Ciencia Latina Revista Científica Multidisciplinar
Noviembre-Diciembre, 2023, Volumen 7, Número 6
https://doi.org/10.37811/cl_rcm.v7i6.9191

An Approach to the State of the art of Variational Thought in First Grades of Schooling

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ABSTRACT

This paper presents a retrospective and documentary study on elements associated with variational thinking, with the purpose of establishing how variational thinking has been developed in children in the first grades of school from the approach of TO. First, a review is made of some of the central categories for the Theory of Objectification (TO), with the purpose of analyzing subsequent research. Then, the main research developed on variational thinking in the early grades of schooling is described, from a Colombian and international perspective. Subsequently, some elements that are considered important to be considered in future research are proposed, in the light of the TO categories. Finally, some elements are proposed that show the importance of studying variational thinking from the early grades of schooling, as well as some elements to consider for such a study.

Keywords: variational thinking; functional thinking; objectification theory; schools

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Una Aproximación al Estado del arte del Pensamiento Variacional en Primeros Grados de Escolaridad

RESUMEN

En este documento se presenta un estudio retrospectivo y documental sobre elementos asociados al pensamiento variacional, con la finalidad de establecer cómo se ha desarrollado el pensamiento variacional en niños de primeros grados de escolaridad desde el enfoque de la TO. En un primer momento, se hace un recorrido por algunas de las categorías centrales para la Teoría de la objetivación (TO), con el propósito de analizar las investigaciones posteriores. Luego, se describen las principales investigaciones desarrolladas sobre pensamiento variacional en los primeros grados de escolaridad, desde una perspectiva colombiana e internacional, Posteriormente, se plantean algunos elementos que se consideran importantes tener en cuenta en futuras investigaciones, a la luz de las categorías de la TO. Finalmente, se proponen algunos elementos que dan cuenta de la importancia de estudiar el pensamiento variacional desde primeros grados de escolaridad, así como algunos elementos a tener en cuenta para dicho estudio.

***Palabras Clave*:** pensamiento variacional; pensamiento funcional; teoría de la objetivación; colegios

Artículo recibido 18 noviembre 2023
Aceptado para publicación: 30 diciembre 2023

INTRODUCTION

The MEN- Ministry of National Education- in Colombia, establishes for the curricular guidelines of the area of mathematics, the need to address the different mathematical thinking (metric, spatial, numerical, random and variational), and the relevance of teaching mathematics in a comprehensive way so that students from low grades of schooling, acquire the capacity and ability to pose, reason, analyze, communicate and model problems (Contreras, Martinez and Prada, 2020). According to the MEN, developing "variational thinking is one of the achievements to be reached in basic education, in which two characteristic and relevant elements such as change and variation are interrelated" (MEN, 1998b p. 51).

Different works framed in the perspectives of the early algebra and sociocultural approach, show the development achieved in variational thinking, but also the need to review the forms of variational thinking in the first years of schooling. This is the case of Vasco's proposal, who specifically points out the need to experience the processes of mathematical modeling and variational thinking from the preschool and elementary school levels. He also points out that this path has been developing with the proposal to focus the teacher's attention on the five types of thinking, among which is variational thinking.

The scope that the development of variational thinking can have in the first years of schooling, hand in hand with school algebra, would allow building from an early age some elements of algebra, such as: the concept of variable, the relation of equality in its multiple meanings, the concept of parameter, of unknown and of equation and inequality, among others. In a broader view, it can be seen how this type of thinking also impacts the others: spatial, metric and random; because the study of each of them implies identifying invariant structures in the midst of variation and change, besides the fact that all of them offer tools to model mathematical situations, by means of variation (Obando et al., 2006, p. 17).

We begin with a characterization of some central categories for the Theory of Objectification, which are considered fundamental to be able to establish an analysis between the way in which variational thinking has been working in Colombia and in the international context.

Categories for Objectification Theory

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Learning

In the theory of objectification, learning refers to both knowing and becoming, that is, to know that a student has learned, it is not only limited to asking about the axis of knowledge, but also addresses the axis of being, of becoming (Radford, 2017; 2021). In this sense, it is possible to visualize a significant difference with other cognitive theories, in which learning focuses only on the cognitive and on the mental structures that the student manages to consolidate and leaves aside the ontological. Another characteristic of the category of learning from TO is that it is a social process, in which the student encounters the other, but also with systems of thought that are already in the culture (Vergel & Miranda, 2020).

Thinking

In this same line, TO conceives thought as historical, which means that everything we know and the way we come to know it, bears the trace, the imprint, not only of what we do now and how we do it, but also of what preceding generations did; of a historical intelligence that rests in social practices, institutions, languages, artifacts, books, monuments (Radford, 2017). Thinking is a category mediated by artifacts (objects, instruments, sign systems, etc.), it is these artifacts that make thinking not only materialize on the mental plane, but also on the social plane, in the interaction with the peer and with the teacher, as pointed out by Vigotsky (1981) and quoted by Radford (2006).

Semiotic methods of objectification

With this view of thinking mediated by signs and by students' actions, it is important to account for the role of semiotic means of objectification. One way to understand them is as those signs, tools, artifacts, gestures and rhythm that students and the teacher use to make mathematical objects appear that are general and that by themselves cannot be materialized (Radford, 2008), that is, when the teacher or students point on the board, on the worksheet or when they count to explain a definition or a process, what they are doing is mobilizing semiotic means of objectification. This implies that the processes of objectification, that process of encountering mathematical objects, is closely related to the semiotic means of objectification.

Subject

A subject from TO is characterized by the fact that it is a concrete being, that is to say, of flesh and blood, a being of needs, reflective, who feels and acts, and who is always becoming, always transforming, which makes it an unfinished and unfinishable project of life (Radford, 2017). The satisfaction of those needs mentioned before is going to find it not within himself, but in something that is not him, from dialectical materialism it is proposed that, only by activating, moving, relating with the other together, is how the subject manages to satisfy his needs and that this crucial activation occurs within the framework of activity (Radford, 2018).

Evolution of variational thinking in early school grades

Taking into account that our research interest in cognitive terms is focused on how variational thinking manifests itself in the first grades of school, we propose a brief overview that begins with a review of the first approaches to variational thinking and then studies the category of functional thinking with which this thinking shares elements in mathematical terms. Subsequently, variational thinking is approached from a socio-epistemological perspective, describing the central elements in this theory. Finally, we will give an account of the studies in Colombia that have been concerned with providing a space in primary education for the development of variational thinking

Early approaches to variational thinking

Research on variational thinking in the last forty years has undergone a significant evolution in its objectives, approaches and the very definition of this thinking. The first studies that were found focused on how the use of multiple representations of functions can reduce the appearance of ambiguities in students' understandings of variational thinking, as well as on the translation of representations that students can establish. It was usual for this research moment to use software to construct and interpret the elements of a function (Confrey & Smith, 1991; Earnest & Balti, 2008; Goldenberg, Harvey, Umiker, West & Zodhiates, 1988; Kaput, 1986).

This group of research posits that "representation and symbolization are at the heart of the content of mathematics and are simultaneously at the heart of the cognitions associated with mathematical activity" (Kaput, 1987, p. 22). This way of looking at the representation and symbolization of mathematics is centered on mental processes of representation; no mention is made of how social, ethical, cultural or historical elements influence them; it was not their focus of study.

Functional thinking in early school grades

With the turn of the millennium, the first works are located under the name of functional thinking, which share elements with variational thinking, but are distant in the way in which some categories are conceptualized and conceived. It is at this time of research that the spotlight turns to the development of functional thinking in the early grades of schooling, an element that is of interest to us.

Functional thinking has a representational nature whose center is the relationship between two (or more) variable quantities (Smith, 2008). This author also provides a definition of function as the integration of those representational systems produced by children to show a generalization of a relationship between quantities involved in the task (Smith, 2008). These definitions account for the representational nature of mathematics in general and functional thinking in particular, from the theoretical framework of these investigations. With the elements cited above, it can be affirmed that these investigations of the 80s, 90s and early 2000s conceive mathematics, variational thinking and its teaching, in terms of multiple representations and that these representations can be transformed among them and this, according to these investigations, constitutes a sign that the student learns. Other works of this era have focused on the most common conceptual obstacles observed in students when they approach function as a concept and not only from the procedural (Carlson & Oehrtman, 2005).

In other works of the time, we begin to talk about functional thinking as a category derived from algebraic thinking (Blanton & Kaput, 2005) or as a path for the development of algebra in the first grades of school (Brizuela et al., 2000; Brizuela & Lara-Roth, 2001; Brizuela, 2004; Carraher et al., 2000; Carraher et al., 2006; Schliemann et al., 2001). Work is developed on algebraic reasoning, specifically on processes such as: symbolizing, finding, describing and justifying relationships between varying quantities. Such processes are central to elementary school mathematics because they lay the conceptual foundation for the more formal functional and variational thinking that occurs in later grades (Blanton & Kaput, 2004; 2005; Warren & Cooper, 2006).

The objective of this research is to determine whether students in the first grades of school can identify functional relationships between quantities and to see how the type of instruction proposed facilitates this process (Warren & Cooper, 2006). Among their findings is that students identified correspondence relationships and to a lesser extent covariation relationships, which shows that students in early grades can identify relationships between two variables (Morales et al., 2018; Stephens, Isler, Marum, Blanton, Knuth, & Murphy, 2012; Warren & Cooper, 2006; Torres et al., 2021; Cañadas et al., 2016; Cañadas & Fuentes, 2015).

Warren & Cooper (2006) specifically suggest that, although looking for patterns in single-variable data sets is common in elementary curricula, early grade school mathematics should extend this thinking to include variation among data sets. In their words, "the results of this research show that young children are not only able to think functionally, but also to represent this thinking in a way that we never think" (Warren & Cooper, 2006, P. 9).

In this period, functional thinking is conceived as "a process of constructing, describing, and reasoning with and about functions. It involves algebraic thinking that includes the construction of a generalization about variables that are related" (Blanton, 2008, p. 30). From this statement the following questions arise: Is functional thinking a category of algebraic thinking or vice versa, is variational thinking a category of algebraic thinking or vice versa, is functional thinking equivalent to variational thinking? In the following section we will approach the answer to this last question.

Subsequently, there are some studies that also work with the category of functional thinking and whose objective is to determine the appropriate type of instruction, specifically in working with patterns, so that students in the first grades of schooling manage to move from recursive thinking or recursive relations to covariational or correspondence relations (Blanton et al., 2011; Blanton et al., 2015; Cañadas & Fuentes, 2015; Cañadas et. al., 2016; Morales et al., 2018; Pinto et al., 2016; Stephens et al., 2012; Stephens et al., 2017; Syawahid et al., 2020). For works such as those of Torres et al., (2019; 2021), the interest is to identify the nature of the functional structure that early grade school students can establish when working with linear and affine functions to model the solution of the proposed tasks. In addition to determining the type of functional relationships that students can establish, it was of interest for this segment of studies to describe the systems of representations on which students rely to account for their understandings.

More recent studies have focused on investigating on the layers of generality that a student can achieve when doing work with patterns and/or facing the solution of problems that require the identification of patterns (Ayala-Altamirano & Molina, 2021; Stephens, 2017; Zapatera 2005; 2017; Vergel, González, & Miranda, 2020). Such layers of generality range from descriptors such as "continue the sequence" to "find the general rule of verbal form function" (Stephens, 2017; Zapatera, 2017) or moving from expressing indeterminacy by means of deictics, gestures, etc. (factual layer) to doing so by means of key phrases (contextual layer) and finally employing alphanumeric symbols from algebra (symbolic layer) (Radford, 2010).

In the research mentioned above, functional thinking is conceived as a cognitive activity that takes place when working on the relationships between quantities involved in the solution of the proposed task and the joint variation between them (Pinto et al., 2016). They also consider that when the accent of algebraic thinking is placed on functions, then we speak of functional thinking (Cañadas, Brizuela and Blanton, 2016). This definition is of a purely cognitive nature, which goes hand in hand with the research objectives that they set and that inquire about processes of the same type in the student. Due to the theories in which these studies are located and the proposed objectives, it is not of interest to inquire about the subject and its historical-cultural nature, nor about the nature of the ethical relationships that are woven between subjects.

From a sociocultural perspective, functional thinking is not synonymous with variational thinking; the former is based on constructing, describing, reasoning and representing elements that constitute functions, whereas, the latter is conceived as a dynamic way of thinking trying to produce systems that associate internal variables that covary.

According to Morales et al. (2018), in different investigations in this area, two specific purposes have been pursued: 1) to determine the ability of students in the first grades of school to identify functional relationships when solving problems and in turn, 2) to identify the strategies they use to establish such relationships. Thus, functional thinking studies the way in which the subject identifies functions and builds strategies, which leads to characterize him/her as a psychological subject focusing on what is in the subject's knowledge, and on the characterization of the ways in which students build their own knowledge (Radford, 2020a).

Variational thinking from a socio-epistemological perspective.

From the socio-epistemological approach, there are works that address variational thinking. Although these investigations are not developed in the first grades of schooling, but in calculus courses in the first semesters of university, it is considered important to address the conceptualization of variational thinking and to provide some elements in which it distances itself from the theory of objectification, with respect to some central categories. In Cantoral (2005), it is defined as follows:

This variational thought and language studies phenomena of teaching, learning and communication of mathematical knowledge of variation and change in the educational system and in the social environment that accommodates it, pays particular attention to the study of the different cognitive and cultural processes with which people assign and share senses and meanings using for this purpose different variational structures and languages (p. 1).

Regarding the way of conceiving the learning subject (student or teacher), socioepistemology characterizes it as a subject that is "individual, collective or historical" (Cantoral et al., 2014, p. 99). The attribute of individual conflicts with the way the subject is conceived from the theory of objectification because it is seen as inescapably relational, defining its identity from the relationship with the other, with the non-self in a permanent way. In terms of Bakhtin (1997) there is no identity without otherness, that is to say, there is no subject without the face of the other that questions him, that forces him to leave his axiomatic axis to reach a frontier zone (Bakhtin, 1997) from which he can see the other (student, teacher, ideas expressed in a school text, etc.). The student produces forms of thought that are in history and what he produces will also be read, reflected upon and questioned by others, that is to say, the student writes for others.

Finally, we want to talk about the way in which learning is conceptualized from both approaches. In the socioepistemology it is stated that students learn by mathematizing, giving meanings, organizing their reality; all that reality that is imaginable for the students themselves (Cantoral, 2014). In this same line, it states that learning is a complex process of shared meaning that occurs in specific contexts and is therefore a process located in the game of socially shared practices in the world of the learner's experiences.

From this definition it is possible to see how social interaction is seen as a means, as a basis for learning, and is not seen as an end, as an objective in itself of the activity. It can also be affirmed that the social aspect described in socioepistemology seems to be within the framework of institutionalizing meanings and processes, of sharing understandings reached in order to try to make them as similar as possible to each other and to the mathematical object studied; however, how the subject relates to the other and how this relationship of otherness, these ethical relationships that are formed, affect the way in which variational thinking emerges, is not the object of study. For the current research this is a cornerstone. We want to see how these ethical relationships can influence the forms of variational thinking that emerge. It is of interest to us to study classroom interaction, not as a means for learning, but to study the emergence of a community-oriented ethic and how this emergence can give nuances of sophistication to the forms of variational thinking that students encounter.

Variational thinking in the first grades of school, at the Colombian level

Vasco (2002), and the MEN guidelines (1998a) point out the need to experience the processes of mathematical modeling and variational thinking from the preschool and elementary school levels. They also point out that this path has been developing with the proposal to focus the teacher's attention on the five types of thinking, among which is variational thinking.

Vasco (2002), who has been behind important modifications to the curricular standards in the area of mathematics, proposes the following definition of variational thinking:

A dynamic way of thinking, which attempts to mentally produce systems that relate their internal variables in such a way that they covary in a manner similar to the patterns of covariation of quantities of the same or different magnitudes in subprocesses cut out of reality (Vasco, 2002, p. 70).

This researcher locates the purpose of the development of variational thinking in the modeling of everyday processes. In this sense, the focus of this definition is on the generation of mental models that allow the establishment of correlations between variables. This way of conceiving variational thinking in ways of thinking and generating mental models is cognitively strong. This definition does not account for thinking as a practice and/or a social construction.

Obando & Posada (2006) speak of variational thinking in terms of the systematic study of the notion of change and variation present in different contexts: in the natural sciences, in everyday life and in mathematics itself.

Regarding curricular documents in Colombia, the guidelines of the (Ministry of National Education) MEN 1998a propose:

The study of variation can be initiated early in the mathematics curriculum. The meaning and sense about variation can be established from the problematic situations whose scenarios refer to phenomena of change and variation in practical life (Ministry of National Education, p. 73).

In the context of the early grades of schooling, there are works such as Maury, Palmezano & Cárcamo (2012) and Doncel, Rodríguez, & Melo, (2022), who work with fifth grade students, and focus on the type of tasks suitable for promoting the development of variational thinking in fifth grade students. The design was quasi-experimental, which includes a pre-test and a post-test that inquired about elements associated with deduction of variation patterns, calculation of the unknown magnitude, interpretation of variation through graphs, identification of variables, etc.

(2020) and Doncel et al. (2022) identify the need to analyze thinking in general and variational thinking in particular, from a multimodal perspective, that is, taking into account the semiotic resources (signs, gestures, deictics) that the student uses and adopts to account for the forms of thinking that he/she encounters.

CONCLUSIONS

In order to answer the question "Why study variational thinking in the first grades of school? we can base ourselves on what has been described above, and deduce that, on the one hand, variational thinking develops supported by other thoughts and that this also provides a richness of analysis and perspectives regarding mathematical activity, and it is precisely this relationship between thoughts that places variational thinking as an important axis of study to enrich mathematical processes necessary for the student, on the other hand, the study of variation and the behavior of functions in the first grades of school is a topic that has been on the table since the beginning of the century by the international mathematical community and, timidly, at the national level.

There is a marked interest in determining layers of generality regarding functional thinking, but hand in hand with being able to identify and make the unknown explicit, which is always linked to early algebra and cognitive aspects. Other research concentrates its efforts on how to make the transition from the recurrence relations that a student performs to relations more centered on variational thinking, such as correspondence and covariation relations. Finally, a segment of research is identified that is concerned with the type of instruction that is appropriate for fostering the development of variational ways of thinking based on generalization.

Radford (2020b) mentions that both thinking and learning cannot be restricted to the mental plane, but also occur in the social plane; in this sense, language, signaling, rhythm, perception, gestures and in general all forms of relationship to the other (ethics), etc., are elements of thinking in the social plane. are elements of thought in this social plane, therefore, to propose an investigation dismissing these elements of the social plane of thought, especially in students of early ages, is to leave aside a valuable source of information, not only of how the student is encountering these forms of thought and how his semiotic means of objectification become more sophisticated in doing so (going from gestures to key phrases and the use of algebraic symbolism), but also of how subjectivity is also transformed and enters into a dialectical relationship with the other. In this sense, it is considered important to account for what happens to the subject and his or her way of relating to the other, as he or she engages in the development of these pattern generalization tasks. In synthesis, an interest in this research is to approach learning both in its dimension of knowing, as well as in that of becoming (the subject).

It is necessary to propose a characterization of variational thinking from a cultural-historical approach, linking the epistemological (forms of variational thinking), the historical and cultural (the way in which these emerging forms of variational thinking bear the imprint of the historical moment and the culture in which they are framed, but also that they have been produced by previous generations and that the subject transforms them according to his/her needs (Leontiev, 1969), the social and ethical (how variational thinking is permeated by the relationship with the other, the ways in which knowledge moves in the classroom, that is, how knowledge is conveyed by these relationships).

Limitations

In this research some limitations can be identified regarding the generalization of the results obtained, the study was focused only on a group of third grade students, some of the recordings made presented auditory distortion. Some of these limitations could be addressed in order not to significantly bias the analyses.

Conflict of interest

All authors declare having no conflict of interest to disclose.

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