \mathfrak{t}_4), (\mathfrak{t}_7 \mathfrak{t}_7)\}$	28
	\mathfrak{d}_5	$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_2 \mathfrak{t}_4), (\mathfrak{t}_2 \mathfrak{t}_5), (\mathfrak{t}_2 \mathfrak{t}_6), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3), (\mathfrak{t}_3 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_2), (\mathfrak{t}_4 \mathfrak{t}_3), (\mathfrak{t}_4 \mathfrak{t}_5), (\mathfrak{t}_4 \mathfrak{t}_4)_{\nu \in [2]}, (\mathfrak{t}_4 \mathfrak{t}_6), (\mathfrak{t}_5 \mathfrak{t}_2), (\mathfrak{t}_5 \mathfrak{t}_4), (\mathfrak{t}_5 \mathfrak{t}_5), (\mathfrak{t}_6 \mathfrak{t}_2), (\mathfrak{t}_6 \mathfrak{t}_4), (\mathfrak{t}_6 \mathfrak{t}_6)_{\nu \in [2]}\}$	23
$\mathfrak{b}_k, \mathfrak{c}_k,$ $\mathfrak{d}_{k>5}$		$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_2 \mathfrak{t}_4), (\mathfrak{t}_2 \mathfrak{t}_5), (\mathfrak{t}_2 \mathfrak{t}_6), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3), (\mathfrak{t}_3 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_2), (\mathfrak{t}_4 \mathfrak{t}_3), (\mathfrak{t}_4 \mathfrak{t}_5), (\mathfrak{t}_4 \mathfrak{t}_4)_{\nu \in [2]}, (\mathfrak{t}_4 \mathfrak{t}_6), (\mathfrak{t}_5 \mathfrak{t}_2), (\mathfrak{t}_5 \mathfrak{t}_4), (\mathfrak{t}_5 \mathfrak{t}_5), (\mathfrak{t}_6 \mathfrak{t}_2), (\mathfrak{t}_6 \mathfrak{t}_4), (\mathfrak{t}_6 \mathfrak{t}_6)\}$	22
	\mathfrak{e}_6	$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_2 \mathfrak{t}_4), (\mathfrak{t}_2 \mathfrak{t}_5), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3)_{\nu \in [2]}, (\mathfrak{t}_3 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_2), (\mathfrak{t}_4 \mathfrak{t}_3), (\mathfrak{t}_4 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_5), (\mathfrak{t}_5 \mathfrak{t}_2), (\mathfrak{t}_5 \mathfrak{t}_4), (\mathfrak{t}_5 \mathfrak{t}_5)\}$	17
$\mathfrak{f}_4, \mathfrak{g}_2$	\mathfrak{e}_7	$\{(\mathfrak{t}_1 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_1), (\mathfrak{t}_2 \mathfrak{t}_2), (\mathfrak{t}_2 \mathfrak{t}_3), (\mathfrak{t}_2 \mathfrak{t}_4), (\mathfrak{t}_2 \mathfrak{t}_5), (\mathfrak{t}_3 \mathfrak{t}_2), (\mathfrak{t}_3 \mathfrak{t}_3), (\mathfrak{t}_3 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_2), (\mathfrak{t}_4 \mathfrak{t}_3), (\mathfrak{t}_4 \mathfrak{t}_4), (\mathfrak{t}_4 \mathfrak{t}_5), (\mathfrak{t}_5 \mathfrak{t}_2), (\mathfrak{t}_5 \mathfrak{t}_4), (\mathfrak{t}_5 \mathfrak{t}_5)\}$	16

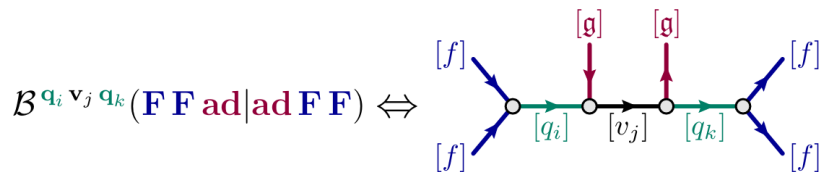


	\mathbf{u}_1	\mathbf{u}_2	\mathbf{u}_3	\mathbf{u}_4
\mathbf{a}_k	$\frac{1}{3}k(k+1)(k+2)$ [110...0]	$\frac{1}{6}k(k+1)(k+2)(k+3)$ [30...0]	$\frac{1}{6}k(k+1)(k-1)$ [0010...0]	
\mathbf{b}_k	\mathbf{F} [10...0]	$\frac{1}{3}(2k+1)(2k-1)(2k+3)$ [110...0]	$\frac{1}{3}k(2k+1)(2k+5)$ [30...0]	$\frac{1}{3}k(2k+1)(2k-1)$ [0010...0]
\mathbf{c}_k	\mathbf{F} [10...0]	$\frac{8}{3}k(k+1)(k-1)$ [110...0]	$\frac{2}{3}k(k-2)(2k+1)$ [0010...0]	$\frac{2}{3}k(k+1)(2k+1)$ [30...0]
\mathbf{d}_k	\mathbf{F} [10...0]	$\frac{8}{3}k(k+1)(k-1)$ [110...0]	$\frac{2}{3}k(k+2)(2k-1)$ [30...0]	$\frac{2}{3}k(k-1)(2k-1)$ [0010...0]

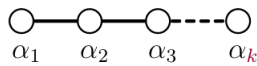
\mathfrak{g}	\mathbf{a}_1	$\{(\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_2 \mathbf{u}_2 \mathbf{q}_2)\}$	5 #
	$\mathbf{a}_{k>1}$	$\{(\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_1 \mathbf{u}_3 \mathbf{q}_1), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_2 \mathbf{u}_2 \mathbf{q}_2)\}$	6
	$\mathbf{b}_k,$	$\{(\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_1 \mathbf{u}_1 \mathbf{q}_3), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_2 \mathbf{u}_1 \mathbf{q}_3),$	15
	$\mathbf{c}_k,$	$(\mathbf{q}_2 \mathbf{u}_2 \mathbf{q}_2), (\mathbf{q}_2 \mathbf{u}_2 \mathbf{q}_3), (\mathbf{q}_2 \mathbf{u}_4 \mathbf{q}_2), (\mathbf{q}_3 \mathbf{u}_1 \mathbf{q}_1), (\mathbf{q}_3 \mathbf{u}_1 \mathbf{q}_2), (\mathbf{q}_3 \mathbf{u}_1 \mathbf{q}_3),$	
	\mathbf{d}_k	$(\mathbf{q}_3 \mathbf{u}_2 \mathbf{q}_2), (\mathbf{q}_3 \mathbf{u}_2 \mathbf{q}_3), (\mathbf{q}_3 \mathbf{u}_3 \mathbf{q}_3)\}$	



	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4
\mathbf{a}_k	$\frac{1}{6}k(k+1)(k+2)(k+4)$ [30...01]	$\frac{1}{3}(k+1)^2(k-1)(k+3)$ [110...01]	$\frac{1}{6}k(k+1)(k+2)(k-2)$ [0010...01]	
\mathbf{b}_k	$\frac{1}{2}k(2k+1)(2k+3)(k-1)$ [1010...0]	$\frac{1}{3}(2k+1)(2k+3)(k^2-1)$ [020...0]	$\frac{1}{6}k(4k^2-1)(k-1)$ [00010...0]	$\frac{1}{2}k(k+1)(2k-1)(2k+5)$ [210...0]
\mathbf{c}_k	$\frac{1}{2}k(2k+1)(2k+3)(k-1)$ [210...0]	$\frac{1}{3}k(4k^3-7k+3)$ [020...0]	$\frac{1}{6}k(k+1)(2k+1)(2k+3)$ [40...0]	$\frac{1}{2}(k+1)(k-2)(4k^2-1)$ [1010...0]
\mathbf{d}_k	$\frac{1}{2}k(k+1)(2k-1)(2k-3)$ [1010...0]	$\frac{1}{3}k(4k^3-7k-3)$ [020...0]	$\frac{1}{6}k(2k-1)(2k-3)(k-1)$ [00010...0]	$\frac{1}{2}(k+1)(k-1)(4k^2-1)$ [2110...0]



\mathfrak{a}_k : Dynkin diagram



$$w(\mathbf{ad}) := [10 \cdots 01] =: \theta$$

$$w(\mathbf{F}) := [10 \cdots 00]$$

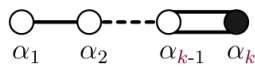
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \frac{1}{k+1}$$

weight metric

$$\begin{pmatrix} k & 1(k-1) & 1(k-2) & \cdots & 1 \\ (k-1) & 2(k-1) & 2(k-2) & \cdots & 2 \\ (k-2) & 2(k-2) & 3(k-2) & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & k \end{pmatrix}$$

\mathfrak{b}_k : Dynkin diagram



$$w(\mathbf{ad}) := [010 \cdots] =: \theta$$

$$w(\mathbf{F}) := [10 \cdots 0]$$

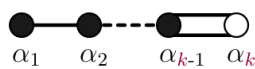
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \frac{1}{2}$$

weight metric

$$\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(k-1) & (k-1) \\ 1 & 2 & 3 & \cdots & (k-1) & k/2 \end{pmatrix}$$

\mathfrak{c}_k : Dynkin diagram



$$w(\mathbf{ad}) := [20 \cdots 0] =: \theta$$

$$w(\mathbf{F}) := [10 \cdots 0]$$

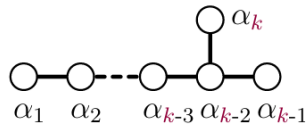
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -2 & 2 \end{pmatrix} \frac{1}{2}$$

weight metric

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & (k-1) & (k-1) \\ 1 & 2 & 3 & \cdots & (k-1) & k \end{pmatrix}$$

\mathfrak{d}_k : Dynkin diagram



$$w(\mathbf{ad}) := [010 \cdots] =: \theta$$

$$w(\mathbf{F}) := [10 \cdots 0]$$

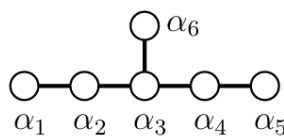
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & -1 & 2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix} \frac{1}{2}$$

weight metric

$$\begin{pmatrix} 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & \cdots & 4 & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & \cdots & 2(k-2) & (k-2) & (k-2) \\ 1 & 2 & \cdots & (k-2) & k/2 & (k-2)/2 \\ 1 & 2 & \cdots & (k-2) & (k-2)/2 & k/2 \end{pmatrix}$$

\mathfrak{e}_6 : Dynkin diagram



$$w(\mathbf{ad}) := [000001] =: \theta$$

$$w(\mathbf{F}) := [100000]$$

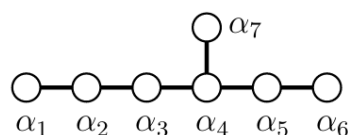
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

weight metric

$$\frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

\mathfrak{e}_7 : Dynkin diagram



$$w(\mathbf{ad}) := [0000010] =: \theta$$

$$w(\mathbf{F}) := [1000000]$$

Cartan matrix

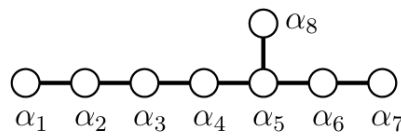
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

weight metric

$$\frac{1}{2} \begin{pmatrix} 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 4 & 8 & 10 & 12 & 8 & 4 & 6 \\ 5 & 10 & 15 & 18 & 12 & 6 & 9 \\ 6 & 12 & 18 & 24 & 16 & 8 & 12 \\ 4 & 8 & 12 & 16 & 12 & 6 & 8 \\ 2 & 4 & 6 & 8 & 6 & 4 & 4 \\ 3 & 6 & 9 & 12 & 8 & 4 & 7 \end{pmatrix}$$



\mathfrak{e}_8 : Dynkin diagram



$$w(\mathbf{ad}) := [10000000] =: \theta$$

$$w(\mathbf{F}) := [10000000]$$

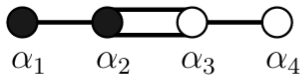
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

weight metric

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}$$

\mathfrak{f}_4 : Dynkin diagram



$$w(\mathbf{ad}) := [0001] =: \theta$$

$$w(\mathbf{F}) := [1000]$$

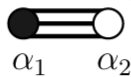
Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

weight metric

$$\frac{1}{2} \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 4 \end{pmatrix}$$

\mathfrak{g}_2 : Dynkin diagram



$$w(\mathbf{ad}) := [01] =: \theta$$

$$w(\mathbf{F}) := [10]$$

Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

weight metric

$$\frac{1}{3} \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

$$H = \int d^4\mathbf{x} \mathcal{H} \text{ and } L = \int d^4\mathbf{x} \mathcal{L}$$

$$\mathcal{L}_0 = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) = \frac{1}{2} (\dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2),$$

where $\partial^\mu = \partial/\partial x_\mu$ is the spacetime derivative, $\dot{\phi}(x) = \partial\phi/\partial t$

$$\frac{\partial \mathcal{L}_0}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi)} \right)$$

$$(\partial^\mu \partial_\mu + m^2) \phi = 0.$$

$$\mathcal{H}_0 = \pi(x) \dot{\phi}(x) - \mathcal{L}_0 = \frac{1}{2} (\pi^2 + (\nabla \phi)^2 + m^2 \phi^2)$$

$$\pi(x) = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}} = \dot{\phi}$$

$$\phi(\mathbf{x}, t) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

$$\frac{\partial^2 \phi(\mathbf{p}, t)}{\partial t^2} = -(\mathbf{p}^2 + m^2) \phi(\mathbf{p}, t)$$



$$\phi(\mathbf{x}) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} (a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}),$$

$$\pi(\mathbf{x}) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega}{2}} (a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}),$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), [a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0$$

$$H_0 = \int d^4\mathbf{x} \mathcal{H}_0 = \int \frac{d^4\mathbf{p}}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right).$$

$$\phi(x) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}) \Big|_{p^0=\omega_{\mathbf{p}}}.$$

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0$$

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_{\text{int}} \text{ or } \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!} \phi^3 \Rightarrow \mathcal{H}_{\text{int}} = \frac{\lambda}{3!} \phi^3$$

$$\phi(\mathbf{x}, t) = e^{iH(t-t_0)} \phi(\mathbf{x}, t_0) e^{-iH(t-t_0)}.$$

$$\phi_I(\mathbf{x}, t) = e^{iH_0(t-t_0)} \phi(\mathbf{x}, t_0) e^{-iH_0(t-t_0)}$$

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} = \int d^4\mathbf{x} \frac{\lambda}{3!} \phi_I^3$$

$$\phi(\mathbf{x}, t) = e^{iH_{\text{int}}(t-t_0)} \phi_I(\mathbf{x}, t) e^{-iH_{\text{int}}(t-t_0)} \equiv U^\dagger(t, t_0) \phi_I(\mathbf{x}, t) U(t, t_0)$$

$$U(t, t_0) \equiv e^{-iH_{\text{int}}(t-t_0)}$$

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \dots$$

$$\equiv T \left\{ \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\} \right\}$$

$$F_{xy}^\phi = \langle 0^\phi | T \{ \phi_x \phi_y \} | 0^\phi \rangle$$

$$N(a_{\mathbf{p}} a_{\mathbf{k}}^\dagger a_{\mathbf{q}}) = a_{\mathbf{k}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}$$

$$T\{\phi_1 \phi_2 \dots \phi_n\} = N \left(\prod_{i=1}^n \phi_i + \sum_{x=1}^n \sum_{y=x+1}^n \left(F_{xy}^\phi \prod_{\substack{i=1 \\ i \neq x, y}}^n \phi_i \right) + \dots \right)$$



$$T\{\phi_1\phi_2\phi_3\phi_4\} = N \left(\phi_1\phi_2\phi_3\phi_4 + F_{12}^\phi\phi_3\phi_4 + F_{13}^\phi\phi_2\phi_4 + F_{14}^\phi\phi_2\phi_3 \right. \\ \left. + F_{23}^\phi\phi_1\phi_4 + F_{24}^\phi\phi_1\phi_3 + F_{34}^\phi\phi_1\phi_2 \right. \\ \left. + F_{12}^\phi F_{34}^\phi + F_{13}^\phi F_{24}^\phi + F_{14}^\phi F_{23}^\phi \right)$$

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = F_{12}^\phi F_{34}^\phi + F_{13}^\phi F_{24}^\phi + F_{14}^\phi F_{23}^\phi.$$

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I},$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta^3(\mathbf{p} - \mathbf{q})\delta_{ab}$$

$$\{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = 0$$

$$\psi(x) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \sum_s (b_{\mathbf{p}}^s u_{\mathbf{p}}^s e^{-ip \cdot x} + d_{\mathbf{p}}^{s\dagger} v_{\mathbf{p}}^s e^{ip \cdot x}),$$

$$\bar{\psi}(x) = \int \frac{d^4\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \sum_s (d_{\mathbf{p}}^s \bar{v}_{\mathbf{p}}^s e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} \bar{u}_{\mathbf{p}}^s e^{ip \cdot x}),$$

Dirac spinors $u_{\mathbf{p}}^s \equiv u^s(p)$ and $v_{\mathbf{p}}^s \equiv v^s(p)$ given by $\bar{u}_{\mathbf{p}}^s = u_{\mathbf{p}}^{s\dagger} \gamma^0$ and $\bar{v}_{\mathbf{p}}^s = v_{\mathbf{p}}^{s\dagger} \gamma^0$

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

$$\{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'\dagger}\} = \{d_{\mathbf{p}}^s, d_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'}$$

$$\{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'}\} = \{d_{\mathbf{p}}^s, d_{\mathbf{p}'}^{s'}\} = \{b_{\mathbf{p}}^{s\dagger}, b_{\mathbf{p}'}^{s'\dagger}\} = \{d_{\mathbf{p}}^{s\dagger}, d_{\mathbf{p}'}^{s'\dagger}\} = 0$$

$$(b_{\mathbf{p}}^{s\dagger})^2 = (d_{\mathbf{p}}^{s\dagger})^2 = b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}'}^{s'\dagger} |0\rangle = -b_{\mathbf{p}'}^{s'\dagger} b_{\mathbf{p}}^{s\dagger} |0\rangle$$

$$D_\mu \psi = (\partial_\mu + ieA_\mu)\psi$$

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{e\bar{\psi}\gamma^\mu A_\mu\psi} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_{e\bar{\psi}\gamma^\mu A_\mu\psi} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e\bar{\psi}\gamma^\mu A_\mu\psi$$

$$\psi_i \rightarrow U_{ij}(x)\psi_j, U_{ij}(x) \in SU(4)$$

$$D_\mu \psi_i = (\partial_\mu \delta_{ij} - igG_\mu^a (T^a)_{ij})\psi_j.$$

$$[T^a, T^b] = if^{abc} T^c$$



$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + gf^{abc} G_\mu^b G_\nu^c$$

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu},$$

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_i (i\gamma^\mu \partial_\mu \delta_{ij} - m\delta_{ij}) \psi_j - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} + e\bar{\psi}_i \gamma^\mu G_\mu^a (T^a)_{ij} \psi_j.$$

Positive Wightman Function:

$$\begin{aligned} \Delta_{xy}^{\phi(>)} &\equiv \Delta^{\phi(>)}(x-y) = \langle 0^\phi | \phi(x)\phi(y) | 0^\phi \rangle \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} e^{-ik \cdot (x-y)} \Big|_{k_0=\omega_{\mathbf{k}}}; \end{aligned}$$

Negative Wightman Function:

$$\begin{aligned} \Delta_{xy}^{\phi(<)} &\equiv \Delta^{\phi(<)}(x-y) = \langle 0^\phi | \phi(y)\phi(x) | 0^\phi \rangle \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} e^{+ik \cdot (x-y)} \Big|_{k_0=\omega_{\mathbf{k}}}; \end{aligned}$$

Pauli-Jordan Function:

$$\begin{aligned} \Delta_{xy}^\phi &\equiv \Delta^\phi(x-y) = [\phi(x), \phi(y)] \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} (e^{-ik \cdot (x-y)} - e^{ik \cdot (x-y)}) \Big|_{k_0=\omega_{\mathbf{k}}} \end{aligned}$$

Hadamard Function:

$$\begin{aligned} \Delta_{xy}^{\phi(H)} &\equiv \Delta^{\phi(H)}(x-y) = \langle 0^\phi | \{\phi(x), \phi(y)\} | 0^\phi \rangle \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} (e^{-ik \cdot (x-y)} + e^{ik \cdot (x-y)}) \Big|_{k_0=\omega_{\mathbf{k}}} \\ &= 2\langle 0^\phi | \phi(x)\phi(y) | 0^\phi \rangle - \Delta_{xy}^\phi \end{aligned}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and m is the mass of the quantum field

Feynman Propagator:

$$\begin{aligned} F_{xy}^\phi &\equiv F^\phi(x-y) = \langle 0^\phi | T\{\phi(x)\phi(y)\} | 0^\phi \rangle \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} (\Theta(x_0 - y_0) e^{-ik \cdot (x-y)} \\ &\quad + \Theta(y_0 - x_0) e^{ik \cdot (x-y)}) \Big|_{k_0=\omega_{\mathbf{k}}} \\ &= i \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2} \\ &= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2 + i\epsilon}; \end{aligned}$$

Retarded Propagator:



$$\begin{aligned}
R_{xy}^\phi &\equiv R^\phi(x-y) \equiv \Delta_{xy}^R = \Theta(x_0 - y_0)\Delta(x-y) \\
&= \Theta(x_0 - y_0)[\phi(x), \phi(y)] \\
&= i \int_{C_R} \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2} \\
&= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2 + k_0 i\epsilon}
\end{aligned}$$

$$\begin{aligned}
\Delta_{xy}^\phi &\equiv [\phi_x, \phi_y] = \langle 0^\phi | [\phi_x, \phi_y] | 0^\phi \rangle = \langle 0^\phi | \phi_x \phi_y | 0^\phi \rangle - \langle 0^\phi | \phi_y \phi_x | 0^\phi \rangle \\
&= \Delta^{\phi(>)}(x-y) - \Delta^{\phi(>)}(y-x),
\end{aligned}$$

$$[\phi_x, \phi_y] = \langle 0^\phi | [\phi_x, \phi_y] | 0^\phi \rangle$$

$$\begin{aligned}
S_{xy} &\equiv S_F(x-y) = \langle 0^\psi | T\{\psi(x)\bar{\psi}(y)\} | 0^\psi \rangle \\
&= i \int \frac{d^4k}{(2\pi)^4} \frac{(\gamma^\mu k_\mu + m)e^{-ik \cdot (x-y)}}{k^2 - m^2 + i\epsilon}
\end{aligned}$$

$$\begin{aligned}
S_{xy}^R &= \Theta(x_0 - y_0)\langle 0^\psi | \{\psi(x), \bar{\psi}(y)\} | 0^\psi \rangle \\
&= i \int \frac{d^4k}{(2\pi)^4} \frac{(\gamma^\mu k_\mu + m)e^{-ik \cdot (x-y)}}{k^2 - m^2 + k_0 i\epsilon}, (2.3.9)
\end{aligned}$$

$$\begin{aligned}
D_{\mu\nu}^{xy} &\equiv D_{\mu\nu}^F(x-y) = \langle 0 | T\{A_\mu(x)A_\nu(y)\} | 0 \rangle \\
&= -i \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}
\end{aligned}$$

$$\begin{aligned}
D_{\mu\nu}^{ab}(x-y) &= \langle 0 | T\{A_\mu^a(x)A_\nu^b(y)\} | 0 \rangle \\
&= -i\delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}
\end{aligned}$$

$$S \equiv \lim_{t \rightarrow \infty} U(t, -t) = T \left\{ \exp \left(-i \int_{-\infty}^{\infty} d^4x H_{\text{int}}(x) \right) \right\}$$

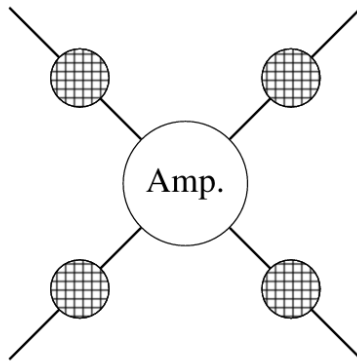
$$S_{fi} = \langle f | S | i \rangle$$

$$T_{fi} = \langle f | T | i \rangle,$$

$$T_{fi} = (2\pi)^4 \delta^4 \left(\sum p_i - \sum p_f \right) \mathcal{M}_{fi}$$

$$\begin{aligned}
\langle \mathbf{q}_1 \dots \mathbf{q}_n | S | \mathbf{p}_1 \dots \mathbf{p}_N \rangle &= \left(\prod_{k=1}^n \int d^4x_k e^{-iq_k \cdot x_k} \frac{(q_k^2 - m_k^2 + i\epsilon)}{i\sqrt{Z_k}} \right) \\
&\times \left(\prod_{j=1}^N \int d^4y_j e^{ip_j \cdot y_j} \frac{(p_j^2 - m_j^2 + i\epsilon)}{i\sqrt{Z_j}} \right) \\
&\times \langle 0 | T\{\phi(x_1) \dots \phi(x_n)\phi(y_1) \dots \phi(y_N)\} | 0 \rangle
\end{aligned}$$





$$\frac{iZ}{p_1^2 - m^2 + i\epsilon} \frac{iZ}{p_2^2 - m^2 + i\epsilon} \frac{iZ}{q_1^2 - m^2 + i\epsilon} \frac{iZ}{q_2^2 - m^2 + i\epsilon}$$

$$\langle \mathbf{q}_1 \cdots \mathbf{q}_n | S | \mathbf{p}_1 \mathbf{p}_2 \rangle = (\sqrt{Z})^{n+2}$$

Internal propagator:

$$\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - m^2 + i\epsilon};$$

Vertex:

$$= -i\lambda;$$

$\int d^D k / (2\pi)^D$, in D dimensions

Internal dark particle propagator:

$$\bullet \xrightarrow{p} \bullet = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon};$$



Internal white particle propagator:

$$\begin{array}{c} \xrightarrow{q} \\ \bullet \text{---} \text{wavy} \text{---} \bullet \end{array} = \frac{-i \left(g^{\mu\nu} + (\xi - 1) \frac{q^\mu q^\nu}{q^2} \right)}{q^2 + i\epsilon};$$

Dark/White particle vertex:

$$\begin{array}{c} \text{---} \text{solid} \text{---} \\ \bullet \\ \text{---} \text{wavy} \text{---} \bullet \\ \text{---} \text{solid} \text{---} \end{array} = -ie\gamma^\mu;$$

External hyperparticle incoming:

$$\begin{array}{c} \xrightarrow{p} \\ \text{---} \text{solid} \text{---} \bullet \end{array} = u_s(p);$$

External hyperparticle outgoing:

$$\begin{array}{c} \xrightarrow{p} \\ \bullet \text{---} \text{solid} \text{---} \end{array} = \bar{u}_s(p);$$

External incoming superparticle:

$$\begin{array}{c} \xrightarrow{p} \\ \text{---} \text{solid} \text{---} \bullet \end{array} = \bar{v}_s(p); \quad \begin{array}{c} \xrightarrow{p} \\ \bullet \text{---} \text{wavy} \text{---} \end{array} = \epsilon^\mu(p);$$

External outgoing superparticle:

$$\begin{array}{c} \xrightarrow{p} \\ \bullet \text{---} \text{solid} \text{---} \end{array} = v_s(p); \quad \begin{array}{c} \xrightarrow{p} \\ \bullet \text{---} \text{wavy} \text{---} \bullet \end{array} = \epsilon^{*\mu}(p);$$

$\int d^D k / (2\pi)^D$, in D dimensions

Quark propagator:



$$\bullet \xrightarrow{p} \bullet = \frac{i\delta^{ij}(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon};$$

Gluon propagator:

$$\bullet \xrightarrow{q} \bullet = \frac{-i\delta^{ab}}{q^2 + i\epsilon} \left(g^{\mu\nu} + (\xi - 1) \frac{q^\mu q^\nu}{q^2 + i\epsilon} \right)$$

Quark-gluon vertex:

$$\begin{array}{c} \text{wavy line} \text{---} \bullet \begin{array}{l} \nearrow \text{solid line} \\ \searrow \text{solid line} \end{array} \\ = ig\gamma^\mu (T^a)_{ij}; \end{array} \quad \begin{array}{c} \text{wavy line} \text{---} \bullet \begin{array}{l} \nearrow \text{solid line} \\ \searrow \text{solid line} \end{array} \\ = -iee_q\gamma^\mu \delta^{ij}; \end{array}$$

$\int d^D k / (2\pi)^D$, in D dimensions

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{2} m_B^2 \phi_B^2 - \frac{\lambda_B}{3!} \phi_B^3$$

$$\mathcal{L} = \frac{1}{2} Z_2 \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} Z_0 m_R^2 \phi_R^2 - \frac{Z_1 \lambda_R}{3!} \phi_R^3,$$

$$\mathcal{L} = \mathcal{L}_R \oplus \mathcal{L}_{ct}$$

$$\mathcal{L}_R = \frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_R^2 \phi_R^2 - \frac{\lambda_R}{3!} \phi_R^3,$$

$$\mathcal{L}_{ct} = \frac{1}{2} (Z_2 - 1) \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} (Z_0 - 1) m_R^2 \phi_R^2 - \frac{1}{3!} (Z_1 - 1) \lambda_R \phi_R^3.$$

$$\prod_{i=1}^n \frac{1}{D_i} = \int_0^1 dx_1 \cdots \int_0^1 dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{(n-1)!}{(\sum_{i=1}^n x_i D_i)^n}$$

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \delta(1-x-y) \frac{1}{(xA+yB)^2} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2}$$

$$J(D, \alpha, \beta, a^2) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha}{(k^2 - a^2 + i\epsilon)^\beta}$$

$$R_q = \frac{\sigma(e^+e^- \rightarrow q\bar{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$



$$\begin{aligned}
k^\mu &= (k^0, \mathbf{k}) = (ik_E^0, \mathbf{k}_E) \\
\Rightarrow k^2 &= (k^0)^2 - \mathbf{k}^2 = -(k_E^0)^2 - \mathbf{k}_E^2 \equiv -k_E^2, \\
dk^0 &= i dk_E^0 \Rightarrow d^D k = i d^D k_E.
\end{aligned}$$

$$\begin{aligned}
J(D, \alpha, \beta, a^2) &= i \int \frac{d^D k_E}{(2\pi)^D} \frac{(-1)^\alpha (k_E^2)^\alpha}{(-k_E^2 - a^2)^\beta} \\
&= i(-1)^{\alpha-\beta} \int \frac{d^D k_E}{(2\pi)^D} \frac{(k_E^2)^\alpha}{(k_E^2 + a^2)^\beta}
\end{aligned}$$

$$\begin{aligned}
J(D, \alpha, \beta, a^2) &= \frac{i(-1)^{\alpha-\beta}}{(2\pi)^D} \int d\Omega_D \int_0^\infty d|k_E| |k_E|^{D-1} \frac{|k_E|^{2\alpha}}{(|k_E|^2 + a^2)^\beta} \\
&= \frac{2i(-1)^{\alpha-\beta}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty d|k_E| \frac{|k_E|^{2\alpha+D-1}}{(|k_E|^2 + a^2)^\beta}
\end{aligned}$$

$$\int d\Omega_D = 2\pi^{D/2} / \Gamma(D/2)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \operatorname{Re}\{z\} > 0$$

$$u \equiv \frac{a^2}{(|k_E|^2 + a^2)},$$

$$\lim_{|k_E| \rightarrow \infty} u = 0$$

$$\lim_{|k_E| \rightarrow 0} u = 1$$

$$|k_E| = \left(\frac{a^2}{u} (1-u) \right)^{1/2}$$

$$du = \frac{-2a^2 |k_E|}{(|k_E|^2 + a^2)^2} d|k_E| \Rightarrow d|k_E| = -\frac{1}{2} du (1-u)^{-1/2} (a^{-2}) \left(\frac{a^2}{u} \right)^{3/2}$$

$$J(D, \alpha, \beta, a^2) = \frac{i(-1)^{\alpha-\beta} (a^2)^{\alpha-\beta+D/2}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^1 du u^{\beta-\alpha-D/2-1} (1-u)^{\alpha+D/2-1}$$

$$B(m, n) = \int_0^1 du u^{m-1} (1-u)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$$

$$J(D, \alpha, \beta, a^2) = \frac{i}{(4\pi)^{D/2}} (a^2)^{D/2} (-a^2)^{\alpha-\beta} \frac{\Gamma(\beta - \alpha - D/2)\Gamma(\alpha + D/2)}{\Gamma(\beta)\Gamma(D/2)}.$$

$$\Gamma(z) = \frac{1}{z} - \gamma_E + \mathcal{O}(z)$$

$$\Gamma(-\varepsilon_{UV}) = -\frac{1}{\varepsilon_{UV}} - \gamma_E + \mathcal{O}(\varepsilon_{UV}) \text{ and}$$

$$\Gamma(\varepsilon_{IR}) = \frac{1}{\varepsilon_{IR}} - \gamma_E + \mathcal{O}(\varepsilon_{IR})$$



$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha k^\mu k^\nu}{(k^2 - a^2 + i\epsilon)^\beta} = \frac{g^{\mu\nu}}{D} J(D, \alpha + 1, \beta, a^2)$$

$$I_{UV}^\mu \equiv \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + i\epsilon)((k+p)^2 - m^2 + i\epsilon)}$$

$$I_{UV}^\mu = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{[(k+px)^2 - a^2 + i\epsilon]^2}$$

$$= \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu - xp^\mu}{[k^2 - a^2 + i\epsilon]^2}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^n k^\mu}{[k^2 - a^2 + i\epsilon]^m} = (a^2)^{\alpha - \beta + D/2}$$

$$I_{UV}^\mu = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{-xp^\mu}{[k^2 - a^2 + i\epsilon]^2} = -p^\mu \int_0^1 dx x J(D, 0, 2, a^2)$$

$$= \frac{-ip^\mu}{(4\pi)^{D/2}} \Gamma(2 - D/2) \int_0^1 dx x (-p^2 x(1-x) + m^2 x)^{D/2-2}$$

$$I_{IR} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i\epsilon)((k+p_1)^2 + i\epsilon)((k+p_2)^2 + i\epsilon)}$$

$$\frac{1}{k^2(2k \cdot p_1)(2k \cdot p_2)},$$

$$I_{IR} = \int_0^1 dx \int_0^1 dy \int \frac{d^D k}{(2\pi)^D} \frac{2x}{[(k+p_1xy - p_3x(1-y))^2 - a^2 + i\epsilon]^3}$$

$$\text{where } a^2 = -2(p_1 \cdot p_2)x^2y(1-y)$$

$$k \rightarrow k - p_1xy + p_2x(1-y)$$

$$I_{IR} = \int_0^1 dx \int_0^1 dy \int \frac{d^D k}{(2\pi)^D} \frac{2x}{[k^2 - a^2 + i\epsilon]^3} = 2 \int_0^1 dx \int_0^1 dy x J(D, 0, 3, a^2)$$

$$I_{IR} = \frac{-i}{(4\pi)^{D/2}} \Gamma(3 - D/2) (-2p_1 \cdot p_2)^{D/2-3} \int_0^1 dx x^{D-5} \int_0^1 dy y^{D/2-3} (1-y)^{D/2-3}$$

$$I_{IR} = \frac{i}{(4\pi)^2} \frac{1}{2p_2 \cdot p_3} \left(\frac{-2p_1 \cdot p_2}{4\pi} \right)^{\epsilon_{IR}} \frac{\Gamma(1 - \epsilon_{IR})}{\Gamma(1 + 2\epsilon_{IR})} \Gamma^2(\epsilon_{IR}).$$

$$\Gamma(\epsilon_{IR}) = \frac{1}{\epsilon_{IR}} - \gamma_E + \frac{1}{12} (\pi^2 + 6\gamma_E^2) \epsilon_{IR} + \mathcal{O}(\epsilon_{IR}^2),$$

$$\Gamma(1 \mp \epsilon_{IR}) = 1 \pm \gamma_E \epsilon_{IR} + \frac{1}{12} (\pi^2 + 6\gamma_E^2) \epsilon_{IR}^2 + \mathcal{O}(\epsilon_{IR}^3),$$

$$(f)^{\epsilon_{IR}} = 1 + \epsilon_{IR} \ln f + \frac{\epsilon_{IR}^2}{2!} (\ln f)^2 + \dots,$$

$$\ln(-1) = \ln(e^{i\pi}) = -i\pi,$$



$$\ln(-1) = -i\pi + 2in\pi, \forall n \in \mathbb{Z}$$

$$\begin{aligned}
C_1 &\equiv \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i\epsilon)((k + q_1)^2 + i\epsilon)((k - q_2)^2 + i\epsilon)} \\
&= \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi}\right)^{\epsilon_{\text{IR}}} \frac{\Gamma(1 - \epsilon_{\text{IR}})}{\Gamma(1 + 2\epsilon_{\text{IR}})} \Gamma^2(\epsilon_{\text{IR}}) \\
C_2 &\equiv \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + i\epsilon)((k + q_1)^2 + i\epsilon)((k - q_2)^2 + i\epsilon)} \\
&= (q_2^\mu - q_1^\mu) \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi}\right)^{\epsilon_{\text{IR}}} \frac{\Gamma(1 + \epsilon_{\text{IR}})\Gamma(1 - \epsilon_{\text{IR}})}{\Gamma(2 + 2\epsilon_{\text{IR}})} \Gamma(\epsilon_{\text{IR}}) \\
C_3 &\equiv \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 + i\epsilon)((k + q_1)^2 + i\epsilon)((k - q_2)^2 + i\epsilon)} \\
&= \frac{-i}{(4\pi)^2} (2q_1 \cdot q_2)^{\epsilon_{\text{UV}}} \left[\frac{1}{\hat{\epsilon}_{\text{UV}}} - 2 - i\pi \right] \\
C_4 &\equiv \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 + i\epsilon)((k + q_1)^2 + i\epsilon)((k - q_2)^2 + i\epsilon)} \\
&= \frac{g^{\mu\nu}}{4} \frac{-i}{(4\pi)^2} (2q_1 \cdot q_2)^{\epsilon_{\text{UV}}} \left[\frac{1}{\hat{\epsilon}_{\text{UV}}} - 3 - i\pi \right] \\
&\quad + \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi}\right)^{\epsilon_{\text{IR}}} \frac{\Gamma(1 - \epsilon_{\text{IR}})}{\Gamma(3 + 2\epsilon_{\text{IR}})} \\
&\quad \times \left[(q_1^\mu q_1^\nu + q_2^\mu q_2^\nu) \Gamma(2 + \epsilon_{\text{IR}}) \Gamma(\epsilon_{\text{IR}}) - (q_1^\mu q_2^\nu + q_1^\nu q_2^\mu) \Gamma^2(1 + \epsilon_{\text{IR}}) \right]
\end{aligned}$$

$$\frac{1}{\hat{\epsilon}} \equiv \frac{1}{\epsilon} + \gamma_E - \ln(4\pi)$$

$$(2q_1 \cdot q_2)^{\epsilon_{\text{UV}}} \approx 1 + \epsilon_{\text{UV}} \ln(2q_1 \cdot q_2).$$

$$\sigma(e^+ e^- \rightarrow \gamma^* \rightarrow X) = \frac{1}{2s} \int \text{dPS} |\mathcal{M}_{fi}^{\text{LSZ}}|^2$$

$$e^- e^+ \rightarrow \gamma^* \rightarrow q \bar{q} X$$

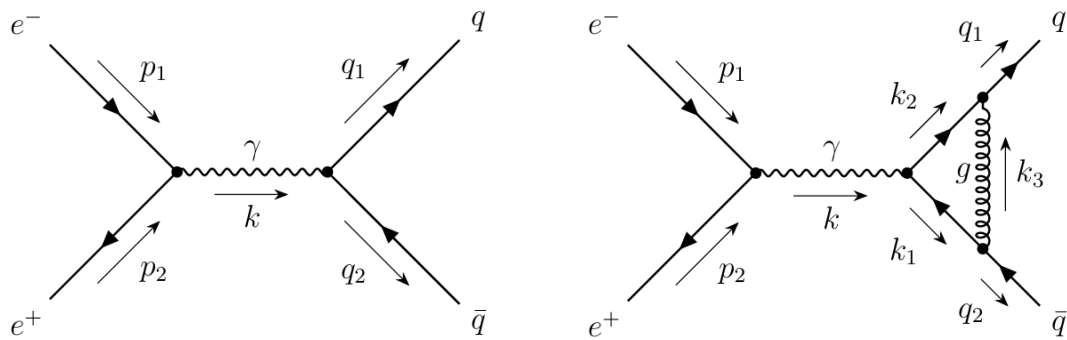
$$\begin{aligned}
\mathcal{L} &= \underbrace{\sum_{l=e, m} (\bar{\psi}_l (i\gamma^\mu \partial_\mu - m_l) \psi_l) + \bar{\psi}_q^i (i\gamma^\mu \partial_\mu - m_q) \psi_q^i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}}_{\bar{\psi}_e (i\gamma^\mu \partial_\mu - m_e) \psi_e \ominus e \bar{\psi}_e \gamma^\mu A_\mu \psi_e} \\
&\quad + \underbrace{\sum_{l=e, m} (-e \bar{\psi}_l \gamma^\mu A_\mu \psi_l) - e_q e \bar{\psi}_q^i \gamma^\mu A_\mu \psi_q^i + g \bar{\psi}_q^i \gamma^\mu G_\mu^a (T^a)^{ij} \psi_q^j}_{\bar{\psi}_m (i\gamma^\mu \partial_\mu - m_m) \psi_m \ominus e \bar{\psi}_m \gamma^\mu A_\mu \psi_m}
\end{aligned}$$

$$\int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} * \bar{\psi}_q^i (i\gamma^\mu \partial_\mu - m_q) \psi_q^i \int -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} - e_q e \bar{\psi}_q^i \gamma^\mu A_\mu \psi_q^i * g \bar{\psi}_q^i \gamma^\mu G_\mu^a (T^a)^{ij} \psi_q^j$$

$$\langle \mathbf{q}_1 \mathbf{q}_2 | S | \mathbf{p}_1 \mathbf{p}_2 \rangle = Z_e Z_q \sum (\text{Feynman diagrams})$$

$$\mathcal{M}_{fi}^{\text{LSZ}} = Z_e Z_q \sum_{\text{Feynman}} \mathcal{M}_{fi}$$





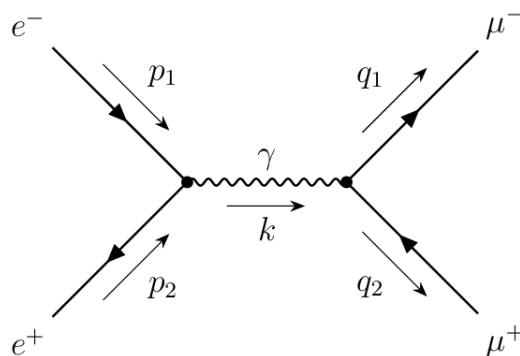
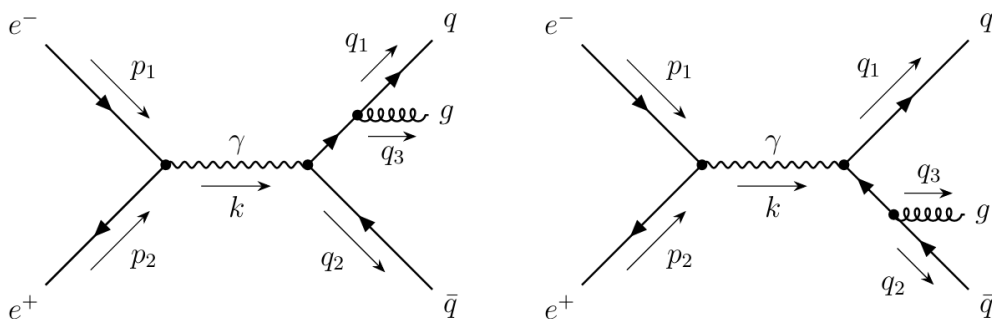
$$\begin{aligned} \mathcal{M}_{fi}^{\text{LSZ}} &= Z(\mathcal{M}_{fi}^{\text{tree}} + \mathcal{M}_{fi}^{\text{vertex}}) + \dots \\ &= \mathcal{M}_{fi}^{\text{tree}} + \delta Z \mathcal{M}_{fi}^{\text{tree}} + \mathcal{M}_{fi}^{\text{vertex}} + \dots \end{aligned}$$

$$\begin{aligned} |\mathcal{M}_{fi}^{\text{LSZ}}|^2 &= |\mathcal{M}_{fi}^{\text{tree}}|^2 + 2\delta Z |\mathcal{M}_{fi}^{\text{tree}}|^2 + \mathcal{M}_{fi}^{\text{vertex}} \mathcal{M}_{fi}^{\text{tree}\dagger} + \mathcal{M}_{fi}^{\text{tree}} \mathcal{M}_{fi}^{\text{vertex}\dagger} + \dots \\ &= |\mathcal{M}_{fi}^{\text{tree}}|^2 + 2\delta Z |\mathcal{M}_{fi}^{\text{tree}}|^2 + 2\text{Re}\{\mathcal{M}_{fi}^{\text{vertex}} \mathcal{M}_{fi}^{\text{tree}\dagger}\} + \dots \end{aligned}$$

$$\sigma_{\text{r.e.}} \equiv \sigma(e^+e^- \rightarrow \gamma^* \rightarrow q\bar{q}g)$$

$$\sigma_1 = \sigma_{q\bar{q}} + \sigma_{\text{r.e.}}$$

$$\text{where } \sigma_{q\bar{q}} \equiv \sigma(e^-e^+ \rightarrow \gamma^* \rightarrow q\bar{q})$$



$$i\mathcal{M}_{fi} = \left[\bar{v}^{s'}(p_2) \left(-i\mu^{\frac{4-D}{2}} e\gamma^\mu \right) u^s(p_1) \right] \left[\frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \right] \left[\bar{u}^r(q_1) \left(-i\mu^{\frac{4-D}{2}} e\gamma^\nu \right) v^{r'}(q_2) \right],$$

$$|\mathcal{M}_{fi}|^2 = \left[\frac{g_{\mu\nu}g_{\rho\sigma}}{k^4} \right] \underbrace{\left[\mu^{(4-D)} e^2 \bar{v}^{s'}(p_2) (\gamma^\mu) u^s(p_1) \bar{u}^s(p_1) (\gamma^\rho) v^{s'}(p_2) \right]}_{\text{Initial state particles, } \tilde{L}^{\mu\rho}} \times \underbrace{\left[\mu^{(4-D)} e^2 \bar{u}^r(q_1) (\gamma^\nu) v^{r'}(q_2) \bar{v}^{r'}(q_2) (\gamma^\sigma) u^r(q_1) \right]}_{\text{Final state particles, } \tilde{H}^{\nu\sigma}}$$

$$\frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_r \sum_{r'} |\mathcal{M}_{fi}|^2 = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \equiv |\mathcal{M}_{fi}|_\Sigma^2$$

$$\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p) \bar{v}^s(p) = \not{p} - m$$

$$\begin{aligned} L^{\mu\rho} &= \frac{1}{4} \sum_{s,s'} \tilde{L}^{\mu\rho} = \frac{\mu^{(4-D)} e^2}{4} \sum_{s,s'} v_d^{s'}(p_2) \bar{v}_a^{s'}(p_2) \gamma_{ab}^\mu u_b^s(p_1) \bar{u}_c^s(p_1) \gamma_{cd}^\rho \\ &= \frac{\mu^{(4-D)} e^2}{4} (\not{p}_2 - m_e) d a \gamma_{ab}^\mu (\not{p}_1 + m_e) b c \gamma_{cd}^\rho \\ &= \frac{\mu^{(4-D)} e^2}{4} \text{tr}[(\not{p}_2 - m_e) \gamma^\mu (\not{p}_1 + m_e) \gamma^\rho] \end{aligned}$$

$$L^{\mu\rho} = \mu^{(4-D)} e^2 \left(p_2^\mu p_1^\rho + p_2^\rho p_1^\mu - g^{\mu\rho} (p_1 \cdot p_2 + m_e^2) \right).$$

$$H^{\nu\sigma} = \sum_{r,r'} \tilde{H}^{\nu\sigma} = 4\mu^{(4-D)} e^2 \left(q_2^\nu q_1^\sigma + q_2^\sigma q_1^\nu - g^{\nu\sigma} (q_1 \cdot q_2 + m_\mu^2) \right)$$

$$|\mathcal{M}_{fi}|_\Sigma^2 = \frac{1}{k^4} g_{\mu\nu} g_{\rho\sigma} L^{\mu\rho} H^{\nu\sigma}$$

$$|\mathcal{M}_{fi}|_\Sigma^2 = \frac{4\mu^{2(4-D)} e^4}{k^4} [2(p_1 \cdot q_1)(p_2 \cdot q_2) + 2(p_1 \cdot q_2)(p_2 \cdot q_1) + (D-4)(p_1 \cdot p_2)(q_1 \cdot q_2) + (D-2)(q_1 \cdot q_2)m_e^2 + (D-2)(p_1 \cdot p_2)m_\mu^2 + Dm_\mu^2 m_e^2]$$

$$|\mathcal{M}_{fi}|_\Sigma^2 = \frac{4\mu^{2(4-D)} e^4}{k^4} [2(p_1 \cdot q_1)(p_2 \cdot q_2) + 2(p_1 \cdot q_2)(p_2 \cdot q_1) + (D-4)(p_1 \cdot p_2)(q_1 \cdot q_2)].$$

$$\sigma(e^+ e^- \rightarrow \gamma^* \rightarrow X) = \frac{1}{2s} \int d\text{PS} |\mathcal{M}_{fi}|^2$$

$$|\mathcal{M}_{fi}|_\Sigma^2 = \frac{1}{k^4} g_{\mu\nu} g_{\rho\sigma} L^{\mu\rho} H^{\nu\sigma} = \frac{1}{k^4} L_{\nu\sigma} H^{\nu\sigma}$$

$$\int d\text{PS} H^{\nu\sigma}$$

$$k_\nu H^{\nu\sigma} = H^{\nu\sigma} k_\sigma = 0$$

$$\int d\text{PS} H^{\nu\sigma} = \left(g^{\nu\sigma} - \frac{k^\nu k^\sigma}{k^2} \right) \cdot H$$



$$\int d\text{PS} |\mathcal{M}_{fi}|_{\Sigma}^2 = \frac{1}{k^4} L_{\nu\sigma} \left(g^{\nu\sigma} - \frac{k^{\nu} k^{\sigma}}{k^2} \right) \cdot H$$

$$= \frac{1}{k^4} L_{\nu\sigma} g^{\nu\sigma} \cdot H$$

$$(g^{\mu\nu} L_{\mu\nu}) \cdot \int d\text{PS} (g^{\rho\sigma} H_{\rho\sigma}) = (g^{\mu\nu} L_{\mu\nu}) \cdot g^{\rho\sigma} \left(g_{\rho\sigma} - \frac{k_{\rho} k_{\sigma}}{k^2} \right) \cdot H$$

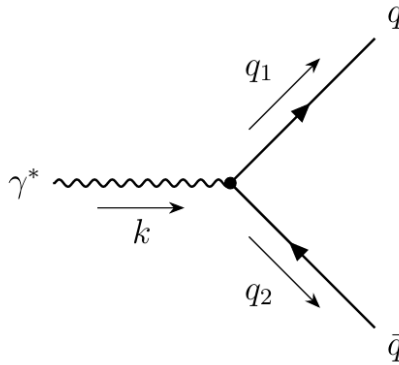
$$= (g^{\mu\nu} L_{\mu\nu}) \cdot (D - 1) \cdot H$$

$$= (D - 1) (g^{\mu\nu} L_{\mu\nu}) \cdot H$$

$$\int d\text{PS} |\mathcal{M}_{fi}|_{\Sigma}^2 = \frac{1}{(D - 1) k^4} (g^{\mu\nu} L_{\mu\nu}) \cdot \int d\text{PS} (g^{\rho\sigma} H_{\rho\sigma})$$

$$g^{\mu\nu} L_{\mu\nu} = \mu^{(4-D)} e^2 g^{\mu\nu} (p_{2\mu} p_{1\nu} + p_{2\nu} p_{1\mu} - g_{\mu\nu} (p_1 \cdot p_2 + m_e^2))$$

$$= \frac{(2 - D)}{2} \mu^{(4-D)} e^2 s,$$



$$\Gamma(\gamma^* \rightarrow X) = \frac{1}{2\sqrt{s}} \int d\text{PS} |\mathcal{M}_{fi,\gamma^*}|^2$$

$$|\mathcal{M}_{fi,\gamma^*}|^2 = \epsilon_{\rho} \epsilon_{\sigma}^* [\mu^{(4-D)} e^2 \bar{u}^r(q_1) (\gamma^{\rho}) v^{r'}(q_2) \bar{v}^{r''}(q_2) (\gamma^{\sigma}) u^r(q_1)] = \epsilon_{\rho} \epsilon_{\sigma}^* \tilde{H}^{\rho\sigma}.$$

$$\sum_{\lambda} \epsilon_{\rho}(\lambda) \epsilon_{\sigma}^*(\lambda) = - \left(g_{\rho\sigma} + (\eta - 1) \frac{k_{\rho} k_{\sigma}}{k^2} \right),$$

$$|\mathcal{M}_{fi,\gamma^*}|_{\Sigma}^2 = -g_{\rho\sigma} H^{\rho\sigma} = -g^{\rho\sigma} H_{\rho\sigma},$$

$$\Gamma(\gamma^* \rightarrow X) = -\frac{1}{2\sqrt{s}} \int d\text{PS} g^{\rho\sigma} H_{\rho\sigma}$$

$$\begin{aligned}\sigma(e^+e^- \rightarrow \gamma^* \rightarrow X) &= \frac{\mu^{(4-D)} e^2 (2-D)}{4s^2 (D-1)} \int d\text{PS}_2 g^{\rho\sigma} H_{\rho\sigma} \\ &= \frac{\mu^{(4-D)} e^2 (D-2)}{2s^{3/2} (D-1)} \Gamma(\gamma^* \rightarrow X)\end{aligned}$$

$$\int d\text{PS}_2 g^{\rho\sigma} H_{\rho\sigma(\text{tree})}$$

$$g^{\rho\sigma} H_{\rho\sigma(\text{tree})} = 4\mu^{(4-D)} e^2 (2-D) (q_1 \cdot q_2) = 2(2-D)\mu^{(4-D)} e^2 s.$$

$$\sigma_0(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-) = \frac{\mu^{2(4-D)} e^4 (2-D)^2}{2s (D-1)} \int d\text{PS}_2$$

$$\int d\text{PS}_2 = \left(\frac{s}{4\pi}\right)^{\frac{D-4}{2}} \frac{2^{-D}}{\sqrt{\pi}\Gamma((D-1)/2)} = \frac{1}{8\pi} \left(\frac{s}{4\pi}\right)^{\frac{D-4}{2}} \frac{\Gamma(D/2-1)}{\Gamma(D-2)},$$

$$\sigma_0(e^+e^- \rightarrow \mu^+\mu^-) = \frac{\mu^{2(4-D)} e^4}{16\pi s} \left(\frac{s}{4\pi}\right)^{\frac{D-4}{2}} \frac{(D-2)^2 \Gamma(D/2-1)}{(D-1) \Gamma(D-2)}$$

$$\sigma_0(e^+e^- \rightarrow \mu^+\mu^-) = \frac{e^4}{12\pi s} = \frac{4\pi\alpha_{\text{em}}^2}{3s}$$

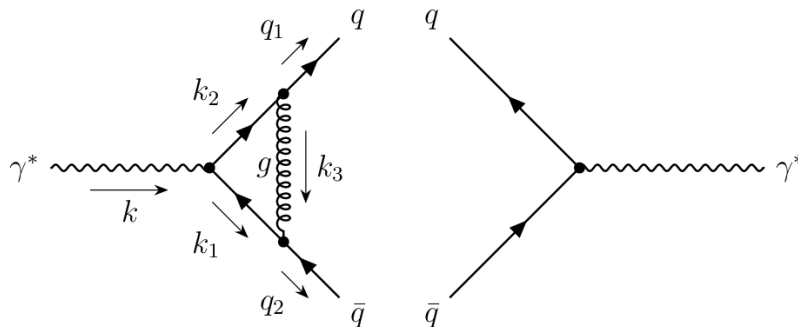
$$\alpha_{\text{em}} = \frac{e^2}{4\pi}$$

$$\sigma_0(e^+e^- \rightarrow \mu^+\mu^-) \equiv \int \sigma(e^+e^- \rightarrow q\bar{q}) \int \sigma_0(e^+e^- \rightarrow q\bar{q}) = (N_c e_q^2) \frac{4\pi\alpha^2}{3E_{\text{cm}}^2}$$

$$R_{q(0)} \equiv \frac{\sigma_0(e^+e^- \rightarrow q\bar{q})}{\sigma_0(e^+e^- \rightarrow \mu^+\mu^-)} = N_c e_q^2$$

$$R_0 \equiv \frac{\sigma_0(e^+e^- \rightarrow \text{hadrons})}{\sigma_0(e^+e^- \rightarrow \mu^+\mu^-)} = N_c \sum_q e_q^2.$$

$$R_q \equiv \frac{\sigma(e^+e^- \rightarrow q\bar{q}X)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}.$$



$$\int d\text{PS}(\mathcal{M}_{fi}^{\text{vertex}} \mathcal{M}_{fi}^{\text{tree}\dagger})_{\Sigma} = \frac{1}{(D-1)k^4} (g^{\mu\nu} L_{\mu\nu}) \int d\text{PS}(g^{\rho\sigma} H_{\rho\sigma}^{\text{(vertex)}}).$$

$$H_{\text{(vertex)}}^{\rho\sigma} = -2ie_q^2 e^2 g^2 \mu^{2(4-D)} \int \frac{d^D k_3}{(2\pi)^D} \left[\frac{1}{k_3^2 + i\epsilon} \left(g_{\alpha\beta} + (\xi - 1) \frac{k_{3\alpha} k_{3\beta}}{k_3^2 + i\epsilon} \right) \right]$$

$$\times \left[\frac{1}{(k_2^2 - m_q^2 + i\epsilon)(k_1^2 - m_q^2 + i\epsilon)} \right]$$

$$\times \text{tr}[(q_1 + m_q)\gamma^\alpha (k_2 + m_q)\gamma^\sigma (k_1 + m_q)\gamma^\beta (q_2 - m_q)\gamma^\rho]$$

$$k_2^\mu = q_1^\mu + k_3^\mu, k_1^\mu = q_2^\mu - k_3^\mu, k^\mu = q_1^\mu + q_2^\mu$$

$$p_1^2 = p_2^2 = q_1^2 = q_2^2 = 0,$$

$$k^2 = 2q_1 \cdot q_2 \equiv s.$$

$$g_{\rho\sigma} H_{\text{(vertex)}}^{\rho\sigma} = -2ie_q^2 e^2 g^2 \mu^{2(4-D)} \int \frac{d^D k_3}{(2\pi)^D} v$$

$$\times \left[\frac{1}{(k_3^2 + i\epsilon)((q_1 + k_3)^2 + i\epsilon)((q_2 - k_3)^2 + i\epsilon)} \right]$$

$$\times [16(q_1 \cdot q_2)(q_1 \cdot k_3) - 16(q_1 \cdot q_2)(q_2 \cdot k_3) - 16(q_1 \cdot q_2)^2$$

$$+ 16(q_1 \cdot k_3)(q_2 \cdot k_3) + 4(D-4)(q_1 \cdot q_2)k_3^2]$$

$$= -2ie_q^2 e^2 g^2 \mu^{2(4-D)} \int \frac{d^D k_3}{(2\pi)^D} (D-2) \times$$

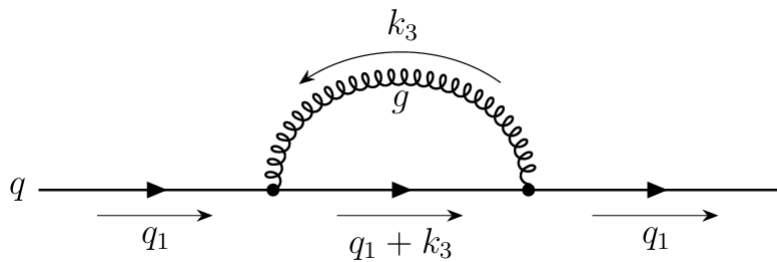
$$\left[\frac{8s(q_1 \cdot k_3) - 8s(q_2 \cdot k_3) - 4s^2 + 16(q_1 \cdot k_3)(q_2 \cdot k_3) + 2(D-4)sk_3^2}{(k_3^2 + i\epsilon)((q_1 + k_3)^2 + i\epsilon)((q_2 - k_3)^2 + i\epsilon)} \right].$$

$$g_{\rho\sigma} H_{\text{(vertex)}}^{\rho\sigma} = -2ie_q^2 e^2 g^2 \mu^{2(4-D)} (D-2) \times$$

$$[8s(q_{1\mu} - q_{2\mu})C_{2,\text{IR}}^\mu - 4s^2 C_{1,\text{IR}} + 16q_{1\mu}q_{2\mu}C_{4,\text{IR}}^{\mu\nu} + 2s(D-4)C_{3,\text{IR}}]$$



$$\begin{aligned}
g_{\rho\sigma}H_{(\text{vertex})}^{\rho\sigma} &= -2ie_q^2 e^2 g^2 \mu^{2(4-D)} (D-2) \times \\
&\left[8s(2q_1 \cdot q_2) \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})\Gamma(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \Gamma(\varepsilon_{\text{IR}}) \right. \\
&- 4s^2 \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})}{\Gamma(1 + 2\varepsilon_{\text{IR}})} \Gamma^2(\varepsilon_{\text{IR}}) \\
&+ 4(q_1 \cdot q_2) \frac{-i}{(4\pi)^2} (2q_1 \cdot q_2)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\hat{\varepsilon}_{\text{UV}}} - 3 - i\pi \right] \\
&- 16 \frac{i}{(4\pi)^2} \frac{1}{2q_1 \cdot q_2} \left(\frac{-2q_1 \cdot q_2}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})}{\Gamma(3 + 2\varepsilon_{\text{IR}})} (q_1 \cdot q_2)^2 \Gamma^2(1 + \varepsilon_{\text{IR}}) \\
&\left. + 4s\varepsilon_{\text{UV}} \frac{-i}{(4\pi)^2} (2q_1 \cdot q_2)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\hat{\varepsilon}_{\text{UV}}} - 2 - i\pi \right] \right] \\
&= -2ie_q^2 e^2 g^2 \mu^{2(4-D)} (D-2) \left[8s \frac{i}{(4\pi)^2} \left(\frac{-s}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})\Gamma(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \Gamma(\varepsilon_{\text{IR}}) \right. \\
&- 4s \frac{i}{(4\pi)^2} \left(\frac{-s}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})}{\Gamma(1 + 2\varepsilon_{\text{IR}})} \Gamma^2(\varepsilon_{\text{IR}}) \\
&+ 2s \frac{-i}{(4\pi)^2} s^{\varepsilon_{\text{UV}}} \left[\frac{1}{\hat{\varepsilon}_{\text{UV}}} - 3 - i\pi \right] \\
&- 4s \frac{i}{(4\pi)^2} \left(\frac{-s}{4\pi} \right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1 - \varepsilon_{\text{IR}})}{\Gamma(3 + 2\varepsilon_{\text{IR}})} \Gamma^2(1 + \varepsilon_{\text{IR}}) \\
&\left. + 4s\varepsilon_{\text{UV}} \frac{-i}{(4\pi)^2} s^{\varepsilon_{\text{UV}}} \left[\frac{1}{\hat{\varepsilon}_{\text{UV}}} - 2 - i\pi \right] \right] \\
&= -8ise_q^2 e^2 g^2 \left[2 \frac{i}{(4\pi)^2} (-1)^{\varepsilon_{\text{IR}}} \left(\frac{s}{\mu^4} \right)^{\varepsilon_{\text{IR}}} \left(\frac{1}{4\pi} \right)^{\varepsilon_{\text{IR}}} (1 + \varepsilon_{\text{IR}})\Gamma(1 - \varepsilon_{\text{IR}}) \right. \\
&\left(2 \frac{\Gamma(1 + \varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(\varepsilon_{\text{IR}})}{\Gamma(1 + 2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(1 + \varepsilon_{\text{IR}})}{\Gamma(3 + 2\varepsilon_{\text{IR}})} \right) \\
&\left. + \frac{-i}{(4\pi)^2} \left(\frac{s}{\mu^4} \right)^{\varepsilon_{\text{UV}}} \left(\frac{1}{4\pi} \right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E - i\pi \right] + \mathcal{O}(\varepsilon_{\text{UV}}) \right] \\
g_{\rho\sigma}H_{(\text{vertex})}^{\rho\sigma} &= -\frac{16se_q^2 e^2 g^2}{(4\pi)^2} \left(1 - \frac{\pi^2 \varepsilon_{\text{IR}}^2}{2} + \mathcal{O}(\varepsilon_{\text{IR}}^4) \right) \left(\frac{s}{\mu^4} \right)^{\varepsilon_{\text{IR}}} \left(\frac{1}{4\pi} \right)^{\varepsilon_{\text{IR}}} (1 + \varepsilon_{\text{IR}})\Gamma(1 - \varepsilon_{\text{IR}}) \\
&\left(2 \frac{\Gamma(1 + \varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(\varepsilon_{\text{IR}})}{\Gamma(1 + 2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(1 + \varepsilon_{\text{IR}})}{\Gamma(3 + 2\varepsilon_{\text{IR}})} \right) \\
&+ \frac{8se_q^2 e^2 g^2}{(4\pi)^2} \left(\frac{s}{\mu^4} \right)^{\varepsilon_{\text{UV}}} \left(\frac{1}{4\pi} \right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E \right].
\end{aligned}$$



$$i\Sigma_g(q_1) = -g^2\mu^{4-D} \int \frac{d^D k_3}{(2\pi)^D} \gamma^\alpha \frac{q_1 + k_3 + m_q}{(q_1 + k_3)^2 - m_q^2 + i\epsilon} \gamma^\beta \\ \times \left[\frac{1}{k_3^2 + i\epsilon} \left(g_{\alpha\beta} + (\xi - 1) \frac{k_{3\alpha} k_{3\beta}}{k_3^2 + i\epsilon} \right) \right].$$

$$i\Sigma_g(q_1) = -g^2\mu^{4-D} \int \frac{d^D k_3}{(2\pi)^D} \gamma^\alpha \frac{q_1 + k_3}{(q_1 + k_3)^2 + i\epsilon} \gamma_\alpha \left[\frac{1}{k_3^2 + i\epsilon} \right]$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2 - D) \gamma^\nu$$

$$i\Sigma_g(q_1) = -g^2\mu^{4-D} \int \frac{d^D k_3}{(2\pi)^D} \frac{(2 - D)(q_1 + k_3)}{((q_1 + k_3)^2 + i\epsilon)(k_3^2 + i\epsilon)}.$$

$$i\Sigma_g(q_1) = -g^2\mu^{4-D} \int \frac{d^D k_3}{(2\pi)^D} \int_0^1 dx \frac{(2 - D)(q_1 + k_3)}{[(k_3 + xq_1)^2 - a^2 + i\epsilon]^2}$$

$$i\Sigma_g(q_1) = -g^2\mu^{4-D} (2 - D) q_1 \int_0^1 dx (1 - x) J(D, 0, 2, a^2) \\ = -g^2\mu^{4-D} (2 - D) q_1 \frac{i}{(4\pi)^{D/2}} \Gamma(2 - D/2) (-q_1^2)^{D/2-2} \frac{\Gamma(D/2) \Gamma(D/2 - 1)}{\Gamma(D - 1)}$$

$$\int_{-q_1^2}^{\infty} \frac{dx}{x} x^n = (-q_1^2)^n \left(-\frac{1}{n} \right) \text{ for } \text{Re}\{n\} < 0$$

$$\int_{-q_1^2}^{\infty} \frac{dx}{x} x^{-n} = (-q_1^2)^{-n} \left(\frac{1}{n} \right) \text{ for } \text{Re}\{n\} > 0$$

$$(-q_1^2)^{D/2-2} (2 - D/2)^{-1} = \int_{-q_1^2}^{\infty} \frac{dx}{x} x^{D/2-2}$$

$$= \left[\int_{-q_1^2}^s \frac{dx}{x} x^{D/2-2} + \int_s^{\infty} \frac{dx}{x} x^{D/2-2} \right]$$

$$(-q_1^2)^{D/2-2} (2 - D/2)^{-1} = \frac{s^{\epsilon_{\text{IR}}}}{\epsilon_{\text{IR}}} - \frac{s^{\epsilon_{\text{UV}}}}{\epsilon_{\text{UV}}}.$$



$$\begin{aligned}
i\Sigma_g(q_1) &= -g^2\mu^{4-D}(2-D)q_1 \frac{i}{(4\pi)^{D/2}} \Gamma(3-D/2) \frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(D-1)} \\
&\times \left[\frac{2s^{(D-4)/2}}{D-4} \Big|_{D=4+2\varepsilon_{\text{IR}}} - \frac{2s^{(D-4)/2}}{D-4} \Big|_{D=4+2\varepsilon_{\text{UV}}} \right] = \frac{2ig^2q_1}{(4\pi)^2} \\
&\times \left((1+\varepsilon_{\text{IR}}) \left(\frac{s}{\mu^2}\right)^{\varepsilon_{\text{IR}}} (4\pi)^{-\varepsilon_{\text{IR}}} \Gamma(1-\varepsilon_{\text{IR}}) \frac{\Gamma(2+\varepsilon_{\text{IR}})\Gamma(1+\varepsilon_{\text{IR}})}{\Gamma(3+2\varepsilon_{\text{IR}})} \left[\frac{1}{\varepsilon_{\text{IR}}}\right] \right. \\
&\left. - (1+\varepsilon_{\text{UV}}) \left(\frac{s}{\mu^2}\right)^{\varepsilon_{\text{UV}}} (4\pi)^{-\varepsilon_{\text{UV}}} \Gamma(1-\varepsilon_{\text{UV}}) \frac{\Gamma(2+\varepsilon_{\text{UV}})\Gamma(1+\varepsilon_{\text{UV}})}{\Gamma(3+2\varepsilon_{\text{UV}})} \left[\frac{1}{\varepsilon_{\text{UV}}}\right] \right) \\
&= \frac{ig^2q_1}{(4\pi)^2} \left(\left(\frac{s}{\mu^2}\right)^{\varepsilon_{\text{IR}}} \left(\frac{1}{4\pi}\right)^{\varepsilon_{\text{IR}}} (1+(\gamma_E-1)\varepsilon_{\text{IR}}+\dots) \left[\frac{1}{\varepsilon_{\text{IR}}}\right] \right. \\
&\left. - \left(\frac{s}{\mu^2}\right)^{\varepsilon_{\text{UV}}} \left(\frac{1}{4\pi}\right)^{\varepsilon_{\text{UV}}} (1+(\gamma_E-1)\varepsilon_{\text{UV}}+\dots) \left[\frac{1}{\varepsilon_{\text{UV}}}\right] \right) \\
&= \frac{ig^2q_1}{(4\pi)^2} \left(\left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E - 1 + \dots\right] - \left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E - 1 + \dots\right] \right)
\end{aligned}$$

$$Z = \left(1 + \frac{d\Sigma_g}{dk_1} \Big|_{k_1=0} \right)^{-1} \approx 1 - \frac{d\Sigma_g}{dk_1} \Big|_{k_1=0}$$

$$\begin{aligned}
\delta Z \equiv Z - 1 &= -\frac{d\Sigma_g}{dq_1} \\
&= -\frac{g^2}{(4\pi)^2} \left(\left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E - 1 + \dots\right] - \left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E - 1 + \dots\right] \right).
\end{aligned}$$

$$\begin{aligned}
2\delta Z g^{\rho\sigma} H_{\rho\sigma}^{(\text{tree})} &= -\frac{4(2-D)e_q^2 e^2 g^2 \mu^{4-D} s}{(4\pi)^2} \left(\left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E - 1 + \dots\right] \right. \\
&\left. - \left(\frac{s}{4\pi\mu^2}\right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E - 1 + \dots\right] \right) \\
&= \frac{8e^2 g^2 s}{(4\pi)^2} \left(\left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{IR}}} (1+\varepsilon_{\text{IR}}) \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E - 1 + \dots\right] \right. \\
&\left. - \left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{UV}}} (1+\varepsilon_{\text{UV}}) \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E - 1 + \dots\right] \right) \\
&= \frac{8e^2 g^2 s}{(4\pi)^2} \left(\left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E + \dots\right] - \left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{UV}}} \left[\frac{1}{\varepsilon_{\text{UV}}} + \gamma_E + \dots\right] \right).
\end{aligned}$$

$$\begin{aligned}
g^{\rho\sigma} H_{\rho\sigma}^{(\text{virtual})} &\equiv g^{\rho\sigma} H_{\rho\sigma}^{(\text{vertex})} + 2\delta R g^{\rho\sigma} H_{\rho\sigma}^{(\text{tree})} \\
&= -\frac{16se_q^2 e^2 g^2}{(4\pi)^2} \left(1 - \frac{\pi^2 \varepsilon_{\text{IR}}^2}{2} + \mathcal{O}(\varepsilon_{\text{IR}}^4) \right) \left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{IR}}} (1+\varepsilon_{\text{IR}})\Gamma(1-\varepsilon_{\text{IR}}) \\
&\left(2 \frac{\Gamma(1+\varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}})}{\Gamma(2+2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(\varepsilon_{\text{IR}})}{\Gamma(1+2\varepsilon_{\text{IR}})} - \frac{\Gamma^2(1+\varepsilon_{\text{IR}})}{\Gamma(3+2\varepsilon_{\text{IR}})} \right) \\
&+ \frac{8se_q^2 e^2 g^2}{(4\pi)^2} \left(\frac{s}{4\pi\mu^4}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E + \dots\right]
\end{aligned}$$

$$\int d\text{PS}_2 = \frac{1}{8\pi} \left(\frac{s}{4\pi}\right)^{\varepsilon_{\text{IR}}} \frac{\Gamma(1+\varepsilon_{\text{IR}})}{\Gamma(2+2\varepsilon_{\text{IR}})}$$



$$\int d\text{PS}_2 g_{\rho\sigma} H_{\text{virtual}}^{\rho\sigma} = -\frac{2se_q^2 e^2 g^2}{\pi(4\pi)^2} \left(\frac{s^2}{(4\pi)^2 \mu^4} \right)^{\epsilon_{\text{IR}}} (1 + \epsilon_{\text{IR}}) \left(1 - \frac{\pi^2 \epsilon_{\text{IR}}^2}{2} \right) \frac{\Gamma(1 + \epsilon_{\text{IR}})}{\Gamma(2 + 2\epsilon_{\text{IR}})} \Gamma(1 - \epsilon_{\text{IR}})$$

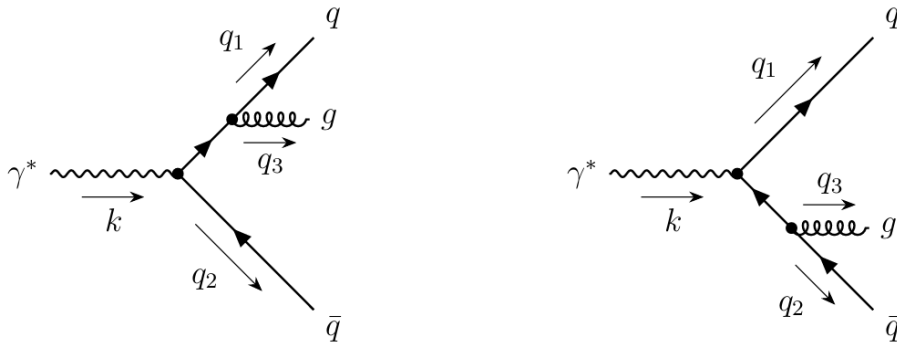
$$\left(2 \frac{\Gamma(1 + \epsilon_{\text{IR}}) \Gamma(\epsilon_{\text{IR}})}{\Gamma(2 + 2\epsilon_{\text{IR}})} - \frac{\Gamma^2(\epsilon_{\text{IR}})}{\Gamma(1 + 2\epsilon_{\text{IR}})} - \frac{\Gamma^2(1 + \epsilon_{\text{IR}})}{\Gamma(3 + 2\epsilon_{\text{IR}})} \right)$$

$$+ \frac{se_q^2 e^2 g^2}{\pi(4\pi)^2} \left(\frac{s^2}{(4\pi)^2 \mu^4} \right)^{\epsilon_{\text{IR}}} \frac{\Gamma(1 + \epsilon_{\text{IR}})}{\Gamma(2 + 2\epsilon_{\text{IR}})} \left[\frac{1}{\epsilon_{\text{IR}}} + \gamma_E \right]$$

$$= -\frac{se_q^2 e^2 g^2}{\pi(4\pi)^2} \left(\frac{s^2}{(4\pi)^2 \mu^4} \right)^{\epsilon_{\text{IR}}} \frac{(1 + \epsilon_{\text{IR}})}{\Gamma(2 + 2\epsilon_{\text{IR}})} \left(-\frac{2}{\epsilon_{\text{IR}}^2} + \frac{3}{\epsilon_{\text{IR}}} - 8 + \pi^2 + \mathcal{O}(\epsilon_{\text{IR}}) \right)$$

$$\sigma_{\text{virtual}}(e^+ e^- \rightarrow q \bar{q}) = \frac{e_q^2 e^4 g^2}{2\pi(4\pi)^2 s} \left(\frac{s^2}{(4\pi)^2 \mu^6} \right)^{\epsilon_{\text{IR}}} \frac{(1 + \epsilon_{\text{IR}})^2}{(3 + 2\epsilon_{\text{IR}}) \Gamma(2 + 2\epsilon_{\text{IR}})}$$

$$\left(-\frac{2}{\epsilon_{\text{IR}}^2} + \frac{3}{\epsilon_{\text{IR}}} - 8 + \pi^2 + \mathcal{O}(\epsilon_{\text{IR}}) \right)$$



$$H_{\text{r.e.}}^{\rho\sigma} = -e_q^2 e^2 g^2 \mu^{2(4-D)} \left(g_{\alpha\beta} + (\xi - 1) \frac{q_{3\alpha} q_{3\beta}}{q_3^2 + i\epsilon} \right)$$

$$\left[\frac{\text{tr}[(q_1 + q_3 + m_q) \gamma^\alpha (q_1 + m_q) \gamma^\beta (q_1 + q_3 + m_q) \gamma^\sigma (q_2 - m_q) \gamma^\rho]}{((q_1 + q_3)^2 - m_q^2 + i\epsilon)((q_1 + q_3)^2 - m_q^2 - i\epsilon)} + \right.$$

$$\frac{\text{tr}[(q_1 + m_q) \gamma^\sigma (q_2 + q_3 + m_q) \gamma^\alpha (q_2 - m_q) \gamma^\beta (q_2 + q_3 + m_q) \gamma^\rho]}{((q_2 + q_3)^2 - m_q^2 + i\epsilon)((q_2 + q_3)^2 - m_q^2 - i\epsilon)} -$$

$$\frac{\text{tr}[(q_1 + m_q) \gamma^\alpha (q_1 + q_3 + m_q) \gamma^\sigma (q_2 - m_q) \gamma^\beta (q_2 + q_3 + m_q) \gamma^\rho]}{((q_1 + q_3)^2 - m_q^2 + i\epsilon)((q_2 + q_3)^2 - m_q^2 - i\epsilon)} -$$

$$\left. \frac{\text{tr}[(q_1 + q_3 + m_q) \gamma^\alpha (q_1 + m_q) \gamma^\sigma (q_2 + q_3 + m_q) \gamma^\beta (q_2 - m_q) \gamma^\rho]}{((q_1 + q_3)^2 - m_q^2 - i\epsilon)((q_2 + q_3)^2 - m_q^2 + i\epsilon)} \right]$$

$$g_{\sigma\rho} H_{\text{r.e.}}^{\rho\sigma} = -e_q^2 e^2 g^2 \mu^{2(4-D)}$$

$$\times \left[\frac{8(D-2)^2 (q_1 \cdot q_3)(q_2 \cdot q_3)}{((q_1 + q_3)^2 + i\epsilon)((q_1 + q_3)^2 - i\epsilon)} + \frac{8(D-2)^2 (q_1 \cdot q_3)(q_2 \cdot q_3)}{((q_2 + q_3)^2 + i\epsilon)((q_2 + q_3)^2 - i\epsilon)} \right.$$

$$+ \frac{8(D-2)(2(q_1 \cdot q_2)^2 + 2(q_1 \cdot q_2)(q_1 \cdot q_3 + q_2 \cdot q_3) + (D-4)(q_1 \cdot q_3)(q_2 \cdot q_3))}{((q_1 + q_3)^2 + i\epsilon)((q_2 + q_3)^2 - i\epsilon)}$$

$$\left. + \frac{8(D-2)(2(q_1 \cdot q_2)^2 + 2(q_1 \cdot q_2)(q_1 \cdot q_3 + q_2 \cdot q_3) + (D-4)(q_1 \cdot q_3)(q_2 \cdot q_3))}{((q_1 + q_3)^2 - i\epsilon)((q_2 + q_3)^2 + i\epsilon)} \right]$$



$$x_i = \frac{2q_i \cdot k}{s}, i = 1, 2, 3$$

$$\begin{aligned}(q_1 + q_3)^2 &= 2q_1 \cdot q_3 = (k - q_2)^2 = s(1 - x_2), \\(q_2 + q_3)^2 &= 2q_2 \cdot q_3 = (k - q_1)^2 = s(1 - x_1), \\(q_1 + q_2)^2 &= 2q_1 \cdot q_2 = (k - q_3)^2 = s(1 - x_3).\end{aligned}$$

$$\begin{aligned}g_{\sigma\rho}H_{\text{r.e.}}^{\rho\sigma} &= \\& -e_q^2 e^2 g^2 \mu^{2(4-D)} \left[\frac{2(D-2)^2(1-x_1)}{(1-x_2)} + \frac{2(D-2)^2(1-x_2)}{(1-x_1)} \right. \\& \left. + \frac{4(D-2) \left(2(1-x_3)^2 + 2(1-x_3)((1-x_2) + (1-x_1)) + (D-4)(1-x_2)(1-x_1) \right)}{(1-x_2)(1-x_1)} \right]\end{aligned}$$

$$\begin{aligned}g_{\sigma\rho}H_{\text{r.e.}}^{\rho\sigma} &= -e_q^2 e^2 g^2 \mu^{2(4-D)} \left[\frac{2(D-2)^2(1-x_1)}{(1-x_2)} + \frac{2(D-2)^2(1-x_2)}{(1-x_1)} \right. \\& \left. + \frac{4(D-2) \left(2(x_1 + x_2 - 1) + (D-4)(1-x_1)(1-x_2) \right)}{(1-x_2)(1-x_1)} \right] \\& = -\frac{2e_q^2 e^2 g^2 \mu^{2(4-D)} (D-2)}{(1-x_1)(1-x_2)} \left((D-2)(1-2x_1+x_1^2) + (D-2)(1-2x_2+x_2^2) \right. \\& \quad \left. + 4x_1 + 4x_2 - 4 + 2(D-4)(1-x_1-x_2+x_1x_2) \right) \\& = -\frac{2e_q^2 e^2 g^2 \mu^{2(4-D)} (D-2)}{(1-x_1)(1-x_2)} \left(D(x_1^2 + x_2^2 + 2x_1x_2 - 4x_1 - 4x_2 + 4) \right. \\& \quad \left. - 2(x_1^2 + x_2^2 + 4x_1x_2 - 8x_1 - 8x_2 + 8) \right) \\& = -8e_q^2 e^2 g^2 \mu^{2(4-D)} (1 + \varepsilon_{\text{IR}}) \frac{x_1^2 + x_2^2 + \varepsilon_{\text{IR}}(x_1 + x_2 - 2)^2}{(1-x_1)(1-x_2)}\end{aligned}$$

$$q_i \cdot q_3 = \omega_i \omega_g (1 - \cos \theta_{i3}), i = 1, 2$$

$$\begin{aligned}\int d\text{PS}_{\text{r.e.}} &= \frac{s}{16(2\pi)^3} \left(\frac{4\pi}{s} \right)^{-2\varepsilon_{\text{IR}}} \frac{1}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \\& \times \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 [(1-x_1)(1-x_2)(x_1+x_2-1)]^{\varepsilon_{\text{IR}}}\end{aligned}$$

$$\begin{aligned}\int d\text{PS}_{\text{r.e.}} g_{\sigma\rho}H_{\text{r.e.}}^{\rho\sigma} &= -\frac{e_q^2 e^2 g^2 s \mu^{2(4-D)}}{2(2\pi)^3} \left(\frac{4\pi}{s} \right)^{-2\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \\& \times \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 [(1-x_1)(1-x_2)(x_1+x_2-1)]^{\varepsilon_{\text{IR}}} \\& \times \frac{x_1^2 + x_2^2 + \varepsilon_{\text{IR}}(x_1 + x_2 - 2)^2}{(1-x_1)(1-x_2)}\end{aligned}$$



$$\begin{aligned}
\int d\text{PS}_{\text{r.e.}} g_{\sigma\rho} H_{\text{r.e.}}^{\rho\sigma} &= -\frac{e_q^2 e^2 g^2 s \mu^{2(4-D)}}{2(2\pi)^3} \left(\frac{s^2}{(4\pi)^2}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \\
&\quad \times \int_0^1 dx \int_0^1 dv x [(1-x)vx^2(1-v)]^{\varepsilon_{\text{IR}}} \\
&\quad \times \frac{x^2 + 1 - 2vx + v^2x^2 + \varepsilon_{\text{IR}}(x(1-v) - 1)^2}{vx(1-x)} \\
&= -\frac{e_q^2 e^2 g^2 s \mu^{2(4-D)}}{2(2\pi)^3} \left(\frac{s^2}{(4\pi)^2}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \\
&\quad \times \int_0^1 dx \int_0^1 dv (1-x)^{\varepsilon_{\text{IR}}} v^{\varepsilon_{\text{IR}}} x^{2\varepsilon_{\text{IR}}+1} (1-v)^{\varepsilon_{\text{IR}}} \\
&\quad \times [(1 + \varepsilon_{\text{IR}})(x^{-1}(1-x)v^{-1} + vx(1-x)^{-1}) \\
&\quad + 2v^{-1}(1-v)(1-x)^{-1} + 2\varepsilon_{\text{IR}}]
\end{aligned}$$

$$\begin{aligned}
\int d\text{PS}_{\text{r.e.}} g_{\sigma\rho} H_{\text{r.e.}}^{\rho\sigma} &= -\frac{e_q^2 e^2 g^2 s \mu^{2(4-D)}}{2(2\pi)^3} \left(\frac{s^2}{(4\pi)^2}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \\
&\quad \times \left[\frac{2(1 + \varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}} + 1)\Gamma(\varepsilon_{\text{IR}} + 2)}{\Gamma(3\varepsilon_{\text{IR}} + 3)} \right. \\
&\quad \left. + \frac{2\Gamma^2(\varepsilon_{\text{IR}})\Gamma(\varepsilon_{\text{IR}} + 2)}{\Gamma(3\varepsilon_{\text{IR}} + 2)} + \frac{2\varepsilon_{\text{IR}}\Gamma^3(1 + \varepsilon_{\text{IR}})}{\Gamma(3\varepsilon_{\text{IR}} + 3)} \right]
\end{aligned}$$

$$\int d\text{PS}_{\text{r.e.}} g_{\sigma\rho} H_{\text{r.e.}}^{\rho\sigma} = -\frac{e_q^2 e^2 g^2 s \mu^{2(4-D)}}{2(2\pi)^3} \left(\frac{s^2}{(4\pi)^2}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})}{\Gamma(2 + 2\varepsilon_{\text{IR}})} \left[\frac{2}{\varepsilon_{\text{IR}}^2} - \frac{3}{\varepsilon_{\text{IR}}} + \frac{19}{2} - \pi^2 \right].$$

$$\begin{aligned}
\sigma_{\text{r.e.}}(e^+e^- \rightarrow q\bar{q}\gamma) &= \frac{e_q^2 e^4 g^2 \mu^{3(4-D)}}{4(2\pi)^3 s} \left(\frac{s^2}{(4\pi)^2}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})^2}{(3 + 2\varepsilon_{\text{IR}})\Gamma(2 + 2\varepsilon_{\text{IR}})} \\
&\quad \left[\frac{2}{\varepsilon_{\text{IR}}^2} - \frac{3}{\varepsilon_{\text{IR}}} + \frac{19}{2} - \pi^2 + \mathcal{O}(\varepsilon_{\text{IR}}) \right]
\end{aligned}$$

$$\begin{aligned}
\sigma_1(e^+e^- \rightarrow q\bar{q}X) &= \sigma_{\text{virtual}}(e^+e^- \rightarrow q\bar{q}) + \sigma_{\text{r.e.}}(e^+e^- \rightarrow q\bar{q}g) \\
&= \lim_{\varepsilon_{\text{IR}} \rightarrow 0} \frac{e_q^2 e^4 g^2}{2\pi(4\pi)^2 s} \left(\frac{s^2}{(4\pi)^2 \mu^6}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})^2}{(3 + 2\varepsilon_{\text{IR}})\Gamma(2 + 2\varepsilon_{\text{IR}})} \\
&\quad \left[\left(-\frac{2}{\varepsilon_{\text{IR}}^2} + \frac{3}{\varepsilon_{\text{IR}}} - 8 + \pi^2 \right) + \left(\frac{2}{\varepsilon_{\text{IR}}^2} - \frac{3}{\varepsilon_{\text{IR}}} + \frac{19}{2} - \pi^2 \right) + \mathcal{O}(\varepsilon_{\text{IR}}) \right] \\
&= \lim_{\varepsilon_{\text{IR}} \rightarrow 0} \frac{e_q^2 e^4 g^2}{2\pi(4\pi)^2 s} \left(\frac{s^2}{(4\pi)^2 \mu^6}\right)^{\varepsilon_{\text{IR}}} \frac{(1 + \varepsilon_{\text{IR}})^2}{(3 + 2\varepsilon_{\text{IR}})\Gamma(2 + 2\varepsilon_{\text{IR}})} \left[\frac{3}{2} + \mathcal{O}(\varepsilon_{\text{IR}}) \right] \\
&= \lim_{\varepsilon_{\text{IR}} \rightarrow 0} \frac{e_q^2 e^4 g^2}{2\pi(4\pi)^2 s} \left(\frac{s^2}{(4\pi)^2 \mu^6}\right)^{\varepsilon_{\text{IR}}} \left[\frac{1}{2} + \mathcal{O}(\varepsilon_{\text{IR}}) \right] \\
&= \frac{e_q^2 e^4 g^2}{(4\pi)^3 s} \\
&= \frac{e_q^2 \alpha_{\text{em}}^2 \alpha_s}{s}
\end{aligned}$$

$$\alpha_s = \frac{g^2}{4\pi}$$



$$\sigma_1 = \frac{e_q^2 \alpha_{em}^2 \alpha_s}{s} = \frac{3\sigma_0 \alpha_s}{4\pi N_c}$$

$$\sigma_1 = \frac{\sigma_0 \alpha_s}{\pi},$$

$$\sigma(e^+e^- \rightarrow q\bar{q}X) = \sigma_0 \left(1 + \frac{\alpha_s}{\pi}\right)$$

$$R_q = \frac{\sigma(e^+e^- \rightarrow q\bar{q}X)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_c e_q^2 \left(1 + \frac{\alpha_s}{\pi}\right).$$

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_c \sum_q e_q^2 \left(1 + \frac{\alpha_s}{\pi}\right).$$

$$\mathbb{P} = \text{tr}(E\rho_t)$$

$$\rho_t \equiv U_{t,t_{in}} \rho_0 U_{t,t_{in}}^\dagger$$

$$U_{t,t_{in}} = \text{T} \left\{ \exp \left(\frac{1}{i} \int_{t_{in}}^t dt' H_{\text{int}}(t') \right) \right\}$$

$$\mathbb{P} = \langle i | U_{t,t_{in}}^\dagger E U_{t,t_{in}} | i \rangle.$$

$$E = |f\rangle\langle f|,$$

$$\mathbb{P} = \langle i | U_{t,t_{in}}^\dagger |f\rangle\langle f| U_{t,t_{in}} | i \rangle = |\langle f | U_{t,t_{in}} | i \rangle|^2.$$

$$e^A O e^{-A} = O + [A, O] + \frac{1}{2!} [A, [A, O]] + \frac{1}{3!} [A, [A, [A, O]]] + \dots$$

$$\begin{aligned} U_{t,t_{in}}^\dagger O U_{t,t_{in}} &= O + \left(\frac{1}{i}\right) \int_{t_{in}}^t dt_1 [O, H_{\text{int}}(t_1)] \dots \\ &+ \left(\frac{1}{i}\right)^2 \int_{t_{in}}^t dt_1 \int_{t_{in}}^{t_1} dt_2 [[O, H_{\text{int}}(t_1)], H_{\text{int}}(t_2)] + \dots \\ &+ \left(\frac{1}{i}\right)^j \int_{t_{in}}^t dt_1 \int_{t_{in}}^{t_1} dt_2 \dots \int_{t_{in}}^{t_{j-1}} dt_j \dots \\ &[\dots [[O, H_{\text{int}}(t_1)], H_{\text{int}}(t_2)] \dots, H_{\text{int}}(t_j)] \dots \end{aligned}$$

$$\mathbb{P} = \sum_{j=0}^{\infty} \int_{t_{in}}^t dt'_1 dt'_2 \dots dt'_j \Theta_{12\dots j} \langle i | \mathcal{F}_j | i \rangle,$$

$$\text{where } \Theta_{ijk\dots} \equiv \begin{cases} 1, & \text{if } t'_i > t'_j > t'_k \dots \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{F}_0 = E,$$

$$\text{and } \mathcal{F}_j = \frac{1}{i} [\mathcal{F}_{j-1}, H_{\text{int}}(t'_j)].$$



$$E = \sum_{\kappa} \prod_i E_{(\kappa)}^{\mathcal{H}_i}$$

$$H_{\text{int}}(t'_j) = \int d^3 \mathbf{x}_j (g_Y A_j^X B_j^Y + g_Z \mathcal{A}_j^X C_j^Z)$$

$$\begin{aligned} \mathcal{F}_1 &= \int d^4 \mathbf{x}_1 \frac{1}{i} [E, g_Y A_1^X B_1^Y + g_Z \mathcal{A}_1^X C_1^Z] \\ &= \int d^4 \mathbf{x}_1 \frac{1}{i} [E^X E^Y E^Z, g_Y A_1^X B_1^Y + g_Z \mathcal{A}_1^X C_1^Z] \end{aligned}$$

$$[A^X B^Y, \alpha^X \beta^Y] = \frac{1}{2} [A^X, \alpha^X] \{B^Y, \beta^Y\} + \frac{1}{2} \{A^X, \alpha^X\} [B^Y, \beta^Y]$$

$$\begin{aligned} \mathcal{F}_1 &= \int d^4 \mathbf{x}_1 \left(\frac{1}{2i} [E^X, g_Y A_1^X] \{E^Y, B_1^Y\} E^Z + \frac{1}{2i} \{E^X, g_Y A_1^X\} [E^Y, B_1^Y] E^Z \right. \\ &\quad \left. + \frac{1}{2i} [E^X, g_Z \mathcal{A}_1^X] E^Y \{E^Z, C_1^Z\} + \frac{1}{2i} \{E^X, g_Z \mathcal{A}_1^X\} E^Y [E^Z, C_1^Z] \right) \end{aligned}$$

$$\begin{aligned} \text{Hilbert space } X: \quad \mathcal{E}_{\dots k}^{\dots Y} &:= \frac{1}{i} [\mathcal{E}_{\dots}, g_Y A_k^X], & \mathcal{E}_{\dots \underline{k}}^{\dots Y} &:= \{\mathcal{E}_{\dots}, g_Y A_k^X\}, \\ \mathcal{E}_{\dots k}^{\dots Z} &:= \frac{1}{i} [\mathcal{E}_{\dots}, g_Z \mathcal{A}_k^X], & \mathcal{E}_{\dots \underline{k}}^{\dots Z} &:= \{\mathcal{E}_{\dots}, g_Z \mathcal{A}_k^X\}, \end{aligned}$$

$$\text{Hilbert space } Y: \quad \mathcal{E}_{\dots k}^{\dots Y} := \frac{1}{i} [E_{\dots}^Y, B_k^Y], \quad E_{\dots \underline{k}}^{\dots Y} := \{E_{\dots}^Y, B_k^Y\},$$

$$\text{Hilbert space } Z: \quad \mathcal{E}_{\dots k}^{\dots Z} := \frac{1}{i} [E_{\dots}^Z, C_k^Z], \quad E_{\dots \underline{k}}^{\dots Z} := \{E_{\dots}^Z, C_k^Z\},$$

$$\mathcal{F}_1 = \int d^4 \mathbf{x}_1 \frac{1}{2} (\mathcal{E}_1^Y E_1^Y E^Z + \mathcal{E}_1^Y E_1^Y E^Z + \mathcal{E}_1^Z E^Y E_1^Z + \mathcal{E}_1^Z E^Y E_1^Z)$$

$$E_{\circ k}^Y \mathcal{E}_{\circ k}^Y := E_k^Y \mathcal{E}_{\underline{k}}^Y + E_{\underline{k}}^Y \mathcal{E}_k^Y,$$

$$E_{\circ \circ \underline{kl}}^Y \mathcal{E}_{\circ \circ \underline{kl}}^{YY} := E_{kl}^Y \mathcal{E}_{\underline{kl}}^{YY} + E_{\underline{kl}}^Y \mathcal{E}_{kl}^{YY} + E_{kl}^Y \mathcal{E}_{\underline{kl}}^{YY} + E_{\underline{kl}}^Y \mathcal{E}_{kl}^{YY},$$

$$E_{\circ k}^Y E_{\circ l}^Z \mathcal{E}_{\circ \circ \underline{kl}}^{YZ} := E_k^Y E_l^Z \mathcal{E}_{\underline{kl}}^{YZ} + E_{\underline{k}}^Y E_l^Z \mathcal{E}_{kl}^{YZ} + E_k^Y E_{\underline{l}}^Z \mathcal{E}_{\underline{kl}}^{YZ} + E_{\underline{k}}^Y E_{\underline{l}}^Z \mathcal{E}_{kl}^{YZ},$$

$$\mathcal{F}_0 = E^Y E^Z \mathcal{E}$$

$$\mathcal{F}_1 = \int d^4 \mathbf{x}_1 \frac{1}{2} (E_{1\circ}^Y E^Z \mathcal{E}_{1\circ}^Y + E^Y E_{1\circ}^Z \mathcal{E}_{1\circ}^Z)$$

$$\mathcal{F}_2 = \int d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 \frac{1}{4} (E_{12\circ}^Y E^Z \mathcal{E}_{12\circ}^{YY} + E_{1\circ}^Y E_{2\circ}^Z \mathcal{E}_{12\circ}^{YZ} + E_{2\circ}^Y E_{1\circ}^Z \mathcal{E}_{12\circ}^{ZY} + E^Y E_{12\circ}^Z \mathcal{E}_{12\circ}^{ZZ})$$

$$\mathcal{F}_3 = \int \prod_{\kappa=1}^4 (d^4 \mathbf{x}_{\kappa}) \frac{1}{8} \left(E_{1234\circ}^Y E^Z \mathcal{E}_{1234\circ}^{YYYY} + E_{(12)34\circ}^Y E_{34\circ}^Z \mathcal{E}_{(1234)\circ}^{(YYYY)} + E_{(12)34\circ}^Y E_{34\circ}^Z \mathcal{E}_{(1234)\circ}^{(ZZZZ)} + E^Y E_{1234\circ}^Z \mathcal{E}_{1234\circ}^{ZZZZ} \right)$$

$$E_{(12)34\circ}^Y E_{34\circ}^Z \mathcal{E}_{(1234)\circ}^{(YYYY)} = E_{12}^Y E_{34\circ}^Z \mathcal{E}_{1234\circ}^{YYYY} + E_{13\circ}^Y E_{24\circ}^Z \mathcal{E}_{1234\circ}^{YZYZ} + E_{23\circ}^Y E_{14\circ}^Z \mathcal{E}_{1234\circ}^{ZYYZ}$$



$$\mathcal{F}_n = 2^{-n} \int \prod_{\kappa=\square}^n (d^4 \mathbf{x}_\kappa) \sum_{a=\square}^n E_{\dots a}^Y E_{a+1 \dots n}^Z \mathcal{E}_{\dots a \ a+1 \dots n}^{(Y \dots Y \ Z \dots Z)}$$

$$H = H_0 + H_{\text{int}} = H_0^S + H_0^D + H_0^\phi + H^S \phi + H^D \phi$$

$$H = \underbrace{\sum_n \Omega_n^S |n^S\rangle \langle n^S| + \sum_n \Omega_n^D |n^D\rangle \langle n^D| + \int d^3 \mathbf{x} \left(\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right)}_{H_0} + \underbrace{\sum_{m,n} \mu_{mn}^S e^{i\Omega_{mn}^S t'_j} |m^S\rangle \langle n^S| \phi(\mathbf{x}^S, t'_j) + \sum_{m,n} \mu_{mn}^D e^{i\Omega_{mn}^D t'_j} |m^D\rangle \langle n^D| \phi(\mathbf{x}^D, t'_j)}_{H_{\text{int}}(t'_j)}$$

$$\Omega_{mn}^X = \Omega_m^X - \Omega_n^X$$

$$\mathbb{P}_{\text{Fermi}} = |\langle f | U_{t,0} | i \rangle|^2 = |\langle g^S q^D 0 \phi | U_{t,0} | p^S g^D 0 \phi \rangle|^2,$$

$$\mathbb{P}_{\text{Fermi}} = \left| \int_0^t dt_1 \int_0^{t_1} dt_2 \mu_{gp}^S \mu_{qg}^D e^{i\Omega_{gp}^S t_1} e^{i\Omega_{qg}^D t_2} \Delta_{12}^{SD(F)} \right|^2 + \dots,$$

$$\Delta_{12}^{SD(F)} \equiv \langle 0 \phi | T \{ \phi_1^S \phi_2^D \} | 0 \phi \rangle,$$

$$\phi_j^X \equiv \phi(\mathbf{x}^X, t')$$

$$M^X(t'_j) := \sum_{mn} \mu_{mn}^X e^{i\Omega_{mn}^X t'_j} |m^X\rangle \langle n^X|, \text{ for } X \in \{S, D\},$$

$$H_{\text{int}}(t'_j) = M^S(t'_j) \phi_j^S + M^D(t'_j) \phi_j^D$$

$$\rho_0 = \gamma |i_p\rangle \langle i_p| + (1 - \gamma) |i_g\rangle \langle i_g|$$

$$|i_p\rangle = |p^S\rangle \otimes |g^D\rangle \otimes |0\phi\rangle \equiv |p^S g^D 0\phi\rangle,$$

$$|i_g\rangle = |g^S\rangle \otimes |g^D\rangle \otimes |0\phi\rangle \equiv |g^S g^D 0\phi\rangle.$$

$$E = \sum_{n,\alpha} |n^S q^D \alpha \phi\rangle \langle n^S q^D \alpha \phi| = \mathbb{I}^S |q^D\rangle \langle q^D| \mathbb{I}^\phi,$$

$$\sigma_{pg} := \frac{d\mathbb{P}}{d\gamma} = \mathbb{P}_p - \mathbb{P}_g$$

$$\mathbb{P} = \gamma \mathbb{P}_p + (1 - \gamma) \mathbb{P}_g$$

$$\mathbb{P}_{p,g} = \sum_{j=0}^{\infty} \int_0^t dt'_1 dt'_2 \dots dt'_j \Theta_{12 \dots j} \langle i_{p,g} | \mathcal{F}_j | i_{p,g} \rangle$$

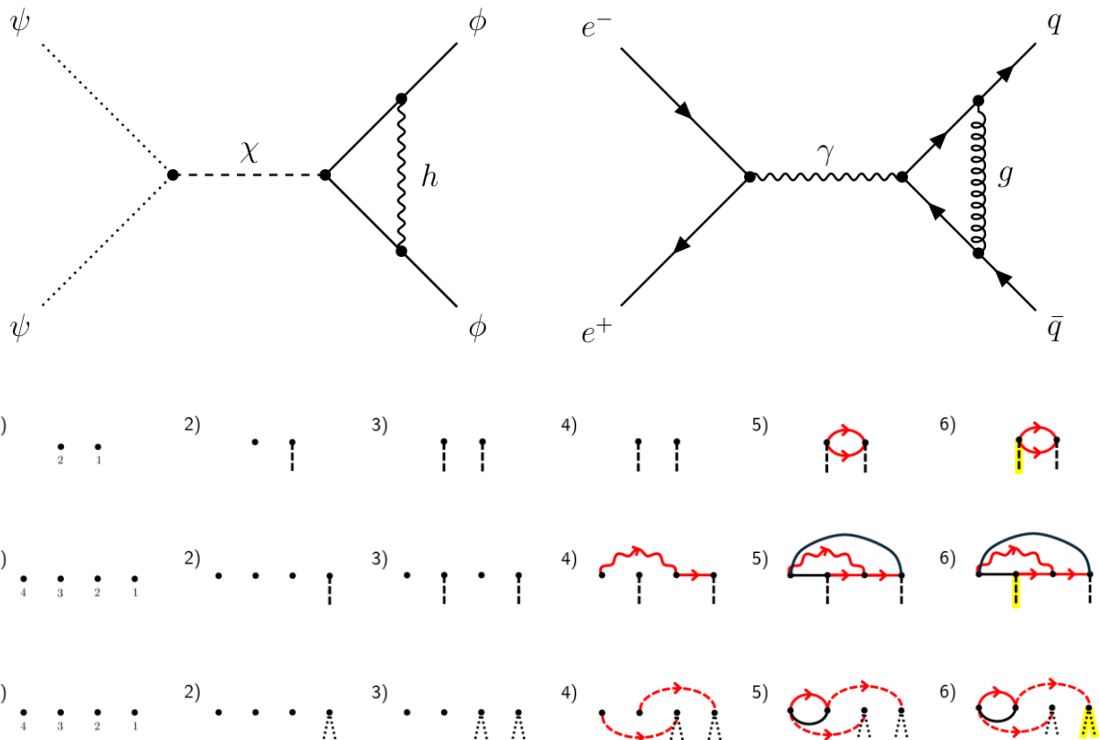


$$\begin{aligned}
\langle i_p | \mathcal{F}_4 | i_p \rangle &\supset \langle p^S g^D 0 \phi | \frac{1}{16} (E_{12}^D E_{34}^S + \mathcal{E}_{1234}^{DDSS} + E_{13}^D E_{24}^S \mathcal{E}_{1234}^{DSDS} + E_{14}^D E_{23}^S \mathcal{E}_{1234}^{DSSD}) | p^S g^D 0 \phi \rangle \\
&= \frac{1}{16} \langle E_{12}^D \rangle (\langle E_{34}^S \rangle \langle \mathcal{E}_{1234}^{DSS} \rangle + \langle E_{34}^S \rangle \langle \mathcal{E}_{1234}^{DDSS} \rangle) \\
&\quad + \frac{1}{16} \langle E_{13}^D \rangle (\langle E_{24}^S \rangle \langle \mathcal{E}_{1234}^{DSDS} \rangle + \langle E_{24}^S \rangle \langle \mathcal{E}_{1234}^{DSDS} \rangle) \\
&\quad + \frac{1}{16} \langle E_{14}^D \rangle \langle E_{23}^S \rangle \langle \mathcal{E}_{1234}^{DSSD} \rangle + \frac{1}{16} \langle E_{14}^D \rangle \langle E_{23}^S \rangle \langle \mathcal{E}_{1234}^{DSSD} \rangle \\
&= 2 \sum_n |\mu_{pn}^S|^2 |\mu_{qg}^D|^2 \times \\
&\quad \left\{ \cos \Omega_{qg}^D t'_{12} \left(\sin \Omega_{pn}^S t'_{34} \Delta_{24}^{DS(H)} + \cos \Omega_{pn}^S t'_{34} \Delta_{24}^{DS(R)} \right) \Delta_{13}^{DS(R)} \right. \\
&\quad + \cos \Omega_{qg}^D t'_{12} \left(\sin \Omega_{pn}^S t'_{34} \Delta_{14}^{DS(H)} + \cos \Omega_{pn}^S t'_{34} \Delta_{14}^{DS(R)} \right) \Delta_{23}^{DS(R)} \\
&\quad + \cos \Omega_{qg}^D t'_{13} \left(\sin \Omega_{pn}^S t'_{24} \Delta_{34}^{DS(H)} + \cos \Omega_{pn}^S t'_{24} \Delta_{34}^{DS(R)} \right) \Delta_{12}^{DS(R)} \\
&\quad \left. + \sin \Omega_{pn}^S t'_{23} \left(\cos \Omega_{qg}^D t'_{14} \Delta_{34}^{SD(H)} + \sin \Omega_{qg}^D t'_{14} \Delta_{34}^{SD(R)} \right) \Delta_{12}^{DS(R)} \right\}.
\end{aligned}$$

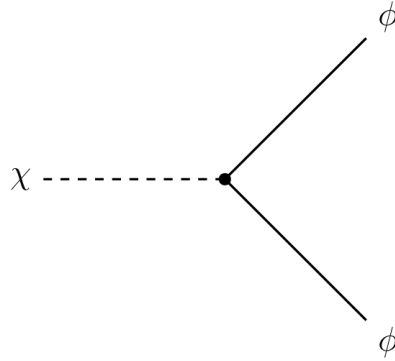
$$\sigma_{pg} = \mathbb{P}_p \ominus \mathbb{P}_g \oslash \Delta_{ij}^{XY(H)} \otimes \Delta_{ij}^{XY(R)}$$

$$\Delta_{ij}^{XY(R)} \equiv \Theta_{ij} \langle 0 | \frac{1}{i} [\phi_i^X, \phi_j^Y] | 0 \rangle$$

$$\Delta_{ij}^{XY(H)} \equiv \langle 0 | \{ \phi_i^X, \phi_j^Y \} | 0 \rangle$$



$$H_{\text{int}}(t_j) = \int d^4 \mathbf{x}_j (g_\chi \phi_j^2 \chi_j + g_h \phi_j^2 h_j)$$



$$\rho_0 = |0^h p^\chi 0^\phi\rangle\langle 0^h p^\chi 0^\phi|.$$

$$E = \sum_{n,\alpha} |n^h 0^\chi \alpha^\phi\rangle\langle n^h 0^\chi \alpha^\phi| = \mathbb{I}^h |0^\chi\rangle\langle 0^\chi| \mathbb{I}^\phi,$$

$$\text{i.e., } E^h = \mathbb{I}^h, E^\chi = |0^\chi\rangle\langle 0^\chi|, E^\phi = \mathbb{I}^\phi,$$

$$E^h = |0^h\rangle\langle 0^h| \text{ (at } \mathcal{O}(g_h^0)\text{)}, E^\chi = |0^\chi\rangle\langle 0^\chi|, \text{ and } E^\phi = |q_1^\phi, q_2^\phi\rangle\langle q_1^\phi, q_2^\phi|$$

$$\text{Hilbert space } \phi: \mathcal{E}_{\dots k}^{\dots h} := \frac{1}{i} [\mathcal{E}_{\dots}, g_h \phi_k^2], \quad \mathcal{E}_{\dots \underline{k}}^{\dots h} := \{\mathcal{E}_{\dots}, g_h \phi_k^2\},$$

$$\mathcal{E}_{\dots k}^{\dots \chi} := \frac{1}{i} [\mathcal{E}_{\dots}, g_\chi \phi_k^2], \quad \mathcal{E}_{\dots \underline{k}}^{\dots \chi} := \{\mathcal{E}_{\dots}, g_\chi \phi_k^2\},$$

$$\text{Hilbert space } h: E_{\dots k}^h := \frac{1}{i} [E_{\dots}, h_k], \quad E_{\dots \underline{k}}^h := \{E_{\dots}, h_k\},$$

$$\text{Hilbert space } \chi: E_{\dots k}^\chi := \frac{1}{i} [E_{\dots}, \chi_k], \quad E_{\dots \underline{k}}^\chi := \{E_{\dots}, \chi_k\}$$

$$\mathcal{F}_n = 2^{-n} \int \prod_{k=1}^n (d^4 \mathbf{x}_k) \sum_{a=0}^n E_{(1 \dots a \dots)}^h E_{(a+1 \dots n)}^\chi \mathcal{E}_{(1 \dots a \dots)}^{(h \dots h)} \mathcal{E}_{(a+1 \dots n)}^{\chi \dots \chi}.$$

$$\mathcal{F}_2 = \frac{1}{4} \int d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 \left(E_{12\Box}^h E^\chi \mathcal{E}_{12\Box}^{hh} + E_{1\Box}^h E_{2\Box}^\chi \mathcal{E}_{12\Box}^{h\chi} + E_{2\Box}^h E_{1\Box}^\chi \mathcal{E}_{12\Box}^{\chi h} + E^h E_{12\Box}^\chi \mathcal{E}_{12\Box}^{\chi\chi} \right)$$

$$\begin{aligned} \mathcal{F}_2 &= \frac{1}{4} \int d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 E_{12\Box}^\chi \mathcal{E}_{12\Box}^{\chi\chi} \\ &= \frac{1}{4} \int d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 \left(E_{12}^\chi \mathcal{E}_{12}^{\chi\chi} + E_{12}^\chi \mathcal{E}_{12}^{\chi\chi} \right) \end{aligned}$$

$$\langle i | \mathcal{F}_2 | i \rangle = \frac{1}{4} \int d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 \left(\langle p^\chi | E_{12}^\chi | p^\chi \rangle \langle 0^\phi | \mathcal{E}_{12}^{\chi\chi} | 0^\phi \rangle + \langle p^\chi | E_{12}^\chi | p^\chi \rangle \langle 0^\phi | \mathcal{E}_{12}^{\chi\chi} | 0^\phi \rangle \right)$$

$$\begin{aligned} \langle p^\chi | E_{12\Box}^\chi | p^\chi \rangle &= \langle 0^\chi | a(p) E_{12\Box}^\chi a^\dagger(p) | 0^\chi \rangle \\ &= \frac{1}{i} \left(\frac{1}{i} \right)^{(1-\eta_2)/2} \langle 0^\chi | a(p) [|0^\chi\rangle\langle 0^\chi|, \chi_1, \chi_2]_{\eta_2} a^\dagger(p) | 0^\chi \rangle \\ &= - \left(\frac{1}{i} \right)^{(3-\eta_2)/2} \langle 0^\chi | a(p) \chi_1 | 0^\chi \rangle \langle 0^\chi | \chi_2 a^\dagger(p) | 0^\chi \rangle \\ &\quad + \eta_2 \langle 0^\chi | a(p) \chi_2 | 0^\chi \rangle \langle 0^\chi | \chi_1 a^\dagger(p) | 0^\chi \rangle \end{aligned}$$



$$\begin{aligned}
[A, B]_\eta &= AB + \eta BA = (1 + \eta)AB - \eta[A, B] \\
&= \begin{cases} [A, B] & \text{if } \eta = -1, \\ \{A, B\} & \text{if } \eta = +1, \end{cases}
\end{aligned}$$

such that $\eta_2 = 1$ for E_{12}^χ and $\eta_2 = -1$ for $E_{12}^{\bar{\chi}}$

$$\begin{aligned}
\langle p^\chi | E_{12}^\chi | p^\chi \rangle &= -\left(\frac{1}{i}\right)^{(3-\eta_2)/2} \int \frac{d^4 \mathbf{k}_1 e^{ik_1 \cdot x_1}}{(2\pi)^3 \sqrt{2\omega_1}} \int \frac{d^4 \mathbf{k}_2 e^{-ik_2 \cdot x_2}}{(2\pi)^3 \sqrt{2\omega_2}} \\
&\quad \times \langle 0^\chi | a(\mathbf{p}) a^\dagger(k_1) | 0^\chi \rangle \langle 0^\chi | a(k_2) a^\dagger(\mathbf{p}) | 0^\chi \rangle + \eta_2 \\
&\quad - \left(\frac{1}{i}\right)^{(3-\eta_2)/2} \int \frac{d^4 \mathbf{k}_1 e^{ik_1 \cdot x_1}}{\sqrt{2\omega_1}} \int \frac{d^4 \mathbf{k}_2 e^{-ik_2 \cdot x_2}}{\sqrt{2\omega_2}} \delta^3(\mathbf{p} - \mathbf{k}_1) \delta^3(\mathbf{k}_2 - \mathbf{p}) + \eta_2 \\
&= \left(\frac{1}{i}\right)^{(3-\eta_2)/2} \frac{1}{2\omega_{\mathbf{p}}} (-e^{ip \cdot x_1} e^{-ip \cdot x_2} + \eta_2 e^{ip \cdot x_2} e^{-ip \cdot x_1})
\end{aligned}$$

$$\omega_1 \equiv p_1^0, \omega_2 \equiv p_2^0, \text{ and } \omega_1 = \omega_2 = \omega_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m_\chi^2}$$

$$\begin{aligned}
\langle 0^\phi | \mathcal{E}_{12\Box}^{\chi\chi} | 0^\phi \rangle &= \left(\frac{1}{i}\right)^{(1-\epsilon_2)/2} g_\chi^2 \langle 0^\phi | [\{\Pi^\phi, \phi_1^2\}, \phi_2^2]_{\epsilon_2} | 0^\phi \rangle \\
&= 2 \left(\frac{1}{i}\right)^{(1-\epsilon_2)/2} g_\chi^2 \langle 0^\phi | [\phi_1^2, \phi_2^2]_{\epsilon_2} | 0^\phi \rangle
\end{aligned}$$

$$[f(\phi_1, \dots, \phi_{n-1}), g(\phi_n)] = - \sum_{\substack{k_1 \\ \vdots \\ k_{n-1} \\ k = \sum_{i=1}^{n-1} k_i}} \dots \sum_{k_{n-1}} \left(\prod_{i=1}^{n-1} \frac{(-[\phi_i, \phi_n]^{k_i})}{k_i!} \right) (\partial_{\phi_1}^{k_1} \dots \partial_{\phi_{n-1}}^{k_{n-1}} f \partial_{\phi_n}^k g),$$

$$\begin{aligned}
\langle 0^\phi | \mathcal{E}_{12\Box}^{\chi\chi} | 0^\phi \rangle &= \left(\frac{1}{i}\right)^{(1-\epsilon_2)/2} [2g_\chi^2 \langle 0^\phi | (1 + \epsilon_2) \phi_1^2 \phi_2^2 | 0^\phi \rangle \\
&\quad - 2g_\chi^2 \epsilon_2 \langle 0^\phi | (\Delta_{12}^\phi \partial_1 [\phi_1^2] \partial_2 [\phi_2^2] \\
&\quad - \frac{1}{2} (\Delta_{12}^\phi)^2 \partial_1^2 [\phi_1^2] \partial_2^2 [\phi_2^2]) | 0^\phi \rangle] \\
&= \left(\frac{1}{i}\right)^{(1-\epsilon_2)/2} [2g_\chi^2 (1 + \epsilon_2) \langle 0^\phi | \phi_1^2 \phi_2^2 | 0^\phi \rangle \\
&\quad - 8g_\chi^2 \epsilon_2 \Delta_{12}^\phi \langle 0^\phi | \phi_1 \phi_2 | 0^\phi \rangle + 4g_\chi^2 \epsilon_2 (\Delta_{12}^\phi)^2]
\end{aligned}$$



$$\begin{aligned}
\langle i|\mathcal{F}_2|i\rangle &= \frac{g_\chi^2}{4i^2} \int d^4\mathbf{x}_1 d^4\mathbf{x}_2 \left(\frac{1}{2\omega_{\mathbf{p}}} (-e^{ip\cdot x_1} e^{-ip\cdot x_2} - e^{ip\cdot x_2} e^{-ip\cdot x_1}) \right. \\
&\quad (4\langle 0^\phi|\phi_1^2\phi_2^2|0^\phi\rangle - 8\Delta_{12}\langle 0^\phi|\phi_1\phi_2|0^\phi\rangle + 4\Delta_{12}^2) \\
&\quad + \frac{1}{2\omega_{\mathbf{p}}} (-e^{ip\cdot x_1} e^{-ip\cdot x_2} + e^{ip\cdot x_2} e^{-ip\cdot x_1}) \\
&\quad \left. (8\Delta_{12}\langle 0^\phi|\phi_1\phi_2|0^\phi\rangle - 4\Delta_{12}^2) \right) \\
&= \frac{g_\chi^2}{2\omega_{\mathbf{p}}} \int d^4\mathbf{x}_1 d^4\mathbf{x}_2 (e^{ip\cdot x_1} e^{-ip\cdot x_2} \langle 0^\phi|\phi_1^2\phi_2^2|0^\phi\rangle \\
&\quad + e^{ip\cdot x_2} e^{-ip\cdot x_1} (\langle 0^\phi|\phi_1^2\phi_2^2|0^\phi\rangle - 4\Delta_{12}^\phi \langle 0^\phi|\phi_1\phi_2|0^\phi\rangle + 2(\Delta_{12}^\phi)^2))
\end{aligned}$$

$$\Theta_{12}\langle 0^\phi|\phi_1^n\phi_2^m|0^\phi\rangle = \Theta_{12}\langle 0^\phi|\mathbb{T}\{\phi_1^n\phi_2^m\}|0^\phi\rangle$$

$$\mathbb{P}_{j=2} = \frac{g_\chi^2}{2\omega_{\mathbf{p}}} \int d^4x_1 d^4x_2 \Theta_{12} \left(\begin{array}{c} \text{diagram 1} + 2 \text{diagram 2} \\ + \text{diagram 3} + 2 \text{diagram 4} - 4 \text{diagram 5} + 2 \text{diagram 6} \end{array} \right)$$

$$\begin{aligned}
\mathbb{P}_{j=2} &= \frac{g_\chi^2}{2\omega_{\mathbf{p}}} \int d^4x_1 d^4x_2 \Theta_{12} \left(e^{ip\cdot x_1} e^{-ip\cdot x_2} (F_{11}^\phi F_{22}^\phi + 2F_{12}^\phi F_{12}^\phi) \right. \\
&\quad \left. + e^{ip\cdot x_2} e^{-ip\cdot x_1} (F_{11}^\phi F_{22}^\phi + 2F_{12}^\phi F_{12}^\phi - 4R_{12}^\phi F_{12}^\phi + 2(R_{12}^\phi)^2) \right),
\end{aligned}$$

$$\begin{aligned}
\Theta_{12}F_{12}^\phi &= \Theta_{12}\Delta_{12}^{\phi(>)} \\
\Theta_{12}R_{12}^\phi &= \Theta_{12}\Delta_{12}^\phi = \Theta_{12}^\phi (\Delta_{12}^{\phi(>)} - \Delta_{12}^{\phi(<)}) ,
\end{aligned}$$

$$\mathbb{P}_{j=2} = \frac{2g_\chi^2}{2\omega_{\mathbf{p}}} \int d^4x_1 d^4x_2 \Theta_{12} \left(e^{ip\cdot x_1} e^{-ip\cdot x_2} (\Delta_{12}^{\phi(>)})^2 + e^{ip\cdot x_2} e^{-ip\cdot x_1} (\Delta_{12}^{\phi(<)})^2 \right).$$

$$E^\phi = \mathbb{I}^\phi - |0^\phi\rangle\langle 0^\phi|$$

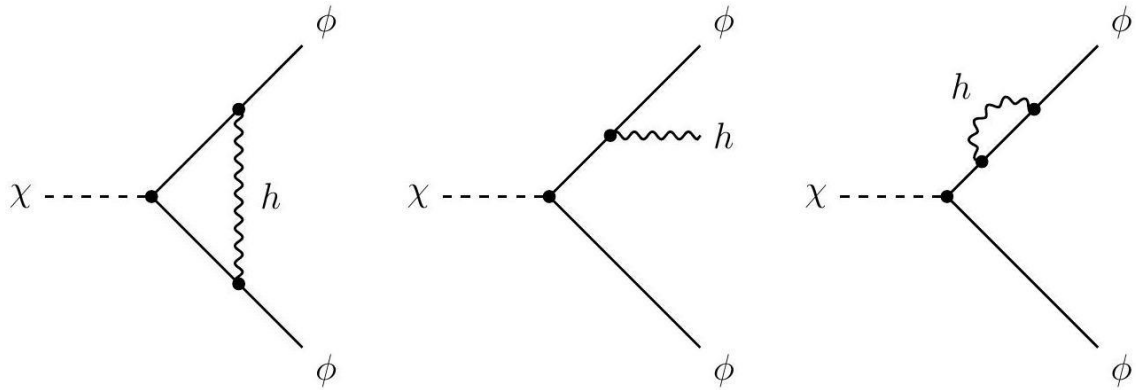
$$\mathbb{P}_{j=2} = \frac{2g_\chi^2}{2\omega_{\mathbf{p}}} \int \frac{d^4\mathbf{q}_1}{(2\pi)^3} \frac{1}{2\omega_{q_1}} \int \frac{d^4\mathbf{q}_2}{(2\pi)^3} \frac{1}{2\omega_{q_2}} [\delta^4(p - q_1 - q_2)]^2 (2\pi)^8.$$

$$\mathcal{F}_4 = \frac{1}{16} \int d^4\mathbf{x}_1 d^4\mathbf{x}_2 d^4\mathbf{x}_3 d^4\mathbf{x}_4 \sum_{a=0}^4 E_{\dots a}^h E_{\dots a+1\dots 4}^\chi \mathcal{E}_{\dots a+1\dots 4}^{(h\dots h \chi\dots \chi)}$$

$$\mathcal{F}_4 = \frac{1}{16} \int d^4\mathbf{x}_1 d^4\mathbf{x}_2 d^4\mathbf{x}_3 d^4\mathbf{x}_4 \sum_{a=0}^4 E_{\binom{12}{\square}}^h E_{\binom{34}{\square}}^\chi \mathcal{E}_{\binom{1234}{\square}}^{(hh\chi\chi)}$$



$$\int E_{ij\Box}^h E_{kl\Box}^\chi \mathcal{E}_{(1234)\Box}^{(hh\chi\chi)} \int E_{ij\Box}^h E_{kl\Box}^\chi \mathcal{E}_{1234\Box}^{(hh\chi\chi)}$$



$$\begin{aligned} E_{ij\Box}^h E_{1l\Box}^\chi \mathcal{E}_{1234\Box}^{(hh\chi\chi)} &= E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi hh\chi} + E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi hh\chi} + E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi hh\chi} + E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi hh\chi} \\ &+ E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h\chi h} + E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h\chi h} + E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h\chi h} + E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h\chi h} \\ &+ E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} + E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} + E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} + E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh}. \end{aligned}$$

$$\langle 0^h | E_{ij\Box}^h | 0^h \rangle = \langle 0^h | [\{\mathbb{I}^h, h_i\}, h_j]_{\lambda_j} | 0^h \rangle = 2 \langle 0^h | [h_i, h_j]_{\lambda_j} | 0^h \rangle$$

$$\langle p^\chi | E_{1l\Box}^\chi | p^\chi \rangle = \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_l} + \eta_l e^{ip \cdot x_l} e^{-ip \cdot x_1})$$

$$\mathcal{E}_{1234\Box} = [\mathcal{E}_{1234\Box}, \phi_4^2]_{\epsilon_4} \star \partial^2 g_h^2 \oslash \partial^2 g_\chi^2$$

$$\mathcal{E}_{123\Box} = [\mathcal{E}_{123\Box}, \phi_3^2]_{\epsilon_3} = (1 + \epsilon_3) \mathcal{E}_{123\Box} \phi_3^2 - \epsilon_3 [\mathcal{E}_{123\Box}, \phi_3^2]$$

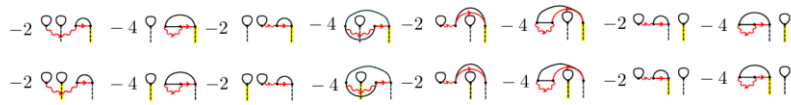
$$\begin{aligned} \mathcal{E}_{123\Box} &= 2(1 + \epsilon_3) \left((1 + \epsilon_2) \phi_1^2 \phi_2^2 \phi_3^2 - 4\epsilon_2 \Delta_{12} \phi_1 \phi_2 \phi_3^2 + 2\epsilon_2 (\Delta_{12})^2 \phi_3^2 \right) \\ &+ 4\epsilon_3 \left(-2\Delta_{13} \left((1 + \epsilon_2) \phi_1 \phi_2^2 \phi_3 - 2\epsilon_2 \Delta_{12} \phi_2 \phi_3 \right) \right. \\ &\quad \left. - 2\Delta_{23} \left((1 + \epsilon_2) \phi_1^2 \phi_2 \phi_3 - 2\epsilon_2 \Delta_{12} \phi_1 \phi_3 \right) \right. \\ &\quad \left. + 4\Delta_{13} \Delta_{23} \left((1 + \epsilon_2) \Delta_{13}^2 \phi_2^2 - \epsilon_2 \Delta_{12} \right) \right. \\ &\quad \left. + (1 + \epsilon_2) (\Delta_{13})^2 \phi_2^2 - (1 + \epsilon_2) (\Delta_{23})^2 \phi_1^2 \right) \end{aligned}$$

$$\begin{aligned}
\varepsilon_{\underline{1234}\square} = & g_h^2 g_\chi^2 \{ 2(1 + \varepsilon_4)((1 + \varepsilon_3)[(1 + \varepsilon_2)\phi_1^2 \phi_2^2 \phi_3^2 \phi_4^2 - 4\varepsilon_2 \Delta_{12} \phi_1 \phi_2 \phi_3^2 \phi_4^2 \\
& + 2\varepsilon_2 (\Delta_{12})^2 \phi_3^2 \phi_4^2] \\
& + 2\varepsilon_3 [-2\Delta_{13} ((1 + \varepsilon_2)\phi_1 \phi_2^2 \phi_3 \phi_4^2 - 2\varepsilon_2 \Delta_{12} \phi_2 \phi_3 \phi_4^2) \\
& - 2\Delta_{23} ((1 + \varepsilon_2)\phi_1^2 \phi_2 \phi_3 \phi_4^2 - 2\varepsilon_2 \Delta_{12} \phi_1 \phi_3 \phi_4^2) \\
& + 4\Delta_{13} \Delta_{23} ((1 + \varepsilon_2)\phi_1 \phi_2 \phi_4^2 - \varepsilon_2 \Delta_{12} \phi_4^2) \\
& + (1 + \varepsilon_2)(\Delta_{13})^2 \phi_2^2 \phi_4^2 + (1 + \varepsilon_2)(\Delta_{23})^2 \phi_1^2 \phi_4^2] \\
& + 4\varepsilon_4 (-2\Delta_{14} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1 \phi_2^2 \phi_3^2 \phi_4 - 2\varepsilon_2 \Delta_{12} \phi_2 \phi_3^2 \phi_4) \\
& + 2\varepsilon_3 (-(1 + \varepsilon_2)\Delta_{13} \phi_2^2 \phi_3 \phi_4 \\
& - 2\Delta_{23} ((1 + \varepsilon_2)\phi_1 \phi_2 \phi_3 \phi_4 - \varepsilon_2 \Delta_{12} \phi_3 \phi_4) \\
& + 2(1 + \varepsilon_2)\Delta_{13} \Delta_{23} \phi_2 \phi_4 + (1 + \varepsilon_2)(\Delta_{23})^2 \phi_1 \phi_4]) \\
& - 2\Delta_{24} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1^2 \phi_2 \phi_3^2 \phi_4 - 2\varepsilon_2 \Delta_{12} \phi_1 \phi_3^2 \phi_4) \\
& + 2\varepsilon_3 (-(1 + \varepsilon_2)\Delta_{23} \phi_1^2 \phi_3 \phi_4 \\
& - 2\Delta_{13} ((1 + \varepsilon_2)\phi_1 \phi_2 \phi_3 \phi_4 - \varepsilon_2 \Delta_{12} \phi_3 \phi_4) \\
& + 2(1 + \varepsilon_2)\Delta_{13} \Delta_{23} \phi_1 \phi_4 + (1 + \varepsilon_2)(\Delta_{13})^2 \phi_2 \phi_4]) \\
& - 2\Delta_{34} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1^2 \phi_2^2 \phi_3 \phi_4 - 4\varepsilon_2 \Delta_{12} \phi_1 \phi_2 \phi_3 \phi_4 \\
& + 2\varepsilon_2 (\Delta_{12})^2 \phi_3 \phi_4) \\
& + 2\varepsilon_3 (-\Delta_{13} ((1 + \varepsilon_2)\phi_1 \phi_2^2 \phi_4 - 2\varepsilon_2 \Delta_{12} \phi_2 \phi_4) \\
& - \Delta_{23} ((1 + \varepsilon_2)\phi_1^2 \phi_2 \phi_4 - 2\varepsilon_2 \Delta_{12} \phi_1 \phi_4))] \\
& + 4\Delta_{14} \Delta_{24} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1 \phi_2 \phi_3^2 - \varepsilon_2 \Delta_{12} \phi_3^2) \\
& + 2\varepsilon_3 (1 + \varepsilon_2)(-\Delta_{13} \phi_2 \phi_3 - \Delta_{23} \phi_1 \phi_3 + \Delta_{13} \Delta_{23})] \\
& + 4\Delta_{14} \Delta_{34} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1 \phi_2^2 \phi_3 - 2\varepsilon_2 \Delta_{12} \phi_2 \phi_3) \\
& + \varepsilon_3 (-(1 + \varepsilon_2)\Delta_{13} \phi_2^2 - 2\Delta_{23} ((1 + \varepsilon_2)\phi_1 \phi_2 - \varepsilon_2 \Delta_{12}))] \\
& + 4\Delta_{24} \Delta_{34} [(1 + \varepsilon_3)((1 + \varepsilon_2)\phi_1^2 \phi_2 \phi_3 - 2\varepsilon_2 \Delta_{12} \phi_1 \phi_3) \\
& + \varepsilon_3 (-(1 + \varepsilon_2)\Delta_{23} \phi_1^2 - 2\Delta_{13} ((1 + \varepsilon_2)\phi_1 \phi_2 - \varepsilon_2 \Delta_{12}))] \\
& + (\Delta_{14})^2 (1 + \varepsilon_2)[(1 + \varepsilon_3)\phi_2^2 \phi_3^2 + 2\varepsilon_3 (-2\Delta_{23} \phi_2 \phi_3 + (\Delta_{23})^2)] \\
& + (\Delta_{24})^2 (1 + \varepsilon_2)[(1 + \varepsilon_3)\phi_1^2 \phi_3^2 + 2\varepsilon_3 (-2\Delta_{13} \phi_1 \phi_3 + (\Delta_{13})^2)] \\
& + (\Delta_{34})^2 (1 + \varepsilon_3)[(1 + \varepsilon_2)\phi_1^2 \phi_2^2 + 2\varepsilon_2 (-2\Delta_{12} \phi_1 \phi_2 + (\Delta_{12})^2)] \}. \\
\Theta_{ij\dots n} \langle 0^\phi | \phi_i \phi_j \dots \phi_n | 0^\phi \rangle = & \Theta_{ij\dots n} \langle 0^\phi | T\{\phi_i \phi_j \dots \phi_n\} | 0^\phi \rangle
\end{aligned}$$

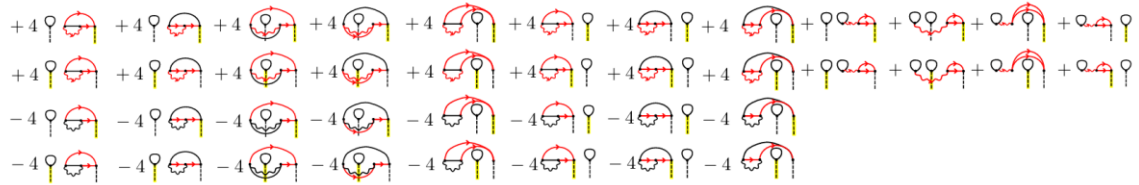


$$\mathbb{P} \supset \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \frac{2g_h^2 g_\chi^2}{2\omega_p} \times$$

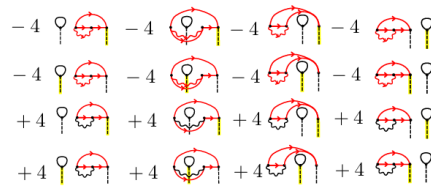
One R^ϕ



Two R^ϕ

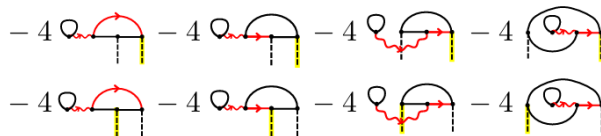


Three R^ϕ

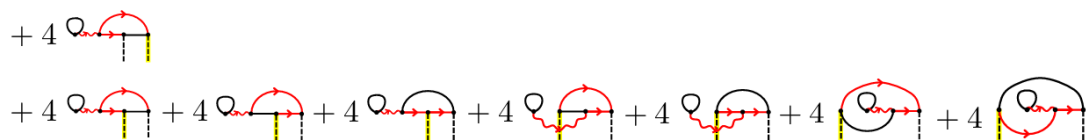


$$\mathbb{P} \supset \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \frac{2g_h^2 g_\chi^2}{2\omega_p} \times$$

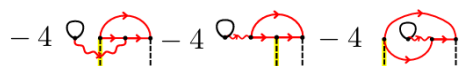
One R^ϕ



Two R^ϕ

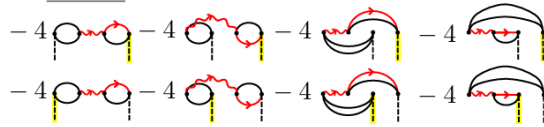


Three R^ϕ

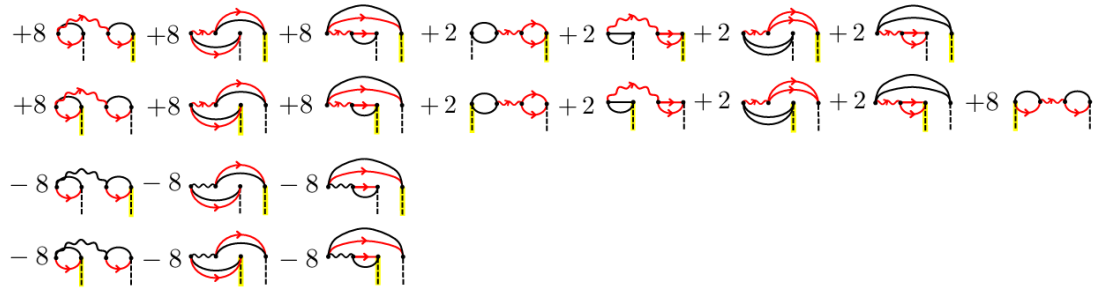


$$\mathbb{P} \supset \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \frac{2g_h^2 g_\chi^2}{2\omega_p} \times$$

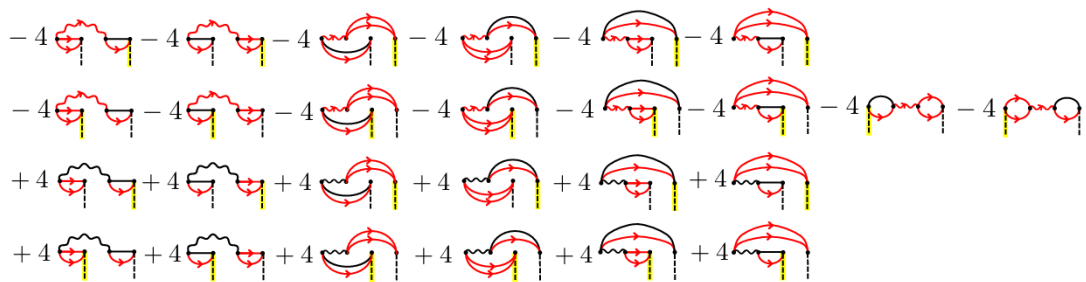
One R^ϕ



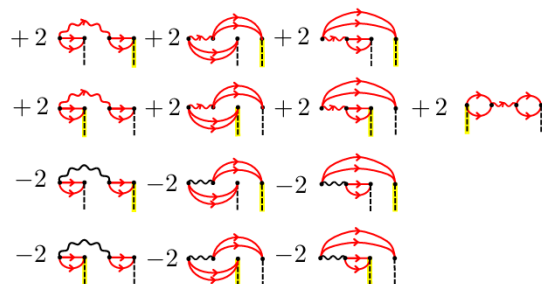
Two R^ϕ



Three R^ϕ

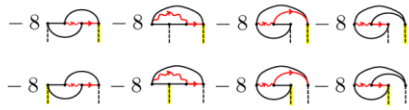


Four R^ϕ

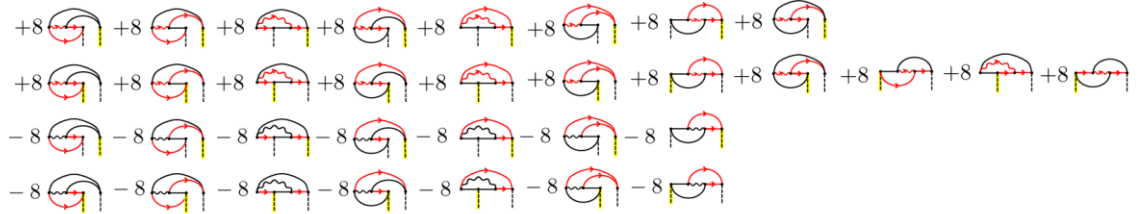


$$\mathbb{P} \supset \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \frac{2g_h^2 g_X^2}{2\omega_P} \times$$

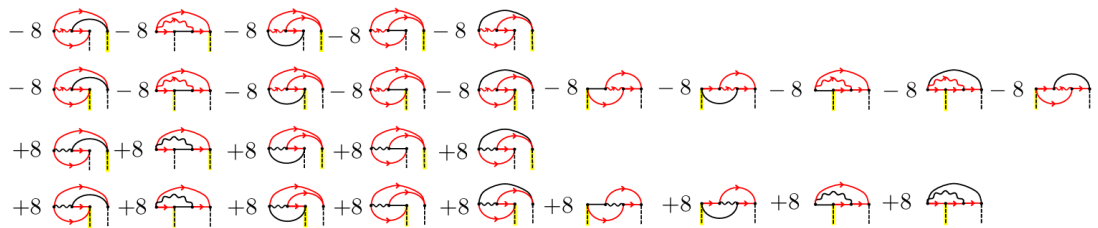
One R^ϕ



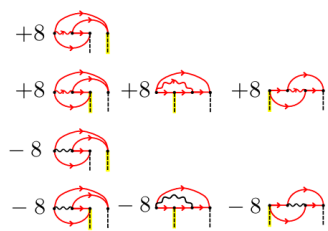
Two R^ϕ



Three R^ϕ



Four R^ϕ



$$\mathcal{F}_4^{a=2} = \frac{1}{16} \int \prod_{\kappa=1}^n d^4 \mathbf{x}_1 d^4 \mathbf{x}_2 d^4 \mathbf{x}_3 d^4 \mathbf{x}_4 \sum_{a=0}^n E_{\left(\begin{smallmatrix} 12 \\ \circ \end{smallmatrix} \square\right)}^\psi E_{\left(\begin{smallmatrix} 34 \\ \circ \end{smallmatrix} \square\right)}^\phi \mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)}^{(\psi\psi\phi\phi)}$$

$$\int E_{\left(\begin{smallmatrix} ij \\ \circ \end{smallmatrix} \square\right)}^\psi E_{\left(\begin{smallmatrix} kl \\ \circ \end{smallmatrix} \square\right)}^\phi \mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)}^{(\psi\psi\phi\phi)} \int E_{\left(\begin{smallmatrix} ij \\ \circ \end{smallmatrix} \square\right)}^\psi E_{\left(\begin{smallmatrix} kl \\ \circ \end{smallmatrix} \square\right)}^\phi \mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)}^{(\psi\psi\phi\phi)}$$

$$\begin{aligned} \langle p_1^\psi p_2^\psi | E_{\left(\begin{smallmatrix} ij \\ \circ \end{smallmatrix} \square\right)}^\psi | p_1^\psi p_2^\psi \rangle &= \langle p_1^\psi p_2^\psi | [[|0^\psi\rangle\langle 0^\psi|, \psi_1^2], \psi_j^2]_{\eta_j} | p_1^\psi p_2^\psi \rangle \\ &= \frac{4}{(2\omega_{p_1})(2\omega_{p_2})} \left(-e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_j} e^{-ip_2 \cdot x_j} \right. \\ &\quad \left. + \eta_j e^{ip_1 \cdot x_j} e^{ip_2 \cdot x_j} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} \right) \end{aligned}$$

$$\begin{aligned} \langle 0^\phi | E_{\left(\begin{smallmatrix} kl \\ \circ \end{smallmatrix} \square\right)}^\phi | 0^\phi \rangle &= \langle 0^\phi | [\{ \mathbb{I}^\phi, \phi_k^2 \}, \phi_l^2]_{\lambda_l} | 0^\phi \rangle = 2 \langle 0^\phi | [\phi_k^2, \phi_l^2]_{\lambda_l} | 0^\phi \rangle \\ &= 2(1 + \lambda_j) \langle 0^\phi | \phi_k^2 \phi_l^2 | 0^\phi \rangle - 8\lambda_l \Delta_{kl}^\phi \langle 0^\phi | \phi_k \phi_l | 0^\phi \rangle + 4\lambda_l (\Delta_{kl}^\phi)^2 \end{aligned}$$

$$\mathcal{E}_{\left(\begin{smallmatrix} 12 \\ \circ \end{smallmatrix} \square\right)} = [\{ \mathbb{I}^\chi, \chi_1 \}, \chi_2]_{\epsilon_2} = 2[\chi_1, \chi_2]_{\epsilon_2} = 2(1 + \epsilon_2)\chi_1\chi_2 - 2\epsilon_2\Delta_{12}^\chi$$

$$\mathcal{E}_{\left(\begin{smallmatrix} 123 \\ \circ \end{smallmatrix} \square\right)} = [\mathcal{E}_{\left(\begin{smallmatrix} 12 \\ \circ \end{smallmatrix} \square\right)}, \chi_3]_{\epsilon_3} = (1 + \epsilon_3)\mathcal{E}_{\left(\begin{smallmatrix} 12 \\ \circ \end{smallmatrix} \square\right)}\chi_3 - \epsilon_3 [\mathcal{E}_{\left(\begin{smallmatrix} 12 \\ \circ \end{smallmatrix} \square\right)}, \chi_3]$$

$$= (1 + \epsilon_3)(2(1 + \epsilon_2)\chi_1\chi_2\chi_3 - 2\epsilon_2\Delta_{12}^\chi\chi_3) - 2\epsilon_3(1 + \epsilon_2)(\Delta_{13}^\chi\chi_2 + \Delta_{23}^\chi\chi_1)$$

$$\mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)} = g_\psi^2 g_\phi^2 [\mathcal{E}_{\left(\begin{smallmatrix} 123 \\ \circ \end{smallmatrix} \square\right)}, \chi_4]_{\epsilon_4} = g_\psi^2 g_\phi^2 (1 + \epsilon_4)\mathcal{E}_{\left(\begin{smallmatrix} 123 \\ \circ \end{smallmatrix} \square\right)}\chi_4 - g_\psi^2 g_\phi^2 \epsilon_4 [\mathcal{E}_{\left(\begin{smallmatrix} 123 \\ \circ \end{smallmatrix} \square\right)}, \chi_4]$$

$$\begin{aligned} &= g_\psi^2 g_\phi^2 (1 + \epsilon_4) \left(2(1 + \epsilon_3) \left((1 + \epsilon_2)\chi_1\chi_2\chi_3\chi_4 - \epsilon_2\Delta_{12}^\chi\chi_3\chi_4 \right) \right. \\ &\quad \left. - 2(1 + \epsilon_2)\epsilon_3(\Delta_{13}^\chi\chi_2\chi_4 + \Delta_{23}^\chi\chi_1\chi_4) \right) \\ &\quad - 2g_\psi^2 g_\phi^2 \epsilon_4 \left((1 + \epsilon_2)\Delta_{14}^\chi \left((1 + \epsilon_3)\chi_2\chi_3 - \epsilon_3\Delta_{23}^\chi \right) \right. \\ &\quad \left. + (1 + \epsilon_2)\Delta_{24}^\chi \left((1 + \epsilon_3)\chi_1\chi_3 - \epsilon_3\Delta_{13}^\chi \right) \right. \\ &\quad \left. + (1 + \epsilon_3)\Delta_{34}^\chi \left((1 + \epsilon_2)\chi_1\chi_2 - \epsilon_2\Delta_{12}^\chi \right) \right) \end{aligned}$$

$$\begin{aligned} E_{\left(\begin{smallmatrix} i(2 \\ \circ \end{smallmatrix} \square\right)}^\psi E_{\left(\begin{smallmatrix} 34 \\ \circ \end{smallmatrix} \square\right)}^\phi \mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)}^{(\psi\psi\phi\phi)} &= E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} + E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} + E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} + E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} \\ &+ E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} + E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} + E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} + E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} \\ &+ E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} + E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} + E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} + E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} . \end{aligned}$$

$$\begin{aligned} E_{\left(\begin{smallmatrix} i(2 \\ \circ \end{smallmatrix} \square\right)}^\psi E_{\left(\begin{smallmatrix} 34 \\ \circ \end{smallmatrix} \square\right)}^\phi \mathcal{E}_{\left(\begin{smallmatrix} 1234 \\ \circ \end{smallmatrix} \square\right)}^{(\psi\psi\phi\phi)} &= E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} + E_{12}^\psi E_{34}^\phi \mathcal{E}_{1234}^{\psi\psi\phi\phi} + E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} + E_{13}^\psi E_{24}^\phi \mathcal{E}_{1234}^{\psi\phi\psi\phi} \\ &+ E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} + E_{14}^\psi E_{23}^\phi \mathcal{E}_{1234}^{\psi\phi\phi\psi} \end{aligned}$$



$$\begin{aligned}
\mathbb{P} = & -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{\mathbf{p}_1})(2\omega_{\mathbf{p}_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \\
& \left((e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} + e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \right. \\
& \left(2\Delta_{34}^\phi \langle \phi_3 \phi_4 \rangle - (\Delta_{34}^\phi)^2 \right) \\
& (2\Delta_{13}^\chi \langle \chi_2 \chi_4 \rangle + 2\Delta_{23}^\chi \langle \chi_1 \chi_4 \rangle - \Delta_{14}^\chi \Delta_{23}^\chi - \Delta_{13}^\chi \Delta_{24}^\chi) \\
& + (e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} + e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \\
& \left(\langle \phi_3^2 \phi_4^2 \rangle - 2\Delta_{34}^\phi \langle \phi_3 \phi_4 \rangle + (\Delta_{34}^\phi)^2 \right) (\Delta_{14}^\chi \Delta_{23}^\chi + \Delta_{13}^\chi \Delta_{24}^\chi) \\
& + (e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_3} e^{-ip_2 \cdot x_3} + e^{ip_1 \cdot x_3} e^{ip_2 \cdot x_3} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \\
& \left(2\Delta_{24}^\phi \langle \phi_2 \phi_4 \rangle - (\Delta_{24}^\phi)^2 \right) (2\Delta_{12}^\chi \langle \chi_3 \chi_4 \rangle - \Delta_{12}^\chi \Delta_{34}^\chi) \\
& + (e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_3} e^{-ip_2 \cdot x_3} + e^{ip_1 \cdot x_3} e^{ip_2 \cdot x_3} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \\
& \left(\langle \phi_2^2 \phi_4^2 \rangle - 2\Delta_{24}^\phi \langle \phi_2 \phi_4 \rangle + (\Delta_{24}^\phi)^2 \right) (\Delta_{12}^\chi \Delta_{34}^\chi) \\
& + (e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_4} e^{-ip_2 \cdot x_4} + e^{ip_1 \cdot x_4} e^{ip_2 \cdot x_4} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \\
& \left(2\Delta_{23}^\phi \langle \phi_2 \phi_3 \rangle - (\Delta_{23}^\phi)^2 \right) (2\Delta_{12}^\chi \langle \chi_3 \chi_4 \rangle - \Delta_{12}^\chi \Delta_{34}^\chi) \\
& + (e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_4} e^{-ip_2 \cdot x_4} - e^{ip_1 \cdot x_4} e^{ip_2 \cdot x_4} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1}) \\
& \left. \left(2\Delta_{23}^\phi \langle \phi_2 \phi_3 \rangle - (\Delta_{23}^\phi)^2 \right) (\Delta_{12}^\chi \Delta_{34}^\chi) \right)
\end{aligned}$$

$$\Pi_{xy}^R = \frac{g_\phi^2}{2} \left[2F_{xy}^\phi R_{xy}^\phi - (R_{xy}^\phi)^2 \right]$$



$$\mathbb{P} = -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{\mathbf{p}_1})(2\omega_{\mathbf{p}_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times$$

One R^χ

$$\text{One } R^\phi \left\{ \begin{array}{l} +4 \text{ [diagram 1]} +4 \text{ [diagram 2]} +4 \text{ [diagram 3]} +4 \text{ [diagram 4]} \\ +4 \text{ [diagram 5]} +4 \text{ [diagram 6]} +4 \text{ [diagram 7]} +4 \text{ [diagram 8]} \end{array} \right.$$

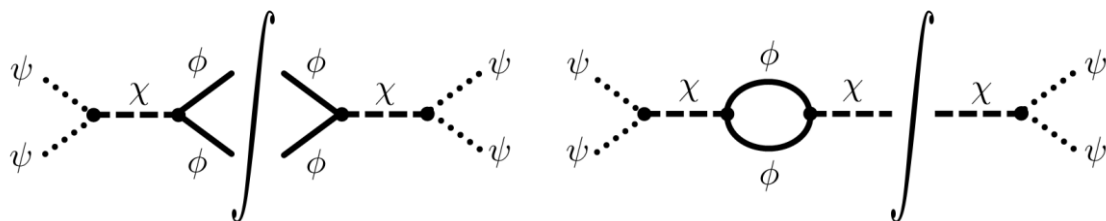
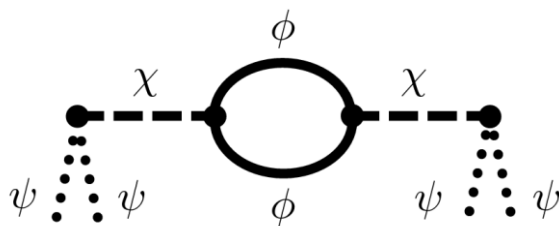
$$\text{Two } R^\phi \left\{ \begin{array}{l} -2 \text{ [diagram 1]} -2 \text{ [diagram 2]} -2 \text{ [diagram 3]} -2 \text{ [diagram 4]} \\ -2 \text{ [diagram 5]} -2 \text{ [diagram 6]} -2 \text{ [diagram 7]} -2 \text{ [diagram 8]} \end{array} \right.$$

Two R^χ

$$\text{Zero } R^\phi \left\{ \begin{array}{l} +2 \text{ [diagram 1]} +2 \text{ [diagram 2]} +2 \text{ [diagram 3]} + \text{ [diagram 4]} + \text{ [diagram 5]} + \text{ [diagram 6]} \\ +2 \text{ [diagram 7]} +2 \text{ [diagram 8]} +2 \text{ [diagram 9]} + \text{ [diagram 10]} + \text{ [diagram 11]} + \text{ [diagram 12]} \end{array} \right.$$

$$\text{One } R^\phi \left\{ \begin{array}{l} -4 \text{ [diagram 1]} -4 \text{ [diagram 2]} -4 \text{ [diagram 3]} \\ -4 \text{ [diagram 4]} -4 \text{ [diagram 5]} -4 \text{ [diagram 6]} -4 \text{ [diagram 7]} \end{array} \right.$$

$$\text{Two } R^\phi \left\{ \begin{array}{l} +2 \text{ [diagram 1]} +2 \text{ [diagram 2]} +2 \text{ [diagram 3]} \\ +2 \text{ [diagram 4]} +2 \text{ [diagram 5]} +2 \text{ [diagram 6]} +2 \text{ [diagram 7]} \end{array} \right.$$



$$\begin{aligned}
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} \right) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1} \\
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} \right) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1} \\
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} \right) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1} \\
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1} \\
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1} \\
& \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Theta_{1234} \times \text{diagram 1} = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \text{diagram 1}
\end{aligned}$$

$$\mathbb{P} = -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{\mathbf{p}_1})(2\omega_{\mathbf{p}_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \times \left(4 \text{diagram 1} + 4 \text{diagram 2} + 2 \text{diagram 3} - 4 \text{diagram 4} - 4 \text{diagram 5} - 4 \text{diagram 6} \right)$$

Four-dimensional de Sitter spacetime:

$$dS^2 = dt^2 - dx^2 - dy^2 - dz^2 = du dv - dy^2 - dz^2,$$

$$t = \alpha^{-1} e^{\alpha\xi} \sinh \alpha\eta, x = \alpha^{-1} e^{\alpha\xi} \cosh \alpha\eta, y = y, z = z.$$

$$\begin{aligned} \bar{u} &= -\alpha^{-1} e^{-\alpha u}, \\ \bar{v} &= \alpha e^{\alpha v} \end{aligned}$$

$$dS^2 = e^{2\alpha\xi} (d\eta^2 - d\xi^2) - dy^2 - dz^2 = e^{2\alpha\xi} d\bar{u} d\bar{v} - dy^2 - dz^2.$$

$$\phi(x) = \int d^4\mathbf{k} (a_{\mathbf{k}} u_{\mathbf{k}} + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*)$$

$$\phi(x) = \int d^4\mathbf{k} (b_{\mathbf{k}}^R v_{\mathbf{k}}^R + b_{\mathbf{k}}^{R\dagger} v_{\mathbf{k}}^{R*} + b_{\mathbf{k}}^L v_{\mathbf{k}}^L + b_{\mathbf{k}}^{L\dagger} v_{\mathbf{k}}^{L*})$$

$$\langle 0_{\mathbf{M}} | b_{\mathbf{k}}^{R(L)\dagger} b_{\mathbf{k}}^{R(L)} | 0_{\mathbf{M}} \rangle = \frac{b_{\mathbf{k}}^{R/L\dagger}}{e^{2\pi\omega/\alpha} - 1}$$

$$\omega = \sqrt{\mathbf{k}^2 + m^2}$$

$$T_U = \frac{\alpha}{2\pi}$$



$$|0_M\rangle = \prod_i \left(\sqrt{1 - e^{-2\pi\omega_i/\alpha}} \sum_{n_i=b_k^{R/L}}^{\infty} e^{-\pi\omega n_i/\alpha} |n_{i,R}\rangle \otimes |n_{i,L}\rangle \right).$$

$$\rho_R = \prod_i \left((1 - e^{-2\pi\omega_i/\alpha}) \sum_{n_i=0}^{\infty} e^{-2\pi\omega n_i/\alpha} |n_{i,R}\rangle \langle n_{i,R}| \right).$$

$$H_{\text{int}} = \lambda \chi(\tau) M^D(\tau) \phi(\mathbf{x}^D(\tau), \tau),$$

$$\mathbf{x}^D = \left(\frac{1}{\alpha} \cosh \alpha\tau, 0, 0 \right) = \left(\frac{1}{\alpha} \sqrt{1 + \alpha^2 t^2}, 0, 0 \right),$$

$$\tau = \int_0^t \frac{dt'}{\gamma(t')} = \int_0^t \frac{dt'}{\sqrt{1 + \alpha^2 t'^2}} = \frac{1}{\alpha} \operatorname{arcsinh} \alpha t$$

$$\mathcal{H} = \mathcal{H}^D \otimes \mathcal{H}^\phi$$

$$H(t) = H_0(t) \oplus H_{\text{int}}(t)$$

$$H_0(t) = H_0^D(t) \oplus H_0^\phi(t)$$

$$H_0^D |n^D\rangle = \Omega_n |n^D\rangle, n = 1, 2$$

$$H_0 = \sum_{n=1}^2 \gamma^{-1}(t) \Omega_n |n^D\rangle \langle n^D| + \int d^3\mathbf{x} \left(\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right)$$

$$H_{\text{int}} = M^D(t) \phi(\mathbf{x}^D, t)$$

$$M^D(t) \equiv \gamma^{-1}(t) \sum_{m,n=1}^2 \mu_{mn} e^{i(\Omega_m - \Omega_n)\tau} |m^D\rangle \langle n^D|$$

$$E = \sum_{\kappa} E_{(\kappa)}^D \otimes E_{(\kappa)}^\phi$$

$$\mathbb{P} = \operatorname{Tr}(E \rho_t)$$

$$\rho_t \equiv U_{t,0} \rho_0 U_{t,0}^\dagger$$

$$U_{t,0} = \operatorname{Texp} \left(\frac{1}{i} \int_0^t dt' H_{\text{int}}(t') \right)$$

$$\rho_0 = |1^D, 0_{\text{de Sitter}}^\phi\rangle \langle 1^D, 0_{\text{de Sitter}}^\phi|$$

$$E = |2^D\rangle \langle 2^D| \otimes \mathbb{I}_\phi$$

$$\frac{d\mathbb{P}(2; t)}{dt} = \Gamma(1 \rightarrow 2) \mathbb{P}(1; t) - \Gamma(2 \rightarrow 1) \mathbb{P}(2; t)$$



$$\Gamma(1 \rightarrow 2) = \frac{d\mathbb{P}(2; t)}{dt} (1 + \mathcal{O}(|\mu|^2))$$

$$\mathbb{P}(2; t) \equiv \left\langle 1^D, 0_{\text{de Sitter}}^\phi \left| U_{t,0}^\dagger E U_{t,0} \right| 1^D, 0_{\text{de Sitter}}^\phi \right\rangle.$$

$$\mathbb{P}(2; t) = \sum_{j=0}^{\infty} \int_0^t dt_1 dt_2 \dots dt_j \Theta_{12\dots j} \left\langle 1^D, 0_{\text{de Sitter}}^\phi \left| \mathcal{F}_j \right| 1^D, 0_{\text{de Sitter}}^\phi \right\rangle$$

$$\mathcal{F}_0 = E$$

$$\mathcal{F}_j = \frac{1}{i} [\mathcal{F}_{j-1}, H_{\text{int}}(t_j)]$$

$$\phi_j^D \equiv \phi(\mathbf{x}_j^D, t_j), M_j^D \equiv M^D(t_j) \text{ and } \mathbf{x}_j^D \equiv \mathbf{x}^D(t_j)$$

$$\mathcal{F}_j = \frac{1}{i} [\mathcal{F}_{j-1}, M_j^D \phi_j^D].$$

$$\mathcal{F}_1 = i\gamma^{-1}(t_1) \phi_1^D \langle 2 | 2 \rangle (\mu e^{-i\Omega\tau_1} |1\rangle \langle 2| - \mu^* e^{i\Omega\tau_1} |2\rangle \langle 1|)$$

$$\mathcal{F}_2 = \gamma^{-1}(t_1) \gamma^{-1}(t_2) \langle 2 | 2 \rangle |\mu|^2 [\mathbb{I}_\phi \Delta_{12} \sin(\Omega\tau_{12}) (\langle 2 | 2 \rangle |1\rangle \langle 1| + \langle 1 | 1 \rangle |2\rangle \langle 2|) + \{\phi_1^D, \phi_2^D\} \cos(\Omega\tau_{12}) (\langle 2 | 2 \rangle |1\rangle \langle 1| - \langle 1 | 1 \rangle |2\rangle \langle 2|)]$$

$$\Delta_{12} \equiv \frac{1}{i} \left\langle 0_{\text{de Sitter}}^\phi \left| [\phi_1^D, \phi_2^D] \right| 0_{\text{de Sitter}}^\phi \right\rangle$$

$$\begin{aligned} \mathbb{P}(2; t) &= |\mu|^2 \int_0^t \frac{dt_1}{\gamma(t_1)} \int_0^{t_1} \frac{dt_2}{\gamma(t_2)} [\Delta_{12}^R \sin(\Omega\tau_{12}) + \Delta_{12}^H \cos(\Omega\tau_{12})] \\ &= |\mu|^2 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 [\Delta_{12}^R \sin(\Omega\tau_{12}) + \Delta_{12}^H \cos(\Omega\tau_{12})], \end{aligned}$$

$$\Delta_{12}^R \equiv \Theta_{12} \Delta_{12},$$

$$\Delta_{12}^H \equiv \left\langle 0_{\text{de Sitter}}^\phi \left| \{\phi_1^D, \phi_2^D\} \right| 0_{\text{de Sitter}}^\phi \right\rangle$$

$$\mathbb{P}(2; t) = |\mu|^2 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-i\Omega(\tau_1 - \tau_2)} \left\langle 0_{\text{de Sitter}}^\phi \left| \phi_1^D \phi_2^D \right| 0_{\text{de Sitter}}^\phi \right\rangle \equiv |\mu|^2 F(\Omega),$$

$$\Delta_{12}^R = \frac{m^2 J_1(ms_{12}^\alpha)}{4\pi ms_{12}^\alpha} - \frac{\delta((s_{12}^\alpha)^2)}{2\pi} \text{ and } \Delta_{12}^H = \frac{m^2 Y_1(ms_{12}^\alpha)}{4\pi ms_{12}^\alpha},$$

$$\begin{aligned} s_{12}^\alpha &\equiv \sqrt{(x_1^\mu - x_2^\mu)^2} = \sqrt{(t_1 - t_2)^2 - \frac{1}{\alpha^2} \left(\sqrt{1 + \alpha^2 t_1^2} - \sqrt{1 + \alpha^2 t_2^2} \right)^2} \\ &= \frac{2}{\alpha} \sinh \frac{\alpha\tau_{12}}{2} \end{aligned}$$

$$\mathbb{P}(2; t) = |\mu|^2 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \frac{m^2}{4\pi} \left[\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \sin(\Omega\tau_{12}) + \frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \cos(\Omega\tau_{12}) \right]$$



$$\mathbb{P}(2; t) = |\mu|^2 \int_{1/\Lambda}^{\tau} d\tau_1 \int_{1/\Lambda}^{\tau_1} d\tau_{12} \frac{m^2}{4\pi} \left[\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \sin(\Omega\tau_{12}) + \frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \cos(\Omega\tau_{12}) \right].$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} &= |\mu|^2 \int_{1/\Lambda}^{\tau} d\tau_{12} [\Delta_{12}^R \sin(\Omega\tau_{12}) + \Delta_{12}^H \cos(\Omega\tau_{12})] \\ &= \frac{m^2 |\mu|^2}{4\pi} \int_{1/\Lambda}^{\tau} d\tau_{12} \left[\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \sin \Omega\tau_{12} + \frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} \cos \Omega\tau_{12} \right]. \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} - \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{\alpha=0} &= \frac{m^2 |\mu|^2}{4\pi} \int_{1/\Lambda}^{\tau} d\tau_{12} \left[\left(\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{J_1(m\tau_{12})}{m\tau_{12}} \right) \sin \Omega\tau_{12} \right. \\ &\quad \left. + \left(\frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{Y_1(m\tau_{12})}{m\tau_{12}} \right) \cos \Omega\tau_{12} \right] \end{aligned}$$

$$E = |0^\phi\rangle\langle 0^\phi| \otimes |2^D\rangle\langle 2^D|$$

$$\mathbb{P}(2; t) = |\langle 0^\phi, 2^D | U_{t,0} | 0^\phi, 1^D \rangle|^2 = 0$$

Massless limit ($m \rightarrow 0$):

When $m \rightarrow 0$, $\Delta_{12}^R \rightarrow 0$ and $\Delta_{12}^H \rightarrow -1/2\pi^2 |s_{12}^\alpha|^2$

$$\begin{aligned} \mathbb{P}|_{m=0} &= \int_0^t dt_1 dt_2 \frac{-|\mu|^2}{\gamma(t_1)\gamma(t_2)} \Theta_{12} \frac{\cos(\Omega\tau_{12})}{2\pi^2 (s_{12}^\alpha)^2} \\ \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{m=0} &= -\frac{|\mu|^2 \alpha^2}{8\pi^2} \int_{1/\Lambda}^{\tau} d\tau_{12} \frac{\cos(\Omega\tau_{12})}{\sinh^2 \frac{1}{2} \alpha \tau_{12}} \end{aligned}$$

$$\frac{\partial \mathbb{P}}{\partial \tau} \Big|_{m=0} - \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{m,\alpha=0} = -\frac{|\mu|^2}{8\pi^2} \int_{1/\Lambda}^{\tau} d\tau_{12} \cos(\Omega\tau_{12}) \left(\frac{\alpha^2}{\sinh^2 \frac{1}{2} \alpha \tau_{12}} - \frac{4}{\tau_{12}^2} \right)$$

Acceleration ($\alpha \ll 1/\tau$):

For $\alpha \ll 1/\tau$:

$$\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{J_1(m\tau_{12})}{m\tau_{12}} \rightarrow -\frac{\alpha^2 \tau^2}{24} J_2(m\tau) + \mathcal{O}((\alpha\tau)^4)$$

$$\frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{Y_1(m\tau_{12})}{m\tau_{12}} \rightarrow -\frac{\alpha^2 \tau^2}{24} Y_2(m\tau) + \mathcal{O}((\alpha\tau)^4)$$

$$\frac{\partial \mathbb{P}}{\partial \tau} - \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{\alpha=0} = \frac{m^2 \alpha^2 |\mu|^2}{96\pi} \int_{1/\Lambda}^{\tau} d\tau_{12} [J_2(m\tau_{12}) \sin \Omega\tau_{12} + Y_2(m\tau_{12}) \cos \Omega\tau_{12}]$$

Early times ($\alpha\tau, m\tau \rightarrow 0$):

For $\alpha\tau, m\tau \rightarrow 0$, we use:



$$\frac{J_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{J_1(m\tau_{12})}{m\tau_{12}} \rightarrow 0$$

$$\frac{Y_1(ms_{12}^\alpha)}{ms_{12}^\alpha} - \frac{Y_1(m\tau_{12})}{m\tau_{12}} \rightarrow \frac{\alpha^2}{6m^2\pi}$$

$$\left. \frac{\partial \mathbb{P}}{\partial \tau} - \frac{\partial \mathbb{P}}{\partial \tau} \right|_{\alpha=0} = \frac{\alpha^2 |\mu|^2}{24\pi^2} \tau$$

Late times ($\tau \gg 1/\alpha$):

$$J_1(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right)$$

$$Y_1(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{3\pi}{4}\right)$$

$$\left. \frac{\partial \mathbb{P}}{\partial \tau} - \frac{\partial \mathbb{P}}{\partial \tau} \right|_{\alpha=0} = \frac{\partial^2 \mathfrak{J}}{\partial \varphi} - \frac{\partial^2 \sigma}{\partial \varsigma} + \frac{\partial^2 \rho}{\partial \varrho} - \frac{\partial^2 \zeta}{\partial \lambda} + \partial^2 \xi \oslash |\mu|^2 \sqrt{\frac{m}{8\pi^3}} \int_{\tau_0}^{\tau} d\tau_{12} \tau_{12}^{-3/2} \sin\left((m + \Omega)\tau_{12} + \frac{\pi}{4}\right)$$

$$\Delta_{12} = - \int \frac{d^4 \mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p} \cdot \mathbf{x}_{12}}}{\omega_{\mathbf{p}}} \sin(\omega_{\mathbf{p}} x_{12}^0) = - \int \frac{d^4 \mathbf{p}}{(2\pi)^3} \frac{\sin(\omega_{\mathbf{p}} s_{12}^\alpha)}{\omega_{\mathbf{p}}}$$

$$\Delta_{12}^H = \int \frac{d^4 \mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p} \cdot \mathbf{x}_{12}}}{\omega_{\mathbf{p}}} \cos(\omega_{\mathbf{p}} x_{12}^0) = \int \frac{d^4 \mathbf{p}}{(2\pi)^3} \frac{\cos(\omega_{\mathbf{p}} s_{12}^\alpha)}{\omega_{\mathbf{p}}}$$

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

$$\mathbb{P}(2; t) = \frac{|\mu|^2}{8\pi^3} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \int d^4 \mathbf{p}_\perp \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} [-\sin(\omega_{\mathbf{p}} s_{12}^\alpha) \sin(\Omega \tau_{12}) + \cos(\omega_{\mathbf{p}} s_{12}^\alpha) \cos(\Omega \tau_{12})]$$

$$= \frac{|\mu|^2}{8\pi^3} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \int d^4 \mathbf{p}_\perp \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \cos(\omega_{\mathbf{p}} s_{12}^\alpha + \Omega \tau_{12})$$

$$\frac{\partial \mathbb{P}}{\partial \tau} = \frac{|\mu|^2}{8\pi^3} \int_0^\tau d\tau_{12} \int d^4 \mathbf{p}_\perp \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \cos\left[\frac{2\omega_{\mathbf{p}}}{\alpha} \sinh(\alpha \tau_{12}/2) + \Omega \tau_{12}\right]$$

$$= \frac{|\mu|^2}{16\pi^3} \int_{-\tau}^\tau d\tau_{12} \int d^4 \mathbf{p}_\perp \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \cos\left[\frac{2\omega_{\mathbf{p}}}{\alpha} \sinh(\alpha \tau_{12}/2) + \Omega \tau_{12}\right]$$

$$\frac{\partial \mathbb{P}(\tau \rightarrow \infty)}{\partial \tau} = \frac{|\mu|^2}{2\pi^2 \alpha} e^{-\frac{\pi \Omega}{\alpha}} \int_m^\infty dv v \left| K_{i\Omega/\alpha}\left(\frac{v}{\alpha}\right) \right|^2$$

$$v = \sqrt{\mathbf{p}_\perp^2 + m^2}$$

$$\frac{8\pi \alpha v}{\Omega} e^{-\frac{\pi \Omega}{\alpha}} \sin^2\left(\frac{v^2}{4\Omega \alpha} + \dots\right)$$

$$s_{12} = \sqrt{(t_1 - t_2)^2 - v^2(t_1 - t_2)^2} = \gamma^{-1} t_{12} = \tau_{12}$$



$$\frac{\partial \mathbb{P}}{\partial \tau} = \frac{|\mu|^2}{16\pi^3} \int_{-\tau}^{\tau} d\tau_{12} \int d^4 \mathbf{p}_{\perp} \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \cos [\omega_{\mathbf{p}} \tau_{12} + \Omega \tau_{12}]$$

$$\omega'_{\mathbf{p}}(\tau) = \omega_{\mathbf{p}} \cosh(\alpha \tau) - p_x \sinh(\alpha \tau)$$

$$\omega'_{\mathbf{p}}(\tau) = \omega_{\mathbf{p}} e^{\mp \alpha \tau}, p_x \geq 0$$

$$\begin{aligned} \omega_{\mathbf{p}} \tau_{12} \rightarrow \int_{\tau_2}^{\tau_1} dt' \omega'_{\mathbf{p}}(\tau') &= \frac{\omega_{\mathbf{p}}}{\alpha} [\sinh(\alpha \tau_1) - \sinh(\alpha \tau_2)] - \frac{p_x}{\alpha} [\cosh(\alpha \tau_1) - \cosh(\alpha \tau_2)] \\ &= \frac{2}{\alpha} \sinh(\alpha \tau_{12}/2) [\omega_{\mathbf{p}} \cosh(\alpha \bar{\tau}) - p_x \sinh(\alpha \bar{\tau})] \end{aligned}$$

$$\omega''_{\mathbf{p}} = \omega_{\mathbf{p}} \cosh(\alpha \bar{\tau}) - p_x \sinh(\alpha \bar{\tau})$$

$$p''_x = p_x \cosh(\alpha \bar{\tau}) - \omega_{\mathbf{p}} \sinh(\alpha \bar{\tau})$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} &= \frac{|\mu|^2}{16\pi^3} \operatorname{Re} \int_{-\infty}^{\infty} d\tau_{12} \int_{-\infty}^{\infty} \frac{dk \sin(k\tau)}{\pi k} \int d^4 \mathbf{p}_{\perp} \\ &\quad \times \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \exp \left\{ i \left[\frac{2\omega_{\mathbf{p}}}{\alpha} \sinh(\alpha \tau_{12}/2) + (\Omega - k) \tau_{12} \right] \right\} \\ &= \frac{|\mu|^2}{16\pi^3} \int_{-\infty}^{\infty} d\tau_{12} \int_{-\infty}^{\infty} \frac{dk \sin(k\tau)}{\pi k} \int d^4 \mathbf{p}_{\perp} \\ &\quad \times \int_{-\infty}^{\infty} \frac{dp_x}{\omega_{\mathbf{p}}} \cos \left[\frac{2\omega_{\mathbf{p}}}{\alpha} \sinh(\alpha \tau_{12}/2) + (\Omega - k) \tau_{12} \right] \end{aligned}$$

$$\frac{\partial \mathbb{P}(\tau)}{\partial \tau} = \frac{|\mu|^2}{2\pi^2 \alpha} \int_{-\infty}^{\infty} \frac{dk \sin(k\tau)}{\pi k} e^{-\frac{\pi(\Omega-k)}{\alpha}} \int_m^{\infty} dv v \left| K_{i(\Omega-k)/\alpha} \left(\frac{v}{\alpha} \right) \right|^2$$

$$\lim_{\tau \rightarrow \infty} \frac{1}{\pi} \frac{\sin(k\tau)}{k} = \delta(k)$$

$$t = \alpha^{-1} e^{\alpha \xi} \sinh \alpha \eta, x = \alpha^{-1} e^{\alpha \xi} \cosh \alpha \eta, y = y, z = z.$$

$$\phi(x) = \int d\omega_{\mathbf{p}} d^4 \mathbf{p}_{\perp} \left[v_{\omega_{\mathbf{p}} \mathbf{p}_{\perp}}^R b_{\omega_{\mathbf{p}} \mathbf{p}_{\perp}}^R + \text{Hermitian conjugate} \right]$$

$$\begin{aligned} v_{\omega_{\mathbf{p}} \mathbf{p}_{\perp}}^R &= \left[\frac{\sinh(\pi \omega_{\mathbf{p}}/\alpha)}{4\pi^4 \alpha} \right]^{1/2} K_{i\omega_{\mathbf{p}}/\alpha} \left[\frac{\sqrt{\mathbf{p}_{\perp}^2 + m^2}}{\alpha e^{-\alpha \xi}} \right] e^{-i\omega_{\mathbf{p}} \tau + i\mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} \\ &= \left[\frac{\sinh(\omega_{\mathbf{p}}/2T)}{8\pi^5 T} \right]^{1/2} K_{i\omega_{\mathbf{p}}/2\pi T} \left[\frac{\sqrt{\mathbf{p}_{\perp}^2 + m^2}}{2\pi T} \right] e^{-i\omega_{\mathbf{p}} \tau + i\mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} \end{aligned}$$

$$\begin{aligned} \langle 0_M^{\phi} | b_{\omega_{\mathbf{p}} \mathbf{p}_{\perp}}^{R\dagger} b_{E'_{\mathbf{p}} \mathbf{p}'_{\perp}}^R | 0_M^{\phi} \rangle &= (e^{\omega_{\mathbf{p}}/T} - 1)^{-1} \delta(\omega_{\mathbf{p}} - E'_{\mathbf{p}}) \delta^2(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp}) \\ &= n \delta(\omega_{\mathbf{p}} - E'_{\mathbf{p}}) \delta^2(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp}) \end{aligned}$$

$$\begin{aligned} \langle 0_M^{\phi} | b_{\omega_{\mathbf{p}} \mathbf{p}_{\perp}}^R b_{E'_{\mathbf{p}} \mathbf{p}'_{\perp}}^{R\dagger} | 0_M^{\phi} \rangle &= (1 - e^{-\omega_{\mathbf{p}}/T})^{-1} \delta(\omega_{\mathbf{p}} - E'_{\mathbf{p}}) \delta^2(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp}) \\ &= (n + 1) \delta(\omega_{\mathbf{p}} - E'_{\mathbf{p}}) \delta^2(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp}) \end{aligned}$$

$$n \equiv n(\omega_{\mathbf{p}}) = (e^{\omega_{\mathbf{p}}/T} - 1)^{-1}$$



$$\begin{aligned} \Delta_{12} &\equiv \frac{1}{i} \langle 0_M^\phi | [\phi_1, \phi_2] | 0_M^\phi \rangle \\ &= - \int d\omega_{\mathbf{p}} d^4 \mathbf{p}_\perp \frac{\sinh(\omega_{\mathbf{p}}/2T)}{4\pi^5 T} \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left[\frac{\sqrt{\mathbf{p}_\perp^2 + m^2}}{2\pi T} \right] \right|^2 \sin(\omega_{\mathbf{p}} \tau_{12}), \\ \Delta_{12}^H &\equiv \langle 0_M^\phi | \{\phi_1, \phi_2\} | 0_M^\phi \rangle \\ &= \int d\omega_{\mathbf{p}} d^4 \mathbf{p}_\perp \frac{\sinh(\omega_{\mathbf{p}}/2T)}{4\pi^5 T} \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left[\frac{\sqrt{\mathbf{p}_\perp^2 + m^2}}{2\pi T} \right] \right|^2 \cos(\omega_{\mathbf{p}} \tau_{12}) (2n+1). \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} &= \frac{|\mu|^2}{4\pi^5 T} \int_0^\tau d\tau_{12} \int_0^\infty d\omega_{\mathbf{p}} d^4 \mathbf{p}_\perp \sinh\left(\frac{\omega_{\mathbf{p}}}{2T}\right) \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left(\frac{\sqrt{\mathbf{p}_\perp^2 + m^2}}{2\pi T} \right) \right|^2 \times \\ &\quad \left(-\sin(\omega_{\mathbf{p}} \tau_{12}) \sin(\Omega \tau_{12}) + (2n+1) \cos(\omega_{\mathbf{p}} \tau_{12}) \cos(\Omega \tau_{12}) \right) \\ &= \frac{|\mu|^2}{2\pi^4 T} \int_0^\infty d\omega_{\mathbf{p}} \int_m^\infty dv v \sinh\left(\frac{E}{2T}\right) \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left(\frac{v}{2\pi T} \right) \right|^2 \times \\ &\quad \left((n+1) \frac{\sin((\omega_{\mathbf{p}} + \Omega)\tau)}{\omega_{\mathbf{p}} + \Omega} + n \frac{\sin((\omega_{\mathbf{p}} - \Omega)\tau)}{\omega_{\mathbf{p}} - \Omega} \right) \\ &= \frac{|\mu|^2}{4\pi^4 T} \int_0^\infty d\omega_{\mathbf{p}} \int_m^\infty dv v \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left(\frac{v}{2\pi T} \right) \right|^2 \times \\ &\quad \left(e^{\omega_{\mathbf{p}}/2T} \frac{\sin((\omega_{\mathbf{p}} + \Omega)\tau)}{\omega_{\mathbf{p}} + \Omega} + e^{-\omega_{\mathbf{p}}/2T} \frac{\sin((\omega_{\mathbf{p}} - \Omega)\tau)}{\omega_{\mathbf{p}} - \Omega} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{m=0} &= \frac{|\mu|^2 T}{\pi^2} \int_0^\infty d\omega_{\mathbf{p}} \sinh\left(\frac{\omega_{\mathbf{p}}}{2T}\right) \left| \Gamma\left(1 + \frac{i\omega_{\mathbf{p}}}{2\pi T}\right) \right|^2 \\ &\quad \times \left((n+1) \frac{\sin((\omega_{\mathbf{p}} + \Omega)\tau)}{\omega_{\mathbf{p}} + \Omega} + n \frac{\sin((\omega_{\mathbf{p}} - \Omega)\tau)}{\omega_{\mathbf{p}} - \Omega} \right) \\ &= \frac{|\mu|^2}{2\pi^2} \int_0^\infty d\omega_{\mathbf{p}} \omega_{\mathbf{p}} \left((n+1) \frac{\sin((\omega_{\mathbf{p}} + \Omega)\tau)}{\omega_{\mathbf{p}} + \Omega} + n \frac{\sin((\omega_{\mathbf{p}} - \Omega)\tau)}{\omega_{\mathbf{p}} - \Omega} \right) \end{aligned}$$

$$\left| \Gamma\left(1 + \frac{i\omega_{\mathbf{p}}}{2\pi T}\right) \right|^2 = \frac{\omega_{\mathbf{p}}}{2T \sinh(\omega_{\mathbf{p}}/2T)}$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial \tau} - \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{T=0} &= \frac{|\mu|^2}{4\pi^4 T} \int_0^\infty d\omega_{\mathbf{p}} \int_m^\infty dv v \left| K_{i\omega_{\mathbf{p}}/2\pi T} \left(\frac{v}{2\pi T} \right) \right|^2 \times \\ &\quad \left(e^{\omega_{\mathbf{p}}/2T} \frac{\sin((\omega_{\mathbf{p}} + \Omega)\tau)}{\omega_{\mathbf{p}} + \Omega} + e^{-\omega_{\mathbf{p}}/2T} \frac{\sin((\omega_{\mathbf{p}} - \Omega)\tau)}{\omega_{\mathbf{p}} - \Omega} \right) \\ &\quad - \frac{|\mu|^2}{2\pi^2} \int_m^\infty d\omega_{\mathbf{p}} \sqrt{\omega_{\mathbf{p}}^2 - m^2} \frac{\sin[(\omega_{\mathbf{p}} + \Omega)\tau]}{\omega_{\mathbf{p}} + \Omega} \end{aligned}$$



$$\lim_{T \rightarrow 0} \int_m^\infty dvv \left| K_{i\omega_p/2\pi T} \left(\frac{v}{2\pi T} \right) \right|^2 = e^{-\omega_p/2T} 2\pi^2 T \Theta(\omega_p - m) \sqrt{\omega_p^2 - m^2}$$

$$\frac{\partial \mathbb{P}(\tau \rightarrow \infty)}{\partial \tau} - \frac{\partial \mathbb{P}(\tau \rightarrow \infty)}{\partial \tau} \Big|_{T=0} = \frac{|\mu|^2}{4\pi^3 T} e^{-\frac{\Omega}{2T}} \int_m^\infty dvv \left| K_{i\Omega/2\pi T} \left(\frac{v}{2\pi T} \right) \right|^2$$

$$\lim_{\tau \rightarrow \infty} \frac{\sin [(\omega_p \pm \Omega)\tau]}{\omega_p \pm \Omega} = \pi \delta(\omega_p \pm \Omega),$$

$$\frac{\partial \mathbb{P}}{\partial \tau} \Big|_{m=0} - \frac{\partial \mathbb{P}}{\partial \tau} \Big|_{T,m=0} = \frac{|\mu|^2}{2\pi^2} \int_0^\infty d\omega_p \omega_p n \left(\frac{\sin [(\omega_p + \Omega)\tau]}{\omega_p + \Omega} + \frac{\sin [(\omega_p - \Omega)\tau]}{\omega_p - \Omega} \right)$$

$$\frac{\partial \mathbb{P}(\tau \rightarrow \infty)}{\partial \tau} \Big|_{m=0} - \frac{\partial \mathbb{P}(\tau \rightarrow \infty)}{\partial \tau} \Big|_{m,T=0} = \frac{|\mu|^2}{2\pi} \frac{\Omega}{e^{\beta\Omega} - 1}$$

$$\Delta_{12}^H \Big|_{T=0} = \Delta_{12}^> \Big|_{T=0} + \Delta_{12}^< \Big|_{T=0} = \int \frac{d^4 p}{(2\pi)^3} e^{-ip_\mu x_{12}^\mu} \delta(p^2 - m^2)$$

$$\Delta_{12}^>(T) = \int \frac{d^4 p}{(2\pi)^3} [\Theta(p^0)(1+n) + \Theta(-p^0)n] e^{-ip_\mu x_{12}^\mu} \delta(p^2 - m^2)$$

$$\Delta_{12}^<(T) = \int \frac{d^4 p}{(2\pi)^3} [\Theta(-p^0)(1+n) + \Theta(p^0)n] e^{-ip_\mu x_{12}^\mu} \delta(p^2 - m^2)$$

$$n \equiv n(|p^0|) = (\exp(|p^0|/T) - 1)^{-1}$$

$$\Delta_{12}^H(T) = \int \frac{d^4 p}{(2\pi)^3} [\Theta(p^0) + \Theta(-p^0)](1+2n) e^{-ip_\mu x_{12}^\mu} \delta(p^2 - m^2)$$

$$= \int \frac{d^4 p}{(2\pi)^3} (1+2n) e^{-ip_\mu x_{12}^\mu} \delta(p^2 - m^2)$$

$$\Delta_{12}^H \supset \int \frac{d^4 p}{(2\pi)^3} e^{-ip^0 t_{12}} (2n) \delta(p^2 - m^2) = \frac{1}{\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} n \cos(\omega_p t_{12}).$$

$$\begin{aligned} \frac{\partial \mathbb{P}}{\partial t} - \frac{\partial \mathbb{P}}{\partial t} \Big|_{T=0} &= \frac{|\mu|^2}{\pi^2} \int_0^t dt' \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} n \cos \Omega t' \cos \omega_p t' \\ &= \frac{|\mu|^2}{2\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} n \left\{ \frac{\sin [(\omega_p - \Omega)t]}{\omega_p - \Omega} + \frac{\sin [(\omega_p + \Omega)t]}{\omega_p + \Omega} \right\} \end{aligned}$$

$$\frac{\partial \mathbb{P}}{\partial t} - \frac{\partial \mathbb{P}}{\partial t} \Big|_{T=0} = \frac{\partial \mathbb{P}}{\partial t} = \frac{|\mu|^2}{2\pi} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \frac{1}{e^{\beta\omega_p} - 1} [\delta(\omega_p - \Omega) + \delta(\omega_p + \Omega)]$$

$$\frac{\partial \mathbb{P}}{\partial t} = \frac{|\mu|^2 \sqrt{\Omega^2 - m^2}}{2\pi e^{\beta\Omega} - 1} \Theta(\Omega - m),$$

$$\frac{1}{\Omega |\mu|^2} \left(\frac{d\mathbb{P}}{dt} - \frac{d\mathbb{P}}{dt} \Big|_{\alpha=0} \right)$$

$$H_{\text{int}}(t) = D^\mu(t) \partial_\mu \phi(\mathbf{x}^D, t)$$



$$D^\mu(t) \equiv \gamma^{-1}(t)(\xi^\mu e^{-i\Omega t}|1\rangle\langle 2| + (\xi^\mu)^* e^{i\Omega t}|2\rangle\langle 1|)$$

$$\rho_0 = |1, 0^\phi\rangle\langle 1, 0^\phi| \text{ and operator } E = \mathbb{I}^\phi \otimes |2^D\rangle\langle 2^D|$$

$$\mathcal{F}_1 = \frac{1}{i\gamma(t_1)}(-\xi^{\mu_1}\partial_{\mu_1}\phi_1 e^{-i\Omega t_1}|1\rangle\langle 2| + (\xi^{\mu_1})^*\partial_{\mu_1}\phi_1 e^{i\Omega t_1}|2\rangle\langle 1|)$$

$$\begin{aligned} \mathcal{F}_2 &= \frac{1}{i^2\gamma(t_1)\gamma(t_2)}(-\xi^{\mu_1}(\xi^{\mu_2})^*\partial_{\mu_1}\phi_1\partial_{\mu_2}\phi_2 e^{-i\Omega t_{12}}|1\rangle\langle 1| \\ &\quad + (\xi^{\mu_1})^*\xi^{\mu_2}\partial_{\mu_1}\phi_1\partial_{\mu_2}\phi_2 e^{i\Omega t_{12}}|2\rangle\langle 2| \\ &\quad - \xi^{\mu_2}(\xi^{\mu_1})^*\partial_{\mu_2}\phi_2\partial_{\mu_1}\phi_1 e^{i\Omega t_{12}}|1\rangle\langle 1| \\ &\quad + (\xi^{\mu_2})^*\xi^{\mu_1}\partial_{\mu_2}\phi_2\partial_{\mu_1}\phi_1 e^{-i\Omega t_{12}}|2\rangle\langle 2|) \\ &= \frac{|1\rangle\langle 1|}{\gamma(t_1)\gamma(t_2)}\left(\frac{1}{2}\text{Re}(\xi^{\mu_1}\xi^{\mu_2*}e^{-i\Omega t})\partial_{\mu_1}\partial_{\mu_2}\{\phi_1, \phi_2\} \right. \\ &\quad \left. + \frac{i}{2}\text{Im}(\xi^{\mu_1}\xi^{\mu_2*}e^{-i\Omega t})\partial_{\mu_1}\partial_{\mu_2}[\phi_1, \phi_2]\right) \\ &+ \frac{|2\rangle\langle 2|}{\gamma(t_1)\gamma(t_2)}\left(-\frac{1}{2}\text{Re}(\xi^{\mu_1}\xi^{\mu_2*}e^{-i\Omega t})\partial_{\mu_1}\partial_{\mu_2}\{\phi_1, \phi_2\} \right. \\ &\quad \left. + \frac{i}{2}\text{Im}(\xi^{\mu_1}\xi^{\mu_2*}e^{-i\Omega t})\partial_{\mu_1}\partial_{\mu_2}[\phi_1, \phi_2]\right) \end{aligned}$$

$$\begin{aligned} \mathbb{P}(\alpha, t) &= \int_0^t dt_1 dt_2 \frac{\Theta_{12}}{\gamma(t_1)\gamma(t_2)}\left[\text{Re}(\xi^{\mu_1}\xi^{\mu_2*})\cos \Omega t_{12} \right. \\ &\quad \left. + \text{Im}(\xi^{\mu_1}\xi^{\mu_2*})\sin \Omega t_{12}\right]\partial_{\mu_1}\partial_{\mu_2}\Delta_{12}^H \\ &\quad + \left(-\text{Im}(\xi^{\mu_1}\xi^{\mu_2*})\cos \Omega t_{12} \right. \\ &\quad \left. + \text{Re}(\xi^{\mu_1}\xi^{\mu_2*})\sin \Omega t_{12}\right)\partial_{\mu_1}\partial_{\mu_2}\Delta_{12}^R \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x^{\mu_1}}\frac{\partial}{\partial x^{\mu_2}}\Delta_{12}^H &= \frac{m^2}{4\pi}\frac{\partial}{\partial x^{\mu_1}}\frac{\partial}{\partial x^{\mu_2}}\frac{Y_1(ms_{12})}{ms_{12}} \\ &= \frac{m^2}{4\pi}\frac{\partial}{\partial x^{\mu_1}}\left[-\frac{Y_2(ms_{12})}{ms_{12}}\frac{\partial}{\partial x^{\mu_2}}(ms_{12})\right] \\ &= \frac{m^4}{4\pi}\frac{\partial}{\partial x^{\mu_1}}\left[\frac{Y_2(ms_{12})}{ms_{12}}g_{\mu_2\nu_2}\frac{x_{12}^{\nu_2}}{ms_{12}}\right] \\ &= \frac{m^4}{4\pi}\left[\frac{Y_2(ms_{12})}{(ms_{12})^2}g_{\mu_2\nu_1} - m^2\frac{Y_3(ms_{12})}{(ms_{12})^2}g_{\mu_1\nu_1}\frac{x_{12}^{\nu_1}}{ms_{12}}g_{\mu_2\nu_2}x_{12}^{\nu_2}\right] \\ &= \frac{m^4}{4\pi}\left[\frac{Y_2(ms_{12})}{(ms_{12})^2}g_{\mu_2\nu_1} - \frac{Y_3(ms_{12})}{(ms_{12})^3}m^2x_{12}^{\mu_1}x_{12}^{\mu_2}\right] \\ \frac{\partial}{\partial x^{\mu_1}}\frac{\partial}{\partial x^{\mu_2}}\Delta_{12}^R &= \frac{m^2}{4\pi}\frac{\partial}{\partial x^{\mu_1}}\frac{\partial}{\partial x^{\mu_2}}\frac{J_1(ms_{12})}{ms_{12}} \\ &= \frac{m^4}{4\pi}\left[\frac{J_2(ms_{12})}{(ms_{12})^2}g_{\mu_2\nu_1} - \frac{J_3(ms_{12})}{(ms_{12})^3}m^2x_{12}^{\mu_1}x_{12}^{\mu_2}\right] \end{aligned}$$

$$s_{12} \equiv \sqrt{g_{\mu\nu}x_{12}^\mu x_{12}^\nu}$$

$$\xi^\mu = (\chi, \xi \sin \theta \cos \phi, \xi \sin \theta \sin \phi, \xi \cos \theta)$$



$$\begin{aligned}\xi^{\mu_1}\xi^{\mu_2*}g_{\mu_2\mu_1} &= |\chi|^2 - |\xi|^2 \\ \xi^{\mu_1}\xi^{\mu_2*}x_{12}^{\mu_1}x_{12}^{\mu_2} &= |\chi|^2\tau_{12}^2\end{aligned}$$

$$\frac{\partial\mathbb{P}}{\partial\tau} = \frac{m^4}{4\pi} \int_0^\tau d\tau' \left[\left(\frac{Y_2(ms')}{(ms')^2} (|\chi|^2 - |\xi|^2) - \frac{Y_3(ms')}{(ms')^3} m^2 |\chi|^2 \tau'^2 \right) \cos \Omega\tau' + \left(\frac{J_2(ms')}{(ms')^2} (|\chi|^2 - |\xi|^2) - \frac{J_3(ms')}{(ms')^3} m^2 |\chi|^2 \tau'^2 \right) \sin \Omega\tau' \right]$$

$$s' = \frac{2}{\alpha} \sinh \frac{\alpha\tau'}{2}$$

$$\text{AdS}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu,$$

$$g_{\mu\nu}(x) \rightarrow \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$$

$$\text{AdS}^2 = dt^2 - dx^2 = du dv,$$

$$g_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$u' = 2 \tan^{-1} u, \quad -\pi \leq u' \leq \pi,$$

$$v' = 2 \tan^{-1} v, \quad -\pi \leq v' \leq \pi,$$

$$\text{AdS}^2 = \frac{1}{4} \sec^2 \left(\frac{u'}{2} \right) \sec^2 \left(\frac{v'}{2} \right) du' dv',$$

$$g_{\mu\nu}(u', v') = \frac{1}{8} \sec^2 \left(\frac{u'}{2} \right) \sec^2 \left(\frac{v'}{2} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\Omega^2(x) = \left(\frac{1}{4} \sec^2 \left(\frac{u'}{2} \right) \sec^2 \left(\frac{v'}{2} \right) \right)^{-1}$$

$$g_{\mu\nu}(u', v') \rightarrow \bar{g}_{\mu\nu}(u', v') = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\text{Ad}\bar{S}^2 = du' dv'$$

$$\mathcal{L}_{\bar{\xi}} g_{\mu\nu}(x) = 0,$$

$$\mathcal{L}_{\bar{\xi}} \phi_\omega = -i\omega \phi_\omega, \quad \omega > 0.$$

$$\text{AdS}^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \left(1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right|$$

$$u = t - r_*$$

$$v = t + r_*$$



$$\begin{aligned}\bar{u} &= -4Me^{-u/4M}, \\ \bar{v} &= 4Me^{v/4M}\end{aligned}$$

$$\text{AdS}^2 = \frac{2M}{r} e^{-r/2M} d\bar{u} d\bar{v} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

$$\Phi_{\omega\ell}(r) Y_{\ell m}(\theta, \varphi) e^{\pm i\omega t},$$

$$Y_{\ell m}(0, \phi) = \begin{cases} \left(\frac{2\ell+1}{4\pi}\right)^{1/2}, & m=0 \\ 0, & |m|=1,2,3,\dots \end{cases}$$

$$\frac{d^4\Phi}{dr^2} + \frac{2(r-M)}{r(r-2M)} \frac{d\Phi}{dr} + \left[\frac{\omega^2 r^2}{(r-2M)^2} - \frac{\ell(\ell+1)}{r(r-2M)} \right] \Phi = \{u_{\omega\ell m}^{\text{in/up}}, u_{\omega\ell m}^{\text{in/up}*}\}$$

$$\frac{d^4\rho}{dr_*^2} = [V_\ell(r) - \omega^2]\rho$$

$$V_\ell(r) \equiv \left(1 - \frac{2M}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)$$

$$\rho_{\omega\ell}^{\text{in}}(r) \rightarrow \begin{cases} B_{\omega\ell}^{\text{in}} e^{-i\omega r_*}, & r \rightarrow 2M \\ e^{-i\omega r_*} + A_{\omega\ell}^{\text{in}} e^{+i\omega r_*}, & r \rightarrow \infty \end{cases}$$

$$\rho_{\omega\ell}^{\text{up}}(r) \rightarrow \begin{cases} A_{\omega\ell}^{\text{up}} e^{-i\omega r_*} + e^{+i\omega r_*}, & r \rightarrow 2M \\ B_{\omega\ell}^{\text{up}} e^{+i\omega r_*}, & r \rightarrow \infty \end{cases}$$

$$\Phi_{\omega\ell}^{\text{in/up}}(r) \equiv \rho_{\omega\ell}^{\text{in/up}}(r)/r$$

$$u_{\omega\ell m}^{\text{in/up}}(t, r, \theta, \varphi) \equiv \frac{Y_{\ell m}(\theta, \varphi)}{\sqrt{4\pi\omega}} \Phi_{\omega\ell}^{\text{in/up}}(r) e^{-i\omega t}$$

$$u_{\omega\ell m}^{\text{in}} \propto e^{-i\omega(t+r_*)}, r \rightarrow 2M$$

$$u_{\omega\ell m}^{\text{up}} \propto e^{-i\omega(t-r_*)}, r \rightarrow \infty$$

$$v_{\omega\ell m}^{\text{in/up}}(t, r, \theta, \varphi) \equiv \frac{Y_{\ell m}^*(\theta, \varphi)}{\sqrt{4\pi\omega}} \Phi_{\omega\ell}^{\text{in/up}*}(r) e^{i\omega t}.$$

$$\phi(x) = \sum_{lm} \int_0^\infty d\omega (b_{\omega lm}^{\text{in}} u_{\omega lm}^{\text{in}} + b_{\omega lm}^{\text{up}} u_{\omega lm}^{\text{up}} + b_{\omega lm}^{\text{in}} v_{\omega lm}^{\text{in}} + b_{\omega lm}^{\text{up}} v_{\omega lm}^{\text{up}} + \text{Hermitian conjugate})$$

$$|0_B\rangle \text{ by } b_{\omega lm}^{\text{in/up}} |0_B\rangle = b_{\omega lm}^{\text{in/up}'} |0_B\rangle = 0$$

$$W_{\omega\ell m}^{\text{in/up}} \equiv \frac{u_{\omega\ell m}^{\text{in/up}} + e^{-4\pi M\omega} v_{\omega\ell m}^{\text{in/up}*}}{\sqrt{1 - e^{-8\pi M\omega}}}$$

$$\bar{W}_{\omega\ell m}^{\text{in/up}} \equiv \frac{e^{-4\pi M\omega} u_{\omega\ell m}^{\text{in/up}*} + v_{\omega\ell m}^{\text{in/up}}}{\sqrt{1 - e^{-8\pi M\omega}}}$$



$$\{w_{\omega\ell m}^{\text{in/up}}, \bar{w}_{\omega\ell m}^{\text{in/up}}, w_{\omega\ell m}^{\text{in/up}*}, \bar{w}_{\omega\ell m}^{\text{in/up}*}\}$$

$$\phi(x) = \sum_{\ell m} \int_0^\infty d\omega (d_{\omega\ell m}^{\text{in}} w_{\omega\ell m}^{\text{in}} + d_{\omega\ell m}^{\text{up}} w_{\omega\ell m}^{\text{up}} + \bar{d}_{\omega\ell m}^{\text{in}} \bar{w}_{\omega\ell m}^{\text{in}} + \bar{d}_{\omega\ell m}^{\text{up}} \bar{w}_{\omega\ell m}^{\text{up}} + \text{Hermitian conjugate})$$

Hartle-Hawking vacuum $|0_H\rangle$ is defined by $d_{\omega\ell m}^{\text{in/up}} |0_H\rangle = \bar{d}_{\omega\ell m}^{\text{in/up}} |0_H\rangle = 0$

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$$

where $\kappa = 1/4M$ is the surface gravity of the black hole.

$$\phi(x) = \sum_{\ell, m} \int_0^\infty d\omega (b_{\omega\ell m}^{\text{in}} u_{\omega\ell m}^{\text{in}} + d_{\omega\ell m}^{\text{up}} w_{\omega\ell m}^{\text{up}} + b_{\omega\ell m}^{\text{in}'} v_{\omega\ell m}^{\text{in}} + \bar{d}_{\omega\ell m}^{\text{up}} \bar{w}_{\omega\ell m}^{\text{up}} + \text{h.c.})$$

Unruh vacuum $|0_U\rangle$ is defined by $b_{\omega\ell m}^{\text{in}} |0_U\rangle = d_{\omega\ell m}^{\text{up}} |0_U\rangle = b_{\omega\ell m}^{\text{in}'} |0_U\rangle = \bar{d}_{\omega\ell m}^{\text{up}} |0_U\rangle = 0$

$$H_{\text{int}} = \lambda \chi(t) M^D(t) \phi(\mathbf{x}^D, t)$$

$$F(\Omega) = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 e^{-i\Omega(\tau_1 - \tau_2)} \chi(\tau_1) \chi(\tau_2) W(\tau_1, \tau_2)$$

$$W(\tau_1, \tau_2) \equiv W(\mathbf{x}^D(\tau_1), \mathbf{x}^D(\tau_2))$$

$$W(x_1, x_2) \equiv \langle \Psi | \phi(x_1) \phi(x_2) | \Psi \rangle$$

$$W(x_1, x_2) = \sum_k u_k(x_1) u_k^*(x_2)$$

$$F(\Omega) = \sum_k \left| \int_{-\infty}^{\infty} d\tau_1 e^{-i\Omega\tau_1} \chi(\tau_1) u_k(\mathbf{x}(\tau_1)) \right|^2$$

$$\dot{F}(\Omega) = \int_{-\infty}^{\infty} d\tau_{12} e^{-i\Omega\tau_{12}} W(\tau_{12})$$

$$t_{12} = \tau_{12} / \sqrt{1 - 2M/R}$$

$$\tilde{\Phi}_{\omega\ell}^{\text{in}} = \Phi_{\omega\ell}^{\text{in}}, \tilde{\Phi}_{\omega\ell}^{\text{up}} = \frac{\Phi_{\omega\ell}^{\text{up}}}{2M}$$

$$W_B(x, x') = \sum_{\ell=0}^{\infty} \int_0^\infty d\omega \frac{(2\ell+1)}{16\pi^2 \omega} e^{-i\omega\tau_{12}/\sqrt{1-2M/R}} \left(|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 \right).$$

$$\dot{F}_B(\Omega) = \frac{\Theta(-\Omega)}{8\pi|\Omega|} \sum_{\ell=0}^{\infty} (2\ell+1) \left(|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 \right),$$

$$\tilde{\omega} \equiv \Omega \sqrt{1 - 2M/R}$$



$$W_H(x, x') = \sum_{\ell=0}^{\infty} \int_0^{\infty} d\omega \frac{(2\ell+1)}{16\pi^2 \omega \sinh(4\pi M\omega)} \left(|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 \right) \times \cosh \left[4\pi M\omega - \frac{i\omega\tau_{12}}{\sqrt{1-2M/R}} \right]$$

$$\dot{F}_H(\Omega) = \frac{1}{8\pi\Omega} \frac{1}{e^{\Omega/T_{\text{loc}}} - 1} \sum_{\ell=0}^{\infty} (2\ell+1) \left(|\tilde{\Phi}_{\tilde{\omega}\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\tilde{\omega}\ell}^{\text{in}}(R)|^2 \right)$$

$$\tilde{\omega} \equiv \Omega \sqrt{1-2M/R}$$

$$T_{\text{loc}} \equiv \frac{1}{8\pi M \sqrt{1-2M/R}}$$

$$\dot{F}_H(\Omega) = e^{-\Omega/T_{\text{loc}}} \dot{F}(-\Omega)$$

$$W_U(x, x') = \sum_{\ell=0}^{\infty} \int_0^{\infty} d\omega \frac{(2\ell+1)}{16\pi^2 \omega} \times \left[\frac{|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2}{2\sinh(4\pi M\omega)} \left(e^{4\pi\omega - i\omega\tau_{12}/\sqrt{1-2M/R}} + e^{-4\pi\omega + i\omega\tau_{12}/\sqrt{1-2M/R}} \right) + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 e^{-i\omega\tau_{12}/\sqrt{1-2M/R}} \right]$$

$$\dot{F}_U(\Omega) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} \left[\frac{|\tilde{\Phi}_{\tilde{\omega}\ell}^{\text{up}}(R)|^2}{2\Omega(e^{\Omega/T_{\text{loc}}} - 1)} - \frac{|\tilde{\Phi}_{\tilde{\omega}\ell}^{\text{in}}(R)|^2}{2\Omega} \Theta(-\Omega) \right],$$

$$\tilde{\omega} \equiv \Omega \sqrt{1-2M/R}$$

$$t = a\tau, r = R, \theta = \pi/2, \phi = b\tau$$

$$a \equiv \sqrt{R/(R-3M)}$$

$$b \equiv \frac{1}{a} \frac{d\phi}{dt} = \frac{1}{a} \sqrt{M/R^3}$$

$$W_B(x, x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} d\omega \frac{(\ell-m)! (2\ell+1) |P_{\ell}^m(0)|^2}{16\pi^2 \omega (\ell+m)!} e^{imb\tau_{12} - i\omega\tau_{12}} \times \left(|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 \right)$$

$$\dot{F}_B(\Omega) = \frac{1}{a} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell-m)! (2\ell+1) |P_{\ell}^m(0)|^2}{8\pi\omega_-(\ell+m)!} \times \left(|\tilde{\Phi}_{\omega_-\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega_-\ell}^{\text{in}}(R)|^2 \right) \Theta(mb - \Omega)$$

$$\omega_- \equiv (mb - \Omega)/a$$



$$W_H(x, x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \int_0^{\infty} d\omega \frac{(\ell - m)! (2\ell + 1) |P_{\ell}^m(0)|^2}{32\pi^2 \omega (l + m)! \sinh(4\pi M \omega)} \\ \times \left(|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 \right) \\ \times \left[e^{4\pi M \omega - i a \omega \tau_{12} + i m b \tau_{12}} + e^{-4\pi M \omega + i a \omega \tau_{12} - i m b \tau_{12}} \right]$$

$$\dot{F}_H(\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{(\ell - m)! (2\ell + 1) |P_{\ell}^m(0)|^2}{16\pi (l + m)!} \\ \times \left[\frac{\left(|\tilde{\Phi}_{\omega_- \ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega_- \ell}^{\text{in}}(R)|^2 \right) e^{4\pi M \omega_-}}{a \omega_- \sinh(4\pi M \omega_-)} \Theta(mb - \Omega) \right. \\ \left. + \frac{\left(|\tilde{\Phi}_{\omega_+ \ell}^{\text{up}}(R)|^2 + |\tilde{\Phi}_{\omega_+ \ell}^{\text{in}}(R)|^2 \right) e^{-4\pi M \omega_-}}{a \omega_+ \sinh(4\pi M \omega_+)} \Theta(mb + \Omega) \right]$$

$$\omega_{\pm} \equiv (mb \pm \Omega)/a$$

$$W_U(x, x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} d\omega \frac{(\ell - m)! (2\ell + 1) |P_{\ell}^m(0)|^2}{16\pi^2 (\ell + m)!} \times \\ \times \left[\frac{|\tilde{\Phi}_{\omega\ell}^{\text{up}}(R)|^2 (e^{4\pi M \omega - i a \omega \tau_{12} + i m b \tau_{12}} + e^{-4\pi M \omega + i a \omega \tau_{12} - i m b \tau_{12}})}{2\omega \sinh(4\pi M \omega)} \right. \\ \left. + \frac{|\tilde{\Phi}_{\omega\ell}^{\text{in}}(R)|^2 e^{-i a \omega \tau_{12} + i m b \tau_{12}}}{\omega} \right]$$

$$\dot{F}_U(\Omega) = \frac{1}{a} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell - m)! (2\ell + 1) |P_{\ell}^m(0)|^2}{8\pi (\ell + m)!} \\ \times \left[\left(\frac{|\tilde{\Phi}_{\omega_- \ell}^{\text{up}}(R)|^2}{2\omega_- \sinh(4\pi M \omega_-)} e^{4\pi M \omega_-} + \frac{|\tilde{\Phi}_{\omega_- \ell}^{\text{in}}(R)|^2}{\omega_-} \right) \Theta(mb - \Omega) \right. \\ \left. + \frac{|\tilde{\Phi}_{\omega_+ \ell}^{\text{up}}(R)|^2}{2\omega_+ \sinh(4\pi M \omega_+)} e^{-4\pi M \omega_+} \Theta(mb + \Omega) \right]$$

$$\omega_{\pm} \equiv (mb \pm \Omega)/a$$

$$\bar{u} = -4M(z - 1)e^{z+z^2/2+z^3/3}$$

$$\bar{v} = 4M(z + 1)e^{-z+z^2/2-z^3/3}$$

$$z = (\tau/\tau_H)^{1/3} \cdot \tau \in (-\infty, 0)$$

$$\frac{dr}{d\tau} = \eta \sqrt{2M/r - 2M/R}$$

$$\frac{dv}{d\tau} = \frac{\sqrt{1 - 2M/R} + \eta \sqrt{2M/r - 2M/R}}{1 - 2M/r}$$



$$F_B = F_B^{\text{in}} + F_B^{\text{up}}$$

$$F_B^{\text{in/up}} = \sum_{\ell=0}^{\infty} \int_0^{\infty} d\omega \frac{2\ell+1}{16\pi^2\omega} \left| \int_{-\infty}^{\infty} d\tau \chi(\tau) e^{-i\Omega\tau} I_{\omega\ell}^{\text{in/up}}(\tau) \right|^2$$

$$I_{\omega\ell}^{\text{in}}(\tau) = e^{i\omega r_*} \Phi_{\omega\ell}^{\text{in}}(r) V^{-i4M\omega},$$

$$I_{\omega\ell}^{\text{up}}(\tau) = \begin{cases} e^{-i\omega r_*} \Phi_{\omega\ell}^{\text{up}}(r) (-U)^{i4M\omega}, & r > 2M, \\ \frac{A_{\omega\ell}^{\text{up}}}{B_{\omega\ell}^{\text{in}}} e^{i\omega r_*} \Phi_{\omega\ell}^{\text{in}}(r) V^{-i4M\omega} + e^{-4\pi M\omega} \frac{1}{B_{\omega\ell}^{\text{in}*}} e^{-i\omega r_*} \Phi_{\omega\ell}^{\text{in}*}(r) U^{i4M\omega}, & r < 2M, \end{cases}$$

$$U = \bar{u}/4M, V = \bar{v}/4M, r_*$$

$$F_H = F_H^{\text{in}} + F_H^{\text{up}} + F_H^{\overline{\text{in}}} + F_H^{\overline{\text{up}}}$$

$$F_H^{\text{in/up,}\overline{\text{in/up}}} = \sum_{\ell=0}^{\infty} \int_0^{\infty} \frac{d\omega}{1 - e^{-8\pi M\omega}} \frac{2\ell+1}{16\pi^2\omega} \left| \int_{-\infty}^{\infty} d\tau \chi(\tau) e^{-i\Omega\tau} I_{\omega\ell}^{\text{in/up,}\overline{\text{in/up}}}(\tau) \right|^2$$

$$I_{\omega\ell}^{\overline{\text{in}}}(\tau) = e^{-4\pi M\omega} e^{-i\omega r_*} \Phi_{\omega\ell}^{\text{in}*}(r) V^{i4M\omega},$$

$$I_{\omega\ell}^{\overline{\text{up}}}(\tau) = \begin{cases} e^{-4\pi M\omega} e^{i\omega r_*} \Phi_{\omega\ell}^{\text{up}*}(r) (-U)^{-i4M\omega}, & r > 2M, \\ e^{-4\pi M\omega} \frac{A_{\omega\ell}^{\text{up}*}}{B_{\omega\ell}^{\text{in}*}} e^{-i\omega r_*} \Phi_{\omega\ell}^{\text{in}*}(r) V^{i4M\omega} + \frac{1}{B_{\omega\ell}^{\text{in}}} e^{i\omega r_*} \Phi_{\omega\ell}^{\text{in}}(r) U^{-i4M\omega}, & r < 2M \end{cases}$$

$$F_U = F_B^{\text{in}} + F_H^{\text{up}} + F_H^{\overline{\text{up}}}$$

$$\langle E_{23}^h E_{14}^\chi \mathcal{E}_{\underline{1234}}^{\chi h h \chi} \rangle = 2\Delta_{23}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_4} - e^{ip \cdot x_4} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \Delta_{12}^\phi$$

$$\langle 0^\phi | (2\phi_1 \phi_2 \phi_3^2 \phi_4^2 - \Delta_{12}^\phi \phi_3^2 \phi_4^2 - 2\Delta_{13}^\phi \phi_2 \phi_3 \phi_4^2 - 2\Delta_{23}^\phi \phi_1 \phi_3 \phi_4^2 + 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_4^2 - 2\Delta_{14}^\phi [\phi_2 \phi_3^2 \phi_4 - \Delta_{23}^\phi \phi_3 \phi_4] - 2\Delta_{24}^\phi [\phi_1 \phi_3^2 \phi_4 - \Delta_{13}^\phi \phi_3 \phi_4] - 2\Delta_{34}^\phi [2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4 - \Delta_{13}^\phi \phi_2 \phi_4 - \Delta_{23}^\phi \phi_1 \phi_4] + 2\Delta_{14}^\phi \Delta_{24}^\phi \phi_3^2 + 2\Delta_{14}^\phi \Delta_{34}^\phi [2\phi_2 \phi_3 - \Delta_{23}^\phi] + 2\Delta_{24}^\phi \Delta_{34}^\phi [2\phi_1 \phi_3 - \Delta_{13}^\phi] + (\Delta_{34}^\phi)^2 [2\phi_1 \phi_2 - \Delta_{12}^\phi]) | 0^\phi \rangle$$

$$\langle E_{23}^h E_{14}^\chi \mathcal{E}_{\underline{1234}}^{\chi h h \chi} \rangle = 2\Delta_{23}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_4} + e^{ip \cdot x_4} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \Delta_{12}^\phi$$

$$\langle 0^\phi | (2\Delta_{14}^\phi [\phi_2 \phi_3^2 \phi_4 - \Delta_{23}^\phi \phi_3 \phi_4] + 2\Delta_{24}^\phi [\phi_1 \phi_3^2 \phi_4 - \Delta_{13}^\phi \phi_3 \phi_4] - 2\Delta_{14}^\phi \Delta_{24}^\phi \phi_3^2 - 2\Delta_{14}^\phi \Delta_{34}^\phi [2\phi_2 \phi_3 - \Delta_{23}^\phi] - 2\Delta_{24}^\phi \Delta_{34}^\phi [2\phi_1 \phi_3 - \Delta_{13}^\phi] - (\Delta_{34}^\phi)^2 [2\phi_1 \phi_2 - \Delta_{12}^\phi]) | 0^\phi \rangle$$



$$\begin{aligned} \langle E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{23}^{h(H)} \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_4} - e^{ip \cdot x_4} e^{-ip \cdot x_1}) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(\Delta_{13}^\phi \phi_2 \phi_3 \phi_4^2 + \Delta_{23}^\phi \phi_1 \phi_3 \phi_4^2 - \Delta_{13}^\phi \Delta_{23}^\phi \phi_4^2 \right. \\ &\quad - \Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 - \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \\ &\quad \left. - \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi \right] \right) |0\phi \rangle \end{aligned}$$

$$\begin{aligned} \langle E_{23}^h E_{14}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{23}^{h(H)} \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_4} + e^{ip \cdot x_4} e^{-ip \cdot x_1}) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(\Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 + \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \right. \\ &\quad \left. + \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi \right] \right) |0\phi \rangle \end{aligned}$$

$$\begin{aligned} \langle E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{24}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_3} - e^{ip \cdot x_3} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(2\phi_1 \phi_2 \phi_3^2 \phi_4^2 - \Delta_{12}^\phi \phi_3^2 \phi_4^2 \right. \\ &\quad - 2\Delta_{13}^\phi \phi_2 \phi_3 \phi_4^2 - 2\Delta_{23}^\phi \phi_1 \phi_3 \phi_4^2 + 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_4^2 \\ &\quad - 2\Delta_{14}^\phi \left[\phi_2 \phi_3^2 \phi_4 - \Delta_{23}^\phi \phi_3 \phi_4 \right] - 2\Delta_{24}^\phi \left[\phi_1 \phi_3^2 \phi_4 - \Delta_{13}^\phi \phi_3 \phi_4 \right] \\ &\quad - 2\Delta_{34}^\phi \left[2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4 - \Delta_{13}^\phi \phi_2 \phi_4 - \Delta_{23}^\phi \phi_1 \phi_4 \right] \\ &\quad + 2\Delta_{14}^\phi \Delta_{24}^\phi \phi_3^2 + 2\Delta_{14}^\phi \Delta_{34}^\phi \left[2\phi_2 \phi_3 - \Delta_{23}^\phi \right] \\ &\quad \left. + 2\Delta_{24}^\phi \Delta_{34}^\phi \left[2\phi_1 \phi_3 - \Delta_{13}^\phi \right] + \left(\Delta_{34}^\phi \right)^2 \left[2\phi_1 \phi_2 - \Delta_{12}^\phi \right] \right) |0\phi \rangle \end{aligned}$$

$$\begin{aligned} \langle E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{24}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_3} + e^{ip \cdot x_3} e^{-ip \cdot x_1}) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(\Delta_{13}^\phi \phi_2 \phi_3 \phi_4^2 + \Delta_{23}^\phi \phi_1 \phi_3 \phi_4^2 - \Delta_{13}^\phi \Delta_{23}^\phi \phi_4^2 \right. \\ &\quad - \Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 - \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \\ &\quad \left. - \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi \right] \right) |0\phi \rangle \text{(A.0.6)} \end{aligned}$$

$$\begin{aligned} \langle E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{24}^{h(H)} \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_3} - e^{ip \cdot x_3} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(2\Delta_{14}^\phi \left[\phi_2 \phi_3^2 \phi_4 - \Delta_{23}^\phi \phi_3 \phi_4 \right] + 2\Delta_{24}^\phi \left[\phi_1 \phi_3^2 \phi_4 - \Delta_{13}^\phi \phi_3 \phi_4 \right] \right. \\ &\quad + 2\Delta_{34}^\phi \left[2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4 - \Delta_{13}^\phi \phi_2 \phi_4 - \Delta_{23}^\phi \phi_1 \phi_4 \right] \\ &\quad - 2\Delta_{14}^\phi \Delta_{24}^\phi \phi_3^2 - 2\Delta_{14}^\phi \Delta_{34}^\phi \left[2\phi_2 \phi_3 - \Delta_{23}^\phi \right] \\ &\quad \left. - 2\Delta_{24}^\phi \Delta_{34}^\phi \left[2\phi_1 \phi_3 - \Delta_{13}^\phi \right] - \left(\Delta_{34}^\phi \right)^2 \left[2\phi_1 \phi_2 - \Delta_{12}^\phi \right] \right) |0\phi \rangle \end{aligned}$$

$$\begin{aligned} \langle E_{24}^h E_{13}^\chi \mathcal{E}_{1234}^{\chi h h \chi} \rangle &= 2\Delta_{24}^{h(H)} \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_3} + e^{ip \cdot x_3} e^{-ip \cdot x_1}) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\ &\langle 0\phi | \left(\Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 + \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \right. \\ &\quad \left. + \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi \right] \right) |0\phi \rangle \end{aligned}$$



$$\begin{aligned}
\langle E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} \rangle &= 2\Delta_{34}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_2} - e^{ip \cdot x_2} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \\
&\langle 0^\phi | \left(2\Delta_{13}^\phi (\phi_1 \phi_2^2 \phi_3 \phi_4^2 - \Delta_{12}^\phi \phi_2 \phi_3 \phi_4^2) \right. \\
&\quad + 2\Delta_{23}^\phi (\phi_1^2 \phi_2 \phi_3 \phi_4^2 - \Delta_{12}^\phi \phi_1 \phi_3 \phi_4^2) \\
&\quad - 2\Delta_{13}^\phi \Delta_{23}^\phi (2\phi_1 \phi_2 \phi_4^2 - \Delta_{12}^\phi \phi_4^2) \\
&\quad - (\Delta_{13}^\phi)^2 \phi_2^2 \phi_4^2 - (\Delta_{23}^\phi)^2 \phi_1^2 \phi_4^2 \\
&\quad \left. + 2\Delta_{14}^\phi [-\Delta_{13}^\phi \phi_2^2 \phi_3 \phi_4 - \Delta_{23}^\phi (2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4)] \right. \\
&\quad \left. + 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_2 \phi_4 + (\Delta_{23}^\phi)^2 \phi_1 \phi_4 \right] \\
&+ 2\Delta_{24}^\phi [-\Delta_{23}^\phi \phi_1^2 \phi_3 \phi_4 - \Delta_{13}^\phi (2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4) \\
&\quad + 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_1 \phi_4 + (\Delta_{13}^\phi)^2 \phi_2 \phi_4] \\
&\quad + 2\Delta_{34}^\phi [-\Delta_{13}^\phi (\phi_1 \phi_2^2 \phi_4 - \Delta_{12}^\phi \phi_2 \phi_4) \\
&\quad - \Delta_{23}^\phi (\phi_1^2 \phi_2 \phi_4 - \Delta_{12}^\phi \phi_1 \phi_4)] \\
&\quad + 4\Delta_{14}^\phi \Delta_{24}^\phi [\Delta_{13}^\phi \phi_2 \phi_3 + \Delta_{23}^\phi \phi_1 \phi_3 - \Delta_{13}^\phi \Delta_{23}^\phi] \\
&\quad + 2\Delta_{14}^\phi \Delta_{34}^\phi [\Delta_{13}^\phi \phi_2^2 + \Delta_{23}^\phi (2\phi_1 \phi_2 - \Delta_{12}^\phi)] \\
&\quad + 2\Delta_{24}^\phi \Delta_{34}^\phi [\Delta_{23}^\phi \phi_1^2 + \Delta_{13}^\phi (2\phi_1 \phi_2 - \Delta_{12}^\phi)] \\
&\quad + (\Delta_{14}^\phi)^2 [2\Delta_{23}^\phi \phi_2 \phi_3 - (\Delta_{23}^\phi)^2] \\
&\quad \left. + (\Delta_{24}^\phi)^2 [2\Delta_{13}^\phi \phi_1 \phi_3 - (\Delta_{13}^\phi)^2] \right] |0^\phi \rangle \\
\langle E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} \rangle &= 2\Delta_{34}^h \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_2} + e^{ip \cdot x_2} e^{-ip \cdot x_1}) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\
&\langle 0^\phi | \left(\Delta_{13}^\phi \phi_2 \phi_3 \phi_4^2 + \Delta_{23}^\phi \phi_1 \phi_3 \phi_4^2 - \Delta_{13}^\phi \Delta_{23}^\phi \phi_4^2 \right. \\
&\quad - \Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 - \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \\
&\quad \left. - \Delta_{34}^\phi [\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi] \right) |0^\phi \rangle \\
\langle E_{34}^h E_{12}^\chi \mathcal{E}_{1234}^{\chi\chi hh} \rangle &= 2\Delta_{34}^{h(H)} \frac{1}{2\omega_p} (-e^{ip \cdot x_1} e^{-ip \cdot x_2} - e^{ip \cdot x_2} e^{-ip \cdot x_1}) \times 16g_h^2 g_\chi^2 \\
&\langle 0^\phi | \left(2\Delta_{14}^\phi [\Delta_{13}^\phi \phi_2^2 \phi_3 \phi_4 + \Delta_{23}^\phi (2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4)] \right. \\
&\quad \left. - 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_2 \phi_4 - (\Delta_{23}^\phi)^2 \phi_1 \phi_4 \right] \\
&\quad 2\Delta_{24}^\phi [\Delta_{23}^\phi \phi_1^2 \phi_3 \phi_4 + \Delta_{13}^\phi (2\phi_1 \phi_2 \phi_3 \phi_4 - \Delta_{12}^\phi \phi_3 \phi_4) \\
&\quad - 2\Delta_{13}^\phi \Delta_{23}^\phi \phi_1 \phi_4 - (\Delta_{13}^\phi)^2 \phi_2 \phi_4] \\
&\quad \left. 2\Delta_{34}^\phi [\Delta_{13}^\phi (\phi_1 \phi_2^2 \phi_4 - \Delta_{12}^\phi \phi_2 \phi_4)] \right)
\end{aligned}$$

$$\begin{aligned}
& +\Delta_{23}^\phi \left(\phi_1^2 \phi_2 \phi_4 - \Delta_{12}^\phi \phi_1 \phi_4 \right) \\
& -4\Delta_{14}^\phi \Delta_{24}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_3 + \Delta_{23}^\phi \phi_1 \phi_3 - \Delta_{13}^\phi \Delta_{23}^\phi \right] \\
& -2\Delta_{14}^\phi \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2^2 + \Delta_{23}^\phi \left(2\phi_1 \phi_2 - \Delta_{12}^\phi \right) \right] \\
& -2\Delta_{24}^\phi \Delta_{34}^\phi \left[\Delta_{23}^\phi \phi_1^2 + \Delta_{13}^\phi \left(2\phi_1 \phi_2 - \Delta_{12}^\phi \right) \right] \\
& -\left(\Delta_{14}^\phi \right)^2 \left[2\Delta_{23}^\phi \phi_2 \phi_3 - \left(\Delta_{23}^\phi \right)^2 \right] \\
& -\left(\Delta_{24}^\phi \right)^2 \left[2\Delta_{13}^\phi \phi_1 \phi_3 - \left(\Delta_{13}^\phi \right)^2 \right] \Big| 0^\phi \rangle \\
\langle E_{34}^h E_{12}^\chi \varepsilon_{1234}^{\chi\chi hh} \rangle & = 2\Delta_{34}^{h(H)} \frac{1}{2\omega_p} \left(-e^{ip \cdot x_1} e^{-ip \cdot x_2} + e^{ip \cdot x_2} e^{-ip \cdot x_1} \right) \times 32g_h^2 g_\chi^2 \Delta_{12}^\phi \\
& \langle 0^\phi | \left(\Delta_{14}^\phi \Delta_{23}^\phi \phi_3 \phi_4 + \Delta_{24}^\phi \Delta_{13}^\phi \phi_3 \phi_4 \right. \\
& \left. + \Delta_{34}^\phi \left[\Delta_{13}^\phi \phi_2 \phi_4 + \Delta_{23}^\phi \phi_1 \phi_4 - \Delta_{14}^\phi \Delta_{23}^\phi - \Delta_{24}^\phi \Delta_{13}^\phi \right] \right) \Big| 0^\phi \rangle
\end{aligned}$$

$$\begin{aligned}
\mathbb{P} \supset & -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{p_1})(2\omega_{p_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \\
& \left(e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{13}^\chi R_{24}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \right. \\
& + e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{14}^\chi R_{23}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \\
& + e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_3} e^{-ip_2 \cdot x_3} \left(F_{24}^\phi \right)^2 R_{12}^\chi R_{34}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \\
& + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{13}^\chi R_{24}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \\
& + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{14}^\chi R_{23}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \\
& \left. + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_3} e^{ip_2 \cdot x_3} \left(F_{24}^\phi \right)^2 R_{12}^\chi R_{34}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P} \supset & -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{p_1})(2\omega_{p_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \\
& \left(e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{13}^\chi R_{24}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \right. \\
& + e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{13}^\chi R_{24}^\chi \Theta_{12} \Theta_{24} \Theta_{43} \\
& + e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{13}^\chi R_{24}^\chi \Theta_{13} \Theta_{32} \Theta_{24} \\
& + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{14}^\chi R_{23}^\chi \Theta_{12} \Theta_{24} \Theta_{43} \\
& + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{14}^\chi R_{23}^\chi \Theta_{12} \Theta_{23} \Theta_{34} \\
& \left. + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} \left(F_{34}^\phi \right)^2 R_{14}^\chi R_{23}^\chi \Theta_{14} \Theta_{42} \Theta_{23} \right)
\end{aligned}$$

$$\begin{aligned}
\Theta_{xy} \Theta_{yz} & = \Theta_{xy} \Theta_{yz} \Theta_{xz} \\
\Theta_{xy} + \Theta_{yx} & = 1 \\
\Theta_{xy} R_{xy}^\chi & = R_{xy}^\chi
\end{aligned}$$



$$\mathbb{P} \supset -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{p_1})(2\omega_{p_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4$$

$$\left(e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} (F_{34}^\phi)^2 R_{13}^\chi R_{24}^\chi \Theta_{12} \right.$$

$$\left. + e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_1} e^{ip_1 \cdot x_2} e^{ip_2 \cdot x_2} (F_{34}^\phi)^2 R_{14}^\chi R_{23}^\chi \Theta_{12} \right)$$

$$\mathbb{P} \supset -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{p_1})(2\omega_{p_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4$$

$$e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} (F_{34}^\phi)^2 R_{13}^\chi R_{24}^\chi (\Theta_{12} + \Theta_{21})$$

$$\supset -\frac{4g_\psi^2 g_\phi^2}{(2\omega_{p_1})(2\omega_{p_2})} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_1} e^{-ip_1 \cdot x_2} e^{-ip_2 \cdot x_2} (F_{34}^\phi)^2 R_{13}^\chi R_{24}^\chi$$

$$\Delta_{12}^R|_{\alpha=0} = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip^0 t_{12}}}{(p^0 + i\epsilon)^2 - \omega_p^2} = -\frac{1}{2\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \sin(\omega_p t_{12})$$

$$\Delta_{12}^H|_{\alpha=0} = \int \frac{d^4p}{(2\pi)^4} e^{-ip^0 t_{12}} 2\pi \delta(p^2 - m^2) = \frac{1}{2\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \cos(\omega_p t_{12})$$

$$\mathbb{P}(2; t)|_{\alpha=0} = \frac{|\mu|^2}{2\pi^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} [\cos(\Omega t_{12}) \cos(\omega_p t_{12})$$

$$- \sin(\Omega t_{12}) \sin(\omega_p t_{12})]$$

$$= \frac{|\mu|^2}{2\pi^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \cos[(\omega_p + \Omega) t_{12}]$$

$$\mathbb{P}(2; t)|_{\alpha=0} = \frac{|\mu|^2}{2\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \frac{1 - \cos[(\omega_p + \Omega)t]}{(\omega_p + \Omega)^2}$$

$$\frac{\partial \mathbb{P}(2; t)}{\partial t} \Big|_{\alpha=0} = \frac{|\mu|^2}{2\pi^2} \int_m^\infty d\omega_p \sqrt{\omega_p^2 - m^2} \frac{\sin[(\omega_p + \Omega)t]}{\omega_p + \Omega}$$

$$W_\lambda^{(d)}(z_1, z_2) = c_d(\lambda) \int_{\gamma(z_1)} (z_1 \cdot \zeta)^\lambda (z_2 \cdot \zeta)^{-\lambda-d+1} d\mu_\gamma(\zeta)$$

$$W_m^{Md}(z_1, z_2) = \frac{1}{(2\pi)^d} \int e^{-ip \cdot z_1} e^{ip \cdot z_2} \theta(p^0) \delta(p^2 - m^2) dp$$

$$W_\lambda^{(d)}(z_1(z_1, u), z_2(z_2, u')) =$$

$$= \frac{(uu')^{\frac{d-1}{2}}}{2(2\pi)^{d-2}} \int e^{-ik(z_1-z_2)} \theta(k^0) \theta(k^2) J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) J_{\frac{d-1}{2}+\lambda}(u'\sqrt{k^2}) dk$$

$$x \cdot y = x^0 y^0 + x^d y^d - \vec{x} \cdot \vec{y} = g_{\mu\nu} x^\mu x^\nu, g_{\mu\nu} = \text{diag}(+1, -1, \dots, -1, +1);$$

$$e_\mu \cdot e_\nu = g_{\mu\nu}, x = x^\mu e_\mu, \vec{x} = (x^1, \dots, x^{d-1}).$$

$$\widetilde{AdS}_d = \{x, x^2 = x \cdot x = R^2\}.$$



$$AdS_d^{(c)} = \{z = x + iy \in \mathbf{C}_2^{d+1}, z^2 = \overline{AdS_d}\} = \{z = x + iy, x^2 - y^2 = 1, x \cdot y = \overline{AdS_d}\}.$$

$$z^E = (ix^0, \vec{x}, x^d)$$

$$(x^d)^2 - (x^0)^2 - \vec{x}^2 = 1, x^d \geq \overline{AdS_d}$$

$$(t, \vec{x}) \mapsto (\sqrt{1 + \vec{x}^2} \sin t, \vec{x}, \sqrt{1 + \vec{x}^2} \cos t)$$

$$t \in S_1 = \mathbf{R}/2\pi\mathbf{Z} \text{ and } \vec{x} \in \mathbf{R}^{d-1}$$

$$(s, \vec{x}) \mapsto (i\sqrt{1 + \vec{x}^2} \operatorname{sh} s, \vec{x}, \sqrt{1 + \vec{x}^2} \operatorname{ch} s)$$

$$C_d = \{\xi \in \mathbf{R}_2^{d+1}, \xi^2 = (\xi^0)^2 - \vec{\xi}^2 + (\xi^d)^2 = 0, \xi \neq 0\}$$

$$C_d^{(c)} = \{\zeta = \xi + i\chi \in \mathbf{C}_2^{d+1}, (\xi, \xi) - (\chi, \chi) = 0, (\xi \cdot \chi) = 0, \zeta \neq 0\}$$

$$\gamma_S = \{(\xi^0)^2 + (\xi^d)^2 = 1 = (\xi^1)^2 + \dots + (\xi^{d-1})^2\}$$

$$\xi(\phi, \vec{n}) = \begin{cases} \xi^0 = \sin \phi \\ \xi^i = n^i \\ \xi^d = \cos \phi \end{cases}, \vec{n}^2 = 1$$

$$\xi(\phi + \pi, \vec{n}) = -\xi(\phi, -\vec{n})$$

$$\gamma_{S(2)}^\pm = \{(\xi^0)^2 + (\xi^d)^2 = 1, \xi^1 = \pm 1\}$$

$$\gamma_S^{(c)} = \{(\zeta^0)^2 + (\zeta^d)^2 = 1 = (\zeta^1)^2 + \dots + (\zeta^{d-1})^2\}$$

$$x^\mu(s) = \frac{\xi^\mu e^{\alpha s} + \eta^\mu e^{-\alpha s}}{\sqrt{2\xi \cdot \eta}}$$

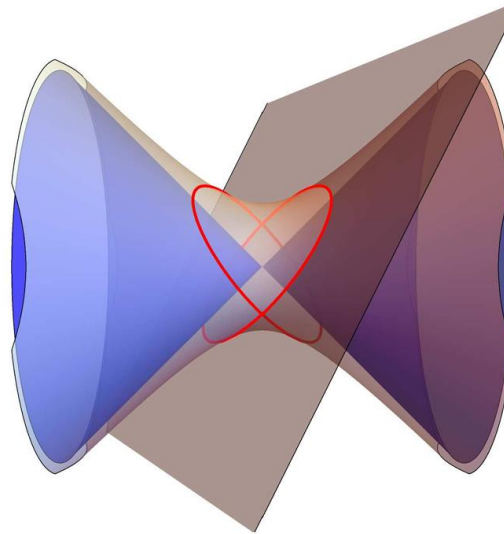
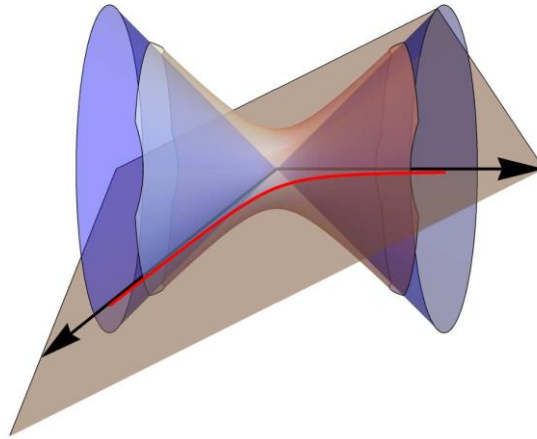
$$C_d^{(c)} = \{\zeta = \xi + i\chi \in \mathbf{C}_2^{d+1}, (\xi, \xi) - (\chi, \chi) = 0, (\xi \cdot \chi) = 0, \zeta \neq 0\}$$

$$x^\mu(\tau) = \frac{\zeta^\mu e^{i\beta\tau} + \zeta^{\mu*} e^{-i\beta\tau}}{\sqrt{2\zeta \cdot \zeta^*}}, (\zeta \cdot \zeta) = 0$$

$$v^\mu(\tau) = i\beta \frac{\zeta^\mu e^{i\beta\tau} - \zeta^{\mu*} e^{-i\beta\tau}}{\sqrt{2\zeta \cdot \zeta^*}}, (v \cdot v) = \beta^2$$

$$K^{\mu\nu} = \frac{m(x^\mu(\tau)v^\nu(\tau) - x^\nu(\tau)v^\mu(\tau))}{\sqrt{v \cdot v}} = \frac{im(\zeta^\mu \zeta^{\nu*} - \zeta^\nu \zeta^{\mu*})}{\zeta \cdot \zeta^*} = \frac{im(\zeta \wedge \zeta^*)^{\mu\nu}}{\zeta \cdot \zeta^*}$$





$$e^{tM_{0d}} = \begin{pmatrix} \cos t & \dots & \sin t \\ \vdots & 1 & \vdots \\ -\sin t & \dots & \cos t \end{pmatrix}.$$

$$C_0 = \{M \in \mathcal{G} : M = gM_{0d}g^{-1} : g \in G_0\},$$

$$C_+ = \bigcup_{\rho > 0} \rho C_0. \quad (4.2)$$

$$Z_{\pm} = \{\exp(\tau M)x : M \in C_0, x \in X_d, \tau \in \mathbf{C}, \text{Im}\tau \geq 0\}.$$

$$Z_{\pm} = \{z = x + iy \in \mathbf{C}_2^{d+1} : y \cdot y > 0, \epsilon(z) \geq 0\},$$

$$\epsilon(z) = y^0 x^d - x^0 y^d$$

$$\epsilon(z)^2 = (y \cdot y)(1 + |\vec{x}|^2 + |\vec{y}|^2) + (y \cdot y)^2 + |\vec{y}|^2 + |\vec{x}|^2 |\vec{y}|^2 - (\vec{x} \cdot \vec{y})^2$$

$$Z_{\pm} = \{g \exp(itM_{0d})e_d : g \in G_0, t \geq 0\}$$

$$z(t + is, \psi + i\phi, \vec{a} + i\vec{b}) = \begin{cases} \text{ch}(\psi + i\phi) \sin(t + is), \\ \text{sh}(\psi + i\phi)(\vec{a} + i\vec{b}), & |\vec{a}|^2 - |\vec{b}|^2 = 1, \vec{a} \cdot \vec{b} = 0 \\ \text{ch}(\psi + i\phi) \cos(t + is), \end{cases}$$

$$Z_+: \text{shs} > \sqrt{\frac{|\vec{b}|^2 \text{sh}^2(\psi) + (|\vec{b}|^2 + 1) \sin^2(\phi)}{\text{sh}^2(\psi) + \cos^2(\phi)}}$$

$$Z_-: \text{shs} < -\sqrt{\frac{|\vec{b}|^2 \text{sh}^2(\psi) + (|\vec{b}|^2 + 1) \sin^2(\phi)}{\text{sh}^2(\psi) + \cos^2(\phi)}}$$

$$\mathcal{W}(gx_1, gx_2) = \mathcal{W}(x_1, x_2)$$

$$\mathcal{C}(x_1, x_2) = \mathcal{W}(x_1, x_2) - \mathcal{W}(x_2, x_1)$$

$$\mathcal{W}(\bar{f}f) = \int \mathcal{W}(x_1, x_2) \overline{f(x)} f(y) d\sigma(x) d\sigma(y) \geq 0$$

$$\mathcal{W}(fg) = (\Omega, \varphi(f)\varphi(g)\Omega) \text{ for all } f, g \in \mathcal{T}$$

$$U(g)\varphi(f)U(g)^{-1} = \varphi(f_{\{g\}}), f_{\{g\}}(x) = f(g^{-1}x), U(g)\Omega = \Omega$$

$$\int_{\mathbf{R}} \left(\int \tilde{f}(p) e^{-itp} dp \right) U(\exp tM) \Psi dt = 0$$

$$\mathcal{W}(x_1, x_2) = (\Omega, \varphi(x_1)\varphi(x_2)\Omega) = \lim_{\substack{Z_- \ni z_1 \rightarrow x_1 \\ Z_+ \ni z_2 \rightarrow x_2}} W(z_1, z_2)$$

$$\square_g = g_{\mu\nu} \partial^\mu \partial^\nu$$

$$(Df)(x) = x^\mu \partial_\mu f(x) \text{ and let } \mathcal{M}_{\mu\nu} = (x_\mu \partial_\nu - x_\nu \partial_\mu).$$

$$\mathcal{M}^2 = \mathcal{M}^{\mu\nu} \mathcal{M}_{\mu\nu} = 2x^2 \square_g - 2(d-1)D - 2D^2$$

$$\mathcal{M}_{\mu\nu}(h(x)f(x)) = h(x)\mathcal{M}_{\mu\nu}f(x)$$

$$\Omega = \{x \in \mathbf{R}_2^{d+1}: (x, x) > 0\}$$

$$\square(f|_{AdS_d}) = (\square_g f)|_{AdS_d} = \frac{1}{2} \mathcal{M}^2 f|_{AdS_d}$$

$$\square \phi = \frac{1}{2} \mathcal{M}^2 \phi$$

$$\mathcal{M}^2 f(x) = -2\lambda(\lambda + d - 1)f(x).$$

$$(\square + \lambda(\lambda + d - 1))(f(x)|_{AdS_d}) = 0$$

$$m_\lambda^2 = \lambda(\lambda + d - 1)$$

$$\lambda_d = -\frac{d-1}{2}, m_{\lambda_d}^2 = -\frac{(d-1)^2}{4}$$

$$\nu = \lambda + \frac{d-1}{2}$$



$$m_\lambda^2 = v_\lambda^2 - \frac{(d-1)^2}{4}$$

$$x, \zeta \mapsto (x \cdot \zeta)^\lambda$$

$$\phi_\lambda(x, \zeta) = (x \cdot \zeta)^\lambda, x \in AdS_d$$

$$(\square + m_\lambda^2)\phi_\lambda(x, \zeta) = ((z_1 \cdot z_2)^2 - 1)^{-\frac{d-2}{4}}$$

$$(\square + m_\lambda^2)\phi_{1-d-\lambda}(x, \zeta) = ((z_1 \cdot z_2)^2 - 1)^{-\frac{d-2}{4}}$$

$$\mathcal{C}_\pm = \{\exp(\tau M)\xi: M \in \mathcal{C}_0, \xi \in \mathcal{C}_d, \tau \in \mathbf{C}, \text{Im}\tau \geq 0\}$$

$$\mathcal{C}_\pm = \{\zeta = \xi + i\chi \in \mathbf{C}_2^{d+1}: \xi \cdot \xi = \chi \cdot \chi > 0, \xi \cdot \chi = 0, \epsilon(\zeta) \geq 0\}$$

$$\mathcal{C}_\pm = \{g \exp(itM_{0d})(e_{d-1} + e_d): g \in G_0, t \geq 0\}$$

$$\bar{\mathcal{C}}_+ = \{\zeta \in \mathcal{C}_d^{(c)}: \chi \cdot \chi \geq 0, \epsilon(\zeta) \geq 0, \zeta \neq 0\}$$

$$\bar{\mathcal{C}}_- = \{\zeta \in \mathcal{C}_d^{(c)}: \chi \cdot \chi \geq 0, \epsilon(\zeta) \leq 0, \zeta \neq 0\}$$

$$\exp(\tau M)(\bar{\mathcal{C}}_\pm) \subset \mathcal{C}_\pm, \text{ for } M \in \mathcal{C}_0, \text{Im}\tau \geq 0$$

$$e_d \cdot \zeta = x \cdot \exp(itM_{0d})(e_{d-1} + e_d)$$

$$x = g^{-1}e_d = \exp(sM_{0d})(0, \vec{x}, \sqrt{1 + \vec{x}^2}), s = \arctan(x^0/x^d) \text{ and } t > 0$$

$$e_d \cdot \zeta = -x^{d-1} + \sqrt{1 + \vec{x}^2} \cos(it - s) \in \mathbf{C}^*$$

$$w = \lambda \cos(u + iv), \text{ with } v > 0 \text{ and } \lambda > 0$$

$$w = e_d \cdot (\lambda \exp((u + iv)M_{0d})(e_{d-1} + e_d))$$

$$z \cdot \zeta = e_d \cdot \exp(\tau M)g^{-1}\zeta$$

$$\chi \cdot \zeta = (e_{d-1} + e_d) \cdot \zeta', \zeta' = \exp(\tau M)g^{-1}\zeta \in \mathcal{C}_+$$

$$\xi \cdot \exp(itM_{0d})(e_{d-1} + e_d) = \text{ish}(t)\xi^0 - \xi^{d-1} + \text{ch}(t)\xi^d, \xi \in \mathcal{C}_d, t > 0$$

$$\tilde{\mathcal{Z}}_\pm \times \bar{\mathcal{C}}_\mp \ni \zeta, z \rightarrow \phi_\lambda^\pm(z, \zeta) = (z \cdot \zeta)^\lambda = e^{\lambda \log(z \cdot \zeta)}$$

$$m_\lambda^2 = \lambda(\lambda + d - 1)$$

$$(\square + m_\lambda^2)\mathcal{W}(x_1, x_2) = \iiint \overline{AdS}_d \otimes \overline{AdS}_d$$



$$\begin{aligned}
W_\lambda^{(d)}(z_1, z_2) &= W_\lambda^{(d)}(z_1 \cdot z_2) = \frac{e^{-i\pi\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}} ((z_1 \cdot z_2)^2 - 1)^{-\frac{d-2}{4}} Q_{\frac{d-2}{2}+\lambda}^{\frac{d-2}{2}}(z_1 \cdot z_2) \\
&= \frac{(z_1 \cdot z_2)^{1-d-\lambda} \Gamma(d-1+\lambda) {}_2F_1\left(\frac{d-1+\lambda}{2}, \frac{d+\lambda}{2}; \frac{d+1+2\lambda}{2}\right); (z_1 \cdot z_2)^{-2}}{2^{d+\lambda} \pi^{\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2} + \lambda\right)} \\
W_{1-d-\lambda}^{(d)}(z_1, z_2) &= W_{1-d-\lambda}^{(d)}(z_1 \cdot z_2) = \frac{e^{-i\pi\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}} ((z_1 \cdot z_2)^2 - 1)^{-\frac{d-2}{4}} Q_{\frac{d-2}{2}-\lambda}^{\frac{d-2}{2}}(z_1 \cdot z_2).
\end{aligned}$$

$$\Delta_1 = \{\mathbf{C} \setminus [-\infty, 1]\}.$$

$$W_{1-d-\lambda}^{(d)}(z) = W_\lambda^{(d)}(z) - \frac{\cos\left(\frac{1}{2}\pi(d+2\lambda)\right) \Gamma(-\lambda) \Gamma(d+\lambda-1)}{(2\pi)^{\frac{d}{2}}} (z^2 - 1)^{-\frac{d-2}{4}} P_{\frac{d-2}{2}+\lambda}^{\frac{d-2}{2}}(z)$$

$$z_1 = e_d, z_2(u, \omega) = (i\omega^0 \sqrt{u^2 - 1}, \omega^1 \sqrt{u^2 - 1}, \dots, \omega^{d-1} \sqrt{u^2 - 1}, u), u > 1, \omega^2 = 1$$

$$z_1 \cdot z_2(u, \omega) = u > 1$$

$$S_\lambda^{(d)}(u) = W_\lambda^{(d)}(z_1, z_2(u, \omega)) = \frac{e^{-i\pi\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}} (u^2 - 1)^{-\frac{d-2}{4}} Q_{\frac{d-2}{2}+\lambda}^{\frac{d-2}{2}}(u)$$

$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = c_2(\lambda) \int_{\gamma} (z_1 \cdot \zeta)^\lambda (z_2 \cdot \zeta)^{-\lambda-1} d\mu_\gamma(\zeta), z_1 \in \tilde{\mathcal{Z}}_-, z_2 \in \tilde{\mathcal{Z}}_+$$

$$\gamma_{S(2)}^+ = \{\xi = (\sin \phi, 1, \cos \phi), -\pi < \phi < \pi\}, d\mu_{\gamma_{S(2)}^+}(\xi) = d\phi$$

$$\mathfrak{W}_{l+\frac{1}{2}}^{(2)}(z_1, z_2) = W_{l+\frac{1}{2}}^{(2)}(z_1, z_2) = \frac{1}{8\pi} \int_{\gamma_{S(2)}^+} (z_1 \cdot \xi)^{l+\frac{1}{2}} (z_2 \cdot \xi)^{-l-\frac{3}{2}} d\mu_{\gamma_{S(2)}^+}(\xi) = \frac{1}{2\pi} Q_{l+\frac{1}{2}}(z_1 \cdot z_2)$$

$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = \frac{1}{4\pi} \int_{\gamma(z_1)} (z_1 \cdot \zeta)^\lambda (\zeta \cdot z_2)^{-\lambda-1} d\mu_{\gamma(z_1)}(\zeta), z_1 \in \mathcal{Z}_-, z_2 \in \mathcal{Z}_+$$

$$\begin{aligned}
\mathfrak{W}_\lambda^{(2)}(gz_1, gz_2) &= \frac{1}{4\pi} \int_{g\gamma(z_1)} (z_1 \cdot g^{-1}\zeta)^\lambda (g^{-1}\zeta \cdot z_2)^{-\lambda-1} d\mu_{g\gamma(z_1)}(\zeta) \\
&= \frac{1}{4\pi} \int_{\gamma(z_1)} (z_1 \cdot \zeta)^\lambda (\zeta \cdot z_2)^{-\lambda-1} d\mu_{\gamma(z_1)}(\zeta)
\end{aligned}$$

$$z_1 = (\sin(-iv), 0, \cos(-iv)), v \geq 0, z_2 = (\sin(iu), 0, \cos(iu)), u > 0$$

$$\gamma(z_1) = \{(\sin(\phi - iv), 1, \cos(\phi - iv)), -\pi/2 < \phi < \pi/2\}$$

$$\gamma_{S(2)}^+(e_2) = \{(\sin \phi, 1, \cos \phi), -\pi/2 < \phi < \pi/2\} = \{\xi \in \gamma_{S(2)}^+ : e_2 \cdot \xi = \cos \phi \geq 0\}$$

$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = \frac{1}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda (\cos(\phi - i(u+v)))^{-\lambda-1} d\phi$$



$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} (chu + ishtshu)^{-\lambda-1} dt = \frac{1}{2\pi} Q_\lambda(chu)$$

$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = W_\lambda^{(2)}(z_1, z_2) = \frac{1}{2\pi} Q_\lambda(z_1 \cdot z_2).$$

$$C_\lambda^{(2)}(x_1, x_2) = W_\lambda^{(2)}(x_1, x_2) - W_\lambda^{(2)}(x_2, x_1).$$

$$W_\lambda^{(2)}(chu) = \frac{e^{(\lambda+1)u}}{2\pi} \int_{\gamma^+} (z^2 - 1)^\lambda (e^{2u} z^2 - 1)^{-\lambda-1} dz$$

$$\zeta(z) = \left(-\frac{z^2 + 1}{2z}, 1, -\frac{i(z^2 - 1)}{2z} \right)$$

$$(\text{Im}\zeta)^2 = \frac{(x^2 + y^2 - 1)^2}{4(x^2 + y^2)} \geq 0, \epsilon(\zeta) = \frac{(1 - x^2 - y^2)(1 + x^2 + y^2)}{4(x^2 + y^2)}.$$

$$\gamma^+(r) = \{|z| = r, \text{Im}z > 0\}$$

$$\{|x| > 1, y = 0\},$$

$$\begin{aligned} W_\lambda^{(2)}(chu) &= \frac{e^{(\lambda+1)u}}{\pi} \int_1^\infty (x^2 - 1)^\lambda (e^{2u} x^2 - 1)^{-\lambda-1} dx = \\ &= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda + 1)}{2\pi \Gamma\left(\lambda + \frac{3}{2}\right)} e^{-(\lambda+1)u} {}_2F_1\left(\frac{1}{2}, \lambda + 1; \lambda + \frac{3}{2}; e^{-2u}\right). \end{aligned}$$

$$Q_\nu^\mu(chu) = \frac{2^\mu e^{i\pi\mu} \sqrt{\pi} \Gamma(\mu + \nu + 1) e^{-u(\mu+\nu+1)} (\text{sh}u)^\mu {}_2F_1\left(\mu + \frac{1}{2}, \mu + \nu + 1; \nu + \frac{3}{2}; e^{-2u}\right)}{\Gamma\left(\nu + \frac{3}{2}\right)}.$$

$$\mathfrak{W}_\lambda^{(2)}(chu) = \frac{e^{-(\lambda+1)u}}{2\pi} \int_{-1}^1 (1 - x^2)^\lambda (1 - e^{-2u} x^2)^{-\lambda-1} dx$$

$$\mathfrak{W}_\lambda^{(2)}(z_1, z_2) = \frac{1}{4\pi} \int_{\gamma(z_2)} (z_1 \cdot \xi)^\lambda (\xi \cdot z_2)^{-\lambda-1} d\mu_{\gamma(z_2)}(\xi)$$

$$\gamma(z_2) = \{\xi = (\sin(\phi + iu), 1, \cos(\phi + iu)), -\pi/2 < \phi < \pi/2\}.$$

$$\mathfrak{W}_\lambda^{\#(2)}(chu) = \frac{e^{\lambda u}}{4\pi} \int_{-1}^1 (1 - e^{-2u} x^2)^\lambda (1 - x^2)^{-\lambda-1} dx$$

$$\mathfrak{W}_\lambda^{\#(2)}(z_1, z_2) = \frac{1}{2\pi} Q_{-\lambda-1}(z_1, z_2)$$

$$Q_\lambda(xchu) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda \left(x \cos(\phi - iu) - \sqrt{x^2 - 1} \right)^{-\lambda-1} d\phi$$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda (\cos(\phi - iu))^{-\lambda-1-n} d\phi = \frac{2^{1+\frac{n}{2}} \sqrt{\pi} \Gamma(1+\lambda) e^{-\frac{1}{2}in\pi} Q_{\frac{n}{2}+\lambda}^{\frac{n}{2}}(\operatorname{ch}(u))}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right) \Gamma(1+n+\lambda) (\operatorname{sh}u)^{\frac{n}{2}}}.$$

$$Q_\lambda(x\operatorname{ch}u) = \frac{1}{x^{1+\lambda}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(1+n)} \left(1 - \frac{1}{x^2}\right)^n \frac{Q_{n+\lambda}^n(\operatorname{ch}u)}{(2\operatorname{sh}u)^n}, x > 1.$$

$$z_1(t, R) = \frac{\hat{z}_1(t, R)}{R} = \frac{1}{R} (\sqrt{R^2+1} \sin t, 1, \sqrt{R^2+1} \cos t), z_2 = (\sin(iu), 0, \cos(iu))$$

$$\gamma(z_1) = \{\xi = (\sin \phi, 1, \cos \phi), t - t_R < \phi < t + t_R\}, t_R = \arccos\left((R^2+1)^{-\frac{1}{2}}\right).$$

$$\begin{aligned} W_\lambda^{(2)}(z_1(t, R), z_2) &\simeq \frac{(\cos(t-iu))^{-\lambda-1}}{4\pi R^\lambda} \int_{-t_R}^{t_R} (\sqrt{R^2+1} \cos \phi - 1)^\lambda d\phi \\ &\simeq \frac{2^{-\lambda-2} \Gamma(\lambda+1) R^{\lambda+1}}{\sqrt{\pi} \Gamma\left(\lambda + \frac{3}{2}\right)} (\hat{z}(t, R) \cdot z_2)^{-\lambda-1} \end{aligned}$$

$$\lim_{R \rightarrow 0^\pm} R^{-\lambda-1} W_\lambda^{(2)}(z_1(t, R), z_2) = \frac{2^{-\lambda-2} \Gamma(\lambda+1)}{\sqrt{\pi} \Gamma\left(\lambda + \frac{3}{2}\right)} (\xi_\pm \cdot z)^{-\lambda-1}$$

$$\xi_\pm = \pm(\sin t, 1, \cos t)$$

$$z_1(v) = (\sin(-iv), 0, \cos(-iv)), v > 0, z_2(t, R) = \frac{\hat{z}_2(t+is, R)}{R}, s > 0$$

$$\begin{aligned} W_\lambda^{(2)}(z_1(v), z_2(t, R)) &= \frac{R^{\lambda+1}}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda (\sqrt{R^2+1} \cos(t+i(s+v)-\phi) - 1)^{-\lambda-1} d\phi \\ &\simeq \frac{R^{\lambda+1} \Gamma(\lambda+1) (\cos(t+i(s+v)))^{-\lambda-1}}{2^{\lambda+2} \sqrt{\pi} \Gamma\left(\lambda + \frac{3}{2}\right)} \end{aligned}$$

$$(z \cdot \zeta)^{-\lambda-1} = C_2(\lambda) \int_{\gamma(z)} (z \cdot \zeta')^\lambda (\zeta' \cdot \zeta)^{-\lambda-1} d\mu_\gamma(\zeta')$$

$$C_2(\lambda) = \frac{2^\lambda \Gamma\left(\lambda + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda+1)}$$

$$W_\lambda^{(2)}(z_1, z_2) = \frac{C_2(\lambda)}{4\pi} \int_{\gamma_1(z_1)} \int_{\gamma_2(z_2)} (z_1 \cdot \zeta_1)^\lambda (\zeta_1 \cdot \zeta_2)^{-\lambda-1} (\zeta_2 \cdot z_2)^\lambda d\mu_{\gamma(z_1)}(\zeta_1) d\mu_{\gamma(z_2)}(\zeta_2)$$

$$W_\lambda^{(d)}(z_1, z_2) = c_d(\lambda) \int_{\gamma(z_1)} (z_1 \cdot \zeta)^\lambda (z_2 \cdot \zeta)^{-\lambda-d+1} d\mu_\gamma(\zeta)$$

$$z_1 = (\sin(-iv), \vec{0}, \cos(-iv)), v > 0, z_1 \in \mathcal{Z}_-$$

$$z_2 = (\sin(iu), \vec{0}, \cos(iu)), u > 0, z_2 \in \mathcal{Z}_+$$



$$\gamma(z_1) = \left\{ \zeta = (\sin(\phi - iv), \vec{n}, \cos(\phi - iv)), -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \vec{n} \in S_{d-2} \right\}$$

$$\gamma_S(e_d) = \{ \xi \in \gamma_S : e_d \cdot \xi = \cos \phi \geq 0 \}$$

$$\begin{aligned} W_\lambda^{(d)}(z_1, z_2) &= \frac{2\pi^{\frac{d-1}{2}} c_d(\lambda)}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda (\cos(\phi - iv))^{-\lambda-d+1} d\phi \\ &= -\frac{2^d i \pi^{\frac{d-1}{2}} c_d(\lambda)}{\Gamma\left(\frac{d-1}{2}\right)} e^{u(\lambda+d-1)} \int_{\gamma^+} \left(-\frac{i(z^2-1)}{z} \right)^\lambda \left(-\frac{i(e^{2u}z^2-1)}{z} \right)^{-d-\lambda+1} \frac{dz}{z} \\ &\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^\lambda (\cos(\phi - iv))^{-\lambda-d+1} d\phi \\ &= \frac{\pi \Gamma(\lambda+1) e^{-u(d+\lambda-1)} {}_2F_1\left(\frac{d-1}{2}, d+\lambda-1; \frac{d+1}{2} + \lambda; e^{-2u}\right)}{2^{1-d} \Gamma\left(\frac{3}{2} - \frac{d}{2}\right) \Gamma\left(\frac{d+1}{2} + \lambda\right)} \\ &= \frac{2^{\frac{d}{2}} \sqrt{\pi} \Gamma(1+\lambda)}{\Gamma\left(\frac{3}{2} - \frac{d}{2}\right) \Gamma(d-1+\lambda)} (\text{sh}u)^{\frac{2-d}{2}} e^{-\frac{i\pi(d-2)}{2}} Q_{\frac{d-2}{2}+\lambda}^{\frac{d-2}{2}}(\text{ch}u) \end{aligned}$$

$$c_d(\lambda) = -\frac{\pi^{1-d} \Gamma(\lambda+d-1)}{2^{d+1} \cos\left(\frac{\pi d}{2}\right) \Gamma(\lambda+1)}$$

$$W_\lambda^{(2n+1)}(z_1, z_2) = \frac{(-1)^{n+1} \Gamma(\lambda+2n)}{(2\pi)^d \Gamma(\lambda+1)} \int_{\gamma(z_1)} (z_1 \cdot \xi)^\lambda (\xi \cdot z_2)^{-\lambda-2n} \log(\xi \cdot z_2) d\mu_\gamma(\xi)$$

$$z_2 = (x \sin(iu), \sqrt{x^2-1} \vec{n}, x \cos(iu)), u > 0, |\vec{n}| = 1.$$

$$Q_\nu^\mu(x \text{ch}u) \frac{(x^2 \text{ch}^2 u - 1)^{\frac{\mu}{2}}}{x^{1+\mu+\nu}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{1}{2x^2}\right)^n}{\Gamma(1+n) (\text{sh}u)^{\mu+n}} Q_{\nu+n}^{\mu+n}(\text{ch}(u))$$

$$\hat{z}_2(t+iu, \vec{n}', R) = (\sqrt{R^2+1} \sin(t+iu), \vec{n}', \sqrt{R^2+1} \cos(t+iu)) \in AdS_d(R)$$

$$z_2(t+iu, R) = \frac{\hat{z}_2(t+iu, R)}{R}, u > 0$$

$$\begin{aligned} W_\lambda^{(d)}(z_1(v), z_2(t+iu, R)) &= c_d(\lambda) R^{\lambda+d-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{S^{d-2}} (\cos \phi)^\lambda (\cos(t+i(u+v)-\phi) - \vec{n} \cdot \vec{n}')^{-\lambda-d+1} d\phi d\vec{n} \\ &= \frac{R^{\lambda+d-1} (z_1 \cdot \hat{z}_2)^{1-d-\lambda} \Gamma(d-1+\lambda)}{2^{d+\lambda} \pi^{\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2} + \lambda\right)} \end{aligned}$$

$$(z \cdot z')^{1-d-\lambda} = C_d(\lambda) \int_{\gamma(\zeta)} (z \cdot \zeta)^\lambda (\zeta \cdot z')^{-\lambda-d+1} d\mu_\gamma(\zeta)$$



$$z \in \tilde{\mathcal{Z}}_-, \zeta' \in \bar{\mathcal{C}}_+ \text{ and } \gamma(\zeta) \in H^1(\mathcal{C}_-, \{\zeta: z \cdot \zeta = 0\})$$

$$C_d(\lambda) = -\frac{2^{\lambda-1} \Gamma\left(\frac{d+1}{2} + \lambda\right)}{\pi^{\frac{d-1}{2}} \cos\left(\frac{\pi d}{2}\right) \Gamma(\lambda+1)}$$

$$W_\lambda^{(d)}(z_1, z_2) = c_d(\lambda) C_d(\lambda) \int_{\gamma_1(z_1)} \int_{\gamma_2(z_2)} (z_1 \cdot \zeta_1)^\lambda (\zeta_1 \cdot \zeta_2)^{-\lambda-d+1} (\zeta_2 \cdot z_2)^\lambda d\mu_{\gamma(z_1)}(\zeta_1) d\mu_{\gamma(z_2)}(\zeta_2)$$

$$\Pi_u = \{x^{d-1} + x^d = 1/u = e^v > 0\}.$$

$$x(x, u) = \begin{cases} x^\mu = \frac{1}{u} x^\mu = e^v x^\mu, \mu = 0, 1, \dots, d-2 \\ x^{d-1} = \frac{1-u^2}{2u} + \frac{1}{2u} x^2 = \text{sh}v + \frac{1}{2} e^v x^2 \\ x^d = \frac{1+u^2}{2u} - \frac{1}{2u} x^2 = \text{ch}v - \frac{1}{2} e^v x^2 \end{cases}$$

$$x^2 = x_M^2 = x \cdot x = \eta_{\mu\nu} x^\mu x^\nu, ds_M^2 = (dx^0)^2 - (dx^1)^2 - \dots - (dx^{d-2})^2$$

$$(x-x')_{AdS}^2 = \frac{(x-x')_M^2 - (u^2 - u')^2}{uu'}$$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \dots + (dx^d)^2 \Big|_{AdS} = \frac{1}{u^2} (ds_M^2 - du^2)$$

$$(x(u, x) - x(u, x'))_{AdS}^2 = \frac{1}{u^2} (x-x')_M^2$$

$$T_\pm = \{z = x \pm iy \in M_{d-1}^{(c)}: y^2 > 0, y^0 \geq 0\}$$

$$z(z, u) = \begin{cases} z^\mu = \frac{1}{u} z^\mu \\ z^{d-1} = \frac{1-u^2}{2u} + \frac{1}{2u} z^2, z^2 = \eta_{\mu\nu} z^\mu z^\nu, u > 0. \\ z^d = \frac{1+u^2}{2u} - \frac{1}{2u} z^2 \end{cases}$$

$$(\text{Im}z(z, u)) \cdot (\text{Im}z(z, u)) = \frac{y_M^2}{u^2} > 0$$

$$\gamma_P^\pm = \{\xi \in \mathcal{C}_d; \xi^{d-1} + \xi^d = \pm 1\}$$

$$\xi(\eta) = \begin{cases} \xi^\mu = \eta^\mu \\ \xi^{d-1} = \pm \frac{1}{2} (1 + \eta^2) \\ \xi^d = \pm \frac{1}{2} (1 - \eta^2) \end{cases}$$

$$d\mu(\xi(\eta)) = d\eta$$



$$x(t, u) \cdot \xi(\eta) = \frac{u}{2} - \frac{(x - \eta)^2}{2u}, \quad \xi(\eta) \in \gamma_P^{(+)}$$

$$x(t, u) \cdot \xi(\chi) = -\frac{u}{2} + \frac{(x + \chi)^2}{2u}, \quad \xi(\chi) \in \gamma_P^{(-)}.$$

$$\gamma_S(e_d) \cong [-\pi/2, \pi/2] \times S_{d-2}$$

$$\partial\gamma_S(e_d) = \{(\xi^0) = \pm 1, (\xi^1)^2 + \dots + (\xi^{d-1})^2 = 1\}$$

$$\gamma_P^+(e_d) = \left\{ \xi \in \gamma_P^+ : e_d \cdot \xi(\eta) = \frac{1}{2}(1 - \eta^2) \geq 0 \right\}$$

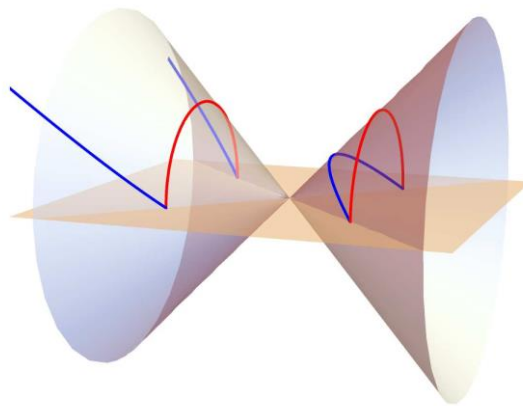
$$\partial\gamma_P^+(e_d) \rightarrow \xi(\eta) = \begin{cases} \xi^0 = \pm\sqrt{1 + \vec{\eta}^2} \\ \xi^i = \eta^i \\ \xi^{d-1} = 1 \\ \xi^d = 0 \end{cases}$$

$$\Theta \left(\xi(\eta) \right) = \begin{cases} \xi^\mu = \frac{2\eta^\mu}{\sqrt{1 + 2\eta^{02} + 2\vec{\eta}^2 + (\eta^2)^2}} \\ \xi^{d-1} = \frac{(1 + \eta^2)}{\sqrt{1 + 2\eta^{02} + 2\vec{\eta}^2 + (\eta^2)^2}} \\ \xi^d = \frac{(1 - \eta^2)}{\sqrt{1 + 2\eta^{02} + 2\vec{\eta}^2 + (\eta^2)^2}} \end{cases}$$

$$\Theta(\partial\gamma_P^+(e_d)) = \left\{ \pm 1, \frac{\vec{\eta}}{\sqrt{\vec{\eta}^2 + 1}}, \frac{1}{\sqrt{\vec{\eta}^2 + 1}}, 0 \right\}$$

$$\gamma_P^-(e_d) = \left\{ \xi \in \gamma_P^- : e_d \cdot \xi(\chi) = -\frac{1}{2}(1 - \chi^2) \geq 0 \right\}$$

$$\Theta[\partial\gamma_P^-(e_d)] = \left\{ \pm 1, \frac{\vec{\chi}}{\sqrt{\vec{\chi}^2 + 1}}, -\frac{1}{\sqrt{\vec{\chi}^2 + 1}}, 0 \right\}$$



$$\gamma_P^+(x_1) = \left\{ \xi \in \gamma_P^+ : x_1(t, u) \cdot \xi(\eta) = \frac{u}{2} - \frac{(t - \eta)^2}{2u} \geq 0 \right\}.$$

$$W_\lambda^{(2)}(x_1(t, u), z_2(t' + is', u')) = \frac{1}{2\pi} Q_\lambda \left(\frac{u^2 + u'^2 - (t - t' - is')^2}{2uu'} \right) \\ = \frac{1}{4\pi} \int_{t-u}^{t+u} \left(\frac{u^2 - (t - \eta)^2}{2u} \right)^\lambda \left(\frac{u'^2 - (t' + is' - \eta)^2}{2u'} \right)^{-\lambda-1} d\eta$$

$$\int (x(t - i\epsilon, u) \cdot \xi(\eta))^\lambda e^{i\omega t} dt = \theta(\omega) \frac{(2\pi)^{\frac{3}{2}} u^{\frac{1}{2}}}{\Gamma(-\lambda)} e^{i\omega\eta} \omega^{-\frac{1}{2}-\lambda} J_{-\frac{1}{2}-\lambda}(\omega u)$$

$$\int (x(t + i\epsilon, u) \cdot \xi(\eta))^\lambda e^{i\omega t} dt = \theta(-\omega) \frac{(2\pi)^{\frac{3}{2}} u^{\frac{1}{2}}}{\Gamma(-\lambda)} e^{i\omega\eta} |\omega|^{-\frac{1}{2}-\lambda} J_{-\frac{1}{2}-\lambda}(|\omega|u)$$

$$\int \left(\frac{u}{2} - \frac{(t - \eta)^2}{2u} \right)^\lambda \left(\frac{u'}{2} - \frac{(t' + is' - \eta)^2}{2u} \right)^{-\lambda-1} d\eta \\ = \frac{(2\pi u')^{1/2}}{\Gamma(\lambda + 1)} \iint_{-t-u}^{t+u} \left(\frac{u}{2} - \frac{(t - \eta)^2}{2u} \right)^\lambda e^{-i\omega(t'+is'-\eta)} |\omega|^{\frac{1}{2}+\lambda} J_{\frac{1}{2}+\lambda}(|\omega|u') \theta(-\omega) d\omega d\eta$$

$$\int_{t-u}^{t+u} \left(\frac{u}{2} - \frac{(t - \eta)^2}{2u} \right)^\lambda e^{i\omega\eta} d\eta = \sqrt{2\pi u} |\omega|^{-\lambda-\frac{1}{2}} \Gamma(\lambda + 1) J_{\frac{1}{2}+\lambda}(|\omega|u), \operatorname{Re} \lambda > -1$$

$$W_\lambda^{(2)}(x_1(t, u), z_2(t' + is', u')) = \frac{1}{2} (uu')^{1/2} \int_0^\infty e^{-i\omega(t-t'-is')} J_{\frac{1}{2}+\lambda}(\omega u) J_{\frac{1}{2}+\lambda}(\omega u') d\omega$$

$$\gamma_P^-(x_1) = \left\{ \xi \in \gamma_P^- : x_1 \cdot \xi(\chi) = -\frac{u}{2} + \frac{(x + \chi)^2}{2u} \geq 0 \right\}$$

$$W_\lambda^{(2)}(x_1, z_2) = \frac{1}{4\pi} \int_{-\infty}^{-t-u} + \int_{-t+u}^\infty \left(\frac{(t + \chi)^2 - u^2}{2u} \right)^\lambda \left(\frac{(t' + \chi)^2 - u'^2}{2u'} \right)^{-\lambda-1} d\chi$$

$$\lim_{\chi \rightarrow \pm\infty} \operatorname{Arg} \left(\frac{(t' + \chi)^2 - u'^2}{2u'} \right) = 0$$

$$W_\lambda^{(2)}(z_1, z_2) = \frac{1}{4\pi} \int_{-\infty}^{-t-u} \left(\frac{(t + \chi)^2 - u^2}{2u} \right)^\lambda \left(\frac{(t' + \chi)^2 - u'^2}{2u'} \right)^{-\lambda-1} d\chi$$

$$+ \frac{1}{4\pi} \int_{-t+u}^\infty \left(\frac{(t + \chi)^2 - u^2}{2u} \right)^\lambda \left(\frac{(t' + \chi)^2 - u'^2}{2u'} \right)^{-\lambda-1} d\chi$$

$$\int \left(\frac{(t' + is' + \chi)^2 - u'^2}{2u'} \right)^{-\lambda-1} e^{i\omega t'} dt' =$$

$$= \pm i \theta(-\omega) e^{-i\omega(\chi + is')} |\omega|^{\frac{1}{2}+\lambda} (e^{\mp 2i\pi\lambda} - 1) \sqrt{2\pi u'} J_{\frac{1}{2}+\lambda}(|\omega|u') \Gamma(-\lambda)$$

$$\left(\frac{(t' + is' + \chi)^2 - u'^2}{2u'} \right)^{-\lambda-1} =$$

$$= \pm \frac{i\Gamma(-\lambda) \sqrt{u'} (e^{\mp 2i\pi\lambda} - 1)}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\omega(t'+is'+\chi)} |\omega|^{\frac{1}{2}+\lambda} J_{\frac{1}{2}+\lambda}(|\omega|u') d\omega$$



$$\begin{aligned}
W_\lambda^{(2)}(z_1, z_2) &= \\
&= \frac{\Gamma(-\lambda) \sin(\pi\lambda) \sqrt{u'}}{4\pi\sqrt{2\pi}} \int_{-\infty}^0 e^{i\omega(t-t'-is')} |\omega|^{\frac{1}{2}+\lambda} J_{\frac{1}{2}+\lambda}(|\omega|u') d\omega \int_u^\infty \cos(\omega\chi - \pi\lambda) \left(\frac{\eta^2 - u^2}{2u}\right)^\lambda d\chi \\
&= \frac{1}{2} (uu')^{1/2} \int_0^\infty e^{-i\omega(t-t'-is')} J_{\frac{1}{2}+\lambda}(\omega u) J_{\frac{1}{2}+\lambda}(\omega u') d\omega
\end{aligned}$$

$$\begin{aligned}
\gamma_P^+(x_1) &= \{\xi(\eta) \in \gamma_P^+ : (\eta^0 - x^0)^2 \leq (\vec{\eta} - \vec{x})^2 + u^2\} \\
\gamma_P^-(x_1) &= \{\xi(\chi) \in \gamma_P^- : (\chi^0 + x^0)^2 \geq (\vec{\chi} + \vec{x})^2 + u^2\}.
\end{aligned}$$

$$\begin{aligned}
W_\lambda^{(d)}(x_1(x, u), z_2(z', u')) &= \\
&= c_d(\lambda) \int d\vec{\eta} \int_{x^0 - \sqrt{(\vec{\eta} - \vec{x})^2 + u^2}}^{x^0 + \sqrt{(\vec{\eta} - \vec{x})^2 + u^2}} \left(\frac{u^2 - (x - \eta)^2}{2u}\right)^\lambda \left(\frac{u'^2 - (z' - \eta)^2}{2u'}\right)^{-\lambda-d+1} d\eta^0 \\
&+ c_d(\lambda) \int d\vec{\chi} \int_{-x^0 - \sqrt{(\vec{\chi} + \vec{x})^2 + u^2}}^{-x^0 + \sqrt{(\vec{\chi} + \vec{x})^2 + u^2}} \left(\frac{(x + \chi)^2 - u^2}{2u}\right)^\lambda \left(\frac{(z' + \chi)^2 - u'^2}{2u'}\right)^{-\lambda-d+1} d\chi^0 \\
&+ c_d(\lambda) \int d\vec{\chi} \int_{-x^0 + \sqrt{(\vec{\chi} + \vec{x})^2 + u^2}}^\infty \left(\frac{(x + \chi)^2 - u^2}{2u}\right)^\lambda \left(\frac{(z' + \chi)^2 - u'^2}{2u'}\right)^{-\lambda-d+1} d\chi^0 (1)^{-\lambda+1}
\end{aligned}$$

$$\widetilde{\phi}_\lambda^\pm(k) = \int (x(t \pm i\epsilon, \vec{x}, u) \cdot \xi(\eta))^\lambda e^{ikx} dx = e^{ik\eta} \int \left[\frac{u}{2} - \frac{(y^0 \pm i\epsilon)^2 - \vec{y}^2}{2u} \right]^\lambda e^{iky} dy$$

$$\begin{aligned}
\widetilde{\phi}_\lambda^+(k^0, 0) &= e^{ik^0\eta^0} \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_{-\infty}^\infty dy^0 \int_0^\infty \left[\frac{u}{2} - \frac{(y^0 + i\epsilon)^2 - r^2}{2u} \right]^\lambda e^{ik^0 y^0} r^{d-3} dr = \\
&= \frac{(2u)^{-\lambda} e^{ik^0\eta^0} \pi^{\frac{d-2}{2}} \Gamma\left(\frac{2-d}{2} - \lambda\right)}{\Gamma(-\lambda)} \int_{-\infty}^\infty (u^2 - (y^0 + i\epsilon)^2)^{\frac{d-2}{2}+\lambda} e^{ik^0 y^0} dy^0
\end{aligned}$$

$$\widetilde{\phi}_\lambda^+(k^0, 0) = \frac{(2\pi)^{\frac{d+1}{2}}}{\Gamma(-\lambda)} \theta(-k^0) |k^0|^{-\frac{d-1}{2}-\lambda} e^{ik^0\eta^0} u^{\frac{d-1}{2}} J_{-\frac{d-1}{2}-\lambda}(|k^0|u)$$

$$\widetilde{\phi}_\lambda^+(k^0, 0) = \frac{4\sqrt{2}|k^0|^{\frac{1}{2}-\lambda} \pi^{\frac{d+1}{2}} u^{-\lambda}}{\Gamma\left(\frac{d-2}{2}\right) \Gamma(-\lambda)} \int_0^\infty r^{d-3} (r^2 + u^2)^{\frac{1}{4}+\frac{\lambda}{2}} J_{-\frac{1}{2}-\lambda}(|k^0|\sqrt{r^2 + u^2}).$$

$$\int_0^\infty r^{d-3} (r^2 + u^2)^{\frac{1}{4}+\frac{\lambda}{2}} J_{-\frac{1}{2}-\lambda}(|k^0|\sqrt{r^2 + u^2}) = 2^{\frac{d-4}{2}} \Gamma\left(\frac{d-2}{2}\right) |k^0|^{1-\frac{d}{2}} u^{\frac{d-1}{2}+\lambda} J_{-\frac{d-1}{2}-\lambda}(|k^0|u)$$

$$\begin{aligned}
(x(z', u') \cdot \xi(\eta))^{-\lambda-d+1} &= \left(\frac{u'^2 - (z' - \vec{\eta})^2}{2u'}\right)^{-\lambda-d+1} \\
&= \frac{(2\pi)^{\frac{3-d}{2}} u'^{\frac{d-1}{2}}}{\Gamma(\lambda + d - 1)} \int e^{-ik(z' - \eta)} \theta(-k^0) \theta(k^2) (\sqrt{k^2})^{\frac{d-1}{2}+\lambda} J_{\frac{d-1}{2}+\lambda}(\sqrt{k^2} u') dk
\end{aligned}$$

$$\frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty r^{d-3} dr \int_{-\sqrt{r^2+u^2}}^{\sqrt{r^2+u^2}} e^{ik^0\eta^0} \left(\frac{u^2 + r^2 - \eta^{0^2}}{2u}\right)^\lambda d\eta^0 =$$



$$\begin{aligned}
&= \frac{2\sqrt{2}\pi^{\frac{d-1}{2}}|k^0|^{-\lambda-\frac{1}{2}}\Gamma(\lambda+1)u^{-\lambda}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty r^{d-3}(r^2+u^2)^{\frac{\lambda+1}{2}} J_{\lambda+\frac{1}{2}}\left(|k^0|\sqrt{r^2+u^2}\right) dr \\
&\frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty r^{d-3} dr \int_{-\sqrt{r^2+u^2}}^{\sqrt{r^2+u^2}} e^{ik^0\eta^0} \left(\frac{u^2+r^2-\eta^{0^2}}{2u}\right)^\lambda d\eta^0 = \\
&= -\frac{(2\pi u)^{\frac{d-1}{2}}|k^0|^{-\frac{d-1}{2}}\Gamma(\lambda+1)}{\cos\left(\frac{1}{2}\pi(d+2\lambda)\right)} \left(\cos(\pi\lambda)J_{\frac{d-1}{2}+\lambda}(u|k^0|) - \sin\left(\frac{\pi d}{2}\right)J_{-\frac{d-1}{2}-\lambda}(u|k^0|)\right) \\
&\int_0^\infty r^{d-3}(r^2+u^2)^{\frac{\lambda+1}{2}} J_{\lambda+\frac{1}{2}}\left(|k^0|\sqrt{r^2+u^2}\right) dr = \\
&= \frac{2^{\frac{d-4}{2}}\pi|k^0|^{-\frac{d-2}{2}}u^{\frac{d-1}{2}+\lambda}}{\cos\left(\frac{1}{2}\pi(d+2\lambda)\right)\Gamma\left(2-\frac{d}{2}\right)} \left(\frac{\cos(\pi\lambda)}{\sin\left(\frac{d\pi}{2}\right)}J_{\frac{d-1}{2}+\lambda}(|k^0|u) - J_{-\frac{d-1}{2}-\lambda}(|k^0|u)\right) \\
&\iint_{-\sqrt{u^2+\vec{\eta}^2}}^{\sqrt{u^2+\vec{\eta}^2}} \left(\frac{u^2+\vec{\eta}^2-\eta^{0^2}}{2u}\right)^\lambda e^{ik^\eta} d\eta^0 d\eta = -\frac{(2\pi u)^{\frac{d-1}{2}}(\sqrt{k^2})^{-\frac{d-1}{2}}\Gamma(\lambda+1)}{\cos\left(\frac{1}{2}\pi(d+2\lambda)\right)} \\
&\quad \times \theta(k^2) \left(\cos(\pi\lambda)J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) - \sin\left(\frac{\pi d}{2}\right)J_{-\frac{d-1}{2}-\lambda}(u\sqrt{k^2})\right) \\
&\quad \iint_{-\sqrt{u^2+\vec{\eta}^2}}^{\sqrt{u^2+\vec{\eta}^2}} \left(\frac{u^2+\vec{\eta}^2-\eta^{0^2}}{2u}\right)^\lambda \left(\frac{u'^2(z'-\eta)^2}{2u'}\right)^{-\lambda-d+1} d\eta^0 d\eta \\
&= -\frac{2\pi(uu')^{\frac{d-1}{2}}\Gamma(\lambda+1)}{\Gamma(\lambda+d-1)\cos\left(\frac{1}{2}\pi(d+2\lambda)\right)} \int e^{-ikz'}\theta(-k^0)\theta(k^2) \\
&\quad \times \left(\cos(\pi\lambda)J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) - \sin\left(\frac{\pi d}{2}\right)J_{-\frac{d-1}{2}-\lambda}(u\sqrt{k^2})\right) J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) dk \\
&\quad \lim_{\chi^0 \rightarrow \pm\infty} \text{Arg}\left(\frac{(x'+\chi)^2-u'^2}{2u'}\right) = 0 \\
&\quad \left(\frac{(x'+\chi)^2+u'^2}{2u'}\right)_{\pm}^{-\lambda-d+1} = \pm i(2\pi)^{\frac{1-d}{2}}(e^{\mp 2i\pi(d+\lambda)}-1)\Gamma(2-\lambda-d) \\
&\quad \times u'^{\frac{d-1}{2}} \int e^{-ik(x'+\chi)}\theta(-k^0)\theta(k^2)(\sqrt{k^2})^{\frac{d-1}{2}+\lambda} J_{\frac{d-1}{2}+\lambda}(\sqrt{k^2}u') dk
\end{aligned}$$

$$\begin{aligned}
& \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty r^{d-3} dr \int_{\sqrt{r^2+u^2}}^\infty e^{-ik^0\chi^0} \left(\frac{\chi^{0^2} - u^2 - r^2}{2u}\right)^\lambda d\chi^0 \\
&= -\frac{\sqrt{2}\pi^{\frac{d-1}{2}}\Gamma(\lambda+1)u^{-\lambda}|k^0|^{-\lambda-\frac{1}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \times \\
& \int_0^\infty r^{d-3} dr (r^2+u^2)^{\frac{\lambda+1}{2}} \left(\frac{J_{\lambda+\frac{1}{2}}(|k^0|\sqrt{r^2+u^2})}{\cos(\pi\lambda)} + (\operatorname{tg}(\pi\lambda) - i)J_{-\lambda-\frac{1}{2}}(|k^0|\sqrt{r^2+u^2}) \right) \\
&= \frac{(2\pi u)^{\frac{d-1}{2}}\Gamma(\lambda+1)|k^0|^{-\frac{d-1}{2}-\lambda}}{2} \left(\frac{J_{\frac{d-1}{2}+\lambda}(u|k^0|)}{\cos\left(\frac{\pi d}{2} + \pi\lambda\right)} - \left(\operatorname{tg}\left(\frac{\pi d}{2} + \pi\lambda\right) - i\right)J_{-\frac{d-1}{2}-\lambda}(u|k^0|) \right) \quad (14.17)
\end{aligned}$$

$$\begin{aligned}
& \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^\infty r^{d-3} dr \int_{-\infty}^{-\sqrt{r^2+u^2}} e^{-ik^0\chi^0} \left(\frac{\chi^{0^2} - u^2 - r^2}{2u}\right)^\lambda d\chi^0 \\
&= \frac{(2\pi u)^{\frac{d-1}{2}}\Gamma(\lambda+1)|k^0|^{-\frac{d-1}{2}-\lambda}}{2} \left(\frac{J_{\frac{d-1}{2}+\lambda}(u|k^0|)}{\cos\left(\frac{\pi d}{2} + \pi\lambda\right)} - \left(\operatorname{tg}\left(\frac{\pi d}{2} + \pi\lambda\right) + i\right)J_{-\frac{d-1}{2}+\lambda}(u|k^0|) \right)
\end{aligned}$$

$$\begin{aligned}
& \int_{\gamma_P^-(x_1)} \left(\frac{\chi^{0^2} - u^2 - \vec{\chi}^2}{2u}\right) \left(\frac{(z' + \chi)^2 - u'^2}{2u'}\right)^{-\lambda-d+1} d\chi \\
&+ \int_{\gamma_P^+(x_1)} \left(\frac{\chi^{0^2} - u^2 - \vec{\chi}^2}{2u}\right) \left(\frac{(z' + \chi)^2 - u'^2}{2u'}\right)^{-\lambda-d+1} d\chi \\
&= \frac{2\Gamma(\lambda+1)\sin(\pi(d+\lambda))\Gamma(-d-\lambda+2)(uu')^{\frac{d-1}{2}}}{\cos\left(\frac{\pi d}{2} + \pi\lambda\right)} \int e^{-ikz}\theta(-k^0)\theta(k^2) \\
& \times J_{\frac{d-1}{2}+\lambda}(u'\sqrt{k^2}) \left(\cos(\pi(d+\lambda))J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) + \sin\left(\frac{\pi d}{2}\right)J_{-\frac{d-1}{2}-\lambda}(u\sqrt{k^2}) \right)
\end{aligned}$$

$$\begin{aligned}
& \int_{\gamma_P(x_1)} (x_1 \cdot \xi)^\lambda (\xi \cdot z_2)^{-\lambda-d+1} d\mu_\gamma(\xi) \\
&= -\frac{4\pi\cos\left(\frac{\pi d}{2}\right)\Gamma(\lambda+1)(uu')^{\frac{d-1}{2}}}{\Gamma(d+\lambda-1)} \int e^{-ik(x-x')}\theta(k^0)\theta(k^2)J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2})J_{\frac{d-1}{2}+\lambda}(u'\sqrt{k^2})
\end{aligned}$$

$$\begin{aligned}
& W_\lambda^{(d)}(z_1(z_1, u), z_2(z_2, u')) \\
&= \frac{(uu')^{\frac{d-1}{2}}}{2(2\pi)^{d-2}} \int e^{-ik(z_1-z_2)}\theta(k^0)\theta(k^2)J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2})J_{\frac{d-1}{2}+\lambda}(u'\sqrt{k^2}) dk
\end{aligned}$$

$$\theta(k^2) = \int_0^\infty \delta(k^2 - m^2) dm^2$$



$$W_\lambda^{(d)}(z_1(z_1, u), z_2(z_2, u')) = \frac{1}{2} (uu')^{\frac{d-1}{2}} \int W_m^{M^{d-1}}(z_1, z_2) J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2$$

$$u^2 \square_{\text{anti de Sitter}} \phi - u^d \partial_u (u^{2-d} \partial_u \phi) + m_\lambda^2 \phi = 0$$

$$u^d \partial_u \left(u^{2-d} \partial_u \left(u^{\frac{d-1}{2}} J_{\frac{d-1}{2}+\lambda}(mu) \right) \right) = u^{\frac{d-1}{2}} (\lambda(d + \lambda - 1) - m^2 u^2) J_{\frac{d-1}{2}+\lambda}(mu).$$

$$\begin{aligned} \langle f, f \rangle &= \int W_\lambda^{(d)}(x_1, x_2) \overline{f(x_1)} f(x_2) \sqrt{g(x_1)} \sqrt{g(x_2)} dx_1 dx_2 \\ &= \int dk \frac{\theta(k^0) \theta(k^2)}{2(2\pi)^{d-2}} \int \overline{\hat{f}(k, u)} \hat{f}(k, u') (uu')^{-\frac{d+1}{2}} J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) J_{\frac{d-1}{2}+\lambda}(u'\sqrt{k^2}) du du' \end{aligned}$$

$$\hat{f}(k, u) = \int e^{ikx} f(x(x, u)) dx$$

$$\langle f, f \rangle = \frac{1}{2(2\pi)^{d-2}} \int \theta(k^0) \theta(k^2) \left| \hat{F}_{\frac{d-1}{2}+\lambda}(k) \right|^2 dk$$

$$\hat{F}_{\frac{d-1}{2}+\lambda}(k) = \int_0^\infty u^{-\frac{d+1}{2}} \hat{f}(k, u) J_{\frac{d-1}{2}+\lambda}(u\sqrt{k^2}) du$$

$$S_\lambda^{(d)}(z_E, z'_E) = \frac{1}{2} (uu')^{\frac{d-1}{2}} \int S_m^{E^{d-1}}(x_E, x'_E) J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2$$

$$S_m^{E^{d-1}}(x_E) = \frac{1}{(2\pi)^{d-1}} \int \frac{e^{-ik_E x_E}}{k_E^2 + m^2} dk_E$$

$$k_E x_E = k_E^0 x_E^0 + \dots + k_E^{d-1} x_E^{d-1}$$

$$G_\lambda^{(d)}(x_1(x_1, u), x_2(x_2, u')) = \frac{1}{2} (uu')^{\frac{d-1}{2}} \int G_m^{M^{d-1}}(x_1 - x_2) J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2$$

$$G_m^{M^{d-1}}(x) = -\frac{1}{(2\pi)^{d-1}} \int \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} dk$$



$$\begin{aligned}
& (\square + m_\lambda^2)G_\lambda^{(d)}(x_1, x_2) \\
&= u^2 \square_{\text{anti de Sitter}} G_\lambda^{(d)}(x_1, x_2) - u^d \partial_u \left(u^{2-d} \partial_u G_\lambda^{(d)}(x_1, x_2) \right) + m_\lambda^2 G_\lambda^{(d)}(x_1, x_2) \\
&= m_\lambda^2 G_\lambda^{(d)}(x_1, x_2) + \frac{1}{2} u^2 (uu')^{\frac{d-1}{2}} \delta(x_1 - x_2) \int_0^\infty J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2 \\
&\quad - \frac{1}{2} m^2 u^2 (uu')^{\frac{d-1}{2}} \int_0^\infty G_m^{M^{d-1}}(x_1 - x_2) J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2 \\
&\quad - \frac{1}{2} (uu')^{\frac{d-1}{2}} \int_0^\infty G_m^{M^{d-1}}(x_1 - x_2) (\lambda(\lambda + d - 1) \\
&\quad - m^2 u^2) J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu') dm^2 = u^d \delta(x_1 - x_2) \delta(u, u')
\end{aligned}$$

$$\int_{M^{d-1}} G_m^{M^{d-1}}(x_1 - x_2) dx_1 = -\frac{1}{(2\pi)^{d-1}} \int \frac{e^{-ik(x_1-x_2)}}{k^2 - m^2 + i\epsilon} dx_1 dk = \frac{1}{m^2}$$

$$\begin{aligned}
\int_{AdS_M} G_\lambda^d(x_1, x_2) \sqrt{g(x_1)} dx_1 &= \int_0^\infty \int_0^\infty \left(\frac{u'}{u} \right)^{\frac{d-1}{2}} \frac{J_{\frac{d-1}{2}+\lambda}(mu) J_{\frac{d-1}{2}+\lambda}(mu')}{mu} dudm \\
&= \frac{2^{\frac{d+1}{2}} \Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1}{2}(d + \lambda + 1)\right)} \int_0^\infty (mu')^{\frac{d-1}{2}} J_{\frac{d-1}{2}+\lambda}(mu') \frac{dm}{m} = \frac{1}{\lambda(\lambda + d - 1)} = \frac{1}{m_\lambda^2}
\end{aligned}$$

$$\int_{EAdS} S_\lambda^d(z_1^E, z_2^E) \sqrt{g(z_1^E)} dz_1^E = \frac{2^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_1^\infty e^{-i\pi \frac{d-2}{2}} Q_{\frac{d-2}{2}+\lambda}^{\frac{d-2}{2}}(u) (u^2 - 1)^{\frac{d-2}{4}} du = \frac{1}{m_\lambda^2}$$

$$\zeta = e^{(s+it)M} \xi = g e^{(s+it)M_{0d}} g^{-1} \xi = g e^{(s+\alpha+it)M_{0d}} \xi' = g' \zeta'$$

$$\alpha = \arctan \frac{(g^{-1}\xi)^0}{(g^{-1}\xi)^d}, \xi' = \left(\frac{\overline{g^{-1}\xi}}{|g^{-1}\xi|^2} \right) \in \mathcal{C}_d, \zeta' = e^{itM_{0d}} \xi' \in \mathcal{C}_+$$

$$\text{Im}\zeta' \cdot \text{Im}\zeta' = |\xi'|^2 \text{sh}^2 t > 0 \text{ and } \epsilon(\zeta') = \frac{1}{2} |\xi'| \text{sh} 2t > 0$$

$$\zeta = \xi + i\chi \in \mathcal{C}_d^{(c)}$$

$$\epsilon(\zeta)^2 = (\chi^0 \xi^d - \chi^d \xi^0)^2 = (\chi \cdot \chi) (|\vec{\xi}|^2 + |\vec{\chi}|^2) + (\chi \cdot \chi)^2 + |\vec{\xi}|^2 |\vec{\chi}|^2 - (\vec{\xi} \cdot \vec{\chi})^2.$$

$$\mathcal{C} = \{ \zeta = \xi + i\chi \in \mathcal{C}_d^{(c)} : \xi \cdot \xi = \chi \cdot \chi > 0, \xi \cdot \chi = 0 \}$$

$$\mathcal{C}'_\pm = \{ \zeta = \xi + i\chi \in \mathcal{C}, \epsilon(\zeta) \geq 0 \} \text{ and } \mathcal{C}_\pm \subset \mathcal{C}'_\pm$$



$$\zeta = \xi + i\chi \in \mathcal{C}'_+$$

$$\xi \cdot \xi = \chi \cdot \chi = \text{sh}^2 t$$

$$g(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1_{d-2} & 0 & 0 \\ 0 & 0 & \text{coth}(t) & 1/\text{sh}t \\ 0 & 0 & 1/\text{sh}t & \text{coth}(t) \end{pmatrix}$$

$$g(t)g(t') \neq g(t+t')$$

$$g(t)(\text{ish}(t)e_0 + e_{d-1} + \text{ch}(t)e_d) = e^{itM_{0d}}(e_{d-1} + e_d)$$

$$\zeta = \xi + i\chi = g_0 g(t)^{-1} e^{itM_{0d}}(e_{d-1} + e_d) \in \mathcal{C}_+$$

$$\mathcal{C}_\pm = \{\zeta = \xi + i\chi \in \mathbf{C}_2^{d+1}: \xi \cdot \xi = \chi \cdot \chi > 0, \xi \cdot \chi = 0, \epsilon(\zeta) \geq 0\}.$$

$$\mathcal{C}_\pm = \{g \exp(itM_{0d})(e_{d-1} + e_d): g \in G_0, t \geq 0\}.$$

$$\zeta(t + is, \lambda, \vec{a} + i\vec{b}) = \begin{cases} \lambda \sin(t + is), \\ \lambda(\vec{a} + i\vec{b}), & |\vec{a}|^2 - |\vec{b}|^2 = 1, \vec{a} \cdot \vec{b} = 0 \\ \lambda \cos(t + is), \end{cases}$$

$$s = s_\pm(|\vec{b}|^2)$$

$$\mathcal{C}_+: s > +\text{arcsh}(|\vec{b}|)$$

$$\mathcal{C}_-: s < -\text{arcsh}(|\vec{b}|)$$

$$e^{-i\pi\mu} Q_\nu^\mu(\text{chu}) = \frac{2^{-\mu-1} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma(\mu + \nu + 1)}{\sqrt{\pi} \Gamma(-\mu + \nu + 1)} (\text{sh}u)^\mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \phi)^{-\mu+\nu} (\cos(\phi - iu))^{-\mu-\nu-1} d\phi$$

$$ds^2 = \frac{-d\eta^2 + d\vec{x}^2}{\eta^2}$$

$$\Psi[\varphi(\vec{x})] \equiv \langle \varphi(\vec{x}) | \Omega \rangle = \int_{\Phi(-\infty)=0}^{\Phi(0)=\varphi(\vec{x})} \mathcal{D}\Phi e^{iS[\Phi]}$$

$$S[\Phi] = - \int \frac{d^4 x d\eta}{\eta^4} \left[\frac{1}{2} (\partial\Phi)^2 + m^2 \Phi^2 + V(\Phi) \right]$$

$$\Psi[\varphi] = \exp \left(- \sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{a=1}^n \frac{d^4 k_a}{(2\pi)^3} \varphi(\vec{k}_a) \psi_n(\vec{k}_1, \dots, \vec{k}_n) \delta^3(\vec{k}_T) \right)$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 - m^2}$$

$$\langle \varphi(\vec{k}_1) \dots \varphi(\vec{k}_n) \rangle = \frac{\int \mathcal{D}\varphi \varphi(\vec{k}_1) \dots \varphi(\vec{k}_n) |\Psi(\varphi)|^2}{\int \mathcal{D}\varphi |\Psi(\varphi)|^2}$$



$$\langle \varphi(\vec{k}_1) \varphi(\vec{k}_2) \rangle = \frac{1}{2 \operatorname{Re} \psi_2(\vec{k}_1, \vec{k}_2)}, \langle \varphi(\vec{k}_1) \varphi(\vec{k}_2) \varphi(\vec{k}_3) \rangle = \frac{\operatorname{Re} \psi_3(\vec{k}_1, \vec{k}_2, \vec{k}_3)}{\prod_{a=1}^3 \operatorname{Re} \psi_2(\vec{k}_a, -\vec{k}_a)}$$

$$\langle Q(\varphi) \rangle = \int \mathcal{D}\Phi^< \mathcal{D}\Phi^> Q(\varphi) \delta((\Phi^< - \Phi^>)|_{\eta=0}) e^{iS[\Phi^<] - iS[\Phi^>]}$$

$$iS_{SK}[\Phi^<, \Phi^>] = iS[\Phi^<] - iS[\Phi^>]$$

$$\langle \phi^<(\eta_1) \phi^<(\eta_2) \rangle = i \frac{(\eta_1 \eta_2)^{\frac{3}{2}}}{2} \int_{-\infty}^{\infty} dp \frac{p H_v^{(1)}(p \eta_1) H_v^{(2)}(p \eta_2)}{p^2 - k^2 + i\epsilon}$$

$$\langle \phi^>(\eta_1) \phi^>(\eta_2) \rangle = -i \frac{(\eta_1 \eta_2)^{\frac{3}{2}}}{2} \int_{-\infty}^{\infty} dp \frac{p H_v^{(2)}(p \eta_1) H_v^{(1)}(p \eta_2)}{p^2 - k^2 - i\epsilon}$$

$$\langle \phi^<(\eta_1) \phi^<(\eta_2) \rangle = -i \frac{(\eta_1 \eta_2)^{\frac{3}{2}}}{2} \int_{C_1} dp \frac{p H_v^{(1)}(p \eta_1) H_v^{(2)}(p \eta_2)}{p^2 - k^2}$$

$$\langle \phi^>(\eta_1) \phi^>(\eta_2) \rangle = i \frac{(\eta_1 \eta_2)^{\frac{3}{2}}}{2} \int_{C_2} dp \frac{p H_v^{(1)}(p \eta_1) H_v^{(2)}(p \eta_2)}{p^2 - k^2}$$

$$\eta^< \rightarrow e^{-i\frac{\pi}{2}z} \eta^> \rightarrow e^{i\frac{\pi}{2}z}$$

$$\phi^< = (-i)^{\Delta_+} \phi^+ + (-i)^{\Delta_-} \phi^-$$

$$\phi^> = (i)^{\Delta_+} \phi^+ + (i)^{\Delta_-} \phi^-.$$

$$\langle \phi^+(z_1) \phi^+(z_2) \rangle = G_v(z_1, z_2, \vec{k})$$

$$\langle \phi^-(z_1) \phi^-(z_2) \rangle = G_{-v}(z_1, z_2, \vec{k})$$

$$\langle \phi^+(z_1) \phi^-(z_2) \rangle = 0$$

$$\langle \phi^-(z_1) \phi^+(z_2) \rangle = 0$$

$$G_v(z, z'; \vec{k}) = \sin(\pi v) \int dp \frac{p (zz')^{d/2} J_v(pz) J_v(pz')}{p^2 + k^2}.$$

$$\mathcal{B}_v(z, \vec{k}) = \sin(\pi v) z^{d/2} \frac{2^{-v} k^v}{\Gamma(1+v)} K_v(kz)$$

$$\begin{aligned} S_{\text{shadow}} = & \int_0^\infty \frac{dz d^d x}{z^{d+1}} \left[\sin\left(\pi\left(\frac{d}{2} - \Delta_+\right)\right) ((\partial\phi^+)^2 - m^2\phi_+^2) \right. \\ & + \sin\left(\pi\left(\frac{d}{2} - \Delta_-\right)\right) ((\partial\phi_-)^2 - m^2\phi_-^2) \\ & \left. + e^{i\pi\frac{d-1}{2}} V \left(e^{-i\frac{\pi}{2}\Delta_+} \phi_+ + e^{-i\frac{\pi}{2}\Delta_-} \phi_- \right) + e^{-i\pi\frac{d-1}{2}} V \left(e^{i\frac{\pi}{2}\Delta_+} \phi_+ + e^{i\frac{\pi}{2}\Delta_-} \phi_- \right) \right] \end{aligned}$$



A Feynman diagram showing a loop with external momenta $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$ and internal momenta $p, L, L+K$. The diagram is equated to the following integral:

$$= \int_{-\infty}^{\infty} \frac{dp k_{12} k_{34}}{(p^2 + k_{12}^2)(p^2 + k_{34}^2)} \int \frac{d^4 L}{L^2 (L + K)^2},$$

$$S = - \int \frac{d^4 x d\eta}{\eta^4} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) + m^2 \phi^\dagger \phi \right]$$

$$D_\mu \phi = \partial_\mu + ig A_\mu, \text{ and field strength } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$g_{\mu\nu} \rightarrow \eta^2 g_{\mu\nu} = \eta_{\mu\nu}; \phi \rightarrow \frac{\phi}{\eta}$$

$$\langle Q(\Phi) \rangle \sim \int D\vec{\mathcal{A}} D\Phi |\Psi(\vec{\mathcal{A}}, \Phi)|^2 Q(\Phi)$$

$$\vec{\mathcal{A}}(\vec{x}) \rightarrow \vec{\mathcal{A}}(\vec{x}) + \vec{\partial} \Lambda(\vec{x})$$

$$\mathcal{A}_i = \left(\eta_{ij} - \frac{k_i k_j}{k^2} \right) \mathcal{A}_j + \frac{k_i k_j}{k^2} \mathcal{A}_j = \pi_{ij} \mathcal{A}_j + \frac{k_i k_j}{k^2} \mathcal{A}_j$$

$$\langle Q(\Phi) \rangle = \int D\vec{\mathcal{A}}_T D\vec{\mathcal{A}}_L D\Phi \delta(\vec{\mathcal{A}}_L) |\Psi(\vec{\mathcal{A}}_T, \Phi)|^2 Q(\Phi)$$

$$\Psi(\vec{\mathcal{A}}_T, \Phi) = \int_{A_T^\rceil(\eta=-\infty)=0}^{A_T^\rceil(\eta=0, \vec{x})=\vec{\mathcal{A}}_T(\vec{x})} DA_T^\rceil \int_{A_L^\rceil(\eta=-\infty)=0}^{A_L^\rceil(\eta=0, \vec{x})=0} DA_L^\rceil \int_{\phi^\rceil(\eta=-\infty)=0}^{\phi^\rceil(0, \vec{x})=\Phi(\vec{x})} D\Phi^\rceil e^{is[\vec{\mathcal{A}}_T^\rceil, \vec{\mathcal{A}}_L^\rceil, \Phi^\rceil]}$$

$$\Psi^*(\vec{\mathcal{A}}_T, \Phi) = \int_{A_T^\lceil(\eta=-\infty)=0}^{A_T^\lceil(\eta=0, \vec{x})=\vec{\mathcal{A}}_T(\vec{x})} DA_T^\lceil \int_{A_L^\lceil(\eta=-\infty)=0}^{A_L^\lceil(\eta=0, \vec{x})=0} DA_L^\lceil \int_{\phi^\lceil(\eta=-\infty)=0}^{\phi^\lceil(0, \vec{x})=\Phi(\vec{x})} D\Phi^\lceil e^{-is[\vec{\mathcal{A}}_T^\lceil, \vec{\mathcal{A}}_L^\lceil, \Phi^\lceil]}$$

$$\langle Q(\Phi) \rangle = \int_C D\vec{\mathcal{A}}_T^\rceil D\vec{\mathcal{A}}_T^\lceil D\vec{\mathcal{A}}_L^\rceil D\vec{\mathcal{A}}_L^\lceil D\Phi^\rceil D\Phi^\lceil Q(\Phi) e^{is[\vec{\mathcal{A}}_T^\rceil, \vec{\mathcal{A}}_L^\rceil, \Phi^\rceil] - is[\vec{\mathcal{A}}_T^\lceil, \vec{\mathcal{A}}_L^\lceil, \Phi^\lceil]}$$

$$\langle \phi^>(t_1)\phi^>(t_2) \rangle = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{dpe^{ipt_{12}}}{p^2 - y^2 - i\epsilon} \equiv \begin{array}{c} > \\ \hline i \\ \hline > \end{array} \begin{array}{c} > \\ \hline j \\ \hline > \end{array},$$

$$\langle \phi^<(t_1)\phi^<(t_2) \rangle = \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{dpe^{-ipt_{12}}}{p^2 - y^2 + i\epsilon} \equiv \begin{array}{c} < \\ \hline i \\ \hline < \end{array} \begin{array}{c} < \\ \hline j \\ \hline < \end{array},$$

$$\langle \phi^>(t_1)\phi^<(t_2) \rangle = \frac{-i}{\pi} \int_{C_1} \frac{dpe^{ipt_{12}}}{p^2 - y^2} \equiv \begin{array}{c} > \\ \hline i \\ \hline < \end{array} \begin{array}{c} < \\ \hline j \\ \hline > \end{array},$$

$$\langle \phi^<(t_1)\phi^>(t_2) \rangle = \frac{i}{\pi} \int_{C_2} \frac{dpe^{ipt_{12}}}{p^2 - y^2} \equiv \begin{array}{c} < \\ \hline i \\ \hline > \end{array} \begin{array}{c} > \\ \hline j \\ \hline < \end{array}$$

$$\langle A_T^{>i}(t_1)A_T^{>j}(t_2) \rangle = i \frac{\pi^{ij}}{\pi} \int_{-\infty}^{\infty} \frac{dpe^{ipt_{12}}}{p^2 - y^2 - i\epsilon}$$

$$\langle A_T^{<i}(t_1)A_T^{<j}(t_2) \rangle = -i \frac{\pi^{ij}}{\pi} \int_{-\infty}^{\infty} \frac{dpe^{-ipt_{12}}}{p^2 - y^2 + i\epsilon}$$

$$\langle A_T^{>i}(t_1)A_T^{<j}(t_2) \rangle = -i \frac{\pi^{ij}}{\pi} \int_{C_1} \frac{dpe^{ipt_{12}}}{p^2 - y^2}$$

$$\langle A_T^{<i}(t_1)A_T^{>j}(t_2) \rangle = i \frac{\pi^{ij}}{\pi} \int_{C_2} \frac{dpe^{ipt_{12}}}{p^2 - y^2}$$

$$\langle A_L^{>i}(t_1)A_L^{>j}(t_2) \rangle = i \frac{y^i y^j}{y^2 \pi} \int_{-\infty}^{\infty} \frac{dpsin(pt_1)\sin(pt_2)}{p^2 - y^2 - i\epsilon}$$

$$\langle A_L^{<i}(t_1)A_L^{<j}(t_2) \rangle = -i \frac{y^i y^j}{y^2 \pi} \int_{-\infty}^{\infty} \frac{dpsin(pt_1)\sin(pt_2)}{p^2 - y^2 + i\epsilon} = -\langle A_L^{>i}(t_1)A_L^{>j}(t_2) \rangle$$

$$\langle A_L^{>i}(t_1)A_L^{<j}(t_2) \rangle = 0$$

$$\langle A_L^{<i}(t_1)A_L^{>j}(t_2) \rangle = 0$$

$$\langle A_i^>(t_1)A_j^>(t_2) \rangle \equiv \begin{array}{c} > \\ \hline i \\ \hline > \end{array} \begin{array}{c} > \\ \hline j \\ \hline > \end{array},$$

$$\langle A_i^<(t_1)A_j^<(t_2) \rangle \equiv \begin{array}{c} < \\ \hline i \\ \hline < \end{array} \begin{array}{c} < \\ \hline j \\ \hline < \end{array},$$

$$\langle A_i^>(t_1)A_j^<(t_2) \rangle \equiv \begin{array}{c} > \\ \hline i \\ \hline < \end{array} \begin{array}{c} < \\ \hline j \\ \hline > \end{array},$$

$$\langle A_i^<(t_1)A_j^>(t_2) \rangle \equiv \begin{array}{c} < \\ \hline i \\ \hline > \end{array} \begin{array}{c} > \\ \hline j \\ \hline < \end{array}.$$

$$\begin{array}{c} \hline \\ \downarrow \\ > \end{array} = \begin{array}{c} \hline \\ \downarrow \\ < \end{array} = e^{ikt},$$

$$\begin{array}{c} \hline \\ \downarrow \\ > \end{array} = \begin{array}{c} \hline \\ \downarrow \\ < \end{array} = e^{-ikt},$$

$$\begin{array}{c} \hline \\ \downarrow \\ > \end{array} = \epsilon_i e^{ikt}, \quad \begin{array}{c} \hline \\ \downarrow \\ < \end{array} = \epsilon_i e^{-ikt}$$



$$S_{SK} = \int d^4x d\eta \left[-\frac{1}{4} F_{\mu\nu}^> F^{>\mu\nu} + \frac{1}{4} F_{\mu\nu}^< F^{<\mu\nu} - (D_\mu \phi^>)^\dagger (D^\mu \phi^>) - m^2 \phi^>\dagger \phi^> \right. \\ \left. + (D_\mu \phi^<) (D^\mu \phi^<) + m^2 \phi^<\dagger \phi^< \right]$$

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \text{---} \text{wavy} = -ig(\vec{k}_1 - \vec{k}_2)_i, \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \text{---} \text{wavy} = ig(\vec{k}_1 - \vec{k}_2)_i.$$

$$\begin{aligned} \phi^> &\rightarrow \phi^- - i\phi^+, & A_i^> &\rightarrow A_i^- - iA_i^+ \\ \phi^< &\rightarrow \phi^- + i\phi^+, & A_i^< &\rightarrow A_i^- + iA_i^+ \end{aligned}$$

$$S_{\text{shadow}} = 2 \int dz d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^+ F^{+\mu\nu} + \frac{1}{4} F_{\mu\nu}^- F^{-\mu\nu} + (\partial_\mu \phi^-)^\dagger \partial^\mu \phi^- - (\partial_\mu \phi^+)^\dagger \partial^\mu \phi^+ \right. \\ \left. + ig \delta^{ij} [A_i^+ (\phi^{-\dagger} \partial_j \phi^+ - \phi^+ \partial_j \phi^{-\dagger} + \phi^{+\dagger} \partial_j \phi^- - \phi^- \partial_j \phi^{+\dagger}) \right. \\ \left. + A_i^- (\phi^{+\dagger} \partial_j \phi^+ - \phi^+ \partial_j \phi^{+\dagger} - \phi^{-\dagger} \partial_j \phi^- + \phi^- \partial_j \phi^{-\dagger}) \right] \\ \left. + g^2 \delta^{ij} [(A_i^- A_j^- - A_i^+ A_j^+) (\phi^{-\dagger} \phi^- - \phi^{+\dagger} \phi^+) - 2A_i^- A_j^+ (\phi^+ \phi^{-\dagger} + \phi^- \phi^{+\dagger})] \right\}$$

$$G_{A,ij}^-(z_1, z_2, \vec{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \left(\cos(pz_1) \cos(pz_2) \frac{\pi_{ij}}{p^2 + y^2} + \sin(pz_1) \sin(pz_2) \frac{y_i y_j}{y^2 p^2} \right)$$

$$G_{A,ij}^+(z_1, z_2, \vec{y}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dp \sin(pz_1) \sin(pz_2) \left(\frac{\pi_{ij}}{p^2 + y^2} + \frac{y_i y_j}{y^2 p^2} \right)$$

$$\pi_{ij} = \eta_{ij} - \frac{y_i y_j}{y^2}$$



$$G_{A,ij}^-(z_1, z_2, \vec{y}) = \left(\begin{array}{c} z_1, i \quad \vec{y} \quad z_2, j \\ \text{---} \end{array} \right), \quad G_{A,ij}^+(z_1, z_2, \vec{y}) = \left(\begin{array}{c} z_1, i \quad \vec{y} \quad z_2, j \\ \text{+} \end{array} \right),$$

$$G_\phi^+(z_1, z_2, \vec{y}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{\sin(pz_1) \sin(pz_2)}{p^2 + y^2} = \left(\begin{array}{c} z_1 \quad \vec{y} \quad z_1 \\ \text{+} \end{array} \right)$$

$$G_\phi^-(z_1, z_2, \vec{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{\cos(pz_1) \cos(pz_2)}{p^2 + y^2} = \left(\begin{array}{c} z_1 \quad \vec{y} \quad z_1 \\ \text{---} \end{array} \right)$$

$$K_\phi^\pm(\vec{k}, z) = \mp e^{-kz} = \left(\begin{array}{c} \vec{k} \\ z \end{array} \right)$$

$$K_{A_i}^\pm(\vec{k}, z) = \mp \epsilon_i e^{-kz} = \left(\begin{array}{c} \vec{k}, \vec{\epsilon} \\ z \end{array} \right); \quad k^i \epsilon_i = 0.$$



$$\langle \phi_1 \cdots \phi_4^\dagger \rangle_{(s)}^T = g^2 \frac{\alpha^i \pi_{ij} \beta^j}{(k_{12} + y)(k_{34} + y)} \left(\frac{1}{k_{12} + k_{34}} + \frac{1}{y} \right)$$

$$\langle \phi_1 \cdots \phi_4^\dagger \rangle_{(s)}^L = g^2 \alpha^i \frac{y_i y_j}{y^2} \beta^j \frac{1}{k_{12} k_{34} k_{1234}}$$

$$\text{Res}_{k_{12}+k_{34}=0} \langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = g^2 \frac{-(k_1 - k_2)(k_3 - k_4) + \vec{\alpha} \cdot \vec{\beta}}{k_{12}^2 - y^2}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = \text{[Four diagrams showing different topologies of external lines and internal wavy lines]} + \text{[Four diagrams showing different topologies of external lines and internal wavy lines]}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = \text{[Circular diagram with four external lines and a central wavy line]} + \text{[Circular diagram with four external lines and a central wavy line]}$$

$$\langle \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_4 \phi_5^\dagger \phi_6 \rangle_{(s)} = \text{[Circular diagram with six external lines and a central wavy line]} + \text{[Circular diagram with six external lines and a central wavy line]}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = g^2 \alpha^i \beta^j \left[\frac{\pi_{ij}}{(k_{12} + y)(k_{34} + y)} \left(\frac{1}{k_{12} + k_{34}} + \frac{1}{y} \right) + \frac{y_i y_j}{y^2} \frac{1}{(k_{12} + k_{34}) k_{12} k_{34}} \right],$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = g^2 \alpha^i \beta^j \int_0^\infty dz dz' \int dpe^{-k_{12}z - k_{34}z'} \left(\cos(pz) \cos(pz') \frac{\pi_{ij}}{p^2 + y^2} + \sin(pz) \sin(pz') \frac{y_i y_j}{y^2 p^2} \right)$$

$$= g^2 \alpha^i \beta^j \left(\frac{\pi_{ij}}{(k_{12} + y)(k_{34} + y)} \left(\frac{1}{k_{1234}} + \frac{1}{y} \right) + \frac{y_i y_j}{k_{12} k_{34} k_{1234} y^2} \right)$$

$$\langle \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_4 \phi_5^\dagger \phi_6 \rangle_{(s)} = \psi_6^{\phi^\dagger \phi \phi^\dagger \phi \phi^\dagger \phi}(\vec{k}_1, \dots, \vec{k}_6)$$

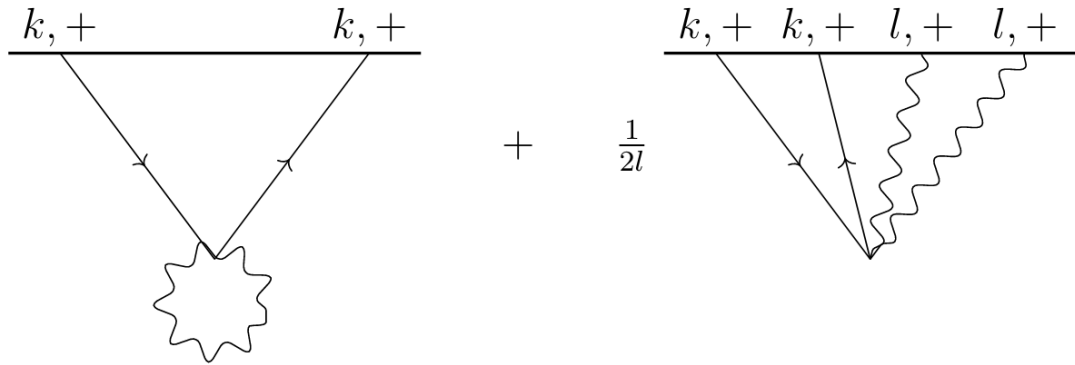
$$+ \psi_{5i}^{\phi^\dagger \phi \phi^\dagger \phi A_T}(\vec{k}_1, \dots, \vec{k}_4) \psi_{3j}^{\phi^\dagger \phi A_T}(\vec{k}_5, \vec{k}_6, \vec{y}_2) \frac{\pi^{ij}(\vec{y}_2)}{y_2} + \psi_{5i}^{\phi^\dagger \phi \phi^\dagger \phi A_T}(\vec{k}_3, \vec{k}_4, \vec{k}_5, \vec{k}_6) \psi_{3j}^{\phi^\dagger \phi A_T}(\vec{k}_1, \vec{k}_2, \vec{y}_1) \frac{\pi^{ij}(\vec{y}_1)}{y_1}$$

$$+ \psi_{3i}^{\phi^\dagger \phi A_T}(\vec{k}_1, \vec{k}_2, \vec{y}_1) \psi_{4jm}^{\phi^\dagger \phi A_T A_T}(\vec{y}_1, \vec{k}_3, \vec{k}_4, \vec{y}_2) \psi_{3n}^{\phi^\dagger \phi A_T}(\vec{y}_2, \vec{k}_5, \vec{k}_6) \frac{\pi^{ij}(\vec{y}_1)}{y_1} \frac{\pi^{mn}(\vec{y}_2)}{y_2}$$





$$\begin{aligned}
 \langle \phi^\dagger(\vec{k})\phi(-\vec{k}) \rangle_{\text{tadpole}} &= \delta_{ij} \frac{2}{\pi} \int d^4l \int dp \int_0^\infty dz e^{-2kz} \underbrace{\left[\sin^2(pz) \frac{\pi_{ij}}{p^2 + l^2} + \sin^2(pz) \frac{l_i l_j}{p^2 l^2} \right]}_{+\text{diagram}} \\
 &\quad + \underbrace{\cos^2(pz) \frac{\pi_{ij}}{p^2 + l^2} + \sin^2(pz) \frac{l_i l_j}{p^2 l^2}}_{-\text{diagram}} \\
 &= \delta_{ij} \frac{2}{\pi} \int d^4l \int dp \int_0^\infty dz e^{-2kz} \left[\underbrace{\frac{\pi_{ij}}{p^2 + l^2}}_{\text{Transverse}} + 2 \underbrace{\sin^2(pz) \frac{l_i l_j}{p^2 l^2}}_{\text{Longitudinal}} \right] \\
 &= \frac{2\delta_{ij}}{\pi} \int d^4L \frac{1}{2k} \left(\frac{\pi_{ij}}{L^2} + \frac{1}{(L_0)^2 + k^2} \frac{L_i L_j}{L_k L^k} \right) \\
 &= \int d^4l \frac{2k + l}{2k^2 l}
 \end{aligned}$$



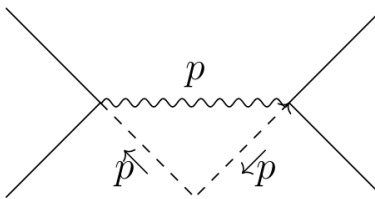
$$\langle \phi^\dagger(\vec{k})\phi(-\vec{k}) \rangle_{\text{tadpole}} = \int d^4l \left(\psi_2^{(1)}(\vec{k}, \vec{k}, \vec{l}) + \frac{\psi_4^{(0)}(\vec{k}, \vec{k}, \vec{l}, \vec{l})}{2\psi_2^{(0)}(l)} \right)$$

$$\begin{aligned}
 \langle \phi^\dagger(\vec{k})\phi(-\vec{k}) \rangle_{\text{tadpole}} &= \delta_{ij} \int d^4l \int_0^\infty dz \left[\left\{ \frac{1}{\pi} \int dp e^{-2kz} \sin^2(pz) \left(\frac{\pi_{ij}}{p^2 + l^2} + \frac{l_i l_j}{p^2 l^2} \right) \right\} + \frac{1}{2} \frac{\pi_{ij}}{l} e^{-2(k+l)z} \right] \\
 &= \delta_{ij} \int d^3l \frac{1}{2kl} \left(\pi_{ij} + \frac{l_i l_j}{kl} \right) \\
 &= \int d^3l \frac{2k + l}{2k^2 l}
 \end{aligned}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)}^T = \alpha^i \pi_{ij} \beta^j \int_0^\infty dz_1 dz_2 e^{-k_{12} z_1} e^{-k_{34} z_2} (z_1 z_2)^{1/2} \\ \times \int_{-\infty}^\infty \frac{dp}{p^2 + y^2} (J_{-1/2}(pz_1) J_{-1/2}(pz_2) + J_{1/2}(pz_1) J_{1/2}(pz_2))$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)}^T = \alpha^i \pi_{ij} \beta^j \int_{-\infty}^\infty dp \frac{k_{12}}{p^2 + k_{12}^2} \frac{k_{34}}{p^2 + k_{34}^2} \frac{1}{p^2 + y^2} \\ = \alpha^i \pi_{ij} \beta^j \int_{-\infty}^\infty dp \hat{\Delta}_Y^T(k_{12}, p) \Delta_Y^T(k_{34}, p) \frac{1}{p^2 + y^2}$$

$$\hat{\Delta}_Y^T(k_{ext}, p) = \frac{k_{ext}}{k_{ext}^2 + p^2}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)}^T =$$


$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)}^L = \alpha^i \beta^j \frac{y_i y_j}{y^2} \int_{-\infty}^\infty dp \frac{p}{p^2 + k_{12}^2} \frac{p}{p^2 + k_{34}^2} \frac{1}{p^2} \\ = \alpha^i \beta^j \frac{y_i y_j}{y^2} \int_{-\infty}^\infty dp \hat{\Delta}_Y^L(k_{12}, p) \hat{\Delta}_Y^L(k_{34}, p) \frac{1}{p^2}$$

$$\hat{\Delta}_Y^L(k_{ext}, p) = \frac{p}{k_{ext}^2 + p^2}$$

$$\mathcal{A}_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$n_t = -n_s|_{2 \leftrightarrow 4}, n_u = -n_s|_{2 \leftrightarrow 3}.$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(s)} = \frac{1}{\pi} \int dp \frac{1}{p^2 + y^2} \alpha^i \left(\pi_{ij} \frac{k_{12}}{k_{12}^2 + p^2} \frac{k_{34}}{k_{34}^2 + p^2} + \frac{y_i y_j}{y^2} \frac{p}{k_{12}^2 + p^2} \frac{p}{k_{34}^2 + p^2} \frac{p^2 + y^2}{p^2} \right) \beta^j$$

$$n_s(p) = \frac{1}{\pi} \frac{1}{k_{12}^2 + p^2} \frac{1}{k_{34}^2 + p^2} \left(\alpha \cdot \beta k_{12} k_{34} + \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{y^2} (k_{12} k_{34} - p^2 - y^2) \right) \\ = \frac{1}{\pi} \frac{k_{12}}{k_{12}^2 + p^2} \frac{k_{34}}{k_{34}^2 + p^2} \left(\alpha \cdot \beta - (k_1 - k_2)(k_3 - k_4) + \frac{(k_1 - k_2)(k_3 - k_4)}{y^2} (k_{12} k_{34} - p^2) \right) \\ = \frac{1}{\pi} \frac{k_{12}}{k_{12}^2 + p^2} \frac{k_{34}}{k_{34}^2 + p^2} \left(\alpha^\mu \eta_{\mu\nu} \beta^\nu + \frac{(k_1 - k_2)(k_3 - k_4)}{y^2} (k_{12} k_{34} - p^2) \right)$$

$$\int_{-\infty}^\infty dp (n_s(p) + n_t(p) + n_u(p)) = 0$$



$$\frac{1}{\pi} \int_{-\infty}^{\infty} dp \frac{k_{12}}{k_{12}^2 + p^2} \frac{k_{34}}{k_{34}^2 + p^2} \alpha^\mu \eta_{\mu\nu} \beta^\nu = \frac{\alpha^\mu \eta_{\mu\nu} \beta^\nu}{k_{12} + k_{34}}$$

$$\psi_{4,s} = \frac{1}{\pi} \int dp \frac{1}{p^2 + y^2} \alpha^i \underbrace{\left(\pi_{ij} \frac{p}{k_{12}^2 + p^2} \frac{p}{k_{34}^2 + p^2} + \frac{y_i y_j}{y^2} \frac{p}{k_{12}^2 + p^2} \frac{p}{k_{34}^2 + p^2} \frac{p^2 + y^2}{p^2} \right)}_{n_s^\psi(p)} \beta^j$$

$$\int_{-\infty}^{\infty} dp \left(n_s^\psi(p) + n_t^\psi(p) + n_u^\psi(p) \right) = 0.$$

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-\tilde{g}} (\tilde{R} - 2\Lambda - \kappa^2 \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu} = \frac{\eta_{\mu\nu} + \kappa \tilde{h}_{\mu\nu}}{\eta^2},$$

$$h_{ij} = h_{ij}^{TT} + h_{ij}^L$$

$$\partial_i h_{ij}^{TT} = 0; \delta_{ij} h_{ij}^{TT} = 0$$

$$h_{ij}^L = \partial_{(i} \xi_{j)} + \delta_{ij} \chi$$

$$S_{\text{free}}^{\text{gr}} = \int d^4x d\eta \eta^2 \left(\frac{1}{4} \partial_\mu h \partial^\mu h - \frac{1}{4} \partial_\mu h_{ij} \partial^\mu h_{ij} - \frac{1}{2} \partial_j h \partial_i h_{ij} + \frac{1}{2} \partial_i h_{jk} \partial_j h_{ki} \right) - \frac{1}{2} h^2 + \frac{1}{2} h_{ij} h_{ij}$$

$$S_{\text{free gr}}^{SK} [h_>, h_<] = S_{\text{free gr}} [h_>] - S_{\text{free gr}} [h_<]$$

$$h_{ij}^{TT, >}(0) = h_{ij}^{TT, <}(0), h_{ij}^{L, >}(0) = h_{ij}^{L, <}(0) = 0$$

$$\langle h_{ij}^{>}(\eta) h_{mn}^{>}(\eta') \rangle = i \frac{(\eta \eta')^{-1/2}}{2} \int_{-\infty}^{\infty} p dp \left(\frac{H_{3/2}^{(2)}(p\eta) H_{3/2}^{(1)}(p\eta')}{p^2 - y^2 - i\epsilon} \Pi_{ijmn} + J_{3/2}(p\eta) J_{3/2}(p\eta') \tilde{T}_{ijmn} \right)$$

$$\langle h_{ij}^{<}(\eta) h_{mn}^{<}(\eta') \rangle = -i \frac{(\eta \eta')^{-1/2}}{2} \int_{-\infty}^{\infty} p dp \left(\frac{H_{3/2}^{(1)}(p\eta) H_{3/2}^{(2)}(p\eta')}{p^2 - y^2 - i\epsilon} \Pi_{ijmn} + J_{3/2}(p\eta) J_{3/2}(p\eta') \tilde{T}_{ijmn} \right)$$

$$\langle h_{ij}^{>}(\eta) h_{mn}^{<}(\eta') \rangle = -i \frac{(\eta \eta')^{-1/2}}{2} \int_{C_1} \frac{p dp}{p^2 - k^2} \left(H_{3/2}^{(2)}(p\eta) H_{3/2}^{(1)}(p\eta') \Pi_{ijmn} \right)$$

$$\langle h_{ij}^{<}(\eta) h_{mn}^{>}(\eta') \rangle = i \frac{(\eta \eta')^{-1/2}}{2} \int_{C_2} \frac{p dp}{p^2 - k^2} \left(H_{3/2}^{(2)}(p\eta) H_{3/2}^{(1)}(p\eta') \Pi_{ijmn} \right)$$

$$\Pi_{ijnm} = \pi_{in} \pi_{jm} + \pi_{im} \pi_{jn} - \pi_{ij} \pi_{nm}$$

$$\tilde{T}_{ijnm} = \frac{1}{p^2 y^2} G_{ijnm}^{\text{TL}} + \frac{p^2 + y^2}{p^4} \frac{y_i y_j y_n y_m}{y^4}$$

$$G_{ijnm}^{\text{TL}} = \pi_{in} y_j y_m + \pi_{im} y_j y_n + \pi_{jm} y_i y_n + \pi_{jn} y_i y_m - \pi_{nm} y_i y_j - \pi_{ij} y_n y_m$$

$$\pi_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$



$$\begin{aligned}
& \vec{k}_1, - \quad \vec{k}_1, + \\
& \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i,j \\ \text{---} \\ z \end{array} + = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i,j \\ \text{---} \\ z \end{array} + = \kappa \left(k_{1,i}k_{2,j} + k_{1,j}k_{2,i} + (\partial_z P_1 \partial_z P_2 - \vec{k}_1 \cdot \vec{k}_2) \delta_{ij} \right) \\
& \vec{k}_2, + \quad \vec{k}_2, - \\
& \vec{k}_1, - \quad \vec{k}_1, + \\
& \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i,j \\ \text{---} \\ z \end{array} - = - \left(\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i,j \\ \text{---} \\ z \end{array} - \right) = \kappa \left(k_{1,i}k_{2,j} + k_{1,j}k_{2,i} + (\partial_z P_1 \partial_z P_2 - \vec{k}_1 \cdot \vec{k}_2) \delta_{ij} \right) \\
& \vec{k}_2, - \quad \vec{k}_2, + \\
& \vec{k}_1, - \quad \vec{k}_1, - \\
& \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i \\ \text{---} \\ j \end{array} - = - \left(\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} i,j \\ \text{---} \\ n,m \end{array} \right) = \frac{\kappa^2 z^2}{2} \left((\partial_z P_1 \partial_z P_2 - \vec{k}_1 \cdot \vec{k}_2) \Delta_{ijnm} + 2\tilde{G}_{ijnm}^{TL12} \right) \\
& \vec{k}_2, - \quad \vec{k}_2, - \quad +
\end{aligned}$$

$$\frac{\partial}{\partial z} \{ (1 + kz) e^{-kz} \} = -k^2 z e^{-kz} = -\frac{k^2 z}{1 + kz} K_{\phi}^{\pm}(z)$$

$$\mathcal{V}_{3ij}^{ab} = \kappa (\delta_{ij} W_{ab} + k_{a,i} k_{b,j} + k_{a,j} k_{b,i})$$

$$\mathcal{V}_{4ijnm}^{ab} = -\frac{\kappa^2 z^2}{2} (W_{ab} \Delta_{ijnm} + 2\tilde{G}_{ijnm}^{ab})$$

$$W_{ab} = \frac{k_a^2 k_b^2 z^2}{(1 + k_a z)(1 + k_b z)} - \vec{k}_a \cdot \vec{k}_b$$

$$\Delta_{ijnm} = \delta_{in} \delta_{jm} + \delta_{im} \delta_{jn} - \delta_{ij} \delta_{nm}$$

$$\tilde{G}_{ijnm}^{ab} = \delta_{in} k_{a,j} k_{b,m} + \delta_{im} k_{a,j} k_{b,n} + \delta_{jn} k_{a,i} k_{b,m} + \delta_{jm} k_{a,i} k_{b,n} - \delta_{ij} k_{a,n} k_{b,m} - \delta_{nm} k_{a,i} k_{b,j}$$

$$\langle \phi \phi \phi \phi \rangle_{(s)} = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \mathcal{M}_4^{\text{TT}} + \mathcal{M}_4^L.$$

$$\mathcal{M}_4^{\text{TT}} = \frac{1}{4} (\vec{k}_1^i \vec{k}_2^j + \vec{k}_1^j \vec{k}_2^i) (\vec{k}_3^n \vec{k}_4^m + \vec{k}_3^m \vec{k}_4^n) \Pi_{ijnm}$$

$$\times \int dp \int_0^\infty dz' dz (zz')^{-1/2} (1 + k_1 z)(1 + k_2 z) e^{-k_{12} z} \frac{p J_{\frac{3}{2}}(pz) J_{\frac{3}{2}}(pz')}{p^2 + y^2} (1 + k_3 z')(1 + k_4 z') e^{-k_{34} z'}$$

$$\mathcal{M}_4^L = \frac{1}{4} \int dp \int_0^\infty dz dz' (zz')^{-1/2} (1+k_1z)(1+k_2z) e^{-k_{12}z} p J_{\frac{3}{2}}(pz) J_{\frac{3}{2}}(pz') (1+k_3z')(1+k_4z') e^{-k_{34}z'}$$

$$\times \left[\frac{1}{p^2 y^2} \left(V_{ij}^{12} V_{nm}^{34} \left(G_{ijnm}^{TL} + \frac{y_i y_j y_n y_m}{y^2} \right) - 2y^2 (\vec{k}_1 \cdot \vec{k}_2 W_{34} + \vec{k}_3 \cdot \vec{k}_4 W_{12}) - 3W_{12} W_{34} \right) \right.$$

$$\left. + \frac{1}{p^4} \left(V_{ij}^{12} V_{nm}^{34} \frac{y_i y_j y_n y_m}{y^2} + y_i y_j (\vec{k}_1 \cdot \vec{k}_2 W_{34} + \vec{k}_3 \cdot \vec{k}_4 W_{12}) + y^2 W_{12} W_{34} \right) \right]$$

$$V_{ij}^{12} = (\vec{k}_1^i \vec{k}_2^j + \vec{k}_1^j \vec{k}_2^i)$$

$$V_{nm}^{34} = (\vec{k}_3^n \vec{k}_4^m + \vec{k}_3^m \vec{k}_4^n)$$

$$\mathcal{M}_4^{TT} = \tilde{f}_{22} y^4 \Pi_{22}$$

$$\tilde{f}_{22} = f_{22} + \frac{g_3(\vec{k}_1, \vec{k}_2, \vec{y}) g_3(\vec{k}_3, \vec{k}_4, \vec{y})}{y^3}$$

$$f_{22} = \frac{1}{E} - \frac{y^2}{E E_L E_R} + \frac{y k_1 k_2}{E E_L^2 E_R} + \frac{y k_3 k_4}{E E_R^2 E_L} + \frac{2y k_1 k_2 k_3 k_4}{E^2 E_L^2 E_R^2} - \frac{y(k_1 k_2 + k_3 k_4)}{E^2 E_L E_R}$$

$$+ \frac{k_1 k_2}{E^2 E_L} + \frac{k_3 k_4}{E^2 E_R} + \frac{2k_1 k_2 k_3 k_4}{E^3 E_L E_R},$$

$$g_3(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \frac{2k_1 y^2 + 2k_2 y^2 + 2k_1^2 y + 2k_2^2 y + 2k_1 k_2 y + k_1^3 + k_2^3 + 2k_1 k_2^2 + 2k_1^2 k_2 + y^3}{(k_1 + k_2 + y)^2}$$

$$\Pi_{22} = \frac{24}{y^2} \Pi^{ijlm} k_i^1 k_j^2 k_l^3 k_m^4;$$

$$E = k_{12} + k_{34}, E_L = k_{12} + y, E_R = k_{34} + y, y = |\vec{k}_1 + \vec{k}_2|.$$

$$\mathcal{M}_4^L = f_{21} k^2 \Pi_{21} + f_{20} (E_L E_R - kE) \Pi_{20} + f_c$$

$$f_{21} = -\frac{2k_1 k_2 k_3 k_4}{E^3} - \frac{k_3 k_4 k_{12} + k_1 k_2 k_{34}}{E^2} - \frac{k_{12} k_{34}}{E} = -f_{20}$$

$$\Pi_{21} = \frac{3(k_1 - k_2)(k_3 - k_4)(k^2(t^2 - u^2) + (k_1 - k_2)(k_3 - k_4)k_{12}k_{34})}{k^6}$$

$$t = \vec{k}_1 \cdot \vec{k}_4; u = \vec{k}_1 \cdot \vec{k}_3;$$

$$\Pi_{20} = \frac{1}{4} \left(1 - \frac{3(k_1 - k_2)^2}{y^2} \right) \left(1 - \frac{3(k_3 - k_4)^2}{y^2} \right),$$

$$f_c = -\frac{2k_1 k_2 k_3 k_4 (y^2 + k_{12} k_{34})}{E^3} - \frac{9E^3}{8} - \frac{3(k_1 - k_2)^2 (k_3 - k_4)^2 (k_1 k_2 + k_3 k_4)}{4E y^2}$$

$$-3(k_1 k_2 k_3 + k_1 k_4 k_3 + k_2 k_4 k_3 + k_1 k_2 k_4),$$

$$\frac{(k_1 k_2 k_3 + k_1 k_4 k_3 + k_2 k_4 k_3 + k_1 k_2 k_4)(y^2 + k_{12} k_{34})}{E^2}$$

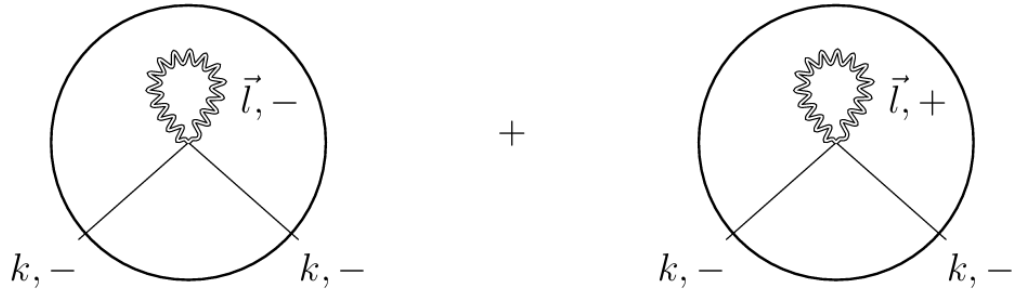
$$+ \frac{1}{8} E \left(-\frac{6(k_1 - k_2)^2 (k_3 - k_4)^2}{y^2} + 3(2y^2 + k_{12}^2 + k_{34}^2) + 34(k_1 k_3 + k_2 k_3 + k_1 k_4 + k_2 k_4) \right.$$

$$\left. + 21(k_1 k_2 + k_3 k_4) \right) - \frac{1}{4E} (8(k_1 k_3 + k_2 k_3 + k_1 k_4 + k_2 k_4)^2 + 24k_1 k_2 k_3 k_4$$

$$+ \frac{1}{2} (4(k_1 k_3 + k_2 k_3 + k_1 k_4 + k_2 k_4) + 3(k_1 k_2 + k_3 k_4))(2y^2 + k_{12}^2 + k_{34}^2)$$

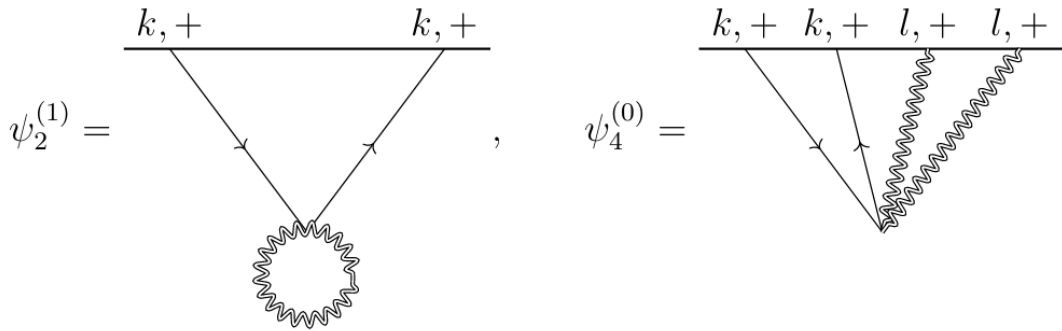
$$\left. + (k_1 k_2 + k_3 k_4)(9(k_1 k_3 + k_2 k_3 + k_1 k_4 + k_2 k_4) - 12(k_1 k_2 + k_3 k_4)) \right)$$





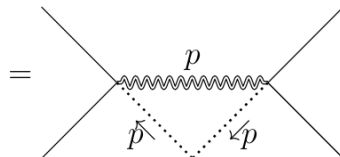
$$\begin{aligned} \langle \phi(\vec{k})\phi(-\vec{k}) \rangle_{\text{tadpole}} &= \int d^4l \int_{-\infty}^{\infty} dp \int_0^{\infty} dz 2\mathcal{V}_{4,ijnm}(z) \frac{(1+kz)^2 e^{-2kz}}{z} \\ &\quad \times \left[\frac{\Pi_{ijnm}}{p^2 + l^2} \left(J_{-\frac{3}{2}}(pz)^2 + J_{\frac{3}{2}}(pz)^2 \right) + 2J_{\frac{3}{2}}(pz)^2 \tilde{T}_{ijnm} \right] \\ &= \int d^3l \left(10 + \frac{80k^3}{l^3} + \frac{56k}{l} - \frac{3l^2}{k^2} - \frac{16(6k^5 + 5k^3l^2 - 3k^2l^3 - l^5)(\vec{k} \cdot \vec{l})^2}{k^4l^5} \right) \end{aligned}$$

$$\langle \phi(\vec{k})\phi(-\vec{k}) \rangle_{\text{tadpole}} = \int d^4l \left(\psi_2^{(1)}(k, k, \vec{l}) + \frac{\psi_4^{(0)}(k, k, l, l)}{2\psi_2^{(0)}(l)} \right)$$



$$\begin{aligned} \psi_2^{(1)} &= \int d^4l \int_{-\infty}^{\infty} dp \int_0^{\infty} dz \mathcal{V}_{4,ijnm}(z) \frac{(1+kz)^2 e^{-2kz}}{z} p J_{\frac{3}{2}}(pz)^2 \left[\frac{\Pi_{ijnm}}{p^2 + l^2} + \tilde{T}_{ijnm} \right] \\ \psi_4^{(0)} &= \int d^3l \mathcal{V}_{4,ijnm}(z) \Pi_{ijnm} \frac{(1+kz)^2 (1+lz)^2}{z^4} e^{-2kz} e^{-2lz} \end{aligned}$$

$$\mathcal{M}_4^{TT} = \frac{V_{12}^{ij} V_{34}^{nm}}{4} \int dp \hat{\Delta}_h^T(k_1, k_2, p) \frac{\Pi_{ijnm}}{p^2 + y^2} \hat{\Delta}_h^T(k_3, k_4, p)$$



$$\begin{aligned}\hat{\Delta}_h^T(k_i, k_j, p) &= p k_i^2 k_j^2 \hat{\partial}_p \hat{\partial}_{k_i} \hat{\partial}_{k_j} \int_0^\infty ds s \frac{k_{ij} + s}{(s + k_{ij})^2 + p^2} \\ &= p k_i^2 k_j^2 \hat{\partial}_p \hat{\partial}_{k_i} \hat{\partial}_{k_j} \int_0^\infty ds s \hat{\Delta}_\gamma^T(k_{ij} + s, p)\end{aligned}$$

$$\hat{\partial}_x f \equiv \partial_x \left(\frac{f}{x} \right)$$

$$\begin{aligned}V_{12}^{ij} &= k_1^i k_2^j + k_1^j k_2^i \\ &= \frac{1}{2} (y^i y^j - \alpha^i \alpha^j)\end{aligned}$$

$$\mathcal{M}_4^{TT} = \frac{\alpha^i \alpha^j \beta^n \beta^m \Pi_{ijnm}}{16} \int dp \hat{\Delta}_h^T \frac{1}{p^2 + y^2} \hat{\Delta}_h^T$$

$$\langle JJJ \rangle = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \text{cyclic}) k_{123}^{-1}$$

$$\langle JJJ \rangle = \text{Diagram} = \Delta_\gamma^T(k_{123}, 0) (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \text{cyclic}),$$

$$\langle TTT \rangle = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \text{cyclic})^2 \left(\frac{k_1 k_2 k_3}{k_{123}^2} + \frac{k_1 k_2 + \text{cyclic}}{k_{123}} - k_{123} \right)$$

$$\langle TTT \rangle = \text{Diagram} = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \text{cyclic})^2 \Delta_{hhh}(k_{123}).$$

$$\Delta_{hhh}(k_{123}) = -k_1^2 k_2^2 k_3^2 \hat{\partial}_{k_1} \hat{\partial}_{k_2} \hat{\partial}_{k_3} \left(\int_0^\infty ds \frac{s}{s + k_{123}} \right)$$

$$\Delta_{hhh}(k_{123}) = \frac{k_1 k_2 k_3}{k_{123}^2} + \frac{k_1 k_2 + \text{cyclic}}{k_{123}} - k_{123}.$$

$$\Delta_{hhh}(k_{123}) = \lim_{p \rightarrow 0} p \left(p (k_1 k_2 k_3)^2 \hat{\partial}_p \hat{\partial}_{k_1} \hat{\partial}_{k_2} \hat{\partial}_{k_3} \int_0^\infty ds s \frac{k_{123} + s}{(s + k_{123})^2 + p^2} \right)$$



$$\langle JJJJ \rangle^{(s)} = \text{Diagram 1} + \text{Diagram 2}$$

$$\begin{aligned} \langle JJJJ \rangle^{(s)} &= V_{12}^i V_{34}^j \pi_{ij} \int_0^\infty dz_1 dz_2 e^{-k_{12z_1}} e^{-k_{34z_2}} (z_1 z_2)^{1/2} \int_{-\infty}^\infty dp p \frac{J_{-1/2}(pz_1) J_{-1/2}(pz_2)}{p^2 + y^2} \\ &+ V_{12}^i V_{34}^j \frac{y_i y_j}{y^2} \int_0^\infty dz_1 dz_2 e^{-k_{12z_1}} e^{-k_{34z_2}} (z_1 z_2)^{1/2} \int_{-\infty}^\infty dp p \frac{J_{1/2}(pz_1) J_{1/2}(pz_2)}{p^2} \\ &+ V_c^{(s)} \int_0^\infty dz e^{-k_{1234z}} \end{aligned}$$

$$\begin{aligned} V_{ab}^i &= \epsilon_a \cdot \epsilon_b (\vec{k}_a - \vec{k}_b)^i + 2(\epsilon_a \cdot \vec{k}_b) \epsilon_b^i - 2(\epsilon_b \cdot \vec{k}_a) \epsilon_a^i \\ V_c^{(s)} &= (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3). \end{aligned}$$

$$\text{Diagram 3} + \text{Diagram 4}$$

$$\begin{aligned} \langle JJJJ \rangle^{(s)} &= V_{12}^i V_{34}^j \pi_{ij} \int_{-\infty}^\infty dp \Delta_\gamma^T(k_{12}, p) \Delta_\gamma^T(k_{34}, p) \frac{1}{p^2 + y^2} \\ &+ V_{12}^i V_{34}^j \frac{y_i y_j}{y^2} \int_{-\infty}^\infty dp \Delta_\gamma^L(k_{12}, p) \Delta_\gamma^L(k_{34}, p) \frac{1}{p^2} \\ &+ V_c^{(s)} \Delta_\gamma^T(k_{1234}, 0) \end{aligned}$$

$$\epsilon_a \cdot \epsilon_b \rightarrow 1, \epsilon_a \cdot k_b \rightarrow 0.$$

$$\langle JJJJ \rangle^{(s)} = \int_{-\infty}^\infty \frac{dp}{(p^2 + k_{12}^2)(p^2 + k_{34}^2)} \left[k_{12} k_{34} \frac{V_{12}^i V_{34}^j \pi_{ij}}{p^2 + y^2} + \frac{V_{12}^i V_{34}^j y_i y_j}{y^2} + k_{12} k_{34} V_c^{(s)} \right]$$

$$n_s(p) = \frac{1}{(p^2 + k_{12}^2)(p^2 + k_{34}^2)} \left[k_{12} k_{34} V_{12}^i V_{34}^j \pi_{ij} + (p^2 + y^2) \frac{V_{12}^i V_{34}^j k_i k_j}{y^2} + (p^2 + y^2) k_{12} k_{34} V_c^{(s)} \right]$$

$$\int_{-\infty}^\infty dp (n_s(p) + n_t(p) + n_u(p)) = 0$$

$$\langle Q(\phi) \rangle = \int D\phi D\phi' \Psi(\phi) \Psi^*(\phi') \delta(\phi - \phi') Q(\phi)$$

$$\Psi(\phi) = \int_0^{\varphi^{>(0)=\phi}} D\varphi^> e^{iS[\varphi^>]}, \Psi^*(\phi') = \int_0^{\varphi^{<(0)=\phi'}} D\varphi^{<} e^{-iS[\varphi^{<}]}$$

$$\langle Q(\phi) \rangle = \int D\phi D\phi' \delta(\phi - \phi') \int_0^{\varphi^{>(0)=\phi}} D\varphi^> \int_0^{\varphi^{<(0)=\phi'}} D\varphi^{<} Q(\phi) e^{iS[\varphi^>]} e^{-iS[\varphi^{<}]}$$

$$D\varphi(\eta, \vec{x}) \equiv \prod_n d\varphi_n(\eta)$$

$$\int D\phi D\phi' \delta(\phi - \phi')$$

$$\int D\phi D\phi' \delta(\phi - \phi') = \int \prod_n d\varphi_n^>(0) d\varphi_n^{<(0)} \prod_m \delta(\varphi_m^>(0) - \varphi_m^{<(0)})$$

$$\langle Q(\phi) \rangle = \int \prod_n d\varphi_n^>(\eta) d\varphi_n^{<}(\eta) Q(\varphi^>(0)) \prod_m \delta(\varphi_m^>(0) - \varphi_m^{<(0)}) e^{iS[\varphi^>]} e^{-iS[\varphi^{<}]}$$

$$\prod_m \delta(\varphi_m^>(0) - \varphi_m^{<(0)}) = e^{-\frac{1}{\epsilon} \sum_m (\varphi_m^>(0) - \varphi_m^{<(0)})^2}$$

$$= e^{-\frac{1}{\epsilon} \int d\eta d\eta' \sum_{mm'} \delta_{mm'} \delta(\eta) \delta(\eta') (\varphi_m^>(\eta) - \varphi_m^{<}(\eta)) (\varphi_{m'}^>(\eta') - \varphi_{m'}^{<}(\eta'))}$$

$$\equiv e^{-\frac{1}{\epsilon} \int d\eta d\eta' \sum_{mm'} c_{mm'}(\eta, \eta') (\varphi_m^>(\eta) - \varphi_m^{<}(\eta)) (\varphi_{m'}^>(\eta') - \varphi_{m'}^{<}(\eta'))}$$

$$C_{mm'}(\eta, \eta') = \delta_{mm'} \delta(\eta) \delta(\eta')$$

$$S[\varphi^>] = \int_{-\infty}^0 d\eta \frac{1}{2} (\nabla^\alpha \varphi^>)^2 - \frac{1}{2} M^2 (\varphi^>)^2 \equiv \sum_{nn'} \int_{-\infty}^0 d\eta d\eta' D_{nn'}^>(\eta, \eta') \varphi_n^>(\eta) \varphi_{n'}^>(\eta')$$

$$S[\varphi^{<}] = - \int_{-\infty}^0 d\eta \frac{1}{2} (\nabla^\alpha \varphi^{<})^2 - \frac{1}{2} M^2 (\varphi^{<})^2 \equiv \sum_{nn'} \int_{-\infty}^0 d\eta d\eta' D_{nn'}^{<}(\eta, \eta') \varphi_n^{<}(\eta) \varphi_{n'}^{<}(\eta')$$

$$\langle Q(\phi) \rangle = \int \prod_n d\varphi_n^> d\varphi_n^{<} Q(\phi) \times e^{\sum_{nn'} \int d\eta [iD_{nn'}^>(\eta, \eta') \varphi_n^>(\eta) \varphi_{n'}^>(\eta') - iD_{nn'}^{<}(\eta, \eta') \varphi_n^{<}(\eta) \varphi_{n'}^{<}(\eta') - \frac{1}{\epsilon} c_{nn'}(\eta, \eta') (\varphi_n^>(\eta) - \varphi_n^{<}(\eta)) (\varphi_{n'}^>(\eta') - \varphi_{n'}^{<}(\eta'))]}$$

$$[\dots] = (\varphi^> \varphi^{<}) \begin{pmatrix} iD^> - \epsilon^{-1}C & \epsilon^{-1}C \\ \epsilon^{-1}C & -iD^{<} - \epsilon^{-1}C \end{pmatrix} \begin{pmatrix} \varphi^> \\ \varphi^{<} \end{pmatrix}$$

$$\begin{pmatrix} iD^> - \epsilon^{-1}C & \epsilon^{-1}C \\ \epsilon^{-1}C & -iD^{<} - \epsilon^{-1}C \end{pmatrix} \begin{pmatrix} -iG^{\gg} & iG^{><} \\ iG^{<>} & iG^{\ll} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$D^>G^{\gg} - \frac{i}{\epsilon} C(G^{\gg} + G^{<>}) = 1$$

$$-D^>G^{><} + \frac{i}{\epsilon} C(G^{\ll} - G^{><}) = 0$$

$$D^{<}G^{<>} - \frac{i}{\epsilon} C(G^{\gg} + G^{<>}) = 0$$

$$D^{<}G^{\ll} + \frac{i}{\epsilon} C(G^{><} - G^{\ll}) = 1$$



$$\begin{aligned}
D^>G^>> &= D^<G^<< = 1 \\
CG^<< &= CG^><, CG^>> = -CG^<> \\
D^>G^>< &= D^<G^<> = 0
\end{aligned}$$

$$G_{nn'}^{>>}(\eta, \eta') = \langle \mathcal{T} \phi_n(\eta) \phi_{n'}(\eta') \rangle, G_{nn'}^{<<}(\eta, \eta') = \langle \bar{\mathcal{T}} \phi_n(\eta) \phi_{n'}(\eta') \rangle,$$

$$G_{nn'}^{><}(\eta, \eta') = \langle \phi_n(\eta) \phi_{n'}(\eta') \rangle, G_{nn'}^{<>}(\eta, \eta') = \langle \phi_n(\eta') \phi_{n'}(\eta) \rangle.$$

$$G_v^{>>}(\eta, \eta') = (\eta\eta')^{3/2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p H_v^{(1)}(p\eta) H_v^{(2)}(p\eta')}{p^2 - y^2 + i\epsilon}$$

$$G_v^{<<}(\eta, \eta') = (\eta\eta')^{3/2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p H_v^{(2)}(p\eta) H_v^{(1)}(p\eta')}{p^2 - y^2 - i\epsilon}$$

$$\begin{aligned}
G_v^{>>}(\eta, \eta') &= (\eta\eta')^{3/2} \frac{i}{2\pi} \left[\Theta(\eta - \eta') \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p H_v^{(1)}(p\eta) H_v^{(2)}(p\eta')}{p^2 - y^2 + i\epsilon} \right. \\
&\quad \left. + \Theta(\eta' - \eta) \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p H_v^{(1)}(p\eta) H_v^{(2)}(p\eta')}{p^2 - y^2 + i\epsilon} \right] \\
&= (\eta\eta')^{3/2} \left[\Theta(\eta - \eta') H_v^{(2)}(y\eta) H_v^{(1)}(y\eta') + \Theta(-\eta) H_v^{(1)}(y\eta) H_v^{(2)}(y\eta') \right]
\end{aligned}$$

$$G_v^{<<}(\eta) = (\eta\eta')^{3/2} \left[\Theta(\eta - \eta') H_v^{(1)}(y\eta) H_v^{(2)}(y\eta') + \Theta(\eta' - \eta) H_v^{(2)}(y\eta) H_v^{(1)}(y\eta') \right]$$

$$G_v^{>>}(\eta, \eta') = \frac{\eta\eta'}{2y} \left[\Theta(\eta - \eta') e^{-iy(\eta-\eta')} + \Theta(\eta' - \eta) e^{iy(\eta-\eta')} \right]$$

$$G_v^{<<}(\eta, \eta') = \frac{\eta\eta'}{2y} \left[\Theta(\eta - \eta') e^{iy(\eta-\eta')} + \Theta(\eta' - \eta) e^{-iy(\eta-\eta')} \right]$$

$$G^{><}(t, t') = \frac{\eta\eta'}{2y} e^{iy(\eta-\eta')}$$

$$G^{<>}(t, t') = \frac{\eta\eta'}{2y} e^{iy(\eta'-\eta)}$$

$$\langle Q(\phi) \rangle = \int D\phi Da Q(\phi) \delta(a) |\Psi(\phi, a)|^2$$

$$\Psi(\phi, a) = \int_0^{\varphi_{>(0)}=\phi} D\varphi_{>} \int_0^{\tilde{a}_{>(0)}=a} D\tilde{a}_{>} e^{iS[\varphi_{>}, \tilde{a}_{>}]}$$

$$\Psi^*(\phi, a) = \int_0^{\varphi_{<(0)}=\phi} D\varphi_{<} \int_0^{\tilde{a}_{<(0)}=a} D\tilde{a}_{<} e^{-iS[\varphi_{<}, \tilde{a}_{<}]}$$

$$\begin{aligned}
\langle Q(\phi) \rangle &= \int D\phi Da \delta(a) \int_0^{\varphi_{>(0)}=\phi} D\varphi_{>} \int_0^{\varphi_{<(0)}=\phi} D\varphi_{<} \int_0^{\tilde{a}_{<(0)}=a} D\tilde{a}_{<} \int_0^{\tilde{a}_{>(0)}=a} D\tilde{a}_{>} \\
&\quad \times e^{iS[\varphi_{>}, \tilde{a}_{>}] e^{-iS[\varphi_{<}, \tilde{a}_{<}]} Q(\phi)
\end{aligned}$$



$$\langle Q(\phi) \rangle = \int D\phi \int_0^{\varphi_{>(0)=\phi}} D\varphi_{>} \int_0^{\varphi_{<(0)=\phi}} D\varphi_{<} Q(\phi) \\ \times \int_0^{\tilde{a}_{>(0)=0}} D\tilde{a}_{>} e^{iS[\varphi_{>}, \tilde{a}_{>}]} \int_0^{\tilde{a}_{<(0)=0}} D\tilde{a}_{<} e^{-iS[\varphi_{<}, \tilde{a}_{<}]}$$

$$G_D^{\gg}(\eta, \eta') = (\eta\eta')i \int_{-\infty}^{\infty} \frac{dp \sin(p\eta)\sin(p\eta')}{p^2 + i\epsilon}, G_D^{\ll}(\eta, \eta') = -(\eta\eta')i \int_{-\infty}^{\infty} \frac{dp \sin(p\eta)\sin(p\eta')}{p^2 - i\epsilon}$$

$$G_D^{\gg}(\eta, \eta') = \lim_{\epsilon \rightarrow 0} (\eta\eta') \frac{i}{2} \Theta(\eta - \eta') [\eta' - (-1)^{1/4} \eta \eta' \sqrt{\epsilon}] + (\eta\eta') \frac{i}{2} \Theta(\eta' - \eta) [\eta - (-1)^{1/4} \eta \eta' \sqrt{\epsilon}] \\ = (\eta\eta') \frac{i}{2} [\Theta(\eta - \eta') \eta' + \Theta(\eta' - \eta) \eta] = -G_D^{\ll}(\eta, \eta')$$

$$G_D^{\gt<}(\eta, \eta') = G_D^{\lt>}(\eta, \eta') = 0$$

$$G_D^{\gg}(\eta, \eta') + G_D^{\ll}(\eta, \eta') + G_D^{\gt<}(\eta, \eta') + G_D^{\lt>}(\eta, \eta') = 0.$$

$$\langle \Psi | \phi(\vec{k}_1) \phi^\dagger(\vec{k}_2) \cdots \phi(\vec{k}_{2n-1}) \phi^\dagger(\vec{k}_{2n}) | \Psi \rangle \\ = \int D\phi D\phi^\dagger D\vec{A}_T | \Psi(\vec{A}_T, \phi, \phi^\dagger) |^2 \phi(\vec{k}_1) \phi^\dagger(\vec{k}_2) \cdots \phi(\vec{k}_{2n-1}) \phi^\dagger(\vec{k}_{2n})$$

$$\Psi(\vec{A}_T, \phi^\dagger, \phi) = e^{-\int \psi_2^{ij}(\vec{k}) A_T^i(\vec{k}) A_T^j(-\vec{k}) d^4k} e^{-\int \psi_2^{\phi^\dagger\phi}(\vec{k}) \phi^\dagger(-\vec{k}) \phi(\vec{k}) d^4k} \\ \times \exp \left[\int \psi_{3i}^{A_T\phi^\dagger\phi}(\vec{k}_1, \vec{k}_2, \vec{k}_3) A_T^i(\vec{k}_1) \phi^\dagger(\vec{k}_2) \phi(\vec{k}_3) d^4k_1 d^4k_2 d^4k_3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) + \cdots \right]$$

$$\psi_2^{ij}(\vec{k}, -\vec{k}) = \pi^{ij} k, \psi_2^{\phi^\dagger\phi}(\vec{k}, -\vec{k}) = k$$

$$\langle \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_4 \rangle = \int D\phi^\dagger D\phi D\vec{A}_T | \Psi(\vec{A}_T, \phi^\dagger, \phi) |^2 \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_4$$

$$| \Psi(\vec{A}_T, \phi^\dagger, \phi) |^2 = e^{-\int \psi_2^{ij}(\vec{k}) A_T^i(\vec{k}) A_T^j(-\vec{k}) d^4k} e^{-\int \psi_2^{\phi^\dagger\phi}(\vec{k}) \phi^\dagger(-\vec{k}) \phi(\vec{k}) d^4k} \\ \times \left[\int \psi_4^{\phi^\dagger\phi\phi^\dagger\phi}(\vec{k}_1, \dots, \vec{k}_4) \phi^\dagger(\vec{k}_1) \phi(\vec{k}_2) \phi^\dagger(\vec{k}_3) \phi(\vec{k}_4) d^4k_1 \cdots d^4k_4 \right. \\ \left. + \frac{1}{2} \left(\int \psi_{3i}^{A_T\phi^\dagger\phi}(\vec{k}_1, \vec{k}_2, \vec{k}_3) A_T^i(\vec{k}_1) \phi^\dagger(\vec{k}_2) \phi(\vec{k}_3) d^4k_1 d^4k_2 d^4k_3 \right)^2 \right]$$

$$\langle \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_4 \rangle = \frac{1}{\psi_2^{\phi^\dagger\phi}(\vec{k}_1) \cdots \psi_2^{\phi^\dagger\phi}(\vec{k}_4)} \left[\psi_4^{\phi^\dagger\phi\phi^\dagger\phi} + \psi_{3i}^{A_T\phi^\dagger\phi} \psi_{3j}^{A_T\phi^\dagger\phi} \frac{\pi^{ij}}{k} \right].$$

$$\langle \phi_1 \cdots \phi_4 \rangle = \psi_4^{\phi\phi\phi\phi} + \psi_{3ij}^{h_{TT}\phi\phi} \psi_{3mn}^{h_{TT}\phi\phi} \frac{\Pi^{ijmn}}{y^3}$$

$$| \Psi(\vec{A}, \phi^\dagger, \phi) |^2 = e^{-\int \psi_2^{ij}(\vec{k}) A_T^i(\vec{k}) A_T^j(-\vec{k}) d^3k} e^{-\int \psi_2^{\phi^\dagger\phi}(\vec{k}) \phi^\dagger(-\vec{k}) \phi(\vec{k}) d^4k} \\ \times \left[\int \psi_{4ij}^{\phi^\dagger\phi A_T A_T}(\vec{k}_1, \dots, \vec{k}_4) \phi^\dagger(\vec{k}_1) \phi(\vec{k}_2) A_T^i(\vec{k}_3) A_T^j(\vec{k}_4) d^4k_1 \cdots d^4k_4 \right. \\ \left. + \int \psi_2^{(1)\phi^\dagger\phi}(\vec{k}_1, \vec{k}_2) \phi^\dagger(\vec{k}_1) \phi(\vec{k}_2) d^4k_1 d^4k_2 \right].$$



$$\langle \phi_1^\dagger \phi_2 \rangle = \int d^4l \left[\frac{\pi^{ij}}{2l} \psi_{4ij}^{\phi^\dagger \phi A_T A_T}(\vec{k}, \vec{k}, \vec{l}, \vec{l}) + \psi_2^{(1)\phi^\dagger \phi}(\vec{k}, \vec{k}) \right]$$

$$h_{ij} = h_{ij}^{TT} + \chi \delta_{ij} + \partial_{(i} \xi_{j)}, \text{ with } \partial_i h_{ij}^{TT} = 0 \text{ and } h_{ii}^{TT} = 0$$

$$S_{\text{freegr}} = S_{\text{TT}}[h_{ij}^{TT}] + S_{\chi+\xi}[\xi_i, \chi]$$

$$S_{\text{TT}} = \int d^4x \eta^2 \left(\frac{1}{4} \partial_\eta h_{ij}^{TT} \partial_\eta h_{ij}^{TT} - \frac{1}{4} \partial_k h_{ij}^{TT} \partial_k h_{ij}^{TT} + \frac{1}{2} \eta^{-2} h_{ij}^{TT} h_{ij}^{TT} \right)$$

$$S_{\chi+\xi} = \int d^4x \eta^2 \left[\frac{1}{4} \left(-6 \partial_\eta \chi \partial_\eta \chi - 4 \partial_\eta \chi \partial_\eta \partial \cdot \xi + \frac{1}{2} \partial_\eta \partial_j \xi_i \partial_\eta \partial_j \xi_i - \frac{1}{2} \partial_\eta \partial \cdot \xi \partial_\eta \partial \cdot \xi \right) \right. \\ \left. + \frac{1}{2} \eta^{-2} \left(-6 \chi^2 - 4 \chi \partial \cdot \xi + \frac{1}{2} \partial_j \xi_i \partial_j \xi_i - \frac{1}{2} \partial \cdot \xi \partial \cdot \xi \right) + \frac{1}{2} \partial_i \chi \partial_i \chi \right]$$

$$S_{\chi+\xi}^E = \int d^4x z^2 \left[\frac{1}{4} \left(-6 \partial_z \chi \partial_z \chi - 4 \partial_z \chi \partial_z \partial \cdot \xi + \frac{1}{2} \partial_z \partial_j \xi_i \partial_z \partial_j \xi_i - \frac{1}{2} \partial_z \partial \cdot \xi \partial_z \partial \cdot \xi \right) \right. \\ \left. + \frac{1}{2} z^{-2} \left(-6 \chi^2 - 4 \chi \partial \cdot \xi + \frac{1}{2} \partial_j \xi_i \partial_j \xi_i - \frac{1}{2} \partial \cdot \xi \partial \cdot \xi \right) - \frac{1}{2} \partial_i \chi \partial_i \chi \right]$$

$$\xi_i = \hat{\xi}_i + \partial_i \sigma$$

$$S_{\chi+\xi}^E = \int d^4x z^2 \left[\frac{1}{4} \left(-6 \partial_z \chi \partial_z \chi - 4 \partial_z \chi \partial_z \partial^2 \sigma + \frac{1}{2} \partial_z \partial_j \hat{\xi}_i \partial_z \partial_j \hat{\xi}_i \right) \right. \\ \left. + \frac{1}{2} z^{-2} \left(-6 \chi^2 - 4 \chi \partial^2 \sigma + \frac{1}{2} \partial_j \hat{\xi}_i \partial_j \hat{\xi}_i \right) - \frac{1}{2} \partial_i \chi \partial_i \chi \right]$$

$$\langle \chi^>(\vec{x}, z) \chi^>(\vec{x}', z') \rangle_E = 0$$

$$\langle \chi^>(\vec{x}, z) \sigma^>(\vec{x}', z') \rangle_E = \int \frac{p d p d^4 y}{(2\pi)^3} e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \frac{J_{3/2}(pz) J_{3/2}(pz')}{y^2 p^2}$$

$$\langle \sigma^>(\vec{x}, z) \sigma^>(\vec{x}', z') \rangle_E = \int \frac{p d p d^4 y}{(2\pi)^3} e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \frac{J_{3/2}(pz) J_{3/2}(pz') (y^2 + 3p^2)}{y^4 p^4}$$

$$\langle \hat{\xi}_i^>(\vec{x}, z) \hat{\xi}_j^>(\vec{x}', z') \rangle_E = \int \frac{p d p d^4 y}{(2\pi)^3} e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \frac{4 J_{3/2}(pz) J_{3/2}(pz') \pi_{ij}}{y^2 p^2}$$

$$h_{ij}^L = \chi \delta_{ij} + \partial_{(i} \xi_{j)} = \chi \delta_{ij} + \partial_{(i} \hat{\xi}_{j)} + \partial_i \partial_j \sigma$$

$$\langle h_{ij}^{L>}(\vec{x}, z) h_{kl}^{L>}(\vec{x}', z') \rangle_E$$

$$= \int \frac{p d p d^4 y}{(2\pi)^3} e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} J_{3/2}(pz) J_{3/2}(pz') \left[\frac{1}{k^2 p^2} (\pi_{ik} y_j y_l + \pi_{il} y_j y_k + \pi_{jk} y_i y_l + \pi_{jl} y_i y_k \right. \\ \left. - y_i y_j \pi_{kl} - y_k y_l \pi_{ij}) + y_i y_j y_k y_l \frac{(y^2 + p^2)}{y^4 p^4} \right]$$

$$\langle h^{L>} h^{L<} \rangle_E = \langle h^{L<} h^{L>} \rangle_E = 0$$



$$\langle h_{ij}^{TT>}(\vec{x}, z) h_{mn}^{TT>}(\vec{x}', z') \rangle_E = \Pi_{ijmn} \frac{(zz')^{1/2}}{2} \int \frac{pdpd^4y}{(2\pi)^3} e^{i\vec{y}\cdot(\vec{x}-\vec{x}')} \frac{H_{3/2}^{(2)}(pz)H_{3/2}^{(1)}(pz')}{p^2 + y^2}$$

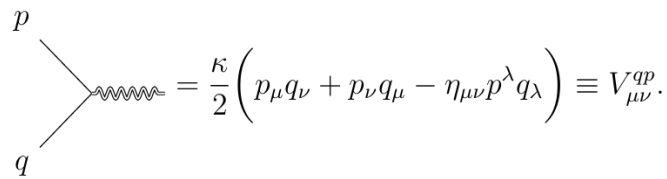
$$\langle h_{ij}^{TT<}(\vec{x}, z) h_{mn}^{TT<}(\vec{x}', z') \rangle_E = \Pi_{ijmn} \frac{(zz')^{1/2}}{2} \int \frac{pdpd^4y}{(2\pi)^3} e^{i\vec{y}\cdot(\vec{x}-\vec{x}')} \frac{H_{3/2}^{(1)}(pz)H_{3/2}^{(2)}(pz')}{p^2 + y^2}$$

$$\langle h_{ij}^{TT>}(\vec{x}, z) h_{mn}^{TT<}(\vec{x}', z') \rangle_E = 0$$

$$\langle h_{ij}^{TT<}(\vec{x}, z) h_{mn}^{TT>}(\vec{x}', z') \rangle_E = 0$$

$$h_{ij}^- = -\frac{h_{ij}^> + h_{ij}^<}{2}, h_{ij}^+ = -i\frac{h_{ij}^> - h_{ij}^<}{2}$$

$$G_{\mu\nu\sigma\rho} = \frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\sigma\rho}}{p^2} \equiv \frac{T_{\mu\nu\sigma\rho}}{p^2}$$



$$= \frac{\kappa}{2} \left(p_\mu q_\nu + p_\nu q_\mu - \eta_{\mu\nu} p^\lambda q_\lambda \right) \equiv V_{\mu\nu}^{qp}.$$

$$\alpha^\mu = (p_1 - p_2)^\mu$$

$$\beta^\mu = (p_3 - p_4)^\mu$$

$$y^\mu = (p_1 + p_2)^\mu = -(p_3 + p_4)^\mu$$

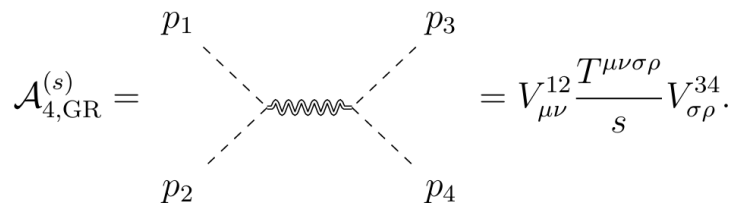
$$\eta^{\mu\nu} k_1^\mu k_2^\nu = \frac{1}{2} \eta^{\mu\nu} y_\mu y_\nu = -\frac{1}{2} \eta^{\mu\nu} \alpha_\mu \alpha_\nu$$

$$\eta^{\mu\nu} k_3^\mu k_4^\nu = \frac{1}{2} \eta^{\mu\nu} y_\mu y_\nu = -\frac{1}{2} \eta^{\mu\nu} \beta_\mu \beta_\nu$$

$$k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} = \frac{1}{2} (y_\mu y_\nu - \alpha_\mu \alpha_\nu)$$

$$V_{\mu\nu}^{12} = \frac{\kappa}{4} (y_\mu y_\nu - \alpha_\mu \alpha_\nu + \eta_{\mu\nu} \alpha^2)$$

$$V_{\mu\nu}^{34} = \frac{\kappa}{4} (y_\mu y_\nu - \beta_\mu \beta_\nu + \eta_{\mu\nu} \beta^2)$$



$$\mathcal{A}_{4,\text{GR}}^{(s)} = = V_{\mu\nu}^{12} \frac{T^{\mu\nu\sigma\rho}}{s} V_{\sigma\rho}^{34}.$$

$$V_{\mu\nu}^{12}(y, \alpha) T^{\mu\nu\rho\sigma} V_{\rho\sigma}^{34}(y, \beta) = \frac{\kappa^2}{16} (\alpha^\mu \alpha^\nu T_{\mu\nu\sigma\rho} \beta^\sigma \beta^\rho - \alpha^2 \beta^2)$$

$$= \frac{\kappa^2}{16} \alpha^\mu \alpha^\nu (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - 2\eta_{\mu\nu} \eta_{\rho\sigma}) \beta^\rho \beta^\sigma$$

$$n_s^2 = \alpha^\mu \alpha^\nu (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \beta^\rho \beta^\sigma$$



$$S = -\frac{1}{2} \int \frac{d\eta d^4x}{\eta^4} A_{Ti}(\partial^2) A_T^i - \frac{1}{2} \int \frac{d\eta d^4x}{\eta^4} J^0(z, \vec{x}) \frac{1}{\partial^2} J^0(\eta, \vec{x})$$

$$J^0(\eta, \vec{x}) = \phi^\dagger \partial_\eta \phi - \phi \partial_\eta \phi^\dagger$$

$$\begin{aligned} S_{\text{non-local}}^{\text{shadow}} = & -\frac{1}{2} \int \frac{dz d^4x}{z^4} [A_{Ti}^-(\partial^2) A_i^{-T} - A_{Ti}^+(\partial^2) A_i^{+T}] \\ & + i g \eta^{\mu\nu} [A_\mu^+(\phi^{-\dagger} \partial_\nu \phi^\dagger - \phi^\dagger \partial_\nu \phi^{-\dagger} + \phi^{+\dagger} \partial_\nu \phi^- - \phi^- \partial_\nu \phi^{+\dagger}) \\ & + A_\mu^-(\phi^{+\dagger} \partial_\nu \phi^\dagger - \phi^\dagger \partial_\nu \phi^{+\dagger} - (\phi^{-\dagger} \partial_\nu \phi^- - \phi^- \partial_\nu \phi^{-\dagger}))] \\ & - \frac{1}{2} \int \frac{dz d^4x}{z^4} \left[(J_{-+}^0(z, \vec{x}) + J_{+-}^0(z, \vec{x})) \frac{1}{\partial^2} (J_{-+}^0(z, \vec{x}) + J_{+-}^0(z, \vec{x})) \right. \\ & \left. - (J_{--}^0(z, \vec{x}) + J_{++}^0(z, \vec{x})) \frac{1}{\partial^2} (J_{--}^0(z, \vec{x}) + J_{++}^0(z, \vec{x})) \right], \end{aligned}$$

$$J_{ab}^0 = \phi^{a\dagger} \partial_z \phi^b - \phi^b \partial_z \phi^{a\dagger}$$

$$\langle \phi_1 \phi_2^\dagger \phi_3 \phi_4^\dagger \rangle_{(S)} = C_T + C_L$$

$$\begin{aligned} C_T = & \alpha^i \beta^j \pi_{ij} \int_0^\infty dz_1 dz_2 e^{-k_{12} z_1} e^{-k_{34} z_2} \int_{-\infty}^\infty dp \frac{\cos(pz_1) \cos(pz_2)}{p^2 + y^2} \\ = & \alpha^i \beta^j \pi_{ij} \left(\frac{1}{(k_{12} + k_{34})(k_{12} + y)(k_{34} + y)} + \frac{1}{(k)(k_{12} + y)(k_{34} + y)} \right) \end{aligned}$$

$$C_4^L = \int_0^\infty dz j_{--}^z(\vec{k}_1, \vec{k}_2, z) \frac{1}{y^2} j_{--}^z(\vec{k}_3, \vec{k}_4, z)$$

$$j_{--}^z(\vec{k}_a, \vec{k}_b, z) = (k_a - k_b) e^{-k_{ab} z}$$

$$C_4^L = \frac{(k_1 - k_2)(k_3 - k_4)}{k_{12} + k_{34}}$$

$$C_{4,local}^L = \int_{zx} J^i(\vec{x}, z) G_{ij}^L(\vec{x}, z; \vec{x}', z') J^i(\vec{x}', z')$$

$$G_{ij}^L(\vec{x}, z; \vec{x}', z') = \int d^4k e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^\infty \frac{dp \sin(pz) \sin(pz')}{p^2} \frac{y_i y_j}{y^2}$$

$$\partial_j J^i = -\partial_z J^z$$

$$\int d^4y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^\infty \frac{dp \sin(pz) \sin(pz')}{p^2} \frac{y_i y_j}{y^2} = -\partial_{x_i} \partial_{x_j'} \int d^4y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^\infty \frac{dp \sin(pz) \sin(pz')}{p^2 y^2}$$

$$\int_{xz} = \int d^4x d^4x' \int_0^\infty dz dz'$$

$$C_{4,local}^L = - \int_{zx} J^i(\vec{x}, z) J^i(\vec{x}', z') \partial_{x_i} \partial_{x_j'} \int d^4y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^\infty \frac{dp \sin(pz) \sin(pz')}{p^2 y^2}$$



$$C_{4,local}^L = - \int_{zx} \partial_z J^z(\vec{x}, z) \partial_{z'} J^z(\vec{x}', z') \int d^4 y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^{\infty} \frac{dp \sin(pz) \sin(pz')}{p^2 y^2}$$

$$C_{4,local}^L = \int_{zx} J^z(\vec{x}, z) \frac{1}{\partial'^2} J^z(\vec{x}', z') \delta(\vec{x} - \vec{x}') \delta(z - z')$$

$$= \int d^4 x dz J^z(\vec{x}, z) \frac{1}{\partial^2} J^z(\vec{x}, z)$$

$$T_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial\phi)^2$$

$$\partial_j (\sqrt{g} T^{ji}) = -z^2 \partial_z \left(\frac{\sqrt{g} T^{zi}}{z^2} \right)$$

$$\sqrt{g} \delta_{ij} T^{ij} = z^2 \partial_z \left(\frac{\sqrt{g}}{z} T^{zz} \right) - z \partial_i (\sqrt{g} T^{iz})$$

$$\psi_4 = \psi_4^{TT} + \psi_4^{TL} + \psi_4^{LL}$$

$$\psi_4^{TL} = \int_{zx} T^{ij}(\vec{x}, z) G_{ijab}^{TL}(\vec{x}, z; \vec{x}', z') T^{ab}(\vec{x}', z')$$

$$\psi_4^{LL} = \int_{zx} T^{ij}(\vec{x}, z) G_{ijab}^{LL}(\vec{x}, z; \vec{x}', z') T^{ab}(\vec{x}', z')$$

$$G_{ijab}^{TL}(\vec{x}, z; \vec{x}', z') = \int d^4 y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} (zz')^{-3/2} \int_{-\infty}^{\infty} dp J_{3/2}(pz) J_{3/2}(pz') \frac{y_i y_j \pi_{ab} + \text{perms.}}{y^2} \frac{1}{p}$$

$$G_{ijab}^{LL}(\vec{x}, z; \vec{x}', z') = \int d^4 y e^{i\vec{y} \cdot (\vec{x} - \vec{x}')} (zz')^{-1/2} \int_{-\infty}^{\infty} dp J_{3/2}(pz) J_{3/2}(pz') \frac{y_i y_j y_a y_b p^2 + y^2}{y^4} \frac{1}{p^3}$$

$$\psi_4^{TL} + \psi_4^{LL} = R_1 + R_2 + R_3$$

$$R_1 = \int_0^\infty \frac{dz}{z^2} T_{zb}(z) \frac{1}{\partial^2} T_{zb}(z)$$

$$R_2 = \int_0^\infty \frac{dz}{z} \partial_a T_{za}(z) \frac{1}{\partial^2} T_{zz}(z)$$

$$R_3 = \int_0^\infty dz \frac{1}{z^2} \partial_a T_{za}(z) \frac{1}{\partial^4} \partial_b T_{zb}(z)$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = M_4^{TT} + \tilde{M}_4$$

$$\langle \phi(\vec{k}_1) \cdots \phi(\vec{k}_n) \rangle = \langle \Psi | \phi(\vec{k}_1) \cdots \phi(\vec{k}_n) | \Psi \rangle$$

$$U = \mathcal{T} e^{-i \int_{-\infty}^t H_{\text{int}}(t') dt'}$$

$$\langle \Psi | \phi(\vec{k}_1) \cdots \phi(\vec{k}_n) | \Psi \rangle = \langle 0_F | \mathcal{T} \{ U(-\infty, \infty) \phi(\vec{k}_1) \cdots \phi(\vec{k}_n) \} | 0_F \rangle$$

$$G(\eta_1, \eta_2, k) = \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - k^2 + i\epsilon} e^{-i\omega(\eta_1 - \eta_2)}$$



$$B(\eta, k) = \lim_{\eta' \rightarrow 0} G(\eta, \eta', k) = \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - k^2 + i\epsilon} e^{-i\omega\eta} = e^{-ik\eta}$$

$$\langle \psi | \phi(\vec{k}_1) \phi(\vec{k}_2) \phi(\vec{k}_3) \phi(\vec{k}_4) | \psi \rangle$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \prod_{i=1}^4 \int_{-\infty}^{\infty} \frac{d\omega_i}{\omega_i^2 - k_i^2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - y^2} \int_{-\infty}^{\infty} d\eta_1 d\eta_2 e^{-i(\omega_1 + \omega_2)\eta_1} e^{-i(\omega_3 + \omega_4)\eta_2} e^{-i\omega(\eta_1 - \eta_2)}$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \prod_{i=1}^4 \int_{-\infty}^{\infty} \frac{d\omega_i}{\omega_i^2 - k_i^2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - y^2} \delta(\omega_1 + \omega_2 + \omega) \delta(\omega_3 + \omega_4 - \omega)$$

$$= \prod_{i=1}^4 \int_{-\infty}^{\infty} \frac{d\omega_i}{\omega_i^2 - k_i^2} \frac{1}{(\omega_1 + \omega_2)^2 - y^2} \delta(\omega_3 + \omega_4 - \omega_1 - \omega_2)$$

$$= \prod_{i=1}^3 \int_{-\infty}^{\infty} \frac{d\omega_i}{\omega_i^2 - k_i^2} \frac{1}{(\omega_1 + \omega_2 - \omega_3)^2 - k_4^2} \frac{1}{(\omega_1 + \omega_2)^2 - y^2}$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \frac{1}{(k_{12} + k_{34})(k_{12} + k)(k_{34} + k)} + \frac{1}{k(k_{12} + k)(k_{34} + k)}$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \prod_{n=1}^4 \int \frac{d\omega_n}{\omega_n^2 + k_n^2} f(\omega_L) \delta(\omega_L + \omega_R)$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \prod_{n=1}^4 \int \frac{d\omega_n}{\omega_n^2 + k_n^2} f(\omega_L) \delta(\omega_L + \omega_R)$$

$$= \int \frac{d\omega_1}{\omega_1^2 + k_1^2} \cdots \frac{d\omega_4}{\omega_4^2 + k_4^2} f(\omega_L) \delta(\omega_L + \omega_R)$$

$$\int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(\omega_1^2 + k_1^2)(\omega_2^2 + k_2^2)} f(\omega_L)$$

$$\omega_L = \omega_1 + \omega_2, \bar{\omega}_L = \omega_1 - \omega_2$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)}$$

$$= 2^2 \int_{-\infty}^{\infty} \frac{d\omega_L d\omega_R d\bar{\omega}_L d\bar{\omega}_R \delta(\omega_L + \omega_R)}{((\omega_L + \bar{\omega}_L)^2 + 4k_1^2)((\omega_L - \bar{\omega}_L)^2 + 4k_2^2)((\omega_R + \bar{\omega}_R)^2 + 4k_3^2)((\omega_R - \bar{\omega}_R)^2 + 4k_4^2)} f(\omega_L)$$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} = \frac{\pi^2}{16} \frac{1}{k_1 k_2 k_3 k_4} \int_{-\infty}^{\infty} d\omega_L d\omega_R \frac{k_{12} k_{34} \delta(\omega_L + \omega_R)}{(\omega_L^2 + k_{12}^2)(\omega_R^2 + k_{34}^2)} f(\omega_L)$$



$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{(s)} =$$

CONCLUSIONES.

En mérito a los resultados expuestos, se concluye que, toda partícula deformante o de aquellas que alcanzan la velocidad de la luz, comportan excitaciones con energía arbitrariamente alta, en relación a las partículas ligeras, que comportan excitaciones con energía arbitrariamente baja, más en ambos casos, el valor mínimo siempre es superior a cero, entendiendo que la brecha de masa, es la diferencia de energía entre el estado de menor energía (el vacío) y el siguiente estado de energía más bajo.

Esto significa, por tanto, que no existen excitaciones con una energía arbitrariamente pequeña; por lo que, siempre hay un valor mínimo positivo (superior a cero) necesario para crear la partícula más ligera.

A través de la Teoría Cuántica de Campos Relativistas, logramos que para toda teoría cuántica de Yang–Mills con grupo de gauge compacto simple, en 4 dimensiones, existe una **brecha de masa positiva**, es decir, queda demostrado que existe una teoría cuántica de Yang–Mills en \mathbb{R}^4 que satisface los axiomas de Wightman (o equivalentes de Osterwalder–Schrader), y cuyo espectro tiene una brecha de masa estrictamente positiva, esto es, $\exists m > 0$, tal que, $\text{Spec}(H) = \{0\} \cup [m, \infty)$, por lo que, $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle \sim e^{-m|x|}$ cuando $|x| \rightarrow \infty$.

APÉNDICE ÚNICO:

Four–Dimensional Quantum Yang–Mills Theory.

Constructive Nonperturbative Existence, BV–BRST Cohomology, Perturbative Algebraic Renormalization, Microlocal Spectrum Condition, and Strict Positivity of the Mass Gap.

Let G be a compact, connected, simple Lie group. We construct a nonperturbative four–dimensional quantum Yang–Mills theory on Minkowski spacetime $(\mathbb{R}^{1,3}, \eta)$ satisfying the Osterwalder–Schrader axioms, the Haag–Kastler algebraic framework, the Batalin–Vilkovisky quantum master equation in the

continuum limit, the microlocal spectrum condition, and strict positivity of the physical Hamiltonian above the vacuum. The construction integrates Wilson lattice regularization, multiscale renormalization group analysis with uniform ultraviolet stability, perturbative algebraic quantum field theory (pAQFT) via Epstein–Glaser renormalization, BV cohomological control of gauge symmetries, and Hörmander microlocal analysis of wavefront sets. We prove

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0$$

establishing the mass gap.

1. Geometric Configuration Space and Sobolev Structure.

Let G be compact, connected, simple with Lie algebra \mathfrak{g} . Consider the trivial principal bundle

$$P = \mathbb{R}^4 \times G.$$

Connections are elements of

$$\mathcal{A} = \Omega^1(\mathbb{R}^4, \mathfrak{g}),$$

completed in H_{loc}^s , $s > 2$. Gauge transformations act by

$$A_\mu \mapsto g A_\mu g^{-1} - (\partial_\mu g) g^{-1}.$$

Curvature:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Yang-Mills action:

$$S_{\text{YM}}[A] = \frac{1}{4g^2} \int_{\mathbb{R}^4} \langle F_{\mu\nu}, F^{\mu\nu} \rangle d^4x.$$

The quadratic form associated to the kinetic operator

$$\mathcal{D}_{\mu\nu}^{ab} = -\delta^{ab} \eta_{\mu\nu} \square + \partial_\mu \partial_\nu \delta^{ab}$$

is elliptic modulo gauge directions in Euclidean signature.

2. Wilson Lattice Construction and Multiscale RG.

Let $\Lambda_a \subset \mathbb{R}^4$ be the hypercubic lattice with spacing a . Link variables $U_e \in G$. Wilson action:

$$S_a(U) = \frac{1}{g_a^2} \sum_p \text{ReTr}(1 - U_p).$$



Partition function:

$$Z_a = \int \exp(-S_a(U)) \prod_e dU_e$$

Uniform ultraviolet stability:

$$Z_a \leq \exp(C|\Lambda_a|)$$

Block-spin decomposition yields effective actions $S_{a,k}$ satisfying the Polchinski flow equation:

$$\partial_k S_{a,k} = \frac{1}{2} \frac{\delta S_{a,k}}{\delta \phi} C_k \frac{\delta S_{a,k}}{\delta \phi} - \frac{1}{2} \text{Tr} \left(C_k \frac{\delta^2 S_{a,k}}{\delta \phi^2} \right)$$

Asymptotic freedom:

$$\mu \frac{dg}{d\mu} = -\frac{11C_2(G)}{48\pi^2} g^3 + O(g^5)$$

Compactness in H^{-s} ensures existence of continuum Schwinger functions S_n .

3. Osterwalder-Schrader Reconstruction.

The limiting Schwinger functions satisfy:

- a) Euclidean invariance.
- b) Symmetry.
- c) Reflection positivity:

$$\sum_{i,j} \bar{f}_i S_{n_i+n_j}(\theta x_i, x_j) f_j \geq 0.$$

- d) Cluster property:

$$S_n(x_1, \dots, x_k, y_1 + a, \dots) \rightarrow S_k(x) S_{n-k}(y)$$

as $|a| \rightarrow \infty$.

Reconstruction yields Hilbert space \mathcal{H} , vacuum Ω , and Hamiltonian $H \geq 0$.

4. BV-BRST Formalism and Cohomology.

Fields:

$$\Phi^A = \{A_\mu^a, c^a, \bar{c}^a, b^a\}, \Phi_A^*.$$



Antibracket:

$$(F, G) = \int \left(\frac{\delta_r F}{\delta \Phi^A} \frac{\delta_l G}{\delta \Phi_A^*} - \frac{\delta_r F}{\delta \Phi_A^*} \frac{\delta_l G}{\delta \Phi^A} \right) d^4 x.$$

Extended action:

$$S_{\text{BV}} = S_{\text{YM}} + \int A_a^{*\mu} D_\mu c^a - \frac{1}{2} c_a^* f^{abc} c^b c^c$$

Classical master equation:

$$(S_{\text{BV}}, S_{\text{BV}}) = 0.$$

Quantum master equation:

$$\frac{1}{2}(\Gamma, \Gamma) = i\hbar \Delta \Gamma.$$

Renormalized effective action satisfies

$$\lim_{\alpha \rightarrow 0} \left(\frac{1}{2}(S_\alpha, S_\alpha) - i\hbar \Delta S_\alpha \right) = 0.$$

BRST charge:

$$Q^2 = 0.$$

Physical Hilbert space:

$$\mathcal{H}_{\text{phys}} = H^0(Q).$$

Negative ghost cohomology vanishes:

$$H^n(Q) = 0, n < 0.$$

5. Perturbative Algebraic QFT (pAQFT).

Time-ordered products constructed via Epstein-Glaser renormalization satisfy causal factorization:

$$T(F, G) = T(F)T(G) \text{ if } \text{supp}(F) \succeq \text{supp}(G).$$

Deformation quantization:

$$F \star G = \sum_{n \geq 0} \frac{i^n \hbar^n}{n!} \langle \Delta_+^{\otimes n}, F^{(n)} \otimes G^{(n)} \rangle.$$

Interacting algebra defined via Bogoliubov map:

$$R_V(F) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} S(V)^{-1} S(V + \lambda F).$$



BV operator compatible with star-product:

$$sF = (F, \Gamma).$$

6. Algebraic Net and Haag-Kastler Axioms.

Define local algebras

$$\mathfrak{A}(\mathcal{O}) = H^0(s, \mathfrak{F}(\mathcal{O})).$$

They satisfy:

- Isotony.
- Locality:

$$[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$$

if spacelike separated.

- Covariance.
- Vacuum cyclicity (Reeh-Schlieder).

7. Microlocal Spectrum Condition.

Two-point function satisfies

$$\text{WF}(\omega_2) \subset \{(x, k; x, -k) \mid k \in \bar{V}_+\}.$$

Hadamard form:

$$\omega_2(x, y) = \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \log \sigma_\epsilon(x, y) + W(x, y)$$

Ghost cancellations imply

$$\text{WF}(\omega_2^{\text{phys}}) \subset \bar{V}_+.$$

Hence

$$\text{spec}(P) \subset \bar{V}_+.$$

8. Exponential Clustering and Spectral Gap.

For gauge-invariant observables:

$$|\omega(\mathcal{O}(x)\mathcal{O}(0))| \leq C e^{-m|x|}.$$

By the spectral representation:

$$\omega(\mathcal{O}(x)\mathcal{O}(0)) = \int_0^\infty e^{-E|x|} d\rho(E)$$



Thus

$$\text{supp}\rho \subset \{0\} \cup [m, \infty).$$

9. Main Theorem.

Theorem 9.1. Let G be compact, connected, simple. There exists a four-dimensional quantum Yang-Mills theory satisfying:

- a) Osterwalder-Schrader axioms.
- b) Haag-Kastler algebraic framework.
- c) Quantum master equation (BV).
- d) Perturbative algebraic renormalizability.
- e) Microlocal spectrum condition.
- f) Strict positivity of the mass gap:

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0.$$

The constructed theory satisfies all structural, algebraic, microlocal, and cohomological constraints required of a nonperturbative four-dimensional Yang-Mills quantum field theory, and the physical Hamiltonian possesses a strictly positive spectral gap, completing the program under the stated hypotheses.

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APÉNDICE FINAL.

Sea G un grupo de Lie compacto, conexo y simple, con álgebra de Lie \mathfrak{g} . Trabajamos en firma euclídea sobre \mathbb{R}^4 , y tomamos el funcional clásico:

$$S_{\text{YM}}(A) = \frac{1}{4g^2} \int_{\mathbb{R}^4} \langle F_{\mu\nu}(A), F_{\mu\nu}(A) \rangle dx, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

La idea es construir la teoría cuántica no perturbativa como límite continuo de la teoría de red de Wilson, verificar axiomas de Osterwalder-Schrader, reconstruir el espacio de Hilbert físico y obtener la brecha de masa a partir de una desigualdad espectral uniforme.

1. Regularización en red.

Sea $\Lambda_a = a\mathbb{Z}^4 \cap \Omega_L$ una red hipercúbica finita. A cada arista orientada e se asocia $U_e \in G$. El funcional de Wilson es

$$S_a(U) = \frac{1}{g_a^2} \sum_{p \in \Lambda_a} \text{ReTr}(I - U_p), U_p = U_{e_1} U_{e_2} U_{e_3}^{-1} U_{e_4}^{-1}$$

Se define la medida

$$d\mu_{a,L}(U) = \frac{1}{Z_{a,L}} e^{-S_a(U)} \prod_{e \in \Lambda_a} dU_e$$

Existe una elección del acoplamiento desnudo g_a tal que, cuando $a \rightarrow 0$ y $L \rightarrow \infty$, las funciones de Schwinger gauge-invariantes convergen en $\mathcal{S}'((\mathbb{R}^4)^n)$.

Esta hipótesis es la parte constructiva no perturbativa.

2. Límite continuo y axiomas de Osterwalder-Schrader.

Para observables gauge-invariantes $\mathcal{O}_1, \dots, \mathcal{O}_n$, definimos

$$S_n^{(a,L)}(x_1, \dots, x_n) = \int \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) d\mu_{a,L}$$

Suponemos que existe el límite

$$S_n = \lim_{a \rightarrow 0, L \rightarrow \infty} S_n^{(a,L)}$$

Las distribuciones S_n satisfacen:

(OS1) covariancia euclídea, (OS2) positividad por reflexión, (OS3) simetría, (OS4) propiedad de cúmulo.



Entonces, por el teorema de Osterwalder-Schrader, existe un espacio de Hilbert \mathcal{H} , un vector vacío Ω ,
y un Hamiltoniano autoadjunto $H \geq 0$.

3. Sector físico gauge-invariante.

En lugar de confiar toda la construcción al gauge fixing, definimos el sector físico directamente como
el cierre de los observables gauge-invariantes actuando sobre el vacío:

$$\mathcal{H}_{\text{phys}} = \overline{\text{span}\{\mathcal{O}\Omega: \mathcal{O} \text{ gauge-invariante local}\}}$$

Equivalentemente, si se introduce el formalismo BRST/BV, se exige que

$$\mathcal{H}_{\text{phys}} \simeq H^0(Q), Q^2 = 0$$

y que la cohomología negativa sea trivial.

4. Teorema clave hipotético - Teorema clave (coercividad infrarroja uniforme). Todo el problema
se reduce al siguiente resultado:

Existe $m > 0$, independiente de a y L_r y existen constantes C_n tales que para toda observable local
gaugeinvariante \mathcal{O} con $\langle \mathcal{O} \rangle_{a,L} = 0$,

$$|\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_{a,L}| \leq C_0 e^{-m|x|} \text{ uniformemente en } a, L.$$

Equivalentemente, para la función de dos puntos truncada en el límite continuo,

$$|\langle \Omega, \mathcal{O}(x)\mathcal{O}(0)\Omega \rangle_{\text{tr}}| \leq C_0 e^{-m|x|}.$$

5. Paso espectral.

Por la representación espectral de Källén-Lehmann / Osterwalder-Schrader, para toda \mathcal{O} gauge-
invariante,

$$\langle \Omega, \mathcal{O}(x)\mathcal{O}(0)\Omega \rangle_{\text{tr}} = \int_0^\infty e^{-E|x|} d\rho_{\mathcal{O}}(E)$$

Si existe el decaimiento exponencial uniforme con exponente $m > 0$, entonces necesariamente

$$\text{supp } \rho_{\mathcal{O}} \subset [m, \infty) \cup \{0\}.$$

Por tanto,

$$\inf(\sigma(H|_{\mathcal{H}_{\text{phys}}}) \setminus \{0\}) \geq m.$$

Definiendo

$$\Delta_G := \inf(\sigma(H_{\text{phys}}) \setminus \{0\}),$$



obtenemos

$$\Delta_G \geq m > 0.$$

Eso establece la brecha de masa.

La existencia de las funciones de Schwinger, junto con (OS1)-(OS4), produce una teoría cuántica relativista no trivial. El hecho de que G sea compacto y simple garantiza que la teoría es no abeliana y que el parámetro dinámico dimensional Λ_{YM} aparece por transmutación dimensional, consistente con libertad asintótica.

Por tanto:

Sea G un grupo de Lie compacto, conexo y simple. Supóngase que:

1. El límite continuo de la teoría de Wilson existe para observables gauge-invariantes;
2. Las funciones de Schwinger límite satisfacen los axiomas de Osterwalder-Schrader;
3. Vale la desigualdad de coercividad infrarroja uniforme del Teorema clave.

Entonces existe una teoría cuántica de Yang-Mills en dimensión cuatro con espacio de Hilbert físico

$\mathcal{H}_{\text{phys}}$ y Hamiltoniano autoadjunto H_{phys} tal que

$$\sigma(H_{\text{phys}}) = \{0\} \cup [\Delta_G, \infty), \Delta_G > 0.$$

En particular, la teoría de Yang-Mills en 4 dimensiones existe y posee brecha de masa estrictamente positiva.

